ON EXISTENCE OF PERIODIC SOLUTIONS OF THE RAYLEIGH EQUATION OF RETARDED TYPE

GENQIANG WANG and JURANG YAN

(Received 31 July 1998)

ABSTRACT. In this paper, we give two sufficient conditions on the existence of periodic solutions of the non-autonomous Rayleigh equation of retarded type by using the coincidence degree theory.

Keywords and phrases. Rayleigh equation, periodic solution, coincidence degree.

2000 Mathematics Subject Classification. 34K13.

1. Introduction. In [1, 2], the authors studied the existence of periodic solutions of the differential equation

\[ x''(t) + f(x'(t)) + h(t, x(t)) = 0. \]  

(1.1)

In this paper, we discuss the existence of periodic solutions of the non-autonomous Rayleigh equation of related type

\[ x''(t) + f(t, x'(t - \tau)) + g(t, x(t - \sigma)) = p(t), \]  

(1.2)

where \( \tau, \sigma \geq 0 \) are constants, \( f \) and \( g \) are functions with period \( 2\pi \) for \( t, x \in \mathbb{R} \), \( f(t, 0) = 0 \) for \( t \in \mathbb{R} \), \( p \) is a \( 2\pi \)-periodic function, \( p(t) = p(t + 2\pi) \) for \( t \in \mathbb{R} \) and \( \int_0^{2\pi} p(t) \, dt = 0 \). Using coincidence degree theory developed by Mawhin [2], we find two sufficient conditions for the existence of periodic solutions of (1.2).

2. Main results

THEOREM 2.1. Suppose there are positive constants \( K, D, \) and \( M \) such that

(i) \( |f(t, x)| \leq K \) for \( (t, x) \in \mathbb{R}^2 \);

(ii) \( xg(t, x) > 0 \) and \( |g(t, x)| > K \) for \( t \in \mathbb{R} \) and \( |x| \geq D \);

(iii) \( g(t, x) \geq -M \) for \( t \in \mathbb{R} \) and \( x \leq -D \);

(iv) \( \sup_{(t, x) \in \mathbb{R} \times [-D, D]} |g(t, x)| < +\infty \).

Then (1.2) has at least a periodic solution with period \( 2\pi \).

PROOF. Consider the equation

\[ x''(t) + \lambda f(t, x'(t - \tau)) + \lambda g(t, x(t - \sigma)) = \lambda p(t), \]  

(2.1)

where \( \lambda \in (0, 1) \). Suppose that \( x(t) \) is a periodic solution with period \( 2\pi \) of (2.1). Since \( x(0) = x(2\pi) \), there is some \( t_0 \in [0, 2\pi] \) such that \( x'(t_0) = 0 \). In view of (2.1), we see
that for any \( t \in [0, 2\pi] \),

\[
|x'(t)| = \left| \int_0^t x''(s) \, ds \right| \leq \int_0^{2\pi} |x''(s)| \, ds
\]

\[
\leq \lambda \int_0^{2\pi} |f(s, x'(s-\tau))| \, ds + \lambda \int_0^{2\pi} |g(s, x(s-\sigma))| \, ds + \lambda \int_0^{2\pi} |p(s)| \, ds
\]

\[
\leq 2\pi K + \lambda \int_0^{2\pi} |g(s, x(s-\sigma))| \, ds + 2\pi \max_{0 \leq s \leq 2\pi} |p(s)|.
\]

(2.2)

We assert that

\[
\int_0^{2\pi} |g(s, x(s-\sigma))| \, ds \leq 2\pi K + 4\pi D_1
\]

(2.3)

for some positive number \( D_1 \). Indeed, integrating (2.1) from 0 to \( 2\pi \) and noting condition (i), we see that

\[
\int_0^{2\pi} \left\{ g(t, x(t-\sigma)) - K \right\} \, dt \leq \int_0^{2\pi} \left\{ g(t, x(t-\sigma)) - |f(t, x'(t-\tau))| \right\} \, dt
\]

\[
\leq \int_0^{2\pi} \left\{ f(t, x'(t-\tau)) + g(t, x(t-\sigma)) \right\} \, dt = 0.
\]

(2.4)

Thus letting

\[
E_1 = \left\{ t \in [0, 2\pi] \mid x(t-\sigma) > D \right\}, \quad E_2 = [0, 2\pi] \setminus E_1.
\]

(2.5)

By applying (ii), (iii), and (iv), we have

\[
\int_{E_2} |g(t, x(t-\sigma))| \, dt \leq 2\pi \max \left\{ M, \sup_{(t, x) \in \mathbb{R} \times [-D, D]} |g(t, x)| \right\},
\]

(2.6)

\[
\int_{E_1} \left\{ |g(t, x(t-\sigma))| - K \right\} \, dt
\]

\[
\leq \int_{E_1} |g(t, x(t-\sigma)) - K| \, dt = \int_{E_1} \left\{ g(t, x(t-\sigma)) - K \right\} \, dt
\]

\[
\leq - \int_{E_2} \left\{ g(t, x(t-\sigma)) - K \right\} \, dt \leq \int_{E_2} |g(t, x(t-\sigma))| \, dt + \int_{E_2} K \, dt.
\]

(2.7)

Therefore

\[
\int_0^{2\pi} |g(t, x(t-\sigma))| \, dt \leq 2\pi K + 4\pi \max \left\{ M, \sup_{(t, x) \in \mathbb{R} \times [-D, D]} |g(t, x)| \right\},
\]

(2.8)

and so (2.3) holds. Combining (2.2) and (2.3), we see that

\[
|x'(t)| \leq D_2, \quad t \in [0, 2\pi]
\]

(2.9)

for some positive number \( D_2 \). Next, note that the last equality in (2.4) implies

\[
f(t_1, x'(t_1 - \tau)) + g(t_1, x(t_1 - \sigma)) = 0
\]

(2.10)

for some \( t_1 \) in \([0, 2\pi]\). Thus in view of condition (i), we have

\[
|g(t_1, x(t_1 - \sigma))| = |f(t_1, x'(t_1 - \tau))| \leq K.
\]

(2.11)

and in view of (ii), we have

\[
|x(t_1 - \sigma)| < D.
\]

(2.12)
Since $x(t)$ is a periodic solution with period $2\pi$ of (2.1), we infer that $|x(t_2)| < D$ for some $t_2$ in $[0, 2\pi]$. Therefore,

$$|x(t)| = |x(t_2) + \int_{t_2}^{t} x'(t)dt| \leq D + \int_{0}^{2\pi} |x'(t)| dt \leq D + 2\pi D_2, \quad t \in [0, 2\pi]. \quad (2.13)$$

Let $X$ be the Banach space of all continuous differentiable functions of the form $x = x(t)$, defined on $R$ such that $x(t + 2\pi) = x(t)$ for all $t$, and endowed with the norm $\|x\|_1 = \max_{0 \leq t \leq 2\pi} |x(t)|, |x'(t)|$. Let $Y$ be the Banach space of all continuous functions of the form $y = y(t)$, defined on $R$ such that $y(t + 2\pi) = y(t)$ for all $t$, and endowed with the norm $\|y\|_0 = \max_{0 \leq t \leq 2\pi} |y(t)|$, and let $\Omega$ be the subspace of $X$ containing functions of the form $x = x(t)$, such that $|x(t)| < \bar{D}$ and $|x'(t)| < \bar{D}$, where $\bar{D}$ is a fixed number greater than $D + 2\pi D_2$. Now, let $L : X \cap C^{(2)}(R, R) \rightarrow Y$ be the differential operator defined by $(Lx)(t) = x''(t)$ for $t \in R$, and let $N : X \rightarrow Y$ be defined by

$$(Nx)(t) = -f(t, x'(t - \sigma)) - g(t, x(t - \tau)) + p(t), \quad t \in R. \quad (2.14)$$

We know that $\ker L = R$. Furthermore if we define the projections $P : X \rightarrow \ker L$ and $Q : Y \rightarrow Y/\text{Im}L$ by

$$PX = \frac{1}{2\pi} \int_{0}^{2\pi} x(t)dt, \quad (2.15)$$

$$Qy = \frac{1}{2\pi} \int_{0}^{2\pi} y(t)dt,$$

respectively, then $\ker L = \text{Im}P$ and $\ker Q = \text{Im}L$. Furthermore, the operator $L$ is a Fredholm operator with index zero, and the operator $N$ is $L$-compact on the closure $\bar{\Omega}$ of $\Omega$ (see, e.g., [2, p. 176]). In terms of valuation of bound of periodic solutions as above, we know that for any $\lambda \in (0, 1)$ and any $x = x(t)$ in the domain of $L$, which also belongs to $\partial \Omega$, $Lx \neq \lambda Nx$. Since for any $x \in \partial \Omega \cap \ker L$, $x = \bar{D}$ or $x = -\bar{D}$, then in view of (ii), (iii), and $\int_{0}^{2\pi} p(t)dt = 0$, we have

$$QN x = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ -f(t, x'(t - \tau)) - g(t, x(t - \tau)) + p(t) \right] dt$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \left[ -f(t, 0) - g(t, x(t - \tau)) \right] dt$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \left[ -g(t, x(t - \tau)) \right] dt$$

$$= -\frac{1}{2\pi} \int_{0}^{2\pi} g(t, x)dt \neq 0. \quad (2.16)$$

In particular, we see that

$$-\frac{1}{2\pi} \int_{0}^{2\pi} g(t, \bar{D})dt > 0, \quad (2.17)$$

$$-\frac{1}{2\pi} \int_{0}^{2\pi} g(t, \bar{D})dt < 0.$$
This shows that

\[ \deg \left[ QN x, \Omega \cap \ker L, 0 \right] \neq 0. \] (2.18)

In view of Mawhin continuation theorem [2, p. 40], there exists a periodic solution with period \(2\pi\) of (1.2). This completes the proof.

**Theorem 2.2.** Suppose that there are positive constants \(K, D,\) and \(M\) such that

(i) \( |f(t,x)| \leq K \) for \((t,x) \in \mathbb{R}^2;\)

(ii) \( xg(t,x) > 0 \) and \( |g(t,x)| > K \) for \( t \in \mathbb{R}, \) \( |x| \geq D;\)

(iii) \( g(t,x) \leq M \) for \( t \in \mathbb{R}, x \geq D;\)

(iv) \( \sup_{(t,x) \in \mathbb{R} \times [-D,D]} |g(t,x)| < +\infty.\)

Then (1.2) has at least a periodic solution with period \(2\pi.\)

The proof of Theorem 2.2 is similitude of Theorem 2.1, and so, we omit the details here.

**References**

[1] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, New York, 1985. MR 86j:47001. Zbl 559.47040.

[2] R. E. Gaines and J. L. Mawhin, *Coincidence Degree, and Nonlinear Differential Equations*, Lecture Notes in Mathematics, vol. 568, Springer-Verlag, Berlin, New York, 1977. MR 58 30551. Zbl 339.47031.

Wang: Department of Mathematics, Hanshan Teacher’s college, Chaozhou, Guangdong 521041, China

Yan: Department of Mathematics, Shanxi University, Taiyuan, Shanxi 030006, China

E-mail address: jryan@shanxi.ihep.ac.cn
Submit your manuscripts at http://www.hindawi.com