Three–loop renormalization group analysis
of a complex model with stable fixed point:
Critical exponents up to $\epsilon^3$ and $\epsilon^4$

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Abstract

The complete analysis of a model with three quartic coupling constants associated
with an $O(2N)$–symmetric, a cubic, and a tetragonal interactions is carried out within
the three–loop approximation of the renormalization–group (RG) approach in $D = 4 - 2\epsilon$ dimensions. Perturbation expansions for RG functions are calculated using
dimensional regularization and the minimal subtraction (MS) scheme. It is shown
that for $N \geq 2$ the model does possess a stable fixed point in three dimensional
space of coupling constants, in accordance with predictions made earlier on the base
of the lower-order approximations. Numerical estimate for critical (marginal) value of
the order parameter dimensionality $N_c$ is given using Padé–Borel summation of the
Corresponding $\epsilon$–expansion series obtained. It is observed that two–fold degeneracy of
the eigenvalue exponents in the one–loop approximation for the unique stable fixed
point leads to the substantial decrease of the accuracy expected within three loops
and may cause powers of $\sqrt{\epsilon}$ to appear in the expansions. The critical exponents $\gamma$
and $\eta$ are calculated for all fixed points up to $\epsilon^3$ and $\epsilon^4$, respectively, and processed
by the Borel summation method modified with a conformal mapping. For the unique
stable fixed point the magnetic susceptibility exponent $\gamma$ for $N = 2$ is found to differ
in third order in $\epsilon$ from that of an $O(4)$–symmetric point. Qualitative comparison of
the results given by $\epsilon$–expansion, three–dimensional RG analysis, non–perturbative RG
arguments, and experimental data is performed.

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1 Introduction

There are numerous complicated models with more than two independent quartic coupling constants. They describe phase transitions in a variety of systems and are actively studied within the $\epsilon$–expansion as well as by the field–theoretical renormalization group method in three dimensions [1–10]. Critical fluctuations in anisotropic systems with several quartic coupling constants are known to destroy, as usual, continuous transitions converting them into first–order ones. This fact, however, has not been strictly proved. On the contrary, it is possible to construct models with a large number of coupling constants whose RG equations have stable fixed points [11]. It means that the presence of three and more coupling constants in the Landau–Wilson Hamiltonian does not forbid continuous phase transitions in the system. Nevertheless, complicated models with stable fixed points are quite rare. One of such models describing certain antiferromagnetic phase transitions and the structural transition in $\text{NbO}_2$ crystal will be studied in the paper.

We consider the critical behavior of a model given by the fluctuation Landau-Wilson Hamiltonian with three quartic interaction terms:

$$H = \int d^Dx \left[ \frac{1}{2} \left( m_0^2 \varphi_i^\alpha \varphi_i^\alpha + \partial_\mu \varphi_i^\alpha \partial_\mu \varphi_i^\alpha \right) + \frac{1}{4!} \left( u_0 G_1^{\alpha\beta\mu\nu}_{ijkl} + v_0 G_2^{\alpha\beta\mu\nu}_{ijkl} + 2z_0 G_3^{\alpha\beta\mu\nu}_{ijkl} \right) \varphi_i^\alpha \varphi_j^\beta \varphi_k^\mu \varphi_l^\nu \right].$$

(1)

Here $\varphi_i^\alpha$, $i = 1, \ldots, N$, $\alpha = 1, 2$, is the real $2N$–component order parameter field in $D = 4 - 2\epsilon$ dimensions and $m_0$, $u_0$, $v_0$, $z_0$ are the bare mass and coupling constants, respectively. The squared bare mass $m_0^2$ can be thought of as proportional to the deviation from the mean–field transition point (line). The field $\varphi_i^\alpha$ is regarded as consisting of two sets of components, even and odd, each of them may be considered as a real $N$–component vector. Tensors $G_1$, $G_2$, and $G_3$ in the Hamiltonian (1) corresponding to isotropic, cubic, and tetragonal interactions have the following symmetrized form:

$$G_1^{\alpha\beta\mu\nu}_{ijkl} = \frac{1}{3} \left( \delta^{\alpha\beta} \delta^\mu^\nu \delta_{ij} \delta_{kl} + \delta^{\alpha\mu} \delta^{\beta\nu} \delta_{ik} \delta_{jl} + \delta^{\alpha\nu} \delta^{\beta\mu} \delta_{il} \delta_{jk} \right),$$

$$G_2^{\alpha\beta\mu\nu}_{ijkl} = \delta^{\alpha\beta} \delta^\mu^\nu \delta_{ij} \delta_{kl},$$

$$G_3^{\alpha\beta\mu\nu}_{ijkl} = \frac{1}{3} \left( \delta^{\alpha\beta} \delta^{\mu\nu} \delta_{ij} \delta_{kl} + \delta^{\alpha\mu} \delta^{\beta\nu} \delta^2_{ij} \delta^2_{kl} + \delta^{\alpha\nu} \delta^{\beta\mu} \delta^2_{ij} \delta^2_{kl} + \delta^{\alpha\mu} \delta^{\beta\nu} \delta^2_{ij} \delta^2_{kl} \right).$$

(2)

When $u$ is equal to zero, the Hamiltonian (1) describes $N$ identical non–interacting anisotropic $XY$ models [12], while for $z = 0$ it turns into the Hamiltonian of the well–known hypercubic model.
The Hamiltonian (1) governs the critical thermodynamics in a number of interesting physical systems. So, for example, when \( N = 2 \) it describes the structural phase transition in \( NbO_2 \) crystal and, for \( v = z \), the antiferromagnetic transitions in \( TbAu_2 \) and \( DyC_2 \). Another physically important case \( N = 3 \) is relevant to the antiferromagnetic phase transition in \( K_2IrCl_6 \) crystal and, for \( v = z \), to those in \( TbD_2 \) and \( Nd \) [1, 3]. The detailed analysis of these systems along the line of the Landau phenomenological theory can be found in [1, 2, 13] with references to the experimental works therein.

For the first time the renormalization group analysis of the model (1) was performed to second order in \( \epsilon \) by Mukamel and Krinsky in Refs. [1, 2, 3]. On this ground, it was shown that the \( 2N \)-component real anisotropic model (1) has a unique (three-dimensionally) stable fixed point for each \( N \geq 2 \). The corresponding critical exponents were recorded and for \( n = 2N = 4 \) they were found to coincide with those of the Heisenberg fixed point. On the other hand, the critical behavior of this model was studied within the two–loop approximation by the alternative RG approach in three dimensions [14]. The calculations made provided the same qualitative predictions, although for the physically interesting cases \( N = 2 \) and \( N = 3 \) the critical exponents were found to be numerically close to those of the 3D XY model rather than the Heisenberg ones.

It is well known, however, that low–order approximations lead to rather crude quantitative and, sometimes, contradictory qualitative results, especially for systems with nontrivial symmetry (see, for instance, Refs. [7, 8, 15]). To make more definite conclusion concerning the unique fixed point stability and obtain more accurate values of the critical exponents one has to consider long enough perturbation theory series. Such series are known to have the zero radius of convergence and therefore are, at best, asymptotic. To extract reliable information from them a proper resummation procedure must be applied. Recently such work for the model under consideration was done within the field–theoretical RG approach in three dimensions, where the three–loop expansions for \( \beta \)–functions and critical exponents were calculated for arbitrary \( N \) [14]. Using the generalized Padé–Borel resummation technique, the coordinates of all the fixed points were found. It was shown that the unique stable fixed point did exist on the three–dimensional RG flow diagram when \( N \geq 2 \).

It should be noted that, assuming \( v = z \) the model (1) formally turns into that with generalized cubic anisotropy and the complex order parameter field. The latter is a specific case \((m = 2)\) of the well–known \( mn \)–component model. The critical behavior of this model was investigated in Refs. [14, 15]. Two– and three–loop calculations done for the case \( m = 2, n \geq 2 \) predict stability of the mixed fixed point, the analog of the unique stable fixed point of the model (1).

At the same time, there are general non–perturbative arguments in favor of the unique stable fixed point should not be in the physical space although its existence
is not forbidden at $D > 3$ \[19\]. According to those considerations the only three-
dimensionally stable fixed point may be the Bose one and it is that point which governs
the critical thermodynamics in the phase transitions mentioned. The point is that the
model \((1)\) describes $N$ interacting Bose systems when $v = z$. As was shown by J. Sak
\[20\], the interaction term can be represented as the product of the energy operators
of various two–component subsystems. It was also found that one of the eigenvalue
exponents characterizing the evolution of this term under the renormalization group in
a neighborhood of the Bose fixed point is proportional to the specific heat exponent $\alpha$.
Since $\alpha$ is believed to be negative at this point (that is confirmed by highly precise up-
to-date mesurements of the specific heat exponent of liquid Helium \[21\] including those
in outer space \[22\] and the high–loop RG computations carried out for the simple $O(n)$–
symmetric model in three dimensions \[23, 24\] the interaction is irrelevant. Therefore,
The Bose fixed point should be stable in three dimensions.

Renormalization group approach, however, when directly applied to the model \((1)\)
and to the relative $m n$–component one, has not still confirmed that non–perturbative
conclusion. On the contrary, all calculations performed up to now indicate existence
of the unique stable fixed point in the physical space, while the Bose point appears to
be three–dimensionally unstable \[1–3,14,16–18\]. This may be a consequence of rather
crude approximations used, and the higher order being taken into account the closer
the perturbative results could be to the precise ones. So, the aim of the paper is to
investigate the critical behavior of the three coupling constants model \((1)\) in the next,
third order in $\epsilon$ and verify compatibility of predictions given by the $\epsilon$–expansion method
with predictions based on the other techniques.

The main results of our study to be discussed below are as follows.

- The $\beta$–functions of the record length for the model \((1)\) are obtained by the $\epsilon$–
expansion method. To calculate tensor convolutions associated with the Feyn-
man’s graphs an algorithm was developed and a specially designed computer
application package was written.

- Coordinates of all fixed points and their eigenvalue exponents are calculated in
general form within the three–loop approximation. The problem of stability of
the fixed points is analyzed. The unique fixed point rather than the Bose one is
found to be three–dimensionally stable in the frame of given approximation. Nu-
merical estimate of the critical dimensionality of the order parameter $N_c$ obtained
confirms this conclusion.

- It is observed that one–loop degeneracy of the eigenvalue exponents of the unique
stable fixed point leads to a certain complication in calculating their $\epsilon$–expansion
series. This problem is investigated in detail. It is shown that such a degeneracy
substantially reduces the accuracy expected from given approximation and may result in appearance of the powers of $\sqrt{\epsilon}$ in corresponding series.

- Perturbation series for the critical exponents $\gamma$ and $\eta$ are expanded to $\epsilon^3$ and $\epsilon^4$, respectively. For $N = 2$ the magnetic susceptibility exponent series of the unique stable fixed point and the $O(4)$–symmetric point are found to be different (up to second order in $\epsilon$ they exactly coincide [3]). The numerical values of the critical exponents are estimated by resumming the series using the Borel transformation with a conformal mapping.

The results of our investigation are discussed in comparison with conclusions given by other theoretical approaches and experimental data.

The set up of the paper is as follows. In Section 2 the renormalization scheme is formulated and three–loop expansions for the $\beta$–functions and critical exponents are presented. Specific symmetry properties of the initial Hamiltonian (I) are revealed and used as criteria of correctness of the equations deduced. In Section 3 RG expansions for coordinates of the fixed points are written out for arbitrary $N$ and the problem of their stability is studied. The numerical estimate of the critical dimensionality $N_c$, at which the topology of flow diagrams changes, is obtained in Section 4 by resummation of its series using Padé–Borel method. The RG expansions of the critical exponents for the physically interesting cases $N = 2, N = 3$ and their numerical estimates are given therein. Conclusion is devoted to discussions of the results of the investigation. The paper has two appendices. Appendix A contains $\epsilon$–expansions for the eigenvalue exponents of the fixed points for arbitrary $N$. In Appendix B we analyze the problem of degeneracy of the eigenvalue exponents of the unique stable fixed point and its implications.

2 RG expansions and symmetries

To calculate the $\beta$–functions and critical exponents normalizing conditions must be imposed on renormalized one–particle irreducible inverse Green’s functions $\Gamma_R^{(2)}$ and vertices $\Gamma_R^{(4)}$ given by corresponding Feynman’s diagrams. Within the massless theory
they are normalized in a conventional way:

\[
\Gamma^{(2)}_R\left(\{p\}; \mu, u, v, z\right) \bigg|_{p^2=0} = 0 , \\
\frac{\partial}{\partial p^2}\Gamma^{(2)}_R\left(\{p\}; \mu, u, v, z\right) \bigg|_{p^2=\mu^2} = 1 ,
\]

\[
\Gamma^{(4)}_{UR}\left(\{p\}; \mu, u, v, z\right) = \mu \epsilon u , \\
\Gamma^{(4)}_{VR}\left(\{p\}; \mu, u, v, z\right) = \mu \epsilon v , \\
\Gamma^{(4)}_{ZR}\left(\{p\}; \mu, u, v, z\right) = \mu \epsilon z ,
\]

with one more condition on the \(\varphi^2\) insertion

\[
\Gamma^{(1,2)}_R\left(\{p\}, \{q\}; \mu, u, v, z\right) \bigg|_{p^2=q^2=\mu^2} = 1 .
\]

Here \(m\), \(u\), \(v\), and \(z\) are the renormalized mass and dimensionless coupling constants, with \(\mu\) being an arbitrary mass parameter introduced for dimensional regulariztion.

The vertices \(\Gamma^{(4)}_u\), \(\Gamma^{(4)}_v\), \(\Gamma^{(4)}_z\) are connected with the vertex function without external lines normalized in the following way:

\[
\Gamma^{(4)}_{ijkl} = \Gamma^{(4)}_u \cdot G^{(4)}_{ijkl} + \Gamma^{(4)}_v \cdot G^{(4)}_{ijkl} + \Gamma^{(4)}_z \cdot G^{(4)}_{ijkl}.
\]

From renormalization conditions (3) and (4) the expansions for the renormalization constants \(Z_{\phi}, Z_u, Z_v, Z_z,\) and \(Z_{\varphi^2}\) may be obtained. These constants relate the bare mass \(m_0\) and three coupling constants \(u_0, v_0, z_0\) of the Hamiltonian (1) to the corresponding physical parameters:

\[
m^2 = \frac{Z_{\varphi^2}}{Z_{\varphi}} m^2, \quad u_0 = \mu \frac{Z_u}{Z_{\varphi}^2} u, \quad v_0 = \mu \frac{Z_v}{Z_{\varphi}^2} v, \quad z_0 = \mu \frac{Z_z}{Z_{\varphi}^2} z.
\]

Thus, with relations (5) taken into account, the \(\beta\)-functions and critical exponents can be calculated via the formulas

\[
\frac{\partial \ln u_0}{\partial u} \beta_u + \frac{\partial \ln v_0}{\partial v} \beta_v + \frac{\partial \ln z_0}{\partial z} \beta_z = -\epsilon,
\]

\[
\frac{\partial \ln u_0}{\partial u} \beta_u + \frac{\partial \ln v_0}{\partial v} \beta_v + \frac{\partial \ln z_0}{\partial z} \beta_z = -\epsilon,
\]

\[
\eta(u, v, z) = 2 \frac{\partial \ln Z_u}{\partial u} \beta_u + 2 \frac{\partial \ln Z_v}{\partial v} \beta_v + 2 \frac{\partial \ln Z_z}{\partial z} \beta_z,
\]

\[
\eta_2(u, v, z) = 2 \frac{\partial \ln Z_{\varphi}}{\partial u} \beta_u + 2 \frac{\partial \ln Z_{\varphi}}{\partial v} \beta_v + 2 \frac{\partial \ln Z_{\varphi}}{\partial z} \beta_z,
\]

where \(\beta_g \equiv \frac{\partial g}{\partial |\ln \mu|^g}, g = u, v, z\). The critical exponents \(\eta\) and \(\eta_2\) are found by substituting zeros of the \(\beta\)-functions into expressions (6). The critical exponent \(\gamma\) is given by the well known scaling relation \(\gamma^{-1} = 1 + \frac{\eta}{2-\eta}\).
The contribution of a Feynman’s graph into an RG–function comprises three factors: the combinatorial coefficient, the result of tensor convolution and the integral value associated to the diagram. The combinatorial factors, and the values of integrals are known from Ref. [25], while evaluating tensor convolutions for vertex and mass diagrams is the problem to be solved. To do it we have developed a computer application package written in PASCAL. The algorithm is based upon two quite natural assumptions:

1. Tensor convolution algebra is closed, i.e. each monomial

\[ G_{i_1} \star \ldots \star G_{i_{l+1}} \]

contributes to a vertex function is a linear combination of the basic tensors \( G_1, G_2, \) and \( G_3 \):

\[ G_{i_1} \star \ldots \star G_{i_{l+1}} = a(N)G_1 + b(N)G_2 + c(N)G_3. \] (8)

2. Dependence of the coefficients \( a(N), b(N), \) and \( c(N) \) upon \( N \) is of polynomial character. The degree of the polynomials does not exceed the number \( l \) of loops in the Feynman’s graph.

The first condition means that one has no new interactions generated in the model (1). The second proposition becomes evident upon analyzing the particular form of the tensors \( G_i \).

Since a polynomial of degree \( l \) is fully determined by its values in \( l + 1 \) different points, it is sufficient to compute convolutions consecutively assuming \( N = 2, \ldots l + 2 \) (the reason to start from 2 is linear dependence between \( G_i \) when \( N = 1 \)). In order to evaluate three indeterminates \( a(N), b(N), \) and \( c(N) \) we compare both sides of expression (8), having assigned values \( (1122), (1111), \) and \( (1112) \) to the multi–index \( (\alpha\beta\mu\nu) \). It provides a non–degenerate system of linear equations whose \( 3 \times 3 \)–matrix does not depend on \( N \). From this system the coefficients of decomposition (8) are found. Similar procedure was applied to the mass diagrams. The results of our computations recover those achieved within the four–loop approximation for simple \( O(n) \)–symmetric model [25].

After some work, we obtain the expressions for the RG–functions within the three–loop approximation (Fisher’s exponent \( \eta \) is calculated up to four loops) using dimensional regularization [26] and the MS scheme [27]:

\[ \beta_u = \epsilon u - u^2 - \frac{1}{2(N+4)} \left( 6uv + 2uz \right) + \frac{1}{4(N+4)^2} \left[ 12u^3(3N + 7) + 132u^2v + 44u^2z + 30uv^2 + 10uz^2 \right] - \frac{1}{16(N+4)^3} \left[ 4u^4(48\zeta(3)(5N + 11) + 33N^2 + 461N + 740) + 12u^3v(384\zeta(3) + 79N + 659) + 4u^3z(384\zeta(3) + 79N + 659) + 18u^2v^2(96\zeta(3) + N + 321) + 1380u^2vz + 2u^2z^2(288\zeta(3) + 3N + 733) + 1512uv^3 + 18uv^2z + 504uvz^2 + 222uz^3 \right], \]
\[ \beta_v = \epsilon v - \frac{1}{2(N+1)}(12uv + 9v^2 + z^2) + \]
\[
\frac{1}{4(N+4)^2}\left[4u^2v(5N + 41) + 276uv^2 + 20uvw + 24uz^2 + 102v^3 + 10vz^2 + 8z^3\right] - \frac{1}{16(N+4)^3}\left[8u^3v(96\zeta(3)(N + 7) - 13N^2 + 184N + 821) + 18u^2v^2(768\zeta(3) + 17N + 975) + 12u^2vz(96\zeta(3) - 13N + 154) + 2u^2z^2(576\zeta(3) + 43N + 667) + 108uv^3(96\zeta(3) + 131) + 306uvz^2 + 12u^2vz^2(96\zeta(3) + 187) + 2uz^3(384\zeta(3) + 395) + 27v^4(96\zeta(3) + 145) + 162v^2z^2 + 8v^3z(48\zeta(3) + 101) + 3z^4\right].
\]

\[ \beta_z = \epsilon z - \frac{1}{2(N+1)}(12uz + 6vz + 4z^2) + \]
\[
\frac{1}{4(N+4)^2}\left[4u^2z(5N + 41) + 204uvw + 116uz^2 + 30v^2z + 72vz^2 + 18z^3\right] - \frac{1}{16(N+4)^3}\left[8u^3z(96\zeta(3)(N + 7) - 13N^2 + 184N + 821) + 12u^2vz(864\zeta(3) + 4N + 1129) + 4u^2z^2(1440\zeta(3) + 47N + 1796) + 18uw^2(192\zeta(3) + 391) + 72uvw^2(96\zeta(3) + 103) + 2uz^3(96\zeta(3) + 1517) + 1512v^3z + 36v^2z^2(48\zeta(3) + 35) + 72vz^3(16\zeta(3) + 25) + 4z^4\right].
\]

\[ \gamma^{-1} = 1 - \frac{1}{2(N+4)}(2u(N+1) + 3v + z) + \]
\[
\frac{1}{2(N+4)^2}\left[6u^2(N+1) + 18uv + 6uz + 9v^2 + 3z^2\right] - \frac{1}{16(N+4)^3}\left[12u^3(N+1)(11N+39) + 54u^2v(11N+39) + 18u^2z(11N+39) + 6uv^2(5N+398) + 564uvw + 2uz^2(5N+304) + 801v^3 + 15v^2z + 267vz^2 + 117z^3\right] + \]
\[
\frac{1}{2(N+4)^2}\left[2u^2(N+1) + 6uv + 2uz + 3v^2 + z^2\right] - \frac{1}{8(N+4)^3}\left[4u^3(N+1)(N+4) + 18u^2v(N+4) + 6u^2z(N+4) + 81uv^2 + 18uvw + 21uz^2 + 27v^3 + 9vz^2 + 4z^3\right].
\]
where \( \zeta \) is the Riemann \( \zeta \)-function: \( \zeta(3) = 1.20206 \). Expressions (9) – (11) are in accordance with those obtained earlier in Ref. [3], where corresponding calculations for RG functions were carried out to \( \epsilon^2 \), and, assuming \( v = z \equiv 0 \) and \( N = n^2 \), with results of Ref. [25], in which the critical exponents of the well–known \( O(n) \)–symmetric model were found up to \( \epsilon^4 \). If \( u \equiv 0 \) and \( v = z \), the right–hand side of the second (third) equation (9) goes over into the \( \beta \)–function for Bose–like systems, the coupling constants being normalized properly. The latter, obviously, coincides with that of the \( O(n) \)–symmetric model when \( n = 2N = 2 \).

In conclusion of this section, let us formulate a criterion to check the correctness of the results obtained. It relies on a specific symmetry property of the Hamiltonian (1) of the system under consideration [14]. It occurs that transformation

\[
\begin{align*}
\varphi_{2N-1} & \rightarrow \frac{1}{\sqrt{2}} \left( \varphi_{2N-1} + \varphi_{2N} \right), \\
\varphi_{2N} & \rightarrow \frac{1}{\sqrt{2}} \left( \varphi_{2N-1} - \varphi_{2N} \right),
\end{align*}
\]

combined with substitution of quartic coupling constants

\[
u \rightarrow u, \quad v \rightarrow \frac{1}{2}(v + z), \quad z \rightarrow \frac{1}{2}(3v - z)
\]

does not change the structure of the Hamiltonian itself.

Similar situation takes place for \( N = 1 \) and \( z = 0 \) in the case of another field transformation

\[
\begin{align*}
\varphi_1 & \rightarrow \frac{1}{\sqrt{2}} \left( \varphi_1 + \varphi_2 \right), \\
\varphi_2 & \rightarrow \frac{1}{\sqrt{2}} \left( \varphi_1 - \varphi_2 \right),
\end{align*}
\]

which does not affect the Hamiltonian resulting only in the following replacement of \( u \) and \( v \):

\[
u \rightarrow u + \frac{3}{2} v, \quad v \rightarrow -v.
\]

It is well known that the RG equations should be invariant with respect to any transformation conserving the structure of the Hamiltonian [29]. It means that for
every $N$, in the case of symmetry (13), functions $\beta_u$, $\beta_v$, and $\beta_z$ should obey special relations which may be readily written down:

\[
\begin{align*}
\beta_u(u, v, z) &= \beta_u\left(u, \frac{1}{2} (v + z), \frac{1}{2} (3v - z)\right), \\
\beta_v(u, v, z) + \beta_z(u, v, z) &= 2\beta_v\left(u, \frac{1}{2} (v + z), \frac{1}{2} (3v - z)\right), \\
3\beta_v(u, v, z) - \beta_z(u, v, z) &= 2\beta_z\left(u, \frac{1}{2} (v + z), \frac{1}{2} (3v - z)\right).
\end{align*}
\] (16)

For $N = 1$ and $z = 0$ the other symmetry (17) results in

\[
\begin{align*}
\beta_u(u, v, 0) + \frac{3}{2} \beta_v(u, v, 0) &= \beta_u\left(u + \frac{3}{2} v, - v, 0\right), \\
\beta_v(u, v, 0) &= -\beta_v\left(u + \frac{3}{2} v, - v, 0\right).
\end{align*}
\] (17)

At last, the critical exponents are invariant under the transformations (13) and (15). So, the first symmetry gives

\[
\begin{align*}
\gamma^{-1}(u, v, z) &= \gamma^{-1}\left(u, \frac{1}{2} (v + z), \frac{1}{2} (3v - z)\right), \\
\eta(u, v, z) &= \eta\left(u, \frac{1}{2} (v + z), \frac{1}{2} (3v - z)\right).
\end{align*}
\] (18)

Similar relations should take place in the case of the symmetry (15). It can be easily verified that conditions (16)–(18) are satisfied indeed.

Symmetries of the initial Hamiltonian like those described above (such symmetries do not exist always and to find them requires certain efforts) play, in some cases, an extremely important role. Namely, the series being obtained within the field–theoretical RG approach in 3$D$ are necessarily processed with the use of some resummation procedure (e.g. Padé, Padé–Borel, Chisholm–Borel etc.), and satisfaction of the numerical results to the exact symmetry relations serves as a criterion to estimate the accuracy expected from the approximation scheme employed [7].

3 Fixed points and stability

Two critical exponents $\gamma$ and $\eta$ are known to completely specify the critical behavior of a system [30]. They are determined from RG functions by going to the infrared–stable fixed points $g_c = (u_c, v_c, z_c)$, which are found as zeros of the $\beta$–functions in the form of series in powers of $\epsilon$:

\[
g_c = g_c(\epsilon) = \sum_{k=1}^{\infty} g_k \epsilon^k.
\]
There exist eight fixed points in the model under consideration \[^3\] \[^4\], one of them (Gaussian) is trivial:

1. **Gaussian fixed point**
   
   \[ u_c = v_c = z_c = 0. \]

2. **\( O(2N) \)-symmetric or Heisenberg fixed point**

   \[ u_c = \epsilon + \frac{3(3N+7)}{(N+4)^2} \epsilon^2 - \left( \frac{12(3)(5N+11)}{(N+4)^3} + \frac{33N^3-55N^2-440N-568}{4(N+4)^3} \right) \epsilon^3, \]
   
   \[ v_c = z_c = 0. \]

3. **Ising fixed point**

   \[ u_c = z_c = 0, \]
   
   \[ v_c = \frac{2(N+4)}{9} \epsilon + \frac{68(N+4)}{243} \epsilon^2 + \left( \frac{709(N+4)}{6561} - \frac{32(N+4) \zeta(3)}{81} \right) \epsilon^3. \]

4. **Cubic fixed point**

   \[ u_c = \frac{N+4}{3N} \epsilon + \frac{N+4}{81N^3} \left( 1 - 2N \right) \left( 19N - 53 \right) \epsilon^2 + \frac{4(N+4)}{27N^4} \zeta(3) (8N^3 - 12N^2 - 7N + 7) - \frac{N+4}{8748N^7} (3910N^4 + 41971N^3 - 114987N^2 + 90160N - 22472) \epsilon^3, \]
   
   \[ v_c = \frac{2(N+4)}{9N} \left( N - 2 \right) \epsilon + \frac{2(N+4)}{243N^3} \left( 2N - 1 \right) \left( 17N^2 + 55N - 106 \right) \epsilon^2 + \frac{16(N+4)}{81N^4} \zeta(3) (2N^4 + 8N^3 - 10N^2 - 9N + 7) + \frac{N+4}{13122N^9} (1418N^5 + 11713N^4 + 90281N^3 - 247414N^2 + 187528N - 44944) \epsilon^3, \]
   
   \[ z_c = 0. \]

5. **Bose fixed point**

   \[ u_c = 0, \]
   
   \[ v_c = \frac{N+4}{5} \epsilon + \frac{6(N+4)}{25} \epsilon^2 + \frac{N+4}{1250} \left( 103 - 384 \zeta(3) \right) \epsilon^3, \]
   
   \[ z_c = \frac{N+4}{5} \epsilon + \frac{6(N+4)}{25} \epsilon^2 + \frac{N+4}{1250} \left( 103 - 384 \zeta(3) \right) \epsilon^3. \]

6. **VZ–cubic fixed point**

   \[ u_c = 0, \]
   
   \[ v_c = \frac{N+4}{9} \epsilon + \frac{34(N+4)}{243} \epsilon^2 + \frac{N+4}{13122} \left( 709 - 2592 \zeta(3) \right) \epsilon^3, \]
   
   \[ z_c = \frac{N+4}{3} \epsilon + \frac{34(N+4)}{81} \epsilon^2 + \frac{N+4}{4374} \left( 709 - 2592 \zeta(3) \right) \epsilon^3. \]
7. I-tetragonal fixed point

\[ u_c = \frac{N+1}{3N} \epsilon + \frac{N+4}{81N^4} (70N^2 - 205N + 139) \epsilon^2 + \]
\[ \left( \frac{12(N+1)}{(5N-4)^2} \right) \zeta(3)(64N^3 - 188N^2 + 151N - 23) + \]
\[ \frac{N+4}{4(5N-4)^2} (6370N^4 + 24149N^3 - 144719N^2 + 197208N - 83256) \right) \epsilon^3, \]

\[ v_c = \frac{N+1}{9N} (N-2) \epsilon + \frac{N+4}{243N^4} (2N - 1)(17N^2 + 55N - 106) \epsilon^2 - \]
\[ \left( \frac{8(N+1)}{81N^2} \right) \zeta(3)(2N^4 + 8N^3 - 10N^2 - 9N + 7) - \frac{N+4}{20944N^4} (1418N^5 + 11713N^4 + 90281N^3 - 247414N^2 + 187528N - 44944) \right) \epsilon^3, \]

\[ z_c = \frac{N+1}{3N} (N-2) \epsilon + \frac{N+4}{81N^4} (2N - 1)(17N^2 + 55N - 106) \epsilon^2 - \]
\[ \left( \frac{8(N+1)}{81N^2} \right) \zeta(3)(2N^4 + 8N^3 - 10N^2 - 9N + 7) - \frac{N+4}{8748N^4} (1418N^5 + 11713N^4 + 90281N^3 - 247414N^2 + 187528N - 44944) \right) \epsilon^3. \]

8. II-tetragonal fixed point

\[ u_c = \frac{N+1}{5N-4} \epsilon + \frac{N+4}{(5N-4)^3} (70N^2 - 205N + 139) \epsilon^2 + \]
\[ \left( \frac{12(N+1)}{(5N-4)^2} \right) \zeta(3)(64N^3 - 188N^2 + 151N - 23) + \]
\[ \frac{N+4}{4(5N-4)^2} (370N^4 + 24149N^3 - 144719N^2 + 197208N - 83256) \right) \epsilon^3, \]

\[ v_c = \frac{N+1}{5N-4} (N-2) \epsilon + \frac{N+4}{(5N-4)^3} (30N^3 + 25N^2 - 217N + 166) \epsilon^2 - \]
\[ \left( \frac{24(N+1)}{(5N-4)^2} \right) \zeta(3)(8N^4 + 16N^3 - 88N^2 + 75N - 9) - \frac{N+4}{4(5N-4)^2} (1030N^5 + 2751N^4 + 46033N^3 - 207590N^2 + 267336N - 109808) \right) \epsilon^3, \]

\[ z_c = \frac{N+1}{5N-4} (N-2) \epsilon + \frac{N+4}{(5N-4)^3} (30N^3 + 25N^2 - 217N + 166) \epsilon^2 - \]
\[ \left( \frac{24(N+1)}{(5N-4)^2} \right) \zeta(3)(8N^4 + 16N^3 - 88N^2 + 75N - 9) - \frac{N+4}{4(5N-4)^2} (1030N^5 + 2751N^4 + 46033N^3 - 207590N^2 + 267336N - 109808) \right) \epsilon^3. \]

From these expressions it is seen that for the physically interesting case \( N = 2 \) the coordinates of the fixed points 2, 4, 7, and 8 coincide in the one–loop approximation, i.e. the Heisenberg point \( u_c = \epsilon \), \( v_c = z_c = 0 \) is four–fold degenerate. Such strong degeneracy is occasional, however, and lifted out in higher orders of the perturbation theory. So, the two–loop approximation splits those points apart. This situation is typical for a number of complicated models (see, for example, Refs. [7, 9]).

One can also notice from the above list that the Heisenberg and the Ising fixed points coincide at \( N = \frac{1}{2} \), while for the components of the cubic and the Ising fixed
points the relation \(-v_c^c = v_I^c\) holds at \(N = 1\). With \(N\) increasing, the cubic fixed point approaches the Heisenberg point from below and crosses it at \(N = N_c\), changing the sign of its \(v\)-coordinate. Further, when \(N \to \infty\) it moves towards the Ising point. Note, that the II–tetragonal fixed point is getting close to the Bose one when \(N\) grows. Such a behavior of the fixed points is in accordance with results obtained within the RG analysis in three dimensions [14, 15].

Since the symmetry transformations (13), (15) do not affect the form of the RG equations, they can only rearrange the fixed points. This observation may be used as an additional criterion for verification of our results. For example, points 1, 2, 5, and 8 stay untouched under transformation (13), while points 3 and 4 turn into 6 and 7, respectively (and vice versa).

Now let us discuss the character of the stability of the fixed points found. It is known to be determined by the signs of the eigenvalues \(\lambda_1, \lambda_2, \text{and } \lambda_3\) of the matrix

\[
M_{ij} = \begin{pmatrix}
\frac{\partial^2 \beta_u}{\partial u \partial u} & \frac{\partial^2 \beta_v}{\partial u \partial v} & \frac{\partial^2 \beta_z}{\partial u \partial z} \\
\frac{\partial^2 \beta_u}{\partial v \partial u} & \frac{\partial^2 \beta_v}{\partial v \partial v} & \frac{\partial^2 \beta_z}{\partial v \partial z} \\
\frac{\partial^2 \beta_u}{\partial z \partial u} & \frac{\partial^2 \beta_v}{\partial z \partial v} & \frac{\partial^2 \beta_z}{\partial z \partial z}
\end{pmatrix}
\]

evaluated at \(u = u_c, v = v_c, \text{and } z = z_c\). If the real parts of all the eigenvalue exponents are negative, the corresponding fixed point is infrared stable in three dimensional \((u, v, z)\)-space. Besides, the "saddle–knot" type fixed points may occur on the phase diagram, provided their eigenvalue exponents are of opposite signs. General expressions for the eigenvalue exponents are written out for arbitrary \(N\) in Appendix A. For the interesting cases \(N = 2\) and \(N = 3\) relevant to the substances of concern they are presented in Table 1 and Table 2, where \(\epsilon = \frac{1}{2}\) corresponds to the physical case.

It is seen from the tables that the Ising point has single negative eigenvalue, therefore it is stable only on the \(v\)-axis. The Heisenberg point is stable on the axis too if \(N > N_c\), becoming stable within the plane \((u, v)\) for \(N < N_c\). The cubic fixed point has the critical behavior opposite to that of the Heisenberg one; they interchange their stability at \(N = N_c\). The Bose point is stable within the plane \((v, z)\), being of the "saddle–knot" type in the three–parameter space. Note that the eigenvalue exponents of points 3 and 6 are the same as well as those of 4 and 7. It is a consequence of the symmetry (13) of the initial Hamiltonian.
Table 1: Eigenvalue exponents for $N = 2$ to third order in $\epsilon$.

| No | Type of fixed point | Eigenvalues |
|----|---------------------|-------------|
| 1  | Gaussian            | $\lambda_u = \lambda_v = \lambda_z = \epsilon$ |
| 2  | Heisenberg          | $\lambda_u = -\epsilon + \frac{13}{12} \epsilon^2 - \frac{84\zeta(3)+65}{36} \epsilon^3$  
|   |                     | $\lambda_v = \lambda_z = \frac{1}{3} \epsilon^2 + \frac{5\zeta(3)}{6} \epsilon^3$ |
| 3  | Ising               | $\lambda_u = \lambda_z = \frac{1}{3} \epsilon - \frac{38}{81} \epsilon^2 + \frac{2592\zeta(3)-937}{2187} \epsilon^3$  
|   |                     | $\lambda_v = -\epsilon + \frac{34}{27} \epsilon^2 - \frac{2592\zeta(3)+1605}{729} \epsilon^3$ |
| 4  | Cubic               | $\lambda_1 = -\epsilon + \frac{13}{12} \epsilon^2 - \frac{84\zeta(3)+65}{36} \epsilon^3$  
|   |                     | $\lambda_2 = -\frac{1}{3} \epsilon^2 + \frac{15\zeta(3)+11}{18} \epsilon^3$  
|   |                     | $\lambda_z = \frac{1}{3} \epsilon^2 - \frac{15\zeta(3)-7}{18} \epsilon^3$ |
| 5  | Bose                | $\lambda_u = \frac{1}{5} \epsilon - \frac{14}{25} \epsilon^2 + \frac{768\zeta(3)-311}{625} \epsilon^3$  
|   |                     | $\lambda_1 = -\frac{1}{5} \epsilon + \frac{33}{50} \epsilon^2 - \frac{768\zeta(3)+29}{125} \epsilon^3$  
|   |                     | $\lambda_2 = -\epsilon + \frac{6}{5} \epsilon^2 - \frac{384\zeta(3)+257}{125} \epsilon^3$ |
| 6  | VZ–cubic            | $\lambda_u = \lambda_1 = \frac{1}{5} \epsilon - \frac{38}{81} \epsilon^2 + \frac{2592\zeta(3)-937}{2187} \epsilon^3$  
|   |                     | $\lambda_2 = -\epsilon + \frac{34}{27} \epsilon^2 - \frac{2592\zeta(3)+1605}{729} \epsilon^3$ |
| 7  | I-tetragonal        | $\lambda_1 = \frac{1}{3} \epsilon^2 - \frac{15\zeta(3)-7}{18} \epsilon^3$  
|   |                     | $\lambda_2 = -\frac{1}{3} \epsilon^2 + \frac{15\zeta(3)+11}{18} \epsilon^3$  
|   |                     | $\lambda_3 = -\epsilon + \frac{13}{12} \epsilon^2 - \frac{84\zeta(3)+65}{36} \epsilon^3$ |
| 8  | II–tetragonal       | $\lambda_1 = \lambda_2 = -\frac{1}{3} \epsilon^2$  
|   |                     | $\lambda_3 = -\epsilon + \frac{17}{12} \epsilon^2 - \frac{84\zeta(3)+65}{36} \epsilon^3$ |
The most intriguing is the II–tetragonal fixed point proving to be absolutely stable in 3D, as it follows from the tables. Obviously, simple resummation procedures, such as Padé and Padé–Borel methods, applied to λ’s do not dismiss this conclusion. The presence of such a stable point is extremely important. It implies that the critical fluctuations do not destroy the second–order phase transitions, at least, if the anisotropy of the initial Hamiltonian is not too strong. Since the stable fixed point is located on the plane v = z it is certainly relevant to the critical behavior of TbAu₂, DyC₂, TbD₂, and Nd.

Let us note, that the ε–expansions of the eigenvalue exponents λ₁ and λ₂ for the II–tetragonal point substantially differ from the others. Namely, their series prove to be shorter by one order (see Table 1 and Table 2). This phenomenon originates from the two–fold degeneracy of the roots of the characteristic polynomial in the one–loop approximation. As a consequence, the eigenvalue exponents should be expanded in √ε rather than ε. It can be shown, however, that for almost all N non–integer powers drop from λ₁ and λ₂ for the eighth fixed point in every order of the perturbation theory. For the special case N = 2, significant from the physical viewpoint, one cannot make such a statement within three–loop approximation. To answer that question one should take into account at least four–loop contributions. Apart from whether or not non–integer powers of ε appear in the expansions, one–loop degeneracy of λ’s results in reduction of information available from a given approximation. So, assuming that √ε will not appear in the series for N = 2, evaluating coefficients of λ’s in third order in ε would require accounting five–loop terms. To understand the structure of the eigenvalue exponent series with one–loop degeneracy, we conduct detailed analysis of the problem in Appendix B.

4 Marginal dimensionality and critical exponents

We have shown in the previous section that for the physically interesting cases N = 2 and N = 3 the II–tetragonal fixed point is three–dimensionally stable in 3D. The question may be put forward whether this point is stable for all N. To answer it the critical dimensionality of the order parameter Nc needs to be calculated. It separates two different regimes of critical behavior of the model. When N > Nc the II–tetragonal rather than the Bose fixed point is three–dimensionally stable in 3D. At N = Nc they interchange their stability so that when N < Nc the stable fixed point is the Bose one. The ε–expansion for Nc can be found from the condition v_c = z_c = 0 imposed on the coordinates of the eighth fixed point (see Sec. 3). Three–loop approximation gives

\[ N_c = 2 - 2\epsilon + \frac{5}{6}(6\zeta(3) - 1)\epsilon^2 + O(\epsilon^3), \]

(19)
Table 2: Eigenvalue exponents for $N = 3$ to third order in $\epsilon$.

| No | Type of fixed point | Eigenvalues |
|----|---------------------|-------------|
| 1  | Gaussian            | $\lambda_u = \lambda_v = \lambda_z = \epsilon$ |
| 2  | Heisenberg          | $\lambda_u = -\epsilon + \frac{48}{49} \epsilon^2 - \frac{2(2184\zeta(3)+1931)}{10807} \epsilon^3$  
  $\lambda_v = \lambda_z = \frac{1}{9} \epsilon + \frac{104}{343} \epsilon^2 - \frac{2(5208\zeta(3)-2311)}{16807} \epsilon^3$ |
| 3  | Ising               | $\lambda_u = \lambda_z = \frac{1}{3} \epsilon - \frac{38}{81} \epsilon^2 + \frac{2592\zeta(3)-937}{2187} \epsilon^3$  
  $\lambda_v = -\epsilon + \frac{34}{27} \epsilon^2 - \frac{2592\zeta(3)+1603}{729} \epsilon^3$ |
| 4  | Cubic               | $\lambda_1 = -\epsilon + \frac{250}{2187} \epsilon^2 - \frac{2(46888\zeta(3)+165287)}{118098} \epsilon^3$  
  $\lambda_2 = -\frac{1}{9} \epsilon - \frac{250}{2187} \epsilon^2 + \frac{5(36936\zeta(3)+41611)}{1062882} \epsilon^3$  
  $\lambda_3 = \frac{1}{9} \epsilon + \frac{520}{2187} \epsilon^2 - \frac{2(120528\zeta(3)-16033)}{531441} \epsilon^3$ |
| 5  | Bose                | $\lambda_u = \frac{1}{5} \epsilon - \frac{14}{25} \epsilon^2 + \frac{768\zeta(3)-311}{625} \epsilon^3$  
  $\lambda_1 = -\frac{1}{5} \epsilon + \frac{38}{81} \epsilon^2 - \frac{768\zeta(3)+29}{625} \epsilon^3$  
  $\lambda_2 = -\epsilon + \frac{6}{5} \epsilon^2 - \frac{384\zeta(3)+257}{125} \epsilon^3$ |
| 6  | VZ–cubic            | $\lambda_u = \lambda_1 = \frac{1}{3} \epsilon - \frac{38}{81} \epsilon^2 + \frac{2592\zeta(3)-937}{2187} \epsilon^3$  
  $\lambda_2 = -\epsilon + \frac{34}{27} \epsilon^2 - \frac{2592\zeta(3)+1603}{729} \epsilon^3$ |
| 7  | I-tetragonal        | $\lambda_1 = \frac{1}{9} \epsilon + \frac{520}{2187} \epsilon^2 - \frac{2(120528\zeta(3)-16033)}{531441} \epsilon^3$  
  $\lambda_2 = -\frac{1}{9} \epsilon - \frac{250}{2187} \epsilon^2 + \frac{5(36936\zeta(3)+41611)}{1062882} \epsilon^3$  
  $\lambda_3 = -\epsilon + \frac{58}{55} \epsilon^2 - \frac{3(123600\zeta(3)+71621)}{166375} \epsilon^3$ |
| 8  | II–tetragonal       | $\lambda_1 = -\frac{1}{11} \epsilon - \frac{2}{605} \epsilon^2$  
  $\lambda_2 = -\frac{1}{11} \epsilon + \frac{1}{605} \epsilon^2$  
  $\lambda_3 = -\epsilon + \frac{58}{55} \epsilon^2 - \frac{3(123600\zeta(3)+71621)}{166375} \epsilon^3$ |
where $\epsilon = \frac{1}{2}$ corresponds to the physical space dimensionality $D = 3$. The same expansion holds within the plane $(u, v)$. Note, that expression (19) coincides with that found for the cubic model with the complex order parameter [8]. Such a coincidence is not occasional because, as was already emphasized, for $v = z$ the model (1) goes over into the complex cubic model.

Unfortunately, RG expansion (19) is known to be divergent. Nevertheless, the physical information may be extracted from it, provided some resummation method is applied. Since the series of $N_c$ is alternating, the Borel transformation combined with its proper analytical continuation may play a role of such method. To perform analytical continuation the Padé approximant only of the type $[1/1]$ may be used within given approximation. The Padé–Borel summation of the expansion (19) gives

$$N_c = a - \frac{2b^2}{c} + \frac{4b^3}{c^2 \epsilon} \exp \left( -\frac{2b}{c\epsilon} \right) E_i \left( \frac{2b}{c\epsilon} \right),$$

(20)

where $a, b, c$ are the coefficients before $\epsilon^0, \epsilon^1, \epsilon^2$ in Eq. (13), respectively, and $E_i(x)$ is the exponential integral. Setting $\epsilon = \frac{1}{2}$ in Eq. (20), we obtain the value of the critical dimensionality

$$N_c = 1.50.$$  

(21)

This number is close to $N_c = 1.47$ found within the three dimensional RG approach [7]. Since $N_c$ lies below two, the critical behavior of antiferromagnets ($N = 2, N = 3$) and NbO$_2$ ($N = 2$) must be governed by the II–tetragonal fixed point.

Now let us turn to calculating the critical exponents. To this end, substitute the coordinates of fixed points (see Sec. 3) into the expressions for $\gamma^{-1}$ and $\eta$ (Eqs. (14) and (11)). For the stable fixed point 8 it gives

$$\gamma^{-1} = 1 + \frac{\epsilon}{(5N - 4)}3(1 - N) + \frac{\epsilon^2}{(5N - 4)^2}(N - 1)(40N^2 - 214N + 205)$$

$$+ \frac{\epsilon^3}{(5N - 4)^3}(1 - N)(12\zeta(3)(5N - 4)(32N^3 - 156N^2 + 159N - 13)$$

$$- 940N^4 - 6748N^3 + 42681N^2 - 67102N + 32558),$$

(22)

$$\eta = \frac{\epsilon^2}{(5N - 4)^2}(N - 1)(2N - 1) + \frac{\epsilon^3}{2(5N - 4)^4}(N - 1)(190N^3 - 535N^2$$

$$+ 652N - 324) + \frac{\epsilon^4}{4(5N - 4)^6}(1 - N)(96\zeta(3)(5N - 4)(32N^4 - 128N^3$$

$$+ 212N^2 - 153N + 33) - 10570N^5 + 22691N^4 + 68527N^3$$

$$- 280399N^2 + 326888N - 127676).$$

16
From Eqs. (22) we find for $N = 2$

$$\gamma^{-1} = 1 - \frac{\epsilon}{2} - \frac{7\epsilon^2}{24} + \frac{\epsilon^3(84\zeta(3) - 1)}{144},$$

$$\eta = \frac{\epsilon^2}{12} + \frac{5\epsilon^3}{36} + \frac{\epsilon^4(13 - 21\zeta(3))}{108}$$

(23)

and for $N = 3$

$$\gamma^{-1} = 1 - \frac{6\epsilon}{11} - \frac{14\epsilon^2}{121} + \frac{2\epsilon^3(912\zeta(3) + 3905)}{14641},$$

$$\eta = \frac{10\epsilon^2}{121} + \frac{177\epsilon^3}{1331} + \frac{\epsilon^4(50083 - 59328\zeta(3))}{322102}$$

(24)

where $\epsilon = \frac{1}{2}$ as before. Other critical exponents are found from the well known scaling relations.

We will focus first on qualitative discussion of the results obtained. As was found in Ref. [3], the critical exponents of the Heisenberg and the II–tetragonal fixed points coincide within the two–loop approximation. Three–loop analysis yields for the Heisenberg fixed point at $N = 2$

$$\gamma^{-1} = 1 - \frac{\epsilon}{2} - \frac{7\epsilon^2}{24} + \frac{\epsilon^3(28\zeta(3) - 11)}{48},$$

$$\eta = \frac{\epsilon^2}{12} + \frac{5\epsilon^3}{36} + \frac{\epsilon^4(13 - 21\zeta(3))}{108}$$

(25)

Comparing (23) with (25) we see that the critical exponent $\gamma$ of the Heisenberg and the II–tetragonal fixed points is different in third order in $\epsilon$, although that difference is not too strong. This is one of the results of our investigation.

It is known that RG series for critical exponents are badly divergent. However they contain important physical information which can be extracted provided some procedure making them convergent is applied. Unfortunately, it is impossible to use simple Padé–Borel summation to process series (23)–(25) because their coefficients have irregular signs, in contrast to the critical dimensionality $N_c$. The most appropriate resummation scheme known for now is a modification of the Borel technique. Principal underlying ideas of this method are the analytical continuation of the Borel transform beyond its circle of convergence over the cut–plane and a conformal mapping sending the cut–plane onto the circle. Such an operation leads to integration of a holomorphic function represented by an absolutely convergent series and allows to perform integration prior to summation, thus substantiating the perturbation theory approach. The algorithm just mentioned incorporates both exactly calculated first several terms and
high order asymptotic behavior of perturbation series. For the simple $O(n)$–symmetric model the coefficients at large order $k$ were shown to look like $(-1)^k k! a^{-k} k^b$ \[31, 32\]. It can be expected that in complex models with more than one coupling constants asymptotics of RG series will comprise such a factor, at least. Parameters $a$ and $b$ play an essential role in the modified Borel method. For a given series

$$F(\epsilon) = \sum f_k \epsilon^k$$

transformation

$$F(\epsilon) \sim \int_0^\infty e^{-\frac{t}{a \epsilon}} \left( \frac{t}{a \epsilon} \right)^b B(t) \frac{dt}{a \epsilon},$$

where $B(t) = \sum k \frac{f_k}{a^{b+1} k^k} t^k$, is followed by the conformal mapping

$$\omega = \frac{\sqrt{t+1} - 1}{\sqrt{t+1} + 1}.$$ 

Function $B(t)$ is represented by the series in $\omega$

$$B(t(\omega)) = \left( \frac{2}{1-\omega} \right)^{2\lambda} \sum_k A_k(\lambda) \omega^k$$

where the additional parameter $\lambda$ is introduced to eliminate possible singularity at $\omega = 1$. Since the type of that singularity is unknown $\lambda$ is chosen so as to ensure the most rapid convergence of the series \[28\].

The main obstruction for application of the method just outlined to the model (1) as well as to a great deal of other complex anisotropic systems is unknown asymptotic parameters $a$ and $b$. Evaluating them requires enormous efforts. In the case of the $n$–vector model with one coupling constant the parameters $a$ and $b$ have been exactly calculated that allowed to obtain accurate numerical estimates for the critical exponents \[24, 28, 33, 34\]. Attempts to find asymptotic parameters for the cubic model also were made \[35\]. They proved to be successful, however, only within the assumption of very weak anisotropy. Despite there is no information about asymptotic parameters of the model (1) available at the moment we chose to resort to the resummation scheme of \[24, 28\], in view of the following arguments. Although the asymptotic parameters for the isotropic model are explicitly calculated, in Ref. \[24\] parameter $b$ was varied in a neighborhood of the exact value. It is justified by that exact $a$ and $b$ determine large order behavior of $F(\epsilon)$ while actually one deals with only few terms of perturbation series. We believe therefore that, in connection with the model (1), similar manipulations may be valid not only with respect to parameter $b$ but to parameter $a$ as well. Variation of $a$ and $b$ in a range containing exact asymptotic parameter values of the
Table 3: Critical exponents $\eta$ and $\gamma$ of the model (1.1) for $N = 2$ and $N = 3$ calculated within the three–loop approximation

| Type of fixed point | $N = 2$       |               | $N = 3$       |               |
|---------------------|---------------|---------------|---------------|---------------|
|                     | $\eta$        | $\gamma$      | $\eta$        | $\gamma$      |
| Heisenberg          | 0.0285 ± 0.0002 | 1.368 ± 0.004 | 0.0271 ± 0.0002 | 1.440 ± 0.005 |
| Bose                | 0.0279 ± 0.0002 | 1.265 ± 0.011 | 0.0279 ± 0.0002 | 1.265 ± 0.011 |
| II-tetragonal       | 0.0285 ± 0.0002 | 1.355 ± 0.015 | 0.0281 ± 0.0002 | 1.380 ± 0.008 |

$O(n)$–symmetric fixed point and using $\lambda$ as an optimizing parameter result in values of the critical exponents displayed in the Table 3. Here we suppose that unknown exact asymptotic values $a$ and $b$ of the model (1) are not much distant from those of the $O(n)$–symmetric model. The error of the numerical estimates is established through the dispersion of the output due to the variation of $a$, $b$, and $\lambda$.

As may be seen from the table, the critical exponents of the II–tetragonal fixed point appear to be close to those of the Heisenberg point. Unfortunately, we cannot compare the critical exponent values of the II–tetragonal fixed point with their two–loop analogs and therefore decide how far they shift from the Heisenberg ones with higher–loop terms being taken into account. The point is that the estimates of the critical exponents were done in Ref. [4] by direct summation of the $\epsilon$–expansion terms setting $\epsilon = 1$, that was illegal for the asymptotic series. Under such circumstances, let us compare the results obtained with predictions given for the investigated model by the RG procedure in three dimensions. Two– and three–loop calculations carried out in Refs. [14, 16] shown that the critical exponents of the II–tetragonal point turned out to be close not to those of the $O(2N)$–symmetric model, as in the case of the $\epsilon$–expansion method, but to the exponents of the $3D$ XY (Bose) model. Within the RG analysis in 3D it is a consequence of the closeness of the stable fixed point 8 and the Bose point 5 on the three dimensional RG flow diagram. Despite of such a distinction in estimates of the critical exponents given by these two RG approaches, one can hope that involving higher perturbation orders and using an appropriate resummation technique will soften this discrepancy. Not so strong difference between the critical exponent values for the II–tetragonal fixed point obtained within the $\epsilon$–expansion and the $3D$ RG methods may serve as a possible confirmation to this conjecture. Indeed, $3D$ RG analysis of the model (1) yields the following estimates for the critical exponents of the II–tetragonal fixed point: $\gamma = 1.336$, $\eta = 0.0261$ at $N = 2$ and $\gamma = 1.329$, $\eta = 0.0261$ at $N = 3$. Comparing these numbers with their analogs from Table 3, we conclude that the relative deviation does not exceed 4% for $\gamma$ and 8% for $\eta$, that is...
not so bad for the three–loop approximation. An additional stimulus for our hope is the beautiful agreement of numerical estimates of the critical exponents for the simple $O(n)$–symmetric model achieved in sufficiently high orders of the perturbation theory between the 3$D$ RG [24, 33] and $\epsilon$–expansion [34] approaches. So, for the Ising model the magnetic susceptibility exponent was found to be $\gamma = 1.241$ in the frame of 3$D$ RG and $\gamma = 1.239$ within the $\epsilon$–expansion method. The relative deviation of these values is about 0.1%.

At last, we would like to emphasize, that although the accuracy of the estimates of the critical exponents achieved in the paper cannot be regarded as satisfactory the numerical values of the critical exponents for the II–tetragonal fixed point presented here are, in our opinion, the most realistic among those so far obtained on the base of the $\epsilon$–expansion method.

5 Conclusion

The complete RG analysis of a model with three quartic coupling constants and 2$N$–component real order parameter field describing phase transitions in certain cubic and tetragonal antiferromagnets as well as the structural phase transition in $NbO_2$ crystal has been carried out within the three–loop approximation in $D = 4 - 2\epsilon$ dimensions. Perturbation expansions for the $\beta$–functions of the record length were obtained using dimensional regularization and the minimal subtraction scheme. Coordinates of the fixed points and their eigenvalue exponents were calculated for arbitrary $N$. The analysis performed for the eigenvalue exponents has shown that for $N \geq 2$ the II–tetragonal rather than the Bose fixed point is absolutely stable in the physical space within given approximation. The three–loop $\epsilon$–expansion for the critical dimensionality of the order parameter $N_c$ was found and processed by the Padé–Borel resummation technique. The numerical estimate $N_c = 1.50$ obtained confirms the conclusion about the stability of the II–tetragonal fixed point. Consequently, the phase transitions in the $NbO_2$ crystal and antiferromagnets $TbAu_2$, $DyC_2$, $K_2IrCl_6$, $TbD_2$, and $Nd$ are of second order and their critical thermodynamics should be controlled by this point, in the frame of given approximation.

It was observed that the degeneracy of the eigenvalue exponents in the one–loop approximation for the II–tetragonal fixed point resulted in certain difficulties in calculating their $\epsilon$–series. According to the analysis carried out, two–fold degenerate eigenvalue exponents should be expanded not in $\epsilon$ but in $\sqrt{\epsilon}$. Although non–integer powers of $\epsilon$ was shown to drop from the expansions for all $N$ excepting $N = 1, 2$, such a degeneracy led to reduction in length of the RG series for eigenvalue exponents and therefore to the loss of accuracy expected from given approximation. Indeed, within
the three–loop approximation we actually obtain two–loop–like pieces of the series and
evaluation of the term of order $\epsilon^3$ may require to account the five–loop contributions. To
understand the structure of the eigenvalue exponent series for the special case $N = 2$,
important physically, one has to consider at least four–loop approximation.

Perturbation expansions for the critical exponents $\gamma$ and $\eta$ were calculated up to
$\epsilon^3$ and $\epsilon^4$, respectively. For $N = 2$ the magnetic susceptibility exponents for the II–
tetragonal and Heisenberg fixed points were found to be different in third order in $\epsilon$. For the first time within the $\epsilon$–expansion method the numerical estimates of the
critical exponents of the model under consideration were given on the base of the
Borel summation technique modified with a conformal mapping. For the physically
interesting cases $N = 2$ and $N = 3$ the critical exponents of the II–tetragonal fixed
point turned out to be numerically close to those of the Heisenberg one. On the
contrary, in the frame of the field–theoretical RG approach in three dimensions the
critical exponents of the Bose and the unique stable fixed points are close to each
other. Possibly, these two alternative RG approaches will be in better agreement,
provided the higher–loop contributions are taken into account.

The results achieved in our study seem to be self–consistent although there is defi-
nite discrepancy with the non–perturbative theoretical predictions. We believe that it
is the effect of insufficiently high approximation employed and the problem of bring-
ing the results given by the $\epsilon$–expansion method into accordance with those of other
theoretical approaches and experimental data needs to be solved.

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A Appendix

In this appendix we present the eigenvalue exponents of all the fixed points for arbitrary
$N$.

1. Gaussian fixed point

$$\lambda_u = \lambda_v = \lambda_z = \epsilon.$$
2. $O(2N)$–symmetric or Heisenberg fixed point

\[
\lambda_u = -\epsilon + \frac{3}{(N+1)^2} (3N + 7)\epsilon^2 - \frac{1}{2(N+4)^3} \left(48\zeta(3)(N+4)(5N+11) + 33N^3 + 269N^2 + 1072N + 1196\right)\epsilon^3,
\]
\[
\lambda_v = \lambda_z = \frac{1}{N+3} (N - 2)\epsilon + \frac{1}{(N+4)^2} (5N^2 + 7N + 38)\epsilon^2 - \frac{1}{2(N+4)^3} \left(48\zeta(3)(N+4)(2N^2 + 7N + 23) - 13N^4 - 199N^3 - 2(183N^2 + 98N - 532)\right)\epsilon^3.
\]

3. Ising fixed point

\[
\lambda_u = \lambda_z = \frac{1}{3} \epsilon - \frac{48}{81} \epsilon^2 + \frac{1}{2187} (2592\zeta(3) - 937)\epsilon^3,
\]
\[
\lambda_v = -\epsilon + \frac{34}{27} \epsilon^2 - \frac{1}{729} (2592\zeta(3) + 1603)\epsilon^3.
\]

4. Cubic fixed point

\[
\lambda_1 = -\epsilon + \frac{2N-1}{27N(N+1)} (17N^2 - 2N + 53)\epsilon^2 - \frac{1}{1458N^4(N+1)^4} \left(1296\zeta(3)N(4N^6 + 4N^4 + 27N^3 + 15N^2 - 11N - 7) + 3206N^7 - 11683N^6 + 48012N^5 + 34522N^4 - 111830N^3 + 71205N^2 + 3452N - 11236\right)\epsilon^3,
\]
\[
\lambda_2 = \frac{2N}{3N} \epsilon + \frac{2N-1}{81N(N+1)} (19N^3 - 36N^2 - 165N + 106)\epsilon^2 + \frac{1}{4374N^4(N+1)^4} \left(1296\zeta(3)N(2N^6 + 3N^5 - 22N^4 - 39N^3 - 3N^2 + 20N + 7) - 937N^7 + 7850N^6 - 40674N^5 + 6832N^4 + 146287N^3 - 99642N^2 - 27196N + 22472\right)\epsilon^3,
\]
\[
\lambda_z = \frac{N-2}{3N} \epsilon + \frac{1}{3N^3} (19N^2 - 127N + 106)\epsilon^2 + \frac{1}{4374N^4} \left(1296\zeta(3)N(4N^4 - 20N^3 + 4N^2 + 21N - 7) - 1874N^5 - 9997N^4 + 94159N^3 - 168626N^2 + 109028N - 22472\right)\epsilon^3.
\]

5. Bose fixed point

\[
\lambda_u = \frac{1}{5} \epsilon - \frac{14}{25} \epsilon^2 + \frac{1}{625} (768\zeta(3) - 311)\epsilon^3,
\]
\[
\lambda_1 = -\frac{1}{5} \epsilon + \frac{2}{5} \epsilon^2 - \frac{1}{625} (768\zeta(3) + 29)\epsilon^3,
\]
\[
\lambda_2 = -\epsilon + \frac{6}{5} \epsilon^2 - \frac{1}{125} (384\zeta(3) + 257)\epsilon^3.
\]
6. VZ–cubic fixed point

\[ \lambda_u = \lambda_1 = \frac{1}{3} \epsilon - \frac{38}{81} \epsilon^2 + \frac{1}{2187} (2592 \zeta(3) - 937) \epsilon^3 , \]
\[ \lambda_2 = -\epsilon + \frac{34}{27} \epsilon^2 - \frac{1}{729} (2592 \zeta(3) + 1603) \epsilon^3 . \]

7. I-tetragonal fixed point

\[ \lambda_1 = \frac{N-2}{3N} \epsilon + \frac{1-2N}{81N^3} (19N^2 - 127N + 106) \epsilon^2 + \frac{1}{4374N^7} (1296 \zeta(3) N (4N^4 - 20N^3 + 4N^2 + 21N - 7) - 1874N^6 - 9997N^4 + 94159N^3 - 168626N^2 - 109028N - 22472) \epsilon^3 , \]
\[ \lambda_2 = \frac{2-N}{3N} \epsilon + \frac{2N-1}{81N^9(N+1)} (19N^3 - 36N^2 - 165N + 106) \epsilon^2 + \frac{1}{4374N^{12}(N+1)^7} (1296 \zeta(3) N (2N^6 + 3N^5 - 22N^4 - 39N^3 - 3N^2 + 20N + 7) - 937N^7 + 7850N^6 - 40674N^5 + 6832N^4 + 146287N^3 - 99642N^2 - 27196N + 22472) \epsilon^3 , \]
\[ \lambda_3 = -\epsilon + \frac{2N-1}{27N^2(N+1)} (17N^2 - 2N + 53) \epsilon^2 - \frac{1}{1458N^{15}(N+1)^9} (1296 \zeta(3) N (4N^6 + 4N^4 + 27N^3 + 15N^2 - 11N - 7) + 3206N^7 - 11683N^6 + 48012N^5 + 34522N^4 - 111830N^3 + 71205N^2 + 3452N - 11236) \epsilon^3 . \]

8. II-tetragonal fixed point

\[ \lambda_1 = \frac{2-N}{5N-4} \epsilon + \frac{1-N}{(5N-4)(2N-1)} (4 \text{sign}(N-1) |5N^3 + 6N^2 - 48N + 32| - 3(40N^3 - 208N^2 + 253N - 66))\epsilon^2 , \]
\[ \lambda_2 = \frac{2-N}{5N-4} \epsilon + \frac{(N-1)}{(5N-4)^2(2N-1)} (4 \text{sign}(N-1) |5N^3 + 6N^2 - 48N + 32| + 3(40N^3 - 208N^2 + 253N - 66))\epsilon^2 , \]
\[ \lambda_3 = -\epsilon + \frac{1}{(5N-4)^2(2N-1)} (60N^3 - 160N^2 + 181N - 85) \epsilon^2 + \frac{1}{2(5N-4)^6(1-2N)^2} (48 \zeta(3)(2N-1)^2(5N-4)(32N^4 - 128N^3 + 212N^2 - 153N + 33) + 20560N^7 - 165328N^6 + 644392N^5 - 1406864N^4 + 1756745N^3 - 1224341N^2 + 433704N - 59052) \epsilon^3 . \]

Here \( \epsilon = \frac{1}{2} \) corresponds to the physical space.
As was noted in Sec. 3, the II–tetragonal fixed point has an unusual structure of the series of the $\epsilon$–expansion for the eigenvalues of the stability matrix. Namely, those series are shorter by one order, comparing to their analogs for the other fixed points. This is, actually, a consequence of the multiplicity of the roots of the characteristic equation in the one–loop approximation, that may cause non–integer powers of $\epsilon$ to contribute to the expansions. Such a conclusion seems so exotic that deserves thorough investigation, to which the present Appendix is devoted. The result of the analysis is that for $N \neq 1, 2$ non–integer powers do not appear in the eigenvalue expansions series in all orders of the perturbation theory. As to the physically important case $N = 2$, we cannot make such a statement within the three–loop approximation. To answer the question whether or not non–integer powers of $\epsilon$ will appear in the expansions the higher–loop (at least four–loop) approximations need to be considered.

An eigenvalue $\lambda$ of the stability matrix is a root of its characteristic polynomial. It is convenient, rather, to deal with the quantity $y = \frac{1}{\epsilon} \lambda$ which is a root of the corresponding reduced polynomial denoted hereafter $P(y, \epsilon)$. In every order of the perturbation theory its coefficients are also polynomials in $\epsilon$, therefore a piece of the series of $y(\epsilon)$ determined within corresponding approximation coincides with that of some algebraic function. Such a function is not analytical in those points on the complex plane where the defining polynomial ($P(y, \epsilon)$ in our case) has multiple roots. Instead, it has branching of an order not greater than the multiplicity of the root. As to the II–tetragonal point, at $\epsilon = 0$ (one–loop approximation) the reduced characteristic polynomial has two equal roots of the three. It leads to the conclusion that $y(\epsilon)$ should be expanded not in $\epsilon$ but in $\sqrt{\epsilon}$ as a Puiseux series \[36\]. That is how half–integer powers of $\epsilon$ may occur in the series of eigenvalue exponents. Let us show, however, that in the model (1) they are absent at least for $N \neq 1, 2$. Consider the reduced characteristic equation

$$-y^3 + ay^2 - by + c = 0 \quad (26)$$

and assume

$$a = a_0 + a_1 \epsilon + a_2 \epsilon^2 + \ldots, \quad (27)$$
$$b = b_0 + b_1 \epsilon + b_2 \epsilon^2 + \ldots, \quad (27)$$
$$c = c_0 + c_1 \epsilon + c_2 \epsilon^2 + \ldots, \quad (27)$$
$$y = y_0 + y_1 \epsilon^{\frac{1}{2}} + y_2 \epsilon + y_3 \epsilon^{\frac{3}{2}} + \ldots.$$  

Here we have omitted higher terms relevant to higher than three loops. The coefficients
in (27) are rational functions in $N$:

\[ a_0 = \left(\frac{1}{5N-4}\right)(7N - 8), \]
\[ a_1 = \left(\frac{1}{5N-4}\right)^2(270N^3 - 1129N^2 + 1591N - 736), \]
\[ a_2 = \left(\frac{1}{5N-4}\right)^3(48\zeta(3)(5N - 4)(144N^4 - 720N^3 + 1289N^2 - 947N + 230) + 10030N^5 - 104229N^4 + 429747N^3 - 804632N^2 + 691620N - 222720), \]
\[ b_0 = \left(\frac{1}{5N-4}\right)^2(N - 2)(11N - 10), \]
\[ b_1 = \left(\frac{1}{5N-4}\right)^2(510N^4 - 3157N^3 + 6615N^2 - 5832N + 1868), \]
\[ b_2 = \left(\frac{1}{5N-4}\right)^3(48\zeta(3)(5N - 4)(272N^5 - 1824N^4 + 4455N^3 - 5095N^2 + 2754N - 558) + 25890N^6 - 338437N^5 + 1547050N^4 - 3437182N^3 + 4044203N^2 - 2430752N + 589412), \]
\[ c_0 = \left(\frac{1}{5N-4}\right)^2(N - 2)^2, \]
\[ c_1 = \left(\frac{1}{5N-4}\right)^2(N - 2)(150N^3 - 809N^2 + 1229N - 566), \]
\[ c_2 = \left(\frac{1}{5N-4}\right)^3(48\zeta(3)(N - 2)(5N - 4)(80N^4 - 464N^3 + 865N^2 - 641N + 164) + 13950N^6 - 184745N^5 + 887705N^4 - 2072060N^3 + 2541094N^2 - 1575640N + 389512). \]

(28)

Substituting (27) into (26) and letting $\epsilon = 0$ we find that $y_0$ is two–fold degenerate taking the value $-1$ once and $\frac{-N}{5N-4}$ twice. Comparing factors before equal powers of $\epsilon$ in (28), we recursively evaluate next terms. The first appearance of a coefficient $y_l$, $l > 0$, occurs at the $l$-th step, where it has the multiplier $\partial_y P(y_0, 0)$. Factor $\partial_y P(y_0, 0)$ vanishes due to the multiplicity of $y_0$, hence $y_l$ actually cannot be determined from the $l$-th order. So, for $y_\frac{1}{2}$ from first order in $\epsilon$ we have the quadratic equation

\[ y^2_{\frac{1}{2}} = \frac{a_1 y^2_0 - b_1 y_0 + c_1}{3y_0 - a_0}. \]

(29)

The denominator on the right–hand side is non–zero because it is proportional to $\partial^2_y P(y_0, 0)$ and only two of the three roots coincide. Substitution of (28) into (29) gives $y_{\frac{1}{2}} = 0$. The next order ($\epsilon^2$) does not provide $y_1$, as it might be expected, because at this step equation (27) vanishes identically. Considering factors before $\epsilon^2$, we come to the quadratic equation for $y_1$:

\[ y^2_1(-3y_0 + a_0) + y_1(2a_1 y_0 - b_1) + a_2 y^2_0 - b_2 y_0 + c_2 = 0. \]

(30)

The highest coefficient $-3y_0 + a_0$ is proportional to the non–zero quantity $\partial^2_y P(y_0, 0)$, therefore $y_1$ is explicitly determined:

\[ y_1 = \frac{3(N-1)(10N^3 - 208N^2 + 253N - 66)}{(2N-1)(5N-4)^3}. \]
\[ \pm \frac{4|N-1(N-2)(N+4)(5N-4)|}{(2N-1)(5N-4)^3}. \] (31)

Factors before \( \epsilon^3 \) in Eq. (26) obey the equation

\[ By_{\frac{3}{2}} = 0, \]

where

\[ B = -6y_0y_1 + 2a_0y_1 + 2a_1y_0 - b_1. \] (32)

We have come to the crucial point of the consideration. Supposing \( B \neq 0 \) we have \( y_{\frac{3}{2}} = 0 \). Further, this step can be put into the base of mathematical induction in proving disappearance of non–integer powers of \( \epsilon \). For \( m \geq \frac{3}{2} \) we have

\[ 0 = \partial_\epsilon P(y_0, 0)y_{m+1} + (-6y_0 + 2a_0)y_{\frac{3}{2}}y_{m+\frac{1}{2}} + By_m + ..., \]

were the terms depending only on \( y_l \) with \( l < m \) have been suppressed. The first two terms turn into zero while \( B \neq 0 \), so the coefficients \( y_m, m \geq \frac{3}{2} \), can be calculated recurrently. Suppose now that \( m = \frac{2k+1}{2} \) with \( k \) integer. The equation on \( y_{\frac{2k+1}{2}} \) can be written in the form

\[ By_{\frac{2k+1}{2}} = F(y_{\frac{3}{2}}, y_{\frac{5}{2}}, \ldots, y_{\frac{2k+1}{2}}), \]

where \( F \) is some polynomial with zero absolute term in it (because \( a(\epsilon), b(\epsilon), \) and \( c(\epsilon) \) does not depend upon non–integer powers). Recursively we have \( y_{\frac{2k+1}{2}} = 0 \) as required.

Thus, the multiplicity of the roots does not give rise to non–integer powers of \( \epsilon \) unless expression (32) vanishes. The important fact is that (32) is the derivative of the quadratic polynomial (30) with respect to \( y_{\frac{3}{2}} \). It implies that (32) turns into zero if and only if \( y_{\frac{3}{2}} \) is two–fold degenerate. Formula (31) shows that this possibility is realized only for \( N = 1 \) and \( N = 2 \). Concerning these two special cases, one can see, that \( y_{\frac{3}{2}} \) is determined from a quadratic equation \( \partial^2_\epsilon P(y_0, 0)(y_{\frac{3}{2}})^2 = \ldots \) arising from comparing factors before \( \epsilon^3 \) in Eq. (26). The right–hand side depends on the four–loop contributions. So, if it does not vanish, the expansions would contain non–integer powers. Otherwise, \( y_{\frac{3}{2}} = 0 \) and the five–loop approximation gives, in its turn, a quadratic equation for \( y_2 \).

In summary, let us have a look at the structure of the one–loop degenerate eigenvalue exponent series in general. For every \( l \) let \( d(l) \) be the order of the expansion of the reduced characteristic equation in which \( y_l \) is determined. It ranges from \( l + 1 \) to \( 2l \). If \( d(l) = 2l \) then \( y_l \) is found from a quadratic equation with the non–zero highest
coefficient $\partial_y^2 P(y_0, 0)$. We shall say that the solution $y(\epsilon)$ splits at the step $l_s$ if that equation gives two different values of $y_{l_s}$. If $y(\epsilon)$ does not split at all, it is convenient to assign $l_s \equiv \infty$. Let us formulate the resulting theorem as the set of four propositions.

**Theorem**

1. Either the characteristic equation has two equal roots in every order of the perturbation theory or its solution splits at a finite step $l_s$. For every half-integer number $l$ from the interval $[0, l_s]$ coefficient $y_l$ is determined in the order $d(l) = 2l$.

2. In the case of finite $l_s$ coefficient $y_{l_s+m}$ is determined in the order $d(l_s + m) = 2l_s + m$ for all $m \geq \frac{1}{2}$.

3. Coefficients $y_l$ with non-integer numbers $l < l_s$ are equal to zero.

4. Non-integer powers of $\epsilon$ contribute to the expansions of the eigenvalue exponents if and only if $l_s$ is a non-integer number.

We have demonstrated how the theorem works in the frame of the model under consideration, and its full proof will be given elsewhere.

**References**

[1] D. Mukamel, Phys. Rev. Lett. 34, 481 (1975).
[2] D. Mukamel and S. Krinsky, J. Phys. C 8, L496 (1975).
[3] D. Mukamel and S. Krinsky, Phys. Rev. B 13, 5078 (1976).
[4] P. Bak and D. Mukamel, Phys. Rev. B 13, 5086 (1976).
[5] S. A. Brazovskii, I. E. Dzyaloshinskii, and B. G. Kukharenko, Zh. Eksp. Teor. Fiz. 70, 2257 (1976) [Sov. Phys. JETP 43, 1178 (1976)].
[6] E. J. Blagoeva et al., Phys. Rev. B 42, 6124 (1990).
[7] S. A. Antonenko and A. I. Sokolov, Phys. Rev. B 49, 15901 (1994).
[8] S. A. Antonenko, A. I. Sokolov, and K. B. Varnashev, Phys. Lett. A 208, 161 (1995).

[9] S. A. Antonenko, A. I. Sokolov, and K. B. Varnashev, Mol. Mat. 8, 175 (1996).

[10] J.-C. Toledano et al., Phys. Rev. B 31, 7171 (1985).

[11] G. Grinstein and D. Mukamel, J. Phys. A: Math. Gen. 15, 233 (1982).

[12] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).

[13] D. Mukamel and S. Krinsky, Phys. Rev. B 13, 5065 (1976)

[14] K. B. Varnashev and A. I. Sokolov, Fizika Tverdogo Tela (St.Petersburg) 38, 3665 (1996) [Phys. Sol. State 38, 1996 (1996)].

[15] H. Kawamura, Phys. Rev. B 38, 960 (1988); Phys. Rev. B 38, 4916 (1988).

[16] A. I. Sokolov, K. B. Varnashev, and A. I. Mudrov, Critical exponents for the model with unique stable fixed point from three–loop RG expansions, report at the International Conference ”Renormalization Group’96” (Dubna, Russia, August 1996), to appear in the Conference Proceedings (World Scientific, Singapore).

[17] D. K. De’Bell and D. J. W. Geldart, Phys. Rev. B 32, 4763 (1985).

[18] N. A. Shpot, Phys. Lett. A 133, 125 (1988); Phys. Lett. A 142, 474 (1989).

[19] K. A. Cowley and A. D. Bruce, J. Phys. C 11, 3577 (1978).

[20] J. Sak, Phys. Rev. B 10, 3957 (1974).

[21] L. S. Goldner and G. Ahlers, Phys. Rev. B 45, 13129 (1992).

[22] J. A. Lipa et. al., Phys. Rev. Lett. 76, 944 (1996).

[23] G. A. Baker, B. G. Nickel, and D. I. Meiron, Phys. Rev. B 17, 1365 (1978).

[24] J. C. Le Guillou and J. Zinn–Justin, Phys. Rev. Lett. 39, 95 (1977).

[25] D. I. Kazakov, O. V. Tarasov, and A. A. Vladimirov, Preprint JINR E2–12249 Dubna (1979).

[26] G. ’t Hooft and M. Veltman, Nucl. Phys. B 44, 189 (1972).

[27] G. ’t Hooft, Nucl. Phys. B 61, 455 (1973).
[28] A. A. Vladimirov, D. I. Kazakov, and O. V. Tarasov, Zh. Eksp. Teor. Fiz. 77, 1035 (1979) [Sov. Phys. JETP 50, 521 (1979)].

[29] A. L. Korzhenevskii, Zh. Eksp. Teor. Fiz. 71, 1434 (1976) [Sov. Phys. JETP 44, 751 (1976)].

[30] E. Brezin, Le J. C. Guillou, and J. Zinn–Justin, Phase Transitions and Critical Phenomena Vol. 6, edited by C.Domb and M.S.Green, Academic Press, New York, 1976.

[31] L. N. Lipatov, Zh. Eksp. Teor. Fiz. 72, 411 (1977) [Sov. Phys. JETP 45, 216 (1977)].

[32] E. Brezin, Le J. C. Guillou, and J. Zinn–Justin, Phys.Rev. D 15, 1544 (1977).

[33] J. C. Le Guillou and J. Zinn–Justin, Phys. Rev. B 21, 3976 (1980).

[34] J. C. Le Guillou and J. Zinn–Justin, J. Phys. (Paris) Lett. 46, L137 (1985).

[35] H. Kleinert and S. Thoms, Phys.Rev. D 52, 5926 (1995).

[36] B. V. Shabat, Introduction to the complex analysis (Nauka, Moscow, 1985).