K$^F$-INORMALS IN IRREDUCIBLE
REPRESENTATIONS OF $G^F$, WHEN $G = GL_n$

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Abstract. Using a general result of Lusztig, we give explicit formulas for
the dimensions of $K^F$-invariants in irreducible representations of $G^F$, when
$G = GL_n$, $F : G \to G$ is a Frobenius map, and $K$ is an $F$-stable subgroup
of finite index in $G^\theta$ for some involution $\theta : G \to G$ commuting with $F$. The
proofs use some combinatorial facts about characters of symmetric groups.

INTRODUCTION

Let $G$ be a connected reductive group defined over a finite field $\mathbb{F}_q$ of odd character-
istic. Let $F$ be the Frobenius morphism on $G$, whose fixed-point subgroup
$G^F = G(\mathbb{F}_q)$ is finite. Let $\theta : G \to G$ be an involution of algebraic groups com-
muting with $F$, and $K$ an $F$-stable subgroup of the fixed-point subgroup $G^\theta$ which
contains $(G^\theta)$. The homogeneous variety $G/K$ is a symmetric space, and the set
of cosets $G^F/K^F$ might reasonably be called a finite symmetric space.

The classification of irreducible representations of $G^F$ (in characteristic 0) was
completed by Lusztig in the mid-1980s (see [6] for a general state-
ment, and [8] and [9] for more details). A roughly analogous problem for symmetric spaces is that of
determining the dimension of the $K^F$-invariants in each irreducible representa-
tion of $G^F$; equivalently, calculating the multiplicities $\langle \chi, \text{Ind}_{G^F}^G(1) \rangle$ for every irreducible
character $\chi$ of $G^F$. A large first step towards solving this problem was Lusztig’s
calculation, in [10], of $\langle \text{tr}(\cdot, R_\lambda^T), \text{Ind}_{G^F}^G(1) \rangle$ for every Deligne-Lusztig virtual repre-
sentation $R_\lambda^T$ of $G^F$. In [11], Lusztig proceeded to solve the problem completely in
the case when $G^F = (G')^F$, $K^F = (K')^F$, and $G'$ has connected centre.

This paper is devoted to the solution of this problem when $G$ is a general linear
group (with either split or non-split $F$, so that $G^F$ is either $GL_n(\mathbb{F}_q)$ or $U_n(\mathbb{F}_{q^2})$)
and $\theta$ is arbitrary. (The solution for the case $G^F = GL_n(\mathbb{F}_q)$, $K^F = Sp_n(\mathbb{F}_q)$
was found by Bannai, Kawanaka, and Song in [3] §4.] For such $G$, the functions
$\text{tr}(\cdot, R_\lambda^T)$ form a basis of the class functions, and the transition matrix from this
basis to that of the irreducible characters is known. So Lusztig’s result gives a
formula for $\langle \chi, \text{Ind}_{K^F}^G(1) \rangle$. All that remains is to manipulate this formula until it is
manifestly a nonnegative integer, a straightforward (though not entirely easy)
matter. Two justifications for presenting it in detail are the potential interest of
the answers, and the pleasantness of the symmetric group combinatorics involved.

In §1 we recall Lusztig’s formula and the character theory of the finite gen-
eral linear and unitary groups as well as introducing some vital notation. Then
we traverse the various cases in §§2-4, which could be thought of as a theme
and variations: the theme, or underlying pattern, is stated in its simplest form
in §2.1 (the case already known from [3]), and successive subsections follow the
same pattern with progressively more elaborate alterations. In §2 the involution is symplectic, so \( n \) must be even and the possible symmetric spaces are \( GL_n(\mathbb{F}_q)/Sp_n(\mathbb{F}_q) \) and \( U_n(\mathbb{F}_q^2)/Sp_n(\mathbb{F}_q) \). In §3 the involution is inner, so the possible symmetric spaces are \( GL_n(\mathbb{F}_q)/(GL_n(\mathbb{F}_q) \times GL_n(\mathbb{F}_q)) \), \( GL_n(\mathbb{F}_q)/GL_n/2(\mathbb{F}_q^2) \), \( U_n(\mathbb{F}_q^2)/(U_n(\mathbb{F}_q^2) \times U_n(\mathbb{F}_q^2)) \), and \( U_n(\mathbb{F}_q^2)/U_n/2(\mathbb{F}_q^2) \). In §4 the involution is orthogonal, so \( G^\theta \) is not connected. However as noted in Lemma 1.0.1, it is enough to solve the problem when \( K = G^\theta \), in which case the possible symmetric spaces are \( GL_n(\mathbb{F}_q)/O_n(\mathbb{F}_q) \) (\( n \) odd), \( GL_n(\mathbb{F}_q)/O_n^+(\mathbb{F}_q) \) (\( n \) odd), \( U_n(\mathbb{F}_q^2)/O_n(\mathbb{F}_q) \) (\( n \) odd), and \( U_n(\mathbb{F}_q^2)/O_n^+(\mathbb{F}_q) \) (\( n \) even).

The key combinatorial results we need along the way are all proved in §5. A reader interested only in these results could skip all of §§1-4 except §1.2.

To give some idea of how the formulas in §§2-4 connect with previously known results, we here extract the answers for \( \text{unipotent} \) irreducible characters. For both \( GL_n(\mathbb{F}_q) \) and \( U_n(\mathbb{F}_q^2) \), these are parametrized by partitions of \( n \), say \( \rho \mapsto \chi^\rho \in \hat{G}^F \).

(For our convention \( \chi^{(n)} \) is the trivial character and \( \chi^{(1^n)} \) is the Steinberg character.)

Recall that a signed tableau of shape \( \mu \) is a signed Young diagram of shape \( \mu \) where signs alternate across rows, modulo permutations of rows of equal length. For this and all other combinatorial notation, see §1.2. We have:

\[
\langle \chi^\rho, \text{Ind}^{GL_n(\mathbb{F}_q)}_{Sp_n(\mathbb{F}_q)}(1) \rangle = \begin{cases} 
1, & \text{if } \rho \text{ is even} \\
0, & \text{otherwise}
\end{cases}
\]

(for the general \( GL_n(\mathbb{F}_q)/Sp_n(\mathbb{F}_q) \) case, see Theorem 2.1.1);

\[
\langle \chi^\rho, \text{Ind}^{U_n(\mathbb{F}_q^2)}_{Sp_n(\mathbb{F}_q)}(1) \rangle = \begin{cases} 
1, & \text{if } \rho \text{ is even} \\
0, & \text{otherwise}
\end{cases}
\]

(for the general \( U_n(\mathbb{F}_q^2)/Sp_n(\mathbb{F}_q) \) case, see Theorem 2.2.1);

\[
\langle \chi^\rho, \text{Ind}^{GL_n(\mathbb{F}_q)}_{GL_n^+(\mathbb{F}_q) \times GL_n^-(\mathbb{F}_q)}(1) \rangle = \text{the number of signed tableaux of shape } \rho' \text{ and signature } (n^+, n^-)
\]

(for the general \( GL_n(\mathbb{F}_q)/(GL_n^+(\mathbb{F}_q) \times GL_n^-(\mathbb{F}_q)) \) case, see Theorem 3.1.1);

\[
\langle \chi^\rho, \text{Ind}^{GL_n(\mathbb{F}_q)}_{GL_n/2(\mathbb{F}_q^2)}(1) \rangle = \begin{cases} 
1, & \text{if } \rho \text{ is even} \\
0, & \text{otherwise}
\end{cases}
\]

(for the general \( GL_n(\mathbb{F}_q)/GL_n/2(\mathbb{F}_q^2) \) case, see Theorem 3.2.1);

\[
\langle \chi^\rho, \text{Ind}^{U_n(\mathbb{F}_q^2)}_{U_n(\mathbb{F}_q^2) \times U_n(\mathbb{F}_q^2)}(1) \rangle = \text{the number of signed tableaux of shape } \rho' \text{ and signature } (n^+, n^-), \text{stable under inverting all rows}
\]
(for the general $U_n(F_g)/U_{n+1}(F_g) \times U_{n-1}(F_g^2)$) case, see Theorem 3.3.1):

\[
\langle \chi^\rho, \text{Ind}_{U_n(F_g \times U_{n-1}(F_g^2))}^{U_n(F_g^2)}(1) \rangle = \begin{cases} 
\prod_i (m_2(\rho') + 1), & \text{if } 2 \mid m_{2i+1}(\rho'), \forall i \\
0, & \text{otherwise}
\end{cases}
\]

(\text{even})

(\text{odd})

\[
\left\langle \chi^\rho, \text{Ind}_{GL_n(F^g)}^{O_n(F^g)}(1) \right\rangle = \frac{1}{2} \prod_i (m_i(\rho) + 1)
\]

(\text{even})

(\text{odd})

\[
\left\langle \chi^\rho, \text{Ind}_{O_n(F^g)}^{GL_n(F^g)}(1) \right\rangle = \begin{cases} 
\frac{1}{2} \prod_i (m_2(\rho) + 1), & \text{if } \rho' \text{ is even, } \epsilon = + \\
\frac{1}{2} \prod_i (m_2(\rho) + 1), & \text{if } \rho' \text{ is even, } \epsilon = - \\
\frac{1}{2} \prod_i (m_2(\rho) + 1), & \text{otherwise}
\end{cases}
\]

(\text{even})

(\text{odd})

\[
\left\langle \chi^\rho, \text{Ind}_{O_n(F^g)}^{U_{n-1}(F_g)}(1) \right\rangle = \begin{cases} 
\left\lceil \frac{1}{2} \prod_i (m_2(\rho) + 1) \right\rceil, & \text{if } \rho' \text{ is even, } \epsilon = + \\
\left\lceil \frac{1}{2} \prod_i (m_2(\rho) + 1) \right\rceil, & \text{if } \rho' \text{ is even, } \epsilon = - \\
\left\lceil \frac{1}{2} \prod_i (m_2(\rho) + 1) \right\rceil, & \text{if } 2 \mid m_{2i+1}(\rho'), \forall i, \text{ but } \rho' \text{ not even} \\
0, & \text{otherwise}
\end{cases}
\]

(\text{even})

(\text{odd})

Readers experienced in the theory of cells for the symmetric group will find these answers familiar. In fact, most of the above facts about unipotent characters can be obtained by a more direct method than the one used in this paper. For instance, suppose that $G^F \cong GL_n(F^g)$. Then the unipotent irreducible characters are the constituents of $\text{Ind}_{G^F(B^F)}^{G^F}(1)$ where $B$ is an $F$-stable Borel subgroup. The Hecke algebra $\mathcal{H}(G^F, B^F)$ is the specialization at $q$ of the abstract Hecke algebra $\mathcal{H}$ of $S_n$; let $V_\rho$ be the simple $\mathcal{H}$-module indexed by $\rho$.

It is trivial to show that the above multiplicity $\left\langle \chi^\rho, \text{Ind}_{G^F}^{G^F}(1) \right\rangle$ equals the multiplicity of $(V_\rho)_q$ in the $\mathcal{H}(G^F, B^F)$-module $C(B^F \setminus G^F / K^F)$ of functions on $G^F$. 


which are constant on the \( B^F - K^F \) double cosets. Assuming that \( K \) is connected and split over \( \mathbb{F}_q \), this module is the specialization at \( q \) of the \( \mathcal{H} \)-module \( M^K \) defined in \([3]\). So \( \langle \chi^\rho, \text{Ind}_{K^F}^G(1) \rangle \) is the multiplicity of \( V^\rho \) in \( M^K \). In the current type-\( A \) case, this equals the number of \textit{cells} of \( M^K \) which afford the representation \( V^\rho \), for which there is a combinatorial formula. For example, our answer in the case of \( GL_n(\mathbb{F}_q)/\text{Sp}_n(\mathbb{F}_q) \) could be deduced from the results in \([2]\).

When \( G \) and \( K \) are non-split, this argument must be refined to incorporate folding involutions, in the manner of \([7, \S 10]\).

In principle, such arguments apply to \( \langle \chi^\rho, \text{Ind}_{K^F}^G(1) \rangle \) whenever \( \chi^\rho \) is a constituent of \( \text{Ind}_{B^F}^G(\lambda) \), since the generalized Hecke algebras \( \text{End}_{G^F} \text{Ind}_{B^F}^G(\lambda) \) have been completely described. But the requisite facts about cells for these Hecke algebras are somewhat diffuse in the literature, and usually quoted in the slightly different context of real Lie groups. I hope that the results of this paper, which are deduced from \([10]\) in an independent way, will in fact shed further light on the theory of cells.

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1. Review of Known Results

In §1.1 we recall the theorem of Lusztig which underpins all our results, and in §1.3 the relevant parts of the character theory of the finite general linear and unitary groups in a convenient form. In §1.2 and §1.4, we introduce some combinatorial notation to be used throughout the paper.

1.1. Lusztig’s Formula. Let $k$ be the algebraic closure of a finite field $\mathbb{F}_q$ of odd cardinality $q$. Let $G$ be a connected reductive group over $k$ defined over $\mathbb{F}_q$, with Frobenius map $F : G \to G$. Let $\theta : G \to G$ be an involution of algebraic groups commuting with $F$, and $K$ an $F$-stable subgroup of the fixed-point subgroup $G^\theta$ which contains $(G^\theta)^0$. Fix a prime $l$ not dividing $q$. All representations and characters of finite groups in this paper will be over $\overline{\mathbb{Q}}_l$.

A pair $(T, \lambda)$ means an $F$-stable maximal torus $T$ and a character $\lambda : T^F \to \overline{\mathbb{Q}}_l^\times$. We have a conjugation action of $G^F$ on the set of pairs. In [10], Deligne and Lusztig attached to each pair a virtual representation $R_T^\lambda$ of $G^F$, depending only on the $G^F$-orbit of $(T, \lambda)$. (In general, “most” of the $R_T^\lambda$ are irreducible up to sign, and every irreducible representation occurs in some $R_T^\lambda$; when $G = GL_n$ the situation is even better, as we will see below.) The main result of [10] is a formula for

$$\frac{1}{|K^F|} \sum_{k \in K^F} \text{tr}(k, R_T^\lambda) = \langle \text{tr}(\cdot, R_T^\lambda), \text{Ind}_{K^F}^{G^F}(1) \rangle.$$ 

After some trivial adjustments, it reads as follows. Define

$$\Theta_T = \{ f \in G | \theta(f^{-1}Tf) = f^{-1}Tf \}.$$ 

Then $T$ acts on $\Theta_T$ by left multiplication and $K$ acts by right multiplication. If $B$ is a Borel subgroup containing $T$, the obvious map $T \setminus \Theta_T / K \to B \setminus G / K$ is a bijection (see [10] Proposition 1.3), so $T \setminus \Theta_T / K$ is in bijection with the set of $K$-orbits on the flag variety. For any $f \in \Theta_T^F$, define $\epsilon_{T,f} : (T \cap fKf^{-1})^F \to \{ \pm 1 \}$ by

$$\epsilon_{T,f}(t) = (-1)^{\mathbb{F}_q\text{-rank}(Z_G((T \cap fKf^{-1})^F)) + \mathbb{F}_q\text{-rank}(Z_G((T \cap fKf^{-1})^F))}.$$ 

It follows from [10], Proposition 2.3] that $\epsilon_{T,f}$ is a group homomorphism which factors through $(T \cap fKf^{-1})^F / ((T \cap fKf^{-1})^F)^F$. Finally, define

$$\Theta_{E,T}^F = \{ f \in \Theta_T^F | \lambda | (T \cap fKf^{-1})^F = \epsilon_{T,f} \},$$

a union of $T^F - K^F$ double cosets.

Theorem 1.1.1. (Lusztig, [10, Theorem 3.3])

$$\langle \text{tr}(\cdot, R_T^\lambda), \text{Ind}_{K^F}^{G^F}(1) \rangle = \sum_{f \in T^F \setminus \Theta_{E,T}^F / K^F} (-1)^{\mathbb{F}_q\text{-rank}(T) + \mathbb{F}_q\text{-rank}(Z_G((T \cap fKf^{-1})^F))}.$$ 

1.2. Combinatorial Notation. In general, our combinatorial notation always follows [10]. For instance, $\mu \vdash n$ means that $\mu$ is a partition of $n$. The size of a partition $\mu$ is written $|\mu|$ and its length $\ell(\mu)$; it has parts $\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}$. The transpose partition is $\mu^t$. We define

$$n(\mu) = \sum_i (i - 1)\mu_i = \sum_i \left( \frac{\mu_i^t}{2} \right).$$

The multiplicity of $i$ as a part of $\mu$ is written $m_i(\mu)$. We say that $\mu$ is even if all its parts are even, or equivalently if $2|m_i(\mu')$, $\forall i$. 

K^{F}\text{-invariants in irreducible representations of } G^{F}, \text{ when } G = GL_{n}$
It will be useful to have, for any partition \( \nu \), a concrete realization of the symmetric group \( S_\nu \) which includes a canonical element of cycle-type \( \nu \). Let \( \Lambda(\nu) \) be a set indexing the parts of \( \nu \); usually, \( \Lambda(\nu) = \{1, \ldots, \ell(\nu)\} \). Then the set

\[
\Omega(\nu) = \{(j, i) \mid j \in \Lambda(\nu), i \in \mathbb{Z}/\nu_j\mathbb{Z}\}
\]

has \(|\nu|\) elements, and we will write \( S_{\nu|\nu} \) for the group of permutations of \( \Omega(\nu) \) (a slight abuse of notation, since it is not canonically associated to \( |\nu| \)). Let \( w_\nu \) be the permutation \( (j, i) \mapsto (j, i + 1) \). By construction this has cycle-type \( \nu \), and in fact we can identify \( \Lambda(\nu) \) with the set of cycles of \( w_\nu \). Write \( Z^\nu \) for the centralizer \( Z^\nu_{S_{\nu|\nu}}(w_\nu) \). The sign of \( w_\nu \) is written \( \epsilon_\nu \), and the size of \( Z^\nu \) is \( z_\nu \).

For any \( w \in Z^\nu \), let \( \bar{w} \) be the induced permutation of \( \Lambda(\nu) \). Note that \( \nu_{\bar{w}(j)} = \nu_j \) always. For \( w \in Z^\nu \) and \( j \in \Lambda(\nu) \), there is a unique \( i(w, j) \in \mathbb{Z}/\nu_j\mathbb{Z} \) such that

\[
w(j, i) = (\bar{w}(j), i + i(w, j)), \quad \forall i \in \mathbb{Z}/\nu_j\mathbb{Z}.
\]

We will write \( \nu_{\bar{w}(j)} = \nu_j \) always. For \( w \in Z^\nu \) and \( j \in \Lambda(\nu) \), there is a unique \( i(w, j) \in \mathbb{Z}/\nu_j\mathbb{Z} \) such that

\[
w(j, i) = (\bar{w}(j), i + i(w, j)), \quad \forall i \in \mathbb{Z}/\nu_j\mathbb{Z}.
\]

Note that any \( w \in Z^\nu_{\text{inv}} \), divides the set \( \Lambda(\nu) \) of cycles of \( w_\nu \) into three disjoint subsets \( \Lambda^1_w(\nu) \), \( \Lambda^2_w(\nu) \), and \( \Lambda^3_w(\nu) \), according to whether \( w \) fixes the cycle pointwise, fixes the cycle but not pointwise, or does not fix the cycle. Explicitly,

\[
\Lambda^1_w(\nu) = \{j \in \Lambda(\nu) \mid \bar{w}(j) = j, i(w, j) = 0\},
\]

\[
\Lambda^2_w(\nu) = \{j \in \Lambda(\nu) \mid \bar{w}(j) = j, i(w, j) = 1\},
\]

\[
\Lambda^3_w(\nu) = \{j \in \Lambda(\nu) \mid \bar{w}(j) \neq j\}.
\]

We will write \( \ell^1_w(\nu) \), \( \ell^2_w(\nu) \), and \( \ell^3_w(\nu) \) for \( |\Lambda^1_w(\nu)| \), \( |\Lambda^2_w(\nu)| \), and \( |\Lambda^3_w(\nu)| \), so that

\[
\ell^1_w(\nu) + \ell^2_w(\nu) + \ell^3_w(\nu) = \ell(\nu).
\]

Note that \( \ell^3_w(\nu) \) is always even, and \( w \in Z^\nu_{\text{inv}} \Leftrightarrow \ell^1_w(\nu) = 0 \).

We will use the notations \( \ell(\nu)_0 \) and \( \ell(\nu)_1 \) for the number of even and odd parts respectively. For instance, \( \epsilon_\nu = (-1)^{\ell(\nu)_0} \). On occasion we will need to further analyse \( \ell(\nu)_0 \mod 2 \) and \( \ell(\nu)_1 \mod 2 \). We will also combine these notations in the obvious way, e.g. \( \ell^1_w(\nu)_1 \) means the number of odd cycles of \( w_\nu \) moved by \( w \), and \( \ell^3_w(\nu)_1 = 0 \) always.

In \( \S 3 \) we will need to consider involutions with signed fixed points. Let \( (S_{\nu|\nu})_{\pm \text{--inv}} \) be the set of pairs \( (w, \epsilon) \) where \( w \in (S_{\nu|\nu})_{\text{inv}} \) and \( \epsilon : \{(j, i) \mid w(j, i) = (j, i)\} \mapsto \{+, -\} \) is a way of signing the fixed points of \( w \). We define the signature of \( (w, \epsilon) \) to be \( |\epsilon^{-1}(+)\mid, |\epsilon^{-1}(-)\mid \). We declare that signatures are considered as elements of \( \mathbb{Z}/\mathbb{Z}(1, 1) \), e.g. \( (0, 1) \) and \( (2, 1) \) are the same. Let \( (S_{\nu|\nu})_{(p^+, p^-)} \text{--inv} \) be the set of \( (w, \epsilon) \in (S_{\nu|\nu})_{\pm \text{--inv}} \) with signature \( (p^+, p^-) \). Now define

\[
Z^\nu_{\pm \text{--inv}} = \{(w, \epsilon) \in (S_{\nu|\nu})_{\pm \text{--inv}} \mid w \in Z^\nu, \epsilon \circ w_\nu = \epsilon\}.
\]

Note that

\[
|Z^\nu_{\pm \text{--inv}}| = \sum_{w \in Z^\nu_{\text{inv}}} 2^{\ell^1_w(\nu)}.
\]

Define \( Z^\nu_{(p^+, p^-)} \text{--inv} \) similarly, and also

\[
Z^\nu_{-\text{inv}} = \{(w, \epsilon) \in (S_{\nu|\nu})_{\pm \text{--inv}} \mid w \in Z^\nu, \epsilon \circ w_\nu = -\epsilon\}.
\]
Note that any \((w, e) \in Z_{\nu}^{-\text{inv}}\) must have signature \((0,0)\). Also

\[
|Z_{\nu}^{-\text{inv}}| = \sum_{w \in Z_{\nu}^{-\text{inv}}} 2t_{w}(\nu).
\]

Let \(T_{\pm}(\mu)\) be the set of signed tableaux of shape \(\mu\). These are ways of labelling the boxes of the Young diagram of \(\mu\) with a sign, in such a way that signs alternate across each row (so all signs in a row are determined by that of the last box), with the proviso that two labellings which differ by a permutation of rows of equal length are not distinguished. Hence

\[
|T_{\pm}(\mu)| = \prod_{i} (m_{i}(\mu) + 1).
\]

For \(T \in T_{\pm}(\mu)\), the signature of \(T\), again in \(\mathbb{Z}^{2}/\mathbb{Z}(1,1)\), is defined as

\[
(|\{\text{boxes of } T \text{ signed } +\}|, |\{\text{boxes of } T \text{ signed } -\}|),
\]

or equivalently

\[
(|\{\text{odd rows of } T \text{ ending } \Box\}|, |\{\text{odd rows of } T \text{ ending } \square\}|).
\]

Write \(T_{(p^{+},p^{-})}(\mu)\) for the set of \(T \in T_{\pm}(\mu)\) with signature \((p^{+}, p^{-})\). There are two important involutions on \(T_{\pm}(\mu)\): \(\varphi\) which changes all signs, and \(\psi\) which reverses all rows. Write \(T_{\pm}(\mu)^{\varphi}\) etc. for the fixed-point sets. Note that any \(T \in T_{\pm}(\mu)^{\varphi}\) or \(T_{\pm}(\mu)^{\psi}\) must have signature \((0,0)\). Also

\[
|T_{\pm}(\mu)^{\varphi}| = \begin{cases} 
1, & \text{if } 2 \mid m_{i}(\mu), \forall i \\
0, & \text{otherwise},
\end{cases}
\]

\[
|T_{\pm}(\mu)^{\psi}| = \begin{cases} 
\prod_{i} (m_{2i+1}(\mu) + 1), & \text{if } 2 \mid m_{2i}(\rho'), \forall i \\
0, & \text{otherwise, and}
\end{cases}
\]

\[
|T_{\pm}(\mu)^{\psi|}\rangle = \begin{cases} 
\prod_{i} (m_{2i}(\mu) + 1), & \text{if } 2 \mid m_{2i+1}(\mu), \forall i \\
0, & \text{otherwise.}
\end{cases}
\]

There are no similar formulas for \(|T_{(p^{+},p^{-})}(\mu)|\) or \(|T_{(p^{+},p^{-})}(\mu)^{\psi}|\).

We label the irreducible characters of \(S_{|\nu|}\) as \(\{\chi^{\rho} | \rho \vdash |\nu|\}\) as in\([14, 1.7]\), and write \(\chi_{\nu}^{\rho}|\) for the value of \(\chi^{\rho}\) at an element of cycle-type \(\nu\), so that \(\chi_{\nu}^{(|\nu|)}\) is the trivial character and \(\chi_{\nu}^{\rho}| = \epsilon_{\nu}\chi_{\nu}^{\rho}|\).

1.3. Character Theory of \(GL_{n}(\mathbb{F}_{q})\) and \(U_{n}(\mathbb{F}_{q}^{2})\). For the remainder of the paper, we specialize the context of §1.1 drastically, to the case when \(G \cong GL_{n}\), for some positive integer \(n\). More concretely, let \(V\) be a vector space over \(k\) of dimension \(n\) and let \(G = GL(V)\). There are two kinds of \(\mathbb{F}_{q}\)-structures on \(G\), split and non-split. A Frobenius map \(F: G \to G\) is split if it is induced by some Frobenius map \(F_{V}: V \to V\), in the sense that

\[
F_{V}(gv) = F(g)F_{V}(v), \ \forall g \in G, v \in V.
\]

Then \(G^{F} \cong \text{Aut}_{\mathbb{F}_{q}}(F^{V}) \cong GL_{n}(\mathbb{F}_{q})\), the finite general linear group. If \(F\) is a non-split Frobenius map, there exists some outer involution \(\theta'\) of \(G\) commuting with \(F\), and for any such \(\theta'\), \(\theta'F\) is a split Frobenius map. In this case \(G^{F} \cong U_{n}(\mathbb{F}_{q}^{2})\), the finite unitary group.
As in [14, Chapter IV], we will need to consider the system of maps \( \hat{F}^e_q \to \hat{F}^{e'}_q \), for \( e \mid e' \) (the transpose of the norm map), and its limit \( L = \text{colim} \hat{F}^e_q \). Let \( \sigma \) denote the \( q \)-th power map on both \( k^\times \) and \( L \), so that \((k^\times)^{\sigma^e} \cong F^e_q \), \( L^{\sigma^e} \cong F^e_q \) for all \( e \geq 1 \). Write \( \langle \cdot, \cdot \rangle^{\sigma^e} : (k^\times)^{\sigma^e} \times L^{\sigma^e} \to \overline{Q}^{\times} \) for the canonical pairing. Let \( \iota \) denote the inverse map on \( k^\times \) and \( L \), and write \( \hat{\sigma} \) for \( \iota \sigma \), the \((-q)\)-th power map. (Note that \( \hat{\sigma}^2 = \sigma^2 \).) We also have a canonical pairing \( \langle \cdot, \cdot \rangle^{\sigma^e} : (k^\times)^{\sigma^e} \times L^{\sigma^e} \to \overline{Q}^{\times} \) (the same as \( \langle \cdot, \cdot \rangle^{\sigma^e} \) if \( e \) is even).

We will fix some set of representatives for the orbits of the group \( \langle \sigma \rangle \) generated by \( \sigma \) on \( L \), and call it \( \langle \sigma \rangle \backslash L \). Similarly define \( \langle \hat{\sigma} \rangle \backslash L \). For \( \xi \in \langle \sigma \rangle \backslash L \), let \( m_\xi = \| \langle \sigma \rangle \xi \| \), in other words the smallest \( e \geq 1 \) such that \( \sigma^e(\xi) = \xi \). Let \( d_\xi = (-1, \xi)^{m_\xi} \), which equals 1 if \( L^{\sigma^e} \) contains square roots of \( \xi \), and \(-1\) if it does not. Similarly define \( \hat{m}_\xi \) and \( \hat{d}_\xi \) using \( \hat{\sigma} \) instead of \( \sigma \).

First consider the case when \( F : G \to G \) is a split Frobenius map. Let \( P_n \) be the set of collections of partitions \( \mu = (\mu_\alpha)_{\alpha \in k^\times} \), almost all zero, such that \( \sum_{\alpha \in k^\times} |\mu_\alpha| = n \). Let \( P_n^\sigma \) be the subset of \( P_n \) consisting of all \( \mu \) such that \( \mu_{\sigma(\alpha)} = \mu_\alpha \) for all \( \alpha \). It is well known that there is a natural bijection between \( P_n^\sigma \) and the set of conjugacy classes in \( G^F \). Dually, let \( \hat{P}_n \) be the set of collections of partitions \( \nu = (\nu_\xi)_{\xi \in L} \), almost all zero, such that \( \sum_{\xi \in L} n_\xi = n \). Let \( \hat{P}_n^\sigma \) be the subset of \( \hat{P}_n \) of all \( \nu \) such that \( \nu_{\sigma(\xi)} = \nu_\xi \) for all \( \xi \). Note that for \( \nu \in \hat{P}_n^\sigma \),

\[
\sum_{\xi \in \langle \sigma \rangle \backslash L} m_\xi |\nu_\xi| = n.
\]

For \( \nu, \rho \in \hat{P}_n^\sigma \), we write \( |\nu| = |\rho| \) to mean that \( |\nu_\xi| = |\rho_\xi| \) for all \( \xi \).

We can define a bijection between \( \hat{P}_n^\sigma \) and the set of \( G^F \)-orbits of pairs \( (T, \lambda) \) as in §1.1, so that if \( (T, \lambda) \) is in the orbit corresponding to \( \nu \)

1. the eigenlines of \( T \) can be labelled

\[
\{L_{(\xi,j,i)} \mid \xi \in \langle \sigma \rangle \backslash L, \ 1 \leq j \leq \ell(\nu_\xi), \ i \in \mathbb{Z}/m_\xi(\nu_\xi)\mathbb{Z}\}
\]

so that under the resulting isomorphism

\[
T \cong \prod_{\xi \in \langle \sigma \rangle \backslash L} \prod_{j=1}^{\ell(\nu_\xi)} k^\times / \text{m}_\xi(\nu_\xi) \text{ factors},
\]

\( F|_T \) corresponds to cyclic permutation of each group of factors \( k^\times \), composed with \( \sigma \);

2. consequently,

\[
T^F \cong \prod_{\xi \in \langle \sigma \rangle \backslash L} \prod_{j=1}^{\ell(\nu_\xi)} (k^\times)^{\sigma^{m_\xi(\nu_\xi)}};
\]

3. under this isomorphism, \( \lambda \) corresponds to

\[
\prod_{\xi \in \langle \sigma \rangle \backslash L} \prod_{j=1}^{\ell(\nu_\xi)} \langle \cdot, \xi \rangle^{\sigma^{m_\xi(\nu_\xi)}}.\]

For \( \nu \in \hat{P}_n^\sigma \), let \( B_{\nu} = \text{tr}(\cdot, R^\nu_\nu) \) for \( (T, \lambda) \) in the corresponding \( G^F \)-orbit. As proved by Lusztig in [3], these coincide with the basic characters defined by Green.
\( \chi^\xi := (-1)^{n + \sum_{\epsilon \in (\sigma) \setminus L} |\rho_\epsilon|} \sum_{\rho \in \hat{P}_n^\sigma \atop |\rho| = |\underline{\rho}|} \left( \prod_{\epsilon \in (\sigma) \setminus L} (z_{\nu_\epsilon})^{-1} \chi^\rho_{\nu_\epsilon} \right) B_{\underline{\rho}} \)

is an irreducible character of \( G^F \), and all irreducible characters arise in this way for unique \( \underline{\rho} \in \hat{P}_n^\sigma \). (See also \[4\], Chapter IV] and \[1\] Theorem 1.2.10. Note that Macdonald’s parameters in \( \hat{P}_n^\sigma \) differ from those of \[1\] by transposing all partitions; we are following the convention of \[1\].) In words, the transition matrix between the Macdonald’s parameters in \( \hat{\rho} \) for unique \( \nu \) is very similar, in fact mostly identical once \( \nu \) is an irreducible character of \( G \) well defined. For any \( \eta \) \( \in L^\sigma \), \( B_{\underline{\rho}} \) and \( \chi^\eta_{\underline{\rho}} \) are the result of multiplying \( B_{\underline{\rho}} \) and \( \chi^\eta_{\underline{\rho}} \) by the one-dimensional character \( \langle \det(\cdot), \eta \rangle^\sigma \) of \( G^F \). The \textit{unipotent} irreducible characters referred to in the introduction are those \( \chi^\xi_{\underline{\rho}} \) for which \( \rho_\xi = 0 \) unless \( \xi = 1 \). (In the introduction we parametrized these by \( \rho = \rho_1 \).

The case when \( F : G \to G \) is a non-split Frobenius map is less well known, but very similar, in fact mostly identical once \( \sigma \) is replaced by \( \tilde{\sigma} \), \( m_\xi \) by \( \tilde{m}_\xi \), and so on. Define \( \hat{P}_n^{\tilde{\sigma}} \) in the obvious way. Again, for any \( \nu \in \hat{P}_n^{\tilde{\sigma}} \),

\[ \sum_{\xi \in (\tilde{\sigma}) \setminus L} \tilde{m}_\xi |\nu_\xi| = n. \]

For \( \underline{\nu}, \underline{\rho} \in \hat{P}_n^{\tilde{\sigma}} \), we write \( |\underline{\nu}| = |\underline{\rho}| \) to mean that \( |\nu_\xi| = |\rho_\xi| \) for all \( \xi \).

We can define a bijection between \( \hat{P}_n^{\tilde{\sigma}} \) and the set of \( G^F \)-orbits of pairs \( (T, \lambda) \) as above, so that if \( (T, \lambda) \) is in the orbit corresponding to \( \underline{\nu} \):

1. the eigenlines of \( T \) can be labelled

\[ \{ L_{(\xi,j,i)} \mid \xi \in (\tilde{\sigma}) \setminus L, 1 \leq j \leq \ell(\nu_\xi), i \in \mathbb{Z}/\tilde{m}_\xi(\nu_\xi)\mathbb{Z} \} \]

so that under the resulting isomorphism

\[ T \cong \prod_{\xi \in (\tilde{\sigma}) \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} k^{\times} \times \cdots \times k^{\times}, \]

\( F|_T \) corresponds to cyclic permutation of each group of factors \( k^{\times} \), composed with \( \tilde{\sigma} \);

2. consequently,

\[ T^F \cong \prod_{\xi \in (\tilde{\sigma}) \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} (k^{\times})^{\tilde{m}_\xi(\nu_\xi)}; \]
3. under this isomorphism, $\lambda$ corresponds to

$$\prod_{\xi \in \mathcal{P} \setminus \mathcal{L}} \prod_{j=1}^{\ell(\rho_{\lambda})} \langle \xi, \tilde{\sigma}^m\xi \rangle.$$ 

For $\mu \in \hat{P}^T_n$, let $B_T = \text{tr} (\cdot, R_{n}^\mu)$ for $(T, \lambda)$ in the corresponding $G^{F}$-orbit. The extension of Green’s result to the non-split case was proved by Lusztig and Srinivasan in [12, Theorem 3.2]: in our notation, for any $\rho \in \hat{P}^T_n$,

$$\chi^\ell_B := (-1)^{\left\lfloor \frac{n}{2} \right\rfloor + \sum_{\xi \in \mathcal{P} \setminus \mathcal{L}} \hat{m}_{\xi} n(\rho_{\xi}) + |\rho_{\xi}|} \sum_{\mu \in \hat{P}^T_n} \prod_{\xi \in \mathcal{P} \setminus \mathcal{L}} (z_{\xi})^{-1} \chi_{\rho_{\xi}} B_{\mu}$$

is an irreducible character of $G^{F}$, and all irreducible characters arise in this way for unique $\rho \in \hat{P}^T_n$. Inverting, we see that for any $\mu \in \hat{P}^T_n$,

$$(1.3.2) \quad B_{\mu} = \sum_{\rho \in \hat{P}^T_n} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor + \sum_{\xi \in \mathcal{P} \setminus \mathcal{L}} \hat{m}_{\xi} n(\rho_{\xi}) + |\rho_{\xi}|} \prod_{\xi \in \mathcal{P} \setminus \mathcal{L}} \chi_{\rho_{\xi}} \chi^\ell_B$$

Again, the obvious action of $L^\mu$ on $\hat{P}^T_n$ corresponds to multiplication by one-dimensional characters, and the unipotent irreducible characters of the introduction are those $\chi^\ell_B$ for which $\rho_{\xi} = 0$ unless $\xi = 1$. (In contrast to the case of $GL_n (F_q)$, not all the unipotent characters are constituents of $\text{Ind}^{G^F}_{B} (1)$ for an $F$-stable Borel subgroup $B$.)

1.4. Descriptions of Weyl Groups. Much of this paper deals with the special properties of Weyl groups in $GL_n$, so the following ideas and notation will be crucial. Let $T$ be any maximal torus of $G = GL(V)$. If $\{ L_{\omega} | \omega \in \Omega \}$ is some labelling of the eigenlines of $T$, we can identify the Weyl group $W(T)$ with the group of permutations of $\Omega$. Let $W(T)_{\text{inv}}$ be the set of involutions in $W(T)$, and $W(T)_{\text{inv}}^f$ the set of fixed-point free involutions (note that this means fixed-point free as a permutation of $\Omega$, not as an automorphism of $T$).

If $T$ is $\theta$-stable, there is a special involution $w_1^T \in W(T)_{\text{inv}}$ characterized as follows. If $\theta$ is an inner involution, namely conjugation by $s \in G$, then

$$s(L_{\omega}) = L_{w_1^T(\omega)}, \forall \omega \in \Omega, \text{ and } \theta|_T = w_1^T.$$ 

If $\theta$ is an outer involution, namely adjoint inverse with respect to a nondegenerate symplectic or symmetric form on $V$, then

$$L_\omega^\perp = \bigoplus_{\omega' \neq w_1^T(\omega) \in \Omega} L_{\omega'}, \forall \omega \in \Omega, \text{ and } \theta|_T = w_1^T \circ \iota.$$ 

For any $T$, if $f \in \Theta_T$, then $f^{-1} T f$ is a $\theta$-stable maximal torus, and we obtain

$$w_f^T = \text{Ad}(f^{-1}) \circ w_1^T \circ \text{Ad}(f) \in W(T)_{\text{inv}}.$$ 

When $T$ is fixed, we will write $w_f$ instead of $w_f^T$. Clearly $w_f$ depends only on the double coset $T f K$; the resulting map $T \backslash \Theta_T / K \rightarrow W(T)_{\text{inv}}$ will be crucial in our combinatorial rewritings of Lusztig’s theorem.
Now suppose that $F : G \to G$ is split, and $(T, \lambda)$ is a pair in the $G^{'F}$-orbit corresponding to $\nu \in \widehat{\mathcal{P}}_n$. Apart from the full Weyl group $W(T)$, we will mainly be concerned with the $F$-fixed subgroup $W(T)^F$ and its subsets

$$W(T)^F_\lambda = \{ w \in W(T)^F \mid \lambda \circ w = \lambda \},$$

$$W(T)^F_{\lambda \to \lambda^{-1}} = \{ w \in W(T)^F \mid \lambda \circ w = \lambda^{-1} \}.$$

We can consider these sets in the framework of §1.2 as follows. Define

$$\Lambda(\nu) = \{ (\xi, j) \in \langle \sigma \rangle \setminus \mathbb{Z} \mid 1 \leq j \leq \ell(\nu_\xi) \}.$$

By abuse of notation, write $\nu$ also for the partition of $n$ whose parts are

$$(m_\xi(\nu_\xi), \xi, j) \in \Lambda(\nu).$$

Then $\Lambda(\nu)$ indexes the parts of $\nu$, in accordance with the notation of §1.2; also $\ell(\nu) = |\Lambda(\nu)|$ is the $F$-rank of $T$. Now $\Omega(\nu)$ is the set of triples

$$\{(\xi, j, i) \mid (\xi, j) \in \Lambda(\nu), i \in \mathbb{Z}/m_\xi(\nu_\xi)\mathbb{Z} \},$$

which is precisely the set of labels of the eigenlines of $T$ used in §1.3. So $W(T)$ in the above realization, namely as the group of permutations of $\Omega(\nu)$, coincides with $S_{\nu}$ in the realization of §1.2. As in that section, let $w(\nu) \in W(T)$ be the permutation $(\xi, j, i) \mapsto (\xi, j, i + 1)$. Then by the description of $F|_{\nu}$ given in §1.3, $W(T)^F$ is exactly $Z_{W(T)}(w_{\nu}) = Z_{\nu}$. Note that

$$\epsilon_\nu = \text{sign}(w_{\nu}) = \prod_{\xi \in \langle \sigma \rangle \setminus \mathbb{L}} \frac{\epsilon_{\nu_\xi}}{2m_\xi} \prod_{\xi \in \langle \sigma \rangle \setminus \mathbb{L}} (-1)^{\ell(\nu_\xi)}.$$

For $w \in W(T)^F$, we will use the notation $\tilde{w}, i(w, \xi, j) \in \mathbb{Z}/m_\xi(\nu_\xi)\mathbb{Z}$ of §1.2. So

$$w(\xi, j, i) = (\tilde{w}(\xi, j), i + i(w, \xi, j)), \quad \forall (\xi, j) \in \Lambda(\nu), i \in \mathbb{Z}/m_\xi(\nu_\xi)\mathbb{Z}.$$

With this notation, $w \in W(T)^F$ lies in $W(T)^F_{\lambda}$ if and only if $\tilde{w}(\xi, j) = (\xi, j')$ (for some $j'$) and $m_\xi \mid i(w, \xi, j)$ hold for all $(\xi, j) \in \Lambda(\nu)$. Here it is helpful to consider $\prod_{\xi \in \langle \sigma \rangle \setminus \mathbb{L}} \Omega(\nu_\xi)$, which is the set of triples $(\xi, j, s)$ with $(\xi, j) \in \Lambda(\nu)$ and $s \in \mathbb{Z}/(\nu_\xi)\mathbb{Z}$, and $\prod_{\xi \in \langle \sigma \rangle \setminus \mathbb{L}} S_{\nu_\xi}$, which is the group of permutations of such triples which preserve the first factor. For $w \in W(T)^F_{\lambda}$, we define $\tilde{w} \in \prod_{\xi \in \langle \sigma \rangle \setminus \mathbb{L}} S_{\nu_\xi}$ by

$$\tilde{w}(\xi, j, s) = (\tilde{w}(\xi, j), s + \frac{i(w, \xi, j)}{m_\xi}).$$

Clearly $w \mapsto \tilde{w}$ is an isomorphism between $W(T)^F_{\lambda}$ and $\prod_{\xi \in \langle \sigma \rangle \setminus \mathbb{L}} Z_{\nu_\xi}$.

The analogous description of $W(T)^F_{\lambda \to \lambda^{-1}}$ is as follows. For $\xi \in \langle \sigma \rangle \setminus \mathbb{L}$, let $\xi^\vee$ be the chosen representative in the $\langle \sigma \rangle$-orbit of $\xi^{-1}$. Define $i_0(\xi) \in \mathbb{Z}/m_\xi\mathbb{Z}$ by

$$\xi^\vee = \sigma^{i_0(\xi)}(\xi^{-1}),$$

so that $i_0(\xi^\vee) = -i_0(\xi)$. Assume $\nu_{\xi^\vee} = \nu_\xi$ for all $\xi \in \langle \sigma \rangle \setminus \mathbb{L}$; otherwise $W(T)^F_{\lambda \to \lambda^{-1}}$ is empty. Clearly $w \in W(T)^F$ lies in $W(T)^F_{\lambda \to \lambda^{-1}}$ iff $\tilde{w}(\xi, j) = (\xi^\vee, j')$ (for some $j'$) and $i(w, \xi, j) \equiv i_0(\xi) \mod m_\xi$ hold for all $(\xi, j) \in \Lambda(\nu)$. For every permutation $\tilde{w}$ of $\Lambda(w)$ such that $\tilde{w}(\xi, j) = (\xi^\vee, j')$ (for some $j'$), and every $(\xi, j) \in \Lambda(w)$, lift $i_0(\xi)$ to an element $i_0(\tilde{w}, \xi, j)$ of $\mathbb{Z}/m_\xi(\nu_\xi)\mathbb{Z}$ in an arbitrary way. Consider the set of permutations of $\prod_{\xi \in \langle \sigma \rangle \setminus \mathbb{L}} \Omega(\nu_\xi)$ which interchange $\Omega(\nu_\xi)$ and $\Omega(\nu_{\xi^\vee})$ for all $\xi$. 
Since $\nu_\xi^\prime = \nu_\xi$, we can identify this set with $\prod_{\{\xi, 0\}} S_{1|\nu_\xi}$. For $w \in W(T)_{\lambda^+_1 \rightarrow \lambda^-_1}$, we define $\tilde{w} \in \prod_{\{\xi, \nu_\xi\}} S_{1|\nu_\xi}$ by
\[
\tilde{w}(\xi, j, s) = \left(\tilde{w}(\xi, j), s + \frac{i(w, \xi, j) - i_0(\tilde{w}, \xi, j)}{m_\xi}\right).
\]

Clearly $w \mapsto \tilde{w}$ defines a bijection between $W(T)^F_{\lambda^+_1 \rightarrow \lambda^-_1}$ and $\prod_{\{\xi, \nu_\xi\}} Z^\nu_\xi$.

Now consider involutions in $W(T)^F$. Since $W(T)^{F_{\text{inv}}}_0 = Z^\nu_{\text{inv}}$, we can apply the concepts of §1.2. In particular, any $w \in W(T)^{F_{\text{inv}}}_0$ decomposes $\Lambda(w)$ into
\[
\Lambda^1_w(\nu) = \{(\xi, j) \in \Lambda(\nu) \mid \tilde{w}(\xi, j) = (\xi, j), i(w, \xi, j) = 0\},
\]
\[
\Lambda^2_w(\nu) = \{(\xi, j) \in \Lambda(\nu) \mid \tilde{w}(\xi, j) = (\xi, j), i(w, \xi, j) = \frac{1}{2} m_\xi(\nu_\xi)\},\text{ and}
\]
\[
\Lambda^3_w(\nu) = \{(\xi, j) \in \Lambda(\nu) \mid \tilde{w}(\xi, j) \neq (\xi, j)\}.
\]

If $w \in W(T)^{F_{\text{inv}}}_{\pm \text{inv}}$, then $\Lambda^2_w(\nu)$ is empty. As in §1.2, we write $\ell^I_w(\nu)$ for $|\Lambda^I_w(\nu)|$, and also use the notations $\ell^I_w(\nu)_0$, $\ell^I_w(\nu)_1$, $\ell^I_w(\nu)_{2 \text{ mod } 4}$ etc. Note that $W(T)^{F_{\text{inv}}}_{\pm \text{inv}}$, i.e. the $F$-fixed points of $W(T)_{\pm \text{inv}}$, is identified with $Z^\nu_{\text{inv}}$ as defined in §1.2, and similar statements holds for $W(T)^{F_{\text{inv}}}_{(p^+, p^-) \text{ inv}}$ and $W(T)^{F_{\text{inv}}}_{\lambda^+_1 \rightarrow \lambda^-_1}$.

We will also write $W(T)^{F_{\text{inv}}}_0$ for $W(T)_{\lambda^+_1} \cap W(T)^{F_{\text{inv}}}_0$, and $W(T)^{F_{\text{inv}}}_{\lambda^+_1 \rightarrow \lambda^-_1}$ for $W(T)^{F_{\text{inv}}}_{\lambda^+_1 \rightarrow \lambda^-_1} \cap W(T)^{F_{\text{inv}}}_0$. Under the isomorphism $W(T)_{\lambda^+_1} \cong \prod_{\xi} Z^\nu_\xi$ defined above, $W(T)^{F_{\text{inv}}}_0$ corresponds to $\prod_{\xi} Z^\nu_\xi$. Moreover, as will be crucial later, if $w \in W(T)^{F_{\text{inv}}}_0$ corresponds to $(\nu_\xi)$, then
\[
\ell^I_w(\nu) = \sum_{\xi \in (\sigma) \backslash \mathcal{L}} \ell^I_{w_\xi}(\nu_\xi),
\]
and similarly for $\ell^2$ and $\ell^3$. However, the situation is more complex when we introduce divisibility criteria, because of the division by $m_\xi$; for instance
\[
\ell^I_w(\nu)_0 = \sum_{\xi \in (\sigma) \backslash \mathcal{L}, 2|m_\xi} \ell^I_{w_\xi}(\nu_\xi) + \sum_{\xi \in (\sigma) \backslash \mathcal{L}, 2|m_\xi} \ell^I_{w_\xi}(\nu_\xi).
\]

To get a similar description of $W(T)^{F_{\text{inv}}}_{\lambda^+_1 \rightarrow \lambda^-_1}$, we need to put further constraints on the choice of $i_0(\tilde{w}, \xi, j)$. It is easy to see that we can arrange to have
\[
i_0(\tilde{w}, \tilde{w}(\xi, j)) = -i_0(\tilde{w}, \xi, j)
\]
except in the case when $\xi^\prime = \xi$, $\xi \neq 1, -1, 2|(\nu_\xi)_j$, and $\tilde{w}(\xi, j) = (\xi, j)$. (For instance, when $\xi^\prime = \xi$, $\xi \neq 1, -1, 2|(\nu_\xi)_j$ and $\tilde{w}(\xi, j) = (\xi, j)$, we are forced to set $i_0(\tilde{w}, \xi, j) = \frac{1}{2} m_\xi(\nu_\xi)_j$, since $i_0(\xi) = \frac{1}{2} m_\xi$.) But if $\tilde{w}$ is induced as above from $w \in W(T)^{F_{\text{inv}}}_{\lambda^+_1 \rightarrow \lambda^-_1}$, this case cannot arise, since if $2|(\nu_\xi)_j$ we cannot have both
\[
i(w, \xi, j) = -i(w, \xi, j) \text{ mod } m_\xi(\nu_\xi)_j, \text{ and}
\]
\[
i(w, \xi, j) \equiv i_0(\xi) = \frac{1}{2} m_\xi \text{ mod } m_\xi.
\]

So $w \mapsto \tilde{w}$ with these conventions is a bijection between $W(T)^{F_{\text{inv}}}_{\lambda^+_1 \rightarrow \lambda^-_1}$ and
\[
Z^\nu_\xi \times Z^\nu_{\xi^\prime}_1 \times \prod_{\xi \in (\sigma) \backslash \mathcal{L}, \xi^\prime = \xi, \xi \neq 1, -1} \{\nu_\xi \in Z^\nu_{\text{inv}} \mid \ell^I_{w_\xi}(\nu_\xi)_0 = 0\} \times \prod_{\xi \neq 1, -1} Z^\nu_{\xi^\prime}.
\]
In contrast to the previous situation, if we write \( \tilde{w} \) as \((w_\xi) \in \prod_{\xi} Z_{\text{inv}}^{\nu_\xi} \), then if \( \xi \neq \xi' \), the elements of \( A_1(\nu_\xi) \) contribute not to \( \ell_w^*(\nu) \) but to \( \ell_w^* (\nu) \). Similarly, if \( \xi = \xi' \), \( \xi \neq 1, -1 \), then \( A_1(\nu_\xi) \) contributes to \( \ell_w^*(\nu) \). This subtlety is one factor complicating the formulas below.

Everything we have said applies equally well to the non-split case, with \( \bar{\theta} \) instead of \( \sigma \) and \( \tilde{m}_\xi \) instead of \( m_\xi \) throughout, except for the fact that the \( \mathbb{F}_q \)-rank of \( T \) is not \( \ell(w) \) but \( \ell(w)_0 \). Indeed, \( q - 1 \) (as a polynomial) divides
\[
| (k^\times)^{\tilde{m}_\xi(\nu_\xi)} | = q^{\tilde{m}_\xi(\nu_\xi)} - (-1)^{\tilde{m}_\xi(\nu_\xi)},
\]
once if \( \tilde{m}_\xi(\nu_\xi)_j \) is even and not at all if \( \tilde{m}_\xi(\nu_\xi)_j \) is odd. In general, the main difference between the split and non-split cases below lies in the calculation of \( \mathbb{F}_q \)-ranks, which affects the signs in Lusztig’s formula.

2. Cycles where \( G/K = GL_n/Sp_n \)

In this section, we suppose that \( V \) has a nondegenerate symplectic form \( \langle \cdot, \cdot \rangle \) (so in particular \( n \) is even), and that \( \theta : G \to G \) is the involution defined by
\[
\langle \theta(g)v, v' \rangle = \langle v, g^{-1}v' \rangle, \quad \forall g \in G, v, v' \in V.
\]

Since \( G^\theta = Sp(V, \langle \cdot, \cdot \rangle) \) is connected, \( K \) must be equal to it. So \( G/K \) is the symmetric space \( GL_n/Sp_n \).

Let \( T \) be any maximal torus of \( G \). In \( \S 1.4 \), we associated to any \( f \in \Theta_T \) an involution \( \omega_f \in W(T) \), depending only on the double coset \( TfK \). Since \( \langle \cdot, \cdot \rangle \) is a symplectic form, every line is orthogonal to itself, so \( \omega_f \) is fixed-point free (as a permutation of the eigenlines of \( T \), not as an automorphism of \( T \)). Thus
\[
T \cap fKf^{-1} = \{ t \in T \mid \omega_f(t) = t^{-1} \}
\]
is connected, and \( \mathbb{Z}_G(T \cap fKf^{-1}) = T \). Moreover:

**Proposition 2.0.1.** The map \( f \mapsto \omega_f \) induces a bijection \( T \Theta_T/K \to W(T)_{\text{ff-inv}} \).

**Proof.** This is very well known, especially when translated into the language of flags via the connection mentioned in \( \S 1.1 \). Surjectivity can be proved by an explicit construction, and injectivity is easy by induction. \( \square \)

2.1. The \( GL_n(\mathbb{F}_q)/Sp_n(\mathbb{F}_q) \) Case. In this subsection, let \( F : G \to G \) be a split Frobenius map which commutes with \( \theta \). So \( F \) is induced by a Frobenius map \( F_V \) on \( V \) which respects \( \langle \cdot, \cdot \rangle \). One has \( G^F \cong GL_n(\mathbb{F}_q), K^F \cong Sp_n(\mathbb{F}_q) \). The following result was obtained by a different (and simpler) method in [4, \( \S 4 \]):

**Theorem 2.1.1.** For any \( \rho \in \widehat{\mathbb{P}}_m^G \),
\[
\langle \chi^\rho, \text{Ind}_{Sp_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle = \begin{cases} 
1, & \text{if all } \rho_\xi \text{ are even} \\
0, & \text{otherwise}.
\end{cases}
\]

By the results in \( \S 1.3 \), it is equivalent to prove that for any \( \underline{\rho} \in \widehat{\mathbb{P}}_m^G \),
\[
\langle B_{\underline{\rho}}, \text{Ind}_{Sp_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle = \prod_{\xi \in \mathcal{S}} \sum_{\rho_\xi \text{ even}} \chi_{\rho_\xi}^{\rho_\xi}.
\]

Note that the sign in [1.3.3] disappears since \( n \) is even, and \( |\rho| \) is even for even \( \rho \).

We aim to deduce this from Lusztig’s general formula for the left-hand side (Theorem 1.1.1 above). This entails successively analysing the components of the
formula in our combinatorial terms, until we are reduced to a fact about class functions on the symmetric group (in this case one which is well known). This basic strategy will be repeated in every case; the main reason for including the present case, where the result is not new, is that it serves as the prototype for the following subsections.

Let \( T \) be an \( F \)-stable maximal torus, and \( \lambda : T^F \to \mathbb{Q}_l^\times \) a character, for which \((T, \lambda)\) is in the \( G^F \)-orbit corresponding to \( \nu \in \mathcal{P}_n^+ \). We will use the description of \( W(T)^F \) as \( Z^\nu \) given in \( \S 1.4 \). Proposition \[2.0.1\] implies:

**Lemma 2.1.2.** The map \( f \mapsto w_f \) induces a bijection
\[
T^F \setminus \Theta_T^F/K^F \simeq W(T)^F_{\text{inv}}.
\]

**Proof.** Clearly the map in Proposition \[2.0.1\] is \( \Theta_T^F \)-stable. So we need only note that since \( T, K \), and all \( T \cap fK f^{-1} \) are connected, \( T^F \setminus \Theta_T^F/K^F = (T \setminus \Theta_T/K)^F \). \( \square \)

**Lemma 2.1.3.** For \( f \in \Theta_T^F \), \( f \in \Theta_{T, \lambda}^F \iff w_f \in W(T)_\lambda^F \).

**Proof.** Since \( T \cap fK f^{-1} = \{ t \in T \mid w_f(t) = t^{-1} \} \) is connected, \( \epsilon_{T, f} = 1 \), so
\[
\Theta_{T, \lambda}^F = \{ f \in \Theta_T^F \mid \lambda_{\{ t \in T^F \mid w_f(t) = t^{-1} \}} = 1 \}.
\]
Thus it suffices to show that
\[
\{ t \in T^F \mid w_f(t) = t^{-1} \} = \{ tw_f(t)^{-1} \mid t \in T^F \}.
\]
This follows easily from the fact that \( w_f \in W(T)^F_{\text{inv}} \). \( \square \)

**Corollary 2.1.4.** The map \( f \mapsto w_f \) induces a bijection
\[
T^F \setminus \Theta_{T, \lambda}^F/K^F \simeq W(T)^F_{\text{inv}}.
\]
As noted above, \( Z_G((T \cap fK f^{-1})^\circ) = T \), so Lusztig’s formula becomes
\[
\langle B_{\mathcal{L}}, \text{Ind}_{K^F}^{G^F}(1) \rangle = |T^F \setminus \Theta_{T, \lambda}^F/K^F| = |W(T)^F_{\text{inv}}|.
\]
It is clear that under the isomorphism \( W(T)^F_{\text{inv}} \simeq \prod_{\xi \in (\sigma) \setminus \mathcal{L}} Z^{\nu_{\xi}} \) given in \( \S 1.4 \), \( W(T)^F_{\text{inv}} \) corresponds to \( \prod_{\xi \in (\sigma) \setminus \mathcal{L}} Z^{\nu_{\xi}}_{\text{inv}} \). Now we apply the combinatorial fact (for which see \[14\], VII.(2.4)):
\[
|Z^{\nu_{\xi}}_{\text{inv}}| = \sum_{\rho \in \mathcal{P}_n^+, |\nu_{\xi}|} \chi_{\rho}^{\nu_{\xi}}.
\]
This gives Equation \[2.1.1\] and hence Theorem \[2.1.1\].

2.2. The \( U_n(F_q)/Sp_n(F_q) \) Case. Now we keep the assumptions from before \( \S 2.1 \), but take \( F \) to be instead a non-split Frobenius map which commutes with \( \theta \). So \( \theta F \) is a split Frobenius map as above, induced by a Frobenius map \( F_V \) on \( V \) which respects \( \langle \cdot, \cdot \rangle \). One has \( G^F \cong U_n(F_q^\theta) \), \( K^F \cong Sp_n(F_q) \). In this case the result is:

**Theorem 2.2.1.** For any \( \rho \in \mathcal{P}^+_n \),
\[
\langle \chi_{\rho}, \text{Ind}_{Sp_n(F_q^\theta)}^{U_n(F_q^\theta)}(1) \rangle = \begin{cases} 1, & \text{if all } \rho_{\xi} \text{ are even} \\ 0, & \text{otherwise.} \end{cases}
\]
It is equivalent to prove that for any $\underline{\rho} \in \hat{P}^\sigma_n$,
\[
(2.2.1) \quad \langle B_{\underline{\rho}}, \text{Ind}_{K}^{G^F}(1) \rangle = \prod_{\xi \in (\hat{\sigma}) \setminus L} \sum_{\rho | \xi \text{ even}} \chi_{\rho \xi}^\xi.
\]

Note that the sign in (3.2) disappears because $n(\rho') \equiv \frac{|\rho'|}{2} \mod 2$ for any even $\rho$, so that
\[
\sum_{\xi \in (\hat{\sigma}) \setminus L} \tilde{m}_\xi |\rho'| \equiv \sum_{\xi \in (\hat{\sigma}) \setminus L} \tilde{m}_\xi |\rho| \equiv \frac{n}{2} \mod 2.
\]
Equation (2.2.1) is deduced exactly as in the previous section, using the non-split versions of the notation in §1.4.

3. Cases where $G/K = GL_n/(GL_{n^+} \times GL_{n^-})$

In this section, we suppose that $\theta$ is an inner involution, namely conjugation with respect to $s \in G$ such that $s^2 = 1$. Let $V^+$ be the (+1)-eigenspace and $V^-$ the (−1)-eigenspace of $s$ on $V$, so that $V = V^+ \oplus V^-$. Let $n^+ = \dim V^+$, $n^- = \dim V^-$, so that $n = n^+ + n^-$. Since $G^\theta = GL(V^+) \times GL(V^-)$ is connected, $K$ must be equal to it. So $G/K$ is the symmetric space $GL_n/(GL_{n^+} \times GL_{n^-})$.

Let $T$ be any maximal torus of $G$. For $f \in \Theta_T$, we have $w_f \in W(T)_{\text{inv}}$, not necessarily fixed-point free. Indeed fixed points of $w_f$ correspond to eigenlines of $T$ which are stable under $s$, and therefore lie in either $V^+$ or $V^-$. Let $\epsilon_f : \{\text{fixed points of } w_f\} \to \{+,-\}$ be the resulting map. In the notation of §1.2, $(w_f, \epsilon_f) \in W(T)_{(n^+,n^-)_{\text{inv}}}$.

**Proposition 3.0.1.** The map $f \mapsto (w_f, \epsilon_f)$ induces a bijection $T/\Theta_T/K \cong W(T)_{(n^+,n^-)_{\text{inv}}}$.

**Proof.** As with Proposition 2.0.1, this is well known when translated in terms of $K$-orbits on the flag variety, and easy to prove (see for instance [14]).

3.1. The $GL_n(F_q)/(GL_{n^+}(F_q) \times GL_{n^-}(F_q))$ Case. In this subsection, assume that $F : G \to G$ is a split Frobenius map such that $F(s) = s$. So $F$ is induced by a Frobenius map $F_V$ on $V$ which stabilizes $V^+$ and $V^-$. One has $G^F = GL_n(F_q)$, $K^F = GL_{n^+}(F_q) \times GL_{n^-}(F_q)$. Recall the definition of $T_{(\rho^+,\rho^-)}(\mu)$ from §1.2.

**Theorem 3.1.1.** For any $\underline{\rho} \in \hat{P}^\sigma_n$,
\[
\langle \chi_{\underline{\rho}}, \text{Ind}_{GL_{n^+}(F_q) \times GL_{n^-}(F_q)}^{GL_n(F_q)}(1) \rangle = \left\{
\begin{array}{ll}
|T_{(n^+,n^-)}(\rho_1')|, & \text{if } \rho_\xi = \rho_{\xi-1}, \forall \xi \text{ and } \rho_{\xi-1} \text{ is even,} \\
0, & \text{otherwise.}
\end{array}
\right.
\]
(Note that $|T_{(n^+,n^-)}(\rho_1')|$ could be zero.) By (3.3.1), it is equivalent to say that for any $\underline{\rho} \in \hat{P}^\sigma_n$,
\[
(3.3.1) \quad \langle B_{\underline{\rho}}, \text{Ind}_{K}^{G^F}(1) \rangle = \left( \sum_{\rho_1^+|\nu_1} |T_{(n^+,n^-)}(\rho_1')| \chi_{\rho_1'}^{\rho_1} \right) \prod_{\rho \in \hat{P}^\sigma_n} \chi_{\rho}^\rho \prod_{\xi \in (\hat{\sigma}) \setminus L} \chi_{\xi}^{\xi} \prod_{\xi \neq 1,-1} \delta_{\nu_1,\nu_2} z_{\nu_1,\nu_2}.
\]
Here the sign \((-1)^{n+\sum_{\xi \in \nu} |\nu_\xi|}\) in (1.3.1) is simplified by noting that \(m_1 = m_{-1} = 1, m_0\) is even for \(\xi \neq 1, -1\) such that \(\xi = \xi_1, \) and \(\nu_\xi\) and \(\nu_\xi\) must be equal whenever \(\xi_1 = \xi_2, \xi_1 \neq \xi_2\) in order for the right-hand side to be nonzero.

Let \((T, \lambda)\) be a pair in the \(G^F\)-orbit corresponding to \(\nu \in \widehat{P}^n_\nu\).

**Lemma 3.1.2.** The map \(f \mapsto (w_f, \epsilon_f)\) induces a bijection

\[ T^F \setminus \Theta^F_{T,\lambda}/K^F \cong W(T)^{F_{(n^+,n^-)-\text{inv}}}. \]

**Proof.** Again, this follows from Proposition 3.0.1, since \(T, K\), and \(T \cap K f^{-1} = T^{w_f}\) are all connected.

Note that for \(w \in W(T)_\lambda^{\lambda_{-1,\text{inv}}}, (\xi, j) \in \Lambda^1_{w}(\nu) \Rightarrow \xi = \pm 1\). Define

\[ Y^\mu_{\text{inv}} = \{w \in W(T)_\lambda^{\lambda_{-1,\text{inv}}}, (\xi, j) \in \Lambda^1_{w}(\nu) \Rightarrow \xi = 1\}. \]

**Lemma 3.1.3.** For \(f \in \Theta_{T,\lambda}^F, f \in \Theta_{T,\lambda}^F \iff w_f \in Y^\mu_{\text{inv}}. \)

**Proof.** Since \(T \cap K f^{-1} = T^{w_f}\) is connected, \(\epsilon_{T,f} = 1, \) so

\[ \Theta^F_{T,\lambda} = \{f \in \Theta^F_T | \lambda_{(T^F)^{w_f}} = 1\}. \]

Thus it suffices to show that \((T^F)^{w_f}\) is generated by

\[ \{tw_f(t) | t \in T^F\} \text{ and } \prod_{(\xi, j) \in \Lambda^1_{\nu_f}(\mu)} (k^\times)^{m_{(\nu_\xi)}}. \]

This follows easily from our description of \(T^F.\) 

**Corollary 3.1.4.** The map \(f \mapsto (w_f, \epsilon_f)\) induces a bijection

\[ T^F \setminus \Theta^F_{T,\lambda}/K^F \cong Y^\mu_{(n^+,n^-)-\text{inv}}, \]

where \(Y^\mu_{(n^+,n^-)-\text{inv}} = \{(w, \epsilon) \in Z^\mu_{(n^+,n^-)-\text{inv}} | w \in Y^\mu_{\text{inv}}\}. \)

Now the \(F_q\)-rank of \(T\) is \(\ell(\nu),\) and that of

\[ Z_G((T \cap K f^{-1})^0) = \prod_{(\xi, j) \in \Lambda^1_{\nu_f}(\mu)} GL(L_{(\xi, j, i)}) \times \prod_{\{(\xi, j), (\xi, j, i) \in \Lambda^1_{\nu_f}(\mu)\}} GL(L_{(\xi, j, i)} \oplus L_{(\xi, j, i, j, i) + \frac{1}{2} m_{(\nu_\xi)}}) \times \prod_{\{(\xi, j) \in \Lambda^1_{\nu_f}(\mu)\}} GL(L_{(\xi, j, i) \oplus L_{w_f(\xi, j, i)}}) \]

is \(\ell(\nu) + \ell^2_{w_f}(\nu).\) So Lusztig’s formula becomes

\[ \langle B^G, \text{Ind}^{G^F}_{K^F}(1) \rangle = \sum_{f \in T^F \setminus \Theta^F_{T,\lambda}/K^F} (-1)^{w_f(\nu)} = \sum_{(w, \epsilon) \in Y^\mu_{(n^+,n^-)-\text{inv}}} (-1)^{\ell_{w_f}(\nu)}. \]
Under the bijection \( w \mapsto \hat{w} \) defined in §1.4, \( Y^G_{\text{inv}} \) corresponds to
\[
Z^\nu_{\text{inv}} \times Z^{\nu-1}_{\text{inv}} \times \prod_{\substack{\xi \in (\sigma) \setminus L \\ \xi' = \xi \\ \xi \neq 1, -1}} \{ w_\xi \in Z^\nu_{\text{inv}} \mid \ell^1_{w_\xi}(\nu_\xi)0 = \ell^2_{w_\xi}(\nu_\xi)0 = 0 \} \times \prod_{\substack{\xi_1 \neq \xi_2 \in (\sigma) \setminus L \\ \xi_1 = \xi_2}} Z^{\nu_1}_{\text{inv}}.
\]

Hence
\[
\langle B_{\hat{w}}, \text{Ind}^{G_F}_{K_F}(1) \rangle = \left( \sum_{(w_1, \epsilon_1) \in Z^\nu_{(n+, n-)\text{inv}}} (-1)^{\ell^1_{w_1}(\nu_1)} \right) \left( \epsilon_{\nu-1} |Z^\nu_{\text{inv}}| \right) \\
\times \prod_{\substack{\xi \in (\sigma) \setminus L \\ \xi' = \xi \\ \xi \neq 1, -1}} (-1)^{|\nu_\xi|} \{ w_\xi \in Z^\nu_{\text{inv}} \mid \ell^1_{w_\xi}(\nu_\xi)0 = \ell^2_{w_\xi}(\nu_\xi)0 = 0 \} \\
\times \prod_{\substack{\xi_1 \neq \xi_2 \in (\sigma) \setminus L \\ \xi_1 = \xi_2}} \delta_{\nu_1 \nu_2} Z_{\nu_1}.
\]

In the second factor we have used the fact that if \( w_{-1} \in Z^{\nu-1}_{\text{inv}} \), then
\[
\ell^2_{w_{-1}}(\nu_{-1}) = \ell^2_{w_{-1}}(\nu_{-1})0 \equiv \ell(\nu_{-1})0 \mod 2.
\]

In the third factor, we have noted that both \( \Lambda^1_{w_\xi}(\nu_\xi) \) and \( \Lambda^2_{w_\xi}(\nu_\xi) \) contribute to \( \ell^2_w(\nu) \), and if there exists \( w_\xi \in Z^\nu \) such that \( \ell^1_{w_\xi}(\nu_\xi)0 = \ell^2_{w_\xi}(\nu_\xi)0 = 0 \), then
\[
\ell^1_{w_\xi}(\nu_\xi) + \ell^2_{w_\xi}(\nu_\xi) \equiv \ell(\nu_\xi)1 \equiv |\nu_\xi| \mod 2.
\]

In the fourth factor there is no contribution to \( \ell^2_w(\nu) \), as noted in §1.4. Now we apply (2.1.2) and the following combinatorial facts (for which see §5):
\[
\text{(3.1.2)} \quad \sum_{(w, \epsilon) \in Z^\nu_{(p^+, p^-)\text{inv}}} (-1)^{\ell^1_w(\nu)} = \sum_{\rho^+ | |\nu|} |T_{(p^+, p^-)}(\rho')| \chi^\rho_{\nu},
\]
\[
\text{(3.1.3)} \quad |\{ w \in Z^\nu_{\text{inv}} \mid \ell^1_w(\nu)0 = \ell^2_w(\nu)0 = 0 \}| = \sum_{\rho^+ | |\nu|} \chi^\rho_{\nu}.
\]

These give Equation (3.1.1) and hence Theorem 3.1.

3.2. The \( \text{GL}_n(\mathbb{F}_q) / \text{GL}_{n/2}(\mathbb{F}_q^2) \) Case. Now keep the assumptions from before §3.1, but take \( F \) to be a split Frobenius map such that \( F(s) = -s \). (Then \( F \) still commutes with \( \theta \).) So \( F \) is induced by a Frobenius map \( F_V \) on \( V \) which interchanges \( V^+ \) and \( V^- \), whence \( n \) is even and \( n^+ = n^- = n/2 \). One has \( G^F \cong \text{GL}_n(\mathbb{F}_q) \), \( K^F \cong \text{GL}_{n/2}(\mathbb{F}_q^2) \). The result is:

**Theorem 3.2.1.** For any \( \rho \in \hat{\mathbb{P}}^n \),
\[
\langle \chi, \text{Ind}^{\text{GL}_n(\mathbb{F}_q)}_{\text{GL}_{n/2}(\mathbb{F}_q^2)}(1) \rangle = \begin{cases} 
1, & \text{if } \rho_\xi = \rho_{\xi-1}, \forall \xi, \rho_1 \text{ is even, and } \rho_{-1} \text{ is even} \\
0, & \text{otherwise}.
\end{cases}
\]
By \([\text{Lemma 3.3}]\), it is equivalent to say that for any \(\nu \in \hat{P}_n^\sigma\),

\[
\langle B_{\nu}, \text{Ind}_{K_F}^G(1) \rangle = \left( \sum_{\rho_1^{+} \mid \nu_1} \chi_{\nu_1}^{\rho_1} \right) \left( \sum_{\rho_{-1}^{+} \mid \nu_{-1}} \chi_{\nu_{-1}}^{\rho_{-1}} \right) \\
\times \prod_{\xi \in (\sigma) \setminus L} \left( \sum_{\rho_1^{+} \mid \nu_1} \chi_{\nu_1}^{\rho_1} \right) \prod_{\{\xi, \xi'\} \in (\sigma) \setminus L} \delta_{\nu_1 \nu_2} \delta_{\nu_1 \nu_2}. 
\]

(3.2.1)

(For the signs here, see the comments after Equation (3.1.1).)

Let \((T, \lambda)\) be a pair in the \(G^F\)-orbit corresponding to \(\nu \in \hat{P}_n^\sigma\). Obviously Lemma 3.1.2 becomes:

**Lemma 3.2.2.** The map \(f \mapsto (w_f, \epsilon_f)\) induces a bijection

\[
T^F \setminus \Theta^F_t / K^F \simeq W(T)^{F_{\text{inv}}}. 
\]

Now Lemma 3.1.3 holds again here, with the same \(Y_{\text{inv}}^{\nu}\) and the same proof. So arguing as in §3.1, we get

\[
\langle B_{\nu}, \text{Ind}_{K_F}^G(1) \rangle = \sum_{(w, \epsilon) \in Y_{\text{inv}}^{\nu}} (-1)^{\ell_w(\nu)},
\]

where the definition of \(Y_{\text{inv}}^{\nu}\) is the obvious one. The rest of the proof is also the same as in §3.1, except that Equation (3.1.2) is replaced by:

\[
\sum_{(w, \epsilon) \in Z_{\text{inv}}^{\nu}} (-1)^{\ell_w(\nu)} = \sum_{\rho \mid \nu} \chi_{\nu}^{\rho}. 
\]

(3.2.2)

This too will be proved in §5.

3.3. The \(U_n(\mathbb{F}_q)/(U_n^+(\mathbb{F}_q^2) \times U_n^-(\mathbb{F}_q^2))\) Case. Still under the general assumptions of this section, let \(F : G \rightarrow G\) be a non-split Frobenius map for which \(F(s) = s\). Replacing \(s\) by a \(G^F\)-conjugate if necessary, we may assume that there is some non-degenerate symmetric form \(\langle \cdot, \cdot \rangle\) on \(V\), for which \(V^+\) and \(V^-\) are orthogonal, and such that the associated outer involution \(\theta^F : G \rightarrow G\) commutes with \(F\). Then \(\theta^F\) is the split Frobenius map induced by some \(F_V : V \rightarrow V\) which respects \(\langle \cdot, \cdot \rangle\) and fixes \(V^+\) and \(V^-\). One has \(G^F \cong U_n(\mathbb{F}_q^2), K^F \cong U_n^+(\mathbb{F}_q^2) \times U_n^-(\mathbb{F}_q^2)\).

**Theorem 3.3.1.** For any \(\rho \in \hat{P}_n^\sigma\),

\[
\langle \chi_{\nu}, \text{Ind}_{U_n^+(\mathbb{F}_q^2) \times U_n^-(\mathbb{F}_q^2)}(1) \rangle = \begin{cases} 
|T_{(n^+, n^-)}(\rho_1)|^{\nu}, & \text{if } \rho_1 = \rho_\xi, \forall \xi \text{ and } \rho_{-1} \text{ is even} \\
0, & \text{otherwise}
\end{cases}
\]

where the definition of \(T_{(n^+, n^-)}(\cdot)\) is the obvious one. The rest of the proof is also the same as in §3.1, except that Equation (3.1.2) is replaced by:

\[
\sum_{(w, \epsilon) \in Z_{\text{inv}}^{\nu}} (-1)^{\ell_w(\nu)} = \sum_{\rho \mid \nu} \chi_{\nu}^{\rho}. 
\]

(3.2.2)

This too will be proved in §5.
(Note that \(|T_{(n^+, n^-)}(\rho'_1)^\psi|\) could be zero.) By \([1.3.2]\), it is equivalent to say that for any \(\nu \in \tilde{\mathcal{P}}_n^\sigma\).

\[ (3.3.1) \]
\[
\langle B_\nu, \text{Ind}^{G_F}_{K_F}(1) \rangle = \left( \sum_{\rho_1^+|\nu_1} (-1)^{n(\rho_1)} |T_{(n^+, n^-)}(\rho'_1)^{\psi})| \chi_{\nu_1}^{\rho_1} \right) \sum_{\rho_2|\nu_2 \text{ even}} \chi_{\nu_2}^{\rho_2}
\]
\[
\times \prod_{\xi \in \langle \sigma \rangle \setminus L} \prod_{\xi' = \xi \atop \xi \neq 1, -1} \chi_{\nu_1}^{\rho_1}
\]
\[
\times \prod_{\{\xi_1 \neq \xi_2\} \in \langle \sigma \rangle \setminus L \atop \xi_1 = \xi_2} \delta_{\nu_1 \nu_2} \sum_{\nu_1} \prod_{\{\xi_1 \neq \xi_2\} \in \langle \sigma \rangle \setminus L \atop \xi_1 = \xi_2} (-1)^{\nu_1 \nu_2} \delta_{\nu_1 \nu_2} \sum_{\nu_1}
\]

Here the sign \((-1)^{\nu_1} + \sum_{\xi \in \langle \sigma \rangle \setminus L} m_\xi n(\rho'_1) + |\rho_1|\) of \([1.3.2]\) is simplified as follows. If \(\rho_\xi = \rho_{\xi-1}\) for all \(\xi\) and \(\rho'_1\) is even, then

\[ n = \sum_{\xi \in \langle \sigma \rangle \setminus L} m_\xi |\rho_\xi| \equiv |\rho_1| \mod 2, \]

and the sign can be replaced by

\[ (-1)^{|\mu| + n(\rho'_1)} (-1)^{|\nu_1| + n(\rho'_{-1})} \prod_{\xi \in \langle \sigma \rangle \setminus L \atop \xi \neq 1, -1} \left( \sum_{\nu_1} \prod_{\{\xi_1 \neq \xi_2\} \in \langle \sigma \rangle \setminus L \atop \xi_1 = \xi_2} (-1)^{\nu_1 \nu_2} \delta_{\nu_1 \nu_2} \sum_{\nu_1} \right) \]

Then we observe that if \(\rho'_{-1}\) is even, \(n(\rho'_1)\) is even; and if all even parts of \(\rho'_1\) occur with even multiplicity (which is necessary for \(\tilde{\alpha}(n^+, n^-, \rho'_1) \neq 0\)), then the Young diagram of \(\rho'_1\) (excluding the top left corner if \(|\rho_1|\) is odd) can be tiled by \(2 \times 1\) dominoes, from which we see that \(\tilde{\alpha}(n^+, n^-, \rho'_1) = \tilde{\alpha}(n^+, n^-, \rho_1)\) is even.

Let \((T, \lambda)\) be a pair in the \(G_F\)-orbit corresponding to \(\nu \in \tilde{\mathcal{P}}_n^\sigma\). Proposition \(3.0.1\) has the following corollary in this case:

**Lemma 3.3.2.** The map \(f \mapsto (w_f, \epsilon_f)\) induces a bijection

\[ T^F \setminus \Theta^F_T/K^F \cong W(\mathcal{T}_{(n^+, n^-)}^{-\text{inv}}). \]

Define \(Y^F_{\text{inv}} \subset W(\mathcal{T})^F_{\lambda \rightarrow \lambda - 1, \text{inv}}\) in the same way as in \(3.1.3\), but in the \(\tilde{\sigma}\) version.

**Lemma 3.3.3.** For \(f \in \Theta^F_T, \ f \in \Theta^F_{\lambda, \lambda} \iff w_f \in Y^F_{\text{inv}}\).

**Proof.** The proof is exactly analogous to that of Lemma \(3.1.3\). \(\Box\)

**Corollary 3.3.4.** The map \(f \mapsto (w_f, \epsilon_f)\) induces a bijection

\[ T^F \setminus \Theta^F_{\lambda, \lambda}/K^F \cong Y^F_{(n^+, n^-)^{-\text{inv}}}. \]
The important point of difference from §3.1 is the $\mathbb{F}_q$-ranks involved. The $\mathbb{F}_q$-rank of $T$ is now $\ell(\nu)_0$, and that of

$$Z_G((T \cap K^f)^o) = \prod_{(\xi, j) \in \Lambda^\nu_{w_f}(\nu)} \prod_{(\xi, j) \in \Lambda^\nu_{w_f}(\nu), \{\xi, j\}\in \mathcal{A}^\nu_{w_f}(\nu)} GL(L_{(\xi, j, i)})$$

	$$\times \prod_{(\xi, j) \in \Lambda^\nu_{w_f}(\nu)} GL(L_{(\xi, j, i)} \oplus L_{(\xi, j, i + \delta \hat{m}_\xi(\nu_\xi)_i)})$$

$$\times \prod_{(\xi, j) \in \Lambda^\nu_{w_f}(\nu)} GL(L_{(\xi, j, i)} \oplus L_{w_f(\xi, j)})$$

is $\ell(\nu)_0 + \nu^2(\nu_1)_0 \mod 4 + \frac{1}{2} \nu^3(\nu_1)_0$. (The $GL_2$ factors corresponding to $(\xi, j) \in \Lambda^\nu_{w_f}(\nu)$ are split if $4|\hat{m}_\xi(\nu_\xi)_j$, and those corresponding to $\{(\xi, j) \neq \hat{w}_f(\xi, j)\}$ are split if $2|\hat{m}_\xi(\nu_\xi)_j$.) So Lusztig’s formula becomes

$$\langle B_{\mathbb{L}}, \text{Ind}_{K^F}^{G_F}(1) \rangle = \sum_{f \in T^\Phi \setminus \mathcal{A}^\nu_{f, x}/K^F} (-1)^{\nu^2(\nu_1)_0 + \nu^3(\nu_1)_0 + \frac{1}{2} \nu^3(\nu_1)_0} .$$

Using the same reasoning as in §3.1, we can transform this expression to get:

$$\langle B_{\mathbb{L}}, \text{Ind}_{K^F}^{G_F}(1) \rangle = \sum_{(w, \ell) \in \mathcal{Z}_{(\nu, \nu\ell)^{\text{inv}}}^n} (-1)^{\nu^2(\nu_1)_0 \mod 4 + \frac{1}{2} \nu^3(\nu_1)_1}$$

$$\times (-1)^{\nu^2(\nu_1)_0 \mod 4 + \frac{1}{2} \nu^3(\nu_1)_1} \prod_{\nu < \nu_1} |Z_{\text{inv}}^{\nu-1}|$$

$$\times \prod_{\nu < \nu_1} (\nu^2(\nu_1)_0 \mod 4 + \frac{1}{2} \nu^3(\nu_1)_1)$$

$$\times \prod_{\nu < \nu_1} |Z_{\text{inv}}^{\nu-1}|$$

$$\times \prod_{\nu < \nu_1} \delta_{\nu\xi_1, \nu\xi_1} \zeta_{\nu\xi_1}$$

$$\times \prod_{\nu < \nu_1} \delta_{\nu\xi_1, \nu\xi_1} \zeta_{\nu\xi_1}$$

$$\times \prod_{\nu < \nu_1} \delta_{\nu\xi_1, \nu\xi_1} \zeta_{\nu\xi_1}$$

In the second factor, we have noted that if $w_1 \in Z_{\text{inv}}^{\nu-1}$, then

$$\nu^2(w_1) \mod 4 + \frac{1}{2} \nu^3(w_1) \equiv \nu(\nu_1) \mod 4 + \frac{1}{2} \nu(\nu_1) \equiv \frac{\nu(\nu_1)}{2} + \nu(\nu_1) \mod 2.$$
In the third factor, since \(4 \nmid \hat{m}_\xi\), both \(A^1_{w_2}(\nu \xi)\) and \(A^2_{w_2}(\nu \xi)\) contribute to \(\ell^2_w(\mathfrak{g})_{0 \mod 4}\), and nothing contributes to \(\ell^2_w(\mathfrak{g})_{1}\). In the fourth factor, since \(2 \nmid \hat{m}_\xi\) but \(4 \nmid \hat{m}_\xi\), and \(\ell^1_{w_2}(\nu \xi)_0 = \ell^2_{w_2}(\nu \xi)_0 = 0\), there is no contribution to the sign. So in addition to (2.1.2) and (3.1.3), we need the following fact:

\[
(3.3.2) \quad \sum_{(w, e) \in \mathbb{Z}^{(p^+, p^-)}_{\text{inv}}} (-1)^{\ell^2_w(e)_{0 \mod 4} + \frac{1}{2} \ell^2_w(e)_{1}} = \sum_{\rho \in \mathfrak{g}^+} (-1)^{n(\rho)} |T_{(p^+, p^-)}(\rho^\prime)^\psi | \chi^\rho_{\psi}.
\]

This will be proved in §5.

3.4. The \(U_n(\mathbb{F}_{q^2})/U_{n/2}(\mathbb{F}_{q^4})\) Case. The final case to consider in this section is when \(F : G \to G\) is a non-split Frobenius map for which \(F(s) = -s\). Replacing \(s\) by a \(G^F\)-conjugate if necessary, we may assume that there is a form \((\cdot, \cdot)\) on \(V\) and an involution \(\theta : G \to G\) with the same properties as in §3.3. Then \(\theta^F\) is the split Frobenius map induced by some \(F_V : V \to V\) which respects \((\cdot, \cdot)\) and interchanges \(V^+\) and \(V^-\). In particular, \(n\) is even, and \(n^+ = n^- = \frac{n}{2}\). One has \(G^F \cong U_n(\mathbb{F}_{q^2}), K^F \cong U_{n/2}(\mathbb{F}_{q^4})\). The result is:

**Theorem 3.4.1.** For any \(\rho \in \hat{\mathfrak{g}}^+_n\),

\[
\langle \chi^\rho, \text{Ind}^{U_n(\mathbb{F}_{q^2})}_{U_{n/2}(\mathbb{F}_{q^4})}(1) \rangle = \begin{cases} 
\prod_i (m_{2i}(\rho^\prime_i) + 1), & \text{if } \rho \subseteq \rho^\prime, \forall \xi, \varepsilon, 2|\rho^\prime_i + 1, \forall i, \text{ and } \rho^\prime_{-1} \text{ is even} \\
0, & \text{otherwise.}
\end{cases}
\]

By (1.3.3), it is equivalent to say that for any \(\nu \in \hat{\mathfrak{g}}^+_n\),

\[
(3.4.1) \quad \langle B_{\nu}, \text{Ind}^{G^F}_{K^F}(1) \rangle = \left( \sum_{\rho^0_1 = \rho^0_{-1}} (-1)^{|\rho^0_{-1}|} \prod_{i} (m_{2i}(\rho^0_i) + 1) \chi^{\rho^0_i}_{\nu_i} \right)
\times \left( (-1)^{|\nu_{-1}|} \sum_{\rho^0_{-1} \text{ even}} \chi^{\rho^0_{-1}}_{\nu_{-1}} \right)
\times \prod_{\xi \in \langle \sigma \rangle \setminus L, \xi \neq 1} \left( \sum_{\rho^0_1 = \rho^0_{-1}} \prod_{\rho^0_{-1} \text{ even}} \chi^{\rho^0_{-1}}_{\nu_{-1}} \right)
\times \prod_{\xi \in \langle \sigma \rangle \setminus L, \xi \neq 1} \delta_{\nu_1, \nu_2} \prod_{\xi} (-1)^{|\nu_1|} \delta_{\nu_2, \nu_3} \delta_{\nu_3, \nu_1}.
\]

For the signs here, see the comments after (3.3.1).

Let \((T, \lambda)\) be as in §3.3. We have:

**Lemma 3.4.2.** The map \(f \mapsto (w_f, \epsilon_f)\) induces a bijection

\[
T^F \setminus \Theta_T^{F} / K^F \overset{\sim}{\to} W(T)^F_{\ast - \text{inv}}.
\]
Now Lemma 3.3 holds again here, with the same $Y^{\text{inv}}_\nu$ and the same proof. So arguing as in §3.3, we get

$$\langle B_\nu, \text{Ind}_{K^F}^G(1) \rangle = \sum_{(\nu,v) \in Y^{\text{inv}}_\nu} (-1)^{\ell_\nu(\nu) \mod 4 + \nu_1}.$$

The rest of the proof is also the same as in §3.3, except that Equation (3.3.2) is replaced by:

$$\sum_{(\nu,v) \in Z^{\text{inv}}_\nu} (-1)^{\ell_\nu(\nu) \mod 4 + \nu_1} \chi_{\nu}^\rho = \sum_{\rho \vdash |v|} (-1)^{\iota_\rho(\rho')} (\prod_{i} (m_{2i}(\rho') + 1)) \chi_{\rho'}^\mu.$$

This will be proved in §5.

4. Cases where $G/K = GL_n/O_n$ or $GL_n/SO_n$

In this section, we suppose that $V$ has a nondegenerate symmetric form $\langle , \rangle$, and that $\theta : G \to G$ is the involution defined by

$$\langle \theta(g)v, v' \rangle = \langle v, g^{-1}v' \rangle, \ \forall g \in G, v, v' \in V.$$

Since $G^\theta = O(V,\langle , \rangle)$ has two components, $K$ can be either $O_n$ or $SO_n$. It suffices to solve the problem for $O_n$:

**Lemma 4.0.1.** Let $F : G \to G$ be a Frobenius morphism which commutes with $\theta$. Let $\rho \in \hat{P}_n$ if $F$ is split, $\rho \in \hat{P}_n^\circ$ if $F$ is non-split. If $F$ is split, let $\zeta \in L^\rho$ such that $\langle -1, \zeta \rangle^\sigma = -1$. If $F$ is non-split, let $\zeta \in L^\rho$ be such that $\langle -1, \zeta \rangle^\rho = -1$. Then

$$\langle \chi^{\zeta}, \text{Ind}_{(G^\theta)^F}^{G^\rho}(1) \rangle = \langle \chi^{\zeta}, \text{Ind}_{(G^\rho)^F}^{G^\rho}(1) \rangle + \langle \chi^{\zeta}, \text{Ind}_{(G^\rho)^F}^{G^\rho}(1) \rangle.$$

**Proof.** Clearly

$$\text{Ind}_{(G^\rho)^F}^{G^\rho}(1) = \text{Res}_{(G^\rho)^F}^{G^\rho}(1 + \langle \cdot, \zeta^{-1} \rangle),$$

so

$$\text{Ind}_{(G^\theta)^F}^{G^\rho}(1) = (1 + \langle \cdot, \zeta^{-1} \rangle) \text{Ind}_{(G^\rho)^F}^{G^\rho}(1),$$

which proves the result. \hfill \Box

For the remainder of this section, we will write $K$ for $G^\theta \cong O_n$ and $K^\circ$ for $(G^\theta)^\circ \cong SO_n$.

Let $T$ be a maximal torus of $G$. For $f \in \Theta_T$, we have $w_f \in W(T)_{\text{inv}}$. This time

$$T \cap fKf^{-1} = \{ t \in T | w_f(t) = t^{-1} \}$$

is not necessarily connected.

**Proposition 4.0.2.** The map $f \mapsto w_f$ is a bijection $T \setminus \Theta_T/K \to W(T)_{\text{inv}}$. Moreover, if $w \in W(T)_{\text{inv}}$, the corresponding $T-K$ double coset breaks into two $T-K^\circ$ double cosets if $w \in W(T)_{H-\text{inv}}$, and is a single $T-K^\circ$ double coset otherwise.

**Proof.** As with Propositions 2.0.1 and 0.1, this is better known as a statement about $K$-orbits on the flag variety (see [15, §6]). It is easy to prove. \hfill \Box
4.1. The $GL_n(\mathbb{F}_q)/O_n(\mathbb{F}_q)$ Case ($n$ odd). In this subsection, suppose that $n$ is odd and let $F: G \to G$ be a split Frobenius map which commutes with $\theta$. So $F$ is induced by a Frobenius map $F_V$ on $V$ which respects $\langle \cdot, \cdot \rangle$, such that $\langle \cdot, \cdot \rangle$ has Witt index $\lfloor \frac{n}{2} \rfloor$ on $V^F$. One has $G^F \cong GL_n(\mathbb{F}_q)$, $K^F \cong O_n(\mathbb{F}_q)$. Recall the definition of $d_\xi$ from §1.2. The result is:

**Theorem 4.1.1.** For any $\rho \in \hat{\mathbb{P}}_n^r$,

\[
\langle \chi, \text{Ind}_{O_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle = \begin{cases} 
\frac{1}{2} \prod_{\xi \in \sigma \setminus L} (\prod_{d_\xi = 1} (m_i(\rho_\xi) + 1)), & \text{if } d_\xi = -1 \Rightarrow \rho_\xi \text{ is even} \\
0, & \text{otherwise.}
\end{cases}
\]

By (4.1.1), it is equivalent to say that for any $\nu \in \check{\mathbb{P}}_n^r$,

\[
\langle B_\nu, \text{Ind}_{K^F}^{G^F}(1) \rangle = -\frac{1}{2} \prod_{\xi \in \sigma \setminus L} (\prod_{d_\xi = 1} ((-1)^{\nu_\xi} \sum_{\rho_\xi \uparrow ^{\nu_\xi}} (\prod_{i} (m_i(\rho_\xi) + 1)))\chi_{\nu_\xi}^{\rho_\xi}) \\
\times \prod_{\xi \in \sigma \setminus L} (\sum_{\rho_\xi \uparrow ^{\nu_\xi}} \chi_{\nu_\xi}^{\rho_\xi}).
\]

(4.1.1)

Here the sign in (4.1.1) has been distributed in an obvious way.

Let $(T, \lambda)$ be a pair in the $G^F$-orbit corresponding to $\nu \in \check{\mathbb{P}}_n^r$.

**Lemma 4.1.2.** The map $f \mapsto w_f$ induces a surjection

\[T^F \setminus \Theta_T^F/K^F \to W(T)^\text{inv}.\]

**Proof.** By Proposition 1.0.2 and Lemma 4.1.2, we may identify $W(T)^\text{inv}_{\nu}$ with $(T \setminus \Theta_T/K)^F$. Under this identification, an involution is in the image of $T^F \setminus \Theta_T^F/K^F$ precisely when the corresponding $F$-stable $K$-orbit on $T \setminus \Theta_T$ contains an $F$-fixed point (by connectedness of $T$). Since $n$ is odd, this $K$-orbit is a single $K^\circ$-orbit, so this is automatic.

**Lemma 4.1.3.** For $f \in \Theta_T^F$, there are $2^{\text{inv}}(\nu)^{-1} T^F - K^F$ double cosets in $(T f K)^F$.

**Proof.** Since the image of the Lang map on $K$ is $K^\circ$, the number of $T^F - K^F$ double cosets in $(T f K)^F$ is the same as the number of orbits of $T \cap f K f^{-1}$ on $T \cap f K f^{-1}$ for the action $t t' = t t' (t)^{-1}$. Using Lang’s theorem, one sees that only the “disconnected part” of $T \cap f K f^{-1}$ contributes. So this is the same as the number of orbits of

\[
\prod_{(\xi,j) \in \Lambda_{\nu}^1(\nu)} (\pm^1)^{m_{\xi}(\nu_\xi)} \text{ on } \left\{ (\epsilon_{(\xi,j,i)}') \in \prod_{(\xi,j) \in \Lambda_{\nu}^1(\nu)} (\pm^1)^{m_{\xi}(\nu_\xi)} \right\} \prod_{(\xi,j,i)} \epsilon_{(\xi,j,i)}'(1)
\]

for the action $(\epsilon_{(\xi,j,i)})(\epsilon_{(\xi,j,i)}') = \epsilon_{(\xi,j,i)}\epsilon_{(\xi,j,i)}\epsilon_{(\xi,j,i)}^{-1}\epsilon_{(\xi,j,i)}'$. In other words, it is the index in the latter group of the subgroup where $\prod_{(\xi,j)} \epsilon_{(\xi,j,i)}' = 1$, for all $(\xi, j) \in \Lambda_{\nu}^0(\nu)$. Now since $n$ is odd, $\Lambda_{\nu}^1(\nu)$ is non-empty. The result follows.

Now let $X_{\text{inv}}^F = \{ w \in W(T)^F_{\text{inv}} \mid (\xi,j) \in \Lambda_{\nu}^1(\nu) \Rightarrow (-1, \xi)^{m_{\xi}(\nu_\xi)} = 1\}$. 

Lemma 4.1.4. For \( f \in \Theta_T^F, f \in \Theta_{T,\lambda}^F \Leftrightarrow w_f \in X_{inv}^\nu. \)

Proof. We first note that \( \epsilon_{T,f} = 1, \) as may be seen directly (using formulas for \( F_q \)-rank such as those below) or deduced by the method of [10, Lemma 11.3]. So

\[
\Theta_{T,\lambda}^F = \{ f \in \Theta_T^F | \lambda \{ t \in T^F | w_f(t) = t^{-1} \} = 1 \}.
\]

Thus it suffices to observe that \( \{ t \in T^F | w_f(t) = t^{-1} \} \) is generated by \( \{ tw_f(t) | t \in T^F \} \) and \( \prod_{(\xi,j) \in \Lambda_{w_f}^1(\nu)} (\pm 1) \).

Corollary 4.1.5. The map \( f \mapsto w_f \) induces a surjection \( T^F \setminus \Theta_{T,\lambda}^F/K^F \to X_{inv}^\nu. \)

Now the \( F_q \)-rank of \( T \) is \( \ell(\nu) \), and that of

\[
Z_G((T \cap fKf^{-1})^\nu) = GL \left( \bigoplus_{(\xi,j,i) \in \Lambda_{w_f}^1(\nu)} L(\xi,j,i) \right) \times \prod_{(\xi,j) \in \Lambda_{w_f}^2(\nu)} GL(L(\xi,j,i)) \times \prod_{(\xi,j) \in \Lambda_{w_f}^3(\nu)} GL(L(\xi,j,i))
\]

is

\[
\ell(\nu) + \sum_{(\xi,j,i) \in \Lambda_{w_f}^1(\nu)} (m_{\xi}(\nu_\xi)_j - 1) \equiv \ell(\nu) + \ell_{w^1_f}(\nu) \mod 2.
\]

So Lusztig’s formula gives

\[
\langle B_\nu, \text{Ind}_{K^F}^{G^F}(1) \rangle = (-1)^n \sum_{f \in T^F \setminus \Theta_{T,\lambda}^F/K^F} (-1)^{\ell_{w_f}(\nu)} = - \sum_{w \in X_{inv}^\nu} (-1)^{\ell_{w_f}(\nu)} 2^{\ell_{w_f}(\nu)-1} = - \frac{1}{2} \sum_{w \in X_{inv}^\nu} (-2)^{\ell_{w_f}(\nu)}.
\]

Under the isomorphism \( W(T)^F_\lambda \cong \prod_{\xi \in (\sigma) \setminus L} \mathbb{Z}^{\nu_\xi} \) of §1.4, \( X_{inv}^\nu \) corresponds to

\[
\{(w_\xi) \in \prod_{\xi \in (\sigma) \setminus L} \mathbb{Z}^{\nu_\xi} | d_\xi = -1 \Rightarrow \ell_1^{w_\xi}(\nu_\xi)_1 = 0 \}.
\]

Hence

\[
\langle B_\nu, \text{Ind}_{K^F}^{G^F}(1) \rangle = - \frac{1}{2} \prod_{\xi \in (\sigma) \setminus L} \left( \sum_{w_\xi \in \mathbb{Z}_{inv}^{\nu_\xi}} (-2)^{\ell_{w_\xi}(\nu_\xi)} \right) \left( \sum_{w_\xi \in \mathbb{Z}_{inv}^{\nu_\xi}} (-2)^{\ell_{w_\xi}(\nu_\xi)} \right). \]
Theorem 4.2.1. is induced by a Frobenius map 

\[(4.2.1)\]

Either \(K\) is even and \(F\) is automatic. Suppose \(w\) is a pair in the \(F\)-stabilizer of \(\nu\). By (1.3.1), it is equivalent to say that for any \(\nu\), this map is surjective. If \(\nu = 0\), this \(F\)-orbit is a single \(K\)-orbit, so this \(F\)-orbit is surjective. If \(\nu = 0\), this \(F\)-orbit is surjective. If \(\nu = 0\), this \(F\)-orbit is surjective. If \(\nu = 0\), this \(F\)-orbit is surjective. If \(\nu = 0\), this \(F\)-orbit is surjective.

Lemma 4.2.2. The map \(f \mapsto w_f\) induces a map \(T^F \setminus \Theta_T^F / K^F \to W(T)^F_{\text{inv}}\). If \(\epsilon = \epsilon_f\), this map is surjective. If \(\epsilon = -\epsilon\), the image is \(W(T)^F_{\text{inv}} / W(T)^{F-\text{inv}}\).

Proof. As in the proof of Lemma 4.1.2, an involution is in the image of \(T^F \setminus \Theta_T^F / K^F\) precisely when the corresponding \(F\)-stable \(K\)-orbit on \(T \setminus \Theta_T\) contains an \(F\)-fixed point. If the involution has a fixed point, this \(K\)-orbit is a single \(K\)-orbit, so this is automatic. Suppose \(w \in W(T)^F_{\text{inv}}\). It is in the image of \(T^F \setminus \Theta_T^F / K^F\) precisely when there exists a decomposition of \(V\) into lines \(\{L_i^F \mid 1 \leq i \leq n\}\) such that

1. \((L_i^F)^\perp = \oplus_{i' \neq w(i)} L_i^F\)
2. \(F_V(L_i^F) = L_{w_i(i)}^F\)
Lemma 4.2.3. For $f \in \Theta^F_T$, the number of $T^F-K^F$ double cosets in $(TfK)^F$ is

$$
\begin{cases}
1, & \text{if } w_f \in W(T)^F_{\text{ff-inv}} \\
2^{|w_f(\omega)|-1}, & \text{otherwise.}
\end{cases}
$$

Proof. The method of proof of Lemma 4.1.3 applies again here. \qed

Now define $X^w_{\text{ff-inv}}$ as in §4.1. Let $X^w_{\text{ff-inv}} = X^w_{\text{inv}} \cap W(T)^F_{\text{ff-inv}}$.

Lemma 4.2.4. For $f \in \Theta^F_T$, $f \in \Theta^F_{T,\lambda} \iff w_f \in X^w_{\text{inv}}$.

Proof. The proof is exactly the same as that of Lemma 4.1.4. \qed

Corollary 4.2.5. 1. The map $f \mapsto w_f$ induces a map $T^F/\Theta^F_{T,\lambda}/K^F \to X^w_{\text{inv}}$.

If $\epsilon_w = \epsilon$, this map is surjective; if $\epsilon_w = -\epsilon$, its image is $X^w_{\text{inv}} \setminus X^w_{\text{ff-inv}}$.

2. If $w$ is in the image of $T^F/\Theta^F_{T,\lambda}/K^F \to X^w_{\text{inv}}$, there are

$$
\begin{cases}
1, & \text{if } w \in X^w_{\text{ff-inv}} \\
2^{|w(\omega)|-1}, & \text{otherwise}
\end{cases}
$$

$T^F-K^F$ double cosets in the preimage of $w$.

Proof. This follows by combining Proposition 4.0.2, Lemma 4.2.3, Lemma 4.2.3 and Lemma 4.2.4. \qed

Now as in §4.1, the $F_p$-rank of $T$ is $\ell(\omega)$, and that of $Z_G((T \cap fKf^{-1})^g)$ is congruent to $\ell(\omega) + n + \ell_w(\omega)$ mod 2. So Lusztig’s formula gives

$$
\langle B_\omega, \text{Ind}_{K^F_T}^{\Theta^F_T} (1) \rangle = \sum_{f \in T^F/\Theta^F_{T,\lambda}/K^F} (-1)^{\ell_w(\omega)}
$$

$$
= \sum_{w \in X^w_{\text{inv}}} (-1)^{\ell_w(\omega)} \begin{cases} 
1, & \text{if } w \in X^w_{\text{ff-inv}} \\
2^{|\ell_w(\omega)|-1}, & \text{otherwise}
\end{cases}
$$

$$
- \frac{1}{2} \epsilon_w / X^w_{\text{ff-inv}}
$$

$$
= \frac{1}{2} \sum_{w \in X^w_{\text{inv}}} (-2)^{\ell_w(\omega)} + \frac{1}{2} \epsilon_w / X^w_{\text{ff-inv}}.
$$
We transform this expression as in the previous subsection to obtain:

\[
\langle B_\nu, \text{Ind}^{G^F}_{K^F} (1) \rangle = \frac{1}{2} \prod_{\xi \in \langle \sigma \rangle \backslash L} \left( \sum_{w_\xi \in \mathcal{Z}_{inv}^w} (-2)^{t_\xi (w_\xi)} \right) + \frac{1}{2} \epsilon \prod_{\xi \in \langle \sigma \rangle \backslash L} \epsilon_{w_\xi} \left| \mathcal{Z}_{ff - \text{inv}}^{w_\xi} \right|.
\]

In those factors of the second term for which \( m_\xi \) is even, we have used the fact that if there exists a fixed-point free involution in \( \mathcal{Z}_{\nu_\xi} \), then \( \ell(\nu_\xi)_1 \) is even, so \((-1)^{\ell(\nu_\xi)} = \epsilon_{\nu_\xi} \). So along with (4.1.2) and (4.1.3), we need (2.1.2) multiplied on both sides by \( \epsilon_{\nu} \).

4.3. The \( U_n(\mathbb{F}_{q^2})/O_n(\mathbb{F}_q) \) Case (n odd). In this subsection, suppose that \( n \) is odd and take \( F: G \to G \) to be a non-split Frobenius map commuting with \( \theta \). Thus \( \theta F \) is a split Frobenius map induced by \( F_V \) as in §4.1. One has \( G^F \cong U_n(\mathbb{F}_{q^2}) \), \( K^F \cong O_n(\mathbb{F}_q) \). The result is:

**Theorem 4.3.1.** For any \( \rho \in \hat{P}_n^+ \),

\[
\langle \chi_\nu, \text{Ind}^{U_n(\mathbb{F}_{q^2})}_{O_n(\mathbb{F}_q)} (1) \rangle = \begin{cases} 
\frac{1}{2} \prod_{\xi \in \langle \sigma \rangle \backslash L} \left( \prod_{d_\xi = 1} (m_{2i+1}(\rho_\xi) + 1) \right) \prod_{d_\xi = -1} (m_{2i}(\rho_\xi) + 1) \\
\times \prod_{\xi \in \langle \sigma \rangle \backslash L} \left( \prod_{d_\xi = 1} (m_{i}(\rho_\xi) + 1) \right), \\
\text{if } d_\xi = 1, 2 \nmid \tilde{m}_\xi \Rightarrow 2|m_{2i}(\rho_\xi), \ \forall i, \\
d_\xi = -1, 2 \nmid \tilde{m}_\xi \Rightarrow 2|m_{2i+1}(\rho_\xi), \ \forall i, \\
and \tilde{d}_\xi = -1, 2|m_\xi \Rightarrow \rho'_\xi \text{ is even} \\
0, \text{ otherwise.}
\end{cases}
\]
By (1.3.2), it is equivalent to say that for any \( \nu \in \widehat{P}_n \),

(4.3.1)

\[
\langle B_\nu, \text{Ind}^{G_F}_{K_F}(1) \rangle = \frac{1}{2} \left\langle \sum_{\xi \in \langle \tilde{\sigma} \rangle \cap L} (-1)^n(\rho_\xi)(\prod_i (m_{2i}(\rho_\xi) + 1)) \chi_\rho_\xi^{\rho_\xi} \right\rangle
\times \prod_{\xi \in \langle \tilde{\sigma} \rangle \cap L} \left( \sum_{\rho_\xi \vDash |\nu_\xi|} (-1)^n(\rho_\xi)(\prod_i (m_{2i}(\rho_\xi) + 1)) \chi_\rho_\xi^{\rho_\xi} \right)
\times \prod_{\xi \in \langle \tilde{\sigma} \rangle \cap L} \left( \chi_\rho_\xi^{\rho_\xi} \right).
\]

Here the sign

\[ (-1)^{\frac{n}{2}} \prod_{\xi \in \langle \tilde{\sigma} \rangle \cap L} m_\xi n(\rho_\xi) + |\rho_\xi| = (-1)^{\frac{n}{2}} \prod_{\xi \in \langle \tilde{\sigma} \rangle \cap L} m_\xi n(\rho_\xi) + |\rho_\xi| \]

of (1.3.2) has been distributed in an obvious way.

The proof of these statements is mostly very similar to that of (4.1.1). Let \( (T, \lambda) \) be a pair in the \( G_F \)-orbit corresponding to \( \nu \in \widehat{P}_n \).

**Lemma 4.3.2.** The map \( f \mapsto w_f \) induces a surjection

\[ T^F \backslash \Theta^F_T / K^F \twoheadrightarrow W(T)^F_{\text{inv}}. \]

**Proof.** This is deduced in the same way as Lemma 4.1.2. \( \square \)

**Lemma 4.3.3.** For \( f \in \Theta^F_T \), there are \( 2^{n \frac{\lambda}{\tau} (\omega)^{-1}} T^F - K^F \) double cosets in \( (T f K)^F \).

**Proof.** The proof is the same as that of Lemma 4.1.3. \( \square \)

Now define \( X^F_{\text{inv}} \) as in §4.1, but with \( \tilde{\sigma} \) instead of \( \sigma \) and \( \tilde{m}_\xi \) instead of \( m_\xi \).

**Lemma 4.3.4.** For \( f \in \Theta^F_T \), \( f \in \Theta^F_{T, \lambda} \iff w_f \in X^F_{\text{inv}}. \)
Corollary 4.3.5. The map \( t \mapsto f_K \) is surjective.

Proof. We first prove that \( \epsilon_{T,f} = 1 \). The \( \mathbb{F}_q \)-rank of

\[
Z_G((T \cap fKf^{-1})^o) = GL \left( \bigoplus_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} L_{(\xi,j,i)} \right) \times \prod_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} GL(L_{(\xi,j,i)})
\]

is

\[
\left[ \frac{1}{2} \sum_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j \right] + \frac{1}{2} \sum_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j - \frac{1}{2} \sum_{(\xi,j) \in \Lambda^-_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j.
\]

If \( t \in (T \cap fKf^{-1})^F \) has eigenvalue \( \alpha_{(\xi,j)} \) on \( L_{(\xi,j,0)} \), then

\[
Z_G(t) \cap Z_G((T \cap fKf^{-1})^o) = GL \left( \bigoplus_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} L_{(\xi,j,i)} \right) \times \prod_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} GL(L_{(\xi,j,i)})
\]

has \( \mathbb{F}_q \)-rank which differs from that of \( Z_G((T \cap fKf^{-1})^o) \) by

\[
\left[ \frac{1}{2} \sum_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j + \frac{1}{2} \sum_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j - \frac{1}{2} \sum_{(\xi,j) \in \Lambda^-_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j \right] + \frac{1}{2} \sum_{(\xi,j) \in \Lambda^-_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j - \frac{1}{2} \sum_{(\xi,j) \in \Lambda^-_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j.
\]

Since \( n \) is odd, \( \sum_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j \) is odd, so this difference is zero. Thus \( \epsilon_{T,f} = 1 \). The rest of the proof follows that of Lemma 1.1.4. \( \square \)

Corollary 4.3.5. The map \( f \mapsto w_f \) induces a surjection \( T^F \setminus \Theta^F_{T,\lambda} / K^F \to X^\mu_{inv} \).

Now the \( \mathbb{F}_q \)-rank of \( T \) is \( t(\mathfrak{w})_0 \), and that of \( Z_G((T \cap fKf^{-1})^o) \) is given above, whence

\[
\mathbb{F}_q \text{-rank}(T) + \mathbb{F}_q \text{-rank}(Z_G((T \cap fKf^{-1})^o)) = \left[ \frac{1}{2} \sum_{(\xi,j) \in \Lambda^+_0(\mathfrak{w})} \tilde{m}_\xi(\nu_\xi)_j + \ell^1_{w_f}(\mathfrak{w})_0 \right] + \frac{n}{2} + \ell^1_{w_f}(\mathfrak{w})_0 + \ell^2_{w_f}(\mathfrak{w})_2 \mod 4 + \frac{1}{2} \ell^3_{w_f}(\mathfrak{w})_1 \mod 2.
\]
So Lusztig’s formula gives (compare §4.1):

\[ \langle B_L, \text{Ind}_{K_F}^{G_F}(1) \rangle = \frac{1}{2} (-1)^{\frac{d}{2}} \sum_{w \in \Lambda_{\text{inv}}^\ell} (-1)^{\ell_w(\nu) + \ell_{\theta K}(\nu)\mod 4 + \ell_{\theta K}(\nu) + \ell_{\theta K}(\nu)} \cdot 2^{\ell_w(\nu)}. \]

As in §4.1, we transform this expression to get

\[ \langle B_L, \text{Ind}_{K_F}^{G_F}(1) \rangle = \frac{1}{2} (-1)^{\frac{d}{2}} \prod_{\xi \in \langle \sigma \rangle \setminus L} \left( \sum_{w \in \Lambda_{\text{inv}}^\ell} \sum_{w \in \Lambda_{\text{inv}}^\ell} (-1)^{\ell_w(\nu) + \ell_{\theta K}(\nu)\mod 4 + \ell_{\theta K}(\nu) + \ell_{\theta K}(\nu)} \cdot 2^{\ell_w(\nu)} \right) \]

In the second factor, we have noted that if \( \tilde{m}_\xi \) is even, no element of \( \Lambda_{\text{inv}}^\ell(\nu) \) or \( \Lambda_{\text{inv}}^3(\nu) \) can contribute to \( \ell_w^\ell(\nu)2^{\ell_w(\nu)} \), and every element of \( \Lambda_{\text{inv}}^1(\nu) \) contributes to \( \ell_w(\nu)0 \). So in addition to \( (4.1.2) \) and \( (4.1.3) \), the combinatorial facts we need are:

\[ \sum_{w \in \Lambda_{\text{inv}}^\ell} (-1)^{\ell_w(\nu) + \ell_{\theta K}(\nu)\mod 4 + \ell_{\theta K}(\nu) + \ell_{\theta K}(\nu)} \cdot 2^{\ell_w(\nu)} = \sum_{\rho \in [\nu]} (-1)^{n(\rho')} \left( \prod_{i} (m_{2i+1}(\rho) + 1) \right) \chi_{\rho'}, \]

and

\[ \sum_{w \in \Lambda_{\text{inv}}^\ell, \ell_w(\nu) = 0} (-1)^{\ell_w(\nu) + \ell_{\theta K}(\nu)\mod 4 + \ell_{\theta K}(\nu) + \ell_{\theta K}(\nu)} \cdot 2^{\ell_w(\nu)} = \sum_{\rho \in [\nu]} (-1)^{n(\rho')} \left( \prod_{i} (m_{2i}(\rho) + 1) \right) \chi_{\rho'}. \]

These will be proved in §5.

4.4. The \( U_n(\mathbb{F}_{q^2})/O_n(\mathbb{F}_q) \) Case \((n \text{ even})\). Finally, we suppose that \( n \) is even and \( F: G \rightarrow G \) is non-split. So \( \theta F \) is induced by \( F_V \) as in §4.2, and we have the same dichotomy as to the Witt index of \( \langle \cdot, \cdot \rangle \) on \( V^{F_V} \). Define \( \epsilon \in \{ \pm 1 \} \) as in §4.2, so that \( G^F \cong U_n(\mathbb{F}_{q^2}), K^F \cong O_n(\mathbb{F}_q) \).
Theorem 4.4.1. For any $\rho \in \hat{\mathcal{P}}_n^\sigma$,

\[
\langle \chi, \text{Ind}_{G_n^\sigma}^{U_n(F, \sigma)}(1) \rangle = \left\{ \begin{array}{ll}
\frac{1}{2} \prod_{\xi \in \langle \sigma \rangle \setminus L} \prod_{d_\xi = 1} (m_{2i}(\rho_\xi) + 1) \left( \sum_{\rho_\xi \in \langle \nu \rangle} (-1)^{n(\nu_\xi)} \prod_{i} (m_i(\rho_\xi) + 1) \chi_{\nu_\xi}^\rho \right) \\
\times \prod_{\xi \in \langle \sigma \rangle \setminus L} \prod_{d_\xi = -1} \left( \sum_{\rho_\xi \in \langle \nu \rangle} (-1)^{n(\nu_\xi)} \prod_{i} (m_i(\rho_\xi) + 1) \chi_{\nu_\xi}^\rho \right) \right.
\end{array} \right.
\]

By (4.3.2), it is equivalent to say that for any $\mu \in \hat{\mathcal{P}}_n^\sigma$,

\( (4.4.1) \)

\[
\langle B_\mu, \text{Ind}_{K_F^G}^{G_F}(1) \rangle = \frac{1}{2} (-1)^{\tilde{\mu}} \prod_{\xi \in \langle \sigma \rangle \setminus L} \prod_{d_\xi = 1} \left( \sum_{\rho_\xi \in \langle \nu \rangle} (-1)^{n(\nu_\xi)} \prod_{i} (m_i(\rho_\xi) + 1) \chi_{\nu_\xi}^\rho \right)
\]

Here the sign

\[
(-1)^{\tilde{\mu} + \sum_{\xi \in \langle \sigma \rangle \setminus L} m_\xi n(\rho_\xi) + |\nu_\xi|} = (-1)^{\tilde{\mu} + \sum_{\xi \in \langle \sigma \rangle \setminus L} m_\xi (n(\rho_\xi) + |\nu_\xi|) + |\rho_\xi|}
\]
of \((3.2)\) has been distributed in an obvious way (in the second term it has been rewritten as \(\prod_{\xi \in \langle \theta \rangle} L(-1)^{\widehat{m}_\xi}\)) since \(|\rho|\) and \(n(\rho')\) are even if \(\rho'\) is even.

The proof of this is similar to that of \((4.2.1)\). Let \((T, \lambda)\) be as in §4.3. Lemma 4.2.2 must be modified as follows:

**Lemma 4.4.2.** The proof of Lemma 4.1.3 applies again here.

**Proof.** Clearly this is equivalent to \(w \mapsto w f \in W(T)^{\text{inv}}\).

Proof. The proof is mostly the same as that of Lemma 4.2.2. Note that if \(f \in \Theta_T^F\),

\[
F_V(f^{-1} L_{(\xi,j,i)}) = \theta(f)^{-1} \bigcap_{\langle \xi', j', i' \rangle \neq w_{\xi,j,i}} L_{(\xi', j', i')}^{-1} = \left( \bigoplus_{\langle \xi', j', i' \rangle \neq w_{\xi,j,i}} f^{-1} L_{(\xi', j', i')} \right) - \left( f^{-1} L_{w^{-1} w_{\xi,j,i}} \right).
\]

So in (2) of the proof of Lemma 4.2.2, \(w_{\xi,j,i}\) should be replaced by \(w^{-1} w_{\xi,j,i}\). Since \(\text{sign}(w) = (-1)^{\# f}\) if \(w \in W(T)^{\text{inv}}\), we get the result. 

**Lemma 4.4.3.** For \(f \in \Theta_T^F\), the number of \(T^F - K^F\) double cosets in \((T \, f K)^F\) is

\[
\begin{cases}
1, & \text{if } w_f \in W(T)^{\text{inv}}_{\text{inv}} \\
2^{\ell_{w_f}(\nu) - 1}, & \text{otherwise}.
\end{cases}
\]

Proof. The proof of Lemma 4.1.3 applies again here.

Now we diverge somewhat from §4.2. and define

\[
X_{\text{inv}}^w = \{ w \in W(T)^{\text{inv}}_{\lambda, w} | (\xi, j) \in \Lambda^1_{w}(\nu) \Rightarrow (-1, \xi) \widehat{m}_\xi(\nu_j) = (-1)^{\widehat{m}_\xi(\nu_j)} \}.
\]

**Lemma 4.4.4.** For \(f \in \Theta_T^F\), \(f \in \Theta_{T, \lambda} \Leftrightarrow w_f \in X_{\text{inv}}^w\).

Proof. By the same argument as the proof of Lemma 4.3.4, we see that

\[
\epsilon_{T, f}(t) = (-1)^{|\{ (\xi, j) \in \Lambda^1_{w_f}(\nu) | 2 | \widehat{m}_\xi(\nu_j) \alpha(\xi, j) = -1 \}|}.
\]

Hence \(f \in \Theta_{T, \lambda}^F\) if and only if \(w_f \in W(T)^{\text{inv}}_{\lambda}\) and for all \((\xi, j) \in \Lambda^1_{w_f}(\nu)\),

\[
\hat{d}_\xi = -1, 2 \mid (\nu_j) \Rightarrow 2 \mid \widehat{m}_\xi(\nu_j).
\]

Clearly this is equivalent to \(w_f \in X_{\text{inv}}^w\).

As in §4.3,

\[
F_q^\text{rank}(T) + F_q^\text{rank}(Z_{C}(\langle T \, f K f^{-1} \rangle)) = \frac{n}{2} + f^1_{w_f}(\nu) + f^2_{w_f}(\nu) \pmod{4} + \frac{1}{2} f^3_{w_f}(\nu) \pmod{2}.
\]
So Lusztig’s formula gives (compare §4.2):
\[
\langle B, \text{Ind}_{K^F}^G (1) \rangle = \frac{1}{2} (-1)^{\frac{d}{2}} \sum_{w \in X_{\text{inv}}^F} (-1)^{e^1_w(v_{\xi}) + e^2_{w_2} (v_{\xi}) + e^3_{w_2} (v_{\xi})} \frac{1}{2} e^3_{w_1} (v_{\xi}) + e^4_{w_1} (v_{\xi}) + e^5_{w_1} (v_{\xi}) + e^6_{w_1} (v_{\xi}) + e^7_{w_1} (v_{\xi}) + e^8_{w_1} (v_{\xi})
\]

\[
+ \frac{1}{2} \varepsilon_{w_1} \sum_{w \in X_{\text{inv}}^F} (-1)^{e^1_w(v_{\xi}) + e^2_{w_2} (v_{\xi})} \frac{1}{2} e^3_{w_1} (v_{\xi}) + e^4_{w_1} (v_{\xi}) + e^5_{w_1} (v_{\xi}) + e^6_{w_1} (v_{\xi}) + e^7_{w_1} (v_{\xi}) + e^8_{w_1} (v_{\xi})
\]

Under the isomorphism \( W(T)^F \times \prod_{\ell} \mathbb{K}^{\ell} \) of §1.4, \( X_{\text{inv}}^F \) corresponds to
\[
\{(w_{\xi}) \in \prod_{\xi \in \langle \sigma \rangle \setminus L} Z^{\nu_{\xi}}_{\text{inv}} \mid d_{\xi} = -(-1)^{m_{\ell}} \Rightarrow \ell^1_{\nu_{\xi}} (v_{\xi}) = 0 \}.
\]
Hence
\[
\langle B, \text{Ind}_{K^F}^G (1) \rangle = \frac{1}{2} (-1)^{\frac{d}{2}}
\]
\[
\times \prod_{\xi \in \langle \sigma \rangle \setminus L} \left( \sum_{2^{n_{\ell}}} \sum_{w_{\xi} \in Z^{\nu_{\xi}}_{\text{inv}}} (-1)^{e^1_{w_{\xi}} (v_{\xi}) + e^2_{w_2} (v_{\xi}) + e^3_{w_2} (v_{\xi})} \frac{1}{2} e^3_{w_1} (v_{\xi}) + e^4_{w_1} (v_{\xi}) + e^5_{w_1} (v_{\xi}) + e^6_{w_1} (v_{\xi}) + e^7_{w_1} (v_{\xi}) + e^8_{w_1} (v_{\xi})
\]
\[
+ \frac{1}{2} \varepsilon_{w_1} \sum_{w \in X_{\text{inv}}^F} (-1)^{e^1_w(v_{\xi})} \frac{1}{2} e^3_{w_1} (v_{\xi}) + e^4_{w_1} (v_{\xi}) + e^5_{w_1} (v_{\xi}) + e^6_{w_1} (v_{\xi}) + e^7_{w_1} (v_{\xi}) + e^8_{w_1} (v_{\xi})
\]
\[
\times \prod_{\xi \in \langle \sigma \rangle \setminus L} \left( \sum_{2^{n_{\ell}}} \sum_{w_{\xi} \in Z^{\nu_{\xi}}_{\text{inv}}} (-1)^{e^1_{w_{\xi}} (v_{\xi})} \frac{1}{2} e^3_{w_1} (v_{\xi}) + e^4_{w_1} (v_{\xi}) + e^5_{w_1} (v_{\xi}) + e^6_{w_1} (v_{\xi}) + e^7_{w_1} (v_{\xi}) + e^8_{w_1} (v_{\xi})
\]
\[
+ \frac{1}{2} \varepsilon_{w_1} \sum_{w \in X_{\text{inv}}^F} (-1)^{e^1_w(v_{\xi})} \frac{1}{2} e^3_{w_1} (v_{\xi}) + e^4_{w_1} (v_{\xi}) + e^5_{w_1} (v_{\xi}) + e^6_{w_1} (v_{\xi}) + e^7_{w_1} (v_{\xi}) + e^8_{w_1} (v_{\xi}) \right).
\]

In those factors of the second term for which \( m_{\ell} \) is odd, we have used the fact that if \( w_{\xi} \in Z^{\nu_{\xi}}_{\text{inv}} \), then
\[
\ell^2_{w_2} (v_{\xi}) + \frac{1}{2} \ell^3_{w_1} (v_{\xi}) \equiv \frac{|v_{\xi}|}{2} \mod 2.
\]
So (4.3.1) follows by applying (4.3.2), (4.3.3), (4.1.2), (4.1.3), and (2.1.2).

5. Combinatorics of the Symmetric Group

This section is devoted to the proof of the combinatorial facts invoked in §2-4. The notation introduced in §1.2 will be used. We say that a function \( f \) on the set of partitions is \textit{multiplicative} if
\[
f(\nu) = \prod_i f^{(i_{m_i}(\nu))}, \quad \forall \nu.
\]
Examples of multiplicative functions of \( \nu \) are \( \varepsilon, \chi = |Z_{\nu}|, \text{ and } |Z_{\nu}|\).

Our first starting point is [2] VII.(2.4)], which as noted above is precisely
\[
|Z_{\nu}| = \sum_{\rho \in [\nu]} \chi_{\nu}^\rho.
\]
So to prove (4.1.3), it suffices to prove that
\[
\epsilon_\nu \sum_{w \in Z^\nu_{\text{inv}}} (-2)\ell_1^w(\nu) = |Z^\nu_{\text{inv}}|.
\]

To see this, note that since both sides are multiplicative, it suffices to consider the case when \(\nu\) is of the form \((a^b)\), in which case both sides are
\[
\begin{cases}
0, & \text{if } a \text{ is odd and } b \text{ is odd}, \\
\frac{a^{b/2} b!}{2^{b/2}(b/2)!}, & \text{if } a \text{ is odd and } b \text{ is even, and} \\
\sum_{r=0}^{\lfloor b/2 \rfloor} \binom{b}{2r} a^r (2r)! \frac{1}{2^r r!}, & \text{if } a \text{ is even.}
\end{cases}
\]

This fact also implies
\[
(3.2.2) \quad \sum_{(w,\epsilon) \in Z^\nu_{\text{inv}}} (-1)\ell_2^w(\nu) = \sum_{\rho \vdash |\nu|} \chi^\rho_{p^+},
\]
which we can again prove simply by observing that when \(\nu = (a^b)\) both sides are
\[
\begin{cases}
0, & \text{if } a \text{ is even and } b \text{ is odd,} \\
\frac{a^{b/2} b!}{2^{b/2}(b/2)!}, & \text{if } a \text{ is even and } b \text{ is even, and} \\
\sum_{r=0}^{\lfloor b/2 \rfloor} \binom{b}{2r} a^r (2r)! \frac{1}{2^r r!}, & \text{if } a \text{ is odd.}
\end{cases}
\]

Our second starting point is [14, I.8 Example 11], which can be rewritten:
\[
\sum_{w \in Z^\nu_{\text{inv}}} (-1)\ell_2^w(\nu) = \sum_{\rho \vdash |\nu|} \chi^\rho_{p^+}.
\]

So to prove (3.1.3), it suffices to show that
\[
|\{w \in Z^\nu_{\text{inv}} | \ell^w_1(\nu)_0 = \ell^w_2(\nu)_0 = 0\}| = \sum_{w \in Z^\nu_{\text{inv}}} (-1)\ell_2^w(\nu),
\]
which we can again prove simply by observing that when \(\nu = (a^b)\) both sides are
\[
\begin{cases}
0, & \text{if } a \text{ is even and } b \text{ is odd,} \\
\frac{a^{b/2} b!}{2^{b/2}(b/2)!}, & \text{if } a \text{ is even and } b \text{ is even, and} \\
\sum_{r=0}^{\lfloor b/2 \rfloor} \binom{b}{2r} a^r (2r)! \frac{1}{2^r r!}, & \text{if } a \text{ is odd.}
\end{cases}
\]

The remaining identities require a different approach. We first prove
\[
(3.1.3) \quad \sum_{(w,\epsilon) \in Z^\nu_{(p^+, p^-)_{\text{inv}}}} (-1)^{\ell^w_2(\nu)} = \sum_{\rho \vdash |\nu|} |T_{(p^+, p^-)}(\rho')| \chi^\rho_{p^+}.
\]
By definition of induction product ([14, I.7]), and using (2.1.2),

\[
\sum_{(w,\epsilon)\in Z_\nu^{(p^+,p^-) - \text{inv}}} (-1)^{f_w^\nu(\nu)} = \min\{p^+,p^-\} \left( \sum_{\mu \vdash |\nu| + 2r - p^+ - p^-} \chi_{\mu} \cdot \chi^{(p^+ - r)} \cdot \chi^{(p^- - r)}(w_{\nu}) \right) \]

where, by Pieri’s formula ([14, (5.16)]), \( b(p^+ - r, p^- - r, \rho') \) is the number of ways of removing first a vertical \((p^- - r)\)-strip, then a vertical \((p^+ - r)\)-strip, from the Young diagram of \(\rho'\), to leave a diagram with all rows of even length. Now every signed tableaux \(T \in T_{(p^+,p^-)}(\rho')\) determines uniquely an \(r\) as above and such a way of removing strips, as follows:

- order rows of equal length so that rows ending \(\Box\) are below rows ending \(\triangledown\);
- take the vertical \((p^- - r)\)-strip to consist of all final boxes signed \(\Box\);
- take the vertical \((p^+ - r)\)-strip to consist of all final boxes signed \(\triangledown\) in rows of odd length, including those made odd by removal of the first strip.

This correspondence is clearly bijective, which proves (3.1.2).

The proof of

\[
(3.3.2) \quad \sum_{(w,\epsilon)\in Z_\nu^{(p^+,p^-) - \text{inv}}} (-1)^{f_w^\nu(\nu) \mod 4 + \frac{1}{2}f_w^\nu(\nu)} = \sum_{\rho \vdash |\nu|} (-1)^{n(\rho)} |T_{(p^+,p^-)}(\rho')\psi| \chi_{\nu}^{\rho}
\]

proceeds similarly:
where by the same bijection as before,
\[
c(p^+, p^-, \rho') = \sum_{T \in \mathcal{T}(p^+, p^-)(\rho')} (-1)^{\frac{1}{2}\left| \rho \right| - \left| \text{rows of } T \text{ ending } \mathbb{E} \right|} = (-1)^{\frac{1}{2}(\left| \rho \right| - \ell(\rho'))} \sum_{T \in \mathcal{T}(p^+, p^-)(\rho')} (-1)^{\left| \text{even rows of } T \text{ ending } \mathbb{E} \right|}.
\]

Now
\[
(-1)^{\frac{1}{2}(\left| \rho \right| - \ell(\rho'))} = (-1)^{\ell(\rho')_2 \mod 4 + \ell(\rho')_3 \mod 4} = (-1)^{\sum_i \ell(\rho')_i} = (-1)^{n(\rho)},
\]
and by grouping together signed tableaux which differ only in even rows it is easy to see that the sum equals \(|\mathcal{T}(p^+, p^-)(\rho')|\). So (3.3.2) is proved.

Our next task is to modify this proof of (3.3.2) to derive
\[
\sum_{(w, \epsilon) \in Z_{\nu, \text{ inv}}^*} (-1)^{\ell(\nu)_0 \mod 4 + \frac{1}{2} \ell(\nu)_1} = \sum_{\rho \sim [\nu]} (-1)^{n(\rho)} \left( \prod_i (m_{2i}(\rho') + 1) \right) \chi^\nu_{\rho'}.
\]

We may assume that \(|\nu|\) is even, for otherwise both sides vanish. Following the above pattern, we need to replace \(\chi(\rho^+, -r), \chi(\rho^-, -r)\) with the class function on \(S_{|\nu|-2r}\) defined by
\[
w \mapsto |\{(A^+, A^-) \mid \{1, \cdots, |\nu| - 2r\} = A^+ \bigsqcup A^-, |A^+| = |A^-|, w(A^\pm) = A^\mp\}|.
\]

It is easy to see that this is \(\sum_{i=0}^{\frac{|\nu|}{2} - r} (-1)^i \chi^{(|\nu| - 2r - i, i)}\). Thus
\[
\sum_{(w, \epsilon) \in Z_{\nu, \text{ inv}}^*} (-1)^{\ell(\nu)_0 \mod 4 + \frac{1}{2} \ell(\nu)_1} = \sum_{\rho \sim [\nu]} (-1)^{n(\rho)} \left( \prod_i (m_{2i}(\rho') + 1) \right) \chi^\nu_{\rho'}.
\]

where, analogously to the above, we can write \(d(\rho')\) as the sum, over \(T \in \mathcal{T}(0, 0)(\rho')\), of a sign determined by \(T\). As in the proof of (3.3.2), the \((-1)^r\) contribution to the sign is
\[
(-1)^{n(\rho) + \left| \text{even rows of } T \text{ ending } \mathbb{E} \right|}.
\]
The \((-1)^i\) contribution is trickier to rephrase in terms of \(T\), but an examination of the bijection
\[
\text{Tab}(\rho' - \mu', \lceil |\nu|/2 \rceil , \lfloor |\nu|/2 \rfloor) \sim \prod_{i=0}^{\left| |\nu| - 2r - i, i \right|} \text{Tab}^0(\rho' - \mu', (|\nu| - 2r - i, i))
\]
defined in \[14, (9.4)\] (in the proof of the Littlewood-Richardson Rule) reveals that the correct reformulation is

\[
(1)^{|\{\text{rows of } T \text{ ending } \square\}| - m(T)}
\]

where \(m(T)\) is the maximum, over all rows \(R\) of \(T\), of the quantity

\[
|\{\text{odd rows ending } \Box \text{ below or equal to } R\}| - |\{\text{odd rows ending } \square \text{ below or equal to } R\}|
\]

(assuming that the rows of \(T\) are ordered, as above, so that rows ending \(\Box\) come below rows ending \(\square\) of the same length). Hence

\[
d(\rho') = (1)^{n(\rho)} \sum_{T \in T(0,0)(\rho')} (1)^{|\{\text{odd rows of } T \text{ ending } \square\}| - m(T)},
\]

and we are reduced to proving that

\[
\sum_{T \in T(0,0)(\rho')} (1)^{|\{\text{odd rows of } T \text{ ending } \square\}| - m(T)} = \begin{cases} \prod_i (m_{2i}(\rho') + 1), & \text{if } 2 \mid m_{2i+1}(\rho'), \forall i \\ 0, & \text{otherwise.} \end{cases}
\]

Grouping together signed tableaux which differ only in even rows, we see that we may assume that \(\rho'\) has only odd parts. In lieu of a direct proof, we can deduce this from \[3.2.2\], proved above. It says that

\[
\sum_{\rho' \vdash |\nu|} \epsilon_\nu \sum_{\rho \vdash |\nu|} (1)^{n(\rho)} \sum_{(w,\epsilon) \in Z_{\nu}^* - inv} (-1)\ell_w(\nu)
\]

whence

\[
\sum_{T \in T(0,0)(\rho')} (1)^{|\{\text{rows of } T \text{ ending } \square\}| - m(T)} = \begin{cases} 1, & \text{if } 2 \mid m_1(\rho'), \forall i \\ 0, & \text{otherwise.} \end{cases}
\]

When \(\rho'\) has only odd parts, this is precisely the statement we want.

It is now easy to prove \[4.3.3\], since

\[
\sum_{w \in Z_{\nu}^*_{\text{inv}}} (1)^{\ell_w(\nu)+\ell_w(\nu)_{2 \mod 4}+\ell_w(\nu)_{2 \mod 4}} = (-1)^{\ell(\nu)} \sum_{(w,\epsilon) \in Z_{\nu}^* - inv} (-1)^{\ell_w(\nu)_{2 \mod 4}+\ell_w(\nu)_{2 \mod 4}}
\]

\[
= \epsilon_\nu \sum_{\rho \vdash |\nu|} (1)^{n(\rho)} (\prod_i (m_{2i}(\rho') + 1)) \chi_\nu^\rho
\]

\[
= \sum_{\rho \vdash |\nu|} (1)^{n(\rho)} (\prod_i (m_{2i}(\rho) + 1)) \chi_\nu^\rho
\]
as required. Finally, (4.1.2) follows from (3.1.2) by summing over all signatures:

\[
\sum_{w \in \mathbb{Z}^\text{inv}_\nu} (-1)^{\ell(w)(\nu)} = (-1)^{\ell(\nu)} \sum_{(w, r) \in \mathbb{Z}^\text{inv}_{\nu}} (-1)^{\ell_r(w)(\nu)}
\]

\[
= (-1)^{|\nu|} \epsilon_\nu \sum_{\rho \vdash |\nu|} |T_{\pm}(\rho')| |\chi_\rho^0|
\]

\[
= (-1)^{|\nu|} \sum_{\rho \vdash |\nu|} (\prod_i (m_i(\rho) + 1)) |\chi_\rho^0|
\]

and (4.3.2) follows from (3.3.2) in an analogous way:

\[
\sum_{w \in \mathbb{Z}^\text{inv}_\nu} (-1)^{\ell_r(w)(\nu) + \ell_2(w)(\nu)_{2 \mod 4} + \frac{1}{2} \ell_2(w)(\nu)_{1} - \ell_3(w)(\nu)} = \epsilon_\nu \sum_{\rho \vdash |\nu|} (-1)^{n(\rho)} |T_{\pm}(\rho')| |\chi_\rho^0|
\]

\[
= \sum_{\rho \vdash |\nu|} \frac{(-1)^{n(\rho')}}{2|m_{2i}(\rho)|} (\prod_i (m_{2i+1}(\rho) + 1)) |\chi_\rho^0|
\]

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