Stability and synchronization of a fractional BAM neural network system of high-order type

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Abstract

In this paper, stability and synchronization of a Caputo fractional BAM neural network system of high-order type and neutral delays are examined. A mixture of properties of fractional calculus, Laplace transform, and analytical techniques is used to derive Mittag-Leffler stability and synchronization for two classes of activation functions. A fractional version of Halanay inequality is utilized to deal with the fractional character of the system and some suitable evaluations and handling to cope with the higher order feature. Another feature is the treatment of unbounded activation functions. Explicit examples to validate the theoretical outcomes are shown at the end.

Keywords: Mittag-Leffler stability, full synchronization, higher-order, bidirectional associative memory, Caputo fractional derivative, distributed delay, delay of neutral type

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1. Introduction

The synchronization process involves the coherence between coupled systems over time. It can be in different forms: full synchronization, anti-synchronization, delayed synchronization, generalized synchronization, phase synchronization, projective synchronization or finite time synchronization. This process is very effective in many fields of technology such as computer science [16, 21, 26, 29].

One of the most efficient neural networks system is bidirectional associative memory (BAM) neural networks system introduced by Kosko in 1987. They take the form of recurrent neural networks and widen the single-layer self-associated Hebbian correlator. These networks are effectively used in signal and image processing, pattern recognition, and optimization problems. With the aim of describing and modelling the dynamics of complex neural reactions, the incorporation of information about the past state derivative in systems is necessary. This kind of delay is referred to as neutral delay [11, 12, 32]. One of the negative effects of delays on systems are oscillations, divergences, chaos and bifurcations [6]. On the other hand, owing to the higher approximation...
property, faster convergence rate, greater storage capacity and greater fault tolerance, higher-order neural networks are more advantageous compared to the lower-order neural networks [4,8,25]. Moreover, fractional derivatives reflect the reliance of states on their past history, and therefore fractional models are capable of portraying adequately many complex phenomena and processes [3-8,25].

In this paper, we consider the following fractional higher-order BAM neural network with distributed delays as well as delays of neutral type

\[
\begin{align*}
D^\mu_a[ x_p(t) - cx_p(t) + \sum_{q,s=1}^{n_2} d_{qps} \int_0^\infty k_{qps}(s)g_q(y_q(t-s))ds \\
\times \int_0^\infty h_{qps}(s)g_s(y_s(t-s))ds + I_p, \ t > 0, \ p = 1, \ldots, n_1, \\
D^\mu_a[ y_q(t) - cy_q(t) - \sum_{p,s=1}^{n_1} d_{psr} \int_0^\infty \bar{k}_{psr}(s)\bar{g}_p(x_p(t-s))ds \\
\times \int_0^\infty \bar{h}_{psr}(s)\bar{g}_r(x_r(t-s))ds + J_q, \ t > 0, \ q = 1, \ldots, n_2, \\
x_p(t) = \phi_p(t), \ t \leq 0, \ p = 1, \ldots, n_1, \\
y_q(t) = \varphi_q(t), \ t \leq 0, \ q = 1, \ldots, n_2,
\end{align*}
\]

where \( n_1 + n_2 \) is the number of neurons; \( x_p \) and \( y_q \) correspond to the state of the \( i \)-th neuron and the \( j \)-th neuron at time \( t \); \( a_p > 0 \) and \( \bar{a}_q > 0 \) denote the dissipation coefficients; \( c > 0 \) and \( \bar{c} > 0 \) represent the coefficients of the temporal derivation of the lagged states; \( d_{qps} \) and \( \bar{d}_{psr} \) account for the second order parameters; \( g_q \) and \( \bar{g}_p \) are the activation functions; \( \mu \) corresponds to the neutral delays; \( k_{qps}, h_{qps}, \bar{k}_{psr} \) and \( \bar{h}_{psr} \) represent the distributed delay kernels; \( I_p \) and \( J_q \) refer to external inputs; finally, \( \phi_p \) and \( \varphi_q \) are the history functions of the \( p \)-th and the \( q \)-th state.

Numerous investigations have been carried out on fractional neural network systems. In [5,30,31], stability and synchronization of the following fractional Hopfield NN system were examined

\[
\begin{align*}
\frac{d}{dt} w_k(t) = -\alpha_k w_k(t) + \sum_{i=1}^{p} \beta_{ki} h_i(w_i(t)) + \chi_k, \ k = 1, 2, \ldots, p, \\
w_k(0) = w_{k0}.
\end{align*}
\]

Both Mittag-Leffler stability of the equilibrium and synchronization via linear feedback controls were proved, utilizing an extended second method of Lyapunov for Lipschitz continuous activation functions in [30]. In contrast, for discontinuous activation functions, asymptotic stability results were investigated in [31], whilst Mittag-Leffler synchronization was discussed in [3] through linear feedback controls. The authors in [5,31] used Fillipov theory, Laplace transform technique and fractional differential inequalities. Furthermore, fractional-order bidirectional associative memory (BAM) neural networks with delays were treated in [1,4,8,14,15,20,23,24,27,28]. In [4], sufficient conditions were established ensuring the asymptotic stability for a fractional BAM NN with leakage delays. Yang et al. [27] discussed the asymptotic stability of a fractional BAM NN with discrete delays via a fractional inequality for Lipschitz continuous activation functions. The following fractional
BAM NN with delays depending of time was studied in [15].

\[
\begin{aligned}
\frac{d}{dt} \zeta_k(t) &= -\alpha_k \zeta_k(t) + \sum_{l=1}^{q} \beta_{kl} \eta_l(w_l(t)) + \sum_{l=1}^{q} \delta_{kl} \eta_l(w_l(t - \mu_{kl}(t))) + \chi_k, \quad k = 1, 2, ..., p, \\
\frac{d}{dt} \xi_l(t) &= -\alpha_l \xi_l(t) + \sum_{k=1}^{p} \beta_{lk} \xi_k(z_k(t)) + \sum_{k=1}^{p} \beta_{lk} \xi_k(z_k(t - \nu_{lk}(t))) + \chi_l, \quad l = 1, 2, ..., q,
\end{aligned}
\]

where the asymptotic stability of the stationary state was achieved through a fractional inequality and Lyapunov functionals. In [1], Cao and Bai proved the stability for a fractional BAM NN with distributed delays using the M-matrix theory, Laplace transform and some inequalities such as Gronwall inequality. Besides, the stability for fractional BAM NN with delays and impulses were studied in [20, 23], considering Lipschitz continuous activation functions and linear impulsive operators. Moreover, in [24], a new generalized Gronwall inequality was proved to examine the stability of Mittag-Leffler kind for a fractional BAM NN model, whilst, Mittag-Leffler synchronization results through delayed controls were shown in [28].

In this paper, we examine the Mittag-Leffler stability and synchronization for both bounded and unbounded activation functions. Apart from Hölder continuous functions, no works have been reported on unbounded activation functions so far. For this aim we shall utilize, in addition to these techniques, a new Halanay inequality of fractional order with neutral delay and distributed delay for a wide family of delay kernels defined by an integral condition. The unbounded case is more problematic as it requires more skills. In particular, it requires appropriate manipulations and adequate estimations.

This paper is arranged as follows: In Section 2, we present some notation, definitions, and lemmas. Mittag-Leffler stability of the equilibrium is discussed for the bounded and unbounded cases in Section 3 and Section 4, respectively. The synchronization result is examined in Section 5. Finally, numerical illustrations are provided to confirm the findings in Section 6.

2. Preliminaries

This section contains some specific assumptions, definitions of fractional derivatives and lemmas. For the sake of brevity, we shall omit the ranges of indexes. In all our assumptions and statements, it is understood that the indexes \(p, r\) and \(q, s\) range from 1 to \(n_1\) and from 1 to \(n_2\), respectively.

\textbf{(A1)} The delay kernel functions \(k_{qps}, h_{qps}, \hat{k}_{pqr}\) and \(\hat{h}_{pqr}\) are piecewise continuous and non-negative such that

\[
\hat{k}_{qps} = \int_0^\infty k_{qps}(s)ds < \infty, \quad \hat{h}_{qps} = \int_0^\infty h_{qps}(s)ds < \infty,
\]

\[
\hat{k}_{pqr} = \int_0^\infty k_{pqr}(s)ds < \infty, \quad \hat{h}_{pqr} = \int_0^\infty h_{pqr}(s)ds < \infty.
\]

\textbf{(A2)} The functions \(g_q\) and \(\bar{g}_p\) satisfy for some constants \(G, \bar{G} > 0\)

\[
|g_q(x)| \leq G, \quad |\bar{g}_p(x)| \leq \bar{G}, \quad x \in \mathbb{R}.
\]

\textbf{(A3)} The functions \(g_q\) and \(\bar{g}_p\) are Lipschitz continuous on \(\mathbb{R}\) with Lipschitz constants \(L_q\) and \(M_p\) such that

\[
|g_q(x) - g_q(y)| \leq L_q|x - y|, \quad |\bar{g}_p(x) - \bar{g}_p(y)| \leq M_p|x - y|, \quad \forall x, y \in \mathbb{R}.
\]
Definition 2.1. The point \((x_p^*, y_q^*)\) is an equilibrium of the system (1), if
\[
0 = -a_p x_p^* + \sum_{q,s=1}^{n_2} d_{qps} \hat{k}_{qps} g_q(y_q^*) \hat{h}_{qps} g_s(y_s^*) + I_p,
\]
\[
0 = -\bar{a}_q y_q^* + \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \hat{k}_{pqr} \bar{g}_p(x_p^*) \hat{h}_{pqr} \bar{g}_r(x_r^*) + J_q.
\]

According to the uniform boundedness (A2) and Lipschitz continuity (A3), there exists a unique equilibrium \((x_p^*, y_q^*)\).

Definition 2.2. For any measurable function \(f\), the Riemann-Liouville fractional integral of order \(\delta > 0\) is equal to
\[
I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s) ds, \quad \delta > 0
\]
if the integral term can be found. Notice that \(\Gamma(\delta)\) is the Gamma function.

Definition 2.3. The Caputo fractional derivative of order \(\delta\) is given by
\[
D^\delta_C f(z) = \frac{1}{\Gamma(1-\delta)} \int_0^z (z-w)^{-\delta} f'(w) dw, \quad 0 < \delta < 1
\]
if the integral term can be found.

The one-parametric and two-parametric Mittag-Leffler functions are expressed by
\[
E_\delta(w) := \sum_{\kappa=0}^{\infty} \frac{w^\kappa}{\Gamma(\delta \kappa + 1)}, \quad Re(\delta) > 0,
\]
and
\[
E_{\delta,\rho}(w) := \sum_{\kappa=0}^{\infty} \frac{w^\kappa}{\Gamma(\delta \kappa + \rho)}, \quad Re(\delta) > 0, \quad Re(\rho) > 0,
\]
respectively, with the remark that \(E_{\delta,1}(w) \equiv E_\delta(w)\).

Lemma 2.4. \[17\] For \(\sigma, \gamma, \beta > 0\), we have
\[
I^\sigma t^{\gamma-1} E_{\alpha,\beta}(ct^\beta)(x) = x^{\sigma+\gamma-1} E_{\beta,\sigma+\gamma}(cx^\beta).
\]

Mainardi’s conjecture: \[13\] For any \(t > 0\) and fixed \(\delta, \quad 0 < \delta < 1\), we get
\[
\frac{1}{1 + c\Gamma(1-\delta)t^\delta} \leq E_\delta(-ct^\delta) \leq \frac{1}{1 + c\Gamma(1+\delta)^{-1}t^\delta}, \quad t \geq 0.
\]
This conjecture has been proved in \[3, 18\].

Definition 2.5. The solution \(u(t)\) is globally \(\delta\)-Mittag-Leffler stable \((0 < \delta < 1)\) if for some constants \(\Lambda, \lambda > 0\)
\[
\|u(t)\| \leq \Lambda E_\delta(-\lambda t^\delta), \quad t > 0
\]
for a prescribed norm \(\|.\). A local \(\delta\)-Mittag-Leffler stability is provided for small data.
The result below has been proved in [22]. We report it here with its proof for self-containedness.

**Lemma 2.6.** [22] Assuming that $y(t)$ is a solution of

\[
\begin{cases}
D^\gamma_C [y(t) - cy(t - \mu)] \leq -ry(t) + \int_0^\infty h(s)y(t - s)ds, & 0 < \gamma < 1, \ t, c, \mu > 0, \\
y(t) = \phi(t) \geq 0, & t \leq 0,
\end{cases}
\]

with $r > 0$ and $h$ is a nonnegative summable function. If $c > 0$ and $h$ are such as

\[
\int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma) \left( \int_{-\infty}^s E_{\gamma}(-r\lambda^\gamma)h(s - \lambda)d\lambda \right)ds \leq ME_{\gamma}(-rt^\gamma), \ t > 0
\]

hold for some $M > 0$ with

\[
M < 1 - \left[1 + \Gamma(1 + \gamma)\Gamma(1 - \gamma)\right]Vc, \ \left[1 + \Gamma(1 + \gamma)\Gamma(1 - \gamma)\right]Vc < 1, \ V := \frac{1}{r\mu^\gamma} + 2\Gamma(1 - \gamma).
\]

Then, for some constant $\Lambda > 0$

\[
y(t) \leq \Lambda E_{\gamma}(-rt^\gamma), \ t > 0.
\]

**Proof.** Let $|\phi(s)| < y_0E_{\gamma}(-r(s + \mu)^\gamma)$, $s \in [-\mu, 0]$, $y_0 > 0$. It is obvious that for $0 < c < 1$, we obtain the expression

\[
y(t) - cy(t - \mu) = E_{\gamma}(-rt^\gamma)\left[\phi(0) - c\phi(-\mu)\right] + \int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma) \left( -rcy(s - \mu) + \int_0^\infty h(\sigma)y(s - \sigma)d\sigma \right)ds, \ t > 0.
\]

Then

\[
|y(t)| \leq 2y_0E_{\gamma}(-rt^\gamma) + c|y(t - \mu)| + rc\int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma)|y(s - \mu)|ds + \int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma) \left( \int_0^\infty h(\sigma)|y(s - \sigma)|d\sigma \right)ds, \ t > 0.
\]

For $t \in [0, \mu]$,

\[
\frac{|y(t)|}{E_{\gamma}(-rt^\gamma)} \leq 3y_0 + \frac{rcy_0}{E_{\gamma}(-rt^\gamma)} \int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma)E_{\gamma}(-rs^\gamma)ds + M \sup_{-\infty < \sigma \leq t} \frac{|y(\sigma)|}{E_{\gamma}(-r\sigma^\gamma)}. \quad (6)
\]

Again, as

\[
\int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma)E_{\gamma}(-rs^\gamma)ds \\
\leq \frac{\Gamma(1 + \gamma)}{r} \int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma)s^{-\gamma}ds \\
\leq \frac{\Gamma(1 + \gamma)\Gamma(1 - \gamma)}{r} E_{\gamma,1}(-rt^\gamma), \quad (7)
\]

we may write

\[
\frac{|y(t)|}{E_{\gamma}(-rt^\gamma)} \leq 3y_0 + cy_0\Gamma(1 + \gamma)\Gamma(1 - \gamma) + M \sup_{-\infty < \sigma \leq t} \frac{|y(\sigma)|}{E_{\gamma}(-r\sigma^\gamma)}
\]

or

\[
(1 - M)\frac{|y(t)|}{E_{\gamma}(-rt^\gamma)} \leq \left[3 + c\Gamma(1 + \gamma)\Gamma(1 - \gamma)\right]y_0. \quad (8)
\]

In case $t \in [\mu, 2\mu]$, we first notice that

\[
|y(t - \mu)| \leq \frac{3 + c\Gamma(1 + \gamma)\Gamma(1 - \gamma)}{1 - M} E_{\gamma}(-r(t - \mu)^\gamma)E_{\gamma}(-rt^\gamma) \\
\leq \frac{3 + c\Gamma(1 + \gamma)\Gamma(1 - \gamma)}{1 - M} B_0y_0E_{\gamma}(-rt^\gamma)
\]

\[
|y(t - \mu)| \leq \frac{3 + c\Gamma(1 + \gamma)\Gamma(1 - \gamma)}{1 - M} E_{\gamma}(-r(t - \mu)^\gamma)E_{\gamma}(-rt^\gamma) \\
\leq \frac{3 + c\Gamma(1 + \gamma)\Gamma(1 - \gamma)}{1 - M} B_0y_0E_{\gamma}(-rt^\gamma)
\]
where
\[
\frac{E_v(-r(t-\mu)^\gamma)}{E_v(-r^\gamma)} \leq \frac{1}{E_v(-r(t-\mu)^\gamma)} \leq \frac{1}{E_v(-r^\gamma)} \leq 1 + r\Gamma(1 - \gamma)(2\mu)^\gamma := B. \tag{10}
\]

Using the relations (5) and (9), we entail
\[
|y(t)| \leq 2y_0E_\gamma(-rt^\gamma) + cB_0\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}E_\gamma(-rt^\gamma)
+ rcB_0\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}\int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma)E_\gamma(0)ds
+ \int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma)\left(\int_0^\infty h(\sigma)|y(s - \sigma)|d\sigma\right)ds
\]

Next, we apply (7), to obtain
\[
|y(t)| \leq 2y_0E_\gamma(-rt^\gamma) + cB_0\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}E_\gamma(-rt^\gamma)
+ rcB_0\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}\int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma)E_\gamma(0)ds
+ \int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma)\left(\int_0^\infty h(\sigma)|y(s - \sigma)|d\sigma\right)ds
\]
or
\[
(1 - M)\frac{|y(t)|}{E_\gamma(-rt^\gamma)} \leq 2y_0 + cB_0\left[1 + \Gamma(1 + \gamma)(1 - r\gamma)\right]\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}c + cB_0\frac{\Gamma(1 + \gamma)(1 - r\gamma)^2}{1 - M}c^2. \tag{11}
\]

For \( t \in [2\mu, 3\mu] \), in view of the estimation \( \frac{c_\gamma}{(t-\mu)^\gamma} \leq 2^\gamma \) and the relations (5), we obtain
\[
\frac{E_v(-r(t-\mu)^\gamma)}{E_v(-r^\gamma)} \leq \frac{1 + \Gamma(1 + \gamma)(1 - r\gamma)}{1 + r\Gamma(1 + \gamma)(1 - r\gamma)} \leq \frac{\Gamma(1 + \gamma)}{1 - M} + \frac{\Gamma(1 + \gamma)}{(1 - r\gamma)^\gamma}
\leq \Gamma(1 + \gamma)\left[\frac{1}{1 - M} + 2\Gamma(1 - \gamma)\right]
\leq \frac{1}{1 - M} + 2\Gamma(1 - \gamma) := V > 1. \tag{12}
\]

Notice that (12) holds for all \( t \geq 2\mu \) and, as \( \Gamma(1 + \gamma) \) is very close to (and below) than 1, we may ignore it.

Therefore (12) implies
\[
|y(t)| \leq 2y_0E_\gamma(-rt^\gamma) + cV\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}\left[2y_0 + cB_0\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}c + cB_0\frac{\Gamma(1 + \gamma)(1 - r\gamma)^2}{1 - M}c^2\right]E_\gamma(-rt^\gamma)
+ \int_0^t (t - s)^{\gamma - 1}E_{\gamma,\gamma}(-r(t - s)^\gamma)\left(\int_0^\infty h(\sigma)|y(s - \sigma)|d\sigma\right)ds
\]

where
\[
W := \frac{1 + \Gamma(1 + \gamma)(1 - \gamma)}{1 - M}. \tag{13}
\]

So
\[
(1 - M)\frac{|y(t)|}{E_\gamma(-rt^\gamma)} \leq 2y_0 + cV\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}\left[2y_0 + cB_0\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}c\right]\left[1 + \Gamma(1 + \gamma)(1 - \gamma)\right]
\]
or
\[
(1 - M)\frac{|y(t)|}{E_\gamma(-rt^\gamma)} \leq 2y_0 + cW\left[2y_0 + cB_0\frac{\Gamma(1 + \gamma)(1 - r\gamma)}{1 - M}c\right].
\]

That is
\[
(1 - M)\frac{|y(t)|}{E_\gamma(-rt^\gamma)} \leq 3B_0\left[1 + WVc + (VWc)^2 + (VWc)^3\right]. \tag{14}
\]
Claim: We have
\[
(1 - M) \frac{|y(t)|}{E_\gamma(-rt^\gamma)} \leq 3B y_0 \sum_{l=0}^k (VWc)^l, \quad t \in [(k-1)\mu, k\mu].
\]  
(15)

By (8), (11) and (14), the claim is valid for \( n \) and (15) is fulfilled. If \( VWc < 1 \), then by the relations (5), (7) and (15), it is obvious that
\[
|y(t)| \leq 2y_0 E_\gamma(-rt^\gamma) + 3BcV \frac{y_0}{1 - M} \sum_{l=0}^k (VWc)^lE_\gamma(-rt^\gamma)
\]
By (8), (11) and (14), the claim is valid for \( n \) and (15) is fulfilled. If \( VWc < 1 \), then by the relations (5), (7) and (15), it is obvious that
\[
(1 - M) \frac{|y(t)|}{E_\gamma(-rt^\gamma)} \leq 2y_0 + 3BcV \frac{y_0}{1 - M} \left[ 1 + \Gamma(1 + \gamma)\Gamma(1 - \gamma) \right] \sum_{l=0}^k (VWc)^l.
\]
In view of (13), we deduce that
\[
(1 - M) \frac{|y(t)|}{E_\gamma(-rt^\gamma)} \leq 2y_0 + 3BcV \frac{y_0}{1 - M} \left[ 1 + \Gamma(1 + \gamma)\Gamma(1 - \gamma) \right] \sum_{l=0}^k (VWc)^l
\]
and (15) is fulfilled. If \( VWc < 1 \), then \( \sum_{l=0}^\infty (VWc)^l \) is convergent.

Two classes of kernels satisfying the assumptions in this lemma are provided in [22]. Namely, we shall endorse the following notation in the remainder of this paper

\[
\sum_{p,r,q,s=1}^{n_1;n_2} = \sum_{p,r=1}^{n_1} \sum_{q,s=1}^{n_2}.
\]

3. Bounded activation functions

In this section, the Mittag-Leffler stability of system (3) will be discussed using the Laplace transform and fractional calculus. The boundedness of the activation functions converts the non-linearities resulting from the higher-order terms to linear expressions. We shall study the stability of the origin point of the following transformed system

\[
D_C^\delta \left[ u_p(t) - cu_p(t - \mu) \right] = -a_p u_p(t) + \sum_{q,s=1}^{n_2} d_{qps} \int_0^\infty k_{qps}(\omega) g_q(y_q(t - \omega)) d\omega
\]
\[
\times \int_0^\infty h_{qps}(\omega) g_s(y_s(t - \omega)) d\omega - \sum_{q,s=1}^{n_2} d_{qps} \hat{h}_{qps} g_q(y_q^\ast) \hat{g}_s g_s(y_s^\ast), \quad t > 0,
\]
\[
D_C^\delta \left[ v_q(t) - \bar{v}_q(t - \mu) \right] = -\bar{a}_q v_q(t) + \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \int_0^\infty \bar{k}_{pqr}(\omega) \bar{g}_p(x_p(t - \omega)) d\omega
\]
\[
\times \int_0^\infty \bar{h}_{pqr}(\omega) \bar{g}_r(x_r(t - \omega)) d\omega - \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \bar{\bar{h}}_{pqr} \bar{\bar{g}}_p(x_p^\ast) \bar{\bar{g}}_r(x_r^\ast), \quad t > 0,
\]
where
\[
x_p(t) = u_p(t) + x_p^\ast, \quad y_q(t) = v_q(t) + y_q^\ast,
\]
\[
u_p(t) = \tilde{\phi}_p(t) = \phi_p(t) - x_p^\ast, \quad \nu_q(t) = \tilde{\varphi}_q(t) = \varphi_q(t) - y_q^\ast, \quad t \leq 0.
\]
Notice that this is equivalent to the stability of the equilibrium point of the system (1).

We shall adopt the notation below

\[
F := \frac{1}{\xi^p} + 2\delta(1-\delta), \quad a = \min_{1 \leq p \leq n_1} \{a_p\}, \quad \bar{a} = \min_{1 \leq q \leq n_2} \{\bar{a}_q\}, \\
\xi = \min\{a, \bar{a}\}, \quad c^* = \max\{c, \bar{c}\}, \quad a^* = \max\{a, \bar{a}\}, \\
K(t) = \max \left\{ \sum_{p,q,s=1}^{n_1,n_2} GL_q \left[ d_q ps \hat{h}_{qps}^p y_q (t) + d_{spq} \hat{k}_{spq} h_{spq} (t) \right], \\
\sum_{p,r,q=1}^{n_1,n_2} G M_p \left[ d_{prq} \hat{h}_{prq} p_{rq}^r (t) + d_{rqp} \hat{k}_{rqp} h_{rqp} (t) \right] \right\}.
\]

(A4) Let \(\Omega > 0\), \(0 < c^* < 1\) be constants such that \(\Omega < 1 - \left[1 + \Gamma(1+\delta)(1-\delta)\right] F \left(\frac{a^*}{\xi}\right), \quad \left[1 + \Gamma(1+\delta)(1-\delta)\right] F \left(\frac{a^*}{\xi}\right) < 1, \quad t > 0.

Theorem 3.1. Assume that (A1)-(A4) hold. Then, for some constants \(C, \xi > 0\)

\[
u(t) \leq CE_{\delta}(-\xi t^\delta), \quad \nu(t) \leq CE_{\delta}(-\xi t^\delta), \quad t \geq 0.
\]

where \(u(t) = \sum_{p=1}^{n_1} |u_p(t)|\) and \(v(t) = \sum_{q=1}^{n_2} |v_q(t)|\).

Proof. The non-linear terms may be expressed as

\[
\sum_{q,s=1}^{n_2} d_{qps} \int_0^\infty k_{qps}(\xi) \omega g_q(y_q(t-\omega))d\omega f_0^\infty h_{qps}(\xi) g_s(y_s(t-\omega))d\omega - \sum_{q,s=1}^{n_2} d_{qps} \hat{k}_{qps}^p g_q(y_q^*) \hat{h}_{qps}^p g_s(y_s^*)
\]

and

\[
\sum_{p,r,q=1}^{n_1,n_2} d_{prq} \int_0^\infty \hat{k}_{prq}^p (\xi) g_p(x_p(t-\omega))d\omega f_0^\infty \hat{h}_{prq}^p (\xi) g_r(x_r(t-\omega))d\omega - \sum_{p,r,q=1}^{n_1,n_2} d_{prq} \hat{k}_{prq}^p \tilde{g}_p(x_p^*) \hat{h}_{prq}^p \tilde{g}_r(x_r^*)
\]

where

\[
\tilde{g}_q(v_q(t)) := g_q(y_q(t)) - g_q(y_q^*), \quad \tilde{g}_p(u_p(t)) := \tilde{g}_p(x_p(t)) - \tilde{g}_p(x_p^*).
\]

Next, we add and subtract the expressions \(c a_p u_p(t-\mu)\) and \(\bar{a} \bar{a}_q v_q(t-\mu)\) in the equations 110 and 112, resp.

\[
D_{\mu}^\delta \left[ u_p(t) - cu_p(t) - \bar{a} \bar{a}_q v_q(t-\mu) \right] = -a_p \left[ u_p(t) - cu_p(t) - \bar{a} \bar{a}_q v_q(t-\mu) \right] - c a_p u_p(t-\mu) + \sum_{q,s=1}^{n_2} d_{qps} \int_0^\infty k_{qps}(\xi) \\
\times g_q(y_q(t-\omega))d\omega f_0^\infty h_{qps}(\xi) g_s(y_s(t-\omega))d\omega - \sum_{q,s=1}^{n_2} d_{qps} \hat{k}_{qps}^p g_q(y_q^*) \hat{h}_{qps}^p g_s(y_s^*),
\]

(21)
\[ D_C^\delta \left[ v_q(t) - \bar{c}v_q(t - \mu) \right] = -\bar{a}_q \left[ v_q(t) - \bar{c}v_q(t - \mu) \right] - \bar{c}\bar{a}_q v_q(t - \mu) + \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \int_0^\infty \bar{k}_{pqr}(\omega) \, d\omega \]
\[ \times \bar{g}_p(x_p(t-\omega)) \, d\omega - \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \bar{k}_{pqr} \bar{g}_p(x_p^r) \bar{h}_{pqr}(x_r^r) \] (22)

An application of Laplace transform to (21) and (22) yields

\[ u_p(t) - cu_p(t - \mu) = E_\delta(-a_t t^\delta) \left[ \bar{\phi}_p(0) - \bar{c}\bar{\phi}_p(-\mu) \right] - ca_p \int_0^t (t - \omega)^{\delta-1} E_\delta,\delta(-a_p(t - \omega)^\delta) \]
\[ \times u_p(\omega - \mu) d\omega + \sum_{q,s=1}^{n_2} d_{qsp} \int_0^t (t - \omega)^{\delta-1} E_\delta,\delta(-a_p(t - \omega)^\delta) \int_0^\infty \bar{k}_{qsp}(\lambda) \bar{g}_q(\mu(\omega - \lambda)) d\lambda \]
\[ \times \int_0^\infty \bar{h}_{qsp}(\lambda) g_s(\omega - \lambda) d\lambda d\omega - \sum_{q,s=1}^{n_2} d_{qsp} \bar{h}_{qsp} \bar{g}_s(y_s(\omega - \lambda)) d\lambda d\omega \] (23)

\[ v_q(t) - \bar{c}v_q(t - \mu) = E_\delta(-\bar{a}_q t^\delta) \left[ \bar{\varphi}_q(0) - \bar{c}\bar{\varphi}_q(-\mu) \right] - \bar{c}\bar{a}_q \int_0^t (t - \omega)^{\delta-1} E_\delta,\delta(-\bar{a}_q(t - \omega)^\delta) \]
\[ \times v_q(\omega - \mu) d\omega + \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \int_0^t (t - \omega)^{\delta-1} E_\delta,\delta(-\bar{a}_q(t - \omega)^\delta) \int_0^\infty \bar{k}_{pqr}(\lambda) \bar{g}_p(x_p(t - \omega)) d\lambda \]
\[ \times \int_0^\infty \bar{h}_{pqr}(\lambda) \bar{g}_r(x_r(\omega - \lambda)) d\lambda d\omega - \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \bar{h}_{pqr} \bar{g}_p(x_p^r) \bar{h}_{pqr} \bar{g}_r(x_r^r) \int_0^t (t - \omega)^{\delta-1} E_\delta,\delta(-\bar{a}_q(t - \omega)^\delta) d\omega \] (24)

In light of the assumptions (A2) and (A3), (19) and (20), we obtain after evaluating the last terms in (23) and (24)

\[ u_p(t) - cu_p(t - \mu) \leq E_\delta(-a_t t^\delta) \left[ \bar{\phi}_p(0) - \bar{c}\bar{\phi}_p(-\mu) \right] - ca_p \int_0^t (t - \omega)^{\delta-1} \]
\[ \times E_\delta,\delta(-a_p(t - \omega)^\delta) \bar{g}_p(\mu(\omega - \lambda)) d\lambda \]
\[ + G \sum_{q,s=1}^{n_2} L_q \int_0^t (t - \omega)^{\delta-1} E_\delta,\delta(-a_p(t - \omega)^\delta) \]
\[ \times \int_0^\infty \bar{h}_{qsp}(\lambda) g_s(\omega - \lambda) d\lambda d\omega, \quad t > 0, \]

Therefore

\[ |u_p(t)| \leq c|u_p(t - \mu)| + E_\delta(-a_t t^\delta) \left| \bar{\phi}_p(0) - \bar{c}\bar{\phi}_p(-\mu) \right| \]
\[ \times E_\delta,\delta(-a(t - \omega)^\delta) \bar{g}_p(\mu(\omega - \lambda)) d\lambda \]
\[ + G \sum_{q,s=1}^{n_2} L_q \int_0^t (t - \omega)^{\delta-1} E_\delta,\delta(-a(t - \omega)^\delta) \]
\[ \times \int_0^\infty \bar{h}_{qsp}(\lambda) g_s(\omega - \lambda) d\lambda d\omega, \quad t > 0, \]

and

\[ |v_q(t)| \leq c|v_q(t - \mu)| + E_\delta(-\bar{a} t^\delta) \left| \bar{\varphi}_q(0) - \bar{c}\bar{\varphi}_q(-\mu) \right| \]
\[ \times E_\delta,\delta(-\bar{a}(t - \omega)^\delta) \bar{g}_q(\mu(\omega - \lambda)) d\lambda \]
\[ + G \sum_{p,r=1}^{n_1} M_p \int_0^t (t - \omega)^{\delta-1} E_\delta,\delta(-\bar{a}(t - \omega)^\delta) \]
\[ \times \int_0^\infty \bar{h}_{pqr}(\lambda) \bar{g}_r(\mu(\omega - \lambda)) d\lambda d\omega, \quad t > 0, \]

After summing up, we arrive at

\[ u(t) \leq cu(t - \mu) + E_\delta(-a_t t^\delta) \sum_{p=1}^{n_1} \bar{d}_{pqr} \bar{g}_p(x_p) \bar{k}_{pqr} \bar{g}_r(x_r^r) \bar{h}_{pqr} \]
\[ \int_0^\infty v(\omega - \lambda) d\lambda d\omega, \quad t > 0, \] (25)
\[ v(t) \leq \bar{c}v(t - \mu) + E_\delta(-\bar{\alpha}t^\delta) \sum_{q=1}^{n_2} |\tilde{\varphi}_q(0) - \bar{c}\tilde{\varphi}_q(-\mu)| + \bar{c}\bar{a} \int_0^t (t - \omega)^{\delta - 1} \]
\[ \times E_\delta(-\bar{\alpha}(t - \omega)^\delta) v(\omega - \mu) d\omega + G \sum_{p,r,q=1}^{n_1,n_2} M_p \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\bar{\alpha}(t - \omega)^\delta) \]
\[ \times \int_0^\infty \left[ \tilde{d}_{pq} \tilde{h}_{pq} \tilde{h}_{pp} (\lambda) + \tilde{d}_{rq} \tilde{h}_{rr} \tilde{h}_{qp} (\lambda) \right] u(\omega - \lambda) d\lambda d\omega, \quad t > 0. \]

Notice that we can assume that
\[ |\tilde{\varphi}(\omega)| \leq u_0 E_\delta(-a(\omega + \mu)^\delta) \quad \text{for } \omega \in [-\mu, 0], \quad u_0 > 0, \]
\[ |\tilde{\varphi}(\omega)| \leq v_0 E_\delta(-\bar{\alpha}(\omega + \mu)^\delta) \quad \text{for } \omega \in [-\mu, 0], \quad v_0 > 0. \]

This is always possible because, if \( \tilde{\varphi}(\omega) \) and \( \tilde{\varphi}(\omega) \) are bounded by \( \vartheta \) and \( \tilde{\vartheta} \), resp, then
\[ \vartheta \leq u_0 E_\delta(-a\mu^\delta) \leq u_0 E_\delta(-a(\omega + \mu)^\delta), \]
\[ \tilde{\vartheta} \leq v_0 E_\delta(-\bar{\alpha} \mu^\delta) \leq v_0 E_\delta(-\bar{\alpha}(\omega + \mu)^\delta). \]

We can choose \( u_0 = \frac{\vartheta}{E_\delta(-a\mu^\delta)} \) and \( v_0 = \frac{\tilde{\vartheta}}{E_\delta(-\bar{\alpha}\mu^\delta)} \). Consequently, (26) and (26) become
\[ u(t) \leq 2u_0 E_\delta(-a\mu^\delta) + cu(t - \mu) + ca \int_0^t (t - \omega)^{\delta - 1} E_\delta(-a(t - \omega)^\delta) u(\omega - \mu) d\omega \]
\[ + G \sum_{p,q,r,s=1}^{n_1,n_2} I_q \int_0^t (t - \omega)^{\delta - 1} E_\delta(-a(t - \omega)^\delta) \int_0^\infty \left[ \tilde{d}_{pq} \tilde{h}_{pq} \tilde{h}_{pp} (\lambda) + \tilde{d}_{sp} \tilde{h}_{sp} \tilde{h}_{sp} (\lambda) \right] v(\omega - \lambda) d\lambda d\omega, \]
\[ v(t) \leq 2v_0 E_\delta(-\bar{\alpha} \mu^\delta) + \bar{c}v(t - \mu) + \bar{c}\bar{a} \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\bar{\alpha}(t - \omega)^\delta) v(\omega - \mu) d\omega \]
\[ + G \sum_{p,q,r,s=1}^{n_1,n_2} M_p \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\bar{\alpha}(t - \omega)^\delta) \int_0^\infty \left[ \tilde{d}_{pq} \tilde{h}_{pq} \tilde{h}_{pp} (\lambda) + \tilde{d}_{rp} \tilde{h}_{rp} \tilde{h}_{pp} (\lambda) \right] u(\omega - \lambda) d\lambda d\omega. \]

Let \( V(t) = \max\{u(t), v(t)\} \) and \( V_0 = \max\{u_0, v_0\} \). From the relations (29) and (28), we entail the single inequality
\[ V(t) \leq 2V_0 E_\delta(-\xi t^\delta) + c^* V(t - \mu) + c^* a^* \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\xi(t - \omega)^\delta) V(\omega - \mu) d\omega \]
\[ + \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\xi(t - \omega)^\delta) \int_0^\infty K(\lambda) V(\omega - \lambda) d\lambda d\omega, \quad t > 0. \]

For \( t \in [0, \mu] \) and from (29), we see that
\[ V(t) \leq 3V_0 E_\delta(-\xi t^\delta) + c^* a^* \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\xi(t - \omega)^\delta) V(\omega - \mu) d\omega \]
\[ + \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\xi(t - \omega)^\delta) \int_0^\infty K(\lambda) V(\omega - \lambda) d\lambda d\omega. \]

Dividing by \( E_\delta(-\xi t^\delta) \), we get
\[ \frac{V(t)}{E_\delta(-\xi t^\delta)} \leq 3V_0 + \frac{c^* a^*}{E_\delta(-\xi t^\delta)} \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\xi(t - \omega)^\delta) E_\delta(-\omega^\delta) d\omega \]
\[ + \frac{1}{E_\delta(-\xi t^\delta)} \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\xi(t - \omega)^\delta) \int_0^\infty K(\lambda) - \omega^\delta) d\lambda d\omega \sup_{-\infty < \lambda < t} \frac{V(\lambda)}{E_\delta(-\xi t^\delta)}. \]

By virtue of the estimation (31) and formula (2), we obtain
\[ \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\xi(t - \omega)^\delta) E_\delta(-\omega^\delta) d\omega \leq \frac{\Gamma(1 + \delta)}{\xi} \int_0^t (t - \omega)^{\delta - 1} E_\delta(-\xi(t - \omega)^\delta) \omega^\delta d\omega \]
\[ \leq \frac{\Gamma(1 + \delta)\Gamma(1 - \delta)}{\xi} E_\delta(-\xi^\delta), \]
and from the condition on the kernels, the relation \([31]\) becomes

\[
\frac{V(t)}{E_\delta(-\xi t^\beta)} \leq 3V_0 + \frac{e^{a_+}}{\xi} V_0 \Gamma(1 + \delta) \Gamma(1 - \delta) + \Omega \sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_\delta(-\xi \lambda t^\beta)},
\]

and thus

\[
\sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_\delta(-\xi \lambda t^\beta)} \leq 3V_0 + \frac{e^{a_+}}{\xi} V_0 \Gamma(1 + \delta) \Gamma(1 - \delta) + \Omega \sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_\delta(-\xi \lambda t^\beta)}.
\]  \hspace{1cm} (33)

Therefore,

\[
(1 - \Omega) \frac{V(t)}{E_\delta(-\xi t^\beta)} \leq \left[3 + \frac{e^{a_+}}{\xi} \Gamma(1 + \delta) \Gamma(1 - \delta)\right] V_0.
\]  \hspace{1cm} (34)

In case of \( t \in [\mu, 2\mu] \), we note that

\[
V(t - \mu) \leq W \frac{E_\delta(-\xi(t - \mu)^\beta)}{E_\delta(-\xi t^\beta)} V_0 E_\delta(-\xi t^\beta) \leq W^* B V_0 E_\delta(-\xi t^\beta),
\]

where

\[
W^* = \frac{3 + \frac{e^{a_+}}{\xi} \Gamma(1 + \delta) \Gamma(1 - \delta)}{1 - \Omega},
\]

\[
\frac{E_\delta(-\xi(t - \mu)^\beta)}{E_\delta(-\xi t^\beta)} \leq \frac{1}{E_\delta(-\xi t^\beta)} \leq \frac{1}{E_\delta(-\xi(2\mu)^\beta)} \leq 1 + \xi \Gamma(1 - \delta)(2\mu)^\delta =: B.
\]

In view of the relations \([29], [32] and [55]\), we find

\[
V(t) \leq 2V_0 E_\delta(-\xi t^\beta) + c^* B V_0 W^* E_\delta(-\xi t^\beta) + a^* c^* B V_0 W^* \frac{F \Gamma(1 + \delta) \Gamma(1 - \delta)}{\xi} E_\delta(-\xi t^\beta)
\]

\[
+ \int_0^t (t - \omega)^{\delta - 1} E_{\delta, \delta}(-\xi(t - \omega)^\delta) \int_0^\infty K(\lambda) V(\omega - \lambda) d\lambda d\omega.
\]  \hspace{1cm} (36)

Therefore, as \( \xi \leq a^* \)

\[
(1 - \Omega) \frac{V(t)}{E_\delta(-\xi t^\beta)} \leq 2V_0 + \frac{c^* E_\delta}{\xi} V_0 \left[1 + \Gamma(1 + \delta) \Gamma(1 - \delta)\right] W^*
\]

\[
\leq 2V_0 + 3B V_0 W^2 \left(\frac{\alpha^* c^*}{\xi}\right)^2 + 3B V_0 W^2 \left(\frac{\alpha^* c^*}{\xi}\right)^2,
\]  \hspace{1cm} (37)

where \( W = \frac{1 + \Gamma(1 + \delta) \Gamma(1 - \delta)}{1 - \Omega} \).

For \( t \in [2\mu, 3\mu] \), from the estimation \( \frac{t^\beta}{(t - \mu)^\beta} \leq 2^\beta \) and the relation \( [3] \), we obtain a new estimation of \( E_\delta(-\xi(t - \mu)^\beta)/E_\delta(-\xi t^\beta) \)

\[
\frac{E_\delta(-\xi(t - \mu)^\beta)}{E_\delta(-\xi t^\beta)} \leq \frac{1 + \Gamma(1 - \delta) t^\beta}{1 + \Gamma(1 - \delta)(t - \mu)^\beta} \leq \frac{\Gamma(1 + \delta)}{\xi t^\beta} + \frac{\Gamma(1 - \delta) t^\beta}{(t - \mu)^\beta} \leq \Gamma(1 + \delta) \left[\frac{1}{\xi t^\beta} + 2^\beta \Gamma(1 - \delta)\right]
\]  \hspace{1cm} (38)

or

\[
\frac{E_\delta(-\xi(t - \mu)^\beta)}{E_\delta(-\xi t^\beta)} \leq \frac{1}{\xi t^\beta} + 2^\beta \Gamma(1 - \delta) =: F, \; t \in [2\mu, 3\mu].
\]  \hspace{1cm} (39)

Therefore, \([29]\) implies

\[
V(t) \leq 2V_0 E_\delta(-\xi t^\beta) + \frac{c^* E_\delta}{\xi} \left[2V_0 + 3B V_0 W^2 (\frac{\alpha^* c^*}{\xi})^2 + 3B V_0 W^2 (1 - \Omega) \left(\frac{\alpha^* c^*}{\xi}\right)^2\right] E_\delta(-\xi t^\beta)
\]

\[
+ c^* a^* F \frac{\Gamma(1 + \delta) \Gamma(1 - \delta)}{\xi t^\beta} \left[2V_0 + 3B V_0 W^2 (\frac{\alpha^* c^*}{\xi})^2 + 3B V_0 W^2 (1 - \Omega) \left(\frac{\alpha^* c^*}{\xi}\right)^2\right] E_\delta(-\xi t^\beta)
\]

\[
+ \int_0^t (t - \omega)^{\delta - 1} E_{\delta, \delta}(-\xi(t - \omega)^\delta) \int_0^\infty K(\lambda) V(\omega - \lambda) d\lambda d\omega.
\]  \hspace{1cm} (40)

Hence

\[
(1 - \Omega) \frac{V(t)}{E_\delta(-\xi t^\beta)} \leq 2V_0 + \left(\frac{\alpha^* c^*}{\xi}\right) F W^2 \left[2V_0 + 3B V_0 W^2 (\frac{\alpha^* c^*}{\xi})^2 + 3B V_0 W^2 (1 - \Omega) \left(\frac{\alpha^* c^*}{\xi}\right)^2\right].
\]
Thus
\[(1 - \Omega) \frac{V(t)}{E_\delta(-\xi t^\delta)} \leq 3BV_0 \left[ 1 + FW \left( \frac{a^+c^*}{\xi} \right) + \left( FW \left( \frac{a^+c^*}{\xi} \right) \right)^2 + \left( FW \left( \frac{a^+c^*}{\xi} \right) \right)^3 \right]. \tag{41}\]

We claim that
\[(1 - \Omega) \frac{V(t)}{E_\delta(-\xi t^\delta)} \leq 3BV_0 \sum_{l=0}^{k} \left( FW \left( \frac{a^+c^*}{\xi} \right) \right)^l, \quad t \in [(k-1)\mu, k\mu]. \tag{42}\]

By virtue of the estimations \((33)\), \((37)\) and \((41)\), the relation \((42)\) is valid for \(k = 1, 2\) and \(3\) respectively.

Assuming that it is valid for \(t \in [(k-1)\mu, k\mu]\) and we will prove it for \(t \in [k\mu, (k+1)\mu]\). From the relations \((20)\), \((32)\) and \((42)\), we get
\[
V(t) \leq 2BV_0 E_\delta(-\xi t^\delta) + 3BFc^* \frac{V_{0}}{1-\Omega} \sum_{l=0}^{k} \left( FW \left( \frac{a^+c^*}{\xi} \right) \right)^l E_\delta(-\xi t^\delta)
\]
\[+ 3BF \left( \frac{a^+c^*}{\xi} \right) \frac{V_{0}}{1-\Omega} \Gamma(1+\delta) \Gamma(1-\delta) \sum_{l=0}^{k} \left( FW \left( \frac{a^+c^*}{\xi} \right) \right)^l E_\delta(-\xi t^\delta)
\]
\[+ \int_{0}^{t}(t-\omega)^{\delta-1} E_\delta,\delta(-\xi(t-\omega)^{\delta}) \int_{0}^{t} K(\lambda)V(\omega-\lambda)d\lambda d\omega.
\]
Consequently,
\[(1 - \Omega) \frac{V(t)}{E_\delta(-\xi t^\delta)} \leq 2BV_0 + 3BF \left( \frac{a^+c^*}{\xi} \right) \frac{V_{0}}{1-\Omega} \left[ 1 + \Gamma(1+\delta) \Gamma(1-\delta) \sum_{l=0}^{k} \left( FW \left( \frac{a^+c^*}{\xi} \right) \right)^l \right]. \]

Then, by the definition of \(W\)
\[(1 - \Omega) \frac{V(t)}{E_\delta(-\xi t^\delta)} \leq 3BV_0 \left[ 1 + \left( \frac{a^+c^*}{\xi} \right) FW \sum_{l=0}^{k} \left( FW \left( \frac{a^+c^*}{\xi} \right) \right)^l \right] = 3BV_0 \sum_{l=0}^{k+1} \left( FW \left( \frac{a^+c^*}{\xi} \right) \right)^l. \]

In light of the conditions indicated in Theorem 3.1, the series is convergent.

\[\square\]

4. Unbounded activation functions

Unbounded activation functions in case of higher-order NNs are not easy to deal with because of the nonlinear terms. This is in contrast to the lower-order case. To address this issue, we shall use some analytical techniques based on suitable evaluations and properties of the Mittag-Leffler functions. The notation below will be utilized

\[U := A \frac{\Gamma(1+\delta)\Gamma(1-\delta)}{\xi}, \quad A = \max \left\{ \sum_{p=1}^{n_1} a_p, \sum_{q=1}^{n_2} a_q \right\}, \quad \Lambda := \sum_{k=0}^{+\infty} \left[ 2B^*c^*(1+U) \right]^k, \quad B^* := \max \{ B, F \}, \]
\[\theta := \sum_{p:q:s=1}^{n_1\times n_2\times n_2} d_{bps} [L_q \hat{h}_{bps} g_s(y_s^*) + L_s \hat{k}_{bps} g_q(y_q^*)], \quad \nu := \sum_{p:q:s=1}^{n_1\times n_2\times n_2} d_{bps} L_q L_s,
\]
\[\bar{\theta} := \sum_{p:q:r:q=1}^{n_1\times n_2\times n_2\times n_2} \bar{d}_{pqrs} \left[ M_p \hat{h}_{pqrs} \tilde{g}_r(x_r^*) + M_r \hat{k}_{pqrs} \tilde{g}_p(x_p^*) \right], \quad \bar{\nu} := \sum_{p:q:r:q=1}^{n_1\times n_2\times n_2\times n_2} \bar{d}_{pqrs} M_p M_r,
\]
\[k(t) := \max_{1 \leq p \leq n_1, 1 \leq q, s \leq n_2} \{ k_{bps}(t) \}, \quad h(t) := \max_{1 \leq p \leq n_1, 1 \leq q, s \leq n_2} \{ h_{bps}(t) \}, \quad K(t) := \max \{ k(t), h(t) \}, \]
\[\bar{k}(t) := \max_{1 \leq p, r \leq n_1, 1 \leq q, s \leq n_2} \{ \bar{k}_{pqrs}(t) \}, \quad \bar{h}(t) := \max_{1 \leq p, r \leq n_1, 1 \leq q, s \leq n_2} \{ \bar{h}_{pqrs}(t) \}, \quad H(t) := \max \{ \bar{k}(t), \bar{h}(t) \},
\]
\[\pi := \max \{ \theta, \bar{\theta} \}, \quad \kappa := \max \{ \nu, \bar{\nu} \}, \quad K^*(t) := \max \{ K(t), H(t) \}, \quad k^*(t) := \max \{ k(t), \bar{k}(t) \}, \quad h^*(t) := \max \{ h(t), \bar{h}(t) \}, \quad \tilde{h}^*(t) = \int_{0}^{\infty} h^*(t)dt.\]
Let $\Omega$ and $c^*$ be positive constants such that $\Omega \pi < \frac{1}{q}; c^* < \min\{1, \frac{1}{2\Omega^*(1+\epsilon)}\}$.

$$
\int_0^t (t - \omega)^{\delta-1} E_{\delta,\delta}(-\xi(t - \omega)^{\delta}) \left( \int_{-\infty}^\omega E_{\delta}(-\xi^\lambda) K^*(\omega - \lambda) d\lambda \right) d\omega \leq \Omega E_{\delta}(-\xi^\delta), \> t > 0.
$$

(A5) Theorem 4.1. Assuming that (A1), (A3) and (A5) hold. Then, the solutions of system (4) are locally \(\delta\)-Mittag-Leffler stable, which means that for some constant \(C > 0\)

$$
u(t) \leq CV_0E_{\delta}(-\xi^\delta), \quad v(t) \leq CV_0E_{\delta}(-\xi^\delta), \quad t > 0,
$$

for small \(V_0\) and positive constant \(\xi\).

Proof. It is obvious that

$$
\sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(y_q(t - \omega))d\omega \int_0^\infty h_{qps}(\omega)g_s(y_s(t - \omega))d\omega - \sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(y_q(t - \omega))d\omega \int_0^\infty h_{qps}(\omega)g_s(y_s(t - \omega))d\omega
$$

and

$$
\sum_{p,r=1}^{n_1} \int_0^\infty h_{pqr}(\omega)g_p(x_p(t - \omega))d\omega \int_0^\infty h_{pqr}(\omega)g_r(x_r(t - \omega))d\omega - \sum_{p,r=1}^{n_1} \int_0^\infty h_{pqr}(\omega)g_p(x_p(t - \omega))d\omega \int_0^\infty h_{pqr}(\omega)g_r(x_r(t - \omega))d\omega
$$

These identities are very useful. They will facilitate some evaluations below.

According to (43), (44), (45), and (46), we obtain

$$
\int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda = \sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda + \sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda \left( \int_0^\infty h_{qps}(\lambda)g_q(\omega - \lambda)d\lambda \right) d\omega
$$

$$
\int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda = \sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda + \sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda \left( \int_0^\infty h_{qps}(\lambda)g_q(\omega - \lambda)d\lambda \right) d\omega
$$

$$
\int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda = \sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda + \sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda \left( \int_0^\infty h_{qps}(\lambda)g_q(\omega - \lambda)d\lambda \right) d\omega
$$

$$
\int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda = \sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda + \sum_{q,s=1}^{n_2} \int_0^\infty h_{qps}(\omega)g_q(\omega - \lambda)d\lambda \left( \int_0^\infty h_{qps}(\lambda)g_q(\omega - \lambda)d\lambda \right) d\omega
$$

13
and

\[ v_p(t) - \bar{c}v_q(t - \mu) \leq E_\delta(-\bar{a}_q t^\delta) |\bar{\varphi}(0) - \bar{c}\bar{\varphi}(\mu)| + \bar{c}\bar{a}_q \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\bar{a}_q (t - \omega)^\delta) \] 
\[ \times |v_q(\omega - \mu)| d\omega + \sum_{p,r=1}^{n_1:n_2} \bar{d}_{pq} M_p \bar{h}_{pq} \bar{g}_r(x_r^*) \left( \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\omega \right) \]
\[ \times \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \] 
\[ + \sum_{p,r=1}^{n_1:n_2} \bar{d}_{pq} M_p \bar{h}_{pq} \bar{g}_r(x_r^*) \left( \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \right) \]
\[ \times \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \] 
\[ \times \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \] 

Using the notation in the above, we get

\[ u(t) \leq (1 + c)u_0 E_\delta(-at^\delta) + c v(t - \mu) + c \sum_{p=1}^{n_1:n_2} a_p \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-a(t - \omega)^\delta) u(\omega - \mu) d\omega \]
\[ + \sum_{p,r=1}^{n_1:n_2} d_{pq} \left( L_q \bar{h}_{pq} g_s(y_s^*) + L_s \bar{k}_{pq} g_q(y_q^*) \right) \left( \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \right) \]
\[ \times \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \] 
\[ + \sum_{p,r=1}^{n_1:n_2} d_{pq} L_q L_s \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-a(t - \omega)^\delta) \]
\[ \times \left( \int_0^\infty \bar{k}_{pq}(\lambda)|v(\omega - \lambda)| d\lambda \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \right) \]

and

\[ v(t) \leq (1 + \bar{c})v_0 E_\delta(-\bar{a}t^\delta) + \bar{c} v(t - \mu) + \bar{c} \sum_{q=1}^{n_1:n_2} a_q \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\bar{a}(t - \omega)^\delta) v(\omega - \mu) d\omega \]
\[ + \sum_{p,r,q=1}^{n_1:n_2} \bar{d}_{pq} \left[ M_p \bar{h}_{pq} \bar{g}_r(x_r^*) + M_r \bar{h}_{pq} \bar{g}_q(x_q^*) \right] \left( \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \right) \]
\[ \times \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \] 
\[ + \sum_{p,r,q=1}^{n_1:n_2} \bar{d}_{pq} M_p M_r \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\bar{a}(t - \omega)^\delta) \]
\[ \times \left( \int_0^\infty \bar{k}_{pq}(\lambda)|v(\omega - \lambda)| d\lambda \int_0^\infty \bar{h}_{pq}(\lambda)|u_r(\omega - \lambda)| d\lambda d\omega \right) \]

Let \( V(t) = \max\{u(t), v(t)\}, t \geq 0 \) and \( V_0 = \max\{u_0, v_0\} \), then

\[ u(t) \leq (1 + c) V_0 E_\delta(-at^\delta) + c V(t - \mu) + c \sum_{p=1}^{n_1:n_2} a_p \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-a(t - \omega)^\delta) V(\omega - \mu) d\omega \]
\[ + \bar{c} \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\bar{a}(t - \omega)^\delta) \int_0^\infty K(\omega - \lambda) V(\lambda) d\lambda d\omega \]
\[ + \bar{c} \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\bar{a}(t - \omega)^\delta) \int_0^\infty H(\omega - \lambda) V(\lambda) d\lambda d\omega \] 
\[ \times \left( \int_0^\infty \bar{k}(\lambda) V(\omega - \lambda) d\lambda \int_0^\infty \bar{h}(\lambda) V(\omega - \lambda) d\lambda d\omega \right) \]

\[ v(t) \leq (1 + \bar{c}) V_0 E_\delta(-\bar{a}t^\delta) + \bar{c} v(t - \mu) + \bar{c} \sum_{q=1}^{n_1:n_2} a_q \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\bar{a}(t - \omega)^\delta) v(\omega - \mu) d\omega \]
\[ + \bar{c} \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\bar{a}(t - \omega)^\delta) \int_0^\infty H(\omega - \lambda) V(\lambda) d\lambda d\omega \]
\[ + \bar{c} \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\bar{a}(t - \omega)^\delta) \int_0^\infty H(\omega - \lambda) V(\lambda) d\lambda d\omega \] 
\[ \times \left( \int_0^\infty \bar{k}(\lambda) V(\omega - \lambda) d\lambda \int_0^\infty \bar{h}(\lambda) V(\omega - \lambda) d\lambda d\omega \right) \]

Furthermore,

\[ V(t) \leq (1 + c^*) V_0 E_\delta(-\xi t^\delta) + c^* V(t - \mu) + c^* A \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\xi(t - \omega)^\delta) V(\omega - \mu) d\omega \]
\[ + \kappa \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\xi(t - \omega)^\delta) \int_0^\infty K^*(\omega - \lambda) V(\lambda) d\lambda d\omega + \kappa \int_0^t (t - \omega)^{\delta - 1} E_{\delta,\delta}(-\xi(t - \omega)^\delta) \]
\[ \times \int_0^\infty k^*(\lambda) V(\omega - \lambda) d\lambda \int_0^\infty \bar{h}^*(\lambda) V(\omega - \lambda) d\lambda d\omega \]
For \(\omega \in [0, \mu]\), we have \(-\mu \leq \omega - \mu \leq 0\) and assume
\[
\bar{\varphi}(\omega) := \sum_{q=1}^{n_2} |\bar{\varphi}_q(\omega)| \leq \nu_0 E_{\delta}(-\bar{a}(\omega + \mu)^{\delta}), \quad \omega \in [-\mu, 0],
\]
\[
\tilde{\varphi}(\omega) := \sum_{q=1}^{n_2} |\tilde{\varphi}_q(\omega)| \leq \nu_0 E_{\delta}(-\bar{a}(\omega + \mu)^{\delta}), \quad \omega \in [-\mu, 0].
\]
Choosing \(\eta > 0\) such that \(\Omega \eta \hat{\varphi}^* < \frac{1}{4}\) and \(V_0 \Lambda < \frac{1}{4}\), furthermore, as \(V(t)\) is continuous on \([0, t_*]\), we have \(V(t) \leq \eta\) on \([0, t_*]\) with \(t_* > 0\). If \(t_* \leq \mu\), and \(0 < t \leq t_*\), then
\[
\int_0^t (t - \omega)^{\delta - 1} E_{\delta, \delta}(-\xi(t - \omega)^{\delta}) \int_0^\infty k^*(\lambda)V(\omega - \lambda)d\lambda d\omega \\
\leq \eta \hat{\varphi}^* \int_0^t (t - \omega)^{\delta - 1} E_{\delta, \delta}(-\xi(t - \omega)^{\delta}) \int_0^\infty k^*(\lambda)V(\omega - \lambda)d\lambda.
\]
In view of the estimations (46) and (47), we obtain
\[
V(t) \leq (1 + c^* V_0 E_{\delta}(-\xi t^\delta) + c^* V(t) - \mu) + c^* V_0 A \int_0^t (t - \omega)^{\delta - 1} E_{\delta, \delta}(-\xi(t - \omega)^{\delta}) E_{\delta}(-\xi\omega^\delta) d\omega \\
+ \left(\pi + \eta \hat{\varphi}^*\right) \left(\int_0^t (t - \omega)^{\delta - 1} E_{\delta, \delta}(-\xi(t - \omega)^{\delta}) \int_{-\infty}^\infty K^*(\omega - \lambda) E_{\delta}(-\xi\lambda^\delta) d\lambda d\omega\right).
\]
Therefore
\[
V(t) \leq \left[1 + 2c^* + c^* A \frac{\Gamma(1 + \delta)^{\Gamma(1 - \delta)}}{\xi}\right] V_0 E_{\delta}(-\xi t^\delta) + \Omega \left(\pi + \eta \hat{\varphi}^*\right) \sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_{\delta}(-\xi \lambda^\delta)} E_{\delta}(-\xi t^\delta).
\]
In light of the relation (49), we end up with
\[
V(t) \leq \left[1 + 2c^* + c^* A \frac{\Gamma(1 + \delta)^{\Gamma(1 - \delta)}}{\xi}\right] V_0 E_{\delta}(-\xi t^\delta) + \Omega \left(\pi + \eta \hat{\varphi}^*\right) \sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_{\delta}(-\xi \lambda^\delta)} E_{\delta}(-\xi t^\delta).
\]
Dividing by \(E_{\delta}(-\xi t^\delta)\), (48) yields
\[
\sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_{\delta}(-\xi \lambda^\delta)} \leq \left[1 + 2c^* + A c^* \frac{\Gamma(1 + \delta)^{\Gamma(1 - \delta)}}{\xi}\right] V_0 + \Omega \left(\pi + \eta \hat{\varphi}^*\right) \sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_{\delta}(-\xi \lambda^\delta)} E_{\delta}(-\xi t^\delta),
\]
or
\[
\left[1 - \Omega \left(\pi + \eta \hat{\varphi}^*\right)\right] \sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_{\delta}(-\xi \lambda^\delta)} \leq \left[1 + 2c^* + A c^* \frac{\Gamma(1 + \delta)^{\Gamma(1 - \delta)}}{\xi}\right] V_0 := \Lambda_1 V_0.
\]
with
\[
\Lambda_1 := 1 + 2c^* + A c^* \frac{\Gamma(1 + \delta)^{\Gamma(1 - \delta)}}{\xi}.
\]
In view of the previous assumptions \(\Omega \left(\pi + \eta \hat{\varphi}^*\right) < \frac{1}{4}\), the term \(\left[1 - \Omega \left(\pi + \eta \hat{\varphi}^*\right)\right]\) is positive, and
\[
V(t) \leq 2V_0 \Lambda_1 E_{\delta}(-\xi t^\delta), \quad t \in [0, \mu].
\]
As \(V_0 \Lambda < \frac{1}{4}\) implies \(V(t_* < \frac{1}{2}\), the process can be continued.

For \(t_* \in (\mu, 2\mu]\) and \(\mu \leq t \leq t_*\), \(0 < t - \mu \leq \mu\), notice that (43) gives
\[
\frac{E_{\delta}(-\xi(t - \mu)^\delta)}{E_{\delta}(-\xi t^\delta)} \leq \frac{1}{E_{\delta}(-\xi t^\delta)} \leq \frac{1}{E_{\delta}(-\xi(2\mu)^\delta)} \leq 1 + \xi \Gamma(1 - \delta)(2\mu)^\delta =: B
\]
Therefore and

In accordance with the estimations (50) and (53), we conclude that

\[ V(t - \mu) \leq 2V_0A_1 \frac{E_\delta(-\xi(t-\mu)^\delta)}{E_x(-\xi^\delta)} E_\delta(-\xi^\delta) \leq 2V_0A_1BE_\delta(-\xi^\delta). \]

Returning to (50), we infer that

\[ V(t) \leq (1 + c^*)V_0E_\delta(-\xi^\delta) + 2c^*V_0A_1BE_\delta(-\xi^\delta) + 2c^*V_0A_1BA \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) \times E_\delta(-\xi^\delta)d\omega + \pi \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) \int_\omega^\infty K^*(\omega - \lambda)V(\lambda)d\lambda d\omega \\
+ \hat{h}^*\eta\kappa \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) \int_\omega^\infty k^*(\omega - \lambda)V(\lambda)d\lambda, \]

and therefore

\[ V(t) \leq V_0 \left[ (1 + c^*) + 2c^*A_1B \right] E_\delta(-\xi^\delta) + 2V_0A_1BA \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) \int_\omega^\infty K^*(\omega - \lambda)V(\lambda)d\lambda d\omega. \]

The estimation (51) implies

\[ \frac{V(t)}{E_x(-\xi^\delta)} \leq V_0 \left[ (1 + c^*) + 2c^*A_1B \right] + 2V_0A_1BA \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) \int_\omega^\infty K^*(\omega - \lambda)V(\lambda)d\lambda d\omega. \]

Hence

\[ \left[ 1 - \Omega \left( \pi + \eta\kappa h^* \right) \right] \frac{V(t)}{E_x(-\xi^\delta)} \leq V_0 \left[ (1 + c^*) + 2c^*A_1B \right] + 2V_0A_1BA \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) \int_\omega^\infty K^*(\omega - \lambda)V(\lambda)d\lambda d\omega. \]

As a consequence

\[ V(t) \leq 2V_0A_2E_\delta(-\xi^\delta), \]

where

\[ A_2 := (1 + c^*) + 2c^*A_1B + 2c^*A_1BA \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) \int_\omega^\infty K^*(\omega - \lambda)V(\lambda)d\lambda d\omega. \]

In case \( t_* \in (2\mu, 3\mu) \), and \( 2\mu < t < t_* \), \( \mu < t - \mu < 2\mu \), from (46), we have

\[ V(t) \leq (1 + c^*)V_0E_\delta(-\xi^\delta) + 2c^*V_0A_2E_\delta(-\xi^\delta) + c^*A \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) \times V(\omega - \mu)d\omega + \sup_{-\infty < \lambda < t} \frac{V(\lambda)}{E_x(-\xi^\delta)} \left( \pi + \eta\kappa h^* \right) \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) \int_\omega^\infty K^*(\omega - \lambda)V(\lambda)d\lambda d\omega. \]

In accordance with the estimations (50) and (53), we conclude that

\[ \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) V(\omega - \mu)d\omega \leq V_0 \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) E_\delta(-\xi^\delta)d\omega \\
+ 2V_0A_1 \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) E_\delta(-\xi^\delta)d\omega \\
+ 2V_0A_2 \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) E_\delta(-\xi^\delta)d\omega. \]

Therefore

\[ \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) V(\omega - \mu)d\omega \leq V_0 \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) E_\delta(-\xi^\delta)d\omega \\
+ 2V_0BA_1 \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) E_\delta(-\xi^\delta)d\omega \\
+ 2V_0FA_2 \int_0^t (t - \omega)^{\delta-1}E_\delta,\delta(-\xi(t - \omega)^\delta) E_\delta(-\xi^\delta)d\omega, \]
and
\[ \int_0^t (t-\omega)^{\delta-1} E_{\delta,\delta}(-\xi(t-\omega)^{\delta}) V(\omega - \mu) d\omega \leq V_0 \int_0^t (t-\omega)^{\delta-1} E_{\delta,\delta}(-\xi(t-\omega)^{\delta}) E_\delta(-\xi \omega^{\delta}) d\omega + 2V_0 \Lambda_2 \max\{B, F\} \int_0^t (t-\omega)^{\delta-1} E_{\delta,\delta}(-\xi(t-\omega)^{\delta}) E_\delta(-\xi \omega^{\delta}) d\omega. \]

Moreover
\[ \int_0^t (t-\omega)^{\delta-1} E_{\delta,\delta}(-\xi(t-\omega)^{\delta}) V(\omega - \mu) d\omega \leq 2V_0 \Lambda_2 \max\{B, F\} \int_0^t (t-\omega)^{\delta-1} E_{\delta,\delta}(-\xi(t-\omega)^{\delta}) E_\delta(-\xi \omega^{\delta}) d\omega \]
\[ \leq 2V_0 \Lambda_2 \max\{B, F\} \frac{\Gamma(1+\delta)\Gamma(1-\delta)}{\xi} E_\delta(-\xi t^{\delta}) \leq 2V_0 \Lambda_2 B^* \frac{\Gamma(1+\delta)\Gamma(1-\delta)}{\xi} E_\delta(-\xi t^{\delta}), \tag{55} \]
with \( F := \frac{B}{\xi \mu r} \) and \( \max\{B, F\} = B \max\{1, \frac{1}{\xi \mu r}\} =: B^* \).

In view of the relations (54) and (55), we find
\[ \frac{V(t)}{E_t(-\xi t^{\delta})} \leq V_0 \left[ (1 + c^*) + 2c^* FA_2 \right] + 2V_0 \Lambda_2 B^* A c^* \frac{\Gamma(1+\delta)\Gamma(1-\delta)}{\xi} + \Omega \left( \pi + \eta \kappa h^* \right) \sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_{\delta}(-\xi \lambda^{\delta})}. \tag{56} \]

Then,
\[ \left[ 1 - \Omega \left( \pi + \eta \kappa h^* \right) \right] \sup_{-\infty < \lambda \leq t} \frac{V(\lambda)}{E_{\delta}(-\xi \lambda^{\delta})} \leq V_0 \left[ (1 + c^*) + 2c^* FA_2 \right] + 2V_0 \Lambda_2 B^* A c^* \frac{\Gamma(1+\delta)\Gamma(1-\delta)}{\xi}. \]

Consequently, we obtain
\[ V(t) \leq 2\Lambda_3 V_0 E_\delta(-\xi t^{\delta}) \tag{57} \]
where \( \Lambda_3 := (1 + c^*) + 2c^* B^* A_2 + 2\Lambda_2 B^* A c^* \frac{\Gamma(1+\delta)\Gamma(1-\delta)}{\xi} \).

Recalling that \( U := A \frac{\Gamma(1+\delta)\Gamma(1-\delta)}{\xi} \),
\[ A_1 = (1 + c^*) + c^* (1 + U), \]
\[ A_2 = (1 + c^*) + 2c^* B A \left[ c^* (1 + U) \right] + 2B \left[ c^* (1 + U) \right]^2 \]
and
\[ A_3 = (1 + c^*) + 2c^* B^* A_2 (1 + U) \]
\[ \leq (1 + c^*) \left\{ 1 + \left[ 2B^* c^* (1 + U) \right] + \left[ 2B^* c^* (1 + U) \right]^2 \right\} + \left[ 2B^* c^* (1 + U) \right]^3. \]

We claim that
\[ V(t) \leq 2\Lambda_k V_0 E_\delta(-\xi t^{\delta}), \quad (k - 1) \mu < t \leq t_*, \quad t_* \in ((k - 1) \mu, k \mu], \quad k \geq 1, \tag{58} \]
with
\[ \Lambda_k \leq (1 + c^*) \sum_{l=0}^{k} \left[ 2B^* c^* (1 + U) \right]^l. \]

We proceed by induction to prove (58). It is obvious that (58) is valid for \( k = 1, 2, 3 \). Assume that the claim (58) is valid for \( k \). We want to prove it for \( k + 1 \). Clearly, we obtain
\[ \Lambda_{k+1} := (1 + c^*) + 2B^* c^* \Lambda_k (1 + U) \leq (1 + c^*) + 2B^* c^* (1 + U) (1 + c^*) \sum_{l=0}^{k} \left[ 2B^* c^* (1 + U) \right]^l \]
\[ \leq (1 + c^*) \left\{ 1 + 2B^* c^* (1 + U) \sum_{l=0}^{k} \left[ 2B^* c^* (1 + U) \right]^l \right\} \leq (1 + c^*) \left\{ 1 + \sum_{l=0}^{k} \left[ 2B^* c^* (1 + U) \right]^{(l+1)} \right\} \]
\[ \leq (1 + c^*) \sum_{l=0}^{k+1} \left[ 2B^* c^* (1 + U) \right]^l. \]

17
According to the conditions stated in Theorem 4.1, $\Lambda$ is convergent.

$$V(t) \leq 2A\nu_0E_{\delta}(-\xi t^4), \quad t > 0,$$

with

$$\Lambda := \sum_{l=0}^{+\infty} \left[ 2B^* c^*(1 + U) \right]^l.$$

According to the conditions stated in Theorem 4.1, $\Lambda$ is convergent.

5. Synchronization

From the above stability results, the synchronization of coupled systems can be derived. We refer to system (1) as the uncontrolled system, and the controlled system is given by

$$D^\delta_C \left[ z_p(t) - cz_p(t - \mu) \right] = -a_p z_p(t) + \sum_{q,s=1}^{n_2} d_{qps} \left( \int_0^\infty k_{qps}(\omega)g_q(w_q(t - \omega))d\omega \right)$$

$$\times \int_0^\infty h_{qps}(\omega)g_s(w_s(t - \omega))d\omega + I_p + \chi_p(t), \quad t > 0,$$

$$D^\delta_C \left[ w_q(t) - \bar{c}w_q(t - \mu) \right] = -\bar{a}_q w_q(t) + \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \left( \int_0^\infty \bar{k}_{pqr}(\omega)\bar{g}_p(z_p(t - \omega))d\omega \right)$$

$$\times \int_0^\infty \bar{h}_{pqr}(\omega)\bar{g}_r(z_r(t - \omega))d\omega + J_q + \bar{\chi}_q(t), \quad t > 0,$$

where $\chi_p(t)$ and $\bar{\chi}_q(t)$ are the feedback controls.

Besides, the error system is defined by

$$D^\delta_C \left[ e_p(t) - ce_p(t - \mu) \right] = -a_p e_p(t) + \sum_{q,s=1}^{n_2} d_{qps} \left( \int_0^\infty k_{qps}(\omega)g_q(w_q(t - \omega))d\omega \right)$$

$$\times \int_0^\infty h_{qps}(\omega)g_s(w_s(t - \omega))d\omega - \sum_{q,s=1}^{n_2} d_{qps} \left( \int_0^\infty k_{qps}(\omega)g_q(y_q(t - \omega))d\omega \right)$$

$$\times \int_0^\infty h_{qps}(\omega)g_s(y_s(t - \omega))d\omega + \chi_p(t), \quad t > 0,$$

$$D^\delta_C \left[ \bar{e}_q(t) - \bar{c}\bar{e}_q(t - \mu) \right] = -\bar{a}_q \bar{e}_q(t) + \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \left( \int_0^\infty \bar{k}_{pqr}(\omega)\bar{g}_p(z_p(t - \omega))d\omega \right)$$

$$\times \int_0^\infty \bar{h}_{pqr}(\omega)\bar{g}_r(z_r(t - \omega))d\omega - \sum_{p,r=1}^{n_1} \bar{d}_{pqr} \left( \int_0^\infty \bar{k}_{pqr}(\omega)\bar{g}_p(x_p(t - \omega))d\omega \right)$$

$$\times \int_0^\infty \bar{h}_{pqr}(\omega)\bar{g}_r(x_r(t - \omega))d\omega + \bar{\chi}_q(t), \quad t > 0,$$

where $e_p(t) := z_p(t) - x_p(t)$ and $\bar{e}_q(t) := w_q(t) - y_q(t)$ are the synchronization errors.

Let the controls be

$$\chi_p(t) := -\beta(z_p(t) - x_p(t)), \quad \beta > 0,$$

$$\bar{\chi}_q(t) := -\bar{\beta}(w_q(t) - y_q(t)), \quad \bar{\beta} > 0.$$

Then, the synchronization of systems (1) and (59) boils down to the stability shown in the preceding sections.

Moreover, the convergence rate is enhanced by adding these negative feedbacks to the dissipation coefficients $a_p$ and $\bar{a}_q$. 

18
6. Numerical illustration

Two examples of higher-order fractional BAM NNs will be given to validate the previous theoretical findings.

Example 6.1. We consider the following fractional higher-order BAM NN system

\[
\begin{align*}
D_{\frac{c}{2}}^{\mu}[x_1(t) - cx_1(t - \mu)] &= -a_1 x_1(t) + \sum_{q,s=1}^{2} d_{q,s} \int_{0}^{t} \bar{k}_{q,s}(\omega) g_q(y_q(t - \omega)) d\omega + 1, \\
D_{\frac{c}{2}}^{\mu}[x_2(t) - cx_2(t - \mu)] &= -a_2 x_2(t) + \sum_{q,s=1}^{2} d_{q,s} \int_{0}^{t} \bar{k}_{q,s}(\omega) g_q(y_q(t - \omega)) d\omega + 0.75, \\
D_{\frac{c}{2}}^{\mu}[y_1(t) - \bar{c} y_1(t - \mu)] &= -\bar{a}_1 y_1(t) + \sum_{p,r=1}^{2} \bar{d}_{p,r} \int_{0}^{t} \bar{k}_{p,r}(\omega) \bar{g}_p(x_p(t - \omega)) d\omega + 0.5, \\
D_{\frac{c}{2}}^{\mu}[y_2(t) - \bar{c} y_2(t - \mu)] &= -\bar{a}_2 y_2(t) + \sum_{p,r=1}^{2} \bar{d}_{p,r} \int_{0}^{t} \bar{k}_{p,r}(\omega) \bar{g}_p(x_p(t - \omega)) d\omega + 1, \\
\end{align*}
\]

where the coefficients and functions for \( p, q = 1, 2, t \in [0, 10] \) are taken as

\[
\begin{align*}
a_1 &= 5, \quad a_2 = 7, \quad \bar{a}_1 = 6, \quad \bar{a}_2 = 8, \quad g_q(x) = \tanh(x), \quad \bar{g}_p(x) = \tanh(x), \\
k_{q,s}(t) &= h_{q,s}(t) = e^{-5t}, \quad \bar{k}_{p,r}(t) = \bar{h}_{p,r}(t) = e^{-6t}, \quad r, s = 1, 2, \\
c &= \bar{c} = 0.0001, \quad \mu = 1, \quad b = 0.9, \\
d_{111} &= 1.3, \quad d_{112} = 0.5, \quad d_{211} = 1, \quad d_{212} = 0.25, \\
d_{121} &= 0.75, \quad d_{122} = 1, \quad d_{221} = 0.5, \quad d_{222} = 0.4, \\
\bar{d}_{111} &= 0.6, \quad \bar{d}_{112} = 1, \quad \bar{d}_{211} = 0.5, \quad \bar{d}_{212} = 0.25, \\
\bar{d}_{121} &= 1, \quad \bar{d}_{122} = 1.4, \quad \bar{d}_{221} = 0.75, \quad \bar{d}_{222} = 1.25.
\end{align*}
\]
The controlled system is described by

\begin{align*}
D_c^h[\bar{x}_1(t) - c\bar{x}_1(t - \mu)] &= -a_1\bar{x}_1(t) + \sum_{q,s=1}^{2} d_{q,s} \int_{0}^{t} k_{q,s}(\omega) g_0(\bar{y}_q(t - \omega))d\omega \\
&+ \int_{0}^{t} h_{q,s}(\omega) g_s(\bar{y}_s(t - \omega))d\omega + 1 + \chi_1(\bar{x}_1(t)), \\
D_c^h[\bar{x}_2(t) - c\bar{x}_2(t - \mu)] &= -a_2\bar{x}_2(t) + \sum_{q,s=1}^{2} d_{q,s} \int_{0}^{t} k_{q,s}(\omega) g_0(\bar{y}_q(t - \omega))d\omega \\
&+ \int_{0}^{t} h_{q,s}(\omega) g_s(\bar{y}_s(t - \omega))d\omega + 0.75 + \chi_2(\bar{x}_2(t)), \\
D_c^h[\bar{y}_1(t) - c\bar{y}_1(t - \mu)] &= -\bar{a}_1\bar{y}_1(t) + \sum_{p,r=1}^{2} \bar{d}_{p,r} \int_{0}^{t} \bar{k}_{p,r}(\omega) \bar{g}_p(\bar{x}_p(t - \omega))d\omega \\
&+ \int_{0}^{t} \bar{h}_{p,r}(\omega) \bar{g}_r(\bar{x}_r(t - \omega))d\omega + 0.5 + \bar{\chi}_1(\bar{y}_1(t)), \\
D_c^h[\bar{y}_2(t) - c\bar{y}_2(t - \mu)] &= -\bar{a}_2\bar{y}_2(t) + \sum_{p,r=1}^{2} \bar{d}_{p,r} \int_{0}^{t} \bar{k}_{p,r}(\omega) \bar{g}_p(\bar{x}_p(t - \omega))d\omega \\
&+ \int_{0}^{t} \bar{h}_{p,r}(\omega) \bar{g}_r(\bar{x}_r(t - \omega))d\omega + 1 + \bar{\chi}_2(\bar{y}_2(t)), \\
\bar{x}_1(t) &= -1, \quad \bar{x}_2(t) = -1.75, \quad t \in [-10, 0] \\
\bar{y}_1(t) &= -1, \quad \bar{y}_2(t) = -2, \quad t \in [-10, 0],
\end{align*}

where

\begin{align*}
\chi_p(t) &= -\beta(\bar{x}_p(t) - x_p(t)), \quad \bar{\chi}_q(t) = -\bar{\beta}(\bar{y}_q(t) - y_q(t)), \quad \beta = \bar{\beta} = 2.
\end{align*}

\begin{align*}
D_c^h[e_1(t) - ce_1(t - \mu)] &= -a_1e_1(t) + \sum_{q,s=1}^{2} d_{q,s} \int_{0}^{t} k_{q,s}(\omega) g_0(y_q(t - \omega))d\omega \\
&- \sum_{q,s=1}^{2} d_{q,s} \int_{0}^{t} k_{q,s}(\omega) g_q(\bar{y}_q(t - \omega))d\omega \\
&+ \int_{0}^{t} h_{q,s}(\omega) g_s(\bar{y}_s(t - \omega))d\omega + \chi_1(t), \\
D_c^h[e_2(t) - ce_2(t - \mu)] &= -a_2e_2(t) + \sum_{q,s=1}^{2} d_{q,s} \int_{0}^{t} k_{q,s}(\omega) g_0(y_q(t - \omega))d\omega \\
&- \sum_{q,s=1}^{2} d_{q,s} \int_{0}^{t} k_{q,s}(\omega) g_q(\bar{y}_q(t - \omega))d\omega \\
&+ \int_{0}^{t} h_{q,s}(\omega) g_s(\bar{y}_s(t - \omega))d\omega + \chi_2(t), \\
D_c^h[e_1(t) - c\bar{e}_1(t - \mu)] &= -\bar{a}_1\bar{e}_1(t) + \sum_{p,r=1}^{2} \bar{d}_{p,r} \int_{0}^{t} \bar{k}_{p,r}(\omega) \bar{g}_p(\bar{x}_p(t - \omega))d\omega \\
&- \sum_{p,r=1}^{2} \bar{d}_{p,r} \int_{0}^{t} \bar{k}_{p,r}(\omega) \bar{g}_r(\bar{x}_r(t - \omega))d\omega \\
&+ \int_{0}^{t} \bar{h}_{p,r}(\omega) \bar{g}_r(\bar{x}_r(t - \omega))d\omega + \bar{\chi}_1(t), \\
D_c^h[e_2(t) - c\bar{e}_2(t - \mu)] &= -\bar{a}_2\bar{e}_2(t) + \sum_{p,r=1}^{2} \bar{d}_{p,r} \int_{0}^{t} \bar{k}_{p,r}(\omega) \bar{g}_p(\bar{x}_p(t - \omega))d\omega \\
&- \sum_{p,r=1}^{2} \bar{d}_{p,r} \int_{0}^{t} \bar{k}_{p,r}(\omega) \bar{g}_r(\bar{x}_r(t - \omega))d\omega \\
&+ \int_{0}^{t} \bar{h}_{p,r}(\omega) \bar{g}_r(\bar{x}_r(t - \omega))d\omega + \bar{\chi}_2(t),
\end{align*}

the system (62) is the error system, where

\begin{align*}
e_p(t) &= \bar{x}_p(t) - x_p(t), \quad \bar{e}_q(t) = \bar{y}_q(t) - y_q(t).
\end{align*}
Furthermore, we have
\[
\left[ 1 + \Gamma(1 + b)\Gamma(1 - b) \right] F_{a^* c^*}^{\alpha^*} = 0.02 < 1,
\]
\[
\Omega < 1 - \left[ 1 + \Gamma(1 + b)\Gamma(1 - b) \right] F_{a^* c^*}^{\alpha^*} = 0.98.
\]

Then, Theorem 3.1 is applied. On the other hand, for step \( h = 0.02 \), Figures 1, 2 and 3 illustrate trajectories of the states \( x_1(t), x_2(t), y_1(t) \) and \( y_2(t) \), whilst, Figures 4 and 5 depict trajectories of the error states \( e_1(t), e_2(t), \bar{e}_1(t) \) and \( \bar{e}_2(t) \). The convergence of solutions of system (61) to the equilibrium in \( b\)-Mittag-Leffler manner is shown by Figures 1, 2 and 3. Besides, Figures 4 and 5 describe the convergence of solutions of system (63) to the zero state.

**Example 6.2.** For the unbounded case, we assume
\[
g_q(x) = h_q(x) = \bar{g}_p(x) = \bar{h}_p(x) = \text{argsh}(x), \ p, q = 1, 2,
\]
and we have
\[
e^* < \frac{1}{2B\Gamma(1+U)} = 0.0002, \quad \Omega < \frac{1}{4} = 0.4.
\]

Therefore, the conditions stated in Theorem 4.1 are met. Moreover, for step \( h = 0.02 \), trajectories of the states \( x_1(t), x_2(t), y_1(t) \) and \( y_2(t) \) are illustrated by Figures 6, 7 and 8, and Figures 9 and 10 depict trajectories of \( e_1(t), e_2(t), \bar{e}_1(t) \) and \( \bar{e}_2(t) \) for various data and values of \( b \), respectively. The convergence of solutions of systems (61) and (62) in \( b\)-Mittag Leffler type is depicted by Figures 6, 7, 8, 9 and 10.
7. Conclusion

We have considered a neural network system with some challenging features. Indeed, the system is of fractional order and furthermore it is of higher-order in addition to the presence of neutral delays. Both features are problematic. The difficulties caused by the fractional derivatives are overcome by the use of a fractional version of Halanay inequality. To get around the second obstacle, we have performed some appropriate manipulations and evaluations. The stability is shown to be of Mittag-Leffler type as is expected for fractional differential equations. The synchronization issue is obtained easily from our stability results through linear feedback controls.

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Figure 1: Convergence of the solutions of system \text{(61)} to the stationary state.

Figure 2: Convergence of solutions of system \text{(61)} to the stationary state for various data.
Figure 3: Convergence of solutions of system (61) to the stationary state for different values of $b$.

Figure 4: Convergence of solutions of system (63) to the stationary state for various data.
Figure 5: Convergence of solutions of system (63) to the stationary state for different values of $b$.

Figure 6: Decay of the solutions of system (61) to the stationary state.
Figure 7: Convergence of solutions of system (61) to the stationary state for various data.

Figure 8: Decay of solutions of system (61) to the stationary state for different values of $b$. 
Figure 9: Convergence of solutions of system (63) to the stationary state for various data.

Figure 10: Decay of solutions of system (63) to the stationary state for different values of $b$. 