Duality of string amplitudes in a curved background

Måns Henningson

Institute of Theoretical Physics, S-412 96 Göteborg, Sweden

We initiate a program to study the relationship between the target space, the spectrum and the scattering amplitudes in string theory. We consider scattering amplitudes following from string theory and quantum field theory on a curved target space, which is taken to be the $SU(2)$ group manifold, with special attention given to the duality between contributions from different channels. We give a simple example of the equivalence between amplitudes coming from string theory and quantum field theory, and compute the general form of a four-scalar field theoretical amplitude. The corresponding string theory calculation is performed for a special case, and we discuss how more general string theory amplitudes could be evaluated.

1. Introduction

The purpose of this paper is to study the interplay between the geometry of the target space of string theory, the string spectrum and the symmetry properties of string scattering amplitudes. We will consider string theory and quantum field theory in parallel. In the following we give some reasons why we hope that such a program might be fruitful.

String theory is a remarkably unique theory. In addition to the bosonic string, the Ramond-Neveu-Schwarz string and the Green-Schwarz string, one may also mention the heterotic string, which is really a synthesis of the Ramond-Neveu-Schwarz string and the bosonic string, but this more or less exhausts the list. Furthermore, the Ramond-Neveu-Schwarz string and the Green-Schwarz string are probably equivalent, although formulated in rather different ways. It is true that we have a plethora of (perturbative) vacua, but there are reasons to believe that non-perturbative tunneling effects will eventually select a unique one, or at least narrow down the choice considerably. Anyway, these different vacua should be regarded as different solutions to the same equations, rather than as distinct theories.

We may get a hint of why string theory is so unique already from a popular account of the theory. The original string action for the freely propagating string was simply proportional to the area of the world-sheet swept out by the string $\sqrt{g}$, and thus has a
strong flavour of geometry. This is even more apparent when we consider interactions. In point particle theories, we have distinguished interaction vertices where world-lines meet. Nothing such is possible in string theory, since we cannot give a Lorentz invariant definition of where the interaction takes place. The interactions of a string theory are therefore more or less determined by the propagation of a free string, and we have few coupling constants to adjust. Finally, the quantum mechanical consistency of a string theory seems to be a very delicate issue with many potential anomalies. The proper cancelation of such anomalies puts severe restrictions on the choice of symmetry groups and representations.

Symmetry arguments (in flat space) usually relate particles of the same mass and spin (for example the gauge group of the standard model $SU(3) \times SU(2) \times U(1)$ or flavour $SU(3)$), or exceptionally particles of the same mass but different spins (supersymmetry). String theory seems to go beyond these limitations in that it assembles particles of different masses in multiplets, as in the familiar spectra of the various string theories. If we regard such a spectrum as an “irreducible representation” of a hitherto unknown “string symmetry group”, it is plausible that all scattering amplitudes are related to each other. This suggests that we regard string theory in the following way: The in-data supplied is the target space in which the string lives (for example Minkowski space, superspace, a group manifold,...), a parameter related to the size of a typical string (the string tension, the level,...) and a single string coupling constant. The usual axioms for an $S$-matrix theory, together with some unknown “string principle” should then determine not only the spectrum of physical states of the theory, but also the amplitudes for scattering of those states. Furthermore, it seems that the consistency of string theory puts severe restrictions on the allowed target spaces.

String theory dates its roots back to the days of $S$-matrix theory. The first result in what was to become string theory was the Veneziano amplitude [3], which was thought to be the Born term of a four-point amplitude for strongly interacting particles. An alternative amplitude was proposed in [4]. These amplitudes attracted interest because of a remarkable symmetry property, called duality [5], which relates contributions in different channels. It was only somewhat later that it was realized that these amplitudes may be given an interpretation in terms of a theory of relativistic strings. The trend in the last few years has been to focus more on string theory as a two-dimensional theory defined on the world-sheet swept out by the propagating string, and less on its target space properties. This approach has been remarkably fruitful, but several important questions remain unanswered. It is a common belief that a deep understanding of string theory would require a completely new formulation. To find such a formulation, we should scrutinize string theory from as many different perspectives as possible.

The great interest in string theory stems largely from the hope that it will prove a viable way to quantum gravity [3][7]. Indeed, the string spectrum contains states which may be identified as gravitons, and string theory seems to provide a consistent scheme for calculating perturbative graviton-graviton scattering amplitudes. However, most research in string theory concerns strings propagating in a flat Minkowski space, but if we are to describe gravity, we must also consider more general backgrounds. A truly consistent theory must allow for the string to influence its own background, but this is as yet beyond our understanding. Anyway, a study of strings in any background
different from flat Minkowski space should be worthwhile.

Minkowski space is in many respects the simplest possible background for string theory, but for a study of these questions it is not so well suited. The group theory of the underlying isometry group, the Poincaré group, is rather involved. We hope that something could be learnt by studying simpler homogeneous spaces. Most of this paper will be devoted to bosonic strings on target manifolds which could be equipped with a group structure, in particular $SU(2)$. Our reason for this is largely technical. We feel that there is good hope, though, that the features we are interested in will survive even in such an extremely simplified and unrealistic toy model.

We have seen that string theory has departed rather much from its origin. Our intention is to focus on the properties of the final result, the string amplitudes, rather than on intermediate steps in the calculations. This is in the spirit of the old dual models ideas. A comparatively large part of this paper is devoted to a general discussion of scattering amplitudes from string theory and quantum field theory, but we also give a few concrete examples to illustrate the ideas. We hope to be able to present more realistic examples in forthcoming publications.

This paper is organized as follows: In section 2 we give a brief review of some aspects of string theory in a curved background. Section 3 is devoted to explain how a scattering process in symmetric spaces is described. In section 4 we discuss the calculation of scattering amplitudes in a quantum field theory. This discussion is specialized to quantum field theory on a group manifold in section 7 after a review of some group theory in section 5 and a discussion of the string spectrum in section 6. In sections 8, 9 and 10 we give some examples of how string theory amplitudes could be computed. In section 11 we briefly consider the flat space limit, and finally, in section 12, we discuss the relevance of the present work, and indicate how we intend to continue the programme.

2. String theory in a curved background

String theory is most often treated in a first quantized formalism, i. e. as a two-dimensional quantum field theory defined on the world-sheet of the string. For the string interpretation to be consistent, it is necessary that this quantum field theory is invariant, not only under general reparametrizations of the world sheet, but also under scale changes. It should thus be a conformal field theory.

We will restrict our attention to a purely bosonic string moving on a $D$-dimensional target manifold $\mathcal{M}$. If we introduce (local) coordinates $x^\mu$, $\mu = 1, \ldots, D$ on $\mathcal{M}$, we may write down the most general renormalizable string action:

$$S = \frac{1}{\alpha'} \int d^2 \sigma \sqrt{g} g^{\alpha \beta} \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu \nu}(x) + \frac{1}{\alpha'} \int d^2 \sigma \epsilon^{\alpha \beta} \partial_\alpha x^\mu \partial_\beta x^\nu B_{\mu \nu}(x) + \int d^2 \sigma \sqrt{g} R^{(2)}(x).$$

Here $g^{\alpha \beta}$ and $R^{(2)}$ denote the metric and the curvature scalar of the world-sheet respectively, and $\alpha'$ is a constant of (target space) dimension $[\text{length}]^2$. The functions $G_{\mu \nu}(x)$, $B_{\mu \nu}(x)$ and $\phi(x)$ are interpreted as a metric, an anti-symmetric tensor field and a scalar field (the dilaton) on the target space $\mathcal{M}$ respectively.
The requirement that the theory be conformally invariant at quantum level amounts to the vanishing of the beta-functionals of the couplings $G_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$. To the first non-vanishing order in $\alpha'$ this means \[^8\] that

\[0 = R_{\mu\nu} + \frac{1}{4}H^{\lambda\rho}_{\mu}H_{\nu\lambda\rho} - 2D_{\mu}D_{\nu}\phi\]

\[0 = D_{\lambda}H^{\lambda\mu}_{\nu} - 2(D_{\lambda}\phi)H^{\lambda}_{\mu\nu},\]

where $H_{\mu\nu\rho} = 3D_{[\mu}B_{\nu\rho]}$ and $R_{\mu\nu}$ is the Ricci tensor corresponding to the metric $G_{\mu\nu}(x)$.

A truly consistent string theory also requires the conformal anomaly $c$, including contributions from the Fadeev-Popov ghosts that arise upon gauge fixing, to vanish. This means that the beta-functional of $\phi(x)$ should vanish, which to lowest non-trivial order in $\alpha'$ yields

\[0 = \frac{D - 26}{3\alpha'} + 4D_{\mu}\phi D^\mu\phi - 4D_{\mu}D^\mu\phi + R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}.\]

However, we will only consider string theory at genus zero (tree level), where the conformal anomaly is of no consequence. The toy model that we will describe later in fact has a negative conformal anomaly. The reader who feels uneasy about this may always add other conformal field theories, such as for example free bosons or minimal models, to make up for the missing anomaly.

It is obviously not easy to find solutions to the non-linear equations (2), especially if we include terms of higher orders in $\alpha'$. An interesting possibility would be a string moving in a maximally symmetric space so that $R_{\mu\nu\rho\sigma} \sim g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}$. As we will see in the next section, the description of scattering is facilitated in a maximally symmetric space. However, it should be realized that the fields $B_{\mu\nu}$ and $\phi$ will in general break this symmetry. Namely, if we use the condition that $H_{\mu\nu\rho}$ is a maximally symmetric tensor (see for example \[^9\]),

\[\delta^\tau_\mu H^{\sigma}_{\nu\rho} + \delta^\leftarrow_\nu H^{\sigma}_{\mu\rho} + \delta^\leftarrow_\rho H^{\sigma}_{\mu\nu} = \delta^\sigma_\mu H^\tau_{\nu\rho} + \delta^\sigma_\nu H^\tau_{\rho\mu} + \delta^\sigma_\rho H^\tau_{\mu\nu},\]

and contract with $\delta^\tau_\tau$, we see that we must require that $D = 3$ (so that $H_{\mu\nu\rho} \sim \epsilon_{\mu\nu\rho}$).

An important class of solutions to (2) have $\phi \equiv 0$ and $B_{\mu\nu}$ chosen so that $H_{\mu\nu\rho}$ acts as a parallelizing torsion \[^10\]. Such torsions only exist for manifolds which admit a group structure and for $S_7$ with the round metric \[^1\]\[^2\]. The latter possibility is excluded, however, since the parallelizing torsion on $S_7$ is non-closed and therefore non-exact (even locally).

Although the corrections to (2) are not known to arbitrarily high orders in $\alpha'$, it has been shown that a group manifold with a parallelizing torsion is an exact solution \[^13\]. This is in fact not too astonishing. These models are the familiar Wess-Zumino-Witten models \[^4\]\[^5\]\[^6\]. They are exactly solvable conformal field theories, and thus fixed points of the renormalization group.

A Wess-Zumino-Witten model is completely specified by its symmetry group $G$ and an integer $k$ (the level), which is inversely proportional to $\alpha'$. We will be mostly concerned with $G = SU(2)$. This group is three-dimensional, and the metric is maximally symmetric. Models based on the closely related group $SU(1, 1)$ have attracted much interest recently \[^17\]\[^18\]. They offer the attractive feature of containing a time-direction, and therefore seem closer to physical reality. Many questions concerning
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the $SU(1,1)$ Wess-Zumino-Witten model still remain unanswered, though. We will not consider $SU(1,1)$ in this paper, but we hope that a thorough understanding of $SU(2)$ string amplitudes will prove helpful when investigating other string theories in general and $SU(1,1)$ strings in particular. Namely, if the final result is expressible in purely group theoretical terms, it should be possible to translate it from one group to another.

Strictly speaking, an $S$-matrix theory presupposes that the external states are free, i.e., the fields should obey a free field equation at infinity. On a compact manifold, such as the $SU(2)$ group manifold, we cannot construct such asymptotic states, and one might therefore argue that the whole idea of scattering is ill defined. What we are really calculating in this paper, however, is amputated amplitudes or Green functions. It seems that our prescription for calculating such Green functions is consistent, and this is sufficient for our purposes. We stress once more that we are considering a toy model, which in many respects is far from physical reality, but that our primary object is to investigate certain properties of its scattering amplitudes.

3. Kinematics in a curved space

In this section we will investigate the constraints on particle scattering amplitudes that follow from the geometry of the target space.

The particle concept is in general not well defined in an arbitrarily curved space. (See for example [19].) We will consider a theory invariant under some symmetry group $G$, which may include internal symmetries in addition to space-time isometries, and we will use Wigner’s definition of an elementary particle: An elementary system corresponds to a unitary representation of $G$. This procedure is correct at least if the space is maximally symmetric.

To a particle transforming in the $D_R$ representation under $G$ there is associated a representation space $H_R$. Consider now a scattering situation with $n$ external particles transforming in the $D_{R_1}, \ldots, D_{R_n}$ representations. This configuration transforms in the tensor product representation $D_{tot} = D_{R_1} \otimes \cdots \otimes D_{R_n}$, defined in the space $H_{tot} = H_{R_1} \otimes \cdots \otimes H_{R_n}$, under $G$.

A scattering amplitude $\Gamma$ could now be regarded as a linear functional on this space $H_{tot}$. We should require $\Gamma$ to be invariant under $G$. If we decompose $D_{tot}$ as a direct sum of irreducible representations, it is not difficult to see that this amounts to the vanishing of $\Gamma$ on all subspaces of $H_{tot}$ which transform non-trivially under $G$. The values of $\Gamma$ on the subspace $H_0$ of $G$ invariant states are arbitrary. The scattering amplitude $\Gamma$ is thus completely specified by its values on a set of basis vectors of $H_0$.

This way of viewing scattering may seem unfamiliar, and we will therefore briefly consider the well-known example when $G$ is the four-dimensional Poincaré group. Its unitary irreducible representations are characterized by a mass and a spin $[20]$. We will take the external particles to be spinless and of mass $m$, so that the vectors $|p \mu >$ where $p^2 = m^2$ constitute a basis for the one-particle space. In the case of four external particles the space $H_{tot}$ is thus spanned by the vectors

$$|p_1, \ldots, p_4 >= |p_1 > \otimes \cdots \otimes |p_4 >$$
where $p_1^2 = \ldots = p_4^2 = m^2$, and a general state in $H_{\text{tot}}$ may be written as

$$|f> = \prod_{i=1}^{4} \int d^4 p_i \delta(p_i^2 - m^2) f(p_1, \ldots, p_4) |p_1, \ldots, p_4>$$

(6)

for an arbitrary function $f(p_1, \ldots, p_4)$. The requirement that $|f>$ is $G$ invariant is equivalent to demanding that

$$f(p_1, \ldots, p_4) = \tilde{f}((p_1 + p_2)^2, (p_1 + p_3)^2) \delta^{(4)}(p_1 + \ldots + p_4)$$

(7)

for some function $\tilde{f}(s, t)$. As a basis for $H_0$ we may take the vectors

$$|s, t> = \prod_{i=1}^{4} \int d^4 p_i \delta(p_i^2 - m^2) \delta((p_1 + p_2)^2 - s) \delta((p_1 + p_3)^2 - t) \delta^{(4)}(p_1 + \ldots + p_4) |p_1, \ldots, p_4>,$$

(8)

and the scattering amplitude $\Gamma$ could be described by its values $\Gamma(s, t)$ on these vectors. We have thus recovered the usual description of a four-point amplitude in terms of the Mandelstam variables $s$ and $t$.

Let us now return to a general symmetry group $G$. The Clebsch-Gordan series for its representations provides us with a suitable basis for the subspace $H_0$ of $G$ invariant states, as we will now explain. The tensor product of two irreducible representations is in general reducible, and may be decomposed as a direct sum of irreducible representations. We will assume that to every irreducible representation $D_R$ there is a conjugate irreducible representation $D_{\bar{R}}$ so that the tensor product of $D_R$ and $D_{\bar{R}}$ contains exactly one copy of the trivial one-dimensional representation. Furthermore, we will assume that the tensor product of $D_R$ with any other irreducible representation does not contain the trivial representation. As a basis for $H_0$ for the scattering of four external particles transforming in the $D_{R_1}, \ldots, D_{R_4}$ representations we may now take the orthonormal vectors $|R>$, which transform trivially under $G$ acting on all four external particles, and belong to the representation space of $D_R$ in the Clebsch-Gordan decomposition of $D_{R_1} \otimes D_{R_2}$. Our scattering amplitude $\Gamma$ may thus be described by a function $\Gamma(R)$, where $R$ runs over a set of representations of $G$. By choosing a different pairing of the external particles we may obtain a different basis. The matrix which relates the bases is the $G$ counterpart of the Wigner $6j$ symbol for $SU(2)$.

If two of the external particles are identical, the operation of permuting them is a well-defined map from $H_0$ to itself, and thus also from the dual space $H_0^*$ of scattering amplitudes to itself. It thus makes sense to say that an amplitude is symmetric under permutation of those external particles. An analogous reasoning applies in the case that all four external particles are identical. Such “crossing symmetry” is in fact one of the requirements that is usually imposed on an $S$-matrix theory [21].

4. Field theoretical scattering amplitudes

So far we have only used “kinematical” symmetry arguments to determine the possible form of the scattering amplitude $\Gamma$. We will now impose the “dynamical” constraint that $\Gamma$ follows from a local quantum field theory.
Let us assume that the symmetry group $G$ is the isometry group of the target space manifold $\mathcal{M}$. This means that an element of $G$ acts as a metric preserving map from $\mathcal{M}$ to itself. These maps induce linear transformation laws for tensor fields on $\mathcal{M}$. Each field carries a reducible representation of $G$, which may be decomposed as a direct sum of irreducible components.

In a Lagrangian formalism, the theory is defined by a set of such fields $\phi^i(x)$ and an action functional $S[\phi]$, which should be $G$ invariant. The action $S[\phi]$ is often decomposed as a sum of a kinetic term $S_{\text{kin}}$, which is bilinear in the fields $\phi^i(x)$, and an interaction term $S_{\text{int}}$, which is trilinear or higher. If we denote the part of $\phi^i(x)$ that transforms in the $D_R$ representation under $G$ as $\phi^i_R$, we may write

$$S_{\text{kin}} = \sum_i \sum_R \phi^i_R \phi^i_R C_{RR}K^i_i(R)$$

and

$$S_{\text{int}} = \sum_{ijk} \sum_{R_i R_j R_k} \phi^i_{R_i} \phi^j_{R_j} \phi^k_{R_k} C_{R_i R_j R_k}V_{ijk}(R_i, R_j, R_k) + O(\phi^4),$$

where $C_{RR}$ ($C_{R_i R_j R_k}$) denotes the Clebsch-Gordan coefficient for coupling two (three) representations of $G$ to yield the trivial representation. If a certain representation does not occur in $\phi^i(x)$, or if the three representations may not be coupled together, we will assume that the corresponding $V_{ijk}(R_i, R_j, R_k)$ vanishes.

We should now calculate the tree-level contribution to the scattering amplitude for four external particles transforming in the $R_1, \ldots, R_4$ representations. In a path integral quantization it is easy to see what happens. The amplitude is a sum of contributions in different channels:

![Diagram](image)

To calculate a diagram we need the vertex factor $V_{ijk}(R_i, R_j, R_k)$ and the propagator $K_i^{-1}(R)$. The rest is pure group theory. It is not difficult to see that the $s$-channel contribution to the function $\Gamma(R)$ introduced in the last section is

$$\Gamma_{i_1 \ldots i_4}^{(s)}(R) = \sum_i V_{i_1 i_2 i}(R_1, R_2, R)K_i^{-1}(R)V_{i_3 i_4}(R_3, R_4).$$

The contributions from the other channels are most easily calculated in the corresponding bases of $H_0$ and then transformed to the $s$-channel basis by means of the “$6j$” symbols.

If the external particles are identical, the total amplitude is obviously symmetric under permutation of the external legs, as we described in the last section. However, we may conceive of a theory in which already the $s$-channel contribution is symmetric under exchange of two external particles (planar duality), or the sum of the $s$- and $t$-channel contributions is symmetric under permutation of all four external particles (non-planar duality). This is clearly a non-trivial constraint on the field content and the couplings of the field theory.

There are good reasons to believe that duality is an essential property of string theory, and therefore a plausible candidate for a “string principle”. In fact, there is
no clear distinction between diagrams in different channels in string theory. Using local conformal rescalings of the world-sheet metric, we may relate seemingly different diagrams.

The functions $K_i(R)$ and $V_{i_1i_2i_3}(R_1, R_2, R_3)$ may not be chosen at will, but are constrained by locality of the quantum field theory. To see how this works we will first consider the simplest case, namely a scalar field $\phi(x)$. The most general kinetic term is

$$S_{\text{kin}} = \int d^Dx \; \phi(x) (\Box - \mu^2) \phi(x), \quad (12)$$

where $\Box$ is the d’Alembertian on $\mathcal{M}$ and $\mu$ is a constant. We have assumed at most two derivatives in analogy with flat space field theory, where terms with more derivatives lead to non-unitary theories. As we have already mentioned, $\phi(x)$ may be decomposed into its irreducible components $\phi_R$. The d’Alembertian is a Casimir operator of $G$, so the components $\phi_R$ are eigenfunctions of $\Box$ with some eigenvalue $Q(R)$. The decomposition of $\phi$ into $\phi_R$ thus amounts to doing harmonic analysis on $\mathcal{M}$. The kinetic term (12) may now be rewritten in the form (9) with $K_\phi(R) = Q(R) - \mu^2$.

We should now scatter external $\sigma(x)$ fields by the exchange of $\phi(x)$ quanta. The only trilinear $G$ invariant interaction term without any derivatives is

$$S_{\text{int}} = \int d^Dx \; \lambda \phi(x) \sigma(x) \sigma(x). \quad (13)$$

With the same normalization of the irreducible field components as in the kinetic term this corresponds to an interaction of the form (14) with $V_{\sigma\phi}(R_1, R_2, R) = 1$. According to our previous results the $s$-channel contribution to the four point amplitude is thus

$$\Gamma(R) = (Q(R) - \mu^2)^{-1}.$$

We could now in principle go on and calculate the contributions from exchange of vector fields $A_\mu(x)$ and higher rank tensor fields $A_{\mu_1...\mu_r}(x)$. However, the decomposition of the fields into their irreducible components and the determination of how these representations are coupled together in the action in general require a great deal of knowledge about the differential geometry of the manifold $\mathcal{M}$.

5. Differential geometry on group manifolds

To write down our Lagrangian in a covariant way, so that the general covariance is manifest, we need a machinery for doing tensor algebra on the target space. The actual evaluation of the theory is most often performed in Fourier space, where the derivatives in the Lagrangian are simply algebraic operations. The interpretation of harmonic eigenfunctions as asymptotic states is also easier in Fourier space. Both of these technical questions, tensor algebra and harmonic analysis, are much simpler if the target space may be equipped with a group structure. As is well known, there is a close relationship between harmonic analysis on a group manifold and the representation theory of the group. For tensor analysis, we have a canonical choice for a vielbein, namely the left (or right) invariant vector fields associated with the Lie algebra of the group.

Let us therefore assume $\mathcal{M}$ to be the group manifold of some Lie group $G$. The isometry group is then $\mathcal{G} = G \otimes G$, and acts as

$$\mathcal{G} \ni (u, v) : \quad \mathcal{M} \ni x \mapsto x' = u xv^{-1} \in \mathcal{M}. \quad (14)$$
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Anticipating that $G = SU(2)$, we will denote the unitary irreducible representations of $G$ as $D_j$, and label the states within such a representation as $|j, m\rangle$. A $G$ representation $D_R = (D_{jL}, D_{jR})$ is then specified by giving the representations $D_{jL}$ and $D_{jR}$ of the left and right $G$ factor.

We will often have reason to consider different ways of coupling representations together. Such recouplings are described by the Wigner $3nj$ symbols for the case $G = SU(2)$. In particular we will often use the formula

\[
(j_1 j_2 j_3 j_4) = \sum_{j'} (-1)^{j_2 j_3 j_4 j'} [(2j + 1)(2j' + 1)]^{1/2} \left\{ \begin{array}{ccc} j_1 & j_2 & j \\ j_3 & j_4 & j' \end{array} \right\},
\]

which describes the relation between different ways of coupling $D_{j_1}, \ldots, D_{j_4}$ to yield the trivial representation. (See for example [22].)

It is well-known that the elements $V^j_{\bar{m}m}(x) = \langle j, m|V^j(x)|j, \bar{m}\rangle$ of the representation matrices form an orthogonal basis of $L^2(G)$, or explicitly

\[
\int d\mu(x) \ V^j_{\bar{m}m}(x)V^j_{m'\bar{m}'}(x) = \delta_{jj'}\delta_{mm'}\delta_{\bar{m}\bar{m}'} \frac{1}{\dim D_j},
\]

where $d\mu(x)$ is the $G$ invariant Haar measure on $G$. (See for example [23].) We may thus expand a scalar field $\phi(x)$ on $G$ as

\[
\phi(x) = \sum_j \sum_{m \bar{m}} \phi^j_{m\bar{m}} V^j_{m\bar{m}}(x),
\]

where the Fourier components $\phi^j_{m\bar{m}}$ are given by

\[
\phi^j_{m\bar{m}} = \dim D_j \int d\mu(x) \ \phi(x)V^j_{m\bar{m}}(x).
\]

It is not difficult to verify that the scalar field transformation law for $\phi(x)$ under the isometry group implies that $\phi^j_{m\bar{m}}$ transforms in the $(D_j, D_{\bar{j}})$ representation under $G$ acting as in (14). Here $D_{\bar{j}}$ denotes the conjugate representation of $D_j$. The full representation content of a scalar field $\phi(x)$ is thus

\[
D_0 = \bigoplus_j (D_j, D_{\bar{j}}),
\]

where the sum runs over all unitary representations of $G$.

Before we treat vector fields and higher rank tensor fields we need to introduce some more notation. A group element $x \in G$ may be expanded as $x = \exp(i\theta_a T^a)$, where the $T^a$ span the Lie algebra $g$ of $G$ and fulfil

\[
[T^a, T^b] = if^{abc} T^c.
\]

Two especially important solutions to these commutation relations are the trivial one-dimensional representation $D_{\text{triv}}$, in which $T^a_{\text{triv}} = 0$, and the $\dim G$ dimensional adjoint representation $D_{\text{adj}}$, in which

\[
(T^a_{\text{adj}})_c = -if^{abc}.
\]
We introduce a metric $\eta^{ab}$ in $g$ through $\eta^{ab} = \text{Tr}(T^a T^b)$, where $\text{Tr}$ denotes the (suitably normalized) trace in an arbitrary representation. Lie algebra indices are raised and lowered with $\eta^{ab}$ and its inverse $\eta_{ab}$.

It is straightforward to verify that the matrices $T^a$ in the $D_j$ representation are proportional to the Clebsch-Gordan coefficients for coupling of $D_j$ and $D_{adj}$ to yield $D_j$, or more precisely

$$\left(T^a\right)_{jm} = \sqrt{Q(j)} C^{adj}_{ab} j m m'.$$

(22)

Here $Q(j)$ denotes the eigenvalue of the quadratic Casimir operator $Q = T^a T_a$ in the $D_j$ representation.

To treat tensor fields we introduce a set of (local) coordinates $x^\mu$, $\mu = 1, \ldots, \dim G$ on $\mathcal{M}$ and define the vielbein $e_\mu^a(x)$ as

$$e_\mu^a(x) = \text{Tr}(T^a \partial_\mu xx^{-1}).$$

(23)

Here $x$ denotes an element of $G$, or equivalently a point in $\mathcal{M}$. The $G$ invariant metric on $\mathcal{M}$ is given by

$$g_{\mu\nu}(x) = \eta_{ab} e_\mu^a(x) e_\nu^b(x),$$

(24)

and has inverse $g^{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta^{ab}$, where $e^a_\mu(x)$ is the inverse of $e_\mu^a(x)$. The Haar measure could now be written as $d\mu(x) = d^D x \sqrt{\det g_{\mu\nu}}$. Using

$$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu = i f^{abc} e_{b\mu} e_{c\nu}$$

(25)

we may calculate the affine connection

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) = \frac{1}{2} e^a_\lambda (\partial_\mu e^a_\nu + \partial_\nu e^a_\mu)$$

(26)

and the covariant derivative

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda = \frac{i}{2} f^{abc} e_{b\mu} e_{c\nu}.$$  

(27)

Finally, we note that

$$e^{a\mu}(x) D_\mu V^j_{m\bar{m}}(x) = \sum_n (T^a_j)_{mn} V^j_{\bar{m}n}(x),$$

(28)

and that

$$D_\mu D^\mu V^j_{m\bar{m}}(x) = Q(j) V^j_{m\bar{m}}(x).$$

(29)

The vielbein $e_\mu^a(x)$ should transform as a covariant vector under the isometry (21), i.e. as

$$e_\mu^a(x) \mapsto e_{\mu'}^a(x') = \frac{\partial x'^\nu}{\partial x^\mu} e_\nu^a(x) = (V^{adj}(u))^a b e_{\mu'}^b(x),$$

(30)

where $(V^{adj}(u))^a b = \text{Tr}(uT^a u^{-1} T_b)$ is the representation matrix of $u$ in the adjoint representation. The vielbein $e_\mu^a(x)$ thus transforms as $(D_{adj}, D_{\text{triv}})$ under $G = G \otimes G$ acting as in (21). Equivalently, we could have worked with $\tilde{e}_\mu^a(x) = \text{Tr}(T^a x^{-1} \partial_\mu x)$, which transforms as $(D_{\text{triv}}, D_{adj})$. 

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A general tensorfield $A_{\mu_1...\mu_r}(x)$ may now be expanded as

$$A_{\mu_1...\mu_r}(x) = e_{\mu_1}^{a_1}(x) \cdots e_{\mu_r}^{a_r}(x) A_{a_1...a_r}(x).$$  \((31)\)

We already know that the scalar field $A_{a_1...a_r}(x)$ transforms as $D_0$ given by \((19)\), so $A_{\mu_1...\mu_r}(x)$ transforms in the representation

$$D_r = (D_{\text{adj}}, D_{\text{triv}})^{\otimes r} \otimes D_0 = \bigoplus_j \left( D_{\text{adj}}^{\otimes r} \otimes D_j, D_j \right).$$  \((32)\)

Despite its asymmetric appearance the formula \((32)\) is really symmetric under exchange of the two $G$ factors in $G$.

6. The spectrum of the SU(2) string theory

A conformal field theory may be interpreted as a string theory in the following way. The physical states of the string theory are in a one to one correspondence with the Virasoro primary states of conformal dimension $(h, \bar{h}) = (1, 1)$. The amplitude for scattering of such string states is computed by calculating the conformal field theory correlation function of the corresponding Virasoro primary fields and integrating it over all possible world-sheet configurations \([24]\). In general this implies an integration over the insertion points on the world-sheet as well as an integration over the space of world-sheet geometries. However, conformal invariance reduces the latter infinite dimensional integral to a sum over the number of handles (the genus) of the world-sheet and a finite dimensional integral over the space of conformally inequivalent geometries (the modulispaces) for each genus. Different genera correspond to different orders in the string coupling constant.

The algebraic structure of the SU(2) Wess-Zumino-Witten conformal field theory consists of two SU(2) level $k$ Kac-Moody algebras spanned by the current modes $J^a_n$ and $\bar{J}^a_n$, and two enveloping Virasoro algebras spanned by $L_n$ and $\bar{L}_n$ \([13]\). The commutation relations are

$$[J^a_n, J^b_m] = i f_{ab}^c J^c_{n+m} + \frac{k}{2} n \eta^{ab} \delta_{n+m,0}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0}$$

$$[L_n, J^a_m] = -m J^a_{n+m},$$

and an identical set for the $\bar{J}^a_n$ and $\bar{L}_n$ operators. The Virasoro operators are constructed as normal ordered bilinears in the Kac-Moody currents:

$$L_n = \frac{1}{k+2} \sum_{m \in \mathbb{Z}} : J^n_{n-m} J_{a m} : .$$  \((34)\)

The conformal anomaly is $c = 3k/(k+2)$, but, as we have already mentioned, the value of $c$ does not concern us for the moment.

All fields in the theory could be arranged into families consisting of an ancestor field, which is primary with respect to the extended conformal algebra \((33)\), and an infinite set of descendant fields. A primary field is characterized by the representation
in which it transforms under the isometry group $\mathcal{G} = SU(2) \times SU(2)$ of the target manifold. Not all representations are allowed, however, but only a subset of so called integrable ones \cite{16}.

A representation of $SU(2)$ is labeled by a non-negative integer or half integer spin $j$. These representations are self conjugate, and in the following $j$ and $\bar{j}$ will denote independent representations. The representation space of $D_j$ is spanned by the orthonormal states $|j,m>$. These representations are self conjugate, and in the following $j$ and $\bar{j}$ will denote independent representations. The representation space of $D_j$ is spanned by the orthonormal states $|j,m>$, where $m = -j, -j+1, \ldots, j$. The tensor product of two representations is decomposed as

$$D_{j_1} \otimes D_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D_j.$$  \hfill (35)

The integrable representations of the level $k$ $SU(2)$ Kac-Moody algebra are those with $j \leq k/2$ \cite{16}. They form a closed operator product algebra \cite{16} according to

$$D_{j_1} \times D_{j_2} = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2,k-j_1-j_2)} D_j.$$  \hfill (36)

The Hilbert space is built on groundstates $|j_0; m, \bar{m}>$, where $j_0 \leq k/2$, which are created by the primary fields acting on the vacuum state $|0>$. These groundstates are annihilated by the Kac-Moody currents with positive mode numbers, and transform in the $(D_{j_0}, D_{\bar{j}_0})$ representation under the $SU(2) \times SU(2)$ generated by the zero-modes. The Hilbert space is thus spanned by states of the form

$$|\Psi> = J^a_{-n_1} \ldots J^a_{-n_s} \bar{J}^{\bar{a}1}_{-\bar{n}_1} \ldots \bar{J}^{\bar{a}t}_{-\bar{n}_t} |j_0; m, \bar{m}>.$$  \hfill (37)

The state $|\Psi>$ is an eigenstate of the $L_0$ and $\bar{L}_0$ operators with eigenvalues

$$h = \frac{j_0(j_0+1)}{k+2} + \sum_s n_s, \quad \bar{h} = \frac{j_0(j_0+1)}{k+2} + \sum_t \bar{n}_t.$$  \hfill (38)

It is easy to determine the spectrum of physical states, i.e. the Virasoro primary states of conformal dimension $(h, \bar{h}) = (1, 1)$, from equation (38). The only solutions are

$$|j_0; m, \bar{m}>, \quad j_0(j_0+1) = k+2$$  \hfill (39)

and

$$J^a_{-1} \bar{J}^{\bar{a}}_{-1} |0; 0>.$$  \hfill (40)

We see that for a general integer level $k$, the first set of states is absent, since the spin $j_0$ has to be integer. (Half integer spins would require non-integer levels.) The corresponding Virasoro primary fields are a field which is primary with respect to the left and right Kac-Moody algebras

$$V_{m\bar{m}}^{j_0}(z, \bar{z})$$  \hfill (41)

and a product of a holomorphic and an anti-holomorphic Kac-Moody current

$$V^{ab}(z, \bar{z}) = J^a(z) \bar{J}^b(\bar{z}).$$  \hfill (42)
The physical states (39) and (40) transform in the \((j, \bar{j}) = (j_0, j_0)\) and \((j, \bar{j}) = (1, 1)\) representations respectively under \(\mathcal{G} = G \times G\).

7. Field theory on the \(SU(2)\) group manifold

Our object in this section is to construct a local quantum field theory on the \(SU(2)\) group manifold which reproduces the scattering amplitude of the Wess-Zumino-Witten model interpreted as a string theory.

The first question to settle is what the field content of our theory should be. The quantum numbers of the primary fields suggest that we should introduce a scalar field \(\sigma(x)\) and a rank two tensor field \(A_{\mu\nu}(x)\). It is convenient to decompose the tensor field into its antisymmetric part \(b_{\mu\nu}(x)\), its symmetric trace-less part \(h_{\mu\nu}(x)\) and its trace \(\phi(x)\). This is in analogy with the bosonic string in flat space, where we find a tachyon \(\sigma\), an antisymmetric tensor \(b_{\mu\nu}\), a graviton \(h_{\mu\nu}\) and a dilaton \(\phi\). In flat space string theory we also get an infinite tower of massive states which have no counterparts in the \(SU(2)\) string. It is easy to see that this is related to the target space being Euclidean rather than Minkowskian in signature.

The next step is to determine the representation content of each of these fields under the target space isometry group \(\mathcal{G} = SU(2) \times SU(2)\). As was explained in section 5 we rewrite \(A_{\mu\nu}(x)\) as

\[
A_{\mu\nu}(x) = e^{a}_{\mu}(x)e^{b}_{\nu}(x)A_{ab}(x). \tag{43}
\]

A scalar field such as \(\sigma(x)\), \(\phi(x)\) or \(A_{ab}(x)\) transforms under \(\mathcal{G} = SU(2) \times SU(2)\) as

\[
D_{\sigma} = D_{\phi} = \bigoplus_{j} (D_{j}, D_{j}). \tag{44}
\]

The vielbein \(e^{\alpha}_{\mu}(x)\) transforms as \((D_{adj}, D_{triv}) = (D_{1}, D_{0})\). We see that the antisymmetric, symmetric traceless and trace parts of the product of two vielbeins transform as \((D_{1}, D_{0})\), \((D_{2}, D_{0})\) and \((D_{0}, D_{0})\) respectively. The representation content of \(b_{\mu\nu}(x)\) and \(h_{\mu\nu}(x)\) is thus

\[
D_{b} = \bigoplus_{j} (D_{1} \otimes D_{j}, D_{j}) = \bigoplus_{|j-j| \leq 1} (D_{j}, D_{j})
\]

\[
D_{h} = \bigoplus_{j} (D_{2} \otimes D_{j}, D_{j}) = \bigoplus_{|j-j| \leq 2} (D_{j}, D_{j}). \tag{45}
\]

(The prime on the summation symbols indicates that a few of the lowest lying representations are missing.)

We should now find all possible kinetic terms and interaction terms for these fields and determine how the different representations couple together. Let us begin with the interaction terms. For the moment we will only consider tree level contributions to scattering of external \(\sigma(x)\) fields, so we should write down possible three point couplings between two such on-shell fields and a third arbitrary field which may be off shell. By charge conjugation symmetry there is no possible \(b\sigma^{2}\) interaction. The possible \(\phi\sigma^{2}\) and \(\sigma^{3}\) terms without any derivatives are just

\[
S_{\phi\sigma\sigma}^{\text{int}} = \int d^{3}x \lambda_{\phi}\sigma^{2}\phi(x) \tag{46}
\]
and

\[ S_{\sigma\sigma}^{\text{int}} = \int d^3 x \, \lambda_{\sigma} \sigma^3(x) \]  

respectively. The most general \( h\sigma^2 \) term with two derivatives is

\[ S_{h\sigma}^{\text{int}} = \int d^3 x \, \left( \lambda_1 h^{\mu\nu}(x) D_\mu \sigma(x) D_\nu \sigma(x) + \lambda_2 h^{\mu\nu}(x) \sigma(x) D_\mu D_\nu \sigma(x) \right) \]

\[ = \int d^3 x \, \left( \lambda_1 h^{ab}(x) D_a \sigma(x) D_b \sigma(x) + \lambda_2 h^{ab}(x) \sigma(x) D_a D_b \sigma(x) \right). \]  

We next turn to the kinetic terms. For \( \sigma \) and \( \phi \), the only possible terms which contain at most two derivatives are

\[ S_{\sigma\sigma}^{\text{kin}} = \int d^3 x \, \sigma(x)(D^2 D_\mu - m_\sigma^2) \sigma(x) = \int d^3 x \, \sigma(x)(D^2 D_a - m_\sigma^2) \sigma(x) \]  

and

\[ S_{\phi\phi}^{\text{kin}} = \int d^3 x \, \phi(x)(D^2 D_\mu - m_\phi^2) \phi(x) = \int d^3 x \, \phi(x)(D^2 D_a - m_\phi^2) \phi(x) \]  

respectively. The constants \( m_\sigma \) and \( m_\phi \) are determined by the requirement that the propagators should diverge on shell.

For \( h^{\mu\nu} \) the most general kinetic term is

\[ S_{hh}^{\text{kin}} = \int d^3 x \, \left( h^{\mu\nu}(x) D^2 D_\rho D_\rho h^{\mu\nu}(x) + \kappa D_\mu h^{\mu\nu}(x) D^2 h_{\rho\nu}(x) - m_h^2 h^{\mu\nu}(x) h_{\mu\nu}(x) \right) \]

\[ = \int d^3 x \, \left( h^{ab}(x) D^2 D_c h_{ab}(x) + 2i h^{ab}(x) f^{ce}_a D_a h_{cb}(x) \right. \]

\[ + \kappa D_a h^{ab}(x) D_c h^{c\,b}(x) - (m_h^2 + \frac{3}{4}) h^{ab}(x) h_{ab}(x) \right). \]  

There may also be a kinetic term of \( h\phi \) type:

\[ S_{h\phi}^{\text{kin}} = \int d^3 x \, \rho h^{\mu\nu}(x) D_\mu D_\nu \phi(x) = \int d^3 x \, \rho h^{ab}(x) D_a D_b \phi(x). \]  

The action is invariant under \( G = SU(2) \times SU(2) \). To evaluate the couplings between different representations in the various terms, we need the following rules:

1. An integration of the product of three scalar functions yields a \( 3j m \) symbol for each of the two \( SU(2) \) factors. We depict this as a vertex with three incoming lines and a factor 1:

\[ \int d^3 x \, \sigma_1(x) \sigma_2(x) \sigma_3(x) = \begin{array}{c} \text{j}_1 \\ \text{j}_2 \end{array} \rightarrow \text{j}_3 \]  

A special case is when one of the functions is the identity 1, which only carries the trivial representation \( D_0 \). The representations from the remaining two functions then couple together to yield the trivial representation.

2. The derivative \( D_a \) acts on a scalar function as a multiplication with the Lie algebra generator \( T_a \) from the left. The matrix \( T_a \) in the \( D_j \) representation,
in its turn, is proportional to $\sqrt{j(j+1)}$ times the $3jm$ symbol for coupling of $D_j$ and the adjoint representation $D_1$ to yield $D_j$. The right $SU(2)$ factor is unaffected. A graphic representation is

$$D_\sigma(x) = \begin{array}{c} j \\
\sqrt{j(j+1)} \\
1 \end{array}$$ (54)

3. The d’Alembertian $D_\sigma D^a$ acting on a scalar function amounts to multiplication by the eigenvalue $j(j+1)$ of the quadratic Casimir operator.

$$D^a D_\sigma(x) = \begin{array}{c} j \\\n\sqrt{j(j+1)} \\\n1 \end{array} \begin{array}{c} j \\\n\sqrt{j(j+1)} \\\n1 \end{array}$$ (55)

4. Contracting three Lie algebra indices with the structure constants yields the $3jm$ symbol for coupling three adjoint representations $D_1$ together:

$$f^{abc} = \begin{array}{c} 1 \\
\sqrt{j(j+1)} \\\n1 \end{array}$$ (56)

5. A subdiagram may be rearranged by using the $6j$ symbols as in [13].

The right hand $SU(2)$ factor is trivial in all the terms we have considered. It is just a two- or three-point vertex for the kinetic and interaction terms respectively with no additional factors.

For the left hand factors in the kinetic terms we get

$$S_{\phi\phi}^{kin} \sim K_{\phi\phi}(j,j) \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array}$$

$$S_{\sigma\sigma}^{kin} \sim K_{\sigma\sigma}(j,j) \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array}$$

$$S_{h\phi}^{kin} \sim j(j+1) \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array} \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array} \sim K_{h\phi}(j,j) \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array}$$

$$S_{hh}^{kin} \sim \left( j(j+1) - m_h^2 - \frac{3}{4} \right) \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array}$$

$$+ 2i \sqrt{j(j+1)} \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array} \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array} + \kappa j(j+1) \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array} \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array}$$

$$\sim \sum_{j=j-2}^{j+2} K_{hh}(j,j) \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array} \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array} \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array} \begin{array}{c} j \\
\sqrt{j(j+1)} \\\n1 \end{array} ,$$

where the inverse propagators are given by
\[ K_{\sigma\sigma}(j, j) = j(j + 1) - m_{\sigma}^2 \]
\[ K_{\phi\phi}(j, j) = j(j + 1) - m_{\phi}^2 \]
\[ K_{h\phi}(j, j) = \rho(3 - 4j(j + 1)) \sqrt{\frac{j(j + 1)}{(2j + 3)(2j - 1)}} \]
\[ K_{hh}(j, j) = \sqrt{\frac{2j + 1}{2j + 1}} \left( j(j + 1) + j(j + 1) - 2m_{h}^2 - \frac{15}{2} \right) + \kappa(j + j + 3)(j + j + 1)(j - j + 2)(j - j + 2) \]

For the interaction terms, we are only interested in the case where the external \( \sigma(x) \) fields are on shell, i.e., we need only determine the couplings for their \((D_{j_0}, D_{j_0'})\) components, where \( j_0(j_0 + 1) = k + 2 \). Furthermore, the \( j_0 \) (or equivalently the \( k \)) dependence may be absorbed in a renormalization of the different coupling constants. We thus get

\[ S_{\sigma\sigma}^{\text{int}} \sim V_{\sigma\sigma}(j, j) \]
\[ S_{\phi\sigma}^{\text{int}} \sim V_{\phi\sigma}(j, j) \]
\[ S_{h\sigma}^{\text{int}} \sim \lambda_1 \]
\[ \sim \sum_{j=j-2}^{j+2} V_{h\sigma}(j, j) \]

where the vertex factors are given by

\[ V_{\sigma\sigma}(j, j) = \delta_{j,j}\lambda_{\sigma} \]
\[ V_{\phi\sigma}(j, j) = \delta_{j,j}\lambda_{\phi} \]
\[ V_{h\sigma}(j, j) = \delta_{j,j} \sqrt{\frac{(2j + 2)(2j)}{(2j + 3)(2j - 1)}} \left( (3\lambda_{2} - \lambda_{1})j(j + 1) - 4(\lambda_{1} + \lambda_{2})j_0(j_0 + 1) - 3\lambda_{2} \right) \]
\[ + \delta_{j-1,j} \sqrt{\frac{3(2j)(2j + 1)}{2(2j + 3)(2j - 1)}} \left( (3\lambda_{2} - \lambda_{1})j(j + 1) - 4(\lambda_{1} + \lambda_{2})j_0(j_0 + 1) - 3\lambda_{2} \right) \]
\[ - \delta_{j+1,j} \sqrt{\frac{3((2j + 2)(2j_0 - j)(2j_0 + j + 2)}{2(2j + 4)(2j + 1)(2j)}} \lambda_{2} \]
\[ + \delta_{j-2,j} \sqrt{\frac{6j(j - 1)(2j_0 - j)(2j_0 - j)(2j_0 + j)(2j_0 + j)}{(2j + 1)(2j - 1)}} \left( \lambda_{1} + \lambda_{2} \right) \]
Duality of string amplitudes in a curved background

\[ + \delta_{j+2,j} \sqrt{\frac{6(j+2)(j+1)(2j_0-j)(2j_0-j-1)(2j_0+j+3)(2j_0+j+2)}{(2j+3)(2j+1)}} (\lambda_1 + \lambda_2). \]

The different coupling constants have been renormalized as compared to their original definitions in the action terms. The asymmetry between the left and the right \(SU(2)\) factor is due to our normalization conventions and will disappear when we put the propagators and the vertices together. We get the general form of the \(s\)-channel contribution to four-\(\sigma\) scattering at tree level as

\[
\Gamma^{(s)}(j, \bar{j}) = V^{\sigma\sigma}(j, \bar{j}) K^{-1}_{\sigma\sigma}(j, \bar{j}) V^{\sigma\sigma}(j, \bar{j}) + V^{\phi\sigma}(j, \bar{j}) K^{-1}_{\phi\phi}(j, \bar{j}) V^{\phi\sigma}(j, \bar{j})
\]

As we have previously explained, we should add the \(t\)- and \(u\)-channel contributions.

8. Calculation of tensor particle scattering amplitudes

We should now compute the amplitudes for scattering of the physical string states we found in section 3, and compare them to the field theory amplitudes of the previous section.

To each external physical state corresponds a dimension \((h, \bar{h}) = (1, 1)\) vertex operator, as we have already described. To calculate a string scattering amplitude we integrate the conformal field theory correlation function of the corresponding vertex operators over their insertion points on the world-sheet. The infinite volume of the global conformal group will lead to a divergence, however, which we remove by fixing three of the insertion points. If the genus of the world-sheet is greater than zero, we should also integrate over the appropriate moduli space of conformally inequivalent Riemann surfaces. For genus zero (tree level) we get the \(n\)-point amplitude as

\[
\Gamma = \int \prod_{i=1}^{n} d^2 z_i \delta^2(z_A - z_A^0) \delta^2(z_B - z_B^0) \delta^2(z_B - z_B^0) |(z_A - z_B)(z_B - z_C)(z_C - z_A)|^2 < V_1(z_1, \bar{z}_1) \ldots V_n(z_n, \bar{z}_n) >.
\]

The extra factor is a Jacobian, which arises upon fixing the values of \(z_A, z_B\) and \(z_C\). Note that the conformal dimension of \(d^2 z_i\) is \((h, \bar{h}) = (-1, -1)\), so the amplitude is conformally invariant.

Our vertex operators are the Kac-Moody primary fields (11) and the Kac-Moody current bilinears (12). Correlation functions involving the latter fields are related to correlation functions of Kac-Moody primaries by the current algebra Ward identity (15)

\[
< J^\alpha(z) V_1(z_1, \bar{z}_1) \ldots V_n(z_n, \bar{z}_n) > = \sum_{i=1}^{n} \frac{T_i^\alpha}{z - z_i} < V_1(z_1, \bar{z}_1) \ldots V_n(z_n, \bar{z}_n) >
\]

and its anti-holomorphic counterpart. As a simple example of how this works, we consider scattering of three scalar particles and one tensor particle. According to our
previous discussions, such an amplitude is given by
\[
\Gamma_{m_1 \bar{m}_1} = |(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|^2 \int d^2 z < J^a(z)J^b(\bar{z}) V_{m_1 \bar{m}_1}(z_1, \bar{z}_1)V_{m_2 \bar{m}_2}(z_2, \bar{z}_2)V_{m_3 \bar{m}_3}(z_3, \bar{z}_3)>
\]
\[
= |(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|^2 \int d^2 z \sum_{i=1}^3 T_i^a T_i^b(z - z_i)^{-1}(\bar{z} - \bar{z}_i)^{-1}
\]
\[
= \int d^2 z \sum_{i=1}^3 T_i^a T_i^b(z - z_i)^{-1}(\bar{z} - \bar{z}_i)^{-1} \left( \begin{array}{ccc} j_0 & j_0 & j_0 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} j_0 & j_0 & j_0 \\ \bar{m}_1 & \bar{m}_2 & \bar{m}_3 \end{array} \right),
\]
where we have used the $3jm$ symbols in the last step. We now put $(z_1, z_2, z_3) = (0, 1, \infty)$, and use the property that the $T^a$ and $T^b$ matrices are the Clebsch-Gordan coefficients for coupling spin $j_0$ and spin 1 to yield spin $j_0$ for the left and right $SU(2)$ factor respectively. Our amplitude could then symbolically be written as
\[
\Gamma = \int d^2 z \left[ \begin{array}{ccc} z^{-1} & 1 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right] + \left( z - 1 \right)^{-1} \left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right] \times \left[ \begin{array}{ccc} z^{-1} & 1 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right] + \left( \bar{z} - 1 \right)^{-1} \left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right],
\]
where the first and second bracket corresponds to the left and right $SU(2)$ factors respectively.

The regularization of integrals like those above will be discussed in section 10. We require that
\[
\int d^2 z \ z^{-1} \bar{z}^{-1} = \int d^2 z \ (z - 1)^{-1}(\bar{z} - 1)^{-1}
\]
\[
= 2 \int d^2 z \ z^{-1}(\bar{z} - 1)^{-1} = 2 \int d^2 z \ (z - 1)^{-1}\bar{z}^{-1} = 2\delta,
\]
where the constant $\delta$ diverges as the regulator is turned off. We thus get
\[
\Gamma \sim \left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right] \times \left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right] + \left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right] \times \left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right] + \frac{1}{2} \left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right] \times \left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right].
\]

An evaluation of the $6j$ symbols for the coupling of three spin $j_0$ and one spin 1 yields (see equation [65])
\[
\left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right] = \sum_{j'=j_0-1}^{j_0+1} c_{j_0,j'} \left[ \begin{array}{ccc} 1 & j_0 & j_0 \\ j_0 & j_0 & j_0 \end{array} \right]
\]
where
\[
\left( \begin{array}{ccc} c_{j_0-1,j_0-1} & c_{j_0-1,j_0} & c_{j_0-1,j_0+1} \\ c_{j_0,j_0-1} & c_{j_0,j_0} & c_{j_0,j_0+1} \\ c_{j_0+1,j_0-1} & c_{j_0+1,j_0} & c_{j_0+1,j_0+1} \end{array} \right)
\]

\[
(69)\]
\[
\begin{pmatrix}
    j_0 + 1 & \sqrt{(3j_0 + 1)(2j_0 + 1)} & \sqrt{(3j_0 + 2)(3j_0 + 1)} \\
    \sqrt{(3j_0 + 1)(2j_0 + 1)} & 2j_0 + 1 & -\sqrt{(3j_0 + 2)(2j_0 + 1)} \\
    \sqrt{(3j_0 + 2)(3j_0 + 1)} & -\sqrt{(3j_0 + 2)(2j_0 + 1)} & j_0
\end{pmatrix}
\]

If we also introduce
\[
\begin{pmatrix}
    d_{j_{0-1},j_0} \\
    d_{j_0,j_0} \\
    d_{j_0+1,j_0}
\end{pmatrix}
= \begin{pmatrix}
    1 & 0 & 0 \\
    0 & -1 & 0 \\
    0 & 0 & 1
\end{pmatrix},
\]
we may express our scattering amplitude in the s-channel basis as
\[
\Gamma_{jj} \sim c_{j_0,j}c_{j_0,j} + d_{j_0,j}d_{j_0,j} + \frac{1}{2}c_{j_0,j}d_{j_0,j} + \frac{1}{2}d_{j_0,j}c_{j_0,j},
\]

or in matrix form
\[
\Gamma \sim \begin{pmatrix}
    3j_0 + 1 & 0 & -\sqrt{(3j_0 + 1)(3j_0 + 2)} \\
    0 & 3(2j_0 + 1) & 0 \\
    -\sqrt{(3j_0 + 1)(3j_0 + 2)} & 0 & 3j_0 + 2
\end{pmatrix}.
\]

In a quantum field theory we expect a pole at \( j = \bar{j} = j_0 \) in each channel due to exchange of a scalar field quanta. The divergent part of the s-channel contribution should therefore be proportional to
\[
\Gamma^{(s)} = \begin{pmatrix}
    0 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 0
\end{pmatrix}.
\]

To include the contributions from the other channels, we should symmetrize the s-channel contribution with respect to the permutation group of the external legs. The total amplitude is thus
\[
\Gamma \sim \sum_{p \in P} p\Gamma^{(s)}p^T,
\]
where \( P \) is the matrix group generated by the matrices \( c \) and \( d \). An explicit evaluation of \( (74) \) shows that this amplitude is in fact proportional to \( (72) \). The field theoretical amplitude should also include finite contributions when the exchanged quanta is not on shell. Since our string theory amplitude only contains divergent parts, we must conclude that the finite parts from exchange of scalar and tensor particles cancel. Observe that the kinematics are such, that the latter particles cannot be on shell, so we get no further infinite contributions.

9. Calculation of scalar particle correlation functions

Correlation functions of Kac-Moody primary fields are in principle determined by the Knizhnik-Zamolodchikov equations
\[
\left( \frac{\partial}{\partial z_i} - \frac{2}{k + 2} \sum_{j \neq i} T_i^a T_{aj}(z_i - z_j)^{-1} \right) < V_1(z_1, \bar{z}_1) \ldots V_n(z_n, \bar{z}_n) >
\]
and their anti-holomorphic counterparts when supplemented by the requirement that the correlation functions be well-defined on the punctured Riemann sphere.

We will only consider four-point functions, where these equations amount to ordinary differential equations, since global conformal invariance allows us to fix three of the insertion points \[25\] \[26\]. Indeed, a four-point function takes the form

\[
<V_1(z_1, z_\bar{1}) \ldots V_4(z_4, z_\bar{4})> = |z_1 - z_4|^{-4}|z_2 - z_3|^{-4}f(\eta, \bar{\eta}).
\]  

(76)

Here the crossratio \(\eta\) is defined as

\[
\eta = (z_1 - z_2)(z_3 - z_4)(z_3 - z_2)^{-1}(z_1 - z_4)^{-1},
\]

(77)

and the functions \(f(\eta, \bar{\eta})\) fulfil

\[
\frac{\partial}{\partial \eta} f(\eta, \bar{\eta}) = \left( \frac{A}{\eta} + \frac{B}{1 - \eta} \right) f(\eta, \bar{\eta}),
\]

(78)

where \(A = -2(k + 2)^{-1}T_1 T_2\) and \(B = 2(k + 2)^{-1}T_1 T_3\), and a similar equation for the \(\bar{\eta}\)-dependence. This is a linear matrix differential equation with (regular) singular points in \(\eta = 0, 1, \infty\). (See for example \[27\].)

The functions \(f(\eta, \bar{\eta})\) transform in the \(D_{j_0} \otimes \ldots \otimes D_{j_0}\) representations under the left and right \(SU(2)\) groups. The invariance of the correlation function tells us that the components of \(f(\eta, \bar{\eta})\) that transform non-trivially vanish. The remaining components may conveniently be labeled by the spins \(j\) and \(\bar{j}\), ranging from 0 to \(2j_0\), of the left and right \(SU(2)\) representations which are exchanged in one the “channels”, for example \(1, 2 \rightarrow 3, 4\). In this basis the \(A\) matrix is diagonal with elements

\[
(A)_{jj} = \frac{j(j + 1)}{j_0(j_0 + 1)} - 2
\]

(79)

and \(B\) is tri-diagonal \[28\] with diagonal elements

\[
(B)_{jj} = \frac{j(j + 1)}{2j_0(j_0 + 1)}
\]

(80)

and off-diagonal elements

\[
(B)_{j-1,j} = (B)_{jj-1} = \frac{j(j^2 - (2j_0 + 1)^2)}{2j_0(j_0 + 1)\sqrt{4j^2 - 1}}.
\]

(81)

If we introduce fundamental solutions \(X^{(i)}\) to (82) of the form

\[
X^{(i)}_j(\eta) = \eta^{(A)_{ii}}(\delta^i_j + O(\eta^{j-i})).
\]

(82)

we may write the functions \(f_{jj}\) as

\[
f_{jj}(\eta, \bar{\eta}) = \sum_{i, \bar{i}} c_{i\bar{i}} X^{(i)}_j(\eta) X^{(\bar{i})}_j(\bar{\eta}).
\]

(83)
The coefficients $c_{i\bar{i}}$ are determined by the requirement that the correlation functions be well-defined when we impose that $\eta$ is the complex conjugate of $\eta$. To get a well-defined correlation function as $\eta \to 0$ we choose the matrix $c_{i\bar{i}}$ diagonal, i.e. $c_{i\bar{i}} = \delta_{i\bar{i}}$. (There might also be non-diagonal solutions but they need not concern us here.) If one of the fundamental solutions $X^{(i)}(\eta)$ is continued analytically around one of the singular points $\eta = 0, 1, \infty$ it is transformed into a linear combination of solutions:

$$X^{(i)}(\eta) \mapsto \sum_n \alpha_inX^{(n)}(\eta). \quad (84)$$

The coefficients $\alpha_i$ should now be chosen so that the diagonal form of $c_{i\bar{i}}$ is preserved, i.e. we must require that

$$\sum_k c_{k\bar{k}}\alpha_k\alpha_{km} = 0, \quad n \neq m. \quad (85)$$

This is quite different from string theory in flat space, where the correlation functions are simply a product of a holomorphic and an anti-holomorphic factor.

It is easy to see, that for analytic continuation around $\eta = 0$, the monodromy matrix $\alpha_{i\bar{i}}$ is diagonal with elements $\alpha_{i\bar{i}} = \exp(2\pi i(A)_{i,i})$. For analytic continuation around $\eta = 1$ or $\eta = \infty$ the problem is more difficult, but nevertheless tractable. The fundamental solutions (82) may be expressed in terms of multiple integrals of Euler type [28], which are suitable for analytic continuation [29], and the coefficients $\alpha_{i\bar{i}}$ may be calculated.

The correlation functions may thus be expressed in terms of multiple (2$j_0$-tuple) integrals [28][30]. This form is not well suited for integrating over the insertion points on the world-sheet, however. We have not found any way to perform such integrals analytically, and probably one has to resort to numerical methods. We expect to come back to this issue shortly.

10. The k=4 case

The computation of the $c_{i\bar{i}}$ coefficients is greatly facilitated in the case that $k-j_3-j_4 = j_4-j_3$, so that the operator product expansion of primary fields with spins $j_3$ and $j_4$ only contains fields which are descendant from spin $j_4-j_3$. This means that $c_{i\bar{i}} = 0$ except for $i = \bar{i} = j_4-j_3$.

In particular, if $j_1 = j_2 = j_3 = j_4 = j_0$ and $k = 2j_0$, we have

$$(A)_{jj} = \frac{j(j+1)}{2(j_0+1)} - j_0, \quad j = 0,1,\ldots,2j_0 \quad (86)$$

$$(B)_{jj} = \frac{j(j+1)}{4(j_0+1)}$$

and

$$(B)_{j,j-1} = (B)_{j-1,j} = \frac{j((2j_0+1)^2 - j^2)}{4(j_0+1)\sqrt{4j^2-1}}, \quad j = 1,2,\ldots,2j_0. \quad (87)$$

The relevant solution to the Knizhnik-Zamolodchikov equation is

$$X_j(\eta) = \sqrt{2j+1} \frac{(2j_0-j)!}{j!(2j_0)!} \sum_{s=0}^{2j_0-j} \frac{(j+s)!(2j_0-s)!}{(2j_0-j-s)!s!} \eta^{j-j_0}(1-\eta)^{s-j_0}. \quad (88)$$
In string theory we need $h = 1$ for our primary fields, so according to equation (88) we must take $k = j_0(j_0 + 1) - 2$. To be able to use the simplified solution procedure described in the last paragraph we must also have $k = 2j_0$, which yields $j_0 = 2$ and $k = 4$. The solution (88) is in this case

$$X_0(\eta) = \sqrt{1} \eta^{-2} \left( (1 - \eta)^{-2} + (1 - \eta)^{-1} + 1 + (1 - \eta) + (1 - \eta)^2 \right)$$

$$X_1(\eta) = \sqrt{3} \eta^{-1} \left( (1 - \eta)^{-2} + \frac{3}{2} (1 - \eta)^{-1} + \frac{3}{2} (1 - \eta) \right)$$

$$X_2(\eta) = \sqrt{5} \eta^0 \left( (1 - \eta)^{-2} + \frac{3}{2} (1 - \eta)^{-1} + 1 \right)$$

$$X_3(\eta) = \sqrt{7} \eta^1 \left( (1 - \eta)^{-2} + (1 - \eta)^{-1} \right)$$

$$X_4(\eta) = \sqrt{9} \eta^2 \left( 1 - \eta \right)^{-2}.$$

It will prove convenient to perform a partial fraction decomposition of these functions and write them as

$$X_0(\eta) = 1 + 5\eta^{-2} + 3(1 - \eta)^{-1} + (1 - \eta)^{-2}$$

$$X_1(\eta) = \sqrt{3}(-1 + 5\eta^{-1} + \frac{5}{2}(1 - \eta)^{-1} + (1 - \eta)^{-2})$$

$$X_2(\eta) = \sqrt{5}(1 + \frac{3}{2}(1 - \eta)^{-1} + (1 - \eta)^{-2})$$

$$X_3(\eta) = \sqrt{7}(-1 + (1 - \eta)^{-2})$$

$$X_4(\eta) = 3(1 - 2(1 - \eta)^{-1} + (1 - \eta)^{-2}).$$

The complete correlation function follows from (96):

$$< V_{m_1 \ldots m_4}(z_1, \bar{z}_1) \ldots V_{m_1 \ldots m_4}(z_4, \bar{z}_4) > = \sum_{jj} P_{m_1 \ldots m_4}^j P_{\bar{m}_1 \ldots \bar{m}_4}^{j \dagger} |(z_1 - z_4)(z_2 - z_3)|^{-4} X_j(\eta) X_j(\bar{\eta}),$$

where $P_{m_1 \ldots m_4}^j$ projects on the invariant in $(D_2)^{\otimes 4}$ which has $D_j$ as intermediate representation in the $s$-channel, and $\eta$ is given by (74).

Integrating the correlation function (96) with the measure in equation (62), choosing $(\bar{z}_1^0, z_2^0, \bar{z}_4^0) = (\infty, 1, 0)$ and changing the integration variable to $\eta$ as defined in (77), we get the amplitude

$$\Gamma_{m_1 \ldots m_4 \bar{m}_1 \ldots \bar{m}_4} = \sum_{jj} P_{m_1 \ldots m_4}^j P_{\bar{m}_1 \ldots \bar{m}_4}^{j \dagger} \Gamma_{jj},$$

where the matrix $\Gamma_{jj}$ is given by

$$\Gamma_{jj} = \int \frac{d^2 \eta}{\pi} X_j(\eta) X_j(\bar{\eta}).$$

We note that, formally, this amplitude is invariant under permutation of the $s$-, $t$- and $u$-channels. Namely, such transformations correspond to replacing

$$\Gamma_{jj} \rightarrow \Gamma'_{jj} = (MTM^T)_{jj}.$$
The matrix $M$ is given by \((\mathbf{13})\), and belongs to the set of matrices that constitute the five-dimensional representation of the permutation group of three elements. This group is generated by the matrices

\begin{equation}
\begin{pmatrix}
\frac{1}{5} & \frac{\sqrt{2}}{5} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{5} & \frac{3}{5} \\
\frac{\sqrt{3}}{5} & \frac{1}{2} & \frac{\sqrt{2}}{2\sqrt{3}} & 0 & -\frac{2\sqrt{3}}{5} \\
\frac{\sqrt{2}}{2\sqrt{3}} & -\frac{3}{11} & -\frac{4}{\sqrt{33}} & 1 & -\frac{6}{\sqrt{5}} \\
\frac{1}{2} & \frac{\sqrt{3}}{5} & \frac{1}{2} & -\frac{3}{10\sqrt{7}} & 1 \\
\frac{3}{5} & -\frac{2\sqrt{3}}{5} & \frac{6}{7\sqrt{5}} & -\frac{3}{10\sqrt{7}} & \frac{1}{70} \\
\end{pmatrix},
\end{equation}

\((95)\)

It is straightforward to verify, by changing the integration variable as

\begin{equation}
\eta \rightarrow \eta' = \eta, 1 - \eta, \eta - 1, \eta, \frac{1}{\eta - 1}, \frac{1}{\eta},
\end{equation}

\((96)\)

that $\Gamma_{jj}$ is invariant under \((94)\). Furthermore, by replacing

\begin{equation}
\eta \rightarrow \bar{\eta},
\end{equation}

\((97)\)

we see that

\begin{equation}
\Gamma_{jj} = \Gamma_{jj}.
\end{equation}

The integrals \((93)\) are all divergent and need to be regularized. In doing this we certainly wish to respect the symmetries \((94)\) and \((98)\). If we introduce the (possibly infinite) constants

\begin{align*}
\alpha &= \int \frac{d^2 \eta}{\pi} \\
\beta &= \int \frac{d^2 \eta}{\pi} \eta^{-1} \\
\gamma &= \int \frac{d^2 \eta}{\pi} \eta^{-2} \\
\delta &= \frac{1}{2} \int \frac{d^2 \eta}{\pi} \eta^{-1} \bar{\eta}^{-1},
\end{align*}

\((99)\)

we may calculate all other integrals we need by changing integration variable as in \((96)\) and \((97)\). The result for the amplitude $\Gamma_{jj}$ is

\begin{align*}
\Gamma_{00} &= 27\alpha - 48\beta + 22\gamma + 18\delta \\
\Gamma_{11} &= 6\alpha - 90\beta - 6\gamma + \frac{225}{2}\delta \\
\Gamma_{22} &= 10\alpha + 30\beta + 10\gamma + \frac{15}{2}\delta \\
\Gamma_{33} &= 14\alpha - 14\gamma \\
\Gamma_{44} &= 18\alpha - 72\beta + 18\gamma + 72\delta \\
\Gamma_{20} = \Gamma_{02} &= 2\sqrt{5}\alpha - 6\sqrt{5}\beta + 12\sqrt{5}\gamma + 9\sqrt{5}\delta \\
\Gamma_{40} = \Gamma_{04} &= 6\alpha + 66\beta + 36\gamma - 36\delta \\
\Gamma_{31} = \Gamma_{13} &= 2\sqrt{21}\alpha - 15\sqrt{21}\beta - 2\sqrt{21}\gamma \\
\Gamma_{42} = \Gamma_{24} &= 6\sqrt{5}\alpha - 3\sqrt{5}\beta + 6\sqrt{5}\gamma - 18\sqrt{5}\delta,
\end{align*}

\((100)\)

with all other components vanishing.
To regularize the integrals (99) we modify the integration measure as

$$\int \frac{d^2 \eta}{\pi} \rightarrow \int \frac{d^2 \eta}{\pi} \lambda(\eta),$$

(101)

where $\lambda(\eta)$ should be invariant under (96) and (97), and go to 1 as the regulator is turned off. We will propose two different choices for $\lambda(\eta)$. The first is

$$\lambda(\eta) = \frac{1}{6}(|\eta|^{2\epsilon_1}|1 - \eta|^{2\epsilon_2} + \text{five terms with } \eta \text{ replaced as in (96)}).$$

(102)

The integrals could now be calculated using the formula [31]

$$\int \frac{d^2 \eta}{\pi} |\eta|^{\alpha} |1 - \eta|^{\beta n}(1 - \eta)^m$$

$$= (-1)^{n+m} \frac{\Gamma(-\frac{\alpha}{2} - \frac{n}{2})\Gamma(1 + n + \frac{\alpha}{2})\Gamma(1 + m + \frac{\beta}{2})}{\Gamma(-\frac{\alpha}{2})\Gamma(-\frac{\beta}{2})\Gamma(2 + n + m + \frac{\alpha}{2} + \frac{\beta}{2})},$$

(103)

which converges for $\text{Re}(\alpha + \beta + n + m + 2) < 0$, $\text{Re}(\alpha + n) > -2$, $\text{Re}(\beta + m) > -2$, and is to be understood in the sense of analytic continuation elsewhere. We get

$$\alpha = -\frac{1}{12} \frac{(\epsilon_1 - \epsilon_2)^2}{\epsilon_1 + \epsilon_2} + \frac{1}{3} \frac{\epsilon_2^2}{\epsilon_1} + \frac{1}{3} \frac{\epsilon_1^2}{\epsilon_2},$$

$$\beta = \frac{1}{2},$$

$$\gamma = 0,$$

$$\delta = \frac{1}{3} \frac{1}{\epsilon_1} + \frac{1}{3} \frac{1}{\epsilon_2} - \frac{1}{3} \frac{1}{\epsilon_1 + \epsilon_2}$$

(104)

modulo terms which vanish as $(\epsilon_1, \epsilon_2) \rightarrow (0, 0)$. If we furthermore put $\epsilon_1 = \epsilon_2 = \epsilon$ the result is

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 0, \quad \delta = \frac{1}{2} \frac{1}{\epsilon}.$$

(105)

Our second regulator is

$$\lambda(\eta) = \theta(|\eta| - \epsilon) \times \text{(five factors with } \eta \text{ replaced as in (96))},$$

(106)

where $\theta$ denotes the step function. Now we get

$$\alpha = \frac{1}{\epsilon_2} - \frac{2}{\pi \epsilon}, \quad \beta = \frac{1}{2}, \quad \gamma = 0, \quad \delta = -4 \ln \epsilon$$

(107)

modulo terms which vanish as $\epsilon \rightarrow 0$. This latter regularization will prove less useful, though, since it fails to eliminate the divergence of $\alpha$.

We should now compare the string amplitudes found in this section with the field theory results of section 7. From our discussions in section 7 we expect the $s$-channel contribution from exchange of the scalar particle to be

$$\langle \Gamma_s^{(s)} \rangle_{j \bar{j}} \sim \delta_{j \bar{j}} (j(j + 1) - 6)^{-1},$$

(108)
whereas the contribution \((\Gamma^{(s)}_A)_{jj}\) from tensor particle exchange should vanish for \(|j - \bar{j}| > 2\) and might diverge for \(j = \bar{j} = 1\) since this representation is “on-shell”. If we denote the total s-channel contribution as \(\Gamma^{(s)} = \Gamma^{(s)}_0 + \Gamma^{(s)}_A\) and calculate the complete amplitude, including \(t\)- and \(u\)-channel contributions, we get a result proportional to \([100]\) with the constants \(\alpha, \beta, \gamma\) and \(\delta\) given by

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix} = \begin{pmatrix}
588 & 0 & 720 & 1050 & 162 & -336\sqrt{5} & 0 & 192\sqrt{5} \\
0 & 0 & 240 & 210 & -30 & -84\sqrt{5} & -70\sqrt{5} & -62\sqrt{5} \\
0 & 0 & -240 & -525 & 135 & 336\sqrt{5} & 0 & 48\sqrt{5} \\
0 & 196 & 180 & 84 & 100 & 0 & -56\sqrt{5} & -120\sqrt{5}
\end{pmatrix}
\]

(109)

Our result \([109]\) seems to indicate that all \(\Gamma^{(s)}_{jj}\) components are finite except \(\Gamma^{(s)}_{11}\). We therefore conclude that the effective field theory corresponding to our string theory has no \(\phi^3(x)\) coupling, since otherwise \(\Gamma^{(s)}_{22}\) would be divergent. The divergence of \(\Gamma^{(s)}_{11}\) is of course due to the \(A_{\mu\nu}(x)\) resonance. However, our example is too small to allow for a determination of the coupling constants in the effective quantum field theory.

Finally, let us consider the question whether the amplitude is dual in the sense of section 4. More precisely, is it possible that the s-channel contribution \(\Gamma^{(s)}\) in itself is invariant under (94) and (98)? This means that \(\Gamma^{(s)}\) should have the form \([100]\) with the additional requirement that \(\Gamma^{(s)}_{04} = \Gamma^{(s)}_{40} = 0\) since \(\Gamma^{(s)}_{jj} = 0\) for \(|j - \bar{j}| > 2\). We see that this is not possible if \(\Gamma^{(s)}_{11}\) and/or \(\Gamma^{(s)}_{22}\) are infinite, since then other components of \(\Gamma^{(s)}\) would be infinite as well. We have thus found, that not only must the \(\phi^3(x)\) coupling be absent in a dual model, but the \(A_{\mu\nu}(x)\) field should couple in such a way to the \(\phi(x)\) field that the \((j, \bar{j}) = (1, 1)\) pole from the \(A_{\mu\nu}\) propagator cancels.

### 11. The continuum limit

We have seen that the complexity of the problem appears to increase with the spin \(j_0\). There are reasons to expect simplifications, however, as \(j_0 \to \infty\). This limit means that the wavelength of the incoming particles gets very small compared to the curvature radius of the \(SU(2)\) target space. It should therefore be a good approximation to neglect the curvature altogether. We will not pursue this approach in detail in this publication, but merely indicate what techniques may be used.

Let us therefore introduce the variables \(x = j/j_0\) and \(\bar{x} = \bar{j}/j_0\), where \(j\) and \(\bar{j}\) as before are the spins of the left and right representations exchanged in a four-point scattering respectively. We see that \(0 \leq x, \bar{x} \leq 2\). As \(j_0 \to \infty\) the spacing between the allowed values of \(x\) and \(\bar{x}\) goes to zero. It is therefore natural to take a continuum limit and replace the functions \(f_{jj}(\eta, \bar{\eta})\) by \(f(x, \bar{x}; \eta, \bar{\eta})\). The matrices \(A\) and \(B\) of (78) then turn into differential operators. We see that

\[
(Af)(x) = A(x)f(x),
\]

(110)
and that
\[(Bf)(x) = \left( B_0(x) + B_1(x) \frac{\partial}{\partial x} + B_2(x) \frac{\partial^2}{\partial x^2} \right) f(x) + \mathcal{O}\left(\frac{1}{j_0}\right), \tag{111}\]

where
\[
\begin{align*}
A(x) &= x(x + \frac{1}{j_0})(1 + \frac{1}{j_0})^{-1} - 2 \\
B_0(x) &= (x^2 - 2 + \frac{1}{j_0}x(1 - x) + \frac{1}{j_0^2}(x^2 - x + \frac{3}{16} - \frac{1}{4x^2}) \\
B_1(x) &= \frac{1}{j_0^2} \cdot \frac{x}{2} \\
B_2(x) &= \frac{1}{j_0^2} \cdot \frac{x^2}{4} - 1.
\end{align*}
\tag{112}\]

To lowest order in \(1/j_0\), \(A\) and \(B\) commute, which means that the continuum counterparts of the \(\alpha\)-matrices are diagonal, and we may thus write the correlation function as a global product of a holomorphic and an anti-holomorphic function, just as for a string in flat space:
\[
f(x, \bar{x}; \eta, \bar{\eta}) = c(x)|\eta|^{2A(x)}|1 - \eta|^{-2B_0(x)}. \tag{113}\]

To determine the function \(c(x)\) we must go to the second order in \(1/j_0\) so that the \(A\) and \(B\) operators no longer commute. Our differential equation is then
\[
\frac{\partial f(x; \eta)}{\partial \eta} = \left( \eta^{-1}A(x) + (1 - \eta)^{-1}(B_0(x) + B_1(x) \frac{\partial}{\partial x} + B_2(x) \frac{\partial^2}{\partial x^2}) \right) f(x; \eta) + \mathcal{O}\left(\frac{1}{j_0^3}\right).
\tag{114}\]

We see that as \(\eta \to 0\), the general solution behaves as
\[
f(x; \eta) = f_0(x) \eta^{A(x)}(1 - \eta)^{-B_0(x)}(1 + \mathcal{O}(\eta)) \tag{115}\]
for some function \(f_0(x)\). As the solution is continued analytically around a singular point, the function \(f_0(x)\) undergoes a linear transformation. We see that \(f_0(x) \mapsto \exp(2\pi i A(x))f_0(x)\) under analytic continuation around \(\eta = 0\). To determine the behaviour under analytic continuation around \(\eta = 1\) it is convenient to think of the \(B\) operator as the sum of a “free” term \(B_0(x)\) and an “interaction term” of order \(1/j_0^2\). We then change variables from \(f(x; \eta)\) to \(f_0(x; \eta)\) defined by
\[
f(x; \eta) = f_0(x, \eta) \eta^{A(x)}(1 - z)^{-B_0(x)} \tag{116}\]
and treat the problem in the “interaction picture”. The equation (78) now reads
\[
\frac{\partial f_0(x; \eta)}{\partial \eta} = (1 - \eta)^{-1} \left( B_1(x) \frac{\partial}{\partial x} + B_2(x) \frac{\partial^2}{\partial x^2} \\
+ \ln \eta \left( A'(x)B_1(x) + A''(x)B_2(x) + 2A'(x)B_2(x) \frac{\partial}{\partial x} \right) \\
- \ln(1 - \eta) \left( B_0'(x)B_1(x) + B_0''(x)B_2(x) + 2B_0'(x)B_2(x) \frac{\partial}{\partial x} \right) \\
+ \ln^2 \eta A''(x)B_2(x) - 2 \ln \eta \ln(1 - \eta) A'(x)B_0'(x)B_2(x) \\
+ \ln^2(1 - \eta) B_0''(x)B_2(x) + \mathcal{O}\left(\frac{1}{j_0^3}\right) \right) f_0(x; \eta). \tag{117}\]
The function $f_0(x; \eta)$ on the right hand side could be replaced by $f_0(x) + \mathcal{O}(1/j_0^2)$. Analytic continuation along a contour $C$ thus transforms $f_0(x)$ as

$$f_0(x) \mapsto f_0(x) + \int_C d\eta \frac{\partial f_0(x; \eta)}{\partial \eta}.$$  

By evaluating these integrals, we may determine the counterpart of the $\alpha$-matrices in (84). Finally, we may take the limit $j_0 \to \infty$, and solve condition (85) for the function $c(x)$.

12. Discussion

The main result of this paper is the discussion of how quantum field theory calculations on a group manifold could be performed. The possible amplitudes depend on only a few arbitrary coupling constants. We have also verified that the results from string theory and quantum field theory agree for some simple examples. In the $k = 4$ case, we were able to extract some non-trivial information concerning the possibility of constructing dual string amplitudes. A major obstacle for the interpretation of the results has been that all information in the amplitude is contained in only four divergent constants, the calculation of which requires a somewhat arbitrary regularization procedure. The two limits $k = 4$ and $k \to \infty$, which we have considered in this paper, have the advantage of being exactly solvable, but for our purposes they are not really sufficient. We need results from an intermediate regime, where the curvature of the target space could be expected to play an important role, and the $SU(2)$ representations are big enough so that the result contains more non-trivial information.

As we have already mentioned, the obvious way to continue the program is to evaluate some string scattering amplitudes numerically, and thus determine the coupling constants in the corresponding effective field theory. A problem is that we may expect such amplitudes to be divergent, when interpreted literally, just as in the $k = 4$ case. In flat space string theory, we are used to continue analytically in the external momenta to get a sensible answer, which is equivalent to what we did for $k = 4$ although our momenta are really discrete. Apart from the practical problem of implementing analytical continuation numerically, we may expect this procedure to be insufficient for levels exceeding four, however. The reason is that, as we have already noticed, the conformal field theory correlation functions are not globally a product of a holomorphic and an anti-holomorphic function any longer. Consequently they have singularities with several different exponents simultaneously as we let two insertion points coalesce. The regulator (102) will therefore in general fail to produce a convergent integral. We expect these difficulties to be tractable, though, and hope to come back to this issue shortly.

We have already mentioned the possibility of replacing the group $SU(2)$ with its non-compact relative $SU(1,1)$. Not only does this group provide us with a time direction, which raises interesting questions concerning the unitarity of the theory, but more important for our purposes is that it has a set of representations labeled by a continuous variable $j$. This would allow us to impose the constraint that scattering amplitudes should be analytic in $j$. Analyticity of the $S$-matrix plays an important role in flat space. It might also prove to be a convenient way of regularizing divergent
amplitudes. More general manifolds, with a non-vanishing dilation expectation value, certainly also merit study. A suitable first step should be the coset manifolds, among which is the two-dimensional black hole solution to string theory [18].

Our model may only be studied at tree level, since the total conformal anomaly is different from zero. At the present stage, this is not a serious problem, but eventually it would be interesting to consider higher loop contributions in a fully consistent theory.

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