Towards Bin Packing (preliminary problem survey, models with multiset estimates) *

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The paper described a generalized integrated glance to bin packing problems including a brief literature survey and some new problem formulations for the cases of multiset estimates of items. A new systemic viewpoint to bin packing problems is suggested: (a) basic element sets (item set, bin set, item subset assigned to bin), (b) binary relation over the sets: relation over item set as compatibility, precedence, dominance; relation over items and bins (i.e., correspondence of items to bins). A special attention is targeted to the following versions of bin packing problems: (a) problem with multiset estimates of items, (b) problem with colored items (and some close problems). Applied examples of bin packing problems are considered: (i) planning in paper industry (framework of combinatorial problems), (ii) selection of information messages, (iii) packing of messages/information packages in WiMAX communication system (brief description).

Keywords: combinatorial optimization, bin-packing problems, solving frameworks, heuristics, multiset estimates, application

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1. Introduction

Bin-packing problem is one of the well-known basic combinatorial optimization problems (e.g., 40,70,120,203). The problem is a special case of one-dimensional “cutting-stock” problem 123,261 and the “assembly-line balancing” problem 69. Fig. 1 illustrates the relationship of one-dimensional bin-packing problems and some other combinatorial optimization problems.

The bin packing problem can be described as follows (Fig. 2). Initial information involves the following: (i) a set of items \( A = \{a_1, ..., a_i, ..., a_n\} \), each item \( a_i \) has a weight \( w_i \in (0, 1] \); (ii) a set of bins (or one-dimensional containers, blocks) \( B = \{B_1, ..., B_\kappa, ..., B_m\} \), capacity of each bin \( B_\kappa \) equals 1. The basic (classical) bin packing problem is (e.g., 150,151,152,259):

Find a partition of the items such that: (a) each part of the item set is packed into the same bin while taking into account the bin capacity constraint (i.e., the sum of packed items in each bin \( \leq 1 \)), (b) the total number of used bins is minimized.

This problem is one of basic NP-hard combinatorial optimization problems (e.g., 120,157). Fig. 2 illustrates the classic bin packing (i.e., packing the items into the minimal number of bins): 6 items are packed into 3 bins.

Note the following basic types of items are examined (e.g., 19,24,25,30,67,89,90,115,117,120,193,196,203,261,227,243,265): rectangular items, 2D items, irregular shape items, variable sizes items, composite 2D items (including items with common components), 3D items, multidimensional items, items as cylinders, items as circles, etc. A generalized illustration for bin packing problem is depicted in Fig. 3.
Further, it is necessary to point out binary relations:

I. Binary relations over initial items and bins (items $A = \{a_1, a_2, ..., a_n\}$, bins $B = \{B_1, ..., B_k, ..., B_m\}$):

1.1. correspondence of items to bins or preference (for each item) as binary relation (or weighted binary relation): $R_{A \times B}$.

II. Binary relation over items:

2.1. conflicts as a binary relations for item pairs that can not be assigned into the same bin: $R^{conf|}_{A \times A}$ (this can be considered as a part of the next relation),

2.2. compatibility (e.g., by type/color) as binary relation for items which are compatible (e.g., for assignment to the same bin, to be neighbor in the same bin): $R^{compt}_{L \times L}$, here a weighted binary relation can be useful (e.g., for colors, including non-symmetric binary relation for neighborhood),

2.3. compatibility (e.g., by common components, for multi-component items), close to previous case (this may be crucial for “intersection” of items): $R^{compt-\neg \text{cons}}_{A \times A}$,

2.4. precedence over items (this is important in the case of ordering of items which are assigned into the same bin): $R^{prec|}_{A \times A}$.

2.5. importance (dominance, preference) of items from the viewpoint of the first assignment to bins, as a linear ordering or poset-like structure over items: $G(A, E^{\text{dom}})$ (the poset-like structure may be based on multicriteria estimates or multiset estimates of items).

III. Binary relations over bins:

3.1. importance of bins from the viewpoint of the first usage, as a linear ordering or poset-like structure) over bins: $G(B, E^{\text{imp}})$ (the poset-like structure may be based on multicriteria estimates or multiset estimates of bins).

Numerical examples of the above-mentioned relations are present as follows (on the basis of example from Fig. 2: six items and four bins): (i) correspondence of items to bins $R_{A \times B}$ (Table 1), (ii) relation of item conflict $R^{conf|}_{A \times A}$ (Table 2), (iii) relation of item compatibility $R^{compt}_{A \times A}$ (e.g., by type/color) (Table 3), (iv) precedence relation over items $R^{prec|}_{A \times A}$ (Fig. 4), (v) (relation of dominance over items $G(A, E^{\text{dom}})$ (Fig. 5), and (vi) relation of importance over bins $G(B, E^{\text{imp}})$ (Fig. 6).

Further, the solution of the bin packing problem can be examined as the following (i.e., assignment of items into bins, a Boolean matrix): $S = \{A^1, ..., A^\kappa, ..., A^k, ..., A^m\}$ where $|A^\kappa \cap A^{\kappa+1}| = 0 \ \forall \kappa_1, \kappa_2 = 1, m$ (i.e., the intersection is empty), $A = \bigcup_{\kappa=1}^{m} A^\kappa$.

For classic bin packing problem (i.e., minimization of used bins), $|A^\kappa| = 0 \ \forall \kappa = k + 1, m$ (the first $k$ bins are used) and $A = \bigcup_{\kappa=1}^{m} A^\kappa$.

In inverse bin packing problem (maximization of assigned items into the limited number of bins), a part of the most important items are assigned into $m$ bins: $\bigcup_{\kappa=1}^{m} A^\kappa \subseteq A$.

Additional requirements to packing solutions are the following (i.e., fulfillment of the constraints):

1. **Correspondence of item to bin.** The following has to be satisfied: $a_i \in A_{\kappa}$ if $(a_i, B_{\kappa}) \in R_{A \times B}$.

2. **Importance/dominate of items.** This corresponds to inverse problem: If $(a_i, a_j) \in R^{\text{dom}}_{A \times A}$ (i.e., $a_i \ge a_j$) Then three cases are correct: (a) both $a_i$ and $a_j$ are assigned to bin(s), (b) both $a_i$ and $a_j$ are not assigned to bin(s), (c) $a_i$ is assigned to bin and $a_j$ is not assigned to bin.

3. **Item precedence.** In the case of precedence constraint(s) according the above-mentioned precedence relations over items $R^{\text{prec}}_{A \times A}$, the items have to be linear ordered in each bin (for each bin, i.e., $\forall \kappa$):
If \((a_{i_1}, a_{i_2}) \in R_{A \times A}^{\text{prec}}\) and \(a_{i_1}, a_{i_2} \in A\), then \(a_{i_1} \rightarrow a_{i_2}\).

4. **Item conflicts.** In the case of conflict constraints, the following has to be satisfied:
\[a_{i_1}, a_{i_2} \in A\] If \((a_{i_1}, a_{i_2}) \in R_{A \times A}^{\text{conf}}\).

5. **Item compatibility.** In the case of compatibility constraints, the following has to be satisfied:
\[a_{i_1}, a_{i_2} \in A\] If \((a_{i_1}, a_{i_2}) \in R_{A \times A}^{\text{comp}}\).

In general, it is possible to use some penalty functions in the cases when the constraints are not satisfied.

### Table 1. Correspondence of items to bins \(R_{L \times B}^{L \times B}\)

| Item \(a_i\) | Bin \(B_1\) | Bin \(B_2\) | Bin \(B_3\) | Bin \(B_4\) |
|------------|-----------|-----------|-----------|-----------|
| \(a_1\)    | 3         | 2         | 1         | 0         |
| \(a_2\)    | 3         | 1         | 0         | 0         |
| \(a_3\)    | 1         | 3         | 2         | 0         |
| \(a_4\)    | 3         | 2         | 2         | 0         |
| \(a_5\)    | 1         | 3         | 1         | 1         |
| \(a_6\)    | 2         | 3         | 3         | 1         |

### Table 2. Relation on item conflict \(R_{A \times A}^{\text{conf}}\)

| Item \(a_i\) \(\lor\) Item \(a_j\) | \(a_1\) | \(a_2\) | \(a_3\) | \(a_4\) | \(a_5\) | \(a_6\) |
|-----------------|--------|--------|--------|--------|--------|--------|
| \(a_1\)         | \(*\)  | 1      | 1      | 1      | 0      | 0      |
| \(a_2\)         | 1      | \(*\)  | 1      | 1      | 1      | 0      |
| \(a_3\)         | 1      | 3      | \(*\)  | 4      | 1      | 0      |
| \(a_4\)         | 1      | 1      | 1      | \(*\)  | 1      | 0      |
| \(a_5\)         | 1      | 1      | 1      | 1      | \(*\)  | 0      |
| \(a_6\)         | 1      | 0      | 0      | 0      | 0      | \(*\)  |

### Table 3. Relation on item compatibility \(R_{A \times A}^{\text{comp}}\)

| Item \(a_i\) \(\lor\) Item \(a_j\) | \(a_1\) | \(a_2\) | \(a_3\) | \(a_4\) | \(a_5\) | \(a_6\) |
|-----------------|--------|--------|--------|--------|--------|--------|
| \(a_1\)         | \(*\)  | 1      | 1      | 1      | 0      | 0      |
| \(a_2\)         | 1      | \(*\)  | 1      | 1      | 1      | 0      |
| \(a_3\)         | 1      | 3      | \(*\)  | 4      | 1      | 0      |
| \(a_4\)         | 1      | 1      | 1      | \(*\)  | 1      | 0      |
| \(a_5\)         | 1      | 1      | 1      | 1      | \(*\)  | 0      |
| \(a_6\)         | 1      | 0      | 0      | 0      | 0      | \(*\)  |

Numerous publications have already addressed and analyzed various versions of static and dynamic bin packing problems. Many surveys on BPPs have been published (e.g., \([23,64,65,66,67,68,81,116,120,195,203,252]\)). The basic taxonomies/typologies of bin packing problems have been examined in \([67,89,90,193,265]\).

Some basic versions of bin packing problems (BPPs) are listed in Table 4 and special classes of bin packing problems (e.g., with relations among items) are pointed out in Table 5. Table 6 contains a list of main applications of bin packing problems.
Many surveys on algorithms for bin packing problems have been published (e.g., \[23,63,66,68,81,101,114,138,150,203,231,255\]). Basic algorithmic approaches are listed in Table 7, Table 8, and Table 9.

### Table 4. Main bin packing problem formulations

| No. | Problem                                                                 | Some source(s) |
|-----|-------------------------------------------------------------------------|----------------|
| I.  | Basic bin packing problems:                                            |                |
| 1.1 | Classic one-dimensional bin-packing                                    | \[150,151,152,259\] |
| 1.2 | Bin-packing with discrete item sizes                                   | 62,64          |
| 1.3 | Linear programming formulation                                         | 261            |
| 1.4 | Variable sized bin packing                                             | \[37,72,111,115,155,215,241,243\] |
| 1.5 | Maximum resource bin packing problem                                   | 36             |
| 1.6 | Bin packing with cardinality (maximization, constraints)               | \[81,172,221\] |
| 1.7 | Unrestricted black and white bin packing                               | 14             |
| 1.8 | Bin packing with rejection (including variable sized)                  | 21,78,96       |
| 1.9 | Bin packing with item fragmentation                                    | 45             |
| 1.10| Bin packing games                                                      | 160,170        |
| II. | Multidimensional bin packing problems:                                 |                |
| 2.1 | 2D bin-packing                                                         | \[5,47,77,81,120,139,140\] |
| 2.2 | Oriented 2D bin packing                                                | 192,193        |
| 2.3 | Orthogonal 2D bin packing                                              | 111,178        |
| 2.4 | 2D bin-packing with variable sizes (and costs)                        | 54,157,227     |
| 2.5 | 2D bin packing with due dates                                          | 26             |
| 2.6 | 2D bin-packing with guillotine constraints                             | 47,131,134     |
| 2.7 | 2D irregular shape bin packing                                         | 151,131        |
| 2.8 | 3D bin packing                                                         | \[79,86,201,204,205,254\] |
| 2.9 | Multi-dimensional bin-packing (vector packing)                         | \[18,19,74,117,196\] |
| III. | Online and dynamic bin packing problems:                               |                |
| 3.1 | Online bin packing                                                     | \[15,65,95,118,212,243,264\] |
| 3.2 | Online bin packing with two item sizes                                 | 97,129         |
| 3.3 | Online bin packing with advise                                         | 38             |
| 3.4 | Online variable-sized bin packing                                      | \[72,102,270\] |
| 3.5 | On-line bin-packing with cardinality constraints                       | 8              |
| 3.6 | Dynamic bin packing problems                                           | 61             |
| 3.7 | Bin packing with controllable item sizes                               | 71             |
| 3.8 | Batched bin packing                                                    | 128            |

| IV.  | Dual/inverse bin packing problems:                                     |                |
| 4.1  | Inverse problem (maximizing the number of packed items, maximum cardinality bin packing) | \[73,115,58,52,171,172,221\] |
| 4.2  | Dual bin packing with items of random sizes                            | 232            |
| 4.3  | On-line dual bin packing                                               | 35,73          |
| 4.4  | On-line variable-sized dual problem                                    | 94             |
| 4.5  | “Maximization” of total preference estimate for packed items (preference relation over item set) | 115 |
| 4.6  | Inverse bin packing with multiset estimates                            | This paper     |

| V.  | Multi-stage bin packing problems:                                      |                |
| 5.1 | Three-stage two-dimensional bin packing                                | 229            |
| 5.2 | Multi-stage bin packing                                                |                |

| VI.  | Bin packing problems in game theory perspective:                      |                |
| 6.1  | Bin-packing of selfish items                                          | \[29,100,102,132,163,212,274\] |
| 6.2  | Generalized selfish bin-packing                                       | 82             |
Table 5. Bin packing problems with multi-component items, with binary relations

| No. | Problem                                                                 | Some source(s) |
|-----|--------------------------------------------------------------------------|----------------|
| I.  | Problems with multi-component items/items fragmentation:                |                |
| 1.1 | Bin packing with multi-component items                                  | 115            |
| 1.2 | Packing with item fragmentation                                         | 215            |
| 1.3 | Packet scheduling with fragmentation                                    | 216            |
| II. | Colored bin packing:                                                    |                |
| 2.1 | Offline black and white bin packing                                      | 14             |
| 2.2 | Online black and white bin packing                                       | 15             |
| 2.3 | Colored bin packing                                                      | 51, 257        |
| 2.4 | Offline colored bin packing                                              | 257            |
| 2.5 | Online colored bin packing                                               | 31, 257        |
| 2.6 | Online bin coloring (packing with minimum colors)                       | 167            |
| 2.7 | Composite planning framework in paper production system                 | This paper     |
| III. | Multicriteria/multobjective bin packing, relations over items:          |                |
| 3.1 | Bin packing with conflicts                                              | 121, 98, 107, 149, 237 |
| 3.2 | Bin packing with multicriteria items                                     | 115            |
| 3.3 | Multi-objective bin packing                                              | 189, 217       |
| 3.4 | Multi-objective bin packing with rotations                              | 108            |
| 3.5 | Problems with preference over items                                     | 115            |
| 3.6 | Problems with precedence among items                                     | 80, 239, 222   |

A general classification scheme for bin packing problems has been suggested in [67]:

| arena | objective function | algorithm class | results | constraints |
|-------|--------------------|-----------------|---------|-------------|

where the scheme components are as follows: (a) arena describes types of bins (e.g., sizes, etc), (b) objective function describes types of problem (i.e., minimum of bin, minimum of “makespan”, etc.), (c) algorithm class describes types of algorithm (e.g., offline, online, complexity estimate, greedy-type, etc.), (d) constraints describes quality of solution, e.g., asymptotic worst case ratios, absolute worst case, average case, etc., (e) constraints describes bounds on item sizes, bound on the number of items which can be packed in a bin, binary relation over item set (e.g., items $a_1$ and $a_2$ can not be put into the same bin), etc.

Fig. 7 illustrates the basic trends in modifications of bin packing problems: (1) multicriteria (multi-objective) bin packing, (2) bin packing problems under uncertainty (e.g., fuzzy set usage of estimates), (3) examination of additional relations over items and over bins, (4) dynamic bin packing.

This paper addresses the bin packing problem survey and some new formulations of bin packing problems: (a) with relations over item set, (b) with multiset estimates of items.
Table 6. Some applications of bin packing problems

| No. | Domain(s)/Problem(s)                                                                 | Some source(s) |
|-----|-------------------------------------------------------------------------------------|----------------|
| I.  | Basic applications:                                                                 |                |
| 1.1 | Table formatting                                                                    | 152            |
| 1.2 | Prepaging                                                                           | 152            |
| 1.3 | File allocation, storage allocation                                                 | 60, 152, 259   |
| 1.4 | Processor allocation                                                                | 60, 149        |
| 1.5 | Multi-processor scheduling                                                          | 59, 60, 69, 143, 264 |
| 1.6 | Examination timetabling                                                             | 174            |
| II. | Industrial applications:                                                             |                |
| 2.1 | Packing systems in industry                                                          | 139, 267       |
| 2.2 | Liquid loading problem                                                               | 50             |
| 2.3 | Assembly line balancing                                                              | 222            |
| 2.4 | Filling up containers                                                                | 120            |
| 2.5 | Loading tracks with weights capacity constraints                                     | 120            |
| 2.6 | Vehicle container loading problem                                                    | 91, 130, 149   |
| 2.7 | Loading of tractor trailer trucks                                                   | 189            |
| 2.8 | Loading of cargo airplanes                                                           | 189            |
| 2.9 | Loading of containers into ships                                                    | 125, 136, 161, 189 |
| 2.10| Packing in design automation                                                        | 79, 254        |
| 2.11| Delivery problem                                                                    | 203            |
| 2.12| Configuration of support tools for satellite mission                                 | 115            |
| III. | Applications in distributed computing:                                              |                |
| 3.1 | Assignment of processes to processors                                               | 149            |
| 3.2 | Allocating jobs in distributed computing systems (grids, etc.)                      | 257            |
| 3.3 | Data placement on parallel discs                                                    | 124, 1156     |
| 3.4 | Dynamic resource allocation in cloud data centers                                   | 270            |
| 3.5 | Periodic task scheduling in real-time distributional control systems               | 277            |
|                               | (e.g., automobile electronic control system, satellite control system,               |                |
|                               | medical equipment’s electronic control system)                                     |                |
| IV. | Applications in networking:                                                         |                |
| 4.1 | Routing and wavelength assignment in optical networks                               | 250            |
| 4.2 | Bandwidth allocation (e.g., channel assignment)                                     | 71             |
| 4.3 | Video-on-demand systems                                                             | 271            |
| 4.4 | Creating file backups in media                                                      | 120            |
| 4.5 | Allocating files in P2P networks                                                    | 257            |
| 4.6 | Packet scheduling with fragmentation                                                | 216            |
| 4.7 | Selection of messages/packages in communication system                               | This paper     |
| 4.8 | 2D packing for mobile WiMAX (e.g., data location in IEEE 802.16/OFDMA)              | 55, 66, 57, 197, 206 |
| 4.9 | Resource allocation in multispot MFTDMA satellite networks                          | 4              |
| V.  | Some contemporary applications:                                                     |                |
| 5.1 | Configuration of maintenance devices for satellite mission                          | 115            |
| 5.2 | Balanced combinatorial cooperative games                                             | 103, 104, 1268 |
| 5.3 | Technology mapping in field-programmable gate array                                 | 120            |
| 5.4 | Production scheduling                                                               | 203, 139       |
### Table 7. Main algorithmic approaches, part I: basic methods

| No. | Solving approach | Some source(s) |
|-----|------------------|----------------|
| I.  | Fitting algorithms (i.e., classical ones) and their combinations: |               |
| 1.1 | Next Fit (NF) algorithm | 63, 67, 68, 120 |
| 1.2 | Next-fit (NFD) decreasing algorithm | 277 |
| 1.3 | First-Fit (FF) (on-line) | 12, 84 |
| 1.4 | First-Fit decreasing (FFD) (off-line) | 12, 83, 248 |
| 1.5 | Best-Fit (BF) (on-line) | 26, 84, 277 |
| 1.6 | Best-Fit decreasing (BFD) algorithm (off-line) | 248, 277 |
| 1.7 | Worst Fit (WF) algorithm (makespan context) | 67 |
| 1.8 | Worst Fit decreasing (makespan context) | 67 |
| 1.9 | Shelf algorithms (for 2D bin packing problems) | 13, 75 |

| II. | Exact enumerative methods: |               |
| 2.1 | Surveys | 81 |
| 2.2 | Branch-and-bound algorithms | 91, 171, 214, 222, 240, 260 |
| 2.3 | Branch-and-price algorithms | 221, 262, 263 |
| 2.4 | Exact column generation and branch-and-bound method | 260, 263 |
| 2.5 | Bin completion algorithm (bin-oriented branch-and-bound strategy) | 114 |

| III. | Basic approximation algorithms: |               |
| 3.1 | Surveys | 63, 68, 81, 1234 |
| 3.2 | Near-optimal algorithms for bin packing | 150, 224 |
| 3.3 | Fast algorithms for bin packing | 151 |
| 3.4 | Linear-time approximation algorithms for bin packing | 276 |
| 3.5 | Efficient approximation scheme | 156 |
| 3.6 | Efficient approximation scheme for variable sized bin packing | 215 |
| 3.7 | Asymptotic Polynomial Time Approximation Scheme (APTAS) | 14, 21, 106, 257 |
| 3.8 | Asymptotic Fully Polynomial Time Approximation Scheme (AFPTAS) | 14, 156 |
| 3.9 | Augmented asymptotic PTAS | 71 |
| 3.10 | Robust APTAS (for classical bin packing) | 99 |
| 3.11 | Approximation schemes for multidimensional problems | 181, 19 |

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**Using relations:**
- (a) compatibility
- (b) precedence
- (c) preference

**Usage of multiset estimates** (e.g., [134, 137])

**Design of solution trajectory (restructuring)** (e.g., [181, 186])

- **Multicriteria** (multiobjective) problems
- **Problems with relations over items / bins**
- **Problems under uncertainty**
- **Dynamical (real-time) problems**

**Basic multicontainer packing problems:**
- (a) bin packing problem,
- (b) multiple knapsack problem,
- (c) bin covering problem (basic dual bin packing),
- (d) min-cost covering problem (multiprocessor or makespan scheduling)

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Fig. 7. Examined extension trends in bin packing problems
Table 8. Main algorithmic approaches, part II: heuristics

| No. | Solving approach                                                                 | Some source(s) |
|-----|----------------------------------------------------------------------------------|----------------|
| IV. | Heuristics:                                                                       |                |
| 4.1 | Surveys and heuristics comparison                                                 | 65,86,133      |
| 4.2 | Basic heuristics                                                                 | 79,121,122,140,193,199 |
| 4.3 | Local search algorithms                                                           | 187,220        |
| 4.4 | Greedy procedures                                                                 | 51             |
| 4.5 | Variable neighborhood search procedures                                          | 63,109         |
| 4.6 | Dynamic programming based heuristics                                            | 222            |
| 4.7 | Simulated annealing based algorithms                                            | 221,154,254    |
| 4.8 | Tabu search algorithms                                                           | 61,196,240     |
| 4.9 | GRASP algorithms                                                                  | 175            |
| 4.10| Ant colony algorithms                                                           | 187            |
| 4.11| Average-weight-controlled bin-oriented heuristics                                 | 110            |
| 4.12| Set-covering-based heuristics                                                   | 19,213         |
| 4.13| Bottom-left bin packing heuristic (for 2D problem)                                | 45             |
| 4.14| Heuristic for 2D and 3D large bin packing                                       | 201            |
| V.  | Hybrid approaches, metaheuristics and hyper-heuristics:                         |                |
| 5.1 | Hybrid approach, metaheuristics for 2D bin packing                                | 77,137,140,141,193 |
| 5.2 | Hyper-heuristics, generalized hyper-heuristics                                  | 247,255        |
| 5.3 | Unified hyper-heuristic framework                                                | 200            |
| 5.4 | Combinations of evolutionary algorithms and hyper-heuristics                    | 42,198,236     |
| 5.5 | Combination of Lagrangian relaxation and column generation                      | 92             |

Table 9. Main algorithmic approaches, part III: online and evolutionary methods

| No. | Solving approach                                                                 | Some source(s) |
|-----|----------------------------------------------------------------------------------|----------------|
| VI. | Online and dynamic algorithms for bin packing:                                  |                |
| 6.1 | Survey of online algorithms for bin packing                                      | 101            |
| 6.2 | Fully dynamic algorithms for bin packing                                         | 147            |
| 6.3 | Simple on-line bin-packing algorithm                                            | 177            |
| 6.4 | Online algorithms for variable sized bin packing                                 | 72,241,269     |
| 6.5 | On-line algorithms for dual version of bin packing                               | 73             |
| 6.6 | On-line algorithm for multidimensional bin packing                               | 74             |
| VII.| Evolutionary approaches:                                                         |                |
| 7.1 | Genetic algorithms/evolutionary based heuristics                                | 20,28,42,139,166,251 |
| 7.2 | Genetic algorithms in 2D packing problems                                        | 148            |
| 7.2 | Mixed simulated annealing-genetic algorithm                                      | 178            |
|     | for 2D orthogonal packing                                                        |                |
| 7.3 | Evolutionary particle swam optimization                                          | 189            |
|     | for multiobjective bin packing                                                   |                |
| 7.4 | Hybrid genetic algorithms                                                        | 231            |
| 7.5 | Grouping genetic algorithms                                                       | 238            |
| 7.6 | Grouping genetical algorithm with controlled gene transmission                   | 230            |
| 7.7 | Hybrid grouping genetic algorithms                                               | 105            |
| 7.8 | Nature inspired genetic algorithms                                               | 233            |
| 7.9 | Histogram-matching approach to the evolution of                                 | 228            |
|     | bin-packing strategies for discrete sizes of item/bins                          |                |
| 7.10| Combinations of evolutionary algorithms and hyper-heuristics                    | 42,198,236     |
2. Preliminary information

2.1. Basic problem formulations

The classical formal statement of BPP is the following (e.g., [150,151,152,259]). Given a bin \( S \) of size \( V \) and a list of \( n \) items with sizes \( a_1, \ldots, a_n \) to pack.

Find an integer number of bins \( B \) and a \( B \)-partition \( S_1 \cup \ldots \cup S_B \) of set \( \{1, \ldots, n\} \) such that \( \sum_{i \in S_k} a_i \leq V \) for all \( k = 1, \ldots, B \).

A solution is optimal if it has minimal \( B \). The \( B \)-value for an optimal solution is denoted \( \text{OPT} \) below.

A possible integer linear formulation of the problem is [203]:

\[
\min B = \sum_{i=1}^{n} y_i \\
\text{s.t. } B \geq 1, \quad \sum_{j=1}^{n} a_j x_{ij} \leq V y_i, \quad \forall i \in \{1, \ldots, n\} \quad \sum_{i=1}^{n} x_{ij} = 1, \quad \forall j \in \{1, \ldots, n\} \\
y_i \in \{0,1\}, \quad \forall i \in \{1, \ldots, n\} \quad x_{ij} \in \{0,1\}, \quad \forall i \in \{1, \ldots, n\}, \quad \forall j \in \{1, \ldots, n\}
\]

where \( y_i = 1 \) if bin \( i \) is used and \( x_{ij} = 1 \) if item \( j \) is put into bin \( i \).

2.2. Maximizing the number of packed items (inverse problems)

The inverse bin packing problem is targeted to maximization of the number of packed items. Here, two basic kinds of the problems have been considered:

(i) maximization of the number of packed items (the number of bins is fixed) (e.g., [58]);

(ii) “maximization” of the total preference estimate for packed items (the number of bins is fixed, (preference relation over item set) (e.g., [115]).

The description of inverse bin packing problem will be examined in further section.

2.3. Interval multiset estimates

Interval multiset estimates have been suggested by M.Sh. Levin in [183]. A brief description of interval multiset estimates is the following [183,185]. The approach consists in assignment of elements \((1, 2, 3, \ldots)\) into an ordinal scale \([1, 2, \ldots, l]\). As a result, a multi-set based estimate is obtained, where a basis set involves all levels of the ordinal scale: \( \Omega = \{1, 2, \ldots, l\} \) (the levels are linear ordered: \( 1 > 2 > 3 > \ldots \)) and the assessment problem (for each alternative) consists in selection of a multiset over set \( \Omega \) while taking into account two conditions:

1. cardinality of the selected multiset equals a specified number of elements \( \eta = 1, 2, 3, \ldots \) (i.e., multisets of cardinality \( \eta \) are considered);

2. “configuration” of the multiset is the following: the selected elements of \( \Omega \) cover an interval over scale \([1, l]\) (i.e., “interval multiset estimate”).

Thus, an estimate \( e \) for an alternative \( A \) is (scale \([1, l]\), position-based form or form format): \( e(A) = (\eta_1, \eta_2, \ldots, \eta_l) \), where \( \eta_i \) corresponds to the number of elements at the level \( i \) (i.e. \( i = 1, \ldots, l \)), or \( e(A) = \{1, 2, \ldots, l\} \). The number of multisets of cardinality \( \eta \) with elements taken from a finite set of cardinality \( l \), is called the “multiset coefficient” or “multiset number” ([164,272]): \( \mu_l,\eta = \frac{l(l+1)(l+2)\ldots(l+\eta-1)}{\eta!} \). This number corresponds to possible estimates (without taking into account interval condition 2). In the case of condition 2, the number of estimates is decreased. Generally, assessment problems based on interval multiset estimates can be denoted as follows: \( P^l,\eta \).

A poset-like scale of interval multiset estimates for assessment problem \( P^{3,3} \) is presented in Fig. 8.
Fig. 8 illustrates the scale-poset and estimates for problem $P_{3,3}$ (assessment over scale $[1, 3]$ with three elements, estimates $(2, 0, 2)$ and $(1, 0, 2)$ are not used). For evaluation of multi-component system, multi-component poset-like scale composed from several poset-like scale may be used. Fig. 8b depicts the integrated poset-like scale for tree-component system (ordinal scale for system component compatibility is $[0, 1, 2, 3]$).

The following operations over multiset estimates are used as well: integration, vector-like proximity, aggregation, and alignment.

Integration of estimates (mainly, for composite systems) is based on summarization of the estimates by components (i.e., positions). Let us consider $n$ estimates (position form): $e^i = (\eta^1_1, ..., \eta^i_1, ..., \eta^i_n)$, $\ldots$, estimate $e^n = (\eta^n_1, ..., \eta^n_1, ..., \eta^n_n)$. Then, the integrated estimate is: $e^I = (\eta^I_1, ..., \eta^I_1, ..., \eta^I_n)$, where $\eta^I = \sum_{\kappa=1}^{\kappa} \eta^\kappa \forall \kappa = 1, 7$. In fact, the operation $\{\}$ is used for multiset estimates: $e^I = e^1 \{ \ldots \} e^n$.

Further, vector-like proximity is described. Let $A_1$ and $A_2$ be two alternatives with corresponding interval multiset estimates $e(A_1)$, $e(A_2)$. Vector-like proximity for the alternatives above is: $\delta(e(A_1), e(A_2)) = (\delta^-(A_1, A_2), \delta^+(A_1, A_2))$, where vector components are: (i) $\delta^-$ is the number of one-step changes: element of quality $\kappa + 1$ into element of quality $\kappa + 1$ (this corresponds to “improvement”); (ii) $\delta^+$ is the number of one-step changes: element of quality $\kappa$ into element of quality $\kappa + 1$ (this corresponds to “degradation”). It is assumed: $|\delta(e(A_1), e(A_2))| = |\delta^-(A_1, A_2)| + |\delta^+(A_1, A_2)|$.

Now let us consider median estimates (aggregation) for the specified set of initial estimates (traditional approach). Let $E = \{e_1, ..., e_n\}$ be the set of specified estimates (or a corresponding set of specified alternatives), let $\overline{D}$ be the set of all possible estimates (or a corresponding set of possible alternatives) ($E \subseteq \overline{D}$). Thus, the median estimates (“generalized median” $M^g$ and “set median” $M^*$) are: $M^g = \arg\min_{M \in \overline{D}} \sum_{\kappa=1}^{\kappa} |\delta(M, e_\kappa)|; \ M^* = \arg\min_{M \in E} \sum_{\kappa=1}^{\kappa} |\delta(M, e_\kappa)|$.

In recent decade, the significance of multiset studies and applications has been increased. Some recent studies in multisets and their applications are pointed out in Table 9.
### Table 9. Studies in multisets and their applications

| No. | Research direction(s)                                                                 | Source(s) |
|-----|--------------------------------------------------------------------------------------|-----------|
| I.  | Formal models, definitions:                                                          |           |
| 1.1 | Basic definitions, development of multiset theory                                    | 32, 33, 164, 249, 272 |
| 1.2 | Mathematics of multisets (axiomatic view, operations between multisets)               | 27, 85, 253 |
| 1.3 | Multiset automata (Chomsky-like hierarchy of multiset grammars in terms of multiset automata) | 76       |
| 1.4 | High-level framework for the definition of visual languages (constraint multiset grammars) | 202      |
| 1.5 | Fuzzy multisets, their generalization, soft multisets theory                          | 8, 20, 209, 211 |
| 1.6 | Tolerance multisets                                                                  | 207      |
| 1.7 | Multiset metric spaces                                                                | 145, 225 |
| 1.8 | Framework for multiset merging                                                        | 39       |
| 1.9 | Interval multiset estimates, operations over multisets (e.g., proximity, summarization, aggregation) | 183, 185 |
| 1.10 | Perturbation of multisets (measure of remoteness between multisets)                   | 165      |
| 1.11 | Multiset processing (general)                                                         | 43       |
| I.  | Some applications:                                                                   |           |
| 2.1 | Multisets in database systems                                                         | 173      |
| 2.2 | Neural network processing of multiset data                                            | 208      |
| 2.3 | Programs as multiset transformations                                                  | 10, 17   |
| 2.4 | Multiset rewriting systems                                                            | 2, 88    |
| 2.5 | Proving termination with multiset ordering                                            | 87, 136  |
| 2.6 | Automatic construction of user interfaces                                             | 244      |
| 2.7 | Clustering                                                                            | 210, 223, 225 |
| 2.8 | Classification (e.g., classification of credit cardholders)                          | 224      |
| 2.9 | Applications in decision making (e.g., multicriteria ranking/sorting)                 | 20, 223, 225 |
| 2.10 | Processing of data streams                                                            | 91, 126, 127 |
| 2.11 | Evaluation of composite system(s)/alternative(s)                                     | 179, 180, 183, 185 |
| 2.12 | Knapsack problem                                                                     | 183, 185 |
| 2.13 | Multiple choice knapsack problem                                                     | 183, 185 |
| 2.14 | Combinatorial synthesis (morphological system design)                                | 179, 180, 183, 185 |

#### 2.4. Support model: morphological design with ordinal and interval multiset estimates

A brief description of combinatorial synthesis (Hierarchical Morphological Multicriteria Design - HMMD) with ordinal estimates of design alternatives is the following ([179, 180, 183, 185]). An examined composite (modular, decomposable) system consists of components and their interconnection or compatibility (IC). Basic assumptions of HMMD are the following: (a) a tree-like structure of the system; (b) a composite estimate for system quality that integrates components (subsystems, parts) qualities and qualities of IC (compatibility) across subsystems; (c) monotonic criteria for the system and its components; (d) quality of system components and IC are evaluated on the basis of coordinated ordinal scales. The designations are: (1) design alternatives (DAs) for leaf nodes of the model; (2) priorities of DAs ($i = 1, \ldots, l$; $l$ corresponds to the best one); (3) ordinal compatibility for each pair of DAs ($w = 1, \ldots, v$ corresponds to the best one). Let $S$ be a system consisting of $m$ parts (components): $R(1), \ldots, R(i), \ldots, R(m)$. A set of design alternatives is generated for each system part above. The problem is:

**Find a composite design alternative** $S = S(1) \ast \ldots \ast S(i) \ast \ldots \ast S(m)$ of DAs (one representative design alternative $S(i)$ for each system component/part $R(i), i = 1, \ldots, m$) with non-zero compatibility between design alternatives.

A discrete “space” of the system excellence (a poset) on the basis of the following vector is used: $N(S) = (w(S); e(S))$, where $w(S)$ is the minimum of pairwise compatibility between DAs which correspond to different system components (i.e., $\forall R_{j_1}$ and $R_{j_2}$, $1 \leq j_1 \neq j_2 \leq m$) in $S$, $e(S) = (\eta_1, \ldots, \eta_l)$, where $\eta_i$ is the number of DAs of the $i$th quality in $S$. Further, the problem is described...
as follows:

$$\max e(S), \max w(S), \quad s.t. \ w(S) \geq 1.$$  

As a result, we search for composite solutions which are nondominated by $N(S)$ (i.e., Pareto-efficient). “Maximization” of $e(S)$ is based on the corresponding poset. The considered combinatorial problem is NP-hard and an enumerative solving scheme is used.

Here, combinatorial synthesis is based on usage of multiset estimates of design alternatives for system parts. For the resultant system $S = S(1) \ast \ldots \ast S(i) \ast \ldots \ast S(m)$ the same type of the multiset estimate is examined: an aggregated estimate (“generalized median”) of corresponding multiset estimates of its components (i.e., selected DAs). Thus, $N(S) = (w(S); e(S))$, where $e(S)$ is the “generalized median” of estimates of the solution components. Finally, the modified problem is:

$$\max e(S) = M^g = \arg \min_{M \in \mathcal{M}} \sum_{i=1}^{m} |\delta(M, e(S_i))|, \quad \max w(S), \quad s.t. \ w(S) \geq 1.$$  

Enumeration methods or heuristics can be used (179,180,183,185).
3. Problems with multiset estimates

3.1. Some combinatorial optimization problems with multiset estimates

3.1.1. Knapsack problem with multiset estimates

The basic knapsack problem (i.e., “0−1 knapsack problem”) is (e.g., [120,159,203]): (i) given item set $A = \{1, \ldots, i, \ldots, m\}$ with parameters $\forall i \in A$: profit (or utility) $\gamma_i$, resource requirement (e.g., weight) $a_i$; (ii) given a resource (capacity) of knapsack $b$. Thus, the model is as follows:

$$\text{max} \sum_{i=1}^{m} \gamma_i x_i \quad \text{s.t.} \sum_{i=1}^{m} a_i x_i \leq b, \ x_i \in \{0,1\}, \ i = 1, m$$

where $x_i = 1$ if item $i$ is selected, and $x_i = 0$ otherwise. Often nonnegative coefficients are assumed.

In the case of multiset estimates of item “utility” $e_i$, $i \in \{1, \ldots, i, \ldots, n\}$ (instead of $\gamma_i$), the following aggregated multiset estimate can be used for the objective function (“maximization”) (e.g., [183,185]):

(a) an aggregated multiset estimate as the “generalized median”, (b) an aggregated multiset estimate as the “set median”, and (c) an integrated multiset estimate. Knapsack problem with multiset estimates and the integrated estimate for the solution is (solution $S = \{i| x_i = 1\}$):

$$\text{max} e(S) = \bigcup_{i \in S} e_i, \quad \text{s.t.} \sum_{i=1}^{m} a_i x_i \leq b, \ x_i \in \{0,1\}.$$ 

In the case of objective function based on median estimate for solution, the problem is:

$$\text{max} e(S) = \max M = \arg \min_{M \in D} \bigcup_{i \in S} \delta(M, e_i) \quad \text{s.t.} \sum_{i=1}^{m} a_i x_i \leq b, \ x_i \in \{0,1\}.$$ 

In addition, it is reasonable to consider a new problem formulation while taking into account the number of the selected items (i.e. a special two-objective knapsack problem with multiset estimates) (solution $S = \{i| x_i = 1\}$):

$$\text{max} e(S) = \max M = \arg \min_{M \in D} \bigcup_{i \in S} \delta(M, e_i) \quad \max \sum_{i=1}^{n} x_i$$ 

$$\text{s.t.} \sum_{i=1}^{m} a_i x_i \leq b, \ x_i \in \{0,1\}.$$ 

Fig. 9 depicts the corresponding “two”-dimensional space of solution quality.

3.1.2. Multiple choice problem with interval multiset estimates

In multiple choice problem, items are divided into groups (without intersection) and items are selected in each group under total resource constraint (e.g., [120,159,203]). Here, one item is selected in each group. This version of multiple choice problem is (Boolean variable $x_{i,j}$ equals 1 if item $(i, j)$ is selected):

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n_i} \gamma_{ij} x_{ij} \quad \text{s.t.} \sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij} x_{ij} \leq b, \ \sum_{j=1}^{n_i} x_{ij} = 1, \ i = 1, m, \ x_{ij} \in \{0,1\}.$$
A special case of multiple choice problem is considered \cite{183,185}: (1) multiset estimates of item “utility” \( e_{ij} \) (\( i = \overline{1,m}, \quad j = \overline{1,q_i}, \quad \forall i \)) (instead of \( c_{ij} \)); (2) an aggregated multiset estimate as the “generalized median” (or “set median”) is used for the objective function (“maximization”). The item set is: \( A = \bigcup_{i=1}^{m} A_i, \quad A_i = \{(i,1), (i,2), ..., (i,q_i)\} \). The solution is a subset of the initial item set: \( S = \{(i,j)|x_{ij} = 1\} \).

Formally,

\[
\max e(S) = \max M = \arg \min_{M \in \mathcal{D}} \sum_{(i,j) \in S \cap \{(i,j)|x_{ij} = 1\}} |\delta(M, e_{ij})|
\]

s.t.
\[
\sum_{i=1}^{m} \sum_{j=1}^{q_i} a_{ij}x_{ij} \leq b, \quad \sum_{j=1}^{q_i} x_{ij} = 1, \quad x_{ij} \in \{0,1\}.
\]

Evidently, this problem is similar to the above-mentioned combinatorial synthesis problem without compatibility of the selected items (objects, alternatives) \cite{183,185}.

### 3.1.3. Multiple knapsack problem with multiset estimates

The basic multiple knapsack problem is the following (e.g., \cite{19,20,141,159,203,226}): (i) item set \( A = \{1, ..., i, ..., m\} \); (ii) knapsack set \( B = \{B_1, ..., B_j, ..., B_k\} \) (\( k \leq m \)); (iii) parameters \( \forall i \in A: \) profit \( c_i \), resource requirement (e.g., weight) \( a_i \); and (iv) resource (capacity) of knapsack \( B_j \in B: b_j \). This problem is a special case of generalized assignment problem (multiple knapsack problem contains bin packing problem as special case). The model (i.e., “0 – 1 multiple knapsack problem”) is:

\[
\max \sum_{j=1}^{k} \sum_{i=1}^{m} a_{ij}x_{ij} \quad s.t. \quad \sum_{i=1}^{m} a_{ij}x_{ij} \leq b_j, \quad \forall j = \overline{1,k}, \quad \sum_{j=1}^{k} x_{ij} \leq 1, \quad \forall i = \overline{1,m}, \quad x_{ij} \in \{0,1\}, \quad i = \overline{1,m}, \quad j = \overline{1,k},
\]

where \( x_{ij} = 1 \) if item \( i \) is selected for knapsack \( B_j \), and \( x_{ij} = 0 \) otherwise.

In the case of multiset estimates, item “utility” \( e_i, i = \overline{1,m} \) (instead of \( c_i \)) is considered. Multiple knapsack problem with multiset estimates and the integrated estimate for the solution is (solution \( S = \{(i,j)|x_{ij} = 1\} \)):

\[
\max e(S) = \bigcup_{(i,j) \in S \cap \{(i,j)|x_{ij} = 1\}} e_i,
\]

s.t.
\[
\sum_{i=1}^{m} a_{ij}x_{ij} \leq b_j, \quad \forall j = \overline{1,k}, \quad \sum_{j=1}^{k} x_{ij} \leq 1, \quad \forall i = \overline{1,m}, \quad x_{ij} \in \{0,1\}, \quad i = \overline{1,m}, \quad j = \overline{1,k}.
\]

In the case of objective function based on median estimate for solution, the problem is:

\[
\max e(S) = \max M = \arg \min_{M \in \mathcal{D}} \bigcup_{(i,j) \in S \cap \{(i,j)|x_{ij} = 1\}} |\delta(M, e_i)|
\]

s.t.
\[
\sum_{i=1}^{m} a_{ij}x_{ij} \leq b_j, \quad \forall j = \overline{1,k}, \quad \sum_{j=1}^{k} x_{ij} \leq 1, \quad \forall i = \overline{1,m}, \quad x_{ij} \in \{0,1\}, \quad i = \overline{1,m}, \quad j = \overline{1,k}.
\]

In addition, it is reasonable to consider a new problem formulation while taking into account the number of the selected items (i.e., a special two-objective knapsack problem with multiset estimates) (solution \( S = \{(i,j)|x_{ij} = 1\} \)):

\[
\max e(S) = \max M = \arg \min_{M \in \mathcal{D}} \bigcup_{(i,j) \in S \cap \{(i,j)|x_{ij} = 1\}} |\delta(M, e_i)|
\]

\[
\max \sum_{i=1}^{n} x_{i,j}
\]

s.t.
\[
\sum_{i=1}^{n} a_{ij}x_{ij} \leq b_j, \quad \forall j = \overline{1,k}, \quad \sum_{j=1}^{k} x_{ij} \leq 1, \quad \forall i = \overline{1,m}, \quad x_{ij} \in \{0,1\}, \quad i = \overline{1,m}, \quad j = \overline{1,k}.
\]

Here, “two”-dimensional space of solution quality (Fig. 9) can be considered as well.
3.1.4. Assignment and generalized assignment problems with multiset estimates

The basic assignment problem is the following (e.g., [120, 246]). Simple assignment problem involves nonnegative correspondence matrix \( Y = [\gamma_{ij}] \) \((i = 1, m, j = 1, m)\) where \( c_{ij} \) is a profit (‘utility’) to assign element \( i \) to position \( j \). The problem is (e.g., [120]):

\[
\text{Find the assignment } \pi = (\pi(1), ..., \pi(m)) \text{ of elements } i (i = 1, m) \text{ to positions } \pi(i) \text{ which corresponds to a total effectiveness: } \sum_{i=1}^{m} \gamma_{i\pi(i)} \to \text{max}.
\]

The simplest algebraic problem formulation is:

\[
\max \sum_{i=1}^{m} \sum_{j=1}^{m} \gamma_{ij} x_{ij} \quad \text{s.t.} \quad \sum_{i=1}^{m} x_{ij} \leq 1, \, j = 1, m; \quad \sum_{j=1}^{m} x_{ij} = 1, \, i = 1, m; \quad x_{ij} \in \{0, 1\}, \, i = 1, m, \, j = 1, m.
\]

Here \( x_{ij} = 1 \) if element \( i \) is assigned into position \( j \), \( c_{ij} \) is a profit (‘utility’) of this assignment. The problem can be solved efficiently, for example, on the basis of Hungarian method (e.g., [169]). Note this problem is the matching problem for a bipartite graph (e.g., [120]).

In the generalized assignment problem, each item \( i (i = 1, m) \) can be assigned to \( k (k \leq m) \) positions (knapsacks, bins) and a capacity is considered for each position \( j (j = 1, k) \) (with corresponding capacity constraint \( \leq b_j \)) (Fig. 10).

![Fig. 10. Generalized assignment problem](image)

Formally,

\[
\max \sum_{i=1}^{m} \sum_{j=1}^{m} \gamma_{ij} x_{ij} \quad \text{s.t.} \quad \sum_{i=1}^{m} x_{ij} \leq 1, \, j = 1, k; \quad \sum_{j=1}^{k} x_{ij} \geq 1, \, i = 1, m; \quad x_{ij} \in \{0, 1\}, \, i = 1, m, \, j = 1, k.
\]

In the case of multiset estimates, item “utility” \( e_{ij} \) \((i = 1, m \quad j = 1, k)\) instead of \( c_{ij} \) is considered.

The generalized assignment problem with multiset estimates and the integrated estimate for the solution is (solution \( S = \{(i, j)|x_{ij} = 1\}\)):

\[
\max \sum_{(i,j)\in S} e_{ij}, \quad \text{s.t.} \quad \sum_{i=1}^{m} a_i x_{ij} \leq b_j, \, \forall j = 1, k; \quad \sum_{j=1}^{k} x_{ij} = 1, \, \forall i = 1, m; \quad x_{ij} \in \{0, 1\}, \, i = 1, m, \, j = 1, k.
\]

In the case of objective function based on median estimate for solution, the problem is:

\[
\max \sum_{(i,j)\in S} e_{ij}, \quad \text{s.t.} \quad \sum_{i=1}^{m} a_i x_{ij} \leq b_j, \, \forall j = 1, k; \quad \sum_{j=1}^{k} x_{ij} = 1, \, \forall i = 1, m; \quad x_{ij} \in \{0, 1\}, \, i = 1, m, \, j = 1, k.
\]
In addition, it is reasonable to consider a new problem formulation while taking into account the number of the selected items (i.e., a special two-objective generalized assignment problem with multiset estimates) (solution $S = \{(i, j)|x_{ij} = 1\}$):

$$\max e(S) = \max M = \arg \min_{M \in D} \left\{ \bigcup_{(i, j) \in S = \{(i, j)|x_{ij} = 1\}} \delta(M, e_i) \right\} \max \sum_{i=1}^{n} x_{i,j}$$

s.t. $\sum_{i=1}^{m} a_i x_{ij} \leq b_j, \forall j = 1, k, \sum_{j=1}^{k} x_{ij} = 1, \forall i = 1, m, x_{ij} \in \{0, 1\}, i = 1, m, j = 1, k.$

Here, “two”-dimensional space of solution quality (Fig. 9) can be considered as well.

### 3.2. Inverse bin packing problem with multiset estimates

Generally, the inverse bin packing problem can be formulated as multiple knapsack problem with equal knapsack (i.e., bins).

First, the basic inverse bin packing problem (with maximization of packed items), i.e., maximum cardinality bin packing problem, is considered as follows (e.g., [7,11,15,52,94,171,172,221]). Problem components are: (i) item set $A = \{1, ..., i, ..., m\}$; (ii) set of equal bins $B = \{B_1, ..., B_j, ..., B_k\}$ (usually, $k \leq m$); (iii) parameters $\forall i \in A$: profit $\gamma_i$, resource requirement (e.g., weight) $a_i$; and (iv) equal resource capacity of each bin $B_j \in B$: $b$. The model is:

$$\max \sum_{j=1}^{k} \sum_{i=1}^{m} \gamma_i x_{ij} \quad \text{s.t.} \quad \sum_{i=1}^{m} a_i x_{ij} \leq b, \forall j = 1, k, \sum_{j=1}^{k} x_{ij} \leq 1, \forall i = 1, m, x_{ij} \in \{0, 1\}, i = 1, m, j = 1, k,$$

where $x_{ij} = 1$ if item $i$ is selected for knapsack $B_j$, and $x_{ij} = 0$ otherwise.

In the case of multiset estimates, item “utility” $e_i, i = 1, m$ (instead of $c_i$) is considered. The inverse bin packing problem with multiset estimates and the integrated estimate for the solution is (solution $S = \{(i, j)|x_{ij} = 1\}$):

$$\max e(S) = \bigcup_{(i, j) \in S = \{(i, j)|x_{ij} = 1\}} e_i,$$

s.t. $\sum_{i=1}^{m} a_i x_{ij} \leq b, \forall j = 1, k, \sum_{j=1}^{k} x_{ij} \leq 1, \forall i = 1, m, x_{ij} \in \{0, 1\}, i = 1, m, j = 1, k.$

In the case of objective function based on median estimate for solution, the problem is:

$$\max e(S) = \max M = \arg \min_{M \in D} \left\{ \bigcup_{(i, j) \in S = \{(i, j)|x_{ij} = 1\}} \delta(M, e_i) \right\}$$

s.t. $\sum_{i=1}^{m} a_i x_{ij} \leq b, \forall j = 1, k, \sum_{j=1}^{k} x_{ij} \leq 1, \forall i = 1, m, x_{ij} \in \{0, 1\}, i = 1, m, j = 1, k.$

The problem formulation while taking into account the number of the selected items (i.e., a special two-objective inverse bin packing problem with multiset estimates) (solution $S = \{(i, j)|x_{ij} = 1\}$) is:

$$\max e(S) = \max M = \arg \min_{M \in D} \left\{ \bigcup_{(i, j) \in S = \{(i, j)|x_{ij} = 1\}} \delta(M, e_i) \right\} \max \sum_{i=1}^{n} x_{i,j}$$

s.t. $\sum_{i=1}^{m} a_i x_{ij} \leq b, \forall j = 1, k, \sum_{j=1}^{k} x_{ij} \leq 1, \forall i = 1, m, x_{ij} \in \{0, 1\}, i = 1, m, j = 1, k.$

Here, “two”-dimensional space of solution quality (Fig. 9) can be considered as well.
3.3. Bin packing with conflicts

The bin packing problem with conflict consists in packing items into the minimum number of bins subject to incompatibility constraints. (e.g., [98,107,121,149,237]). The description of the problem is the following. Given a set of \( n \) items \( A \), corresponding their weights \( w_1, w_2, \ldots, w_n \), and a set of identical bins \((k = 1, 2, \ldots)\) with capacity \( b \). It can be assumed: \( w_1 \geq w_2 \geq \ldots \geq w_n \). Given conflict relation over items as conflict graph \( G = (A, E) \), where an edge \((\iota_1, \iota_2) \in E\) exists if and only if items \( \iota_1, \iota_2 \in A \) conflict or \( w_{\iota_1} + w_{\iota_2} \geq b \). Let \( y_k \) be a binary variable: \( y_k = 1 \) if bin \( k \) is used, and \( x_{\iota k} \) be a binary variable: \( x_{\iota k} = 1 \) if item \( \iota \) is assigned to bin \( k \). Formally,

\[
\min z = \sum_{i=1}^{n} y_k
\]

\[
\text{s.t.} \sum_{i=1}^{n} w_{i} x_{i k} \leq b y_k \forall k = 1, n; \quad \sum_{i=1}^{n} x_{i k} = 1 \forall \iota = 1, n; \quad x_{i_1 k} + x_{i_2 k} \leq 1 \forall (\iota_1, \iota_2) \in E, \forall k = 1, n;
\]

\( y_k \in \{0, 1\} \forall k = 1, n; \quad x_{i k} \in \{0, 1\} \forall \iota = 1, n, \forall k = 1, n. \)

Evidently, the problem generalizes the classic bin packing problem and is HP-hard (e.g., [203]).

In inverse bin packing problem (maximization of the number of packed items subject to fixed set of bins), the problem is as follows. Let \( \gamma_\iota \) be an importance (utility, profit) of packing item \( \iota \in A \). Formally,

\[
\max \sum_{i=1}^{n} \sum_{k=1}^{q} \gamma_\iota x_{i k}
\]

\[
\text{s.t.} \sum_{i=1}^{n} w_{i} x_{i k} \leq b \forall k = 1, n; \quad \sum_{i=1}^{n} x_{i k} \leq 1 \forall \iota = 1, n; \quad x_{i_1 k} + x_{i_2 k} \leq 1 \forall (\iota_1, \iota_2) \in E, \forall k = 1, n;
\]

\( x_{i k} \in \{0, 1\} \forall \iota = 1, n, \forall k = 1, n. \)

Let \( e_\iota \) be an importance multiset estimate (utility, profit) of packing item \( \iota \in A \). The inverse bin packing problem with multiset estimates and the integrated estimate for the solution is (solution \( S = \{(\iota, k)|x_{i k} = 1\} \)):

\[
\max e(S) = \bigcup_{(\iota, k) \in S = \{(\iota, k)|x_{i k} = 1\}} e_\iota,
\]

\[
\text{s.t.} \sum_{i=1}^{n} \sum_{k=1}^{q} w_{i, k} x_{i k} \leq b \forall k = 1, n; \quad \sum_{i=1}^{n} x_{i k} \leq 1 \forall \iota = 1, n; \quad x_{i_1 k} + x_{i_2 k} \leq 1 \forall (\iota_1, \iota_2) \in E, \forall k = 1, n;
\]

\( x_{i k} \in \{0, 1\} \forall \iota = 1, n, \forall k = 1, n. \)

In addition, objective function can be examine:

\[
\max \sum_{i=1}^{n} \sum_{k=1}^{q} x_{i k} \forall \iota = 1, n, \forall k = 1, n.
\]
4. Colored bin packing

4.1. Basic colored bin packing

Now consider the basic colored bin packing problem (e.g., [51,257]). A set of items \( A = \{a_1, ..., a_n\} \) of different sizes (e.g., \( w_i \in (0,1) \forall i = 1, n \)) is given. It is necessary to pack the items above into bins of equal size so that a few bins is used in total (at most \( \alpha \) times optimal), and that the items of each color span few bins (at most \( \beta \) times optimal). The obtained allocations are called \( \alpha, \beta \)-approximate.

The colored bin packing problem corresponds to many significant applications, for example (e.g., [257]): (1) allocating files in P2P networks, (2) allocating related jobs (i.e., related jobs are of the same color) to processors, (3) allocating related items in a distributed cache, and (4) allocating jobs in a grid computing system. Fig. 11 illustrates the colored bin packing problem: eleven items, three colors (\( \lambda, \mu, \theta \)). The illustrative solution is: (i) color \( \lambda \) for bin 1, bin 2; (ii) color \( \mu \) for bin 3; and (iii) color \( \theta \) for bin 4, bin 5.

![Initial items](image1)

![Bins (blocks, containers, knapsacks)](image2)

Fig. 11. Illustration for colored bin-packing

Recently, some versions of colored bin packing problem have been examined: (1) basic colored bin packing [51,257], (2) offline colored bin packing [257], (3) online colored bin packing [257], and (4) online bin coloring (packing with minimum colors) [167].

4.2. Two auxiliary graph coloring problems

4.2.1. Auxiliary vertex graph coloring problem with ordinal color proximity

First, the vertex coloring problem is considered as a basic one. The problem can be described as the following (e.g., [44,93,120,134,135,168,218,266]). Given undirected graph \( G = (A,E) \) (a node/vertices set \( A \) and an edge set \( E \), \( |A| = n \)). There is a set of colors (labels, numbers) \( X = \{x_1, ..., x_l\} \). Let \( C(G) = \{C(a_1), ..., C(a_i), ..., C(a_n)\} \) \( (C_{a_i} \in X) \) (or \( < C(a_1) \ast ... \ast C(a_i) \ast ... \ast C(a_n) > \)) be a color configuration (i.e., assignment of a color for each vertex). The problem is:

\[
\text{Assign for each vertex } \forall a_i \in A \text{ label or color (i.e., } C(a_i) \text{) such that no edge connects two identical colored vertices, i.e., } \forall a_i, a_j \in A \text{ if } (a_i, a_j) \in E \text{ (i.e., adjacent vertices) then } C(a_i) \neq C(a_j).
\]

Thus, color configuration (e.g., \( C(G) = \{C(a_1), ..., C(a_i), ..., C(a_n)\} \)) for a given graph \( G = (A,E) \) is searched for. Clearly, \( |C(G)| \) equals the number of used colors (labels). (The minimal number of required colors for a graph \( G \) is called chromatic number of the graph \( \chi(G) \)). Note, other coloring problems can be transformed into the vertex version. Fig. 12 illustrates the vertex coloring problem: \( G = (A,E), A = \{p, q, u, v, w\}, E = \{(p, q), (p, u), (q, v), (u,v), (w, p)(w, q)(w, u)(w, v)\} \) and three colors \( \{x_1, x_2, x_3\} \) (i.e., corresponding indices for colors of vertices).
The resultant color configuration (solution) is: $C(G) = \{P_2, W_1, V_3, Q_3, U_3\}$. The number of possible resultant color configurations (three colors) equals 6:

1. $C^1(G) = \{P_1, W_2, V_1, Q_3, U_3\}$,
2. $C^2(G) = \{P_1, W_3, V_1, Q_2, U_2\}$,
3. $C^3(G) = \{P_3, W_2, V_3, Q_1, U_1\}$,
4. $C^4(G) = \{P_3, W_1, V_3, Q_2, U_2\}$,
5. $C^5(G) = \{P_2, W_1, V_2, Q_3, U_3\}$,
6. $C^6(G) = \{P_2, W_3, V_2, Q_1, U_1\}$.

In addition, an aggregated weight (e.g., additive aggregation function) of used colors (each color has its nonnegative weight $w(x_i) \forall x_i \in X$, $l = 1, k$) can be considered as well. As a result, the following minimization problem formulation can be examined:

$$\begin{align*}
\min_{\{C(G)\}} & \quad |C^* (G = (A, E))| \\
\text{s.t.} & \quad C^*(a_i) \neq C^*(a_j) \forall (a_i, a_j) \in E, \ i \neq j.
\end{align*}$$

This problem is NP-hard (e.g., [93][120][153][266]). Let $C^*(G) = \{c^*_y\}$ be the set of used colors (i.e., $C^*(G) \subseteq C^*(G)$). In the case of weighted colors (and additive aggregation function), the following model can be considered:

$$\begin{align*}
\min & \quad \sum_{c^*_y \in C^*(G)} w(c^*_y) \\
\text{s.t.} & \quad C^*(a_i) \neq C^*(a_j) \forall (a_i, a_j) \in E, \ i \neq j.
\end{align*}$$

Clearly, if $w(x_i) = 1 \forall x_i \in X$ this problem formulation is equivalent to the previous one. In the case of vector-like color weight

$$(w^1(c_y), ..., w^\mu(c_y), ..., w^\lambda(c_y)) \quad \forall c_y \in C$$

and additive aggregation functions, the objective vector function is:

$$(\sum_{c^*_y \in C^*(G)} w^1(c^*_y), ..., \sum_{c^*_y \in C^*(G)} w^\mu(c^*_y), ..., \sum_{c^*_y \in C^*(G)} w^\lambda(c^*_y))$$

and Pareto-efficient solutions by the vector function are searched for.

Generally, it may be prospective to consider a set of objective functions (criteria) as follows (e.g., [179][185]): (i) number of used colors, (ii) an aggregated weight of used colors, (iii) correspondence of colors to vertices (e.g., the worst correspondence, average correspondence) (e.g., [179]); (iv) quality of compatibility of colors, which were assigned to the neighbor (i.e., adjacent) vertices (e.g., the worst case, average case) (e.g., [179]); and (v) conditions at a distance that equals three, four, etc.

The author’s version of graph (vertex) coloring problem (while taking into account color compatibility and correspondence of colors to vertices) is described in [179] (numerical example, Fig. 13). Here, the solving approach is based on morphological clique problem (i.e., HMMD). Six colors are used: $x_1, x_2, x_3, x_4, x_5,$ and $x_6$. Estimates of correspondence of colors to vertices are shown in parentheses in Fig. 13 (1 corresponds to the best level). Table 10 contains compatibility estimates for colors (4 corresponds to the best level).
If the edge between vertices is absent the corresponding compatibility estimates of colors equal to the best level (i.e., 4 for vertex pair \((p, v)\)). Two examples of color combinations (color compositions) and their quality vectors are the following (Fig. 14):

(a) \(C^{*1}(G) = P_2 \ast Q_3 \ast V_3 \ast W_5 \), \(N(C^{*1}(G)) = (4; 1, 3, 0)\);  
(b) \(C^{*2}(G) = P_3 \ast Q_5 \ast V_2 \ast W_4 \), \(N(C^{*2}(G)) = (2; 3, 1, 0)\);  
(c) \(C^{*3}(G) = P_2 \ast Q_5 \ast V_2 \ast W_5 \), \(N(C^{*2}(G)) = (2; 3, 1, 0)\).

Table 10. Compatibility estimates of colors

|   | \(Q_1\) | \(Q_2\) | \(Q_3\) | \(Q_4\) | \(Q_5\) | \(V_1\) | \(V_2\) | \(V_3\) | \(V_4\) | \(V_5\) | \(W_1\) | \(W_2\) | \(W_3\) | \(W_4\) | \(W_5\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(P_1\) | 0 | 1 | 2 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 0 | 1 | 2 | 3 | 3 |
| \(P_2\) | 1 | 0 | 4 | 2 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 1 | 0 | 1 | 2 | 4 |
| \(P_3\) | 2 | 4 | 0 | 1 | 2 | 4 | 4 | 4 | 4 | 4 | 2 | 1 | 0 | 2 | 4 |
| \(P_4\) | 3 | 2 | 1 | 0 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 3 | 2 | 1 | 0 | 2 |
| \(P_5\) | 4 | 3 | 2 | 3 | 0 | 4 | 4 | 4 | 4 | 4 | 4 | 3 | 2 | 3 | 2 |
| \(Q_1\) | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 3 |
| \(Q_2\) | 1 | 0 | 4 | 2 | 3 | 1 | 0 | 1 | 2 | 4 |
| \(Q_3\) | 4 | 1 | 4 | 1 | 2 | 2 | 1 | 0 | 1 | 4 |
| \(Q_4\) | 3 | 2 | 1 | 0 | 3 | 2 | 1 | 0 | 2 |
| \(Q_5\) | 4 | 3 | 2 | 3 | 0 | 4 | 3 | 2 | 3 | 3 |
| \(V_1\) | 0 | 1 | 2 | 3 | 3 |
| \(V_2\) | 1 | 0 | 1 | 2 | 4 |
| \(V_3\) | 2 | 1 | 0 | 1 | 4 |
| \(V_4\) | 3 | 2 | 1 | 0 | 2 |
| \(V_5\) | 4 | 3 | 2 | 3 | 2 |

Fig. 14. Poset-like scale for color configuration
4.2.2. Partition coloring problem

Here the partition coloring problem (i.e., selective graph clustering over clustered graph) is considered as a close auxiliary problem [113, 142, 188, 219]. The problem formulation is as follows. Given a non-directed graph \( G = (V, E) \), where \( V \) is the set of vertices (nodes) and \( E \) is the set of edges. Let \( \{V_1, V_2, ..., V_q\} \) be a partition of \( V \) into \( q \) subsets with \( V = \bigcup_{\iota=1}^{q} V_\iota \) and \( |V_\iota \cap V_{\iota+1}| = 0 \) \( \forall \iota_1, \iota_2 = 1, 2, ..., q \) with \( \iota_1 \neq \iota_2 \). Clearly, \( V_\iota \ (\forall \iota = 1, q) \) is a graph part or a graph component. The partition coloring problem is:

Find a subset \( V' \subseteq V \) such that \( |V' \cap V_\iota| = 1 \) \( \forall \iota = 1, q \) (i.e., \( V' \) contains one vertex from each component \( V_\iota \)), and the chromatic number of the graph induced in \( G \) by \( V' \) is minimum.

Evidently, the problem is a generalization of the graph coloring problem and belongs to class of NP-hard problems (e.g., [188]). Several formal models for this problem have been proposed: (a) binary integer programming problem (e.g., [112, 113, 142]), (b) model based on the independent set problem [142], and (c) two integer programming formulations using representatives [10].

Fig. 15 depicts an instance of partition coloring problem (graph with ten vertices and four graph parts). Here, the resultant colorings are (two colors: \( c_1, c_2 \)):

\[
Q^1 = <2(c_1), 6(c_2), 9(c_1), 5(c_2)>, \quad Q^2 = <2(c_2), 6(c_1), 9(c_2), 5(c_1)>.
\]

Some solving approaches proposed for the partition coloring problem are listed in Table 11.

![Fig. 15. Instance of partition coloring problem](image)

**Table 11. Algorithms for partition coloring problem**

| No. | Approach                 | Source(s)       |
|-----|--------------------------|-----------------|
| 1.  | Branch-and-price approach | [10, 112, 113, 142] |
| 2.  | Tabu search heuristic    | 219             |
| 3.  | Two-phase heuristic      | 219             |
| 4.  | Engineering heuristics   | [188, 191]      |

In real world, this problem corresponds to routing and wavelength assignment in all-optical networks (i.e., computation of alternative routes for the lightpaths, followed by the solution of a partition colorings problem in a conflict graph) (e.g., [188, 191, 219]).

In fact, the partition coloring problem is very close to representative problems (e.g., [10]). Generally, this kind of problems is based on selection of elements from graph parts (components) (e.g., vertices) while taking into account compatibility of the selected elements (i.e., construction of a clique or quasi-clique). In addition, it is possible to examine some preference relation(s) over elements for graph part. Thus, the problem can be considered as a morphological clique problem (i.e., hierarchical morphological design or combinatorial synthesis) [179, 180, 185].

In the future, it may be very interesting to examine a new multistage partition coloring problem with costs of changes of vertex colors as restructuring of partition coloring problem. (i.e., a version of dynamical partition coloring problem).
5. Some applications

5.1. Composite planning framework in paper production system

Here a composite planning framework is described that was prepared by the author for a seminar of Institute for Industrial Mathematics in May 1992 (Beer Sheva, Israel). Fig. 16 depicts an illustrative solution of the composite planning problem for three machines.

In the problem, there are a set of paper horizontal bar for each machine. It is necessary to cut it (by special knifes) to obtain a set of 2D items of the required sizes and colors (by coloring). Seven colors are considered: white ($col_1$), blue ($col_2$), red ($col_3$), green ($col_4$), magenta ($col_5$), brown ($col_6$), and yellow ($col_7$). Table 12 contains ordinal estimates of color change: $col_i \Rightarrow col_j$ ($i = 1, 7, j = 1, 7$). Item parameters are presented in Table 13: twenty five 2D items (required items of required sizes and colors) (the width of the paper horizontal bar equals 20).

Evidently, two objective functions are considered:
(i) minimizing the volume of non-used domain in bins,
(ii) minimizing the total cost of color changes (e.g., as a total sum of color change estimates in the solution) (this function can be transformed to non-used bin domains as well).

Table 12. Ordinal estimates of color change ($col_i \Rightarrow col_j$)

| $col_i \backslash col_j$ | $col_1$ | $col_2$ | $col_3$ | $col_4$ | $col_5$ | $col_6$ | $col_7$ |
|------------------------|--------|--------|--------|--------|--------|--------|--------|
| $col_1$ (white)        | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| $col_2$ (blue)         | 4      | 0      | 4      | 2      | 1      | 3      |        |
| $col_3$ (red)          | 4      | 0      | 4      | 3      | 0      | 3      |        |
| $col_4$ (green)        | 4      | 4      | 4      | 0      | 3      | 0      | 5      |
| $col_5$ (magenta)      | 4      | 0      | 3      | 4      | 0      | 0      | 3      |
| $col_6$ (brown)        | 4      | 4      | 4      | 4      | 0      | 4      |        |
| $col_7$ (yellow)       | 2      | 0      | 2      | 3      | 1      | 0      | 0      |

The following heuristic solving scheme is considered:

Stage 1. Grouping of initial items by colors.
Stage 2. For each color: forming the general items (combinations of items of the same color) as packed bins (bin size equals 20). For the items in the same bin, their heights/lengths are about close. Some initial items can be integrated (as items 3 and 4 in the example, Fig. 16). Here, bin packing problem can be used. As a result, a set of general items (the same color for each item) are obtained. In Fig. 16, the following 8 general items are depicted: (i) items 1, 2, and 3 (color \( \text{col}_1 \)); (ii) items 5, 6, and 7 (color \( \text{col}_2 \)); (iii) items 8, 9, 10, and 11 (color \( \text{col}_3 \)); (iv) items 12, 13, and 14 (color \( \text{col}_4 \)); (v) items 15, 16, and 17 (color \( \text{col}_5 \)); (vi) items 18 and 19 (color \( \text{col}_6 \)); (vii) items 20, 21, and 22 (color \( \text{col}_7 \)); and (viii) items 23, 24, and 25 (color \( \text{col}_7 \)).

Stage 3. Forming the bins for each machine and for one period (from the general items): bin size corresponds to time period). Here bin packing problem can be used.

Stage 4. For each obtained bin: linear ordering of the generalized items while taking into account color changes. Here the traveling salesman problem can be used (while taking into account the ordinal estimates of color change as element distance, Table 12).

| Item | Width | Height/length | Color | General item | Machine | Time interval |
|------|-------|---------------|-------|--------------|---------|---------------|
| 1    | 8     | 43            | \( \text{col}_1 \) | I           | 1       | 1             |
| 2    | 5     | 30            | \( \text{col}_1 \) | I           | 1       | 1             |
| 3    | 6     | 21            | \( \text{col}_1 \) | I           | 1       | 1             |
| 4    | 5     | 21            | \( \text{col}_1 \) | I           | 1       | 1             |
| 5    | 5     | 36            | \( \text{col}_1 \) | II          | 1       | 2             |
| 6    | 7     | 33            | \( \text{col}_1 \) | II          | 1       | 2             |
| 7    | 7     | 28            | \( \text{col}_1 \) | II          | 1       | 2             |
| 8    | 4     | 25            | \( \text{col}_5 \) | III         | 2       | 1             |
| 9    | 5     | 24            | \( \text{col}_5 \) | III         | 2       | 1             |
| 10   | 6     | 23            | \( \text{col}_5 \) | III         | 2       | 1             |
| 11   | 5     | 22            | \( \text{col}_5 \) | III         | 2       | 1             |
| 12   | 5     | 26            | \( \text{col}_2 \) | IV          | 2       | 2             |
| 13   | 8     | 25            | \( \text{col}_2 \) | IV          | 2       | 2             |
| 14   | 5     | 23            | \( \text{col}_2 \) | IV          | 2       | 2             |
| 15   | 8     | 26            | \( \text{col}_6 \) | V           | 2       | 3             |
| 16   | 6     | 25            | \( \text{col}_6 \) | V           | 2       | 3             |
| 17   | 5     | 23            | \( \text{col}_6 \) | V           | 2       | 3             |
| 18   | 10    | 24            | \( \text{col}_3 \) | VI          | 3       | 1             |
| 19   | 9     | 23            | \( \text{col}_3 \) | VI          | 3       | 1             |
| 20   | 6     | 24            | \( \text{col}_3 \) | VII         | 3       | 2             |
| 21   | 5     | 23            | \( \text{col}_3 \) | VII         | 3       | 2             |
| 22   | 7     | 22            | \( \text{col}_3 \) | VII         | 3       | 2             |
| 23   | 6     | 30            | \( \text{col}_7 \) | VIII        | 3       | 3             |
| 24   | 8     | 27            | \( \text{col}_7 \) | VIII        | 3       | 3             |
| 25   | 6     | 25            | \( \text{col}_7 \) | VIII        | 3       | 3             |

Note the considered composite planning framework can be extended/modified to use in communication systems (e.g., multiple channel systems).

5.2. Planning in communication system

The basic multi-processor scheduling problems based on bin packing have been described in [59,60,69,143]. Here, some combinatorial planning problems as 2D bin packing for communications (one-channel communications, telecommunication WiMAX systems). Note, close problems are used in resource allocation in multispot satellite networks (e.g., [4]).

5.2.1. Selection of messages/information packages

First, the basic simplified planning problem can be considered as the well-known secretary problem. Given a set of items \( n \) (e.g., messages) \( A = \{a_1, ..., a_i, ..., a_n\} \), each item \( a_i \) has a weight \( w_i \) (e.g., time for processing). The problem is (Fig. 17):
Find the schedule (i.e., ordering of items as permutation) of the items from set \( A \): 
\[
S = s[1], ..., s[i], ..., s[n] > (s[i]) \text{ corresponds to an item } a_i \text{ that is processed at the } i\text{-th place in schedule } S \text{ such that average completion time for each item } a_i \in A \text{ (i.e., sum of waiting time and processing time) is minimal: }
\[
t(S) = \frac{1}{n} \sum_{i=1}^{n} \tau_s[i], \text{ where the waiting time is as follows } (\tau_s[i] = w_s[i], \
\tau = \frac{T}{n}).
\]

Evidently, the algorithm to obtain the optimal solution is based on ordering of the items by non-decreasing of weight \( w_i \) (i.e., the item with minimal weight has to be processed as the 1st, and so on) (complexity estimate of the algorithm is \( O(n \log n) \)). This is the algorithm: ‘smallest weight first’.

Note, the solution can be defined by Boolean variables: \( x_{a_i,s[i]} \in \{0, 1\} \), where \( x_{a_i,s[i]} = 1 \) if item \( a_i \) is assigned into place \( s[i] \) in the solution. Thus, the solution is defined by Boolean matrix:
\[
X = ||x_{a_i,s[i]}||, \ i = 1, n, \ t = 1, n.
\]

![Fig. 17. Illustration for secretary problem](image)

Usually the described secretary problem is used for planning in one-channel communication system. In this case, there is a time interval (i.e., planning period) \( T \) and the initial set of items \( A \) is ordered to send via the channel. If all messages can be send during period \( T \) (i.e., \( \sum_{i=1}^{n} w_i \leq T \)) the considered algorithm can be successfully used. Unfortunately, if period \( T \) is not sufficient to send all message (i.e., \( \sum_{i=1}^{n} w_i \geq T \)), a subset of items with highest weights have to wait the next period (i.e., a wait set). Here, the problem can be formulated as a knapsack model:

\[
\min t(S) = \frac{1}{n} \sum_{i=1}^{n} \tau_s[i] \ x_{a_i,s[i]}, \quad \max \sum_{i=1}^{n} \sum_{i=1}^{n} x_{a_i,s[i]}
\]

\[
s.t. \sum_{i=1}^{n} x_{a_i,s[i]} w_i \leq T, \quad \sum_{i} x_{a_i,s[i]} \leq 1 \quad \forall i = 1, n, \quad x_{a_i,s[i]} \in \{0, 1\}.
\]

Here, the algorithm above leads to the optimal solution. Note, the first objective function requires linear ordering of items in the solution by non-decreasing of \( w_i \) (as in previous problem).

After using the algorithm the items which do not belong to the solution can be considered as a wait set. Thus, it is reasonable to examine an extension of the problem above. Let each item \( a_i \in A \) has two parameters: (i) the weight (i.e., processing time) \( w_i \) and (ii) the number of wait periods \( \gamma_i = 0, 1, ... \). The problem statement can be considered as two-criteria knapsack model:

\[
\min t(S) = \frac{1}{n} \sum_{i=1}^{n} \tau_s[i] \ x_{a_i,s[i]}, \quad \max \sum_{i=1}^{n} \sum_{i=1}^{n} x_{a_i,s[i]}, \quad \max \sum_{i=1}^{n} \sum_{i=1}^{n} x_{a_i,s[i]} \ \gamma_i
\]

\[
s.t. \sum_{i=1}^{n} x_{a_i,s[i]} w_i \leq T, \quad x_{a_i,s[i]} \in \{0, 1\}.
\]

This problem is NP-hard. The selection of items for sending (i.e., solution) can be based on detection of Pareto-efficient items by two parameters: (a) minimum weight \( w_i \) (rule: smallest weight first) and (b) maximum number \( \gamma_i \) (rule: longest wait first). The following heuristic algorithm can be considered:

1. **Stage 1.** Definition \( \hat{A} = A \).

2. **Stage 2.** Deletion of Pareto-efficient items in \( \hat{A} \) by two parameters weight \( w_i \) (minimum) and importance \( \gamma_i \) (maximum) to obtain the subset \( A' \subseteq \hat{A} \) (the current items layer by Pareto rule).
Stage 3. Assignment of items from $A^P$ to bins.
Stage 4. Definition subset $\hat{A} = A \setminus A^P$. If $|\hat{A}| = 0$ that GO TO Stage 5 Otherwise GO TO Stage 2.
Stage 5. Stop.

Complexity estimates for the above-mentioned version hierarchical clustering algorithm (by stages) is presented in Table 14.

| Stage | Description | Complexity estimate (running time) |
|-------|-------------|-----------------------------------|
| Stage 1 | Definition $\hat{A} = A$. | $O(1)$ |
| Stage 2 | Deletion of current Pareto-efficient items layer $A^P \subseteq \hat{A}$ in $\hat{A}$ (by parameters $w_i$ and $\gamma_i$) | $O(n^2)$ |
| Stage 3 | Assignment of items from $A^P$ to bins. | $O(n)$ |
| Stage 4 | $\hat{A} = A \setminus A^P$. If all items are processed GO TO Stage 2. Otherwise GO TO Stage 5. | $O(n)$ |
| Stage 5 | Stopping | $O(1)$ |

Afterhere, the first objective function $\min t(S) = \frac{1}{n} \sum_{i=1}^{n} \tau_{s[i]} x_{a_i,s[i]}$ will not be considered because items of the solution can be ordered to take into account the objective function.

Evidently, each item (message) can have other parameters, for example, importance (it will leads to an additional objective function in the model above). In this case, the model is:

$$\max \sum_{i=1}^{n} \sum_{i=1}^{n} \beta_{a_i} x_{a_i,s[i]}, \quad \max \sum_{i=1}^{n} \sum_{i=1}^{n} x_{a_i,s[i]}, \quad \max \sum_{i=1}^{n} \sum_{i=1}^{n} x_{a_i,s[i]} \gamma_i$$

s.t. $\sum_{i=1}^{n} \sum_{i=1}^{n} x_{a_i,s[i]} w_i \leq T, \quad x_{a_i,s[i]} \in \{0,1\},$

where $\beta_{a_i}$ is importance parameter of the corresponding item $i$. Note the importance parameter may be dependent on scheduling place $s[i]: \beta_{a_i,s[i]}$.

In the case of multiset estimate of the importance parameter $e_{a_i,s[i]}$, the model is:

$$\max M = \arg \min_{M \in D} \left\{ \bigcup_{i \in \{i | x_{a_i,s[i]} = 1\}} \delta(M,e_i), \quad \max \sum_{i=1}^{n} \sum_{i=1}^{n} x_{a_i,s[i]}, \quad \max \sum_{i=1}^{n} \sum_{i=1}^{n} x_{a_i,s[i]} \gamma_i \right\}$$

s.t. $\sum_{i=1}^{n} \sum_{i=1}^{n} x_{a_i,s[i]} w_i \leq T, \quad x_{a_i,s[i]} \in \{0,1\},$

In addition, precedence binary relation over items can be examined as well. This leads to an additional logical constraint in the model above and corresponding algorithm scheme is based on linear ordering of the selected items while taking account the precedence constraint.

5.2.2. Two-dimensional packing in WiMAX system

In recent decade, two-dimensional packing problems have been used in contemporary telecommunication systems (IEEE 802.16/WiMAX standard). An illustrative structure of WiMAX system is depicted in Fig. 18.
A general description of the above-mentioned approach is presented in [206] as follows. Information transmission process is based on rectangular frames “down link zones”: time (width) × frequency (height). Thus, data packages correspond to 2D items (i.e., rectangular) which are stored in “down link zones” (i.e., bins). In [206], a general three phase solving scheme is examined:

**Phase 1.** Selection of information packages (messages) for the current transmission period.
**Phase 2.** Arranging the selected packets into rectangular regions (as general items).
**Phase 3.** Allocation of the resultant regions to the rectangular frame.

Note, the above-mentioned phase 1 can be based on model and solving approach from the previous section as selection of Pareto-efficient messages (information packages) for the current transmission period. The allocation problem above (i.e., phase 3) is studied in [55,56,57] (including problem statement, complexity, heuristic algorithms, computing experiments). In mobile broadband wireless access systems like IEEE 802.16/WiMAX, Orthogonal Frequency Division Multiple Access (OFDMA) is used in order to exploit frequency and multi-user diversity (i.e., improving the spectral efficiency). MAC (medium access control) frame extends in two dimensions, i.e., time and frequency. At the beginning of each frame, i.e., every 5 ms, the base station is responsible both for scheduling packets, based on the negotiated quality of service requirements, and for allocating them into the frame, according to the restrictions imposed by 802.16 OFDMA.

Here, a two-stage solving scheme for resource allocation is applied (e.g., [55,56,57]): (a) scheduling of packets in a given time frame, (b) allocation of packets across different subcarriers and time slots. The second stage above can be examined as a special 2D bin packing problem [55,56,57].

Evidently, integrated solving scheme for the two above-mentioned stages is a prospective research direction. In general, it is necessary to study the integrated approach for three-phase for planning in WiMAX system from [206]. In addition, it may be prospective to examine ordinal and/or multiset estimates for problem elements including lattice-based quality domain(s) for problem solutions.
6. Conclusion

In this paper, a generalized integrated glance to bin packing problems is suggested. The approach is based on a system structural problem description: (a) element sets (i.e., item set, bin set, item subset assigned to bin), (b) binary relation over the sets above: relation over item set(s) as compatibility, precedence, dominance; relation over items and bins (i.e., correspondence of items to bins). Here, the following objective functions can be examined: (1) traditional functions (i.e., minimizing the number of used bins, maximizing the number of assigned items), (2) weighted and vector versions of the functions above, and (3) the objective functions based on lattices. Some new problem statements with multiset estimates of items are presented. Two applied examples are considered: (i) planning in paper industry, (ii) planning in communication systems (selection of messages, packing of massages in WiMAX).

Generally, it is necessary to point out the following. In recent decades there exists a trend in applied combinatorial optimization (e.g., [182,184,185]):

“FROM basic combinatorial problem TO composite framework consisting of several interconnected combinatorial problems”.

A well-known example of the composite frame is the following: timetabling problem that is usually based a combination of basic combinatorial optimization problems (e.g., assignment, clustering, graph coloring, scheduling). From this viewpoint, bin packing problems and their extensions/modifications can be examined as a basis of various applied composite frameworks. Thus, our material may be useful to build the applied composite frameworks above.

In the future, it may be reasonable to investigate the following research directions:
1. further examination of bin packing problems with multiset estimates;
2. study of various versions of colored bin packing problems (e.g., various color proximities, various objective functions);
3. examination of multi-stage bin packing problems (i.e., models, methods, applications);
4. examination of new applied composite frameworks based on bin packing problems;
5. execution of computing experiments to compare many solving schemes for various bin packing problems with ordinal/multiset estimates;
6. examination of multi-period (or cyclic) multi-channel scheduling problems based on various bin packing models;
7. study of resource allocation in multispot satellite networks on the basis of various bin packing problems;
8. analysis of applied multicriteria bin packing problems and bin packing problems with multiset estimates; and
9. usage of our material in educational courses (e.g., applied mathematics, computer science, engineering, management).
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