On a Novel Application of Wasserstein-Procrustes for Unsupervised Cross-Lingual Learning

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Abstract

The emergence of unsupervised word embeddings, pre-trained on very large monolingual text corpora, is at the core of the ongoing neural revolution in Natural Language Processing (NLP). Initially introduced for English, such pre-trained word embeddings quickly emerged for a number of other languages. Subsequently, there have been a number of attempts to align the embedding spaces across languages, which could enable a number of cross-language NLP applications. Performing the alignment using unsupervised cross-lingual learning (UCL) is especially attractive as it requires little data and often rivals supervised and semi-supervised approaches. Here, we analyze popular methods for UCL and we find that often their objectives are, intrinsically, versions of the Wasserstein-Procrustes problem. Hence, we devise an approach to solve Wasserstein-Procrustes in a direct way, which can be used to refine and to improve popular UCL methods such as iterative closest point (ICP), multilingual unsupervised and supervised embeddings (MUSE) and supervised Procrustes methods. Our evaluation experiments on standard datasets show sizable improvements over these approaches. We believe that our rethinking of the Wasserstein-Procrustes problem could enable further research, thus helping to develop better algorithms for aligning word embeddings across languages. Our code and instructions to reproduce the experiments are available at https://github.com/guillemram97/wp-hungarian.

1 Introduction

Pre-trained word embeddings, which map words to dense vectors of low dimensionality, have been the key enabler of the ongoing neural revolution, and today they serve as the basic building blocks of the vast majority of the contemporary Natural Language Processing (NLP) models. While initially introduced for English only [1–4], pre-trained embeddings quickly emerged for a number of other languages [5], and the idea of cross-language embedding spaces was born. In a cross-language embedding space, two semantically similar (or dissimilar) words would be close to (or far from) each other regardless of whether they are from the same or from different languages. Using such a space is attractive, as for a number of NLP tasks, it enables the application of an NLP model trained for one language on test input from another language. Ideally, such spaces could be trained on parallel bilingual datasets, but such resources are of limited size, e.g., compared to the large-scale monolingual resources typically used to pre-train monolingual word embeddings.

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Thus, it has been more attractive to train monolingual word embeddings for different languages independently, and then to try to align the corresponding embedding spaces in what is commonly known as bilingual lexicon induction. This has been attempted in a supervised [6, 7], in a semi-supervised [8], and in an unsupervised setting [9].

Initial attempts at aligning the spaces used a dictionary of word translation pairs as anchors between the two spaces to infer the nature of the transformation that relates the first language to the second one [6]. This is a supervised setup, where the alignment is typically done according to an orthogonal transformation that minimizes the Frobenius norm in the Procrustes problem, which has a closed-form solution, easily obtainable via SVD, as we describe below.

For the translation of word embeddings, \( W \) is taken to be an orthogonal matrix due to a self-similarity argument [10]. The convenience of using an orthogonal matrix has also been supported empirically [11–13]. The orthogonal Procrustes problem has a closed-form solution \( W = UV^T \), where \( U\Sigma V^T \) is the singular value decomposition (SVD) of \( X^T Y \) as shown by Schönemann [14].

**Procrustes.** Given two ordered clouds of points \( X, Y \in \mathbb{R}^{N \times d} \), each with \( N \) points of dimension \( d \), the orthogonal Procrustes problem finds the orthogonal matrix \( W \in \mathbb{R}^{d \times d} \) that minimizes the following Frobenius norm:

\[
\text{arg min}_{W \in \mathbb{R}^{d \times d}} \|XW - Y\|_2^2.
\]  

A popular unsupervised formulation of the problem is known as the Wasserstein-Procrustes [15, 16], which is more challenging as it needs to optimize a generalization of the Procrustes objective. One-to-one maps are encouraged through a permutation matrix \( P \). The convenience of one-to-one maps is justified for different reasons. First, the hubness problem [17] occurs in high-dimensional vector spaces where certain vectors are the nearest neighbor to a disproportionate number of other vectors, thus reducing the quality of the embedding space [18]. Second, one-to-one maps can be linked to Wasserstein distance and computational optimal transport.

**Wasserstein-Procrustes.** Given two clouds of points \( X, Y \in \mathbb{R}^{N \times d} \), each with \( N \) points of dimension \( d \), the Wasserstein-Procrustes problem finds an orthogonal matrix \( W \in \mathbb{R}^{d \times d} \) and a permutation matrix \( P \in \mathbb{R}^{N \times N} \) that minimize the Frobenius norm:

\[
\text{arg min}_{P \in \pi(N), W \in O(d)} \|XW - PY\|_2^2
\]

where \( \pi(N) \) is the set of \( N \)-dimensional permutation matrices and \( O(d) \) is the set of \( d \)-dimensional orthogonal matrices.

In practice, even though most existing approaches resort to some modification of this objective, they nevertheless yield good accuracy for synthetically generated dictionary induction tasks. Therefore, here we ask the following questions: Can we find approximate solutions to the Wasserstein-Procrustes objective as per Equation 2 that not only minimize the objective, but also yield good accuracy on dictionary induction tasks? Can we take existing methods and improve them further by using refinements that optimize the objective in Equation 2? Can we find natural scenarios for which we find good solutions? We attempt to answer these questions after a thoughtful analysis of the different objective functions used in the literature, following the call from Artetxe et al. [19] for a more fair model comparison.

## 2 Background. Towards a Unifying Framework

Although there have been attempts to compare different methods proposed for unsupervised dictionary induction [20], we are not aware of a unified description of the important existing methods. Different methods are based on different loss functions, which emerge from different hypotheses about the original problem. We believe that such a presentation is necessary for further contributions to this field, as all methods have strong similarities and exhibit important differences. We start by analyzing methods based on optimal transport methods, as they are most relevant to our approach.
2.1 Optimal Transport Methods

There have been some approaches framing the problem of unsupervised dictionary induction as an optimal transport problem, and this is the approach we will adopt in the following sections. Haghighi et al. [21] proposed a self-learning method for bilingual lexicon induction, representing words with orthographic and contextual features and using the Hungarian algorithm [22] to find an optimal one-to-one matching.

With the emergence of word embeddings [1], words were interpreted as vectors in high-dimensional spaces, and concepts such as distance between words started to gain attention. Ruder et al. [23] presented Viterbi EM, where words were mapped following a one-to-one map between subsets $X$ and $Y$ of $X$ and $Y$, respectively, and the isometry was induced by an orthogonal matrix. They deviated from the Wasserstein-Procrustes objective by including a penalization term for unmatched words $Y'_T = Y - Y'$. They did not consider all possible matches, instead imposing a restriction on the $k$ nearest neighbors when running the Jonker-Volgenant algorithm for optimal transport [24].

Zhang et al. [25] proposed two different methods: WGAN (an adversarial network that optimizes the Wasserstein distance) and EMDOT (an iterative procedure that uses Procrustes and solves a linear transport problem). Both methods are inspired by the Earth Mover’s Distance (EMD), which defines a distance between probability distributions, which they applied to frequencies of words. They found that, although EMDOT could converge to bad local minima, it improved the results when used as a refinement tool after first optimizing with WGAN. Alvarez-Melis and Jaakkola [26] used the concept of Gromov-Wasserstein distance to provide an alternative to Wasserstein-Procrustes. This distance does not operate on points but on pairs of points, turning the problem of finding optimal matching $\Gamma^*$ from a linear into a quadratic one. This new loss function can be optimized efficiently with first-order methods, whereby each iteration involves solving a traditional optimal transport problem. Artetxe et al. [27] achieved better results by combining this idea with a refinement method called stochastic dictionary induction, i.e., randomly dropping dimensions out of the similarity matrix when extracting a seed dictionary for the next iteration of the Procrustes analysis.

2.2 Other Methods

Wasserstein-Procrustes is one of the recurring loss functions in the literature, but there have been also deviations from the original problem. Grave et al. [15] suggested an iterative procedure whose initial condition minimizes the convex relaxation $\|X^\top PY\|^2_2$ instead of the original problem. This relaxation is known as the Gold-Rangarajan relaxation and can be solved using the Frank-Wolfe algorithm [28, 29]. The solution to this relaxation is then used as the initial condition for a gradient-based iterative procedure that stochastically samples different subsets of words for which there is not necessarily a direct translation. This deviates strongly from Objective 2: not only the initial condition does not optimize Wasserstein-Procrustes, but also the iterative procedure does not optimize it as it translates words that are not necessarily the optimal matches. Alaux et al. [16] were also inspired by Objective 2 for aligning multiple languages to a common space. However, they minimized a loss function based on the CSLS metric from Lample et al. [30]. In a similar fashion, the entropy regularization of the Gromov-Wasserstein problem [31] has been used for bilingual lexicon induction.

Generative Adversarial Network (GAN) optimization was first introduced for bilingual lexicon induction by Barone and Valerio [32], but its canonical implementation was given by Lample et al. [30], who presented multilingual unsupervised and supervised embeddings (MUSE), an adversarial method in which the transformation matrix $W$ is considered as a generator, and thus is trained by a generative adversarial network, so that the mapped word embeddings $XW$ cannot be distinguished from the set $Y$ via a discriminator [33]. However, a simple thought experiment can convince ourselves that this approach does not minimize distances; see the Supplementary material for more details.

Hoshen and Wolf [34] were inspired by the Iterative Closest Point (ICP) method used in 3D point cloud alignment. Although their transformation matrix is not necessarily orthogonal, this property is enforced using the regularization $L(X, Y, W; \lambda) := \lambda\|XWW^\top - X\|^2_2 + \lambda\|YW^\top W - Y\|^2_2$. Another fundamental difference to Objective 2 is that they do not use a one-to-one mapping for $P$. 
This list is not exhaustive, as there have been methods that do not rely on loss functions, and such that go beyond the geometry of the trained word embeddings. For example, Artetxe et al. [35] used both the word embeddings and the monolingual corpus used to train them.

To sum up, in Table 1, we list the relevant objectives from above using our formalism described above.

Table 1: Objective functions of relevant existing methods in the language of our formalism.

| Method | Objective |
|--------|-----------|
| Grave et al. [15] and Ours | \( \min_{W \in O(d), P \in \pi(N)} \|XW - PY\|_F^2 \) |
| Alvarez-Melis and Jäakkola [26] | \( \min_{U^*} \text{best coupling } W \in O(N) \|XU^* - WY\|_2^2 \) |
| Hoshen and Wolf [34] | \( \min_{W \in O(d)} \|XW - Y\|_2^2 + \|YW^\top - X\|_2^2 + L(X, Y, W; \lambda) \) |
| Ruder et al. [23] | \( \min_{W \in O(d), P' \in \pi(N')} \|X'W - P'Y'\|_2^2 + \|Y'\|_2^2 + \sum_{i,j=1}^{N, N} P_{i,j} ((X_i W)_j - Y_i)^2 \) |
| Lample et al. [9] | \( \min_{W \in O(d), P \in \pi(N)} \sum_{i=1,j=1}^{N, N} P_{i,j} ((X_i W)_j - Y_i)^2 \) |
| Zhang et al. [25] | \( \min_{W \in O(d), P \in \pi(N)} \sum_{i=1,j=1}^{N, N} P_{i,j} ((X_i W)_j - Y_i)^2 \) |

3 Properties of the Wasserstein-Procrustes Problem

We begin by simplifying Objective 2 to arrive at some essential properties, described below.

**Proposition 1.** The Wasserstein-Procrustes problem is equivalent to maximizing the trace norm on the permutation matrix \( X^\top P Y \) over \( P \), described as follows:

\[
\arg \min_{P \in \pi(N)} \|X - PY\|_2^2 = \arg \max_{P \in \pi(N)} \|X^\top P Y\|_*
\]  

where \( \|\cdot\|_* \) denotes the nuclear norm and \( W \) is selected, so that it fulfills that \( U^\top W V = \mathbb{I}_d \), where both \( U(P) \) and \( V(P) \) are evaluated at a matrix \( P^* \) that achieves the optimum of Equation 3.

We demonstrate this equivalence in the Supplementary material.

**Hungarian algorithm** Given two clouds of points \( X, Y \in \mathbb{R}^{N \times d} \), each with \( N \) points of \( d \) dimensions, the Hungarian algorithm finds the permutation matrix \( P \) that gives the correspondence between the different points by solving the following problem:

\[
\arg \min_{P \in \pi(N)} \|X - PY\|_2^2.
\]  

**Proposition 2.** Problem 4 is equivalent to maximizing the trace of \( X^\top P Y \) over \( P \):

\[
\arg \min_{P \in \pi(N)} \|X - PY\|_2^2 = \arg \min_{P \in \pi(N)} \text{Tr} \left( X^\top P Y \right),
\]  

which is the maximum weight matching problem. The latter can be solved using the Hungarian algorithm, which has a complexity of \( O(N^3) \) [22].

We demonstrate this equivalence in the Supplementary material.

**Equivalent problems** One useful property of the trace norm is that \( \|U A\|_* = \|AV\|_* = \|A\|_* \), where \( U \) and \( V \) are orthogonal matrices. Knowing this, and writing \( U_X \Sigma_X V_X^\top \) and \( U_Y \Sigma_Y V_Y^\top \) as the SVD decompositions for \( X \) and \( Y \), respectively, we obtain the following:

\[
\arg \max_{P \in \pi(N)} \|X^\top P Y\|_* = \arg \max_{P \in \pi(N)} \|V_X \Sigma_X U_X^\top P U_Y \Sigma_Y V_Y^\top\|_*
\]
which yields
\[
\arg \max_{P \in \pi(N)} \left\| \Sigma_X U_X^T P U_Y \Sigma_Y \right\|_*,
\]

(6)

Let us define \( \tilde{X} = U_X \Sigma_X \) and \( \tilde{Y} = U_Y \Sigma_Y \). Then, the optimal solution \( P \) would be the same for translations involving all of the following pairs of word embeddings: \((X, Y), (\tilde{X}, \tilde{Y}), (X, \tilde{Y})\) and \((\tilde{X}, Y)\). However, the optimal transformation matrix \( W^* \) will be different for each of these problems.

There is a different, yet interesting way of looking at this: if we follow the iterative procedure that starts from an initial transformation matrix \( X_0 = XW_0 \), and then we want to solve Problem (5), the equivalent problems will induce a set of natural initializations of the transformation \( W \), which we formalize below:

\[
\text{Given the iterative procedure that tries to minimize Wasserstein-Procrustes by first obtaining the permutation matrix } P_n = \arg \min_{P \in \pi(N)} \text{Tr}(X_n^T P Y_n) \text{ and then the transformation matrix } W_n = \arg \min_{W \in R^{N \times N}} \|X_n W - P_n Y_n\|_2^2, \text{ the procedure aims for the same solution } P \text{ as the problems with initial conditions } X_0 = XW_0, X_0 = XV_X W_0, X_0 = XV_X W_0 V_Y^T, X_0 = XV_X W_0 V_Y^T.
\]

The significance of the different natural initialization is that it gives us a starting point for different problems that have the same solution \( P \). It must be noted, however, that these transformations of \( X_0 \) are not the unique ones that will have the same original solution, as the trace norm is invariant to any orthogonal transformation; however, they help to avoid bad local minima as we will show in Section 5 below.

4 Approach

Below, we present a general iterative algorithm to solve the Wasserstein-Procrustes problem.

Joint optimization on \( W \) and \( P \). For the Wasserstein-Procrustes problem from Equation 2, a joint iterative procedure involving the Procrustes problem and the Hungarian algorithm (see Algorithm 1) has been dismissed due to its computational cost and convergence to bad local minima [25]. However, as we will show below, there are a number of situations where such an approach can be extremely beneficial if we apply some improvements based on the discussion in the previous section.

Algorithm 1. Cut Iterative Hungarian (CIH) Algorithm

1. We initialize as follows: \( X \leftarrow XW_0 \).
2. We find \( P \leftarrow \text{Hungarian} \ (X, Y) \) and \( W \leftarrow \text{Procrustes} \ (X, PY) \).
3. If the trace norm has increased, update \( X_{NEW} \leftarrow XW \) and \( Y_{NEW} \leftarrow PY \), repeat Step 2.

Variants of the natural initializations. The first improvement is to consider the different equivalent problems or the natural initialization transformations, mentioned in the previous section. We observe empirically that apart from the four problems that share the same optimal \( P \), it is possible to improve the results by considering the opposite optimization problem: instead of maximizing the costs for the two clouds of points \( (\tilde{X}, \tilde{Y}) \), sometimes minimizing the costs yields a solution with a higher trace norm, and thus the algorithm eventually converges to a better solution. The minimization is achieved by simply considering the cloud \(-X\) instead of \( X \). Algorithm 2 is the most general iterative procedure that we consider here, and it serves as the backbone for our experiments below:

Algorithm 2. Iterative Hungarian (IH) Algorithm. It is the same as Algorithm 1, but in Step 2 we also consider the solutions for four natural initializations: \( X_0 = XW_0, X_0 = XV_X W_0, X_0 = XV_X W_0 V_Y^T, X_0 = XV_X W_0 V_Y^T \), also considering the cloud \(-X\) for the four different initializations.

Supervised translation. We also consider the task of supervised word-to-word translation. There are different ways of doing this, but the procedure that converges the fastest is to fix \( n \) pairs of words when calculating the Hungarian map, where typically \( n \ll N \). We also consider similar approaches, e.g., deciding how to update Algorithm 2, taking into account the accuracy of the maps on a small subset of the data. Choosing among these methods could be motivated by how trustworthy the
initial dictionary is. By trustworthy here we mean how many of the corresponding cloud points are correctly matched.

We use a fast implementation of the Hungarian algorithm\(^2\) for dense matrices based on shortest path augmentation [36]. Relaxations of the original problem can achieve higher speed ups. Cuturi [37] showed how smoothing the classical optimal transport problem with an entropic regularization term results in a problem that can be solved using the Sinkhorn-Knopp’s matrix scaling algorithm [38] at a speed that is several orders of magnitude faster than that of transportation solvers.

**Mapping.** Although our method finds a permutation matrix \(P\), this is not necessarily the best possible mapping as the set of word-to-word translations does not have to represent a one-to-one mapping. Nearest neighbor approaches can be used, but they suffer from the so-called hubness problem: in high-dimensional vector spaces, certain vectors are universal nearest neighbors [18], and this is a common problem for word-embedding-based bilingual lexicon induction [17]. Lample et al. [30] presented cross-domain similarity local scaling (CSLS), which is a method intended to reduce the influence of hubs by expanding high-density areas and condensing low-density ones. Given a source vector \(x_s\), the mean similarity of its transformation \(Wx_s\) to its \(k\) target nearest neighbors \(\mathcal{N}_k^T(Wx_s)\) is defined as \(\mu_k^T(Wx_s) = \frac{1}{k} \sum_{y_t \in \mathcal{N}_k^T(Wx_s)} \cos(Wx_s, y_t)\). Likewise is defined \(\mu_k^S(y_t)\), i.e., the mean similarity of a target word \(y_t\) to its neighborhood of source mapped vectors. Then, the CSLS similarity between a mapped source vector \(x_s\) and a target vector \(y_t\) is calculated as follows:

\[
\text{CSLS}(Wx_s, y_t) = 2 \cos(Wx_s, y_t) - \mu_k^T(Wx_s) - \mu_k^S(y_t).
\]

Intuitively, this mapping increases the similarity associated with isolated word vectors, and it decreases the one for vectors lying in dense areas. In the following experiments, we use the mapping induced by CSLS with \(k = 10\).

## 5 Experiments

In our first set of experiments, we deploy our method on top of well-known methods for cross-lingual learning and we show that it improves their accuracy, meaning that it can be used as a refinement tool. In the second set of experiments, we recreate the benchmarks from [15], and we show that our method can align word embedding spaces without a good initialization matrix.

### 5.1 The Iterative Hungarian Algorithm as a Refinement Tool

The experiments in this section use the Iterative Hungarian (IH) algorithm starting with the initial condition \(W_0\) produced from the following methods:

- The adversarial approach by Lample et al. [9]. This combines the adversarial training described in Section 2 with a refinement step, which consists of creating a dictionary from the best matches and then running the supervised Procrustes algorithm using that dictionary.
- The supervised Procrustes approach.
- The Iterative Closest Point (ICP) method by Hoshen and Wolf [34].

We used the word embeddings, the dictionaries and the evaluation methods from Lample et al. [30]. We trained the transformation matrix obtained from MUSE [30] on 200,000 words. Then we ran the Iterative Hungarian algorithm on a subsample of 45,000 words. Finally, we refined the new transformation matrix following the procedure in Lample et al. [30]. Also, inspired by their work, we induced mappings using CSLS with \(k = 10\) nearest neighbors.

We ran the Iterative Hungarian algorithm after normalizing the word embeddings, which we found to converge faster. It must be noted that, since the adversarial part does not normalize the word embeddings, the \(W_0\) matrices do not match exactly and thus not normalizing them should yield better results at a higher computational cost. Hartmann et al. [20] showed that unit-length normalization makes GAN-based methods more unstable and deteriorates their performance, but supervised alignments or Procrustes refinement are not affected by it.

The results can be seen in Table 2 for MUSE, in Table 3 for Procrustes, and in Table 4 for ICP. We can see that our Iterative Hungarian algorithm improves the accuracy when used as a refinement tool.

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\(^2\)http://github.com/cheind/py-lapsolver
Table 2: The Iterative Hungarian (IH) Algorithm starts with a transformation matrix $W$ from MUSE over three different seeds (1, 2 and 3 represent the different runs of MUSE), and then refines it.

| Method       | en-es | es-en | en-fr | fr-en | en-it | it-en | it-en | de-en | de-en | ru-en | ru-en | mean |
|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| MUSE (1)     | 82.6  | 83.7  | 82.5  | 82.0  | 76.8  | 77.6  | 75.1  | 72.5  | 42.5  | 60.1  | 73.5  |
| MUSE (1) + IH| 82.5  | 84.1  | 82.7  | 82.4  | 78.3  | 77.9  | 74.9  | 73.3  | 44.5  | 60.7  | 74.1  |
| MUSE (2)     | 81.9  | 83.2  | 82.1  | 82.4  | 77.5  | 77.5  | 74.7  | 72.9  | 37.0  | 61.9  | 73.1  |
| MUSE (2) + IH| 82.5  | 84.1  | 82.7  | 82.4  | 78.1  | 78.4  | 74.7  | 73.3  | 42.3  | 62.5  | 74.0  |
| MUSE (3)     | 82.1  | 84.0  | 82.1  | 82.3  | 77.9  | 77.7  | 74.8  | 69.9  | 37.1  | 60.1  | 72.8  |
| MUSE (3) + IH| 82.3  | 83.9  | 82.6  | 82.4  | 77.8  | 77.8  | 75.1  | 72.9  | 38.9  | 62.1  | 73.6  |

Table 3: The Iterative Hungarian (IH) Algorithm starts with a transformation matrix $W$ from the supervised Procrustes, and then refines it.

| Method       | en-es | es-en | en-fr | fr-en | en-it | it-en | it-en | de-en | de-en | ru-en | ru-en | mean |
|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Procrustes   | 81.7  | 83.3  | 82.1  | 81.9  | 77.3  | 77.0  | 73.7  | 72.7  | 49.9  | 60.8  | 74.0  |
| Procrustes + IH| 82.5  | 84.2  | 82.2  | 82.6  | 78.1  | 78.0  | 75.0  | 73.5  | 47.9  | 63.9  | 74.8  |

refinement tool. We believe that this is because the other methods do not try to optimize the Wasserstein-Procrustes objective directly, even though they achieve very good translations without relying on it.

5.2 Aligning Word Embeddings from the Same Doing

The second set of experiments justify that the simple iterative procedure displayed in Algorithm 2 works and we explain under what circumstances it can be relaxed or needs some help in the form of either supervision or a natural initialization matrix $W_0$. For the following controlled experiments, we set the initialization matrix to be the identity. We experiment with the following four approaches:

- **Hungarian.** Running the Hungarian algorithm for only one iteration, and then taking the permutation matrix $P$ as the map.
- **Cut Iterative Hungarian (CIH).** Running the Hungarian algorithm to update $Y \leftarrow PY$ and $X \leftarrow XW$ (see Algorithm 1).
- **Iterative Hungarian (IH).** Running the previous iterative procedure but considering the different natural initializations (see Algorithm 2).
- **Supervised Iterative Hungarian (SIH).** We learn the correct mapping from a random sub-sample of 5% of the words, and then we run the IH algorithm for the remaining words.

The experiments from this subsection recreate those by Grave et al. [15]. We use fastText [3, 4] to train word embeddings on 100M English tokens from the 2007 News Crawl corpus. The different experiments in this section consist of changing the different training conditions and correctly mapping the results. We train the models using Skipgram [39] unless stated otherwise, and using the standard parameters of fastText.

We perform the following four experiments:

- **Seed.** We only change the seed used to generate the word embeddings in our fastText runs. The source and the target are word embeddings trained using the same parameters.
- **Window.** We use window sizes of 2 and 10, respectively. The source and the target correspond to word embeddings trained on the same data but with different window sizes.
- **Algorithm.** We train the first algorithm with Skipgram and the second one with CBOW [39]. The source and the target correspond to word embeddings trained on the same data but using a different method.
- **Data.** We separate the dataset in two different parts of the same length. We train corresponding word embeddings from the two separate parts. The source and the target correspond to word embeddings trained with the same parameters but on different data.

\(^3\)http://statmt.org/wmt14/translation-task.html
\(^4\)https://github.com/facebookresearch/fastText
Table 4: The Iterative Hungarian (IH) Algorithm starts with a transformation matrix \( W \) from ICP over three different seeds (1, 2 and 3 represent the different runs of ICP), and then refines it.

| method         | en-es | es-en | en-fr | fr-en | en-it | it-en | en-de | de-en | en-ru | ru-en | mean |
|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|------|
| ICP (1)        | 81.9  | 82.7  | 81.9  | 81.5  | 76.0  | 75.5  | 72.3  | 72.3  | 46.4  | 56.6  | 72.7 |
| ICP (1) + IH   | 82.5  | 84.1  | 82.1  | 82.7  | 78.1  | 78.0  | 76.6  | 72.7  | 46.2  | 63.2  | 74.6 |
| ICP (2)        | 80.8  | 82.5  | 81.3  | 80.4  | 76.3  | 76.3  | 72.3  | 72.4  | 46.5  | 57.5  | 72.6 |
| ICP (2) + IH   | 82.2  | 84.1  | 82.4  | 82.3  | 78.2  | 77.9  | 76.4  | 73.3  | 46.6  | 63.1  | 74.7 |
| ICP (3)        | 82.0  | 82.6  | 82.0  | 81.8  | 75.7  | 76.6  | 73.1  | 72.6  | 45.1  | 56.2  | 72.8 |
| ICP (3) + IH   | 82.5  | 84.2  | 82.0  | 82.4  | 77.7  | 77.7  | 76.9  | 73.5  | 45.2  | 63.1  | 74.5 |

Table 5: Our method correctly aligns the word embeddings.

| Method | Seed | Window | Algorithm | Data |
|--------|------|--------|-----------|------|
| Hungarian | 99% | 7% | 7% | 1% |
| CIH    | 100% | 100% | 100% | 0% |
| IH     | 100% | 100% | 100% | 0% |
| SIH    | 100% | 100% | 100% | 100% |

We run the above algorithms on the 10,000 most frequent words. Table 5 shows the results for the different algorithms. We perform the final mapping using the nearest neighbor for CSLS with \( k = 10 \), and we calculate the percentage over all words from the model. Some observations follow:

- The supervised approach works well with very little supervision, but all other attempts failed when facing the problem of mapping data from different datasets. This is probably because, by adding some supervision, we improve the initial \( W_0 \). This effect may be similar (although with less impact) to the help introduced in the IH algorithm with the equivalent problems or the natural initial transformations.
- The first three experiments converged in three iterations or less. The SIH algorithm took around twenty iterations to converge for the Data experiment.
- The Hungarian algorithm, which was not designed for the Wasserstein-Procrustes method, correctly finds the mapping for the seed experiment, whereas some other reported iterative experiments failed to achieve good results in this experiment [15].

The proposed iterative procedures do converge, but they usually need good initial conditions or the help of supervision to converge to a good minimum. This suggests that Algorithm 1 could work well as long as we start from an initial transformation matrix \( W_0 \) close enough to the true solution. The importance of the initial condition can be shown by the natural initial conditions. The solution of the four different equivalent problems induce different optimal transformation matrices \( W^* \). In the first iteration of the IH algorithm, a branch among these four is chosen. Table 6 shows the Euclidean distance between each of the four natural initializations (assuming \( W_0 = \mathbb{I} \)) and their respective optimal solution \( W^* \) for the four experiments. These distances are different for the four branches, and being able to choose the best one (the one that minimizes this distance) is key for convergence.

Table 6: Distance between the natural initialization and the optimal solution for the four experiments.

| Method | Seed | Window | Algorithm | Data |
|--------|------|--------|-----------|------|
| \( \mathbb{I} \) | **9.49** | 12.59 | 12.45 | 14.11 |
| \( V_X \) | 14.13 | 14.14 | 14.18 | 14.19 |
| \( V_Y \) | 14.15 | 14.18 | 14.18 | 14.14 |
| \( V_X V_Y \) | 13.95 | 14.10 | 14.09 | 14.16 |

The distances that are too big do not converge to a good solution. For the Seed experiment, such a small distance explains why a single iteration of the Hungarian algorithm was enough for a strong result. The Window and the Algorithm do not converge when running on a branch different from the first one—which has the smallest distance—and when they run on the first branch, they
converge in a few iterations. Hence, being able to provide a good initial transformation matrix $W_0$ and to correctly discriminate what the best branches are is essential for this approach.

Even though our method is designed as a refinement tool starting from a good initialization, we also experimented with supervised translation from English to Spanish; see the Supplementary Material.

6 Conclusion and Future Work

We have underlined some mathematical properties of the Wasserstein-Procrustes problem and hence used the concept of the different natural initialization transformations in an iterative algorithm to achieve improved results for mapping word embeddings between different languages. In particular, we have shown that it is possible to use our algorithm as a refinement tool and we have demonstrated improved results after using the transformation of Lample et al. [30] as the initialization matrix $W_0$

We hope that our rethinking of the Wasserstein-Procrustes problem would enable further research and would eventually help develop better algorithms for aligning word embeddings across languages, especially taking into account that most unsupervised approaches try to minimize loss functions different from Objective 2.

In future work, we plan to study other loss functions. We are further interested to see how well the objectives in Table 1 correlate with CSLS. Finally, we plan combinations with other existing methods.

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Supplementary Material

The Supplementary Material is organized as follows. In Section A, we present a thought experiment on the objective function that the MUSE model optimizes. In Section B, we offer the proofs referred to in Section 3 from the main text. In Section C, we present some experiments on creating a bilingual English–Spanish dictionary. In Section D, we discuss some ideas for future research.

A GANs do not optimize Wasserstein-Procrustes

Multilingual unsupervised and supervised embeddings (MUSE) is an adversarial method in which the transformation matrix $W$ is considered as a generator, and thus it is trained by a generative adversarial network, so that the mapped word embeddings $XW$ cannot be distinguished from the target embeddings $Y$ via a discriminator [33].

Let us consider a setup with very low dimensionality. Suppose that the source language $S$ and the target language $T$ have only two words each, which are represented as two-dimensional vectors. Moreover, let the vectors that correspond to the words in language $S$ be orthogonal, i.e., let the angle between them be 90 degrees, and let the angle between the vectors for the two words in language $T$ be 180 degrees.

On one hand, the orthogonal transformation that minimizes the Wasserstein-Procrustes objective function (the one that minimizes the Frobenius norm between sets) would be a rotation of 45 degrees. On the other hand, an adversarial discriminator that is smart enough would “realize” that all the target points are on the same axis ($x = 0$), and therefore it would achieve 100% accuracy for categorizing source vs. target points. The only way to fool the discriminator is to bring points to that axis, which is equivalent to making points coincide. Since only one point from the source language would coincide with a point from the target, we would expect the discriminator to have an accuracy of around 75%.

Figure 1 shows an illustration, where the words from language $S$ are shown as red triangles, representing vectors at 90 degrees, and the words for language $T$ are depicted as blue circles, representing vectors at 180 degrees. On the left is shown the transformation that an adversarial approach would find, while on the right is shown a rotation that minimizes the Frobenius norm in the Wasserstein-Procrustes problem.

![Figure 1: GANs do not optimize Wasserstein-Procrustes.](image)

We could try to generalize this result: when the discriminantor is smart enough—meaning that it can learn the exact position of the target points because it has enough degrees of freedom—it’s best strategy is to make points coincide, which does not involve a distance measure. As unsupervised bilingual lexicon induction typically involves numerous word vectors of a higher dimensionality, the number of parameters of the discriminator and the way the system is trained play an important role.
B Proofs from Section 3

B.1 Proof of Proposition 1

To begin the proof of the proposition, we convert the nuclear norm to a Frobenius inner product 
\[ \langle A, B \rangle_F = \text{Tr} \left( A^T B \right), \]
as follows:

\[
\arg \min_{P \in \pi(N), W \in O(d)} \| XW - PY \|_F^2 = \\
\arg \min_{P \in \pi(N), W \in O(d)} \langle XW - PY, XW - PY \rangle_F = \\
\arg \min_{P \in \pi(N), W \in O(d)} \langle XW, XW \rangle_F + \langle PY, PY \rangle_F - 2 \langle XW, PY \rangle_F = \\
\arg \min_{P \in \pi(N), W \in O(d)} \| X \|_F^2 + \| Y \|_F^2 - 2 \langle XW, PY \rangle_F = \\
\arg \max_{P \in \pi(N), W \in O(d)} \langle XW, PY \rangle_F
\]

In the above derivation, we used that \( \| XW \|_F^2 = \| X \|_F^2 \) and \( \| PY \|_F^2 = \| Y \|_F^2 \) since \( P \) and \( W \) are orthogonal. Since these two norms are independent of \( P \) and \( W \), we can ignore them for the optimization.

Next, we use the cyclic property of the trace, \( \text{Tr}(ABC) = \text{Tr}(CAB) = \text{Tr}(BCA) \), to develop the above expression further as follows:

\[
\arg \max_{P \in \pi(N), W \in O(d)} \langle XW, PY \rangle_F = \\
\arg \max_{P \in \pi(N), W \in O(d)} \text{Tr} \left( W^T X^T P Y \right) = \\
\arg \max_{P \in \pi(N), W \in O(d)} \langle W^T X^T P Y \rangle_F = \\
\arg \max_{P \in \pi(N), W \in O(d)} \langle W^T X^T P Y \rangle_F = \\
\arg \max_{P \in \pi(N), W \in O(d)} \langle W^T X^T P Y \rangle_F = \\
\arg \max_{P \in \pi(N), W \in O(d)} \langle W^T X^T P Y \rangle_F = \\
\arg \max_{P \in \pi(N), W \in O(d)} \langle W^T X^T P Y \rangle_F = \\
\arg \max_{P \in \pi(N), W \in O(d)} \langle S, \Sigma \rangle_F,
\]

where \( U(P) \Sigma V(P)^T = \text{SVD}(X^T P Y) \) and \( S = U^T W V \) for \( U \equiv U(P) \) and \( V \equiv V(P) \).

Note that \( S \) is orthogonal since it is the product of orthogonal matrices, which implies that it must be the identity matrix \( I_d \) in order to maximize the Frobenius inner product. Therefore, we further develop the above derivation as follows:

\[
\arg \max_{P \in \pi(N), W \in O(d)} \langle S, \Sigma \rangle_F = \arg \max_{P \in \pi(N)} \text{Tr}(\Sigma) = \arg \max_{P \in \pi(N)} \| X^T P Y \|_{*},
\]

\[ \square \]

B.2 Proof of Proposition 2

The proof of this proposition follows directly from the the proof of the previous one after replacing \( W \) with the identity matrix \( I_d \) and noting that \( \langle I_d, X^T P Y \rangle_F = \text{Tr} \left( X^T P Y \right) \), as desired.

\[ \square \]
C Further Experiments on English to Spanish

We used a variant of the Iterative Hungarian (IH) Algorithm for translation from English to Spanish using the same datasets and evaluation measures as in [30]. Our algorithm presents here a supervised variant: in the first iteration, \( W_0 \) is taken from the known dictionary as in the Procrustes problem. Then, for each iteration, the map between the words in the dictionary is not modified by the Hungarian algorithm. Finally, the new transformation matrix \( W_{n+1} \) is calculated by solving the Procrustes problem on all the words.

We ran tests of this setup on 10,000 words with different sizes for the dictionaries. We qualitatively observed that, when the size of the dictionary was large (more than 10% of the data) and it induces a good initialization \( W_0 \), then the algorithm was not able to improve these results by a wide margin. When the size of the dictionary was small (1 – 2% of the data) and induced a bad accuracy score for \( W_0 \) (30%), then the final performance of the algorithm could either maintain that accuracy or it could go down. Finally, there is an intermediate region where this setup can improve the final accuracy. In particular, we observed that with a dictionary of only 5% of the words, it was possible to achieve an accuracy of 68%, which could be refined up to 81% after doing the same refinements as in Lample et al. [30].

The translation from English to Spanish presents one important issue: it is not a one-to-one mapping. Furthermore, the evaluation methods from [30] highly discourage it since the test dictionary has a lot of repeated words. This means that improving these accuracy scores is a hard task, even tough the convenience of one-to-one mapping has been argued for in the past [23]. We have shown in Section 5.1 that our approach in the MUSE framework should be used as a refinement step in this context.

D Alternative Developments of Wasserstein-Procrustes and Ideas for Further Research

Some modifications of our approach induce interesting ideas for future work: In the same way Equation 4 is only dependent on the permutation matrix \( \tilde{P} \), it is possible to develop a similar expression involving \( W \). Using the exact same properties, we obtain the following:

\[
\arg \max_{P \in \pi(N), W \in O(d)} \langle P, X W Y^T \rangle_F = \arg \max_{P \in \pi(N), W \in O(d)} \langle P, U \Sigma V^T \rangle_F = \arg \max_{P \in \pi(N_1, W \in O(d))} \langle S, \Sigma \rangle_F = \arg \max_{W \in O(d)} \|X W Y^T\|_*.
\]

However, here we assume that \( U^T P V = I \), which is generally false. Nevertheless, we can use the orthogonal matrix \( \tilde{P} = U V^T \) and to project it to the space of the permutation matrices. This formulation of the problem has the advantage that the orthogonal matrix \( W \) generally has a lower dimensionality than the permutation matrix \( \tilde{P} \), and thus it could be reduced through PCA.

A completely different approach would consider the Wasserstein-Procrustes objective in the alternative form (3) and would perform an iterative procedure on the trace norm. From the property of the trace norm, we have that \( \|A\|_* = \sup_{\|B\|_F \leq 1} \|\text{Tr}(BA)\| \). Hence, it is possible to design an iterative algorithm that optimizes \( B \) and \( A \) jointly. Given a particular \( B_n \), we can maximize \( \text{Tr}(B_n X^T P Y) \) using the Hungarian algorithm. Given one particular \( P_n \), \( B \) can be calculated in a similar way and then it can be refined using Sinkhorn’s theorem: given one matrix \( A \) with strictly positive elements, there exist diagonal matrices \( D_1 \) and \( D_2 \) with strictly positive diagonal elements such that \( D_1 A D_2 \) is doubly stochastic. The matrices \( D_1 \) and \( D_2 \) are unique modulo multiplying the first matrix by a positive number and dividing the second one by the same number [40]. A very simple application of this theorem is that, given one matrix \( A \) with strictly positive elements, the algorithm that alternatively rescales all rows and all columns of \( A \) to sum to 1 converges into a doubly stochastic matrix [38].