The vector type Maxwell viscoelastic model and its relationship to thixotropy

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Abstract
Thixotropy is a common behavior of soft matters widely used in various industries. Numerous models, both empirical and theoretical, have been proposed for different soft matters. In this paper the Maxwell viscoelastic model has been modified by decoupling the second order tensor type equation into vector type equations, without introducing new material parameters or assuming the dependence of the viscosity and/or elasticity on time and/or shear rate. The proposed Maxwell model is known as the vector type Maxwell model while the original one as the second order tensor type Maxwell model. The vector type Maxwell model was applied to both the stationary and the single step shear rate shear flow and the numeric results were compared with the experiments in the literature. It is found that the vector type Maxwell model predicted better both the shear and the normal stresses in the stationary shear flow than the second order tensor type Maxwell model, and the vector type Maxwell fluids may show thixotropic behavior in the single step shear rate flow as observed in experiments.

Key words: Thixotropy, Maxwell viscoelastic model, Corotational, Stress overshoot, Shear thinning

1. Introduction
Thixotropy is the rheological property of wide range of materials used in various industries. Great efforts from both industry and academy have been devoted to modelling thixotropy since it was experimentally observed about 90 years before by Schalek and Szegavri (1923), and numerous models, both empirical and theoretic, were proposed. The progress in the field are constantly summarized up to date, stage by stage, by the then review papers periodically appeared (Scott-Blair, 1943, Bauer and Collins, 1967, Mewis, 1979, Barnes, 1997, Mewis and Wagner, 2009).

The approaches to modeling thixotropy were classified into viscous theory and viscoelastic theory (Barnes, 1997). In viscous theory the thixotropic fluid is considered as the pure viscous fluid but the viscosity changes with both shear rate (stress) and time, while in viscoelastic theory the thixotropic fluid is assumed to be viscoelastic. The viscous theory is further divided into three categories: direct microstructure, indirect microstructure theories, and continuum theory [5]. In both direct and indirect microstructure theories the thixotropic viscosity change is attributed to the breakdown and buildup of the structure of fluids during shear flow. Direct microstructure theory uses reaction kinetics to model the structure breakdown and buildup by introducing into a structure parameter which is unit for completely built-up structure and is zero for completely broken-down structure (Moore, 1959, Worrall and Tuliani, 1964, Cheng and Evans, 1965, Tiu and Boger, 1974, De Kee and Chan Man Fong, 1994), while the indirect one describes the extent of the structure breakdown and buildup by introducing into a structure parameter which is unit for completely built-up structure and is zero for completely broken-down structure (Moore, 1959, Worrall and Tuliani, 1964, Cheng and Evans, 1965, Tiu and Boger, 1974, De Kee, Code and Turcotte, 1983, Nguyen and Boger, 1985, Toorman, 1997). Continuum theory is started by defining constitutive equation without understanding the physical process underlying (Slibar and Paslay, 1959, 1964, Elliott and Ganz, 1971, 1977, Suetsugu and White, 1984, Powell, 1995, Phan-Thien, Safari-Ardi, and Morales-Patino, 1997).

In this paper a very different approach from all the existing ones, without introducing new rheological properties or assuming dependence of the viscosity or the elasticity on time or shear rate, will be attempted only by decoupling the second order tensors of the Maxwell viscoelastic model into vectors. The theoretical results will be compared with the
experimental results in the literature, and the fact that the vector type Maxwell viscoelastic fluids is thixotropic will be clarified.

2. The Maxwell viscoelastic model

2.1 The Maxwell viscoelastic model of the second order tensor type

The elastic model relating the stress tensor $\sigma$ to the elastic strain tensor $\varepsilon_e$ is given as follows

$$\sigma = K : \varepsilon_e$$  \hspace{1cm} (1)

where $K$ is the 4th order elastic constant tensor and for isotropic linear elastic materials it is simplified as

$$K_{ijkl} = \lambda \delta_{ij} \delta_{kl} + G \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj} \right),$$ \hspace{1cm} (2)

$$\lambda = \frac{E}{(1 + \nu)(1 - 2\nu)},$$ \hspace{1cm} (3)

and

$$G = \frac{E}{2(1 + \nu)},$$ \hspace{1cm} (4)

where $E$ is the Young modulus and $\nu$ the Poisson ratio.

For incompressible isotropic linear elastic materials Eq. (1) is further simplified as

$$\sigma = 2G \varepsilon_e.$$ \hspace{1cm} (5)

Equation (5) shows that the deformation of isotropic linear elastic materials is always in the same direction of the stress.

As to the incompressible Newtonian viscous fluid we have the following constitutional equation

$$\sigma = 2\mu \dot{\varepsilon}_v - pI,$$ \hspace{1cm} (6)

where $\dot{\varepsilon}_v$ is the viscous strain rate tensor. The Maxwell viscoelastic model defines that the total strain rate is the sum of the elastic and viscous strain rates

$$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v.$$ \hspace{1cm} (7)

Therefore we have

$$\frac{\mu}{G} \dot{\sigma} + \sigma + pI = 2\mu \dot{\varepsilon},$$ \hspace{1cm} (8)

where (′) is the Lie time derivative which is different from the conventional total time derivative (′′). The Lie time derivative meets the requirement for all the physical laws of the objectivity principle. Equation (8) is the general Maxwell viscoelastic model of the second order tensor type. For the corotational coordinates the Lie derivative is given by

$$\dot{\sigma} = \dot{\sigma} - \omega \sigma - \sigma \omega^T = \dot{\sigma} - \omega \sigma + \sigma \omega,$$ \hspace{1cm} (9)

where $\omega$ is the spin tensor of the reference fluid particle. Putting Eq. (9) into Eq. (8) gives the corotational Maxwell viscoelastic model of the second order tensor type as follows

$$\frac{\mu}{G} (\dot{\sigma} - \omega \sigma + \sigma \omega) + (\sigma + pI) = 2\mu \dot{\varepsilon}.$$ \hspace{1cm} (8′)

2.2 The Maxwell viscoelastic model of the vector type

Equations (5) and (6) may be rewritten in more detail as follows

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = 2G \begin{pmatrix} \varepsilon_{e,xx} & \varepsilon_{e,xy} & \varepsilon_{e,xz} \\ \varepsilon_{e,yx} & \varepsilon_{e,yy} & \varepsilon_{e,yz} \\ \varepsilon_{e,zx} & \varepsilon_{e,zy} & \varepsilon_{e,zz} \end{pmatrix},$$ \hspace{1cm} (5′)
\[ \begin{pmatrix} \sigma_{xx} + p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} + p & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} + p \end{pmatrix} = 2\mu \begin{pmatrix} \dot{\varepsilon}_{xx} \\ \dot{\varepsilon}_{yx} \\ \dot{\varepsilon}_{zx} \end{pmatrix} \quad (6') \]

It is obvious that Eqs (5) and (6) of the second order tensor type can be further rewritten into the following element equations

\[
\sigma_{im} = 2G\varepsilon_{,im}, \quad \sigma_{im} + \delta_{im}p = 2\mu\dot{\varepsilon}_{,im}. \tag{5''} \]

Now we assume that the selected vectors in the tensors follow the Maxwell model, instead of the second order tensors themselves and the elements either. For example the second columns of the stress tensor, strain tensor and the strain rate tensor in Eqs. (5’) and (6’) are related by applying the Maxwell model as follows

\[
\frac{\mu}{G} \sigma_{xy} + (\sigma_{xy} + p) = 2\mu \dot{\varepsilon}_{,xy}. \tag{10} \]

This operation of column selection may be carried out by multiplying Eqs. (5) and (6) with the unit normal vector \( n \) of the reference surface, that is

\[
\begin{align*}
\mathbf{s} &= \sigma n, \\
\mathbf{d} &= \dot{n}. 
\end{align*} \tag{11, 12}
\]

It is clear that \( \mathbf{s} \) and \( \mathbf{d} \) are the tension vector and strain rate vector of the reference surface, respectively. Therefore equation (10) can be rewritten into

\[
\frac{\mu}{G} s + (s + pn) = 2\mu \dot{d}. \tag{13} \]

Equation (13) is known as the vector type Maxwell model. For corotational coordinates, the Lie derivative of the surface tension vector is given by the definition as follows

\[
\dot{s} = \dot{s} - \omega s. \tag{14} \]

Therefore the vector type Maxwell equation (13) in corotational coordinates may be rewritten as

\[
\frac{\mu}{G} (\dot{s} - \omega s) + (s + pn) = 2\mu \dot{d}. \tag{15} \]

Equation (15) is known as the corotational vector type Maxwell model.

### 2.3 Difference between the second order tensor type and the vector type Maxwell viscoelastic models

Multiplying both sides of the second order tensor type Maxwell Eq. (8) with the unit normal vector of reference surface gives

\[
\frac{\mu}{G} \sigma n + s + pn = 2\mu \dot{d}. \tag{16} \]

For the stationary reference surface it is known

\[
\dot{n} \neq \dot{s} \tag{17} \]

since we have

\[
\dot{n} = (\sigma - \omega \sigma + \sigma \omega)n = \dot{n} - \omega \sigma n + \sigma \omega n = \dot{\sigma} - \omega s + \sigma \omega n = \dot{s} + \sigma \omega n \tag{18} \]

However, if the reference surface is corotational with the reference fluid particle, we have

\[
(\dot{n}) = \dot{\sigma} n + \sigma \dot{n} = \dot{n} + \sigma \omega n \tag{19} \]

and therefore

\[
\dot{n} = \dot{s}. \tag{20} \]

The vector type Maxwell model becomes equivalent to the second order tensor type Maxwell model. Therefore the
The mathematical difference between the vector type and the second order tensor type Maxwell models is whether the reference surface is stationary or corotational with the reference fluid particle. The vector type Maxwell Eq. (13) or (15) contains only the surface stress vector while the Maxwell equation contains the stress tensor. Vector operation is much easier than tensor operation. Now let us see the difference between the second order tensor type and the vector type Maxwell models through analyzing the shear and normal stresses in stationary shear flow. For the shear flow in the x-y plane at shear rate $\dot{\gamma}$ in x direction as shown in Fig. 1 we have

$$\dot{\varepsilon} = \begin{pmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_{zz} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \dot{\gamma} & 0 \\ \frac{1}{2} \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(21)

$$\omega = \begin{pmatrix} 0 & \omega_a & 0 \\ -\omega_a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(22)

$$\sigma = \begin{pmatrix} \sigma_x & \tau & 0 \\ \tau & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix},$$

(23)

and furthermore for the planes normal to x, y, and z axes the normal vectors are as follows

$$n_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad n_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad n_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

(24)

Substituting Eqs. (21), and (23-24) into Eqs. (11-12) gives

$$s_x = \begin{pmatrix} \sigma_x \\ \tau \\ 0 \end{pmatrix}, \quad s_y = \begin{pmatrix} \tau \\ \sigma_y \\ 0 \end{pmatrix}, \quad s_z = \begin{pmatrix} 0 \\ 0 \\ \sigma_z \end{pmatrix},$$

(25)

$$2\dot{d}_x = \begin{pmatrix} \dot{\gamma} \\ 0 \\ 0 \end{pmatrix}, \quad 2\dot{d}_y = \begin{pmatrix} 0 \\ \dot{\gamma} \\ 0 \end{pmatrix}, \quad 2\dot{d}_z = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$  

(26)

Putting Eqs. (22) and (25-26) into Eq. (15) and noting $\dot{s} = 0$ we have

$$\begin{pmatrix} 1 & -\frac{\mu}{G} \omega_a & 0 \\ \frac{\mu}{G} \omega_a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \tau \\ 0 \end{pmatrix} + p \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mu \begin{pmatrix} \dot{\gamma} \\ 0 \\ 0 \end{pmatrix},$$

(27)

$$\begin{pmatrix} 1 & -\frac{\mu}{G} \omega_a & 0 \\ \frac{\mu}{G} \omega_a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau \\ \sigma_y \\ 0 \end{pmatrix} + p \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mu \begin{pmatrix} \dot{\gamma} \\ 0 \\ 0 \end{pmatrix},$$

(28)

$$\begin{pmatrix} 1 & -\frac{\mu}{G} \omega_a & 0 \\ \frac{\mu}{G} \omega_a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \sigma_z \end{pmatrix} + p \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mu \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. $$

(29)

Five independent equations are obtained from Eqs. (27-29) to solve the stresses and pressure, which are given as follows

$$\sigma_x - \frac{\mu}{G} \omega_a \tau + p = 0,$$

(30)

$$\frac{\mu}{G} \omega_a \sigma_x + \tau = \mu \dot{\gamma},$$

(31)

$$\frac{\mu}{G} \omega_a \sigma_y + \tau = \mu \dot{\gamma},$$

(32)

$$\sigma_y + \frac{\mu}{G} \omega_a \tau + p = 0,$$

(33)
\[ \sigma_z + p = 0 . \]  

From Eqs. (30-34), we have

\[ \tau = \frac{1}{1 + \left( \omega_a \frac{\mu}{G} \right)^2} \mu \dot{\gamma}, \]  

\[ \sigma_x = -\sigma_y = \frac{\omega_a \mu}{1 + \left( \omega_a \frac{\mu}{G} \right)^2} \mu \dot{\gamma}, \]  

\[ \sigma_z = 0, \]  

\[ p = 0. \]  

For comparison the shear and normal stresses of the second order tensor type Maxwell model are also given below (see Appendix A)

\[ \tau = \frac{1}{1 + \left( 2\omega_a \frac{\mu}{G} \right)^2} \mu \dot{\gamma} \]  

\[ \sigma_x = -\sigma_y = \frac{2\omega_a \mu}{1 + \left( 2\omega_a \frac{\mu}{G} \right)^2} \mu \dot{\gamma} \]  

\[ \sigma_z = -p \]  

It is known that, from the above results, the vector type and the second order tensor type Maxwell models give different results for all the stresses. For shear stress \( \tau \) and the normal stresses \( \sigma_x \) and \( \sigma_y \), the spin velocity \( \omega_a \) in the vector type Maxwell model is replaced by \( 2\omega_a \) in the second order tensor type Maxwell model. Both the normal stress \( \sigma_z \) and the hydraulic pressure \( p \) are always zero for the vector type Maxwell model, while \( \sigma_z = -p \) for the second order tensor Maxwell model.

It is surprising to find that the stationary shear flow of the vector type Maxwell fluid necessitates zero hydraulic pressure since it is common sense that the hydraulic pressure does not affect the flow. The fact that the common sense does not apply to the vector type Maxwell model is easily understood by noting that the hydraulic pressure \( p \) in the vector type Maxwell Eq. (13) or (15) cannot be eliminated by using new stresses

\[ s' = s + pn . \]  

On the other hand, the second order tensor type Maxwell model gives the normal stresses \( \sigma_x \) and \( \sigma_y \) independent of the hydraulic pressure \( p \), which is also not physically reasonable.

For verification Fig. 2 gives the comparison of the shear stress and the normal stress difference between the experimental measurements for SRM 2941 at three different temperatures (Amis and Rumble, 2002) and the predictions from both the vector type and the second order tensor type Maxwell models. It is clear from Fig. 2 that the vector type Maxwell model agree better with the experimental results than the second order tensor type Maxwell model. The vector type Maxwell model is reasonable.
Fig. 2 Dependence of shear stress and normal stress difference on the shear rate for SRM 2491 (a) T=0 °C, (b) T=25 °C, and (c) T=50 °C. Measurements by Amis and Rumble (2002)
In the following section the zero-hydraulic pressure is assumed and therefore Eq. (13) becomes
\[ \frac{\mu}{G} \dot{s} + s = 2\mu \dot{d}. \] (13')

3 The vector type Maxwell viscoelastic model in plane shear flow

In this paper we confines ourselves to considering the plane shear flow in which almost all the experiments are conducted. For the plane shear flow the vector type Maxwell model Eq. (13') is reduced from three dimension to two dimension, which is rewritten as follows
\[ \frac{\mu}{G} \dot{s} + s = \mu \dot{\gamma}. \] (13'')
where \( \dot{\gamma} = 2\dot{d} \) is the shear strain vector.

For convenience the complex coordinate system is used in this paper in place of the commonly used Cartesian coordinate system. Figure 3 is the illustration of the coordinate systems. In addition to stationary coordinates we also introduce a coordinate system rotating with the reference fluid particle together, the corotational coordinate system. Any vector \( V \) in a stationary coordinate system may be changed to \( V_r \) in the corotational coordinate system as follows
\[ V = Ve^{(-\theta + \theta)i} = V_re^{i\theta}, \] (45)
\[ V_r = Ve^{-i\theta}, \] (46)
where \( \theta \) is the rotational angle. We know that \( e^{i\theta} \) is the base vector of the corotational coordinate system. Applying this transformation to both the surface stress \( s \) and the shear strain rate \( \dot{\gamma} \) we obtain
\[ s = s_re^{i\theta}, \] (47)
and
\[ \dot{\gamma} = \dot{\gamma}_re^{i\theta}, \] (48)

The Lie derivative is obtained simply by conducting the time derivative by fixing the rotating base vector, that is
\[ \dot{s} = \dot{s}_re^{i\theta}. \] (49)

Substituting Eqs. (47-49) into Eq. (13'') we have
\[ \mu \dot{\gamma}_r e^{i\theta} = \frac{\mu}{G} \frac{ds_r}{dt} e^{i\theta} + s_r e^{i\theta}, \] (50)
and eliminating \( e^{i\theta} \) from Eq. (50) gives
\[ \mu \dot{\gamma}_r = \frac{\mu}{G} \frac{ds_r}{dt} + s_r. \] (51)
Equation (51) is exactly the same as Eq. (13") in form, which is a consequence of the objectivity principle. It is easy to show that the general solution to the vector type Maxwell Eq. (51) for constant elasticity \( G \) and constant viscosity \( \mu \) is given as follows

\[
s_v = G \int_{-\infty}^{t} \gamma_v(t') e^{-\frac{G(t-t')}{\mu}} dt',
\]

(52)

Because the physical observation of fluids is almost always conducted in stationary coordinates in practice it will be convenient to rewrite Eq. (52) in stationary coordinates, which is given as follows

\[
s = G e^{i\theta(t)} \int_{-\infty}^{t} \gamma(t') e^{-i\theta(t')} e^{-\frac{G(t-t')}{\mu}} dt' = G \int_{-\infty}^{t} \gamma(t') e^{i\left(\theta(t)-\theta(t')\right)} e^{-\frac{G(t-t')}{\mu}} dt'.
\]

(53)

The real part of the surface stress in Eq. (53) is in the shearing direction, and therefore the shear stress, while the imaginary part, which is in the normal direction of the shearing surface, is the normal stress. Accordingly we have

\[
\tau = G \int_{-\infty}^{t} \gamma(t') e^{-\frac{G(t-t')}{\mu}} \cos(\theta(t) - \theta(t')) dt',
\]

(54)

\[
\sigma = G \int_{-\infty}^{t} \gamma(t') e^{-\frac{G(t-t')}{\mu}} \sin(\theta(t) - \theta(t')) dt'.
\]

(55)

It should be noted that, in Eqs. (54-55), since the shear strain rate is always in the direction of the real axis the vector notation is not necessary anymore and a scalar notation is used instead.

For the stationary plane shear flow the angular velocity \( \omega_a \) is constant, and therefore

\[
\theta = \omega_a t = -\frac{\dot{\gamma}}{2} t,
\]

(56)

where

\[
\omega_a = -\frac{\dot{\gamma}}{2}
\]

(57)

has been used

4. Numerical results for thixotropy in single step shear rate

The step shear rate is given in stationary coordinates as follows

\[
\dot{\gamma} = \begin{cases} \dot{\gamma} & t \geq 0 \\ 0 & t < 0 \end{cases}
\]

(58)

Substituting Eqs. (56) and (58) into Eqs. (54-55) and implementing the integration and simplification give

\[
\tau = \frac{1}{1 + \left(\frac{\omega_a}{\omega_0}\right)^2} \mu \dot{\gamma} \left[1 - \left(\cos \omega_a t - \frac{\mu \omega_a}{G} \sin \omega_a t\right) e^{-\omega_0 t}\right]
\]

\[
= \frac{1}{1 + \left(\frac{\omega_a}{\omega_0}\right)^2} \mu \dot{\gamma} \left[1 - \left[1 + \left(\frac{\omega_a}{\omega_0}\right)^2\right]^{1/2} \cos(\omega_a t + \varphi) e^{-\omega_0 t}\right]
\]

(59)

\[= \tau_s \rho_s(t),\]
Fig. 4 Comparison between the theoretical predictions and the measurements: (a) the dependence of the stable equivalent viscosity on the shear rate; (b) equivalent viscosity variation with time. Experimental data by Suetsugu Y and White J (1984)
and

\[
\sigma = \frac{\omega_0}{\omega_0} \left( 1 + \left( \frac{\omega_0}{\omega_0} \right)^2 \right) \frac{\mu_\gamma}{1 + \left( \frac{\omega_0}{\omega_0} \right)^2} \left[ 1 + \left( \frac{\omega_0}{\omega_0} \right)^2 / 2 \right] \sin(\omega_0 t + \varphi) e^{-\omega_0 t} \right] \right)
\]

where

\[
\omega_0 = \frac{1}{\tau_0} = \frac{G}{\mu}, \quad \varphi = \tan^{-1} \left( \frac{\omega_0}{\omega_0} \right) \]

\[
\tau_s = \frac{1}{1 + \left( \omega_0/\omega_0 \right)^2} \]

\[
\rho_s(t) = 1 - \frac{1}{1 + \left( \omega_0/\omega_0 \right)^2} \cos(\omega_0 t + \varphi) e^{-\omega_0 t} \]

\[
\sigma_s = \frac{1}{1 + \left( \omega_0/\omega_0 \right)^2} \mu_\gamma \]

\[
\rho_s(t) = 1 - \frac{1}{1 + \left( \omega_0/\omega_0 \right)^2} \left( \omega_0 t + \varphi \right) e^{-\omega_0 t} \]

\[
\omega_0 \text{ is defined as the character angular velocity of the fluid, } \tau_0 \text{ the relaxation time, } \tau_s \text{ the stable shear stress, } \sigma_s \text{ the stable normal stress, } \rho_s(t) \text{ and } \rho_s(t) \text{ the transient factors of the shear stress and the normal stress, respectively. The stable shear stress of Eq. (61) and the stable normal stress of Eq. (63) are equal to Eqs. (35) and (36), respectively, as expected.}

Numerical results of the transient shear stress at the single step shear rate are given in Fig. 4, together the experimental results for polystyrene (Suetsugu and White, 1984) for comparison. The viscosity \( \mu \) and the shear elastic modulus \( G \) is obtained by fitting the theoretical stable equivalent viscosity to the measured one. According to Eq. (61) the theoretical stable equivalent viscosity is given by

\[
\mu_{es} = \frac{\tau_s}{\gamma} = \frac{\mu}{1 + \left( \omega_0/\omega_0 \right)^2} \]

It is shown, in Fig. 4 (a), that Eq. (65) agrees well with the experimental results by choosing \( \mu = 2.4 \times 10^4 \text{ Pa} \text{s} \) and \( G = 4 \text{ kPa} \). The theoretical transient equivalent viscosity in Fig. 4 (b) is then calculated by putting \( \mu = 2.4 \times 10^4 \text{ Pa} \text{s} \) and \( G = 4 \text{ kPa} \) into Eq. (59) and noting that

\[
\mu_e = \frac{\tau}{\gamma} \]

From Fig. 4(b) it is known that the theoretical predictions agree well with the experimental results. The equivalent viscosity increases sharply with the shearing time in the beginning, and then reaches different stable values depending on the shear rate. For high shear rate \( \gamma = 0.25 \text{ s} \) both the experiment and the theory show the viscosity overshoot (stress overshoot) which is generally known as a typical phenomenon of thixotropy. It is the first time to show that the Maxwell viscoelasticity itself may give the stress overshoot and cause thixotropy.

The shear stress peak, if any, will appear at time given as follows (Appendix B)

\[
\tau_{sp} = \frac{\pi}{2 |\omega_0|} = \frac{1}{|\gamma|} \]

which is only the function of the angular velocity or the shear rate, and is independent of any properties of the viscoelastic fluid.
5. Conclusion

The vector type Maxwell viscoelastic model was proposed by decoupling the second order tensor type Maxwell viscoelastic model. The numerical results of the vector type Maxwell model for both the stationary plane shear flow and the single step shear rate flow were compared with the results the experiments in the literature. It is found that the vector type Maxwell model agrees well with the experimental results for both the normal and shear stresses in stationary shear flow, and the stress overshoot which is known as a typical thixotropic behavior.

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Appendix A

The second tensor type Maxwell equation for the stationary plane shear flow is given as follows

\[
\begin{pmatrix}
0 & \omega_a & 0 \\
-\omega_a & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\sigma_x & \tau & 0 \\
\tau & \sigma_y & 0 \\
0 & 0 & \sigma_z
\end{pmatrix}
\begin{pmatrix}
0 & \omega_a & 0 \\
-\omega_a & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\sigma_x + p & \tau & 0 \\
\tau & \sigma_y + p & 0 \\
0 & 0 & \sigma_z + p
\end{pmatrix}
\]

\[= 2G
\begin{pmatrix}
0 & 1/2 \dot{\gamma} & 0 \\
1/2 \dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (a1)

Implementing the multiplication of matrices and rearranging gives

\[
\begin{pmatrix}
\frac{G}{\mu} (\sigma_x + p) - 2\omega_a \tau \\
\frac{G}{\mu} \tau + \omega_a (\sigma_x - \sigma_y) \\
0
\end{pmatrix}
\begin{pmatrix}
\frac{G}{\mu} \frac{G}{\mu} (\sigma_y + p) + 2\omega_a \tau \\
0
\end{pmatrix}
\begin{pmatrix}
0 & \frac{G}{\mu} (\sigma_z + p)
\end{pmatrix}
\]

\[= G
\begin{pmatrix}
0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (a2)

Equation (a2) gives four independent equations as follows

\[
\begin{aligned}
\frac{G}{\mu} (\sigma_x + p) - 2\omega_a \tau &= 0 \\
\frac{G}{\mu} (\sigma_y + p) + 2\omega_a \tau &= 0 \\
\frac{G}{\mu} \tau + \omega_a (\sigma_x - \sigma_y) &= G\dot{\gamma} \\
\sigma_z + p &= 0
\end{aligned}
\] (a3) (a4) (a5) (a6)

Finally we obtain

\[
\begin{aligned}
\tau &= \frac{1}{1 + \left(2\omega_a \frac{\mu}{G}\right)^2 \dot{\gamma}^2} \mu \dot{\gamma} \\
\sigma_x - \sigma_y &= \frac{4\omega_a}{G/\mu} \tau = \frac{2\omega_a \mu}{G} \tau = \frac{1}{1 + \left(2\omega_a \frac{\mu}{G}\right)^2 \dot{\gamma}^2} \mu \dot{\gamma} \\
\sigma_y - \sigma_z &= -\frac{2\omega_a}{G/\mu} \tau = -\frac{1}{2} (\sigma_x - \sigma_y) \\
\sigma_z &= -p
\end{aligned}
\] (a7) (a8) (a9) (a10)

Furthermore for incompressible fluids as we have presumed we have

\[
\sigma_x + \sigma_y + \sigma_z = -p
\] (a11)

Therefore we have
\[
\sigma_x = -\sigma_y = \frac{2\omega_a}{G/\mu} = \frac{2\omega_a\mu/G}{1 + (2\omega_a/\mu)^2} \dot{\gamma}
\] (a12)

The normal stresses of \( \sigma_x \) and \( \sigma_y \) are independent of the hydraulic pressure \( p \).

**Appendix B**

The shear stress changes with time as follows

\[
\rho_s(t) = 1 - \left[ 1 + \left( \frac{\omega_a}{\omega_0} \right)^2 \right]^{\frac{1}{2}} \cos(\omega_a t + \varphi) e^{-\omega_0 t}.
\] (b1)

The maximum reaches when

\[
\frac{d\rho_s(t)}{dt} = - \left[ 1 + \left( \frac{\omega_a}{\omega_0} \right)^2 \right]^{\frac{1}{2}} \{ \omega_a \sin(\omega_a t + \varphi) + \omega_0 \cos(\omega_a t + \varphi) \} e^{-\omega_0 t} = 0.
\] (b2)

Noting that

\[ \varphi = \tan^{-1} \left( \frac{\omega_a}{\omega_0} \right), \] (b3)

we can rewrite equation (b2) as follows

\[
\frac{d\rho_s(t)}{dt} = -\omega_0 \left[ 1 + \left( \frac{\omega_a}{\omega_0} \right)^2 \right] \cos(\omega_a t) e^{-\omega_0 t},
\] (b4)

\[
\frac{d\rho_s(t)}{dt} = 0 \quad \text{for} \quad \omega_a t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots.
\] (b5)