The Discrete Geometry of a Small Causal Diamond

Mriganko Roy, Debdeep Sinha and Sumati Surya
Raman Research Institute, Bangalore, India

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Abstract

We study the discrete causal set geometry of a small causal diamond in a curved spacetime using the average abundance $\langle C_k \rangle$ of $k$-element chains or total orders in the underlying causal set $C$. We begin by obtaining the first order curvature corrections to the flat spacetime expression for $\langle C_k \rangle$ using Riemann normal coordinates. For fixed spacetime dimension this allows us to find a new expression for the discrete scalar curvature of $C$ as well as the time-time component of its Ricci tensor in terms of the $\langle C_k \rangle$. We also find a new dimension estimator for $C$ which replaces the flat spacetime Myrheim-Meyer estimator in generic curved spacetimes.

1 Introduction

That there is a profound relationship between order and Lorentzian geometry has been evident ever since the work of Malament, Hawking and others \[1, 2\] where they showed the existence of a bijection between the causal structure, itself a partially ordered set, and the conformal class of the spacetime metric. This is one of the main motivations for the causal set approach to quantum gravity, which assumes that the primitive structure underlying spacetime is a locally finite partially ordered set, or causal set \[3\]. Instead of considering the spacetime metric as the fundamental dynamical variable in causal set theory (CST) it is the causal structure that one wishes to “quantise”. However, to recover the full spacetime geometry from the causal structure, there must be a way to obtain the spacetime volume or equivalently, the conformal factor. This is achieved in CST via the condition of “local finiteness” which implies a fundamental spacetime discreteness: underlying every finite volume region of spacetime is a finite cardinality causal set. Thus, the continuum-discrete correspondence in CST is not exact but approximate, with the continuum being the approximation of the underlying causal set.

In order to maintain the relationship between volume and cardinality in all coordinate systems, a causal set $C$ approximated by a spacetime $(M, g)$ is obtained via the following random, Poisson discretisation of $(M, g)$ \[3, 4\]. Given a fundamental scale $\rho^{-1}$ (which could be the Planck
volume) the probability that a spacetime region of volume \( V \) contains \( N \) elements of \( C \) is given by the Poisson distribution

\[
P_V(N) = \exp^{-\rho V} \frac{(\rho V)^N}{N!},
\]

(1)

for which

\[
\langle N \rangle = \rho V,
\]

(2)

thus establishing the required number to volume correspondence. To give credence to the existence of a fundamental spacetime discreteness CST moreover requires the following conjecture. Namely, if a causal set \( C \) approximates to a spacetime \((M, g)\), then \((M, g)\) is unique, up to modifications to it on scales \(< \rho^{-1}\). In other words, this conjectures that all the meaningful information about the geometry and topology of \((M, g)\) at scales \(\gg \rho^{-1}\) is contained in the causal set; continuum information below these scales is irrelevant since the discrete substructure, i.e., the causal set, is fundamental.

From a purely mathematical point of view, this conjecture is very intriguing. While it has been verified in several different cases a general proof is still not known, though considerable progress has been made in this direction \([5, 6]\). A key question is how to extract continuum topological and geometric properties from \( C \) using purely order theoretic information. Uniqueness of the approximating spacetime with respect to a given geometric or topological property then follows, i.e., any two spacetimes which are approximations to \( C \) must share this property on scales \(\gg \rho^{-1}\). For example, for a causal set \( C \) that is approximated by flat spacetime, the Myrheim-Meyer dimension gives a good estimate of the spacetime dimension \([7, 8]\), while the length of the longest chain or total order between elements in \( C \) gives a good estimate of the time-like distance \([9]\). An estimator for spatial distance in this case has also been obtained \([10]\). Additionally, the homology of spatial hypersurfaces can be constructed from the causal set underlying a globally hyperbolic spacetime \([11, 12]\). A very important recent result is the construction of the scalar curvature from which the causal set action is obtained \([13]\).

A natural question to ask of the flat spacetime results of \([7, 8, 9]\) is how they are modified in the presence of curvature. While it is true in the continuum that a sufficiently small neighbourhood of a point is approximately flat, the corrections from curvature can in fact be well quantified using Riemann normal coordinates in a convex normal neighbourhood of any point. However, we run into the following issue in the discrete case: there is at present no known purely order theoretic definition of a “small” neighbourhood of an element in a causal set which corresponds to a convex normal neighbourhood. At best it may be possible to find approximately flat subsets in a causal set but it is still unclear how to do this in a systematic way \([14]\).

This issue however will not be the focus of our current investigation. Instead we consider only those \( N \)-element causal sets \( C \) which are approximated by a small causal diamond or Alexandrov interval \([p, q]\) in a generic curved spacetime \((M, g)\), where the smallness parameter

\footnote{It is also often referred to as a “fundamental theorem” of CST.}
is given by the proper time $T$ between the events $p$ and $q$. We expand the metric in Riemann normal coordinates (RNC) about an origin $r = (0, \ldots, 0)$

$$g_{ab}(x) = \eta_{ab}(0) - \frac{1}{3} x^c x^d R_{acbd}(0) + O(x^3), \quad (3)$$

where the $O(x)$ correction to the metric vanishes since $\Gamma^c_{ab}(0) = 0$. The RNC has been used to calculate the volume of a small interval $I[p, q]$ \([7, 15, 16]\) for which the first order correction to the volume of $I[p, q]$ due to the effect of curvature occurs at $O(T^{2+n})$.

The starting point of our analysis is Meyer’s work \[8\] in which a general expression was found for the average abundance $\langle C_k \rangle$ of $k$-chains or $k$-element total orders in a causal set $C_0$ which is approximated by an Alexandroff interval $I_0[p, q]$ in flat spacetime. Following Myrheim, Meyer used this to find a dimension estimator for the dimension of $I_0[p, q]$ employing only $\langle C_1 \rangle$, the average abundance of elements in $I_0[p, q]$ and $\langle C_2 \rangle$, the average abundance of relations. Using the RNC expansion and with the help of \[16\] we extend this analysis to the curved spacetime case. We find that the $\langle C_k \rangle$ in $I[p, q]$ satisfy a recursion relation and depend on the scalar curvature $R(0)$ and the time-time component of the Ricci tensor $R_{00}(0)$. It is then an easy exercise to invert these relations and find expressions for $R(0)$ and $R_{00}(0)$ in terms of $\langle C_1 \rangle, \langle C_2 \rangle, \langle C_3 \rangle$ for fixed spacetime dimension. We find that the Myrheim-Meyer dimension estimator is insufficient to determine dimension in a generic curved spacetime and construct a new dimension estimator for generic non-flat spacetimes using $\langle C_1 \rangle, \langle C_2 \rangle, \langle C_3 \rangle, \langle C_4 \rangle$.

As mentioned earlier, the scalar curvature of an element in a causal set was first calculated by Benincasa and Dowker \[13\] using a curved spacetime expression for a non-local D’Alembertian on a causal set. They showed that $R(q)$ for an element $q$ in the $C$ can be expressed in terms of the abundance of $k$-element “inclusive-intervals” which are order theoretically very distinct from $k$-chains. Like $k$-chains they too have a bottom element $e_1$ and a top element $e_k$, but unlike $k$-chains, every element in the interval $I[e_1, e_k]$ which is the intersection of the future of $e_1$ and the past of $e_k$ belongs to the inclusive interval, and has precisely $k - 2$ elements satisfying $e_1 < e_i < e_k$. In contrast, the expression for $R(0)$ that we find depends only on the abundance of $k$-chains. This may suggest that for manifold-like causal sets there are hidden relations between these seemingly different order theoretic entities.

The plan of our paper is as follows. In Section 2 after first presenting some basic definitions, we reproduce Meyer’s results for the $\langle C_k \rangle$ thus setting the notation that we will use in the rest of the paper. In Section 3 we present the main calculation in the paper, where we use the RNC to obtain the lowest order curvature correction to $\langle C_k \rangle$. We show that there is a recursion relation between the coefficients in the expression for the different $\langle C_k \rangle$, but that the form of the dependence on the $R(0)$ and $R_{00}(0)$ is the same for all $k$. In Section 4 for a fixed spacetime dimension we find expressions for $R(0)$ and $R_{00}(0)$ which depend only on $\langle C_1 \rangle, \langle C_2 \rangle, \langle C_3 \rangle$. The coefficients in these expressions again have a simple dependence on dimension. In Section 5 we point out that the Myrheim-Meyer dimension estimator is insufficient in a generic curved spacetime and find a new dimension estimator using $\langle C_1 \rangle, \langle C_2 \rangle, \langle C_3 \rangle, \langle C_4 \rangle$. An important question
in these calculations is how the error decreases with the sprinkling density \( \rho \) (i.e., the inverse of the volume cut-off) – the larger \( \rho \) is the closer one comes to the continuum. In Section 6 using the technique developed in \cite{3} we show that the error in \( \langle C_k \rangle \) grows as \( \rho^{\frac{2k-1}{2}} \) which means that the error in \( R(0), R_{00}(0) \) and \( n \) goes like \( \rho^{-1/2} \), thus going to zero in the continuum limit. We discuss the implications of our results in Section 7 and the questions that need to be addressed in the future. Finally, the Appendix contains explicit calculations of the results that appear in the main body of the paper.

2 The Abundance of \( k \)-Chains in Flat Spacetime

A \( k \)-chain in a causal set \( C \) is a \( k \)-element total order, i.e., a set of elements \( \{e_1, e_2, \ldots, e_k\} \), \( e_i \in C \) such that \( e_i \prec e_{i+1} \) for all \( i \). For any finite element \( C \), the number \( C_k \) of \( k \)-chains is therefore invariant of the choice of labeling of \( C \). This makes \( C_k \) a good observable. Note that the \( e_i \) and \( e_{i+1} \) need not be “linked”, i.e., there could exist an element \( e \in C \) such that \( e_i \prec e \prec e_{i+1} \). Moreover, \( e_1 \prec e \prec e_k \) does not imply that \( e \) belongs to the \( k \)-chain. In contrast, a \( k \)-inclusive interval is defined as \( I_k[e_1, e_k] = \text{Future}(e_1) \cap \text{Past}(e_k) \) \cite{13}. Along with the elements \( e_1, e_k \) it also contains precisely \( k \)-elements. However, every \( e \in C \) such that \( e_1 \prec e \prec e_k \) belongs to \( I_k[e_1, e_k] \), which means that the order theoretic structure of a \( k \)-chain is very different from that of a \( k \)-inclusive interval. The fact that one can express the discrete scalar curvature both in terms of the abundance of the inclusive intervals as shown by Benincasa and Dowker \cite{13} and in terms of the abundance of \( k \)-chains as we will show in Section 4 thus suggests a hidden connection between the two.

We now reproduce Meyer’s results in \( n \)-dimensional flat spacetime \((M_0, \eta)\) using notation that we will find convenient in the curved spacetime generalisation. Let \( p, q \in M_0 \) such that \( p = (-T/2, 0, \ldots, 0) \) and \( q = (T/2, 0, \ldots, 0) \). For a causal set \( C_0 \) that is approximated by \( I_0[p, q] \) for a given sprinkling density \( \rho \), the average abundance of elements, or 1-chains \( \langle C_1 \rangle_\eta \) is given by

\[
\langle C_1 \rangle_\eta = \rho V_0 = \rho \int_{I_0[p,q]} dx_1
\]

\[
= 2\rho \int_0^{\frac{T}{2}} dt \int_0^{\frac{T-t}{2}} dr \ r^{n-2} \int d\Omega_{n-2} = \rho \frac{2A_{n-2}}{n(n-1)} \left( \frac{T}{2} \right)^n = \rho \zeta_0 T^n,
\]  

\( (4) \)

where \( A_{n-2} \) is the volume of the unit \( (n-2) \) sphere \( S^{n-2} \). Next, the average number of 2-chains or relations in \( I_0[p, q] \) is given by the probability of there being a pair of elements \( x_1, x_2 \in I_0[p, q] \) such that \( x_1 \prec x_2 \), or

\[
\langle C_2 \rangle_\eta = \rho^2 \int_{I_0[p,q]} dx_1 \int_{J^+(x_1) \cap I_0[p,q]} dx_2
\]

\( (5) \)
Recognising that the integral over $dx_2$ is simply the volume of the smaller interval $I_0[x_1, q]$ and using Eq (4)

$$\langle C_2 \rangle_\eta = \rho^2 \frac{2A_{n-2}}{2^n n(n-1)} \int_{I_0[p,q]} dx_1 T_1^n = \rho^2 V_0^2 \frac{\Gamma(n+1)\Gamma(\frac{3}{2})}{4\Gamma(\frac{3n}{2})}$$ (6)

Meyer was able to similarly use the nested integral expression for $\langle C_k \rangle_\eta$

$$\langle C_k \rangle_\eta = \rho^k \int_{I_0[p,q]} dx_1 \int_{J^+(x_1)\cap I_0[p,q]} dx_2 \ldots \int_{J^+(x_{k-1})\cap I_0[p,q]} dx_k = \rho^k \int_{I_0[p,q]} dx_1 \langle C_{k-1}(x_1) \rangle_\eta$$ (7)

to find by induction the general form

$$\langle C_k \rangle_\eta = \rho^k \chi_k V_0^k = \rho^k \zeta_k T^{kn},$$ (8)

where

$$\chi_k \equiv \frac{1}{k} \left( \frac{\Gamma(n+1)}{2} \right)^{k-1} \frac{\Gamma(\frac{n}{2})\Gamma(n)}{\Gamma(\frac{k}{2})\Gamma(\frac{(k+1)n}{2})}, \quad \zeta_k \equiv \left( \frac{2A_{n-2}}{2^n n(n-1)} \right)^k \chi_k = \zeta_0^k \chi_k,$$ (9)

with $\zeta_0$ defined as in Eqn (4). Note in particular that $\chi_1 = 1$. We will find it useful to express $\langle C_k \rangle_\eta$ as

$$\langle C_k \rangle_\eta = \rho^k \zeta_{k-1} \int_{I_0[p,q]} dx_1 T_1^{(k-1)n} = \rho^k \zeta_{k-1} I_1((k-1)n)$$ (10)

where $I_1(m)$ is evaluated in Eqn (75) of the Appendix. As discussed above, the average number of chains in a finite element causal set $C$ is itself a covariant observable. In particular, the distribution of the abundance of $k$-chains as a function of $k$ in a finite element causal set can be compared with the distribution of $\langle C_k \rangle_\eta$; if the two distributions agree, it is an indication that the $C$ may be well approximated by flat spacetime and is therefore manifold-like. A similar comparison using $k$-inclusive intervals was found to be useful in determining flat spacetime behaviour in a model of 2d causal set quantum gravity [17]. It is therefore important to find a generalisation of $\langle C_k \rangle_\eta$ to curved spacetime.

Meyer obtained a dimension estimator from $\langle C_k \rangle_\eta$ by observing that the ratio

$$f_0(n) \equiv \frac{\langle C_2 \rangle_\eta}{\langle C_1 \rangle_\eta^2} = \frac{\Gamma(n+1)\Gamma(\frac{3}{2})}{4\Gamma(\frac{3n}{2})}$$ (11)

is only a function of $n$. Thus, one has an expression for the dimension which depends only on order-theoretic information in the causal set. Indeed, $f_0(n)$ is one-half of Myrheim’s ordering fraction

$$f(C) \equiv R \left( \frac{N}{2} \right)^{-1} \approx \frac{2R}{N^2}$$ (12)
where $R = \langle C_2 \rangle$ is the number of relations and $N = \langle C_1 \rangle$. In 2 spacetime dimensions for example, $f(2) = 1/2$, i.e., the inverse of the spacetime dimension. In Section 5 we will show that Eqn (11) does not suffice in curved spacetime and there is need for a new dimension estimator.

3 The Abundance of $k$-Chains in a Small Causal Diamond in Curved Spacetime

The RNC expansion to order $T^2$ gives an expression for $\langle C_1 \rangle$ in $I[p,q]$ \cite{7 15}

$$\langle C_1 \rangle = \rho V = \rho \int_{I[p,q]} \sqrt{-g_1} \, dx_1 = \rho V_0 \left(1 + \alpha_1 R(0) T^2 + \beta_1 R_{00}(0) T^2\right)$$  \hspace{1cm} (13)

where

$$\alpha_1 = -\frac{n}{24(n+1)(n+2)}, \quad \beta_1 = \frac{n}{24(n+1)},$$  \hspace{1cm} (14)

and which uses the RNC expansion

$$\sqrt{-g_1} = 1 - \frac{1}{6} x_1^\mu x_1^\nu R_{\mu\nu}(0) + O(x^3).$$  \hspace{1cm} (15)

Now, the average number of 2-chains or relations is given by the similar generalisation

$$\langle C_2 \rangle = \rho^2 \int_{I[p,q]} \int_{J^+(x_1) \cap I[p,q]} \sqrt{-g_1} \, dx_1 \sqrt{-g_2} \, dx_2 = \rho^2 \int_{I[p,q]} \sqrt{-g_1} \, dx_1 V_1,$$  \hspace{1cm} (16)

where $V_1$ denotes the volume of the region $J^+(x_1) \cap I[p,q]$. Using the covariant form of Eqn (13) we see that

$$\langle C_2 \rangle = \rho^2 \zeta_1 \int_{I[p,q]} dx_1 T^n_1 \left(1 - \frac{1}{6} x_1^\mu x_1^\nu R_{\mu\nu}(0) + \alpha_1 T^2_1 R(y_1) + \beta_1 T_1^\mu T_1^\nu R_{\mu\nu}(y_1)\right) + O(T^{n+3})$$

$$= \rho^2 \zeta_1 \int_{I[p,q]} dx_1 T^n_1$$

$$+ \int_{I_0[p,q]} dx_1 T^n_1 \left(-\frac{1}{6} x_1^\mu x_1^\nu R_{\mu\nu}(0) + \alpha_1 T^2_1 R(y_1) + \beta_1 T_1^\mu T_1^\nu R_{\mu\nu}(y_1)\right) + O(T^{n+3}),$$  \hspace{1cm} (17)

where we have split the integral in the manner of \cite{7} : the first is a flat spacetime integral over the curved spacetime interval $I[p,q]$, whereas the second is the contribution from the curvature...
Figure 1: An Alexandroff interval $I[p, q]$ in flat spacetime. $T_1$ is the proper time between the events $x_1$ and $q$ and $y_1$ is the midpoint of $I[x_1, q]$.

terms over the flat spacetime interval $I_0[p, q]$. $y_1$ is the midpoint of the interval $I[x_1, q]$ as shown in Fig 1. Using the light-cone coordinates $u = t - r$, $v = t + r$ we see from Fig 1 that

$$T_1^\mu = \left(\frac{T}{2}\right)^\mu x_1^\mu, \quad T_1 = \sqrt{(\frac{T}{2} - u_1)\left(\frac{T}{2} - v_1\right)}.$$  \hfill (18)

Thus,

$$T_1^\mu T_1^\nu R_{\mu\nu}(y_1) = \frac{T^2}{4} R_{00}(y_1) - x_1^\mu T R_{0\mu}(y_1) + x_1^\mu x_1^\nu R_{\mu\nu}(y_1).$$  \hfill (19)

The first integral was evaluated in [16] and shown to be of the form

$$\int_{I[p, q]} dx_1 T_1^m = \int_{I_0[p, q]} dx_1 T_1^m \left(1 + \frac{T^2}{24} R_{00}(0)\right)$$  \hfill (20)

for any non-negative integer $m$. Moreover, as can be readily seen, to order $T^2$, $R_{\mu\nu}(y_1)$ can be replaced with $R_{\mu\nu}(0)$. We can moreover simplify the expressions in $\langle C_2 \rangle$ substantially by using the symmetries of $I_0[p, q]$. Expanding the term

$$\int_{I_0[p, q]} dx_1 x_1^i x_1^j R_{\mu\nu}(0) T_1^m = \int_{I_0[p, q]} dx_1 t_1^2 R_{00}(0) T_1^m + 2 \int_{I_0[p, q]} dx_1 t x_1^i R_{0i}(0) T_1^m + \int_{I_0[p, q]} dx_1 x_1^i x_1^j R_{ij}(0) T_1^m,$$  \hfill (21)
for \( m \) a positive integer, we see that the cross-terms do not contribute, so that we are left with

\[
\int_{I_0[p,q]} dx_1 x_1^\mu x_1^\nu R_{\mu\nu}(0) T_1^m = R_{00}(0) \int_{I_0[p,q]} dx_1 t_1^2 T_1^m + \sum_{i=1}^{n-1} R_{ii}(0) \int_{I_0[p,q]} dx_1 (x_1^i)^2 T_1^m, \tag{22}
\]

since the last integral is independent of the spatial direction \( i \), due to the symmetry of \( I_{0[p,q]} \).

Gathering the coefficients of \( R(0) \) and \( R_{00}(0) \)

\[
\langle C_2 \rangle = \rho^2 \xi_1 \left[ I_1(n) + R(0) \left( (\beta_1 - \frac{1}{6}) I_2(n) + \alpha_1 I_1(n+2) \right) \right] + R_{00}(0) \left[ \frac{T^2}{24} I_1(n) + (\beta_1 - \frac{1}{6}) I_2(n) - \frac{1}{6} I_3(n) + \beta_1 I_4(n) \right], \tag{23}
\]

where we have used \( \sum_{i=1}^{n-1} R_{ii}(0) = R_{00}(0) + R(0) \) and we define the general class of integrals

\[
I_1(m) = \int_{I_0[p,q]} dx_1 T_1^m, \quad I_2(m) = \int_{I_0[p,q]} dx_1 T_1^m r_1^2 \cos^2 \theta_1
\]

\[
I_3(m) = \int_{I_0[p,q]} dx_1 T_1^m, \quad I_4(m) = \int_{I_0[p,q]} dx_1 \left( \frac{T}{2} - t_1 \right)^2 T_1^m. \tag{24}
\]

for non-negative integers \( m \). These integrals have been evaluated in the Appendix. Using the terminology defined therein, we will find it useful to re-express the last three integrals in terms of the first

\[
I_2(m) = f_2(m) I_1(m) T^2, \quad I_3(m) = f_3(m) I_1(m) T^2, \quad I_4(m) = f_4(m) I_1(m) T^2
\]

and

\[
I_1(m+2) = g_1(m) I_1(m) T^2,
\]

where \( g_1(m), f_2(m), f_3(m) \) and \( f_4(m) \) are defined in Eqn (87) of the Appendix. Given Eqn (10) we find that

\[
\langle C_2 \rangle = \langle C_2 \rangle_0 \left[ 1 + T^2 \alpha_2 R(0) + T^2 \beta_2 R_{00}(0) \right] \tag{25}
\]

where

\[
\alpha_2 = (\beta_1 - \frac{1}{6}) f_2(n) + \alpha_1 g_1(n)
\]

\[
\beta_2 = \frac{1}{24} + (\beta_1 - \frac{1}{6}) f_2(n) - \frac{1}{6} f_3(n) + \beta_1 f_4(n). \tag{26}
\]
We find that
\[ \alpha_2 = -\frac{4n}{24(2n+2)(3n+2)} \]
\[ \beta_2 = \frac{4n}{24(3n+2)}. \] (27)

We can go one step further and calculate
\[
\langle C_3 \rangle = \rho \int dx_1 \sqrt{-g_1} \langle C_2(x_1) \rangle
\]
\[ = \rho^3 \zeta_2 \left[ I_1(2n) + R(0) \left( (\beta_2 - \frac{1}{6})I_2(2n) + \alpha_2 I_1(2n + 2) \right) + R_{00}(0) \left( \frac{T^2}{24}I_1(2n) + (\beta_2 - \frac{1}{6})I_2(2n) - \frac{1}{6}I_3(2n) + \beta_2 I_4(2n) \right) \right],
\]
\[ = \langle C_3 \rangle_\eta \left[ 1 + T^2 \alpha_3 R(0) + T^2 \beta_3 R_{00}(0) \right]. \] (28)

where again we have used Eqn (10) and
\[ \alpha_3 = (\beta_2 - \frac{1}{6})f_2(2n) + \alpha_2 g_1(2n) = -\frac{6n}{24(3n+2)(4n+2)} \]
\[ \beta_3 = \frac{1}{24} + (\beta_2 - \frac{1}{6})f_2(2n) - \frac{1}{6}f_3(2n) + \beta_2 f_4(2n) = \frac{6n}{24(4n+2)}. \] (29)

This suggests an iterative formula
\[
\alpha_{k+1} = (\beta_k - \frac{1}{6})f_2(kn) + \alpha_k g_1(kn)
\]
\[ \beta_{k+1} = \frac{1}{24} + (\beta_k - \frac{1}{6})f_2(kn) - \frac{1}{6}f_3(kn) + \beta_k f_4(kn), \] (30)

with
\[ \alpha_k = -\frac{nk}{12(kn+2)((k+1)n+2)} \]
\[ \beta_k = \frac{nk}{12((k+1)n+2)}. \] (31)

Lemma 1. To the lowest order correction in the flat spacetime expression, the average number of \( k \)-element chains in a small causal diamond is
\[ \langle C_k \rangle = \langle C_k \rangle_\eta \left[ 1 + T^2 \alpha_k R(0) + T^2 \beta_k R_{00}(0) \right] + O(T^{kn+3}), \] (32)

where \( \alpha_k \) and \( \beta_k \) are given by Eqn (31).
Proof: We will prove this inductively. We have already shown it for \( k = 2 \). Now, let us assume \( \langle C_k \rangle \) is of the form Eqn. (32). Just as in the flat spacetime case, one has nested integrals so that

\[
\langle C_{k+1} \rangle = \rho \int_{I[p,q]} dx_1 \sqrt{-g_1} \langle C_k(x_1) \rangle \\
= \rho^{k+1} \zeta_k \int_{I_0[p,q]} dx_1 T^{kn}_1 \left( 1 + \frac{T^2}{24} R_{00}(0) - \frac{1}{6} x_1^{\mu} x_1^{\nu} R_{\mu\nu}(0) + \alpha_k T^2 R(0) + \beta_k T_1^\mu T_1^\nu R_{\mu\nu}(0) \right) \\
+ O(T^{kn+3})
\]  

(33)

where we have used Eqn (20) to reduce the integral over \( I[p,q] \) to one over \( I_0[p,q] \) to order \( O(T^{kn+2}) \). Using the integrals \( I_{1,2,3,4}(m) \) from the Appendix, we can reduce this to

\[
\langle C_{k+1} \rangle = \rho^{k+1} \zeta_k \left[ I_1(kn) + R(0) \left( (\beta_k - \frac{1}{6}) I_2(kn) + \alpha_k I_1(kn + 2) \right) + \\
\frac{T^2}{24} R_{00}(0) \left( \frac{T^2}{24} I_1(kn) + (\beta_k - \frac{1}{6}) I_2(kn) - \frac{1}{6} I_3(kn) + \beta_k I_4(kn) \right) \right],
\]

\[
= \langle C_{k+1} \rangle_\eta \left[ 1 + T^2 R(0) \left( (\beta_k - \frac{1}{6}) f_2(kn) + \alpha_k g_1(kn + 2) \right) + \\
T^2 R_{00}(0) \left( \frac{1}{24} + (\beta_k - \frac{1}{6}) f_2(kn) - \frac{1}{6} f_3(kn) + \beta_k f_4(kn) \right) \right].
\]

(34)

Writing it in the form Eq (32), with

\[
\alpha_{k+1} = (\beta_k - \frac{1}{6}) f_2(kn) + \alpha_k g_1(kn + 2)
\]

\[
\beta_{k+1} = \frac{1}{24} + (\beta_k - \frac{1}{6}) f_2(kn) - \frac{1}{6} f_3(kn) + \beta_k f_4(kn).
\]

(35)

we find the desired form for \( \langle C_{k+1} \rangle \). It then follows from the expressions for \( g_1(m), f_2(m), f_3(m) \) and \( f_4(m) \) (Eqns. (76), (78), (79), (86)) that

\[
\alpha_{k+1} = \frac{n(k + 1)}{12((k + 1)n + 2)((k + 2)n + 2)}
\]

\[
\beta_{k+1} = \frac{n(k + 1)}{12((k + 2)n + 2)}.
\]

(36)
4 Scalar Curvature from the Abundance of $k$-Chains

For a fixed $n$ $\langle C_k \rangle$ contains three unknowns, $T, R(0)$ and $R_{00}(0)$. Thus, we need at least three values of $k$ in order to determine $R(0)$. For each $\langle C_k \rangle$ the lowest order correction due to curvature is $O(T^{kn+2})$. Hence, as in the flat spacetime calculation of the Myrheim-Meyer dimension Eqn (11) we must take appropriate powers of $\langle C_1 \rangle, \langle C_2 \rangle$ and $\langle C_3 \rangle$ to be able to compare their lowest order corrections. Defining

$$Q_k \equiv \left( \langle C_k \rangle \right)^{3/k} = \frac{1}{\zeta_0} \left( \frac{\langle C_k \rangle}{\rho^k \chi_k} \right)^{3/k}$$

for $k = 1, 2, 3$

$$Q_1 = T^{3n} \left( 1 + 3\alpha_1 R(0)T^2 + 3\beta_1 R_{00}(0)T^2 \right) + O(T^{3n+2})$$

$$Q_2 = T^{3n} \left( 1 + \frac{3}{2} \alpha_2 R(0)T^2 + \frac{3}{2} \beta_2 R_{00}(0)T^2 \right) + O(T^{3n+2})$$

$$Q_3 = T^{3n} \left( 1 + \alpha_3 R(0)T^2 + \beta_3 R_{00}(0)T^2 \right) + O(T^{3n+2})$$

Thus, the $Q_k$ are independent of the sprinkling density $\rho$ and hence can be used to construct continuum geometric parameters. It is useful to gather a few identities and definitions before we proceed:

$$\beta_k \alpha_k^{-1} = -(kn + 2)$$

$$\Psi_k \equiv \alpha_k \beta_{k+1} - \alpha_{k+1} \beta_k = -n \alpha_k \alpha_{k+1}$$

$$\Phi_k \equiv \frac{k}{k+1} \beta_{k+1} - \beta_k$$

$$\Theta_k \equiv \frac{k}{k+1} \alpha_{k+1} - \alpha_k$$

$$K_k \equiv ((k + 1)n + 2)Q_k,$$

$$J_k \equiv (kn + 2)K_k$$

We eliminate the $R_{00}(0)$ term from Eqn (38) and (39) and subsequently from Eqn (39) and (40) to get the pair of equations

$$\left( \frac{\beta_2}{2} Q_1 - \beta_1 Q_2 \right) T^{-3n} = \Phi_1 + \frac{3}{2} \Psi_1 T^2 R(0)$$

$$\left( \frac{2\beta_3}{3} Q_2 - \beta_2 Q_3 \right) T^{-3n} = \Phi_2 + \Psi_2 T^2 R(0).$$
Since both $R(0)$ and $T$ are unknowns, we first eliminate the $R(0)T^2$ term:

$$
\left( \frac{\beta_2}{3} \Psi_2 Q_1 - \frac{2}{3}(\beta_1 \Psi_2 - \beta_3 \Psi_1)Q_2 + \beta_2 \Psi_1 Q_3 \right) T^{-3n} = \frac{2}{3} \Psi_2 \Phi_1 - \Psi_1 \Phi_2.
$$

(44)

We find after some algebraic manipulation that

$$T^{3n} = \frac{1}{2n^2} \left( J_1 - 2J_2 + J_3 \right).$$

(45)

with the $J_i$'s given by Eqn. (41). We thus obtain the expression for the scalar curvature

$$R(0) = -\frac{2(n+2)(2n+2)(3n+2)}{n^3T^{3n+2}} \left( K_1 - 2K_2 + K_3 \right)$$

(46)

or more explicitly

$$R(0) = -2(n+2)(2n+2)(3n+2)2^{\frac{3n+2}{3n+1}}n^{\frac{4}{3n+1}} \frac{(K_1 - 2K_2 + K_3)}{(J_1 - 2J_2 + J_3)^{\frac{2n+2}{3n+1}}}.$$  

(47)

We may additionally solve for $R_{00}(0)$ by eliminating $R(0)$ from Eqn (38), (39), (40):

$$\left( \frac{\alpha_2}{2} Q_1 - \alpha_1 Q_2 \right) T^{-3n} = \Theta_1 - \frac{3}{2} \Psi_1 T^2 R(0)$$

(48)

$$\left( \frac{2\alpha_3}{3} Q_2 - \alpha_2 Q_3 \right) T^{-3n} = \Theta_2 - \Psi_2 T^2 R(0).$$

(49)

Either equation along with Eqn (45) gives

$$R_{00}(0) = -4\frac{(2n+2)(3n+2)}{n^3T^{3n+2}} \left( (n+2)Q_1 - (5n+4)Q_2 + (4n+2)Q_3 \right).$$

(50)

As a check, let us consider the case $R_{00}(0) = 0$ so that from Eqn (38)-(40) we see that

$$R(0) > 0 \Rightarrow Q_1 < Q_2 < Q_3$$

$$R(0) < 0 \Rightarrow Q_1 > Q_2 > Q_3$$

$$R(0) = 0 \Rightarrow Q_1 = Q_2 = Q_3 = T^{3n}.$$

(51)

Moreover,

$$K_1 - 2K_2 + K_3 = -\frac{n^3}{2(n+2)(2n+2)(3n+2)} R(0)T^{3n+2}$$

(52)

which therefore has the opposite sign to $R(0)$ and is zero when $R(0) = 0$.

As a further check, we note that if both $R_{00}(0) = 0$ and $R(0) = 0$, $Q_1 = Q_2 = Q_3 = T^{3n}$ so that not only is $K_1 - 2K_2 + K_3 = (2n+2)Q_1 - 2(3n+2)Q_2 + (4n+2)Q_3 = 0$, but also $(n+2)Q_1 - (5n+4)Q_2 + (4n+2)Q_3 = 0$ which appears in Eqn (50).
5 A New Dimension Estimator for Curved Spacetime

As one can guess by now, the ordering fraction or equivalently the function \( f(n) \) in curved spacetime clearly involves the curvature contribution non-trivially. Expanding to order \( T^2 \) the curved spacetime version of the Myrheim-Meyer dimension estimator is

\[
f(n) = \frac{\langle C_2 \rangle}{\langle C_1 \rangle^2}
\]

\[
= f_0(n) \left( 1 + T^2(\alpha_2 - 2\alpha_1)R(0) + T^2(\beta_2 - 2\beta_1)R_{00}(0) \right) + O(T^3)
\]

\[
= f_0(n) \left( 1 + \frac{n^2 T^2}{12(n+1)(3n+2)} \left( \frac{2}{(n+2)}R(0) - R_{00}(0) \right) \right) + O(T^3). \tag{53}
\]

In the special case that \( R(0) = R_{00}(0) = 0 \), \( f(n) \approx f_0(n) \) up to order \( T^2 \). For a generic spacetime \( f(n) \) it is however insufficient as a dimension estimator and we must find a replacement. Given that along with \( n \) there are 4 unknowns to be solved in terms of the \( \langle C_k \rangle \), the simplest way to do so is to include \( \langle C_4 \rangle \) in our analysis.

We define

\[
S_k = \left( \frac{\langle C_k \rangle}{\rho^k \zeta_k} \right)^{4/k}
\tag{54}
\]

analogous to the \( Q_i \) in the previous section, with \( k = 1, 2, 3, 4 \) which again is independent of the sprinkling density. To the lowest order correction we have the four equations

\[
S_1 = T^{4n} \left( 1 + 4\alpha_1 R(0)T^2 + 4\beta_1 R_{00}(0)T^2 \right)
\]

\[
S_2 = T^{4n} \left( 1 + 2\alpha_2 R(0)T^2 + 2\beta_2 R_{00}(0)T^2 \right)
\]

\[
S_3 = T^{4n} \left( 1 + \frac{4}{3}\alpha_3 R(0)T^2 + \frac{4}{3}\beta_3 R_{00}(0)T^2 \right)
\]

\[
S_4 = T^{4n} \left( 1 + \alpha_4 R(0)T^2 + \beta_4 R_{00}(0)T^2 \right) \tag{55}
\]

Eliminating \( R(0)T^2 \) from the above we get

\[
\left( \frac{1}{2}\alpha_2 S_1 - \alpha_1 S_2 \right) = T^{4n} \left( \Theta_1 - 2\Psi_1 R_{00}T^2 \right)
\]

\[
\left( \frac{2}{3}\alpha_3 S_2 - \alpha_2 S_3 \right) = T^{4n} \left( \Theta_2 - \frac{4}{3}\Psi_2 R_{00}T^2 \right)
\]

\[
\left( \frac{3}{4}\alpha_4 S_3 - \alpha_3 S_4 \right) = T^{4n} \left( \Theta_3 - \Psi_3 R_{00}T^2 \right) \tag{56}
\]

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from which we may eliminate \( R_{00}(0)T^2 \) to get
\[
\frac{4}{3}\left(\frac{1}{2}\alpha_2 S_1 - \alpha_1 S_2\right)\Psi_2 - 2\left(\frac{2}{3}\alpha_3 S_2 - \alpha_2 S_3\right)\Psi_1 = T^{4n}\left(\frac{4}{3}\Theta_1 \Psi_2 - 2\Theta_2 \Psi_1\right)
\]
\[
\frac{3}{4}\left(\frac{2}{3}\alpha_3 S_2 - \alpha_2 S_3\right)\Psi_3 - \left(\frac{3}{4}\alpha_4 S_3 - \alpha_3 S_4\right)\Psi_2 = T^{4n}\left(\frac{3}{4}\Theta_2 \Psi_3 - \Theta_3 \Psi_2\right).
\]

This gives us an expression for \( T^{4n} \) (which we check reduces to Eqn(45)). After some algebra this gives the following implicit form for the dimension:
\[
(n + 2)(2n + 2)S_1 - 3(2n + 2)(3n + 2)S_2 + 3(3n + 2)(4n + 2)S_3 - (4n + 2)(5n + 2)S_4 = 0. \quad (57)
\]

Importantly, in the absence of curvature the \( S_k \) are all equal and the left-hand side is identically zero. Hence this cannot be a replacement for the Myrheim-Meyer dimension in flat spacetime.

Using \( U_k = \left(\frac{n}{k + 2}\right)\left(\frac{(k + 1)n + 2}{(k + 1)n + 2}\right)S_k \) we may write the expression more succinctly as
\[
U_1 - 3U_2 + 3U_3 - U_4 = 0. \quad (58)
\]

It is interesting to note the appearance of the binomial coefficients \((-1)^k \binom{r-1}{k}\) for \( r = 4 \) in the above expression for the dimension estimator, as well as in the expressions for \( R(0) \) and \( T \) for \( r = 3 \).

Since the \( S_k \) themselves explicitly contain dimension information via the \( \zeta_k \) it is more useful to expand the expression to
\[
(n + 2)(2n + 2)\left(\frac{\langle C_1 \rangle}{\chi_1}\right)^4 - 3(2n + 2)(3n + 2)\left(\frac{\langle C_2 \rangle}{\chi_2}\right)^2
\]
\[
+3(3n + 2)(4n + 2)\left(\frac{\langle C_3 \rangle}{\chi_3}\right)^{4/3} - (4n + 2)(5n + 2)\left(\frac{\langle C_4 \rangle}{\chi_4}\right) = 0 \quad (59)
\]
or defining
\[
\omega_k \equiv (-1)^k \left(\frac{3}{k - 1}\right)\left(\frac{kn + 2}{(k + 1)n + 2}\right)\frac{\langle C_k \rangle^{4/k}}{\chi_k} \quad (60)
\]
we get our final expression for the dimension estimator in curved spacetime
\[
\sum_{k=1}^{4} \omega_k(n)\langle C_k \rangle^{4/k} = 0. \quad (61)
\]

6 Calculating the Errors

We expect that as the sprinkling density \( \rho \) increases, our curvature and dimension estimators should do a better job of reproducing the continuum results. While the geometric parameters
themselves do not depend on $\rho$, it is clear that the error will. The (rms) error $\delta C_k = \sqrt{\Delta C_k} = \sqrt{\langle C_k^2 \rangle - \langle C_k \rangle^2}$, where $\Delta C_k$ is the variance. We follow the analysis in [8] to find the dependence of $\Delta C_k$ on $\rho$ in the RNC. Unlike the flat spacetime case however, it is not the continuum volume that must be increased to improve accuracy since this region must still be “small” for the RNC to be valid. Instead, it is the sprinkling density $\rho$ that should be increased for reducing the error.

Let us begin with $k = 2$, so that $\Delta C_2 = \langle C_2^2 \rangle - \langle C_2 \rangle^2$. Now $\langle C_2^2 \rangle$ is the probability of finding two sets of 2-chains in $I[p,q]$, with the possibility that some of the elements can coincide. Let us call the points of these two chains $x, x'$ and $y, y'$, with $x \prec x'$ and $y \prec y'$. Thus, $\langle C_2^2 \rangle$ gets contributions from each type of coincidence. The first is simply that there are no coincidences, i.e., that all 4 points $x, y, x', y'$ are distinct, which gives a contribution $\langle C_2 \rangle^2$. Although this term $\sim \rho^4$, it cancels out in the expression for the variance and therefore plays no role. The next type is the one-coincidence case. For this there are two types: (i) $x = y'$ or $y = x'$, so that the two 2-chains collapse to a single 3-chain, and (ii) $x \neq y$ and $x' \neq y'$, corresponding to the probability for a three element “V” or “Λ” shaped causal set. The contribution from (i) is clearly twice that of $\langle C_3 \rangle$ whose $\rho$ dependence is $\sim \rho^3$, while the contribution from (ii) is

$$2\rho^3 \int_{I[p,q]} dx \sqrt{-g(x)} \int_{J^+(x) \cap I[p,q]} dx' \sqrt{-g(x')} \int_{J^+(x') \cap I[p,q]} dy' \sqrt{-g(y')} = 2\rho \int_{I[p,q]} dx \langle C_1(x) \rangle^2 \sim \rho^3. \quad (62)$$

Finally, there is a contribution from two coincidences $x = y, x' = y'$, which is just $\langle C_2 \rangle$ which goes as $\sim \rho^2$. Thus, $\Delta C_2 \sim \rho^3$ or $\delta C_2 \sim \rho^{3/2}$, with the dominant contribution coming from the one-coincidence case.

In order to calculate the error for all our geometric parameters, we need to perform a similar analysis for $k = 3, 4$. In each such case, the dominant contribution to the error comes from the one-coincidence case, since the no-coincidence contributions simply cancel out.

For $k = 3$, if $x \prec x' \prec x''$ and $y \prec y' \prec y''$, the no-coincidence case is again simply $\langle C_3^2 \rangle$ and again cancels out. The one-coincidence terms include: (i) $x = y''$ or $y = x''$ which is the 5-chain, $\langle C_5 \rangle \sim \rho^5$, (ii) $x = y'$ or $y = x'$ or $x'' = y'$ or $y'' = x'$ which contribute

$$4\rho^2 \int_{I[p,q]} dy \sqrt{-g(y)} \int_{J^+(y) \cap I[p,q]} dx \sqrt{-g(x)} \langle C_2(x) \rangle \langle C_1(x) \rangle \sim \rho^5, \quad (63)$$

(iii) $x = y$ or $x'' = y''$ which contribute

$$2\rho \int_{I[p,q]} dx \sqrt{-g(x)} \langle C_2(x) \rangle^2 \sim \rho^5, \quad (64)$$

and finally (iv) $x' = y'$ which contribute

$$\rho^3 \int_{I[p,q]} dx \sqrt{-g(x)} \int_{I[p,q]} dy \sqrt{-g(y)} \int_{J^+(x) \cap J^+(y) \cap I[p,q]} dx' \sqrt{-g(x')} \langle C_1(x) \rangle^2 \sim \rho^5. \quad (65)$$
Figure 2 shows the contributions to $\langle C_3^2 \rangle$ from the case of no-coincidence and the case of 1-coincidence. Since the no-coincidence term cancels out, $\delta C_3 \sim \rho^{5/2}$.

Figure 2: The contributions to $\langle C_3^2 \rangle$ from no-coincidences and one-coincidence are shown.

A similar analysis shows that for $\langle C_4^2 \rangle$ the no-coincidence term cancels out and the one-coincidence cases lead to a dependence $\sim \rho^7$, so that $\delta C_4 \sim \rho^{7/2}$. This generalises in a straightforward manner to

$$\delta C_k \sim \rho^{2k-1/2}.$$ (66)

We are now in a position to calculate the dependence on $\rho$ of the errors in the $Q_k$ and $S_k$:

$$\delta Q_k \sim \frac{3}{k\rho^3} (C_k)^{3/k-1} \delta C_k \sim \rho^{-1/2}.\hspace{1cm} (67)$$

and

$$\delta S_k \sim \frac{4}{k\rho^4} (C_k)^{4/k-1} \delta C_k \sim \rho^{-1/2}.\hspace{1cm} (68)$$

This immediately means that the errors $\delta T^{3n}$, $\delta R(0)$, $\delta R_{00}(0)$ go as $\rho^{-1/2}$ and hence become smaller as $\rho$ increases. The error in the dimension estimator is similarly given by

$$\delta n \left( (4n+6)S_1 - 3(12n+10)S_2 + 3(24n+14)S_3 - (40n+18)S_4 \right)$$

$$= - \left( (n+2)(2n+2)\delta S_1 - 3(2n+2)(3n+2)\delta S_2 + 3(3n+2)(4n+2)\delta S_3 - (4n+2)(5n+2)\delta S_4 \right)$$

$$\Rightarrow \delta n \sim \rho^{-1/2}.\hspace{1cm} (69)$$

### 7 Conclusions and Remarks

In this work we have found expressions for the proper time $T$ (Eqn (45)), the scalar curvature $R(0)$ (Eqn (47)), the time-time component of the Ricci tensor $R_{00}(0)$ (Eqn (50)), and a new
dimension estimator (Eqn (61)) from a causal set underlying a small causal diamond \( I[p, q] \) in a generic spacetime in arbitrary dimensions. We find that the errors in these estimators goes as \( \rho^{-1/2} \), thus becoming smaller as the sprinkling density is increased, while keeping the volume of \( I[p, q] \) fixed. Our results not only verify the deep relationship between order and Lorentzian geometry, but also provide new observables that can be used to assess whether a causal set is manifold-like or not.

Our calculation moreover brings to light an intriguing connection between two seemingly disparate order theoretic structures in a causal set. While our expression for the scalar curvature is purely in terms of the abundance of \( k \)-chains, the Benincasa-Dowker (BD) scalar curvature \( R_{BD} \) is constructed from the abundance of \( k \)-inclusive intervals. In 4-dimensions for example, their expression for the scalar curvature is

\[
R_{BD}(0) = \frac{2}{6\sqrt{\rho}} (1 - (N_2(0) - 9N_3(0) + 16N_4(0) - 8N_5(0)))
\] (70)

where \( N_k(0) \) are the number of \( k \)-inclusive intervals \( I_k[x, 0] \), where \( x \prec 0 \). As noted in Section \( 2 \), \( k \)-chains and \( k \)-inclusive intervals are very distinct order theoretic structures. The continuum geometry however seems to link them via the scalar curvature. If such a relationship exists, does it for example indicate manifold-likeness in a causal set? This and several related questions remain to be investigated.

Further comparisons between the two expressions for \( R(0) \) are warranted. Both contain alternating sums, although the coefficients differ markedly. For one, the BD expression appears to have a strong dimension dependence in the number of terms required – for \( n = 2 \) the sum is truncated at \( k = 4 \), while for \( n = 4 \) it is truncated at \( k = 5 \). Our expression for \( R(0) \) is in this sense independent of \( n \) – it requires \( \langle C_1 \rangle, \langle C_2 \rangle, \langle C_3 \rangle \) in all dimensions. Moreover, the systematic determination of the coefficients for \( R_{BD} \) for arbitrary \( n \) is fairly involved, whereas the coefficients in Eqn (47) and Eqn (58) are simply the binomial coefficients \( (-1)^k \binom{2}{k} \) and \( (-1)^k \binom{3}{k} \), respectively, in all dimensions.

Apart from these, there are deeper differences in the two expressions for \( R(0) \). \( R_{BD}(0) \) is constructed from all inclusive intervals of the form \( I_k[x, 0] \) in the causal set for \( k = 1, \ldots, 5 \). It is therefore an essentially non-local expression since it depends on the structure of the causal set throughout a past (or future) neighbourhood of the element 0 in the causal set. The expression Eqn (47) on the other hand is only valid in a small causal diamond, and hence is strictly local and dependent on a proper choice of neighbourhood of the element in the causal set. This means that the BD form for \( R(0) \), being “neighbourhood independent”, can be readily used to obtain an action for the entire causal set, which has strong implications for causal set theory. Unless

\footnote{Our notation differs from \cite{13} where \( k \) is replaced by \( k - 1 \), in keeping with our definition of a \( k \)-chain.}

\footnote{As may be already evident to the astute reader, one could replace these three values \( k = 1, 2, 3 \) with any other \( k_1, k_2, k_3 \), to get an expression for \( R(0) \) in terms of \( \langle C_{k_1} \rangle, \langle C_{k_2} \rangle, \langle C_{k_3} \rangle \). What is important is that this choice is not dimension dependent – every choice works for every \( n \).}
the definition of a small neighbourhood can be made entirely order theoretically in the causal set, our expression for $R(0)$ on the other hand cannot be used to obtain an action in a simple manner. Of course, in the specific case when the entire causal set is approximated by a small causal diamond $I[p, q]$ the action is simply

$$S/h = \sum_{s=1}^{N} R(e_s) = NR,$$  \hspace{1cm} (71)

where $e_s$ denotes an element in $C$, since to this approximation $R(x)$ is the same throughout $I[p, q]$.

Nevertheless, the geometric estimators we have found in this work take a big step towards the order-Lorentzian geometry correspondence. We have a new dimension estimator which can determine the manifold dimension for a curved spacetime with greater accuracy than the currently available flat spacetime Myrheim-Meyer estimator. The local nature of the expression for the scalar curvature can also be seen as an advantage since only a small neighbourhood of an element in the causal set is required to determine the curvature, rather than its entire past. An obvious next step is to follow up with a numerical analysis of causal sets which are approximated by different curved spacetimes and to see how well our estimators work in these cases [18].

These observables can also be used as additional tests of manifold-likeness of a causal set. In [17] the expectation values of a range of such observables, including the abundance $N_k$ of $k$-element inclusive intervals was found from Monte Carlo simulations of 2-d causal set quantum gravity. Comparing with the flat spacetime distribution of the $N_k$, these observables were used to demonstrate that the dominant contribution to the causal set path integral in 2d comes from flat spacetime suggesting that manifold-like behaviour is emergent. The estimators we have obtained in this current work could well be employed to calculate more accurately how close to flatness one is – is the 2-d universe just a little positively curved, for example?

The extended hope of the current analysis is also that as more of the order theoretic basis of geometry is uncovered it may be possible to find a meaningful order theoretic definition of locality which translates to our commonly held (albeit Riemannian geometry based) notions of locality in the continuum.
8 Appendix

To evaluate the $\langle C_k \rangle$ we find it convenient to transform from the Cartesian coordinates $x^i$ to spherical polar coordinates $x^i = rf^i(\Omega)$, where

$$f^i(\theta_1, \theta_2 \ldots \theta_{n-2}) = \begin{cases} 
\prod_{k=1}^{n-i-1} \sin \theta_k \cos \theta_{n-i} & (i > 1) \\
\prod_{k=1}^{n-2} \sin \theta_k & (i = 1)
\end{cases}$$

It is also useful to express the radial coordinate for any $x_1 \in I[p, q]$ in light-cone coordinates:

$$r_1 = \frac{v_1 - u_1}{2} = \frac{1}{2} \left( \left( \frac{T}{2} - u_1 \right) - \left( \frac{T}{2} - v_1 \right) \right)$$

$$\Rightarrow \quad r_1^l = \frac{1}{2^l} \sum_{k=0}^{l} (-1)^k \binom{l}{k} \left( \frac{T}{2} - v_1 \right)^k \left( \frac{T}{2} - u_1 \right)^{l-k}$$

Since the metric $\eta(u, v)$ in $I_0[p, q]$ in light cone coordinates is

$$ds^2 = -dudv + \left( \frac{v - u}{2} \right)^2 d\Omega^2,$$  \hspace{1cm} (72)$$

the associated measure of any integral over $I_0[p, q]$ is

$$\sqrt{-\eta(u, v)} = \frac{1}{2} \left( \frac{v - u}{2} \right)^{n-2} \left( \prod_{i=1}^{n-3} \sin^{(n-i-2)} \theta_i \right)$$

$$= \frac{1}{2^{n-1}} \left( \prod_{i=1}^{n-3} \sin^{(n-i-2)} \theta_i \right) \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \left( \frac{T}{2} - v \right)^k \left( \frac{T}{2} - u \right)^{n-2-k}. \hspace{1cm} (73)$$

Finally, we will find it useful to define the following pair of integrals:

$$L(a, v, T) = \int_{-\frac{T}{2}}^{v} du \left( \frac{T}{2} - u \right)^a = \frac{1}{a+1} \left( T^{a+1} - \left( \frac{T}{2} - v \right)^{a+1} \right)$$

$$L(a, b, T) = \int_{-\frac{T}{2}}^{v} du \left( \frac{T}{2} - v \right)^b L(a, v, T) = \frac{T^{a+b+2}}{(b+1)(a+b+2)}.$$

(74)

The above identities will now be used to evaluate the set of integrals $I_{1,2,3,4}(m)$ required for the calculation of $\langle C_k \rangle$ (Eqns (17), (28) (33)).
1.

\[ I_1(m) = \int dx_1 T_1^m \]

\[ = \frac{T}{2} \int_{-T}^{v_1} \int_{-T}^{u_1} \int d\Omega \sqrt{-\eta(u_1,v_1)} T_1^m \]

\[ = \frac{A_{n-2}}{2^{n-1}} \sum_{k=0}^{n-2} \binom{n-2}{k} L\left(\frac{m}{2} + n - k - 2, \frac{m}{2} + k, T\right) \]

\[ = \frac{A_{n-2} T^{n+m}}{2^{n-1}} \frac{(n-2)! \left(\frac{m}{2}\right)!}{(n+m)(n+\frac{m}{2}+1)!}. \] (75)

2. For \( m + 2 \) the above equation gives us the relation

\[ I_1(m + 2) = I_1(m) T^2 \frac{(n + m)(\frac{m}{2} + 1)}{(n + m + 2)(n + \frac{m}{2})} = I_1(m) T^2 g_1(m). \] (76)

3. As noted in Section 3, any integral of the form

\[ I_2(m) = \int dx_1 (x_1^i)^2 T_1^m \]

is independent of the spatial direction \( i \) because of the spatial symmetry of \( I_0[p,q] \). Thus, we can choose \( x^{n-1} = r \cos \theta_1 \) to simplify our calculation so that

\[ I_2(m) = \int dx_1 T_1^m r_1^2 \cos^2 \theta_1 \]

\[ = \frac{T}{2} \int_{-T}^{v_1} \int_{-T}^{u_1} \int d\Omega \cos^2 \theta_1 \left(\frac{T}{2} - u_1\right)^\frac{m}{2} \left(\frac{T}{2} - v_1\right)^\frac{m}{2} (v_1 - u_1)^n \]

\[ = \frac{A_{n-2}}{2^{n+1}} \frac{T^{n+m+2}}{(n-1)(n+m+2)(n+\frac{m}{2}+1)!} \frac{n! \left(\frac{m}{2}\right)!}{n(n+m)} \]

\[ = \frac{I_1(m) T^2}{4(n+m+2)(n+\frac{m}{2}+1)(n+\frac{m}{2})} \]

\[ = I_1(m) T^2 f_2(m). \] (78)

The next integral can be split into three parts

\[ I_3(m) = \int dx_1 t_1^2 T_1^m = I_3^x + I_3^y + I_3^z \] (79)
where, using \( a_k = n - 2 - k + \frac{m}{2} \), \( b_k = k + \frac{m}{2} \)

\[
I_3^a(m) = \frac{1}{4} \int dx_1 u_1^2 T_1^m
\]
\[
= \frac{A_{n-2}}{2^{n+1}} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \left( \frac{T^2}{4} L(a_k, b_k, T) - TL(a_k + 1, b_k, T) + L(a_k + 2, b, T) \right),
\]

\[
(80)
\]

\[
I_3^b(m) = \frac{1}{4} \int dx_1 v_1^2 T_1^m
\]
\[
= \frac{A_{n-1}}{2^{n+1}} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \left( \frac{T^2}{4} L(a_k, b_k, T) - TL(a_k, b_k + 1, T) + L(a_k, b + 2, T) \right),
\]

\[
(81)
\]

and

\[
I_3^c(m) = \frac{2}{4} \int dx_1 u_1 v_1 T_1^m
\]
\[
= \frac{A_{n-2}}{2^n} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \left( \frac{T^2}{4} L(a_k, b_k, T) - \frac{T}{2} L(a_k + 1, b_k, T) - \frac{T}{2} L(a_k, b_k + 1, T) + L(a_k + 1, b + 1, T) \right).
\]

\[
(82)
\]

After some algebra, we find that

\[
I_3(m) = I_1(m) T^2 \frac{8n + m(m + 2)(m + n + 2)}{4(2m + n)(2n + m)(2n + m + 2)} = I_1(m) T^2 f_3(m).
\]

\[
(83)
\]

4. Finally,

\[
I_4(m) = \int dx_1 \left( \frac{T}{2} - t_1 \right)^2 T_1^m
\]
\[
= \frac{T^2}{4} I_1(m) + I_3(m) - T \dot{I}_4(m).
\]

\[
(84)
\]

Evaluating

\[
\dot{I}_4(m) = \int dx_1 t_1 T^m
\]
\[
= I_1(m) \frac{T}{2} \left( 1 - \frac{m + n}{m + n + 1} - \frac{(\frac{m}{2} + 1)(m + n)}{(m + n + 1)(\frac{m}{2} + n)} \right)
\]
\[
= I_1(m) T \dot{f}_4(m).
\]

\[
(85)
\]
We can then express

\[ I_4(m) = I_1(m)T^2f_4(m) \]  \hspace{1cm} (86)

where \( f_4(m) = \frac{1}{4} + \tilde{f}_4(m) + f_3(m) \).

Gathering these expressions

\[ g_1(m) = \frac{(n + m)(\frac{m}{2} + 1)}{(n + m + 2)(n + \frac{m}{2})} \quad \quad f_2(m) = \frac{n(n + m)}{4(n + m + 2)(n + \frac{m}{2} + 1)(n + \frac{m}{2})} \]

\[ f_3(m) = \frac{8n + m(m + 2)(m + n + 2)}{4(2 + m + n)(2n + m)(2n + m + 2)} \quad \quad f_4(m) = \frac{1}{4} - \tilde{f}_4(m) + f_3(m). \]  \hspace{1cm} (87)

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