Vacuum Energy Densities and Multiplicative Anomalies in a Free Bose Gas

T.S. Evans

Theoretical Physics, Blackett Laboratory, Imperial College, Prince Consort Road, London, SW7 2BW, U.K.

Abstract

The vacuum energy density or free energy of a free charged Bose gas at non-zero densities is studied in the context of the debate about Multiplicative Anomalies. Some ζ-function regularised calculations of the free energy in the literature are reexamined, clarified and extended. A range of apparently distinct answers can obtained. Equivalent dimensional regularisation results are also presented for comparison. I conclude that operator ordering and normal ordering are not responsible for these differences. Rather it is an undesirable but unavoidable property of ζ-function regularisation which leads to these different results, making it a bad scheme in general. By comparison I show how dimensional regularisation calculations give a consistent result without any complications, making this a good scheme in this context.

1 Introduction

The vacuum energy density in QFT (quantum field theory) is a non-trivial object which leads to interesting physical phenomena such as inflation or the Casimir effect. When using path integrals to calculate it, one often encounters terms of the form ln det(Δ⁻¹) where Δ is the propagator for some field. However, in QFT this is the determinant of an infinite dimensional matrix, and this is usually infinite. It is therefore extremely interesting to find that the “Multiplicative Anomaly” a(A, B)

\[ a(A, B) := \ln \det(AB) - \ln \det(A) - \ln \det(B), \]  

need not be zero for two pseudo-differential operators A and B, [1, 2, 3]. The term anomaly in this context is used to indicate the failure of a familiar algebraic expression rather than a breakdown of a classical symmetry in the quantum theory. Since the individual terms in (1.1) are usually naively infinite in QFT, one must regulate before any serious discussion. A subscript on a quantity will be used to indicate the regularisation scheme used e.g. \( a_R, [\ln \det(A)]_R \) for some scheme R. The first examples of a non-zero
Multiplicative Anomaly in QFT was given by Elizalde, Vanzo and Zerbini [2] using \( \zeta \)-function regularisation of Dowker and Critchley [4] and two free scalar fields in four or more even dimensional Euclidean space time. The Multiplicative Anomaly \( \alpha_{\zeta} \) of (1.1) is related to the Wodzicki residue [5, 1] which is important for non-commutative geometry [6].

There have been two types of extension to the original work of [2]. Firstly, one can investigate different renormalisation schemes. In an earlier paper I showed that loop momentum cutoffs, including space-time lattices and dimensional regularisation — schemes common in particle physics — are all free of this Multiplicative Anomaly [7]. Rather, non-zero Multiplicative Anomalies are a feature of the whole family of Schwinger proper time regularisation schemes of which \( \zeta \)-function methods are just one example [8]. The second direction has been to show that the original Multiplicative Anomaly of (1.1) is just one of many algebraic identities which \( \zeta \)-function regularisation expressions fail to obey [7, 9, 12, 13, 14, 15, 11, 10]. If one is using \( \zeta \)-function regularisation, the importance of these Multiplicative Anomalies is not in doubt from the point of view of mathematical consistency. Furthermore there are several types of problem where \( \zeta \)-function regularisation methods are probably the best method to use, for example Casimir effect problems or problems of QFT in curved space time. The important debate is whether or not the extra terms due to Multiplicative Anomalies lead to new physics. A more extensive summary of Multiplicative Anomalies is given in appendix B.

One of the best examples to study in the discussion of physical meaning of Multiplicative Anomalies is that of the free Bose gas. This is because even in a free theory with one complex relativistic field, the \( U(1) \) symmetry can be broken and Bose-Einstein condensation then takes place [16]. In this model, the calculations of Elizalde et al [9] (see also [10]) of the free energy, using \( \zeta \)-function regularisation methods, gave a result which has extra terms when compared to standard results found in texts such as Kapusta [16]. The extra terms of [9] were chemical potential but not temperature dependent, and they led to alterations of the critical temperature. Furthermore in [9] these extra terms, which come from the Multiplicative Anomaly, were shown to be essential for mathematical consistency. Elizalde et al. suggested that normal ordering might be relevant in understanding why these extra terms appear. McKenzie-Smith and Toms [12, 13] then looked at the free Bose gas in \( \zeta \)-function regularisation with particular reference to canonical calculations and normal ordering rather than path integral calculations alone. They concluded that there is no problem if one follows canonical definitions.

Unfortunately there are mathematical inconsistencies between some of these \( \zeta \)-function regularisation results for the free Bose gas. I will reexamine the claims and calculations of Elizalde et al. [9], and McKenzie-Smith and Toms [12, 13] within \( \zeta \)-function regularisation, and will show how to fit them into a consistent mathematical framework. However, I confirm the mathematical results of [9] which show that one can reasonably obtain a number of different results for the free energy of the free Bose gas when using \( \zeta \)-function regularisation. However I differ from previous authors in my interpretation of the physics contained in these results. I conclude that canonical methods, and normal ordering in particular, offer no explanation for the genuinely different answers obtained. All of the many formal starting points are equally good. Rather my explanation is that in certain regularisation schemes, such as \( \zeta \)-function regularisation, the UV divergences are controlled by functions whose form depends on physical parameters such as chemical potentials. Equivalently the renormalisation scale in \( \zeta \)-function regularisation depends on some of the physical parameters of the problem. Hence the comparison of results

---

1 See Connes [6], page 307.
calculated in $\zeta$-function regularisation at different values of physical parameters is then extremely difficult. I conclude that there is no new physics of the free Bose gas in the non-standard results provided by some $\zeta$-function regularisation calculations.

However, in order to discuss non-standard results, I must also construct a standard. Thus to compare the $\zeta$-function regularisation results for the free Bose gas of [9, 12, 13] with the free energy results found in the literature such as [16], it is convenient to use a standard particle physics regularisation scheme. Dimensional regularisation, as defined in the particle physics literature\textsuperscript{2} [23, 17], is used here for several reasons. Firstly it is widely used in practical calculations [17]. Secondly it has no Multiplicative Anomalies when used in a sensible and correct manner [7], and so is in the same position as several other standard schemes such as lattice regularisation and simple momentum cutoffs. Lastly, dimensional regularisation and $\zeta$-function regularisation regulate by inserting a non-integer power of a polynomial of loop momentum, so one often finds that the same standard integral is required for both schemes. Therefore I will discuss the relation of $\zeta$-function regularisation calculations of free Bose gas to the standard results obtained with dimensional regularisation and in doing so make contact with results found in the literature such as are given in [16].

In the next section I examine how various formal, and therefore strictly meaningless, expressions for the free Bose gas are derived. I give precise well defined mathematical expressions only in the following section by carefully defining the regularisations. In section 4 I study the relationships between the precise and well defined forms of the free energy. Section 5 is devoted to looking at the physical content of the expressions and my conclusions are given in the final section. I have tried to put as many technical details in the first appendix, but because the confusion in the literature I feel it is important to specify the precise mathematical approach used here. Appendix B puts the multiplicative anomaly found in the free Bose gas in a wider context.

2 Formal expressions for the free energy

The free Bose gas is a heat bath of charged non-interacting scalar particles. The dynamics are described by the usual Klein-Gordon Lagrangian for a free relativistic complex scalar field\textsuperscript{3}

\[ \mathcal{L} = |\partial_\mu \Phi|^2 - m^2 |\Phi|^2. \]  

(2.1)

For simplicity I work in four-dimensional Euclidean space, of which the spatial dimensions have volume $V$, as results such as [2] show that Multiplicative Anomalies are often non-trivial in such space-times. The Lagrangian (2.1) is invariant under a global phase transformation which is associated with the conservation of particle minus anti-particle number\textsuperscript{4}. To describe the statistical average I use the density matrix of an equilibrium grand canonical ensemble, for which the temperature and the chemical potential of the heat bath (intensive variables) are specified rather than the energy and charge (extensive variables). The Euclidean approach to thermal field theory encodes this very simply

\textsuperscript{2}Confusingly, some of the $\zeta$-function regularisation literature, such as [11], refer to some Schwinger proper-time regularisation schemes as "dimensional regularisation". This is not the same as the scheme widely used in particle physics, as noted by Ball [8] and discussed in appendix A.3 around equation (A.20).

\textsuperscript{3}The condensed matter limit was considered separately in [12]. It is contained in this analysis as the $m \sim \mu \gg T \sim m - \mu \geq 0$ limit [19] though regularisation prescriptions may differ from that used in [12].

\textsuperscript{4}The number of particles and the number of anti-particles in each mode is also separately conserved but these symmetries are lost in interacting theories so I do not consider them here.
and this is sufficient for our purposes. I will use the approach in which the temporal direction is made periodic with length $\beta$ while the effects of non-zero chemical potential are included by working with an effective Hamiltonian, corresponding to the Lagrangian density$^5$ [16, 18, 19]

$$\mathcal{L}_\mu = \Phi^\dagger K_\pm \Phi,$$

(2.2)

where

$$K_\pm := (\partial_4 \pm \mu)^2 + \omega^2$$

(2.3)

$$\omega^2 := -\vec{\nabla}^2 + m^2.$$  

(2.4)

The $\vec{\nabla}$ acts only on the three spatial directions. The Euclidean temporal derivatives, $\partial_4$, are just shifted by the chemical potential $\mu$ (a real parameter) corresponding to a real shift in the origin of Minkowski energies [19]$^6$. Consider the partition function $Z$ and the associated free energy density $F$

$$Z := \text{Tr}\{e^{-\beta(H-H_Q)}\}, \quad F := -\frac{1}{\beta V} \ln(Z).$$

(2.5)

I will keep $|\mu| < m$ which, in a free theory, means working in the symmetric phase [16]. Likewise, I keep $T > 0$ as at zero temperature all particles will be in the ground state so $|\mu| = 0$ or $m$. Since the theory is free, one may quickly obtain an expression for $F$ but it is crucial to examine the familiar steps, given that the failure of supposedly familiar algebraic identities such as (1.1) is at the centre of the Multiplicative Anomaly debate.

Before I look at these calculations, note immediately that in this section I am writing formal expressions, by which I mean they have not yet been regularised, so they are naïvely infinite and therefore strictly meaningless. In doing this I am merely reflecting standard QFT procedures and will correct this later sections. This distinction, between formal infinite expressions and their meaningful regularised counterparts, is however not always clearly maintained in the literature. One of the aims of this paper is to indicate clearly when one is performing formal manipulations, and when rigorous manipulations of finite objects is being performed.

### 2.1 Path integral approach

When working with the path integral in this model, one would normally exploit the fact that the field is complex and work in the one-dimensional complex irreducible representation of $U(1)$. The path integral for $Z$ is

$$Z = \int D\Phi D\Phi^\dagger \exp\{-\int d^4x \mathcal{L}_\mu[\Phi, \Phi^\dagger]\}$$

(2.6)

---

$^5$The alternative Euclidean method puts both temperature and chemical potential in the boundary conditions [18, 19].

$^6$The meaning of Hermitian conjugation has been modified in the appropriate manner for Euclidean theories.
so the usual results for complex Gaussian integration give\footnote{One often sees the $\ln \det$ notation in $\zeta$-function regularisation work rather than $\text{Tr} \ln$. However, neither has any proper definition unless regularised, and I am free to choose an appropriate definition, \textit{pace} [20]. I will define them to be equal to the same regularised expression. In any case, all practical regularised expressions I know involve a sum or integral, i.e. they look like a trace and do not contain products which might remind one of a finite determinant. After all, the most familiar definition of a Riemann $\zeta$-function involves a sum over positive integers, rather than the alternative formula of a product over prime numbers. I will therefore tend to use the $\text{Tr} \ln$ notation, though I treat the two as equivalent.} \[ F_{4K^+} := -\frac{1}{\beta V} \ln \det(K_+) = -\frac{1}{\beta V} \text{Tr} \ln(K_+). \] (2.7)

Here the trace $\text{Tr}$ with a capital T and lower case r indicates that the trace is over the four dimensional Euclidean space so that in coordinate representation

\[
\text{Tr} \equiv \int_0^{\beta} dt_4 \int_V d^4 \vec{x}
\] (2.8)

If I work with the reducible complex two dimensional representation of $U(1)$, i.e. use a vector $\vec{\Psi} = (\Phi, \Phi^\dagger)$, then the Lagrangian is given by

\[
\mathcal{L}_\mu = \frac{1}{2} \vec{\Psi}^T D^{-1} \vec{\Psi}
\] (2.9)

where

\[
D^{-1} = \begin{bmatrix} K_+ & 0 \\ 0 & K_- \end{bmatrix}.
\] (2.10)

The path integral for $Z$ is

\[
Z = \int D\vec{\Psi} \exp\{-\int d^4 x \mathcal{L}_\mu[\vec{\Psi}]\}
\] (2.11)

and Gaussian integration gives

\[
F_{4A} := -\frac{1}{2} \ln \det(D^{-1}) = -\frac{1}{2} \text{Tr} \ln(D^{-1})
\] (2.12)

where $\text{Tr}$ and $\det$ are now taken over both the two-dimensional field space and the infinite dimensions coming from the four-dimensional space-time. One can take the determinant over the two-dimensional field space [21] to give

\[
F_{4A} := -\frac{1}{2} \text{Tr} \ln(A)
\] (2.13)

where

\[
A = K_+K_-.
\] (2.14)

One often sees this calculation expressed in terms of two real fields, the real and imaginary parts of the complex field $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$, e.g. [16]. Writing in a real vector notation $\vec{\phi} = (\phi_1, \phi_2)$ gives

\[
\mathcal{L}_\mu = \frac{1}{2} \vec{\phi}^T G^{-1} \vec{\phi}
\] (2.15)
where
\[ G^{-1} = \begin{bmatrix} -\partial_4^2 - \nabla^2 + m^2 - \mu^2 & \mu \partial_4 \\ -\mu \partial_4 & -\partial_4 - \nabla^2 + m^2 - \mu^2 \end{bmatrix}. \] (2.16)

The path integral for \( Z \) is
\[ Z = \int D\vec{\phi} \exp\{-\int d^4x \, L_\mu[\vec{\phi}]\}. \] (2.17)

In this case the propagator has off diagonal elements proportional to \( \mu \), but the eigenvalues are the diagonal entries of \( \mathbf{D} \) in (2.10) and the eigenvectors are \( \Phi \) and \( \Phi^\dagger \). These complex fields are of course the \( \text{U}(1) \) charge eigenstates and are therefore the most appropriate basis for unbroken symmetry problems. In any case one obtains the same result as before (2.13).

One can now look at a strange variation of the calculation of \( F_{4A} \). One can take (2.17) and formally one can factorize it into two integrations
\[ Z = Z_+Z_-, \quad Z_\pm = \int D\Phi D\Phi^\dagger \exp\{-\int d^4x \frac{1}{2} \Phi^\dagger K_\pm \Phi\} \] (2.18)
to give the formula
\[ F_{4K} := -\frac{1}{\beta V} \frac{1}{2} \text{Tr} \ln(K_+) - \frac{1}{\beta V} \frac{1}{2} \text{Tr} \ln(K_-) \] (2.19)

Now I see that this seemingly innocent variation with an initial factorization in the path integral, is producing a result which is nothing more than a factorization of the quartic operator \( A \) of the expression \( F_{4A} \) (2.13) into two quadratic operators. However, it is exactly the success or failure of this sort of factorization which is measured by the Multiplicative Anomaly (1.1), i.e.
\[ a(K_+, K_-) = 2\beta V (F_{4K} - F_{4A}) \] (2.20)

Thus a non-zero Multiplicative Anomaly says that I can not do these trivial maneuvers in the path integral such as described here. Since so much work is based on such formal manipulations of the path integral, Multiplicative Anomalies have potentially very grave implications.

Returning to the formal expressions for the free Bose gas free energy, note that there are many other factorizations of the operator \( A \). I will follow [9] and consider just one other, namely
\[ L_\pm := -\partial_4^2 + (\omega \pm \mu)^2 \] (2.21)
where \( \omega \) was given in (2.4). I could therefore equally well define the free energy density as
\[ F_{4L} := -\frac{1}{\beta V} \frac{1}{2} \text{Tr} \ln(L_+) - \frac{1}{\beta V} \frac{1}{2} \text{Tr} \ln(L_-) \] (2.22)

Again, the Multiplicative Anomaly expresses the failure of algebraic identities needed to relate \( F_{4A} \) to \( F_{4L} \) and we have from (1.1)
\[ a(L_+, L_-) = 2\beta V (F_{4L} - F_{4A}) \] (2.23)
This shows again that the Multiplicative Anomalies encode differences between formally equivalent definitions of the free energy of a free Bose gas.

Several other combinations also exist but I will focus on just the \( L \) and \( K \) factorizations. One common feature of all these expressions is that they involve four dimensional traces, equivalently are evaluated using integrals over energy and momentum, and that they are most naturally obtained within a path integral approach.
2.2 Canonical approach

The free Bose gas is also easily obtained using canonical methods and, as emphasized by McKenzie-Smith and Toms [12, 13], this leads to additional familiar forms. I will start with the Hamiltonian and charge operators for a relativistic free Bose gas. Unlike the path integral with its c-numbered fields, I must specify an operator ordering. My first definition of a Hamiltonian and charge operator are ones which have been normal ordered in the conventional zero temperature sense\(^8\), i.e. annihilation operators to the right, which I will denote with a subscript \(N\), namely

\[
\hat{H}_N := \int \frac{d^3k}{(2\pi)^3} \left( \omega_k \hat{a}^+_k \hat{a}_k + \omega_k \hat{b}^+_k \hat{b}_k \right),
\]

\[
\hat{Q}_N := \int \frac{d^3k}{(2\pi)^3} \left( \hat{a}^+_k \hat{a}_k - \hat{b}^+_k \hat{b}_k \right),
\]

with dispersion relation \(\omega_k\). Right at the start note that I am working on mass shell with integrals over three momenta, not traces over four space-time coordinates. Thus the expressions obtained will have a trace over the three dimensional spatial coordinates, denoted with lower case letters, as

\[
\text{tr} \equiv \int_V d^3\vec{x},
\]

(2.26)

to distinguish it from the full four-dimensional trace \(\text{Tr}\) of (2.8).

The vacuum state of the Fock space associated with the \(\hat{a}_k\) and \(\hat{b}_k\) operators is

\[
\hat{a}_k|0\rangle = \hat{b}_k|0\rangle = 0 \tag{2.27}
\]

This vacuum state will be the true physical vacuum provided we avoid symmetry breaking, i.e. if I choose \(m^2 > 0\) and if I have low charge densities so that \(|\mu| < m\) so there is no Bose-Einstein condensation.

With these definitions, I see that the expectation values of both \(\hat{H}_N\) and \(\hat{Q}_N\) in the vacuum state are zero,

\[
\langle 0|\hat{H}_N|0\rangle = \langle 0|\hat{Q}_N|0\rangle = 0. \tag{2.28}
\]

The density matrix in terms of these normal ordered operators is just

\[
\hat{\rho}_N := \exp\{-\beta(\hat{H}_N - \mu \hat{Q}_N)\} = \exp\{-\beta \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \left( (\omega_k - \mu)\hat{a}^+_k \hat{a}_k + (\omega_k + \mu)\hat{b}^+_k \hat{b}_k \right) \}
\]

(2.29)

Another ordering is the symmetric one, which I will denote with a subscript \(S\), namely

\[
\hat{H}_S := \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{2} \omega_k (\hat{a}^+_k \hat{a}_k + \hat{a}^+_k \hat{a}^+_k) + \frac{1}{2} \omega_k (\hat{b}^+_k \hat{b}_k + \hat{b}^+_k \hat{b}^+_k) \right), \tag{2.30}
\]

\[
\hat{Q}_S := \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{2} (\hat{a}^+_k \hat{a}_k - \hat{a}_k \hat{a}^+_k) - \frac{1}{2} (\hat{b}^+_k \hat{b}_k - \hat{b}_k \hat{b}^+_k) \right). \tag{2.31}
\]

Such a symmetric operator ordering is often the ordering found necessary to match path integral results, though a path integral is written in terms of commuting objects. Note

\(^8\)Interestingly, at \(T > 0\) or \(\mu \neq 0\) the most appropriate normal ordering is different and depends on \(T\) and \(\mu\) explicitly [22].
that, at least formally, the expectation value of this Hamiltonian in the vacuum state is now not zero, but is equal to a half $\omega_{\vec{k}}$ per mode in the system. On the other hand the expectation value of this charge operator is still formally zero,

$$\langle 0 | \hat{H}_N | 0 \rangle = 2 \left( \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} \omega_{\vec{k}} \right), \quad \langle 0 | \hat{Q}_N | 0 \rangle = 0.$$  \hfill (2.32)

The symmetric ordering density matrix is then just

$$\hat{\rho}_S := \exp \{ -\beta (\hat{H}_S - \mu \hat{Q}_S) \}$$  \hfill (2.33)

$$= \exp \{ -\beta \int \frac{d^3 \vec{k}}{(2\pi)^3} \left( \frac{1}{2} (\omega_{\vec{k}} - \mu) (\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \hat{a}_{\vec{k}} \hat{a}_{\vec{k}}^\dagger) + \frac{1}{2} (\omega_{\vec{k}} + \mu) (\hat{b}_{\vec{k}}^\dagger \hat{b}_{\vec{k}} + \hat{b}_{\vec{k}} \hat{b}_{\vec{k}}^\dagger) \right) \}$$  \hfill (2.34)

One can imagine other operator orderings. In particular, the only way to get charge operators with a non-zero vacuum expectation value is to change the ordering from that of $\hat{Q}_N$ by different amounts for the $\hat{a}_{\vec{k}}$ and $\hat{b}_{\vec{k}}$ operators to give a charge operator equal to $\hat{Q}_{AS} = \hat{Q}_N + \alpha$ where $\alpha$ is a c-number. This is equivalent to changing the definition of zero of charge. It will not effect the physics as all such $\hat{Q}_{AS}$ operators still commute with the Hamiltonian, but the labels I give to a given physical state will depend on such decisions. Each (anti-)particle still carries charge $+1$ ($-1$) but now the state with equal numbers of particles and anti-particles has charge $\alpha$. This does not seem useful physically and further I shall show that normal ordering has nothing to do with the anomalous terms of interest here. It is therefore sufficient to focus on just the normal ordered (2.24) and symmetric (2.30) versions.

The canonical calculation runs as follows. Since it is a free theory, the Hilbert space is a direct product of the single mode states, that is a direct product of the Fock spaces associated with the single oscillators $\hat{a}_{\vec{k}}(\vec{k})$ or $\hat{b}_{\vec{k}}(\vec{k})$ for all possible $\vec{k}$. In practice this means that for the symmetric density matrix (2.33)

$$Z_S = \text{Tr}\{ e^{-\beta (\hat{H}_S - \mu \hat{Q}_S)} \}$$  \hfill (2.35)

$$= \prod_{\vec{k}} \text{tr}_{a,\vec{k}} \{ \exp \{ -\beta V(\omega_{\vec{k}} - \mu) \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} \} \} \times \prod_{\vec{k}} \text{tr}_{b,\vec{k}} \{ \exp \{ -\beta V(\omega_{\vec{k}} + \mu) \hat{b}_{\vec{k}}^\dagger \hat{b}_{\vec{k}} \} \}$$  \hfill (2.36)

where here the traces, tr, are over the appropriate single oscillator Fock spaces. The direct product of the Hilbert space has resulted in a factorisation of terms in this formal expression. Standard calculations then give

$$F_{3\mu} := \sum_{\pm} \left[ \frac{1}{2} \text{tr} \{ \omega_{\vec{k}} \pm \mu \} + \frac{1}{2} \text{tr} \{ \ln [1 - \exp(-\beta (\omega_{\vec{k}} \pm \mu)) \} \} \right].$$  \hfill (2.37)

In exactly the same way, the normal ordered form for the density matrix (2.29) gives a second formal expression

$$F_{3N} := \sum_{\pm} \left[ \frac{1}{2} \text{tr} \{ \ln [1 - \exp(-\beta (\omega_{\vec{k}} \pm \mu)) \} \} \right]$$  \hfill (2.38)
with the $\beta$ independent zero-point energy term missing as one would expect.

However, there is yet another formal expression discussed in the literature. This I will call $F_3$ where

$$F_3 := \text{tr}\{\omega_{\vec{k}}\} + \frac{1}{2}\text{tr}\ln[1 - \exp(-\beta(\omega_{\vec{k}} - \mu))] + \frac{1}{2}\text{tr}\ln[1 - \exp(-\beta(\omega_{\vec{k}} + \mu))]$$ (2.39)

This has $\omega_{\vec{k}} \pm \mu$ factors in temperature dependent terms but not in the temperature independent terms, so is perhaps not the most natural form to arrive at by direct calculation though physically it is quite intuitive. The obvious way to obtain this third form $F_3$ is by making algebraic manipulations of the formal form (2.37). Later, inspired by the work of [13], I will find different answers in $\zeta$-function regularisation for these three forms, (2.37), (2.38) and (2.39). Normal ordering gives a genuine formal difference between the formal expressions $F_{3\mu}$ (2.37) and $F_{3N}$ (2.38) but not between $F_{3\mu}$ (2.37) and $F_3$ (2.39). The difference between $F_{3\mu}$ (2.37) and $F_3$ (2.39) is a matter of the Multiplicative Anomaly

$$a_{\text{shift}}(A, \alpha) := 2\text{tr}\{A\} - \text{tr}\{A + \alpha\} - \text{tr}\{A - \alpha\}$$ (2.40)

This expresses the failure of two basic algebraic identities (B.5) and (B.6) discussed in appendix B. In the case of the three dimensional expressions $F_3$ and $F_{3\mu}$, the difference is related to the Multiplicative Anomaly (2.40) through

$$F_3 = F_{3\mu} + \frac{1}{2}a_{\text{shift}}(\omega_{\vec{k}}, \mu)$$ (2.41)

Thus the unexpected differences between various $\zeta$-function regularisation versions of the formally UV divergent canonical calculations $F_3$ and $F_{3\mu}$, as found in the literature and to be discussed below, are not caused by normal ordering but failures of basic algebraic identities. It is the difference between $F_{3N}$ normal ordered UV finite expression and $F_{3\mu}$ UV infinite expressions which is a matter of normal ordering and the zero point energy in this model. In this I differ from both [9] and [12, 13] who emphasise the role of normal ordering and canonical methods in their resolutions for the problems.

### 3 Free Bose gas results

I will now turn from infinite formal expressions to finite regularised ones. Only with the latter can one compare the results for physical quantities obtained using different calculational schemes. I will denote regularised quantities by adding a further subscript: a $\zeta$ to indicate a $\zeta$-regularised expression, or a $\epsilon$ to indicate that dimensional regularisation was used, e.g. $F_{4K\zeta}$ is the $\zeta$-function regularised form of $F_{4K}$.

Before looking at the results in detail, I will make a few comments about the way I approached the four-dimensional finite temperature calculations of (2.13), (2.19) and (2.22) where there are energy variables to be summed over. As noted above, I use a Euclidean approach to thermal field theory where the energies are the discrete Matsubara energies but the chemical potential is encoded directly in the Lagrangian, propagators etc. rather than in the boundary conditions [18, 19]. There are standard methods for performing the Euclidean energy sums, for instance using contour integration methods. In this case the integrands are logarithms or non-integer powers of polynomials of energy so more care is needed than with simple Green functions with their integrals of rationals of polynomials. Nonetheless the discussion in the appendix A.1 shows that, as with Green...
functions, one can separate any calculation into two pieces

\[ F(T, \mu) = \int_{\beta} dk_4 g(k_4) \equiv \frac{1}{\beta} \sum_n g(k_4 = \frac{2\pi n}{\beta}) \]  

(3.1)

\[ F(T, \mu) = F_0(\mu) + F_\beta(T, \mu), \quad F_0(\mu) = \int_{-\infty}^{+\infty} dk_4 g(k_4). \]  

(3.2)

where in (3.1) \( g \) contains all the \( \mu \) dependence. In (3.2) \( F_\beta \) contains all the explicit temperature dependence in factors of \( \exp\{\beta(\omega \pm \mu)\} \)\(^9\). The \( F_\beta \) are also guaranteed to be UV finite. The first piece, \( F_0 \), has no explicit temperature dependence and is just the original expression with the energy sum replaced by a Euclidean energy integral from minus infinity to plus infinity. However, the zero temperature piece \( F_0 \) will be chemical potential dependent. As the problem of Multiplicative Anomalies is all about the behaviour of UV divergences, the focus will be on the temperature independent but possibly \( \mu \) dependent UV divergent parts \( F_0(\mu) \). The major question is how to implement the regularisation of the UV divergences in these temperature independent, chemical potential dependent terms.

Various terms crop up again and again so I will define some useful functions.

\[ f_{-1}(m) := f_{-1} = -\frac{m^4}{32\pi^2}, \]

(3.3)

\[ f_0(m, M) := f_0 = f_{-1}(m) \left[ \ln \left( \frac{M^2}{m^2} \right) + \psi(3) - \psi(1) \right] = \frac{m^4}{32\pi^2} \left[ \ln \left( \frac{m^2}{M^2} \right) - \frac{3}{2} \right], \]

(3.4)

\[ Y(m, \mu) := Y = \frac{1}{16\pi^2} \mu^2 \left( m^2 - \frac{\mu^2}{3} \right), \]

(3.5)

\[ X(m, \mu) := X = \frac{Y(m, \mu)}{2f_{-1}(m)} = -\frac{\mu^2}{m^2} \left( 1 - \frac{1}{3} \frac{\mu^2}{m^2} \right), \]

(3.6)

\[ f_\beta(m, \mu, \beta) := f_\beta = \frac{1}{\beta} \sum_{\pm} \int \frac{d^3k}{(2\pi)^3} \ln \left( 1 - \exp\{-\beta(\omega \pm \mu)\} \right), \]

(3.7)

3.1 4D \( \zeta \)-function regularisation results.

The details about how to implement \( \zeta \)-function regularisation in this simple case are given in appendix A.3. I have chosen to write these \( \zeta \)-function regularised expressions so as to make clear the close relationship to the dimensional regularisation results of later subsections. This will simplify the comparisons I make in section 4 but it means that my \( \zeta \)-function regularisation expressions are not always exactly the same as those in the literature, but the relationship is trivial (see appendix A.3). My \( \zeta \)-function regularisation versions of the formal expressions (2.13), (2.19) and (2.22) are

\[ F_{A\zeta} := \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left( \frac{-2}{s} \right) \left[ \frac{(k_4^2 + \omega_k^2 + \mu^2)^2 - 4\omega_k^2\mu^2}{M^4} \right]^{-s/2} \]

(3.8)

\[ F_{K\zeta} := \frac{1}{2} \sum_{\pm} \int \frac{d^4k}{(2\pi)^4} \left( \frac{-1}{s} \right) \left[ \frac{(k_4 \pm i\mu)^2 + \omega_k^2}{M^2} \right]^{-s} \]

(3.9)

\(^9\)For Bose or Fermi statistics and \( m > |\mu| \) such \( F_\beta \) are always zero at zero temperature. However the charge density is the physical parameter, not \( \mu \), and this makes \( \mu \) an implicit function of temperature. See appendix A.1 for further comments.
\[ F_{4L\zeta} := \frac{1}{2} \sum_{\pm} \int \frac{d^4 k}{(2\pi)^4} \left( \frac{-1}{s} \right) \left[ \frac{k^2 + (\omega_k \pm \mu)^2}{M^2} \right]^{-s} \]  

(3.10)

where I have chosen to ensure that in all expressions the regularisation scale always appears as \( M^2s \) and the first term in a small \( s \) expansion is always \( O(1/s) \) to maintain a close analogy with dimensional regularisation’s \( M^2\epsilon \) and \( O(1/\epsilon) \). Using standard tricks it is straightforward to do these integrals and I find for (3.9) and (3.10)

\[ F_{4K\zeta} = f_{-1}(m) \frac{1}{s} + f_0(m, M) + f_\beta(m, \mu, \beta) \]  

(3.11)

\[ F_{4L\zeta} = f_{-1}(m) \frac{1}{s} + f_0(m, M) + 2Y(m, \mu) + f_\beta(m, \mu, \beta) \]  

(3.12)

Note that both of these are calculated directly from the expressions quadratic in energy-momentum used to define them in (2.19) and (2.22), that is no use of Multiplicative Anomalies was made in their calculation. The surprise is that \( F_{4L\zeta} \) is different from \( F_{4K\zeta} \) by just the sort of \( \beta \) independent, \( \mu \) dependent term I will be talking about in the context of Multiplicative Anomalies. The expressions are essentially the same\(^{10}\) as given in [9].

Now one can consider the expression \( F_{4A\zeta} \) based on quartic energy-momentum terms. The calculation has been done in three ways. First, one can work directly with the quartic terms and [9] obtain

\[ F_{4A\zeta} = f_{-1}(m) \frac{1}{s} + f_0(m, M) + Y(m, \mu) + f_\beta(m, \mu, \beta) \]  

(3.13)

In the second and third approaches to \( F_{4A\zeta} \), I start from the definition of \( F_{4A\zeta} \) in terms of \( \text{Tr} \ln \) over quadratic operators and an additional anomaly term, as given in (2.20) and (2.23). Results for the \( \text{Tr} \ln \) over quadratic operators were given above in (3.11) and (3.12). However, one sees from the definition of the Multiplicative Anomaly (1.1) that to get the result for \( F_{4A\zeta} \) in these cases I must calculate the relevant Multiplicative Anomalies, as indicated in (2.20) and (2.23). These types of Multiplicative Anomaly can be calculated directly from a formula based on the Wodzicki residue [5] of the theory of elliptic pseudo-differential operators. Details are given in appendix A.4. In any case, I find that

\[ a(K_+, K_-) = -a(L_+, L_-) = -2\beta VY(m, \mu) \]  

(3.14)

which agrees with earlier results of Elizalde et al. [9]. It is then clear that all three methods of calculating \( F_{4A\zeta} \) give the same answer and therefore I disagree with the suggestion of an inconsistency made in [13].

The fact that all three approaches give the same answer is fundamental. It is a matter of mathematical definition that all three methods — direct, via \( K_\pm \) and its Multiplicative Anomaly, and finally via \( L_\pm \) and its Multiplicative Anomaly — are calculating the same object. However, if we had forgotten the Multiplicative Anomaly then I would have an inconsistency in our results for \( F_{4A\zeta} \). In this sense the existence of the Multiplicative Anomaly is not a problem but is absolutely essential for mathematical consistency.\(^{11}\)

Note that while there is complete consistency in the \( F_{4A\zeta} \) results, the \( Y \) factor comes up in different ways in different approaches. When using \( F_{4K\zeta} \) (2.20) approach the \( Y \)

\(^{10}\)The only differences are the sign in front of the \( f_0 \) term in (3.11) (equation (44) of [9]) and a factor of two in (3.12) (equation (38) of [9]).

\(^{11}\)Of course, if you accept that Anomalies exist then this shows how to calculate them without knowing the Wodzicki formula. Calculate \( F_{4A\zeta} \), and \( F_{4K\zeta} \) directly as done in [9], and then one can deduce the result for \( a(K_+, K_-) \).
comes solely from the Multiplicative Anomaly \( a(K_+, K_-) \). When starting from the \( L \) based form \( F_{4L}^\zeta \) of (2.23), there is a \( Y \) term intrinsic to \( F_{4L}^\zeta \) and another one from the Multiplicative Anomaly \( a(L_+, L_-) \) which contribute to the final \( Y \) factor in \( F_{4A}^\zeta \) (3.13).

Of course, while mathematical consistency of the \( \zeta \)-function regularisation may require non-zero Multiplicative Anomalies, this appears to cause major problems to the physical interpretation. I will return to this later when all relevant results have been acquired.

### 3.2 3d \( \zeta \)-function regularisation results.

The details of the finite temperature aspects and \( \zeta \)-function regularisation are as for the four-dimensional calculations above (also see appendices A.1 and A.3). Thus I define the \( \zeta \)-function regularisation forms as follows

\[
F_{3}^\zeta := \int \frac{d^3 k}{(2\pi)^3} \omega^{1-2s} M^{2s} \tag{3.15}
\]

\[
F_{3\mu}^\zeta := \frac{1}{2} \sum_{\pm} \int \frac{d^3 k}{(2\pi)^3} (\omega \pm \mu)^{1-2s} M^{2s} \tag{3.16}
\]

and \( F_{3N} \) is UV finite so needs no regularisation. I choose \( s \) so that the renormalisation scale appears to the same power as in the four-dimensional calculations. Direct calculation then gives

\[
F_{3N} = f_\beta \tag{3.17}
\]

\[
F_{3}^\zeta = f_{-1\frac{1}{s}} + f_{-1} \left( \ln \left( \frac{M^2}{m^2} \right) + \psi(3) - \psi(-1/2) \right) + f_\beta \tag{3.18}
\]

\[
= f_{-1\frac{1}{s}} + f_{0} + f_{-1} (\psi(1) - \psi(-1/2)) + f_\beta \tag{3.19}
\]

\[
F_{3\mu}^\zeta = F_{3}^\zeta + 2Y(m, \mu) \tag{3.20}
\]

Thus I see that in \( \zeta \)-function regularisation

\[
a_{\text{shift}, \zeta}(\omega, \mu) = 2\text{tr}(\omega) - \text{tr}(\omega - \mu) - \text{tr}(\omega + \mu) = 2(F_{3\zeta} - F_{3\mu}) = -4\beta VY(m, \mu). \tag{3.21}
\]

The Wodzicki residue formula (A.24) is only useful for \( \text{Tr} \ln \) expressions. Attempts to use it here forces one to try exponentials of well behaved operators in the formula (A.24) but these are not suitable. However, I am able to get the form of this Multiplicative Anomaly directly, if not the overall factor, by using a conjectured generalisation of the Wodzicki residue formula (A.30).

### 3.3 4D dimensional regularisation results.

It is useful to compare the \( \zeta \)-function regularisation results against results of a method more often used in particle physics, so I will use dimensional reduction. When using dimensional regularisation at non-zero temperature it is important to note that the regularisation takes place solely in the spatial integration, as discussed in the appendix A.2, i.e.

\[
\int_\beta d^{4-2\epsilon} k \equiv (2\pi)^{2\epsilon-4} \frac{1}{\beta} \sum_n \int d^{3-2\epsilon} k \tag{3.22}
\]
Let us start with the quartic expression (2.13). The appropriate dimensionally regularised form at $T > 0$ is

$$F_{4\epsilon} := \int_{\beta} d^{4-2\epsilon} k \ M^2 \ln \left( \frac{K_+ K_-}{M^4} \right)$$

(3.23)

The key point here is that this expression is finite while $\epsilon \neq 0$ so I can manipulate the integrand and swap the order of summing integrands with integration, for instance

$$F_{4K \epsilon} := \int_{\beta} d^{4-2\epsilon} k \ M^2 \ln \left( \frac{K_+}{M^2} \right) + \int_{\beta} d^{4-2\epsilon} k \ M^2 \ln \left( \frac{K_-}{M^2} \right) = F_{4\epsilon}$$

(3.24)

Thus it is immediately obvious that there are no Multiplicative Anomalies in dimensional regularisation as noted in [7]. Denoting all calculations in dimensional regularisation based on any of these four-dimensional forms as $F_{4\epsilon}$ I find

$$F_{4\epsilon} = f_{-1}(m) \frac{1}{\epsilon} + f_{-1}(m) \left( \ln \left( \frac{M^2}{m^2} \right) + \ln(4\pi) + \psi(3) \right) + f_{\beta}(m, \mu, \beta)$$

(3.25)

$$= f_{-1}(m) \frac{1}{\epsilon} + f_{-1}(m) \left( \ln \left( \frac{4\pi e^\gamma M^2}{m^2} \right) + \psi(3) - \psi(1) \right) + f_{\beta}(m, \mu, \beta)$$

(3.26)

$$= f_{-1}(m) \frac{1}{\epsilon} + f_0(m, (4\pi e^\gamma)^{1/2} M) + f_{\beta}(m, \mu, \beta)$$

(3.27)

Before moving on, note that it is possible to produce failures of algebraic identities in dimensional regularisation if one fails to implement dimensional reduction correctly. For instance this can be done by regulating the energy sums rather than the three-momentum integrals, see appendix A.2 for further comments.

### 3.4 3d dimensional regularisation results.

Just as I noted for the four-dimensional calculations using dimensional regularisation, if one follows the standard application of the dimensional reduction scheme to one of the three dimensional forms (A.7) one quickly sees that there is no problem with simple algebraic identities such as (B.5) and hence the Multiplicative Anomaly $a_{\text{shift}}$ of (2.40) is zero in dimensional reduction. Thus there is no difference between the two dimensionally regularised expressions based on the symmetric ordering, $F_{3\mu}$ of (2.37) and $F_3$ and (2.39). Likewise, the third expression, $F_{3N\epsilon}$ is just the UV finite $T > 0$ term present in the all the other expressions encountered in dimensional reduction or $\zeta$-function regularisation calculations. The dimensionally regulated form of both (2.39) and (2.37) is then

$$F_{3\mu \epsilon} = F_{3\epsilon} := \int d^{3-2\epsilon} k \ M^2 \ \omega(k), \ F_{3N\epsilon} = f_{\beta}$$

(3.28)

From this I find

$$F_{3\epsilon}(m, M_\epsilon, \mu, T, \epsilon) = f_{-1}(m) \frac{1}{\epsilon} + f_{-1}(m) \left( \ln \left( \frac{M_\epsilon^2}{m^2} \right) + \ln(4\pi^2) + \psi(3) \right) + f_4(3.29)$$

We see that the three- and four-dimensional results (except for the UV finite $F_{3N\epsilon}$) are identical with the same choice of $\epsilon$ and renormalisation scale $M_\epsilon$. From this I find

$$F_{3\epsilon}(m, M_\epsilon, \mu, T, \epsilon) = F_{4\epsilon}(m, M_\epsilon, \mu, T, \epsilon)$$

(3.30)
This is not entirely trivial but comes as a result of the careful implementation of the dimensionally regularised scheme, e.g. before integration free energies are always proportional to $M^2 \epsilon$. This ensures identities such as (A.8) are always satisfied which helps link three- and four-dimensional results. One can also make this connection directly. First, remember that $F_{4\epsilon\epsilon}^A$ is regulated only in the spatial integration, as shown in (3.23). Doing the Euclidean energy sum is thus straightforward using the usual contour integration methods as all the regularisation is in the spatial momentum. This shows that there are no $T = 0 \mu$ dependent terms, just a $T = 0$ UV divergent piece and the UV finite $T > 0$ contribution $f_{\beta}$. More relevant is that the $T = 0$ piece of $F_{4\epsilon}$ is then a three-dimensional integral and is then clearly of the form (3.28), the three-dimensional expression $F_{3\epsilon}$.

4 Comparison of results

I have focused on six different expressions using two different regularisation schemes all purportedly for the same quantity - the free energy of a free charged Bose gas. As always in QFT, comparison of results is complicated by the need to make sure that in two expressions for the same quantity that the physical parameters, upon which the answer depends, are defined in the same manner in all cases. Since it is a free field theory, there are no obvious corrections to the bare mass which is the physical mass, despite the quantum and statistical fluctuations encoded in the calculations. However one also has the quartic UV divergences in the free energy density or equivalently in the zero point energy density. Thus to discuss the physics in these results one must not only discuss the regularisation of the UV divergences, but also the removal of these divergences as part of the renormalisation procedure. I will postpone the discussion of the renormalised results and the physics to the following section and in this section I will compare the full regulated results.

The regulated but unrenormalised $\zeta$-function regularisation and dimensional reduction results can be compared directly by setting $\epsilon = s$ as by inspection one see that all the singular terms are then identical. This is not a miracle but comes from the close relationship between dimensional reduction and $\zeta$-function regularisation and from the way I have chosen various optional factors in such a way as to ensure this behaviour, see the appendices for further details. Also note that the expressions for the Multiplicative Anomalies are themselves naively finite and in principle require no renormalisation to obtain finite answers, though one needs to implement regularisation carefully to ensure this is seen in the final answers.

However, regularisation alone introduces a new scale, $M$ the renormalisation scale. This is not a trivial object. A good practical example of the importance of $M$ is the way that in particle physics, a great deal of effort goes into setting the scale for lattice and $\overline{\text{MS}}$ dimensional regularisation results for the same quantities [17].

The conclusion is that in order to be able to compare the different results for the free energy obtained above I must not assume that the renormalisation scales $M$ appearing in different calculations are the same.

---

12 The $T = 0$ part of $F_{4\epsilon\epsilon\epsilon}$ can be rewritten using the identity (A.8) as the usual four-dimensional zero point energy result for a complex scalar field in dimensional regularisation.

13 The latter process is needed when extracting the physics in QFT whether or not there are UV divergences.

14 These extra factors, functions $g$ in (A.7) or (A.12) in the appendices, can easily be included provided one modifies the relationship to $s = \epsilon/g(\epsilon)$ or similar. Fixing $g = 1$ still leaves the freedom to manipulate the renormalisation scale $M$. Thus I choose to work with $g = 1$ merely to remove irrelevant complications.
4.1 Discussion of dimensional regularised results.

As I will consider these to represent the ‘standard’ results, I will discuss these first. Several aspects are well known but, in view of the complications associated with the $\zeta$-function regularisation results, I will repeat them and emphasize several aspects for later comparison with the $\zeta$-function regularisation results.

The canonical expression for the normal ordered expression $F_{3N}$ leads to the non-zero temperature term $f_\beta$. This is UV finite and hence independent of regularisation. $f_\beta$ is also zero at zero temperature. This accords with the usual idea that normal ordering removes the zero point energy term associated with expectation values in the pure vacuum state. Thus $f_\beta$ appears to be the $T > 0$ UV finite correction to the zero point energy, and indeed we will find this borne out below in other non-normal ordered calculations. The finite temperature literature usually focuses on this term alone and leaves the temperature independent UV divergent zero point energy terms to one side e.g. [16].

The other five expressions for the free energy density, which are all UV divergent, lead to identical results in dimensional regularisation. Specifically the four-dimensional results $F_{4A\epsilon}(3.23)$, $F_{4K\epsilon}(3.24)$ and $F_{4L\epsilon}$ lead to (3.25). The two three-dimensional expressions $F_{3\epsilon}$ and $F_{3\epsilon\mu}$ of (3.28) and (2.37) respectively lead to the single result (3.29), which is identical to the four-dimensional dimensional reduction result (3.25) if the same renormalisation scale $M$ is used\footnote{The fact that the three- and four-dimensional dimensional regularisation results are equal with the same renormalisation scale follows from identities in dimensional regularisation such as (A.8).}. Since equations such as (2.20) show that any differences in results are related to non-zero anomalies, my conclusion is that there are no Multiplicative Anomalies in dimensional regularisation of the free Bose gas model. This agrees with the suggestion made in the introduction, based on the criteria set out in [7], namely that conventional dimensional reduction is always Multiplicative Anomaly free.

The result for these UV divergent expressions comes in two parts. First, the difference between $T > 0$ and $T = 0$ values (given that $|\mu| < m$) is always the same $f_\beta$ term as in $F_{3\epsilon}$. The remaining part, $r f_{-1} + f_0$ (where $r$ is either $s$ or $\epsilon$) is independent of temperature, and contains both finite contributions and UV divergences. An important point is that in dimensional regularisation this $T = 0$ part is also independent of chemical potential. Since the normal ordered form $F_{3N}$ should have removed the zero point energy, $(F_3 - F_{3\mu})$ is the zero point energy in dimensional reduction. The zero point energy in dimensional reduction is made up of the temperature and chemical independent terms (infinite and finite) as one would expect from its definition in terms of a vacuum energy expectation value\footnote{One may easily work at $T = 0$ and $\mu \neq 0$ to confirm this directly, as appendix A.1 shows.}. Thus normal ordering merely removes the UV divergent zero point energy, it is not the source of any Multiplicative Anomaly. Indeed there are no Multiplicative Anomalies in dimensional reduction [7], as discussed earlier, while normal ordering has the expected effect in dimensional reduction, clearly showing there is no link between Multiplicative Anomalies and normal ordering. If any final proof of this is needed, recall that the appropriate normal ordering at $T > 0$, $\mu \neq 0$ is not the usual one considered at zero temperature [22].

One last comment upon the dimensional regularisation results is to note that all of the UV divergent forms in dimensional regularisation are equal when the same renormalisation scale, $M_\epsilon$, is used. This is a direct consequence of the consistency demanded for dimensional regularisation integrals [23] and in particular comes from the result (A.8) which can be used to link the three and four-dimensional results directly with the same scale $M_\epsilon$.
4.2 Discussion of $\zeta$-function regularisation results

The $\zeta$-function regularisation produces the same result as dimensional reduction for the UV finite normal ordered expression $F_{3N}$. After this, things become more complicated.

Let us start with the four dimensional form $F_{4K}$. The $\zeta$-function regularisation result for $F_{4K}$ has no terms which depend on chemical potential other than the UV finite $f_{\beta}$, e.g. no $Y$ factors. This is the same behaviour as all the dimensional regularisation results. One can see this result quickly when using the Schwinger trick

$$\frac{1}{a^s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} e^{-at}$$

on (3.9). There the $T = 0$ term has an energy integral which becomes a Gaussian with a peak at $\mu$ so a simple shift removes all $\mu$ dependence from the $T = 0$ terms. By inspecting the results for $F_{4K}$ it is therefore straightforward to make a link between the dimensional regularisation results and this $\zeta$-function regularisation result. From (3.25) equality is obtained by choosing slightly different renomalisation scales for this $\zeta$-function regularisation calculation and the dimensional reduction calculations

$$F_\epsilon(m, M_\epsilon, \mu, T, \epsilon = s) = F_{4K}(m, M_{4K}, \mu, T, s = \epsilon), \quad (4.2)$$

$$\frac{(4\pi e^\gamma)^{1/2} M_\epsilon}{2} = M_{4K}$$

(4.3)

The problem is with the other two four-dimensional $\zeta$-function regularisation results, $F_{4A}$ of (3.13) and $F_{4L}$ of (3.12). Both differ from $F_{4K}$ by factors of $Y$, which is both mass and chemical potential dependent and which can not be absorbed as constant shifts to the regularisation scale. In fact I see that to obtain equality I must set

$$F_{4A}(m, M_{4A}, \mu, T, \epsilon = s) = F_{4K}(m, M_{4K}, \mu, T, s = \epsilon),$$

$$e^X M_{4A} = M_{4K}$$

(4.4)

$$F_{4L}(m, M_{4L}, \mu, T, \epsilon = s) = F_{4K}(m, M_{4K}, \mu, T, s = \epsilon),$$

$$e^{2X} M_{4L} = M_{4K}$$

(4.5)

where $X(m, \mu)$ is defined in (3.6). Thus as $0 \leq |\mu| \leq m$ I have $1 \geq e^X \geq 0.513$ and $1 \geq e^{2X} \geq 0.264$, sizable shifts in the renormalisation scales if I want to demand equality of free energies in these different $\zeta$-function regularisation schemes.

Having dealt with the four-dimensional forms, let me now turn to the three dimensional results using $\zeta$-function regularisation (3.18) and (3.20). Comparing with previous $\zeta$-function regularised results for the four-dimensional calculations, I see that

$$F_{3\zeta}(m, M_{3\zeta}, \mu, T, s) = F_{4K}(m, M_{4K}, \mu, T, s),$$

$$F_{3\rho\zeta}(m, M_{3\zeta}, \mu, T, s) = F_{4L}(m, M_{4L}, \mu, T, s)$$

(4.8)

provided I shift the renormalisation scale for these three-dimensional $\zeta$-function regularisation results as compared with the four-dimensional scales as

$$M_{3\zeta}^2 = M_{4K}^2 \exp\{\psi(-1/2) - \psi(1)\}, \quad \Rightarrow \quad M_{3\zeta} = \frac{e}{2} M_{4K}. \quad (4.10)$$

In fact one can derive these relationships directly from the four-dimensional forms, and in doing so prove explicitly the relationships (4.8) and (4.9) between the three- and four-dimensional $\zeta$-function regularisation expressions. This can be done by starting with
the four-dimensional form, rewriting the integrand using the Schwinger trick (4.1), then doing the energy integration, and finally inverting the Schwinger trick (4.1). However one tackles the problem, the results (4.8) and (4.9) show that the three-dimensional $\zeta$-function regularisation forms, as used in [12, 13], hold no new lessons as compared to the four-dimensional case, which were the focus in [9]. As in dimensional reduction, normal ordering gives a difference between normal ordered $F_{3N\zeta}$ and symmetric ordered three-dimensional $F_{3\zeta}$ $\zeta$-function regularisation expressions, specifically it removes the UV divergences. However the Multiplicative Anomaly problem comes in comparing $F_{3\zeta}$ and $F_{3\mu\zeta}$, both based on the same symmetric ordering of operators and so any difference is not due to operator ordering. Since the three- and four-dimensional results can be linked by (4.8) and (4.9), the conclusion is that operator ordering can not be responsible for the differences in UV divergent four-dimensional expressions either. The slightly more complicated relationship between the renormalisation scales in three- and four-dimensional calculations is to be expected in general regularisation schemes, and was only avoided in dimensional reduction because of identities such as (A.8).

Still, in $\zeta$-function regularisation it appears that one can obtain a single result for the UV divergent expressions only if we allow a rescaling of the renormalisation scale that depends on both mass and chemical potential. By comparison in dimensional reduction simple equality of all UV divergent results was achieved.

Having shown how one can link all these different results, I will now turn to the most important question, namely are there any physical differences encoded by these results.

5 The physics of the free Bose gas

The results of the previous section are summarised in table 1.

| Quantity | Coeff. of $r^{-1}f_{-1} + f_0$ $(T = 0 \mu \text{ ind.})$ | Coeff. of $Y$ $(T = 0 \mu \text{ dep.})$ | Coeff. of $f_{\beta}$ $(T > 0)$ | Relative scale used |
|----------|-------------------------------------------------|-----------------|----------------|------------------|
| Generic $F$ | a | b | c | $z$ |
| $F_{3N\zeta}$ and $F_{3N\zeta}$ | 0 | 0 | 1 | - |
| All other $F_{\epsilon}$ | 1 | 0 | 1 | $\frac{1}{2}$ |
| $F_{4K\zeta}$ | 1 | 2 | 1 | $e^{2x}$ |
| $F_{4L\zeta}$ | 1 | 1 | 1 | $e^{-x}$ |
| $F_{4A\zeta}$ | 1 | 0 | 1 | $e/2$ |
| $F_{3\zeta}$ | 1 | 2 | 1 | $e^{-2x}/2$ |

Table 1: Table of the different terms appearing in each expression for the free energy. The necessary rescalings of the renormalisation scale relative to the $F_{4K\zeta}$ calculation scale are given. The generic example is therefore $F = a(r^{-1}f_{-1} + f_0) + bY + cf_{\beta}$ using a scale $M = zM_{4K\zeta}$. $r = s$ or $\epsilon$ as appropriate.

However, before any discussion of physics in these results can take place, one must renormalise the free energy density, $F$. I will consider two distinct methods of removing its quartic divergences, first a physical subtraction and then an unphysical minimal subtraction.
5.1 Physical Subtraction

Consider a free energy difference, $\Delta F$, defined with respect to a reference temperature $T_R$ and chemical potential $\mu_R$, all other physical parameters held fixed

$$\Delta F(m, M, T, \mu) = F(m, M, T, \mu) - F(m, M, T_R, \mu_R).$$

(5.1)

Such a subtraction will not alter the thermodynamics and it is common to work with energy differences in many problems. One quickly finds that there are three distinct results for $\Delta F$. For all the dimensional regularisation calculations, and for the $\zeta$-function regularisation calculations based on $F_{4K\zeta}$, $F_{3N\zeta}$ and $F_{3\zeta}$ I have

$$\Delta F_{\epsilon} := \Delta F_{4K\zeta} = \Delta F_{3N\zeta} = \Delta F_{3\zeta} = f_{\beta}(m, T, \mu) - f_{\beta}(m, T_R, \mu_R)$$

(5.2)

where $f_{\beta}$ of (3.7) is the usual $T > 0$ term encountered in the free Bose gas model e.g. in [16]. The $\zeta$-function regularisation result based on the quartic operator, $F_{4A\zeta}$, picks up an extra $Y$ difference term

$$\Delta F_{4A\zeta} = \Delta F_{\epsilon} + Y(m, \mu) - Y(m, \mu_R),$$

(5.3)

while the $L$ factorisation, $F_{4L\zeta}$, and one of the three-dimensional results, $F_{3\mu\zeta}$, in $\zeta$-function regularisation give

$$\Delta F_{4L\zeta} = \Delta F_{3\mu\zeta} = \Delta F_{\epsilon} + 2Y(m, \mu) - 2Y(m, \mu_R).$$

(5.4)

These are all UV finite, though in most interacting models a single subtraction would not normally leave a finite result in this way. They are also all independent of the renormalisation scale $M$, provided we use the same value for $M$ at $T$ and $\mu$ as at the reference point $T_R, \mu_R$. One can check that the physical quantities, various derivatives of $\Delta F$, as a function of the observables, $T$ and the charge density $Q/V$ rather than a function of $T$ and $\mu$, do differ depending which result from (5.2), (5.3) and (5.4) we take. They are though independent of the reference point $T_R, \mu_R$. At the same time, all the different definitions of $F$ given in section (2) seem to be equally good. Thus there appears to be a serious problem in identifying the physics even in this simple free Bose gas model. Before suggesting a resolution, let me now consider a different renormalisation scheme.

5.2 Minimal subtraction

In minimal subtraction, one just drops the divergent terms. This is a common scheme in dimensional reduction and it is also performed implicitly in $\zeta$-function regularisation as explained in appendix A.3. Now I find four distinct results, which I denote as $F^{\text{MS}}$. The UV finite normal ordered results are unchanged by this renormalisation

$$F^{\text{MS}}_{3Ne} = F^{\text{MS}}_{3N\zeta} = f_{\beta}(m, T, \mu).$$

(5.5)

Otherwise, the results fall into three types depending on the calculational scheme, just as they did for the physical subtraction renormalisation used above. Thus

$$F^{\text{MS}}_{3e} = F^{\text{MS}}_{3\mu e} = F^{\text{MS}}_{4e} = F^{\text{MS}}_{4K\zeta} = f_0(m, M) + f_{\beta}(m, T, \mu)$$

(5.6)

$$F^{\text{MS}}_{4L\zeta} = f_0(m, M) + f_{\beta}(m, T, \mu) + Y(m, \mu)$$

(5.7)

$$F^{\text{MS}}_{4A\zeta} = F^{\text{MS}}_{3\mu\zeta} = f_0(m, M) + f_{\beta}(m, T, \mu) + 2Y(m, \mu)$$

(5.8)
All these renormalised free energies based on UV divergent expressions are finite again but now all have a factor of $f_0$ (3.4) unlike in the case of the physical subtraction results. Thus these all depend on the renormalisation scale $M$. However the $f_0$ factors do not alter the thermodynamics, assuming the same $M$ is used at all $T$ and $\mu$, so these minimal subtraction calculations give the same physical results as when the physical subtraction renormalisation is used. The physical results appear to be independent of the renormalisation scheme yet depend on the details of the definition and regularisation of the free energy.

5.3 Resolution of the paradox

The clue comes from the minimal subtraction results which retain some explicit $M$ dependence even though this does not effect the physical thermodynamics. It is a reminder that when comparing results at different $T$ and $\mu$ (or even different $m$), we are naturally inclined to hold this renormalisation scale constant. This is implicit in the physical subtraction calculations (5.2), (5.3) and (5.4) where $M$ is assumed to be the same at $T, \mu$ as at the reference point $T_R, \mu_R$.

But why should the renormalisation scale $M$ be held constant? Well it first appears when we cutoff the UV modes at some scale $\Lambda$ using a regulating function $R(k)$. In the language of this paper, $\Lambda \sim M/r$ where $r = \epsilon$ in dimensional reduction and $r = s$ in $\zeta$-function regularisation. Even though the interesting physics will be happening at scales much less than $\Lambda$ and will be encoded in $O(r^0)$ terms of a small $r$ series, there will also be a memory of the UV modes in the $O(r^0)$ terms and these will be $M$ dependent.

Now if we compare a regularised free energy expression for different physical parameter values, e.g. calculate $\Delta F$, it is absolutely essential that we have subtracted these UV modes in exactly the same manner in both expressions. If not each expression will have a slightly different remnant of the UV modes and the difference in the way the UV sector was treated will appear in the physical results and may be finite. It is though a purely mathematical artifact, a sign that we did not deal with the UV sector carefully enough and that we are not comparing like with like.

For instance, the simplest way to cut off the UV modes is to use a straight cutoff, inserting a regulating function $R_\lambda = \theta(M/\lambda - |\vec{k}|) (\lambda \to 0)$ into all loop integrals\(^\text{17}\). This will give terms such as

$$
F_\lambda = \int \frac{d^4k}{(2\pi)^4} R_\lambda(k) \ln(k^2 + m^2) = cM^4 \frac{1}{\lambda^2} + \ldots 
$$

where $c$ is some number. The UV cutoff scale is at $\Lambda = M/\lambda^{1/2}$ where $\lambda$ is the small parameter playing the same role as $\epsilon$ in dimensional reduction and $s$ in $\zeta$-function regularisation. There is then no doubt that we are removing the UV modes in the same manner for all calculations and, provided that the physics was occurring at scales much less than $\Lambda = M/\lambda^{1/2}$, no physics would be effected. If we were then to compare expressions for the free energy at different temperature or chemical potential in the free Bose gas model, e.g. calculate $\Delta F$, the expected result of [16] would appear, namely it would depend only on $f_0$ factors and no $cM^4\lambda^{-2}$ type term would survive, no extra $T$ or $\mu$ dependent terms.

Suppose though that one chose the cutoff $\Lambda(\mu)$ to have some slight dependence on the chemical potential, e.g. $\Lambda^4(\mu) = M^4/\lambda^2 + \mu^4$ (or indeed any other physical parameter). If

\(^{17}\)At non-zero temperature, the best approach is to cutoff only the three-momentum, but such detail is not relevant to the discussion here.
one compared the same regulated expression at two slightly different \( \mu \) values (e.g. looking at a \( \mu \) derivative of the free energy needed to calculate the physical charge density), terms depending on \( \Lambda \) would not exactly cancel. For instance making a physical subtraction we would find \( \Delta F_{\Lambda} \simeq c(\mu^4 - \mu_1^4) + \ldots \) for the simple example mentioned above. A finite remnant of the UV mode contribution is left, it is chemical potential dependent and it therefore alters the thermodynamics.

Thus in the case of a simple cutoff, the regularisation procedure is obvious. The cutoff \( \Lambda \) must be taken to be independent of physical parameters.

In the case of a simple cutoff, making the cutoff \( \Lambda \), or equivalently \( M \), depend on physical parameters is clearly a bad idea but it is easily noted and avoided. However, what if the form of the cutoff function \( R \) was \( \mu \) dependent rather than the cutoff scale itself? For instance

\[
R_{\Lambda} = 2[\exp\{(|\vec{k}| - M/\lambda)/w\} + 1]^{-1}
\]

is a theta function with its sharp jump at \( M/\lambda \) smoothed over a region of size \( w \). If the width \( w \) varies slightly with \( \mu \) then this will generate spurious \( \mu \) dependent terms from high energy \( k \sim M/\lambda \) modes when comparing free energy calculations at different chemical potentials.

Of course one could try to combine these two ideas. Thus one can compensate for a cutoff function \( R \) which depends on \( \mu \) by choosing the scale \( M \) to be \( \mu \) dependent too, chosen so that these spurious \( \mu \) dependent terms from \( M/\lambda \) modes in any physical result were removed. However it is extremely difficult to do this and \( M \) would have to be a complicated function of \( \mu \). In fact the only easy way to distinguish physical terms from spurious ones coming from \( M/\lambda \) modes is to compare against a calculation done with a regulating function \( R \) and renormalisation scale \( M \), both of which chosen constant. In this case though it is clear that the one may as well just work with the fixed regulator prescription in the first place.

Armed with these simple examples in terms of the cutoff \( \Lambda = M/\lambda^{1/2} \), let me now return to the main calculations of this paper. The discussion in the previous section 4 showed how to use the renormalisation scale \( M \) to relate all the different results but in several cases \( M \) had to depend on the physical parameter \( \mu \). This suggests that what is happening in the different calculations is that the UV modes around the scale \( M/r \) are being removed in different ways for different physical parameter values. Drawing on the comments in the appendix A.2, it is clear that dimensional reduction is not the offender. Roughly speaking, dimensional reduction regulates by inserting a function \( R_\epsilon = (\bar{k}^2/M^2)^{-\epsilon} \), which is completely independent of any physical parameters.

However, \( \zeta \)-function regularisation regulates by altering the integrands. For instance for the \( L \) factorization, the regulator is, crudely speaking, \( R_\zeta = [(k_4^2 + (\omega \pm \mu)^2)/M^2]^{-s} \) in (3.10). This is clearly changing the way the UV modes are cutoff in a manner which depends on \( \mu \) and \( m \). It is therefore no wonder that strange \( \mu \) and \( m \) dependent terms, \( Y \), appear even in the physical results. Equally, it should be no surprise that by choosing \( M \) to be a suitable, but complicated, function of \( \mu \) and \( m \), such terms can be removed.

The lesson from the \( \Lambda \) cutoff examples is that one should not trust dependence on physical parameters in calculations using regulators which involve those same physical parameters. Thus in \( \zeta \)-function regularisation one should not trust the \( \mu \) or \( m \) dependence of the results for the free energy of a free Bose gas. Also, the only way to identify the true physics is to use a regulation scheme which is not sensitive to the physical parameters. Therefore one can not ascribe some fundamental meaning to one or other of the \( \zeta \)-function regularised forms results (pace [12, 13]). In the case of the free Bose gas one may as well just work with a simple constant UV cutoff or dimensional reduction as \( \zeta \)-function
regularisation merely adds unnecessary complications.

6 Conclusions

My first conclusion is to confirm the assertion of [9] that if and only if one includes Multiplicative Anomalies in $\zeta$-function regularisation calculations does one get mathematical consistency in the free Bose gas model. That is if one considers a well defined mathematical object, such as the $\zeta$-function regulated expression for $F_{4A}$ (3.8), it does not matter then how you calculate it, either directly or via other quadratic expressions with the appropriate Multiplicative Anomalies, the same answer is always obtained. This is in contrast to suggestions elsewhere [13] that different calculational approaches to one well defined mathematical object in $\zeta$-function regularisation might give different answers because of some subtle physical effect. This point has been confirmed in several other models, see [2, 7, 14, 15] and appendix B for more details.

The second conclusion of this work is that the $\zeta$-function regularisation calculations of the free Bose gas can not be trusted. Perhaps after a careful study of the regulation of the UV modes, one might be able to be sure that some $\zeta$-function regularisation results are free of spurious finite UV mode contributions. However it is far easier to use an physics independent UV regulation scheme, such as dimensional reduction, to find the correct physics in this case. Thus the thermodynamics of the free Bose gas is described by $f_3(\mu, T, \mu)$ of (3.7) alone, which is the standard result of the literature e.g. [16]. The additional terms, such as $Y(m, \mu)$ of (3.5), found in some results, such as $F_{4A\zeta}$, are genuine enough mathematically but they do not have a simple physical interpretation, they are merely regulation artifacts contrary to suggestions made in [9, 10].

This second conclusion means that one should not worry about the physics obtained when such artifacts are included in the free energy, as was done in [9]. All the different results, $F_{4K\zeta}, F_{4A\zeta}$ etc., all encode the physics of this model exactly but they do so in an extremely convoluted manner. Thus I also disagree with [12, 13] that only some of these expressions are physical, e.g. $F_{4K\zeta}$, and the others, such as $F_{4A\zeta}$, are unphysical. The dimensional regularisation calculations confirm that all the formal starting points are equally good. It is merely that some regularisation schemes are less convenient than others. Thus only by comparing $\zeta$-function regularised calculations against results obtained in good regularisation schemes, such as dimensional reduction, can one see that $F_{4K\zeta}$ is the most convenient, but not more physical, starting point in $\zeta$-function regularisation.

Thirdly, as was noted at the start, the extra terms in the free Bose gas results can sometimes be described in terms of Multiplicative Anomalies. Thus the results here throw some light on the importance of Multiplicative Anomalies in general. My results for this model confirm what has been noted elsewhere, namely that it is essential for mathematical consistency that Multiplicative Anomalies are included when using $\zeta$-function regularisation. However, the interpretation given here, and in the model used in [7], suggest that Multiplicative Anomalies have no novel physical content. They are merely contributions from high energy modes reflecting the way that $\zeta$-function regularisation cuts off these modes in a physical parameter dependent manner.

Finally, these conclusions lead me to question the extraction of physical information from any $\zeta$-function regularisation calculation, even after Multiplicative Anomalies are accounted for. The analysis here was for one of the simplest examples, a free complex scalar field on a simple flat Euclidean $T^1 \times S^3$ space-time\(^\text{18}\). Yet I have shown that it is

\(^{18}\)The split of (3.2) and the role of UV divergences in this problem suggests that only the short distance
almost impossible to identify the correct physics from the $\zeta$-function regularisation results alone. It required a comparison with dimensional regularisation to do that. However, $\zeta$-function regularisation calculations are often used in much more complicated situations where surely it will be even harder to see the true physics from the UV regularisation remnants. I therefore believe that it will be almost impossible to identify the true physics rather than some regularisation remnants in many calculations using $\zeta$-function regularisation.

Acknowledgement

I thank E.Elizalde, A.Filippi, A.Flachi, J. McKenzie-Smith, I.Moss, W.Naylor, R.Rivers, and D.Toms for useful discussions.

References

[1] C.Kassel, Asterisque 177 (1989) 199.

[2] E.Elizalde, L.Vanzo and S.Zerbini, Comm.Math.Phys. 194 (1998) 613 [hep-th/9701060].

[3] M.Kontsevich and S.Vishik, in “Functional Analysis on the Eve of the 21st Century”, edited by S.Gindikin, J.Lepowsky and R.L.Wilson (Birkhäuser, Boston, 1995 (ISBN 0817637559 x v.1)) Vol. 1, pp. 173.

[4] J.S.Dowker and R.Critchley, Phys.Rev. D13 (1976) 3224.

[5] M.Wodzicki, Invent.Math. 75 (1984) 143.

[6] A.Connes, “Non-commutative geometry” (Academic Press, New York, 1994).

[7] T.S.Evans, Phys.Lett.B457 (1999) 127 [hep-th/9803184].

[8] R.D.Ball, Phys.Rep.182 (1989) 1.

[9] E.Elizalde, A.Filippi, L.Vanzo and S.Zerbini, Phys.Rev.D57 (1998) 7430 [hep-th/9710171].

[10] A.Filippi, Multiplicative anomaly and finite charge density, Nucl.Phys. A642 (1998) 222-227 [hep-th/9809050]; ibid New physics in the charged relativistic Bose gas using zeta-function regularization?, in “Strong and Electro-Weak Matter 2000”, June 2000, Marseilles, France. (World Scientific, Singapore, 2001) p.??-?? [hep-th/0102040]; ibid Chemical potential and multiplicative anomaly, in “Proceedings of the 5th International Workshop on Thermal Field Theories and Their Applications”, August 1998, Regensburg, Germany [hep-th/9809098].

[11] G.Cognola and S.Zerbini, On the dimensional reduction procedure, hep-th/0008061.

[12] J.J.McKenzie-Smith and D.J.Toms, Phys.Rev.D58 (1998) 105001.

[13] J.J.McKenzie-Smith and D.J.Toms, Zero-point energies and the multiplicative anomaly, preprint no. hep-th/0005201.

behaviour is important so results will probably be repeated in many other space-times.
A Calculational methods.

A.1 Finite temperature calculations.

First note that doing the Euclidean energy sum can be done using contour integration methods [18]. Suppose I have the following.

\begin{equation}
F = \int_{\beta} dk_{4} g(k_{4}) = \frac{1}{\beta} \sum_{n} g(-iz = \frac{2\pi n}{\beta}). \tag{A.1}
\end{equation}

\begin{equation}
= \frac{1}{2\pi} \int_{C} dz \frac{1}{2} \coth\left(\frac{\beta z}{2}\right)g(-iz). \tag{A.2}
\end{equation}

The function \(g\) varies in this paper but one example is

\begin{equation}
g(k_{4}) = -\frac{1}{s} \int \frac{d^{3}k}{(2\pi)^{3}} [(k_{4} + i\mu)^{2} + \omega_{k}^{2}]^{-s}. \tag{A.3}
\end{equation}
In this form, all the \( \mu \) dependence is in the propagators and hence in the \( g \)'s, and the energy sums are over pure Euclidean discrete values [19]. I use a subscript \( \beta \) on integrals as a short hand for the Matsubara sum

\[
\int_{\beta} dk_4 \ g(k_4) \equiv \frac{1}{\beta} \sum_n g(k_4 = 2\pi n\beta^{-1}) \tag{A.4}
\]

The contour \( C \) in (A.2) is made of several pieces, each a small circle centred on the Matsubara frequencies \( z = 2\pi n/\beta \), running in the positive direction [18]. The one complication in the types of integral I am looking at is that there are cuts which may run along the real \( z \) (Minkowski energy) axis as the integrands are logarithms or polynomials in energy raised to some non-integer power. At the same time for the Bosonic fields being considered I have a Matsubara frequency at zero energy and therefore lying on the real axis. Thus I must distort the cuts to run either slightly above or slightly below the real \( z \) axis at \( z = 0 \). Normally this is trivial but the presence of a poles in the coth function at that point complicates matters there. Luckily, one can prove that bosonic functions, such as those relevant to the free energy expressions I am studying, have zero discontinuity at zero energy [24] so the cuts may be distorted to either side without changing the final answer.

Now having dealt with that technicality, I distort the contour, expanding the contours round each of the poles on the imaginary energy axis until they merge and I am left with a contour running up the right hand side, and down the left hand side of the imaginary energy axis, i.e. from \( -i\infty + \epsilon \) to from \( +i\infty + \epsilon \) and then from \( +i\infty - \epsilon \) to \( -i\infty - \epsilon \) \((0 < \epsilon \ll 1)\). Now I add on semicircular contours running at \( |z| = \infty \) in each of the \( \text{Re}(z) > 0, \text{Re}(z) < 0 \) half-planes. The integrand should be zero on these contours. The coth and tanh functions are constant there and usually the form I am given for \( f(z) \) dies away at large \( z \). I am then sure that I have added nothing to the integral by adding these semicircular pieces to the contour.

To make the result clearer I can rewrite the coth functions using

\[
\frac{1}{2} \coth(\frac{\beta z}{2}) = \frac{1}{2} + \frac{1}{e^{+\beta z} - 1} = -\frac{1}{2} - \frac{1}{e^{-\beta z} - 1}, \tag{A.5}
\]

using the first (second) expression for the \( \text{Re}(z) > (\!<)0 \) part of the curve. This leaves

\[
F = F_0 + F_\beta, \quad F_0 = \int_{-\infty}^{+i\infty} \frac{-i}{2\pi} dz \ g(-iz)
\]

\[
F_\beta = \sum_{\{z_0\}} (-1)^{\text{sgn}(z_0)} \text{Res} \left( \frac{1}{e^{\beta z + \text{sgn}(z_0)} - 1} g(-iz) \right)
\]

\[
+ \frac{1}{2\pi i} \int_{\text{cuts}} dz \ (-1)^{\text{sgn}(z_0)} \text{Disc} \left( g(-iz) \frac{1}{e^{\beta z + \text{sgn}(z_0)} - 1} \right) \tag{A.6}
\]

where \( \{z_0\} \) are the set of poles of \( g \). \( \text{Res}[...] \) and \( \text{Disc}[...] \) indicate that the residues at the poles and the discontinuities along the real \( z \) cuts should be calculated. The \( \text{sgn} \) function is given by \( \text{sgn}(z) = +1 (-1) \) if \( \text{Re}(z) > (\!<)1 \).

The first term \( F_0 \) is the Euclidean zero temperature energy integral. It has no explicit temperature dependence, though \( g \) contains chemical potential terms coming from the propagators. This shows that the UV finite \( T > 0 \) contribution \( F_\beta \) can be pulled off. All the discussion in this paper centres around terms which are independent of temperature,
though not of $\mu$ or $m$. Thus one can just focus on the UV infinite, $T = 0$ terms $F_0$ where one merely replaces the Matsubara energy sums in the original expression by a straight Euclidean integral. The chemical potential remains encoded here in the quadratic operators and propagators, present even in $T = 0 F_0$ expressions.

One point of warning. A free Bose gas with non-zero charge density always has $|\mu| = m$ at $T = 0$ if $Q/V \neq 0$ since all particles must lie in the ground state in this case [16]. One must always remember that $\mu$ is an implicit function of temperature since it is the charge density $Q/V$ and not $\mu$ which is the physical observable. For a free Bose gas it may not always be possible to ensure $|\mu| < m$ for any given temperature and charge density.

### A.2 Dimensional Regularisation

In this appendix, I will summarize the most relevant aspects of the literature and fix my notation. References such as Collins [23] provide a much more detailed and careful derivation of the necessary mathematical theorems. However, I will also comment on how to apply this at finite temperature there is a crucial point which has direct relevance to the discussion of Multiplicative Anomalies.

In dimensional regulation, one adds extra components to the loop momenta vectors in ‘directions’ orthogonal to all other four-vectors. The number of these ‘extra’ dimensions is then taken to be $-2\epsilon$, close to, but not exactly zero. More precisely the calculations are performed at a value of $\epsilon$ where the integrals are well defined and then analytically continued to small finite $\epsilon$. Luckily there is no need to do these stages in detail as provided one follows some basic rules, dimensional regularisation can be implemented using some standard identities which guarantee mathematical consistency, at least for the simple field theories considered here.

Suppose one has a single $d$-dimensional Euclidean integration over a Euclidean loop momentum variable $k$ with no external four vectors in the problem. If working at zero temperature and if the integrand depends only on $K = k^2$, then dimensional regularisation is implemented as follows

$$
\int \frac{d^Dk}{(2\pi)^D} \xrightarrow{\text{Dim.Reg.}} \int \frac{d^d k}{(2\pi)^d} = \int \frac{d^D k}{(2\pi)^D} R_\epsilon k
$$

$$
= g(\epsilon)c(d)M^{2\epsilon} \int dK K^{(d-2)/2}, \quad K = k^2
$$

$$
d := D - 2\epsilon, \quad D \in \mathbb{Z}^+
$$

$$
[c(d)]^{-1} := (4\pi)^{d/2}\Gamma(d/2), \quad g(\epsilon) = 1
$$

where $M$ is the renormalisation scale. Crudely, dimensional regulation controls the UV by inserting a regulating function $R(\epsilon) = 1/\epsilon$ into the integrand. The UV divergences appear as $1/\epsilon$ terms. Note that IR divergences will still appear in the same terms, a drawback for dimensional reduction in general but not an issue in the free Bose gas model.

One useful identity in dimensional regularisation is

$$
\int d^p k' \int d^q k'' f(k'^2 + k''^2) = \int d^{p+q} k f(k^2).
$$

---

19I believe that one could easily make temperature dependent Multiplicative Anomalies given the explanation for Multiplicative Anomalies given elsewhere in this paper. All one needs to do is to make the UV regulating function temperature dependent.

20I am assuming that one has kept the chemical potential in the propagators and not put it in the boundary conditions which is an alternative approach [18, 19].
Demanding that this identity, and others like it, are satisfied, fixes the forms of \(c(d)\) and \(g(\epsilon)\) [23]. For this to hold it is important that \(c(d)\) has the form given in (A.7).

For more complicated integrals, e.g. with external loop momenta, consistency in dimensional reduction requires that one regulates only using components of the loop momenta orthogonal to any external four-vectors in the problem [23]. When working in a heat bath though, there is the velocity of the heat bath with respect to the observer, \(u^\mu\). I have chosen to work in the rest frame of the heat bath where \(u^\mu = (1, 0)\). Thus \(k_4 \equiv k.u\) and, by the rules of dimensional regularisation, \(k_4\) can not be included in the regulating function. So I have in four-dimensions in thermal field theory for integrals with no external momenta and only one loop variable

\[
\int \frac{d^4k}{(2\pi)^4} \xrightarrow{\text{Dim. Reg.}} \int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \rightarrow \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} = c(3 - 2\epsilon)M^{2\epsilon} \frac{1}{\beta} \sum_n \int dK K^{1/2-\epsilon},
\]

(A.9)

\[
k_4 := \frac{2\pi n}{\beta}, \quad n \in \mathbb{Z}
\]

(A.10)

and \(c(d)\) is given in (A.7). One can perform the energy sum in the usual manner assured that the result is finite even after the spatial integration provided one keeps \(\epsilon\) non-integer. One can then separate off the zero temperature and finite temperature parts as discussed in appendix A.1 and shown in (A.6). The former can then be returned to the usual zero-temperature \((4 - 2\epsilon)\)-dimensional form through the use of (A.8). The finite temperature part can have the regulator removed, \(\epsilon = 0\), since it is UV finite.

It is absolutely crucial that no regulation occurs in the energy variable \(k_4 = k.u\). So while the replacement

\[
\int \frac{d^4k}{(2\pi)^4} \rightarrow \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \left( \frac{k_4^2}{M^2} \right)^{-\epsilon} \int \frac{d^{3}k}{(2\pi)^3}, \quad (k_4 = 2\pi n/\beta)
\]

(A.11)

may also regulate the divergent integrals, the resulting expressions will not in general be mathematically consistent. In other words, this type of regulation may have Multiplicative Anomalies!

Finally, note that there is a close relationship between the family of different regularisation schemes and the families of renormalisation schemes. In dimensional reduction we can always change the definition of regularised integrals by multiplying by \(g(\epsilon)\), where \(g(z)\) is a function which is unit valued and analytic at \(z = 0\). However, if one was to implement MS renormalisation but with a \(g\) dependent regularisation, i.e. just subtract the leading \(1/\epsilon\) poles whatever \(g\) is, we would get extra finite terms proportional to \(g'(0)\). If \(g\) is a simple algebraic function, and has no additional dependence on physical variables, then these can be absorbed in constant redefinitions of the renormalisation scale \(M\). For instance, in dimensional reduction it is common to use \(\overline{\text{MS}}\) (Modified Minimal Subtraction scheme) which is equivalent to MS with a \(g(z) = 1 + z(\ln(4\pi) - \gamma) + O(z^2)\) and the relationship between these two schemes is simple. Alternatively, the effects of \(g(\epsilon)\) can be encoded through a conformal transformation on the dimensionless parameter \(\epsilon\).
A.3 ζ-function regularisation

I implement ζ-function regularisation scheme [4, 8, 25] as follows

\[
\text{Tr} \ln \{ A \} \xrightarrow{\text{ζ-func.reg.}} g \left( \frac{2s}{a} \right) \frac{a}{2s} \text{Tr} \left\{ \left( \frac{A}{M^a} \right)^{2s/a} \right\} = \frac{a}{s} \zeta \left( \frac{2s}{a} \right) \frac{A}{M} g \left( \frac{2s}{a} \right) \quad (A.12)
\]

\[
\text{Tr} \{ A \} \xrightarrow{\text{ζ-func.reg.}} g \left( \frac{2s}{a} \right) \text{Tr} \left\{ \left( \frac{A}{M^a} \right)^{1+2s/a} \right\} = \zeta \left( 1 + \frac{2s}{a} \frac{A}{M} \right) g \left( \frac{2s}{a} \right) \quad (A.13)
\]

where \( a \in \mathbb{Z}^+ \) is both the order and dimension of the operator \( A \), i.e. I assume that \( A \sim |k|^a \) in the UV limit. The function \( g \) is set to one in standard calculations and I will do that here, though see below for further comments about this. Roughly speaking, regulation is achieved by

\[
\int \frac{d^Dk}{(2\pi)^D} A \rightarrow \int \frac{d^Dk}{(2\pi)^D} R_\zeta A
\]

(A.14)

where \( R_\zeta = [A/M^a]^{-2s/a} \). Thus both dimensional reduction and ζ-function regularisation use non-integer powers, \( \epsilon \) or \( s \), of \( k \) to achieve their regulation. The key difference for this work is the fact that ζ-function regularisation’s regulating function involves the operator \( A \), and consequently varies with changes in any physical parameters in \( A \). The regulating function of dimensional reduction does not have this dependence. This ensures that dimensional regularisation satisfies certain basic identities [23], while ζ-function regularisation does not as appendix B notes. In fact this is the essence of the Multiplicative Anomalies in ζ-function regularisation, while a great deal of work has been done to ensure mathematical consistency of basic algebraic identities such as (A.8) in dimensional regularisation [23]. A few further points are worth making.

Firstly I have ensured that naively the expressions always have an overall factor of \( (M^2)^{-s} \) to mimic the \( (M^2)^{-s} \) factor in the dimensional regularisation expressions. This aids comparison of the results obtained using the two regularisation schemes. More importantly though, it means that I have made a very specific choice for the form of the regulating power, which is not simply a small parameter \( s \) but a specific multiple of it. This is to ensure that the poles will cancel when calculating different terms in Multiplicative Anomalies as shown in (A.15), i.e. I can use the same \( s \) parameter in all cases. In the same way, when comparing ζ-function regularisation and dimensional regularisation expressions, a simple equality \( s = \epsilon \) will be sufficient to match such expressions. This is to be contrasted with the renomalisation scale \( M \) which will often be rescaled when comparing different expressions.

Secondly, I choose to mimic dimensional regularisation in another way by ensuring that the UV divergence always appears as a \( s^{-1} \) pole with the physics contained in the \( O(s^0) \) term of a small \( s \) expansion. The generalised ζ-function \( \zeta(z|A) \) (for relevant operators \( A \)) is finite at \( z = 0 \) but in general has \( s^{-1} \) poles at \( z = n+s, n = 1, 2, \ldots \). Thus creating \( s^{-1} \) pole is only an issue for the regularisation of logarithm of operators where I have had to add an overall factor of \( \frac{s}{2s} \) in (A.12). In fact this is very natural as the following example will show.

Consider

\[
H = \ln(AB) - \ln(A) - \ln(B) = 0 \quad (A.15)
\]

where \( A \) and \( B \) could be ordinary numbers or more complicated objects. These logarithms can be represented as the second term in the following series

\[
[(A)^{s/a}] = 1 + \frac{s}{a} \ln(A) + \frac{s^2}{2a^2} (\ln(A))^2 + O(s^3) \quad (A.16)
\]
The parameter \( a \) has no special meaning here but one can add it in anticipation of its role in the field theory case where \( A \sim |k| \) as \( |k| \to \infty \). Therefore I can construct a representation of \( H \) through the replacement

\[
\ln(A) \to \frac{a}{s}[(A)^{s/a}]
\]

(A.17)

where \( s \) is small. Similarly for \( \ln(B) \) I use \( s/b \) rather than \( s/a \) where \( b \) is arbitrary in this simple context. I must then use

\[
\ln(AB) \to \frac{a+b}{s}[(AB)^{s/(a+b)}]
\]

(A.18)

to ensure that the \( 1/s \) infinities cancel and I indeed find that

\[
H \to H_s = -\frac{s}{2ab(a+b)} \left[ \ln \left( \frac{A^b}{B^a} \right) \right]^2 + O(s^2)
\]

(A.19)

which is zero as \( s \to 0 \). The first \( O(1/s) \) terms in the expression cancel because of the judicious choice of regularisation parameters for (A.17) and (A.18). The next \( O(s^0) \) terms are just the original simple logarithms which, if we have constructed a sensible representation of the original expression (A.15), must cancel. Thus the first non-zero terms are the \( O(s^1) \) terms but, provided all the terms are finite, this term will be zero in the limit \( s = 0 \) where we expect to recover the expression \( H \). However, if one is taking a trace over some variable upon which \( A \) and \( B \) depend, one can imagine that the trace may be divergent even if \( A \) and \( B \) are not. This may correspond to an additional \( 1/s \) pole being generated which can then combine with these \( O(s^1) \) terms to give a non-zero contribution in the \( s \to 0 \) limit, i.e. an anomalous \( H \neq 0 \) result. Note the close similarity between the form of the expression for \( H_s \) and the Wodzicki formula (A.24) for the logarithmic Multiplicative Anomaly anomaly.

One often hears that one of the advantages of \( \zeta \)-function regularisation is that it is finite. However all that is meant is that \( \zeta(z|A) \) is finite near \( z = 0 \) so one may define the \( \text{Tr} \ln A = \ln \det A = \zeta'(z = 0|A) \). In the context, of QFT this is not especially amazing. The physics is still in the second term of some expansion and one is merely giving a prescription for extracting this term. In my implementation of \( \zeta \)-function regularisation, I have merely multiplied by a factor of \( 1/s \) when regulating \( \text{Tr} \ln \text{expression} \) in order to keep other \( \zeta \)-function regularisation expressions and indeed dimensional reduction expressions on a similar footing. Thus the usual definition of \( \zeta \)-function regularisation, as applied to vacuum energy densities, corresponds to dropping what are the leading \( 1/s \) poles in my expressions, i.e. traditional \( \zeta \)-function regularisation is also a ‘minimal subtraction’ renormalisation scheme. Ignoring the first term in the small \( s \) expansion is merely a particular renormalisation scheme, not just a simple regularisation\(^{21} \). Put another way one can mimic the usual implementation of \( \zeta \)-function regularisation for vacuum energy densities in dimensional regularisation. To do this the usual dimensional regularisation expression is multiplied by \( \epsilon \), which makes the expression finite in the same way that \( \zeta(0|A) \) is. One would then define vacuum energies \( \text{Tr} \ln A = \ln \det A \) to be the \( \epsilon \) derivative of that combination, i.e. I take just the second term in a small epsilon expansion.

\(^{21}\) Regularisation is process where infinities are turned into large but finite terms. Renormalisation is a process which expresses bare parameters in terms of finite values which can be related to physical measurements. The latter must always be performed in QFT, even if the theory is finite, as quantum fluctuations are always present. It will though remove any infinities present in expressions for physical quantities.
and drop the first term. This is of course just the Minimal Subtraction renormalisation scheme.

Finally, one may always multiply the $\zeta$-function regularisation expressions, (A.12) and (A.13), by any function $g(2s/a)$ provided this function $g(z)$ is analytic and of value one at $z = 0$. Its introduction merely moves us through the family Schwinger proper-time regularisations [8], changing the finite terms by an amount proportional to $g'(0)$. The $g(0) = 1$ condition can be imposed without loss of generality as any other value is equivalent to a rescaling of the small regulating $s$ parameter. This freedom is used to ensure that poles cancel in Multiplicative Anomalies, as discussed for equation (A.17) and (A.18). One example of interest is $\bar{g} = g(2s/a) = s\Gamma(s)$ for $\text{Tr} \ln$ problems so that from (A.12)

$$\text{Tr} \ln A \, \zeta^{-f \text{reg.}} \, \bar{g} \left( \frac{2s}{a} \frac{A}{M} \right) \Gamma(s),$$

(A.20)

I will refer to this as SPTDR (Schwinger Proper Time Dimensional Regularisation). This member of the family of Schwinger proper time regularisation schemes is sometimes referred to as plain ‘dimensional regularisation’ in some of the literature on $\zeta$-function methods, as in [11]. However SPTDR is not the dimensional regularisation of the particle physics literature such as [17, 23] and as described in appendix A.2. The review of Ball [8] makes this distinction clear and simple examples confirm this view.

### A.4 The Wodzicki residue.

In the theory of elliptic pseudo-differential operators, there is a unique extension of the Dixmier trace to elliptic pseudo-differential operators and Wodzicki gave the explicit form [5, 26]. This beautiful result of modern mathematics is central to several areas, such as Non-Commutative Geometry (e.g. see [6]). However, there is a simple formula for calculating it and a simple relation between the Multiplicative Anomaly of (1.1) and the Wodzicki residue.

The simple recipe to find the Wodzicki residue is as follows [5, 26, 1]. The “complete symbol” for our operators is $A(x, k) := e^{-ikx} A e^{ikx}$. Extract the UV behaviour as an asymptotic expansion

$$A(x, tk) = \sum_{j=a}^{-\infty} t^j A_j(x, k)$$

(A.21)

where $a$ is the order of the operator. The Wodzicki residue $W$ of an operator $A$ is then

$$\text{res}_W(A) = \int_M d^Dx \int q^D k M^2 \delta(k^2 - M^2) A \, -D(x, k)$$

(A.22)

Note that in many places (all the ones I found) the regularisation scale $M$ is set to one, that is the calculations are done with all dimensionful parameters being measured in units of $M$. While perfectly acceptable mathematically, this does mean that the renomalisation scale is hidden even though it is extremely important when extracting real physical numbers from the mathematical results [17].

There is a simple link to the Cauchy residues of $\zeta$-functions through

$$\text{Res}_{s=\sigma} [\zeta(s) A] = a^{-1} \text{res}_W(A^{-\sigma})$$

(A.23)

Footnote 22: In [5] see sec. 7.13, pp.176. In [1] see section 6.5, pp.225-226, especially his eqn.(9). Also note the comments in section 1.2, eqn.(4) which cites Wodzicki’s 1984 thesis for a residue formula. See also Wodzicki [26].
and finally to the Multiplicative Anomaly through

\[ a_\zeta(A, B) = (2ab(a + b))^{-1}\text{res}_W \left[ \{\ln(A^b B^{-a})\}^2 \right] \]  

(A.24)

Note the strange expressions in (A.24) are easily obtained in a simple analysis of logarithms of numbers which gives some insight into this expression, see (A.19).

A.4.1 Conjecture for a generalised Multiplicative Anomaly formula

The Wodzicki residue formula (A.24) is for Tr ln expressions (1.1), relevant to zero point energy calculations in general and to all the four-dimensional examples considered here. However simple traces over other operators can give rise to Multiplicative Anomalies, as the examples (B.2), (B.5) and (B.6) show. One can try to write these examples in terms of \( \text{tr}\{z\} = \text{tr ln}\{A\} \), with \( A = \exp\{z\} \), and then try to use the Wodzicki residue formula for the Multiplicative Anomaly. Unfortunately, the UV behaviour of the resulting operators \( A \) is then exponentially divergent or suppressed and the formula does not work. However, since direct calculations in \( \zeta \)-function regularisation in these cases does show the existence of an Multiplicative Anomaly, one wonders if there is not a formula similar to the one based on the Wodzicki residue. I have made the following conjecture. Suppose a Multiplicative Anomaly is given by an unregulated expression \( F \)

\[ F = \sum_j \int d^D k \ F_j(k), \]  

(A.25)

where

\[ \sum_j F_j(k) = 0 \ \forall |k| < \infty \]  

(A.26)

The operators \( F_j \) in each term must have the same dimension which, without loss of generality, I will take to be the same as the order \( a \), that is \( F \sim |k|^a \) as \( k \rightarrow \infty \). Let me now write down the \( \zeta \)-function regularisation form

\[ F_\zeta = \sum_j \int d^D k \ [F_j(k)]^{1+s/a} M^{-s} \]  

(A.27)

where for definiteness I choose to regulate the expression so that the UV behaviour is \( |k|^{a+s} \) and the renormalisation scale always appears as the simple \( M^{-s} \) factor. Now suppose I make a double expansion of \( F^{1+s/a} \) in \( s \) and \( 1/k \) and define

\[ [F_j(t\bar{k})]^{1+s/a} = |k|^{a+s} \sum_{m=0}^\infty \sum_{n=0}^\infty s^m t^{-n} \frac{1}{m!n!} \tilde{f}_{j,m,n}(\bar{k}) \]  

(A.28)

The interest is in the UV divergences so it makes sense to separate off the integral over the \( |k| \) through

\[ 1 = \int_0^\infty dt \ 2t \delta(t^2 - k^2/M^2). \]  

(A.29)

By looking at the formula for the Tr ln examples and trying a few examples, I have made the following conjecture

\[ F_\zeta \propto \sum_j 2 \int d^D \bar{k} \ \delta(1 - \bar{k}^2/M^2) M^a \frac{1}{(D + a)!} \tilde{f}_{j,2,D+a}(\bar{k}). \]  

(A.30)
i.e. only the $O(s^2)$ and $|k|^{-D-a}$ term of the $f_j^{1+s/a}$ expansion contributes. The reason for trying the $n = D + a$ term comes from the original Multiplicative Anomaly formula using Wodzicki residue. There the term which appears is the one in the asymptotic expansion of the operator which is right on the boundary between divergence and convergence, i.e. the logarithmic divergent term which in this case is the $n = D + a$ term.

The reason for trying $m = 2$ term in the conjectured form (A.30) is that it is quickly apparent that the $m = 0$ and $m = 1$ terms are zero. That this must be so follows from the two constraints on Multiplicative Anomaly expressions. Firstly it should not itself be divergent so $1/s$ terms should cancel when doing the sum over $j$. In general one must choose the $s$ parameter used to regulate each term in the $j$ sum in just the right way to achieve this, and the definition used here ensures this, as noted in appendix A.3. Secondly, the Multiplicative Anomaly expressions are naively zero in the first place (A.26), and this is linked to the fact that the sum over $j$ of the $O(s^0)$ terms must be zero. Thus it can only be the third term in the $s$ series of $f$ that is relevant, presumably because in a proper calculation it is mixing with divergent terms. See (A.15) for a simple example of this.

**B A collection of Multiplicative Anomalies**

In [7] it was suggested that Multiplicative Anomalies appear only if three criteria hold, namely

1. the terms in an Multiplicative Anomaly must contain infinities,

2. the regularisation scheme used must mix physical parameters (mass, chemical potential, etc.) in the artificial function used to cutoff UV momenta, and

3. the different terms in the Multiplicative Anomaly must contain different combinations of these physical parameters.

Thus regularisation schemes such as dimensional regularisation, physical cutoffs (e.g. lattice) and ones typically used in particle physics never have Multiplicative Anomalies, as the examples in [7] show.

However, $\zeta$-function regularisation and all other members of the family of Schwinger proper time regularisations [8] are plagued by Multiplicative Anomalies. In general with these regularisation schemes regulated expressions will fail to obey many algebraic identities naively satisfied by their unregulated counterparts. Is there any identity which $\zeta$-function regularisation respects?

The criteria above are merely necessary, not sufficient. For instance the simplest example of an Multiplicative Anomaly in QFT found in [2] is non-zero only in even dimensions from four upwards. However, numerous examples of non-zero Multiplicative Anomalies are now known. As some are relevant for the discussion of the free Bose gas, I will give a brief list. Here $A, B$ are suitable operators, $\alpha$ is a c-number.

The original example

$$\text{Tr \ln}(AB) \neq \text{Tr \ln}(A) + \text{Tr \ln}(B)$$

(B.1)

was given by Elizalde, Vanzo and Zerbini for several simple cases in field theory, including two real scalar fields of different masses in flat Euclidean space-time [2]. Physically it is important when considering the vacuum energy of two free scalar fields as in cosmology.
or in the Casimir effect. It also appears in the free Bose gas case, in particular the four-dimensional expressions (2.13), (2.19) and (2.22), which were first given in [9]. The problem of Multiplicative Anomalies appears with Green functions as well as with vacuum energy densities considered in [2]. Consider

\[ \mathrm{Tr}\left\{ \frac{1}{A(A + \alpha)} \right\} \neq \left( \mathrm{Tr}\left\{ \frac{1}{A} \right\} - \mathrm{Tr}\left\{ \frac{1}{A + \alpha} \right\} \right) \frac{1}{\alpha} \quad (B.2) \]

It is formally related to the first Multiplicative Anomaly, as defined in (1.1), by the replacement in (1.1) \( A \rightarrow \exp\{1/A\} \) and \( B \rightarrow \exp\{1/(A+\alpha)\} \). Examples of this failing with \( \zeta \)-function regularisation were shown in [7] for two real scalar fields with \( A = \Delta^{-1}, \alpha = \delta m^2 \). The Multiplicative Anomaly is then related to expectation values of the squares of the two real fields, \( \langle (\phi_1^2 - \phi_2^2) \rangle \neq \langle \phi_1^2 \rangle - \langle \phi_2^2 \rangle \).

Mass shifts are perhaps the most simple interaction possible and often appear in QFT. They are also afflicted by an Multiplicative Anomaly in \( \zeta \)-function regularisation, for instance

\[ \mathrm{Tr}\{\ln(A + \alpha)\} \neq \mathrm{Tr}\{\ln(A)\} - \sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n} \mathrm{Tr}\{A^{-n}\}. \quad (B.3) \]

An example of this Multiplicative Anomaly was given in [7] for a single real scalar field, where one looks at a mass shift of the scalar field, \( A = \Delta^{-1}, \alpha = \delta m^2 \).

A generalisation of the Multiplicative Anomaly in (B.1) is

\[ \mathrm{Tr}\{\ln(\prod_n A_n)\} \neq \sum_n \mathrm{Tr}\{\ln(A_n)\}. \quad (B.4) \]

This version is relevant in the context of dimensional reduction [14, 15, 11].

Simple shifts by c-numbers lead to Multiplicative Anomalies

\[ \mathrm{Tr}\{A + \alpha\} + \mathrm{Tr}\{A - \alpha\} \neq \mathrm{Tr}\{2A\}, \quad (B.5) \]

showing that the linear property of the trace is not preserved in \( \zeta \)-function regularisation pace [20]. The first example of a Multiplicative Anomaly of this type was first noted by McKenzie-Smith and Toms [12, 13] in the context of the free Bose gas using the three-dimensional expressions of (2.37) and (2.39). Again if in (1.1) I make the replacements \( A \rightarrow \exp\{-A - \alpha\} \) and \( B \rightarrow \exp\{-A + \alpha\} \) this links this Multiplicative Anomaly to the original one.

Finally, if one needed any further convincing about the existence of Multiplicative Anomalies, one need look no further than simple c-number multiplication of operators as in \( \zeta \)-function regularisation

\[ \mathrm{Tr}\{\alpha A\} \neq \alpha \mathrm{Tr}\{A\}. \quad (B.6) \]

It is needed as well as (B.5) when relating \( F_3 \) and \( F_{3\mu} \). It is easy to see that if \( \mathrm{Tr}\{A\} \) is regularised using \( \zeta \)-function regularisation and becomes \( \mathrm{Tr}\{A^{1+s}\} = a_{-1}s^{-1} + a_0 + O(s) \), then there is a Multiplicative Anomaly of

\[ a_{\text{rescale}}(A, \alpha) := \mathrm{Tr}\{\alpha A\} - \alpha \mathrm{Tr}\{A\} = \alpha \ln(\alpha)a_{-1}. \quad (B.7) \]

This is non-zero whenever \( \mathrm{Tr}\{A\} \) is infinite, just as required by the criteria in [7]. This is no more than a changing of the regularisation scale which always appears in QFT e.g. let \( A = \Delta \) the propagator for a real scalar field, one will have to remember that I need

\[ \text{The analysis in [9] confirms that of [7] contrary to what is stated in [20]} \]
to introduce a new scale, say $M$, to keep the dimensions of the regularised expression correct, e.g. as in (3.23) and (3.9). Note that this tells us that ‘trivial’ operations such as pulling out the factor of two on the right hand side of (B.5) or pulling a minus sign outside the trace are in fact highly non trivial if Multiplicative Anomalies are non-zero. Yet such operations are performed frequently in formal derivations, and in particular they abound in our canonical derivations of the free energy of the free Bose gas in the last section. Thus the Multiplicative Anomaly, $a_{\text{shift}}$ of (2.40), appearing used in the three-dimensional forms for the free energy encodes a failure in a combination of (B.5) and in (B.6).

The point of giving this summary has been to show how many different algebraic operations fail when using $\zeta$-function regularisation, even in the simplest of QFT problems involving free fields in flat Euclidean space times. One can make these algebraic operations but only if one remembers to include the Multiplicative Anomaly terms as well. It is extremely easy to neglect these Multiplicative Anomalies but this will lead to mathematical inconsistencies. In this sense Multiplicative Anomalies are of vital importance to problems using $\zeta$-function regularisation.

Several of these Multiplicative Anomalies are relevant to the analysis of the free energy of the free Bose gas. In particular, the regularisation scale, here called $M$, is often shifted in the analysis above. The simple example (B.7) can be interpreted in this way. This all gives further weight to the idea in [7] that one can take account of all Multiplicative Anomalies by regarding them as shifts in renormalisation scales.

### C Useful formulae

\[
(1 + x)^\epsilon = \sum_{n=0}^{\infty} \frac{\Gamma(1 + \epsilon)}{\Gamma(1 + n)\Gamma(1 + \epsilon - n)} x^n
\]  

\[
\int_0^\infty dx \frac{x^{\mu-1}}{(1 + \beta x^p)^\nu} = \frac{\beta^{-\mu/p}}{p} \frac{\Gamma(\mu/p)\Gamma(\nu - \mu/p)}{\Gamma(\nu)}
\]

\[
\Gamma(n + \epsilon) = \Gamma(n) (1 + \epsilon \psi(n)), \quad n \in \mathbb{Z}^+
\]

\[
\Gamma(-n + \epsilon) = \frac{1}{\epsilon} \frac{(-1)^n}{\Gamma(1 + n)} (1 + \epsilon \psi(1 + n)), \quad n \in \{0, \mathbb{Z}^+\}
\]

\[
\Gamma(1/2) = \sqrt{\pi}
\]

\[
\Gamma(3/2) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}
\]

\[
\Gamma(-1/2) = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}
\]

\[
\psi(z) := \frac{d \ln(\Gamma(z))}{dz}
\]

\[
\psi(1 + x) = \psi(x) + \frac{1}{x}
\]

\[
\psi(1) = -\gamma \approx -0.5772156649, \quad \gamma \text{ is Euler’s constant},
\]

\[
\psi(1/2) = \psi(1) - 2 \ln(2)
\]
\[ \int d^D K = \frac{c(D - 1)}{2\pi} \int_0^\infty dK K^{D-1} \int_0^\pi d\theta (\sin(\theta))^{D-2} \] (C.12)
\[ = c(D) \int_0^\infty dK K^{D-1}, \] (C.13)
\[ K^i = (\vec{k}, k_d)^2, \ k_d = K \cos(\theta), \ |\vec{k}| = K \sin(\theta) \] (C.14)
\[ c(d) = \left( \frac{(4\pi)^{d/2} \Gamma(d/2)}{d} \right)^{-1} \] (C.15)