Chiral Gauge Theories and Fermion–Higgs Systems

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The status of several proposals for defining a theory of chiral fermions on the lattice is reviewed and some new estimates for the upper bound on the Higgs mass are presented.

1. Introduction

The amount of work on lattice theory relating to chiral gauge theory has dramatically increased this year with some forty papers in the subject, represented by about twenty contributions to this conference. In contrast, only a handful of papers have been written in the broader field of fermion–Higgs systems without an eye toward chiral gauge theory. Therefore, with one exception, I have decided to limit my review to the topic of chiral gauge theories, and within that I will primarily emphasize the conceptual developments rather than numerical simulations. The exception is some work in the pure Higgs sector concerning the regularization dependence of the upper bound for the Higgs mass\[1\] to which I would like to draw your attention.

Last March, a very pleasant workshop was held in Rome, on the topic of non-perturbative approaches to chiral gauge theories. In that workshop, virtually all proposals around at the time were presented, following which some quite intense discussions took place in the proverbial smoke-filled back room\[1\] in an attempt to understand the problems of some of the proposals and the connections between them. Some valuable insights were gained in these discussions and in my talk I will attempt to portray the underlying theme that we arrived at during that meeting. The topics discussed in the Rome workshop are the following:

- the failure of the Wilson–Yukawa approach
- consequences for the Eichten–Preskill approach
- the Rome approach and its relation to the above
- the staggered fermion approach
- mirror fermions
- Zaragoza “replica” fermions
- the topological method
- contribution of doublers to the S–parameter
- the Banks “U(1) problem”

Suggestions that were not represented at the meeting are:

- the Bodwin-Kovacs proposal
- reflection positive fermions

which I would also like to mention, and finally I would like to spend some time to discuss a proposal made after the conference which is perhaps the most interesting proposal presently on the table:

- domain wall fermions

Aside from these, a rather pessimistic contribution concerning the use of the random lattice as a solution to the chiral fermion problem was presented at this conference\[2\], although I am not prepared to discuss this work.

Naturally with the large amount of material represented above, I have been forced to make some subjective choices about what to cover and what to omit. So keeping in mind my attempt

\[1\] Although due to federal regulations the smoke was not allowed
to focus on the conceptual development of the field, I have decided to omit discussion of mirror fermions for the most part with a twofold justification. First, that theory is a vector theory in principle and not an attempt at producing a chiral theory. The chiral theory is modeled at low energy, with the chiral partners of these fermions appearing close to the electroweak scale. Second, and perhaps more to the point, the theory has already been well represented in plenary talks of the last two lattice conferences with little conceptual development since then. I would like to comment though that Montvay gave us a nice talk in Rome indicating that the theory of mirror fermions is not in contradiction to any known experimental phenomena to date, which should force us not to foreclose on the possibility that mirror fermions actually do exist in nature, and the world is not chiral after all. Work goes on in the numerical arena, and as this is the first year that the effects of fermions on the upper bound of the Higgs mass have been reported, I will mention some results obtained from the mirror fermion method when I come to this topic.

I will also say little about the Banks problem other than to point out what it is and how it can be resolved in various of the proposals, and I will not have time to discuss the contribution of doubler fermions to the S–parameter. All other topics mentioned above will find their way into my talk, although in several cases I will have time only to present the barest essentials.

2. HIGGS MASS UPPER BOUND

Before a discussion of chiral fermions I would like to briefly describe some progress on a topic that falls under my jurisdiction: recent results on regularization dependence of the upper bound to the Higgs mass have been reported, I will mention some results obtained from the mirror fermion method when I come to this topic.

I will also say little about the Banks problem other than to point out what it is and how it can be resolved in various of the proposals, and I will not have time to discuss the contribution of doubler fermions to the S–parameter. All other topics mentioned above will find their way into my talk, although in several cases I will have time only to present the barest essentials.

Figure 1. Graphs containing numerical data and results from large $N$ calculations for hypercubic (HC), $F_4$ and Symanzik improved lattice regularization schemes, and with Pauli–Villars (PV) regularization, including results from actions with dimension 6 operators.

have been substantially broadened, with investigations of theories with actions including higher derivative terms, using the $1/N$ approximation as well Monte Carlo calculations.

A typical graph of the results of a calculation of the Higgs mass in units of $f_\pi$ versus the Higgs mass in cutoff units is given in figure 1. These graphs represent calculations on an $F_4$ lattice, a hypercubic lattice with standard action, a hypercubic lattice with a Symanzik improved action, and a calculation with Pauli–Villars (PV) regularization, including results from actions including higher dimension operators. The results from the $1/N$ expansion (extrapolated to $N = 4$) with the standard action and an action including higher dimension operators are given in each case by the dotted and solid lines respectively. The main point I would like to make concerning this graph is that both adding 'irrelevant' terms to the ac-
tion which represent cutoff effects, and looking at a broader class of regularization schemes, allows the upper bound on the Higgs mass to be relaxed. With the usual requirement that the cutoff effects are only a few per cent on pion scattering, from this analysis a conservative estimate for the upper bound is about 750 GeV ± 50 GeV which is somewhat higher than the old bound. In addition, the decay width is found to be about 290 GeV which is somewhat broader than the tree level value of 210 GeV, indicating that in the region in question the Higgs particle may be more strongly coupled than previously believed. I will come back to this topic shortly when I consider the effects of fermions on these numbers.

3. THE BANKS “U(1) PROBLEM”

Up to a little over a year ago, most models which were under investigation as theories of chiral fermions possessed a rigid $U(1)$ symmetry corresponding to fermion number which commutes with the gauge symmetry. Banks pointed out that for a realistic theory which hopes to reproduce the standard model either directly or as a low energy approximation to some other chiral theory obtained from the lattice, this presented a problem from the point of view of anomalies. For in the standard model, baryon number $B$ and lepton number $L$ are each independently violated by instanton processes, and only the difference $B - L$ is conserved, whereas the rigid $U(1)$ symmetry in the lattice theories would insure that both $B$ and $L$ would be conserved independently. Although this problem had been overlooked for the most part in the face of other more severe problems, the comments by Banks did force the rest of us to ‘come clean’ and solutions to the problem for the various models formed a part of our discussion in the Rome workshop. There are several ways around the problem that are either built in explicitly or realized dynamically by the various models which I will point out as we go along.

It should be pointed out that Eichten and Preskill did address this problem directly in their approach.

4. WILSON–YUKAWA, EICHEN–PRESKILL AND ROMA

There is very strong evidence by now that the Wilson–Yukawa approach fails to result in a theory of lattice chiral fermions, and that unfortunately its failure takes down the Eichten–Preskill approach with it. In this section I review why this is so, and indicate how the findings are related to the Rome approach and why the latter may escape the problems that arise in the first two.

4.1. The Wilson–Yukawa approach

To understand the problems involved, it is sufficient to consider a simple case of the Wilson–Yukawa model containing only a single species of fermion, and in which only the left-handed chiral symmetry is gauged, leaving the right–handed chiral symmetry rigid. The model is defined by the fermion action

$$S_{WY} = \sum_{x, \mu} \frac{1}{4} \overline{\psi}_x \gamma_\mu (D_\mu + \bar{D}_\mu) \psi_x + y \sum_x [\overline{\psi}_L x \psi_{Rx} + \overline{\psi}_{Rx} x V_2 \psi_{Lx}]$$

along with the usual actions for the gauge and scalar fields, where $\psi_R$ and $\psi_L$ are right- and left-handed chiral fermion fields $\frac{1}{2}(1 - \gamma_5) \psi$ and $\frac{1}{2}(1 + \gamma_5) \psi$ respectively, $D_\mu$ and $\bar{D}_\mu$ are the forward and backward lattice derivatives respectively, gauged with respect to the gauge field for the left chiral symmetry $U_L$, $\Box$ is the lattice laplacian, and $V$ is a scalar field with frozen radius taking its value in a group $G_L$, which for our purposes is taken to be $U(1)$ or $SU(2)$.

Aside from the term ‘Wilson–Yukawa’ term containing $w$, this action is just that which produces a naive (doubled) theory of gauged chiral fermions coupled to a scalar field via a Yukawa interaction. The only differences between this and the usual standard model are the presence of the doublers, and an extra neutrino with right-handed chiral symmetry. In the broken phase at tree level the Wilson–Yukawa term takes the form $wv \overline{\psi} \Box \psi$ where $v = \langle V \rangle$, so that naively, this term
looks like a Wilson mass term, which could serve as a remedy for the doubling problem.

The action possesses a local chiral gauge symmetry

\[
\psi_{Lx} \rightarrow g_x \psi_{Lx} \\
V_x \rightarrow g_x V_x \\
U_{\mu x}^L \rightarrow g_x U_{\mu x}^L g_{x+\mu}^\dagger
\]

which we will refer to as \(g\)-symmetry.

The phase diagram for this model for small gauge coupling is well known by now \cite{4} and is represented in figure 2. The pure Higgs model exhibits a phase transition for some \(\kappa_c\), below which lies a symmetric phase and above a broken phase. Taking \(y \rightarrow 0\) insures the presence of at least one light fermion\cite{3}, so we will limit our discussion to that case. In the weak \(w\) region, the critical point extends into the \(w - \kappa\) plane, separating a symmetric (PMW) phase from a broken (FM) phase. Perturbation theory applies in this region, and just as in usual standard model perturbation theory, all fermion masses are proportional to the vacuum expectation value of the Higgs particle \(\langle V \rangle = v\), including the masses of the doublers:

\[
m_f = v(y + 2nw) \quad n = 0, 1, 2, 3, 4.
\]

Therefore, in the PMW phase (region I of figure 2), all 16 fermions are massless, and in the broken phase (region II), the degeneracy is lifted due to a non-vanishing value of \(v\). However, as the continuum limit is approached, \(v\) scales to zero, and all doublers appear in the physical spectrum. Thus the weak coupling region is clearly not a region in which to look for a solution to the doubler problem. This has also been checked via Monte Carlo calculations\cite{14}.

To study the strong \(w\) region, it is convenient to change variables to fermion fields, neutral with respect to \(g\)-gauge symmetry:

\[
\psi_{(n)}^{(x)} = V_x^\dagger \psi_x,
\]

in terms of this field, the fermion part of the action becomes

\[
S'_{WY} = \sum_{x,\mu} \left[ \frac{1}{2} \bar{\psi}_{Lx}^{(n)} \gamma_\mu (D_\mu(W_\mu) + D_\mu(W_\mu)^\dagger) \psi_{Lx}^{(n)} \right. \\
+ \left. \frac{1}{2} \bar{\psi}_{Rx}^{(n)} \gamma_\mu (\partial_\mu + \tilde{\partial}_\mu) \psi_{Rx}^{(n)} \right] \\
+ \sum_x \bar{\psi}_{x}^{(n)} (y - \frac{w}{2}) \psi_{x}^{(n)},
\]

where \(D_\mu(W_\mu)\) is a covariant derivative with respect to the composite gauge field \(W_\mu = V_x^\dagger U_{\mu x} V_x + \tilde{\epsilon}_{\mu}\):

\[
D_\mu(W_\mu) \psi_{Lx}^{(n)} = W_{\mu x} \psi_{Lx}^{(n)} - \psi_{Lx}^{(n)}.
\]

We should note that this action looks like that of a chiral gauge theory invariant under the following symmetry \cite{15,16}, (which we will here denote as \(h\)-symmetry):

\[
\psi_{Lx}^{(n)} \rightarrow h_x \psi_{Lx}^{(n)}, \\
W_{\mu x} \rightarrow h_x W_{\mu x} h_{x+\mu}^\dagger
\]

along with a symmetry breaking Wilson mass term. This form of the lagrangian appears as the starting point for the Rome proposal with the relation being made manifest in unitary gauge \((V = 1)\), in which \(\psi\) and \(\psi^{(n)}\) are equivalent as are \(U_\mu\) and \(W_\mu\). We will return to this point later. (See also \cite{17,18}.)

At strong \(w\), the Schwinger–Dyson equations for the right-handed fermion, which happen to
correspond to the Ward identities for a rigid fermion shift symmetry, lead to some conclusions about the effective action of the model\cite{19}. In particular, $\psi^{(n)}_R$ does not appear in any interactions, and the fermion mass spectrum is

$$m_f^{(n)} = \sqrt{Z_2(y + 2nw)}, \quad n = 0, 1, 2, 3, 4,$$  \hspace{1cm} (8)

where $Z_2$ is the wave function renormalization which can be determined by expansion techniques or by Monte Carlo calculations. The important thing is that it is finite and non-vanishing. Thus in the broken (FM) phase all fermions are massive, and a single massless fermion can be arranged by tuning $y$ to zero. In the symmetric phase the spectrum is exactly the same.

In addition, an examination of the propagator for the composite field $\psi^{(c)}_L = V_x \psi^{(n)}_L$ which has non-zero charge with respect to $g$-symmetry reveals that in the absence of gauge fields, apparently no fermion states with non-singlet $g$-quantum numbers appear in the symmetric phase\cite{20,21,22,23}. One can understand this by realizing that the large coupling $w$ has caused the original left-handed charged fermion $\psi_L$ to form a bound state with $V^\dagger$ and this in turn pairs up with $\psi_R$ to form a Dirac fermion, screening the charge of the original state. In the broken phase, $\psi^{(c)}$ mixes with $\psi^{(n)}$ and does not represent an independent state.

With only the neutral fermions as physical states, we next must examine the couplings to these fermions to see if they interact. We have already noted that in the effective action, all bare interactions of the right-handed fermion vanish. For the left-handed fermion, it is a simple matter to see that its Yukawa interaction with the Higgs field is proportional to $v$ which vanishes in the PMS phase and scales to zero in the FM phase near the phase transition (region III). The situation is similar for higher order couplings between this fermion field and the scalar. Finally, for small gauge coupling and hopping parameter $\alpha = 1/(4w + y)$, one can show that the gauge coupling to the neutral field also vanished\cite{23} in the scaling region\cite{19,22}, and this can be demonstrated in the broken phase taking into account the contributions of the scalar field exactly\cite{10}. The interpretation is that once the scalar fields form bound states with the fermions so as to form Dirac fermions which are neutral with respect to the $g$-symmetry, they then screen these fermions from any interaction with the gauge fields.

A quick way to look at this is to notice that in the interaction lagrangian, $W_\mu$ couples to the fermions through the operator $V_\mu D_\mu V - \text{h.c.}$ which is a dimension three operator in contrast to the usual gauge field. Thus in the continuum limit the interaction term is dimension 6 and will need two extra powers of the lattice spacing to compensate. In the broken phase this means that the interaction must be proportional to $v^2 \equiv \nabla^2 a^2$ (where $\nabla$ is dimension 1), and in the symmetric phase a factor of $\Lambda^2 a^2$ will emerge, where $\Lambda$ is the scale associated with the gauge fields (e.g. $\Lambda_{\text{QCD}}$). In both cases, the interaction vanishes in the continuum limit.

Now we can summarized what we learned so far. First, fermions pair up as vectors everywhere in the phase diagram. Second, There are no $g$-charged fermions in the PMS phase. And third, in the strong $w$ region, the $g$-neutral ($h$-charged) fermion is a free particle. It is possible to define a theory with $g$-charged states that appear in the physical spectrum, by changing the form of the Wilson–Yukawa term, and these do couple with the gauge field, but they do so in a vector-like manner\cite{10}. Hence we have learned several valuable lessons from the study of this model which may serve as warnings for future investigations:

- If there is a strong coupling in the game, and chirally opposite bound states can form to pair up as Dirac fermions, they likely will.
- In this model, the scalar field $V$ is the culprit. $V$ screens the chiral gauge interactions and decoupling $V$ by letting its mass diverge (as deep in the PMS phase) doesn’t work.
- It is not a priori clear which states are physical by looking at the lagrangian.
- Anomalies play no role (see also next section).

\footnote{One author is not in agreement\cite{24}, but I believe the arguments referred to here to be correct.}
One final comment: note the dimension dependence of the conclusions regarding the gauge interaction term. In two dimensions, the operator $V_x^\dagger D_\mu V_x - \text{h.c.}$ is dimension one and the quick argument above would not require the interaction to vanish. Indeed there is evidence that in a two dimensional theory one can construct a theory of chiral fermions using the Smit–Swift model\cite{26}. This case also gives us the first scenario for escaping the Banks “$U(1)$ problem”. In two dimensions, apparently the fermion forms a bound state with the scalar to form a vector multiplet, but the left-handed component nevertheless interacts with the composite gauge field in a chiral fashion. Thus $h$-symmetry emerges as the symmetry associated with this coupling. This turns out to be a realization of the Dugan–Manohar solution to this problem\cite{27}, namely that the current associated with the relevant gauge symmetry is not what one would expect a priori. Indeed the $U(1)$ rigid symmetry commutes with $g$-symmetry, but the states are classified according to $h$-symmetry. The current for this symmetry is related to that for $g$-symmetry by a local counterterm, and so the Noether current for the rigid $U(1)$ symmetry is not the $h$-gauge invariant current. In other words, the rigid $U(1)$ symmetry and the $h$-symmetry do not commute. Although the Dugan–Manohar scenario is not realized in four dimensions due to the screening of the interaction, it still teaches us that the way to the continuum can be more subtle than first imagined.

4.2. The Eichten–Preskill Model

Eichten and Preskill assumed from the start that when constructing a lattice theory with any hope of defining a theory of chiral fermions, one must pay careful attention to anomalies\cite{28}. Therefore they proposed to look at a theory that respects all desired symmetries of the target continuum theory, and explicitly breaks any symmetry that should be broken. To satisfy this criterion, they chose to study a lattice version of a chiral $SU(5)$ grand unified theory.

For simplicity and with no loss I will discuss an $SO(10)$ theory with the gauge fields turned off. The $SU(5)$ theory can be obtained by adding explicit symmetry breaking terms to the action. The action for the $SO(10)$ theory can be written as follows.

\begin{equation}
S_{EP} = \sum_{x \mu} \frac{1}{2} [\bar{\psi}_L^x \slashed{D}_\mu \psi_L^x + h.c.] - \sum_x \frac{\lambda}{24} [(\bar{\psi}_L^x \sigma_2 T \psi_L^x)^2 + h.c.] - \sum_x \frac{r}{48} [(\bar{\psi}_L^x \sigma_2 T \psi_L^x)^2 + h.c.],
\end{equation}

where $\psi_L^x$ is a Weyl spinor in the 16 representation of $SO(10)$, $T^{ij}$ is an $SO(10)$ invariant tensor transforming as $\bar{10} \times \bar{10} \times 10 \times 10$, $\sigma = (1, i\sigma)$, where $\sigma$ are the Pauli matrices, and

\begin{equation}
\Delta(\bar{\psi}_i \psi_j \bar{\psi}_k \psi_l) = -\frac{1}{2} \sum_{x, \mu} [\bar{\psi}_i \psi_j + h.c.]_x \psi_k \psi_{l x} + h.c. (10)
\end{equation}

This latter construction allows the term in the action proportional to $r$ to play the role of a Wilson mass term, breaking the degeneracy in the mass spectrum between the fermion and its doublers. Note that the quantity $\bar{\psi}_L^x \sigma_2 T^{ij} \psi_L^x$ explicitly breaks fermion number and so by construction the Eichten–Preskill model does not have the Banks problem.

Studying the model for $r = 0$, Eichten and Preskill found a symmetric phase in the strong $\lambda$ region with massive fermions, and another symmetric phase in the weak coupling region but with massless fermions. They then hoped that at the phase transition between them, the degeneracy of the fermion and its doublers would be lifted by turning on the coupling $r$, and that a region would exist where the one fermion can be tuned to become massless, whereas the doublers have masses of the order of the cutoff. They also pointed out that such a scenario would not be realized if the two symmetric phases were separated by a broken phase.

The model above is rather like the Nambu–Jona-Lasinio model in appearance, and recent work showing the equivalence of a model of this type with the standard model\cite{28} suggests that a similar association can be made with the Eichten–Preskill model. Indeed this is the case. An action
with the same symmetries as the model in question is

\[ S'_{EP} = \sum_{x\mu} \frac{1}{2} [\psi^{\dagger}_{Lx}\sigma_{\mu}\psi_{Lx+\hat{\mu}} - \text{h.c.}] \]  

\[ + \frac{i}{2} \sum_{x} \phi_{x}\phi_{x} - \kappa \sum_{x\mu} \phi_{x}\phi_{x+\hat{\mu}} \]

\[ + \frac{i}{2} y \sum_{x} \psi_{L}^{\dagger}\sigma_{2}T\psi_{L}\phi + \text{h.c.} \]

\[ - \frac{1}{4} w \sum_{x} \psi_{L}^{\dagger}\sigma_{2}T\psi_{L}\phi + \text{h.c.} \]

where we have introduced a scalar \( \phi^{a} \) in the 10 representation of \( SO(10) \). This model with Wilson–Yukawa term obviously has a similar structure as the Smit–Swift model. To make contact with the Eichten–Preskill model, one can invent an additional ‘flavor’ and perform a large \( N \) expansion in the number of flavors. To do this, we studied the Weyl fermion \( \psi_{L} \) (explicit in both models), its right-handed partner \( \psi_{R} \) (formed as a bound state in both models), and the scalar field (a bound state in the Eichten–Preskill model), whose composite forms are

field \quad Eichten–Preskill \quad Wilson–Yukawa

\[ \psi_{L} \quad \psi_{L} \quad \psi_{L} \]

\[ \psi_{R} \quad \sigma_{2}T^{\dagger a}\psi_{L}^{\dagger}(\psi_{L}^{\dagger}\sigma_{2}T^{\dagger a}\psi_{L}^{\dagger}) \quad \phi^{a}\sigma_{2}T^{\dagger a}\psi_{L}^{\dagger} \]  

(12)

\[ \phi^{a} \quad \text{Re}\psi_{L}^{\dagger}\sigma_{2}T^{\dagger a}\psi_{L} \quad \phi^{a} \]

where a sum over \( a \) is implied when repeated. Not surprisingly, the phase diagram of the above Wilson–Yukawa model is similar to that of the Smit–Swift–model (figure 2), with the parameters \( (y, w) \) playing the role of \( (\lambda, \kappa) \) in the Eichten–Preskill model. One can find matching conditions between the couplings of the two models so that all Green functions involving the particles above coincide in the large \( N \) expansion. The physics of the Eichten–Preskill model is realized in the \( \kappa = 0 \) plane, and in particular, the two symmetric phases are separated by a broken phase. In the strong \( \lambda \) region, the right-handed fermion forms as a bound state and pairs with the original fermion to give a theory of vector fermions interacting with gauge fields, and in the weak coupling region the doublers are present in the physical spectrum. Thus all that we learned about the Smit–Swift model applies here, and again no theory of chiral fermions is realized. Finally we should emphasize that even though Eichten and Preskill were careful to pay attention to the anomaly structure of the theory, this plays no role in its failure.

4.3. The Rome proposal

An understanding of the preceding problems, and in particular that the culprit for the failure of the Smit–Swift model to produce a theory of chiral fermions is the scalar field forming bound states with the fermions to create Dirac partners, suggests a direction for proceeding. The scalar field represents just those gauge degrees of freedom which are unphysical, and would be removed by gauge fixing. This is easily seen in a transverse gauge for which the scalar fields represent the longitudinal modes. Since we have learned that decoupling of these modes by keeping their mass at the cutoff does not prevent the bound states from being formed and destroying the chiral structure of the theory, perhaps enforcing gauge fixing on the lattice would do the trick. This is precisely the proposal of the Rome group[31], although historically this proposal was quite independent from the reasoning I have presented here. Originally the motivation of the Rome group was to define a theory motivated by perturbative gauge fixing as a prescription for obtaining a full non-perturbative asymptotically free chiral gauge theory.

The action for the Rome group’s proposal starts with that given in eq. (6), but without the scalar field (as it appears in unitary gauge). Then they add gauge fixing and ghost terms of the form required for a particular gauge choice such as Landau gauge. Thus they start with a theory that violates gauge invariance (here we are talking about \( h \)-gauge invariance, not involving the scalar fields), and the final ingredient is to add all counterterms necessary to allow tuning to impose the satisfying of the BRST identities associated with \( h \)-symmetry:

\[ S_{Roma} = S'_{WY}(V = 1) + S_{\text{g.f.}} + S_{\text{ghosts}} + S_{\text{c.t.}}. \]  

(13)

With this procedure the longitudinal modes are decoupled not by putting their mass at the cut-
off scale, but by decoupling their interaction with other particles of the theory.

Can his decoupling be accomplished? It appears that the prescription should work in principle to all orders in perturbation theory, and indeed there are now some explicit two loop results indicating that things are working at least to that order \[^{[32]}\]. The question arises how well it can work as a non-perturbative prescription, in which the tuning must (at least in part) be done numerically. To address this question, let us take a look at the path integral of the theory. The path integral for the Rome proposal is given by

\[
\int [dU][d\psi][d\bar{\psi}] e^{-S_{\text{Rome}}}, \tag{14}
\]

One procedure to approach this path integral is to multiply by the trivial integral \[\int [dV] = 1\], and then make a transformation of variables to rotate \(V\) back into the action \[^{[17,18]}\], so that the first term looks like \(S_{\text{WY}}\) as given in eq. (1). As Smit has pointed out (see e.g. \[^{[33]}\]) the longitudinal modes are actually present in the Haar measure for the gauge fields, and the above trick is merely making explicit what already is there. Thus in so far as the Rome proposal would work, it is obvious that it depends on how well they are able to decouple the scalars. If the decoupling of the scalars is very sensitive to tuning then values of the coupling that are a little off may not prevent the scalars from forming bound states with the fermions and we would be back to the scenario of the Smit–Swift model. Because it is expensive to tune couplings numerically, this is a potentially serious problem. As to whether the problem is realized, “the proof is in the pudding” so to speak, and we will have to wait for a realistic attempt at doing the non-perturbative problem.

Bodwin and Kovacs \[^{[34]}\] have made an observation that could make the problem more manageable, provided the technical difficulties can be worked out. They observe that the magnitude of the chiral fermion determinant is equal to the positive square root of a vector determinant, and provided a method is found to deal with the phase of the chiral determinant, the calculation could be done with a vector theory plus an extra calculation for the phase. In the abelian case this reduces the number of counterterms needed from seven down to two, which is a substantial savings, and similarly in the non-abelian theory. There is also a proposal on the market for calculating the phase \[^{[35]}\] using topological methods following a definition by Alvarez-Gaumé et. al. \[^{[36]}\], so a combination of all three of these ideas may eventually prove fruitful, if not elegant.

Finally, I would like to point out how the Rome approach gets around the Banks problem. Actually so far two ways have been proposed \[^{[35]}\]. One way around it is to write each chiral multiplet as a charged chiral fermion and a neutral ‘spectator’ fermion, in which the neutral spectator fermion obeys shift symmetry. Then using the decoupling theorem \[^{[13]}\] the spectator fermions should all decouple, and the \(S\)-matrix should factorize into charged particles and spectators. Kikukawa has already shown that the combination of a right and left charged fermion along with the respective spectators can produce the correct anomaly responsible for baryon number violation \[^{[37]}\]. So despite the fact that the full \(S\)-matrix including spectators would be \(U(1)\) invariant, due to the decoupling of the spectators, the baryon number would be carried off by the spectators. The second method is to use Wilson-Yukawa terms that have the form of Majorana mass terms \[^{[38]}\]. These explicitly violate the \(U(1)\) symmetry, and Pryor has shown (in the context of the Smit-Swift type models) that such a term reproduces the correct anomaly \[^{[39]}\].

5. Staggered fermion approach

With the demise of the Wilson–Yukawa approach, this past year has also seen renewed interest in the staggered fermion \[^{[40,41,42]}\] approach toward constructing a lattice standard model \[^{[43,44]}\]. For details of the method, see \[^{[33,44]}\]. First, the basic idea of the staggered fermion theory is to spread out the spin and flavor components of a fermion on the lattice, resulting in the decoupling of the original 16 doubled fermion components into 4 independent 4-plets. Then only one of these 4-plets is kept to formulate the theory, thus reducing the 16 original flavors to 4. Because of this spreading out of the
spin components, the hypercubic symmetry and flavor symmetry are now mixed, and thus for example, a shift on the lattice of one lattice spacing mixes flavors, whereas it takes two shifts to generate a spatial translation. Also \( \gamma_5 \) corresponds to a four link operator. Now the beauty of the staggered fermion approach is that rather than trying to get rid of the remaining doublers, they are to be used as physical flavors in the theory. The important consequence of this is that if the four flavors are to correspond to flavors in the continuum theory, since their components are sitting on different sites of the lattice, the global chiral symmetry is broken and therefore cannot be gauged explicitly on the level of the action. (QCD escapes this by adding an independent color index to each flavor, and the new symmetry is gauged.) This is reminiscent of the Rome proposal in which chiral gauge invariance is broken explicitly on the level of the action and must be restored by tuning, and so it is with staggered fermions.

It is useful to keep track of two remnant \( U(1) \) symmetries that are not broken, one transforming each fermion component by the same phase, relating to fermion number and the second rotating alternate components by opposite phases:

\[
\exp(i\omega \epsilon_x) \in U_r(1), \quad \epsilon_x = (-1)^{x_1 + x_2 + x_3 + x_4}.
\]

These are of course part of a rich lattice symmetry group including many other discrete symmetries which I will not enumerate here. The group goes under the name \( SF \) or staggered fermion symmetry group \([14]\).

If \( \chi \) is the component of a fermion at a particular site, one can cut the number of components in half by letting \( \overline{\chi} = \chi \) on each site (equivalent to letting \( \chi \) live only on even sites and \( \overline{\chi} \) on odd sites). These are referred to as reduced staggered fermions. Each reduced fermion field represents a doublet of fermion flavors, and because of the ‘Majorana’ constraint that defines them, the \( U(1) \) relating to fermion number is broken and only \( U_r(1) \) is left.

It is straightforward now to build a lattice theory with particle content equivalent to the standard model. For example\([13]\), for the first generation of the standard model, the electron and neutrino together can be represented by one \( \chi \) doublet, and the three colors of \( u \) and \( d \) quarks can be represented by three fields \( \chi_a \) with the color index \( a = 1, 2, 3 \). (In this formulation, the \( U_r(1) \) symmetry may lead to fermion number conservation in the scaling limit and presumably the Banks problem arises. A somewhat less elegant embedding of standard model quantum numbers into staggered fermion doublets can be made however that explicitly breaks \( U_r(1) \) and avoids the problem\([33]\).)

To gauge the model, a more useful organization of the fermion components is to define the Grassmann matrices

\[
\Psi_x = \frac{1}{\sqrt{8}} \sum_b \Gamma(x, b) \frac{1}{2}(1 - \epsilon_{x+b}) \chi_{x+b}, \quad (16)
\]

\[
\overline{\Psi} x = \frac{1}{\sqrt{8}} \sum_b \frac{1}{2}\Gamma(x, b)(1 + \epsilon_{x+b}) \chi_{x+b}, \quad (17)
\]

\[
\Gamma(x, b) = \gamma_1 \delta_{b_1} \gamma_2 \delta_{b_2} \gamma_3 \delta_{b_3} \gamma_4 \delta_{b_4} \quad (18)
\]

where \( b \) is summed over all corners of a unit lattice cell. In terms of these fields one can define a chiral model in a notation reminiscent of that of the continuum theory. For example, and \( SU(2) \times SU(2) \) model in which only \( SU(2) \times U(1) \) is gauged, can be defined through the action

\[
S = - \sum_{x\mu} \frac{1}{2} \left[ i \overline{\Psi}_x \gamma_\mu \Psi_{x+\hat{\mu}} U^\dagger_{\mu x} \right]
- \left. \frac{1}{2} \right| \overline{\Psi}_{x+\hat{\mu}} \gamma_\mu \Psi_x U_{\mu x} \right| + \sum_{x\mu} m_{\mu} \rho_x \left[ \overline{\Psi}_x^a \Psi_{x} \gamma_\mu \right], \quad (19)
\]

where the gauge fields are the appropriate ones for gauging the groups desired and the last term is present since no symmetry excludes it. \( \rho_x \) is a spacetime dependent amplitude factor. Keep in mind that because the components of \( \Psi \) lie on different sites, this action is not gauge invariant. However, if one sets \( m_\mu \) to zero, the action does lead to the proper gauge invariant continuum action in the classical continuum limit.

Now recall that in the Smit-Swift model one could freely transform between the \( g \)-charged fields \( \psi^{(c)} \) and the \( g \)-neutral fields \( \psi^{(n)} \), and represent the path integral in terms of either. Such a transformation is not possible here because of the lack of local gauge invariance. One can
However expose the longitudinal degrees of freedom by making the transformation of variables $U_{\mu x} \rightarrow V_x U_{\mu x} V_x^{\dagger}$, then one arrives at the action ($m_\mu = 0$)

$$S = - \sum_{x \mu} \frac{1}{2} \left( \overline{\Psi}_x \gamma_\mu \Psi_{x+\hat{\mu}} V_{x+\hat{\mu}} U_{\mu x} + \frac{1}{2} \overline{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}} V_{x+\hat{\mu}} \right)$$

$$S = - \sum_{x \mu} \frac{1}{2} \left( \overline{\Psi}_x \gamma_\mu \Psi_{x+\hat{\mu}} V_{x+\hat{\mu}} U_{\mu x} + \frac{1}{2} \overline{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}} V_{x+\hat{\mu}} \right)$$

which appears similar in form to the version of the Smit–Swift model written in terms of neutral fields in eq. (3). The obvious question based on our previous experience is whether the corresponding desired ‘charged’ states appear in the physical spectrum, or equivalently, whether the propagator

$$\langle \overline{\Psi}_x \gamma_\mu \psi_{y+\hat{\mu}} V_y \rangle$$

has a pole structure (when the gauge fields are turned off). This is a crucial test as to whether a gauge theory emerges which includes charged states in the usual sense.

Alternatively, one can define another fermion–Higgs model through an action which has the form of eq. (1)

$$S = \sum_{x \mu} \frac{1}{2} \left( \overline{\Psi}_x \gamma_\mu \Psi_{x+\hat{\mu}} + \overline{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}} \right) + \sum_x \overline{\psi}_x \gamma_\mu \psi_x + S_{\text{scalar}},$$

where $\Phi = \sum_{x \mu} \overline{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}}$. Because there is no formal transformation between the models defined by eq. (2) and by eq. (1), it is a non-trivial question whether the two quantum theories are equivalent. Hence it makes sense to first study the two fermion–Higgs systems alone, for if these are seen to be equivalent, one might expect the gauged theories to be also. Then the more pressing question can be addressed: whether they (one or the other or both) give rise to the desired gauge invariant continuum theory (assuming the necessary tuning is performed).

As a first serious investigation of these questions, recently a group has begun to study an SU(2) $\times$ SU(2) invariant version of the latter model numerically[36]. I will summarize the highlights here as befit the flow of my theme; a more complete report can be found elsewhere in this volume[37]. To define the model, the action for the scalar field

$$S_{\text{scalar}} = \kappa \sum_{x \mu} tr(\Phi_{x+\hat{\mu}} \Phi_{x+\hat{\mu}} + \Phi_{x+\hat{\mu}} + \Phi_{x+\hat{\mu}} \Phi_{x+\hat{\mu}})$$

$$S_{\text{scalar}} = \kappa \sum_{x \mu} tr(\Phi_{x+\hat{\mu}} \Phi_{x+\hat{\mu}} + \Phi_{x+\hat{\mu}} + \Phi_{x+\hat{\mu}} \Phi_{x+\hat{\mu}})$$

is added to eq. (23). The case of 2 reduced staggered doublets or 4 flavors and the case of 2 multiplets of naive fermion corresponding to 32 flavors were studied.

First of all, the phase diagram has been mapped out and no surprises were found. For weak values of $y$ and large $\kappa$ a ferromagnetic phase occurs. As one reduces $\kappa$ a second order transition appears to a paramagnetic phase which continues on into the negative $\kappa$ region, after which for large enough negative $\kappa$ an antiferromagnetic phase occurs. For much larger values of $y$ the diagram moves directly from the ferromagnetic to a ferrimagnetic phase for some value of $\kappa$ below zero. As a by-product in the simulation it was discovered that the staggered fermion method appears to offer a very efficient way to simulate such models.

Another question addressed is $O(4)$ symmetry restoration. Because of the explicit breaking of the symmetry, two counterterms $O(1)$ and $O(2)$:

$$O(1) = \sum_{x \mu} \Phi_{x+\hat{\mu}}$$

$$O(2) = \sum_{x \mu} (\Phi_{x+\hat{\mu}} + \Phi_{x+\hat{\mu}})^2, (26)$$

are necessary to restore it as the continuum is approached, which in principle would mean that two couplings need to be tuned. What was found however, is that in the region studied, the $O(4)$ symmetry is already present to a good approximation, and that the breaking effects are only on the order of a few per cent. Two other facets of the calculation deserve mention before going on. The authors have included one loop fermion effects in fitting the scalar parameters, and in order to control systematic errors due to finite size effects, the authors have performed calculations on lattices of different physical sizes, in order to make an extrapolation to large volume. Among results obtained from these models are estimates of the effects of fermions on the upper bound
Figure 3. Estimates for Higgs mass upper bound in pure Higgs theory and with fermions. The standard action pure Higgs theory on a hypercubic lattice is given by the white box, and the results of Heller et. al. are the black box. The circles are those including mirror fermions, and the diamonds are with staggered fermions.

of the Higgs mass, which brings me to the next topic.

5.1. Higgs mass upper bound revisited

We are now back to the topic I mentioned earlier in my talk in the context of pure scalar theory. Estimates of the fermion effects on the Higgs mass upper bound have also been obtained using mirror fermions\[46\]. Figure 3 is a summary of these various results for the upper bound of the Higgs mass. This figure is a plot of the renormalized mass of the Higgs particle, $m_\sigma$, versus the renormalized mass of the fermion, $m_f$, both in units of the renormalized Higgs vacuum expectation value, $v_R \equiv f_\pi$ from section 2). The data comes largely from refs. \[1\],\[46\] and \[43\]. $m_f = 0$ represents the case without fermions. The white box represents the data from previous calculations on a hypercubic lattice with the standard action, and the black box is the number quoted from Heller et. al. \[1\] earlier in the talk. The circles are estimates from the mirror fermion method, and the diamonds come from the infinite volume extrapolation of the calculations with the staggered fermion method. For details of how the numbers were obtained, please see the appropriate references. First of all, notice that the bounds are in fairly good agreement between the two fermion methods. This is reassuring and is a check that the regularization dependent effects are not dominant. In fact for rather large fermion mass, these results are close to the tree level unitarity bound, which indicates that the renormalized couplings are still rather small. Secondly, note that the inclusion of fermions allows a slightly higher (perhaps 30%) upper bound estimate than that found from the pure Higgs sector, perhaps as high as 1 TeV. This, accompanied by the broadening of the decay width, is an indication that finding the Higgs particle might be more difficult than we thought.

6. Domain wall fermions

A new and perhaps the most interesting proposal presently on the market for putting chiral fermions on the lattice, goes off in a different direction (quite literally). Based on work by Callan and Harvey\[47\], showing that a vector fermion theory in $2n + 1$ dimensions in the presence of a domain wall scalar field, admits a $2n$ dimensional chiral fermion solution, Kaplan suggests that this approach might be used in the lattice theory as well\[48\]. The beauty is that because the theory is vector-like in $2n + 1$ dimensions, A Wilson term can be used to make the doublers massive. So we start from the standard Wilson action for a free fermion in 5 dimensions (I will limit myself to $n = 2$ in what follows) with the exception that the ‘mass’ of the fermion is actually a background scalar field in a domain wall configuration:

$$S = \sum_x \frac{1}{2} \sum_{\mu=1}^5 \overline{\psi}(x) \gamma_\mu (\partial_\mu + \tilde{\partial}_\mu) \psi(x),$$

$$+ \sum_x \overline{\psi}(x)(m(x) + \frac{r}{2} \Box) \psi(x),$$

(27)
where

\[
m(x) = \begin{cases} 
    m & x_5 > 0 \\
    0 & x_5 = 0 \\
    -m & x_5 < 0 
\end{cases} \tag{29}
\]

in which \(m > 0\), and \(\Box\) is now the five dimensional lattice laplacian.

Kaplan shows that in this theory, a normalizable solution of the associated Dirac equation exists which is a simultaneously a chiral zero mode of the four dimensional lattice Weyl equation on the domain wall, and no other normalizable zero modes exist. The argument goes as follows. The Dirac equation following from eq. \((28)\) is given by (for \(r = 1\))

\[
P_5^+ \psi(p, x_5 + 1) + P_5^- \psi(p, x_5 - 1) + M(p, x_5)\psi(p, x_5) + i \sum_{\mu=1}^{4} \gamma_{\mu} \sin p_{\mu} \psi(p, x_5) = 0,
\]

with

\[
M(p, x_5) = m(x_5) - 1 - F(p),
\]

\[
F(p) = \sum_{\mu=1}^{4} (1 - \cos p_{\mu}),
\]

and \(P_{\mu}^\pm = \frac{1}{2}(1 \pm \gamma_{\mu})\). Now if we take the Fourier transform in four dimensions, then solutions of the form (\(p\) is the 4-momentum)

\[
\psi_{\pm}(p, x_5) = e^{ip \cdot x} \phi_{\pm}(p, x_5)
\]

satisfy the four dimensional lattice Weyl equation

\[
i \sum_{\mu=1}^{4} \gamma_{\mu} \sin p_{\mu} \psi_{\pm} = 0,
\]

if \(\phi_{\pm}\) satisfy

\[
\phi_{+}(p, x_5 + 1) + M(p, x_5)\phi_{+}(p, x_5) = 0,
\]

\[
\phi_{-}(p, x_5 - 1) + M(p, x_5)\phi_{-}(p, x_5) = 0.
\]

Solutions are easily obtained by assuming a value for \(\phi(p, 0)\) and hopping. The solutions either grow exponentially with \(x_5\) or decay according to the size of \(M(p, x_5)\). Normalizable, left-handed (positive chirality) solutions exist if

\[
|m - 1 - F(p)| > 1, \quad x_5 > 0,
\]

\[
|m + 1 + F(p)| > 1, \quad x_5 < 0.
\]

and the conditions for right-handed (negative chirality) normalizable solutions are similar, but with the inequalities reversed:

\[
|m - 1 - F(p)| > 1, \quad x_5 > 0,
\]

\[
|m + 1 + F(p)| < 1, \quad x_5 < 0.
\]

If \(m < 2\), the conditions for a left-handed chiral solution are satisfied if \(0 < m - F(p) < 2\) or \(F(p) \approx 0\), and one left-handed chiral solution exists. The conditions for a right-handed chiral fermion on the other hand cannot be satisfied. For other values of \(m\) the situation changes somewhat: for the ranges \(2 < m < 4\), \(4 < m < 6\), \(6 < m < 8\) and \(8 < m < 10\) there are four right-handed solutions, six left-handed solutions, four right-handed solutions and one left-handed solution respectively, with no solutions for larger values of \(m\)\([4]\).

To see what has happened to the usual doublers of chiral fermions we should put the theory in a box of size \(L\) in the fifth direction, and impose periodic boundary conditions. Then the domain wall mass becomes

\[
m(x) = \begin{cases} 
    0 & x_5 = 0 \\
    m & 0 < x_5 < L/2 \\
    0 & x_5 = L/2 \\
    -m & L/2 < x_5 < L \equiv 0,
\end{cases} \tag{41}
\]

so that there are now two domain walls. Following similar arguments as above, one finds that for every chiral fermion bound to one domain wall there is one of opposite chirality bound to the other. Thus the doubling is still there, but the chiral fermions are separated by a distance \(L/2\) in the fifth dimension, and communication between them is damped exponentially by the mass of the vector particles of the theory. Thus if \(m\) is taken large (as the cutoff), the four dimensional hypersurfaces decouple and we have two mirror worlds of free chiral fermions which don’t interact. To summarize, for \(m < 2\) say, we have a spectrum of one chiral fermion on each domain wall, and otherwise Dirac fermions with mass \(m\) and doublers with a mass of the order of the cutoff.

Now that we have defined a theory of free chiral fermions, the central question is whether they can be coupled to gauge fields. The first test is to study the theory with background gauge fields
to see whether the correct anomaly structure will be reproduce, as it is in the continuum case. In the latter case, Callan and Harvey follow Goldstone and Wilczek to obtain a contribution to the fermion current which arises from the heavy fermions, as $m$ is taken large. From calculating the relevant triangle diagram they find that

$$\langle \bar{\psi} \gamma_\mu \psi \rangle = -\frac{i}{2} \frac{m(x)}{|m(x)|} \epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}. \quad (42)$$

In particular this term survives as $m \to \infty$. Now the five dimensional theory is exactly gauge invariant, so the five dimensional current is conserved. For large $m$, the factor $m(x)/|m(x)|$ becomes a step function so that

$$\sum_{\mu=1}^4 \partial_\mu J^\mu = -\partial_5 J^5 \quad (43)$$

$$\propto \delta(x_5) \epsilon_{5\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \quad (44)$$

which is the correct anomalous divergence of a left-handed chiral fermion in four dimensions. Thus we see that the anomaly can be interpreted as current flowing off the domain wall. The effective action whose variation is the anomaly can easily be calculated:

$$S_{\text{eff}} \propto \int d^5x \langle J_\mu A_\mu \rangle \quad (45)$$

$$\propto \int d^5x \frac{m(x)}{|m(x)|} A_\mu \epsilon_{\mu\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \quad (46)$$

which is proportional to the Chern–Simons density. Thus the domain wall mass has conspired with the gauge fields to give rise to a Chern–Simons term.

The lattice version of the derivation follows along similar lines, although it differs somewhat due to the richer fermion spectrum. The end result is that the factor $m(x)/m(x)$ must be replaced by the factor

$$\sum_{n=0}^5 (-1)^n \binom{5}{n} \frac{m(x) - 2n}{|m(x) - 2n|}, \quad (47)$$

which is consistent with the zero mode spectrum of ref. [51]. In addition, using the above ideas with regard to a three dimensional domain wall theory, a direct numerical computation has been performed of the anomalous Ward identity in two dimensions [2].

So now we have a theory of chiral fermions that appears to work when coupled to background gauge fields. The real test and the most difficult task is to include the dynamics of the gauge fields. As we have seen, this has also been a problem with previous methods although for somewhat different reasons. In this case, we start with a five dimensional theory which somehow must be made to look four dimensional in the end, so in particular we would not expect that a perturbation theory in all gauge fields could be performed due to the non-renormalizability of the five dimensional theory. Fortunately this is not necessary, for five dimensional euclidean (Lorentz) invariance is already broken, so that the gauge couplings in the four dimensions and the fifth can be taken to differ:

$$S_{\text{gauge}} = \beta \sum_{i,j=1}^4 \text{tr} U_{ij} + \beta_5 \sum_{i=1}^5 \text{tr} U_{i5}, \quad (48)$$

where $U_{ij}(p)$ are plaquette variables. Thus $\beta$ could be taken large as usual for an asymptotically free chiral gauge theory, and $\beta_5$ could be varied, as needed. Alternatively, we may consider a subset of the possible five dimensional gauge fields, as long as gauge invariance in the four dimensional end product theory is maintained.

Another potential pitfall comes from the strange constraints relating different regions in the Brillouin zone to the existence of chiral solutions [52]. These will cause the momentum space propagators to be highly unusual, and they could cause gauge invariance to be broken in the four dimensional sense. Then in order to restore gauge invariance we would have to add counterterms to the theory, bringing us right back to the scenario of the Rome proposal. This brings one to ask, do all roads lead to Rome? One may hope not, but in any case, the crucial test for this model remains to understand how to add dynamical gauge fields.

A final note: assuming that somehow this model would survive the addition of dynamical gauge fields and a lattice version of the standard model could be defined. This model would also es-
cape the Banks $U(1)$ problem, and perhaps in the most novel way. Since only local anomalies need be canceled, the global anomalies would remain present, and proton decay would be permitted by sending baryon number off into the fifth dimension.

7. TWO MORE PROPOSALS

7.1. Reflection positive fermions

Another new proposal this year due to Zenkin is an attempt to define a theory of chiral fermions by enlarging the Hilbert space and enforcing reflection positivity\[54\]. The method borrows a trick from constructive field theorists\[55\], writing the gauge field link variables from constructive field theorists,\[55\], writing the gauge field link variables $U_{\mu x}$ as the product of two different group elements:

$$U_{\mu x} = W_{\mu x} W_{-\mu + \bar{\mu}},$$  \hspace{1cm} (49)

so that each site in a $d$ dimensional theory is associated with $2d$ $W$-fields, one for each direction both forward and backward, rather than the usual $d$ link variables. Operators in the theory are defined as functionals in this larger space, with the path integration over both $W_\mu$ and $W_{-\mu}$ independently. Finally, one can take each $W_{\pm \mu}$ to have components that couple differently to right- and left-handed fields,

$$W_{\pm \mu x} = W^L_{\pm \mu x} P_L + W^R_{\pm \mu x} P_R.$$  \hspace{1cm} (50)

For a vector theory, there is no cost at enlarging the Hilbert space in this way, for a change of variables will remove half of the $W$-fields from the action, and they can be integrated over trivially. This is not true of a chiral theory.

The final step of the proposal is to construct an action that is reflection positive, while using the Wilson mechanism to control the doublers. This action has the form

$$S = B + \Theta[B] + \sum_i C_i \Theta[C_i],$$  \hspace{1cm} (51)

where $B$ and $C_i$ are functionals of fermions $\psi$ and gauge fields $W$ defined for $x_4 > 0$, and $\Theta$ is an antilinear operator such that

$$\Theta[\psi_z \cdots \Gamma \cdots W_{\pm \mu y} \cdots \psi_z] = -\overline{\psi}_r(z) \gamma_0 \cdots W^{\dagger}_{\pm \mu y} \cdots \Gamma \cdots \gamma_0 \psi_r(z),$$  \hspace{1cm} (52)

where $\Gamma$ is a matrix in spin space and $r$ denotes a reflection along the $x_4$ axis. The action is

$$S = \sum_{\mu x} \left[ \frac{i}{2} \overline{\psi}_x \gamma_{\mu}(P_L U^L_{\mu x} + P_R U^R_{\mu x}) \psi_{x+\bar{\mu}} - (P_L U^L_{\mu x-\bar{\mu}} + P_R U^R_{\mu x-\bar{\mu}}) \psi_{x-\bar{\mu}} \right]$$

$$- \sum_{\mu x} \left[ \frac{i}{2} \overline{\psi}_x \gamma_{\mu}(P_L U^{RL}_{\mu x} + P_R U^{LR}_{\mu x}) \psi_{x+\bar{\mu}} - (P_L U^{RL}_{\mu x-\bar{\mu}} + P_R U^{LR}_{\mu x-\bar{\mu}}) \psi_{x-\bar{\mu}} - 2(P_L \Phi^{RL}_{\mu x} + P_R \Phi^{LR}_{\mu x}) \psi_{\bar{\mu}} \right]$$

$$+ \sum_p [\beta_L tr U^L_{\mu x} (p) + \beta_R tr U^R_{\mu x} (p)]$$

$$+ \sum_{\mu x} tr(1 - \Phi^{LR}_{\mu x} \Phi^{RL}_{\mu x})$$

where $U^L_{\mu}$ is the usual link variable, $p$ denotes plaquettes, and

$$U^{LR}_{\mu x} = W^{LR}_{\mu x} W^{\dagger}_{-\mu x+\bar{\mu}},$$  \hspace{1cm} (54)

$$U^{RL}_{\mu x} = W^{RL}_{\mu x} W^{\dagger}_{-\mu x+\bar{\mu}},$$  \hspace{1cm} (55)

$$\Phi^{RL}_{\mu x} = \frac{1}{2}(W^{RL}_{\mu x} W^{\dagger}_{\mu x} + W^{RL}_{\mu x} W^{\dagger}_{-\mu x}) = \Phi^{LR}_{\mu x},$$

are chiral changing link or site variables. As a first test, Zenkin has reproduced the chiral Schwinger model effective action\[56\] by integrating out the extra degrees of freedom in the two dimensional theory. However, the difficult questions are still to be asked, in light of what we have learned from other models. For example, what is the spectrum of the model in four dimensions? Do bound states form that pair up with the chiral fermions to result in a vector-like theory? This seems likely due to the presence of the Wilson term which must be considered a strong coupling in the present context. Further, what is the relation of the gauge invariance in the larger space of states to gauge invariance in the usual sense.

7.2. Zaragosa fermions

Yet another proposal which was discussed at the Rome workshop is the method proposed by the Zaragosa group\[57\]. The basic idea of the proposal is to decouple the doubler or replica fermions by suppressing interactions. The decoupling mechanism is accomplished by replacing all fermion fields in interaction terms of the action by fields which are averaged over a lattice hyper-
In momentum space this becomes

\[ \psi'(p) = \prod_\mu \cos \frac{p_\mu}{2} \psi(p), \]  

(57)

which shows us how the decoupling works. In the latter equation, the cosine is near unity for fermions near the origin in momentum space, but when any component of momentum gets close to a corner of the Brillouin zone, this function vanishes, just when a contribution from doublers would arise. So in principle one can simply write down an action with naive fermions with standard model quantum numbers, and then replace the \( \psi \) fields by \( \psi' \) fields in every interaction term. The desired effect is that the doublers remain present, but simply as free particles. In fact, the action is invariant under a generalized fermion shift symmetry\(^\text{[57]}\) which guarantees the decoupling of the 15 doubler fermions, in the same way that the right-handed neutrino decouples in models with the usual shift symmetry\(^\text{[13]}\).

The introduction of \( \psi' \) into the action breaks chiral symmetry. So just as in other approaches where this occurs, there is a benefit and a drawback. The benefit is that the Banks problem is not present, but the drawback is that as in other models, counterterms need be added to restore gauge invariance. Results to date include some perturbative calculations in chiral Yukawa models and some work on the phase diagram, but so far only limited progress has been made in the direction of including gauge fields into the theory. We will therefore have to be patient to wait for further results.

8. CONCLUSION

We have learned a lot this year concerning attempts to put chiral gauge theories on the lattice. Although some proposals must be abandoned, others have stepped in to take their place and there is still much to be understood. So the field looks promising for the year to come.

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