Random numbers as probabilities of machine behaviour

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Abstract. A fruitful way of obtaining meaningful, possibly concrete, algorithmically random numbers is to consider a potential behaviour of a Turing machine and its probability with respect to a measure (or semi-measure) on the input space of binary codes. In this work we obtain characterizations of the algorithmically random reals in higher randomness classes, as probabilities of certain events that can happen when an oracle universal machine runs probabilistically on a random oracle. Moreover we apply our analysis to several machine models, including oracle Turing machines, prefix-free machines, and models for infinite online computation. We find that in many cases the arithmetical complexity of a property is directly reflected in the strength of the algorithmic randomness of the probability with which it occurs, on any given universal machine. On the other hand, we point to many examples where this does not happen and the probability is a number whose algorithmic randomness is not the maximum possible (with respect to its arithmetical complexity). Finally we find that, unlike the halting probability of a universal machine, the probabilities of more complex properties like totality, cofinality, computability or completeness do not necessarily have the same Turing degree when they are defined with respect to different universal machines.

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1 Introduction

In this work we examine the probabilities of certain outcomes of a universal Turing machine that runs on random input from the point of view of algorithmic complexity. The input of a Turing machine can be a finite binary string that is written on the input tape, or even an infinite binary stream whose bits are either written on the input tape at the start of the computation or are provided upon demand as the computation progresses. Such computations can be interpreted in terms of probabilistic machines and randomized computation or as oracle computations with a random oracle. The study of probabilistic Turing machines goes back to [dLMSS55], where it was shown that if a randomized machine computes a function \( f \) with positive probability, then \( f \) is computable. Similarly, if a randomized machine produces an enumeration of a set \( W \) with positive probability, then \( W \) is computably enumerable, i.e. \( W \) can be enumerated by a deterministic machine without any oracle.

One way to introduce algorithmic randomness to the study of randomized computation is to consider what a machine can do given an algorithmically random oracle of a certain strength. This line of research has produced a large body of work and a sub-discipline in the area between algorithmic randomness and computability; see [BDLP14, BLP15] for a presentation and a bibliography. A common theme in this topic is that algorithmically random oracles of sufficient strength cannot compute useful sets or functions. For example, oracles that are random relative to the halting problem do not compute any complete extension of Peano Arithmetic (Stephan [Ste06]).

In this paper we apply the theory of algorithmic randomness to randomized computation in a different way. Given a property of the outcome of a randomized computation, we consider the probability with which this property occurs on a given universal machine. Then we consider the algorithmic randomness of this probability as a real number in \([0, 1]\). It turns out that in this way we can characterize the real numbers that are probabilities of certain key properties (like halting, totality, computability, etc.), as the algorithmically random numbers of certain well known classes in algorithmic information theory.

1.1 Previous work on the topic and outline of our results in historical context

One can trace this methodology back to Chaitin [Cha75] where it was shown that the probability that a universal self-delimiting machine halts is a Martin-Löf random real. Becher, Daicz, and Chaitin [BDC01] applied this idea to a model for infinite computations which was introduced earlier in Chaitin [Cha76] and exhibited a probability which is 2-random, i.e. Martin-Löf random relative to the halting problem. Becher and Chaitin [BC02] studied a related model for infinite computations in order to exhibit a probability which is 3-random, i.e. Martin-Löf random relative to 2-quantifier sentences in arithmetic. A subsequent series of papers by Becher and Grigorieff [BG05, BG07, BG09] as well as Becher, Figueira, Grigorieff and Miller [BFGM06], followed this idea with the main aim of exhibiting algorithmically random numbers that are concrete, in the sense that they can be expressed as probabilities that a universal machine will have a certain outcome when it is run on a random input. Sureson [Sur15] continues this line of research, showing that many natural open sets related to the universal machine have random probabilities of different algorithmic strength, while his arguments appeal to completeness phenomena as opposed to machine arguments.

A crucial development in the study of halting probabilities was the cumulative work of Solovay [Sol75], Calude, Hertling, Khoussainov, and Wang [CHKW01], and Kučera and Slaman [KS01]. In these papers it was established that a real number is the halting probability of a universal self-delimiting machine if
and only if it is Martin-Löf random and has a computably enumerable (in short, c.e.) left Dedekind cut. Moreover, as we explain in the following, this analysis shows that the same characterization remains true for most commonly used types of Turing machines and is not specific to the self-delimiting model which is sometimes used in algorithmic information theory. A similar approach was followed in [BD12] in order to provide a characterization of the universality probability of a self-delimiting machine, a notion that was introduced in [WD99].

In the present paper we take the aforementioned work as a starting point and develop a methodology for characterizing algorithmically random numbers as probabilities of certain outcomes of a universal machine like totality, computability, and co-finiteness. There are three main ways that our work stands out from previous attempts on this topic. We provide

1. characterizations of algorithmically random numbers as probabilities;
2. results that hold uniformly for several types of Turing machine models;
3. examples of probabilities which are not as random as their arithmetical complexity would suggest;
4. demonstrations that the probability of properties that are not definable with one quantifier is not Turing degree-invariant with respect to different universal machines.

We have a few remarks for each of these clauses. With respect to (1), the only such characterizations in the literature are the two that we mentioned, namely the halting probability and the universality probability. However, even in these cases the characterizations were proved for the specific model of self-delimiting machines although, as we will see, they apply in a much more general class of models. With respect to (2) we would like to note that all examples of random probabilities from the literature that we have encountered refer to self-delimiting models and there is a reason for this preference. Self-delimiting models correspond to prefix-free domains, which means that the probabilities considered turn out to be measures of open sets, which are easily represented and manipulated as sets of strings. In the present paper we free ourselves from this restriction, considering probabilities that are measures of classes that have higher Borel complexity, so they are not necessarily open or closed or even $G_\delta$. Moreover, as we will see, this generality does not introduce an extra burden in our analysis. The reason for this pleasing fact can be traced to the theory of 1-random reals with left Dedekind cut (also known as left-c.e. reals) and the fact that the measures of $\Sigma^0_1$ classes are the same as the measures of $\Sigma^0_1$ classes relative to $0^{(n-1)}$, i.e. the Turing degree of the halting problem iterated $n-1$ times. As a result of this generality, we can look at classic properties of computability theory like totality, cofinality, completeness or computability, and characterize their probabilities in various standard models of Turing machines.

Clause (3) deserves a somewhat more elaborate remark. There is a direct contrast between the definability of a mathematical object and its algorithmic complexity in terms of effective randomness. Indeed, according to Martin-Löf [ML66] a sequence is random if it is not contained in a null set of a certain arithmetical complexity. So, for instance, since each arithmetically definable sequence $x$ belongs to the null set $\{x\}$, which has a arithmetical definition with the same complexity as that of $x$, $x$ cannot be random with respect to null sets of this particular arithmetic complexity. When one looks at properties of the universal machine, it often turns out that the probabilities of the occurrence of these properties are, in a sense, maximally random with respect to the arithmetical class that they belong to. Such is the case with the halting probability of a universal prefix-free machine, also known as Chaitin’s $\Omega$, which is a real with a $\Sigma^0_1$ left Dedekind cut and is $1$-random i.e. random with respect to all Martin-Löf tests, a notion based on $\Sigma^0_1$ definable open sets. Indeed, a real with $\Sigma^0_1$ left Dedekind cut cannot be random with respect to stronger statistical tests
such as Martin-Löf tests relative to the halting problem, or even random with respect to null \( \Pi^0_2 \) classes (which give a slightly stronger notion of randomness than Martin-Löf randomness). Grigorieff, as discussed in [BFGM06], conjectured that in a certain more restricted context, the randomness of a property of a universal machine has strength corresponding to the arithmetical class that it belongs to. This conjecture, in the specialized form that it appeared in, was refuted by Becher, Figueira, Grigorieff, and Miller in [BFGM06], where many examples of properties of every level of the arithmetical hierarchy were given which have probabilities which are not even random relative to the halting problem. Later work by Becher and Grigorieff [BG05, BG07, BG09] salvages this conjecture but in specific contexts (using specialized machines or non-standard notions of universality). In Section 3.6 we point to a rather common but nontrivial reason why a property of a universal machine may fail to have maximal algorithmic complexity with respect to its arithmetical class.

Regarding (4), recall that the halting probabilities with respect to the various universal Turing machines all have the same Turing degree, namely the degree of the halting problem. In other words, given the halting probability of one universal machine as an oracle, we can compute the halting probability of any other universal machine. Thus we may say that the (universal) halting probability is degree-invariant with respect to the choice of the underlying machine. This is no longer true when one considers probabilities with more complex arithmetical complexity. This was demonstrated by Barmpalias and Dowe [BD12] who studied the universality probability, a notion from [WD99] which is only definable with four alternating quantifiers, and showed that its Turing degree depends on the choice of the underlying machine. In the present paper we give several more examples of this phenomenon and observe that it typically occurs when the arithmetical complexity of the probability is above the complexity of the halting problem.

1.2 Random versus algorithmically random

The notion of randomness has several interpretations in different fields of science, e.g. in probability theory, in algorithmic randomness, and in quantum physics. This paper has to do with the first two of these uses, as we examine the algorithmic randomness of certain probabilities associated with running a machine probabilistically on an oracle chosen at random with respect to the uniform distribution on the binary outcomes of a hypothetical experiment. Hence there are two different ways that we use the word ‘random’ in this paper. The first one is random as in probability theory, i.e. an object drawn at random from a given distribution (in our case, the uniform distribution on a binary outcome). The second one is algorithmically random in the sense of algorithmic randomness, i.e. referring to a fixed individual object which may or may not be algorithmically random with respect to a given definition of algorithmic randomness.

The first meaning of the word ‘random’ is indicated by writing the word in italics (like random) and its use has been restricted with reference to the oracle of a Turing machine. When the word is used in the sense of algorithmic randomness, it always refers to the probability of a given outcome of a randomized machine as a real number. For these reasons there should be no confusion regarding the two uses of the word ‘random’ in the following discussion. In addition to this caveat, we note that in formal statements regarding randomness of probabilities, the word ‘random’ appears with a qualification indicating the type or strength of algorithmic randomness in question, e.g. Martin-Löf random or 2-random.


1.3 Turing machines

We do not aim at a comprehensive classification of all properties of universal machines on all models of computation. However our methodology applies rather uniformly to a number of standard models of computation and properties of these models. In order to demonstrate this fact, we have chosen three models of interest and examine the probabilities that certain key properties from classical computability theory occur when computation is performed probabilistically on a random input stream, which can also be thought of as an oracle. In this paper we focus our attention to the following models of computation:

(1) Oracle (Turing) machines computing (partial) functions.

(2) Monotone machines computing strings or streams.

(3) Chaitin’s infinitary self-delimiting machines, computing strings, or streams.

In case (1) we consider a standard notion of computation of a (partial) function \( f : \mathbb{N} \to \mathbb{N} \) from a Turing machine with a random oracle. Such a computation may be infinitary, in the sense that we compute a potential infinite object \( f \), but it is a juxtaposition of countably many finite computations, namely the computation of each \( f(n) \) for \( n \) in the domain of \( f \). In this case we can ask more complex question about the output \( f \), for example if the domain of \( f \) has a certain property or if \( f \) is total. Universality of oracle machines is defined in a standard way.

**Definition 1.1** (Universal oracle machines). Given an effective list \((M_e)\) of all oracle machines, an oracle machine \( U \) is universal if there exists a computable function \( e \mapsto \sigma_e \) from numbers into a prefix-free set of strings, such that \( U(\sigma_e \ast X, n) \equiv M_e(X, n) \) for all \( e, X, n \).

In case (2) we consider monotone machines, which were used by Levin in [Lev71, Lev73] in order to give a definition of the algorithmic complexity of finite objects. Let \( \preceq \) denote the prefix relation between strings.

**Definition 1.2** (Monotone machine). A monotone machine is a Turing machine \( M \) with input/output finite binary strings and the monotonicity property that if \( \sigma \preceq \tau \) and \( M(\sigma), M(\tau) \) both halt, then \( M(\sigma) \preceq M(\tau) \).

A monotone machine \( M \) can be regarded as model for infinitary computations, where \( M(X) \), the output on an infinite input \( X \), is defined as the supremum of all \( M(\sigma) \) for all \( \sigma < X \). In this context we can ask whether the output is finite or infinite, or even if the output belongs to a certain class (e.g. if it has a tail of 1s). The universality of monotone machines is defined in a standard way.

**Definition 1.3** (Universal monotone machines). Given an effective list \((M_e)\) of all monotone machines, an monotone machine \( U \) is universal if there exists a computable function \( e \mapsto \sigma_e \) from numbers into a prefix-free set of strings, such that \( U(\sigma_e \ast \tau) \equiv M_e(\tau) \) for all \( e, \tau \).

We remark that these standard notions of universality are quite different from the notion of optimal machines in the context of Kolmogorov complexity (e.g. see [LV97, Definition 2.1]).

Finally in case (3) we consider a specialized self-delimiting model for infinite computations, which was introduced by Chaitin in [Cha76] (we defer the formal definition of this model to Section 4.1, see Definition 4.1). In Section 4.1 we will see that this model can be seen as an infinitary prefix-free machine with an infinitary notion of halting which is \( \Pi^0_1 \) instead of the usual \( \Sigma^0_1 \) halting that one considers in finitary
computations in Turing machines. For this reason, Chaitin’s self-delimiting infinitary machine model is different from the models in cases (1) and (2). We consider it because it has been discussed in several papers, as we pointed out in Section 1.1.

1.4 Characterizations of the probabilities of key outcomes of randomized machines

Recall that by the cumulative work of Solovay [Sol75], Calude, Hertling, Khoussainov, and Wang [CHKW01], and Kučera and Slaman [KS01], the halting probabilities of universal prefix-free machines are exactly the 1-random left-c.e. reals. Let \( \emptyset' \) denote the halting problem and let \( \emptyset'' \) denote the Turing degree of \( \emptyset' \). More generally, let \( \emptyset^{(n)} \) denote the halting problem iterated \( n \) times and let \( \emptyset^{(n)} \) denote the Turing degree of \( \emptyset^{(n)} \).

A real is called \( \emptyset^{(n)} \)-left-c.e. if its left Dedekind cut is c.e. relative to \( \emptyset^{(n)} \). Similarly, a real is called \( \emptyset^{(n)} \)-right-c.e. if its right Dedekind cut is c.e. relative to \( \emptyset^{(n)} \). In Börjeson and Dowe [BD12], it was shown that the universality probabilities (a notion from [WD99]) of universal prefix-free machines are exactly the 4-random \( \emptyset^{(3)} \)-right-c.e. reals.

In this section we present our main results, which are characterizations of probabilities of certain machine outcomes in the models presented in Section 1.3. We start with the following characterization of 2-random \( \emptyset' \)-right-c.e. reals in terms of the totality probability of oracle Turing machines (case (1) in Section 1.3).

**Theorem 1.4** (Totality for oracle machines). The probability that the function computed by a universal oracle machine is total is a 2-random \( \emptyset' \)-right-c.e. real. Conversely every 2-random \( \emptyset' \)-right-c.e. real in \( (0,1) \) is the probability that a certain universal oracle machine computes a total function.

We will also show that Theorem 1.4 also holds with regard to the probability that the function computed by a universal oracle Turing machine has infinite domain. Moreover, the same argument shows a similar characterization of the probability that a universal monotone machine (case (1) in Section 1.3) has infinite output, as we note below. This probability for an infinitary self-delimiting machine (case (3) in Section 1.3) has the same characterization, but the proof is different, as it depends on the definition of this type of infinitary computation.

**Theorem 1.5** (Infinite output of monotone or infinitary self-delimiting machines). The probability that the output of a universal monotone machine is infinite is a 2-random \( \emptyset' \)-right-c.e. real. Conversely every 2-random \( \emptyset' \)-right-c.e. real in \( (0,1) \) is the probability that a certain universal monotone machine computes a total function. Moreover the same is true for Chaitin’s self-delimiting infinitary machines.

Naturally, in order to obtain characterizations of higher randomness we need to consider properties of higher arithmetical complexity. Recall that a subset of \( \mathbb{N} \) is cofinite if its complement is finite. Moreover recall that the indices of partial computable functions with a cofinite domain is a \( \Sigma^0_3 \)-complete set. The same is true of the partial computable functions with a computable domain. This is an indication that these notions can give the required characterizations of 3-randomness.

Before we demonstrate this intuition, we show that one has to be careful to chose the right model. As we explain in Section 4.1, there is a notion of domain of an infinitary self-delimiting machine, and properties of randomized computations with respect to this model correspond to subsets of the domain of the machine in question. In this sense, the following lemma says that we cannot exhibit 3-randomness by considering properties of computations in this model.
Lemma 1.6 (Limits on randomness of outcomes of infinitary self-delimiting machines). The measure of any subset of the domain of an infinitary self-delimiting machine is not a 3-random real number.

In particular, the property that the output of a universal infinitary self-delimiting machine has a tail of 1s (i.e. is the characteristic sequence of a cofinite subset of ℕ) does not have 3-random probability. The same is true when one considers monotone machines, but for different reasons as will be seen in Section 3.6.

Proposition 1.7 (Limits on randomness of computable outcomes of oracle and monotone machines). The probability that an oracle machine computes the characteristic sequence of a cofinite subset of ℕ is not a 3-random real number. The same is true for monotone machines.

The following result concerning oracle machines (case (1) in Section 1.3) demonstrates the correct notion of cofiniteness and computability which gives the desired characterization of 3-randomness.

Theorem 1.8 (Cofiniteness and computability for oracle machines). The probability that a universal oracle machine computes a function with cofinite domain is a 3-random 0(2)-left-c.e. real. Conversely, every 3-random 0(2)-left-c.e. real in (0, 1) is the probability that a certain universal oracle machine computes a function with cofinite domain. The same holds for ‘computable’ instead of ‘cofinite’.

Since the domain of a partial computable function can be seen as the range of another partial computable function, we will see that Theorem 1.8 also holds for the range instead of the domain of functions. Finally, we note that similar methods can be used in order to obtain the following characterization of 4-randomness.

Theorem 1.9 (Hardness for oracle machines). The probability that a universal oracle machine computes a function whose domain computes the halting problem is a 4-random 0(3)-left-c.e. real. Conversely, every 4-random 0(3)-left-c.e. real in (0, 1) is the probability that a certain universal oracle machine computes a function whose domain computes the halting problem.

The proof that the index set of Turing complete c.e. sets is Σ04-complete plays a crucial role in the proof of Theorem 1.9, just as we use the Σ03-completeness of the index set of the cofinite and computable c.e. sets in order to obtain Theorem 1.8.

1.5 Turing degrees of probabilities of universal machines

The characterization of 1-random left-c.e. reals as halting probabilities of universal prefix-free machines by Solovay [Sol75], Calude, Hertling, Khoussainov, and Wang [CHKW01], and Kučera and Slaman [KS01] relativizes to an arbitrary oracle. In particular, n-random 0(n−1)-left-c.e. reals are exactly the halting probabilities of prefix-free machines with oracle 0(n−1). Hence, given our characterizations of Section 1.4, the probabilities we consider can be viewed as halting or non-halting probabilities of universal prefix-free machines relative to 0(n−1) for some n > 0. We will combine these facts with the work of Downey, Hirschfeldt, Miller, and Nies [DHMN05] on relativizations of Chaitin’s halting probability in order to show the following dependence of the degree of the probabilities that we considered.

Corollary 1.10 (Turing degree variance). Given any machine model from (1)-(3) of Section 1.3 and any probability of that model considered in Theorems 1.4, 1.5, 1.8, and 1.9, the Turing degree of the particular probability depends on the underlying universal machine that is chosen. In fact, in each of these cases, if PM denotes the particular probability with respect to a machine M of the given type, there exist universal machines U, V such that PU and PV have incomparable Turing degrees.
Indeed, Downey, Hirschfeldt, Miller, and Nies [DHMN05] showed that for each $n > 0$ there exist oracle universal prefix-free machines $U, V$ such that the degrees of $\Omega_U^{\Omega_U(n)}$, $\Omega_V^{\Omega_V(n)}$ are incomparable. This fact, along with the above discussion and the results of Section 1.4 give Corollary 1.10. The reader who would like a more detailed discussion of the relativization of the characterization of halting probabilities and the application of the results from [DHMN05] is referred to [BD12] where a similar implication was established, along with a detailed presentation of the necessary background.

More can be said about the Turing degrees of the probabilities we discussed in Section 1.3. By Downey, Hirschfeldt, Miller, and Nies [DHMN05], for each oracle $X$ and each universal oracle prefix-free machine $U$, we have $\Omega_U^{\Omega_U(n)} \oplus \emptyset(n) \equiv_T \emptyset(n+1)$ for all $n > 0$. Moreover, for all $n > 0$, the Turing degrees of $\Omega_U^{\Omega_U(n)}$ and $\emptyset(n)$ form a minimal pair. Hence if $V$ is an oracle machine and $T_V$ is the probability that $T$ computes a total function, then the Turing degree of $T_V$ forms a minimal pair with the degree of the halting problem while $T_V \oplus \emptyset'$ computes $\emptyset(2)$. Similar facts can be deduced about all the probabilities considered in the theorems of Section 1.3.

2 Background material, terminology and notation

A standard reference regarding the interaction of computability theory and algorithmic randomness are Downey and Hirschfeldt [DH10] and Nies [Nie09], while Calude [Cal02] has an information-theoretic perspective. Odifreddi [Odi89, Odi99] is a standard reference in classical computability theory while Li and Vitányi [LV97] is a standard reference in the theory of Kolmogorov complexity.

The arguments in this paper borrow many ideas from the theory of arithmetical complexity and in particular $\Sigma^0_1$-completeness. A compact presentation of this part of computability theory can be found in e.g. [Soa87, Chapter IV]. A more comprehensive source on this topic is [Hin78]. In this section we introduce some notions that are directly related to our analysis, and prove some related results which will be used in the main part of the paper.

Recall the types of machines that we discussed in Section 1.3. If $M$ is an oracle machine then $M(X)$ refers to a (partial) function $n \mapsto M(X, n)$ from $\mathbb{N}$ to $\mathbb{N}$. Let $\text{TOT}(M)$ denote the class of all $X$ such that $M(X)$ is total. We occasionally refer to the measure of $\text{TOT}(M)$ as the totality probability of $M$. Let $\text{COF}(M)$ consist of the reals $X$ such that the domain of $M(X)$ is cofinite. The measure of $\text{COF}(M)$ is sometimes referred to as the cofiniteness probability of $M$. Moreover we let $\text{INF}(M)$ denote the class of all $X$ such that the domain of $M(X)$ is infinite. We refer to the measure of $\text{INF}(M)$ as the infinitude probability of $M$.

A monotone machine $N$ is viewed as a (partial) function $\sigma \mapsto N(\sigma)$ from strings to strings with a monotonicity property (see Section 1.3). Moreover in this case $N(X)$ denotes the supremum of all $N(\sigma)$ where $\sigma$ is a prefix of $X$. Given a monotone machine $N$ we let $\text{INF}(N)$ be the set of all $X$ such that $N(X)$ is infinite, i.e. a stream. Similarly, let $\text{FIN}(N)$ be the set of all $X$ such that $N(X)$ is finite, i.e. a string. We refer to the measure of $\text{INF}(N)$ as the infinity probability of $N$. Also let $\text{COF}(N)$ denote the reals $X$ in $\text{INF}(N)$ such that $N(X)$ is equal to $\sigma \ast 1^\omega$ for some binary string $\sigma$. Finally, let $\text{COM}(N)$ denote the set of reals $X$ such that the domain of $N(X)$ is computable.

Similar notations and terminology apply to the infinitary self-delimiting machines, which we introduce in Section 4.1. A string $\sigma$ is compatible with a string $\tau$ if $\sigma \leq \tau$ or $\tau \leq \sigma$; otherwise we say that $\sigma$ is incompatible with $\tau$. As it is customary in computability theory, the suffix $[s]$ on a parameter which is part
of a construction that takes place in countably many stages indicates the value of the parameter at stage $s$.

2.1 Arithmetical classes and measure

We work with subsets of the Cantor space, the class of infinite binary sequences with the usual topology generated by the basic open sets $\|\sigma\| = \{X \mid \sigma < X\}$ where $<$ denotes the (strict) prefix relation. Given a set of strings $S$, define $\|S\|$ to be the set of all reals that have a prefix in $S$. We use $\ast$ to denote concatenation of strings. Moreover if $P$ is a set of reals and $\sigma$ is a string, then $\sigma \ast P$ denotes the set of reals $Y$ of the form $\sigma \ast X$, where $X \in P$. We consider the uniform Lebesgue product measure on subsets of the Cantor space. Recall that a class of reals is $\Sigma^0_n$ if it is definable by a $\Sigma^0_n$ formula in arithmetic. Moreover $\Pi^0_n$ classes are the complements of $\Sigma^0_n$ classes. These definitions relativize to any oracle $X$, obtaining the $\Sigma^0_n(X)$ and the $\Pi^0_n(X)$ classes which are definable in arithmetic with an extra parameter $X$. The $\Sigma^0_1$ classes are known as the effectively open sets. However note that for $n > 1$ the $\Sigma^0_n$ classes are not necessarily open, so they are not necessarily $\Sigma^0_1(0^{(n-1)})$. The following fact will be used several times in our analysis. In the first part of Lemma 2.1 the qualification uniformly means that from an index of a $\Sigma^0_n$ class we can compute an index for a $0^{(n-1)}$-computable increasing sequence of rationals converging to the measure of the class. Similar meaning is intended by the qualification uniformly in the second part of the lemma, but with respect to decreasing approximations by rationals.

Lemma 2.1 (Measures of arithmetical classes). Let $n > 0$. The measure of a $\Sigma^0_n$ class is uniformly a $0^{(n-1)}$-left c.e. real. Similarly, the measure of a $\Pi^0_n$ class is uniformly a $0^{(n-1)}$-right c.e. real.

Proof. By induction on $n$. For $n = 1$ this clearly holds. Assume that the statement is true for $n = k$ and consider a $\Sigma^0_{k+1}$ class $S$. Then $S = \cup_i P_i$ where $(P_i)$ is a uniform increasing sequence of $\Pi^0_k$ classes. By the hypothesis, the reals $\mu(P_i)$ are uniformly right-c.e. relative to $0^{(k-1)}$. So the reals $\mu(P_i)$ are uniformly $0^{(k)}$-computable. On the other hand, $\mu(S)$ is the supremum of all $\mu(P_i)$. Hence $\mu(S)$ is left-c.e. relative to $0^{(k)}$. A similar argument shows that the measure of a $\Pi^0_{k+1}$ class is right-c.e. relative to $0^{(k)}$. □

Hence, although $\Sigma^0_n$ classes are topologically very different from the $\Sigma^0_1(0^{(n-1)})$ classes for $n > 1$, their measures are the same type of real numbers. This simple fact will be used in our main analysis. Note that $\Sigma^0_1(0^{(n-1)})$ classes can be represented by $0^{(n-1)}$-c.e. sets of binary strings. We denote the set of binary strings by $2^{<\omega}$ and the set of binary streams by $2^{\omega}$. Note that $\Sigma^0_1$ sets of strings are exactly the c.e. sets of strings.

For $\Sigma^0_1$ upward closed sets of strings we will use the following fact:

Lemma 2.2 (Canonical $\Sigma^0_1$ approximations). If $U$ is a $\Sigma^0_1$ upward closed set of strings, there exists a computable sequence $(V_s)$ of finite sets of strings with upward closures $V_s$ respectively such that

(i) for each string $\sigma$ we have $\sigma \in U$ if and only if there exists $s_0$ such that $\sigma \in V_s$ for all $s > s_0$;

(ii) there are infinitely many $s$ such that $V_s \subseteq U$.

We call $(V_s)$ a canonical $\Sigma^0_1$ approximation to $U$ (although it is not an approximation in the literal sense).

Proof. Given a set of strings $S$, let $U$ be the upward closure of $S$. Given $U$, there exists a c.e. operator $W$ such that $W^W = U$. We can modify the enumeration of $W$ (obtaining a modified $W$) with respect to a computable enumeration $(\theta'_s)$ of $\theta'$, so that if $n \in \theta'_s$, then $n \in \theta'_s$ for some number $n$ and stage $s$, any number
The converse of Lemma 2.1 is also true in a strong effective way. This is a consequence of the Kraft-Chaitin theorem (e.g. see [DH10, Section 2.6]) which says that, given a computable sequence of positive integers \( (c_i) \) such that \( \sum_i 2^{-c_i} \leq 1 \), we can effectively produce a computable sequence of binary strings \( (\sigma_i) \) such that \( |\sigma_i| = c_i \) for each \( i \) and the set \( \{\sigma_i \mid i \in \mathbb{N}\} \) is prefix-free. Moreover, the Kraft-Chaitin theorem holds relative to any oracle. Recall that \( \Sigma^0_1(0^{(n-1)}) \) classes of reals \( P \) can be represented by \( \Sigma^0_n \) sets \( S \) of strings, in the sense that \( P = \|S\| \). The measure of a set of binary strings \( S \) is defined to be the measure of \( \|S\| \). Moreover these sets of strings can be chosen to be prefix-free.

**Lemma 2.3 (Measures of arithmetical classes, converse).** Let \( n > 0 \). Given any \( \theta^{(n-1)} \)-left-c.e. real \( \alpha \in [0,1] \), we can effectively produce a \( \Sigma^0_1 \) prefix-free set of strings of measure \( \alpha \). Similarly, if \( \beta \in [0,1] \) is a \( \theta^{(n-1)} \)-right-c.e. real, we can effectively produce a \( \Pi^0_n \) prefix-free set of strings of measure \( \beta \).

**Proof.** By symmetry it suffices to prove the first statement. Note that if \( \alpha = 0 \) then the statement is trivial, so assume that \( \alpha \in (0,1] \) and that \( \alpha \) is a \( \theta^{(n-1)} \)-left-c.e. real. Then there exists a \( \theta^{(n-1)} \)-computable sequence of positive integers \( (c_i) \) such that \( \alpha = \sum_i 2^{-c_i} \). By the Kraft-Chaitin theorem relative to \( \theta^{(n-1)} \), we can obtain a \( \theta^{(n-1)} \)-computable sequence of binary strings \( (\sigma_i) \) such that the set \( S := \{\sigma_i \mid i \in \mathbb{N}\} \) is a prefix-free set and \( |\sigma_i| = c_i \) for each \( i \). Therefore \( \mu(\|S\|) = \alpha \) and this concludes the proof. \( \square \)

Lemma 2.1 and a modified version of Lemma 2.3 will be used in the proofs of most of the results presented in Section 1.4.

### 2.2 Martin-Löf randomness

We use the formulation of algorithmic randomness in terms of effective statistical tests, as introduced by Martin-Löf in [ML66]. A Martin-Löf test is a uniformly c.e. sequence of \( \Sigma^0_1 \) classes \( (U_i) \) such that \( \mu(U_i) \leq 2^{-i} \) for each \( i \). More generally, a Martin-Löf test relative to an oracle \( X \) is a uniformly c.e. relative to \( X \) sequence of \( \Sigma^0_1(X) \) classes \( (U_i) \) such that \( \mu(U_i) \leq 2^{-i} \) for each \( i \). A real \( \alpha \) is Martin-Löf random (also known as \( 1 \)-random) if \( \alpha \notin \cap_i U_i \) for every Martin-Löf test \( (U_i) \). More generally, for each \( n \) we say that a real \( \alpha \) is \( (n+1) \)-random if it is Martin-Löf relative to \( \theta^{(n)} \), i.e. \( \alpha \notin \cap_i U_i \) for every Martin-Löf test \( (U_i) \) relative to \( \theta^{(n)} \). Martin-Löf [ML66] showed that there are universal Martin-Löf tests \( (U_i) \), in the sense that the reals that are not \( 1 \)-random are exactly the reals in \( \cap_i U_i \). A similar statement is true regarding \( n \)-randomness. It is well known, e.g. from [KS01], that the measure \( U_i \) of any member of a universal Martin-Löf test is...
a 1-random real. Similarly, the measure of any member of a universal Martin-Löf test relative to $\theta^n$ is $(n + 1)$-random.

Demuth [Dem75] showed that if $\alpha$ is a 1-random left-c.e. real and $\beta$ is another left-c.e. real then $\alpha + \beta$ is a 1-random left-c.e. real. Downey, Hirschfeldt, and Nies [DHN02] showed that conversely, if $\alpha, \beta$ are left-c.e. reals and $\alpha + \beta$ is $n$-random and is 1-random then at least one of $\alpha, \beta$ is 1-random. These results relativize to an arbitrary oracle. In particular, if $\alpha$ is $n$-random and a $0^{(n-1)}$-left-c.e. real and $\beta$ is another $0^{(n-1)}$-left-c.e. real then $\alpha + \beta$ is $n$-random and is a 0$^{(n-1)}$-left-c.e. real. Downey, Hirschfeldt, and Nies [DHN02] also showed that if $\alpha$ is a 1-random left-c.e. real and $\beta$ is any left-c.e. real then there exists some $c_0$ such that $\alpha - 2^{-c} \cdot \beta$ is a left-c.e. real for each $c > c_0$. This fact relativizes and also applies to right-c.e. reals. In particular, for each $n > 0$,

$$\begin{align*}
\text{if } \alpha, \beta \text{ are } 0^{(n-1)}\text{-left-c.e. reals and } \alpha \text{ is } n\text{-random then there exists some } c_0 \in \mathbb{N} \\
\text{such that } \alpha - 2^{-c} \cdot \beta \text{ is a } 0^{(n-1)}\text{-left-c.e. real for each } c > c_0.
\end{align*}$$

Moreover an analogous statement holds for $0^{(n-1)}$-right-c.e. reals. This result is heavily based on earlier work by Solovay [Sol75] and Calude, Hertling, Khoussainov, and Wang [CHKW01].

### 3 Oracle machines and monotone machines

In this section we prove Theorem 1.4 and the part of Theorem 1.5 which refers to monotone machines. The plan of the proof is as follows. We first show how to construct a machine of the desired type, whose totality probability is 2-random. Then we argue that because of this first result, universal machines of the given type have 2-random totality probabilities. The final step is to prove the converse, i.e. that given any 2-random $0'$-left-c.e. real number $\alpha \in (0, 1)$, we can find a universal machine of the given type, whose totality probability is $\alpha$. We note that the arguments involved in the three steps we just described are interconnected in an essential way. In particular, in order to prove the converses of Theorems 1.4 and 1.5 we need to use the lemmas established while showing the randomness of the totality probability.

#### 3.1 Randomness of totality and infinitude probabilities

We start with showing how to build an oracle machine $M$ such that the set of reals $X$ for which $n \mapsto M(X, n)$ is a total function is based on a given $\Sigma^0_2$ set of strings. Recall from Section 1.3 that if $M$ is an oracle machine then $M(X)$ refers to a (partial) function $n \mapsto M(X, n)$ from $\mathbb{N}$ to $\mathbb{N}$.

**Lemma 3.1 (From a $\Sigma^0_2$ set to an oracle machine).** Given any $\Sigma^0_2$ set $U$ of strings, there exists an oracle machine $M$ such that for each real $X$ the function $M(X)$ is total if and only if it has infinite domain if and only if $X$ does not have a prefix in $U$.

**Proof.** Let $(U_s)$ be a canonical $\Sigma^0_2$ approximation to $U$ as defined in Section 2.1. We define the oracle machine $M$ in stages as a partial computable function from $\mathbb{N}^{<\omega} \times \mathbb{N}$ to $\mathbb{N}$.

Let $M_0$ be the empty machine. At stage $s + 1$, for each string $\sigma$ of length at most $s$ such that $M_s(\sigma, |\sigma|)$ is not defined, check if $\sigma$ has a prefix in $U_{s+1}$. If it does not, then define $M_{s+1}(\sigma, |\sigma|) = 0$. This concludes the construction.
First, note that $M$ is a well defined oracle machine since given $\sigma < \tau$ and $n$ such that $M(\sigma, n)$, $M(\tau, n)$ are defined we have $M(\sigma, n) = M(\tau, n) = 0$. Moreover the oracle use is the identity function, regardless of the oracle. Suppose that $X$ does not have a prefix in $U$ and let $n$ be a number. Since $X \upharpoonright_n$ is not in $U$, there will be a stage $s + 1 > n$ such that $X \upharpoonright_n \notin U_{s+1}$. Hence by the construction we have $M(X \upharpoonright_n, n) = 0$. Hence if $X$ does not have a prefix in $U$ then $M(X)$ is total. Conversely, suppose that $X$ does have a prefix $X \upharpoonright_n$ in $U$. There there is a stage $s_0$ such that $X \upharpoonright_n \in U_{s+1}$ for all $s > s_0$. Hence by the construction, for all $m > s_0$ we have that $M(X \upharpoonright_m, m)$ is undefined. Again by the construction, this means that for all $m > s_0$ we have that the domain of $M(X)$ is finite, so $M(X)$ is not total. Lastly, if $M(X)$ is not total then $X$ has a prefix in $U$, the latter implies that $M(X)$ has finite domain, and this last clause implies that $M(X)$ is not total. This concludes the proof. 

Recall from Section 1.3 that a monotone machine $N$ is viewed as a (partial) function $\sigma \mapsto N(\sigma)$ from strings to strings with the monotonicity property that if $\sigma \leq \tau$, $N(\sigma) \downarrow$, and $N(\tau) \downarrow$, then $N(\sigma) \leq N(\tau)$. Moreover in this case $N(X)$ denotes the supremum of all $N(\sigma)$ where $\sigma$ is a prefix of $X$.

**Lemma 3.2** (From a $\Sigma_2^0$ set to an oracle machine). *Given a $\Sigma_2^0$ set $U$ of strings, there exists a monotone machine $N$ such that for all reals $X$ the output $N(X)$ is infinite if and only if $X$ does not have a prefix in $U$.*

**Proof.** Consider the machine $M$ of Lemma 3.1 and for each string $\sigma$ let $N(\sigma)$ be $0^{\lceil |\sigma| \rceil}$ if $M(\sigma, i) \downarrow$ for all $i < |\sigma|$ and undefined otherwise. Then clearly $N$ is a monotone machine. Moreover since the use function of the oracle machine $M$ is the identity, for each $X$ we have that $M(X)$ is a total function if and only if $N(X)$ is infinite. Then the desired properties of $N$ follow from the properties of $M$ according to Lemma 3.1. 

The next step is to establish the existence of machines with 2-random totality probability. For this we are going to apply Lemmas 3.1 and 3.2 and the fact from Sections 2.1 and 2.2 that the measure of any member of a universal Martin-Löf test relative to $0'$ is a 2-random $0'$-left-c.e. real.

**Lemma 3.3** (Machines with 2-random totality and infinitude probability). *There exist an oracle machine $M$ and a monotone machine $N$ such that the measures of $\text{TOT}(M)$, $\text{INF}(M)$ and $\text{INF}(N)$ are all 2-random $0'$-right-c.e. reals.*

**Proof.** Let $U$ be a member of a universal Martin-Löf test relative to $0'$. Consider the machine $M$ of Lemma 3.1 with respect to $U$. Since $\text{TOT}(M)$ is a $\Pi^0_2$ class, by Lemma 2.1 its measure is a $0'$-right-c.e. real. Moreover $\text{TOT}(M)$ is exactly the complement of the reals that have a prefix in $U$. The latter is 2-random, as $U$ is a member of a universal Martin-Löf test relative to $0'$. Hence $\text{TOT}(M)$ is a 2-random real. A similar application of Lemma 3.2 to the class $U$ shows that the constructed monotone machine $N$ is such that $\text{INF}(N)$ is a 2-random real. 

We are ready to prove the ‘only if’ direction of Theorem 1.4.

**Lemma 3.4** (Randomness of totality and infinitude probabilities for universal oracle machines). *If $U$ is a universal oracle machine then the measures of $\text{TOT}(U)$ and $\text{INF}(U)$ are both 2-random $0'$-right-c.e. reals.*

**Proof.** Consider the machine $M$ of Lemma 3.3, so that $\mu(\text{TOT}(M))$ is a 2-random $0'$-right-c.e. real. Since $U$ is universal, there exists a string $\tau$ such that for all strings $\sigma$ we have $M(\sigma) = U(\tau \ast \sigma)$. We have

$$\text{TOT}(U) = \tau \ast \text{TOT}(M) \cup (\text{TOT}(U) \cap (2^\omega \setminus [\|\tau]\|))$$

and

$$\tau \ast \text{TOT}(M) \cap (\text{TOT}(U) \cap (2^\omega \setminus [\|\tau]\|)) = \emptyset.$$
Let \( P = \text{TOT}(U) \cap (2^\omega - \mathbb{N}) \) and note that this is a \( \Pi^0_2 \) class, so \( \mu(P) \) is a \( \emptyset' \)-right-c.e. real by Lemma 2.1. So \( \mu(\text{TOT}(U)) = 2^{-|U|} \cdot \mu(\text{TOT}(M)) + \mu(P) \) is a 2-random \( \emptyset' \)-right-c.e. real, as it is the sum of a 2-random \( \emptyset' \)-right-c.e. real and another \( \emptyset' \)-right-c.e. real. A similar argument applies to \( \mu(\text{INF}(U)) \). □

A very similar argument provides the ‘only if’ direction of the part of Theorem 1.5 which refers to monotone machines.

**Lemma 3.5** (Randomness of infinitude probabilities for universal monotone machines). If \( U \) is a universal monotone machine then the measure of \( \text{INF}(U) \) is a 2-random \( \emptyset' \)-right-c.e. real.

**Proof.** We consider the monotone machine \( N \) of Lemma 3.2 and a universal monotone machine \( U \). Then the proof is the same as that of Lemma 3.4, with \( M \) replaced by \( N \) and \( \text{TOT} \) replaced by \( \text{INF} \). □

### 3.2 From random reals to totality and infinitude probabilities

In this section we prove the ‘if’ direction of Theorem 1.4 and the ‘if’ direction of the part of Theorem 1.5 which refers to monotone machines. In other words, given a 2-random \( \emptyset' \)-left-c.e. real \( \alpha \) we will build a universal machine \( U \) of the desired type such that the measure of \( \text{TOT}(U) \) or the measure of \( \text{INF}(U) \) is equal to \( \alpha \).

**Lemma 3.6** (Oracle machines). If \( \alpha \in (0, 1) \) is a \( \emptyset' \)-right-c.e. real and \( c \in \mathbb{N} \) is such that \( \alpha + 2^{-c} < 1 \) then there exists a machine \( M \) and a string \( \rho \) of length \( c \) such that \( M(\sigma, n) \) is undefined for any \( n \) and any string \( \sigma \) which is compatible with \( \rho \), and \( \mu(\text{TOT}(M)) = \mu(\text{INF}(M)) = \alpha \).

**Proof.** According to the hypothesis, \( 1 - \alpha - 2^{-c} \) is a \( \emptyset' \)-left-c.e. real in \( (0, 1) \), so there exists a \( \emptyset' \)-computable sequence \( (b_i) \) of positive integers such that \( 1 - \alpha - 2^{-c} = \sum_i 2^{-b_i} \). By the Kraft-Chaitin theorem relative to \( \emptyset' \) there exists a \( \emptyset' \)-computable sequence of strings \( (\sigma_i) \) such that \( |\sigma_0| = c \), \( |\sigma_{i+1}| = b_i \) for each \( i \) and \( S := \{ \sigma_i \mid i \in \mathbb{N} \} \) is a \( \Sigma^0_2 \) prefix-free set of strings. Define \( \rho := \sigma_0 \) and note that \( \mu([S]) = 1 - \alpha \). Then we can choose a canonical \( \Sigma^0_2 \) approximation \( (S_i) \) to \( S \) such that \( \rho \in S_i \) for all \( i \). Then we can apply the construction of Lemma 3.1 to \( S \) with this specific canonical \( \Sigma^0_2 \) approximation \( (S_i) \) and we obtain a machine \( M \) such that \( M(\sigma, n) \) is not defined for any string \( \sigma \) compatible (with respect to the prefix relation) with \( \rho \) and any \( n \). Then we get that

\[
\mu(\text{TOT}(M)) = \mu(\text{INF}(M)) = \mu(2^\omega - [S]) = 1 - \mu([S]) = \alpha,
\]

which concludes the proof. □

Lemma 3.6 has an analogue for monotone machines.

**Lemma 3.7** (Monotone machines). If \( \alpha \in (0, 1) \) is a \( \emptyset' \)-right-c.e. real and \( c \in \mathbb{N} \) is such that \( \alpha + 2^{-c} < 1 \) then there exists a monotone machine \( M \) and a string \( \rho \) of length \( c \) such that \( M(\sigma) \) is undefined for any string \( \sigma \) which is compatible with \( \rho \), and \( \mu(\text{INF}(M)) = \alpha \).

**Proof.** The proof is the same as the proof of Lemma 3.6 only that instead of obtaining the machine \( M \) from the construction of Lemma 3.1 we use Lemma 3.3, which gives a monotone machine. The rest of the parameters, including the \( \Sigma^0_2 \) set \( S \) and its canonical \( \Sigma^0_2 \) approximation \( (S_i) \), remain the same and as before

\[
\mu(\text{INF}(M)) = \mu(2^\omega - [S]) = 1 - \mu([S]) = \alpha,
\]
where $M$ is now the monotone machine from Lemma 3.3 based on $(S_j)$. This concludes the proof. □

We are ready to prove the remaining clauses of Theorems 1.4 and 1.5 which refer to oracle machines and monotone machines, as well as the statement immediately below Theorems 1.4 which refers to the infinity probability of a universal oracle machine.

**Lemma 3.8.** Let $\alpha \in (0, 1)$ be a 2-random $0'$-right-c.e. real. Then there exists a universal oracle machine $M$ such that $\mu(TOT(M)) = \alpha$. Similarly, there exists a universal oracle machine $U$ such that $\mu(INF(U)) = \alpha$. Finally there exists a monotone machine $T$ such that $\mu(INF(T)) = \alpha$.

**Proof.** Let $V$ be a universal oracle machine and let $\gamma$ be the measure of the reals $X$ such that $V(X)$ is total. By Lemma 2.1 the real $\gamma$ is a $0'$-right-c.e. real. By (2.2.1) and the discussion of Section 2.2 there exists $c \in \mathbb{N}$ such that $\alpha + 2^{-c} < 1$ and the real $\beta := \alpha - 2^{-c} \gamma$ is a $0'$-right-c.e. real. By Lemma 3.6 consider an oracle machine $N$ and a string $\rho$ of length $c$ such that the measure of $\mu(TOT(N))$ is $\beta$ and $N(\sigma, n)$ is not defined for any $\sigma$ which is compatible with $\rho$ and any $n$. Define an oracle machine $M$ as follows. For each string $\sigma$ which is incompatible with $\rho$ and any $n$ let $M(\sigma, n) = N(\sigma, n)$. Moreover for each $\tau$ and any $n$ let $M(\rho * \tau, n) \approx V(\tau, n)$. Since $V$ is universal, it follows that $M$ is also a universal oracle machine. Moreover
\[
TOT(M) = \rho * TOT(V) \cup TOT(N) \quad \text{and} \quad \rho * TOT(V) \cap TOT(N) = \emptyset
\]
and so
\[
\mu(TOT(M)) = 2^{-|\rho|} \cdot \mu(TOT(V)) + \mu(TOT(N)) = 2^{-c} \cdot \gamma + \beta = \alpha,
\]
which concludes the proof of the first clause. The argument for the second clause is entirely similar. We let $\delta$ be the measure of the reals $X$ such that the domain of $V(X)$ is infinite and choose $c$ such that such that $\alpha + 2^{-c} < 1$ and the real $\zeta := \alpha - 2^{-c} \delta$ is a $0'$-right-c.e. real. Then by Lemma 3.6 we can choose an oracle machine $F$ and a string $\eta$ of length $c$ such that the measure of $\mu(INF(F))$ is $\beta$ and $F(\sigma, n)$ is not defined for any $\sigma$ which is compatible with $\eta$ and any $n$. Then we define an oracle machine $U$ as follows. For each string $\sigma$ which is incompatible with $\rho$ and any $n$ let $U(\sigma, n) \approx F(\sigma, n)$. Moreover for each $\tau$ and any $n$ let $U(\eta * \tau, n) \approx V(\tau, n)$. Since $V$ is universal, it follows that $U$ is also a universal oracle machine. Moreover as before
\[
\mu(INF(U)) = 2^{-|\rho|} \cdot \mu(INF(V)) + \mu(INF(F)) = 2^{-c} \cdot \delta + \zeta = \alpha,
\]
which concludes the proof of the second clause of the lemma. The proof of the third clause is entirely analogous with the previous two, with the only difference that we use Lemma 3.7 instead of Lemma 3.6. □

### 3.3 Randomness of cofiniteness and computability probabilities

In this section we prove the ‘only if’ direction of Theorem 1.8. The methodology is similar to the one we employed in Section 3.1, but the arguments will be more involved since we are dealing with $\Sigma^0_3$ properties, namely cofiniteness and computability. The first step is coding an arbitrary $\Sigma^0_3$ class into $COF(M)$ and $COP(M)$ for some oracle machine $M$. The proof of the following lemma draws considerably from the proof of the well-known fact that the index set of the cofinite c.e. sets and the index set of the computable c.e. sets are both $\Sigma^0_3$-complete.

**Lemma 3.9** (from a $\Sigma^0_3$ set to an oracle machine for cofiniteness). Given any upward closed $\Sigma^0_3$ set of strings $J$ there exists an oracle machine $M$ such that the following are equivalent for each $X$:
Note that for each $\sigma \in M$, define the oracle machine $A$. At stage 0 we let $m = 0$, and once a term is infinite, all later terms will also be infinite. Let $\langle m, \sigma \rangle$.

For each binary string $\sigma$, such that for each $\sigma$, $\exists M \exists H_0(i, n, s, \sigma)$. Consider the relation $\exists i \leq t \forall n \exists s H_0(i, n, s, \sigma)$. This is a $\Pi^0_2$ relation since bounded quantifiers do not increase arithmetical complexity. So there exists a computable predicate $H_1$ such that $\exists i \leq t \forall n \exists s H_0(t, n, s, \sigma)$ is equivalent to $\forall n \exists s H_1(t, n, s, \sigma)$. Moreover for each $q < p$, if $\forall n \exists s H_1(q, n, s, \sigma)$ then $\forall n \exists s H_1(p, n, s, \sigma)$.

We claim that there exists a computable predicate $H$ such that for each $\sigma, t$,$$\forall n \exists s H(t, n, s, \sigma) \iff \bigvee_{i=0}^{\lfloor \sigma \rfloor} \forall n \exists s H_1(t, n, s, \sigma_{\uparrow i}).$$

This holds because arithmetical complexity above the first level of the arithmetical hierarchy is invariant under bounded disjunctions. By the choice of $H_0, H_1$ and the fact that $J$ is upward closed, the predicate $H$ has the following properties:

(a) for each $q < p$ and each $\sigma$, if $\forall n \exists s H(q, n, s, \sigma)$ then $\forall n \exists s H(p, n, s, \sigma)$;

(b) for each $q$ and each $\sigma < \tau$, if $\forall n \exists s H(q, n, s, \sigma)$ then $\forall n \exists s H(q, n, s, \tau)$;

(c) $\sigma \in J$ if and only if $\exists t \forall n \exists s H(t, n, s, \sigma)$.

Let $(W_e)$ be a universal enumeration of all c.e. sets. Then there exist a computable function $g$ such that

1. $\forall n \exists s H(t, n, s, \sigma) \iff |W_{g(t, \sigma)}| = \infty$; and

2. if $\sigma \leq \tau$, $t \leq k$, and $|W_{g(t, \sigma)}| = \infty$, then $|W_{g(k, \tau)}| = \infty$ for all $t, k, \sigma, \tau$. The second clause above says that the double sequence $(W_{g(t, \sigma)})$ is monotone, in the sense that once a term is infinite, all later terms will also be infinite. Let $(\sigma, n) \mapsto (\sigma, n)$ be a computable bijection from ordered pairs of strings and numbers $(\sigma, n)$ onto $\mathbb{N}$. Moreover define $\mathbb{N}^{[\sigma]} = \{ (\sigma, n) | n \in \mathbb{N} \}$ and let $(0')$ be a computable enumeration of the halting problem $0'$.

For each binary string $\sigma$, we define a movable marker $m(\sigma)[s]$ dynamically to take values in $\mathbb{N}^{[\sigma]}$. In the case where $m(\sigma)[s]$ reaches a limit as $s \to \infty$, that limit is denoted by $m(\sigma)$.

At stage 0 we let $m(\sigma)[0] = (\sigma, 0)$ for all $\sigma$. At stage $s + 1$,

- if $|W_{g(\sigma, \sigma)}| > |W_{g(\sigma, \sigma)}[s]|$ then let $m(\sigma)[s + 1] = (\sigma, s + 1)$;
- if $|\sigma| \in 0'_{s+1} - 0'_s$, let $m(\sigma)[s + 1] = (\sigma, s + 1)$;
- otherwise, let $m(\sigma)[s + 1] = m(\sigma)[s]$.

Note that for each $\sigma$, the marker $m(\sigma)[s]$ reaches a limit if and only if $W_{g(\sigma, \sigma)}$ is infinite. We are ready to define the oracle machine $M$ by induction on the stages $s$, based on the function $g$.

Given any stage $s + 1$, any $\sigma$ of length at most $s + 1$ and any $n \in \mathbb{N}^{[\sigma]}$ such that $n < s + 1$ we define $M(\sigma, n)[s + 1] = n$ for all $n < s + 1$ such that $n \neq m(\sigma)[s + 1]$, and leave $M(\sigma, m(\sigma))[s + 1]$ undefined.
By the definition of $M$ we have:

- if $m(\sigma)$ reaches a limit then $M(\sigma, m(\sigma))$ is undefined and $M(\sigma, i) \downarrow$ for all $i \in \mathbb{N}^{[\sigma]} - \{m(\sigma)\}$;
- if $m(\sigma) \to \infty$ then $M(\sigma, i) \downarrow$ for all $i \in \mathbb{N}^{[\sigma]}$.

Let $X$ be a real. If $X$ has a prefix in $J$ then by the properties of $H$ and $g$, the marker $m(X \uparrow n)$ diverges for any sufficiently large $n$. If, on the other hand, $X$ does not have a prefix in $J$, then for the same reasons $m(X \uparrow n)$ reaches a limit for all $n$. Hence by the construction of $M$ and its properties, as discussed above, we have

- if $X$ has a prefix in $J$, then the domain of $M(X)$ is cofinite;
- otherwise, the domain of $M(X)$ is coinfinite.

Moreover, by construction, the domain of $M(X)$ is equal to its range. In addition, for each real $X$, if the domain of $M(X)$ is not cofinite, then $m(X \uparrow n)$ reaches a limit for each $n$, so the domain of $M(X)$ computes the halting problem $0'$. This is because the final position of $m(X \uparrow n)$ corresponds to the $n$th zero in the characteristic sequence of the domain of $M(X)$. Hence the settling time of $0'(n)$ is bounded above by the settling time of $m(X \uparrow n)$, which can be calculated by the domain of $M(X)$. Hence if $X$ does not have a prefix in $J$, then the domain of $M(X)$ computes the halting problem, so it is not computable. If, on the other hand, $X$ does have a prefix in $J$, then the domain of $M(X)$ is cofinite, hence computable. \qed

The above result allows us to construct a machine whose cofiniteness and computability probability are both 3-random. This will later be used in order to show this property for any universal machine.

**Lemma 3.10** (Machines with 3-random cofiniteness and computability probabilities). *There exists an oracle machine $M$ such that $\text{COF}(M)$ and $\text{COM}(M)$ are equal and have measure a 3-random $0^{(2)}$-left-c.e. real.*

**Proof.** Let $U$ be a member of a universal Martin-Löf test relative to $0^{(2)}$ and consider the machine $M$ of Lemma 3.9 with respect to $U$. Then $\text{COF}(M) = \text{COM}(M)$ and since this is a $\Sigma^0_3$ class, by Lemma 2.1 its measure is a $0^{(2)}$-left-c.e. real. Moreover, by Lemma 3.9 the measure of $\text{COF}(M)$ is equal to the measure of $U$, which is 3-random, as $U$ is a member of a universal Martin-Löf test relative to $0^{(2)}$. \qed

Finally we are ready to prove the ‘only if’ direction of Theorem 1.8.

**Lemma 3.11** (Randomness of cofiniteness and computability probabilities for universal machines). *If $U$ is a universal oracle machine then the measures of $\text{COF}(U)$ and $\text{COM}(U)$ are both 3-random $0^{(2)}$-left-c.e. reals.*

**Proof.** Consider the machine $M$ of Lemma 3.10, so that $\text{COF}(M) = \mu(\text{COM}(M))$ and $\mu(\text{COF}(M))$ is a 3-random $0^{(2)}$-left-c.e. real. Since $U$ is universal, there exists a string $\tau$ such that for all strings $\sigma$ we have $M(\sigma) \approx U(\tau \ast \sigma)$. We have

$$\text{COF}(U) = \tau \ast \text{COF}(M) \cup \left(\text{COF}(U) \cap (2^{\omega} - \|\tau\|)\right) \quad \text{and} \quad \tau \ast \text{COF}(M) \cap \left(\text{COF}(U) \cap (2^{\omega} - \|\tau\|)\right) = \emptyset$$

Let $P = \text{COF}(U) \cap (2^{\omega} - \|\tau\|)$ and note that this is a $\Sigma^0_3$ class, so $\mu(P)$ is a $0^{(2)}$-left-c.e. real by Lemma 2.1. So $\mu(\text{COF}(U)) = 2^{-|\tau|} \cdot \mu(\text{COF}(M)) + \mu(P)$ is a 3-random $0^{(2)}$-left-c.e. real as the sum of a 3-random $0^{(2)}$-left-c.e. real and another $0^{(2)}$-left-c.e. real. \qed
3.4 From random reals to cofiniteness and computability probabilities

In this section we prove the ‘if’ direction of Theorem 1.8. We first show how to obtain a machine with prescribed cofiniteness probability.

Lemma 3.12 (Prescribed cofiniteness probability). If \( \alpha \in (0, 1) \) is a \( 0^{(2)} \)-left-c.e. real and \( c \in \mathbb{N} \) is such that \( \alpha + 2^{-c} < 1 \) then there exists an oracle machine \( M \) and a string \( \rho \) of length \( c \) such that \( M(\sigma, n) \) is undefined for any \( n \) and any string \( \sigma \) which is compatible with \( \rho \), and \( \mu(\text{COF}(M)) = \alpha \).

Proof. Let \( (b_i) \) be a \( 0^{(2)} \)-computable sequence of positive integers such that \( \alpha = \sum_i 2^{-b_i} \). By the Kraft-Chaitin theorem relative to \( 0^{(2)} \) and since \( \alpha + 2^{-c} < 1 \), there exists a \( 0^{(2)} \)-computable sequence of strings \( (\sigma_i) \) such that \( |\sigma_0| = c, |\sigma_{i+1}| = b_i \) for each \( i \) and \( S := \{\sigma_{i+1} \mid i \in \mathbb{N}\} \) is a \( \Sigma^0_3 \) prefix-free set of strings. Define \( \rho := \sigma_0 \) and note that \( \mu(\langle S \rangle) = \alpha \). Let \( J \) be the upward closure of \( S \) and construct an oracle machine \( M \) as in the proof of Lemma 3.9 based on the \( \Sigma^0_3 \) upward closed set of strings \( J \), with the additional restriction that \( M(\sigma, n) \) is not defined for any \( \sigma \) which is compatible with \( \rho \) and any \( n \). Then by the same arguments, all of the properties of \( M(X) \) that are listed in the proof of Lemma 3.9 hold as long as \( X \) is not prefixed by \( \rho \). If \( \rho \) is a prefix of \( X \), then \( M(X) \) is the empty function, so its domain is cofinite. Hence

\[
\mu(\text{COF}(M)) = \mu(\text{COF}(M) \cap (2^\omega - \langle \rho \rangle)) + \mu(\text{COF}(M) \cap \langle \rho \rangle) = \mu(\langle S \rangle) + 0 = \alpha
\]

which concludes the proof.

A minor modification of the above argument gives a similar result regarding the computability probability.

Lemma 3.13 (Prescribed computability probability). If \( \alpha \in (0, 1) \) is a \( 0^{(2)} \)-left-c.e. real and \( c \in \mathbb{N} \) is such that \( 2^{-c} < \alpha \) then there exists an oracle machine \( M \) and a string \( \rho \) of length \( c \) such that \( M(\sigma, n) \) is undefined for any \( n \) and any string \( \sigma \) which is compatible with \( \rho \), and \( \mu(\text{COM}(M)) = \alpha \).

Proof. Let \( (b_i) \) be a \( 0^{(2)} \)-computable sequence of positive integers such that \( \alpha - 2^{-c} = \sum_i 2^{-b_i} \). By the Kraft-Chaitin theorem relative to \( 0^{(2)} \), there exists a \( 0^{(2)} \)-computable sequence of strings \( (\sigma_i) \) such that \( |\sigma_0| = c, |\sigma_{i+1}| = b_i \) for each \( i \) and \( S := \{\sigma_{i+1} \mid i \in \mathbb{N}\} \) is a \( \Sigma^0_3 \) prefix-free set of strings. Define \( \rho := \sigma_0 \) and note that \( \mu(\langle S \rangle) = \alpha - 2^{-c} \). Let \( J \) be the upward closure of \( S \) and construct an oracle machine \( M \) as in the proof of Lemma 3.9 based on the \( \Sigma^0_3 \) upward closed set of strings \( J \), with the additional restriction that \( M(\sigma, n) \) is not defined for any \( \sigma \) which is compatible with \( \rho \) and any \( n \). Then by the same arguments, all of the properties of \( M(X) \) that are listed in the proof of Lemma 3.9 hold as long as \( X \) is not prefixed by \( \rho \). If \( \rho \) is a prefix of \( X \), then \( M(X) \) is the empty function, so its domain is computable. Hence

\[
\mu(\text{COM}(M)) = \mu(\text{COM}(M) \cap (2^\omega - \langle \rho \rangle)) + \mu(\text{COM}(M) \cap \langle \rho \rangle) = \mu(\langle S \rangle) + 2^{-c} = \alpha
\]

which concludes the proof.

We are now ready to prove the ‘if’ direction of Theorem 1.8. We use Lemma 3.12 and Lemma 3.13 in order to produce universal oracle machines with prescribed probabilities.

Lemma 3.14. Let \( \alpha \in (0, 1) \) be a \( 3 \)-random \( 0^{(2)} \)-left-c.e. real. Then there exist universal oracle machines \( M \) and \( N \) such that \( \mu(\text{COF}(M)) = \alpha = \mu(\text{COM}(N)) \).
Proof. Let $V$ be a universal oracle machine and let $\gamma = \mu(\text{COF}(V))$. By Lemma 2.1 the real $\gamma$ is a $\Theta^{(2)}$-left-c.e. real. By (2.2.1) and the discussion of Section 2.2 there exists $c \in \mathbb{N}$ such that $\alpha + 2^{-c} < 1$ and the real $\beta := \alpha - 2^{-c} \gamma$ is a $\Theta^{(2)}$-left-c.e. real. By Lemma 3.12 consider an oracle machine $F$ and a string $\rho$ of length $c$ such that the measure of $\text{COF}(F)$ is $\beta$ and $F(\sigma, n)$ is not defined for any $\sigma$ which is compatible with $\rho$ and any $n$. Define an oracle machine $M$ as follows. For each string $\sigma$ which is incompatible with $\rho$ and any $n$ let $M(\sigma, n) \equiv F(\sigma, n)$. Moreover for each $\tau$ and any $n$ let $M(\rho * \tau, n) \equiv V(\tau, n)$. Since $V$ is universal, it follows that $M$ is also a universal oracle machine. Moreover,

$$\text{COF}(M) = \rho * \text{COF}(V) \cup \text{COF}(F) \quad \text{and} \quad \rho * \text{COF}(V) \cap \text{COF}(F) = \emptyset,$$

and so

$$\mu(\text{COF}(M)) = 2^{-|\rho|} \cdot \mu(\text{COF}(V)) + \mu(\text{COF}(F)) = 2^{-c} \cdot \gamma + \beta = \alpha$$

which concludes the proof of the first equality.

For the second equality, we let $\delta$ be the measure of the reals $X$ such that the domain of $V(X)$ is computable and choose $c$ such that such that $\alpha + 2^{-c} < 1$ and the real $\alpha - 2^{-c} \delta$ is a $\Theta^*$-right-c.e. real. Now let $\zeta = \alpha - 2^{-c} \delta + 2^{-c}$. Then by Lemma 3.13, we can choose an oracle machine $G$ and a string $\eta$ of length $c$ such that the measure of $\text{COM}(G)$ is $\beta$ and $G(\eta, n)$ is not defined for any $\sigma$ which is compatible with $\eta$ and any $n$. Then we define an oracle machine $N$ as follows. For each string $\sigma$ which is incompatible with $\eta$ and any $n$ let $U(\sigma, n) \equiv G(\sigma, n)$. Moreover for each $\tau$ and any $n$ let $U(\eta * \tau, n) \equiv V(\tau, n)$. Since $V$ is universal, it follows that $U$ is also a universal oracle machine. Moreover

$$\mu(\text{COM}(N)) = 2^{-|\eta|} \cdot \mu(\text{COM}(V)) + \mu(\text{COM}(G)) - 2^{-c} = 2^{-c} \cdot \delta + \zeta - 2^{-c} = \alpha.$$

which concludes the proof of the second equality. \qed

### 3.5 Outline of the proof of Theorem 1.9

A classic fact from computability theory says that the property that $W_e$ (the $e$th c.e. set) is $\Sigma^0_4$-complete. Theorem 1.9 is a version of this fact in terms of measures of oracle Turing machines.

**Theorem 1.9.** The probability that a universal oracle machine computes a function whose domain computes the halting problem is a 4-random $\Theta^{(3)}$-left-c.e. real. Conversely, every 4-random $\Theta^{(3)}$-left-c.e. real in $(0, 1)$ is the probability that a certain universal oracle machine computes a function whose domain computes the halting problem.

At this point we have given enough arguments in order to illustrate the methodology of obtaining characterizations of probabilities of universal oracle machines in terms of algorithmic randomness. For this reason, we give a mere outline of the proof of Theorem 1.9, which is entirely along the lines of the proof of Theorem 1.5 and Theorem 1.8 which were presented in excruciating detail. The crucial ingredient is the following lemma, which allows the construction of oracle machines with suitably prescribed probability of computing the halting problem.

**Lemma 3.15 (Domain or range computing the halting problem).** Given a $\Sigma^0_4$ upward closed set of strings, there exists an oracle machine $M$ such that for each $X$ the domain (or range) of $M(X)$ computes the halting problem if and only if $X$ has a prefix in $J$. 

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The proof of 3.15 is based on the classic argument showing that the index set of the complete c.e. sets is $\Sigma^0_4$-complete (e.g. see [Soa87, Corollary XII 1.7]). Given this result and using a member of a universal Martin-Löf test relative to $0^{(3)}$ as a $\Sigma^0_4$ set of strings $J$, we can show that the completeness probability of a universal oracle machine is a 4-random $0^{(3)}$-left-c.e. real. For the converse one follows faithfully the structure of the argument that we developed in Section 3.4. We use the relativized Kraft-Chaitin method along with (2.2.1) and the discussion of Section 2.2 in order to construct a universal oracle machine whose probability of computing a function with domain computing the halting problem equals a given 4-random $0^{(3)}$-left-c.e. real.

3.6 Probabilities for oracle and monotone machines that are not so random

In this section we show that the closely related cofiniteness probability considered in Proposition 1.7 is not as random as one might expect. Given an oracle machine $M$, consider the class of reals such that $M(X)$ is a total function which is the characteristic sequence of a cofinite set. Then this is the intersection of a $\Pi^0_2$ class, imposing totality, and a $\Sigma^0_2$ class. Indeed, this class is equal to

$$\{X \mid \forall n \exists s M(X, n)[s] \downarrow \} \cap \{X \mid \exists t \forall s \forall n > t (M(X, n)[s] \uparrow \lor M(X, n)[s] \downarrow = 1)\}$$

where clearly the first is a $\Pi^0_2$ class and the second is a $\Sigma^0_2$ class. Therefore the measure of this class of reals is the difference of two $0'$-left-c.e. reals. Rettinger (see [ZR04, RZ05] or [DH10, Theorem 9.2.4]) proved that such a real is either $0'$-left-c.e. or $0'$-right-c.e. or it is not 2-random. In any of these cases, the probability that $M(X)$ is a total computable function is not a 3-random real.

The same argument holds for the case of monotone machines where we look at the outcome that the output is a computable stream or a stream with a tail of 1s. We have thus proved Proposition 1.7.

4 Probabilities of infinitary self-delimiting machines

In this section we prove all the results that refer to infinitary self-delimiting machines. We devote Section 4.1 to defining and discussing this special machine model. We emphasize that although the general methodology is the same as the one we developed in Section 3, the arguments are special to the rather different nature of the model of infinitary self-delimiting machines.

4.1 Infinitary self-delimiting machines

We define Chaitin’s infinitary self-delimiting machines which were originally introduced in Chaitin [Cha76] and later studied in [BG05, BFGM06, BG07, BG09] and [BFNP05]. This model is based on a partial computable function $M : 2^{<\omega} \times \mathbb{N} \rightarrow 2^{<\omega}$ with certain properties, which can be used to define a self-delimiting infinitary model $M^*$, which in turn can be seen as a prefix-free infinitary model $M^*$.

The properties we require for the partial computable function $M$ are:

(a) if $M(\sigma, m) \downarrow$ and $n < m$, then $M(\sigma, n) \downarrow$ and $M(\sigma, n) \leq M(\sigma, m)$;

(b) if $M(\sigma, n) \downarrow$, then for all strings $\tau$, $M(\sigma \ast \tau, n) \downarrow$ and $M(\sigma \ast \tau, n) = M(\sigma, n)$;
(c) the relation $M(\sigma, n) \downarrow$ is decidable.

These conditions may seem non-standard from the point of view of classical computability theory. However they can be understood if one considers the following definition of infinitary self-delimiting computations that they facilitate.

**Definition 4.1** (Infinitary self-delimiting machines). If $M : 2^{<\omega} \times \mathbb{N} \to 2^{<\omega}$ is a partial computable function which satisfies conditions (a)-(c) above, define the infinitary machine $M^\omega$ as follows:

(i) $M^\omega(\sigma) \downarrow$ if $M(\sigma, n) \downarrow$ for all $n$;

(ii) If $M^\omega(\sigma) \downarrow$, then $M^\omega(\sigma)$ is the supremum of all $M(\sigma, n), n \in \mathbb{N}$

where $\sigma$ is a binary string and $n \in \mathbb{N}$. Note that $M^\omega(\sigma)$ could be a string or a stream.

We can now see how the properties (a)-(c) on $M$ above give a self-delimiting quality to $M^\omega$. Indeed, the properties on $M$ imply that

$$M^\omega(\sigma) \downarrow \text{ then for any } \tau \text{ we have } M^\omega(\sigma \ast \tau) \downarrow \text{ and the two outputs are equal.}$$

Note that if $M^\omega(\sigma) \downarrow$ and it is finite, i.e. a string, then $M^\omega(\sigma \ast \tau)$ is not allowed to be a proper extension of $M^\omega(\sigma) \downarrow$. This is one of the main differences with the standard oracle Turing machine model and the monotone machines. If $M^\omega(\sigma) \downarrow$ then the values of $M^\omega(\tau)$ for any extension $\tau$ of $\sigma$ are completely determined by $M^\omega(\sigma)$. The other difference is that the relation $M^\omega(\sigma) \downarrow$ is $\Pi^0_1$ and in general not $\Sigma^0_1$ as we might be used to in classical computability theory. The domain of $M^\omega$ is denoted by $\text{DOM}(M^\omega)$ and consists of the strings $\sigma$ such that $M^\omega(\sigma) \downarrow$.

In the same way that we view self-delimiting machines as prefix-free machines, we can do the same with these infinitary self-delimiting machines. Given $M$ and $M^\omega$ which is defined in terms of $M$, we can define the relation $M^\omega(\sigma) \downarrow$ to mean that $\sigma$ is a minimal string such that $M^\omega(\sigma) \downarrow$ (i.e. $M^\omega(\sigma) \downarrow$ and $M^\omega(\tau) \uparrow$ for all proper prefixes of $\sigma$). Moreover if $M^\omega(\sigma) \downarrow$ then let $M^\omega(\sigma) := M^\omega(\sigma) \downarrow$. Then the domain of $M^\omega$, denoted by $\text{DOM}(M^\omega)$ consists of the strings $\sigma$ such that $M^\omega(\sigma) \downarrow$ and is a prefix-free set of strings. In this way, $M^\omega$ may be regarded as a prefix-free machine of a higher type. Formally, $M^* : 2^{<\omega} \to 2^{<\omega}$ and

$$M^*(\sigma) = \begin{cases} M^\omega(\sigma), & \text{if for every } \tau \in 2^{<\omega}, M^\omega(\sigma) \downarrow = M^\omega(\sigma \ast \tau) \downarrow, \\ \uparrow, & \text{otherwise.} \end{cases}$$

Note that $\text{DOM}(M^\omega)$ is a $\Pi^0_1$ set of strings. On the other hand, $\text{DOM}(M^\omega)$ is merely $\Delta^0_1$ since to decide membership in $\text{DOM}(M^\omega)$ one has to ask a finite number of $\Pi^0_1$ questions, namely a finite number of questions about membership in $\text{DOM}(M^\omega)$. Note further that $\text{DOM}(M^\omega) \subseteq \text{DOM}(M^\omega)$ and the corresponding $\Sigma^0_1$ classes are equal, namely $\|\text{DOM}(M^\omega)\| = \|\text{DOM}(M^\omega)\|$, so that

$$\mu(\text{DOM}(M^\omega)) = \mu(\text{DOM}(M^\omega)) = \sum_{\sigma \in \text{DOM}(M^\omega)} 2^{-k_\sigma}.$$ 

The transition from $M^\omega$ to $M^\omega$ is trivial, so we will mainly work with $M^\omega$ in the following. However, the results we present concerning $M^\omega$ also apply to $M^\omega$. Universality for $M^\omega$ is defined as in most machine models. Note that there is an effective list of all partial computable functions $M : 2^{<\omega} \times \mathbb{N} \to 2^{<\omega}$ with the properties (a)-(c) above. Such a list induces an effective list $(M^\omega_\rho)$ of all infinitary self-delimiting machines. The relation $\equiv$ denotes the fact that either the expressions on either side of it are undefined (or do not halt) or both of these expressions are defined and are equal.
Definition 4.2 (Universal infinitary self-delimiting machines). Given an effective list \((M^\omega)\) of all infinitary self-delimiting machines, an infinitary self-delimiting machine \(U^\omega\) is universal if there exists a computable function \(e \mapsto \sigma_e\) such that \(U^\omega(\sigma_e \ast \tau) \approx M^\omega_e(\tau)\) for all \(e, \tau\).

Theorem 1.5 concerns the question of whether the outcome of a universal infinitary self-delimiting machine that runs on a random input is finite or infinite. We conclude the present section with an analysis of these outcomes from a complexity point of view. This discussion will be the basis for the arguments of Section 4. Given an infinitary self-delimiting machine \(M^\omega\) we define:

- \(\mathsf{INF}(M^\omega)\) is the set of strings in \(\mathsf{DOM}(M^\omega)\) such that \(M^\omega(\sigma)\) is a stream;
- \(\mathsf{FIN}(M^\omega) = \mathsf{DOM}(M^\omega) - \mathsf{INF}(M^\omega)\).

We note that \(\mathsf{INF}(M^\omega)\) is a \(\Pi^0_2\) set because to determine membership in \(\mathsf{INF}(M^\omega)\) it is enough to first decide membership \(\mathsf{DOM}(M^\omega)\) and then ask a \(\Pi^0_2\) question about whether the length of the output is infinite or not. Similarly, \(\mathsf{FIN}(M^\omega)\) is a \(\Sigma^0_2\) set. Moreover, by standard manipulations of quantifiers we have that \(\mathsf{INF}(M)\) is a \(\Pi^0_2\) set and \(\mathsf{FIN}(M)\) is a \(\Sigma^0_2\) set. The reader can easily verify that these complexity bounds are the best possible in general and exhibit machines such that these sets are complete for the arithmetical classes that they belong to. Similar remarks can be made about the classes of reals represented by these sets of strings (recall the notation \([S]\) for a set of strings \(S\) from Section 2.1). For example, \([\mathsf{DOM}(M^\omega)]\) is not only an open set but also a \(\Sigma^0_2(\emptyset')\) class, and the same is true of \([\mathsf{FIN}(M^\omega)]\), which of course equals \([\mathsf{FIN}(M)]\). Moreover \([\mathsf{INF}(M^\omega)]\) is also an open set but merely a \(\Sigma^0_1(\emptyset'')\) class since membership in \(\mathsf{INF}(M^\omega)\) requires oracle \(\emptyset''\). Also note that \([\mathsf{INF}(M^\omega)]\) is the difference of two \(\Sigma^0_2(\emptyset')\) classes, namely \([\mathsf{DOM}(M^\omega)] - [\mathsf{FIN}(M^\omega)]\). Similar remarks apply to the measures of these sets. For example, the measures of \(\mathsf{DOM}(M^\omega)\), \(\mathsf{FIN}(M)\) are \(\emptyset'\)-left-c.e. reals while the measure of \(\mathsf{INF}(M^\omega)\) is the difference of two \(\emptyset'\)-left-c.e. reals. Let \(\mathsf{COF}(M^\omega)\) consist of the strings in \(\mathsf{DOM}(M^\omega)\) such that \(M^\omega(\sigma)\) is equal to \(\tau \ast 1^\omega\) for some string \(\tau\).

4.2 Randomness of finiteness probability

In this section we prove the part of Theorem 1.5 which refers to infinitary self-delimiting machines. We emphasize that this is the content of Becher, Daicz, and Chaitin [BDC01]. Our proof here serves one main purpose: it facilitates the proof of the converse, which is one of our results regarding infinitary self-delimiting machines. In addition, our proof is different and more concise than in [BDC01], being based on concepts of definability rather than initial segment complexity.

Due to the nature of this model, we are not able to prove Lemma 3.1 for infinitary self-delimiting machines. Instead, we show a slightly weaker version, namely that given any upward closed \(\Sigma^0_2\) set \(U\) of strings we can define a machine \(M^\omega\) such that for each string \(\sigma\) we have \(M^\omega(\sigma) \downarrow\) if and only if \(\hat{\sigma}\) has a (finite) covering consisting of strings in \(U\). In terms of classes of reals, this means that \(M^\omega(\sigma) \downarrow\) if and only if \(\sigma \in [U]\). In the following we let \(\mathsf{EXT}(\sigma, n)\) denote the set of extensions of string \(\sigma\) of length \(n\).

Lemma 4.3. Given an upward closed \(\Sigma^0_2\) set \(U\) of strings there exists a machine \(M\) which never prints anything on its tape and such that, for all \(\sigma\),

(i) \(\sigma \in U \Rightarrow \exists n \geq |\sigma| \forall \tau \in \mathsf{EXT}(\sigma, n), \, \tau \in \mathsf{DOM}(M^\omega)\)

(ii) \(\sigma \in \mathsf{DOM}(M^\omega) \Rightarrow \sigma \in U\).
Proof. Let \((U_s)\) be a canonical \(\Sigma^0_2\) approximation of \(U\), as defined in Section 2.1. We define \((\sigma, n) \mapsto M(\sigma, n)\) computably in stages. Let \(M(\sigma, 0)\) be defined for all \(\sigma\) and equal to the empty string. At stage \(s + 1\) we first define \(M(\sigma, s + 1)\) for all \(\sigma\) of length at most \(s\) and then define \(M(\tau, s + 1)\) for all \(\tau\) of length \(s + 1\). Therefore by the end of stage \(s\), we have determined exactly all \(M(\tau, t)\) for all \(\tau\) of length at most \(s\) and all \(t \leq s\). Let \(s \in \mathbb{N}\) and suppose inductively that the first \(s\) stages of the construction have been completed. At stage \(s + 1\), and each string \(\sigma\) of length at most \(s\) we first check if \(M(\sigma, s) \uparrow\) and in that case we let \(M(\sigma, s + 1) \uparrow\). Otherwise we check if there exists a prefix of \(\sigma\) in \(U_{s+1}\). In this case we let \(M(\sigma, s + 1) \downarrow\) and equal to the empty string. Otherwise we let \(M(\sigma, s) \uparrow\). For each string \(\tau\) of length \(s + 1\) we perform a similar check. We check if \(\tau\) has a prefix in \(U_{s+1}\), and in that case we define \(M(\tau, t) \downarrow = \lambda\) for each \(t \leq s + 1\), where \(\lambda\) denotes the empty string. Otherwise we let \(M(\tau, t) \uparrow\) for all \(t \leq s + 1\). This completes stages \(s + 1\) and the inductive definition of \(M\).

By a straightforward induction on the steps of the construction we have

1. \(M\) has the properties (a)-(c) of Section 4.1 and it never prints anything on the output tape;
2. for each \(\tau, s\) we have \(M(\tau, s) \downarrow\) if and only if for each \(|\tau| \leq t \leq s\) there exists \(\rho \leq t\) such that \(\rho \in U_t\).

We can now use property (2) in order to establish property (i) of the lemma. Suppose that \(\sigma \in U\). Then there exists a least stage \(s\) such that for all \(t \geq s\) we have \(\sigma \in U_t\). Then by property (2), for all strings \(\tau \in \text{EXT}(\sigma, s)\) we have \(M(\tau, n) \downarrow\) for all \(n \in \mathbb{N}\). In other words, if \(\sigma \in U\) then there exists some \(s\) such that for all \(\tau \in \text{EXT}(\sigma, s)\) we have \(M^n(\tau) \downarrow\). For (ii), assume that \(\sigma \notin U\). Then there certainly exists \(s\) such that \(\sigma \notin U_{s+1}\) and hence \(\tau \notin U_{s+1}\) for any prefix \(\tau\) of \(\sigma\). It follows from the definition of \(M\) that \(M(\sigma, s + 1) \uparrow\) and therefore \(\sigma \notin \text{DOM}(M^\omega)\). This completes the proof of the lemma.

Becher, Daicz, and Chaitin [BDC01] called a program \(\sigma\) of \(M^\omega\) circular if \(M^\omega(\sigma) \downarrow\) and \(M^\omega(\sigma)\) is finite, i.e. a string. Note that all the programs in the domain of \(M^\omega\) of Lemma 4.3 are circular. This shows that the \(\Sigma^0_2\) complexity is not hidden specifically in \(\text{FIN}(M^\omega)\) but rather in the \(\Pi^0_2\) definition of convergence of an infinitary self-delimiting machine, i.e. in the domain \(\text{DOM}(M^\omega)\) itself. In particular, as a direct consequence of Lemma 4.3 we have

\[
\text{Given an upward closed } \Sigma^0_2 \text{ set } S \text{ of strings there exists a machine } M \text{ such that } \text{INF}(M^\omega) \text{ is empty and } \|\text{DOM}(M^\omega)\| = |S| \tag{4.2.1}
\]

We remark that the proof of Lemma 4.3 can be modified in a straightforward way so that the constructed machine \(M^\omega\) has output \(M^\omega(\sigma) = 0^n\) whenever \(M^\omega(\sigma) \downarrow\). This modified construction shows the dual of (4.2.1), namely that given an upward closed \(\Sigma^0_2\) set \(S\) of strings there exists a machine \(M\) such that \(\text{FIN}(M^\omega)\) is empty and \(\|\text{DOM}(M^\omega)\| = |S|\).

The next step is to establish the existence of a machine \(M\) such that \(\text{FIN}(M^\omega)\) is 2-random.

Lemma 4.4. There exists an infinitary self-delimiting machine \(M^\omega\) such that \(\text{INF}(M^\omega)\) is empty and the measure of \(\text{DOM}(M^\omega) = \text{FIN}(M^\omega)\) is a 2-random \(\emptyset'\)-left-c.e. real.

Proof. Consider the member \(S\) of a Martin-Löf test relative to \(\emptyset'\). This can be represented as an upward closed \(\Sigma^0_2\) set of strings, and its measure is 2-random. Therefore the statement follows by a direct application of (4.2.1) to \(S\). \(\square\)

Finally we are ready to deduce the ‘only if’ direction of the part of Theorem 1.5 which refers to infinitary self-delimiting machines, which was originally proved by Becher, Daicz, and Chaitin [BDC01].
Lemma 4.5 (Universal finiteness probability). If $U^\infty$ is a universal infinitary self-delimiting machine then $\text{FIN}(M^\infty)$ is a 2-random 0'-left-c.e. real.

Proof. Consider the machine $M^\infty$ of Lemma 4.4, so that $\mu(\text{FIN}(M))$ is a 2-random 0'-left-c.e. real. Since $U^\infty$ is universal, there exists a string $\tau$ such that for all strings $\sigma$ we have $M^\infty(\sigma) = U^\infty(\tau * \sigma)$. We have

\[ \text{FIN}(U^\infty) = \tau * \text{FIN}(M^\infty) \cup \left( \text{FIN}(U^\infty) \cap (2^\omega - [\tau]) \right) \] and \[ \tau * \text{FIN}(M^\infty) \cap \left( \text{FIN}(U^\infty) \cap (2^\omega - [\tau]) \right) = \emptyset \]

Let $P = \text{FIN}(U) \cap (2^\omega - [\tau])$ and note that this is a $\Sigma^0_2$ class, so $\mu(P)$ is a 0'-left-c.e. real by Lemma 2.1. So $\mu(\text{FIN}(U)) = 2^{-|\tau|} \cdot \mu(\text{FIN}(M)) + \mu(P)$ is a 2-random 0'-left-c.e. real as the sum of a 2-random 0'-left-c.e. real and another 0'-left-c.e. real. \qed

4.3 From random numbers to the finiteness probability

In this section we follow the methodology which were developed in Section 3.2 for a different machine model. We see that this is straightforward, given Lemma 4.3 and the fact that $\text{DOM}(M^\infty)$ and $\text{FIN}(M^\infty)$ are $\Sigma^0_2$ sets.

Lemma 4.6. If $\alpha < 1$ is a 0'-left-c.e. real and $2^{-c} < 1 - \alpha$, then there exists an infinitary self-delimiting machine $M^\infty$ and a string $\rho$ of length $c$ such that $M^\infty(\sigma)$ is undefined for any string $\sigma$ which is a prefix or a suffix of $\rho$, $\text{DOM}(M^\infty) = \text{FIN}(M^\infty)$, and the measure of the domain of $M^\infty$ is $\alpha$.

Proof. Given $\alpha$ and $c$, let $(\alpha_s)$ be a 0'-computable increasing sequence of rationals converging to $\alpha$ with $\alpha_0 = 0$. We apply the Kraft-Chaitin algorithm with sequence of requests $2^{-c}$, $\alpha_1 - \alpha_0$, $\alpha_2 - \alpha_1$, ... and we get a 0'-computable enumeration of a prefix-free set of strings whose measure is $2^{-c} + \alpha$. Let $\rho$ be the first of these strings and let $U$ contain the rest of them. Then we may choose a canonical $\Sigma^0_2$ approximation $(U_s)$ to $U$ such that no string that is compatible with $\rho$ belongs to $U_s$ for any $s$. Then we can apply Lemma 4.3 to $U$ with this specific $\Sigma^0_2$ approximation $(U_s)$ and we obtain a machine $M$ such that $M^\infty(\sigma)$ is not defined for any string $\sigma$ compatible with $\rho$. Moreover we get that the measure of $\text{DOM}(M^\infty)$ is equal to the measure of $U$, which is $\alpha$. \qed

We are now ready to deduce the ‘if’ direction of the part of Theorem 1.5 which refers to infinitary self-delimiting machines. Note that this is the converse of the result of Becher, Daicz, and Chaitin [BDC01].

Lemma 4.7. Let $\alpha$ be a 2-random 0'-left-c.e. real. Then there exists a universal infinitary self-delimiting machine $M^\infty$ such that $\mu(\text{DOM}(M^\infty)) = \alpha$.

Proof. Let $V^\infty$ be a universal infinitary self-delimiting machine and let $\gamma = \mu(\text{DOM}(V^\infty))$. Then $\gamma$ is a 0'-left-c.e. real. Let $c$ be a constant such that the real $\beta = \alpha - 2^{-c} \gamma$ is a 0'-left-c.e. real and $\beta + 2^{-c}$ is less than 1. By Lemma 4.6, consider an infinitary self-delimiting machine $N^\infty$ and a string $\rho$ of length $c$ such that the measure of $\text{DOM}(N^\infty)$ is $\beta$ and $\text{DOM}(N^\infty)$ does not contain any string which is compatible with $\rho$. Define an infinitary self-delimiting machine $M^\infty$ as follows. For each string $\sigma$ which is incompatible with $\rho$ let $M^\infty(\sigma) = V(\sigma)$. Moreover for each $\tau$ let $M^\infty(\rho * \tau) = N(\tau)$. Then

\[ \text{DOM}(M^\infty) = \text{DOM}(V^\infty) \cup \rho * \text{DOM}(N^\infty) \] and \[ \text{DOM}(V^\infty) \cap \rho * \text{DOM}(N^\infty) = \emptyset \]
and so
\[ \mu(\text{DOM}(M^{\omega})) = \mu(\text{DOM}(V^{\omega})) + 2^{-|\omega|} \cdot \mu(\text{DOM}(N^{\omega})) = \beta + 2^{-c} \cdot \gamma = \alpha \]
which concludes the argument.

4.4 Restricted infinite models and higher randomness restrictions

The model of infinitary self-delimiting machines that we described in Section 4.1 was introduced in Chaitin [Cha76] and studied in Becher, Daicz and Chaitin [BDC01]. Although one can exhibit 2-random probabilities in this model, there is a fundamental reason why higher randomness is not attainable as the measure of a subset of the domain of such a machine. The reason for this is that the domain can be seen as a prefix-free $\Delta^0_2$ set of strings. The following lemma can be used in order to give a formal proof of this fact.

**Lemma 4.8.** There is no c.e. prefix-free set of strings that contains a $\Sigma^0_1(\emptyset')$ subset of 2-random measure. More generally, for each $n \in \mathbb{N}$ and any $\Sigma^0_n$ prefix-free set of strings there is no $\Sigma^0_{n+2}$ subset of this set which has $(n+2)$-random measure.

**Proof.** Let $S$ be a c.e. prefix-free set of strings and let $V$ be a $\Sigma^0_1(\emptyset')$ subset of $S$. We construct a Martin-Löf test $(U_n)$ relative to $\emptyset'$ such that $\mu(V) \in \cap_n U_n$. Note that $\mu(V)$ is a $\emptyset'$-left-c.e. real. Given $n$ we use $\emptyset'$ to compute a finite subset $D_n$ of $S$ such that $\mu(S) - \mu(D_n) < 2^{-n-1}$. Then let $\epsilon_n$ be a rational which is less than $2^{-|\rho|}$ for any $\rho \in D_n$. Let $\delta_n = 2^{-n-1} \cdot \epsilon_n/(1 + \epsilon_n)$ and note that the number $d_n$ of strings in $D_n$ is at most $\mu(S)/\epsilon_n \leq 1/\epsilon_n$. Moreover note that $\delta_n < \epsilon_n$. Let $(\tau_i)$ be a $\emptyset'$-computable enumeration of $V$ and let $V_s = \{\tau_i \mid i < s\}$. For each $n, s$ define
\[ J(n, s) = (\mu(V_s), \mu(V_s) + \delta_n) \]
and let $U_n = \cup_s J(n, s)$. Since the intervals $J(n, s)$ have fixed length $\delta_n$ it follows that $\mu(V) \in U_n$ for each $n$. Moreover the intervals $J(n, s)$ are uniformly computable in $\emptyset'$ so $(U_n)$ are uniformly $\Sigma^0_1$ in $n$. It remains to verify that $\mu(U_n) \leq 2^{-n}$ for each $n$.

Now $U_n$ may be decomposed as the union of $J(n, 0)$ together with $\cup_s (J(n, s + 1) \setminus J(n, s))$. We have $\mu(J(n, 0)) = \delta_n$ and, for each $s, \mu(J(n, s + 1) \setminus J(n, s)) = \min(\delta_n, 2^{-|\tau_i|})$. The values of $s$ may be considered in two cases. First, there are $d_n$ values for which $\tau_s \in D_n$; together with the measure $\delta_n$ from $J(n, 0)$, this gives measure $\leq (1 + d_n)\delta_n \leq 2^{-n-1}$, since $d_n \leq 1/\epsilon_n$. Second, there are all the values of $s$ for which $\tau_s \notin D_n$. Here the total measure is $\sum_{\tau_s \in \cap \cup_{n} D_n} 2^{-|\tau_i|} \leq \mu(S) - \mu(D_n) \leq 2^{-n-1}$. It follows that $\mu(U_n) \leq 2^{-n}$, as desired. Hence $(U_n)$ is a Martin-Löf test relative to $\emptyset'$ and $\mu(V) \in U_n$ for all $n$, which means that $\mu(V)$ is not 2-random.

Let us discuss the implications of this result. In the following informal discussion we implicitly assume that probability of a property of a machine with a notion of a domain refers to the measure of a subset of its domain. By the first clause of Lemma 4.8, any prefix-free machine model where the relation of convergence (namely the domain) is $\Sigma^0_1$ cannot exhibit properties whose probability is a second-order $\Omega$ number, i.e. a 2-random $\emptyset'$-left-c.e. real. This holds because the measure of a $\Sigma^0_2$ prefix-free set of strings is always a $\emptyset'$-left-c.e. real. Chaitin’s model for infinite computations, as we discussed, can be seen as a higher type prefix-free machine, where the domain is a $\Pi^0_1$ prefix-free set of strings. This non-standard infinitary feature allowed the presence of properties that occur with 2-random probability, as we saw in the previous sections. However the second clause of Lemma 4.8 for $n = 1$ says that this model cannot exhibit properties whose probability is a 3-random $\emptyset^{(2)}$-left-c.e. real. Indeed, convergence in this model is $\Pi^0_1$ and so it is $\Sigma^0_2$. This
observation gives a limit to the complexity of relative $\Omega$ numbers we can exhibit in this model, this limit being $\Omega$ numbers of the second level, i.e. relative to the second iteration of the halting problem.

Let $\text{COF}(M^{\omega})$ be the set of $\sigma$ in the domain of $M^{\omega}$ such that $M^{\omega}(\sigma)$ is a stream with a tail of 1s. Note that $\text{COF}(M^{\omega})$ is a $\Sigma^0_3$ set of strings, and it is not hard to see that it is $\Sigma^0_3$-complete whenever $M^{\omega}$ is universal. In particular, the measure of $\text{COF}(M^{\omega})$ is always a $\vartheta^{(2)}$-left-c.e. real. So the above observations imply that for every infinitary self-delimiting machine $M^{\omega}$ the measure of $\text{COF}(M^{\omega})$ is never a 3-random real. Similarly, Lemma 4.8 implies Lemma 1.6.

Becher and Chaitin [BC02] study a self-delimiting model for infinite computations which is close to the model of [BDC01], but which avoids the above problem, as the convergence notion is now $\Pi^0_2$. Using this modified model, Becher and Chaitin exhibit probabilities that are random relative to $0'''$.

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