\textit{N}-extended supersymmetric Calogero models

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Abstract

We propose a new $N$-extended supersymmetric $su(n)$ spin-Calogero model. Employing a generalized Hamiltonian reduction adopted to the supersymmetric case, we explicitly construct a novel rational $n$-particle Calogero model with an arbitrary even number of supersymmetries. It features $Nn^2$ rather than $Nn$ fermionic coordinates and increasingly high fermionic powers in the supercharges and the Hamiltonian.
1 Introduction

The original rational Calogero model of \( n \) interacting identical particles on a line \([1]\), pertaining to the roots of \( A_1 \oplus A_{n-1} \) and given by the classical Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i \neq j} g^2 (x_i - x_j)^2 ,
\]

(1.1)

has often been the subject of “supersymmetrization”. In this endeavor, extended supersymmetry has turned out to be surprisingly rich. After the straightforward formulation of \( \mathcal{N}=2 \) supersymmetric Calogero models by Freedman and Mende \([2]\), a barrier was encountered at \( \mathcal{N}=4 \) \([3]\). An important step forward then was the explicit construction of the supercharges and the Hamiltonian for the \( \mathcal{N}=4 \) supersymmetric three-particle Calogero model \([4, 5]\), which introduced a second prepotential \( F \) besides the familiar prepotential \( U \). However, it was found that quantum corrections modify the potential in (1.1), and that \( F \) is subject to intricate nonlinear differential equations, the WDVV equations, beyond the three-particle case. These results were then confirmed and elucidated in a superspace description \([6]\). Finally, extending the system by a single harmonic degree of freedom \((su(2) \text{ spin variables}) \([7]\) it was possible to write down a unique \( osp(4|2) \) symmetric four-particle Calogero model \([8]\). A detailed discussion concerning the supersymmetrization of the Calogero models can be found in the review \([9]\).

It seems that a guiding principle was missing for the construction of extended supersymmetric Calogero models. Indeed, while for \( n \leq 3 \) translation and (super-)conformal symmetry almost completely defines the system, the \( n \geq 4 \) cases admit a lot of freedom which cannot \textit{a priori} be fixed. In the bosonic case, such a guiding principle exists \([10]\). The Calogero model as well as its different extensions (see, e.g. \([11, 12, 13]\) are closely related with matrix models and can be obtained from them by a reduction procedure (see \([14]\) for first results and \([15]\) for a review). If we want to employ this principle also for finding extended supersymmetric Calogero models, then the two main steps are

- supersymmetrization of a matrix model
- supersymmetrization of the reduction procedure or proper gauge fixing.

This idea is not new. It has successfully been employed in \([16, 17, 18, 19]\). The resulting supersymmetric systems feature

- a large number of fermions – far more than the \( 4n \) fermions expected in an \( \mathcal{N}=4 \) \( n \)-particle system within the standard (but unsuccessful!) approach
- a rather complicated structure of the supercharges and the Hamiltonian, with fermionic polynomials of maximal degree
- a variety of bosonic potentials, including \( su(2) \) spin-Calogero interactions

but they do not contain a genuine \( \mathcal{N}=4 \) supersymmetric Calogero model, i.e. one with a mere pairwise inverse-square no-spin bosonic potential.

Here we use the same guiding principle and start with the bosonic \( su(n) \) spin-Calogero model in the Hamiltonian approach. We then provide an \( \mathcal{N} \)-extended supersymmetrization of this system. It is important that we do \textit{not a priori} fix a realization for the \( su(n) \) generators. Finally we generalize the reduction procedure to the \( \mathcal{N} \)-extended system and find the first \( \mathcal{N} \)-extended supersymmetric Calogero model, for \textit{any} even number of supersymmetries.

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Here and in the above history, the goal is a bosonic potential exactly as in (1.1). Models with more general interactions can be found for any number of particles.

1
2 $N$-extended supersymmetric Calogero model

2.1 Bosonic Calogero model from hermitian matrices

It is well known that the rational $n$-particle Calogero model \[^1\] can be obtained by Hamiltonian reduction from the hermitian matrix model \[^10, 14\]. Adapted to our purposes, the procedure reads as follows. One starts from the $su(n)$ spin generalization \[^12\] of the standard Calogero model, as given by

\[
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i \neq j}^{n} \ell_{ij} \ell_{ji} \left( x_i - x_j \right)^2 .
\] (2.1)

The particles are described by their coordinates $x_i$ and momenta $p_i$ together with their internal degrees of freedom encoded in the angular momenta $(\ell_{ij})^\dagger = \ell_{ji}$ with $\sum_i \ell_{ii} = 0$. The non-vanishing Poisson brackets are

\[
\{x^i, p_j\} = \delta^i_j \quad \text{and} \quad \{\ell_{ij}, \ell_{km}\} = i(\delta_{im}\ell_{kj} - \delta_{kj}\ell_{im}) .
\] (2.2)

The Hamiltonian (2.1) follows directly from the free hermitian matrix model (for details see \[^15\]).

To get the standard Calogero Hamiltonian (1.1) from (2.1) one has to reduce the angular sector of the latter, in two steps. Firstly, one (weakly) imposes the constraints

\[
\ell_{11} \approx \ell_{22} \approx \ldots \approx \ell_{nn} \approx 0 .
\] (2.3)

They commute with the Hamiltonian (2.1) and with each other, hence are of first class. To resolve them one introduces auxiliary complex variables $v_i$ and $\bar{v}_i = (v_i)^\dagger$ obeying the Poisson brackets

\[
\{v_i, \bar{v}_j\} = -i \delta^i_j
\] (2.4)

and realizes the $su(n)$ generators $\ell_{ij}$ as

\[
\hat{\ell}_{ij} = -v_i \bar{v}_j + \frac{1}{n} \delta_{ij} \sum_k v_k \bar{v}_k .
\] (2.5)

Secondly, passing to polar variables $r_i$ and $\phi_i$ defined as

\[
v_i = r_i e^{i\phi_i} \quad \text{and} \quad \bar{v}_i = r_i e^{-i\phi_i} \quad \Rightarrow \quad \{r_i, \phi_j\} = \frac{1}{2r_i} \delta_{ij} ,
\] (2.6)

the constraints (2.3) are resolved by putting

\[
r_1 \approx r_2 \approx \ldots \approx r_n .
\] (2.7)

Plugging this solution into the Hamiltonian (2.1) one may additionally fix $n-1$ angles $\phi_i$, say

\[
\phi_1 \approx \phi_2 \approx \ldots \approx \phi_{n-1} \approx 0 .
\] (2.8)

At this stage the $2n$ variables $\{r_i, \phi_i\}$ are reduced to the two variables $r_n$ and $\phi_n$. However, the reduced Hamiltonian does not depend on $\phi_n$ and has the form

\[
H_{\text{red}} = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i \neq j}^{n} \frac{r_n^4}{(x_i - x_j)^2} .
\] (2.9)

Therefore

\[
\{H_{\text{red}}, r_n\} \approx 0 \quad \text{and} \quad r_n^2 \approx \text{const} =: g ,
\] (2.10)

and the reduced Hamiltonian $H_{\text{red}}$ coincides with the standard $n$-particle rational Calogero Hamiltonian. We note that in the bosonic case most reduction steps are not needed, because the Hamiltonian (2.1) does not depend on the angles $\phi_i$ at all. However, in the supersymmetric case all reduction steps will be important.

In what follows we will construct an $N$-extended supersymmetric generalization of the Hamiltonian (2.1) and perform the supersymmetric version of the reduction just discussed, finishing with an $N$-extended supersymmetric Calogero model, for $N = 2M$ and $M = 1, 2, 3, \ldots$.
2.2 $\mathcal{N}$-extended supersymmetric $su(n)$ spin-Calogero model

On the outset we have to clarify what is the minimal number of fermionic variables necessary to realize an $\mathcal{N} = 2M$ supersymmetric extension of the $su(n)$ spin-Calogero model (2.1). Clearly, as partners to the bosonic coordinates $x^i$ one needs $\mathcal{N}n$ fermions $\psi^a_i$ and $\bar{\psi}_i^a$ with $a = 1, 2, \ldots, M$. However, this is not enough to construct $\mathcal{N}$ supercharges $Q^a$ and $\bar{Q}_b$ which must generate the $\mathcal{N}=2M$ superalgebra

$$\{Q^a, \bar{Q}_b\} = -2i \delta^a_b \ H \quad \text{and} \quad \{Q^a, Q^b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0 .$$

The reason is simple: to generate the potential term $\sum_{i \neq j}^n \ell_{ij} (\psi^a_i - \psi^a_j)^2 / (x^i - x^j)^2$ in the Hamiltonian, the supercharges $Q^a$ and $\bar{Q}_b$ must contain the terms

$$i \sum_{i \neq j}^n \frac{\ell_{ij} \rho_{ij}^a}{x^i - x^j} \quad \text{and} \quad -i \sum_{i \neq j}^n \frac{\ell_{ij} \bar{\rho}_{ij}^a}{x^i - x^j} ,$$

respectively, where $\rho_{ij}^a$ and $\bar{\rho}_{ij}^a$ are some additional fermionic variables. These fermions cannot be constructed from $\psi^a_i$ or $\bar{\psi}_i^a$. Hence, we are forced to introduce $\mathcal{N}n(n-1)$ further independent fermions $\rho_{ij}^a$ and $\bar{\rho}_{ij}^a$ subject to $\rho_{ii}^a = \bar{\rho}_{ii}^a = 0$ for each value of the index $i$. In total, we thus utilize $\mathcal{N}n^2$ fermions of type $\psi$ or $\rho$, which we demand to obey the following Poisson brackets,

$$\{\psi_i^a, \bar{\psi}_j^b\} = -i \delta^a_b \delta_{ij} , \quad \{\rho_{ij}^a, \bar{\rho}_{km}^b\} = -i \delta^a_b \delta_{im} \delta_{jk} , \quad \text{with} \quad \{\rho_{ij}^a\} = \bar{\rho}_{ij}^a \quad \text{and} \quad \rho_{ii}^a = \bar{\rho}_{ii}^a = 0 .$$

The next important ingredient of our construction is the composite object

$$\Pi_{ij} = \sum_{a=1}^M \left[ (\psi_i^a - \psi_j^a) \bar{\rho}_{ij}^a + (\bar{\psi}_i^a - \bar{\psi}_j^a) \rho_{ij}^a + \sum_k \left( \rho_{ik}^a \bar{\rho}_{kj}^a + \bar{\rho}_{ik}^a \rho_{kj}^a \right) \right] \quad \Rightarrow \quad \{\Pi_{ij}\} = \Pi_{ij} .$$

One may check that, with respect to the brackets (2.13), the $\Pi_{ij}$ form an $su(n)$ algebra just like the $\ell_{ij}$,

$$\{\Pi_{ij}, \Pi_{km}\} = i (\delta_{im} \Pi_{kj} - \delta_{kj} \Pi_{im}) ,$$

and commute with the our fermions as follows,

$$\{\Pi_{ij}, \psi_k^a\} = i (\delta_{jk} - \delta_{kj}) \rho_{ij}^a , \quad \{\Pi_{ij}, \rho_{km}^a\} = -i \delta_{im} \delta_{jk} (\psi_i^a - \psi_j^a) - i \delta_{kj} \rho_{im}^a + i \delta_{im} \rho_{jk}^a ,$$

$$\{\Pi_{ij}, \bar{\psi}_k^a\} = i (\delta_{jk} - \delta_{kj}) \bar{\rho}_{ij}^a , \quad \{\Pi_{ij}, \bar{\rho}_{km}^a\} = -i \delta_{im} \delta_{jk} (\bar{\psi}_i^a - \bar{\psi}_j^a) - i \delta_{kj} \bar{\rho}_{im}^a + i \delta_{im} \bar{\rho}_{jk}^a .$$

It is a matter of straightforward calculation to check that the supercharges

$$Q^a = \sum_{i=1}^n p_i \psi_i^a + i \sum_{i \neq j}^n \frac{\ell_{ij} + \Pi_{ij}}{x^i - x^j} \quad \text{and} \quad \bar{Q}_b = \sum_{i=1}^n p_i \bar{\psi}_i^b - i \sum_{i \neq j}^n \bar{\rho}_{ij}^b \frac{\ell_{ji} + \Pi_{ji}}{x^i - x^j}$$

obey the $\mathcal{N}=2M$ superalgebra (2.11) with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{\ell_{ij} + \Pi_{ij}}{(x^i - x^j)^2} ,$$

modulo the first-class constraints

$$\chi_i := \ell_{ii} + \Pi_{ii} \approx 0 \quad \forall \ i ,$$

with

$$\{Q^a, \chi_i\} \approx \{\bar{Q}_b, \chi_i\} \approx \{H, \chi_i\} \approx \{\chi_i, \chi_j\} \approx 0 .$$

The supercharges $Q^a$ and $\bar{Q}_b$ in (2.17) and the Hamiltonian $H$ in (2.18) describe the $\mathcal{N}=2M$ supersymmetric $su(n)$ spin-Calogero model.

For $\mathcal{N}=4$ it essentially coincides with the $osp(4|2)$ supersymmetric mechanics constructed in [16] [17]. However, there are a few differences:

- The Hamiltonian (2.18) has no interaction for the center-of-mass coordinate $X = \sum_i x^i$. Correspondingly, the supercharges (2.17) do not include certain terms which appeared in [16] [17].
- Working at the Hamiltonian level, we may keep the $su(n)$ generators $\ell_{ij}$ unspecified. Precisely this enables the minimal realization (2.5) with a minimal number of auxiliary variables $\psi_i, \bar{\psi}_i$. At the Lagrangian level this corresponds to using (2, 4, 2) supermultiplets for the auxiliary bosonic superfields instead of (4, 4, 0) superfields as in [16] [17].

Now we are ready to reduce our $\mathcal{N}=2M$ $su(n)$ spin-Calogero model to a genuine $\mathcal{N}=2M$ Calogero model.

3
2.3 $\mathcal{N}$-extended supersymmetric (no-spin) Calogero models

As we can see from the previous subsection, the supersymmetric analogs (2.19) of the purely bosonic constraints (2.23) appear automatically. These constraints generate $n-1$ local $U(1)$ transformations of the variables $\{v_i, \tilde{v}_i, \rho^{a}_{ij}, \bar{\rho}^{a}_{ji}\}$. In terms of the $2n$ polar variables $r_i$ and $\phi_i$ defined in (2.6), the constraints (2.19) can be easily resolved as

$$r^2_k \approx r^2_n + \Pi_{kk} - \Pi_{nn} \quad \text{for} \quad k = 1, \ldots, n-1. \quad (2.21)$$

After fixing the residual gauge freedom as

$$\phi_1 \approx \phi_2 \approx \cdots \approx \phi_{n-1} \approx 0, \quad (2.22)$$

we obtain the supercharges and Hamiltonian which still obey the $\mathcal{N}=2M$ superalgebra (2.11) and contain only the surviving pair $(r_n, \phi_n)$ of the originally $2n$ “angular” variables. One may check that the supercharges $Q^a$ and $\bar{Q}^a_b$ and the Hamiltonian $H$, with the generators $\ell_{ij}$ replaced by $\bar{\ell}_{ij}$ and with the constraints (2.21) and (2.22) taken into account, perfectly commute with $r^2_n - \Pi_{nn}$. Thus, the final step of the reduction is to impose the constraint

$$r^2_n - \Pi_{nn} \approx \text{const} =: g \quad (2.23)$$

and to fix the remaining $U(1)$ gauge symmetry via

$$\phi_n \approx 0. \quad (2.24)$$

The previous two relations are the supersymmetric analogs of (2.10). We conclude that the full set of the reduction constraints reads

$$r^2_i \approx g + \Pi_{ii} \quad \text{and} \quad \phi_i \approx 0 \quad \text{for} \quad i = 1, \ldots, n. \quad (2.25)$$

With these constraints taken into account, our supercharges $Q^a$ and $\bar{Q}^a_b$ and the Hamiltonian $H$ acquire the form

$$\begin{align*}
\tilde{Q}^a &= \sum_{i=1}^{n} p_i \tilde{v}^a_i - i \sum_{i \neq j} \left( \sqrt{g + \Pi_{ii}} \sqrt{g + \Pi_{jj}} - \Pi_{ij} \right) \tilde{\rho}^a_{ji}, \\
\tilde{Q}^a_b &= \sum_{i=1}^{n} p_i \tilde{v}^a_{ib} + i \sum_{i \neq j} \tilde{\rho}^a_{ib} \left( \sqrt{g + \Pi_{ii}} \sqrt{g + \Pi_{jj}} - \Pi_{ji} \right), \\
\tilde{H} &= \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i \neq j} \left( \sqrt{g + \Pi_{ii}} \sqrt{g + \Pi_{jj}} - \Pi_{ij} \right) \sqrt{g + \Pi_{ii}} \sqrt{g + \Pi_{jj}} - \Pi_{ij} \right) \left( x^i - x^j \right)^2.
\end{align*} \quad (2.26)$$

It is matter of quite lengthy and tedious calculations to check that these supercharges and Hamiltonian form an $\mathcal{N}=2M$ superalgebra (2.11). The main complication arises from the expressions $\sqrt{g + \Pi_{ii}}$ present in the supercharges and the Hamiltonian. Due to the nilpotent nature of $\Pi_{ij}$, the series expansion eventually terminates, but even in the two-particle case with $\mathcal{N}=4$ supersymmetry we encounter a lengthy expression,

$$\sqrt{g + \Pi_{11}} = \sqrt{g} \left( 1 + \frac{1}{2g} \Pi_{11} - \frac{1}{8g^2} \Pi_{11}^2 + \frac{1}{16g^3} \Pi_{11}^3 - \frac{1}{128g^4} \Pi_{11}^4 \right) . \quad (2.27)$$

For $n$ particles the series will end with a term proportional to $(\Pi_{11})^{N(n-1)}$. Clearly, these terms will generate higher-degree monomials in the fermions, both for the supercharges and for the Hamiltonian. We can only speculate that the dread of such complexities impeded an earlier discovery of genuine $\mathcal{N}=4$ Calogero models.

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2 Due to the relation $\sum_{i} x_i = 0$ we have only $n-1$ independent constraints.
2.4 Simplest example: $\mathcal{N}=2$ supersymmetric two-particle Calogero model

For $\mathcal{N}=2$ supersymmetry one has to put $M=1$ in the expressions (2.25) for the supercharges and Hamiltonian. This somewhat reduces their complexity compared to the $\mathcal{N}=4$ case, but the real simplification occurs for two particles. Indeed, for $n=2$ we get

$$\Pi_2 = -\Pi_{11} \quad \text{and} \quad \Pi_1^3 \equiv 0 \quad \Rightarrow \quad \sqrt{g + \Pi_1^1} \sqrt{g - \Pi_1^1} = \left(g - \frac{1}{g} \Pi_1^2\right) \quad \text{for} \quad g \neq 0 . \quad (2.28)$$

Moreover, the term $\Pi_2^1$ is of the maximal possible power in the $\rho$ and $\bar{\rho}$ fermions and, therefore, disappears from the supercharges. Thus, we are left with

$$\hat{Q}^{(2)} = \sum_{i=1}^{2} p_i \psi_i - i \sum_{i \neq j} \frac{\rho_{ij}}{x^i - x^j} \quad \text{and} \quad \hat{Q}^{(2)} = \sum_{i=1}^{2} \bar{p}_i \bar{\psi}_i + i \sum_{i \neq j} \frac{\bar{\rho}_{ij}}{x^i - x^j} , \quad (2.29)$$

which have the standard structure – linear and cubic in the fermions. The Hamiltonian $\hat{H}^{(2)}$ reduces to

$$\hat{H}^{(2)} = \frac{1}{2} \sum_{i=1}^{2} p_i^2 + \frac{1}{2} \frac{g^2 - \Pi_1^2 - g (\Pi_{12} + \Pi_{21}) + \Pi_{12} \Pi_{21}}{(x^1 - x^2)^2} , \quad (2.30)$$

with the explicit expressions

$$\Pi_{11} = \rho_{12} \rho_{21} + \bar{\rho}_{12} \rho_{21} , \quad \Pi_{12} = (\psi_1 - \psi_2) \rho_{12} + (\bar{\psi}_1 - \bar{\psi}_2) \rho_{12} , \quad \Pi_{21} = (\psi_2 - \psi_1) \rho_{21} + (\bar{\psi}_2 - \bar{\psi}_1) \rho_{21} . \quad (2.31)$$

This $\mathcal{N}=2$ supersymmetric two-particle Calogero model has been previously constructed and analyzed in [16] (for details see the review [9]). This demonstrates that our approach perfectly reproduces the unique known $\mathcal{N}=2$ example.

3 Conclusion

We propose a novel $\mathcal{N}$-extended supersymmetric $su(n)$ spin-Calogero model as a direct supersymmetrization of the bosonic $su(n)$ model [12]. In the case of $\mathcal{N}=4$ supersymmetry, our model resembles the one constructed in [16, 17]. However, there are two main differences:

- the center of mass is free
- the $su(n)$ generators are not specified in a particular realization.

Thanks to these features, we were able to generalize the reduction procedure to the no-spin Calogero model from $\mathcal{N}=4$ supersymmetry to any number $\mathcal{N}=2M$ of supersymmetries. This lead to the discovery of a genuine $\mathcal{N}=2M$ supersymmetric rational Calogero model for any number of particles.

Our models belong to same class which was proposed in [16, 17]. Its main features are

- a huge number of fermionic coordinates, namely $\mathcal{N}n^2$ in number rather than the $\mathcal{N}n$ to be expected
- the supercharges and the Hamiltonian contain terms which a fermionic power much larger than three.

Clearly, these features merit a more careful and detailed analysis.

The following further developments come to mind:

- a superspace description of the constructed models, at least for $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetry, presumably with nonlinear chiral supermultiplets
- an extension to the Calogero–Sutherland inverse-sine-square model
- an extension to the Euler–Calogero–Moser system [11] and its reduction to the goldfish system [13], yielding a supersymmetric goldfish model upon reduction, to be compared with recent results from [20].
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