Chapter 1

The orthogonality of Al-Salam-Carlitz polynomials for complex parameters

Howard S. Cohl*, Roberto S. Costas-Santos† and Wenqing Xu‡

*Applied and Computational Mathematics Division, National Institute of Standards and Technology, Gaithersburg, MD 20899-8910, USA
howard.cohl@nist.gov

†Departamento de Física y Matemáticas, Facultad de Ciencias, Universidad de Alcalá, 28871 Alcalá de Henares, Madrid, Spain
rscosa@gmail.com

‡Department of Mathematics and Statistics, California Institute of Technology, CA 91125, USA
williamxuxu@yahoo.com

In this contribution, we study the orthogonality conditions satisfied by Al-Salam-Carlitz polynomials $U_n^{(a)}(x; q)$ when the parameters $a$ and $q$ are not necessarily real nor ‘classical’, i.e., the linear functional $u$ with respect to such polynomial sequence is quasi-definite and not positive definite. We establish orthogonality on a simple contour in the complex plane which depends on the parameters. In all cases we show that the orthogonality conditions characterize the Al-Salam-Carlitz polynomials $U_n^{(a)}(x; q)$ of degree $n$ up to a constant factor. We also obtain a generalization of the unique generating function for these polynomials.

Keywords: $q$-orthogonal polynomials; $q$-difference operator; $q$-integral representation; discrete measure.

MSC classification: 33C45; 42C05

1. Introduction

The Al-Salam-Carlitz polynomials $U_n^{(a)}(x; q)$ were introduced by W. A. Al-Salam and L. Carlitz in 1965 as follows:

$$U_n^{(a)}(x; q) := (-a)^n q^{\binom{n}{2}} \sum_{k=0}^{n} (q^{-n}; q)_{k} (x^{-1}; q)_{k} q^{k} a^{k} x^k.$$  (1)
In fact, these polynomials have a Rodrigues-type formula \[2 (3.24.10)\]

\[
U_n^{(a)}(x; q) = \frac{a^n q^{(n)}(1-q)^n}{q^n w(x; a; q)} D_q^{-1}(w(x; a; q)),
\]

where

\[
w(x; a; q) := (qx; q)_{\infty}(qx/a; q)_{\infty},
\]

the \(q\)-Pochhammer symbol (\(q\)-shifted factorial) is defined as

\[
(z; q)_0 := 1, \quad (z; q)_n := \prod_{k=0}^{n-1} (1 - zq^k),
\]

\[
(z; q)_{\infty} := \prod_{k=0}^{\infty} (1 - zq^k), \quad |z| < 1,
\]

and the \(q\)-derivative operator is defined by

\[
D_q f(z) := \begin{cases} 
\frac{f(qz) - f(z)}{(q-1)z} & \text{if } q \neq 1 \text{ and } z \neq 0, \\
 f'(z) & \text{if } q = 1 \text{ or } z = 0. 
\end{cases}
\]

**Remark 1.** Observe that by the definition of the \(q\)-derivative

\[
D_q^{-1} f(z) = D_q f(qz), \quad \text{and} \quad D_q^{-1} f(z) := D_q^{n-1} \left(D_q^{-1} f(z)\right), \quad n = 2, 3, \ldots
\]

The expression (1) shows us that \(U_n^{(a)}(x; q)\) is an analytic function for any complex value parameters \(a\) and \(q\), and thus can be considered for general \(a, q \in \mathbb{C} \setminus \{0\}\).

The classical Al-Salam-Carlitz polynomials correspond to parameters \(a < 0\) and \(0 < q < 1\). For these parameters, the Al-Salam-Carlitz polynomials are orthogonal on \([a, 1]\) with respect to the weight function \(w\). More specifically, for \(a < 0\) and \(0 < q < 1\) \([2 (14.24.2)\],

\[
\int_a^1 U_n^{(a)}(x; q) U_m^{(a)}(x; q)(qx, qx/a; q)_{\infty} d_q x = d_n^{2} \delta_{n,m},
\]

where

\[
d_n^{2} := (a)^n (1-q)(q; q)_{\infty}(a; q)_{\infty}(q/a; q)_{\infty} q^{(n)},
\]

and the \(q\)-Jackson integral \([2 (1.15.7)\] is defined as

\[
\int_a^b f(x) d_q x := \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,
\]
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where
\[
\int_0^a f(x) \, dq_x := a(1-q) \sum_{n=0}^\infty f(aq^n)q^n.
\]

Taking into account the previous orthogonality relation, it is a direct result that if \(a\) and \(q\) are classical, i.e., \(a, q \in \mathbb{R}\), with \(a \neq 1, 0 < q < 1\) all the zeros of \(U_n^{(a)}(x; q)\) are simple and belong to the interval \([a, 1]\), but this is no longer valid for general \(a\) and \(q\) complex. In this paper we show that for general \(a, q\) complex numbers, but excluding some special cases, the Al-Salam-Carlitz polynomials \(U_n^{(a)}(x; q)\) may still be characterized by orthogonality relations. The case \(a < 0\) and \(0 < q < 1\) or \(0 < aq < 1\) and \(q > 1\) are classical, i.e., the linear functional \(u\) with respect to such polynomial sequence is orthogonal is positive definite and in such a case there exists a weight function \(\omega(x)\) so that
\[
\langle u, p \rangle = \int_a^1 p(x) \omega(x) \, dx, \quad p \in \mathbb{P}[x].
\]

Note that this is the key for the study of many properties of Al-Salam-Carlitz polynomials I and II. Thus, our goal is to establish orthogonality conditions for most of the remaining cases for which the linear form \(u\) is quasi-definite, i.e., for all \(n, m \in \mathbb{N}_0\)
\[
\langle u, p_n p_m \rangle = k_n \delta_{n,m}, \quad k_n \neq 0.
\]

We believe that these new orthogonality conditions can be useful in the study of the zeros of Al-Salam-Carlitz polynomials. For general \(a, q \in \mathbb{C} \setminus \{0\}\), the zeros are not confined to a real interval, but they distribute themselves in the complex plane as we can see in Figure 1. Throughout this paper denote \(p := q^{-1}\).

2. Orthogonality in the complex plane

**Theorem 1.** Let \(a, q \in \mathbb{C}, a \neq 0, 1, 0 < |q| < 1\), the Al-Salam-Carlitz polynomials are the unique polynomials (up to a multiplicative constant) satisfying the property of orthogonality
\[
\int_a^1 U_n^{(a)}(x; q)U_m^{(a)}(x; q)w(x; a; q) \, dq_x = d_n^2 \delta_{n,m}.
\]

**Remark 2.** If \(0 < |q| < 1\), the lattice \(\{q^k : k \in \mathbb{N}_0\} \cup \{aq^k : k \in \mathbb{N}_0\}\) is a set of points which are located inside on a single contour that goes from 1 to 0, and then from 0 to \(a\), through the spirals
\[
S_1 : z(t) = |q|^t \exp(it \arg q), \quad S_2 : z(t) = |a||q|^t \exp(it \arg q + i \arg a),
\]
Fig. 1. Zeros of \( U_{30}^{(1+i)}(x; \frac{4}{5}\exp(\pi i/6)) \)

where \( 0 < |q| < 1 \), \( t \in [0, \infty) \), which we can see in Figure 2. Taking into account (2), we need to avoid the \( a = 1 \) case. For the \( a = 0 \) case, we cannot apply Favard’s result\(^\text{2}\) because in such a case this polynomial sequence fulfills the recurrence relation\(^\text{2}\)

\[
U_{n+1}^{(0)}(x; q) = (x - q^n)U_n^{(0)}(x; q), \quad U_0^{(0)}(x; q) = 1.
\]

**Proof.** Let \( 0 < |q| < 1 \), and \( a \in \mathbb{C}, a \neq 0, 1 \). We are going to express the \( q \)-Jackson integral \(^\text{2}\) as the difference of the two infinite sums and apply the identity

\[
\sum_{k=0}^{M} f(q^k)Q^{-1}g(q^k)q^k = \frac{f(q^M)g(q^M) - f(q^{-1})g(q^{-1})}{q^{-1} - 1} - \sum_{k=0}^{M} g(q^{k-1})Q^{-1}f(q^k)q^k. \tag{3}
\]

Let \( n \geq m \). Then, for one side since \( w(q^{-1}; a; q) = 0 \), and using the
identities [2, (14.24.7), (14.24.9)], one has

\[
\sum_{k=0}^{\infty} U_n^{(a)}(q^k; q) U_m^{(a)}(q^k; q) w(q^k; a; q) q^k
\]

\[
= a(1-q) \lim_{M \to \infty} \frac{1}{q^{2-n}} \sum_{k=0}^{M} \delta_{q^{-1}}[w(q^k; a; q) U_{n-1}^{(a)}(q^k; q)] U_m^{(a)}(q^k; q) q^k
\]

\[
= aq^{n-1} \lim_{M \to \infty} U_m^{(a)}(q^M; q) U_{n-1}^{(a)}(q^M; q) w(q^M; a; q)
\]

\[
+ aq^{n-1}(q^m-1) \lim_{M \to \infty} \sum_{k=0}^{M-1} w(q^k; a; q) U_{n-1}^{(a)}(q^k; q) U_m^{(a)}(q^k; q) q^k.
\]

Following an analogous process as before, and since \(w(aq^{-1}; a; q) = 0\), we
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have

\[
\sum_{k=0}^{\infty} U_m^{(a)}(aq^k; q)U_n^{(a)}(aq^k; q)w(aq^k; a; q)aq^k \\
= aq^{n-1} \lim_{M \to \infty} U_m^{(a)}(aq^M; q)U_n^{(a)}(aq^M; q)w(aq^M; a; q) \\
+ aq^{n-1}(q^m - 1) \lim_{M \to \infty} \sum_{k=0}^{M-1} w(aq^k; a; q)U_m^{(a)}(aq^k; q)U_n^{(a)}(aq^k; q)aq^k.
\]

Therefore, if \( m < n \), and since \( m \) is finite one can first repeat the previous process \( m + 1 \) times obtaining

\[
\sum_{k=0}^{\infty} U_m^{(a)}(q^k; q)U_n^{(a)}(q^k; q)w(q^k; a; q)q^k \\
= \lim_{M \to \infty} \sum_{\nu=1}^{m+1} (-aq^n)^\nu q^{-\nu(\nu+1)/2}(q^{-m+\nu-1}; q)_\nu \\
\times U_{m-\nu+1}^{(a)}(q^M; q)U_{n-\nu}^{(a)}(q^M; q)w(q^M; a; q),
\]

and

\[
\sum_{k=0}^{\infty} U_m^{(a)}(aq^k; q)U_n^{(a)}(aq^k; q)w(aq^k; a; q)aq^k \\
= \lim_{M \to \infty} \sum_{\nu=1}^{m+1} (-aq^n)^\nu q^{-\nu(\nu+1)/2}(q^{-m+\nu-1}; q)_\nu \\
\times U_{m-\nu+1}^{(a)}(aq^M; q)U_{n-\nu}^{(a)}(aq^M; q)w(aq^M; a; q).
\]

Hence since the difference of both limits, term by term, goes to 0 since \( |q| < 1 \), then

\[
\int_a^1 U_n^{(a)}(x; q)U_m^{(a)}(x; q)(qx, qx/a; q)_{\infty}dx = 0.
\]
For \( n = m \), following the same idea, we have
\[
\int_{a}^{1} U_{n}^{(a)}(x; q) U_{n}^{(a)}(x; q) w(x; a; q) d_q x
\]
\[
= \frac{a(q^n - 1)}{q^{1-n}} \sum_{k=0}^{\infty} \left( w(q^k; a; q) \left( U_{n-1}^{(a)}(aq^k; q) \right)^2 q^k - aw(aq^k; a; q) \left( U_{n-1}^{(a)}(aq^k; q) \right)^2 q^k \right)
\]
\[
= (-a)^n(q; q)_{n} q^n \sum_{k=0}^{\infty} \left( w(q^k; a; q)q^k - a \ w(aq^k; a; q)q^k \right)
\]
\[
= (-a)^n(q; q)_{n}(q; q)_{\infty} q^n \sum_{k=0}^{\infty} \left( (q^{k+1}/a; q)_{\infty} - a(aq^{k+1}; q)_{\infty} \right) \frac{q^k}{(q; q)_k},
\]
since it is known that in this case \(2 \) (14.24.2)]

\[
\int_{a}^{1} U_{n}^{(a)}(x; q) U_{n}^{(a)}(x; q) w(x; a; q) d_q x
\]
\[
= (-a)^n(q; q)_{n}(q; q)_{\infty}(a; q)_{\infty}(q/a; q)_{\infty} q^n (2).
\]

Due to the normality of this polynomial sequence, i.e., \( \deg U_{n}^{(a)}(x; q) = n \) for all \( n \in \mathbb{N}_0 \), the uniqueness is straightforward, hence the result holds.

From this result, and taking into account that the squared norm for the Al-Salam-Carlitz polynomials is known, we got the following consequence for which we could not find any reference.

**Corollary 1.** Let \( a, q \in \mathbb{C} \setminus \{0\}, |q| < 1 \). Then
\[
\sum_{k=0}^{\infty} \left( (q^{k+1}/a; q)_{\infty} - a(aq^{k+1}; q)_{\infty} \right) \frac{q^k}{(q; q)_k} = (a; q)_{\infty}(q/a; q)_{\infty}.
\]

The following case, which is just the Al-Salam-Carlitz polynomials for the \(|q| > 1\) case, is commonly called the Al-Salam-Carlitz II polynomials.

**Theorem 2.** Let \( a, q \in \mathbb{C}, a \neq 0,1, |q| > 1 \). Then, the Al-Salam-Carlitz polynomials are unique (up to a multiplicative constant) satisfying the property of orthogonality given by
\[
\int_{a}^{1} U_{n}^{(a)}(x; q^{-1}) U_{m}^{(a)}(x; q^{-1})(q^{-1}x; q^{-1})_{\infty}(q^{-1}x/a; q^{-1})_{\infty} d_{q^{-1}} x
\]
\[
= (-a)^n(1 - q^{-1})(q^{-1}; q^{-1})_{n}(q^{-1}; q^{-1})_{\infty}(a; q^{-1})_{\infty}(q^{-1}/a; q^{-1})_{\infty} q^{-n(2)} \delta_{m,n}. (4)
\]
Proof. Let us denote \( q^{-1} \) by \( p \), then \( 0 < |p| < 1 \). For \( a \in \mathbb{C}, a \neq 0,1 \), then, by using the identity (3) replacing \( q \rightarrow p \), and taking into account that \( w(aq; a; p) = w(q; a; p) = 0 \) and [2] (14.24.9), for \( m < n \) one has
\[
\sum_{k=0}^{\infty} aw(ap^k; a; p)U_m^{(a)}(ap^k; p)U_n^{(a)}(ap^k; p)p^k
= ap^{n-1} \lim_{M \to \infty} U_m^{(a)}(ap^M; p)U_n^{(a)}(ap^M; p)w(ap^M; a; p)
+ ap^{n-1}(1 - p^m) \lim_{M \to \infty} \sum_{k=0}^{M-1} aw(ap^k; a; p)U_n^{(a)}(ap^k; p)U_m^{(a)}(ap^k; p)p^k.
\]

Following the same idea from the previous result, we have
\[
\sum_{k=0}^{\infty} w(p^k; a; p)U_m^{(a)}(p^k; p)U_n^{(a)}(p^k; p)p^k
= ap^{n-1} \lim_{M \to \infty} U_m^{(a)}(p^M; p)U_n^{(a)}(p^M; p)w(p^M; a; p)
+ ap^{n-1}(1 - p^m) \lim_{M \to \infty} \sum_{k=0}^{M-1} w(p^k; a; p)U_n^{(a)}(p^k; p)U_m^{(a)}(p^k; p)p^k.
\]

Therefore, the property of orthogonality holds for \( m < n \). Next, if \( n = m \), we have
\[
\int_{-1}^{1} U_n^{(a)}(x; p)U_n^{(a)}(x; p)w(x; a; p) \, dp \, dx
= \frac{a(p^n - 1)}{p^n - n} \sum_{k=0}^{\infty} \left( aw(ap^n; a; p) \left( U_n^{(a)}(ap^n; p) \right)^2 p^k
- w(p^k; a; p) \left( U_n^{(a)}(p^k; p) \right)^2 p^k \right)
= (-a)^n(p; p)_n p(\frac{1}{2}) \left( \sum_{k=0}^{\infty} aw(ap^k; a; p)p^k - w(p^k; a; p)p^k \right)
= (-a)^n \left( q^{-1}; q^{-1} \right)_n(p; p)_\infty \left( \sum_{k=0}^{\infty} q^k \left( a(p^{k+1}; a; p) \right)_\infty - (p^{k+1}; a; p)_\infty \right)
= (-a)^n \left( q^{-1}; q^{-1} \right)_n(p; p)_\infty (a; p)_\infty (p; a; p)_\infty p(\frac{1}{2}).
\]

Using the same argument as in Theorem 1 the uniqueness holds, so the claim follows.

Remark 3. Observe that in the previous theorems if \( a = q^m \), with \( m \in \mathbb{Z}, a \neq 0 \), after some logical cancellations, the set of points where we need to calculate the \( q \)-integral is easy to compute. For example, if \( 0 < a q < 1 \) and \( 0 < q < 1 \), one obtains the sum [2] p. 537, (14.25.2)].
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Remark 4. The $a = 1$ case is special because it is not considered in the literature. In fact, the linear form associated with the Al-Salam-Carlitz polynomials $u$ is quasi-definite and fulfills the Pearson-type distributional equations

$$D_q[(x - 1)^2 u] = \frac{x - 2}{1 - q} u$$

Moreover, the Al-Salam-Carlitz polynomials fulfill the three-term recurrence relation [2] (14.24.3)

$$xU_n^{(a)}(x; q) = U_{n+1}^{(a)}(x; q) + (a + 1)q^n U_n^{(a)}(x; q) - aq^{n-1}(1 - q^n)U_{n-1}^{(a)}(x; q),$$

where $n = 0, 1, \ldots$, with initial conditions $U_0^{(a)}(x; q) = 1, U_1^{(a)}(x; q) = x - a - 1$.

Therefore, we believe that it will be interesting to study such a case for its peculiarity because the coefficient $q^{n-1}(1 - q^n) \neq 0$ for all $n$, so one can apply Favard’s result.

2.1. The $|q| = 1$ case.

In this section we only consider the case where $q$ is a root of unity. Let $N$ be a positive integer such that $q^N = 1$ then, due to the recurrence relation [5] and following the same idea that the authors did in [4] Section 4.2, we apply the following process:

1. The sequence $(U_n^{(a)}(x; q))_{n=0}^{N-1}$ is orthogonal with respect to the Gaussian quadrature

$$\langle v, p \rangle := \sum_{s=1}^{N} \gamma_1^{(a)} \cdots \gamma_{N-1}^{(a)} \frac{p(x_s)}{U_{a}^{(a)}(x_s)}^2,$$

where $\{x_1, x_2, \ldots, x_N\}$ are the zeros of $U_N^{(a)}(x; q)$ for such value of $q$.

2. Since $\langle v, U_n^{(a)}(x; q)U_n^{(a)}(x; q) \rangle = 0$, we need to modify such a linear form.

Next, we can prove that the sequence $(U_n^{(a)}(x; q))_{n=0}^{2N-1}$ is orthogonal with respect to the bilinear form

$$\langle p, r \rangle_2 = \langle v, pq \rangle + \langle v, D_q^N p D_q^N r \rangle,$$

since $D_q U_n^{(a)}(x; q) = (q^n - 1)/(q - 1)U_{n-1}^{(a)}(x; q)$.

3. Since $(U_{2N}^{(a)}(x; q), U_{2N}^{(a)}(x; q))_2 = 0$ and taking into account what we did before, we consider the linear form

$$\langle p, r \rangle_3 = \langle v, pq \rangle + \langle v, D_q^N p D_q^N r \rangle + \langle v, D_q^{2N} p D_q^{2N} r \rangle.$$

4. Therefore one can obtain a sequence of bilinear forms such that the Al-Salam-Carlitz polynomials are orthogonal with respect to them.
3. A generalized generating function for Al-Salam-Carlitz polynomials

For this section, we are going to assume \(|q| > 1\), or \(0 < |p| < 1\). Indeed, by starting with the generating functions for Al-Salam-Carlitz polynomials [2] (14.25.11-12), we derive generalizations using the connection relation for these polynomials.

**Theorem 3.** Let \(a, b, p \in \mathbb{C} \setminus \{0\}, |p| < 1, a, b \neq 1\). Then

\[
U_n^{(a)}(x; p) = (-1)^n (p; p)_n p^{-\binom{n}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k a^{n-k} (b/a; p)_{n-k}}{(p; p)_{n-k} (p; p)_k} U_k^{(b)}(x; p).
\]  

(6)

**Proof.** If we consider the generating function for Al-Salam-Carlitz polynomials [2] (14.25.11)

\[
\frac{(xt; p)_\infty}{(t; at; p)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p; p)_n} U_n^{(a)}(x; p) t^n,
\]

and multiply both sides by \((bt; p)_\infty/(bt; p)_\infty\), obtaining

\[
\sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p; p)_n} U_n^{(a)}(x; p) t^n = \frac{(bt; p)_\infty}{(at; p)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p; p)_n} U_n^{(b)}(x; p) t^n.
\]

(7)

If we now apply the \(q\)-binomial theorem [2] (1.11.1)

\[
\frac{(az; p)_\infty}{(z; p)_\infty} = \sum_{k=0}^{\infty} \frac{(ap; p)_n}{(p; p)_n} z^n, \quad 0 < |p| < 1, \quad |z| < 1,
\]

to (7), and then collect powers of \(t\), we obtain

\[
\sum_{k=0}^{\infty} t^k \sum_{m=0}^{k} \frac{(-1)^m a^{k-m} (b/a; p)_{k-m} p^{\binom{m}{2}}}{(p; p)_{k-m} (p; p)_m} U_m^{(b)}(x; p)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p; p)_n} U_n^{(a)}(x; p) t^n.
\]

Taking into account this expression, the result follows.

**Theorem 4.** Let \(a, b, p \in \mathbb{C} \setminus \{0\}, |p| < 1, a, b \neq 1, t \in \mathbb{C}, |at| < 1\). Then

\[
(at; p)_\infty \phi_1 \left( x; at^*, p; t \right) = \sum_{k=0}^{\infty} \frac{p^{k(k-1)}}{(p; p)_k} \phi_1 \left( \frac{b/a}{0}; p, at^k \right) U_k^{(b)}(x; p) t^k,
\]

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where

\[ r_\phi (a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; p, z) = \sum_{k=0}^{\infty} \frac{(a_1; p)_k (a_2; p)_k \cdots (a_r; p)_k}{(b_1; p)_k (b_2; p)_k \cdots (b_s; p)_k} \frac{z^k}{(p; p)_k} (-1)^{(1+s-r)k} p^{(1+s-r)(\frac{k}{2})}, \]

is the unilateral basic hypergeometric series.

Proof. We start with a generating function for Al-Salam-Carlitz polynomials [2, (14.25.12)]

\[ (at; q)_{\infty} \phi_1 \left( \frac{x}{at}; q, t \right) = \sum_{k=0}^{\infty} \frac{q^{n(n-1)} (q; q)_n}{(q; q)_n} V_n^{(a)}(x; q)t^n \]

and (3) to obtain

\[ (at; p)_{\infty} \phi_1 \left( \frac{x}{at}; p, t \right) = \sum_{n=0}^{\infty} t^n (-1)^n p^{\binom{n}{2}} \sum_{k=0}^{n} \frac{(-1)^k a^{n-k} (b/a; p)_{n-k}^{(\frac{k}{2})}}{(p; p)_{n-k} (p; p)_k} U_k^{(b)}(x; p). \]

If we reverse the order of summations, shift the \( n \) variable by a factor of \( k \), using the basic properties of the \( q \)-Pochhammer symbol, and [2 (1.10.1)]. Observe that we can reverse the order of summation since our sum is of the form

\[ \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} c_{n,k} U_k^{(a)}(x; p), \]

where

\[ a_n = t^n, \quad c_{n,k} = \frac{(-1)^k a^{n-k} (b/a; p)_{n-k}^{(\frac{k}{2})}}{(p; p)_{n-k} (p; p)_k}. \]

In this case, one has

\[ |a_n| \leq |t|^n, \quad |c_{n,k}| \leq K (1+n)^{\sigma_2} |a|^n, \]

and \( |U_n^{(a)}(x; p)| \leq (1+n)^{\sigma_2} \), where \( K, \sigma_1, \) and \( \sigma_2 \) are positive constants independent of \( n \). Therefore, if \( |at| < 1 \), then

\[ \left| \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} c_{n,k} U_k^{(a)}(x; p) \right| < \infty, \]

and this completes the proof. \( \square \)
As we saw in Section 2, the orthogonality relation for Al-Salam-Carlitz polynomials for $|q| > 1$, $|p| < 1$, and $a \neq 0, 1$ is

$$\int_{\Gamma} U^{(a)}_n(x; p)U^{(a)}_m(x; p)w(x; a; p)d_p x = d_n^2 \delta_{n,m}.$$ 

Taking this result in mind, the following result follows.

**Theorem 5.** Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $t \in \mathbb{C}$, $|at| < 1$, $|p| < 1$, $m \in \mathbb{N}_0$. Then

$$\int_{a}^{1} \phi_1 \left( \frac{q^{-x}}{at} : q, t \right) U^{(b)}_m(q^{-x} ; p)(q^{-1}x ; q^{-1})_\infty (q^{-1}x/a ; q^{-1})_\infty dq^{-1}$$

$$= (-bt)^m \frac{\phi(q^m ; b ; p)_\infty}{\phi(p/b ; p)_\infty} \phi_1 \left( \frac{b/a}{0} ; q, atq^m \right).$$

**Proof.** From (8), we replace $x \mapsto px$ and multiply both sides by $U^{(b)}_m(x; p)w(x; a; p)$, and by using the orthogonality relation (4), the desired result holds.

Note that the application of connection relations to the rest of the known generating functions for Al-Salam-Carlitz polynomials [2, (14.24.11), (14.25.11)] leave these generating functions invariant.

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