Asymptotic series for distributions

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Abstract

Asymptotic expansions for a wide class of distribution are studied. Simple method for the computation of the series coefficients is suggested. The case when regularization parameter of distribution depends on the asymptotic parameter is considered.

1 Introduction

A method of asymptotic series expansion of $e^{\pm i\tau x} f(x)$ for $\tau \to \infty$ was suggested in [1, 2]. An exhaustive investigation of the power type distributions, i.e. distributions of a form $f(x) \sim (x \pm i0)^{\lambda} \ln^m (x \pm i0)$ has been undertaken and the results were presented in [2, 3]. In this paper we extend this method to the family of distributions specified below. Furthermore, we consider the case when the regularization parameter in $f(x)$ depends on $\tau$ and present resulting changes in the series expansion.

Recall that tempered distribution is defined as functional $\langle F(t), \phi(t) \rangle$ on fast decreasing test functions $\phi(t) \in S [4, 5]$. Distribution $F(t)$ is tempered ($F(t) \in S'$), if and only if it is a finite order derivative of some continuous tempered function $G(t)$, i.e. there exists some finite $n$ and $\sigma$ such that

$$|G(t)| < |t|^\sigma; \quad |t| \to \infty,$$

and

$$F(t) = G^{(n)}(t).$$

(1)

Fourier transform of any tempered distribution is a tempered distribution too.

Further it will be necessary to interchange the order of the integration in the Fourier transform and taking the limit. To ensure uniform convergence of the Fourier integral

$$f(x) \triangleq F[F,x] = \int_{-\infty}^{\infty} F(t) \exp \{ixt\} dt; \quad F(t) \in S',$$

(3)

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necessary for such interchange, we apply the Abel-Poisson regularization, i.e.
will be interpreted as
\[ f(x) = \int_0^\infty F(t) \exp\{i(x + i\varepsilon)t\} dt + \int_{-\infty}^0 F(t) \exp\{i(x - i\varepsilon)t\} dt, \]
where \( \varepsilon \) is infinitesimal positive parameter which finally tends to zero unless specified otherwise.

It is known (see e.g. [6]) that any tempered distribution \( F(t) \) allows an analytical representation
\[ f(x) = f_+(x + i\varepsilon) - f_-(x - i\varepsilon), \]
where \( f_\pm(x) \) are analytical functions in the upper/lower complex half-plane \( x \).
In particular, \( f_\pm(x) \) may be chosen as
\[ f_+(x) = \int_0^\infty F(t) \exp\{itx\} dt \]
and
\[ f_-(x) = -\int_{-\infty}^0 F(t) \exp\{itx\} dt. \]

The inverse to (6) and (7) transforms may be written as
\[ \frac{1}{2\pi} \int_{-\infty}^\infty f_+(x) \exp\{-itx\} dx = \begin{cases} F(t) & \text{if } t > 0, \\ 0 & \text{if } t < 0 \end{cases} \]
and
\[ \frac{1}{2\pi} \int_{-\infty}^\infty f_-(x) \exp\{-itx\} dx = \begin{cases} 0 & \text{if } t > 0, \\ -F(t) & \text{if } t < 0. \end{cases} \]

Similarly one may write
\[ F(t) = F_+(t + i\varepsilon) - F_-(t - i\varepsilon) \]
where
\[ F_+(t) = \frac{1}{2\pi} \int_{-\infty}^0 f(x) \exp\{-ixt\} dx; \]
and
\[ F_-(t) = -\frac{1}{2\pi} \int_0^\infty f(x) \exp\{-ixt\} dx. \]

Representations (11) and (12) are not unique. For instance, one may subtract the same entire function \( \Xi(x) \) from \( f_+(x) \) and \( f_-(x) \) [6, 5]. In particular, the analytical properties of \( f_\pm(x) \) are not changed, if one chooses \( \Xi(x) = \int_0^\xi F(t) \exp\{ixt\} dt \). It means that decomposition of the domain of integration is done at arbitrary finite \( t = \xi \). The similar statement is evidently true for \( F_\pm(t) \).

It is evident that if \( \varepsilon \) is finite, \( F(t) \) in (11) and \( f(x) \) in (5) are regular functions of their arguments for all real \( t \) and \( x \). They must be treated as distributions only when \( \varepsilon \to 0 \).
2 Asymptotic Taylor series for distributions

From (6) it follows that

$$e^{-i\tau x}f(x) = \int_{-\infty}^{\infty} F(\tau+t)e^{itx}dt$$

(13)

so taking into account (10) one may write

$$e^{-i\tau x}f(x) = \int_{-\infty}^{\infty} \left[ F_+ (\tau + t + \epsilon) - F_- (\tau + t - \epsilon) \right] e^{itx}dt. \quad (14)$$

Since $F_{\pm}(t)$ are analytical functions in upper/lower complex half-plane $t$, then for any finite $\epsilon$ and real $t$ and $\tau$ they may be expanded in a Taylor series

$$F_{\pm}(\tau + t \pm \epsilon) = \sum_{n=0}^{\infty} \frac{F_{\pm}^{(n)}(\tau \pm \epsilon)}{n!} t^n \quad (15)$$

that we write simply as

$$F(t+\tau) = \sum_{n=0}^{\infty} \frac{t^n}{n!} F^{(n)}(\tau) \quad (16)$$

where

$$F^{(n)}(\tau) \equiv F^{(n)}_+ (\tau + i\epsilon) - F^{(n)}_- (\tau - i\epsilon). \quad (17)$$

If distribution $f(x)$ belongs to linear space $E'$, i.e. it is defined as functional $\langle T(x), \phi(x) \rangle$ on infinitely differentiable test functions $\phi(x) \in E$ with arbitrary support\(^1\), then its Fourier transform $F[f,t] = F(t)$ is an entire function \([4,5]\), so Taylor series for $F(t + \tau)$ converges in the entire complex plane $t$. It is evident that even for exponentially increasing $F_{\pm}^{(n)}(\tau)$ with $n$, the radii of convergence

$$R_{\pm} = \left( \lim_{n \to \infty} \left| \frac{F_{\pm}^{(n)}(\tau \pm i\epsilon)}{n!} \right|^{-\frac{1}{n}} \right) \quad (18)$$

are infinite. Tempered distributions $F_{\pm}^{(n)}(\tau)$ may show quicker increase with $n$ and then radii may appear to be finite. It may happen, when decreasing $F_{\pm}^{(n)}(\tau)$ with $\tau$ is fast enough. Indeed, simple qualitative analysis shows that, e.g.

$$F_{-}^{(n)}(\tau) = -\frac{1}{2\pi} \int_{0}^{\infty} f(x) x^n \exp \{-ix\tau + n \ln x - i\pi n/2\} dx, \quad (19)$$

have the quickest increase with $n$ when the main contribution in the integral comes from large $x$. In this case, however, term $-ix\tau$ introduces oscillation, thus providing a decrease with $\tau$. On the other hand, an increasing $F_{+}^{(n)}(\tau)$

\(^1\)In \([7]\) it is only such distributions that are called **generalized functions**.
with \( \tau \) is the largest when the principal contribution comes from singularity (if any) of \( f(x) \) which is located in the complex upper half-plane, that, in turn, brings an oscillation with increasing \( n \) and reduces contribution from this area. It brings us to a suggestion that

\[
F_{(n+1)}^{(\pm)} (\tau \pm i\varepsilon) = o \left( F_{\pm}^{(n)} (\tau \pm i\varepsilon) \right) \tag{20}
\]

should be considered. Condition (20) means that \( F_{\pm}^{(n)} (\tau \pm i\varepsilon) \) constitutes an asymptotic sequence and Taylor series (15) must be treated as asymptotic, i.e. \( \psi \) must be interpreted as

\[
F_{\pm} (\tau + t \pm i\varepsilon) = \sum_{n=0}^{N} \frac{F_{\pm}^{(n)} (\tau \pm i\varepsilon)}{n!} t^n + o \left( F_{\pm}^{(N)} (\tau \pm i\varepsilon) \right) . \tag{21}
\]

We can not impose a condition similar to (20) on \( F^{(n)} (\tau) \) directly, because \( F^{(n+1)} (\tau) = o \left( F^{(n)} (\tau) \right) \) simply means \( F^{(n+1)} (\tau) / F^{(n)} (\tau) \to 0 \) for \( \tau \to \infty \) (or \( \tau \to -\infty \)). However, neither product nor ratio are defined for arbitrary distribution. Fortunately even such an approach is not hopeless, because many distributions may be treated as regular in some area (see e.g. [6]). In particular, if \( F^{(n)} (\tau) \) is regular, there exists function \( \psi^{(n)} (t) = \frac{\partial^n}{\partial t^n} \psi (t) \) with \( n = 0, ..N \), such that \( \psi^{(N)} (t) \) is continuous and

\[
\left< F^{(n)} (t), \phi_\Delta (t) \right> = \left< \psi^{(n)} (t), \phi_\Delta (t) \right> \tag{22}
\]

for all test functions \( \phi_\Delta, (t) \in S \) with the support located in the interval \( \Delta_\tau = (\tau - \Delta, \tau + \Delta) \), where positive parameter \( \Delta \) may be chosen as arbitrary small, but must remain finite. Functions \( \psi^{(n)} (t) \) are unique and may be treated as asymptotics of \( F^{(n)} (t) \).

Since tempered distribution \( F (\tau) \) is defined by conditions 11 and 2, it is natural to assume that for\(^2 \) \( \tau \to \infty \)

\[
\psi (\tau) \sim C \tau^{-\gamma} \exp \left\{ -\lambda \tau^\alpha \right\} \tag{23}
\]

where constants \( C, \gamma \) and \( \alpha \) are arbitrary and \( \lambda \) is nonnegative. Asymptotic behavior of \( \exp \left\{ -\lambda \tau^\alpha \right\} \) for \( \alpha < 0 \) is trivial. For \( \alpha > 0 \) for \( \tau \to \infty \) we get

\[
\psi^{(n)} (\tau) \sim \left( -\lambda \alpha \tau^{\alpha - 1} \right)^n \psi (\tau) \tag{24}
\]

so increasing \( F^{(n)} (\tau) \) with \( n \) remains exponential and \( R \) is infinite. However, for \( \alpha = 0 \) we obtain

\[
\psi^{(n)} (\tau) \sim \frac{\partial^n}{\partial \tau^n} \tau^{-\gamma} = (-1)^n \frac{\Gamma (n + \gamma)}{\Gamma (\gamma)} \tau^{-\gamma-n} \tag{25}
\]

\(^2\)Proceeding to limit \( \tau \to -\infty \) is very similar, but constants may differ.
and

\[ R = \lim_{n \to \infty} \left| \frac{\psi^{(n)}(\tau)}{n!} \right|^{\frac{1}{n}} = \tau. \]  

(26)

In this case Taylor series do not converge for \(|t| > |\tau|\), but it is not too restrictive because we intend to increase \(|\tau|\) infinitely.

Now, taking into account

\[ \frac{1}{2\pi} \int_{-\tau}^{\infty} t^n e^{itx} dt = \begin{cases} \int_{-\infty}^{\infty} e^{itx} dt & \text{if } \tau \to \infty, \\ 0 & \text{if } \tau \to -\infty. \end{cases} \]  

(27)

(see e.g. [2]), in (24) we may confine ourselves to area \(|t/\tau| < 1\), so

\[ e^{-itx} f_+ (x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(\tau)}{n!} \frac{1}{2\pi} \int_{-\tau}^{\infty} t^n e^{itx} dt \]  

(28)

and we may write

\[ e^{-itx} f_+ (x) = \begin{cases} \sum_{n=0}^{\infty} \frac{F^{(n)}(\tau)}{n!} i^{-n} \delta^{(n)} (x) & \text{if } \tau \to \infty, \\ 0 & \text{if } \tau \to -\infty. \end{cases} \]  

(29)

In the same way, from

\[ e^{-itx} f_- (x) = -\int_{-\infty}^{-\tau} F(\tau + t) e^{itx} dt \]  

(30)

we find

\[ e^{-itx} f_- (x) = \begin{cases} \sum_{n=0}^{\infty} \frac{F^{(n)}(\tau)}{n!} 0 & \text{if } \tau \to \infty, \\ -2\pi \sum_{n=0}^{\infty} \frac{F^{(n)}(\tau)}{n!} i^{-n} \delta^{(n)} (x) & \text{if } \tau \to -\infty. \end{cases} \]  

(31)

From (31), (29) and (22) we also get for \(\tau \to \pm \infty\)

\[ e^{-itx} f(x) = 2\pi \sum_{n=0}^{\infty} \frac{F^{(n)}(\tau)}{n!} i^{-n} \delta^{(n)} (x). \]  

(32)

To finalize this section we wish to suggest a simple interpretation of the condition (20), namely what it means for \(f(x)\). Since (20) is satisfied for \(t' > \tau\), we can write an asymptotic series

\[ F_{\pm} (t + t' \pm i\varepsilon) \exp \{it'x\} = \sum_{n=0}^{\infty} \frac{1}{n!} F_{\pm}^{(n)} (t' \pm i\varepsilon) \exp \{it'x\} t^n. \]  

(33)

The asymptotic series allows termwise integration [8], so

\[ \int_{-\tau}^{\infty} F_{\pm} (t + t' \pm i\varepsilon) \exp \{it'x\} dt' = \sum_{n=0}^{\infty} \frac{1}{n!} t^n s_{\pm}^{[n]} (x, \tau) \]  

(34)
with
\[ s^{[n]}_{\pm}(x, \tau) = \int_{\tau}^{\infty} F^{(n)}_{\pm}(t' \pm i\varepsilon) \exp \{it'x\} dt' \quad (35) \]
is an asymptotic series too and the condition (20) transforms into
\[ s^{[n]}_{\pm}(x, \tau) = o\left(s^{[n]}_{\pm}(x, \tau)\right). \quad (36) \]

One may rewrite (6) as
\[ f_+(x) = \int_{\tau}^{\infty} [F_+(t + i\varepsilon) - F_-(t - i\varepsilon)] \exp \{itx\} dt + f^{reg}_+(x) \quad (37) \]
where
\[ f^{reg}_+(x) = \int_{0}^{\tau} [F_+(t + i\varepsilon) - F_-(t - i\varepsilon)] \exp \{itx\} dt. \quad (38) \]

Since integration in (38) is done over the finite interval, \( f^{reg}_+(x) \) is regular part of \( f_+(x) \), so that \( s^{[0]}_{\pm}(x, \tau) \) reproduces singularity of \( f_+(x) \) at \( x = 0 \), if any. At \( x \neq 0 \) oscillations remove singularity.

It is clear that
\[ s^{[n]}_{\pm}(x, \tau) = -F^{(n-1)}_{\pm}(\tau \pm i\varepsilon) \exp \{i\tau x\} - ixs^{[n-1]}_{\pm}(x, \tau). \quad (39) \]

If we assume \( s^{[n]}_{\pm}(x, \tau) \sim o\left(F^{(n-1)}_{\pm}(\tau \pm i\varepsilon)\right) \) immediately leads to
\[ s^{[n]}_{\pm}(x, \tau) \sim \frac{1}{\tau} F^{(n)}_{\pm}(\tau \pm i\varepsilon) \] and consequently to \( F^{(n)}_{\pm}(\tau \pm i\varepsilon)/x \sim o\left(F^{(n-1)}_{\pm}(\tau \pm i\varepsilon)\right) \); then, since \( \tau \) is fixed, this condition will be broken for small enough \( x \). However, if \( F^{(n-1)}_{\pm}(\tau \pm i\varepsilon) \sim s^{[n]}_{\pm}(x, \tau) \) or \( F^{(n-1)}_{\pm}(\tau \pm i\varepsilon) \sim o\left(s^{[n]}_{\pm}(x, \tau)\right) \) then
\[ s^{[n]}_{\pm}(x, \tau) \sim ixs^{[n-1]}_{\pm}(x, \tau) \] or
\[ s^{[n]}_{\pm}(x, \tau) \simeq (-ix)^n s^{[0]}_{\pm}(x, \tau). \quad (40) \]

Hence, condition (20) looks quite natural, since it means a reduction of singularity degree when \( x^{n-1}f_+(x) \) it is multiplied by \( x \).

### 3 Series for distributions with \( \tau \)-dependent regularization parameter

Until now we dealt with the distributions defined by (5). It implies that \( |\tau| \) may be taken arbitrary large, but transition to the limit \( \tau \to \pm \infty \) must be carried out only after taking the limit \( \varepsilon \to 0 \). Here we consider another way of taking this limit, namely
\[ e^{-i\tau x}f_+ \left(x + \frac{i\nu}{\tau}\right) = e^{-i\tau x}f_+ \left(x + i\frac{\nu}{\tau}\right) = \int_{-\tau}^{\infty} F(\tau + t) e^{it(x+i\frac{\nu}{\tau})} dt; \quad \nu > 0. \quad (41) \]
Entire function \(\exp \{ it (x + i\frac{\nu}{\tau}) \} \) may be expanded in a Taylor series for arbitrary \(\tau \nu/\tau\), but to preserve the uniform convergence of the integral we left infinitesimal parameter \(\varepsilon\) in the exponent. In this case the interchange of summations and integration order is legal and we may write

\[
e^{-i\tau x} f_+ \left( x + i\frac{\nu}{\tau} \right) = \sum_{m=0}^{\infty} \frac{(-\frac{\nu}{\tau} + \varepsilon)^m}{m!} \int_{-\tau}^{\tau} F (\tau + t) t^m e^{it(x+i\varepsilon)} dt.
\] (42)

After taking the limit \(\varepsilon \to 0\) we obtain

\[
e^{-i\tau x} f_+ \left( x + i\frac{\nu}{\tau} \right) = \sum_{m=0}^{\infty} \frac{(-\frac{\nu}{\tau})^m}{m!} \int_{-\tau}^{\tau} F (\tau + t) t^m e^{it(x+i0)} dt
\] (43)

that leads to

\[
e^{-i\tau x} f_+ \left( x + i\frac{\nu}{\tau} \right) = \int_{-\tau}^{\tau} \sum_{m=0}^{\infty} \frac{(-\frac{\nu}{\tau})^m}{m!} t^m \sum_{n=0}^{\infty} \frac{F^{(n)} (\tau)}{n!} t^n e^{it(x+i0)} dt.
\] (44)

Since \(t\) in Taylor series is located in the convergence area, we may write

\[
\sum_{m=0}^{\infty} \frac{(-\frac{\nu}{\tau})^m}{m!} t^m \sum_{n=0}^{\infty} \frac{F^{(n)} (\tau)}{n!} t^n = 2\pi \sum_{m=0}^{\infty} (it)^n C_n (\nu, \tau)
\] (45)

with

\[
C_n (\tau, \nu/\tau) = \frac{i^{-n}}{2\pi} \sum_{n=k}^{\infty} \frac{F^{(n-k)} (\tau)}{(n-k)!} \frac{(-\frac{\nu}{\tau})^k}{k!}
\] (46)

and we finally get

\[
e^{-i\tau x} f_+ \left( x + i\frac{\nu}{\tau} \right) = \sum_{n=0}^{\infty} C_n (\tau, \nu/\tau) \delta^{(n+m)} (x).
\] (47)

In the same way, from

\[
e^{-i\tau x} f_- \left( x - i\frac{\nu}{\tau} \right) = -\int_{-\tau}^{\tau} F (\tau + t) e^{it(x-i\varepsilon)} dt
\] (48)

we find

\[
e^{-i\tau x} f_- (x) = \begin{cases} 
0; & \text{if } \tau \to \infty, \\
-\sum_{n=0}^{\infty} C_n (\tau, \nu/|\tau|) \delta^{(n)} (x); & \text{if } \tau \to -\infty.
\end{cases}
\] (49)

It easy to check that expression (47) may be rewritten as

\[
e^{-i\tau x} \left[ f_+ \left( x + i\frac{\nu}{\tau} \right) - f_- \left( x - i\frac{\nu}{\tau} \right) \right] = \sum_{n=0}^{\infty} C_n (\tau, \nu/|\tau|) \delta^{(n)} (x)
\] (50)

with

\[
C_n (\tau) = 2\pi i^{-n} \sum_{m=0}^{n} \frac{F^{(n-m)} (\tau)}{(n-m)!} \frac{(-\nu/|\tau|)^m}{m!}.
\] (51)

It is clear that with \(\nu \to 0\) we return to (32).
4 Asymptotic expansion for power type distribution

Consider as an example expansion for \( \exp \{-ix\tau\} (x + i\nu/\tau)^{\lambda} \). With
\[
(x + i\nu/\tau)^{\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(-\lambda + n)}{n!\Gamma(-\lambda)} (-i\nu/\tau)^n (x + i\varepsilon \text{sgn} (\tau))^{\lambda-n}
\]
and taking into account [3]8.8(677) one may obtain
\[
F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( x + i\frac{\nu}{\tau} \right)^{\lambda} e^{-ix\tau} \, dx = \begin{cases} 
\frac{e^{-i\nu\frac{\tau}{\lambda}}}{\Gamma(-\lambda)} \sum_{n=0}^{\infty} \frac{(\frac{\tau}{\lambda})^n (-1)^{n+\lambda-1}}{n!}; & \text{if } \tau > 0, \\
\frac{e^{-i\nu\frac{\tau}{\lambda}}}{\Gamma(-\lambda)} \sum_{n=0}^{\infty} \frac{(\frac{\tau}{\lambda})^n e^{-\lambda-1}}{n!}; & \text{if } \tau < 0.
\end{cases}
\]

It is clear that for \( \tau \to \pm\infty \) and \( |t| < |\tau| \) we get
\[
F(t + \tau) = \begin{cases} 
e^{-\nu + \frac{i\tau}{\lambda}} \sum_{m=0}^{\infty} \frac{\lambda^m m^m (\lambda - m)^m}{(\lambda - \nu)^m}; & \text{if } \tau \to \infty, \\
e^{-\nu - \frac{i\tau}{\lambda}} \sum_{m=0}^{\infty} \frac{\lambda^m m^m (\lambda - m)^m}{(\lambda - \nu)^m}; & \text{if } \tau \to -\infty.
\end{cases}
\]

In the area \( |t| < \tau \) one may also write
\[
\frac{|t|^{\lambda-1}}{\Gamma(-\lambda)} = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(-\lambda-n)} (\pm t)^n |\tau|^{-\lambda-1-n}
\]
so
\[
F(t + \tau) = \begin{cases} 
e^{\nu + \frac{i\tau}{\lambda}} \sum_{n,m=0}^{\infty} \frac{\lambda^m m^m (\lambda - m)^m \nu^m |\tau|^{-\lambda-1-n}}{m!}; & \text{if } \tau \to \infty, \\
e^{-\nu - \frac{i\tau}{\lambda}} \sum_{n,m=0}^{\infty} \frac{\lambda^m m^m (\lambda - m)^m \nu^m |\tau|^{-\lambda-1-n}}{m!}; & \text{if } \tau \to -\infty,
\end{cases}
\]

or
\[
F(t + \tau) = \begin{cases} 
e^{\nu + \frac{i\tau}{\lambda}} \sum_{n=0}^{\infty} p_n^\lambda \nu^\lambda |\tau|^{-\lambda-1-n}; & \text{if } \tau \to \infty, \\
e^{-\nu - \frac{i\tau}{\lambda}} \sum_{n=0}^{\infty} p_n^\lambda (-\nu) |\tau|^{-\lambda-1-n}; & \text{if } \tau \to -\infty,
\end{cases}
\]
where
\[
p_n^\lambda (\nu) \equiv \sum_{m=0}^{n} \frac{\lambda^m m^m}{m! (n-m)!\Gamma(m-\lambda-n)}
\]
are polynomials\(^4\) of degree \( n \). So having collected everything we obtain
\[
e^{-ix\tau} \left( x + i\frac{\nu}{\tau} \right)^{\lambda} = \begin{cases} 
e^{\nu + \frac{i\tau}{\lambda}} \sum_{n=0}^{\infty} p_n^\lambda (\nu) \tau^{-\lambda-1-n} i^{-n} \delta^{(n)} (x); & \text{if } \tau \to \infty, \\
e^{-\nu - \frac{i\tau}{\lambda}} \sum_{n=0}^{\infty} p_n^\lambda (-\nu) |\tau|^{-\lambda-1-n} i^{n} \delta^{(n)} (x); & \text{if } \tau \to -\infty.
\end{cases}
\]

\(^3\)An Abel-Poisson regularization is implied.
\(^4\)Up to a constant \( p_n^\lambda (\nu) \) are a finite Kummer series (see [3] 6.1).
It goes without saying that for $\tau \to \pm \infty$

$$e^{\pm i\tau \left(x \pm \frac{i\nu}{\tau}\right)^\lambda} = 0 \quad (60)$$

5 Conclusion

A simple method is suggested to compute a series for $e^{-i\tau x}f(x)$ for $\tau \to \infty$. It is shown that $n$-th term in the series is proportional to $n$-th derivative of the Fourier transform $f(x)$, and namely to $F^{(n)}(\tau)$. If $f(x) \in E'$, i.e. defined as functional $(f(x), \phi(x))$ on infinitely differentiable test functions $\phi(x) \in E$ with arbitrary support, then $F(t + \tau)$ is an entire function and Taylor series converges in the whole complex plane $t$. A more interesting case appears when we consider tempered distributions $f(x)$, such that $F(t + \tau)$ allows asymptotic expansions, i.e., roughly speaking, $F^{(n+1)}(\tau) = O\left(\frac{F^{(n)}(\tau)}{\tau}\right)$ for $\tau \to \pm \infty$. In such case the Taylor series for $F(t + \tau)$ converges only in the bounded area $|t| < |\tau|$, but an asymptotic expansion still exists.

As it is shown in Appendix, if distribution $F(t)$ allows the Mellin transform and obeys conditions specified there, the asymptotic series (29) and (31) may be obtained with the asymptotic expansion for a particular type of power-type distributions $e^{i\tau x} (x \pm i\varepsilon)^{-s}$ computed in [2, 3].

6 Appendix. Modified Mellin transform

Comprehensive study of Mellin transform

$$m(s) = \int_0^\infty t^{s-1} F(t) dt \quad (61)$$

for distributions was undertaken in [7]. It is shown, in particular, that if $m(s)$ is represented by (61) for $s \in \Lambda$, then $m(s)$ is regular in $\Lambda$. Moreover, if $m(s)$ is regular in strip $a < \text{Re} s < b$ and

$$|m(s)| < C_0 |s|^{-2}, \quad C_0 = \text{constant}, \quad (62)$$

then the inverse Mellin transform may be written as

$$F(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} m(s) t^{-s} ds \quad (63)$$

where $a < \sigma < b$.

Integration path in (61) includes the area of positive $t$ only, hence $m(s)$ carries no information about $F(t)$ behavior for $t < 0$. Expression (63) may allow to compute $F(t)$ for some $t < 0$ as an analytical extension, however, it may appear unacceptable for $F(t)$ being nonanalytic for $t < 0$. 9
To represent the distribution $F(t)$ with Mellin transform in the whole definitional domain of $F(t)$, we divide this domain into positive and negative parts. Mellin transform of $F(t)$ for $t > 0$ is

$$m_+(s) = \int_0^\infty t^{s-1} F(t) \, dt = \int_{-\infty}^\infty t_+^{s-1} F(t) \, dt$$

(64)

and, in fact, it coincides with (61). For $t < 0$ we relate $F(t)$ to

$$m_-(s) = \int_0^\infty t^{s-1} F(-t) \, dt = \int_{-\infty}^0 |t|^{s-1} F(t) \, dt = \int_{-\infty}^\infty t_+^{s-1} F(t) \, dt$$

(65)

where by the definition (see e.g. [4])

$$t^\lambda_+ \triangleq \begin{cases} t; & \text{if } t > 0 \\ 0; & \text{if } t < 0 \end{cases} \quad t^\lambda_- \triangleq \begin{cases} 0; & \text{if } t > 0 \\ |t|^\lambda; & \text{if } t < 0 \end{cases}$$

(66)

If we assume that $m_{\pm}(s)$ are regular in the strips $a_{\pm} < \text{Re } s < b_{\pm}$, then inverse Mellin transforms acquire the form

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} m_+(s) t_+^{-s} \, ds = \begin{cases} F(t); & \text{if } t > 0 \\ 0; & \text{if } t < 0 \end{cases}$$

(67)

and

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} m_-(s) t_-^{-s} \, ds = \begin{cases} 0; & \text{if } t > 0 \\ F(t); & \text{if } t < 0 \end{cases}$$

(68)

where $a_{\pm} < \sigma_{\pm} < b_{\pm}$.

If strips $a_{\pm} < \text{Re } s < b_{\pm}$ have the common area $a < \text{Re } s < b$, where $a \geq a_{\pm}$ and $b \leq b_{\pm}$, then we may write

$$F(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left( m_+(s) t_+^{-s} + m_-(s) t_-^{-s} \right) ds$$

(69)

where $a < \sigma < b$.

Recall, that distributions $t_\pm^{-s}$ are nonanalytic in $s$, namely, they have simple poles for positive integer $s$

$$t_\pm^{-s} = \frac{(-1)^n}{s - n} \delta^{(n-1)}(t) + \xi_n(t_\pm, s)$$

(70)

where $\xi_n(t_\pm, s)$ is the regular part of the Laurent series [4]. For our purpose it is convenient to modify Mellin transform by changing $t_\pm^{-s}$ for distribution $(x \pm i\varepsilon)^{s-1}$ which depends on $s$ regularly. From [4]

$$\int_{-\infty}^\infty t_\pm^{-s} \exp\{itx\} \, dt = \exp\left\{ \pm i \frac{\pi}{2} (1 - s) \right\} \Gamma(1 - s) (x \pm i\varepsilon)^{s-1}$$

(71)
we see that it may be done by Fourier transform of (67) and (68) that leads to
\[
\frac{1}{2\pi i} \int_{\sigma_- - i\infty}^{\sigma_+ + i\infty} m_+(s) \exp \left\{ \frac{\pi}{2} (1 - s) \right\} \Gamma (1 - s) (x + i\varepsilon)^{s-1} ds
\]
(72)
\[
= \int_0^\infty F(t) \exp \{itx\} dt
\]
and
\[
\frac{1}{2\pi i} \int_{\sigma_- - i\infty}^{\sigma_+ + i\infty} m_-(s) \exp \left\{ -i \frac{\pi}{2} (1 - s) \right\} \Gamma (1 - s) (x - i\varepsilon)^{s-1} ds
\]
(73)
\[
= \int_0^\infty F(t) \exp \{itx\} dt
\]

Taking into account (6) and (7), and changing \(s \to 1 - s\) in (72), (73) we may write
\[
f_+ (x) = \int_0^\infty F(t) \exp \{itx\} dt = \frac{1}{2\pi i} \int_{c_+ - i\infty}^{c_+ + i\infty} \mu_+ (s) (x + i\varepsilon)^{-s} ds
\]
(74)
and
\[
f_- (x) = -\int_0^\infty F(t) \exp \{itx\} dt = -\frac{1}{2\pi i} \int_{c_- - i\infty}^{c_- + i\infty} \mu_- (s) (x - i\varepsilon)^{-s} ds
\]
(75)
where
\[
\mu_\pm (s) = \exp \{\pm \pi s \pi / 2\} \Gamma (s) m_\pm (1 - s); \quad c_\pm = 1 - \sigma_\pm,
\]
(76)
The regularity of \(m_\pm (s)\) in strips \(a_\pm < \text{Re } s < b_\pm\) causes the regularity of \(\mu_\pm (s)\) in \(1 - b_\pm < \text{Re } s < 1 - a_\pm\) except, possibly, simple poles at negative integer values of \(s = -n\), if such are located in the considered strip. Such poles correspond to the ones in (70). It is clear that if one shifts the path of integration in (72) or (73) in the area \(1 - b_\pm < \text{Re } s < 1 - a_\pm\) with some pole crossing, a result is obtained which differs from the original by the residues in such poles.

Since
\[
\lim_{|\text{Im } s| \to \infty} |\Gamma (s)| = \sqrt{2\pi} \exp \left\{-\frac{\pi}{2} |\text{Im } s| \right\} |\text{Im } s|^{\text{Re } s - \frac{1}{2}}
\]
(77)
(see [3] 1.1.18(6)), we get from (72) a restriction on \(\mu_\pm (s)\) in strips \(1 - b_\pm < \text{Re } s < 1 - a_\pm\)
\[
|\mu_\pm (s)| < C_0 |1 - s|^{-2} |\text{Im } s|^{\text{Re } s - \frac{1}{2}} \exp \{-\pi \theta (\pm \text{Im } s) |\text{Im } s|\}
\]
(78)
that for \(|\text{Im } s| \to \infty\) leads to
\[
|\mu_\pm (s)| < C_0 |\text{Im } s|^{-\sigma_\pm - \frac{3}{2}} \exp \{-\pi \theta (\pm \text{Im } s) |\text{Im } s|\}.
\]
(79)
Fast decrease of $|\mu_\pm(s)|$ for $\text{Im} \, s \to \pm \infty$ compensates the increase of $|x|^{-s} \exp \{\pm \pi \text{Im} \, s\}$ for $x < 0$ and provides convergence of integral in (74).

Expressions (74) and (75) give convenient representations for the functions $f_\pm(x)$ which are analytical in the upper/lower half-plane of a complex variable $x$.

An application of inverse Mellin transform (67) to (74) relates $\mu_\pm(s)$ to $F(t)$

$$\mu_\pm(s) = \exp\left\{\pm \frac{\pi}{2} s\right\} \Gamma(s) \int_{-\infty}^{\infty} t_\pm^{-s} F(t) \, dt \quad (80)$$

so with (74) one may easily get

$$\mu_+(s) = \frac{1}{1-e^{2\pi i s}} \int_{-\infty}^{\infty} f_+(x) (x-i\varepsilon)^{s-1} \, dx \quad (81)$$

and

$$\mu_-(s) = -\frac{1}{1-e^{2\pi i s}} \int_{-\infty}^{\infty} f_-(x) (x+i\varepsilon)^{s-1} \, dx. \quad (82)$$

With (74) one may easily regain the series (29). Indeed, with simple relation

$$\frac{1}{\Gamma(1-s-n)} \tau^{-s-n} = \frac{1}{\Gamma(1-s)} \frac{\partial^n}{\partial \tau^n} \tau^{-s} \quad (83)$$

one may rewrite series for $\tau \to +\infty$

$$(x+i\varepsilon)^{s-1} \exp\{-ix\tau\} = 2\pi \sum_{n=0}^{\infty} \delta^{(n)}(x) \frac{\exp\{i\tau (s-1-n)\}}{n! \Gamma(1-s-n)} \tau^{-s-n} \quad (84)$$

as (see [2])

$$(x+i\varepsilon)^{s-1} \exp\{-ix\tau\} = 2\pi \sum_{n=0}^{\infty} \delta^{(n)}(x) \frac{\exp\{i\tau (s-1-n)\}}{n! \Gamma(1-s)} \frac{\partial^n}{\partial \tau^n} \tau^{-s} \quad (85)$$

and from (72) we obtain

$$f_+(x) \exp\{-ix\tau\} = 2\pi \sum_{n=0}^{\infty} \delta^{(n)}(x) \frac{i^n}{n!} \frac{\partial^n}{\partial \tau^n} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} m_+(s) \tau^{-s} ds \quad (86)$$

that with (67) leads to (29). Mutatis mutandis the similar result may be obtained for $f_-(x) \exp\{-ix\tau\}$.

References

[1] Yu.A. Brychkov and Yu.M. Shirokov, Teor.Mat.Fiz.4(1970)301.

[2] Yu.A. Brychkov, Teor.Mat.Fiz.5(1970)98; 15(1973)375; 23(1975)191.
[3] Yu.A. Brychkov, A.P. Prudnikov 'Integral transforms of generalized functions', "Nauka", Moscow 1977.

[4] I.M. Gelfand and G.E. Shilov. 'Generalized functions', Vol. 1: Properties and Operations, Fizmatgiz, Moscow, 1958, New York: Harcourt Brace, 1977. I. M. Gel’fand, and G. E. Shilov, 'Generalized Functions', Vol. 2: Spaces of Fundamental and Generalized Functions. New York: Harcourt Brace, 1977.

[5] Bremermann, H. 'Distributions, complex variables and Fourier transforms', Addison-Wesley 1965.

[6] V. S. Vladimirov, 'Generalized Functions in Mathematical Physics'. Moscow: Nauka, 1976.

[7] Zemanian, A.H. 'DISTRIBUTION THEORY AND TRANSFORM ANALYSIS: an introduction to generalized functions, with applications'. McGraw-Hill, 1965.

[8] A. Erdelyi, 'The asymptotic Evaluation of Certain Integrals', Arch. Rat. Mech. Anal. 14(1963)217. A. Erdelyi, 'Asymptotic expansions', Dover publications, inc., New York 1956.

[9] H. Bateman and A. Erdélyi, 'Higher Transcendental Functions', MC Graw-Hill, inc. 1953.