Lane formation in a lattice model for oppositely driven binary particles

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Abstract – A lattice model for oppositely driven binary particles with purely repulsive interactions is investigated on the square lattice. Two classes of steady states related to stuck configurations and lane formations have been constructed in systematic ways under certain conditions. A mean-field-type analysis carried out using a percolation problem based on the constructed steady states provides an estimation of the phase diagram, which is qualitatively consistent with numerical simulations. Further, finite-size effects in terms of lane formations are discussed.

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Introduction. – The understanding of the relationship between collective phenomena and microscopic properties of the system has been an important topic in statistical physics. Recently, it has been observed that various systems consisting of oppositely driven binary particles with repulsive interactions exhibit a collective phenomenon called lane formation, where the same type of particles align to the driven direction [1–7]. The observation of lane formations in various systems suggests that certain types of properties are universal in lane formations, irrespectively of the systems. Thus far, a phenomenological theory for determining a special condition to cause lane formations in colloidal suspensions has been presented, which is qualitatively consistent with numerical results [2,3]. However, we still lack the knowledge for the relationship between lane formations and microscopic properties. For example, questions on whether the special condition is related to some nonequilibrium phase transitions or which types of fluctuations, for instance, finite-size effects, appear in lane formations remain unanswered.

As is common in the literature on statistical physics, lattice models are often useful for obtaining insights into universal phenomena, owing to the simplicity of the system and the universality itself. For example, driven lattice gas models and simple exclusion processes have provided many insights into universal aspects of nonequilibrium systems, such as phase transitions or the KPZ universality [8–12]. In this context, a simple lattice model exhibiting a lane formation can be useful for understanding lane formations more comprehensively. Indeed, it has been known that some lattice models motivated by pedestrian dynamics exhibit lane formations [13,14]. In these lattice models, a certain effect motivated by characteristic human behaviors called floor field is essential for lane formations. However, the effect corresponding to the floor field in the context of colloidal suspensions is not necessarily clear. Thus, the answer as to which microscopic properties in lattice models are essential for lane formations is still missing.

In this paper, in order to obtain insights into the universal aspects of lane formations, we propose a lattice model where binary particles with purely repulsive interactions are driven in opposite directions as a toy model of colloidal suspensions on the corresponding condition. We present the exact constructions of two classes of steady states related to stuck configurations and lane formations. On the basis of these constructions, we introduce a percolation problem in a stochastic cellular automaton, which helps us estimate the phase diagram. Further, we discuss finite-size effects in lane formation using the constructed steady states.

Model. – Let us consider a square lattice Λ consisting of site \( i \in \{(i_x, i_y) \in \mathbb{N} \times \mathbb{N} | 1 \leq i_x, i_y \leq L \} \), where \( \mathbb{N} \) is the set of natural numbers. In order to introduce binary (positive and negative) particles on the lattice, we consider an occupation variable \( \sigma_i \in \mathbb{Z} \), where \( \mathbb{Z} \) is the set of integers. In this expression, \( \sigma_i \) indicates that there are \( |\sigma_i| \) positive and negative particles at site \( i \).
number of particles with the sign of \( \sigma_i \) at site \( i \), and no particles if \( \sigma_i = 0 \) as illustrated in fig. 1. Therefore, the density \( \rho \) of the particles in the system is defined as \( \rho \equiv \frac{1}{L^2} \sum_{i \in \Lambda} |\sigma_i| \). We consider a situation where each particle interacts with a soft-core repulsive potential through the following Hamiltonian for \( \sigma \equiv \{\sigma_i\}_{i \in \Lambda} \):

\[
H(\sigma) = V_0 \sum_{i \in \Lambda} |\sigma_i|(|\sigma_i| - 1),
\]

where \( V_0 > 0 \). On the basis of this purely repulsive Hamiltonian, we consider continuous-time Markov processes where a particle hops to another site with some transition rates; this process is expressed as follows. Prelim-
narily, in order to express the hopping of one particle from site \( i \) to site \( j \), we define an operator \( F_{ij} \) that satisfies \( F_{ij}\sigma_i = \sigma_i - \text{sgn}(\sigma_i), \ F_{ji}\sigma_j = \sigma_j + \text{sgn}(\sigma_j), \) and \( F_{ij}\sigma_k = \sigma_k \), where \( \text{sgn}(x) = -1 \) for \( x < 0 \) and \( \text{sgn}(x) = 1 \) for \( x > 0 \), otherwise \( \text{sgn}(x) = 0 \). Here, the destination site \( j \) can be in set \( B_i \equiv \{j \in \Lambda | i - j | \leq l_{\text{hop}} \} \) where \( i - j \equiv \sqrt{ (x_i - x_j)^2 + (y_i - y_j)^2 } \) and \( 1 \leq l_{\text{hop}} < 2 \), as shown in fig. 2. Here, the hopping events along the edge for sites \( i \) and \( j \) occur in proportion to the number of particles along the edge.

Here, we consider an explicit form of \( \Delta E_{ij}^0(\sigma) \). In order to construct a model with universal properties, we develop a guiding principle for determining the forms of \( \Delta E_{ij}^0(\sigma) \) in the following manner. The idea is to consider the systems described by overdamped Langevin equations where particles move in the exact direction of the driving force at the infinite driving force limit [1]. Keeping this in mind, we assume that particles at each site hop only to sites \( k \in B_i \) with maximum \( D_{ik}(\sigma_i)\Theta(\sigma,\sigma_k) \) at the infinite driving field limit, which we refer to as normality of the limit. For example, the following expression provides the normality of the limit:

\[
\Delta E_{ij}^0(\sigma) \equiv \min_{k \in B_{ij}} \{ \min_{\lambda \in \Lambda} \Delta E_{ik}(\sigma), \min_{k \in B_{ij}} \Delta E_{jk}(F_{ij}\sigma) \},
\]

where \( \Delta E_{ik}(\sigma) \equiv \Delta H_{ik}(\sigma) - D_{ik}(\sigma), \) and \( B_{ij} \equiv \{ k \in \Lambda | |k - i|,|k - j| \leq l_{\text{hop}} \} \) with \( l_{\text{hop}} = \sqrt{2} \). This expression also satisfies \( \Delta E_{ij}^0(\sigma) = \Delta E_{ji}^0(\sigma) \leq 0 \) and \( \leq r(\sigma \to F_{ij}\sigma) \leq 1 \). Let us refer to fig. 2 in order to see visually \( \Delta E_{ik} \) required to calculate \( \Delta E_{ij}^0(\sigma) \). In this study, we investigate only this case although other choices for \( \Delta E_{ij}^0(\sigma) \) to hold the normality of the limit are possible. It should be noted that in order to ensure only \( \Delta E_{ij}^0(\sigma) = \Delta E_{ji}^0(\sigma) \), we may set \( \Delta E_{ij}^0(\sigma) \) as a constant, which can be also zero. However, in this case, the model does not follow the normality of the limit. As a result, the arguments mentioned in the rest of this paper do not hold. On the other hand, we expect that the arguments mentioned in the rest of this paper are robust against small changes in the expression for \( \Delta E_{ij}^0(\sigma) \) if the expression provides the normality of the limit.
however the extent of changes for which the arguments hold is uncertain. This type of guiding principle used to determine transition rates with universal properties has been discussed in different nonequilibrium lattice models with a nonconserved variable [15].

In this study, we consider the behaviors of the system by changing the set of parameters $(\rho, f)$. Further, for simplicity, we focus on the case with the zero-temperature limit, the periodic boundary condition, and 50:50 binary mixtures where the densities of positive and negative particles are $\rho/2$. If $f = 0$, the steady states are determined by the canonical distribution of Hamiltonian (1). In particular, one can immediately determine that if $f = 0$ and $\rho \leq 1$, $\sigma_i$ in equilibrium states randomly takes only the values of 1, $-1$, or 0 with a fixed density $\rho$. Obviously, there are no singular points in the parameter space under this equilibrium condition. We consider relaxation behaviors from this equilibrium initial condition. It should be noted that these behaviors are still nontrivial even under these simple conditions owing to purely nonequilibrium effects, as explained later. While performing Monte Carlo simulations for this model, we randomly select a particle. If it is located at site $i$, we select any $j \in B_i$ with probability $\delta_{ij} / (2 + \sqrt{2})$. Then, both the directions from $i$ to $j$ and from $j$ to $i$ are adopted with a probability of 1/2. Finally, a particle hops to the adopted direction (if $j \rightarrow i$) with probability $\Theta(\sigma_i, \sigma_j) r(\sigma \rightarrow F_j, \sigma)$, and if there are no particles at site $j$, nothing occurs. This process is repeated and time $t = 1$ corresponds to the repeated $L^2 / \rho / \tau_0$ steps where $\tau_0 = 1 / (4 + 2 \sqrt{2})$.

Exact arguments and the related conjectures. – Let us confirm some nontrivial arguments for the model. Preliminarily, we consider a current-like quantity as follows. Let us define a colored current $J(t)$ at time $t$ with $\sigma$ as

$$J(t) \equiv \frac{1}{L^2 \rho} \sum_{x \in A} \sum_{y \in \pm 1} \text{sgn}(\sigma_y) \Theta(\sigma, \sigma_y) r(\sigma \rightarrow F_y, \sigma),$$

where $-1 \leq J(t) \leq 1$ by definition. We consider $J = 1$ such that $J(t) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \sum_{t' = \tau}^t J(t')$. It is trivial that if $f = 0$, $J$ should be absolutely zero with $\tau \rightarrow \infty$ owing to the symmetry of the directions. By using this quantity, we define the maximum-current (MC) steady states as $J = 1$ and the stuck steady states as $J = 0$. Next, using the colored current, we construct the microscopic characterizations of such steady states in some parameter regions. After these constructions, we present some conjectures related to the constructed steady states.

CA-stuck steady states as one explicit class of stuck steady states. Let us construct one explicit class of the stuck steady states at 0 < $f < 2V_0$ for $\rho < 1$. Here, we assume the boundary condition where $\sigma_{x_y, i_y}(t) \neq 0$ for arbitrary time $t$ and values of $i_y$ with sites $i_y = 0$ and $L$ to simplify the argument, but one may construct a similar case also for the periodic boundary condition with small modifications based on the following. Preliminarily, let us consider an auxiliary variable $\tilde{\sigma}_i \in \{-1, 0, 1, 2\}$ satisfying

$$\{\tilde{\sigma}_i = 2\} \equiv \{\sigma_{i_x, i_y} = 1, \sigma_{i_x+1, i_y} = -1\},$$

and

$$\{\tilde{\sigma}_i \equiv 1\} \equiv \{\tilde{\sigma}_i = \tilde{\sigma}_j\}. $$

In this description, $\tilde{\sigma}_{x_y, i_y} = 2$ is equivalent to $\tilde{\sigma}_{x_y, i_y} = 1$. For the basis of this variable $\{\tilde{\sigma}_i\}$, we consider a cellular automaton for determining the value of $\tilde{\sigma}_i$ under the initial condition $\tilde{\sigma}_{x_1} = 2$ and $\tilde{\sigma}_i = 0$ for all other sites $i$ except for boundary sites; the rule to update the values for all the other sites is explained in the following sentences. First, we set $\tilde{\sigma}_i \equiv 3$ for $\tilde{\sigma}_y, +2 = 2$ or $\tilde{\sigma}_y, -1, 2 = 2$. Next, if $\tilde{\sigma}_y, +2 = 2$, we set $\tilde{\sigma}_i \equiv 2$ or $\tilde{\sigma}_i, 1, 3 = 2$ or $\tilde{\sigma}_i, +1 = 3$. This process is repeated until reaching the site with $i_y = L - 1$. Then, we set $\tilde{\sigma}_i = -1$ for $i_y + 1 < i_y < i'$, where $i_y \equiv i'_y$ such that $\tilde{\sigma}_y = 2$ for a given value of $i'_y$ and $i'^2 = \max_y i'^2 y + 1$. Similarly, we set $\tilde{\sigma}_i = 1$ for $i' < i_y < i'_y$, where $i' \equiv \min_y i'^2 y$. Thus, we obtain a sequence $\{\tilde{\sigma}_i\}_1$. Finally, by determining the value of $\tilde{\sigma}_i$ for all sites with $\tilde{\sigma}_y = 0$ under the condition $H(\sigma) = 0$ and $\rho = \delta_{ij} \sum |\tilde{\sigma}_i|$, this system reaches a steady state with $J \equiv 0$ at the long time limit because $\{\tilde{\sigma}_i\}_1$ are independent of $t$ in this construction. This corresponds to the fact that there exist the stick steady states characterized by $\{\tilde{\sigma}_i\}_1$, with $H(\sigma) = 0$ for $f < 2V_0$ and $\rho = \delta_{ij} \sum |\tilde{\sigma}_i|$. Further, a state constructed by using $\{\tilde{\sigma}_i\}_1$ and $\{\tilde{\sigma}_i\}_2$, with $\tilde{\sigma}_2 = 0$ and $H(\sigma) = 0$ can also be a stuck steady state. We define the stuck steady states constructed in this way as CA-stuck steady states. However, there are no principles to determine whether CA-stuck steady states appear as steady states under the equilibrium initial condition, which is discussed later.

Global-MC steady states as one explicit class of maximum-current steady states. Let us construct one explicit class of the maximum-current steady states with sufficiently large values of $f$ for any density. First, let us define $N_{y, y} \equiv \sum_{i \in \pm 1} |\tilde{\sigma}_i(\tilde{\sigma}_y \delta_{i_y, y})$, and $N_{y, \Lambda} \equiv \sum_{i \in \pm 1} |\tilde{\sigma}_i| \Theta(\sigma, \tilde{\sigma}_y) \delta_{i_y, y}$. For each value of $y$, we set $N_{y, y}$ and $N_{y, \Lambda}$ such that $N_{y, y} \equiv 0$ under the condition $\sum_y N_{y, y} = \sum_y N_{y, \Lambda} = \rho L^2 / 2$. Then, if we set $f$ such that

$$f \equiv \frac{2\sqrt{2}V_0}{\sqrt{2} - 1} \max(N_{y, y}, N_{y, \Lambda}),$$

can easily find that this system reaches a steady state $J = 1$ at the long time limit because $N_{y, y}$ and $N_{y, \Lambda}$ are independent of $t$. This corresponds to the fact that the set $\{N_{y, y}, N_{y, \Lambda}\}_{0 \leq y < L}$ under the condition mentioned above determines a maximum-current steady state for $f$ satisfying (9) and $\rho = \delta_{ij} \sum_{y = 0}^L (N_{y, y} + N_{y, \Lambda})$. We define the maximum-current steady state constructed in this way as global-MC steady states. However, there are no principles to determine whether global-MC steady states appear as steady states under the equilibrium initial condition, which is also discussed later.

Conjecture A: mean-field-type analysis of phase transitions. So far, we have microscopically constructed the CA-stuck steady states and the global-MC steady states,
which lead us to expect that singular points of $\mathcal{J}$ exist in parameter space $(\rho, f)$ where the steady state changes qualitatively. With this background, let us assume that there is a line $f_c(\rho)$ where $\mathcal{J}$ is singular in driving field $f$ at each fixed $\rho$. On the basis of this assumption, we attempt to approximately estimate $f_c(\rho)$ as follows.

In low-density regions, particles appear to be driven independently. Therefore, a nonzero value of $f_c(\rho)$ would be observed only in sufficiently high-density regions, and there should be a threshold density $\rho_{th}$ below which $f_c = 0$. We estimate $\rho_{th}$ as follows. First, it should be noted that it is plausible for a type of seed to exist under the initial condition for realizing the CA-stuck steady states. One reasonable candidate for such a seed is the particles present at the sites indicated by the one-step run of the cellular automaton from a focused particle, which we refer to as an automaton-pointed segment of the focused particle, as discussed in the construction of the CA-stuck steady states. We assume that if there exists at least one sequence of automaton-pointed segments starting from a focused particle, which connects with a site with distance $L$ from the focused particle under the initial condition, the system can reach a CA-stuck steady state after the dynamical particle-exchange processes. As mentioned in the construction of the CA-stuck steady states, this consideration is applied in the case of $f < 2$, which leads to $f_c(\rho_{th}) = 2V_0$. On this assumption, we consider the probability $P_y$ that there exists at least one sequence of automaton-pointed segments starting from a focused particle at $i_y = y$, which connects with a site with $i_y = L$ under the initial condition. This is a typical percolation problem in a stochastic cellular automaton with a percolation point $\rho_p$ [16]. Reminding that the number of sites in such an automaton-pointed segment is 4, if each site in an automaton-pointed segment is assumed to be independent, we can obtain the recursive equation $P_y = \rho(1 - (1 - P_{y+1})^4)$. Thus, we can estimate a lower bound $\rho_p = 1/4$ of the percolation point $\rho_p$, which also gives an approximate value of $\rho_{th}$.

On the basis of this consideration, let us estimate $f_c(\rho)$ at $\rho > \rho_{th}$. Here, we consider the condition with which the percolation breaks owing to the hopping of a particle through an automaton-pointed segment, where only the same type of particles exist. Let $\delta n$ denote the increment in the number of particles in an automaton-pointed segment when $\rho$ is increased by $\delta \rho$ from $\rho_{th}$. Then, we can estimate $\delta n = 4 \delta \rho$ approximately where 4 is the number of sites in one automaton-pointed segment. Indeed, if $f$ is increased by $\delta f$ from $f = f_c(\rho_{th})$ satisfying $\delta f > V_0((n + \delta n)(n + \delta n - 1) - n(n - 1)) = V_0(\delta n + \delta n^2)$, where $n = 1$ is the maximum particle number per one site at $f < f_c(\rho_{th})$, a particle can pass through an automaton-pointed segment in the driven direction even for the worst case. Thus, we obtain $f_c(\rho) \approx f_{MF}(\rho)$, where

$$f_{MF}(\rho) \equiv 4V_0(\rho - \rho_p^1) + 16V_0(\rho - \rho_p^1)^2 + f_c(\rho_{th}) = 0,$$

for $\rho > \rho_p^1$, otherwise $f_{MF}(\rho) = 0$.

Further, using (9), we can obtain the minimum possible value of $f = f_0 \equiv \frac{2\sqrt{2}V_0}{\sqrt{3}} \approx 6.83$ to realize MC-global steady states at the dilute limit. At $f \geq f_0$, microscopic events change considerably, where a particle can overlap with the front particle in the driven direction without being affected by other particles. As a result, fluctuations in the local density could increase considerably, possibly causing another singularity in $\mathcal{J}$ even in finite density regions. Thus, we can predict that there is a cross-point $\rho_0 \approx 0.69$, determined by $f_{MF}(\rho_0) = f_0$, where the percolation effects and the large fluctuations of local densities become comparable.

**Conjecture B: finite-size effects due to noncommutativity between $t \to \infty$ and $L \to \infty$.** Let us consider the situation with a sufficiently large driving field, where $N_{p,y}N_{n,y} \neq 0$ for some values of $y$ at a finite time $t$, and focus on such a line-$y$, which is a set of sites $(i_x, y, i_x)$. We consider the conditions where such all line-$y$ can be divided into finite number of segments, each of which satisfies $N_{p,y}N_{n,y} = 0$, where $N_{p,y}N_{n,y}$ is the number of positive (negative) particles in segment $x$ at $i_y = y$. Clearly, there are no positive contributions to $J(t)$ at the contact points between such segments. However, owing to the finite numbers of segments, we can neglect the effects brought about by such contact points with $L \to \infty$. Hence, in the thermodynamic limit $L \to \infty$, $\mathcal{J} = 1$ can be realized in other states besides global-MC steady states if we assume that the number of segments divided by $L$ goes to zero at any time. Furthermore, when we consider the limit $L \to \infty$ at a fixed large value of $t$, such states appear not to relax to any global-MC steady states. This is because, in simple terms, the time required for the displacement of particles to form global-MC steady states would be at least of the order of $L$. Therefore, owing to the uniform configurations under the initial conditions, this type of transient state should be dominant for sufficiently large driving fields, as compared to the global-MC steady states, in the procedure where $t \to \infty$ after $L \to \infty$.

**Numerical simulations.** Next, in order to verify the plausibility of the obtained results, we perform numerical simulations with $t = 50000\tau_0$ Monte Carlo steps and practically set $\tau = 40000\tau_0$. Further, we set $V_0 = 1$ without loss of generality. We have verified that the following results are qualitatively unchanged with $\tau = 20000\tau_0$. As shown in the left-hand side of fig. 3, we have observed a global-MC steady state at $\rho = 0.5$, $f = 100.0$, and $L = 50$. It should be noted that the observation of global-MC steady states becomes considerably difficult for larger system sizes, which is discussed later. On the other hand, as shown in the right-hand side of fig. 3, although we have observed a stuck steady state at $\rho = 0.5$, $f = 1.0$, and $L = 50$, the stuck steady state is not identical to the CA-stuck steady states. Generally, some parts of CA-stuck steady states and other complicated configurations exist in the observed stuck steady state. Nevertheless, maximum-current steady
states and stuck steady states exist in different parameter regions, which indicates the existence of phase transitions.

In order to investigate the expected phase transitions explicitly, we change the external field by an increment of $\delta f = 0.5$ at a fixed density $\rho$ and measure $\Delta J(f) \equiv J(f) - J(f - \delta f)$. Then, we attempt to detect a point

$$f_n(\rho) \equiv \arg \max_f \Delta J(f),$$

which can be a reasonable candidate for the singular points $f_c(\rho)$ of $J$. It should be noted that the following observation is made by one simulation run under an initial condition with $L = 200$. As shown in fig. 4, the numerical simulations demonstrate that it is easy to determine $f_n(\rho)$ at sufficiently low densities because $J$ jumps discontinuously at some values of $f$. As shown in fig. 5, using $f_n(\rho)$, we can approximately estimate $f_c(\rho)$, which operationally defines a flowing phase and a blocked phase. It should be noted that another weak jump around $f_0$ was observed independently of $\rho$ at sufficiently low densities, as expected by the Conjecture A. However, it is not easy to estimate the location of $f_n(\rho)$ for $\rho > 0.5$ because of sample-to-sample fluctuations in $f_n(\rho)$. For this region, we need another quantity for the plausible estimation of $f_n(\rho)$, as discussed later. Further, discontinuous jumps of $J$ seem to disappear at a density between $\rho = 0.8$ and 0.9.

Next, we investigate finite-size effects in the system, in particular, focusing on lane formations. Lane formations indicate that particles tend to be aligned to the direction of the driving field. The lane formations can be characterized by at least two approaches. The first approach, referred to as local lane formation, involves estimating how often the same type of particles are nearest neighbors of the focused particle in the driven direction [7]. The second approach, referred to as global lane formation, involves estimating the number of times the same type of particles appear in all sites of the driven direction. [1–3]. As one possible way to quantify such local lane formations, we focus on

$$\Phi_1 \equiv \phi_X(l_0) - \phi_Y(l_0),$$

where $\phi_X(l_0) \equiv \frac{1}{\rho L} \sum_{i \in \Lambda} |\sigma_1 X^+_i(l_0) - X^-_i(l_0) \sigma_1|$ with denoting by $X^\pm_i$ the number of the same (opposite) type of the particles in sites with distance $l_0$ from site $i$ along the $x$-axis at $i_y$ and $\phi_Y(l_0) \equiv \frac{1}{\rho L} \sum_{i \in \Lambda} |\sigma_1 Y^+_i(l_0) - Y^-_i(l_0) \sigma_1| + Y^+_i(l_0) + Y^-_i(l_0)$ with denoting by $Y^\pm_i$ the number of the same (opposite) type of the particles in sites with distance $l_0$ from site $i$ along the $y$-axis at $i_x$. In precise terms, $X^\pm_i(l_0) \equiv \sum_{l_0=0}^{\rho L} |\sigma_1 X^+_i(l_0) - \sigma_1 X^-_i(l_0) \sigma_1|$ and $Y^\pm_i(l_0) \equiv \sum_{l_0=0}^{\rho L} |\sigma_1 Y^+_i(l_0) - \sigma_1 Y^-_i(l_0) \sigma_1|$. Here, in order to determine nearest-neighbor particles for the focused particle, we set $l_0 = \lfloor \rho^{-1} \rfloor$ for $L/2 \geq \lfloor \rho^{-1} \rfloor$ under the assumption that two particles exist in the segment with an average distance of approximately $2l_0$ to the driven direction. Further, we can also quantify global lane formations by focusing on

$$\Phi_2 \equiv \phi_X(L/2),$$

As shown in the left-hand side of fig. 6, $\Phi_1$ shows a clear discontinuous jump between a negative value and a positive value around the obtained $f_n(\rho)$ even at $\rho > 0.5$. Further, another singularity around $f_0$ is clearer than that of $J$. In high-density regions such as $\rho = 0.9$ or 1.0, it is also clear that there is one large discontinuous jump between a negative value and a positive value whose critical fields $f_i$ are plotted in fig. 5 as a reference. More importantly, the global tendencies of $\Phi_1$ do not depend on the system sizes, which indicates that behaviors
of local lane formations are independent of the system sizes. In our simulations, global tendencies of $\mathcal{J}$ also do not depend on the system sizes at $f \geq f_1(\rho)$, which is reasonable because $\mathcal{J}$ is also a local quantity. Therefore, we used $f_1(\rho)$ as the value of $f_0(\rho)$ for $0.5 \leq \rho \leq 0.8$ in the thermodynamic limit. Thus, as shown in fig. 5, $f_{\text{MF}}(\rho)$ and $\rho_0$ obtained by the mean-field-type analysis in Conjecture A provide qualitatively consistent behaviours with $f_0(\rho)$, although the mean-field-type analysis appears to overestimate the parameter regions of the blocked phase. These deviations are rather natural for this mean-field-type analysis because of the existence of spatial correlations that we have ignored. On the other hand, as shown in the right-hand side of fig. 6, $\Phi_{\text{g}}$ shows strong dependence on the system sizes. In particular, the tendency of these finite-size effects indicates that global lane formations disappear in the procedure where $t \to \infty$ after $L \to \infty$, which is consistent with Conjecture B. It should be noted that although another large jump is observed near $f = 14$ as shown in fig. 6, it is reasonable to guess that it occurs at $2f_0$ by the reason similar to that for the jump at $f_0$.

Concluding remarks. — We have proposed a simple lattice model for oppositely driven binary particles with purely repulsive interactions, where CA-stuck steady states at small driving fields for $\rho < 1$ and global-MC steady states at sufficiently large driving field for any density are exactly constructed. A mean-field-type analysis leads us to estimate singular points in the colored current, which are qualitatively consistent with numerical simulations. This strongly suggests that such a singularity originates from percolation in a stochastic cellular automaton buried in the equilibrium configuration. Further, we have presented a conjecture for the absence of the relaxation toward global-MC steady states (global lane formations) from the equilibrium initial conditions in the procedure where $t \to \infty$ after $L \to \infty$, which is also consistent with numerical simulations.

Here, we present some remarks on the universality of the observed phenomena in the proposed model. First, we describe the singular point $f_0$ in low-density regions, above which the driving field is large enough to allow a particle to enter inside a soft-core repulsive potential via a two-body collision, leading to the strong fluctuations in the local density. This phenomenon might be related to instabilities of homogeneous density profiles in a Langevin system [3] and a two-lane traffic model [17]. On the other hand, the Langevin system exhibits a lane formation also in high-density regions [3], which might be related to the singularities above $\rho_0$ in the proposed model. Thus, the knowledge obtained by the proposed model might shed new light on lane formations in other systems including more realistic ones with various effects such as inertial effects [18], attractive interactions [19], hydrodynamic interactions [20], and anisotropic dissipations [21]. Related to this, the elucidation of the robustness of the obtained knowledge will be an interesting future issue.

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