EVALUATION OF THE CONVOLUTION SUMS \( \sum_{(l,m)\in\mathbb{N}_0^2 \atop \alpha l + \beta m = n} \sigma(l)\sigma(m) \), WHERE

\[\alpha\beta = 44,52\]

EBÉNÉZER NTIENJEM

ABSTRACT. The convolution sum, \( \sum_{(l,m)\in\mathbb{N}_0^2 \atop \alpha l + \beta m = n} \sigma(l)\sigma(m) \), where \( \alpha\beta = 44,52 \), is evaluated for all natural numbers \( n \). We then use these convolution sums to determine formulae for the number of representations of a natural number by the octonary quadratic forms

\[a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2)\],

where \( (a,b) = (1,11) \), \( (1,13) \).

1. INTRODUCTION

The sets of natural numbers, non-negative integers, integers, rational numbers, real numbers and complex numbers, are denoted by \( \mathbb{N} \), \( \mathbb{N}_0 \), \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \), respectively.

Suppose that \( k,n \in \mathbb{N} \). We define the sum of positive divisors of \( n \) to the power of \( k \), \( \sigma_k(n) \), by

\[\sigma_k(n) = \sum_{0 < d \mid n} d^k.\]

We write \( \sigma(n) \) as a synonym for \( \sigma_1(n) \) and we set \( \sigma_k(m) = 0 \) if \( m \notin \mathbb{N} \).

The convolution sum, \( W_{(\alpha,\beta)}(n) \), is defined for all \( \alpha,\beta \in \mathbb{N} \) such that \( \alpha \leq \beta \) as follows:

\[W_{(\alpha,\beta)}(n) = \sum_{(l,m)\in\mathbb{N}_0^2 \atop \alpha l + \beta m = n} \sigma(l)\sigma(m)\]

We write \( W_{\beta}(n) \) as a short hand for \( W_{(1,\beta)}(n) \).

For those convolution sums \( W_{(\alpha,\beta)}(n) \) that have so far been evaluated, the values of \( (\alpha,\beta) \) are given in Table 1. We evaluate the convolution sums for \( (\alpha,\beta) = (1,44) \), \( (4,11) \), \( (1,52) \), \( (4,13) \), i.e., \( \alpha\beta = 44 \) and \( \alpha\beta = 52 \). The evaluation of these convolution sums have not been done yet according to Table 1.

Let \( a, b, c, d \in \mathbb{N} \) be such that \( \gcd(a, b) = 1 \) and \( \gcd(c, d) = 1 \). The convolution sums are generally used to determine explicit formulae for the number of representations of a positive integer \( n \) by the octonary quadratic forms

\[a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2),\]

and

\[c(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_3x_4 + x_3^2) + d(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2),\]

respectively.

2010 Mathematics Subject Classification. 11A25, 11E20, 11E25, 11F11, 11F20, 11F27.

Key words and phrases. Sums of Divisors function; Convolution Sums; Dedekind eta function; Modular Forms; Eisenstein Series; Cusp Forms; Octonary quadratic Forms; Number of Representations.
We use the evaluated convolution sums and other known convolution sums to determine formulae for the number of representations of a positive integer \( n \) by the octonary quadratic form Equation 1.3 for which \( (a,b) = (1,11), (1,13) \). These number of representations are also new according to Table 2 which displays known explicit formulae for the number of representations of \( n \) by the octonary form Equation 1.3.

We have organized this paper as follows. In Section 2 we briefly discuss modular forms and define eta functions and convolution sums. Then in Section 3 we discuss our main results on the evaluation of the convolution sums; the main results on the formulae for the number of representations of a positive integer \( n \) are given in Section 4.

We use a software for symbolic scientific computation to obtain the results of this paper. The open source software packages GiNaC, Maxima, REDUCE, SAGE and the commercial software package MAPLE build this software.

2. Preliminaries

We consider the upper half-plane, \( \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \), and the group \( G = \text{SL}_2(\mathbb{R}) \) of \( 2 \times 2 \)-matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that \( a, b, c, d \in \mathbb{R} \) and \( ad - bc = 1 \). Let \( \Gamma = \text{SL}_2(\mathbb{Z}) \) be a subset of \( G \) and let \( N \in \mathbb{N} \). Then

\[
\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}
\]

is a subgroup of \( \Gamma \). The subgroup \( \Gamma(N) \) is called a principal congruence subgroup of level \( N \). If a subgroup \( H \) of \( G \) contains \( \Gamma(N) \), then it is a congruence subgroup of level \( N \).

For our purpose we consider the congruence subgroup \( \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\} \).

Let \( k \in \mathbb{Z}, \gamma \in \Gamma \) and \( f^{\gamma}(z) : \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \rightarrow \mathbb{C} \cup \{ \infty \} \) be the function whose value at \( z \) is \( f^{\gamma}(z) = (cz+d)^{-k}f(\gamma(z)) \). The following definition is based on N. Koblitz’s textbook [15] p. 108.

**Definition 2.1.** Let \( N \in \mathbb{N}, k \in \mathbb{Z}, f \) be a meromorphic function on \( \mathbb{H} \) and \( \Gamma' \subset \Gamma \) be a congruence subgroup of level \( N \).

(a) \( f \) is a modular function of weight \( k \) for \( \Gamma' \) if

(a1) \( f^{\gamma}(z) = f \) for all \( \gamma \in \Gamma' \),

(a2) for any \( \delta \in \Gamma \) it holds that \( f^{\delta\gamma}(z) \) can be expressed in the form \( \sum_{n \in \mathbb{Z}} a_n e^{2\pi inz/N} \),

wherein \( a_n \neq 0 \) for finitely many \( n \in \mathbb{Z} \) such that \( n < 0 \).

(b) \( f \) is a modular form of weight \( k \) for \( \Gamma' \) if

(b1) \( f \) is a modular function of weight \( k \) for \( \Gamma' \),

(b2) \( f \) is holomorphic on \( \mathbb{H} \),

(b3) \( a_n = 0 \) for all \( \delta \in \Gamma \) and for all \( n \in \mathbb{Z} \) such that \( n < 0 \).

(c) \( f \) is a cusp form of weight \( k \) for \( \Gamma' \) if

(c1) \( f \) is a modular form of weight \( k \) for \( \Gamma' \),

(c2) \( a_0 = 0 \) for all \( \delta \in \Gamma \).

Let \( k, N \in \mathbb{N} \). We denote by \( \mathcal{M}_k(\Gamma_0(N)) \) the space of modular forms of weight \( k \) for \( \Gamma_0(N) \), \( S_k(\Gamma_0(N)) \) the subspace of cusp forms of weight \( k \) for \( \Gamma_0(N) \), and \( \mathcal{E}_k(\Gamma_0(N)) \) the subspace of Eisenstein forms of weight \( k \) for \( \Gamma_0(N) \). In W. A. Stein’s book (online version) [25] p. 81 it is shown that \( \mathcal{M}_k(\Gamma_0(N)) = \mathcal{E}_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)) \).
According to Section 5.3 of W. A. Stein’s book [25] p. 86] $E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$ where $B_k$ are the Bernoulli numbers, if the primitive Dirichlet characters are trivial and $2 \leq k$ is even.

We only consider trivial primitive Dirichlet characters and $4 \leq k$ even in the sequel. Based on this consideration Theorems 5.8 and 5.9 in Section 5.3 of [25, p. 86] also hold.

2.1. Eta Functions. On the upper half-plane $\mathbb{H}$ the Dedekind eta function, $\eta(z)$, is defined by $\eta(z) = e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi i nz})$. When we set $q = e^{2\pi i z}$, then

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} F(q), \quad \text{where} \quad F(q) = \prod_{n=1}^{\infty} (1 - q^n).$$

L. J. P. Kilford’s book [14] p. 99] and G. Köhler’s book [16] p. 37] have a proof of the following theorem which we will apply to determine eta functions which belong to $M_k(\Gamma_0(N))$, and particularly those eta functions that belong to $S_k(\Gamma_0(N))$. As noted by A. Alaca et al. [1] credit to this theorem also goes to M. Newman [20, 21].

Theorem 2.2 (M. Newman and G. Ligozat). Let $N \in \mathbb{N}$ and let $f(z) = \prod_{1 \leq \delta | N} \eta^\delta(\delta z)$ be an eta function which satisfies the following conditions:

(i) $\sum_{1 \leq \delta | N} \delta r_\delta \equiv 0 \pmod{24},$  
(ii) $\sum_{1 \leq \delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$

(iii) $\prod_{1 \leq \delta | N} \delta^\alpha$ is a square in $\mathbb{Q}$.

(iv) $k = \frac{1}{2} \sum_{1 \leq \delta | N} r_\delta$ is an even integer.

(v) for each positive divisor $d$ of $N$, the inequality $\sum_{1 \leq \delta | N} \frac{gcd(\delta, d)^2}{\delta} r_\delta \geq 0$ holds.

Then $f(z) \in M_k(\Gamma_0(N))$.

If (v) is replaced by

(v’) for each positive divisor $d$ of $N$ the inequality $\sum_{1 \leq \delta | N} \frac{gcd(\delta, d)^2}{\delta} r_\delta > 0$ holds

then $f(z) \in S_k(\Gamma_0(N))$.

2.2. Evaluating $W_{(\alpha, \beta)}(n)$. Let $\alpha, \beta \in \mathbb{N}$ be such that $\alpha \leq \beta$. We let the convolution sum, $W_{(\alpha, \beta)}(n)$, be defined as in [Equation 1.2]

Following the observation by A. Alaca et al. [2], we assume that $gcd(\alpha, \beta) = 1$. Suppose that $q \in \mathbb{C}$ is such that $|q| < 1$. We define the Eisenstein series $L(q)$ and $M(q)$ by

$$L(q) = E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n,$$

$$M(q) = E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n.$$ 

The following two results whose proofs are given by A. Alaca et al. [1] are essential for the sequel of this work

Lemma 2.3. Let $\alpha, \beta \in \mathbb{N}$. Then

$$(\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in M_4(\Gamma_0(\alpha \beta)),$$
Theorem 2.4. Let \( \alpha, \beta \in \mathbb{N} \) be such that \( \alpha \) and \( \beta \) are relatively prime and \( \alpha < \beta \). Then

\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 = (\alpha - \beta)^2 + \sum_{n=1}^{\infty} \left( 240 \alpha^2 \sigma_3 \left( \frac{n}{\alpha} \right) + 240 \beta^2 \sigma_3 \left( \frac{n}{\beta} \right) - 48 \alpha (\beta - 6n) \sigma_1 \left( \frac{n}{\alpha} \right) - 48 \beta (\alpha - 6n) \sigma_1 \left( \frac{n}{\beta} \right) 
- 1152 \alpha \beta W(\alpha, \beta)(n) \right) q^n.
\]

(2.3)

3. Evaluation of the Convolution Sums \( W(\alpha, \beta)(n) \), where \( \alpha \beta = 44, 52 \)

We give explicit formulae for the convolution sums \( W(1,44)(n), W(4,11)(n), W(1,52)(n) \) and \( W(4,13)(n) \).

3.1. Bases for \( E_4(\Gamma_0(\alpha \beta)) \) and \( S_4(\Gamma_0(\alpha \beta)) \) with \( \alpha \beta = 44, 52 \). We apply the dimension formulae for the space of Eisenstein forms and the space of cusp forms in T. Miyake’s book [19, Thrm 2.5.2, p. 60] or W. A. Stein’s book [25, Prop. 6.1, p. 91] to compute the divisors of 44 and 52, respectively.

Theorem 3.1. (a) The sets \( B_{E,44} = \{ M(q^t) \mid t \in D(44) \} \) and \( B_{E,52} = \{ M(q^t) \mid t \in D(52) \} \) are bases of \( E_4(\Gamma_0(44)) \) and \( E_4(\Gamma_0(52)) \), respectively.

(b) Let \( 1 \leq i \leq 15 \) and \( 1 \leq j \leq 18 \) be positive integers.

Let \( \delta_1 \in D(44) \) and \( (r(i, \delta_1))_{\delta_1} \) be the set of the powers of \( \eta(\delta_1 z) \).

Let \( \delta_2 \in D(52) \) and \( (r(j, \delta_2))_{\delta_2} \) be the set of the powers of \( \eta(\delta_2 z) \).

Let furthermore \( A_i(q) = \prod_{\delta_1 \in \delta_1^{44}} \eta^{r(i, \delta_1)}(\delta_1 z) \) and \( B_j(q) = \prod_{\delta_2 \in \delta_2^{52}} \eta^{r(j, \delta_2)}(\delta_2 z) \) be selected elements of \( S_4(\Gamma_0(44)) \) and \( S_4(\Gamma_0(52)) \), respectively.

Then the sets \( B_{S,44} = \{ A_i(q) \mid 1 \leq i \leq 15 \} \) and \( B_{S,52} = \{ B_j(q) \mid 1 \leq j \leq 18 \} \) are bases of \( S_4(\Gamma_0(44)) \) and \( S_4(\Gamma_0(52)) \), respectively.

(c) The sets \( B_{M,44} = B_{E,44} \cup B_{S,44} \) and \( B_{M,52} = B_{E,52} \cup B_{S,52} \) constitute bases of \( M_4(\Gamma_0(44)) \) and \( M_4(\Gamma_0(52)) \), respectively.

For \( 1 \leq i \leq 15 \) and \( 1 \leq j \leq 18 \) the eta quotients \( A_i(q) \) and \( B_j(q) \) can be expressed in the form \( \sum_{n=1}^{\infty} a_i(n) q^n \) and \( \sum_{n=1}^{\infty} b_j(n) q^n \), respectively.

Proof. We only prove the case \( \alpha \beta = 44 \). The case \( \alpha \beta = 52 \) is proved similarly.

(a) When we apply Theorem 5.8 in Section 5.3 of W. A. Stein [25, p. 86], it follows that \( M(q^t) \) belongs to \( M_4(\Gamma_0(t)) \) for each \( t \in D(44) \). Since \( E_4(\Gamma_0(44)) \) has a finite dimension, it is sufficient to show that the set of \( M(q^t) \) such that \( t \in D(44) \) is linearly independent. Suppose that \( \chi_t \in \mathbb{C} \) with \( t | 44 \). Then

\[
\sum_{t | 44} \chi_t M(q^t) = \sum_{t | 44} \chi_t + 240 \sum_{n \geq 1} \left( \sum_{t | 44} \chi_t \sigma_3 \left( \frac{n}{t} \right) \right) q^n = 0.
\]
We compare the coefficients of $q^n$ for $n \in D(44)$ to obtain the following homogeneous system of 6 equations in 6 unknowns:

$$\sum_{u|44} \tau(u) x_u = 0, \quad t \in D(44).$$

The matrix of this homogeneous system of equations is triangular with positive integer values, 1, on the diagonal. Hence, the solution is $x_i = 0$ for all $t \in D(44)$. Therefore, the set $B_2$ is linearly independent and hence is a basis of $E_4(\Gamma_0(44))$.

(b) As mentioned above, the $A_i(q)$ with $1 \leq i \leq 15$ are obtained from an exhaustive search using Theorem 2.2. Hence, each $A_i(q)$ is in the space $S_4(\Gamma_0(44))$.

Since the dimension of $S_4(\Gamma_0(44))$ is 15, it is sufficient to show that the set $\{ A_i(q) \mid 1 \leq i \leq 15 \}$ is linearly independent. For that suppose that $x_i \in \mathbb{C}$ and $\sum_{i=1}^{15} x_i A_i(q) = 0$. Then

$$\sum_{i=1}^{15} x_i A_i(q) = \sum_{n=1}^{15} \left( \sum_{i=1}^{15} x_i a_i(n) \right) q^n = 0$$

which gives the following homogeneous system of equations in 15 unknowns

$$\sum_{i=1}^{15} a_i(n) x_i = 0, \quad 1 \leq n \leq 15.$$

A computation using a software package for (symbolic) scientific computation shows that the determinant of the matrix of this homogeneous system of equations is non-zero. So, $x_i = 0$ for all $1 \leq i \leq 15$. Hence, the set $\{ A_i(q) \mid 1 \leq i \leq 15 \}$ is linearly independent and therefore a basis of $S_4(\Gamma_0(44))$.

(c) Since $M_4(\Gamma_0(44)) = E_4(\Gamma_0(44)) \oplus S_4(\Gamma_0(44))$, the result follows from (a) and (b).

We observe that the basis elements

(01) $A_i(q)$, $1 \leq i \leq 5$, come from $S_4(\Gamma_0(22))$ which is the space of cusp forms necessary for the evaluation of the convolution sums $W_{22}(n)$ and $W_{(2,11)}(n)$ given by A. Alaca et al. [1]. The element $A_2(q)$ is inherited from $S_4(\Gamma_0(11))$ which is part of $M_4(\Gamma(11))$; the convolution sum $W_{11}(n)$ is evaluated by E. Royer [24].

(02) $A_2(q) = A_i(q^2)$, for $i = 2, 3, 4, 5$. Therefore, $a_{22}(n) = a_i(\frac{2}{5})$, for $i = 2, 3, 4, 5$.

(03) $B_2(q)$, $1 \leq j \leq 7$, $B_{15}(q)$ and $B_{17}(q)$ are imported from $S_4(\Gamma_0(26))$ which is the space of cusp forms required for the evaluation of the convolution sums $W_{26}(n)$ and $W_{(2,13)}(n)$ given by A. Alaca et al. [1].

(04) $B_{2j}(q) = B_j(q^2)$, for $4 \leq j \leq 7$, $B_{16}(q) = B_{15}(q^2)$ and $B_{18}(q) = B_{17}(q^2)$. Consequently, $b_{2j}(n) = b_j(\frac{2}{5})$, for $4 \leq j \leq 7$, $b_{16}(n) = b_{15}(\frac{2}{5})$ and $b_{18}(n) = b_{17}(\frac{2}{5})$.

The above observation is based on the fact that

$$M_4(\Gamma_0(11)) \subset M_4(\Gamma_0(22)) \subset M_4(\Gamma_0(44)) \quad \text{and} \quad M_4(\Gamma_0(13)) \subset M_4(\Gamma_0(26)) \subset M_4(\Gamma_0(52)).$$

As mentioned in (01) above, the eta quotient $A_2(q)$ is a basis element of $S_4(\Gamma_0(11))$. Hence, basis elements of $S_4(\Gamma_0(11))$ can be determined using Theorem 2.2. There is no basis element of $S_4(\Gamma_0(13))$ in the space of cusp forms $S_4(\Gamma_0(26))$; this is an indication that there is no basis element of $S_4(\Gamma_0(13))$ that can be determined using Theorem 2.2.
3.2. **Evaluation of** $W_{(\alpha, \beta)}(n)$ **when** $\alpha \beta = 44, 52$.

**Lemma 3.2.** We have

\begin{equation}
(3.2) \quad (L(q) - 44L(q^{44}))^2 = 1849 + \sum_{n=1}^{\infty} \left( \frac{124464}{61} \sigma_3(n) - \frac{577662336}{40565} \sigma_3\left(\frac{n}{2}\right) \right)
\end{equation}

\begin{equation}
+ \frac{68986368}{5795} \sigma_3\left(\frac{n}{4}\right) - \frac{174240}{61} \sigma_3\left(\frac{n}{11}\right) + \frac{62064288}{5795} \sigma_3\left(\frac{n}{22}\right) + \frac{2525690112}{5795} \sigma_3\left(\frac{n}{44}\right)
\end{equation}

\begin{equation}
+ \frac{1440}{61} a_1(n) - \frac{82927872}{5795} a_2(n) - \frac{887345568}{5795} a_3(n) - \frac{1676429568}{5795} a_4(n)
\end{equation}

\begin{equation}
- \frac{2804007168}{5795} a_5(n) + \frac{3753380736}{5795} a_6(n) - \frac{13356288}{19} a_7(n) + \frac{4226609664}{5795} a_8(n)
\end{equation}

\begin{equation}
- \frac{633600}{19} a_9(n) - \frac{527332608}{1159} a_{10}(n) + \frac{7679232}{19} a_{11}(n) - \frac{12595968}{95} a_{12}(n)
\end{equation}

\begin{equation}
- \frac{131079168}{95} a_{13}(n) + \frac{317952}{19} a_{14}(n) - \frac{131079168}{95} a_{15}(n) \right) q^n,
\end{equation}

\begin{equation}
(3.3) \quad (4L(q^4) - 11L(q^{11}))^2 = 49 + \sum_{n=1}^{\infty} \left( - \frac{110880}{61} \sigma_3(n) + \frac{80121888}{5795} \sigma_3\left(\frac{n}{2}\right) \right)
\end{equation}

\begin{equation}
- \frac{48338688}{5795} \sigma_3\left(\frac{n}{4}\right) + \frac{1817904}{61} \sigma_3\left(\frac{n}{11}\right) - \frac{98480448}{5795} \sigma_3\left(\frac{n}{22}\right) + \frac{27320832}{5795} \sigma_3\left(\frac{n}{44}\right)
\end{equation}

\begin{equation}
+ \frac{110880}{61} a_1(n) + \frac{174857472}{5795} a_2(n) + \frac{1169427168}{5795} a_3(n) - \frac{2114189568}{5795} a_4(n)
\end{equation}

\begin{equation}
+ \frac{3025513728}{5795} a_5(n) - \frac{3511806576}{5795} a_6(n) - \frac{13318272}{19} a_7(n) + \frac{3641762304}{5795} a_8(n)
\end{equation}

\begin{equation}
+ \frac{633600}{19} a_9(n) + \frac{663913728}{1159} a_{10}(n) - \frac{7679232}{19} a_{11}(n) + \frac{15231744}{95} a_{12}(n)
\end{equation}

\begin{equation}
+ \frac{131079168}{95} a_{13}(n) - \frac{317952}{19} a_{14}(n) + \frac{12595968}{95} a_{15}(n) \right) q^n,
\end{equation}
(3.4) \((L(q) - 52L(q^{52}))^2 = 2601 + \sum_{n=1}^{\infty} \left( \frac{6109008}{1243}\sigma_3(n) - \frac{456504088416}{6064597}\sigma_3\left(\frac{n}{2}\right) \right. \\
+ \frac{254592}{41}\sigma_3\left(\frac{n}{4}\right) - \frac{7361952}{1243}\sigma_3\left(\frac{n}{13}\right) - \frac{4829528827344}{6064597}\sigma_3\left(\frac{n}{26}\right) \\
+ \frac{434738304}{41}\sigma_3\left(\frac{n}{52}\right) - \frac{3066144}{1243}b_1(n) + \frac{498157179048}{6064597}b_2(n) \\
+ \frac{927327070704}{6064597}b_3(n) - \frac{442577500560}{6064597}b_4(n) - \frac{8530413669648}{6064597}b_5(n) \\
- \frac{10161699732288}{6064597}b_6(n) - \frac{10388366352}{1243}b_7(n) + \frac{1040832}{41}\sigma_3\left(\frac{n}{2}\right) \\
+ \frac{7488b_9(n) + 329100929664}{147917}b_10(n) + 27456b_11(n) \\
- \frac{15249288510144}{6064597}b_{12}(n) + 17472b_{13}(n) + \frac{47009664}{41}b_{14}(n) \\
- \frac{25166713896}{551327}b_{15}(n) + \frac{4167031826880}{6064597}b_{16}(n) - \frac{126425023920}{6064597}b_{17}(n) \\
+ \frac{868608}{41}b_{18}(n) \left) q^n, \right.

(3.5) \((4L(q^4) - 13L(q^{13}))^2 = 81 + \sum_{n=1}^{\infty} \left( \frac{3066144}{1243}\sigma_3(n) - \frac{240061230672}{6064597}\sigma_3\left(\frac{n}{2}\right) \right. \\
+ \frac{139392}{41}\sigma_3\left(\frac{n}{4}\right) + \frac{45798672}{1243}\sigma_3\left(\frac{n}{13}\right) - \frac{53922031824}{6064597}\sigma_3\left(\frac{n}{26}\right) \\
+ \frac{20290176}{41}\sigma_3\left(\frac{n}{52}\right) - \frac{3066144}{1243}b_1(n) + \frac{212735819880}{6064597}b_2(n) \\
+ \frac{251848851024}{6064597}b_3(n) - \frac{400561037808}{6064597}b_4(n) - \frac{5152459820400}{6064597}b_5(n) \\
- \frac{5489355312}{1243}b_6(n) - \frac{6064597}{540874831292}b_7(n) + \frac{150336}{41}b_8(n) \\
- \frac{7488b_9(n) + 151016538432}{147917}b_{10}(n) - 27456b_{11}(n) \\
- \frac{8224832431680}{6064597}b_{12}(n) - 17472b_{13}(n) - \frac{544896}{41}b_{14}(n) \\
- \frac{11115614088}{551327}b_{15}(n) + \frac{2056953609600}{6064597}b_{16}(n) - \frac{64745693328}{6064597}b_{17}(n) \\
- \frac{2304}{41}b_{18}(n) \left) q^n. \right.

**Proof.** We just prove the case \((4L(q^4) - 11L(q^{11}))^2\). The other cases are proved similarly.

From Lemma 2.3 it follows that \((4L(q^4) - 11L(q^{11}))^2 \in M_4(\Gamma_0(44))\). Hence, by Theorem 3.1(c), there exist \(X_8, Y_j \in \mathbb{C}, 1 \leq j \leq 15\) and \(\delta \in D(44)\), such that

(3.6) \((4L(q^4) - 11L(q^{11}))^2 = \sum_{\delta | 44} X_\delta M(q^\delta) + \sum_{j=1}^{15} Y_j A_j(q) \\
= \sum_{\delta | 44} X_\delta + \sum_{n=1}^{\infty} \left( 240 \sum_{\delta | 44} \sigma_3\left(\frac{n}{\delta}\right) X_\delta + \sum_{j=1}^{m_5} a_j(n) Y_j \right) q^n. \)
We compare the right hand side of Equation 3.6 with that of Equation 2.3 when we have set \((\alpha, \beta) = (4, 11)\) to obtain

\[
\sum_{n=1}^{\infty} \left( 240 \sum_{j=1}^{15} a_j(n) Y_j \right) q^n = \sum_{n=1}^{\infty} \left( 3840 \sigma_3 \left( \frac{n}{4} \right) + 29040 \sigma_3 \left( \frac{n}{11} \right) + 192 (11 - 6n) \sigma_1 \left( \frac{n}{4} \right) + 528 (4 - 6n) \sigma_1 \left( \frac{n}{11} \right) - 50688 W_{(4,11)}(n) \right) q^n.
\]

We then take the coefficients of \(q^n\) for which \(n\) is in

\[\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 22, 44\}\]

to obtain a system of linear equations whose resolution using a software package for symbolic scientific computation yields the unique solution which determines the values of the unknowns \(X_\delta\) for all \(\delta \in D(44)\) and the values of the unknowns \(Y_j\) for all \(1 \leq j \leq 15\). Therefore, we get the stated result.

\[\square\]

Our main result of this section will now be stated and proved.

**Theorem 3.3.** Let \(n\) be a positive integer. Then

\[
W_{(4,44)}(n) = -\frac{13}{366} \sigma_3(n) + \frac{501443}{1784860} \sigma_3 \left( \frac{n}{2} \right) - \frac{1361}{5795} \sigma_3 \left( \frac{n}{4} \right) + \frac{55}{976} \sigma_3 \left( \frac{n}{11} \right) - \frac{19591}{92720} \sigma_3 \left( \frac{n}{22} \right) + \frac{9878}{17385} \sigma_3 \left( \frac{n}{44} \right) + \left( \frac{1}{24} - \frac{1}{176} \right)n \sigma(n)
\]

\[\sum_{j=1}^{15} a_j(n) Y_j = \frac{3081061}{1019920} a_3(n) + \frac{66147}{11590} a_4(n) + \frac{1217017}{127490} a_5(n) - \frac{3258143}{254980} a_6(n) + \frac{527}{38} a_7(n) - \frac{917233}{63745} a_8(n) + \frac{25}{38} a_9(n) + \frac{20807}{2318} a_{10}(n)
\]

\[\frac{303}{38} a_{11}(n) + \frac{601}{190} a_{12}(n) + \frac{2586}{95} a_{13}(n) - \frac{69}{209} a_{14}(n) + \frac{497}{190} a_{15}(n).
\]

(3.7)

\[
W_{(4,11)}(n) = \frac{35}{976} \sigma_3(n) - \frac{25291}{92720} \sigma_3 \left( \frac{n}{2} \right) + \frac{4178}{17385} \sigma_3 \left( \frac{n}{4} \right) - \frac{11}{732} \sigma_3 \left( \frac{n}{11} \right) + \frac{15543}{46360} \sigma_3 \left( \frac{n}{22} \right) + \frac{539}{5795} \sigma_3 \left( \frac{n}{44} \right) + \left( \frac{1}{24} - \frac{1}{44} \right) n \sigma(n)
\]

\[\sum_{j=1}^{15} a_j(n) Y_j = -\frac{4060511}{1019920} a_3(n) - \frac{917617}{127490} a_4(n) - \frac{1313157}{127490} a_5(n) + \frac{3047813}{254980} a_6(n) - \frac{497}{76} a_7(n) + \frac{790313}{63745} a_8(n) - \frac{25}{38} a_9(n) - \frac{288157}{254980} a_{10}(n)
\]

\[\frac{303}{38} a_{11}(n) - \frac{601}{190} a_{12}(n) - \frac{2586}{95} a_{13}(n) + \frac{69}{209} a_{14}(n) - \frac{497}{190} a_{15}(n).
\]

(3.8)
We prove the case 
\[ W_{(1,52)}(n) = - \frac{97}{1243} \sigma_3(n) + \frac{731577059}{582201312} \sigma_3\left(\frac{n}{2}\right) - \frac{17}{164} \sigma_3\left(\frac{n}{4}\right) + \frac{5899}{59664} \sigma_3\left(\frac{n}{13}\right) + \frac{582201312}{7739629531} \sigma_3\left(\frac{n}{26}\right) - \frac{1}{7} \sigma_3\left(\frac{n}{32}\right) + \frac{81757}{492} b_1(n) + \left(\frac{1}{24} - \frac{1}{208}\right) n \sigma(n)
\]
\[ + \left(\frac{1}{24} - \frac{7}{4} n\right) \sigma(n) + \frac{31939}{775632} b_1(n) - \frac{1931913973}{7568617056} b_5(n) + \frac{236419605}{194067104} b_4(n) + \frac{4556844909}{194067104} b_5(n) - 1357064601 b_6(n) + 5549341 b_7(n) - \frac{139}{328} b_6(n)
\]
\[ - \frac{1}{8} b_9(n) - \frac{1775004}{349537693} b_{10}(n) - \frac{23}{24} b_1(n) + \frac{2036496863}{48516776} b_{12}(n)
\]
\[ - \frac{7}{24} b_{13}(n) - \frac{31939}{164} b_{14}(n) + \frac{556494635}{48516776} b_{16}(n) + \frac{67534735}{194067104} b_{17}(n) - \frac{29}{82} b_{18}(n),
\]

\( (3.9) \)

\[
W_{(4,13)}(n) = - \frac{31939}{775632} \sigma_3(n) + \frac{4501275639}{7568617056} \sigma_3\left(\frac{n}{2}\right) + \frac{47}{6396} \sigma_3\left(\frac{n}{4}\right) + \frac{24049}{387816} \sigma_3\left(\frac{n}{13}\right) + \frac{1123375663}{7568617056} \sigma_3\left(\frac{n}{26}\right) - \frac{17613}{2132} \sigma_3\left(\frac{n}{52}\right) + \left(\frac{1}{24} - \frac{1}{52}\right) n \sigma(n)
\]
\[ + \left(\frac{1}{24} - \frac{1}{52}\right) n \sigma(n) + \frac{31939}{775632} b_1(n) - \frac{2954664165}{5045744704} b_2(n)
\]
\[ - \frac{5246851063}{7568617056} b_3(n) + \frac{8345021621}{7568617056} b_4(n) + \frac{3578097095}{2522872352} b_5(n)
\]
\[ + \frac{4695094021}{315359044} b_6(n) + \frac{38120523}{517088} b_7(n) - \frac{261}{4264} b_8(n)
\]
\[ + \frac{7}{8} b_9(n) - \frac{8}{46150104} b_{10}(n) + \frac{11}{24} b_{11}(n) + \frac{42837668915}{1892154264} b_{12}(n)
\]
\[ + \frac{7}{24} b_{13}(n) + \frac{473}{2132} b_{14}(n) + \frac{154383529}{458704064} b_{15}(n)
\]
\[ - \frac{5356650025}{946077132} b_{16}(n) + \frac{1348868611}{7568617056} b_{17}(n) + \frac{1}{1066} b_{18}(n).
\]

\( (3.10) \)

**Proof.** We prove the case \( W_{(4,13)}(n) \) as the other cases are proved similarly.
We compare the right hand side of Equation 3.5 with that of Equation 2.3 when we have set \((\alpha, \beta) = (4, 13)\), namely

\[
\sum_{n=1}^{\infty} \left( 3840\sigma_3\left(\frac{n}{4}\right) + 40560\sigma_3\left(\frac{n}{13}\right) + 192 (13 - 6n) \sigma\left(\frac{n}{4}\right) \\
+ 624 (4 - 6n) \sigma\left(\frac{n}{13}\right) - 59904 W_{(4,13)}(n) \right) q^n = \\
\sum_{n=1}^{\infty} \left( \frac{3066144}{1243} \sigma(n) - \frac{240061230672}{6064597} \sigma_3\left(\frac{n}{2}\right) + \frac{139392}{41} \sigma_3\left(\frac{n}{4}\right) \\
+ \frac{45798672}{1243} \sigma_3\left(\frac{n}{13}\right) - \frac{53922031824}{6064597} \sigma_3\left(\frac{n}{26}\right) + \frac{20290176}{41} \sigma_3\left(\frac{n}{52}\right) \\
- \frac{3066144}{1243} b_1(n) + \frac{212735819880}{6064597} b_2(n) + \frac{251848851024}{6064597} b_3(n) - \frac{400561037808}{6064597} b_4(n) \\
- \frac{5152459820400}{6064597} b_5(n) - \frac{5408748312192}{6064597} b_6(n) - \frac{5489355312}{1243} b_7(n) \\
+ \frac{150336}{41} b_8(n) - 7488 b_9(n) + \frac{151016358432}{147917} b_{10}(n) - 27456 b_{11}(n) \\
- \frac{8224832431680}{6064597} b_{12}(n) - 17472 b_{13}(n) - \frac{544896}{41} b_{14}(n) - \frac{11115614088}{551327} b_{15}(n) \\
+ \frac{2056953609600}{6064597} b_{16}(n) - \frac{64745693328}{6064597} b_{17}(n) - \frac{2304}{41} b_{18}(n) \right) q^n.
\]

We obtain the stated result when we solve for \(W_{(4,13)}(n)\). □

4. Number of Representations of a Positive Integer \(n\) by the Octonary Quadratic Form Using \(W_{(\alpha,\beta)}(n)\) when \(\alpha \beta = 44, 52\)

Let \(n \in \mathbb{N}_0\) and then assume that \(r_4(n)\) denote the number of representations of \(n\) by the quaternary quadratic form \(x_1^2 + x_2^2 + x_3^2 + x_4^2\) which is defined by

\[
 r_4(n) = \text{card}\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = x_1^2 + x_2^2 + x_3^2 + x_4^2 \}.
\]

Obviously \(r_4(0) = 1\). The Jacobi’s identity

\[
(4.1) \quad \forall n \in \mathbb{N} \quad r_4(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right).
\]

is proved in K. S. Williams’ book [28, Thrm 9.5, p. 83]; it will be very useful in the following.

Let \(a, b \in \mathbb{N}\) and let \(N_{(a,b)}(n)\) denote the number of representations of \(n\) by the octonary quadratic form

\[
a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2)
\]

which is defined by

\[
N_{(a,b)}(n) = \text{card}\{ (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid n = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2) \}.
\]

We then deduce the following result.
Theorem 4.1. Let \( n \in \mathbb{N} \) and \((a, b) = (1, 11), (1, 13)\). Then
\[
N_{(1,11)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{11}\right) - 32\sigma\left(\frac{n}{52}\right)
+ 64 W_{(1,11)}(n) + 1024 W_{(1,11)}\left(\frac{n}{4}\right) - 256 \left(W_{(4,11)}(n) + W_{(1,44)}(n)\right),
\]
\[
N_{(1,13)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{13}\right) - 32\sigma\left(\frac{n}{52}\right)
+ 64 W_{(1,13)}(n) + 1024 W_{(1,13)}\left(\frac{n}{4}\right) - 256 \left(W_{(4,13)}(n) + W_{(1,52)}(n)\right).
\]

Proof. We only prove \(N_{(1,11)}(n)\) since that for \(N_{(1,13)}(n)\) is done similarly.

From the definition of \(N_{(1,11)}(n)\) it follows that
\[
N_{(1,11)}(n) = \sum_{(l,m) \in \mathbb{N}^2 \atop l+11m=n} r_4(l)r_4(m) = r_4(n) + r_4(0) + \sum_{(l,m) \in \mathbb{N}^2 \atop l+11m=n} r_4(l) r_4(m).
\]

We apply Equation 4.1 to derive
\[
N_{(1,11)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{11}\right) - 32\sigma\left(\frac{n}{52}\right)
+ \sum_{(l,m) \in \mathbb{N}^2 \atop l+11m=n} (8\sigma(l) - 32\sigma\left(\frac{l}{4}\right))(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)).
\]

From this previous identity we observe that
\[
(8\sigma(l) - 32\sigma\left(\frac{l}{4}\right))(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)) = 64\sigma(l)\sigma(m) - 256\sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right)
- 256\sigma(l)\sigma\left(\frac{m}{4}\right) + 1024\sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right).
\]

E. Royer [24] Thm 1.3 has shown the evaluation of
\[
W_{(1,11)}(n) = \sum_{(l,m) \in \mathbb{N}^2 \atop l+11m=n} \sigma(l)\sigma(m).
\]

When we assign \(4l\) to \(l\), then we infer
\[
W_{(4,11)}(n) = \sum_{(l,m) \in \mathbb{N}^2 \atop l+11m=n} \sigma\left(\frac{l}{4}\right)\sigma(m) = \sum_{(l,m) \in \mathbb{N}^2 \atop 4l+11m=n} \sigma(l)\sigma(m).
\]
The evaluation of \(W_{(4,11)}(n)\) is given in Equation 3.8. When we next assign \(4m\) to \(m\), we conclude that
\[
W_{(1,44)}(n) = \sum_{(l,m) \in \mathbb{N}^2 \atop l+11m=n} \sigma(l)\sigma\left(\frac{m}{4}\right) = \sum_{(l,m) \in \mathbb{N}^2 \atop 4l+4m=n} \sigma(l)\sigma(m).
\]
The evaluation of \(W_{(1,44)}(n)\) is provided by Equation 3.7. When we simultaneously assign \(4l\) to \(l\) and \(4m\) to \(m\), we deduce that
\[
\sum_{(l,m) \in \mathbb{N}^2 \atop l+11m=n} \sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right) = \sum_{(l,m) \in \mathbb{N}^2 \atop l+11m=n} \sigma(l)\sigma(m) = W_{(1,11)}\left(\frac{n}{4}\right).
\]

Again, E. Royer [24] Thrm 1.3 has proved the evaluation of \(W_{(1,11)}(n)\).

We then bring these evaluations together to obtain the stated result for \(N_{(1,11)}(n)\). □
REFERENCES

[1] A. Alaca, Ş. Alaca, and E. Ntienjem. Evaluation of the Convolution Sum involving the Sum of Divisors Function for 14, 22 and 26. ArXiv e-prints, apr 2016.

[2] A. Alaca, Ş. Alaca, and K. S. Williams. Evaluation of the convolution sums \( \sum_{l \mid 12m=n} \sigma(l)\sigma(m) \) and \( \sum_{3l \mid 4m=n} \sigma(l)\sigma(m) \). Adv Theor Appl Math, 1(1):27–48, 2006.

[3] A. Alaca, Ş. Alaca, and K. S. Williams. Evaluation of the convolution sum \( \sum_{l \mid 15m=n} \sigma(l)\sigma(m) \) and \( \sum_{2l \mid 3m=n} \sigma(l)\sigma(m) \). Int Math Forum, 2(2):45–68, 2007.

[4] A. Alaca, Ş. Alaca, and K. S. Williams. Evaluation of the convolution sums \( \sum_{l \mid 24m=n} \sigma(l)\sigma(m) \) and \( \sum_{3l \mid 5m=n} \sigma(l)\sigma(m) \). Math J Okayama Univ, 49:93–111, 2007.

[5] A. Alaca, Ş. Alaca, and K. S. Williams. The convolution sum \( \sum_{m \leq \frac{n}{m}} \sigma(m)\sigma(n-16m) \). Canad Math Bull, 51(1):3–14, 2008.

[6] Ş. Alaca and Y. Kesicioglu. Evaluation of the convolution sums \( \sum_{l \mid 27m=n} \sigma(l)\sigma(m) \) and \( \sum_{l \mid 32m=n} \sigma(l)\sigma(m) \). Int J Number Theory, 12(1):1–13, 2016.

[7] Ş. Alaca and K. S. Williams. Evaluation of the convolution sums \( \sum_{l \mid 6m=n} \sigma(l)\sigma(m) \) and \( \sum_{2l \mid 3m=n} \sigma(l)\sigma(m) \). J Number Theory, 124(2):490–510, 2007.

[8] M. Besge. Extrait d’une lettre de M Besge à M Liouville. J Math Pure Appl, 4:7, 1852.

[9] H. H. Chan and S. Cooper. Powers of theta functions. Proc J Math, 235:1–14, 2008.

[10] S. Cooper and P. C. Toh. Quintic and septic Eisenstein series. Ramanujan J, 19:163–181, 2009.

[11] S. Cooper and D. Ye. Evaluation of the convolution sums \( \sum_{l \mid 28m=n} \sigma(l)\sigma(m) \), \( \sum_{4l \mid 5m=n} \sigma(l)\sigma(m) \) and \( \sum_{2l \mid 3m=n} \sigma(l)\sigma(m) \). Int J Number Theory, 10(6):1386–1394, 2014.

[12] J. W. L. Glaisher. On the square of the series in which the coefficients are the sums of the divisors of the exponents. Messenger Math, 14:156–163, 1862.

[13] J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams. Elementary evaluation of certain convolution sums involving divisor functions. Number Theory Millenium, 7:229–274, 2002. A K Peters, Natick, MA.

[14] L. J. P. Kilford. Modular forms: A classical and computational introduction. Imperial College Press, London, 2008.

[15] N. Koblitz. Introduction to Elliptic Curves and Modular Forms, volume 97 of Graduate Texts in Mathematics. Springer Verlag, New York, 2 edition, 1993.

[16] G. Köhler. Eta Products and Theta Series Identities, volume 3733 of Springer Monographs in Mathematics. Springer Verlag, Berlin Heidelberg, 2011.

[17] M. Lemire and K. S. Williams. Evaluation of two convolution sums involving the sum of divisors function. Bull Aust Math Soc, 73:107–115, 2006.

[18] G. Ligozat. Courbes modulaires de genre 1. Bull Soc Math France, 143:5–80, 1975.

[19] T. Miyake. Modular Forms. Springer monographs in Mathematics. Springer Verlag, New York, 1989.

[20] M. Newman. Construction and application of a class of modular functions. Proc Lond Math Soc, 7(3):334–350, 1957.

[21] M. Newman. Construction and application of a class of modular functions II. Proc Lond Math Soc, 9(3):373–387, 1959.

[22] B. Ramakrishnan and B. Sahu. Evaluation of the convolution sums \( \sum_{l \mid 15m=n} \sigma(l)\sigma(m) \) and \( \sum_{3l \mid 5m=n} \sigma(l)\sigma(m) \). Int J Number Theory, 9(3):799–809, 2013.

[23] S. Ramanujan. On certain arithmetical functions. T Cambridge Phil Soc, 22:159–184, 1916.

[24] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. Int J Number Theory, 3(2):231–261, 2007.

[25] W. A. Stein. Modular Forms, A Computational Approach, volume 79. American Mathematical Society, Graduate Studies in Mathematics, 2011. http://wstein.org/books/modform/modform/.

[26] K. S. Williams. The convolution sum \( \sum_{m \leq \frac{n}{m}} \sigma(m)\sigma(n-9m) \). Int J Number Theory, 1(2):193–205, 2005.

[27] K. S. Williams. The convolution sum \( \sum_{m \leq \frac{n}{m}} \sigma(m)\sigma(n-8m) \). Pac J Math, 228:387–396, 2006.
[28] K. S. Williams. *Number Theory in the Spirit of Liouville*, volume 76 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2011.

[29] E. X. W. Xia, X. L. Tian, and O. X. M. Yao. Evaluation of the convolution sum \( \sum_{l+25m=n} \sigma(l)\sigma(m) \). *Int J Number Theory*, 10(6):1421–1430, 2014.

[30] D. Ye. Evaluation of the convolution sums \( \sum_{l+36m=n} \sigma(l)\sigma(m) \) and \( \sum_{4l+9m=n} \sigma(l)\sigma(m) \). *Int J Number Theory*, 11(1):171–183, 2015.

### Tables

| \((\alpha, \beta)\) | Authors | References |
|------------------|---------|------------|
| (1,1)            | M. Besge, J. W. L. Glaisher, S. Ramanujan | [8, 12, 23] |
| (1,2),(1,3),(1,4) | J. G. Huard & Z. M. Ou & B. K. Spearman & K. S. Williams | [13] |
| (1,5),(1,7)      | M. Lemire & K. S. Williams, S. Cooper & P. C. Toh | [17, 19] |
| (1,6),(2,3)      | S. Alaca & K. S. Williams | [7] |
| (1,8), (1,9)     | K. S. Williams | [27, 26] |
| (1,10), (1,11),(1,13), (1,14) | E. Royer | [24] |
| (1,12),(1,16),(1,18), (1,24),(2,9),(3,4), (3,8) | A. Alaca & S. Alaca & K. S. Williams | [2, 3, 4, 5] |
| (1,15),(3,5)     | B. Ramakrishman & B. Sahu | [22] |
| (1,20),(2,5),(4,5) | S. Cooper & D. Ye | [11] |
| (1,23)           | H. H. Chan & S. Cooper | [9] |
| (1,25)           | E. X. W. Xia & X. L. Tian & O. X. M. Yao | [29] |
| (1,27),(1,32)    | S. Alaca & Y. Kesicioğlu | [6] |
| (1,36),(4,9)     | D. Ye | [30] |
| (1,22),(1,26),(2,7), (2,11),(2,13) | A. Alaca & Ş. Alaca & E. Ntienjem | [1] |

Table 1: Known convolution sums \( W_{(\alpha, \beta)}(n) \)

| \((a, b)\) | Authors | References |
|-----------|---------|------------|
| (1,1),(1,3), (1,9),(2,3) | A. Alaca & Ş. Alaca & E. Ntienjem | [1] |
| (1,2)     | K. S. Williams | [27] |
| (1,4)     | A. Alaca & S. Alaca & K. S. Williams | [3] |
| (1,5)     | S. Cooper & D. Ye | [11] |
| (1,6)     | B. Ramakrishman & B. Sahu | [22] |
| (1,8)     | S. Alaca & Y. Kesicioğlu | [6] |

Table 2: Known representations of \( n \) by the form \( \text{Equation 1.3} \)

### 1 | 2 | 4 | 13 | 26 | 52
Table 3: Power of \( \eta \)-functions being basis elements for \( S_4(\Gamma_0(52)) \)

|   | 1   | 2   | 3   | 4   | 5   | 6   |
|---|-----|-----|-----|-----|-----|-----|
| 1 | 1   | 5   | 0   | 3   | -1  | 0   |
| 2 | 3   | 3   | 0   | 1   | 1   | 0   |
| 3 | 1   | 3   | 0   | 3   | 1   | 0   |
| 4 | 3   | 1   | 0   | 1   | 3   | 0   |
| 5 | 1   | 1   | 0   | 3   | 3   | 0   |
| 6 | 3   | -1  | 0   | 1   | 5   | 0   |
| 7 | 1   | -1  | 0   | 3   | 5   | 0   |
| 8 | 0   | 3   | 1   | 0   | 1   | 3   |
| 9 | 2   | 1   | 1   | -2  | 3   | 3   |
|10 | 0   | 1   | 1   | 0   | 3   | 3   |
|11 | 2   | -1  | 1   | -2  | 5   | 3   |
|12 | 0   | 3   | -1  | 0   | 1   | 5   |
|13 | 2   | 1   | -1  | -2  | 3   | 5   |
|14 | 0   | 1   | -1  | 0   | 3   | 5   |
|15 | -1  | 5   | 0   | 5   | -1  | 0   |
|16 | 0   | -1  | 5   | 0   | 5   | -1  |
|17 | 7   | -3  | 0   | -3  | 7   | 0   |
|18 | 0   | 7   | -3  | 0   | -3  | 7   |

Table 4: Power of \( \eta \)-functions being basis elements for \( S_4(\Gamma_0(44)) \)

|   | 1   | 2   | 4   | 11  | 22  | 44  |
|---|-----|-----|-----|-----|-----|-----|
| 1 | 6   | -2  | 0   | 6   | -2  | 0   |
| 2 | 4   | 0   | 0   | 4   | 0   | 0   |
| 3 | 2   | 2   | 0   | 2   | 2   | 0   |
| 4 | 0   | 4   | 0   | 0   | 4   | 0   |
| 5 | -2  | 6   | 0   | -2  | 6   | 0   |
| 6 | 0   | 2   | 2   | 0   | 2   | 2   |
| 7 | 0   | -3  | 5   | 0   | 5   | 1   |
| 8 | 0   | 0   | 4   | 0   | 0   | 4   |
| 9 | 3   | 0   | 1   | -1  | 0   | 5   |
|10 | 0   | -2  | 6   | 0   | -2  | 6   |
|11 | 1   | -3  | 4   | -3  | 5   | 4   |
|12 | 2   | 0   | 0   | 2   | -4  | 8   |
|13 | 0   | 2   | 0   | 0   | -2  | 8   |
|14 | -3  | 9   | 0   | 1   | 1   | 0   |
|15 | 0   | 0   | 2   | 0   | -4  | 10  |