Abstract

A relativistic quantum mechanics is studied for bound hadronic systems in the framework of the Point Form Relativistic Hamiltonian Dynamics. Negative energy states are introduced taking into account the restrictions imposed by a correct definition of the Poincaré group generators. We obtain nonpathological, manifestly covariant wave equations that dynamically contain the contributions of the negative energy states. Auxiliary negative energy states are also introduced, specially for studying the interactions of the hadronic systems with external probes.

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1. Introduction

The study of hadronic few-body systems in terms of constituent particles represents a very important tool for the determination of their intrinsic properties and their interactions with external probes. We use the term constituent particle to mean a system that transforms as an irreducible representation of the Poincaré group with definite mass, spin and internal symmetry quantum number.

In this respect we recall that historically the investigation started with the study of light nuclei in terms of constituent nucleons. Later, many efforts have been devoted to the study of the hadrons in terms of constituent quarks.
For both cases nonrelativistic models were initially considered, later on, relativistic corrections were added, improving the reproduction of the experimental data.

Nowadays, the construction of hadronic covariant constituent models must be considered strictly necessary for an accurate description of these systems and for the study of their interactions with electroweak probes. In particular, the present work, that represents a generalization of the Relativistic Hamiltonian Dynamics, is focussed on the study of quark (and quark-di-quark) models, but its formal developments can be also applied to the study of few-body nuclear systems.

As for the relativistic covariant quark models, they represent effective models that should be related to quantum chromodynamics (QCD), incorporating in a nonperturbative way its symmetries and dynamical features. On the other hand, the nuclear models rely on a phenomenological meson exchange theory.

1.1 General remarks

From a theoretical point of view, we note that, for building few-body relativistic models, two slightly different approaches can be followed. The first one, is denominated Relativistic Hamiltonian Dynamics (RHD). The second one is represented by the Relativistic Wave Equations (RWE).

The aim of the first approach, that is the RHD, is to satisfy the Poincaré Group commutation rules by defining the generators of that group in terms of the constituent particle operators.

We now syntetically recall some technical aspects related to RHD. In the case of free particles the total generators of the Poincaré group are given by the sum of the single particle generators. The main problem of RHD, not found in the nonrelativistic case, consists in fulfilling the Poincaré group commutation rules when the interaction is introduced in the generators. In this respect, RHD can be formulated in different ways, three of which have been introduced in the pionieristic work by Dirac [1]. They are called the Instant Form (IF), the Front Form (FF) and the Point Form (PF).

The most relevant difference among them is represented by the way in which the generators depend on the interaction.

In the IF both the Hamiltonian, i.e. the time translation generator, and
the boost generators are modified (with respect to the free case) by the interaction [2-4].
In the FF, linear combinations of the components of the four-vectors are considered and, in consequence, not standard Lorentz transformations must be introduced. In this form of RHD, the interaction modifies pieces of both the boost and of the four-momentum [3].
Finally, in the PF [5-7], that is the scheme adopted in the present work, the interaction modifies the total four-momentum, i.e. the Hamiltonian and the three-momentum of the system, that are given not only by the sum of the four-momenta of the constituents, but also receive a contribution that depends on the interaction operator. On the other hand, the boost is left free from interaction. Due to this last property, PF RHD has been defined as manifestly covariant [5]. This point will be analyzed in the next subsect.1.2, considering the definition of manifest covariance adopted in the present work.

Theoretically, the relevant aspect of RHD is that its quantum mechanical properties are well defined, in analogy to nonrelativistic case.
All the generators of the Poincaré algebra are represented by hermitic operators so that the corresponding transformations are performed by unitary operators, satisfying Poincaré covariance and allowing, at the same time, to use the standard quantum mechanical procedures for the calculation of physical observables. The interaction operator is given by a quasipotential that is, in general, momentum dependent.
Note that some aspects that are typical of quantum field theories have been completely excluded in RHD: in particular, the possibility of creating or destroying particles and the presence of negative energy states in the interaction amplitudes. In this work we shall analyze and overcome this second problem.

The quark models based on RHD reproduce some general hadronic features related to QCD, like colour global symmetry, confinement and isospin invariance for the $u,d$ quark sector.
Furthermore, in all the three forms of RHD very encouraging quantitative calculations have been done and are still in progress both for the hadronic spectra and for the electroweak form factors [8].
On the other hand, the dynamics of these models is intrinsically phenomenological and partially unsatisfactory, because, as discussed above, some fundamental aspects of QCD considered as field theory, are not taken into account by standard RHD. Moreover, the parameters that appear in the mass oper-
ator of these models are usually fitted to the experimental data in order to reproduce the hadronic spectra.

The electroweak form factors are calculated in the relativistic impulse (or spectator) approximation by using a current operator that, in the case of the electromagnetic interaction, is not dynamically conserved. The discrepancies that remain between the theoretical calculations and the form factor experimental data are usually cured by inserting phenomenological quark form factors and/or vector meson exchange factors at the quark interaction vertex.

The second possible approach for the relativistic study of few-body bound hadronic systems, is represented by the use of integro-differential three-dimensional RWE.

We leave aside from our discussion the four-dimensional Bethe-Salpeter equation because, if it is not reduced to a three-dimensional form, its formalism cannot be easily interpreted in a physical way and also the practical solution of the equation presents serious difficulties.

In general, the procedure to obtain the RWE (for two-body interacting systems) starts from an underlying field theory. Historically, quantum electrodynamics was considered.

From the perturbative expansion of the field theory, a set of relevant Feynman graphs is selected, excluding, in general, the contributions due to the poles of the bosonic propagators. Usually, the box and (in some cases) the crossed box graphs are taken. The infinite series of these graphs, denoted respectively as ladder and crossed ladder, is summed up by means of a Lippman-Schwinger equation for the scattering matrix, in which a suitable Green function is used. Finally, from that equation, an integro-differential, three-dimensional RWE for the bound states is derived.

The origin of the differences among the various RWE is due to which Feynman graphs are selected and to the approximations done to sum them up in a three-dimensional form. The procedure outlined above establishes the connection between the RWE relativistic model and the underlying field theory. Generally, the solutions of the RWE cannot be interpreted as standard quantum mechanical wave functions. Also, discussing some specific examples we shall highlight some difficulties of the RWE approach.

Historically, RWE have been introduced to study electromagnetically bound systems, like atoms and positronium, taking quantum electrodynamics as the
fundamental theory. Later, they have been applied to the study of few-body nuclear systems. In this case it was used a phenomenological interaction operator that represents the exchange of pions and heavier mesons.

The simplest case of RWE is represented by the positive energy states Blackenbecler-Sugar equation [9]. In the remainder of this section we shall not further discuss this case that is not specifically relevant for the main objective of this work. The Blackenbecler-Sugar equation will be considered again in sect.4 (as a special example with no negative energy states) showing that it can be directly interpreted by means of the standard PF RHD formalism and re-written in a manifestly covariant way.

Considering the Dirac-like equation [10], we recall that, in this case, a theoretical difficulty was found because this equation presents an unphysical pole for a vanishing value of the total mass of the system. As a consequence, a correct normalization of the wave function is not possible. This problem has been analyzed and denoted as \textit{continuum dissolution} and \textit{cockroach nest} in refs. [11] and [12], respectively. An analysis of this problem will be also performed in the present work by according to the requirements of PF RHD.

The Gross equation [13] represents a relevant example of a manifestly covariant approach for the study of interacting systems. It has been obtained by means of a covariant procedure starting from the \textit{box} and \textit{crossed box} Feynman graphs. Even though also in this equation a pole for a vanishing total mass is present, this singularity is easily removed requiring the (phenomenological) kernel to vanish faster than the total mass. In any case the solutions of the equation are normalizable if the total mass is not equal to zero. The Gross equation is not manifestly symmetric, as such, under particle interchange, but it has been symmetrized in all its applications. Within this approach, the mass of the bound system is obtained as a pole \textit{below threshold} of the interacting scattering matrix.

We also consider the so-called Breit equation in the form originally given for the study of electromagnetically bound systems [14]. This equation is obtained from the four-dimensional Bethe-Salpeter equation approximating the electromagnetic interaction with an instantaneous quasipotential. Only \textit{box} Feynman graphs are taken. This equation can be directly interpreted in the scheme of RHD that will be developed in the present work.

Finally, the Mandelzweig-Wallace integro-differential equation [11] is structurally similar to the previous one, but also includes, in the eikonal approximation, the \textit{crossed box} graphs, improving the correspondence with the
underlying field theory and obtaining the so called \textit{one-body limit} when the mass of one particle is set equal to infinity. This equation can be also written in a covariant form [15, 16]. However, a standard definition of the Poincaré group generators in terms of PF RHD quantum mechanical operators is not directly feasible within this model. Such problem arises because the Mandelzweig-Wallace equation is \textit{not} an eigenvalue equation for the mass of the system. In other words, the mass operator is not defined explicitly.

More recently, the Mandelzweig-Wallace formalism has been revised and denoted as \textit{equal time} reduction. In a clever work [17] concerning electron scattering on the deuteron, considered as a two-nucleon relativistic bound system, that equation has been written in a Hamiltonian form and the boost generator is constructed, with some approximations, in the instant form RHD. With respect to this problem, in sect.4 we shall show that, with some handling, an equation equivalent to the Mandelzweig-Wallace one can be written in the framework of our PF RHD model, obtaining \textit{exact} relativistic covariance. By means of this procedure we shall introduce the \textit{auxiliary states} that represent another original aspect of our work.

\subsection*{1.2 Motivations and structure of the work}

Having discussed the theoretical framework of PF RHD and RWE, we introduce the motivations of the present study. The long term objective of the relativistic quark model investigation would be the construction of a covariant Hamiltonian model for the study of the hadronic systems. This model should reproduce, with a limited number of parameters, the hadronic spectra and the response of the hadronic particles to electroweak probes, that is elastic and inelastic form factors, Compton scattering amplitude, etc.

Relativistic covariance, possibly in manifest form, should represent the formal framework of the model. As for the \textit{dynamics}, we point out that QCD is assumed to be the \textit{ultimate} physical description for these system. For this reason, the covariant Hamiltonian model should represent a solvable approximation of QCD in the sense that its (generalized) wave equation should be able to \textit{sum up} the relevant QCD graphs, allowing to treat perturbatively other effects not included in the sum.

The connection with QCD should allow to relate the parameters of the model
with those of QCD. In particular, we refer to the effective quark masses. Also, the quark-(anti)quark interaction should be possibly derived from QCD.

These objectives are, obviously, ambitious and lie beyond the present understanding of nonperturbative aspects of field theories. Deep and long investigations are required. However, the present work, as a preliminary step, can help to solve some specific theoretical problems, also improving the understanding of the hadronic phenomenology.

As for the main objective of the present work, that is focused on the study of spin $1/2$, identical, interacting quarks, we want to include, with some approximations, the negative energy states in PF RHD, in order to represent the dynamical relativistic effects that have not been taken into account by standard RHD models. We recall that in quantum field theories, like QCD, when the perturbative series is considered, the negative energy states appear in the propagators of the intermediate particles. For this reason, the procedure of the present work can help to improve the understanding of the link between RHD and the underlying field theory. Also in the case of few-body nuclear systems, the discussion about RWE shows that the contributions of the negative energy states are considered necessary for constructing a dynamically consistent model.

Technically, the insertion of negative energy states in the mass operator of the model, is obtained, in sects.2 and 3, by generalizing the PF RHD construction of the Poincaré group generators, fulfilling, also in presence of negative energy states, the commutation rules of the algebra. As shown in sect.3, this procedure is possible if all the particles of the state have the same energy sign. These states represent the dynamical states of our generalized PF RHD. Such condition is strictly necessary in order to define the four-velocity operator $V^\mu$, that, in turn, is essential to introduce the total four-momentum of the system $P^\mu$.

Our wave equation, written in sect.4, is a coupled equation that involves positive and negative energy dynamical states and explicitly satisfies charge conjugation invariance.

Another original result of this work is the introduction of the auxiliary states that represent the states in which particles with different energy signs are present. For this reason (as discussed before) they do not directly participate in the dynamics of the model. The auxiliary states are covariantly defined,
by means of the interaction operator, in eqs. (4.14b) and (4.18b) for two and three-body systems, respectively. In this way, the definition of the *auxiliary states* does not require the introduction of new parameters. Their expression is derived by using RWE formalism as a link with field theory.

In order to understand in more detail the physical meaning of the auxiliary states, we recall that, in perturbative expansions, the *negative energy* terms of the fermionic propagators give rise to the so-called *z-graphs*, corresponding to intermediate states with one quark and a *quark-antiquark pair* [17-19]. The contributions of such states to the electroweak currents have been extensively studied in the nonrelativistic limit, within a constituent chiral quark model. In particular, the two-body (or exchange) electromagnetic four-currents have been derived [18], obtaining significant contributions for the inelastic transition amplitudes of all nucleon resonances. In the same context it has been also shown that the exchange contributions are necessary to satisfy Partial Conservation of Axial Current (PCAC) condition for the weak interactions, also leading to a possible interpretation of the missing nucleon spin as angular momentum carried by the nonvalence degrees of freedom of the nucleon [19].

Note that, while the electromagnetic exchange currents could be in part also derived by means of a minimal coupling substitution in the nonrelativistic Hamiltonian, on the other hand the derivation of the weak (axial) exchange currents strictly requires the use of the *z-graphs*.

For the reasons discussed above, in our generalized PF RHD the *auxiliary states*, that represent the pair terms in effective way, are expected to give relevant contributions to the interactions of the hadronic system with external probes. In subsequent works it will be studied at numerical level their effect on electroweak form factors.

Furthermore, as for the possible relevance of both *dynamical* and *auxiliary* negative energy states for the electromagnetic interactions of hadronic systems, we point out that Compton scattering amplitude, if studied by means of the Feynman graphs of quantum electrodynamics, requires the presence of the negative energy states in the fermionic propagators, in order to obtain the correct expansion up to the second order in photon energy [20]. For this reason, photon scattering on nucleons (considered as bound quark states) can represent a very helpful tool for the study of some nontrivial
aspects of the hadronic dynamics that should be reproduced by RHD quark models.

As for the strong interaction of the quarks, the negative energy states appear in the covariant wave equation of the model, given in eq.(4.6), at two levels. First, as discussed previously, we have the dynamical negative energy states. Furthermore, the connection established with RWE and, in turn, with the underlying field theory, gives rise to quadratic terms in the effective interaction operators of eqs.(4.13b) and (4.17b) for two and three-body systems, respectively. These terms reproduce, in the wave equation, the effects due to intermediate states (of the scattering matrix expansion) with different energy signs. By means of the same arguments used for the auxiliary states, the quadratic terms, representing the z-graphs [17], can be interpreted as contributions of quark-antiquark pairs.

Note that (as in the case of the auxiliary states) these (extra) quadratic terms do not introduce new parameters, being related to the linear terms of the quasipotential. The effects of the quadratic terms on the constituent interaction have been studied in the context of few-body nuclear physics [13, 21]. In the case of quark models, the small value of the quark masses can give rise to highly nonrelativistic effects that were not present in the interaction of the nuclear systems.

The kinetic term of our wave equation given in eq.(4.6) is similar to that of the Breit equation of ref.[14], where a perturbative technique has been also studied for the case of positronium, that is bound by the electromagnetic interaction. Taking into account that the properties of the wave equation are strictly related to the structure of the interaction operator, we note that in the present case, that is quark models, such operator is not directly known and should be determined taking into account the underlying field theory. The total effective interaction (linear + quadratic terms) is in any case highly momentum dependent and a careful formal and numerical examination of the wave equation should be performed to highlight its properties in connection with nonperturbative hadronic phenomenology.

Another objective of this work is to write the wave equations of our generalized PF RHD in a manifestly covariant way. We define here as manifest covariance the property of an equation of being written in terms of quantities that (a) transform as tensors under Lorentz transformations and (b) are not
related to a specific reference frame.

In this sense, the Bethe-Salpeter and Gross equations are *manifestly covariant*. On the other hand, the wave equations of PF RHD, written in the standard formalism, do not fully satisfy the requirement (b) due to the use of the zero-momentum (rest) frame for the definition of the velocity states [5]. However, it is well-known that, if a theory is really covariant, with some handling it can be written in a *manifestly covariant* form, as it will be done for PF RHD in the present work.

Obviously, *manifest covariance* only represents a formal property of the equations and does not lead, as such, to improve the knowledge of the hadronic dynamics.

Technically, *manifest covariance* is obtained by means of (a) the definition of the projection states of eq.(3.9), where the three-momenta of \( N - 1 \) particles and spatial part of the four-velocity of the system are selected as spatial variables, and (b) the choice of the normalization of the wave function given in eq.(4.5), leading as result to a great clarification the formalism of PF RHD. Note that the structure of the covariant integration in eq.(4.6) is similar to that originally introduced by the Gross equation [13].

We recall that the methods of PF RHD allow to define a consistent framework for the study of the electroweak interactions of the hadronic systems (elastic and inelastic form factors), specially by introducing as a starting point the so-called relativistic impulse (or spectator) approximation [5]. Our *manifestly covariant* formalism also allows to calculate in a much more direct and clear way the matrix elements of the electroweak current in that approximation, as done in ref. [22] for a model with only positive energy states. Furthermore, in this way it is possible to introduce a minimal coupling procedure to derive a conserved electromagnetic current [23] for a model containing positive energy states.

The formal developments of the paper are organized as follows.

In sect.2, we define the Poincaré Group generators for the case of *free particles* also introducing the negative energy states. This objective is reached considering that in the three-momentum Lorentz transformation (and, in general, in the Lorentz boost operators), the relevant parameter is the product \( \lambda v \),
introduced in eq. (2.2b), being \( \lambda \) the energy sign of the particle and \( \mathbf{v} \) the spatial part of the four-velocity boost parameter.
We also study the connection between the state representation of PF RHD with that given by the standard Dirac equation spinors, which is currently used in the developments of quantum field theory. In particular we show the equivalence of the use of the RHD Wigner rotations with standard Dirac boosts for the calculation of the relevant matrix elements of the model.

In sect. 3, we construct the generators of the Poincaré algebra for *interacting particles* with negative energy states, verifying that the commutation rules of the Poincaré Group are still fulfilled. To this aim, we introduce the definition of *dynamical states* and *auxiliary states*. Only the former enter in the wave equation of the model. We also introduce the projection states of eq. (3.9). These states depend on the spatial variables that are used to obtain the manifestly covariant form of the model. Furthermore, the interaction operator is written in terms of Dirac spinors and matrices highlighting its covariant character and allowing to derive its expression from field theories.

In subsect. 4.1, we write in eq. (4.6) the wave equation of the model in a manifestly covariant way. We show that relativistic covariance, realized by the Poincaré Group commutation rules, as a dynamical consequence, automatically avoids, in our formalism, the continuum dissolution disease. 
In subsect. 4.2 we use the techniques of the RWE (referring to the Mandelzweig-Wallace model) to establish a *link* with field theories. In particular we introduce, starting from the same interaction operator, both the *auxiliary states* in eqs. (4.14b) and (4.18b), and the *quadratic terms* in eqs. (4.13b) and (4.17b) for the effective quasipotential of the wave equation.

In the appendix we also show explicitly that the interaction operator is invariant under Lorentz transformations.

### 2. Poincaré group transformations for single particle states

As a starting point we recall the form of a Lorentz transformation (boost) of a four-vector \( f^\mu = (f^0, \mathbf{f}) \). In all this work we consider canonical boosts.
The boosted four-vector $f^\mu_b$ has the form

$$ f^0_b = v^0 f^0 + vf \tag{2.1a} $$

$$ f_b = f + v(vf\frac{1}{v^0 + 1} + f^0) \tag{2.1b} $$

These equations can be summarized as follows

$$ f^\mu_b = L^\mu_\nu(v)f^\nu \tag{2.1c} $$

In the previous equations we have introduced the relative four-velocity $v^\mu = (v^0, v)$ that satisfies $v_\mu v^\mu = 1$ and, in consequence

$$ v^0 = v^0(v) = [1 + v^2]^{1/2} \tag{2.1d} $$

The well-known relations $v = \gamma u$ and $v^0 = \gamma = [1 - u^2]^{-1/2}$ easily connect the four-velocity to the standard physical velocity $u$ of the initial reference frame measured from the boosted one.

We now consider the Lorentz transformation for the on-shell four-momentum of a single particle. In the quantum-mechanical model that will be studied in the following, the particle three-momentum $p$ will be used as spatial variable. In order to study also negative energy states, we introduce the notation $p^\mu(\lambda, p) = (\lambda \epsilon(p), p)$ being $\lambda$ the energy sign of the state and

$$ \epsilon(p) = [p^2 + m^2]^{1/2} $$

the absolute value of the energy; also, $m$ represents the mass of the particle. We emphasize that in our model the free energy of a particle $p^0(\lambda, p)$ is always considered as a function of the energy sign $\lambda$ and of the three-momentum $p$ of the state. According to eqs.(2.1 a,b) the Lorentz transformation is

$$ p^0_b(\lambda, p; v) = v^0 \lambda \epsilon(p) + v p = \lambda \epsilon(p_b(\lambda; \lambda v)) \tag{2.2a} $$

$$ p_b(\lambda, p; v) = p_b(p; \lambda v) = p + \lambda v(p\frac{1}{v^0 + 1} + \epsilon(p)) \tag{2.2b} $$

where $v^0$, that is given by eq.(2.1d), takes the same value for $\lambda = +1, -1$.

Being the particle on-shell, the independent transformation is that of the three-momentum $p$, displayed in eq.(2.2b). For the development of the model we highlight the two following properties of this transformation:
(i) the transformation is not linear with respect to this variable $p$ due to the presence of $\epsilon(p)$ in the r.h.s of that equation;

(ii) the transformation only depends on the product $\lambda v$.

For the construction of the PF RHD we shall introduce the infinitesimal generator of the Lorentz boost, denoted in the present work as $t$. To this aim one has, first, to expand eq.(2.2b) up to the first order in the physical velocity $u$ and, second, to construct the operator $t$ that transforms the momentum eigenstates according to that expansion [1-4].

Furthermore, we point out that we shall construct a unitary representation of the boost generator in order to represent the Lorentz transformations according to the standard rule of quantum mechanics, i.e. analogously to nonrelativistic quantum mechanics.

This procedure seems suitable for the study of systems composed of particles whose (strong) interaction is described by means of a quasipotential operator. On the other hand, we recall that the spinors of the Dirac equation are transformed by pseudounitary operators, that will be shown explicitly in eq.(2.22). In the Dirac equation formalism, unitarity is recovered in the context of field theory [24], whose construction is beyond the scope of the present work.

However, we shall show that the two representations are connected by the transformation of eq.(2.18). Finally, we remark that the price paid for using a unitary representation of the boost operator is the nonlocal character of both the Hamiltonian operator and of the boost generator, that will be explicitly defined in eqs. (2.4 b-e) for positive energy states and, in general, in eqs.(2.10) and (2.11).

We now introduce for a single particle all the 10 infinitesimal generators of the Poincaré group, collectively denoted as $[g^I]$ ($I=1,...,10$). In more detail, these generators are: the 3-momentum $p$, i.e. the generator of the spatial translation; the angular momentum $j$, i.e. the generator of the spatial rotations; $t$, i.e. the generator of the Lorentz transformations; finally, the Hamiltonian $h$, i.e. the generator of the time translations. Their commutation rules are

\[
[p^\alpha, p^\beta] = [p^\alpha, h] = [j^\alpha, h] = 0 \tag{2.3a}
\]

\[
[j^\alpha, j^\beta] = i\epsilon^{\alpha\beta\gamma} j^\gamma \tag{2.3b}
\]
By introducing the operator \( r \) canonically conjugated to \( p \), that is
\[
[r, p] = -i\delta^{\alpha\beta}
\]
and the spin operator \( s \), the generators \( j, t \) and \( h \) can be put, for the positive energy states \((\lambda = +1)\), in the following form that satisfies the commutation rules of eqs.\((2.3a-g)\).

\[
\begin{align}
[j^\alpha, p^\beta] &= i\epsilon^{\alpha\beta\gamma} p^\gamma \\
[j^\alpha, t^\beta] &= i\epsilon^{\alpha\beta\gamma} t^\gamma \\
[t^\alpha, p^\beta] &= i\delta^{\alpha\beta} h \\
[t^\alpha, t^\beta] &= -i\epsilon^{\alpha\beta\gamma} j^\gamma \\
[t^\alpha, h] &= ip^\alpha
\end{align}
\]

As anticipated, the boost is represented by a unitary operator that, for a finite transformation, has the following form

\[
B(v) = \exp(itw(v)) \simeq 1 + itu
\]

with

\[
\begin{align}
\text{d} &= \frac{1}{2}(r\epsilon(p) + \epsilon(p)r) \\
g &= \frac{p \times s}{\epsilon(p) + m}
\end{align}
\]

Considering a state of three-momentum \( p \), positive energy \((\lambda = +1)\) and \( z \)-projection of the spin \(\sigma\), the action of the boost operator on such state is

\[
B(v)\lvert_{p, \lambda = +1, \sigma} = \left[\frac{\epsilon(p_0(p; v))}{\epsilon(p)}\right]^{1/2} R(p; v)\lvert_{p_0(p; v), \lambda = +1, \sigma} >
\]

For these states we adopt the delta normalization shown below in eq.\((2.8)\). In consequence, the numerical factor in the r.h.s. of the previous equation,
whose origin is due to the nonlinearity of the Lorentz transformation with respect to \( p \) (see eq.(2.2a)), guarantees the correct normalization of the boosted state, being \( B(v) \) a unitary operator; finally, the rotation operator \( R(p;v) \) that is function of the numerical vector \( p \), has been introduced to represent the spin rotation produced by the operator \( g \) of eq.(2.4c).

For completeness we also introduce the standard two components spinors \( w_\sigma \) to represent the spin states. In this way the wave function corresponding \( |p,\sigma> \) is written as

\[
\psi_{p\sigma}(q) = \langle q|p,\sigma> = w_\sigma \delta(q - p) \tag{2.8}
\]

and the matrix elements of the boost operator of eq.(2.7) take the form

\[
\langle q,\lambda = +1,\mu|B(v)|p,\lambda = +1,\sigma> = \left( \frac{\epsilon(p_b(p;v))}{\epsilon(p)} \right)^{1/2} R_{\mu\sigma}(p;v)\delta(q - p_b(p;v)) \tag{2.9}
\]

Here \( R_{\mu\sigma}(p;v) \) is the \( 2 \times 2 \) matrix representation of the operator \( R(p;v) \) acting in the space of the spinors \( w_\sigma \).

We shall now generalize the procedure outlined above in order to include in the theory also the negative energy states. These states, as it is shown by the study of the Dirac equation and by the development of the field theories, are introduced for a consistent relativistic treatment of interacting particles. To this aim we have to replace, in eq.(2.4b), the positive energy Hamiltonian with

\[
h = h(\lambda, p) = \lambda \epsilon(p) \tag{2.10}
\]

where \( \lambda \) represents here an operator with the eigenvalues \( \lambda = +1 \) and \( \lambda = -1 \) for positive and negative energy, respectively. In consequence, one can immediately verify that the Poincaré group commutation rules of eqs(2.3a-g) can be satisfied with the Hamiltonian of eq.(2.10) and by replacing, in eq.(2.4c)

\[
t = \lambda(d + g) \tag{2.11}
\]

By using the previous expression, the generalization of the finite boost for including the negative energy states is easily found: in eq.(2.5) the argument \( v \) must be replaced by \( \lambda v \), in agreement with the Lorentz transformation of
the three-momentum that was discussed before. For the generalized boost operator we shall keep using the notation $B(v)$.

In summary, for an Hamiltonian with negative eigenvalues ($\lambda = -1$), also the boost generator must take a minus sign to give the correct commutation rules. The other generators, $p$ and $j$ of eq.(2.4a), remain unchanged. Having introduced for the free particle state the following ket $\vert p, \lambda, \sigma \rangle$ we now need, to represent these states, a 4-component spinorial wave function of the following form

$$
\psi^{P}_{p, \lambda \sigma}(q) = \langle q \vert p, \lambda, \sigma \rangle = u^{P}(\lambda)w_{\sigma}\delta(q - p) \quad (2.12)
$$

that satisfies standard orthonormality properties. We denote this representation of the states as Poincaré representation. In the previous equation we have introduced the following $4 \times 2$ components block spinors

$$
u^{P}(+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.13a)
$$

$$
u^{P}(-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.13b)
$$

for positive and negative energy, respectively. These spinors act, in eq.(2.12), on the standard two component spinor $w_{\sigma}$. In the Poincaré representation the operator $\lambda$ is a $4 \times 4$ block matrix of the form

$$
\lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.14)
$$

One straightforwardly obtains the expressions for the Hamiltonian $h$ and for the boost generator $k$ replacing the previous expression of $\lambda$ in eqs.(2.10) and (2.11). Boosting the state $\vert p, \lambda, \sigma \rangle$ represents a generalization of eq.(2.7)

$$
B(v)\vert p, \lambda, \sigma \rangle = \left[ \frac{\epsilon(p;\lambda v)}{\epsilon(p)} \right]^{1/2} R(p; \lambda v)\vert p_0(p; \lambda v), \lambda, \sigma \rangle \quad (2.15a)
$$

In consequence, the boosted wave function is of the form

$$
\psi^{P}_{b, p, \lambda \sigma}(q) = \left[ \frac{\epsilon(q)}{\epsilon(p)} \right]^{1/2} u^{P}(\lambda)R(p; \lambda v)w_{\sigma}\delta(q - p_0(p; \lambda v)) \quad (2.15b)
$$

where the equality $q = p_0(p; \lambda v)$ given by the delta function has been used in the normalization factor.
For the matrix elements of the boost operator, eq.(2.9) is generalized in the following way

\[
\langle q, \lambda', \mu | B(v) | p, \lambda, \sigma \rangle = \delta_{\lambda \lambda'} \left[ \frac{\epsilon(p_b(p; \lambda v))}{\epsilon(p)} \right]^{1/2} R_{\mu \sigma} (p; \lambda v) \delta(q - p_b(p; \lambda v))
\]  

(2.15c)

We now study the connection of the Poincaré representation with the one given by the solution of the standard Dirac equation. In particular this study is very useful in order to construct, in the following, invariant interaction operators with respect to boost transformations. We show that the Poincaré representation is completely equivalent to the Dirac one. To this aim we recall that the solutions of the Dirac equation for a free particle, in the momentum space, with the same notations introduced before, have the following form

\[
\psi^D_{\lambda p, \sigma}(q) = u^D_{\lambda p, \sigma} \delta(q - p)
\]  

(2.16)

with

\[
u^D_{\lambda p, \sigma}(q) = u^D_{\lambda p, \sigma} w_\sigma \delta(q - p)
\]

(2.17a)

where we are introducing the Dirac matrices \( \gamma^\mu = (\gamma^0, \tilde{\gamma}) \) in the standard representation. With straightforward handling one verifies the following properties of the FW transformation [25] given in the following equation

\[
U(p) = \left[ \frac{\epsilon(p) + m}{2\epsilon(p)} \right]^{1/2} + \left[ \frac{\epsilon(p) - m}{2\epsilon(p)} \right]^{1/2} \frac{(p \gamma)}{|p|}
\]  

(2.18)

where we are introducing the Dirac matrices \( \gamma^\mu = (\gamma^0, \tilde{\gamma}) \) in the standard representation. With straightforward handling one verifies the following properties

\[
U^{-1}(p) = U^+(p) = U(-p)
\]  

(2.19)
The transformation for the spinors have the form

\[ U(p)u^D(\lambda, p)w_\sigma = u^P(\lambda)w_\sigma \quad (2.20a) \]

Note that, due to the unitarity of the FW transformation, the spinors \( u^D(\lambda, p) \) and \( u^P(\lambda) \) have the same normalization to unity. Also, for the complete wave functions, defined in eq.(2.12), one has the relation

\[ U(q)u^D(\lambda, p)w_\sigma \delta(q-p) = u^P(\lambda)w_\sigma \delta(q-p) \quad (2.20b) \]

Applying the FW transformation as in the previous equation to the boosted Poincaré wave function given in eq.(2.15b), the boost transformation of a Dirac wave function is easily found in the form

\[ \psi_{b,\lambda p\sigma}^D(q) = U^+(q)\psi_{b,\lambda p\sigma}^P(q) = \]

\[ = \left[ \frac{\epsilon(q)}{\epsilon(p)} \right]^{1/2} u^D(\lambda, q)R(p; \lambda v)w_\sigma \delta(q - p_b(p; \lambda v)) \quad (2.21) \]

where, analogously to eq.(2.15b), the equality \( q = p_b(p; \lambda v) \) given by the delta function has been used both in the normalization factor and in the argument of the Dirac spinor.

On the other hand it is well known that in the Dirac theory the spinor boost is introduced in the form

\[ B_D(v) = B_D^+(v) = \left[ \frac{1}{2}(v^0 + 1) \right]^{1/2} + \left[ \frac{1}{2}(v^0 - 1) \right]^{1/2} (v^0 \gamma^0 \gamma^0) \]

\[ \approx 1 + \frac{1}{2}(u^0 \gamma^0 \gamma^0) \quad (2.22) \]

where the time component \( v^0 \) of the four-velocity, given in eq.(2.1d), has been used. Standard calculations show that

\[ B_D(v)u^D(\lambda, p)w_\sigma = \left[ \frac{\epsilon(p_b(p; \lambda v))}{\epsilon(p)} \right]^{1/2} u^D(\lambda, p_b(p; \lambda v))R(p; \lambda v)w_\sigma \quad (2.23) \]

and, consequently, the boosted wave function of eq.(2.21) can be written as

\[ \psi_{b,\lambda p\sigma}^D(q) = B_D(v)u^D(\lambda, p)w_\sigma \delta(q - p_b(p; \lambda v)) \quad (2.24) \]
For further developments, we note that eq.(2.23) can be simplified by introducing the covariantly normalized Dirac spinors

\[ u^{DC}(\lambda, p) = \left[ \frac{\ell(p)}{m} \right]^{1/2} u^D(\lambda, p) \]  

(2.25)

satisfying the condition \( \bar{u}^{DC}(\lambda, p) u^{DC}(\lambda, p) = \lambda \). The Dirac boost for these spinors is

\[ B_D(v) u^{DC}(\lambda, p) w_\sigma = u^{DC}(\lambda, (p_b(p; \lambda v))) R(p; \lambda v) w_\sigma \]  

(2.26)

We recall that by using Dirac boosts and Dirac matrices one can construct Lorentz covariant operators. From standard algebra of the Dirac matrices one has

\[ B_D(v) \gamma^0 B_D(v) = \gamma^0 \]  

(2.27a)

and

\[ B_D(v) \gamma^0 \gamma^\mu B_D(v) = L^\mu(v) \gamma^0 \gamma^\nu \]  

(2.27b)

In consequence, by using eq.(2.26) and eq.(2.27a), one obtains

\[ w^+_\sigma R^+(p; \lambda v) \bar{u}^{DC}(\lambda, (p_b(p; \lambda v))) u^{DC}(\lambda', (p_b(p'; \lambda' v))) R(p'; \lambda' v) w_{\sigma'} = \]

\[ = w^+_\sigma \bar{u}^{DC}(\lambda, p) u^{DC}(\lambda', p') w_{\sigma'} \]  

(2.28a)

for the scalar matrix element.

Also, by using eq.(2.26) and eq.(2.27b), one has

\[ w^+_\sigma R^+(p; \lambda v) \bar{u}^{DC}(\lambda, (p_b(p; \lambda v))) \gamma^\mu u^{DC}(\lambda', (p_b(p'; \lambda' v))) R(p'; \lambda' v) w_{\sigma'} = \]

\[ = L^\mu(v) w^+_\sigma \bar{u}^{DC}(\lambda, p) \gamma^\nu u^{DC}(\lambda', p') w_{\sigma'} \]  

(2.28b)

for the vector matrix element. Similar equations hold for the other Dirac covariants, namely, the pseudoscalar, axial-vector and tensor matrix elements. Eqs.(2.28a,b), are very important for the construction of invariant interaction operators that will be done in the next section.
From eq.(2.26) one obtains the following useful expression for the spin rotation matrix

\[ R(p; v) = \bar{u}^{DC}(+, p_b(p; v))B(v)u^{DC}(+, p) \]  

(2.29)

From the previous expression, with standard Dirac algebra one also finds the following relation

\[ R^+(p; v) = R(p_b(p; v); -v) \] 

(2.30)

that will be used in the appendix to show the covariance of the interaction.

3. Systems of relativistic interacting particles

In this section we shall construct a Relativistic Hamiltonian Dynamical model for \( N \) interacting spin 1/2 particles also considering negative energy states. This objective will be achieved by defining the 10 total generators of the Poincaré group, denoted with the capital letters \([G^I]\), in terms of single particle operators. Obviously, the total generators must satisfy the same commutation rules given in eqs.(2.3a-g) for the single particle generators. If the interaction were not present, one could easily define the total generators as the sum of the single particle ones:

\[ G^I = \sum_{i=1}^{N} g_i^I \] 

(3.1)

automatically satisfying the commutation rules.

As anticipated in the introduction, the way in which the interaction is introduced makes the difference among various models of RHD. For this problem, that has been mainly faced considering positive energy states, different solutions have been proposed as explained in subsect.1.1.

In the present work we want to keep using standard Lorentz transformations of the four-vectors (canonical boosts), as given in eqs.(2.1 a-c) and (2.2a,b), so we can only choose IF RHD (i) or PF RHD (ii).

(i) As for the IF RHD, the interaction is added to the sum of the free Hamiltonians but not to the momenta. In consequence, an interaction operator
must be also added to the sum of the free boost operators in order to satisfy the Poincaré algebra commutation rules. This method, that is also adopted for the quantization of the relativistic field theories, has been widely used to introduce relativistic corrections [2] both to the Hamiltonians of bound systems and to the operators that describe the interaction of these systems with external electromagnetic fields, significantly improving, for the quark models, the reproduction of the experimental data [26], specially for the low energy observables.

The difficulty of this approach consists in finding the exact expression of the interaction dependent operator that modifies the boost generator. For this reason we do not follow this method in the present work.

(ii) In the PF RHD, the interaction modifies both the Hamiltonian and the total momentum of the system, leaving the boost free of the interaction [5-7]. As shown in the following, the form of the interacting four-momentum operator can be directly determined.

A relevant consequence of the properties of PF RHD is that it is possible to study the dynamics of the composite system in terms of explicitly covariant integro-differential wave equations that will be derived in the next section.

We revise the Point Form procedure considering the possibility of introducing also negative energy states.

Given a (bound) system of \(N\) interacting constituent particles, it is possible to observe this system both in its rest reference frame (RF) and in a generic reference frame (GF). It is convenient to introduce the observable quantity

\[ V^\mu = (V^0, \mathbf{V}) \]

that represents the four-velocity of the RF measured from a GF. It means that \(\mathbf{V}\) is the parameter that, inserted in eqs.(2.2a,b), allows to transform the momenta observed in the RF of the system to the corresponding quantities in the GF. The relation between \(V^0\) and \(\mathbf{V}\) is the same as in eq.(2.1d).

Furthermore, for a system of mass \(M\), considering \(V^\mu\) as a classical quantity, one has

\[ V^\mu = \left( \frac{E}{M}, \frac{\mathbf{P}}{M} \right) \]  \hspace{1cm} (3.2)

where \(E = \sqrt{\mathbf{P}^2 + M^2}\) and \(\mathbf{P}\) respectively represent its energy and three-momentum measured in a GF.
The procedure to construct the generators of the Poincaré group requires to define $V$ as an operator, that is as a dynamical variable of the system. This definition will be given in eq.(3.7). As first step we introduce the RF four-momentum of the $i$-th particle

$$p_i^*(\lambda_i) = (\lambda_i \epsilon(p_i^*), p_i^*)$$

(3.3)

where the asterisk denotes the quantities observed in the RF. The sum of these four-momenta over the $N$ constituents, by definition of the RF (that is also called zero momentum frame), is given by the following equation

$$\sum_{i=1}^{N} p_i^*(\lambda_i) = (\sum_{i=1}^{N} \lambda_i \epsilon(p_i^*) = M_F, 0)$$

(3.4)

where we have also introduced $M_F$ that represents the free mass operator of the system. By applying the Lorentz transformation of eq.(2.2b) as function of the parameter $V$ to the $p_i^*(\lambda_i)$, also using eq.(3.4), one can write the sum of the four-momenta of the particles in a GF as

$$\sum_{i=1}^{N} p_i^\mu(\lambda_i) = V^\mu M_F$$

(3.5)

If $M_F$ is nonvanishing one can solve the previous equation with respect to $V^\mu$; then by writing $M_F$ in terms of the $p_i^\mu(\lambda_i)$, one can express $V^\mu$ as a function the $p_i^\mu(\lambda_i)$, or, more precisely, of the $p_i$ and $\lambda_i$, that are chosen as dynamical variables of the relativistic model. The condition $M_F \neq 0$ is fulfilled by the states in which all the particles have the same energy sign, that is $\lambda_i = \Lambda$ for $i = 1, ..., N$. These states will be denoted as dynamical states. On the other hand, the states in which the particles do not have all the same energy sign can give a vanishing value of $M_F$, not allowing for a definition of $V^\mu$ in terms of the particle momenta. For this reason, these states will be treated separately as auxiliary states.

For the dynamical states the expression of $M_F$ as a function of the momenta in a GF is the following

$$M_F = M_F(\Lambda, \{p\}) = \Lambda \left[ \sum_{i,j=1}^{N} p_i^\mu(\Lambda) p_j^\nu(\Lambda) g_{\mu\nu} \right]^{\frac{1}{2}}$$

(3.6)

where we have introduced the collective shorthand notation $\{p\} = p_1, ..., p_N$. Analogously we introduce the notation $\{\lambda\} = \lambda_1, ..., \lambda_N$. For the dynamical
states one has $\{\lambda\} = \lambda_1 = \lambda_2 = \ldots = \lambda_N = \Lambda$. In consequence, we can also write

$$V^\mu(\{\lambda\} = \Lambda, \{p\}) = [M_F(\Lambda, \{p\})]^{-1} \sum_{i=1}^{N} p_i^\mu(\Lambda) \quad (3.7)$$

Let us note that the observable four-vector $V^\mu$, as given in the previous expression, for both $\Lambda = +1$ and $\Lambda = -1$, transforms in the same way as a positive energy four-momentum, that is replacing $p$ with $V$, $\epsilon(p)$ with $V^0(V)$ and setting $\lambda = +1$ in eq.(2.2b). In this way we introduce

$$V^0_b = V^0_b(V; v) = V^0_b(V_b(V; v)) \quad (3.8a)$$

$$V_b = V_b(V; v) \quad (3.8b)$$

This result, that is consistent with the definitions of eqs.(3.2a-c), can be easily derived by transforming, with the help of eq.(2.2b), the $p_\mu^i(\Lambda)$ that appear in eq.(3.7).

We shall now choose the complete set of commuting operators that will be used for the quantum-mechanical description of the system. To this aim we note that, due to its definition in eq.(3.7), the operator $V^\mu$ commutes with the momenta of all the particles. In consequence, it is possible to choose the following operators: the three-momenta of $N-1$ particles $p_1, \ldots, p_{N-1} = \{q\}$, the spatial components of the four-velocity $V$, the energy signs $\{\lambda\}$, and, finally, the spin projections on the z axis $\sigma_1, \ldots, \sigma_N = \{\sigma\}$.

With this set of commuting operators, the representation states that will be used to write down the wave functions of the model, are of the form

$$|\psi_r > = |\{q\}, V, \{\lambda\}, \{\sigma\} > \quad (3.9)$$

Their normalization is

$$<\psi_r | \psi'_r >= <\{q\}, V, \{\lambda\}, \{\sigma\} | \{q'\}, V', \{\lambda'\}, \{\sigma'\} > =$$

$$= \delta^3(p_1 - p'_1) \ldots \delta^3(p_{N-1} - p'_{N-1}) \delta^3(V - V') \delta(\sigma - \sigma') \delta(\lambda - \lambda') \quad (3.10)$$

We choose these representation states in order to derive in a simple way the manifestly covariant wave equation of the model and, in turn, to obtain a dynamically conserved current, as it will be studied in subsequent works.
A different type of representation states, currently denoted as *velocity states* can be advantageously used to study the relativistic bound state wave functions. In the velocity states the spatial variables are represented by $V$ and by the N (not independent) rest frame momenta $\{p^*\}$ or better by the N-1 (independent) Jacobi momenta $\{k\}$. As shown in ref. [5], the Lorentz transformation of these states is given by the standard boost of $V$, as in eq.(3.8b), and by a Wigner rotation for the $\{p^*\}$ or for the $\{k\}$.

In general, we point out that, in our relativistic model, as it will be shown in the following, the total three-momentum $P$ is interaction dependent, so it does not commute with the three-momenta of the constituent particles $p_i$. For this reason $P$ cannot be diagonalized simultaneously with them and, as discussed before, $V$ is conveniently chosen from the beginning.

We note that the momentum of the $N^{th}$ particle, when it appears in the calculations, can be expressed as function of $\{q\}$, $V$ and $\{\lambda\}$. To this aim, we firstly introduce the four-vector

$$Q^\mu = \sum_{i=1}^{N-1} p_i^\mu (\lambda_i) \quad (3.11)$$

We also recall the standard relation

$$\lambda_i \epsilon(p_i^*) = V_\mu p_i^\mu (\lambda_i) \quad (3.12)$$

Then, we write eq.(3.5) with the definition of $M_F$ given in eq.(3.4) in the form

$$Q^\mu = -p_N^\mu (\lambda_N) + V_\mu \sum_{i=1}^{N} \lambda_i \epsilon(p_i^*) \quad (3.13a)$$

Squaring both sides, with the help of eq. (3.12), we find

$$\epsilon(p_N^*) = \epsilon_N^* (\{\lambda\}, \{q\}, V) = \left[ (Q^\mu V_\mu)^2 + m^2 - Q^\mu Q_\mu \right]^{\frac{1}{2}} \quad (3.13b)$$

Then

$$M_F (\{\lambda\}, \{q\}, V) = V^\mu Q_\mu + \lambda_N \epsilon_N^* (\{\lambda\}, \{q\}, V) \quad (3.13c)$$

and finally, by means of eq.(3.5)

$$p_N^\mu (\{\lambda\}, \{q\}, V) = -Q^\mu + V^\mu M_F (\{\lambda\}, \{q\}, V) \quad (3.13d)$$
The previous expression can be also used for the auxiliary states. In fact, when $M_F(\{\lambda\}, \{q\}, V) = 0$, one has

\[ p^\mu_N(\{\lambda\}, \{q\}, V) = -Q^\mu \]

In the case of dynamical states, for the developments of the next section it is convenient to introduce

\[ M_F(\Lambda = -1, \{q\}, V) = -M_F(\Lambda = +1, \{q\}, V) = -\bar{M}_F(\{q\}, V) \quad (3.13e) \]

Explicit expressions for the positive free mass $\bar{M}_F(\{q\}, V)$ will be given in eqs.(4.8) and (4.16a,b) for the two and three-body case, respectively.

We can now take advantage of eq.(3.2) to define the total momentum operator as

\[ P^\mu = MV^\mu \quad (3.14a) \]

Here $M$ represents the invariant mass operator of the model, defined as

\[ M = M_F(\{\lambda\}, \{q\}, V) + W \quad (3.14b) \]

where $W$ represents the Lorentz invariant interaction operator. Our procedure for introducing the interaction represents the generalization of the Bakamjian-Thomas construction [5,27] to a theory with negative energy states. In order to obtain a Lorentz invariant operator, we require the following commutation rule of $W$ with the boost generator

\[ [W, T] = 0 \quad (3.15) \]

We also require

\[ [W, V] = 0 \quad (3.16) \]

From the previous equation, recalling that $V^0 = [V^2 + 1]^{1/2}$ one straightforwardly has

\[ [W, V^0] = 0 \quad (3.17) \]

ensuring that no commutation problem arises when defining the total momentum $P^\mu$ in eq.(3.14a). Furthermore, from the definition of eq.(3.14a) and the requirement of eqs.(3.16) and (3.17) one has

\[ [P^\mu, M] = 0 \quad (3.18) \]

ensuring the space and time translational invariance of the model.
In order to satisfy eqs. (3.15) and (3.16) we choose the interaction operator in the form

$$W = \sum_{\{\lambda\}/\{\lambda'\}} W_{\{\lambda\}/\{\lambda'\}}$$  \hspace{1cm} (3.19a)

with

$$W_{\{\lambda\}/\{\lambda'\}} = \sum_{\{\sigma\}/\{\sigma'\}} \int d^3\{q\} d^3\{q'\} d^3V F(\{q\}, \{q'\})$$

$$<\{q\}, V, \{\lambda\}, \{\sigma\}|W^C|\{q'\}, V, \{\lambda'\}, \{\sigma'\}>$$

$$|\{q\}, V, \{\lambda\}, \{\sigma\}> <\{q'\}, V, \{\lambda'\}, \{\sigma'\}]$$  \hspace{1cm} (3.19b)

where the integration \(d^3\{q\}\) symbolizes the integration over the momenta \(p_1, \ldots, p_{N-1}\) and analogously for the primed variables; \(F(\{q\}, \{q'\})\) is a function of the momenta, that will be explicitly given in eq. (3.20), that ensures the Lorentz covariance of the total operator. Also, we introduce the \textit{manifestly covariant interaction amplitude}, in the form

$$<\{q\}, V, \{\lambda\}, \{\sigma\}|W_C|\{q'\}, V, \{\lambda'\}, \{\sigma'\}> =$$

$$\sum_K \sum_{i>j=1}^N V^K_{ij}(\{q\}, \{q'\}, \{\lambda\}, \{\lambda'\}, V)$$

$$w^+_{\{\sigma\}} \bar{u}^{DC}(\{\lambda\}, \{p\}) \Gamma^K_{ij} \bar{u}^{DC}(\{\lambda'\}, \{p'\}) w_{\{\sigma'\}}$$  \hspace{1cm} (3.19c)

in the previous equation

$$V^K_{ij}(\{q\}, \{q'\}, \{\lambda\}, \{\lambda'\}, V)$$

is a \textit{Lorentz invariant} function, i.e. depending on the scalar products of the four-momenta, that expresses the spatial part of the two-body interaction of the model;

\(\Gamma^K_{ij}\) can represent:

a) the product of the covariant Dirac matrices (introduced in the previous section) for the particle i and j; more explicitly, for \(K = 1\) one has a \textit{scalar} interaction with \(I_i I_j\) , being \(I_i\) the identity Dirac matrix of the \(i^{th}\) particle; for \(K = 2\) one has a \textit{vector} interaction with \(\gamma^\nu \gamma^\mu g_{\mu\nu}\) and so on for the \textit{pseudoscalar}, \textit{axial-vector} and \textit{tensor} interactions; but also, for \(K > 5\)
b) terms containing Lorentz invariant products of the Dirac matrices with the four-momenta of the $N-1$ particles, or with $V^\mu$, that is $V_\mu \gamma^\mu_i$, as we shall see in the next section for a specific model.

Furthermore, the notation $u^{DC}(\{\lambda\}, \{p\})w(\sigma)$ represents the direct product of the Dirac spinors (see eq.(2.25)), for all the particles. Here and in the following, $p'_N$ and the corresponding primed quantity are given by eq.(3.13d).

The structure of eq.(3.19b) immediately shows that the interaction operator $W$ satisfies the commutation rule of eq.(3.16). In fact, the interaction amplitude of eq.(3.19c) depends on $V$ but the operator $W$ of eq.(3.19b) has vanishing matrix elements between states with different values of $V$. Furthermore, the Lorentz invariance of the interaction, that is the commutation of the interaction operator with the boost generator, expressed by eq.(3.15), is ensured by the invariance of the Dirac spinor matrix elements that appear in eq.(3.19c) taking, in eq.(3.19b), the function $F(\{q\}, \{q'\})$ in the form

$$F(\{q\}, \{q'\}) = \frac{\epsilon(\{q\})\epsilon(\{q'\})}{\epsilon(\{q\})\epsilon(\{q'\})^{-1/2}}$$

The details of the demonstration are given in the appendix. According to eq.(A.2), here and in the following subsection (in particular when considering the covariant wave equations) the three-momentum of each particle must be transformed according to its energy sign as in eq.(2.2b).

We have now all the elements to prove that the generators $[G^I]$ of our relativistic model satisfy the Poincare algebra commutation rules of eqs.(2.3a-f). The only generators that contain the interaction are $P^0 = H$ and $P$ defined by means of eqs.(3.14a,b). The other ones, being free of the interaction, are given by the sum of the single particle generators. For this reason eqs.(2.3b),(2.3d) and (2.3f) are automatically satisfied.

From the definition of $(P^0 = H, P)$ given in eqs.(3.14a,b) one immediately obtains the first two relations of eq.(2.3a).

From the rotational invariance of $V^0, M_F$ and $W$, one obtains the last relation of eq.(2.3a).

Taking also into account the vector character of $V$, one verifies eq.(2.3c).

Finally, considering the definition of $V$ in terms of single particle operators as given in eq.(3.7), with $[T, M_F] = 0$ and eq.(A.1) for the interaction term, one simply derives eqs.(2.3e,g) completing the verification of the Poincaré invariance of the model.
4. The wave equation of the model

In this section we study explicitly the eigenvalue wave equation of the model according to the properties discussed in the previous section. In the subsect. 4.1 we shall analyze the general structure of the equation, highlighting its manifest covariance, while in the subsect. 4.2, by using some techniques developed by RWE, we shall introduce the auxiliary states and the quadratic terms of the quasipotential for two and three-body systems.

4.1 General structure of the wave equation

The mass eigenvalue equation, by means of the mass operator definition of eq.(3.14b), may be written in the general form

\[ D(M, \{q\}, \Lambda, V)\Psi = W\Psi \quad (4.1a) \]

with

\[ D(M, \{q\}, \Lambda, V) = M - M_F(\{q\}, \Lambda, V) \quad (4.1b) \]

Here we use the free mass operator \( M_F \) introduced in eq.(3.13c) only for the dynamical states; the structure of the interaction operator \( W \) has been given in eqs.(3.19a-c) and (3.20); the operator \( D(M, \{q\}, \Lambda, V) \) has been introduced here only to simplify the comparison with RWE models.

Considering only dynamical states, one has

\[ |\Psi > = |\Psi, \Lambda = +1, V > + |\Psi, \Lambda = -1, V > \quad (4.2) \]

Projecting eq.(4.1) onto the states defined in eq.(3.9) gives the following set of coupled equations

\[ [M - \Lambda \bar{M}_F(\{q\}, V)]\Psi(\{q\}, V, \Lambda, \{\sigma\}) = \]

\[ \sum_{\Lambda'(\sigma')} \int d^3\{q'\} < \{q\}, V, \Lambda, \{\sigma\}|W|\{q'\}, V, \Lambda', \{\sigma'\} > \Psi(\{q'\}, V, \Lambda', \{\sigma'\}) \]

(4.3)

where also the definition of the positive free mass operator given in eq.(3.13e) has been used.
In order to find a numerical solution, the previous equation can be conveniently written in the RF of the bound system, then the obtained wave function is standardly boosted to any GF. In the RF the sum of the \( N \) three-momenta of the particles gives zero, as shown in eq.(3.4). This allows to introduce the \( N - 1 \) independent Jacobi momenta \( \{k\} \) that are used to study in a clear way the symmetries of the spatial part of the wave function. In this way eq.(4.3) takes the following form (for brevity, here and in the remainder of the paper, we do not write the RF eigenvalue \( V = 0 \)):

\[
[M - \Lambda \bar{M}_F(\{k\})] \Psi(\{k\}, \Lambda, \{\sigma\}) =
\sum_{\Lambda', \{\sigma'\}} \int d^3k' \langle \{k\}, \Lambda, \{\sigma\}|W|\{k'\}, \Lambda', \{\sigma'\} > \Psi(\{k'\}, \Lambda', \{\sigma'\}) \quad (4.4)
\]

We now turn to write eq.(4.3) in a manifestly covariant way. By introducing

\[
\Phi(\{q\}, V, \Lambda, \{\sigma\}) = [\epsilon(\{q\})]^{1/2} \Psi(\{q\}, V, \Lambda, \{\sigma\}) \quad (4.5)
\]

and with the definition of the manifestly covariant interaction amplitude of eq.(3.19c), eq.(4.3) is written as

\[
[M - \Lambda \bar{M}_F(\{q\}, V)] \Phi(\{q\}, V, \Lambda, \{\sigma\}) =
\sum_{\Lambda', \{\sigma'\}} \int \frac{d^3q'}{\epsilon(\{q'\})} \langle \{q\}, V, \Lambda, \{\sigma\}|W^C|\{q'\}, V, \Lambda', \{\sigma'\} > \Phi(\{q'\}, V, \Lambda', \{\sigma'\}) \quad (4.6)
\]

Note that the integrations over the \( N - 1 \) particle momenta is performed in a covariant way by means of the factor \( \epsilon(\{q'\}) \) in the denominator of the r.h.s. of the previous equation. A similar structure of covariant integration was firstly used in the RWE proposed by Gross [13].

We point out that, as manifest covariance explicitly shows, PF RHD allows to boost in an unambiguous way the wave function of the model for calculating physical observables.

The covariant interaction amplitude of eq.(4.6) should be determined in order to reproduce, with the best possible approximation, the dynamics of
the underlying field theory. In particular, the inclusion of the *z-graphs* will be performed in the next subsect.4.2 obtaining the amplitudes of $W_{\text{eff}}^C$ in eqs.(4.13b) and (4.17b) for the two and three-body case, respectively.

For completeness, note that the standard form of PF RHD could be recovered by completely excluding in eq.(4.6) the negative energy states, that is taking only the first term in eq.(4.2). Obviously, this choice does not violate the relativistic invariance of the model. In this way, as anticipated in the introduction, the so called Blackenbecler-Sugar wave equation [9] is obtained without difficulties and the Poincaré algebra commutation rules are satisfied by generators that only act onto positive energy states.

Another possible choice is to include the states with $\Lambda = -1$ and to use an interaction operator without *z-graphs*. It corresponds to the Breit equation.

In order to make a comparison with other relativistic models, we consider explicitly the case of a two-body system, that is particularly relevant in nuclear physics for the study of the deuteron. In this case the only Jacobi variable is the relative momentum $k = p_1^* = -p_2^*$ and eq.(4.4) for the RF takes the form

\[ [M - 2\Lambda \epsilon(k)]\Psi(k, \Lambda, \{\sigma\}) = \sum_{\Lambda'(\sigma')} \int d^3k' < k, \Lambda, \{\sigma\}|W|k', \Lambda', \{\sigma'\} > \Psi(k', \Lambda', \{\sigma'\}) \quad (4.7a) \]

The *manifestly covariant* form of the previous equation (to be used in a GF), derived directly from eq.(4.6), is

\[ [M - \Lambda \bar{M}_F(p_1, V)]\Phi(p_1, V, \Lambda, \{\sigma\}) = \sum_{\Lambda'(\sigma')} \int \frac{d^3p_1'}{\epsilon(p_1')} < p_1, V, \Lambda, \{\sigma\}|W^C|p_1', V, \Lambda', \{\sigma'\} > \Phi(p_1', V, \Lambda', \{\sigma'\}) \quad (4.7b) \]

where, by means of eqs.(3.13a-e), we can express the positive free mass operator as

\[ \bar{M}_F(p_1, V) = 2p_1^\mu V_\mu = 2\epsilon(k) \quad (4.8) \]

In eqs.(4.7a,b) the two particles have the same energy sign as in the Breit equation that is obtained by reducing to a three-dimensional form the Bethe-Salpeter equation in the case of a static interaction between the particles. In
particular, in ref. [14], the Coulomb interaction was considered to derive the Breit wave equation in the RF of the bound system. To reproduce that result in our model, the following covariant interaction amplitude must be inserted in eq.(4.7b)

\[
\begin{align*}
< p_1, V, \{ \lambda \}, \{ \sigma \}| W_{Br}^C | p'_1, V, \{ \lambda' \}, \{ \sigma' \}> &= \\
&= C_{Br}(\lambda_1, \lambda_2, \lambda'_1, \lambda'_2) U_{Br}(V, p_1, p'_1) \\
&\quad w^+_{\sigma_1} \bar{u}^{DC}(\lambda_1, p_1) w^+_{\sigma_2} \bar{u}^{DC}(\lambda_2, p_2) \\
&\quad (V_{\mu} \gamma^\mu_1)(V_{\nu} \gamma^\nu_2) u^{DC}(\lambda'_1, p'_1) w_{\sigma'_1} u^{DC}(\lambda'_2, p'_2) w_{\sigma'_2}
\end{align*}
\]

(4.9a)

with

\[
C_{Br}(\{ \lambda \}, \{ \lambda' \}) = \lambda_1 \delta_{\lambda_1, \lambda'_2} \delta_{\lambda'_1, \lambda'_2}
\]

(4.9b)

and the covariant function

\[
U_{Br}(V, p_1, p'_1, \{ \lambda \}, \{ \lambda' \}) = -m^2 e^2 \frac{e^2}{2\pi^2} \left[ |V_{\mu} p^\mu_1(\lambda_1) V_{\nu} p^\nu_1'(\lambda'_1)| \right]^{-1/2} \\
\left[ |V_{\mu} (p^\mu_1(\lambda_1) - p^\mu_1'(\lambda'_1))| \right]^2 - (p^\mu_1(\lambda_1) - p^\mu_1'(\lambda'_1))(p^\nu_1(\lambda_1) - p^\nu_1'(\lambda'_1)) g_{\mu\nu}\right]^{-1}
\]

(4.9c)

In the RF the two-body Breit equation is obtained in the standard form of eq.(4.7a) with

\[
< k, \Lambda, \{ \sigma \}| W_{Br} | k', \Lambda', \{ \sigma' \}> = \\
&= -m^2 e^2 \frac{e^2}{2\pi^2} C_{Br}(\{ \lambda \}, \{ \lambda' \}) \\
w^+_{\sigma_1} u^{D+}(\lambda_1, k) w^+_{\sigma_2} u^{D+}(\lambda_2, -k) \frac{1}{(k - k')^2} u^D(\lambda'_1, k') w_{\sigma'_1} u^D(\lambda'_2, -k') w_{\sigma'_2}
\]

(4.10)

Summarizing, we note that our procedure based on the commutation rules of the Poincaré algebra, naturally introduces the covariant general expressions of eqs.(4.6); eqs.(4.7a,b) are simply obtained by specializing eq.(4.6) for a two-body system. On the other hand, the form of the interaction is not determined by Poincaré algebra and must be chosen according to a specific dynamical model for the bound system. As an example, in the case of the Breit equation discussed above, the instantaneous approximation is
performed to sum up in a three-dimensional form the ladder, or uncrossed, Feynman graphs introduced in the Bethe-Salpeter equation.

As for the states with different energy signs, we can analyze in more detail the point shown in the previous section. For these states, the free mass operator $M_F(\{q\}, \{\lambda\}, V)$ can be, in general, vanishing and in particular, for two-body systems with $\{\lambda\} = (+, -)$ and $\{\lambda\} = (-, +)$ it is always vanishing, being $\epsilon(p^*_1) = \epsilon(-p^*_2) = \epsilon(k)$. In consequence, it is not possible to define, by means of eq.(3.7), the dynamical variable $V$ in terms of the three-momenta of the $N$ particles. It means that Poincaré invariance, in the form discussed in sect.3, only allows to introduce the dynamical states of eq.(4.2).

Such prohibition given by the Poincaré algebra naturally leads to the dynamical consequence of excluding an unphysical pole at $M = 0$ in the Green function of the model, avoiding the continuum dissolution or cockroach nest disease [11,12].

4.2 Auxiliary states and quadratic terms

We first consider the two-body case (i), then we generalize the model to the three-body case (ii).

(i) Two-body case.

The difficulty of the $M = 0$ pole was overcome in the Mandelzweig-Wallace (MW) model, by performing, for two particle bound systems, a three-dimensional reduction of crossed and uncrossed Feynman photon exchange graphs. This procedure, also denoted as equal time reduction, making use of the eikonal approximation for the $z$-graphs, is more accurate than the one of the Breit equation and correctly reproduces the scattering T matrix up to the terms of second order in the interaction operator [11,15,17].

One has a wave equation written by means of the inverse of the Green function that in the RF ($V = 0$) takes the form

$$D_{MW}(M, k, \{\lambda\}) = \delta_{\lambda_1, \lambda_2}M + d(\lambda_1, \lambda_2)2\epsilon(k)$$

with $d(+1, +1) = -1$ and $d(+1, -1) = d(-1, +1) = d(-1, -1) = +1$.

The MW equation can be used with different kinds of interaction operators. Without entering here into details, one can use an operator of the same type.
as that given in eqs.(4.9a-c), replacing $C_{Br}(\{\lambda\}, \{\lambda'\})$ with

$$C_{MW}(\{\lambda\}, \{\lambda'\}) = 1 - 2\delta_{\lambda_1\lambda_2}\delta_{\lambda'_1\lambda'_2}\delta_{\lambda_{1-1}\lambda'_{1-1}} \quad (4.12)$$

allowing for matrix elements with the states that have different energy signs. However, in that model, it is not possible to write the wave equation by defining the mass operator, as required by the Poincaré group commutation rules in the framework of the Relativistic Hamiltonian Dynamics.

Considering as starting point a two-body system, we propose to keep the definition of the mass operator given in the present work and to use eqs.(4.7a,b) as the dynamical wave equation of the model. We include the states with different energy signs, that represent the $z$-graphs, by means of a re-definition of the interaction operator and by the explicit introduction of the auxiliary states.

We introduce the following effective interaction operator that contains the $z$-graphs in the second quadratic term, in the form

$$< \mathbf{k}, \Lambda, \{\sigma\}|W_{eff}|\mathbf{k}', \Lambda', \{\sigma'\}> =$$

$$< \mathbf{k}, \Lambda, \{\sigma\}|W_{MW}|\mathbf{k}', \Lambda', \{\sigma'\}> +$$

$$\sum_{\{\bar{\lambda}'\}, \{\sigma''\}}\int d^3k'' < \mathbf{k}, \Lambda, \{\sigma\}|W_{MW}|\mathbf{k}'', \{\bar{\lambda}'\}, \{\sigma''\}> \frac{1}{2\epsilon(k'')} < \mathbf{k}'', \{\bar{\lambda}'\}, \{\sigma''\}|W_{MW}|\mathbf{k}', \Lambda', \{\sigma'\}> \quad (4.13a)$$

where, in the last term, the states with mixed energy signs are represented by $\{\bar{\lambda}'\} = (+1, -1), (-1, +1)$. The corresponding manifestly covariant amplitude is:

$$< \mathbf{p}_1, \mathbf{V}, \Lambda, \{\sigma\}|W_{eff}^C|\mathbf{p}_1', \mathbf{V}', \Lambda', \{\sigma'\}> =$$

$$< \mathbf{p}_1, \mathbf{V}, \Lambda, \{\sigma\}|W_{MW}^C|\mathbf{p}_1', \mathbf{V}, \Lambda', \{\sigma'\}> +$$

$$\sum_{\{\bar{\lambda}'\}, \{\sigma''\}}\int \frac{d^3p_1''}{\epsilon(p_1'')} < \mathbf{p}_1, \mathbf{V}, \Lambda, \{\sigma\}|W_{MW}^C|\mathbf{p}_1'', \mathbf{V}, \{\bar{\lambda}'\}, \{\sigma''\}>$$

$$[\bar{M}_F(p_1'', V)]^{-1} < \mathbf{p}_1'', \mathbf{V}, \{\bar{\lambda}'\}, \{\sigma''\}|W_{MW}^C|\mathbf{p}_1', \mathbf{V}, \Lambda', \{\sigma'\}> \quad (4.13b)$$
where we have used the factor of eq.(3.20) and we have expressed the denominator of eq.(4.13a) by means of eq.(4.8).

For the calculation of the bound state properties, the effective interaction matrix elements of eqs.(4.13a) and (4.13b) must be inserted in the eigenvalue equations eqs.(4.7a) and (4.7b), respectively.

Also, assuming that the states with equal energy signs represent the dominant contributions in the MW equation, we can define, in our model, the auxiliary states as

$$
\Psi(k, \{\bar{\lambda}\}, \{\sigma\}) = \frac{1}{2\epsilon(k)} \sum_{\Lambda', \{\sigma'\}} \int d^3k' < k, \{\bar{\lambda}\}, \{\sigma\}|W_{MW}|k', \Lambda', \{\sigma'\} > \Psi(k', \Lambda', \{\sigma'\}) \quad (4.14a)
$$

The corresponding covariant definition is

$$
\Phi(p_1, V, \{\bar{\lambda}\}, \{\sigma\}) = [\tilde{M}_F(p, V)]^{-1} \sum_{\Lambda', \{\sigma'\}} \int d^3p' < p_1, V, \{\bar{\lambda}\}, \{\sigma\}|W_{MW}^{c}|p_1', V, \Lambda', \{\sigma'\} > \Phi(p_1', V, \Lambda', \{\sigma'\}) \quad (4.14b)
$$

Considering a perturbative expansion up to the second order in the interaction, one has the same result that is given by the MW equation. In this sense, our model is dynamically equivalent to that by MW.

Furthermore, as explained in subsect.1.2, the explicit introduction of the auxiliary states, even if they do not participate directly in the dynamics, can be very important for the calculation of the electroweak interaction matrix elements of the bound system.

(ii) Three-body case.

We now propose the structure of a possible generalization of the model for a bound system with $N = 3$, that is relevant for baryonic quark models, including the states with mixed energy signs. The same procedure can be straightforwardly generalized to the case of $N > 3$ particles.

First, we specialize eqs.(4.4) and (4.6) to the case of a three-body system, obtaining in the RF

$$
[M - \Lambda \tilde{M}_F(p, p)]\Psi(p_\rho, p_\lambda, \Lambda, \{\sigma\}) =
$$
\[ \sum_{\Lambda'\{\sigma'\}} \int d^3 p'_\rho d^3 p'_\lambda < p_\rho, p_\lambda, \Lambda, \{\sigma\}|W| p'_\rho, p'_\lambda, \Lambda', \{\sigma'\} > \Psi(p'_\rho, p'_\lambda, \Lambda', \{\sigma'\}) \]  

(4.15a)

where we have introduced the standard RF three-body Jacobi momenta
\[ p_\rho = \sqrt{\frac{1}{2}} (p_1^* - p_2^*) \] and \[ p_\lambda = \sqrt{\frac{3}{2}} (p_1^* + p_2^*). \]

The manifestly covariant form of the previous equation (to be used in a GF), is derived directly from eq.(6.4) in the form
\[ [M - \Lambda \tilde{M}_F(p_1, p_2, V)] \Phi(p_1, p_2, V, \Lambda, \{\sigma\}) = \]
\[ \sum_{\Lambda'\{\sigma'\}} \int \frac{d^3 p'_1}{\epsilon(p'_1)} \frac{d^3 p'_2}{\epsilon(p'_2)} < p_1, p_2, V, \Lambda, \{\sigma\}|W^C| p'_1, p'_2, V, \Lambda', \{\sigma'\} > \Phi(p'_1, p'_2, V, \Lambda', \{\sigma'\}) \]

(4.15b)

The positive free mass operator in eq.(4.15) is
\[ \tilde{M}_F(p_\rho, p_\lambda) = \epsilon(p_1^*) + \epsilon(p_2^*) + \epsilon(p_3^*) = \]
\[ \left[ \left( \sqrt{\frac{1}{6}} p_\lambda + \sqrt{\frac{1}{2}} p_\rho \right)^2 + m^2 \right]^{1/2} + \left[ \left( \sqrt{\frac{1}{6}} p_\lambda - \sqrt{\frac{1}{2}} p_\rho \right)^2 + m^2 \right]^{1/2} \]
\[ + \left[ \frac{2}{3} p_\lambda^2 + m^2 \right]^{1/2} \]

(4.16a)

With the help of the eqs.(3.11), (3.12),(3.13b) and (3.13e), the positive free mass operator in eq.(4.15) is
\[ \tilde{M}_F(p_1, p_2, V) = (p_1^* + p_2^*) V_{\mu} \left[ \left((p_1^* + p_2^*) V_{\mu} \right)^2 + m^2 \right] + (p_1^* + p_2^*) (p_1^* + p_2^*) g_{\mu\nu} \right]^{1/2} \]

(4.16b)

For the effective interaction, in order to take into account the effects of the states with mixed energy signs, we use the following expression
\[ < p_\rho, p_\lambda, \Lambda, \{\sigma\}|W_{eff}| p'_\rho, p'_\lambda, \Lambda', \{\sigma'\} > = \]
\[ < p_\rho, p_\lambda, \Lambda, \{\sigma\}|W| p'_\rho, p'_\lambda, \Lambda', \{\sigma'\} > + \]
\[ \sum_{\{\lambda''\}\{\sigma''\}} \int d^3 p''_\rho d^3 p''_\lambda < p_\rho, p_\lambda, \Lambda, \{\sigma\}|W| p''_\rho, p''_\lambda, \lambda'', \{\sigma''\} > \]
The auxiliary states are expressed in general form as

$$\frac{1}{B(p^\prime, p^{\prime\prime}, \{\lambda''\})} < p^\prime, p^{\prime\prime}, \{\lambda''\}, \{\sigma''\}|W|p^\prime, p^{\prime\prime}, \lambda', \{\sigma'\}> \quad (4.17a)$$

For the corresponding covariant amplitude we have

$$< p_1, p_2, V, \lambda, \{\sigma\}|W_{eff}^C|p'_1, p'_2, V, \lambda', \{\sigma'\} =$$

$$< p_1, p_2, V, \lambda, \{\sigma\}|W^C|p'_1, p'_2, V, \lambda', \{\sigma'\} > +$$

$$\sum_{\{\lambda''\}\{\sigma''\}} \int \frac{d^3p''_1 d^3p''_2}{\epsilon(p''_1) \epsilon(p''_2)} < p_1, p_2, V, \lambda, \{\sigma\}|W^C|p''_1, p''_2, \{\lambda''\}, \{\sigma''\} >$$

$$\frac{1}{B(p''_1, p''_2, V, \{\lambda''\})} < p''_1, p''_2, V, \{\lambda''\}, \{\sigma''\}|W^C|p'_1, p'_2, V, \lambda', \{\sigma'\} > \quad (4.17b)$$

As in the two-body case the second quadratic term represents the z-graphs of the model. The auxiliary states are expressed in general form as

$$\Psi(p^\rho, p^\lambda, \{\lambda\}, \{\sigma\}) = \frac{1}{B(p^\rho, p^\lambda, \{\lambda\})} \sum_{\Lambda', \{\sigma'\}} \int d^3p^\rho d^3p^\lambda < p^\rho, p^\lambda, \{\lambda\}, \{\sigma\}|W|p^\rho, p^\lambda, \lambda', \{\sigma'\} > \Psi(p^\rho, p^\lambda, \lambda', \{\sigma'\})$$

$$\Phi(p_1, p_2, \{\lambda\}, \{\sigma\}) = \frac{1}{B(p_1, p_2, \{\lambda\})} \sum_{\Lambda', \{\sigma'\}} \int d^3p_1 d^3p_2 \epsilon(p_1) \epsilon(p_2) < p_1, p_2, \{\lambda\}, \{\sigma\}|W_1^C|p'_1, p'_2, \lambda', \{\sigma'\} > \Phi(p'_1, p'_2, \lambda', \{\sigma'\})$$

where the notation \{\lambda\} refers, as before, to the states with mixed energy signs. The specific form of the interaction amplitude should be fixed on the base of dynamical considerations that go beyond the scope of the present
work that essentially concerns Poincaré invariance. For the \textit{invariant, non-vanishing} quantity $B(p_\rho, p_\lambda, \{\bar{\lambda}\})$ or $B(p_1, p_2, V, \{\bar{\lambda}\})$ in a previous study devoted to three body systems \cite{28}, we proposed a model in which they take the form

\begin{equation}
B(p_\rho, p_\lambda, +1, +1, -1) = 2 \left[ \frac{2}{3} p_\lambda^2 + m^2 \right]^{1/2} \tag{4.19a}
\end{equation}

\begin{equation}
B(p_1, p_2, V, +1, +1, -1) = 2 \left[ \bar{M}_F(p_1, p_2, V) - V_\mu (p_{1\mu}^\mu + p_{2\mu}^\mu) \right] \tag{4.19b}
\end{equation}

and

\begin{equation}
B(p_\rho, p_\lambda, +1, -1, -1) = \left[ \left( \sqrt{\frac{1}{6}} p_\lambda + \sqrt{\frac{1}{2}} p_\rho \right)^2 + m^2 \right]^{1/2} - \bar{M}_F(p_\rho, p_\lambda) \tag{4.20a}
\end{equation}

\begin{equation}
B(p_1, p_2, V, +1, -1, -1) = V_\mu p_{1\mu}^\mu - \bar{M}_F(p_1, p_2, V) \tag{4.20b}
\end{equation}

The results for other values of $\{\bar{\lambda}\}$ are determined by symmetry reasons. This model consists in a generalization of the MW equation to three body systems that gives the correct nonrelativistic limit. Further investigations must be developed on the \textit{dynamics} of the model in order to determine the relevant Feynman graphs (and their three-dimensional reduction) for a three-body system.

We conclude remarking that Poincaré invariance, implemented by means of the Relativistic Hamiltonian Dynamics, offers a reliable framework to study in a covariant way the strongly interacting bound systems, also including negative energy states. The wave functions of the model can be exactly boosted to any reference frame. Manifestly covariant wave equations can be written avoiding some pathologies encountered in other models. Numerical investigations are required to study the effects of negative energy states on the electroweak form factors.

At a more fundamental level, it would be of great interest to study in detail the connection between the Relativistic Hamiltonian Dynamics and the formalism of an underlying field theory, that, for the quark models, is assumed to be the QCD.
Appendix

Instead of proving directly eq. (3.15), we study the finite boost transformation

$$B(v)WB^+(v) = W$$  \hspace{1cm} (A.1)

that, expanded at the first order in \(v\), is equivalent to eq. (3.15). Furthermore, in this way we also learn how the quantities that appear in the model are transformed under finite boost. The procedure can be summarized in the following steps.

(i) We write down in the next equation the explicit expression of the action of the boost operator onto the representation states of eq. (3.9). By considering the boost of eq. (2.15a) for a single particle, one obtains for the total boost

$$B(v)|\{q\}, V, \{\lambda\}, \{\sigma\} >= \left[ \frac{\epsilon(\{q_b\}) V_b^0}{\epsilon(\{q\}) V^0} \right]^{1/2}$$

$$R(\{p\}, \{\lambda\}; v)|\{q_b\}, V_b, \{\lambda\}, \{\sigma\} >$$  \hspace{1cm} (A.2)

In the previous expression, \(\{q_b\} = \{q_b\}(\{q\}, \{\lambda\}; v)\) denotes the set of the standardly boosted momenta for the particles \(1, ..., N - 1\) as functions of the unboosted momenta, of the energy signs and of the parameter of the Lorentz transformation, respectively; \(\epsilon(\{q_b\})\) represents the product of the corresponding boosted absolute values of the energies; recalling that \(V\) must be considered as a function of the particle momenta as shown in eq. (3.7), when a boost operator is applied to eigenstate of \(V\), one obtains an eigenstate of \(V^b = V_b(V; v)\). Its expression and that of \(V_b^0\) have been given in eqs. (3.8a,b); finally, \(R(\{p\}, \{\lambda\}; v)\) represents the product of the spin rotation matrices for all the particles.

(ii) We calculate \(B(v)WB^+(v)\) by using the operatorial relation of eq. (A.2) onto the interaction defined in eqs. (3.19a,b) and then replace the integration variables \(\{q\}\) with \(\{q_b\}\) and analogously for the primed momenta; we also replace \(V\) with \(V_b\). These change of variables introduce the following transformation factors

$$d^3\{q\} = \frac{\epsilon(\{q\})}{\epsilon(\{q_b\})} d^3\{q_b\}$$  \hspace{1cm} (A.3)
and the analogous one for the primed momenta; also

\[ d^3\mathbf{V} = \frac{V^0}{V_b^0} d^3\mathbf{V}_b \]  

(A.4)

At this point we re-define the integration variables eliminating the index \( b \) in \( \{q_b\}, \{q'_b\} \) and \( \mathbf{V}_b \), but now the arguments \( \{q\}, \{q'\} \) and \( \mathbf{V} \) appearing in eq.(3.19b) must be expressed in terms of inverse Lorentz transformations (obtained with the boost parameter \(-v\)) of the new integration variables. The result is

\[
B(v)W_{(\lambda)(\lambda')}B^+(v) = \sum_{\{\sigma\},\{\sigma'\}} \int d^3\{q\} d^3\{q'\} d^3\mathbf{V} \\
F(\{q^-_b\}, \{q'^-_b\}) \\
\sum_K \sum_{i>j=1}^N V^K_{ij}(\{q\}, \{q'\}, \{\lambda\}, \{\lambda'\}, \mathbf{V})
\]

\[
w^+_{\{\sigma\}}R_{DC}(\{\lambda\}, \{p^-_b\})\Gamma^K_{ij}R_{DC}(\{\lambda'\}, \{p'^-_b\})w_{\{\sigma'\}}
\]

\[
R(\{p^-_b\}, \{\lambda\}; -v)|\{q\}, \mathbf{V}, \{\lambda\}, \{\sigma\}> \\
< \{q'\}, \mathbf{V}, \{\lambda'\}, \{\sigma'\}|R^+(\{p'^-_b\}, \{\lambda'\}; -v)
\]

\[
\left[ \frac{\epsilon(\{q^-_b\})\epsilon(\{q'^-_b\})}{\epsilon(\{q\})\epsilon(\{q'\})} \right]^{1/2}
\]

(A.5)

where the shorthand notation \( \{p^-_b\} = \{p_b\}(\{p_b\}, \{\lambda\}; -v) \) and the analogous for the other momentum variables have been introduced.

(iii) By considering the sum over the complete set of the spin variables, the rotation operators can act on the spinors \( w_{\{\sigma'\}} \) and \( w^+_{\{\sigma\}} \). Then one uses for these operators the property of eq.(2.30) with the boost parameter equal to \(-v\) and the transformation of the Dirac covariant matrix elements of eqs.(2.28a) and (2.28b).

Finally, by using the function defined in eq.(3.20) one verifies eq.(A.1) for each \( W_{(\lambda)(\lambda')} \)
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