LONG-RANGE DEPENDENCE IN THE VOLATILITY OF
URUGUAYAN SOVEREIGN DEBT INDICES

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Abstract. One consequence of the fact that a large number of agents with
different behaviors operate in financial systems is the emergence of certain sta-
tistical properties in some time series. Some of these properties contradict the
hypotheses that are established in the traditional models of efficient market
and portfolio optimization. Among them is the long-range dependence that
is the objective of this work. The approach is proposed by fractional calcu-
lus, as a generalization of the classic approach to financial markets through
semi-martingales. This paper study the existence of this property in variables
dependent on the term structure curves of Uruguayan sovereign debt after the
2002 economic crisis.

1. Introduction. In current financial systems, there are a large number of inter-
connections between the participating agents that generate consequences for the
dynamics of macroeconomic variables. This work study the behavior of an index
related to the fixed income market in Uruguay. The index depends on a large num-
ber of variables such as the Gross Domestic Product, international interest rates
and domestic interest rates.

The dynamics of some variables (mainly observed during economic crises) ex-
pose the need to include certain additional hypotheses that are not considered in
traditional theoretical models. The most relevant statistical properties are studied
in many works. In them, the properties generated in the models that have Brownian motion as the basis of randomness are discussed. Besides, empirical per-
formance of market data is made with simulations derived from the models. On
several occasions, it is concluded that this approach may turn out to be a not
entirely satisfactory simplification for modeling. This is shown in several works
through empirical study of financial series, and among them are [15], [27], [33] and
Some of these approaches are established within the theoretical framework of economic complexity. Relevant work in economic complexity are [1], [3] and [31]. This paper uses fractional calculus because it is the natural generalization of the property called long-range dependence in the models. This property allows to study persistent behavior within the series (a more precise mathematical definition will be provided in Section 2). In [16] and [21] the long-range dependence property is studied.

The objective of this work is to present the possible existence of long-range dependence property in financial series linked to fixed income in Uruguay. The work is organized as follows. Section 1 is the introduction to the problem. Section 2 provides a brief historical overview of the advancement of modeling financial markets using stochastic processes. Fractional calculus is introduced as the basis for the approach and the incidence of the Hurst parameter ($H$) in the property is analyzed. This section ends with a description of the methods used in the literature to model the volatility of financial series. In Section 3, the two financial series linked to sovereign debt in Uruguay are presented, and the estimations using the index returns are presented. In Section 4, we present some alternatives for the study of series with the long-range dependence property. Among these, econometric models and stochastic models that apply fractional calculus stand out. Our concluding remarks are given in Section 5.

2. Mathematical modeling of financial markets. The analysis of economic problems from scientific perspectives such as mathematics and statistics dates from the seventeenth century in the academic work of Sir William Petty (see [40]). In financial markets in particular, progress in using quantitative methods has been very important. It is possible to establish that a new modeling trend started with Louis Bachelier's doctoral thesis in 1900 (see [4]). Bachelier introduced Brownian motion for the asset dynamic price. In addition, the development of the mathematical area called stochastic calculus (with the contributions of Norbert Wiener, Paul Levy, Joseph Doob and Kiyosi Ito) generated in the mid-20th century an increase in the investigation of financial markets where the random effect became a relevant phenomenon. In 1973, Fisher Black, Myron Scholes and Robert Merton established an arbitrage-free option valuation method which it is the reference in the financial markets currently (see [7] and [36]). This result is the foundation of an approach to the study of financial markets, which expanded rapidly by using the Brownian motion (see [37]). The properties that stand out in the stochastic process and are assumed in the models are that it has continuous trajectories but is not differentiable at any time (a property that is motivated by the irregularity of financial series that is observed empirically). There are also properties of independence (independent random variables) and stationary in the increments (there is no effect in terms of temporary transfers). Also, the processes satisfies that its finite-dimensional distributions are fitted to Gaussian random vectors. The mathematical techniques used are sophisticated, and it is possible to explain financial phenomena of different categories.

However, the interaction in financial systems induces temporal evolution in some macroeconomic variables that are not reflected when using Brownian motion for
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modeling (see for example [10]). Therefore, there are some hypotheses in this approach that several academic works have tried to refute. Among them, it is worth noting that the prices of financial assets show a non-stationary distribution (see [42]), the existence of clusters of volatility in certain periods of time (see [38]) and different ways of dependence on their historical events (see [32]). These characteristics can be analyzed empirically in the economy at various levels.

New approaches derived from mathematical complexity analysis can be applied to the study of financial markets. However, the methodology to apply in several aspects is in debate. Different alternatives that have been proposed are derived from the financial complexity perspective. In the study of time series in financial markets, these are different concerns that derive from the strong structures of the systems. Among them, the properties of dependence and non-locality stand out, which are motivated by several reasons. The expectations and the decisions of the participating agents strongly influence the series and can be linked to the historical behavior (see [12]).

In a stationary time series, long-range dependence or long memory implies that there is a non-negligible relationship between the present and the historical evolution. The determination of the existence of such property in financial series is of great importance because most of the econometric models usual do not satisfy it. In the case that this property is detected, it may be inconvenient to use models in the analysis that do not consider it. In the literature, there are several works that are based on the applications of this property in certain economic cycles, especially in financial bubbles. For more information see [22].

2.1. Fractional calculus. The mathematical theory called fractional calculus was developed in the 1960’s in the work of Benoit Mandelbrot (see [34]), but from the end of 20th century the techniques were applied to the analysis of financial markets. The aim of this work is the application of these methods on Uruguayan series. The objective is to use the tools of fractional calculus that allow modeling the dynamics in series associated with finance. Actually, some historical series do not present the dynamic behavior that Brownian motion theory is based on. The stochastic process that is used for the construction of the theory is called fractional Brownian motion, which was introduced by Andrei Kolmogorov in [30].

Definition 2.1. A fractional Brownian motion with Hurst parameter $H \in (0,1)$, is an almost surely continuous centered Gaussian process $\{B_H^t\}_{t \in \mathbb{R}}$ such that its auto-covariance function is

$$C_{B_H}(s, t) = \frac{1}{2} (|s|^{2H} + |t|^{2H} - |t-s|^{2H}) \quad \text{for all } s, t \in \mathbb{R}. \quad (1)$$

When analyzing the properties that this process fulfills, it is obtained that several are conserved in relation to Brownian motion. Among them, it stands out that it has continuous and not differentiable trajectories, it has a finite-dimensional Gaussian distribution, it is self-similar and has stationary increments. From equation (1), we observe that the covariance of the increments is positive in the case of $H > \frac{1}{2}$ (persistent process), the covariance is negative when $H < \frac{1}{2}$ (antipersistent process), and when $H = \frac{1}{2}$ the fractional Brownian motion become the standard Brownian motion (see [6]). Therefore, $\{B_H^t\}_{t \in \mathbb{R}}$ is not a semi-martingale when $H \neq \frac{1}{2}$, so the Ito’s formula can not be applied; the mathematical problem is exposed in [2].
This property becomes important because the process does not satisfy the Markov property. Fractional Brownian motion, as a generalization of Brownian motion, has the important advantage that in a certain values of parameters $H$ it captures the long-range dependence property and in a certain values of parameters $H$ it captures the short-range dependence property. Below, we establish the definition that a stationary sequence of random variables has long-range dependence.

**Definition 2.2.** A sequence of stationary random variables \( \{X_n\}_{n \in \mathbb{N}} \) has long-range dependence if the sequence of its auto-covariance \( \rho(n) = \text{Cov}(X_k, X_{k+n}) \) for \( k, n \in \mathbb{N} \) satisfies

\[
\lim_{n \to \infty} \frac{\rho(n)}{c n^{-\alpha}} = 1; \quad (2)
\]

for certain constants \( c \in \mathbb{R} \) and \( \alpha \in (0, 1) \).

From Definition 2.2, it follows that \( \rho(n) \) slowly goes to zero as \( n \) goes to infinity. It is important to note that in the literature there are other definitions of long-range dependence, not all equivalent to each other, but in all of them the definition has to do with a slow convergence to zero of \( \rho(n) \) (see for example [5], [20], [24] and [39]). The interpretation of this property is immediate, even though two values are separated in time, the correlation between those values is greater in a long-range dependence models than in a short-range dependence models. Therefore, observations distant in time have a greater incidence on current observations if it presents the long-range dependence property.

Theorem 2.3 establishes the conditions that the Hurst parameter $H$ must satisfy so that the increments in fractional Brownian motion are a sequence of random variables that have long-range dependence.

**Theorem 2.3.** If \( \{B^H_t\}_{t \in \mathbb{R}} \) is a fractional Brownian motion where \( H > \frac{1}{2} \), then the sequence of increments \( \{X_k := B^H_k - B^H_{k-1}\}_{k \in \mathbb{N}} \) has long-range dependence.

**Proof 1.**

\[
\rho_H(n) = \text{Cov}(X_k, X_{k+n}) = \text{Cov}(B^H_k - B^H_{k-1}, B^H_{k+n} - B^H_{k+n-1})
\]

\[
= \text{Cov}(B^H_k, B^H_{k+n}) - \text{Cov}(B^H_k, B^H_{k+n-1}) - \text{Cov}(B^H_{k-1}, B^H_{k+n}) + \text{Cov}(B^H_{k-1}, B^H_{k+n-1})
\]

\[
= \frac{1}{2} \left( (n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \right)
\]

\[
= \frac{1}{2} \left( n^{2H} \left( 1 + \frac{1}{n} \right)^{2H} + n^{2H} \left( 1 - \frac{1}{n} \right)^{2H} - 2n^{2H} \right)
\]

\[
= \frac{n^{2H-2}}{2} \left( n^{2H} \left( 1 + \frac{1}{n} \right) + n^{2H} \left( 1 - \frac{1}{n} \right) - 2n^{2H} \right)
\]

\[
= \frac{n^{2H-2}}{2} \left( (1 + \frac{1}{n})^{2H} + (1 - \frac{1}{n})^{2H} - 2 \right).
\]

Applying L'Hôpital rule (two times) and taking limits when \( n \) goes to infinity, we obtain that

\[
\frac{(1 + \frac{1}{n})^{2H} + (1 - \frac{1}{n})^{2H} - 2}{\frac{1}{n^2}} \to 2H(2H - 1)
\]
then,
\[
\rho_H(n) \approx n^{2H-2}H(2H-1) \to 0 \quad \forall H \in (0,1).
\]
Therefore the auto-covariance of the increments has order \(n^{2H-2}\) as \(n\) goes to infinity.

Also
\[
\lim_{n \to \infty} \frac{\rho_H(n)}{(2H-1)Hn^{2H-2}} = 1;
\]
if we consider \(c = (2H-1)H\) and \(\alpha = 2 - 2H\) in equation (2), we obtain that if \(H \in (\frac{1}{2},1)\) then \(\alpha \in (0,1)\). This concludes that the fractional Brownian motion presents long-range dependence when \(H > \frac{1}{2}\).

Theorem 2.3 establishes a foundation of focusing on the problems associated with the complexity in financial series related to long-range dependence. Therefore, when analyzing some financial series it becomes relevant to estimate the Hurst parameter \(H\) (see [26]). In the literature there are several estimators. Particularly, a description of some methods and empirical study are found in [41]. In this work, we use the estimator called the scaled range, proposed by Benoit Mandelbrot and James Wallis in [35].

In order to estimate \(H\), it is necessary to estimate the statistic called the adjusted range \(R/S\). For the construction, given the observations \(X_1, X_2, ..., X_n\), we denote \(Y_t = \sum_{j=1}^t X_j\) for \(t + k \leq n\) and define two statistics
\[
R(t, k) = \max_{0 \leq i \leq k} \left\{ Y_{t+i} - Y_t - \frac{i}{k}(Y_{t+i} - Y_t) \right\} - \min_{0 \leq i \leq k} \left\{ Y_{t+i} - Y_t - \frac{i}{k}(Y_{t+i} - Y_t) \right\},
\]
\[
S^2(t, k) = \frac{1}{k} \sum_{j=t+1}^{t+k} \left( X_j - \bar{X}_{t,k} \right)^2 \quad \text{where} \quad \bar{X}_{t,k} = \frac{1}{k} \sum_{j=t+1}^{t+k} X_j.
\]
The expected value for the quotient satisfies
\[
E \left( \frac{R(t, k)}{S(t, k)} \right) \approx Ck^H \quad \text{as} \; k \to \infty.
\]

Therefore, the logarithm of the adjusted range statistic \(R/S\) as a function of the logarithm of \(k\) is approximately a linear function with slope \(H\). It is possible to estimate \(H\) by least squares methods. For more information see for example [6].

### 2.2. Volatility in the market.

The uncertainty in the evolution in financial variables is of great importance in the valuation. In one hand, volatility is defined as a measure of the intensity of random or unpredictable changes in a series. In general, it is possible to declare that volatility is not constant over time and consequently, traditional models that assume constant variance (homoscedasticity) are not suitable for modeling. On the other hand, the concept of long-range dependence appears in several areas of the economy. In this work, it is proposed to relate these two concepts when analyzing the existence of long-range dependence on the volatility in two Uruguayan financial series associated with sovereign debt.

Two different approaches are highlighted in the analysis of volatility using discrete econometric models: the GARCH models and the stochastic volatility (SV)
models. For the first approach, Robert Engle introduced a new class of processes called ARCH models in [17]. The processes satisfies that the variance conditional on past information is not constant and depends on the square of past innovations. Bollerslev generalized the ARCH models in [8], when proposed GARCH models that satisfies that conditional variance depends on the squares of the perturbations and the conditional variances of previous periods. In the second approach, the difference is that volatility in the model is considered as a latent variable that is not observed. Relevant papers in this approach are based on [18].

In the analysis of volatility in continuous time models, the most widely used in the literature are the stochastic volatility models. These processes generally follow a stochastic differential equation based on Brownian motion. These models are important for the existence of the implied volatility curves based on the application of the Black-Scholes-Merton model. The property involves modeling the diffusion process by non-constant function over time. The most widely used is the model proposed by Steven Heston in [25].

The aim of the paper is to question whether it should use models for the analysis of volatility that consider the existence of long-range dependence in some financial series. We use two important financial series in the sovereign debt (mainly emerging countries) that summarize the information in a single value of the sovereign term structure curves. This modeling can be established in several ways.

3. Application in the sovereign debt of Uruguay. In the analysis of sovereign debt there are multiple financial risks that should be studied to understand the assets valuation. A relevant characteristic in the financial market is the increasing exposure to foreign investment portfolios. Therefore, the expectations of the agents play a very important role in behavior, which are related to the historical evolution of some macroeconomic variables. In this context, good mathematical modeling of the variables associated with sovereign debt can help to understand the evolution.

Term structure curves provide information on an asset’s rate of return with respect to the maturity. With the curve information it is possible to establish an index that analyzes in average the evolution the assets that compose it. These indices are defined as the weighted average of the rates of return in the maturities in which there are assets that satisfies certain circulation requirements. The volatility in the indices generates information relevant to the market. Because it is associated with the return in a generic investment strategies presented by financial agents, it is observed that the evolution of these indexes influences their future decisions. We use the notion of log-returns defined by

\[ r_t = \ln \left( \frac{I_t}{I_{t-1}} \right), \quad t = 1, 2, \ldots, T; \]

where \( I_t \) is the value of certain index at time \( t \). In this paper, the sovereign debt in Uruguay is analyzed in two of its three main issue currencies (Uruguayan pesos and dollars) after the financial crisis in 2002. To examine the debt in Uruguayan pesos, we use the ITLUP index (Index of Performance of Uruguayan Debt Issued in Current Pesos) published daily by Bolsa Electrónica de Valores del Uruguay Sociedad Anónima from April 2005 to August 2019 daily (3556 data). On the other hand, to examine sovereign debt in dollars we use the country risk index called UBI
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(Uruguay Bond Index) published daily by República AFAP from January 2005 to December 2018 (3470 data). Figure 1 shows the log returns of the indices throughout the period.

In order to analyze the dynamics in the volatility of the log-returns of the indices, the squared returns and absolute returns are used. These two series are related to the variability in the historical evolution of the financial series. Figure 2 and Figure 3 show the autocorrelations and partial autocorrelations of the log-returns series of the index, absolute returns and squared returns in ITLUP and UBI, respectively.

When analyzing the results in Figure 2 and Figure 3, it is observed that in the case of log-returns index the dependence is not significant in the long term. However in the squared and absolute value series (which are related to volatility) the dependence on these variables is significantly different from zero when analyzing several lags. In addition, Figure 4 studies the dependence on the squared and absolute returns, in which the relationship that exists in these series is observed. This analysis allows us to conclude that it is important to study whether the series have long-range dependence. If so, any model that is proposed for the volatility dynamics in financial series should be able to adapt to this property.

We propose to use models in which the possibility of long-range dependence in its evolution is considered. As stated in Section 2.1, the use of fractional calculus
allows a way to model this property in the case that the Hurst parameter $H$ is greater than 0.5. In this series, we are going to apply the estimator proposed in equation (3) to analyze whether it satisfies the long-range dependence property. The estimates are made in Project R and the results are found in Table 1 and Table 2.

| Method Estimation        | Parameter (square) | Parameter (absolute) |
|--------------------------|--------------------|----------------------|
| Simple $R/S$             | 0.7308             | 0.7489               |
| Corrected R over S       | 0.8407             | 0.8633               |
| Empirical Hurst exponent | 0.7703             | 0.7876               |
| Corrected empirical Hurst| 0.7158             | 0.7342               |

Table 1. ITLUP index.

Figure 2. Autocorrelation Function (left) and Partial Autocorrelation Function (right) in ITLUP returns. Top to bottom: log returns index, absolute returns, square returns.
Figure 3. Autocorrelation Function (left) and Partial Autocorrelation Function (right) in UBI returns. Top to bottom: log-returns index, absolute returns, square returns.

Figure 4. Time-lagged relations between absolute and squared returns in the indices.

The estimates of parameter $H$ are greater than 0.5 in all the cases. Therefore, it is recommended to use models that consider long-range dependence to estimate the volatility of this index in sovereign debt in Uruguay.
4. Models that consider long-range dependence. In the empirical application, the Hurst parameter are larger than 0.5 in the two indices, therefore it is necessary to present models that contemplate this property. Two alternatives are proposed. First, a possible tool is to apply the generalization of discrete econometric models called fractional econometric models. The two discrete approaches established in Section 2.2 of volatility analysis are extended using fractional calculus. There is the so-called FIGARCH (fractionally integrated generalized autoregressive conditional heteroscedasticity) that is the extension of GARCH and there is the so-called LMSV (long memory stochastic volatility) that is the extension of SV models. Currently, there are several academic works that apply FIGARCH models in research on financial markets. Among them, one of the pioneering works can be found in [9]. There are several unresolved questions regarding its theoretical development and the relationship with other stochastic processes. In LMSV models, it is proposed that the logarithm of volatility is an ARFIMA process. The novel papers that use these models are [11] and [23]. Currently, the mathematical problems associated with these models arise in the way of estimating the parameters due to the complex construction of the likelihood function.

Second, is to develop stochastic volatility models in continuous time using fractional calculus. This approach is the generalization of diffusion processes in financial markets in order to value financial derivatives. In general, diffusion processes are used to model the evolution of volatility based on stochastic differential equations that is driven by fractional Brownian motion. The difficulty is the great variety of resulting processes, which in general do not have closed formulas and must be calculated using a numerical approximation. In addition, if the models are parametric, the inference must be made using discrete data, especially in the presence of non-observable and serially correlated state variables. Several works study the volatility processes with high persistence, such as [14] and [19].

From our perspective, a good strategy is to generate stochastic volatility models by using the processes called fractional Ornstein-Uhlenbeck processes. These processes are defined as the solutions of the stochastic differential equation

\[
dX_t = -\lambda X_t dt + \sigma dB_t^H
\]

where \( \lambda, \sigma > 0 \).

The parameter \( \sigma \) governs the dispersion of fractional Brownian motion, while \( \lambda \) is interpreted similarly to the parameter that exists in a traditional AR(1) model. These processes are defined in [13] and the authors proved that the equation has a unique stationary solution defined in all \( \mathbb{R} \) which is given by

\[
X_t = \int_{-\infty}^t e^{-\lambda(t-s)} dB_s^H, \quad \forall t.
\]
It is proved that the process, that we call FOU($\lambda, \sigma, H$), satisfies the long-range dependence property if and only if $H > 0.5$. Furthermore the case $H=0.5$ is the standard Ornstein–Uhlenbeck process (see [43]). More recently in [28], a generalization of the FOU($\lambda, \sigma, H$) process is defined. In this work, the FOU($\lambda_1, \lambda_2, \sigma, H$) processes are defined by means of composition of the integral operator defined in equation (4) for different values of $\lambda$. In this work, it is shown that for the case in which the operator is composed with two different values of $\lambda$ ($\lambda_1 \neq \lambda_2$), the FOU($\lambda_1, \lambda_2, \sigma, H$) processes have short-range dependence for every value of $H$, but FOU($\lambda_1, \lambda_2, \sigma, H$) → FOU($\lambda_2, \sigma, H$) (as $\lambda_1 \to 0$) that has long-range dependence in the case in which $H > 0.5$. Therefore, these processes have the interesting property of going from short-range dependence to long-range dependence as it is modeled with parameters $\lambda_2$ near to zero. In addition, a way to work with these models when the process is observed in a discretized and equispaced interval of time can be found in in [29].

5. Conclusions. In this paper, the analysis of the property called long-range dependence that is observed empirically in Uruguayan financial series is carried out. The principal mathematical models in the analysis of financial markets use Brownian motion. These processes satisfies the property that are semi-martingales, which allows the use of the area called stochastic calculus. In the last decades, a large number of applications and advances in financial engineering have been generated. However, some empirical properties in financial series cannot be explained by this theory. The aim of the paper is to propose another modeling approach by fractional calculus for Uruguayan financial series and to establish the important role that the Hurst parameter $H$ has in the associated stochastic processes. The volatility of an important macroeconomic variable in sovereign debt in Uruguay is analyzed. The results obtained show of the need to use long-range dependence models. Finally, we suggest possible approaches to be used in both discrete and continuous time that have this property.

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Received February 2020; revised April 2020.

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