DIMENSION FORMULAS FOR SOME MODULAR REPRESENTATIONS OF THE SYMPLECTIC GROUP IN THE NATURAL CHARACTERISTIC

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Abstract. We compare the dimensions of the irreducible $\text{Sp}(2g, K)$-modules over a field $K$ of characteristic $p$ constructed by Gow [Go] with the dimensions of the irreducible $\text{Sp}(2g, F_p)$-modules that appear in the first approximation to representations of mapping class groups of surfaces in Integral Topological Quantum Field Theory [GMS]. For this purpose, we derive a trigonometric formula for the dimensions of Gow’s representations. This formula is equivalent to a special case of a formula contained in unpublished work of Foulle [F1, F2]. Our direct proof is simpler than the proof of Foulle’s more general result, and is modeled on the proof of the Verlinde formula in TQFT.

Let $g \geq 1$ be an integer. The irreducible representation of the symplectic group $\text{Sp}(2g, \mathbb{C})$ with fundamental highest weight $\omega_k$ ($1 \leq k \leq g$) can be constructed inside the $k$-th exterior power of the standard representation of $\text{Sp}(2g, \mathbb{C})$ as the kernel of a certain contraction operator (see for example Fulton and Harris [FH, §17]). The dimension of this representation is

\[ \binom{2g}{k} - \binom{2g}{k-2}. \]

Assume now that $p$ is an odd prime and $K$ is an algebraically closed field of characteristic $p$. Premet and Suprunenko [PS] showed that if one performs the same construction over the field $K$, the resulting representation of the symplectic group $\text{Sp}(2g, K)$ is again irreducible with fundamental highest weight $\omega_k$ provided $p > g$. But if $p \leq g$, this representation may no longer be irreducible, although it is still a Weyl module for $\omega_k$, and so it has a unique irreducible quotient which will be an irreducible $\text{Sp}(2g, K)$-module with highest weight $\omega_k$ (see for example Humphreys [H, §3.1]). Gow considered precisely this situation in [Go], except that he did not assume that $K$ is algebraically closed. For every integer $g \geq p - 1$, and for the $p - 1$ values of $k$ in the range $[g - p + 2, g]$, he considered natural subquotients of $\Lambda^k V$, where $V \simeq K^{2g}$ is the standard representation of the symplectic group $\text{Sp}(2g, K)$, and showed that these subquotients are irreducible representations of $\text{Sp}(2g, K)$. Following Gow, we denote these representations by

\[ V(g, k) \quad (g - p + 2 \leq k \leq g). \]

Gow also proved by an explicit computation that if $K$ is algebraically closed, then $V(g, k)$ indeed has fundamental highest weight $\omega_k$ [Go, Corollary 2.4].
In this note, we are interested in the dimensions of Gow’s representations $V(g, k)$. For $g = p - 1$ these dimensions are still given by the classical formula (1). But for $g > p - 1$ this is no longer the case, and the dimension of $V(g, k)$ now depends on the characteristic $p$ of the field $K$. Gow gave a recursion formula for this dimension, but stopped short of solving the recursion except for $p \leq 5$. We solved Gow’s recursion for general primes $p$ and thereby found an explicit trigonometric formula for the dimension of $V(g, k)$ which we will state in Theorem 2 below. Our proof of this dimension formula is analogous to the proof of the famous Verlinde formula in Topological Quantum Field Theory (TQFT) which computes the dimensions of the TQFT vector spaces (see Formula (3) below). In fact, we came to our formula while trying to compare the dimensions of certain irreducible modular representations of $\text{Sp}(2g, \mathbb{F}_p)$ arising from our theory of Integral TQFT [G1, GM1, GM3] with Gow’s representations $V(g, k)$. Experimentally, we found that for $p = 5$, the dimensions coincide, whereas for $p \geq 7$ they do not. The aim of this note is to explain why this is so.

Gow’s recursion formula expresses the dimension of $V(g, k)$ as a matrix coefficient, as follows. Let $C$ be the $(p-1) \times (p-1)$ matrix whose non-zero entries are given by $C_{ii} = 2$ for $1 \leq i \leq p-1$ and $C_{j,j-1} = C_{j-1,j} = 1$ for $2 \leq j \leq p-1$.

**Theorem 1** (Gow [Go]). Let $p$ be an odd prime and $g \geq p - 1$. For the $p-1$ values of $k$ in the range $[g - p + 2, g]$, one has

$$\dim V(g, k) = (C^g)_{g+1-k,1}.$$  

**Remark.** This formula holds also for $g < p - 1$ if we interpret the left hand side as given by the classical formula (1) in that case.

We will prove the following:

**Theorem 2.** Let $p$ be an odd prime and $g \geq 1$. For $1 \leq n \leq p - 1$ one has

$$(C^g)_{n,1} = \frac{2^{2g+1}}{p} \sum_{j=1}^{(p-1)/2} \left( \sin \frac{2\pi j}{p} \right) \left( \cos \frac{\pi j}{p} \right)^{2g} \left( -1 \right)^n \left( \sin \frac{\pi j}{p} \right)^{2g}.$$  

To get an explicit formula for the dimension of $V(g, k)$ from this theorem, it suffices to put $n = g + 1 - k$.

**Remark.** After completing the first version of this note, we have learned\(^1\) that a trigonometric formula for the dimensions of the $V(g, k)$ had been obtained previously by S. Foulle. In fact, Foulle in his 2004 thesis [F1] (see also his 2005 preprint [F2] which apparently was never published) presents a more general trigonometric dimension formula which works also for fundamental highest weights outside the range $[g - p + 2, g]$ considered by Gow. Foulle’s formula is based on Premet and Suprunenko’s work [PS] and is in general much more complicated than the expression in Theorem 2. But we have checked that Foulle’s formula when specialized to fundamental highest weights $\omega_k$, for $k$ in the range $[g - p + 2, g]$, is equivalent to the formula given in Theorem 2. See [F2, Example 4.6]. Still we believe that our direct proof based on Gow’s recursion formula is of interest, as the proof is analogous to the proof of the Verlinde formula in TQFT and seems much simpler than Foulle’s proof of his more general result.

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\(^1\)We thank Alexander Premet for telling us about Foulle’s formula.
As already mentioned, our main motivation for proving Theorem 2 was that we need it to identify the dimensions of the four representations $V(g, k)$ of $\text{Sp}(2g, \mathbb{F}_p)$ in the case $p = 5$ with the dimensions of the irreducible modular representations constructed in [GM3]. The answer will be stated in Corollary 3 below. First, let us briefly review the construction of these representations.

Let $\Sigma$ be an oriented surface of genus $g$ with at most one boundary component. The orientation-preserving mapping class group of $\Sigma$ has a natural surjective homomorphism to the symplectic group $\text{Sp}(2g, \mathbb{Z})$, and hence to $\text{Sp}(2g, \mathbb{F}_p)$ for every $p$. Using this homomorphism, the modular representations of [GM3] are constructed from representations of the mapping class group of $\Sigma$ arising in Integral TQFT, as follows. Let $p \geq 5$ be a prime, and $2c$ an even integer with $0 \leq 2c \leq p - 3$. Integral TQFT [GI] [GM1] provides a representation of a certain central extension of the mapping class group of $\Sigma$ on a free module of finite rank over the ring of cyclotomic integers $\mathbb{Z}[\zeta_p]$ (here $\zeta_p$ is a primitive $p$-th root of unity). These “quantum” representations are denoted by $S(\Sigma_g(2c))$ in [GM3]. We think of these representations as an integral version of the Reshetikhin-Turaev TQFT associated to the Lie group $\text{SO}(3)$, a version of which is obtained if one extends coefficients from $\mathbb{Z}$ to the quotient field $\mathbb{Q}(\zeta_p)$. The advantage of having coefficients in $\mathbb{Z}[\zeta_p]$ is that one can reduce coefficients by dividing out by some ideal in $\mathbb{Z}$.

This also holds for $g \leq 1$ we have identified the modular representation $F(\Sigma_g(2c))$ for any $p$ and $c$ in [GM2]: in this case, one has $\epsilon_g^{(2c)}(p) = 0$ and $\alpha_g^{(2c)}(p) = (p - 1)/2 - c$, and $F(\Sigma_g(2c))$ is isomorphic to the space of homogeneous polynomials over $\mathbb{F}_p$ in two variables of total degree $(p - 3)/2 - c$. It is well-known that these representations of $\text{Sp}(2, \mathbb{F}_p) = \text{SL}(2, \mathbb{F}_p)$ are the unique (up to isomorphism) irreducible representations with these dimensions.

Corollary 3. For $p = 5$ and $g \geq 4$, we have the following equality of dimensions of irreducible $\text{Sp}(2g, \mathbb{F}_p)$-representations:

$$(\dim V(g, g - m))_{m=0,1,2,3} = (\epsilon_g^{(0)}(5), \epsilon_g^{(2)}(5), \alpha_g^{(2)}(5), \alpha_g^{(0)}(5)) .$$

This also holds for $g < 4$ if we interpret the left hand side as given by the classical formula (1) with $k = g - m$ in that case.

At present, we don’t know whether these equalities of dimensions come from isomorphisms of the corresponding representations. We remark that for $g = 1$ we have identified the modular representation $F(\Sigma_1(2c))$ for any $p$ and $c$ in [GM2]: in this case, one has $\epsilon_1^{(2c)}(p) = 0$ and $\epsilon_1^{(2c)}(p) = (p - 1)/2 - c$, and $F(\Sigma_1(2c))$ is isomorphic to the space of homogeneous polynomials over $\mathbb{F}_p$ in two variables of total degree $(p - 3)/2 - c$. It is well-known that these representations of $\text{Sp}(2, \mathbb{F}_p) = \text{SL}(2, \mathbb{F}_p)$ are the unique (up to isomorphism) irreducible representations with these dimensions.
The proof of Corollary 3 is based on explicit formulas for the dimensions $c_g^{(2c)}(p)$ and $o_g^{(2c)}(p)$ which are similar to the expression in Theorem 2. These formulas are obtained as follows. First of all, the sum

$$D_g^{(2c)}(p) = c_g^{(2c)}(p) + o_g^{(2c)}(p)$$

is the dimension of $F(\Sigma_g^{(2c)})$, which is the same as the rank of $S(\Sigma_g^{(2c)})$, and thus the same as the dimension of the $\text{SO}(3)$-TQFT vector space $S(\Sigma_g^{(2c)}) \otimes \mathbb{C}$.

This dimension is given by the celebrated Verlinde formula

$$D_g^{(2c)}(p) = \left(\frac{p}{4}\right)^{g-1} \frac{(p-1)/2}{p} \sum_{j=1}^{\sin(\pi j(p+1)/p) \sin(\pi j/p) 1-2g}$$

We like to think of our expression for the dimensions of Gow’s irreducible representations in Theorem 2 as an analogue of the Verlinde formula.

Next, we showed in [GM3] that the difference

$$\delta_g^{(2c)}(p) = c_g^{(2c)}(p) - o_g^{(2c)}(p)$$

is given by the following formula:

$$\delta_g^{(2c)}(p) = (-1)^{c_g^{(2c)}(p)} \sum_{j=1}^{(p-3)/2} (-1)^k \sum_{k=1}^{(p-1)/2} \left(\frac{\sin \pi j(p+1)/p}{\sin \pi j/p}\right) \Lambda_j^g$$

where

$$\Lambda_j = \left(\frac{p-1}{4}\right) + 2 \sum_{k=1}^{(p-3)/2} (-1)^k \left(\frac{p-2k-1}{4}\right) \cos(2kj\pi/p)$$

Here, $[x]$ is the smallest integer $\geq x$. It is now easy to convert formulas (3) and (5) into explicit formulas for the dimensions $c_g^{(2c)}(p)$ and $o_g^{(2c)}(p)$ of our irreducible $\text{Sp}(2g, \mathbb{F}_p)$-representations coming from TQFT.

**Remark.** We used these formulas in [GM3] to show that the dimensions $c_g^{(2c)}(p)$ and $o_g^{(2c)}(p)$ are given by (inhomogeneous) polynomials in $p$ and $c$ of total degree $3g-2$, whose leading coefficients can be expressed in terms of Bernoulli numbers.

**Proof of Theorem 2.** Let $q$ be a primitive $p$-th root of unity. Let $S = (S_{ij})$ be the $(p-1) \times (p-1)$ matrix defined by

$$S_{ij} = (-1)^{i-j} \left( q^{ij} - q^{-ij} \right)$$

Let $Q = \text{diag}(2 + (-1)^i(q^1 + q^{-i}))_{j=1,2,\ldots,p-1}$, and let $I$ denote the identity matrix.

**Lemma 4.** We have

$$S^2 = -2pI$$

$$C = SQS^{-1} = -\frac{1}{2p} S$$

**Remark.** This lemma (or some variants of it) is well-known in TQFT, where it is used to prove the Verlinde formula [3]. In fact, our matrix $S$ is (up to some rescaling) the $S$-matrix of the $\text{SU}(2)$-TQFT at level $p-2$. This TQFT corresponds to the $V_{2p}$-theory of [BHMV]. In the notation of that paper, the matrix $C$ is the matrix of multiplication by $2 + z$ in the standard basis of the Verlinde algebra $V_{2p}(\text{torus})$, and the lemma follows from the diagonalization procedure on page 913.
of [BHMV]. However, it is also easy to prove the lemma by a direct computation, which we include for the convenience of the reader.

**Proof of Lemma 4.** We have

\[
(S^2)_{ij} = \sum_{k=1}^{p-1} S_{ik} S_{kj} = \sum_{k=1}^{p-1} (-1)^{(i+j)k}(q^{ik} - q^{-ik})(q^{kj} - q^{-kj})
\]

\[
= \sum_{k=1}^{p-1} (-1)^{(i+j)k} (q^{(i+j)k} + q^{-(i+j)k} - q^{(i-j)k} - q^{-(i-j)k})
\]

(8)

This implies that the diagonal terms of \(S^2\) are \((S^2)_{ii} = -2p\), since

\[
\sum_{k=1}^{p-1} q^{2ik} = -1 \quad (1 \leq i \leq p-1).
\]

(9)

It remains to show that \(S_{ij} = 0\) if \(i \neq j\). If \(i + j\) is even, this follows again from (9), while if \(i + j\) is odd, this follows from

\[
\sum_{k=1}^{p-1} (-1)^k (q^{nk} + q^{-nk}) = 0 \quad (n \in \mathbb{Z})
\]

(10)

(To prove (10), observe that the \(k\)-th summand cancels the \((p-k)\)-th summand.) Thus we have proved the first statement of the lemma. To prove the second statement (7), it suffices to observe that

\[
(CS)_{ij} = \sum_{k=1}^{p-1} C_{ik} S_{kj} = 2S_{ij} + S_{i+1,j} + S_{i-1,j}
\]

\[
= (2 + (-1)^j(q^j + q^{-j})) S_{ij}
\]

\[
= (SQ)_{ij}
\]

(This computation is also valid for \(i = 1\) and \(i = p-1\) since \(S_{0j} = 0 = S_{pj}\).) Thus we have \(CS = SQ\), which implies (7). \(\square\)

Using the lemma, we now have

\[
(C^g)_{n,1} = -\frac{1}{2p} \sum_{j=1}^{p-1} S_{nj} \left(2 + (-1)^j(q^j + q^{-j})\right)^g S_{j1}
\]

\[
= -\frac{1}{2p} \sum_{j=1}^{p-1} (-1)^{(n+1)j}(q^{nj} - q^{-nj})(q^j - q^{-j}) \left(2 + (-1)^j(q^j + q^{-j})\right)^g
\]

\[
= -\frac{1}{2p} \sum_{j=1}^{(p-1)/2} (q^{nj} - q^{-nj})(q^j - q^{-j}) \left(2 + q^j + q^{-j}\right)^g - (-1)^n(2 - q^j - q^{-j})^g
\]

(In the last step, we have grouped together the term with \(j\) and the term with \(p-j\). It helps to consider separately the cases where \(j\) is even and where \(j\) is odd.) Substituting now \(q = e^{2\pi i/p}\) into this formula gives Theorem 2. \(\square\)
Proof of Corollary 3. Let $G_j$ be the Galois automorphism of the cyclotomic field $\mathbb{Q}(q)$ defined by $G_j(q) = q^j$. Formulas (3) and (5) can be expressed as follows:\[GM3:\]

\begin{align*}
D_g(2c) &= -\frac{1}{p} \sum_{j=1}^{(p-1)/2} G_j\left((q^{2c+1} - q^{-2c-1})(q - q^{-1})\left(-\frac{p}{(q-q^{-1})^2}\right)^g\right) \tag{11} \\
\delta_g(2c) &= -\frac{1}{p} \sum_{j=1}^{(p-1)/2} G_j\left((q^{2c+1} - q^{-2c-1})(q - q^{-1})(-1)^c\Lambda^g\right) \tag{12}
\end{align*}

where

\[
\Lambda = \left\lceil \frac{(p-1)/4}{(p-2)/4} \right\rceil + \sum_{k=1}^{(p-3)/2} \left\lceil (p-2k+1)/4 \right\rceil (q^{2k} + q^{-2k})
\]

Now let $p = 5$ so that $1 + q + q^2 + q^3 + q^4 = 0$. Thus

\[
\Lambda = 1 - q^2 - q^{-2} = 2 + q + q^{-1}
\]

and one checks that $(q - q^{-1})^2(2 - q - q^{-1}) = -5$ so that

\[
\frac{-5}{(q-q^{-1})^2} = 2 - q - q^{-1}
\]

Therefore

\[
(C^g)_{n,1} = \frac{-1}{10} \sum_{j=1}^{2} G_j\left((q^n - q^{-n})(q - q^{-1})\left(\Lambda^g - (-1)^n\left(-\frac{5}{(q-q^{-1})^2}\right)^g\right)\right)
\]

Comparing this with (11) and (12) proves the corollary. \(\square\)

Remark. We see that for $p \geq 7$, there is no obvious relation between $2 + q + q^{-1}$ and $2 - q - q^{-1}$ on the one hand, and $\Lambda$ and $-p/((q - q^{-1})^2)$ on the other hand. This explains why for $p \geq 7$ we found no relation between the dimensions of Gow's representations $V(g, k)$ on the one hand, and the dimensions $\epsilon_g(2c)(p)$ and $\delta_g(2c)(p)$ of our irreducible $\text{Sp}(2g, \mathbb{F}_p)$-representations coming from TQFT on the other hand. It is, however, intriguing that we get as many irreducible representations as Gow (namely $p - 1$, since we have two of them for each $0 \leq c \leq (p - 3)/2$). We wonder about how to characterize our representations among the irreducible representations of $\text{Sp}(2g, \mathbb{F}_p)$ in characteristic $p$.

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