INFINITESIMAL DEFORMATIONS OF NATURALLY GRADED FILIFORM LEIBNIZ ALGEBRAS

KHUDOYBERDIYEV A.KH. AND OMirov B.A.

Abstract. We describe infinitesimal deformations of complex naturally graded filiform Leibniz algebras. It is known that any $n$-dimensional filiform Lie algebra can be obtained by a linear integrable deformation of the naturally graded algebra $F_3^n(0)$. We establish that in the same way any $n$-dimensional filiform Leibniz algebra can be obtained by an infinitesimal deformation of the filiform Leibniz algebras $F_1^n$, $F_2^n$, and $F_3^n(\alpha)$. Moreover, we describe the linear integrable deformations of above-mentioned algebras with a fixed basis of $HL^2$ in the set of all $n$-dimensional Leibniz algebras.

Among these deformations we found one new rigid algebra.

Mathematics Subject Classification 2010: 17A32, 17A70, 17B30, 13D10, 16S80.

Key Words and Phrases: Leibniz algebra, the second group of cohomology, naturally gradation, filiform algebra, infinitesimal deformation, linear integrable deformation, rigid algebra.

1. Introduction.

Deformations of different algebraic and analytic objects are important aspects if one studies their properties. They characterize the local behavior in a small neighborhood in the variety of a given type objects. A geometric picture of the deformations is obtained by considering the variety $M$ of all those bilinear maps (products) of the underlying vector space into itself which satisfy the conditions defining of the variety. An algebra structure of the variety represents a point $m$ of $M$, and deformations of the structure are represented by points of $M$ near $m$. Thus the study of deformations of these algebras is a special case of a study of local geometric properties of varieties.

Classical deformation theory of associative and Lie algebras began with the works of Gerstenhaber [8] and Nijenhuis-Richardson [16] in the 1960s. They studied one-parameter deformations and established the connection between Lie algebra cohomology and infinitesimal deformations. After these works formal deformation theory was generalized in different categories. In fact, in the last fifty years, deformation theory has played an important role in algebraic geometry. The main goal is the classification of families of geometric objects when the classifying space (the so called moduli space) is a reasonable geometric space. In particular, each point of our moduli space corresponds to one geometric object (class of isomorphism). The theory of deformations is one of the effective approach in investigating of solvable and nilpotent Lie algebras (see for example, [6, 10, 15]).

In this paper we study infinitesimal deformations of some nilpotent Leibniz algebras. Recall, that Leibniz algebras are generalization of Lie algebras [13, 14] and it is natural to apply the theory of deformations to the study of these algebras. Particularly, the problems which were studied in [6, 10, 15] and others can be considered from point of Leibniz algebras view.

From algebraic geometry it is known that an algebraic variety is a union of irreducible components. The closures of orbits of rigid algebras give irreducible components of the variety. That is why the finding of rigid algebras is crucial problem from the geometrical point of view.

Due to [3] we can apply the general principles for deformations and rigidity of Leibniz algebras. Namely, it is proved that nullity of the second cohomology group $(HL^2(L, L) = 0)$ gives a sufficient condition for rigidity. In addition, it is established that Leibniz algebras for which every formal deformation is equivalent to a trivial deformation are rigid.

One of the inherent properties of finite-dimensional Leibniz algebras consist of the existence of nilpotent single-generated Leibniz algebras (so-called null-filiform algebras), which are Leibniz algebras of maximal nilindex. It is known that in each dimension all of those algebras are isomorphic to the algebra $NF_n$ [2] and this algebra is rigid in the variety of $n$-dimensional nilpotent Leibniz algebras. In [12] infinitesimal deformations of the algebra $NF_n$ are studied. It was proved that any single-generated Leibniz algebra (which is solvable) is a linear integrable deformation of $NF_n$. Moreover, it is shown that the closure of the set of all single generated Leibniz algebras forms an irreducible component of the variety of $n$-dimensional Leibniz algebras (denoted by $Leib_n$).
Firstly the notion of filiform algebra was introduced by M.Vergne in \[\text{[18]}\] as an algebra of maximal nilindex in the variety of Lie algebras. Namely, naturally graded filiform Lie algebras are classified and it is proved that any filiform Lie algebra is represented by linear integrable deformation of special filiform Lie algebra.

In \[\text{[2]}\] for Leibniz algebras by the approach of M.Vergne similar description was obtained. In particular, up to isomorphism there are only three naturally graded filiform Leibniz algebras \((F^1_n, F^2_n\) and \(F^3_α, α ∈ \{0; 1\})\).

The structure of the paper is as follows: In Section 2 we give the necessary definitions and facts. Section 3 is divided into three subsections: Subsection 3.1 deals with the second group Leibniz cohomology of the algebra \(F^1_n\) and the description of some linear integrable deformations of \(F^1_n\) in the variety of \(Leib_n\). Among these deformations we indicate unknown till now rigid Leibniz algebra. In Subsection 3.2 we describe infinitesimal deformations of the algebra \(F^2_n\) and its linear integrable deformations with respect to chosen basis of \(HL^2(F^2_n, F^3_n)\). In Subsection 3.3 for the algebra \(F^3_n(0)\) we establish that Lie infinitesimal deformations together with three indicated Leibniz infinitesimal deformations form the space of all Leibniz infinitesimal deformations.

Throughout the paper we consider finite-dimensional vector spaces and algebras over the field of complex numbers. Moreover, in the multiplication table of a Leibniz algebra the omitted products and in the expansion of 2-cocycles the omitted values are assumed to be zero.

2. Preliminaries.

In this section we give necessary definitions and known results.

**Definition 2.1.** A Leibniz algebra over \(F\) is a vector space \(L\) equipped with a bilinear map, called bracket,

\[[-,-]: L × L → L\]

satisfying the Leibniz identity:

\([x, [y, z]] = [[x, y], z] - [x, [z, y]],\]

for all \(x, y, z ∈ L\).

The set \(\text{Ann}_r(L) = \{x ∈ L : [y, x] = 0, \ ∀y ∈ L\}\) is called the right annihilator of a Leibniz algebra \(L\). Note that \(\text{Ann}_r(L)\) is an ideal of \(L\) and for any \(x, y ∈ L\) the elements \([x, [x, y] + [y, x] \in \text{Ann}_r(L)\).

We call a vector space \(M\) a module over \(L\) if there are two bilinear maps:

\[[-,-]: L × M → M\quad \text{and}\quad [-,-]: M × L → M\]

satisfying the following three axioms

\([m, [x, y]] = [[m, x], y] - [[m, y], x],\]
\([[x, m], y] = [x, [m, y]] - [x, [y, m]],\]
\([[x, y, m]] = [[x, y], m] - [[x, m], y],\]

for any \(m ∈ M, x, y ∈ L\).

Given a Leibniz algebra \(L\), let \(C^n(L, M)\) be the space of all \(F\)-linear homogeneous mappings \(L^⊗n → M, n ≥ 0\) and \(C^0(L, M) = M\).

Let \(d^n : C^n(L, M) → C^{n+1}(L, M)\) be an \(F\)-homomorphism defined by

\((d^n f)(x_1, \ldots, x_{n+1}) := [x_1, f(x_2, \ldots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}), x_i] + \sum_{1 ≤ i < j ≤ n+1} (-1)^{i+j+1} f(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_{n+1}),\)

where \(f ∈ C^n(L, M)\) and \(x_i ∈ L\). Since the derivative operator \(d = \sum_{i ≥ 0} d^i\) satisfies the property \(d ◦ d = 0\), the \(n\)-th cohomology group is well defined and

\(HL^n(L, M) = ZL^n(L, M) / BL^n(L, M),\)

where the elements \(ZL^n(L, M)\) and \(BL^n(L, M)\) are called \(n\)-cocycles and \(n\)-coboundaries, respectively.

The elements \(f ∈ BL^2(L, L)\) and \(φ ∈ ZL^2(L, L)\) are defined as follows

\(f(x, y) = [d(x), y] + [x, d(y)] - d([x, y])\) for some linear map \(d\) \hspace{1cm} (2.1)
and
\[(d^2\varphi)(x, y, z) = [x, \varphi(y, z)] - \varphi(x, y, z) + [\varphi(x, z), y] + \varphi(x, [y, z]) - \varphi([x, y], z) + \varphi([x, z], y) = 0. \tag{2.2}\]

Usually a 2-cocycle is called infinitesimal deformation.

A deformation of a Leibniz algebra \(L\) is a one-parameter family \(L_t\) of Leibniz algebras with the bracket
\[\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \cdots,\]
where \(\varphi_i\) are 2-cochains, i.e., elements of \(\text{Hom}(L \otimes L, L) = C^2(L, L)\).

Two deformations \(L_t, L'_t\) with corresponding laws \(\mu_t, \mu'_t\) are equivalent if there exists a linear automorphism \(f_t = id + f_1t + f_2t^2 + \cdots\) of \(L\), where \(f_i\) are elements of \(C^1(L, L)\) such that the following equation holds
\[\mu'_t(x, y) = f_t^{-1}(\mu_t(f_t(x), f_t(y)))\quad \text{for } x, y \in L.\]

The Leibniz identity for the algebras \(L_t\) implies that the 2-cochain \(\varphi_1\) is an infinitesimal deformation, i.e. \(d^2\varphi_1 = 0\). If \(\varphi_1\) vanishes identically, then the first non vanishing \(\varphi_i\) is an infinitesimal deformation.

If \(\mu'_t\) is an equivalent deformation with cochains \(\varphi'_i\), then \(\varphi'_i - \varphi_i = d^1f_i\), hence every equivalence class of deformations defines uniquely an element of \(HL^2(L, L)\).

Note that the linear integrable deformation \(\varphi\) satisfies the condition
\[\varphi(x, \varphi(y, z)) - \varphi(\varphi(x, y), z) + \varphi(\varphi(x, z), y) = 0. \tag{2.3}\]

The linear reductive group \(GL_n(F)\) acts on \(Leib_n\) via change of basis, i.e.,
\[(g \ast \lambda)(x, y) = g\left(\lambda(g^{-1}(x), g^{-1}(y))\right), \quad g \in GL_n(F), \quad \lambda \in Leib_n.\]

The orbits \(\text{Orb}(-)\) under this action are the isomorphism classes of algebras. Recall, Leibniz algebras with open orbits are called rigid. Note that solvable (respectively nilpotent) Leibniz algebras of the same dimension also form an invariant subvariety of the variety of Leibniz algebras under the mentioned action. We give a definition of degeneration.

**Definition 2.2.** It is said that an algebra \(\lambda\) degenerates to an algebra \(\mu\), if \(\text{Orb}(\mu)\) lies in the Zariski closure of \(\text{Orb}(\lambda), \text{Orb}(\lambda)\). We denote this by \(\lambda \to \mu\).

In the case of the field \(\mathbb{F}\) be the complex numbers \(\mathbb{C}\), we give an equivalent definition of degeneration.

**Definition 2.3.** Let \(g : (0, 1) \rightarrow GL_n(V)\) be a continues mapping. We construct a parameterized family of the Leibniz algebras \(g_t = (V, [-, -]), t \in (0, 1)\) isomorphic to \(L\). For each \(t\) the new Leibniz bracket \([-,-]_t\) on \(V\) is defined via the old one as follows: \([x, y]_t = g_t[g_t^{-1}(x), g_t^{-1}(y)], \forall x, y \in V. If for any x, y \in V there exists the limit\]
\[
\lim_{t \to 0^+} [x, y]_t = \lim_{t \to 0^+} g_t[g_t^{-1}(x), g_t^{-1}(y)] =: [x, y]_0,
\]
then \([-,-]_0\) is a well-defined Leibniz bracket. The Leibniz algebra \(L_0 = (V, [-, -]_0)\) is called a degeneration of the algebra \(L\).

For a Leibniz algebra \(L\) consider the following central lower series:
\[L^1 = L, \quad L^{k+1} = [L^k, L^1], \quad k \geq 1.\]

**Definition 2.4.** An \(n\)-dimensional Leibniz algebra is said to be filiform if \(\dim L^i = n - i, 2 \leq i \leq n.\)

Now let us define a natural graduation for a filiform Leibniz algebra.

**Definition 2.5.** Given a filiform Leibniz algebra \(L\), put \(L_i = L^i/L^{i+1}, 1 \leq i \leq n - 1, \text{ and } \text{gr}(L) = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}.\) Then \([L_i, L_j] \subseteq L_{i+j}\) and we obtain the graded algebra \(\text{gr}(L)\). If \(\text{gr}(L)\) and \(L\) are isomorphic, then we say that the algebra \(L\) is naturally graded.

In the following theorem we resume the classification of the naturally graded filiform Leibniz algebras given in \[2\] and \[18\].

**Theorem 2.6.** Any complex \(n\)-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non isomorphic algebras:

\[
\begin{align*}
F^1_n &: [x_i, x_1] = x_{i+1}, \quad 2 \leq i \leq n - 1, \\
F^2_n &: [x_1, x_i] = x_{i+1}, \quad 1 \leq i \leq n - 2, \\
F^3_n(\alpha) &: \begin{cases} 
[x_i, x_1] = -[x_1, x_i] = x_{i+1}, & 2 \leq i \leq n - 1, \\
[x_i, x_{n+1-i}] = -[x_{n+1-i}, x_i] = \alpha(-1)^{i+1}x_n, & 2 \leq i \leq n - 1.
\end{cases}
\end{align*}
\]
where \( \alpha \in \{0, 1\} \) for even \( n \) and \( \alpha = 0 \) for odd \( n \).

The following theorem decomposes all \( n \)-dimensional filiform Leibniz algebras into three families of algebras.

**Theorem 2.7.** \([17]\) Any complex \( n \)-dimensional filiform Leibniz algebra admits a basis \( \{x_1, x_2, \ldots, x_n\} \) such that the table of multiplication of the algebra have one of the following forms:

\[
F_1 = \begin{cases} 
[x_i, x_1] = x_{i+1}, & 2 \leq i \leq n-1, \\
x_1, x_2 = \theta x_n, \\
x_j, x_2 = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \cdots + \alpha_{n+2-j} x_n, & 2 \leq j \leq n-2, \\
x_i, x_1 = x_{i+1}, & 1 \leq i \leq n-2, \\
x_i, x_2 = \gamma x_{n-1}, & 2 \leq i \leq n-1, \\
x_1, x_i = -x_{i+1}, & 3 \leq i \leq n-1, \\
x_1, x_1 = \theta_1 x_n, \\
x_2, x_2 = \theta_3 x_n, \\
x_i, x_j = -[x_i, x_j] = \begin{cases} 
\text{lin} < x_{i+j+1}, x_{i+j+2}, \ldots, x_n >, & 2 \leq i < j \leq n-1, \\
x_{i+1} - x_n, & 2 \leq i \leq n-1.
\end{cases}
\end{cases}
\]

where \( \alpha \in \{0, 1\} \) for even \( n \) and \( \alpha = 0 \) for odd \( n \).

In \([12]\) we obtain that any single-generated Leibniz algebra has the following multiplication:

\[
\bar{\mu}(\alpha_2, \alpha_3, \ldots, \alpha_n) = \begin{cases} 
[x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1, \\
x_i, x_1 = \sum_{k=2}^{n} \alpha_k x_k.
\end{cases}
\]

Note that any algebra of the family \( \bar{\mu}(\alpha_2, \alpha_3, \ldots, \alpha_n) \) is linear integrable deformation of the algebra \( NF_n \).

Let us introduce denotation

\[
X = \bigcup_{\alpha_2, \ldots, \alpha_n} \text{Orb}(\bar{\mu}(\alpha_2, \alpha_3, \ldots, \alpha_n)).
\]

**Theorem 2.8.** \([12]\) \( X \) is an irreducible component of the variety \( \text{Leib}_n \).

3. **DEFORMATIONS OF THE NATURALLY GRADED FILIFORM LEIBNIZ ALGEBRAS**

In this section we calculate infinitesimal deformations of the naturally graded filiform Leibniz algebras.

3.1. **Infinitesimal deformations of the algebra \( F_n^1 \)**

In order to achieve the purpose of the subsection we need the matrix form of a derivation of the filiform Leibniz algebra \( F_n^1 \) \([5]\):

\[
\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & \ldots & 0 & \alpha_n \\
0 & \beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{n-1} & \beta_n \\
0 & 0 & \alpha_1 & \beta_2 & \beta_3 & \ldots & \beta_{n-2} & \beta_{n-1} \\
0 & 0 & 0 & 2\alpha_1 & \beta_2 & \ldots & \beta_{n-3} & \beta_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & (n-2)\alpha_1 + \beta_2
\end{pmatrix}
\]

(3.1)

Due to \([21]\) it is easy to see that \( \dim BL^2(F_n^1, F_n^1) = n^2 - n - 1 \).

The following proposition presents the general form of the Leibniz infinitesimal deformation of the algebra \( F_n^1 \).
Proposition 3.1. An arbitrary infinitesimal deformation $\varphi$ of $F^n_1$ has the following form:

$$
\begin{align*}
\varphi(x_i, x_1) &= \sum_{k=2}^{n} \alpha_{i,k} x_k, \quad \varphi(x_j, x_1) = \sum_{k=1}^{n} \alpha_{j,k} x_k, \quad 2 \leq j \leq n - 1, \\
\varphi(x_n, x_1) &= \sum_{k=2}^{n} \alpha_{n,k} x_k, \quad \varphi(x_1, x_2) = \gamma_1 x_1 + \gamma_n x_n, \\
\varphi(x_i, x_2) &= ((i - 2) \gamma_1 + \beta_2) x_i + \sum_{k=3}^{n+2-i} \beta_k x_{k+i-2}, \quad 2 \leq i \leq n, \\
\varphi(x_i, x_3) &= -(\alpha_{2,1} + \gamma_1) x_{i+1}, \quad 2 \leq i \leq n - 1, \\
\varphi(x_j, x_{j+1}) &= -\alpha_{j,1} x_{j+1}, \quad 2 \leq i \leq n - 1, \ 3 \leq j \leq n - 1.
\end{align*}
$$

Proof. Using the property of infinitesimal deformations for $(d^2 \varphi)(x_i, x_j, x_k) = 0$ with $2 \leq j, k \leq n$, we obtain $[x_i, \varphi(x_j, x_k)] = 0$, which implies $\varphi(x_j, x_k) \in < x_2, x_3, \ldots, x_n >$. Similarly, the equation $(d^2 \varphi)(x_i, x_1, x_1) = 0$ leads to $[x_i, \varphi(x_1, x_1)] = 0$, consequently we have $\varphi(x_1, x_1) \in < x_2, x_3, \ldots, x_n >$.

From the condition $(d^2 \varphi)(x_i, x_j, x_1) = 0$ with $2 \leq j \leq n$, we derive

$$
[x_i, \varphi(x_1, x_1)] - [\varphi(x_i, x_j), x_1] + \varphi(x_i, [x_j, x_1]) + \varphi([x_i, x_1], x_j) = 0. \quad (3.2)
$$

Similarly, from the condition $(d^2 \varphi)(x_i, x_1, x_k) = 0$ with $2 \leq k \leq n$, we have

$$
[x_i, \varphi(x_1, x_k)] + [\varphi(x_i, x_k), x_1] - \varphi([x_i, x_1], x_k) = 0. \quad (3.3)
$$

The equality (3.3) with $i = 1, k = 2$ deduce $[\varphi(x_1, x_2), x_1] = \varphi([x_1, x_1], x_2) - [x_1, \varphi(x_1, x_2)] = 0$, hence we can assume $\varphi(x_1, x_2) = \gamma_1 x_1 + \gamma_n x_n$ for some parameters $\gamma_1, \gamma_2$.

From equality (3.2) with $i = 1, 2 \leq j \leq n - 1$, we have $\varphi(x_1, x_{j+1}) = [\varphi(x_1, x_j), x_1] = 0$.

Summarizing equalities (3.2) and (3.3) we obtain

$$
\begin{align*}
\varphi(x_i, x_3) &= -[x_i, \varphi(x_1, x_2) + \varphi(x_2, x_1)], \\
\varphi(x_i, x_{j+1}) &= -[x_i, \varphi(x_j, x_1)], \\
[x_i, \varphi(x_n, x_1)] &= 0.
\end{align*}
\quad (3.4)
$$

We set

$$
\varphi(x_j, x_1) = \sum_{k=1}^{n} \alpha_{j,k} x_k, \quad 1 \leq j \leq n, \quad \varphi(x_2, x_2) = \sum_{k=2}^{n} \beta_k x_k.
$$

Applying the equations (3.2), (3.3) and (3.4) we derive $a_{1,1} = a_{n,1} = 0$ and

$$
\begin{align*}
\varphi(x_i, x_2) &= ((i - 2) \gamma_1 + \beta_2) x_i + \sum_{k=3}^{n+2-i} \beta_k x_{k+i-2}, \quad 2 \leq i \leq n, \\
\varphi(x_i, x_3) &= -(\alpha_{2,1} + \gamma_1) x_{i+1}, \quad 2 \leq i \leq n - 1, \quad \varphi(x_i, x_{j+1}) = -\alpha_{j,1} x_{j+1}, \quad 2 \leq i \leq n - 1, \ 3 \leq j \leq n - 1.
\end{align*}
$$

Using Proposition 3.1 we indicate a basis of the space $ZL^2(F^n_1, F^n_1)$.

Theorem 3.2. The following cochains:

$$
\begin{align*}
\varphi_{j,1}(2 \leq j \leq n - 1) : & \begin{cases} \\
\varphi_{j,1}(x_j, x_1) = x_1, \\
\varphi_{j,1}(x_j, x_{j+1}) = -x_{j+1},
\end{cases} \quad 2 \leq i \leq n - 1, \\
\varphi_{j,k}(1 \leq j \leq n, 2 \leq k \leq n) : & \begin{cases} \\
\varphi_{j,k}(x_j, x_1) = x_k, \\
\psi_j(2 \leq j \leq n) : & \begin{cases} \\
\psi_j(x_i, x_2) = x_{j+i-2}, \quad 2 \leq i \leq n - j + 2,
\end{cases}
\end{cases}
\end{align*}
$$

form a basis of the space $ZL^2(F^n_1, F^n_1)$.

Corollary 3.3. dim($ZL^2(F^n_1, F^n_1)$) = $n^2 + n - 1$.

Below, we describe a basis of the subspace $BL^2(F^n_1, F^n_1)$ in terms of $\varphi_{j,k}, \psi_j, \xi_1$ and $\xi_2$. 


Proposition 3.4. The cocycles
\[ \eta_{j,k} : \left\{ \begin{array}{ll}
\eta_{1,k-1} = \varphi_{1,k}, & 3 \leq k \leq n, \\
\eta_{2,1} = \psi_3, & \\
\eta_{j,1} = \varphi_{j-1,1}, & 3 \leq j \leq n, \\
\eta_{j,k} = \varphi_{j-1,k}, & 3 \leq j \leq k \leq n, \\
\eta_{j,k} = \varphi_{j-1,k} - \varphi_{j,k+1}, & 3 \leq j < k \leq n,
\end{array} \right. \]
form a basis of \( BL^2(F^1_n, F^1_n) \).

Proof. Consider the endomorphisms \( f_{j,k} \) defined as follows:
\[
\begin{align*}
 f_{2,1}(x_2) &= x_1, \\
 f_{1,k}(x_1) &= x_k, & 2 \leq k \leq n - 1, \\
 f_{j,k}(x_j) &= x_k, & 3 \leq j \leq n, & 1 \leq k \leq n.
\end{align*}
\]
According to (3.1) it implies that \( f_{j,k} \) are complemented linear maps to derivations in \( C^1(F^1_n, F^1_n) \).

Therefore, \( d^1 f_{j,k} \) form a basis of the space \( BL^2(F^1_n, F^1_n) \), where \( d^1 f_{j,k} = f_{j,k}([x,y]) - [f_{j,k}(x), y] - [x, f_{j,k}(y)] \).

It should be noted that
\[
\begin{align*}
 d^1 f_{1,k} &= -\varphi_{1,k+1}, & 2 \leq k \leq n - 1, \\
 d^1 f_{2,1} &= -\psi_3, \\
 d^1 f_{j,1} &= \varphi_{j-1,1}, & 3 \leq j \leq n, \\
 d^1 f_{j,k} &= \varphi_{j-1,k} - \varphi_{j,k+1}, & 3 \leq j \leq k \leq n - 1, \\
 d^1 f_{j,n} &= \varphi_{j-1,n}, & 3 \leq j \leq n.
\end{align*}
\]
From the condition \( d^1 f_{j,k} + d^1 f_{j+1,k+1} + \cdots + d^1 f_{n+j-k,n} = \varphi_{j-1,k} \) for \( 3 \leq j \leq k \leq n \), we conclude that the maps \( \eta_{j,k} \) form a basis of \( BL^2(F^1_n, F^1_n) \).

Corollary 3.5. The adjoint classes \( \overline{\varphi_2}, \overline{\xi_1}, \overline{\xi_2}, \overline{\varphi_{1,2}}, \overline{\varphi_{1,k}} \) \((2 \leq k \leq n)\) and \( \overline{\psi_j} \) \((4 \leq j \leq n)\) form a basis of \( HL^2(F^1_n, F^1_n) \). Consequently, \( \dim HL^2(F^1_n, F^1_n) = 2n \).

Since every non-trivial equivalence class of deformations defines uniquely an element of \( HL^2(L, L) \), due to Corollary 3.3 it is sufficient to consider a linear deformation
\[
\mu_t = F^1_n + t\varphi,
\]
where \( \varphi = c_1\xi_1 + c_2\xi_2 + a_1\varphi_{1,2} + \sum_{k=2}^{n} a_k\varphi_{n,k} + b_2\psi_2 + \sum_{k=4}^{n} b_k\psi_k \).

If \( t \neq 0 \), then we can assume \( t = 1 \) and the linear deformation \( \mu_1 \) we shall denote by \( \mu \):

\[
\mu : \left\{ \begin{array}{ll}
[x_1, x_1] &= a_1 x_2, & 2 \leq i \leq n - 1, \\
[x_i, x_1] &= x_{i+1}, & \\
[x_n, x_1] &= \sum_{k=2}^{n} a_k x_k, & \\
[x_1, x_2] &= c_1 x_1 + c_n x_n, & \\
[x_i, x_2] &= ((i - 2)c_1 + b_2)x_i + \sum_{k=4}^{n+2-i} b_k x_{k+i-2}, & 2 \leq i \leq n, \\
[x_i, x_3] &= -c_1 x_{i+1}, & 2 \leq i \leq n - 1.
\end{array} \right.
\]

In the next proposition we clarify under which conditions on parameters \( a_i, b_i, c_1 \) and \( c_n \) the algebra of the family \( \mu \) is a Leibniz algebra.

Proposition 3.6. Linear integrable deformation of the algebra \( F^1_n \) consist of the first class of filiform Leibniz algebra \( F_1 \) and following Leibniz algebras

\[
\lambda(a_1, \ldots, a_n) : \left\{ \begin{array}{ll}
[x_1, x_1] &= a_1 x_2, & 2 \leq i \leq n - 1, \\
[x_i, x_1] &= x_{i+1}, & \\
[x_n, x_1] &= \sum_{k=2}^{n} a_k x_k, & \\
[x_i, x_2] &= ((i - 2)c_1 + b_2)x_i + \sum_{k=4}^{n+2-i} b_k x_{k+i-2}, & 2 \leq i \leq n, \\
[x_i, x_3] &= -c_1 x_{i+1}, & 2 \leq i \leq n - 1.
\end{array} \right.
\]
Proposition 3.8. Any derivation of the algebra $\mu$ we obtain the following restrictions:

$$b_2 = 0, \quad c_1a_k = 0, \quad c_n a_k = 0, \quad b_i a_k = 0, \quad 4 \leq i \leq n, \quad 1 \leq k \leq n.$$  

If $a_i \neq 0$ for some $i$, then we deduce $c_1 = c_n = 0, b_i = 0, 4 \leq i \leq n$. So, the family of algebras $\lambda(a_1, a_2, \ldots, a_n)$ is obtained.

If $a_i = 0$ for all $i$, then table of multiplication of the family $\mu$ have the form:

$$
\begin{align*}
[x_i, x_1] &= x_{i+1}, & 2 \leq i \leq n-1, \\
[x_1, x_2] &= c_1 x_1 + c_n x_n, \\
[x_i, x_2] &= (i-2)c_1 x_i + \sum_{k=4}^{n+2-i} b_k x_{k+i-2}, & 2 \leq i \leq n, \\
[x_i, x_3] &= -c_i x_{i+1}, & 2 \leq i \leq n-1.
\end{align*}
$$

In the case of $c_1 = 0$ we get the family of filiform Leibniz algebras $F_1$.

If $c_1 \neq 0$, then taking the basis transformation in the following form:

$$x'_1 = x_1 + \frac{c_n}{c_1(3-n)} x_n, \quad x'_i = \frac{1}{c_1} x_i + \sum_{j=i+2}^{n} A_{j-i+2} x_j, \quad 2 \leq i \leq n-2, \quad x'_{n-1} = \frac{1}{c_1} x_{n-1}, \quad x'_n = \frac{1}{c_1} x_n,$$

with

$$A_4 = -\frac{b_4}{2c_1^2}, \quad A_5 = \frac{b_5}{3c_1^3}, \quad A_i = -\frac{1}{(i-2)c_1} \left( \frac{b_i}{c_1} + \sum_{j=4}^{i-2} A_j b_{i-j-2} \right), \quad 6 \leq i \leq n,$$

we obtain the algebra $R$.  

Below we establish in which irreducible component belongs the family of algebras $\lambda$.

**Proposition 3.7.** $\lambda(a_1, a_2, \ldots, a_n) \in X$ for any values of parameters $a_i$.

**Proof.** If in the family $\lambda$ the parameter $a_1 \neq 0$, then by taking the change of basis elements as follows:

$$x'_1 = x_1, \quad x'_i = a_1 x_i, \quad 2 \leq i \leq n,$$

we can assume $a_1 = 1$ and the family $\lambda(1, a_2, \ldots, a_n)$ is nothing else but $\tilde{\mu}(a_2, a_3, \ldots, a_n)$.

If $a_1 = 0$, then in the case of $a_2 \neq 0$ by changing of basis in the following way:

$$x'_1 = x_1 + \frac{1}{a_2} (x_n - \sum_{k=3}^{n} a_k x_{k-1}), \quad x'_i = x_i, \quad 2 \leq i \leq n,$$

we have $\lambda(0, a_2, \ldots, a_n) \simeq \tilde{\mu}(a_2, a_3, \ldots, a_n)$.

Let suppose $a_1 = a_2 = 0$, then by choosing the transformations $g_t$ as follows $g_t(x_1) = x_1, \ g_t(x_i) = t x_i, \ 2 \leq i \leq n$, we derive

$$\lim_{t \to 0} g_t \ast \tilde{\mu}(0, a_3, \ldots, a_n) = \lambda(0, 0, a_3, \ldots, a_n),$$

which implies $\lambda(0, 0, a_3, \ldots, a_n) \in X$.  

The interesting properties of the algebra $R$ are given in the following assertions.

**Proposition 3.8.** Any derivation of the algebra $R$ has the matrix form:

$$
\begin{pmatrix}
\alpha & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \beta & 0 & \ldots & 0 & 0 \\
0 & 0 & \alpha & \beta & \ldots & 0 & 0 \\
0 & 0 & 0 & 2\alpha & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & (n-3)\alpha & \beta \\
0 & 0 & 0 & 0 & \ldots & 0 & (n-2)\alpha
\end{pmatrix}.
$$

Taking into account that operators of right multiplications $R_{x_1}$ and $R_{x_2}$ of the algebra $R$ are linear independent inner derivations, we conclude that any derivation of the algebra $R$ is inner.
Proposition 3.9. Any infinitesimal deformation of the algebra $R$ has the following form:

$$
\begin{align*}
\varphi(x_1, x_1) &= a_{1,1}x_1 + a_{1,1}x_3 + \sum_{k=4}^{n} a_{1,k}x_k, \\
\varphi(x_i, x_1) &= \sum_{k=1}^{n} a_{i,k}x_k, & 2 \leq i \leq n-1, \\
\varphi(x_n, x_1) &= -\frac{1}{2}(n-2)a_{1,1}x_n + \sum_{k=2}^{n-1} a_{k,k}x_n, \\
\varphi(x_1, x_2) &= b_{1,1}x_1 - a_{1,1}x_2 + \sum_{k=4}^{n-1} (n-3)a_{1,k+1}x_k, \\
\varphi(x_2, x_2) &= b_{2,1}x_1 + b_{2,1}x_3 + \sum_{k=4}^{n} b_{2,k}x_k, \\
\varphi(x_3, x_2) &= a_{2,2}x_2 + b_{1,1}x_3 + \sum_{k=4}^{n} (b_{2,k-1} - (k-3)a_{2,k})x_k, \\
\varphi(x_i, x_2) &= (i-3)a_{i-1,1}x_1 + (i-2)b_{1,1}x_i - \left(\frac{(i-2)(i-3)}{2}a_{1,1} - \sum_{k=2}^{i-1} a_{k,k}\right)x_{i-1} + \\
&+ \sum_{k=2}^{i-2} (i-k) a_{i-j,k+1}x_{j} + \sum_{j=i+1}^{n} \left(b_{2,k+2-i} + (i-k) \sum_{j=2}^{i-1} a_{j,k+1+j-1}\right)x_{k}, & 4 \leq i \leq n, \\
\varphi(x_1, x_3) &= -a_{2,2}x_1 - a_{1,1}x_3 - \sum_{k=4}^{n} a_{1,k}x_k, \\
\varphi(x_2, x_3) &= -a_{2,1}x_1 - a_{2,2}x_2 - (b_{1,1} + a_{2,1})x_3 - \sum_{k=4}^{n} a_{2,k}x_k, \\
\varphi(x_i, x_3) &= (i-2)(a_{1,1} - a_{2,2})x_i + (a_{2,3} - a_{2,1} - b_{1,1})x_{i+1} - \sum_{k=4}^{n} a_{i,k}x_k, & 3 \leq i \leq n-1, \\
\varphi(x_n, x_3) &= \frac{1}{2}(n-2)(n-3)a_{1,1} + \sum_{j=2}^{n-1} a_{k,k}x_n, \\
\varphi(x_1, x_j) &= -a_{j-1,2}x_1, & 4 \leq j \leq n, \\
\varphi(x_2, x_j) &= (a_{j-2,2} - a_{j-1,1} + a_{j-1,3})x_3, & 4 \leq j \leq n, \\
\varphi(x_i, x_j) &= -(i-2)a_{j-1,2}x_i + (a_{j-2,2} - a_{j-1,1} + a_{j-1,3})x_{i+1}, & 3 \leq i \leq n-1, 4 \leq j \leq n, \\
\varphi(x_n, x_j) &= -(n-2)a_{j-1,2}x_n, & 4 \leq j \leq n.
\end{align*}
$$

Corollary 3.10. The algebra $R$ is rigid.

Proof. Due to Proposition 3.3 we have $\dim Der R = 2$. Therefore, $\dim BL^2(R, R) = n^2 - 2$. From Proposition 3.9 we conclude $\dim ZL^2(R, R) = n^2 - 2$, hence $HL^2(R, R) = 0$. Applying the result of the paper [3] on rigidity of Leibniz algebras which satisfy the condition $HL^2(R, R) = 0$ we complete the proof. \hfill $\square$

3.2. Infinitesimal deformations of the algebra $F_n^2$.

Further we shall use the result of [3] on description of derivations of the filiform Leibniz algebra $F_n^2$. Namely, any derivation of $F_n^2$ has the following matrix form:

$$
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n-1} & \alpha_n \\
0 & 2\alpha_1 & \alpha_2 & \ldots & \alpha_{n-2} & 0 \\
0 & 0 & 3\alpha_1 & \ldots & \alpha_{n-2} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & (n-1)\alpha_1 & 0 \\
0 & 0 & 0 & \ldots & \beta_1 & \beta_2
\end{pmatrix}
\quad (3.6)
$$

This matrix form of derivations implies $\dim Der(F_n^2) = n + 2$ and $\dim BL^2(F_n^2, F_n^2) = n^2 - n - 2$. 

Proposition 3.11. An arbitrary infinitesimal deformation \( \varphi \) of \( F^2_n \) has the following form:

\[
\begin{align*}
\varphi(x_j, x_1) &= \sum_{k=1}^{n} \alpha_{j,k} x_k, & 1 \leq j \leq n - 2, \\
\varphi(x_{n-1}, x_1) &= \sum_{k=2}^{n} \alpha_{n-1,k} x_k, & \varphi(x_n, x_1) = \sum_{k=1}^{n} \alpha_{n,k} x_k, \\
\varphi(x_i, x_{j+1}) &= -\alpha_{j+1,i} x_{i+1}, & 1 \leq i \leq n - 2, 1 \leq j \leq n - 2, \\
\varphi(x_1, x_n) &= -\alpha_{n,1} x_1 + \sum_{k=2}^{n} \beta_{k} x_k, \\
\varphi(x_i, x_{n}) &= -i \alpha_{n,1} x_i + \sum_{k=2}^{n} \beta_{k} x_{k+i-1}, & 2 \leq i \leq n - 1, \\
\varphi(x_n, x_{n}) &= \gamma_{n} x_{n} + \gamma_{n} x_{n}.
\end{align*}
\]

Proof. The proof of this proposition is carrying out by applying similar arguments as in the proof of Proposition 3.11. □

Using the assertion of Proposition 3.11 we indicate a basis of the space \( ZL^2(F^2_n, F^2_n) \).

Theorem 3.12. The following cochains:

\[
\varphi_{j,i}(1 \leq j \leq n - 2): \begin{cases} 
\varphi_{j,1}(x_j, x_1) = x_1, \\
\varphi_{j,i}(x_i, x_{j+1}) = -x_{i+1}, & 1 \leq i \leq n - 2,
\end{cases}
\]

\[
\varphi_{j,k}(1 \leq j \leq n, 2 \leq k \leq n): \begin{cases} 
\varphi_{j,k}(x_j, x_1) = x_k, \\
\psi_1: \begin{cases} 
\psi_1(x_n, x_1) = x_1, \\
\psi_1(x_i, x_n) = -ix_i, & 1 \leq i \leq n - 1,
\end{cases} \\
\psi_2(2 \leq j \leq n - 1): \begin{cases} 
\psi_2(x_j, x_n) = x_{j+i-1}, & 1 \leq i \leq n - j,
\end{cases} \\
\psi_n: \begin{cases} 
\psi_n(x_1, x_n) = x_n, \\
\psi_{n+1}: \begin{cases} 
\psi_{n+1}(x_n, x_n) = x_{n-1}, \\
\psi_{n+2}: \psi_{n+2}(x_n, x_n) = x_n.
\end{cases}
\end{cases}
\]

form a basis of \( ZL^2(F^2_n, F^2_n) \).

Corollary 3.13. \( \dim(ZL^2(F^2_n, F^2_n)) = n^2 + n \).

Below we describe a basis of the subspace \( BL^2(F^2_n, F^2_n) \) by means of \( \varphi_{j,k}, \psi_j \).

Proposition 3.14. The cocycles

\[
\eta_{j,k}: \begin{cases} 
\eta_{j,1} = \varphi_{j-1,1} - \varphi_{j,2}, & 2 \leq j \leq n - 1, \\
\eta_{j,k} = \varphi_{j-1,k}, & 2 \leq j \leq k \leq n - 1, \\
\eta_{j,k} = \varphi_{j-1,k} - \varphi_{j,k+1}, & 2 \leq k < j \leq n - 1, \\
\eta_{j,n} = \varphi_{j-1,n}, & 2 \leq j \leq n - 1, \\
\eta_{n,1} = \varphi_{n,2} + \psi_2, \\
\eta_{n,k} = \varphi_{n,k+1}, & 2 \leq k \leq n - 2,
\end{cases}
\]

form a basis of \( BL^2(F^2_n, F^2_n) \).

Corollary 3.15. The adjoint classes \( \overline{\varphi_{n,n}}, \overline{\varphi_{n-1,n}} (2 \leq k \leq n) \) and \( \overline{\psi_j} (1 \leq j \leq n + 2) \) form a basis of \( HL^2(F^2_n, F^2_n) \). Consequently, \( \dim HL^2(F^2_n, F^2_n) = 2n + 2 \).

In the next proposition we clarify that basis elements of \( ZL^2(F^2_n, F^2_n) \) satisfies the condition (2.3).

Proposition 3.16. The infinitesimal deformations \( \varphi_{j,k} \) (1 \leq j \leq n, 2 \leq k \leq n), \( \psi_j \) (1 \leq j \leq n - 1) and \( \psi_{n+1} \) satisfy the condition (2.3), however \( \psi_n, \psi_{n+2}, \) and \( \varphi_{j,1} \) (1 \leq j \leq n - 2) do not satisfy the condition (2.3).

Proof. The proof of this proposition is straightforward. □

Since every non-trivial equivalence class of deformations defines uniquely an element of \( HL^2(L, L) \), due to Corollary 3.15 it is sufficient to consider the linear deformation

\[
\nu_t = F^2_n + t \varphi,
\]
such that its table of multiplication has the form of the families and Proposition 3.17.

Proof. Then we have the table of multiplications

\[
\nu : \begin{cases}
[x_i, x_1] = x_{i+1}, & 1 \leq i \leq n - 2, \\
[x_{n-1}, x_1] = \sum_{k=2}^{n} a_k x_k, & \\
[x_n, x_1] = b_1 x_1 + c_1 x_n, & \\
x_1, x_n] = -b_1 x_1 + \sum_{k=2}^{n} b_k x_k, & 2 \leq i \leq n - 1, \\
x_i, x_n] = -ib_1 x_i + \sum_{k=i}^{n-i} b_k x_{k+i-1}, & 2 \leq i \leq n - 1, \\
x_n, x_n] = b_{n+1} x_{n-1} + b_{n+2} x_n.
\end{cases}
\]

From the equalities

\[
0 = [x_n, [x_n, x_n]] = [x_n, b_{n+1} x_{n-1} + b_{n+2} x_n] = b_{n+2} (b_{n+1} x_{n-1} + b_{n+2} x_n)
\]

we get \(b_{n+2} = 0\).

**Proposition 3.17.** Any linear integrable deformation of the algebra \(F^2_n\) admits a basis \(\{x_1, x_2, \ldots, x_n\}\) such that its table of multiplication has the form of the families \(F_1, F_2, \tilde{\mu}(a_2, \ldots, a_n), \tilde{\mu}(a_2, \ldots, a_{n-1}) \oplus \mathbb{C}\) and

\[
\nu_1(a_2, a_3, \ldots, a_{n-1}) : \begin{cases}
[x_i, x_1] = x_{i+1}, & 1 \leq i \leq n - 2, \\
[x_{n-1}, x_1] = \sum_{k=2}^{n-1} a_k x_k, & \sum_{k=2}^{n-1} a_k = 1, \\
x_n, x_1] = x_n,
\end{cases}
\]

\[
\nu_2 : \begin{cases}
x_i, x_1 = x_{i+1}, & 1 \leq i \leq n - 2, \\
x_n, x_1 = x_1, & \\
x_i, x_n = -ix_i, & 1 \leq i \leq n - 1,
\end{cases}
\]

\[
\nu_3(b_2, b_3, \ldots, b_{n-1}) : \begin{cases}
x_i, x_1 = x_{i+1}, & 1 \leq i \leq n - 2, \\
x_{n-1}, x_1 = -x_{n-1}, & \\
x_n, x_1 = -x_n, & \\
x_1, x_n = x_n + \sum_{k=2}^{n-1} b_k x_k, & 2 \leq i \leq n - 2, \\
x_i, x_n = \sum_{k=2}^{n-i} b_k x_{k+i-1}, & 2 \leq i \leq n - 2.
\end{cases}
\]

where the first non-zero element of the vector \((b_2, b_3, \ldots, b_{n-1})\) can be assumed to be equal to 1,

\[
\nu_4 : \begin{cases}
x_i, x_1 = x_{i+1}, & 1 \leq i \leq n - 2, \\
x_{n-1}, x_1 = -2x_{n-1}, & \\
x_n, x_1 = -x_n, & \\
x_1, x_n = x_n, & \\
x_n, x_n = x_{n-1}, & 
\end{cases}
\]

\[
\nu_5(a_2, a_3, \ldots, a_{n-1}) : \begin{cases}
x_i, x_1 = x_{i+1}, & 1 \leq i \leq n - 2, \\
x_{n-1}, x_1 = \sum_{k=2}^{n-1} a_k x_k, & \\
x_n, x_1 = -x_n, & \\
x_1, x_n = x_n.
\end{cases}
\]

**Proof.** Note that \(\{x_2, x_3, \ldots, x_{n-1}\} \in \text{Ann}_r(v)\). If \(x_n \in \text{Ann}_r(v)\), then we have \(b_k = 0, 1 \leq k \leq n + 1\).

If \(a_n \neq 0\) then we have the class of single generated algebras \(\tilde{\mu}(a_2, \ldots, a_n)\).

If \(a_n = 0\) and \(c_1 = 0\), then we have the split algebra \(\tilde{\mu}(a_2, \ldots, a_{n-1}) \oplus \mathbb{C}\).
Comparing the coefficients at the basic elements, we deduce
\[ c_1 = \begin{cases} 1 & \text{if } i = n - 2, \\ b_{n-1}a_k = -b_n b_k & \text{if } 2 \leq k \leq n - 1. \end{cases} \]

Consider the Leibniz identity
\[ [x_1, [x_n, x_1]] = [x_1, b_1 x_1 - b_n x_n] = b_1 x_2 - b_n \sum_{k=2}^{n-1} b_k x_k + b_n (b_1 x_1 - b_n x_n). \]

On the other hand
\[ [x_1, [x_n, x_1]] = [x_1, b_1 x_1 - b_n x_n] = b_1 x_2 - b_n \sum_{k=2}^{n-1} b_k x_k + b_n (b_1 x_1 - b_n x_n). \]

Comparing the coefficients at the basic elements, we derive
\[ b_{n-2} a_k = 0, \quad 2 \leq i \leq n - 2, \quad 2 \leq k \leq i - 1, \]
\[ b_{n-1} a_i = i b_n b_1, \quad 2 \leq i \leq n - 2, \]
\[ b_{n-2} a_k = -b_n b_{k-1}, \quad 2 \leq i \leq n - 2, \quad i + 1 \leq k \leq n - 1. \]
Similarly, from equalities

\[ [x_{n-1}, [x_n, x_1]] = [[x_{n-1}, x_n], x_1] - [[x_{n-1}, x_1], x_n] = -(n-1)b_1[x_{n-1}, x_1] - \sum_{k=2}^{n-1} a_k x_k, x_n] = \]

\[-(n-1)b_1 \sum_{k=2}^{n-1} a_k x_k - \sum_{k=2}^{n-1} a_k (-kb_1 x_k + \sum_{s=2}^{n-k} b_s x_{s+k-1}) = b_1 \sum_{k=2}^{n-2} (-n+1+k) a_k b_1 x_k - (\sum_{k=2}^{n-2} a_k b_{n-k}) x_{n-1} = \]

\[ b_1 \sum_{k=2}^{n-2} (-n+1+k) a_k b_1 x_k - \sum_{k=2}^{n-2} i b_1 b_n x_{n-1} = b_1 \sum_{k=2}^{n-2} (-n+1+k) a_k b_1 x_k - \frac{n(n-3)b_1 b_n}{2} x_{n-1} \]

and equalities

\[ [x_{n-1}, [x_n, x_1]] = [x_{n-1}, b_1 x_1 - b_n x_n] = b_1 \sum_{k=2}^{n-1} a_k x_k + (n-1)b_n b_1 x_{n-1}, \]

we obtain

\[
\begin{cases}
(n-k)a_k b_1 = 0, & 2 \leq k \leq n-2, \\
(a_{n-1} + \frac{(n+1)(n-2)b_n}{2})b_1 = 0.
\end{cases}
\]

Consider

\[ [x_n, [x_n, x_1]] = [[x_n, x_n], x_1] - [[x_n, x_1], x_n] = b_{n+1}[x_{n-1}, x_1] - [b_1 x_1 - b_n x_n, x_n] = \]

\[ b_{n+1} \sum_{k=2}^{n-1} a_k x_k - b_1 (-b_1 x_1 + \sum_{k=2}^{n} b_k x_k) + b_n b_{n+1} x_{n-1} = \]

\[ b_1 b_1 x_1 + \sum_{k=2}^{n-2} (b_{n+1} a_k - b_1 b_k) x_k + (b_{n+1} a_{n-1} - b_1 b_{n-1} + b_n b_{n+1}) x_{n-1} - b_1 b_n x_n. \]

On the other hand, we have

\[ [x_n, [x_n, x_1]] = [x_n, b_1 x_1 - b_n x_n] = b_1^2 x_1 - b_n b_{n+1} x_{n-1} - b_1 b_n x_n. \]

Comparing the appropriate coefficients, we conclude

\[
\begin{cases}
b_1 b_k = b_{n+1} a_k, & 2 \leq k \leq n-2, \\
b_1 b_{n-1} = b_{n+1} (a_{n-1} + 2b_n).
\end{cases}
\]

Let us summarize the above restrictions:

\[
\begin{align*}
b_{n-1} a_k &= -b_n b_k, & 2 \leq k \leq n-1, \\
b_{n-i} a_k &= 0, & 2 \leq i \leq n-2, & 2 \leq k \leq i-1, \\
b_{n-i} a_i &= i b_n b_1, & 2 \leq i \leq n-2, \\
b_{n-i} a_k &= -b_n b_{k-i+1}, & 2 \leq i \leq n-2, & i+1 \leq k \leq n-1, \\
(n-k) a_k b_1 &= 0, & 2 \leq k \leq n-2, \\
(a_{n-1} + \frac{(n+1)(n-2)b_n}{2}) b_1 &= 0, \\
b_1 b_k &= b_{n+1} a_k, & 2 \leq k \leq n-2, \\
b_1 b_{n-1} &= b_{n+1} (a_{n-1} + 2b_n).
\end{align*}
\]

We need to consider the following distinguish cases.

**Case 1.** Let \( b_1 \neq 0 \). Then \( a_k = 0, 2 \leq k \leq n-1 \) and \( b_k = 0, 2 \leq k \leq n \). In this case case we have the table of multiplication

\[
\begin{align*}
[x_i, x_1] &= x_{i+1}, & 1 \leq i \leq n-2, \\
x_n, x_1] &= b_1 x_1, \\
[x_i, x_n] &= -ib_1 x_i, & 1 \leq i \leq n-1, \\
x_n, x_n] &= b_{n+1} x_{n-1}, & 1 \leq i \leq n-1.
\end{align*}
\]

Taking the change \( x' = \frac{1}{b_i} x_n + \frac{b_{n-1}}{b_i (n-1)} x_{n-1} \) we obtain the algebra \( \nu_2 \).
Case 2. Let $b_1 = 0$. Then the above restrictions are reduced to the following one:

\[
\begin{align*}
\{b_{n-1}a_k &= -b_nb_k, & 2 \leq k \leq n - 1, \\
b_{n-i}a_k &= 0, & 2 \leq i \leq n - 2, \ 2 \leq k \leq i, \\
b_{n-i}a_k &= -b_nb_{k-i+1}, & 2 \leq i \leq n - 2, \ i + 1 \leq k \leq n - 1, \\
b_{n+1}a_k &= 0, & 2 \leq k \leq n - 2, \\
b_{n+1}(a_{n-1} + 2b_n) &= 0. 
\end{align*}
\] (3.7)

- If there exists some $b_i \neq 0$, where $2 \leq i \leq n - 1$, then from (3.7) we obtain $a_i = 0$, $2 \leq i \leq n - 1$, $a_{n-1} = -b_n$, $b_nb_{n+1} = 0$.

- Let us suppose $b_n = 0$. Then we have the multiplication

\[
\begin{align*}
[x_i, x_1] &= x_{i+1}, & 1 \leq i \leq n - 2, \\
[x_i, x_n] &= \sum_{k=2}^{n-i} b_k x_{k+i-1}, & 1 \leq i \leq n - 2, \\
[x_n, x_n] &= b_{n+1} x_{n-1}.
\end{align*}
\]

This family of algebras represents families of filiform Leibniz algebras $F_1$ and $F_2$. Namely, if $b_2 \neq 0$, then we obtain the family $F_1$ and if $b_2 = 0$, then we get the family $F_2$.

- Let us assume now that $b_n \neq 0$. Then $b_{n+1} = 0$ and by scaling the basis elements, one can assume $b_n = 1$. Thus, we obtain the algebra $\nu_3(b_2, b_3, \ldots, b_{n-1})$.

- If $b_i = 0$ for $2 \leq i \leq n - 1$.

- Let $b_{n+1} \neq 0$. Then $a_i = 0$, $2 \leq i \leq n - 1$ and $a_{n-1} = -2b_n$. In the case of $b_n = 0$, we have a filiform Leibniz algebra of the family $F_2$ and in the case of $b_n \neq 0$, by scaling of appropriate basis elements, we can suppose $b_n = b_{n+1} = 1$. Thus, we get the algebra $\nu_4$.

- Let $b_{n+1} = 0$. Since $x_n \notin Ann(\nu)$ one can conclude $b_n \neq 0$. By scaling the basis elements, we can assume $b_n = 1$ and the algebra $\nu_5(a_2, a_3, \ldots, a_{n-1})$ is obtained.

Below we give some remarks concerning algebras $\nu_1 - \nu_5$.

**Remark 3.18.**

1) Since the following single-generated Leibniz algebra:

\[
\begin{align*}
[x_i, x_1] &= x_{i+1}, & 1 \leq i \leq n - 2, \\
[x_{n-1}, x_1] &= x_n + \sum_{k=2}^{n-1} a_k x_k, \\
[x_n, x_1] &= x_n.
\end{align*}
\]

degenerates to the algebra $\nu_1(a_2, a_3, \ldots, a_{n-1})$ via family of transformations $g_2$, which are given as follows

\[
g_1(x_n) = tx_n, \quad g_i(x_i) = x_i, \ 1 \leq i \leq n - 1,
\]

we conclude $\nu_1(a_2, \ldots, a_{n-1}) \in X$.

2) Note that the algebra $\nu_2$ is the unique (up to isomorphism) solvable Leibniz algebra with null-filiform nilradical $[1]$. Due to work [1] the algebra $\nu_2$ is rigid.

3) The algebra $\nu_3$ is a solvable Leibniz algebra with nilradical $N = \langle x_2, x_3, \ldots, x_n \rangle$, which has the table of multiplication:

\[
[x_1, x_n] = \sum_{k=2}^{n-i} b_k x_{k+i-1}, \ 2 \leq i \leq n - 2.
\]

In particular, if $b_2 \neq 0$, then $N$ is a filiform algebra.

4) The algebra $\nu_4$ is a solvable Leibniz algebra with nilradical $N = \langle x_2, x_3, \ldots, x_n \rangle$, which is isomorphic to the direct sum of two-dimensional Leibniz algebra $[x_n, x_n] = x_{n-1}$ and $C^{n-3}$. It should be noted that algebra $N$ is one of the fourth algebras of level one $[11]$.

5) The algebra $\nu_5$ is a solvable Leibniz algebra with abelian nilradical $\langle x_2, x_3, \ldots, x_n \rangle$. 
3.3. On the description of Leibniz infinitesimal deformations of the algebra $\mathcal{F}_n^3$.

In this subsection give some additional information on Leibniz infinitesimal deformations of $\mathcal{F}_n^3(0)$. Recall, a Lie infinitesimal deformation $\varphi$ is defined as bilinear map which satisfies the equality \[ \varphi(x, y) = -\varphi(y, x) \] and skew-symmetric condition $\varphi(x, y) = -\varphi(y, x)$ [9].

Thanks to the works [18] and [10], where the infinitesimal deformations of the algebra $\mathcal{F}_n^3(0)$ in the varieties of nilpotent and all Lie algebras are described, respectively, it is sufficient to study the infinitesimal Leibniz deformations which do not satisfy skew-symmetric condition.

Denote by $\mathcal{ZL}^2(\mathcal{F}_n^3(0), \mathcal{F}_n^3(0))$ and $\mathcal{Z}^2(\mathcal{F}_n^3(0), \mathcal{F}_n^3(0))$ the space of Leibniz and Lie infinitesimal deformations, respectively.

Theorem 3.19. [10] If $\mathcal{Z}^2(\mathcal{F}_n^3(0), \mathcal{F}_n^3(0))$ is a vector space of the infinitesimal deformations of $\mathcal{F}_n^3(0)$ on the $n$-dimensional Lie algebra laws Lie$_n$, then

\[
\dim \mathcal{Z}^2(\mathcal{F}_n^3(0), \mathcal{F}_n^3(0)) = \begin{cases} 
8, & n = 3, \\
15, & n = 4, \\
\frac{(n-1)(3n-5)}{8} + n^2 - n - 1, & n \text{ odd and } n \geq 5, \\
\frac{n(3n-10)}{8} + n^2 - n + \lfloor \frac{n}{4} \rfloor, & n \text{ even and } n \geq 5.
\end{cases}
\]

In the next proposition we present a general form of non-Lie Leibniz infinitesimal deformations of the algebra $\mathcal{F}_n^3(0)$.

Proposition 3.20. If $\varphi \in \mathcal{ZL}^2(\mathcal{F}_n^3(0), \mathcal{F}_n^3(0))$, then

\[
\varphi(x_1, x_1) = \alpha x_n, \quad \varphi(x_1, x_2) + \varphi(x_2, x_1) = \beta x_n, \quad \varphi(x_2, x_2) = \gamma x_n,
\]

\[
\varphi(x_i, x_j) = -\varphi(x_j, x_i) \quad \text{for other } i \text{ and } j.
\]

Proof. Let $\varphi \in \mathcal{ZL}^2(\mathcal{F}_n^3(0), \mathcal{F}_n^3(0))$. We shall consider the equation (2.2) for elements $x_i, x_j, x_k$.

For $j = k = 1$ we have $[x_i, \varphi(x_1, x_1)] = 0$, consequently one can suppose

\[
\varphi(x_1, x_1) = \alpha x_n.
\]

Similarly, for $j = k = 2$ we obtain $[x_i, \varphi(x_2, x_2)] = 0$, which deduces

\[
\varphi(x_2, x_2) = \gamma x_n.
\]

For $i = j = 1$ and any $k$ with $2 \leq k \leq n - 1$ we conclude $\varphi(x_1, [x_1, x_k]) + \varphi([x_1, x_k], x_1) = 0$, which implies

\[
\varphi(x_1, x_{k+1}) = -\varphi(x_{k+1}, x_1), \quad 2 \leq k \leq n - 1. \tag{3.8}
\]

For $i = k = 1, j = 2$ we derive $[x_i, \varphi(x_2, x_1)] - [\varphi(x_1, x_2), x_1] = 0$ from which we can assume

\[
\varphi(x_1, x_2) + \varphi(x_2, x_1) = \beta_1 x_1 + \beta_n x_n. \tag{3.9}
\]

Considering the condition $(d^2 \varphi)(x_1, x_1, x_k) = 0$ with $2 \leq i \leq n, 2 \leq k \leq n - 1$ we have

\[
[x_i, \varphi(x_1, x_k)] - [\varphi(x_1, x_1), x_k] + [\varphi(x_1, x_k), x_1] - \varphi(x_1, x_{k+1}) - \varphi([x_i, x_1], x_k) = 0. \tag{3.10}
\]

Analogously, from $(d^2 \varphi)(x_1, x_k, x_k) = 0$ with $2 \leq j \leq n, 2 \leq k \leq n - 1$ we obtain

\[
[x_1, \varphi(x_j, x_k)] - [\varphi(x_1, x_j), x_k] + [\varphi(x_1, x_k), x_j] - \varphi([x_1, x_j], x_k) - \varphi(x_{k+1}, x_j) = 0. \tag{3.11}
\]

Combining the equalities (3.10) and (3.11) we derive

\[
\varphi(x_i, x_{k+1}) + \varphi(x_{k+1}, x_i) = [x_k, \varphi(x_1, x_1) + \varphi(x_1, x_1)], \quad 2 \leq i \leq n, 2 \leq k \leq n - 1.
\]

Thanks to above equality and (3.8)–(3.9) we obtain

\[
\varphi(x_2, x_{k+1}) + \varphi(x_{k+1}, x_2) = \beta_1 x_{k+1}, \quad \varphi(x_i, x_{k+1}) = -\varphi(x_{k+1}, x_i)
\]

with $3 \leq i \leq n, 2 \leq k \leq n - 1$.

The restriction $\beta_1 = 0$ follows from $(d^2 \varphi)(x_2, x_2, x_1) = 0$. \hfill \Box

Theorem 3.21. The cochains $\psi_1, \psi_2, \psi_3$ defined as

\[
\psi_1(x_1, x_1) = x_n, \quad \psi_2(x_1, x_2) = x_n, \quad \psi_3(x_2, x_2) = x_n
\]

complement the subspace $\mathcal{Z}^2(\mathcal{F}_n^3(0), \mathcal{F}_n^3(0))$ to the space $\mathcal{ZL}^2(\mathcal{F}_n^3(0), \mathcal{F}_n^3(0))$. 

Corollary 3.22.

\[ \dim ZL^2(F_n^3(0), F_n^3(0)) = \begin{cases} 11, & n = 3, \\ 18, & n = 4, \\ \frac{(n-1)(3n-5)}{8} + n^2 - n + 2, & n \text{ is odd and } n \geq 5, \\ \frac{n(3n-10)}{8} + n^2 - n + 3 + \left[ \frac{n}{4} \right], & n \text{ is even and } n \geq 5. \end{cases} \]

It is known that any derivation of algebra \( F_n^3(0) \) with \( n = 3 \) and \( n \geq 4 \) has the following matrix form [9]:

\[
\begin{pmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  0 & 0 & a_1 + b_2
\end{pmatrix},
\]

\[
\begin{pmatrix}
  \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{n-1} & \alpha_n \\
  0 & \beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{n-1} & \beta_n \\
  0 & 0 & \alpha_1 + \beta_2 & \beta_3 & \ldots & \beta_{n-2} & \beta_{n-1} \\
  0 & 0 & 0 & 2\alpha_1 + \beta_2 & \ldots & \beta_{n-3} & \beta_{n-2} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \ldots & 0 & (n-2)\alpha_1 + \beta_2
\end{pmatrix}.
\]

Therefore,

\[
\dim Der(F_n^3(0)) = \begin{cases} 6, & n = 3, \\ 2n - 1, & n \geq 4. \end{cases}
\]

\[
\dim BL^2(F_n^3(0), F_n^3(0)) = \begin{cases} 3, & n = 3, \\ (n-1)^2, & n \geq 4. \end{cases}
\]

Consequence of Theorem 3.19 is the following result.

Corollary 3.23.

\[
\dim HL^2(F_n^3(0), F_n^3(0)) = \begin{cases} 9, & n = 3, \\ 10, & n = 4, \\ \frac{(n-1)(3n-5)}{8} + n + 2, & n \text{ is odd and } n \geq 5, \\ \frac{n(3n-10)}{8} + n + 3 + \left[ \frac{n}{4} \right], & n \text{ is even and } n \geq 5. \end{cases}
\]

REFERENCES

[1] Ancochea Bermúdez J.M., Campoamor-Stursberg R. On a complete rigid Leibniz non-Lie algebra in arbitrary dimension. Linear Algebra and its Applications 438(8) (2013), p. 3397–3407.

[2] Ayupov Sh.A., Omirov B.A. On some classes of nilpotent Leibniz algebras, Siberian Math. J. 42(1) (2001), p. 18–29.

[3] Balavoine D. Déformations et rigidité géométrique des algèbres de Leibniz, Comm. Algebra 24 (1996), p. 1017–1034.

[4] Casas J. M., Ladra M., Omirov B.A., Karimjanov I.K. Classification of solvable Leibniz algebras with null-filiform nilradical. Linear and Multilinear Algebra 61(6) (2013), p. 758–774.

[5] Casas J. M., Ladra M., Omirov B.A., Karimjanov I.K. Classification of solvable Leibniz algebras with naturally graded filiform nilradical. Linear Algebra and its Applications. 438 (7) (2013), p. 2973–3000.

[6] Fialowski A., Millionschikov D.V. Cohomology of Graded Lie Algebras of Maximal Class, J. Algebra, 296 (2006), p. 157–176.

[7] Fialowski, A., Penkava, M. Formal deformations, contractions and moduli spaces of Lie algebras, Internat. J. Theoret. Phys. 47 (2008), p. 561–582.

[8] Gerstenhaber M. On the deformation of rings and algebras, I, III. Ann. of Math. (2) 79 (1964), p. 59–103; 88 (1968), p. 1–34.

[9] Goze M., Khakimdjanov Yu. Nilpotent Lie algebras, Kluwer Academic Publishers, Dordrecht, (1996) vol. 361, 336 pp.

[10] Khakimdjanov Yu., Navarro R.M. Deformations of filiform Lie algebras and superalgebras, J. Geom. Phys. 60 (2010), p. 1156–1169.

[11] Khudoyberdiyev A.Kh., Omirov B.A. The classification of algebras of level one. [arXiv:1301.5777]

[12] Khudoyberdiyev A.Kh., Omirov B.A., Infinitesimal deformations of null-filiform Leibniz superalgebras, [arXiv:1304.0120]

[13] Loday J.-L. Une version non commutative des algèbres de Lie: les algèbres de Leibniz, Enseign. Math. (2) 39 (1993), p. 269–293.

[14] Loday J.-L., Pirashvili T. Universal enveloping algebras of Leibniz algebras and (co)homology, Math. Ann. 296 (1993), p. 139–158.

[15] Millionschikov D. V. Deformations of filiform Lie algebras and symplectic structures, Proc. Steklov Inst. Math. 252 (1) (2006), p. 182–204.

[16] Nijenhuis A., Richardson R. W. Cohomology and deformations in graded Lie algebras, Bull. Amer. Math. Soc. 72 (1966), p. 1–29.
[17] Omirov B.A., Rakhimov I.S. *On Lie-like complex filiform Leibniz algebras*, Bull. Aust. Math. Soc. 79 (3) (2009), p. 391–404.

[18] Vergne M. *Cohomologie des algèbres de Lie nilpotentes. Application à l’étude de la variété des algèbres de Lie nilpotentes*. Bull. Soc. Math. France, 98 (1970), p. 81 - 116.

[A. Kh. Khudoyberdiyev and B. A. Omirov] Institute of Mathematics, National University of Uzbekistan, Tashkent, 100125, Uzbekistan.

E-mail address: khabror@mail.ru, omirovb@mail.ru