SIGNATURE of ROTORS

MIECZYSŁAW DĄBKOWSKI
Department of Mathematics, University of Texas at Dallas
Richardson, Texas 75083-0688, USA
e-mail: mdab@utdallas.edu

MAKIKO ISHIWATA
Department of Mathematics, Tokyo Woman’s Christian University
Zempukuji 2-6-1, Suginamiku, Tokyo 167-8585, Japan
e-mail: mako@lab.twcu.ac.jp

JÓZEF H. PRZYTYCKI
Department of Mathematics, The George Washington University
Washington, DC 20052, USA
e-mail: przytyck@gwu.edu

AKIRA YASUHARA
Department of Mathematics, Tokyo Gakugei University
Nukuikita 4-1-1, Koganei, Tokyo 184-8501, Japan
e-mail: yasuhara@u-gakugei.ac.jp

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Abstract

Rotors were introduced as a generalization of mutation by Anstee, Przytycki and Rolfsen in 1987. In this paper we show that Tristram-Levine signature is preserved by orientation-preserving rotations. Moreover, we show that any link invariant obtained from the characteristic polynomial of Goeritz matrix, including Murasugi signature, is not changed by rotations. In 2001, P. Traczyk showed that the Conway polynomials of any pair of orientation-preserving rotants coincide. But it was still an open problem if an orientation-reversing rotation preserves Conway polynomial. We show that there is a pair of orientation-reversing rotants with different Conway polynomials. This provides a negative solution to the problem.

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1 Introduction

Rotors were introduced in graph theory by W. Tutte [2], [17] and [18]. The concept was adapted to knot theory in [1] as a generalization of Conway’s mutation. For the orientation of the boundary of an oriented rotor, we have two basic possibilities:
(a) inputs and outputs alternate as in Fig.2.2 (a). Such a rotor is called an orientation-preserving rotor, or
(b) we have the pattern in-in, out-out as in Fig.2.2 (b). Such a rotor is called an orientation-reversing rotor.

In Section 3 (resp. Section 4), we show, in particular, that the Murasugi’s unoriented version of the classical signature [4, 10, 11] (Theorem 3.1) (resp. Tristram-Levine signature) is preserved by any rotations (resp. any orientation-preserving rotations).

It was shown in [1] that rotations of order three and four preserve the Homflypt polynomial, and in particular, the Conway polynomial of links. In 2001, P. Traczyk [14] showed that Conway polynomials of a pair of any orientation-preserving rotants coincide, solving in this case, the Jin-Rolfsen Conjecture [6]. But it was inconclusive if orientation-reversing rotations preserve Conway polynomials for \( n \geq 6 \). In the last section, we present an example of orientation-reversing rotants which do not share the same Conway polynomial. This provides a negative answer for the Jin-Rolfsen Conjecture in the orientation-reversing case [6, 12].

In general, it is not true that a rotation preserves the first homology of the double branched cover, \( M^{(2)}_L \), of \( S^3 \) branched along \( L \). Necessary conditions for preserving the homology are given in [3, 13]. Figure 1.1 taken from [3] shows rotants with different \( H_1(M^{(2)}_{L_k}; \mathbb{Z}) \) and \( H_1(M^{(2)}_{L_k}; \mathbb{Z}_5) \). For the link \( L_1 \) in Fig. 1.1(a), \( H_1(M^{(2)}_{L_1}; \mathbb{Z}) = \mathbb{Z}_{15} \oplus \mathbb{Z}_{30} \) and \( H_1(M^{(2)}_{L_1}; \mathbb{Z}_5) = \mathbb{Z}_5 \oplus \mathbb{Z}_5 \), and for its orientation preserving rotant \( L_2 \) in Fig.1.1(b) we obtain \( H_1(M^{(2)}_{L_2}; \mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_{150} \), \( H_1(M^{(2)}_{L_2}; \mathbb{Z}_5) = \mathbb{Z}_5 \). All the homology groups were

\(^1\)The terminology used in here is explained in Section 2.
calculated using K. Kodama’s program KNOT [7].

However, if we assume that a given pair of oriented rotants can be put into the “special” periodic disk-band form then the first homology groups of the corresponding double branched covers of $S^3$ branched along this pair of rotants are isomorphic (Corollary 2.3).

2 Definitions and basic properties of rotors

For an oriented link $L$ of $k$-components $K_1, \cdots, K_k$ we form the linking matrix $A_L$ with entries $a_{ij} = \text{lk}(K_i, K_j)$, where $i \neq j$. We put $a_{ii} = 0$ unless $L$ is a framed link. In this case we define $a_{ii}$ to be the framing of the $i$th component $K_i$ of $L$ ($a_{ii}$ measures the difference with respect to the standard framing). The linking matrix $A_L$, up to the order of components of $L$, is a link invariant. One half of the sum $\sum_{i<j} a_{ij}$ of entries of $A_L$ outside the diagonal is the total linking number of $L$, denoted by $\ell_k(L)$. The trace of $A_L$ for a framed link $L$ is denoted by $\text{tr}(L)$. Note that $\text{tr}(L)$ does not depend on the orientation of $L$, so $\text{tr}(L)$ is an invariant of an unoriented framed link $L$.

Consider a link $L$ in $S^3$ decomposed into two $n$-tangles ($n > 2$) $S$ and $R$ (Fig. 2.1), where by $n$-tangle we mean any 1-dimensional manifold properly embedded into a three-ball and consisting of $n$-arcs and, possibly, closed components. Let $\phi$ be a rotation of $B^3 = B^2 \times I$ by the angle $\frac{2\pi}{n}$ along the $z$ axis. Assume that $R$, called the rotor part of $L$, satisfies $\phi(R) = R$. The other tangle part, $S$, of $L$ is called the stator. Equivalently, $L$ admits a projection decomposed into the projections of the rotor and
the stator (these projections will also be denoted by $S$ and $R$) such that $R$ lies in the regular $n$-gon and intersects its boundary in $2n$ points, and that $\phi(R) = R$ (Fig. 2.1).

The regular $n$-gon has a dihedral group of symmetry $D_{2n}$. This group is generated by the $2\pi/n$ rotation along the $z$ axis $\phi$ and the dihedral flype $d_0$ which corresponds to the rotation by $\pi$ along the $y$ axis. The group $D_{2n}$ has a presentation, $D_{2n} = \{ \phi, d_0 \mid \phi^n = d_0^2 = 1, d_0\phi d_0 = \phi^{-1} \}$. Let $d_{k/2} = \phi^k d_0$. Note that $d_{k/2}$ is the dihedral flype along the axis obtained from the $y$ axis by rotating it counterclockwise by the angle $\frac{2\pi k}{2n}$.

A rotant of a link $L_1$ is the link $L_2$ (Fig. 1.1 and Fig. 2.1) obtained from $L_1$ by a dihedral flype of its rotor part. Note that $L_2$ is independent of the choice of a dihedral flype. We say that $L_2$ is obtained from $L_1$ by a rotation.

\[\text{Fig. 2.1}\]

If a link is equipped with additional structures such as orientation or a blackboard framing, we also assume that the rotation preserves these structures. In the oriented case, we allow the global change of the orientation of the rotor part. More precisely, for an oriented rotor we have two basic choices of directions of arcs at its boundary points: inputs and outputs alternate as in Fig. 2.2(a), we call such a rotor the orientation-preserving rotor, or we have the pattern in-in, out-out, · · · , in-in, out-out for an even $n$ as in Fig. 2.2(b); we call such a rotor the orientation-reversing rotor. For an oriented rotor $R$ of an oriented link $L$ and a dihedral flype $d$, the orientations of $d(R)$ and the stator parts do not always necessarily match. If they do not match, then by reversing
the orientation of $d(R)$, we obtain an oriented link $L_2 = d(R) \cup S$ that we also call the oriented rotant of $L_1$.

![Fig. 2.2](image)

The following theorem describes basic properties of rotors.

**Theorem 2.1**  
(i) Any rotation preserves the number of components of a link.

(ii) If two oriented links are related by a rotation of an oriented rotor, then the total linking numbers are the same.

(iii) If two oriented framed links are related by a rotation of an oriented rotor, and the rotor part has no closed components, then their linking matrices are the same.

(iv) If $L$ is an unoriented framed link, then $tr(L)$ is preserved by any rotation.

**Proof** Let $R$ be an unoriented rotor with boundary points $a_0, b_0, a_1, b_1, \ldots, a_{n-1}, b_{n-1}$, as in Figure 2.3(a). Consider the connection of $a_0$, that is, the boundary point connected to $a_0$ by an arc in $R$. Initially, we have two cases: $a_0$ connects to either $a_m$ or $b_m$ for some $m$. 

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If \( n > 2 \) then \( a_0 \) cannot be connected to \( a_m \). To prove this claim let us assume, by contradiction, that \( a_0 \) connects to \( a_m \) then \( \phi^m(a_0) = a_m \) connects to \( \phi^m(a_m) = a_{2m} \) which must be the same as \( a_0 \). Therefore, \( 2m = n \). This implies that \( a_i \) connects to \( a_{i+\frac{m}{2}} \) and \( b_i \) to \( b_{i+\frac{m}{2}} \). The arc \( \gamma(x_i) \) of \( R \) connecting \( x_i \) to \( x_{i+\frac{m}{2}} \), where the symbol \( x \) may stand for \( a \) or \( b \), is setwise preserved by the rotation \( \phi^{\frac{n}{2}} \). Therefore the arc \( \gamma(x_i) \) has one fixed point, namely the point of the intersection with the \( z \)-axis. For \( n > 2 \) we have at least two arcs of the type \( \gamma(a_i) \). Such arcs cut the \( z \)-axis at different heights, say \( h_i \). On the other hand \( \phi(\gamma(a_i)) = \gamma(a_{i+1}) \), so \( h_i = h_{i+1} \), which gives a contradiction. So in this case, we have \( n = 2 \), and in this case Theorem 2.1 follows easily.

Suppose \( a_0 \) is connected to \( b_m \) for some \( m \). Let \( \gamma_i = \gamma(a_i) \) denote the arc connecting the point \( a_i \) with \( b_{i+m} \) in \( R \). Consider the dihedral flype \( d_{\frac{m}{2}+i} \) exchanging \( a_i \) with \( b_{i+m} \). The image \( d_{\frac{m}{2}+i}(\gamma_i) \) connects the same points on the boundary as \( \gamma_i \) that is \( a_i \) and \( b_{i+m} \) (Fig. 2.3(b)), so two boundary points of \( R \) are connected in \( R \) if and only if they are connected in \( d_0(R) = d_{\frac{m}{2}+i}(R) \). In particular, the link \( L_1 = S \cup R \) and its rotant \( L_2 = S \cup d_0(R) \) have the same number of components.

By observations similar to the above, we have

**Claim 2.2**  
(i) For an unoriented rotor \( R \) choose any orientation (directions) of its arcs (e.g. from \( a_j \) to \( b_{j+m} \)). Let \( I(\gamma_j,\gamma_k) \) denote the sum of sign of crossings \( \gamma_j \) and \( \gamma_k \), possibly \( j = k \), then \( I(\gamma_j,\gamma_k) = I(d_{\frac{m}{2}+j+k+m}(\gamma_j),d_{\frac{m}{2}+j+k+m}(\gamma_k)) \).

(ii) For an oriented rotor \( R \) and a closed component \( \alpha \) of \( R \), \( I(\gamma_i,\alpha) = I(d_{\frac{m}{2}+i}(\gamma_i),d_{\frac{m}{2}+i}(\alpha)) \).
Notice that $\partial \gamma_j = \partial(d_{j+k+m}(\gamma_k))$, $\partial \gamma_k = \partial(d_{j+k+m}(\gamma_j))$ and $\partial \gamma_i = \partial(d_{i+k+m}(\gamma_i))$.

**Proof** (i) The dihedral flype $d_{j+k+m}$ of $R$ sends $a_j$ to $b_{k+m}$ and $a_k$ to $b_{j+m}$, thus it sends the arc $\gamma_j$, connecting $a_j$ with $b_{j+m}$ in $R$ (resp. $a_k$ with $b_{k+m}$) to the arc $d_{j+k+m}(\gamma_k)$ connecting $b_{k+m}$ with $a_k$ in $d_0(R)$ (resp. $d_{j+k+m}(\gamma_j)$ connecting $b_{j+m}$ with $a_k$) (Fig. 2.4). Therefore $I(\gamma_j, \gamma_k) = I(d_{j+k+m}(\gamma_k), d_{j+k+m}(\gamma_j))$, as required. 

(ii) Since $\gamma_i$ in $R$ and $d_{i+k+m}(\gamma_i)$ in $d_0(R)$ connect the same boundary points $a_i$ and $b_{i+m}$, we have the conclusion.

Theorem 2.1 (ii), (iii) and (iv) follows from Claim 2.2 and the fact that $L_1$ and $L_2$ have the same stator. 

We use Theorem 2.1 to show that with some technical assumptions, that are explained below, the double branched covers of $S^3$ branched along rotant links have isomorphic first homology groups. We do not use later in the paper the result of Corollary 2.3, however, we would like to contrast it with the example in Fig. 1.1 of rotant links with different first homology groups.

In the proof of Corollary 2.3 we use Montesinos method [9] of finding surgery description of the double branched covers of $S^3$ branched along links, when a surface (possibly unoriented) bounding the link is given. We closely follow, in this part of the paper, notation used in [5].

Let $T_0$ be a trivial $n$-tangle diagram as in Fig. 2.5(a). Let $D_1 \cup \cdots \cup D_n$ be a disjoint union of disks bounded by $T_0$ and a disjoint union of arcs in $\partial B^3$ con-
necting ∂T₀. Let b₁, ..., bₘ be mutually disjoint disks (ribbons) in B³ such that 
bᵢ ∩ ⋃_j D_j = ∂bᵢ ∩ T₀ are two disjoint arcs in ∂bᵢ (i = 1, ..., m), Fig. 2.5(b). We
denote by Ω(T₀; {D₁, ..., Dₙ}, {b₁, ..., bₘ}) the tangle T₀ ∪ ⋃ᵢ ∂bᵢ ∪ int(T₀ ∩ ⋃ᵢ ∂bᵢ) to-
gether with the surface ⋃ Dᵢ ∪ ⋃ bⱼ and its decomposition into disks Dᵢ and bᵢ. We
call such a structure a disk-band representation of a tangle [5].

If a rotor part has a rotationally symmetric disk-band representation, then the
following corollary of Theorem 2.1 holds.

Corollary 2.3 Let L₁ and L₂ be a pair of unoriented n-rotants such that n-rotor
R₁ of L₁ admits a rotational symmetric disk-band representation with the number of
ribbon disks in the representation equal to n. Then H₁(Mₙ⁻¹(2), Z) = H₁(Mₙ⁺¹(2); Z) where
Mₙ⁻¹(2) denotes the double branched cover of S³ branched along a link L.

Proof Let Ω(T₀; {D₁, ..., Dₙ}, {b₁, ..., bₘ}) be the the disk-band representations of R₁
and R₂ = d₀(R₁) respectively, related by the dihedral flype d₀. Let B³ be the 3-ball
such that B³ ∩ Lₖ is the tangle ingredient of Ω(T₀; {D₁, ..., Dₙ}, {b₁, ..., bₘ}) (k = 1, 2)
and B₀ = B³ ∪ ⋃ Dᵢ ∪ b₁ ∪ ... ∪ bₘ ∪ b₁ ∪ ... ∪ bₘ in B³. There are compact, connected, possibly non-orientable surfaces
F₁ (k = 1, 2) in S³ such that F₁ ∩ B³ = D₁ ∪ ... ∪ Dₙ ∪ b₁ ∪ ... ∪ bₘ and the
surface $F_k \cap (S^3 - B^3)$ is connected. We follow [5] in constructing a surgery description of the double branched cover using a surface $F_k$. We work with $F_1$ and and $L_1$, the construction for $F_2$ and $L_2$ is related by a dihedral flype.

Choose a point $v_i$ in $D_i \cap \partial B^3 (i = 1, \ldots, n)$. Let $G_k$ be a spine of $F_k$ with the vertex set $\{v_1, \ldots, v_n\}$ such that $G_k \cap B^3$ is a spine of $D_1 \cup \ldots \cup D_n \cup b_{k1} \cup \ldots \cup b_{kn}$. Let $T_k \subset S^3 - \text{int} B^3$ be a spanning tree of $G_k$ and $G_k/T_k$ a spine obtained from $G_k$ by contracting $T_k$ into a point $v$. We may assume that $N(G_k/T_k) \cap F_k$ consists of a disk $D_{k0}$ containing $v$ and mutually disjoint disks $b'_{k1}, \ldots, b'_{km}$ such that $b'_{k1} \cap D_{k0} = \partial b'_{k1} \cap \partial D_{k0}$ are two disjoint arcs in $\partial b'_{ki}$ $(i = 1, \ldots, m)$, $(D_{k0} \cup b'_{k1} \cup \cdots \cup b'_{km}) \cap B^3_0 = (b_{k1} \cup \cdots \cup b_{kn}) \cap B^3_0$, and that $(D_{10} \cup b'_{11} \cup \cdots \cup b'_{1m}) - B^3_0 = (D_{20} \cup b'_{21} \cup \cdots \cup b'_{2m}) - B^3_0$. Let $\varphi : S^3 \to S^3$ be the double branched cover branched along $\partial D_{k0}$. Then $M^{(2)}_{L_k}$ is obtained from $S^3$ by surgery along a framed link $\varphi^{-1}(b'_{k1} \cup \cdots \cup b'_{km})$. Note that $\varphi^{-1}((b'_{k1} \cup \cdots \cup b'_{km}) \cap B^3_0) = \varphi^{-1}((b_{k1} \cup \cdots \cup b_{kn}) \cap B^3_0)$ are two $n$-rotors and $\varphi^{-1}((b'_{11} \cup \cdots \cup b'_{1m}) - B^3_0) = \varphi^{-1}((b'_{21} \cup \cdots \cup b'_{2m}) - B^3_0)$. Since each $\varphi^{-1}(b'_{ki})$ is a component of $\varphi^{-1}(b'_{k1} \cup \cdots \cup b'_{km})$, it is not hard to see that there is a blackboard framed, oriented link $c_{k1} \cup \cdots \cup c_{km}$ such that each $c_{ki}$ corresponds to $b'_{ki}$ and the both components of $(c_{k1} \cup \cdots \cup c_{km}) \cap \varphi^{-1}(B^3_0)$ are oriented $n$-rotors. So $c_{21} \cup \cdots \cup c_{2m}$ is obtained from $c_{11} \cup \cdots \cup c_{1m}$ by two oriented $n$-rotations. By Theorem 2.1 (iii), the linking matrices of $c_{11} \cup \cdots \cup c_{1m}$ and $c_{21} \cup \cdots \cup c_{2m}$ coincide. Since the linking matrix of $c_{k1} \cup \cdots \cup c_{km}$ is a relation matrix of the first homology group of $M^{(2)}_{L_k}$, we have the conclusion.

Corollary 2.3 and the example in Fig 1.1 allow us to conclude that not every $n$-rotor has a symmetric disk-band representation with $n$ bands.

Let $F_L$ be a Seifert surface of an oriented link $L$. Denote by $\psi : H_1(F_L; \mathbb{Z}) \times H_1(F_L; \mathbb{Z}) \to \mathbb{Z}$ the Seifert form associated with $F_L$ (i.e. $\psi(x, y) = \text{lk}(x^+, y)$, where $x^+$ denotes a curve pushed $x$ slightly off $F_L$ into the positive direction). Choosing an ordered basis for $H_1(F_L; \mathbb{Z})$ allows us to describe the form $\psi$ by the corresponding Seifert matrix. Let $\mathcal{A}_L$ be the Seifert matrix of the form $\psi$ with respect to some ordered basis of $H_1(F_L; \mathbb{Z})$. 

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Let $F_L$ be a spanning surface, possibly nonorientable, of an unoriented link $L$. We use the following generalization of Seifert\cite{Goeritz} and Goeritz forms defined by Gordon and Litherland in \cite{Gordon_Litherland}. For the spanning surface $F_L$ consider regular neighborhood, $N(F_L)$, of $F_L$ in $S^3 - L$. Then $N(F_L)$ is the $I$-bundle over $F_L$ and the $\partial I$-bundle $\widetilde{F}_L$ is a double cover of $F_L$ (possibly disconnected) with the projection map $p : \widetilde{F}_L \rightarrow F_L$. The bilinear form $G_{F_L} : H_1(F_L; \mathbb{Z}) \times H_1(F_L; \mathbb{Z}) \rightarrow \mathbb{Z}$ defined by $G_{F_L}(x, y) = \text{lk}(p^{-1}x, y)$, where $x$ and $y$ are oriented loops in $F_L$, is called the Goeritz form associated to the surface $F_L$.

For an ordered basis of $H_1(F_L; \mathbb{Z})$ we have the matrix $G_{F_L}$ representing the Goeritz form $G_{F_L}$. The matrix $G_{F_L}$ is called the Goeritz matrix of $F_L$ with respect to a basis of $H_1(F_L; \mathbb{Z})$.

The form $G_{F_L}$ defined over $\mathbb{Z}$ can be extended to the form $\hat{G}_{F_L}$ over $\mathbb{C}$. We view the form $\hat{G}_{F_L}$ as the Hermitian form represented in a basis by the Hermitian matrix $\hat{G}_{F_L}$ (i.e. $\hat{G}_{F_L} = \overline{G_{F_L}^t}$).

For a spanning surface $F_{L_k}$ of $L_k = K_{k1} \cup K_{k2} \cup \cdots \cup K_{km}$, the framing of $L_k$ is uniquely determined by $F_{L_k}$ as follows\cite{Murasugi}: Let $K_{ki}^{F_{L_k}}$ be a parallel copy of $K_{ki}$ that misses $F_{L_k}$. We define the framing $K_{ki}$ to be $\text{lk}(K_{ki}, K_{ki}^{F_{L_k}})$. We put $c(F_{L_k}) = -\sum_i \text{lk}(K_{ki}, K_{ki}^{F_{L_k}}) = -\text{tr}(L_k)$.

We recall the definition of the Tristram-Levine signature of an oriented link.

**Definition 2.4** \cite{Tristram-Levine, Levine} Let $L$ be an oriented link in $S^3$ and let $\omega$ be a complex number with $|\omega| = 1$, $\omega \neq 1$. The Tristram-Levine signature of $L$, denoted by $\sigma_\omega(L)$, is the signature of the Hermitian matrix $(1 - \omega)A_L + (1 - \overline{\omega})A_L^t$, where $A_L$ is a Seifert matrix of $L$.

**Definition 2.5** \cite{Gordon_Litherland, Murasugi} Let $L$ be an unoriented link in $S^3$, and let $\hat{L}$ be the link obtained from $L$ by a choice of an orientation. The Murasugi signature $\hat{\sigma}(L)$ of an unoriented link $L$ is defined to be $\hat{\sigma}(L) = \sigma(\hat{L}) + \ell k(\hat{L})$.

**Remark 2.6** Murasugi showed in \cite{Murasugi} that $\sigma(\hat{L}) + \ell k(\hat{L})$ does not depend on the

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\footnote{It is a generalization of the symmetrization of the Seifert form.}

\footnote{The regular neighborhood of $K_{ki}$ in $F_k$ is the frame knot associated to $K_{ki}$. Its framing, when compared to the standard framing, is given by $\text{lk}(K_{ki}, K_{ki}^{F_{L_k}})$.}
choice of an orientation of $L$. So $\hat{\sigma}(L)$ is an invariant of unoriented links. We shall use later the fact that $\hat{\sigma}(L) = \text{sign}(G_{F_L}) + \frac{1}{2}e(F_L)$ [4].

3 Unoriented rotation and Murasugi signature

In this section we prove that the Murasugi signature of unoriented links is preserved by any rotation. The result follows from a more general statement (see Theorem 3.2) that the rotation preserves the characteristic polynomial of the Goeritz matrix (with the special choices of surfaces). In particular Theorem 3.2 allows us to obtain the result mentioned first in [12] that was also proven by Traczyk that the determinant of an unoriented link is preserved by any rotation.

Theorem 3.1 Let $L_1$ and $L_2$ be a pair of unoriented $n$-rotants (no restrictions on $n$). Then $\hat{\sigma}(L_1) = \hat{\sigma}(L_2)$.

The main result of this section is Theorem 3.2 from which Theorem 3.1 follows.

Let $L_1$ and $L_2$ be a pair of unoriented rotant links. Consider projections of the links $L_1$ and $L_2$ onto $\mathbb{R}^2$ with rotor parts $R_1$ and $R_2$ contained in disks $D_1$ and $D_2$, respectively. We can deform the stator parts $S_1$ and $S_2$ of the diagrams of $L_1$ and $L_2$ into the position shown in Figure 3.1.

We color the regions on $\mathbb{R}^2$ bounded by the diagrams of $L_k$ in a checkerboard manner as in Figure 3.2. Using the black regions we form the spanning surface $F_{L_k}$ for $k = 1, 2$. We choose for a basis of $H_1(F_{L_k}; \mathbb{Z})$ the anti-clockwise oriented boundary curves of the bounded white regions, and we refer to this basis as the standard basis.
We also may assume that the framed links $L_1$ and $L_2$ obtained from $F_1$ and $F_2$ respectively, form a pair of rotants. By Theorem 2.1, $\text{tr}(L_1) = \text{tr}(L_2)$, so we have $e(F_{L_1}) = e(F_{L_2})$. This fact, Remark 2.6 and the following theorem imply Theorem 3.1.

With the choices for $F_k$’s and bases of $H_1(F_k;\mathbb{Z})$’s, made above, we can formulate the main result of this section.

**Theorem 3.2** Let $G_{F_k}$ $(k = 1, 2)$ be the Goeritz matrices with respect to the standard basis. Then $\det(G_{F_{L_1}} - \lambda E) = \det(G_{F_{L_2}} - \lambda E)$.

**Proof** Let $X_{S_k}$ and $X_{R_k}$ be the subsets of the standard basis of $H_1(F_{L_k};\mathbb{Z})$ which live entirely in the stator and rotor part respectively, and let $X_{M_k}$ be the complement of $X_{S_k} \cup X_{R_k}$ in the standard basis. $X_{M_k}$ is composed of boundaries of white regions intersecting the boundary of the rotor. We can have $n$ such regions or just one region. We can, however, always assume, modifying the rotor part of the diagram if necessary, that $X_{M_k}$ has $n$ different elements. Consider submodules $S_k$, $R_k$ and $M_k$ of $H_1(F_{L_k};\mathbb{Z})$ generated by $X_{S_k}$, $X_{R_k}$ and $X_{M_k}$. We have the following decomposition into the direct sum of $\mathbb{Z}$-modules: $H_1(F_{L_k};\mathbb{Z}) = S_k \oplus M_k \oplus R_k$. Let $v$ denote the generator of $M_1$ intersecting the $y$ axis of the dihedral flype $d$ (Fig. 3.3). There is an action of the cyclic group $\mathbb{Z}_n = \langle \alpha \mid \alpha^n = 1 \rangle$ on $R_1 \oplus M_1$ induced by the $\frac{2\pi}{n}$-rotation around the center of $D_1$. Thus the ordered set $X_{M_1} : v, \alpha(v), \alpha^2(v), \cdots, \alpha^{n-1}(v)$ can be assumed to be a basis of $M_1$. Let $X^*_{R_1}$ be a set of generators of $R_1$ formed by choosing one representative from each orbit of $\mathbb{Z}_n$-action on standard generators of $R_1$ (i.e. $X^*_{R_1} = X_{R_k}/\mathbb{Z}_n$). We construct a bijection $\eta$ between the set of standard generators

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\[ \text{We adjust here the Traczyk’s method [14] to the case of unoriented rotors and Goeritz matrices.} \]
of $H_1(F_{L_1};\mathbb{Z})$ and $H_1(F_{L_2};\mathbb{Z})$. First, define $\eta|_{X_{S_1}}: X_{S_1} \to X_{S_2}$ to be the identity map since the stator part is unchanged by rotation. The map $\eta|_{X_{M_1}}: X_{M_1} \to X_{M_2}$ is given by $\eta(\alpha^j(v)) = \alpha^j(d(v))$ (i.e. $\alpha^j(v)$ and $\eta(\alpha^j(v))$ have the same stator parts). Finally, $\eta|_{X_{R_1}}: X_{R_1} \to X_{R_2}$ is given by $\eta(\alpha^j(x)) = d(\alpha^j(x))$ for $x \in X_{R_1}$. The bijection $\eta$ extends to the isomorphism, $H_1(F_{L_1};\mathbb{Z}) \to H_1(F_{L_2};\mathbb{Z})$, that is also denoted by $\eta$. We use the isomorphism $\eta$ to identify $H_1(F_{L_1};\mathbb{Z})$ with $H_1(F_{L_2};\mathbb{Z})$. This identification allows us to drop the indices in $S_k, M_k$ and $R_k$ and write $S, M$ and $R$.

Let us consider forms $G_1 = G_{F_{L_1}}$ and $G_2 = G_{F_{L_2}}$ on the same space $S \oplus M \oplus R$.

We have the following properties of $G_1$ and $G_2$.

1. $G_2(x, y) = G_1(x, y)$ for all $x, y \in S \oplus M$.
2. $G_2(x, y) = G_1(x, y)$ for all generators $x, y \in R$ and
3. $G_1(x, y) = G_1(\alpha^l(x), \alpha^l(y))$ for all generators $x, y \in M \oplus R,$

   $G_2(x, \alpha^l(v)) = G_1(x, \alpha^{-l}(v))$ for every generator $x$ of $R,$

   $G_2(\alpha^l(x), v) = G_1(\alpha^l(x), v)$ for every generator $x$ of $R,$ and

   $G_2(x, v) = G_2(\alpha^l(x), \alpha^{-l}(v))$ for every generator $x \in R$.

4. $G_k(x, y) = 0$ for all $x \in S, y \in R, (k = 1, 2)$. 

Fig. 3.3
Let $S, M$ and $R$ be the subspaces of $(S \oplus M \oplus R) \otimes C$ complexifying $S, M$ and $R$, respectively. We have the involution $\bar{\cdot} : S \oplus M \oplus R \to S \oplus M \oplus R$ corresponding to the conjugation in the factor $C$ of the tensor product. The image of $x \in S \oplus M \oplus R$ under this involution is denoted by $\bar{x}$. Using the rotational symmetry of the rotor part we conveniently change the basis of $M$ and the generating set of $R$ in the following way. Let $\omega_j$ be an $n$th root of unity, $\omega_j = e^{2\pi i/j}$. We replace the basis $\{\alpha^j(v) \mid j = 0, 1, \cdots, n-1\}$ of $M$ by $\{v_j \mid v_j = \sum_{l=0}^{n-1} \omega^j_l \alpha^l(v), j = 0, 1, \cdots, n-1\}$. For $R$ we consider two choices of generating sets that are related by the involution $\bar{\cdot}$ as follows. We either replace the set $\{\alpha^j(y_p) \mid y_p \in X_{\bar{R}}^*, j = 0, 1, \cdots, n-1\}$, by $\{y_{j,p} \mid y_{j,p} = \sum_{l=0}^{n-1} \omega^j_l \alpha^l(y_p), y_p \in X_{\bar{R}}^*, j = 0, 1, \cdots, n-1\}$ or by $\{\overline{y}_{j,p} \mid \overline{y}_{j,p} = \sum_{l=0}^{n-1} \omega^j_l \alpha^l(y_p), y_p \in X_{\bar{R}}^*, j = 0, 1, \cdots, n-1\}$.

Let us consider the Hermitian forms $\hat{G}_1 = \hat{G}_{F_{L1}}$ and $\hat{G}_2 = \hat{G}_{F_{L2}}$, induced by $\mathcal{G}_1$ and $\mathcal{G}_2$, on the same space $S \oplus M \oplus R$.

These new generating sets for $M \oplus R$ satisfy the following conditions.

1. $\hat{G}_k(v_j, v_m) = 0$ for $j \neq m$, where $v_j, v_m \in M$ and $k = 1, 2$,

$$\hat{G}_1(x_{j,p}, v_m) = \hat{G}_2(\overline{x}_{j,p}, v_m) = 0 \text{ for } j \neq m \text{ where } x_{j,p} \in R_1, \overline{x}_{j,p} \in R_2, v_m \in M,$$

$$\hat{G}_1(x_{j,p}, y_{m,q}) = \hat{G}_2(\overline{x}_{j,p}, \overline{y}_{m,q}) \text{ for } j \neq m \text{ where } x_{j,p}, y_{m,q} \in R_1, \overline{x}_{j,p}, \overline{y}_{m,q} \in R_2.$$

2. $\hat{G}_1(x, y_{j,p}) = \hat{G}_2(x, \overline{y}_{j,p}) = 0$ for any $x \in S, y_{j,p} \in R_1, \overline{y}_{j,p} \in R_2$.

3. $\hat{G}_1(y_{j,p}, y_{j,q}) = \hat{G}_2(\overline{y}_{j,p}, \overline{y}_{j,q})$, for any $y_{j,p}, y_{j,q} \in R_1, \overline{y}_{j,p}, \overline{y}_{j,q} \in R_2$.

4. $\hat{G}_1(v_j, y_{j,p}) = \hat{G}_2(v_j, \overline{y}_{j,p})$ for any $v_j \in M, y_{j,p} \in R_1, \overline{y}_{j,p} \in R_2$.

$$\hat{G}_1(v_j, v_j) = \hat{G}_2(v_j, \overline{v}_j) \text{ for any } v_j \in M.$$

5. $\hat{G}_1(x, v_j) = \hat{G}_2(x, v_j)$ for any $x \in S, v_j \in M$.

For a given $\omega_j$, $0 \leq j \leq n-1$, let $W_j$ be the subspace of $M \oplus R$ defined by choosing its ordered basis in the following way. Take $v_j$ from $M$ first and $y_{j,p}$ from $R$ in any
order. To obtain the ordered basis of $M \oplus R$ we place the basis of $W_j$ before the basis of $W_{j+1}$ for $j = 0, 1, \cdots, n - 1$. Finally we add the ordered basis of $S$. We obtain, in this way, an ordered basis of $H_1(F_{L_1}; \mathbb{C})$. Notice that we can construct an ordered basis of $H_1(F_{L_2}; \mathbb{C})$ by replacing each $y_{j,p}$ with $\overline{y_{j,p}}$.

Let $\hat{G}_1$ and $\hat{G}_2$ be the matrices of the forms $\hat{G}_1$ and $\hat{G}_2$ respectively, in the ordered bases of $S \oplus M \oplus R$, chosen before.

$$\hat{G}_{L_1} = \begin{pmatrix} B_{10} & 0 & \overline{S_0} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{1n-1} \overline{S_{n-1}} \end{pmatrix}$$ and $$\hat{G}_{L_2} = \begin{pmatrix} B_{20} & 0 & \overline{S_0} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{2n-1} \overline{S_{n-1}} \end{pmatrix}.$$ 

In these bases, $B_{1j}$ (respectively $B_{2j}$), where $j = 0, 1, \cdots, n - 1$, is the matrix of the restriction of the form $\hat{G}_1$ (and $\hat{G}_2$ respectively) to the subspace $W_j$ generated by $\{v_j\} \cup \{y_{j,p} \mid y_p \in X_{R_1}^*\}$ (respectively $\{\overline{y_{j,p}} \mid y_p \in X_{R_1}^*\}$). Finally, the restrictions of $\hat{G}_1$ and $\hat{G}_2$ to the stator part are the same for $\hat{G}_1$ and $\hat{G}_2$ and denoted by $S$. Notice that $B_{1k}^\dagger = B_{2k}$, $S_l = (s_{l1} \ 0 \ \cdots \ 0)$, and $s_{l1}$ is the first column of each matrix $S_l$.

The matrices $M_k = (\hat{G}_{L_k} - \lambda E)$ ($k = 1, 2$) satisfy the conditions of Traczyk’s Proposition 2.9 for any real number $\lambda$, [14]. Thus $\det(M_1) = \det(M_2)$ for any real $\lambda$. So the determinants are equal for any complex $\lambda$ as well. \hfill \Box

4 Oriented rotation and Tristram-Levine signature

In this section we extend the method developed by Traczyk in [14] in order to show that orientation-preserving rotations (see Fig. 2.2(a)) preserve Conway polynomial. We show that the characteristic polynomial of the Hermitian form associated with the Seifert form of appropriately chosen Seifert surface is invariant under orientation-preserving rotations. In particular we prove the following result.

**Theorem 4.1** Let $L_1$ and $L_2$ be a pair of orientation-preserving $n$-rotants. Then $\sigma_\omega(L_1) = \sigma_\omega(L_2)$.
The main result of this section is Theorem 4.2 from which Theorem 4.1 follows.

Let $S^2$ be the sphere of a projection of a link $L$, and $F_L$ the Seifert surface of $L$ obtained from the diagram of $L$ by the Seifert algorithm. Let $H$ be a trivalent graph that consist of the Seifert circles and the cores of the bands. Let $R_1, R_2, \cdots, R_m$ be the components of $S^2 - H$ which are not bounded by Seifert circles. Assign the anti-clockwise orientation to each boundary curve of the regions $R_i (i = 1, \cdots, m)$; then these curves are generators of $H_1(F_L; \mathbb{Z})$. Whenever we refer to generators of $H_1(F_L; \mathbb{Z})$, we mean this particular set of standard generators for Seifert surface $F_L$.

Let $L_1$ and $L_2$ be a pair of orientation-preserving $n$-rotant diagrams.

We deform the diagrams $L_k \ (k = 1, 2)$ on $S^2$ into the position for which our computation is feasible, as it was done in [14]. Let $D_k$ be a disk in $S^2$ such that $D^r_k = D_k \cap L_k$ is the rotor part of the diagram $L_k \ (k = 1, 2)$, and $D^s_k = \overline{D_k \cap L_k}$ the stator part ($\overline{D_k} = S^2 - \text{int} D_k$). Rotors and stators constructed above are all $n$-tangles. We deform the stator part $D^s_1 = D^s_2$ to the form shown in Fig. 4.1. By doing so we obtain an outermost Seifert circle $C$ in $\overline{D}$ that is parallel to $\partial D_k$. Let $\overline{D_C}$ be the region which is bounded by $C$ and $\partial D_k$ in $\overline{D}$. We extend the rotational symmetries of the rotor parts $D^r_k \ (k = 1, 2)$ to the parts embedded in $D_k \cup \overline{D_C}$, i.e., we may assume that $D_k \cup \overline{D_C} \ (k = 1, 2)$ contain $n$-rotors.

![Fig. 4.1](image)

Let $F_{L_k} \ (k = 1, 2)$ be the Seifert surface for $L_k$ (Fig. 4.2), and let $A_{L_k}$ be the corresponding Seifert matrix of $L_k, k = 1, 2$. 

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Let $\xi$ be a complex number and let $X_L^k = \xi A_L^k + \bar\xi A_L^k \dagger$ be the Hermitian matrix that represents the Hermitian form $\theta(x, y) = \xi \psi(x, y) + \bar\xi \psi(y, x)$, $x, y \in H_1(F_L^k; \mathbb{Z})$.

With the choices for Seifert surfaces $F_k$ and the bases of $H_1(F_k)$ made above, we can formulate the main result of this section.

**Theorem 4.2** The characteristic polynomials of the Hermitian matrices $X_{L_1}$ and $X_{L_2}$ coincide.

**Proof** We consider three submodules $S_k, R_k$ and $M_k$ of $H_1(F_L^k; \mathbb{Z})$, where $S_k, R_k$ and $M_k$ are generated by the sets $X_{S_k}, X_{R_k}$, and $X_{M_k}$ of the standard generators of $H_1(F_L^k; \mathbb{Z})$ which live entirely in the stator part $\bar{D}$, rotor part $D_k$, and partially in $\bar{D}$ and $D_k$ ($k = 1, 2$), respectively. We have the following decomposition of the module $H_1(F_L^k; \mathbb{Z})$ into the direct sum of its submodules, $H_1(F_L^k; \mathbb{Z}) = S_k \oplus (M_k + R_k)$ ($k = 1, 2$).

Let $v$ denote the generator of $M_1$ intersecting the axis $y$ of the dihedral flype $d$ (Fig. 4.3). There is an action of the cyclic group $\mathbb{Z}_n = \langle \alpha \mid \alpha^n = 1 \rangle$ on $M_1 + R_1$ induced by the $\frac{2\pi}{n}$-rotation around the center of $D_1$. The set $X_{M_1} = \{ v, \alpha(v), \alpha^2(v), \cdots, \alpha^{n-1}(v) \}$ is a generating set of $M_1$ (not necessary a basis). We also identify $\alpha^j(v)$ with the generator of $M_2$ that coincides with $\alpha^j(v)$ of $M_1$ in $\bar{D}_C$. The submodule $R_1$ is generated by the set $\{ \alpha^j(x) \mid x \in X_{R_1}, j = 0, 1, \cdots, n-1 \}$. Since $D_2$ is the image of $D_1$ by the dihedral flip $d$ around the axis $y$ which crosses $v$, $R_2$ is generated by $\{ d(\alpha^j(x)) \mid x \in X_{R_1}, j = 0, 1, \cdots, n-1 \}$ (Fig. 4.3). In order to compare $\psi_1$ with $\psi_2$, we identify the generator $\alpha^j(x)$ of $R_1$ with the generator $d(\alpha^j(x)) \in R_2$ ($j = 0, 1, 2, \cdots, n-1$).
Using these identifications we can consider both forms $\psi_1$ and $\psi_2$ on the same sub-modules $S$, $M$ and $R$ (indices are no more needed) and derive the following relationship between them.

1. $\psi_2(x, y) = \psi_1(x, y)$ for all $x, y \in S + M$.
2. $\psi_2(x, y) = \psi_1(y, x)$ for all $x, y \in R$.
3. $\psi_2(x, \alpha^j(v)) = \psi_1(\alpha^{-j}(v), x)$, and
   $$\psi_2(\alpha^j(v), x) = \psi_1(x, \alpha^{-j}(v))$$
   for all $x \in R$ ($j = 0, 1, \cdots, n - 1$).
4. $\psi_1(x, y) = \psi_1(y, x) = 0 = \psi_2(x, y) = \psi_2(y, x)$ for all $x \in S, y \in R$.

Using relations (1),(2),(3),(4), we obtain the corresponding relations between $\theta_1$ and $\theta_2$. Let $S$, $M$ and $R$ be the complexifications of subspaces $S, M$ and $R$ of $S \oplus (M + R) \otimes \mathbb{C}$ respectively. There is a well defined involution $\overline{\cdot} : S \oplus (M + R) \to S \oplus (M + R)$ corresponding to the conjugation in the factor $\mathbb{C}$ of the tensor product. We denote by $\overline{x}$ the image of $x \in S \oplus (M + R)$ under this involution. The following identities follow from the identities (1)-(4) given before.

1. $\theta_2(x, y) = \theta_1(x, y)$ for all $x, y \in S \oplus M$.
2. $\theta_2(x, y) = \theta_1(y, x) = \overline{\theta_1(x, y)}$ for all generators $x, y \in R$, and
\[ \theta_2(x, y) = \overline{\theta_1(x, y)} \] for all \( x, y \in \mathbb{R} \).

(3) \( \theta_1(x, y) = \theta_1(\alpha^j(x), \alpha^j(y)) \) for all generator \( x, y \in \mathbb{M} + \mathbb{R} \),

\[ \theta_2(x, \alpha^j(v)) = \theta_1(\alpha^{-j}(v), x) \] for every generator \( x \) of \( \mathbb{R} \),

\[ \theta_2(\alpha^j(x), v) = \overline{\theta_1(\alpha^j(x), v)} \] for every generator \( x \) of \( \mathbb{R} \), and

\[ \theta_2(x, v) = \theta_2(\alpha^j(x), \alpha^{-j}(v)) \] for every generator \( x \in \mathbb{R} \).

(4) \( \theta_k(x, y) = 0 \) for all \( x \in \mathbb{S}, y \in \mathbb{R}, k = 1, 2 \).

In order to define Hermitian matrices \( H_{L_k} \) representing \( \theta_k \), \( k = 1, 2 \), we first choose a basis of \( H_1(F_{L_k}; \mathbb{C}) \) that is formed using the generators of \( H_1(F_{L_k}; \mathbb{Z}) \) in the following way. Set \( \omega_j = e^{2\pi i j} \) \((j = 1, \ldots, n)\). We replace the generating set \( \{\alpha^j(v) | j = 0, 1, \ldots, n - 1\} \) of \( \mathbb{M} \) by \( \{v_j | v_j = \sum_{l=0}^{n-1} \omega_j^l \alpha^l(v), j = 0, 1, \ldots, n - 1\} \). For \( \mathbb{R} \) we consider two choices of generating sets related by involution \( \overline{\cdot} \). We either replace \( \{\alpha^j(y_p) | y_p \in X_\mathbb{R}, j = 0, 1, \ldots, n - 1\} \) or by \( \{y_{j,p} | y_{j,p} = \sum_{l=0}^{n-1} \omega_j^l \alpha^l(y_p), y_p \in X_\mathbb{R}, j = 0, 1, \ldots, n - 1\} \).

We obtain in this way the new generating set for \( \mathbb{M}_k + \mathbb{R}_k \). The following relationships hold:

(1) \( \theta_k(v_j, v_m) = 0 \) for \( j \neq m \), where \( v_j, v_m \in \mathbb{M}, k = 1, 2 \),

\[ \theta_1(x_{j,p}, v_m) = \theta_2(\overline{x}_{j,p}, v_m) = 0 \] for \( j \neq m \), where \( x_{j,p} \in \mathbb{R}_1, \overline{x}_{j,p} \in \mathbb{R}_2, v_m \in \mathbb{M} \),

\[ \theta_1(x_{j,p}, y_{m,q}) = \theta_2(\overline{x}_{j,p}, \overline{y}_{m,q}) \] for \( j \neq m \) where \( x_{j,p}, y_{m,q} \in \mathbb{R}_1, \overline{x}_{j,p}, \overline{y}_{m,q} \in \mathbb{R}_2 \).

(2) \( \theta_1(x, y_{j,p}) = \theta_2(x, \overline{y}_{j,p}) = 0 \) for any \( x \in \mathbb{S}, y_{j,p} \in \mathbb{R}_1, \overline{y}_{j,p} \in \mathbb{R}_2 \).

(3) \( \theta_1(y_{j,p}, y_{j,q}) = \overline{\theta_2(y_{j,p}, \overline{y}_{j,q})} \) for any \( y_{j,p}, y_{j,q} \in \mathbb{R}_1, \overline{y}_{j,p}, \overline{y}_{j,q} \in \mathbb{R}_2 \).

(4) \( \theta_1(v_j, y_{j,p}) = \overline{\theta_2(\overline{v}_j, y_{j,p})} \) for any \( v_j \in \mathbb{M}, y_{j,p} \in \mathbb{R}_1, \overline{y}_{j,p} \in \mathbb{R}_2 \).
\[ \theta_1(v_j, v_j) = \theta_2(v_j, v_j) \text{ for any } v_j \in M. \]

(5) \[ \theta_1(x, v_j) = \theta_2(x, v_j) \text{ for any } x \in S, v_j \in M. \]

Take the subspace \( W_j \) of \( M \oplus R \) corresponding to \( \omega_j \), and choose its ordered basis by taking \( v_j \) from \( M \) first\(^5\) and the rest of a basis of \( W_j \) from the generating set \( y_{j,p} \) of \( R \) in any order. To obtain the ordered basis of \( M \oplus R \) we place the basis of \( W_j \) before the basis of \( W_{j+1} \) for \( j = 0, 1, \ldots, n - 1 \). Finally, we add ordered basis of \( S \). Then we have an ordered basis of \( H_1(F_L; \mathbb{C}) \). We also obtain an ordered basis of \( H_1(F_{L_2}; \mathbb{C}) \) by replacing each \( y_{j,p} \) with \( \overline{y}_{j,p} \).

We obtain the matrices of forms \( \theta_1 \) and \( \theta_2 \) in ordered bases of \( S \oplus (M + R) \) as described below.

\[
H_{L_1}' = \begin{pmatrix}
B_{10} & 0 & S_0 \\
\vdots & \ddots & \vdots \\
0 & B_{1n-1} & S_{n-1} \\
S_0 & \cdots & S_{n-1}
\end{pmatrix},
H_{L_2}' = \begin{pmatrix}
B_{20} & 0 & S_0 \\
\vdots & \ddots & \vdots \\
0 & B_{2n-1} & S_{n-1} \\
S_0 & \cdots & S_{n-1}
\end{pmatrix},
\]

In those bases, \( B_{1j} \) (respectively \( B_{2j} \)), where \( j = 0, 1, \ldots, n - 1 \), is the matrix of the restriction of the form \( \theta_1 \) (and \( \theta_2 \) respectively) to the subspace \( W_j \) generated by \( \{v_j\} \cup \{y_{j,p} \mid y_p \in X_{R_1}\} \) (\( \{v_j\} \cup \{\overline{y}_{j,p} \mid y_p \in X_{R_1}\} \) respectively). Finally the restriction to the stator part, \( S \), is the same for both \( \theta_1 \) and \( \theta_2 \). Notice that \( B_{1k}^t = B_{2k} \), \( S_l = ([s_{l1} \ 0 \ \cdots \ 0]) \), and \( s_{l1} \) is the first column of each matrix \( S_l \).

Matrices \( M_k = (H_{L_k}' - \lambda I) \) \((k = 1, 2)\) satisfy the conditions\(^6\) of Traczyk’s Proposition 2.9 for any real number \( \lambda \), [14]. Thus \( \det(M_1) = \det(M_2) \) for any real \( \lambda \). So the determinants are equal for any complex number \( \lambda \) as well.

\[\square\]

\(^5\)If \( v_j = 0 \), what can happen if the generating set \( \{v, \alpha(v), \alpha^2(v), \ldots, \alpha^{n-1}(v)\} \) is not a basis of \( M \), we skip this element when building basis of \( H_1(F_L; \mathbb{C}) \).

\(^6\)We can use Proposition 2.9, even if some vectors \( w_j \in W_{i,j} \) may be equal to 0. In such a case the block \( W_{i,j} \) is orthogonal to other factors (\( S \) and \( W_{i,j'} \), \( j' \neq j \)).
5 Counterexamples

It was proven in [1] that any pair of oriented 3- or 4-rotant links share the same Homflypt polynomials (in particular, Conway polynomials). In [14] Traczyk showed that a pair of orientation-preserving \( n \)-rotant links share the same Conway polynomial. On the other hand, for orientation-reversing \( n \)-rotants \( (n \geq 6) \), the invariance was still an open question. We present, in this section, an example of a pair of 6-rotant knots with different Conway polynomials and different Jones polynomials. Therefore, the invariance in [1] of Conway polynomial and the Jones polynomial for the orientation-reversing rotant links is the best possible. We should also stress that rotants described in Fig. 5.1 have different Jones and Conway polynomials, however they share the same determinant and the same homology of the corresponding double branched covers.

Let \( L_1 \) and \( L_2 \) be the knots (6-rotants) illustrated in Fig. 5.1. Using program KNOT [7], we have the following.

Conway polynomials (with the skein relation \( \nabla_{L_+} - \nabla_{L_-} = z \nabla_{L_0} \)) are different:

\[
\nabla_{L_1}(z) = 1 + 3z^2 - 37z^4 + 17z^6 - 3z^8 - 2z^{10} - 59z^{12} - 34z^{14} - 55z^{16} - 48z^{18} - 10z^{20} - 4z^{22} - z^{24},
\]

and

\[
\nabla_{L_2}(z) = 1 + 3z^2 - 25z^4 - 116z^6 - 57z^8 - 174z^{10} - 157z^{12} - 159z^{14} - 119z^{16} - 102z^{18}
\]

Fig.5.1
\[-37z^{20} - 8z^{22} - z^{24}.\]

Jones polynomials (with the skein relation \( t^{-1}V_{L_+} - tV_{L_-} = (\sqrt{t} - 1/\sqrt{t})V_{L_0}\)) are different:

\[
V_{L_1} = t^{23} - 16t^{22} + 131t^{21} - 713t^{20} + 2881t^{19} - 9193t^{18} + 24058t^{17} - 52926t^{16} \\
+ 99534t^{15} - 161854t^{14} + 229195t^{13} - 283357t^{12} + 304679t^{11} - 280476t^{10} \\
+ 211413t^9 - 112418t^8 + 7697t^7 + 77824t^6 - 127092t^5 + 136195t^4 - 114114t^3 \\
+ 77214t^2 - 41391t + 16087 - 2934 - t^{-8},
\]

and

\[
V_{L_2} = t^{23} - 16t^{22} + 131t^{21} - 713t^{20} + 2881t^{19} - 9193t^{18} + 24057t^{17} - 52919t^{16} \\
+ 99503t^{15} - 161752t^{14} + 228932t^{13} - 282808t^{12} + 303730t^{11} - 279098t^{10} \\
+ 209727t^9 - 110701t^8 + 6314t^7 + 78540t^6 - 126958t^5 + 135242t^4 \\
- 112578t^3 + 75451t^2 - 39756t + 14823 - 2118t^{-1} - 1933t^{-2} + 1941t^{-3} \\
- 1010t^{-4} + 354t^{-5} - 85t^{-6} + 13t^{-7} - t^{-8}.
\]

Their homology groups are the same: \( H_1(M_{L_1}; \mathbb{Z}) = H_1(M_{L_2}; \mathbb{Z}) = \mathbb{Z}/3 \oplus \mathbb{Z}/397449.\)
Their determinants coincide as well: \( \Delta_{L_1}(-1) = \Delta_{L_2}(-1) = -1192347 \) (here \( \Delta_L(t) = \nabla_L(z) \) for \( z = \sqrt{t} - \frac{1}{\sqrt{t}} \)).

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