RAEDEMACHER FUNCTIONS IN WEIGHTED SYMMETRIC SPACES

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Abstract

The closed span of Rademacher functions is investigated in the weighted spaces $X(w)$, where $X$ is a symmetric space on $[0,1]$ and $w$ is a positive measurable function on $[0,1]$. By using the notion and properties of the Rademacher multiplicator space of a symmetric space, we give a description of the weights $w$ for which the Rademacher orthogonal projection is bounded in $X(w)$.

1 Introduction

We recall that the Rademacher functions on $[0,1]$ are defined by $r_k(t) = \text{sign}(\sin 2^k \pi t)$ for every $t \in [0,1]$ and each $k \in \mathbb{N}$. It is well known that $\{r_k\}$ is an incomplete orthogonal system of independent random variables. This system plays a prominent role in the modern theory of Banach spaces and operators (see, e.g., [11], [12], [17] and [19]).

A classical result of Rodin and Semenov [20] states that the sequence $\{r_k\}$ is equivalent in a symmetric space $X$ to the unit vector basis in $\ell^2$, i.e.,

$$\|\sum_{k=1}^{\infty} a_k r_k\|_X \asymp \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}, \quad (a_k) \in \ell^2,$$

if and only if $G \subset X$, where $G$ is the closure of $L_\infty[0,1]$ in the Zygmund space $\text{Exp} L^2[0,1]$. When this condition is satisfied, the span $[r_k]$ of Rademacher functions is complemented in $X$ if and only if $X \subset G'$, where the Köthe dual space $G'$ to $G$ coincides (with equivalence of norms) with another well-known Zygmund space $L \log^{1/2} L[0,1]$. This was proved independently by Rodin and Semenov [21] and Lindenstrauss and Tzafriri [15, pp.138-138]. Moreover, the condition $G \subset X \subset G'$ (equivalently, complementability of $[r_k]$ in $X$) is equivalent to the boundedness in $X$ of the orthogonal projection

$$Pf(t) := \sum_{k=1}^{\infty} c_k(f) r_k(t),$$

where $c_k(f) := \int_0^1 f(u) r_k(u) \, du$, $k = 1, 2, \ldots$. The main purpose of this paper is to investigate the behaviour of Rademacher functions and of the respective projection $P$ in the weighted spaces $X(w)$ consisting of all measurable functions $f$ such that $fw \in X$ with the norm $\|f\|_{X(w)} := \|fw\|_X$. Here, $X$ is a symmetric space on $[0,1]$ and $w$ is a positive measurable function on $[0,1]$. We make use of the notion of the Rademacher multiplicator space $\mathcal{M}(X)$ of a symmetric space $X$, which originally arose from the study of vector measures and scalar functions integrable with respect to them (see [8] and [10]). For the first time a connection between the space $\mathcal{M}(X)$ and the behavior
of Rademacher functions in the weighted spaces \( X(w) \) was observed in \([4]\) when proving a weighted version of inequality \([1]\) (under more restrictive conditions in the case of \( L_p \)-spaces it was proved in \([23]\)).

To ensure that the operator \( P \) is well defined, we have to guarantee that the Rademacher functions belong both to \( X(w) \) and to its Köthe dual space \((X(w))' = X'(1/w)\). For this reason, in what follows we assume that

\[
L_\infty \subset X(w) \subset L_1. \tag{3}
\]

This assumption allows us to find necessary and sufficient conditions on the weight \( w \) under which the orthogonal projection \( P \) is bounded in the weighted space \( X(w) \). Moreover, extending above mentioned result of Rodin and Semenov from \([20]\) to the weighted symmetric spaces, we show that, in contrast to the symmetric spaces, the embedding \( X(w) \supseteq G \) is a stronger condition, in general, than equivalence of the sequence of Rademacher functions in \( X(w) \) to the unit vector basis in \( \ell_2 \).

In the final part of the paper, answering a question from \([10]\), we present a concrete example of a function \( f \in M(L_1) \), which does not belong to the symmetric kernel of the latter space.

## 2 Preliminaries

Let \( E \) be a Banach function lattice on \([0,1]\), i.e., if \( x \) and \( y \) are measurable a.e. finite functions on \([0,1]\) such that \( x \in E \) and \( |y| \leq |x| \), then \( y \in E \) and \( \|y\|_E \leq \|x\|_E \). The Köthe dual of \( E \) is the Banach function lattice \( E' \) of all functions \( y \) such that \( \int_0^1 |x(t)y(t)| \, dt < \infty \), for every \( x \in E \), with the norm

\[
\|y\|_E' := \sup \left\{ \int_0^1 x(t)y(t) \, dt : x \in E, \|x\|_E \leq 1 \right\}.
\]

\( E' \) is a subspace of the topological dual \( E^* \). If \( E \) is separable we have \( E' = E^* \). A Banach function lattice \( E \) has the Fatou property, if from \( 0 \leq x_n \searrow x \) a.e. on \([0,1]\) and \( \sup_{n \in \mathbb{N}} \|x_n\|_E < \infty \) it follows that \( x \in E \) and \( \|x_n\|_E \nearrow \|x\|_E \).

Suppose a Banach function lattice \( E \supseteq L_\infty \). By \( E_0 \) we will denote the closure of \( L_\infty \) in \( E \). Clearly, \( E_0 \) contains the absolutely continuous part of \( E \), that is, the set of all functions \( x \in E \) such that \( \lim_{m(A) \to 0} \|x \cdot \chi_A\|_E = 0 \). Here and next, \( m \) is the Lebesgue measure on \([0,1]\) and \( \chi_A \) is the characteristic function of a set \( A \subset [0,1] \).

Throughout the paper a symmetric (or rearrangement invariant) space \( X \) is a Banach space of classes of measurable functions on \([0,1]\) such that from the conditions \( y* \leq x* \) and \( x \in X \) it follows that \( y \in X \) and \( \|y\|_X \leq \|x\|_X \). Here, \( x* \) is the decreasing rearrangement of \( x \), that is, the right continuous inverse of its distribution function: \( n_x(\tau) = m\{t \in [0,1] : |x(t)| > \tau \} \) Functions \( x \) and \( y \) are said to be equimeasurable if \( n_x(\tau) = n_y(\tau) \), for all \( \tau > 0 \). The Köthe dual \( X' \) is a symmetric space whenever \( X \) is symmetric. In what follows we assume that \( X \) is isometric to a subspace of its second Köthe dual \( X'' := (X')' \). In particular, this holds if \( X \) is separable or it has the Fatou property. For every symmetric space \( X \) the following continuous embeddings hold: \( L_\infty \subset X \subset L_1 \).

If \( X \) is a symmetric space, \( X \neq L_\infty \), then \( X_0 \) is a separable symmetric space.

Important examples of symmetric spaces are Marcinkiewicz, Lorentz and Orlicz spaces. Let \( \varphi : [0,1] \to [0,\infty) \) be a quasi-concave function, that is, \( \varphi \) increases, \( \varphi(t)/t \) decreases and \( \varphi(0) = 0 \). The Marcinkiewicz space \( M(\varphi) \) is the space of all measurable functions \( x \) on \([0,1]\) for which the norm

\[
\|x\|_{M(\varphi)} = \sup_{0 < t \leq 1} \frac{\varphi(t)}{t} \int_0^t x^*(s) \, ds < \infty.
\]
If $\varphi : [0,1] \to [0, +\infty)$ is an increasing concave function, $\varphi(0) = 0$, then the Lorentz space $\Lambda(\varphi)$ consists of all measurable functions $x$ on $[0,1]$ such that

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(s) \, d\varphi(s) < \infty.$$  

For arbitrary increasing convex function $\varphi$ we have $\Lambda(\varphi)' = M(\tilde{\varphi})$ and $M(\varphi)' = \Lambda(\check{\varphi})$, where $\tilde{\varphi}(t) := t/\varphi(t)$ \[14\] Theorems II.5.2 and II.5.4.

Let $M$ be an Orlicz function, that is, an increasing convex function on $[0, \infty)$ with $M(0) = 0$. The norm of the Orlicz space $L_M$ is defined as follows

$$\|x\|_{L_M} = \inf \left\{ \lambda > 0 : \int_0^1 M\left(\frac{|x(s)|}{\lambda}\right) \, ds \leq 1 \right\}.$$  

In particular, if $M(u) = u^p$, $1 \leq p < \infty$, we have $L_M = L_p$ isometrically. Next, by $\|f\|_p$ we denote the norm $\|f\|_{L_p}$.

The fundamental function of a symmetric space $X$ is the function $\phi_X(t) := \|X_{[0,t]}\|_X$. In particular, we have $\phi_M(\varphi)(t) = \phi_M(\varphi)(t) = \varphi(t)$, and $\phi_M(t) = 1/M^{-1}(1/t)$, respectively. The Marcinkiewicz $M(\varphi)$ and Lorentz $\Lambda(\varphi)$ spaces are, respectively, the largest and the smallest symmetric spaces with the fundamental function $\varphi$, that is, if the fundamental function of a symmetric space $X$ is equal to $\varphi$, then $\Lambda(\varphi) \subset X \subset M(\varphi)$.

If $\psi$ is a positive function defined on $[0,1]$, then its lower and upper dilation indices are

$$\gamma_\psi := \lim_{t \to 0^+} \frac{\log \left( \sup_{0 < s \leq 1} \frac{\psi(st)}{\psi(s)} \right)}{\log t} \quad \text{and} \quad \delta_\psi := \lim_{t \to +\infty} \frac{\log \left( \sup_{0 < s \leq 1/t} \frac{\psi(st)}{\psi(s)} \right)}{\log t},$$

respectively. Always we have $0 \leq \gamma_\psi \leq \delta_\psi \leq 1$.

In the case when $\delta_\varphi < 1$ the norm in the Marcinkiewicz space $M(\varphi)$ satisfies the equivalence

$$\|x\|_{M(\varphi)} \asymp \sup_{0 < t \leq 1} \varphi(t)x^*(t).$$

[14] Theorem II.5.3. Here, and throughout the paper, $A \asymp B$ means that there exist constants $C > 0$ and $c > 0$ such that $cA \leq B \leq CA$.

The Orlicz spaces $L_{N_p}$, $p > 0$, where $N_p$ is an Orlicz function equivalent to the function $\exp(t^p) - 1$, will be of major importance in our study. Usually these are referred as the Zygmund spaces and denoted by $\exp L^p$. The fundamental function of $\exp L^p$ is equivalent to the function $\varphi_p(t) = \log^{-1/p}(c/t)$. Since $N_p(u)$ increases at infinity very rapidly, $\exp L^p$ coincides with the Marcinkiewicz space $M(\varphi_p)$ \[16\]. This, together with the equality $\delta_{\varphi_p} = 0 < 1$, gives

$$\|x\|_{\exp L^p} \asymp \sup_{0 < t \leq 1} x^*(t) \log^{-1/p}(c/t).$$

In particular, for every $x \in \exp L^p$ and $0 < t \leq 1$ we have

$$x^*(t) \leq C \|x\|_{\exp L^p} \log^{1/p}(c/t). \quad (4)$$

Hence, for a symmetric space $X$, the embedding $\exp L^p \subset X$ is equivalent to the condition $\log^{1/p}(c/t) \in X$.

Recall that the Rademacher functions are $r_k(t) := \text{sign} \sin(2^k \pi t)$, $t \in [0,1]$, $k \geq 1$. The famous Khintchine inequality \[13\] states that, for every $1 \leq p < \infty$, the sequence $\{r_k\}$ is equivalent in $L_p$.
Moreover, if \( X \) that Sym \((g)\), defined as follows
\[ g, \text{ arbitrary function which are bounded from the subspace } [r_M] \]
\( X \) is a symmetric space different from \( X \). The opposite situation is when the Rademacher multiplicator space \( M(X) \) is not symmetric. This result was extended in [3] to include all symmetric spaces such that the lower dilation index \( \gamma_M \) of their fundamental function \( \varphi_X \) is positive. This result motivated the study of the symmetric kernel Sym \((X)\) of the space \( M(X) \). The space Sym \((X)\) consists of all functions \( f \in M(X) \) such that an arbitrary function \( g, \text{ equimeasurable with } f, \) belongs to \( M(X) \) as well. The norm in Sym \((X)\) is defined as follows
\[ \|f\|_{\text{Sym}(X)} = \sup \{ \|f \cdot \sum_{k=1}^{\infty} a_k r_k\|_X : \|\sum_{k=1}^{\infty} a_k r_k\|_X \leq 1 \}, \]
\( M(X) \) can be viewed as the space of operators given by multiplication by a measurable function, which are bounded from the subspace \([r_k]\) in \( X \) into the whole space \( X \).

The Rademacher multiplicator space \( M(X) \) was firstly considered in [9], where it was shown that for a broad class of classical symmetric spaces \( X \) the space \( M(X) \) is not symmetric. This result was extended in [3] to include all symmetric spaces such that the lower dilation index \( \gamma_M \) of their fundamental function \( \varphi_X \) is positive. This result motivated the study of the symmetric kernel Sym \((X)\) of the space \( M(X) \). The space Sym \((X)\) consists of all functions \( f \in M(X) \) such that an arbitrary function \( g, \text{ equimeasurable with } f, \) belongs to \( M(X) \) as well. The norm in Sym \((X)\) is defined as follows
\[ \|f\|_{\text{Sym}(X)} = \sup \|g\|_{\text{Sym}(X)}, \]
where the supremum is taken over all \( g \) equimeasurable with \( f \). From the definition it follows that Sym \((X)\) is the largest symmetric space embedded into \( M(X) \) (see also [3, Proposition 2.4]). Moreover, if \( X \) is a symmetric space such that \( X'' \supset \text{Exp} L^2 \), then
\[ \|f\|_{\text{Sym}(X)} \asymp \|f^*(t) \log^{1/2}(e/t)\|_{X''} \]
(see [3, Proposition 3.1 and Corollary 3.2]). The opposite situation is when the Rademacher multiplicator space \( M(X) \) is symmetric. The simplest case of this situation is when \( M(X) = L_\infty \). It was shown in [3] that \( M(X) = L_\infty \) if and only if \( \log^{1/2}(e/t) \notin X_o \). Regarding the case when \( M(X) \) is a symmetric space different from \( L_\infty \) see the paper [5].

We will denote by \( \Delta_n^k \) the dyadic intervals of \([0,1]\), that is, \( \Delta_n^k = [(k-1)2^{-n}, k2^{-n}] \), where \( n = 0, 1, \ldots, k = 1, \ldots, 2^n \); we say that \( \Delta_n^k \) has rank \( n \). For any undefined notions we refer the reader to the monographs [7], [14], [15].

3 Rademacher sums in weighted spaces

First, we find necessary and sufficient conditions on the symmetric space \( X \), under which there is a weight \( w \) such that the sequence of Rademacher functions spans \( \ell_2 \) in \( X(w) \). We prove the following refinement of the nontrivial part of above mentioned Rodin–Semenov theorem.

\[ \frac{1}{\sqrt{2}} \|\{a_k\}\|_{\ell_2} \leq \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{1} \leq \|\{a_k\}\|_{\ell_2} \]

(see [22]), where \( \|\{a_k\}\|_{\ell_2} := (\sum_{k=1}^{\infty} |a_k|^2)^{1/2} \).

The Rademacher multiplicator space of a symmetric space \( X \) is the space \( M(X) \) of all measurable functions \( f : [0,1] \to \mathbb{R} \) such that \( f \cdot \sum_{k=1}^{\infty} a_k r_k \in X \), for every Rademacher sum \( \sum_{k=1}^{\infty} a_k r_k \in X \). It is a Banach function lattice on \([0,1]\) when endowed with the norm
\[ \|f\|_{M(X)} = \sup \{ \|f \cdot \sum_{k=1}^{\infty} a_k r_k\|_X : \|\sum_{k=1}^{\infty} a_k r_k\|_X \leq 1 \}. \]
Proposition 3.1. For every symmetric space $X$ the following conditions are equivalent:

(i) there exists a set $D \subset [0,1]$ of positive measure such that

$$\left\| \sum_{k=1}^{\infty} a_k r_k \cdot \chi_D \right\|_X \leq M \|a_k\|_{\ell_2},$$

for some $M > 0$ and arbitrary $(a_k) \in \ell_2$;

(ii) $X \supset G$.

Proof. Since implication $(ii) \Rightarrow (i)$ is an immediate consequence of the fact that the sequence $\{r_k\}$ spans $\ell_2$ in the space $G$ (see [18] or [24, Theorem V.8.16]), we need to prove only that $(i)$ implies $(ii)$.

Assume that $(i)$ holds. By Lebesgue’s density theorem, for sufficiently large $m \in \mathbb{N}$, we can find a dyadic interval $\Delta := \Delta_m = [(k_0 - 1)2^{-m}, k_02^{-m}]$ such that

$$2^{-m} = m(\Delta) \geq m(\Delta \cap D) > 2^{-m-1}.$$

Let us consider the set $E = \bigcup_{k=1}^{2m} E_m^k$, where $E_m^k$ is obtained by translating the set $\Delta \cap D$ to the interval $\Delta_m^k$, $k = 1, 2, \ldots, 2^m$, (in particular, $E_m^m = \Delta \cap D$). Denote $f_i = r_i \cdot \chi_E, i \in \mathbb{N}$. It follows easily that $|f_i(t)| \leq 1, t \in [0,1], \|f_i\|_2 \geq 1/\sqrt{2}$, and $f_i \to 0$ weakly in $L_2[0,1]$ when $i \to \infty$.

Therefore, by [11, Theorem 5], the sequence $(f_i)_{i=1}^{\infty}$ contains a subsequence $(f_{i_j})$, which is equivalent in distribution to the Rademacher system. The last means that there exists a constant $C > 0$ such that

$$C^{-1}m \left\{ t \in [0,1] : \left| \sum_{j=1}^{l} a_j r_j(t) \right| > Cz \right\} \leq m \left\{ t \in [0,1] : \left| \sum_{j=1}^{l} a_j f_{i_j}(t) \right| > z \right\} \leq Cm \left\{ t \in [0,1] : \left| \sum_{j=1}^{l} a_j r_j(t) \right| > C^{-1}z \right\}$$

for all $l \in \mathbb{N}, a_j \in \mathbb{R},$ and $z > 0$. Hence, by the definition of $r_j$ and $f_j$, for every $n \in \mathbb{N}$ we have

$$C^{-1}m \left\{ t \in [0,1] : \left| \sum_{j=m+1}^{m+n} r_j(t)\chi_{[0,2^{-m}]}(t) \right| > Cz \right\} \leq m \left\{ t \in [0,1] : \left| \sum_{j=m+1}^{m+n} f_{i_j}(t)\chi_{\Delta}(t) \right| > z \right\} \leq Cm \left\{ t \in [0,1] : \left| \sum_{j=m+1}^{m+n} r_j(t)\chi_{[0,2^{-m}]}(t) \right| > C^{-1}z \right\},$$

whence

$$\left\| \sum_{j=m+1}^{m+n} r_{i_j} \chi_{\Delta \cap D} \right\|_X \geq \alpha \left\| \sum_{j=m+1}^{m+n} r_j \chi_{[0,2^{-m}]} \right\|_X,$$

where $\alpha > 0$ depends only on the constant $C$ and on the space $X$.

Now, assume that $(ii)$ fails, i.e., $X \not\supset G$. Then, by [11] inequality (2) in the proof of Theorem 1, there exists a constant $\beta > 0$, depending only on $X$, such that for every $m \geq 0$ there exists $n_0 \geq 1$
such that, if \( n \geq n_0 \) and \( \Delta \) is an arbitrary dyadic interval of rank \( m \), we have

\[
\left\| \chi_\Delta \sum_{i=m+1}^{m+n} r_i \right\|_X \geq \beta \left\| \sum_{i=1}^{n} r_i \right\|_X.
\]

From this inequality with \( \Delta = [0, 2^{-m}] \) and inequality (7) it follows that, for \( n \) large enough,

\[
\left\| \sum_{j=m+1}^{m+n} r_i \chi_D \right\|_X \geq \alpha \beta \left\| \sum_{j=1}^{n} r_j \right\|_X.
\]

Combining the latter inequality together with (6) we deduce

\[
\frac{1}{\sqrt{n}} \left\| \sum_{j=1}^{n} r_j \right\|_X \leq \frac{M}{\alpha \beta}
\]

for all \( n \in \mathbb{N} \) large enough. At the same time, as it follows from the proof of Rodin–Semenov theorem [20], the last condition is equivalent to the embedding \( X \supset G \). This contradiction concludes the proof.

**Corollary 3.1.** Suppose \( X \) is a symmetric space. Then, \( X \supset G \) if and only if there exists a weight \( w \) such that the sequence \( \{r_k\} \) spans \( \ell_2 \) in \( X(w) \).

**Proof.** If \( \{r_k\} \) spans \( \ell_2 \) in \( X(w) \) for some weight \( w \), we have

\[
\left\| \sum_{k=1}^{\infty} a_k r_k \cdot w \right\|_X \leq C \|(a_k)\|_\ell_2.
\]

Since \( w(t) > 0 \) a.e. on \( [0, 1] \), there is a set \( D \subset [0, 1] \) of positive measure such that inequality (6) holds for some \( M > 0 \) and arbitrary \( (a_k) \in \ell_2 \). Applying Proposition 3.1 we obtain that \( X \supset G \). The converse is obvious, and so the proof is completed.

Corollary 3.1 shows the necessity of the condition \( X \supset G \) in the following main result of this part of the paper.

**Theorem 3.1.** Let \( X \) be a symmetric space such that \( X \supset G \) and let a positive measurable function \( w \) on \( [0, 1] \) satisfy condition (8). Then we have

(i) The sequence \( \{r_k\} \) spans \( \ell_2 \) in \( X(w) \) if and only if \( w \in \mathcal{M}(X) \), where \( \mathcal{M}(X) \) is the Rademacher multiplicator space of \( X \);

(ii) \( X(w) \supset G \) if and only if \( w \in \text{Sym}(X) \), where \( \text{Sym}(X) \) is the symmetric kernel of \( \mathcal{M}(X) \).

The part (i) of this theorem was actually obtained in [6, p. 240]. However, for the reader’s convenience we provide here its proof. But we begin with the following technical result, which will be needed us to prove the part (ii).

**Lemma 3.1.** Let \( Y \) be a symmetric space and \( w \) be a positive measurable function on \( [0, 1] \). Suppose the weighted function lattice \( Y(w^*) \) contains an unbounded decreasing positive function \( a \) on \( (0, 1] \). Then \( (Y(w))_o = Y_o(w) \).
Proof. Since \((wa)^*(t) \leq w^*(t/2)a(t/2), 0 < t \leq 1\), [14, §II.2] and, by assumption, \(w^*a \in Y\), we have \(wa \in Y\). Equivalently, \(a \in Y(w)\).

Let \(y \in (Y(w))_0\). By definition, there is a sequence \(\{y_k\} \subset L_\infty\) such that

\[
\lim_{k \to \infty} \|y_k - y\|_Y = 0.
\] (8)

Show that \(y_kw \in Y_0\) for every \(k \in \mathbb{N}\).

Since \(a\) decreases, for arbitrary \(A \subset [0, 1]\) and every (fixed) \(k \in \mathbb{N}\) we have

\[
\|y_kw\chi_A\|_Y \leq \|y_k\|_\infty \|w^*\chi_{(0, m(A))}\|_Y \leq \frac{\|y_k\|_\infty}{a(m(A))}\|w^*a\|_Y.
\]

Hence, \(y_kw \in (Y(w))_0\), \(k \in \mathbb{N}\). Since \(\|y_k/w - y\|_{Y(w)} = \|y_k - yw\|_Y\), from \((8)\) it follows that \(y \in (Y(w))_0\).

To prove the opposite embedding, assume that \(y \in Y_0\). Then

\[
\lim_{k \to \infty} \|y_k - yw\|_Y = 0
\] (9)

for some sequence \(\{y_k\} \subset L_\infty\). From hypothesis of lemma it follows that \(Y \neq L_\infty\). Therefore, for arbitrary \(A \subset [0, 1]\) and each \(k \in \mathbb{N}\)

\[
\|y_kw\chi_A\|_{Y(w)} = \|y_k\chi_A\|_Y \to 0 \text{ as } m(A) \to 0.
\]

Hence, \(y_k/w \in (Y(w))_0\), \(k \in \mathbb{N}\). Since \(\|y_k/w - y\|_{Y(w)} = \|y_k - yw\|_Y\), from \((9)\) it follows that \(y \in (Y(w))_0\).

Proof of Theorem 2.14 (i) Since \(X \supset G\), equivalence \((1)\) holds. At first, assume that \(w \in M(X)\). Then, by definition of the norm in \(M(X)\), we have

\[
\|w\|_{M(X)} \asymp \sup \left\{ \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X : \|(a_k)\|_{\ell_2} \leq 1 \right\}.
\] (10)

Therefore,

\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X(w)} = \left\| w \cdot \sum_{k=1}^{\infty} a_k r_k \right\|_X \leq \|w\|_{M(X)} \|(a_k)\|_{\ell_2}
\]

for every \((a_k) \in \ell_2\). On the other hand, from embeddings \((3)\) and inequality \((5)\) it follows that

\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X(w)} \geq c \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_1 \geq \frac{c}{\sqrt{2}} \|(a_k)\|_{\ell_2}.
\]

As a result we deduce that \(\{r_k\}\) spans \(\ell_2\) in \(X(w)\).

Conversely, if

\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X(w)} \asymp \|(a_k)\|_{\ell_2},
\]

from \((10)\) we obtain that \(\|w\|_{M(X)} < \infty\), i.e., \(w \in M(X)\).
(ii) Assume that \( w \in \text{Sym}(X) \). Then, taking into account the properties of the symmetric kernel \( \text{Sym}(X) \) (see Preliminaries or [5, Corollary 3.2]) we have \( w^*(t) \log^{1/2}(e/t) \in X'' \). Let us prove that 
\[
\text{Exp} L_2 \subset X''(w). \tag{11}
\]
Given \( x \in \text{Exp} L_2 \), by [7, Theorem 2.7.5], there exists a measure-preserving transformation \( \sigma \) of \( (0,1] \) such that \( |x(t)| = x^*(\sigma(t)) \). Applying inequality (4) and a well-known property of the rearrangement of a measurable function (see e.g. [14, §II.2]), we have
\[
(wx)^*(t) = (wx^*(\sigma))^*(t) \leq C \left( w \log^{1/2}(e/\sigma) \right)^*(t) \leq Cw^*(t/2) \log^{1/2}(2e/t), \quad 0 < t \leq 1.
\]
Therefore, \( wx \in X'' \) or, equivalently, \( x \in X''(w) \), and (11) is proved. Hence, \( G = (\text{Exp} L_2)_o \subset (X''(w))_o \). Since \( \log^{1/2}(e/t) \in X''(w) \), we can apply Lemma 3.1 and so, by [2, Lemma 3.3],
\[
G \subset (X''(w))_o = X_o(w) \subset X(w).
\]

Now, let \( X(w) \supseteq G \). We show that \( X(w^*) \supseteq G \).
In fact, let \( \tau \) be a measure-preserving transformation of \( (0,1] \) such that \( w(t) = w^*(\tau(t)) \) [7, Theorem 2.7.5]. Suppose \( x \in G \). Since \( x(\tau) \) and \( x \) are equimesurable functions, we have \( x(\tau) \in G \) and \( \|x(\tau)\|_G = \|x\|_G \). Therefore,
\[
\|x(\tau)w^*(\tau)\|_X = \|x(\tau)w\|_X \leq C\|x\|_G.
\]
Then, \( \|x(\tau)w^*(\tau)\|_X = \|xw^*\|_X \), because \( X \) is a symmetric space, and from the preceding inequality we infer that \( \|xw^*\|_X \leq C\|x\|_G \). Thus, \( x \in X(w^*) \), and the embedding \( X(w^*) \supseteq G \) is proved. Passing to the second Kōthe dual spaces, we obtain: \( X''(w^*) \supseteq G'' = \text{Exp} L^2 \). Hence, \( \log^{1/2}(e/t) \in X''(w^*) \) or, equivalently, \( w \in \text{Sym}(X) \) (as above, see Preliminaries or [5, Corollary 3.2]), and the proof is complete.

By Rodin-Semenov theorem [20], the sequence \( \{r_k\} \) is equivalent in a symmetric space \( X \) to the unit vector basis in \( \ell_2 \) if and only if \( X \supseteq G \). In contrast to that from Theorem 3.1 we immediately deduce the following result.

**Corollary 3.2.** Suppose \( X \) is a symmetric space such that \( \text{Sym}(X) \neq \mathcal{M}(X) \). Then, for every \( w \in \mathcal{M}(X) \setminus \text{Sym}(X) \) the Rademacher functions span \( \ell_2 \) in \( X(w) \) but \( X(w) \not\supseteq G \).

By [3, Theorem 2.1], \( \text{Sym}(X) \neq \mathcal{M}(X) \) (and therefore there is \( w \in \mathcal{M}(X) \setminus \text{Sym}(X) \)) whenever the lower dilation index of the fundamental function \( \phi_X \) is positive. In particular, it is fulfilled for \( L_p \)-spaces, \( 1 \leq p < \infty \). The condition \( \gamma_{\phi_X} > 0 \) means that the space \( X \) is situated “far” from the minimal symmetric space \( L_\infty \). Now, consider the opposite case when a symmetric space is “close” to \( L_\infty \). Then the Rademacher multiplicator space \( \mathcal{M}(X) \) may be symmetric (equivalently, it coincides with its symmetric kernel). Since the space \( \text{Sym}(X) \) has an explicit description (see Preliminaries), in this case we are able to state a sharper result. For simplicity, let us consider only Lorentz and Marcinkiewicz spaces (for more general results of such a sort see [3]).

Recall [5] that a function \( \varphi(t) \) defined on \( [0,1] \) satisfies the \( \Delta^2 \)-condition (briefly, \( \varphi \in \Delta^2 \)) if it is nonnegative, increasing, concave, and there exists \( C > 0 \) such that \( \varphi(t) \leq C \cdot \varphi(t^2) \) for all \( 0 < t \leq 1 \). By [5, Corollary 3.5], if \( \varphi \in \Delta^2 \), then \( \mathcal{M}(\Lambda(\varphi)) = \text{Sym}(\Lambda(\varphi)) \) and \( \mathcal{M}(M(\varphi)) = \text{Sym}(M(\varphi)) \). Moreover, it is known [3, Example 2.15 and Theorem 4.1] that \( \text{Sym}(\Lambda(\varphi)) = \Lambda(\psi) \) (resp. \( \text{Sym}(M(\varphi)) = M(\psi) \)), where \( \psi(t) = \varphi'(t) \log^{1/2}(e/t) \), whenever \( \log^{1/2}(e/t) \in \Lambda(\varphi) \) (resp. \( \log^{1/2}(e/t) \in M(\varphi) \)). Therefore, we get
Corollary 3.3. Let \( \varphi \in \Delta^2 \) and \( \log^{1/2}(e/t) \in \Lambda(\varphi) \) (resp. \( \log^{1/2}(e/t) \in M(\varphi) \)). If \( w \) is a positive measurable function on \([0,1]\) satisfying condition (3), then the sequence \( \{r_k\} \) is equivalent in the space \( \Lambda(\varphi)(w) \) (resp. \( M(\varphi)(w) \)) to the unit vector basis in \( \ell_2 \) if and only if \( w \in \Lambda(\psi) \) (resp. \( w \in M(\psi) \)), where \( \psi'(t) = \varphi'(t) \log^{1/2}(e/t) \).

In particular, if \( 0 < p \leq 2 \), the sequence \( \{r_k\} \) is equivalent in the Zygmund space \( \text{Exp} L^p(w) \) to the unit vector basis in \( \ell_2 \) if and only if \( w \in \text{Exp} L^q \), where \( q = 2p/(2 - p) \) (here, we set \( \text{Exp} L^\infty = L^\infty \)).

4 Rademacher orthogonal projection in weighted spaces

Proposition 4.1. Let \( E \) be a Banach function lattice on \([0,1]\) that is isometrically embedded into \( E'' \), \( L^\infty \subseteq E \subseteq L^1 \). Then the projection \( P \) defined by (2) is bounded in \( E \) if and only if there are constants \( C_1 \) and \( C_2 \) such that for all \( a = (a_k) \in \ell_2 \)

\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_E \leq C_1 \|a\|_{\ell_2} \tag{12}
\]

and

\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{E'} \leq C_2 \|a\|_{\ell_2}. \tag{13}
\]

Proof. Firstly, assume that inequalities (12) and (13) hold. Then, denoting, as above, \( c_k(f) := \int_0^1 f(u)r_k(u) \, du \), \( k = 1, 2, \ldots \), for every \( n \in \mathbb{N} \), by (13), we have

\[
\sum_{k=1}^{n} c_k(f)^2 = \int_0^1 f(u) \sum_{k=1}^{n} c_k(f)r_k(u) \, du \leq \|f\|_E \left\| \sum_{k=1}^{n} c_k(f)r_k \right\|_{E'} \leq C_2 \|f\|_E \left( \sum_{k=1}^{n} c_k(f)^2 \right)^{1/2},
\]

whence

\[
\left( \sum_{k=1}^{\infty} c_k(f)^2 \right)^{1/2} \leq C_2 \|f\|_E, \quad f \in E.
\]

Therefore, by (12), we obtain

\[
\|Pf\|_E \leq C_1 \left( \sum_{k=1}^{\infty} c_k(f)^2 \right)^{1/2} \leq C_1 C_2 \|f\|_E
\]

for all \( f \in E \).

Conversely, suppose that the projection \( P \) is bounded in \( E \). Let us consider the following sequence of finite dimensional operators

\[
P_n f(t) := \sum_{k=1}^{n} c_k(f)r_k(t), \quad n \in \mathbb{N}.
\]

Clearly, \( P_n \) is bounded in \( E \) for every \( n \in \mathbb{N} \). Furthermore, by assumption, the series \( \sum_{k=1}^{\infty} c_k(f)r_k \) converges in \( E \) for each \( f \in E \). Therefore, by the Uniform Boundedness Principle,

\[
\|P_n\|_{E \to E} \leq B, \quad n \in \mathbb{N}. \tag{14}
\]
Moreover, since $L_\infty \subset E \subset L_1$, then $L_\infty \subset E' \subset L_1$ as well, and hence, by the $L_1$-Khintchine inequality \[5\],
\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{E'} \geq c \|a\|_{\ell_2} \quad \text{and} \quad \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{E} \geq c \|a\|_{\ell_2}.
\]
Therefore, for all $f \in E$, $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$, we have
\[
\int_0^1 f(t) \cdot \sum_{k=1}^{n} a_k r_k(t) \, dt = \sum_{k=1}^{n} a_k c_k(f) \leq \|a\|_2 \left( \sum_{k=1}^{n} c_k(f)^2 \right)^{1/2} \leq c^{-1} \|a\|_{\ell_2} \cdot \|P_n f\|_E \leq Bc^{-1} \|a\|_{\ell_2} \cdot \|f\|_E.
\]
Taking the supremum over all $f \in E$, $\|f\|_E \leq 1$, we get
\[
\left\| \sum_{k=1}^{n} a_k r_k \right\|_{E'} \leq Bc^{-1} \|a\|_{\ell_2}, \quad n \in \mathbb{N}.
\]
Applying the latter inequality to Rademacher sums $\sum_{k=n}^{m} a_k r_k$, $1 \leq n < m$, with $a = \{a_k\}_{k=1}^{\infty} \in \ell_2$, we deduce that the series $\sum_{k=1}^{\infty} a_k r_k$ converges in the space $E'$ and
\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{E'} \leq Bc^{-1} \|a\|_{\ell_2}.
\]
Thus, \[13\] is proved. Let us prove similar inequality for $E$.

By Fubini theorem and \[14\], for arbitrary $f \in E$, $g \in E'$ and every $n \in \mathbb{N}$ we have
\[
\int_0^1 f(u) \cdot \sum_{k=1}^{n} c_k(g) r_k(u) \, du = \int_0^1 g(t) \cdot \sum_{k=1}^{n} c_k(f) r_k(t) \, dt \leq \|P_n f\|_E \|g\|_{E'} \leq B \|f\|_E \|g\|_{E'},
\]
whence
\[
\left\| \sum_{k=1}^{n} c_k(g) r_k \right\|_{E'} \leq B \|g\|_{E'}, \quad n \in \mathbb{N}.
\]
Applying this inequality instead of \[14\], as above, we get
\[
\left\| \sum_{k=1}^{n} a_k r_k \right\|_{E''} \leq Bc^{-1} \|a\|_{\ell_2}.
\]
Since $L_\infty \subset E$ and $E$ is isometrically embedded into $E''$, from the last inequality it follows that
\[
\left\| \sum_{k=1}^{n} a_k r_k \right\|_E \leq Bc^{-1} \|a\|_{\ell_2}
\]
for all $n \in \mathbb{N}$. Hence, if $a = \{a_k\}_{k=1}^{\infty} \in \ell_2$, the series $\sum_{k=1}^{\infty} a_k r_k$ converges in $E$ and
\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_E \leq Bc^{-1} \|a\|_{\ell_2}.
\]
Thus, inequality \[12\] holds, and the proof is complete.
From Proposition 4.1 Corollary 3.1 and Theorem 3.1 we obtain the following results.

**Theorem 4.1.** Let a symmetric space $X$ and a positive measurable function $w$ on $[0,1]$ satisfy condition (3). Then, the projection $P$ defined by (2) is bounded in $X(w)$ if and only if $G \subset X \subset G'$, $w \in \mathcal{M}(X)$ and $1/w \in \mathcal{M}(X')$.

In particular, $P$ is bounded in $X(w)$ whenever $w^*(t) \log^{1/2}(e/t) \in X''$ and $(1/w)^*(t) \log^{1/2}(e/t) \in X'$.

As above, the result can be somewhat refined for Lorentz and Marcinkiewicz spaces whose fundamental function satisfies the $\Delta^2$-condition.

**Corollary 4.1.** Let $\varphi \in \Delta^2$ and let $w$ be a positive measurable function on $[0,1]$ satisfying condition (3) for $X = \Lambda(\varphi)$ (resp. $X = M(\varphi)$). Then the projection $P$ defined by (2) is bounded in $\Lambda(\varphi)(w)$ (resp. $M(\varphi)(w)$) if and only if $G \subset \Lambda(\varphi) \subset G'$, $w \in \Lambda(\psi)$ and $1/w \in \mathcal{M}(M(\varphi))$ (resp. $G \subset M(\varphi) \subset G'$, $w \in M(\psi)$ and $1/w \in \mathcal{M}(\Lambda(\varphi)))$, where $\psi'(t) = \varphi'(t) \log^{1/2}(e/t)$ and $\varphi(t) = t/\varphi(t)$.

**Remark 4.1.** It is easy to see that the orthogonal projection $P$ is bounded in the space $X(w)$ if and only if the projection

$$P_w f(t) := \sum_{k=1}^{\infty} \int_0^1 f(s) r_k(s) \frac{ds}{w(s)} \cdot r_k(t) w(t), \quad 0 \leq t \leq 1,$$

(on the subspace $[r_kw]$) is bounded in $X$.

5 Example of a function from $\mathcal{M}(L_1) \setminus \text{Sym}(L_1)$

Answering a question from [10], we present here a concrete example of a function $f \in \mathcal{M}(L_1)$, which does not belong to the symmetric kernel $\text{Sym}(L_1)$, that is,

$$\int_0^1 f^*(t) \log^{1/2}(e/t) \, dt = \infty.$$  

Since the latter space is symmetric, it is sufficient to find a function $f \in \mathcal{M}(L_1)$, for which there exists a function $g \notin \mathcal{M}(L_1)$ equimeasurable with $f$. We will look for $f$ and $g$ in the form

$$f = \sum_{k=1}^{\infty} \alpha_k \chi_{B_k}, \quad g = \sum_{k=1}^{\infty} \alpha_k \chi_{D_k},$$

(15)

where $\{B_k\}$ and $\{D_k\}$ are sequences of pairwise disjoint subsets of $[0,1]$, $m(B_k) = m(D_k)$, $\alpha_k \in \mathbb{R}$, $k = 1, 2, \ldots$. Next, we will make use of some ideas of the paper [9].

Let $n = 2^m$ with $m \in \mathbb{N}$ and let $J$ be a subset of $\{1, 2, \ldots, 2^n\}$ with cardinality $n$. We define the set $A = \bigcup_{j \in J} \Delta_n^j$ associated with $J$ (as above, $\Delta_n^j$ are the dyadic intervals of $[0,1]$). Clearly, $m(A) = n2^{-n}$.

For arbitrary sequence $(b_i) \in \ell_2$ we have

$$\left\| \chi_A \sum_{i=1}^{n} b_i r_i \right\|_1 \leq \left\| \chi_A \sum_{i=1}^{n} b_i r_i \right\|_1 + \left\| \chi_A \sum_{i=n+1}^{\infty} b_i r_i \right\|_1.$$  

(16)
Firstly, we estimate the tail term from the right hand side of this inequality. It is easy to see that the functions
\[ \chi_A(t) \cdot \sum_{i=n+1}^{\infty} b_i r_i(t) \quad \text{and} \quad \chi_{[0,n2^{-n}]}(t) \cdot \sum_{i=n+1}^{\infty} b_i r_i(t) \]
are equimeasurable on \([0,1]\) and
\[
\chi_{[0,n2^{-n}]}(t) \sum_{i=n+1}^{\infty} b_i r_i(t) = \sum_{i=n+1}^{\infty} b_i r_{i+m-n}(n2^{-n}t), \quad 0 < t \leq 1.
\]
Therefore,
\[
\left\| \chi_A \sum_{i=n+1}^{\infty} b_i r_i \right\|_1 = \left\| \chi_{[0,n2^{-n}]} \sum_{i=n+1}^{\infty} b_i r_i \right\|_1 = n2^{-n} \left\| \sum_{i=n+1}^{\infty} b_i r_{i+m-n} \right\|_1 \leq n2^{-n} \left( \sum_{i=n+1}^{\infty} b_i^2 \right)^{1/2}. \tag{17}
\]

Now, choosing a set \(A\) in a special way, estimate the first term from the right hand side of (16). Denote by \(c_{ij}^n\) the value of the function \(r_i, i = 1, 2, \ldots, n,\) on the interval \(\Delta_{j,n}^n, 1 \leq j \leq 2^n.\) Since \(n = 2^m,\) we can find a set \(J_1(n) \subset \{1, 2, \ldots, 2^n\}, \) \(\text{card } J_1(n) = n,\) such that the \(n \times n\) matrix \(n^{-1/2} \cdot (c_{ij}^n)_{1 \leq i \leq n, j \in J_1(n)}\) is orthogonal. Then, if \(c_j := n^{-1/2} \sum_{i=1}^{n} c_{ij}^n b_i, j \in J_1(n),\) we have \(\|(c_j)_{j \in J_1(n)}\|_2 = \|(b_i)_{i=1}^{n}\|_2.\) Therefore, setting \(B(n) := \bigcup_{j \in J_1(n)} \Delta_{j,n}^n,\) we obtain
\[
\left\| \chi_{B(n)} \sum_{i=1}^{n} b_i r_i \right\|_1 = \left\| \sum_{j \in J_1(n)} \left( \sum_{i=1}^{n} b_i r_i \right) \chi_{\Delta_{j,n}^n} \right\|_1 = \left\| \sum_{j \in J_1(n)} c_j \chi_{\Delta_{j,n}^n} \right\|_1 = n^{1/2} \left\| \sum_{j \in J_1(n)} c_j \chi_{\Delta_{j,n}^n} \right\|_1 = n^{1/2} n^{2^{-n}} \|b_i\|_{i=1}^{n} \|_2.
\]
Combining this inequality with (16), (17) for \(A = B(n)\) and (15), by definition of the norm in the space \(\mathcal{M}(L_1),\) we have
\[
\left\| \chi_{B(n)} \right\|_{\mathcal{M}(L_1)} \leq 2 \sqrt{2} n 2^{-n}. \tag{18}
\]
Let \(\{n_k\}_{k=1}^{\infty}\) be an increasing sequence of positive integers, \(n_k = 2^{m_k}, m_k \in \mathbb{N},\) satisfying the condition
\[
n_k^{1/8} \geq 2^{n_1 + \cdots + n_{k-1}}, \quad k = 2, 3, \ldots \tag{19}
\]
At first, we construct a sequence of sets \(\{B_k\}.\) Setting \(J_1^k := J_1(n_k)\) and \(B_1 := B(n_1),\) in view of (18) we have
\[
\left\| \chi_{B_k} \right\|_{\mathcal{M}(L_1)} \leq 2 \sqrt{2} n_k 2^{-n_1}.
\]
To define \(B_2,\) we take for \(I_1\) any interval \(\Delta_{j_1}^1,\) such that \(j \notin J_1^1.\) Now, we can choose a set \(J_1^2 \subset \{1, 2, \ldots, 2^{n_1+n_2}\}\) satisfying the conditions: \(\text{card } J_1^2 = n_2, \Delta_{n_1+n_2}^2 \subset I_1\) for every \(j \in J_1^2\) and the \(n_2 \times n_2\) matrix \(n_2^{-1/2} \cdot (c_{ij}^{n_1+n_2})_{n_1 < i \leq n_1+n_2, j \in J_1^2}\) is orthogonal. We set \(B_2 := \bigcup_{j \in J_1^2} \Delta_{j_1}^{n_1+n_2} ,\) Clearly, \(m(B_2) = n_2 2^{-(n_1+n_2)}\) and \(B_1 \cap B_2 = \emptyset,\) because of \(B_2 \subset I_1.\) As in the case of \(B(n)\) we have
\[
\left\| \chi_{B_2} \sum_{i=1}^{n_2} b_i r_i \right\|_1 = \left\| \sum_{j \in J_1^2} \left( \sum_{i=1}^{n_2} b_i r_i \right) \chi_{\Delta_{j_1}^{n_1+n_2}} \right\|_1 \leq \left\| \sum_{j \in J_1^2} \left( \sum_{i=1}^{n_1} b_i r_i \right) \chi_{\Delta_{j_1}^{n_1+n_2}} \right\|_1 + \left\| \sum_{j \in J_1^2} \left( \sum_{i=1}^{n_2} b_i r_i \right) \chi_{\Delta_{j_1}^{n_1+n_2}} \right\|_1 \leq \left( n_1^{1/2} + 1 \right) n_2 2^{-(n_1+n_2)} \|b_i\|_{i=1}^{n_1+n_2} \|_2 \leq n_2 2^{-n_2} \|b_i\|_{i=1}^{n_1+n_2} \|_2.\]
Therefore, from (16), (17) and (5) it follows that
\[
\|\chi_{B_2}\|_{\mathcal{M}(L_1)} \leq \sqrt{2} \left((n_1 + n_2)2^{-(n_1+n_2)} + n_22^{-n_2}\right) \leq 2\sqrt{2}n_22^{-n_2}.
\]

Proceeding in the same way, we get a sequence \( \{B_k\} \) of pairwise disjoint subsets of \([0, 1]\) such that \( m(B_k) = n_2^{-(n_1+\ldots+n_k)} \) and
\[
\|\chi_{B_k}\|_{\mathcal{M}(L_1)} \leq 2\sqrt{2}n_22^{-n_k}, \ k = 1, 2, \ldots
\]

Now, define the sets \( D_k, \ k = 1, 2, \ldots \). Select a set \( J^2_2 \subset \{1, 2, \ldots, 2^{n_1}\} \), \( \text{card} \ J^2_2 = n_1 \), such that each column of the \( n_1 \times n_1 \) matrix \( (\varepsilon_{ij}^{n_1})_{1 \leq i \leq n_1, j \in J^2_2} \) has exactly one entry equal to \(-1\) and the rest are equal to \(1\). Setting \( D_1 := \bigcup_{j \in J^2_2} \Delta^j_{n_1} \), we have \( m(D_1) = n_12^{-n_1} \). Furthermore, from the inequality \( \|n_1^{-1/2} \sum_{i=1}^{n_1} r_i\|_1 \leq 1 \) (see (5)) and the definition of \( D_1 \) it follows that
\[
\|\chi_{D_1}\|_{\mathcal{M}(L_1)} \geq \left\| \sum_{j \in J^2_2} \left(n_1^{-1/2} \sum_{i=1}^{n_1} r_i\right) \chi_{\Delta^j_{n_1}} \right\|_1
\]
\[
= \left\| \sum_{j \in J^2_1} \left(n_1^{-1/2} \sum_{i=1}^{n_1} \varepsilon_{ij}^{n_1}\right) \chi_{\Delta^j_{n_1}} \right\|_1
\]
\[
= (n_1^{1/2} - 2n_1^{1/2})n_12^{-n_1} \geq \frac{1}{2}n_1^{3/2}2^{-n_1}
\]
if \( n_1 \) is large enough.

Similarly, we can define the set \( D_2 \). Let \( I_2 \) be any interval \( \Delta^j_{n_1} \) with \( j \notin J^1_2 \). Choose the set \( J^2_2 \subset \{1, 2, \ldots, 2^{n_1+n_2}\} \) such that \( \text{card} \ J^2_2 = n_2 \), \( \Delta^j_{n_1+n_2} \subset I_2 \) for every \( j \in J^2_2 \) and each column of the \( n_2 \times n_2 \) matrix \( (\varepsilon_{ij}^{n_1+n_2})_{n_1 \leq i \leq n_1+n_2, j \in J^2_2} \) has exactly one entry equal to \(-1\) and the rest are equal to \(1\). Then, if \( D_2 := \bigcup_{j \in J^2_2} \Delta^j_{n_1+n_2} \), then \( m(D_2) = n_22^{-(n_1+n_2)} \) and \( D_1 \cap D_2 = \emptyset \). Moreover, we have
\[
\|\chi_{D_2}\|_{\mathcal{M}(L_1)} \geq \left\| \sum_{j \in J^2_2} \left(n_2^{-1/2} \sum_{i=n_1+1}^{n_1+n_2} r_i\right) \chi_{\Delta^j_{n_1+n_2}} \right\|_1
\]
\[
= \left\| \sum_{j \in J^2_2} \left(n_2^{-1/2} \sum_{i=n_1+1}^{n_1+n_2} \varepsilon_{ij}^{n_1+n_2}\right) \chi_{\Delta^j_{n_1+n_2}} \right\|_1
\]
\[
= (n_2^{1/2} - 2n_2^{1/2})n_22^{-(n_1+n_2)} \geq \frac{1}{2}n_2^{3/2}2^{-(n_1+n_2)}
\]

Arguing in the same way, we construct a sequence \( \{D_k\} \) of pairwise disjoint subsets of \([0, 1]\) such that \( m(D_k) = n_2^{-(n_1+\ldots+n_k)} \) and
\[
\|\chi_{D_k}\|_{\mathcal{M}(L_1)} \geq \frac{1}{2}n_k^{3/2}2^{-(n_1+\ldots+n_k)}, \ k = 1, 2, \ldots
\]

Since \( m(B_k) = m(D_k), \ k = 1, 2, \ldots \), the functions \( f \) and \( g \) defined by (15) are equimeasurable ones for arbitrary \( \alpha_k \in \mathbb{R}, \ k = 1, 2, \ldots \). Setting \( \alpha_k = 2^{n_k}n_k^{-5/4} \), by (20), we obtain
\[
\|f\|_{\mathcal{M}(L_1)} \leq \sum_{k=1}^{\infty} \alpha_k \|\chi_{B_k}\|_{\mathcal{M}(L_1)} \leq 2\sqrt{2} \sum_{k=1}^{\infty} n_k^{-1/4} < \infty,
\]
because of $n_k = 2^{m_k}$, $m_1 < m_2 < \ldots$. Thus, $f \in \mathcal{M}(L_1)$.

On the other hand, since $\mathcal{M}(L_1)$ is a Banach function lattice, for every $k = 1, 2, \ldots$ from (21) and (19) it follows that

$$\|g\|_{\mathcal{M}(L_1)} \geq \alpha_k \|\chi_{D_k}\|_{\mathcal{M}(L_1)} \geq \frac{1}{2} n_k^{1/4} 2^{-(n_1 + \ldots + n_{k-1})} \geq \frac{1}{2} n_k^{1/8}.$$ 

Hence, $g \notin \mathcal{M}(L_1)$.

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