ON THE COMPRESSIBLE NAVIER-STOKES EQUATIONS IN THE WHOLE SPACE: FROM NON-ISENTROPIC FLOW TO ISENTROPIC FLOW

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Abstract. The present work aims at the mathematical derivation of the equations for the isentropic flow from those for the non-isentropic flow for perfect gases in the whole space. Suppose that the following things hold for the entropy equation: (1). both conduction of heat and its generation by dissipation of mechanical energy are sufficiently weak (with the order of $\varepsilon$); (2). initially the entropy $S_N^N$ is around a constant $c_S$, that is, $S_N^N|_{t=0} = c_S + O(\varepsilon)$. Then the non-isentropic compressible Navier-Stokes equations admit a unique and global solution $(\rho_N^N, u_N^N, S_N^N)$ with the initial data $(\rho_0, u_0, c_S + \varepsilon S_0)$, which is a perturbation of the equilibrium $(1, 0, c_S)$. Moreover, $(\rho_N^N, u_N^N)$ can be approximated by $(\rho_I^I, u_I^I)$, the solution to the associated isentropic compressible Navier-Stokes equations equipped with the initial data $(\rho_0, u_0)$, in the sense that

$$(\rho_N^N(t), u_N^N(t)) = (\rho_I^I(t), u_I^I(t)) + O(\varepsilon),$$

which holds globally in the so-called critical Besov spaces for the compressible Navier-Stokes equations.

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1. Introduction. It is well-known that before the formation of the shock, the compressible isentropic Euler equations can be derived by the non-isentropic compressible Euler equations if initially the entropy is a constant. In this paper, we want to show that such kind of the result can be generalized to the compressible Navier-Stokes equations. More precisely, for the perfect gas in the whole space, if both conduction of heat and its generation by dissipation of mechanical energy are small enough, we prove that the non-isentropic compressible Navier-Stokes equations can be approximated by the associated isentropic compressible Navier-Stokes equations if initially the entropy is near a constant. This gives the mathematical derivation of the equations for the isentropic flow from those for the non-isentropic flow.

1.1. Setting of the problem. The full compressible Navier-Stokes equations read as follows

$$\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) &= \div T, \\
\partial_t \bar{S} + u \cdot \nabla \bar{S} + \frac{\varepsilon}{\rho} \div (\frac{q}{\theta}) &= \varepsilon \frac{S(u)}{\rho \theta} - \varepsilon \frac{q \cdot \nabla \theta}{\rho \theta^2},
\end{align*}$$

(1.1)

where $\rho = \rho(t,x) \in \mathbb{R}^+$, $u = u(t,x) \in \mathbb{R}^3$, $\bar{S} = \bar{S}(t,x) \in \mathbb{R}$ are density, velocity and entropy of the fluid respectively, and $\varepsilon$ is a parameter which is sufficiently small. For perfect gases,

$$T = S(u) - p I_{3 \times 3}, \quad S(u) = 2 \mu \left( \nabla u + \nabla u^T - \frac{2}{3} \div u I_{3 \times 3} \right) + \lambda \div u I_{3 \times 3},$$

$$\bar{S} = C_v \log \theta - R \log \rho, \quad p = R \rho \theta, \quad q = -\kappa(\theta) \nabla \theta, \quad \kappa(\theta) \geq 0.$$

Here $\mu > 0$, $\lambda > 0$ are the shear viscosity and bulk viscosity, $C_v$, $R$ are two positive constants, $p$, $\theta$ are the pressure and the temperature, and $\kappa(\theta) \geq 0$ is the heat conductivity. The system is equipped with the initial data as follows

$$\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \quad \bar{S}|_{t=0} = c_S + \varepsilon S_0.$$

The main purpose of our paper aims at the investigation of the asymptotic limit in which $\varepsilon$ goes to zero. We refer readers to [16] for more details on the physical background. In order to get the global result, we restrict our study to the perturbation framework. Without loss of generality, we assume that the solution $(\rho, u, S)$ is near the equilibrium $(1, 0, c_S)$.

1.1.1. Reformulation of (1.1) and the limiting equations. To derive the limiting equations, we first reformulate the full system (1.1). Because of the special form of the initial data for the entropy, it is natural to set $\bar{S} = c_S + \varepsilon S$. Without loss of generality, let $c_S = 0$. Thanks to the fact $\bar{S} = C_v \log \theta - R \log \rho$, we first obtain that

$$\theta = \rho^{\frac{C_v}{R}} e^{\varepsilon \frac{C_v}{R}},$$

(1.2)

and

$$\partial_t S + u \cdot \nabla S + \frac{1}{\rho} \div (\frac{q}{\theta}) = \frac{S(u)}{\rho \theta} - \frac{q \cdot \nabla \theta}{\rho \theta^2}.$$
For simplicity, we impose that \( \kappa(\theta) = 1 \) which yields that
\[
q = -\nabla \theta.
\]
From this, we derive that
\[
\frac{1}{\rho} \text{div} \left( q \frac{\nabla \theta}{\rho \theta^2} \right) + \frac{q \cdot \nabla \theta}{\rho \theta} = \frac{1}{\rho \theta} \text{div} q,
\]
which implies
\[
\partial_t S + u \cdot \nabla S - \frac{1}{\rho \theta} \Delta \theta = \frac{S(u) : \nabla u}{\rho \theta},
\]
(1.3)
Thanks to (1.2) and (1.3), the full compressible Navier-Stokes equations (1.1) can be reformulated by
\[
\begin{cases}
\partial_t \rho + \text{div} (\rho u) = 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) - \text{div} S(u) + \nabla \left( R \rho \gamma e^{\frac{\mu}{\rho \gamma}} \right) = 0, \\
\partial_t S + u \cdot \nabla S - \frac{1}{\rho \theta} \Delta \theta = \frac{S(u) : \nabla u}{\rho \theta},
\end{cases}
\]
(1.4)
where
\[
S(u) = 2\mu (\nabla u + \nabla u^T) - \frac{2}{3} \text{div} u I_{3 \times 3} + \lambda \text{div} u I_{3 \times 3}, \quad \theta = \rho \gamma e^{\frac{\mu}{\rho \gamma}}.
\]
The system is supplemented with the initial data as follows
\[
\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \quad S|_{t=0} = S_0.
\]
(1.5)
From (1.4), when the parameter \( \varepsilon \) tends to zero, we formally obtain the associated isentropic compressible Navier-Stokes equations which read:
\[
\begin{cases}
\partial_t \rho + \text{div} (\rho u) = 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) - \text{div} S(u) + \nabla \left( R \rho \gamma \right) = 0,
\end{cases}
\]
(1.6)
where \( \gamma = 1 + \frac{p}{p \gamma} > 1. \)

Suppose \((\rho^N_\varepsilon, u^N_\varepsilon, S^N_\varepsilon)\) and \((\rho^I, u^I)\) are the solutions to (1.4) and (1.6) with the initial data \((\rho_0, u_0, S_0)\) and \((\rho_0, u_0)\) respectively. Then the asymptotic problem from non-isentropic flows to isentropic flows is formulated to the study of the limit in which \( \varepsilon \) tends to zero. In this paper, we will focus not only on the convergence but also on the asymptotic expansion for the solutions to (1.4) and (1.6).

1.1.2. Short review of the existing results. The local well-posedness for the system (1.4) or (1.6) was proved by Nash [24] for the smooth initial data which is away from vacuum. For the global smooth solutions in Sobolev spaces, we refer readers to [20, 22] for the recent progress.

To catch the scaling invariance property of the system (1.6), Danchin first introduced in his series papers [8, 11] the “Critical Spaces” which were inspired by the results for the incompressible Navier-Stokes. More precisely, he proved the local well-posedness of (1.6) in the critical Besov spaces \( \dot{B}_{p,1}^{\frac{3}{p}} \times \dot{B}_{p,1}^{\frac{3}{p} - 1} \) with \( 2 \leq p < 6 \), and global well-posedness of (1.6) for the initial data close to a stable equilibrium in spaces \( (\dot{B}_{2,1}^{\frac{3}{2}} \cap \dot{B}_{2,1}^{\frac{3}{2}}) \times \dot{B}_{2,1}^{\frac{3}{2}} \). We refer readers to [3, 4, 5, 6, 12, 14, 15, 18, 19] for the recent progress, in particular, on the solutions which allow to have high oscillation. For the full system (1.4), we refer readers to [7, 6, 9, 10, 12, 13, 17, 21, 23] and refererences therein for well-posedness or ill-posedness results.
1.2. Notations and Function spaces. Before stating the main results, we first introduce notations and function spaces.

1.2.1. Notations and definitions. The notation \( f \sim g \) means that there exists a constant \( C \) such that \( \frac{1}{C} f \leq g \leq Cf \). \( f \lesssim g \) and \( g \gtrsim f \) mean that there exists a constant \( C \) such that \( f \leq Cg \). We shall use \( C \) to denote a universal constant which may change from line to line. The notation \( \| \cdot \|_{L^p} \) stands for the \( L^p(\mathbb{R}^3) \) norm for \( 1 \leq p \leq \infty \).

Let \( X_1, X_2 \) be Banach spaces. We define the norms as follows

\[
\| \cdot \|_{L^p(X_1)} \overset{\text{def}}{=} \| \cdot \|_{L^p((0,t); X_1)}, \quad \| \cdot \|_{L^p(X_1 \cap X_2)} \overset{\text{def}}{=} \| \cdot \|_{L^p(X_1)} + \| \cdot \|_{L^p(X_2)},
\]

for \( p \in [1, \infty] \).

We shall denote by \( \langle f | g \rangle \) the \( L^2(\mathbb{R}^3) \) inner product of \( f \) and \( g \). If \( A, B \) are two operators, \( [A, B] = AB - BA \) denotes their commutator.

1.2.2. Littlewood-Paley theory and function spaces. Because of the weak dissipation of the entropy, the control of \( \| \nabla u \|_{L^1(\mathbb{R}^3)} \) seems compulsory in order to get the global result. Unfortunately this is not so easy to derive in the framework of Sobolev spaces. Thus we try to solve the equations (1.4) in the so-called critical Besov space. For the convenience of the reader, we state some basic facts on the Littlewood-Paley theory. To do so, we first introduce the dyadic decomposition in the frequency space. We fix an smooth function \( \varphi : \mathbb{R} \to [0, 1] \) supported in \( [0, \frac{3}{4}] \) and equals to 1 in \( [0, \frac{5}{4}] \). We also denote by \( \varphi(x) = \varphi(|x|) \) for any \( x \in \mathbb{R}^3 \). For any \( k \in \mathbb{Z} \), we define

\[
\varphi_k(x) \overset{\text{def}}{=} \varphi\left( \frac{x}{2^k} \right) - \varphi\left( \frac{x}{2^{k-1}} \right), \quad \varphi_{[k_1, k_2]}(x) \overset{\text{def}}{=} \sum_{k = k_1}^{k_2} \varphi_k(x),
\]

\[
\varphi_{\leq k}(x) \overset{\text{def}}{=} \varphi\left( \frac{x}{2^k} \right) = \sum_{l \leq k} \varphi_l(x), \quad \varphi_{\geq k}(x) \overset{\text{def}}{=} 1 - \varphi_{\leq k-1}(x).
\]

For any \( x \in \mathbb{R}^3 \),

\[
\sum_{k \in \mathbb{Z}} \varphi_k(x) = 1 \quad \text{and} \quad \text{supp} \varphi_k(\cdot) \subset \{ x \in \mathbb{R} \mid |x| \in \left[ \frac{5}{8} 2^k, \frac{3}{2} 2^k \right] \}.
\]

For \( u \in S'(\mathbb{R}^3) \), we set

\[
\Delta_j u \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi_j(\xi)\hat{u}(\xi)), \quad S_j u \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi_{\leq j}(\xi)\hat{u}(\xi)),
\]

where \( \mathcal{F}u \) and \( \hat{u} \) denote the Fourier transform of the distribution \( u \). We also regard \( \Delta_j \) and \( S_j \) as follows

\[
\Delta_j = \varphi_j(D), \quad S_j = \varphi_{\leq j}(D).
\]

Due to (1.7), \( (\Delta_j)_{j \in \mathbb{Z}} \) is an homogeneous Littlewood-Paley decomposition. Thanks to (1.7) and (1.8), we have the property of almost orthogonality:

\[
\Delta_k \Delta_j u \equiv 0 \quad \text{if} \quad |k - j| \geq 2 \quad \text{and} \quad \Delta_k (S_{j-1} u \Delta_j v) \equiv 0 \quad \text{if} \quad |k - j| \geq 5.
\]

**Definition 1.1.** [see [1]] For any \( s \in \mathbb{R} \), we define the homogeneous Besov space

\[
\dot{B}_{2, 1}^s(\mathbb{R}^3) \overset{\text{def}}{=} \{ f \in S'(\mathbb{R}^3) \mid \| f \|_{\dot{B}_{2, 1}^s} < +\infty \quad \text{and} \quad \lim_{j \to -\infty} S_j f = 0 \},
\]

equipped with the norm

\[
\| f \|_{\dot{B}_{2, 1}^s} \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_{L^2}.
\]
We will also use the Chemin-Lerner type spaces $\tilde{L}^p_t(\dot{B}^{s}_{2,1}(\mathbb{R}^3))$ (see [2] for more details) to catch the parabolic estimates in Besov spaces.

**Definition 1.2.** Let $p \in [1, +\infty]$ and $T \in (0, +\infty]$. We define the $\tilde{L}^p_t(\dot{B}^{s}_{2,1}(\mathbb{R}^3))$ by

$$
\|u\|_{\tilde{L}^p_t(\dot{B}^{s}_{2,1})} = \sum_{j \in \mathbb{Z}} 2^{js} \|\Delta_j u\|_{L^p_T([0,T];L^2)}.
$$

**Remark 1.1.** By Definition 1.2 and Minkovski inequality, we have

$$
\|u\|_{\tilde{L}^p_t(\dot{B}^{s}_{2,1})} = \|u\|_{L^p_t(\dot{B}^{s}_{2,1})}, \quad \|u\|_{\tilde{L}^p_t(\dot{B}^{s}_{2,1})} \lesssim \|u\|_{L^p_T(\dot{B}^{s}_{2,1})} \quad \text{for } p > 1. \quad (1.10)
$$

1.3. **Main results and the outline of the proof.** Before stating our main results, we first introduce the energy spaces and the dissipation norm for the equations (1.4).

**Definition 1.3.** For any $s \in \mathbb{R}$, the energy norm $E_s(t)$ and the dissipation norm $D_s(t)$ can be defined by

$$
E_s(t) \overset{\text{def}}{=} \|\rho(t) - 1\|_{\dot{B}^{-s}_{2,1}} + \|u(t)\|_{\dot{B}^{-s}_{2,1}} + \|S(t)\|_{\dot{B}^{-s}_{2,1}},
$$

$$
D_s(t) \overset{\text{def}}{=} \|\rho - 1\|_{\dot{B}^{-s}_{2,1}} + \|\rho - 1\|_{\dot{B}^{-s}_{2,1}} + \|u(t)\|_{\dot{B}^{-s}_{2,1}} + \varepsilon \|S(t)\|_{\dot{B}^{-s}_{2,1}},
$$

where

$$
\|f\|_{\dot{B}^{-s}_{2,1}} = \sum_{j \leq 0} 2^{js} \|\Delta_j f(t)\|_{L^2}, \quad \|f\|_{\dot{B}^{s}_{2,1}} = \sum_{j \geq 1} 2^{js} \|\Delta_j f(t)\|_{L^2}.
$$

We also define the energy norm $E_s(t)$ in the Chemin-Lerner type spaces $\tilde{L}^\infty_t(\dot{B}^{s}_{2,1}(\mathbb{R}^3))$ as follows

$$
E_s(t) \overset{\text{def}}{=} \|\rho - 1\|_{\tilde{L}^\infty_t(\dot{B}^{s}_{2,1})} + \|u\|_{\tilde{L}^\infty_t(\dot{B}^{s}_{2,1})} + \|S\|_{\tilde{L}^\infty_t(\dot{B}^{s}_{2,1})},
$$

(1.14)

Now we are in a position to state the main results. The first result is about the global wellposedness of the system (1.4).

**Theorem 1.1.** There exists a sufficiently small constant $\eta > 0$, independent of $\varepsilon$, such that if $E_2(0) + \varepsilon E_2(0) \leq \eta$, then (1.4)-(1.5) admits a global and unique solution $(\rho(t,x), u(t,x), S(t,x))$. Moreover,

$$
\sup_{t \in \mathbb{R}^+} (E_2'(t) + \varepsilon E_2(t)) + \int_0^\infty (D_2(t) + \varepsilon D_2(t)) dt \lesssim E_2(0) + \varepsilon E_2(0) \leq \eta; \quad (1.15)
$$

$$
\sup_{t \in \mathbb{R}^+} E_2(t) + \int_0^\infty D_2(t) dt \lesssim E_2(0); \quad (1.16)
$$

$$
E_2(\infty) + \varepsilon E_2(\infty) \lesssim E_2(0) + \varepsilon E_2(0); \quad (1.17)
$$

$$
E_2(\infty) \lesssim E_2(0). \quad (1.18)
$$

The second result is concerning with the asymptotic expansion for the solutions to (1.4) and (1.6).

**Theorem 1.2.** Let $(\rho^N_t, u^N_t, S^N)$ and $(\rho^I_t, u^I_t)$ be the solutions to (1.4) and (1.6) with the initial data $(\rho_0, u_0, S_0)$ and $(\rho_0, u_0)$ respectively. Suppose that $E_2(0) + \varepsilon E_2(0) \leq \eta$. Then

$$
\sup_{t \in \mathbb{R}^+} \left( \|\rho^N_t - \rho^I_t\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{1}{2}_{2,1} \cap B^{\frac{3}{2}_{2,1}}})} + \|u^N_t - u^I_t\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{1}{2}_{2,1}})} \right) \lesssim \varepsilon. \quad (1.19)
$$
Some remarks are in order:

**Remark 1.2.** Our energy space \( E \) is consistent with the critical function for the isentropic Navier-Stokes equations (see [8]) when \( \varepsilon \) tends to zero.

**Remark 1.3.** (1.15-1.16) show that the regularity of the solution \((\rho, u, S)\) can be propagated globally and uniformly with respect to the parameter \( \varepsilon \). In particular, we have the control of the key quantity \( \|\nabla u\|_{L^1(L^\infty)} \) which plays an important role on the propagation of the regularity for the entropy. This is main reason that we adopt the Besov spaces instead of the Sobolev spaces. If additional assume that \( E_s(0) < \infty \) with \( s \in \left(-\frac{1}{2}, \frac{1}{2}\right) \), then we have

\[
\sup_{t \in \mathbb{R}^+} E_s(t) + \int_0^\infty D_s(\tau)d\tau \lesssim E_s(0). 
\] (1.20)

**Remark 1.4.** We explain that the estimate (1.19) is non trivial. If we set

\[
\rho^N(t) = \rho^I(t) + \varepsilon \rho_1(t), \quad u^N(t) = u^I(t) + \varepsilon u_1(t),
\]

then it is easy to see that \((\rho_1, u_1)\) satisfies the following system

\[
\begin{cases}
\partial_t \rho_1 + T_u \cdot \nabla \rho_1 + \text{div} u_1 = N_{\rho_1}, \\
\partial_t u_1 + T_u \cdot \nabla u_1 - 2\mu \Delta u_1 - \frac{2}{3} \mu \lambda \nabla \text{div} u_1 + R(1 + \frac{R}{C_v}) \nabla \rho_1 \\
\rho_1|_{t=0} = 0, \quad u_1|_{t=0} = 0,
\end{cases}
\] (1.21)

where \( N_{\rho_1} \) and \( N_{u_1} \) are nonlinear terms. At the first glance, the estimate \( \|(\rho_1(t), u_1(t))\| \sim O(t) \) is expected because for the linear term \( \nabla S \) in \( u_1 \)-equation we only have \( \varepsilon S \in L^1_t(\dot{B}_{2,1}^{\frac{3}{2}}) \) and \( S \in L^\infty_t(\dot{B}_{2,1}^{\frac{3}{2}} \cap \dot{B}_{2,1}^{\frac{5}{2}}) \). But (1.19) shows that we still can get the globally uniform-in-time estimate for \((\rho_1, u_1)\). Roughly speaking, we obtain it by using the localization techniques and the parabolic estimates in the Chemin-Lerner type spaces. More details will be given in the next subsection.

### 1.3.1. Difficulty and the strategy of the proof

The main difficulties of the problem lie in the following two aspects: 1). How to prove the propagation of the regularity for the system (1.4) uniformly with respect to the parameter \( \varepsilon \); 2). How to prove the globally uniform-in-time estimate for the system (1.21).

We first address the first difficulty that is related to the features of the system (1.4):

(1). Due to the setting of the problem, the full Navier-Stokes equations (1.4) are written in terms of \((\rho, u, S)\) instead of \((\rho, u, \theta)\). This induces the new linearized system which is different from the previous work.

(2). The dissipation of the temperature \( \theta \) is sufficiently weak. More precisely, it is only of order \( \varepsilon \). It means that the hyperbolic structure will dominate the behavior of the entropy for a long time, that is, before the critical time \( O(\varepsilon^{-1}) \), the entropy propagates along the trajectory which is determined by the velocity.

These two special properties require the new analysis of the linearized system and the control of \( \|\nabla u\|_{L^1(L^\infty)} \) which are crucial to get the global result. Our strategy is based on the fact that the system is decoupled into two sub systems in the limiting process. It is unfolded as follows.
(1). We first try to catch the weak dissipation of the entropy. To do that, we introduce the good unknown function, that is motivated by the effective viscous flux, to catch the coupling effect of the system. It plays the role of diagonalizing the system. Thanks to this observation, we succeed to get the desired result.

(2). Once we have the weak dissipation of the entropy, we can use it to decouple the system and recover the strong dissipation of the density and the velocity. In particular, we can control \( \|\nabla u\|_{L^1_t(L^\infty)} \).

It seems that we almost solve the problem in the perturbation framework. However, when we focus on the nonlinear equations, a new difficulty occurs, that is, we cannot use the energy and its associated dissipation to control the nonlinear term \( \varepsilon^2 |\nabla S|^2 \), in particular, in the energy space \((B^\frac{1}{2}_{2,1} \cap B^\frac{3}{4}_{2,1}) \times B^\frac{-\frac{5}{2}}{2,1} \times B^\frac{-\frac{5}{2}}{2,1} \) for \((\rho-1, u, S)\).

More precisely, we have
\[
\varepsilon^2 \|\nabla S\|^2_{L^1_t(B^\frac{-\frac{5}{2}}{2,1})} \lesssim \varepsilon \|S\|_{L^1_t(B^\frac{3}{4}_{2,1})} \cdot \varepsilon \|S\|_{L_t^\infty(B^\frac{3}{4}_{2,1})}.
\]

It is obvious that \( \varepsilon \|S\|_{L_t^\infty(B^\frac{3}{4}_{2,1})} \) is out of control. To overcome this difficulty, our strategy directly comes from the above estimate. The idea is to take \( \varepsilon \|S\|_{B^\frac{3}{4}_{2,1}} \) as the part of the energy functional. Thanks to this observation, we introduce the new energy functional \( \mathcal{E}_k + \varepsilon \mathcal{E}_k \) (see the definition (1.11)) for \((\rho-1, u, S)\). Fortunately in the new function spaces we can close the energy argument.

Now we turn to the second difficulty. The main problem lies in the control of the linear term \( \nabla S \) in the \( u_t \)-equation of (1.21). To make it precisely, let us first describe the behavior of the entropy. If we additional assume that \( \mathcal{E}_S(0) < \infty \) with \( s \in (-\frac{1}{2}, \frac{1}{2}) \), by standard interpolation, we may derive that
\[
\|S(t)\|_{B^\frac{3}{4}_{2,1}} \lesssim (\mathcal{E}_S(0)^{-\frac{4}{5} + \varepsilon t})^{-\frac{5-2s}{5}}.
\]

Moreover, if
\[
\|S_0\|_{B^\frac{3}{4}_{2,1}} = \|S_0\|_{B^\frac{-\frac{5}{2}}{2,1}} = O(\eta),
\|\rho_0 - 1\|_{B^\frac{1}{2}_{2,1} \cap B^\frac{3}{4}_{2,1}} + \|u_0\|_{B^\frac{1}{2}_{2,1}} = O(\eta^2), \quad \mathcal{E}_S(0) \sim 1.
\]

Then for \( t \leq O(\eta^2 \varepsilon^{-1}) \),
\[
\|S(t)\|_{B^\frac{3}{4}_{2,1}} \geq (1 - C\eta)\|S_0\|_{B^\frac{-\frac{5}{2}}{2,1}}.
\]

In particular, if \( S_0 = \eta \mathcal{F}_{\geq 3}(D)g \) where \( g \) is a Schwartz function, then for \( t \leq O(\eta^2 \varepsilon^{-1}) \),
\[
\|S(t)\|_{B^\frac{3}{4}_{2,1}} \geq C(1 - C\eta)\|S_0\|_{B^\frac{-\frac{5}{2}}{2,1}}.
\]

We refer readers to the proof of (1.22-1.25) in Section 3. These estimates show that at least for \( t \leq O(\eta^2 \varepsilon^{-1}) \), it holds that \( S(t) \sim S_0 \) which looks dangerous to the globally uniform-in-time estimates for the solution \((\rho_1, u_1)\) of (1.21). To overcome this difficulty, we develop a new method which can be stated clearly for the following toy model. Suppose that \( u \) is a solution to the heat equation with a source term \( f \), that is,
\[
\begin{cases}
\partial_t u - \Delta u = f; \\
u|_{t=0} = u_0.
\end{cases}
\]
By basic energy estimate, we have
\[ \frac{d}{dt} \| \Delta_{j} u \|_{L^2} + 2c_1 2^j \| \Delta_{j} u \|_{L^2} \leq \| \Delta_{j} f \|_{L^2}, \]
which implies that for \( j \leq 0 \),
\[ \| \Delta_{j} u(t) \|_{L^2} + c_1 \int_{0}^{t} e^{-c_1 2^j (t-s) 2^j} \| \Delta_{j} u(s) \|_{L^2} ds \leq \| \Delta_{j} u(0) \|_{L^2} + \int_{0}^{t} e^{-c_1 2^j (t-s)} \| \Delta_{j} f(s) \|_{L^2} ds, \]
and for \( j \geq 1 \),
\[ \| \Delta_{j} u(t) \|_{L^2} + c_1 \int_{0}^{t} e^{-c_1 2^j (t-s) 2^j} \| \Delta_{j} u(s) \|_{L^2} ds \leq \| \Delta_{j} u(0) \|_{L^2} + \int_{0}^{t} e^{-c_1 (t-s)} \| \Delta_{j} f(s) \|_{L^2} ds. \]

Then
\[ \left\| u \right\|_{L_{loc}^{\infty} \left( \mathbb{B}_{\frac{3}{2},1}^{\frac{1}{2}} \right)} + \sum_{j \geq 0} c_1 \int_{0}^{t} e^{-c_1 (t-s)} 2^{\frac{3}{2} j} \| \Delta_{j} u(s) \|_{L^2} ds \]
\[ + \sum_{j \leq 0} c_1 \int_{0}^{t} e^{-c_1 2^{j} (t-s) 2^{j}} \| \Delta_{j} u(s) \|_{L^2} ds \]
\[ \leq \left\| u \right\|_{B_{\frac{3}{2},1}^{\frac{1}{2}}} + \| f \|_{L_{loc}^{\infty} \left( \mathbb{B}_{\frac{3}{2},1}^{\frac{3}{2}}, \mathbb{B}_{\frac{3}{2},1}^{\frac{1}{2}} \right)}. \]

In this way, on one hand, we can get the uniform-in-time estimate for the solution. On the other hand, we still can catch the dissipation which is crucial for the propagation of the regularity of the solution to the transport-diffusion equation. We refer readers to the proof of Theorem 1.2 for more details.

1.3.2. Organization of the paper. In Section 2, we focus on the a priori estimates for the linearized system of (1.4). Then in the next section, we give the detailed proofs for the main theorems. As a byproduct, we obtain (1.22-1.25). In the appendix, we list some basic facts on the Littlewood-Paley theory and the product estimates.

2. A priori estimates for the linearized system. In this section, we first linearize the system of (1.4) around the equilibrium \((1,0,0)\) and then give the corresponding a priori estimates to catch the dissipation structure of the system. It is not difficult to check that the linearized system of (1.4) around the equilibrium \((1,0,0)\) can be concluded as follows:

\[
\begin{aligned}
&\partial_t \rho + T_u \cdot \nabla \rho + \text{div } u = N_{\rho}, \\
&\partial_t u + T_u \cdot \nabla u - 2\mu \Delta u - \left( \frac{2}{3} \mu + \lambda \right) \nabla \text{div } u + R(1 + \frac{R}{C_v}) \nabla \rho + \frac{\varepsilon R}{C_v} \nabla S = N_u, \\
&\partial_t S + T_u \cdot \nabla S - \frac{\varepsilon}{C_v} \Delta S - \frac{R}{C_v} \Delta \rho = N_S,
\end{aligned}
\]

where

\[
\begin{aligned}
N_{\rho} &\overset{\text{def}}{=} -T_{\nabla \rho} \cdot u - R(u, \nabla \rho) - (\rho - 1) \text{div } u; \\
N_u &\overset{\text{def}}{=} -T_{\nabla u} \cdot u - R(u, \nabla u) + 2\mu(\frac{1}{\rho} - 1) \Delta u + \left( \frac{2}{3} \mu + \lambda \right) \left( \frac{1}{\rho} - 1 \right) \nabla \text{div } u \\
&- R(1 + \frac{R}{C_v}) \left( \rho - \frac{\rho - 1}{\rho} \right) e^{\frac{\varepsilon}{C_v} - 1} \nabla \rho - \frac{\varepsilon R}{C_v} \left( \rho e^{\frac{\varepsilon}{C_v} - 1} \right) \nabla S; \\
\end{aligned}
\]
where as follows

The main purpose of this section is to give the a priori estimates for \((2.1)\) with initial data \((1.5)\).

Proposition 2.1. For \((2.1)\), there exists an energy functional \(X_s(t)\) (see \((2.75)\) for the definition) with \(s \in \mathbb{R}\), satisfying \(X_s(t) \sim E_s(t)\), and a constant \(c\) such that

\[
\frac{d}{dt} X_s(t) + cD_s(t) \lesssim \|\nabla u\|_{L^\infty} X_s(t) + N_s(t),
\]

(2.5)

where

\[
N_s(t) \overset{\text{def}}{=} \|N_\rho(t)\|_{\dot{B}_{2,1}^s \cap \dot{B}_{2,1}^{s+1}} + \|N_u(t)\|_{\dot{B}_{2,1}^s} + \|N_S(t)\|_{\dot{B}_{2,1}^{s+1}}.
\]

(2.6)

Moreover, we have

\[
E_s(t) + \int_0^t D_s(\tau)d\tau \lesssim E_s(0) + \|\nabla u\|_{L^1(L^\infty)} E_s(t) + \int_0^t N_s(\tau)d\tau.
\]

(2.7)

Here \(E_s(t), D_s(t)\) and \(E_s(t)\) are defined in Definition 1.3.

Proof. Since the proof is a little bit longer, we divide it into several steps.

**Step 1. Localization of the system (2.1).** By setting

\[
d \overset{\text{def}}{=} |D|^{-1} \text{div } u, \quad g \overset{\text{def}}{=} |D|^{-1} \text{curl } u,
\]

we easily derive that

\[
\|u\|_{\dot{B}_{2,1}^s} \sim \|g\|_{\dot{B}_{2,1}^s} + \|d\|_{\dot{B}_{2,1}^s}.
\]

(2.8)

Then the equation for \(u\) in \((1.4)\) is equivalent to

\[
\partial_t d + T_u \cdot \nabla d - \left(\frac{8}{3} \mu + \lambda\right) \Delta d - R \left(1 + \frac{R}{C_v}\right)|D|\rho - \frac{\varepsilon R}{C_v}|D|S = N_d,
\]

\[
\partial_t g + T_u \cdot \nabla g - 2\mu \Delta g = N_g,
\]

(2.9)

where

\[
N_d = |D|^{-1} \text{div } N_u - |||D|^{-1} \text{div } |T_u| \cdot \nabla u|,
\]

\[
N_g = |D|^{-1} \text{curl } N_u - |||D|^{-1} \text{curl } T_u| \cdot \nabla u|.
\]

(2.10)

We address that due to \((4.9)\), for any \(s > -\frac{3}{2}\),

\[
\|N_d\|_{\dot{B}_{2,1}^s} + \|N_g\|_{\dot{B}_{2,1}^s} \lesssim \|N_u\|_{\dot{B}_{2,1}^s} + \|\nabla u\|_{L^\infty} \|u\|_{\dot{B}_{2,1}^s}.
\]

(2.11)

In what follows, we set \(a \overset{\text{def}}{=} \rho - 1\). The equations for \((a, d, S, g)\) can be localized as follows
using (4.9) and (4.14), we have

\[
\begin{aligned}
\partial_t (\Delta_j a) + \Delta_j (T_u \cdot \nabla a) + |D| \Delta_j d &= \Delta_j N_p, \\
\partial_t \Delta_j d + \Delta_j (T_u \cdot \nabla d) - \left( \frac{8}{3} \mu + \lambda \right) \Delta \Delta_j d - R(1 + \frac{R}{C_v}) |D| \Delta_j a - \frac{\varepsilon R}{C_v} |D| \Delta_j S \\
&= \Delta_j N_d, \\
\partial_t (|D|^{-1} \Delta_j S) + |D|^{-1} \Delta_j (T_u \cdot \nabla S) + \frac{\varepsilon}{C_v} |D| \Delta_j S + \frac{R}{C_v} |D| \Delta_j a &= |D|^{-1} \Delta_j N_S, \\
\partial_t \Delta_j g + \Delta_j (T_u \cdot \nabla g) - 2\mu \Delta \Delta_j g &= \Delta_j N_g.
\end{aligned}
\]  

(2.12)

**Step 2.** Catching the dissipation of $S$. To do so, we investigate the coupling effect of the system. We split the estimates into several steps.

**Step 2.1:** Basic energy estimates for $(a, d)$. Taking $L^2$-inner products of the first two equations in (2.12) with $\Delta_j a$ and $\Delta_j d$ respectively, we have

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\Delta_j a\|_{L^2}^2 + (\Delta_j a \cdot |D| \Delta_j a) &= -(\Delta_j (T_u \cdot \nabla a) \cdot \Delta_j a) + (\Delta_j N_p \cdot \Delta_j a), \\
\frac{1}{2} \frac{d}{dt} \|\Delta d\|_{L^2}^2 + \left( \frac{8}{3} \mu + \lambda \right) ||\nabla \Delta d||_{L^2}^2 - R(1 + \frac{R}{C_v}) \|\Delta_j d\|_{L^2} &
\end{aligned}
\]

\[
- \frac{\varepsilon R}{C_v} \|\Delta_j S \cdot \Delta_j d\) 
\]

\[
= -(\Delta_j (T_u \cdot \nabla d) \cdot \Delta_j d) + (\Delta_j N_d \cdot \Delta_j d),
\]

which along with (4.14) implies

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left[R(1 + \frac{R}{C_v}) \|\Delta_j a\|_{L^2}^2 + \|\Delta_j d\|_{L^2} \right] + \left( \frac{8}{3} \mu + \lambda \right) ||\nabla \Delta d||_{L^2}^2 - \frac{\varepsilon R}{C_v} \|\Delta_j S \cdot \Delta_j d\) 
\]

\[
\lesssim \|\nabla u\|_{L^\infty} \left( \|\Delta_j a\|_{L^2} \right. \sum_{|j'| \leq 4} \|\Delta_j a\|_{L^2} + \|\Delta_j d\|_{L^2} \sum_{|j'| \leq 4} \|\Delta_j d\|_{L^2} 
\]

\[
+ \|\Delta_j N_p\|_{L^2} \|\Delta_j a\|_{L^2} + \|\Delta_j N_d\|_{L^2} \|\Delta_j d\|_{L^2}.
\]

(2.13)

**Step 2.2:** Basic energy estimate for $S$. Taking $L^2$-inner product of the third equation in (2.12) with $|D|^{-1} \Delta_j S$, we have

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\Delta_j S\|_{L^2}^2 + \frac{\varepsilon}{C_v} \|\Delta_j S\|_{L^2}^2 + \frac{R}{C_v} \|\Delta_j a \cdot \Delta_j S\) 
\]

\[
= -(|D|^{-1} \Delta_j (T_u \cdot \nabla S) \cdot |D|^{-1} \Delta_j S) + (|D|^{-1} \Delta_j N_S \cdot |D|^{-1} \Delta_j S).
\]

(2.14)

Since

\[
|(|D|^{-1} \Delta_j (T_u \cdot \nabla S) \cdot |D|^{-1} \Delta_j S)| 
\]

\[
\leq |(\Delta_j |D|^{-1}, T_u \cdot \nabla S) \cdot |D|^{-1} \Delta_j S)| + |(\Delta_j (T_u \cdot \nabla |D|^{-1} S) \cdot \Delta_j |D|^{-1} S)|, 
\]

using (4.9) and (4.14), we have

\[
\begin{aligned}
|(|D|^{-1} \Delta_j (T_u \cdot \nabla S) \cdot |D|^{-1} \Delta_j S)| \lesssim 2^{-2j} \|\nabla u\|_{L^\infty} \sum_{|j'| \leq 4} \|\Delta_j \nabla S\|_{L^2} \|\Delta_j S\|_{L^2} 
\]

\[
+ \|\nabla u\|_{L^\infty} \|\Delta_j |D|^{-1} S\|_{L^2} \sum_{|j'| \leq 4} \|\Delta_j S\|_{L^2} 
\]

\[
\lesssim 2^{-2j} \|\nabla u\|_{L^\infty} \|\Delta_j S\|_{L^2} \sum_{|j'| \leq 4} \|\Delta_j S\|_{L^2}.
\]

(2.15)
Thanks to (2.14) and (2.15), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \frac{1}{|D|^{\frac{1}{2}}} \Delta_j S \right)^2_{L^2} + \frac{\varepsilon}{C_0} \left\| \Delta_j S \right\|_{L^2}^2 + \frac{R}{C_0} \left( \Delta_j a \right. \left| \Delta_j S \right)
\lesssim 2^{-2j} \left\| \nabla u \right\|_{L^{\infty}} \left\| \Delta_j S \right\|_{L^2} \sum_{|j'-j| \leq 4} \left| \Delta_{j'} S \right|_{L^2} + 2^{-2j} \left\| \Delta_j N_S \right\|_{L^2} \left\| \Delta_j S \right\|_{L^2}.
\] (2.16)

**Step 2.3:** Basic energy estimate for good unknown function $h$. Motivated by the effective viscous flux, we introduce the good unknown function $h$ defined by
\[
h \overset{\text{def}}{=} d - \left( \frac{8}{3} \mu + \lambda \right) |D| a + R |D|^{-1} S.
\] (2.17)

Thanks to (2.12), we have
\[
\partial_t \Delta_j h + \Delta_j (T_u \cdot \nabla h) - R |D| \Delta_j a = \Delta_j N_h,
\] (2.18)

where
\[
N_h = N_d - \left( \frac{8}{3} \mu + \lambda \right) |D| N_a + R |D|^{-1} N_S + \left( \frac{8}{3} \mu + \lambda \right) |D| |T_u| \cdot \nabla a - R |D|^{-1} T_u \cdot \nabla S.
\] (2.19)

Taking $L^2$-inner product of (2.18) with $\Delta_j h$, we have
\[
\frac{1}{2} \frac{d}{dt} \left\| \Delta_j h \right\|_{L^2}^2 - R (|D| \Delta_j a | \Delta_j h) = - (\Delta_j (T_u \cdot \nabla h) | \Delta_j h) + (\Delta_j N_h | \Delta_j h).
\] (2.20)

Using the definition of $h$ in (2.17) and the first equation of (2.12), we have
\[
-R (|D| \Delta_j a | \Delta_j h)
= -R (\Delta_j a | |D| \Delta_j d) + \left( \frac{8}{3} \mu + \lambda \right) R \left\| \Delta_j a \right\|_{L^2}^2 - R^2 (\Delta_j a | \Delta_j S)
= \frac{R}{2} \frac{d}{dt} \left\| \Delta_j a \right\|_{L^2}^2 + \left( \frac{8}{3} \mu + \lambda \right) R \left\| \nabla \Delta_j a \right\|_{L^2}^2 - R^2 (\Delta_j a | \Delta_j S)
+ R (\Delta_j (T_u \cdot \nabla a) | \Delta_j a) - R (\Delta_j N_p | \Delta_j a),
\]
which along with (2.20) and (4.14) implies
\[
\frac{1}{2} \frac{d}{dt} (\left\| \Delta_j h \right\|_{L^2}^2 + R \left\| \Delta_j a \right\|_{L^2}^2) + \left( \frac{8}{3} \mu + \lambda \right) R \left\| \nabla \Delta_j a \right\|_{L^2}^2 - R^2 (\Delta_j a | \Delta_j S)
\lesssim \left\| \nabla u \right\|_{L^{\infty}} \left( \left\| \Delta_j h \right\|_{L^2} + \sum_{|j'-j| \leq 4} \left| \Delta_{j'} h \right|_{L^2} + \left\| \Delta_j a \right\|_{L^2} \sum_{|j'-j| \leq 4} \left| \Delta_{j'} a \right|_{L^2} \right)
(2.21)
\]
Multiplying (2.16) by $RC_v$, and plus (2.21), we have
\[
\frac{1}{2} \frac{d}{dt} (\left\| \Delta_j h \right\|_{L^2}^2 + R \left\| \Delta_j a \right\|_{L^2}^2 + RCR_v \left\| D \right|^{-1} \Delta_j S \right\|_{L^2}^2) + \left( \frac{8}{3} \mu + \lambda \right) R \left\| \nabla \Delta_j a \right\|_{L^2}^2
+ R \varepsilon \left\| \Delta_j S \right\|_{L^2}^2
\lesssim \left\| \nabla u \right\|_{L^{\infty}} (2^{-2j} \left\| \Delta_j S \right\|_{L^2} + \left\| \Delta_j h \right\|_{L^2} + \left\| \Delta_j a \right\|_{L^2})
+ \sum_{|j'-j| \leq 4} (2^{-2j} \left\| \Delta_{j'} S \right\|_{L^2} + \left\| \Delta_{j'} h \right\|_{L^2} + \left\| \Delta_{j'} a \right\|_{L^2})
+ 2^{-2j} \left\| \Delta_j N_S \right\|_{L^2} \left\| \Delta_j S \right\|_{L^2}
+ \left\| \Delta_j N_a \right\|_{L^2} \left\| \Delta_j h \right\|_{L^2} + \left\| \Delta_j N_p \right\|_{L^2} \left\| \Delta_j a \right\|_{L^2}.
\] (2.22)
Step 2.4: Getting the dissipation of $S$. Let $A$ be a constant satisfying $A > \frac{1}{(\frac{2}{3} \mu + \lambda)^2 \lambda}$. Thanks to (2.13) and (2.22), we introduce

$$F_j^2 \overset{\text{def}}{=} [R(1 + \frac{R}{C_v}) + AR]\|\Delta_j a\|_{L^2}^2 + A\|\Delta_j h\|_{L^2}^2 + \|\Delta_j d\|_{L^2}^2 + ARC_v\|D\|^{-1}\Delta_j S\|_{L^2}^2,$$

$$H_j^2 \overset{\text{def}}{=} \left(\frac{8}{3}\mu + \lambda\right)\|\nabla \Delta_j d\|_{L^2}^2 + A\left(\frac{8}{3}\mu + \lambda\right)R\|\nabla \Delta_j a\|_{L^2}^2 + AR\|\Delta_j S\|_{L^2}^2 - \frac{\varepsilon R}{C_v} (|D|\Delta_j S \cdot \Delta_j d).$$

By virtue of $F_j$ and $H_j$, multiplying (2.22) by $A$, and plus (2.13), we get

$$\frac{1}{2} \frac{d}{dt} F_j^2 + H_j^2 \lesssim \|\nabla u\|_{L^\infty} \sum_{|j'-j| \leq 4} F_{j'} \cdot F_j$$

$$+ (2^{-j}\|\Delta_j N_S\|_{L^2} + \|\Delta_j N_d\|_{L^2} + \|\Delta_j N_h\|_{L^2} + \|\Delta_j N_p\|_{L^2}) \cdot F_j.$$  (2.24)

It is easy to check that

$$F_j^2 \sim \|\Delta_j a\|_{L^2}^2 + \|\Delta_j \nabla S\|_{L^2}^2 + \|\Delta_j d\|_{L^2}^2 + \||D|^{-1}\Delta_j S\|_{L^2}^2,$$

$$H_j^2 \sim \|\nabla \Delta_j d\|_{L^2}^2 + \|\nabla \Delta_j a\|_{L^2}^2 + \varepsilon \|\Delta_j S\|_{L^2}^2.$$  (2.25)

Thanks to (2.19) and (4.9), we have

$$\|\Delta_j N_h\|_{L^2} \lesssim \|\Delta_j N_d\|_{L^2} + 2^j \|\Delta_j N_h\|_{L^2} + 2^{-j} \|\Delta_j N_p\|_{L^2}$$

$$+ \|\nabla u\|_{L^\infty} \sum_{|j'-j| \leq 4} (2^j \|\Delta_j a\|_{L^2} + 2^{-j'} \|\Delta_j S\|_{L^2}).$$  (2.26)

Due to (2.25) and (2.27), we deduce from (2.24) that

$$\frac{1}{2} \frac{d}{dt} F_j^2 + H_j^2 \lesssim \|\nabla u\|_{L^\infty} \sum_{|j'-j| \leq 4} F_{j'} \cdot F_j$$

$$+ \left((1 + 2^j)\|\Delta_j N_p\|_{L^2} + \|\Delta_j N_h\|_{L^2} + 2^{-j} \|\Delta_j N_S\|_{L^2}\right) \cdot F_j.$$  (2.28)

Now, we consider (2.28) by splitting the frequency space into high and low parts. (1). For low frequency, i.e., $j \leq 0$, (2.25) and (2.26) yield

$$F_j^2 \sim (\|\Delta_j a\|_{L^2} + \|\Delta_j d\|_{L^2} + 2^{-j} \|\Delta_j S\|_{L^2})^2,$$

$$H_j^2 \gtrsim \varepsilon 2^j \left(\frac{1}{\varepsilon} \|\Delta_j a\|_{L^2}^2 + \frac{1}{\varepsilon} \|\Delta_j d\|_{L^2}^2 + 2^{-2j} \|\Delta_j S\|_{L^2}^2 \right) \gtrsim \varepsilon 2^j F_j^2.$$  (2.29)

Thanks to (2.28) and (2.29), there exists a constant $c_1' > 0$ such that

$$\frac{d}{dt} F_j + c_1' \varepsilon 2^j F_j \lesssim \|\nabla u\|_{L^\infty} \sum_{|j'-j| \leq 4} F_{j'} + \|\Delta_j N_p\|_{L^2} + \|\Delta_j N_h\|_{L^2} + 2^{-j} \|\Delta_j N_S\|_{L^2},$$

Since (2.29) shows that

$$F_j \gtrsim 2^{-j} \|\Delta_j S\|_{L^2},$$

there exists a constant $c_1 > 0$ such that

$$\frac{d}{dt} [2^{j's} F_j] + c_1 \varepsilon 2^{(s+1)} \|\Delta_j S\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} \sum_{|j'-j| \leq 4} 2^{j's} F_{j'},$$

$$+ 2^{j's} (\|\Delta_j N_p\|_{L^2} + \|\Delta_j N_h\|_{L^2} + 2^{-j} \|\Delta_j N_S\|_{L^2}).$$  (2.30)
Since (2.32) gives rise to 
\[ c \] there exists a constant 
\[ c \] such that 
\[ \| \nabla u \|_{L^\infty} \left( \| a \|_{B_{2,1}^s} + \| d \|_{B_{2,1}^s} + \| S \|_{B_{2,1}^{s-1}} \right) + \| N_p \|_{B_{2,1}^s} + \| N_d \|_{B_{2,1}^s} + \| N_S \|_{B_{2,1}^{s-1}}. \]
(2.31)

(2). For high frequency, i.e., \( j \geq 1 \), (2.25) and (2.26) yield
\[ F_j^2 \sim \left( 2^j \| \Delta_j a \|_{L^2} + \| \Delta_j d \|_{L^2} + 2^{-j} \| \Delta_j S \|_{L^2} \right)^2, \]
\[ H_j^2 \gtrsim 2^{2j} \| \Delta_j a \|_{L^2}^2 + \epsilon 2^{2j} \cdot \frac{1}{\epsilon} \| \Delta_j d \|_{L^2}^2 + \epsilon 2^{2j} \cdot 2^{-2j} \| \Delta_j S \|_{L^2}^2 \gtrsim \min \{ \epsilon 2^{2j}, 1 \} F_j^2. \]
(2.32)

Due to (2.32), we split the high frequency into the following two parts:
(i) if \( \epsilon 2^{2j} \leq 1 \), (2.32) gives rise to 
\[ H_j^2 \gtrsim c_2 \epsilon 2^{2j} F_j^2, \quad \text{for some} \quad c_2 > 0. \]
(2.33)

With (2.33), we deduce from (2.28) that
\[ \frac{d}{dt} F_j + c_2 \epsilon 2^{2j} F_j \]
\[ \lesssim \| \nabla u \|_{L^\infty} \sum_{|j' - j| \leq 4} F_{j'} + 2^j \| \Delta_j N_p \|_{L^2} + \| \Delta_j N_d \|_{L^2} + 2^{-j} \| \Delta_j N_S \|_{L^2}. \]

Since (2.32) yields that
\[ F_j \gtrsim 2^{-j} \| \Delta_j S \|_{L^2}, \]
there exists a constant \( c_2 > 0 \) such that
\[ \frac{d}{dt} [2^{js} F_j] + c_2 \epsilon 2^{j(s+1)} \| \Delta_j S \|_{L^2} \lesssim \mathcal{N}_j(t), \]
(2.34)

where
\[ \mathcal{N}_j(t) \overset{\text{def}}{=} \| \nabla u \|_{L^\infty} \sum_{|j' - j| \leq 4} 2^{js} F_{j'} + 2^j \| \Delta_j N_p \|_{L^2} + 2^j \| \Delta_j N_d \|_{L^2} \]
\[ + 2^{j(s-1)} \| \Delta_j N_S \|_{L^2}. \]
(2.35)

(ii) if \( \epsilon 2^{2j} > 1 \), (2.32) gives rise to 
\[ H_j^2 \gtrsim c'_2 F_j^2, \quad \text{for some} \quad c'_2 > 0. \]
(2.36)

With (2.36), we deduce from (2.28) that
\[ \frac{d}{dt} F_j + c'_2 F_j \lesssim \| \nabla u \|_{L^\infty} \sum_{|j' - j| \leq 4} F_{j'} + 2^j \| \Delta_j N_p \|_{L^2} + \| \Delta_j N_d \|_{L^2} + 2^{-j} \| \Delta_j N_S \|_{L^2}. \]

Since (2.32) gives rise to
\[ F_j \gtrsim 2^j \| \Delta_j a \|_{L^2}, \]
there exists a constant \( c_3 > 0 \) such that
\[ \frac{d}{dt} [2^{js} F_j] + c_3 2^{j(s+1)} \| \Delta_j a \|_{L^2} \lesssim \mathcal{N}_j(t). \]
(2.37)

Going back to the third equation of (2.12), we have
\[ \partial_t \left( |D|^{-1} \Delta_j S \right) + |D|^{-1} \Delta_j (T_u \cdot \nabla S) + \frac{\epsilon}{C_v} |D| \Delta_j S = - \frac{R}{C_v} |D| \Delta_j a + |D|^{-1} \Delta_j N_S, \]
which gives rise to
\[
\frac{1}{2} \frac{d}{dt} \left\| D^{-1} \Delta_j S \right\|_{L^2}^2 + \frac{\varepsilon}{C_v} \left\| \Delta_j S \right\|_{L^2}^2 = -\left( |D|^{-1} \Delta_j (T_u \cdot \nabla S) \right) \left| D \right|^{-1} \Delta_j S \\
- \frac{R}{C_v} \left( |D| \Delta_j a \right) |D|^{-1} \Delta_j S + (|D|^{-1} \Delta_j N_S \right) \left| D \right|^{-1} \Delta_j S.
\]
(2.38)

Using (2.15), we deduce from (2.38)
\[
\frac{d}{dt} \left( |D|^{-1} \Delta_j S \right) \left| L^2 \right| + \frac{\varepsilon}{4C_v} 2^{2j} \left( |D|^{-1} \Delta_j S \right) \left| L^2 \right|
\]
\[
\lesssim \left\| \nabla u \right\|_{L^\infty} \sum_{|j' - j| \leq 4} \left| D \right|^{-1} \Delta_j S \left| L^2 \right| + \left\| D \right| \left\| \Delta_j a \right\|_{L^2} + \left| D \right|^{-1} \Delta_j N_S \left| L^2 \right|
\]
Then there exists a constant $C_1 > 0$ such that for any $t \in (0, T)$
\[
\frac{d}{dt} \left( |D|^{-1} \Delta_j S \right) \left| L^2 \right| + \frac{\varepsilon}{8C_v} 2^{2j} \left( |D|^{-1} \Delta_j S \right) \left| L^2 \right| \leq C_1 2^{j} \left| D \right| \left\| \Delta_j a \right\|_{L^2} + C_1 N_j(t),
\]
(2.39)

where we use the fact $F_j \geq 2^{-j} \left| \Delta_j S \right|_{L^2}$.

Multiplying (2.39) with $\frac{\left( |D|^{-1} \Delta_j S \right) \left| L^2 \right|}{2^{2j} C_1}$, and plus (2.37), we have
\[
\frac{d}{dt} \left( |D|^{-1} \Delta_j S \right) \left| L^2 \right| + \frac{\varepsilon}{8C_v} 2^{2j} \left( |D|^{-1} \Delta_j S \right) \left| L^2 \right| \lesssim C_1 2^{j} \left| D \right| \left\| \Delta_j a \right\|_{L^2} + C_1 N_j(t).
\]
(2.40)

Thanks to (2.34) and (2.40), for $j \geq 1$, we have
\[
\frac{d}{dt} \left( |D|^{-1} \Delta_j S \right) \left| L^2 \right| \cdot 1_{c2^{j} > 1} + c_4 \varepsilon 2^{j} \left| D \right| \left\| \Delta_j S \right\|_{L^2} \lesssim N_j(t),
\]
(2.41)

where $c_4 = \min \{ c_2, \frac{c_3}{16C_v C_1} \}$.

Summing up $j \geq 1$ for (2.41), and using (2.32) and (2.35), we have
\[
\frac{d}{dt} \left( \sum_{j \geq 1} 2^{j} \left| D \right|^{-1} \Delta_j S \right) \left| L^2 \cdot 1_{c2^{2j} > 1} \right| + c_4 \varepsilon \left( \sum_{j \geq 1} 2^{j} \left| D \right| \left\| \Delta_j S \right\|_{L^2} \right)
\]
\[
\lesssim \left\| \nabla u \right\|_{L^\infty} \left( \left\| a \right\|_{B_2^{+1}} + \left\| d \right\|_{B_2^{-1}} + \left\| S \right\|_{B_2^{-1}} + \left\| N_\rho \right\|_{B_2^{+1}} + \left\| N_d \right\|_{B_2^{+1}} + \left\| N_S \right\|_{B_2^{-1}} \right).
\]
(2.42)

Combining (2.31) and (2.42), and using (1.11), (2.6) and (2.11), we have
\[
\frac{d}{dt} \left( \sum_{j \in \mathbb{Z}} 2^{j} \left( F_j \right) \left| D \right|^{-1} \Delta_j S \right) \left| L^2 \cdot 1_{c2^{2j} > 1} \right| + c_5 \varepsilon \left( S \right)_{B_2^{+1}}
\]
\[
\lesssim \left\| \nabla u \right\|_{L^\infty} E_s(t) + N_s(t),
\]
(2.43)

where $c_5 = \min \{ c_1, c_4 \}$. We address here that due to (2.8), it is easy to check that
\[
\sum_{j \in \mathbb{Z}} 2^{j} \left( F_j(t) + \left\| \Delta_j g \right\|_{L^2} \right) \sim E_s(t).
\]
(2.44)

Moreover, steady of (2.31) and (2.42), firstly integrating (2.30) and (2.41) over time interval $(0, t)$, then summing up the resulting inequalities for $j \leq 0$ and $j \geq 1$
respectively, we obtain by using (2.29) and (2.32) that for any \( t > 0, \)
\[
\|a\|_{L_t^\infty(B_{r_2}^\alpha \cap B_{r_2}^{-\alpha})} + \|d\|_{L_t^\infty(B_{r_2}^\alpha)} + \|S\|_{L_t^\infty(B_{r_2}^{-\alpha})} + \|S\|_{L_t^1(B_{r_2}^{-\alpha})} 
\lesssim E_a(0) + \|\nabla u\|_{L_t^1(L_\infty)} (\|a\|_{L_t^\infty(B_{r_2}^\alpha \cap B_{r_2}^{-\alpha})} + \|d\|_{L_t^\infty(B_{r_2}^\alpha)}) + \int_0^t N_a(\tau)d\tau. \tag{2.45}
\]

**Step 3. Estimates for \( a \) and \( d \).** Going back to the equations for \( a \) and \( d \) in (2.12), we have
\[
\begin{cases}
\partial_t \Delta_j a + \Delta_j (T_u \cdot \nabla a) + |D|\Delta_j d = \Delta_j N_p, \\
\partial_t \Delta_j d + \Delta_j (T_u \cdot \nabla d) - (\frac{8}{3}\mu + \lambda)\Delta\Delta_j d - R(1 + \frac{R}{C_v})|D|\Delta_j a \\
+ \frac{\varepsilon R}{C_v} |D|\Delta_j S + \Delta_j N_d.
\end{cases} \tag{2.46}
\]

Taking \( L^2 \)-inner products of (2.46) with \((R(1 + \frac{R}{C_v})\Delta_j a, \Delta_j d)^T\), we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} [R(1 + \frac{R}{C_v})\|\Delta_j a\|_{L^2}^2 + \|\Delta_j d\|_{L^2}^2] &+ (\frac{8}{3}\mu + \lambda)\|\nabla \Delta_j d\|_{L^2}^2 \\
&= -R(1 + \frac{R}{C_v}) (\Delta_j (T_u \cdot \nabla a) | \Delta_j a) - (\Delta_j (T_u \cdot \nabla d) | \Delta_j d) \\
&+ \frac{\varepsilon R}{C_v} (|D|\Delta_j S | \Delta_j d) + R(1 + \frac{R}{C_v}) (\Delta_j N_p | \Delta_j a) + (\Delta_j N_d | \Delta_j d),
\end{align*}
\]
which along with (4.14) implies
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} [R(1 + \frac{R}{C_v})\|\Delta_j a\|_{L^2}^2 + \|\Delta_j d\|_{L^2}^2] &+ (\frac{8}{3}\mu + \lambda)\|\nabla \Delta_j d\|_{L^2}^2 \\
&\lesssim \|\nabla u\|_{L_\infty} \left(\|\Delta_j a\|_{L^2} \sum_{|j' - j| \leq 4} \|\Delta_{j'} a\|_{L^2} + \|\Delta_j d\|_{L^2} \sum_{|j' - j| \leq 4} \|\Delta_{j'} d\|_{L^2}\right) \tag{2.47} \\
&+ \varepsilon 2^j \|\Delta_j S\|_{L^2} \|\Delta_j d\|_{L^2} + \|\Delta_j N_p\|_{L^2} \|\Delta_j a\|_{L^2} + \|\Delta_j N_d\|_{L^2} \|\Delta_j d\|_{L^2}.
\end{align*}
\]
To derive the diffusion of \( a \), we consider the following function
\[
\tilde{h} = d - (\frac{8}{3}\mu + \lambda)|D|a. \tag{2.48}
\]
Thanks to (2.46), we have
\[
\partial_t \Delta_j \tilde{h} + \Delta_j (T_u \cdot \nabla \tilde{h}) - R(1 + \frac{R}{C_v})|D|\Delta_j a = \Delta_j N_{\tilde{h}}, \tag{2.49}
\]
where
\[
N_{\tilde{h}} \overset{\text{def}}{=} (\frac{8}{3}\mu + \lambda)|D|, T_u \cdot \nabla a + \frac{\varepsilon R}{C_v} |D|S + N_d - (\frac{8}{3}\mu + \lambda)|D|N_p. \tag{2.50}
\]
Using (4.9), we have
\[
\|N_{\tilde{h}}\|_{B_{r_2}^{-\alpha}} \lesssim \|\nabla u\|_{L_\infty} \|a\|_{B_{r_2}^{\alpha}} + \varepsilon \|S\|_{B_{r_2}^{\alpha}} + \|N_d\|_{B_{r_2}^{-\alpha}} + \|N_p\|_{B_{r_2}^{-\alpha}}. \tag{2.51}
\]
Taking \( L^2 \)-inner product of (2.49) with \( \Delta_j \tilde{h} \), and using (2.48), we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\Delta_j \tilde{h}\|_{L^2}^2 &+ (\frac{8}{3}\mu + \lambda)R(1 + \frac{R}{C_v})\|\Delta_j a\|_{L^2}^2 \\
&= - (\Delta_j (T_u \cdot \nabla \tilde{h}) | \Delta_j \tilde{h}) + (\Delta_j N_{\tilde{h}} | \Delta_j \tilde{h}). \tag{2.52}
\end{align*}
\]

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Using the first equation of (2.46), we have

$$- \langle \Delta_j a | D \Delta_j d \rangle = \frac{1}{2} \frac{d}{dt} \| \Delta_j a \|^2_{L^2} + \langle \Delta_j (T_u \cdot \nabla a) | \Delta_j a \rangle - \langle \Delta_j N_{\rho} | \Delta_j a \rangle,$$

which along with (2.52) and (4.14) implies that

$$\frac{1}{2} \frac{d}{dt} \| \Delta_j \tilde{h} \|^2_{L^2} + R(1 + \frac{R}{C_u}) \| \Delta_j a \|^2_{L^2} + (\frac{8}{3} \mu + \lambda) R(1 + \frac{R}{C_v}) \| \nabla \Delta_j a \|^2_{L^2} \lesssim \| \nabla u \|_{L^\infty} \sum_{|j'-j| \leq 4} \| \Delta_j a \|_{L^2} + \| \Delta_j \tilde{h} \|_{L^2} \sum_{|j'-j| \leq 4} \| \Delta_j \tilde{h} \|_{L^2}$$

(2.53)

By the definition of \( \tilde{h} \), which along with (2.55) implies that

$$\langle \frac{d}{dt} \tilde{h} \rangle \lesssim \| \nabla u \|_{L^\infty} \sum_{|j'-j| \leq 4} \tilde{F}_{j'}$$

(2.55)

We define

$$\tilde{F}^2_j \overset{\text{def}}{=} 4 R(1 + \frac{R}{C_u}) \| \Delta_j a \|^2_{L^2} + 3 \| \Delta_j d \|^2_{L^2} + \| \Delta_j \tilde{h} \|^2_{L^2},$$

(2.54)

Multiplying (2.47) by 3, and plus (2.53), we obtain by (2.54)

$$\frac{1}{2} \frac{d}{dt} \tilde{F}^2_j + \tilde{H}^2_j \lesssim \| \nabla u \|_{L^\infty} \sum_{|j'-j| \leq 4} \tilde{F}_{j'}$$

(2.55)

$$+ \left( \varepsilon 2^j \| \Delta_j S \|_{L^2} + \| \Delta_j N_{\rho} \|_{L^2} + \| \Delta_j N_d \|_{L^2} + \| \Delta_j N_{\tilde{h}} \|_{L^2} \right) \tilde{F}_j.$$
(2). For \( j \leq 0 \), (2.66) and (2.57) yield
\[
\tilde{F}_j^2 \sim (\|\Delta_j a\|_{L^2} + \|\Delta_j d\|_{L^2})^2, \quad \tilde{H}_j^2 \geq c_7^2 2^{2j} \tilde{F}_j^2
\]
for some \( c_7 > 0 \). (2.62)

which along with (2.55) implies that
\[
\frac{d}{dt} \tilde{F}_j + c_7^2 2^{2j} \tilde{F}_j \lesssim \|\nabla u\|_{L^\infty} \sum_{|j' - j| \leq 4} \tilde{F}_{j'} + \varepsilon 2^j \|\Delta_j S\|_{L^2} + \|\Delta_j N_p\|_{L^2} + \|\Delta_j N_d\|_{L^2} + \|\Delta_j N_\delta\|_{L^2}.
\]

Summing up (2.63) for all \( j \leq 0 \), using (2.62) and (2.51), we obtain
\[
\frac{d}{dt} \sum_{j \leq 0} 2^{js} \tilde{F}_j + c_7 \sum_{j \leq 0} 2^{j(s+2)} \|\Delta_j a\|_{L^2} + c_7 \sum_{j \leq 0} 2^{j(s+2)} \|\Delta_j d\|_{L^2} \lesssim \varepsilon \|S\|_{L^{2,+1}}
\]
\[
+ \|\nabla u\|_{L^\infty} \sum_{|j' - j| \leq 4} 2^{j's} \tilde{F}_{j'} + 2^{js} \big(\|\Delta_j N_p\|_{L^2} + \|\Delta_j N_d\|_{L^2} + \|\Delta_j N_\delta\|_{L^2}\big).
\]

(2.64)

Summing up (2.63) for all \( j \leq 0 \), using (2.62) and (2.51), we obtain
\[
\frac{d}{dt} \sum_{j \leq 0} 2^{js} \tilde{F}_j + c_7 \sum_{j \leq 0} 2^{j(s+2)} \|\Delta_j a\|_{L^2} + c_7 \sum_{j \leq 0} 2^{j(s+2)} \|\Delta_j d\|_{L^2} \lesssim \varepsilon \|S\|_{L^{2,+1}}
\]
\[
+ \|\nabla u\|_{L^\infty} \sum_{|j' - j| \leq 4} 2^{j's} \tilde{F}_{j'} + 2^{js} \big(\|\Delta_j N_p\|_{L^2} + \|\Delta_j N_d\|_{L^2} + \|\Delta_j N_\delta\|_{L^2}\big).
\]

(2.65)

Moreover, integrating (2.60) and (2.63) over time interval \((0, t]\), using (2.50) and (4.9), we have for any \( j \in \mathbb{Z} \),
\[
\sup_{(0, t]} [2^{j's} \tilde{F}_j(\tau)] + c_6 \int_0^t 2^{j(s+1)} \|\Delta_j a\|_{L^2} \cdot 1_{j \geq 1} d\tau
\]
\[
+ c_7 \int_0^t 2^{j(s+2)} (\|\Delta_j a\|_{L^2} + \|\Delta_j d\|_{L^2}) \cdot 1_{j \leq 0} d\tau
\]
\[
\leq C_2 2^{j's} \tilde{F}_j(0) + C_2 \varepsilon \int_0^t 2^{j(s+1)} \|\Delta_j S(\tau)\|_{L^2} d\tau + C_2 \|\nabla u\|_{L^4(\Lambda^\infty)} \sum_{|j' - j| \leq 4} \sup_{(0, t]} [2^{j's} \tilde{F}_j(\tau)] + C_2 \int_0^t \left[ 2^{j's} + 2^{j(s+1)} \|\Delta_j N_p(\tau)\|_{L^2} \right] d\tau
\]
\[
+ 2^{j's} \|\Delta_j N_d(\tau)\|_{L^2} d\tau.
\]

Step 4. Improving estimate for \( d \) in high frequency. Going back to the equation for \( d \) in (2.12), we have
\[
\partial_t \Delta_j d + \Delta_j (T_u \cdot \nabla d) - \left( \frac{8}{3} \mu + \lambda \right) \Delta \Delta_j d
\]
\[
= R(1 + \frac{R}{C_v}) |D| \Delta_j a + \varepsilon R \frac{R}{C_v} |D| \Delta_j S + \Delta_j N_d.
\]

(2.67)
Taking $L^2$-inner product of (2.67) with $\Delta_j d$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\Delta_j d\|_{L^2}^2 + \left(\frac{8}{3} \mu + \lambda\right) \|\nabla \Delta_j d\|_{L^2}^2 = -\langle \Delta_j (T_u \cdot \nabla d) | \Delta_j d \rangle \\
+ R(1 + \frac{R}{C_v})(|D|\Delta_j a | \Delta_j d) + \varepsilon R(\|D|\Delta_j S | \Delta_j d) + (\Delta_j N_d | \Delta_j d),
\]
which along with (4.14) implies
\[
\frac{d}{dt} \|\Delta_j d\|_{L^2}^2 + \left(\frac{8}{3} \mu + \lambda\right) 2^{2j} \|\Delta_j d\|_{L^2}^2 \\
\lesssim \|\nabla u\|_{L^\infty} \sum_{|j'-j| \leq 4} \|\Delta_{j'} d\|_{L^2} + \|\Delta_j a\|_{L^2} + \varepsilon 2^j \|\Delta_j S\|_{L^2} + \|\Delta_j N_d\|_{L^2}.
\]
(2.68)

Multiplying (2.68) by $2^{j*}$, then summing up the resulting inequality for all $j \geq 1$, we obtain that there exists a constant $C_3 > 0$ such that
\[
\frac{d}{dt} \left( \sum_{j \geq 1} 2^{j*} \|\Delta_j d\|_{L^2} \right) + \frac{1}{4} \left(\frac{8}{3} \mu + \lambda\right) \sum_{j \geq 1} 2^{j(s+2)} \|\Delta_j d\|_{L^2} \\
\leq C_3 \sum_{j \geq 1} 2^{j(s+1)} \|\Delta_j a\|_{L^2} + C_3 \varepsilon \|S\|_{\bar{B}_{2,1}^{s+1}} + C_3 \|\nabla u\|_{L^\infty} \left( \|\mathcal{B}_{B_{2,1}^s} \cap \bar{B}_{2,1}^{s+1} + \|d\|_{\bar{B}_{2,1}^{s+1}} \right) \\
+ C_3 \left( \|N_d\|_{\bar{B}_{2,1}^s} \cap \bar{B}_{2,1}^{s+1} + \|N\|_{\bar{B}_{2,1}^{s+1}} \right) \\
\leq C_3 \sum_{j \geq 1} 2^{j(s+1)} \|\Delta_j d\|_{L^2} + C_3 \varepsilon \|S\|_{\bar{B}_{2,1}^{s+1}} + C_3 \|\nabla u\|_{L^\infty} \mathcal{E}_s(t) + C_3 N_s(t),
\]
(2.69)

where we use (1.11) and (2.11).

Moreover, multiplying (2.68) by $2^{j*}$ and integrating the resulting inequality over time interval $(0, t)$, we have for $j \geq 1$
\[
\sup_{(0, t)} 2^{j*} \|\Delta_j d(\tau)\|_{L^2} + \frac{1}{4} \left(\frac{8}{3} \mu + \lambda\right) \int_0^t 2^{j(s+2)} \|\Delta_j d\|_{L^2} d\tau \\
\leq C_3 2^{j*} \|\Delta_j d(0)\|_{L^2} + C_3 \int_0^t 2^{j(s+1)} \|\Delta_j a\|_{L^2} d\tau + C_3 \varepsilon \int_0^t 2^{j(s+1)} \|\Delta_j S\|_{L^2} d\tau \\
+ C_3 \|\nabla u\|_{L^1(\infty)} \sum_{|j'-j| \leq 4} \sup_{(0, t)} 2^{j*} \|\Delta_j^\prime d(\tau)\|_{L^2} + C_3 \int_0^t 2^{j*} \|\Delta_j N_d\|_{L^2} d\tau.
\]
(2.70)

**Step 5. Energy estimate for $g$.** Taking $L^2$-inner product of the last equation in (2.12) with $\Delta_j g$ gives rise to
\[
\frac{1}{2} \frac{d}{dt} \|\Delta_j g\|_{L^2}^2 + 2\mu \|\nabla \Delta_j g\|_{L^2}^2 = -\langle \Delta_j (T_u \cdot \nabla g) | \Delta_j g \rangle + (\Delta_j N_g | \Delta_j g),
\]
which along with (4.14) implies
\[
\frac{d}{dt} \|\Delta_j g\|_{L^2}^2 + \frac{\mu}{2} 2^{j(s+2)} \|\Delta_j g\|_{L^2} \\
\lesssim \|\nabla u\|_{L^\infty} \sum_{|j'-j| \leq 4} 2^{j*} \|\Delta_j^\prime g\|_{L^2} + 2^{j*} \|\Delta_j N_g\|_{L^2}.
\]
(2.71)
By similar derivation as previous, using (2.11), we obtain
\[
\frac{d}{dt} \|g\|_{L^2} + \frac{\mu}{2} \|u\|_{L^2} \leq C_4 \|\nabla u\|_{L^\infty} \|g\|_{L^2} + C_4 \|N_g\|_{L^2},
\]
(2.72)
for some constant $C_4 > 0$. Moreover, we have for $j \in \mathbb{Z}$
\[
\sup_{(0,t)} \|2^{j\alpha} \Delta_j g(\tau)\|_{L^2} + \frac{\mu}{2} \int_0^t \|2^{j\alpha+2} \Delta_j g\|_{L^2} d\tau \leq C_4 \int_0^t \|2^{j\alpha} \Delta_j g(0)\|_{L^2} \nabla u\|_{L^\infty} \|g\|_{L^2} \leq C_4 \|N_g\|_{L^2},
\]
(2.73)
Step 6. The priori estimate (2.5). Combining (2.43), (2.65), (2.69) and (2.72), we have
\[
\frac{d}{dt} X_s(t) + Y_s(t) \leq \|\nabla u\|_{L^\infty} E_s(t) + N_s(t),
\]
(2.74)
where
\[
X_s(t) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{j\alpha} F_j + \frac{C_s}{2C_1} \sum_{j \geq 1} 2^{j\alpha} \|1 \Delta_j S\|_{L^2} \cdot 1_{\alpha \geq 2} \|g\|_{L^2},
\]
(2.75)
and
\[
Y_s(t) \overset{\text{def}}{=} \frac{C_5}{4} \|S\|_{L^2} + \frac{C_5}{8C_2} \left( C_6 \|a\|_{L^2} + C_7 \|d\|_{L^2} \right)
\]
(2.76)
By virtue of (2.8), (2.44) and (2.56), we have
\[
X_s(t) \sim E_s(t),
\]
(2.77)
and
\[
Y_s(t) \geq C \Delta_s(t), \quad \text{for some } C > 0.
\]
Due to (2.76) and (2.77), we derive the desired result (2.5).
Moreover, combining (2.45), (2.66), (2.70) and (2.73), we arrive at (2.7). It ends the proof of the proposition.

3. Proof of Theorem 1.1 and Theorem 1.2. In this section, we shall prove the global existence of (1.4) with initial data (1.5).

3.1. Estimates of nonlinear terms. In this subsection, we shall drive the bounds of $N_1(t)$ and $N_2(t)$. The main result of this subsection is stated as follows

**Proposition 3.1.** Assume that $(\rho, u, S)$ is smooth on $(0, T) \times \mathbb{R}^3$. Then for any $s \in (-\frac{1}{2}, \frac{1}{2})$
\[
N_1(t) \leq C \|\rho - 1\|_{L^\infty(B_{\frac{3}{2}})} \|S\|_{L^\infty(B_{\frac{3}{2}})} \cdot \left( E_1(t) + \epsilon \|S\|_{L^\infty(B_{\frac{3}{2}})} \right) \cdot D_s(t),
\]
(3.1)
and
\[
N_2(t) \leq C \|\rho - 1\|_{L^\infty(B_{\frac{3}{2}})} \|S\|_{L^\infty(B_{\frac{3}{2}})} \cdot \left( E_1(t) + \epsilon \|S\|_{L^\infty(B_{\frac{3}{2}})} \right) \cdot D_s(t)
\]
(3.2)
Here and in what follows, $C(\lambda_1, \lambda_2, \cdots)$ is a constant depending on $\lambda_1, \lambda_2, \cdots$. 
Proof. Firstly, we recall that \( N_s(t) = \|N_\rho\|_{\dot{B}^1_{2,1}} + \|N_u\|_{\dot{B}^1_{2,1}} + \|N_\tau\|_{\dot{B}^1_{2,1}} \). We divide the proof into two steps.

**Step 1. Bound of \( N_s(t) \) for \( s \in (-\frac{1}{2}, \frac{1}{2}] \).** Since \( s \in (-\frac{1}{2}, \frac{1}{2}) \), then \( s - 1, s + 1 \in (-\frac{3}{2}, \frac{3}{2}] \).

(1). **Bound of \( \|N_\rho\|_{\dot{B}^1_{2,1}} \).** Firstly, we recall the expression of \( N_\rho \) in (2.2),

\[
N_\rho = -T_{\nu \rho} \cdot u - R(u, \nabla \rho) - (\rho - 1) \text{div} \ u.
\]

By virtue of (4.5) and the definitions of \( T_\theta \) and \( R(a, b) \), we have

\[
\|T_{\nu \rho} \cdot u + R(u, \nabla \rho)\|_{\dot{B}^1_{2,1}} \lesssim \|\rho - 1\|_{\dot{B}^1_{2,1}} \|\nabla u\|_{\dot{B}^1_{2,1}}.
\]

Using (4.5) again, we have

\[
\|\text{div} \ u\|_{\dot{B}^1_{2,1}} \lesssim \|\rho - 1\|_{\dot{B}^1_{2,1}} \|\nabla u\|_{\dot{B}^1_{2,1}}.
\]

Finally we get that

\[
\|N_\rho\|_{\dot{B}^1_{2,1}} \lesssim \|\rho - 1\|_{\dot{B}^1_{2,1}} \|\nabla u\|_{\dot{B}^1_{2,1}} \lesssim E(t) \cdot D_s(t).
\]

(2). **Bound of \( \|N_u\|_{\dot{B}^1_{2,1}} \).** We recall the expression of \( N_u \) in (2.3),

\[
N_u \overset{\text{def}}{=} -T_{\nu u} \cdot u - R(u, \nabla u) + 2\mu \left( \frac{1}{\rho} - 1 \right) \Delta u + \frac{2}{3} \mu + \lambda \left( \frac{1}{\rho} - 1 \right) \nabla \text{div} \ u
\]

\[
- R(1 + \frac{\nu}{C_v})(\rho \frac{\nu}{C_v} - e^{\frac{s}{2}}) - 1) \nabla \rho - \frac{\nu}{C_v}(\rho \frac{\nu}{C_v} - e^{\frac{s}{2}} - 1) \nabla S.
\]

By virtue of (4.5) and the definitions of \( T_\theta \) and \( R(a, b) \), we have

\[
\|T_{\nu u} \cdot u + R(u, \nabla u)\|_{\dot{B}^1_{2,1}} \lesssim \|u\|_{\dot{B}^1_{2,1}} \|u\|_{\dot{B}^1_{2,1} + 2}.
\]

Using (4.5) and (14.5), we have

\[
\|2\mu \left( \frac{1}{\rho} - 1 \right) \Delta u + \frac{2}{3} \mu + \lambda \left( \frac{1}{\rho} - 1 \right) \nabla \text{div} \ u\|_{\dot{B}^1_{2,1}} \lesssim \|\rho - 1\|_{\dot{B}^1_{2,1}} \|\nabla u\|_{\dot{B}^1_{2,1}} \lesssim C(\|\rho - 1\|_{L^\infty}) \|\rho - 1\|_{\dot{B}^1_{2,1}} \|\nabla u\|_{\dot{B}^1_{2,1} + 2}.
\]

For term \( R(1 + \frac{\nu}{C_v})(\rho \frac{\nu}{C_v} - e^{\frac{s}{2}} - 1) \nabla \rho \), using (4.5) again, we have

\[
\|\rho \frac{\nu}{C_v} - e^{\frac{s}{2}} - 1\|_{\dot{B}^1_{2,1}} \lesssim \|\rho \frac{\nu}{C_v} - e^{\frac{s}{2}} - 1\|_{\dot{B}^1_{2,1}} \|\nabla \rho\|_{\dot{B}^1_{2,1}} + \|\rho \frac{\nu}{C_v} - e^{\frac{s}{2}} - 1\|_{\dot{B}^1_{2,1}} \|\nabla \rho\|_{\dot{B}^1_{2,1} + 1}.
\]

Since

\[
\rho \frac{\nu}{C_v} - e^{\frac{s}{2}} - 1 = (\rho \frac{\nu}{C_v} - 1)(e^{\frac{s}{2}} - 1) + (\rho \frac{\nu}{C_v} - 1) + (e^{\frac{s}{2}} - 1),
\]

using (4.5) and (14.5), for any \( r \in (-\frac{3}{2}, \frac{3}{2}] \), we have

\[
\|\rho \frac{\nu}{C_v} - e^{\frac{s}{2}} - 1\|_{\dot{B}^1_{2,1}} \lesssim (1 + \|\rho \frac{\nu}{C_v} - 1\|_{\dot{B}^1_{2,1}}) \|\nabla S\|_{L^2} + \|\rho \frac{\nu}{C_v} - 1\|_{\dot{B}^1_{2,1}} \|\nabla S\|_{L^2} \lesssim C(\|\rho - 1\|_{L_{x,t}^\infty(\dot{B}^1_{2,1})}, \|\nabla S\|_{L^2} + \|\rho - 1\|_{\dot{B}^1_{2,1}}),
\]

(3.7)
where we used the fact \( \|a\|_{L^n} \lesssim \|a\|_{\dot{B}^{\frac{n}{2}}_{n,1}} \). Then we get

\[
\| (\rho \frac{\partial}{\partial t} e^{\frac{\varepsilon}{\rho^2}} - 1) \nabla \rho \|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leq C \left( \| \rho - 1 \|_{L^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}, \varepsilon \| S \|_{L^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} \right) \cdot (\varepsilon \| S \|_{\dot{B}^{\frac{3}{2}}_{2,1}}, \| \rho - 1 \|_{\dot{B}^{\frac{3}{2}}_{2,1}}) + \| \rho - 1 \|_{\dot{B}^{\frac{3}{2}}_{2,1} \cap \dot{B}^{\frac{3}{2}}_{2,1}} \cdot (\| \varphi_{\geq 1}(D)(\rho - 1) \|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \| \varphi_{\leq 0}(D)(\rho - 1) \|_{\dot{B}^{\frac{3}{2}}_{2,1}}).
\]

(3.8)

After similar derivation as (3.8), we have

\[
\| \frac{\varepsilon R}{C_v} (\rho \frac{\partial}{\partial t} e^{\frac{\varepsilon}{\rho^2}} - 1) \nabla S \|_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim \| \rho \frac{\partial}{\partial t} e^{\frac{\varepsilon}{\rho^2}} - 1 \|_{\dot{B}^{\frac{3}{2}}_{2,1}} \cdot \| \nabla S \|_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim C \left( \| \rho - 1 \|_{L^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}, \varepsilon \| S \|_{L^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} \right) (\varepsilon \| S \|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \| \rho - 1 \|_{\dot{B}^{\frac{3}{2}}_{2,1}}) \cdot \| S \|_{\dot{B}^{\frac{3}{2}}_{2,1}}.
\]

(3.9)

Thanks to (3.4), (3.5), (3.8) and (3.9), using the following inequalities

\[
\varepsilon \| S \|_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim \| S \|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \varepsilon \| S \|_{\dot{B}^{\frac{3}{2}}_{2,1}}, \quad \text{for } \varepsilon \lesssim 1,
\]

\[
\| \varphi_{\geq 1}(D)(\rho - 1) \|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \| \varphi_{\leq 0}(D)(\rho - 1) \|_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim \| \rho - 1 \|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \| \rho - 1 \|_{\dot{B}^{\frac{3}{2}}_{2,1}}
\]

(3.10)

we obtain

\[
\| N_u \|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leq C \left( \| \rho - 1 \|_{L^\infty(\dot{B}^{3/2}_{2,1})}, \varepsilon \| S \|_{L^\infty(\dot{B}^{3/2}_{2,1})} \right) \left( \varepsilon \| S \|_{\dot{B}^{3/2}_{2,1}} \right) \cdot \mathcal{D}(t). \quad \text{(3.11)}
\]

(3. Bound of \( \| N_S \|_{\dot{B}^{3/2}_{2,1}} \)). We recall the expression of \( N_S \) in (2.4),

\[
N_S \overset{\text{def}}{=} -T_{VS} \cdot u - R(u, \nabla S) + \frac{\varepsilon}{C_v} (1 - 1) \Delta S + \frac{R}{C_v} (1 - 1) \Delta \rho
\]

\[
+ \left( \frac{R^2}{C_v^2} - \frac{R}{C_v} \right) \frac{1}{\rho^2} |\nabla \rho|^2 + \frac{\varepsilon^2}{C_v^2} \cdot \nabla S \right)^2 + \frac{2 \varepsilon R}{C_v^2} \cdot \rho \cdot \nabla S + \rho^2 \cdot \rho \cdot \nabla S
\]

\[
+ \rho^{-1} (1 + \frac{\rho}{C_v}) e^{-\frac{\varepsilon}{C_v}} S(u) : \nabla u.
\]

Using (4.5) and the definitions of \( T_u b \) and \( R(a, b) \), we have

\[
\| T_{VS} \cdot u + R(u, \nabla S) \|_{\dot{B}^{3/2}_{2,1}} \lesssim \| S \|_{\dot{B}^{3/2}_{2,1}} \cdot \| u \|_{\dot{B}^{3/2}_{2,1}}. \quad \text{(3.12)}
\]

After similar derivation as (3.5) and (3.8), due to (4.5) and (4.15), we have

\[
\| \frac{\varepsilon}{C_v} \left( \frac{1}{\rho} - 1 \right) \Delta S + \frac{R}{C_v} \left( \frac{1}{\rho^2} - 1 \right) \Delta \rho \|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leq C \left( \| \rho - 1 \|_{L^\infty(\dot{B}^{3/2}_{2,1})}, \| \rho - 1 \|_{\dot{B}^{3/2}_{2,1}} \cdot (\| \varphi_{\geq 1}(D)(\rho - 1) \|_{\dot{B}^{3/2}_{2,1}} + \| \varphi_{\leq 0}(D)(\rho - 1) \|_{\dot{B}^{3/2}_{2,1}}).
\]

(3.13)

Whereas, we also have

\[
\| \frac{1}{\rho} |\nabla \rho|^2 \|_{\dot{B}^{3/2}_{2,1}} \leq C \left( \| \rho - 1 \|_{L^\infty(\dot{B}^{3/2}_{2,1})}, \| \rho - 1 \|_{\dot{B}^{3/2}_{2,1}} \cdot (\| \varphi_{\geq 1}(D)(\rho - 1) \|_{\dot{B}^{3/2}_{2,1}} + \| \varphi_{\leq 0}(D)(\rho - 1) \|_{\dot{B}^{3/2}_{2,1}}).
\]

(3.14)

Using (4.5) and (4.15) once again, we have

\[
\| \frac{\varepsilon^2}{C_v^2} \cdot \nabla S \|_{\dot{B}^{3/2}_{2,1}} \leq C \left( \| \rho - 1 \|_{L^\infty(\dot{B}^{3/2}_{2,1})}, \| \rho - 1 \|_{\dot{B}^{3/2}_{2,1}} \cdot \varepsilon \| \nabla S \|_{\dot{B}^{3/2}_{2,1}} \cdot \nabla S \|_{\dot{B}^{3/2}_{2,1}} \right)
\]

\[
\leq C \left( \| \rho - 1 \|_{L^\infty(\dot{B}^{3/2}_{2,1})}, \| \rho - 1 \|_{\dot{B}^{3/2}_{2,1}} \cdot \varepsilon \| \nabla S \|_{\dot{B}^{3/2}_{2,1}} \cdot \nabla S \|_{\dot{B}^{3/2}_{2,1}} \right).
\]
which implies

$$\| \frac{\varepsilon^2}{C_v^\frac{1}{2}} \| \nabla S \|_{B_{2,1}^\frac{2}{3}} \| \leq C(\| \rho - 1 \|_{L_\infty^2(B_{2,1}^\frac{2}{3})}) \| \| S \|_{B_{2,1}^\frac{3}{2}} \| \cdot \| S \|_{B_{2,1}^\frac{3}{2}}. \quad (3.15)$$

Similarly, we have

$$\| \frac{2\varepsilon R}{C_v \rho^2} \| \nabla \rho \cdot \nabla S \|_{B_{2,1}^\frac{2}{3}} \| \leq C(\| \rho - 1 \|_{L_\infty^2(B_{2,1}^\frac{2}{3})}) \| \| S \|_{B_{2,1}^\frac{3}{2}} \| \cdot \| S \|_{B_{2,1}^\frac{3}{2}}. \quad (3.16)$$

For the last term of $N_S$, due to (4.5), we first have

$$\| \rho^{-\left(1 + \frac{\beta}{20} \right)} e^{-\frac{\varepsilon}{20}} S(u) : \nabla u \|_{B_{2,1}^\frac{2}{3}} \leq \left(1 + \| \rho^{-\left(1 + \frac{\beta}{20} \right)} e^{-\frac{\varepsilon}{20}} - 1 \right) \cdot \| S(u) \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}}. \quad (3.17)$$

Following similar derivation as (3.7), we have

$$\| \rho^{-\left(1 + \frac{\beta}{20} \right)} e^{-\frac{\varepsilon}{20}} S(u) : \nabla u \|_{B_{2,1}^\frac{2}{3}} \leq C(\| \rho - 1 \|_{L_\infty^2(B_{2,1}^\frac{2}{3})}, \| S \|_{L_\infty^2(B_{2,1}^\frac{2}{3})}) (\| S \|_{B_{2,1}^\frac{3}{2}} + \| \rho - 1 \|_{B_{2,1}^\frac{3}{2}}).$$

Then we get

$$\| \rho^{-\left(1 + \frac{\beta}{20} \right)} e^{-\frac{\varepsilon}{20}} S(u) : \nabla u \|_{B_{2,1}^\frac{2}{3}} \leq C(\| \rho - 1 \|_{L_\infty^2(B_{2,1}^\frac{2}{3})}, \| S \|_{L_\infty^2(B_{2,1}^\frac{2}{3})}) (\| S \|_{B_{2,1}^\frac{3}{2}} + \| \rho - 1 \|_{B_{2,1}^\frac{3}{2}}). \quad (3.18)$$

Thanks to (3.12)-(3.17), using (3.10), we have

$$\| N_S \|_{B_{2,1}^\frac{2}{3}} \leq C(\| \rho - 1 \|_{L_\infty^2(B_{2,1}^\frac{2}{3})}, \| S \|_{L_\infty^2(B_{2,1}^\frac{2}{3})}) (E_\frac{1}{2}(t) + \| S \|_{B_{2,1}^\frac{3}{2}}) \cdot \mathcal{D}_t(t). \quad (3.18)$$

Combining (3.3), (3.11) and (3.18), we finally obtain (3.1).

**Step 2. Bound of $N_1^2(t)$.** Estimate of $N_1^2(t)$ is similar to $N_1(t)$ with $s \in (-\frac{1}{2}, \frac{1}{2}]$.

1. **Bound of $\| N_1 \|_{B_{2,1}^\frac{3}{2}}$.** By virtue of (4.4) and the definitions of $T_0b$ and $R(a, b)$, we have

$$\| T \nabla \rho \cdot u + R(u, \nabla \rho) \|_{B_{2,1}^\frac{3}{2}} \|_{B_{2,1}^\frac{3}{2}} \| \leq \| \rho - 1 \|_{B_{2,1}^\frac{3}{2}} \cdot \| u \|_{B_{2,1}^\frac{3}{2}} \cdot \| \rho - 1 \|_{B_{2,1}^\frac{3}{2}}.$$

Using (4.4) again, we have

$$\| \rho - 1 \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}} \| \leq \| \rho - 1 \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}} \| \cdot \| \rho - 1 \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}} \|.$$

Then we obtain

$$\| N_1 \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}} \| \leq E_\frac{1}{2}(t) \cdot \mathcal{D}_t(t) + E_\frac{1}{2}(t) \cdot \mathcal{D}_t(t). \quad (3.19)$$

2. **Bound of $\| N_2 \|_{B_{2,1}^\frac{3}{2}}$.** By virtue of (4.4) and the definitions of $T_0b$ and $R(a, b)$, we have

$$\| T \nabla u \cdot u + R(u, \nabla u) \|_{B_{2,1}^\frac{3}{2}} \|_{B_{2,1}^\frac{3}{2}} \| \leq \| u \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}} \cdot \| u \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}} \|.$$

Using (4.4) again, we have

$$\| \rho - 1 \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}} \| \leq \| \rho - 1 \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}} \| \cdot \| \rho - 1 \|_{B_{2,1}^\frac{3}{2}} \cdot \| \nabla u \|_{B_{2,1}^\frac{3}{2}} \|.$$
For term $R(1 + \frac{B}{\sqrt{c_v}}) (\rho^{\frac{1}{\gamma}} - 1) \frac{e^{\frac{1}{2} \frac{B}{\sqrt{c_v}}} - 1}{\frac{1}{2}} \nabla \rho$, using (4.4) again, we have

$$
\|(\rho^{\frac{1}{\gamma}} - 1) \frac{e^{\frac{1}{2} \frac{B}{\sqrt{c_v}}} - 1}{\frac{1}{2}} \nabla \rho\|_{B_{T,1}^{\frac{3}{2}}} \leq C(\|\rho - 1\|_{L^p(L^\infty)}(\|\rho - 1\|_{B_{T,1}^{\frac{3}{2}}} \cdot \|\nabla \rho\|_{B_{T,1}^{\frac{3}{2}}} + \|\rho - 1\|_{B_{T,1}^{\frac{3}{2}}} \cdot \|\nabla \rho\|_{B_{T,1}^{\frac{3}{2}}} )).
$$

which along with (3.7) for $r = \frac{1}{2}$ implies

$$
\|(\rho^{\frac{1}{\gamma}} - 1) \frac{e^{\frac{1}{2} \frac{B}{\sqrt{c_v}}} - 1}{\frac{1}{2}} \nabla \rho\|_{B_{T,1}^{\frac{3}{2}}} \leq C(\|\rho - 1\|_{L^p(L^\infty)}(\|\rho - 1\|_{B_{T,1}^{\frac{3}{2}}} \cdot \|\nabla \rho\|_{B_{T,1}^{\frac{3}{2}}} + \|\rho - 1\|_{B_{T,1}^{\frac{3}{2}}} \cdot \|\nabla \rho\|_{B_{T,1}^{\frac{3}{2}}} )).
$$

After similar derivation as (3.22), we have

$$
\|(\rho^{\frac{1}{\gamma}} - 1) \frac{e^{\frac{1}{2} \frac{B}{\sqrt{c_v}}} - 1}{\frac{1}{2}} \nabla \rho\|_{B_{T,1}^{\frac{3}{2}}} \leq C(\|\rho - 1\|_{L^p(L^\infty)}(\|\rho - 1\|_{B_{T,1}^{\frac{3}{2}}} \cdot \|\nabla \rho\|_{B_{T,1}^{\frac{3}{2}}} + \|\rho - 1\|_{B_{T,1}^{\frac{3}{2}}} \cdot \|\nabla \rho\|_{B_{T,1}^{\frac{3}{2}}} )).
$$

and
\[
\| R (\rho \frac{\partial}{\partial t} e^\frac{\partial}{\partial t} - 1) \nabla S \|_{B^\frac{3}{2}_2} \leq C \|\rho - 1\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|\nabla S\|_{L^\infty(B^\frac{3}{2}_2)} \\
\cdot (\|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \|\nabla S\|_{B^\frac{3}{2}_2} + \varepsilon \|\nabla S\|_{B^\frac{3}{2}_2} + \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \|\nabla S\|_{B^\frac{3}{2}_2}).
\] (3.24)

Thanks to (3.21)-(3.24) and (3.10), we obtain
\[
\| N_u \|_{B^\frac{3}{2}_2} \leq C \|\rho - 1\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|\nabla S\|_{L^\infty(B^\frac{3}{2}_2)} \cdot (E^\frac{3}{2}_\infty(t) \cdot D^\frac{3}{2}_\infty(t) \\
+ \varepsilon E^\frac{3}{2}_\infty(t) \cdot D^\frac{3}{2}_\infty(t) + E^\frac{3}{2}_\infty(t) \cdot D^\frac{3}{2}_\infty(t)).
\] (3.25)

(3). Bound of \( N_S \|_{B^\frac{3}{2}_2} \). We first have
\[
\| T \nabla S \cdot u + R(u, \nabla S) \|_{B^\frac{3}{2}_2} \lesssim \|S\|_{B^\frac{3}{2}_2} \cdot \|u\|_{B^\frac{3}{2}_2}.
\] (3.26)

Using (4.5) and (4.15), we get
\[
\| \frac{\varepsilon}{C_v} (1 - 1) \Delta S \|_{B^\frac{3}{2}_2} \leq C \|\rho - 1\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \|\nabla S\|_{B^\frac{3}{2}_2}.
\] (3.27)

By the similar argument as that for (3.22) or (3.23), we have
\[
\| R (\frac{1}{\rho^2} - 1) \Delta \rho \|_{B^\frac{3}{2}_2} \leq C \|\rho - 1\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \||\varphi \geq 1(D)\|_{B^\frac{3}{2}_2} \\
+ \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \||\varphi \leq 0(D)\|_{B^\frac{3}{2}_2} + \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \||\varphi \leq 0(D)\|_{B^\frac{3}{2}_2},
\] (3.28)

and
\[
\| \frac{1}{\rho} |\nabla \rho|^2 \|_{B^\frac{3}{2}_2} \leq C \|\rho - 1\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \||\varphi \geq 1(D)\|_{B^\frac{3}{2}_2} \\
+ \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \||\varphi \leq 0(D)\|_{B^\frac{3}{2}_2} + \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \||\varphi \leq 0(D)\|_{B^\frac{3}{2}_2}.
\] (3.29)

By virtue of (4.4) and (4.15), we have
\[
\| \frac{\varepsilon^2}{C_v} \rho \|_{B^\frac{3}{2}_2} \leq C \|\rho - 1\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \|\nabla S\|_{B^\frac{3}{2}_2} \cdot \|\nabla S\|_{B^\frac{3}{2}_2}.
\] (3.30)

Similarly, we have
\[
\| \frac{2C_R \varepsilon}{C_v^2} \rho \|_{B^\frac{3}{2}_2} \leq C \|\rho - 1\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \|\nabla S\|_{B^\frac{3}{2}_2} \\
+ \|\rho - 1\|_{B^\frac{3}{2}_2} \cdot \|\nabla S\|_{B^\frac{3}{2}_2}.
\] (3.31)

For the last term of \( N_S \), similarly as (3.17), we have
\[
\| \rho^{-1} (1 - \frac{\partial}{\partial t}) e^{-\frac{\partial}{\partial t}} \cdot \nabla u \|_{B^\frac{3}{2}_2} \\
\leq C \|\rho - 1\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|\nabla S\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|u\|_{B^\frac{3}{2}_2} \cdot \|u\|_{B^\frac{3}{2}_2}.
\] (3.32)

Thanks to (3.26)-(3.32), we have
\[
\| N_S \|_{B^\frac{3}{2}_2} \leq C \|\rho - 1\|_{L^\infty(B^\frac{3}{2}_2)} \cdot \|\nabla S\|_{L^\infty(B^\frac{3}{2}_2)} \cdot (E^\frac{3}{2}_\infty(t) \cdot D^\frac{3}{2}_\infty(t) \\
+ \varepsilon E^\frac{3}{2}_\infty(t) \cdot D^\frac{3}{2}_\infty(t) + E^\frac{3}{2}_\infty(t) \cdot D^\frac{3}{2}_\infty(t)).
\] (3.33)

We complete the proof of the proposition.  \( \square \)
3.2. Proof of Theorem 1.1. Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into several steps.

Step 1: Proof of (1.15-1.20). We first prove (1.15). Since the local well-posedness is easy to derive, we assume that

\[ T_M = \sup_{t \geq 0} \{ t \mid \sup_{s \in [0,t]} (X_\frac{1}{2}(s) + \varepsilon X_\frac{1}{2}(s)) \leq \eta^{\frac{1}{2}} \}, \]

where \( X_\varepsilon \) is defined in (2.75) and \( X_\varepsilon(t) \sim \mathcal{E}_\varepsilon(t) \).

By continuity, it is easy to see that \( T_M \geq 1 \). The next we want to show that \( T_M = \infty \). Thanks to (2.5) and Proposition 3.1, we obtain that

\[
\frac{d}{dt} (X_\frac{1}{2}(t) + \varepsilon X_\frac{1}{2}(t)) + c(D_{\frac{1}{2}}(t) + \varepsilon D_{\frac{1}{2}}(t)) \leq \mathcal{C}D_{\frac{1}{2}}(t) \cdot (X_\frac{1}{2}(t) + \varepsilon X_\frac{1}{2}(t)) + C\left( \| \rho - 1 \|_{L^\infty(\mathbb{R}^3)}, \varepsilon \| S \|_{L^\infty(\mathbb{R}^3)} \right) \left[ \mathcal{E}_{\frac{1}{2}}(t) \cdot D_{\frac{1}{2}}(t) - \mathcal{E}_{\frac{1}{2}}(t) \cdot D_{\frac{1}{2}}(t) + \varepsilon \mathcal{E}_{\frac{1}{2}}(t) \cdot D_{\frac{1}{2}}(t) \right],
\]

From this together with the definition of \( T_M \), we get that for \( t \leq T_M \),

\[
\frac{d}{dt} (X_\frac{1}{2}(t) + \varepsilon X_\frac{1}{2}(t)) + (c - C\eta^{\frac{1}{2}})(D_{\frac{1}{2}}(t) + \varepsilon D_{\frac{1}{2}}(t)) \leq 0,
\]

which along with the assumption \( X_\frac{1}{2}(0) + \varepsilon X_\frac{1}{2}(0) \leq C\eta \) shows that \( X_\frac{1}{2}(t) + \varepsilon X_\frac{1}{2}(t) \leq C\eta \). It contradicts with the definition of \( T_M \) if \( T_M < \infty \). Thus \( T_M = \infty \) and moreover (1.15) holds.

Next we want to show (1.16) and (1.20). For \( s \in (-\frac{1}{2}, \frac{1}{2}) \), (2.5) and Proposition 3.1 yield that

\[
\frac{d}{dt} X_s(t) + cD_s(t) \leq \mathcal{C}D_{\frac{1}{2}}(t) \cdot X_s(t) + C\left( \| \rho - 1 \|_{L^\infty(\mathbb{R}^3)}, \varepsilon \| S \|_{L^\infty(\mathbb{R}^3)} \right) \cdot \left( \mathcal{E}_{\frac{1}{2}}(t) \cdot D_{\frac{1}{2}}(t) + \mathcal{E}_{\frac{1}{2}}(t) \cdot D_{\frac{1}{2}}(t) \right),
\]

and

\[
\frac{d}{dt} X_{\frac{1}{2}}(t) + cD_{\frac{1}{2}}(t) \leq \mathcal{C}D_{\frac{1}{2}}(t) \cdot X_{\frac{1}{2}}(t) + C\left( \| \rho - 1 \|_{L^\infty(\mathbb{R}^3)}, \varepsilon \| S \|_{L^\infty(\mathbb{R}^3)} \right) \cdot \left( \mathcal{E}_{\frac{1}{2}}(t) \cdot D_{\frac{1}{2}}(t) + \mathcal{E}_{\frac{1}{2}}(t) \cdot D_{\frac{1}{2}}(t) \right).
\]

Thanks to (1.15) and Gronwall inequality, we get (1.16) and (1.20). In particular, we can obtain that

\[
\frac{d}{dt} X_{\frac{1}{2}}(t) + c_5D_{\frac{1}{2}}(t) \leq c_5D_{\frac{1}{2}}(t) \cdot X_{\frac{1}{2}}(t),
\]

for some constants \( c_9 > 0 \) and \( C_5 > 0 \).

Furthermore, using (2.7) and (1.10), we arrive at (1.17) and (1.18).

Step 2: Proof of (1.22). We recall that (1.20) holds. The interpolation inequality gives rise to

\[
\| u \|_{B_{\frac{3}{2},1}^\frac{1}{2}} \lesssim \| u \|_{B_{\frac{2}{2},1}^\frac{5}{2}} \| u \|_{B_{\frac{2}{2},1}^\frac{4}{2}}, \quad \| S \|_{B_{\frac{3}{2},1}^\frac{1}{2}} \lesssim \| S \|_{B_{\frac{2}{2},1}^\frac{5}{2}} \| S \|_{B_{\frac{2}{2},1}^\frac{4}{2}},
\]

\[
\| \rho - 1 \|_{B_{\frac{3}{2},1}^\frac{1}{2}} \lesssim \left( \| \rho - 1 \|_{B_{\frac{2}{2},1}^\frac{5}{2}} \right)^{\frac{5-4}{5}} \cdot \left( \| \rho - 1 \|_{B_{\frac{2}{2},1}^\frac{4}{2}} \right)^{\frac{4-4}{4}},
\]
from which together with (1.20), we have
\[ \|u\|_{\tilde{B}^3_{2,1}}^{\frac{9}{2} - \frac{2}{3}} \leq \|S\|_{\tilde{B}^3_{2,1}}^{\frac{9}{2} - \frac{2}{3}} \leq \mathcal{E}_S(t) \frac{2}{3 - 2\rho} \mathcal{D}_2(t) \leq \mathcal{D}_2(t); \]
\[ \left(\|\rho - 1\|^h_{\tilde{B}^3_{2,1}}\right)^{\frac{9}{2} - \frac{2}{3}} \leq \mathcal{E}_S(t) \frac{2}{3 - 2\rho} \mathcal{D}_2(t) \leq \mathcal{D}_2(t). \]
Then we have
\[ \varepsilon(X_2(t))^{\frac{9}{2} - \frac{2}{3}} \sim \varepsilon(\mathcal{E}_2(t))^{\frac{9}{2} - \frac{2}{3}} \leq \mathcal{D}_2(t). \] (3.35)
Thanks to (3.34) and (3.35), we deduce that there exists a constant \(c_{10} > 0\) such that
\[ \frac{d}{dt} X_{\frac{2}{3}}(t) + c_{10} \varepsilon(X_{\frac{2}{3}}(t))^{\frac{9}{2} - \frac{2}{3}} \leq C_5 \mathcal{D}_2(t) \cdot X_{\frac{2}{3}}(t). \] (3.36)
Since \( \int_0^\infty \mathcal{D}_2(\tau)d\tau \leq 1 \), if \( Y(t) = e^{-C_5 \int_0^t \mathcal{D}_2(\tau)d\tau} \cdot X_\frac{2}{3}(t) \), then \( X_\frac{2}{3}(t) \sim Y(t) \) and
\[ \frac{d}{dt} Y(t) + c_{11} \varepsilon Y(t)^{\frac{9}{2} - \frac{2}{3}} \leq 0, \] for some \(c_{11} > 0\),
from which, we get the desired result (1.22).

**Step 3:** Proof of (1.24) and (1.25). We first emphasize that under the condition (1.23), (1.15) and (1.16) still hold for the solution \((\rho, u, S)\). Now we give new estimates to \((\rho - 1, u)\). Thanks to (2.65), Proposition 3.1, (2.56), (1.15) and (1.16), we infer that for \( t \leq O(\eta^2 \varepsilon^{-1}) \)
\[ \|\rho(t) - 1\|^h_{\tilde{B}^3_{2,1}} + \int_0^t \|\rho(\tau) - 1\|^h_{\tilde{B}^3_{2,1}} d\tau \lesssim \varepsilon^2 + \varepsilon t \sup_{(0,t)} \|S\|^{\frac{9}{2} - \frac{2}{3}}_{\tilde{B}^3_{2,1}} \lesssim \eta^2 + \varepsilon t \lesssim \eta^2. \]
Now we are in a position to prove (1.24). Recalling (2.38), using Proposition 3.1 and (2.15), we easily derive that
\[ \frac{d}{dt} \left( \sum_{j \geq 1} 2^{-\frac{j}{2}} \|\Delta_j S(t)\|_{L^2} \right) \geq -\|\nabla u\|_{L^\infty} \|S\|_{\tilde{B}^3_{2,1}} \frac{1}{2} - \|\rho - 1\|^h_{\tilde{B}^3_{2,1}} - \varepsilon \|S\|^{\frac{9}{2} - \frac{2}{3}}_{\tilde{B}^3_{2,1}} \]
\[ - C(\|\rho - 1\|^h_{\tilde{B}^3_{2,1}} \cdot \varepsilon \|S\|_{\tilde{B}^3_{2,1}}^{\frac{9}{2} - \frac{2}{3}}) \cdot \mathcal{D}_\frac{2}{3}(t), \]
from which and the condition (1.23), we deduce that for \( t \leq O(\eta^2 \varepsilon^{-1}) \),
\[ \|S(t)\|^h_{\tilde{B}^3_{2,1}} \geq \|S_0\|^h_{\tilde{B}^3_{2,1}} - C_6 \eta^2, \] for some \(C_6 > 0\).
This is enough to get (1.24) and (1.25). For (1.25), we use the fact that if \( S_0 = \eta \varphi \geq 3(D)g \) with \( g \) being a Schwartz function,
\[ \|S_0\|_{\tilde{B}^3_{2,1}}^{\frac{1}{2}} \sim \|S_0\|_{\tilde{B}^3_{2,1}}^{\frac{1}{2}}. \]
We complete the proof of Theorem 1.1. \( \square \)

3.3. **Proof of Theorem 1.2.** Now we are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** We divide the proof into several steps.

**Step 1. Equations for the difference between the solutions to (1.4) and (1.6).** Suppose that \((\rho_0^N, u_0^N, S)\) is a solution to (1.4) with the initial data \((\rho_0, u_0, S_0)\). Thanks to Theorem 1.1, we have
\[ E_\frac{1}{2}(\infty) + \int_0^\infty \mathcal{D}_\frac{2}{3}(\tau)d\tau \lesssim \eta, \quad E_\frac{2}{3}(\infty) + \int_0^\infty \mathcal{D}_2(\tau)d\tau \lesssim 1. \] (3.37)
It is not difficult to check that there exist functions $\rho^I(t, x)$ and $u^I(t, x)$ such that these two functions are the limit of the sequence $\{\rho^N_n\}_{n \in \mathbb{N}}$ and $\{u^N_n\}_{n \in \mathbb{N}}$ as $\varepsilon_n \to 0$. Moreover $(\rho^I, u^I)$ is a global solution to (1.6) with the initial data $(\rho_0, u_0)$. If for any $r > -\frac{3}{2}$,

$$
E_r(\rho - 1, u)(t) \overset{\text{def}}{=} \|\rho - 1\|_{L^\infty(B_{r+1}^n)} + \|u\|_{L^\infty(B_{r+1}^n)},
$$

$$
D_r(\rho - 1, u)(t) \overset{\text{def}}{=} \|\rho(t) - 1\|_{B_{r+1}^1} + \|\rho(t) - 1\|_{B_{r+1}^{r+2}} + \|u(t)\|_{B_{r+1}^{r+2}},
$$

we have

$$
E_\frac{1}{2}(\rho^I - 1, u^I)(\infty) + \int_0^\infty D_\frac{1}{2}(\rho^I - 1, u^I)dt \lesssim \eta. \tag{3.39}
$$

For any $t \in (0, T)$, we set

$$
\rho^N(t) = \rho^I(t) + \varepsilon P_1(t), \quad u^N(t) = u^I(t) + \varepsilon u_1(t). \tag{3.40}
$$

It is easy to see that $(\rho_1, u_1)$ satisfies the following system

$$
\begin{aligned}
\partial_t \rho_1 + T_u \cdot \nabla \rho_1 + \text{div} \ u_1 &= N_{\rho_1}, \\
\partial_t u_1 + T_u \cdot \nabla u_1 - 2\mu \Delta u_1 - \left(\frac{2}{3} \mu + \lambda\right) \nabla \text{div} \ u_1 + R(1 + \frac{R}{C_v}) \nabla \rho_1 &= - R \nabla S + N_{u_1}, \\
\rho_1|_{t=0} = 0, \quad u_1|_{t=0} = 0,
\end{aligned} \tag{3.41}
$$

where $N_{\rho_1} = \tilde{N}_{\rho_1} + \tilde{N}_u_1$, and $N_{u_1} = \tilde{N}_{u_1} + \tilde{N}_u_1$ with

$$
\begin{aligned}
\tilde{N}_{\rho_1} &\overset{\text{def}}{=} - T \nabla \rho_1 \cdot u^I - R(u^I, \nabla \rho_1) - \rho_1 \text{div} \ u^I, \\
\tilde{N}_{u_1} &\overset{\text{def}}{=} - \text{div} ((\rho^N_1 - 1) u_1), \\
N_{\rho_1} &\overset{\text{def}}{=} - \nabla \rho_1 \cdot u^I - R(u^I, \nabla u_1) - u_1 \cdot \nabla u^N_1 + \frac{2\mu}{\varepsilon} \left(\frac{1}{\rho^N_1} - \frac{1}{\rho^N}\right) \Delta u^N_1 \\
+ \left(\frac{2}{3} \mu + \lambda\right) \frac{1}{\rho^N_1} \nabla \text{div} u^N_1 - \frac{1}{\varepsilon} R(1 + \frac{R}{C_v}) ((\rho^N_1)^{\frac{\mu}{\rho^N_1} - 1} e^{\frac{\varepsilon}{\rho^N_1}} - (\rho^I)^{\frac{\mu}{\rho^N_1} - 1}) \nabla \rho^N_1, \\
N_{u_1} &\overset{\text{def}}{=} 2\mu \left(\frac{1}{\rho^N_1} - 1\right) \Delta u_1 + \left(\frac{2}{3} \mu + \lambda\right) \left(\frac{1}{\rho^N_1} - 1\right) \nabla \text{div} u_1 \\
&- R(1 + \frac{R}{C_v}) ((\rho^I)^{\frac{\mu}{\rho^N_1} - 1} - 1) \nabla \rho_1 - \frac{R}{C_v} ((\rho^N_1)^{\frac{\mu}{\rho^N_1} e^{\frac{\varepsilon}{\rho^N_1}} - 1}) \nabla S.
\end{aligned} \tag{3.42}
$$

Denoting by

$$
d_1 \overset{\text{def}}{=} |D|^{-1} \text{div} \ u_1, \quad g_1 \overset{\text{def}}{=} |D|^{-1} \text{curl} \ u_1,
$$

the equation for $u_1$ in (3.41) is equivalent to

$$
\begin{aligned}
\partial_t d_1 + T_u \cdot \nabla d_1 - \left(\frac{8}{3} \mu + \lambda\right) \Delta d_1 - R(1 + \frac{R}{C_v}) |D| \rho_1 &= \frac{R}{C_v} |D| S + N_{d_1}, \\
\partial_t g_1 + T_u \cdot \nabla g_1 - 2\mu \Delta g_1 &= N_{g_1},
\end{aligned} \tag{3.43}
$$

where

$$
\begin{aligned}
N_{d_1} &= |D|^{-1} \text{div} \ N_{u_1} - |D|^{-1} \text{div} \ T_u \cdot \nabla u_1, \\
N_{g_1} &= |D|^{-1} \text{curl} \ N_{u_1} - |D|^{-1} \text{curl} \ T_u \cdot \nabla u_1.
\end{aligned}$$
Due to (4.9), we have
\[
\|\Delta_j N_{d_1}\|_{L^2} + \|\Delta_j N_{g_1}\|_{L^2} \lesssim \|\Delta_j N_{u_1}\|_{L^2} + \|\nabla u^f\|_{L^\infty} \sum_{|j'| - j| \leq 4} \|\Delta_{j'} u_1\|_{L^2}.
\] (3.44)

**Step 2. Estimates for \((\rho_1, u_1)\).** Similar derivations as (2.66), (2.70) and (2.73) yield
\[
\frac{d}{dt} X_j^{(1)} + Y_j^{(1)} \lesssim 2^{\frac{3}{2}j} \|\Delta_j S\|_{L^2} + \|\nabla u^f\|_{L^\infty} \sum_{|j'| - j| \leq 4} X_{j'}^{(1)}
\]
\[
+ (2^{\frac{1}{2}j} + 2^{\frac{3}{2}j}) \|\Delta_j N_{\rho_1}\|_{L^2} + 2^{\frac{1}{2}j} \|\Delta_j N_{d_1}\|_{L^2} + 2^{\frac{1}{2}j} \|\Delta_j N_{g_1}\|_{L^2},
\] (3.45)
where
\[
X_j^{(1)} \overset{\text{def}}{=} 2^{\frac{1}{2}j} F_j^{(1)} + \frac{c_6}{C_3} 2^{\frac{3}{2}j} \|\Delta_j d_1\|_{L^2} \cdot 1_{j \geq 1} + 2^{\frac{3}{2}j} \|\Delta_j g_1\|_{L^2},
\]
\[
Y_j^{(1)} \overset{\text{def}}{=} \frac{c_6}{2} 2^{\frac{3}{2}j} \|\Delta_j \rho_1\|_{L^2} \cdot 1_{j \geq 1} + c_7 2^{\frac{3}{2}j} \|\Delta_j \rho_1\|_{L^2} \cdot 1_{j \leq 0} + c_7 2^{\frac{3}{2}j} \|\Delta_j d_1\|_{L^2} \cdot 1_{j \leq 0}
\]
\[
+ \frac{c_6}{2C_3} \cdot \frac{1}{4} \left( \frac{8}{3} \mu + \lambda \right) 2^{\frac{3}{2}j} \|\Delta_j d_1\|_{L^2} \cdot 1_{j \geq 1} + \frac{\mu}{2} 2^{\frac{3}{2}j} \|\Delta_j \rho_1\|_{L^2}.
\]
Here \(F_j^{(1)}\) is defined in the same way as \(F_j\), that is,
\[
(F_j^{(1)})^2 \overset{\text{def}}{=} 4R(1 + \frac{R}{C_v}) \|\Delta_j \rho_1\|^2_{L^2} + 3 \|\Delta_j d_1\|^2_{L^2} + \|\Delta_j (d_1 - (\frac{8}{3} \mu + \lambda)D|\rho_1|)\|^2_{L^2}.
\]
Similar derivation as (2.56) yields
\[
2^{\frac{1}{2}j} F_j^{(1)} \sim (2^{\frac{1}{2}j} + 2^{\frac{3}{2}j}) \|\Delta_j \rho_1\|_{L^2} + 2^{\frac{1}{2}j} \|\Delta_j d_1\|_{L^2}
\]
which along with the following fact
\[
\|\Delta_j u_1\|_{L^2} \sim \|\Delta_j d_1\|_{L^2} + \|\Delta_j g_1\|_{L^2},
\] (3.46)
implies
\[
X_j^{(1)} \sim \bar{X}_j^{(1)} = (2^{\frac{1}{2}j} + 2^{\frac{3}{2}j}) \|\Delta_j \rho_1\|_{L^2} + 2^{\frac{1}{2}j} \|\Delta_j u_1\|_{L^2},
\]
\[
Y_j^{(1)} \gtrsim 2^{\frac{1}{2}j} \|\Delta_j \rho_1\|_{L^2} \cdot 1_{j \geq 1} + 2^{\frac{1}{2}j} \|\Delta_j \rho_1\|_{L^2} \cdot 1_{j \leq 0} + 2^{\frac{1}{2}j} \|\Delta_j u_1\|_{L^2}.
\] (3.47)

We study (3.45) in high and low frequencies separately.

**Step 2.1. For high frequencies \(j \geq 1\),** by virtue of (3.47), we have
\[
X_j^{(1)} \sim 2^{\frac{3}{2}j} \|\Delta_j \rho_1\|_{L^2} + 2^{\frac{3}{2}j} \|\Delta_j u_1\|_{L^2},
\]
\[
Y_j^{(1)} \geq 2c_{15} (2^{\frac{3}{2}j} \|\Delta_j \rho_1\|_{L^2} + 2^{\frac{3}{2}j} \|\Delta_j u_1\|_{L^2}) \geq 2c_{15} X_j^{(1)},
\] (3.48)
for some constants \(c_{15} > 0\) and \(c_{15} > 0\).

Due to (3.48) and (3.44), we deduce from (3.45) that
\[
\frac{d}{dt} X_j^{(1)} + c_{15} X_j^{(1)} + c_{15} (2^{\frac{3}{2}j} \|\Delta_j \rho_1\|_{L^2} + 2^{\frac{3}{2}j} \|\Delta_j u_1\|_{L^2})
\]
\[
\lesssim 2^{\frac{3}{2}j} \|\Delta_j S\|_{L^2} + \|\nabla u^f\|_{L^\infty} \sum_{|j'| - j| \leq 4} X_{j'}^{(1)} + 2^{\frac{3}{2}j} \|\Delta_j N_{\rho_1}\|_{L^2} + 2^{\frac{3}{2}j} \|\Delta_j N_{u_1}\|_{L^2}.
\] (3.49)
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Multiplying (3.49) by \( e^{\alpha t} \) (for \( \alpha \in (0, c_{15}) \) being determined in the later), then integrating the resulting inequality over \((0, t)\), and noticing that \( X_j^{(1)}(0) = 0 \), we have

\[
\sup_{(0,t)} X_j^{(1)} + c'_{15} \int_0^t e^{-\alpha(t-\tau)} (2^{\frac{j}{2}} \| \Delta_j \rho_1(\tau) \|_{L^2} + 2^{\frac{j}{2}} \| \Delta_j u_1(\tau) \|_{L^2}) d\tau \\
\lesssim \int_0^t e^{-\alpha(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j S(\tau) \|_{L^2} d\tau + \int_0^t e^{-\alpha(t-\tau)} \| \nabla u'(\tau) \|_{L^\infty} \sum_{|j'-j| \leq 4} X_{j'}^{(1)}(\tau) d\tau \\
+ \int_0^t e^{-\alpha(t-\tau)} (2^{\frac{j}{2}} \| \Delta_j N_{\rho_1}(\tau) \|_{L^2} + 2^{\frac{j}{2}} \| \Delta_j N_{u_1}(\tau) \|_{L^2}) d\tau,
\]

Since \( \int_0^t e^{-\alpha(t-\tau)} d\tau \lesssim 1 \), we have

\[
\int_0^t e^{-\alpha(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j S(\tau) \|_{L^2} d\tau \lesssim \sup_{(0,t)} \int_0^t e^{-\alpha(t-\tau)} d\tau \\
\lesssim \sup_{(0,t)} 2^{\frac{j}{2}} \| \Delta_j S \|_{L^2},
\]

\[
\int_0^t e^{-\alpha(t-\tau)} \| \nabla u'(\tau) \|_{L^\infty} \sum_{|j'-j| \leq 4} X_{j'}^{(1)}(\tau) d\tau \lesssim \| \nabla u' \|_{L^1(\sup)} \sum_{|j'-j| \leq 4} \sup_{(0,t)} X_{j'}^{(1)}.
\]

**Step 2.2. For low frequencies** \( j \leq 0 \), by virtue of (3.47), we have

\[
X_j^{(1)} \sim 2^{\frac{j}{2}} \| \Delta_j \rho_1 \|_{L^2} + 2^{\frac{j}{2}} \| \Delta_j u_1 \|_{L^2},
\]

\[
Y_j^{(1)} \geq 2c'_{16} 2^{\frac{j}{2}} (\| \Delta_j \rho_1 \|_{L^2} + \| \Delta_j u_1 \|_{L^2}) \geq 2c_{16} 2^{2j} X_j^{(1)},
\]

for some constants \( c'_{16} > 0 \) and \( c_{16} > 0 \). By virtue of (3.52) and (3.44), we deduce from (3.45) that

\[
\frac{d}{dt} X_j^{(1)} + c_{16} 2^{2j} X_j^{(1)} + c'_{16} 2^{\frac{j}{2}} \left( \| \Delta_j \rho_1 \|_{L^2} + \| \Delta_j u_1 \|_{L^2} \right) \\
\lesssim 2^{\frac{j}{2}} \| \Delta_j S \|_{L^2} + \| \nabla u' \|_{L^\infty} \sum_{|j'-j| \leq 4} X_{j'}^{(1)} + 2^{\frac{j}{2}} \| \Delta_j N_{\rho_1} \|_{L^2} + 2^{\frac{j}{2}} \| \Delta_j N_{u_1} \|_{L^2}.
\]

Taking \( \alpha = \min\{c_{15}, c_{16}\} \), then multiplying (3.49) by \( e^{\alpha 2^j t} \), we have

\[
\sup_{(0,t)} X_j^{(1)} + c_{16} \int_0^t e^{-\alpha 2^j(t-\tau)} (2^{\frac{j}{2}} \| \Delta_j \rho_1(\tau) \|_{L^2} + \| \Delta_j u_1(\tau) \|_{L^2}) d\tau \\
\lesssim \int_0^t e^{-\alpha 2^j(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j S(\tau) \|_{L^2} d\tau \\
+ \int_0^t e^{-\alpha 2^j(t-\tau)} \| \nabla u'(\tau) \|_{L^\infty} \sum_{|j'-j| \leq 4} X_{j'}^{(1)}(\tau) d\tau \\
+ \int_0^t e^{-\alpha 2^j(t-\tau)} (2^{\frac{j}{2}} \| \Delta_j N_{\rho_1}(\tau) \|_{L^2} + 2^{\frac{j}{2}} \| \Delta_j N_{u_1}(\tau) \|_{L^2}) d\tau,
\]
Since \( \int_0^t e^{-\alpha_2 \delta(t - \tau)} 2^{2j} d\tau \leq 1 \), we have
\[
\int_0^t e^{-\alpha_2 \delta(t - \tau)} 2^{2j} \|\Delta_j S(\tau)\|_{L^2} d\tau \lesssim \sup_{(0, t)} 2^{-\frac{j}{2}} \|\Delta_j S\|_{L^2},
\]
\[
\int_0^t e^{-\alpha_2 \delta(t - \tau)} \|\nabla u^I(\tau)\|_{L^\infty} \sum_{|j'| - j| \leq 4} X^{(j)}_{j'}(\tau) d\tau \lesssim \|\nabla u^I\|_{L^1(0, t)} \sum_{|j'| - j| \leq 4} X^{(j)}_{j'}.
\]
(3.55)

**Step 2.3. Estimate for \((\rho_1, d_1)\).** Summing up (3.50) for all \(j \geq 1\) while summing up (3.54) for all \(j \leq 0\), then adding the resulting inequalities together, by using (3.47), (3.51) and (3.55), we arrive at
\[
E_1^1(\rho_1, u_1)(t) + D_2^h(\rho_1, u_1)(t) + D_1^f(\rho_1, u_1)(t)
\]
\[
\lesssim \|S\|_{L^\infty(0, t; L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))} + \|\bar{S}\|_{L^\infty(0, t; L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))} + \|u^I\|_{L^1(0, t; L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))} \cdot E_1(\rho_1, u_1)(t) + \|\bar{N}_{\rho_1}\|_{L^1(0, t; L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))} + \|\bar{N}_{u_1}\|_{L^1(0, t; L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))} + \sum_{j \geq 1} \int_0^t e^{-\alpha_2 \delta(t - \tau)} (2^{\frac{1}{2}j} \|\Delta_j \bar{N}_{\rho_1}(\tau)\|_{L^2} + 2^{\frac{3}{2}j} \|\Delta_j \bar{N}_{u_1}(\tau)\|_{L^2}) d\tau
\]
\[
+ \sum_{j \leq 0} \int_0^t e^{-\alpha_2 \delta(t - \tau)} (2^{\frac{1}{2}j} \|\Delta_j \bar{N}_{\rho_1}(\tau)\|_{L^2} + 2^{\frac{3}{2}j} \|\Delta_j \bar{N}_{u_1}(\tau)\|_{L^2}) d\tau,
\]
(3.56)

where
\[
D_2^h(\rho_1, u_1)(t) \overset{\text{def}}{=} \sum_{j \geq 1} \int_0^t e^{-\alpha_2 \delta(t - \tau)} (2^{\frac{1}{2}j} \|\Delta_j \rho_1(\tau)\|_{L^2} + 2^{\frac{3}{2}j} \|\Delta_j u_1(\tau)\|_{L^2}) d\tau,
\]
\[
D_1^f(\rho_1, u_1)(t) \overset{\text{def}}{=} \sum_{j \leq 0} \int_0^t e^{\alpha_2 \delta(t - \tau)} 2^{\frac{1}{2}j} (\|\Delta_j \rho_1(\tau)\|_{L^2} + \|\Delta_j u_1(\tau)\|_{L^2}) d\tau.
\]
(3.57)

**Step 3. The bound of the nonlinear terms.** In this step, we give the bound of the source terms in the r.h.s of (3.56).

(1) Bound of \(\|\bar{N}_{\rho_1}\|_{L^1(0, t; L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))}\). By the definitions of \(T_a b\) and \(R(a, b)\), using (4.5), we have
\[
\|T_{\nabla \rho_1} \cdot u^I + R(u^I, \nabla \rho_1) + \rho_1 \text{div} u^I\|_{L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}})} \lesssim \|\rho_1\|_{L^\infty(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}})} \|u^I\|_{B^\frac{1}{2}_{\frac{1}{2}}}
\]
which along with (3.39) and (1.10) implies that
\[
\|\bar{N}_{\rho_1}\|_{L^1(0, t; L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))} \lesssim \|\rho_1\|_{L^\infty(0, t; L^\infty(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))} \|u^I\|_{L^1(0, t; L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))} \lesssim \eta E_1^1(\rho_1, u_1)(t).
\]
(3.58)

(2) Bound of \(\|\bar{N}_{u_1}\|_{L^1(0, t; L^2(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}))}\). We first deduce from (3.37) and (3.39) that
\[
\|\rho_1 - 1\|_{L^\infty(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}})} + \|\rho_1^N - 1\|_{L^\infty(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}})} + \varepsilon \|S\|_{L^\infty(B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}})} \lesssim \eta + \varepsilon \lesssim 1.
\]
Similarly as (3.58), using (4.5) and (4.15), we have
\[
\|T_{\nabla u_1} \cdot u^I + R(u^I, \nabla u_1) + u_1 \cdot \nabla u^N\|_{B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}}
\]
\[
\lesssim \|u_1\|_{B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}} \|u^I\|_{B^\frac{1}{2}_{\frac{1}{2}}} + \|u_1\|_{B^3_{\frac{1}{2}} \cap B^\frac{1}{2}_{\frac{1}{2}}} \|u_N^N\|_{B^\frac{1}{2}_{\frac{1}{2}}}.
\]
\[
\frac{2\mu}{\varepsilon} \left( \frac{1}{\rho_e^N} - \frac{1}{\rho_f^N} \right) \Delta u_e^N + \left( \frac{3}{2} \mu + \lambda \right) \left( \frac{1}{\rho_e^N} - \frac{1}{\rho_f^N} \right) \nabla \div u_e^N \bigg|_{B_{2,1}^N} \lesssim \| \rho_e^N \|_{B_{2,1}^N} \cdot \| \rho_1 \|_{B_{2,1}^N} \lesssim \| \rho_1 \|_{B_{2,1}^N} \| u_e^N \|_{B_{2,1}^N},
\]

and
\[
\frac{1}{\varepsilon} R(1 + \frac{R}{C}) \left( (\rho_e^N)_{\varepsilon e}^{-1} e^{\varepsilon \varepsilon} - (\rho_f^N)_{\varepsilon e}^{-1} \right) \nabla \rho_e^N \bigg|_{B_{2,1}^N} \lesssim \frac{1}{\varepsilon} (\| \rho_e^N \|_{B_{2,1}^N} (e^{\varepsilon \varepsilon} - 1)) \nabla \rho_e^N \bigg|_{B_{2,1}^N} + \| (\rho_f^N)_{\varepsilon e}^{-1} - (\rho_f^N)_{\varepsilon e}^{-1} \rho_e^N - \rho_f^N \|_{B_{2,1}^N} \cdot \rho_1 \cdot \nabla \rho_e^N \bigg|_{B_{2,1}^N} \lesssim \| S \|_{B_{2,1}^N}^\frac{1}{2} + \| \rho_1 \|_{B_{2,1}^N} \| \phi \|_{B_{2,1}^N} \bigg( \| \varphi_{\geq 1}(D)(\rho_e^N - 1) \|_{B_{2,1}^N} + \| \varphi_{\leq 0}(D)(\rho_e^N - 1) \|_{B_{2,1}^N} \bigg).
\]

Using the interpolation inequality
\[
\| S \|_{B_{2,1}^N} \lesssim \| S \|_{B_{2,1}^N}^\frac{1}{2} + \| S \|_{B_{2,1}^N}^\frac{1}{2},
\]

we have
\[
\| \tilde{N}u_e \|_{L^1_t(\tilde{B}_{1,1}^\frac{3}{2}, \tilde{B}_{1,1}^\frac{3}{2})} \lesssim \| S \|_{L^\infty_t(\tilde{B}_{1,1}^\frac{3}{2}, \tilde{B}_{1,1}^\frac{3}{2})} + \| E_2(\rho_e^N - 1, u_e^N) \|_{L^2_t} \sum_{j \geq 1} \int_0^t e^{-\alpha(t-t_j)} 2^j \| \Delta_j \tilde{N}u_e \|_{L^2_t} \]
\[
\lesssim \sum_{|j'-j| \leq 4} \int_0^t e^{-\alpha(t-\tau)} \|u_1(\tau)\|_{L^2(B_{\frac{1}{2}})} \|u_1(\tau)\|_{L^2(B_{\frac{1}{2}})} \cdot (2^{\frac{3}{2}j'} \|\Delta_j'(\rho_N^N - 1)\|_{L^2})^2 \\
(2^{\frac{3}{2}j'} \|\Delta_j'(\rho_N^N - 1)\|_{L^2})^{\frac{1}{2}} d\tau
\]
\[
\lesssim \sum_{|j'-j| \leq 4(0,t)} \sup_{0,t} \|u_1(\tau)\|_{L^2(B_{\frac{1}{2}})} \left(2^{\frac{3}{2}j'} \|\Delta_j'(\rho_N^N - 1)\|_{L^2}ight)^{\frac{1}{2}} \left(\int_0^t e^{-\alpha(t-\tau)} \|u_1(\tau)\|_{L^2} d\tau\right)^{\frac{1}{2}} \\
\cdot \left(\int_0^t e^{-\alpha(t-\tau)} 2^{\frac{3}{2}j'} \|\Delta_j'(\rho_N^N - 1)\|_{L^2} d\tau\right)^{\frac{1}{2}},
\]
which implies
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha(t-\tau)} 2^{\frac{3}{2}j} \|\Delta_j \text{div} (T_{u_1}(\rho_N^N - 1))\|_{L^2} d\tau 
\lesssim \|u_1\|_{L^\infty(B_{\frac{1}{2}})} \|\rho_N^N - 1\|_{L^\infty(B_{\frac{1}{2}})} \left(\int_0^t \sum_{j \geq -3} 2^{\frac{3}{2}j} \|\Delta_j(\rho_N^N - 1)\|_{L^2} d\tau\right)^{\frac{1}{2}}.
\]
Since
\[
\int_0^t e^{-\alpha(t-\tau)} \|u_1(\tau)\|_{L^2} d\tau 
\lesssim \sum_{j \geq 1} \int_0^t e^{-\alpha(t-\tau)} 2^{\frac{3}{2}j} \|\Delta_j u_1(\tau)\|_{L^2} d\tau + \sum_{j \leq 0} \int_0^t e^{-\alpha 2^{2j}(t-\tau)} 2^{\frac{3}{2}j} \|\Delta_j u_1(\tau)\|_{L^2} d\tau,
\]
\[
\int_0^t \sum_{j \geq -3} 2^{\frac{3}{2}j} \|\Delta_j(\rho_N^N - 1)\|_{L^2} d\tau 
\lesssim \int_0^t \sum_{j \geq 1} 2^{\frac{3}{2}j} \|\Delta_j(\rho_N^N - 1)\|_{L^2} d\tau + \int_0^t \sum_{j \leq 0} 2^{\frac{3}{2}j} \|\Delta_j(\rho_N^N - 1)\|_{L^2} d\tau
\]
using (3.37) and (3.57), we obtain
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha(t-\tau)} 2^{\frac{3}{2}j} \|\Delta_j \text{div} (T_{u_1}(\rho_N^N - 1))\|_{L^2} d\tau 
\lesssim \sqrt{\eta} \left(E_{\frac{1}{2}}(\rho_1, u_1)(t) + D_{\frac{1}{2}}(\rho_1, u_1)(t) + D_{\frac{3}{2}}(\rho_1, u_1)(t)\right).
\] (3.62)

For \(j \leq 0\), since \(\int_0^t e^{-\alpha 2^{2j}(t-\tau)} 2^{\frac{3}{2}j} d\tau \lesssim 1\), using Bernstein inequality (4.2), we have
\[
\int_0^t e^{-\alpha(t-\tau)} 2^{\frac{3}{2}j} \|\Delta_j \text{div} (T(\rho_N^N - 1)u_1 + R(\rho_N^N - 1, u_1))\|_{L^2} d\tau 
\lesssim \|\rho_N^N - 1\|_{L^\infty(B_{\frac{1}{2}})} \sum_{j' \geq j-4} 2^{\frac{3}{2}(j-j')} \sup_{(0,t)} \|\Delta_j' u_1\|_{L^2},
\]
and
\[
\int_0^t e^{-\alpha 2^{2j}(t-\tau)} 2^{\frac{3}{2}j} \|\Delta_j \text{div} (T_{u_1}(\rho_N^N - 1))\|_{L^2} d\tau
\]
Then we have
\[ \sum_{j \leq 0} \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j \div (\rho_e^N - 1) \|_{L^2} d\tau \]
\[ \lesssim \| u_1 \|_{L_t^\infty(B_{2,1}^2)} \sum_{|j-j'| \leq 4} \int_0^t 2^{\frac{j}{2}} \| \Delta_j (\rho_e^N - 1) \|_{L^2} d\tau. \]

which along with (3.60) and (3.62) implies
\[ \sum_{j \leq 0} \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j \div (\rho_e^N - 1) \|_{L^2} d\tau \lesssim \eta E_{\frac{1}{2}}(\rho_1, u_1)(t). \] (3.63)

Thanks to (3.60), (3.62) and (3.63), we obtain
\[ \sum_{j \geq 1} \int_0^t e^{-\alpha (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j \tilde{N}_{\rho_1}(\tau) \|_{L^2} d\tau \]
\[ + \sum_{j \leq 0} \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j \Delta_j \tilde{N}_{\rho_1}(\tau) \|_{L^2} d\tau \]
\[ \lesssim \sqrt{\eta} (E_{\frac{1}{2}}(\rho_1, u_1)(t) + D^h_{\frac{1}{2}}(\rho_1, u_1)(t) + D^k_{\frac{1}{2}}(\rho_1, u_1)(t)). \] (3.64)

(4). Bound of \( \sum_{j \geq 1} \int_0^t e^{-\alpha (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j \tilde{N}_{u_1}(\tau) \|_{L^2} d\tau + \sum_{j \leq 0} \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j \tilde{N}_{u_1}(\tau) \|_{L^2} d\tau \). For term 2\( \mu(\frac{1}{\rho^2} - 1) \Delta u_1 \), we first have for \( j \geq 1 \)
\[ \int_0^t e^{-\alpha (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j (T \frac{1}{\rho^2} - 1) \Delta u_1 + R(\frac{1}{\rho^2} - 1, \Delta u_1) \|_{L^2} d\tau \]
\[ \lesssim \| \frac{1}{\rho^2} - 1 \|_{L_t^\infty(B_{2,1}^2)} \sum_{j-j' \geq -4} 2^{\frac{j}{2} - j'} \int_0^t e^{-\alpha (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j' u_1 \|_{L^2} d\tau, \]
\[ \int_0^t e^{-\alpha (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j (T \Delta u_1 (\frac{1}{\rho^2} - 1)) \|_{L^2} d\tau \]
\[ \lesssim \sum_{|j' - j| \leq 4} \sup_{(0, t)} |\Delta_j' (\frac{1}{\rho^2} - 1)|_{L^2} \int_0^t e^{-\alpha (t-\tau)} \| u_1 \|_{B_{2,1}^2} d\tau, \]

which along with (4.15) and (3.61) implies
\[ \sum_{j \geq 1} \int_0^t e^{-\alpha (t-\tau)} 2^{\frac{j}{2}} \| \Delta_j \left( \frac{1}{\rho^2} - 1 \right) \Delta u_1 \|_{L^2} d\tau \]
\[ \lesssim \| \rho^t - 1 \|_{L_t^\infty(B_{2,1}^2)} \int_0^t e^{-\alpha (t-\tau)} \| u_1 \|_{B_{2,1}^2} d\tau \]
\[ \lesssim \eta (D^h_{\frac{1}{2}}(\rho_1, u_1)(t) + D^k_{\frac{1}{2}}(\rho_1, u_1)(t)). \] (3.65)
For $j \leq 0$, since \( \int_0^t e^{-\alpha 2j(t-\tau)} \overline{\Delta_j (T_{\frac{1}{\rho^j}} - 1) \Delta u_1} \, d\tau \lesssim 1 \), we have
\[
\int_0^t e^{-\alpha 2j(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j (T_{\frac{1}{\rho^j}} - 1) \Delta u_1 \|_{L^2} \, d\tau 
\lesssim \| \frac{1}{\rho^j} - 1 \|_{L^\infty(B_{2,1}^j)} \sum_{|j' - j| \leq 4} \sup_{0 \leq t < 1} 2^{\frac{j}{2} j'} \| \Delta_j \Delta u_1 \|_{L^2},
\]
which along with (4.15) shows that
\[
\sum_{j \leq 0} \int_0^t e^{-\alpha 2j(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j (T_{\frac{1}{\rho^j}} - 1) \Delta u_1 \|_{L^2} \, d\tau \lesssim \| \rho^j - 1 \|_{L^\infty(B_{2,1}^j)} \| u_1 \|_{L^\infty(B_{2,1}^j)}.
\]
(3.66)

And we also have
\[
\int_0^t e^{-\alpha 2j(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j (T_{\frac{1}{\rho^j}} - 1) + R(\frac{1}{\rho^j} - 1, \Delta u_1) \|_{L^2} \, d\tau 
\lesssim \| u_1 \|_{L^\infty(B_{2,1}^j)} \left( \sum_{j \geq 1} 2^{\frac{j}{2}} \int_0^t 2^{\frac{j}{2} j'} \| \Delta_j (\frac{1}{\rho^j} - 1) \|_{L^2} \, d\tau \right)
\]
which implies
\[
\sum_{j \leq 0} \int_0^t e^{-\alpha 2j(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j (T_{\frac{1}{\rho^j}} - 1) + R(\frac{1}{\rho^j} - 1, \Delta u_1) \|_{L^2} \, d\tau 
\lesssim \| u_1 \|_{L^\infty(B_{2,1}^j)} \left( \sum_{j' \geq 1} \int_0^t 2^{\frac{j}{2} j'} \| \Delta_j (\frac{1}{\rho^j} - 1) \|_{L^2} \, d\tau \right)
\]
\[
\quad + \sum_{j \leq 0} 2^{\frac{j}{2}} \int_0^t 2^{\frac{j}{2} j'} \| \Delta_j (\frac{1}{\rho^j} - 1) \|_{L^2} \, d\tau \right).
\]
Using (4.15), we get
\[
\sum_{j \leq 0} \int_0^t e^{-\alpha 2j(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j (T_{\frac{1}{\rho^j}} - 1) + R(\frac{1}{\rho^j} - 1, \Delta u_1) \|_{L^2} \, d\tau 
\lesssim \| u_1 \|_{L^\infty(B_{2,1}^j)} \left( \int_0^t (\| \rho^j - 1 \|_{B_{2,1}^j}^h + \| \rho^j - 1 \|_{B_{2,1}^j}^\alpha) \, d\tau \right).
\]
(3.67)

Thanks to (3.39), we deduce from (3.66) and (3.67) that
\[
\sum_{j \leq 0} \int_0^t e^{-\alpha 2j(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j (\frac{1}{\rho^j} - 1) \|_{L^2} \, d\tau \lesssim \eta E_k(\rho_1, u_1)(t),
\]
(3.68)
which along with (3.65) implies
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha(t-t')} 2^{\frac{3}{2}j} \| \Delta_j \left( \left( \frac{1}{\rho} - \frac{\mu}{\rho^2} - 1 \right) \Delta u_1 \right) \|_{L^2} d\tau
\]
\[+ \sum_{j \leq 0} \int_0^t e^{-\alpha 2^{2j}(t-t')} 2^{\frac{3}{2}j} \| \Delta_j \left( \left( \frac{1}{\rho} - \frac{\mu}{\rho^2} - 1 \right) \Delta u_1 \right) \|_{L^2} d\tau \]
\[
\lesssim \eta \left( E_{\frac{1}{2}}(\rho_1, u_1)(t) + D_{\frac{1}{2}}^h(\rho_1, u_1)(t) + D_{\frac{1}{2}}^\ell(\rho_1, u_1)(t) \right) .
\]
(3.69)

The same estimate holds for term \((\frac{3}{2}\mu + \lambda)(\frac{1}{\rho^2} - 1)\nabla \text{div} u_1\).

For the third term \(-R(1 + \frac{\mu}{\rho^2})((\rho')^{\frac{\mu}{\rho^2} - 1} - 1)\nabla \rho_1\), we first have for \(j \geq 1\),
\[
\int_0^t e^{-\alpha(t-t')} \frac{2^j}{2} \| \Delta_j \left( T_{(\rho')}^{\frac{\mu}{\rho^2} - 1} - 1 \right) \nabla \rho_1 \|_{L^2} d\tau
\]
\[
\lesssim \|(\rho')^{\frac{\mu}{\rho^2} - 1} - 1\|_{L^\infty(B_{2^j}^\frac{3}{2})} \sum_{j' \geq j-4} \int_0^t e^{-\alpha(t-t')} \frac{2^{j'}}{2} \| \Delta_j' \rho_1 \|_{L^2} d\tau
\]
\[
\int_0^t e^{-\alpha(t-t')} \frac{2^j}{2} \| \Delta_j \left( T_{\nabla \rho_1} \left( ((\rho')^{\frac{\mu}{\rho^2} - 1} - 1) \right) \right) \|_{L^2} d\tau
\]
\[
\lesssim \|\rho_1\|_{L^\infty(B_{2^j}^\frac{3}{2})} \sum_{j' = j-4} \int_0^t \frac{2^{j'}}{2} \| \Delta_j' \left( ((\rho')^{\frac{\mu}{\rho^2} - 1} - 1) \right) \|_{L^2} d\tau
\]
which along with (4.15) implies
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha(t-t')} \frac{2^j}{2} \| \Delta_j \left( \left( ((\rho')^{\frac{\mu}{\rho^2} - 1} - 1) \nabla \rho_1 \right) \|_{L^2} d\tau
\]
\[
\lesssim \|(\rho')^{\frac{\mu}{\rho^2} - 1} - 1\|_{L^\infty(B_{2^j}^\frac{3}{2})} \sum_{j' \geq j-3} \int_0^t e^{-\alpha(t-t')} \frac{2^{j'}}{2} \| \Delta_j' \rho_1 \|_{L^2} d\tau + \|\rho_1\|_{L^\infty(B_{2^j}^\frac{3}{2})}
\]
\[
\sum_{j' = j-3} \int_0^t \frac{2^{j'}}{2} \| \Delta_j' (\rho') - 1 \|_{L^2} d\tau
\]
\[
\lesssim \|(\rho')^{\frac{\mu}{\rho^2} - 1} - 1\|_{L^\infty(B_{2^j}^\frac{3}{2})} \left( D_{\frac{3}{2}}^h(\rho_1, u_1)(t) + D_{\frac{3}{2}}^\ell(\rho_1, u_1)(t) \right) + \|\rho_1\|_{L^\infty(B_{2^j}^\frac{3}{2})}
\]
\[
\int_0^t \|(\rho')^{\frac{\mu}{\rho^2} - 1} - 1\|_{L^\infty(B_{2^j}^\frac{3}{2})}^h + \|(\rho')^{\frac{\mu}{\rho^2} - 1} - 1\|_{L^\infty(B_{2^j}^\frac{3}{2})}^\ell d\tau.
\]

Thanks to (3.39), we get
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha(t-t')} \frac{2^j}{2} \| \Delta_j \left( \left( ((\rho')^{\frac{\mu}{\rho^2} - 1} - 1) \nabla \rho_1 \right) \|_{L^2} d\tau
\]
\[
\lesssim \eta \left( E_{\frac{1}{2}}(\rho_1, u_1)(t) + D_{\frac{1}{2}}^h(\rho_1, u_1)(t) + D_{\frac{1}{2}}^\ell(\rho_1, u_1)(t) \right) .
\]
(3.70)

For \(j \leq 0\), since \(\int_0^t e^{-\alpha 2^{2j}(t-t')} 2^{2j} d\tau \lesssim 1\), similarly as the derivation of (3.66), we have
\[
\sum_{j \leq 0} \int_0^t e^{-\alpha 2^{2j}(t-t')} \frac{2^j}{2} \| \Delta_j \left( T_{(\rho')}^{\frac{\mu}{\rho^2} - 1} - 1 \right) \nabla \rho_1 \|_{L^2} d\tau
\]
\[
\lesssim \|\rho' - 1\|_{L^\infty(B_{2^j}^\frac{3}{2})} \|\rho_1\|_{L^\infty(B_{2^j}^\frac{3}{2})}
\]
(3.71)
And we also have
\[
\int_0^t e^{-\alpha 2^j(t-\tau)2^{\frac{j}{2}}} ||\Delta_j(T\nabla \rho_1((\rho^t)^{\frac{N}{e}} - 1) + R((\rho^t)^{\frac{N}{e}} - 1), \nabla \rho_1)||_{L^2} d\tau \\
\lesssim ||\rho_1||_{L^\infty(B^j_{2,1})} 2^{\frac{j}{2}} \left( \sum_{j' \geq 1} + \sum_{0 \leq j' \leq j-4} \right) \int_0^t 2^{\frac{j}{2}} ||\Delta_j((\rho^t)^{\frac{N}{e}} - 1)||_{L^2} d\tau,
\]
which implies
\[
\sum_{j \leq 0} \int_0^t e^{-\alpha 2^j(t-\tau)2^{\frac{j}{2}}} ||\Delta_j(T\nabla \rho_1((\rho^t)^{\frac{N}{e}} - 1) + R((\rho^t)^{\frac{N}{e}} - 1), \nabla \rho_1)||_{L^2} d\tau \\
\lesssim ||\rho_1||_{L^\infty(B^j_{2,1})} \left( \sum_{j \leq 0} 2^j \sum_{j' \geq 1} \int_0^t 2^{\frac{j}{2}} ||\Delta_j((\rho^t)^{\frac{N}{e}} - 1)||_{L^2} d\tau \\
+ \sum_{j \leq 0} \sum_{0 \leq j' \leq j-4} 2^{2(j-j')} \int_0^t 2^{\frac{j}{2}} ||\Delta_j((\rho^t)^{\frac{N}{e}} - 1)||_{L^2} d\tau \right) \\
\lesssim ||\rho_1||_{L^\infty(B^j_{2,1})} \left( \sum_{j' \geq 1} \int_0^t 2^{\frac{j}{2}} ||\Delta_j((\rho^t)^{\frac{N}{e}} - 1)||_{L^2} d\tau \\
+ \sum_{j' \leq 0} \int_0^t 2^{\frac{j}{2}} ||\Delta_j((\rho^t)^{\frac{N}{e}} - 1)||_{L^2} d\tau \right).
\]
Using (4.15), together with (3.71), we get
\[
\sum_{j \leq 0} \int_0^t e^{-\alpha 2^j(t-\tau)2^{\frac{j}{2}}} ||\Delta_j((\rho^t)^{\frac{N}{e}} - 1)\nabla \rho_1)||_{L^2} d\tau \\
\lesssim ||\rho_1||_{L^\infty(B^j_{2,1})} \left( \int_0^t ||\rho^t(\tau) - 1||_{B^j_{2,1}}^h + ||\rho^t(\tau) - 1||_{B^j_{2,1}}^0 + ||\rho^t - 1||_{L^\infty(B^j_{2,1})} \right). \\
(3.72)
\]
Thanks to (3.39) and (3.70), we get
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha(t-\tau)2^{\frac{j}{2}}} ||\Delta_j((\rho^t)^{\frac{N}{e}} - 1)\nabla \rho_1)||_{L^2} d\tau \\
+ \sum_{j \leq 0} \int_0^t e^{-\alpha 2^j(t-\tau)2^{\frac{j}{2}}} ||\Delta_j((\rho^t)^{\frac{N}{e}} - 1)\nabla \rho_1)||_{L^2} d\tau \\
\lesssim \eta(E^1(\rho_1, u_1)(t) + D^h(\rho_1, u_1)(t) + D^0(\rho_1, u_1)(t)). \\
(3.73)
\]
For the last term \(\frac{R}{c^e}((\rho^N)^{\frac{N}{e}} - 1)\nabla S\), using (4.15), we have
\[
||\Delta_j((\rho^N)^{\frac{N}{e}} - 1)\nabla S||_{L^2} \lesssim ||\Delta_j((\rho^N - 1)\nabla S)||_{L^2} + ||\epsilon \Delta_j(S\nabla S)||_{L^2}.
\]
Firstly, we get by using (4.5)
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha(t-\tau)2^{\frac{j}{2}}}||\epsilon \Delta_j(S\nabla S)||_{L^2} d\tau + \sum_{j \leq 0} \int_0^t e^{-\alpha 2^j(t-\tau)2^{\frac{j}{2}}}||\epsilon \Delta_j(S\nabla S)||_{L^2} d\tau \\
\lesssim \epsilon \int_0^t ||S\nabla S||_{B^j_{2,1}}^1 d\tau \lesssim ||S||_{L^\infty(B^j_{2,1})} \cdot \epsilon ||S||_{L^1(B^j_{2,1})}^3.
\]
which along with (3.37) implies
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha(t-\tau)} 2^{j+1} \| \varepsilon \Delta_j (S \nabla S) \|_{L^2} d\tau + \sum_{j \geq 0} \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{j+1} \| \varepsilon \Delta_j (S \nabla S) \|_{L^2} d\tau \lesssim \eta. \tag{3.74}
\]

For \( j \geq 1 \), since \( \int_0^t e^{-\alpha (t-\tau)} d\tau \lesssim 1 \), using (3.37), we have
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha (t-\tau)} 2^{j+1} \| \Delta_j ((\rho_\varepsilon^N - 1) \nabla S) \|_{L^2} d\tau \lesssim \| \rho_\varepsilon^N - 1 \|_{L^\infty_t(B^\frac{3}{2}_2)} \| S \|_{L^\infty_t(B^\frac{3}{2}_2)} \lesssim \eta. \tag{3.75}
\]

For \( j \leq 0 \), since \( \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{j+1} d\tau \lesssim 1 \), using (3.37), we have
\[
\sum_{j \geq 0} \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{j+1} \| \Delta_j (T_\varepsilon (\rho_\varepsilon^N - 1) \nabla S + T_{\mathbf{v}^S} (\rho_\varepsilon^N - 1)) \|_{L^2} d\tau \lesssim \| \rho_\varepsilon^N - 1 \|_{L^\infty_t(B^\frac{3}{2}_2)} \| S \|_{L^\infty_t(B^\frac{3}{2}_2)}, \tag{3.76}
\]

Whereas for \( j \leq 0 \), we have
\[
\int_0^t e^{-\alpha 2^j (t-\tau)} 2^{j+1} \| \Delta_j (R((\rho_\varepsilon^N - 1), \nabla S)) \|_{L^2} d\tau \leq 2^j \left( \sum_{j' \geq 1} + \sum_{0 \leq j' \leq j-4} \right) \int_0^t \| \Delta_{j'} (\rho_\varepsilon^N - 1) \|_{L^2} \| \tilde{\Delta}_{j'} S \|_{L^2} d\tau \leq 2^j \left( \sum_{j' \geq 1} \int_0^t 2^{j' \Delta_j (\rho_\varepsilon^N - 1) \|_{L^2} d\tau \right) \| S \|_{L^\infty_t(B^\frac{3}{2}_2)}, \]

which implies
\[
\sum_{j \geq 0} \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{j+1} \| \Delta_j (R((\rho_\varepsilon^N - 1), \nabla S)) \|_{L^2} d\tau \leq \int_0^t \| \rho_\varepsilon^N - 1 \|_{L^\infty_t(B^\frac{3}{2}_2)} d\tau \lesssim \left( \sum_{j \geq 0} \int_0^t \| \rho_\varepsilon^N - 1 \|_{L^\infty_t(B^\frac{3}{2}_2)} d\tau \right) \| S \|_{L^\infty_t(B^\frac{3}{2}_2)}, \tag{3.77}
\]

Since
\[
\| S \|_{L^\infty_t(B^\frac{3}{2}_2)} \lesssim \| S \|_{L^\infty_t(B^\frac{1}{2}_2)} + \| S \|_{L^\infty_t(B^\frac{3}{2}_1)},
\]

using (3.37), we deduce from (3.76) and (3.77) implies
\[
\sum_{j \geq 0} \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{j+1} \| \Delta_j ((\rho_\varepsilon^N - 1) \nabla S) \|_{L^2} d\tau \lesssim \eta. \tag{3.78}
\]

Thanks to (3.74), (3.75) and (3.78), we obtain
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha (t-\tau)} 2^{j+1} \| \Delta_j ((\rho_\varepsilon^N - 1) \nabla S) \|_{L^2} d\tau + \sum_{j \leq 0} \int_0^t e^{-\alpha 2^j (t-\tau)} 2^{j+1} \| \Delta_j ((\rho_\varepsilon^N - 1) \nabla S) \|_{L^2} d\tau \lesssim \eta \lesssim 1. \tag{3.79}
\]
By virtue of (3.69), (3.73) and (3.79), we arrive at
\[
\sum_{j \geq 1} \int_0^t e^{-\alpha(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j \tilde{N}_{u_1}(\tau) \|_{L^2} d\tau + \sum_{j \leq 0} \int_0^t e^{-\alpha 2^{2j}(t-\tau)} 2^{\frac{j}{2}} \| \Delta_j \tilde{N}_{u_1}(\tau) \|_{L^2} d\tau \\
\lesssim 1 + \eta \left( E_\frac{1}{2}(\rho_1, u_1)(t) + D_h^b(\rho_1, u_1)(t) + D_\frac{1}{2}^b(\rho_1, u_1)(t) \right).
\]
(3.80)

**Step 4. The final energy estimate.** Combining (3.58), (3.59), (3.64) and (3.80), using (3.37) and (3.39), for sufficiently small \( \eta > 0 \), we deduce from (3.56) that
\[
E_\frac{1}{2}(\rho_1, u_1)(t) + D_h^b(\rho_1, u_1)(t) + D_\frac{1}{2}^b(\rho_1, u_1)(t) \lesssim 1.
\]
(3.81)
Then (3.81) implies (1.19). We complete the proof of Theorem 1.2. \( \square \)

4. Appendix. In this appendix, we will prove some technical lemmas. For the convenience of the readers, we recall the following Bernstein type lemma from [1]:

**Lemma 4.1.** Let \( B \) and \( C \) be a ball and a circle of \( \mathbb{R}^3 \) respectively, and let \( 1 \leq p \leq q < \infty \). Then there holds

- if the support of \( \tilde{u} \) is included in \( 2^k B \), then for any \( \alpha \in (\mathbb{Z}_{\geq 0})^3 \),
  \[
  \| \partial^\alpha u \|_{L^q} \lesssim 2^{k\left(2\alpha \cdot \frac{1}{2} - \frac{1}{4}\right)} \| u \|_{L^p};
  \]
  (4.1)

- if the support of \( \tilde{u} \) is included in \( 2^k C \), then for any \( N \in \mathbb{Z} \),
  \[
  \| u \|_{L^q} \lesssim 2^{-kN} \| \partial^\alpha u \|_{L^q} \lesssim 2^{-kN + 3\left(\frac{1}{2} - \frac{1}{4}\right)} \| \partial^\alpha u \|_{L^p};
  \]
  (4.2)

We also use para-differential decomposition of Bony from [2]: let \( a, b \in \mathcal{S}'(\mathbb{R}^3) \),
\[
ab = T_a b + T_b a + R(a, b),
\]
(4.3)
where
\[
T_a b \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} S_{j-1} a \Delta_j b, \quad R(a, b) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \Delta_j a \tilde{\Delta}_j b, \quad \text{with} \quad \tilde{\Delta}_j b \overset{\text{def}}{=} \sum_{\ell = j - 1}^{j+1} \Delta_\ell b.
\]

With the Bony decomposition (4.3), we obtain the following product estimates.

**Lemma 4.1.** Assume that \( s > -\frac{3}{2}, s_1, s_2 \leq \frac{3}{2} \). For any \( a \in \dot{B}^{s_1+\frac{3}{2}-s_2}_{2,1}(\mathbb{R}^3) \cap \dot{B}^{s_2+\frac{3}{2}-s_1}_{2,1}(\mathbb{R}^3) \) and \( b \in \dot{B}^{s_1+\frac{3}{2}-s_2}_{2,1}(\mathbb{R}^3) \), we have \( ab \in \dot{B}^{s_1}_{2,1}(\mathbb{R}^3) \) and
\[
\| ab \|_{\dot{B}^{s_1}_{2,1}} \lesssim \| a \|_{\dot{B}^{s_1}_{2,1}} \| b \|_{\dot{B}^{s_1}_{2,1}} + \| a \|_{\dot{B}^{s_2+\frac{3}{2}-s_1}_{2,1}} \| b \|_{\dot{B}^{s_2+\frac{3}{2}-s_2}_{2,1}}.
\]
(4.4)
Moreover, if \(-\frac{3}{2} < s < \frac{3}{2} \) and \( s_1 + s_2 = s + \frac{3}{2} \), we have
\[
\| ab \|_{\dot{B}^{s_1}_{2,1}} \lesssim \| a \|_{\dot{B}^{s_1}_{2,1}} \| b \|_{\dot{B}^{s_1}_{2,1}}.
\]
(4.5)

**Proof.** The inequality (4.5) follows by (4.4). Thanks to (1.9) and Hölder inequality, we have
\[
\| \Delta_j(T_a b) \|_{L^2} \lesssim \sum_{|j' - j| \leq 4} \| S_{j'-1} a \|_{L^\infty} \| \Delta_j b \|_{L^2}.
\]
By virtue of (4.2), we have for \( s_1 \leq \frac{3}{7} \)
\[
\| S_{j'-1} a \|_{L^\infty} \lesssim \sum_{j'' \leq j'-1} 2^{\frac{3}{2}j''} \| \Delta_j a \|_{L^2} \lesssim 2^{\frac{3}{2}j''} \sum_{j'' \leq j'-1} 2^{s_1 j''} \| \Delta_j a \|_{L^2} \lesssim 2^{\frac{3}{2}j''} \| a \|_{\dot{B}^{s_1}_{2,1}}.
\]
Then we have
\[
\sum_{j \in \mathbb{Z}} 2^{js} \| \Delta_j (T_a b) \|_{L^2} \lesssim \|a\|_{\dot{B}^{s+\frac{1}{2}-s_2}} \|b\|_{\dot{B}^{s+\frac{1}{2}-s_2}}.
\]
(4.6)

Similarly, we have for \(s_2 \leq \frac{3}{2}\)
\[
\sum_{j \in \mathbb{Z}} 2^{js} \| \Delta_j (T_b a) \|_{L^2} \lesssim \|a\|_{\dot{B}^{s+\frac{1}{2}-s_2}} \|b\|_{\dot{B}^{s+\frac{1}{2}-s_2}}.
\]
(4.7)

For \(R(a,b)\), using (1.9) and (4.2), we have
\[
2^{js} \| \Delta_j (R(a,b)) \|_{L^2} \lesssim 2^{(s+\frac{1}{2})j} \sum_{j' \geq j-4} \| \Delta_j a \|_{L^2} \| \Delta_j b \|_{L^2} \lesssim 2^{(s+\frac{1}{2})j} \sum_{j' \geq j-4} 2^{-j's_2} \| \Delta_j a \|_{L^2} \| \Delta_j b \|_{L^2} \lesssim \sum_{j' \geq j-4} 2^{(s+\frac{1}{2})(j-j')} \cdot 2^{j'(s+\frac{1}{2}-s_2)} \| \Delta_j a \|_{L^2} \| \Delta_j b \|_{L^2}.
\]
(4.8)

Since \(s > -\frac{3}{2}\), using Young inequality, we have
\[
\sum_{j \in \mathbb{Z}} 2^{js} \| \Delta_j (R(a,b)) \|_{L^2} \lesssim \|a\|_{\dot{B}^{s+\frac{1}{2}-s_2}} \|b\|_{\dot{B}^{s+\frac{1}{2}-s_2}}.
\]
(4.9)

Combining (4.6), (4.7) and (4.8), we obtain (4.4). The lemma is proved. \(\Box\)

The following lemma is involving the commutator estimates.

**Lemma 4.2.** Assume that \(f(\xi) \in \mathbb{R}^3 \setminus \{0\}\) and for any \(\lambda \in \mathbb{R} \setminus \{0\}\)
\[
f(\lambda \xi) = \lambda^k f(\xi), \quad \text{for some } k \in \mathbb{Z}.
\]

We have for any \(s > -\frac{3}{2}\)
\[
\| \Delta_j ([f(D), T_a b]) \|_{L^2} \lesssim \| \nabla a \|_{L^\infty} \cdot 2^{j(k-1)} \sum_{|j'| \leq 4} \| \Delta_{j'} b \|_{L^2},
\]
(4.9)

\[
\| ([f(D), T_a b])_{\dot{B}^{s+\frac{1}{2}}_{2,1}} \| \lesssim \| \nabla a \|_{L^\infty} \| b \|_{\dot{B}^{s+\frac{1}{2}}_{2,1}}.
\]

**Proof.** Firstly, by the definition of \(T_a b\) we have
\[
\mathcal{F} \left( \Delta_j ([f(D), T_a b]) \right)(\xi) = \sum_{|j' - j| \leq 4} \int_{\mathbb{R}^3} \varphi_j(\xi)(f(\eta) - f(\xi)) \hat{S}_{j'-1} a(\xi - \eta) \hat{\Delta}_{j'} b(\eta) d\eta
\]
(4.10)

\[
= \int_{|\eta| \leq 2^{j'} |\xi|} \varphi_j(\xi)(f(\eta) - f(\xi)) \hat{S}_{j'-1} a(\xi - \eta) \hat{\Delta}_{j'} b(\eta) d\eta,
\]
where we used the fact \(|\xi| \in (2^{j'-1}, 2^{j'+1})\) and \(|\eta| \in (2^{j'-1}, 2^{j'+1})\).

Now, we define \(h \in \mathcal{S}(\mathbb{R}^3)\) as follows
\[
h(x) = \mathcal{F}^{-1} (f(\cdot) \varphi_{[-6,6]}(\cdot))(x).
\]
(4.11)

Then we have
\[
\varphi_{[-6,6]}(D)f(D) a = h \ast a
\]
Lemma 4.4. For any $a, b \in S'(\mathbb{R}^3)$ with $\lim_{k \to -\infty} S_k a = \lim_{k \to -\infty} S_k b = 0$, we have
$$\Delta_j([f(D), T_a] b) = 2^k \Delta_j([f(2^{-j} D), T_a] b) = 2^{j(k+3)} \Delta_j \left( \sum_{|j' - j| \leq 4} \int_{\mathbb{R}^3} h(2^j (x - y)) (S_{j'-1} a(y) - S_{j'-1} a(x)) \Delta_j b(y) \, dy \right),$$
which implies that
$$\|\Delta_j ([f(D), T_a] b)\|_{L^2} \lesssim 2^{j(k+2)} \sum_{|j' - j| \leq 4} \| \int_{\mathbb{R}^3} |2^j (x - y) \cdot h(2^j (x - y))| \cdot |\Delta_j b(y)| \, dy\|_{L^2} \| \Delta_j S_{j'-1} a\|_{L^\infty} \lesssim 2^{j(k-1)} \|xh(x)\|_{L^1} \sum_{|j' - j| \leq 4} \|\Delta_j b\|_{L^2} \|\Delta_j S_{j'-1} a\|_{L^\infty} \lesssim 2^{j(k-1)} \sum_{|j' - j| \leq 4} \|\Delta_j S_{j'-1} a\|_{L^\infty} \|\Delta_j b\|_{L^2}. \tag{4.12}$$
That is
$$\|\Delta_j ([f(D), T_a] b)\|_{L^2} \lesssim \|\nabla a\|_{L^\infty} \cdot 2^{j(k-1)} \sum_{|j' - j| \leq 4} \|\Delta_j b\|_{L^2}. \tag{4.13}$$
Multiplying (4.12) by $2^{js_1}$, then summing up the resulting inequality for all $j \in \mathbb{Z}$, we obtain the second inequality in (4.9). We complete the proof of the lemma. \hfill \Box

As a consequence of the proof to Lemma 4.2, we obtain the following commutator lemma.

Lemma 4.3. For any $a, b \in S'(\mathbb{R}^3)$ with $\lim_{k \to -\infty} S_k a = \lim_{k \to -\infty} S_k b = 0$, we have for any $j \in \mathbb{Z}$
$$\|([\varphi_j(D), T_a] b)\|_{L^2} \lesssim 2^{-j} \sum_{|j' - j| \leq 4} \|\nabla S_{j'-1} a\|_{L^\infty} \|\Delta_j b\|_{L^2}. \tag{4.14}$$
Moreover, we have
$$\sum_{|j' - j| \leq 4} \|([\varphi_j(D), S_{j-1} a] \Delta_j b)\|_{L^2} \lesssim 2^{-j} \|\nabla S_{j-1} a\|_{L^\infty} \sum_{|j' - j| \leq 4} \|\Delta_j b\|_{L^2}. \tag{4.15}$$

With Lemma 4.3, we have the following lemma.

Lemma 4.4. For any $u$ with $\nabla u \in L^\infty(\mathbb{R}^3)$, $f \in S'(\mathbb{R}^3)$ with $\lim_{k \to -\infty} S_k f = 0$, we have
$$\|([\Delta_j (T_u \cdot \nabla f) \cdot \Delta_j f])\| \lesssim \|\nabla u\|_{L^\infty} \|\Delta_j f\|_{L^2} \sum_{|j' - j| \leq 4} \|\Delta_j f\|_{L^2}. \tag{4.16}$$

Proof. Firstly, we have
$$\Delta_j (T_u \cdot \nabla f) = \sum_{|j' - j| \leq 4} (\varphi_j(D)(S_{j'-1} u \cdot \nabla \Delta_j f) \varphi_j(D)f) = \sum_{|j' - j| \leq 4} (\varphi_j(D)(S_{j'-1} u - S_{j-1} u) \cdot \nabla \Delta_j f) \varphi_j(D)f)$$
$$+ \sum_{|j' - j| \leq 4} ([\varphi_j(D), S_{j-1} u] \cdot \nabla \Delta_j f) \varphi_j(D)f)$$
$$+ (S_{j-1} u \cdot \nabla \varphi_j(D)f \varphi_j(D)f) \overset{\text{def}}{=} A_1 + A_2 + A_3.$$
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For $A_1$, using Bernstein inequality (4.2), we have
\[
|A_1| \lesssim \sum_{j-j' \leq 4} 2^j \| S_{j-1} u - S_{j-1} f \|_{L^\infty} \cdot \| \Delta_j f \|_{L^2} \cdot \| \Delta_j f \|_{L^2} \\
\lesssim \| \nabla u \|_{L^\infty} \| \Delta_j f \|_{L^2} \cdot \sum_{j-j' \leq 4} \| \Delta_j f \|_{L^2}.
\]

For $A_2$, using Lemma 4.3, we obtain
\[
|A_2| \lesssim \| \nabla u \|_{L^\infty} \| \Delta_j f \|_{L^2} \cdot \sum_{|j-j'| \leq 4} \| \Delta_j f \|_{L^2}.
\]

For $A_3$, integrating by parts, we have
\[
|A_3| = |(\nabla \cdot S_{j-1} u \varphi_j(D) f) \varphi_j(D) f| \lesssim \| \nabla u \|_{L^\infty} \| \Delta_j f \|_{L^2}^2.
\]

Thus, we obtain (4.14). The lemma is proved.

The last technical lemma is about the estimate of composition function.

Lemma 4.5. Assume that $f(\cdot) \in C^\infty(\mathbb{R})$ with $f(0) = 0$, and $u \in L^\infty(\mathbb{R}^3) \cap B^{s+k}_{2,1}(\mathbb{R}^3)$ with $s \in (-\frac{3}{2}, \frac{3}{2})$ and $k \in \mathbb{Z}_{\geq 0}$. Then we have
\[
\| f \circ u \|_{B^{s+k}_{2,1}} \leq C(\| f \|_{C^k}, \| u \|_{L^\infty}) \| u \|_{B^{s+k}_{2,1}}. \tag{4.15}
\]

We refer the reader to the proof of Theorem 2.61 in [1] for case $s \in (-\frac{3}{2}, \frac{3}{2})$. By the Chain rule for the composition function and the product estimates, we could obtain (4.15) for any $s \in (-\frac{3}{2}, \frac{3}{2})$ and $k \in \mathbb{Z}_{\geq 0}$. We omit the details here.

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