Interaction of dark–bright solitons in two-component Bose–Einstein condensates

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Abstract
We study the interaction of dark–bright solitons in two-component Bose–Einstein condensates by suitably tailoring the trap potential, atomic scattering length and atom gain or loss. We show that the coupled Gross–Pitaevskii equation can be mapped onto the Manakov model. An interesting class of matter wave solitons and their interaction are identified with time-independent and periodically modulated trap potentials, which can be experimentally realized in two-component condensates. These include periodic collapse and revival of solitons, and snake-like matter wave solitons as well as different kinds of two-soliton interactions.

The advent of solitons in ultra-cold quantum gases has provided a new insight into the understanding of localized wave packets that travel over long distances without attenuation. In this context, recent explorations of solitons in Bose–Einstein condensates (BECs) have paved the way for new developments in manipulating coherent matter waves for applications, including atom interferometry, coherent atom transport and quantum information processing or quantum computation. In particular, experimental observations of matter wave solitons of the dark [1, 2], and bright [3–5] types in BECs have attracted a great deal of attention in connection with the dynamics of nonlinear matter waves, including soliton propagation [6, 7], vortex dynamics [8], interference patterns [9] and domain walls in binary BECs [10].

In this connection, recently much effort has been given to the study of matter wave solitons in BECs with time-varying control parameters such as (i) variation of atomic scattering length which can be achieved through Feshbach resonance [11–15], (ii) inclusion of the appropriate time-dependent gain or loss term which can be phenomenologically incorporated to account for the interaction of the atomic cloud or thermal cloud and (iii) periodic modulation of trap frequencies [16]. Bright solitons created in the experiments are themselves condensates and propagate over much larger distances than dark solitons which, on the other hand, can only exist as notches or holes within the condensates [1, 2]. Recently, dark solitons, their oscillations and interaction have been demonstrated in experiments with single component BECs [17–19]. Many studies on the bright/dark soliton formation and propagation of attractive/repulsive BECs have focused mainly on single-species systems.

Multi-component generalization of the soliton dynamics is very natural in the context of atomic BECs because of the several ways to create such systems, for example as mixtures with two different atomic species/hyperfine states [20–22] and as internal degrees of freedom liberated under an optical trap and atom–molecule BECs [23]. Multi-component BECs, far from being a trivial extension of the single-component ones, present novel and fundamentally different scenarios for their ground states and excitations [24, 25]. Matter wave solitons in multi-component BECs hold promise for a number of applications, including the multi-channel signals and their switching, coherent storage and processing of optical fields.

Since multi-component BECs are of greater interest for the aforesaid reasons, in this paper we investigate the interaction of dark and bright solitons in two-component BECs. In the context of cold atomic gases, the two vector components which evolve under the Gross–Pitaevskii (GP) equation are the macroscopic wavefunctions of Bose condensed atoms in two different internal states, which we shall denote as |1⟩ and |2⟩. If one considers condensates of, for instance, ²³Na and ⁸⁷Rb atoms, the nonlinear interactions are due to elastic s-wave scattering among the atoms, and are effectively repulsive (positive scattering length) for both
systems in which multi-component condensates have been realized [26]. Very recently, there has been a tremendous interest in studying the dynamics of two-component Bose–Einstein condensates coupled to the environment using both experimental and theoretical means [27–32]. Here we bring out exact bright–dark solitons in repulsively interacting two-component BECs coupled with the thermal clouds, which introduce gain/loss of condensate atoms. In particular, by considering both time-independent and periodically modulated trap potentials, we show fascinating interactions of matter wave solitons.

For the above purpose, we consider a trapped BEC with two components, where the dynamics takes place only in one dimension due to the strong trap confinement in the transverse direction. We have also included the interaction of external thermal clouds which is described by the gain or loss term. The properties of BECs that are prepared in two hyperfine states can be described at sufficiently low temperatures by the dimensionless form of two-coupled GP equations as [33]

\[
\frac{1}{i} \frac{\partial \psi_j}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi_j}{\partial x^2} + \left[ R(t) \sum_{k=1}^2 g_{jk} |\psi_k|^2 + V(x,t) - \mu_j + \frac{\gamma(t)}{2} \right] \psi_j, \quad j = 1, 2.
\]

(1)

Here \( V(x,t) = \frac{1}{2} \Omega^2(t)x^2 \) is the external potential, \( \Omega^2(t) = \omega_0^2/x^2 \), \( \omega_0 \) is the trap frequency in the axial direction, \( \omega_x \) is the radial trap frequency (\( \Omega^2(t) < 0 \) for expulsive potential and \( \Omega^2(t) > 0 \) for confining potential), \( \mu \) is the chemical potential, \( R(t) = 2\omega_x/\omega_0 \), \( g_{ij} \) is the s-wave scattering length, \( a_B \) is the Bohr radius and \( \gamma(t) = \gamma_0/\omega_0 \), \( \Gamma(t) \) is the gain/loss term, which is the phenomenologically incorporated interaction of the thermal cloud [34–36]. Here we have considered the case of a two-component condensate with \(^{87}\)Rb atoms prepared in two different hyperfine states for which the atomic masses, intra- and inter-component atomic scattering lengths are equal as described in [17]. Note that in equation (1), the variable \( x \) actually represents \( \frac{x}{a_\perp} \), where \( a_\perp = \sqrt{\frac{2}{\mu g_{11}}} \), and similarly \( t \) stands for \( \omega_\perp t \).

In equation (1) when the gain/loss term \( \gamma(t) \) is positive, it leads to the mechanism of loading external atoms (thermal clouds) into the BEC by optical pumping while \( \gamma(t) < 0 \) describes a BEC that is continuously depleted (loss) of atoms. If the condensate is fed by a surrounding thermal cloud, then the condensate undergoes an appropriate growth/collapse. In actual BEC experiments, for example, with \(^7\)Li and \(^{85}\)Rb atoms, the atomic scattering length can be varied by suitably tuning the external magnetic field through the Feshbach resonance as [3, 4, 15]

\[
a_{ij}(t) = a_0 \left( 1 - \frac{\Delta}{B(t) - B_0} \right),
\]

(2)

where \( a_0 \) is scattering length of condensed atoms, \( B(t) \) is the external time varying magnetic field, \( B_0 \) is the resonance magnetic field and \( \Delta \) is the resonance width. Similarly, tuning of scattering length should be possible for two-component condensates as well, for example in the case of hyperfine states of \(^{87}\)Rb.

Using the following point transformation [37–40]:

\[
\psi_j(x,t) = \Delta_j(x,t) q_j(x,T), \quad j = 1, 2.
\]

(3)

equation (1) can be reduced to the set of two coupled nonlinear Schrödinger equations (2CNLS) of the form

\[
\frac{d}{dt} \frac{\partial q_j}{\partial T} = -\frac{1}{2} \frac{\partial^2 q_j}{\partial x^2} + q_j \sum_{k=1}^2 g_{jk} |q_k|^2 + V(x,t) - \mu_j + \frac{\gamma(t)}{2} q_j, \quad j = 1, 2,
\]

(4)

where \( \Delta_j(x,t) = \tau_0 \sqrt{2} R(t) \exp \left[ i(\theta(x,t) + \mu_j t + \int \frac{\gamma(t)}{2} dt) \right] \),

\[
R(t) = R(t) \exp \left[ \int \gamma(t) dt \right] \theta = -\frac{\pi}{2} x^2 + 2c_1 r_0^2 \kappa - 2c_1 r_0^2 \int \frac{R^2 dt}{\Omega_1}, \quad \Omega_1 = \sqrt{2} r_0 \tilde{R} x - 2(\sqrt{2} c_1 r_0^3 \int \tilde{R}^2 dt, \quad T = \sqrt{2} r_0^2 \int \tilde{R}^2 dt \left( c_1, r_0: \text{ constants} \right) \text{ and } \tilde{R}(t) \text{ and } \Omega(t) \text{ have to satisfy the condition}
\]

\[
\frac{d}{dt} \left( \frac{\tilde{R}}{\tilde{R}} \right) = -\left( \frac{\tilde{R}}{\tilde{R}} \right)^2 - \Omega_1^2(t) = 0,
\]

(5)

which is a Riccati type equation for \( \tilde{R} / \tilde{R} \). Equation (4) is the so-called defocusing Manakov system, which exhibits interesting one-, two- and N-soliton solutions of bright–bright, dark–dark and dark–bright types [41–43]. Concentrating for the present only on dark–bright solitons, from the solutions of equation (4), one can straightforwardly construct the one, two and N dark–bright soliton solutions for equation (1), provided \( R(t), \gamma(t) \) and \( \Omega(t) \) satisfy equation (5). We may note here that because of the complicated transformation involved above, the resultant soliton parameters are no longer constants but are functions of \( x \) and \( t \). Therefore, the resultant soliton solutions may be considered as generalized solitons and not simple standard solitons.

**One-soliton dynamics.** The dark–bright components of the one-soliton solution of the defocusing Manakov system (4) can be given as

\[
q_1(X, T) = \frac{\tau - \frac{X}{2\pi} \chi e^{\kappa n + \eta} - n e^{\kappa X - i[\kappa^2/2 + 2\pi \tau^2]} T}{1 + \chi e^{\kappa n + \eta}},
\]

(6)

\[
q_2(X, T) = \frac{e^{\kappa n + \eta}}{1 + \chi e^{\kappa n + \eta}},
\]

(7)

where \( n = \kappa X + i(\frac{1}{2} \kappa^2 - \tau^2) T, \kappa = a + ib, \rho = \kappa - ic, \chi = \left[ (\kappa + ic)^2 + (\frac{1}{2} \kappa^2 - \tau^2)^2 \right]^{1/2}. \) The parameters \( a \) and \( b \) correspond to the amplitude and velocity of the soliton envelope, respectively, and \( c \) refers to the phase.

Now using the transformation (3), the corresponding dark–bright one-soliton solution of the two coupled GP equation can be written as

\[
\psi_j(x,t) = r_0 \sqrt{2 R(t)} q_j(x,T) e^{i(\theta(x,t) + \mu_j t + \int \frac{\gamma(t)}{2} dt) \}, \quad j = 1, 2.
\]

(8)

**Two-soliton dynamics.** The dark–bright two-soliton solution of the defocusing Manakov system (4) can be written as

\[
q_1(X, T) = \frac{\tau}{d} e^{i(\kappa X - i[\kappa^2/2 + 2\pi \tau^2] T)} \left( 1 - \sum_{j,k=1}^2 \frac{\rho_j \rho_k}{\rho_1 \rho_2} f e^{i(\eta_j + \eta_k) T} + \frac{\rho_1 \rho_2}{\rho_1^2 \rho_2^2} f e^{i(\eta_j + \eta_k) T} \right)
\]

(9)
(This figure is in colour only in the electronic version)

Figure 1. Choice of the atomic scattering length \( a_s(t) \) and gain or loss term \( \Gamma(t) \) as a function of time that exhibits dark–bright solitons for (a) time-independent expulsive trap potential, \( V(x,t) = -\frac{1}{2} \Omega_0^2 x^2 \) and (b) periodic modulated trap potential, \( V(x,t) = \omega^2 [1 - 4 \sum_{j,k} \Re \chi_{jk} \Re \chi_{jk}^* (\sin \omega t + \sin \omega (t + \Delta))] x^2 \).

Figure 2. Dynamics of dark (left) and bright (right) components of the one soliton in an expulsive trap potential for (a) \( \gamma = \Omega_a \tanh(\Omega_0 t) \) and (b) \( \gamma = \Omega' \sin(\Omega' t) \). The parameters are fixed at \( a = 0.7, b = 0.5, r_0 = 1/\sqrt{2}, c_1 = c = 0, \Omega' = 2.5 \) and \( \Omega_0 = 0.001 \).

\[ q_2(X, T) = \frac{1}{d} \left( \sum_{j,k=1}^2 e^{\eta_t} - \sum_{j,k=1}^2 v_{jk} X_{jk}^2 \chi_{jk}^* \chi_{jk} e^{\eta_1 \eta_t^* + \eta_2 \eta_t^*} \right), \quad (10) \]

where

\[ d = 1 + \sum_{j,k=1}^2 \chi_{jk} e^{\eta_1 \eta_t^*} + f e^{\eta_1 \eta_t^* \eta_2 \eta_t^*} \]

\[ \eta_j = \kappa_j X + i \left( \frac{1}{2} \kappa_j^2 - \tau^2 \right) T, \quad \kappa_j = a_j + i \rho_j, \quad \rho_j = \kappa_j - i c, \quad \chi_{jk} = \left[ (\kappa_j + \kappa_k)^2 \left( \frac{1}{\rho_j \rho_k} + 1 \right) \right]^{-1}, \quad v_{jk} = (\kappa_j - \kappa_k)^2 \left( \frac{|\tau|}{\rho_j \rho_k} + 1 \right), \quad f = \chi_{11} X_{12}^2 |\chi_{12}|^2 \chi_{11}^2 \chi_{12}^2. \]

The dark–bright two-soliton solution of the two-coupled GP equation (1) can be again represented by equations (8) with \( q_1 \) and \( q_2 \) of the forms (9) and (10). The procedure can be extended to \( N \)-soliton solutions also. However, we do not present their forms here. Depending on the form of the trap potential, gain/loss and interatomic interaction novel type of dark–bright matter wave solitons can be deduced using the above forms (8)–(10). In the following, we demonstrate them for two simple trap potentials. For the other choices, results will be presented elsewhere. In the present study we fixed the trap parameters similar to that used in a recent experiment on dark–bright solitons in two-component \(^{87}\text{Rb} \) condensates [17].

Time-independent trap potential. First let us consider, as an example, the case of the time-independent expulsive parabolic trap potential, \( \Omega^2(t) = -\Omega_0^2 \) for which the integrability condition (5) gives \( \dot{R}(t) = \text{sech}(\Omega_0 t + \delta) \). The amplitude of the wave packet is given by \( r_0 \sqrt{2 \Gamma(t)} e^{\gamma(t)/2} dt \). Different types of soliton solutions can be constructed for a suitably chosen gain term \( \gamma(t) \). In figure 1(a), we plot the gain \( \gamma(t) = \Omega_a \tanh(\Omega_0 t) \), and the corresponding choice of atomic scattering length \( a_s(t) = \frac{1}{2} a_B R(t) = \frac{1}{2} a_B \text{sech}^2(\Omega_0 t + \delta) \), which may be realized by tuning the external magnetic field as

\[ B(t) = B_0 + \frac{a_0^2 \Delta}{a_0^2 - \frac{1}{2} a_B \text{sech}^2(\Omega_0 t + \delta)}. \quad (11) \]

We point out here that such a form of scattering length has been realized in \(^7\text{Li} \) and \(^{85}\text{Rb} \) atoms [3, 4, 15]. Figure 2(a) shows the dark–bright one-soliton solution for the above gain term where the amplitude of the wave packet remains constant. On the other hand, if we choose a periodic gain term, \( \gamma(t) = \Omega' \sin(\Omega' t) \), the amplitude shows oscillatory
behaviour leading to periodic collapse and revival phenomena, see figure 2(b).

We have also investigated different types of two-soliton interactions for the gain term \( \gamma(t) = \Omega(t) \tan[\Omega(t)t] \) by suitably choosing the parameters \( a_1, b_1, a_2, b_2 \), and \( \Omega(t) \) in equations (9) and (10). Figure 3(a) shows the interaction of dark and bright two solitons with the strong beating effect without interaction while figure 3(b) illustrates interaction of two solitons with the beating effect. Figure 3(c) shows the soliton interactions without beating. These types of soliton interactions are well known in the fibre optics context [42].

**Periodically modulated potential.** If we choose \( R(t) = 1 + \omega \cos(\omega t + \delta) \) and \( \gamma(t) = \frac{\omega_0^2 \sin(\omega_0 t + \delta)}{[1 + \cos(\omega_0 t + \delta)]^2} \), \( \omega < 1 \) we get \( \dot{R}(t) = \sqrt{1 + \omega \cos(\omega t + \delta)} \) and the integrability condition (5) gives \( \Omega^2(t) = \alpha^2 \left[ 1 - \frac{4 \pi [\cos(\omega_0 t + \delta)]^2}{[1 + \cos(\omega_0 t + \delta)]^4} \right] \) which is a periodically modulated trap. For the above case, we sketch the gain \( \Gamma(t) = \frac{\alpha_0^2}{2[1 + \cos(\omega_0 t + \delta)]} \) and the corresponding choice of atomic scattering length \( a_0(t) = \frac{\alpha_0}{2}[1 + \omega \cos(\omega t + \delta)] \) in figure 1(b), which may be realized by periodically tuning the external magnetic field as

\[
B(t) = B_0 + \frac{a_0^2 \Delta}{\alpha_0^2 - \frac{1}{2} \alpha_0^2 [1 + \omega \cos(\omega t + \delta)]}.
\]

Figure 4(a) shows the snake-like effect of the one-soliton solution for the periodically modulated trap potential with \( R(t) = 1 + \omega \cos(\omega t + \delta) \) and \( \gamma(t) = \frac{\omega_0^2 \sin(\omega_0 t + \delta)}{[1 + \cos(\omega_0 t + \delta)]^2} \). A similar snake effect has been demonstrated recently in scalar coupled nonautonomous NLS equations in BEC [39] and in the context of optical solitons [44]. Next we analyse different types of two-soliton interactions for the periodically modulated potential and suitable choice of other parameters.

Figure 5(a) shows the dark and bright non-interacting two solitons again with the strong beating effect in the bound state, where the velocities are zero \( (b_1 = b_2 = 0) \) for \( a_1 = 0.7, a_2 = 0.8 \). Figure 5(b) shows non-interacting moving two solitons again with the strong beating effect for \( a_1 = 0.7, a_2 = 0.8, b_1 = b_2 = 0.1 \). Figure 5(c) depicts the interacting two solitons with the beating effect for \( a_1 = 0.7, a_2 = 0.85, b_1 = b_2 = 0.12 \). Finally figure 5(d) depicts the two-soliton interaction without the beating effect for \( a_1 = 0.7, a_2 = 0.85, b_1 = 0.1, b_2 = 0.17 \).

Recent experimental studies show the above types of dark and dark–bright soliton oscillations and interactions in single- and two-component BECs [17–19]. However these experiments do not consider the interaction of the thermal cloud or time-dependent interatomic interaction. Our studies clearly suggest the possibility of observing them in experiments.

In summary, we have investigated the exact dark–bright one- and two-soliton solutions of the two-component BECs with time-varying parameters such as s-wave scattering length and the gain/loss term. On mapping the two-coupled GP equation onto the coupled NLS equation under certain conditions, we have deduced different kinds of dark–bright one-soliton solutions and interaction of two solitons for time-independent and periodically modulated trap potentials. The present study provides an understanding of the possible mechanism for soliton excitations in multi-component BECs. These excitations can be realized in experiments by suitable
control of time-dependent trap parameters, atomic interaction and interaction with the thermal cloud.

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References

[1] Burger S, Bongs K, Dettmer S, Ertmer W, Sengstock K, Sanpera A, Shlyapnikov G V and Lewenstein M 1999 Phys. Rev. Lett. 83 5198–201
[2] Denschlag J et al 2000 Science 287 97–101
[3] Strecker K E, Partridge G B, Truscott A G and Hulet R G 2002 Nature 417 120–3
[4] Khaykovich L, Schreck F, Ferrari G, Bourdel T, Cubizolles J, Carr L D, Castin Y and Salomon C 2002 Science 296 1290–3
[5] Cornish S L, Thompson T and Wieman C E 2006 Phys. Rev. Lett. 96 170401
[6] Busch T and Anglin J R 2000 Phys. Rev. Lett. 84 2298–301
[7] Salasnich L 2004 Phys. Rev. A 70 053617
[8] Rosenbusch P, Bretin V and Dalibard J 2002 Phys. Rev. Lett. 89 200403
[9] Liu W M, Wu B and Niu Q 2000 Phys. Rev. Lett. 84 2294–7
[10] Malomed B A, Nistazakis H E, Frantzeskakis D J and Kevrekidis P G 2004 Phys. Rev. A 70 043616
[11] Moerdijk A J, Verhaar B J and Axelsson A 1995 Phys. Rev. A 51 4852–61
[12] Roberts J L, Claussen N R, Burke J P, Greene C H, Cornell E A and Wieman C E 1998 Phys. Rev. Lett. 81 5109–12
[13] Stenger J, Inouye S, Andrews M R, Miesner H J, Stamper-Kurn D M and Ketterle W 1999 Phys. Rev. Lett. 82 2422–5
[14] Cornish S L, Claussen N R, Roberts J L, Cornell E A and Wieman C E 2000 Phys. Rev. Lett. 85 1795–8
[15] Courtiller P, Freeland R S, Heinzen D J, van Abeel F A and Verhaar B J 1998 Phys. Rev. Lett. 81 69
[16] Janis J, Banks M and Bigelow N P 2005 Phys. Rev. A 71 013422
[17] Backer C, Stellmer S, Soltan-Panahi P, Dorsch S, Baumert M, Richter E M, Kronjäger J, Bongs K, Sengstock and Klaus 2008 Nature Phys. 4 496–501
[18] Weller A, Ronzheimer J P, Gross C, Frantzeskakis D J, Theoscharis G, Kevrekidis P G, Esteve J and Oberthaler M K 2008 Phys. Rev. Lett. 101 130401
[19] Stellmer S, Becker C, Soltan-Panahi P, Richter E M, Dörsch S, Baumert M, Kronjäger J, Bongs K and Sengstock K 2008 Phys. Rev. Lett. 101 120406
[20] Myatt C J, Burt E A, Ghrist R W, Cornell E A and Wieman C E 1997 Phys. Rev. Lett. 78 586–9
[21] Stenger J, Inouye S, Stamper-Kurn D M, Miesner H J, Chikkatur A P and Ketterle W 1998 Nature 396 345–8
[22] Papp S B, Pino J M and Wieman C E 2008 Phys. Rev. Lett. 101 040402
[23] Woo S J, Park Q H and Bigelow N P 2008 Phys. Rev. Lett. 100 120403
[24] Emary D and Greene C H 1998 Phys. Rev. A 57 1265–71
[25] Busch T, Cirac J I, Pérez-García V M and Zoller P 1997 Phys. Rev. A 56 2978–83
[26] Bongs K, Burger S, Dettmer S, Hellweg D, Arlt J, Ertmer W and Sengstock K 2001 Phys. Rev. A 63 031602
[27] Syassen N, Bauer D, Lettner M, Volz Tand Dietze D, Garca-Ripoll J I, Cirac J I, Rempe G and Drr S 2008 Science 320 1329–31
[28] Anglin J 1997 Phys. Rev. Lett. 79 6–9
[29] Ruostekoski J and Walls D F 1998 Phys. Rev. A 58 R50–3
[30] Vardi A and Anglin J R 2001 Phys. Rev. Lett. 86 568–71
[31] Ponomarev A V, nero J M, Kolovsky A R and Buchleitner A 2006 Phys. Rev. Lett. 96 050404
[32] Wang W, Fu L B and Yi X X 2007 Phys. Rev. A 75 045601
[33] Busch T and Anglin J R 2001 Phys. Rev. Lett. 87 010401
[34] Köhl M, Davis M J, Gardiner C W, Hänsch T W and Esslinger T 2002 Phys. Rev. Lett. 88 080402
[35] Miesner H J, Stamper-Kurn D M, Andrews M R, Durfee D S, Inouye S and Ketterle W 1998 Science 279 1005–7
[36] Gerton J M, Strekalov D, Prodan I and Hulet R G 2000 Nature 408 692–5
[37] Giürses M 2007 Integrable nonautonomous nonlinear Schrödinger equations (arXiv:0704.2435)
[38] Serkin V N, Hasegawa A and Belyaeva T L 2007 Phys. Rev. E 75 045601
[39] Rajendran S, Muruganandam P and Lakshmanan M 2008 Bright and dark solitons in Bose–Einstein condensates (arXiv:0812.5073)
[40] Kundu A 2009 Phys. Rev. E 79 015601 (R)
[41] Radhakrishnan R and Lakshmanan M 1995 J. Phys. A: Math. Gen. 28 2683
[42] Sheppard A P and Kivshar Y S 1997 Phys. Rev. E 55 4773–82
[43] Vijayajayanthi M, Kann A and Lakshmanan M 2008 Phys. Rev. A 77 013820
[44] Serkin V N and Hasegawa A 2000 Phys. Rev. Lett. 85 4502–5