Sufficient conditions on cycles that make planar graphs 4-choosable

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Abstract

Xu and Wu proved that if every 5-cycle of a planar graph $G$ is not simultaneously adjacent to 3-cycles and 4-cycles, then $G$ is 4-choosable. In this paper, we improve this result as follows. Let $\{i, j, k, l\} = \{3, 4, 5, 6\}$. For any chosen $i$, if every $i$-cycle of a planar graph $G$ is not simultaneously adjacent to $j$-cycles, $k$-cycles, and $l$-cycles, then $G$ is 4-choosable.

1 Introduction

Every graph in this paper is finite, simple, and undirected graph. The concept of choosability was introduced by Vizing in 1976 [12] and Erdős, Rubin, and Taylor in 1979 [5], independently. A $k$-list assignment $L$ of a graph $G$ assigns a list $L(v)$ (a set of colors) and $|L(v)| = k$ to each vertex $v$. A graph $G$ is $L$-colorable if there is a proper coloring $f$ where $f(v) \in L(v)$. If $G$ is $L$-colorable for any $k$-assignment $L$, then we say $G$ is $k$-choosable.

It is known that every planar graphs is 4-colorable [1] [2]. Thomassen [11] proved that every planar graph is 5-choosable. In contrast, Voight [13] presented an example of non 4-choosable planar graph. Additionally, Gutner [8] showed that determining whether a given

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planar graph 4-choosable is NP-hard. Since every planar graph without 3-cycle always has a vertex of degree at most 3, it is 4-choosable. More conditions for a planar graph to be 4-choosable are investigated. It is shown that a planar graph is 4-choosable if it has no 4-cycles \[10\], 5-cycles \[15\], 6-cycles \[7\], 7-cycles \[6\], intersecting 3-cycles \[10\], intersecting 5-cycles \[9\], or 3-cycles adjacent to 4-cycles \[3, 4\]. Xu and Wu \[14\] proved that if every 5-cycle of a planar graph \(G\) is not simultaneously adjacent to 3-cycles and 4-cycles, then \(G\) is 4-choosable. In this paper, we improve this result as follows.

**Theorem 1.** Let \(\{i, j, k, l\} = \{3, 4, 5, 6\}\). For any chosen \(i\), if every \(i\)-cycle of a planar graph \(G\) is not simultaneously adjacent to \(j\)-cycles, \(k\)-cycles, and \(l\)-cycles, then \(G\) is 4-choosable.

### 2 Structure

First, we introduce some notations and definitions. A \(k\)-vertex (face) is a vertex (face) of degree \(k\), a \(k^+\)-vertex (face) is a vertex (face) of degree at least \(k\), and a \(k^-\)-vertex (face) is a vertex (face) of degree at most \(k\). A \((d_1, d_2, \ldots, d_k)\)-face \(f\) is a face of degree \(k\) where all vertices on \(f\) have degree \(d_1, d_2, \ldots, d_k\). A \((d_1, d_2, \ldots, d_k)\)-vertex \(v\) is a vertex of degree \(k\) where all faces incident to \(v\) have degree \(d_1, d_2, \ldots, d_k\). A wheel graph \(W_n\) is an \(n\)-vertex graph formed by connecting a single vertex (hub) to all vertices (external vertices) of an \((n - 1)\)-cycle.

Some basic properties are collected in the following proposition.

**Proposition 1.** Let \(\{i, j, k, l\} = \{3, 4, 5, 6\}\) and \(G\) be a planar graph such that every \(i\)-cycle is not simultaneously adjacent to \(j\)-cycles, \(k\)-cycles, and \(l\)-cycles. Embed \(G\) into the plane, then

1. \(G\) does not contain a 4-cycle with one chord that shares exactly one edge in a cycle with a 4-cycle or a 5-cycle.
2. \(G\) does not contain a 4-face that shares exactly one edge simultaneously with a 3-cycle and a 4\(^-\)-cycle.
3. \(G\) does not contain 5-cycle with one chord that shares exactly one edge with two 5-cycles.
4. \(G\) does not contain \(W_5\) that shares exactly one edge with a 6\(^-\)-cycle.

A \((3, 3, 5, 5^+)\)-vertex is called a flaw 4-vertex. A 5-face is called a poor 5-face if it is adjacent to at least four 3-faces, and incident with either five 4-vertices or four 4-vertices and one 5-vertex. Let \(P\) be a face and \(T\) a 3-face in \(G\). If \(P\) and \(T\) share exactly one edge. The
vertex $v$ on $T$ but not on $P$ is called a source.

In this part, we consider a minimal non 4-choosable planar graph $G$ embedded into the plane.

**Lemma 2.** [11] Every vertex has degree at least 4.

**Lemma 3.** [3] Every source of poor 5-face is a $5^+$-vertex.

**Lemma 4.** [3] Let $f$ and $g$ be two faces in $G$. If $f$ shares exactly one edge $x_ix_j$ with $g$ where $d(x_i) \leq 5$, then at least one vertex in $V(f) \cup V(g) - \{x_i\}$ is a $5^+$-vertex.

**Theorem 2.** Let $v$ be a 5-vertex in $G$. Then at least three incident faces of $v$ are incident to at least two $5^+$-vertices.

**Proof.** Suppose that $v$ has at most two incident faces that are incident to at least two $5^+$-vertices. Then there are two incident faces of $v$, $f_1$ and $f_2$ where $f_1$ and $f_2$ are adjacent and are $(4, 4, \ldots, 4, 5)$-faces, a contradiction to Lemma 4. \hfill \Box

**Theorem 3.** Let $v$ be a 6-vertex in $G$. Then at least two incident faces of $v$ are incident at least two $5^+$-vertices.

**Proof.** Let $v$ be incident to six faces $f_1, f_2, \ldots, f_6$. Suppose that at most one $f_i$ is a $(4^+, 4^+, \ldots, 5^+, 6)$-face. By minimality of $G$ and for any subgraph $Z$, the graph $G - V(Z)$ has an $L'$-coloring. If we show that $Z$ has an $L''$-coloring for any $|L''(x_i)| = 4 - |N(x_i) - Z|$, then we obtain $G$ is 4-choosable, a contradiction. Let each $f_i$ be a $(4, 4, \ldots, 4, 6)$-face and $Z$ be a graph $f_1 \cup f_2 \cup \cdots \cup f_6$. Then $L''(v) = 4$, $L''(x) = 3$ if $x$ is adjacent to $v$, and $L''(x) = 2$ for otherwise. We choose one adjacent vertex of $v$ given $w$. There is a color $a$ in $L''(v) - L''(w)$ and we color $v$ with a color $a$. Moreover, $|L''(w) - \{a\}| = 3$ and $|L''(x) - \{a\}| \geq 2$ for each $x$. Thus $Z$ has an $L''$-coloring since each cycle is 2-choosable if there are two lists that are not equal, a contradiction. Let $v$ be incident exactly one $(4^+, 4^+, \ldots, 5^+, 6)$-face given $f_6$, $Z$ be a graph $f_1 \cup f_2 \cup \cdots \cup f_5$, and $x_1$ and $x_2$ be two vertices that adjacent to $v$ and incident to $f_6$. Then $|L''(x)| = 2$ for each $x$. Moreover, $L''(v) = 4$ and $L''(x) = 3$ if $x$ is adjacent to $v$ except to $x_1$ and $x_2$. There is a color $a \in L''(v) - (L''(x_1) \cap L''(x_2))$, WLOG, we let $a \in L''(x_1)$. We color $v$ with a color $a$. Moreover, $|L''(x_1) - \{a\}| = 1$ and $|L''(x) - \{a\}| \geq 2$ for the others vertices. Thus It is easy that $Z$ has an $L''$-coloring, a contradiction. \hfill \Box

**Theorem 4.** If each vertex of $W_5$ in $G$ is a $5^+$-vertex, then $W_5$ has at least three 5-vertices.
Proof. Let $V(W_5) = x_1, x_2, x_3, x_4, x_5$ where $x_5$ be a hub and $d(x_i) \leq 5$ for each $i$. WLOG, we suppose that $d(x_1), d(x_2) = 5$ and the others vertices are 4-vertices. If $x_1$ is not adjacent to $x_2$, then it's a contradiction by Lemma 4. Let $L$ be a 4-list assignment of $G$. By minimality of $G$, the graph $G - V(W_5)$ has an $L'$-coloring. If we show that $W_5$ has an $L''$-coloring for any $|L''(x_i)| = 4 - |N(x_i) - W_5|$, then $G$ is 4-choosable, a contradiction. Then, we have $L''(x_1) = 2$, $L''(x_2) = 2$, $L''(x_3) = 3$, $L''(x_4) = 3$, and $L''(x_5) = 4$. If there is a color $a$ from $L''(x_5)$ such that $a \notin L''(x_1) \cup L''(x_2)$, then $|L''(x_i) - \{a\}| \geq 2$ for each $i \neq 5$. It's complete since a 4-cycle is 2-choosable. Otherwise, we have $L''(x_1) \cap L''(x_2) = \emptyset$. WLOG, we let $L''(x_1) = \{a, b\}$. First, we choose $a$ from $L''(x_5)$. Thus there is an $L''$-coloring $f$ where $f(x_5) = a$, $f(x_1) = b$, $f(x_3) \in L''(x_3) - \{a, b\}$, $f(x_4) \in L''(x_4) - \{a, f(x_3)\}$, and $f(x_2) \in L''(x_2) - \{f(x_4)\}$, a contradiction. Thus $W_5$ has at least three 5-vertices. \[\Box\]

3 Proof of Theorem 1

Embed a minimal counterexample graph $G$ into the plane. Let the initial charge of a vertex $u$ in $G$ be $\mu(u) = 2d(u) - 6$ and the initial charge of a face $f$ in $G$ be $\mu(f) = d(f) - 6$. Then by Euler’s formula $|V(G)| - |E(G)| + |F(G)| = 2$ and by the Handshaking lemma, we have

$$\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.$$  

Now we design the discharging rule transferring charge from one element to another to provide a new charge $\mu^*(x)$ for all $x \in V(G) \cup F(G)$. The total of new charges remains $-12$. If the final charge $\mu^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and the proof is completed.

Before we establish a discharging rule, some definitions are required. A cluster of three 3-faces isomorphic to a graph consist of a vertex set of five elements, namely $\{u, v, w, x, y\}$, and an edge set $\{xy, xu, xv, yv, yw, uw, uv, vw\}$ is called a trio. A vertex that is not in any trio is called a good vertex. We call a vertex $s$ on a face $f$ in a trio a bad vertex of $f$ if $f$ is the only 3-cycle containing $s$ on that trio, a worst vertex of $f$ if $s$ is a vertex of all three 3-cycles in a trio, otherwise $s$ is called a worse vertex of $f$. We call a face $f$ is a bad (worse, or worst, respectively) face of a vertex $v$ if $v$ is a bad (worse, or worst, respectively) vertex of $f$. Note that each external vertex of $W_5$ formed by four 3-faces is a worse vertex of some trio.

Let $w(v \rightarrow f)$ be the charge transferred from a vertex $v$ to an incident face $f$. The dis-
charging rules are as follows.

(R1) Let $f$ be a 3-face that is not adjacent to the others 3-faces.

(R1.1) For a 4-vertex $v$,

$$w(v \rightarrow f) = \begin{cases} 
0.6, & \text{if } v \text{ is flaw where } f \text{ is a } (4, 5^+, 5^+)-\text{face}, \\
0.8, & \text{if } v \text{ is flaw where } f \text{ is a } (4, 4, 5^+)-\text{face}, \\
1, & \text{otherwise.} 
\end{cases}$$

(R1.2) For a $5^+$-vertex $v$,

$$w(v \rightarrow f) = \begin{cases} 
1.4, & \text{if } f \text{ is a } (4, 4, 5^+)-\text{face where each 4-vertex is flaw}, \\
1.2, & \text{if } f \text{ is a } (4, 4^+, 5^+)-\text{face where exactly one 4-vertex is flaw}, \\
1, & \text{otherwise.} 
\end{cases}$$

(R2) Let $f$ be a 3-face that is adjacent to the others 3-faces.

(R2.1) For a 4-vertex $v$,

$$w(v \rightarrow f) = \begin{cases} 
0.5, & \text{if } v \text{ is a hub of } W_5, \\
1, & \text{if } f \text{ is a good, bad, or worse face of } v, \\
2/3, & \text{if } f \text{ is a worst face of } v. 
\end{cases}$$

(R2.2) For a 5-vertex $v$,

$$w(v \rightarrow f) = \begin{cases} 
1, & \text{if } f \text{ is a good or worst face of } v, \\
1.5, & \text{if } f \text{ is a bad face of } v, \\
1.25, & \text{if } f \text{ is a worse face of } v. 
\end{cases}$$

(R2.3) For a $6^+$-vertex $v$,

$$w(v \rightarrow f) = \begin{cases} 
1, & \text{if } f \text{ is a good or worst face of } v, \\
1.5, & \text{if } f \text{ is a bad or worse face of } v. 
\end{cases}$$

(R3) Let $f$ be a 4-face.

(R3.1) For a 4-vertex $v$, $w(v \rightarrow f) = 1/3$.

(R3.2) For a $5^+$-vertex $v$,

$$w(v \rightarrow f) = \begin{cases} 
1, & \text{if } f \text{ is a } (4, 4, 4, 5^+)-\text{face}, \\
2/3, & \text{otherwise.} 
\end{cases}$$

(R4) Let $f$ be a 5-face.

(R4.1) For a 4-vertex $v$,
SUBCASE 1.1: Let $w$ be adjacent to exactly one 5-face, the we obtain $w(v - f) = 1/3$, if $v$ is a flaw 4-vertex.

(R4.2) For a 5-vertex $v$,

$$w(v - f) = \begin{cases} 
0.4, & \text{if } f \text{ is a (4, 4, 4, 5, 5)} \text{-face where both incident 5-vertices are adjacent}, \\
1/3, & \text{if } f \text{ is a (4, 4, 4, 5)} \text{-face}, \\
0.3, & \text{otherwise}.
\end{cases}$$

(R4.3) For a 6-vertex $v$,

$$w(v - f) = \begin{cases} 
0.8, & \text{if } f \text{ is a (4, 4, 4, 4, 6)} \text{-face}, \\
0.4, & \text{otherwise}.
\end{cases}$$

(R4.4) For a 7$^+$-vertex $v$, $f w(v - f) = 0.8$.

(R5) Let $f$ be a 7$^+$-face and $g$ be a 3-face. If $g$ and other three 3-faces form $W_5$ and $f$ shares exactly one edge with $g$, let $w(f - g) = 1/8$.

(R6) After (R1) to (R6), redistribute the total of charges of 3-faces in the same cluster of adjacent 3-faces (trio or $W_5$) equally among its 3-faces.

It remains to show that resulting $\mu^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

CASE 1: Consider a 4-vertex $v$.

We use (R1.1), (R2.1), (R3.1), and (4.1) to prove this case.

SUBCASE 1.1: Let $v$ be flaw 4-vertex.

Then $v$ is a (3, 3, 5, 5$^+$)-vertex. If each adjacent vertex of $v$ is a 4-vertex, then we obtain $\mu^*(v) = \mu(v) - (2 \cdot 1) = 0$. If $v$ is adjacent to exactly one 5$^+$-vertex, then we obtain $\mu^*(v) = \mu(v) - (1 + 0.8 + 2 \cdot (0.1)) \geq 0$. If $v$ is adjacent to at least two 5$^+$-vertices, the we obtain $\mu^*(v) = \mu(v) - (2 \cdot (0.8) + 2 \cdot (0.2)) \geq 0$ for two 5$^+$-vertices are not in the same 3-face. Otherwise, we obtain $\mu^*(v) = \mu(v) - (1 + 0.6 + 2 \cdot (0.2)) \geq 0$.

SUBCASE 1.2: Let $v$ be not flaw 4-vertex.

If $v$ is incident at most one 3-face, then we obtain $\mu^*(v) \geq \mu(v) - (1 + 3 \cdot (1/3)) \geq 0$. If $v$ is incident to two 3-faces, then $v$ is a (3, 3, 6$^+$, 6$^+$)-vertex. Thus we obtain $\mu^*(v) = \mu(v) - (2 \cdot 1) = 0$. If $v$ is incident to three 3-faces, then $v$ is a worst vertex of these faces and its remaining incident face is a 6$^+$-face. Thus we obtain $\mu^*(v) = \mu(v) - (3 \cdot (2/3)) = 0$. If $v$ is incident to four 3-faces, then $v$ is a hub of $W_5$. Thus $\mu^*(v) = \mu(v) - 4 \cdot (1/2) = 0$.

CASE 2: Consider a 5-vertex $v$.
**SUBCASE 2.1**: Let \( v \) be incident to an adjacent triangle or a bad face.

Then \( v \) is incident to at least two 6\(^+\)-faces by Proposition \( \text{II}(1) \). Additionally, a 5-vertex \( v \) has at most two bad faces. We use (R2.2), (R3.2), and (R4.2) to prove the following cases.

We obtain \( \mu^*(v) = \mu(v) - (3 \cdot (1.25)) > 0 \) if there is no any bad face, \( \mu^*(v) = \mu(v) - (1.5 + 2 \cdot (1.25)) \geq 0 \) if there is one bad face, and \( \mu^*(v) = \mu(v) - 2 \cdot (1.5) > 0 \) if there are two bad faces.

**SUBCASE 2.2**: A vertex \( v \) has neither adjacent triangles nor bad faces.

Then \( v \) is incident to at most two 3-faces. We use (R1.2), (R3.2), and (R4.2) to prove the following cases.

Let \( v \) be incident to at least one 6\(^+\)-face. If \( v \) is not incident to any 3-face, then \( \mu^*(v) = \mu(v) - (4 \cdot (1)) \geq 0 \). If \( v \) is incident exactly one 3-face, then \( \mu^*(v) = \mu(v) - (1.4 + 2 \cdot (1) + 0.4) > 0 \). If \( v \) is incident to two 3-faces, then \( v \) is incident to at most one 4-face by Proposition \( \text{II}(2) \). Then we obtain \( \mu^*(v) = \mu(v) - (1.4 + 1.2 + 1 + 0.4) > 0 \).

Next, a 5-vertex \( v \) is not incident to any 6\(^+\)-face.

Let \( v \) be not incident to any 3-face. If \( v \) is a \((4^+, 4^+, 4^+, 5, 5)\)-vertex, then \( \mu^*(v) = \mu(v) - (3 \cdot (1) + 2 \cdot (0.4)) > 0 \). If \( v \) is a \((4, 4, 4, 4, 5)\)-vertex, then at least two incident 4-faces of \( v \) are \((4^+, 4^+, 5, 5^+)\)-faces by Theorem \( \text{II} \). Thus we obtain \( \mu^*(v) = \mu(v) - (2 \cdot (1) + 2 \cdot (2/3)) + 0.4 > 0 \).

If \( v \) is a \((4, 4, 4, 4, 4)\)-vertex, then at least three incident 4-faces of \( v \) are \((4^+, 4^+, 5, 5^+)\)-faces by Theorem \( \text{II} \). Thus we obtain \( \mu^*(v) = \mu(v) - (2 \cdot (1) + 3 \cdot (2/3)) > 0 \).

Let \( v \) be incident to one 3-face. If \( v \) is a \((3, 4^+, 5, 5, 5)\)-vertex, then \( \mu^*(v) = \mu(v) - (1.4 + 1 + 3 \cdot (0.4)) \geq 0 \). Let \( v \) be a \((3, 4, 4, 5, 5)\)-vertex with two 4-faces, \( f_1 \) and \( f_2 \). If \( f_1 \) is adjacent to \( f_2 \), then either \( f_1 \) or \( f_2 \) is a \((4^+, 4^+, 5, 5^+)\)-face by Lemma \( \text{I} \). We obtain \( \mu^*(v) = \mu(v) - (1.4 + 1 + 2/3 + 2 \cdot (0.4)) > 0 \). If \( f_1 \) is not adjacent to \( f_2 \), then an incident 3-face of \( v \) is adjacent to \( f_1 \) and \( f_2 \) by Proposition \( \text{II}(3) \). Then we obtain \( \mu^*(v) = \mu(v) - (3 \cdot (1) + 2 \cdot (0.4)) \geq 0 \).

Let \( v \) be incident to two 3-faces. By Proposition \( \text{II}(2) \), \( v \) is not incident to at least two 4-faces and by Proposition \( \text{II}(3) \), \( v \) is not incident to one 4-face. Thus \( v \) is a \((3, 3, 5, 5, 5)\)-vertex and we obtain \( \mu^*(v) = \mu(v) - (2 \cdot (1.4) + 3 \cdot (0.4)) \geq 0 \).

**CASE 3:** Consider a 6-vertex \( v \).

**SUBCASE 3.1**: Let \( v \) be incident to an adjacent triangle or a bad face.

Then \( v \) is incident to at least two 6\(^+\)-faces by Proposition \( \text{II}(1) \). Thus we obtain \( \mu^*(v) = \mu(v) - 4 \cdot (1.5) \geq 0 \) since \( v \) sends charge at most 1.5 to each incident face by (R1.2), (R2.3), (R3.2), and (R4.3).

**SUBCASE 3.2**: A vertex \( v \) has neither adjacent triangles nor bad faces.

Then \( v \) is incident to at most three 3-faces. We use (R1.2), (R3.2), and (R4.3) to prove the
following cases.

Let $v$ be incident to at least one $6^+$-face. Then $\mu^*(v) = \mu(v) - (2 \cdot (1.4) + 3 \cdot (1)) > 0$ if there are at most two 3-faces and we obtain $\mu^*(v) = \mu(v) - (3 \cdot (1.4) + 2 \cdot (0.8)) > 0$ if there are three 3-faces.

Next, a 6-vertex $v$ is not incident to any $6^+$-face.

Let $v$ be not incident to any 3-face. Then $\mu^*(v) = \mu(v) - (6 \cdot (1)) \geq 0$.

Let $v$ be incident to one 3-face. By Proposition [1](2), $v$ is incident to at most three 4-faces. If $v$ is a $(3, 4, 4, 4, 5, 5)$-vertex, then $\mu^*(v) = \mu(v) - (1.4 + 3 \cdot (1) + 2 \cdot (0.8)) \geq 0$.

Let $v$ be incident to two 3-faces. By Proposition [1](2) and [1](3), $v$ is incident to at most one 4-face. If $v$ is a $(3, 3, 4, 5, 5, 5)$-vertex, then an incident 4-face of $v$ and an incident 3-faces of $v$ are not adjacent by Proposition [1](3). By Theorem [1] at least two incident faces of $v$ are incident at least two $5^+$-vertices. Then one of incident 5-face of $v$ is incident to two $5^+$-vertices. We obtain $\mu^*(v) = \mu(v) - (2 \cdot (1.4) + 1 + 2 \cdot (0.8) + 0.4) > 0$. If $v$ is a $(3, 3, 5, 5, 5, 5)$-vertex, then $\mu^*(v) = \mu(v) - (2 \cdot (1.4) + 4 \cdot (0.8)) \geq 0$.

Let $v$ be incident to three 3-faces. By Proposition [1](2), $v$ is not incident to any 4-face. Thus $v$ is a $(3, 3, 3, 5, 5, 5)$-vertex. By Theorem [1] at least two of incident faces $f_1$ and $f_2$ of $v$ are incident to at least two $5^+$-vertices. Then we obtain $\mu^*(v) = \mu(v) - (3 \cdot (1.4) + 0.8 + 2 \cdot (0.4)) \geq 0$ if $f_1$ and $f_2$ are $5^+$-faces and $\mu^*(v) = \mu(v) - (2 \cdot (1.4) + 1.2 + 2 \cdot (0.8) + 0.4) \geq 0$ if a 3-face is either $f_1$ or $f_2$.

CASE 4: Consider a $k$-vertex $v$ with $k \geq 7$.

SUBCASE 4.1: Let $v$ be incident to an adjacent triangle or a bad face.

Then $v$ is incident to at least two $6^+$-faces by Proposition [1](1). Then $v$ sends charge to at most $k - 2$ faces. We obtain $\mu^*(v) = \mu(v) - (k - 2) \cdot (1.5) > 0$ since $(2 \cdot (k) - 6)/k - 2 \geq 1.5$ for $k \geq 6$ by (1.2), (R2.3), (R3.2), and (R4.4).

SUBCASE: 4.2 A vertex $v$ has neither adjacent triangles nor bad faces.

We use (R1.2), (R3.2), and (R4.4) to prove the following cases.

Let $v$ be a 7-vertex. If $v$ is incident to at most two 3-faces, then $\mu^*(v) = \mu(v) - (2 \cdot (1.4) + 5 \cdot (1)) \geq 0.2$. If $v$ is incident to three 3-faces, then $v$ is incident to at most one 4-face by Proposition [1](2). Thus we obtain $\mu^*(v) = \mu(v) - (3 \cdot (1.4) + 1 \cdot (1) + 3 \cdot (0.8)) \geq 0.4$.

Let $v$ be a $8^+$-vertex. Then $v$ is incident to at most $d(v)/2$ 3-faces. $\mu^*(v) \geq 0.4$ since $1.4(d(v)/2) + d(v)/2 \leq 2d(v) - 6$ for $d(v) \geq 8$. Thus $\mu^*(v) \geq 0.4$ for each vertex $v$.

It is clear that $\mu^*(f) = \mu(f) \geq 0$ for each $f$ is a $6^+$-face since $(1/8)d(f) < d(f) - 6$ for $d(f) \geq 7$.

CASE 5: Consider a 3-face $f$ that is not adjacent to the others 3-faces.
We use (R1.1) and (R1.2) to prove the following cases.

Let $f$ be not incident to each flaw 4-vertex. Then $\mu^*(f) = \mu(f) + (3 \cdot (1)) = 0$.

Next at least one incident 4-vertex of $f$ is flaw. If $f$ is a $(4, 4, 5^+)$-face, then $\mu^*(f) = \mu(f) + (1.4 + 2 \cdot (0.8)) = 0$ when both incident 4-vertices of $f$ are flaw and $\mu^*(v) = \mu(v) + (1.2 + 1 + 0.8) = 0$ when exactly one of incident 4-vertex of $v$ is flaw. If $f$ is a $(4, 5^+, 5^+)$-face, then $\mu^*(f) = \mu(f) + (2 \cdot (1.2) + 0.6) = 0$.

**CASE 6:** Consider a 3-face $f$ that is adjacent to the others 3-faces.

We use (R2.1), (R2.2), (R2.3), (R5), and (R6) to prove the following cases.

**SUBCASE 6.1:** If $f$ is not in a trio, then $\mu^*(f) = \mu(f) + (3 \cdot (1)) = 0$ since each incident vertex sends charge at least one to $f$.

**SUBCASE 6.2:** Let $f$ be in a trio.

Let $f_1$, $f_2$, and $f_3$ be 3-faces in a same trio $T$. Define $\mu(T) := \mu(f_1) + \mu(f_2) + \mu(f_3) = -9$ and $\mu^*(T) := \mu^*(f_1) + \mu^*(f_2) + \mu^*(f_3)$ by (R6).

If $f$ is in a trio that a worst vertex is not a 4-vertex, then each 3-face of trio $h$ that

$$\mu^*(h) = \mu(h) + (3 \cdot (1)) \geq 0.$$

If $T$ is a trio that a worst vertex is a 4-vertex, then there are many cases as follows.

If each worse vertex is a 4-vertex, then two bad vertices are $5^+$-vertices by Lemma [4]. Then $\mu^*(T) = -9 + 3 \cdot (2/3) + 2 \cdot (1.5) + 4 \cdot (1) \geq 0$.

If one of worse vertex is a 5-vertex, then either the other worse vertex or at least one bad vertex is a $5^+$-vertex by Lemma [4]. Then $\mu^*(T) = -9 + 3 \cdot (2/3) + 2 \cdot (1.25) + 1.5 + 3 \cdot (1) \geq 0$ or $\mu^*(T) = -9 + 3 \cdot (2/3) + 2 \cdot (1.25) + 1.5 + 3 \cdot (1) \geq 0$, respectively.

If a worse vertex is a $6^+$-vertex, then $\mu^*(T) = -9 + 3 \cdot (2/3) + 2 \cdot (1.5) + 4 \cdot (1) \geq 0$.

**SUBCASE 6.3:** Let $f$ be in $W_5$.

If each vertex of $W_5$ is not a $6^+$-vertex, then at least three vertices are $5^+$-vertices by Theorem [4]. Thus we obtain $\mu^*(W_5) = -12 + 6 \cdot (1.25) + 2 \cdot (1) + 4 \cdot (0.5) + 4 \cdot (1/8) \geq 0$.

If exactly one vertex of $W_5$ is a $6^+$-vertex, then one of the others vertices is a $5^+$-vertex by Lemma [4]. Thus we obtain $\mu^*(W_5) = -12 + 2 \cdot (1.5) + 2 \cdot (1.25) + 4 \cdot (1) + 4 \cdot (0.5) + 4 \cdot (1/8) \geq 0$.

If at least two vertices of $W_5$ are $6^+$-vertices, then we obtain $\mu^*(W_5) = -12 + 4 \cdot (1.5) + 4 \cdot (1) + 4 \cdot (0.5) + 4 \cdot (1/8) > 0$.

**CASE 7:** Consider a 4-face $f$.

Then every vertex on $f$ has degree at least 4 and one of them has degree at least 5. If $f$ is a $(4, 4, 4, 5^+)$-face, then $\mu^*(f) \geq \mu(f) + 3(1/3) + 1 = 0$ and we obtain $\mu^*(f) \geq \mu(f) + 2(1/3) + 2 \cdot (2/3) \geq 0$ if $f$ is a $(4^+, 4^+, 5^+, 5^+)$-face.

**CASE 8:** Consider a 5-face $f$. 
We use (R4.1), (R4.2), (R4.3), and (R4.4) to prove the following cases.

**SUBCASE 8.1:** Let $f$ be incident to at least three $5^+$-vertices.

Then, each incident 4-vertex of $f$ is adjacent to at least one $5^+$-vertex. Thus we obtain

$$
\mu^*(f) = \mu(f) + 3 \cdot (0.3) + 2 \cdot (0.1) \geq 0.
$$

**SUBCASE 8.2:** Let $f$ be incident to two $5^+$-vertices.

If $x$ is not adjacent to $y$, then each incident 4-vertex of $f$ is adjacent to at least one $5^+$-vertex. Moreover, one of these is adjacent to at least two $5^+$-vertices. Thus we obtain

$$
\mu^*(f) = \mu(f) + 2 \cdot (0.3) + 2 \cdot (0.1) + 0.2 \geq 0.
$$

If $x$ is adjacent to $y$, then two of incident 4-vertices of $f$ is adjacent to at least one $5^+$-vertex. Thus we obtain $\mu^*(f) = \mu(f) + 2 \cdot (0.4) + 2 \cdot (0.1) \geq 0$.

**SUBCASE 8.3:** Let $f$ be incident to at most one $5^+$-vertex.

If $f$ is a $(4, 4, 4, 4, 6^+)$-face, then two of incident 4-vertices of $f$ are adjacent to at least one $6^+$-vertex. Thus we obtain

$$
\mu^*(f) = \mu(f) + 0.8 + 2 \cdot (0.1) \geq 0.
$$

Now, it remains to show $f$ is a $(4, 4, 4, 4, 5^-)$-face.

If $f$ is adjacent to at most one $4^+$-face, then $f$ is a poor 5-face. Thus each incident flaw 4-vertex of $f$ is adjacent to at least two $5^+$-vertices by Lemma 3. Thus we obtain

$$
\mu^*(f) = \mu(f) + 5 \cdot (0.2) \geq 0.
$$

If $f$ is adjacent to at least two $4^+$-faces, then at least three incident vertices of $f$ are not flaw 4-vertices. Thus we obtain

$$
\mu^*(f) = \mu(f) + 3 \cdot (1/3) \geq 0.
$$

This completes the proof.

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