Distinguished Riemann-Hamilton geometry in the polymomentum electrodynamics

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Abstract

In this paper we develop the distinguished (d-) Riemannian differential geometry (in the sense of d-connections, d-torsions, d-curvatures and some geometrical Maxwell-like and Einstein-like equations) for the polymomentum Hamiltonian which governs the multi-time electrodynamics.

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1 Introduction

Let $M^n$ be a smooth real manifold of dimension $n$, whose local coordinates are $x = (x^i)_{i=1,n}$, having the physical meaning of "space of events". In order to justify the "electrodynamics" terminology used in this paper, we recall that, in the study of classical electrodynamics, the Lagrangian function $L : TM \rightarrow \mathbb{R}$ that governs the movement law of a particle of mass $m \neq 0$ and electric charge $e$ placed concomitantly into a gravitational field and an electromagnetic one, is given by

$$L(x, y) = mc \varphi_{ij}(x)y^iy^j + \frac{2e}{m}A_i(x)y^i + \mathcal{P}(x),$$  \hspace{1cm} (1)

where the semi-Riemannian metric $\varphi_{ij}(x)$ represents the gravitational potentials of the space $M$, $A_i(x)$ are the components of an 1-form on $M$ representing the electromagnetic potential, $\mathcal{P}(x)$ is a smooth potential function on $M$ and $c$ is the velocity of light in vacuum. The Lagrange space $L^n = (M, L(x, y))$, where $L$ is given by (1), is known in the literature of specialty as the autonomous Lagrange space of electrodynamics. A deep geometrical study of the Lagrange space $L^n$ is now completely done by Miron and Anastasiei in the book [6]. More general, we point out that, in the study of classical rheonomic (time-dependent) electrodynamics, a central role is played by the autonomous time-dependent Lagrangian function of electrodynamics expressed by

$$L(t, x, y) = mc \varphi_{ij}(x)y^iy^j + \frac{2e}{m}A_i(t, x)y^i + \mathcal{P}(t, x),$$  \hspace{1cm} (2)

where $L : \mathbb{R} \times TM \rightarrow \mathbb{R}$. Note that the non-dynamical character (i.e., the independence on the temporal coordinate $t$) of the spatial semi-Riemannian
metric $\varphi_{ij}(x)$ determines the usage of the term "autonomous" in the preceding definition.

Let $(T^m, h_{ab}(t))$ be a "multi-time" smooth Riemannian manifold of dimension $m$ (please do not confuse with the mass $m \neq 0$), having the local coordinates $t = (t^c)_{c=1,m}$, and let $J^1(T, M)$ be the 1-jet space produced by the manifolds $T$ and $M$. By a natural extension of the preceding examples of electrodynamics Lagrangian functions, we can consider the jet multi-time Lagrangian function

$$
L(t^c, x^k, x^k_c) = mch_{ab}(t)\varphi_{ij}(x)x^i_ax^j + \frac{2e}{m}A^{(a)}_{(ij)}(t, x)x^a_a + \mathcal{P}(t, x),
$$

(3)

where $A^{(a)}_{(ij)}(t, x)$ is a d-tensor on $J^1(T, M)$ and $\mathcal{P}(t, x)$ is a smooth function on the product manifold $T \times M$.

**Remark 1** Throughout this paper, the indices $a, b, c, \ldots$ run from 1 to $m$, while the indices $i, j, k, \ldots$ run from 1 to $n$. The Einstein convention of summation is also adopted all over this work.

The pair $\mathcal{E}DM^n_m = (J^1(T, M), L)$, where $L$ is given by (3), is called the autonomous multi-time Lagrange space of electrodynamics. The distinguished Riemannian geometrization of the multi-time Lagrange space $\mathcal{E}DM^n_m$ is now completely developed in the Neagu’s works [8] and [9].

Via the classical Legendre transformation, the jet multi-time Lagrangian function of electrodynamics (3) leads us to the Hamiltonian function of polymomenta

$$
H = \frac{1}{4mc}h_{ab}\varphi^{ij}p^a_ip^b_j - \frac{e}{m2c}h_{ab}\varphi^{ij}A^{(b)}_{(ij)}p^a + \frac{e^2}{m3c}\|A\|^2 - \mathcal{P},
$$

(4)

where $H : J^{1*}(T, M) \to \mathbb{R}$, and

$$
\|A\|^2(t, x) = h_{ab}\varphi^{ij}A^{(a)}_{(ij)}A^{(b)}_{(ij)}.
$$

**Definition 2** The pair $\mathcal{E}DM^n_m = (J^{1*}(T, M), H)$, where $H$ is given by (4), is called the autonomous multi-time Hamilton space of electrodynamics.

But, using as a pattern the Miron’s geometrical ideas from [7], the distinguished Riemannian geometry for quadratic Hamiltonians of polymomenta (geometry in the sense of d-connections, d-torsions, d-curvatures and geometrical Maxwell-like and Einstein-like equations) is constructed on dual 1-jet spaces in the Oană-Neagu’s paper [11]. Consequently, in what follows, we apply the general geometrical result from [11] for the particular Hamiltonian function of polymomenta [4], which governs the multi-time electrodynamics.

## 2 The geometry of the autonomous multi-time Hamilton space of electrodynamics $\mathcal{E}DM^n_m$

To initiate our Hamiltonian geometrical development for multi-time electrodynamics, let us consider on the dual 1-jet space $E^* = J^{1*}(T, M)$ the fundamental
vertical metrical d-tensor
\[ \Phi^{(i)(j)}_{(a)(b)} = \frac{1}{2} \frac{\partial^2 H}{\partial p_a \partial p_b} = h^*_{ab}(t^c) \varphi^{ij}(x^k), \]
where \( h^*_{ab}(t) := (4mc)^{-1} h_{ab}(t) \). Let \( \chi^a_{bc}(t) \) (respectively \( \gamma^k_{ij}(x) \)) be the Christoffel symbols of the metric \( h_{ab}(t) \) (respectively \( \varphi_{ij}(x) \)). Obviously, if \( \tilde{\chi}^a_{bc} \) are the Christoffel symbols of the Riemannian metric \( h^*_{ab}(t) \), then we have \( \tilde{\chi}^a_{bc} = \chi^a_{bc} \).

Using a general result from the geometrical theory of multi-time Hamilton spaces (see [2] and [11]), by direct computations, we find

**Theorem 3** The pair of local functions \( N_{ED} = \left( N_{1(1)ib}, N_{2(2)ij} \right) \) on the dual 1-jet space \( E^* \), which are given by
\[ N^{(a)}_{1(1)ib} = \chi^a_{bf} p_f, \]
\[ N^{(a)}_{2(2)ij} = \gamma^r_{ij} \left[ \frac{2e}{m} A^{(a)}_{(r)} - p_f^r \right] - \frac{e}{m} \left[ \frac{\partial A^{(a)}_{(i)}}{\partial x^j} + \frac{\partial A^{(a)}_{(j)}}{\partial x^i} \right], \]
represents a nonlinear connection on \( E^* \). This nonlinear connection is called the canonical nonlinear connection of the multi-time Hamilton space of electrodynamics \( E\Gamma M^H_m \).

Now, let
\[ \left\{ \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_a^r} \right\} \subset \chi^a (E^*), \quad \left\{ dt^a, dx^i, \delta p_a^r \right\} \subset \chi^a (E^*) \]
be the adapted bases produced by the nonlinear connection \( N_{ED} \), where
\[ \frac{\delta}{\delta t^a} = \frac{\partial}{\partial t^a} - N^{(f)}_{1(1)ia} \frac{\partial}{\partial p_f^r}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{(f)}_{2(2)ia} \frac{\partial}{\partial p_f^r}, \]
\[ \delta p_a^r = dp_a^r + N^{(a)}_{1(1)i} dt^f + N^{(a)}_{2(2)i} dx^f. \]

Working with these adapted bases, by direct computations, we can determine the adapted components of the generalized Cartan canonical connection of the space \( E\Gamma M^H_m \), together with its local d-torsions and d-curvatures (for details, see the general formulas from [11]).

**Theorem 4** (1) The generalized Cartan canonical linear connection of the autonomous multi-time Hamilton space of electrodynamics \( E\Gamma M^H_m \) is given by
\[ CT(N) = \left( \chi^a_{bc}, A^a_{jc}, H^i_{jk}, C^{(k)}_{j(c)} \right), \]
where its adapted components are
\[ H^c_{ab} = \chi^c_{ab}, \quad A^a_{jc} = 0, \quad H^i_{jk} = \gamma^i_{jk}, \quad C^{(k)}_{j(c)} = 0. \]
3 Electromagnetic-like model on the multi-time Hamilton space of electrodynamics $\mathcal{EDM}^n_m$

In order to describe our geometrical electromagnetic-like theory (depending on polymomenta) on the multi-time Hamilton space of electrodynamics $\mathcal{EDM}^n_m$, we underline that, by a simple direct calculation, we obtain (see [11])

(2) The torsion $T$ of the generalized Cartan canonical linear connection of the space $\mathcal{EDM}^n_m$ is determined by three effective adapted components:

$$
R^{(f)}_{(r)ab} = \chi_{gab}^d P_r^g,
$$

$$
R^{(f)}_{(r)a} = -\frac{2e}{m} \gamma_{gaj} A^{(f)}_{(s)a} + \frac{e}{m} \left[ \frac{\partial A^{(f)}_i}{\partial x^j} + \frac{\partial A^{(f)}_j}{\partial x^i} \right]_{i,a},
$$

$$
R^{(f)}_{(r)ij} = \mathcal{R}_{\gamma_{ij}} \left[ \frac{2e}{m} A^{(f)}_s - p^f_s \right] - \frac{e}{m} \left[ \frac{\partial A^{(f)}_i}{\partial x^j} - \frac{\partial A^{(f)}_j}{\partial x^i} \right],
$$

(7)

where $\chi_{gab}^d(t)$ (respectively $\mathcal{R}^{k}_{rij}(x)$) are the classical local curvature tensors of the Riemannian metric $h_{ab}(t)$ (respectively semi-Riemannian metric $\varphi_{ij}(x)$), and ”$a$” and ”$k$” represent the following generalized Levi-Civita covariant derivatives:

- the $T$-generalized Levi-Civita covariant derivative:

$$
T^{bi(d)(r)...}_{cjl(f)...;a} \equiv \frac{\partial T^{bi(d)(r)...}_{cjl(f)...}}{\partial x^a} + T^{bi(d)(r)...}_{cjl(f)...} \chi_{ga} + T^{bi(g)(r)...}_{cjl(f)...} \chi_{ga} + ...
$$

$$
- T^{bi(d)(r)...}_{cjl(f)...} \chi_{ga} - T^{bi(d)(r)...}_{cjl(g)...} \chi_{ja} - ...
$$

- the $M$-generalized Levi-Civita covariant derivative:

$$
T^{bi(d)(r)...}_{cjl(f)...;k} \equiv \frac{\partial T^{bi(d)(r)...}_{cjl(f)...}}{\partial x^k} + T^{bi(d)(r)...}_{cjl(f)...} \gamma_{sk} + T^{bi(d)(s)...}_{cjl(f)...} \gamma_{sk} + ...
$$

$$
- T^{bi(d)(r)...}_{cjl(s)...} \gamma_{sk} - T^{bi(d)(r)...}_{cjl(s)...} \gamma_{lk} - ...
$$

(3) The curvature $\mathcal{R}$ of the Cartan canonical connection of the space $\mathcal{EDM}^n_m$ is determined by the following four effective adapted components:

$$
H^{d}_{abc} = \chi_{abc}^d, \quad R^l_{ijk} = \mathcal{R}^l_{ijk}
$$

and

$$
-R^{l(d)(i)}_{(i)(a)bc} = \delta^l_{[a} \chi_{abc]^d}, \quad -R^{l(d)(i)}_{(i)(a)jk} = -\delta^l_{[a} \mathcal{R}^l_{ijk].
$$

3 Electromagnetic-like model on the multi-time Hamilton space of electrodynamics $\mathcal{EDM}^n_m$

In order to describe our geometrical electromagnetic-like theory (depending on polymomenta) on the multi-time Hamilton space of electrodynamics $\mathcal{EDM}^n_m$, we underline that, by a simple direct calculation, we obtain (see [11])
Proposition 5 The metrical deflection d-tensors of the space \( \mathcal{EDMH}_m \) are expressed by the formulas:

\[
\Delta^{(i)}_{(a)b} = \left[ h_a^f \varphi^{ir} p_f^l \right] / b = 0,
\]

\[
\Delta^{(i)}_{(a)j} = \left[ h_a^f \varphi^{ir} p_f^l \right]_{ij} = \frac{e}{4mc} h_a^f \varphi^{ir} \left[ A^{(f)}_{(r)j} + A^{(f)}_{(l)r} \right],
\]

\[
\psi^{(i)(j)}_{(a)(b)} = \left[ h_a^f \varphi^{ir} p_f^l \right]_{(a)(b)} = \frac{1}{4mc} h_{abc} \varphi^{ij},
\]

where "\( / b \)”, "\( , \)" and "\( (a)(b) \)” are the local covariant derivatives induced by the generalized Cartan canonical connection \( \mathcal{CT} (N) \) (see [10] and [11]).

Moreover, taking into account some general formulas from [11], we introduce

Definition 6 The distinguished 2-form on \( J^1 (T, M) \), locally defined by

\[
F = F^{(i)}_{(a)(j)} \delta p^a_t \wedge dx^j + f^{(i)(j)}_{(a)(b)} \delta p^a_t \wedge \delta p^b_t,
\]

where

\[
F^{(i)}_{(a)(j)} = \frac{1}{2} \left\{ \Delta^{(i)}_{(a)(j)} - \Delta^{(j)}_{(a)(i)} \right\} = \frac{e}{8mc} \cdot A \left\{ h_a^f \varphi^{ir} \left[ A^{(f)}_{(r)j} + A^{(f)}_{(l)r} \right] \right\},
\]

\[
f^{(i)(j)}_{(a)(b)} = \frac{1}{2} \left\{ \varphi^{(i)(j)}_{(a)(b)} - \varphi^{(j)(i)}_{(a)(b)} \right\} = 0,
\]

is called the polymomentum electromagnetic field attached to the multi-time Hamilton space of electrodynamics \( \mathcal{EDMH}_m \).

Now, particularizing the generalized Maxwell-like equations of the polymomentum electromagnetic field that govern a general multi-time Hamilton space \( MH_n \), we obtain the main result of the polymomentum electromagnetism on the space \( \mathcal{EDMH}_m \) (for more details, see [11]):

Theorem 7 The polymomentum electromagnetic components \( \mathcal{EDMH}_m \) of the autonomous multi-time Hamilton space of electrodynamics \( \mathcal{EDMH}_m \) are governed by the following geometrical Maxwell-like equations:

\[
\begin{align*}
F^{(i)}_{(a)j/b} &= \frac{e \cdot h_a^f}{8mc} A \left( \varphi^{ir} \left[ \frac{\partial A^{(f)}_{(r)j}}{\partial x^j} + \frac{\partial A^{(f)}_{(l)r}}{\partial x^r} \right] \right)_{jb} - 2\varphi^{ir} \gamma_{rjk} A^{(f)}_{(s)k} \\
\sum_{\{i,j,k\}} F^{(i)}_{(a)j/k} &= -\frac{h_a^f}{8mc} \sum_{\{i,j,k\}} \left\{ \varphi^{ir} \mathcal{R}_{rjk}^{i} - \varphi^{ir} \mathcal{R}_{rjk}^{j} \right\} p_f^l + \frac{e}{m} \\
\cdot \varphi^{ir} \left[ 2\mathcal{R}_{rjk} A^{(f)}_{(s)k} - \left( \frac{\partial A^{(f)}_{(j)l}}{\partial x^k} - \frac{\partial A^{(f)}_{(k)l}}{\partial x^j} \right) \right]_{rj} \\
\sum_{\{i,j,k\}} F^{(i)}_{(a)j/k} &= 0,
\end{align*}
\]
where $A_{(i,j)}$ represents an alternate sum and $\sum_{(i,j,k)}$ represents a cyclic sum.

4 Gravitational-like geometrical model on the multi-time Hamilton space of electrodynamics

To expose our geometrical Hamiltonian polymomentum gravitational theory on the autonomous multi-time Hamilton space of electrodynamics $\mathcal{EDMH}_m^n$, we recall that the fundamental vertical metrical d-tensor

$$\Phi_{(a)(b)}^{(i)(j)}(x) = h_{ab}^{*}(t)\varphi^{ij}(x)$$

and the canonical nonlinear connection

$$N_{\mathcal{ED}} = \left( N_{\mathcal{D}H}^{(a)1}(i,j), N_{\mathcal{D}H}^{(a)2}(i,j) \right)$$

do the multi-time Hamilton space $\mathcal{EDMH}_m^n$ produce a polymomentum gravitational $h^{*}$-potential $G$ on $E^{*} = J^{1*}(T,M)$, locally expressed by

$$G = h_{ab}^{*}dt^a \otimes dt^b + \varphi_{ij}dx^i \otimes dx^j + h_{ab}^{*}\varphi^{ij}\delta p^a_i \otimes \delta p^b_j. \quad (12)$$

We postulate that the geometrical Einstein-like equations, which govern the multi-time gravitational $h^{*}$-potential $G$ of the multi-time Hamilton space of electrodynamics $\mathcal{EDMH}_m^n$, are the abstract geometrical Einstein equations attached to the Cartan canonical connection $CT(N)$ and to the adapted metric $G$ on $E^*$, namely

$$\text{Ric}(CT) - \frac{\text{Sc}(CT)}{2} G = \kappa T, \quad (13)$$

where $\text{Ric}(CT)$ represents the Ricci tensor of the Cartan connection, $\text{Sc}(CT)$ is the scalar curvature, $\kappa$ is the Einstein constant and $T$ is an intrinsic d-tensor of matter, which is called the stress-energy d-tensor of polymomenta.

In order to describe the local geometrical Einstein-like equations (together with their generalized conservation laws) in the adapted basis

$$\{X_A\} = \left\{ \delta \frac{\delta}{\delta t^a}, \delta \frac{\delta}{\delta x^i}, \delta \frac{\delta}{\delta p^a_i} \right\},$$

let $CT(N) = (\chi^{c}_{ab}, 0, \gamma^{j}_{jk}, 0)$ be the generalized Cartan canonical connection of the space $\mathcal{EDMH}_m^n$. Taking into account the expressions of its adapted curvature d-tensors on the space $\mathcal{EDMH}_m^n$, we immediately find (see [11]):

**Theorem 8** The Ricci tensor $\text{Ric}(CT)$ of the autonomous multi-time Hamilton space of electrodynamics $\mathcal{EDMH}_m^n$ is characterized by two effective local Ricci d-tensors:

$$\chi_{ab} = \chi_{ab}^{if}, \quad \mathcal{R}_{ij} = \mathcal{R}_{ij}^{tr}.$$  

These are exactly the classical Ricci tensors of the Riemannian temporal metric $h_{ab}(t)$ and the semi-Riemannian spatial metric $\varphi_{ij}(x)$.  

6
Consequently, using the notations \( \chi = h^{ab} \chi_{ab} \) and \( R = \varphi^{ij} R_{ij} \), we get

**Theorem 9** The scalar curvature \( \text{Sc}(CT) \) of the generalized Cartan connection \( CT \) of the space \( EDMH^m_n \) has the expression (for details, see [11]):

\[
\text{Sc}(CT) = (4mc) \cdot \chi + R,
\]

where \( \chi \) and \( R \) are the classical scalar curvatures of the semi-Riemannian metrics \( h_{ab}(t) \) and \( \varphi_{ij}(x) \).

Particularizing the generalized Einstein-like equations and the generalized conservation laws of an arbitrary multi-time Hamilton space \( MH^m_n \), we can establish the main result of the generalized polymomentum gravitational theory on the autonomous multi-time Hamilton space of electrodynamics \( EDMH^m_n \) (for more details, see [11]):

**Theorem 10**

1. The local geometrical Einstein-like equations, that govern the polymomentum gravitational potential of the space \( EDMH^m_n \), have the form

\[
\begin{align*}
\chi_{ab} - \frac{(4mc) \cdot \chi + R}{8mc} h_{ab} &= k T_{ab} \\
R_{ij} - \frac{(4mc) \cdot \chi + R}{2} \varphi_{ij} &= k T_{ij} \\
- \frac{(4mc) \cdot \chi + R}{8mc} h_{ab} \varphi_{ij} &= k T^{(i)(j)}_{(a)(b)},
\end{align*}
\]

(14)

\[
\begin{align*}
0 &= T_{ai}, & 0 &= T_{ja}, & 0 &= T^{(i)}_{(a)b} \\
0 &= T_{a(b)}, & 0 &= T_{i(b)}, & 0 &= T^{(i)}_{(a)j},
\end{align*}
\]

(15)

where \( T_{AB}, A, B \in \{ a, i, (i) \} \), are the adapted components of the polymomentum stress-energy \( \delta \)-tensor of matter \( T \).

2. The polymomentum conservation laws of the geometrical Einstein-like equations of the space \( EDMH^m_n \) are expressed by the formulas

\[
\begin{align*}
\left(4mc \cdot \chi^f_h - \frac{(4mc) \cdot \chi + R}{2} \delta^f_h \right)_{/f} &= 0 \\
\left[R^r_j - \frac{(4mc) \cdot \chi + R}{2} \delta^r_j \right]_{/r} &= 0,
\end{align*}
\]

(16)

where \( \chi^f_h = h^{ld} \chi_{db} \) and \( R^r_j = \varphi^{rs} R_{sj} \).

**Open Problem.** There exist real physical interpretations for previous geometrical polymomentum field-like theories, which to be relevant for the physical domain of electrodynamics?
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