Fedosov and Riemannian supermanifolds

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Generalizations of symplectic and metric structures for supermanifolds are analyzed. Two types of structures are possible according to the even/odd character of the corresponding quadratic tensors. In the even case one has a very rich set of geometric structures: even symplectic supermanifolds (or, equivalently, supermanifolds with non-degenerate Poisson structures), even Fedosov supermanifolds and even Riemannian supermanifolds. The existence of relations among those structures is analyzed in some details. In the odd case, we show that odd Riemannian and Fedosov supermanifolds are characterized by a scalar curvature tensor. However, odd Riemannian supermanifolds can only have constant curvature.

1 Introduction

The formulation of fundamental physical theories, either classical or quantum, in terms of differential geometric methods is now well established and presents many conceptual advantages. Probably, the most prominent examples are the formulation of general relativity on Riemannian manifolds and the geometric formulation of gauge field theories of the fundamental forces on fiber bundles. Another essential connection was opened by the formulation of classical mechanics (see, for example, [1]) – and also classical field theories – on symplectic manifolds. The properties of these manifolds are now well understood.

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Deformation quantization \[2\] was formulated in terms of symplectic manifolds with a symmetric connection compatible with the given symplectic structure (the so-called Fedosov manifolds \[3\]). The geometrical formulation of supersymmetric field theories and the quantization of general gauge theories introduced a number of applications of differential geometry based on the concept of supermanifold, first introduced and studied by Berezin \[4\] (see also \[5, 6\]). In these cases, supermanifolds appear equipped with appropriate symplectic structures or (and) symmetric connections. On the other hand the geometrical formulation of the Batalin-Vilkovisky quantization \[7\] is based on the so-called antisymplectic supermanifolds which are supermanifolds equipped with an antibracket \[8\]. Finally, in some approaches to modern gauge field theory \[9\], flat even Fedosov supermanifolds (in the terminology adopted here) have been used. In summary, the geometry of manifolds and supermanifolds percolates all fundamental physical theories.

The paper addresses the study of possible extensions of symplectic and metric structures to supermanifolds by means of graded symmetric and antisymmetric second-order tensor fields of types \((2, 0)\) and \((0, 2)\). In particular, we analyze the cases of even and odd symplectic supermanifolds and even and odd Riemannian supermanifolds. Symplectic supermanifolds coincide with graded non-degenerate Poisson supermanifolds. If, in addition, a graded symplectic supermanifold is equipped with a symmetric connection compatible with the symplectic structure, it becomes a (even or odd) Fedosov supermanifold. In the even case it can be considered as a straightforward generalization of Fedosov manifold \[3\]. The scalar curvature tensor for any such a Fedosov supermanifold vanishes, as for standard Fedosov manifolds. We prove that a graded metric supermanifold with a compatible symmetric connection also leads to a (even or odd) Riemannian supermanifold with a unique torsionless connection. The scalar curvature tensor is, in general, non trivial for both odd Riemannian and odd Fedosov supermanifold.

The paper is organized as follows. In Sect. 2, we study the basic tensor field operations on supermanifolds: multiplication, contraction and symmetry properties. In Sect. 3, we consider scalar structures which can be used for the construction of symplectic and metric supermanifolds. The properties of symmetric affine connections on supermanifolds and their curvature tensors are analyzed in Sect. 4. In Sect. 5, we introduce the concepts of even and odd Fedosov supermanifolds and even and odd Riemannian supermanifolds are analyzed in Sect. 6. In Sect. 7, we summarize the main results.

We use the condensed notation suggested by DeWitt \[10\] and definitions and notations adopted in \[11\]. Derivatives with respect to the coordinates \(x^i\) are understood as acting from the left and are denoted by \(\partial A/\partial x^i\). Right derivatives with respect to \(x^i\) are labeled by the subscript “\(r\)” or by \(A_{,i} = \partial_r A/\partial x^i\). The Grassmann parity of any quantity \(A\) is
denoted by $\epsilon(A)$.

2 Tensor fields

Let $\mathcal{M}$ be a supermanifold with dimension $\dim \mathcal{M} = N$ and let $T_p\mathcal{M}$ and $T^*_p\mathcal{M}$ be the tangent and cotangent spaces at a point $p \in \mathcal{M}$ respectively. We assume that each element $X \in T_p\mathcal{M}$ has a certain Grassmann parity $\epsilon(X)$ and each element $\omega \in T^*_p\mathcal{M}$ has a dual Grassmann parity $\epsilon(\omega)$. Consider the Cartesian product $\Pi^m_n$ of $T_p\mathcal{M}$ and $T^*_p\mathcal{M}$:

$$\Pi^m_n = T^*_p \times \cdots \times T^*_p \times T_p \times \cdots \times T_p.$$  

Let $T$ be a map $T : \Pi^m_n \to \Lambda$, that maps each element $(\omega^1, \ldots, \omega^n, X_1, \ldots, X_m) \in \Pi^m_n$ into a certain supernumber $T(\omega^1, \ldots, \omega^n, X_1, \ldots, X_m) \in \Lambda$, where $\Lambda$ is a Berezin algebra, such that $\epsilon(T(\omega^1, \ldots, \omega^n, X_1, \ldots, X_m)) = \epsilon(T) + \epsilon(\omega^1) + \ldots + \epsilon(\omega^n) + \epsilon(X_1) + \ldots + \epsilon(X_m)$. $T$ is an even map if $\epsilon(T) = 0$ or an odd map if $\epsilon(T) = 1$. A map $T$ is a tensor of type $(n,m)$ and rank $n + m$ at a point $p$, if for all $\omega, \sigma \in T^*_p\mathcal{M}$, all $X, Y \in T_p\mathcal{M}$ and all $\alpha \in \Lambda$, it satisfies the multilinear laws:

$$T(...\omega + \sigma...) = T(...\omega...) + T(...\sigma...),$$
$$T(...X + Y...) = T(...X...) + T(...Y...),$$
$$T(...\omega\alpha, \sigma...) = T(...\omega, \alpha\sigma...),$$
$$T(...\omega\alpha, X...) = T(...\omega, \alpha X...),$$
$$T(...X\alpha, Y...) = T(...X, \alpha Y...),$$
$$T(...X\alpha) = T(...X)\alpha.$$  

Let the variables $\{x^i\}, \epsilon(x^i) = \epsilon_i$ be local coordinates on $\mathcal{M}$ in the vicinity of a point $p \in \mathcal{M}$ and $\{e_i\}$ and $\{e^i\}$ be the corresponding coordinate bases in the tangent space $T_p\mathcal{M}$ and the cotangent space $T^*_p\mathcal{M}$, respectively. Under the change of of local coordinates $\bar{x}^i = \bar{x}^i(x)$ the basis vectors in $T_p\mathcal{M}$ and $T^*_p\mathcal{M}$ transform as

$$\bar{e}_i = e_j \frac{\partial x^j}{\partial \bar{x}^i}, \quad \bar{e}^i = e^j \frac{\partial x^i}{\partial \bar{x}^j}.$$  

The transformation matrices satisfy the following relations:

$$\frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^k}{\partial \bar{x}^j} = \delta^i_j, \quad \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^i}{\partial \bar{x}^j} = \delta_j^i, \quad \frac{\partial x^i}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^j} = \delta_i^j, \quad \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^i} = \delta_i^j.$$  

3
A tensor $T$ can be described in local components with respect to the chosen bases \{\epsilon^i\}, \{e_i\):

$$T^i_{j_1...j_m} = T(e^{i_1}, ..., e^{i_n}, e_{j_1}, ..., e_{j_m}),$$
$$\epsilon(T^i_{j_1...j_m}) = \epsilon(T) + \epsilon_{i_1} + \cdots + \epsilon_{i_n} + \epsilon_{j_1} + \cdots + \epsilon_{j_m}.$$ (5)

Then a tensor field of type $(n, m)$ and rank $n + m$ is by definition a geometric object that can be given by a set of functions with $n$ upper and $m$ lower indices in each local coordinate system $(x) = (x^1, ..., x^N)$ with the following transformation laws under a change of coordinates $x \to \bar{x}$

$$\bar{T}^i_{j_1...j_m} = T^{i_1...i_n}_{k_1...k_m} \frac{\partial \bar{x}^{k_m}}{\partial x^{k_m}} \cdots \frac{\partial \bar{x}^{k_1}}{\partial x^{k_1}} \frac{\partial \bar{x}^{i_n}}{\partial x^{i_n}} \cdots \frac{\partial \bar{x}^{i_1}}{\partial x^{i_1}} (-1)^{\epsilon_{i_1}+\epsilon_{i_n}} P^{i_1...i_n}_{j_1...j_m}.$$ (6)

where

$$P^{i_1...i_n}_{j_1...j_m} = \sum_{s=1}^{n} \sum_{p=1}^{m} \epsilon_{jp}(\epsilon_{is} + \epsilon_{ls}) + \sum_{s=1}^{n-1} \sum_{p=s+1}^{m} \epsilon_{ip}(\epsilon_{is} + \epsilon_{ls}) + \sum_{s=1}^{m-1} \sum_{p=s+1}^{m} \epsilon_{jp}(\epsilon_{js} + \epsilon_{ks}).$$ (7)

In the simplest case, the relations for vector fields $T^i$ and co-vector fields $T_i$ have the form

$$\bar{T}^{i} = T^{i} \frac{\partial \bar{x}^{j}}{\partial x^{j}}, \quad \bar{T}_{i} = T_{n} \frac{\partial x^{j}}{\partial \bar{x}^{j}}$$ (8)

and for second-rank tensor fields of different types Eq. (6) become

$$\bar{T}^{ij} = T^{mn} \frac{\partial \bar{x}^{j}}{\partial x^{n}} \frac{\partial \bar{x}^{i}}{\partial x^{m}} (-1)^{\epsilon_{j}+\epsilon_{m}},$$ (9)
$$\bar{T}_{ij} = T_{mn} \frac{\partial x^{j}}{\partial \bar{x}^{j}} \frac{\partial x^{m}}{\partial \bar{x}^{i}} (-1)^{\epsilon_{j}+\epsilon_{m}},$$ (10)
$$\bar{T}^{i}_{j} = T^{mn} \frac{\partial x^{j}}{\partial \bar{x}^{j}} \frac{\partial \bar{x}^{i}}{\partial x^{m}} (-1)^{\epsilon_{j}+\epsilon_{m}}.$$ (11)

In definitions (3), (6) we follow the rule that places all transformation matrices to the right. In particular, it means that we adopted the following representation of elements in the tangent and cotangent spaces

$$X = X^{i} \epsilon_{i} (-1)^{\epsilon_{i}}, \quad \omega = \omega_{i} \epsilon^{i}.$$ (12)

Note that the unit matrix $\delta_{j}^{i}$ is related to the unit tensor field $E_{j}^{i}$ which transforms according to the law (11) as

$$E_{j}^{i} = \delta_{j}^{i}.$$ (13)

From a tensor field of type $(n, m)$ and rank $n + m$ with $n \neq 0$, $m \neq 0$, one can construct a tensor field of type $(n-1, m-1)$ and rank $n + m - 2$ by contracting an upper with a
lower index by the rules (for details, see [11]). In particular, for tensor fields of type \((1, 1)\), the contraction gives the supertrace,

\[
\text{str} T = T^i_i (-1)^{\epsilon_i}.
\] (14)

Using the multiplication operation, from two tensor fields of types \((n, 0)\) and \((0, m)\), one can construct new tensor fields of type \((n - 1, m - 1)\). In particular, from a vector \(U^i\) and a covector \(V_i\) field one can built a scalar field

\[
(-1)^{\epsilon_i(\epsilon(V) + 1)} U^i V_i = (-1)^{\epsilon(U)\epsilon(V) + \epsilon_i\epsilon(U)} V_i U^i,
\] (15)

which is invariant with respect to any choice of local coordinates. Two second-rank tensor fields \(U^{ij}\) and \(V_{ij}\) yield tensor fields

\[
(-1)^{\epsilon_i + \epsilon_k} U^{ik} V_{kj}, \quad (-1)^{\epsilon_i + \epsilon_k} U^{ki} V_{jk},
\] (16)

transforming according \([11]\). The four fields are not independent, in fact there are only two independent ones due to the relations

\[
(-1)^{\epsilon_i + \epsilon_k} U^{ik} V_{kj} = (-1)^{\epsilon(U)\epsilon(V) + \epsilon_i + \epsilon_k} U^{ik} V_{kj},
\] (17)

\[
(-1)^{\epsilon_i + \epsilon_k} U^{ki} V_{jk} = (-1)^{\epsilon(U)\epsilon(V) + \epsilon_i + \epsilon_k} U^{ki} V_{jk}.
\] (18)

Further contractions of indices yield the scalar field

\[
(-1)^{(\epsilon_i + \epsilon_k)\epsilon(V) + 1} U^{ik} V_{ki} = (-1)^{\epsilon(U)\epsilon(V) + \epsilon_i + \epsilon_k} V_{ik} U^{ki}.
\] (19)

Later on we shall construct tensor fields from tensor fields of type \((1, 2)\) and vector or co-vector tensor fields. The corresponding rules

\[
U^i V_{kj} (-1)^{(\epsilon(V) + 1)\epsilon_i}, \quad U^i V_{ij} (-1)^{\epsilon(V)\epsilon_i}
\] (20)

give \((1, 1)\) and \((0, 2)\) tensor fields transforming in accordance with \([11]\) and \([10]\) respectively.

Now recalling \([13]\), \([16]\), \([17]\), \([18]\) and \([19]\), the inverse tensor field \(T_{ij}\) for a non-degenerate second-rank tensor field \(T^{ij}\) of type \((2, 0)\) should be defined via the relations

\[
(-1)^{(\epsilon_i + \epsilon_k)\epsilon(T) + \epsilon_k} T^{ik} T_{kj} = \delta^i_j,
\] (21)

\[
(-1)^{(\epsilon_j + \epsilon_k)\epsilon(T) + \epsilon_j} T_{jk} T^{ki} = \delta^j_i,
\] (22)

\[
\epsilon(T_{ij}) = \epsilon(T^{ij}) = \epsilon(T) + \epsilon_i + \epsilon_j.
\]
and similarly for tensor fields of type (0, 2). In particular, in the even case (\(\epsilon(T) = 0\)) from (21) and (22) it follows

\[
(-1)^{\epsilon_k} T^{ik} T_{kj} = \delta^i_j, \quad (23)
\]

\[
(-1)^{\epsilon_j} T_{jk} T^{ki} = \delta^i_j, \quad (24)
\]

and, in its turn, in the odd case (\(\epsilon(T) = 1\)) we have

\[
(-1)^{\epsilon_i} T^{ik} T_{kj} = \delta^j_i, \quad (25)
\]

\[
(-1)^{\epsilon_k} T_{jk} T^{ki} = \delta^j_i. \quad (26)
\]

Notice that the definitions (21) and (22) are in agreement with the relation (19). Indeed, contracting indices in (21) to obtain a scalar (see (14)) we have

\[
(-1)^{(\epsilon_i + \epsilon_k)(\epsilon(T) + 1)} T^{ik} T_{ki} = \delta^i_i(-1)^{\epsilon_i} = (-1)^{(\epsilon_i + \epsilon_k) + \epsilon(T)} T^{ik} T_{ki}
\]

and doing the same in (22) we find

\[
(-1)^{(\epsilon_i + \epsilon_k)\epsilon(T)} T^{ki} = \delta^i_i(-1)^{\epsilon_i} = (-1)^{(\epsilon_i + \epsilon_k)(\epsilon(T) + 1) + \epsilon(T)} T^{ik} T_{ki},
\]

that proves our statement. Notice that in contrast with previous definitions of inverse tensor fields used in [11], the definitions (21) and (22) lead to coincidence of left and right inverse tensor fields to a given one.

It is well known that in the tensor calculus on manifolds, an important role is played by symmetric and antisymmetric tensor fields. In the supersymmetric case, supermatrices have more possible symmetry properties (eight types, see, for example, [12]), and a natural question is whether these properties are compatible with the tensor transformation laws. Among the eight types of supermatrices with possible symmetry properties there exist only two which satisfy the tensor transformation laws. In our definition of tensor fields on supermanifolds, only the supermatrices having the generalized symmetry or antisymmetry properties satisfy the tensor transformation rules. Indeed, let us consider a second-rank supermatrix of type (2, 0) with the generalized symmetry (antisymmetry)

\[
T^{ij} = (-1)^{\epsilon_i\epsilon_j} T^{ji} \quad (T^{ij} = -(-1)^{\epsilon_i\epsilon_j} T^{ji}). \quad (27)
\]

The symmetry is invariant under the transformation law (9),

\[
\tilde{T}^{ij} = T^{mn} \frac{\partial \tilde{x}^j}{\partial x^m} \frac{\partial \tilde{x}^i}{\partial x^n} (-1)^{\epsilon_j(\epsilon_i + \epsilon_m)} = T^{mn} \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial \tilde{x}^j}{\partial x^n} (-1)^{\epsilon_i\epsilon_n} = (-1)^{\epsilon_i\epsilon_j} \tilde{T}^{ji}
\]

and similarly for antisymmetric tensor fields. The other six possible symmetry types of supermatrices do not survive verification of the compatibility with adopted tensor transformation laws. It is very important to note that for non-degenerate symmetric and
antisymmetric tensor fields, their inverse tensor fields also have the necessary symmetry properties. For example, if we consider a second-rank tensor field $T^{ij}$ with symmetric (antisymmetric) properties ($\epsilon(T^{ij}) = \epsilon(T) + \epsilon_i + \epsilon_j$), from definitions (21) and (22) we can then find that the inverse tensor field also satisfies

$$T_{ij} = (-1)^{\epsilon(T)+\epsilon_i+\epsilon_j} T_{ji}$$

with symmetric (antisymmetric) properties, from definitions (21), and we can then find that the inverse tensor field also satisfies

$$T_{ij} = (-1)^{\epsilon(T)+\epsilon_i+\epsilon_j} T_{ji}$$

The results mean that for any even symmetric (antisymmetric) tensor field the inverse tensor field is also symmetric (antisymmetric), while for any odd symmetric (antisymmetric) tensor field the inverse tensor field is antisymmetric (symmetric). In what follows, we shall use symmetric and antisymmetric tensor fields only to construct scalar invariant fields defined on supermanifolds.

## 3 Scalar Fields

Let us analyze the most relevant scalar structures on supermanifolds which can be defined in terms of graded symmetric and antisymmetric tensor fields.

In general, there exist eight types of second rank tensor fields with the required symmetry properties

$$\omega^{ij} = (-1)^{\epsilon_i+\epsilon_j} \omega^{ji}, \quad \epsilon(\omega^{ij}) = \epsilon(\omega) + \epsilon_i + \epsilon_j,$$

$$\Omega^{ij} = (-1)^{\epsilon_i+\epsilon_j} \Omega^{ji}, \quad \epsilon(\Omega^{ij}) = \epsilon(\Omega) + \epsilon_i + \epsilon_j,$$

$$E_{ij} = (-1)^{\epsilon_i+\epsilon_j} E_{ji}, \quad \epsilon(E_{ij}) = \epsilon(E) + \epsilon_i + \epsilon_j,$$

$$g_{ij} = (-1)^{\epsilon_i+\epsilon_j} g_{ji}, \quad \epsilon(g_{ij}) = \epsilon(g) + \epsilon_i + \epsilon_j.$$  

Using these tensor fields (29)-(32) it is not difficult to built eight scalar structures on a supermanifold:

$$\{A, B\} = \frac{\partial A}{\partial x^i} (-1)^{\epsilon(\omega)} \omega^{ij} \frac{\partial B}{\partial x^j}, \quad \epsilon(\{A, B\}) = \epsilon(\omega) + \epsilon(A) + \epsilon(B),$$

$$(A, B) = \frac{\partial A}{\partial x^i} (-1)^{\epsilon(\Omega)} \Omega^{ij} \frac{\partial B}{\partial x^j}, \quad \epsilon((A, B)) = \epsilon(\Omega) + \epsilon(A) + \epsilon(B),$$

$$E = E_{ij} dx^j \wedge dx^i, \quad \epsilon(E) = \epsilon(E_{ij} dx^j \wedge dx^i),$$

$$g = g_{ij} dx^j \wedge dx^i, \quad \epsilon(g) = \epsilon(g_{ij} dx^j \wedge dx^i).$$

where $A$ and $B$ are any superfunctions.

The bilinear operation $\{A, B\}$ (33) obeys the following symmetry property

$$\{A, B\} = -(-1)^{\epsilon(\omega)+(\epsilon(A)+\epsilon(\omega))(\epsilon(B)+\epsilon(\omega))} \{B, A\}$$  

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which in the even case \((\epsilon(\omega) = 0)\) reduces to
\[
\{ A, B \} = -(-1)^{\epsilon(A)\epsilon(B)} \{ B, A \}
\] (38)
and in the odd case \((\epsilon(\omega) = 1)\) to
\[
\{ A, B \} = (-1)^{(\epsilon(A)+1)(\epsilon(B)+1)} \{ B, A \}.
\] (39)

On the other hand, the bilinear operation \((A, B)\) has the symmetry property
\[
(A, B) = (-1)^{(\epsilon(A)+\epsilon(\omega))(\epsilon(B)+\epsilon(\omega))}(B, A)
\] (40)
which in the even case \((\epsilon(\omega) = 0)\) reduces to
\[
(A, B) = (-1)^{\epsilon(A)\epsilon(B)}(B, A)
\] (41)
and in the odd case \((\epsilon(\omega) = 1)\) to
\[
(A, B) = -(-1)^{(\epsilon(A)+1))}(\epsilon(B)+1)(B, A).
\] (42)

Let us now try verify the Jacobi identity for both operations \(\{ A, B \}\) and \((A, B)\). Indeed, we have
\[
\{ A, \{ B, C \} \}(1(-1)^{(\epsilon(A)+\epsilon(\omega))(\epsilon(C)+\epsilon(\omega))} \text{ + cyclic perms. } (A, B, C)
= (1 - (-1)^{(\epsilon(\omega))}(\partial_i A \omega^{ij} \frac{\partial^2 B}{\partial x^k \partial x^j} \omega^{kl} \frac{\partial C}{\partial x^l} (-1)^{(\epsilon_i + \epsilon_k)\epsilon(\omega) + \epsilon_j (\epsilon(B)+\epsilon_k+1) + (\epsilon(A)+\epsilon(\omega))(\epsilon(C)+\epsilon(\omega)))
+ \text{cyclic perms. } (A, B, C) + (\lambda^{ij} \frac{\partial \omega^{kl}}{\partial x^j} (-1)^{(\epsilon_i + \epsilon(\omega))} + \text{cyclic perms. } (i, k, l))
(\epsilon(C)(\epsilon(A)+\epsilon_i+\epsilon_k)+(\epsilon(A)+\epsilon(B)+\epsilon(C)+1)\epsilon(\omega)+\epsilon_i (\epsilon(B)+\epsilon_k)+\epsilon_k \epsilon_i.

We see that in the even case \((\epsilon(\omega) = 0)\) the bilinear operation \(\{ A, B \}\) satisfies the Jacobi identity
\[
\{ A, \{ B, C \} \}(-1)^{\epsilon(A)\epsilon(C)} + \text{cyclic perms. } (A, B, C) \equiv 0
\] (43)
if and only if \(\omega\) satisfies
\[
\omega^{ij} \frac{\partial \omega^{kl}}{\partial x^j} (-1)^{\epsilon_i \epsilon_l} + \text{cyclic perms. } (i, k, l) \equiv 0.
\] (44)
In the odd case there is no possibility of satisfying the Jacobi identity.
The product \((A, B)\) associated to \(\Omega\) can also satisfy the Jacobi identity because

\[
\begin{align*}
(A, (B, C))(-1)^{(\epsilon(A)+\epsilon(\Omega))(\epsilon(C)+\epsilon(\Omega))} + \text{cyclic perms. } (A, B, C) \\
= (1 + (-1)^{\epsilon(\Omega)}) \left( \frac{\partial A}{\partial x^i} \Omega^{ij} \frac{\partial^2 B}{\partial x^k \partial x^j} \Omega^{kl} \frac{\partial C}{\partial x^l} (-1)^{(\epsilon_i+\epsilon_k)(\epsilon(A)+\epsilon(B)+\epsilon(C)+1)} + \text{cyclic perms. } (A, B, C) \right) \\
+ (\Omega^{ij} \frac{\partial \Omega^{kl}}{\partial x^j} (-1)^{\epsilon_i+\epsilon(\Omega)}) + \text{cyclic perms. } (i, k, l) \\
+ \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^k} \frac{\partial C}{\partial x^l} \left( -1 \right)^{\epsilon_i+\epsilon(\Omega)} + \text{cyclic perms. } (A, B, C) \\
+ \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^k} \frac{\partial C}{\partial x^l} \left( -1 \right)^{\epsilon_i+\epsilon(\Omega)} + \text{cyclic perms. } (A, B, C).
\end{align*}
\]

In the odd case \((\epsilon(\Omega) = 1)\) the Jacobi’s identity can be satisfied

\[
(A, (B, C))(-1)^{\epsilon(A)+\epsilon(\Omega)} + \text{cyclic perms. } (A, B, C) \equiv 0 \quad (45)
\]

if and only if \(\Omega\) satisfies

\[
\Omega^{ij} \frac{\partial \Omega^{kl}}{\partial x^j} (-1)^{\epsilon_i+\epsilon(\Omega)} + \text{cyclic perms. } (i, k, l) \equiv 0. \quad (46)
\]

Therefore, when identities \((44)\) and \((46)\) hold, one can identify \(\{A, B\} \ (\epsilon(\{A, B\}) = \epsilon(A) + \epsilon(B))\) and \((A, B) \ (\epsilon((A, B)) = \epsilon(A) + \epsilon(B) + 1)\) with the Poisson bracket and the antibracket respectively.

It is also possible to combine the Poisson bracket associated to \(\omega\) and the antibracket into the so-called graded Poisson bracket (see, for example, \([13, 14, 15, 16]\)). in the following bilinear operation

\[
\{A, B\}_g = \frac{\partial A}{\partial x^i} (-1)^{\epsilon_i(\omega)} \omega^{ij} \frac{\partial B}{\partial x^j}, \quad \epsilon(\{A, B\}_g) = \epsilon(\omega) + \epsilon(A) + \epsilon(B). \quad (47)
\]

From \((47)\) it follows the following symmetry property

\[
\{A, B\}_g = \{B, A\}_g. \quad (48)
\]

If additionally tensor fields \(\omega^{ij}\) satisfy the identities

\[
\omega^{ij} \frac{\partial \omega^{kl}}{\partial x^j} (-1)^{\epsilon_i(\omega)} + \text{cyclic perms. } (i, k, l) \equiv 0, \quad (49)
\]

then \(\{A, B\}_g\) satisfies the Jacobi identity

\[
\{A, \{B, C\}_g\}_g + \text{cyclic perms. } (A, B, C) \equiv 0 \quad (50)
\]

and plays a role of a graded Poisson bracket.

A supermanifold \(\mathcal{M}\) equipped with a Poisson bracket is called a Poisson supermanifold, \((\mathcal{M}, \{,\})\). Usually a manifold \(\mathcal{M}\) equipped with an non-degenerate antibracket is called
an antisymplectic supermanifold \((\mathcal{M}, (\cdot, \cdot))\) or, sometimes, an odd Poisson supermanifold (see, for example, [15, 16]).

In Eq. (31) \(E\) is any graded differential 2-form. If \(E\) is closed

\[
dE = E_{ij,k} dx^k \wedge dx^j \wedge dx^i = 0
\]

and non-degenerate, then it defines a graded (even or odd) symplectic supermanifold \((\mathcal{M}, E)\) [5]. In terms of tensor fields \(E_{ij}\) the closure condition (51) can be expressed as

\[
E_{ij,k}(-1)^{\epsilon_{i}\epsilon_{k}} + \text{cyclic perms.} (i,j,k) = 0, \quad E_{ij} = -(-1)^{\epsilon_{i}\epsilon_{j}} E_{ji}
\]

and in terms of inverse tensor fields \(E^{ij}\) Eqs. (52) can be rewritten in the form

\[
E^{il} \partial E^{jk} \frac{\partial}{\partial x^l}(-1)^{\epsilon_{i}(\epsilon_{k} + \epsilon(E))} + \text{cyclic perms.} (i,j,k) = 0, \quad E^{ij} = -(-1)^{\epsilon(E) + \epsilon_{i}\epsilon_{j}} E^{ji}.
\]

Identifying \(E^{ij}\) with tensor fields \(\omega^{ij}\) in (33), we obtain in the even case \(\epsilon(E) = 0\) the Poisson bracket for which the Jacobi identity (43) follows from (53). Therefore, in the even case there is one-to-one correspondence between non-degenerate Poisson supermanifolds and an even symplectic supermanifolds. In the odd case \(\epsilon(E) = 1\), if we assume \(E^{ij} = \Omega^{ij}\) in (34) then \(E^{ij}\) defines an antibracket for which the Jacobi identity (46) follows from (53). Therefore antisymplectic supermanifolds can be identified with odd symplectic manifolds.

If the tensor field \(g_{ij}\) in (36) is non-degenerate, one has a graded metric that can provide the supermanifold \(\mathcal{M}\) with a graded (even or odd) metric structure, giving rise to a Riemannian supermanifold \((\mathcal{M}, g)\). On the other hand, the inverse tensor field \(g^{ij}\) also defines a bilinear operation with symmetry properties (39) or (41) but it does not satisfy the Jacobi identity.

4 Connections in Supermanifolds

Let us introduce a covariant derivative \(\nabla\) (or an affine connection \(\Gamma\)) on a supermanifold \(\mathcal{M}\). In each local coordinate system \(\{x\}\) the covariant derivative \(\nabla\) is described by its components \(\nabla_i (\epsilon(\nabla_i) = \epsilon_i)\), which are related to the components the affine connection \(\Gamma_{jk}^i, (\epsilon(\Gamma_{jk}^i) = \epsilon_i + \epsilon_j + \epsilon_k)\) by

\[
e^i \nabla_j = e^k \Gamma_{kj}^i (-1)^{\epsilon_{k}(\epsilon_{i}+1)}, \quad e_i \nabla_j = -e_k \Gamma_{kj}^i
\]

where \(\{e_i\}\) and \(\{e^i\}\) are the associated bases of the tangent \(T\mathcal{M}\) and cotangent \(T^*\mathcal{M}\) spaces respectively. The choice of factors in (54) is dictated by the rules (20). From
It follows that the action of covariant derivative on scalar, vector and co-vector tensor fields is

\[ T \nabla_i = T_i \quad (55) \]

\[ T^i \nabla_j = T^i_j + T^k \Gamma^i_{kj} (-1)^{\epsilon_k (\epsilon_i + 1)} \quad (56) \]

\[ T_i \nabla_j = T_{ij} - T_k \Gamma^k_{ij} \quad (57) \]

and on second-rank tensor fields of type (2,0), (0,2) and (1,1)

\[ T^{ij} \nabla_k = T^{ij}_{,k} + T^{il} \Gamma^{i}_{lk} (-1)^{\epsilon_l (\epsilon_i + 1)} + T^{lj} \Gamma^{l}_{ik} (-1)^{\epsilon_i (\epsilon_l + \epsilon_j + 1)} \quad (58) \]

\[ T_{ij} \nabla_k = T_{ij,k} - T_{il} \Gamma^{l}_{jk} - T_{lj} \Gamma^{l}_{ik} (-1)^{\epsilon_i (\epsilon_l + \epsilon_j) + \epsilon_l (\epsilon_i + \epsilon_j)} \quad (59) \]

\[ T^i_j \nabla_k = T^i_{jk} + T^l_i \Gamma^l_{jk} + T^l_j \Gamma^l_{ik} (-1)^{\epsilon_i (\epsilon_l + \epsilon_j) + \epsilon_l (\epsilon_i + \epsilon_j + 1)} \quad (60) \]

Similarly, the action of the covariant derivative on a tensor field of any rank and type is given in terms of the tensor components, the ordinary derivatives and the connection components.

The components of the affine connection do not transform as components of a mixed tensor

\[ \bar{\Gamma}^i_{jk} = (-1)^{\epsilon_n (\epsilon_m + \epsilon_j)} \frac{\partial}{\partial x^i} \bar{x}^m \frac{\partial}{\partial x^j} \Gamma^{m}{}_{mn} \frac{\partial}{\partial x^k} + \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} \bar{x}^n. \quad (61) \]

In general, the connection components \( \Gamma^i_{jk} \) are not (generalized) symmetric w.r.t. the lower indices. The obstruction to this symmetry is given by the torsion,

\[ T^i_{jk} := \Gamma^i_{jk} - (-1)^{\epsilon_j \epsilon_k} \Gamma^i_{kj}, \quad (62) \]

which also transforms as a tensor field. A connection \( \nabla \) is torsionless if \( T^i_{jk} = 0 \), i.e. if obeys the relation

\[ \Gamma^i_{jk} = (-1)^{\epsilon_j \epsilon_k} \Gamma^i_{kj}. \quad (63) \]

Here on, we shall consider only symmetric connections.

The curvature tensor field \( R^i_{mjk} \) is defined in a coordinate basis in terms of the commutator of covariant derivatives, \( [\nabla_i, \nabla_j] = \nabla_i \nabla_j - (-1)^{\epsilon_i \epsilon_j} \nabla_j \nabla_i \), whose action on a vector field \( T^i \) is

\[ T^i [\nabla_j, \nabla_k] = -(-1)^{\epsilon_m (\epsilon_i + 1)} T^m R^i_{mjk}. \quad (64) \]

The choice of factor in r.h.s \( (64) \) is dictated by the requirement that the contraction of tensor fields of types (1,0) and (1,3) yield a tensor field of type (1,2). A straightforward calculation yields

\[ R^i_{mjk} = -\Gamma^i_{mj,k} + \Gamma^i_{mk,j} (-1)^{\epsilon_j \epsilon_k} + \Gamma^i_{jn} \Gamma^m_{nk} (-1)^{\epsilon_j \epsilon_m} - \Gamma^i_{kn} \Gamma^m_{nj} (-1)^{\epsilon_k (\epsilon_m + \epsilon_j)}. \quad (65) \]
The curvature tensor field is generalized antisymmetric,
\[ R^i_{mjk} = -(-1)^{\epsilon_j \epsilon_k} R^i_{mkj} ; \]  
(66)
and satisfies the Jacobi identity,
\[ (-1)^{\epsilon_m \epsilon_k} R^i_{mjk} + (-1)^{\epsilon_j \epsilon_m} R^i_{jkm} + (-1)^{\epsilon_k \epsilon_j} R^i_{kmj} \equiv 0 . \]  
(67)
Using the Jacobi identity for the covariant derivatives,
\[ [\nabla_i , [\nabla_j, \nabla_k]](-1)^{\epsilon_i \epsilon_k} + [\nabla_k , [\nabla_i, \nabla_j]](-1)^{\epsilon_k \epsilon_j} + [\nabla_j , [\nabla_k, \nabla_i]](-1)^{\epsilon_j \epsilon_i} \equiv 0 , \]  
(68)
one obtains the Bianchi identity,
\[ (-1)^{\epsilon_i \epsilon_j} R^n_{mjk;i} + (-1)^{\epsilon_i \epsilon_k} R^n_{mi;j;k} + (-1)^{\epsilon_k \epsilon_j} R^n_{mk;i,j} \equiv 0 , \]  
(69)
with the notation \( R^n_{mjk;i} := R^n_{mjk} \nabla_i \).

5 Fedosov supermanifolds

Let us consider a symplectic supermanifold \((\mathcal{M}, \omega)\), i.e. a supermanifold \(\mathcal{M}\) with a closed non-degenerate graded differential 2-form \(\omega\). A symmetric connection \(\Gamma\) (covariant derivative \(\nabla\)) with components \(\Gamma^i_{jk}\) in each local coordinate system \(\{x^i\}\). \(\Gamma\) is compatible with the symplectic structure \(\omega\) if \(\omega \nabla = 0\). In local coordinates the compatibility condition is
\[ \omega_{ij} \nabla_k = \omega_{ij,k} - \Gamma_{ijk}(-1)^{\epsilon_i \epsilon_j} = 0 , \quad \omega_{ij} = -(-1)^{\epsilon_i \epsilon_j} \omega_{ji} \]  
(70)
where we use the notation
\[ \Gamma_{ijk} = \omega_{in} \Gamma^n_{jk} , \quad \epsilon(\Gamma_{ijk}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k . \]  
(71)
Notice that for a given symplectic structure \(\omega\) there exists a large family of connections satisfying (70). For instance, for any symplectic connection we have that
\[
\Gamma_{ki} = \frac{1}{2} \left( \omega_{ki,j} + \omega_{jk,i} \right)(-1)^{(\epsilon_i + \epsilon_k)\epsilon_j} - \omega_{ij,k}(-1)^{(\epsilon_i + \epsilon_j)\epsilon_k} \\
+ \left( \Pi_{ki} + \Pi_{kj} \right)(-1)^{\epsilon_k \epsilon_j} - \Pi_{jk}(-1)^{(\epsilon_i + \epsilon_k)\epsilon_j} \\
= -\omega_{ij,k}(-1)^{(\epsilon_i + \epsilon_j)\epsilon_k} + \left( \Pi_{ki} + \Pi_{kj} \right)(-1)^{\epsilon_k \epsilon_j} - \Pi_{jk}(-1)^{(\epsilon_i + \epsilon_k)\epsilon_j} \\
= -\omega_{ij,k}(-1)^{(\epsilon_i + \epsilon_j)\epsilon_k} + \Pi_{ki} + \left( \Pi_{kj}(-1)^{\epsilon_k \epsilon_j} - \Pi_{jk}(-1)^{(\epsilon_i + \epsilon_k)\epsilon_j} \right),
\]  
(72)
where $\Pi_{kij}$ is the symmetric part of $\Gamma_{ijk}$:

$$\Pi_{kij} = \frac{1}{2}(\Gamma_{kij} + \Gamma_{kji}(-1)^{\epsilon_i\epsilon_j}), \quad (73)$$

which is not a tensor field. If the connection is also symmetric we have that $\Gamma_{kij}$:

$$\Gamma_{kij} = -\omega_{ij,k}(-1)^{(\epsilon_i+\epsilon_j)\epsilon_k} + \Pi_{kij}, \quad (74)$$

The presence of $\Pi_{kij}$ in (74) is very important because if we only had

$$\Gamma_{kij} = -\omega_{ij,k}(-1)^{(\epsilon_i+\epsilon_j)\epsilon_k}, \quad (75)$$

then $\Gamma_{kij}$ in (75) will not transform according the corresponding rules for connections (61). Since $\Pi_{ijk}$ is arbitrary this shows that there is not a unique symmetric connection compatible with a given symplectic structure. In Darboux coordinates $\omega_{ij,k} = 0$ [17].

A symplectic supermanifold $(\mathcal{M}, \omega)$ equipped with a symmetric symplectic connection $\Gamma$ is called a Fedosov supermanifold $(\mathcal{M}, \omega, \Gamma)$.

Consider now curvature tensor $R_{ijkl}$ of a symplectic connection

$$R_{ijkl} = \omega_{in} R^n_{jkl}, \quad \epsilon(R_{ijkl}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l, \quad (76)$$

where $R^n_{jkl}$ is defined in (65). It is obvious that

$$R_{ijkl} = -(\epsilon_k^{\epsilon l}) R_{ijlk}, \quad (77)$$

and, using (65) and (67), one deduces the Jacobi identity for $R_{ijkl}$,

$$-(\epsilon_j^{\epsilon l}) R_{ijkl} + (\epsilon_k^{\epsilon l}) R_{iljk} + (\epsilon_i^{\epsilon l}) R_{iklj} = 0. \quad (78)$$

The curvature tensor $R_{ijkl}$ is generalized symmetric w.r.t. the first two indices (for details, see [11, 18]),

$$R_{ijkl} = -(1)^{(\epsilon_j^{\epsilon l})} R_{jikl} \quad (79)$$

and satisfies the identity

$$R_{ijkl} + (\epsilon_i^{(\epsilon_l+\epsilon_k+\epsilon_j)}) R_{ltijk} + (\epsilon_k^{(\epsilon_l+\epsilon_j+\epsilon_i)}) R_{kli} + (\epsilon_l^{(\epsilon_i+\epsilon_j+\epsilon_k)}) R_{jkl} = 0. \quad (80)$$

The last statement is proved by using the Jacobi identity (78) together with a cyclic change of indices [11]. The identity (80) involves components of the curvature tensor with cyclic permutation of all indices, but the sign factors depending on the Grassmann parities of the indices do not follow from a cyclic permutation, as is the case, for example, of the
Jacobi identity, but are defined by the permutation of the indices that maps a given set into the original one.

With the curvature tensor, $R_{ijkl}$, and the inverse tensor field $\omega^{ij}$ of the symplectic structure $\omega_{ij}$, one can construct the only tensor field of type $(0,2)$, 

$$K_{ij} = \omega^{kn} R_{mkij} (-1)^{\epsilon_i \epsilon_k + (\epsilon(\omega)+1)(\epsilon_k + \epsilon_n)} = R^k_{ij} (-1)^{\epsilon_k (\epsilon_i + 1)}, \quad \epsilon(K_{ij}) = \epsilon_i + \epsilon_j. \quad (81)$$

This tensor satisfies the relations [18]

$$[1 + (-1)^{\epsilon(\omega)}] (K_{ij} - (-1)^{\epsilon_i \epsilon_j} K_{ji}) = 0 \quad (82)$$

and is called the Ricci tensor. In the even case this tensor is symmetric whereas in the odd case there are not restrictions on its (generalized) symmetry properties.

Now we can define the scalar curvature tensor $K$ by the formula

$$K = \omega^{ji} K_{ij} (-1)^{\epsilon_i \epsilon_j} = \omega^{ji} \omega^{kn} R_{mkij} (-1)^{\epsilon_i + \epsilon_j + \epsilon_k + (\epsilon(\omega)+1)(\epsilon_k + \epsilon_n)} \quad (83)$$

From the symmetry properties of $R_{ijkl}$, it follows that

$$[1 + (-1)^{\epsilon(\omega)}] K = 0. \quad (84)$$

Therefore we have proved that as in the case of Fedosov manifolds [3] the following proposition holds

**Proposition:** Even Fedosov supermanifolds have vanishing scalar curvature $K$.

However, for odd Fedosov supermanifolds this curvature is, in general, not vanishing. This fact was quite recently used in Ref. [13] to generalize the BV formalism [7].

Consider the Bianchi identity (69) in the form

$$R^m_{mij;k} - R^m_{mk;j;i} (-1)^{\epsilon_k \epsilon_j} + R^m_{mj;k;i} (-1)^{\epsilon_i \epsilon_j + \epsilon_k} \equiv 0. \quad (85)$$

Contracting indices $i$ and $n$ with the help of (81) we obtain

$$K_{mj;k} - K_{mk;j;i} (-1)^{\epsilon_k \epsilon_j} + R^n_{mj;k;n} (-1)^{\epsilon_n (\epsilon_m + \epsilon_j + \epsilon_k + 1)} \equiv 0. \quad (86)$$

Now using the relations

$$K^i_{\cdot j} = \omega^{ik} K_{kj} (-1)^{\epsilon_k}, \quad K^i_{\cdot j;m} = \omega^{ik} K_{kj;m} (-1)^{\epsilon_k} \quad (87)$$

$$K^i_{\cdot j;\cdot i} (-1)^{\epsilon_i (\epsilon_i + 1)} = \omega^{ik} K_{kj;i;\cdot i} (-1)^{\epsilon_k + \epsilon_i (\epsilon_i + 1)}, \quad (88)$$

from (86) it follows that

$$K_{\cdot i} - K^j_{\cdot i;j} (-1)^{\epsilon_j (\epsilon_i + 1)} + \omega^{jm} R^n_{mj;i;n} (-1)^{\epsilon_j + \epsilon_n (\epsilon_m + \epsilon_j + \epsilon_i + 1)} \equiv 0. \quad (89)$$
Since
\[
\omega^{jm} R_{mji;n} (-1)^{\epsilon_j + \epsilon_m + \epsilon_n (\epsilon_m + \epsilon_j + 1)} = \omega^{jm} \omega^np R_{pmj;i} (-1)^{\epsilon_n + \epsilon_p} + \epsilon_j + \epsilon_m + \epsilon_n (\epsilon_m + \epsilon_j + 1)
\]
\[
= \omega^{jp} \omega^{jm} R_{mpji;n} (-1)^{\epsilon_j + \epsilon_m (\epsilon_j + 1) + \epsilon_j + \epsilon_p + \epsilon_n}
\]
\[
= K^j_{i;j} (-1)^{\epsilon_j + \epsilon_j (\epsilon_j + 1)},
\]
we have
\[
K_{;i} = [1 - (-1)^{\epsilon_j} \epsilon_{j} K^j_{i;j}] (-1)^{\epsilon_j (\epsilon_j + 1)}.
\]

In the odd case this implies that
\[
K_{;i} = 2K^j_{i;j} (-1)^{\epsilon_j (\epsilon_j + 1)}.
\]

In the even case $K_{;i} = 0$ but the relation (94) does not provides any new information because in this case $K = 0$.

6 Riemannian supermanifolds

Let $\mathcal{M}$ be a supermanifold is equipped both with a metric structure $g$
\[
g = g_{ij} \, dx^i dx^j, \quad g_{ij} = (-1)^{\epsilon_i \epsilon_j} g_{ji}, \quad \epsilon(g_{ij}) = \epsilon (g) + \epsilon_i + \epsilon_j , \quad (96)
\]
and $\Delta$ a symmetric connection with a covariant derivative $\nabla$ compatible with the super-Riemannian metric $g$
\[
g_{ij} \nabla_k = g_{ij, k} - g_{im} \Delta^m_{jk} - g_{jm} \Delta^m_{ik} (-1)^{\epsilon_i \epsilon_j} = 0. \quad (97)
\]

It is easy to show that as in the case of Riemannian geometry there exists the unique symmetric connection $\Delta^i_{jk}$ which is compatible with a given metric structure. Indeed, repeating calculations analogous to usual Riemannian geometry we obtain the generalization of famous Christoffel formula for the connection in supersymmetric case
\[
\Delta^i_{ki} = \frac{1}{2} g^{ij} (g_{ij,k} (-1)^{\epsilon_k \epsilon_i} + g_{jk,i} (-1)^{\epsilon_k \epsilon_j} - g_{ki,j} (-1)^{\epsilon_k \epsilon_j} (-1)^{\epsilon_j \epsilon_i + \epsilon_j (+ \epsilon_i)}) \quad (98)
\]
It is straightforward to show that the symbols $\Delta^i_{ki}$ in (98) are transformed according with transformation laws (61) for connections. A metric supermanifold $(\mathcal{M}, g)$ equipped with a (even or odd) symmetric connection $\Delta$ compatible with a given metric structure $g$ can be refereed as a (even or odd) Riemannian supermanifold $(\mathcal{M}, g, \Delta)$. 

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The curvature tensor of the connection is \((98)\)

\[ R_{ijkl} = g_{in} R_{jkl}^n, \quad \epsilon(R_{ijkl}) = \epsilon(g) + \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l, \tag{99} \]

where \(R_{jkl}^n\) is given by \((65)\) with natural replacement \(\Gamma^i_{jk}\) for \(\Delta^i_{jk}\), which leads to the following representation,

\[ R_{ijkl} = -\Delta_{ijk,l} + \Delta_{jik,l}(-1)^{\epsilon_k \epsilon_l} + \Delta_{nil}^n \Delta_{jkl}^n(-1)^{\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l} - \Delta_{nik} \Delta_{jil}^n(-1)^{\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l} \tag{100} \]

where

\[ \Delta_{ijk} = g_{in} \Delta_{jkl}^n, \quad \epsilon(\Delta_{ijk}) = \epsilon(g) + \epsilon_i + \epsilon_j + \epsilon_k. \tag{101} \]

In this representation the relations \((97)\) and \((98)\) read

\[ g_{ij,k} = \Delta_{ijk} + \Delta_{jik}(-1)^{\epsilon_k \epsilon_j}, \tag{102} \]

\[ \Delta_{ijk} = -\frac{1}{2}\left(g_{ki,j}(-1)^{\epsilon_k \epsilon_j} + g_{ij,k}(-1)^{\epsilon_i \epsilon_k} - g_{jki}(-1)^{\epsilon_j \epsilon_k}\right)(-1)^{\epsilon_i \epsilon_k}. \tag{103} \]

Furthermore, from Eq. \((100)\) it is follows that

\[ R_{ijkl} = -(-1)^{\epsilon_k \epsilon_l} R_{ijlk}, \tag{104} \]

and, using \((99)\) and \((107)\), one obtains the Jacobi identity for \(R_{ijkl}\),

\[ (-1)^{\epsilon_i \epsilon_j} R_{ijkl} + (-1)^{\epsilon_i \epsilon_k} R_{iljk} + (-1)^{\epsilon_i \epsilon_l} R_{iklj} = 0. \tag{105} \]

In addition, the curvature tensor \(R_{ijkl}\) is generalized antisymmetric w.r.t. the first two indices,

\[ R_{ijkl} = -(-1)^{\epsilon_i \epsilon_j} R_{jikl}. \tag{106} \]

In order to prove this, let us consider

\[ g_{ij,kl} = \Delta_{ijk,l} + \Delta_{jik,l}(-1)^{\epsilon_k \epsilon_l}. \tag{107} \]

Then, using the relations

\[ \Delta_{ijk,l} = g_{in} \Delta_{jkl}^n + g_{in,l} \Delta_{jkl}^n(-1)^{\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l} \tag{108} \]

and the definitions \((100)\) and \((107)\), we get

\[ 0 = g_{ij,kl} - (-1)^{\epsilon_k \epsilon_l} g_{ij,kl} = \Delta_{ijk,l} + \Delta_{jik,l}(-1)^{\epsilon_k \epsilon_j} - \Delta_{ijl,k}(-1)^{\epsilon_k \epsilon_l} - \Delta_{jil,k}(-1)^{\epsilon_i \epsilon_j + \epsilon_k \epsilon_l} = -R_{ijkl} - (-1)^{\epsilon_i \epsilon_j} R_{jikl}. \tag{109} \]
The tensor $\mathcal{R}_{ijkl}$ obeys the generalized symmetry property w.r.t permutation of pair indices

$$\mathcal{R}_{ijkl} = \mathcal{R}_{klij}(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}. \quad (110)$$

Indeed, from the Jacobi identity (105) one has

$$\mathcal{R}_{ijkl}(-1)^{\epsilon_i\epsilon_l} + \mathcal{R}_{iljk}(-1)^{\epsilon_k\epsilon_i} + \mathcal{R}_{iklj}(-1)^{\epsilon_j\epsilon_k} = 0, \quad (111)$$

$$\mathcal{R}_{jklm}(-1)^{\epsilon_l\epsilon_m} + \mathcal{R}_{jkil}(-1)^{\epsilon_k\epsilon_i} + \mathcal{R}_{jlik}(-1)^{\epsilon_l\epsilon_k} = 0 \quad (112)$$

Now, multiplying relation (111) by $(-1)^{\epsilon_i(\epsilon_j+\epsilon_k)}$, multiplying Eq. (112) by $(-1)^{\epsilon_j\epsilon_l}$, and subtracting the results, we obtain

$$2\mathcal{R}_{ijkl}(-1)^{\epsilon_j\epsilon_l+\epsilon_i(\epsilon_l+\epsilon_j)} + \mathcal{R}_{iljk}(-1)^{\epsilon_k\epsilon_i+\epsilon_j(\epsilon_i+\epsilon_j)} + \mathcal{R}_{iklj}(-1)^{\epsilon_j\epsilon_k+\epsilon_i(\epsilon_l+\epsilon_j)}$$

$$-\mathcal{R}_{jklm}(-1)^{\epsilon_l\epsilon_m+\epsilon_i(\epsilon_l+\epsilon_j)} - \mathcal{R}_{jkil}(-1)^{\epsilon_k\epsilon_i+\epsilon_j(\epsilon_i+\epsilon_j)} - \mathcal{R}_{jlik}(-1)^{\epsilon_l\epsilon_k+\epsilon_i(\epsilon_l+\epsilon_j)} = 0. \quad (113)$$

and by the permutation of pair indices

$$2\mathcal{R}_{klij}(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_j)} + \mathcal{R}_{lijk}(-1)^{\epsilon_l\epsilon_j+\epsilon_k(\epsilon_l+\epsilon_j)} + \mathcal{R}_{lijk}(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_i)}$$

$$-\mathcal{R}_{lijk}(-1)^{\epsilon_l\epsilon_j+\epsilon_k(\epsilon_l+\epsilon_i)} - \mathcal{R}_{lijk}(-1)^{\epsilon_l\epsilon_j+\epsilon_k(\epsilon_l+\epsilon_i)} = 0. \quad (114)$$

Multiplying the relation (113) by $(-1)^{\epsilon_i(\epsilon_j+\epsilon_k)}$, subtracting the results and using

$$\mathcal{R}_{ijkl} = -\mathcal{R}_{jikl}(-1)^{\epsilon_i\epsilon_j}, \quad \mathcal{R}_{ijkl} = -\mathcal{R}_{ijkl}(-1)^{\epsilon_k\epsilon_i},$$

we obtain the property (110).

From the curvature tensor $\mathcal{R}_{ijkl}$ and the inverse tensor field $g^{ij}$ of the metric $g_{ij}$

$$g^{ij} = (-1)^{\epsilon(g)+\epsilon_i\epsilon_j}g_{ji}, \quad \epsilon(g^{ij}) = \epsilon(g) + \epsilon_i + \epsilon_j, \quad (115)$$

one can define the following three tensor field of type $(0,2)$:

$$K_{ij} = \mathcal{R}^k_{kij}(-1)^{\epsilon_k} = g^{kn}\mathcal{R}_{nki}(-1)^{(\epsilon_k+\epsilon_n)(\epsilon(g)+1)}, \quad (116)$$

$$\mathcal{R}_{ijkl} = \mathcal{R}^k_{ikj}(-1)^{\epsilon_k+\epsilon_j} = g^{kn}\mathcal{R}_{nikj}(-1)^{(\epsilon_k+\epsilon_n)(\epsilon(g)+1)+\epsilon_i\epsilon_k}, \quad (117)$$

$$Q_{ij} = \mathcal{R}^k_{ijk}(-1)^{\epsilon_k(\epsilon_i+\epsilon_j+1)} = g^{kn}\mathcal{R}_{nijk}(-1)^{(\epsilon_k+\epsilon_n)(\epsilon(g)+1)\epsilon_i\epsilon_k+\epsilon_k}, \quad (118)$$

$$\epsilon(K_{ij}) = \epsilon(\mathcal{R}_{ij}) = \epsilon(Q_{ij}) = \epsilon_i + \epsilon_j.$$

Taking into account the definitions (116)-(118) and the symmetry properties (104), (106), (110) and (115), one can easily find the symmetry properties of $K_{ij}$, $\mathcal{R}_{ij}$ and $Q_{ij}$

$$K_{ij} = -(-1)^{\epsilon_i\epsilon_j}K_{ji}, \quad \mathcal{R}_{ij} = (-1)^{\epsilon(g)+\epsilon_i\epsilon_j}\mathcal{R}_{ji}, \quad Q_{ij} = (-1)^{\epsilon(g)+\epsilon_i\epsilon_j}Q_{ji}. \quad (119)$$
Moreover from (116) and (106), (115) it follows

\[ [1 + (-1)^{\epsilon(g)}] K_{ij} = 0. \]  

(120)

This implies that in the even case \((\epsilon(g) = 0)\) \(K_{ij} = 0\). In a similar way one obtains

\[ Q_{ij} = -R_{ij}. \]  

(121)

From the Jacobi identity (105) the relations among tensors \(K_{ij}, R_{ij}, Q_{ij}\) can be obtained

\[ R_{ij} + K_{ji}(-1)^{\epsilon_i \epsilon_j} + Q_{ji}(-1)^{\epsilon_i \epsilon_j} = 0. \]  

(122)

Therefore

\[ K_{ij} = [1 - (-1)^{\epsilon(g)}] R_{ij} \]  

(123)

and \(R_{ij}\) is the only independent second-rank tensor field which can be constructed from the curvature tensor \(R_{ijkl}\). It is the generalized Ricci tensor.

A further contraction between the metric and Ricci tensors define the scalar curvature

\[ R = g^{ij} R_{ij} (-1)^{\epsilon_i + \epsilon_j}, \quad \epsilon(R) = \epsilon(g) \]  

(124)

which, in general, is not equal to zero. Notice that for an odd metric structure the scalar curvature tensor squared is identically equal to zero, \(R^2 = 0\).

Consider now relations which follow from the Bianchi identity (69). Repeating all arguments given in the end of previous Section one can derive the following relation between the scalar curvature and the Ricci tensor

\[ R_{,i} = [1 + (-1)^{\epsilon(g)}] R_{ij} (-1)^{\epsilon_j (\epsilon_i + 1)}. \]  

(125)

In the even case we have

\[ R_{,i} = 2 R_{ij} (-1)^{\epsilon_j (\epsilon_i + 1)}, \]  

(126)

which is a supersymmetric generalization of known relation in Riemannian geometry [19].

In the odd case \(R_{,i} = 0\) and the relation (125) implies that \(R = \text{const.}\).

Therefore we have proved the following proposition.

**Proposition:** Odd Riemann supermanifolds have constant scalar curvature \(R = \text{const.}\).
7 Discussion

We have analyzed the natural geometric structures of supermanifolds defined symmetric and antisymmetric graded tensor fields of the second rank and types $(2,0)$ and $(0,2)$. It was shown that a Poisson bracket can be associated with an antisymmetric even tensor field of type $(2,0)$ while an antibracket is related to a symmetrical odd tensor field of type $(2,0)$. We have shown that all properties and relations for both even and odd symplectic supermanifolds equipped with a symmetric connection compatible with a given symplectic structure (even and odd Fedosov supermanifolds) have a similar form. In a similar way both even and odd metric supermanifolds equipped with a (unique) symmetric connection compatible with a given metric structure (even and odd Riemannian supermanifolds) have the same algebraic properties except that in the odd case a scalar curvature tensor squared is identically equal to zero and the Ricci tensor is antisymmetric. It was shown that an antisymplectic supermanifold underlying the Batalin-Vilkovisky quantization method in general coordinates is just an odd Fedosov supermanifold. It was proven that in the odd case the scalar curvature tensor for both Riemannian and Fedosov supermanifolds is, in general, non-zero. Odd Riemannian supermanifolds are however strongly constrained by the fact that their scalar curvature has to be constant.

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