NULL SECTIONAL CURVATURES OF WARPED PRODUCTS

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ABSTRACT. In this paper, we investigate the null (light-like) sectional curvatures of Lorentzian warped product manifolds. We derive the formulas for the null sectional curvature of many well-known warped product space-time models such as multiply generalized Robertson-Walker space-times, generalized Kasner space-times and standard static space-times.

CONTENTS

1. Introduction 1
2. Preliminaries 3
3. Null Sectional Curvature 6
   3.1. Null Sectional Curvature of a multiply GRW space-time 6
   3.2. Null Sectional Curvature of a GRW space-time 8
   3.3. Null Sectional Curvature of a Generalized Kasner Space-time 9
4. Four-dimensional Space-time Models 10
   4.1. Type I: Null Sectional Curvature of a Kasner Space-time with fiber of dimension (3) 10
   4.2. Type II: Null Sectional Curvature of a Kasner Space-time with fiber of dimension (1, 2) 10
   4.3. Type III: Null Sectional Curvature of a Kasner Space-time with the fiber of dimension (1, 1, 1) 11
5. Null sectional curvature of a SSS-T 12
Appendix A. Null Sectional Curvature of MGRW space-time 13
References 16

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1. Introduction

Warped product manifolds were first introduced to the literature by R. Bishop and B. O’Neill in [7] to construct complete Riemannian manifolds with negative sectional curvature everywhere. After that, Beem, Ehrlich and Powell established that a wide class of well known exact solutions of the Einstein’s field equations can be expressed as a Lorentzian warped product in [4, 5]. Two important examples of warped products are the generalized Robertson-Walker space-times (GRW, for short) and the standard static space-times (SSS-T, for short). The former are obviously a generalization of Robertson-Walker space-times and the latter a generalization of the Einstein static universe. In addition to these space-times, we can list generalized Kasner and Reissner-Nordstr¨ om space-time models.

In the current paper, we focus on the null (light-like) sectional curvatures of Lorenztian warped products. The concept of null sectional curvature were first defined by S. G. Harris in [14] to study sectional curvatures of null plane sections of Lorenztian manifolds. In [15], it is proved that Robertson-Walker metrics can be locally characterized as those for which the null sectional curvature denoted by $K_N$ at each point is a constant for all the null planes at that point.

In [24], J. P. Palomo proved the smoothness of the null sectional curvature. Moreover, in a not necessarily complete manifold, if the null sectional curvature is not zero for any degenerate planes at a point $p$, then there is at most one local GRW structure in a neighborhood of $p$ (see [12]). Moreover he further showed that for an $n(\geq 4)$-dimensional manifold the $U-$ normalized null sectional curvature function is constant on every pencil of degenerate planes belonging to a certain null direction, if and only if, the Lorentzian manifold is conformally flat.

In [15, 19], it is shown that if $(M,g)$ is a time orientable Lorentzian manifold with $\text{dim}(M) \geq 3$, and $U$ is a globally defined unitary time-like vector field, then the $U-$ normalized null sectional curvature is isotropic, i.e, it is only a point function. Moreover,

$$K^U(p, \Pi) = K^U(p)$$

for all null planes $\Pi$ if and only if the curvature tensor satisfies:

(a) $R(X, Y)Z = k(X \wedge g Y)Z$, for any $X, Y, Z \in U^\perp$

(b) $R(X, U)U = \mu X$, for any with $\kappa, \mu: M \to \mathbb{R}$ and $K^U = k + \mu$.

If the $U-$ normalized null sectional curvature $K^U$ is isotropic (it is said to be spatially constant) if it is constant on the space of orthogonal to the chosen time-like vector field $U$, i.e, $K^U = 0$ for every $X \in U^\perp$ (see [19]).

Let $(M, g)$ be an $n \geq 4$- dimensional Lorentzian manifold and let $U$ be a unitary time-like vector field on $M$. Suppose that the $U$-normalized null sectional curvature is non-zero, isotropic and spatially constant. Then $g$ is locally a Friedmann-Lemaitre-Robertson-Walker metric (see [15, 19]).
In [12], to characterize global decomposition of a manifold as a generalized Robertson-Walker space-time, the null sectional curvature is applied.

The null sectional curvature has been used in the study of conjugate points along null geodesics (see [13, 14, 25]).

We recall that a warped product can be defined as follows [5, 23]. Let \((B, g_B)\) and \((F, g_F)\) be pseudo-Riemannian manifolds and also let \(b: B \to (0, +\infty)\) be a smooth function. Then the (singly) warped product, \(B \times_b F\) is the product manifold \(B \times F\) furnished with the metric tensor \(g = g_B \oplus b^2 g_F\), more precisely

\[
g = \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_F),
\]

where \(\pi: B \times F \to B\) and \(\sigma: B \times F \to F\) are the usual projection maps and \(*\) denotes the pull-back operator on tensors.

A standard static space-time can be considered as a Lorentzian warped product where the warping function is defined on a Riemannian manifold called the fiber and acting on the negative definite metric on an open interval of real numbers, called the base. More precisely, a SSS-T, \(f(a, b) \times F\) is a Lorentzian warped product furnished with the metric \(g = -f^2 dt^2 \oplus g_F\), where \((F, g_F)\) is a Riemannian manifold, \(f: F \to (0, +\infty)\) is smooth, and \(-\infty \leq a < b \leq +\infty\). In [23], it was shown that any static space-time is locally isometric to a SSS-T.

Standard static space-times have been previously studied by many authors. Kobayashi and Obata [18] stated the geodesic equation for this class of space-times and the causal structure and geodesic completeness was considered in [3], where sufficient conditions on the warping function for nonspacelike geodesic completeness of the SSS-T was obtained (see also [26]). In [2], conditions are found which guarantee that SSS-Ts either satisfy or else fail to satisfy certain curvature conditions from general relativity. The existence of geodesics in SSS-Ts have been studied by several authors. Sánchez [27] gives a good overview of geodesic connectedness in semi-Riemannian manifolds, including a discussion for SSS-Ts.

Two of the most famous examples of SSS-Ts are Minkowski space-times and the Einstein static universe [5, 17] which is \(\mathbb{R} \times S^3\) equipped with the metric

\[
g = -dt^2 + (dr^2 + \sin^2 r d\theta^2 + \sin^2 r \sin^2 \theta d\phi^2)
\]

where \(S^3\) is the usual 3-dimensional Euclidean sphere and the warping function \(f \equiv 1\). Another well-known example is the universal covering space of the anti-de Sitter space-time, a SSS-T of the form \(f\mathbb{R} \times \mathbb{H}^3\) where \(\mathbb{H}^3\) is the 3-dimensional hyperbolic space with constant negative sectional curvature and the warping function \(f: \mathbb{H}^3 \to (0, +\infty)\) defined as \(f(r, \theta, \phi) = \cosh r\) [5, 17]. Finally, we can also mention the Exterior Schwarzschild space-time [5, 17], a SSS-T of the form \(f\mathbb{R} \times (2m, +\infty) \times S^2\), where \(S^2\) is the 2-dimensional Euclidean sphere, the warping function \(f: (2m, +\infty) \times S^2 \to (0, +\infty)\) is given by \(f(r, \theta, \phi) = \sqrt{1 - 2m/r}\) and the
line element on \((2m, +\infty) \times S^2\) is
\[
ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

2. Preliminaries

In this section, we give the formal definitions of several types of warped product space-time models such as GRWs, SSS-Ts and multiply generalized Robertson-Walker space-times (MGRW, for short) and their related geometric structure formulas (see [5, 23]).

Throughout the article \(I\) will denote an open real interval of the form \(I = (t_1, t_2)\), where \(-\infty \leq t_1 < t_2 \leq +\infty\).

**Definition 1.** Let \((F, g_F)\) be an \(s\)-dimensional Riemannian manifold and \(b: I \to (0, +\infty)\) be a smooth function. Then the \(n(= 1 + s)\)-dimensional product manifold \(I \times F\) furnished with the metric tensor \(g = -dt^2 \oplus b^2 g_F\) is called a generalized Robertson-Walker space-time and is denoted by \(I \times_b F\), where \(dt^2\) is the Euclidean metric tensor on \(I\).

**Definition 2.** Let \((F, g_F)\) be an \(s\)-dimensional Riemannian manifold and \(f: F \to (0, +\infty)\) be a smooth function. Then the \(n(= 1 + s)\)-dimensional product manifold \(I \times F\) furnished with the metric tensor \(g = -f^2 dt^2 \oplus g_F\), where \(dt^2\) is the Euclidean metric tensor on \(I\), is called a standard static space-time and is denoted by \(I \times F\).

**Definition 3.** Let \((B, g_B)\) and \((F_i, g_{F_i})\) be pseudo-Riemannian manifolds and also let \(b_i: B \to (0, \infty)\) be smooth functions for any \(i \in \{1, 2, \cdots, m\}\). The product manifold \(M = B \times F_1 \times F_2 \times \cdots \times F_m\) furnished with the metric tensor \(g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \cdots \oplus b_m^2 g_{F_m}\) is called a multiply warped product and is denoted as

\[
B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m.
\]

More precisely
\[
g = \pi^*(g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus \cdots \oplus (b_m \circ \pi)^2 \sigma_m^*(g_{F_m}),
\]
where the \(\pi\) and \(\sigma_i\) are the usual projection maps and \(^*\) denotes the pull-back operator on tensors. \((B, g_B), b_i: B \to (0, \infty)\) and \((F_i, g_{F_i})\) are called base, \(i\)-fiber and \(i\)-warping function of the multiply warped product, respectively.

- If \(m = 1\), then we obtain a singly warped product.
- If all \(b_i \equiv 1\), then we have a (trivial) product manifold.
- If \((B, g_B)\) and all the \((F_i, g_{F_i})\) are Riemannian manifolds, then \((M, g)\) is also a Riemannian manifold.
- The multiply warped product \((M, g)\) is a Lorentzian multiply warped product if \((F_i, g_{F_i})\) are all Riemannian for any \(i \in \{1, 2, \cdots, m\}\) and either \((B, g_B)\) is Lorentzian or else \((B, g_B)\) is a one-dimensional manifold with a negative definite metric \(-dt^2\).
If $B$ is an open interval $I$ equipped with the negative definite metric $g_B = -dt^2$ and all the $(F_i, g_{F_i})$ manifolds are Riemannian, then the Lorentzian multiply warped product $(M, g)$ is called a multiply generalized Robertson-Walker space-time. In particular, a MGRW is called a generalized Reissner-Nordström space-time when $m = 2$.

Throughout the paper the fiber(s) $(F, g_{F_i})$ of a warped product space-time model is always assumed to be connected. We denote the set of lifts of vector fields on $B$ and $F$ to $B \times F$ by $\mathcal{L}(B)$ and $\mathcal{L}(F)$, respectively and use the same notation for a vector field and for its lift (see page 205 of [23]).

From now on, we follow the convention applied in [5] (note the difference with [23]) for the definition and sign of the Riemann curvature tensor $R$, namely. For any $n$-dimensional pseudo-Riemannian manifold $(N, h)$,

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,
\]

where $\nabla$ is the $h$-Levi-Civita connection and $X, Y, Z$ are vector fields on $N$. Besides, $\text{Ric}$ will denote the Ricci tensor (see [23]).

Furthermore, we will apply the sign convention for the Laplacian in [23], i.e, $\Delta = \text{tr}(H) = \text{div \, grad}$, where $\text{tr}$ denotes the $h$-trace, $H$ the Hessian tensor respect to the Levi-Civita connection, $\text{div}$ the $h$-divergence and $\text{grad}$ the $h$-gradient (see page 85 of [23]).

Eventually, we will put a sub- or super-index in the corresponding operator indicating the manifold on which it is acting, for instance $\nabla^B$ if the connection is that on the manifold $B$.

The following basic formulas about the geometry of multiply warped product can be found in [10]

**Proposition 1.** Let $M = B \times b_1 F_1 \times \cdots \times b_m F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ also let $X, Y \in \mathcal{L}(B)$ and $V, W \in \mathcal{L}(F_i)$. Then

1. $\nabla_X Y = \nabla^B_X Y$
2. $\nabla_X V = \nabla_Y X = \frac{X(b_i)}{b_i} V$
3. $\nabla_V W = \begin{cases} 0 & \text{if } i \neq j, \\ \nabla^F_V W - \frac{g(V, W)}{b_i} \text{grad}_B b_i & \text{if } i = j, \end{cases}$

where $\nabla = \nabla^M$.

One can compute the gradient and the Laplace-Beltrami operator on $M$ in terms of the gradient and the Laplace-Beltrami operator on $B$ and $F_i$, respectively.

**Proposition 2.** Let $M = B \times b_1 F_1 \times \cdots \times b_m F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ and $\phi: B \to \mathbb{R}$ and $\psi_i: F_i \to \mathbb{R}$ be smooth functions for any $i \in \{1, \cdots, m\}$. Then
Proposition 3. Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ also let $X, Y, Z \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F_i), W \in \mathfrak{L}(F_j)$ and $U \in \mathfrak{L}(F_k)$. Then

1. $R(X, Y)Z = R_B(X, Y)Z$
2. $R(V, X)Y = -\frac{H^B_{b_i}(X, Y)}{b_i} V$
3. $R(X, V)W = R(V, W)X = R(V, X)W = 0$ if $i \neq j$.
4. $R(X, Y)V = 0$
5. $R(V, W)X = 0$ if $i = j$.
6. $R(V, W)U = 0$ if $i = j$ and $i, j \neq k$.
7. $R(U, V)W = -g(V, W) \frac{g_B(\nabla_B b_i, \nabla_B b_k)}{b_i b_k} U$ if $i = j$ and $i, j \neq k$.
8. $R(X, V)W = -\frac{g(V, W)}{b_i} \nabla_X(\nabla_B b_i) \text{ if } i = j$.
9. $R(V, W)U = R_{F_i}(V, W)U + \frac{\|\nabla_B b_i\|^2}{b_i^2} (g(V, U)W - g(W, U)V)$ if $i, j = k$.

where $\Delta = \Delta_M$ and $\text{grad} = \text{grad}_M$.

Proposition 4. Let $M = B \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$, also let $X, Y, Z \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F_i)$ and $W \in \mathfrak{L}(F_j)$. Then

1. $\text{Ric}(X, Y) = \text{Ric}_B(X, Y) - \sum_{i=1}^m \frac{s_i}{b_i} H^B_{b_i}(X, Y)$
2. $\text{Ric}(X, V) = 0$
3. $\text{Ric}(V, W) = 0$ if $i \neq j$.
4. $\text{Ric}(V, W) = \text{Ric}_{F_i}(V, W) - \left(\frac{\Delta_B b_i}{b_i} + (s_i - 1) \frac{\|\nabla_B b_i\|^2}{b_i^2}\right) g(V, W) \text{ if } i = j$,

where $\text{Ric} = \text{Ric}_M$. 
3. Null Sectional Curvature

Suppose that $\Pi$ is a null plane, that is $\Pi$ consists of a one-dimensional subspace of null vectors and of space-like vectors perpendicular to that subspace. Let $L$ be one of the null vectors and $S$ be one of the space-like vectors. Since $Q(\Pi) = 0$, sectional curvature is not defined for a null plane. Then S. Harris introduces the null sectional curvature in \cite{14} for degenerate planes as follows (see also \cite{1}):

In order to define null sectional curvature, first we need to fix a choice of null vector.

**Definition 4.** Let $M$ be an $n$-dimensional Lorentzian manifold. A null congruence on $M$ is a submanifold $C$ of the tangent bundle $T_0M$ of nonzero null vectors on $M$ such that for all $N$ in $T_0M$, there is exactly one scalar $\alpha$ satisfying $\alpha N \in C$.

Then, null sectional curvature of $\Pi$ with respect to $N$ is given by

$$K_N(\Pi) = \frac{g(R(L,S),L)}{g(S,S)}$$

where $R$ is the Riemannian curvature tensor.

**Remark 1.** Note that, this formula is independent of the choice of the space-like vector $S$ in $\Pi$. However, $K_N(p,\Pi)$ depends quadratically on the null vector $L$.

Null sectional curvature can be normalized via the help of a time-like vector field $U$ in the following way: Assume that $U$ is a time-like vector field on a Lorentzian manifold $M$. Then the null congruence $C(U)$ associated with $U$ is given by

$$C(U) = \{L \in T_0M | g(L,L) = 0, g(L,U) = -1\}.$$

**Remark 2.** The null congruence lies in the future null cone due to the -1 in the definition.

Then $U$-normalized null sectional curvature is

$$K_U(\Pi) = \frac{g(R(L,S),L)}{g(S,S)}$$

where

$$C_U(M) = \{L \in TM | g(L,L) = 0 \text{ and } g(L, U_{\Pi(L)}) = -1\}.$$

3.1. Null Sectional Curvature of a multiply GRW space-time. Let $M = I \times_{b_1} F_1 \times_{b_2} \ldots \times_{b_m} F_m$ be a multiply GRW space-time with the metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \ldots \oplus b_m^2 g_{F_m}$. Let $L = -\partial_t + V$ and $S = Y + W$ where $V = \Sigma V_i$ $W = \Sigma W_j$. Then,
Theorem 1. Let $M = I \times_{b_i} F_1 \ldots \times_{b_{n}} F_m$ be a multiply GRW space-time with the metric $g = -dt^2 \oplus b_i^2 g_{F_i} \oplus \ldots \oplus b_{m}^2 g_{F_m}$. Assume that $L = -\partial_t + \mathbf{V}$ and $S = Y + \mathbf{W}$ where $\mathbf{V} = \Sigma V_i$ and $\mathbf{W} = \Sigma W_j$. Then null sectional of $(M, g)$ is given by

$$P(p, \Pi) = \Sigma b_k g_{F_k}(W_k, V_k) H_B^{b_k}(\partial_t, Y)$$
$$+ \Sigma b_k g_{F_k}(V_k, V_k) H_B^{b_k}(Y, Y)$$
$$- \Sigma b_j b_j g_{F_j}(W_j, W_j)$$
$$+ \Sigma b_IG_{F_i}(W_i, V_i) H_B^{b_i}(\partial_t, Y)$$
$$- \Sigma_j b_j b_j g_{F_j}(V_j, V_j) g_{F_j}(W_j, W_j)$$
$$+ \Sigma b_k^2 g_{F_k}(R_{F_k}(W_i, V_i) V_i, W_i)$$
$$- \Sigma b_j^2 (b_j')^2 [g_{F_j}(W_i, V_i) g_{F_j}(W_i, V_i)]$$

$$g(S, S) = g(Y + \Sigma W_j, Y + \Sigma W_j)$$
$$= g_B(Y, Y) + \Sigma (b_j')^2 g_{F_j}(W_j, W_j)$$

By taking $Y$ as $h\partial_t$ in Theorem 1, we obtain the following result.

$$g(R(L, S), L) = g(R(-\partial_t + \mathbf{V}, Y + \mathbf{W}), Y + \mathbf{W}, -\partial_t + \mathbf{V})$$
$$= g(R(-\partial_t, Y) Y, -\partial_t) + g(R(\Sigma V_i) Y, -\partial_t)$$
$$+ g(R(-\partial_t, \Sigma W_j) Y, -\partial_t) + g(R(\Sigma V_i, \Sigma W_j) Y, -\partial_t)$$
$$+ g(R(-\partial_t, Y) \Sigma W_j, -\partial_t) + g(R(\Sigma V_i, Y) \Sigma W_j, -\partial_t)$$
$$+ g(R(-\partial_t, \Sigma W_j) \Sigma W_j, -\partial_t) + g(R(\Sigma V_i, \Sigma W_j) \Sigma W_j, -\partial_t)$$

$$= \Sigma_{i \neq j} \frac{1}{b_i^2} b_j'' g(W_j, W_i) g(V_i, V_i)$$
$$+ \Sigma g(R_{F_i}(V_i, W_i), W_i, V_i)$$
$$+ \Sigma \frac{1}{b_i^2} b_j'' g(g(V_i, W_i) W_i - g(W_i, W_i) V_i, V_i)$$

$$= \Sigma_{i \neq j} - (b_j')^2 g_{F_j}(W_j, W_j) b_i'' g_{F_i}(V_i, V_i)$$
$$+ \Sigma b_i^2 g_{F_i}(R_{F_i}(V_i, W_i), W_i, V_i)$$
$$+ \Sigma b_i^2 (b_i')^2 [g_{F_i}(V_i, W_i)^2 - g_{F_i}(V_i, V_i) g_{F_i}(W_i, W_i)]$$
Corollary 1. Let $M = I \times b_1 F_1 \times \ldots \times b_m F_m$ be a MGRW space-time with the metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \ldots \oplus b_m^2 g_{F_m}$. Suppose that $L = -\partial_t + V$ and $S = h\partial_t + W$ where $V = \Sigma V_i$ and $W = \Sigma W_j$ and $h : I \to \mathbb{R}$ is a smooth function. Then,

$$
P(p, \Pi) = \Sigma h_b b_k g_{F_k}(W_k, V_k) + \Sigma h^2 b_k b_k' g_{F_k}(V_k, V_k) - \Sigma b_j b_j g_{F_j}(W_j, W_j) + \Sigma h b_k b_k' g_{F_k}(W_i, V_i) - \Sigma_j \delta k (b_j)^2 g_{F_k}(V_k, V_k) g_{F_j}(W_j, W_j) + \Sigma b_i^2 g_{F_i}(R F_i(W_i, V_i) V_i, W_i) - \Sigma b_i (b_i')^2 g_{F_i}(W_i, V_i))^2 - g_{F_i}(V_i, V_i) g_{F_i}(W_i, W_i))
$$

$$
g(S, S) = g(h\partial_t + \Sigma W_j, h\partial_t + \Sigma W_j) = -h'' + \Sigma (b_j)^2 g_{F_j}(W_j, W_j)
$$

3.2. Null Sectional Curvature of a GRW space-time. Assume that $M = I \times b F$ is a GRW space-time with the metric $g = g_t \oplus b^2 g_F$ where $g_t = -dt^2$. Let $\Pi$ be degenerate null plane at $p \in M$ spanned by a null vector $L$ and space-like vector $S$, i.e., $g(L, L) = 0$ and $g(S, S) > 0$ where $U = \partial_t$ is a reference frame since $g(U, U) = g(\partial_t, \partial_t) = -1$.

(i) Let $L = h\partial_t + V, U = \partial_t$

$$
g(L, U) = g(h\partial_t + V, \partial_t) = -dt^2(h\partial_t, \partial_t) + b^2 g_F(V, 0)
$$

This implies $g(L, U) = 1$ if and only if $h = -1$. Then we have $L = -\partial_t + V$

(ii) $g(L, L) = g(-\partial_t + V, -\partial_t + V) = -dt^2(-\partial_t, V) + b^2 g_F(V, V) = -1 + b^2 g_F(V, V) = -1 + g_F(V, V)$

By imposing $L$ to be null, i.e., $g(L, L) = 0$, we obtain $g_F(V, V) = \frac{1}{b^2}$.

(iii) Let $E = \text{span}(\{L, S\})$ be a degenerate plane section, i.e., $Q(L, S) = 0$

$$
Q(L, S) = Q(-\partial_t + V, Y + W) = g(-\partial_t - \partial_t)g(Y + W, Y + W) - g(-\partial_t, Y + W)^2 = -g_t(\partial_t, Y + b^2 g_F(V, W))^2 = 0
$$

This implies $g_F(V, W) = \frac{1}{b^2} g_t(\partial_t, Y)$

Corollary 2. Let $M = I \times b F$ be a GRW space-time with the metric $g = -dt^2 \oplus b^2 g_F$ where $b : I \to (0, \infty)$ is a smooth function. Then, null sectional curvature of $(M, g)$ is as follows:

$$
g(R(L, S), S, L) = -bb'' g_F(W, W) + b^2 g_F(R_F(W, V) V, W)) + b g_F(V, V) H^b(Y, Y) + b^2(b')^2 [g_F(V, W)^2 - \frac{1}{b^2} g_F(W, W)]
$$

(3.2)
Remark 3. As a special case assume that \( Y = 0 \). Then
\[
Q_F(V, W) = g_F(V, V)g_F(W, W) - g_F(V, W)^2
\]
(3.4) 
\[
= \frac{1}{b^2}g_F(W, W)
\]
In this case we have,
\[
K_U(\Pi) = \frac{1}{b^2}K_F(V, W) + \frac{b''}{b} - \left(\frac{b'}{b}\right)^2
\]
(3.5)
Using the above equation we establish that \( K_U(\Pi) = \frac{1}{b^2}K_F(V, W) \) if and only if the warping function \( b(t) = ce^{kt} \) where \( c \) and \( k \) are arbitrary constants.

3.3. Null Sectional Curvature of a Generalized Kasner Space-time.

Definition 5. A generalized Kasner space-time \( (M, g) \) is a Lorentzian multiply warped product of the form \( M = I \times \varphi_{p_1}F_1 \times \ldots \times \varphi_{p_m}F_m \) with the metric \( g = -dt^2 \oplus \varphi_{p_1}^2g_{F_1} \oplus \ldots \oplus \varphi_{p_m}^2g_{F_m} \) where \( \varphi : I \to (0, \infty) \) is smooth and \( p_i \in \mathbb{R} \) for \( i = 1, \ldots, m \) and also \( I = (t_1, t_2) \) with \( -\infty \leq t_1 < t_2 \leq \infty \).

Corollary 3. Let \( L = -\partial_t + V \) and \( S = Y + W \) where \( \varphi \to (0, \infty) \) with \( -\infty \leq t_1 < t_2 \leq \infty \) and \( V = \sum V_i, W = \sum W_j \) Then null sectional of a multiply Kasner space-time of the form above is given by

\[
P(p, \Pi) = \sum \varphi^{p_k}g_{F_k}(W_k, V_k)H_I^{\varphi_{p_k}}(\partial_t, Y)
+ \sum \varphi^{p_j}g_{F_j}(V_j, V_j)H_I^{\varphi_{p_j}}(Y, Y)
- \sum \varphi^{p_j}p_j(p_j - 1)\varphi^{(p_j - 2)}g_{F_j}(W_j, W_j)
+ \sum \varphi^{p_j}g_{F_j}(W_i, V_i)H_I^{\varphi_{p_j}}(\partial_t, Y)
- \sum_{j \neq k}^{p_k}p_j^2\varphi^{2(p_j - 1)}g_{F_k}(V_k, V_k)g_{F_j}(W_j, W_j)
+ \sum \varphi^{p_j}g_{F_i}(R_{F_i}(W_i, V_i)V_i, W_i)
- \sum \varphi^{p_j}p_j^2\varphi^{2(p_j - 1)}p_i(p_i - 1)\varphi^{(p_i - 2)}[g_{F_i}(W_i, V_i)]^2 - g_{F_i}(V_i, V_i)g_{F_i}(W_i, W_i)]
\]
\[
g(S, S) = g(Y + \sum W_j, Y + \sum W_j)
= \sum \varphi^{2p_j}g_{F_j}(Y, Y) + g_{F_j}(W_j, W_j)
\]
4. Four-dimensional Space-time Models

4.1. Type I: Null Sectional Curvature of a Kasner Space-time with fiber of dimension (3).

**Corollary 4.** Let $M = I \times_b F$ and $g = -dt^2 \oplus b^2g_F$, $L = -\partial_t + V$ and $S = h\partial_t + W$ where $\Pi$ be a degenerate null plane which is spanned by the tangent vectors $L$ and $S$. Then,

\[
P(\Pi) = f^2bb''g_F(V,V) - bb''g_F(W,W) + bb''g_F(V,W) + b^2g_F(R_F(W,V)V,W) - b^2(b')^2[g_F(V,W)^2 - g_F(V,V)g_F(V,W)]
\]

(4.1)

4.2. Type II: Null Sectional Curvature of a Kasner Space-time with fiber of dimension (1,2).

**Corollary 5.** Let $\Pi$ be degenerate null plane at $p \in M$ spanned by a null vector $L$ and space-like vector $S$, i.e., $g(L, L) = 0$ and $g(S, S) > 0$.

Let $L = -\partial t + f_1\partial x + V$ and $S = f\partial t + h_1\partial x + W$ where

\[
V_1 = f_1\partial x, \\
W_1 = h_1\partial x
\]

Then the null sectional curvature of $\Pi$ with respect to $L$ is given by

\[
K_U(\Pi) = \frac{R((L, S)S, L)}{g(S, S)}
\]

where $U = \partial_t$ is a reference frame since $g(U, U) = g(\partial_t, \partial_t) = -1$ and

\[
g(R(\Sigma V_i, \Sigma W_j)\Sigma W_j, \Sigma V_i) = b_1f_1h_1f b''_1 + b_2f^2b''_2g_F(V,V) - b_1b''_1h_1^2 - b_2b''_2g_F(W,W) + b_1f_1h_1f b''_1 + b_2b''_2g_F(V,W) + b^2g_F(R_F(W,V)V,W) - b^2(b')^2[g_F(V,W)^2 - g_F(V,V)g_F(V,W)]
\]

(4.2)

\[
g(S, S) = g(f\partial t + \Sigma W_j, f\partial t + \Sigma W_j) = -f^2 + b^2_1h_1^2 + b^2_2g_F(W,W)
\]

(4.3)
Remark 4. For the case $Y = h \partial_t$ and $V_1 = W_1 = 0$, $P(E)$ becomes as follows:

$$P(\Pi) = \varphi^{p_2} p_2 (p_2 - 1) \varphi^{p_2 - 2} g_{F_2}(W_2, V_2)$$

$$+ h^2 \varphi^{p_2} p_2 (p_2 - 1) \varphi^{p_2 - 2} g_{F_2}(V_2, V_2)$$

$$- \varphi^{p_2} p_2 (p_2 - 1) \varphi^{p_2 - 2} g_{F_2}(W_2, V_2)$$

$$+ h \varphi^{p_2} p_2 (p_2 - 1) \varphi^{p_2 - 2} g_{F_2}(W_2, V_2)$$

$$+ \varphi^{2p_2} g_{F_2}(R_{F_2}(W_2, V_2) V_2, W_2)$$

$$- p_2^4 \varphi^{6p_2 - 4} [g_{F_2}(W_2, V_2)^2 - g_{F_2}(V_2, V_2) g_{F_2}(W_2, W_2)]$$

(4.4)

Similarly we can obtain $P(E)$ for the case $V_2 = W_1 = 0$ by using the symmetry properties.

4.3. Type III: Null Sectional Curvature of a Kasner Space-time with the fiber of dimension $(1, 1, 1)$. A Kasner space-time with the fiber of dimension $(1, 1, 1)$ is a Lorentzian multiply warped product $(M, g)$ of the form

$$M = (0, \infty) \times \varphi_{p_1} \mathbb{R} \times \varphi_{p_2} \mathbb{R} \times \varphi_{p_3} \mathbb{R}$$

with the metric

$$g = -dt^2 + \varphi^{2p_1} dx^2 + \varphi^{2p_2} dy^2 + \varphi^{2p_3} dz^2$$

where $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$.

Corollary 6. Let $\Pi$ be degenerate null plane at $p \in M$ spanned by a null vector $L$ and spacelike vector $S$, i.e., $g(L, L) = 0$ and $g(S, S) > 0$.

Let $L = -\partial_t + \Sigma V_i$ and $S = f \partial_t + \Sigma W_j$ where

$$V_1 = f_1 \partial_x, V_2 = f_2 \partial_y$$

$$V_3 = f_3 \partial_z, W_1 = h_1 \partial_x$$

$$W_2 = h_2 \partial_y, W_3 = h_3 \partial_z$$

Then the null sectional curvature of $\Pi$ with respect to $L$ is is given by

$$K_U(\Pi) = \frac{R((L, S) S, L)}{g(S, S)}$$

where $U = \partial_t$ is a reference frame since $g(U, U) = g(\partial_t, \partial_t) = -1$.

$$g(R(\Sigma V_i, \Sigma W_j) \Sigma W_j, \Sigma V_i) = -\Sigma \varphi^{p_k} f_i h_i$$

$$+ \Sigma \varphi^{p_k} \left( f_k \right)^2 f^2 p_k (p_k - 1) \varphi^{p_k - 2}$$

$$- \Sigma p_j (p_j - 1) \varphi^{p_j - 2} (\varphi^p h_j^2$$

$$+ \Sigma \varphi^{p_i} f_i h_i f p_i (p_i - 1) \varphi^{p_i - 2}$$

$$- \Sigma j \neq k \varphi^{p_k} f_k p_j \varphi^{2p_j - 2} h_j^2$$

(4.5)
5. Null sectional curvature of a SSS-T

Let $M = f_I \times F$ be a SSST with the metric $g = -f^2 g_I \oplus g_F$ where $-dt^2$ is the negative definite metric on $I$.

Let $\Pi$ be degenerate null plane at $p \in M$ spanned by a null vector $L$ and spacelike vector $S$, i.e., $g(L, L) = 0$ and $g(S, S) > 0$ with the reference frame $U = f^{-1} \partial_t$.

The following calculations will be useful in Corollary 7:

(i) Let $L = h \partial_t + V$, $U = f^{-1} \partial_t$

$$g(L, U) = g(h \partial_t + V, f^{-1} \partial_t) = -f^2 dt^2 (h \partial_t, f^{-1} \partial_t) + g_F(V, 0)$$

$$= -f^2 h f^{-1}$$

This implies $g(L, U) = 1$ iff $h = -f^{-1}$. Then we have $L = -f^{-1} \partial_t + V$

(ii) $g(L, L) = g(-f^{-1} \partial_t + V, -f^{-1} \partial_t + V) = -f^2 dt^2 (-f^{-1} \partial_t, v) + g_F(V, V) = -f^2 \frac{1}{f^2} + g_F(V, V) = -1 + g_F(V, V)$

If we want to make $g(L, L) = 0$, i.e., $L$ is null, then we must have $g_F(V, V) = 1$.

(iii) Let $E = \text{span}(\{L, S\})$ be a degenerate plane section, i.e., $Q(L, S) = 0$

$$Q(L, S) = Q(-f^{-1} \partial_t + V, Y + W) = g(-f^{-1} \partial_t, -f^{-1} \partial_t)g(Y + W, Y + W) - g(-f^{-1} \partial_t, Y + W)^2 = [-f g_I(\partial_t, Y) + g_F(V, W)]^2 = 0$$

This implies $g_F(V, W) = f g_I(\partial_t, Y)$

**Corollary 7.** Then the null sectional curvature of SSST is as follows:

$$g(R(L, S)S, L) = -g_F(\nabla_F f, \nabla_F f)(g_I(\partial_t, Y)^2 + g_I(Y, Y))$$

$$- f g_I(Y, Y)H^f_F(V, V)$$

$$- g_I(Y, \partial_t)H^f_F(V, W)$$

$$+ g_I(Y, \partial_t)H^f_F(W, W)$$

$$+ \frac{1}{f} H^f_F(W, W)$$

$$+ g_F(R_F(V, W)V, W)$$

(5.1)

If we take $Y = h \partial_t$, we obtain that:

$$g(R(L, S)S, L) = fh^2 H^f_F(V, V)$$

(5.2)
Since null sectional curvature is independent from the choice of $S$ (space-like vector), without loss of generality we can assume that $g(S, S) = 1$. Then

$$g(R(L, S)S, L) = -g_F(\nabla_F f, \nabla_F f)(g_I(Y, \partial_t)^2 + g_I(Y, Y))$$

$$- f g_I(Y, Y)H^F_F(V, V)$$

$$- \frac{1}{f} H^F_F(W, W)$$

$$+ g_F(R_F(V, W)V, W)$$

(5.3)

Remark 5. As a special case assume that $Y = 0$. Then

$$Q_F(V, W) = g_F(W, W)$$

(5.4)

In this case we have

$$K_U(\Pi) = K_F(V, W) - \frac{H^F_F(W, W)}{fg_F(W, W)}$$

(5.5)

If we assume that $H^F_F = kf g_F(W, W)$ for some $k \in \mathbb{R}$, then

$$K_U(\Pi) = K_F(V, W) - k$$

Appendix A. Null Sectional Curvature of MGRW space-time

Here we give main steps of the proof of Theorem 1:

$$g(R(L, S)S, L) = g(R(\partial_t + \Sigma V_i, Y + \Sigma W_j)Y + \Sigma W_j - \partial_t + \Sigma V_i)$$

$$= g(R(\partial_t, Y)Y, -\partial_t) + g(R(\Sigma V_i, Y)Y, -\partial_t)$$

$$+ g(R(\partial_t, \Sigma W_j)Y, -\partial_t) + g(R(\Sigma V_i, \Sigma W_j)Y, -\partial_t)$$

$$+ g(R(\partial_t, \Sigma W_j)\Sigma W_j, -\partial_t) + g(R(\Sigma V_i, \Sigma W_j)\Sigma W_j, -\partial_t)$$

$$+ g(R(\partial_t, \Sigma W_j)\Sigma W_j, \Sigma V_i) + g(R(\Sigma V_i, \Sigma W_j)\Sigma W_j, \Sigma V_i)$$

$$+ g(R(\partial_t, \Sigma W_j)\Sigma W_j, \Sigma V_i) + g(R(\Sigma V_i, \Sigma W_j)\Sigma W_j, \Sigma V_i)$$

$$+ g(R(\partial_t, \Sigma W_j)\Sigma W_j, \Sigma V_i) + g(R(\Sigma V_i, \Sigma W_j)\Sigma W_j, \Sigma V_i)$$

$$= 0$$

(A.1)
\[ g(R(\Sigma V_i, Y) Y, -\partial_t) = \Sigma g(-\frac{1}{b_i} H^b_i (Y, Y)V_i, -\partial_t) \]
\[ = \Sigma \frac{1}{b_i} H^b_i (Y, \partial_t) g(V_i, -\partial_t) \]
\[ = 0 \]

\[ g(R(-\partial_t, \Sigma W_j) Y, -\partial_t) = \Sigma g(-\frac{1}{b_j} H^b_j (-\partial_t, Y)W_j, -\partial_t) \]
\[ = \Sigma \frac{1}{b_j} H^b_j (\partial_t, Y) g(W_j, \partial_t) \]
\[ = 0 \]

(A.4)

\[ g(R(\Sigma V_i, \Sigma W_j) Y, -\partial_t) = 0 \]

(A.5)

\[ g(R(-\partial_t, Y) \Sigma W_j, -\partial_t) = 0 \]

\[ g(R(\Sigma V_i, Y) \Sigma W_j, -\partial_t) = \Sigma g(-\frac{1}{b_j} g(V_i, W_i) \nabla^I (\nabla^I b_i), -\partial_t) \]
\[ = \Sigma \frac{-b^2_j}{b_i} g_{F_i}(V_i, W_i) g(\nabla^I (\nabla^I b_i), -\partial_t) \]
\[ = \Sigma b_i g_{F_i}(V_i, W_i) H^b_i (Y, \partial_t) \]

(A.6)

\[ g(R(-\partial_t, \Sigma W_j) \Sigma W_j, -\partial_t) = \Sigma g(-\frac{1}{b_j} g(V_i, W_i) \nabla^I (\nabla^I b_i), Y) \]
\[ = \Sigma \frac{-1}{b_j} g_{F_j}(W_j, W_j) g(\nabla^I (\nabla^I b_j), -\partial_t) \]
\[ = \Sigma - b_j g_{F_j}(W_j, W_j) H^b_j (-\partial_t, -\partial_t) \]
\[ = \Sigma - b_j b^\prime_j g_{F_j}(W_j, W_j) \]

(A.7)

\[ g(R(\Sigma V_i, \Sigma W_j) \Sigma W_j, -\partial_t) = \Sigma_{i \neq j} g(g(W_j, W_j) \frac{g_{F_i}(\nabla^I b_j, \nabla^I b_i)}{b_i b_j} V_i, \partial_t) \]
\[ + g(R_{F_i}(V_i, W_i) W_i, -\partial_t) \]
\[ + \frac{g_{F_i}(\nabla^I b_i, \nabla^I b_i)}{b_i b_i} g(V_i, W_i) W_i - g(W_i, W_i) V_i, -\partial_t) \]
\[ = 0 \]

(A.8)

\[ g(R(-\partial_t, Y) Y, \Sigma V_i) = g(R_{I} (\partial_t, Y) Y, \Sigma V_i) \]
\[ = 0 \]

(A.9)
$$g(R(\Sigma V_i, Y)Y, \Sigma V_i) = \Sigma g(-\frac{H^b_i(Y, -\partial_t)}{b_i}V_i, V_i)$$  
(A.10)  
$$= \Sigma b_i^2 \frac{H^b_i(Y, Y)}{b_i}g_{F_i}(V_i, V_i)$$  
$$= \Sigma b_i H^b_i(Y, Y)g_{F_i}(V_i, V_i)$$

$$g(R(-\partial_t, \Sigma W_j)Y, \Sigma V_i) = \Sigma g(-\frac{H^b_i(-\partial_t, Y)}{b_i}W_j, V_i)$$  
(A.11)  
$$= \Sigma b_i^2 \frac{H^b_i(-\partial_t, Y)}{b_i}g_{F_i}(V_i, W_i)$$  
$$= \Sigma b_i H^b_i(-\partial_t, Y)g_{F_i}(V_i, W_i)$$

(A.12)  
$$g(R(\Sigma V_i, \Sigma W_j)Y, \Sigma V_i) = 0$$

(A.13)  
$$g(R(-\partial_t, Y)\Sigma W_j, \Sigma V_i) = 0$$

$$g(R(\Sigma V_i, Y)\Sigma W_j, \Sigma V_i) = \Sigma g(\frac{g(V_i, W_i)}{b_i}\nabla^I_Y(\nabla^I b_i), V_i)$$  
(A.14)  
$$= \Sigma b_i g_{F_i}(V_i, W_i)g(\nabla^I_Y(\nabla^I b_i), V_i)$$  
$$= \Sigma \frac{-g_{F_i}(V_i, W_i)}{b_i}H^b_i(Y, V_i)$$  
$$= 0$$

$$g(R(-\partial_t, \Sigma W_j)\Sigma W_j, \Sigma V_i) = \Sigma g(\frac{g(W_j, W_j)}{b_j}\nabla^I_{-\partial_t}(\nabla^I b_i), V_i)$$  
(A.15)  
$$= \Sigma - b_j g_{F_j}(W_j, W_j)g(\nabla^I_{-\partial_t}(\nabla^I b_i), V_i)$$  
$$= 0$$
\begin{aligned}
g(R(\Sigma V_i, \Sigma W_j)\Sigma W_j, \Sigma V_i) &= \Sigma_{ij}g(-g(W_j, W_j)\frac{g_I(\nabla^I b_j, \nabla^I b_j)}{b_j^2}V_i, V_i) \\
&\quad+ \Sigma g(R_{V_i}(V_i, W_i), W_i, V_i) \\
&\quad+ g\left(\frac{g_I(\nabla^I b_i, \nabla^I b_i)}{b_i^2}(g(V_i, W_i)W_i - g(W_i, W_i)V_i), V_i\right) \\
&= \Sigma_{ij} - (b_j^2)^2 g_{F_j}(W_j, W_j)b_i^2 g_{F_i}(V_i, V_i) \\
&\quad+ \Sigma b_i^2 g_{F_i}(R_{V_i}(V_i, W_i), W_i, V_i) \\
&\quad+ \Sigma b_i^2 (b_i^2)^2 [g_{F_i}(V_i, W_i)^2 - g_{F_i}(V_i, V_i)g_{F_i}(V_i, W_i)]
\end{aligned}

(A.16)

Then we obtain the following formula for null sectional curvature of Multiply Generalized Robertson Walker Spacetimes in Theorem 1.

\begin{align*}
g(S, S) &= g(Y + \Sigma W_j, Y + \Sigma W_j) \\
&= \Sigma (b_j^2)^2 g_B(Y, Y)g_{F_j}(V_j, W_j)
\end{align*}

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