Chiral Anomaly in Ideal Fluid

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Abstract

Ideal or Eulerian fluid is shown to possess a chiral anomaly. The anomalous equations of motion (continuity and Euler or force equation) are derived in a Hamiltonian framework. The anomalous fluid variable algebra is generated from the extended Poisson brackets in phase space, with Berry curvature corrections.

1 Introduction

Fluid dynamics, one of the earliest disciplines in earth science is increasingly becoming relevant in topical theoretical physics. In very general terms, it provides a universal description of long wavelength physics that deals with low energy effective degrees of freedom of a field theory, classical or quantum. The success of fluid models rests on the reasonable assumption that at sufficiently high energy densities local equilibrium prevails in an interacting field theory so that the discrete Lagrangian system is smoothed out to a continuum Eulerian fluid model. The constitutive relations provide the map between the fundamental (Lagrangian) and the continuous (Eulerian) degrees of freedom. In the fluid picture we have the continuity equation and the Euler force equation. Hence there will be immediate non-trivial consequences if these equations are modified. One such possibility is the introduction of anomalies.

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Anomalies are a manifestation of short distance singularity of quantum field theory where not all symmetries (and corresponding conservation laws) of a classical field theory survive upon quantization. In particular the celebrated chiral anomaly, discovered by Adler [1] and by Bell and Jackiw [2] shows that although vector and axial vector current conservation laws in a massless fermionic theory simultaneously exist, one has to be broken in the quantum theory. The vector law is kept intact as this represents the electric charge conservation and the axial current conservation becomes anomalous. In fact this result explains the $\pi_0 \rightarrow \gamma + \gamma$ decay and constrains the Standard Model multiplet structure. However, very recently it has been observed by Son and Spivak [3] large classical negative magneto resistance of Weyl metals is connected to the triangle anomaly; but interestingly enough it occurs in the classical regime, characterized by short mean free path of the electron compared to the magnetic length. Berry curvature and induced magnetic field in momentum space is responsible for the anomalous velocity in quantum Hall effect [4]. Nonzero fluxes of the Berry curvature in electron Fermi surface plays an important role. In the present work we will show that classical Eulerian fluid can possess chiral anomaly in the presence of external electromagnetic field where Berry curvature plays a vital role. In earlier work [5] quantum anomalies for global currents in hydrodynamic limit and its novel consequences were revealed. Gauge anomalies present in in hydrodynamics were also studied in Hamiltonian formulation in [6].

Let us outline our work. The Eulerian fluid dynamics is generated by fluid variable Poisson brackets and a Hamiltonian where the fluid equations of motion are given Hamilton’s equations of motion. Following the review by Jackiw et. al. [7], we exploit the map between Lagrangian degrees of freedom $x_i, \dot{x}_i$ and Eulerian density and velocity field variables $\rho(x, t), v_i(x, t)$ respectively. The canonical fluid brackets are induced by the canonical Poisson brackets. However a non-canonical extension of the latter will naturally yield a generalized fluid algebra. Similar formalism was adopted in previous works [8][9] where a non-commutative generalized Poisson brackets between Lagrangian degrees of freedom was used to develop a non-commutative extension of fluid dynamics. In the present work we will exploit the well known non-canonical brackets appearing in [10][11][12].

The paper is divided in to the following sections: In Section 2 we provide derivation of the anomalous fluid model. The paper ends with a discussion of present results in Section 3.

2 Anomalous fluid dynamics

We will use the non-canonical bracket relations:

$$\{X^i(x), X^j(x')\} = \frac{1}{\rho_0} \epsilon^{ijk} \mathcal{F}_k \delta(x - x'); \quad \{X^i(x), \dot{X}^j(x')\} = \frac{\delta^i_j + eB^i\Omega_j}{\rho_0 A} \delta(x - x');$$
\[ \{\dot{X}^i(x), \dot{X}^j(x')\} = -\frac{\epsilon_{ijk}B^k}{\rho_0A} \delta(x - x') \quad (1) \]

where
\[ F_i = \frac{\Omega_i}{1 + eB_0}. \quad A = 1 + eB_0. \]

In the above, \( \rho_0 \) is a dimensionful numerical parameter, \( e \) is the electronic charge, \( B \) the external magnetic field and \( \Omega \) the Berry curvature term. For \( \Omega = 0, B = 0 \) one recovers the free canonical Poisson brackets. The semiclassical electron dynamics in a magnetic Bloch band, (generated in presence of a periodic potential and a magnetic field), is described by (1) in a Hamiltonian framework [10, 11, 12].

The map connecting the above Lagrangian degrees of freedom to the Euler variables is,
\[ \rho(r) = \rho_0 \int dx \, \delta(X(x) - r), \quad v_i(r) = \frac{\int dx \, \dot{X}_i(x) \delta(X(x) - r)}{\int dx \, \delta(X(x) - r)}. \quad (2) \]

It is now straightforward to construct the brackets between the fluid variables,
\[ \{\rho(r), \rho(r')\} = \epsilon_{ijk} \partial^r_i (\rho(r)F_k), \partial^r_j \delta(r - r') \quad (3) \]
\[ \{\rho(r), v_i(r')\} = \frac{\partial^r_i \delta(r - r')}{A(r')} + eB_j F_i(r') \partial^r_j \delta(r - r') + \epsilon^{ikj} F_i(r') \partial^r_j v_i \partial^r_k \delta(r - r') \quad (4) \]
\[ \{v_i(r), v_j(r')\} = \left\{ \frac{\partial_i v_j - \partial_j v_i}{\rho A} - \epsilon_{ijk} \frac{B_k}{\rho A} + \frac{eB_i}{\rho} \left( F_i \partial_l v_j - F_j \partial_l v_i \right) + 2 \epsilon^{km} v_i \frac{\partial_m v_j \partial_k (F_i \rho)}{\rho^2} \right. \]
\[ + \epsilon^{lmk} \frac{1}{\rho} \partial_m v_i \partial_k v_j F_l + 2 \epsilon^{km} \frac{\partial_k \rho}{\rho^2} \partial_l (v_i v_j) F_m \} \delta(r - r') - 2 \epsilon^{km} \frac{\partial_k (r - r')}{\rho} \partial_l^r (v_i v_j) F_m (r). \quad (5) \]

We emphasize that this set of generalized fluid algebra is different from the general structure derived from the results in [13].

**Ideal barotropic fluid:** Notice that for the ideal fluid \( B = \Omega = 0 \), the above reduces to a simple form
\[ \{\rho(r), \rho(r')\} = 0, \quad \{\rho(r), v_i(r')\} = \partial^r_i \delta(r - r'), \quad \{v_i(r), v_j(r')\} = \frac{(\partial_i v_j - \partial_j v_i)}{\rho} \delta(r - r'). \quad (6) \]

The structure \( \omega_{ij} = \partial_i v_j - \partial_j v_i \) is known as the (canonical) vorticity. Note that even if \( \omega_{ij} = 0 \), vorticity can be generated by the external interaction terms as seen in (5). Together with
the Hamiltonian for a barotropic fluid (with the pressure \( P = \rho(dV)/(d\rho) - V \) depending only on density \( \rho \)) in a non relativistic framework

\[
H_0 = \int d\mathbf{x} \left( \frac{1}{2} \rho \mathbf{v}^2 + V(\rho) \right)
\]

the continuity and Euler equations are obtained as

\[
\dot{\rho}(\mathbf{x}) = \{\rho(\mathbf{x}), H_0\} = -\nabla(\rho \mathbf{v}), \quad \dot{\mathbf{v}}(\mathbf{x}) = \{\mathbf{v}(\mathbf{x}), H_0\} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{\nabla P}{\rho}.
\]

**Anomalous fluid:** Returning to the full theory, the fluid Hamiltonian is now given by

\[
\mathcal{H} = \int d\mathbf{x} \left( \frac{1}{2} \rho \mathbf{v}^2 + V(\rho) - e\rho \Phi \right)
\]

where the third term is the energy density associated with background electric field satisfying \(-\nabla \Phi = \mathbf{E}\). The effect of magnetic field \( \mathbf{B} \) has already been taken into account through the extended symplectic bracket structure. The continuity equation gets modified to

\[
\dot{\rho} + \nabla \cdot \mathbf{J} = e\mathbf{E} \cdot (\nabla \times (\rho \mathcal{F}))
\]

where \( \mathbf{J} = (\frac{\rho \mathbf{v}}{\rho A}) + e\{\rho(\mathcal{F} \cdot \mathbf{v}) \mathbf{B} \} + \rho \frac{\nabla P}{\rho} \times \mathcal{F} + \rho \nabla \mathbf{v}^2 \times \mathcal{F} \). Notice that the term on the RHS of (10) can be written as \( \nabla.(e\Phi \nabla \times (\rho \mathcal{F})) \) such that a conserved current \( \rho, \tilde{\mathbf{J}} \) can be defined, which however will depend explicitly on \( \Phi \) and hence will not be gauge invariant. This is our first encounter with anomalous conservation law. Indeed, the anomalous term will comprise of electromagnetic field but an interesting generic feature to be noticed throughout is that the anomalous term will always include the Berry curvature term \( A \).

The Euler equation is derived as

\[
\dot{\mathbf{v}} + \frac{(\mathbf{v} \cdot \nabla)\mathbf{v}}{A} = -\nabla P \frac{\rho}{\rho A} - e\frac{\rho \mathbf{v} \times \mathbf{B}}{\rho A} - e\frac{\mathbf{B} \cdot \nabla P}{\rho} \mathcal{F} - e(\mathbf{v} \cdot \mathcal{F}) (\mathbf{B} \cdot \nabla)\mathbf{v} + \{\mathcal{F},(\nabla P \frac{\rho}{\rho} \times \nabla)\mathbf{v} - e\{\frac{\mathbf{E}}{A} + e(\mathbf{E} \cdot \mathbf{B}) \mathcal{F} + (\mathcal{F} \cdot (\mathbf{E} \times \nabla))\mathbf{v}\}
\]

The most interesting term in the RHS of (11) is obviously \( e^2(\mathbf{E} \cdot \mathbf{B}) \mathcal{F} \) since it has the classic chiral anomaly form.

The time derivative of energy density is given by

\[
\dot{\mathcal{E}} = (\frac{1}{2} \mathbf{v}^2 + V'(\rho))\dot{\rho} + \rho \mathbf{v} \dot{\mathbf{v}}
\]
An important property of fluid dynamics is the vorticity \( \omega = \nabla \times \mathbf{v} \) and the related pseudo-scalar quantity helicity \( \Sigma = \int d^3x \ h = \int d^3x \ \mathbf{v} \cdot \omega \). In Newtonian fluid helicity is conserved and satisfies a local conservation law. In the present scenario, the modified time-evolution for helicity can be expressed as

\[
\dot{h} = \dot{v} \cdot \omega + v \cdot \dot{\omega} = -\nabla \cdot (\mathbf{v} \times \dot{\mathbf{v}}) + 2\dot{\mathbf{v}} \cdot \omega.
\]

Ignoring the divergence term that will drop off upon space integration and using (11) we find

\[
\frac{\dot{h}}{2} = -e^2(\mathbf{E} \cdot \mathbf{B})(\omega \cdot \mathbf{F}) - e \left( \frac{\omega \cdot \mathbf{E}}{\mathbf{A}} \right) - e \left( \frac{\mathbf{B} \cdot \nabla P}{\rho} \right) (\omega \cdot \mathbf{F}) - e \left( \frac{\mathbf{v} \cdot (\nabla \times \mathbf{B})}{\mathbf{A}} \right) - e (\mathbf{v} \cdot \mathbf{F}) [\omega \cdot (\mathbf{B} \cdot \nabla) \mathbf{v}]
\]

\[
+ \Psi h + \omega \cdot [\mathbf{F} \cdot (\mathbf{C} \times \nabla)] \mathbf{v}
\]

where

\[
\Psi = -\nabla v^2 \cdot (\nabla \times \mathbf{F}) + \nabla v^2 \cdot (\mathbf{F} \times \frac{\nabla \rho}{\rho}) ; \quad \mathbf{C} = -\frac{\nabla P}{\rho} + \nabla v^2 - e\mathbf{E}.
\]

Let us look a little closely to the anomaly term in (14) for the conservation of helicity and consider some simple and restrictive cases; (i) Only the first three terms in the RHS are of \( O(v) \) with rest of the terms being of higher order in \( v \). Hence in a low energy case these terms can be ignored.

(ii) We can consider pressure \( P \) to be absent.

(iii) Rewriting \( \omega \cdot \mathbf{E} = (\nabla \times \mathbf{v}) \cdot \mathbf{E} = \nabla \cdot (\mathbf{v} \times \mathbf{E}) + \mathbf{v} \cdot (\nabla \times \mathbf{E}) \) we can consider time independent external \( \mathbf{E}, \mathbf{B} \) configurations so that \( \nabla \times \mathbf{E} = 0 \) and hence the second term also will not contribute. Thus we are left with the chiral anomaly term

\[
\dot{h} = -2e^2(\mathbf{E} \cdot \mathbf{B})(\omega \cdot \mathbf{F}).
\]

This is the cherished form of anomaly in Eulerian fluid (in low energy limit) and constitute our principal result. Probably a better option for isolating the \( e^2(\mathbf{E} \cdot \mathbf{B})(\omega \cdot \mathbf{F}) \) is to modify the canonical definition of helicity by adding extra terms such that only the above term
survive in the modified anomaly equation.

3 Discussion

In this paper, we have developed an extended classical fluid model incorporated with Berry phase effects that generates an $e^2 E B$ form of chiral anomaly contribution. The approach is semiclassical where we have based our fluid variables bracket structure on Poisson brackets augmented with (quantum mechanical) Berry phase effect. The latter is applicable for electrons moving in magnetic Bloch bands. Hence our anomalous fluid model can have relevance in hydrodynamic equations describing electron gas models subject to spin-orbit-like interactions in condensed matter systems, such as graphene [14].

Previously, $O(h^2)$ corrections were introduced in classical fluid equations [15] from a moment expansion of the Wigner-Boltzmann equation. Interestingly in the present work, the Berry curvature plays an essential role in inducing $O(h)$ correction. This can be seen by comparing a classical model Lagrangian

$$ L = \frac{1}{2} m v^2 - e\Phi + eA \cdot v \quad (16) $$

with $e$ dimensionless and $[B] = \frac{M}{L}$, $[E] = \frac{ML^2}{t}$ and the definitions of Lorentz force and electromagnetic fields respectively

$$ F = \nu(E + v \times B), \quad E = -\nabla\Phi - \frac{\partial A}{\partial t}, \quad B = \nabla \times A $$

with a quantum Lagrangian

$$ L = \hbar k \dot{\mathbf{r}} - e\mathbf{v} \times \mathbf{A} + \hbar \dot{k} A_\beta + e\Phi - W \quad (17) $$

$[\hbar k] = \frac{ML}{t}$, $[A_B] = L$, $[\mathcal{F}] = \frac{I}{ML^2} = \frac{I}{M}$ with $[B, \Omega]$ being dimensionless. In the above $A_B$ is the Berry potential and $\Omega = \nabla_p \times A_B$ the Berry curvature. Coming back to the fluid variables, its dimensions are $[\rho] = \frac{M}{L^3}$, $[v] = \frac{L}{t}$.

After completing our work we came across a paper by Abanov and Wiegmann [16] that has also shown the presence of chiral anomaly in a generalized helicity conservation equation. However the framework is entirely different from ours and this is reflected in the difference between the explicit anomaly equations. In [16] the anomaly appears to be induced as a non-inertial effect. On the other hand we have followed throughout a systematic Hamiltonian approach from first principles, starting from a semi-classical Poisson algebra with Berry phase corrections that induces a generalized (anomalous) fluid algebra. Subsequently the Hamiltonian equations of motion yields the generalized continuity and
Euler equations leading to the anomaly.

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