Weak decay constant of pseudoscalar mesons in a QCD-inspired model*

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We show that a linear scaling between the weak decay constants of pseudoscalars and the vector meson masses is supported by the available experimental data. The decay constants scale as \( f_m/f_\pi = M_V/M_m \) (\( f_m \) decay constant and \( M_V \) vector meson ground state mass). This simple form is justified within a renormalized light-front QCD-inspired model for quark-antiquark bound states.

I. INTRODUCTION

Effective theories used to describe hadrons, which are inspired by Quantum Chromodynamics\(^1\)\(^2\)\(^3\) can be useful in indicating direct correlations between observables of different hadrons. Therefore, it is possible to pin down the relevant dependence of the observables with some physical scales that otherwise would have no simple reason to present a direct relation, besides being properties of the same underlying theory. For example, a systematic dependence of a hadron observable with its mass can offer useful guide for presenting results obtained in Lattice QCD. Systematic correlations between different meson properties with mass scales are also found from the solution of Dyson-Schwinger equations\(^4\)\(^5\).

One intriguing aspect is the dependence of the weak decay constant of the pseudoscalar meson with its mass. For light mesons up to \( D \), the weak decay constant tends to increase with the mass, while numerical simulations of quenched lattice-QCD indicate that \( f_D > f_B \) \(^6\)\(^7\), which is still maintained with two flavor sea quarks \(^8\)\(^9\). General arguments, within Dyson-Schwinger formalism for QCD in the heavy quark limit, says that the weak decay constant should be inversely proportional to \( \sqrt{M_m} \) \(^10\) (\( M_m \) is the pseudoscalar mass). Effective QCD inspired models valid for low energy scales can also be called to help to investigate this subtle point. In these models \(^2\)\(^3\)\(^10\), the interaction is flavor independent, while the masses of constituent quarks can be changed, which naturally implies in correlations between observables and masses.

Our aim here, is to investigate the pseudoscalar weak decay constant within a QCD inspired model \(^8\). The effective mass operator equation for the lowest Light-Front Fock-state component of a bound system of a constituent quark and antiquark of masses \( m_1 \) and \( m_2 \), obtained in the effective one-gluon-exchange interaction approximation \(^1\) and simplified in the \( \uparrow\downarrow \) model\(^2\)\(^3\) to

\[
M_m^2 \psi_m(x, \vec{k}_\perp) = \left[ \frac{k_1^2 + m_1^2}{x} + \frac{k_2^2 + m_2^2}{1-x} \right] \psi_m(x, \vec{k}_\perp)
\]

\[
- \int dx' d\vec{k}_\perp' \xi(x, x') \left( \frac{4m_1m_2\alpha}{3\pi^2 Q^2} - \lambda \right) \psi_m(x', \vec{k}_\perp'), \quad (1)
\]

where the phase space factor is

\[
\xi(x, x') = \frac{\theta(x')\theta(1-x')}{\sqrt{x(1-x)x'(1-x')}},
\]

and \( \psi_m \) is the projection of the light-front wave-function in the quark-antiquark Fock-state. The mean square momentum transfer \( \langle (k_1' - k_1)^2 + (k_2' - k_2)^2 \rangle / 2 \) gives \( Q^2 \) \((k_1' \text{ and } k_2' \text{ are the quark four-momenta})\). The coupling constant \( \alpha \) defines the strength of the Coulomb-like potential and \( \lambda \) is the bare coupling constant of the Dirac-delta hyperfine interaction. The energy transfer in \( Q^2 \) is left out. Confinement comes through the binding of the constituents in the meson, which in practice keeps the quarks inside the mesons.

The mass operator equation \(^1\) needs to be regularized and renormalized in order to give physical results, such development has been performed in Ref.\(^8\). In that work, it was obtained the renormalized form of the equation for the bound state mass, which is \( i ) \) invariant under renormalization group transformations, \( ii \) the physical input is given by the pion mass and radius \( iii \) no regularization parameter.

In the work of Ref.\(^8\), the quark mass was changed to allow the study of mesons with one light antiquark plus a strange, charm or bottom quark. The masses of the constituent quarks were within the range of 300 up to 5000 MeV. The up-down quarks a mass of 384 MeV is found from the rho meson mass, which in the model is weakly bound. The Dirac-delta interaction comes from an effective hyperfine interaction which splits the pseudoscalar and vector meson states. In the singlet channel the hyperfine interaction is attractive, which is not valid for the spin one mesons. In the model, the Dirac-delta interaction mock up short-range physics which are brought by the empirical value of the pion mass, and a reasonable description of the binding energies of the constituent

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quarks forming the pseudoscalar mesons is found \[8\]. The model, without the Coulomb like interaction, was also able to describe the binding energies of the ground state of spin 1/2 baryons containing two light quarks and a heavy one \[8\].

Within the effective model of Eq. (1), the low-lying vector mesons are weakly bound systems of constituent quarks while the pseudo-scalars are more strongly bound \[8\]. This allows to calculate the masses of the constituent quarks directly from the masses of the vector mesons ground states \[8\]:

\[
\begin{align*}
m_u &= \frac{1}{2} M_\rho = 384 \text{ MeV}, \\
m_s &= M_{K^*} - \frac{1}{2} M_\rho = 508 \text{ MeV}, \\
m_c &= M_{D^*} - \frac{1}{2} M_\rho = 1623 \text{ MeV}, \\
m_b &= M_{B^*} - \frac{1}{2} M_\rho = 4941 \text{ MeV},
\end{align*}
\]

where it is used the values of 768 MeV, 892 MeV, 2007 MeV and 5325 MeV for the \(\rho\), \(K^*\), \(D^*\) and \(B^*\) masses, respectively \[10\].

Here, we use the effective model to predict a physical property directly related to the wave-function of the ground state of the pseudo scalar mesons. We calculate the weak decay constants \(f_m\) of \(K^+, D^+, D_s^+\), for which experimental values are known \[10\]. Besides the constituent quark masses from Eq. (2) and the pion mass, our calculation needs as input the pion weak decay constant, \(f_\pi = 92.4 \pm 0.07 \pm 0.25 \text{ MeV}\) \[10\]. The eigenfunction of the interacting mass squared operator from Eq. (1) for large transverse momentum behaves as the asymptotic wave-function, which decreases slowly as \(p_\perp^2\). Therefore, in the calculation of the weak decay constants it is necessary to regulate the logarithmic divergence in the transverse momentum integration and take care of the cut-off dependence to be able to give an unique answer. One has to consider that the pion decay constant provides the short-range information contained in the pion wave function, which we suppose to be the same for all pseudo-scalars. Here, we just write the divergent integral in the transverse momentum in terms of \(f_\pi\) and from that obtain the other decay constants.

\[\psi_m(x, \vec{k}_\perp) = \frac{1}{\sqrt{x(1-x)}} \frac{G_m}{M_m^2 - M_0^2} \left[1 - \int \frac{d^2k'}{4\pi^2} \frac{\theta(x')\theta(1-x')}{\sqrt{x'(1-x')}} \left(\frac{4m_2}{3\pi^2} \frac{\alpha}{Q^2}\right) \right], \]

where

\[M_m^2 = \frac{\vec{k}_\perp^2 + m_2^2}{x} + \frac{\vec{k}_\perp^2 + m_3^2}{1-x}, \]

in the frame in which the meson has zero transverse momentum. \((M_0^2\) is obtained from \(M_m^2\) by substitution of \(\vec{k}_\perp\) and \(x\) by \(\vec{k}_\perp'\) and \(x'\), respectively.) The overall normalization of the \(\bar{q}q\) Fock-component of the meson wave-function \[8\] is \(G_m\).

In this first calculation of the decay constant within this model, we are going to assume the dominance of the asymptotic form of the meson wave function and simply use

\[\psi_m(x, \vec{k}_\perp) = \frac{1}{\sqrt{x(1-x)}} \frac{G_m}{M_m^2 - M_0^2} \quad (5)\]

To obtain the pseudoscalar decay constants, we follow Ref. \[11\]. To construct the observables in terms of the meson wave function, one has to account for the coupling of the quark spins, which is described by an effective Lagrangian density with a pseudo-scalar coupling between the vacuum state \(|0\rangle\) and the meson state \(|\Phi_m(\vec{x})\rangle\) fields \[11\]:

\[\mathcal{L}_{eff}(\vec{x}) = -i G_m \bar{\psi}_m(\vec{x}) \gamma^5 q_2(\vec{x}) + h.c.,\]

the coupling constant is \(G_m\). From the effective Lagrangian above one can derive meson observables and write them in terms of the light-front asymptotic wave function, Eq. \[11\]. To achieve this goal, it is necessary to eliminate the relative \(x^+\)-time \((x^+ = t + z\) between the constituents in the physical amplitude, which then allows to write the meson observable in terms of the wave function \[11\].

### III. RESULTS FOR THE WEAK DECAY CONSTANT OF PSEUDOSCALAR MESONS

The pseudoscalar meson weak decay constant is calculated from the matrix element of the axial current \(A^\mu(0)\), between the vacuum state \(|0\rangle\) and the meson state \(|q_m\rangle\) with four momentum \(q_m\) \[11\]:

\[\langle 0 | A^\mu(0) | q_m \rangle = i \sqrt{2} f_m \eta_m^\mu, \]

where \(A^\mu(\vec{x}) = \bar{\psi}(\vec{x})\gamma^\mu\gamma^5 q(\vec{x})\).

Using the pseudoscalar Lagrangian, Eq. \[11\], one can calculate the matrix element of the axial current, which is expressed by a one-loop diagram and can be written as:

\[i \sqrt{2} M_m f_m = N_c G_m \int \frac{d^4k}{(2\pi)^4} Tr \left[\gamma^+ \gamma^5 S_2(k) \gamma^5 S_1(k - q_m)\right],\]

where \(\gamma^+ = \gamma^0 + \gamma^3\), \(N_c = 3\) is number of colors and \(S_i(p) = i/(p - m_i + i\epsilon)\) is the propagator of the quark field.


By integration over \( k^− \) in Eq. (5), the relative light-front time between the quarks is eliminated and one obtains the expression of \( f_m \) suitable for the introduction of the meson light front wave function. So, performing the Dirac algebra and integrating analytically over \( k^− \), one obtains

\[
f_m = \frac{-\sqrt{2}}{8\pi^3} N_c \int_0^1 dx \left( (1-x)m_2 + xm_1 \right)
\times \int dk^2_\perp \frac{G_m}{x(1-x)M_m^2 - k^2_\perp - m_1 x - m_2(x)} ,
\]

in the meson rest-frame. We have used the momentum fraction \( x = k^+/q^m \).

One can write Eq. (9) in terms of the valence component of the pseudoscalar meson wave function as:

\[
f_m = \frac{\sqrt{2}}{8\pi^3} N_c \int_0^1 dx \frac{G_m}{\sqrt{x(1-x)}} \left( (1-x)m_2 + xm_1 \right)
\times \int dk^2_\perp \psi_m(x, \vec{k}_\perp) .
\]

The above expression is general and one can use it to calculate the decay constant of any pseudoscalar meson state, and as well one can use it to normalize the eigenfunction of the squared mass operator from the solution of Eq. (4).

We observe that Eq. (10), written in terms of the asymptotic part of the valence wave function has a logarithmic divergence in the transverse momentum integration due to the slow decrease of the wave function. From a physical point of view, one could think that the regularization scale is larger than the masses of the quarks and the divergent transverse momentum integration, will be defined through the value of \( f_\pi \), for example. Therefore, one has:

\[
f_m = \text{const.} \int_0^1 dx \left( (1-x)m_2 + xm_1 \right) ,
\]

and const. determined by \( f_\pi \). One observe as well that, \( f_m \propto m_1 + m_2 \), which in our model is the vector meson mass, thus one immediately gets:

\[
f_m = \frac{M_v}{M_p} .
\]

The numerical results of Eq. (12) are shown in Table I. It is verified that a reasonable description of the weak decay constants of the pseudoscalar mesons is possible within the effective light-front model. However, we have made use only of the asymptotic form of the wave function and one needs to investigate the decay constant with more refined wave functions, eigenstates of the squared mass operator, Eq. (8), which includes the dynamics of the effective quarks. Therefore, the results which are overestimating the heavier meson decay constants, can be an indication that a more elaborated wave function is needed, although one cannot discard that other mechanisms could be relevant.

Also, we intend to perform the evaluation of the weak decay constants using a more sophisticated version of the model, where confinement is included, which so far was shown to describe the meson spectrum.

In summary, we have shown the existence of a direct proportionality between the weak decay constants and the masses of the vector mesons ground states, which can be an useful tool in the systematic study of these quantities.

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| \( q^m \) (map) | \( M_m [10] \) | \( M_v [10] \) | \( f_{\text{model}}[meV] \) | \( f_{\text{exp}}[meV] \) |
|----------------|----------------|----------------|----------------|----------------|
| \( \pi^+ (u\bar{d}) \) | 140 | 771 (\rho) | 92.4 | 92.4 ± 0.7 ± 0.25 |
| \( K^+(u\bar{c}) \) | 494 | 892 (\( K^+ \)) | 107 | 113.0 ± 1.0 ± 0.31 |
| \( D^+(c\bar{u}) \) | 1869 | 2010 (\( D^+ \)) | 241 | 212.2±127±6 |
| \( D_s^+(c\bar{u}) \) | 1969 | 2112 (\( D_s^+ \)) | 253 | 201 ± 13 ± 28 |

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