On sequential analytic groups

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Abstract

We answer a question of S. Todorčević and C. Uzcátegui from [15] by showing that the only possible sequential orders of sequential analytic groups are 1 and $\omega_1$. Other results on the structure of sequential analytic spaces and their relation to other classes of spaces are given as well. In particular, we provide a full topological classification of sequential analytic groups by showing that all such groups are either metrizable or $k_\omega$-spaces, which, together with a result by Zelenyuk, implies that there are exactly $\omega_1$ non-homeomorphic analytic sequential group topologies.

Keywords: analytic space, topological group, sequential space

1 Introduction

Spaces with ‘definable’ topologies are ubiquitous in mathematics. They often appear as examples when the topology construction does not use the axiom of choice and frequently show up inside function spaces (see [15] for references and further motivation). To make the notion of ‘definable’ more precise recall that a family of subsets of some countable set $X$ viewed as a subset of $2^X$ in the natural product topology is called analytic (see [6]) if it is a continuous image of the irrationals $\mathbb{N}_\omega$. A variety of reasons to study analytic spaces, i.e. countable topological spaces whose topology is analytic is given in [15], [16], and [14]. The authors of [15] coined the term effective topology for the research involving such spaces and presented a number of questions whose answers depend on various set-theoretic assumptions in the realm of general topological spaces (such as the Malykhin problem, see [3]) that have effective counterparts that can be resolved in ZFC alone.

Recall that a space $X$ is called sequential if whenever $X \setminus A \neq \emptyset$ for some $A \subseteq X$ there exists a convergent sequence $S \subseteq A$ such that $S \rightarrow x \in X \setminus A$. If $X$ is sequential one naturally defines the sequential closure of a subset $A$ of $X$ as the set $[A]'$ of all the limits of all the convergent sequences in $A$. Recursively putting $[A]_0 = A$, $[A]_{\alpha+1} = [[A]_{\alpha}]'$, and taking unions at the limit stages, one

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arrives at the concept of an iterated sequential closure. It is well known that for any $A \subseteq X$ where $X$ is sequential, there exists an $\alpha \leq \omega_1$ such that $[A]_\alpha = A$. This observation naturally leads to the definition of the sequential order $so(X)$ of a sequential space $X$ as the smallest ordinal $\gamma \leq \omega_1$ such that $[A]_\gamma = A$ for every $A \subseteq X$. Fréchet spaces are defined as sequential spaces of sequential order $\leq 1$.

In [13] S. Todorčević and C. Uzcátegui show among other results that a countable topological group is metrizable if and only if it is analytic and Fréchet thus solving the effective version of Malykhin’s question on the existence of a non metrizable countable Fréchet group (the non effective version of this question was answered in [4]). In the same paper they pose a question about the sequential orders of sequential analytic groups which can be considered an effective version of a question of Nyikos (see [9] and [11]).

In this paper we answer this question by showing that the only possible sequential orders of sequential analytic groups are 1 and $\omega_1$. In addition, we show that such groups have a very well defined topological structure and their topologies are completely described by an ordinal invariant that measures the scatteredness of their compact subspaces (see below for a more precise discussion).

We assume that all topological spaces appearing below are regular and use the standard set-theoretic notation and terminology (see [7] and [6]). We proceed by defining some of the less common concepts.

A space $X$ is called a $k_\omega$-space if there exists a $\{K_n : n \in \omega\}$ where each $K_n$ is a compact subset of $X$ such that a set $U \subseteq X$ is open if and only if each $U \cap K_n$ is relatively open. The class of $k_\omega$-spaces is stable under taking products, i.e. the product of two (or any finite number of) $k_\omega$-spaces is again $k_\omega$.

Countable $k_\omega$-spaces are sequential and analytic (more precisely, their topology is $F_\sigma\delta$), and form a subclass of $\aleph_0$-spaces (see [8]). Instead of the original definition we shall use the following characterization that describes $\aleph_0$-spaces in the narrow case when $X$ is sequential.

**Lemma 1.** A sequential space $X$ is an $\aleph_0$-space if and only if there exists a countable collection $\{A_n : n \in \omega\}$ of subsets of $X$ such that for any open $U \subseteq X$ and any converging sequence $S \subseteq U$ such that $S \to x \in U$ there is an $n \in \omega$ such that $K_n \subseteq U$ and $K_n \cap S$ is infinite.

Sequential $\aleph_0$-spaces are exactly the quotient images of separable metric spaces ([8]). Even in the case of a countable $X$, not every $\aleph_0$-space is a $k_\omega$-space, however, when $X$ is a countable sequential non Fréchet topological group, a corollary of a more general result in [14] implies that $X$ is a $k_\omega$-space if and only if $X$ is an $\aleph_0$-space. A result in [13] shows that for each such group $so(X) = \omega_1$. Perhaps the most surprising property of the class of all $k_\omega$ countable group topologies is that there are exactly $\omega_1$ of them, moreover, the topological type of such group is uniquely described by the supremum of Cantor-Bendixson ranks of its compact subspaces (see [17] and [5]).
Recall that a collection of open subsets of a topological space is called a
\( \pi \)-base if every open subset of the space contains a member of the collection.
Furthermore, a collection of open subsets is called a local \( \pi \)-base at \( x \in X \) if
every neighborhood of \( x \) contains a set in the collection. It is an easy observation
that a collection of open subsets of \( X \) that is a local \( \pi \)-base at every point in
some dense subset of \( X \) is a \( \pi \)-base of \( X \). The following lemma is well known
(the second statement is the famous Birkhoff-Kakutani metrization theorem).

**Lemma 2.** Every topological group with a countable local \( \pi \)-base at any point
is first countable and every first countable topological group is metrizable.

The countable sequential fan \( S(\omega) \) is defined as the set \( \omega^2 \cup \{ \omega \} \) equipped with
the topology in which every \( (n, i) \in \omega^2 \) is isolated and the basic neighborhoods
of \( \omega \) are \( U_f = \{ (n, i) : i \geq f(n) \} \) where \( f : \omega \to \omega \).

**Definition 1.** Let \( X \) be a topological space. Let \( x \in X \) and \( \langle D_n : n \in \omega \rangle \) be a
collection of infinite countable closed discrete subsets of \( X \) such that for every
open \( U \subseteq X \) such that \( x \in U \) there exists an \( n \in \omega \) such that \( U \cap D_n \) is infinite.
Then \( Y = \cup \{ D_n : n \in \omega \} \cup \{ x \} \) is called a pseudo-\( \mathbb{Q} \) subspace of \( X \).

The utility of the previous definition is illustrated by the following lemma.

**Lemma 3.** If \( X \) has a pseudo-\( \mathbb{Q} \) subspace, \( X \times S(\omega) \) is not sequential.

**Proof.** Let \( Y = \cup \{ D_n : n \in \omega \} \cup \{ x \} \) be a pseudo-\( \mathbb{Q} \) subset of \( X \) where \( x \in X \)
and \( D_n = \{ d^n_i : i \in \omega \} \) be as in Definition 1. Define
\[
A = \cup \{ \cup \{ (d^n_i, (n, i)) : i \in \omega \} : n \in \omega \} \subseteq X \times S(\omega).
\]
Now \( (x, \omega) \in A \setminus A \) but there is no infinite \( S \subseteq A \) such that \( S \rightarrow y \) for some
\( y \in X \). Indeed, otherwise the projection \( \pi_2(S) \subseteq S(\omega) \) contains an infinite
‘diagonal’ convergent subsequence in \( S(\omega) \) or one of the closed discrete subspaces
\( \{ (d^n_i, (n, i)) : i \in \omega \} \) contains an infinite convergent subsequence. \( \square \)

The following lemma is a corollary of Lemma 16 and Corollary 2 of [11].

**Lemma 4.** Let \( \tau \) be a sequential group topology on \( \mathbb{N} \) such that \( \text{so}(\tau) < \omega_1 \). If
\( \{ N_i : i \in \omega \} \) is a collection of nowhere dense subsets of \( \mathbb{N} \) there exists an \( S \subseteq \mathbb{N} \)
such that \( S \rightarrow x \) for some \( x \in \mathbb{N} \) and each \( S \cap N_i \) is finite.

## 2 Analytic and other classes of spaces

In the arguments below, we shall assume that \( \tau \) stands for some analytic topology
on a countable set \( X \). To simplify the notation we will assume that \( X = \mathbb{N} \)
whenever it is convenient. All the references to topological operations and properties
such as convergence, etc. are relative to this topology.

We shall also fix a subtree \( T \) of \( [\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega} \) (see [11] for the definition of
the tree order) that defines \( \tau \), i.e. such that \( U = \pi_1(f) \) for some branch \( f \) of

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Let $T$ whenever $U \in \tau$ is infinite. Given a $P \subseteq 2^\mathbb{N}$ and a $\sigma \in T$ we will use the notation
\[ \sigma \uparrow P = \cap \{ \pi_1(f) : f \text{ is a branch of } T \text{ that extends } \sigma \text{ such that } \pi_1(f) \in P \} \]
if such $f$ exist; otherwise we put $\sigma \uparrow P = \emptyset$.

**Lemma 5.** Let $P \subseteq 2^\mathbb{N}$ and $S \to x$ for some infinite $S \subseteq \mathbb{N}$. Suppose there is an open $U \ni x$ such that $U \in P$. Then there exists a $\sigma \in T$ such that $S \subseteq^* \sigma \uparrow P$.

**Proof.** Let $f$ be a branch of $T$ such that $\pi_1(f) = U$. Pick $\sigma_{-1} \in T$ such that $f$ extends $\sigma_{-1}$ and $x \in \pi_1(\sigma_{-1})$. Suppose no $\sigma \in T$ with the property stated in the lemma exists. Using this one can inductively construct a sequence $\langle \sigma_i : i \in \omega \rangle$ of elements of $T$ and a strictly increasing sequence $\langle n_i : i \in \omega \rangle \subseteq S$ such that for every $i \in \omega$

1. $\sigma_{i+1}$ extends $\sigma_i$;
2. $n_i < \max \pi_1(\sigma_i)$ and $n_i \notin \pi_1(\sigma_i)$;
3. there exists a branch $f_i$ of $T$ that extends $\sigma_i$ such that $\pi_1(f_i) \in P$.

Put $f_{-1} = f$. Let $i \in \omega \cup \{-1\}$ and note that (3) holds for $i = -1$. By the assumption and (3) $S \not\subseteq^* \sigma_i \uparrow P \neq \emptyset$ so one can pick a branch $f_{i+1}$ of $T$ that extends $\sigma_i$ such that $n_{i+1} \notin \pi_1(f_i)$ for some $n_{i+1} \in S$, $n_{i+1} > n_i$. Let $\sigma_{i+1}$ be such that $f_{i+1}$ extends $\sigma_{i+1}$ and $\max \pi_1(f_{i+1}) > \max \pi_1(f_i)$, $\max \pi_1(f_{i+1}) > n_{i+1}$.

Put $f' = \cup \sigma_i : i \in \omega \rangle$. Then $S \not\subseteq^* \pi_1(f') \ni x$, a contradiction. \[ \square \]

As usual, a set function $F : 2^X \to 2^X$ will be called **monotone** if $F(A) \subseteq F(B)$ whenever $A \subseteq B$.

**Lemma 6.** Let $\{ Q_\alpha : \alpha \in \omega_1 \}$ be such that each $Q_\alpha \subseteq 2^\mathbb{N}$ and $Q_\beta \subseteq Q_\alpha$ when $\beta \leq \alpha$. Put $P_\alpha = \{ B \subseteq \mathbb{N} : q \not\subseteq F(B) \forall q \in Q_\alpha \}$ where $F : 2^\mathbb{N} \to 2^\mathbb{N}$ is a monotone set function. Then there exists a $\gamma \in \omega_1$ such that $q \not\subseteq F(\sigma \uparrow P_\gamma)$ for any $q \in \cup \{ Q_\alpha : \alpha \in \omega_1 \}$ and any $\gamma' \geq \gamma$.

**Proof.** Since $T$ is countable, it is enough to show that $\sigma \uparrow P_\alpha = \emptyset$ whenever there is a $q \in Q_\alpha$ such that $q \subseteq F(\sigma \uparrow P_\beta)$ for some $\beta < \omega_1$. Assuming such a $q$ exists, suppose $\sigma \uparrow P_\alpha \neq \emptyset$. Then $\alpha > \beta$, otherwise $q \in Q_\alpha \subseteq Q_\beta$ and $F(\sigma \uparrow P_\beta) \setminus q \neq \emptyset$, since each $P_\beta$ is closed under taking subsets, which contradicts $q \subseteq F(\sigma \uparrow P_\beta)$.

There exists a branch $f$ of $T$ that extends $\sigma$ such that $\pi_1(f) \in P_\alpha \subseteq P_\beta$ so $q \setminus F(\pi_1(f)) \neq \emptyset$. Now $F(\pi_1(f)) \supseteq F(\sigma \uparrow P_\beta \cap \pi_1(f)) = F(\sigma \uparrow P_\beta) \supseteq q$, a contradiction. \[ \square \]

**Lemma 7.** Let $\tau$ be an analytic sequential topology on $\mathbb{N}$. Then there exists a countable family $U$ of open in $\tau$ sets and a countable family $\Xi$ of nowhere dense subsets of $\mathbb{N}$ such that at least one of the following alternatives holds for every $x \in \mathbb{N}$:
for any infinite sequence $S \subseteq \mathbb{N}$ such that $S \to x$ there is a $\xi \in \Xi$ such that $S \subseteq {}^* \xi$;

(2) $U$ is a local $\pi$-base at $x$.

Proof. Put $F(B) = \overline{B}$ and define $Q_\alpha = \{ \text{Int}(\overline{\sigma \uparrow P_\beta}) : \sigma \in T, \beta < \alpha \}$ where $P_\beta$ is defined as in Lemma 6. Find $\gamma \in \omega_1$ as in Lemma 6. It follows from the construction of $Q_\alpha$ that every $\sigma \uparrow P_\gamma$ is nowhere dense.

Put $P = P_\gamma$, $\Xi = \{ \sigma \uparrow P_\gamma : \sigma \in T \}$, $U = Q_\gamma$, and let $S \to x$ for some $x \in \mathbb{N}$. If $U$ is not a local $\pi$-base at $x$ there exists an open $U \ni x$ such that $q \setminus U \neq \emptyset$ for every $q \in Q_\gamma$. Thus $U \in P_\gamma$ and Lemma 6 implies that there is a $\sigma \in T$ such that $S \subseteq {}^* \sigma \uparrow P_\gamma \in \Xi$.

The following is an immediate corollary of Lemma 6, Lemma 2 and the lemma above. It answers Question 7.1 from [15]. Theorem 1 below together with a result in [13] can also be used to obtain this statement.

Corollary 1. A countable topological group is metrizable if and only if it has a sequential analytic topology with the sequential order less than $\omega_1$.

A lemma in [2] and a simple argument result in the following corollary to Lemma 7.

Corollary 2. Every analytic Fréchet space has a countable $\pi$-base.

Proof. Let $X$ be an analytic Fréchet space and put $U = X \setminus P$ where $P$ is the set of all isolated points of $X$. Observe that a result in [2] shows that the first alternative of Lemma 7 does not hold in Fréchet spaces without isolated points so $U$ has a countable $\pi$-base $B$. Adding all the singletons from $P$ to $B$ one obtains a countable $\pi$-base for $X$.

A similar proof shows that the conclusion of Lemma 7 can be sharpened for homogeneous spaces.

Corollary 3. Let $X$ be a homogeneous analytic sequential space. Then $X$ has either a countable $\pi$-base or a countable collection $\Xi$ of nowhere dense subsets such that property (1) of Lemma 7 holds at every $x \in X$.

A quick observation reveals that a disjoint union $Q \cup S_\omega$ of a copy of the rationals and the Arkhangel’skii-Franklin space $S_\omega$ (see [15] for a nice definition of $S_\omega$ and further references) does not satisfy the dichotomy of Corollary 3. Therefore the restrictions in the corollaries above cannot be removed.

Lemma 8. Let $X$ be an analytic sequential space. Then $X$ is a $k_\omega$-space or there exists a pseudo-$Q$ subspace of $X$.

Proof. Put $F(B) = \overline{B}$ and define $Q_\alpha = \{ \overline{\sigma \uparrow P_\beta} : \sigma \in T, \sigma \uparrow P_\beta \text{ is not compact} \}$ where $P_\beta$ is defined as in Lemma 6. Let $\gamma \in \omega_1$ be as in Lemma 5. The construction of $Q_\alpha$ implies that every $\sigma \uparrow P_\gamma$ is compact.
Suppose $X$ has no pseudo-$Q$ subspace. Put $P = P_\gamma$, define a countable family $K = \{ \sigma \uparrow P_\gamma : \sigma \in T, \sigma \uparrow P_\gamma \text{ is compact} \}$, and let $S \to x$ for some $x \in X$. For each $q \in Q_\gamma$, pick a closed infinite discrete subset $D_q \subseteq q$. Call the collection just constructed $\mathcal{D}$. The case of finite $\mathcal{D}$ is immediate so we can assume that $\mathcal{D}$ is infinite. Since $\cup \mathcal{D} \cup \{ x \}$ is not a pseudo-$Q$ subspace of $\mathbb{N}$ there exists an open $U \ni x$ such that $D_q \setminus U \subseteq q \setminus U \neq \emptyset$ for every $q \in Q_\gamma$. Thus $U \in P_\gamma$ and it follows from Lemma 5 that there exists a $\sigma \in T$ such that $S \subseteq^* \sigma \uparrow P_\gamma \subseteq \sigma \uparrow P_\gamma \in K$. 

The next corollary follows from the lemma above and Lemma 4.

**Corollary 4.** Let $X$ be an analytic space. Then $X \times S(\omega)$ is sequential if and only if $X$ is a $k_\omega$-space.

Let $C$ be a closed copy of $S(\omega)$ in $X$. If $X$ is a topological group, it is convenient to assume that $1_X$ is the limit point of $C$ and write $C = \cup \{ C_n : n \in \omega \} \cup \{ 1_X \}$ where $C_n = \{ c_i^p : i \in \omega \} \to 1_X$ are disjoint subsets of $X$ that do not contain $1_X$, such that each $A \subseteq \cup C_n$ satisfying $|A \cap C_n| < \omega$ for every $n \in \omega$ is closed in $X$. We will refer to this representation of $C$ as a natural closed copy of $S(\omega)$ in $X$ and will use the notation above for the sake of brevity below.

**Lemma 9.** Let $X$ be an analytic group and let $\cup \{ C_n : n \in \omega \}$ be a natural closed copy of $S(\omega)$ in $X$. There exists a countable family $\Xi$ of subsets of $X$ with the following properties:

1. for each $p \in \Xi$ there exists an $M_p \in \omega$ such that $p \cap a \cdot C_n = \omega$ implies $n \leq M_p$ for any $a \in X$;

2. for each infinite $S \subseteq X$ where $S \to x$ for some $x \in X$ there exists a $p \in \Xi$ such that $S \subseteq^* p$.

**Proof.** Put $F(B) = (\overline{B})^{-1}\overline{B}$ and define

$$Q_\alpha = \{ F(\sigma \uparrow P_\beta) : \sigma \in T, \beta < \alpha, \{ n \in \omega : F(\sigma \uparrow P_\beta) \cap C_n \neq \emptyset \} = \omega \}$$

for $\alpha < \omega_1$, where $P_\beta$ is as in Lemma 6.

Let $\gamma < \omega_1$ be the index provided by Lemma 6 and $\Xi = \{ \sigma \uparrow P_\gamma : \sigma \in T \}$. Note that for $p = \sigma \uparrow P_\gamma \in \Xi$ the set $\{ n \in \omega : F(p) \cap C_n \neq \emptyset \}$ is finite. Otherwise $q = F(p) \in Q_{\gamma + 1}$ contrary to the choice of $\gamma$. Pick $M_p \in \omega$ so that $F(p) \cap C_n = \emptyset$ for $n \geq M_p$. Now if $|\gamma \cap a \cdot C_n| = \omega$ for some $n \in \omega$ and $a \in X$ then $a \in \gamma$ thus $F(p) \cap C_n = (\overline{\gamma})^{-1}\overline{\gamma} \cap C_n \ni a^{-1} \cdot \overline{\gamma} \cap C_n \neq \emptyset$ so $n \leq M_p$.

Let $S \to x \in X$. Put $P = P_\gamma$. One can construct a set $D \subseteq \cup \{ C_n : n \in \omega \}$ by induction such that $|D \cap C_n| \leq 1$ for each $n \in \omega$ and $D \cap q \neq \emptyset$ for each $q \in Q_\gamma$. Note that $D$ is a closed discrete subset of $X$ and $x^{-1}x = 1_X \notin D$. Therefore there exists an open neighborhood $U$ of $x$ such that $F(U) \cap D = (\overline{U})^{-1}\overline{U} \cap D = \emptyset$ and thus $U \in P$. Now Lemma 5 implies the existence of a $\sigma \in T$ such that $S \subseteq^* \sigma \uparrow P \in \Xi$. 

**Lemma 10.** Let $X$ be an analytic non Fréchet group. If $X$ contains a pseudo-$Q$ subspace then $X$ is not sequential.
Proof. Suppose X is sequential. Since X is not Fréchet, X contains a closed copy of $S(\omega)$ (see [9]) so let $\bigcup \{ C_n : n \in \omega \}$ be a natural closed copy of $S(\omega)$ in X. Let $\Xi = \{ p_n : n \in \omega \}$ be the family provided by Lemma [9] and put $M^n = \max \{ M_i : i \leq n \} + n + 1$ where $M_i$ have the property of Lemma [9](2).

Let $D = \bigcup \{ D_n : n \in \omega \} \cup \{ 1_X \}$ be a pseudo-Q subspace of X where $D_n = \{ d_i^n : i \in \omega \}$ are disjoint closed discrete subspaces of X. Now $M^n$ is a strictly increasing sequence and by the construction the set $a \cdot C_{M^n} \cap (\bigcup \{ p_i : i \leq n \})$ is finite for every $n \in \omega$ and $a \in X$. Pick a strictly increasing sequence $\langle r_i^n : i \in \omega \rangle \subseteq \omega$ such that $e_i^n = d_i^n \cdot c_i^{M^n} \notin \bigcup \{ p_k : k \leq n \}$ and all $e_i^n \neq 1_X$ are distinct. Define $E = \{ e_i^n : n, i \in \omega \}$ and suppose there is an infinite $S \subseteq E$ such that $S \rightarrow x$ for some $x \in X$.

Note that for every $n \in \omega$ the set $\{ e_i^n : i \in \omega \} \subseteq D_n \cdot C_{M^n}$ is closed and discrete in X so we can assume that $S = \{ e_m^{n_i} : i \in \omega \}$ for some strictly increasing $\langle n_i : i \in \omega \rangle \subseteq \omega$. Now $S \subseteq^* p_n$ for some $p_n \in \Xi$ by Lemma [9](2) but $e_m^{n_i} \notin p_n$ for $n_i > n$, a contradiction. Thus $E$ is sequentially closed.

Let V be an open neighborhood of $1_X$ in X and $U \ni 1_X$ be an open subset of X such that $U \cdot U \subseteq V$. Pick $n \in \omega$ such that $D_n \cap U$ is infinite and choose $k \in \omega$ large enough so that $c_k^{M^n} \in U$ and $d_k^n \in D_n \cap U$. Now $e_k^n = d_k^n \cdot c_k^{M^n} \in U \cdot U \subseteq V$. Thus $1_X \in \overline{U} \setminus E$, a contradiction. \qed

Lemmas [10] and [8] imply the following theorem

Theorem 1. Let X be a countable sequential group. Then the following are equivalent:

1) the topology of X is analytic;
2) the topology of X is $F_\sigma$;
3) X is either first countable or $k_\omega$.

The result of Zelenyuk (see [17]) mentioned in the introduction together with the above theorems imply

Corollary 5. There are exactly $\omega_1$ non homeomorphic analytic sequential group topologies. Moreover, if X is an infinite analytic sequential group then all finite powers $X^n$ are such and are homeomorphic to each other.

3 Examples and questions

It has been demonstrated by a number of authors that sequential $R_0$-spaces have a number of properties resembling those of separable metric spaces.

On the other hand, the properties that differentiate between the two classes of spaces are strikingly similar to those that separate analytic spaces and countable metrizable ones. As an example, it is easy to show that a sequential $R_0$-space with a weak diagonal sequence property is first countable (see [8] and [16]). The following example shows that analytic sequential spaces are not necessarily $R_0$-spaces, thus the statement of Theorem [1] is limited to topological groups.
Example 1. Consider the following basis for a topology on \([\mathbb{N}]^{<\omega}\), viewed as a tree with the usual order of end extension. Every point \(x \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}\) is isolated and the basis of neighborhoods of \(\emptyset\) consists of complements of finite unions of branches together with \(\emptyset\). It is shown in [16], Example 5.6 (see also [15], Remark 4.8) that the resulting topology is \(F_\sigma\) and Fréchet with the weak diagonal sequence property but not first countable. Thus the space constructed is not an \(\aleph_0\)-space.

A partial result going in the opposite direction is possible. The next lemma is an easy corollary of the result that each quotient image of the rationals is determined (see [8]) by a countable family of metrizable subspaces.

Lemma 11. If \(X\) is a quotient image of a countable metric space (equivalently, the rationals \(\mathbb{Q}\)) then \(X\) is analytic (more precisely, \(X\) has an \(F_{\sigma\delta}\) topology).

One might hope that due to their ‘tame’ convergence structure, \(\aleph_0\)-spaces form a subfamily of all analytic spaces. The next simple example shows that this is not the case.

Example 2. There are \(2^\mathfrak{c}\) countable Fréchet \(\aleph_0\)-spaces with a single non isolated point. In particular, there are non analytic spaces of such kind.

Proof. Let \(A \subseteq \mathbb{R}\) be an arbitrary subset of the real line. Put

\[
M(A) = (\{0\} \times A) \cup \{(1/n, q) : n \in \mathbb{N}, q \in \mathbb{Q}\}
\]

Define a topology on \(M(A)\) by making the Euclidean neighborhoods of points \((0, a), a \in A\) the new basic neighborhoods and making all other points isolated. The space just constructed is a separable metrizable one. Consider the quotient map that sends \(\{0\} \times A\) into a single point \(\infty\) and is 1-1 on the rest of \(M(A)\). Its image is a Fréchet \(\aleph_0\)-space \(P(A)\) with a single non isolated point.

Let \(A \subseteq \mathbb{R}\) and \(B \subseteq \mathbb{R}\) be two different subsets of the real line, and let, say, \(a \in A\) be such that \(a \notin B\). Given any sequence of rationals \(\langle q_n : n \in \omega\rangle \subseteq \mathbb{Q}\) that converges to \(a\) the set \(\{1/n, q_n : n \in \omega\}\) is a convergent sequence in \(P(A)\) and is a closed discrete subset of \(P(B)\). Hence, the topologies of \(P(A)\) and \(P(B)\) differ, resulting in exactly \(2^\mathfrak{c}\) different Fréchet \(\aleph_0\)-topologies with a single non isolated point. Noting that there are at most \(\mathfrak{c}\) possible homeomorphisms between topologies on a given countable set, one can pick \(2^\mathfrak{c}\) pairwise non homeomorphic spaces \(P(A)\). Given that there are at most \(\mathfrak{c}\) analytic topologies on any countable set, most \(P(A)\) are non analytic. \(\Box\)

Remark 1. The rather crude construction of the example above does not produce any ‘explicit’ \(A \subseteq \mathbb{R}\) such that \(P(A)\) is not analytic. A more precise proof is possible that shows that \(A\) is a projection of a Borel subset of the product of the irrationals and the topology of \(P(A)\) viewed as the subset of the irrationals giving one more control over the complexity of \(P(A)\).

The final example shows that the property established in Lemma 7 is not enough to show that the group is a \(k_\omega\)-space.
Example 3 (CH). There exists a countable sequential group $G$ and a countable collection $\Xi$ of nowhere dense subsets of $G$ such that $G$ is not a $k_\omega$-space and for every convergent sequence $S \subseteq G$ there is a $\xi \in \Xi$ such that $S \subseteq \xi$.

Proof. The full details of the construction are somewhat tedious and are of limited interest. We therefore present just a sketch of the proof. A number of similar arguments can be found in [12].

One starts with a non discrete first countable topology $\tau_0$ on $\mathbb{Q}$ (any topologizable countable group would suffice; it is easy to see that a similar construction just as readily gives an example of a topological field with these properties). Pick a compact subset $K$ of $\mathbb{Q}$ such that $0 \in K$ has Cantor-Bendixson rank $\omega$ in $K$. Pick a countable collection of convergent sequences in $\tau_0$ that witness the Cantor-Bendixson rank of each point of $K$. Let $\eta_0$ be the finest group topology on $\mathbb{Q}$ in which each of these sequences converges. The existence of such a topology can be established by an easy argument (see, for example [12]). Let $\{A_\alpha : \alpha \in \omega_1\} = 2^\mathbb{Q}$.

The construction proceeds by induction on $\alpha \in \omega_1$ where at stage $\alpha$ one defines a pair of topologies $\tau_\alpha \subseteq \eta_\alpha$ such that $\tau_\alpha$ is first countable and $\eta_\alpha$ is determined by countably many compact subsets of finite Cantor-Bendixson rank. At limit stages the construction proceeds in a natural (and trivial) way.

At stage $\alpha + 1$ one picks $\tau_{\alpha+1} \supseteq \tau_\alpha$ such that $\tau_{\alpha+1}$ contains enough open in $\eta_\alpha$ subsets to show that a given $A_\alpha \subseteq \mathbb{Q}$ is closed in $\tau_{\alpha+1}$ provided it is closed in $\eta_\alpha$ and not compact in $\tau_{\alpha+1}$ provided it is not compact in $\eta_\alpha$. Now $\eta_{\alpha+1} \subseteq \eta_\alpha$ is chosen as the finest topology coarser than $\eta_\alpha$ in which $S \rightarrow 0$ for some $S \subseteq K$ such that $S \rightarrow 0$ in $\tau_{\alpha+1}$ and $S$ is discrete in $\eta_\alpha$. Such an $S$ can be built inductively by first finding a compact (in $\tau_{\alpha+1}$) $K' \subseteq K$ of infinite Cantor-Bendixson rank.

Define

$$\tau = \bigcup \{\tau_\alpha : \alpha \in \omega_1\} = \bigcap \{\eta_\alpha : \alpha \in \omega_1\}$$

and put

$$\Xi = \{d + (-1)^{\delta_0} K + \cdots + (-1)^{\delta_n} K : \delta_i \in \{0, 1\}, d \in [\mathbb{Q}]^{<\omega}, n \in \omega\}$$

It is easy to see that the choice of $\tau_\alpha$ ensures that $\tau$ is sequential and the choice of $\eta_\alpha$ and $\tau_{\alpha+1}$ prevents $\tau$ from being $k_\omega$. Moreover, each compact in $\tau$ subset of $\mathbb{Q}$ is compact in some $\eta_\alpha$ and therefore resides in some ‘monomial’ over $K$. Thus the family $\Xi$ has the desired property. \qed

Finally, it seems natural to ask whether Lemma [11] can be generalized to non analytic groups.

Question 1. Do there exist (countable) sequential non Fréchet groups that contain a (closed) pseudo-$\mathbb{Q}$ subspace?
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