High accurate sine and cosine interpolation with repeated differentiation ability obtained by fast expansions technique

A D Chernyshov, M I Popov and D A Litvinov

Department of Higher Mathematics and Information Technologies, Voronezh State University of Engineering Technologies, 19, Revolution Avenue, Voronezh, 394036, Russia

E-mail: mihail_semilov@mail.ru

Abstract. Fast trigonometric interpolations on a finite interval with different basis functions and increased accuracy are provided. Sine interpolation exactly coincides with this function at the interpolation points, and derivatives of cosine interpolation of the second example exactly coincide with derivatives of the said function at the interpolation points. In all cases interpolations allow differentiation for a given number of times being equal to the order of a certain boundary function. The applied Fourier series converge quickly and hence they can be limited to a small number of terms, thus greatly saving computing time. These special features allow their use in solving complex applied multi-dimensional problems of integro-differential type. The compact formulas for trigonometric interpolations in an explicit form are given which facilitate their application. An error estimate is provided and, algorithms for applying the described interpolations are setup.

1. Introduction

Fast trigonometric interpolations can be successfully applied to solve nonlinear problems in mechanics, as well as in cases of curvilinear bodies. Classical trigonometric interpolations on a finite interval have two important drawbacks, such as: they do not allow differentiation and have a large error between interpolation points in the general case. For these reasons, their application in various engineering integro-differential type problems is challenging. [1] provides the basis for classical theory of trigonometric interpolation and [2] represents interpolation by a complete Fourier series; however, formulas are awk-ward, thus complicating their use. Also, other authors prefer trigonometric interpolation with a full Fourier series [3-5] et al. [3] obtains an expression for calculating the linear distortion coefficient when trigonometric interpolations are applied. Formation of effective multidimensional interpolation formulas for periodic functions that are accurate on the classes of Fourier polynomials is proposed in [4]. The use of trigonometric interpolation in discrete measurement problems is studied in [5], discussing the problem of an unrecoverable error.

Research publications discuss issues of interpolation differentiation of in the cases of Lagrange, Chebyshev, Newton polynomials and others [6-8] et al. Research on trigonometric interpolation with differentiation has not been known yet, which is a particularly important case of studying differential problems. [1] proposes term-by-term differentiation of the trigonometric interpolation sum, but validity analysis of such operation is absent, since a large error can occur in the course of differentiation. It is noted that the function takes exact values at the interpolation points, but large
deviations may appear between the points, thus generating an error, especially in determination of derivatives, and thus creating the prerequisites for a fatal error. [9] also applies a full classical Fourier series to reconstruct a periodic signal, and slow convergence of the series is observed.

In the paper, fast expansion is used in trigonometric interpolations, where the Fourier series converges fast and allows multiple differentiation [10]. In the cases of interpolations, where differentiation is not provided, derivatives cannot be calculated in the general case because this leads to accumulation of error and physically implausible results.

2. Notes on method applicability

Trigonometric interpolation with various basis functions in Hilbert space $H$ on a finite interval will be considered for applied purposes. Trigonometric interpolation is especially convenient when considering nonlinear and multidimensional boundary value problems for curved-linear domains. These examples will be studied in future. Fast expansions allow to obtain an approximate solution to the problem analytically with minimal numerical PC consumption and high accuracy theoretically proven by [10] and verified by comparison with test problems [11-17] et al.

Note 1. The sine interpolation method is applicable when setting values $f(x_j)$ at nodes $x_j$ of a uniform grid within the interval $[0,a]$. The system of points $\{x_j\}$ should be such that changing of values of continuous $f(x)$ between any two adjacent interpolation points does not exceed certain specified value $\Delta^*$ that has been selected experimentally, i.e. $|f(x_j) - f(x_{j+1})| \leq \Delta^*$, $\forall \ j = 1 + 2K$.

When $f(x)$ varies greatly between two adjacent points, interpolation is highly doubtful [1]. Cosine interpolations are applicable when specifying derivatives $f'(x_j)$ at nodes $x_j$ of the uniform grid within the interval $[0,a]$. Information about quite smooth behavioral nature of $f(x)$ and its derivatives $f'(x_j)$ between interpolation points can be obtained either from experimental and navigation data or past experience, physical grounds, etc.

Note 2. An interpolation sum $S_{uw}(x)$ approximating $f(x)$ differs from the partial sum of Fourier series $S_{fo}(x)$ having the same basis functions and the same number of terms in the trigonometric sum in that $S_{uw}(x)$ or $S_{uw}'(x)$ exactly equals respectively to the values of $f(x)$ or $f'(x)$ at the interpolation points, while $S_{fo}(x)$ is approximately equal. However, in the integral average, sine interpolation $S_{fo}(x)$ approximates more accurately to $f(x)$ compared to $S_{uw}(x)$. In the case of cosine interpolation, derivative $S'_{fo}(x)$ in the integral mean approximates more accurately to $f'(x)$ comparing with $S'_{uw}(x)$ [18].

Note 3. The main reason for impossibility of term-by-term differentiation of an interpolation is explained by the fact that even if the difference $f(x) - S_{uw}(x) = \delta(x)$ is small, it may turn out to be rapidly oscillating in classical interpolations, and hence the derivative will change strongly. Term-by-term differentiation of such interpolation is impractical.

Note 4. If the number of points $x_j$ tends to infinity in the uniform interpolation, we get an interpolation series. Then, the problem about convergence of such a series and the possibility of its term-by-term differentiation arises. If the interpolation series diverges, the increase of the number of points will not lead to the accuracy increase, since a fatal error occurs.

Note 5. When choosing an interpolation, it is necessary to study whether the applied interpolation system of functions is complete. Otherwise, a fatal error will occur, and increase of the number of interpolation terms will not lead to an error decrease.
Note 6. If there is no information about behavioral nature of \( f(x) \) between the interpolation points, there is a legitimacy risk of applying this method.

Failure to meet at least one of 6 mentioned conditions will lead to an unsatisfactory result and interpolation will be impractical.

Interpolation series on a finite interval \( x \in [0,a] \) are supposed to be used for applied purposes, in this regard the next three conditions arise. They are:

Condition 1) The selected interpolation system of functions must be complete. Otherwise, a fatal error may occur.

Condition 2) It is necessary to analyze convergence of the interpolation series to the values of \( f(x) \) or to the value of its derivatives inside and at the ends of the interval. If the assumed series does not converge to \( f(x) \) or to its derivatives inside the interval and at its ends as well, an unrecoverable error occurs and the boundary conditions at the ends of the interval of the given problem cannot be fulfilled.

Condition 2 can be realized when setting \( f(x) \) as a continuous experimental curve, when it becomes possible to obtain any finite number of interpolation points.

Condition 3) On possibility of term-by-term differentiation of a series.

Using a simple example of the classical sine Fourier series for \( f(x) = \sin(m \pi x / a) \) in some basis functions \( R_m(x) \), one can make sure that differentiation will lead to a diverging series.

Fulfillment of Condition 3) allows to apply trigonometric interpolation to solve integrodifferential problems.

In this regard, for simultaneous fulfillment of all three conditions, the most efficient fast expansions are proposed below.

Let \( f(x) \in L^2_{\text{s+2}}([0,a]) \) be classes of Sobolev-Liouville functions [19]. To define fast expansion \( f(x) \) in \( H[0,a] \) the following terms are applied: \( M_q(x) = \sum_{j=1}^{q/2} A_j P_j(x) \) boundary function to increase the convergence rate of Fourier series, fast polynomials \( P_i(x) \), and a Fourier series for difference \( f(x) - M_q(x) = \psi(x) = \sum_{n=1}^{\infty} \psi_n R_n(x) \) in some basis functions \( R_m(x) \), where \( q \) is the order of boundary function \( M_q(x) \) equal to the order of the highest derivative used in the definition of special construction \( M_q(x) \) [10].

Choice of the fast polynomials \( P_i(x) \), a boundary function \( M_q(x) \) form and choice of the basis functions \( R_m(x) \) used in the Fourier series for \( \psi(x) \) depend on the boundary conditions of the assumed problem for the given \( f(x) \) at the ends of interval \([0,a]\). In this paper, this will be proved with two interpolation examples - with \( R_m(x) = \sin(m \pi x / a) \) and \( R_m(x) = \cos(m \pi x / a) \).

Definition of fast expansion. Fast expansion \( f(x) \) in \( H[0,a] \) for the given basis functions \( R_m(x) \) means a sum of boundary function \( M_q(x) = \sum_{j=1}^{q/2} A_j P_j(x) \) and the Fourier series for the difference \( f(x) - M_q(x) = \psi(x) = \sum_{n=1}^{\infty} \psi_n R_n(x) \), i.e.

\[
\begin{align*}
 f(x) &= M_q(x) + \psi(x), \\
 M_q(x) &= \sum_{j=1}^{q/2} A_j P_j(x), \\
 \psi(x) &= \sum_{n=1}^{\infty} \psi_n R_n(x).
\end{align*}
\]
\( A_m, \psi_m \) coefficients are fast expansion coefficients.

The main purpose of boundary function \( M_q(x) \) and fast polynomials \( P_i(x) \) is to increase the convergence rate of the Fourier series for \( \psi(x) \) by rapid decrease of the coefficients \( \psi_m \) with increasing number \( m \). Below is the description of this possibility and the way of defining \( \psi_m \) and \( \psi_m \sim m^{-q+\alpha} \), i.e. as the number \( m \) grows, Fourier coefficients \( \psi_m \) tends to zero rapidly.

When specifying the Dirichlet conditions at the ends of interval \([0,a]\), a sine series is used. When specifying the Neumann conditions, a cosine series is used. For these cases, [10] provides recursive integral expressions of the fast polynomials \( P_i(x) \) and the corresponding form of boundary functions.

The name \( M_q(x) \) of the boundary function is justified by the fact that the coefficients \( A_i \) in \( M_q(x) \) definition are determined from \( f(x) \) values and its derivatives up to the \( q-\)th order including boundaries of the interval \([0,a]\).

3. Sine-interpolation

3.1 Fast expansions differentiability

Sine interpolation is one of the simplest trigonometric interpolations, which is convenient to use when function \( f(x_j) \) values are given at the interpolation points \( x_j \), while even order derivatives are used in differential equations and boundary conditions of a certain problem. In applied problems of continuum mechanics, physics, mathematical methods of fast trigonometric expansion are especially valuable for bodies of finite dimensions.

Let there exist a function \( f(x) \in L^q_{[0,a]} \), where \( L^q_{[0,a]} \) - identifies classes of Sobolev-Liouville functions [17], and let \( f(x_j) \) values be known only at discrete points of the interval \([0,a]\) when it is uniformly divided by an even number \( 2K \) of intervals with the use of points \( x_j = ja/2K \), \( j = 0, 2K-1 \)

\[
\left. f(x) \right|_{x=x_j} = f(x_j) , \quad x_j = ja/2K \a , \quad j = 0, 2K-1 \) \quad f(x) \in L^q_{[0,a]}
\]

We represent \( f(x) \) \( x \in [0,a] \) by a fast sine expansion in the form of a sum of the second-order boundary function \( M_2(x) \) (here \( q = 2 \)) and a certain \( \psi(x) \), written with sine Fourier series [10]:

\[
f(x) = M_2(x) + \psi(x), \quad \psi(x) = \sum_{m=1}^{\infty} \psi_m \sin m\pi \frac{x}{a}, \quad f(x) \in L^q_{[0,a]}, \quad M_2(x) = f(0) \left( 1 - \frac{x^2}{a} \right) + f(a) \frac{x}{a} + f^\prime(0) \left( \frac{x^2}{2} - \frac{x^4}{6a} - \frac{ax}{3} \right) + f^\prime\prime(a) \left( \frac{x^3}{6a} - \frac{ax}{6} \right).
\]  

\( \text{Theorem 1. If } f(x) \in L^q_{[0,a]}, \text{ fast expansion of } f(x) \text{ in terms of sines in (1) can be differentiated in the term-by-term manner 2 times, remaining in the space of fast decomposition, and the total number of differentiation is 4.} \)

Proof. Let \( \psi(x) \) be written in the following form from (1)

\[
\psi(x) = f(x) - M_2(x) = \sum_{m=1}^{\infty} \psi_m \sin m\pi \frac{x}{a}.
\]
Coefficients \( \psi_m \) for the difference \( f(x) - M_2(x) \) are calculated by integrals

\[
\psi_m = \frac{2}{a} \int_0^a \left( f(x) - M_2(x) \right) \sin m \pi \frac{x}{a} \, dx
\]

We obtain the following by integrating by parts

\[
\psi_m = \frac{2}{a} \int_0^a \left( f(x) - M_2(x) \right) \sin m \pi \frac{x}{a} \, dx = \frac{2}{m \pi} \left( \int_0^a \left( f(x) - M_2(x) \right) \cos m \pi \frac{x}{a} \, dx \right)
\]

\[
+ \frac{2}{m \pi} \left( \int_0^a f'(x) \cos m \pi \frac{x}{a} \, dx + \frac{f(0) - f(a)}{a} \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx - f'(0) \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx - f''(0) \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx - f'''(0) \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx - f''''(0) \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx \right)
\]

\[
= \frac{2}{m \pi} \left( \int_0^a f'(x) \cos m \pi \frac{x}{a} \, dx + \frac{f(0) - f(a)}{a} \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx - f'(0) \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx - f''(0) \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx - f'''(0) \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx - f'\prime\prime(0) \int_0^a \frac{x}{a} \cos m \pi \frac{x}{a} \, dx \right)
\]

Applying integration by parts three more times, we obtain

\[
\psi_m = \frac{2}{a} \left( \frac{a}{m \pi} \right)^4 \int_0^a f^{(4)}(x) \sin m \pi \frac{x}{a} \, dx
\]

(3)

With the use of the last expression of \( \psi_m \) of (3) a Fourier series for \( \psi(x) \) in (2) can be expressed as

\[
\psi(x) = \sum_{m=1}^{\infty} \psi_m \sin m \pi \frac{x}{a} = \frac{2}{a} \sum_{m=1}^{\infty} \left( \frac{a}{m \pi} \right)^4 \left( \int_0^a f^{(4)}(t) \sin m \pi \frac{t}{a} \, dt \right) \sin m \pi \frac{x}{a}
\]

(4)

Since \( f(x) \in L^4_a \) and factor \( (a/m \pi)^4 \) is before the integral in (4), which decreases rapidly with growth of \( m \), the fifth summand, for instance, is less than the first one in 625 times, the series in (4) is majorised by the power series

\[
\frac{2}{a} \sum_{m=1}^{\infty} \left( \frac{a}{m \pi} \right)^4 \left( \int_0^a f^{(4)}(t) \sin m \pi \frac{t}{a} \, dt \right) \sin m \pi \frac{x}{a} \leq \frac{2}{a} M \sum_{m=1}^{\infty} \left( \frac{a}{m \pi} \right)^4
\]

\[
M^* = \sup_{0 \leq t \leq a} \left| \int_0^a f^{(4)}(t) \sin m \pi \frac{t}{a} \, dt \right| \sin m \pi \frac{x}{a}
\]

Hence, we obtain a proof of the absolute convergence of the series and about the Fourier series for \( \psi(x) \) of (4) admitting term-by-term differentiation two times, remaining in the fast expansion space. In general it can be differentiated four times.

**Corollary 1.** If the boundary function \( M_q(x) \) of higher order \( q \) is used in (1), the Fourier series for \( \psi(x) \) after \( q \)-fold differentiation remains in the space of fast decomposition, and \( \psi(x) \) can be differentiated in the term-by-term manner only \( q + 2 \) times. The convergence rate of such series is extremely high similarly to the case of \( q = 2 \) since \( \psi_m - m^{-(q+2)} \). All derivatives inclusive the ones with the \( q \)-th order remain in the space of fast expansion and therefore their decomposition converge quickly and absolutely.
Corollary 2. If \( f(x) \in L_2^2([0,a]) \), Fourier sine series \( \psi(x) \) of the fast expansion (1) and the derivatives \( \psi'(x), \psi''(x) \) with up to the second order inclusively on the interval \([0,a]\) converge absolutely.

The proof follows from the fact that coefficients \( (a/m\pi)^4 \) under the sign of the sum in (4) have the fourth degree. The second derivative will have coefficients \( (a/m\pi)^2 \) in the second degree, thus providing absolute convergence of a series for the second-order derivative.

3.2 Setting up the sine interpolation problem

Let \( f(x), x \in [0,a] \) be an unknown function for which sine Fourier series coefficients are unknown and let it’s known that \( f(x) \in L_2^2([0,a]) \) - a smooth function, \( f(x) \in L_2^2([0,a]) \) - a smooth function, and \( f(x_j) \) values are known within uniform grid \( x_j \) composed of even number \( 2K \) of intervals with length \( \Delta \) each. \( f(x) \) values are unknown between any two interpolation points \( x_j \).

The sine interpolation problem has been formulated in [1], and the present paper has changed it due to the use of fast expansions: it is required to determine approximately smooth function \( f(x) \) at \( H \) having basis functions \{\sin m\pi x/a\} in the following form

\[
 f(x) \approx \tilde{f}(x) = M_z(x) + \sum_{m=1}^{2K-1} \tilde{\psi}_m \sin m\pi x/a, \quad f(x) \in L_2^2([0,a])
\]

Interpolation coefficients \( \tilde{\psi}_m \) shall be determined from the closed system:

\[
 \psi(x_j) = f(x_j) - M_z(x_j) = \sum_{m=1}^{2K-1} \tilde{\psi}_m \sin m\pi x_j/a, \quad x_j = j a/2K, \quad j = 1 \pm 2K - 1
\]

In the sums (1.11) and (7), the summand at \( m = 0 \) is not written down because it is identically zero. In fact, the sine interpolation problem is reduced to define interpolation coefficients \( \tilde{\psi}_m \) in the explicit form from the solution of the closed algebraic system (7).

To keep the orthogonality property in the sums of (6) and (7) the rightmost point \( x = a \) of this interval is not used at \( m = 2K \). However, fast expansion (6) and equalities (7) are identically fulfilled at the ends of the interval at \( x = 0, x = a \). Fulfillment of the system (7) means that exact equalities occur at the interpolation points \( x = x_j, j = 0 \pm 2K - 1 \)

\[
 f(x_j) = \tilde{f}(x_j), \quad j = 0 \pm 2K
\]

If \( (f(x), \tilde{f}(x)) \in L_2^2([0,a]) \) and equalities are satisfied with quite large \( K \), the following approximate equalities can be considered legitimate [1]

\[
 f(x) \approx \tilde{f}(x), \quad x \in [0,a]
\]

And two limit equalities

\[
 \lim_{K \to \infty} \tilde{\psi}_m = \psi_m, \quad m = 0,1,... \quad \lim_{K \to \infty} \tilde{f}(x) = f(x) \text{ with } K \to \infty \text{ and } \Delta \to 0
\]
To use \( f(x) \) expression of (6) for applied purposes, it is necessary to define \( \psi_m \) from the system of (7), which is appropriate for two cases:

1) It’s impossible to compute Fourier coefficients \( \Phiur \) \( \psi_m \) in the finite form by elementary functions according to the last integral expression of (3). The situation of this kind arises when considering nonlinear problems, as well as in the cases of curvilinear domains.

2) When the analytical form of \( \psi(x) \) is unknown, and \( \psi(x_j) \) values are known at discrete points from some experimental data, which is commonly found in conventional publications and engineering practice. In this case it is necessary to have additional information (5) about a sufficiently smooth behavior of \( \psi(x) \) between points \( x_j \), that can be justified if \( 2K \) is large enough.

In the space \( H([0,a]) \), the system of continuous functions \( \{\sin n\pi x/a\}_m \) is orthogonal, but the system of discrete functions \( \{\sin n\pi x_j/a\}_m \) within an interval \( [0,a(1-1/2K)] \) shouldn’t be orthogonal too. However, in this case, the property is valid. To prove orthogonality of the discrete system \( \{\sin n\pi x_j/a\}_m \) (orthogonality definition is given in [1]) the left and right sides of (7) are subject to scalar multiplication by \( \sin n\pi x_j/a \):

\[
\sum_{j=0}^{2K-1} \psi(x_j) \sin n\pi x_j/a = \sum_{m=0}^{2K-1} \psi_m \sum_{j=0}^{2K-1} \sin m\pi x_j/a \sin n\pi x_j/a , \quad x_j = ja/2K , \quad n = 0 \pm 2K - 1
\]

Now prove that the inner sum on the right side equals zero with \( m \neq n \)

\[
S_m = \sum_{j=0}^{2K-1} \sin \left( n\pi \frac{x_j}{a} \right) \sin \left( m\pi \frac{x_j}{a} \right) = 0 , \quad x_j = ja/2K , \quad \forall m \neq n \quad (8)
\]

The orthogonality property (8) will allow to simplify greatly the interpolation formulas and solve the sine interpolation problem. Proving the property (8) requires to express the product of sines in terms of cosines:

\[
\sum_{j=0}^{2K-1} \sin \left( n\pi \frac{j}{2K} \right) \sin \left( m\pi \frac{j}{2K} \right) = \frac{1}{2} \sum_{j=0}^{2K-1} \left( \cos \left( (m-n)\pi \frac{j}{2K} \right) - \cos \left( (m+n)\pi \frac{j}{2K} \right) \right)
\]

Now applies the complex Euler formula for cosines

\[
2 \sum_{j=0}^{2K-1} \cos \left( (m-n)\pi \frac{j}{2K} \right) - \cos \left( (m+n)\pi \frac{j}{2K} \right) = \sum_{j=0}^{2K-1} \exp \left( i(m-n)\pi \frac{j}{2K} \right) + \sum_{j=0}^{2K-1} \exp \left( -i(m-n)\pi \frac{j}{2K} \right)
\]

\[
+ \sum_{j=0}^{2K-1} \exp \left( i(m+n)\pi \frac{j}{2K} \right) - \sum_{j=0}^{2K-1} \exp \left( -i(m+n)\pi \frac{j}{2K} \right) \quad (9)
\]

Here \( i \) is an imaginary one. The following notation is introduced:

\[
q_{m,n}^j = \exp \left( i(m \pm n)\pi j/2K \right)
\]

The superscript \( j \) is considered to be a degree. Using this notation, we represent the right side of (9) as a sum of four geometric progressions
\[
S_{\sin} = \sum_{j=0}^{2K-1} \left( \exp \left( i(m-n)\frac{\pi j}{2K} \right) + \exp \left( -i(m-n)\frac{\pi j}{2K} \right) \right)
\]
\[
- \sum_{j=0}^{2K-1} \left( \exp \left( i(m+n)\frac{\pi j}{2K} \right) + \exp \left( -i(m+n)\frac{\pi j}{2K} \right) \right) = (1 + q_{m+n}^1 + \ldots + q_{m+n}^{2K-1})
\]
\[
+ \left( 1 + q_{m-n}^1 + \ldots + q_{m-n}^{2K-1} \right) - \left( 1 + q_{m+n}^1 + \ldots + q_{m+n}^{2K-1} \right) - \left( 1 + q_{m-n}^1 + \ldots + q_{m-n}^{2K-1} \right)
\]
(10)

In this case, \( m, n \) can be both even and odd. Let us prove that in both cases the right side of (10) is equal to zero, that is the set of functions \( \sin(n\pi j/2K) \) is orthogonal at any \( m \neq n \).

Let \( m \) and \( n \) be simultaneously even and odd at the beginning. We sum up the progressions (10) in the following manner:

\[
\text{If } (m \pm n) \text{ is even, then } q_{m\pm n}^{2K} = 1, \text{ with } m \neq n \quad \Rightarrow \quad q_{m\pm n}^{\pm 1} \neq 1,
\]
\[
\left( 1 + \ldots + q_{m-n}^{2K-1} \right) + \left( 1 + q_{m+n}^1 + \ldots + q_{m+n}^{2K-1} \right) - \left( 1 + \ldots + q_{m+n}^{2K-1} \right) - \left( 1 + q_{m-n}^1 + \ldots + q_{m-n}^{2K-1} \right)
\]
\[
= \frac{1 - q_{m-n}^{2K}}{1 - q_{m-n}} + \frac{1 - q_{m+n}^{2K}}{1 - q_{m+n}} - \frac{1 - q_{m+n}^{2K}}{1 - q_{m+n}} - \frac{1 - q_{m-n}^{2K}}{1 - q_{m-n}} = 0
\]

Since \( (m \pm n) \) is even in this case, we obtain the proof of equality (8) with uniform partitioning of the interval \([0,a]\) with a step \( a/2K \), that is, orthogonality of the functions \( \sin(m \pi x/a) \) within the interval \([0,a-a/2K]\).

Now let \( m \) be even, or \( m \) be odd and \( n \) be even. In both cases the sum and difference of \( (m \pm n \neq 0) \) is odd. The equality \( q_{m\pm n}^{2K} = -1, m \neq n \) is fulfilled for any integer \( m \) and \( n \).

Taking into account that \( q_{m\pm n}^{2K} = -1, q_{m\pm n}^{1} \neq -1, m \neq n \) in the studied case, the following transformations have to be done to prove the equality of (8):

\[
\frac{1 - q_{m-n}^{2K}}{1 - q_{m-n}} + \frac{1 - q_{m+n}^{2K}}{1 - q_{m+n}} - \frac{1 - q_{m+n}^{2K}}{1 - q_{m+n}} - \frac{1 - q_{m-n}^{2K}}{1 - q_{m-n}} = \frac{2}{1 - q_{m-n}} + \frac{2}{1 - q_{m+n}} - \frac{2}{1 - q_{m-n}} - \frac{2}{1 - q_{m+n}}
\]
\[
= \left( \frac{2}{1 - q_{m-n}} + \frac{2}{1 - q_{m+n}} \right) - \left( \frac{2}{1 - q_{m-n}} + \frac{2}{1 - q_{m+n}} \right) = 2 \left( 1 - q_{m-n}^1 \right) + \left( 1 - q_{m-n}^1 \right) - \frac{2}{\left( 1 - q_{m-n}^1 \right) \left( 1 - q_{m-n}^1 \right)}
\]
\[
= 2 \left( \frac{2 - q_{m-n}^1 - q_{m-n}^1}{1 - q_{m-n}^1 \left( 1 - q_{m-n}^1 \right)} \right) - \left( \frac{2 - q_{m-n}^1 - q_{m-n}^1}{1 - q_{m-n}^1 \left( 1 - q_{m-n}^1 \right)} \right) = 2 - 2 = 0
\]

Which was to be proved. The squared norm \( N^2 \) for the functions \( \sin(m \pi j/2K) \) has to be computed:

\[
N^2 = \sum_{j=1}^{2K-1} \sin^2 \left( \frac{m \pi j}{2K} \right) = \sum_{j=0}^{2K-1} \sin^2 \left( \frac{m \pi j}{2K} \right)
\]

The notation \( H \) won’t be used further. The sum is written using the complex Euler formula

\[
N^2 = -\frac{1}{4} \sum_{j=0}^{2K-1} \left( \exp \left( \frac{im \pi j}{K} \right) - 2 + \exp \left( -\frac{im \pi j}{K} \right) \right)
\]
An integer \(-2\) under the sum sign is repeated in each of \(2K\) summands. Since \(q_m^{2K} = \exp(2m\pi) = 1\) and \(q_m^{2K} \neq 1\), geometric progressions in (10) are summed up:

\[
N^2 = -\frac{1}{4} \sum_{j=0}^{2K-1} \left( \exp\left(2im\frac{\pi j}{K}\right) - 2 + \exp\left(-2im\frac{\pi j}{K}\right) \right) = K - \frac{1}{4} \left(1 + \cdots + q_m^{-2K}\right) = \frac{1}{4} \left(1/q_m^{-1} + \cdots + q_m^{-2K}\right) + K = K
\]

Note the difference between the squared norm in (11) and the expression of squared norm in [1]: if the squared norm \(N^2\) of [1] is equal to the number of summands in trigonometric interpolation, according to the approach of (6) the squared norm \(N^2\) of (11) is equal to half the number of summands under the sum sign. Joint properties of (8) and (11) allow to write a useful equality

\[
\sum_{j=0}^{2K-1} \sin\left(n\pi \frac{x_j}{a}\right) \sin\left(m\pi \frac{x_j}{a}\right) = K\delta^m_n,
\]

where \(\delta^m_n\) is the Kronecker delta function. The system orthogonality \(\{\sin m\pi j/2K\}\) results in that the determinant of system (7) is not equal to zero, and therefore solution of the system (7) is valid and unique [1]. Solution of the system (7) takes an explicit form in the final form

\[
\tilde{\psi}_m = \frac{1}{K} \sum_{j=0}^{2K-1} \psi(x_j) \sin m\pi \frac{\pi j}{2K}, \quad \psi(x_j) = f(x_j) - M_q(x_j), \quad m = 0 \div 2K - 1
\]

Now the sine interpolation formula (1.11), where \(\tilde{\psi}_m\) should be taken from (13) and considering fast expansion (1), can be written in the compact form:

\[
\tilde{f}(x) = M_0(x) + \frac{1}{K} \sum_{m=1}^{2K-1} \left( \sum_{j=0}^{2K-1} \left( \tilde{\psi}_m \sin m\pi \frac{\pi j}{2K} \right) \right) \sin m\pi \frac{x_j}{a}
\]

Let us check whether the proposed solution of sine interpolation (14) satisfies condition (7). Place \(x = x_j\) in the left and right sides of (14) and then this equality should turn into the identity:

\[
f(x_j) = M_0(x_j) + \frac{1}{K} \sum_{m=1}^{2K-1} \left( \sum_{j=0}^{2K-1} \left( \tilde{\psi}_m \sin m\pi \frac{\pi j}{2K} \right) \right) \sin m\pi \frac{x_j}{a}.
\]

In the summand with a double sum, swap the signs of the sums

\[
\sum_{j=0}^{2K-1} \sin\left(n\pi \frac{x_j}{a}\right) \sin\left(m\pi \frac{x_j}{a}\right) = \sum_{j=0}^{2K-1} \sin\left(n\pi \frac{mj}{2K}\right) \sin\left(m\pi \frac{nj}{2K}\right) = \sum_{m=1}^{2K-1} \sin m\pi \frac{x_j}{a} \sin m\pi \frac{x_j}{a} = K\delta^m_j.
\]

After substituting (16) into (15), we get an identity.

The first and second derivatives are defined by differentiating (14) by \(x\) :
\[ f'(x) = \left( \frac{f(a) - f(0)}{a} + f^*(0)\left(x - \frac{x^2}{2a} - \frac{a}{3}\right) + f^*(a)\left(x - \frac{x^2}{2a} - \frac{a}{6}\right) \right) \]

\[ + \sum_{m=1}^{2K-1} \frac{m\pi}{Ka} \left( \sum_{j=1}^{2K-1} \left( f(x_j) - M_2(x_j) \right) \sin \frac{m\pi j}{2K} \right) \cos \frac{m\pi x}{a} \]

\[ f^*(x) = \left[ f^*(0)(1 - \frac{x}{a}) + f^*(a) \frac{x}{a} \right] - \frac{1}{K} \sum_{m=1}^{2K-1} \left( \frac{m\pi}{a} \right)^2 \left( \sum_{j=1}^{2K-1} \left( f(x_j) - M_2(x_j) \right) \sin \frac{m\pi j}{2K} \right) \sin \frac{m\pi x}{a}. \]

(17)

When \( K \to \infty \), the series of (17) converge absolutely by virtue of the property (3), that is, differentiation is meaningful.

### 3.3 Application algorithm of fast sine interpolation

1. Determine the length of interval \([0, a]\), divide it by the even number \(2K\) of equal parts using the points \( x_j = ja/2K\), \( j = 0 \div 2K - 1 \) with the division step \( \Delta = a/2K \) in such a way that the difference \( |f(x_{j+}) - f(x_j)| \leq \epsilon \), where \( \epsilon \) is a permissible error.

2. Determine the values at the calculated points \( x_j \) taken from the experimental data and represent \( f(x) \) with \( q = 2 \) by fast expansion (1).

3. The values \( f(0), f(a), f^*(0), f^*(a) \) in (14) to determine \( M_2(x) \) are taken either from the boundary conditions of the problem under consideration or from a given differential equation. If there is neither one nor the other, they can be calculated by finite differences.

4. Make up the algebraic system (7) and write down its solution by the equality (13).

5. The sine interpolation expression is represented by the compact formula (14), its first and second derivatives are expressed by (17).

Note 1. When replacing the boundary function \( M_2(x) \) of (1) with \( M_q(x) \) of a higher order \( q \), the approximation accuracy of (1) increases provided that all even derivatives up to the \( q \)-th order inclusive used to write \( M_q(x) \) are determined with sufficient accuracy.

Note 2. In the simplest case, when using the zero-order boundary function \( M_0(x) \) in (1.1) instead of \( M_q(x) \)

\[ M_0(x) = f(0)(1 - \frac{x}{a}) + f(a) \frac{x}{a} \]

the problem of calculating the derivatives of \( f(x) \) at the ends of the interval disappears, and therefore the sine interpolation is simplified. At the same time, quite high accuracy and possibility of double differentiating trigonometric interpolation (14) remains.

### 3.4 Error estimate

In the classical literature, Rolle's theorem is used to get an error estimate of trigonometric interpolation. The resulting error formula is expressed in terms of the derivatives of the \((2K-1)\) order of the interpolation polynomial, but calculation of the derivative of a high order is wrong. In this regard, we propose a different way of error estimate.
Assume \( f(x) \in L^2([0,a]) \). An interpolation error estimate (14) is obtained as an estimate of the remainder \( R_{\sin} \) of interpolation (6) with \( K \to \infty \). The notation is introduced

\[
M_{\sin}^{(4)} = \sup \left\{ \| f^{(4)}(x) \|, x \in [0,a] \right\}
\]

Let’s write the expression (6) in the form of an interpolation series with an infinite upper limit

\[
\tilde{f}(x) = M_2(x) + \sum_{m=0}^{2K-1} \tilde{\psi}_m \sin m\pi \frac{x}{a} + \sum_{m=2K}^{\infty} \tilde{\psi}_m \sin m\pi \frac{x}{a}
\]

(18)

When \( 2K \to \infty \) the discrete Fourier series changes to the classical Fourier series and therefore the equality sign is placed in (18). The second sum in (18) can be considered as an interpolation error, since it is discarded. Taking \( \psi_m \approx \tilde{\psi}_m \) approximately and using the final expression (3) for \( \psi_m \), the following estimate is obtained as an error

\[
R_{\sin} = \sum_{m=2K}^{\infty} \psi_m \sin m\pi \frac{x}{a} = \frac{2}{a} \sum_{m=2K}^{\infty} \left( \frac{a}{m\pi} \right)^4 \left( \int_{0}^{a} f^{(4)}(t) \sin m\pi \frac{t}{a} \ dt \right) \sin m\pi \frac{x}{a}
\]

\[
\leq \frac{2}{a} \sum_{m=2K}^{\infty} \left( \frac{a}{m\pi} \right)^4 \left( \int_{0}^{a} f^{(4)}(t) \sin m\pi \frac{t}{a} \ dt \right) \leq 2M_{\sin}^{(4)} \left( \frac{a}{\pi} \right)^4 \sum_{m=2K}^{\infty} \frac{1}{m^4}
\]

\[
= 2M_{\sin}^{(4)} \left( \frac{a}{\pi} \right)^4 \left( \frac{1}{(2K)^4} + \sum_{m=2K+1}^{\infty} \frac{1}{m^4} \right) \leq 2M_{\sin}^{(4)} \left( \frac{a}{\pi} \right)^4 \left( \frac{1}{(2K)^4} + \int_{2K}^{\infty} \frac{1}{\zeta^4} d\zeta \right)
\]

\[
= 2M_{\sin}^{(4)} \left( \frac{a}{\pi} \right)^4 \left( \frac{1}{(2K)^4} + \frac{1}{3(2K)^3} \right) = M_{\sin}^{(4)} \left( \frac{a}{\pi} \right)^4 \left( \frac{1}{8K^4} + \frac{1}{12K^3} \right)
\]

This follows that the sine interpolation error decreases rapidly with increasing the number \( K \). The resulting estimate allows to determine the value of \( K \) in advance to obtain the required accuracy when interpolation (14) is applied.

3.5 Examples

Figure 1 shows graphs of absolute interpolation errors (14), its first and second derivatives for function \( \exp(3x) \) in the interval \([0,1]\) when \( K=10 \). When its distance from point \( x=0 \) and approaching \( x=a \) the absolute error increases, which can be explained by an increase in the function itself. This interpolation allows two-fold differentiation, and the accuracy decreases when the derivative order increases. The smallest absolute error of the first and second derivatives is observed in the neighborhood of the point \( x=0 \). The largest errors are in the neighborhood of the point \( x=a-\Delta \).
Figure 1. Absolute interpolation errors and its first and second derivatives.

The fast interpolation errors and their derivatives, compared with the fast expansion error, when the coefficients are calculated by Fourier integrals, are approximately the same and have orders for the function, the first and second derivatives $10^{-5}, 10^{-3}, 10^{-1}$, respectively. The Fourier coefficients $\psi_m$ and the corresponding interpolation coefficients $\tilde{\psi}_m$ are also approximately the same. High accuracy with a small number of interpolation points $2K = 20$ shows that the proposed fast sine interpolation method is one of the most efficient and economical methods, has high accuracy and is convenient for solving nonlinear engineering problems for curved regions.

Let us denote error estimates for the sine Fourier series $f(x)$ and sine interpolation $\tilde{f}(x)$ as a difference between the exact value and the corresponding expansion:

$$\delta(x) = f(x) - M_2(x) = \sum_{m=1}^{2K} \psi_m \sin m\pi \frac{x}{a}, \quad \tilde{\delta}(x) = \tilde{f}(x) - M_2(x) = \sum_{m=1}^{2K} \tilde{\psi}_m \sin m\pi \frac{x}{a}$$

Let us examine the sine interpolation problem for a complex rapidly oscillating function $f(x) = \exp(-x) + 0.2 \cos(b\pi x^2)$.

Table 1 shows interpolation errors for both point-to-point calculation of the coefficients and for the integral one with $b = 5$. There are also indicated errors of the first and second derivatives at the point-to-point calculation of the coefficients. The point-to-point calculation error of the coefficients increases insignificantly in comparison with the integral calculation. When the number of interpolation nodes is doubled, the accuracy increases by an order of magnitude.

Table 1. Error vs. number of interpolation nodes.

| Number of nodes | Point-to-point coefficient error | First derivative error | Second derivative error | Integral coefficient calculation error |
|-----------------|---------------------------------|------------------------|------------------------|---------------------------------------|
| 20              | $3.8 \cdot 10^{-4}$             | $3 \cdot 10^{-1}$      | 18                     | $2.3 \cdot 10^{-3}$                   |
| 40              | $2 \cdot 10^{-4}$               | $3 \cdot 10^{-2}$      | 4                      | $1.2 \cdot 10^{-4}$                   |
| 60              | $4 \cdot 10^{-5}$               | $9 \cdot 10^{-3}$      | 1.7                    | $2 \cdot 10^{-5}$                     |
| 80              | $1.2 \cdot 10^{-5}$             | $4 \cdot 10^{-3}$      | 0.95                   | $7 \cdot 10^{-6}$                     |
| 100             | $5 \cdot 10^{-6}$               | $2 \cdot 10^{-3}$      | 0.6                    | $2.9 \cdot 10^{-6}$                   |

Table 2 shows similar results for $K = 60$ with different $b$. When the number of oscillations "$b$" increases by one, the error increases twice, thus indicating complexity of the interpolation problem of rapidly oscillating functions.

Table 2. Error vs. number of oscillations.

| Number of oscillations $b$ | Point-to-point coefficient | First derivative error | Second derivative error | Integral coefficient calculation error |
|---------------------------|---------------------------|------------------------|------------------------|---------------------------------------|
The above examples clearly show advantages of the fast sine interpolation with high accuracy and possibility of their differentiation in comparison with known interpolations.

4. Cosine-interpolation

Scientific publications do not provide cosine interpolation formulas, even in the simplest form. It is convenient to use cosine interpolations when at the points \( x_j = ja/2K \), \( j = 0 \div 2K-1 \) the derivatives \( f'(x_j) \) are given from the speed registration in moving objects by the speedometer readings, or when measuring deformations and stresses in solids by the strain gauge readings, etc.

We write fast functional cosine expansion in \( H(x \in [0,a]) \) with a boundary function \( M_q(x) \) of th odd order \( q \) [10]:

\[
f(x) = M_q(x) + \psi(x), \quad \psi(x) = f(x) - M_q(x) = \psi_0 + \sum_{m=1}^\infty \psi_m \cos m\pi \frac{x}{a}, \quad x \in [0,a] \quad (19)
\]

In some cases, for computation convenience the coefficient \( \psi_0 \) is entered under the sum sign, and then summation starts from the value \( m = 0 \). When writing \( M_q(x) \), derivatives of the odd order \( q \) are used at the ends of the interval \([0,a]\) up to the \( q \)-th order inclusive. For \( q = 3 \), the Fourier series of (19) allows calculation of the derivative up to the 3\(^{rd}\) order inclusive, thus allowing fast expansion (19) to be applied to solve the differential equation up to the 3\(^{rd}\) order inclusive.

The expression \( \psi(x) \) in (19) is defined by such two important ways as the difference \( f(x) - M_q(x) \) and the Fourier cosine series for this difference. Fast expansion with the applied \( M_3(x) \) gets the form of [10]

\[
f(x) = f'(0) \left( x - \frac{x^2}{2a} \right) + f'(a) \frac{x^2}{2a} + f''(0) \left( \frac{x^4}{6} - \frac{x^4}{24a} - \frac{ax^2}{6} \right) + f'''(a) \left( \frac{x^4}{24a} - \frac{ax^2}{12} \right) + \psi_0 + \sum_{m=1}^\infty \psi_m \cos m\pi \frac{x}{a}, \quad x \in [0,a] \quad (20)
\]

The structure of boundary function \( M_3(x) \) in (20) ensures fast convergence of the Fourier series. To verify this fact, the integral expression of \( \psi_m \) and is written and then integrated by parts five times. As a result we obtain:

\[
\psi_m = -\frac{2}{a} \left( \frac{a}{m\pi} \right)^5 \int_0^a f^{(5)}(x) \sin m\pi \frac{x}{a} \, dx, \quad \Rightarrow \ \psi_m \sim m^{-5} \quad (21)
\]

proves that \( \psi_m \sim m^{-5} \), i.e. as the serial number \( m \) increases, the coefficients \( \psi_m \) rapidly decrease and therefore the Fourier series converges rapidly. If (20) applies the boundary function \( M_q(x) \) of a higher order \( q > 3 \), we would have the expression below instead of (21).
The interval \(0, \pi\) composed of even numbers \(2K\). Hence, below is the proof of \(\sin x\) in the interval \(a, 23\).

Theorem 2. \(q\) consists of two statements:

1) \(\tilde{f}(x) \in L^2([0,a])\), 2) \(\tilde{f}'(x_j) = f'(x_j), j = 0 \div 2K - 1\) \(22\)

In the second condition \(22\), we have a closed system composed for derivatives with respect to the interpolation coefficients \(\psi_m\) that are applied in interpolation \(23\) below. As compared to sine interpolation, the preset derivatives \(f'(x_j)\) must coincide with the derivatives of the interpolation function \(\tilde{f}'(x_j)\) at the interpolation points.

To solve the problem, we choose a complete system \(\{1, \cos m\pi x/a\}\) as a basis within the interval \([0,a]\) and use cosine interpolation \(\tilde{f}(x)\) in \(H([0,a])\) from the class of Sobolev-Liouville functions \(L^2([0,a])\) \(19\), which can be represented by a fast cosine expansion \(10\) similarly to \(19\),

\[
\tilde{f}(x) = M_q(x) + \sum_{m=1}^{2K-1} \psi_m \cos m\pi x/a, \tilde{f}(x) \in L^2([0,a])
\]

When studying the cosine interpolation \(23\), in contrast to the sine interpolation, the following exception occurs: if the system of continuous functions \(\{1, \cos m\pi x/a\}\) within the interval \([0,a]\) is orthogonal, the corresponding discrete cosine interpolation system \(\{1, \cos m\pi x_j/a\}, j = 0 \div 2K - 1\) in \(H(j = 0 \div 2K - 1)\) is not orthogonal unlike the sine interpolation system \(\{\sin m\pi x_j/a\}\). Even the discrete cosine system

\[
\{1, \cos m\pi x_j/a\}, x_j \in [0, a(1 - \Delta)], j = 0 \div 2K - 1
\]

is complete, but it is orthogonal only for even values \(m\) or only for odd \(m\) in the interval \(x \in [0, a(1 - \Delta)]\). However, for even or only odd \(m\) the system of \(24\) is not complete. In the cases of incomplete systems a fatal error may occur when an increase of the number \(2K\) does not lead to a decrease of the interpolation error. If \(m\) takes both even and odd values, the set of functions \(24\) being complete is not orthogonal. This contradiction significantly complicates derivation of compact formulas when determining coefficients \(\psi_m\). Hence, below is the proof of Theorem 2 on the exceptions of orthogonal properties of the cosine interpolation system \(24\).

**Theorem 2.** Let \(f(x) \in L^p([0,a])\). The interval \([0,a]\) is uniformly divided into an even number of \(2K\) parts with the step \(\Delta = a/2K\). The theorem consists of two statements:
1) For random integers \( m \) (even and odd) the cosine interpolation system in \( H\left( j = 0 \div 2K - 1\right) \)
\[ \{1, \cos(m \pi x/a), x_j = j a/2K, j = 0 \div 2K - 1\} \] is not orthogonal;

2) If the index \( m \) takes only even or only odd values, the cosine interpolation system will be orthogonal in such a case.

Below is the proof of the first theorem part.

For algebraic system (22) in \( H \) the scalar product is written:
\[
\sum_{j=0}^{2K-1} \cos m \pi x_j/a \cos \pi x_j/a, \; m \neq t, \; (m,t) = 0 \div 2K - 1
\] (25)

Since the indices \( (m,t) \) are random integers, let \( m \) be even and \( t \) be odd or vice versa odd \( m \) and even \( t \). In both cases \( m \pm t \) is an odd number. Let us prove that the following inequality is satisfied in this case
\[
\sum_{j=0}^{2K-1} \cos m \pi x_j/a \cos \pi x_j/a \neq 0, \; m \neq t, \; (m,t) = 0 \div 2K - 1
\]

Meaning that the discrete system (23) is not orthogonal.

The sum (25) is transformed using the complex Euler formula:
\[
\sum_{j=0}^{2K-1} \cos m \pi x_j/a \cos \pi x_j/a = 2 \sum_{j=0}^{2K-1} \left( \cos (m+t) \pi x_j/a + \cos (m-t) \pi x_j/a \right)
\]

\[
= \sum_{j=0}^{2K-1} \left( \expi (m+t) \pi x_j/a + \expi (m-t) \pi x_j/a \right)
\]

\[
= \sum_{j=0}^{2K-1} \left( \expi (m+t) \pi j/2K + \expi (m-t) \pi j/2K \right) (26)
\]

Here, \( i \) is an imaginary one. Let us introduce the following notation
\[
\expim \pi j/2K = q^j_{mst}, \; \expi (m \pm t) \pi j/2K = q^j_{mst}, \; (m,t) = 0 \div 2K - 1
\]

When applying this notation the presentation of scalar product (26) is simplified
\[
4 \sum_{j=0}^{2K-1} \cos m \pi x_j/a \cos \pi x_j/a = \sum_{j=0}^{2K-1} (q^j_{m+st} + q^j_{m-st} + q^{-j}_{m+st} + q^{-j}_{m-st}) = (1 + q^1_{m+st} + q^2_{m+st} + \ldots + q^{2K-1}_{m+st})
\]

\[+ (1 + q^1_{m-st} + q^2_{m-st} + \ldots + q^{2K-1}_{m-st}) + (1 + q^1_{m+st} + q^2_{m+st} + \ldots + q^{2K-1}_{m+st}) + (1 + q^1_{m-st} + q^2_{m-st} + \ldots + q^{2K-1}_{m-st}) \] (27)

We have four geometric progressions, which are summed up on the right side of (27)
\[
(1 + q_{m+1}^K + q_{m+2}^K + \ldots + q_{m+2K-1}^K) + (1 + q_{m+1}^{-1} + q_{m+2}^{-1} + \ldots + q_{m+2K-1}^{-1}^K)
\]
\[
= \frac{(1 - q_{m+1}^K)(1 - q_{m+1}^{-1})}{(1 - q_{m+1})} + \frac{(1 - q_{m+2}^K)(1 - q_{m+2}^{-1})}{(1 - q_{m+2})} = 2 - 2q_{m+1}^K - q_{m+1}^{-1} + q_{m+2}^K - q_{m+2}^{-1} + 2K
\]
\[
= \frac{2(1 - q_{m+1}^K) - q_{m+1}^{-1} + q_{m+2}^K - q_{m+2}^{-1} + 2K}{(1 - q_{m+1})(1 - q_{m+1}^{-1})} = \frac{2 - q_{m+1}^{-1} - q_{m+2}^{-1} + 2K}{(1 - q_{m+1})(1 - q_{m+1}^{-1})} = 1 - q_{m+1}^K
\]

Obtaining of (28) assumes satisfying the condition \( m \neq t \), whence it follows that \( q_{m+1}^K \neq 1 \). Therefore, the cosine sum (27) has the following form with the use of (28)

\[
\sum_{j=0}^{2K-1} \cos \pi x_j/a \cos \pi x_j/a = \frac{1}{2} \left( 1 - q_{m+1}^K \right) = \frac{1}{2} \left( 1 - (-1)^{m+1} \right)
\]

(29)

In the general case, when \( m, t \) are independent and can take not only even, but also odd values, it follows from (29) that we have \( \Rightarrow q_{m+1}^K \neq 1 \), \( q_{m+1}^{-1} = -1 \) when \( m \pm t \) is odd. Therefore, we get the following inequality in (29)

\[
\sum_{j=0}^{2K-1} \cos \pi x_j/a \cos \pi x_j/a = 1 \neq 0
\]

The first part of Theorem 2 is proved.

To prove the second part of Theorem 2, we note that the sum and difference of \( (m \pm t) \) will always be an even number with simultaneously even \( m \) and even \( t \) or simultaneously odd \( m \) and odd \( t \). Hence, \( q_{m+1}^{2K} = 1 \). Then, when \( (m \pm t) \) is even and \( m \neq t \), \( (m, t) = 0 \div 2K - 2 \), (29) results in

\[
\sum_{j=0}^{2K-1} \cos \pi x_j/a \cos \pi x_j/a = \frac{1}{2} \left( 1 - (-1)^{m+1} \right) = 0
\]

Theorem 2 is completely proved.

When compiling the algebraic system (22), it has been assumed that the interval \([0, a]\) is uniformly divided into even \( 2K \) number of parts. If the interval is divided into odd number of parts, the compact cosine interpolation (23) cannot be constructed.

To solve the cosine interpolation problem of determining \( 2K \) interpolation coefficients \( \Psi_a, \Psi_m \) from the solution of the closed system (22) in a finite compact form, we proceed as follows.

First, the interpolation system of \( 2K \) equations (22) is differentiated and a sine interpolation system is obtained. After that the sine derivative system is re-written in a convenient form

\[
\psi'(x_j) = f'(x_j) - M_{q_{m+1}}(x_j) = -\pi \frac{2K-1}{a} \sum_{j=1}^{2K-1} m \Psi_m \sin m \pi x_j/a, \quad x_j = j \frac{a}{2K}, \quad j = 1 \div 2K - 1
\]

(30)

Writing (30) actually reduces the problem of finding the interpolation coefficients \( \Psi_m, m = 1 \div 2K - 1 \) in the explicit form to solving the closed algebraic sine system (30), which solution has been previously obtained in a compact form (14). The coefficient \( \Psi_0 \) does not belong to the system (30) and it’s determined from (23) assuming \( x = 0 \):
\[ \tilde{\psi}_0 = f(0) - \sum_{m=1}^{2K-1} \tilde{\psi}_m, \quad M_q(0) = 0 \]  \hspace{1cm} (31)

In obtaining (31), it has been assumed that the boundary functions \( M_q(x) \) with odd indices \( q \) are equal to zero \( M_q(0) = 0 \) when \( x = 0 \).

To find \( \tilde{\psi}_0 \), \( \tilde{\psi}_m \) is first defined. To do this, the left and right sides of (30) are multiplied by \( \sin n\pi x / a \) and summed up over \( j \):

\[ \sum_{j=0}^{2K-1} \psi'(x_j) \sin n\pi j / 2K = -\pi \sum_{m=1}^{2K-1} m\tilde{\psi}_m \sum_{j=0}^{2K-1} \sin n\pi j / 2K \sin m\pi j / 2K \]

The product of sines is transformed by the feature \( (16) \), and after that the equation takes the following form

\[ \sum_{j=0}^{2K-1} \psi'(x_j) \sin m\pi j / 2K = -K\pi a \sum_{j=0}^{2K-1} \psi'(x_j) \sin m\pi j / 2K \]

(32)

By means of (32) from (31) we get

\[ \tilde{\psi}_0 = \tilde{f}(0) + \sum_{m=1}^{2K-1} \frac{a}{Km\pi} \sum_{j=0}^{2K-1} \psi'(x_j) \sin m\pi j / 2K \]

(33)

Applying \( \tilde{\psi}_m \) of (32) and \( \tilde{\psi}_0 \) of (33), the cosine interpolation formula \( (23) \) is written in the resulting form

\[ \tilde{f}(x) = M_q(x) + \tilde{f}(0) + \sum_{m=1}^{2K-1} \frac{a}{Km\pi} \sum_{j=0}^{2K-1} \psi'(x_j) \sin m\pi j / 2K \left( 1 - \cos m\pi x / a \right) \]

(34)

### 4.2 Application algorithm of fast cosine interpolation

1. Determine the interval \([0,a]\) and divide it to even number of equal parts \( 2K \) by points \( x_j = j a / 2K \), \( j = 0, 2K-1 \) having the division step \( \Delta = a / 2K \) so that to satisfy the inequality \( |f'(x_{j+1}) - f'(x_j)| \leq \varepsilon \), where \( \varepsilon \) is the preset computation accuracy.

2. Values of \( f'(0), f'(a) \) and other odd derivatives at the ends of the interval \([0,a]\) in (34) are taken either from the boundary conditions of this problem or from the considered differential equation. If there is neither one nor the other, they can be calculated by finite differences using navigation data.

Figure 2 shows graphs of the fast cosine interpolation errors (34), its first and second derivatives for the function \( \exp(3x) \) in the interval \([0,1]\) when \( M_q(x) = M_1(x) \) and \( K = 10 \). The greatest error is achieved at \( x = a - \Delta \).
4.3 Cosine interpolation error estimate

Assume \( f(x) \in L^2_2([0,a]) \). As an example fast cosine expansion (34) is used where the Fourier coefficients \( \psi_m \) are calculated by formulas (21) with the third-order boundary function.

Trigonometric interpolation (23) has two limiting features [1]:
\[
\lim_{K \to \infty} \psi_m^\text{tr} = \psi_m^\text{tr} \quad \text{and} \quad \lim_{K \to \infty} \tilde{f}(x) = f(x)
\]

When \( 2K \to \infty \) the discrete Fourier series (23) is transformed to conventional Fourier series
\[
\tilde{f}(x) = M_3(x) + \psi_0 + \sum_{m=1}^{2K} \psi_m \cos m\pi x/a + \sum_{m=2K}^{\infty} \psi_m \cos m\pi x/a, \quad \tilde{f}(x) \in L^2_2([0,a])
\]

The second sum can be interpreted as the interpolation error \( R_{\cos} \) at \( 2K \to \infty \) since it is discarded in interpolation (23). Therefore, \( R_{\cos} = \sum_{m=2K}^{\infty} \psi_m \cos m\pi x/a \) can be accepted. The following notation is added
\[
M_{\cos}^{(5)} = \sup \left| f^{(5)}(x) \right|, x \in [0,a]
\]

The interpolation error estimate (23) is obtained as an estimate of the remainder, denoted by \( R_{\cos} \) at \( 2K \to \infty \). Assuming approximately \( \psi_m \approx \psi_m^\text{tr} \) the following error estimate is obtained from (21)
\[
R_{\cos} = \sum_{m=2K}^{\infty} \psi_m \cos m\pi x/a = -\frac{2}{a} \sum_{m=2K}^{\infty} \left( \frac{a}{m\pi} \right)^5 \left( \int_0^a f^{(5)}(t) \sin m\pi t/a \, dt \right) \cos m\pi x/a
\]
\[
\leq 2 \sum_{m=2K}^{\infty} \left( \frac{a}{m\pi} \right)^5 \left( \int_0^a f^{(5)}(t) \sin m\pi t/a \, dt \right) \leq 2M_{\cos}^{(5)} \left( \frac{a}{\pi} \right)^5 \sum_{m=2K}^{\infty} \frac{1}{m^5}
\]
\[
= 2M_{\cos}^{(5)} \left( \frac{a}{\pi} \right)^5 \left( \frac{1}{(2K)^5} + \sum_{m=2K+1}^{\infty} \frac{1}{m^5} \right) \leq 2M_{\cos}^{(5)} \left( \frac{a}{\pi} \right)^5 \left( \frac{1}{(2K)^5} + \int_{2K}^{\infty} \frac{1}{z^5} \, dz \right)
\]
\[
= 2M_{\cos}^{(5)} \left( \frac{a}{\pi} \right)^5 \left( \frac{1}{(2K)^5} + \frac{1}{4(2K)^5} \right) = M_{\cos}^{(5)} \left( \frac{a}{2\pi K} \right)^5 (2 + K)
\]

Figure 2. Cosine interpolation errors and its first and second derivatives.
It follows from the above that the cosine interpolation error decreases rapidly when using a fast expansion and increasing K. Formula (35) allows to determine beforehand the value of K and number of interpolation points \( x_j \) to obtain the required accuracy.

5. Conclusion

All considered examples of trigonometric interpolations apply fast expansions, thus allowing the possibility of differentiating the interpolation a preset number of times. The resulting interpolation series converge. The proposed method is especially efficient when considering nonlinear boundary problems in continuum mechanics for non-standard curvilinear multidimensional domains, when other known methods cannot be applied.

The fast expansion method has the following features:
1. The method provided here is analytical and allows analytical research with the use of the obtained solution;
2. The approximate interpolation (34) allows differentiation of a preset \( q \) number of times;
3. Fast and absolute discrete Fourier series convergence is proved;
4. Applied problems require to use a small number of terms of the Fourier series due to its fast convergence, thus contributing cost-effective numerical programs to implement the method;
5. There is a method error estimate in an explicit form.

Taking into account these features, it can be affirmed that this fast expansion method is considerably superior to all known approximate analytical methods.

References
[1] Bakhvalov N S, Zhidkov N P and Kobelkov G M 1987 Numerical methods (Moscow: Science)
[2] Krylov V I 1967 Approximate calculation of integrals (Moscow: Science)
[3] Ziatdinov S I 2008 Analysis of errors in trigonometric interpolation. Journal of Instrument Engineering 51(5) 42-45
[4] Chubarikov V N and Sharapova M L 2017 Interpolation of functions of many variables. Chebyshev collection 18(4) 338-46
[5] Naats I E 1969 Application of trigonometric interpolation in discrete measurement problems Bulletin of the Tomsk Polytechnic Institute 168 71-4.
[6] Efimov V M and Reznik A L 2005 On the reference functions to recover a periodic signal and error variance of trigonometric interpolation Avtometriya 41(4) 3-14.
[7] Samarsky A A and Gulin A V 1989 Numerical methods (Moscow: Science)
[8] Kalitkin N N 1978 Numerical methods (Moscow: Science)
[9] Zalipaev V V and Gulevich D R 2020 Numerical methods in physics and technology (Saint Petersburg: ITMO University)
[10] Chernyshov A D 2014 Fast expansion method for solving nonlinear differential equations Computational Mathematics and Mathematical Physics 54(1) 13-24
[11] Chernyshov A D, Marchenko A N and Goryainov V V 2013 Analytical solution of the temperature field problem in a rectangular plate with a variable internal source by means of fast decomposition Questions of atomic science and technology. Series: Mathematical modeling of physical processes No. 1 53-8
[12] Chernyshov A D, Popov V M, ShakhoV A S and Goryainov V V 2014 Increased solving accuracy of the contact thermal resistance problem between compressed balls by means of the fast decomposition method Thermal processes in technology 6(4) 179-91
[13] Chernyshov A D and Goryainov V V 2016 On the relationship of number of function graph inflection points and number of Fourier series terms with fast decomposition Theoretical & Applied Science 33(1) 137-41
[14] Chernyshov A D 2018 Solution of the nonlinear thermal conductivity equation for a curvilinear domain by the fast decomposition method Journal of Engineering Physics and Thermophysics 91(2) 433-44
[15] Chernyshov A D, Goryainov V V, Leshonkov O V, Soboleva E A and Nikiforova O Yu 2019 Comparison of the fast decomposition convergence rate to conventional Fourier series decomposition Bulletin of Voronezh State University. Series: System analysis and information technology No. 1 27-34

[16] Chernyshov A D, Goryainov V V and Danshin A A 2018 Analysis of the stress field in a wedge using the fast expansions with pointwise determined coefficients J. Phys.: Conf. Ser. 973 012002

[17] Chernyshov A D, Goryainov V V and Chernyshov O A 2015 Application of the fast expansion method for spacecraft trajectory calculation Russian Aeronautics 58(2) 180-6.

[18] Piskunov N S 1962 Differential and integral calculus (Moscow: State Physics and Mathematics publishing house)

[19] Ilyin V A 1991 Spectral theory of differential operators (Moscow: Science)