MULTI-INDEX ENSEMBLE KALMAN FILTERING

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ABSTRACT. In this work we combine ideas from multi-index Monte Carlo and ensemble Kalman filtering (EnKF) to produce a highly efficient filtering method called multi-index EnKF (MIEnKF). MIEnKF is based on independent samples of four-coupled EnKF estimators on a multi-index hierarchy of resolution levels, and it may be viewed as an extension of the multilevel EnKF (MLEnKF) method developed by the same authors in 2020. Multi-index here refers to a two-index method, consisting of a hierarchy of EnKF estimators that are coupled in two degrees of freedom: time discretization and ensemble size. Under certain assumptions, when strong coupling between solutions on neighboring numerical resolutions is attainable, the MIEnKF method is proven to be more tractable than EnKF and MLEnKF. Said efficiency gains are also verified numerically in a series of test problems.

Key words: Monte Carlo, multilevel, multi-index, convergence rates, Kalman filter, ensemble Kalman filter
AMS subject classification: 65C30, 65Y20.

1. INTRODUCTION

The ensemble Kalman filter (EnKF) is a widely used data assimilation method for high-dimensional state-space problems with nonlinear dynamics. Owing to its simple implementation and efficiency, ensemble-based filtering methods have rapidly gained popularity in geophysical sciences with applications, for example, in weather forecasting [40], atmosphere-ocean/lake simulations [37, 7, 31], and oil reservoir management [1, 56]. The EnKF method was originally proposed by Evensen [20]. Subsequently, several variants were developed [36, 15, 3]. EnKF approximates the filtering distribution using the empirical measure of its ensemble members. The $L^p$-convergence of the EnKF method with perturbed observations [36] has been studied in the literature [49, 46].

A considerable challenge in numerical filtering methods is the increase in simulation cost as the numerical resolution gets finer. This challenge can be overcome by the multilevel Monte Carlo method (MLMC) [24], which achieves substantial variance reduction by simulating pairwise coupled realizations on a hierarchy of temporal discretization levels. MLMC is a flexible methodology that has been combined with many other methods and successfully implemented in various fields: quasi-Monte Carlo [25, 43, 55], sequential Monte Carlo [13, 12, 51, 44], inverse problems and experimental design [10, 57, 47, 26, 58], differential equations with randomness [39, 19, 8, 42, 9, 5], limit theorems [2, 33], importance sampling [32, 41, 21], and machine learning [48].

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The multilevel EnKF (MLEnKF) method was introduced by Hoel et al. [34] for stochastic differential equation models with discrete-time observations, and an alternative version based on a sample average of independent pairwise coupled EnKF estimators was subsequently developed [35]. The main difference between these two versions of MLEnKF is that [34] uses one universal Kalman gain to update the ensemble members on all hierarchy levels, while [35] employs one Kalman gain per independent EnKF sample in the full MLEnKF estimator. The latter approach introduces less correlation between all particle members of the MLEnKF estimator, which particularly simplifies convergence analysis and paves the way for extending MLEnKF to the multi-index EnKF (MIEnKF) method introduced in this work. The MLEnKF method was extended to spatiotemporal (infinite-dimensional state space) models [17]. Similar multilevel techniques have been combined with other ensemble-based filtering methods, such as particle filters [38, 6], transform particle filters [28, 27], multigrid [50] and the recent extension to the continuous-time (Kalman–Bucy) filter [16].

The successful implementation of MLMC depends on a strong pairwise coupling between realizations on neighboring hierarchy levels, meaning a coupling that leads to substantial variance reduction. For stochastic differential equations with sufficiently smooth coefficients, this is achieved quite easily, but in more realistic problems with low-regularity features this can be extremely challenging if at all possible. See [22] for multilevel data assimilation applied to reservoir history matching, and [23, 53, 54] for applications of MLEnKF using sampling resolution constraints (so-called multi-fidelity methods).

Another important method is the multi-index Monte Carlo method (MIMC) [29], which forms the basis of this work. MIMC consists of a multi-index hierarchy of coupled realizations on neighboring resolutions, and can be regarded as an extension of MLMC. Many concepts related to particle-wise coupling in the proposed MIEnKF method are common in the MIMC method for McKean-Vlasov dynamics [30].

The contributions of this work are to develop the MIEnKF method with a subtle variance-reducing coupling idea for realizations on neighboring resolutions, and to numerically verify the asymptotic efficiency gains that MIEnKF achieves over EnKF and MLEnKF. The MIEnKF method extends the recent MLEnKF method [35] by treating not only the numerical discretization but also the EnKF ensemble size as degrees of freedom – resolution parameters. MIEnKF introduces a four-coupling of EnKF estimators (i.e., a coupling in both degrees of freedom) that produces a stronger variance reduction than the pairwise coupling in MLEnKF. Under certain assumptions, MIEnKF is also shown theoretically to achieve efficiency gains over counterparts for weak approximations of quantities of interest (QoI) in the classic and more robust setting of $\alpha = 1$ and $\beta = 2$ defined in [35], cf. Table 1.

| Methods          | EnKF     | MLEnKF  | MIEnKF  |
|------------------|----------|---------|---------|
| Mean-squared error | $O(\epsilon^2)$ | $O(\epsilon^2)$ | $O(\epsilon^2)$ |
| Computational cost | $O(\epsilon^{-3})$ | $O(\epsilon^{-2} \log(\epsilon))$ | $O(\epsilon^{-2})$ |

Table 1. Comparison of computational costs versus errors for ensemble Kalman filtering (EnKF), multilevel EnKF (MLEnKF) and multi-index EnKF (MIEnKF) methods, cf. Section 4.
The rest of this work is organized as follows. In Section 2, the setting and notation for filtering problem are introduced and a brief overview of the EnKF, mean-field EnKF (MFEEnKF), and MLEnKF methods is presented. Section 3 describes the framework of the MIEnKF method. Section 4 presents theory on the performance of the MIEnKF method, including a theorem on approximation error versus computational cost. Section 5 compare the performance of MIEnKF to MLEnKF and EnKF in a series of numerical examples, and we wrap up with concluding remarks in Section 6.

2. Problem setting

In this section, we introduce the filtering problem of interest and give a brief overview of relevant ensemble-based filtering methods.

Let \((\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t \geq 0})\) be a complete probability space equipped with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) of sub-\(\sigma\)-algebras of \(\mathcal{F} = \mathcal{F}_\infty\). We denote by \(L^p(\Omega, \mathbb{R}^k)\) the space of \(\mathcal{F}_t \setminus \mathcal{B}^k\)-measurable functions \(u : \Omega \rightarrow \mathbb{R}^k\) with \(\mathbb{E}[|u|^p] < \infty\). Given the initial value \(u_0 \in \bigcap_{p \geq 2} L^p(\Omega, \mathbb{R}^d)\), we consider the discrete-time filtering problem for a system of stochastic dynamics defined by a sequence of random maps \(\Psi_n : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d\) and observations with additive noise:

\[
\begin{aligned}
  u_{n+1}(\omega) &= \Psi_n(u_n, \omega), \\
  y_{n+1}(\omega) &= Hu_{n+1}(\omega) + \eta_{n+1},
\end{aligned}
\]

where \(\omega \in \Omega\), \(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\), \(H \in \mathbb{R}^{m \times d}\) is an observation operator, \(\{\eta_k\}_{k \in \mathbb{N}}\) is an independent and identically distributed (i.i.d.) sequence with \(\eta_1 \sim N(0, \Gamma)\) and with the independence property \(\{\eta_k\}_{k \in \mathbb{N}} \perp \{u_k\}_{k \in \mathbb{N}_0}\). When confusion is not possible, we shall not indicate the dependence on \(\omega\) for random variables.

Let \(Y_n := (y_1, y_2, \ldots, y_n)\) denote the accumulated observation data up to time \(n\) using the convention that \(Y_0 := \emptyset\). The main objective of a filtering method is to track the underlying signal \(u_n\) given \(Y_n\) through computing the conditional distribution of \(u_n\) given \(Y_n\). The exact filter density for this problem – the so-called Bayes filter – satisfies the following iterative equations:

**Prediction**

\[
\rho_{u_n|Y_{n-1}}(u) \propto \int_{\mathbb{R}^d} \rho_{u_n|u_{n-1}}(u) \rho_{u_{n-1}|Y_{n-1}}(u) \, dv
\]

**Update**

\[
\rho_{u_n|Y_n}(u) \propto \exp \left( - \frac{1}{2} (y_n - Hu)^T \Gamma^{-1} (y_n - Hu) / 2 \right) \rho_{u_n|Y_{n-1}}(u).
\]

We will refer to the posterior distribution in the above update step as the true filter. In the linear-Gaussian setting, the Kalman filter is an exact algorithm that tracks the mean and covariance of the true filter. When \(\Psi\) is nonlinear the true filter becomes non-Gaussian, and the Kalman filter does not apply anymore. Therefore, approximation methods are needed. Among such, particle filters converge to the true filter in the large particle limit, but they are conjectured to perform poorly in high dimensions [11]. The EnKF performs more robustly than particle filters in high dimensions, but it has poorer convergence properties. In the large-ensemble limit, EnKF converges to the so-called mean-field EnKF in the large-ensemble limit [46, 35]. However, due to the application of a biased Gaussian ansatz in the update step of EnKF [20], the mean-field EnKF is not equal to the true filter in nonlinear problem settings. Despite this disparity, the EnKF is a robust and efficient method.

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1. The function \(u\) is \(\mathcal{F}_t \setminus \mathcal{B}^k\)-measurable iff \(u^{-1}(B) \in \mathcal{F}_t\) for all \(B \in \mathcal{B}^k\), where \(\mathcal{B}^k\) denotes the Borel \(\sigma\)-algebra on \(\mathbb{R}^k\).
that is popular among practitioners. Connections between the mean-field EnKF and the true filter are discussed in [45, 35], but there are many open questions that remain to be studied, such as the convergence properties of EnKF in the large-ensemble and long-time limit.

The main objective of this paper is to construct an efficient MIEnKF method that converges weakly to the mean-field EnKF in the large-ensemble limit. In other words, for a given QoI $\phi : \mathbb{R}^d \to \mathbb{R}$, our method approximates
\[
\mathbb{E} \bar{\mu}_n[\phi(u)] = \int_{\mathbb{R}^d} \phi(u) \bar{\mu}_n(du),
\]
where $\bar{\mu}_n$ denotes the mean-field EnKF measure at time $n$, cf. Section 2.2.

Notation.

- For $f, g : (0, \infty) \to [0, \infty)$ the notation $f \lesssim g$ implies that there exists a $C > 0$ such that
  \[ f(x) \leq Cg(x), \quad \forall x \in (0, \infty). \]

- The notation $f \asymp g$ implies that $f \lesssim g$ and $g \lesssim f$.

- The expectation operator is defined by $\mathbb{E} [\cdot]$ and the variance operator (applicable to scalar-valued rv) is denoted by $\text{Var} [\cdot]$.

- For $d \in \mathbb{N}$, $|x|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^d$. For $\mathcal{F}\mathcal{B}^d$-measurable functions $u : \Omega \to \mathbb{R}^d$ and $p \geq 1$,
  \[ \|u\|_p := \|u\|_{L^p(\Omega, \mathbb{R}^d)} = \left( \int_{\Omega} |u(\omega)|^p \mu(\omega) \right)^{1/p}. \]

- $\lceil x \rceil := \min \{ z \in \mathbb{Z} \mid z \geq x \}$.

Let $\Psi^N_n$ denote the numerical discretization of the dynamics $\Psi_n$ using $N \geq 1$ uniform timesteps over every observation-time interval. The following assumption ensures that the mean-field EnKF measure $\mu_n$ is well-defined cf. [35, Appendix A] and Section 2.2:

**Assumption 1.** Let $u, v \in \cap_{p \geq 2} L^p_n(\Omega, \mathbb{R}^d)$ for any $n \in \mathbb{N}_0$, $p \geq 2$, then there exists a constant $c_p > 0$ such that for all $N \geq 1$:

1. $\|\Psi^N_n(u)\|_p \leq c_p(1 + \|u\|_p)$,
2. $\|\Psi^N_n(u) - \Psi^N_n(v)\|_p \leq c_p \|u - v\|_p$.

2.1. EnKF. The EnKF method is an ensemble-based nonlinear filtering method that is an extension of the Kalman filter. For an EnKF ensemble of size $P$, let $v_{n,i} := v_n(\omega_i)$ and $\hat{v}_{n,i} := \hat{v}_n(\omega_i)$, respectively, denote the $i$-th particle of the prediction and updated ensemble at time $n$. Then, the EnKF algorithm with perturbed observations and numerical dynamics $\Psi^N$ comprises the following steps:

\[
\begin{align*}
\text{Prediction} & \quad \begin{cases}
  v_{n+1,i} = \Psi^N_n(\hat{v}_{n,i}), \quad i = 1, 2, \ldots, P, \\
  m_{n+1} = \frac{1}{P} \sum_{i=1}^P v_{n+1,i}, \\
  C_{n+1} = \frac{1}{P-1} \sum_{i=1}^P (v_{n+1,i} - m_{n+1}) (v_{n+1,i} - m_{n+1})^T.
\end{cases}
\end{align*}
\]
\[ \begin{align*}
\dot{y}_{n+1,i} &= y_{n+1} + \eta_{n+1,i}, \quad i = 1, 2, \ldots, P, \\
K_{n+1} &= C_{n+1}H^T(HC_{n+1}H^T + \Gamma)^{-1}, \\
\hat{v}_{n+1,i} &= (I - K_{n+1}H)\hat{v}_{n+1,i} + K_{n+1}\hat{y}_{n+1,i},
\end{align*} \]

where \( \eta_{n+1,i} \) are i.i.d. draws from \( N(0, \Gamma) \).

The updated EnKF empirical measure is defined by
\[ \mu_{N,P}^N(dv) = \frac{1}{P} \sum_{i=1}^P \delta (dv; \hat{v}_{n,i}), \]
where \( \delta \) is the Dirac measure centered at \( \hat{v}_{n,i} \), and the expectation of a QoI \( \varphi : \mathbb{R}^d \to \mathbb{R} \) with respect to the EnKF empirical measure is expressed as
\[ \mu_{N,P}^N[\varphi] = \frac{1}{P} \sum_{i=1}^P \varphi (\hat{v}_{n,i}). \]

Note that \( \mu_{N,P}^N[\varphi] \) is a random variable that depends on parameters \( N \) and \( P \). Under sufficient regularity, \( \mu_{N,P}^N[\varphi] \to \bar{\mu}_n[\varphi] \) as \( N, P \to \infty \), cf. [46, 35, 17], where \( \bar{\mu}_n[\varphi] \) denotes the expectation of \( \varphi \) with respect to the mean-field EnKF measure that is introduced in the next section.

### 2.2. MFEnKF

The MFEnKF is the large-ensemble and fine-discretization limit of EnKF. In the large-ensemble limit, the Kalman gain becomes a deterministic matrix. Consequently, one may view MFEnKF as an ensemble of i.i.d. noninteracting particles, so that it suffices to represent the resulting filtering distribution by one particle. Let \( \bar{v}_n \) and \( \hat{v}_n \) denote the prediction and updated state of a mean-field particle at time \( n \), respectively. The following algorithm defines the MFEnKF for fully non-Gaussian models:

\[
\begin{align*}
\text{Prediction} & \quad \bar{v}_{n+1} = \Psi_n (\hat{v}_n) \\
& \quad \bar{m}_{n+1} = \mathbb{E} [\hat{v}_{n+1}], \\
& \quad \bar{C}_{n+1} = \mathbb{E} \left[ (\hat{v}_{n+1} - \bar{m}_{n+1}) (\hat{v}_{n+1} - \bar{m}_{n+1})^T \right].
\end{align*}
\]

\[
\begin{align*}
\text{Update} & \quad \hat{y}_{n+1} = y_{n+1} + \bar{\eta}_{n+1}, \\
& \quad K_{n+1} = \bar{C}_{n+1}H^T (HC_{n+1}H^T + \Gamma)^{-1}, \\
& \quad \hat{v}_{n+1,i} = (I - K_{n+1}H)\hat{v}_{n+1,i} + K_{n+1}\hat{y}_{n+1,i},
\end{align*}
\]

where \( \bar{\eta}_{n+1} \) is i.i.d. draws from \( N(0, \Gamma) \).

The expectation of a QoI \( \varphi : \mathbb{R}^d \to \mathbb{R} \) with respect to the updated mean-field EnKF measure is given by
\[ \bar{\mu}_n[\varphi] := \mathbb{E}^{\hat{\mu}_n}[\varphi(v)] = \int_{\mathbb{R}^d} \varphi(v) \bar{\mu}_n (dv). \]

**Remark 1.** For the EnKF filter with the numerical-solution dynamics \( \Psi_n \) (instead of \( \Psi \)), Assumption 1 ensures that the analogous mean-field EnKF measure \( \bar{\mu}_n \) is well-defined for any observation time \( n \geq 0 \) and numerical resolution \( N \geq 1 \), cf. [35, Appendix A].
2.3. MLEnKF. The recently developed MLEnKF method [35] is a natural stepping stone on the way from EnKF to explaining all of the complexities in the MIEnKF method. MLEnKF is a filtering method based on a sample average of independent and pairwise coupled samples of EnKF estimators at different resolution levels.

Let \( L \in \mathbb{N} \) denote the finest resolution level of the estimator, and let the sequences \( N_{\ell} = N_0 \times 2^\ell \) and \( P_{\ell} = P_0 \times 2^\ell \) with \( N_0, P_0 \in \mathbb{N}, \ \ell = 0, 1, \ldots, L \) respectively denote the numerical resolution and ensemble size.

**Pairwise coupling of EnKF estimators.** For a level \( \ell \geq 0 \), let
\[
\hat{v}_{n, i}^\ell := \hat{v}_{n}^\ell (\omega_{i}^\ell) \quad i = 1, \ldots, P_{\ell}
\]
denote \( i \)-th particle of the updated ensemble at time \( n \) in size \( P_{\ell} \) corresponding to the fine-level numerical resolution \( N_{\ell} \). Each \( \hat{v}_{n, i}^\ell \) is *coupled pairwise* to the respective \( i \)-th particle of the coarser-level updated ensemble at time \( n \) computed with the numerical resolution \( N_{\ell-1} \) via shared driving noise \( \omega_{i}^\ell \). To obtain a \( 1 \leftrightarrow 1 \) coupling between ensemble-members/particles on the fine- and coarse level, the total size of the coarse-level ensemble is set to \( P_{\ell} \) with the relation \( P_{\ell} = 2P_{\ell-1} \), meaning that the coarse-level ensemble can be viewed as a union of two ensembles in size \( P_{\ell-1} \):
\[
\hat{v}_{n, i}^{\ell-1, 1} := \hat{v}_{n}^{\ell-1, 1} (\omega_{i}^{\ell}) \quad i = 1, \ldots, P_{\ell-1}
\]
and
\[
\hat{v}_{n, i}^{\ell-1, 2} := \hat{v}_{n}^{\ell-1, 2} (\omega_{P_{\ell-1}+i}^{\ell}) \quad i = 1, \ldots, P_{\ell-1},
\]
with the convention \( \hat{v}^{-1, i} := 0 \).

It is important to note here that the particle-wise pairs share the same realization of driving noise within a level and the superscript \( \ell \) in \( \omega_{i}^{\ell} \) indicates an independence of underlying noise between levels. In addition to this, the pairwise coupling is imposed under the particle-wisely shared initial condition:
\[
\hat{v}_{0, i}^{\ell} = \begin{cases} 
\hat{v}_{0, i}^{\ell-1, 1} & \text{if } i \in \{1, \ldots, P_{\ell-1}\} \\
\hat{v}_{0, P_{\ell-1}+1}^{\ell-2} & \text{if } i \in \{P_{\ell-1}+1, \ldots, P_{\ell}\}, 
\end{cases}
\]
and the perturbed observations are also shared particle-wisely (see the below update step). Iterative simulation of pairwise coupled ensemble-members on the \( \ell \)-th resolution level of the MLEnKF filter consists of the following prediction and update steps:
\[
\begin{aligned}
\hat{v}_{n+1, i}^{\ell-1, 1} &= \Psi_{n}^{N_{\ell-1}} (\hat{v}_{n, i}^{\ell-1, 1}), \quad i = 1, \ldots, P_{\ell-1}, \\
\hat{v}_{n+1, i}^{\ell-1, 2} &= \Psi_{n}^{N_{\ell-1}} (\hat{v}_{n, i}^{\ell-1, 2}), \quad i = 1, \ldots, P_{\ell-1}, \\
\hat{v}_{n+1, i}^{\ell} &= \Psi_{n}^{N_{\ell}} (\hat{v}_{n, i}^{\ell}), \quad i = 1, \ldots, P_{\ell}, \\
C_{n+1}^{\ell-1, 1} &= \text{Cov} [\hat{v}_{n+1, 1: P_{\ell-1}}^{\ell-1, 1}], \\
C_{n+1}^{\ell-1, 2} &= \text{Cov} [\hat{v}_{n+1, 1: P_{\ell-1}}^{\ell-1, 2}], \\
C_{n+1}^{\ell} &= \text{Cov} [\hat{v}_{n+1, 1: P_{\ell}}^{\ell}].
\end{aligned}
\]
\[
\text{Cov}[v_{n,1}^\ell : := \sum_{i=1}^{P_\ell} \left( v_{n,i}^\ell \right)^T \left( P_\ell^{-1} \right) \left( \sum_{i=1}^{P_\ell} v_{n,i}^\ell \right)^T, \\
\text{Cov}[v_{n,1}^{\ell-1,k} := \sum_{i=1}^{P_{\ell-1}} \left( v_{n,i}^{\ell-1,k} \right)^T \left( P_{\ell-1}^{-1} \right) \left( \sum_{i=1}^{P_{\ell-1}} v_{n,i}^{\ell-1,k} \right)^T, \quad k = 1, 2. 
\]

\[
\begin{align*}
\text{Update} \quad & \quad \{ \text{y}_{n+1,i}^\ell = y_{n+1} + \eta_{n+1,i}^\ell, \quad i = 1, \ldots, P_\ell, \\
K_{n+1}^{\ell-1,1} = C_{n+1}^{\ell-1,1} H^T (HC_{n+1}^{\ell-1,1} H^T + \Gamma)^{-1}, \quad & \quad \text{and} \\
K_{n+1}^{\ell-1,2} = C_{n+1}^{\ell-1,2} H^T (HC_{n+1}^{\ell-1,2} H^T + \Gamma)^{-1}, \\
\text{or} \quad & \quad \{ \text{v}_{n,i}^{\ell-1,1} = (I - K_{n+1}^{\ell-1,1} H) v_{n,i}^{\ell-1,1} + K_{n+1}^{\ell-1,1} \eta_{n+1,i}^\ell, \quad i = 1, \ldots, P_\ell, \\
\text{or} \quad & \quad \{ \text{v}_{n,i}^{\ell-1,2} = (I - K_{n+1}^{\ell-1,2} H) v_{n,i}^{\ell-1,2} + K_{n+1}^{\ell-1,2} \eta_{n+1,i}^\ell + \gamma, \quad i = 1, \ldots, P_\ell - 1, \\
\text{or} \quad & \quad \{ \text{v}_{n,i}^\ell = (I - K_{n+1} H) v_{n,i}^\ell + K_{n+1} \eta_{n+1,i}^\ell, \quad i = 1, \ldots, P_\ell, \\
\end{align*}
\]

where \{\eta_{n+1,i}^\ell \}_{i=1}^{P_\ell} is a sequence of independent \(N(0, \Gamma)\)-distributed random variables. In the above notation, the coupling between the fine-level EnKF estimator

\[
\mu_n^{N_{\ell},P_\ell}[\varphi] := \sum_{i=1}^{P_\ell} \frac{\varphi(v_{n,i}^\ell)}{P_\ell},
\]

and the two coarse-level estimators

\[
\mu_n^{N_{\ell-1},P_{\ell-1}}[\varphi] := \sum_{i=1}^{P_{\ell-1}} \frac{\varphi(v_{n,i}^{\ell-1,k})}{P_{\ell-1}} , \quad k = 1, 2
\]
is obtained through particle-wise coupling

\[
\hat{v}_{n,i}^\ell \xleftarrow{\text{coupling}} \begin{cases} 
\hat{v}_{n,i}^{\ell-1,1} & \text{if } i \in \{1, \ldots, P_\ell\}, \\
\hat{v}_{n,i}^{\ell-1,2} & \text{if } i \in \{P_\ell + 1, \ldots, P_\ell\}.
\end{cases}
\]

Finally, the updated MLEnKF estimator at time \(n\) assumes the following form:

\[
\mu_n^{ML}[\varphi] = \sum_{\ell=0}^{L} \sum_{m=1}^{M_\ell} \left( \mu_n^{N_{\ell},P_{\ell}m}[\varphi] - \left( \mu_n^{N_{\ell-1},P_{\ell-1},1,m}[\varphi] + \mu_n^{N_{\ell-1},P_{\ell-1},2,m}[\varphi] \right) / 2 \right) / M_\ell
\]

where \(\{M_\ell\}_{\ell=0}^{L} \subset \mathbb{N}\) is a decreasing sequence with \(M_\ell\) representing the number of i.i.d. and pairwise coupled EnKF estimators on level \(\ell\):

\[
\left\{ \mu_n^{N_{\ell-1},P_{\ell-1}}[\varphi], \mu_n^{N_{\ell-1},P_{\ell-1},1,m}[\varphi], \mu_n^{N_{\ell-1},P_{\ell-1},2,m}[\varphi] \right\}_{m=1}^{M_\ell},
\]

where \(\mu_n^{N_{\ell-1},P_{\ell-1},m}[\varphi] := 0\). For the configuration of \(L\) and \(M_\ell\), we refer the reader to [35, Corollary 2].
3. MIEnKF

In this section, we develop the MIEnKF method by extending the MLEnKF method from the previous section.

To define a set of discretization levels for the MIEnKF, we first introduce the 2-index \( \ell := (\ell_1, \ell_2) \in \mathbb{N}_0^2 \) with the shorthands \( e_1 := (1, 0) \), \( e_2 := (0, 1) \), and \( 1 := (1, 1) \). Similarly as for MLEnKF, we associate sequences of natural numbers \( N_{\ell_1} = N_0 \times 2^{\ell_1} \) and \( P_{\ell_2} = P_0 \times 2^{\ell_2} \) with \( N_0, P_0 \in \mathbb{N} \) to the number of timesteps and particles on the 2-index “level” \( \ell \).

Seeking to approximate \( \bar{\mu}_n[\varphi] \) (the expectation of the QoI \( \varphi \) with respect to the mean-field measure \( \bar{\mu}_n \)), we denote the discrete approximation corresponding to the 2-index \( \ell \) by \( \mu_n[\varphi] := \mu_{N_0, P_0}[\varphi] \). In other words, \( \mu_n[\varphi] \) is the EnKF estimator (3) computed with \( N_{\ell_1} \) timesteps and \( P_{\ell_2} \) ensemble-members/particles. We define first-order difference operators for numbers of timesteps and particles as follows:

\[
\Delta_1 \mu_n[\varphi] = \begin{cases} 
(\mu_n^\ell - \mu_n^{\ell-e_1})[\varphi], & \text{if } \ell_1 > 0, \\
\mu_n^\ell[\varphi], & \text{if } \ell_1 = 0
\end{cases}
\]

\[
\Delta_2 \mu_n[\varphi] = \begin{cases} 
(\mu_n^\ell - (\mu_n^{\ell-e_2,1} + \mu_n^{\ell-e_2,2}) / 2)[\varphi], & \text{if } \ell_2 > 0, \\
\mu_n^\ell[\varphi], & \text{if } \ell_2 = 0
\end{cases}
\]

where \( \mu_n^{\ell-e_2,1}[\varphi] \) and \( \mu_n^{\ell-e_2,2}[\varphi] \) are two i.i.d. copies of \( \mu_n^\ell[\varphi] \). Note that \( \mu_n[\varphi] \) comprises \( P_{\ell_2} \) ensemble members, whereas \( \mu_n^{\ell-e_2}[\varphi] \) has half as many ensemble members, \( P_{\ell_2-1} \). Therefore, the pair \( (\mu_n^{\ell-e_2,1}[\varphi], \mu_n^{\ell-e_2,2}[\varphi]) \) are introduced to achieve a \( 1 \leftrightarrow 1 \) coupling of ensemble-members/particles on the “levels” /2-indices \( \ell \) and \( \ell - e_2 \).

We define the four-coupled EnKF estimator using the first-order mixed difference as follows:

\[
\Delta \mu_n^\ell[\varphi] := \Delta_1(\Delta_2 \mu_n^\ell[\varphi]) = \Delta_2(\Delta_1 \mu_n^\ell[\varphi]) = \Delta_2(\mu_n^\ell - \mu_n^{\ell-e_1})[\varphi]
\]

\[
= \left( \mu_n^\ell - \left( \mu_n^{\ell-e_2,1} + \mu_n^{\ell-e_2,2} \right) / 2 \right. \\
- \left. \mu_n^{\ell-e_1} + \left( \mu_n^{\ell-1,1} + \mu_n^{\ell-1,2} \right) / 2 \right)[\varphi],
\]

where the pair \( (\mu_n^{\ell-1,1}[\varphi], \mu_n^{\ell-1,2}[\varphi]) \) of i.i.d. copies of \( \mu_n^{\ell-1}[\varphi] \) is also introduced to achieve a \( 1 \leftrightarrow 1 \) coupling of ensemble-members/particles on the “levels” /2-indices \( \ell \) and \( \ell - 1 \). If it holds that

\[
\mathbb{E} [\mu_n^\ell[\varphi]] \to \bar{\mu}_n[\varphi] \quad \text{as} \quad (\ell_1, \ell_2) \to (\infty, \infty),
\]

and the sequence \( \{\mathbb{E} [\Delta \mu_n^\ell[\varphi]]\}_{\ell \in \mathbb{N}_0^2} \) is absolutely summable (both of these conditions hold under Assumption 2 (A1), which is presented below), then the linearity of the expectation operator implies that

\[
\bar{\mu}_n[\varphi] = \sum_{\ell \in \mathbb{N}_0^2} \mathbb{E} [\Delta \mu_n^\ell[\varphi]] = \sum_{\ell \in \mathcal{I}} \mathbb{E} [\Delta \mu_n^\ell[\varphi]] + \sum_{\ell \notin \mathcal{I}} \mathbb{E} [\Delta \mu_n^\ell[\varphi]],
\]

for any index set \( \mathcal{I} \subset \mathbb{N}_0^2 \).
Remark 2. The magnitude of the second term on the right-hand side, the truncated region, relates to the bias error of the MIEEnKF estimator. Accurate information on how this magnitude varies with $I$ can and should be used to determine an index set such that the resulting MIEEnKF method meets the bias-error constraint. In the last part of the proof of Theorem 1 below, we have indeed used Assumption 2 (A1) to control the bias error through bounding the magnitude of said right-hand-side term in (9).

For a given index set $I$, which we will specify later in Section 4, the MIEEnKF estimator is defined as the sample-average estimator of the first term on the right-hand side of (9):

$$
\mu_n^{MII}[\varphi] := \sum_{\ell \in I} \sum_{m=1}^{M_\ell} \frac{\Delta \mu_n^{\ell, m}[\varphi]}{M_\ell},
$$

where $\{\Delta \mu_n^{\ell, m}[\varphi]\}_{m=1}^{M_\ell}$ are i.i.d. copies of $\Delta \mu_n^{\ell, m}[\varphi]$, and $\{\Delta \mu_n^{\ell, m}[\varphi]\}_{(\ell, m)}$ are mutually independent.

The primary motivation for sampling four-coupled EnKF estimators in the MIEEnKF estimator is that it leads to a substantial variance reduction that improves the tractability of the sampling method. Similar to multilevel Monte Carlo estimators, the tractability of (10) is optimized through careful selection of the index set $I$ and the number of samples $M_\ell$. Provided that convergence rates for the multi-index hierarchy are available or approximable, this can be achieved by solving a constrained optimization problem [29, 30].

3.1. Four-coupled EnKF estimators. To describe the coupling between the EnKF estimators $(\mu_n^{\ell, e_2}, \mu_n^{\ell, e_1}, \mu_n^{\ell, -1})[\varphi]$, we introduce the four-coupled updated-state ensembles at time $n$:

$$
\{(\hat{\nu}^\ell_{n,i}, \hat{\nu}^\ell_{n,i}^\ell, \hat{\nu}^\ell_{n,i+1}^\ell, \hat{\nu}^\ell_{n,i+1}^\ell)\}_{i=1}^{P_{\ell_2}} := \{(\hat{\nu}^\ell_{n,i}, \hat{\nu}^\ell_{n,i}^\ell, \hat{\nu}^\ell_{n,i+1}^\ell, \hat{\nu}^\ell_{n,i+1}^\ell)\}_{i=1}^{P_{\ell_2}}.
$$

The set of particles on index $\ell-2$ is a union of two EnKF ensembles:

$$
\hat{\nu}^\ell_{n,i} := \hat{\nu}^\ell_{n,i}^\ell, \quad i = 1, \ldots, P_{\ell_2-1},
$$

and

$$
\hat{\nu}^\ell_{n,i} := \hat{\nu}^\ell_{n,i+1}^\ell, \quad i = 1, \ldots, P_{\ell_2-1},
$$

and the set of particles on index $\ell-1$ is also a union of two EnKF ensembles:

$$
\hat{\nu}^{\ell-1}_{n,i} := \hat{\nu}^{\ell-1}_{n,i}, \quad i = 1, \ldots, P_{\ell_2-1},
$$

and

$$
\hat{\nu}^{\ell-1}_{n,i} := \hat{\nu}^{\ell-1}_{n,i+1}, \quad i = 1, \ldots, P_{\ell_2-1}.
$$

Similarly to MLEnKF (Section 2.3), the MIEEnKF employs a $1 \leftrightarrow 1$ coupling between particles on all four levels associated to one index $\ell$. We defer further details on how this is achieved to Section 3.2, and are now ready to properly define the MIEEnKF estimator.

The empirical estimator $\mu_n^{\ell}[\varphi]$ is induced by the ensemble $\hat{\nu}_{n,1:P_{\ell_2}} := \{\hat{\nu}_{n,i}\}_{i=1}^{P_{\ell_2}}$, meaning that it equals the sample average of $\{\varphi(\hat{\nu}_{n,i})\}_{i=1}^{P_{\ell_2}}$. Similarly, $\mu_n^{\ell, e_2}[\varphi]$ is induced by $\hat{\nu}_{n,1:P_{\ell_2}}^{\ell, e_2} := \{\hat{\nu}_{n,i}^{\ell, e_2}\}_{i=1}^{P_{\ell_2}}$:

$$
\mu_n^{\ell, e_2}[\varphi] := \frac{(\mu_n^{\ell, e_2, 1} + \mu_n^{\ell, e_2, 2})[\varphi]}{2}
$$
is induced by the union of two ensembles

\[ \hat{v}_{n,1}^{\ell-2} := \{ \hat{v}_{n,1,k}^{\ell-2} \}_{k=1}^{P_{\ell-2}} \cup \{ \hat{v}_{n,2}^{\ell-2} \}_{k=1}^{P_{\ell-2}} \]

and

\[ \mu_n^{\ell-1}[\varphi] := \left( \frac{\mu_n^{\ell-1,1} + \mu_n^{\ell-1,2}}{2} \right)[\varphi] \]

is induced by \( \hat{v}_{n,1}^{\ell} := \{ \hat{v}_{n,1,k}^{\ell} \}_{k=1}^{P_{\ell}} \). For consistency with (7), we impose the condition that \( \mu_n^{\ell-1,1}[\varphi] = \mu_n^{\ell-1}[\varphi] = 0 \) when \( \ell_1 = 0 \), and \( \mu_n^{\ell-2}[\varphi] = 0 \) when \( \ell_2 = 0 \). Figure 1 shows a visual description of all the couplings of the MIEnKF estimator.

Then, the MIEnKF estimator (10) can also be written

\[ \mu_n^{\ell M}[\varphi] := \sum_{\ell \in I} \sum_{m=1}^{M_{\ell}} \left( \mu_n^{m} - \mu_n^{\ell,1,m} - \mu_n^{\ell,2,m} + \mu_n^{\ell,1-m} \right)[\varphi] \]

where \( \{ (\mu_n^{m}, \mu_n^{\ell,1,m}, \mu_n^{\ell,2,m}, \mu_n^{\ell,1-m})[\varphi] \}_m \) are independent copies of the estimators \( (\mu_n^{m}, \mu_n^{\ell,1,m}, \mu_n^{\ell,2,m}, \mu_n^{\ell,1-m})[\varphi] \) and \( \{ (\mu_n^{m}, \mu_n^{\ell,1,m}, \mu_n^{\ell,2,m}, \mu_n^{\ell,1-m})[\varphi] \}_m \) are mutually independent.

**Remark 3.** For comparison, the MLEnKF estimator (6) takes the following form when represented in the above 2-index notation

\[ \mu_n^{\ell M}[\varphi] = \sum_{\ell=0}^{L} \sum_{m=1}^{M_{\ell}} \left( \mu_n^{(\ell,m)} \right)[\varphi] \]

3.2. Particle-wise four-coupling for MIEnKF. We now describe how the four-coupling of EnKF estimators manifests itself particle by particle. At time \( n = 0 \), the fine-index update ensemble \( \{ \hat{v}_{0,i}^{\ell} \}_{i=1}^{P_{\ell}} \) comprises independent \( P_{w_0|y_0} \)-distributed samples that are particle-wisely coupled to three other ensembles by

\[ \hat{v}_{0,i}^{\ell} = \hat{v}_{0,i}^{\ell-e_1} = \hat{v}_{0,i}^{\ell-e_2} = \hat{v}_{0,i}^{\ell-1} \quad \text{for} \quad i = 1, 2, \ldots, P_{\ell}. \]

To describe how the coupling enters in the prediction-update iterations of the particles, let us consider the update state of a foursome \( \{ \hat{v}_{n+1,i}^{\ell-e_1}, \hat{v}_{n+1,i}^{\ell-e_2}, \hat{v}_{n+1,i}^{\ell-1} \} \) at time \( n \geq 0 \). The next-time prediction state of the foursome is given by

\[ \begin{align*}
\hat{v}_{n+1,i}^{\ell-e_1} &= \Psi_n^{N_{\ell}^{-1}}(\hat{v}_{n+1,i}^{\ell-e_1}), \\
\hat{v}_{n+1,i}^{\ell-e_2} &= \Psi_n^{N_{\ell}^{-1}}(\hat{v}_{n+1,i}^{\ell-e_2}), \\
\hat{v}_{n+1,i}^{\ell-1} &= \Psi_n^{N_{\ell}^{-1}}(\hat{v}_{n,i}^{\ell-1}),
\end{align*} \]

where the four particles share the same driving noise in the dynamics. The sample covariance matrices and Kalman gains are expressed as follows:

\[ \begin{align*}
C_{n+1}^{\ell} &= \text{Cov} \{ \hat{v}_{n+1,i}^{\ell} \}, & K_{n+1}^{\ell} &= C_{n+1}^{\ell} H^T (HC_{n+1}^{\ell} H^T + \Gamma)^{-1}, \\
C_{n+1}^{\ell-e_1} &= \text{Cov} \{ \hat{v}_{n+1,i}^{\ell-e_1} \}, & K_{n+1}^{\ell-e_1} &= (HC_{n+1}^{\ell-e_1} H^T + \Gamma)^{-1}, \\
C_{n+1}^{\ell-e_2} &= \text{Cov} \{ \hat{v}_{n+1,i}^{\ell-e_2} \}, & K_{n+1}^{\ell-e_2} &= (HC_{n+1}^{\ell-e_2} H^T + \Gamma)^{-1}, \\
C_{n+1}^{\ell-1} &= \text{Cov} \{ \hat{v}_{n+1,i}^{\ell-1} \}, & K_{n+1}^{\ell-1} &= (HC_{n+1}^{\ell-1} H^T + \Gamma)^{-1},
\end{align*} \]
where we recall that index $\ell - e_2$ and index $\ell - 1$ both consist of two EnKF ensembles of size $P_{\ell_2 - 1}$, cf. Section 3.1. The perturbed observations are also particle-wisely coupled, so that one obtains the updated states:

\[
\begin{align*}
\hat{y}_{n+1,i}^\ell &= y_{n+1} + \eta_{n+1,i}^\ell, \\
\hat{v}_{n+1,i}^\ell &= (I - K_{n+1}^\ell H)\hat{v}_{n+1,i}^\ell + K_{n+1}^\ell \hat{y}_{n+1,i}^\ell, \\
\hat{v}_{n+1,i}^{\ell - e_1} &= (I - K_{n+1}^{\ell - e_1} H)\hat{v}_{n+1,i}^{\ell - e_1} + K_{n+1}^{\ell - e_1} \hat{y}_{n+1,i}^{\ell - e_1}, \\
\end{align*}
\]

and

\[
\begin{align*}
\hat{v}_{n+1,i}^{\ell - e_2,1} &= (I - K_{n+1}^{\ell - e_2,1} H)\hat{v}_{n+1,i}^{\ell - e_2,1} + K_{n+1}^{\ell - e_2,1} \hat{y}_{n+1,i}^{\ell - e_2,1}, \\
\hat{v}_{n+1,i}^{\ell - e_2,2} &= (I - K_{n+1}^{\ell - e_2,2} H)\hat{v}_{n+1,i}^{\ell - e_2,2} + K_{n+1}^{\ell - e_2,2} \hat{y}_{n+1,i}^{\ell - e_2,2}, \\
\hat{v}_{n+1,i}^{\ell - 1,1} &= (I - K_{n+1}^{\ell - 1,1} H)\hat{v}_{n+1,i}^{\ell - 1,1} + K_{n+1}^{\ell - 1,1} \hat{y}_{n+1,i}^{\ell - 1,1}, \\
\hat{v}_{n+1,i}^{\ell - 1,2} &= (I - K_{n+1}^{\ell - 1,2} H)\hat{v}_{n+1,i}^{\ell - 1,2} + K_{n+1}^{\ell - 1,2} \hat{y}_{n+1,i}^{\ell - 1,2}, \\
\end{align*}
\]

where $\{\eta_{n+1,i}^\ell \}_{i=1}^{P_{\ell_2}}$ are i.i.d. with $\eta_{n+1,1}^\ell \sim N(0, \Gamma)$.

To summarize, four ensembles are particle-wisely coupled by sharing the initial condition, the driving noise, and perturbed observations. A sketch of one prediction-update iteration of the MIEEnKF method and the composition of the MIEEnKF estimator is provided in Figure 1 and Algorithm 1 describes the essential steps of the MIEEnKF method.

4. MIEEnKF Complexity

This section presents a cost-versus-accuracy result for the MIEEnKF method, and compares the performance of MIEEnKF to MLenKF and EnKF.

Let us first recall that we restrict ourselves to resolutions of the form

\[
N_{\ell_1} = N_0 \times 2^{\ell_1} \quad \text{and} \quad P_{\ell_2} = P_0 \times 2^{\ell_2} \quad \forall \ell_1 \in \mathbb{N}_0,
\]

for some $N_0, P_0 \in \mathbb{N}$, and proceed with defining the notion of admissible QoIs:

**Definition 1** (Admissible QoI). A Borel-measurable mapping $\varphi : \mathbb{R}^d \to \mathbb{R}$ is said to be an admissible QoI if it satisfies the following two integrability conditions for all $n \geq 0$:

\[
\tilde{\mu}_n[\varphi] < \infty \quad \text{and} \quad \mu_n^\ell[\varphi] \in L^2(\Omega) \quad \forall \ell \in \mathbb{N}_0.
\]

For any admissible QoI $\varphi$ and $\ell \in \mathbb{N}_0$, the definition implies that $\Delta \mu_n^\ell[\varphi] \in L^2(\Omega)$, and we impose the additional assumptions to ensure good performance for MIEEnKF:

**Assumption 2.** For any admissible QoI $\varphi$ and any $n \geq 0$, the four-coupled EnKF estimator $\Delta \mu_n^\ell[\varphi]$ satisfies the following conditions:

\[
\begin{align*}
\mathbb{E}[\Delta \mu_n^\ell[\varphi]] &\lesssim N_{\ell_1}^{-1} P_{\ell_2}^{-1}, \\
\forall [\Delta \mu_n^\ell[\varphi]] &\lesssim N_{\ell_1}^{-2} P_{\ell_2}^{-2}, \\
\text{and} \\
\text{Cost}(\Delta \mu_n^\ell[\varphi]) &\approx N_{\ell_1} P_{\ell_2}. \quad \text{2}
\end{align*}
\]

The constraint (A3) could be stated as a property rather than an assumption, as it holds by construction. Every prediction-update iteration of the coupled EnKF ensembles relating to $\Delta \mu_n^\ell$ costs $\mathcal{O}(N_{\ell_1} P_{\ell_2})$, cf. Section 3.2.
Algorithm 1: MIEnKF

Input: The model parameters, the QoI $\varphi$, the final time $N$, the observation operator $H$, observations $\{y_n\}_{n=1}^N$, $m_0$, $\Sigma_0$, $\Gamma$, $L$, $M_\ell$, $N_\ell$, $P_\ell$.

Output: The MIEnKF estimator $\mu_n^{MI}[\varphi]$.

for $\ell \in \mathcal{I}$ do
  for $m = 1 : M_\ell$ do
    Initialize the ensembles at time $n = 0$ by sampling $v_{0,i}^{\ell,m} \sim N(m_0, \Sigma_0)$ and setting $\tilde{v}_{0,i}^{\ell,m} = \tilde{v}_{0,i}^{\ell-1,m}$ for $i = 1, ..., P_\ell$.
  end for
  for $n = 1 : N$ do
    for $\ell \in \mathcal{I}$ do
      if $\ell_1 = 0$ and $\ell_2 = 0$ then
        Compute the EnKF prediction states for $i = 1, ..., P_\ell$
        $v_{n,i}^{\ell,m} = \text{Prediction}(v_{n-1,i}^{\ell,m})$ similar to (1).
        Compute the EnKF updated states for $i = 1, ..., P_\ell$
        $\tilde{v}_{n,i}^{\ell,m} = \text{Update}(v_{n,i}^{\ell,m})$ similar to (2).
        Compute the EnKF estimator $\Delta\mu_n^{\ell,m}[\varphi] = \sum_{i=1}^{P_\ell} \frac{\varphi(\tilde{v}_{n,i}^{\ell,m}) - \varphi(v_{n,i}^{\ell,m})}{P_\ell}$.
      else if $\ell_1 > 0$ and $\ell_2 = 0$ then
        Compute pairwise coupled prediction states for $i = 1, ..., P_\ell$
        $v_{n,i}^{\ell,m}, v_{n,i}^{\ell-e_1,m} = \text{Prediction}(v_{n-1,i}^{\ell,m}, v_{n-1,i}^{\ell-e_1,m})$ similar to (4).
        Compute pairwise coupled updated states for $i = 1, ..., P_\ell$
        $\tilde{v}_{n,i}^{\ell,m}, \tilde{v}_{n,i}^{\ell-e_1,m} = \text{Update}(v_{n,i}^{\ell,m}, v_{n,i}^{\ell-e_1,m})$ similar to (5).
        Compute the EnKF estimator pairwise coupled in $N_{\ell_1}$
        $\Delta\mu_n^{\ell,m}[\varphi] = \sum_{i=1}^{P_\ell} \frac{\varphi(\tilde{v}_{n,i}^{\ell,m}) - \varphi(v_{n,i}^{\ell,m})}{P_\ell}$.
      else if $\ell_1 = 0$ and $\ell_2 > 0$ then
        Compute pairwise coupled prediction states for $i = 1, ..., P_\ell$
        $v_{n,i}^{\ell,m}, v_{n,i}^{\ell-e_2,m} = \text{Prediction}(v_{n-1,i}^{\ell,m}, v_{n-1,i}^{\ell-e_2,m})$ similar to (12)-(14).
        Compute pairwise coupled updated states for $i = 1, ..., P_\ell$
        $\tilde{v}_{n,i}^{\ell,m}, \tilde{v}_{n,i}^{\ell-e_2,m} = \text{Update}(v_{n,i}^{\ell,m}, v_{n,i}^{\ell-e_2,m})$ similar to (15)-(16).
        Compute the EnKF estimator pairwise coupled in $P_{\ell_2}$
        $\Delta\mu_n^{\ell,m}[\varphi] = \sum_{i=1}^{P_\ell} \frac{\varphi(\tilde{v}_{n,i}^{\ell,m}) - \varphi(v_{n,i}^{\ell,m})}{P_\ell}$.
      else if $\ell_1 > 0$ and $\ell_2 > 0$ then
        Compute the four-coupled prediction states for $i = 1, ..., P_\ell$
        $v_{n,i}^{\ell,m}, v_{n,i}^{\ell-e_1,m}, v_{n,i}^{\ell-e_2,m}, \tilde{v}_{n,i}^{\ell-1,m} = \text{Prediction}(v_{n-1,i}^{\ell,m}, v_{n-1,i}^{\ell-e_1,m}, v_{n-1,i}^{\ell-e_2,m}, \tilde{v}_{n-1,i}^{\ell-1,m})$ by (12)-(14).
        Compute the four-coupled updated states for $i = 1, ..., P_\ell$
        $\tilde{v}_{n,i}^{\ell,m}, \tilde{v}_{n,i}^{\ell-e_1,m}, \tilde{v}_{n,i}^{\ell-e_2,m}, \tilde{v}_{n,i}^{\ell-1,m} = \text{Update}(v_{n,i}^{\ell,m}, v_{n,i}^{\ell-e_1,m}, v_{n,i}^{\ell-e_2,m}, \tilde{v}_{n,i}^{\ell-1,m})$ by (15)-(16).
        Compute the four-coupled EnKF estimator
        $\Delta\mu_n^{\ell,m}[\varphi] = \sum_{i=1}^{P_\ell} \frac{\varphi(\tilde{v}_{n,i}^{\ell,m}) - \varphi(v_{n,i}^{\ell,m}) - \varphi(\tilde{v}_{n,i}^{\ell-e_2,m}) + \varphi(\tilde{v}_{n,i}^{\ell-1,m})}{P_\ell}$.
      end if
    end for
  end for
end for

Compute the MIEnKF estimator $\mu_n^{MI}[\varphi] = \sum_{\ell \in \mathcal{I}} \sum_{m=1}^{M_\ell} \frac{\Delta\mu_n^{\ell,m}[\varphi]}{M_\ell}$. 
Figure 1. One prediction-update iteration of the multi-index ensemble Kalman filtering (MIEnKF) estimator described in Section 3.2. The ovals represent four-coupled prediction-state particles, sharing the same driving noise $\omega^\ell$ and coupled initial conditions. The respective squares represent updated-state particles sharing the same perturbed observations.

**Theorem 1 (MIEnKF complexity).** Let Assumptions 1 and 2 hold, and for any $\epsilon > 0$ consider the MIEnKF method with triangular index set $\mathcal{I} = \{\ell \in \mathbb{N}_0^2 \mid \ell_1 + \ell_2 \leq L\}$, where

$$L = \max \left( \lceil \log \epsilon^{-1} + \log \log \epsilon^{-1} \rceil - L_0, 1 \right) \quad \text{for some} \quad L_0 \in \mathbb{N}_0$$

and the number of samples

$$M_\ell \approx \epsilon^{-2} N_{\ell_1}^{-3/2} P_{\ell_2}^{-3/2} \quad \ell \in \mathcal{I}.$$
For any admissible QoI \( \phi \) and \( n \geq 0 \), it then holds that
\[
E \left[ (\mu_n^{M1}[\phi] - \bar{\mu}_n[\phi])^2 \right] \lesssim \epsilon^2,
\]
and the computational cost of the MIEnKF estimator satisfies that
\[
\text{Cost}(\mu_n^{M}[\phi]) \approx \epsilon^{-2}.
\]

Proof. Adding and subtracting \( E [\mu_n^{M1}[\phi]] \) in the mean-squared error, we obtain
\[
E \left[ (\mu_n^{M1}[\phi] + E [\mu_n^{M1}[\phi]] - \bar{\mu}_n[\phi])^2 \right] = V[\mu_n^{M1}[\phi]] + E [\mu_n^{M1}[\phi] - \bar{\mu}_n[\phi])^2.
\]
For the variance term, the independence of the random variables \( \{\mu_n^{M1}[\phi]\}_{j,m} \) and \( (A1) \) yield
\[
V[\mu_n^{M1}[\phi]] = \sum_{\ell \in \mathcal{I}} \sum_{m=1}^{M_\ell} E \left[ \Delta \mu_n^{\ell,m}[\phi] \right]^2 \lesssim \sum_{\ell \in \mathcal{I}} M_\ell^{-1} N_{\ell_1}^{-2} P_{\ell_2}^{-2} \lesssim \epsilon^2.
\]
For the squared bias term, \( (A1) \) and the multi-index telescoping properties of the MIEnKF estimator imply that
\[
\left( E [\mu_n^{M1}[\phi]] - \bar{\mu}_n[\phi] \right)^2 \lesssim \left( \sum_{\ell \notin \mathcal{I}} E \left[ \Delta \mu_n^{\ell}[\phi] \right] \right)^2 \lesssim \left( \sum_{\ell \notin \mathcal{I}} N_{\ell_1}^{-1} P_{\ell_2}^{-1} \right)^2.
\]
The mean-squared error bound (17) follows by
\[
\sum_{\ell \notin \mathcal{I}} N_{\ell_1}^{-1} P_{\ell_2}^{-1} \lesssim \sum_{k=L+1}^{\infty} (k+1) 2^{-k} \approx 2^{-L} L \approx \epsilon
\]
and
\[
\text{Cost}(\mu_n^{M1}[\phi]) = \sum_{\ell \in \mathcal{I}} M_\ell \text{Cost}(\Delta \mu_n^{\ell}[\phi]) \approx \sum_{\ell \in \mathcal{I}} M_\ell N_{\ell_1} P_{\ell_2} \approx \epsilon^{-2}.
\]

Remark 4. For comparison, we briefly recall the cost-versus-accuracy results for EnKF and MLEnKF. For any \( \epsilon > 0 \) and sufficient regularity
\[
\|(\mu_n^{N,P}[\phi] - \bar{\mu}_n)[\phi]\|_p \lesssim \epsilon, \quad \text{EnKF}
\]
\[
\|(\mu_n^{ML}[\phi] - \bar{\mu}_n)[\phi]\|_p \lesssim \epsilon, \quad \text{MLEnKF}
\]
with the computational cost bounded by
\[
\text{Cost}(\mu_n^{N,P}[\phi]) \approx \epsilon^{-3}, \quad \text{Cost}(\mu_n^{ML}[\phi]) \approx \epsilon^{-2} \|\log(\epsilon)\|^3.
\]
For more details, see [35].

Remark 5. An alternative to \( (A2) \) that is more aligned with assumption made for the existing convergence results for MLEnKF is to assume that
\[
\|(\Delta \mu_n^{\ell}[\phi]\|_p \lesssim N_{\ell_1}^{-1} P_{\ell_2}^{-1}.
\]

\(^3\)Optimal choices for \( L \) and \( M_\ell \) in Theorem 1 can be obtained by solving a constrained optimization problem, analogously as elaborated on for the MLMC method in the seminal paper [24].
for \( p \geq 2 \). Then, using the same aforementioned index set \( \mathcal{I} \) and \( \mathcal{L} \) and with a slight change in the sample size \( M_\ell \approx \epsilon^{-2}N_{\ell_1}^{-4/3}P_{\ell_2}^{-4/3} \), the MIEnKF estimator satisfies

\[
\| (\mu_{n,M}^{\ell} - \bar{\mu}_n) [\varphi] \|_p \lesssim \epsilon,
\]

with the asymptotic MIEnKF cost bounded by \( O(\epsilon^{-2}) \). This can be proved similarly as the case of Theorem 1, where the \( L_p \)-norm of the statistical error can be bounded using the Marcinkiewicz-Zygmund inequality:

\[
\| (\mu_{n,M}^{\ell} - \bar{\mu}_n) [\varphi] \|_p \leq \| \mu_{n,M}^{\ell} [\varphi] - \mathbb{E}[\mu_{n,M}^{\ell} [\varphi]] \|_p + \| \mathbb{E}[\mu_{n,M}^{\ell} [\varphi]] - \bar{\mu}_n [\varphi] \|_p \\
\lesssim \sum_{\ell \in \mathcal{I}} M_\ell^{-1/2} \| \Delta \mu_{n,M}^{\ell,m} [\varphi] \|_p + \sum_{\ell \notin \mathcal{I}} \| \mathbb{E}[\Delta \mu_{n,M}^{\ell,m} [\varphi]] \|.
\]

We note that (A2) is a weaker assumption than (A2') since

\[
\forall \| \Delta \mu_{n,M}^{\ell} [\varphi] \|_p \leq \| \Delta \mu_{n,M}^{\ell} [\varphi] \|_2^2 \leq \| \Delta \mu_{n,M}^{\ell} [\varphi] \|_p^2.
\]

**Remark 6.** Under more general settings, Assumption 2 may be transformed into

\[
\text{(A1)} \quad \| \mathbb{E}[\Delta \mu_{n,M}^{\ell} [\varphi]] \| \leq N_{\ell_1}^{-\alpha_1}P_{\ell_2}^{-\alpha_2},
\]

\[
\text{(A2)} \quad \| \Delta \mu_{n,M}^{\ell} [\varphi] \|_p \leq N_{\ell_1}^{-\beta_1}P_{\ell_2}^{-\beta_2},
\]

\[
\text{(A3)} \quad \text{Cost}(\Delta \mu_{n,M}^{\ell} [\varphi]) \approx N_{\ell_1}^{\gamma_1}P_{\ell_2}^{\gamma_2},
\]

for some \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0 \). The construction of an efficient MIEnKF estimator may then lead to a differently shaped (possibly even non-triangular) index set \( \mathcal{I} \), a different sequence of number of samples \( \{M_\ell\}_{\ell \in \mathcal{I}} \), and other common ratios for the geometric sequences \( \{N_{\ell_1}\} \) and \( \{P_{\ell_2}\} \). The problem of optimizing the set \( \mathcal{I} \) may be recast as a knapsack problem, which is a well-studied optimization problem with many available solution algorithms, cf. [29] and [30, equation (21)]. For instance, the approach developed for approximations of multi-index Monte Carlo applied to McKean-Vlasov dynamics in [30] defines the set by

\[
\mathcal{I} = \{(\ell_1, \ell_2) \in \mathbb{N}_0^2 : (\alpha_1 + \gamma_1)\ell_1 + (\alpha_2 + \gamma_2)\ell_2 \leq L \}.
\]

### 5. Numerical examples

This section presents a numerical comparison of MIEnKF with the EnKF and MLEnKF methods outlined in Section 2. Three problems will be considered: the Ornstein-Uhlenbeck (OU) process, a stochastic differential equation (SDE) with a double-well (DW) potential, and Langevin dynamics [34, 35, 6, 4, 18].

We consider SDE on the general form

\[
du = -U'(u)dt + \sigma dW_t,
\]

with a constant diffusion coefficient \( \sigma = 0.5 \) and two types of potential functions:

(i) \( U(u) = u^2/2 \),  \quad \text{(OU)}  
(ii) \( U(u) = u^2/4 + 1/(4u^2 + 2) \),  \quad \text{(DW)}.

The numerical discretizations of (20) are computed using the Milstein numerical scheme\(^4\) with uniform timestep \( \Delta t = 1/N \) for any \( N \geq 1 \). The observations of

---

\(^4\)Note that in all three examples considered, the SDEs are with constant diffusion terms. For such SDEs, the Milstein scheme coincides with the Euler-Maruyama scheme.
the process \( u \) are equally spaced with observation time interval \( \tau = 1 \), observation operator \( H = 1 \), \( \Gamma = 0.1 \) and the QoI \( \varphi(x) = x \).

To numerically verify assumptions (A1) and (A2\( ^* \)), the following rates are estimated from \( S \) independent copies of \( \Delta \mu_n^\ell[\varphi] \):

\[
\left| \mathbb{E}[\Delta \mu_n^\ell[\varphi]] \right| \approx \left| \frac{\sum_{i=1}^S \Delta \mu_{n,i}^\ell[\varphi]}{S} \right|,
\]

\[
\| \Delta \mu_n^\ell[\varphi] \|_2 \approx \sqrt{\frac{1}{S} \sum_{i=1}^S \| \Delta \mu_{n,i}^\ell[\varphi] \|^2}.
\]

We analyze the convergence rates of the methods by computing the time-averaged root-mean-squared error (RMSE).

\[
\text{RMSE} := \sqrt{\frac{1}{S(N+1)} \sum_{i=1}^S \sum_{n=0}^{N'} \left| \mu_{n,i}^\ell[\varphi] - \bar{\mu}_n[\varphi] \right|^2},
\]

where \( \{ \mu_{n,i}^\ell[\varphi] \}_{i=1}^S \) are independent copies of \( \mu^\ell[\varphi] \) for the specific methods (EnKF, MLEnKF, and MIEnKF).

5.1. Reference solutions and computer architecture. Since dynamics \( \Psi \) is linear for the OU problem, the reference solution \( \bar{\mu}_n[\varphi] \) can be computed exactly using the Kalman filter. However, the reference solution for the DW problem, which involves nonlinear dynamics, must be approximated. This solution is computed using the deterministic mean-field EnKF algorithm, cf. [35, Appendix C]. A pseudoreference solution for the final test problem based on Langevin dynamics is computed by the sample average of \( S = 180 \) independent simulations of the MIEnKF estimator at the tolerance \( \epsilon = 2^{-11} \) using the following parameters:

\[
L = \lfloor L_\star + \log_2(L_\star) \rfloor - 1, \quad \text{with} \quad L_\star = \lfloor \log_2(\epsilon^{-1}) \rfloor - 1,
\]

\[
N_{\ell_1} = 4 \times 2^{\ell_1}, \quad P_{\ell_2} = 30 \times 2^{\ell_2},
\]

\[
M_\ell = \begin{cases} 
6 \times \lfloor \epsilon^{-2}N_{\ell_1}^{-3/2}P_{\ell_2}^{-3/2} \rfloor & \text{if} \quad \ell_1 = 0 \text{ and } \ell_2 = 0, \\
90 \times \lfloor \epsilon^{-2}N_{\ell_1}^{-3/2}P_{\ell_2}^{-3/2} \rfloor & \text{if} \quad 1 \leq \ell_1 + \ell_2 \leq L.
\end{cases}
\]

The numerical simulations were computed in parallel on 18 cores on an Intel(R) Xeon(R) CPU E5-2680 v2 20-core processor with 128 GB RAM. The computer code was written in the Julia programming language [14], and it can be downloaded from https://github.com/GaukharSH/mienkf.

5.2. Ornstein-Uhlenbeck process. We consider the SDE (20) with the (OU) potential function and initial condition \( u(0) \sim N(0, \Gamma) \). Convergence rates (A1) and (A2\( ^* \)) shown in Figure 2 were estimated by the Monte Carlo method using \( S = 10^6 \) independent samples of \( \Delta \mu_n^\ell[\varphi] \). In the figure, the left panel shows the weak and \( L_2 \) convergence rates over \( N' = 10 \) observation times with \( (\ell_1 + \ell_2) \in [0, 7] \), and the right panel shows the ratio of the rates to \( N_{\ell_1}^{-1}P_{\ell_2}^{-1} \). The plane-like flatness of the right panel for \( (\ell_1 + \ell_2) \in [1, 7] \) validates the said rate assumptions.
When conducting runtime-versus-accuracy convergence tests for an input tolerance \( \epsilon > 0 \), we set the parameters of the respective methods as follows:

\[
\text{EnKF:} \quad P = \lfloor 15\epsilon^{-2} \rfloor \quad \text{and} \quad N = \lfloor \epsilon^{-1} \rfloor, \\
L = \lfloor \log_2(\epsilon^{-1}) \rfloor - 1, \\
N_\ell = 2 \times 2^\ell, \\
M_\ell = \begin{cases} 
2 \times \lfloor \epsilon^{-2}L^22^{-3} \rfloor & \text{if } \ell = 0, \\
\lfloor \epsilon^{-2}L^22^{-2\ell-3} \rfloor & \text{if } 1 \leq \ell \leq L,
\end{cases}
\]

\[
\text{MLEnKF:} \\
P_\ell = 10 \times 2^\ell, \\
M_\ell = \begin{cases} 
2 \times \lfloor \epsilon^{-2}L^22^{-3} \rfloor & \text{if } \ell = 0, \\
\lfloor \epsilon^{-2}L^22^{-2\ell-3} \rfloor & \text{if } 1 \leq \ell \leq L,
\end{cases}
\]

\[
\text{MIEnKF:} \\
P_{t_2} = 30 \times 2^{t_2}, \\
M_\ell = \begin{cases} 
6 \times \lfloor \epsilon^{-2}N_{t_1}^{-3/2}P_{t_2}^{-3/2} \rfloor & \text{if } \ell_1 = 0 \text{ and } \ell_2 = 0, \\
120 \times \lfloor \epsilon^{-2}N_{t_1}^{-3/2}P_{t_2}^{-3/2} \rfloor & \text{if } 1 \leq \ell_1 + \ell_2 \leq L.
\end{cases}
\]

For a sequence of predefined tolerances \( \epsilon = [2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}] \) for EnKF and MLEnKF, and \( \epsilon = [2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10}, 2^{-11}] \) for MIEnKF, Figure 3 shows the runtime against the RMSE for the three methods over observation times \( \mathcal{N} = 10 \) and \( \mathcal{N} = 100 \) estimated using \( S = 100 \) independent runs. MIEnKF outperforms EnKF and MLEnKF for sufficiently small tolerances, and the complexity rate agrees with the theory.

5.3. Double-well SDE. We consider the SDE (20) with the DW potential function \( u(0) \sim N(0, \Gamma) \). Similar to the OU case, Figure 4 provides numerical evidence of the conjecture rates under assumptions (A1) and (A2'). For the same predefined \( \epsilon \)-inputs with the same degrees of freedom setting as in the example of OU, the performance of the three methods were compared in terms of runtime against RMSE for observation times \( \mathcal{N} = 10 \) and \( \mathcal{N} = 100 \) and estimated over \( S = 100 \) independent runs (Figure 5). We observe that MIEnKF outperforms EnKF and MLEnKF for small RMSE.

5.4. Langevin SDE. In the last example, we consider the two-dimensional stochastic Langevin dynamics

\[
\begin{align*}
\text{EnKF:} \quad P &= \lfloor 10\epsilon^{-2} \rfloor \quad \text{and} \quad N = \lfloor \epsilon^{-1} \rfloor, \\
dX_t &= V_t dt, \\
dV_t &= -U'(X_t)dt - \kappa V_t dt + (2\kappa T)^{1/2}dW_t,
\end{align*}
\]

where \( X_t \) and \( V_t \) denotes the particle position and velocity, respectively, \( U(X) \) is the previously introduced DW potential, \( \kappa = 2^{-5} \times \pi^2 \) is the viscosity and \( T = 1 \) is the temperature. To improve the pairwise coupling between particles, we used the first-order symplectic Euler splitting scheme [52]. The initial conditions are provided by \( X_0 \sim N(0, \Gamma) \) and \( V_0 \sim N(0, \Gamma) \) with \( X_0 \) and \( V_0 \) being independent. Further, based on initial test runs, the method parameters are set to
Figure 2. Ornstein-Uhlenbeck problem. Estimates based on $S = 10^6$ independent runs (Section 5.2). Top row: Numerical evidence of assumption (A1) over $N = 20$ observation times when using $N_{\ell_1} = 4 \times 2^{\ell_1}$ and $P_{\ell_2} = 20 \times 2^{\ell_2}$. Bottom row: Similar plots for the verification of assumption (A2$^\star$).

MLEnKF:
\[
\begin{align*}
L &= \lceil \log_2(\epsilon^{-1}) \rceil - 1, \\
N_{\ell} &= 2 \times 2^\ell, \\
P_{\ell} &= 8 \times 2^\ell, \\
M_{\ell} &= \begin{cases} 
2 \times \lceil \epsilon^{-2}L^22^{-2}\rceil & \text{if } \ell = 0, \\
\lceil \epsilon^{-2}L^22^{-2\ell-2}\rceil & \text{if } 1 \leq \ell \leq L,
\end{cases}
\end{align*}
\]

and

MIEnKF:
\[
\begin{align*}
L &= \lceil L_\star + \log_2(L_\star) \rceil - 1, \quad \text{with } L_\star = \lceil \log_2(\epsilon^{-1}) \rceil - 1 \\
N_{\ell_1} &= 4 \times 2^{\ell_1}, \\
P_{\ell_2} &= 20 \times 2^{\ell_2}, \\
M_{\ell} &= \begin{cases} 
6 \times \lceil \epsilon^{-2}N_{\ell_1}^{-3/2}P_{\ell_2}^{-3/2}\rceil & \text{if } \ell_1 = 0 \text{ and } \ell_2 = 0, \\
50 \times \lceil \epsilon^{-2}N_{\ell_1}^{-3/2}P_{\ell_2}^{-3/2}\rceil & \text{if } 1 \leq \ell_1 + \ell_2 \leq L.
\end{cases}
\end{align*}
\]

To shed some light on the importance of the temperature parameter, Figure 6 illustrates the phase-portrait time evolution of the realization of Langevin dynamics up to the final time $N = 50$ for different temperatures $T = [0, 0.01, 0.1, 1.0]$. Damping causes a rapid decay of the velocity from the initial value to zero when
Figure 3. Ornstein-Uhlenbeck problem. Estimates based on $S = 100$ independent runs (Section 5.2). Comparison of the runtime versus root-mean-squared error (RMSE) for mean over observation times $\mathcal{N} = 10$ (left) and $\mathcal{N} = 100$ (right). The solid-crossed line represents MLEnKF and the dot-dashed line is a fitted $\mathcal{O}(\log(10 + \text{Runtime})^{1/3}\text{Runtime}^{-1/2})$ reference line. The solid-asterisk line represents the MIEnKF and the dotted line is a fitted $\mathcal{O}(\text{Runtime}^{-1/2})$ reference line. The solid-bulleted line represents EnKF and the dashed line is a fitted $\mathcal{O}(\text{Runtime}^{-1/3})$ reference line.

$T = 0$. For positive temperatures, thermal fluctuation leads to more diffusive dynamics. Figure 7 shows the signal-tracking performance of MIEnKF for the full observation operator

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the partial observation operators $H = [1 \ 0]$ or $H = [0 \ 1]$, all computed at the tolerance $\epsilon = 2^{-7}$. The method is tracking the true state of the observed components well in all cases, but, as is to be expected, it does not track the true state of unobserved components with the same level of accuracy. The numerical verification of assumptions (A1) and (A2*) with respect to different observation operators is shown in Figures 8 and 9, respectively. For a sequence of predefined tolerances, $\epsilon = [2^{-4}, 2^{-5}, \ldots, 2^{-9}]$ for EnKF and MLEnKF and $\epsilon = [2^{-4}, 2^{-5}, \ldots, 2^{-10}]$ for MIEnKF, we compare the performance of the three methods in terms of runtime versus RMSE. We consider $\mathcal{N} = 10$ and $\mathcal{N} = 20$ observation times, the QoI $\varphi(X,V) = X$ and $\varphi(X,V) = V$, and we use $S = 90$ independent runs of each method to estimate both RMSE and runtime. Figures 10 and 11 show the results for the observation operators

$$H = [1 \ 0] \quad \text{and} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

respectively. The observed complexity rates for MIEnKF are close to the theory, and the method is more efficient than the alternatives for small tolerances in both cases.
6. Conclusion

We have developed a hierarchical ensemble-based filtering method called the MIEnKF method. MIEnKF is based on independent samples of four-coupled EnKF estimators on a multi-index hierarchy of resolution levels. Under Assumptions 1 and 2, we proved that the method is highly efficient and that it will asymptotically outperform the comparable methods EnKF and MLEnKF. For instance, when the weak convergence rate $\alpha = 1$ and the strong convergence rate $\beta = 2$, which is a more robust setting of the EnKF and MLEnKF methods considered in [35], the computational cost of reaching $O(\epsilon^2)$ MSE is $O(\epsilon^{-2})$ for MIEnKF, $O(\epsilon^{-2} |\log(\epsilon)|^3)$ for MLEnKF, and $O(\epsilon^{-3})$ for EnKF.

In this work we have constructed a multi-index EnKF method with two resolution parameters: $N_{\ell_1}$ relating to the time-discretization, and the ensemble-size $P_{\ell_2}$, a 2-index MIEnKF method. For more complicated high-dimensional filtering problems, it is an open question if it is possible to extend MIEnKF to having more resolution parameters, and whether that would lead to further performance
Figure 5. Double Well problem. Estimates based on $S = 100$ independent runs (Section 5.3). Similar plots as those shown in Figure 3.

Figure 6. Time evolution of a solution to Langevin dynamics with different temperature $T$ values. The symplectic Euler scheme is used up to final time $N = 50$. The red dot represents the initial value.

... gains. One extension we currently working on is a 3-index MIEnKF for spatiotemporal models that are discretized in both space and time, e.g., reaction-diffusion stochastic partial differential equations (SPDE) [17].
Another interesting direction would be MIEnKF for filtering problems with high-frequency or continuous-time observations. Here, the new challenge is that low-resolution levels have to be updated – has to assimilate observations – at a lower frequency than high-resolution levels, but strong coupling still has to be preserved. The recent work on MLEnKF for Kalman-Bucy filters [16] would be a good starting point for developing an MIEnKF method for such problem settings.

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Figure 8. Langevin dynamics with partial observations, $H = [1 \ 0]$. Estimates based on $S = 10^6$ independent runs (Section 5.4). Top row in each subfigure: Numerical evidence of assumption (A1) for $N = 10$ observation times when using $N_{\ell_1} = 4 \times 2^\ell_1$ and $P_{\ell_2} = 20 \times 2^\ell_2$. Bottom row in each subfigure: Similar plots for verifying assumption (A2').

(a) The particle position $X_t$

(b) The particle velocity $V_t$
Figure 9. Langevin dynamics with full observations, $H = [1 \ 0 \ 0 \ 1]$. Estimates based on $S = 10^6$ independent runs (Section 5.4). Similar plots as those shown in Figure 8.
Figure 10. Langevin dynamics with partial observations, $H = [1 \ 0]$. Estimates based on $S = 90$ independent runs (Section 5.4). Top row: Comparison of the runtime versus RMSE for the mean of the component $X$ (left) and the component $V$ (right) over $N = 10$ observation times. The solid-crossed line represents MLEnKF and the dot-dashed line is a fitted $O(\log(10 + \text{Runtime})^{1/3}\text{Runtime}^{-1/2})$ reference line. The solid-asterisk line represents MIEnKF and the dotted line is a fitted $O(\text{Runtime}^{-1/2})$ reference line. The solid-bulleted line represents EnKF and the dashed line is a fitted $O(\text{Runtime}^{-1/3})$ reference line. Bottom row: Similar plots for $N = 20$ observation times.
Figure 11. Langevin dynamics with full observations, $H = [1 \ 0 \ 0 \ 1]$. Estimates based on $S = 90$ independent runs (Section 5.4). Similar plots as those shown in Figure 10.