CONVERGENCE THEOREMS FOR BARYCENTRIC MAPS

FUMIO HIAI AND YONGDO LIM

Abstract. We first develop a theory of conditional expectations for random variables with values in a complete metric space $\mathcal{M}$ equipped with a contractive barycentric map $\beta$, and then give convergence theorems for martingales of $\beta$-conditional expectations. We give the Birkhoff ergodic theorem for $\beta$-values of ergodic empirical measures and provide a description of the ergodic limit function in terms of the $\beta$-conditional expectation. Moreover, we prove the continuity property of the ergodic limit function by finding a complete metric between contractive barycentric maps on the Wasserstein space of Borel probability measures on $\mathcal{M}$. Finally, the large derivation property of $\beta$-values of i.i.d. empirical measures is obtained by applying the Sanov large deviation principle.

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1. Introduction and preliminaries

The main purpose of the present paper is to establish several convergence theorems for random variables with values in a complete metric space $(\mathcal{M}, d)$ equipped with a contractive barycentric map $\beta : \mathcal{P}^p(\mathcal{M}) \to \mathcal{M}$, where $\mathcal{P}^p(\mathcal{M})$ is the Wasserstein space of Borel probability measures with finite $p$th moment. This important class of metric spaces with contractive barycentric maps contains all Banach spaces, metric spaces that are nonpositively curved in the weak sense of Busemann, including global NPC spaces, and convex metric spaces [17, 16]. For instance, a typical convex metric space is the Banach-Finsler manifold of positive invertible operators on a Hilbert space equipped with the Thompson metric. We need no extra condition on the underlying space $\mathcal{M}$, like separability or local compactness, except only the existence of a contractive barycentric map $\beta : \mathcal{P}^p(\mathcal{M}) \to \mathcal{M}$ for some $p \in [1, \infty)$.

As usual, a barycentric map is useful to define expectations of $p$th integrable $\mathcal{M}$-valued random variables via push-forward measures. However, defining conditional expectations of random variables with values in a metric space is non-trivial, as previously discussed by Es-Sahib and Heinich [10], Sturm [23] and others (as referenced in [10, 23]). In Section 2, when a probability space is standard Borel, we introduce, by
using the disintegration theorem, the $\beta$-conditional expectation and derive its fundamental properties including the contractive and projective properties. We show that our conditional expectation coincides with Sturm’s conditional expectation \cite{23} when restricted to the canonical barycentric map on a global NPC space. In Section 3, motivated by Sturm’s martingale convergence theorem \cite{23} on a global NPC space, we obtain the convergence theorem in the sense of $L^p$ and almost everywhere convergence for $\beta$-martingales of regular type. We also discuss filtered $\beta$-martingales of Sturm’s type.

The most natural problem for contractive barycentric maps is an extension of the classical Birkhoff ergodic theorem. Ergodic type results were formerly given in \cite{10, 24} for $L^1$ or $L^2$ i.i.d. random variables in nonpositively curved spaces. More recently, Austin \cite{1} obtained an $L^2$-ergodic theorem for the canonical barycentric map on a global NPC space, and Navas \cite{21} obtained an $L^1$-ergodic theorem for a specific contractive barycentric map on a metric space of nonpositive curvature in the sense of Busemann. The paper \cite{20} contains an extension of Navas’ ergodic theorem to the parametrized version of the Cartan barycenter. In Section 4 we review the $L^p$-ergodic theorem in \cite{1} \cite{21} for the $\beta$-expectation values of the ergodic empirical measures in the setting of a general barycentric space $(M, d, \beta)$. We also provide the description of the ergodic limit function in terms of the $\beta$-conditional expectation.

There exists many distinct contractive barycentric maps on a fixed barycentric space $(M, d, \beta)$; for instance, see Remark 6.4 and Example 6.5 of \cite{24}. In Section 5 we study perturbations for the ergodic convergence theorem varying over contractive barycentric maps. We introduce a complete metric on the set of all $p$-contractive barycentric maps on $M$ and then show the continuity of the ergodic limit function varying over the pairs of barycentric maps and $p$th integrable random variables. For the global NPC space case, we construct a semiflow of contractive barycentric maps such that the canonical barycentric map plays as a global attractor fixed point. The convergence of ergodic limits along any trajectory of barycentric maps to that of the canonical barycentric map is established as an application of our $\beta$-convergence theorems.

Finally, in Section 6 we present the large derivation principle for the $\beta$-values of the empirical measures of $M$-valued i.i.d. random variables, which is a stronger version of Sturm’s empirical law of large numbers \cite{24}.

In order to give precise formulations of the above results, one needs to recall some backgrounds on measurable $M$-valued functions, Borel probability measures on $M$, and so on, which are summarized in the rest of this introductory section.

Let $(M, d)$ be a complete metric space and $\mathcal{B}(M)$ be the $\sigma$-algebra of Borel subsets of $M$. Let $\mathcal{P}(M)$ be the set of all probability measures on $\mathcal{B}(M)$ with full support, and $\mathcal{P}_0(M)$ be the set of $\mu \in \mathcal{P}(M)$ of the form $\mu = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j}$ with some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in M$. We note \cite{13} that every $\mu \in \mathcal{P}(M)$ has separable support and is the
week limit of a sequence of finitely supported measures. For \(1 \leq p < \infty\) let \(\mathcal{P}^p(M)\) be the set of \(\mu \in \mathcal{P}(M)\) such that \(\int_M d^p(x, y) \, d\mu(y) < \infty\) for some (equivalently, for all) \(x \in M\), and \(\mathcal{P}^\infty(M)\) be the set of \(\mu \in \mathcal{P}(M)\) with bounded support, i.e., \(\mu\) is supported on \(\{y \in M : d(x, y) \leq \alpha\}\) for some \(x \in M\) and some \(\alpha < \infty\). Obviously,

\[
\mathcal{P}^1(M) \supset \mathcal{P}^p(M) \supset \mathcal{P}^q(M) \supset \mathcal{P}^\infty(M) \quad \text{for} \quad 1 < p < q < \infty.
\]

(1.1)

For \(1 \leq p < \infty\) the \(p\)-Wasserstein distance on \(\mathcal{P}^p(M)\) is defined as

\[
d^W_p(\mu, \nu) := \left[ \inf_{\pi \in \Pi(\mu, \nu)} \int_{M \times M} d^p(x, y) \, d\pi(x, y) \right]^{1/p}, \quad \mu, \nu \in \mathcal{P}^p(M),
\]

where \(\Pi(\mu, \nu)\) denotes the set of \(\pi \in \mathcal{P}(M \times M)\) such that \(\pi(B \times M) = \mu(B)\) and \(\pi(M \times B) = \nu(B)\) for all \(B \in \mathcal{B}(M)\). Moreover, for \(p = \infty\) we define

\[
d^W_\infty(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \sup \{d(x, y) : (x, y) \in \text{supp}(\pi)\}, \quad \mu, \nu \in \mathcal{P}^\infty(M).
\]

Note that

\[
d^W_1 \leq d^W_p \leq d^W_\infty \quad \text{for} \quad 1 < p < \infty.
\]

(1.2)

It is well-known [24] that \(d^W_p\) is a complete metric on \(\mathcal{P}^p(M)\) for \(1 \leq p \leq \infty\) and \(\mathcal{P}_0(M)\) is dense in \(\mathcal{P}^p(M)\) for \(1 \leq p < \infty\).

Let \((\Omega, \mathcal{A}, \mathcal{P})\) be a probability space. A Borel measurable function \(\varphi : \Omega \to M\) (i.e., measurable with respect to \(\mathcal{A}\) and \(\mathcal{B}(M)\) or \(M\)-valued random variable) is strongly measurable if there exists a sequence \(\{\varphi_n\}\) of \(M\)-valued simple functions, i.e., \(\varphi_n(\omega) = \sum_{j=1}^{K_n} A_{n,j} x_{n,j}\) with \(A_{n,j} \in \mathcal{A}\) and \(x_{n,j} \in M\), such that \(d(\varphi_n(\omega), \varphi(\omega)) \to 0\) for a.e. \(\omega \in \Omega\). From the definition it follows that if \(\varphi : \Omega \to M\) is strongly measurable, then there exists a \(\mathcal{P}\)-null set \(N \in \mathcal{A}\) for which \(\{\varphi(\omega) : \omega \in \Omega \setminus N\}\) is a separable subset of \(M\) and for any \(x \in M\) the function \(\omega \in \Omega \setminus N \mapsto d(x, \varphi(\omega))\) is \(\mathcal{A}\)-measurable. Hence the integral \(\int_{\Omega} d^p(x, \varphi(\omega)) \, d\mathcal{P}(\omega)\) makes sense for any \(p \in (0, \infty)\). For each \(p \in [1, \infty)\), we say that a function \(\varphi : \Omega \to M\) is \(p\)th Bochner integrable if \(\varphi\) is strongly measurable and \(\int_{\Omega} d^p(x, \varphi(\omega)) \, d\mathcal{P}(\omega) < \infty\) for some (equivalently, for all) \(x \in M\). We denote by

\[
L^p(\Omega; M) = L^p(\Omega, \mathcal{A}, \mathcal{P}; M)
\]

the set of all \(M\)-valued \(p\)th Bochner integrable functions. We also denote by

\[
L^\infty(\Omega; M) = L^\infty(\Omega, \mathcal{A}, \mathcal{P}; M)
\]

the set of all strongly measurable functions \(f : \Omega \to M\) such that \(d(x, f(\omega))\) is essentially bounded for some (equivalently, for all) \(x \in M\). As usual, for \(\varphi, \psi \in L^p(\Omega; M)\) we consider \(\varphi = \psi\) whenever \(\varphi(\omega) = \psi(\omega)\) a.e. Obviously,

\[
L^1(\Omega; M) \supset L^p(\Omega; M) \supset L^q(\Omega; M) \supset L^\infty(\Omega; M) \quad \text{for} \quad 1 < p < q < \infty.
\]

(1.3)

The theory of Bochner integrable functions mostly treats measurable functions with values in a Banach space (see, e.g., [7]), but basic definitions and results are valid for
measurable functions with values in a complete metric space as well. For instance, a standard argument gives:

**Lemma 1.1.** For every $1 \leq p \leq \infty$, the set $L^p(\Omega; M)$ is a complete metric space with the usual $L^p$-distance

$$d_p(\varphi, \psi) := \left[ \int_{\Omega} d^p(\varphi(\omega), \psi(\omega)) dP(\omega) \right]^{1/p} \quad \text{if } 1 \leq p < \infty,$$

and $d_\infty(\varphi, \psi) := \text{ess sup}_{\omega \in \Omega} d(\varphi(\omega), \psi(\omega))$ for $p = \infty$. The set of $M$-valued simple functions is dense in $L^p(\Omega; M)$ for $1 \leq p < \infty$, and the set of countably valued functions in $L^\infty(\Omega; M)$ is dense in $L^\infty(\Omega; M)$.

**Lemma 1.2.** Let $1 \leq p \leq \infty$.

1. If $\varphi \in L^p(\Omega; M)$, then the push-forward measure $\varphi_* P$ by $\varphi$ belongs to $\mathcal{P}^p(M)$.
2. If $\varphi, \psi \in L^p(\Omega; M)$, then $d^W(\varphi_* P, \psi_* P) \leq d_p(\varphi, \psi)$.

**Proof.** (1) Let $\varphi \in L^p(\Omega; M)$. There exists a separable closed set $M_0 \subset M$ such that $\varphi(\omega) \in M_0$ for a.e. $\omega \in \Omega$. Since $(\varphi_* P)(M_0) = P(\varphi^{-1}(M_0)) = 1$, supp$(\varphi_* P) \subset M_0$ and so $\varphi_* P \in \mathcal{P}(M)$. Moreover, when $1 \leq p < \infty$,

$$\int_M d^p(x, y) d(\varphi_* P)(y) = \int_{\Omega} d^p(x, \varphi(\omega)) dP(\omega) < \infty.$$

When $p = \infty$, we have $d(x, \varphi(\omega)) \leq \alpha$ a.e. for some $\alpha < \infty$, and hence supp$(\varphi_* P) \subset \{y \in M : d(x, y) \leq \alpha\}$.

(2) Let $\varphi, \psi \in L^p(\Omega; M)$. Set $\pi := (\varphi \times \psi)_* P$, the push-forward of $P$ by the map $\omega \in \Omega \mapsto (\varphi(\omega), \psi(\omega)) \in M \times M$. As in the proof of (1), we have $\pi \in \mathcal{P}(M \times M)$. For any $B \in \mathcal{B}(M)$, $\pi(B \times M) = P(\varphi^{-1}(B))$ and $\pi(M \times B) = P(\psi^{-1}(B))$, so $\pi \in \Pi(\varphi_* P, \psi_* P)$. Therefore, when $1 \leq p < \infty$,

$$d^W(\varphi_* P, \psi_* P) \leq \left[ \int_{M \times M} d^p(x, y) d(\varphi \times \psi)_* P(x, y) \right]^{1/p} = \left[ \int_{\Omega} d^p(\varphi(\omega), \psi(\omega)) dP(\omega) \right]^{1/p} = d_p(\varphi, \psi).$$

When $p = \infty$, let $\alpha := d_\infty(\varphi, \psi)$ and $\Delta := \{(x, y) \in M \times M : d(x, y) \leq \alpha\}$. Then $\pi(\Delta) = 1$, and we have supp$(\pi) \subset \Delta$, so $d^W(\varphi_* P, \psi_* P) \leq \alpha$. $\square$

The following lemma will play an essential role for our purpose. In fact, a similar inequality follows by specializing [26] Proposition 7.10 to $\mu = \sum_{i=1}^K \alpha_i \delta_{x_i}$ and $\nu = \sum_{i=1}^K \beta_i \delta_{x_i}$. The following proof is a modification (in the specialized situation) of that in [26].
Lemma 1.3. Assume that \(1 \leq p < \infty\). Let \(x_1, \ldots, x_K \in M\), and \((\alpha_1, \ldots, \alpha_K)\) and \((\beta_1, \ldots, \beta_K)\) be probability vectors. Then
\[
d_p^W \left( \sum_{i=1}^{K} \alpha_i \delta_{x_i}, \sum_{i=1}^{K} \beta_i \delta_{x_i} \right) \leq \Delta \left[ \frac{1}{2} \sum_{i=1}^{K} |\alpha_i - \beta_i| \right]^{1/p},
\]
where \(\Delta := \text{diam}\{x_1, \ldots, x_K\}\), the diameter of \(\{x_1, \ldots, x_K\}\).

Proof. Let \(\gamma_i := \min\{\alpha_i, \beta_i\}\) for \(1 \leq i \leq K\), \(I := \{i : \alpha_i > \gamma_i\}\) and \(J := \{i : \beta_i > \gamma_i\}\).

It is clear that \(I \cap J = \emptyset\) and
\[
\sum_{i \in I} (\alpha_i - \gamma_i) = \sum_{i=1}^{K} (\alpha_i - \gamma_i) = \sum_{j \in J} (\beta_j - \gamma_j) = \frac{1}{2} \sum_{j=1}^{K} (\alpha_j - \beta_j).
\]

Let \(\rho_{ij} := (\alpha_i - \gamma_i) (\beta_j - \gamma_j)/\sum_{k \in J} (\beta_k - \gamma_k)\) for \(i \in I\) and \(j \in J\); then it is immediate to check that \(\alpha_i - \gamma_i = \sum_{j \in J} \rho_{ij}\) (\(i \in I\)) and \(\beta_j - \gamma_j = \sum_{i \in I} \rho_{ij}\) (\(j \in J\)). One can define \(\pi \in \Pi(\sum_{i} \alpha_i \delta_{x_i}, \sum_{i} \beta_i \delta_{x_i})\) by \(\pi := \sum_{i=1}^{K} \gamma_i \delta_{(x_i, x_i)} + \sum_{i \in I, j \in J} \rho_{ij} \delta_{(x_i, x_j)}\).

Therefore,
\[
d_p^W \left( \sum_{i=1}^{K} \alpha_i \delta_{x_i}, \sum_{i=1}^{K} \beta_i \delta_{x_i} \right) \leq \left[ \int_{M \times M} d_p^P(x, y) d\pi(x, y) \right]^{1/p} = \left[ \sum_{i \in I, j \in J} \rho_{ij} d_p^P(x_i, x_j) \right]^{1/p} \leq \Delta \left[ \sum_{i \in I} (\alpha_i - \gamma_i) \right]^{1/p} = \Delta \left[ \frac{1}{2} \sum_{i=1}^{K} |\alpha_i - \beta_i| \right]^{1/p}.\]

\(\Box\)

Let \((X, \leq)\) be a partially ordered set. For a nonempty subset \(A\) of \(X\), let \(\uparrow A := \{y \in X : x \leq y\}\) for some \(x \in A\). We say that \(A\) is an upper set if \(\uparrow A = A\). Assume that a complete metric space \(M\) is equipped with a closed partial order \(\leq\); i.e., \(\{(x, y) : x \leq y\}\) is closed in \(M \times M\) equipped with the product topology. The stochastic order on \(\mathcal{P}(M)\) introduced in [13] is defined by \(\mu \leq \nu\) if \(\mu(U) \leq \nu(U)\) for every open upper set \(U\), several equivalent conditions of \(\mu \leq \nu\) were given in [13]. We note from [15, 13] that for \(\mu = \frac{1}{n} \sum_{j=1}^{n} \delta_{a_j}\) and \(\nu = \frac{1}{n} \sum_{j=1}^{n} \delta_{b_j}\), \(\mu \leq \nu\) if and only if there exists a permutation \(\sigma\) on \(\{1, \ldots, n\}\) such that \(a_j \leq b_{\sigma(j)}\) for all \(j = 1, \ldots, n\).

Assume that \(E\) is a real Banach space containing an open convex cone \(C\) such that \(\overline{C}\) is a normal cone (cf. [5]). The cone \(\overline{C}\) defines a closed partial order on \(E\) (hence on \(C\)) by \(x \leq y\) if \(y - x \in \overline{C}\). Moreover, \(C\) is a complete metric space with the Thompson metric [23, 22] defined by \(d_T(x, y) := \max\{\log M(x/y), \log M(y/x)\}\), where \(M(x/y) := \inf\{\lambda > 0 : x \leq \lambda y\}\). Note that the \(d_T\)-topology on \(C\) coincides with the relative topology inherited from \(E\). Hence we may consider \(\mathcal{P}(C)\) on \((C, d_T)\). Then it was shown in [13] that the stochastic order on \(\mathcal{P}(C)\) is a partial order. This is typically the case when \(E\) is the algebra \(S(H)\) with the operator norm, consisting of self-adjoint bounded linear operators on a Hilbert space \(H\), and \(C\) is the cone \(\mathbb{P}(H)\) of positive invertible operators on \(H\).
Now, assume that a complete metric space \( M \) is equipped with a closed partial order. For strongly measurable \( M \)-values functions \( \varphi, \psi \) on \( \Omega \), we define \( \varphi \leq \psi \) if \( \varphi(\omega) \leq \psi(\omega) \) a.e. (The definition makes sense since \( \{ \omega : \varphi(\omega) \leq \psi(\omega) \} \) is measurable up to a \( P \)-null set.)

**Lemma 1.4.** If \( \varphi, \psi : \Omega \to M \) are strongly measurable and \( \varphi \leq \psi \), then \( \varphi_* P \leq \psi_* P \).

**Proof.** Assume that \( \varphi(\omega) \leq \psi(\omega) \) for all \( \omega \in \Omega \setminus N \) with a \( P \)-null set \( N \). Let \( U \) be an open upper set. If \( \omega \in \varphi^{-1}(U) \cap (\Omega \setminus N) \), then \( \varphi(\omega) \in U \) and \( \varphi(\omega) \leq \psi(\omega) \), so \( \psi(\omega) \in U \). Hence \( \varphi^{-1}(U) \cap (\Omega \setminus N) \subset \psi^{-1}(U) \), so that \( P(\varphi^{-1}(U)) \leq P(\psi^{-1}(U)) \), implying \( \varphi_* P \leq \psi_* P \). \qed

2. **Conditional expectations**

In this section, let \( 1 \leq p \leq \infty \) be fixed, and assume that \( \beta : \mathcal{P}^p(M) \to M \) is a \( p \)-contractive barycentric map, that is, \( \beta(\delta_x) = x \) for all \( x \in M \) and

\[
\delta(\beta(\mu), \beta(\nu)) \leq d_p^W(\mu, \nu), \quad \mu, \nu \in \mathcal{P}^p(M).
\]  

**Definition 2.1.** Let \( \varphi \in L^p(\Omega; M) \).

1. Define the \( \beta \)-expectation \( E^\beta(\varphi) \in M \) of \( \varphi \) by

\[
E^\beta(\varphi) := \beta(\varphi_* P).
\]

This is well defined by Lemma 1.2(1).

2. For every \( A \in \mathcal{A} \) with \( P(A) > 0 \), consider the reduced probability space

\[
(A, \mathcal{A} \cap A, P_A) \quad \text{where} \quad P_A := P(A) \mathcal{P}|_{\mathcal{A} \cap A}.
\]

Let \( E^\beta(\varphi|_A) \) be the \( \beta \)-expectation of \( \varphi|_A \) on \( (A, \mathcal{A} \cap A, P_A) \), i.e.,

\[
E^\beta(\varphi|_A) := \beta((\varphi|_A)_* P_A).
\]

**Proposition 2.2.** Let \( \varphi, \psi \in L^p(\Omega; M) \).

1. \( d(E^\beta(\varphi), E^\beta(\psi)) \leq d_p(\varphi, \psi) \).
2. \( E^\beta(1_\Omega x) = x \) for all \( x \in M \).
3. Assume that \( M \) is equipped with a closed partial order and \( \beta \) is monotone, that is, for each \( \mu, \nu \in \mathcal{P}^p(M) \), \( \mu \leq \nu \) implies \( \beta(\mu) \leq \beta(\nu) \). If \( \varphi \leq \psi \), then \( E^\beta(\varphi) \leq E^\beta(\psi) \).

**Proof.** (1) By (2.1) and Lemma 1.2(2),

\[
d(E^\beta(\varphi), E^\beta(\psi)) = d(\beta(\varphi_* P), \beta(\psi_* P)) \leq d_p^W(\varphi_* P, \psi_* P) \leq d_p(\varphi, \psi).
\]

(2) Since \( (1_\Omega x)_* P = \delta_x \), \( E^\beta(1_\Omega x) = \beta(\delta_x) = x \).

(3) is obvious from Lemma 1.4 \( \square \)
Proposition 2.3. Let $\varphi, \psi \in L^p(\Omega; M)$. If $E^\beta(\varphi |A) = E^\beta(\psi |A)$ for all $A \in \mathcal{A}$ with $P(A) > 0$, then $\varphi = \psi$.

Proof. Assume that $\varphi \neq \psi$; then there exists a $\delta > 0$ such that

$$P\{\omega \in \Omega : d(\varphi(\omega), \psi(\omega)) > \delta\} > 0.$$  

One can choose a sequence $\{x_n\}_{n=1}^\infty$ in $M$ such that $\varphi(\omega), \psi(\omega) \in \{x_n\}$ a.e. For $m, n = 1, 2, \ldots$, let

$$A_{m,n} := \{\omega \in \Omega : d(\varphi(\omega), \psi(\omega)) > \delta, d(x_m, \varphi(\omega)) \leq \delta/4, d(x_n, \psi(\omega)) \leq \delta/4\}.$$  

Since $P(\bigcup_{m,n=1}^\infty A_{m,n}) = P\{d(\varphi(\omega), \psi(\omega)) > \delta\} > 0$, one can choose $m, n$ such that $P(A_{m,n}) > 0$. For $\omega \in A_{m,n}$ we have

$$\delta < d(\varphi(\omega), \psi(\omega)) \leq d(\varphi(\omega), x_m) + d(x_m, x_n) + d(x_n, \psi(\omega)) < d(x_m, x_n) + \delta/2,$$

so that $d(x_m, x_n) > \delta/2$. Let $\varphi_n := 1_\Omega x_m$ and $\psi_n := 1_\Omega x_n$ be constant functions. For $A = A_{m,n}$ we have $E^\beta(\varphi_0 |A) = x_m$ and $E^\beta(\psi_0 |A) = x_n$. Moreover, by Proposition 2.2 (1),

$$d(E^\beta(\varphi |A), x_m) = d(E^\beta(\varphi |A), E^\beta(\varphi_0 |A)) \leq d_p(\varphi |A, \varphi_0 |A) \leq \delta/4,$$

$$d(E^\beta(\psi |A), x_n) = d(E^\beta(\psi |A), E^\beta(\psi_0 |A)) \leq d_p(\psi |A, \psi_0 |A) \leq \delta/4.$$  

Since $E^\beta(\varphi |A) = E^\beta(\psi |A)$ by assumption, we have $d(x_m, x_n) \leq \delta/2$, a contradiction. \qed

Recall that $(\Omega, \mathcal{A})$ is a standard Borel space if it is isomorphic to $(X, \mathcal{B}(X))$ of a Polish space $X$ and its Borel $\sigma$-algebra $\mathcal{B}(X)$. In the rest of this section, unless otherwise stated, we assume that $(\Omega, \mathcal{A}, P)$ is a probability space over a standard Borel space $(\Omega, \mathcal{A})$ and $\mathcal{B}$ is a sub-$\sigma$-algebra of $\mathcal{A}$. To introduce the notion of the $\beta$-conditional expectation with respect to $\mathcal{B}$, we utilize the disintegration theorem, which we state as a lemma for convenience. For details see [11 Theorem 5.8] (where a probability measure space on a standard Borel space is called a regular measure space).

Remark 2.4. It is known [3 Corollary 10.4.6] that if $X$ is a Souslin space (i.e., a continuous image of a Polish space), then for any probability measure $P$ on $\mathcal{B}(X)$ and every sub-$\sigma$-algebra $\mathcal{B}$ of $\mathcal{B}(X)$ there exists a disintegration of $P$ with respect to $\mathcal{B}$. Thus, the results of this paper when $(\Omega, \mathcal{A})$ is a standard Borel space are also true with a bit weaker assumption that $(\Omega, \mathcal{A})$ is isomorphic to $(X, \mathcal{B}(X))$ of a Souslin space $X$.

Lemma 2.5. There exists a family $(P_\omega)_{\omega \in \Omega}$ of probability measures on $(\Omega, \mathcal{A})$ such that for every $A \in \mathcal{A}$,

- (i) $\omega \in \Omega \mapsto P_\omega(A)$ is $\mathcal{B}$-measurable, and
- (ii) $\omega \mapsto P_\omega(A)$ is a conditional expectation of $1_A$ with respect to $\mathcal{B}$.

Such a family $(P_\omega)_{\omega \in \Omega}$ is unique up to a $P$-null set, and moreover it satisfies the following:
(iii) for every \( f \in L^1(\Omega; \mathbb{R}) \), \( f \in L^1(\Omega, A, \mathbb{P}_\omega; \mathbb{R}) \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( \omega \mapsto \int_\Omega f(\tau) \, d\mathbb{P}_\omega(\tau) \) is a conditional expectation of \( f \) with respect to \( \mathcal{B} \). In particular,
\[
\int_\Omega f \, d\mathbb{P} = \int_\Omega \left[ \int_\Omega f(\tau) \, d\mathbb{P}_\omega(\tau) \right] \, d\mathbb{P}(\omega).
\]

The family \((\mathbb{P}_\omega)_{\omega \in \Omega}\) given in the above lemma is called a **disintegration** of \( \mathbb{P} \) with respect to \( \mathcal{B} \). The next lemma is easily seen from the primary property (ii) of the above lemma, while we supply the proof for completeness.

**Lemma 2.6.** Let \((\mathbb{P}_\omega)_{\omega \in \Omega}\) be a disintegration of \( \mathbb{P} \) with respect to \( \mathcal{B} \). For every \( \psi \in L^1(\Omega, \mathcal{B}, \mathbb{P}; M) \), there is a \( \mathbb{P} \)-null set \( N \in \mathcal{B} \) such that for every \( \omega \in \Omega \setminus N \), \( \psi(\tau) \) is constant for \( \mathbb{P}_\omega \)-a.e. \( \tau \in \Omega \).

**Proof.** Note that \( \psi(\tau) \) is constant for \( \mathbb{P}_\omega \)-a.e. if and only if \( \psi \circ \mathbb{P}_\omega \) is singly supported. Choose a countable set \( \{x_i\}_{i=1}^\infty \in M \) such that \( \psi(\omega) \in \{x_i\} \) a.e. For \( x \in M \) and \( k \in \mathbb{N} \), set \( U_{1/k}(x) := \{y \in M : d(y, x) < 1/k\} \), the open ball of center \( x \) and radius \( 1/k \). Let \( \{(U_n, V_n)\}_{n=1}^\infty \) be an enumeration of all pairs \((U_{1/k}(x_i), U_{1/k}(x_j))\) such that \( U_{1/k}(x_i) \cap U_{1/k}(x_j) = \emptyset \) with \( i, j \in \mathbb{N} \). Then it is easy to see that \( \psi \circ \mathbb{P}_\omega \) is singly supported if and only if \((\psi \circ \mathbb{P}_\omega)(U_n) \cdot (\psi \circ \mathbb{P}_\omega)(V_n) = 0 \) for all \( n \), that is, \( \mathbb{P}_\omega(\psi^{-1}(U_n)) \cdot \mathbb{P}_\omega(\psi^{-1}(V_n)) = 0 \) for all \( n \). Since \( \psi^{-1}(U_n), \psi^{-1}(V_n) \in \mathcal{B} \), it follows from Lemma 2.5(ii) that
\[
\mathbb{P}_\omega(\psi^{-1}(U_n)) = 1_{\psi^{-1}(U_n)}(\omega), \quad \mathbb{P}_\omega(\psi^{-1}(V_n)) = 1_{\psi^{-1}(V_n)}(\omega) \quad \text{a.e.}
\]
so that \( \mathbb{P}_\omega(\psi^{-1}(U_n)) \cdot \mathbb{P}_\omega(\psi^{-1}(V_n)) = 0 \) a.e. Hence there is a \( \mathbb{P} \)-null set \( N \in \mathcal{B} \) such that for every \( \omega \in \Omega \setminus N \) we have \( \mathbb{P}_\omega(\psi^{-1}(U_n)) \cdot \mathbb{P}_\omega(\psi^{-1}(V_n)) = 0 \) for all \( n \), so \( \psi(\tau) \) is constant for \( \mathbb{P}_\omega \)-a.e. \( \square \)

**Lemma 2.7.** Let \((\mathbb{P}_\omega)_{\omega \in \Omega}\) be a disintegration of \( \mathbb{P} \) with respect to \( \mathcal{B} \). Let \( 1 \leq p < \infty \).

1. If \( \varphi \in L^p(\Omega; M) \), then there is a \( \mathbb{P} \)-null \( N \in \mathcal{B} \) such that \( \varphi \in L^p(\Omega, A, \mathbb{P}_\omega; M) \) and \( \varphi \circ \mathbb{P}_\omega \in \mathcal{P}^p(M) \) for all \( \omega \in \Omega \setminus N \).
2. If \( \varphi, \psi \in L^p(\Omega; M) \), then there is a \( \mathbb{P} \)-null \( N \in \mathcal{B} \) such that
\[
d(\beta(\varphi \circ \mathbb{P}_\omega), \beta(\psi \circ \mathbb{P}_\omega)) \leq \left[ \int_\Omega d^p(\varphi(\tau), \psi(\tau)) \, d\mathbb{P}_\omega(\tau) \right]^{1/p}, \quad \omega \in \Omega \setminus N. \tag{2.2}
\]

**Proof.** (1) Let \( \varphi \in L^p(\Omega; M) \) and \( x \in M \). Since \( \omega \mapsto d^p(x, \varphi(\omega)) \) is in \( L^1(\Omega; \mathbb{R}) \), it follows from Lemma 2.5(iii) that
\[
\int_\Omega \left[ \int_\Omega d^p(x, \varphi(\tau)) \, d\mathbb{P}_\omega(\tau) \right] \, d\mathbb{P}(\omega) = \int_\Omega d^p(x, \varphi(\omega)) \, d\mathbb{P}(\omega) < \infty.
\]
Hence there is a \( \mathbb{P} \)-null \( N \in \mathcal{B} \) such that for every \( \omega \in \Omega \setminus N \) we have
\[
\int_\Omega d^p(x, \varphi(\tau)) \, d\mathbb{P}_\omega(\tau) < \infty, \quad \text{i.e.,} \quad \varphi \in L^p(\Omega, A, \mathbb{P}_\omega; M)
\]
so that \( \varphi \circ \mathbb{P}_\omega \in \mathcal{P}^p(M) \) by Lemma 1.2(1).
(2) Let \( \varphi, \psi \in L^p(\Omega; M) \). By (1) there is a \( P \)-null \( N \in \mathcal{B} \) such that \( \varphi_*, P_\omega, \psi_*, P_\omega \in \mathcal{P}^p(M) \) for all \( \omega \in \Omega \setminus N \). For such \( \omega \), by (2.1) and Lemma 1.2 (2) (applied to \( P_\omega \) in place of \( P \)) we have

\[
d(\beta(\varphi_*, P_\omega), \beta(\psi_*, P_\omega)) \leq d_p^w(\varphi_*, P_\omega, \psi_*, P_\omega) \leq \left[ \int_\Omega d_p(\varphi(\tau), \psi(\tau)) dP_\omega(\tau) \right]^{1/p}.
\]

\[\Box\]

Now, assume that \( 1 \leq p < \infty \) and \( \beta : \mathcal{P}^p(M) \to M \) is a \( p \)-contractive barycentric map.

**Definition 2.8.** By using the disintegration \( (P_\omega)_{\omega \in \Omega} \) of \( P \) with respect to \( \mathcal{B} \), for each \( \varphi \in L^p(\Omega, \mathcal{A}, P; M) \), define the \( \beta \)-conditional expectation of \( \varphi \) with respect to \( \mathcal{B} \) by

\[
E^\beta_B(\varphi)(\omega) := \beta(\varphi_*, P_\omega), \quad \omega \in \Omega.
\]

The above definition makes sense by Lemma 2.7 (1) but the \( \mathcal{B} \)-strong measurability of \( E^\beta_B(\varphi) \) is proved in (1) of the next theorem. This implies that the left-hand side of (2.2) is a \( \mathcal{B} \)-measurable function of \( \omega \), while the measurability of the right-hand side is seen from Lemma 2.3 (iii). The following shows in particular that the conditional expectation \( E^\beta_B : L^p(\Omega, \mathcal{A}, P; M) \to L^p(\Omega, \mathcal{B}, P; M) \) is well defined and is a contractive retraction.

**Theorem 2.9.** Let \( \varphi, \psi \in L^p(\Omega; M) \).

1. \( E^\beta_B(\varphi) \in L^p(\Omega, \mathcal{B}, P; M) \).
2. \( d_p(E^\beta_B(\varphi), E^\beta_B(\psi)) \leq d_p(\varphi, \psi) \).
3. \( \varphi \in L^p(\Omega, \mathcal{B}, P; M) \) if and only if \( E^\beta_B(\varphi) = \varphi \). Hence \( E^\beta_B(E^\beta_B(\varphi)) = E^\beta_B(\varphi) \).
4. When \( \mathcal{B} = \{\emptyset, \Omega\} \), \( E^\beta_B(\varphi) = E^\beta(\varphi) \).
5. Assume that \( M \) is equipped with a closed partial order and \( \beta \) is monotone. If \( \varphi \leq \psi \), then \( E^\beta_B(\varphi) \leq E^\beta_B(\psi) \).

**Proof.** (1) First, assume that \( \varphi \) is a simple function, i.e., \( \varphi = \sum_{j=1}^K 1_{A_j} x_j \), where \( \{A_1, \ldots, A_K\} \) is a measurable partition of \( \Omega \). Since \( \varphi_* P_\omega = \sum_{j=1}^K P_\omega(A_j) \delta_{x_j} \), one has

\[
E^\beta_B(\varphi)(\omega) = \beta \left( \sum_{j=1}^K P_\omega(A_j) \delta_{x_j} \right) \quad (2.3)
\]

By Lemma 2.3 (ii) one has \( \sum_{j=1}^K P_\omega(A_j) = 1 \) for all \( \omega \in \Omega \setminus N \) with a \( P \)-null set \( N \in \mathcal{B} \). For each \( k \in \mathbb{N} \) and \( \omega \in \Omega \setminus N \), approximating \( P_\omega(A_j) \) \( (1 \leq j \leq K) \) with numbers \( l/k \) \( (0 \leq l \leq k) \) one can choose sequences \( \{\xi_{jk}\}_{k=1}^\infty \) \( (1 \leq j \leq K) \) of \( \mathcal{B} \)-simple functions \( \xi_{jk} : \Omega \to [0, 1] \) such that \( \sum_{j=1}^K \xi_{jk}(\omega) = 1 \) for all \( \omega \in \Omega \) and \( k \in \mathbb{N} \), and \( \operatorname{ess sup}_{\omega \in \Omega} |\xi_{jk}(\omega) - P_\omega(A_j)| \leq 1/k \) for \( 1 \leq j \leq K \). Then one has by (2.3), (2.1) and
Lemma 1.3
\[
d^\left(\sum_{j=1}^{K} \xi_{jk}(\omega) \delta_{x_j}, E_B^\beta(\varphi)(\omega)\right) \leq d^W_p\left(\sum_{j=1}^{K} \xi_{jk}(\omega) \delta_{x_j}, \sum_{j=1}^{K} P_{\omega}(A_j) \delta_{x_j}\right)
\]
\[
\leq \Delta \left[\sum_{j=1}^{K} |\xi_{jk}(\omega) - P_{\omega}(A_j)|\right]^{1/p} \rightarrow 0 \quad \text{a.e. (2.4)}
\]
as \(k \rightarrow \infty\), where \(\Delta := \text{diam}\{x_1, \ldots, x_K\}\). It is clear that \(\beta(\sum_{j=1}^{K} \xi_{jk}(\omega) \delta_{x_j})\)'s are \(\mathcal{B}\)-simple functions. Hence \(E_B^\beta(\varphi)\) is \(\mathcal{B}\)-strongly measurable. Moreover, for each \(k, l \in \mathbb{N}\), by (2.1) and Lemma 1.3 again,
\[
d^\left(\sum_{j=1}^{K} \xi_{jk}(\omega) \delta_{x_j}, \sum_{j=1}^{K} \xi_{jl}(\omega) \delta_{x_j}\right) \leq d^W_p\left(\sum_{j=1}^{K} \xi_{jk}(\omega) \delta_{x_j}, \sum_{j=1}^{K} \xi_{jl}(\omega) \delta_{x_j}\right)
\]
\[
\leq \Delta \left[\sum_{j=1}^{K} |\xi_{jk}(\omega) - \xi_{jl}(\omega)|\right]^{1/p},
\]
so that for whichever \(p \in [1, \infty)\),
\[
d^p_\mathcal{P}\left(\sum_{j=1}^{K} \xi_{jk}(\cdot) \delta_{x_j}, \sum_{j=1}^{K} \xi_{jl}(\cdot) \delta_{x_j}\right) \leq \Delta \left[\sum_{j=1}^{K} \text{ess sup}_{\omega \in \Omega} |\xi_{jk}(\omega) - \xi_{jl}(\omega)|\right]^{1/p} \rightarrow 0
\]
as \(k, l \rightarrow \infty\). Therefore, \(\beta(\sum_{j=1}^{K} \xi_{jk}(\cdot) \delta_{x_j})\) converges in \(d^p_\mathcal{P}\) as \(k \rightarrow \infty\) to an element of \(L^p(\Omega, \mathcal{B}, \mathbf{P}; M)\). Since the limit must be \(E_B^\beta(\varphi)\) due to (2.4), it follows that \(E_B^\beta(\varphi)\) is a simple function.

Next, for general \(\varphi \in L^p(\Omega; M)\) choose a sequence \(\{\varphi_k\}_{k=1}^\infty\) of simple functions in \(L^p(\Omega; M)\) such that \(d^p_\mathcal{P}(\varphi_k, \varphi) \rightarrow 0\), due to the denseness of \(M\)-valued simple functions. Then \(E_B^\beta(\varphi_k) \in L^p(\Omega, \mathcal{B}, \mathbf{P}; M)\) as proved above. By Lemmas 2.7(2) and 2.5(iii),
\[
d^p_\mathcal{P}(E_B^\beta(\varphi_k), E_B^\beta(\varphi_l)) = \int_\Omega d^p(\beta((\varphi_k)_*, \mathbf{P}_\omega), \beta((\varphi_l)_*, \mathbf{P}_\omega)) d\mathbf{P}(\omega)
\]
\[
\leq \int_\Omega \left[\int_\Omega d^p(\varphi_k(\tau), \varphi_l(\tau)) d\mathbf{P}_\omega(\tau)\right] d\mathbf{P}(\omega)
\]
\[
= d^p_\mathcal{P}(\varphi_k, \varphi_l) \rightarrow 0 \quad \text{as} \ k, l \rightarrow \infty. \quad (2.5)
\]
Moreover, by Lemma 2.7(2) there is a \(\mathbf{P}\)-null \(N_0 \in \mathcal{B}\) such that
\[
d(E_B^\beta(\varphi_k)(\omega), E_B^\beta(\varphi_l)(\omega)) = d(\beta((\varphi_k)_*, \mathbf{P}_\omega), \beta((\varphi_l)_*, \mathbf{P}_\omega))
\]
\[
\leq \left[\int_\Omega d^p(\varphi_k(\tau), \varphi_l(\tau)) d\mathbf{P}_\omega(\tau)\right]^{1/p}, \quad \omega \in \Omega \setminus N_0.
\]
Now, let \(\zeta_k(\omega) := \int_\Omega d^p(\varphi_k(\tau), \varphi(\tau)) d\mathbf{P}_\omega(\tau)\) for \(\omega \in \Omega\). Then using Lemma 2.5(iii) to the function \(d^p(\varphi_k(\omega), \varphi(\omega))\) we have \(\zeta_k \in L^1(\Omega, \mathcal{B}, \mathbf{P}; \mathbb{R})\) and
\[
\int_\Omega \zeta_k(\omega) d\mathbf{P}(\omega) = d^p_\mathcal{P}(\varphi_k, \varphi) \rightarrow 0 \quad \text{as} \ k \rightarrow \infty.
\]
Hence, by choosing a subsequence of \( \{ \zeta_k \} \) we may assume that \( \zeta_k(\omega) \to 0 \) a.e. (see \cite[p.93, Theorem D]{12}), so there is a \( \mathbf{P} \)-null \( N_1 \in \mathcal{B} \) such that \( \lim_{k \to \infty} \zeta_k(\omega) = 0 \) for all \( \omega \in \Omega \setminus N_1 \). Furthermore, for every \( k \in \mathbb{N} \), since \( E^\beta_B(\varphi_k) \) is \( \mathcal{B} \)-strongly measurable, one can choose a \( \mathcal{B} \)-simple function \( \psi_k \) and a \( B_k \in \mathcal{B} \) such that \( \mathbf{P}(B_k) < 1/k^2 \) and \( d(E^\beta_B(\varphi_k), \psi_k(\omega)) < 1/k \) for all \( \omega \in \Omega \setminus B_k \). Set \( N := N_0 \cup N_1 \cup (\limsup_k B_k) \in \mathcal{B} \). Then \( \mathbf{P}(N) = 0 \) by the Borel-Cantelli lemma, and for every \( \omega \in \Omega \setminus N \), we have \( \omega \in \Omega \setminus N_0, \omega \in \Omega \setminus N_1 \) and \( \omega \in \Omega \setminus B_k \) for all \( k \) sufficiently large, so that

\[
d(E^\beta_B(\varphi)(\omega), \psi_k(\omega)) \leq d(E^\beta_B(\varphi)(\omega), E^\beta_B(\varphi_k)(\omega)) + d(E^\beta_B(\varphi_k)(\omega), \psi_k(\omega)) \\
\leq \zeta_k(\omega)^{1/p} + \frac{1}{k} \quad \to \quad 0 \quad \text{as} \quad k \to \infty.
\]

This implies that \( E^\beta_B(\varphi) \) is \( \mathcal{B} \)-strongly measurable and \( E^\beta_B(\varphi_k)(\omega) \to E^\beta_B(\varphi)(\omega) \) a.e.

From this and (2.5) we find that \( E^\beta_B(\varphi) \) is the \( d_p \)-limit of \( E^\beta_B(\varphi_k) \) and hence (1) follows.

\[\text{(2) The proof is similar to that of the inequality in (2.5).}\]

\[\text{(3) If} \quad E^\beta_B(\varphi) = \varphi, \quad \text{then} \quad \varphi \in \mathcal{L}^p(\Omega, \mathcal{B}, \mathbf{P}; M) \quad \text{by (1).} \]

Conversely, assume that \( \varphi \in \mathcal{L}^p(\Omega, \mathcal{B}, \mathbf{P}; M) \). By approximation, we may assume that \( \varphi \) is a \( \mathcal{B} \)-simple function, i.e., \( \varphi = \sum_{j=1}^K 1_{B_j} x_j \) with a \( \mathcal{B} \)-partition \( \{ B_1, \ldots, B_n \} \) of \( \Omega \), so \( E^\beta_B(\varphi) = \beta(\sum_{j=1}^K \mathbf{P}_\omega(B_j) \delta_{x_j}) \).

Since \( \mathbf{P}_\omega(B_j) = 1_{B_j}(\omega) \) a.e. by Lemma 2.5(ii), we have

\[
E^\beta_B(\varphi)(\omega) = \sum_{j=1}^K 1_{B_j}(\omega) \beta(\delta_{x_j}) = \sum_{j=1}^K 1_{B_j}(\omega) x_j = \varphi(\omega) \quad \text{a.e.}
\]

(4) is obvious and (5) follows from Lemma 1.4. \( \square \)

**Remark 2.10.** The last paragraph of the above proof of (1) may be a bit complicated. A simpler way to construct the map \( \varphi \in \mathcal{L}^p(\Omega, M) \mapsto E^\beta_B(\varphi) \in \mathcal{L}^p(\Omega, \mathcal{B}, \mathbf{P}; M) \) is as follows: For a simple function \( \varphi : \Omega \to M, E^\beta_B(\varphi) \in \mathcal{L}^p(\Omega, \mathcal{B}, \mathbf{P}; M) \) is well defined as above. For every simple functions \( \varphi, \psi \) we have \( d_p(E^\beta_B(\varphi), E^\beta_B(\psi)) \leq d_p(\varphi, \psi) \) as in (2.5). Hence the map \( E^\beta_B \) on the simple functions can uniquely extend to \( E^\beta_B \) on \( \mathcal{L}^p(\Omega, M) \) by continuity. However, this abstract definition does not imply the \( \mathcal{B} \)-strong measurability of \( \omega \mapsto \beta(\varphi, \mathbf{P}_\omega) \), so the expression \( E^\beta_B(\varphi)(\omega) = \beta(\varphi, \mathbf{P}_\omega) \) (Definition 2.3) is not clear.

**Remark 2.11.** Assume that \( 1 \leq p_0 < \infty \) and \( \beta : \mathcal{P}^{p_0}(M) \to M \) is a \( p_0 \)-contractive barycentric map. Then in view of (1.1) and (1.2) we note that for every \( p \in [p_0, \infty] \), \( \beta |_{\mathcal{P}^p(M)} : \mathcal{P}^p(M) \to M \) is a \( p \)-contractive barycentric map. It follows from this and (1.3) that Theorem 2.9 holds for every \( p \in [p_0, \infty) \). Moreover, when \( \varphi \in L^\infty(\Omega, M) \) and \( \varphi_0 := 1_{\Omega} x_0 \in M \), one has

\[
d_p(E^\beta_B(\varphi), \varphi_0) = d_p(E^\beta_B(\varphi), E^\beta_B(\varphi_0)) \leq d_p(\varphi, \varphi_0), \quad p_0 \leq p < \infty,
\]

whose limit as \( p \to \infty \) gives \( d_\infty(E^\beta_B(\varphi), \varphi_0) \leq d_\infty(\varphi, \varphi_0) < \infty \) so that \( E^\beta_B(\varphi) \in L^\infty(\Omega, \mathcal{B}, \mathbf{P}; M) \). Also, for \( \varphi, \psi \in L^\infty(\Omega, M) \),

\[
d_\infty(E^\beta_B(\varphi), E^\beta_B(\psi)) = \lim_{p \to \infty} d_p(E^\beta_B(\varphi), E^\beta_B(\psi)) \leq \lim_{p \to \infty} d_p(\varphi, \psi) = d_\infty(\varphi, \psi).
\]
Therefore, Theorem 2.9 holds for \( p = \infty \) as well in this situation. However, it is not clear whether Theorem 2.9 holds for \( p = \infty \) when an \( \infty \)-contractive barycentric map \( \beta : \mathcal{P}^\infty(M) \to M \) is given. Note that the proof of the theorem heavily relies on Lemma 1.3 and the assumption \( 1 \leq p < \infty \) is essential for Lemma 1.3. So, when \( p = \infty \), it does not seem easy to prove that the function \( \omega \mapsto \beta(\varphi, \mu_\omega) \) is \( \mathcal{B} \)-strongly measurable.

**Example 2.12.** An important property of the conventional conditional expectation is the associativity \( E_C \circ E_B = E_C \) for sub-\( \sigma \)-algebras \( C \subset B \subset A \). However, this fails to hold for the \( \beta \)-conditional expectation. To give a counter-example, let \( M = \mathbb{P}_n \) be the Cartan-Hadamard manifold of \( n \times n \) positive definite matrices equipped with the trace metric \( ds = \|A^{-1/2} dA A^{-1/2}\|_2 = \| \text{tr}(A^{-1} dA)^2 \|^{1/2} \), and \( \beta = G \) be the Cartan barycenter (or the Karcher mean):

\[
G(\mu) = \arg\min_{\tilde{Z} \in \mathbb{P}_n} \int_{\mathbb{P}_n} \left[ d^2(Z, X) - d^2(Y, X) \right] d\mu(X).
\]

Let \( \Omega = \{1, 2, 3\} \), \( A = 2^\Omega \), and \( \mathbf{P} = (p_1, p_2, p_3) \). Let \( B = \{\emptyset, \{1\}, \{2, 3\}, \Omega\} \). Let \( \varphi = \sum_{j=1}^3 1_{(j)} A_j \) with \( A_j \in \mathbb{P}_n \). Then we have for \( S \in A \),

\[
P_1(S) = \frac{\mathbf{P}(S \cap \{1\})}{p_1}, \quad P_2(S) = \frac{p_2}{p_2 + p_3}, \quad P_3(S) = \frac{p_3}{p_2 + p_3}.
\]

Therefore,

\[
\varphi_\ast P_1 = \delta_{A_1}, \quad \varphi_\ast P_2 = \varphi_\ast P_3 = \frac{p_2}{p_2 + p_3} \delta_{A_2} + \frac{p_3}{p_2 + p_3} \delta_{A_3},
\]

so that \( E_B^G(\varphi)(1) = G(\delta_{A_1}) = A_1 \) and

\[
E_B^G(\varphi)(2) = E_B^G(\varphi)(3) = G \left( \frac{p_2}{p_2 + p_3} \delta_{A_2} + \frac{p_3}{p_2 + p_3} \delta_{A_3} \right) = A_2 \#_{p_3/(p_2 + p_3)} A_3,
\]

where \( t \mapsto A \#_t B := A^{t/2} (A^{-1/2} B A^{-1/2})^{t/2} A^{1/2} \) is the unique (up to parametrization) geodesic joining \( A \) and \( B \) (cf. [2]). Now we show that \( E_B^G \neq E_C \circ E_B^G \) (note that \( E_B^G = E_C^G \) with \( C = \{\emptyset, \Omega\} \) by Theorem 2.9(4)). Assume on the contrary that \( E_B^G = E_C \circ E_B^G \); then

\[
G(p_1 \delta_{A_1} + p_2 \delta_{A_2} + p_3 \delta_{A_3}) = A_1 \#_{p_2 + p_3} (A_2 \#_{p_3/(p_2 + p_3)} A_3),
\]

holds for all \( A_1, A_2, A_3 \in \mathbb{P}_n \) and all probabilities \( (p_1, p_2, p_3) \). Then we must have

\[
A_1 \#_{p_2 + p_3} (A_2 \#_{p_3/(p_2 + p_3)} A_3) = G(p_1 \delta_{A_1} + p_2 \delta_{A_2} + p_3 \delta_{A_3}) = G(p_2 \delta_{A_2} + p_1 \delta_{A_1} + p_3 \delta_{A_3}) = A_2 \#_{p_1 + p_3} (A_1 \#_{p_3/(p_1 + p_3)} A_3).
\]

In particular, when \( p_1 = p_2 = p_3 = 1/3 \) and \( A_3 = I \), the above becomes \( A_1 \#_{2/3} A_2^{1/2} = A_2 \#_{2/3} A_1^{1/2} \) or \( A_1 \#_{2/3} A_2^{1/2} = A_1^{1/2} \#_{1/3} A_2 \). Since this certainly fails to hold, we have a contradiction.

In view of Theorem 2.13 below, Sturm’s example in [23, Example 3.2] on the 3-spider serves as another counter-example to the associativity of the \( \beta \)-conditional expectation.
From Example 2.12 we find that the following characterization of $E^3_B(\varphi)$ of $\varphi \in L^p(\Omega; M)$ like the conventional conditional expectation is not valid:

$$\psi = E^3_B(\varphi) \iff \begin{cases} \psi \in L^p(\Omega, B, P; M) \text{ and } \\ E^3(\psi|_B) = E^3(\varphi|_B) \text{ for all } B \in B \text{ with } P(B) > 0. \end{cases}$$

Finally, we specialize our conditional expectation to the case of a global NPC space (alternatively, CAT(0) or Hadamard space). Let $(M, d)$ is a global NPC space. The canonical barycentric map $\lambda$ on $P^1(M)$ defined in [24] is

$$\lambda(\mu) := \arg\min_{z \in M} \int_M \left[d^2(z, x) - d^2(y, x)\right] d\mu(x) \tag{2.6}$$

for each $\mu \in P^1(M)$ independently of the choice of $y \in M$. If $\mu \in P^2(M)$, then $\lambda(\mu)$ is more simply given by

$$\lambda(\mu) = \arg\min_{z \in M} \int_M d^2(z, x) d\mu(x).$$

Assume that $(\Omega, A, P)$ be a general probability space and $B$ is a sub-$\sigma$-algebra of $A$. In [23] Sturm introduced, for each $\varphi \in L^2(\Omega; M)$, the conditional expectation $E_B(\varphi) \in L^2(\Omega, B, P; M)$ of $\varphi$ with respect to $B$ as

$$E_B(\varphi) := \arg\min_{\psi \in L^2(\Omega, B, P; M)} d_2(\varphi, \psi).$$

He then proved that for every $\varphi, \psi \in L^2(\Omega; M)$,

$$d(E_B(\varphi)(\omega), E_B(\psi)(\omega)) \leq E_B[d(\varphi, \psi)](\omega) \quad \text{a.e.,}$$

where $E_B[d(\varphi, \psi)]$ is the usual conditional expectation of the function $d(\varphi(\omega), \psi(\omega))$ with respect to $B$. From this he showed that $E_B$ extends continuously from $L^2(\Omega; M)$ to $L^1(\Omega; M)$ and that for every $p \in [1, \infty]$, $E_B$ maps $L^p(\Omega; M)$ into $L^p(\Omega, B, P; M)$ and

$$d_p(E_B(\varphi), E_B(\psi)) \leq d_p(\varphi, \psi), \quad \varphi, \psi \in L^p(\Omega; M). \tag{2.7}$$

Now, we assume that $(\Omega, A)$ is a standard Borel space. Our definition then provides the conditional expectation $E^\lambda_B(\varphi) \in L^1(\Omega, B, P; M)$ for every $\varphi \in L^1(\Omega; M)$, and by Remark 2.11 for every $p \in [1, \infty]$, $E_B^\lambda$ maps $L^p(\Omega; M)$ into $L^p(\Omega, B, P; M)$ and

$$d_p(E^\lambda_B(\varphi), E^\lambda_B(\psi)) \leq d_p(\varphi, \psi), \quad \varphi, \psi \in L^p(\Omega; M). \tag{2.8}$$

Sturm’s conditional expectation is restricted to a global NPC space $(M, d)$ while $(\Omega, A, P)$ is general. On the other hand, our definition needs a restriction on $(\Omega, A)$ to guarantee the existence of a disintegration, while it can be applied to a general contractive barycentric map. The next theorem says that Sturm’s conditional expectation and ours are the same, in the situation where both can be defined.

**Theorem 2.13.** Assume that $(\Omega, A, P)$ is a standard Borel probability space. Let $(M, d)$ be a global NPC space, and $\lambda$ be given as above. Then for every $p \in [1, \infty]$ and every $\varphi \in L^p(\Omega; M)$,

$$E_B(\varphi) = E^\lambda_B(\varphi).$$
First, assume that \( p = 2 \) and \( \varphi \in L^2(\Omega; M) \). By Lemma 2.6 there is a \( \mathbb{P} \)-null set \( N \in \mathcal{B} \) such that for every \( \omega \in \Omega \setminus N \), both \( E_B^\lambda(\tau) \) and \( E_B(\varphi)(\omega) \) are constant \( \mathbb{P}_\omega \)-a.e. \( \tau \in \Omega \). Hence, for every \( \omega \in \Omega \setminus N \), letting

\[
E_B^\lambda(\tau) = \lambda(\varphi, \mathbb{P}_\omega) = x, \quad E_B(\varphi)(\tau) = z \quad \mathbb{P}_\omega \text{-a.e.,}
\]

we have

\[
\int_\Omega d^2(E_B^\lambda(\tau), \varphi(\tau)) d\mathbb{P}_\omega(\tau) = \int_\Omega d^2(x, \varphi(\tau)) d\mathbb{P}_\omega(\tau) = \int_M d^2(\lambda(\varphi, \mathbb{P}_\omega), y) d\varphi, \mathbb{P}_\omega(y)
\]

\[
\leq \int_M d^2(z, y) d\varphi, \mathbb{P}_\omega(y) = \int_\Omega d^2(z, \varphi(\tau)) d\mathbb{P}_\omega(\tau)
\]

\[
= \int_\Omega d^2(E_B(\varphi)(\tau), \varphi(\tau)) d\mathbb{P}_\omega(\tau).
\]

Therefore, we have by Lemma 2.5(iii)

\[
d^2_2(E_B^\lambda(\varphi), \varphi) = \int_\Omega \left[ \int_\Omega d^2(E_B^\lambda(\tau), \varphi(\tau)) d\mathbb{P}_\omega(\tau) \right] d\mathbb{P}(\omega)
\]

\[
\leq \int_\Omega \left[ \int_\Omega d^2(E_B(\varphi)(\tau), \varphi(\tau)) d\mathbb{P}_\omega(\tau) \right] d\mathbb{P}(\omega) = d^2_2(E_B(\varphi), \varphi).
\]

Hence \( E_B(\varphi) = E_B^\lambda(\varphi) \) follows by definition of \( E_B(\varphi) \).

Next, let \( p \in [1, \infty] \) be arbitrary and \( \varphi \in L^p(\Omega; M) \). One can choose a sequence \( \{\varphi_k\} \) in \( L^\infty(\Omega; M) \) \((\subset L^2(\Omega; M))\) such that \( d_p(\varphi_k, \varphi) \to 0 \). Since \( E_B(\varphi_k) = E_B^\lambda(\varphi_k) \) for all \( k \) by the above case, one has by (2.7) and (2.8)

\[
d_p(E_B(\varphi), E_B^\lambda(\varphi)) \leq d_p(E_B(\varphi), E_B(\varphi_k)) + d_p(E_B^\lambda(\varphi_k), E_B(\varphi))
\]

\[
\leq 2d_p(\varphi_k, \varphi) \to 0,
\]

and hence \( E_B(\varphi) = E_B^\lambda(\varphi) \). \( \square \)

3. Martingale convergence theorem

Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space on a standard Borel space \((\Omega, \mathcal{A})\). Let \( \{\mathcal{B}_n\}_{n=1}^\infty \) be a sequence of sub-\(\sigma\)-algebras of \( \mathcal{A} \) such that either \( \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \) or \( \mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots \). Then let \( \mathcal{B}_\infty \) be the sub-\(\sigma\)-algebra of \( \mathcal{A} \) generated by \( \bigcup_{n=1}^\infty \mathcal{B}_n \) in the increasing case and \( \mathcal{B}_\infty := \bigcap_{n=1}^\infty \mathcal{B}_n \) in the decreasing case. Let \( 1 \leq p < \infty \) and \( \beta : \mathcal{P}^p(M) \to M \) be a \( p \)-contractive barycentric map. For every \( \varphi \in L^p(\Omega; M) \) we have a sequence \( \{E_B^\beta_n(\varphi)\}_{n=1}^\infty \) of \( \beta \)-conditional expectations, which we call a \( \beta \)-martingale of regular type with respect to \( \{\mathcal{B}_n\} \). (A different and more intrinsic definition will be given in Definition 3.5.)

A main result of this section is the martingale convergence theorem for \( \beta \)-martingales of regular type. To prove this, we follow the idea of the proof of Banach’s theorem given in [9, IV.11.3]. So we treat the space \( \mathcal{M}(\Omega; \mathbb{R}) \) of measurable real functions on \( \Omega \), where \( f = g \) in \( \mathcal{M}(\Omega; \mathbb{R}) \) is as usual understood as \( f(\omega) = g(\omega) \) a.e. As is well-known
\[ M(\Omega, \mathbb{R}) \] is a Fréchet space with the complete metric \( \rho(f, g) = |f - g|_p \), where
\[
|f|_p := \inf_{\alpha > 0} \left[ \alpha + P\{\omega : |f(\omega)| > \alpha\} \right], \quad f \in M(\Omega; \mathbb{R}). \tag{3.1}
\]
Note that the topology induced by \( |\cdot|_p \) on \( M(\Omega; \mathbb{R}) \) coincides with the topology of convergence in measure \( \mathcal{P} \).

**Theorem 3.1.** Assume that \((\Omega, \mathcal{A}, \mathcal{P})\) be a standard Borel probability space. Let \( \mathcal{B}_n, \ n \in \mathbb{N} \cup \{\infty\} \), be sub-\( \sigma \)-algebras of \( \mathcal{A} \), either increasing or decreasing, and let \( 1 \leq p < \infty \) and \( \beta : \mathcal{P}^p(M) \to M \) be as above. Then for every \( \varphi \in L^p(\Omega; M) \), as \( n \to \infty \),
\[
d_p\left(E_{\mathcal{B}_n}^\beta(\varphi), E_{\mathcal{B}_\infty}^\beta(\varphi)\right) \to 0 \quad \text{and} \quad d\left(E_{\mathcal{B}_n}^\beta(\varphi)(\omega), E_{\mathcal{B}_\infty}^\beta(\varphi)(\omega)\right) \to 0 \ a.e.
\]

**Proof.** First, assume that \( \varphi \) is a simple function, so \( \varphi = \sum_{j=1}^K 1_{A_j} x_j \) with \( x_j \in M \) and a measurable partition \( \{A_1, \ldots, A_K\} \) of \( \Omega \). By (2.3) we can write
\[
E_{\mathcal{B}_n}^\beta(\varphi)(\omega) = \beta \left( \sum_{j=1}^K \xi_{j,n}(\omega) \delta_{x_j} \right), \quad \omega \in \Omega, \ n \in \mathbb{N} \cup \{\infty\},
\]
where \( \xi_{j,n}(\omega) = E_{\mathcal{B}_n}(1_{A_j})(\omega) \), the usual conditional expectation of \( 1_{A_j} \) with respect to \( \mathcal{B}_n \). Here we may assume that \( \xi_{j,n}(\omega) \geq 0 \) and \( \sum_{j=1}^K \xi_{j,n}(\omega) = 1 \) for all \( \omega \in \Omega \) and \( n \in \mathbb{N} \cup \{\infty\} \). The classical martingale convergence theorem (see, e.g., [8]) says that \( \xi_{j,n} \to \xi_{j,\infty} \) in \( L^1 \)-norm and for a.e. \( \omega \in \Omega \) as \( n \to \infty \). With \( \Delta := \text{diam}\{x_1, \ldots, x_K\} \) we have by Lemma 1.3
\[
d\left(E_{\mathcal{B}_n}^\beta(\varphi)(\omega), E_{\mathcal{B}_\infty}^\beta(\varphi)(\omega)\right) \leq d_p\left(\sum_{j=1}^K \xi_{j,n}(\omega) \delta_{x_j}, \sum_{j=1}^K \xi_{j,\infty}(\omega) \delta_{x_j}\right)
\leq \Delta \left[ \frac{1}{2} \sum_{j=1}^K \left| \xi_{j,n}(\omega) - \xi_{j,\infty}(\omega) \right| \right]^{1/p} \to 0 \ a.e. \quad \text{as} \ n \to \infty,
\]
and
\[
d_p\left(E_{\mathcal{B}_n}^\beta(\varphi), E_{\mathcal{B}_\infty}^\beta(\varphi)\right) \leq \Delta \left[ \frac{1}{2} \sum_{j=1}^K \int_{\Omega} \left| \xi_{j,n}(\omega) - \xi_{j,\infty}(\omega) \right| d\mathcal{P} \right]^{1/p} \to 0 \ a.e. \quad \text{as} \ n \to \infty.
\]

For general \( \varphi \in L^p(\Omega; M) \) choose a sequence \( \{\varphi_k\} \) of simple functions such that \( d_p(\varphi, \varphi_k) \to 0 \). By Theorem 2.9(2) we have
\[
d_p\left(E_{\mathcal{B}_n}^\beta(\varphi), E_{\mathcal{B}_\infty}^\beta(\varphi)\right) \leq d_p\left(E_{\mathcal{B}_n}^\beta(\varphi), E_{\mathcal{B}_n}^\beta(\varphi_k)\right) + d_p\left(E_{\mathcal{B}_n}^\beta(\varphi_k), E_{\mathcal{B}_\infty}^\beta(\varphi_k)\right)
+ d_p\left(E_{\mathcal{B}_\infty}^\beta(\varphi_k), E_{\mathcal{B}_\infty}^\beta(\varphi)\right)
\leq 2d_p(\varphi, \varphi_k) + d_p\left(E_{\mathcal{B}_n}^\beta(\varphi_k), E_{\mathcal{B}_\infty}^\beta(\varphi_k)\right)
\]
so that
\[
\limsup_{n \to \infty} d_p\left(E_{\mathcal{B}_n}^\beta(\varphi), E_{\mathcal{B}_\infty}^\beta(\varphi)\right) \leq 2d_p(\varphi, \varphi_k) \to 0 \quad \text{as} \ k \to \infty.
\]
Hence \( d_p\left(E_{\mathcal{B}_n}^\beta(\varphi), E_{\mathcal{B}_\infty}^\beta(\varphi)\right) \to 0 \) as \( n \to \infty \).
It remains to prove the a.e. convergence. Choose an \( x_0 \in M \) and let \( \varphi_0 := 1_{\Omega} x_0 \). Let \( (P^{(n)}_\omega)_{\omega \in \Omega} \) be a disintegration of \( P \) with respect to \( B_n \) (see Section 2). Since \( x_0 = \beta((\varphi_0), P^{(n)}_\omega) \) for all \( \omega \in \Omega \) and \( n \in \mathbb{N} \cup \{\infty\} \), note that

\[
d(\beta(\varphi, P^{(m)}_\omega), \beta((\varphi_0), P^{(n)}_\omega)) + d(\beta(\psi, P^{(m)}_\omega), \beta((\varphi_0), P^{(n)}_\omega))
\]

for a.e. \( \omega \in \Omega \), where we have used Lemma 2.7 (2). Therefore, we find that

\[
\sup_{m,n \geq 1} d\left(E^\beta_{B_n}(\varphi), E^\beta_{B_n}(\varphi)(\omega)\right) \leq 2 \left[ \sup_{n \geq 1} E_{B_n}[d^p(\varphi, x_0)](\omega) \right]^{1/p} \text{ a.e. } \omega,
\]

where \( \{E_{B_n}[d^p(\varphi, x_0)]\}_{n=1}^\infty \) is the usual martingale for the function \( \omega \mapsto d^p(\varphi(\omega), x_0) \) in \( L^1(\Omega; \mathbb{R}) \). For each \( \varphi \in L^p(\Omega; M) \), since the classical a.e. martingale convergence gives

\[
\sup_{n \geq 1} E_{B_n}[d^p(\varphi, x_0)](\omega) < \infty \text{ a.e. } \omega,
\]

we can define a function \( W(\varphi) \in \mathcal{M}(\Omega; \mathbb{R}) \) by

\[
W(\varphi)(\omega) := \lim_{k \to \infty} \sup_{m,n \geq k} d\left(E^\beta_{B_m}(\varphi)(\omega), E^\beta_{B_n}(\varphi)(\omega)\right) \text{ a.e. } \omega.
\]

Then it is obvious that \( \lim_{n \to \infty} E^\beta_{B_n}(\varphi)(\omega) \) exists a.e. if and only if \( W(\varphi) = 0 \) as an element of \( \mathcal{M}(\Omega; \mathbb{R}) \). In this case, the a.e. limit of \( E^\beta_{B_n}(\varphi)(\omega) \) must be \( E^\beta_{B_\infty}(\varphi)(\omega) \) for a.e. \( \omega \) since \( E^\beta_{B_n}(\varphi) \to E^\beta_{B_\infty}(\varphi) \) in \( L^p \) sense as already shown above. Furthermore, we have shown above that \( \lim_{n \to \infty} E^\beta_{B_n}(\varphi)(\omega) \) exists a.e. if \( \varphi \) is a simple function. From the denseness of the simple functions in \( L^p(\Omega; M) \), it suffices to prove that \( W \) is a continuous map from \( L^p(\Omega; M) \) into \( \mathcal{M}(\Omega; \mathbb{R}) \) equipped with topology of convergence in measure \( P \).

To prove the last statement, note that for every \( \varphi, \psi \in L^p(\Omega; M) \) and every \( m, n \geq 1 \),

\[
d(\beta(\varphi, P^{(m)}_\omega), \beta(\psi, P^{(n)}_\omega)) \leq d\left(E^\beta_{B_m}(\varphi)(\omega), E^\beta_{B_n}(\psi)(\omega)\right) + d\left(E^\beta_{B_m}(\psi)(\omega), E^\beta_{B_n}(\psi)(\omega)\right) + d\left(E^\beta_{B_n}(\psi)(\omega), E^\beta_{B_n}(\varphi)(\omega)\right),
\]

which implies that

\[
\sup_{m,n \geq k} d\left(E^\beta_{B_m}(\varphi)(\omega), E^\beta_{B_n}(\varphi)(\omega)\right)
\]

\[
\leq \sup_{m,n \geq k} d\left(E^\beta_{B_m}(\psi)(\omega), E^\beta_{B_n}(\psi)(\omega)\right) + 2 \sup_{n \geq 1} d\left(E^\beta_{B_n}(\varphi)(\omega), E^\beta_{B_n}(\psi)(\omega)\right).
\]

We hence have

\[
|W(\varphi)(\omega) - W(\psi)(\omega)| \leq 2 \sup_{n \geq 1} d\left(E^\beta_{B_n}(\varphi)(\omega), E^\beta_{B_n}(\psi)(\omega)\right)
\]

\[
\leq 2 \sup_{n \geq 1} \left[ \int_\Omega d^p(\varphi(\tau), \psi(\tau)) dP^{(n)}_\omega(\tau) \right]^{1/p} \text{ a.e. } \omega.
\]
by Lemma 2.7(2). Therefore,

\[ |W(\varphi)(\omega) - W(\psi)(\omega)| \leq 2 \left[ \sup_{n \geq 1} E_{B_n}[d^p(\varphi, \psi)](\omega) \right]^{1/p}, \tag{3.2} \]

where \( \{ E_{B_n}[d^p(\varphi, \psi)] \}_{n=1}^{\infty} \) is the usual martingale for the function \( \omega \mapsto d^p(\varphi(\omega), \psi(\omega)) \) in \( L^1(\Omega; \mathbb{R}) \). Note that the function \( \omega \mapsto \sup_{n \geq 1} E_{B_n}[d^p(\varphi, \psi)](\omega) \) belongs to \( \mathcal{M}(\Omega; \mathbb{R}) \) since this supremum is finite for a.e. \( \omega \). From the proof of [9, IV.11.3] it follows that

\[ f \in L^1(\Omega; \mathbb{R}) \mapsto \sup_{n \geq 1} |E_{B_n}(f)(\omega)| \in \mathcal{M}(\Omega; \mathbb{R}) \tag{3.3} \]

is continuous at \( f = 0 \). If \( \varphi, \varphi_k \in L^p(\Omega; M) \) and \( d^p(\varphi, \varphi_k) \to 0 \), then \( d^p(\varphi, \varphi_k) \to 0 \) in \( L^1 \)-norm, and from (3.2) and the continuity of (3.3), we obtain \( W(\varphi_k) \to W(\varphi) \) in \( \mathcal{M}(\Omega; \mathbb{R}) \), as desired. \( \square \)

Sturm [23] showed a convergence theorem for martingales with locally compact range in a global NPC space, where martingales were introduced from the viewpoint of stochastic processes differently from those discussed above. In the rest of this section we consider Sturm’s type martingales in our general setting.

Assume that \((\Omega, A, P)\) and \( \beta : P^p(M) \to M \) are as in Theorem 3.1 and let \( B_n, n \in \mathbb{N} \cup \{ \infty \} \), be an increasing sequence of sub-\( \sigma \)-algebras of \( A \). Following [23], for \( \varphi \in L^p(\Omega; M) \) and \( m \geq k \geq 1 \), we define

\[ E^\beta[\varphi](B_n)_{m \geq n \geq k} := E^\beta_B \circ E^\beta_{B_k+1} \circ \cdots \circ E^\beta_{B_m}(\varphi), \]

which is an element of \( L^p(\Omega, B_k; P; M) \). The proof of the next lemma is based on Theorem 3.1.

**Lemma 3.2.** For every \( \varphi \in L^p(\Omega; M) \) and \( k \geq 1 \) the following equal limits exist:

\[ \lim_{m \to \infty} E^\beta[\varphi](B_n)_{m \geq n \geq k} = \lim_{m \to \infty} E^\beta[B^\infty_{B^\infty}(\varphi)](B_n)_{m \geq n \geq k} \quad \text{in metric } d_p. \tag{3.4} \]

**Proof.** For notational simplicity, for any \( \varphi \in L^p(\Omega; M) \) write \( \varphi_\infty := E^\beta_{B^\infty_{B^\infty}}(\varphi) \) and \( \varphi_{m,k} := E^\beta[\varphi](B_n)_{m \geq n \geq k} \) for \( m \geq k \geq 1 \). For \( l > m \geq k \) we have

\[ d_p(\varphi_{m,k}, \varphi_{l,k}) \leq d_p(\varphi_{m,k}, \varphi_{\infty,m,k}) + d_p(\varphi_{\infty,m,k}, \varphi_{\infty,l,k}) + d_p(\varphi_{\infty,l,k}, \varphi_{l,k}). \tag{3.5} \]

Moreover, by Theorem 2.7(2) and Theorem 3.1,

\[ d_p(\varphi_{m,k}, \varphi_{\infty,m,k}) \leq d_p(E^\beta_{B^\infty_{B^\infty}}(\varphi), E^\beta_{B^\infty_{B^\infty}}(\varphi)) \]
\[ \leq d_p(E^\beta_{B^\infty_{B^\infty}}(\varphi), \varphi_{\infty}) + d_p(\varphi_{\infty}, E^\beta_{B^\infty_{B^\infty}}(\varphi)) \]
\[ \to 0 \quad \text{as } m \to \infty, \tag{3.6} \]

and similarly \( d_p(\varphi_{l,k}, \varphi_{\infty,l,k}) \to 0 \) as \( l \to \infty \). Since \( \bigcup_{p=1}^{\infty} L^p(\Omega, B_n, P; M) \) is \( d_p \)-dense in \( L^p(\Omega, B^\infty_{\infty}, P; M) \), for every \( \varepsilon > 0 \) one can choose a \( \psi \in L^p(\Omega, B_n, P; M) \) for some \( n \geq 1 \) such that \( d_p(\psi, \varphi_{\infty}) < \varepsilon \). One has

\[ d_p(\varphi_{m,k}, \varphi_{\infty,l,k}) = d_p(E^\beta_{B^\infty_{B^\infty}} \circ \cdots \circ E^\beta_{B^\infty_{B^\infty}}(\varphi), E^\beta_{B^\infty_{B^\infty}} \circ \cdots \circ E^\beta_{B^\infty_{B^\infty}}(\varphi)) \]
\[ \leq d_p(\varphi_\infty, E_{B_{m+1}}^\beta \circ \cdots \circ E_{B_1}^\beta(\varphi_\infty)) \]
\[ \leq d_p(\varphi_\infty, \psi) + d_p(\psi, E_{B_{m+1}}^\beta \circ \cdots \circ E_{B_1}^\beta(\varphi_\infty)). \]

For \( l > m \geq \max\{k, n\} \), since \( \psi = E_{B_{m+1}}^\beta \circ \cdots \circ E_{B_1}^\beta(\psi) \), one has \( d_p((\varphi_\infty)_{m,k}, (\varphi_\infty)_{l,k}) \leq 2\varepsilon \), which implies that \( d_p((\varphi_\infty)_{m,k}, (\varphi_\infty)_{l,k}) \to 0 \) as \( l, m \to \infty \). Hence it follows from (3.5) that \( d_p(\varphi_{m,k}, \varphi_{l,k}) \to 0 \) as \( l, m \to \infty \), so that \( \varphi_{m,k} \) converges in \( d_p \) to some element of \( L^p(\Omega, B_k, P; M) \) as \( m \to \infty \). Thanks to (3.3), \( (\varphi_\infty)_{m,k} \) also converges to the same limit as \( m \to \infty \). \( \square \)

**Definition 3.3.** For every \( \varphi \in L^p(\Omega; M) \) and \( k \geq 1 \), we write

\[ E^\beta[\varphi\|(B_n)_{n \geq k}] \]

for the equal limits in (3.4), which is an element of \( L^p(\Omega, B_k, P; M) \) and we call the filtered \( \beta \)-conditional expectation of \( \varphi \) with respect to \((B_n)_{n \geq k}\).

The associativity in (4) below is a merit of filtered \( \beta \)-conditional expectations, which is not satisfied for those in Theorem 3.1 (see Example 2.12).

**Proposition 3.4.** Let \( \varphi, \psi \in L^p(\Omega; M) \).

1. \( E^\beta[\varphi\|(B_n)_{n \geq k}] \in L^p(\Omega, B_k, P; M) \) for all \( k \geq 1 \).
2. For every \( k \geq 1 \), \( \varphi \in L^p(\Omega, B_k, P; M) \) if and only if \( E^\beta[\varphi\|(B_n)_{n \geq k}] = \varphi \).
3. \( d_p(E^\beta[\varphi\|(B_n)_{n \geq k}], E^\beta[\psi\|(B_n)_{n \geq k}]) \leq d_p(\varphi, \psi) \) for all \( k \geq 1 \).
4. For every \( l \geq k \geq 1 \),

\[ E^\beta[E^\beta[\varphi\|(B_n)_{n \geq l}]\|(B_n)_{n \geq k}] = E^\beta[\varphi\|(B_n)_{n \geq k}] \]

**Proof.** (1) is obvious.

(2) If \( \varphi \in L^p(\Omega, B_k, P; M) \), then \( E_{B_k}^\beta \circ \cdots \circ E_{B_1}^\beta(\varphi) = \varphi \) for all \( m \geq k \) and hence \( E^\beta[\varphi\|(B_n)_{n \geq k}] = \varphi \). The converse is obvious from (1).

(3) For every \( m \geq k \geq 1 \) we have by Theorem 2.9(2)

\[ d_p(E_{B_k}^\beta \circ \cdots \circ E_{B_1}^\beta(\varphi), E_{B_k}^\beta \circ \cdots \circ E_{B_1}^\beta(\psi)) \leq d_p(\varphi, \psi), \]

whose limit as \( m \to \infty \) is the asserted inequality.

(4) For simplicity, for \( \varphi \in L^p(\Omega; M) \) write \( \varphi_{\infty,k} := E^\beta[\varphi\|(B_n)_{n \geq k}] \) for \( k \geq 1 \). Since \( \varphi_{\infty,l} \in L^p(\Omega, B_l, P; M) \) by (1), we have for every \( m \geq l > k \)

\[ E_{B_k}^\beta \circ \cdots \circ E_{B_m}^\beta(\varphi_{\infty,l}) = E_{B_k}^\beta \circ \cdots \circ E_{B_{l-1}}^\beta(\varphi_{\infty,l}) \]
\[ = \lim_{m \to \infty} E_{B_k}^\beta \circ \cdots \circ E_{B_{l-1}}^\beta \circ E_{B_l}^\beta \circ \cdots \circ E_{B_m}^\beta(\varphi) = \varphi_{\infty,k}. \]

Therefore,

\[ E^\beta[\varphi_{\infty,l}\|(B_n)_{n \geq k}] = \lim_{m \to \infty} E_{B_k}^\beta \circ \cdots \circ E_{B_m}^\beta(\varphi_{\infty,l}) = \varphi_{\infty,k}, \]

as required. \( \square \)

Following [23] Definition 4.1] we define:
Definition 3.5. A sequence \( \{ \varphi_k \}_{k=1}^{\infty} \) in \( L^p(\Omega; M) \) is called a **filtered \( \beta \)-martingale** with respect to \( \{ \mathcal{B}_n \}_{n=1}^{\infty} \) if \( \varphi_k \in L^p(\Omega, \mathcal{B}_k, \mathbb{P}; M) \) for every \( k \geq 1 \) and

\[
E^\beta[\varphi_{k+1} \vert (\mathcal{B}_n)_{n\geq k}] = \varphi_k, \quad k \geq 1. \tag{3.7}
\]

By associativity in Proposition 3.4(4), property (3.7) is equivalent to

\[
E^\beta[\varphi_l \vert (\mathcal{B}_n)_{n\geq k}] = \varphi_k, \quad l \geq k \geq 1.
\]

For any \( \varphi \in L^p(\Omega; M) \), it is clear that the sequence \( \varphi_k := E^\beta[\varphi \vert (\mathcal{B}_n)_{n\geq k}] \), \( k \geq 1 \), is a filtered \( \beta \)-martingale with respect to \( \{ \mathcal{B}_n \} \). The next theorem includes its \( d_p \)-convergence.

**Theorem 3.6.** Let \( \{ \varphi_k \}_{k=1}^{\infty} \) be a filtered \( \beta \)-martingale with respect to \( \{ \mathcal{B}_n \} \). Then the following are equivalent:

(i) there exists a \( \varphi \in L^p(\Omega; M) \) such that \( \varphi_k = E^\beta[\varphi \vert (\mathcal{B}_n)_{n\geq k}] \) for all \( k \geq 1 \);

(ii) \( \varphi_k \) converges to some \( \varphi_\infty \in L^p(\Omega, \mathcal{B}_\infty, \mathbb{P}; M) \) in metric \( d_p \) as \( k \to \infty \).

**Proof.** (i) \( \implies \) (ii). Let \( \varphi \in L^p(\Omega; M) \) be as stated in (i). Let \( \varphi_\infty := E^\beta_{\mathcal{B}_\infty}(\varphi) \in L^p(\Omega, \mathcal{B}_\infty, \mathbb{P}; M) \). For every \( \varepsilon > 0 \), by Theorem 3.1 one can choose a \( \psi \in L^p(\Omega, \mathcal{B}_l, \mathbb{P}; M) \) for some \( l \geq 1 \) such that \( d_p(\psi, \varphi_\infty) < \varepsilon \). For every \( k \geq l \), since \( E^\beta[\psi \vert (\mathcal{B}_n)_{n\geq k}] = \psi \) by Proposition 3.4(2), one has

\[
d_p(\varphi_k, \varphi_\infty) \leq d_p(E^\beta[\varphi_\infty \vert (\mathcal{B}_n)_{n\geq k}], E[\psi \vert (\mathcal{B}_n)_{n\geq k}]) + d_p(\psi, \varphi_\infty) \leq 2d_p(\psi, \varphi_\infty) < 2\varepsilon
\]

by Proposition 3.4(3). Hence (ii) follows.

(ii) \( \implies \) (i). For \( l \geq k \geq 1 \) one has by Proposition 3.4 again

\[
d_p(\varphi_k, E^\beta[\varphi_\infty \vert (\mathcal{B}_n)_{n\geq k}]) = d_p(E^\beta[\varphi_\infty \vert (\mathcal{B}_n)_{n\geq k}], E^\beta[\varphi_\infty \vert (\mathcal{B}_n)_{n\geq k}]) \leq d_p(\varphi_l, \varphi_\infty) \to 0 \quad \text{as} \quad l \to \infty.
\]

Hence \( \varphi_k = E^\beta[\varphi_\infty \vert (\mathcal{B}_n)_{n\geq k}] \). \( \square \)

**Remark 3.7.** When \((M, d)\) is a locally compact global NPC space and \( \beta \) is a canonical barycentric map \( \lambda \), it follows from [23, Theorem 4.11] (and Theorem 2.13) that if \( \{ \varphi_k \} \) in \( L^p(\Omega; M) \) is a filtered martingale and \( \sup_k d_p(\varphi_k, \varphi_k) < \infty \) for some \( z \in M \), then there exists a \( \mathcal{B}_\infty \)-measurable function \( \varphi_\infty : \Omega \to M \) such that \( \varphi_k(\omega) \to \varphi_\infty(\omega) \) \( \mathbb{P} \)-a.e. From the Hopf-Rinow theorem (cf. [4]) we see that this result holds more generally when \((M, d)\) is a locally compact and complete length space and \( \beta : \mathcal{P}^d(M) \to M \) is any contractive barycentric map. But it does not seem easy to extend the \( \mathbb{P} \)-a.e. martingale convergence of filtered \( \beta \)-martingales to our general setting. Although the details are omitted here, the same result holds under an even more general situation that \((M, d)\) satisfies finite-compactness with respect to \( \beta \) in the sense that for any finite set \( Q_0 \) in \( M \) the closure of \( \bigcup_{n=1}^{\infty} Q_n \) is compact, where

\[
Q_n := \{ \beta(\mu) : \mu \in \mathcal{P}_0(M), \text{supp}(\mu) \subset Q_{n-1} \}, \quad n \in \mathbb{N}.
\]
This finite-compactness property clearly holds in the case of Banach spaces with the arithmetic mean map. But it is unknown whether it holds in the case where $M = \mathbb{P}(\mathcal{H})$ on an infinite-dimensional Hilbert space $\mathcal{H}$ and $\beta$ is the Karcher barycenter $G$ (see Example 4.5(b) below).

4. Ergodic theorem

Let $T$ be a $\mathbb{P}$-preserving measurable transformation on $(\Omega, \mathcal{A}, \mathbb{P})$. It is clear that the map $\varphi \mapsto \varphi \circ T$ is a $d_p$-isometric transformation on $L^p(\Omega; M)$. (Although we may treat a measure-preserving action of an amenable group $G$ as in [21], we consider the case $G = \mathbb{Z}$ for the sake of simplicity.)

Let $1 \leq p < \infty$ and $\beta : \mathcal{P}^p(M) \to M$ be a $p$-contractive barycentric map. For each $\varphi \in L^p(\Omega; M)$ we define the empirical measures (random probability measures) of $\varphi$ as

$$\mu_n^\varphi(\omega) := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\varphi(T^k\omega)}, \quad n \in \mathbb{N},$$

i.e., for Borel sets $B \subset M$,

$$\mu_n^\varphi(\omega)(B) = \frac{\#\{k \in \{0, 1, \ldots, n-1\} : \varphi(T^k\omega) \in B\}}{n},$$

and consider the sequence of $M$-valued functions $\beta(\mu_n^\varphi)(\omega) := \beta(\mu_n^\varphi(\omega)), \, \omega \in \Omega$, for $n \in \mathbb{N}$.

**Lemma 4.1.** For every $\varphi, \psi \in L^p(\Omega; M)$ we have $\beta(\mu_n^\varphi) \in L^p(\Omega; M)$ and

$$d_p(\beta(\mu_n^\varphi), \beta(\mu_n^\psi)) \leq d_p(\varphi, \psi), \quad n \in \mathbb{N}.$$

**Proof.** Let $n \in \mathbb{N}$ be arbitrarily fixed. First, assume that $\varphi$ is a simple function so that $\varphi = \sum_{i=1}^K 1_{A_i} \delta_{x_i}$, where $x_1, \ldots, x_K \in M$ and $\mathcal{F} = \{A_1, \ldots, A_K\}$ is a measurable partition of $\Omega$. Then, as easily seen, we can write

$$\mu_n^\varphi(\omega) = \sum_{A \in \bigvee_{k=0}^{n-1} T^{-k}\mathcal{F}} 1_A(\omega)\mu_A$$

with $\mu_A \in \mathcal{P}_0(M)$, where $\bigvee_{k=0}^{n-1} T^{-k}\mathcal{F}$ is the finite partition generated by $T^{-k}\mathcal{F}$, $0 \leq k \leq n-1$. Therefore,

$$\beta(\mu_n^\varphi(\omega)) = \sum_{A \in \bigvee_{k=0}^{n-1} T^{-k}\mathcal{F}} 1_A(\omega)\beta(\mu_A)$$

so that $\omega \mapsto \beta(\mu_n^\varphi(\omega))$ is a simple function.

Next, let $\varphi, \psi$ be arbitrary elements in $L^p(\Omega; M)$. For each fixed $\omega \in \Omega$, since

$$\pi := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{(\varphi(T^k\omega), \psi(T^k\omega))} \in \mathcal{P}(M \times M)$$
is in $\Pi(\mu_n^\varphi(\omega), \mu_n^\psi(\omega))$, we have
\[
d^p_p(\mu_n^\varphi(\omega), \mu_n^\psi(\omega)) \leq \left[ \frac{1}{n} \sum_{k=0}^{n-1} d^p(\varphi(T^k \omega), \psi(T^k \omega)) \right]^{1/p}.
\]
From this and the $p$-contractivity of $\beta$ we find that
\[
d(\beta(\mu_n^\varphi(\omega)), \beta(\mu_n^\psi(\omega))) \leq \left[ \frac{1}{n} \sum_{k=0}^{n-1} d^p(\varphi(T^k \omega), \psi(T^k \omega)) \right]^{1/p}, \quad \omega \in \Omega.
\]
Now, choose $M$-valued simple functions $\varphi_l (l \in \mathbb{N})$ such that $d(\varphi(\omega), \varphi_l(\omega)) \to 0$ a.e. as $l \to \infty$. Letting $\psi = \varphi_l$ in (4.1) we have $d(\beta(\mu_n^\varphi(\omega)), \beta(\mu_n^\psi(\omega))) \to 0$ a.e. as $l \to \infty$. Since $\beta(\mu_n^\varphi)$'s are simple functions as proved above, it follows that $\beta(\mu_n^\varphi)$ is a strongly measurable function on $\Omega$. Letting $\psi = 1_\Omega x (x \in M)$ in (4.1), since $\beta(\mu_n^\varphi)(\omega) = x$ for all $\omega \in \Omega$, we have
\[
\int_\Omega d^p(\beta(\mu_n^\varphi(\omega)), x) d\mathcal{P}(\omega) \leq \int_\Omega \frac{1}{n} \sum_{k=0}^{n-1} d^p(\varphi(T^k \omega), x) d\mathcal{P}(\omega) = \int_\Omega d^p(\varphi(\omega), x) d\mathcal{P}(\omega) < \infty
\]
so that $\beta(\mu_n^\varphi) \in L^p(\Omega; M)$. Finally, it follows from (4.1) again that
\[
\int_\Omega d^p(\beta(\mu_n^\varphi(\omega)), \beta(\mu_n^\psi(\omega))) d\mathcal{P}(\omega) \leq \int_\Omega \frac{1}{n} \sum_{k=0}^{n-1} d^p(\varphi(T^k \omega), \psi(T^k \omega)) d\mathcal{P}(\omega)
\]
so that $d_p(\beta(\mu_n^\varphi), \beta(\mu_n^\psi)) \leq d_p(\varphi, \psi)$. \hfill \qed

In [1] Austin obtained an $L^2$-ergodic theorem for the canonical barycentric map on a global NPC space. In [21] Navas established an $L^1$-ergodic theorem for a specific contractive barycentric map on a metric space of nonpositive curvature in the sense of Busemann (a weaker notion than that of a global NPC space). In [20], Navas’ ergodic theorem was proved for parametrized barycentric maps extending the Cartan (or Karcher) barycenter on the positive definite matrices.

In this section we give an $L^p$-ergodic theorem for $1 \leq p < \infty$ on a general complete metric space with a general $p$-contractive barycentric map $\beta$. Moreover, we give the description of the ergodic limit function in terms of the $\beta$-conditional expectation. Since the proof of the next theorem is along the essentially same lines as [1] [21], we shall only present its sketchy version.

**Theorem 4.2.** Let $1 \leq p < \infty$ and $\beta : \mathcal{P}^p(M) \to M$ be a $p$-contractive barycentric map. Then there exists a map $\Gamma$ from $L^p(\Omega; M)$ onto $\{ \varphi \in L^p(\Omega; M) : \varphi \circ T = \varphi \}$ such that for every $\varphi, \psi \in L^p(\Omega; M)$,

(i) $d(\beta(\mu_n^\varphi(\omega)), \Gamma(\varphi)(\omega)) \to 0$ a.e. as $n \to \infty$,
(ii) $d_p(\beta(\mu_n^\varphi), \Gamma(\varphi)) \to 0$ as $n \to \infty$,
(iii) $d_p(\Gamma(\varphi), \Gamma(\psi)) \leq d_p(\varphi, \psi)$.
Furthermore, if $T$ is ergodic, then $\Gamma(\varphi)$ is constant with value $E^\beta(\varphi)$, the $\beta$-expectation of $\varphi$ (see Definition 2.1).

**Proof.** Let $\varphi, \psi \in L^p(\Omega; M)$. Applying the maximal ergodic theorem to the function $f(\omega) := d^p(\varphi(\omega), \psi(\omega)) \in L^1(\Omega; \mathbb{R})$ we have for every $\lambda > 0$,

$$P \left\{ \omega \in \Omega : \sup_{n \geq 1} \frac{1}{n} \sum_{k=0}^{n-1} d^p(\varphi(T^k \omega), \psi(T^k \omega)) > \lambda \right\} \leq \frac{1}{\lambda} d^p_\beta(\varphi, \psi),$$

which together with (4.1) implies that

$$P \left\{ \omega \in \Omega : \sup_{n \geq 1} d^p(\beta(\mu^n_\varphi(\omega)), \beta(\mu^n_\psi(\omega))) > \lambda \right\} \leq \frac{1}{\lambda} d^p_\beta(\varphi, \psi). \quad (4.2)$$

Now, assume that $\psi$ is a simple function so that $\psi = \sum_{i=1}^K 1_{A_i} x_i$ with $x_i \in M$ and a measurable partition $\{A_1, \ldots, A_K\}$ of $\Omega$. Note that we can write

$$\mu_\psi^n(\omega) = \sum_{i=1}^K \left( \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_i}(T^k \omega) \right) \delta_{x_i}. \quad (4.3)$$

By the usual Birkhoff ergodic theorem, there are $T$-invariant functions $\xi_i \in L^1(\Omega; \mathbb{R})$ ($1 \leq i \leq K$) such that $\frac{1}{n} \sum_{k=0}^{n-1} 1_{A_i}(T^k \omega) \to \xi_i(\omega)$ a.e. as $n \to \infty$. Therefore, with $\mu_\psi^n(\omega) := \sum_{i=1}^K \xi_i(\omega) \delta_{x_i}$ we have

$$d(\beta(\mu_\psi^n(\omega)), \beta(\mu_\psi^n(\omega))) \leq d_W(\mu_\psi^n(\omega), \mu_\psi^n(\omega)) \to 0 \quad \text{a.e. as } n \to \infty \quad (4.4)$$

from Lemma 1.3. Choose simple functions $\psi_l$ ($l \in \mathbb{N}$) such that $d(\varphi(\omega), \psi_l(\omega)) \to 0$ a.e. and $d_p(\varphi, \psi_l) \to 0$ as $l \to \infty$. For every $\varepsilon \in (0, 1)$ choose a $\psi_l$ such that $d_p(\varphi, \psi_l) < \varepsilon^2$.

Furthermore, from (4.4) with $\psi = \psi_l$, one can choose an $n_\varepsilon \in \mathbb{N}$ and an $N_\varepsilon \in \mathcal{A}$ such that $P(N_\varepsilon) < \varepsilon$ and $d(\beta(\mu_\psi^n(\omega)), \beta(\mu_\psi^n(\omega))) \leq \varepsilon$ for $\omega \in \Omega \setminus N_\varepsilon$. Let

$$\tilde{N}_\varepsilon := \left\{ \omega \in \Omega : \sup_{n \geq 1} d(\beta(\mu_\psi^n(\omega)), \beta(\mu_\psi^n(\omega))) > \varepsilon^2 \right\} \cup N_\varepsilon.$$

Then, from (4.2) with $\psi = \psi_l$ and $\lambda = \varepsilon^n$, one has $P(\tilde{N}_\varepsilon) < 2\varepsilon$ and for every $\omega \in \Omega \setminus \tilde{N}_\varepsilon$ and $n \geq n_\varepsilon$, $d(\beta(\mu_\psi^n(\omega)), \beta(\mu_\psi^n(\omega))) \leq 2\varepsilon$, so that

$$d(\beta(\mu_\psi^n(\omega)), \beta(\mu_\psi^n(\omega))) \leq 4\varepsilon, \quad \omega \in \Omega \setminus \tilde{N}_\varepsilon, \ n, m \geq n_\varepsilon.$$

Letting $\varepsilon_k := k^{-2}$ and $N := \limsup_{k \to \infty} \tilde{N}_{\varepsilon_k}$, one has $P(N) = 0$ by the Borel-Cantelli lemma and

$$\lim_{n,m \to \infty} d(\beta(\mu_\psi^n(\omega)), \beta(\mu_\psi^n(\omega))) = 0, \quad \omega \in \Omega \setminus N,$$

which implies that there exists a strongly measurable function $\Gamma(\varphi) : \Omega \to M$ for which property (i) holds, though $\Gamma(\varphi) \in L^p(\Omega; M)$ as well as $\Gamma(\varphi) \circ T = \Gamma(\varphi)$ will be proved below.

To prove (ii), note by Lemma 4.1 that

$$d_p(\beta(\mu_\psi^n(\omega)), \beta(\mu_\psi^n(\omega))) \leq 2d_p(\varphi, \psi_l) + d_p(\beta(\mu_\psi^n(\omega)), \beta(\mu_\psi^n(\omega))). \quad (4.5)$$
For every \( \varepsilon > 0 \) choose a \( \psi_l \) such that \( d_p(\varphi, \psi_l) < \varepsilon \), and write \( \psi_l = \sum_{j=1}^{K} 1_{A_j} x_j \). Then it follows from (4.3) and Lemma 1.3 that with \( \Delta := \text{diam}\{x_1, \ldots, x_K\} \),

\[
d_p(\beta(\mu_n^\psi), \beta(\mu_m^\psi)) \leq \Delta^p \sum_{i=1}^{K} \left\| \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_i} \circ T^k - \frac{1}{m} \sum_{k=0}^{m-1} 1_{A_i} \circ T^k \right\|_1.
\]

Hence, by the usual mean ergodic theorem, one can choose an \( n \) such that \( d_p(\beta(\mu_n^\psi), \beta(\mu_m^\psi)) < \varepsilon \) for all \( n, m \geq n^\varepsilon \). Therefore, since property (i) implies that \( \psi_l \) is \( \sigma \)-invariant, for a simple function \( \psi \), choose simple functions \( n, m \) such that \( \psi \) is \( \sigma \)-invariant sets. Then for \( \omega \),

\[
\int_{\Omega} d_p(\beta(\mu_n^\psi(\omega)), \Gamma(\varphi)(\omega)) dP(\omega) \leq (3\varepsilon)^p, \quad n \geq n^\varepsilon,
\]

which implies that \( \Gamma(\varphi) \in L^p(\Omega; M) \) and property (ii) holds.

When \( \varphi, \psi \in L^p(\Omega; M) \), property (iii) follows from Lemma 1.3 and Fatou’s lemma since property (i) implies that \( d(\beta(\mu_n^\psi(\omega)), \beta(\mu_m^\psi(\omega))) \to d(\Gamma(\varphi)(\omega), \Gamma(\psi)(\omega)) \) a.e. as \( n \to \infty \).

Next, we confirm that \( \Gamma(\varphi) \) is \( T \)-invariant. For a simple function \( \psi \) it follows from (4.3) and Lemma 1.3 that \( \Gamma(\psi)(T\omega) = \Gamma(\psi)(\omega) \) a.e. For an arbitrary \( \varphi \in L^p(\Omega; M) \), choose a sequence \( \psi_l \) as above. Since \( \Gamma(\psi_l) \circ T = \Gamma(\psi_l) \) as verified just above, we have

\[
d_p(\Gamma(\varphi), \Gamma(\varphi) \circ T) \leq 2d_p(\Gamma(\varphi), \Gamma(\psi_l)) \leq 2d_p(\varphi, \psi_l) \to 0 \quad \text{as} \ l \to \infty
\]

thanks to property (iii). Hence \( \Gamma(\varphi) = \Gamma(\varphi) \circ T \), as desired.

Finally, assume that \( T \) is ergodic. For a simple function \( \psi = \sum_{i=1}^{K} 1_{A_i} x_i \), since \( \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_i} (T^k \omega) \to P(A_i) \) a.e. as \( n \to \infty \) due to the ergodicity of \( T \), it follows from (4.3) that \( \Gamma(\psi) = \beta(\sum_{i=1}^{K} P(A_i) \delta_{x_i}) = \beta(\psi, P) = E^\beta(\psi) \). For general \( \varphi \in L^p(\Omega; M) \) choose simple functions \( \psi_l \) such that \( d_p(\varphi, \psi_l) \to 0 \) as \( l \to \infty \). Since \( d_p(\Gamma(\varphi), \Gamma(\psi_l)) \to 0 \) by (iii) and \( d(E^\beta(\varphi), E^\beta(\psi_l)) \to 0 \) by Proposition 2.2, \( \Gamma(\varphi) = E^\beta(\varphi) \) follows.

When \( (\Omega, \mathcal{A}) \) is a standard Borel space and \( T \) is not necessarily ergodic, the limit \( \Gamma(\varphi) \) in Theorem 4.2 can be specified in terms of the \( \beta \)-conditional expectation of \( \varphi \) as follows.

**Theorem 4.3.** In Theorem 3.2 assume that \( (\Omega, \mathcal{A}) \) is a standard Borel space. Let \( \mathcal{I} := \{ A \in \mathcal{A} : T^{-1} A = A \} \), the sub-\( \sigma \)-algebra consisting of \( T \)-invariant sets. Then for every \( \varphi \in L^p(\Omega; M) \), \( \Gamma(\varphi) \) is the \( \beta \)-conditional expectation \( E^\beta_Z(\varphi) \) of \( \varphi \) with respect to \( \mathcal{I} \) (see Definition 2.8).

**Proof.** Let \( (P_\omega)_{\omega \in \Omega} \) be a disintegration of \( P \) with respect to \( \mathcal{I} \), as stated in Lemma 2.3. First, let \( \psi \) be a simple function as \( \psi = \sum_{i=1}^{K} 1_{A_i} \delta_{x_i} \), with a measurable partition \( \{A_1, \ldots, A_K\} \) of \( \Omega \). From (4.3) we write

\[
\beta(\mu_n^\psi(\omega)) = \beta \left( \sum_{i=1}^{K} \left( \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_i} (T^k \omega) \right) \delta_{x_i} \right), \quad n \in \mathbb{N}.
\]

(4.6)
Moreover, note that $P_{\omega}(A_j)$ is the usual conditional expectation of $1_{A_j}$ with respect to $\mathcal{I}$ (see Lemma 2.3). Hence the usual individual ergodic theorem gives

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_i}(T^k\omega) = P_{\omega}(A_i) \text{ a.e.}$$

Moreover, as in (2.3) we write

$$E^\beta_T(\psi)(\omega) = \beta \left( \sum_{i=1}^{K} P_{\omega}(A_i)\delta_{x_i} \right).$$

(4.7)

Therefore, from (4.6) and (4.7) together with (2.1) we have

$$d(\beta(\mu_n^\psi(\omega), E_T^\beta(\psi)(\omega))) \leq d_p^W \left( \sum_{i=1}^{K} \left( \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_i}(T^k\omega) \right) \delta_{x_i} - \sum_{i=1}^{K} P_{\omega}(A_i)\delta_{x_i} \right)$$

$$\leq \Delta \left[ \sum_{i=1}^{K} \left( \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_i}(T^k\omega) - P_{\omega}(A_i) \right) \right]^{1/p} \to 0 \text{ a.e.,}$$

where we have used Lemma 1.3. This implies that $\Gamma(\psi) = E_T^\beta(\psi)$.

For general $\varphi \in L^p(\Omega; M)$ choose simple functions $\psi_l$ such that $d_p(\varphi, \psi_l) \to 0$ as $l \to \infty$. By the above case, $\Gamma(\psi_l) = E_T^\beta(\psi_l)$ for all $l$. Since Theorems 4.2(iii) and 2.9(2) give $d_p(\Gamma(\varphi), \Gamma(\psi_l)) \to 0$ and $d_p(E_T^\beta(\varphi), E_T^\beta(\psi_l)) \to 0$, we obtain $\Gamma(\varphi) = E_T^\beta(\varphi)$. \hfill \Box

**Theorem 4.4.** Assume that $M$ is equipped with a closed partial order and $\beta$ is monotone. Then $\Gamma$ is monotone, that is, for $\varphi, \psi \in L^p(\Omega; M)$, $\varphi \leq \psi$ (i.e., $\varphi(\omega) \leq \psi(\omega)$ a.e.) implies $\Gamma(\varphi) \leq \Gamma(\psi)$.

**Proof.** Let $\varphi, \psi \in L^p(\Omega; M)$ and assume that $\varphi(\omega) \leq \psi(\omega)$ a.e. Then $\mu_n^\psi(\omega) \leq \mu_n^\psi(\omega)$ a.e. and hence $\beta(\mu_n^\psi(\omega)) \leq \beta(\mu_n^\psi(\omega))$ a.e. for all $n$. By closedness of the partial order, letting $n \to \infty$ gives $\Gamma(\varphi(\omega)) \leq \Gamma(\psi(\omega))$ a.e. (When $(\Omega, \mathcal{A})$ is a standard Borel space, the result also follows from Theorems 2.9(5) and 1.3) \hfill \Box

**Example 4.5.** (a) Consider the space $M_N$ of all $N \times N$ complex matrices with any norm $||| \cdot |||$ (typically, the Hilbert-Schmidt norm), so $(M_N, d)$ with $d(X, Y) := |||X-Y|||$ is a Banach space. One can define the arithmetic mean map $A : \mathcal{P}^1(M_N, ||| \cdot |||) \to M_N$ by

$$A(\mu) := \int_{M_N} X \, d\mu(X), \quad X \in M_N.$$

(4.8)

Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}^p(M_N)$. For every $\pi \in \Pi(\mu, \nu)$ we have

$$\left[ \int_{M_N \times M_N} |||X-Y|||^p \, d\pi(X,Y) \right]^{1/p} \geq \int_{M_N \times M_N} |||X-Y||| \, d\pi(X,Y) \geq \left[ \int_{M_N \times M_N} (X-Y) \, d\pi(X,Y) \right] \geq |||A(\mu) - A(\nu)|||,$$
which implies that \(||A(\mu) - A(\nu)||| \leq d_P^W(\mu, \nu)\). Thus, \(A\) restricted on \(P^p(M, || \cdot ||)\) is a \(p\)-contractive barycentric map. For every \(\varphi \in L^p(\Omega; M)\) where \(1 \leq p < \infty\), since
\[
A(\mu_n^\varphi(\omega)) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k \omega), \quad \omega \in \Omega,
\]
Theorem 4.2 in this case is the classical individual and mean ergodic theorems for \(M\)-valued functions, where \(\Gamma(\varphi) = E_T(\varphi)\), the usual conditional expectation of \(\varphi\) with respect to \(T\).

(b) Let \(P = P(\mathcal{H})\) be the set of positive invertible operators on a Hilbert space \(\mathcal{H}\). A natural metric on \(P\) is the Thompson metric \(d_T(A, B) := || \log A^{-1/2} BA^{-1/2} ||\), where \(\cdot ||\) is the operator norm. Note that \((P, d_T)\) is a complete metric space. It turns out [13, 19] that there exists a contractive barycentric map \(G : P^1(\mathcal{P}) \to \mathcal{P}\), called the Karcher barycenter, which is uniquely determined by
\[
X = G(\frac{1}{n} \sum_{j=1}^{n} \delta_{A_j}) \iff \sum_{j=1}^{n} \log(A_j^{-1/2} X A_j^{-1/2}) = 0
\]
for all \(n \in \mathbb{N}\) and \((A_1, \ldots, A_n) \in P^n\). The Löwner ordering \(A \leq B\) is defined if \(B - A\) is a positive semidefinite operator on \(\mathcal{H}\), which is a closed partial order on \(P\). It turns out [13] that the Karcher barycenter is monotone for the Löwner ordering. For \(\varphi \in L^p(\Omega, \mathcal{P})\) with \(1 \leq p < \infty\), note that
\[
G(\mu_n^\varphi(\omega)) = G(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{\varphi(T^k \omega)}) = G(\varphi(\omega), \varphi(T\omega), \ldots, \varphi(T^{n-1}\omega)), \quad \omega \in \Omega,
\]
which is the Karcher mean of \(\varphi(T^k \omega)\) \((0 \leq k \leq n-1)\). Theorem 4.2 says that
\[
\lim_{n \to \infty} G(\varphi, \varphi \circ T, \ldots, \varphi \circ T^{n-1}) = \Gamma(\varphi) \text{ a.e. and in metric } d_P,
\]
where \(\Gamma\) is a map from \(L^p(\Omega; \mathcal{P})\) onto \(\{ \varphi \in L^p(\Omega, \mathcal{P}) : \varphi \circ T = \varphi \}\). When \((\Omega, \mathcal{A})\) is a standard Borel space, it follows from Theorem 4.3 that \(\Gamma(\varphi) = E_T^G(\varphi)\), the \(G\)-conditional expectation of \(\varphi\) with respect to \(T\). By Theorem 4.4 the monotonicity of \(\Gamma\) follows from that of \(G\). Moreover, \(\Gamma\) is monotone and \(\Gamma(\varphi^{-1}) = \Gamma(\varphi)^{-1}\) as seen from \(G(\mu^{-1}) = G(\mu)^{-1}\) where \(\mu^{-1}\) is the push-forward of \(\mu\) by \(A \mapsto A^{-1}\) on \(\mathcal{P}\).

5. Barycentric metric spaces and semiflows

In this section, let \(T\) be, as in Section 4, a \(P\)-preserving measurable transformation on \((\Omega, \mathcal{A}, \mathcal{P})\). Let \(1 \leq p < \infty\) be fixed and denote by \(C_p(M)\) the set of all \(p\)-contractive barycentric maps on the complete metric space \(M\). We note that a metric space equipped with a contractive barycentric map is called a barycentric metric space and that there are many (distinct) contractive barycentric maps on a metric space. For every \(\beta \in C_p(M)\), let \(\Gamma_\beta : L^p(\Omega; M) \to \{ \varphi \in L^p(\Omega; M) : \varphi \circ T = \varphi \}\) be the map given
in Theorem 4.2. This naturally defines a two-variable map

\[ \Gamma : \mathcal{C}_p(M) \times L^p(\Omega; M) \to \{ \varphi \in L^p(\Omega; M) : \varphi \circ T = \varphi \}, \quad (\beta, \varphi) \mapsto \Gamma_\beta(\varphi). \] 

(5.1)

By Theorem 4.2 (iii), \( \Gamma \) is continuous in variable \( \varphi \in L^p(\Omega; M) \). We will construct a complete metric on \( \mathcal{C}_p(M) \) such that \( \Gamma \) is continuous on \( \mathcal{C}_p(M) \times L^p(\Omega; M) \) with respect to the product metric.

For \( x = (x_1, \ldots, x_n) \in M^n \) and \( \beta \in \mathcal{C}_p(M) \), we write \( \Delta(x) \) for the diameter of \( \{x_1, \ldots, x_n\} \), and \( \beta(x) := \beta\left(\frac{1}{n} \sum_{j=1}^{n} \delta_{x_j}\right) \).

**Proposition 5.1.** For \( \beta_1, \beta_2 \in \mathcal{C}_p(M) \), define

\[ d_p(\beta_1, \beta_2) = \sup_{x \in M^n, n \in \mathbb{N}} \frac{d(\beta_1(x), \beta_2(x))}{\Delta(x)}. \]

Then \( d_p(\beta_1, \beta_2) \leq 1 \) for all \( \beta_1, \beta_2 \in \mathcal{C}_p(M) \) and \( d_p \) is a complete metric on \( \mathcal{C}_p(M) \).

**Proof.** Let \( z := \beta_1(x) \). We have

\[ d(\beta_1(x), \beta_2(x)) = d\left(\beta_2(z), \beta_2\left(\sum_{j=1}^{n} \delta_{x_j}\right)\right) = d^W_p\left(\delta_z, \sum_{j=1}^{n} \delta_{x_j}\right) \leq \left[\frac{1}{n} \sum_{j=1}^{n} d^p(\beta_1(x), x_j)\right]^{1/p} \leq \left[\frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} d^p(x_i, x_j)\right]^{1/p} \leq \Delta(x), \]

and hence \( d_p(\beta_1, \beta_2) \) is well defined with \( d_p(\beta_1, \beta_2) \leq 1 \). It is straightforward to see that \( d_p \) satisfies the metric properties. In particular, \( d_p(\beta_1, \beta_2) = 0 \) if and only if \( \beta_1 = \beta_2 \), by denseness of \( \mathcal{P}_0(M) \) in \( \mathcal{P}^p(M) \).

Assume that \( \{\beta_k\} \) is a Cauchy sequence in \( \mathcal{C}_p(M) \). Let \( \mu \in \mathcal{P}^p(M) \) and let \( \varepsilon > 0 \). By denseness of \( \mathcal{P}_0(M) \) in \( \mathcal{P}^p(M) \), one can find an \( x = (x_1, \ldots, x_n) \in M^n \) such that \( d^W_p(\mu, \mu_0) < \varepsilon \), where \( \mu_0 := \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j} \). For every \( k, l \in \mathbb{N} \) one has

\[ d(\beta_k(\mu), \beta_l(\mu)) \leq d(\beta_k(\mu), \beta_k(x)) + d(\beta_k(x), \beta_l(x)) + d(\beta_l(x), \beta_l(\mu)) \leq d(\beta_k(x), \beta_l(x)) + 2d^W_p(\mu, \mu_0) \leq d(\beta_k(x), \beta_l(x)) + 2\varepsilon. \]

There is a \( k_0 \in \mathbb{N} \) such that \( d(\beta_k(x), \beta_l(x)) \leq d_p(\beta_k, \beta_l)\Delta(x) \leq \varepsilon \) for all \( k, l \geq k_0 \). Hence \( d(\beta_k(\mu), \beta_l(\mu)) \leq 3\varepsilon \) for all \( k, l \geq k_0 \), so \( \{\beta_k(\mu)\} \) is Cauchy in \( M \). Therefore, one can define \( \beta : \mathcal{P}^p(M) \to M \) by

\[ \beta(\mu) := \lim_{k \to \infty} \beta_k(\mu) \in M. \]

For every \( x \in M \), since \( \beta_k(\delta_x) = x \) for all \( k \), we have \( \beta(\delta_x) = x \). For every \( \mu, \nu \in \mathcal{P}^p(M) \),

\[ d(\beta(\mu), \beta(\nu)) = \lim_{k \to \infty} d(\beta_k(\mu), \beta_k(\nu)) \leq d^W_p(\mu, \nu). \]

Hence \( \beta \in \mathcal{C}_p(M) \).
Next, we show that \( d_p(\beta_k, \beta) \to 0 \). For every \( \varepsilon > 0 \) choose a \( k_0 \in \mathbb{N} \) such that 
\[
\frac{d(\beta_k(x), \beta_l(x))}{\Delta(x)} \leq d_p(\beta_k, \beta_l) \leq \varepsilon, \quad k, l \geq k_0.
\]
Since \( d(\beta_l(x), \beta(x)) \to 0 \) as \( l \to \infty \), one has \( d(\beta_k(x), \beta(x))/\Delta(x) \leq \varepsilon \) for \( k \geq k_0 \), which implies that \( d_p(\beta_k, \beta) \) is close to 0 for all \( k \geq k_0 \). Hence \( d_p(\beta_k, \beta) \to 0 \).

\[ \square \]

**Theorem 5.2.** The map \( \Gamma \) is continuous on \( C_p(M) \times L^p(\Omega; M) \) with respect to the product metric of \( d_p \) and \( d_p \), that is, for sequences \( \{\beta_k\} \) in \( C_p(M) \) and \( \{\varphi_k\} \) in \( L^p(\Omega; M) \), if \( \beta_k \to \beta \in C_p(M) \) in \( d_p \) and \( \varphi_k \to \varphi \in L^p(\Omega; M) \) in \( d_p \), then \( \Gamma_{\beta_k}(\varphi_k) \to \Gamma_{\beta}(\varphi) \) in \( d_p \) as \( k \to \infty \). In particular, if \( T \) is ergodic, then \( \lim_{k \to \infty} E^{\beta_k}(\varphi_k) = E^{\beta}(\varphi) \).

Furthermore, assume that \((\Omega, A)\) is a standard Borel space. If \( \beta_k, \beta \in C_p(M) \) and \( \beta_k \to \beta \) in \( d_p \), then for every \( \varphi \in L^p(\Omega; M) \),
\[
\lim_{k \to \infty} \Gamma_{\beta_k}(\varphi)(\omega) = \Gamma_{\beta}(\varphi)(\omega) \quad \text{a.e.}
\]

**Proof.** First, assume that \( \varphi \) is a simple function with values \( x_1, \ldots, x_K \), and let \( A_j := \varphi^{-1}(x_j) \) for \( 1 \leq j \leq K \). From the proof of Theorem 4.2 (see the paragraph containing (4.4)), we recall that for any \( \beta \in C_p(M) \),
\[
\Gamma_{\beta}(\varphi)(\omega) = \beta \left( \sum_{i=1}^{K} \xi_i(\omega) \delta_{x_i} \right) \quad \text{a.e.} \quad \omega \in \Omega, \tag{5.2}
\]
where \( \xi_i(\omega) := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_i}(T_k^k \omega) \). Assume that \( \{\beta_k\} \) is a sequence in \( C_p(M) \) converging to \( \beta \). Since \( \xi_i(\omega) \geq 0 \) and \( \sum_{i=1}^{K} \xi_i(\omega) = 1 \) a.e., one can choose, for any \( N \in \mathbb{N} \) and for a.e. \( \omega \in \Omega, m_1(\omega), \ldots, m_n(\omega) \in \mathbb{N} \cup \{0\} \) such that \( m_i(\omega) \)'s are measurable and
\[
\sum_{i=1}^{n} m_i(\omega) = N, \quad \left| \xi_i(\omega) - \frac{m_i(\omega)}{N} \right| \leq \frac{1}{N} \quad \text{a.e.} \quad \omega \in \Omega.
\]

By Lemma 1.3 with \( \Delta := \text{diam}\{x_1, \ldots, x_K\} \),
\[
d_p^W \left( \sum_{i=1}^{K} \xi_i(\omega) \delta_{x_i}, 1 - \frac{1}{N} \sum_{i=1}^{K} m_i(\omega) \delta_{x_i} \right) \leq \Delta \left( \frac{1}{2} \sum_{i=1}^{K} \left| \xi_i(\omega) - \frac{m_i(\omega)}{N} \right| \right)^{1/p} \leq \Delta \left( \frac{K}{2N} \right)^{1/p} \quad \text{a.e.} \tag{5.3}
\]

Then, from (5.2) and (5.3) it follows that
\[
d(\Gamma_{\beta_k}(\varphi)(\omega), \Gamma_{\beta}(\varphi)(\omega)) \leq d \left( \beta_k \left( \sum_{i=1}^{K} \xi_i(\omega) \delta_{x_i} \right), \beta_k \left( \frac{1}{N} \sum_{i=1}^{K} m_i(\omega) \delta_{x_i} \right) \right) + d \left( \beta_k \left( \frac{1}{N} \sum_{i=1}^{K} m_i(\omega) \delta_{x_i} \right), \beta \left( \frac{1}{N} \sum_{i=1}^{K} m_i(\omega) \delta_{x_i} \right) \right)
\]
Therefore, letting $N \to \infty$ gives

$$d(\Gamma_{\beta_k}(\varphi), \Gamma_\beta(\varphi)) \leq \Delta d_p(\beta_k, \beta) \quad \text{a.e.}$$

Since $d_p(\beta_k, \beta) \to 0$, we find that $d(\Gamma_{\beta_k}(\varphi), \Gamma_\beta(\varphi)) \to 0$ a.e. and $\Gamma_{\beta_k}(\varphi) \to \Gamma_\beta(\varphi)$ in $d_p$ as $k \to \infty$.

Now, let $\varphi \in L^p(\Omega, M)$ be arbitrary, and pick a sequence of simple functions $\varphi_m : \Omega \to M$ such that $d_p(\varphi_m, \varphi) \to 0$. Since

$$d_p(\Gamma_{\beta_k}(\varphi), \Gamma_\beta(\varphi)) \leq d_p(\Gamma_{\beta_k}(\varphi), \Gamma_{\beta_k}(\varphi_m)) + d_p(\Gamma_{\beta_k}(\varphi_m), \Gamma_\beta(\varphi_m)) + d_p(\Gamma_\beta(\varphi_m), \Gamma_\beta(\varphi)),$$

one has $d_p(\Gamma_{\beta_k}(\varphi), \Gamma_\beta(\varphi)) \to 0$ from the preceding paragraph and Theorem 4.2(iii). When $\varphi_k, \varphi \in L^p(\Omega; M)$ and $d_p(\varphi_k, \varphi) \to 0$, one has

$$d_p(\Gamma_{\beta_k}(\varphi_k), \Gamma_\beta(\varphi)) \leq d_p(\Gamma_{\beta_k}(\varphi_k), \Gamma_{\beta_k}(\varphi)) + d_p(\Gamma_{\beta_k}(\varphi), \Gamma_\beta(\varphi))$$

$$\leq d_p(\varphi_k, \varphi) + d_p(\Gamma_{\beta_k}(\varphi), \Gamma_\beta(\varphi)) \to 0 \quad \text{as } k \to \infty.$$

If $T$ is ergodic, then by Theorem 4.2, $d(E^{\beta_k}(\varphi_k), E^\beta(\varphi)) = d_p(\Gamma_{\beta_k}(\varphi_k), \Gamma_\beta(\varphi)) \to 0$.

Next, assume that $(\Omega, \mathcal{A})$ is a standard Borel space, and let $(P_\omega)_{\omega \in \Omega}$ be a disintegration of $P$ with respect to $\mathcal{I} := \{A \in \mathcal{A} : T^{-1}A = A\}$. The following proof of the a.e. convergence is based on the same method as that of Theorem 3.1. Choose an $x_0 \in M$ and let $\varphi_0 := 1_{\Omega}x_0$. Since $x_0 = \beta_k(\varphi_0 \ast P_\omega)$ for all $k$, note that for every $\varphi \in L^p(\Omega; M)$,

$$d(\Gamma_{\beta_k}(\varphi)(\omega), \Gamma_{\beta_l}(\varphi)(\omega)) \leq d(\Gamma_{\beta_k}(\varphi)(\omega), x_0) + d(\Gamma_{\beta_l}(\varphi)(\omega), x_0)$$

$$= d(\beta_k(\varphi \ast P_\omega), \beta_k(\varphi_0 \ast P_\omega)) + d(\beta_l(\varphi \ast P_\omega), \beta_l(\varphi_0 \ast P_\omega))$$

$$\leq 2 \left( \int_\Omega d_p(\varphi(\tau), x_0) dP_\omega(\tau) \right)^{1/p} \quad \text{a.e. } \omega,$$

where we have used Theorem 4.3 and Lemma 2.7(i). Since Lemma 2.5(iii) gives

$$\int_\Omega \left( \int_\Omega d_p(\varphi(\tau), x_0) dP_\omega(\tau) \right) dP_\omega(\omega) = d_p^p(\varphi, \varphi_0) < \infty,$$

one can define $W(\varphi) \in L^p(\Omega; \mathbb{R})$ by

$$W(\varphi)(\omega) := \lim_{m \to \infty} \sup_{k,l \geq m} d(\Gamma_{\beta_k}(\varphi)(\omega), \Gamma_{\beta_l}(\varphi)(\omega)) \quad \text{a.e.}$$

When $\varphi$ is a simple function, since $\lim_{k \to \infty} \Gamma_{\beta_k}(\varphi)(\omega)$ exists a.e. as shown in the first paragraph of the proof, we have $W(\varphi) = 0$ as an element of $L^p(\Omega; \mathbb{R})$. Now it suffices to prove that $W$ is continuous from $L^p(\Omega; M)$ into $L^p(\Omega; \mathbb{R})$. Indeed, it then follows that for every $\varphi \in L^p(\Omega; M)$, $W(\varphi) = 0$ and hence $\lim_{k \to \infty} \Gamma_{\beta_k}(\varphi)(\omega)$ exists a.e.
To prove the above stated continuity of $W$, note that for every $\varphi, \psi \in L^p(\Omega; M)$ and every $k, l \geq 1$,

$$d(\Gamma_{\beta_k}(\varphi)(\omega), \Gamma_{\beta_l}(\varphi)(\omega)) \leq d(\Gamma_{\beta_k}(\varphi)(\omega), \Gamma_{\beta_k}(\psi)(\omega)) + d(\Gamma_{\beta_k}(\psi)(\omega), \Gamma_{\beta_l}(\psi)(\omega)) + d(\Gamma_{\beta_l}(\psi)(\omega), \Gamma_{\beta_l}(\varphi)(\omega)),$$

from which we find that

$$|W(\varphi)(\omega) - W(\psi)(\omega)| \leq 2 \sup_{k \geq 1} d(\Gamma_{\beta_k}(\varphi)(\omega), \Gamma_{\beta_k}(\psi)(\omega))$$

$$= 2 \sup_{k \geq 1} d(\beta_k(\varphi_* P_\omega), \beta_k(\psi_* P_\omega))$$

$$\leq 2 \left[ \int_{\Omega} d^p(\varphi(\tau), \psi(\tau)) d\mu_\omega(\tau) \right]^{1/p}$$

due to Lemma 2.7(2). Therefore, by Lemma 2.5(iii), $W(\varphi) - W(\psi)$ is in $L^p(\Omega; \mathbb{R})$ and

$$\|W(\varphi) - W(\psi)\|_p \leq 2d_p(\varphi, \psi),$$

implying the desired continuity of $W$. \qed

**Remark 5.3.** Assume that $(\Omega, \mathcal{A})$ is a standard Borel space and $\mathcal{B}$ is a sub-$\sigma$-algebra of $\mathcal{A}$. In view of Theorem 2.9 we have two-variable map

$$E_B : \mathcal{C}_p(M) \times L^p(\Omega; M) \to L^p(\Omega, \mathcal{B}, P; M), \quad (\beta, \varphi) \mapsto E_B^\beta(\varphi).$$

In the same way as in the proof of Theorem 5.2 one can see that $E_B$ is continuous on $\mathcal{C}_p(M) \times L^p(\Omega; M)$ with respect to the product metric and that if $\beta_k \to \beta$ in $\mathcal{C}_p(M)$ then $E_B^{\beta_k}(\varphi)(\omega) \to E_B^\beta(\varphi)(\omega)$ a.e. for every $\varphi \in L^p(\Omega; M)$. When $\mathcal{B} = \mathcal{I}$, this is the latter assertion of Theorem 5.2.

In the remaining of this section we assume that $(M, d)$ is a global NPC space. For any $x, y \in M$, there exists a unique minimal geodesic $\gamma_{x,y} : [0, 1] \to M$ such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. Denote $x\# t := \gamma_{x,y}(t)$, $t \in [0, 1]$. We note that $x\# y := x\#_{1/2} y$ is the unique midpoint between $x$ and $y$. One can see that

$$x\#_{r}(x\#_r y) = x\#_{(1-r)+rt} y, \quad x = x\#_t y \iff x = y. \quad (5.4)$$

Every global NPC space satisfies the following uniform convexity (cf. [14]): for $x, y, z \in M, t \in [0, 1]$ and $q \geq 2$,

$$d^q(z, x\#_t y) \leq (1 - t)d^q(z, x) + td^q(z, y) - \frac{k_q}{2} t(1 - t)d^q(x, y), \quad (5.5)$$

where $k_2 = 2$ and for $q > 2$, $k_q = \frac{8}{2q} \frac{1 + \tau_q^{q-1}}{1 + \tau_q}$ and $\tau_q \in (1, \infty)$ is the unique solution to $x^{q-1} + (1 - q)x + 2 - q = 0$.

**Definition 5.4.** Let $1 \leq p < \infty$. For $x \in M, t \in [0, 1]$ and $\mu \in \mathcal{P}^p(M)$, define $x\#_{t\mu} \in \mathcal{P}^p(M)$ by $x\#_{t\mu} := f_*(\mu)$, where $f : M \to M$ is the contraction mapping $f(a) = x\#_{t} a$.  

Theorem 5.5. Let $x\#0\mu = \delta_x$ and $x\#1\mu = \mu$, and $x\#\mu = \frac{1}{n}\sum_{j=1}^{n}\delta_{x\#t\alpha_j}$ for $\mu = \frac{1}{n}\sum_{j=1}^{n}\delta_{\alpha_j}$. One can directly see that $x\#t(x\#s\mu) = x\#s\mu$ for $s, t \in [0, 1]$ and $\mu \in \mathcal{P}^p(M)$, which implies that for any fixed $x \in M$, $(t, \mu) \mapsto x\#t\mu$ is a semiflow on $\mathcal{P}^p(M)$ under the multiplicative semigroup on $[0, 1]$. We note that $1$ is the identity on the semigroup. The map $t \mapsto e^{-t}$ is a homeomorphic isomorphism from the additive semigroup $\mathbb{R}_+ := [0, \infty)$ onto $(0, 1]$, so $(t, \mu) \mapsto x\#e^{-t}\mu (t \geq 0)$ becomes an additive semiflow.

Recall the metric space $C_p(M)$ of $p$-contractive barycentric maps in Proposition 5.1.

**Theorem 5.5.** Let $1 \leq p < \infty$. There exists a continuous semiflow $\Phi_p : (0, 1] \times C_p(M) \rightarrow C_p(M)$ satisfying

$$x = \Phi_p(t, \beta)(\mu) \iff x = \beta(x\#t\mu)$$

for $t \in (0, 1]$, $\beta \in C_p(M)$ and $\mu \in \mathcal{P}^p(M)$. Furthermore, for every $\beta \in C_p(M)$,

$$d_p(\Phi_p(t, \beta), \Phi_p(s, \beta)) \leq \left[ \frac{k_2p(s + t) + 2(2 - k_2p)}{4} \right]^{1/p}.$$

In particular, $\lim_{t \rightarrow 0^+} \Phi_1(t, \beta) = \lambda$ for every $\beta \in C_1(M)$, where $\lambda : \mathcal{P}^1(M) \rightarrow M$ is the canonical barycentric map on the global NPC space $M$ given in (2.6).

**Proof.** Let $t \in (0, 1]$ and $\beta \in C_p(M)$. Let $\mu \in \mathcal{P}^p(M)$. Define $F : M \rightarrow M$ by $F(x) := \beta(x\#t\mu)$. We shall show that $F$ is a strict contraction on $M$. If $\mu = \frac{1}{n}\sum_{j=1}^{n}\delta_{\alpha_j}$,

$$d(F(x), F(y)) = d\left( \beta\left( \frac{1}{n}\sum_{j=1}^{n}\delta_{x\#t\alpha_j} \right), \beta\left( \frac{1}{n}\sum_{j=1}^{n}\delta_{y\#t\alpha_j} \right) \right) \leq \left[ \frac{1}{n}\sum_{j=1}^{n}d^p(x\#t\alpha_j, y\#t\alpha_j) \right]^{1/p} \leq (1 - t)d(x, y),$$

where the last inequality follows from $d(z\#t\alpha_j, x\#t\alpha_j) \leq (1 - t)d(z, x)$. For general $\mu \in \mathcal{P}^p(M)$, pick a sequence $\{\mu_n\} \subset \mathcal{P}_0(M)$ converging to $\mu$ in $\mathcal{P}^p(M)$. Then

$$d(\beta(x\#t\mu_n), \beta(x\#t\mu)) \leq d^W_p(x\#t\mu_n, x\#t\mu) \leq d^W_p(\mu_n, \mu) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and hence $d(F(x), F(y)) = \lim_{n \rightarrow \infty} d(\beta(x\#t\mu_n), \beta(y\#t\mu_n)) \leq (1 - t)\lim_{n \rightarrow \infty} d(x, y) = (1 - t)d(x, y)$ which shows that $F$ is a strict contraction and hence $x = F(x)$ has a unique solution. Let $\Phi_p(t, \beta)(\mu)$ denote the unique fixed point of $F$.

We will show that $\Phi_p(t, \beta) \in C_p(M)$ for all $t \in (0, 1]$ and $\beta \in C_p(M)$. By definition, $\Phi_p(1, \beta) = \beta$ for all $\beta \in C_p(M)$. Fix $t \in (0, 1)$. Let $z \in M$ and put $x = \Phi_p(t, \beta)(\delta_z)$. Then $x = \beta(x\#t\delta_z) = \beta(\delta_{x\#t\alpha_z}) = x\#z$ and hence $\Phi_p(t, \beta)(\delta_z) = x = z$ by (5.4). Let $x = \Phi_p(t, \beta)(\mu)$ and $y = \Phi_p(t, \beta)(\nu)$. Then

$$d(x, y) = d(\beta(x\#t\mu), \beta(y\#t\nu)) \leq d^W_p(x\#t\mu, y\#t\nu) \leq [(1 - t)d^p(x, y) + td^W_p(\mu, \nu)^p]^{1/p},$$
which implies that \(d(x, y) \leq d^W_p(\mu, \nu)\). Therefore, \(\Phi_p(t, \beta) : \mathcal{P}^p(M) \to M\) is a p-contractive barycentric map.

We will prove that \(\Phi_p\) is a continuous semiflow on \(\mathcal{C}_p(M)\). Let \(x = \Phi_p(t, \Phi_p(s, \beta)) (\mu)\). Then \(x = \Phi_p(s, \beta)(x \#_s \mu)\), which is equivalent to \(x = \beta(x \# s (x \# t \mu)) = \beta(x \# st \mu)\), that is, \(x = \Phi_p(st, \beta)(\mu)\). To see the continuity of \(\Phi_p\), let \(a = (a_1, \ldots, a_n) \in M^n\), and set \(\mu_a := \frac{1}{n} \sum_{j=1}^n \delta_{a_j}\) for notational simplicity. Let \(x = \Phi_p(t, \beta_1)(\mu_a)\) and \(y = \Phi_p(s, \beta_2)(\mu_a)\). Then by the triangle inequality and [16, Proposition 3.8 (2)],

\[
d(x, y) = d(\beta_1(x \#_1 \mu_a), \beta_2(y \#_1 \mu_a)) \\
\leq d(\beta_1(x \#_1 \mu_a), \beta_2(x \#_1 \mu_a)) + d(\beta_2(x \#_1 \mu_a), \beta_2(y \#_1 \mu_a)) \\
\leq d_p(\beta_1, \beta_2) \Delta(x \#_1 a_1, \ldots, x \#_1 a_n) + \frac{1}{n} \sum_{j=1}^n d(x \#_1 a_j, y \#_1 a_j) \\
\leq t d_p(\beta_1, \beta_2) \Delta(a) + \frac{1}{n} \sum_{j=1}^n [(1-t) d(x, y) + |t-s| d(y, a_j)] \\
= t d_p(\beta_1, \beta_2) \Delta(a) + (1-t) d(x, y) + \frac{|t-s|}{n} \sum_{j=1}^n d(y, a_j),
\]

where \(\Delta(x \#_1 a_1, \ldots, x \#_1 a_n) \leq t \Delta(a)\) follows from \(d(x \#_1 a_1, x \#_1 a_j) \leq td(a_1, a_j)\). Moreover, since \(d(y, a_j) = d(\Phi_p(s, \beta_2)(\mu_a), \Phi_p(s, \beta_2)(\delta_{a_j})) \leq d^W_p(\mu_a, \delta_{a_j}) \leq \Delta(a)\), we find that \(d(x, y) \leq d_p(\beta_1, \beta_2) \Delta(a) + \frac{|t-s|}{t} \Delta(a)\). This implies that

\[
d(\Phi_p(t, \beta_1), \Phi_p(s, \beta_2)) = \sup_{a \in M^n, n \in \mathbb{N}} \frac{d(\Phi_p(t, \beta_1)(\mu_a), \Phi_p(s, \beta_2)(\mu_a))}{\Delta(a)} \\
\leq d_p(\beta_1, \beta_2) + \frac{|t-s|}{t},
\]

which shows continuity of \(\Phi_p\).

Next, we shall show (5.7). Let \(x = \Phi_p(t, \beta)(\mu_a)\). For any \(z \in M\), from (5.5) we have

\[
d^{2p}(z, x) = d^{2p}\left(\beta(\delta_z), \beta\left(\frac{1}{n} \sum_{j=1}^n \delta_{x \#_1 a_j}\right)\right) \\
\leq d^W_p\left(\delta_z, \left(\frac{1}{n} \sum_{j=1}^n \delta_{x \#_1 a_j}\right)\right)^{2p} \\
\leq \frac{1}{n} \sum_{i=1}^n d^{2p}(z, x \#_1 a_i) \\
\leq \frac{1}{n} \sum_{i=1}^n \left[(1-t)d^{2p}(z, x) + td^{2p}(z, a_i) - \frac{k_{2p}}{2}(1-t)td^{2p}(x, a_i)\right] \\
= (1-t)d^{2p}(z, x) + \frac{t}{n} \sum_{i=1}^n d^{2p}(z, a_i) - \frac{k_{2p}(1-t)t}{2n} \sum_{i=1}^n d^{2p}(x, a_i),
\]
and hence
\[ d^{2p}(z, x) \leq \frac{1}{n} \sum_{i=1}^{n} d^{2p}(z, a_i) - \frac{k_{2p}(1-t)}{2n} \sum_{i=1}^{n} d^{2p}(x, a_i). \quad (5.8) \]

Applying this with \( y = \Phi_p(s, \beta)(\mu_a) \) leads to
\[ d^{2p}(y, x) \leq \frac{1}{n} \sum_{i=1}^{n} d^{2p}(y, a_i) - \frac{k_{2p}(1-t)}{2n} \sum_{i=1}^{n} d^{2p}(x, a_i), \]
\[ d^{2p}(x, y) \leq \frac{1}{n} \sum_{i=1}^{n} d^{2p}(x, a_i) - \frac{k_{2p}(1-s)}{2n} \sum_{i=1}^{n} d^{2p}(y, a_i). \]

Summing these yields
\[ 2d^{2p}(x, y) \leq \frac{2 - k_{2p}(1-s)}{2n} \sum_{j=1}^{n} d^{2p}(y, a_j) + \frac{2 - k_{2p}(1-t)}{2n} \sum_{j=1}^{n} d^{2p}(x, a_j) \]
\[ \leq \frac{k_{2p}(s + t) + 2(2 - k_{2p})}{2} \Delta(a)^{2p}. \]

Finally, assume that \( p = 1 \). Since \( k_2 = 2 \), it follows from (5.7) that \( \{\Phi_1(t, \beta)\}_{t \in (0,1]} \) is a Cauchy net in the complete metric space \( C_1(M) \). Let
\[ \beta_0 := \lim_{t \to 0^+} \Phi_1(t, \beta). \]

By (5.8) we have for every \( z \in M \)
\[ d^2(z, \Phi_1(t, \beta)(\mu_a)) \leq \frac{1}{n} \sum_{i=1}^{n} d^2(z, a_i) - \frac{1-t}{n} \sum_{i=1}^{n} d^2(\Phi_1(t, \beta)(\mu_a), a_i). \]

By taking the limit of both sides of the above as \( t \to 0^+ \), we have
\[ d^2(z, \beta_0(\mu_a)) \leq \frac{1}{n} \sum_{i=1}^{n} d^2(z, a_i) - \frac{1}{n} \sum_{i=1}^{n} d^2(\beta_0(\mu_a), a_i), \]
which implies that \( \frac{1}{n} \sum_{i=1}^{n} d^2(\beta_0(\mu_a), a_i) \leq \frac{1}{n} \sum_{i=1}^{n} d^2(z, a_i) \) for any \( z \in M \). This shows that \( \beta_0(\mu_a) = \lambda(\mu_a) \) for all \( a = (a_1, \ldots, a_n) \in M^n \) and \( n \in \mathbb{N} \). By continuity and denseness of \( \mathcal{P}_0(M) \) in \( \mathcal{P}^1(M) \), we have \( \beta_0(\mu) = \lambda(\mu) \) for all \( \mu \in \mathcal{P}^1(M) \). \( \square \)

**Remark 5.6.** The canonical barycenter \( \lambda : \mathcal{P}^1(M) \to M \) is then the global attractor of the semiflow \( \Phi_p \) on \( C_p(M) \) for \( p = 1 \).

By Theorems 5.2 and 5.3

**Corollary 5.7.** Let \( \beta \in C_1(M) \) and set \( \beta_t := \Phi_1(t, \beta) \). Then for every \( \varphi \in L^1(\Omega; M) \),
\[ d_1(\Gamma_{\beta_t}(\varphi), \Gamma_\lambda(\varphi)) \to 0 \quad \text{and} \quad d(\Gamma_{\beta_t}(\varphi)(\omega), \Gamma_\lambda(\varphi)(\omega)) \to 0 \quad \text{a.e.} \]
as \( t \to 0^+ \). If \( T \) is ergodic, then \( \lim_{t \to 0^+} E^{\beta_t}(\varphi) = E^\lambda(\varphi) \).
Remark 5.8. (1) It does not seem easy to show the convergence of the net \( \{ \Phi_p(t, \beta) \}_{t \in (0,1]} \) as \( t \to 0^+ \) for general \( p > 1 \). Although the minimizer
\[
\lambda_p(\mu_a) := \arg \min_{x \in M} \sum_{i=1}^{n} d_{2p}(x, a_i)
\]
eexists uniquely for every \( a = (a_1, \ldots, a_n) \in M^n \) and \( n \in \mathbb{N} \) from the uniform convexity in (5.3) and from [24, Proposition 1.7], to the best of our knowledge, its \( p \)-contractive property is unknown.

(2) For \( \beta_1, \beta_2 \in C_p(M) \) and \( t \in [0,1] \), define \( (\beta_1 \#_t \beta_2)(\mu) := \beta_1(\mu) \#_t \beta_2(\mu) \) for \( \mu \in \mathcal{P}^p(M) \). Then it is direct to see that the map \( t \mapsto \beta_1 \#_t \beta_2 \) is a minimal geodesic in \( C_p(M) \) with respect to the complete metric \( d_p \), and also that \( (\beta_1 \#_s \beta_2) \#_r (\beta_1 \#_r \beta_2) = \beta_1 \#_{(s+r)\beta_2} \) and \( d_p(\beta_1 \#_t \beta_2, \beta_3 \#_t \beta_4) \leq (1-t)d_p(\beta_1, \beta_3) + td_p(\beta_2, \beta_4) \). This shows that \( (C_p(M), d_p) \) is a convex metric space [17, 16], or a Busseman space without the uniqueness of geodesics or midpoints. Navas’ approach in [21] allows us to define on \( C_p(M) \) the contractive barycentric map of Es-Sahib and Heinich [10].

6. LARGE DEVIATION PRINCIPLE

First, recall the general formulation of the large deviation principle (LDP) (cf. [6]). Let \( \mathcal{X} \) be a metric space and \( \mathcal{B}(\mathcal{X}) \) the Borel \( \sigma \)-algebra on \( \mathcal{X} \). Let \( (\mu_n)_{n=1}^{\infty} \) be a sequence of Borel probability measures on \( \mathcal{X} \). A function \( I : \mathcal{X} \to [0,\infty) \) is called a rate function if \( I \) is lower semicontinuous, that is, for every \( \alpha \in [0,\infty) \) the level set \( \{ x \in \mathcal{X} : I(x) \leq \alpha \} \) is closed. A good rate function is a rate function \( I : \mathcal{X} \to [0,\infty) \) whose level sets are compact for all \( \alpha \in [0,\infty) \). It is said that \( (\mu_n) \) satisfies the LDP (in the scale \( 1/n \)) with a rate function \( I \), if for every \( \Gamma \in \mathcal{B}(\mathcal{X}) \),
\[
- \inf_{x \in \Gamma} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(\Gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(\Gamma) \leq - \inf_{x \in \overline{\Gamma}} I(x),
\]
where \( \overline{\Gamma} \) and \( \Gamma^o \) denote the interior and the closure of \( \Gamma \), respectively.

Let \( (\Omega, \mathcal{A}, \mathbb{P}) \) be a probability space. Let \( \Sigma \) be a Polish space and \( \mathcal{P}(\Sigma) \) be the set of Borel probability measures on \( \Sigma \) equipped with the weak topology. Note that the weak topology on \( \mathcal{P}(\Sigma) \) is metrizable with the Lévy-Prokhorov metric \( \rho \) and \( (\mathcal{P}(\Sigma), \rho) \) becomes a Polish space. Let \( X = (X_1, X_2, \ldots) \) be a sequence of i.i.d. \( \Sigma \)-valued random variables and \( \mu_0 \in \mathcal{P}(\Sigma) \) be their equal distribution, i.e., \( \mu_0(B) = \mathbb{P}(X_i^{-1}(B)) \) for all \( i \in \mathbb{N} \) and \( B \in \mathcal{B}(\Sigma) \). We define the empirical measure
\[
\mu_n^X(\omega) := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(\omega)}, \quad n \in \mathbb{N},
\]
and consider the distribution \( \tilde{\mu}_n \) of \( \mu_n^X : \Omega \to \mathcal{P}(\Sigma) \), i.e., for Borel sets \( \Gamma \subset \mathcal{P}(\Sigma) \),
\[
\tilde{\mu}_n(\Gamma) := \mathbb{P}(\mu_n^X \in \Gamma) = \mu_0^{\times n}\left( \left\{ (x_1, \ldots, x_n) \in \Sigma^n : \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \in \Gamma \right\} \right).
\]
Then the celebrated Sanov theorem is

**Theorem 6.1.** The distributions \((\hat{\mu}_n)\) of the empirical measures \((\mu_n^X)\) satisfies the LDP with the relative entropy functional \(S(\cdot||\mu_0)\) as the good rate function, where the relative entropy (or the Kullback-Leibler divergence) \(S(\mu||\mu_0)\) is defined by

\[
S(\mu||\mu_0) := \begin{cases} 
\int_\Sigma \log \frac{d\mu}{d\mu_0} \, d\mu & \text{if } \mu \ll \mu_0 \text{ (absolutely continuous),} \\
\infty & \text{otherwise.}
\end{cases}
\]

Now, assume that \((M,d)\) be a complete metric space and let \(X = (X_1, X_2, \ldots)\) be a sequence of i.i.d. \(M\)-valued random variables. Assume that the distribution \(\mu_0\) of \(X_i\) is in \(P^\infty(M)\), i.e., \(X_i \in L^\infty(\Omega; M)\). Since a strongly measurable \(M\)-valued function has a separable range except on a \(P\)-null set, one can choose a separable closed subset \(M_0\) of \(M\) such that \(X_i(\omega) \in M_0\) for all \(i \in \mathbb{N}\) and a.e. \(\omega \in \Omega\) (or \(\mu_0\) is supported on \(M_0\)). Moreover, choose an \(x_0 \in M\) and let \(\alpha := \text{ess sup}_{\omega \in \Omega} d(X_1(\omega), x_0) < \infty\). Then \(X_i(\omega) \in \Sigma := \{x \in M_0 : d(x, x_0) \leq \alpha\}\) for all \(i \in \mathbb{N}\) and a.e. \(\omega \in \Omega\). Note that \(\Sigma\) is a Polish space, and we may assume that \(X_i\)'s are \(\Sigma\)-valued random variables. Hence the Sanov LDP holds for the sequence \(X = (X_1, X_2, \ldots)\).

Let \(1 \leq p < \infty\) and \(\beta : P^p(M) \rightarrow M\) be a \(p\)-contractive barycentric map. Note that \(P(\Sigma)\) is a subset of \(P^p(M)\). Since \(\Sigma\) is bounded, it follows from [26, Theorem 7.12] that the \(d^W_p\)-topology on \(P(\Sigma)\) coincides with the weak topology on \(P(\Sigma)\). Hence \(\mu \in P(\Sigma) \mapsto \beta(\mu)\) is a continuous map from \(P(\Sigma)\) equipped with the weak topology to \((M,d)\). Note that the push-forward of \(\hat{\mu}_n\) by \(\beta|_{P(\Sigma)}\) is the distribution of \(\beta(\mu_n^X)\), i.e., for every \(\Gamma \in B(M)\),

\[
\hat{\mu}_n\{\mu \in P(\Sigma) : \beta(\mu) \in \Gamma\} = P(\beta(\mu_n^X) \in \Gamma) = P\left(\left\{\omega \in \Omega : \beta\left(\frac{1}{n} \sum_{i=1}^n \delta_{X_i(\omega)}\right) \in \Gamma\right\}\right).
\]

Therefore, from Theorem 6.1 applying the contraction principle for LDP (see [6, Theorem 4.2.1]) with the continuous map \(\beta : P(\Sigma) \rightarrow M\), we have the following:

**Theorem 6.2.** With the above definitions and assumptions, the distribution of the \(M\)-valued random variable \(\beta(\mu_n^X) = \beta(\frac{1}{n} \sum_{i=1}^n \delta_{X_i})\) satisfies the LDP with the good rate function

\[
I(x) := \inf\{S(\mu||\mu_0) : \mu \in P(\Sigma), x = \beta(\mu)\}, \quad x \in M. \tag{6.1}
\]

That is, for every \(\Gamma \in B(M)\),

\[
- \inf_{x \in \Gamma} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P\left(\beta(\mu_n^X) \in \Gamma\right) \leq \limsup_{n \to \infty} \frac{1}{n} \log P\left(\beta(\mu_n^X) \in \Gamma\right) \leq - \inf_{x \in \Gamma} I(x). \tag{6.2}
\]

The above LDP is a stronger version of the strong law of large numbers for the \(\beta\)-value \(\beta(\frac{1}{n} \sum_{i=1}^n \delta_{X_i})\) of the empirical measure, given in [24, Proposition 6.6]. Let
$x_0 := \beta(\mu_0)$. Since $S(\cdot \| \mu_0)$ is a good rate function on $\mathcal{P}(\Sigma)$, for every $x \in M$ with $I(x) < \infty$ there is a $\mu \in \mathcal{P}(\Sigma)$ such that $x = \beta(\mu)$ and $I(x) = S(\mu \| \mu_0)$. Therefore, from the strict positivity of the relative entropy, we see that $I(x) > 0$ whenever $x \neq x_0$.

For any $\varepsilon > 0$ take a closed set $F := \{ x \in M : d(x, x_0) \geq \varepsilon \}$; then the LDP upper bound in [6.2] gives

$$\limsup_{n \to \infty} \frac{1}{n} \log P(\beta(\mu_n^X) \in F) \leq -\inf_{x \in F} I(x) < -\alpha$$

for some $\alpha > 0$, since $I$ is a good rate function. This implies that

$$\sum_{n=1}^{\infty} P(\{ \omega : d(\beta(\mu_n^X)(\omega), x_0) \geq \varepsilon \}) < \infty,$$

so the Borel-Cantelli lemma yields that

$$P\left( \limsup_{n \to \infty} \{ \omega : d(\beta(\mu_n^X)(\omega), x_0) \geq \varepsilon \} \right) = 0,$$

which implies that $d(\beta(\mu_n^X)(\omega), x_0) \to 0$ as $n \to \infty$. We thus have the strong law of large numbers in [24, Proposition 6.6].

**Corollary 6.3.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $M$-valued random variables having the distribution $\mu_0 \in \mathcal{P}^\infty(M)$. Then

$$\beta\left( \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(\omega)} \right) \to \beta(\mu_0) \text{ a.e. as } n \to \infty.$$

**Remark 6.4.** A point of the above argument is that although the Sanov LDP is concerned with the weak topology on $\mathcal{P}(M)$, the contractive barycentric map $\beta$ on $\mathcal{P}^p(M)$ is continuous with respect to the Wasserstein distance $d_p^W$ so that $\beta$ is not necessarily continuous with respect to the weak topology. This is the reason why we have to assume that the i.i.d. random variables $X_1, X_2, \ldots$ have a bounded support, i.e., the distribution measure is in $\mathcal{P}^\infty(M)$.

**Example 6.5.** Let $X_1, X_2, \ldots$ be a sequence i.i.d. random variables with values in a finite set $\{ A_1, \ldots, A_K \}$ in $\mathcal{P} = \mathcal{P}(\mathcal{H})$, whose distribution is $\mu_0 = \sum_{j=1}^{K} w_j \delta_{A_j}$, where $w_j > 0$ and $\sum_{j=1}^{K} w_j = 1$. Let $G$ be the Karcher barycenter on $\mathcal{P}$, and consider the $G$-value of the empirical measure

$$G\left( \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(\omega)} \right) = G(X_1(\omega), \ldots, X_n(\omega)).$$

By Theorem [6.2] the distribution of $\mathcal{P}$-valued random variable $G(X_1(\omega), \ldots, X_n(\omega))$ satisfies the LDP with the good rate function

$$I(A) := \inf \left\{ \sum_{j=1}^{K} p_j \log \frac{p_j}{w_j} : A = G\left( \sum_{j=1}^{K} p_j \delta_{A_j} \right) \right\} \text{ for } A \in \mathcal{P}.$$
Let $\Delta_K$ be the set of all $K$-dimensional probability vectors, and let

$$\Gamma_G(A_1, \ldots, A_K) := \left\{ G \left( \sum_{j=1}^{K} p_j \delta_{A_j} \right) : (p_1, \ldots, p_K) \in \Delta_K \right\}.$$ 

Assume that $A_1, \ldots, A_K$ are “in general position” with respect $G$ in the sense that $(p_1, \ldots, p_K) \in \Delta_K \mapsto G(\sum_{j=1}^{K} p_j \delta_{A_j}) \in \mathbb{P}$ is one-to-one. In this case, the above rate function is written as

$$I(A) = \begin{cases} \sum_{j=1}^{K} p_j \log \frac{p_j}{w_j} & \text{if } A \in \Gamma_G(A_1, \ldots, A_K) \text{ and } A = G(\sum_{j=1}^{K} p_j \delta_{A_j}), \\ \infty & \text{if } A \notin \Gamma_G(A_1, \ldots, A_K). \end{cases}$$

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Tohoku University (Emeritus), Hakusan 3-8-16-303, Abiko 270-1154, Japan

*E-mail address*: hiai.fumio@gmail.com

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea

*E-mail address*: ylim@skku.edu