ELLIPTIC DIFFERENTIAL INCLUSIONS ON NON-COMPACT RIEMANNIAN MANIFOLDS

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Dedicated to Professor Siegfried Carl on the occasion of his 70th birthday

Abstract. We investigate a large class of elliptic differential inclusions on non-compact complete Riemannian manifolds which involves the Laplace-Beltrami operator and a Hardy-type singular term. Depending on the behavior of the nonlinear term and on the curvature of the Riemannian manifold, we guarantee non-existence and existence/multiplicity of solutions for the studied differential inclusion. The proofs are based on nonsmooth variational analysis as well as isometric actions and fine eigenvalue properties on Riemannian manifolds. The results are also new in the smooth setting.

1. Introduction

Various geometric/physical phenomena can be reduced to finding solutions for the problem

\[ Lu(x) = \alpha(x)f(u(x)), \quad x \in \Omega, \quad (P) \]

where \( \Omega \) is an open domain in an ambient metric measure space, \( L \) is an elliptic-type operator, \( \alpha : \Omega \to \mathbb{R} \) is a measurable potential, and \( f : \mathbb{R} \to \mathbb{R} \) is a nonlinear function having certain regularity and growth properties. Such problems arise from the Yamabe problem on compact/non-compact Riemannian manifolds, the standing Schrödinger equation in \( \mathbb{R}^n \) (\( n \geq 2 \)), Dirichlet and Neumann problems on bounded/unbounded domains, etc. Wide range of strategies and theories have been applied in the last century in order to investigate problem (P), as variational methods, fixed point arguments, sub- and super-solution techniques, etc.

An important class of problems within (P) appears when the nonlinear term \( f(x, \cdot) \) is not necessarily continuous; such a relevant example appears in the description of the von Kármán adhesive plates, see Panagiotopoulos [39]. Due to the jumping effect of \( f(x, \cdot) \), as a first approach, problem (P) need not has any solution. However, from physical reasons, we expect to obtain certain equilibrium states of the phenomena described by means of problem (P). Accordingly, a natural way to handle the aforementioned discontinuity situation is to ‘fill the gaps’, defining a differential inclusion associated with problem (P). More precisely, if \( f \) is essentially bounded on \( \mathbb{R} \), we consider instead of the value \( f(t) \) the interval \( [\underline{f}(t), \overline{f}(t)] \), where

\[ \underline{f}(t) = \lim_{\delta \to 0^+} \text{essinf}_{|s-t|<\delta} f(s), \quad \overline{f}(t) = \lim_{\delta \to 0^+} \text{esssup}_{|s-t|<\delta} f(s); \]

here, \( \text{essinf}_A f = \sup\{a \in \mathbb{R} : f(x) \geq a \text{ for a.e. } x \in A\} \) and \( \text{esssup}_A f = -\text{essinf}_A (-f) \) whenever \( A \neq \emptyset \). In this way, we replace (P) by the differential inclusion problem

\[ Lu(x) \in \alpha(x)\partial F(u(x)), \quad x \in \Omega, \quad (DI) \]

where \( F(t) = \int_0^t f(s)ds \) is a locally Lipschitz function, and \( \partial F(t) = [\underline{f}(t), \overline{f}(t)], \quad t \in \mathbb{R} \). Hereafter, \( \partial F \) stands for the subdifferential of \( F \) at \( t \in \mathbb{R} \) in the sense of Clarke [11].

Differential inclusion problems, similar to (DI), may appear on not necessarily Euclidean structures; indeed, in certain circumstances the domain \( \Omega \) can be a subset of a curved space (Riemannian or Finsler manifolds, sub-Riemannian structures as Heisenberg or Carnot groups, etc.), while the operator \( L \) may reflect the geometric feature of the ambient space.

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In the present paper we consider the variational differential inclusion
\[ L_{x_0}u(x) = -\Delta_g u(x) - \mu \frac{u(x)}{d_g^2(x_0, x)} + u(x) \in \lambda \alpha(x) \partial F(u(x)), \quad x \in M, \] (1.1)
where \((M, g)\) is an \(n\)-dimensional complete Riemannian manifold, \(n \geq 3\) (endowed with its canonical measure \(d_v\)), \(\Delta_g\) is the Laplace-Beltrami operator on \((M, g)\), \(d_g : M \times M \to \mathbb{R}\) is the distance function associated with the Riemannian metric \(g\), \(x_0 \in M\) is a fixed point, \(\mu, \lambda, \alpha : M \to \mathbb{R}\) is a measurable potential, \(F : \mathbb{R} \to \mathbb{R}\) is a locally Lipschitz function and \(\partial F\) stands for the Clarke subdifferential of \(F\). An element \(u \in H^1(M)\) is a solution of (1.1) if there exists a measurable selection \(x \mapsto \xi_x \in \partial F(u(x))\) such that the map \(x \mapsto \alpha(x)\xi_x w(x)\) belongs to \(L^1(M)\) for every test-function \(w \in H^1(M)\) and one has
\[ \int_M \nabla_g u(x) \nabla_g w(x) d\nu_g - \mu \int_M \frac{u(x)w(x)}{d_g^2(x_0, x)} d\nu_g + \int_M u(x)w(x) d\nu_g = \lambda \int_M \alpha(x)\xi_x w(x) d\nu_g. \] (1.2)
One can readily observe that (1.2) reduces to the fact that \(u\) is a weak solution of
\[ -\Delta_g u(x) - \mu \frac{u(x)}{d_g^2(x_0, x)} + u(x) = \lambda \alpha(x) f(u(x)), \quad x \in M, \] (1.3)
whenever \(f\) is continuous (and consequently, \(F\) is of class \(C^1\) and \(\partial F(t) = f'(t) = f(t)\)).

On one hand, variational elliptic differential inclusions as (1.1) – or slightly different versions of them formulated in terms of variational-hemivariational inequalities – have been deeply studied in the last three decades, mostly in Euclidean spaces (both for bounded and unbounded domains), see e.g. Bonanno, D’Agui and Winkert [4], Candito and Livrea [6], Carl and Le [7, 8], Carl, Le and Motreanu [9], Costea, Kristály and Varga [12], Gasiński and Papageorgiou [16], Kristály and Varga [26], Liu, Liu and Motreanu [30], Liu, Livrea, Motreanu and Zeng [31], Migórski, Ochal and Sofonea [32], Motreanu and Papaghiuc [36], Papageorgiou [39], Varga [43], etc. On the other hand, various forms of (1.3) have been investigated both on compact and non-compact Riemannian manifolds (mostly without the singular term), see e.g. Berchio, Ferrero and Grillo [2], Bonanno, Molica Bisci and Rădulescu [5], Jaber [18], Lisei and Varga [28], Liu and Liu [29], Molica Bisci and Pucci [33], Molica Bisci and Secchi [34], Molica Bisci and Vilasi [35], etc. As expected, on non-compact manifolds additional restrictions and approaches are needed to compensate the lack of compactness.

We shall focus to a broad class of non-compact Riemannian manifolds and prove various non-existence, existence and multiplicity results for the differential inclusion problem (1.1), by assuming certain curvature hypotheses and growths for the function \(F\) (at the origin and at infinity). In fact, we consider two classes of Riemannian manifolds having different curvature restrictions; namely, we assume that a complete, non-compact Riemannian manifold \((M, g)\) satisfies one of the conditions:

(i) \(\mathbf{K} \leq -\kappa\) for some \(\kappa \geq 0\), where \(\mathbf{K}\) is the sectional curvature of the Cartan-Hadamard manifold\(^1\) \((M, g)\);

(ii) \(\text{Ric}_{(M, g)} \geq 0\), where \(\text{Ric}_{(M, g)}\) is the Ricci curvature on \((M, g)\).

The ‘clash’ of (i) and (ii) is precisely the Euclidean space \(\mathbb{R}^n\) endowed with the usual metric. In the case (ii), i.e., when \(\text{Ric}_{(M, g)} \geq 0\), a crucial role is played by the asymptotic volume ratio
\[ \text{AVR}_{(M, g)} = \lim_{r \to \infty} \frac{V_g(B_x(r))}{\omega_n r^n}, \]
where \(V_g\) stands for the volume in \((M, g)\), \(B_x(r) = \{ y \in M : d_g(x, y) < r \}\) is the ball of radius \(r > 0\) and center \(x \in M\), while \(\omega_n = \pi^{n/2}/\Gamma(1 + n/2)\) is the volume of the Euclidean unit ball in \(\mathbb{R}^n\). By Bishop-Gromov comparison principle it turns out that the asymptotic volume ratio is well-defined (i.e., independent of the choice of \(x \in M\)) and \(\text{AVR}_{(M, g)} \in [0, 1]\).

We assume on the potential \(\alpha : M \to \mathbb{R}\) that

\[ \mathbf{(H)_\alpha} : \alpha \geq 0 \quad \text{and} \quad \alpha \in L^1(M) \cap L^\infty(M) \setminus \{0\}. \]

For the locally Lipschitz function \(F : \mathbb{R} \to \mathbb{R}\) we require

\(^1\)Complete, simply connected Riemannian manifold with non-positive sectional curvature.
(H)\(_0\) : there exists \(C_0 > 0\) such that
\[|\xi| \leq C_0|t|, \quad \forall \xi \in \partial F(t), \ t \in \mathbb{R}.\]

The first result of the paper reads as follows.

**Theorem 1.1.** (Non-existence) Let \((M, g)\) be an \(n\)-dimensional complete non-compact Riemannian manifold, \(n \geq 3\), and assume that the potential \(\alpha : M \to \mathbb{R}\) and the locally Lipschitz function \(F : \mathbb{R} \to \mathbb{R}\) satisfy assumptions \((H)_\alpha\) and \((H)_0\), respectively. Assume in addition that one of the following curvature conditions holds:

(i) \(K \leq - \kappa\) for some \(\kappa \geq 0\), \((M, g)\) is simply connected and
\[
\text{either } \kappa = 0, \mu \leq \frac{\lambda C_0}{\mathcal{L}^\infty} \text{ and } |\lambda| C_0 \mathcal{L}^\infty \leq 1, \\
\text{or } \kappa > 0, \mu \leq \frac{\lambda C_0}{\mathcal{L}^\infty} \text{ and } (n - 2)^2 |\lambda| C_0 \mathcal{L}^\infty - 1 \leq (n - 1)^2 \left(\frac{(n - 2)^2}{4} - \mu_+\right) \kappa,
\]
where \(\mu_+ = \max(\mu, 0)\);

(ii) \(\text{Ric}_{(M, g)} \geq 0, \mu \leq \frac{\lambda C_0}{\mathcal{L}^\infty} \text{ and } |\lambda| C_0 \mathcal{L}^\infty \leq 1.\)

Then the differential inclusion (1.1) has only the zero solution.

The assertions in (i) show that there is a balance in the sense that when a stronger curvature restriction occurs, the analytic assumption can be relaxed. The proof of Theorem 1.1 is based on a direct computation combined with Hardy-type inequalities and sharp spectral gap estimates on Riemannian manifolds; the ingredients to the proof are recalled in §2.1.

In order to produce existence or even multiplicity of non-zero solutions to (1.1), we require on the locally Lipschitz function \(F : \mathbb{R} \to \mathbb{R}\) the following assumptions:

\[
(H)_1 : \lim_{t \to 0} \frac{\max\{|\xi| : \xi \in \partial F(t)\}}{t} = 0;
\]

\[
(H)_2 : \lim_{|t| \to \infty} \frac{\max\{|\xi| : \xi \in \partial F(t)\}}{t} = 0;
\]

\[
(H)_3 : F(0) = 0 \text{ and there exist } t_0^- < 0 < t_0^+ \text{ such that } F(t_0^+) > 0.
\]

Note that \((H)_1\) and \((H)_2\) mean that the function \(t \mapsto \max\{|\xi| : \xi \in \partial F(t)\}\) is superlinear at the origin and sublinear at infinity, respectively; in particular, by using Lebourg’s mean value theorem, we observe that \(F\) is sub-quadratic at infinity. In addition, by the upper semicontinuity of the set-valued function \(t \mapsto \partial F(t)\) and conditions \((H)_1\) and \((H)_2\), it turns out that the hypothesis \((H)_0\) is also valid for a suitably large value of \(C_0 > 0\); in particular, Theorem 1.1 can be applied (under the assumptions \((H)_1\) and \((H)_2\)), and for sufficiently ‘small’ values of \(|\lambda|\) only the zero solution exists for the differential inclusion (1.1). However, for ‘large’ values of \(\lambda > 0\), we can guarantee the existence of multiple non-zero solutions for (1.1) by requiring further assumptions on the behavior of the isometric group of the Riemannian manifold \((M, g)\).

In fact, the latter assumptions are destined to balance the lack of compactness of the Riemannian manifolds we are dealing with.

To state the second result of the paper, we denote by \(\text{Isom}_g(M)\) the group of isometries of the complete Riemannian manifold \((M, g)\). Let \(G\) be a subgroup of \(\text{Isom}_g(M)\) and
\[
\text{Fix}_M(G) = \{x \in M : \sigma(x) = x, \forall \sigma \in G\}
\]be the set of fixed points of the isometry group \(G\) in \(M\). The \(G\)-orbit of a point \(x \in M\) is \(O_G = \{\sigma(x) : \sigma \in G\}\). The continuous action of the group \(G\) on \(M\) is coercive if for every \(t > 0\) the set \(O_t := \{x \in M : \text{diam}(O_G) \leq t\}\) is bounded, see Skrzypczak and Tintarev \[40, 41\]; here \(\text{diam}(S)\) denotes the diameter of \(S \subset M\). A function \(u : M \to \mathbb{R}\) is \(G\)-invariant if \(u(x) = u(\sigma(x))\) for every \(x \in M\) and \(\sigma \in G\).

**Theorem 1.2.** (Multiplicity: sub-quadratic nonlinearity at infinity) Let \((M, g)\) be an \(n\)-dimensional complete non-compact Riemannian manifold, \(n \geq 3\), and \(G\) be a compact connected subgroup of \(\text{Isom}_g(M)\) such that \(\text{Fix}_M(G) = \{x_0\}\) for the same \(x_0 \in M\) as in problem (1.1). Let \(\alpha : M \to \mathbb{R}\) be a potential satisfying \((H)_\alpha\) which depends only on \(d_g(x_0, \cdot)\) and the locally Lipschitz function \(F : \mathbb{R} \to \mathbb{R}\) satisfying assumptions \((H)_i\), \(i \in \{1, 2, 3\}\), respectively. In addition, we assume that one of the following curvature assumptions holds:
(i) $(M, g)$ is of Cartan-Hadamard-type and $0 \leq \mu < \frac{(n-2)^2}{4};$

(ii) $\text{Ric}(M,g) \geq 0$, $\text{AVR}(M,g) > 0$, $0 \leq \mu < \frac{\text{AVR}}{(n-2)^2} \frac{(n-2)^2}{4}$ and $G$ is coercive.

Then there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ the differential inclusion (1.1) has at least four non-zero $G$-invariant solutions in $H^1(M)$.

The proof of Theorem 1.2 is based on truncation and variational arguments, combined with careful isometry actions on $H^1(M)$. The key ingredients are the nonsmooth principle of symmetric criticality and mountain pass theorem (with the Palais-Smale condition) and the compact embedding of $G$-invariant functions of $H^1(M)$ into appropriate Lebesgue spaces over $M$, which are valid in both geometric contexts (i) and (ii). Examples of Riemannian manifolds with the above curvature restrictions and isometric actions are presented in Kristály [22] and Farkas, Kristály and Mester [15] in the setting (i), and Balogh and Kristály [3] in the framework (ii).

As we already noticed, assumptions $(H)_{1}$ and $(H)_{2}$ imply that $F$ is sub-quadratic at infinity.

In the sequel, we establish a counterpart of Theorem 1.2 whenever $F$ is super-quadratic at infinity.

More precisely, we assume that the locally Lipschitz function $F: \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

$(H)_{4}$: $F(0) = 0$ and there exist $\nu > 2$ and $C > 0$ such that

$$2F(t) + F^{0}(t; -t) \leq -C|t|^\nu, \quad \forall t \in \mathbb{R}; \quad (1.5)$$

$(H)_{5}$: there is $q \in (2, 2 + \frac{4}{n})$ such that $\max \{ |\xi| : \xi \in \partial F(t) \} = O(|t|^{q-1})$ as $|t| \to \infty$.

Here, $F^{0}(t; s)$ is the generalized directional derivative of $F$ at the point $t \in \mathbb{R}$ and direction $s \in \mathbb{R}$, see §2.2. Note that by $(H)_{1}$ and $(H)_{4}$, $F$ is super-quadratic at infinity, see §4.

**Theorem 1.3.** (Existence/Multiplicity: super-quadratic nonlinearity at infinity) Let $(M, g)$ be an $n$-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and $G$ be a compact connected subgroup of $\text{Isom}_{g}(M)$ such that $\text{Fix}_{M}(G) = \{ x_0 \}$ for the same $x_0 \in M$ as in problem (1.1). Let $\alpha \in \mathcal{L}^{\infty}(M)$ be a potential which depends only on $d_{g}(x_{0}, \cdot)$ and $\text{essinf}_{x \in M} \alpha(x) = \alpha_{0} > 0$, while the locally Lipschitz function $F: \mathbb{R} \to \mathbb{R}$ satisfies the assumptions $(H)_{i}, i \in \{ 1, 4, 5 \}$, respectively.

If one of the curvature assumptions (i) or (ii) holds from Theorem 1.2, then for every $\lambda > 0$ the differential inclusion (1.1) has at least a non-zero $G$-invariant solution in $H^1(M)$. In addition, if $F$ is an even function, (1.1) has infinitely many distinct $G$-invariant solutions in $H^1(M)$.

The proof is based on the same geometric arguments as in Theorem 1.2 (curvature constraints, isometric actions), combined with the nonsmooth mountain pass or fountain theorem involving the Cerami compactness condition.

The paper is organized as follows. In §2 we collect those results that are indispensable in our proofs. Namely, we first recall certain functional inequalities and spectral estimates on Riemannian manifolds; then we recall some elements from the nonsmooth calculus of Clarke [11] including also the nonsmooth principle of symmetric criticality. In §3 we prove the non-existence results, established within Theorem 1.1. In §4 we discuss our first existence/multiplicity results in the sub-quadratic case, by proving Theorem 1.2. Finally, Section §5 is devoted to handle the super-quadratic case, i.e., Theorem 1.3.

2. Preliminaries

In this section we recall those notions and results that are crucial to carry out our proofs. Before to do this, we fix some notations. If $(M, g)$ is a complete Riemannian manifold, the Sobolev space $H^1(M)$ over $M$ is the completion of $C_{0}^{\infty}(M)$ with respect to the norm

$$\|u\|_{H^1} = \left( \int_M |\nabla_g u|^2 dv_g + \int_M u^2 dv_g \right)^{1/2},$$

while the $L^q$-Lebesgue norm $(q \geq 1)$ is

$$\|u\|_{L^q} = \left( \int_M |u|^q dv_g \right)^{1/q},$$

with the supremum-norm for $q = +\infty$. 

2.1. Functional inequalities and spectral estimates on Riemannian manifolds.

2.1.1. Cartan-Hadamard manifolds. Throughout this subsection, let \((M, g)\) be an \(n\)-dimensional Cartan-Hadamard manifold, \(n \geq 3\). We notice that in this geometric context, there exists \(C_n > 0\) such that

\[
\|u\|_{L^{2^*}} \leq C_n \left( \int_M |\nabla_g u|^2 \, dv_g \right)^{1/2}, \quad \forall u \in C^\infty_0(M),
\]

see e.g. Hebey [17, Chapter 8], where \(2^* = 2n/(n-2)\) is the critical Sobolev exponent. Moreover, the best Sobolev embedding constant \(C_n\) is precisely its Euclidean counterpart \(\text{AT}_n\), provided by Aubin [1] and Talenti [42], whenever the Cartan-Hadamard conjecture holds on \((M, g)\) (e.g. in dimensions 3 and 4). In high-dimensions, the sharp constant \(C_n > 0\) is not known; however, a non-optimal form can be given by means of the Croke-constant as in Hebey [17, p. 239].

A density argument combined with a simple interpolation shows that the Sobolev space \(H^1(M)\) is continuously embedded into \(L^q(M)\) for every \(q \in [2, 2^*]\); more precisely, there exists \(K_q^- > 0\) such that

\[
\|u\|_{L^q} \leq K_q^- \|u\|_{H^1}, \quad \forall u \in H^1(M). \tag{2.1}
\]

Let \(x_0 \in M\) be fixed. Then the Hardy inequality holds on \((M, g)\), which reads as

\[
\frac{(n-2)^2}{4} \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \, dv_g \leq \int_M |\nabla_g u|^2 \, dv_g, \quad \forall u \in H^1(M), \tag{2.2}
\]

where \(\frac{(n-2)^2}{4}\) is sharp and never achieved, see e.g. D’Ambrosio and Dipierro [14], and Kristály [21].

In addition, if the sectional curvature has the property \(K \leq -\kappa\) for some \(\kappa > 0\), then McKeans spectral gap theorem asserts that

\[
\gamma(M, g) := \inf_{u \in H^1(M) \setminus \{0\}} \frac{\int_M |\nabla_g u|^2 \, dv_g}{\int_M u^2 \, dv_g} \geq \frac{(n-1)^2}{4} \kappa. \tag{2.3}
\]

The inequality (2.3) is sharp, see e.g. on the \(n\)-dimensional hyperbolic space \(\mathbb{H}^n\) with constant sectional curvature \(K = -\kappa\); we also notice that the infimum in (2.3) is not achieved by any function \(u \in H^1(M)\).

2.1.2. Riemannian manifolds with non-negative Ricci curvature. In this subsection we consider an \(n\)-dimensional \((n \geq 3)\) complete non-compact Riemannian manifold \((M, g)\) with \(\text{Ric}(M, g) \geq 0\). As we already noticed in the Introduction, the asymptotic volume ratio \(\text{AVR}_{(M, g)} \in [0, 1]\) provides deep geometric information about the manifold; for instance, \(\text{AVR}_{(M, g)} = 1\) if and only if \((M, g)\) is isometric to the Euclidean space \(\mathbb{R}^n\). Quantitatively speaking, closer value of \(\text{AVR}_{(M, g)}\) to 1 implies topologically closer manifold \((M, g)\) to the Euclidean space \(\mathbb{R}^n\), expressed in terms of the trivialization of higher homotopy groups of \(M\), see Munn [37].

In the geometric context when \((M, g)\) is a complete non-compact Riemannian manifold with \(\text{Ric}(M, g) \geq 0\), a necessarily and sufficient condition to have the Sobolev embedding is the fact that \(\text{AVR}_{(M, g)} > 0\), see Coulhon and Saloff-Coste [13] and Hebey [17]. Moreover, a recent result of Balogh and Kristály [3] asserts that if \(\text{AVR}_{(M, g)} > 0\) then

\[
\|u\|_{L^{2^*}} \leq \text{AVR}^{-\frac{1}{2}}_{(M, g)} \text{AT}_n \left( \int_M |\nabla_g u|^2 \, dv_g \right)^{1/2}, \quad \forall u \in H^1(M),
\]

where the constant \(\text{AVR}^{-\frac{1}{2}}_{(M, g)} \text{AT}_n\) is sharp; here, as in §2.1.1, \(\text{AT}_n\) stands for the best Sobolev embedding constant in the Euclidean Sobolev inequality on \(\mathbb{R}^n\), see Aubin [1] and Talenti [42]. In particular, \(H^1(M)\) is continuously embedded into \(L^q(M)\) for every \(q \in [2, 2^*]\); more precisely, there exists \(K_q^+ > 0\) such that

\[
\|u\|_{L^q} \leq K_q^+ \|u\|_{H^1}, \quad \forall u \in H^1(M). \tag{2.4}
\]
Given \( x_0 \in M \) fixed, the Hardy inequality on \((M, g)\) is verified as
\[
\text{AVR}_{(M, g)}^\frac{2}{n} \frac{(n-2)^2}{4} \int_M \frac{u^2(x)}{d^2_g(x_0, x)} \, dv_g \leq \int_M |\nabla_g u|^2 \, dv_g, \quad \forall u \in H^1(M),
\]
see Kristály, Mester and Mezei [24]. The sharpness of the constant in (2.5) is not known unless we are in the classical Euclidean setting. However, if we assume that there exists a non-zero function realizing the equality in (2.5), from the proof in [24] (based on a Pólya-Szegő inequality involving the asymptotic volume ratio) we would obtain that \( \text{AVR}_{(M, g)} = 1 \), i.e., \((M, g)\) is isometric to the Euclidean space \( \mathbb{R}^n \), which is a contradiction (since no non-zero extremal function exists in the Hardy inequality in \( \mathbb{R}^n \)).

2.1.3. Compact embeddings via isometric actions. According to the previous subsections, the Sobolev space \( H^1(M) \) is continuously embedded into \( L^q(M) \), \( q \in [2, 2^*] \), whenever \((M, g)\) is either a Cartan-Hadamard manifold or a complete non-compact Riemannian manifold with \( \text{Ric} \geq 0 \); however, none of them is compact, which represents an impediment to apply variational arguments on \( H^1(M) \). To handle the lack of compactness, we use certain symmetrization à la Lions [27] by means of isometries of \( M \).

Let \((M, g)\) be an \( n \)-dimensional complete non-compact Riemannian manifold, \( n \geq 3 \), and – as in the Introduction – \( \text{Isom}_g(M) \) be the group of isometries of \((M, g)\). Let \( G \) be a subgroup of \( \text{Isom}_g(M) \) and \( \text{Fix}_M(G) \) be the set of fixed points of the isometry group \( G \) in \( M \), see (1.4). Let
\[
H_G^1(M) = \{ u \in H^1(M) : u \circ \sigma = u, \quad \forall \sigma \in G \}
\]
be the closed \( G \)-invariant subspace of \( H^1(M) \). The consequences of the main results in the paper by Farkas, Kristály and Mester [15] state that if one of the following assumptions hold, i.e.,
\begin{itemize}
  \item \((M, g)\) is a Cartan-Hadamard manifold and \( \text{Fix}_M(G) \) is a singleton, or
  \item \( \text{Ric} \geq 0 \), \( \text{AVR}_{(M, g)} > 0 \) and \( G \) is coercive,
\end{itemize}
then the space \( H_G^1(M) \) can be compactly embedded into \( L^q(M) \) for every \( q \in (2, 2^*) \).

2.2. Non-smooth analysis.

2.2.1. Locally Lipschitz functions. In this subsection we recall those basic properties of locally Lipschitz functions which are used in our proofs; for details, see Clarke [11].

Let \( X \) be a real Banach space with the norm \( \| \cdot \| \). A function \( h : X \to \mathbb{R} \) is locally Lipschitz if every point \( u \in X \) possesses a neighborhood \( U_u \subset X \) such that
\[
|h(u_1) - h(u_2)| \leq K \|u_1 - u_2\|, \quad \forall u_1, u_2 \in U_u,
\]
for a constant \( K > 0 \) depending on \( U_u \). The generalized directional derivative of the locally Lipschitz function \( h : X \to \mathbb{R} \) at \( u \in X \) in the direction \( v \in X \) is given by
\[
h^0(u; v) := \limsup_{t \searrow 0} \frac{h(u + tv) - h(u)}{t}.
\]
If \( h : X \to \mathbb{R} \) is a function of class \( C^1 \) on \( X \), then \( h^0(u; v) = \langle h'(u), v \rangle \) for all \( u, v \in X \). Hereafter, \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) stand for the duality mapping on \((X^*, X)\) and the norm on \( X^* \), respectively. The Clarke subdifferential \( \partial h(u) \) of \( h \) at a point \( u \in X \) is the subset of the dual space \( X^* \) given by
\[
\partial h(u) := \{ \zeta \in X^* : \langle \zeta, v \rangle \leq h^0(u; v), \quad \forall v \in X \}.
\]
An element \( u \in X \) is a critical point of \( h \) if \( 0 \in \partial h(u) \), see Chang [10, Definition 2.1].

**Proposition 2.1.** (Clarke [11]) Let \( h : X \to \mathbb{R} \) be a locally Lipschitz function. The following assertions hold:

(i) For every \( u \in X \), \( \partial h(u) \) is a nonempty, convex and weak*-compact subset of \( X^* \). Moreover, \( \| \zeta \|_* \leq K \) for all \( \zeta \in \partial h(u) \), with \( K > 0 \) from (2.6).

(ii) For every \( u \in X \), \( h^0(u; \cdot) \) is the support function of \( \partial h(u) \), i.e.,
\[
h^0(u; v) = \max \{ \langle \zeta, v \rangle : \zeta \in \partial h(u) \}, \quad \forall v \in X.
\]
(iii) The set-valued map $\partial h : X \rightrightarrows X^*$ is closed from $s - X$ into $w^* - X^*$. In particular, if $X$ is finite dimensional, then $\partial h$ is an upper semicontinuous set-valued map.

(iv) (Lebourg’s mean value theorem) Let $U$ be an open subset of a Banach space $X$ and $u, v$ be two points of $U$ such that the line segment $[u, v] = \{(1 - t)u + tv : 0 \leq t \leq 1\} \subset U$. If $h : U \to \mathbb{R}$ is a Lipschitz function, then there exist $w \in (u, v)$ and $\zeta \in \partial h(w)$ such that $h(v) - h(u) = \langle \zeta, v - u \rangle$.

(v) If $j : X \to \mathbb{R}$ is of class $C^1$ on $X$, then $\partial (j + h)(u) = j'(u) + \partial h(u)$ and $(j + h)^0(u; v) = \langle j'(u), v \rangle + \partial h^0(u; v)$ for every $u, v \in X$.

(vi) $-h^0(u) = \partial h^0(u, v)$ for every $u, v \in X$.

(vii) $\partial (sh)(u) = s \partial h(u)$ for every $s \in \mathbb{R}$ and $u \in X$.

2.2.2. Principle of symmetric criticality for locally Lipschitz functionals. Let $G$ be a compact Lie group acting linear isometrically on the real Banach space $(X, \| \cdot \|)$, i.e., the action $G \times X \to X$, $(\sigma, u) \mapsto \sigma u$ is continuous and for every $\sigma \in G$ the map $u \mapsto \sigma u$ is linear such that $\|\sigma u\| = \|u\|$ for every $u \in X$. A function $h : X \to \mathbb{R}$ is $G$-invariant if $h(\sigma u) = h(u)$ for all $\sigma \in G$ and $u \in X$.

Let

$$\text{Fix}_X(G) = \{u \in X : \sigma u = u, \forall \sigma \in G\} \tag{2.7}$$

be the set of fixed points of $G$ over $X$. According to Kravcewicz and Marzantowicz [20] (see also Costea, Kristály and Varga [12, Section 3.4]), the principle of symmetric criticality for locally Lipschitz functionals can be stated as follows.

**Proposition 2.2.** (Kravcewicz and Marzantowicz [20]) Let $G$ be a compact Lie group acting linear isometrically on the real Banach space $(X, \| \cdot \|)$ and $h : X \to \mathbb{R}$ be a $G$-invariant, locally Lipschitz functional. If $h|_G$ denotes the restriction of $h$ to $\text{Fix}_X(G)$ and $u \in \text{Fix}_X(G)$ is a critical point of $h|_G$ then $u$ is also a critical point of $h$.

The smooth version of the principle of symmetric criticality has been provided by Palais [38] and later extended to various nonsmooth settings.

3. Non-existence of solutions: Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let $u \in H^1(M)$ be a solution of (1.1), i.e., relation (1.2) holds for every $v \in H^1(M)$. Let us choose $v = u$ in (1.2), obtaining

$$\int_M |\nabla g u(x)|^2 dv_g - \mu \int_M \frac{u^2(x)}{d^2_j(x_0, x)} dv_g + \int_M u^2(x) dv_g = \lambda \int_M \alpha(x) \xi_x u(x) dv_g, \tag{3.1}$$

where $\xi_x \in \partial F(u(x))$ is a suitable selection, $x \in M$, such that $x \mapsto \alpha(x) \xi_x v(x)$ belongs to $L^1(M)$.

By assumptions (H)$_{\alpha}$, (H)$_0$ and relation (3.1) we obtain that

$$\int_M |\nabla g u(x)|^2 dv_g - \mu \int_M \frac{u^2(x)}{d^2_j(x_0, x)} dv_g + \int_M u^2(x) dv_g \leq |\lambda| C_0 \|\alpha\|_{L^\infty} \int_M u^2(x) dv_g. \tag{3.2}$$

Assume by contradiction that $u \neq 0$.

**Proof of (i): $K \leq -\kappa$ for some $\kappa \geq 0$.**

Let $\kappa = 0$. If $\mu \leq \frac{(n - 2)^2}{4}$, by the Hardy inequality (2.2) and relation (3.2), it turns out that

$$\int_M u^2(x) dv_g < |\lambda| C_0 \|\alpha\|_{L^\infty} \int_M u^2(x) dv_g;$$

here we used the fact that equality cannot occur in the Hardy inequality (2.2) unless $u = 0$. Consequently, if $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$, we arrive to a contradiction, i.e., we necessarily have $u = 0$, concluding the proof of (i).

Let $\kappa > 0$. Assume first that $0 < \mu \leq \frac{(n - 2)^2}{4}$. Then by the Hardy inequality (2.2) we have that

$$\mu \int_M \frac{u^2(x)}{d^2_j(x_0, x)} dv_g < \frac{4\mu}{(n - 2)^2} \int_M |\nabla g u(x)|^2 dv_g,$$
where we again used the fact that no equality occurs in (2.2) for non-zero functions. Thus, by (3.2) it follows that
\[
\left(1 - \frac{4\mu}{(n-2)^2}\right) \int_M |\nabla_g u(x)|^2 dv_g < (|\lambda|C_0\|\alpha\|_{L^\infty} - 1) \int_M u^2(x) dv_g. \tag{3.3}
\]
First, if $|\lambda|C_0\|\alpha\|_{L^\infty} \leq 1$, since $\mu \leq \frac{(n-2)^2}{4}$, relation (3.3) gives a contradiction. Second, if $|\lambda|C_0\|\alpha\|_{L^\infty} > 1$, by our assumption $(n-2)^2(|\lambda|C_0\|\alpha\|_{L^\infty} - 1) \leq (n-1)^2 \left(\frac{(n-2)^2}{4} - \mu\right)\kappa$ we obtain that $\mu < \frac{(n-2)^2}{4}$; moreover, relation (3.3) and the assumption imply that
\[
\int_M |\nabla_g u(x)|^2 dv_g < \frac{(n-1)^2}{4}\kappa \int_M u^2(x) dv_g.
\]
The latter inequality is in contradiction to McKean’s spectral gap theorem, see (2.3). Therefore, we necessarily have $u = 0$, concluding the proof of (i2) for $\mu > 0$.

If $\mu \leq 0$, then our assumption reduces to $|\lambda|C_0\|\alpha\|_{L^\infty} - 1 \leq \frac{(n-1)^2}{4}\kappa$ and by (3.2) one has that
\[
\int_M |\nabla_g u(x)|^2 dv_g \leq (|\lambda|C_0\|\alpha\|_{L^\infty} - 1) \int_M u^2(x) dv_g.
\]
Therefore, we obtain that
\[
\int_M |\nabla_g u(x)|^2 dv_g \leq \frac{(n-1)^2}{4}\kappa \int_M u^2(x) dv_g.
\]
Since no equality occurs in McKean’s spectral gap estimate (2.3) for any non-zero function $u \in H^1(M)$, we arrive to a contradiction. In conclusion, we necessarily have that $u = 0$, which ends the proof of (i2) also for $\mu \leq 0$.

Proof of (ii): $\text{Ric}_{(M,g)} \geq 0$. Since $\mu \leq \text{AVP}_{(M,g)}^2 (n-2)^2 \frac{(n-2)^2}{4}$, the Hardy inequality from (2.5) (together with the fact that no non-zero function realizes the equality) and relation (3.2) imply that
\[
\int_M u^2(x) dv_g < |\lambda|C_0\|\alpha\|_{L^\infty} \int_M u^2(x) dv_g.
\]
Consequently, if $|\lambda|C_0\|\alpha\|_{L^\infty} \leq 1$, we arrive to a contradiction; thus $u = 0$. This ends the proof of (ii). \[\square\]

4. Sub-quadratic case: proof of Theorem 1.2

Throughout this section we assume the assumptions in Theorem 1.2 are satisfied. The proof is divided into several steps.

Step 1. (Truncation and nonsmooth energy functional) Since $F(0) = 0$ by (H3), we consider the truncated locally Lipschitz function $F^+(t) = F(t_+), t \in \mathbb{R}$. The energy functional $\mathcal{E}^+: H^1(M) \to \mathbb{R}$ to the slightly modified problem (1.1), considering $F^+$ instead of $F$, is defined as
\[
\mathcal{E}^+(u) = \frac{1}{2} \mathcal{N}_\mu(u) - \lambda \mathcal{F}^+(u),
\]
where
\[
\mathcal{N}_\mu(u) = \int_M |\nabla_g u(x)|^2 dv_g - \mu \int_M \frac{u^2(x)}{d_g^2(x_0, x)} dv_g + \int_M u^2(x) dv_g
\]
and
\[
\mathcal{F}^+(u) = \int_M \alpha(x)F^+(u(x)) dv_g.
\]

On one hand, it is clear that $\mathcal{N}_\mu$ is of class $C^1$ on $H^1(M)$ and due to the Hardy inequalities (i.e., (2.2) and (2.5)), for the corresponding values of $\mu$ from the statement of the theorem, $\mathcal{N}^{1/2}_\mu$ turns out to be equivalent to the usual norm $\|\cdot\|_{H^1}$ on $H^1(M)$, i.e.,
\[
c_{\mu}\|u\|_{H^1}^2 \leq \mathcal{N}_\mu(u) \leq \|u\|_{H^1}^2, \quad \forall u \in H^1(M), \tag{4.1}
\]
where

\[ 0 < c_\mu = \begin{cases} 
1 - \frac{4\mu}{(n-2)^2}, & \text{in the case (i)}; \\
1 - \frac{\operatorname{AVR}}{(M,g)} \frac{4\mu}{(n-2)^2} & \text{in the case (ii)}. 
\end{cases} \]

On the other hand, one can prove that \( \mathcal{F}^+ \) is well-defined and locally Lipschitz on \( H^1(M) \). To see this, we first observe that by \((H)_1\) and \((H)_2\), for every \( \epsilon > 0 \) there exists \( \delta_\epsilon \in (0,1) \) such that

\[ |\xi| \leq \epsilon t, \forall \xi \in \partial \mathcal{F}^+(t), \forall 0 < t < \delta_\epsilon \text{ and } t > \delta_\epsilon^{-1}. \tag{4.2} \]

Fix \( \epsilon_0 > 0 \). Since \( \partial \mathcal{F}^+ \) is an upper semicontinuous set-valued map with non-empty compact values, see Proposition 2.1/(i), we also have for some \( K_{\epsilon_0} > 0 \) that \( |\xi| \leq K_{\epsilon_0} t \) for every \( \xi \in \partial \mathcal{F}^+(t) \) and \( t \in [\delta_\epsilon, \delta_\epsilon^{-1}] \). The latter fact with (4.2) implies that

\[ |\xi| \leq C_{\epsilon_0} t, \forall \xi \in \partial \mathcal{F}^+(t), \forall t > 0, \]

where \( C_{\epsilon_0} = \max\{\epsilon_0, K_{\epsilon_0}\} \). Now, let \( u \in H^1(M) \) and \( U_u \) be any open bounded neighborhood of \( u \) in \( H^1(M) \), i.e., for some \( K > 0 \) we have \( \|u\|_{H^1} \leq K \) for every \( w \in U_u \). If \( u_1, u_2 \in U_u \), then by Lebourg’s mean value theorem, see Proposition 2.1/(iv), for a.e. \( x \in M \) there exist \( \gamma \in [0,1] \) and \( \xi^2 \in \partial \mathcal{F}^+((1 - \gamma)u_1(x) + \gamma u_2(x)) \) such that

\[ |F^+(u_1(x)) - F^+(u_2(x))| = |\xi^2_u|u_1(x) - u_2(x)| \leq C_{\epsilon_0}(|u_1(x)| + |u_2(x)|)|u_1(x) - u_2(x)|. \]

By Hölder’s inequality and the trivial embedding \( H^1(M) \subset L^2(M) \), we have that

\[ |F^+(u_1) - F^+(u_2)| \leq \int_M \alpha(x)|F^+(u_1(x)) - F^+(u_2(x))|d\nu_g \leq 2C_{\epsilon_0}\|\alpha\|_{L^\infty K} \|u_1 - u_2\|_{H^1}, \]

which means that \( \mathcal{F}^+ \) is Lipschitz on \( U_u \). The fact that \( \mathcal{F}^+ \) is well-defined follows in a similar way. Having these properties, a similar argument as in Clarke [11, Section 2.7] (see also Costea, Kristály and Varga [12]) shows that for every closed subspace \( W \) of \( H^1(M) \) we have that

\[ \partial(\mathcal{F}^+|_W)(u) \subseteq \int_M \alpha(x)\partial \mathcal{F}^+(u(x))d\nu_g, \forall u \in W; \]

here, \( \mathcal{F}^+|_W \) is the restriction of the functional \( \mathcal{F}^+ \) to the subspace \( W \) and the latter inclusion has the following interpretation: to every \( \xi \in \partial(\mathcal{F}^+|_W)(u) \) there exists a measurable selection \( x \mapsto \xi_x \in \partial \mathcal{F}^+(u(x)) \) such that the map \( x \mapsto \alpha(x)\xi_x w(x) \) belongs to \( L^1(M) \) for every \( w \in W \) and

\[ \langle \xi, w \rangle = \int_M \alpha(x)\xi_x w(x)d\nu_g. \]

By using Fatou’s lemma, Lebourg’s mean value theorem, Lebesgue’s dominated convergence theorem, and a careful limiting argument, see e.g. Kristály [23] in the Euclidean setting, it turns out that

\[ (\mathcal{F}^+|_W)^0(u;w) \leq \int_M \alpha(x)(F^0(u);w)d\nu_g, \forall u, w \in W. \tag{4.3} \]

Let \( u \in H^1(M) \) be a critical point of \( \mathcal{E}^+ \), i.e., \( 0 \in \partial \mathcal{E}^+(u) \). We are going to prove that \( u \) is a non-negative solution to the differential inclusion (1.1). First, by Proposition 2.1/(v)&(vii), we have that

\[ \frac{1}{2}N'_\mu(u) = \lambda \partial \mathcal{F}^+(u), \]

i.e., for every test-function \( w \in H^1(M) \) one has

\[ \int_M \nabla_\mu u(x) \nabla_\nu w(x)d\nu_g - \mu \int_M \frac{u(x)w(x)}{d^2_\mu(x,0)}d\nu_g + \int_M u(x)w(x)d\nu_g = \lambda \int_M \alpha(x)\xi_x w(x)d\nu_g, \]

with the above interpretation for the right hand side. Let \( u_- = 0 \) be the non-positive part of \( u \) and note that it belongs to the space \( H^1(M) \), see Hebey [17, Proposition 2.5]. If we put \( v = u_- \) into the latter relation, we obtain that \( \xi_x u_-(x) = 0 \) for a.e. \( x \in M \) since \( \xi_x \in \partial \mathcal{F}^+(u(x)) \) (thus \( \xi_x = 0 \) whenever \( u(x) < 0 \)). In consequence, \( N'_\mu(u_-) = 0 \), thus \( u_- = 0 \), i.e., \( u \geq 0 \). In particular, \( \xi_x \in \partial \mathcal{F}^+(u(x)) = \partial \mathcal{F}(u(x)) \), therefore the latter relation is precisely (1.2), which means that \( u \in H^1(M) \) is a non-negative solution of (1.1).
Step 2. (Isometry actions) Let $G$ be a compact connected subgroup of $\text{Isom}_g(M)$ with the property that $\text{Fix}_M(G) = \{x_0\}$ for the same $x_0 \in M$ as in problem (1.1). The action of $G$ on $H^1(M)$, i.e., $G \times H^1(M) \to H^1(M)$, is defined by

$$(\sigma u)(x) = u(\sigma^{-1}(x)), \quad \forall \sigma \in G, \ u \in H^1(M), \ x \in M.$$  \hspace{1cm} (4.4)

It is standard to prove that $G$ acts continuously and linearly on $H^1(M)$. For instance, if $\sigma_1, \sigma_2 \in G$, it turns out that for every $u \in H^1(M)$ and $\sigma \in G$ we have

$$(\sigma_1 \circ \sigma_2)u(x) = u((\sigma_1 \circ \sigma_2)^{-1}(x)) = u(\sigma_2^{-1}(\sigma_1^{-1}(x))) = (\sigma_2u)(\sigma_1^{-1}(x)) = (\sigma_1(\sigma_2u))(x).$$

Moreover, since $G$ contains isometries of $(M,g)$, the functionals $u \mapsto \int_M |\nabla_g u(x)|^2 dv_g$ and $u \mapsto \int_M u^2(x)dv_g$ are both $G$-invariant; in particular, $\|\sigma u\|_{H^1} = \|u\|_{H^1}$ for every $\sigma \in G$ and $u \in H^1(M)$, i.e., $G$ acts isometrically on $H^1(M)$.

Since $\text{Fix}_M(G) = \{x_0\}$, it turns out that for every $\sigma \in G$ and $y \in M$, we have $d_g(x_0, \sigma(y)) = d_g(x_0, y)$; therefore, a change of variables implies that

$$\int_M \frac{(\sigma u)^2(x)}{d^2_g(x_0, x)} dv_g(x) = \int_M \frac{u^2(\sigma^{-1}(x))}{d^2_g(x_0, x)} dv_g(x) = \int_M \frac{u^2(y)}{d^2_g(x_0, \sigma(y))} dv_g(\sigma(y))$$

In particular, the functional $u \mapsto N_\mu(u)$ is $G$-invariant on $H^1(M)$.

Furthermore, since $\alpha : M \to \mathbb{R}$ depends only on $d_g(x_0, \cdot)$, it is also $G$-invariant, and one can prove by a change of variables that for every $\sigma \in G$ and $u \in H^1(M)$,

$$\mathcal{E}^+(\sigma u) = \int_M \alpha(x)F^+((\sigma u)(x))dv_g(x) = \int_M \alpha(x)F^+(u(\sigma^{-1}(x)))dv_g(x)$$

i.e., $\mathcal{E}^+$ is $G$-invariant on $H^1(M)$. In conclusion, the energy functional $\mathcal{E}^+ = N_\mu/2 - \lambda \mathcal{F}^+$ is $G$-invariant on $H^1(M)$.

The set of fixed points of $G$ over $H^1(M)$, i.e., $\text{Fix}_{H^1(M)}(G)$, is nothing but the closed subset of $G$-invariant functions of $H^1(M)$. Now, according to the principle of symmetric criticality, see Proposition 2.2, if $u \in \text{Fix}_{H^1(M)}(G) =: H^1_G(M)$ is a critical point of the restricted energy functional $\mathcal{E}_G^+ := \mathcal{E}^+|_{H^1_G(M)}$ then $u$ is also a critical point of the initial energy functional $\mathcal{E}^+$.

Step 3. (Spectral gap estimate for $\mathcal{F}^+ / N_\mu$ on $H^1_G(M)$.) We are going to prove that for every admissible $\mu$ from the statement of the theorem, one has

$$0 < \sup_{u \in H^1_G(M) \setminus \{0\}} \frac{\mathcal{F}^+(u)}{N_\mu(u)} < +\infty. \hspace{1cm} (4.5)$$

Let $q \in (2, 2^*)$ and fix arbitrarily $\epsilon > 0$ together with the number $\delta_\epsilon > 0$ appearing in (4.2). By the boundedness of the function $t \mapsto \max_{[\delta_\epsilon, \delta_\epsilon^{-1}]} |\partial F^+(t)|$ on $[\delta_\epsilon, \delta_\epsilon^{-1}]$ and due to (4.2), there exists $\ell_\epsilon > 0$ such that

$$0 \leq |\xi| \leq \epsilon t + \ell_\epsilon t^{q-1}, \quad \forall t \geq 0, \ \xi \in \partial F^+(t) = \partial F(t). \hspace{1cm} (4.6)$$

Note that we have

$$0 \leq |F^+(t)| \leq \epsilon t^q + \ell_\epsilon |t|^q, \quad \forall t \in \mathbb{R}. \hspace{1cm} (4.7)$$

Indeed, on one hand, by definition $F^+(t) = F(0) = 0$ for every $t \leq 0$, thus the latter relation trivially holds. On the other hand, for $t \geq 0$, the estimate (4.6) and Lebourg’s mean value theorem immediately imply the required estimate.
Consequently, the estimate (4.7) shows that for every $u \in H^1_G(M)$ we have
\[
0 \leq |F^+(u)| = \left| \int_M \alpha(x)F^+(u(x))dv_g \right| \leq \int_M \alpha(x)|F^+(u(x))| dv_g \\
\leq ||\alpha||_{L^\infty} \left( \epsilon \|u\|^2_{H^1_G} + L_\epsilon (K^+_q)^q \|u\|_{H^1_G}^q \right),
\]
where $K^+_q > 0$ are the embedding constants from (2.1) and (2.4), respectively. Accordingly, for every $u \in H^1_G(M) \setminus \{0\}$ one has that
\[
0 \leq \frac{|F^+(u)|}{N_\mu(u)} \leq c_\mu^{-1} ||\alpha||_{L^\infty} \left( \epsilon + L_\epsilon (K^+_q)^q \|u\|_{H^1_G}^q \right),
\]
where $c_\mu > 0$ is the constant from (4.1). Due to the fact that $q > 2$ and $\epsilon > 0$ is arbitrarily fixed, it turns out that
\[
\frac{F^+(u)}{N_\mu(u)} \to 0 \quad \text{as} \quad \|u\|_{H^1} \to 0, \quad u \in H^1_G(M). \tag{4.8}
\]
The counterpart of (4.8) at ‘infinity’ reads as
\[
\frac{F^+(u)}{N_\mu(u)} \to 0 \quad \text{as} \quad \|u\|_{H^1} \to +\infty, \quad u \in H^1_G(M). \tag{4.9}
\]
Indeed, combining the boundedness of $t \mapsto \frac{\max|\partial F^+(t)|}{|\partial F^+(t)|}$ on $[\delta_\epsilon, \delta_\epsilon^{-1}]$ with the estimate (4.2), one can find $L_\epsilon > 0$ such that
\[
0 \leq |\xi| \leq \epsilon t + L_\epsilon t^{1/2}, \quad \forall t \geq 0, \quad \xi \in \partial F^+(t) = \partial F(t). \tag{4.10}
\]
Due to hypothesis $(H)_{\alpha_0}$, one has that $\alpha \in L^4(M)$. Then using Lebourg’s mean value theorem and Hölder’s inequality, we can proceed as before, obtaining
\[
0 \leq |F^+(u)| \leq \int_M \alpha(x)|F^+(u(x))| dv_g \leq \epsilon \|\alpha\|_{L^\infty} \|u\|^2_{H^1} + L_\epsilon \|\alpha\|_{L^4} \|u\|^2_{H^1}. \tag{4.11}
\]
Consequently, for every $u \in H^1_G(M) \setminus \{0\}$ we have
\[
0 \leq \frac{|F^+(u)|}{N_\mu(u)} \leq c_\mu^{-1} \left( \epsilon \|\alpha\|_{L^\infty} + L_\epsilon \|\alpha\|_{L^4} \|u\|_{H^1}^{2/3} \right).
\]
This estimate together with the arbitrariness of $\epsilon > 0$ immediately imply (4.9).

In particular, (4.8) and (4.9) imply that the second inequality in (4.5) holds. In order to check the first inequality in (4.5), we recall by $(H)_{\delta_0}$ that there exists $t_0^0 > 0$ such that $F(t_0^0) > 0$. Moreover, by $(H)_{\alpha_0}$, since $\alpha \neq 0$ and it depends only on $d_g(x_0, \cdot)$, there exists an open $x_0$-centered annulus on $M$ with radii $0 \leq r < R$, i.e. $A_{\delta_0}(r, R) = \{x \in M : r < d_g(x_0, x) < R\}$, such that $\text{essinf}_{A_{\delta_0}(r, R)} \alpha = \alpha_0 > 0$. For sufficiently small $\epsilon > 0$ (e.g. $\epsilon < (R - r)/3$), we consider the function $w_\epsilon : M \to \mathbb{R}$ defined by
\[
w_\epsilon(x) = \begin{cases} \frac{\epsilon}{t_0^0}(d_g(x_0, x) - r) & \text{if } d_g(x_0, x) \in (r, r + \epsilon), \\ \frac{\epsilon}{t_0^0}(R - d_g(x_0, x)) & \text{if } d_g(x_0, x) \in (R - \epsilon, R), \\ 0 & \text{if } x \notin A_{\delta_0}(r, R). \end{cases}
\]
Note that $w_\epsilon \in H^1_G(M)$ and $w_\epsilon \geq 0$. Moreover,
\[
F^+(w_\epsilon) = \int_M \alpha(x)F(w_\epsilon(x))dv_g \geq \int_{A_{\delta_0}(r, R)} \alpha(x)F(w_\epsilon(x))dv_g \\
\geq \alpha_0 F(t_0^0) V_g(A_{\delta_0}(r + \epsilon, R - \epsilon)) \\
- \|\alpha\|_{L^\infty} \max_{t \in [0, t_0^0]} |F(t)| \left( V_g(A_{\delta_0}(r, r + \epsilon)) + V_g(A_{\delta_0}(R - \epsilon, R)) \right).
\]
By continuity reason, there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$,
\[
F^+(w_\epsilon) \geq \alpha_0 F(t_0^0) V_g(A_{\delta_0}(r, R))/2 > 0.
\]
On the other hand, by (4.1) and the eikonal equation ($|\nabla g d_g(x_0, \cdot)| = 1$ a.e. on $M$) we have the estimate

$$\mathcal{N}_\mu(w_\epsilon) \leq \|w_\epsilon\|^2_{H^1} \leq (t_0^+)^2(1+\epsilon^{-2})V_g(Ax_0(r, R)) < +\infty.$$ 

Consequently, it turns out that

$$0 < \frac{\mathcal{F}^+(w_{x_0/2})}{\mathcal{N}_\mu(w_{x_0/2})} \leq \sup_{u \in H^1_G(M) \setminus \{0\}} \frac{\mathcal{F}^+(u)}{\mathcal{N}_\mu(u)},$$

which shows the validity of the first inequality in (4.5).

**Step 4.** (Analytic properties of $\mathcal{E}_G^+$) We shall prove three basic properties of $\mathcal{E}_G^+$ on $H^1_G(M)$, namely, coercivity and boundedness from below, as well as the validity of the nonsmooth Palais-Smale condition.

Let $\lambda > 0$ be arbitrarily fixed and $\mu$ be in the admissible range (cf. the statement of the theorem). First, we observe by (4.1) and (4.11) that for every $u \in H^1_G(M)$ we have

$$\mathcal{E}_G^+(u) = \frac{1}{2} \mathcal{N}_\mu(u) - \lambda \mathcal{F}^+(u) \geq \left( \frac{c_\mu}{2} - \epsilon \lambda \alpha \|\alpha\|_{L^\infty} \right) \|u\|^2_{H^1} - \lambda \alpha \|\alpha\|_{L^1} \|u\|_{H^1}^2.$$ 

In particular, for sufficiently small $\epsilon > 0$, e.g. $0 < \epsilon < \frac{c_\mu}{4} \lambda^{-1} \|\alpha\|_{L^\infty}^{-1}$, it follows that $\mathcal{E}_G^+$ is bounded from below and coercive, i.e., $\mathcal{E}_G^+(u) \to +\infty$ whenever $\|u\|_{H^1} \to +\infty$.

Let $\{u_k\} \subset H^1_G(M)$ be a Palais-Smale sequence for $\mathcal{E}_G^+$, i.e., for some $M > 0$, one has $|\mathcal{E}_G^+(u_k)| \leq M$ and $m(u_k) \to 0$ as $k \to \infty$, where $m(u) = \min\{\|\xi\|_s : \xi \in \partial \mathcal{E}_G^+(u)\}$. We want to prove that, up to a subsequence, $\{u_k\}$ strongly converges to some element in $H^1_G(M)$. Being $\mathcal{E}_G^+$ coercive, the sequence $\{u_k\} \subset H^1_G(M)$ is bounded in $H^1_G(M)$. Therefore, due to the fact that $H^1_G(M)$ can be compactly embedded into $L^q(M)$, $q \in (2, 2^*)$, see §2.1.3, up to a subsequence, one has that

$$u_k \to u \text{ weakly in } H^1_G(M); \quad u_k \to u \text{ strongly in } L^q(M), \quad q \in (2, 2^*).$$

(4.12) (4.13)

By Proposition 2.1/(v) and the definition of $\mathcal{E}_G^+$ we have that

$$(\mathcal{E}_G^+)^0(u_k; u - u_k) = \frac{1}{2} \langle \mathcal{N}_\mu'(u_k), u - u_k \rangle + \lambda(-\mathcal{F}^+)^0(u_k; u - u_k);$$

$$(\mathcal{E}_G^+)^0(u; u_k - u) = \frac{1}{2} \langle \mathcal{N}_\mu'(u), u_k - u \rangle + \lambda(-\mathcal{F}^+)^0(u; u_k - u).$$

Note that

$$\frac{1}{2} \langle \mathcal{N}_\mu'(u_k), u - u_k \rangle + \frac{1}{2} \langle \mathcal{N}_\mu'(u), u_k - u \rangle = -\mathcal{N}_\mu(u_k - u).$$

By adding the above relations and using Proposition 2.1/(vi), it turns out that

$$\mathcal{N}_\mu(u_k - u) = \lambda ((\mathcal{F}^+)^0(u_k; -u + u_k) + (\mathcal{F}^+)^0(u; -u_k + u)) - (\mathcal{E}_G^+)^0(u_k; u - u_k) - (\mathcal{E}_G^+)^0(u; u_k - u).$$

(4.14)

In the sequel, we are going to estimate the terms in the right hand side of (4.14). First, by inequality (4.3), Proposition 2.1/(ii) and (4.6) together with the fact that $\partial \mathcal{F}^+(t) = \{0\}$ for
\[ t \leq 0, \text{ we have} \]
\[ I_k^1 := (\mathcal{F})^0(u_k; -u + u_k) + (\mathcal{F}^+)^0(u; -u_k + u) \]
\[ \leq \int_M \alpha(x) \left[ (\mathcal{F}^+)^0(u_k(x); u_k(x) - u(x)) + (\mathcal{F}^+)^0(u(x); u(x) - u_k(x)) \right] \, dv_g \]
\[ = \int_M \alpha(x) \left[ \max\{ \xi_k(u_k(x) - u(x)) : \xi_k \in \partial \mathcal{F}^+(u_k(x)) \} \right. \]
\[ \left. + \max\{ \xi(u(x) - u_k(x)) : \xi \in \partial \mathcal{F}^+(u(x)) \} \right] \, dv_g \]
\[ \leq \| \alpha \|_{L^\infty} \int_M [e(|u_k(x)| + |u(x)|) + l_k(|u_k(x)|^{q-1} + |u(x)|^{q-1})] |u(x) - u_k(x)| \, dv_g \]
\[ \leq 2\varepsilon \alpha \| u \|_{L^\infty} (\|u_k\|^2_{H^1} + \|u\|^2_{H^1}) + l_k \| \alpha \|_{L^\infty} (\|u_k\|^2_{L^q} + \|u\|^2_{L^q}) \|u_k - u\|_{L^q}. \]

By the arbitrariness of \( \varepsilon > 0 \) and the convergence property (4.13), the latter estimate shows that
\[
\limsup_{k \to \infty} I_k^1 \leq 0. \tag{4.15}
\]

Let \( \xi_k \in \partial \mathcal{E}_G^+(u_k) \) be such that \( m(u_k) = \|\xi_k\|_* \). Thus, we have that
\[
I_k^2 := (\mathcal{E}_G^+)^0(u_k; u - u_k) \geq \langle \xi_k, u - u_k \rangle \geq -\|\xi_k\|_* \|u - u_k\|_{H^1}. \tag{4.16}
\]

Consequently, since \( m(u_k) = \|\xi_k\|_* \to 0 \) as \( k \to \infty \), we have that
\[
\liminf_{k \to \infty} I_k^2 \geq 0. \tag{4.17}
\]

Moreover, for every \( \xi \in \partial \mathcal{E}_G^+(u) \), we also have that \( I_k^3 := (\mathcal{E}_G^+)^0(u; u_k - u) \geq \langle \xi, u_k - u \rangle \); thus, by the weak convergence property (4.12) we have that
\[
\liminf_{k \to \infty} I_k^3 \geq 0. \tag{4.17}
\]

By the estimates (4.15)-(4.17) and relation (4.14) we have that
\[
0 \leq \limsup_{k \to \infty} \mathcal{N}_\mu(u_k - u) \leq \limsup_{k \to \infty} I_k^1 - \liminf_{k \to \infty} I_k^2 - \liminf_{k \to \infty} I_k^3 \leq 0,
\]
i.e., \( \mathcal{N}_\mu(u_k - u) \to 0 \) as \( k \to \infty \). Due to (4.1), it turns out that \( u_k \to u \) strongly in the \( H^1 \)-norm as \( k \to \infty \), which is the desired property.

**Step 5.** (Local minimum point for \( \mathcal{E}_G^+ \): first solution) Let
\[
\lambda_0^+ := \inf_{u \in H^1_{G}(M), \mathcal{F}^+(u) > 0} \frac{\mathcal{N}_\mu(u)}{2\mathcal{F}^+(u)}. \tag{4.5}
\]

Due to Step 3, see (4.5), one has that \( 0 < \lambda_0^+ < \infty \).

If we fix \( \lambda > \lambda_0^+ \), one can find \( \tilde{w}_\lambda \in H^1_{G}(M) \) with \( \mathcal{F}^+(\tilde{w}_\lambda) > 0 \) such that
\[
\lambda > \frac{\mathcal{N}_\mu(\tilde{w}_\lambda)}{2\mathcal{F}^+(\tilde{w}_\lambda)} \geq \lambda_0^+. \tag{4.6}
\]

Thus, by the latter inequality we have
\[
C_1^1 := \inf_{H^1_{G}(M)} \mathcal{E}_G^+(\tilde{w}_\lambda) = \frac{1}{2} \mathcal{N}_\mu(\tilde{w}_\lambda) - \lambda \mathcal{F}^+(\tilde{w}_\lambda) < 0.
\]

Due to the fact that \( \mathcal{E}_G^+ \) is bounded from below and verifies the nonsmooth Palais-Smale condition (see Step 4), \( C_1^1 \) is a critical value of \( \mathcal{E}_G^+ \), see Chang [10, Theorem 3.4], i.e., there exists \( u_\lambda^1 \in H^1_{G}(M) \) such that \( \mathcal{E}_G^+(u_\lambda^1) = C_1^1 < 0 \) and \( 0 \in \partial \mathcal{E}_G^+(u_\lambda^1) \). In particular, \( u_\lambda^1 \neq 0 \) (since \( \mathcal{E}_G^+(u_\lambda^1) < 0 = \mathcal{E}_G^+(0) \)), and by the principle of symmetric criticality, \( u_\lambda^1 \) is a critical point also for the initial energy functional (see Step 2), i.e., \( 0 \in \partial \mathcal{E}_G^+(u_\lambda^1) \). According to (the final part of) Step 1, \( u_\lambda^1 \in H^1_{G}(M) \) is a non-negative solution to the differential inclusion (1.1).
Step 6. (Minimax-type critical point for $\mathcal{E}_G^+$: second solution) Let $\lambda > \lambda_0^+$. Due to (4.7), for sufficiently small $\epsilon > 0$ (e.g., $\frac{c_0}{2} > \epsilon \lambda \|\alpha\|_{L^\infty}$) and for every $u \in H^1_G(M)$ one has that

$$\mathcal{E}_G^+(u) = \frac{1}{2} \mathcal{N}_\mu(u) - \lambda \mathcal{F}^+(u) \geq \left(\frac{c_0}{2} - \epsilon \lambda \|\alpha\|_{L^\infty}\right) \|u\|_{H^1}^2 - \lambda \|\alpha\|_{L^\infty} l_i(K_\mathcal{G})\|u\|_{H^1}^q,$$

where $q \in (2, 2^*)$ and $K_\mathcal{G}^+ > 0$ are the embedding constants from (2.1) and (2.4), respectively. Let

$$\rho_\lambda = \min \left\{ \|\tilde{u}\|_{H^1}, \left(\frac{c_0}{2} - \epsilon \lambda \|\alpha\|_{L^\infty}\right) \frac{1}{2} \right\}.$$

The choice of $\rho_\lambda > 0$ and Step 4 show that

$$\inf_{\|u\|_{H^1}=\rho_\lambda; u \in H^1_G(M)} \mathcal{E}_G^+(u) \geq \frac{1}{2} \left(\frac{c_0}{2} - \epsilon \lambda \|\alpha\|_{L^\infty}\right) \rho_\lambda^2 > 0 = \mathcal{E}_G^+(0) > \mathcal{E}_G^+(\tilde{\omega} \lambda).$$

The latter estimate shows that the functional $\mathcal{E}_G^+$ has the mountain pass geometry. On account of Step 4, since $\mathcal{E}_G^+$ satisfies the nonsmooth Palais-Smale condition, we can apply the mountain pass theorem for locally Lipschitz functions, see e.g. Kourogenis and Papageorgiou [19] or Kristály, Motreanu and Varga [25, Theorem 2], guaranteeing the existence of $u_\lambda^2 \in H^1_G(M)$ with the properties $0 \in \partial \mathcal{E}_G^+(u_\lambda^2)$ and

$$\mathcal{E}_G^+(u_\lambda^2) = C_\lambda^2 = \inf_{\gamma \in \Gamma; t \in [0,1]} \max_{\gamma(t)} \mathcal{E}_G^+(\gamma(t)), $$

where

$$\Gamma = \{ \gamma \in C([0,1]; H^1_G(M)) : \gamma(0) = 0, \gamma(1) = \tilde{\omega} \lambda \}. $$

Since

$$C_\lambda^2 \geq \inf_{\|u\|_{H^1}=\rho_\lambda; u \in H^1_G(M)} \mathcal{E}_G^+(u) > 0,$$

it is clear that $0 \neq u_\lambda^2 \neq u_\lambda^1$. The rest of the proof is similar to the end of Step 5, which shows that $u_\lambda^2 \in H^1_G(M)$ is a non-negative solution to the differential inclusion (1.1), different from $u_\lambda^1$.}

Step 7. (Repetition of Steps 1-6 for $\mathcal{E}^-$) Let $F^{-} (t) = F(t_-)$, $t \in \mathbb{R}$, where $t_- = \min(t, 0)$. The locally Lipschitz energy functional $\mathcal{E}^- : H^1(M) \to \mathbb{R}$ is defined as

$$\mathcal{E}^-(u) = \frac{1}{2} \mathcal{N}_\mu^-(u) - \lambda \mathcal{F}^-(u),$$

where

$$\mathcal{F}^-(u) = \int_M \alpha(x) F^-(u(x)) dv_g.$$

One can show that if $u \in H^1(M)$ is a critical point of $\mathcal{E}^-$, i.e., $0 \in \partial \mathcal{E}^-(u)$, then it is a non-positive solution of (1.1), cf. Step 1.

By using the isometry action (4.4), one can prove that $\mathcal{F}^-$ is $G$-invariant on $H^1_G(M)$, and if $u \in \text{Fix}_{H^1_G}(G) =: H^1_G(M)$ is a critical point of $\mathcal{E}_G^- := \mathcal{E}^-|_{H^1_G(M)}$ then $0 \in \partial \mathcal{E}^-(u)$ as well, cf. Step 2.

Instead of the spectral gap estimate (4.5), one can prove

$$0 < \sup_{u \in H^1_G(M) \setminus \{0\}} \frac{\mathcal{F}^-(u)}{\mathcal{N}_\mu^-(u)} < +\infty,$$

cf. Step 3, and similar analytic properties are valid for $\mathcal{E}_G^-$ as in Step 4 (i.e, coercivity, boundedness from below, and the validity of the nonsmooth Palais-Smale condition). Here, we use again the compact embedding results from $\S 2.1.3$. Finally, if

$$\lambda_0^- := \inf_{u \in H^1_G(M)} \frac{\mathcal{N}_\mu^-(u)}{\mathcal{F}^-(u)}.$$
by the previous part we know that $0 < \lambda_0 < \infty$ and similarly to Steps 5 and 6, we can guarantee for every $\lambda > \lambda_0$ a local minimum point $u^3_\lambda \in H^1_G(M)$ of $E^{-}_G$ with $\mathcal{E}^{-}(u^3_\lambda) < 0$ and a minimax-type point $u^4_\lambda \in H^1_G(M)$ of $E^{-}_G$ with $\mathcal{E}^{-}_G(u^4_\lambda) > 0$; in particular, $u^3_\lambda \neq u^4_\lambda$ and none of them is trivial. These elements are $G$-invariant, non-positive solutions to the differential inclusion (1.1).

If we choose $\lambda_0 = \max(\lambda^*_G, \lambda_0)$, we can apply the above arguments, providing four different, non-zero $G$-invariant solutions to the differential inclusion (1.1) for every $\lambda > \lambda_0$, two of them being non-negative and the other two being non-positive. The proof is complete. □

5. Super-quadratic case: proof of Theorem 1.3

We assume in the sequel that the assumptions of Theorem 1.3 are fulfilled. We again divide the proof into some steps.

Step 1. (Functional setting) In view of the previous section, this part is standard. Indeed, the energy functional $E : H^1(M) \rightarrow \mathbb{R}$ is defined as

$$\mathcal{E}(u) = \frac{1}{2}N(u) - \lambda F(u),$$

where

$$F(u) = \int_M \alpha(x) F(u(x)) dv_g.$$ 

Note that by (H) and (H)$_5$, for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\xi| \leq \epsilon|t| + C_\epsilon|t|^q - 1, \ \forall t \in \mathbb{R}, \ \xi \in \partial F(t).$$

Consequently, one has

$$|F(t)| \leq \epsilon t^2 + C_\epsilon |t|^q, \ \forall t \in \mathbb{R}. \quad (5.1)$$

Since $2 < q < 2 + \frac{4}{n} < 2^*$, by using Lebourg’s mean value theorem and (5.1), one can prove that $F$ is well-defined and locally Lipschitz on $H^1(M)$. It is now standard to show that any critical point $u \in H^1(M)$ of $E$ is a solution of (1.1).

Step 2. (Isometry actions) By using the action (4.4), one can prove in a similar way as in §4 that $E$ is $G$-invariant on $H^1(M)$. Moreover, the principle of symmetric criticality (Proposition 2.2) implies that if $u \in \text{Fix}_{H^1(M)}(G) =: H^1_G(M)$ is a critical point of $\mathcal{E}_G := \mathcal{E}|_{H^1_G(M)}$ then $u$ is also a critical point of $\mathcal{E}$.

Step 3. (Super-quadraicity of $F$ at infinity) We are going to prove that

$$F(t) \geq \frac{C}{\nu - 2} |t|^\nu, \ \forall t \in \mathbb{R}, \quad (5.3)$$

where $\nu > 2$ and $C > 0$ come from hypothesis (H)$_4$; this means in particular that $F$ is super-quadratic at infinity (as $\nu > 2$). To do this, let $h : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$h(t) = t^{-2}F(t) - \frac{C}{\nu - 2} |t|^\nu - 2, \ t \neq 0,$$

and $h(0) = 0$. Note that $h$ is well-defined and locally Lipschitz (indeed, by (H) and $F(0) = 0$ we have that $F(t) = o(t^2)$ as $t \rightarrow 0$). By Proposition 2.1/(v), one has

$$\partial h(t) = -2t^{-3}F(t) + t^{-2} \partial F(t) - C |t|^{\nu - 4} t, \ \forall t \in \mathbb{R} \setminus \{0\}.$$ 

We shall prove (5.3) for $t \geq 0$, the case $t \leq 0$ being similar. Let $t > 0$; then by Lebourg’s mean value theorem, there exist $\theta \in (0, t)$ and $\xi_\theta \in \partial h(\theta)$ such that $h(t) = h(t) - h(0) = \xi_\theta t$. In turn, there exists $\xi_F \in \partial F(\theta)$ such that $\xi_\theta = -20^{-3}F(\theta) + \theta^{-2} \xi_F - C \theta^{\nu - 3}$ and by (H)$_4$ we have that

$$h(t) = \xi_\theta t = (-20^{-3}F(\theta) + \theta^{-2} \xi_F - C \theta^{\nu - 3})t = -\theta^{-3}(2F(\theta) + \xi_F(\theta) - C \theta^\nu + \xi_\theta t)^t \geq -\theta^{-3}(2F(\theta) + F(\theta; -\theta) + C \theta^\nu)^t \geq 0,$$

which concludes the proof. In particular, combining (5.2) with (5.3), we necessarily have that $\nu \leq q$. \quad (5.4)
Step 4. (Nonsmooth Cerami compactness condition for $\mathcal{E}_G$) Let \( \{u_k\}_k \subset H^1_G(M) \) be a Cerami sequence for $\mathcal{E}_G$, i.e., for some \( M > 0 \), one has $|\mathcal{E}_G(u_k)| \leq M$ and $1 + \|u_k\|_{H^1}m(u_k) \to 0$ as $k \to \infty$, where $m(u) = \min\{\|\xi\| : \xi \in \partial \mathcal{E}_G(u)\}$. Our objective is to prove that, up to a subsequence, \( \{u_k\}_k \) strongly converges to some element in $H^1_G(M)$.

We first prove that \( \{u_k\}_k \) is bounded in $L^\nu(M)$. For every \( k \in \mathbb{N} \), let \( \xi_k \in \partial \mathcal{E}_G(u_k) \) be such that $\|\xi_k\| = m(u_k)$. We observe that

\[
\mathcal{E}_G^0(u_k; u_k) \geq \langle \xi_k, u_k \rangle - \|\xi_k\|_\star u_k \geq -(1 + \|u_k\|_{H^1})m(u_k).
\]

Since $1 + \|u_k\|_{H^1}m(u_k) \to 0$ as $k \to \infty$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ one has that $\mathcal{E}_G^0(u_k; u_k) \geq -1$. Consequently, Proposition 2.1/(v), inequality (4.3) (which is also valid due to (5.1)) and hypothesis (H4) imply for every $k \in \mathbb{N}$ that

\[
2M + 1 \geq 2\mathcal{E}_G(u_k) - \mathcal{E}_G^0(u_k; u_k) = N_\mu(u_k) - 2\lambda F(u_k) - \frac{1}{2}(\mathcal{N}_\mu(u_k); u_k) - \lambda(-\mathcal{F})^0(u_k; u_k)
\]

\[
\geq -\lambda \int_M \alpha(x)(2F(u_k(x)) + F^0(u_k(x); -u_k(x))) \, dv_g
\]

\[
\geq \lambda C \int_M \alpha(x)|u_k(x)|^\nu \, dv_g.
\]

Since $\alpha \in L^\infty(M)$ and $\text{essinf}_{x \in M} \alpha(x) = \alpha_0 > 0$, the latter estimate implies that

\[
2M + 1 \geq \lambda C \alpha_0 \|u_k\|_{L^\nu}^\nu, \quad \forall k \in \mathbb{N}.
\]

Consequently, \( \{u_k\}_k \) is bounded in $L^\nu(M)$.

Now, we prove that \( \{u_k\}_k \) is bounded in $H^1_G(M)$. By (5.2), for every small $\epsilon > 0$ there exists $\tilde{C}_\epsilon > 0$ such that for every $k \in \mathbb{N}$,

\[
M \geq \mathcal{E}_G(u_k) = \frac{1}{2}N_\mu(u_k) - \lambda F(u_k) \geq \left(\frac{c_\mu}{2} - \epsilon \lambda \|\alpha\|_{L^\infty}\right) \|u_k\|_{H^1}^2 - \lambda \tilde{C}_\epsilon \|\alpha\|_{L^\infty} \|u_k\|_{L^q}^q.
\]

In particular, if $\frac{c_\mu}{4} > \epsilon \lambda \|\alpha\|_{L^\infty}$, then there exists $M_\epsilon > 0$ and $\tilde{C}_\epsilon > 0$ such that

\[
\|u_k\|_{H^1}^2 \leq M_\epsilon + \tilde{C}_\epsilon \|u_k\|_{L^q}^q, \quad \forall k \in \mathbb{N}.
\]  

(5.5)

On account of (5.4), we distinguish two cases:

a) $\nu = q$. Since $\{u_k\}_k$ is bounded in $L^q(M)$ and $\nu = q$, by (5.5) we also have that $\{u_k\}_k$ is bounded in $H^1_G(M)$.

b) $\nu < q$. Let $\eta \in (0, 1)$ be such that $\frac{1}{q} = \frac{1-\eta}{\nu} + \frac{\eta}{2q}$. By (5.5) and a standard interpolation inequality we have that

\[
\|u_k\|_{H^1}^2 \leq M_\epsilon + \tilde{C}_\epsilon \|u_k\|_{L^q}^q \leq M_\epsilon + \tilde{C}_\epsilon \|u_k\|_{L^\nu}^\nu \|u_k\|_{L^{q'}}^{q'} \leq M_\epsilon + \tilde{C}_\epsilon (K_q^\frac{\nu}{\eta q}) \|u_k\|_{L^\nu}^\nu \|u_k\|_{H^1}^\eta q',
\]

(5.6)

where $K_q^\frac{\nu}{\eta q} > 0$ are the embedding constants from (2.1) and (2.4), respectively. Since $q < 2 + \frac{4}{n}$, we have that $\nu > 2 > \frac{n(q-2)}{2}$. We observe that $\nu > \frac{n(q-2)}{2}$ together with $\frac{1}{q} = \frac{1-\eta}{\nu} + \frac{\eta}{2q}$ is equivalent to $\eta q < 2$. The latter inequality and (5.6) imply that $\{u_k\}_k$ is bounded in $H^1_G(M)$.

Now, we can proceed as in Step 4, see §4; in this way we conclude that $\{u_k\}_k$ strongly converges (up to a subsequence) to some element in $H^1_G(M)$.

Step 5. (Existence/multiplicity of critical points for $\mathcal{E}_G$) Under the assumptions of the theorem, one can prove as above that $\mathcal{E}_G$ has the mountain pass geometry. By Step 4 and on account of the mountain pass theorem for locally Lipschitz functions, see e.g. Kourogenis and Papageorgiou [19], we conclude the existence of a non-zero critical point for $\mathcal{E}_G$. When $F$ is even, we may apply the nonsmooth fountain theorem involving the Cerami compactness condition, see e.g. Kristály [23], guaranteeing the existence of a sequence of critical points for the functional $\mathcal{E}_G$. All these points are $G$-invariant solutions to the differential inclusion (1.1), which concludes the proof.\[\square\]
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