ABSTRACT

In this paper, we analyze the effects of expansion on large scale structure formation in our Universe. We do that by incorporating a cosmological constant term in the gravitational partition function. This gravitational partition function with a cosmological constant is used for analyzing the thermodynamics of this system. We analyze the virial expansion for this system, and obtain its equation of state. It is observed that the generalization of this equation of state is like the Van der Waals equation. We also analyze a gravitational phase transition in this system using the mean field theory. We construct the cosmic energy equation for this system of galaxies, and discuss its consequences. We obtain and analyze the distribution function for this system, using the gravitational partition function. We also compare the results obtained in this paper with the observational data.

Key words: Dark energy, Thermodynamics and Statistics, Clustering of Galaxies.

1 INTRODUCTION

The clustering of galaxies is responsible for forming large scale structure in our Universe (Peebles 1980; Peebles 1993; Voit 2005). So, it is very important to analyze the clustering of galaxies, and this can be done using simulation (Frenk et al. 1999; Barnes et al. 2017), or by relating various physical parameters to observational data (Ponman et al. 2003; Voit 2005). In these works, the local matter distribution in galaxies or clusters is analyzed (either by simulation or using observation). However, for analyzing the formation of large scale structure in the Universe, it is more appropriate to approximate each galaxy as a point, analogous to a point particle in a statistical mechanical system. This approximation is valid as the size of galaxies is much smaller than the distance between them. Thus, it is possible to write a statistical mechanical gravitational partition function for this system, and use it for analyzing the large scale structure formation (Saslaw 1986; Saslaw et al. 1990).

It may be noted that this formalism has been used to investigate the thermodynamic description of the cosmological many-body problem and galaxy clustering (Itoh et al. 1993; Saslaw et al., 2011). Here the thermodynamic quantities, like temperature, are obtained using the kinetic theory of gases, with each galaxy acting as a particle analog. It has been observed that such a system can be analyzed using a quasi-equilibrium description, as the macroscopic quantities change slowly compared to local relaxation time scales (Saslaw et al. 2011; Saslaw et al. 1990; Saslaw et al. 1993; Rahmani et al. 2009). As the extended structure of a galaxy is approximated by a point particle, this gravitational partition function can diverge. However, these divergences can be removed by using a softening parameter (Ahmad & Hameeda 2010). This softening parameter incorporates the effects of having an extended structure in the gravitational partition function. This softening parameter also modifies the thermodynamic fluctuations in this system. So, the correct thermodynamics for this system has to be analyzed using this gravitational partition function modified by the softening parameter (Saslaw & Hamilton 1984). The clustering of a system of galaxies has been studied using the grand canonical ensemble of galaxies (Ahmad et al. 2002). This has been done by analyzing the distribution functions and moments of distributions, such as their skewness and
The study of such distribution functions for galaxies has indicated that galaxy clusters are surrounded by individual halos (Sivakoff & Saslaw 2005; Rahmani et al. 2009). The gravitational partition function has also been used to obtain the specific heats and isothermal compressibility for such a system (Ahmad et al. 2006).

It may be noted that the clustering occurs due to gravitational interaction between different galaxies. However, as the gravity pulls the galaxies towards each other (causing clustering), the expansion of the Universe is expected to move the galaxies away from each other. It is now known that the Universe is undergoing accelerated expansion. This is based on observations of Type Ia Supernovae (SNeIa) (Riess et al. 1998; Perlmutter et al. 1998). It may be noted that even though the effects from the cosmological constant are usually neglected on astrophysical scales, it is possible for the cosmological constant to have an impact at the scale of galaxy clusters (Gurzadyn et al. 2020; Gurzadyn et al. 2018; Gurzadyn et al. 2019; Gurzadyn et al. 2013; Gurzadyn et al. 2019). Such effects have been studied in models of interacting dark matter and dark energy (Gurzadyn et al. 2020; Gurzadyn et al. 2018). It has been observed in such interacting models that the dark energy can be constrained from equilibrium states of local galaxy clusters (Delliou et al. 2019). In fact, in such models, it has been argued that local effects from cosmological constant at the scale of galaxy clusters can have important astrophysical consequences (Bonserm et al. 2020). It has been demonstrated that cosmological constant can change the dynamics of a group of galaxies (Peirani et al. 2008). This analysis has been done using five groups of galaxies and the Virgo cluster. It has been argued that cosmological constant can play an important role in the formation of galaxy clusters (Iliev et al. 2001). The effect of dark energy on local galaxy clusters has been studied using the Fisher matrix formalism (Stark et al. 2001). So, the local galaxy clusters contain information about the cosmological constant. In fact, it has been argued that the value of the cosmological constant can be constrained from an accurate measurement of the statistical properties of galaxy clusters (Aguena et al. 2018).

Thus, it is important to analyze the effects of the cosmological constant on the statistical mechanical approach to the clustering of galaxies. The effect of the cosmological constant on the statistical distribution function of galaxies has been studied, and this analysis has been used to constrain the value of the cosmological constant (Wen et al. 2020). As such a statistical distribution function can be obtained from the gravitational partition function, the effects of a cosmological constant term have been also incorporated in the gravitational partition function (Hameeda et al. 2016a).

This gravitational partition function has been used to obtain the Helmholtz free energy for a system of galaxies. This Helmholtz free energy has in turn been used to obtain the entropy of this system. The thermodynamics of this system is used to obtain the clustering parameter for this system, and is used for analyzing the effects of the cosmological constant on the clustering of galaxies. As the internal energy of this system depends on the clustering parameter, which in turn depends on the cosmological constant, the dependence of the internal energy on the cosmological constant has also been studied for this system. Finally, the distribution function for a system of galaxies in an expanding Universe is obtained using the grand canonical gravitational partition. As it is possible to analyze cosmological models with a dynamical time-dependent cosmological constant (Novello et al. 2002, 2001), it is important to analyze the effects of such a time-dependent cosmological constant on the clustering of galaxies. Thus, the gravitational partition function with a time-dependent cosmological constant has also been constructed, and it has been used to study the effects of dynamical dark energy on structure formation in our Universe (Pourhassan et al. 2017). For this model with dynamical dark energy, Helmholtz free energy has been used to obtain the entropy of the system, which in turn has been used to obtain the dependence of the clustering parameter on the dynamical dark energy. The correlation function between galaxies has also been calculated and found to be consistent with observations.

It is possible to modify general relativity by adding higher powers of the curvature tensor to obtain $f(R)$ gravity, and this modification of general relativity also modifies the large-distance behavior of the gravitational potential (Sotiriou & Faraoni 2010). Such a model of $f(R)$ gravity has also been used to study the expansion of the Universe (Sotiriou & Faraoni 2010). The gravitational partition function for $f(R)$ gravity has been used to analyze the effects of $f(R)$ gravity on the large scale structure formation (Capozziello et al. 2018). It was observed that this modification of the gravitational partition function is consistent with observations. The thermodynamics of such a system of galaxies interacting through the modified gravitational potential of $f(R)$ gravity, has also been studied. It has also been demonstrated that $f(R)$ gravity can be constrained using the PLANCK data on galaxy clusters (De Martino et al. 2014). This was done by calculating the pressure profiles of different galaxy clusters. It was assumed that this gas of galaxies was in hydrostatic equilibrium within the $f(R)$ gravitational potential well. It was observed that the profile of this system of galaxies fits the observation data, without requiring dark matter. The thermodynamics of a system of galaxies has also been analyzed using MOND, and this was done by analyzing the modifications to the gravitational partition function from MOND (Upadhyay et al. 2018). It was observed that the modification of the gravitational partition function from MOND also modified the thermodynamic potential for this system, which in turn modified the formation of large scale structure. The modification to the gravitational partition function from MOG has also been studied, and it was observed that the clustering of galaxies depends on the large scale modifications to Newtonian potential in MOG (Hameeda et al. 2019). It has been done by analyzing the thermodynamics of a system of galaxies interacting through MOG Newtonian potential (Hameeda et al. 2019). The clustering in brane-world models has also been studied using a modification to gravitational partition function (Hameeda et al. 2016b). This was done by analyzing the modification to Newtonian potential from super-light brane world perturbative modes. This modified Newtonian potential modifies the thermodynamics of the system, and this changes the large scale structure formation. Thus, it was possible to analyze the effects of super-light brane world perturbative modes on the large scale structure formation in our Universe.
It may be noted that it is also possible to study the gravitational phase transition for a system of galaxies using gravitational partition function (Saslaw et al. 2010; Upadhyay et al. 2019; Khan & Malik 2012). It was observed that a first order phase transition occurs when this system of galaxies starts to cluster from an initial homogeneous phase to a phase with a large scale structure. The phase transition in a system of galaxies has also been analyzed using the gravitational partition function as a function of complex fugacity (Khan & Malik 2013). This was done by extending the Yang-Lee theory to the gravitational phase transition. It was observed that masses of individual galaxies can affect the formation of large scale structure. As it is important to consider the effects of the cosmic expansion on the structure formation, we will analyze the gravitational phase transition for a system of galaxies, with a cosmological constant term. We would like to point out that it is possible to study the clustering in a system of galaxies using cosmic energy equation (Hameeda et al. 2018). Even though the cosmic energy equation is obtained by assuming galaxies as point particles (Wahid et al. 2011), the modification to the cosmic energy equation from the extended structure of galaxies has also been studied (Ahmad et al. 2009). The cosmic energy equation can be used for analyzing the dependence of clustering on the gravitational potential. In fact, the effects of a large distance modification to gravitational potential on clustering have also been analyzed using a modified cosmic energy equation (Hameeda et al. 2018). As it is important to use the gravitational partition function modified by the cosmological constant, in this paper, we will also analyze the cosmic energy equation using such a modified gravitational partition function.

2 GRAVITATIONAL PARTITION FUNCTION

It is known that galaxies interacting via gravitation are not in equilibrium. However, it is possible to analyze such a cosmological system using quasi-equilibrium (Saslaw et al. 2011; Saslaw et al. 1996; Saslaw et al. 1993; Rahmani et al. 2009). This is because for such a cosmological system, the macroscopic quantities change slowly compared to local relaxation timescales, and hence they can be studied using quasi-equilibrium relations. These macroscopic quantities include the average temperature, which is obtained by using the kinetic theory of gases, with each galaxy represented by a particle analog. The pressure and density are also such macroscopic quantities. Now the average density of the Universe is $3.5 \times 10^{19} m_\odot Mpc^{-3}$, whereas most galaxies have an average mass of about $10^{11} m_\odot$ (Spergel et al. 2007). The dynamical timescale of the Universe is $\tau = 25 Gyr$. There are about 0.35 galaxies per cubic megaparsec, with an average peculiar velocity of $Mpc Gyr^{-1}$. So, the time for a galaxy to cross a cell of 3000 galaxies would be around 20 Gyr (Saslaw et al. 2011). However, local time scales for such a system will be much shorter. The density of a cluster of galaxies, like our local group of galaxies, is about $1.5 \times 10^{12} m_\odot Mpc^{-3}$. This is several times greater than the density of the Universe, and so the local group has a dynamical timescale $\tau = 4 Gyr$. As most galaxy clusters have a diameter of 2 – 4 Mpc, the time for a galaxy to cross them is about 3 Gyr (Saslaw et al. 2011). Thus, the microscopic perturbations on local scales will relax much faster than the macroscopic scale (Saslaw et al. 1996). So, this system can evolve from one equilibrium state to another, and can be analyzed using such a quasi-equilibrium description (Saslaw et al. 1993; Rahmani et al. 2009).

So, we can analyze this system, using a large number of galaxies distributed in an ensemble of cells, all of them with same volume $V$, or radius $R$, and average density $\rho$. Both the number of galaxies and their total energies vary among these cells and hence it can be appropriately represented by a grand canonical ensemble. These galaxies within the system have pairwise gravitational interactions generated by the modified Newtonian potential. It is further assumed that the distribution is statistically homogeneous over large regions. Now the temperature for such a system can be obtained from the average kinetic energy of galaxies, using the kinetic theory of gases, with each galaxy representing a particle analog of such a gas. We can thus obtain an average temperature $T$ from a kinetic theory of galaxies, and use it to construct the gravitational partition function. So, the gravitational partition function of a system of $N$ galaxies of mass $m$ interacting through the modified gravitational potential energy $\Phi$ can be written as (Pourhassan et al. 2017; Capozziello et al. 2018; Upadhyay et al. 2018; Hameeda et al. 2019),

$$Z(T, V) = \frac{1}{\Lambda_1 N!} \int d^N p \ d^N r \ \exp \left( - \frac{\sum_{i<j} \phi_{ij} |r_i - r_j|}{T} + \phi(r_1, r_2, r_3, \ldots, r_N, t) \right),$$

(1)

where $\Lambda_1$ is the mean thermal wavelength which is defined as $(2 \pi m T)^{-\frac{1}{2}}$, and $p_i$ is the momentum for different galaxies. Integrating momentum space integral for the phase space, we obtain the following expression

$$Z_N(T, V) = \frac{1}{\Lambda_1^N} \left( \frac{1}{\Lambda_1} \right)^{3N/2} Q_N(T, V).$$

(2)

Here $Q_N(T, V)$ is the configurational integral of the system, and is given by

$$Q_N(T, V) = \int \ldots \int \prod_{1 \leq i < j \leq N} \exp \left[ - \frac{\phi_{ij}}{T} |d^N r| \right].$$

(3)

The gravitational potential energy $\phi_{ij} = \Phi(r_1, r_2, \ldots, r_N, t)$ is a function of the relative position vector $r_{ij} = |r_i - r_j|$ (because the two-body force is a central one), and the cosmological time (because of its dependence on the cosmological expansion $a(t)$). The total potential energy is the sum of the potential energies of all pairs.

This potential energy $\Phi(r_1, r_2, \ldots, r_N, t)$ can be expressed as

$$\Phi(r_1, r_2, \ldots, r_N, t) = \sum_{1 \leq i < j \leq N} \Phi(r_{ij}, t) = \sum_{1 \leq i < j \leq N} \Phi(r, t).$$

(4)

We can use a two-particle Mayer function $f_{ij} = e^{-\phi_{ij}/T} - 1$ to analyze the non-ideal case. This function vanishes in absence of interactions (ideal case), and is non-zero for interacting galaxies. Thus, the configurational integral can be
written as

\[ Q_N(T, V) = \int \cdots \int (1 + f_{12})(1 + f_{13})(1 + f_{23})(1 + f_{14}) \]
\[ \cdots (1 + f_{N-1, N}) d^{3}r_1 d^{3}r_2 \cdots d^{3}r_N. \]  

(5)

It is known that the cosmological constant can change the statistical properties of the system of galaxies (Aguena et al. 2018), such as the statistical distribution function (Wen et al. 2020). As the statistical distribution function can be obtained from the gravitational partition function, it is important to incorporate cosmological constant term in the gravitational partition function. So, we modify the above gravitational potential energy by introducing a cosmological constant term \( \Lambda \) along with an extra parameter that depends on the change in the scale factor \( a \).

Thus, we write the modified potential energy for this system of galaxies as (Shtanov & Sahni 2010)

\[ \Phi(r_{ij}, t) = \frac{Gm^2}{r_{ij}} - \frac{m\Delta r_{ij}^2}{6} + \frac{m\bar{a}r_{ij}^2}{2a}, \]  

(6)

where \( G \) is Newton’s gravitational constant. Moreover, due to the extended nature of galaxies, the softening parameter \( \epsilon \) (Ahmad & Hameeda 2010; Saslaw & Hamilton 1984; Ahmad et al. 2002; Sivakoff & Saslaw 2005; Rahmani et al. 2009; Ahmad et al. 2006) is incorporated in this potential energy term as

\[ \Phi(r_{ij}, t) = \frac{Gm^2}{r_{ij}} + \frac{m\Delta r_{ij}^2}{6} + \frac{m\bar{a}r_{ij}^2}{2a}. \]  

(7)

Now the two-particle Mayer function can be obtained from this expression for the modified potential term. Here, using an approximation for the weak interaction, the effects from the last term for the Mayer function can be neglected. Thus, by retaining only the first term, the two-particle Mayer function modified by the cosmological constant term can be expressed as

\[ f_{ij} = \left( \frac{Gm^2}{T(r_{ij}^2 + \epsilon^2)^{1/2}} + \frac{m\Delta r_{ij}^2}{6T} - \frac{m\bar{a}r_{ij}^2}{2aT} \right). \]  

(8)

In Fig. 1, we can see the behavior of potential energy (7) and Mayer function in terms of \( r_{ij} \). We observe from the blue dashed line that \( f_{ij} \) is bounded in all regions, which produces minus one at infinity and has finite positive values at the origin. It is observed that both \( \Phi \) and \( f_{ij} \) are zero at a single point, as was physically expected.

Now we can analyze the case with \( N = 2 \). For this case, we can write the \( Q_2(T, V) \) as

\[ Q_2(T, V) = 4\pi V \left( \int_0^{R_1} r^2 dr \right) \]
\[ + 4\pi V \left( \frac{Gm^2}{T} \int_0^{R_1} r^2 dr \right) \]
\[ + 4\pi V \left( \left( \frac{\Lambda m}{6T} - \frac{m\bar{a}}{2a} \right) \int_0^{R_1} r^4 dr \right). \]  

(9)

Solving the above integral, we obtain the following expression for \( Q_2(T, V) \)

\[ Q_2(T, V) = V^2 (1 + \alpha x), \]  

(10)

where we defined

\[ \alpha = \sqrt{1 + \frac{\epsilon^2}{R_1^2} + \frac{\epsilon^2}{R_1^2} \ln \left( \frac{\epsilon}{\epsilon + \sqrt{\epsilon^2 + \epsilon^2}} \right)} \]
\[ + 2R_1^2 \left( \frac{\Lambda}{6} \frac{\Lambda}{2a} \right). \]  

(11)

Here \( \alpha \) depends on the free parameters, such as the softening parameter and radius of the cell \( R_1 \). We also observe that a term in \( \alpha \) depends explicitly on the value of the cosmological constant. So, the effect of cosmic expansion on the clustering of galaxies is incorporated through \( \alpha \). We will observe that \( \alpha \) plays an important role in clustering, and as it depends explicitly on the cosmological constant, the clustering will also depend on the rate of cosmic expansion, as is physically expected. Here \( x \) is obtained by first noting that \( 3Gm^2/2R_1 T = 3Gm^2/2\rho^{-1/3} T \). Then by using the scale invariance, \( \rho \to \lambda^{-1} \rho, T \to \lambda^{-1} T \) and \( r \to \lambda r \), we obtain the expression for \( x \) as

\[ x = \frac{3}{2} (Gm^2)^{3/2} \rho^{-3} = \beta \rho^{-3}, \]  

(12)

where \( \beta = \frac{3}{2} (Gm^2)^{3/2} \). Thus, the configurational integral is a function of the average temperature due to the dependence of \( x \) on \( T \). This expression for \( x \) can be used to obtain an explicit expression for any configurational integral. It may be noted that even though such an expression has been derived for \( Q_2(T, V) \), the same procedure can be used to evaluate any other configurational integral.

Thus, by following the same procedure, we can write the general configurational integral as

\[ Q_N(T, V) = V^N (1 + \alpha x)^{N-1}. \]  

(13)

Here we again observe that the general configurational integral also depends on the value of the cosmological constant, due to the dependence of \( \alpha \) on the cosmological constant. Using this general configurational integral, we can write the gravitational partition function for a system of galaxies as

\[ Z_N(T, V) = \frac{1}{N!} (2\pi mT)^{N/2} V^N (1 + \alpha x)^{N-1}. \]  

(14)

This gravitational partition function depends on the cosmo-
The large scale structure formation in an expanding Universe

3 THERMODYNAMICS

The gravitational partition function obtained in the previous section can be used to analyze the thermodynamic behavior of this system of interacting galaxies in an expanding Universe. We can also analyze the dependence of such a system on the scale factor for power law cosmology. We first write the internal energy of this system of galaxies as

\[
U = T^2 \frac{d \ln Z_N}{dT},
\]

where \(Z_N\) is given by equation (14). Using the Eq. (15), we obtain the following expression (assuming \(m = G = 1\)),

\[
U = \frac{3T}{3aR^2_1 - aU_N}.
\]

where \(U_N\) and \(U_d\) are given by

\[
U_N = 10NR_1^5T^3 + \Lambda R_1^5 + 15R_1 \sqrt{R_1^5 + \epsilon^2},
\]

\[
U_d = 10R_1^5T^3 + \Lambda R_1^5 + 15R_1 \sqrt{R_1^5 + \epsilon^2}.
\]

It may be noted that the internal energy of this system depends on the softening parameter, which has been introduced to incorporate the extended nature of the galaxies. This is expected as the internal energy of a thermodynamic system of extended structures is expected to be different from the internal energy of a similar thermodynamic system of point-like structures. The internal energy of the system also depends on the cosmological constant, which is again expected, as we expect the internal energy of such a system of galaxies to depend on the rate of cosmic expansion.

We can analyze the behavior of the Helmholtz free energy using the gravitational partition function modified by the cosmological constant term, \(Z_N\). Thus, using \(Z_N\) from Eq. (14), we can write the Helmholtz free energy modified by the cosmological constant term as

\[
F = -T \ln Z_N.
\]

We observe that for \(N = 1\), the internal energy is related to temperature as \(U = 3T\). So, the internal energy of this system resembles the internal energy of three dimensional harmonic oscillator (in units of Boltzmann constant), and this behavior is expected from the equipartition theorem. In Fig. 3 (a), we analyze the dependence of the internal energy \(U\) on the scale factor \(a\). We can observe that there is a singular point for a small value of the scale factor. The internal energy is initially negative, and then becomes positive, indicating a phase transition. This phase transition corresponds to the maximum value of Helmholtz free energy (see Fig. 3 (b)), which diverges at the initial stage. Thus, it seems from the thermodynamic behavior of this system, that there is a phase transition in it. We would like to point out that it has been suggested that clustering can be regarded as a phase transition (Saslaw et al. 2010; Upadhyay et al. 2019). Here we observe that to be the case from the behavior of its internal energy and Helmholtz free energy. We can also observe from Fig. 3 (b), the Helmholtz free energy becomes minimum at late times. This indicates that the resulting configuration is stable. Thus, with the increasing scale factor, galaxy clusters tend to stabilize. This is expected as the galaxies are expected to cluster, with the expansion of the Universe.

The negative value of the internal energy at the small scale factor or singular point at the initial stage is related to the distance between galaxies. It indicates that there is a minimum value for \(R_1\) where the system will be stable. We can understand this better by analyzing the entropy \(S\) of the system. Entropy \(S\) for a system of galaxies in an expanding Universe can be obtained as

\[
S = U \frac{T}{T} + \ln Z_N.
\]
Using Fig. 4, we can analyze the behavior of the entropy in terms of $R_1$, for various values of the scale factor $a$. We observe that the entropy is positive, indicating that the resulting configuration is physical. However, for the smaller values of $R_1$, entropy is a decreasing function of time. We obtain a critical value of $R_1$ as $R_c \approx 0.8$, and for any $R_1 \geq R_c$, the entropy is an increasing function of time, indicating that the approximation is valid for $R_1 \geq R_c$. Thus, it seems that this approximation is only valid up to a critical size of the cell, and below such a value, the physical system cannot be expressed by this approximation. A stable phase for the system is attained due to the accelerating expansion of the Universe. This can also be seen by analyzing the heat capacity of this system. The heat capacity at constant volume $C_V$ of this system can be expressed as

$$C_V = \left( \frac{dU}{dT} \right)_V.$$  

The behavior of the specific heat at constant volume is given by Fig. 5. We can observe that the specific heat is negative at the early stages. Also, we can observe that the heat capacity rises to a maximum, which is like a Schottky anomaly (appears in some of the two-level systems).

Now we discuss the special cases, by choosing specific forms of time-dependence for the scale factor. We can consider the special case of power law dependence of the scale factor,

$$a = a_0 t^n,$$  

where $a_0$ is a constant and $n$ is a real number. In a Friedmann-Lemaître-Robertson-Walker Universe, the scale factor $a$ gives us the value of $n$, so that in the radiation-dominated Universe, $n = \frac{1}{2}$, while in a matter-dominated Universe, $n = \frac{2}{3}$. In those cases, one can obtain Hubble expansion parameter as $H \propto \frac{1}{t}$. On the other hand, holographic dark energy models suggest that Hubble parameter $H$ is proportional to the length, $H \propto \frac{1}{R_1}$. Under this assumption, equation (21) changes to

$$a = a_1 R_1^n,$$  

where $a_1$ is another constant. Hence, we can obtain, $\ddot{a} = \textit{MNRAS} 000, 000–000 (0000)$
Figure 6. Behavior of $\alpha$ for $m = G = \Lambda = 1$. (a) In terms of $n$ for $R_1 = 1$; (b) In terms of $R_1$ for $\epsilon = 0.1$.

Figure 7. Behavior of $F$ for $m = G = \Lambda = 1$. (a) In terms of $n$ for $R_1 = 1$; (b) In terms of $R_1$ for $n = 0.5$.

$n(n - 1)R_1^{-2}a$. Now for this case, the Eq. (11) can be expressed as

$$
\alpha = \sqrt{1 + \epsilon^2 R_1^2 + \epsilon^2 R_1^2 \ln \left( \frac{\epsilon}{R_1 + \sqrt{R_1^2 + \epsilon^2}} \right)} + \frac{2R_1^3}{5} \left( \frac{\Lambda}{6} - \frac{n(n - 1)}{2R_1^2} \right),
$$

(23)

where we assumed $G = m = 1$. As the gravitational partition function is proportional to $\alpha$, it is important to analyze the behavior of $\alpha$ in such models. This is because $\alpha$ depends on the cosmological constant, and this cosmological constant can be used to analyze the effects of the dark matter and dark energy, in models of interacting dark matter and dark energy (Gurzadyan et al. 2020; Gurzadyan et al. 2018; Gurzadyan et al. 2019; Gurzadyan et al. 2013; Gurzadyan et al. 2019). In the plots of Fig. 6, we observe the behavior of $\alpha$ for various values of $n$ and $R_1$. From Eq. (23), we find that $\frac{dn}{d\alpha} = 0$ yields $n = \frac{1}{2}$. Hence, maximum value of $\alpha$ is obtained in the radiation-dominated Universe, as illustrated in Fig. 6 (a). Also, in Fig. 6 (b), we can see that $\alpha$ increases with increasing $R_1$.

The maximum of $\alpha$ corresponds to the minimum of the Helmholtz free energy. This behavior can be obtained from Eq. (18) and is illustrated in Fig. 7. In fact, we observe from Fig. 7 (a), that the minimum of the Helmholtz free energy, which is equilibrium state of a given system, corresponds to $n = \frac{1}{2}$. Also it is obvious from Fig. 7 (b) that the Helmholtz free energy increases by increasing $R_1$. Moreover, we can see that for $n < 3$ the Helmholtz free energy is negative.

4 VIRIAL EXPANSION

It is possible to study virial expansion for this system of galaxies interacting through a gravitational potential in an expanding Universe (Ahmad et al. 2002; Ahmad et al. 2006). This virial expansion can be used to obtain the equation of state for this system. Now we will use the gravitational partition function modified by the cosmological constant term to analyze the effect of cosmological constant term on the equation of state. Thus, for the case of large galaxy clustering (in the limit $V \rightarrow \infty$), with fugacity $z = \exp(\mu/T)$ ($\mu$ as the chemical potential of the given
system), one can write
\[
\frac{P}{T} = \frac{1}{\Lambda_1} \sum_{\nu=1}^{\infty} I_\nu z^\nu, \quad \frac{N}{V} = \frac{1}{\Lambda_1} \sum_{\nu=1}^{\infty} \nu I_\nu z^{\nu-1}.
\] (24)

It may be noted that here \(I_\nu\) is the clustering integral, which is a dimensionless parameter, and is given by
\[
I_\nu = \frac{1}{\nu \Lambda_1^{\nu-1} V} \int \left( \sum_{ij,kl} f_{ij} f_{kl} + \prod_{r=1}^{\nu} f_{rj} \right) d^3 r_1 \cdots d^3 r_\nu. \quad (25)
\]

It is easy to find that \(I_1 = 1\). This clustering integral can be written in terms of the Mayer function, and it has been observed that such Mayer function depends on the cosmological constant. So the value of this clustering integral would also depend on the cosmological constant. This can be seen, in the simplest case of \(\nu = 2\), as we can write
\[
I_2 = \frac{m}{2 \Lambda_1^{\nu-1} V} \int_0^{R_1} \left( \frac{G m}{T (r^2 + \epsilon^2)^{1/2}} + \frac{N r^2}{6 T} - \frac{\bar{a} r^2}{2 a T} \right) r^2 dr
\]
\[
= \frac{G m^2 R_1}{4 T \Lambda_1^2} \sqrt{R_1^2 + \epsilon^2}
- \frac{G m^2 \epsilon^2}{4 T \Lambda_1^2} \left( \ln (R_1 + \sqrt{R_1^2 + \epsilon^2}) - \ln \epsilon \right)
+ \frac{m}{4 T \Lambda_1^2} \left( \frac{R_1^5}{2} \left( \frac{\Lambda}{3} - \frac{\bar{a}}{a} \right) \right). \quad (26)
\]

Now we can write the case of \(\nu = 3\) as follows
\[
I_3 = \frac{1}{6 \Lambda_1^{1/2} V} \int f_{123} d^3 r_1 d^3 r_2 d^3 r_3, \quad (27)
\]
where \(f_{123}\) can be expressed as \(f_{123} \equiv f_{12} f_{13} + f_{12} f_{23} + f_{13} f_{23} + f_{12} f_{13} f_{23}\). Here we observe that \(f_{123}\) can be expressed using \(f_{ij}\), and the dependence of \(f_{ij}\) on the cosmological constant is known. Thus, we can see that \(f_{123}\) explicitly depends on the value of the cosmological constant. We can repeat this procedure for higher values of \(\nu\).

Eliminating fugacity \(z\) from Eq. (24), one can obtain the clustering equation of state. It has the following virial expansion
\[
\frac{PV}{N T} = \sum_{\nu=1}^{\infty} c_\nu(T) \left( \frac{\Lambda_1^{1/2} N}{V} \right)^{\nu-1}, \quad (28)
\]
where \(c_\nu(T)\) is called the virial coefficient. In case of \(\nu = 1\), we obtain first virial coefficient \(c_1 = I_1 = 1\), hence the Eq. (28) reduces to the equation of state for an ideal gas.

Other virial coefficients can also be expressed in terms of the clustering integral. For example, one can obtain
\[
c_2 = -I_2,
\]
\[
c_3 = 4 I_2^2 - 2 I_3,
\]
\[
c_4 = -20 I_2^3 + 18 I_2 I_3 - 3 I_4. \quad (29)
\]

In Fig. 8, we see the behavior of the second virial coefficient \(c_2\) for the model parameters. In Fig. 8 (a), the effect of the softening parameter \(\epsilon\) on different values of the scale factor \(a\) is shown. It is observed that the \(c_2\) is an increasing function of \(\epsilon\) with both positive and negative values (depending upon scale factor \(a\)). Now for a small value of \(\epsilon\), we find that parameter \(c_2\) is approximately a constant. Fig. 8 (b) depicts that \(c_2\) increases as the separation between galaxies increases. In this case, we find that infinitesimal value of \(R_1\) produces negative value for second virial coefficient. Fig. 8 (c) shows variation of \(c_2\) in terms of scale factor \(a\). We observe that it is a decreasing function of \(a\), which approaches a constant value for the larger \(a\) (late time behavior). Finally, looking at Fig. 8 (d), we see that \(c_2\) is a decreasing function of temperature \(T\). It is observed that low \(T\) behavior is similar for various values of \(\epsilon\).

Now using the first order approximation (reasonably for \(V \to \infty\)), we obtain
\[
\frac{PV}{N T} = 1 - \left( \frac{\Lambda_1^{1/2} N}{V} \right) I_2 + O(\frac{1}{\sqrt{T}^2}). \quad (30)
\]
where \(I_2\) is given by Eq. (26). We thus rewrite Eq. (26) in the following form,
\[
I_2 = \frac{f(R_1)}{2 \Lambda_1^{1/2} T}, \quad (31)
\]
where we can write \(f(R_1)\) as
\[
f(R_1) = \frac{G m^2}{2} R_1 \sqrt{R_1^2 + \epsilon^2}
- \frac{G m^2 \epsilon^2}{2} \left( \ln (R_1 + \sqrt{R_1^2 + \epsilon^2}) - \ln \epsilon \right)
+ \frac{m}{2} \left( \frac{R_1^5}{5} \left( \frac{\Lambda}{3} - \frac{\bar{a}}{a} \right) \right). \quad (32)
\]

This clustering integral depends explicitly on the cosmological constant and the softening parameter. It may be noted that the softening parameter was introduced to incorporate the extended structure of galaxies in the gravitational partition function (Ahmad & Hameeda 2010; Saslaw & Hamilton 1984; Ahmad et al. 2002; Sivakoff & Saslaw 2005; Rahmani et al. 2009; Ahmad et al. 2006). This modification of the gravitational partition function in turn modified the value of the clustering integral. We will now observe that this modification of the cluster
modifies the equation of state for this system. So, with this new definition, we write the equation of state (30) as

$$\left( P + \frac{a_v}{v^3} \right)^2 = T$$

(33)

where $v = V/N$ is the volume per number of galaxies, and $a_v \equiv f(R_t)/2$, which is proportional to the clustering integral. This plays the role of the interaction strength of galaxies (comparing with the Van der Waals gas equation of state). Hence, we can interpret $a_v$ as a measure of the strength of gravitational interaction between galaxies. It may be noted that the value of $a_v$ depends on the cosmological constant, and the behavior of the system would change due to the cosmological constant term. Furthermore, in absence of any interactions, such terms would vanish, and the system of galaxies would be approximated by an ideal gas of galaxies. So, the equation of state for this gas of galaxies would resemble the equation of state for an ideal gas. In fact, as $b_v$ depends on the extended structure of galaxies, it would also vanish if we neglect the effects due to such extended structure. Thus, using such an approximation, we note that $b_v = 0$. However, due to interactions, the equation of state should resemble the Van der Waals. Thus, we observe that Eq. (33) is the usual Van der Waals equation of state for a system of interacting galaxies. This can be used for studying clustering as a phase transition.

5 MEAN FIELD THEORY OF CLUSTERING PHASE TRANSITION

In the previous section, by using virial expansion, we derived Van der Waals equation of state for a system of interacting galaxies. This was done by analyzing such a system of interacting galaxies in an expanding Universe. Furthermore, we also analyzed the effect of the cosmological constant and the scale factor $a$ on clustering. It was also observed that the thermodynamic behavior of this system indicates that there is a gravitational phase transition in it. Now we will analyze such a gravitational phase transition using mean field theory. Such a mean field theory for phase transition in this system can be analyzed (Huang 1987), using the Van der Waals equation. It has been argued that the gravitational clustering of galaxies can be regarded as a form of phase transition (Saslaw et al. 2010; Upadhyay et al. 2019). The clustering of galaxies occurs by passing through a mixed phase regime, in which some parts of the system have clustered, and others have not clustered. So, we can analyze this system as a first order phase transition. To analyze the clustering of galaxies, we extend the previous analysis to a non-zero $b_v$, and use the mean field theory to analyze the phase transition. We note that it is possible to find Landau free energy that can produce the required behavior, when it is minimized with respect to the order parameter. Here we choose $v$ as the order parameter and $P$ as the conjugate field. Then we can minimize the Landau free energy $\psi(v, P, T)$. Thus, we can write

$$\frac{\partial \psi}{\partial v} = 0.$$  

(34)

We can also use $\psi(v, P, T) = Pv - v^{-1}a_v - T\ln(v-b_v)$. Now minimizing this expression of $\psi(v, P, T)$, we can write

$$P = \frac{T}{v-b_v} - \frac{a_v}{v^2}.$$  

(35)

This is the behavior of the pressure that we had obtained from the Van der Waals equation of state. Now as the average kinetic energy of galaxies would remain the same, the temperature of this system would not change. Thus, we need to analyze the change in the volume for such a system which can be measured using the isothermal compressibility. Thus, we can write the isothermal compressibility of this system as

$$\beta T = -\frac{1}{v} \left( \frac{\partial v}{\partial P} \right)_T = \frac{(1-b_v)^2}{v(T-2a_v(1-b_v)^2)}.$$  

(36)

Approaching the critical point from $T > 1$, along $v = 1$, we write the compressibility as

$$\beta T = \frac{(1-b_v)^2}{(1-b_v)^2}.$$  

(37)

Here we have denoted the critical temperature by $T_c$. Let us analyze the compressibility in the limit $\tau \to 0$. We observe that this gives a finite $\beta T$

$$\beta T = \frac{(1-b_v)^2}{1-2a_v(1-b_v)^2}.$$  

(39)

Thus, in the limit $\tau \to 0$, this thermodynamic quantity should have a singular part in addition to the regular part. For the phase transition to take place as $\tau \to 0$, $\beta T$ should be infinite. So, the phase transitions can take place at $\tau = 2a_v(1-b_v)^2 - 1$. Thus, we can write

$$T_c = T_c(2a_v(1-b_v)^2).$$  

(40)

So, in our case, critical temperature changes from $T_c$ to $T_c'$. It may be noted that this critical temperature depends on $b_v$, and hence on the extended structure of galaxies. The critical temperature represents the phase transition from a homogeneous phase to a clustered phase.

Corresponding to a mean field theory in the region of a first order phase transition, Landau free energy $\psi$ must have two minima at volume $v_1$ and $v_2$. The two conditions (continuity of functions and their derivatives) have to be satisfied as

$$\left( \frac{\partial \psi}{\partial v} \right)_{v=v_1} = \left( \frac{\partial \psi}{\partial v} \right)_{v=v_2},$$  

(41)

$$\psi(v_1) = \psi(v_2).$$  

(42)

This leads to the following equation for temperature

$$T\left( \frac{1}{v_1 - \epsilon} - \frac{1}{v_2 - \epsilon} - a_v \frac{1}{v_1^2} - \frac{1}{v_2^2} \right) = 0.$$  

(43)

We can also write the following equation for the pressure of this system

$$P(v_1 - v_2) = \int_{v_2}^{v_1} Pdv.$$  

(44)

The Maxwell construction is required here as the volume continues to increase, but the pressure remains constant. In the Maxwell construction, two coexisting volumes
\(v_1\) and \(v_2\), are symmetrically placed around the critical volume, say \(v\). We can define these two volumes as \(v_1 \equiv 1 + \delta\), and \(v_2 \equiv 1 - \delta\), with \(\delta\) as a small parameter. Hence, by using the Maxwell construction and results obtained in the previous section, we can write
\[
\frac{2\delta(1 + \tau)}{(1 - b_0)^2 - \delta^2} + a_0 \left(\frac{1}{(1 + \delta)^2} - \frac{1}{(1 - \delta)^2}\right) = 0.
\]
A possible solution to the Eq. (45) can be expressed as
\[
\delta \approx \sqrt{(1 + \tau) - 2a_0(1 - b_0)^2},
\]
where we neglected the terms of \(O(\delta^4)\). There is a coexisting volume at phase transition when \(\delta \to 0\) corresponding to \(\tau = 2a_0(1 - b_0)^2 - 1\). Thus, Maxwell construction provides valuable information about the phase transition. Now by neglecting the extended structure of galaxies, we can use \(b_0 = 0\), and write \(\tau\) as
\[
\tau = 2a_0 - 1 = f(R_1) - 1 = 2\Lambda \frac{\delta^2}{T_2} - 1.
\]
Hence, under assumption \(T = m = 1\), we can express \(\tau\) as \(\tau = \left(I_2/\sqrt{2\tau^2}\right) - 1\). Therefore, we can observe that \(\tau\) is proportional to \(T_2\)
\[
\tau \propto I_2.
\]
In that case, the behavior of \(\tau\) is represented by the dotted green line of Fig. 8 (b). Thus, \(\tau \to 0\) as \(R_1 \to 0\), as pointed out before. It may be noted that it was known that the clustering can be analyzed as a gravitational phase transition in absence of the modification of the gravitational partition function by a cosmological constant term (Saslaw et al. 2010; Upadhyay et al. 2019). However, in this paper, we have demonstrated that it can still be viewed as a phase transition, even after the gravitational partition function was modified by the cosmological constant term. Furthermore, in this paper, this was done using the Maxwell construction.

6 COSMIC ENERGY EQUATION

In this section, we discuss cosmic energy equation, as it can provide important information about clustering of galaxies (Voit 2005; Peebles 1980; Peebles 1993). The cosmic energy equation has been used to analyze the effects of cosmic expansion on a large ensemble of pressure-less galaxies interacting via a Newtonian gravitational potential (Voit 2005; Peebles 1980; Peebles 1993). The extended structure of galaxies has been incorporated into the cosmic energy equation using the softening parameter (Ahmad et al. 2009), and this has been done for other non-point like masses (Wahid et al. 2011). A large distance modification to the Newtonian potential has also been used to study a modification of the cosmic energy equation (Hameeda et al. 2018). In the present work, we analyze the modification of the cosmic energy equation by a cosmological constant term. For a system of galaxies with the internal energy \(U\), pressure \(P\), and scale factor \(a(t)\), the first law of thermodynamics can be written as
\[
\frac{d(Ua^3)}{dt} + P\frac{da^3}{dt} = 0.
\]
Writing the equations for energy \(U\) and pressure \(P\) in terms of the potential modified by a cosmological constant term, we obtain
\[
U = \frac{3}{2}NT + \frac{N\rho}{2} \int_v \Phi(r) (\xi(r)) 4\pi r^2 dr,
\]
\[
P = \frac{NT}{V} - \frac{\rho^2}{6} \int_v r \frac{d\Phi(r)}{dr} (\xi(r)) 4\pi r^2 dr,
\]
where \(\rho\) is density number and \(\xi (r)\) is the correlation function which gives the probability of finding another object in a given radius. The integral for the correlation function over a certain volume is obtained using the mean square number fluctuation as
\[
\int \xi dV = \frac{(2 - bb)}{(1 - b)^2},
\]
where we have used \(\partial b/\partial V = -(x/V)(\partial b/\partial x) = -(b/(1-b))/V\) and the Eq. (12). Here \(b\) is the clustering parameter, which measures the clustering in the system (Ahmad et al. 2002; Hameeda et al. 2016a). Now by using Eq. (6), we can write the above parameters as
\[
U = \frac{3}{2}NT + W_e + W_M,
\]
\[
P = \frac{3NT}{3V} + W_e + c^2 W'_e - 2W_M.
\]
Here we can write \(W_e, W_M, W'_e\) as
\[
W_e = \frac{GN\rho m^2}{2} \int \frac{\xi(r)}{(r^2 + c^2)^2} 4\pi r^2 dr,
\]
\[
W_M = \frac{N\rho m^2}{2} \left(\frac{2a}{\Lambda} - \frac{\Lambda}{6}\right) \int r^2 \xi(r) 4\pi r^2 dr,
\]
\[
W'_e = \frac{GN\rho m^2}{2} \int \frac{\xi(r)}{(r^2 + c^2)^2} 4\pi r^2 dr.
\]
It may be noted that the contribution from the cosmological constant comes from \(W_M\), and the original contributions come from \(W_e\) and \(W'_e\) (Ahmad et al. 2009). Now it is known that we can write the conservation of energy for such a system using the cosmic energy equation (Hameeda et al. 2018; Hameeda et al. 2016b). So, we can write the conservation of energy for this system modified by the cosmological constant term as
\[
\frac{d(K + W_\Lambda)}{dt} + \frac{\dot{a}}{a} \left(2K + W_\Lambda(1 + \eta)\right) = 0,
\]
where \(K\) is the kinetic energy and \(W_\Lambda = W_e + W_M\) is the total correlation energy in the presence of cosmological constant \(\Lambda\). This is the cosmic energy equation for \(\Lambda\)CDM model.

The cosmic energy equation derived above can be simplified by using the definition of clustering parameter \(b_\Lambda\), which is the ratio of gravitational correlation energy \(W_\Lambda\) to kinetic energy \(K\), for a given value of \(\Lambda\) (Ahmad & Hameeda 2010)
\[
b_\Lambda = \frac{W_\Lambda}{2K}.
\]
It may be noted that we can use \(y_\Lambda(t) = 1/b_\Lambda(t)\) to simplify the cosmic energy equation (Saslaw 1986; Saslaw et al. 1990). So, we can write the cosmic energy equation for a system modified by the cosmological constant as
\[
\frac{dy_\Lambda(t)}{dt} - \frac{2 - y_\Lambda}{W_\Lambda} \frac{dW_\Lambda}{dt} - \frac{2a}{a} (1 - y_\Lambda + \eta) = 0.
\]
We use the power law form of the correlation energy \( W_\Lambda(t) \propto t^c \), where \( \omega \) is a real number. The power law form of the scale factor given in Eq. (21) can be used to solve Eq. (60), and we can write such a solution as

\[
y_\Lambda(t) = y_c + (y_0 - y_c) \left( \frac{a(t)}{a_0} \right)^j,
\]

where \( j = -(3\omega/2 + n)/n \) and \( y_0 = 1/b_0 \). Here \( y_c = 1/b_0 \) is critical value of \( b \) at which the system is virialized. We can also write \( \eta = c^2 W'/3W_\Lambda/W_\Lambda \). We can use it to determine the critical value of the clustering parameter. So, using \( \eta \), we can express this critical value as \( y_\Lambda = (2\omega + 2n + 2n_j)/(\omega + 2n) \). Now as for the power law, we have \( \omega \sim 1 - \bar{n}/3 \), so we can write this critical value as

\[
b_c = \frac{5 - \bar{n}}{6 - 2n + 4\eta}.
\]

Thus, it is possible to obtain an explicit expression for \( b_c \) in terms of \( \bar{n} \) and \( \eta \). However, as \( \eta \) is very small, it can be ignored. So, the extended structure of galaxies does not contribute to \( b_c \). This indicates that \( b_c \) is independent of such local modifications to the gravitational potential. It only depends on the value of \( \bar{n} \). Now we can obtain the expression for \( b_{\Lambda} \) from Eq. (61) as

\[
b_{\Lambda} = \frac{b_c}{1 + \left( \frac{\bar{n}}{m} - 1 \right) \left( \frac{\bar{n}}{m} \right)^j}.
\]

We can use this expression for \( b_{\Lambda} \) to analyze the relation between the redshift and clustering. So, using the relation \( z = a_0/a \), where \( z \) is the red shift and \( a_0 \) is current value of the scale factor, we can study the variation of \( b_{\Lambda} \) with \( z \) for different models i.e., for different values of \( \bar{n} : 1, 0, -1, -2 \). These different values of \( \bar{n} \) correspond to different values of \( \omega \) because \( \omega \) is related to \( \bar{n} \) as \( 1 - \bar{n}/3 \). So, for \( \bar{n} = 2, 1, 0, -1, -2, -3 \), the corresponding values of \( b_c \) are 1.5, 1.0, 0.83, 0.75, 0.67, 0.66, respectively. Now for \( \bar{n} = 3 \), \( b_c \) diverges. It is possible to analyze the variation of \( b_{\Lambda}(z) \) with \( z \), by fixing \( b_0 \) at a fixed minimum value.

In Fig. 9, we plot \( b_{\Lambda} \) against \( z \) to understand its time-dependence. This is done by fixing \( b_0 = 0.6 \) (with \( G = m = 1 \)). In Fig. 9 (a), we analyze the behavior of \( b_{\Lambda} \) for \( \bar{n} = 2 \). Here \( \omega = -1/3 \) and so \( b_c = 1.5 \). In a matter-dominated Universe, \( b_{\Lambda} \) decreases suddenly and becomes a constant. In Fig. 9 (b), we analyze the case \( \bar{n} = 1 \), with \( \omega = 0 \), and so \( b_c = 1 \). Here \( b_{\Lambda} \) is a decreasing function of \( z \). However, it becomes a constant at high values of the redshift. In all these three plots, the behavior of \( b_{\Lambda} \) corresponding to the radiation-dominated Universe (\( n = 1/2 \)) is represented by solid red line, that of a matter-dominated Universe is represented by dashed blue line (\( n = 2/3 \)) and dash dotted green line for \( n = 1/3 \). Then, we study the case of \( \bar{n} = 0 \), for which \( \omega = 1/3 \) and so \( b_c = 0.83 \). It is illustrated in Fig. 9 (c).

For all other values mentioned above, we can observe similar behavior i.e., \( b_{\Lambda} \) is a decreasing function of redshift for \( \omega > \omega_r \). We also observe that \( \omega_r \) has a negative value and is independent of the redshift \( (\omega = \omega_r \) is a singular point \).

It may be noted that using the N-body simulation (Farieta et al. 2019), it was observed that \( b_{\Lambda} \) is an increasing function of the redshift. This is possible if we chose \( \omega < \omega_c \). This is illustrated in Fig. 10, which depicts behavior of \( b_{\Lambda} \) in terms of \( \omega \). Now \( n = 1/2 \) produces \( \omega_r \approx -0.34 \). In Fig. 10 (a), we show that \( \omega_r \) is increasing (decreasing) function of redshift for \( \omega < \omega_r \) (\( \omega > \omega_r \)). We also observe that there is a singular point for \( \omega = \omega_c \). We see that \( b_{\Lambda} \) at \( z = 0 \) is constant. It is equal to initial value of \( b_0 = 0.6 \). The value of \( \omega_r \) only depends on \( \bar{n} \), which is shown in Fig. 10 (b). We can observe that for the larger values of \( n \), \( \omega_r \) is smaller.

Now we can compare the results of this paper with observations using redshifts for (Hong et al. 2012). The value of \( b_{\Lambda} \) varies from 0 \( \leq b_{\Lambda} \leq 1 \), as expected from physical considerations. However, in order to check validity of our method, we can use some specific values of the temperature. These values are chosen to numerically analyze the behavior of the system. We can use our result from Eqs. (63) and (62) to write \( b_0 \) in terms of the temperature. Then, we obtain a relation for the temperature in terms of the redshift. In order to do that, we use the approximation, \( \eta = AT^2 \), where \( A \) is a constant. Now we can plot the correlation function given by Eq. (52), and compare it with the observational data (Hong et al. 2012). Two-side arrows of Fig.

![Figure 9. Behavior of \( b_{\Lambda} \) in terms of the redshift for \( b_0 = 0.6 \) and \( \omega > \omega_r \).](image-url)
11, obtained from (Hong et al. 2012), are consistent with our calculations, for $T \approx 0.7$ (see solid red line of Fig. 11). Thus, it is possible to fix the temperature from observations. It may be noted that this temperature is obtained from a kinetic theory of gases, with each galaxy representing a point like particle of such a gas. This temperature has been fixed from observations of redshifts of galaxies. It has been also shown that the correlation function behaves as a power law on small scales ($R_1 < 5$). Such a behavior has been previously observed for the correlation function (Bahcall & Soneira 1983). In fact, we find the general behavior of the correlation function (see Fig. 11) in agreement with the best-fit Λ-CDM model (Hong et al. 2012). It also coincides with 11103 and 13904 clusters (see Two-side arrows of Fig. 11 which are obtained from Hong et al. 2012), with known redshifts (Wen et al. 2009).

7 DISTRIBUTION FUNCTION

It is also possible to calculate the distribution function for this system. This distribution function can then be used to analyze the kinetic energy fluctuations for a system of galaxies. These kinetic energy fluctuations can be used to relate this model to the observational data. Now as the galaxies behave as point particles, we can obtain such distribution functions for this system using the standard methods of statistical mechanics. Thus, we can write the probability of finding $N$-galaxies in the grand canonical ensemble as

$$F(N) = \frac{\sum e^{\frac{N\mu}{T}} e^{-\frac{\mathcal{H}_G}}}{Z_G(T,V,z)},$$

(64)

where $Z_G$ is the grand partition function defined by (Huang 1987):

$$Z_G(T,V,z) = \sum_{N=0}^{\infty} z^N Z_N(V,T),$$

(65)

and $z = \exp(\mu/T)$ is the activity.

Now this grand partition function can be expressed as

$$\ln Z_G = \frac{PV}{T} = \bar{N}(1 - b),$$

where $\bar{N}$ is the average number of galaxies in a cell. So, we can express the probability of finding $N$-galaxies as

$$\ln Z_G = \frac{PV}{T} = \bar{N}(1 - b),$$

(66)

This is in agreement with the previous works (Ahmad et al. 2002; Saslaw & Hamilton 1984). The basic assumption for a quasi-equilibrium is that the fluctuations in potential energy over a given volume are proportional to the fluctuations of local kinetic energy. Thus, for $N$ galaxies and assuming $N$ to be very large, we obtain

$$\frac{Gm^2N(N-1)}{2} \left\langle \frac{1}{r^2} \right\rangle = \alpha_1 \frac{Nm^2}{2} v^2,$$

(67)

where $\left\langle \frac{1}{r^2} \right\rangle$ is given by

$$\left\langle \frac{1}{r^2} \right\rangle = \left\langle \left( \frac{1}{r^2 + \epsilon^2} \right)^2 \right\rangle = \frac{1}{Gm} \left( \frac{\Lambda}{6} - \frac{a}{2m} \right) \langle r \rangle.$$

(68)

Assuming $\langle N \rangle = 1$, with $G = m = R = 1$, we obtain

$$\alpha_1 = \left\langle \frac{1}{r^2} \right\rangle \left\langle \frac{\Lambda}{6} - \frac{a}{2m} \right\rangle \langle r \rangle,$$

(69)

where $v$ is the peculiar velocity.

Now, we rescale $F(N)$ from density fluctuations to kinetic energy fluctuations. This is done by replacing $N$ with $N(\frac{\Lambda}{6})$ and replacing the average number $\bar{N}$ with $\bar{N}(\frac{\Lambda}{6})$. 

Figure 10. Behavior of $b_\Lambda$ in terms of the $\omega$ for $b_0 = 0.6$.

Figure 11. Correlation function in unit volume. Two-side arrows show observational data.
Furthermore, we substitute $\alpha_1 v^2$ for $N\langle v \rangle$ and $\alpha_1 \langle v^2 \rangle$ for $N\langle \frac{1}{2} v' \rangle$. Finally, using $N! = \Gamma(N + 1)$, we can express the kinetic energy fluctuations as velocity fluctuations using the Jacobian $2\alpha_1 v$.

$$f(v) = 2\alpha_1^2 v^2 (1 - b) \times \exp \left( -\alpha_1 \langle v^2 \rangle (1 - b) - \alpha_1 b v^2 \right) v \quad (71)$$

Even though the modification of the gravitational distribution function from cosmological constant term has been studied (Wen et al. 2020), here we have explicitly obtained it from the modified gravitational partition function. We also note that the results obtained here are in agreement with the earlier results (Saslaw et al. 1990; Saslaw & Yang 2009). Thus, a Gaussian-like distribution (see Fig. 12) is in agreement with earlier observations (Raychaudhury & Saslaw 1996). Here we have been able to show that this behavior still holds for the system even after the gravitational partition function has been modified by a cosmological constant term. Though this was physically expected, here it has been explicitly demonstrated.

8 SUMMARY

In this paper, we have analyzed the effects of expansion of the Universe on the structure formation in the Universe. This was done by using the gravitational partition function. As the distance between galaxies is much larger than the size of the galaxies, it is possible to approximate galaxies as point particles in this gravitational partition function. These point particles interact through a gravitational potential. The gravitational force pulls these galaxies towards each other, leading to the formation of large scale structure in our Universe. However, the expansion of the Universe moves these galaxies away from each other. Thus, it is important to analyze the effects of the expansion of the Universe on the structure formation in our Universe. This can be done by incorporating a cosmological constant term in this gravitational partition function. We have also used this gravitational partition function with a cosmological constant term to study the thermodynamics for this system.

Then, we have used the virial expansion to obtain equation of state for this system. We have modeled this system of galaxies as a Van der Waals gas. Here the Van der Waals term was obtained from the interaction of galaxies with each other. It is interesting to note that Van der Waals behavior in a different context has been studied for clustering of galaxies (Baldal et al. 2013). We have also analyzed a gravitational phase transition in this system. This was done by using the mean field theory for this system of galaxies. We have also analyzed the effect of cosmological constant on the cosmic energy equation, which was later used for analyzing the time evolution of the clustering parameter. We have also compared our model with the observational data and used it to constrain the free parameters of our model. This was done using both the cosmic energy equation and distribution function.

It may be noted that the modification of gravitational partition function from $f(R)$ gravity has also been analyzed (Capozziello et al. 2018). It was observed that this modified partition function was consistent with the observations. It would be interesting to analyze the gravitational phase transition for this system. The details of such a gravitational phase transition would depend on the specific kind of $f(R)$ gravity model. It would also be interesting to obtain the cosmic energy equation for this modified gravitational partition function and use it for analyzing the effects of $f(R)$ gravity on the time evolution of the clustering parameter. By choosing different models of $f(R)$ gravity, it is plausible to analyze clustering and the dependence of clustering parameter on those models. Furthermore, the correlation between galaxies can be studied in those models, which can later be compared with observations and used to constrain the free parameters in $f(R)$ gravity models. As it is possible to use the gravitational partition function for analyzing clustering in MOND (Upadhyay et al. 2018), it would be interesting to analyze gravitational phase transition using MOND. It would also be possible to perform such a calculation for MOG, as MOG predicts a large scale modification of gravitational potential (Hameeda et al. 2019). It would be interesting to obtain the distribution of different galaxies for such modified theories of gravity, and then compare it with observations. These observations can then be used to constrain certain free parameters in these models of modified gravity. It may be noted that it is possible to obtain large scale correction to the gravitational potential using brane-world models and then use this modified gravitational potential to analyze gravitational partition function for brane-world models (Hameeda et al. 2016b). These large scale corrections to the gravitational potential are obtained from the super-light brane-world perturbative modes. This gravitational partition function can then be used to obtain the dependence of the clustering on large extra dimensions. These effects could be observed in a system of galaxies, and so clustering of galaxies can be used to constrain the size of such large extra dimensions.

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REFERENCES

Ahmad, F., & Hameeda, M. 2010, Ap&SS, 330, 227
Ahmad, F., & Saslaw, W.C., & Bhat, N. I. 2002, ApJ, 571, 576
Ahmad, F., Saslaw, W.C., & Malik, M. A. 2006, ApJ, 645, 940
Ahmad, F., Wahid A., Malik M. A., & Masood, S. 2009, IJMPD, 18, 119
Bahcall, N. A. & Soneira, R. M. 1983, ApJ, 270, 20
Baldauf, T., Seljak, U., Smith, R.E., Hamaus, N., & Desjacques, V. 2013, PhRvD, 88, 083507
Capozziello, S., Faizal, M., Hameeda, M., Pourhassan, B., Salzano, V., & Upadhyay, S. 2018, MNRAS, 474, 2430
De Martino, I., De Laurentis, M., Atrio-Barandela, F., & Capozziello, S. 2014, MNRAS, 442, 921
Farieta, J. E. G., et al. 2019, MNRAS, 488, 1987
Hameeda, M., Upadhyay, S., Faizal, M., & Ali, A. F., 2016a, MNRAS, 463, 3699
Hameeda, M., Faizal, M., & Ali, A. F., 2016b, GReGr, 48, 47
Hameeda, M., Upadhyay S., Faizal M., Ali, A. F., & Pourhassan, B. 2018, PDU, 19, 137
Hameeda, M., Pourhassan, B., Faizal, M., Maasroor, C. P., Ansari, R. U. H., & Suvesh P. K. 2019, EPJC, 79, 769
Hong T., Han J. L., Wen Z. L., Sun L., & Zhan, H. 2012, ApJ, 749, 81
Huang, K., 1987, Statistical Mechanics, John Wiley and Sons
Khan, M., & Malik, M., 2012, MNRAS, 421, 2629
Khan, M., & Malik, M., 2013, Ap&SS, 211, 348
Milgrom, M. 2008, The MOND paradigm, arXiv:0801.3133
Pourhassan, B., Upadhyay S., Hameeda M., Faizal M. 2017, MNRAS, 468, 3166
Peebles P. J. E., 1980, The Large-Scale Structure of the Universe, Princeton University Press, Princeton, NJ
Peebles, P. J. E. 1993, Principles of Physical Cosmology, Princeton University Press, Princeton, NJ
Perlmutter S., et al., 1998, Natur, 391, 51
Raychaudhury S. and Saslaw W. C. (1996). ApJ, 461, 514
Raychaudhury S. and Saslaw W. C. (1996). ApJ, 461, 514
Rahmani H., Saslaw W.C., Tavasoli S., 2009, ApJ, 695, 1121
Riess, A.G., et al., 1998, AJ, 116, 1009
Saslaw, W.C. 1986, ApJ, 304, 11
Saslaw, W.C., & Hamilton, A. J. S., 1984, ApJ, 276, 13
Saslaw, W.C., Chitre, S.M., Itoh, M., & Inagaki S., 1990, ApJ, 365, 419
Saslaw, W.C., & Yang, A., 2009, arXiv:0902.0747
Sivakoff, G.R. & Saslaw, W.C., 2005, ApJ, 626, 795
Shtanov, Y., & Sahni, V., 2010, PhRvD, 82, 101503
Sotiriou, T.P, & Faraoni V. 2010, RvMP, 82, 451
Upadhyay, S., Pourhassan B., & Capozziello, S. 2018, IJMPD, 28, 1950027
Voit, G. M., 2005, RvMP, 77, 207
Wen, Z. L., Han, J. L., & Liu, F. S., 2009, ApJS, 183, 197
Wahid, A., Ahmad F., & Nazir, A. 2011, Ap&SS, 333, 241
Frenk, C. S., et al. 1999, ApJ, 525, 554
Barnes, D. J., et al. 2011, MNRAS, 417, 1088
Ponomarenko, T. J., Sanderson, A. J. R., & Finoguenov, A. 2003, MNRAS, 343, 331
Voit, G. M. 2005, RvMP, 77, 207
Itoh, M., Inagaki, S., & Saslaw, W. C. 1993, ApJ, 403, 476
Saslaw, & Chitre, S. M. 1990, ApJ, 365, 419
Saslaw, W.C. & Yang, A. Proceedings of the Les Houches Summer School, Oxford University : ISBN 978-0-19-957462-9; Page 377-398 of Les Houches 2008 Session XC: Long Range Interacting Systems, eds Dauxois, Ruffo & Cugliandolo, Oxford University Press, Oxford (2009)
Saslaw, W.C., & Fang, F. 1996, ApJ, 460, 16
Saslaw, W. C., & Sheth, R. K. 1993, ApJ, 409, 504
Rahmani, H,Saslaw, W. C., & Tavasoli, S. 2009, ApJ, 695, 2
Spergel, D. N. 2007, ApJS, 170, 377
Gurzadyan, V. G., Kocharyan, A. A., & Stepianian, A. 2020, EPJC, 80, 24
Gurzadyan, V. G., & Stepianian, A. 2018, EPJC, 78, 632
Gurzadyan, V. G., & Stepianian, A. 2019, EPJC, 79, 169
Gurzadyan, V. G., & Penrose, R. 2013, EPJC, 128, 22
Gurzadyan, V. G., & Stepianian, A. 2019, EPJC, 79, 568
Dellitou, M. L, Marcondes, R. J. F., & Lima Neto, G. B. 2019, MNRAS, 490, 1944
Bonsemer, P., Ngampitipan, T., Simpson, T., & Visser, M. 2020, PhRvD, 101, 024050
Peirani, S., & Pacheco, J. A. D. F. 2008, A&A, 488, 845
Iliev, I. T., & Shapiro, P. R. 2001, MNRAS, 325, 468
Stark, A., Miller, C. J., & Huterer, D. 2017, PhRvD, 96, 023543
Agueuza, M., & Lima, M. 2018, PhRvD, 98, 123529
Wen, D., Kemball, A. J., & Saslaw, W. C. 2020, ApJ, 890, 160
Novello, M., Neto, J. B., & Salim, J. M. 2002, CQGra, 19, 3107
Novello, M., Neto, J. B., & Salim, J. M. 2001, CQGra, 18, 1261
Saslaw, W. C., & Ahmad, F., ApJ, 720, 1246
Upadhyay, S., Pourhassan, B., & Capozziello, S. 2019, IJMPD, 28, 1950027

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