1. Introduction

In [RS07, RS08], Reimann and Slaman raise the question “For which infinite binary sequences $X$ do there exist continuous probability measures $\mu$ such that $X$ is effectively random relative to $\mu$?”

1.1. Randomness relative to continuous measures. We begin by reviewing the basic definitions needed to precisely formulate this question.

Notation 1.1. • For $\sigma \in 2^{<\omega}$, $[\sigma]$ is the basic open subset of $2^\omega$ consisting of those $X$’s which extend $\sigma$. Similarly, for $W$ a subset of $2^{<\omega}$, let $[W]$ be the open set given by the union of the basic open sets $[\sigma]$ such that $\sigma \in W$.
• For $U \subseteq 2^\omega$, $\lambda(U)$ denotes the measure of $U$ under the uniform distribution. Thus, $\lambda([\sigma])$ is $1/2^\ell$, where $\ell$ is the length of $\sigma$.

Definition 1.2. A representation $m$ of a probability measure $\mu$ on $2^\omega$ provides, for each $\sigma \in 2^{<\omega}$, a sequence of intervals with rational endpoints, each interval containing $\mu([\sigma])$, and with lengths converging monotonically to 0.

Definition 1.3. Suppose $Z \in 2^\omega$. A test relative to $Z$, or $Z$-test, is a set $W \subseteq \omega \times 2^{<\omega}$ which is recursively enumerable in $Z$. For $X \in 2^\omega$, $X$ passes a test $W$ if and only if there is an $n$ such that $X \notin [W_n]$.

Definition 1.4. Suppose that $m$ represents the measure $\mu$ on $2^\omega$ and that $W$ is an $m$-test.
• $W$ is correct for $\mu$ if and only if for all $n$, $\sum_{\sigma \in W_n} \mu([\sigma]) \leq 2^{-n}$.
• $W$ is Solovay-correct for $\mu$ if and only if $\sum_{n \in \omega} \mu([W_n]) < \infty$.

Definition 1.5. $X \in 2^\omega$ is 1-random relative to a representation $m$ of $\mu$ if and only if $X$ passes every $m$-test which is correct for $\mu$. When $m$ is understood, we say that $X$ is 1-random relative to $\mu$.

By an argument of Solovay, see [Nie09], $X$ is 1-random relative to a representation $m$ of $\mu$ if and only if for every $m$-test which is Solovay-correct for $\mu$, there are infinitely many $n$ such that $X \notin [W_n]$.

Definition 1.6. $X \in \text{NCR}_1$ if and only if there is no representation $m$ of a continuous measure $\mu$ such that $X$ is 1-random relative to the representation $m$ of $\mu$. 

1

K-TRIVIALS ARE NCR

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In [RS08], Reimann and Slaman show that if \( X \) is not hyperarithmetic, then there is a continuous measure \( \mu \) such that \( X \) is 1-random relative to \( \mu \). Conversely, Kjøs-Hanssen and Montalbán, see [Mon05], have shown that if \( X \) is an element of a countable \( \Pi^0_1 \)-class, then there is no continuous measure for which \( X \) is 1-random. As the Turing degrees of the elements of countable \( \Pi^0_1 \)-classes are cofinal in the Turing degrees of the hyperarithmetic sets, the smallest ideal in the Turing degrees that contains the degrees represented in \( \text{NCR}_1 \) is exactly the Turing degrees of the hyperarithmetic sets.

In (author?) [RSte], Reimann and Slaman pose the problem to find a natural \( \Pi^1_1 \)-norm for \( \text{NCR}_1 \) and to understand its connection with the natural norm mapping a hyperarithmetic set \( X \) to the ordinal at which \( X \) is first constructed. As of the writing of this paper, this problem is open in general, but completed in [RSte] for \( X \in \Delta^0_2 \).

Suppose that \( X \in \Delta^0_2 \) and that for all \( n \), \( X(n) = \lim_{t \to \infty} X_t(n) \), where \( X_t(n) \) is a computable function of \( n \) and \( t \). Let \( g_X \) be the convergence function for this approximation, that is for all \( n \), \( g_X(n) \) is the least \( s \) such that for all \( t \geq s \) and all \( m \leq n \), \( X_t(m) = X(m) \). Let \( f_X \) be function obtained by iterated application of \( g_X \): \( f_X(0) = g_X(0) \) and \( f_X(n + 1) = g_X(f_X(n)) \).

For a representation \( m \) of a continuous measure \( \mu \), the granularity function \( s_m \) maps \( n \in \omega \) to the least \( \ell \) found in the representation of \( \mu \) by \( m \) such that for all \( \sigma \) of length \( \ell \), \( \mu([\sigma]) < 1/2^m \). Note that, \( s_m \) is well-defined by the compactness of \( 2^\omega \).

**Theorem 1.7** (Reimann and Slaman [RSte]). If \( X \) is 1-random relative the representation \( m \) of \( \mu \), then the granularity function \( s_m \) for \( \mu \) is eventually bounded by \( f_X \).

Thus, there is a continuous measure relative to which \( X \) is 1-random if and only if there is a continuous measure whose granularity is eventually bounded by \( f_X \). The latter condition is arithmetic, again by a compactness argument.

1.2. **K-triviality.** K-triviality is a property of sequences which characterizes another aspect of their being far from random. We briefly review this notion and the results surrounding it. A full treatment is given in Nies [Nie09].

For \( \sigma \in 2^{<\omega} \), let \( K(\sigma) \) denote the prefix-free Kolmogorov complexity of \( \sigma \). Intuitively, given a universal computable \( U \) with domain an antichain in \( 2^{<\omega} \), \( K(\sigma) \) is length of the shortest \( \tau \) such that \( U(\tau) = \sigma \). Similarly, for \( X \in 2^\omega \), let \( K^X(\sigma) \) denote the prefix-free Kolmogorov complexity of \( \sigma \) relative to \( X \). That is, \( K^X \) is determined by a function universal among those computable relative to \( X \).

**Definition 1.8.** A sequence \( X \in 2^\omega \) is **K-trivial** if and only if there is a constant \( k \) such that for every \( \ell \), \( K(X \upharpoonright \ell) \leq K(0^\ell) + k \), where \( 0^\ell \) is the sequence of 0’s of length \( \ell \).
By early results of Chaitin and Solovay and later results of Nies and others, there are a variety of equivalents to \(K\)-triviality and a variety of properties of the \(K\)-trivial sets. For example, \(X\) is \(K\)-trivial if and only if for every sequence \(R\), \(R\) is 1-random for \(\lambda\) if and only if \(R\) is 1-random for \(\lambda\) relative to \(X\).

In the next section, we will apply the following.

**Theorem 1.9** (Nies [Nie09], strengthening Chaitin [Cha76]). If \(X\) is \(K\)-trivial, then there is a computably enumerable and \(K\)-trivial set which computes \(X\).

The following theorem follows from the work of Nies and others [Nie09]. Some versions of this property have been used by Kučera extensively, e.g. in [Kuc85].

**Theorem 1.10.** Suppose \(X\) is \(K\)-trivial and \(\{U_e^X : e \in \omega\}\) a uniformly \(\Sigma^0_X\) family of sets. Then, there is a computable function \(g\) and a \(\Sigma^0_1\) set \(V\) of measure less than 1 such for every \(e\), if \(\lambda(U_e^Z) < 2^{-g(e)}\) for every oracle \(Z\), then \(U_e^X \subseteq V\).

**Proof.** (George Barmpalias) Let \((E_i^e)_{e \in \mathbb{N}}\) be a uniform sequence of all oracle Martin-Löf tests. A standard construction of a universal oracle Martin-Löf test \((T_i)\) (e.g. see [Nie09]) gives a recursive function \(f\) such that \(\forall Z \subseteq \omega (E_{f(i,e)}^e \subseteq T_i^Z)\) for all \(e, i \in \mathbb{N}\). Let \(T := T_2\) and \(f(e) := f(2,e)\) for all \(e \in \mathbb{N}\), so that \(\mu(T^Y) \leq 2^{-2}\) for all \(Y \in 2^\omega\) and \(E_{f(e)}^e \subseteq T\) for all \(e \in \mathbb{N}\). In [KH07] it was shown that \(X\) is \(K\)-trivial iff for some member \(T\) of a universal oracle Martin-Löf test, there is a \(\Sigma^0_1\) class \(V\) with \(T^X \subseteq V\) and \(\mu(V) < 1\).

Now given a uniform enumeration \((U_e)\) of oracle \(\Sigma^0_1\) classes we have the following property of \(T\):

There is a recursive function \(g\) such that for each \(e\),

either \(\exists Z \subseteq \omega (\mu(U_e^Z) \geq 2^{-g(e)-1})\), or \(\forall Z \subseteq \omega (U_e^Z \subseteq T^Z)\).

To see why this is true, note that every \(U_e\) can be effectively mapped to the oracle Martin-Löf test \((M_i)\) where \(M_i^Z = U_e^Z[s_i]\) and \(s_i\) is the largest stage such that \(\mu(U_e^Z[s_i]) < 2^{-i-1}\) (which could be infinity). Effectively in \(e\) we can get an index \(n\) of \((M_i)\). It follows that if \(\mu(U_e^Z) < 2^{-f(n)-1}\) for all \(Z\), then \(U_e^X = M_{f(n)}^X = E_{f(n)}^{n,X} \subseteq T^X \subseteq V\). So \(g(e) = f(n) + 1\) as wanted. \(\square\)

1.3. \(X\) is \(K\)-trivial implies \(X \in \text{NCR}_1\). Intuitively, \(X \in \text{NCR}_1\) asserts that \(X\) is not effectively random relative to any continuous measure and \(X\) is \(K\)-trivial asserts that relativizing to \(X\) does change the evaluation of randomness relative to the uniform distribution. In the next section, we connect the two notions by showing that if \(X\) is \(K\)-trivial then \(X \in \text{NCR}_1\).

2. The Main Theorem

**Theorem 2.1.** Every \(K\)-trivial set belongs to \(\text{NCR}_1\).
Proof. Let $Y$ be $K$-trivial and let $\mu$ be a continuous measure with representation $m$; we want to show $Y$ is not $\mu$-random. By Theorem 1.9, let $X$ be a computably enumerable $K$-trivial sequence that computes $Y$. Let $f$ be the iterated convergence function as defined above for the computable approximation to $Y$ given by approximating $X$’s computation of $Y$. Since $X$ is computably enumerable, $X$ can compute the convergence function for its own enumeration and hence $f$ is computable from $X$.

Let $s_m$ be the granularity function for $\mu$ as represented by $m$. By Theorem 1.7, $f$ eventually dominates $s_m$. By changing finitely many values of $f$, we may assume that $f$ dominates $s_m$ everywhere. So, we have that for every $n$

$$\mu([Y \upharpoonright f(n)]) \leq 2^{-n}.$$ 

Further, we may assume that $f$ can be obtained as the limit of a computable function $f(n, s)$ such that for all $s$, $f(n-1, s) \leq f(n, s) \leq f(n, s+1)$.

We will build an $m$-test $\{S_i : i \in \omega\}$ which is Solovay-correct for $\mu$ and which $Y$ does not pass, thereby concluding that $Y$ is not $\mu$-random. That is, we plan to build $\{S_i : i \in \omega\}$ to be a uniformly $\Sigma^0_m$ sequence of sets such that $\sum_{i \in \omega} \mu(S_i)$ is bounded and such that there are co-finitely $i$ for which $Y \in [S_i]$. Our construction will not be uniform.

$X$’s $K$-triviality is exploited in the form of Theorem 1.10. Let $V$ and $g$ be given by Theorem 1.10 where $\{U_e^X : e \in \omega\}$ is a listing of all $\Sigma^0_X$ sets. We will build an oracle $\Sigma^0_1$ class $U$ along the construction. We use the recursion theorem to assume that in advance we know an index $e$ such that $U = U_e$. During the construction we will make sure that for every oracle $Z$, $\lambda(U^Z) < 2^{-g(e)}$. Theorem 1.10 then implies that $U^X \subseteq V$ where $V$ is a $\Sigma^0_1$ class of measure less than $1$. To simplify our notation, let $a$ denote $g(e)$. Furthermore, assume $a$ is large enough so that $\lambda(V) + 2^{-a} < 1$.

We use the approximation to $X$ as a computably enumerable set to enumerate approximations to initial segments of $Y$ into the sets $S_i$; we rely on the $K$-triviality of $X$ to keep the total $\mu$-measure of the $S_i$’s bounded.

For each $n > a$ we have a requirement $R_n$ whose task is to enumerate $Y \upharpoonright f(n)$ into $S_n$. Let $y_{n,s} = Y_s \upharpoonright f(n, s)$ the stage $s$ approximation to $Y \upharpoonright f(n)$. Let $x_{n,s}$ be the initial segment of $X_s$ necessary to compute $y_{n,s}$ and $f(n, s)$. So, if $y_{n,s+1} \neq y_{n,s}$, it is because $x_{n,s+1} \neq x_{n,s}$. In this case, $x_{n,s+1}$ is not only different than $x_{n,s}$, but also incomparable. At stage $s$, $R_n$ would like to enumerate $y_{n,s}$ into $S_n$, but before doing that it will ask for confirmation using the fact that $U^X \subseteq V$. Since we are constrained to keep $\lambda(U^X)$ less than or equal to $2^{-a}$, we will restrict $R_n$ to enumerate at most $2^{-n}$ measure into $U^X$. The reason why we need a bit of security before enumerating a string in $S_n$ is that we have to ensure that $\sum_i \mu(S_i)$ is bounded. For this purpose, we will only enumerate mass into $S_n$ when we see an equivalent mass going into $V$.

**Action of requirement $R_n$:**
(1) The first time after $R_n$ is initialized, $R_n$ chooses a clopen subset of $2^\omega$, $\sigma_n$, of $m$-measure $2^{-n}$, that is disjoint from $V_s$ and $U^X_s$. Note that since $V$ and $U^X_s$ have measure less than $\lambda(V) + 2^{-a} < 1$, we can always find such a clopen set. Furthermore we can choose $\sigma_n$ to be different from the $\sigma_i$ chosen by other requirements $R_i$, $i > a$. We note the value of $\sigma_n$ might change if $R_n$ is initialized.

(2) To confirm $x_{n,s}$, requirement $R_n$ enumerates $\sigma_n$ into $U^{x_{n,s}}$. Requirement $R_n$ will not be allowed to enumerate anything else into $U^X_s$ unless $X_s$ changes below $x_{n,s}$. This way $R_n$ is always responsible for at most $2^{-n}$ measure enumerated in $U^X_s$.

(3) Then, we wait until a stage $t > s$ such that

(a) either $x_{n,s} \not\subseteq x_{n,t}$ (as strings),
(b) or $\sigma_n \subseteq V_t$.

Observe that if $x_{n,s}$ is actually an initial segment of $X$, then we will have $\sigma_n \subseteq U^X \subseteq V$. So, we will eventually find such a stage $t$.

- In Case 3(a), we start over with $R_n$. Note that in this case $\sigma_n$ has come out of $U^X_t$, and hence $R_n$ is responsible for no measure inside $U^X$ at stage $t$.

- In Case 3(b), if $\mu ([y_{n,t}]) \leq 2^{-n}$, enumerate $y_{n,t}$ into $S_n$. (Recall that we are allowed to use the representation of $\mu$ as an oracle when enumerating $S_n$.)

Since we only enumerate $y_{n,t}$ of $\mu$-measure less than $2^{-n}$ when $\sigma_n$ is enumerated in $V$, we have that

$$\sum_i \mu(S_i) \leq \lambda(V) < 1.$$  

It is not hard to check that $\lambda(U^X) \leq \sum_{n=a+1}^{\infty} 2^{-n} = 2^{-a}$, so we actually have that $U^X \subseteq V$. Also notice that once $x_{n,s}$ is a initial segment of $X$, we will eventually enumerate $\sigma_n$ into $V$ and an initial segment of $Y$ into $S_n$. □

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