JACOB’S LADDERS AND NEW FAMILIES OF $\zeta$-KINDRED REAL CONTINUOUS FUNCTIONS.

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ABSTRACT. In this paper we obtain, by our method of crossbreeding in certain set of $\zeta$-factorization formulas, the corresponding complete hybrid formulas. These are playing the role of criterion for selection of new families of $\zeta$-kindred real continuous functions.

1. INTRODUCTION

1.1. In this paper we use the following notions:
(A) crossbreeding in the set of $\zeta$-factorization formulas;
(B) complete hybrid formula,
(C) definition of $\zeta$-kindred elements in the set of real continuous functions,
that we have introduced in the paper [11].

Let us present a short survey of these notions.

(a) We begin with the set of functions

\[ f_m(t) \in \tilde{C}_0[T, T + U], \quad U = o \left( \frac{T}{\ln T} \right), \quad T \to \infty \]
\[ m = 1, \ldots, M, \quad M \in \mathbb{N}, \]

where $M$ is arbitrary and fixed.

(b) Next we obtain, by application of the operator $\hat{H}$ (introduced in the paper [8], (3.6)), the vector-valued functions

\[ \hat{H}f_m(t) = \left( \alpha_m^{m,k_0}, \ldots, \alpha_m^{m,k_m}, \beta_m^{k_1}, \ldots, \beta_m^{k_{k_0}} \right), \]
\[ m = 1, \ldots, M, \quad 1 \leq k_m \leq k_0, \quad k_0 \in \mathbb{N}, \]

where $k_0$ is arbitrary and fixed. Simultaneously, we obtain by our algorithm (for the short survey of this one see [8], (3.1) – (3.11)), also the following set of $\zeta$-factorization formulas

\[ \prod_{r=1}^{k_m} \left| \frac{\zeta \left( \frac{T}{T_r} + i\alpha_{m,k_m}^{r,m} \right)}{\zeta \left( \frac{T}{T_r} + i\beta_{k_m}^{r,m} \right)} \right|^2 \sim E_m(U, T)F_m\left[ f_m(\alpha_m^{m,k_m}) \right], \]
\[ m = 1, \ldots, M, \quad L \to \infty. \]

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(c) Further, we will suppose that we have obtained the following complete hybrid formula
\[ F \left\{ \prod_{r=1}^{k_1} (\ldots), \prod_{r=1}^{k_2} (\ldots), \ldots, \prod_{r=1}^{k_M} (\ldots), F_1[f_1(\alpha_{0,1}^{k_1})], \ldots, F_M[f_M(\alpha_{0,M}^{M,k_M})] \right\} = 1 + O \left( \frac{\ln \ln T}{\ln T} \right) \sim 1, \ T \to \infty \]
(1.4)

after the finite number of stages of crossbreeding (every member of (1.3) is the participant on this process of crossbreeding) in the set (1.3) - that is:

after the finite number of eliminations of the external functions

(1.5)

E_m(U, T), m = 1, \ldots, M

from the set (1.3),

(d) Now we see that the complete hybrid formula (1.4) expresses the functional dependence of the set of vector-valued functions (1.2). Consequently, the back-projection of this functional dependence of the set (1.2) into the generating set (1.1) leaves us with the following (see [1])

Definition. We will call the subset
\[ \{ f_1(t), \ldots, f_M(t) \}, \ t \in [T, T+U] \]

(1.6)
of the real continuous functions (comp. (1.1)), for which there is the complete hybrid formula (1.4), as the family of \( \zeta \)-kindred functions.

1.2. In this paper we obtain the following new families of \( \zeta \)-kindred real continuous functions:

\[ \left\{ \frac{1}{\cos^2 t} \sin^2 t \cos^2 t \cdot (t - \pi L)^\Delta \right\}, \]
\[ t \in [\pi L, \pi L + U], \ U \in (0, \pi/2 - \epsilon], \Delta > 0, \ L \to \infty, \]
and

\[ \left\{ \frac{1}{\cos^2 t} \cos t, \cos^2 t, \cos^3 t, \sin^2 t, (t - 2\pi L)^\Delta \right\}, \]
\[ t \in [2\pi L, 2\pi L + U], \ U \in (0, \pi/2 - \epsilon], \Delta > 0, \ L \to \infty. \]

Let us remind we have introduced in the papers [1] – [11] new notions in the theory of the Riemann zeta-function based on Jacob’s ladders (see [1]). The present paper contains new results in this direction.

2. THE FIRST CLASS OF LEMMAS

By making use of our algorithm for generating \( \zeta \)-factorization formulas (see [8], (3.1) – (3.11)) we obtain the following results.

Lemma 1. For the function
\[ f_1(t) = \frac{1}{\cos^2 t} \in \mathcal{C}_0[\pi L, \pi L + U], \ U \in (0, \pi/2 - \epsilon] \]
(2.1)

there are the vector-valued functions
\[ (\alpha_{0,1}^{k_1}, \ldots, \alpha_{0,1}^{k_1}, \beta_1^{k_1}, \ldots, \beta_{k_1}^{k_1}), \]
\[ 1 \leq k_1 \leq k_0, \ k_0 \in \mathbb{N} \]
such that the following factorization formula

\[
\prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha^{1,k_1}_r \right)}{\zeta \left( \frac{1}{2} + i\beta^{1,k_1}_r \right)} \right|^2 \sim \frac{\tan U}{U} \cos^2(\alpha^{1,k_1}_0), \quad L \to \infty
\]

(2.3)

holds true, where

\[
\alpha^{1,k_1}_r = \alpha_r(U, L, k_1; f_1), \quad r = 0, 1, \ldots, k_1,
\]

\[
\beta^{1,k_1}_r = \beta_r(U, L, k_1), \quad r = 1, \ldots, k_1,
\]

\[
\pi L < \alpha^{1,k_1}_0 < 2\pi L + U \Rightarrow 0 < \alpha^{1,k_1}_0 - \pi L < U.
\]

Lemma 2. For the function

\[
f_2(t) = \frac{\sin^2 t}{\cos^4 t} \in \tilde{C}_0[\pi L, \pi L + U], \quad U \in (0, \pi/2 - \epsilon]
\]

(2.5)

such that the following factorization formula

\[
\prod_{r=1}^{k_2} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha^{2,k_2}_r \right)}{\zeta \left( \frac{1}{2} + i\beta^{2,k_2}_r \right)} \right|^2 \sim \frac{1}{3} \frac{\tan^3 U \cos^4(\alpha^{2,k_2}_0)}{\sin^2(\alpha^{2,k_2}_0)}, \quad L \to \infty
\]

(2.6)

holds true, where

\[
\alpha^{2,k_2}_r = \alpha_r(U, L, k_2; f_2), \quad r = 0, 1, \ldots, k_2,
\]

\[
\beta^{2,k_2}_r = \beta_r(U, L, k_2), \quad r = 1, \ldots, k_2,
\]

\[
0 < \alpha^{2,k_2}_0 - \pi L < U.
\]

Lemma 3. For the function

\[
f_\Delta(t, L) = f_\Delta(t) = (t - \pi L)^\Delta \in \tilde{C}_0[\pi L, \pi L + U], \quad U \in (0, \pi/2), \quad \Delta > 0
\]

(2.9)

there are vector-valued functions

\[
(\alpha^{\Delta,k_\Delta}_0, \alpha^{\Delta,k_\Delta}_1, \ldots, \alpha^{\Delta,k_\Delta}_{k_\Delta}),
\]

\[
1 \leq k_\Delta \leq k_0, \quad k_0 \in \mathbb{N}
\]

(2.10)

such that the following factorization formula

\[
\prod_{r=1}^{k_\Delta} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha^{\Delta,k_\Delta}_r \right)}{\zeta \left( \frac{1}{2} + i\beta^{\Delta,k_\Delta}_r \right)} \right|^2 \sim \frac{1}{1 + \Delta} \left( \frac{U}{\alpha^{\Delta,k_\Delta}_0 - \pi L} \right)^\Delta, \quad L \to \infty
\]

(2.11)

holds true, where

\[
\alpha^{\Delta,k_\Delta}_r = \alpha_r(U, L, k_\Delta; f_\Delta), \quad r = 0, 1, \ldots, k_\Delta,
\]

\[
\beta^{\Delta,k_\Delta}_r = \beta_r(U, L, k_\Delta), \quad r = 1, \ldots, k_\Delta,
\]

\[
0 < \alpha^{\Delta,k_\Delta}_0 - \pi L < U.
\]

(see \cite{9}, (2.1) - (2.5), \( L \to \pi L \)).
3. Theorem 1

3.1. **The first stage of the crossbreeding.** It is the result of the crossbreeding between the $\zeta$-factorization formula (2.3) and (2.7):

$$\prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r \right)}{\zeta \left( \frac{1}{2} + i\beta_r \right)} \right|^2 \sim$$

$$\sim \frac{3}{U^2} \cos^6(\alpha_0^{1,2}) \sin^2(\alpha_0^{2,2}) \prod_{r=1}^{k_2} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r \right)}{\zeta \left( \frac{1}{2} + i\beta_r \right)} \right|^2, \ L \to \infty.$$  

(3.1)

3.2. **The second stage of the crossbreeding.** Now, the crossbreeding between the formula (3.1) and the formula (see (2.11))

$$U^\Delta \sim (1 + \Delta)(\alpha_0 - \pi L) \prod_{r=1}^{k_\Delta} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r \right)}{\zeta \left( \frac{1}{2} + i\beta_r \right)} \right|^2$$

(3.2)

gives the following

**Complete Hybrid Formula 1.**

$$\prod_{r=1}^{k_\Delta} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r \right)}{\zeta \left( \frac{1}{2} + i\beta_r \right)} \right|^2 \sim$$

$$\sim \frac{3^{\Delta/2}}{1 + \Delta} \left[ \frac{\cos^6(\alpha_0^{1,k_1}) \sin(\alpha_0^{2,k_2})}{(\alpha_0^{1,k_\Delta} - \pi L) \cos^2(\alpha_0^{2,k_2})} \right]^{\Delta} \times$$

$$\times \left\{ \prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r \right)}{\zeta \left( \frac{1}{2} + i\beta_r \right)} \right|^2 \right\} \times \left\{ \prod_{r=1}^{k_2} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r \right)}{\zeta \left( \frac{1}{2} + i\beta_r \right)} \right|^2 \right\}^{\Delta/2},$$

(3.3)

$$1 \leq k_\Delta, k_1, k_2 \leq k_0, \ \Delta > 0, \ L \to \infty.$$

3.3. Now, we obtain from (3.3) by Definition the following

**Theorem 1.** The subset

$$\left\{ \frac{\sin^2 t}{\cos^2 t}, (t - \pi L)^\Delta \right\},$$

(3.4)

$$t \in [\pi L, \pi L + U], \ U \in (0, \pi/2 - \varepsilon], \Delta > 0, \ L \to \infty$$

is the family of $\zeta$-kindred elements in the class of real continuous functions.

4. **The second class of lemmas**

4.1. If we put in (3.4), (4.1) – (4.10)

$$L \to 2L, \ \mu = 0$$

then we obtain the following result.

**Lemma 4.** For the function

$$f_3(t) = \sin^2 t \in \mathcal{C}_0[2\pi L, 2\pi L + U], \ U \in (0, \pi/2)$$

(4.1)
there are vector-valued functions
\[(\alpha^{3,k}_0, \ldots, \alpha^{3,k}_k, \beta^{k_1}_1, \ldots, \beta^{k_3}_k), \ 1 \leq k_3 \leq k_0\]
such that the following factorization formula
\[
\prod_{r=1}^{k_3} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha^{3,k}_r \right)}{\zeta \left( \frac{1}{2} + i\beta^{k_3}_r \right)} \right|^2 \sim \\
\sim \left\{ \frac{1}{2} - \frac{1}{2} \sin \frac{U}{U} \cos U \right\} \frac{1}{\sin^2 \alpha^{3,k}_0}, \ L \to \infty
\]
holds true, where
\[\alpha^{3,k}_r = \alpha_r(U, 2L, k_3; f_3), \ r = 0, 1, \ldots, k_3,\]
\[\beta^{k_3}_r = \beta_r(U, 2L, k_3), \ r = 1, \ldots, k_3,\]
\[0 < \alpha^{3,k}_0 - 2\pi L < U.\]

**Lemma 5.** For the function
\[(4.5) \quad f_4(t) = \cos^2 t \in \tilde{C}_0[2\pi L, 2\pi L + U], \ U \in (0, \pi/2)\]
there are vector-valued functions
\[(4.6) \quad (\alpha^{4,k_4}_0, \ldots, \alpha^{4,k_4}, \beta^{k_4}_1, \ldots, \beta^{k_4}_k), \ 1 \leq k_4 \leq k_0\]
such that the following factorization formula
\[
\prod_{r=1}^{k_4} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha^{4,k_4}_r \right)}{\zeta \left( \frac{1}{2} + i\beta^{k_4}_r \right)} \right|^2 \sim \\
\sim \left\{ \frac{1}{2} + \frac{1}{2} \sin \frac{U}{U} \cos U \right\} \frac{1}{\cos^2 \alpha^{4,k_4}_0}, \ L \to \infty
\]
holds true, where
\[\alpha^{4,k_4}_r = \alpha_r(U, 2L, k_4; f_4), \ r = 0, 1, \ldots, k_4,\]
\[\beta^{k_4}_r = \beta_r(U, 2L, k_4), \ r = 1, \ldots, k_4,\]
\[0 < \alpha^{4,k_4}_0 - 2\pi L < U.\]

**Lemma 6.** For the function
\[(4.9) \quad f_5(t) = \frac{1}{\cos^2 t} \in \tilde{C}_0[2\pi L, 2\pi L + U], \ U \in (0, \pi/2 - \epsilon)\]
there are vector-valued functions
\[(4.10) \quad (\alpha^{5,k_5}_0, \ldots, \alpha^{5,k_5}, \beta^{k_5}_1, \ldots, \beta^{k_5}_k), \ 1 \leq k_5 \leq k_0\]
such that the following factorization formula
\[
\prod_{r=1}^{k_5} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha^{5,k_5}_r \right)}{\zeta \left( \frac{1}{2} + i\beta^{k_5}_r \right)} \right|^2 \sim \\
\sim \frac{\sin U \cos \frac{1}{2} \sin \frac{1}{2} \cos \alpha^{5,k_5}_0}{\cos U}, \ L \to \infty
\]
holds true, where
\[
\begin{align*}
\alpha_r^{5,k_5} &= \alpha_r(U, 2L, k_5; f_5), \quad r = 0, 1, \ldots, k_5, \\
\beta_r^{5,k_5} &= \beta_r(U, 2L, k_5), \quad r = 1, \ldots, k_5,
\end{align*}
\] (4.12)

\[0 < \alpha_0^{5,k_5} - 2\pi L < U,
\]

(comp. (2.1) – (2.4) at \(L \to \infty\)).

4.2. Next, we have the following results (see [10], (2.1) – (2.6)).

Lemma 7. For the function
\[
f_6(t) = \cos^3 t \in \tilde{C}_0[2\pi L, 2\pi L + U], \; U \in (0, \pi/2)
\]
there are vector-valued functions
\[
(a_0^{6,k_6}, \ldots, a_{k_6}^{6,k_6}, \beta_1^{k_6}, \ldots, \beta_{k_6}), \; 1 \leq k_6 \leq k_0
\]
such that the following factorization formula
\[
\prod_{r=1}^{k_6} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{6,k_6} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{k_6} \right)} \right|^2 \sim \frac{1}{\cos^3(\alpha_0^{6,k_6})}, \; L \to \infty
\]
holds true, where
\[
\begin{align*}
\alpha_r^{6,k_6} &= \alpha_r(U, L, k_6; f_6), \quad r = 0, 1, \ldots, k_6, \\
\beta_r^{k_6} &= \beta_r(U, L, k_6), \quad r = 1, \ldots, k_6, \\
0 &< \alpha_0^{6,k_6} - \pi L < U.
\end{align*}
\] (4.16)

Lemma 8. For the function
\[
f_7(t) = \cos t \in \tilde{C}_0[2\pi L, 2\pi L + U], \; U \in (0, \pi/2)
\]
there are vector-valued functions
\[
(a_0^{7,k_7}, \ldots, a_{k_7}^{7,k_7}, \beta_1^{k_7}, \ldots, \beta_{k_7}), \; 1 \leq k_7 \leq k_0
\]
such that the following factorization formula
\[
\prod_{r=1}^{k_7} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{7,k_7} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{k_7} \right)} \right|^2 \sim \frac{\sin U}{U} \frac{1}{\cos(\alpha_0^{7,k_7})}, \; L \to \infty
\]
holds true, where
\[
\begin{align*}
\alpha_r^{7,k_7} &= \alpha_r(U, L, k_7; f_7), \quad r = 0, 1, \ldots, k_7, \\
\beta_r^{k_7} &= \beta_r(U, L, k_7), \quad r = 1, \ldots, k_7, \\
0 &< \alpha_0^{7,k_7} - 2\pi L < U.
\end{align*}
\] (4.20)
5. Theorem 2

5.1. The first stage of the crossbreeding. The crossbreeding between the formula (4.13) and (4.17) gives the following formula

\[
\frac{\cos^2(\alpha_0, k_1)}{\cos U} \prod_{r=1}^{k_4} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_1 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_1 \right)} \right|^2 - \frac{\sin^2(\alpha_0, k_3)}{\cos U} \prod_{r=1}^{k_3} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_3 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_3 \right)} \right|^2 \\
\sim \frac{\sin U}{U}, \ L \to \infty.
\]

(5.1)

5.2. The second stage of the crossbreeding. The crossbreeding between the formula (4.15) and (4.19) gives the following formula

\[
\prod_{r=1}^{k_6} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_6 k_6 \right)}{\zeta \left( \frac{1}{2} + i\beta_6 k_6 \right)} \right|^2 \\
\sim \frac{\cos(\alpha_0, k_7)}{\cos^3(\alpha_0, k_6)} \prod_{r=1}^{k_7} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_7 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_7 \right)} \right|^2 \\
- \frac{U^2}{3 \cos^3(\alpha_0, k_6)} \times \\
\left\{ \cos^2(\alpha_0, k_4) \prod_{r=1}^{k_4} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_4 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_4 \right)} \right|^2 - \sin^2(\alpha_0, k_3) \prod_{r=1}^{k_3} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_3 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_3 \right)} \right|^2 \right\}^3 \\
L \to \infty.
\]

(5.2)

5.3. The third stage of the crossbreeding. The crossbreeding between the formula (4.11) and (5.1) gives the following formula

\[
\frac{1}{\cos^2 U} \sim \frac{1}{\cos^2(\alpha_0, k_5)} \prod_{r=1}^{k_5} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_5 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_5 \right)} \right|^2 \\
\times \\
\left\{ \cos^2(\alpha_0, k_4) \prod_{r=1}^{k_4} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_4 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_4 \right)} \right|^2 - \sin^2(\alpha_0, k_3) \prod_{r=1}^{k_3} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_3 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_3 \right)} \right|^2 \right\}^{-1} \\
L \to \infty.
\]

(5.3)

In the next stage of the crossbreeding we use the following variant of the formula (5.3).

\[
\frac{1}{\cos^3 U} \sim \frac{1}{\cos^3(\alpha_0, k_5)} \left\{ \prod_{r=1}^{k_5} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_5 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_5 \right)} \right|^2 \right\}^{3/2} \\
\times \\
\left\{ \cos^2(\alpha_0, k_4) \prod_{r=1}^{k_4} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_4 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_4 \right)} \right|^2 - \sin^2(\alpha_0, k_3) \prod_{r=1}^{k_3} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r k_3 \right)}{\zeta \left( \frac{1}{2} + i\beta_r k_3 \right)} \right|^2 \right\}^{-3/2} \\
L \to \infty.
\]

(5.4)
5.4. **The fourth stage of the crossbreeding.** The crossbreeding between the formula (5.2) and (5.4) gives the following formula

\[
\frac{\prod_{r=1}^{k_r} \left( \frac{1}{2} + i\alpha_{r,k_r}^{0} \right) \zeta \left( \frac{1}{2} + i\beta_{r,k_r}^{0} \right)}{\prod_{r=1}^{k_r} \left( \frac{1}{2} + i\alpha_{r,k_r}^{0} \right) \zeta \left( \frac{1}{2} + i\beta_{r,k_r}^{0} \right)} \sim \frac{\cos(\alpha_{0,k_r}^{7,k_r})}{\cos^{3}(\alpha_{0,k_r}^{6,k_r})} \prod_{r=1}^{k_r} \left| \frac{\frac{1}{2} + i\alpha_{r,k_r}^{5,k_r}}{\frac{1}{2} + i\beta_{r,k_r}^{5,k_r}} \right|^{2} - \frac{U^2}{3} \frac{1}{\cos^{3}(\alpha_{0,k_r}^{4,k_r}) \cos^{3}(\alpha_{0,k_r}^{3,k_r})} \left\{ \prod_{r=1}^{k_r} \left| \frac{\frac{1}{2} + i\alpha_{r,k_r}^{3,k_r}}{\frac{1}{2} + i\beta_{r,k_r}^{3,k_r}} \right|^{2} \sin^{2}(\alpha_{0,k_r}^{3,k_r}) \prod_{r=1}^{k_r} \left| \frac{\frac{1}{2} + i\alpha_{r,k_r}^{1,k_r}}{\frac{1}{2} + i\beta_{r,k_r}^{1,k_r}} \right|^{2} \right\} \times L \to \infty.
\]

(5.5)

5.5. **The fifth stage of the crossbreeding.** In this stage we use the formula (2.11) in the case

\[ L \to 2L; \pi L \to \pi2L, \]

i.e. we use the following formula

\[
U^2 \sim (1 + \Delta)^{2/\Delta} \left( \alpha_{0}^{\Delta,k_{\Delta}} - 2\pi L \right)^{2} \left\{ \prod_{r=1}^{\kappa_{\Delta}} \left| \frac{\frac{1}{2} + i\alpha_{r,k_{\Delta}}^{\Delta,k_{\Delta}}}{\frac{1}{2} + i\beta_{r,k_{\Delta}}^{\Delta,k_{\Delta}}} \right|^{2} \right\} ^{2/\Delta}, \quad \Delta > 0, \quad L \to \infty,
\]

(5.6)

where (comp. (2.12))

\[
\alpha_{0}^{\Delta,k_{\Delta}} = \alpha_{r}(U, 2L, \bar{k}_{\Delta}; \bar{f}_{\Delta}), \quad r = 0, 1, \ldots, \kappa_{\Delta},
\]

(5.7)

\[
\beta_{r}^{\Delta} = \beta_{r}(U, 2L, \bar{k}_{\Delta}), \quad r = 1, \ldots, \kappa_{\Delta},
\]

\[
\bar{f}_{\Delta} = (t - 2\pi L)^{\Delta} \in \tilde{C}_{0}[2\pi L, 2\pi L + U].
\]

Finally, we obtain by the crossbreeding between the formulas (5.5) and (5.6) the following
Complete Hybrid Formula 2.

\[
\prod_{r=1}^{k_6} \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{6,k_6} \right)^2 \cos^2 (\alpha_0^{6,k_6})}{\zeta \left( \frac{1}{2} + i\beta_r^{6,k_6} \right)^2 \cos^3 (\alpha_0^{6,k_6})} \prod_{r=1}^{k_7} \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{7,k_7} \right)^2 \cos^2 (\alpha_0^{7,k_7})}{\zeta \left( \frac{1}{2} + i\beta_r^{7,k_7} \right)^2 \cos^3 (\alpha_0^{7,k_7})} - \\
\frac{(1 + \Delta)^{2/\Delta}}{3} \frac{\frac{s_\Delta \cdot k_\Delta}{\alpha_0^{6,k_6}} - 2\pi L^2}{\cos^3 (\alpha_0^{6,k_6}) \cos^3 (\alpha_0^{6,k_6})} \left\{ \prod_{r=1}^{k_\Delta} \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{\Delta,k_\Delta} \right)^2 \cos^2 (\alpha_0^{6,k_6})}{\zeta \left( \frac{1}{2} + i\beta_r^{\Delta,k_\Delta} \right)^2 \cos^3 (\alpha_0^{6,k_6})} \right\}^{2/\Delta} \times \\
\times \left\{ \prod_{r=1}^{k_s} \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{5,k_s} \right)^2 \cos^2 (\alpha_0^{5,k_s})}{\zeta \left( \frac{1}{2} + i\beta_r^{5,k_s} \right)^2 \cos^3 (\alpha_0^{5,k_s})} \right\}^{3/2}
\right\}^{2/\Delta} - \sin^2 (\alpha_0^{3,k_3}) \prod_{r=1}^{k_\Delta} \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{3,k_3} \right)^2 \cos^2 (\alpha_0^{3,k_3})}{\zeta \left( \frac{1}{2} + i\beta_r^{3,k_3} \right)^2 \cos^3 (\alpha_0^{3,k_3})}
\right\}^{3/2},
\]

1 ≤ k_3, k_4, k_5, k_6, k_7, k_\Delta ≤ k_0, \Delta > 0, L \to \infty.

5.6. Now, we obtain from (5.8) by Definition the following

**Theorem 2.** The set of functions

\[
\left\{ \sin^2 t, \cos^2 t, \frac{1}{\cos^2 t}, \cos^3 t, \cos t, (1 - 2\pi L)^\Delta \right\},
\]

t ∈ [2\pi L, 2\pi L + U], U ∈ (0, \pi/2 - \epsilon], \Delta > 0, L \to \infty

is the family of \(\zeta\)-kindred elements in the class of real continuous functions.

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