The transmission of the charge through the normal metal-superconductor interface occurs via the electron-hole conversion known as the Andreev reflection process: an electron incident from the metal side with an energy smaller than the energy gap in the superconductor is converted into a hole which moves backward with respect to the electron. The missing charge $2e$ (an electron has charge $-e$ and a hole $+e$) propagates as an electron pair into the superconductor and joins the Cooper pair condensate $|1\rangle$. Correspondingly, a Cooper pair transfer from the superconductor is described as the Andreev reflection of a hole. This Andreev transport channel is characterized by the so-called Andreev interface contact resistance. Since transport current is introduced into a superconductor via normal leads, the Andreev reflection phenomenon is a foundation for most applications of superconductors (see Ref. [2] for a review).

There exists however an important experimental situation of the hopping insulator coupled to a measuring circuit via superconducting leads (see, for example, [3]), where the conventional Andreev reflection picture does not apply. The transport in hopping semiconductors occurs via localized (non-propagating) single particle states $|\Psi\rangle$ with undefined momentum and therefore a Cooper pair on the superconductor side cannot form. A single particle transport through the interface is exponentially suppressed, $\propto e^{-\Delta/T}$, where $\Delta$ is the superconductor gap, the temperature, $T$, being measured in energy units; therefore to explain the finite conductivity observed in experiments one needs Andreev-type processes capable to facilitate two particle transfer through the hopping insulator/superconductor interface allowing for Cooper pair formation. The possibility of such a transfer through the hopping-superconductor interface was discussed in Ref. [4] but no quantitative theory of hopping transport - supercurrent conversion was presented.

In this Letter we develop a theory for the transport through the hopping insulator-superconductor interface and derive the corresponding contact resistance. We show that the low-temperature charge transfer occurs via the correlated processes mediated by the pairs of hopping centers located near the interface. We demonstrate that this process resembles the conventional Andreev electron-hole reflection into a normal metal, the exponential suppression of transport specific to a single-particle processes being lifted. Thus, despite the limitation in the number of coherent hopping centers that can carry Andreev transport, the resulting contact resistance can become low as compared to the resistance of the hopping insulator. However in mesoscopic structures the interface resistance can be comparable or even exceed the hopping resistance. The proposed mechanism resembles the so-called crossed Andreev charge transfer $\mathcal{G}$, discussed recently in connection with a superconductor-dot entangler $\mathcal{G}$. The difference is that in $\mathcal{G}$ the transport mediated by artificial quantum dots was considered. In our case, the transport occurs via randomly located sites in the hopping insulator (HI), and the main problem one has to solve is finding the optimal configuration of the sites responsible for the charge transfer. Hereafter we will refer to the proposed charge transfer mechanism as to the time reversal reflection.

Let a superconductor (S) and an HI to occupy the adjacent 3D semi-spaces separated by a tunneling barrier (B). The presence of the barrier simplifies calculations which will be made in the lowest non-vanishing approximation in the tunneling amplitude $T_0$. This models the Schottky barrier usually presenting at a semiconductor-metal interface. In the linear response theory the conductance is determined by the Kubo formula $G$ for the susceptibility,

$$\chi(\omega) = i \int_0^\infty \langle \left[ \hat{I}^+(t), \hat{I}(0) \right] \rangle e^{i\omega t} dt \quad (1)$$

as $G = \lim_{\omega \to 0} \omega^{-1} \text{Im} \chi(\omega)$. Here the current operator
\[ \hat{I}(t) \text{ is defined as: } \]
\[ \hat{I}(t) = ie d T_0 \int d^2 r [a^+(r, t)b(r, t) - \text{h.c.}] , \]
where \( r \) is the coordinate in the interface plane, \( a^+(r, t) \) and \( b(r, t) \) are creation and annihilation operators in the semiconductor and superconductor, respectively, \( d \) is the electron localization length under barrier. The susceptibility, \( \chi(\omega) \), is calculated by analytical continuation of the Matsubara susceptibility \[ 11],
\[ \chi_M(\Omega) = \int_0^\beta \langle T_\tau \hat{I}(\tau)\hat{I}(0) \rangle \exp(i\Omega \tau) d\tau. \] (2)
Here \( T_\tau \) means ordering in the imaginary time, \( \beta \equiv 1/T \).
In the expression for \( \langle T_\tau \hat{I}(\tau)\hat{I}(0) \rangle \) one should expand to the second order with respect to the tunneling Hamiltonian,
\[ H_T(\tau) = d T_0 \int d^2 r [a^+(r, \tau)b(r, \tau) + \text{h.c.}] . \] (3)
Keeping only those second order terms that contain \( \langle T_\tau b(r, \tau)b(r_0, 0) \rangle \langle T_\tau b^+(r_1, \tau_1)b^+(r_2, \tau_2) \rangle \) products and thus represent the time reversal scattering which we are interested in, one arrives at the expression
\[ \langle T_\tau \hat{I}(\tau)\hat{I}(0) \rangle = e^2|T_0|^4 \int d\tau_1 d\tau_2 \prod_i d^2 r_i(A + B); \]
\[ A(\{x_i\}) = F(x - x_0)F^+(x_1 - x_2)G(x_1, x)G(x_2, x_0), \]
\[ B(\{x_i\}) = F(x - x_1)F^+(x_2 - x_0)G(x_0, x)G(x_2, x_1) \]
\[ - G(x_0, x_1)G(x_2, x), \]
where \( x \equiv \{ r, \tau \}, x_0 \equiv \{ r_0, 0 \}, x_i \equiv \{ r_i, \tau_i \} ; \) \( F(x - x') = (T_\tau b(r, \tau)b(r', \tau')) \) is the anomalous Green function in the superconductor while \( G(x, x') = - (T_\tau a(r, \tau)a^+(r', \tau')) \) is the Green function in the hopping insulator. One can show that the Andreev-type process we are interested in is given by the first term of \( B(\{x_i\}) \) in Eq. (4). The relevant diagram is shown in Fig. 1. Keeping only this term and using the Matsubara frequency representation one obtains
\[ \chi_M(\Omega) = 2 T e^2 |T_0|^4 \int \prod_i d^2 r_i \sum_n F(r - r_1, \omega_n) \]
\[ \times F^+(r_0 - r_2, \omega_n)G(r_0, \omega_n - \Omega_m)G(r_2, r_1, -\omega_n) \] (5)
where \( \Omega_m = 2\pi m T \) and \( \omega_n = (2n + 1)\pi T \). The normal Green’s functions can be expressed through the wave functions of the localized states, \( \varphi_s(r) = (\pi n^3)^{-1/2} \exp(-|r - r_s|/a) \), as
\[ G(r, r', \omega_n) = \sum_s \frac{\varphi^*_s(r)\varphi_s(r')}{i\omega_n - \xi_s} . \] (6)
We have assumed that for all of the sites under consideration the voltage drops between the site and the superconductor are the same. This is true when the partial interface resistance due to a time reversal pair is much larger than the typical resistance of the bond forming the percolation cluster. This situation resembles that considered by Larkin and Shklovskii for the tunnel resistance between the hopping conductors \[ 12].

The anomalous Green function, \( F(R, \omega_n) \), is
\[ F(R, \omega_n) = \int \frac{d^3 p}{(2\pi)^3} \frac{\Delta}{\Delta^2 + \xi_p^2 + \omega_n^2} \exp(i\mathbf{p} \cdot \mathbf{R}/\hbar) \]
\[ = \frac{\pi g_m \Delta}{2\sqrt{\Delta^2 + \omega_n^2}} \sin(Rk_F) \exp(-i\sqrt{\Delta^2 + \omega_n^2} 
\[ \times Rk_F). \] (7)
Here \( \xi_p = (p^2 - p_F^2)/2m, g_m = m p_F^2/\pi^2 \hbar^3 \) is the density of states in a metal, \( k_F = p_F/\hbar \), while \( \xi \) is the coherence length in a superconductor. Since \( F(R) \) oscillates with the period \( 2\pi/k_F \) over spatial coordinates along the interface yields the factor \( a^4/k_F^2|\rho_0|^2 \). Here \( \rho_0 \) is projection of the vector \( \mathbf{R}_{\mathbf{d}_s} \) connecting the centers on the interface plane. Note that the dependencies on \( R \) and \( \xi \) are similar to that given by Eq. (21) of the paper \[ 6] for the pair of the quantum dots near the superconducting interface. However the latter equation does not specify the dependences of the transmission coefficients on the real physical parameters of the interface and the localized centers.

The summation over the Matsubara frequencies, \( \omega_n \), is standard,
\[ T \sum_{\omega_n} f(\omega_n) = \int \frac{d\varepsilon}{4\pi i} f(\varepsilon) \tanh(\varepsilon/2T). \]
The contour of integration closes the cuts \( |\varepsilon| > \Delta \) along the real axis. Upon analytical continuation one arrives
at the following expression for the conductance:
\[
G = \frac{\pi e^2 g_{\text{HI}}^2 |T_0|^4 d^4}{2kT k^3 a^2} \sum_{s \neq l} n(\varepsilon_s) n(\varepsilon_l) \frac{\Delta^2}{|\rho_{s}\|\Delta^2 - \varepsilon_s^2|} e^{-2(z_s + z_l)/a}
\times e^{-2\rho_{s,l} / \sqrt{\Delta - \varepsilon_s^2 / \pi \xi \delta(\varepsilon_s + \varepsilon_l + U_c)}}, \tag{8}
\]
Here \( n(\varepsilon) = (e^{\varepsilon/T} + 1)^{-1} \) is the Fermi distribution, \( U_c \) is the energy of the inter-site Coulomb repulsion, and \( z \)-axis is perpendicular to the interface.

In what follows we will replace \( \sum_{s,l} \) by \( g^2 \int d^3 r d^3 r_s d\varepsilon_l d\varepsilon_s \) where \( g \) is the the effective density of states in the hopping insulator. This is the density of states in the layer adjacent to the interface. Due to screening by the superconductor it is not affected by the Coulomb gap and can be considered as constant. Since we are dealing with the pairs close to the interface, the Coulomb repulsion is suppressed by screening. This screening can be conveniently regarded as an interaction of the charged particle with its image having the opposite charge. Thus the Coulomb correlation manifest themselves as the dipole-dipole interaction and for \( \rho_{s,l} \gg a \) one arrives at \( U_c = e^2 a^2 / k \rho_{s,l}^3 \). Requiring it to me smaller than \( T \) one obtains a cut-off \( \rho_{s,l} \geq \rho = a(e^2 / k \alpha T)^{1/3} \).

As a crude estimate, we take \( d^4 \sim k^{-1} \), while \( T_0 \approx T e^{-\Delta} \) with \( T_e \sim \varepsilon_F \). Bearing this in mind one finds \( g^2 n^2 T_0^2 / k^2 \sim g^2 \varepsilon_F^2 / k^2 \sim 1 \). Since the ratio \( T_0 / (akF)^2 \) is of the order of the typical energy of the localized state, \( \varepsilon_d \sim h^2 / ma^2 \), one arrives at the estimate
\[
\frac{G}{G_n} \sim g a \rho^2 \varepsilon_d e^{-2\Lambda}, \quad G_n \sim \frac{e^2}{h} g a S \varepsilon_d e^{-2\Lambda}. \tag{9}
\]
Here \( G_n \) is the conductance of a boundary between a normal metal and a hopping insulator, while \( S \) is the contact area. The product \( g a S \varepsilon_d \) is nothing but the number of localized centers within the layer of a thickness \( a \) near the interface and the factor \( g a \rho^2 \varepsilon_d \) expresses the probability of finding a critical pair, i. e., a pair of nearly located hopping centers that dominates the time reversal reflection processes discussed above.

The above approach holds, as we have already mentioned, only if the resistance of the typical time reversal resistor (TRR) is much larger then that of the critical hopping resistor, \( R_k = (h / e^2 \gamma) \varepsilon^6 \), where \( \gamma \) is a dimensionless factor depending on the mechanism of electron-phonon interaction and \( \xi \) is the hopping exponent, i. e., with the exponential accuracy, as long as \( 4\Lambda > \xi \).

There are many realistic situations where the barrier strength, \( \Lambda \), is not too large; the Schottky barrier at the natural interface is certainly the case like that. Consequently, if \( \Lambda > 1, \) i. e., if the system is far from the metal-to-insulator transition point, the procedure of summation over the localized states should be modified. Namely, the choice of the pairs facilitating the charge transfer depends on the structure of the bonds connecting critical pairs to the rest of percolation cluster.

According to the above considerations the voltage drops mainly on the bond connecting percolation cluster in the HI the critical TRR for which the distance between its pair components is less than the correlation length, \( \xi \), of the backbone cluster. The incoherent electron transport can be ensured by a single one bond connecting the cluster to any of the TRR sites. Thus, the ratio \( G / G_n \) is the probability to find a TRR contacting the percolation cluster.

To estimate this probability let us consider the layer with the thickness of the typical hopping distance, \( r_h \), near the interface where all the bonds of the backbone cluster have necessarily have a site within this layer. The total number of states in this layer is \( g S \rho \varepsilon_h \) where \( \varepsilon_h = T \zeta \) is the width of the hopping band. This product can be estimated as \( (\beta / 8)(S / r_h)^3 \), where \( \beta \) is a numerical constant. For the case of Mott variable range hopping (VRH), \( \beta \approx 20 \).

The number of TRRs in this layer can be estimated as follows. Let us note first that the conserving energy, \( \varepsilon_s + \varepsilon_l + U_c \), from \( \delta \)-function in Eq. (8) is associated with the band given by the broadening \( \nu = v_0 \exp(-2r_d / a) \). Here \( r_d \) is distance to the nearest neighbor in HI. Indeed, the most natural source for the broadening of the resonance is coupling of the localized states. Secondly, since both electrons escape from the TRR through a single bond, the in-plane distance, \( r_d \), should not exceed the typical distance between the hopping sites, \( r_h = a \zeta / 2 < \xi \). Keeping the exponential accuracy we arrive at the following criterion that the resistance of TRR is less than the resistance of a typical hopping resistor:
\[
4\Lambda + \ln \left( \frac{T}{\nu_0} \right) \frac{\max(|\varepsilon_s|, |\varepsilon_l|)}{T} + \frac{2r_d}{a} + \frac{2(z_s + z_l)}{a} < \zeta. \tag{10}
\]

One may consider this equation as a generalization of the “connectivity criterion” to include the TRR. Here we deal with the independent variables \( \varepsilon_s, \varepsilon_l, r_d, z_s, z_l \) over which the averaging procedure should be done with an account of the restriction of Eq. (10). Thus the number of the relevant TRRs is
\[
8\pi^2 g^3 S \int d\varepsilon_l \int d\varepsilon_s \int_0^{r_s} d\varepsilon_d \int r_d dr_d \int d\varepsilon_d d\varepsilon_l \int_r^{r_h} \rho d\rho \times \Theta \left[ \max(|\varepsilon_s, |\varepsilon_l|) + \frac{2r_d}{a} + \frac{2(z_s + z_l)}{a} - \alpha \zeta \right]. \tag{11}
\]
where \( \alpha \equiv 1 - [4\Lambda + \ln(T / \nu_0)] / \zeta < 1 \). Let us now measure the energies in units of \( a \varepsilon_h \) and lengths in units of \( a r_h \), where \( \varepsilon_h \equiv T \zeta \). Again, the product \( g^2 \varepsilon_h \) can be estimated as \( \beta / 8 \), and we obtain the number of effective TRRs as \( \tilde{N}_A \sim A a^2 S / r_h^3 \), where
\[
A = 4\pi^2 (\beta / 8)^2 \int d\varepsilon_l \int d\varepsilon_s \int_0^e d\varepsilon_d \int d\varepsilon_d d\varepsilon_d d\varepsilon_l d\varepsilon_l d\rho \times \Theta \left[ \max(|\varepsilon_s, |\varepsilon_l|) + \eta_d + \eta_s + \eta_l - 1 \right] \approx 0.1. \tag{12}
\]
Consequently, \( \mathcal{G}/\mathcal{G}_n \sim A\alpha^7 \ll 1 \). One concludes that the difference between the “contact” resistances in normal and superconducting states is dominated by the contribution of TRRs. Since \( \mathcal{G}_N \approx R_h^{-1} S / L^2 \),

\[
\delta R \equiv \mathcal{G}_N^{-1} - \mathcal{G}_n^{-1} \approx -\mathcal{G}_n^{-1} = -R_h \left( L^2 / S A \alpha^7 \right). \quad (13)
\]

Note that the interface resistance is of the order of the resistance of HI layer with the thickness \( \sim L/(A\alpha^7) \gg L \). Since \( L \sim \alpha c^2 \), one concludes that for \( \zeta > 10 \) the interface resistance can be comparable or even exceed the hopping resistance if the thickness of the sample (or of the contact) is \( \lesssim 10 \mu m \).

The resistance estimated above can be experimentally measured as a magnetoresistance in magnetic fields higher than the critical field for superconductivity (similar effect for quasiparticle channel was studied in [3]).

We were implicitly assuming so far that the variable range hopping occurs according to the Mott’s law. This assumption certainly holds near the interface where the Coulomb gap is screened by a superconductor. However in the bulk of HI the Efros-Shklovskii (ES) law can become the dominant hopping mechanism. Then the value of \( \zeta \) in the connectivity criterion will be controlled by the Coulomb gap, \( \zeta \rightarrow \zeta_{cs} = (\beta_1 e^2 / \kappa aT)^{1/2} \), where \( \beta_1 \) is a numerical constant [13, 14]. In this case, it turns out that each bond of the ES backbone cluster finds some TRR ensuring charge transfer. Thus in the limiting case of a weak tunneling interface barrier the contact resistance will be the same for both normal metal and superconductor leads. This fact can be used to discriminate between Mott and Efros-Shklovskii laws in the situation when it is difficult to do so from temperature dependence.

Note that, in principle, the charge transfer involving double occupied localized states is possible. However, such a process would require an additional activation exponential factor, \( \propto e^{-U/T} \), where \( U \) is the on-site correlation energy. One can also consider processes where a double occupied center (so-called \( D^- \)-center) serves as an intermediate state for the phonon-assisted two-electron tunneling. This channel is unfavorable (at least in the case of a large interface barrier) because of the above-mentioned exponential factor and a small pre-exponential factor due to phonon-assisted tunneling. For the weak tunnel barrier the conductance is controlled by “typical” hopping sites. In this case \( D^- \) channel is suppressed either by the additional tunneling exponential, \( \propto e^{-4\kappa a/\alpha} \), or by a small probability to form a close triad of hopping sites. Therefore the \( D^- \) channel can be also neglected.

To conclude, we have developed a theory of the low-temperature charge transfer between a superconductor and a hopping insulator and calculated the interface resistance. This resistance is dominated by time reversal reflection processes involving localized states in the insulator. It is the time reversal reflection process that allows the low-temperature measurements of hopping transport utilizing superconducting electrodes in the experimental setups. In the Efros-Shklovskii VRH regime, the corresponding interface resistance is small as compared to the bulk hopping resistance and is nearly equal to the resistance at the interface between the HI and normal metal. On the contrary, in the Mott hopping regime (relevant, in particular, for 2D gated structures), the interface resistance grows much larger and becomes commensurate (or even exceeds) to the bulk hopping resistance. This effect is especially pronounced in the mesoscopic samples. The contribution from the interface resistance can be detected by application of the external magnetic field: the relatively weak magnetic field will drive the superconductor into the normal state, but will not affect the hopping transport eliminating thus time reversal reflection process. This effect holds even in the case where the interface contribution is less than the typical resistance of the hopping system itself.

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