Gaussian lower bounds on the Dirichlet heat kernel and non-existence of local solutions for semilinear heat equations of Osgood type

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Abstract

We give a simple proof of a lower bound for the Dirichlet heat kernel in terms of the Gaussian heat kernel. Using this we establish a non-existence result for semilinear heat equations with zero Dirichlet boundary conditions and initial data in $L^q(\Omega)$ when the source term $f$ is non-decreasing and $\limsup_{s \to \infty} s^{-\gamma}f(s) = \infty$ for some $\gamma > q(1 + 2/n)$. This allows us to construct a locally Lipschitz $f$ satisfying the Osgood condition $\int_1^\infty 1/f(s) \, ds = \infty$, which ensures global existence for bounded initial data, such that for every $q$ with $1 \leq q < \infty$ there is an initial condition $u_0 \in L^q(\Omega)$ for which the corresponding semilinear problem has no local-in-time solution.

\textbf{Keywords:} Semilinear heat equation, Dirichlet problem, non-existence, instantaneous blow-up, Osgood condition, Dirichlet heat kernel.

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1. Introduction

In a previous paper [5] we showed that for locally Lipschitz \( f \) with \( f > 0 \) on \((0, \infty)\), the Osgood condition

\[
\int_1^\infty \frac{1}{f(s)} \, ds = \infty,
\]

which ensures global existence of solutions of the scalar ODE \( \dot{x} = f(x) \), is not sufficient to guarantee the local existence of solutions of the Cauchy problem

\[
u_t = \Delta u + f(u)
\]

for initial data in \( L^q(\mathbb{R}^n) \), \( 1 \leq q < \infty \). This is in stark contrast to the case of bounded initial data, for which (1) implies that any solution of (2) exists globally in time; see [6], for example.

In [5] we considered the PDE (2) on the whole space \( \mathbb{R}^n \), which allowed us to use in our calculations the explicit form of the Gaussian heat kernel,

\[
G_n(x, y; t) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}.
\]

The main result there was that for each \( q \) with \( 1 \leq q < \infty \) one can find a non-negative, locally Lipschitz and Osgood \( f \) such that there are initial data in \( L^q(\mathbb{R}^n) \) for which there is no local-in-time integrable solution of (2).

In this paper we obtain a similar result for the equation posed with Dirichlet boundary conditions on a bounded domain, by using Gaussian lower bounds on the Dirichlet heat kernel. Indeed, in Section 2 (Theorem 2.1) we give a lower bound for the Dirichlet heat kernel on a bounded domain \( \Omega \):

\[
K_\Omega(x, y; t) \geq \beta^n G_n(x, y; t) \quad \text{for} \quad t \leq \epsilon^2/n,
\]

whenever \([x, y]\), the line segment joining \( x \) and \( y \), is contained in the interior of \( \Omega \) and is always at least a distance \( \epsilon \) from the boundary of \( \Omega \). Here, \( \beta > 0 \) is an explicit constant. Based on the argument of van den Berg [8] we also provide in the appendix a proof of a result valid for all \( t > 0 \)

\[
K_\Omega(x, y; t) \geq e^{-n^2t/4\epsilon^2} G_n(x, y; t),
\]

but (4) is sufficient for our purposes and has a significantly simpler proof.
More explicitly, we focus throughout the paper on the following problem (P), posed on a smooth bounded domain $\Omega \subset \mathbb{R}^n$:

$$
(P) \begin{cases}
    u_t = \Delta u + f(u) & \text{in } \Omega, \\
    u = 0 & \text{on } \partial \Omega, \\
    u(x, 0) = u_0(x) & \text{in } \Omega.
\end{cases}
$$

The source term $f : [0, \infty) \to [0, \infty)$ is non-decreasing and satisfies the asymptotic growth condition

$$
\limsup_{s \to \infty} s^{-\gamma} f(s) = \infty.
$$

We show in Theorem 4.1 that if (5) holds for some $\gamma > q(1 + 2/n)$ then one can find a non-negative $u_0 \in L^q(\Omega)$ such that there is no solution of (P) that is in $L^1_{\text{loc}}(\Omega)$ for $t > 0$.

We finish (see Corollary 5.1) by constructing a function $f$ that grows quickly enough such that (5) holds for every $\gamma \geq 0$, but nevertheless still verifies the Osgood condition (1). This example shows that there are functions $f$ for which (P) is well posed in $L^\infty(\Omega)$ but not in any $L^p(\Omega)$ with $1 \leq p < \infty$.

One can see this result as in some sense dual to that of Fila et al. [3] (see also Section 19.3 of [7]), who show that there exists an $f$ such that all positive solutions of $\dot{x} = f(x)$ blow up in finite time while all solutions of (P) with Dirichlet boundary conditions are global and bounded.

2. A Gaussian lower bound for the Dirichlet heat kernel

For any smooth domain $D$ in $\mathbb{R}^n$ (i.e. $D$ is smooth, open, and connected), we denote by $K_D(x, y, t)$ the Dirichlet heat kernel associated with the Dirichlet heat semigroup $S_D(t)$, i.e.

$$
w_D(x, t) = (S_D(t)w_0)(x) := \int_D K_D(x, y; t)w_0(y) \, dy
$$

is the classical solution of the linear heat equation

$$
\begin{align*}
w_t &= \Delta w & \text{in } D, \\
w &= 0 & \text{on } \partial D, \\
w &= w_0 & \text{in } D,
\end{align*}
$$
In the special case where $D = \mathbb{R}^n$, we will denote the Gaussian heat kernel on the whole space by $G_n(x, y; t)$, as given by (3).

In this section we provide a proof of a particular case of a result due to van den Berg [8], which shows that away from the boundary the Dirichlet heat kernel is bounded below by a multiple of the Gaussian kernel for the heat equation on the whole space. In this context the result for $\Omega \subset \mathbb{R}^n$ is an easy corollary of the result in $\mathbb{R}$; in the one-dimensional case our proof significantly simplifies that of [8].

**Theorem 2.1.** Let $\Omega$ be a domain in $\mathbb{R}^n$, and denote by $K_\Omega(x, y; t)$ the Dirichlet heat kernel on $\Omega$. Suppose that

$$
\epsilon := \inf_{z \in [x,y]} \text{dist}(z, \partial \Omega) > 0,
$$

where $[x,y]$ denotes the line segment joining $x$ and $y$ (so in particular $[x,y]$ is contained in the interior of $\Omega$). Then for $t \leq \epsilon^2/n$

$$
K_\Omega(x, y; t) \geq \beta^n G_n(x, y; t),
$$

where $\beta = 1 - 2/e > 0$.

Note that if $\Omega$ is convex then $\epsilon$ in (7) is simply given by

$$
\epsilon = \min(\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)).
$$

We delay the proof of Theorem 2.1 for a moment. Following [8] we begin with the corresponding result for an interval in $\mathbb{R}$. Our proof is somewhat simpler than that of Lemma 8 in [8], and non-probabilistic (cf. [10]), since we are able to write down directly the Dirichlet kernel in terms of a sum of Gaussian kernels on the whole line.

We write $K_a$ for the one-dimensional heat kernel on $(-a,a)$.

**Lemma 2.1.** Take $a > 0$. Then for any $x, y \in \Omega = (-a,a)$

$$
K_a(x, y; t) \geq G_1(x, y; t) \left[1 - 2e^{-\epsilon^2/t}\right],
$$

where $\epsilon = \text{dist}([x,y], \partial \Omega)$. In particular for $t \leq \epsilon^2$

$$
K_a(x, y; t) \geq \beta G_1(x, y; t),
$$

where $\beta = 1 - 2/e > 0$. 

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Figure 1: For a particular $u_0$ defined on $[0, 2a]$, an illustration of the periodic extension that is anti-symmetric about $x = 0$ and $x = 2a$. Dots indicate positions and signs of the delta functions that give rise to $K_{(0, 2a)}(x, y; t)$.

Note that Corollary 6.1 in the Appendix improves the lower bound in (8) to $e^{-\pi^2 t/4a^2} G_1(x, y; t)$ for all $t > 0$, but the result of this lemma is sufficient for the arguments in the main body of this paper.

**Proof.** For notational reasons it is simpler to treat the problem on $(0, 2a)$ rather than $(-a, a)$, but since the equations and the resulting lower bound are translation invariant this does not effect the result. We write down the Dirichlet heat kernel on $(0, 2a)$ by reflection. The essential idea is shown in Figure 1: the action of the heat equation on $[0, 2a]$ with initial data $u_0$ is the same as the action of the heat equation on $\mathbb{R}$ with the periodically extended initial data as illustrated, since this extension is antisymmetric about 0 and $a$ and the Gaussian kernel $G_1(x, y; t)$ is symmetric about $x$ for any $x \in \mathbb{R}$.

The contribution to the heat kernel for $x \in [0, 2a]$ from a source at $y \in (0, 2a)$ will be the sum of the Gaussian kernels with positive point sources at $y + 4ka$ and negative point sources at $-y + 4ka$ (see Figure 1, again), yielding

$$K_{(0, 2a)}(x, y; t) = \frac{1}{\sqrt{4\pi t}} \sum_{k \in \mathbb{Z}} e^{-|x-(y+4ka)|^2/4t} - e^{-|x-(-y+4ka)|^2/4t},$$

see Figure 2. [Even if one has doubts about the above derivation, it is clear that $K(x, y, t)$ in (9) satisfies the heat equation, $K(0, y; t) = K(2a, y; t) = 0$, and $K(x, y; 0) = \delta(y)$ for $x, y \in (0, 2a)$.]
Now we simply rewrite the sum:

$$\sqrt{4\pi t}K_{(0,2a)}(x, y; t) = \sum_{k \in \mathbb{Z}} e^{-|x-(y+4ka)|^2/4t} - e^{-|x-(-y+4ka)|^2/4t}$$

$$= e^{-|x-y|^2/4t} - e^{-|x+y|^2/4t} - e^{-|x+y-4a|^2/4t}$$

$$+ \sum_{k=1}^{\infty} \left\{ e^{-|x-(y+4ka)|^2/4t} + e^{-|x-(y-4ka)|^2/4t} - e^{-|(x-y+4a(k+1))|^2/4t} \right\}$$

$$= e^{-|x-y|^2/4t} \left[ 1 - e^{-xy/t} - e^{-(2a-x)(2a-y)/t} \right]$$

$$+ \sum_{k=1}^{\infty} e^{-|x-y-4ka|^2/4t} + e^{-|x+y+4ka|^2/4t} - e^{-|x+y-4(k+1)a|^2/4t} - e^{-|x+y+2ka|^2/4t}.$$

Noting that

$$|x+y+4ka| > |x-y+4ka| \quad \text{and} \quad |4(k+1)a - (x+y)| > |4ka - (x-y)|$$

for $k \geq 1$ and $x, y \in (0, 2a)$, it follows that

$$\sqrt{4\pi t}K_{(0,2a)}(x, y; t) \geq e^{-|x-y|^2/4t} \left[ 1 - 2e^{-\epsilon^2/t} \right].$$

Finally note that for $t \leq \epsilon^2$ the term in the square brackets is at least $\beta = 1 - 2/e$. \hfill \Box

The idea of the proof of Theorem 2.1, inspired by that of Lemma 9 in [8], is illustrated in Figure 3. We bound the Dirichlet heat kernel on $\Omega$ below
by the kernel on the parallelepiped $\Pi$, which is simply the product of one-dimensional kernels which we can control using Lemma 2.1. In this way the proof uses the monotonicity of the Dirichlet heat kernel with respect to the domain:

$$\Omega \subset U \quad \Rightarrow \quad K_\Omega(x, y; t) \leq K_U(x, y; t).$$

A probabilistic proof can be found in [8]; an analytic proof using the theory of semigroups can be found in the notes by Arendt [1].

![Diagram](image.png)

Figure 3: The parallelepiped $\Pi$ with $n - 1$ sides of length $2\epsilon/\sqrt{n}$ when $\text{dist}(x, \partial \Omega) = \epsilon$.

**Proof of Theorem 2.1.** By the definition of $\epsilon$, the line segment joining $x$ and $y$ is entirely contained in a parallelepiped $\Pi$ that lies entirely within $\Omega$, with one side of length $|x - y| + 2\epsilon/\sqrt{n}$ and $n - 1$ sides of length $2\epsilon/\sqrt{n}$, see Figure 3. Note that $x$ and $y$ are at least a distance $\epsilon/\sqrt{n}$ from all faces of $\Pi$. By monotonicity of the Dirichlet heat kernel with respect to the domain

$$K_\Omega(x, y, t) \geq K_\Pi(x, y, t).$$

If we now refer points in $\Pi$ to coordinate axes aligned along $[x, y]$ and in the perpendicular directions, so that $x = (\bar{x}, 0, \ldots, 0)$ and $y = (\bar{y}, 0, \ldots, 0)$, we
can use the separation of variables property to write

\[ K_\Pi(x, y; t) = K_{\frac{1}{2}|x-y|^2/4t}^\Pi(\tilde{x}, \tilde{y}; t) = K_{\frac{1}{2}|x-y|^2/4t}^\Pi(0, 0, t)]^{n-1} \]

\[ \geq \beta(4\pi t)^{-1/2} e^{-|x-y|^2/4t} \beta^{n-1}(4\pi t)^{-1} \]

\[ = (4\pi t)^{-n/2} e^{-|x-y|^2/4t} \beta^n \]

\[ = \beta^n G_n(x, y, t), \]

for all \( t \leq \epsilon^2/n \), using the one-dimensional lower bound from Lemma 2.1.

\[ \square \]

3. A lower bound for the heat equation

Without loss of generality we henceforth assume that \( \Omega \) contains the origin. For \( r > 0 \), \( B(x, r) \) will denote the Euclidean ball in \( \mathbb{R}^n \) of radius \( r \) centred at \( x \), and in an abuse of notation we write \( B(r) \) for \( B(0, r) \).

As an ingredient in the proof of our main result, we want to show that the action of the heat equation on the singular initial condition

\[ w_0(x) = |x|^{-\alpha} \chi_R := \left\{ \begin{array}{ll}
|x|^{-\alpha}, & |x| \leq R, \\
0, & |x| > R
\end{array} \right. \]  

(10)

does not have too pronounced an effect for short times. It is easy to see that

\[ w_0(x) > \phi \quad \text{for} \quad |x| < \phi^{-1/\alpha}; \]

we now show that such a lower bound holds for a similar set of \( x \) for sufficiently small times.

**Proposition 3.1.** Fix \( \alpha \in (0, n) \) and pick \( R > 0 \) such that \( B(R) \subset \Omega \). If \( w_\Omega(x, t) \) denotes the solution of the linear heat equation on \( \Omega \) with initial condition\(^1\) \( w_0 = |x|^{-\alpha} \chi_R \), as represented by (6), then there exist constants \( \sigma = \sigma(R, \alpha, n) > 0 \) and \( \phi_* = \phi_*(R, \alpha, n) > 0 \) such that

\[ w_\Omega(x, t) \geq \phi \quad \text{for all} \quad |x| \leq \sigma \phi^{-1/\alpha} \quad \text{and} \quad 0 \leq t \leq \sigma \phi^{-2/\alpha} \]

(11)

for any \( \phi > \phi_* \).

\(^1\)Strictly speaking \( w_0 \) is defined on the whole of \( \mathbb{R}^n \); we take as initial condition the function in (10) restricted to \( \Omega \).
Proof. Let $w$ denote the solution of the linear heat equation on $\mathbb{R}^n$ with the same initial condition $w_0 = |x|^{-\alpha}\chi_R$. Let $\epsilon = \inf_{x \in B(R)} \text{dist}(x, \partial \Omega) > 0$; then it follows from Theorem 2.1 that with $T = \epsilon^2/n$ we have

$$K_\Omega(x, y, t) \geq \beta^n G_n(x, y, t), \quad \forall x, y \in B(R), \quad t \in (0, T].$$

From here on $c$ will denote any generic constant, and may change from line to line.

Taking $|\hat{x}| = R$, $t \in (0, T]$ and $\psi > 1$, we have

$$w_\Omega(\hat{x}/\psi, t) = \int _\Omega K_\Omega(\hat{x}/\psi, y, t)w_0(y) \, dy = \int _{B(R)} K_\Omega(\hat{x}/\psi, y, t)|y|^{-\alpha} \, dy$$

$$\geq c \int _{B(R)} G_n(\hat{x}/\psi, y, t)|y|^{-\alpha} \, dy$$

$$\geq c(4\pi t)^{-n/2} \int _{B(R)} e^{-(\hat{x}/\psi - \psi^2 t)|y|^{-\alpha}} \, dy$$

$$= c(4\pi t)^{-n/2} \int _{B(R)} e^{-|\hat{x} - \psi y|^2/4\psi^2 t}|y|^{-\alpha} \, dy$$

$$= c\psi^n(4\pi \psi^2 t)^{-n/2} \int _{B(\psi R)} e^{-|\hat{x} - z|^2/4\psi^2 t}|z|^{-\alpha} \, dz \geq c\psi^n w(\hat{x}, \psi^2 t).$$

Defining $M = M(R, \alpha, n) > 0$ by

$$M = \inf \{w(x, t) : |x| = R, \quad 0 \leq t \leq T\} \quad (12)$$

it follows that $w_\Omega(x, t) \geq cM\phi$ for all $|x| = R\psi^{-1}$ and $t \in (0, \psi^{-2}T]$. Furthermore, $w_\Omega(x, 0) = |x|^{-\alpha} \geq R^{-\alpha}\phi$ for all $|x| \leq R\psi^{-1}$. Consequently, by the parabolic maximum principle,

$$w_\Omega(x, t) \geq \phi \psi \quad \text{for all} \quad |x| \leq R\psi^{-1} \quad \text{and} \quad 0 \leq t \leq \psi^{-2}T,$$

where $\phi_* := \min(cM, R^{-\alpha}) > 0$. With $\sigma = \min(R\phi_*^{1/\alpha}, T\phi_*^{2/\alpha}) > 0$ and $\phi = \phi_\psi > \phi_*$, one therefore obtains

$$w_\Omega(x, t) \geq \phi \quad \text{for all} \quad |x| \leq \sigma\phi^{-1/\alpha} \quad \text{and} \quad 0 \leq t \leq \sigma\phi^{-2/\alpha}. \quad \square$$

4. Non-existence of local solutions

In this section we prove the non-existence of local solutions, taking the following as our (essentially minimal) definition of such a solution. Note that the non-existence of a solution in the sense of Definition 4.1 implies the non-existence of mild solutions and of classical solutions [7, p. 77–78].
Definition 4.1. [7, p. 78] Given $f \geq 0$ and $u_0 \geq 0$ we say that $u$ is a local integral solution of $(P)$ on $[0,T)$ if $u : \Omega \times [0,T) \rightarrow [0,\infty]$ is measurable, finite almost everywhere, and satisfies

$$u(t) = S_{\Omega}(t)u_0 + \int_0^t S_{\Omega}(t-s)f(u(s))\,ds$$  \hfill (13)

almost everywhere in $\Omega \times [0,T)$.

We now prove our main result, in which we obtain instantaneous blow-up in $L^1_{\text{loc}}(\Omega)$ for certain initial data in $L^q(\Omega)$, $1 \leq q < \infty$, under the asymptotic growth condition (14) when $f$ is non-decreasing.

Theorem 4.1. Let $q \in [1,\infty)$. Suppose that $f : [0,\infty) \rightarrow [0,\infty)$ is non-decreasing. If

$$\limsup_{s \to \infty} s^{-\gamma}f(s) = \infty$$  \hfill (14)

for some $\gamma > q(1 + \frac{2}{n})$, then there exists $u_0 \in L^q(\Omega)$ such that $(P)$ possesses no local integral solution. Indeed, any solution $u(t)$ that satisfies (13) is not in $L^1_{\text{loc}}(\Omega)$ for any $t > 0$.

Proof. We show that for small $t > 0$, $u(t) \notin L^1_{\text{loc}}(\Omega)$ and hence, arguing as in [5, Theorem 4.1], there can be no local integral solution of $(P)$.

Choose $B(R)$ as in Proposition 3.1. Set $\alpha = (n+2)/\gamma < n/q$, so that

$$\limsup_{s \to \infty} s^{-(n+2)/\alpha} f(s) = \infty.$$

Then in particular there exists a sequence $\phi_i \to \infty$ such that

$$f(\phi_i)\phi_i^{-(n+2)/\alpha} \to \infty \quad \text{as} \quad i \to \infty.$$

Now take $u_0 = |x|^{-\alpha} \chi_R \in L^q(\Omega)$. Defining $T$ as in the proof of Proposition 3.1, fix $t < \min(T,1)$ and choose $i$ sufficiently large such that $\phi_i > \phi_*$, $\sigma \phi_i^{1/\alpha} < R/2$ and $\sigma \phi_i^{-2/\alpha} \leq t$. Clearly, by comparison, $u \geq w_\Omega \geq 0$. Hence
by monotonicity of \( f \) and Theorem 2.1,

\[
I := \int_{B(R)} u(t) \, dx \geq \int_{B(R)} \int_0^t [S_\Omega(t-s)f(w_\Omega(\cdot, s))] (x) \, ds \, dx \\
= \int_0^t \int_{B(R)} \int_\Omega K_\Omega(t-s, x, y) f(w_\Omega(y, s)) \, dy \, dx \, ds \\
\geq c \int_0^{\sigma \phi_i^{-2/\alpha}} \int_{B(R)} \int_{B(\sigma \phi_i^{-1/\alpha})} G_n(t-s, x, y) f(\phi_i) \, dy \, dx \, ds \\
= cf(\phi_i) \int_0^{\sigma \phi_i^{-2/\alpha}} \int_{B(\sigma \phi_i^{-1/\alpha})} (4\pi(t-s))^{-n/2} \int_{B(R)} e^{-|x-y|^2/4(t-s)}\, dx \, dy \, ds.
\]

Let \( z = x - y \). Since \(|y| \leq \sigma \phi_i^{-1/\alpha} < R/2\), it follows that

\[
\{z = x - y : x \in B(R)\} \supset B(R/2).
\]

Thus

\[
I \geq cf(\phi_i) \int_0^{\sigma \phi_i^{-2/\alpha}} \int_{B(\sigma \phi_i^{-1/\alpha})} (4\pi(t-s))^{-n/2} \int_{B(R/2)} e^{-|z|^2/4(t-s)} \, dz \, dy \, ds
\]

\[
= cf(\phi_i) \int_0^{\sigma \phi_i^{-2/\alpha}} \int_{B(\sigma \phi_i^{-1/\alpha})} \int_{B(R/\sqrt{t-s})} e^{-|v|^2} \, dv \, dy \, ds \quad (v = z/2\sqrt{t-s})
\]

\[
\geq cf(\phi_i) \int_0^{\sigma \phi_i^{-2/\alpha}} \int_{B(\sigma \phi_i^{-1/\alpha})} \int_{B(R)} e^{-|v|^2} \, dv \, dy \, ds \quad (\sqrt{t-s} \leq 1)
\]

\[
= cf(\phi_i) \phi_i^{-(n+2)/\alpha} \, ds
\]

\[
= cf(\phi_i) \phi_i^{-(n+2)/\alpha} \to \infty \quad \text{as} \quad i \to \infty. \quad \Box
\]

Note that this result also guarantees instantaneous blow-up of solutions of

\[
u_t = \Delta u + g(u)
\]

for any \( g \) such that \( g(s) \geq f(s) \), where \( f \) satisfies the conditions of Theorem 4.1, even if \( g \) is not monotonic. In particular, for the canonical Fujita equation

\[
u_t = \Delta u + u^p,
\]

(15)
our argument shows the non-existence of local solutions when \( p > q(1 + \frac{2}{n}) \).
The sharp result in this case is known to be \( p > 1 + \frac{2}{n}q \) \([12, 13]\) with equality allowed if \( q = 1 \) \([2]\).

The existence of a finite limit in (14) implies that \( f(s) \leq c(1 + s^\gamma) \), and hence by comparison with (15) is sufficient for the local existence of solutions provided that \( \gamma < 1 + \frac{2}{n}q \) \([11]\). We currently, therefore, have an indeterminate range of \( \gamma \),

\[
1 + \frac{2}{n}q \leq \gamma \leq q(1 + \frac{2}{n})
\]

for which we do not know whether (14) characterises the existence or non-existence of local solutions.

5. A very ‘bad’ Osgood \( f \)

To finish, using a variant of the construction in \([5]\), we provide an example of an \( f \) that satisfies the Osgood condition (1) but for which

\[
\limsup_{s \to \infty} s^{-\gamma} f(s) = \infty, \quad \text{for every } \gamma \geq 0. \tag{16}
\]

**Theorem 5.1.** There exists a locally Lipschitz function \( f : [0, \infty) \to [0, \infty) \) such that \( f(0) = 0 \), \( f \) is non-decreasing, and \( f \) satisfies the Osgood condition

\[
\int_1^\infty \frac{1}{f(s)} \, ds = \infty,
\]

but nevertheless (16) holds. Consequently, for this \( f \), for any \( 1 \leq q < \infty \) there exists a \( u_0 \in L^q(\Omega) \) such that (P) has no local integral solution.

**Proof.** Fix \( \phi_0 = 1 \) and define inductively the sequence \( \phi_i \) via

\[
\phi_{i+1} = e^{\phi_i}.
\]

Clearly, \( \phi_i \to \infty \) as \( i \to \infty \). Now define \( f : [0, \infty) \to [0, \infty) \) by

\[
f(s) = \begin{cases} (e - 1)s, & s \in J_0 := [0, 1], \\
\phi_i - \phi_{i-1}, & s \in I_i := [\phi_{i-1}, \phi_i/2], \ i \geq 1, \\
\ell_i(s), & s \in J_i := (\phi_i/2, \phi_i), \ i \geq 1,
\end{cases} \tag{17}
\]

where \( \ell_i \) interpolates linearly between the values of \( f \) at \( \phi_i/2 \) and \( \phi_i \). By construction \( f(0) = 0 \), \( f \) is non-decreasing, and \( f \) is Osgood since

\[
\int_1^\infty \frac{1}{f(s)} \, ds \geq \sum_{i=1}^{\infty} \int_{I_i} \frac{1}{f(s)} \, ds = \sum_{i=1}^{\infty} \frac{\phi_i/2 - \phi_{i-1}}{\phi_i - \phi_{i-1}} = +\infty.
\]
However, $f(\phi_i) = e^{\phi_i} - \phi_i$, and so for any $\gamma \geq 0$
\[
\lim_{i \to \infty} \phi_i^{-\gamma} f(\phi_i) \to \infty \quad \text{as} \quad i \to \infty,
\]
which shows that (16) holds. \qed

This example shows that there exist semilinear heat equations that are globally well-posed in $L^\infty(\Omega)$, yet ill-posed in every $L^q(\Omega)$ for $1 \leq q < \infty$.

6. Appendix: Gaussian lower bound on the heat kernel for all $t > 0$

For the sake of completeness we now follow [8] (see also [9]) and use the result of Lemma 2.1 to obtain a lower bound on $2K_a(x, y; t)$ in terms of $K_\epsilon(0, 0; t)$. We then bound $K_\epsilon(0, 0; t)$ below by supplementing the bound from Lemma 2.1 with information from the eigenfunction expansion of the kernel. This will allow us a simple proof of a similar form of lower bound on a general domain $\Omega$ when $[x, y] \subset \Omega$.

The main idea in the proof is to use repeatedly the semigroup property of the heat semigroup in the form
\[
K_a(x, y; t) = \int_{(-a,a)} K_a(x, z; t)K_a(z, y; t) \, dz.
\]

**Proposition 6.1.** The one-dimensional heat kernel on $\Omega = (-a, a)$ satisfies
\[
K_a(x, y; t) \geq e^{-|x-y|^2/4t} K_\epsilon(0, 0; t)
\]
for all $x, y \in (-a, a)$ and $t > 0$, where $\epsilon = \text{dist}([x, y], \partial \Omega)$.

**Proof.** Take $x, y \in (-a, a)$, $t > 0$, and $m \in \mathbb{N}$ with $m$ sufficiently large that $1 - 2e^{-mc^2/t} > 0$. For $j = 0, 1, \ldots, m$ set $x_j = x + jz$, where $z = (y - x)/m$. Then using the semigroup property
\[
K_a(x, y; t)
\]
\[
= \int_{\Omega^{m-1}} K_a(x, y_1; t/m) \left\{ \prod_{j=1}^{m-2} K_a(y_j, y_{j+1}; t/m) \right\} K_a(y_{m-1}, y; t/m) \, d^{m-1}y,
\]

\[\tag{18}\]

\footnote{Recall that we use the notation $K_a(x, y; t)$ for the one-dimensional heat kernel on $(-a, a)$.}
writing $d^{m-1}y$ for $dy_1 \cdots dy_{m-1}$.

Now note that $B(x_j, \epsilon) \subset \Omega$ for every $j = 0, \ldots, m$, and so

$$K_a(x, y; t) \geq \int_{B(\epsilon)^{m-1}} \prod_{j=0}^{m-1} K_a(x_j + w_j, x_{j+1} + w_{j+1}; t/m) d^{m-1}w,$$

setting $w_0 = w_m = 0$ and $w_j = y_j - x_j$ for $j = 1, \ldots, m - 1$. Using Lemma 2.1 we obtain the lower bound

$$K_a \geq C_{m,t} \int_{B(\epsilon)^{m-1}} \prod_{j=0}^{m-1} G_1(x_j + w_j, x_{j+1} + w_{j+1}; t/m) d^{m-1}w$$

$$= C_{m,t} \int_{B(\epsilon)^{m-1}} (4\pi t/m)^{-m/2} \exp \left( -\frac{m \sum_{j=0}^{m-1} |z + w_{j+1} - w_j|^2}{4t} \right) d^{m-1}w,$$

where $C_{m,t} = [1 - 2e^{-m\epsilon^2/t}]^m$.

Elementary algebra gives

$$m \sum_{j=0}^{m-1} |z + w_{j+1} - w_j|^2 = |x - y|^2 + m \sum_{j=0}^{m-1} |w_{j+1} - w_j|^2,$$

and therefore

$$K_a \geq C_{m,t} e^{-|x - y|^2/4t} \int_{B(\epsilon)^{m-1}} (4\pi t/m)^{-m/2} \exp \left( -\frac{m \sum_{j=0}^{m-1} |w_{j+1} - w_j|^2}{4t} \right) d^{m-1}w$$

$$= C_{m,t} e^{-|x - y|^2/4t} \int_{B(\epsilon)^{m-1}} \prod_{j=0}^{m-1} G_1(w_j, w_{j+1}; t/m) d^{m-1}w.$$

Now we can use monotonicity of the heat kernel, $G_1 \geq K_\epsilon$, to obtain

$$K_a(x, y; t) \geq C_{m,t} e^{-|x - y|^2/4t} \int_{B(\epsilon)^{m-1}} \prod_{j=0}^{m-1} K_\epsilon(w_j, w_{j+1}; t/m) d^{m-1}w$$

$$= C_{m,t} e^{-|x - y|^2/4t} K_\epsilon(0, 0, t),$$

using the semigroup property of $K_\epsilon$ and recalling that $w_0 = w_m = 0$. Finally, noting that $C_{m,t} \to 1$ as $m \to \infty$, we obtain (18). \qed
We now obtain a lower bound on $K_a(0, 0; t)$ using the eigenfunction expansion of the kernel.

**Lemma 6.1.** For all $t > 0$

$$K_a(0, 0; t) \geq \frac{1}{\sqrt{4\pi t}} e^{-\pi^2 t/4a^2}. \quad (19)$$

**Proof.** The eigenfunctions of $u_{xx} = \lambda u$ with $u(0) = u(2a) = 0$ are $\sin k\pi x/2a$ with corresponding eigenvalues $-k^2 \pi^2 / 4a^2$: the kernel is therefore

$$K_{(0, 2a)}(x, y; t) = \frac{1}{a} \sum_{k=1}^{\infty} e^{-k^2 \pi^2 t/4a^2} \sin(k\pi x/2a) \sin(k\pi y/2a).$$

Since $K_a(0, 0; t) = K_{(0, 2a)}(a, a; t)$ we obtain

$$K_a(0, 0; t) = \frac{1}{a} \sum_{k=1}^{\infty} e^{-k^2 \pi^2 t/4a^2} \sin^2(k\pi/2)$$

$$= \frac{1}{a} \sum_{k=0}^{\infty} e^{-(2k+1)^2 \pi^2 t/4a^2} \geq \frac{1}{a} e^{-\pi^2 t/4a^2},$$

from which (19) follows for $a \leq (4\pi t)^{1/2}$. For $t \leq a^2 / 4\pi$, we use Lemma 2.1 to give

$$K_a(0, 0; t) \geq 1 - 2e^{-t^2/4},$$

now simply observe that $e^{-s} < s/4$ and $e^{-s} < 1 - (s/2)$ for $0 < s < 1/3$, and so certainly $1 - 2e^{-s} \geq e^{-s}$ for $0 < s \leq 1/(4\pi) < 1/3$, and thus the bound in (19) holds for all $t > 0$ as claimed.

Combining the results of Proposition 6.1 and Lemma 6.1 finally yields the required lower bound in one dimension.

**Corollary 6.1.** If $\Omega = (-a, a)$, $[x, y] \subset \Omega$, and $\epsilon = \text{dist}([x, y], \partial \Omega)$ then

$$K_a(x, y; t) \geq e^{-\pi^2 t/4\epsilon^2} G_1(x, y; t) \quad \text{for all } t > 0.$$

For $\Omega \subset \mathbb{R}^n$ the result follows using the argument in the proof of Theorem 2.1, in particular the inequality

$$K_\Omega(x, y, t) \geq K_{\Pi}(x, y, t) \geq K_{\frac{1}{2}|x-y| + \epsilon/\sqrt{n}}(\tilde{x}, \tilde{y}, t)[K_{\Omega}(0, 0, t)]^{n-1}.$$
Corollary 6.2. If $\Omega \subset \mathbb{R}^n$, $[x, y] \subset \Omega$, and $\epsilon = \text{dist}([x, y], \partial \Omega)$ then

$$K_a(x, y; t) \geq e^{-n^2\pi^2 t/4\epsilon^2}G_n(x, y; t) \quad \text{for all} \quad t > 0.$$ 

We note that the argument in [8] does not require the line segment $[x, y]$ to be contained in $\Omega$, leading to a lower bound that depends on the curvature of the geodesic joining $x$ and $y$.

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