A converse comparison theorem for backward stochastic differential equations with jumps

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Abstract

This paper establishes a converse comparison theorem for real-valued decoupled forward backward stochastic differential equations with jumps.

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1 Introduction

A backward stochastic differential equation (BSDE) is an equation of the type

\[ Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \]  

where \( \xi \) is called the terminal condition, \( T \) the terminal time, \( g \) the generator, and the pair of processes \( (Y, Z) \) the solution of the equation. In 1990, Pardoux and Peng (8) proved that there exists a unique adapted and square integrable solution to BSDE (1) as soon as the generator is Lipschitz with respect to \( y \) and \( z \) and the terminal condition is square integrable. Since then, BSDE theory has become an important field of research, with applications, e.g., in stochastic optimal control, mathematical finance or partial differential equations (PDEs).

An important feature of BSDE theory is the comparison theorem (Peng (9), El Karoui et al. (5)), which plays the same role as the maximum principle for PDEs. The comparison theorem allows to compare the solutions of two

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real-valued BSDEs whenever we can compare the terminal conditions and the generators. The converse, however, is not true in general. Hence, converse comparison theorems for BSDEs have been concerned with the following question: If one can compare the solutions of two real-valued BSDEs with the same terminal condition, for all terminal conditions, can one compare the generators?

Previous works on the subject (Chen (3), Briand et al. (2), Coquet et al. (4), Jiang (6), (7)) deal with classical BSDEs (i.e., without jumps). This paper provides a converse comparison theorem for class of BSDEs with jumps, namely infinite horizon decoupled forward backward stochastic differential equations with jumps.

2 BSDEs with jumps

2.1 Notation and assumptions

Suppose that $W_t^\tau = (W_t^1, ... , W_t^d)$, $t \geq 0$, is a $d$-dimensional standard Brownian Motion and $K^\tau(t) = (K_1(t), ..., K_l(t))$, $t \geq 0$, is an $l$-dimensional stationary Poisson point process taking values in a measurable space $(B, \mathcal{B})$ where $B = \mathbb{R}_0$ is equipped with its Borel field $\mathcal{B}$. Denote by $\mu^\tau(dt, de) = (\mu_1(dt, de), ..., \mu_l(dt, de))$ the Poisson random measure induced by $K^\tau$ equipped with its $\sigma$-field $(\mathcal{F}_t)_{t \geq 0}$-stopping time and define the following spaces:

- $S^2$ denotes the set of all $F_t$-progressively measurable RCLL processes $Y$ valued in $\mathbb{R}^n$ such that $E\left[\sup_{0 \leq t \leq \tau} |Y_t|^2\right] < \infty$;
- $L^2(W)$ denotes the set of all predictable processes $Z$ valued in $\mathbb{R}^{n \times d}$ such that $E\left[\int_0^\tau \|Z_t\|^2 \, dt\right] < \infty$;
- $L^2(\tilde{\mu})$ denotes the set of all $P \otimes \mathcal{B}$-measurable processes $U_t(e)$ valued in $\mathbb{R}^{n \times l}$ such that $E\left[\int_0^\tau \int_B |U_t(e)|^2 \lambda(de) \, dt\right] := E\left[\int_0^\tau \|U_t(\cdot)\|^2 \, dt\right] < \infty$,

where $\mathbb{P}$ denotes the $\sigma$-algebra generated by all predictable subsets.

\footnote{By predictable, we mean predictable with respect to the filtration $(F_t)_{0 \leq t \leq \tau}$.}
• $L^2(B, \mathcal{B}; \mathbb{R}^{n \times l})$ denotes the set of all $\mathcal{B}$-measurable functions $\Phi(\cdot)$ valued in $\mathbb{R}^{n \times l}$ such that

$$\|\Phi\|^2 = \int_B |\Phi(e)|^2 \lambda(de) < \infty.$$ 

The (adapted) solution of a random horizon BSDE with jumps is defined as a triple of processes $(Y_t, Z_t, U_t)$, $t \in [0, \tau]$, belonging to $S^2 \times L^2(W) \times L^2(\tilde{\mu})$ such that

$$Y_t = \xi + \int_\tau^t f(\omega, s, Y_s, Z_s, U_s)ds - \int_\tau^t Z_s dW_s - \int_\tau^t \int_B U_s(e)\tilde{\mu}(ds, de),$$

where the terminal condition $\xi \in L^2(\Omega, F_\tau, \mathbb{P})$ and the generator $f : \Omega \times [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times L^2(B, \mathcal{B}, \lambda; \mathbb{R}^{n \times l}) \to \mathbb{R}^n$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{n \times d}) \otimes \mathcal{B}(L^2(B, \mathcal{B}, \lambda; \mathbb{R}^{n \times l}))$-measurable.

We will denote by $Y_t(f, \xi_t)$ the first component of the solution of BSDE (2).

In the sequel, we consider the following assumptions on generator $f$:

(A1) \[E\left[\int_0^\tau |f(\omega, t, 0, 0, 0)|^2 dt\right] < \infty;\]

(A2) $P$-a.s., for any $Y, Y' \in \mathbb{R}^n$, $Z, Z' \in \mathbb{R}^{n \times d}$, $U, U' \in L^2(B, \mathcal{B}, \lambda; \mathbb{R}^{n \times l})$, $t \geq 0$,

$$|f(\omega, t, Y, Z, U) - f(\omega, t, Y', Z', U')| \leq u_1(t) |Y - Y'| + u_2(t) (|Z - Z'| + \|U - U'\|),$$

where $u_1(t)$ and $u_2(t)$ are nonnegative, deterministic functions satisfying

$$\int_0^\infty u_1(t) dt + \int_0^\infty u_2(t)^2 dt < \infty. \quad (3)$$

(A3) $P$-a.s., $\forall (Y, Z, U) \in \mathbb{R}^n \times \mathbb{R}^{n \times d} \times L^2(B, \mathcal{B}, \lambda; \mathbb{R}^{n \times l})$, $t \to f(\omega, t, Y, Z, U)$ is continuous in $t \in [0, \tau]$.

(A4) $P$-a.s., $\forall Y \in \mathbb{R}^n, \forall Z \in \mathbb{R}^{n \times d}, \forall U, U' \in L^2(B, \mathcal{B}, \lambda; \mathbb{R}^{n \times l})$, we have

$$|f(\omega, t, Y, Z, U) - f(\omega, t, Y, Z, U')| \leq \int_B (U(e) - U'(e)) \nabla_t(\omega, e)^\top \lambda(de),$$

where $\nabla : \Omega \times [0, \infty) \times B \to \mathbb{R}^{n \times l}$ is $\mathcal{P} \otimes \mathcal{B}$-measurable and satisfies

$$\int_B |\nabla_t(\omega, e)|^2 \lambda(de) \leq u_2(t)^2,$$

and where $\nabla_t(\omega, e)$, $i = 1, ..., l$, is the $i$th component of $\nabla_t(\omega, e)$ satisfying $\nabla_t^i(\omega, e) \geq -1$, $P$-a.s.
2.2 Preliminary results

This section presents existence, uniqueness and comparison results for adapted solutions to BSDEs with jumps and with random terminal times.

**Theorem 1 (Yin and Situ [13])** Let $\xi$ and $f$ be the terminal condition and the generator of BSDE [2], respectively, and let $f$ satisfy (A1) and (A2). Then there exists a unique solution to BSDE [2].

In preparation of the main theorem, we recall the comparison theorem for one-dimensional BSDEs with jumps. Therefore, from now on, we assume that $n = 1$. In order to get a comparison result, one must impose a control on the size of the jumps. This is the reason why (A4) must hold in addition to (A1) and (A2).

**Theorem 2 (Yin and Mao [12])** Consider two generators $f_1$ and $f_2$ verifying (A1), (A2) and (A4). Let $\xi^1, \xi^2 \in L^2(\Omega, F_\tau, P)$ be two terminal conditions for BSDEs driven respectively by $f_1$ and $f_2$. Denote by $(Y^1_t, Z^1_t, U^1_t)$ and $(Y^2_t, Z^2_t, U^2_t)$, $t \in [0, \tau]$, the respective solutions of these equations. If $\xi^1_t < \xi^2_t$ $P$-a.s. and $f_1(t, Y^1_t, Z^1_t, U^1_t) \leq f_2(t, Y^1_t, Z^1_t, U^1_t)$ $P$-a.s., then almost surely $Y^1_t < Y^2_t$, for all $t \in [0, \tau]$.

Before stating the main theorem, we also need the following lemma. It is a strict comparison theorem for BSDEs with jumps and with the same terminal condition.

**Lemma 3** Let the generators $f_1$ and $f_2$ verify (A1), (A2) and (A4), and let the terminal condition $\xi^1 \in L^2(\Omega, F_\tau, P)$. Denote by $(Y^1_t, Z^1_t, U^1_t)$ and $(Y^2_t, Z^2_t, U^2_t)$, $t \in [0, \tau]$, the respective solutions of the BSDEs with generators $f_1$ and $f_2$ and terminal conditions $\xi$. If $f_2(t, Y^1_t, Z^1_t, U^1_t) < f_1(t, Y^1_t, Z^1_t, U^1_t)$ $P$-a.s., then almost surely, $Y^2_t < Y^1_t$, for all $t \in [0, \tau]$.

**Proof.** The proof follows from the same argument as in the proof of the comparison theorem for classical BSDEs of El Karoui et al. [5] (for the jump diffusion case, see the proofs of theorem 3.1 and corollary 3.1 of Yin and Mao [12]).

3 A converse comparison theorem

We now introduce the infinite horizon decoupled forward backward stochastic differential equations (FBSDEs, for short) for which we will prove a converse comparison result.

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2See Royer [10] for a counter-example when only Lipschitz continuity is assumed.
For any given \((t, x) \in [0, \infty) \times \mathbb{R}^m\), consider the following FBSDEs

\[
X_t^{t,x} = x + \int_t^s a(r, X_r^{t,x})dr + \int_t^s b(r, X_r^{t,x})dW_r + \int_t^s \int_B c(r, X_r^{t,x}, e)\tilde{\mu}(dr, de),
\]

\[
Y_s = h(X_s^{t,x}) + \int_s^\infty f_1(r, X_r^{t,x}, Y_r, Z_r, \int_B U_r(e)\gamma_r(e)^\top \lambda(de))dr - \int_s^\infty Z_r dW_r - \int_s^\infty \int_B U_r(e)\tilde{\mu}(dr, de),
\]

where \(s \in [t, \infty)\), \(\gamma\) is assumed to be deterministic and \(\gamma : [0, \infty) \times B \to \mathbb{R}^{1 \times l}\) is \(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}\)-measurable and satisfies

\[
\int_B |\gamma_t(e)|^2 \lambda(de) \leq u_2(t)^2, \text{ where } u_2(t) \text{ is as in } (3) \text{ and } \gamma_t(e), i = 1, \ldots, l, \text{ is the } i\text{th component of } \gamma_t(e) \text{ satisfying } \gamma_t(e) \geq -1.
\]

From now on, we assume that \(a, b, c, f_1\) and \(h\) in (4) are deterministic. We assume that \(a : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^m\), \(b : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^{m \times d}\) and \(c : [0, \infty) \times \mathbb{R}^m \times B \to \mathbb{R}^{m \times l}\) are continuous and such that they guarantee the existence and uniqueness of a strong solution to (4). We also assume that \(X_{s}^{t,x}\) exists and that \(E\left[X_{\infty}^{t,x} \right]<\infty\).

Regarding (5), we will suppose that the real valued functions \(h : \mathbb{R}^m \to \mathbb{R}\) and \(f_1 : [0, \infty) \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R} \to \mathbb{R}\) are \(\mathcal{B}(\mathbb{R}^m)\)-measurable and \(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^{1 \times d}) \otimes \mathcal{B}(\mathbb{R})\)-measurable, respectively, and that they are Lipschitz continuous with respect to \(X\), with Lipschitzian function satisfying (3). Moreover, we assume that for every \(X \in \mathbb{R}^m\), \(f(t, X, Y, Z, U) = f_1(t, X, Y, Z, \int_B U(e)\gamma_t(e)^\top \lambda(de))\) satisfies assumptions (A1), (A2), (A3) and (A4), and we assume that there exists a constant \(C\) such that for every \(X \in \mathbb{R}^m\), \(|h(X)| \leq C(1 + |X|)\), so that \(h(X_{\infty}^{t,x}) \in L^2(\Omega, F, P)\) and (5) admits a unique solution. We denote the solution by \((Y_{t,x}^{t,x,1}, Z_{t,x}^{t,x,1}, U_{t,x}^{t,x,1})\), where \(Y_{t,x}^{t,x,1}\) is adapted and \((Z_{t,x}^{t,x,1}, U_{t,x}^{t,x,1})\) are predictable with respect to the filtration \((\mathcal{F}_r, r \geq t)\), where \(\mathcal{F}_r = \sigma[W_s - W_t, \mu((t, s), A)]; \quad t \leq s \leq r, A \in \mathcal{B} \vee N\). Then,

\[
(u^1(t, x), \chi^1(t, x), \zeta^1(t, x)) := (Y_{t,x}^{t,x,1}, Z_{t,x}^{t,x,1}, \int_B U_{t,x}^{t,x,1}(e)\gamma_t(e)^\top \lambda(de))
\]

is deterministic, and from the uniqueness of solution to (5), it is known (see for example Barles et al. (11), p. 69) that for any \(s \in [t, \infty)\),

\[
Y_s^{t,x,1} = Y_{s,x}^{t,x,1} = u^1(s, X_{s}^{t,x}).
\]

Since the same reasoning holds for \(\chi\) and \(\zeta\), we similarly have

\[
Z_s^{t,x,1} = \chi^1(s, X_{s}^{t,x}),
\]

\[
\int_B U_{s,x}^{t,x,1}(e)\gamma_s(e)^\top \lambda(de) = \zeta^1(s, X_{s}^{t,x}).
\]
Theorem 4 (Converse comparison theorem) Suppose that $f_1$ is continuous and let $a, b, c, f_1$ and $h$ satisfy the above assumptions, so that there exists a unique solution to (4) and (5). Denote by $Y^{t,x,1}_s(\xi_v)$ the first component of the solution of (5) with generator $f_1$ and terminal condition $\xi_v$ at time $v$, where $v$ is a stopping time and $t \leq s < v$. Assume further that $Y^{t,x,1}_t(h(Y^{\infty,t}_t)) = u^1(t, x) \in C^{1,2}(0, \infty) \times \mathbb{R}^m$. If, for any given $(t, x) \in [0, \infty) \times \mathbb{R}^m$, there exists $0 < \delta > 0$ such that

$$Y^{t,x,1}_t(u^1(v, X^{t,x}_v)) < Y^{t,x,2}_t(u^1(v, X^{t,x}_v)),$$

for every stopping time $v$ such that $t \leq v < t + \delta$, and where $Y^{t,x,2}_t(u^1(v, X^{t,x}_v))$ is the first component of the time $t$ solution of a FBSDE with driver $f_2$ satisfying the same assumptions as $f_1$ and terminal condition $u^1(v, X^{t,x}_v)$ at time $v$, then

$$f_1(t, x, u^1(t, x), \chi^1(t, x), \zeta^1(t, x)) \leq f_2(t, x, u^1(t, x), \chi^1(t, x), \zeta^1(t, x)). \tag{9}$$

**Proof.** By contradiction, suppose that

$$f_1(t, x, u^1(t, x), \chi^1(t, x), \zeta^1(t, x)) > f_2(t, x, u^1(t, x), \chi^1(t, x), \zeta^1(t, x)). \tag{10}$$

By assumption, $f, a, b$ and $c$ are continuous, and $u^1$ is assumed to be continuous and smooth, so it follows from the Feynman-Kac formula (see also theorem 3.4 in Barles et al. (11)) that $\chi^1$ and $\zeta^1$ are continuous functions, since $\chi^1(t, x) = u^1_1(t, x) b(t, x)$ and $\zeta^1(t, x) = \int_B(u^1(t, x + c(t, x, e)) - u^1(t, x)) g(e) e \lambda(de)$ (the letter $x$ as lower indice indicates differentiation, as usual). As a consequence, there exists $0 < \eta < \infty$ such that for any $(s, k) \in [t, \infty) \times \mathbb{R}^m$ satisfying $t \leq s \leq t + \delta$, $|x - k| \leq \eta$,

$$f_1(s, k, u^1(s, k), \chi^1(s, k), \zeta^1(s, k)) > f_2(s, k, u^1(s, k), \chi^1(s, k), \zeta^1(s, k)). \tag{11}$$

Define now the stopping time

$$\tau := \inf\{s > t : |X^{t,x}_s - x| > \eta\} \land (t + \eta) \land (t + \delta),$$

and let

$$(\Upsilon^1_s, Z^1_s, U^1_s) := (Y^{t,x,1}_s, Z^{t,x,1}_s, U^{t,x,1}_s), \quad t \leq s \leq \tau.$$ 

Then, $(\Upsilon^1_s, Z^1_s, U^1_s)$ is solution of the following FBSDE:

$$\begin{align*}
Y^1_s &= u^1(\tau, X^{t,x}_\tau) + \int_s^\tau f_1(r, X^{t,x}_r, \Upsilon^1_r, Z^1_r, U^1_r) g_r(e) e \lambda(de) dr \\
&\quad - \int_s^\tau Z^1_r dW_r - \int_s^\tau \int_B U^1_r(e) \tilde{\mu}(dr, de).
\end{align*}$$

Now, consider the FBSDE

$$\begin{align*}
Y^2_s &= u^1(\tau, X^{t,x}_\tau) + \int_s^\tau f_2(r, X^{t,x}_r, \Upsilon^2_r, Z^2_r, U^2_r) g_r(e) e \lambda(de) dr \\
&\quad - \int_s^\tau Z^2_r dW_r - \int_s^\tau \int_B U^2_r(e) \tilde{\mu}(dr, de).
\end{align*}$$

(12)
where
\[(Y_{s}^{2}, Z_{s}^{2}, U_{s}^{2}) := (Y_{s}, Z_{s}, U_{s}), \quad t \leq s \leq \tau.\]

\(u^{1}(\tau, X_{\tau}^{t,x})\) is the first component of the time \(\tau\) solution of a BSDE with coefficient \(f_1\) and terminal condition \(h(X_{\infty}^{t,x})\); therefore, \(u^{1}(\tau, X_{\tau}^{t,x})\) is square integrable, that is, \(E\left[ u^{1}(\tau, X_{\tau}^{t,x})^2 \right] < \infty\). By assumption on \(f_2\), it follows that there exists a unique solution to (12).

Taking (6), (7) and (8) into account, together with (11), we can apply the comparison theorem of lemma 3 and get
\[Y_{t}^{1}(u^{1}(\tau, X_{\tau}^{t,x})) > Y_{t}^{2}(u^{1}(\tau, X_{\tau}^{t,x})).\]
which, by uniqueness, yields that

\[Y_{t}^{1,1}(u^{1}(\tau, X_{\tau}^{t,x})) > Y_{t}^{1,2}(u^{1}(\tau, X_{\tau}^{t,x})),\]

which contradicts (1). Therefore, we must have that
\[f_1(t, x, u^{1}(t, x), \chi^{1}(t, x), \zeta^{1}(t, x)) \leq f_2(t, x, u^{1}(t, x), \chi^{1}(t, x), \zeta^{1}(t, x)).\]

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