Bootstrapping the effect of the twist operator in the D1D5 CFT

Bin Guo\textsuperscript{1} and Shaun Hampton\textsuperscript{2}

Institut de Physique Théorique, Université Paris-Saclay, CNRS, CEA, Orme des Merisiers, Gif-sur-Yvette, 91191 CEDEX, France

Abstract

In the D1D5 CFT the twist operator of order 2 can twist together two copies in the untwisted sector into a single joined copy in the twisted sector. Traditionally, this effect is computed by using the covering map method. Recently, a new method was developed using the Bogoliubov ansatz and conformal symmetry to compute this effect in a toy model of one free boson. In this paper, we use this method with superconformal symmetry to compute the effect of the twist operator in the D1D5 CFT. This may provide more effective tools for computing correlation functions of twist operators in this system.

\textsuperscript{1}bin.guo@ipht.fr
\textsuperscript{2}shaun.hampton@ipht.fr
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1 Introduction

The D1D5 CFT is widely used in AdS$_3$/CFT$_2$ as a boundary theory to understand the bulk physics [1–12]. It has yielded many interesting results related to black holes [13–16]. It provides a system in which to count black hole microstates [2,3] of the corresponding gravitational theory in the holographic dual. In certain cases these microstates are explicitly known [17–28] and correspond to particular states in the CFT [29–36]. A simple description of the CFT exists at what is known as ‘orbifold’ point where the theory is a free symmetric product orbifold \( \mathcal{M}^N / S_N \) where \( N = N_1 N_5 \) with \( N_1 \) the number of \( N_1 \) branes and \( N_5 \) the number of \( N_5 \) branes and \( S_N \) is the permutation group. However, the supergravity description lives far away from this point in the moduli space. Therefore, only quantities which are protected, as one flows from the free theory into the supergravity regime, can be reliably computed. To gain more understanding about their relationship one can add a deformation [43–50] which connects the two points. More recently, there has been some debate about which operator flows to the supergravity point from the study of tensionless string in AdS$_3$ [12,50–55]. In any case, it is clear that there is one ingredient which is critical in this process. That is the twist operator of order 2. The D1D5 system is composed of component strands with a total length of \( N = N_1 N_5 \). The twist operator can join and separate these strands. A challenge, from the CFT perspective, is using this twist operator to move far away from the free point. This is because computations typically involve introducing covering maps, which quickly become complicated as one adds more twists. For additional works involving deformations away from the orbifold point see [56–82].

Alternative methods to using just the covering maps to compute the effects of the twist operator, is to bootstrap the computations by using Bogoliubov transformations and conformal symmetry. There have been several works along these lines. In [57] the authors used a Bogoliubov transformation to compute various coefficients coming from the effects of the twist operator. This method involved inverting infinite dimensional matrices. The authors in [49] used a combination of conformal symmetry and covering maps to compute various effects produced by the twist operator. Recently, in [83], a method was developed for orbifold CFTs of a single boson, which used a weak Bogoliubov ansatz and conformal symmetry to determine the effect of the twist operator completely. This paper is a continuation of this method to include fermions and thereby compute the full effect of the twist operator in the D1D5 CFT purely by using the symmetries of the theory. It is hopeful that these bootstrap techniques can be applied to higher orders in the twist operator which in turn can help to describe effects and processes which are far from the orbifold point. This would provide insight into nonperturbative processes in the D1D5 CFT which would correspond to processes in the supergravity regime.

This paper is organized as follows. In section 2 we review the D1D5 CFT. In section 3 we describe the effect of the twist operator and introduce the ‘weak’ Bogoliubov ansatz. In section 4 we compute the effect for fermionic modes. In section 5 we compute the effect for bosonic modes by using supersymmetry. In section 6 we summarize all the coefficients.
we obtained for the effect of the twist operator. In section 7 we discuss our results and outlook.

2 The D1D5 CFT

In this section we review the D1D5 CFT at the orbifold point. Consider type IIB string theory compactified on $M_{4,1} \times S^1 \times T^4$ with $N_1$ D1 branes wrapping $S^1$ and $N_5$ D5 branes wrapping $S^1 \times T^4$. The torus is taken to be much smaller than $S^1$ and thus the low energy limit of this configuration reduces to a 1+1 $\mathcal{N} = 4$ Super Conformal Field Theory (SCFT) living on a circle of radius $R$, with central charge $c = 6N$ where $N = N_1N_5$. The theory has $SO_E(4) \cong SU(2)_L \times SU(2)_R$ symmetry in the noncompact directions where $E$ is stands for ‘external’. The torus has an $SO_I(4) \cong SU(1)(2) \times SU(2)$ symmetry where $I$ stands for ‘internal’. Upon compactification, this symmetry is broken however it serves as a useful organizing principle. The base space is a cylinder parameterized by the rescaled coordinates.

$$\tau = it/R, \quad -\infty < \tau < \infty$$
$$\sigma = y/R, \quad 0 \leq \sigma \leq 2\pi$$

Here $t$ is the physical Lorenztian time and $y$ is the physical coordinate of $S^1$. We can group these coordinates into a single complex coordinate

$$w = \tau + i\sigma$$

The theory contains four bosons and four fermions. The bosonic fields can be grouped together according to the $SU(1)(2) \times SU(2)$ symmetry as $X_{A\hat{A}}$ where $A, \hat{A} = +, -$. The left moving fermions can be grouped together according to the combined symmetries $SU_L(2) \times SU(1)(2)$ as $\psi^{\alpha A}$ where $\alpha = +, -$. The right moving fields are grouped in a similar manner.

This CFT is a symmetric product orbifold $\mathcal{M}^N/S_N$, of $N$ copies of a ‘seed’ CFT $\mathcal{M}$ where $\mathcal{M}$ is the target space and $S_N$ is the permutation group. This CFT contains various twisted sectors which are obtained by applying a twist operator which joins and splits copies of the CFT by altering their boundary conditions. To better illustrate this idea consider $N$ bosons $X_{AA}^{(j)}$ where $j = 1, 2, \ldots, N$. Consider a twist operator of order $k$ which will twist together $k$ out of $N$ copies, e.g. the first $k$ copies. As you rotate by an angle $2\pi$ around the cylinder the action of the twist on the bosons gives

$$X_{AA}^{(1)} \rightarrow X_{A\hat{A}}^{(2)} \rightarrow X_{A\hat{A}}^{(3)} \rightarrow \ldots \rightarrow X_{A\hat{A}}^{(k)} \rightarrow X_{AA}^{(1)}$$

for copies 1 to $k$. For the remaining copies we have

$$X_{AA}^{(j)} \rightarrow X_{A\hat{A}}^{(j)}, \quad k + 1 \leq j \leq N$$

In this paper, we take the fermions to be in the Ramond sector. Under the rotation, we have

$$\psi^{\alpha A(1)} \rightarrow \psi^{\alpha A(2)} \rightarrow \psi^{\alpha A(3)} \rightarrow \ldots \rightarrow \psi^{\alpha A(k)} \rightarrow \psi^{\alpha A(1)}$$
for copies 1 to \( k \) and for the remaining copies we have

\[
\psi^{\alpha A(j)} \rightarrow \psi^{\alpha A(j)}, \quad k + 1 \leq j \leq N
\]  

(2.6)

Thus it is convenient to define a single field \( X_{A\dot{A}} \) on the \( k \)-wound copy. The field \( X_{A\dot{A}} \) equals to \( X^{(j)}_{A\dot{A}} \) on the \( j \)-th segment of the \( k \)-wound copy, i.e.

\[
X_{A\dot{A}}(\tau, \sigma) = X^{(j)}_{A\dot{A}}(\tau, \sigma), \quad 2\pi(j - 1) \leq \sigma < 2\pi j
\]  

(2.7)

The field \( X_{A\dot{A}} \) has the boundary condition

\[
X_{A\dot{A}}(w + 2\pi ki) = X_{A\dot{A}}(w)
\]  

(2.8)

Similarly we define a single field \( \psi^{\alpha A} \) on the \( k \)-wound copy as follows

\[
\psi^{\alpha A}(\tau, \sigma) = \psi^{\alpha A(j)}(\tau, \sigma), \quad 2\pi(j - 1) \leq \sigma < 2\pi j
\]  

(2.9)

with the boundary condition

\[
\psi^{\alpha A}(w + 2\pi ki) = \psi^{\alpha A}(w)
\]  

(2.10)

Because of these boundary conditions, one can define modes on the \( k \)-wound copy on the cylinder for bosons on a constant time slice as

\[
\alpha_{A\dot{A}, \dot{\tau}} = \frac{1}{2\pi} \int_{\sigma=0}^{2\pi k} dw \, e^{\dot{\tau} w} \partial X_{A\dot{A}}(w)
\]  

(2.11)

and for fermions in the R sector as

\[
d^{\alpha A}_{\tau} = \frac{1}{2\pi i} \int_{\sigma=0}^{2\pi k} dw \, e^{\dot{\tau} w} \psi^{\alpha A}(w)
\]  

(2.12)

where \( n \) is an integer. In the next section we discuss the effect of the twist operator in the D1D5 CFT.

### 3 The effect of the twist operator

Here we outline the effect of the twist operator \( \sigma^+_2 \) in the D1D5 CFT. The operator \( \sigma^+_2 \) has twist order 2 and is a chiral primary with dimension and charge \( h = j = 1/2 \). In the rest of the paper we will omit the subscript 2 and label it as \( \sigma^+ \) for brevity. The twist operator changes the vacuum of the CFT. Anytime such a change happens, one can introduce a Bogoliubov transformation which relates modes defined with respect to the original vacuum to modes defined with respect to the modified vacuum. For more work on the twist operator in the D1D5 CFT see [8, 84–88].
In this paper we take $N = 2$ where the initial vacuum is composed of two singly wound copies of the D1D5 CFT in the Ramond sector on the cylinder. Consider an initial state, $|\phi\rangle$ which contains an arbitrary number of bosonic and fermionic modes acting on this vacuum

$$|\phi\rangle = \alpha_{A_1 A_1, -m_1}^{(i_1)} \alpha_{A_2 A_2, -m_2}^{(i_2)} \ldots \alpha_{A_k A_k, -m_k}^{(i_k)} d_{-n_1}^{\beta_1 B_1(j_1)} d_{-n_2}^{\beta_2 B_2(j_2)} \ldots d_{-n_l}^{\beta_l B_l(j_l)} |0_R\rangle |0_R\rangle$$

where $i_k', j_l' = 1, 2$ are copy labels and $A_k', \dot{A}_k', B_l', \dot{B}_l' = +, -$ and $\beta_l' = +, -$. Each copy of the singly wound vacuum carries the quantum numbers $h = 1/4$ and $j = -1/2$. We introduce the twist, $\sigma^+(w)$ at some point $w$ on the cylinder. Acting it on the state $|\phi\rangle$ will join the two singly wound vacua into a doubly wound vacuum and produce three basic effects.

(i) Contraction: The action of the twist can cause any two bosonic modes,

$$\alpha_{AA, -m}^{(i)} \text{ and } \alpha_{BB, -n}^{(j)}$$

and any two fermionic modes

$$d_{-m}^{\alpha A(i)} \text{ and } d_{-n}^{\beta B(j)}$$

with the appropriate charges to ‘Wick’ contract producing a number

$$C_{AACC}^{ij}[n_1, n_2] \equiv C[\alpha_{A,-n_1}^{(i)} \alpha_{C,-n_2}^{(j)}] \equiv \varepsilon_{AC} \varepsilon_{\dot{A}C} C_{B}^{ij}[n_1, n_2]$$

$$C_F^{ij, \alpha A\beta B}[n_1, n_2] \equiv C[d_{-n_1}^{\alpha A(i)} d_{-n_2}^{\beta B(j)}] \equiv \varepsilon_{\alpha\beta} \varepsilon^{AB} C_F^{ij}[n_1, n_2]$$

For the contraction we consider all possible pairs of bosons and all possible pairs of fermions. They can contract together as described above but if not they can also pass through the twist as we will describe below.

(ii) Propagation: Any mode remaining after contraction produces a linear combination of modes on a doubly wound string weighted by a certain set of coefficients

$$\alpha_{A_A, -n}^{(i)} \rightarrow \sum_{p>0} f_{i}[-n, -p] \alpha_{A_A, -p}, \quad n > 0$$

$$d_{-n}^{\alpha A(i)} \rightarrow \sum_{p>0} f_{i}^{+}[-n, -p] d_{-p}^{\alpha A}, \quad n \geq 0$$

$$d_{-n}^{\alpha A(i)} \rightarrow \sum_{p>0} f_{i}^{-}[-n, -p] d_{-p}^{-A}, \quad n > 0$$

where $n$ is an integer. In appendix B, we show that

$$f_{i}[-n, -n] = f_{i}^{+}[-n, -n] = \frac{1}{2}$$
\( f_i[-n, -p], f_i^\pm[-n, -p] \neq 0 \) for \( p \neq n \) and \( p \) a positive half integer \hspace{1cm} (3.6)

(iii) Pair creation: After the previous two steps, the twist acts on two copies of the singly wound vacuum. The effect is given below

\[
|\chi\rangle \equiv \sigma^+(w)\langle 0_{R}^{(1)}|0_{R}^{(1)}\rangle = \exp\left(\sum_{m,n>0} \gamma^B_{mn}[\alpha_{++,-m}\alpha_{--,-n} + \alpha_{+-,-m}\alpha_{-+,-n}]\right) \exp\left(\sum_{m,n>0} \gamma^F_{mn}[d_{-m}^{++}d_{-n}^{--} - d_{-m}^{+-}d_{-n}^{--}]\right)|0_{R}^{2-}\rangle
\] \hspace{1cm} (3.7)

Where \( \gamma^B_{mn}, \gamma^F_{mn} \) are the Bogoliubov coefficients describing pair creation for bosons and fermions respectively. In appendix B, we show that

\[
\gamma^B_{mn}, \gamma^F_{mn} \neq 0, \quad m, n > 0 \quad \text{and} \quad m, n \text{ half integer} \hspace{1cm} (3.8)
\]

Since the coefficients in the contraction, propagation and pair creation are independent of each other, we call it ‘weak’ the Bogoliubov ansatz as in [83]. To better understand these rules, we give two examples in the following. Applying the twist operator to a single fermionic mode in the initial state gives

\[
\sigma^+(w)d^{A(i)}_{n}d^{A(j)}_{n_{1}}d^{B(j)}_{n_{2}}|0_{R}^{1}\rangle|0_{R}^{1}\rangle = \sum_{p>0} f_{i}^{-}[n_{1}, -p]d_{-p}^{-A_{j}}d_{-p}^{-B_{j}}|\chi\rangle
\] \hspace{1cm} (3.9)

where the initial mode passes through the twist operator using the rule (3.5). Then the twist operator acts on the untwisted vacuum to produce pairs of modes using the rule (3.7). Applying the twist operator to two fermionic modes in the initial state, we obtain

\[
\sigma^+(w)d_{-n_{1}}^{A(i)}d_{-n_{2}}^{B(j)}|0_{R}^{1}\rangle|0_{R}^{1}\rangle = \sum_{p>0} f_{i}^{+}[-p]d_{-p}^{A_{j}}d_{-p}^{B_{j}}\sum_{q>0} f_{j}^{-}[-q]d_{-q}^{B_{j}} + e^{+-}\epsilon^{AB}C_{F}^{ij}[n_{1}, n_{2}]|\chi\rangle
\] \hspace{1cm} (3.10)

The first term comes from the propagation of the two initial modes while the second term comes from the contraction. The state \( |\chi\rangle \) comes from the pair creation.

4 Bootstrapping the effect of the twist operator

In this section we will take the weak Bogoliubov ansatz and apply the superconformal generators to get recursion relations for the coefficients in the ansatz. We will start with the fermions and compute for the bosons in the next section using supersymmetry.
4.1 Pair creation

In this subsection, we will derive the pair creation coefficients, $\gamma_{mn}^F$. Starting with the state

$$ (L_{-1} + J_{-1}^3) |0_R^{(1)}\rangle |0_R^{(2)}\rangle = 0 \quad (4.1) $$

we then apply the twist operator $\sigma^+$ to find some recursion relations for $\gamma_{mn}^F$. Consider the state

$$ 0 = \sigma^+(w)(L_{-1} + J_{-1}^3) |0_R^{(1)}\rangle |0_R^{(2)}\rangle $$

$$ = ((L_{-1} + J_{-1}^3)\sigma^+(w) - [L_{-1} + J_{-1}^3, \sigma^+(w)]) |0_R^{(1)}\rangle |0_R^{(2)}\rangle \quad (4.2) $$

In the second term, the commutator is given by

$$ [L_{-1}, \sigma^+(w)] = \oint_{c_{w}} \frac{dw'}{2\pi i} e^{-w'} T(w') \sigma^+(w) $$

$$ = e^{-w} \oint_{c_{w}} \frac{dw'}{2\pi i} e^{-(w'-w)} T(w') \sigma^+(w) $$

$$ = e^{-w} \oint_{c_{w}} \frac{dw'}{2\pi i} (1 - (w'-w) + \frac{1}{2} (w'-w)^2 + \ldots) T(w') \sigma^+(w) $$

$$ = e^{-w} (L_{-1}^{(w)} - L_{0}^{(w)} + \frac{1}{2} L_{1}^{(w)} + \ldots) \sigma^+(w) $$

$$ = e^{-w} \left( \theta - \frac{1}{2} \right) \sigma^+(w) \quad (4.3) $$

and

$$ [J_{-1}^3, \sigma^+(w)] = \oint_{c_{w}} \frac{dw'}{2\pi i} e^{-w'} J_{-1}^3(w') \sigma^+(w) $$

$$ = e^{-w} \oint_{c_{w}} \frac{dw'}{2\pi i} e^{-(w'-w)} J_{-1}^3(w') \sigma^+(w) $$

$$ = e^{-w} \oint_{c_{w}} \frac{dw'}{2\pi i} (1 - (w'-w) + \ldots) J_{-1}^3(w') \sigma^+(w) $$

$$ = e^{-w} (J_{0}^{3(w)} - J_{1}^{3(w)} + \ldots) \sigma^+(w) $$

$$ = e^{-w} \frac{1}{2} \sigma^+(w) \quad (4.4) $$

where we have used

$$ L_{-1}^{(w)} \sigma^+(w) = \partial \sigma^+(w), \quad L_{0}^{(w)} \sigma^+(w) = \frac{1}{2} \sigma^+(w), \quad L_{n>0}^{(w)} \sigma^+(w) = 0 $$

$$ J_{0}^{(w)} \sigma^+(w) = \frac{1}{2} \sigma^+(w), \quad J_{n>0}^{(w)} \sigma^+(w) = 0 \quad (4.5) $$

Inserting the relations (4.3) and (4.4) into (4.2) and using the pair creation ansatz (3.7) gives

$$ 0 = \sigma^+(w)(L_{-1} + J_{-1}^3) |0_R^{(1)}\rangle |0_R^{(2)}\rangle $$

$$ = (L_{-1} + J_{-1}^3 - e^{-w} \partial) \exp \left[ \sum_{m,n \geq 0} \gamma_{m+1/2,n+1/2} \frac{d_{m+1/2,n+1/2}}{d_{m+1/2,n+1/2}} \frac{d_{-m-1/2,n-1/2}}{d_{-m-1/2,n-1/2}} \right] |0_R^{2-}\rangle \quad (4.6) $$
where \( m \) and \( n \) are non-negative integers. Note that in the exponential we only keep the fermionic modes in the \( ++, -- \) sector because here they are the only modes which are necessary to derive the coefficient \( \gamma_{m+1/2,n+1/2}^F \). We will show that this equation determines the \( \gamma_{m+1/2,n+1/2}^F \) completely. Let us look for the following term in (4.6)

\[
d^{++}_{-(m+1/2)}d^{--}_{-(n+1/2)}|0^-_R\rangle
\]

(4.7)

4.1.1 The recursion relations

For \( m, n > 0 \), we obtain the relation

\[
\gamma_{m-1/2,n+1/2}^F \left[ L_{-1} + J^3_{-1}, d^{++}_{-(m-1/2)} \right] d^{--}_{-(n+1/2)} \\
+ \gamma_{m+1/2,n-1/2}^F d^{++}_{-(m+1/2)} \left[ L_{-1} + J^3_{-1}, d^{--}_{-(n-1/2)} \right] \\
- e^{-w} \partial \gamma_{m+1/2,n+1/2}^F d^{++}_{-(m+1/2)} d^{--}_{-(n+1/2)} = 0
\]

(4.8)

Using the commutators

\[
\left[ L_{-1} + J^3_{-1}, d^{++}_{-(m-1/2)} \right] = (m + 1/2) d^{++}_{-(k+3/2)} \\
\left[ L_{-1} + J^3_{-1}, d^{--}_{-(n-1/2)} \right] = (n - 1/2) d^{--}_{-(k+3/2)}
\]

(4.9)

we obtain a recursion relation

\[
\gamma_{m-1/2,n+1/2}^F (m + 1/2) + \gamma_{m+1/2,n-1/2}^F (n - 1/2) = e^{-w} \partial \gamma_{m+1/2,n+1/2}^F \quad m, n > 0
\]

(4.10)

For the case \( m = 0, n > 0 \), we do not have the first term in (4.8). Thus we have

\[
\gamma_{1/2,n-1/2}^F (n - 1/2) = e^{-w} \partial \gamma_{1/2,n+1/2}^F \quad m = 0, n > 0
\]

(4.11)

For the case \( m > 0, n = 0 \), we do not have the second term in (4.8). Thus we have

\[
\gamma_{m-1/2,1/2}^F (m + 1/2) = e^{-w} \partial \gamma_{m+1/2,1/2}^F \quad m > 0, n = 0
\]

(4.12)

For the case \( m = n = 0 \), we have

\[
(L_{-1} + J^3_{-1}) |0^-_R\rangle - e^{-w} \partial \gamma_{1/2,1/2}^F d^{++}_{-1/2}d^{--}_{-1/2} |0^-_R\rangle = 0
\]

(4.13)

Using

\[
(L_{-1} + J^3_{-1}) |0^-_R\rangle = -d^{++}_{-1/2}d^{--}_{-1/2} |0^-_R\rangle
\]

(4.14)

we obtain

\[
\partial \gamma_{1/2,1/2}^F = -\frac{e^w}{4}
\]

(4.15)
4.1.2 The solution

To determine the initial condition for this system of differential equations, we consider

\[ |0_R^- \rangle = \sigma^+ (w \to -\infty) |0_R^- \rangle \]

(4.16)

since the twist operator \( \sigma^+ \) is the lowest dimension operator that changes the untwisted sector to the twisted sector. Thus from the ansatz (3.7) we have

\[ \gamma_{m+1/2,n+1/2}^F (w \to -\infty) = 0 \]

(4.17)

The different equations can be solved using these initial conditions. The solution to (4.15) is

\[ \gamma_{1/2,1/2}^F = -\frac{e^w}{4} \]

(4.18)

We can find all other \( \gamma_{m+1/2,n+1/2} \)'s by using the relations (4.10), (4.11), and (4.12). The \( \gamma_{m+1/2,n+1/2} \)'s form an inverted triangle, where the space between each lattice point is an integer and the bottom lattice point is \( \gamma_{1/2,1/2} \). The relation (4.11) moves you along the left edge, (4.12) moves you along the right edge and (4.10) moves you within the interior as follows.

\[ \gamma_{1/2,5/2}^F \]
\[ \gamma_{3/2,3/2}^F \]
\[ \gamma_{5/2,1/2}^F \]
\[ \gamma_{1/2,3/2}^F \]
\[ \gamma_{3/2,1/2}^F \]

(4.19)

The solution is

\[ \gamma_{m+\frac{1}{2},n+\frac{1}{2}}^F = -\frac{e^{(m+n+1)w} \Gamma[\frac{3}{2}+m] \Gamma[\frac{3}{2}+n]}{(2n+1)(m+n+1) \pi[m+1] \Gamma[n+1]} \]

(4.20)

where \( m \) and \( n \) are non-negative integers.

4.2 Propagation

In this subsection, we derive the expressions for propagation \( f_i^\pm [-n, -p] \) which correspond to a fermionic mode passing through the twist operator.
4.2.1 Relations using $L_0 + J_0^3$

Here we use the generator $L_0 + J_0^3$ to find the $w$-dependence of $f_i^{-}[-1,-p]$. We start with

$$(L_0 + J_0^3)\omega_{-1}^{-}[-1,-p] = 0$$

(4.21)

Then applying the twist operator, we obtain the relation

$$0 = \sigma_2^+(w)(L_0 + J_0^3)\omega_{-1}^{-}[-1,-p]$$

(4.22)

Now commuting $L_0 + J_0^3$ to the left gives

$$0 = ((L_0 + J_0^3)\sigma_2^+(w) - [L_0 + J_0^3, \sigma_2^+(w)]) \omega_{-1}^{-}[-1,-p]$$

(4.23)

The commutator is given by

$$[L_0 + J_0^3, \sigma_2^+(w)] = \left( \oint \frac{dw'}{2\pi i} T(w') + \oint \frac{dw'}{2\pi i} J_3(w') \right) \sigma_2^+(w)$$

$$= \left( L_{-1}^3(w) + J_0^3(w) \right) \sigma_2^+(w)$$

$$= \left( \partial + \frac{1}{2} \right) \sigma_2^+(w)$$

(4.24)

Inserting this into (4.23) gives

$$0 = \left( L_0 + J_0^3 - \left( \partial + \frac{1}{2} \right) \right) \sigma_2^+(w) \omega_{-1}^{-}[-1,-p]$$

(4.25)

Using the ansatz (3.9) and keeping terms with only one $d^{-}$ mode we get

$$0 = \left( L_0 + J_0^3 - \left( \partial + \frac{1}{2} \right) \right) \sum_{p>0} f_i^{-}[-1,-p] \omega_{-p}^{-}[-1,-p]$$

(4.26)

Acting with $L_0 + J_0^3$, the partial derivative and matching coefficients we obtain

$$(p-1)f_i^{-}[-1,-p] = \partial f_i^{-}[-1,-p]$$

(4.27)

This relation implies that $f_i^{-}[-1,-p]$ will take the following functional form

$$f_i^{-}[-1,-p] \propto e^{(p-1)w}$$

(4.28)

Using this result in the following section we compute the exact form of $f_i^{-}[-1,-p]$ which includes the proportionality constant.
4.2.2 Relations using $J^3_1$

Here we use $J^3_1$ to compute the full expression for $f^{-}_i[-1, -p]$ upto an over constant $C$. We begin with the state

$$J^3_1 d^{-}_1(0^R) = 0$$  \hspace{1cm} (4.29)

Applying the twist operator gives

$$0 = \sigma^+(w) J^3_1 d^{-}_1(0^R)$$  \hspace{1cm} (4.30)

Commuting $J^3_1$ through the twist gives the relation

$$0 = (J^3_1 \sigma^+(w) - [J^3_1, \sigma^+(w)]) d^{-}_1(0^R)$$  \hspace{1cm} (4.31)

where the commutator is given by

$$[J^3_1, \sigma^+(w)] = \int_{C^w} \frac{dw'}{2\pi i} e^{w'J^3_1}(w')\sigma^+(w)$$

$$= e^w \int_{C^w} \frac{dw'}{2\pi i} e^{w'J^3_1}w'\sigma^+(w)$$

$$= e^w \int_{C^w} \frac{dw'}{2\pi i} (1 + (w' - w) + \ldots) J^3_1 \sigma^+(w)$$

$$= \frac{1}{2} e^w \sigma^+(w)$$  \hspace{1cm} (4.32)

where we have used (4.5). Inserting this into (4.31)

$$0 = (J^3_1 - \frac{1}{2} e^w) \sigma^+(w) d^{-}_1(0^R)$$  \hspace{1cm} (4.33)

Again using the ansatz (3.7) and keeping only the fermionic modes we find

$$0 = (J^3_1 - \frac{1}{2} e^w) \sum_{p>0} f^{-}_i[-1, -p] d^{-}_p$$

$$\exp \left[ \sum_{m,n\geq 0} \gamma^F_{m+1/2, n+1/2} \left\{ [J^3_1, d^{-}_m] d^{-}_n - d^{-}_m [J^3_1, d^{-}_n] \right\} \right] 0^2_R$$  \hspace{1cm} (4.34)

Looking for the term $d^{-}_{(n+1/2)} 0^2_R$, we obtain

$$0 = f^{-}_i[-1, -1/2] \gamma^F_{i/2, 1} \left\{ [J^3_1, d^{-}_{-1/2}] d^{-}_{-1/2} \right\} d^{-}_{(n+1/2)}$$

$$+ f^{-}_i[-1, -(n+1/2)] \gamma^F_{i/2, 1/2} d^{-}_{-(n+1/2)} \left\{ [J^3_1, d^{-}_{1/2}] d^{-}_{-1/2} \right\}$$

$$- f^{-}_i[-1, -(n+1/2)] \gamma^F_{i/2, 1/2} d^{-}_{-(n+1/2)} \left\{ [J^3_1, d^{-}_{1/2}] d^{-}_{1/2} \right\}$$
\[ + f_i^- [-1, -(n + 3/2)] [J_1^3, d_{-(n+3/2)}^-] - \frac{1}{2} e^w f_i^- [-1, -(n + 1/2)] d_{-(n+1/2)}^- \] 

(4.35)

which gives

\[ 0 = f_i^- [-1, -1/2] \gamma_{i/2, n+1/2}^F - 2 f_i^- [-1, -(n + 1/2)] \gamma_{i/2, 1/2}^F \]

\[ - \frac{1}{2} f_i^- [-1, -(n + 3/2)] - \frac{1}{2} e^w f_i^- [-1, -(n + 1/2)] \] 

(4.36)

Notice that the second and fourth terms on the RHS cancel each other because of the value of \( \gamma_{i/2, 1/2}^F \). Plugging in \( \gamma_{i/2, n+1/2}^F \) from (4.18), we obtain

\[ f_i^- [-1, -(n + 3/2)] = -\frac{2e^{(n+1) w} \Gamma[\frac{3}{2}] \Gamma[\frac{3}{2} + n]}{(2n + 1) \pi \Gamma[n + 2]} f_i^- [-1, -1/2], \quad n \geq 0 \] 

(4.37)

We note that copy 2 quantities are related to copy 1 quantities by a shift of \( w \to w + 2\pi i \). Therefore using (4.28) for \( f_i^- [-1, -1/2] \) we get the expression

\[ f_i^- [-1, -1/2] = C(-1)^{i+1} e^{-w/2} \] 

(4.38)

For higher modes we find that

\[ f_i^- [-1, -(n + 1/2)] = C(-1)^i \frac{2e^{(n-1/2) w} \Gamma[\frac{3}{2}] \Gamma[\frac{1}{2} + n]}{\pi (2n - 1) \Gamma[n + 1]}, \quad n \geq 0 \] 

(4.39)

In appendix C we compute the full expression for \( f_i^- [-n, -(m + 1/2)] \) which is given by

\[ f_i^- [-n, -(m + 1/2)] = C(-1)^i \frac{2\Gamma[\frac{1}{2} + n] \Gamma[\frac{1}{2} + m]}{\pi \Gamma[n] \Gamma[1 + m]} \frac{e^{(m-n+1/2) w}}{2m - 2n + 1}, \quad n > 0, m \geq 0 \] 

(4.40)

where the constant \( C \) is computed in (4.61).

### 4.2.3 Relations using \( J_1^+ \)

In the previous section we derived the propagation \( f_i^- [-n, -p] \). We can derive \( f_i^+ [-n, -p] \) in a similar way by replacing \( d^- \) in (4.30) by \( d^+ \). However, this will introduce another undetermined constant similar to the constant \( C \) in (4.40). In this section, we will derive \( f_i^+ [-n, -p] \) by relating it to \( f_i^- [-n, -p] \) using the mode \( J_1^+ \). In this way, no extra undetermined constant will be introduced. We begin with the relation

\[ \sigma^+(w) d_+^{(n+i)} [0_{R_1}^-]^{(1)} [0_{R_2}^-]^{(2)} = \sigma^+(w) J_1^+ d_-^{(n-i)} [0_{R_1}^-]^{(1)} [0_{R_2}^-]^{(2)} = J_1^+ \sigma^+(w) d_-^{(n+i)} [0_{R_1}^-]^{(1)} [0_{R_2}^-]^{(2)} \] 

(4.41)

where we have used the commutator similar to the commutator (4.32)

\[ [J_1^+, \sigma^+(w)] = e^w (J_0^+, w) + J_1^+, w + \ldots) \sigma^+(w) = 0 \] 

(4.42)
Looking for the term $d_{-(m+1/2)}^{++}$ for $m \geq 0$ from both sides of (4.41), we have
\[
\begin{align*}
  f_i^+[-n, -(m + 1/2)] & d_{-(m+1/2)}^{++} \\
  &= f_i^-[-(n+1), -(m + 3/2)] [J_i^+, d_{-(m+3/2)}] \\
  &- f_i^-[-(n+1), -1/2] \gamma_{m+1/2,1/2} \{ [J_1^+, d_{-1/2}^+], d_{-1/2}^- \} d_{-(m+1/2)}^{++}
\end{align*}
\]
which gives
\[
\begin{align*}
  f_i^+[-n, -(m + 1/2)] &= f_i^-[-(n+1), -(m + 3/2)] + 2 f_i^-[-(n+1), -1/2] \gamma_{m+1,1/2} \\
\end{align*}
\]
Plugging in $f_i^-$, (4.40), and $\gamma_{m+1,1/2}$, (6.1), we find
\[
\begin{align*}
  f_i^+[-n, -(m + 1/2)] &= C(-1)^i \frac{2 \Gamma[\frac{1}{2} + n] \Gamma[\frac{3}{2} + m]}{\pi \Gamma[1 + n] \Gamma[1 + m]} e^{(m-n+1/2)w} \\
\end{align*}
\]
4.3 Contraction

In this section we derive the expression for the contraction $C_F^{i\bar{j}}[n_1, n_2]$ in terms of the propagation $f_i^+[-n, -p]$. Let’s start with the following state
\[
|\psi\rangle \equiv \sigma_2^+(w) d_{-n}^{++(i)} d_{-1}^{-(j)} |0_R^{(1)}\rangle |0_R^{(2)}\rangle
\]
where $n \geq 0$.

Using our ansatz we can write $|\psi\rangle$ as
\[
|\psi\rangle = \left( \sum_{p > 0} f_i^+[-n, -p] d_{-p}^{++} \sum_{p' > 0} f_j^-[-1, -p'] d_{-p'}^- + C_F^{i\bar{j}}[n, 1] \right) |\chi\rangle
\]
Expanding $|\psi\rangle$ and keeping only terms which contain no fermionic modes we obtain
\[
|\psi\rangle = C_F^{i\bar{j}}[n, 1] |0_2^{-}\rangle + \ldots
\]
Let’s again consider the state $|\psi\rangle$
\[
|\psi\rangle = \sigma_2^+(w) d_{-n}^{++(i)} d_{-1}^{-(j)} |0_R^{(1)}\rangle |0_R^{(2)}\rangle
\]
We can rewrite this as
\[
|\psi\rangle = 2 \sigma_2^+(w) \omega_i d_{-(n+1)}^{++(i)} d_{-1}^{-(j)} |0_R^{(1)}\rangle |0_R^{(2)}\rangle
\]
\[
= 2 (\omega_i \sigma_2^+(w) - [\omega_i, \sigma_2^+(w)]) d_{-(n+1)}^{++(i)} d_{-1}^{-(j)} |0_R^{(1)}\rangle |0_R^{(2)}\rangle
\]
\[
= 2 \left( \omega_i - \frac{1}{2} \gamma \right) \sigma_2^+(w) d_{-(n+1)}^{++(i)} d_{-1}^{-(j)} |0_R^{(1)}\rangle |0_R^{(2)}\rangle
\]

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we derive the full expression for \( D \)

\[
\sum_{\rho > 0} f^+_{\rho}[-(n + 1), -p] d^+_{\rho} \sum_{\rho' > 0} f^-_{\rho'}[-1, -p'] d^-_{\rho'} + C^i_j[n + 1, 1] \bigg| \chi \bigg>
\]

(4.50)

where in the third equality we have used (4.32). Expanding the above and again keeping only terms which contain no modes we obtain

\[
|\psi\rangle = (2f^+_{i}[-(n + 1), -1/2]f^-_{j}[-1, -1/2]\{[J^3_1, d^+_{-1/2}], d^-_{-1/2}\}
+ 2C^i_j[n + 1, 1]\gamma_{1/2, 1/2}\{[J^3_1, d^+_1], d^-_{1/2}\} - \{[J^3_1, d^+_{-1/2}], d^-_{-1/2}\})
- e^w C^i_j[n + 1, 1]|0_{R}^{2-}\rangle + \ldots
\]

(4.51)

Notice that the second term is zero because of the value of \( \gamma_{1/2, 1/2} \) (4.18). Comparing (4.48) and (4.51) we obtain the relation

\[
C^i_j[n, 1] = -2f^+_{i}[-(n + 1), -1/2]f^-_{j}[-1, -1/2]
\]

(4.52)

For the above expression we insert \( f^+_{i}[-(n + 1), -1/2] \) which is computed in (4.45) and the expression for \( f^-_{j}[-1, -1/2], (4.40) \). Therefore (4.52) becomes

\[
C^i_j[n, 1] = -C^2(-1)^{i+j}e^{-(n+1)w} \frac{\Gamma[\frac{1}{2} + n]}{\sqrt{\pi}(1 + n)\Gamma[1 + n]}
\]

(4.53)

In appendix D we derive the full expression for \( C^i_j[n, n_2] \) and it is given by

\[
C^i_j[n, n_2] = -C^2(-1)^{i+j} \frac{2\Gamma[\frac{1}{2} + n_1]\Gamma[\frac{1}{2} + n_2]}{\pi n_1\Gamma[n_1]\Gamma[n_2]} e^{-(n_1+n_2)w} \frac{n_1 + n_2}{n_1 + n_2}
\]

(4.54)

### 4.4 Relations using \( J^3_{-1} \)

We have derived all the functions in the effect of a twist operator with an undetermined constant \( C \). Here we will compute this constant by using a relation from the mode \( J^3_{-1} \). We start with following

\[
\sigma^+(w)J^3_{-1}|0_{R}^{1-}\rangle|0_{R}^{2-}\rangle
= -\sigma^+(w)\frac{1}{2}(d^{+(1)}_1d^{-(1)}_1 - d^{+(1)}_0d^{-(1)}_1 + d^{+(2)}_0d^{-(2)}_1 - d^{-(2)}_0d^{+(2)}_1)|0_{R}^{1-}\rangle|0_{R}^{2-}\rangle
\]

(4.55)

Bringing the modes through the twist, two fermionic modes can contract with each other and leave terms which contain no modes. Using the contraction (6.4) we obtain

\[
-(C^{11}[0, 1] + C^{22}[0, 1])|0_{R}^{2-}\rangle = 2C^2 e^{-w}|0_{R}^{2-}\rangle
\]

(4.56)
Again looking at the state
\[\sigma^+(w)J_{-1}^3|0_R^-(1)|0_R^-(2) = (J_{-1}^3\sigma^+(w) - [J_{-1}^3, \sigma^+(w)]]|0_R^-(1)|0_R^-(2)\] (4.57)

Inserting the commutator (4.4) into the above gives
\[\sigma^+(w)J_{-1}^3|0_R^-(1)|0_R^-(2) = \left(J_{-1}^3 - \frac{1}{2}e^{-w}\right)\sigma^+(w)|0_R^-(1)|0_R^-(2)\]
\[= \left(J_{-1}^3 - \frac{1}{2}e^{-w}\right)|\chi\rangle\] (4.58)

Expanding and only keeping terms which do not contain any modes we get
\[-\frac{1}{2}e^{-w}|0_R^-(1)|0_R^-(2)\] (4.59)

Comparing (4.56) and (4.59), we obtain
\[C^2 = -\frac{1}{4}\] (4.60)
which gives
\[C = \pm \frac{i}{2}\] (4.61)

This is the constant appearing in the propagation (4.40) and (4.45). The two choices of sign correspond to the two different conventions of labeling the copies.

5 Relations from supersymmetry

In the previous sections we derived the effect of the twist operator for fermionic modes. A similar method can be applied to bosonic modes. See [83] for a toy model. Since the D1D5 CFT has supersymmetry, we will use this to derive the effects for the bosonic modes from the effects of the fermionic modes.

Let us first find the propagation. We start with
\[G_{A,0}^+ d_{-n}^{-B(i)}|0_R^-(1)|0_R^-(2) = -i\epsilon^{AB}\alpha^{(i)}_{AA_{-n}}|0_R^-(1)|0_R^-(2)\] (5.1)
where we have used the commutator in (A.5) and the fact that \(G_{A,0}^+\) annihilates any Ramond ground state
\[G_{A,0}^+|0_R^-(1)|0_R^-(2) = 0\] (5.2)

Applying the twist operator, we obtain the relation
\[\sigma^+(w)G_{A,0}^+ d_{-n}^{-B(i)}|0_R^-(1)|0_R^-(2) = -i\epsilon^{AB}\sigma^+(w)\alpha^{(i)}_{AA_{-n}}|0_R^-(1)|0_R^-(2)\] (5.3)
Commuting the mode $G_{A,0}^+$ through the twist operator using

$$[G_{A,0}^+, \sigma^+(w)] = 0 \quad (5.4)$$

the LHS becomes

$$G_{A,0}^+ \sigma^+(w) d_{-n}^B |0_R^-(1)\rangle |0_R^-(2)\rangle = G_{A,0}^+ \sum_{p>0} f_i^- [-n, -p] d_p^B |0_R^2\rangle$$

$$= -i\epsilon^{AB} \sum_{p>0} f_i^- [-n, -p] \alpha_{A, -p} |0_R^2\rangle \quad (5.5)$$

Using the ansatz (3.9), the RHS of (5.3) becomes

$$-i\epsilon^{AB} \sum_{p>0} f_i^- [-n, -p] \alpha_{A, -p} |0_R^-(1)\rangle |0_R^-(2)\rangle \quad (5.6)$$

Comparing (5.5) and (5.6), we find

$$f_i^- [-n, -p] = f_i^- [-n, -p] \quad (5.7)$$

Let us now find the relation for pair creation. We start with

$$G_{A,0}^+ |0_R^-(1)\rangle |0_R^-(2)\rangle = 0 \quad (5.8)$$

Applying the twist operator we obtain

$$\sigma^+(w) G_{A,0}^+ |0_R^-(1)\rangle |0_R^-(2)\rangle = 0 \quad (5.9)$$

Commuting the mode $G_{A,0}^+$ through the twist operator using (5.4) and taking a specific choice $\dot{A} = +$ we find

$$0 = G_{+,0}^+ \sigma^+(w) |0_R^-(1)\rangle |0_R^-(2)\rangle$$

$$= G_{+,0}^+ \sum_{m,n>0} \left( \gamma_{mn}^B (-\alpha_{+, -m, -n} + \alpha_{+, -m, +, n}) + \gamma_{mn}^F (d_{-m}^+ d_{-n}^- - d_{-m}^- d_{-n}^+) |0_R^2\rangle \right) + \ldots$$

$$= \sum_{m,n>0} \left( \gamma_{mn}^B (-i\alpha_{+, -m, d_{-n}^+} + imd_{-m}^+ \alpha_{+, n}) + \gamma_{mn}^F (-id_{-m}^+ \alpha_{+, n} - id_{-m}^- \alpha_{+, n}) \right) |0_R^2\rangle + \ldots \quad (5.10)$$

which implies

$$\gamma_{mn}^B = -\frac{\gamma_{mn}^F}{m} \quad (5.11)$$
Let us now find the contraction. We start with
\[
\sigma^+(w)\alpha^{(i)}_{++,-n}\alpha^{(j)}_{--,-m}|0^-_{R}\rangle^{(1)}|0^-_{R}\rangle^{(2)} = -C^{ij}_{B}[n, m]|0^-_{R}\rangle + \ldots
\] (5.12)
where we have kept only the term without any mode. This term can also be computed as follows
\[
\begin{align*}
\sigma^+(w)\alpha^{(i)}_{++,-n}\alpha^{(j)}_{--,-m}|0^-_{R}\rangle^{(1)}|0^-_{R}\rangle^{(2)} & = -i\sigma^+(w)\{G^+_{+,0}d^{--(i)}_{-n}\alpha^{(j)}_{--,-m}|0^-_{R}\rangle^{(1)}|0^-_{R}\rangle^{(2)} \\
& = -i\sigma^+(w)G^+_{+0}d^{--(i)}_{-n}\alpha^{(j)}_{--,-m}|0^-_{R}\rangle^{(1)}|0^-_{R}\rangle^{(2)} - i\sigma^+(w)d^{--(i)}_{-n}G^+_{+,0}\alpha^{(j)}_{--,-m}|0^-_{R}\rangle^{(1)}|0^-_{R}\rangle^{(2)} \\
& = -iG^+_{+0}\sigma^+(w)d^{--(i)}_{-n}\alpha^{(j)}_{--,-m}|0^-_{R}\rangle^{(1)}|0^-_{R}\rangle^{(2)} - i\sigma^+(w)d^{--(i)}_{-n}(im)d^{++(j)}_{-m}|0^-_{R}\rangle^{(1)}|0^-_{R}\rangle^{(2)} \\
& = -mC^{ij}_{F}[m, n]|0^-_{R}\rangle + \ldots
\end{align*}
\] (5.13)

To get the fourth line, we commute $G^+_{+,0}$ through $\sigma^+(w)$ in the first term and commute $G^+_{+0}$ with $\alpha^{(j)}_{--,-m}$ in the second term. To obtain the last line, notice that the first term in the fourth line does not contain a term without any mode. We have also used the fact that $C^{ij}_{F}[n_1, n_2]$ is symmetric under the interchange of the copy labels $i$ and $j$. For the second term we have kept only terms without any modes. Comparing (5.12) and (5.13) we obtain
\[
C^{ij}_{B}[n, m] = mC^{ij}_{F}[m, n]
\] (5.14)

Thus by using (5.7), (5.11) and (5.14) we get the effect of the twist operator for bosons from fermions. The result is summarized in the next section.

6 Solutions

In this section we record the full expressions for pair creation, propagation, and contraction for both bosons and fermions. Our results agree with the expressions computed in [46] and [47] which use the covering map method.

6.1 Fermions

Pair creation

The expression for pair creation is given by
\[
\gamma^F_{m+n+\frac{1}{2}, n+\frac{1}{2}} = -\frac{e^{(m+n+1)w}\Gamma[\frac{3}{2} + m]\Gamma[\frac{3}{2} + n]}{(2n + 1)\pi(m + n + 1)\Gamma[m + 1]\Gamma[n + 1]} \quad m, n \geq 0
\] (6.1)
where $m$ and $n$ are integers.
Propagation

Choosing the value of the constant $C$ with a negative sign in (4.61) and inserting this into (4.40) and (4.45) we obtain the expressions

\[
\begin{align*}
f_{j}^{-}[n, -(m + 1/2)] &= -(-1)^{j} \frac{i \Gamma[\frac{1}{2} + n] \Gamma[\frac{1}{2} + m]}{\pi \Gamma[n] \Gamma[1 + m]} \frac{e^{(m-n+1/2)w}}{2m - 2n + 1}, & n > 0, m \geq 0 \\
f_{j}^{+}[n, -(m + 1/2)] &= -(-1)^{j} \frac{i \Gamma[\frac{1}{2} + n] \Gamma[\frac{3}{2} + m]}{\pi \Gamma[1 + n] \Gamma[1 + m]} \frac{e^{(m-n+1/2)w}}{2m - 2n + 1}, & n \geq 0, m \geq 0
\end{align*}
\]

(6.2)

where $m$ and $n$ are integers. As shown in appendix B, when the initial and final mode are equal, we have

\[
f_{i}^{\pm}[-n, -n] = \frac{1}{2}
\]

(6.3)

where $n$ is a strictly positive integer for $f_{i}^{-}$ and a non-negative integer for $f_{i}^{+}$.

Contraction

Inserting the value $C$ chosen above into (6.4) (Since $C^{2}$ appears in the contraction term, you get the same solution when taking either sign) we obtain

\[
C_{k}^{ij}[n_{1}, n_{2}] = (-1)^{i+j} \frac{\Gamma[\frac{1}{2} + n_{1}] \Gamma[\frac{1}{2} + n_{2}] \Gamma[\frac{3}{2} + n_{1}] \Gamma[\frac{3}{2} + n_{2}]}{2\pi n_{1} \Gamma[n_{1}] \Gamma[n_{2}]} \frac{e^{-(n_{1} + n_{2})w}}{n_{1} + n_{2}}, & n_{1} \geq 0, n_{2} \geq 1
\]

(6.4)

and therefore

\[
C^{ij, \alpha\beta AB} = \epsilon^{\alpha\beta} \epsilon^{AB} C_{k}^{ij}[n_{1}, n_{2}]
\]

\[
= \epsilon^{\alpha\beta} \epsilon^{AB} (-1)^{i+j} \frac{\Gamma[\frac{1}{2} + n_{1}] \Gamma[\frac{1}{2} + n_{2}] \Gamma[\frac{3}{2} + n_{1}] \Gamma[\frac{3}{2} + n_{2}]}{2\pi n_{1} \Gamma[n_{1}] \Gamma[n_{2}]} \frac{e^{-(n_{1} + n_{2})w}}{n_{1} + n_{2}}, & n_{1} \geq 0, n_{2} \geq 1
\]

(6.5)

6.2 Bosons

Using the relations derived in section 5 we record the expressions for the bosonic quantities.

Pair creation

For pair creation we use relation (5.11) and obtain

\[
\gamma_{m+\frac{1}{2}, n+\frac{1}{2}}^{B} = \frac{2e^{(m+n+1)w} \Gamma[\frac{3}{2} + m] \Gamma[\frac{3}{2} + n]}{(2m + 1)(2n + 1)\pi(m + n + 1) \Gamma[m + 1] \Gamma[n + 1]}, & m, n \geq 0
\]

(6.6)
Propagation

For propagation we use relation (5.7). This gives

\[
f_j[-n, -(m + 1/2)] = \frac{\pi \Gamma[n] \Gamma[1 + m]}{\Gamma[n + 1] \Gamma[1 + m]} e^{(m-n+1/2)w}, \quad n > 0, m \geq 0
\]

(6.7)

where \( m, n \) are integers. As shown in appendix B, when the initial and final mode are equal, we have

\[
f_i[-n, -n] = \frac{1}{2}
\]

(6.8)

where \( n \) is a positive integer.

Contraction

For contraction, using relation (5.14), we obtain

\[
C_B^{ij}[n_1, n_2] = (-1)^{i+j} \frac{\Gamma[\frac{1}{2} + n_1] \Gamma[\frac{1}{2} + n_2]}{2\pi \Gamma[n_1] \Gamma[n_2]} \frac{e^{-(n_1+n_2)w}}{n_1 + n_2}, \quad n_1 \geq 1, n_2 \geq 1
\]

(6.9)

and therefore

\[
C^{ij}_{A\bar{A}C\bar{C}}[n_1, n_2] = \epsilon_{AC} \epsilon_{A\bar{C}} C_B^{ij}[n_1, n_2]
\]

\[
= \epsilon_{AC} \epsilon_{A\bar{C}} (-1)^{i+j} \frac{\Gamma[\frac{1}{2} + n_1] \Gamma[\frac{1}{2} + n_2]}{2\pi \Gamma[n_1] \Gamma[n_2]} \frac{e^{-(n_1+n_2)w}}{n_1 + n_2}, \quad n_1 \geq 1, n_2 \geq 1
\]

(6.10)

where we have used the symmetry between the exchange of bosonic modes.

7 Discussion

In this paper we have used the bootstrap method developed in [83] to compute the effects of the twist operator in the D1D5 CFT. The majority of the paper has focused on computing the effects involving fermionic modes. In section 5 the bosonic quantities are then derived from the fermionic ones using supersymmetry relations. The effects involving the fermions are fully captured by three quantities: pair creation \( \gamma_{mn}^F \), propagation \( f_i^\pm[-n, -p] \), and contraction \( C_F^{ij}[n_1, n_2] \). Using the weak Bogoliubov ansatz and superconformal symmetry we were able to derive expressions for these quantities. We were able to compute the expression for pair creation \( \gamma_{mn}^F \) using the generator \( L_{-1} + J_3^- \). Knowing the expression for pair creation, we then computed the expression for propagation \( f_i^\pm[-n, -p] \) using the generator \( J_3^\pm \) and then the expression for propagation \( f_i^+[n, p] \) from \( f_i^-[n, -p] \) using the generator \( J_3^+ \). Furthermore, using the generator \( J_1^\pm \), we were able to compute
the expression for contraction $C^i_j[n_1,n_2]$ by knowing $f^+_i[-n,-p]$ and $f^-_i[-n,-p]$. From
the fermionic quantities, we then used the supersymmetric generator $G^\dot{A}_{\dot{A},0}$ to derive the
expressions for bosons.

All results in this paper were derived for a single twist operator. A major goal of this
program is to eventually compute effects and correlators which contain an arbitrary number
of twist operators. This would help to obtain a CFT description of certain quantities in
the supergravity regime. Under some approximations, it seems very promising to compute
some coefficients of the effect for an arbitrary number of twist operators. Here, we also
considered only two singly wound copies in the initial state, twisting them into a doubly
wound copy in the final state. We would like to use the bootstrap approach to derive
effects of the twist operator with multiwound copies in the initial state. There has been
much recent work in developing an exact correspondence between the string worldsheet
with NSNS flux and the orbifold CFT. The bootstrap techniques developed in this and
the previous paper may be helpful in better understanding this correspondence. We hope
to return to this in a future work.

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**A  \( \mathcal{N} = 4 \) superconformal algebra**

In this appendix we record the $\mathcal{N} = 4$ superconformal algebra $[89,90]$ for $c = 6k$. We start
with the basic commutation relations which are composed of 4 bosons and 4 fermions

\begin{align}
[\alpha_{A,A},\alpha_{B,B}] &= -km\epsilon_{A\dot{A}}\epsilon_{BB} \delta_{m+n,0} \\
\{d^A_r, d^B_s\} &= -k\epsilon^{\alpha\beta} \epsilon^{AB} \delta_{r+s,0} \\
(A.1)
\end{align}

The superconformal algebra is composed of Virasoro generators $L_m$, SU(2) current generators $J^a$ which form a Kac-Moody algebra where $a = 1, 2, 3$, and superconformal generators $G^{\dot{A}}_A$ where $\alpha = +, -$ and $\dot{A} = +, -$. For $J^1$ and $J^2$ we define $J^\pm$.

\begin{equation}
J^\pm_n = J^1_n \pm i J^2_n \\
(A.2)
\end{equation}

The current-current commutation relations are given by

\begin{align}
[L_m, L_n] &= \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} + (m - n) L_{m+n} \\
[J^a_m, J^b_n] &= \frac{c}{12} m \delta^{ab} \delta_{m+n,0} + i \epsilon^{ab} \epsilon^c J^c_{m+n} \\
[L_m, J^a_n] &= -n J^a_{m+n} \\
\end{align}
\[ [L_m, G^\alpha_{A,r}] = (\frac{m}{2} - r)G^\alpha_{A,m+r} \]
\[ [J^\alpha_m, G^\beta_{A,r}] = \frac{1}{2}(\sigma^a T)^\alpha_\beta G^\beta_{A,m+r} \]
\[ \{G^\alpha_{A,r}, G^\beta_{B,s}\} = \epsilon_{\hat{A}\hat{B}} \left[ \epsilon^{\alpha\beta} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0} + (\sigma^a T)^\alpha_\gamma \epsilon^{\gamma\beta}(r-s)J^a_{r+s} + \epsilon^{\alpha\beta} L_{r+s} \right] \]
\[ [J^3_m, J^+_n] = J^+_{m+n}, \quad [J^3_m, J^-_n] = -J^-_{m+n} \]
\[ [J^+_m, J^-_n] = \frac{\epsilon}{6} m \delta_{m+n,0} + 2J^3_{m+n} \]
\[ [J^3_m, G^\pm_{A,r}] = \pm \frac{1}{2} G^\pm_{A,m+r} \]
\[ [J^+_m, G^+_{A,r}] = 0, \quad [J^-_m, G^+_{A,r}] = G^-_{A,m+r} \]
\[ [J^+_m, G^-_{A,r}] = G^+_{A,m+r}, \quad [J^-_m, G^-_{A,r}] = 0 \]  \hspace{1cm} (A.3)

Using the free field realization, we can expand the generators in terms of bosonic and fermionic modes

\[ J^a_m = \frac{1}{4k} \sum_r \epsilon_{AB} \epsilon_{\alpha\gamma} (\sigma^a T)^\alpha_\beta d^{\beta A}_{m-r}, \quad a = 1, 2, 3 \]
\[ J^3_m = -\frac{1}{2k} \sum_r d^{++}_{r-}d^{-+}_{r-} - \frac{1}{2k} \sum_r d^+_{r-}d^+_{r-} \]
\[ J^+_m = \frac{1}{k} \sum_r d^{++}_{r-}d^{-+}_{r-}, \quad J^-_m = \frac{1}{k} \sum_r d^+_{r-}d^+_{r-} \]
\[ G^\alpha_{A,r} = -\frac{i}{k} \sum_r d^{\alpha A}_{r-n}a_{A,n} \]
\[ L_m = -\frac{1}{2k} \sum_n \epsilon_{AB} \epsilon_{\hat{A}\hat{B}} \alpha_{A\hat{A},n} \alpha_{BB, m-n} - \frac{1}{2k} \sum_r (m-r + \frac{1}{2}) \epsilon_{\alpha\beta} \epsilon_{AB} d^{\alpha A}_{r} d^{\beta B}_{m-r} \]  \hspace{1cm} (A.4)

Below we record the commutation relations between the generators and bosonic and fermionic modes

\[ [L_m, a_{A,\hat{A},n}] = -n a_{A\hat{A},m+n} \]
\[ [L_m, d^{\alpha A}_{r}] = -\left( \frac{m}{2} + r \right) d^{\alpha A}_{m+r} \]
\[ \{G^\alpha_{A,r}, d^{\beta B}_{s}\} = i \epsilon^{\alpha\beta} \epsilon_{AB} \alpha_{A\hat{A},r+s} \]
\[ [G^\alpha_{A,r}, \alpha_{BB, m}] = -im \epsilon_{AB} d^{\alpha A}_{r} d^{A}_{m+r} \]
\[ [J^\alpha_m, d^{\alpha A}_{r}] = \frac{1}{2}(\sigma^a T)^\alpha_\beta d^{\beta A}_{m+r} \]  \hspace{1cm} (A.5)

\[ [J^3_m, d^{++}_{r}] = \pm \frac{1}{2} d^{++}_{m+r} \]
\[ [J^+_m, d^{++}_{r}] = 0, \quad [J^-_m, d^{++}_{r}] = d^{-+}_{m+r} \]
\[ [J^+_m, d^{+A}_{r}] = d^{+A}_{m+r}, \quad [J^-_m, d^{+A}_{r}] = 0 \]  \hspace{1cm} (A.6)
B Global modes

In this appendix we will show (3.6) and (3.8). Let us consider, for an integer \( n \),

\[
d^{\alpha A}_n \sigma^+(w) - \sigma^+(w)(d^{\alpha A(1)}_n + d^{\alpha A(2)}_n)
\]

\[
= \frac{1}{2\pi i} \oint_{C_{(w)}} dw' e^{nw'} \psi^{\alpha A}(w') \sigma^+(w)
\]

\[
= e^{nw} \frac{1}{2\pi i} \oint_{C_{(w)}} dw' e^{n(w'-w)} \psi^{\alpha A}(w') \sigma^+(w)
\]

\[
= e^{nw} \frac{1}{2\pi i} \oint_{C_{(w)}} dw'(1 + n(w' - w) + \ldots) \psi^{\alpha A}(w') \sigma^+(w)
\]

\[
= e^{nw} (d^{\alpha A(w)}_{1/2} + n d^{\alpha A(w)}_{3/2} + \ldots) \sigma^+(w)
\]  \hspace{1cm} (B.1)

To get the second line, we join the two contours, one before and one after the twist operator \( \sigma(w) \), into a single contour around the twist operator. We denote the modes centered around \( w \) by a superscript \( (w) \), e.g. \( d^{\alpha A(w)}_{1/2} \). Since there is no local operator with dimension \( h = 0 \) and a nonzero \( A \) charge. Furthermore, there is no operator with \( h < 0 \), we have

\[
d^{\alpha A(w)}_{m+1/2} \sigma^+(w) = 0, \quad m \geq 0
\] \hspace{1cm} (B.2)

where \( m \) is an integer. Thus (B.1) implies

\[
d^{\alpha A}_n \sigma^+(w) - \sigma^+(w)(d^{\alpha A(1)}_n + d^{\alpha A(2)}_n) = 0
\] \hspace{1cm} (B.3)

Similarly, we have for bosons

\[
\alpha_{A\tilde{A},n} \sigma^+(w) - \sigma^+(w)(\alpha_{A\tilde{A},n}^{(1)} + \alpha_{A\tilde{A},n}^{(2)}) = e^{nw}(\alpha_{A\tilde{A},0}^{(w)} + n \alpha_{A\tilde{A},1}^{(w)} + \ldots) \sigma^+(w) = 0
\] \hspace{1cm} (B.4)

where we have used the fact that

\[
\alpha_{A\tilde{A},n}^{(w)} \sigma^+(w) = 0, \quad n \geq 0
\] \hspace{1cm} (B.5)

Consider the following state where \( n \) is a positive integer if \( \beta = - \) and a non-negative integer if \( \beta = + \). Using (B.3), we get

\[
\langle 0^2_R | d^{\alpha A}_m \sigma^+(w) d^{\beta B(i)}_{-n} | 0^1_R \rangle^{(1)} | 0^1_R \rangle^{(2)} = \langle 0^2_R | \sigma^+(w)(d^{\alpha A(1)}_m + d^{\alpha A(2)}_m) d^{\beta B(i)}_{-n} | 0^1_R \rangle^{(1)} | 0^1_R \rangle^{(2)} = -\epsilon^{\alpha\beta} \epsilon^{AB} \delta_{m,n}
\] \hspace{1cm} (B.6)

which is nonzero only if \( m \) and \( n \) are equal. Because

\[
\langle 0^2_R | d^{\alpha A}_m d^{\beta B(i)}_{-n} | 0^2_R \rangle = -2 \epsilon^{\alpha\beta} \epsilon^{AB} \delta_{m,n}
\] \hspace{1cm} (B.7)

we find

\[
\sigma^+(w) d^{\beta B(i)}_{-n} | 0^1_R \rangle^{(1)} | 0^1_R \rangle^{(2)} = \left( \frac{1}{2} d^{\beta B}_{-n} + \text{half integer modes} \right) | 0^2_R \rangle
\] \hspace{1cm} (B.8)
Similarly for bosons, consider the following state where \( n \) is a positive integer. Using (B.4), we get
\[
\langle 0^2_R | \alpha_{AA,m} \sigma^+(w) \sigma_{BB,-n}^+ | 0^2_R \rangle = 0
\]
which is nonzero only if \( m \) and \( n \) are equal. Because
\[
\langle 0^2_R | \alpha_{AA,m} \sigma_{BB,-n}^+ | 0^2_R \rangle = -2m \epsilon_{AB} \epsilon_{AB} \delta_{m,n}
\]
we find
\[
\sigma^+(w) \alpha_{BB,-n}^+ | 0^2_R \rangle = \left( \frac{1}{2} \alpha_{BB,-n} + \text{half integer modes} \right) | 0^2_R \rangle
\]
Thus we have shown (3.6).

To show (3.8), consider the following state where \( m \) is a positive integer for \( \alpha = + \) and a non-negative integer for \( \alpha = - \)
\[
d^\alpha_A \chi = d^\alpha_m \sigma^+(w) | 0^2_R \rangle = 0
\]
Similarly for bosons, consider the following state where \( m \) is a positive integer
\[
\alpha_{AA,m} \sigma^+(w) | 0^2_R \rangle = 0
\]
Thus the state \( \chi \) from pair creation should not have any integer mode excitations, which is stated in (3.8).

C Propagation: higher modes

In section 4.2, we have derived \( f_i^[-1,-(p+1/2)] \). In this appendix we derive the expression for \( f_i^[-n,-(p+1/2)] \) for any \( n > 0 \). We start with the following state
\[
d_{-n}^{-1} | 0^2_R \rangle = \frac{1}{(n-1)!} (L_{-1} + J_{-1}^3)^{n-1} d_{-n}^{-1} | 0^2_R \rangle
\]
where \( n > 0 \). Applying the twist operator gives
\[
\sigma^+(w) d_{-n}^{-1} | 0^2_R \rangle = \frac{1}{(n-1)!} \sigma^+(w) (L_{-1} + J_{-1}^3)^{n-1} d_{-n}^{-1} | 0^2_R \rangle
\]
Commuting \((L_1 + J_{-1}^3)^{n-1}\) through the twist and using (4.3) and (4.4) we find the relation
\[
\frac{1}{(n-1)!} \sigma^+(w) (L_{-1} + J_{-1}^3)^{n-1} d_{-n}^{-1} | 0^2_R \rangle
\]
where we have kept terms with only one $d^{-}$ mode. We note that $n^{-1}C_k$ is the binomial coefficient

$$n^{-1}C_k = \frac{(n-1)!}{k!(n-k-1)!} \quad \text{(C.4)}$$

We want to compare the expression (C.3) to the left hand side of (C.2) containing only one $d^{-}$ which is given by

$$\sigma^+(w)d^{-}(1)|0_R^{(1)}|0_R^{(2)} = \left( \frac{1}{2}d^{-} + \sum_{p \geq 0} f_i[-1, -(p+1/2)]d^{-}_{(p+1/2)} + \ldots \right)|0_R^{2-} \quad \text{(C.5)}$$

To match both (C.3) and (C.5) we pick the $m$'th term out of the sum over half integer modes, which is the state

$$d^{-}_{(m+1/2)}|0_R^{2-} \quad \text{(C.6)}$$

For the expression (C.3), this requires us to relabel the indices taking

$$p + k = m \quad \text{(C.7)}$$

This is because each $L_{-1} + J^3_{-1}$ increases the dimension by 1 unit. The limits of the sum are determined by

$$k \geq 0 \implies m \geq p \quad \text{and} \quad p \geq 0 \text{ and } k \leq n - 1 \implies p \geq \max(m - (n - 1), 0) \quad \text{(C.8)}$$

Therefore the relevant term in (C.3) is

$$\frac{1}{(n-1)!}\sum_{p=\max(m-(n-1),0)}^{m-1} n^{-1}C_k(L_{-1} + J^3_{-1})^{m-p}(e^{-w\partial})^{n-(m-p)-1} f_i[-1, -(p+1/2)]d^{-}_{(p+1/2)}|0_R^{2-} \quad \text{(C.9)}$$

Let’s determine the action of $L_{-1} + J^3_{-1}$. We have

$$(L_{-1} + J^3_{-1})d^{-}_{(p+1/2)}|0_R^{2-} = \left( p + \frac{1}{2} \right) d^{+}_{(p+3/2)}|0_R^{2-} + \ldots \quad \text{(C.10)}$$
This implies that
\[
(L_{-1} + J_{-1}^3)^{n'} d_{-(p+1/2)}^- |0_R^2\rangle \\
= \left( p + \frac{1}{2} \right) \left( p + \frac{3}{2} \right) \cdots \left( p + n' - \frac{1}{2} \right) d_{-(p+n'+1/2)}^- |0_R^2\rangle + \cdots \\
= \frac{(p+n' - \frac{1}{2})!}{(p - \frac{1}{2})!} d_{-(p+n'+1/2)}^- |0_R^2\rangle + \cdots \tag{C.11}
\]
where we have kept the terms with only one \(d^-\) mode. Let us also determine the action of \(e^{-w\partial}\). To do so we notice that \(f_i^-[1, -(p+1/2)] \propto e^{(\frac{1}{2})w}\) so we have
\[
(e^{-w\partial}) f_i^-[-1, -(p+1/2)] = f_i^-[-1, -(p+1/2)]_{w=0} \left( e^{-w\partial} e^{(\frac{1}{2})w} \right) \\
= f_i^-[-1, -(p+1/2)]_{w=0} e^{-w \left( p - \frac{1}{2} \right)} e^{(p-1-\frac{1}{2})w} \tag{C.12}
\]
and therefore
\[
(e^{-w\partial})^{n''} f_i^-[-1, -(p+1/2)] \\
= f_i^-[-1, -(p+1/2)]_{w=0} \left( p - \frac{1}{2} \right) \left( p - \frac{3}{2} \right) \cdots \left( p - (n'' - 1) - \frac{1}{2} \right) e^{(p-n''-\frac{1}{2})w} \\
= \frac{(p - \frac{1}{2})!}{(p-n'' - \frac{1}{2})!} e^{(p-n''-\frac{1}{2})w} f_i^-[-1, -(p+1/2)]_{w=0} \tag{C.13}
\]
Looking at \(C.9\) we set \(n' = m - p\) in \(C.10\) and \(n'' = n - (m - p) - 1\) in \(C.13\). This gives the expressions
\[
(L_{-1} + J_{-1}^3)^{m-p} d_{-(p+1/2)}^- |0_R\rangle = \frac{(m-\frac{1}{2})!}{(p-\frac{1}{2})!} d_{-(m+1/2)}^- |0_R\rangle \\
(e^{-w\partial})^{n-(m-p)-1} f_i^-[-1, -(p+1/2)] = \frac{(p - \frac{1}{2})!}{(m-n - \frac{1}{2})!} e^{(m-n-\frac{1}{2})w} f_i^-[-1, -(p+1/2)]_{w=0} \tag{C.14}
\]
Finally \(C.3\) becomes
\[
\frac{1}{(n-1)!} \sigma^+(w) (L_{-1} + J_{-1}^3)^{n-1} d_{-1}^- (i) |0_R^{(1)}\rangle |0_R^{(2)}\rangle \\
= e^{(m-n+\frac{1}{2})w} \frac{1}{(n-1)!} \sum_{p=\text{max}(m-(n-1),0)}^{m} n^{-1} C_{m-p}^{n-p} \frac{(m-\frac{1}{2})!}{(m-n + \frac{1}{2})!} (-1)^{n-(m-p)-1} f_i^-[-1, -(p+1/2)]_{w=0} d_{-(m+1/2)}^- |0_R\rangle + \cdots \tag{C.15}
\]
Comparing \(C.15\) with the term \(d_{-(m+1/2)}^- |0_R^2\rangle\) in \(C.5\) we obtain the relation
\[
f_i^-[-n, -(m+1/2)]
\]
4.39\right)\] for \(|\psi\rangle = \sigma^+(w)d^{+(i)}d^{-(j)}_{-n_2}|0_R\rangle^{(1)}|0_R\rangle^{(2)} \equiv C^{ij}_{-n_1,n_2}|0_R\rangle + \ldots \) (D.1)

where in the second line we have kept only the unexcited \(|0_R^{(2)}\rangle\). Returning back to the original expression of \(|\psi\rangle\) and using \(L_1 + J_3\) we can write this as

\[
|\psi\rangle = \sigma^+_2(w)\frac{1}{(n_2-1)!}d^{+(i)}d^{-}_{-n_1}(L_1 + J_3^{(2)})^{n_2-1}d^{-}_{-n_1}|0_R\rangle^{(1)}|0_R\rangle^{(2)} \equiv C^{ij}_{-n_1,n_2}|0_R\rangle \] (D.2)

To compute the above expression we notice that

\[
d^{+(i)}d^{-}_{-n_1}(L_1 + J_3^{(2)})^{n_2-1} = (L_1 + J_3^{(2)})^{n_2-1}d^{+(i)}d^{-}_{-n_1} \equiv (L_1 + J_3^{(2)}) \circ O_{-n} \] (D.3)

where for some operator \(O_{-n}\)

\[
(L_1 + J_3^{(2)}) \circ O_{-n} \equiv [L_1 + J_1^{(2)}, O_{-n}] \] (D.4)

Thus we have

\[
d^{+(i)}d^{-}_{-n_1}(L_1 + J_3^{(2)})^{n_2-1} = \sum_{k=0}^{n_2-1} C_k(L_1 + J_3^{(2)})^{k}(-1)^{n_2-1-k}d^{+(i)}d^{-}_{-n_1}= \sum_{k=0}^{n_2-1} C_k(L_1 + J_3^{(2)})^{k}(-1)^{n_2-1-k}\frac{(n_1 + n_2 - k - 1)!}{n_1!}d^{+(i)}d^{-}_{-n_1} \] (D.5)

Inserting this into (D.2) gives

\[
|\psi\rangle = \sum_{k=0}^{n_2-1} C_k(-1)^{n_2-k-1}\frac{(n_1 + n_2 - k - 1)!}{n_1!}d^{+(i)}d^{-}_{-n_1} \]
\( \sigma_2^+(w)(L_{-1} + J_{-1}^3)^k d_{- (n_1 + n_2 - k - 1)}^{\pm(i)} d_{-1}^{-(j)} |0_R \rangle^{(1)} |0_R \rangle^{(2)} \) 

This can be written as

\[
|\psi\rangle = \frac{1}{n_1!(n_2 - 1)!} \sum_{k=0}^{n_2-1} n_2-1 \sum_{k=0}^{n_2-1} C_k (-1)^{n_2-k-1} (n_1 + n_2 - k - 1)! \\
(L_{-1} + J_{-1}^3 - e^{-w} \partial)^k \sigma_2^+(w) d_{- (n_1 + n_2 - k - 1)}^{\pm(i)} d_{-1}^{-(j)} |0_R \rangle^{(1)} |0_R \rangle^{(2)} \\
= \frac{1}{n_1!(n_2 - 1)!} \sum_{k=0}^{n_2-1} C_k (-1)^{n_2-k-1} (n_1 + n_2 - k - 1)! \\
(L_{-1} + J_{-1}^3 - e^{-w} \partial)^k (\sum_{p \geq 0} f_1^{p+}[-(n_1 + n_2 - k - 1), -p] d_{-p}^{\pm} \sum_{p' \geq 0} f_j^{p-}[1, -p'] d_{-p'}^{\pm} \\
+ C_F^{ij}[n_1 + n_2 - k - 1, 1]) |\chi\rangle 
\]

(D.7)

Keeping only terms which are not proportional to any fermionic modes and comparing with (D.1) we obtain the relation

\[
C_F^{ij}[n_1, n_2] = \frac{1}{n_1!(n_2 - 1)!} \sum_{k=0}^{n_2-1} C_k (-1)^{n_2-k-1} (n_1 + n_2 - k - 1)! \\
(-e^{-w} \partial)^k C_F^{ij}[n_1 + n_2 - k - 1, 1] 
\]

(D.8)

Using the expression in (4.53) we compute the following term

\[
(-e^{-w} \partial)^k C_F^{ij}[n_1 + n_2 - k - 1, 1] \\
= C_F^{ij}[n_1 + n_2 - k - 1, 1] w=0 (n_1 + n_2 - 1)! \frac{(n_1 + n_2 - 1)!}{(n_1 + n_2 - k - 1)!} e^{-(n_1 + n_2)w} 
\]

(D.9)

Inserting this into (D.8) yields

\[
C_F^{ij}[n_1, n_2] = e^{-(n_1 + n_2)w} \frac{(n_1 + n_2 - 1)!}{n_1!(n_2 - 1)!} \sum_{k=0}^{n_2-1} C_k (-1)^{n_2-k-1} C_F^{ij}[n_1 + n_2 - k - 1, 1] w=0 
\]

(D.10)

Performing the sum yields

\[
C_F^{ij}[n_1, n_2] = -C_2 (-1)^{i+j} \frac{2 \Gamma[1/2 + n_1] \Gamma[1/2 + n_2]}{\pi n_1 \Gamma[n_1] \Gamma[n_2]} \frac{e^{-(n_1 + n_2)w}}{n_1 + n_2} 
\]

(D.11)

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