A new massive vector field theory

Zhiyong Wang\textsuperscript{1} and Ailin Zhang\textsuperscript{2}

\textsuperscript{1} P. O. Box 1, Xindu, Chengdu, Sichuan, 610500, P. R. China
\textsuperscript{2} Institute of theory Physics, P. O. Box 2735, Beijing, 100080, P. R. China

Abstract

In this paper, we put forth a new massive spin-1 field theory. In contrast to the quantization of traditional vector field, the quantization of the new vector field is carried out in a natural way. The Lorentz invariance of the theory is discussed, where owing to an interesting feature of the new vector field, the Lorentz invariance has a special meaning. In term of formalism analogical to QED(i. e. spinor QED), we develop the quantum electrodynamics concerning the new spin-1 particles, say, vector QED, where the Feynman rules are given. The renormalizability of vector QED is manifest without the aid of Higgs mechanism. As an example, the polarization cross section $\sigma_{\text{polar}}$ for $e^+e^- \rightarrow f^+f^-$ is calculated in the lowest order. It turns out that $\sigma_{\text{polar}} \sim 0$ and the momentum of $f^+$ and $f^-$ is purely longitudinal.

1 Introduction

After the introduction of Dirac’s equation\cite{1}, the search began for similar equations for higher spins. In the past, various approaches have been tried→ equations describing many masses and spins particle, non-Lagrangian theories and theories with indefinite metric\textsuperscript{2} et al. At first, it was observed that, apart from spin-1/2, none of the other spins obeys a single-particle relativistic wave equation. For example, it was generally believed that for spins 0 and 1, the Klein-Gordon\cite{3,4} and Proca equations\cite{5} were unique, respectively. However, more than 60 years ago, it was found that the Kemmer-Duffin-Petiau\cite{6,7} equations(KDF equations) can describe both spin-0 and spin-1 objects. Since then, many more systems of equations for arbitrary spins, which originate from different assumptions after considering their invariance under Lorentz group, have been found.

Unfortunately, it has been known for a long time that there still exist many difficulties in the construction of higher spins field theories, which has turned out to be the most intriguing and challenging in theoretical physics. Especially, such a theory has touched upon some of the most basic ingredients of present-day physical theory. For example, in those theories, either the usual connection between spin and statics is violated, or the law of causation do not hold, or the negative energy difficulty is still encountered after second quantization having been accomplished, or, in the presence of interactions, the complex energy eigenvalues, superluminal propagation of waves and many other undesirable features\textsuperscript{2} et al were found too. In particular, these behaviors were exhibited by both the KDP and the Proca equations\cite{11}. 

1
On the other hand, both the Maxwell equations for electromagnetic field and Yang-Mills equations for non-Abelian gauge field can be restated in terms of a spinor notation resembling the one for Dirac equation, which motivates people to extend the idea to a massive system with arbitrary spin. In this paper, we set up a new massive vector field equation (called as Dirac-like equation), which takes a form similar to the Dirac equation but involves some six-by-six matrices. The equation is no longer equivalent to any other existed ones such as the KDP equation (in the spin-1 case), the Proca equation and the Weinberg equation, etc. Moreover, it is observed, on one hand, the general solution of a relativistic wave equation can be, not only the sum of positive-frequency and negative-frequency parts (denoted by \( \varphi_1 \)), but also the difference of them (denoted by \( \varphi_2 \)). On the other hand, the positive-frequency and negative-frequency solutions of an equation are linearly independent such that \( \varphi_1 \) and \( \varphi_2 \) transform in the same way under the Lorentz and the gauge transformations. Therefore, these two types of general solutions (\( \varphi_1 \) and \( \varphi_2 \)) are simultaneously used to construct the Lagrangian of the vector field. As a result, all those difficulties mentioned above are swept away.

The paper is organized as follows. In Sec. II, the Dirac-like equation is put forward and the corresponding plane wave solution is discussed. In Sec. III, by choosing a suitable Lagrangian, from which the Dirac-like equation can be derived, we quantilize the Dirac-like field naturally according to Bose-Einstein statistics, where the energy is positive-definite too. In Sec. IV, an interesting feature of the field is shown. From the Lagrangian, we construct the Feynman propagator for the field in Sec. V, with the causality being preserved. In Sec. VI, the Lorentz invariance of the theory is discussed, where the action of the transversal field is proved to be Lorentz invariant. In Sec. VII, we develop the Feynman rules for the vector QED, where, as an example, the polarization cross section for the process \( e^+e^- \rightarrow f^+f^- \) is calculated. At last, in Sec. VIII, we show that vector QED is a renormalizable theory. The system of natural units and Bjorken conventions are used throughout in the paper.

## 2 A new massive vector field equation

In analogy with the construction of Dirac equation, we can set up the following free relativistic “Dirac-like” equation as follow:

\[
(i\beta^\mu \partial_\mu - m)\varphi(x) = 0,
\]

where \( m \) is the mass and \( \beta^\mu = (\beta^0, \vec{\beta}) \) satisfies

\[
\begin{align*}
(\vec{\beta} \cdot \vec{p})^3 &= -\vec{p}^2 (\vec{\beta} \cdot \vec{p}) \\
\beta^0 \beta^i + \beta^i \beta^0 &= 0 \\
(\beta^0)^2 &= 1,
\end{align*}
\]

where \( \vec{p} \) is an arbitrary three dimensional vector such as the spatial component of a 4-momentum. To express the \( \beta \) matrix explicitly, we choose

\[
\beta^0 = \begin{pmatrix}
I_{3\times3} & 0 \\
0 & -I_{3\times3}
\end{pmatrix},
\quad
\vec{\beta} = \begin{pmatrix}
0 & \vec{\tau} \\
-\vec{\tau} & 0
\end{pmatrix},
\]

\[2\]
where $I_{3 \times 3}$ is the $3 \times 3$ unit matrix and $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$, in which
\[
\tau_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

It is easy to check that $\beta^\mu$ given above satisfy the relation (3). Let $\vec{\alpha} = \beta^0 \vec{\beta}$, Eq. (1) can be rewritten as
\[
i \partial_t \varphi(x) = (\vec{\alpha} \cdot \vec{p} + \beta^0 m) \varphi(x),
\]
where $\hat{p} = -i \nabla$ is the momentum operator. Accordingly, the Hamiltonian is $\hat{H} = \vec{\alpha} \cdot \hat{p} + \beta^0 m$. Let $\hat{L} = \vec{r} \times \hat{p}$ represent the operator of orbital angular momentum, we have
\[
[\hat{H}, \hat{L}] = -i \vec{\tau} \times \hat{p}, \quad [\hat{H}, \hat{L} + \vec{S}] = 0,
\]
where
\[
\vec{S} = \begin{pmatrix} \vec{\tau} & 0 \\ 0 & \vec{\tau} \end{pmatrix}
\]
is the spin matrix. Since $\vec{S}^2 = 2$, the corresponding particle (called Dirac-like particle) has spin 1, which will be demonstrated further in Sec. VI. Namely, Eq. (1) represents the equation of a massive vector field. As we will see later, the Dirac-like field is different from any other vector fields, and hence is a completely new one.

Substituting the plane wave solution $\varphi(p) e^{-ipx}$ into Eq. (5), we obtain the representation of Eq. (5) in momentum space as follow:
\[
(E - \vec{\alpha} \cdot \hat{p} - \beta^0 m) \varphi(p) = 0,
\]
where $E = p^0$ is the energy. When $\{\hat{H}, \hat{p}, \hat{S}^\mu / p^\mu\}$ are chosen as dynamical completeness operators, the fundamental solutions of Eq. (8) are derived
\[
|\vec{p}, s\rangle_+ = \sqrt{E_s + m} \left( \eta_s \begin{pmatrix} \eta_s \\ \vec{\tau} \eta_s / E_s + m \eta_s \end{pmatrix} \right), \quad |\vec{p}, s\rangle_- = \sqrt{E_s + m} \left( -\vec{\tau} \eta_s / E_s + m \eta_s \right),
\]
where $|\vec{p}, s\rangle_+$ and $|\vec{p}, s\rangle_-$ correspond to the positive and the negative energy solutions, respectively. $s = -1, 0, 1$ and
\[
E_{-1} = E_1 = E_\perp = \sqrt{\vec{p}^2 + m^2}, \quad E_0 = E_L = m,
\]

\[
\eta_1 = \frac{1}{\sqrt{2|\vec{p}|}} \begin{pmatrix} p_1 p_3 - ip_2 |\vec{p}| \\ p_1 - ip_2 \\ p_2 + ip_1 |\vec{p}| \\ -p_1 + ip_2 \end{pmatrix}, \quad \eta_{-1} = \eta_1^*, \quad \eta_0 = \frac{1}{|\vec{p}|} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix},
\]
where $\eta_1^*$ is the complex conjugate of $\eta_1$. Let $\eta_s^\dagger$ stands for the Hermitian conjugate of $\eta_s$, we have
\[
\left\{ \begin{array}{l}
\eta_1^\dagger \eta_s^\dagger = \delta_{ss'} \\
\sum_s \eta_s \eta_s^\dagger = I_{3 \times 3},
\end{array} \right.
\]
(12)
which means that \{\eta_s\} form a complete orthonormal basis. It can be verified that

\[
\frac{1}{|\vec{p}|} \vec{p} \eta_s = \lambda_s \eta_s,
\]

where \( \lambda_1 = 1, \lambda_{-1} = -1 \) and \( \lambda_0 = 0 \). Obviously, the solutions with \( s = \pm 1 \) correspond to the transversal polarization solutions of Eq. (8), while another one corresponds to the longitudinal polarization solution. Besides, an interesting result can be read from Eq. (10): the transversal polarization particles have energy \( \sqrt{p^2 + m^2} \) while the longitudinal one has energy \( m \), whose physical reason will be explained in Sec. IV.

As for the wave functions, the ones in momentum space are chosen as

\[
\chi(\vec{p}, s) = |\vec{p}, s\rangle_+, \quad \gamma(\vec{p}, s) = | - \vec{p}, s\rangle_- = \sqrt{\frac{E_{s+1} + m}{2m}} \left( \frac{\varphi_{-p,s}^\dagger \eta_s}{\eta_s} \right).
\]

and the corresponding ones in position space are

\[
\varphi_{p,s}(x) = \sqrt{\frac{m}{vE_s}} \chi(\vec{p}, s)e^{-ipx}, \quad \varphi_{-p,s}(x) = \sqrt{\frac{m}{vE_s}} \gamma(\vec{p}, s)e^{ipx}.
\]

Taking into account Eq. (12), we have (here \( \tilde{\chi} = \chi^\dagger \beta^0 \), and so on)

\[
\begin{align*}
\bar{\chi}(\vec{p}, s) \chi(\vec{p}, s') & = -\bar{\gamma}(\vec{p}, s) \gamma(\vec{p}, s') = \delta_{ss'}, \\
\bar{\chi}(\vec{p}, s) \gamma(\vec{p}, s') & = \bar{\gamma}(\vec{p}, s) \chi(\vec{p}, s') = 0.
\end{align*}
\]

\[
\begin{align*}
\sum_s \chi(\vec{p}, s) \bar{\chi}(\vec{p}, s) & = \frac{\beta p_{+1} + m}{2m} (I_{2 \times 2} \otimes \sum_{s= \pm 1} \eta_s \eta_s^\dagger) + \frac{\beta p_{-1} + m}{2m} (I_{2 \times 2} \otimes \eta_0 \eta_0^\dagger) \\
\sum_s \gamma(\vec{p}, s) \bar{\gamma}(\vec{p}, s) & = \frac{\beta p_{+1} - m}{2m} (I_{2 \times 2} \otimes \sum_{s= \pm 1} \eta_s \eta_s^\dagger) + \frac{\beta p_{-1} - m}{2m} (I_{2 \times 2} \otimes \eta_0 \eta_0^\dagger).
\end{align*}
\]

where \( \otimes \) is the direct product symbol. Obviously, Eq. (14) and Eq. (17) complete the orthonormality relations and the spin summation relations for the massive vector field, from which the completeness relation can be derived too.

## 3 Quantization of the Dirac-like field

In this section, we will quantize the Dirac-like field in a manner similar to that used for the Dirac field but obeying the Bose-Einstein statistics. Two types of general solutions of Eq. (11) can be constructed simultaneously as follows:

\[
\begin{align*}
\varphi_1(x) & \equiv \varphi_+(x) + \varphi_-(x) = \sum_{\vec{p}, s} [a(\vec{p}, s) \varphi_{p,s}(x) + b^\dagger(\vec{p}, s) \varphi_{-p,s}(x)], \\
\varphi_2(x) & \equiv \varphi_+(x) - \varphi_-(x) = \sum_{\vec{p}, s} [a(\vec{p}, s) \varphi_{p,s}(x) - b^\dagger(\vec{p}, s) \varphi_{-p,s}(x)],
\end{align*}
\]

where

\[
\begin{align*}
\varphi_+(x) & = \sum_{\vec{p}, s} a(\vec{p}, s) \varphi_{p,s}(x), \quad \varphi_-(x) = \sum_{\vec{p}, s} b^\dagger(\vec{p}, s) \varphi_{-p,s}(x),
\end{align*}
\]
are the positive and the negative frequency parts of the general solutions, respectively, and $a(p, s), b^\dagger(p, s)$ are coefficients.

To deduce the free Dirac-like equation, the free Lagrangian is chosen as

$$L = \bar{\varphi}_2(x)(i\beta^\mu \partial_\mu - m)\varphi_1(x).$$

Then the canonical momentum conjugates to $\varphi_1(x)$ is

$$\pi(x) = \partial L/\partial \dot{\varphi}_1 = i\varphi_2^\dagger(x),$$

and thus, the Hamiltonian $H$ and the momentum $\vec{p}$ are, respectively,

$$H = \int [\pi(x)\dot{\varphi}_1(x) - L]d^3x = \int \varphi_2^\dagger(-i\alpha \cdot \nabla + \beta^0 m)\varphi_1(x)d^3x,$$
$$\vec{p} = -\int \pi(x)\nabla \varphi_1 d^3x = -i \int \varphi_2^\dagger(x)\nabla \varphi_1 d^3x.$$

Now, we promote $\varphi_1(x)$ and $\pi(x)$ to operators and the canonical equal time commutation relations become

$$[\varphi_1\alpha(\vec{x}, t), \pi_\beta(\vec{x}', t)] = i\delta_{\alpha\beta}\delta^3(\vec{x} - \vec{x}'),$$

with the others vanishing. In term of $a(p, s)$ and $b(p, s)$, we get the following commutation relations

$$[a(p, s), a^\dagger(p', s')] = [b(p, s), b^\dagger(p', s')] = \delta_{pp'}\delta_{ss'},$$

and all other commutators vanish. Making use of Eq. (16), (18), (19) and (25), Eq. (23) transforms into

$$H = \sum_{\vec{p}, s} E_s[a^\dagger(p, s)a(p, s) + b^\dagger(p, s)b(p, s) + 1],$$
$$\vec{p} = \sum_{\vec{p}, s}[a^\dagger(p, s)a(p, s) + b^\dagger(p, s)b(p, s)],$$

where $a^\dagger(p, s), a(p, s)$ are the creation and annihilation operators of particles, respectively, while $b^\dagger(p, s), b(p, s)$ are the corresponding ones of antiparticles. Obviously, the Dirac-like field obeys the Bose-Einstein statics and the corresponding energy is positive-definite, which is just what we desire.

4 Character of the new vector field

Let us turn our attention to the expectation value $\vec{v}$ of the velocity operator $\dot{\vec{x}} = i[\hat{H}, \vec{x}] = \vec{\alpha}$

$$\vec{v} = \int \varphi_1(x)\dot{\vec{x}} \varphi_1(x)d^3x = \int \varphi_2\vec{\alpha} \varphi_1 d^3x.$$

In fact, $\varphi_2\vec{\alpha} \varphi_1$ can be regarded as the probability current density and hence $\vec{v}$ corresponds to the probability current. Substituting Eq. (22), (23) into Eq. (27), we obtain

$$\vec{v} = \vec{v}_\perp + \vec{v}_L,$$
where $\vec{v}_L$ represents the current related to transversal particle only, while $\vec{v}_\perp$ corresponds to the current with the contribution of longitudinal particles involved in too. The explicit expression of them are

$$\vec{v}_L = \sum_{\vec{p}} \sum_{s=\pm 1} \frac{E}{E^2} [a^\dagger(\vec{p}, s)a(\vec{p}, s) - b^\dagger(\vec{p}, s)b(\vec{p}, s)]$$

$$+ \sum_{\vec{p}} \sum_{s=\pm 1} \frac{1}{2E^2 + m^2} \eta \eta^\dagger \cdot [a^\dagger(\vec{p}, s)b^\dagger(\vec{p}, s)e^{i(\vec{p}, s), s} - a(\vec{p}, s)b(\vec{p}, s)e^{-i(\vec{p}, s), s}],$$

$$\vec{v}_\perp = \sum_{\vec{p}} \sum_{s=\pm 1} \sqrt{\frac{E + m^2}{2E^2}} \{[a^\dagger(\vec{p}, 0)a(\vec{p}, 0)e^{i(m-E)t} + b^\dagger(\vec{p}, s)b(\vec{p}, 0)e^{-i(m-E)t}]\eta^\dagger - h.c.,}$$

respectively, in which $E = E_\perp = \sqrt{\vec{p}^2 + m^2}$ and $\eta$ is the vector representation of Eq. (11).

Before going on, we will give some discussions about Eq. (29):

1. As far as $\vec{v}_L$ is concerned, the first term corresponds to the classic current with group velocity $\frac{E}{\vec{p}}$ of the wave packet, while the second term corresponds to the zitterbewegung current of the transversal particle. The zitterbewegung current does not vanish as $\vec{p} \to 0$, which infers that it is intrinsic and independent of macroscopic classic motion.

2. As for $\vec{v}_\perp$, the first term corresponds to the current resulting from the interference between the transversal and the longitudinal particle, and the second term corresponds to the zitterbewegung current containing the contribution of longitudinal particle.

3. $\eta_0$ is parallel to $\vec{p}$ while $\eta_{\perp 0}$ is vertical to $\vec{p}$ (denoted by $\eta_0 \parallel \vec{p}$ and $\eta_0 \perp \vec{p}$, respectively), thus $\vec{v}_L \parallel \vec{p}$ while $\vec{v}_\perp \perp \vec{p}$ (or, $\vec{v}_\perp \cdot \vec{p} = 0$). On the other hand, the longitudinal particle does not contribute to $\vec{v}_L$ (which contributes only to $\vec{v}_\perp$). Therefore, the longitudinal particle makes no contribution to the current in the direction of momentum $\vec{p}$, which is consistent with the statement that $E_L = m$ obtained earlier (see Eq. (10)). As a consequence, we can regard the longitudinal particle as the one corresponding to standing wave. In fact, the behaviors of the new vector field are similar to those of the electromagnetic wave propagating in hollow metallic waveguide, where the longitudinal component of the electromagnetic wave makes no contribution to the flow (or the Poynting vector) along the waveguide.

5 The free propagator of the Dirac-like field

The free propagator of Dirac-like field is defined as

$$iR_f(x_1 - x_2) \equiv \langle 0|T\varphi_1(x_1)\varphi_2(x_2)|0\rangle,$$

where $T$ is the time order symbol. Considering Eq. (17), we obtain

$$iR_f(x_1 - x_2) = iR_{f\perp}(x_1 - x_2) + iR_{fL}(x_1 - x_2),$$

where $iR_{f\perp}(x_1 - x_2)$ is the free propagator of transversal field and $iR_{fL}(x_1 - x_2)$ is the longitudinal one, which read

$$iR_{f\perp}(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4} \frac{i\Omega_\perp}{p^2 - E_\perp^2 + i\varepsilon} e^{-ip(x_1 - x_2)}$$

$$iR_{fL}(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4} \frac{i\Omega_L}{p^2 - E_L^2 + i\varepsilon} e^{-ip(x_1 - x_2)},$$
respectively, where $\varepsilon$ is an infinitesimal real quantity, $p_{\mu} = (p_0, -\vec{p})$ and
\[
\Omega_\perp = (i\beta \cdot \partial + m)A_\perp \\
\Omega_L = (i\beta \cdot \partial + m)A_L,
\] (33)
in which $A_\perp = I_{2 \times 2} \otimes \sum s \epsilon \eta_s \eta_s^\dagger$ and $A_L = I_{2 \times 2} \otimes \eta_0 \eta_0^\dagger$. By applying $p_0^2 - E_\perp^2 = p^2 - m^2$, $(\beta \cdot p)^2 A_\perp = p^2 A_\perp$ and $p_0^2 - E_L^2 = p_0^2 - m^2$, $(\beta \cdot p)^2 A_L = p^2 A_L$, we have
\[
iR_{f\perp}(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4} \frac{iA_\perp}{\beta \cdot p - m + i\varepsilon} e^{-ip(x_1 - x_2)}
\] (34)
\[
iR_{fL}(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4} \frac{iA_L}{\beta \cdot p - m + i\varepsilon} e^{-ip(x_1 - x_2)}.
\]
Due to $A_\perp + A_L = 1$,
\[
iR_f(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\beta \cdot p - m + i\varepsilon} e^{-ip(x_1 - x_2)},
\] (35)
which takes a form similar to the free propagator of Dirac field but involves the matrices $\beta^\mu$ instead of Dirac matrices $\gamma^\mu$.

The representation of $iR_f(x_1 - x_2)$ in momentum space is
\[
iR_f(p) = iR_{f\perp}(p) + iR_{fL}(p) = \frac{i}{\beta \cdot p - m + i\varepsilon},
\] (36)
where
\[
iR_{f\perp}(p) = \frac{i}{\beta \cdot p - m + i\varepsilon} A_\perp, iR_{fL}(p) = \frac{i}{\beta \cdot p - m + i\varepsilon} A_L.
\] (37)
It is easy to find that
\[
(i\beta \cdot \partial - m)R_f(x_1 - x_2) = \delta^4(x_1 - x_2).
\] (38)
Namely, $R_f(x_1 - x_2)$ is the Green’s function of free Dirac-like equation. To make Eq. (37) more explicit, we choose a frame in which $\vec{p} = (0, 0, p_3)$, then
\[
A_\perp = I_{2 \times 2} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_L = I_{2 \times 2} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\] (39)

6 Lorentz invariance of the theory

Considering the fact that the positive and the negative frequency parts of Dirac-like field are linearly independent, one can readily verify that $\varphi_1(x)$ and $\varphi_2(x)$ (given by Eq. (18) and (19), respectively) transform in the same way under a Lorentz transformation.

In following special cases, Lorentz boost along the direction of the Dirac-like field’s motion or Lorentz boost $L(\vec{p})$ which makes a Dirac-like particle from rest to momentum $\vec{p}$, the Lagrangian $L_\varphi$ given by Eq. (21) is easily proved to be Lorentz invariant.
Now, we consider the variation of Lagrangian under an arbitrary Lorentz transformation. Before embarking on the process, the following fact should be noted: the longitudinal field makes no contribution to the current in the direction of the momentum of Dirac-like field and the energy of it is $E_L = m$, so we regard it as the one that exists in a standing wave form or in a virtual form. Meanwhile, for the independence of the longitudinal field with the interaction related to the current, the longitudinal field is taken as an unobservable one. Certainly, the unobservable longitudinal field in one frame can be turned into the observable transversal field in another frame and vice versa. However, the action of the transversal field is Lorentz invariant (just as we will show later). As far as the noncovariance of the transversal condition for the transversal field is concerned, it won’t do any hurt to the theory. The Lorentz invariant transition amplitude\(^23\) still can be obtained from a noncovariant Hamiltonian formulation of field theory, just as what we have done in electromagnetic field theory with the noncovariant Coulomb gauge. For the reasons mentioned above, we will demonstrate only the invariance of the action of transversal field in the following.

For generality, let us consider the case that the Lagrangian contains an interaction term, in which the Dirac-like field is coupled to the electromagnetic field in the minimal form, namely

$$L_\varphi = \varphi_2(x)[i\beta D_\mu - m]\varphi_1(x),$$

(40)

where $\varphi_1$ and $\varphi_2$ contain only the transversal parts, $D_\mu = \partial_\mu - ieA_\mu$, $e$ is a dimensionless coupling constant and $A_\mu$ is the electromagnetic field. The Lorentz invariant free term for $A_\mu$ is omitted. Under Lorentz transformation

$$x^\mu \rightarrow x'^\mu = a^{\mu\nu}x_\nu, \quad D_\mu \rightarrow D'_\mu = a_{\mu\nu}D^\nu, \quad d^4x \rightarrow d^4x' = d^4x,$$

(41)

$\varphi_1$ and $\varphi_2$ transform linearly in the same way,

$$\varphi_1(x) \rightarrow \varphi'_1(x') = \Lambda \varphi_1(x), \quad \varphi_2(x) \rightarrow \varphi'_2(x') = \varphi_2 \beta^0 \Lambda^\dagger \beta^0,$$

(42)

and thus the Lagrangian transforms as

$$L_\varphi \rightarrow L'_\varphi = \varphi_2(x)[\beta^0 \Lambda^\dagger \beta^0][i\beta_{\mu\nu}a_{\mu\nu}D^\nu - m]\Lambda \varphi_1(x).$$

(43)

Owing to the invariance of mass term $m\varphi_2(x)\varphi_1(x)$, $\beta^0 \Lambda^\dagger \beta^0 = \Lambda^{-1}$. Therefore,

$$T(x) \equiv L'_\varphi - L_\varphi = \varphi_2(x)i[\Lambda^{-1} \beta_{\mu\nu}a_{\mu\nu}D^\nu - \beta^{\mu}D_{\mu}]\varphi_1(x).$$

(44)

Obviously, the conclusion of Lorentz invariance of the action $\int L_\varphi d^4x$ can be drawn once

$$\int T(x)d^4x = \int L'_\varphi d^4x' - \int L_\varphi d^4x = 0.$$

(45)

Under the infinitesimal Lorentz transformation, we have

$$a_{\mu\nu} = g_{\mu\nu} + \varepsilon_{\mu\nu}, \quad \Lambda = 1 - \frac{i}{2}\varepsilon^{\mu\nu}s_{\mu\nu},$$

(46)

where $g_{\mu\nu}$ is the metric tensor, $\varepsilon_{\mu\nu}$ is the infinitesimal antisymmetric tensor and $s_{\mu\nu}$ is the unknown coefficient. With the help of $\varepsilon_{\mu\nu}\beta^{\nu}D^\nu = \frac{1}{2}\varepsilon_{\rho\tau}(\beta_\rho D_\tau - \beta_\tau D_\rho)$, $T(x)$ becomes

$$T(x) = \frac{i}{2}\varepsilon^{\rho\tau}\varphi_2(x)[(\beta_\rho D_\tau - \beta_\tau D_\rho) - i[\beta^\sigma, s_{\rho\tau}]D_\sigma]\varphi_1(x).$$

(47)
To prove Eq. (45), all possible cases of Eq. (48) are discussed as follows:

(1) As \( \rho = \tau \), \( \varepsilon^{\rho \tau} = 0 \) and hence \( T(x) = 0 \);

(2) As \( \rho = l \) and \( \tau = m \), namely, in the spatial rotation case, we have \( s_{lm} = -s_{ml} = \epsilon_{lmn}\left(\tau_n 0 \right) \), where \( \tau_n \) has been given by Eq. (33) and \( \epsilon_{lmn} \) is the full antisymmetric tensor\((\epsilon_{123} = 1)\), so \( T(x) = 0 \). As we can see, \( S_{23}^2 + S_{31}^2 + S_{12}^2 = 2 \), the Dirac-like field thus has spin 1.

(3) As \( \rho = l \) and \( \tau = 0 \), that is, in the general Lorentz boost case, the situation is a little more complicated compared with that in previous cases. In the present case, \( s_{l0} = -s_{0l} = \frac{i}{2} \beta^0 \beta^l \equiv \frac{i}{2} \alpha^l \), and \( T(x) \) becomes

\[
T(x) = \frac{i}{2} \varepsilon^{0l} \varphi_2(x)[-D_l + \frac{1}{2}(\alpha^m \alpha^l + \alpha^l \alpha^m)D_m] \varphi_1(x)
\]

\[
= -\frac{i}{4} \varepsilon^{0l} [\nabla \cdot (\vec{F}^\dagger F_l) + \nabla \cdot (\vec{G}^\dagger G_l)],
\]

where \( \vec{F}' = (F'_1, F'_2, F'_3) \) (it is similar for \( \vec{G}' \)), \( \vec{F} \) and \( \vec{G} \), the definition of them are

\[
\varphi_1(x) = \left( \begin{array}{c} F(x) \\ iG(x) \end{array} \right), \quad \varphi_2(x) = \left( \begin{array}{c} F'(x) \\ iG'(x) \end{array} \right),
\]

in which \( F = \left( \begin{array}{c} F_1 \\ F_2 \\ F_3 \end{array} \right) \), \( G, F' \) and \( G' \) are in a similar form. Since \( \varphi_1(x) \) and \( \varphi_2(x) \) contain only the transversal parts, they satisfy transversal conditions

\[
\vec{D} \cdot \vec{F} = \vec{D} \cdot \vec{G} = \vec{D} \cdot \vec{F}' = \vec{D} \cdot \vec{G}' = 0
\]

or

\[
\vec{D}^\dagger \cdot \vec{F}^\dagger = \vec{D}^\dagger \cdot \vec{G}^\dagger = \vec{D}^\dagger \cdot \vec{F}' = \vec{D}^\dagger \cdot \vec{G}' = 0.
\]

In fact, by using Eq. (9) and Eq. (11), we can obtain Eq. (50) and Eq. (51) with \( A_\mu = 0 \).

It can be verified also that

\[
(\alpha^m \alpha^l + \alpha^l \alpha^m)D_m = (2D_l - M_l - M_l^T),
\]

where \( M_l^T \) is the transpose of \( M_l \) and

\[
M_1 = I_{2 \times 2} \otimes \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_1 \end{pmatrix}, \quad M_2 = I_{2 \times 2} \otimes \begin{pmatrix} 0 & D_1 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix}, \quad M_3 = I_{2 \times 2} \otimes \begin{pmatrix} 0 & 0 & D_1 \\ 0 & 0 & D_2 \\ 0 & 0 & D_3 \end{pmatrix}.
\]

Taking use of Eq. (50), (51) and (52), we obtain Eq. (48). As we know, the physical field vanishes as \( |\vec{x}| \to \infty \), so

\[
\int T(x) d^4 x = -\frac{i}{4} \varepsilon^{0l} \int [\nabla \cdot (\vec{F}^\dagger F_l) + \nabla \cdot (\vec{G}^\dagger G_l)] d^4 x = 0.
\]

After all the possible cases (1), (2) and (3) have been considered, the conclusion \( \int T(x) d^4 x \equiv 0 \) becomes obvious. That is to say, the action of transversal Dirac-like field is Lorentz invariant.

Since the action \( \int L_\phi d^4 x \) is Lorentz invariant while the corresponding Lagrangian \( L_\phi \) is not, the Lorentz invariance has a special implication in our theory. We will discuss it further in the next section.
7 Feynman rules and Polarization cross section for $e^+e^- \rightarrow f^+f^-$

Under the gauge transformation $\varphi_1(x) \rightarrow e^{i\theta(x)}\varphi_1(x)$ (where $\theta(x)$ is a real parameter), the field quantity $\varphi_2(x)$ transforms in the same way for the reason that the positive and the negative frequency parts of $\varphi_1(x)$ or $\varphi_2(x)$ are linearly independent. Obviously, the Lagrangian given by Eq. (21) is invariant under the global gauge transformation and the corresponding conserved charge is

$$Q = \int \varphi_2^\dagger(x)\varphi_1(x)d^3x = \sum_{\vec{p},s}[a^\dagger(\vec{p},s)a(\vec{p},s) - b^\dagger(\vec{p},s)b(\vec{p},s)],$$

just as we expect. Minimal electromagnetic coupling is easily introduced into Eq. (21) by making the replacement $\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu$ and Eq. (21) turns into Eq. (40). One can verify that Eq. (40) is invariant also under local gauge transformation

$$\begin{cases} \\
\varphi_1(x) \rightarrow e^{i\theta(x)}\varphi_1(x) \text{(hence } \varphi_2(x) \rightarrow e^{i\theta(x)}\varphi_2(x)) \\
A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x).
\end{cases}$$

Now, we develop the relevant perturbative theory (called ”vector QED” or ”VQED”) in adiabatic approximation, where the S-matrix can be calculated from Dyson’s formula

$$S = \sum_{n=0}^{\infty} S^{(n)} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n T\{H_1(x_1) \cdots H_1(x_n)\},$$

where $H_1(x)$ is the interaction Hamiltonian written in interaction picture. After a tedious calculation, we give the Feynman rules for VQED in momentum space as follows (for simplicity, the Dirac-like particle or antiparticle is denoted by DL or anti-DL, respectively):

1. Propagators:
   - Photon’s $\langle TA_\mu(x_1)A_\nu(x_2)\rangle = \frac{-ig_{\mu\nu}}{q^2 + ie}$,
   - DL’s $\langle T\varphi_1(x_1)\varphi_2(x_2)\rangle = \frac{iA_{\mu\nu}}{\beta\vec{p} - m + ie}$.

2. External lines:
   - Photon annihilation $= \epsilon_\mu(p)$, Photon creation $= \epsilon_\mu^\ast(p)$.
   - DL annihilation $= \frac{m}{\sqrt{VE}}\chi(p,s)$, DL creation $= \frac{m}{\sqrt{VE}}\chi(p,s)$.
   - anti-DL annihilation $= \frac{m}{\sqrt{VE}}\bar{\chi}(p,s)$, anti-DL creation $= \frac{m}{\sqrt{VE}}\bar{\chi}(p,s)$.

3. Vertex (DL with photon): $-ie\beta_\mu$.
4. Impose momentum conservation at each vertex.
5. Integrate over each free loop momentum: $\int \frac{d^4p}{(2\pi)^4}$.
6. Divide by symmetry factor.

As an application, we will calculate the polarization cross section for process $e^-(p, s) + e^+(q, t) \rightarrow f^-(p', s') + f^+(q', t')$: the annihilation of an electron with a positron to create a pair of Dirac-like particles $f^+$ and $f^-$.

In our case, the initial state is $|e\rangle = c^\dagger(p, s)d^\dagger(q, t)|0\rangle$ and the final state is $|f\rangle = a^\dagger(p', s')b^\dagger(q', t')|0\rangle$, where $c^\dagger$, $d^\dagger$ are the creation operators of electron and positron and
\[ s, t(=1, 2) \text{ are their spin indices. } s', t'(= \pm 1) \text{ are the transversal polarization indices while } p, q, p' \text{ and } q' \text{ are the 4-momentum. The interaction Hamiltonian related to the process is} \]

\[ H_I(x) = e\tilde{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) + e\varphi_2(x)\beta^\mu\varphi_1(x)A_\mu(x), \tag{58} \]

where \( \gamma^\mu \) is Dirac matrix and \( \tilde{\psi}(x) = \psi^\dagger(x)\gamma^0 \), in which

\[ \psi(x) = \sum_{\vec{p}, s} \sqrt{\frac{m_0}{vE_0}}[e(p, s)u(p, s)e^{-ipx} + d^\dagger(p, s)\nu(p, s)e^{ipx}] \tag{59} \]

is the electronic field, \( m_0, E_0 \) are the mass and the energy of electron, respectively. Then the corresponding transition amplitude is

\[ S^{(2)}_{fe} = \langle f|S^{(2)}|e \rangle \]

\[ = (2\pi)^4\delta^4(p + q - p' - q')\sqrt{\frac{m_0}{vE_0}}\tilde{\chi}(p', s')(\mp i\epsilon^\mu\nu)\sqrt{\frac{m_0}{vE_0}}y(q', t') \]

\[ \frac{-i\epsilon^\mu\nu}{(p+q)^2}\sqrt{\frac{m_0}{vE_0}}\bar{\nu}(q, t)(\mp i\epsilon^\mu\nu)\sqrt{\frac{m_0}{vE_0}}u(p, s), \tag{60} \]

where \( S^{(2)} \) is defined by Eq. (37).

However, since \( e^+ \) and \( e^- \) have spin 1/2 while \( f^+ \) and \( f^- \) have spin 1, conservation of angular momentum requires that both the total spin of \( e^+e^- \) and that of \( f^+f^- \) are zero, which implies that both the initial state current (denoted by \( J^\mu_i \)) and final state current (denoted by \( J^\mu_f \)) must be spin singlets, so the relevant cross section is a polarization one (denoted by \( \sigma_{\text{polar}} \)). In the CM frame,

\[ J^\mu_0 = \frac{1}{\sqrt{2}}[\bar{\nu}(q, 1)\gamma^\mu u(p, 1) - \bar{\nu}(q, 2)\gamma^\mu u(p, 2)], \tag{61} \]

\[ J^\mu = \frac{1}{\sqrt{2}}[\bar{\chi}(p', 1)\beta^\mu y(q', 1) - \bar{\nu}(p', -1)\beta^\mu y(p, -1)]. \tag{62} \]

It is not difficult to infer that

\[ \sigma_{\text{polar}} = \int (2\pi)^4\delta^4(p + q - p' - q')\frac{m_0^2m^2}{2(p+q)^2}e^4\frac{d^\dagger q'}{(2\pi)^3}d^\dagger q' M^2, \tag{63} \]

where

\[ M^2 = |J_\mu J^\mu_0|^2. \tag{64} \]

Obviously, \( \frac{M}{|p+q|} \) corresponds to the invariant transition amplitude. As a result, \( M^2 \) must be Lorentz invariant. After a cumbersome calculation, we obtain

\[ M^2 = 4\cos^2\theta, \tag{65} \]

where \( \theta \) stands for the angle between the incoming electrons and the outgoing Dirac-like particles. Lorentz invariance of \( M^2 \) requires that \( \theta = 0 \) or \( \pi \) (as we know, only in these cases, the \( \cos^2\theta \) is Lorentz invariant). Namely, the final state momentum is purely longitudinal. Therefore, the Lorentz invariance has an incidental meaning for our theory (i.e. VQED): it provides an additional constraint for a possible physical process, just as the conservation of quantum number provides a selection rule for a possible particle reaction. In other words,
VQED contains not only the Lorentz invariant process (and hence this process is allowed), but also the process violating the Lorentz invariance and hence being forbidden. In a sense, this paper provides a new approach to construct certain quantum field theories still unknown to us.

In a word, \( M^2 = 4 \). Seeing that \( E_0 = E \) in the CM frame and \( |\vec{p}| \approx E_0 \) in the high energy approximation, we obtain

\[
\sigma_{\text{polar}} = \frac{\pi \alpha^2 m_0^2 m^2}{E_0^6} v'^3,
\]

where \( \alpha = \frac{e^2}{4\pi} \) and \( v' \) is the velocity of \( f^+ \) (or \( f^- \)). It is found that \( v' = \frac{\sqrt{3}}{3} \) maximizes \( \sigma_{\text{polar}} \), that is

\[
(\sigma_{\text{polar}})_{\text{max}} = \frac{2\sqrt{3} \pi \alpha^2 m_0^2}{243} m^4.
\]

While in the high energy limit, \( v' \approx 1 \), and

\[
\sigma_{\text{polar}} = \frac{\pi \alpha^2 m_0^2 m^2}{E_0^6}.
\]

To sum up, the following conclusions follows:

(1), In the high energy approximation, \( m, m_e \ll E_0 = E \), \( \sigma_{\text{polar}} \approx 0 \).

(2), The momentum of \( f^+ \) and \( f^- \) is purely longitudinal.

In the light of (1) and (2), we can explain the fact that none of the Dirac-like particles have been found so far.

8 Renormalizability of VQED

After the introduction of VQED in Sec. VII, the question whether VQED is renormalizable then arises. Before proceeding discussions, let us pause a moment to mention the following fact: Let the Dirac-like field quantity \( \varphi = \begin{pmatrix} F \\ iG \end{pmatrix} \) and set \( m = 0 \), where

\[
F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}.
\]

In term of \( F \) and \( G \), Eq. (1) or Eq. (3) can be reexpressed as Maxwell equations written in free vacuum, where \( F \) and \( G \) correspond to the electric field intensity \( E' \) and the magnetic field intensity \( H' \), respectively. However, we can’t regard \( F \) and \( G \) directly as \( E' \) and \( H' \) correspondingly, for two reasons:

(1), Under an infinitesimal Lorentz boost parametrized by the infinitesimal velocity \( v \), the transformation properties of \( F \) (or \( G \)) are different from those of \( E' \) (or \( H' \)). The transformation of \( F \) (or \( G \)) contains the factor of \( \frac{v}{2} \) while the transformation of \( E' \) (or \( H' \)) contains the factor \( v \).

(2), The dimension of \( F \) (or \( G \)) is \( \frac{3}{2} \) while that of \( E' \) (or \( H' \)) is 2. In fact, the canonical dimension of the Dirac-like vector is different from that of electromagnetic field.

In a word, the Dirac-like field is a new kind of vector field.
Now, let us pay attention to the renormalizability mentioned above. Fortunately, VQED can be developed in term of the same formalism as those in the electron-photon interaction theory, namely, spinor QED (denoted by SQED). In particular, the free propagator and the vertex in VQED take the same forms as those in SQED except for involving matrices $\beta^\mu$ (or $\beta^\mu A_\perp$) instead of the Dirac matrices $\gamma^\mu$. As a result, starting from the Feynman integral (i.e. the loop momentum integral) of SQED, we can obtain the corresponding Feynman integral of VQED by replacing the $\gamma^\mu$ with $\beta^\mu$ (or $\beta^\mu A_\perp$). Owing to these facts, the discussions of the renormalization of VQED is just a step by step business.

Let us consider an n-vertex one-particle-irreducible (1PI) diagram in VQED, in which there are $N$ external Dirac-like particle lines and $N'$ external photon lines. Taking use of the similarity between VQED and SQED \cite{24}, we obtain the superficial degree of divergence of the 1PI diagram, say $D$, as follow:

$$D = 4 - N' - \frac{3}{2} N,$$

from which we draw the following conclusions:

1. $D$ is independent of the number $n$ of the vertices (i.e. the order $n$ of perturbation).
2. $D$ depends only on the number of external lines of the 1PI diagram, and there are only a finite number of external lines with $D \geq 0$.

From (1) and (2), we come to the conclusion that VQED is renormalizable.

References

[1] P. A. M. Dirac, Proc. R. Soc. London, Ser. A155, 447(1936).

[2] H. J. Bhabha, Rev. Mod. Phys. 17, 200(1945); \textit{ibid}, 21, 451(1949); S. Weinberg, Phys. Rev. 133, B1318(1964); \textit{ibid}, 134, B882(1964); D. L. Pursey, Ann. Phys(N. Y)32, 157(1965); W. K. Tung, Phys. Rev. Lett. 16, 763(1966); Phys. Rev. 156, 1385(1967); W. J. Hurley, Phys. Rev. Lett. 29, 1475(1972).

[3] O. Klein, Z. Phys. 37, 895(1926).

[4] W. Gordon, Z. Phys. 40, 117(1926).

[5] A. Proca, C. R. Acad, Sci(Paris)202, 1420(1936).

[6] N. Kemmer, Proc. R. Soc. London. 173, 91(1939).

[7] D. J. Duffin, Phys. Rev. 54, 1114(1938).

[8] G. Petiau, Ph. D. thesis, University of Paris, 1936; Acad. R. Belg. Cl. Sci. Mem. Collect. 8, 16(1936).

[9] G. Velo and D. Zwanziger, Phys. Rev. 186, 1337(1969); \textit{ibid}, 188, 2218(1969).

[10] J. V. Pereira, Int. J. Theor. Phys. 5, 447(1972); D. V. Ahluwalla \textit{et al}, Mod. Phys. Lett. A7, 1967(1992).
[11] M. Seetharaman, I. Prabhakaran and P. M. Matthews, Phys. Rev. D12, 458(1975).

[12] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191(1954).

[13] J. R. Oppenheimer, Phys. Rev. 38, 725(1931); R. H. Good, Phys. Rev. 105, 1914(1957); H. E. Moses, Phys. Rev. 113, 1670(1959); J. S. Lomont, Phys. Rev. 111, 1710(1958); E. Giannetto, Lett. Nuovo Cimento, 44, 140(1985).

[14] K. Moriyasu, An elementary primer for gauge theory, World Scientific, 1983.

[15] E. R. Caianiello and W. Guz, Lett. Nuovo Cimento, 43, 1(1985).

[16] Kishor C. Tripathy, Phys. Rev. D2, 2955(1970).

[17] W. Krolikowski, Phys. Rev. D45, 3222(1992); ibid, D46, 5188(1992); IL. Cimento, A, 107, 69(1994).

[18] W. J. Hurley, Phys. Rev. D10, 1185(1974).

[19] H. Joos, Fortschr. Phys. 10, 65(1962).

[20] J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, McGraw-Hill, New York, 1965.

[21] E. Schrodinger, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. 24, 418(1930); ibid, 3, 1(1931); K. Huang, Am. J. Phys. 20, 479(1952); H. Jehle, Phys, Rev. D3, 306(1971); G. A. Perkins, Found. Phys. 6, 237(1976); J. A. Lock, Am. J. Phys. 47, 797(1979); A. O. Barut et al, Phys. Rev. D23, 2454(1981); ibid, D24, 3333(1981); ibid, D31, 1386(1985); Phys. Rev. Lett. 52, 2009(1984).

[22] D. Jackson, Classical Electrodynamics, Second Edition, John Wiley & Sons, Inc, 1975.

[23] S. Weinberg, Gravitation and Cosmology, Wiley, New York, 1972.

[24] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley Publishing Company, 1995.