Theory and simulations of spherical and cylindrical Langmuir probes in non-Maxwellian plasmas

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Abstract

The collected current by spherical and cylindrical Langmuir probes immersed in an unmagnetized and collisionless non-Maxwellian plasma at rest are theoretically studied, and analytical expressions for the currents of attracted and repelled plasma particles are presented. We consider Kappa, Cairns and the generalized Kappa–Cairns distributions as possible models for the velocity field in the plasma. The current–voltage characteristics curves are displayed and discussed. Furthermore, comparisons with the collected currents in Maxwellian plasmas are given. The results of Particle-in-Cell (PIC) simulations of spherical and cylindrical probes in non-Maxwellian plasmas are also presented, and compared with the theoretical expressions. The results for the collected currents by the Langmuir probes obtained by PIC simulations are in good agreement with the corresponding analytical expressions.

Keywords: spherical and cylindrical Langmuir probes, non-Maxwellian plasmas, particle-in-cell simulations

(Some figures may appear in colour only in the online journal)

1. Introduction

Plasmas that are in stationary states far from thermal equilibrium often follow power-law distributions [1, 2]. The charged-particle velocity distributions in space plasmas are quite commonly shown to be non-Maxwellian with superthermal tails decreasing as a power-law of the velocity. Such velocity distributions have been measured [3–5], where it was also shown that a good fit to the measurements can be achieved for the Kappa/Vasyliunas distribution, which is a generalization of the Maxwellian distribution.

Another velocity distribution function, known as the Cairns distribution, has been introduced [6] in order to model superthermal particles observed by the Viking spacecraft [7] and the Freja satellite [8]. Similar to the Kappa distribution, the Cairns distribution is also a generalization of the Maxwellian distribution. As will be explained in more detail in section 2, by taking certain limits of Kappa and Cairns distributions functions, these distributions will reduce to a Maxwellian distribution. A generalized Kappa–Cairns velocity distribution, that effectively blends the former two, has also been introduced [9].

There have been many theoretical investigations of the basic plasma properties described by non-Maxwellian velocity distributions, such as the dust charging process [10–12]. It has been found [13] that plasmas with superthermal particles have higher level of electrostatic fluctuations and shorter Debye lengths when compared with Maxwellian plasmas. For plasmas described by a Kappa distribution, it has been shown [14] that the Debye length is always smaller than the shielding distance in a Maxwellian plasma. Comparisons between the Debye lengths in certain non-Maxwellian plasmas have been given in [15, 16]. As the first task in this paper, we complement those findings and derive the corresponding expression for the Debye
length for both the Cairns and the generalized Kappa–Cairns distributions, and discuss how it behaves compared with the corresponding shielding distance in a Maxwellian plasma.

The most common experimental approach for studying the properties of plasma is by inserting an electrostatic probe, which collects electron and ion currents that depend on the actual plasma velocity distribution. Electrostatic probes, usually called Langmuir probes, are used to study different properties of a plasma, such as temperature, density and electric potential. Due to the geometrical simplicity, which allows for analytical treatment, planar, spherical and cylindrical Langmuir probes are the most commonly utilized probes. The current collected by Langmuir probes in Maxwellian plasmas are extensively studied in the literature [17–19]. The early works by Mott-Smith and Langmuir [17], Kagan and Perel [18] and Laframboise and Parker [19], introduced the Orbital-Motion-Limited (OML) theory, which allows to determine the collected electron and ion currents by spherical and cylindrical probes from the conservation of energy and angular momentum along the particle trajectories, and OML solutions for spherical and cylindrical probes. In other works, the range of applicability of OML theory [20] has been determined, the collected currents beyond the OML regime has also been computed [21, 22], the OML theory is also extended to include collisions [23], relativistic conditions [24], and electron emission [25–27]. Langmuir probes in isotropic and anisotropic non-Maxwellian plasmas have been previously investigated and general formulas for the collected currents are well examined [28, 29]. The main goal of this paper is to further extend the OML theory by presenting analytical expressions, and the results of Particle-In-Cell (PIC) simulations, for the currents collected by spherical and cylindrical Langmuir probes in plasmas with velocity distributions given by the Kappa–Cairns distribution functions.

This paper is organized as follows: in section 2, the Kappa, Cairns and the generalized Kappa–Cairns distributions are introduced and the Debye length corresponding to each distribution is determined. The electric currents due to charged plasma particles collected by spherical and cylindrical Langmuir probes are obtained analytically for these distributions, for both repelled and attracted plasma particle species, and a comparison with the Maxwellian plasmas is given in section 3. The numerical method is briefly described in section 4, and the results of PIC simulations both for spherical and for cylindrical probes are presented in section 5. The validity of the analytical expressions are also examined. Finally, a discussion of the results is given in section 6, with the conclusion in section 7.

2. The Kappa–Cairns distribution and its properties

In this section, we will present the theory of spherical and cylindrical Langmuir probes in an unmagnetized and collisionless plasma, in which the velocity distribution function (VDF) of each plasma species is given in section 3. The numerical method is briefly described in section 4, and the results of PIC simulations both for spherical and for cylindrical probes are presented in section 5. The validity of the analytical expressions are also examined. Finally, a discussion of the results is given in section 6, with the conclusion in section 7.

Figure 1. One-dimensional Kappa–Cairns VDF with $v_{th,s} = 1$, for different values of $\kappa$ and $\alpha$, compared with the Maxwellian distribution.

Kappa–Cairns distribution

$$f_s(v; v_{th,s}, \kappa, \alpha) = A_{\kappa, \alpha} \left(1 + \alpha \frac{v^4}{v_{th,s}^4}\right) \left(1 + \frac{\nu^2}{2(\kappa - \frac{D}{2})v_{th,s}^2}\right)^{-(\kappa+1)}, \quad (1)$$

where

$$A_{\kappa, \alpha} = \frac{\Gamma(\kappa + 1)}{(2\pi v_{th,s}^2(\kappa - \frac{D}{2}))^{\frac{3}{2}}} \left[1 + D(D + 2)\alpha\frac{\nu}{\kappa - \frac{D}{2}}\right].$$

is the normalization factor, $\nu = |v|$, $D = 1, 2, 3$ for 1D, 2D, and 3D systems, $\kappa$ and $\alpha$ are the spectral indices of the distribution function, $v_{th,s} = \sqrt{\frac{kT}{m_s}}$ is the thermal velocity defined for the plasma species $s$ at thermal equilibrium with mass $m_s$ and temperature $T_s$. Here, $\Gamma(\alpha)$ is the gamma function ([30], equation (5.4.1)). We note that the thermal velocity as it is given here is only valid for a Maxwellian plasma, and one should keep in mind that proper definitions of temperature and thermal velocity are needed for the non-Maxwellian plasmas.

A visual comparison between the Kappa–Cairns and the Maxwellian distributions is given in figure 1 for different values of the spectral indices $\kappa$ and $\alpha$. The high-thermal tails of the distribution are clearly seen from this figure.

As it was mentioned in the introduction, the Kappa–Cairns VDF is a generalization of the Kappa and Cairns distributions, which are in turn generalizations of the Maxwellian distribution. The expressions for these VDFs are given in table 1. With $\alpha = 0$, the Kappa–Cairns distribution reduces to the Kappa VDF. Taking the limit $\kappa \to \infty$, the Kappa VDF reduces to the Maxwellian VDF. The Cairns distribution is obtained from (1) by taking the limit $\kappa \to \infty$. By setting $\alpha = 0$, the Cairns distribution simply reduces to the Maxwellian VDF.
The Debye length is the characteristic shielding distance in a plasma, and it depends on the velocity distribution of the plasma species. For instance, the electric fields in the vicinity of an electrostatic probe immersed in a plasma are usually shielded over a distance of the order of a few Debye lengths. In the following, we will obtain an appropriate shielding distance for a plasma given by the Kappa–Cairns VDF, and investigate how it compares to the Debye length in a Maxwellian plasma.

When the electrons and ions follow Maxwellian velocity distributions in a two component (electron–ion) plasma, the effective Debye length is given by [31]

$$\lambda^M_{Di} = \left( \frac{1}{\lambda^M_{De}} \right)^2 \left( \frac{1}{\lambda^M_{De}} \right)^2,$$  \hspace{1cm} (2)

where $\lambda^M_{De} = \lambda^M_{De} = \sqrt{\varepsilon_0 k T e / n_e e^2}$, and $\lambda^M_{Di} = \sqrt{\varepsilon_0 k T i / n_i q_i^2}$ are the electron and ion Debye lengths, respectively. Here, $e > 0$ is the elementary charge, $q_i$ is the ion charge, which for simplicity we will take it as $q_i = e$ in the following, $\varepsilon_0$ is the electric constant, $n_i$ is the volumetric number density of the ambient plasma, and $T_e$ and $T_i$ are the electron and ion temperatures, respectively.

To derive the Debye length in a plasma with the Kappa–Cairns distribution, we follow the approach given in [31, p 151], and introduce an immobile test charge $Q$ into a plasma. In the close vicinity of $Q$, the electron and ion velocity distribution functions will be modified depending on whether the test charge is positive or negative. Without loss of generality, we assume that $Q$ is positive, and that the electrons and ions follow the same kind of VDF, i.e. the Kappa–Cairns distribution. The presence of the test charge results in a local perturbation of the plasma potential, $\phi(r)$, which we assume is only dependent on the radial distance, $r$, from the charge. The electric potential around the test charge can be found by solving Poisson’s equation

$$-\varepsilon_0 \Delta \phi(r) = -e n_e + e n_i + Q \delta(r - r_0),$$ \hspace{1cm} (3)

where $\delta(\cdot)$ is the Dirac delta function, $n_e$ and $n_i$ are the electron and ion number densities, respectively, and $r_0$ is the position of the test charge. The electric potential, $\phi(r)$, affects the spatial distribution of particle velocities, and therefore, in the expression for the velocity distribution function one must take into account both the kinetic and potential energies of the particles. Including the potential energy, $q_i \phi(r)$, in the Kappa–Cairns VDF, (1) takes the form

$$f_r(r, \mathbf{v}; v_{th,e}, \kappa, \alpha) = \frac{1}{(2 \pi)^{3/2} \sigma} \left( \frac{e}{\kappa} \right) \exp \left( -\frac{e^2}{2 \kappa} \right) \left( \frac{m_e}{m_i} \right)^{(\kappa + 1)/2} \left( \frac{m_i}{m_e} \right)^{(\kappa - 1)/2} \int_0^\infty f_e(r, \mathbf{v}; v_{th,e}, \kappa, \alpha) v^2 \sin \theta d\theta d\phi d\psi,$$ \hspace{1cm} (4)

where $q_i$ and $m_i$ are the charge and the mass of species $s \in \{e, i\}$ ($e$ and $i$ stands for electron and ion, respectively). The electron and ion number densities are given by

$$n_e = n_0 \int_0^\pi \int_0^{2\pi} \int_{v_{th,e}(r)}^\infty f_e(r, \mathbf{v}; v_{th,e}, \kappa, \alpha) v^2 \sin \theta d\theta d\phi d\psi,$$ \hspace{1cm} (5)

and

$$n_i = n_0 \int_0^\pi \int_0^{2\pi} \int_{v_{th,i}(r)}^\infty f_i(r, \mathbf{v}; v_{th,i}, \kappa, \alpha) v^2 \sin \theta d\theta d\phi d\psi,$$ \hspace{1cm} (6)

where $v_{th,e}(r) = 2 e \phi(r) / m_e$. In order to be able to solve (3) analytically, we assume that the potential perturbations are small, such that the solutions of (5) and (6) can be approximated by linear functions of $\phi(r)$. For the Kappa–Cairns distribution, integrating (5) and (6), and linearizing the solutions, we end up with

$$n_e(r) = n_0 \left( 1 - B_{e,\alpha} \frac{q_i \phi(r)}{k T_e} \right),$$ \hspace{1cm} (7)

where

$$B_{e,\alpha} = \frac{1}{2} \left( \frac{\kappa}{\kappa - 1} + 3 \alpha \frac{\kappa - 2}{\kappa - 1} \right).$$ \hspace{1cm} (8)

Inserting (7), for both electrons and ions, into (3), and expressing Poisson’s equation in spherical coordinates, yields

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi(r)}{\partial r} \right) = \frac{n_0 e^2}{\varepsilon_0 k} B_{e,\alpha} \left( \frac{1}{T_e} + \frac{1}{T_i} \right) \times 3 \frac{\kappa \phi(r)}{\varepsilon_0 (r - r_0)},$$ \hspace{1cm} (9)
whose solution is the Debye–Hückel potential [31]

$$\phi(r) = \frac{Q}{4\pi\varepsilon_0 \|r - r_0\|} \exp\left(\frac{-\|r - r_0\|}{\lambda_D}\right),$$

(10)

where $$\lambda_D$$ is the effective Debye shielding distance in a Kappa–Cairns plasma, given by

$$\lambda_D = \left(\frac{\varepsilon_0 k}{n_0 e^2 B_{c,\alpha}}\left(\frac{1}{T_e} + \frac{1}{T_i}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$ (11)

Hence, the Debye length for each species s can be defined as

$$\lambda_{Ds} = \frac{\varepsilon_0 k T_s}{n_0 e^2 B_{c,\alpha}} = \frac{\lambda_{Dm}}{B_{c,\alpha}}, \quad \kappa > \frac{5}{2}.$$ (12)

The corresponding expressions for the Debye length for Kappa, Cairns and Maxwellian distributions are given in Table 1. These expressions are found by taking the appropriate limits of (12).

In Figure 2, the normalized electron Debye length, given by (12), is plotted as a function of spectral indices $$\kappa$$ and $$\alpha$$. As we can see from this figure, the Debye length is crucially dependent on $$\kappa$$ and $$\alpha$$. When $$\alpha = 0$$, we have

$$\lambda_{De} = \frac{\kappa - \frac{5}{2}}{\kappa - \frac{5}{2}} \lambda_{Dm}, \quad \kappa > \frac{5}{2}.$$ (13)

Interestingly, as $$\kappa$$ approaches 5/2, the Debye length gets smaller than the Maxwellian Debye length. Increasing $$\kappa$$ leads to larger $$\lambda_{De}$$ and in the limit $$\kappa \to \infty$$, as we would expect, it approaches the Maxwellian Debye length. For $$\alpha > 0$$, in the limit $$\kappa \to 5/2$$, the Debye length becomes infinite. As $$\kappa$$ increases, the Debye length decreases, and eventually, in the limit $$\kappa \to \infty$$, the Debye length takes the form

$$\lambda_{De} = \frac{1 + 15\alpha}{1 + 3\alpha} \lambda_{Dm}.$$ (14)

From this expression, it is apparent that when $$\alpha = 0$$, the Maxwellian Debye length is recovered, and in the limit $$\alpha \to \infty$$, we have $$\lambda_{De} = \sqrt{5} \lambda_{Dm}$$. Thus, for a given $$\kappa > 5/2$$, increasing $$\alpha$$ results in a larger Debye length, which, in the limit $$\alpha \to \infty$$, approaches $$\sqrt{5} \lambda_{Dm}$$.

3. OML-theory for probes in Kappa–Cairns distributed plasmas

Before we present the theory and analytical expressions for the currents of plasma particles to a Langmuir probe in a plasma given by the Kappa–Cairns distribution, we will recall the standard assumptions that we make about the ambient plasma and its interaction with the spherical or cylindrical probe.

In the following, we consider an electron–ion, two component plasma that is quasineutral, stationary, homogeneous and isotropic. The plasma is unmagnetised and collisionless, so that we do not account for the Lorentz force and particles interact only through the Coulomb force. The probe only disturbs the plasma within the sheath surrounding it, and which has a well defined size and boundary. The effect of trapped particles in the sheath is ignored. Outside the sheath, the electric potential of the ambient plasma, denoted as $$\phi_{pl}$$, is uniform. The length $$L_p$$ of a cylindrical probe is much larger than the probe radius $$r_p$$ as well as the Debye length $$\lambda_{De}$$. The electric potential around the probe depends only on the radial distance to the probe, and it is a monotonically decreasing or increasing function, depending on the sign of the probe potential with respect to the local plasma potential. This will be the case for $$\lambda_{De} \gg r_p$$, i.e. in the thick-sheath limit, and the potential can be approximated by the Debye–Hückel potential. Finally, all plasma particles that reach the probe surface are collected by the probe, and contribute to the current to the probe. There is no secondary electron emission from the probe surface.

Although there is a long and extensive history of work dealing with the theory of spherical and cylindrical Langmuir probes, e.g. [17, 32–38], for the sake of completeness, and to make this paper a stand-alone self-supporting document, all the necessary steps required to derive the analytical expressions for the collected current by spherical and cylindrical probes are provided in the appendices of this paper.

3.1. Spherical probes

3.1.1. Collected current of repelled particles—retarding field.

The general expression for the collected current of repelled particles is given by (see appendix A)

$$I_s = 4\pi^2 T_s^2 q_s n_0 \int_{v_{min}}^{\infty} v^3 f_s(v) \left(1 - \frac{q_s e \Phi_p}{\frac{1}{2} m_s v^2}\right) dv,$$ (15)

where, $$n_0$$ is the plasma density, $$r_p$$ and $$\Phi_p$$ are the radius and the bias potential of the spherical probe, and $$v_{min}$$ is defined in appendix A.

Integrating (15) for the Kappa–Cairns distribution, given by (1) with $$D = 3$$ (i.e. 3D case), the following expression is
obtained for the collected current of repelled particles

\[ I_\kappa(\eta; \kappa, \alpha) = I_0^\text{rep} C_\kappa^\text{rep} D_\kappa^\text{rep} \left( 1 + \left( \frac{\eta}{\kappa - \frac{3}{2}} \right)^{1-\kappa} \right) \times \left( 1 + E_\kappa^\text{rep} \left( \eta + 4 \left( \frac{\kappa - \frac{3}{2}}{\kappa - 1} \right) \right) \right), \]  

where

\[ \eta = \frac{q_l \phi_p}{kT}, \]

\[ I_0^\text{rep} = 2\sqrt{2\pi} r_p^2 q_l n_0 v_{th,l}, \]

\[ C_\kappa^\text{rep} = \frac{\sqrt{\kappa - \frac{3}{2}}}{\Gamma(\kappa - 1)} \left( \frac{4\alpha}{\kappa - 2(\kappa - 3) + \frac{3}{2}} \right), \]

\[ D_\kappa^\text{rep} = \frac{1 + 24\alpha}{1 + 15\alpha \left( \frac{\kappa - 3}{2} \right)^3}. \]

and

\[ E_\kappa^\text{rep} = \frac{4\alpha \kappa(\kappa - 1)}{(\kappa - 2)(\kappa - 3) + 24\alpha \left( \frac{\kappa - 3}{2} \right)^3}. \]

For the sake of completeness, the corresponding expressions for the Kappa, Cairns, and Maxwellian distributions are given by

\[ \text{Kappa: } I_\kappa(\eta; \kappa) = I_0^\text{rep} C_\kappa^\text{rep} \left( 1 + \left( \frac{\eta}{\kappa - \frac{3}{2}} \right)^{1-\kappa} \right), \]  

\[ \text{Cairns: } I_\kappa(\eta; \alpha) = \frac{I_0^\text{rep}}{1 + 15\alpha} \exp \left( -\eta \right) \times \left( 1 + 24\alpha + 4\alpha \eta(\eta + 4) \right). \]

Maxwellian: \[ I_\kappa(\eta) = I_0^\text{rep} \exp \left( -\eta \right). \]  

In figure 3, we have plotted the collected current of repelled particles, normalized with \( I_0^\text{rep} \), as a function of normalized probe potential \( \eta \), for different values of \( \kappa \) and \( \alpha \). As we can see from this figure, increasing \( \kappa \), the current decreases, and approaches the corresponding current for a Maxwellian plasma. For a given finite value of \( \kappa \), increasing \( \alpha \) results in a higher current. Independent of \( \kappa \) and \( \alpha \), increasing the probe potential, the current approaches zero. However, the current approaches zero more slowly for lower values of \( \kappa \) and higher values of \( \alpha \).

If the spherical probe is biased at plasma potential, i.e. \( \Phi_p = \phi_p - \phi_{pl} = 0 \), the collected current in (16) reduces to

\[ I_e = I_0^\text{rep} C_\kappa^\text{rep} D_\kappa^\text{rep}. \]  

This expression simply gives the current due to random flux of plasma particles collected by the spherical probe.

### 3.1.2. Collected current of attracted particles—accelerating field

In this case there are two types of currents (see appendix A), namely, the sheath limited current (SL) and the orbit motion limited current (OML). As it is shown in appendix A, the expressions for both SL and OML currents depend on the distance to the sheath edge, \( r_e \), which is unknown but can be determined from the solution of Poisson’s equation for the electric potential. There are, however, two limiting cases in which the expressions for the collected current become independent of the distance to the sheath edge.

#### 3.1.2.1. Thin sheath approximation

If the sheath size is small compared to the size of the probe, then \( r_e \approx r_p \), and we obtain the thin sheath limit. In appendix A, the integral expression for the collected current in the thin sheath approximation is found to be

\[ I_s^\text{SL} = 4\pi^2 r_p^2 q_l n_0 \int_0^\infty v^3 f_i(v) dv, \]  

which, for Kappa–Cairns VDF gives

\[ I_s(\kappa, \alpha) = I_0^\text{rep} C_\kappa^\text{rep} D_\kappa^\text{rep}. \]  

This expression is the same as (20), and it gives the current due to random flux of plasma particles.

#### 3.1.2.2. Thick sheath approximation

If, on the other hand, the sheath size is much bigger than the size of the probe, i.e. \( r_e \gg r_p \), or \( r_p \ll \lambda_D \), we are in the thick sheath limit and the integral expression for the collected current becomes (see appendix A)

\[ I_t = 4\pi^2 r_p^2 q_l n_0 \int_0^\infty v^3 f_i(v) \left( 1 + \frac{|q_l \phi_p|}{2m_v v^2} \right) dv. \]  

For the Kappa–Cairns distribution, the integral above gives the following expression for the current in the thick sheath.
The corresponding expressions for the Kappa, Cairns and Maxwellian distributions are

Kappa: \[ I(\eta; \kappa, \alpha) = I_0^{\kappa \alpha} C_{v_{\min}}^{\kappa \alpha} (1 + F_{\kappa \alpha}(|\eta|)) \]
Cairns: \[ I(\eta; \alpha) = \frac{I_0^{\alpha}}{1 + 15\alpha} (1 + 24\alpha + (1 + 8\alpha)|\eta|) \]
Maxwellian: \[ I(\eta) = I_0^{\kappa}(1 + |\eta|) \]

In figure 4(a), we have plotted the normalized current, \( I/I_0^p \), given by (24), as a function of normalized potential \( \eta \) for different values of \( \kappa \) and \( \alpha \). The current for the Maxwellian distribution is also plotted in the same figure. As we can see from this figure, the current is a linear function of the probe potential, and the slope of this linear function decreases with increasing values of \( \kappa \) and \( \alpha \). In the limit \( \kappa \to \infty \), the current approaches \( \frac{8}{15} (3 + |\eta|) \).

3.2. Cylindrical probes

3.2.1. Collected current of repelled particles—retarding field. In appendix B, the general expression for the collected current of repelled particles for a cylindrical probe is found to be

\[ I_I = 4\pi n_0 L_p q_i n_0 \int_{v_{\min}}^{\infty} \left( 1 - \frac{2q_i D_p}{m_i v_r^2} \right) dv_r dv_r \]

where, \( L_p \) is the length of the cylindrical probe, and \( v_r, v_z \), and \( v_{\min} \) are defined in appendix B.

Inserting the Kappa–Cairns distribution (1) into (28), and evaluating the double integral, yields an expression similar to (16), except for \( I_0^{\kappa \alpha} \), which depends on the geometry of the probe, and therefore, must be replaced with its cylindrical counterpart

\[ I_0^{\kappa \alpha} = \sqrt{2\pi} n_0 L_p q_i n_0 v_{th,<} \]

3.2.2. Collected current of attracted particles—accelerating field. Similar to the case of a spherical probe, the collected current of attracted particles can be divided into sheath limited and orbit motion limited currents, i.e. \( I_{\text{SL}} \) and \( I_{\text{OML}} \), respectively. Again, analytical expressions for the current that are independent on the distance to the sheath edge can be obtained in the thin and thick sheath limits.

3.2.2.1. Thin sheath approximation. The general expression for the current in the thin sheath limit is (see appendix B)

\[ I_I = 4\pi n_0 L_p q_i n_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_r^2 f_i(v) dv_r dv_r \]
which, again similar to (20), gives the current due to random flux of plasma particles
\[ I_\eta(\kappa, \alpha) = I_\eta^0 C^{\kappa}_{\eta,\alpha} D^{\kappa}_{\eta,\alpha}. \]  

(30)

3.2.2. Thick sheath approximation. The integral expression in the thick sheath limit is given by (see appendix B)
\[ I_e = 4\pi p L p E_0 \int_0^\infty \int_{-\infty}^\infty v^2 f_\omega(v) \left(1 + \frac{|q_d q_f|}{w^2 v^2}ight) dv_1 dv_2. \]  

(31)

For the generalized Kappa–Cairns VDF the collected current in the thick sheath limit becomes
\[ I_\eta(\eta; \kappa, \alpha) = \frac{2}{\sqrt{\pi}} I_\eta^0 C^{\kappa}_{\eta,\alpha} D^{\kappa}_{\eta,\alpha} \left(\frac{\eta}{\kappa - \frac{3}{2}}\right)^{1-\kappa} \]
\[ \times \left\{ \left[ \eta E^{\kappa}_{\eta,\alpha} \left( \frac{\kappa - 3}{2} + \frac{2(\kappa - 1)}{|\eta|} \right) \right] \right. \]
\[ \left. + \left[ 1 + E^{\kappa}_{\eta,\alpha} \left( \frac{\kappa - \frac{3}{2}}{\kappa - 1} \right) \right] \right. \]
\[ \times \left. 2 F_1 \left( \kappa - 2, \kappa - \frac{1}{2}; \kappa - 1; 1 - \frac{\kappa - \frac{3}{2}}{|\eta|} \right) \right\}, \]  

(32)

where
\[ C^{\kappa}_{\eta,\alpha} = \frac{(\kappa - \frac{1}{2})^{\kappa - \frac{3}{2}}}{\kappa - 1}, \]
\[ D^{\kappa}_{\eta,\alpha} = \frac{1 + 3\alpha^{\frac{\kappa - 3}{2}}}{1 + 15\alpha^{\frac{\kappa - 3}{2}}}, \]
and
\[ E^{\kappa}_{\eta,\alpha} = \frac{4\alpha(\kappa - 1)}{\left(\kappa - \frac{1}{2}\right)^{\kappa - \frac{3}{2}} + 3\alpha(\kappa - \frac{3}{2})^2}. \]

Here, \( 2 F_1 \) is the Gaussian hypergeometric function [30, equation (15.2.1)]. For the sake of completeness, the corresponding expressions for the Kappa (K), Cairns (C) and the Maxwellian (M) VDFs are
\[ \text{K: } I_\eta(\eta; \kappa) = \frac{2}{\sqrt{\pi}} I_\eta^0 C^{\kappa}_{\kappa} \left(\frac{\eta}{\kappa - \frac{3}{2}}\right)^{1-\kappa} \]
\[ 2 F_1 \left( \kappa - 1, \kappa + 1; \kappa; 1 - \frac{\kappa - \frac{3}{2}}{|\eta|} \right), \]  

(33)

\[ \text{C: } I_\eta(\eta; \eta, \alpha) = I_\eta^0 \left( \frac{1 + 24\alpha}{1 + 15\alpha} \right) \left( \frac{2}{\sqrt{\pi}} \left( 1 - \frac{2\alpha|\eta|}{1 + 24\alpha} \right) \right) \]
\[ \times \sqrt{|\eta|} + \exp(|\eta|) \left( 1 + \frac{4\alpha|\eta|(|\eta| - 4)}{1 + 24\alpha} \right) \text{erfc}(\sqrt{|\eta|}), \]  

(34)

\[ \text{M: } I_\eta(\eta) = I_\eta^0 \left( \frac{2}{\sqrt{\pi}} \sqrt{|\eta|} + \exp(|\eta|) \text{erfc}(\sqrt{|\eta|}) \right), \]  

(35)

where \( \text{erfc}(*) \) is the complementary error function [30, equation (7.2.2)].

It has been demonstrated [39] that for a Maxwellian plasma, the current in the thick sheath limit can be approximated with a much simpler expression. By finding how (32) behaves in the limits \( \eta \rightarrow 0 \) and \( \eta \rightarrow -\infty \), we can construct the following expression
\[ I_\eta(\eta; \kappa, \alpha) = I_\eta^0 \left( C^{\kappa}_{\eta,\alpha} D^{\kappa}_{\eta,\alpha} \right)^2 + 4|\eta| \frac{2}{\pi}, \]  

(36)

which gives a very good approximation for the collected current in the thick sheath limit for a Kappa–Cairns plasma. Equivalent expressions for Kappa, Cairns, and Maxwellian plasma can be obtained by taking the appropriate limits of the spectral indices. We emphasize that this expression gives an even better approximation for small values of \( |\eta| \), in the case of a Maxwellian plasma, than the one given in [39], which is commonly used in the literature.

In figure 5, the normalized thick sheath current is plotted as a function of \( \eta \) for different values of the \( \kappa \) and \( \alpha \). As we can see, the collected current for the Kappa distribution is very close to the Maxwellian current. For higher values of \( \eta \), these currents are indistinguishable. However, for small values of \( \eta \), it is clear that increasing \( \kappa \), the current increases and gets closer to the Maxwellian current. On the other hand, if we increase \( \alpha \), the current increases beyond the Maxwellian current. However, the current decreases as \( \kappa \) increases.

3.3. Current–voltage (IV) characteristics

Based on the analytical expressions obtained above, it is possible to visualize the IV curves for plasmas following Kappa–Cairns VDF. In figure 6, the IV curves for different values of \( \kappa \) and \( \alpha \) are displayed, and compared with the corresponding curve for a Maxwellian plasma. The collected current by the probe, \( I_\eta = I_e + I_i \), is composed of electron and ion currents, i.e. \( I_e \) and \( I_i \), respectively. The plasma density and electron and ion temperatures are set to \( n = 10^{11} \text{ m}^{-3}, T_e = 2000 \text{ K}, \) and \( T_i = 1200 \text{ K}, \) respectively, which are typical values obtained from the International Reference Ionosphere model at an altitude of 300 km. We choose this example for illustration for a hydrogen plasma, as
these could be relevant for many spacecraft based measurements of ionospheric plasma density. Note that the selected example does not limit the discussion.

From figure 6, the first important observation is that for both spherical and cylindrical probes the floating potential, defined at \( I_p = 0 \), decreases for higher values of \( \alpha \), and it increases and gets closer to the corresponding value for a Maxwellian VDF, as \( \kappa \) increases. In the case of a cylindrical probe, the current for a Kappa–Cairns VDF is higher than for the Maxwellian VDF, provided that \( \alpha > 0 \). And the current increases as \( \kappa \) decreases. For a Kappa VDF, the current is relatively close to the Maxwellian current. For the spherical probe, however, the IV characteristics is somewhat different. For negative probe voltages, while the electrons are repelled, the ions for such voltages are attracted by the probe. The ion current is limited by the electric shielding of the probe, and it decreases slowly as the probe voltage becomes more negative. When \( \alpha = 0 \), the ion current, beyond a certain voltage, becomes higher than the Maxwellian current for lower values of \( \kappa \). Increasing \( \alpha \), the current remains above the Maxwellian current, albeit closer to the Maxwellian current for higher values of \( \kappa \). For positive probe voltages, the situation is reversed. Ions are repelled and the electrons are attracted by the probe. While the electron current is higher than the Maxwellian current for a Kappa VDF, the current is lower for a Kappa–Cairns VDF with \( \alpha > 0 \).

The normalized thick sheath current, \( I_s/I^0 \), as a function of normalized potential \( \eta \) for different values of \( \kappa \) and \( \alpha \).

3.4. Floating potential

Figure 7 displays the floating potential for both spherical and cylindrical probes for a hydrogen plasma as a function of spectral indices \( \kappa \) and \( \alpha \). As we clearly see from this figure, when \( \alpha = 0 \), the floating potential is rather close to zero. Increasing \( \kappa \), \( e\Phi_0/k_BT_e \) approaches the corresponding floating potential for a Maxwellian plasma. For \( \alpha > 0 \), for both geometries, \( e\Phi_0/k_BT_e \) decreases as \( \alpha \) increases, and decreasing \( \kappa \), results in a lower floating potential. The major difference between the floating potential for the spherical and cylindrical probes is that \( e\Phi_0/k_BT_e \) decreases faster for the cylindrical probe for smaller values of \( \kappa \).

4. Finite element particle-in-cell simulations

The theoretical expressions for collected current are valid in the thick sheath and thin sheath limits, where the Debye length is very large or very small, respectively, compared to the probe radius. Using numerical simulations, we are also able to study the current collected by probes with a radius comparable to the Debye length. This is useful amongst others to investigate the range of radii for which the analytical approximations are valid. We have chosen to use PIC simulations with an unstructured mesh since these meshes can be finer near the probe and more accurately approximate spherical and cylindrical geometries than structured meshes.

The PIC method works by following the trajectories of an ensemble of particles through time under influence of their mutual forces. Operating in the electrostatic regime without background magnetic flux density, the force on a particle \( p \) can be expressed as

\[
m_p \mathbf{x}_p = q_p \mathbf{E}(\mathbf{x}_p),
\]

where \( \mathbf{x}_p, q_p \) and \( m_p \) denotes the position, charge and mass of the particle, and \( \mathbf{E} \) is the electric field. This is integrated numerically to follow the particles. To obtain the electric field, the charge of all particles are weighted onto a mesh to get a charge density \( \rho \), and subsequently, the Poisson equation is solved:

\[
-\varepsilon_0 \Delta \phi = \rho, \quad \mathbf{E} = -\nabla \phi.
\]

Because the charge density changes as the particles move, the electric field must be recalculated between each time-step. Moreover, it is customary to lump several physical particles into one simulation particle to reduce the computational cost. More information about the PIC method can be found in [40, 41].

We use the openly available computer program PUNC++ (Particles-in-UnStructured-Cells, C++ version) for these computations. PUNC++ integrates (37) using the Leapfrog algorithm [41], and (38) is solved on 1st order linear, continuous finite elements using the third party problem solving environment FEniCS [42, 43]. The particle charges are weighted linearly to surrounding mesh vertices. More information on the Finite Element Method is found in for instance [44, 45], and for PIC simulations using FEM, see [46].

4.1. Modelling Langmuir probes in non-Maxwellian plasmas

Spherical Langmuir probes have been modelled in 3D using a domain with two concentric spheres as Dirichlet boundaries. Likewise, cylindrical Langmuir probes have been modelled in 2D using a domain with two concentric circles as Dirichlet boundaries. The inner boundary represents the probe itself, and its potential is fixed to the potential of the probe with respect to the background plasma, \( \phi_{pb} \). Particles crossing the inner boundary are removed from the simulation, but are counted for each time-step to compute collected current. The current is simply recorded as \( \Delta Q/\Delta t \) for a net charge \( \Delta Q \) collected during a time-step of length \( \Delta t \).
The outer boundary is supposed to be sufficiently far away from the probe that it can be counted as being in the background plasma, and the potential is accordingly set to be zero. Since this is to be an open boundary, there is a free outward flux of particles which are deleted from the simulations, and there must be an inward flux of particles according to the velocity distribution of the background plasma. Support for arbitrary velocity distributions has been implemented in PUNC++, as will be briefly described in the following.

Consider \( f_s(v) \) to be the velocity distribution of species \( s \). The inward flux through the outer boundary per unit area and time due to this velocity distribution is then

\[
\left( n \hat{n} \right) \cdot \int_{\vec{v} \cdot \hat{n} < 0} f_s(v) \, dv, \tag{39}
\]

where \( \hat{n} \) is the inwards-directed unit normal vector, \( dv \) is the volume element of the velocity space, and the integration must be performed only on inwards-directed velocities, i.e. for \( v \cdot \hat{n} > 0 \). Let us define a new distribution function,

\[
g_s(v) = \begin{cases} 
  f_s(v) \cdot \hat{n}, & v \cdot \hat{n} > 0, \\
  0, & v \cdot \hat{n} \leq 0.
\end{cases} \tag{40}
\]

The number of particles of species \( s \) to inject into the domain through a boundary facet with surface area \( \Delta S \) during a time-step of duration \( \Delta t \) is then given by

\[
n_s \Delta S \Delta t \int_{\mathbb{R}^D} g_s(v) \, dv, \tag{41}
\]

where \( D \) is the number of geometric dimensions (2 for the cylinder and 3 for the sphere). The number of particles given by (41) may not be an integer number, but whether to round up or down is decided by chance with a probability according to the fractional part of this number. Each of these particles are then given a random velocity \( v_p \) by sampling the distribution \( g_s(v) \). The injected particles are also given a random, uniform position \( x_p' \) on the boundary facet. To emulate a continuous inward flux of particles throughout the time-step they are also advanced a random fraction of the time-step according to

\[
x_p = x_p' + \xi_p \cdot \Delta t v_p, \tag{40}
\]

where \( \{ \xi_p \} \) are uniformly distributed numbers in \([0,1]\).

PUNC++ also fills the domain uniformly with particles distributed according to \( f_s(v) \) in velocity space prior to the simulation. This will not be correct in vicinity of the probe, but reduces the computational cost as steady-state is reached faster than if all particles were to be injected through the outer boundary.

Figure 6. The current–voltage characteristics for a spherical (left) and a cylindrical probe (right) for a hydrogen plasma with volumetric number density \( n = 10^{11} \text{ m}^{-3} \), and \( T_e = 2000 \text{ K}, T_i = 1200 \text{ K} \), for different values of \( \kappa \) and \( \alpha \). Here, the electron current is chosen to be positive. The probe radius is set to \( r_p = 10^{-3} \text{ m} \) and the length of the cylindrical probe is \( L_p = 25 \times 10^{-3} \text{ m} \).
4.2. Simulation parameters and post-processing

It is interesting that the expressions for collected current for the Kappa–Cairns distribution can be made a function of only one variable through proper normalization, similarly as Laframboise did for the Maxwellian distribution [47]. Rathert than having the current as a function of many independent parameters, e.g. density, temperature, charge and mass of the species considered, and radius of the probe, the normalized currents \( I_s / I_0 \) are only functions of the normalized voltage \( \eta = q_e \Phi / kT_e \). This means that we do not have to carry out simulations of all combinations of densities, temperatures, etc. Instead, we only simulate electron–proton plasma with \( n = 10^{11} \text{ m}^{-3} \) and \( T_e = 2000 \text{ K}, T_i = 1200 \text{ K} \). The probe radius is set to \( r_p = 10^{-3} \text{ m} \) and the length of the cylindrical probe is \( L_p = 25 \times 10^{-3} \text{ m} \).

The meshes are created using Gmsh [48], and parameters used for simulations of different probe radii (inner boundary radii) are shown in tables 2 and 3 for spherical and cylindrical probes, respectively. The statistical weight is the number of physical particles each simulation particle corresponds to.

All the simulations are run in two phases, each with a different time-step. In the first phase, a large time-step is used to quickly reach a steady-state. When the steady-state is reached in the first phase, a smaller time-step is used in the second phase, and the simulation is continued until the system reaches a more accurate steady-state. For the spherical probe, in the first phase, the time-step is \( \Delta t = 0.05 \omega_{pe}^{-1} \) where \( \omega_{pe} \) is the electron plasma frequency:

\[
\omega_{pe} = \sqrt{\frac{e^2 n_e}{\varepsilon_0 m_e}},
\]

and \( \Delta t = 0.01 \omega_{pe}^{-1} \) in the second phase. For the cylindrical probe, the time-steps are: \( \Delta t = 0.4 \omega_{pe}^{-1} \) in the first phase, and \( \Delta t = 0.01 \omega_{pe}^{-1} \) in the second phase.

Because the recorded currents are noisy, we apply an Exponential Moving Average (EMA) [49] filter before taking the last data-point as the steady-state current. EMA filtering a series \( \{I_0, I_1, ..., I_N\} \) of currents sampled with a period \( \Delta t \) is

\[
I_p(t) = \alpha I(t) + (1 - \alpha) I_p(t - 1),
\]

where \( \alpha \) is the smoothing factor (0 < \( \alpha \) < 1).
achieved recursively through

\begin{equation}
I_0 = I_0, \\
I_k = wI_{k-1} + (1 - w)I_k, \quad k = 1, \ldots, N,
\end{equation}

where \( w = \exp(-\Delta t/\tau) \) and \( \tau \) is the relaxation time. In all the simulations, the relaxation time is set to \( \tau = 1.8 \times 10^{-12} \omega_{pe}^{-1} \).

5. Results

In the following we present the results of PIC simulations for the collected current by positively biased spherical and cylindrical probes. In both cases, a set of reference simulations are carried out for a Maxwellian VDF (where, \( \kappa = \infty \) and \( \alpha = 0 \)), and are compared with the corresponding results for attracted species presented in [47]. It is worth noting that

| Table 3. Simulation parameters for cylindrical probes. |
|-----------------------------------------------|
| Inner boundary radius (\( \lambda_{De} \)) | 1.0 | 2.0 | 3.0 | 5.0 | 10.0 |
| Outer boundary radius (\( \lambda_{De} \)) | 30  | 40  | 60  | 100 | 200 |
| Inner boundary resolution (\( \lambda_{De} \)) | 0.1 | 0.2 | 0.3 | 0.6 | 0.6 |
| Outer boundary resolution (\( \lambda_{De} \)) | 1.5 | 2   | 3   | 5   | 10  |
| Statistical weight (dimensionless)—phase 1 | 591 945 799 | 1214 862 831 | 2733 441 369 | 7603 209 123 | 21 446 336 381 |
| Statistical weight (dimensionless)—phase 2 | 36 996 612 | 75 928 927 | 170 840 086 | 475 200 570 | 1340 396 024 |

Figure 8. The simulations results for the spatial variations of the electric potential from the probe surface to the exterior boundary along the radial direction (upper panel), and the time evolution of the collected current by the probes (lower panel). In these simulations, the radius of the spherical probe is \( r_p/\lambda_{De} = 0.2 \), and \( r_p/\lambda_{De} = 1.0 \) for the cylindrical probe. Both probes are positively biased with \( e\phi_p/kT_e = 5 \). The plasma species have a Kappa–Cairns velocity distribution with \( \kappa = 6 \) and \( \alpha = 0.2 \).
the collected current from the PIC simulations include both electron and proton currents, and that for e\(\Phi_p/kT_e\) = 0 the proton current becomes more significant. In this case, the error between the total current from the PIC simulations and the predicted current of attracted species (i.e. electron) by OML theory becomes larger. The proton current is not detectable for higher bias voltages, however.

To demonstrate the quality of the PIC simulations, we have in figure 8, for both spherical (with \(r_p/\lambda_{De} = 0.2\)) and cylindrical (with \(r_p/\lambda_{De} = 1.0\)) probes with bias potential \(e\Phi_p/kT_e = 5\), displayed the simulation results for the electric potential along the radial direction and the time evolution of the normalized collected current by the probes for a Kappa–Cairns velocity distribution with \(\kappa = 0\) and \(\alpha = 0.2\). In the upper panel of figure 8, the simulation data are also compared with 1/(\(r/\lambda_{De}\)) and 1/(\(r/\lambda_{De}\))^2. As we can see from these figures, the electric potential for the spherical probe drops faster than 1/(\(r/\lambda_{De}\)) in the sheath. For the cylindrical probe, however, the electric potential drops slower than 1/(\(r/\lambda_{De}\)). The time evolution of the normalized collected currents are shown in the lower panel of figure 8. The two simulations phases are clearly seen in these figures, with the statistical noise being higher in the second phase where the time-step is smaller. The exponential moving average and the corresponding analytical results from OML calculations are also displayed in the same figures. Although the noise in the collected current is somewhat high, the EMA filter gives rather good results when compared to the analytical calculations from OML theory.

### 5.1. Spherical probe

The normalized collected currents, \(I/I_0^{sp}\), by the spherical probes with different radii are given in tables 4–9. The results in table 4, where \(\kappa = \infty\) and \(\alpha = 0.0\), correspond to a Maxwellian VDF. The relative difference between the results for the Maxwellian VDF obtained from PIC simulations and table 5c in [47] range between 0.1%–1.9% for \(e\Phi_p/kT_e > 0\), and less than 3.5% for \(e\Phi_p/kT_e = 0\).

The normalized collected current, \(I/I_0^{sp}\), for a Cairns VDF, which is given in table 5, shows similar behavior to the Maxwellian VDF. For a given probe radius, increasing the bias voltage, the normalized collected current increases. And for a given normalized bias potential \(e\Phi_p/kT_e > 1\), increasing the probe radius, results in a reduction of the normalized collected current. In most of the cases, the normalized collected current for a Cairns VDF is less than the Maxwellian counterpart, except for the case with probe radius \(r_p/\lambda_{De} = 10\), where the opposite is seen to be the case. It should be noted that although the normalized current decreases with increasing radius, the current actually increases with increasing radius, since the charge-collecting area...
The results of normalized collected currents, \( I/I_{0}^{sp} \), for cylindrical probe with \( \kappa = 6 \) and \( \alpha = 0.0 \).

| \( \frac{e\Phi_{ \alpha}}{eT_{e}} \) | \( r_{p}/\lambda_{De} \) |
|---|---|
| 0.2 | 1.0 | 2.0 | 3.0 | 5.0 | 10.0 |
| 0 | 0.960 | 0.946 | 0.952 | 0.965 | 0.968 | 0.969 |
| 1 | 2.027 | 2.016 | 1.982 | 1.927 | 1.829 | 1.744 |
| 2 | 3.147 | 3.043 | 2.910 | 2.758 | 2.492 | 2.238 |
| 3 | 4.189 | 4.032 | 3.745 | 3.457 | 3.061 | 2.551 |
| 5 | 6.381 | 5.965 | 5.348 | 4.773 | 3.974 | 3.132 |
| 10 | 11.858 | 10.506 | 8.798 | 7.541 | 6.011 | 4.175 |
| 15 | 17.293 | 14.840 | 11.990 | 9.935 | 7.297 | 4.989 |
| 20 | 22.569 | 18.968 | 14.941 | 12.084 | 8.837 | 5.642 |
| 25 | 28.061 | 22.968 | 17.595 | 14.172 | 10.146 | 6.414 |

As we can see from tables 6 and 8, apart from the case with zero bias voltage, the normalized current for the Kappa VDF decreases as the value of \( \kappa \) increases. This is in accordance with what we have obtained with OML theory in (24) and figure 4. It is clear from tables 7 and 9, that the normalized currents further decrease, as the value of the spectral index \( \alpha \) increases from zero to 0.2.

**5.2. Cylindrical probe**

The results of normalized collected currents, \( I/I_{0}^{sp} \), by the cylindrical probes for the Maxwellian VDF are given in table 10. Compared with the corresponding results presented in table 6c in [47], the relative differences range between 0.1%–2.6% for \( e\Phi_{\alpha}/kT_{e} > 0 \). When the bias voltage is zero, the relative differences are mostly higher. The maximum relative difference is, in this case, 6%. Again, this is because the simulations also include collected ions.

Similar to the spherical probe, the results of the normalized collected current by the cylindrical probe, tables 10–15, show that for a given normalized bias potential \( e\Phi_{\alpha}/kT_{e} > 1 \), the normalized current decreases as the probe radius increases, and for a given radius, increasing the bias potential, the normalized current increases. However, unlike the spherical probe, by increasing \( \alpha \) and decreasing \( \kappa \), the normalized collected current increases. Interestingly, for the Kappa VDF, independent of the value of \( \kappa \), the normalized current is very close to the corresponding normalized current by a Maxwellian VDF.
Table 13. Attracted-species normalized current, $I/kT_p$, for cylindrical probe with $\kappa = 4$ and $\alpha = 0.2$.

| $r_p/\lambda_{De}$ | 1.0 | 2.0 | 3.0 | 5.0 | 10.0 |
|-------------------|-----|-----|-----|-----|-----|
| 0                 | 2.219 | 2.217 | 2.198 | 2.198 | 2.159 |
| 1                 | 2.559 | 2.530 | 2.547 | 2.551 | 2.495 |
| 2                 | 2.849 | 2.809 | 2.810 | 2.841 | 2.800 |
| 3                 | 3.113 | 3.059 | 3.043 | 3.074 | 2.986 |
| 5                 | 3.567 | 3.509 | 3.465 | 3.490 | 3.351 |
| 10                | 4.379 | 4.347 | 4.341 | 4.264 | 3.948 |
| 15                | 5.082 | 5.033 | 4.976 | 4.866 | 4.454 |
| 20                | 5.692 | 5.614 | 5.530 | 5.359 | 4.894 |
| 25                | 6.219 | 6.185 | 6.006 | 5.848 | 5.174 |

Table 14. Attracted-species normalized current, $I/kT_p$, for cylindrical probe with $\kappa = 6$ and $\alpha = 0.0$.

| $r_p/\lambda_{De}$ | 1.0 | 2.0 | 3.0 | 5.0 | 10.0 |
|-------------------|-----|-----|-----|-----|-----|
| 0                 | 0.946 | 0.935 | 0.934 | 0.944 | 0.931 |
| 1                 | 1.540 | 1.540 | 1.509 | 1.541 | 1.486 |
| 2                 | 1.937 | 1.938 | 1.889 | 1.849 | 1.748 |
| 3                 | 2.257 | 2.241 | 2.213 | 2.145 | 1.958 |
| 5                 | 2.790 | 2.758 | 2.723 | 2.568 | 2.131 |
| 10                | 3.770 | 3.691 | 3.552 | 3.239 | 2.640 |
| 15                | 4.460 | 4.338 | 4.098 | 3.813 | 2.932 |
| 20                | 5.115 | 4.975 | 4.815 | 4.276 | 3.232 |
| 25                | 5.692 | 5.350 | 5.213 | 4.635 | 3.525 |

Table 15. Attracted-species normalized current, $I/kT_p$, for cylindrical probe with $\kappa = 6$ and $\alpha = 0.2$.

| $r_p/\lambda_{De}$ | 1.0 | 2.0 | 3.0 | 5.0 | 10.0 |
|-------------------|-----|-----|-----|-----|-----|
| 0                 | 1.827 | 1.804 | 1.797 | 1.825 | 1.797 |
| 1                 | 2.209 | 2.190 | 2.165 | 2.185 | 2.154 |
| 2                 | 2.541 | 2.504 | 2.503 | 2.503 | 2.430 |
| 3                 | 2.791 | 2.768 | 2.751 | 2.791 | 2.730 |
| 5                 | 3.242 | 3.251 | 3.203 | 3.208 | 3.082 |
| 10                | 4.153 | 4.132 | 4.058 | 4.033 | 3.682 |
| 15                | 4.880 | 4.844 | 4.772 | 4.540 | 4.175 |
| 20                | 5.483 | 5.440 | 5.332 | 5.202 | 4.327 |
| 25                | 6.050 | 5.961 | 5.875 | 5.523 | 4.753 |

6. Discussion

For probe radii less than or comparable to the Debye length it is possible to compare the results for the collected current from the PIC simulations with the analytical results obtained from OML theory. In figure 9, the collected currents, from both the simulations and the OML theory, for the spherical probe with radius $r_p = 0.2\lambda_{De}$ (left) and the cylindrical probe with radius $r_p = 1.0\lambda_{De}$ (right), are displayed. It is clearly seen that there is a rather good agreement between the PIC simulations and the OML theory. Apart from one of the cases for the cylindrical probe (with $\kappa = 4$ and $\alpha = 0.2$), which show relative differences up to 9%, all the other cases give relative differences less than 3%. The results for the cylindrical probe with $\kappa = 4$ and $\alpha = 0.2$ are not as accurate as we had hoped, and it might be because the Kappa–Caïns VDF with these particular spectral indices has much higher velocity tails than the other cases, which might require a more careful simulation setup to obtain more accurate results.

The obtained analytical expressions for the OML theory for spherical and cylindrical probes are in principle only valid when $r_p < \lambda_{De}$. In the upper panel of figure 10, the predicted OML current versus the results from PIC simulations for the Kappa–Caïns VDF with $\kappa = 6.0$ and $\alpha = 0.2$ for increasing probe radii have been displayed. From these figures it is apparent that OML theory remains a good fit for $e\theta_p/kT_e < 5$ even for large probe radii compared to Debye length. The predicted collected currents by OML theory become less accurate for increasing normalized bias potentials and increasing ratios between the probe radius and Debye length, as expected due to violation of underlying OML assumptions, including an increasing chance of particle trapping. While the OML current for the spherical probe already at $r_p = 1\lambda_{De}$ shows relative differences up to 8%, for the cylindrical probe, however, the highest relative difference for $r_p = 2\lambda_{De}$ is less than 1%, and for $r_p = 3\lambda_{De}$ it is 2.3%.

The maximum radius of the cylindrical probe (with high positive bias voltages) to operate under OML conditions was found by Sammartin et al [20], and Estes et al [21] found the current drop for cylindrical probes (with high positive bias voltages) operating beyond the OML regime. In order to make a qualitative comparison with these studies, we have in the lower panel of figure 10, shown the $I_{\text{PIC}}/I_{\text{OML}}$ versus the probe radius $r_p/\lambda_{De}$ for different values of $e\theta_p/kT_e$ for the Kappa–Caïns VDF with $\kappa = 6.0$ and $\alpha = 0.2$. From these figures, we see that the current drop becomes higher as both the probe radius and the potential bias increase, which is in agreement with the results obtained by Estes et al [21]. Moreover, we also see that the current drop is significantly higher for the spherical probe than the cylindrical probe. When it comes to maximum probe radius to operate under OML conditions, we see from figure 10 that for the cylindrical probe the OML theory gives rather good results for radii up to $3\lambda_{De}$ even for as high bias voltage as $e\theta_p/kT_e = 25$. However, for the spherical probe, when the bias voltage increases, only for $r_p < \lambda_{De}$ the OML results remain in good agreement with the results of PIC simulations. In order to quantify the range of applicability of the OML theory for non-Maxwellian plasmas, more detailed analysis of the particle trapping and potential barriers in numerical simulations is required. This is left as future work.

We have shown that the OML current collected by a positively biased spherical probe remains a good approximation only for low voltages or small radii compared to the plasma Debye length. The validity of OML theory for a spherical body with a negative voltage has been previously reviewed in the case of Maxwellian plasmas [50], and it was shown that not even for small bodies, i.e. those with $r_p < \lambda_{De}$, the OML theory is satisfied. To address this issue, a modified OML theory for Maxwellian plasmas has been suggested [51], and its predictions have been compared with
PIC simulations [52]. It has been shown that the revised OML theory gives good approximations for radii up to $10\lambda_{De}$. A corresponding revision of OML theory for Kappa, Cairns, and Kappa–Cairns VDFs is beyond the scope of this paper, but it will be pursued as part of future work.

With the theoretical formulas derived in this paper it is possible to measure the type of distribution in a plasma by means of Langmuir probes, or more precisely, to measure the spectral indices $\kappa$ and $\alpha$. Consider for instance fixed-bias probes [53] of a spacecraft having an unknown floating potential $\Phi_0$. A thin Langmuir probe $p$ on this spacecraft is designed with a fixed bias $\Phi_{0p}$ with respect to the spacecraft, and will accordingly have a potential $\Phi_p = \Phi_0 + \Phi_{0p}$ with respect to the background plasma. $\Phi_p$ is assumed to be positive, such that only the electrons contribute to the collected current. The current collected by this probe will then be according to (24) for a sphere, or (32) for a cylinder, where the density, temperature and spectral indices of the electrons are all unknown. Having $N_p$ probes with different biases $[\Phi_{0p}]$, it is possible to write a set of equations for the currents $[I_p]$ collected by these probes:

$$I_p = I(\Phi_0 + \Phi_{0p}; n_e, T_e, \kappa_e, \alpha_e),$$

$p = 1, \ldots, N_p.$

This is a set of $N_p$ equations of the five unknowns $\Phi_0$, $n_e$, $T_e$, $\kappa_e$ and $\alpha_e$. If at least five probes are used (less if some parameters are assumed known, e.g. if $\alpha = 0$), this system of equations can in principle be solved, as long as it is not singular or nearly singular. Further investigations are required to examine the practical aspects of this method.

7. Conclusion

Velocity distribution functions measured in space plasmas are commonly known to be non-Maxwellian. Some of the velocity distribution functions that have been suggested to describe the plasma in near-Earth space environments are the Kappa, Cairns and the generalized Kappa–Cairns distributions. In this paper, we have derived an expression describing the Debye shielding distance in a plasma described by a Kappa–Cairns VDF. The Debye length varies depending on the model parameters $\kappa$ and $\alpha$. If $\alpha = 0$, the Debye length is always less than the corresponding Maxwellian plasma independent of the value of $\kappa$. As $\kappa \to 3/2$, the Debye length becomes zero independent of any plasma parameters. For any finite value of $\kappa$, increasing $\alpha$, the shielding distance quickly becomes larger than the Maxwellian counterpart.

The main achievements of this work are the derived analytical expressions for the collected current by spherical and cylindrical Langmuir probes for a generalized Kappa–Cairns VDF for both repelled and attracted plasma species.
Figure 10. Upper panel: normalized collected current from OML theory (solid line) and the corresponding results obtained by PIC simulations for increasing probe radius for the Kappa–Cairns VDF with $\kappa = 6.0$ and $\alpha = 0.2$. Lower panel: the ratio between the collected current obtained by PIC and OML theory, i.e. $I_{PIC}/I_{OML}$ versus the probe radius $r_p/\lambda_{De}$ for different values of $e\Phi_p/kT_e$ for the Kappa–Cairns VDF with $\kappa = 6.0$ and $\alpha = 0.2$. 
Since the Kappa–Cairns VDF is a generalization of the Kappa, Cairns and Maxwellian VDFs, the analytical expressions for the collected current for these distributions are easily obtained from the corresponding Kappa–Cairns expressions by taking appropriate limits of $\kappa$ and $\alpha$. In the case of repelled species, it is found that for both the spherical and cylindrical probes the collected current for Kappa–Cairns VDF is higher than the Maxwellian VDF, provided that $\alpha > 0$. For the attracted species the situation is somewhat different. For the spherical probe, when $q_i\phi_p kT_s < -3$, the collected current decreases with increasing $\alpha$ and decreasing $\kappa$ values. With $\alpha = 0$, increasing $\kappa$, the collected current reduces and eventually approaches the collected current for a Maxwellian VDF. The collected current by the cylindrical probe, on the other hand, increases with increasing $\alpha$ and decreasing $\kappa$ values. When $\alpha = 0$, independent of the value of $\kappa$, the collected current is very close to the corresponding current by a Maxwellian VDF. In general, the collected current by the cylindrical probe is less than the spherical probe.

Particle-in-Cell simulations have been carried out that show good agreement with these analytical expressions. The simulations also include finite radii which is useful for evaluating the error imposed by OML theory when using larger probes. Based on the simulations, it is found that the OML theory is a good approximation for radii less than 0.2 Debye lengths (spheres) or 1 Debye length (cylinders)\(^3\).

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**Appendix A. General expressions for the collected current by spherical probes**

We consider the motion of a single particle of species $i$, with charge $q_i$ and mass $m_i$. The particle reaches the probe surface, whose radius is $r_p$, with velocity $\mathbf{v}'$, and comes from a distance $r > r_p$ with an initial velocity $\mathbf{v}$, see figure A1. Since there are no other external forces exerted on the particle, the plasma is collisionless, and stationary conditions are assumed, conservation of particle energy gives [35]

$$\frac{1}{2}m_iv^2 + q_i\phi_p = \frac{1}{2}m_isv'^2 + q_i\phi_p,$$  \hspace{1cm} (45)

\(^3\) The results from this paper, both analytical formula and tabulated values, are available as functions in the Python library Langmuir ([https://pypsi.org/project/langmuir](https://pypsi.org/project/langmuir)).

where $v = ||\mathbf{v}||$ and $v' = ||\mathbf{v}'||$, are the particle speeds at the position $r$ and $r_p$, respectively. Here, $\phi_p$ and $\phi_{pl}$ are the probe and plasma potentials, respectively. Due to spherical symmetry, the particle motion remains within a plane defined by its velocity vector, which can be decomposed into $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$, where $\mathbf{v}_\parallel$ is parallel to the radial direction, and $\mathbf{v}_\perp$ is perpendicular to the radial direction, see figure A1. The angular momentum $L = r \times \mathbf{v}_\perp$, which is perpendicular to the orbital plane, is also conserved [35]. Therefore,

$$r_{\parallel} = r_p v'_\parallel,$$  \hspace{1cm} (46)

Denoting the angle between the radial direction and the velocity vector $\mathbf{v}$ as $\varphi$, see figure A1, and using $v_\parallel = v \sin \varphi$ and $v_\perp = v \cos \varphi$, (48) reduces to

$$\sin^2 \varphi \leq \frac{r_p^2}{r^2} \left(1 - \frac{q_i\phi_p}{\frac{1}{2}m_isv'^2}\right),$$  \hspace{1cm} (49)

which restricts $\varphi$ to the range $[0, \varphi_{\text{max}}(v)]$, where $\varphi_{\text{max}}(v)$ is the maximum allowed angle the particle velocity vector can have in order for the particle to reach the probe surface.

The current into the probe comes from the component of the current density, $\mathbf{j}$, parallel to the radial direction, which is given by [35]

$$d\mathbf{j}_\parallel = q_i v_\parallel d\mathbf{n}_r,$$  \hspace{1cm} (50)
Here, \( p \)

values of \( p \) are obtained from (49). The maximum allowed angle increases as \( p \) increases, and it eventually approaches \( \sin^{-1}(r_p/r) \).

For the case of attracted particles \( q \Phi_p < 0 \). For the sheath limited current (SL), where \( v \leq v_e \), (56) is always satisfied, and all angles within the range \([0, \pi/2]\) are allowed. In the orbit motion limited (OML) region, \( v > v_e \) and the maximum allowed \( \varphi \) is given by (56), and as we can see, it decreases as \( v \) increases, and eventually approaches \( \sin^{-1}(r_p/r) \).

where

\[
d_n = n_0 f_s(v) \sin \varphi \, dv \, d\varphi,
\]

is the number density of particles within a narrow velocity range, \([v, v + dv]\). Here, \( f_s(v) = v^2 \sin \varphi \, dv \, d\varphi \), is the volume element of the velocity space (expressed in spherical coordinates), \( n_0 \) is the ambient volume number density of species \( s \), and \( f_s(v) \) is the velocity distribution function. Inserting (51) into (50), and using spherical coordinates, we obtain

\[
d_{ji} = q_s n_0 f_s(v) \sin \varphi \, dv \, d\varphi.
\]

The infinitesimal current into surface element \( dS \) of the spherical probe is [35]

\[
dl = dS \, d_{ji},
\]

where \( dS \) evaluates to \( 4\pi r^2 \) upon integration of a sphere, for a given distance \( r \) to the probe. Before integrating (53), the lower and upper integration limits in each direction in velocity space must be specified. These limits will change depending on whether the particles are repelled or attracted by the probe. These two cases will be handled separately in the following subsections.

A1. Collected current of repelled particles—retarding field

In a retarding field \( q \Phi_p > 0 \), and particles are repelled by the probe. Therefore, only those particles that have enough kinetic energy to overcome the potential barrier can reach the probe surface. This means

\[
\frac{1}{2} m_i v^2 \geq q \Phi_p \quad \Rightarrow \quad v \geq v_{\text{min}} = \sqrt{\frac{2 q \Phi_p}{m_i}},
\]

where \( v_{\text{min}} \) is the minimum particle speed needed to reach the probe. With reference to figure A2(a), only for \( v \geq v_{\text{min}} \), will (49) lead to real values for \( \varphi \). Hence, \( \varphi \in [0, \varphi_{\text{max}}(v)] \), where \( \varphi_{\text{max}}(v) \) is given by (49). The angle \( \theta \), represents all the planes through the spherical probe, and since there are no restrictions on \( \theta \), we have \( \theta \in [0, 2\pi] \). Therefore, the current of the repelled particles to the spherical probe must be integrated with the following limits

\[
I_r = 4\pi r^2 q_s n_0 \int_{v_{\text{min}}}^{\infty} f_s(v) \sin \varphi \, d\varphi \, d\theta \, dv.
\]

Assuming the velocity distribution function is isotropic, i.e. independent of \( \theta \) and \( \varphi \), integration of \( \theta \) and \( \varphi \) yields

\[
I_r = 4\pi r^2 q_s n_0 \int_{v_{\text{min}}}^{\infty} f_s(v) \left(1 - \frac{q \Phi_p}{\frac{2}{7} m_i v^2}\right) dv.
\]

Expression (55) is general and valid for any isotropic VDF.

A2. Collected current of attracted particles—accelerating field

In an accelerating field \( q \Phi_p < 0 \), and (49) can be rewritten as

\[
\sin^2 \varphi \leq \frac{r^2}{r^2} \left(1 + \frac{q \Phi_p}{\frac{2}{7} m_i v^2}\right).
\]

Since the particles are attracted to the probe, if the particles enter the sheath region, they will be accelerated towards the surface of the probe. Let \( r_s \) denote the distance to the edge of the sheath region, which is unknown at this point, and must be determined from the solution of Poisson’s equation. Now, let’s consider the motion of a particle that has entered the sheath. We define a critical particle velocity at \( r = r_s \) for the maximum allowed \( \varphi \)-angle, for which the particle does not miss the probe surface. The maximum allowed \( \varphi \)-angle, as it is illustrated in figure A1 for a spherical probe, is \( \pi/2 \). From (56), this critical velocity can be defined as

\[
v_c^2 = \frac{2|q \Phi_p|}{m_i \left(\frac{2}{7} - 1\right)}.
\]
Particles entering the sheath with \( v \leq v_e \) are collected by the probe, simply because they don’t have enough kinetic energy to escape the electric force in the sheath. The current due to such particles is referred to as sheath limited current (SL). We note that, since \( v \leq v_e \), (56) is always satisfied, hence \( \varphi \in [0, \pi/2] \), see figure A2(b). The sheath limited current is, therefore, given by

\[
I_{s}^{SL} = 4\pi r_0^2 q_s n_0 \int_{0}^{\infty} \int_{0}^{2\pi} v^3 f_s(v) \sin \varphi \cos \varphi \, d\varphi \, d\theta \, dv. \tag{58}
\]

If, however, the particles enter the sheath with \( v > v_e \), only those particles that have a velocity vector with an \( \varphi \)-angle that satisfies (56), are collected, see figure A2(b). In this case, the current is referred to as orbit motion limited current (OML), and it is given by

\[
I_{s}^{OML} = 4\pi r_0^2 q_s n_0 \int_{0}^{\infty} \int_{0}^{2\pi} v^3 f_s(v) \sin \varphi \cos \varphi \, d\varphi \, d\theta \, dv. \tag{59}
\]

The total current is the sum of sheath limited and orbit motion limited currents, i.e. \( I_s = I_{s}^{SL} + I_{s}^{OML}. \) Integrating over \( \theta \) and \( \varphi \), (58) and (59) reduce to

\[
I_{s}^{SL} = 4\pi r_0^2 q_s n_0 \int_{0}^{\infty} v^3 f_s(v) \, dv, \tag{60}
\]

\[
I_{s}^{OML} = 4\pi r_0^2 q_s n_0 \int_{0}^{\infty} v^3 f_s(v) \left( 1 + \frac{|q_s \Phi_p|}{\sqrt{2}m_e v^2} \right) \, dv, \tag{61}
\]

respectively.

For the Kappa–Cairns VDF, evaluating the integrals in (60) and (61) for the spherical probe, yield

\[
I_{s}^{SL}(\eta; \kappa, \alpha, r_e) = I_{0}^{SP} C_{\kappa,0}^{SP} \left( \frac{r_e}{r_p} \right)^2 \left[ 1 - \left( 1 + \frac{\kappa^2 v_e^2}{\kappa - 2} \right)^{-\kappa} \right] \times \left\{ 1 + \frac{\kappa^2 v_e^2}{\kappa - 2} \right\} \left[ 3 + \frac{\kappa - 2}{\kappa - \frac{3}{2}} \right] \right]^{\alpha}, \tag{62}
\]

for the sheath limited current, and

\[
I_{s}^{OML}(\eta; \kappa, \alpha, r_e) = I_{0}^{SP} C_{\kappa,0}^{SP} \left( \frac{r_e}{r_p} \right)^2 \left[ 1 + \frac{\kappa^2 v_e^2}{\kappa - 2} \right] \times \left\{ 1 + \frac{\kappa^2 v_e^2}{\kappa - 2} \right\} \left[ 3 + \frac{\kappa - 2}{\kappa - \frac{3}{2}} \right] \right]^{\alpha}, \tag{63}
\]

for the orbit limited current, respectively. Here

\[
\hat{v}_e^2 = \frac{|\eta|}{\left( \frac{c_e}{r_p} - 1 \right)},
\]

and the other parameters are the same as in the text.

From the expressions in (62) and (63), it is clear that the current is dependent on the distance to the sheath edge \( r_e \), which,
as mentioned above, can be determined from the solution of Poisson’s equation for the electric potential. However, there exist two limits of the size of the sheath, for which the current becomes independent of the distance to the sheath edge.

A.2.1. Thin sheath approximation

In the thin sheath limit, the distance to the sheath edge is comparable to the probe radius, i.e. \( r_s \approx r_p \), or since \( r_s \approx \lambda_{De} \), \( r_p \approx \lambda_{De} \), hence (57) gives \( v_s \to \infty \). Therefore

\[
\lim_{v_s \to \infty} I_{\infty}^{SL} = 4\pi r_p^2 q_i n_0 \int_0^\infty v^3 f_s(v) dv,
\]

\[
\lim_{v_s \to \infty} I_{\infty}^{OML} = 0.
\]  

A.2.2. Thick sheath approximation

If, on the other hand, the sheath size is much bigger than the size of the probe, i.e. \( r_s \gg r_p \), or \( r_p \ll \lambda_{De} \), from (57) \( v_s \to 0 \), and therefore

\[
\lim_{v_s \to 0} I_{\infty}^{SL} = 0,
\]

\[
\lim_{v_s \to 0} I_{\infty}^{OML} = 4\pi r_p^2 q_i n_0 \int_0^\infty v^3 f_s(v) \left(1 + \frac{|q_i \Phi_p|}{2 m_e v_s^2}\right) dv.
\]  

Appendix B. General expressions for the collected current by cylindrical probes

We will now consider a cylindrical probe with radius \( r_p \) and length \( L_p \). In order to be able to derive analytical expression for the current to the probe, we assume the length of the cylinder is much larger than its radius, i.e. \( L_p \gg r_p \), and then the Debye length. Just as in the case of a spherical probe, we will denote the velocity of a plasma particle at a radial distance \( r \), as \( \mathbf{v} \), and its velocity at the surface of the probe \( r_p \), as \( \mathbf{v}' \). Without loss of generality, we take the \( z \)-axis to be aligned with the probe axis. In cylindrical coordinates, the velocity can be decomposed into \( \mathbf{v} = \mathbf{v}_r + \mathbf{v}_z \), where \( \mathbf{v}_r \) and \( \mathbf{v}_z \) are parallel and perpendicular to the probe axis, respectively, see figure A3. The perpendicular component of the velocity can be further decomposed into \( \mathbf{v}_z = v_z + v_\parallel \), where \( v_z \) is along the radial direction towards the probe, and \( v_\parallel \) is perpendicular to the radial direction. We note that the notation used here for the decomposition of the velocity vector might be different from what is found in the literature. Since the length of the cylindrical probe is much larger than its radius, and there are no external forces on the charge particle, the component of the velocity vector parallel to probe axis remains constant, i.e. \( \mathbf{v}_z = \mathbf{v}_z' \). Conservation of energy yields \([35]\)

\[
\frac{1}{2} m_e (v_z'^2 + v_\parallel^2 + v_r^2) + q_i \Phi_p
\]

\[
= \frac{1}{2} m_e (v_z'^2 + v_\parallel^2 + v_r^2) + q_i \Phi_p,
\]  

where \( v_z = ||v_z||, v_\parallel = ||v_\parallel|| \), and \( v_r = ||v_r|| \). The \( z \)-component of the angular momentum is also conserved, which gives \([35]\)

\[
v_z = v_z'.
\]  

Combining these two equations, we obtain

\[
v_\parallel^2 = v_z'^2 + v_r^2 \left(1 - \frac{r}{r_p}\right)^2 - \frac{2 q_i \Phi_p}{m_e}.
\]  

In order for the plasma particles to reach the probe surface, we must have \( v_\parallel^2 \geq 0 \), which gives the following constraint on the perpendicular velocity component

\[
v_\parallel^2 \leq \frac{v_z'^2}{1 - \left(\frac{r}{r_p}\right)^2} - \frac{2 q_i \Phi_p}{m_e}.
\]  

Denoting the angle between \( \mathbf{v}_z \) and \( \mathbf{v}_\parallel \) by \( \varphi \), we have \( v_\parallel = v_z \cos \varphi \) and \( v_\parallel = v_z \sin \varphi \). Hence, (71) reduces to

\[
sin^2 \varphi \leq \frac{r_p^2}{r^2} \left(1 - \frac{2 q_i \Phi_p}{m_e v_z'^2}\right).
\]  

This means that, for the particle to be able to reach the probe surface, the angle between \( \mathbf{v}_z \) and the radial direction must be in the range \([ -\varphi_{\max}(v_z), \varphi_{\max}(v_z) ] \), where \( \varphi_{\max}(v_z) \) is the maximum allowed angle that satisfies (72), see figure A3.

In the case of cylindrical probe, the component of current density towards the probe is given by

\[
d_\parallel = q_i v_z \frac{d\omega}{d\theta}, \quad d_\parallel = q_i \frac{d\omega}{d\theta} v_z d\varphi.
\]  

where the volume element in cylindrical coordinates is given by \( d\omega = v_r \, dv_r \, d\varphi \, dv_z \).

The infinitesimal current into the probe expressed in cylindrical coordinates, takes the form

\[
dI = d\Omega \, d_\parallel = d\Omega \, q_i n_0 v_z^2 \cos \varphi f_s(v) \, dv_r \, d\varphi \, dv_z,
\]  

where \( d\Omega \) is an infinitesimal cylindrical surface element that evaluates to \( 2\pi r_p dr \), for a given distance \( r \) to the probe.

Similar to the case of a spherical probe, the integration limits of each component will be dependent on whether the plasma particles are repelled or attracted by the probe.

B.1. Collected current of repelled particles—retarding field

In this case \( q_i \Phi_p > 0 \), and the minimum required velocity (perpendicular to probe axis) that the particle must have in order to reach the probe surface is

\[
v_{\min} = \sqrt{\frac{2 q_i \Phi_p}{m_e}}.
\]  

Only those particles that have \( v_z \geq v_{\min} \) are collected by the probe, provided that the \( \varphi \)-angle is within the range \([ -\varphi_{\max}(v_z), \varphi_{\max}(v_z) ] \). Therefore, the integration limits
of (74) are

\[
I_r = 2\pi r_p q_r n_0 \int_{v_{th}}^{\infty} \int_{-\infty}^{v_{th}} v^2 \cos \varphi f_r(v) dv_\varphi dv_r.
\]  

(76)

Again, assuming \( f_r(v) \) is independent of \( \varphi \), and integrating over \( \varphi \), (76) reduces to

\[
I_r = 4\pi r_p q_r n_0 \int_{v_{th}}^{\infty} \int_{-\infty}^{\infty} v^2 f_r(v) \sqrt{1 - \frac{2q_r^2 \Phi_p}{m_r v_r^2}} dv_\varphi dv_r.
\]  

(77)

B.2. Collected current of accelerated particles—accelerating field

In this case \( q_r \Phi_p < 0 \), and similar to the spherical case, the collected current is the sum of SL and OML currents, i.e. \( I_r = I_r^{SL} + I_r^{OML} \). The critical velocity, \( v_{cr} \), with which particles enter the sheath, at \( r = r_c \), without missing the probe, has the same form as (57). This critical velocity is defined for maximum \( \varphi \)-angle, which is either \( \pi/2 \) or \(-\pi/2\). Hence, in the case of SL current, where \( v \leq v_{cr} \), we have \( \varphi \in [-\pi/2, \pi/2] \). The SL current is, therefore, given by

\[
I_r^{SL} = \frac{2}{\sqrt{\pi}} I_{0r}^{SL} C_{0r}^{SL} D_{0r}^{SL} \frac{\tilde{v}^2 + |\eta|}{(\kappa - \frac{3}{2})^2} \bigg\{ \frac{2F_1}{\kappa - \frac{3}{2}} \bigg( 2F_1 \bigg( \frac{\kappa - \frac{3}{2}}{2} \bigg) - \frac{2(\kappa - \frac{3}{2})}{\kappa - 1} |\eta| \bigg) \bigg\},
\]  

(82)

For the velocity distribution in (1), the expression for the sheath limited current for the cylindrical probe, obtained from the integral in (80), is given by

\[
I_r^{SL}(\eta; \kappa, \alpha) = \frac{4\pi r_p}{3\sqrt{\pi}} \frac{\kappa - \frac{3}{2}}{\kappa - \frac{3}{2}} \frac{\kappa - \frac{3}{2}}{\kappa - 1} \bigg\{ \frac{2F_1}{\kappa - \frac{3}{2}} \bigg( 2F_1 \bigg( \frac{\kappa - \frac{3}{2}}{2} \bigg) - \frac{2(\kappa - \frac{3}{2})}{\kappa - 1} |\eta| \bigg) \bigg\},
\]  

(83)

and the corresponding expression for the orbit motion limited current, obtained by evaluating the integral in (81), is

\[
I_r^{OML}(\eta; \kappa, \alpha) = \frac{2}{\sqrt{\pi}} I_{0r}^{OML} C_{0r}^{OML} D_{0r}^{OML} \frac{\tilde{v}^2 + |\eta|}{(\kappa - \frac{3}{2})^2} \bigg\{ \frac{2F_1}{\kappa - \frac{3}{2}} \bigg( 2F_1 \bigg( \frac{\kappa - \frac{3}{2}}{2} \bigg) - \frac{2(\kappa - \frac{3}{2})}{\kappa - 1} |\eta| \bigg) \bigg\},
\]

(84)

where

\[
C_{0r}^{SL} = \frac{}{(\kappa - \frac{1}{2})^2},
\]

\[
D_{0r,\alpha}^{SL} = \frac{}{(1 + 15\alpha)(\kappa - \frac{3}{2})^2},
\]

(85)

and

\[
E_{0r,\alpha}^{SL} = \frac{}{(\kappa - \frac{1}{2})^2(\kappa - \frac{3}{2})^2 + 3\alpha(\kappa - \frac{3}{2})^2}.
\]

(86)

The other parameters are the same as in the text.
B.2.1. Thin sheath approximation
In the thin sheath limit, we have \( r_\epsilon \approx r_p \), hence \( v_{\perp,\epsilon} \to 0 \). Therefore,
\[
\lim_{v_{\perp,\epsilon} \to 0} I_{\epsilon,\text{SL}} = 4\pi r_\epsilon L_{\epsilon} q_{\text{e}} n_\epsilon \int_0^\infty \int_{-\infty}^\infty v_\perp^2 f_\epsilon(v) dv_{\perp} dv_{\parallel},
\]
(84)
\[
\lim_{v_{\perp,\epsilon} \to 0} I_{\epsilon,\text{OML}} = 0.
\]
(85)

B.2.2. Thick sheath approximation
In the thick sheath limit, the sheath size is much bigger than the size of the probe, i.e. \( r_\epsilon \gg r_p \), hence \( v_{\perp,\epsilon} \to 0 \). The integrals become
\[
\lim_{v_{\perp,\epsilon} \to 0} I_{\epsilon,\text{SL}} = 0,
\]
(86)
\[
\lim_{v_{\perp,\epsilon} \to 0} I_{\epsilon,\text{OML}} = 4\pi r_\epsilon L_{\epsilon} q_{\text{e}} n_\epsilon \int_0^\infty \int_{-\infty}^\infty v_\perp^2 f_\epsilon(v) \left(1 + \frac{|Q_\text{p}|}{m_\parallel v_{\parallel}^2}\right) dv_{\perp} dv_{\parallel},
\]
(87)

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