The forcing monophonic and the forcing geodetic numbers of a graph

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Abstract

For a connected graph $G = (V, E)$, let a set $S$ be a $m$-set of $G$. A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique $m$-set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing monophonic number of $S$, denoted by $f_m(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing monophonic number of $G$, denoted by $f_m(G)$, is $f_m(G) = \min\{f_m(S)\}$, where the minimum is taken over all minimum monophonic sets in $G$. We know that $m(G) \leq g(G)$, where $m(G)$ and $g(G)$ are monophonic number and geodetic number of a connected graph $G$ respectively. However there is no relationship between $f_m(G)$ and $f_g(G)$, where $f_g(G)$ is the forcing geodetic number of a connected graph $G$. We give a series of realization results for various possibilities of these four parameters.

Keywords: geodetic number, monophonic number, forcing geodetic number, forcing monophonic number
Mathematics Subject Classification : 05C12, 05C38
DOI: 10.19184/ijc.2020.4.2.5

1. Introduction

By a graph $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology, we refer to Harary [1]. The distance $d(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of shortest $u-v$ path in $G$. An $u-v$ path of length $d(u,v)$ is called an $u-v$...
geodesic. A vertex \( x \) is said to be lie a \( u - v \) geodesic \( P \) if \( x \) is a vertex of \( P \) including the vertices \( u \) and \( v \). A geodesic set of \( G \) is a set \( S \subseteq V \) such that every vertex of \( G \) is contained in geodesic joining some pair of vertices in \( S \). The geodetic number \( g(G) \) of \( G \) is the minimum order of its geodetic sets and any geodetic set of order \( g(G) \) is a minimum geodetic set or simply a \( g \)-set of \( G \). The geodetic number of a graph was introduced in [1] and further studied in [3, 4, 5, 7, 8, 9, 16, 17, 18, 20, 23, 25]. A subset \( T \subseteq S \) is called a forcing subset for \( S \) if \( S \) is the unique \( g \)-set of \( G \) containing \( T \). A forcing subset for \( S \) of minimum cardinality is a minimum forcing subset of \( S \). The forcing geodetic number of \( S \), denoted by \( f_g(S) \), is the cardinality of a minimum forcing subset of \( S \). The forcing geodetic number of \( G \), denoted by \( f_g(G) \), is \( f_g(G) = \min \{ f_g(S) \} \), where the minimum is taken over all minimum \( g \)-sets of \( G \). The forcing geodetic number of a graph was introduced in [3] and further studied in [19, 21, 22]. A chord of the path \( P \) is an edge joining to non-adjacent vertices of \( P \). An \( u - v \) path \( P \) is called monophonic path if it is a chordless path. A monophonic set of \( G \) is a set \( S \subseteq V \) such that every vertex of \( G \) is contained in a monophonic path joining some pair of vertices in \( M \). The monophonic number \( m(G) \) of \( G \) is the minimum order of its monophonic sets and any monophonic set of order \( m(G) \) is a minimum monophonic set or simply a \( m \)-set of \( G \). The monophonic number of a graph was introduced in [6] and further studied in [2, 6, 10, 11, 12, 13, 14, 15, 19, 24]. A vertex \( v \) is said to be monophonic vertex of \( G \) if \( v \) belongs to every minimum monophonic set of \( G \). A vertex \( v \) is an extreme vertex of a graph \( G \) if the sub graph induced by its neighbours is complete. A vertex \( v \) is said to be geodetic(monophonic) vertex if \( v \) belongs to every \( g \)-set (\( m \)-set) of \( G \). Every extreme vertices are geodetic(monophonic) vertices of \( G \). In fact there are monophonic (monophonic) vertices which are not extreme vertices of \( G \). Let \( G \) be a connected graph and \( S \) a \( m \)-set of \( G \). A subset \( T \subseteq S \) is called a forcing subset for \( S \) if \( S \) is the unique \( m \)-set of \( G \) containing \( T \). A forcing subset for \( S \) of minimum cardinality is a minimum forcing subset of \( S \). The forcing monophonic number of \( S \), denoted by \( f_m(S) \), is the cardinality of a minimum forcing subset of \( S \). The forcing monophonic number of \( G \), denoted by \( f_m(G) \) is defined by \( f_m(G) = \min \{ f_m(S) \} \), where the minimum is taken over all \( m \)-sets \( S \) in \( G \). The forcing monophonic number of a graph was introduced in [11]. The Throughout the following \( G \) denotes a connected graph with at least two vertices. The following theorems are used in the sequel.

**Theorem 1.1.** [4, 12] If \( v \) is an extreme vertex of a connected graph \( G \), then \( v \) belongs to every geodetic (monophonic) set of \( G \).

**Theorem 1.2.** [1, 12] For a connected graph \( G \), \( g(G) = p \ (m(G) = p) \) if and only if \( G = K_p \).

**Theorem 1.3.** [3, 11] Let \( G \) be a connected graph, then
\[ a) \quad f_g(G) = 0 = f_m(G) = 0 \] if and only if \( G \) has a unique minimum geodetic (monophonic) set.
\[ b) \quad f_g(G) = g(G) - |W|, \ (f_m(G) = m(G) - |W|) \], where \( W \) is the set of all geodetic (monophonic) vertices of \( G \).

**Theorem 1.4.** [3, 11] For the complete graph \( G = K_p \), \( f_g(G) = f_m(G) = 0 \).

2. The Forcing Monophonic and the Forcing Geodetic Numbers of a Graph

We know that \( m(G) \leq g(G) \). From the following examples, we observe that there is no relationship between \( f_m(G) \) and \( f_g(G) \).
Example 2.1. For the graph $G$ given in Figure 2.1, $M = \{v_1, v_3\}$ is the unique $m$-set of $G$ so that $f_m(G) = 0$ and $m(G) = 2$. Also $S_1 = \{v_1, v_5, v_6\}$ and $S_2 = \{v_1, v_5, v_7\}$ are the only two $g$-sets of $G$ such that $f_g(S_1) = f_g(S_2) = 1$ so that $f_g(G) = 1$ and $g(G) = 3$. Thus $f_m(G) < f_g(G) < m(G) < g(G)$.

Example 2.2. For the graph $G$ given in Figure 2.2, $M_1 = \{v_1, v_8, v_{12}\}$, $M_2 = \{v_1, v_9, v_{12}\}$ and $M_3 = \{v_1, v_{10}, v_{12}\}$ are the only three $m$-set of $G$ so that $f_m(M_1) = f_m(M_2) = f_m(M_3) = 1$ so that $f_m(G) = 1$ and $m(G) = 3$. Also $S_1 = \{v_1, v_7, v_9, v_{12}\}$ is the unique $g$-set of $G$ so that $f_g(G) = 0$ and $g(G) = 4$. Thus $f_g(G) < f_m(G) < m(G) < g(G)$.

3. Special graphs

In this section, we present some graphs from which various graphs arising in theorem are generated using identification.

Let $P_i : u_i, v_i$ be a copy of paths on two vertices. Let $G_a$ be the graph given in Figure 3.1 obtained from $P_i$ ($i \leq a$) by introducing new vertices $s, t$ and joining each $u_i$ ($1 \leq i \leq a$) with $s$ and joining each $v_i$ ($1 \leq i \leq a$) with $t$ and join $s$ with $t$. 
Let $P_i : n_i, p_i$ ($1 \leq i \leq b$) be a copy of path on two vertices and $P : l, m, n$ be a path on three vertices. Let $Z_b$ be the graph given in Figure 3.2 obtained from $P_i$ ($1 \leq i \leq b$) and $P$ by joining each $n_i$ ($1 \leq i \leq b$) with $l$, each $p_i$ ($1 \leq i \leq b$) with $q$.

Let $P_i : r_i, h_i, k_i$ ($1 \leq i \leq c$) be a copy of path on three vertices and let $P : e, f, g$ be a path on three vertices. Let $H_c$ be the graph given in Figure 3.3 obtained from $P_i$ ($1 \leq i \leq c$) and $P$ by joining $e$ and $f$ with each $h_i$ and $r_i$ ($1 \leq i \leq c$), joining $g$ with each $k_i$ ($1 \leq i \leq c$), joining $h_i$ ($1 \leq i \leq c$) with $k_i$ ($1 \leq i \leq c$), and joining $r_i$ ($1 \leq i \leq c$) with $k_i$ ($1 \leq i \leq c$).
4. Some realization results

**Theorem 4.1.** For every pair $a, b$ of integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph $G$ such that $f_m(G) = f_g(G) = 0$, $m(G) = a$ and $g(G) = b$.

*Proof.* If $a = b$, let $G = K_a$. Then by Theorem 1.2, $m(G) = g(G) = a$. Also by Theorem 1.3(a), $f_m(G) = f_g(G) = 0$. For $1 \leq a < b$, let $G$ be the graph obtained from $H_{b-a}$ by adding new
vertices \( x, z_1, z_2, \ldots, z_{a-1} \) and joining the edges \( xe, g_{z_1}, g_{z_2}, \ldots, g_{z_{a-1}} \). Let \( Z = \{ x, z_1, z_2, \ldots, z_{a-1} \} \) be the set of all end-vertices of \( G \). Then it is clear that \( Z \) is a monophonic set of \( G \) and so by Theorem 1.1, \( Z \) is the unique \( m \)-set of \( G \) so that \( m(G) = a \) and hence by Theorem 1.3(a), \( f_m(G) = 0 \). Since the vertices \( h_i, k_i \) and \( r_i \) \( (1 \leq i \leq b - a) \) does not lie on any geodesic joining a pair of vertices in \( Z \), we see that \( Z \) is not a geodetic set of \( G \). It is easily verified that every \( g \)-set of \( G \) contains each \( h_i \) \( (1 \leq i \leq b - a) \) and so \( g(G) \geq b \). Now it is easily seen that \( W = Z \cup \{ h_1, h_2, \ldots, h_{b-a} \} \) is the unique \( g \)-set of \( G \) and hence by Theorem 1.1 and Theorem 1.3(a) \( g(G) = b \) and \( f_g(G) = 0 \).

\[ \square \]

Theorem 4.2. For every integers \( a, b \) and \( c \) with \( 0 \leq a < b < c \) and \( c > a + b \), there exists a connected graph \( G \) such that \( f_m(G) = 0, f_g(G) = a, m(G) = b \) and \( g(G) = c \).

Proof. Case 1. \( a = 0 \). Then the graph \( G \) constructed in Theorem 4.1 satisfies the requirements of this theorem.

Case 2. \( a \geq 1 \). Let \( G \) be the graph obtained from \( Z_a \) and \( H_{c-(a+b)} \) by identifying the vertex \( q \) of \( Z_a \) and \( e \) of \( H_{c-(a+b)} \) and then adding new vertices \( x, z_1, z_2, \ldots, z_{b-1} \) and joining the edges \( xl, g_{z_1}, g_{z_2}, \ldots, g_{z_{b-1}} \). It is clear that \( Z \) is a monophonic set of \( G \) and by Theorem 1.1, \( Z \) is the unique \( m \)-set of \( G \) so that \( m(G) = b \) and hence by Theorem 1.3(a), \( f_m(G) = 0 \). Next we show that \( g(G) = c \). Let \( S \) be any geodetic set of \( G \). Then by Theorem 1.1, \( Z \subseteq S \). It is clear that \( Z \) is not a geodetic set of \( G \). For \( 1 \leq i \leq a \), let \( Q_i = \{ n_i, p_i \} \). We observed that every \( g \)-set of \( G \) must contain at least one vertex from each \( Q_i \) \( (1 \leq i \leq a) \) and each \( h_i \) \( (1 \leq i \leq c - b - a) \) so that \( g(G) \geq b + a + c - a - b = c \). Now \( W = Z \cup \{ h_1, h_2, \ldots, h_{c-a-b} \} \cup \{ n_1, n_2, \ldots, n_a \} \) is a geodetic set of \( G \) so that \( g(G) \geq b + a + c - a - b = c \). Thus \( g(G) = c \). Since every \( g \)-set contains \( W_1 = Z \cup \{ h_1, h_2, \ldots, h_{c-a-b} \} \) it follows from that from Theorem 1.3 (b) that \( f_g(G) \leq g(G) - | W_1 | = c - (c-a) = a \). Now, since \( g(G) = c \) and every \( g \)-set of \( G \) contains \( W_1 \), it is easily seen that every \( g \)-set \( S \) is of the form \( W_1 \cup \{ d_1, d_2, \ldots, d_a \} \) where \( d_i \in Q_i \) \( (1 \leq i \leq a) \). Let \( T \) be any proper subset of \( S \) with \( | T | < a \). Then it is clear that there exists some \( j \) such that \( T \cap Q_j = \emptyset \), which shows that \( f_j(G) = a \).

\[ \square \]

Theorem 4.3. For every integers \( a, b \) and \( c \) with \( 0 \leq a < b \leq c \) and \( b > a + 1 \) there exists a connected graph \( G \) such that \( f_g(G) = 0, f_m(G) = a, m(G) = b \) and \( g(G) = c \).

Proof. Case 1. \( a = 0 \). Then the graph \( G \) constructed in Theorem 4.1 satisfies the requirements of this theorem.

Case 2. \( a \geq 1 \).

Subcase 2a. \( b = c \). Let \( G \) be the graph obtained from \( R_a \) by adding new vertices \( x, z_1, z_2, \ldots, z_{b-a-1} \) and joining the edges \( xu, v_{z_1}, v_{z_2}, \ldots, v_{z_{b-a-1}} \). Let \( Z = \{ x, z_1, z_2, \ldots, z_{b-a-1} \} \) be the set of all end-vertices of \( G \). Let \( S \) be any geodetic set of \( G \). Then by Theorem 1.1, \( Z \subseteq S \). It is clear that \( Z \) is not a geodetic set of \( G \). For \( 1 \leq i \leq a \), let \( H_i = \{ x_i, w_i \} \). We observe that every \( g \)-set of \( G \) must contain only the vertex \( y_i \) from each \( H_i \) \( (1 \leq i \leq a) \) and so \( g(G) \geq b - a + a = b \). Now \( S = Z \cup \{ y_1, y_2, y_3, \ldots, y_a \} \) is a geodetic set of \( G \) so that \( g(G) \leq b - a + a = b \). Thus \( g(G) = b \). Also it is easily seen that \( W \) is the unique \( g \)-set of \( G \) and so \( f_g(G) = 0 \). Now it is clear that \( Z \) is not a monophonic set of \( G \). We observe that every \( m \)-set of \( G \) must contain at least one vertex from each \( H_i \) \( (1 \leq i \leq a) \). Hence by Theorem 1.1, \( m(G) \geq b - a + a = b \). Now
$W_1 = Z \cup \{y_1, y_2, y_3, \ldots, y_a\}$ is a monophonic set of $G$ so that $m(G) \leq b - a + a = b$. Thus $m(G) = b$. Next we show that $f_m(G) = a$. Since every $m$-set contains $Z$, it follows from Theorem 1.3 (b) that $f_m(G) \leq m(G) - |Z| = b - (b - a) = a$. Now, since $m(G) = b$ and every $m$-set of $G$ contains $Z$, it is easily seen that every $m$-set $S$ is of the form $Z \cup \{d_1, d_2, d_3, \ldots, d_a\}$, where $d_i \in H_i (1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then it is clear that there exists some $j$ such that $T \cap H_j = \emptyset$, which shows that $f_m(G) = a$.

**Subcase 2b.** $b < c$. Let $G$ be the graph obtained from $R_a$ and $H_{c-b}$ by identifying the vertex $v$ of $R_a$ and $g$ of $H_{c-b}$ and then adding the new vertices $x, z_1, z_2, \ldots, z_{b-a-1}$ and joining the edges $xu, gz_1, gz_2, \ldots, g(z_{b-a-1})$. Let $Z = \{x, z_1, z_2, \ldots, z_{b-a-1}\}$ be the set of end vertices of $G$. Let $S$ be any geodetic set of $G$. Then by Theorem 1.1 $Z \subseteq S$. It is clear that $Z$ is not a geodetic set of $G$. For $1 \leq i \leq a$, let $H_i = \{x_i, y_i, w_i\}$. We observe that every $g$-set of $G$ must contain only the vertex $y_i$ $(1 \leq i \leq a)$ from each $H_i (1 \leq i \leq a)$ and each $h_i$ $(1 \leq i \leq c - b)$ and so $g(G) \geq b - a + a + c - b = c$. Now $W = Z \cup \{y_1, y_2, y_3, \ldots, y_a\} \cup \{h_1, h_2, h_3, \ldots, h_{c-b}\}$ is a geodetic set of $G$ so that $g(G) \leq b - a + a + c - b = c$. Thus $g(G) = c$. Also it is easily seen that $W$ is the unique $g$-set of $G$ and so $f_g(G) = 0$. It is clear that $Z$ is not a monophonic set of $G$. We observe that every $m$-set of $G$ must contain at least one vertex from each $H_i (1 \leq i \leq a)$ and so $m(G) \geq b - a + a = b$. Now, $S_1 = Z \cup \{y_1, y_2, y_3, \ldots, y_a\}$ is a monophonic set of $G$ so that $m(G) \leq b - a + a = b$. Thus $m(G) = b$. Next we show that $f_m(G) = a$. Since every $m$-set contains $Z$, it follows from Theorem 1.3 (b) that $f_m(G) \leq m(G) - |Z| = b - (b - a) = a$. Now, since $m(G) = b$ and every $m$-set of $G$ contains $Z$, it is easily seen that every $m$-set $S$ is of the form $Z \cup \{d_1, d_2, d_3, \ldots, d_a\}$, where $d_i \in H_i (1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then it is clear that there exists some $j$ such that $T \cap H_j = \emptyset$, which shows that $f_m(G) = a$.

**Theorem 4.4.** For every pair $a, b$ and $c$ of integers with $0 \leq a \leq b \leq c$, $b > a + 1$ there exists a connected graph $G$ such that $f_g(G) = f_m(G) = a$, $m(G) = b$ and $g(G) = c$.

**Proof.** Case 1. $a = 0$, then the graph $G$ constructed in Theorem 4.1 satisfies the requirements of this theorem.

Case 2. $a \geq 1$,

**Subcase 2a.** $b = c$. Let $G$ be the graph obtained from $G_a$ by adding new vertices $x, z_1, z_2, \ldots, z_{b-a-1}$ and joining the edges $xs, tz_1, tz_2, \ldots, tz_{b-a-1}$. Let $Z = \{x, z_1, z_2, \ldots, z_{b-a-1}\}$ be the set of end-vertices of $G$. First we show that $m(G) = b$. Let $M$ be any monophonic set of $G$. Then by Theorem 1.1, $Z \subseteq M$. It is clear that $Z$ is not a monophonic set of $G$. Let $F_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$. We observe that every $m$-set of $G$ must contain at least one vertex from each $F_i (1 \leq i \leq a)$. Thus $m(G) \geq b - a + a = b$. On the other hand since the set $W = Z \cup \{v_1, v_2, \ldots, v_a\}$ is a monophonic set of $G$, it follows that $m(G) \leq |W| = b$. Hence $m(G) = b$. Next we show that $f_m(G) = a$. By Theorem 1.1, every monophonic set of $G$ contains $Z$ and so it follows from Theorem 1.3(b) that $f_m(G) \leq m(G) - |Z| = a$. Now, since $m(G) = b$ and every $m$-set of $G$ contains $Z$, it is easily seen that every $m$-set $S$ is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in F_i (1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then it is clear that there exists some $j$ such that $T \cap H_j = \emptyset$, which shows that $f_m(G) = a$. By similar way we can prove $g(G) = b$ and $f_g(G) = a$.

**Subcase 2b.** $b < c$. Let $G$ be the graph obtained from $G_a$ and $H_{c-b}$ by identifying the vertex $t$ of
First we show that \( m(G) = b \). Let \( Z = \{ z_1, z_2, \ldots, z_{b-a-1} \} \) be the set of all end-vertices of \( G \). Since the vertices \( u_i, v_i \) do not lie on any monophonic path joining a pair of vertices of \( Z \), it is clear that \( Z \) is not a monophonic set of \( G \). Let \( F_i = \{ u_i, v_i \} \) (\( 1 \leq i \leq a \)). We observe that every \( m \)-set of \( G \) must contain at least one vertex from each \( F_i \) (\( 1 \leq i \leq a \)). Thus \( m(G) \geq b - a + a = b \). On the other hand since the set \( W = Z \cup \{ v_1, v_2, v_3, \ldots, v_a \} \) is a monophonic set of \( G \), it follows that \( m(G) \leq |W| = b \). Hence \( m(G) = b \). Next, we show that \( f_m(G) = a \).

By Theorem 1.1, every monophonic set of \( G \) contains \( Z \) and so it follows from Theorem 1.3(b) that \( f_m(G) \leq m(G) - |Z| = a \). Now, since \( m(G) = b \) and every \( m \)-set of \( G \) contains \( Z \), it is easily seen that every \( m \)-set \( S \) is of the form \( Z \cup \{ c_1, c_2, c_3, \ldots, c_a \} \), where \( c_i \in F_i \) (\( 1 \leq i \leq a \)). Let \( T \) be any proper subset of \( S \) with \( |T| < a \). Then it is clear that there exists some \( j \) such that \( T \cap H_j = \emptyset \), which shows that \( f_m(G) = a \). Next we show that \( g(G) = c \). Since the vertices \( u_i, v_i, h_i \) (\( 1 \leq i \leq a \)) do not lie on any geodesic joining a pair of vertices of \( Z \), it is clear that \( Z \) is not a geodetic set of \( G \). We observe that every \( g \)-set of \( G \) must contain each \( H_i \) (\( 1 \leq i \leq a \)) and each \( h_i \) (\( 1 \leq i \leq c - b \)) so that \( g(G) \geq b - a + a + c - b = c \). On the other hand, since the set \( S_1 = Z \cup \{ h_1, h_2, h_3, \ldots, h_{c-b} \} \cup \{ u_1, u_2, \ldots, u_a \} \) is a geodetic set of \( G \), so that \( g(G) \leq |S_1| = c \). Hence \( g(G) = c \). Next we show that \( f_g(G) = a \). By Theorem 1.1, every geodetic set of \( G \) contains \( W_1 = Z \cup \{ h_1, h_2, h_3, \ldots, h_{c-b} \} \) and so it follows from Theorem 1.3(b) that \( f_g(G) \leq g(G) - |W_1| = a \). Now, since \( g(G) = c \) and every \( g \)-set of \( G \) contains \( Z \), it is easily seen that every \( g \)-set \( S \) is of the form \( W_1 \cup \{ c_1, c_2, c_3, \ldots, c_a \} \), where \( c_i \in F_i \) (\( 1 \leq i \leq a \)). Let \( T \) be any proper subset of \( S \) with \( |T| < a \). Then it is clear that there exists some \( j \) such that \( T \cap H_j = \emptyset \), which shows that \( f_g(G) = a \). This is true for all \( g \)-sets of \( G \) so that \( f_g(G) = a \). □

**Theorem 4.5.** For every integers \( a, b, c \) and \( d \) with \( 2 \leq c < d \), \( 0 \leq a \leq b \leq d \) and \( d > c - a + b \), there exists a connected graph \( G \) such that \( f_m(G) = a \), \( f_g(G) = b \), \( m(G) = c \) and \( g(G) = d \).

**Proof.**

**Case 1.** \( a = b = 0 \). Then the graph \( G \) constructed in Theorem 4.1 satisfies the requirements of this theorem.

**Case 2.** \( a = 0, b > 1 \). Then the graph \( G \) constructed in Theorem 4.2 satisfies the requirements of this theorem.

**Case 3.** \( 1 \leq a = b \). Then the graph \( G \) constructed in Theorem 4.4 satisfies the requirements of this theorem.

**Case 4.** \( 1 \leq a < b \). Let \( G_1 \) be the graph obtained from \( G_a \) and \( Z_{b-a} \) by identifying the vertex \( t \) of \( G_a \) and the vertex \( l \) of \( Z_{b-a} \). Now let \( G \) be the graph obtained from \( G_1 \) and \( H_{d-(c-a+b)} \) by identifying the vertex \( q \) of \( G_1 \) and the vertex \( e \) of \( H_{d-(c-a+b)} \) and adding new vertices \( x, z_1, z_2, \ldots, z_{c-a-1} \) and joining the edges \( xs, gz_1, gz_2, \ldots, gz_{c-a-1} \). Let \( Z = \{ x, z_1, z_2, \ldots, z_{c-a-1} \} \) be the set of end vertices of \( G \). For \( 1 \leq i \leq a \) let \( F_i = \{ u_i, v_i \} \). It is clear that any \( m \)-set is of the form \( S = Z \cup \{ c_1, c_2, c_3, \ldots, c_a \} \) where \( c_i \in F_i \) (\( 1 \leq i \leq a \)). Then as in earlier theorems it can be seen that \( f_m(G) = a \) and \( m(G) = c \). For \( 1 \leq i \leq a \) let \( Q_i = \{ n_i, p_i \} \). It is clear that any \( g \)-set is of the form \( W = Z \cup \{ h_1, h_2, h_3, \ldots, h_{d-(c-a+b)} \} \cup \{ c_1, c_2, c_3, \ldots, c_a \} \cup \{ d_1, d_2, d_3, \ldots, d_{b-a} \} \), where \( c_i \in F_i \) (\( 1 \leq i \leq a \)) and \( d_j \in Q_j \) (\( 1 \leq j \leq b - a \)). Then as in earlier theorems it can be seen that \( f_g(G) = b \) and \( g(G) = d \). □
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**Theorem 4.6.** For every integers \(a, b, c\) and \(d\) with \(0 \leq a \leq b < c \leq d\) and \(c \geq b + 1\) and \(c, d \geq 2\) there exists a connected graph \(G\) such that \(f_g(G) = a, f_m(G) = b, m(G) = c\) and \(g(G) = d\).

**Proof.**

**Case 1.** \(a = b = 0\). Then the graph \(G\) constructed in Theorem 4.1 satisfies the requirements of this theorem.

**Case 2.** \(a = 0, b \geq 1\). Then the graph \(G\) constructed in Theorem 4.2 satisfies the requirements of this theorem.

**Case 3.** \(1 \leq a = b\). Then the graph \(G\) constructed in Theorem 4.4 satisfies the requirements of this theorem.

**Case 4.** \(1 \leq a < b\).

**Subcase 4a.** \(c = d\). Let \(G\) be the graph obtained from \(G_a\) and \(R_{b-a}\) by identifying the vertex \(t\) of \(G_a\) and the vertex \(q\) of \(R_{b-a}\) and then adding the new vertices \(x, z_1, z_2, \ldots, z_{c-b-1}\) and joining the edges \(xs, qz_1, qz_2, \ldots, qz_{c-b-1}\). First we show that \(m(G) = c\). Let \(Z = \{x, z_1, z_2, \ldots, z_{c-b-1}\}\) be the set of end vertices of \(G\). Let \(F_i = \{u_i, v_i\} (1 \leq i \leq a)\) and \(H_i = \{x_i, y_i, w_i\} (1 \leq i \leq b - a)\). It is clear that any \(m\)-set of \(G\) is of the form \(S = Z \cup \{c_1, c_2, c_3, \ldots, c_a\} \cup \{d_1, d_2, d_3, \ldots, d_{b-a}\}\) where \(c_i \in F_i (1 \leq i \leq a)\) and \(d_j \in H_j (1 \leq j \leq b - a)\). Then as in earlier theorems it can be seen that \(f_g(G) = b\) and \(m(G) = c\). It is clear that any \(g\)-set is of the form \(W = Z \cup \{y_1, y_2, y_3, \ldots, y_{b-a}\} \cup \{c_1, c_2, c_3, \ldots, c_a\}\), where \(c_i \in F_i (1 \leq i \leq a)\). Then as in earlier theorems it can be seen that \(f_g(G) = a\) and \(g(G) = d\).

In the realization results we have given some restrictions on the parameters. So we leave the following as open question.

**Problem 1.** For any four positive integers \(a, b, c\) and \(d\) with \(a \geq 0, b \geq 0\) and \(2 \leq c \leq d\), does there exist a connected graph \(G\) with \(f_m(G) = a, f_g(G) = b, m(G) = c\) and \(g(G) = d\).

5. The Upper Forcing Monophonic number of a graph

In [25], P. Zhang introduced the concept of the upper geodetic number of a graph. In the similar manner we define the upper forcing monophonic number of a graph as follows.

**Definition 5.1.** Let \(G\) be a connected graph and \(S\) a \(m\)-set of \(G\). A subset \(T \subseteq S\) is called a forcing subset for \(S\) if \(S\) is the unique \(m\)-set containing \(T\). A forcing subset for \(S\) of minimum cardinality is a minimum forcing subset of \(S\). The **forcing monophonic number** of \(S\), denoted by \(f_m(S)\), is
the cardinality of a minimum forcing subset of $S$. The forcing monophonic number of $G$, denoted by $f_m(G)$ is defined by $f_m(G) = \min \{ f_m(S) \}$, where the minimum is taken over all $m$-set $S$ in $G$ and the upper forcing monophonic number of $G$, denoted by $f_m^+(G) = \max \{ f_m(S) \}$, where the maximum is taken over all $m$-sets $S$ in $G$.

**Theorem 5.2.** For every connected graph $G$, $0 \leq f_m(G) \leq f_m^+(G) \leq m(G)$.

**Example 5.3.** The bounds in Theorem 5.2 is sharp. For $G = K_{1,p-1}$, $f_m(G) = 0$. For $G = C_5$, $f_m(G) = f_m^+(G) = 2$. Also the inequalities in Theorem 5.2 can be strict. For the graph $G$ given in Figure 5.1, $M_1 = \{v_1, v_4, v_5\}$, $M_2 = \{v_1, v_4, v_6\}$ and $M_1 = \{v_1, v_3, v_6\}$ are only three $m$-sets of $G$ so that $f_m(M_1) = 2$, $f_m(M_2) = 1$ and $f_m(M_3) = 2$ so that $f_m(G) = 2$, $f_m^+(G) = 2$ and $m(G) = 3$. Thus $0 < f_m(G) < f_m^+(G) < m(G)$.

So we leave the following as a open question.

**Problem 2.** For any three positive integers $a, b$ and $c$ with $0 \leq a \leq b \leq c$, does there exists a connected graph $G$ with $f_m(G) = a$, $f_m^+(G) = b$ and $m(G) = c$.

**References**

[1] H.A. Ahangara, S. Kosarib, S.M. Sheikholeslamib, and L. Volkmannnc, Graphs with large geodetic number, *Filomat*. 29:6 (2015), 1361 – 1368.

[2] F. Buckley and F. Harary, Distance in Graphs, Addison-Wesley, Redwood City, CA, 1990.

[3] J. Caceres, O. Oellermann, and M. Puertas, Minimal trees and monophonic convexity, *Discuss. Math. Graph Theory*, 32, (2012), 685 – 704.

[4] G. Chartrand and P. Zhang, The forcing geodetic number of a graph, *Discuss. Math. Graph Theory*, 19, (1999), 45 – 58.
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[5] G. Chartrand, F. Harary, and P. Zhang, On the geodetic number of a graph, *Networks*, (2002), 1 – 6.

[6] M.C. Dourado, F. Protti, D. Rautenbach, and J.L. Szwarcfiter, Some remarks on the geodetic number of a graph, *Discrete Math.*, 310, (2010), 832 – 837.

[7] M.C. Dourado, F. Protti, and J.L. Szwarcfiter, Algorithmic aspects of monophonic convexity, *Electron. Notes Discrete Math.*, 30, (2008), 177 – 182.

[8] F. Harary, E. Loukakis, and C. Tsouros, The geodetic number of a graph, *Math. Comput. Modeling*, 17(11), (1993), 89 – 95.

[9] C. Hernando, T. Jiang, M. Mora, I.M. Pelayo, and C. Seara, On the Steiner, geodetic and hull number of graphs, *Discrete Math.* 293, (2005), 139 – 154.

[10] J. John and P.A.P. Sudhahar, The forcing edge monophonic number of a graph , *SCIENTIA Series A: Mathematical Sciences*, 23, (2012), 87-98

[11] J. John and S. Panchali, The forcing monophonic number of a graph, *IJMA*-3 (3), (2012), 935 – 938.

[12] J. John and S. Panchali, The upper monophonic number of a graph, *Int. J. Math. Combin.* 4, (2010), 46 – 52

[13] J. John and K.U. Samundeswari, The forcing edge fixing edge-to-vertex monophonic number of a graph, *Discrete Math. Algorithms Appl.* 5(4), (2013), 1 – 10.

[14] J. John and K.U. Samundeswari, The edge fixing edge-to-vertex monophonic number of a graph, *Appl. Math. E-Notes*, 15, (2015), 261 – 275.

[15] J. John and K.U. Samundeswari, Total and forcing total edge-to-vertex monophonic numbers of graph, *J. Comb. Optim.* 34, (2017), 1 – 14.

[16] J. John and D. Stalin, Edge geodetic self decomposition in graphs, *Discrete Math. Algorithms Appl.* 12(5), (2020), 2050064, 7 pages.

[17] J. John and D. Stalin,The edge geodetic self decomposition number of a graph, *RAIRO Oper. Res.*, DOI:10.1051/ro/2020073.

[18] J. John and D. Stalin, Distinct edge geodetic decomposition in graphs, *Commun. Comb. Optim.*, DOI: 10.22049/CCO.2020.26638.1126

[19] E.M. Paluga and S.R. Canoy, Jr, Monophonic numbers of the join and composition of connected graphs, *Discrete Math.* 307, (2007), 1146 – 1154.

[20] I.M. Pelayo, Geodesic Convexity in Graphs, Springer Briefs in Mathematics, 2013.

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[21] A.P. Santhakumaran and J. John, On the forcing geodetic and forcing Steiner numbers of a graph, *Discuss. Math. Graph Theory*, 31, (2011), 611 – 624.

[22] Li-Da Tong, Geodetic sets and Steiner sets in graphs, *Discrete Math.* 309(12), (2009), 3733 – 4214.

[23] Li-Da Tong, The forcing hull and forcing geodetic numbers of graphs, *Discrete Appl. Math.* 157(5), (2009), 875 – 1164.

[24] Li-Da Tong, The \((a, b)\)-forcing geodetic graphs, *Discrete Math.* 309(6), (2009), 1199 – 1792.

[25] P. Zhang, The upper forcing geodetic number of a graph, *Ars Combin.*, DOI: 10.7151/dmgt.1084.