Noncommutative Multi-Instantons on $\mathbb{R}^{2n} \times S^2$

Tatiana A. Ivanova$\dagger$ and Olaf Lechtenfeld$\ast$

$\dagger$Bogoliubov Laboratory of Theoretical Physics, JINR
141980 Dubna, Moscow Region, Russia
Email: ita@thsun1.jinr.ru

$\ast$Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
Email: lechtenf@itp.uni-hannover.de

Abstract

Generalizing self-duality on $\mathbb{R}^2 \times S^2$ to higher dimensions, we consider the Donaldson-Uhlenbeck-Yau equations on $\mathbb{R}^{2n} \times S^2$ and their noncommutative deformation for the gauge group $U(2)$. Imposing SO(3) invariance (up to gauge transformations) reduces these equations to vortex-type equations for an abelian gauge field and a complex scalar on $\mathbb{R}^{2n}_\theta$. For a special $S^2$-radius $R$ depending on the noncommutativity $\theta$ we find explicit solutions in terms of shift operators. These vortex-like configurations on $\mathbb{R}^{2n}_\theta$ determine SO(3)-invariant multi-instantons on $\mathbb{R}^{2n}_\theta \times S^2_R$ for $R = R(\theta)$. The latter may be interpreted as sub-branes of codimension $2n$ inside a coincident pair of noncommutative Dp-branes with an $S^2$ factor of suitable size.
1 Introduction

Noncommutative deformation is a well established framework for stretching the limits of conventional (classical and quantum) field theories [1, 2]. On the nonperturbative side, all celebrated classical field configurations have been generalized to the noncommutative realm. Of particular interest thereof are BPS configurations, which are subject to first-order nonlinear equations. The latter descend from the 4d Yang-Mills (YM) self-duality equations and have given rise to instantons [3], monopoles [4] and vortices [5], among others. Their noncommutative counterparts were introduced in [6], [7] and [8], respectively, and have been studied intensely for the past five years (see [9] for a recent review).

String/M theory embeds these efforts in a higher-dimensional context, and so it is important to formulate BPS-type equations in more than four dimensions. In fact, noncommutative instantons in higher dimensions and their brane interpretations have recently been considered in [10, 11, 12]. Yet already 20 years ago, generalized self-duality equations for YM fields in more than four dimensions were proposed [13, 14] and their solutions investigated e.g. in [14, 15]. For U(k) gauge theory on a Kähler manifold these equations specialize to the Donaldson-Uhlenbeck-Yau (DUY) equations [16, 17]. They are the natural analogues of the 4d self-duality equations.

In this letter we generalize the DUY equations to the noncommutative spaces $\mathbb{R}^{2n}_\theta \times S^2$ and construct explicit U(2) multi-instanton solutions even though these equations are not integrable. The key lies in a clever ansatz for the gauge potential, due to Taubes [5], which we generalize to higher dimensions and to the noncommutative setting. This SO(3)-invariant ansatz reduces the U(2) DUY equations to vortex-type equations on $\mathbb{R}^{2n}_\theta$. For $n=1$ the latter are the standard vortex equations on $\mathbb{R}^{2}_\theta$, while for $n=2$ they are intimately related to the Seiberg-Witten monopole equations on $\mathbb{R}^{4}_\theta$ [18].

2 Donaldson-Uhlenbeck-Yau equations on $\mathbb{R}^{2n}_\theta \times S^2$

Manifold $\mathbb{R}^{2n}_\theta \times S^2$. We consider the manifold $\mathbb{R}^{2n}_\theta \times S^2$ with the Riemannian metric

$$ds^2 = \sum_{\mu,\nu=1}^{2n} \delta_{\mu\nu} dx^\mu dx^\nu + R^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) = \sum_{i,j=1}^{2n+2} g_{ij} \, dx^i dx^j,$$  (2.1)

where $x^1, \ldots, x^{2n}$ are coordinates on $\mathbb{R}^{2n}$ while $x^{2n+1} = \vartheta$ and $x^{2n+2} = \varphi$ parametrize the standard two-sphere $S^2$ with constant radius $R$, i.e. $0 \leq \varphi \leq 2\pi$ and $0 \leq \vartheta \leq \pi$. The volume two-form on $S^2$ reads

$$\sqrt{\det(g_{ij})} \, d\vartheta \wedge d\varphi =: \omega_{\vartheta\varphi} \, d\vartheta \wedge d\varphi = \omega \quad \implies \quad \omega_{\vartheta\varphi} = -\omega_{\varphi\vartheta} = R^2 \sin \vartheta \, .$$  (2.2)

The manifold $\mathbb{R}^{2n}_\theta \times S^2$ is Kähler, with local complex coordinates $z^1, \ldots, z^n, y$ where

$$z^a = x^{2a-1} - i x^{2a} \quad \text{and} \quad \bar{z}^\bar{a} = x^{2a-1} + i x^{2a} \quad \text{with} \quad a = 1, \ldots, n$$  (2.3)

and

$$y = \frac{R \sin \vartheta}{(1 + \cos \vartheta)} \exp(-i\varphi), \quad \bar{y} = \frac{R \sin \vartheta}{(1 + \cos \vartheta)} \exp(i\varphi) \, .$$  (2.4)
so that \( 1 + \cos \vartheta = \frac{2R^2}{R^2 + y^2} \). In these coordinates, the metric takes the form\(^1\)

\[
ds^2 = \delta_{ab} \, dz^a \, dz^b + \frac{4R^4}{(R^2 + y^2)^2} \, dy \, d\bar{y}
\]

with \( \delta_{aa} = \delta^{aa} = 1 \) (other entries vanish), and the Kähler two-form reads

\[
\Omega = -i \frac{1}{2} \{ \delta_{ab} \, dz^a \wedge d\bar{z}^b + \frac{4R^4}{(R^2 + y^2)^2} \, dy \wedge d\bar{y} \} = -i \frac{1}{2} \delta_{ab} \, dz^a \wedge d\bar{z}^b + \omega_{\vartheta \varphi} \, d\vartheta \wedge d\varphi .
\]

For later use, we also note here the derivatives

\[
\partial_{z^a} = \frac{1}{2}(\partial_{2a-1} - i \partial_{2a}) \quad \text{and} \quad \partial_{\bar{z}^a} = \frac{1}{2}(\partial_{2a-1} + i \partial_{2a}),
\]

where \( \partial_{\mu} \equiv \partial/\partial x^\mu \) for \( \mu = 1, \ldots, 2n \).

Classical field theory on the noncommutative deformation \( R^2_n \theta \) of \( R^2 \) may be realized in a star-product formulation or in an operator formalism. While the first approach alters the product of functions on \( R^2 \) the second one turns these functions into linear operators \( \hat{f} \) acting on the \( n \)-harmonic-oscillator Fock space \( \mathcal{H} \). The noncommutative space \( R^2_n \theta \) may then be defined by declaring its coordinate functions \( \hat{x}^1, \ldots, \hat{x}^{2n} \) to obey the Heisenberg algebra relations

\[
[\hat{x}^a , \hat{x}^b] = i \theta^{ab}
\]

with a constant antisymmetric tensor \( \theta^{ab} \). The coordinates can be chosen in such a way that the matrix \( (\theta^{ab}) \) will be block-diagonal with non-vanishing components

\[
\theta^{2a-1 \, 2a} = -\theta^{2a \, 2a-1} =: \theta^a .
\]

We assume that all \( \theta^a \geq 0 \); the general case does not hide additional complications. For the noncommutative version of the complex coordinates (2.3) we have

\[
[\hat{z}^a , \hat{\bar{z}}^b] = -2\delta^{ab} \theta^a =: \theta^{\hat{a} \hat{b}} = -\theta^{\hat{b} \hat{a}} \leq 0 , \quad \text{and all other commutators vanish} .
\]

The Fock space \( \mathcal{H} \) is spanned by the basis states

\[
|k_1, k_2, \ldots, k_n \rangle = \prod_{a=1}^n (2\theta^a k_a!)^{-1/2} (\hat{z}^a)^{k_a} |0 \rangle \quad \text{for} \quad k_a = 0, 1, 2, \ldots ,
\]

which are connected by the action of creation and annihilation operators subject to

\[
\left[ \frac{\hat{z}^b}{\sqrt{2\theta^b}} , \frac{\hat{z}^a}{\sqrt{2\theta^a}} \right] = \delta^{ab} .
\]

We recall that, in the operator realization \( f \mapsto \hat{f} \), derivatives of \( f \) get mapped according to

\[
\partial_{z^a} f \mapsto \theta_{ab} [\hat{z}^b, \hat{f}] =: \partial_{\hat{z}^a} \hat{f} \quad \text{and} \quad \partial_{\bar{z}^a} f \mapsto \theta_{ab} [\hat{\bar{z}}^b, \hat{f}] =: \partial_{\hat{\bar{z}}^a} \hat{f} ,
\]

\(^1\)From now on we use the Einstein summation convention for repeated indices.
where \( \theta_{\bar{a}b} \) is defined via \( \theta_{b\bar{a}} \delta^{ab} = \delta_{\bar{b}}^{\bar{a}} \) so that \( \theta_{\bar{a}b} = -\theta_{b\bar{a}} = \frac{\delta_{\bar{a}}^{\bar{b}}}{2\theta} \). Finally, we have to replace
\[
\int_{\mathbb{R}^{2n}} d^{2n} f \mapsto \left( \prod_{a=1}^{n} 2\pi \theta^{a} \right) \text{Tr}_H \hat{f} . \tag{2.14}
\]

Tensoring \( \mathbb{R}^{2n}_q \) with a commutative \( \mathbb{S}^2 \) means extending the noncommutativity matrix \( \theta \) by vanishing entries in the two new directions. A more detailed description of noncommutative field theories can be found in the review papers [2].

**Donaldson-Uhlenbeck-Yau equations.** Let \( M_{2q} \) be a complex \( q=n+1 \) dimensional Kähler manifold with some local real coordinates \( x = (x^i) \) and a tangent space basis \( \partial_i := \partial/\partial x^i \) for \( i, j = 1, \ldots, 2q \), so that a metric and the Kähler two-form read \( ds^2 = g_{ij} dx^i dx^j \) and \( \Omega = \Omega_{ij} dx^i \wedge dx^j \), respectively. Consider a rank \( k \) complex vector bundle over \( M_{2q} \) with a gauge potential \( A = A_i dx^i \) and the curvature two-form \( F = dA + A \wedge A \) with components \( F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \). Both \( A_i \) and \( F_{ij} \) take values in the Lie algebra \( u(k) \). The Donaldson-Uhlenbeck-Yau (DUY) equations [16, 17] on \( M_{2q} \) are
\[
* \Omega \wedge F = 0 \quad \text{and} \quad F^{0,2} = 0 , \tag{2.15}
\]
where \( \Omega \) is the Kähler two-form, \( F^{0,2} \) is the \((0,2)\) part of \( F \), and \( * \) is the Hodge operator. In our local coordinates \((x^i)\) we have \( q!(\star \Omega \wedge F) = (\Omega, F) \Omega^q = \Omega^{ij} F_{ij} \Omega^q \) where \( \Omega^{ij} \) are defined via \( \Omega^{ij} \Omega_{jk} = \delta^i_k \). Due to the antihermiticity of \( F \), it follows that also \( F^{2,0} = 0 \). For \( q=2 \) the DUY equations (2.15) coincide with the anti-self-dual Yang-Mills (ASDM) equations
\[
* F = - F \tag{2.16}
\]
introduced in [3].

Specializing now \( M_{2q} \) to be \( \mathbb{R}^{2n} \times \mathbb{S}^2 \), the DUY equations (2.15) in the local complex coordinates \((z^a, y)\) take the form
\[
d^{ab} F_{z^a z^b} + \frac{(R^2 + y^2)^2}{4R^4} F_{yy} = 0 , \quad F_{z^a z^b} = 0 \quad \text{and} \quad F_{z^a y} = 0 , \tag{2.17}
\]
where \( a, b = 1, \ldots, n \). Using formulæ (2.4), we obtain
\[
F_{z^a y} = F_{z^a \varphi} \frac{\partial \varphi}{\partial y} + F_{z^a \varphi} \frac{\partial}{\partial y} = \frac{1}{y} (\sin \varphi F_{z^a \varphi} - i F_{z^a \varphi}) , \tag{2.18}
\]
\[
F_{y y} = F_{\varphi \varphi} \frac{\partial (\varphi, \varphi)}{\partial (y, y)} = \frac{1}{2i} \sin \varphi \frac{\partial}{\partial y} F_{\varphi \varphi} = \frac{1}{2i} \frac{(1 + \cos \varphi)^2}{R^2 \sin \varphi} \frac{\partial}{\partial \varphi} F_{\varphi \varphi} \tag{2.19}
\]
and finally the Donaldson-Uhlenbeck-Yau equations on \( \mathbb{R}^{2n} \times \mathbb{S}^2 \) in the alternative form
\[
2i d^{ab} F_{z^a z^b} + \frac{1}{R^2 \sin \varphi} F_{\varphi \varphi} = 0 , \quad F_{z^a z^b} = 0 , \quad \sin \varphi F_{z^a y} - i F_{z^a \varphi} = 0 . \tag{2.20}
\]

The transition to the noncommutative DUY equations is trivially achieved by going over to operator-valued objects everywhere. In particular, the field strength components in (2.20) then read \( \hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + [\hat{A}_i, \hat{A}_j] \), where e.g. \( \hat{A}_i \) are simultaneously \( u(k) \) and operator valued. To avoid a cluttered notation, we drop the hats from now on.
3 Generalized vortex equations on $\mathbb{R}^{2n}_g$

Noncommutative generalization of Taubes’ ansatz. Considering the particular case (2.16) of the SU(2) DUY equations on $\mathbb{R}^2 \times S^2$, Taubes introduced an SO(3)-invariant ansatz\(^2\) for the gauge potential $A$ which reduces the ASDYM equations (2.16) to the vortex equations on $\mathbb{R}^2$ [5] (see also [21]). Here we extend Taubes’ ansatz to the higher-dimensional manifold $\mathbb{R}^{2n} \times S^2$ and reduce the noncommutative\(^3\) U(2) Donaldson-Uhlenbeck-Yau equations (2.20) to generalized vortex equations on $\mathbb{R}^{2n}_g$, including their commutative ($\theta=0$) limit. In section 4, we will write down explicit solutions of the generalized noncommutative vortex equations on $\mathbb{R}^{2n}$ which determine multi-instanton solutions of the noncommutative YM equations on $\mathbb{R}^{2n} \times S^2$.

We begin with the $u(2)$-valued operator one-form $A$ on $\mathbb{R}^{2n} \times S^2$. Imposing SO(3) invariance up to a gauge transformation, Taubes [5] found for $n=1$ and $\theta=0$ that the $S^2$ dependence of $A$ must be collected in the $su(2)$ matrix

$$Q = i \begin{pmatrix} \cos \vartheta & e^{-i\varphi} \sin \vartheta \\ e^{i\varphi} \sin \vartheta & -\cos \vartheta \end{pmatrix} = i (\sin \vartheta \cos \varphi \sigma_1 + \sin \vartheta \sin \varphi \sigma_2 + \cos \vartheta \sigma_3)$$

\(3.1\)

and its differential $dQ$. Note that $Q^2 = -1$ and $\frac{\partial Q}{\partial \varphi} = -\sin \vartheta \frac{\partial Q}{\partial \vartheta}$. Our slight generalization of his ansatz to $\mathbb{R}^{2n} \times S^2$ reads (1 = (1, 0))

$$A = \frac{1}{2} \left\{(iQ - \gamma 1)A + (\phi_1 - 1)QdQ + \phi_2 dQ\right\},$$

\(3.2\)

where the constant $\gamma$ parametrizes the additional $u(1)$ piece. The one-form $A = A_\mu(x)dx^\mu$ with $A_\mu \in u(1) \cong i\mathbb{R}$ and $\mu=1,\ldots,2n$ is antihermitian while $\phi_{1,2} = \phi_{1,2}(x) \in \mathbb{R}$ are hermitian, all being operators in $\mathcal{H}$ only. Note that this form reduces the nonabelian connection $A$ to the abelian objects $(A, \phi_1, \phi_2)$ whose noncommutative character thus does not interfere with the $u(2)$ structure. Calculation of the curvature

$$\mathcal{F} = dA + A \wedge A = \frac{1}{2} \mathcal{F}_{ij} dx^i \wedge dx^j = \frac{1}{2} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu + \mathcal{F}_{\mu\nu} dx^\mu \wedge d\vartheta + \mathcal{F}_{\mu\varphi} dx^\mu \wedge d\varphi + \mathcal{F}_{\varphi \varphi} d\vartheta \wedge d\varphi$$

\(3.3\)

for $A$ of the form (3.2) yields

$$2\mathcal{F}_{\mu\nu} = iQ \left(\partial_\nu A_\mu - \partial_\mu A_\nu - \gamma[A_\mu, A_\nu]\right) - \gamma 1 \left(\partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1+\gamma^2}{2\gamma} [A_\mu, A_\nu]\right),$$

\(3.4\)

$$4\mathcal{F}_{\mu\varphi} = \left\{Q (2\partial_\mu \phi_1+iA_\mu \phi_2+i\phi_2 A_\mu - \gamma[A_\mu, \phi_1]) + 1 (2\partial_\mu \phi_2-iA_\mu \phi_1-i\phi_1 A_\mu - \gamma[A_\mu, \phi_2])\right\} \frac{\partial Q}{\partial \varphi},$$

\(3.5\)

$$4\mathcal{F}_{\varphi \varphi} = \left\{Q (2\partial_\mu \phi_1+iA_\mu \phi_2+i\phi_2 A_\mu - \gamma[A_\mu, \phi_1]) + 1 (2\partial_\mu \phi_2-iA_\mu \phi_1-i\phi_1 A_\mu - \gamma[A_\mu, \phi_2])\right\} \frac{\partial Q}{\partial \varphi},$$

\(3.6\)

$$2\mathcal{F}_{\varphi \varphi} = \left\{Q (1-\phi_1^2-\phi_2^2) + 1 [\phi_1, \phi_2]\right\} \sin \vartheta.$$  

\(3.7\)

In the complex coordinates (2.3) with $A_z = \frac{1}{2}(A_{2n-1} + i A_{2n})$ and $A_z^\dagger = -A_{z^*}$ we have

$$\mathcal{F}_{2n-1} = -Q \left(\partial_{z^*} A_{z^*} - \partial_{\bar{z}^*} A_{z^*} - \gamma[A_{z^*}, A_{z^*}]\right) - i \gamma 1 \left(\partial_{z^*} A_{z^*} - \partial_{\bar{z}^*} A_{z^*} - \frac{1+\gamma^2}{2\gamma} [A_{z^*}, A_{z^*}]\right)$$

\(3.8\)

which agrees with $2i \mathcal{F}_{z^* \bar{z}^*}$.

\(^2\)Similarly, Witten’s ansatz [19] for gauge fields on $\mathbb{R}^4$ reduces (2.16) to the vortex equations on the hyperbolic space $H^2$ (cf. [20] for the noncommutative $\mathbb{R}^3$).

\(^3\)As it is well known [2], in the noncommutative case one should use U(2) instead of SU(2).
Vortex-type equations in $\mathbb{R}^{2n}_\theta$. Introducing $\phi := \phi_1 + i\phi_2$ and substituting (3.7) and (3.8) into the first equation from (2.20), we obtain
\[
-\delta^{ab}\left\{Q(\partial_{z^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{z^a} - \gamma[A_{z^a}, A_{\bar{z}^b}]) + i\gamma 1(\partial_{z^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{z^a} - \frac{1+\gamma^2}{2\gamma}[A_{z^a}, A_{\bar{z}^b}])\right\} + 
\frac{1}{4R^2}(Q(2-\phi\phi^\dagger - \phi^\dagger\phi) + i1[\phi, \phi^\dagger]) = 0 \tag{3.9}
\]
which splits into the two equations
\[
\delta^{ab}\left\{\partial_{z^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{z^a} - \gamma[A_{z^a}, A_{\bar{z}^b}]\right\} = \frac{1}{4R^2}(2-\phi\phi^\dagger - \phi^\dagger\phi) \tag{3.10}
\]
\[
\gamma\delta^{ab}\left\{\partial_{z^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{z^a} - \frac{1+\gamma^2}{2\gamma}[A_{z^a}, A_{\bar{z}^b}]\right\} = \frac{1}{4R^2}[\phi, \phi^\dagger] \tag{3.11}
\]
after separating into the $su(2)$ (proportional to $Q$) and $u(1)$ (proportional to $i1$) components.

The second equation from (2.20) can be written as
\[
Q(\partial_{\bar{z}^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{\bar{z}^a} - \gamma[A_{\bar{z}^a}, A_{\bar{z}^b}]) + i\gamma 1(\partial_{\bar{z}^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{\bar{z}^a} - \frac{1+\gamma^2}{2\gamma}[A_{\bar{z}^a}, A_{\bar{z}^b}]) = 0 . \tag{3.12}
\]
After some algebra, using (3.5) and (3.6), we find that the third equation from (2.20) is equivalent to
\[
2\partial_{\bar{z}^a}\phi + (1-\gamma)A_{\bar{z}^a}\phi + (1+\gamma)\phi A_{\bar{z}^a} = 0 . \tag{3.13}
\]

Let us consider the commutative case $\theta^{\mu\nu} = 0$ and put $\gamma = 0$. Then the Donaldson-Uhlenbeck-Yau equations on $\mathbb{R}^{2n}\times S^2$ for $A$ defined in (3.2) reduce to
\[
\delta^{ab}\left\{\partial_{z^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{z^a}\right\} = \frac{1}{2R^2}(1-\phi\bar{\phi}) , \tag{3.14}
\]
\[
\partial_{\bar{z}^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{\bar{z}^a} = 0 , \tag{3.15}
\]
\[
\partial_{\bar{z}^a}\phi + A_{\bar{z}^a}\phi = 0 , \tag{3.16}
\]
where $\bar{\phi}$ is the complex conjugate of the scalar field $\phi$. Equations (3.14)–(3.16) generalize the vortex equations [5] on $\mathbb{R}^2$ to the higher-dimensional space $\mathbb{R}^{2n}$.

For the noncommutative case $\theta^{\mu\nu} \neq 0$ we choose $\gamma = -1$. Comparing (3.10) and (3.11), we obtain a constraint equation on the field $\phi$,
\[
2 - \phi\phi^\dagger - \phi^\dagger\phi = -[\phi, \phi^\dagger] \quad \implies \quad \phi^\dagger\phi = 1 , \tag{3.17}
\]
and the following noncommutative generalization of the vortex equations in $2n$ dimensions:
\[
\delta^{ab}F_{\bar{z}^a\bar{z}^b} := \delta^{ab}\left\{\partial_{\bar{z}^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{\bar{z}^a} + [A_{\bar{z}^a}, A_{\bar{z}^b}]\right\} = \frac{1}{4R^2}(1-\phi\phi^\dagger) , \tag{3.18}
\]
\[
F_{\bar{z}^a\bar{z}^b} := \partial_{\bar{z}^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{\bar{z}^a} + [A_{\bar{z}^a}, A_{\bar{z}^b}] = 0 , \tag{3.19}
\]
\[
\partial_{\bar{z}^a}\phi + A_{\bar{z}^a}\phi = 0 . \tag{3.20}
\]
These equations and their antecedent DUY equations on $\mathbb{R}^{2n}_\theta\times S^2$ are not integrable even for $n=1$. Therefore, neither dressing nor splitting approaches, developed in [22] for integrable equations on noncommutative spaces, can be applied. The modified ADHM construction [6] also does not work in this case. However, some special solutions can be obtained by choosing a proper ansatz as we shall see next.
4 Multi-instanton solutions on $\mathbb{R}^{2n}_\theta \times S^2$

Solutions of the constrained vortex-type equations. We are going to present explicit solutions to the noncommutative generalized vortex equations (3.18) – (3.20) subject to the constraint (3.17). The latter can be solved by putting

$$\phi = S_N \quad \text{and} \quad \phi^\dagger = S_N^\dagger,$$

where $S_N$ is an order-$N$ shift operator acting on the Fock space $\mathcal{H}$, i.e.

$$S_N^\dagger S_N = 1 \quad \text{while} \quad S_N S_N^\dagger = 1 - P_N,$$

with $P_N$ being a hermitean rank-$N$ projector: $P_N^2 = P_N = P_N^\dagger$.

It is convenient to introduce the operators

$$X_{z^a} = A_{z^a} + \theta_{ab} \bar{z}^b \quad \text{and} \quad \bar{X}_{\bar{z}^\dagger} = A_{\bar{z}^\dagger} + \theta_{ab} \bar{z}^b$$

in terms of which

$$F_{z^a \bar{z}^b} = [X_{z^a}, X_{\bar{z}^b}] + \theta_{ab} \quad \text{and} \quad \bar{F}_{\bar{z}^\dagger \bar{z}^b} = [X_{\bar{z}^\dagger}, X_{\bar{z}^b}].$$

We now employ the shift-operator ansatz (see e.g. [7, 23])

$$X_{z^a} = \theta_{ab} S_N \bar{z}^b S_N^\dagger \quad \text{and} \quad \bar{X}_{\bar{z}^\dagger} = \theta_{ab} S_N \bar{z}^b S_N^\dagger$$

for which

$$F_{z^a \bar{z}^b} = \theta_{ab} P_N = \delta_{ab} \frac{P_N}{2\theta^a} \quad \text{and} \quad \bar{F}_{\bar{z}^\dagger \bar{z}^b} = 0$$

since $\theta_{ab} = \delta_{ab}$. After substituting (4.1) and (4.6) into the first vortex equation (3.18), we obtain the condition

$$\delta_{ab} \theta_{ab} P_N = \frac{1}{4R^2} P_N \iff \frac{1}{\theta^1} + \ldots + \frac{1}{\theta^n} = \frac{1}{2R^2}.$$ (4.7)

The remaining vortex equations (3.19) and (3.20) are identically satisfied by (4.1) and (4.6).

Hence, for $\gamma = -1$ we have established on $\mathbb{R}^{2n}_\theta$ a whole class of noncommutative constrained vortex-type configurations

$$A_{z^a} = \theta_{ab} (S_N \bar{z}^b S_N^\dagger - \bar{z}^b) \quad \text{and} \quad \phi = S_N,$$ (4.8)

parametrized by shift operators $S_N$. Our particular form (3.2) for $A$ then yields a plethora of solutions to the noncommutative DUY equations on $\mathbb{R}^{2n}_\theta \times S^2$. These configurations generalize U(2) multi-instantons from $\mathbb{R}^2 \times S^2$ to $\mathbb{R}^{2n}_\theta \times S^2$. To substantiate this interpretation we finally calculate their topological charge.
Topological charge. For $\gamma = -1$, from (3.7) and (3.8) we get

$$\mathcal{F}_{\vartheta \varphi} = \frac{1}{4} (Q - i \mathbf{1}) \sin \vartheta \mathcal{P}_N \quad \text{and} \quad \mathcal{F}_{2a-1 \ 2a} = (i \mathbf{1} - Q) F_{z \bar{z} a} = (Q - i \mathbf{1}) \frac{P_N}{2\theta^a}. \quad (4.9)$$

Employing $(Q - i \mathbf{1})^{n+1} = (-2i)^n (Q - i \mathbf{1})$ and $\text{tr}_{2 \times 2} (Q - i \mathbf{1}) = -2i$ \hspace{1cm} (4.10)
we have

$$\text{tr}_{2 \times 2} \mathcal{F} \wedge \ldots \wedge \mathcal{F} = (n+1)! \text{tr}_{2 \times 2} \mathcal{F}_{12} \mathcal{F}_{34} \ldots \mathcal{F}_{2n-1 \ 2n} \mathcal{F}_{\vartheta \varphi} \ d\vartheta^{1} \wedge d\vartheta^{2} \wedge \ldots \wedge d\vartheta^{2n} \wedge d\vartheta \wedge d\varphi$$

$$= (n+1)! \left( \frac{-2i}{2^{n+2}} \right) \frac{P_N}{\prod_{a=1}^{n} \theta^a} \ d\vartheta^{1} \wedge d\vartheta^{2} \wedge \ldots \wedge d\vartheta^{2n} \wedge \sin \vartheta \ d\vartheta \wedge d\varphi. \quad (4.11)$$

With this, the topological charge indeed becomes

$$Q := \frac{1}{(n+1)!} \left( \frac{i}{2\pi} \right)^{n+1} \left( \prod_{a=1}^{n} 2\pi \theta^a \right) \text{Tr}_H \int_{S^2} \text{tr}_{2 \times 2} \mathcal{F} \wedge \ldots \wedge \mathcal{F}$$

$$= \left( \frac{i}{2\pi} \right)^{n+1} \left( \frac{-2i}{2^{n+2}} \right) \left( \prod_{a=1}^{n} 2\pi \theta^a \right) \left( \text{Tr}_H \frac{P_N}{\prod_{a=1}^{n} \theta^a} \right) \int_{S^2} \sin \vartheta \ d\vartheta \wedge d\varphi$$

$$= \frac{1}{4\pi} (\text{Tr}_H P_N) \int_{S^2} \sin \vartheta \ d\vartheta \wedge d\varphi = N. \quad (4.12)$$

5 Concluding remarks

By solving the noncommutative Donaldson-Uhlenbeck-Yau equations we have presented explicit
U(2) multi-instantons on $\mathbb{R}^{2n} \times S^2$ which are uniquely determined by abelian vortex-type configurations on $\mathbb{R}^{2n}$. The existence of these solutions required the condition (4.7) relating the $S^2$-radius $R$ to $\vartheta$ via $R = (2 \sum_{a=1}^{n} \frac{1}{\theta^a})^{-1/2}$. We see that any commutative limit $(\theta^a \to 0)$ forces $R \to 0$ as well, and the configuration becomes localized in $\mathbb{R}^{2n}$ (for $n=1$) or disappears (for $n>1$). The moduli space of our $N$-instanton solutions is that of rank-$N$ projectors in the $n$-oscillator Fock space.

Since standard instantons localize all compact coordinates in the ambient space they have been interpreted as sub-branes inside Dp-branes [1, 2, 9, 10, 11, 12]. The presence of an NS background $B$-field deforms such configurations noncommutatively. In the same vein, the solutions presented in this letter may be viewed as a collection of $N$ sub-branes of codimension $2n$, i.e. as D($p-2n$)-branes located inside two coincident D$p$-branes, with all branes sharing a common two-sphere $S^2_{R}$.

Acknowledgements. The authors are grateful to A.D. Popov for fruitful discussions and for reading the manuscript. O.L. wishes to thank B.-H. Lee for discussions. T.A.I. acknowledges the Heisenberg-Landau Program for partial support and the Institut für Theoretische Physik der Universität Hannover for its hospitality. This work was partially supported by grant LE-838/7 within the framework of the DFG priority program (SPP 1096) in string theory.
References

[1] N. Seiberg and E. Witten, JHEP 9909 (1999) 032 [hep-th/9908142].
[2] J.A. Harvey, hep-th/0102076;
   M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. 73 (2002) 977 [hep-th/0106048];
   A. Konechny and A. Schwarz, Phys. Rept. 360 (2002) 353 [hep-th/0107251].
[3] A.A. Belavin, A.M. Polyakov, A.S. Schwarz and Y.S. Tyupkin, Phys. Lett. B 59 (1975) 85.
[4] E.B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976) 449;
   M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760.
[5] C.H. Taubes, Commun. Math. Phys. 72 (1980) 277; Commun. Math. Phys. 75 (1980) 207.
[6] N. Nekrasov and A. Schwarz, Phys. Lett. B 256 (1990) 97.
[7] D.J. Gross and N.A. Nekrasov, JHEP 0103 (2001) 044 [hep-th/0010090].
[8] D.P. Jatkar, G. Mandal and S.R. Wadia, JHEP 0009 (2000) 018 [hep-th/0007078];
   D. Bak, Phys. Lett. B 495 (2000) 251 [hep-th/0008204];
   D. Bak, K.M. Lee and J.H. Park, Phys. Rev. D 63 (2001) 125010 [hep-th/0011099].
[9] M. Hamanaka, hep-th/0303256.
[10] M. Mihailescu, I.Y. Park and T.A. Tran, Phys. Rev. D 64 (2001) 046006 [hep-th/0011079];
    E. Witten, JHEP 0204 (2002) 012 [hep-th/0201054].
[11] P. Kraus and M. Shigemori, JHEP 0206 (2002) 034 [hep-th/0110035];
    M. Hamanaka, Y. Imaizumi and N. Ohta, Phys. Lett. B 529 (2002) 163 [hep-th/0112050];
    D.S. Bak, K.M. Lee and J.H. Park, Phys. Rev. D 66 (2002) 025021 [hep-th/0204221];
    Y. Hiraoka, hep-th/0205283; hep-th/0301176.
[12] N.A. Nekrasov, hep-th/0203109.
[13] E. Corrigan, C. Devchand, D.B. Fairlie and J. Nuyts, Nucl. Phys. B 214 (1983) 452.
[14] R.S. Ward, Nucl. Phys. B 236 (1984) 381.
[15] D.B. Fairlie and J. Nuyts, J. Phys. A 17 (1984) 2867;
    S. Fubini and H. Nicolai, Phys. Lett. B 155 (1985) 369;
    T.A. Ivanova and A.D. Popov, Lett. Math. Phys. 24 (1992) 85;
    E.G. Floratos and G.K. Leontaris, math-ph/0011027.
[16] S.K. Donaldson, Proc. Lond. Math. Soc. 50 (1985) 1; Duke Math. J. 54 (1987) 231.
[17] K. Uhlenbeck and S.-T. Yau, Commun. Pure Appl. Math. 39 (1986) 257.
[18] A.D. Popov, A.G. Sergeev and M. Wolf, hep-th/0304263.
[19] E. Witten, Phys. Rev. Lett. 38 (1977) 121.
[20] D.H. Correa, E.F. Moreno and F.A. Schaposnik, Phys. Lett. B 543 (2002) 235 [hep-th/0207180].
[21] P. Forgacs and N.S. Manton, Commun. Math. Phys. 72 (1980) 15.
[22] O. Lechtenfeld and A.D. Popov, JHEP 0111 (2001) 040 [hep-th/0106213];
    Phys. Lett. B 523 (2001) 178 [hep-th/0108118]; JHEP 0203 (2002) 040 [hep-th/0109209];
    Z. Horvath, O. Lechtenfeld and M. Wolf, JHEP 0212 (2002) 060 [hep-th/0211041].
[23] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, JHEP 0104 (2001) 001 [hep-th/0009142];
    J.A. Harvey, P. Kraus and F. Larsen, JHEP 0012 (2000) 024 [hep-th/0010060].