KOLMOGOROV-TYPE SYSTEMS WITH REGIME-SWITCHING JUMP DIFFUSION PERTURBATIONS

Fuke Wu*
School of Mathematics and Statistics
Huazhong University of Science and Technology
Wuhan, Hubei 430074, China

George Yin
Department of Mathematics
Wayne State University
Detroit, MI 48202, USA

Zhuo Jin
Centre for Actuarial Studies, Department of Economics
The University of Melbourne
VIC 3010, Australia

Abstract. Population systems are often subject to different types of environmental noises. This paper considers a class of Kolmogorov-type systems perturbed by three different types of noise including Brownian motions, Markovian switching processes, and Poisson jumps, which is described by a regime-switching jump diffusion process. This paper examines these three different types of noises and determines their effects on the properties of the systems. The properties to be studied include existence and uniqueness of global positive solutions, boundedness of this positive solution, and asymptotic growth property, and extinction in the senses of the almost sure and the $p$th moment. Finally, this paper also considers a stochastic Lotka-Volterra system with regime-switching jump diffusion processes as a special case.

1. Introduction. Populations of biological species are often subject to different types of noises that have significant impact on the evolution and biodiversity; see for example, Gard [19–21], Lungu and Øksendal [30,31], among others. These noises include the continuous and discrete perturbations as well as random influence of the environments. They are often modeled by Brownian motions, Poisson processes, and Markovian chains. Aiming at understanding the fundamental underpinning of the different noise effects, this paper reveals their influence on the populations dynamics.

2010 Mathematics Subject Classification. Primary: 60H10, 60J70, 60J28; Secondary: 92D25, 93E03.

Key words and phrases. Kolmogorov-type system, Brownian motion, regime switching, jump process, global solution, boundedness, extinction.

The research of the first author was supported in part by the National Natural Science Foundations of China (NSFC) (No. 11422110 and No. 61473125), and in part by Wayne State University’s Grants Plus. The research of the second author was supported in part by the Army Research Office under grant W911NF-15-1-0218.

* Corresponding author: Fuke Wu.
Kolmogorov-type systems of differential equations have been used to model the evolution of many biological and ecological systems because they are more general and contain many classes of the well known population models such as the Lotka–Volterra model; see for instance, [4–6, 8, 15, 16, 18, 22, 23, 25, 36, 37, 41, 42, 44, 45, 47–49, 53, 54]. The n-dimensional Kolmogorov-type system for n interacting species is described by the following n-dimensional differential equation

\[ \dot{x}(t) = \text{diag}(x_1(t), \ldots, x_n(t)) f(x(t)), \quad (1.1) \]

where \( x = (x_1, \ldots, x_n) \), \( \text{diag}(x_1, \ldots, x_n) \) represents the \( n \times n \) matrix with diagonal entries \( x_1, \ldots, x_n \) and elsewhere 0, and \( f = (f_1, \ldots, f_n)' \). This system can be rewritten as

\[ \frac{\dot{x}_k(t)}{x_k(t)} = f_k(x(t)) \quad \text{for} \quad k = 1, \ldots, n, \]

which shows that \( f_k(x(t)) \) represents the net growth rate of the \( k \)th species on the time \( t \). Owing to various unpredictable factors such as continuous random processes represented by Brownian motions, discrete random process represented by jump processes taking value in a finite or countable set, and environmental changes represented by continuous-time Markov chains. We can therefore replace the net growth rate of the \( k \)th species \( f_k(x(t)) \) by

\[ f_k(x(t)) \to f_k(x(t), r(t)) + g_k(x(t), r(t)) \dot{w}(t) + \int_{\Xi} h_k(y, x(t-), r(t-)) N(1, dy), \]

where \( w(t) \) is a Brownian motion, \( r(t) \) is a continuous-time Markov chain taking value in \( \mathbb{S} \), \( N(t, \Xi) \) is a Poisson measure counting the events happened up to time \( t \), \( g_k(x(t)) \) represents the intensity of the noise, \( h_k(y, x(t-), r(t-)) \) gives the intensity of the jump, and \( N(1, \Xi) \) is a counting measure for the set \( \Xi \) in a unit time. We therefore have the following stochastic Kolmogorov-type system with regime-switching jump diffusion processes

\[ dx(t) = \text{diag}(x_1(t), \ldots, x_n(t)) \left[ f(x(t), r(t)) dt + g(x(t), r(t)) dw(t) \right. \]

\[ \left. + \int_{\Xi} h(y, x(t-), r(t-)) N(dt, dy) \right], \quad (1.2) \]

where \( g = (g_1, \ldots, g_n)' : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n \), \( h = (h_1, \ldots, h_n) : \Xi \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n \) are Borel measurable.

Stochastic population dynamics have received more and more attentions in recent years since they seem to be in good agreement with the reality in the nature. By introducing the Brownian noise from the interactions between the species into the classical Lotka-Volterra system, [4, 32] revealed that the environmental noise may suppress the potential population explosion and guarantee the global positive solution to stochastic Lotka-Volterra system, and moreover, also shows that the stochastic Lotka-Volterra model produces many desired properties, for example, stochastically ultimate boundedness and the moment boundedness. Under the Brownian noise from the birth rate into the deterministic Lotka-Volterra system, [5, 38] revealed that the stochastic Lotka-Volterra system behaves similarly to the corresponding deterministic system. [33] reviewed these two classes of the models and indicates that different structures of environmental noise may have different effects on the population dynamics. By introducing the more general stochastic perturbations including the interactions between species and the net birth rate into the system (1.2) and the corresponding Lotka-Volterra system, [45, 47, 48] obtained
conditions under which the different noise structures have different effects on the asymptotic properties of the population dynamics. These conditions show that if the environmental noise intensity is strongly dependent on the population size (for example, the Brownian noise from the interactions between species), this noise may suppress the population explosion and guarantee the global positive solution and the asymptotic properties of the model will be determined by the noise. When the environmental noise intensity depends weakly on the population size (for example, the net birth rate noise), the stochastic system behaves similarly to the deterministic one and asymptotic properties are also independent of the noise. [49] further showed that although the interactions between the species plays the crucial roles for the existence of the global positive solution of the stochastic Lotka-Voltera systems and some asymptotic properties, the most important factor of the extinction of the species is still from the noise of the birth rate.

It is obvious that the Brownian noise can model the perturbation from continuous random processes or approximately model a sufficiently large number of iterates of independent random factors according the central limit theorem. The population may suffer sudden environmental shocks, e.g., earthquakes, hurricanes, epidemics, which cannot be modeled by the Brownian noise. To model these phenomena, [6,7] introduced the jump process into the underlying population dynamics and considered the effects of the Lévy processes on the system and examined the existence of the global solution and the asymptotic pathwise estimation and uniform boundedness. [27] was also concerned with stochastic Lotka-Volterra models perturbed by Lévy noise and examined stochastic permanence and extinction.

Besides the continuous and discrete stochastic factors mentioned above, the population growth is also different in the different environments. For instance, the growth rates of some species in the rainy season will be much different from those in the dry season. Moreover, the carrying capacities often vary according to the changes in nutrition and/or food resources. The Markov chain offers a suitable tool to describe these environmental changes. Actually, about forty years ago, the Markov chain was used to model the environmental changes in the population dynamics, for example, by using probabilistic technique. [40] examined the stationary distribution of the population system in a Markovian switching environment including two states. [42] considered the evolution of a system composed of two predator-prey deterministic systems described by Lotka-Volterra equations in a Markovian switching environment. [28] discussed a (2-dimensional) predator-prey Lotka-Volterra model in a switching diffusion environment, and then this two dimensional model was generalized as a \(n\)-dimensional Lotka-Volterra system in [29]. [53,54] were concerned with competitive Lotka-Volterra model in a switching diffusion environment and examined the certain long-run-average limits of the solution from several angles.

This paper focuses on system (1.2) and examines how the three different types of noises to determine its properties including

(i) existence and uniqueness of the global positive solution;
(ii) boundedness including the \(p\)th moment boundedness, stochastic ultimate boundedness, and the moment average boundedness in time;
(iii) asymptotic pathwise growth estimation;
(iv) extinction including the almost sure extinction and the \(p\)th moment extinction.

Finally, we also apply these results to examine the Lotka-Volterra system with the regime-switching jump diffusion processes as a special case.
The rest of the paper is organized as follows. Section 2 provides the notation needed and preliminary results. Section 3 examines existence and uniqueness of the global positive solution, which shows that the Brownian noise plays a crucial role to yield the global solution. Section 4 investigates the boundedness of this global positive solution, including the $p$th moment boundedness, stochastic ultimate boundedness, and the moment average boundedness in time. Section 5 shows that system (1.2) grows at most polynomially. In Section 6, we analyze two classes of extinctions including the almost sure extinction and the $p$th moment extinction and examine the effects of these three types of noises on the extinction. Finally, we apply these results to examine the Lotka-Volterra system under regime-switching jump diffusion process formulation as a special case.

2. Preliminaries and notation. Throughout this paper, unless otherwise specified, we use the following notation. Let $| \cdot |$ be the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^n = (0, \infty)$. If $A$ is a vector or matrix, its transpose is denoted by $A'$. If $A$ is a matrix, we use the norm defined by $|A| = \sqrt{\text{trace}(A'A)}$. The $a \lor b$ denotes $\max\{a, b\}$ and $a \land b$ denotes $\min\{a, b\}$. Denote by $C^2(\mathbb{R}^n; \mathbb{R}_+)$ the family of all nonnegative functions $V(x)$ on $\mathbb{R}^n$ that have continuous partial derivatives w.r.t. $x$ up to the second order.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. That is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t)$ be a scalar Brownian motion, $\Xi$ be a subset of $\mathbb{R}^n \setminus \{0\}$ that is the range space of the impulsive jumps, and $N(\cdot, \cdot)$ defined on $\mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\}$ is an $\mathcal{F}_t$-adapted Poisson random measure with compensator $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$, where $\nu$ is a Lévy measure on $\Xi$ with $\nu(\Xi) = \lambda$. Denote by $\mathcal{E}_0$ the family of all bounded positive functions $h(\cdot)$ on $\Xi$. Let $r(t)$ be a Markov chain taking values in $\mathbb{S} = \{1, 2, \ldots, m\}$ on the space $(\Omega, \mathcal{F}, \mathbb{P})$. The corresponding generator is denoted by $\Gamma = (\gamma_{ij})_{m \times m}$, so that for sufficiently small $\delta > 0$,

$$
\mathbb{P}\{r(t + \delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij} \delta + o(\delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii} \delta + o(\delta), & \text{if } i = j. \end{cases}
$$  \hspace{1cm} (2.1)

Here $\gamma_{ij}$ is the transition rate from $i$ to $j$ and $\gamma_{ij} > 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. As a standing hypothesis, this paper assumes that the Markov chain $r(t)$ is irreducible, that is, the system of equations

$$
\begin{cases}
\pi \Gamma = 0 \\
\sum_{i \in \mathbb{S}} \pi_i = 1
\end{cases}
$$

has a unique solution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ satisfying $\pi_i > 0$ for each $i = 1, \ldots, m$.

Throughout the paper, we assume that $w(t)$, $N(t, \cdot)$, and $r(t)$ are independent. It is well known (see e.g., [1]) that almost every sample path of the Markov chain $r(\cdot)$ is a right continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R}_+$. Consequently, there is a sequence of stopping times $0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$ such that

$$
r(t) = \sum_{k=0}^{\infty} r(\tau_k) \mathbb{1}_{[\tau_k, \tau_{k+1})}(t), \quad t \geq 0.
$$  \hspace{1cm} (2.2)

Let $\mathcal{P}$ be the set of all probability measures on the state space $\mathbb{S}$. Then the Markov chain $r(t)$ has the rate function (see [9, 11–14]) given by
where \( p = (p_1, \ldots, p_m) \in \mathcal{P} \) is a probability vector and \( u = (u_1, \ldots, u_m) \in \mathbb{R}^m \). It is known that \( I(p) \geq 0 \) is lower semicontinuous and \( I(p) = 0 \) if and only if \( p = \pi \).

Applying Theorem A.2 in Appendix A into Markov chain \( r(t) \) gives

\[
\Lambda(a) := \lim_{t \to \infty} \frac{1}{t} \log \left( \mathbb{E} \exp \left[ \int_0^t a(r(s))ds \right] \right) = \sup_{p \in \mathcal{P}} \left\{ \sum_{i \in S} a_i p_i - I(p) \right\},
\]

where \( a = (a_1, \ldots, a_m) \in \mathbb{R}^m \). When extinction in the sense of the \( p \)th moment is examined, \( \Lambda(\cdot) \) plays an important role. It is easily seen that

\[
\sum_{i \in S} a_i \pi_i \leq \Lambda(a) \leq \max_{i \in S} \{a_i\}.
\]

To examine the properties of system (1.2), we need the following assumptions:

**Assumption 2.1.** Assume that \( f, g, h \) satisfy the local Lipschitz condition. That is, for each integer \( \theta \geq 1 \), there exists a constant \( K_\theta > 0 \), such that

\[
|f(x, i) - f(\bar{x}, i)| + |g(x, i) - g(\bar{x}, i)| + |h(y, x, i) - h(y, \bar{x}, i)| |v(dy) \leq K_\theta |x - \bar{x}|
\]

for all \( i \in S \) and \( x, \bar{x} \in \mathbb{R}^n \) with \( |x| \vee |\bar{x}| \leq \theta \).

We also need further conditions for \( f, g, \) and \( h \) as follows: for any \( x = (x_1, x_2, \ldots, x_n)' \in \mathbb{R}^n_+ \), \( i \in S \) and \( k = 1, \ldots, n \),

1. **(H1)** there exist \( \alpha > 0 \), \( a_k(i), b_k(i) \in \mathbb{R} \) such that
   \[
   |f_k(x, i)| \leq a_k(i)|x|^\alpha + b_k(i);
   \]
2. **(H2)** there exist \( \beta > 0 \), \( \phi_k(i) > 0 \), \( \xi_k(i) > 0 \) such that
   \[
   |g_k(x, i)| \leq \phi_k(i)|x|^\beta + \xi_k(i);
   \]
3. **(H3)** \( h_k(y, x, i) \neq -1 \) and there exist \( \zeta_k(\cdot, i), \zeta_k(\cdot, i) \in \mathbb{E}_0 \) such that for any \( y \in \Xi \),
   \[
   \zeta_k(y, i) \leq 1 + h_k(y, x, i) \leq \zeta_k(y, i).
   \]

**Remark 1.** When \( h_k = -1 \), if the jump takes place at \( t_0 \), \( x_k(t) = 0 \) for all \( t \geq t_0 \), so this is a trivial case.

**Remark 2.** In this paper, (H3) may be replaced by the following more general condition:

**(H3a)** \( h_k(y, x, i) \neq -1 \) and there exist \( \theta > 0 \), \( \zeta_k(\cdot, i), \phi_k(\cdot, i), \zeta_k(\cdot, i), \zeta_k(\cdot, i) \in \mathbb{E}_0 \) such that

\[
\psi_k(y, i)|x|^\theta + \zeta_k(y, i) \leq 1 + h_k(y, x, i) \leq \varphi_k(y, i)|x|^\theta + \zeta_k(y, i).
\]

In this paper, we often use the function

\[
V_p(x) = \sum_{k=1}^n x_k^p
\]

for any \( p > 0 \) and \( x = (x_1, \ldots, x_n)' \in \mathbb{R}^n_+ \). We also need the following inequality about \( V_p(x) \). Let us write it as the following lemma.

**Lemma 2.1.** For any \( x = (x_1, \ldots, x_n)' \in \mathbb{R}^n_+ \) and \( p > 0 \),

\[
V_p(x) \leq n(1 - \frac{d}{2})^{\nu_0} x^p, \quad |x|^p \leq n(\frac{d}{2} - 1)^{\nu_0} V_p(x).
\]
We also write the following Young inequality as a lemma.

**Lemma 2.2.** For any \( \alpha, \beta \geq 0 \) satisfying \( \alpha + \beta > 0 \), for any \( x, y \geq 0 \),

\[
x^\alpha y^\beta \leq \frac{\alpha}{\alpha + \beta} x^{\alpha + \beta} + \frac{\beta}{\alpha + \beta} y^{\alpha + \beta}.
\]

3. **Global positive solutions.** To examine existence and uniqueness of the global positive solutions, let us first present the following definition including the local solution and the global solution as well as the explosion time:

**Definition 3.1.** Let \( x(t), 0 \leq t < \rho_e \) be a continuous \( \mathbb{R}^n \)-valued \( \mathcal{F}_t \)-adapted process. It is called a local strong solution of Eq. (1.2) with initial value \( x(0) \in \mathbb{R}^n \) if for any \( t \geq 0 \) and some positive integer \( k_0 \),

\[
x(t \wedge \rho_k) = x(0) + \int_0^{t \wedge \rho_k} \text{diag}(x_1(s), \ldots, x_n(s)) f(x(s), r(s)) ds + \int_0^{t \wedge \rho_k} \text{diag}(x_1(s), \ldots, x_n(s)) g(x(s), r(s)) dw(s) + \int_0^{t \wedge \rho_k} \int_{\Xi} \text{diag}(x_1(s-), \ldots, x_n(s-)) h(y, x(s-), r(s-)) \nu(ds, dy)
\]

for each \( k \geq k_0 \), where \( \{\rho_k\}_{k \geq k_0} \) is a nondecreasing sequence of stopping times such that \( \rho_k \to \rho_e \) almost surely as \( k \to \infty \). If, moreover, \( \limsup_{t \to \rho_e} |x(t)| = \infty \) a.s. when \( \rho_e < \infty \), \( x(t) \) is called a maximal local strong solution on \( [0, \rho_e] \) and \( \rho_e \) is called the explosion time. When \( \rho_e = \infty \), it is called a global solution. A maximal local strong solution \( x(t), 0 \leq t < \rho_e \) is said to be unique if for any other maximal local strong solution \( \tilde{x}(t), 0 \leq t < \tilde{\rho}_e \), we have \( \rho_e = \tilde{\rho}_e \) and \( x(t) = \tilde{x}(t) \) for \( 0 \leq t < \rho_e \) a.s.

For any \( i \in S \), define \( F(x, i) = \text{diag}(x_1, \ldots, x_n) f(x, i) \), \( G(x, i) = \text{diag}(x_1, \ldots, x_n) g(x, i) \) and \( H(y, x, i) = \text{diag}(x_1, \ldots, x_n) h(y, x, i) \). By Assumption 2.1, it is easy to observe that \( F(x, i), G(x, i) \) and \( H(y, x, i) \) also satisfy the local Lipschitz condition. The definitions of \( F, G \) and \( H \) implies that \( F(0, i) = 0, G(0, i) = 0 \) and \( H(0, 0, i) = 0 \). This, together with the local Lipschitz conditions of \( F, G, H \), implies the local linear growth condition, namely,

\[
|F(x, i)| \vee |G(x, i)| \vee \int_{\Xi} |H(y, x, i)| \nu(dy) \leq K_\theta |x|,
\]

for all \( i \in S \) and \( x \in \mathbb{R} \) satisfying \( |x| \leq \theta \), where \( K_\theta \) is a constant which may be different from \( K_\theta \). By [34, Chapter 3] and [2, Chapter 6], the local Lipschitz condition, together with the local linear growth condition yields a local solution, which can be expressed as follows:

**Lemma 3.2.** Under Assumption 2.1, there exists a unique, càdlàg, and adapted, local solution \( x(t) \) on \( 0 \leq t < \rho_e \) for system (1.2), where \( \rho_e \) is the explosion time.

By slightly modifying the proof of [3, Lemma 3.3], Assumption 2.1 and condition (3.1) can also guarantee that this local solution can never reach the origin if the initial value is nonzero, namely, the following result holds.

**Lemma 3.3.** Under Assumption 2.1, the solution of system (1.2) can never reach the origin provided that \( x_0 \neq 0 \). This implies that if \( x_0 \neq 0 \), \( \mathbb{P}\{x(t) = 0 \text{ for all } t \in (0, \rho_e)\} = 1 \).
By [34, Chapter 3] and [2, Chapter 6], \((x(t), r(t))\) is a two-component Markov process. According to the regularity of Markov processes, we can establish the existence of global solutions. Let us give the definition of regularity of Markov processes.

**Definition 3.4.** The two-component Markov process \((x(t), r(t))\) is said to be regular if for any \(0 < T < \infty\), \(\mathbb{P}\{\sup_{0 \leq t \leq T}|x(t)| = \infty\} = 0\).

This definition implies that the regular Markov process has no finite explosion time, which is equivalent to the existence of the global solution. Let \(k_0\) be a sufficiently large positive number such that \(|x_0| < k_0\). For each integer \(k > k_0\), define the stopping time

\[\rho_k = \inf\{t \in (0, \rho_\infty) : |x(t)| > k\}\]

with the traditional setting \(\inf\{\emptyset\} = \infty\), where \(\emptyset\) denotes the empty set. Clearly, \(\{\rho_k\}_{k \geq k_0}\) is increasing as \(k \uparrow \infty\) with \(\rho_k \to \rho_\infty \leq \rho_\infty\) a.s. If we can show that \(\rho_\infty = \infty\), then \((x(t), r(t))\) defined by regime-switching jump diffusion process (1.2) is regular.

To examine the existence and uniqueness of global solutions for the Kolmogorov-type system (1.2), we need more notation. For any \(V \in C^2(\mathbb{R}^n; \mathbb{R}_+\), let us define the operator \(\mathcal{G}\) from \(\mathbb{R}^n \times S\) to \(\mathbb{R}\) by

\[\mathcal{G}V(x)(x, i) = \mathcal{L}V(x)(x, i) + \int_S [V(x + H(y, x, i)) - V(x)]\nu(dy)\]

for each \(i \in S\), where \(\mathcal{L}\) is the operator from \(\mathbb{R}^n \times S\) to \(\mathbb{R}\) for a switching diffusion process given by (also see [51])

\[\mathcal{L}V(x)(x, i) = V_x(x)F(x, i) + \frac{1}{2}G''(x, i)V_{xx}(x)G(x, i)\]

Note that here \(V_x(x)F(x, i)\) is the inner product between the vectors \(V_x(x)\) and \(F(x, i)\). If \(V(x, i) \in C^2(\mathbb{R}^n \times S; \mathbb{R}_+)\), namely, \(V\) depends on the state of the Markov Chain \(r(t)\), there is an additional term \(\Gamma V(x, \cdot)(i)\) in \(\mathcal{G}V(x)(x, i)\), where

\[\Gamma V(x, \cdot)(i) = \sum_{j=1}^m \gamma_{ij}V(x, j)\]

However, we are using a function \(V\) that is independent of \(i\), as a result, this term is 0 since \(\sum_j \gamma_{ij} = 0\). In the following, for the purpose of simplicity, we write \(\mathcal{G}V(x)(x, i)\) and \(\mathcal{L}V(x)(x, i)\) as \(\mathcal{G}V(x, i)\) and \(\mathcal{L}V(x, i)\), respectively.

Then for any \(V \in C^2(\mathbb{R}^n; \mathbb{R}_+)\), applying the generalized Itô formula to system (1.2) with regime-switching jump diffusion gives (also see [50])

\[V(x(t)) = V(x_0) + \int_0^t \mathcal{G}V(x(s), r(s))ds + \int_0^t V_x(x(s))G(x(s), r(s))dw(s)\]

\[+ \int_0^t \int_S [V(x(s)) + H(y, x(s), r(s))] - V(x(s))\mathcal{N}(ds, dy)\]

where \(V_x(x)G(x, i)\) is the inner product between the vectors \(V_x(x)\) and \(G(x, i)\). Again, normally for switching diffusions, there will be another martingale term attributes to the Markov chain (converted to a Poisson process). However, since we choose \(V\) to be independent of \(r(t)\), this term is 0. To proceed, let us present the sufficient conditions for the regularity of \((x(t), r(t))\) defined by system (1.2), which can be found in [50] and [51].
**Lemma 3.5.** Assume that $f, g, h$ satisfy Assumption 2.1 and there is a nonnegative function $V(\cdot)$ that is twice continuously differentiable in $U^c_R = \{ x : |x| > R \}$ for some $R > 0$ sufficiently large such that there is a $\gamma_0 > 0$ satisfying $\mathcal{G} V(x, i) \leq \gamma_0 V(x)$, $\inf_{|x| > R} V(x) \to \infty$, as $R \to \infty$. Then process $(x(t), r(t))$ is regular.

Applying this lemma may establish the existence and uniqueness of the global positive solutions for the Kolmogorov-type system (1.2) with the regime-switching jump diffusion process.

**Theorem 3.6.** Let Assumption 2.1, (H1), (H2), and (H3) hold. For any initial value $x(0) > 0$, if $2\beta > \alpha$, then $(x(t), r(t))$ determined by Eq. (1.2) is regular with $x(t) > 0$ for all $t \geq 0$, namely, system (1.2) admits a unique global positive solution almost surely.

**Proof.** Lemma 3.2 shows that system (1.2) admits a unique local solution on $[0, \tau_e)$, where $\tau_e$ is the explosion time. Eq. (1.2) also shows that the $k$th component $x_k(t)$ of $x(t)$ holds the form

$$dx_k(t) = x_k(t) \left[ f_k(x(t), r(t))dt + g_k(x(t), r(t))dw(t) + \int_{\Xi} h_k(y, x(t), r(t))N(dt, dy) \right].$$

By the generalized Itô formula, for any $t \in [0, \tau_e)$, $x_k(t)$ can be expressed as

$$x_k(t) = x_k(0) \exp \left[ \int_0^t \left( f_k(x(s), r(s)) - \frac{1}{2} |g_k(x(s), r(s))|^2 \right) ds + \int_0^t \int_{\Xi} \log(1 + h_k(y, x(s), r(s))) \nu(dy) ds + \int_0^t g_k(x(s), r(s)) dw(s) \right].$$

Noting that the initial value $x_k(0) > 0$, (3.7) implies that $x_k(t) > 0$ for all $t \in [0, \tau_e)$ for all $k = 1, \ldots, n$. This, together with Lemma 3.3, yields $x(t) > 0$ for all $t \in [0, \tau_e)$. Now let us show that this positive local solution is actually global by using Lemma 3.5, that is, $(x(t), r(t))$ is regular. For any $p \in (0, 1)$, letting us apply the operator (3.3) to the function $V_p(x)$ defined by (2.6) gives

$$\mathcal{G} V_p(x, i) = \sum_{k=1}^n x_k^p \left[ p f_k(x, i) + p(p-1)2 \gamma_0 2 \beta g_k(x, i) + \int_{\Xi} \left( 1 + h_k(y, x, i) \right)^p \nu(dy) \right].$$

Note that (H2) gives $|g_k(x, i)|^2 \geq \phi_k^2(i) x_k^{2\beta} + \xi_k^2(i)$. This, together with (H1) and (H3), yields

$$\mathcal{G} V_p(x, i) \leq \sum_{k=1}^n x_k^p \left[ - \frac{p(1-p)}{2} \phi_k^2(i) x_k^{2\beta} + p a_k(i) |x|^\alpha + p b_k(i) \xi_k^2(i) \right].$$

By Lemmas 2.1 and 2.2,

$$\sum_{k=1}^n a_k(i) x_k |x|^{\alpha} \leq n^{\frac{\alpha}{2}(\alpha-1)} \sum_{k=1}^n \sum_{j=1}^n a_k(i) x_k^p x_j^\alpha.$$
Substituting this inequality into (3.8) yields

\[
\begin{align*}
\mathcal{G}_p(x, i) &\leq \sum_{k=1}^{n} x_k^p \left[ pK_{k,p}(x_k, i) - \frac{p(1-p)}{2} \xi_k^2(i) + \int_{\Xi} [\gamma_k^p(y, i) - 1] \nu(dy) \right], \\
&= \sum_{k=1}^{n} \left( \frac{pm^{\frac{q}{q+1}}}{\alpha+p} \sum_{i=1}^{n} a_k(i) x_k^{\alpha+p} + \frac{\alpha n^\alpha (\frac{q}{q-1})^{\nu_0}}{\alpha+p} \sum_{j=1}^{n} a_j(i) x_j^{\alpha+p} \right) \cdot \sum_{k=1}^{n} x_k^p \left( \frac{pm^{\frac{q}{q+1}}}{\alpha+p} a_k(i) + \frac{\alpha n^\alpha (\frac{q}{q-1})^{\nu_0}}{\alpha+p} a_j(i) \right) x_k^p.
\end{align*}
\]

Substituting this inequality into (3.8) yields

\[
\mathcal{G}_p(x, i) \leq \sum_{k=1}^{n} x_k^p \left[ pK_{k,p}(x_k, i) - \frac{p(1-p)}{2} \xi_k^2(i) + \int_{\Xi} [\gamma_k^p(y, i) - 1] \nu(dy) \right],
\]

where

\[
K_{k,p}(x, i) = -\frac{1}{2} p \phi_k^2(i) x^{2\beta} + \frac{1}{\alpha+p} \left( \frac{pm^{\frac{q}{q+1}}}{\alpha+p} a_k(i) + \frac{\alpha n^\alpha (\frac{q}{q-1})^{\nu_0}}{\alpha+p} a_j(i) \right) x^\alpha + b_k(i).
\]

Note that \( p \in (0, 1) \) and \( \alpha < 2\beta \). There must be an upper bound for the polynomial function \( K_k(x, i) \) as follows:

\[
\max_{x \geq 0} \{K_{k,p}(x, i)\} \leq \frac{2\beta - \alpha}{2} \left( \frac{pm^{\frac{q}{q+1}} a_k(i) + \alpha n^\alpha (\frac{q}{q-1})^{\nu_0} a_j(i)}{\beta (\alpha+p)} \right)^{\frac{2\beta}{\alpha}} \times \left( \frac{\alpha}{(1-p) \phi_k^2(i)} \right)^{\frac{\alpha}{\alpha}} + b_k(i)
\]

Noting that \( \xi_k(\cdot, i) \in \Xi_0 \), we have

\[
\int_{\Xi} [\gamma_k^p(y, i) - 1] \nu(dy) < \infty.
\]

For \( i \in S \), define

\[
\eta_{k,p}(i) = pK_{k,p}(i) - \frac{p(1-p)}{2} \xi_k^2(i) + \int_{\Xi} [\gamma_k^p(y, i) - 1] \nu(dy)
\]

and

\[
\eta_p(i) = \max_{(k=1, \ldots, n)} \{\eta_{k,p}(i)\}.
\]

We therefore have

\[
\mathcal{G}_p(x, i) \leq \eta_p(i) V_p(x) \leq \eta_{\max} V_p(x),
\]

where \( \eta_{\max} = \max_{i \in S} \{\eta_p(i)\} \). By Lemma 2.1, noting that \( p \in (0, 1) \),

\[
\inf \left\{ V_p(x) : x > R \right\} = \inf \sum_{k=1}^{n} x_k^p \geq \inf \sum_{k=1}^{n} |x|^p \geq |R|^p \to \infty \text{ as } |R| \to \infty.
\]

Thus by virtue of Lemma 3.5, the two component process \((x(t), r(t))\) is regular. This also shows that \( x(t) \) is a global positive solution. \( \square \)
Remark 3. According to this proof, the bounded property of the polynomial function $K_{k,p}(x,i)$ plays a crucial role to guarantee the regularity of $(x(t),r(t))$, in which condition $2\beta > \alpha$ guarantees this bounded property. This implies that this regularity is induced by the Brownian motion $w(t)$. In our previous papers, for example, [46], it is shown that the Brownian motion may suppress the potential explosion. In [52], we also shows that although the Markov switching may change the explosion time, it cannot change the trend of the explosion for the explosion system. In (3.9), although $\int_{\Xi}[\varsigma_k^p(y,i)-1]\nu(dy) < \infty$ can change the value of $\eta_k,p(i)$, it has no effect on the bounded property of $K_{k,p}(x,i)$. This implies that similar to the Markov chain, the jump process under condition (H3) may change the explosion time, but it cannot change the trend of the explosion.

Remark 4. In this theorem, if (H3) is replaced by (H3a), the proof process is the same. We only need to change the estimation of $\int_{\Xi}[(1 + h_k(y,x,i))^{p-1}]\nu(dy)$. (H3a) implies that

$$\int_{\Xi}[\varphi_k^p(y,i)|x|^\theta + \varsigma_k(y,i)]^p\nu(dy) \leq \int_{\Xi}[\varphi_k^p(y,i)\nu(dy)]^{(\frac{p-1}{\theta})\vee 0} \sum_{k=1}^n x_k^\theta + \int_{\Xi}[\varsigma_k(y,i)\nu(dy)],$$

which can be added into the polynomial function $K_k(x,i)$ and have no effect on the bounded property for any $\theta > 0$ since we can choose sufficiently small $p > 0$. In the following, we can deal with (H3a) similar to this technique.

4. Boundedness. Theorems 3.6 shows that the solutions of Eq. (1.2) will remain in the positive cone $\mathbb{R}_+^n$. This property enables us to further examine how the solution varies in this cone in more detail. Comparing with the existence and uniqueness of the global positive solutions, boundedness is more interesting from the biological point of view. In the following, we will discuss the $p$th moment boundedness, stochastic ultimate boundedness and the moment average boundedness in time.

Theorem 4.1. Under the conditions of Theorem 3.6, for any $p \in (0,1)$, there exists a constant $K_p$ independent of the initial value such that the global positive solution $x(t)$ of Eq. (1.2) satisfies

$$\lim_{t \to \infty} \|x(t)\|_p \leq K_p.$$  \hspace{1cm} (4.1)

Proof. By Theorem 3.6, the solution $x(t)$ of (1.2) remains in $\mathbb{R}_+^n$ almost surely for all $t \geq 0$. Applying the generalized Itô formula to $e^tV_p(x(t))$ gives

$$e^tE[V_p(x(t))] = E[V_p(x(0))] + E \int_0^t e^{s} [V_p(x(s)) + \mathcal{G}V_p(x(s),r(s))]ds,$$  \hspace{1cm} (4.2)

where $\mathcal{G}V_p$ is given and estimated by (3.8). According to (3.9), we therefore have

$$V_p(x) + \mathcal{G}V_p(x,i) \leq \sum_{k=1}^n K_{k,p}(x_k,i),$$
where
\[
K_{k,p}(x,i) = -\frac{p(1-p)}{2} \phi_k^2(i)x^{2\beta+p} + \frac{px^{\alpha+p}}{\alpha} + \left(1 + \frac{p(1-p)}{2} \xi_k^2(i) + \int_{\Xi} [c_k^p(y,i) - 1] \nu(dy) \right) x^p.
\]

Note that \( p \in (0,1), \ 2\beta > \alpha \) and \( \int_{\Xi} [(1 + h_k(y,x,i))^p - 1] \nu(dy) \) is bounded. Therefore, there exists a constant \( \tilde{K}_{k,p}(i) \) such that \( \max_{r \geq 0} \{K_{k,p}(x,i)\} \leq \tilde{K}_{k,p}(i) \). Substituting this estimation into (4.2) gives

\[
e^t \mathbb{E} V_p(x(t)) = \mathbb{E} V_p(x(0)) + \sum_{k=1}^{n} \max_{\{i \in S\}} \tilde{K}_{k,p}(i) \mathbb{E} \int_0^t e^s ds,
\]

which implies that

\[
\mathbb{E} V_p(x(t)) \leq e^{-t} \mathbb{E} V_p(x(0)) + \sum_{k=1}^{n} \max_{\{i \in S\}} \tilde{K}_{k,p}(i)(1 - e^{-t}).
\]

Clearly,

\[
\limsup_{t \to \infty} \mathbb{E} V_p(x(t)) \leq \sum_{k=1}^{n} \max_{\{i \in S\}} \tilde{K}_{k,p}(i).
\]

Lemma 2.1 gives that

\[
\limsup_{t \to \infty} \mathbb{E} |x(t)|^p \leq \limsup_{t \to \infty} \mathbb{E} V_p(x(t)) \leq \sum_{k=1}^{n} \max_{\{i \in S\}} \tilde{K}_{k,p}(i).
\]

Choosing \( K_p = \sum_{k=1}^{n} \max_{\{i \in S\}} \tilde{K}_{k,p}(i) \) gives the desired result.

By the \( p \)th moment boundedness, the stochastically ultimate boundedness will follow directly. We describe it as a theorem below. This proof is from the Chebyshev inequality. We omit it (please see [45]).

**Theorem 4.2.** Under the conditions of Theorem 3.6, for any \( \varepsilon \in (0,1) \), there is a positive constant \( \hat{K} = \hat{K}(\varepsilon) \) such that for any initial value, the solution \( x(t) \) of Eq. (1.2) satisfies

\[
\limsup_{t \to \infty} \mathbb{P} \{|x(t)| \leq \hat{K}\} \geq 1 - \varepsilon.
\]

Using similar techniques as those of Theorems 3.6 and 4.1, we can also examine the moment average in time as follows:

**Theorem 4.3.** Under the conditions of Theorem 3.6, for any \( p \in (0,1) \), there exists a constant \( \bar{K}_p \) dependent on \( p \) such that

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E} |x(s)|^{2\beta+p} ds \leq \bar{K}_p.
\]

**Proof.** Applying the Itô formula to \( V_p(x(t)) \) gives

\[
\mathbb{E} V_p(x(t)) = \mathbb{E} V_p(x(0)) + \int_0^t \mathcal{G} V_p(x(s), r(s)) ds,
\]

where \( \mathcal{G} V \) is given and estimated by (refsoleq1). According to (3.9), we therefore have

\[
\mathcal{G} V_p(x, i) \leq \sum_{k=1}^{n} \left[ -\frac{p(1-p)}{4} \phi_k^2(i)x^{2\beta+p} + \tilde{K}_{k,p}(x_k, i) \right],
\]
where
\[
\bar{K}_{k,p}(x, i) = - \frac{p(1-p)}{4} \phi_k^2(i) x^{2\beta+p} + \left( p n^{\frac{\beta}{\alpha}} a_k(i) + \alpha n^{(\frac{\beta}{\alpha} - 1)0} \sum_{j=1}^{n} a_j(i) \right) \frac{p x^{\alpha+p}}{\alpha + p} + \left( p b_k(i) + \frac{p(1-p)}{2} \right) (\int_E |x_k(y, i) - 1| d(y)) x^{p}
\]

By the similar techniques to Theorems 3.6 and 4.1, there exists a constant \( \bar{K}_{k,p}(i) \) such that \( \max_{\{x > 0\}} \{ \bar{K}_{k,p}(x, i) \} \leq \bar{K}_{k,p}(i) \). This, together with (4.6) gives
\[
0 \leq \mathbb{E} V_p(x(t)) \leq \mathbb{E} V_p(x(0)) - \frac{p(1-p)}{4} \sum_{k=1}^{n} \mathbb{E} \int_{0}^{t} \phi_k^2(r(s)) x_k^{2\beta+p}(s) ds + \sum_{k=1}^{n} \max_{\{i \in \mathbb{S}\}} \{ \bar{K}_{k,p}(i) \} t,
\]
which implies that
\[
\frac{p(1-p)}{4} \sum_{k=1}^{n} \int_{0}^{t} \phi_k^2(r(s)) x_k^{2\beta+p}(s) ds \leq \frac{1}{t} \mathbb{E} V_p(x(0)) + \sum_{k=1}^{n} \max_{\{i \in \mathbb{S}\}} \{ \bar{K}_{k,p}(i) \}. \quad (4.7)
\]

By Lemma 2.1,
\[
\frac{1}{t} \sum_{k=1}^{n} \int_{0}^{t} \phi_k^2(r(s)) x_k^{2\beta+p}(s) ds \geq \min_{\{i \in \mathbb{S}\}} \{ \phi_k^2(i) \} \frac{1}{t} \mathbb{E} \int_{0}^{t} \sum_{k=1}^{n} x_k^{2\beta+p}(s) ds
\geq \min_{\{i \in \mathbb{S}\}} \{ \phi_k^2(i) \} n^{(1-2\beta+p)0} \frac{1}{t} \int_{0}^{t} \mathbb{E} |x(s)|^{2\beta+p} ds,
\]
where \( \phi_k^2(i) = \min_{\{k=1, \ldots, n\}} \{ \phi_k^2(i) \} \). Substituting this inequality into (4.7) and letting \( t \to \infty \) give
\[
\limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} \mathbb{E} |x(s)|^{2\beta+p} ds \leq \frac{p(1-p)}{4} \sum_{k=1}^{n} \max_{\{i \in \mathbb{S}\}} \{ \bar{K}_{k,p}(i) \}
\]
\[
\frac{1}{4 \min_{\{i \in \mathbb{S}\}} \{ \phi_k^2(i) \} n^{(1-2\beta+p)0}}.
\]
Choosing \( \bar{K}_p = [p(1-p) \sum_{k=1}^{n} \max_{\{i \in \mathbb{S}\}} \{ \bar{K}_{k,p}(i) \}] / [4 \min_{\{i \in \mathbb{S}\}} \{ \phi_k^2(i) \} n^{(1-2\beta+p)0}] \) gives the desired result.

**Remark 5.** In Theorems 4.1 and 4.3, the boundedness properties of the functions \( \bar{K}_{k,p}(x, i) \) and \( \tilde{K}_{k,p}(x, i) \) play crucial roles, in which, similar to Theorem 3.6, \( 2\beta > \alpha \) determines their bounded properties. This shows that the Brownian motion plays the most important role in these bounded properties. According to \( \bar{K}_p \) and \( \tilde{K}_p \), the Markov chain \( r(t) \) and the jump process \( N(t, \cdot) \) can have effect on the boundedness, but they cannot determine the bounded properties of \( \bar{K}_{k,p}(x, i) \) and \( \tilde{K}_{k,p}(x, i) \).

### 5. Asymptotic pathwise estimates
This section is devoted to deriving asymptotic pathwise estimates of the solution of (1.2), which shows the solution of (1.2) how to grow in \( \mathbb{R}_+^n \) pathwise.

**Theorem 5.1.** Under the conditions of (3.6), for any initial value \( x(0) > 0 \), the global positive solution \( x(t) \) of (1.2) has the property
\[
\limsup_{t \to \infty} \frac{\log |x(t)|}{\log t} \leq 2, \quad \text{a.s.} \quad (5.1)
\]
Proof. For any $\varepsilon \in (0, 1)$ and $p \in (0, 1)$, applying the generalized Itô formula to $e^{\varepsilon t} \log V_p(x(t))$ gives

$$
\log V_p(x(t)) = e^{-\varepsilon t} \log V_p(x(0)) + e^{-\varepsilon t} \int_0^t e^{\varepsilon s} \left[ \varepsilon \log V_p(x(s)) + \int_{\Xi} Q(y, x(s), r(s)) \nu(dy) + \frac{p}{V_p(x(s))} \left( \sum_{k=1}^n x_k^p(x(s)) \left( f_k(x(s), r(s)) + \frac{p-1}{2} g_k^2(x(s), r(s)) \right) \right) - \frac{1}{2} Z^2(x(s), r(s)) \right] ds 
+ e^{-\varepsilon t} \int_0^t e^{\varepsilon s} Z(x(s), r(s)) dw(s) + e^{-\varepsilon t} \int_0^t e^{\varepsilon s} \int_{\Xi} Q(y, x(s), r(s)) \tilde{N}(ds, dy),
$$

where

$$
Z(x, i) = \frac{1}{V_p(x)} \sum_{k=1}^n p x_k^p g_k(x, i) \quad \text{and} \quad Q(y, x, i) = \log \left[ \frac{\sum_{k=1}^n \left[ 1 + h_k(y, x, i) \right] p x_k^p}{\sum_{k=1}^n x_k^p} \right].
$$

Define

$$
M_1(t) = \int_0^t e^{\varepsilon s} Z(x(s), r(s)) dw(s),
$$
$$
M_2(t) = \int_0^t e^{\varepsilon s} \int_{\Xi} Q(y, x(s), r(s)) \tilde{N}(ds, dy).
$$

Clearly, $M_1(t)$ and $M_2(t)$ are two local martingale with the quadratic variations

$$
\langle M_1, M_1 \rangle_t = \int_0^t e^{2\varepsilon s} Z^2(x(s), r(s)) ds,
$$
$$
\langle M_2, M_2 \rangle_t = \int_0^t \int_{\Xi} e^{2\varepsilon s} Q^2(y, x(s), r(s)) \nu(dy) ds.
$$

For any $\delta > 0$, $\vartheta > 1$ and each integer $n > 0$, for $j = 1, 2$, the exponential martingale inequality yields

$$
P\left\{ \sup_{0 \leq t \leq n} \left[ M_j(t) - \frac{\delta e^{-\varepsilon n}}{2} \langle M_j, M_j \rangle_t \right] \geq \frac{\vartheta \varepsilon n \log n}{\delta} \right\} \leq \frac{1}{n^\vartheta}.
$$

By the Borel–Cantelli lemma, there exists an $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$, there exists an integer $n_0(\omega)$, when $n > n_0(\omega)$, and $n - 1 \leq t \leq n$, according to the definitions of $(M_j, M_j)_t$,

$$
M_j(t) \leq \frac{\delta}{2} e^{-\varepsilon t} (M_j, M_j)_t + \frac{\vartheta}{\delta} e^{(t+1)} \log(t + 1),
$$

which implies that

$$
M_1(t) + M_2(t) \leq \frac{\delta}{2} \int_0^t e^{\varepsilon s} Z^2(x(s), r(s)) ds + \frac{\delta}{2} \int_0^t \int_{\Xi} e^{\varepsilon s} Q^2(y, x(s), r(s)) \nu(dy) ds 
+ \frac{2\vartheta}{\delta} e^{(t+1)} \log(t + 1).
$$
We therefore have
\[
\log V_p(x(t)) \leq e^{-\varepsilon t} \log V_p(x(0)) + e^{-\varepsilon t} \int_0^t e^{\varepsilon s} \left[ \varepsilon \log V_p(x(s)) \right. \\
+ \left. \int_{\Xi} \left( Q(y, x(s), r(s)) + \frac{\delta}{2} Q^2(y, x(s), r(s)) \right) \nu(dy) \right] ds + \frac{2\varepsilon}{\delta} e^{-\varepsilon} \log(t + 1).
\]

Let us now consider the function
\[
\Gamma(x, i) = \varepsilon \log V_p(x) + \int_{\Xi} \left[ Q(y, x, i) + \frac{\delta}{2} Q^2(y, x, i) \right] \nu(dy) \\
+ \frac{p}{V_p(x)} \left[ \sum_{k=1}^n x_k^p \left( f_k(x, i) + \frac{p-1}{2} g_k^2(x, i) \right) \right] ds + \frac{2\varepsilon}{\delta} e^{-\varepsilon} \log(t + 1),
\]

According to the definition of \(Q\), by (H3),
\[
p \log \zeta_{\min}(y, i) \leq \log \left[ \frac{\sum_{k=1}^n \zeta^p_k(y, i) x_k^p}{\sum_{k=1}^n x_k^p} \right] \leq Q(y, x, i) \leq \sum_{k=1}^n \zeta^p_k(y, i) x_k^p \leq p \log \zeta_{\max}(y, i).
\]

where \(\zeta_{\min}(y, i) = \min_{\{k=1, \ldots, n\}} \{\zeta_k(y, i)\}\) and \(\zeta_{\max}(y, i) = \max_{\{k=1, \ldots, n\}} \{\zeta_k(y, i)\}\).

By (H3), it can be observed that there exist functions \(K_1(y, i)\) and \(K_2(y, i)\) such that
\[
Q(y, x, i) \leq p \log \zeta_{\max}(y, i) =: K_1(y, i), \tag{5.2}
\]
\[
Q^2(y, x, i) \leq p^2 [\log^2 \zeta_{\max}(y, i) \lor \log^2 \zeta_{\min}(y, i)] =: K_2(y, i). \tag{5.3}
\]

Therefore there exists a constant \(K_1(i)\) such that
\[
\int_{\Xi} Q(y, x, i) + \frac{\delta}{2} Q^2(y, x, i) \nu(dy) \leq \int_{\Xi} K_1(y, i) + \frac{\delta}{2} K_2(y, i) \nu(dy) \leq K_1(i).
\]

By (H1),
\[
\frac{1}{V_p(x)} \sum_{k=1}^n x_k^p f_k(x, i) \leq \frac{1}{V_p(x)} \sum_{k=1}^n x_k^p [a_k(i)|x|\alpha + b_k(i)] \leq a_{\max}(i)|x|\alpha + b_{\max}(i), \tag{5.4}
\]
where \( a_{\max}(i) = \max\{k=1,\ldots,n\} \{a_k(i)\} \) and \( b_{\max}(i) = \max\{k=1,\ldots,n\} \{b_k(i)\} \). By (H3),
\[
\frac{1}{V_p(x)} \sum_{k=1}^{n} x_k^p g_k^2(x, i) \geq \frac{1}{V_p(x)} \sum_{k=1}^{n} x_k^p \phi_k^2(i) x_k^{2\beta} + \xi_k^2(i)
\]
\[
\geq \phi_{\min}^2(i) \frac{1}{V_p(x)} \sum_{k=1}^{n} x_k^{2\beta + p} + \xi_{\min}^2(i),
\]
(5.5)
where \( \xi_{\min}^2(i) = \min\{k=1,\ldots,n\} \{\xi_k^2(i)\} \). By Lemma 2.1,
\[
\frac{1}{V_p(x)} \sum_{k=1}^{n} x_k^{p+2\beta} = \frac{\sum_{k=1}^{n} x_k^{p+2\beta}}{\sum_{k=1}^{n} x_k^p} \geq n^{(1-\beta-\xi)\lambda_0} x^{2\beta + p} = n^{-(\beta\vee(1-\xi))}\xi^2.
\]
We therefore have
\[
\frac{1}{V_p(x)} \sum_{k=1}^{n} x_k^p g_k^2(x, i) \geq n^{-(\beta\vee(1-\xi))}\phi_{\min}^2(i) |x|^{2\beta} + \xi_{\min}^2(i).
\]
Note that from Lemma 2.1,
\[
V_p(x) = \sum_{k=1}^{n} x_k^p \leq n^{1-\xi} |x|^p,
\]
which implies that
\[
\log V_p(x) \leq \left(1 - \frac{p}{2}\right) \log n + p \log |x|.
\]
Note that there exist \( c_1 \) and \( c_2 > 0 \) such that \( \log |x| \leq c_1 + c_2 |x|^\alpha \). We therefore have
\[
\Gamma(x, i) \leq \epsilon \left(1 - \frac{p}{2}\right) \log n + \epsilon p c_1 + \lambda K_1(i) + p(\epsilon c_2 + a_{\max}(i)) |x|^\alpha + pb_{\max}(i)
\]
\[
- \frac{p(1-p)}{2} n^{-\beta\vee(1-\xi)} \phi_{\min}^2(i) |x|^{2\beta} - \frac{p(1-p)}{2} \xi_{\min}^2(i).
\]
Note that \( 2\beta > \alpha \) and \( p \in (0,1) \), so there exists a constant \( \Gamma(i) \) such that
\[
\max_{\{x>0\}} \{\Gamma(x, i)\} \leq \Gamma(i).
\]
We therefore have
\[
\log V_p(x(t)) \leq e^{-\epsilon t} \log V_p(\xi(0)) + \int_0^t e^{-\epsilon(t-s)} \Gamma(r(s)) ds + \frac{2\theta}{\delta} e^\epsilon \log(t+1)
\]
\[
= e^{-\epsilon t} \log V_p(\xi(0)) + \frac{1}{\epsilon} \max_{\{i: \beta\neq 0\}} \{\Gamma(i)\} (1-e^{-\epsilon t}) + \frac{2\theta}{\delta} e^\epsilon \log(t+1)
\]
which yields
\[
\limsup_{t \to \infty} \frac{\log V_p(x(t))}{\log t} \leq \frac{2\theta e^\epsilon}{\delta}.\]
By Lemma 2.1,
\[
\limsup_{t \to \infty} \frac{\log |x(t)|}{\log t} = \limsup_{t \to \infty} \frac{\log |x(t)|^p}{p \log t} \leq \limsup_{t \to \infty} \frac{\log V_p(x(t))}{p \log t} \leq \frac{2\theta e^\epsilon}{p \delta}.
\]
Letting \( \epsilon \to 0, \theta \to 1, \delta \to 1, p \to 1 \) gives the desired result. \( \square \)
This theorem shows that the trajectory of system (1.2) grows at most polynomially. By virtue of this result, for any \( \epsilon > 0 \), there is a positive random time \( T_\epsilon > 0 \) such that for any \( t \geq T_\epsilon \), \( |x(t)| \leq t^{2+\epsilon} \) with probability 1. In other words, with probability one, the solution will not grow faster than \( t^{2+\epsilon} \).

**Remark 6.** According to the proof of this theorem, the bounded property of the polynomial function \( \Gamma(x,i) \) plays an important role. It can be observed that condition \( 2\beta > \alpha \) determine this bounded property. This shows that the Brownian motion mainly contributes to this asymptotic pathwise estimation.

6. Extinction. This section reveals the role of the stochastic factors that lead to extinction of the population. We discuss two classes of extinction including the \( p \)th moment extinction and the almost sure extinction. Let us give the definition of these two classes of extinction.

**Definition 6.1.** The population system (1.2) is said to almost surely reach the extinction, if

\[
\lim_{t \to \infty} x(t, x(0)) = 0
\]

with probability 1 for any initial value \( x(0) > 0 \).

The population system (1.2) is said to reach the extinction in the sense of the \( p \)th moment, if

\[
\lim_{t \to \infty} \mathbb{E}[|x(t, x(0))|^p] = 0
\]

for any initial value \( x(0) > 0 \).

6.1. Almost sure extinction. The following theorem presents the conditions under which the Brownian motion, the Poisson process, and Markovian switching, contribute to the almost sure extinction of the system (1.2).

**Theorem 6.2.** Under the conditions of Theorem 3.6, define

\[
\gamma := \sum_{i \in S} \pi_i \left[ \tilde{\Gamma}(i) - \frac{1}{2} \xi_{\min}(i) + \int_{\Xi} \log s_{\max}(y, i) \nu(dy) \right],
\]

(6.1)

where

\[
\tilde{\Gamma}(i) = \frac{2\beta - \alpha}{2} \left( \frac{\alpha_1}{\phi_{\min}(i)} \right)^{\frac{\beta}{\alpha - \beta}} \left( \frac{a_{\max}(i)}{\beta} \right)^{\frac{\beta}{\alpha - \beta}} + b_{\max}(i).
\]

Then the regular solution of system (1.2) satisfies

\[
\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq \gamma \quad a.s.
\]

(6.2)

Thus, if \( \gamma < 0 \), system (1.2) becomes extinct almost surely.

**Proof.** For any \( p \in (0, 1) \), applying the generalized Itô formula to \( \log V_p(x) \). This is equivalent to choosing \( \epsilon = 0 \) in the proof of Theorem 5.1 and giving

\[
\log V_p(x(t)) = \log V_p(x(0)) + \int_0^t \left[ \int_{\Xi} Q(y, x(s), r(s)) \nu(dy) - \frac{1}{2} Z^2(x(s), r(s)) \right. \\
+ \frac{p}{V_p(x(s))} \left( \sum_{k=1}^n x_k^p(s) \left( f_k(x(s), r(s)) + \frac{p-1}{2} g_k^2(x(s), r(s)) \right) \right) ds \\
\left. + M^0_1(t) + M^0_2(t), \right)
\]
where \( Z \) and \( Q \) are defined by the proof of Theorem 5.1 and
\[
M_1^0(t) = \int_0^t Z_0(x(s), r(s))dw(s) \quad \text{and} \quad M_2^0(t) = \int_0^t \int_{\Xi} Q(y, x(s), r(s))\tilde{N}(ds, dy).
\]
Clearly, \( M_1^0(t) \) and \( M_2^0(t) \) are two local martingale with the quadratic variations
\[
\langle M_1^0, M_1^0 \rangle_t = \int_0^t Z^2(x(s), r(s))ds,
\]
\[
\langle M_2^0, M_2^0 \rangle_t = \int_0^t \int_{\Xi} Q^2(y, x(s), r(s))\nu(dy)ds.
\]
According to (5.3),
\[
\frac{1}{t} \langle M_2^0, M_2^0 \rangle_t \leq \frac{1}{t} \int_0^t \int_{\Xi} K_2(y, r(s))\nu(dy)ds \leq \max_{\{i \in \mathbb{S}\}} \{ K_2(i) \},
\]
where \( K_2(i) = \int_{\Xi} K_2(y, i)\nu(dy) \). Applying the large number law gives (see [35])
\[
\lim_{t \to \infty} \frac{M_2^0(t)}{t} = 0 \quad \text{a.s.} \quad (6.3)
\]
For any \( \varepsilon \in (0, 1) \) and each integer \( k > 0 \), the exponential martingale inequality gives
\[
\mathbb{P}\left\{ \sup_{1 \leq t \leq n} [M_1^0(t) - \frac{\varepsilon}{2} \langle M_1^0, M_1^0 \rangle_t] \geq \frac{2}{\varepsilon} \log n \right\} \leq \frac{1}{n^2}.
\]
Similar to the proof of Theorem 5.1, by the Borel–Cantelli lemma, there exists an \( \Omega_0 \subseteq \Omega \) with \( \mathbb{P}(\Omega_0) = 1 \) such that for any \( \omega \in \Omega_0 \), there exists an integer \( n(\omega) \), when \( n > n(\omega) \), and \( n - 1 \leq t \leq n \), according to the definitions of \( \langle M_1^0, M_1^0 \rangle_t \),
\[
M_1^0(t) \leq \frac{\varepsilon}{2} \langle M_1^0, M_1^0 \rangle_t + \frac{2}{\varepsilon} \log(t + 1)
\]
We therefore have
\[
\log V_p(x(t)) \leq \log V_p(x(0)) + \int_0^t \left[ \int_{\Xi} Q(y, x(s), r(s))\nu(dy) - \frac{1 - \varepsilon}{2} Z^2(x(s), r(s)) + \frac{2}{\varepsilon} \log(t + 1)
\right.
\]
\[
+ \frac{p}{V_p(x(s))} \left( \sum_{k=1}^n x_k^p \left( f_k(x(s), r(s)) + \frac{p - 1}{2} g_k^2(x(s), r(s)) \right) \right) ds + M_2^0(t),
\]
By (H2),
\[
Z^2(x, i) = \left[ \frac{1}{V_p(x)} \sum_{k=1}^n p x_k^p g_k(x, i) \right]^2 = p^2 \left[ \frac{1}{V_p(x)} \sum_{k=1}^n x_k^p g_k(x, i) \right]^2
\]
\[
\geq p^2 \left[ \sum_{k=1}^n x_k^p \left( \phi_k(i) x_k^{\beta+p} + \xi_k(i) x_k^p \right) \right]^2 \geq p^2 \left[ \sum_{k=1}^n x_k^{\beta+p} \right] + \left[ \sum_{k=1}^n x_k^p \xi_{\min}(i) \right]^2.
\]
By Lemma 2.1,

\[
\sum_{k=1}^{n} x_k^{\beta + p} \geq \frac{n^{1 - \frac{\beta + p}{\beta}} \|x\|^{\beta + p}}{n^{1 - \frac{\beta + p}{\beta} \|x\|^p}} = n^{-\frac{2}{\beta} \|x\|^\beta}.
\]

We therefore have

\[
Z^2(x, i) \geq p^2 n^{-\left(\frac{2}{\beta} \|x\|^\beta\right)} \phi_{\min}^2(i) |x|^{2\beta} + p^2 \varepsilon_{\min}^2(i).
\]

According to (5.4) and (5.5), we have

\[
\frac{1}{V_p(x)} \sum_{k=1}^{n} x_k^{p} f_k(x, i) \leq \alpha_{\max}(i) |x|^\alpha + b_{\max}(i),
\]

\[
\frac{1}{V_p(x)} \sum_{k=1}^{n} x_k^{p} g_k(x, i) \geq n^{-\left(\frac{2}{\beta} \|x\|^\beta\right)} \phi_{\min}^2(i) |x|^{2\beta} + \varepsilon_{\min}^2(i).
\]

These, together with (5.2), give

\[
\log V_p(x(t)) \leq \log V_p(x(0)) + p \int_0^t \left[ \Gamma_1(x(s), r(s), p, \varepsilon) - \frac{(1 - \varepsilon)p + 1 - p}{2} \varepsilon_{\min}(r(s)) \right] ds + \frac{2}{\varepsilon} \log(t + 1) + M_2^0(t),
\]

where

\[
\Gamma_0(x, i, p, \varepsilon) = -\frac{1}{2} \left(1 - \varepsilon\right)n^{-\left(\frac{2}{\beta} \|x\|^\beta\right)} + (1 - p)n^{-\left(\frac{2}{\beta} \|x\|^\beta\right)} \phi_{\min}^2(i) |x|^{2\beta}
+ \alpha_{\max}(i) |x|^\alpha + b_{\max}(i).
\]

Note that $2\beta > \alpha$. When

\[
x^* = \left[ \frac{\alpha \alpha_{\max}(i)}{\beta \left(1 - \varepsilon\right)n^{-\left(\frac{2}{\beta} \|x\|^\beta\right)} + (1 - p)n^{-\left(\frac{2}{\beta} \|x\|^\beta\right)} \phi_{\min}^2(i)} \right]^{\frac{1}{\beta - \alpha}},
\]

we have $\Gamma_0(x^*, i, p, \varepsilon) := \max_{x > 0} \{ \Gamma_0(x, i, p, \varepsilon) \}$. Hence, by virtue of the ergodicity of the Markov chain, we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \left[ \Gamma_0(x(s), r(s), p, \varepsilon) - \frac{(1 - \varepsilon)p + 1 - p}{2} \varepsilon_{\min}(r(s)) \right] ds
+ \int_\Xi \log \varepsilon_{\max}(y, r(s)) \nu(dy) ds
\leq \lim_{t \to \infty} \frac{1}{t} \int_0^t \left[ \Gamma_0(r(s), p, \varepsilon) - \frac{(1 - \varepsilon)p + 1 - p}{2} \varepsilon_{\min}(r(s)) \right] ds
+ \int_\Xi \log \varepsilon_{\max}(y, r(s)) \nu(dy) ds
= \sum_{i \in \mathcal{A}} \pi_i \left[ \Gamma_0(i, p, \varepsilon) - \frac{(1 - \varepsilon)p + 1 - p}{2} \varepsilon_{\min}(i) + \int_\Xi \log \varepsilon_{\max}(y, i) \nu(dy) \right]
=: \gamma_0(p, \varepsilon) \text{ a.s.}
\]
By (6.3), dividing both sides of (6.4) by \( t \) and then letting \( t \to \infty \), in the sense of almost sure,
\[
\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq \limsup_{t \to \infty} \frac{1}{pt} \log V_p(x)
\leq \gamma_0(p, \varepsilon) + \lim_{t \to \infty} \frac{2 \log(t + 1)}{\varepsilon pt} + \lim_{t \to \infty} \frac{M^0_2(t)}{t}
= \gamma_0(p, \varepsilon).
\] (6.5)

Note that letting \( \varepsilon, p \to 0 \) gives
\[
\lim_{p \to 0, \varepsilon \to 0} \gamma_0(p, \varepsilon) = \sum_{i \in S} \pi_i \left[ \Gamma_0(i) - \frac{1}{2} \xi_{\min}^2(i) + \int_{\Xi} \log s_{\max}(y, i) \nu(dy) \right],
\] (6.6)
where
\[
\Gamma_0(i) = -\frac{1}{2} n^{-(\beta \wedge 1)} \phi_{\min}^2(i) \left[ \frac{\alpha n^{\beta \wedge 1} a_{\max}(i)}{\beta \phi_{\min}^2(i)} \right]^{\frac{2 \beta}{2n - \beta}} + a_{\max}(i) \left[ \frac{\alpha n^{\beta \wedge 1} a_{\max}(i)}{\beta \phi_{\min}^2(i)} \right]^{\frac{2 \beta}{2n - \beta}} + b_{\max}(i)
= \frac{2 \beta - \alpha}{2} \left( \frac{\alpha n^{\beta \wedge 1}}{\phi_{\min}^2(i)} \left( \frac{a_{\max}(i)}{\beta} \right) \right)^{\frac{2 \beta}{2n - \beta}} + b_{\max}(i)
= \bar{\Gamma}(i).
\]
Choosing \( \gamma = \lim_{p \to 0, \varepsilon \to 0} \gamma_0(p, \varepsilon) \), (6.5) gives the desired assertion (6.2).

**Remark 7.** In this proof, the boundedness property of the polynomial \( \Gamma_0(x, i, p, \varepsilon) \) plays an important role. It can be observed that the condition \( 2\beta > \alpha \) determines this bounded property, so high order term of the diffusion term \( g \) contributes to this almost sure extinction. Moreover, \( \gamma < 0 \) determines the almost sure extinction of the population. We can easily observe that \(-\xi_{\min}^2(i)\) contributes to \( \gamma < 0 \). This shows that the sufficiently large \( \xi \) will lead to the almost sure extinction of the population. This implies the constant term of the diffusion term \( g \) can play an important role for the almost sure extinction of the population. Note that if \( s_{\max}(y, i) \in (0, 1) \), \( \int_{\Xi} \log s_{\max}(y, i) \nu(dy) \) contributes to \( \gamma < 0 \). This shows that the jump process can also lead to almost sure extinction of the population. The definition of \( \gamma \) shows that when some \( \gamma(i) = \bar{\Gamma}(i) - \xi_{\min}^2(i)/2 + \int_{\Xi} \log s_{\max}(y, i) \nu(dy) > 0 \), but some \( \gamma(i) < 0 \), the whole population can become extinct almost surely. This shows that the Markov chain \( r(t) \) also contributes the almost sure extinction of the population.

### 6.2. \( p \)th-moment extinction

Let us now explore the effects of the Brownian motion \( \omega(t) \), the jump process \( \bar{N}(t, \cdot) \), and the Markovian chain \( r(t) \) on the \( p \)th moment extinction.

**Theorem 6.3.** Under the conditions of Theorem 3.6, for any \( p \in (0, 1) \), the regularity solution of (1.2) satisfies
\[
\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) \leq \Lambda(\eta_p),
\] (6.7)
where \( \eta_p = (\eta_p(1), \ldots, \eta_p(m)) \) is defined by (3.12). If \( \Lambda(\eta_p) < 0 \), system (1.2) is \( p \)th-moment extinct.
Proof. Let $n_0$ be a sufficiently large number. For each $n > n_0$, let $\{\rho_n\}_{n \geq 0}$ be a sequence of the stopping times defined by (3.2). Since the solution $x(t)$ of system (1.2) is regular, $\rho_n \to \infty$ as $n \to \infty$. Recall another stopping time sequence $\{\tau_k\}_{k \geq 0}$ defined by (2.2) on the Markov chain $r(t)$. For any $t \in [\tau_k, \tau_{k+1})$ and $p \in (0, 1)$, applying the generalized Itô formula to function $e^{-\eta_p(r(t))}V_p(x(t))$ yields that

\[
e^{-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(t \wedge \rho_n))}
= e^{-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(\tau_k \wedge \rho_n))}
+ \int_{\tau_k \wedge \rho_n}^{\tau_{k+1} \wedge \rho_n} e^{-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(\tau_k \wedge \rho_n))} X_p(x(s)) G(x(s), r(s)) dw(s)
+ \int_{\tau_k \wedge \rho_n}^{\tau_{k+1} \wedge \rho_n} e^{-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(\tau_k \wedge \rho_n))} \left[-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(s)) + \mathcal{F}V_p(x(s), r(s))\right] ds
+ \int_{\tau_k \wedge \rho_n}^{\tau_{k+1} \wedge \rho_n} \mathbb{E} e^{-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(s) - H(y, x(s), r(s))) - V_p(x(s))\right] \tilde{N}(ds, dy).
\]

Define

\[
l_1^{\tau_k \wedge \rho_n}(s) = e^{-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(s)) G(x(s), r(s)),}
\]

\[
l_2^{\tau_k \wedge \rho_n}(s, y) = e^{-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(s) + H(y, x(s), r(s))) - V_p(x(s))}.\]

Note that $r(s) = r(\tau_k \wedge \rho_n)$ when $s \in [\tau_k \wedge \rho_n, \tau_{k+1} \wedge \rho_n)$. By the definition of $\eta_p$, we know $\mathcal{F}V_p(x, i) - \eta_p(i)V_p(x) \leq 0$ for any $i \in S$. This implies that

\[
e^{-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(\tau_k \wedge \rho_n))} \leq e^{-\eta_p(r(\tau_k \wedge \rho_n)))V_p(x(\tau_k \wedge \rho_n))} + M_1^{\tau_k \wedge \rho_n}(t \wedge \rho_n) + M_2^{\tau_k \wedge \rho_n}(t \wedge \rho_n), \tag{6.8}\]

where

\[
M_1^{\tau_k \wedge \rho_n}(t) = \int_0^t 1_{[\tau_k \wedge \rho_n, \infty)}(s) l_1^{\tau_k \wedge \rho_n}(s) ds,
\]

\[
M_2^{\tau_k \wedge \rho_n}(t) = \int_0^t \mathbb{E} 1_{[\tau_k \wedge \rho_n, \infty)}(s) l_2^{\tau_k \wedge \rho_n}(s, y) \tilde{N}(ds, dy).
\]

Let $\mathcal{D}_t = \sigma\{r(u)_{u \geq 0}, \{w(s)\}_{s \leq t}, \{N(s, \cdot)\}_{s \leq t}\}$ be a $\sigma$-algebra generated by $\{r(u)_{u \geq 0}\}$, $\{w(s)\}_{s \leq t}$ and $\{N(s, \cdot)\}_{s \leq t}$. Then it is easy to show that $\mathcal{D}_t$ is right continuous (see [39, Chapter 1, Theorem 31]). This implies that the filtration $\mathcal{D}_t$ satisfies the usual condition. Clearly, the solution $x(t)$ to (1.2) is adapted for the filtration $\mathcal{D}_t$. Due to the right continuity of the solution $x(t)$, $\rho_n$ is $\mathcal{D}_t$-stopping time by Lemmas 7.2 and 7.6 in [24]. Hence, $1_{[\tau_k \wedge \rho_n, \infty)}(s) l_1^{\tau_k \wedge \rho_n}(s)$ and $1_{[\tau_k \wedge \rho_n, \infty)}(s) l_2^{\tau_k \wedge \rho_n}(s, y)$ are $\mathcal{D}_t$-adapted processes. Moreover, by the regularity of $x(t)$, we have for all $t \in [0, \infty)$

\[
P\left\{ \int_0^t 1_{[\tau_k \wedge \rho_n, \infty)}(s) l_1^{\tau_k \wedge \rho_n}(s) ds < \infty \right\} = 1
\]

\[
P\left\{ \int_0^t \mathbb{E} 1_{[\tau_k \wedge \rho_n, \infty)}(s) l_2^{\tau_k \wedge \rho_n}(s, y) \tilde{N}(ds, dy) ds < \infty \right\} = 1.
\]

Therefore, both $M_1^{\tau_k \wedge \rho_n}(t)$ and $M_2^{\tau_k \wedge \rho_n}(t)$ are local martingales with respect to the filtration $\mathcal{D}_t$ (see [2, Theorem 4.2.12]). Noting that $\tau_k \wedge \rho_n$ is a $\mathcal{D}_t$-stopping time, then by the Doob’s optional sampling theorem, we obtain $\mathbb{E}\{M_i^{\tau_k \wedge \rho_n}(t \wedge \rho_n)\mid \mathcal{D}_{\tau_k \wedge \rho_n}\} = 0$ for $i = 1, 2$. Letting $n \to \infty$, we obtain from (6.8)

\[
\mathbb{E}[e^{-\eta_p(r(\tau_k)))V_p(x(t))] \mid \mathcal{D}_{\tau_k}] \leq e^{-\eta_p(r(\tau_k)))\tau_k} V_p(x(\tau_k)),
\]
which implies that $\mathbb{E}[V_p(x(t)) | \mathcal{D}_{\tau_k}] \leq e^{\eta_p(r(\tau_k))(t-\tau_k)}V_p(x(\tau_k))$. Furthermore, we compute

$$
\mathbb{E}[V_p(x(t)) | \mathcal{D}_{\tau_k-1}] \leq e^{\eta_p(r(\tau_k))(t-\tau_k)}\mathbb{E}[V_p(x(\tau_k)) | \mathcal{D}_{\tau_k-1}] \leq e^{\int_{\tau_k-1}^{\tau_k} \eta_p(r(s))ds}V_p(x(\tau_k-1)).
$$

Repeating this procedure gives

$$
\mathbb{E}[V_p(x(t)) | \mathcal{D}_0] \leq e^{\int_0^t \eta_p(r(s))ds}V_p(x(0)),
$$

which implies that

$$
\mathbb{E}V_p(x(t)) \leq \mathbb{E}[e^{\int_0^t \eta_p(r(s))ds}V_p(x(0))].
$$

By Lemma 2.1, we have

$$
\mathbb{E}[x(t)]^p \leq n^{(\frac{p}{2} - 1)\gamma_0} \mathbb{E}V_p(x) \leq n^{(\frac{p}{2} - 1)\gamma_0} \mathbb{E}[e^{\int_0^t \eta_p(r(s))ds}V_p(x(0))] \mathbb{E}[e^{\int_0^t \eta_p(r(s))ds}V_p(x(0))].
$$

Therefore

$$
\lim_{t \to \infty} \frac{1}{t} \log(\mathbb{E}[x(t)]^p) \leq \lim_{t \to \infty} \frac{1}{t} \log(\mathbb{E}[e^{\int_0^t \eta_p(r(s))ds}V_p(x(0))]) = \Lambda(\eta).
$$

This completes this proof. \(\square\)

**Remark 8.** Since the definition of $\eta_p$ is from the boundedness of the polynomial function $\bar{K}_{k,p}(x,i)$, in which the condition $2\beta > \alpha$ plays a crucial role, so the high order term of the diffusion term contributes to the $p$th moment extinction of the population. By the definition of $\eta_p$ and $\Lambda(\eta_p)$, if a factor can contribute to $\eta_p < 0$ contributing to $\Lambda(\eta_p) < 0$, it will induce the $p$th moment extinction of the population. According to this standard, for $p \in (0,1)$, the constant term of the diffusion term contributes to the $p$th moment extinction since $\eta_p(i)$ contains $-p(1-p)\xi_k^2(i)/2$. According to (3.11), if $\zeta_k(y,i) \in (0,1)$, the jump process also contributes to $\eta_p < 0$ contributing to $\Lambda(\eta_p) < 0$, so it will also induce the $p$th moment extinction of the population.

Because $\Lambda(\eta_p)$ is a functional, we cannot examine the contribution of $r(t)$ for $\Lambda(\eta_p) < 0$ directly. To examine the role of the Markov chain in the $p$th moment extinction of the population, let us introduce the following lemma.

**Lemma 6.4.** Let $\eta_p = (\eta_p(1), \ldots, \eta_p(m))$ and $\Lambda(\eta_p)$ be defined by (3.12) and (2.4), respectively. Then

$$
\lim_{p \to 0} \frac{\Lambda(\eta_p)}{p} = \bar{\eta},
$$

where $\bar{\eta} = \sum_{i \in S} \pi_i \eta(i)$ and

$$
\eta(i) = \max_{k=1, \ldots, n} \left\{ \Gamma_k(i) - \frac{1}{2} \xi_k^2(i) + \int_{\Xi} \log \zeta_k(y,i) \nu(dy) \right\}
$$

with

$$
\Gamma_k(i) := \frac{2\beta - \alpha}{2} \left[ \frac{\alpha}{\phi_k^2(i)} \right]^{\frac{\alpha}{2\beta - \alpha}} \left[ n^{(\frac{p}{2} - 1)\gamma_0} \sum_{j=1}^n a_j(i) \right]^{\frac{2\beta - \alpha}{2(\beta - \alpha)}} + b_k(i).
$$

**Proof.** By the definition of $\Lambda(\eta_p)$ in (2.4), for any $p \in (0,1)$, there exists a $\bar{p}(p) \in \mathcal{P}$ dependent on $p$ such that

$$
\Lambda(\eta_p) = \sum_{i \in S} \eta_p(i) \bar{p}_i(p) - I(\bar{p}(p)).
$$

(6.10)
By using the definition of \( \eta_p(i) \), for each \( i \in \mathbb{S} \),
\[
\lim_{p \to 0} \frac{\eta_p(i)}{p} = \lim_{p \to 0} \max_{k=1,\ldots,n} \left\{ K_{k,p}(i) - \frac{1-p}{2} \xi_k^2(i) + \int_{\Xi} \frac{\zeta_k(y,i)}{p} \nu(dy) \right\} = \eta(i)
\]
which implies that
\[
\lim_{p \to 0} \frac{1}{p} \sum_{i \in \mathbb{S}} \pi_i \eta_p(i) = \sum_{i \in \mathbb{S}} \pi_i \eta(i) = \bar{\eta}.
\]  
(6.12)

Note that \( \sum_{i \in \mathbb{S}} \pi_i \eta_p(i) \leq \Lambda(\eta_p) \leq \max_{i \in \mathbb{S}} \{ \eta_p(i) \} \). By (6.11) and (6.12), we have
\[
\gamma = \lim_{p \to 0} \frac{1}{p} \sum_{i \in \mathbb{S}} \pi_i \eta_p(i) \leq \lim_{p \to 0} \frac{\Lambda(\eta_p)}{p} \leq \lim_{p \to 0} \frac{1}{p} \max_{i \in \mathbb{S}} \{ \eta_p(i) \} = \max_{i \in \mathbb{S}} \{ \eta(i) \}.
\]
By (6.10),
\[
\lim_{p \to 0} \frac{\Lambda(\eta_p)}{p} = \lim_{p \to 0} \sum_{i \in \mathbb{S}} \frac{\eta_p(i)}{p} \bar{p}_i(p) - \lim_{p \to 0} \frac{I(\bar{p}(p))}{p} \in [\bar{\eta}, \max_{i \in \mathbb{S}} \eta(i)].
\]  
(6.13)

(6.11) shows that \( \lim_{p \to 0} \sum_{i \in \mathbb{S}} \frac{\eta_p(i)}{p} \bar{p}_i(p) \) is bounded since \( \bar{p}_i(p) \in (0,1) \) for all \( i \in \mathbb{S} \). The definition (A.1) of rate function \( I(p) \), together with (6.13) gives
\[
0 \leq \lim_{p \to 0} \frac{I(\bar{p}(p))}{p} \leq \lim_{p \to 0} \sum_{i \in \mathbb{S}} \frac{\eta_p(i)}{p} \bar{p}_i(p) - \gamma,
\]
which implies that \( \lim_{p \to 0} I(\bar{p}(p))/p \) is bounded. This shows that \( \lim_{p \to 0} I(\bar{p}(p)) = 0 \). By the lower semicontinuity of the rate function \( I \), we have
\[
0 = \lim_{p \to 0} I(\bar{p}(p)) \geq \lim \inf_{p \to 0} I(\bar{p}(p)) = I(\lim_{p \to 0} \bar{p}(p)) \geq 0,
\]
which implies \( I(\lim_{p \to 0} \bar{p}(p)) = 0 \). Note that Theorem A.1 shows that \( I(p) = 0 \) if and only if \( p = \pi \). This implies that \( \lim_{p \to 0} \bar{p}(p) = \pi \). This, together with (6.11), gives
\[
\lim_{p \to 0} \sum_{i \in \mathbb{S}} \frac{\eta_p(i)}{p} \bar{p}_i(p) = \sum_{i \in \mathbb{S}} \eta(i) \pi_i = \bar{\eta},
\]
By (6.13)
\[
\lim_{p \to 0} \frac{\Lambda(\eta_p)}{p} = \bar{\eta} - \lim_{p \to 0} \frac{I(\bar{p}(p))}{p} \in [\bar{\eta}, \max_{i \in \mathbb{S}} \eta(i)].
\]
Note that \( I(p) \geq 0 \) for any \( p \in \mathcal{P} \). We therefore have
\[
\lim_{p \to 0} \frac{I(\bar{p}(p))}{p} = 0,
\]
which implies that
\[
\lim_{p \to 0} \frac{\Lambda(\eta_p)}{p} = \bar{\eta},
\]
as required.

This lemma leads to the following theorem.

**Theorem 6.5.** Let \( \bar{\eta} \) be defined by (6.9). Under the conditions of Theorem 5.1, the solution of system (1.2) satisfies
\[
\lim_{p \to 0} \left[ \frac{1}{p} \lim_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) \right] = \bar{\eta}.
\]  
(6.14)

Thus, for sufficiently small \( p > 0 \), system (1.2) is \( p \)-th moment extinct if \( \bar{\eta} < 0 \).
Remark 9. This theorem shows that for sufficiently small $p > 0$, the Brownian motion and the jump process contribute to the $p$th moment extinction similar to the case with Theorem 6.3, but it can show that the Markov chain $r(t)$ plays the same role as the almost sure extinction, namely, the Markov chain can contribute to the $p$th moment extinction.

According to the stability theory (for example, [35]), under the linear growth condition, the $p$th moment stability implies the almost sure stability. However, the Kolmogorov-type system (1.2) doesn’t satisfy the linear growth condition, we cannot employ this existing result. In this paper, since $\gamma$ and $\tilde{\gamma}$ have the similar expressions, we can look for conditions which leads to these two classes of extinctions. For example, let us define

$$
\mu = \sum_{i \in S} \pi_i \left[ \hat{\Gamma}(i) - \frac{1}{2} \hat{\xi}_{\min}^2(i) + \int_{\Xi} \log \min \{y, i\} \nu(dy) \right],
$$

where

$$
\hat{\Gamma}(i) = \frac{2\beta - \alpha}{2} \left( \frac{\alpha}{\phi_{\min}^2(i)} \right)^{\frac{2\beta}{\alpha - \beta}} \left( \frac{n^{\beta}}{\beta} \right)^{\frac{2\beta}{\alpha - \beta}} + b_{\max}(i).
$$

Clearly, $\mu \geq \gamma$ and $\mu \geq \tilde{\gamma}$, so if $\mu < 0$, system (1.2) becomes extinct almost surely and $p$th moment.

7. Lotka-Volterra system under regime-switching jump diffusion perturbations. As a special case of the Kolmogorov-type population dynamic system, the Lotka-Volterra system is examined more widely, see for example [17,22,25,26,41]. This section will apply the results of the Kolmogorov-type system to the Lotka-Volterra systems. Let us consider the following Lotka-Volterra system with regime-switching jump diffusion perturbations as follows:

$$
dx(t) = \text{diag}(x_1(t), \ldots, x_n(t)) \left[ (A(r(t))x(t) + b(r(t)))dt + (\Phi(r(t))x(t) + \xi(r(t)))dw(t) + \int_{\Xi} h(y, r(t-))N(dt, dy) \right],
$$

where for any $i \in S$, $A(i) = [\alpha_{ij}(i)]_{n \times n}$, $b(i) = (b_1(i), \ldots, b_n(i))'$, $\Phi(i) = [\phi_{ij}(i)]_{n \times n}$, $\xi(i) = (\xi_1(i), \ldots, \xi_n(i))'$ and $h(\cdot, i) = (h_1(\cdot, i), \ldots, h_n(\cdot, i))'$ with $h_k(y, i) \neq -1$ for any $y \in \Xi$. Let us assume that $\Phi(i) = [\phi_{ij}(i)]_{n \times n}$ satisfies

$$
\phi_{kj} \geq 0, \quad \phi_{kk} > 0 \quad \text{for} \quad 0 \leq k, j \leq n.
$$

This Lotka-Volterra system can be seen as Kolmogorov-type system (1.2) with $\alpha = 1$, $\beta = 1$, $f(x, i) = A(i)x + b(i)$, $g(x, i) = \Phi(i)x + \xi(i)$ and $h(y, x, i) = h(y, i)$.

It is obvious that (H3) holds with $\zeta_k(y, i) = \zeta_k(y, i) = 1 + h_k(y, i)$. Let us check (H1) and (H2) for system (3.4). By using Lemma 2.1, for any $k = 1, \ldots, n$, $i \in S$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^{n \circ},$

$$
f_k(x, i) = \sum_{j=1}^n a_{kj}(i)x_j + b_k(i)
\leq \sum_{j=1}^n |a_{kj}(i)|x_j + b_k(i)
\leq \sqrt{n} \max_{j=1,\ldots,n} \{|a_{kj}(i)| + b_k(i)\},
$$

(7.3)
which implies that (H1) holds with \( a_k(i) = \sqrt{n} \max_{j=1,...,n} \{a_{kj}(i)\} \).

Under condition (7.2), for any \( k = 1, \ldots, n, i \in S \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \),

\[
|g_k(x, i)| = \sum_{j=1}^n \phi_{kj}(i)x_j + \xi_k(i) \\
\geq \phi_{kk}(i)x_k + \xi_k(i),
\]

which implies that (H2) holds with \( \phi_k(i) = \phi_{kk}(i) \).

Applying the results of Kolmogorov-type system (1.2) to the Lotka-Volterra system (7.1) gives the following theorem.

**Theorem 7.1.** Under condition (7.2), Lotka-Volterra system (7.1) has a global positive solution \( x(t) \) for any \( t \geq 0 \), and the following properties hold:

(i) for any \( p \in (0, 1) \), there exists a constant \( K_p \) independent of the initial value such that

\[
\limsup_{t \to \infty} \mathbb{E}[x(t)]^p \leq K_p; \quad (7.5)
\]

(ii) for any \( \epsilon \in (0, 1) \), there is a positive constant \( \tilde{K} = \tilde{K}(\epsilon) \) such that

\[
\limsup_{t \to \infty} \mathbb{P}\{ |x(t)| \leq \tilde{K} \} \geq 1 - \epsilon; \quad (7.6)
\]

(iii) for any \( p \in (0, 1) \), there exists a constant \( \tilde{K}_p \) depending on \( p \) such that

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}[x(s)]^{2+p} ds \leq \tilde{K}_p; \quad (7.7)
\]

(iv) for any initial value \( x(0) > 0 \),

\[
\limsup_{t \to \infty} \frac{\log |x(t)|}{\log t} \leq 2, \quad \text{a.s.}; \quad (7.8)
\]

(v) if

\[
\gamma := \sum_{i \in S} \pi_i \left[ \frac{n \phi_{max}^2(i)}{2 \phi_{min}^2(i)} + b_{max}(i) - \frac{1}{2} \xi_{min}^2(i) + \int_{\Xi} \log(1 + h_{max}(y, i)) \nu(dy) \right] < 0,
\]

then system (7.1) is almost surely extinct;

(vi) for any \( p \in (0, 1) \), let

\[
\eta_p(i) = \max_{\{k=1, \ldots, n\}} \left\{ \frac{p}{2n(1-p)(1+p)^2 \phi_{kk}^2(i)} \left[ pa_k(i) + \sum_{j=1}^n a_j(i) \right]^2 + pb_k(i) \right. \]

\[
- \frac{p(1-p)}{2} \xi_k^2(i) + \int_{\Xi} \left[ (1 + h_k(y, i))^p - 1 \right] \nu(dy) \\right\}
\]

and \( \eta_p = (\eta_p(1), \ldots, \eta_p(m)) \). If \( \Lambda(\eta_p) < 0 \), system (7.1) is pth moment extinct;

(vii) let

\[
\eta(i) = \max_{\{k=1, \ldots, n\}} \left\{ \frac{1}{2n \phi_{kk}^2(i)} \left[ \sum_{j=1}^n a_j(i) \right]^2 + b_k(i) - \frac{1}{2} \xi_k^2(i) + \int_{\Xi} \log(1 + h_k(y, i)) \nu(dy) \right\}
\]

If \( \bar{\eta} = \sum_{i \in S} \pi_i \eta(i) < 0 \), for sufficiently small \( p > 0 \), system (7.1) is pth moment extinct.

In this theorem, it is obvious that \( \bar{\eta} \leq \gamma \), so \( \gamma < 0 \) implies that Lotka-Volterra system (7.1) becomes extinct almost surely and pth moment for the sufficiently small \( p > 0 \).
Appendix A. Appendix. We recall some large deviations results, which can be found in [9] and [10]. Let \((X, B, ||\cdot||)\) be a polish space, and \(p(t, x, dy)\) be the transition probability of \(X\)-valued a continuous time Markov process \(X(t)\) with \(X(0) = x\). Denote by \(T_t\) the strong continuous Markovian semigroup with respect to \(X(t)\) such that \(T_t : C(X) \rightarrow C(X)\) (Feller semigroup). Let \(L\) be the infinitesimal generator of the semigroup \(T_t\) having domain \(D \subset C(X)\). Then \(D\) is dense in \(C(X)\).

Let \(\mathcal{P}\) be the space of all probability measures on the state space \(X\). For any \(p \in \mathcal{P}\), define the rate function by

\[
I(p) = -\inf_{u > 0, u \in D} \int_X \left( \frac{L u}{u} \right) p(dx).
\]  

(A.1)

Then the rate function is non-negative and holds the following property (see [14]):

**Theorem A.1.** \(I(p) = 0\) if and only if \(p\) is the invariant measure of the transition probability function \(p(t, x, dy)\).

Let \(\Omega_x\) be the space of \(X\)-valued càdlàg \(X(t), 0 \leq t < \infty\), with \(X(0) = x\). For each \(t > 0\) \(\omega \in \Omega_x\) and Borel set \(A \subset X\), define the occupation time measure by

\[
L_t(\omega, A) = \frac{1}{t} \int_0^t 1_{\{X(s) \in A\}} ds,
\]  

(A.2)

which is the proportion of time up to time \(t\) that a particular sample path \(\omega = X(\cdot)\) spends in the set \(A\). Note that for each \(t > 0\) and each \(\omega\), \(L_t(\omega, \cdot)\) is a probability measure.

**Remark A.1.** For the Markov chain \(r(t)\) defined in this paper, we have the following assertions (see [10])

\[
L_t(\omega, i) = \frac{1}{t} \int_0^t 1_{\{r(s) = i\}} ds, \quad i \in \mathbb{S}
\]  

(A.3)

and

\[
I(p) = -\inf_{u_1, \ldots, u_m > 0, u_i \in \mathbb{S}} \sum_{i,j \in \mathbb{S}} \frac{p_i \gamma_{ij} u_j}{u_i}.
\]  

(A.4)

By Theorem A.1, \(I(\pi) = 0\), where \(\pi = (\pi_1, \ldots, \pi_m)\) is the stationary distribution of \(r(t)\).

**Theorem A.2.** If \(\Phi\) is a real-valued weakly continuous functional on \(\mathcal{P}\), then

\[
\Lambda(\Phi) := \lim_{t \to \infty} \frac{1}{t} \log \{\mathbb{E} \exp[t\Phi(L_t(\omega, \cdot))]\} = \sup_{p \in \mathcal{P}} \{\Phi(p) - I(p)\},
\]  

(A.5)

where \(L_t(\omega, \cdot)\) and \(I(p)\) are defined by (A.2) and (A.4), respectively.

REFERENCES

[1] W. J. Anderson, *Continuous-Time Markov Chains*, Springer-Verlag, New York, 1991.
[2] D. Applebaum, *Levy Processes and Stochastic Calculus*, 2nd Edition, Cambridge University Press, New York, 2009.
[3] D. Applebaum and M. Siakalli, *Asymptotic stability properties of stochastic differential equations driven by Lévy noise*, *Journal of Applied Probability*, 46 (2009), 1116–1129.
[4] A. Bahar and X. Mao, *Stochastic delay Lotka-Volterra model*, *Journal of Mathematical Analysis and Applications*, 292 (2004), 364–380.
[5] A. Bahar and X. Mao, *Stochastic delay population dynamics*, *International Journal of pure and applied mathematics*, 11 (2004), 377–400.
[6] J. Bao, X. Mao, G. Yin and C. Yuan, *Competitive Lotka-Volterra population dynamics with jumps*, *Nonlinear Analysis: Theory, Methods & Applications*, 74 (2011), 6601–6616.
[7] J. Bao and C. Yuan, Stochastic population dynamics driven by Lévy noise, *Journal of Mathematical Analysis and Applications*, 391 (2012), 363–375.
[8] N. H. Dang, N. H. Du and G. Yin, Existence of stationary distributions for Kolmogorov systems of competitive type under telegraph noise, *Journal of Differential Equations*, 257 (2014), 2078–2101.
[9] A. Dembo and O. Zeitouni, *Large Deviation Techniques and Applications*, 2nd ed., Jones and Bartlett, Boston, 1998.
[10] J. D. Deuschel and D. W. Stroock, *Large Deviations*, Academic Press, Boston, 1989.
[11] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time, I, *Communications on Pure and Applied Mathematics*, 28 (1975), 1–47.
[12] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time, II, *Communications on Pure and Applied Mathematics*, 28 (1975), 279–301.
[13] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time, III, *Communications on Pure and Applied Mathematics*, 29 (1976), 389–461.
[14] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time, IV, *Communications on Pure and Applied Mathematics*, 36 (1983), 183–212.
[15] N. H. Du and N. H. Dang, Dynamics of Kolmogorov systems of competitive type under the telegraph noise, *Journal of Differential Equations*, 250 (2011), 386–409.
[16] N. H. Du, D. H. Nguyen and G. Yin, Conditions for permanence and ergodicity of certain stochastic predator-prey models, *to appear in Journal of Applied Probability*.
[17] T. Faria and J. J. Oliveira, Local and global stability for Lotka-Volterra systems with distributed delays and instantaneous negative feedbacks, *Journal of Differential Equations*, 244 (2008), 1049–1079.
[18] B. M. Gary, A functional equation characterizing monomial functions used in permanence theory for ecological differential equation, *Universitatis Iagellonicae acta mathematica*, 42 (2004), 69–76.
[19] T. C. Gard, Persistence in stochastic food web models, *Bulletin of Mathematical Biology*, 46 (1984), 357–370.
[20] T. C. Gard, Stability for multispecies population models in random environments, *Nonlinear Analysis*, 10 (1986), 1411–1419.
[21] T. C. Gard, *Introduction to Stochastic Differential Equations*, Dekker, New York, 1988.
[22] K. Gopalsamy, Global asymptotic stability in a periodic Lotka-Volterra system, *The Journal of the Australian Mathematical Society. Series B, Applied Mathematics*, 27 (1985), 66–72.
[23] X. Han, Z. Teng and D. Xiao, Persistence and average persistence of a nonautonomous Kolmogorov system, *Chaos Solitons Fractals*, 30 (2006), 748–758.
[24] O. Kallenberg, *Foundations of Modern Probability*, 2nd Edition, Springer, 2002.
[25] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic press, Boston, 1993.
[26] Y. Li and Y. Kuang, Periodic solutions of periodic delay Lotka-Volterra equations and systems, *Journal of Mathematical Analysis and Applications*, 255 (2001), 260–280.
[27] M. Liu and K. Wang, Stochastic Lotka-Volterra systems with Lévy noise, *Journal of Mathematical Analysis and Applications*, 410 (2014), 750–763.
[28] Q. Luo and X. Mao, Stochastic population dynamics under regime switching, *Journal of Mathematical Analysis and Applications*, 334 (2007), 69–84.
[29] Q. Luo and X. Mao, Stochastic population dynamics under regime switching II, *Journal of Mathematical Analysis and Applications*, 355 (2009), 577–593.
[30] E. Lungu and B. Øksendal, Optimal harvesting from a population in a stochastic crowded environment, *Mathematical Biosciences*, 145 (1997), 47–75.
[31] E. Lungu and B. Øksendal, Optimal Harvesting from Interacting Populations in a Stochastic Environment, *Bernoulli*, 7 (2001), 527–539.
[32] X. Mao, G. Marion and E. Renshaw, Environmental noise suppresses explosion in population dynamics, *Stochastic Processes and their Applications*, 97 (2002), 95–110.
[33] X. Mao, Delay population dynamics and environmental noise, *Stochastics and Dynamics*, 5 (2005), 149–162.
[34] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, 2006.
[35] X. Mao, *Stochastic Differential Equations and Applications*, 2nd Edition, Horwood, Chichester, UK, 2008.
[36] J. D. Murray, *Mathematical Biology, I. an Introduction*, 3rd Edition, Springer, 2002.
[37] H. D. Nguyen, N. H. Du and G. Yin, Existence of stationary distributions for Kolmogorov systems of competitive type under telegraph noise, *Journal of Differential Equations*, 257 (2014), 2078–2101.
[38] S. Pang, F. Deng and X. Mao, Asymptotic properties of stochastic population dynamics, *Dynamics of Continuous Discrete and Impulsive Systems Series A*, 15 (2008), 603–620.
[39] P. E. Protter, *Stochastic Integration and Differential Equations*, 2nd Edition, Springer, 2004.
[40] M. Slatkin, The dynamics of a population in a Markovian environment, *Ecology*, 59 (1978), 249–256.
[41] Y. Takeuchi, *Global Dynamical Properties of Lotka-Volterra Systems*, World Scientific, Singapore, 1996.
[42] Y. Takeuchi, N. H. Du, N. T. Hieu and K. Sato, Evolution of predator-prey systems described by a Lotka-Volterra equation under random environment, *Journal of Mathematical Analysis and applications*, 323 (2006), 938–957.
[43] B. Tang and Y. Kuang, Permanence in Kolmogorov-type systems of nonautonomous functional differential equations, *Journal of Mathematical Analysis and Applications*, 197 (1996), 427–447.
[44] Z. Teng, *The almost periodic Kolmogorov competitive systems*, *Nonlinear Analysis*, 42 (2000), 1221–1230.
[45] F. Wu and S. Hu, Stochastic functional Kolmogorov-type population dynamics, *Journal of Mathematical analysis and applications*, 347 (2008), 534–549.
[46] F. Wu and S. Hu, Suppression and stabilisation of noise, *International Journal of Control*, 82 (2009), 2150–2157.
[47] F. Wu and Y. Xu, Stochastic Lotka-Volterra population dynamics with infinite delay, *SIAM Journal on Applied Mathematics*, 70 (2009), 641–657.
[48] F. Wu, S. Hu and Y. Liu, Positive solution and its asymptotic behaviour of stochastic functional Kolmogorov-type system, *Journal of Mathematical Analysis and Applications*, 364 (2010), 104–118.
[49] F. Wu and G. Yin, Environmental noise impact on regularity and extinction of population systems with infinite delay, *Journal of Mathematical Analysis and Applications*, 396 (2012), 772–785.
[50] G. Yin and F. Xi, Stability of regime-switching jump diffusions, *SIAM Journal on Control and Optimization*, 48 (2010), 4525–4549.
[51] G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*, Springer, New York, 2010.
[52] G. Yin, G. Zhao and F. Wu, Regularization and stabilization of randomly switching dynamic systems, *SIAM Journal on Applied Mathematics*, 72 (2012), 1361–1382.
[53] C. Zhu and G. Yin, On hybrid competitive Lotka-Volterra ecosystems, *Nonlinear Analysis: Theory, Methods & Applications*, 71 (2009), e1370–e1379.
[54] C. Zhu and G. Yin, On competitive Lotka-Volterra model in random environments, *Journal of Mathematical Analysis and Applications*, 357 (2009), 154–170.
[55] X. Zong, F. Wu, G. Yin and Z. Jin, Almost sure and pth-moment stability and stabilization of regime-switching jump diffusion systems, *SIAM Journal on Control and Optimization*, 52 (2014), 2595–2622.

Received August 2015; revised November 2015.

E-mail address: wufuke@hust.edu.cn
E-mail address: gyin@math.wayne.edu
E-mail address: zjin@unimelb.edu.au