Beta Reduction is Invariant, Indeed
(Long Version)

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Abstract
Slot and van Emde Boas’ weak invariance thesis states that reasonable machines can simulate each other within a polynomially overhead in time. Is l-calculus a reasonable machine? Is there a way to measure the computational complexity of a l-term? This paper presents the first complete positive answer to this long-standing problem. Moreover, our answer is completely machine-independent and based over a standard notion in the theory of l-calculus: the length of a leftmost-outermost derivation to normal form is an invariant cost model. Such a theorem cannot be proved by directly relating l-calculus with Turing machines or random access machines, because of the size explosion problem: there are terms that in a linear number of steps produce an exponentially long output. The first step towards the solution is to shift to a notion of evaluation for which the length and the size of the output are linearly related. This is done by adopting the linear substitution calculus (LSC), a calculus of explicit substitutions modelled after linear logic proof nets and admitting a decomposition of leftmost-outermost derivations with the desired property. Thus, the LSC is invariant with respect to, say, random access machines. The second step is to show that LSC is invariant with respect to the l-calculus. The size explosion problem seems to imply that this is not possible: having the same notions of normal form, evaluation in the LSC is exponentially longer than in the l-calculus. We solve such an impasse by introducing a new form of shared normal form and shared reduction, deemed useful. Useful evaluation avoids those steps that only unshare the output without contributing to \(\beta\)-redexes, \textit{i.e.} the steps that cause the blow-up in size. The main technical contribution of the paper is indeed the definition of useful reductions and the thorough analysis of their properties.

1 Introduction

Theoretical computer science is built around algorithms, computational models, and machines: an algorithm describes a solution to a problem with respect to a fixed computational model, whose role is to provide a handy abstraction of concrete machines. The choice of the model reflects a tension between different needs. For complexity analysis, one expects a neat relationship between the model’s primitives and the way in which they are effectively implemented. In this respect, random access machines are often taken as the reference model, since their definition closely reflects the von Neumann architecture. The specification of algorithms unfortunately lies at the other end of the spectrum, as one would like them to be as machine-independent as possible. In this case the typical model is provided by programming languages. Functional programming languages, thanks to their higher-order nature, provide very concise and abstract specifications. Their strength is of course also their weakness: the abstraction from physical machines is pushed to a level where it is no longer clear how to measure the complexity of an algorithm. Is there a way in which such a tension can be solved?

The tools for stating the question formally are provided by complexity theory and by Slot and van Emde Boas’ invariance thesis [SvEB84].

\textit{Reasonable} computational models simulate each other
with polynomially bounded overhead in time, and constant factor overhead in space.

The weak invariance thesis is the variant where the requirement about space is dropped, and it is the one we will actually work with in this paper. The idea behind the thesis is that for reasonable models the definition of every polynomial or super-polynomial class such as \( P \) or \( \text{EXP} \) does not rely on the chosen model. On the other hand, it is well-known that sub-polynomial classes depend very much on the model, and thus it does not make much sense to pursue a linear rather than polynomial relationship.

A first refinement of our question then is: are functional languages invariant with respect to standard models like random access machines or Turing machines? Such an invariance has to be proved via an appropriate measure of time complexity for programs, i.e. a cost model.

The natural answer is to consider the unitary cost model, i.e. take the number of evaluation steps as the cost of the underlying term. However, this is not well-defined. The evaluation of functional programs, indeed, depends very much on the evaluation strategy chosen to implement the language, as the \( \lambda \)-calculus — the reference model for functional languages — is so machine-independent that it does not even come with a deterministic evaluation strategy. And which strategy, if any, gives us the most natural, or canonical cost model (whatever that means)? These questions have received some attention in the last decades. The number of optimal parallel \( \beta \)-steps (in the sense of Lévy [Lév78]) to normal form has been shown not to be a reasonable cost model: there exists a family of terms which reduces in a polynomial number of parallel \( \beta \)-steps, but whose complexity is non-elementary [LM96, AM98]. If one considers the number of sequential \( \beta \)-steps (in a given strategy, for a given notion of reduction), the literature offers some partial positive results, all relying on the use of sharing (see below for more details). Some quite general results [DLM12, AM10] have been obtained through graph rewriting, itself a form of sharing, when only first order symbols are considered.

Sharing is indeed a key ingredient, for one of the issues here is due to the representation of terms. The ordinary way of representing terms indeed suffers from the size explosion problem: even for the most restrictive notions of reduction (e.g. Plotkin’s weak reduction), there is a family of terms \( \{t_n\}_{n \in \mathbb{N}} \) such that \( |t_n| \) is linear in \( n \), \( t_n \) evaluates to its normal form in \( n \) steps, but at \( i \)-th step a term of size \( 2^i \) is copied, so that the size of the normal form of \( t_n \) is exponential in \( n \). Put differently, an evaluation sequence of linear length can possibly produce an output of exponential size. At first sight, then, there is no hope that evaluation lengths may provide an invariant cost model. The idea is that such an impasse can be avoided by sharing common subterms along the evaluation process, in order to keep the representation of the output compact. But is appropriately managed sharing enough? The literature offers some positive, but partial, answers to this question. The number of steps is indeed known to be an invariant cost model for weak reduction [DLM08, DLM12] and for head reduction [ADL12].

If the problem at hand consists in computing the normal form of an arbitrary \( \lambda \)-term, however, no positive answer is known. We believe that not knowing whether the \( \lambda \)-calculus in its full generality is a reasonable machine is embarrassing for the \( \lambda \)-calculus community. In addition, this problem is relevant in practice: proof assistants often need to check whether two terms are convertible, itself a problem that can be reduced to the one under consideration.

In this paper, we give a positive answer to the question above, by showing that leftmost-outermost (LO, for short) reduction to normal form indeed induces an invariant cost model. Such an evaluation strategy is standard, in the sense of the standardisation theorem, one of the central theorems in the theory of \( \lambda \)-calculus, first proved by Curry and Feys [CF58]. The relevance of our cost model is given by the fact that LO reduction is an abstract concept from rewriting theory which at first sight is totally unrelated to complexity analysis. In particular, our cost model is completely machine-independent.

Another view on this problem comes in fact from rewriting theory itself. It is common practice to specify the operational semantics of a language via a rewriting system, whose rules always employ some form of substitution, or at least of copying, of subterms. Unfortunately, this practice is very far away from the way languages are implemented. Indeed, actual interpreters perform
copying in a very controlled way (see, e.g., [Wad71, PJ87]). This discrepancy induces serious doubts about the relevance of the computational model. Is there any theoretical justification for copy-based models, or more generally for rewriting theory as a modelling tool? In this paper we give a very precise answer, formulated within rewriting theory itself.

As in our previous work [ADL12], we prove our result by means of the linear substitution calculus (see also [Acc12, ABKL14]), a simple calculus of explicit substitutions (ES, for short) arising from linear logic and graphical syntaxes and similar to calculi studied by De Bruijn [DB77], Nederpelt [Ned92], and Milner [Mil07]. A peculiar feature of the linear substitution calculus (LSC) is the use of rewriting rules at a distance, i.e. rules defined by means of contexts, that are used to closely mimic reduction in linear logic proof nets. Such a framework — whose use does not require any knowledge of these areas — allows an easy management of sharing and, in contrast to previous approaches to ES, admits a theory of standardisation and a notion of LO evaluation [ABKL14]. The proof of our result indeed is a tour de force based on a fine quantitative study of the relationship between LO derivations for the $l$-calculus and a variation over LO derivations for the LSC. Roughly, the latter avoids the size explosion problem while keeping a polynomial relationship with the former.

Let us point out that invariance results usually have two directions, while we here study only one of them (namely that the $l$-calculus can be efficiently simulated by, say, Turing machines). The missing half is a much simpler problem already solved in [ADL12]: there is an encoding of Turing machines into $l$-terms s.t. their execution is simulated by weak head $\beta$-reduction with only a linear overhead.

On Invariance and Complexity Analysis. Before proceeding, let us stress some crucial points:

1. **ES Are Only a Tool.** Although ES are an essential tool for the proof of our result, the result itself is about the usual, pure, $\lambda$-calculus. In particular, the invariance result can be used without any need to care about ES: we are allowed to measure the complexity of problems by simply bounding the number of LO $\beta$-steps taken by any $\lambda$-term solving the problem.

2. **Complexity Classes in the $l$-Calculus.** The main consequence of our invariance result is that every polynomial or super-polynomial class, like $P$ of $EXP$, can be defined using $l$-calculus (and LO $\beta$-reduction) instead of Turing machines.

3. **Our Cost Model is Unitary.** An important point is that our cost model is unitary, and thus attributes a constant cost to any LO step. One could argue that it is always possible to reduce $\lambda$-terms on abstract or concrete machines and take that number of steps as the cost model. First, such a measure of complexity would be very machine-dependent, against the very essence of $l$-calculus. Second, these cost models invariably attribute a more-than-constant cost to any $\beta$-step, making the measure much harder to use and analyse. It is not evident that a computational model enjoys a unitary invariant cost model. As an example, if multiplication is a primitive operation, random access machines need to be endowed with a logarithmic cost model in order to obtain invariance.

The next section explains why the problem at hand is hard, and in particular why iterating our previous results on head reduction [ADL12] does not provide a solution.

## 2 Why is The Problem Hard?

In principle, one may wonder why sharing is needed at all, or whether a relatively simple form of sharing suffices. In this section, we will show that sharing is unavoidable and that a new subtle notion of sharing is necessary.

If we stick to explicit representations of terms, in which sharing is not allowed, counterexamples to invariance can be designed in a fairly easy way. Let $u$ be the lambda term $yx$ and consider the sequence $\{t_n\}_{n\in \mathbb{N}}$ of $\lambda$-terms defined as $t_0 = u$ and $t_{n+1} = (\lambda x.t_n)ju$ for every $n \in \mathbb{N}$. The term $t_n$ has size linear in $n$, and $t_n$ rewrites to its normal form $r_n$ in exactly $n$ steps, following the LO
reduction order; as an example:

\[
\begin{align*}
t_0 &= u = r_0; \\
t_1 &\rightarrow yuu = yr_0r_0 = r_1; \\
t_2 &\rightarrow (\lambda x.t_0)(yuu) = (\lambda x.u) r_1 \rightarrow yr_1r_1 = r_2.
\end{align*}
\]

For every \( n \), however, \( r_{n+1} \) contains two copies of \( r_n \), hence the size of \( r_n \) is exponential in \( n \). As a consequence, the unitary cost model is not invariant: in a linear number of \( \beta \)-steps we reach an object which cannot even be written down in polynomial time.

The solution the authors proposed in [ADL12] is based on ES, and allows to tame the size explosion problem in a satisfactory way when head reduction suffices. In particular, the head steps above become the following linear head steps:

\[
\begin{align*}
t_0 &= u = p_0; \\
t_1 &\rightarrow (yx)x[x\leftarrow u] = u[x\leftarrow u] = p_1; \\
t_2 &\rightarrow ((\lambda x.t_0)u)[x\leftarrow u] = ((\lambda x.u)[x\leftarrow u] \\
&\rightarrow u[x\leftarrow u][x\leftarrow u] = p_2.
\end{align*}
\]

As one can easily verify, the size of \( p_n \) is linear in \( n \). More generally, linear head reduction (LHR) has the subterm property, i.e. it only duplicates subterms of the initial term. This fact implies that the size of the result and the length of the derivation are linearly related. In other words, the size explosion problem has been solved. Of course one needs to show that 1) the compact results unfold to the expected result (that may be exponentially bigger), and 2) that compact representations can be managed efficiently (typically they can be tested for equality in time polynomial in the size of the compact representation), see [ADL12] or below for more details.

It may seem that one is then forced to use ES to measure complexity. In [ADL12] we also showed that LHR is at most quadratically longer than head reduction, so that the polynomial invariance of LHR lifts to head reduction. This is how we exploit sharing to circumvent the size explosion problem: we are allowed to take the length of the head derivation as a cost model, even if it suffers of the size explosion problem, because the actual implementation is meant to be done via LHR and be only polynomially (actually quadratically) longer.

There is a natural candidate for extending the approach to reduction to normal form: just iterate the (linear) head strategy on the arguments, obtaining the (linear) LO strategy, that does compute normal forms [ABKL14]. As we will show, for linear LO derivations the subterm property holds. The size of the output is still under control, being linearly related to the length of the LO derivation. Unfortunately, when computing normal forms this is not enough.

One of the key points in our previous work was that there is a notion of linear head normal form that is a compact representation for head normal forms. The generalisation of such an approach to normal forms has to face a fundamental problem: what is a linear normal form? Indeed, terms with and without ES share the same notion of normal form. Consider again the family of terms \( \{t_n\}_{n\in\mathbb{N}} \): if we go on and unfold all substitutions in \( p_n \), we end up in \( r_n \). Thus, by the subterm property, the linear LO strategy takes an exponential number of steps, and so it cannot be polynomially related to the LO strategy.

Summing up, we need a strategy that 1) implements the LO strategy, 2) has the subterm property and 3) never performs useless substitution steps, i.e. those steps whose role is simply to explicit the normal form, without contributing in any way to \( \beta \)-redexes. The main contribution of this work is the definition of such a linear useful strategy, and the proof that it is indeed polynomially related to both the LO strategy and a concrete implementation model.

This is not a trivial task, actually. One may think that it is enough to evaluate a term \( t \) in a LO way, stopping as soon as the unfolding \( u \) of the current term \( u \) — the term obtained by expanding the ES of \( u \) — is a \( \beta \)-normal form. Unfortunately, this simple approach does not work, because the exponential blow-up may be caused by ES lying between two \( \beta \)-redexes, so that proceeding in a LO way would unfold the problematic substitutions anyway.
Our notion of useful step will elaborate on this idea, by computing partial unfoldings, to check if a substitution step contributes or will contribute to some future $\beta$-redex. Of course, we will have to show that such tests can be themselves performed in polynomial time, and that the notion of LO useful reduction retains all the good properties of LO reduction.

3 The Calculus

We assume familiarity with the l-calculus (see [Bar84]). The language of the linear substitution calculus (LSC for short) is given by the following grammar for terms:

$$t, u, r, p ::= x \mid lx.t \mid tu \mid t[x\leftarrow u].$$

The constructor $t[x\leftarrow u]$ is called an explicit substitution (of $u$ for $x$ in $t$, the usual (implicit) substitution is instead noted $t[x\leftarrow u]$). Both $lx.t$ and $t[x\leftarrow u]$ bind $x$ in $t$, and we silently work modulo $\alpha$-equivalence of these bound variables, e.g. $(xy)[y\leftarrow t][x\leftarrow y] = (yz)[z\leftarrow t]$. We use $fv(t)$ for the set of free variables of $t$.

Contexts. The operational semantics of the LSC is parametric in a notion of (one-hole) context. General contexts are defined by:

$$C ::= \langle \cdot \rangle \mid lx.C \mid Ct \mid C[x\leftarrow t] \mid t[x\leftarrow C],$$

and the plugging of a term $t$ into a context $C$ is defined as $\langle \cdot \rangle(t) := t$, $(lx.C)(t) := lx.(C(t))$, and so on. As usual, plugging in a context can capture variables, e.g. $((\langle \cdot \rangle)[y\leftarrow t])(y) = (yy)[y\leftarrow t]$. The plugging $C(D)$ of a context $D$ into a context $C$ is defined analogously.

Along most of the paper, however, we will not need such a general notion of context. In fact, our study takes a simpler form if the operational semantics is defined with respect to shallow contexts, defined as (note the absence of the production $t[x\leftarrow S]$):

$$S, P, T, V ::= \langle \cdot \rangle \mid lx.S \mid St \mid tS \mid S[x\leftarrow t].$$

In the following, whenever we refer to a context without further specification it is implicitly assumed that it is a shallow context. A special class of contexts is that of substitution contexts:

$$L ::= \langle \cdot \rangle \mid L[x\leftarrow t].$$

Operational Semantics. The (shallow) rewriting rules $\rightarrow_{dB}$ ($dB = \beta$ at a distance) and $\rightarrow_{1s}$ (linear substitution) are given by the closure by (shallow) contexts of the following rules:

$$L(lx.t)u \rightarrow_{dB} L(t[x\leftarrow u]);$$
$$S(x)[x\leftarrow u] \rightarrow_{1s} S(u)[x\leftarrow u].$$

The union of $\rightarrow_{dB}$ and $\rightarrow_{1s}$ is simply noted $\rightarrow$. The rewriting rules are assumed to use on-the-fly $\alpha$-equivalence to avoid variable capture. For instance,

$$(lx.t)[y\leftarrow u]y \rightarrow_{dB} t[y\leftarrow z][x\leftarrow y][z\leftarrow u] \quad \text{for } z \notin fv(t);$$
$$(ly.(xy))[x\leftarrow y] \rightarrow_{1s} (lz.(yz))[x\leftarrow y].$$

Moreover, in rule $1s$ the context $S$ is assumed to not capture $x$, so that $(lx.x)[x\leftarrow y] \not\rightarrow_{1s} (lx.y)[x\leftarrow y]$.

The just defined shallow fragment simply ignores garbage collection (that in the LSC can always be postponed [Acc12]) and lacks some of the nice properties of the LSC (obtained simply by replacing shallow contexts by general contexts). Its relevance is the fact that it is the smallest fragment implementing linear LO reduction. The following are examples of shallow steps:

$$(lx.x)y \rightarrow_{dB} x[x\leftarrow y];$$
$$(xx)[x\leftarrow t] \rightarrow_{1s} (xt)[x\leftarrow t];$$
while the following steps are not
\[ t[z\leftarrow (lx.x)y] \rightarrow_{\text{db}} t[z\leftarrow x[x\leftarrow y]]; \]
\[ x[x\leftarrow y][y\leftarrow t] \rightarrow_{\text{ls}} x[x\leftarrow t][y\leftarrow t]. \]

Taking the external context into account, a substitution step has the following explicit form:
\[ P(S(t)[x\leftarrow u]) \rightarrow_{\text{ls}} P(S(u)[x\leftarrow u]). \] We shall often use a compact form, writing \( T\langle x \rangle \rightarrow_{\text{ls}} T\langle u \rangle \) where it is implicitly assumed that \( T = P(S[x\leftarrow u]) \). We use \( R \) and \( Q \) as metavariables for redexes. A derivation \( \rho : t \rightarrow^k u \) is a finite sequence of reduction steps, sometimes given as \( R_1; \ldots; R_k \), i.e. as the sequence of reduced redexes. We write \(|t|\) for the size of \( t \), \(|t|\) for the number of substitutions in \( t \), \(|\rho|\) for the length of \( \rho \), and \(|\rho|_{\text{db}}\) for the number of \( \text{db} \)-steps in \( \rho \).

(Relative) Unfoldings. The unfolding \( t\downarrow \) of a term \( t \) is the \( l \)-term obtained from \( t \) by turning its explicit substitutions into implicit ones:
\[
\begin{align*}
x\downarrow & := x; \\
(lx.t)\downarrow & := lx.t\downarrow; \\
(tu)\downarrow & := t\downarrow u\downarrow; \\
(t[x\leftarrow u])\downarrow & := t[x\leftarrow u]\downarrow.
\end{align*}
\]

We will also need a more general notion, the unfolding \( t\downarrow_S \) of \( t \) in a context \( S \):
\[
\begin{align*}
t_{\downarrow \langle \rangle} & := t\downarrow; \\
t_{\downarrow S} & := t\downarrow_S; \\
t_{\downarrow S[x\leftarrow u]} & := t\downarrow_S[x\leftarrow u]\downarrow; \\
t_{\downarrow Su} & := t\downarrow_S.
\end{align*}
\]
For instance,
\[
\begin{align*}
(xyz)[y\leftarrow x][x\leftarrow z]\downarrow & = z(zz); \\
(xyz)[y\leftarrow x][x\leftarrow z]\downarrow & = (lz)(zz)lz.(zz).
\end{align*}
\]

We extend implicit substitutions and unfoldings to contexts by setting \( \langle \rangle_{\gamma} t := \langle \rangle \) and \( \langle \rangle_{\downarrow} := \langle \rangle \) (all other cases are defined as expected, e.g. \( S[x\leftarrow t]\downarrow := S\downarrow_{\gamma} \)). We also write \( S \prec_{\rho} t \) if there is a term \( u \) s.t. \( S\langle u \rangle = t \), call it the prefix relation. We have the following properties, that only hold because our contexts are shallow (implying that the hole cannot be duplicated during the unfolding).

**Lemma 3.1.** Let \( S \) be a shallow context. Then:
\begin{enumerate}
\item \( S\downarrow \) is a shallow context;
\item \( S(t)[x\leftarrow u] = S(x\leftarrow u)\langle t[x\leftarrow u] \rangle \); 
\item \( S(t)\downarrow = S\downarrow_{\gamma} \langle t \rangle_S \), in particular if \( S \prec_{\rho} t \) then \( S\downarrow \prec_{\rho} t\downarrow \).
\end{enumerate}

**Proof.** All points are by induction on \( S \). In particular, Point 3 uses Point 2 when \( S = P[x\leftarrow u]. \) □

Given a derivation \( \rho : t \rightarrow^* u \) in the LSC, we often consider the \( \beta \)-derivation \( \rho\downarrow : t\downarrow \rightarrow^*_\beta u\downarrow \) obtained by projecting \( \rho \) via unfolding.

**Reduction Combinatorics.** Given any calculus, a deterministic strategy \( \rightarrow \) for it, and a term \( t \), the expression \( \#_{\rightarrow}(t) \) stands for the number of reduction steps necessary to reach the normal form of \( t \) along \( \rightarrow \), or \( \infty \) if \( t \) diverges. Similarly, given a natural number \( n \), the expression \( \rightarrow^n(t) \) stands for the term \( u \) such that \( t \rightarrow^n u \), if \( n \leq \#_{\rightarrow}(t) \), or for the normal form of \( t \) otherwise.

## 4 The Proof, Made Abstract

Our proof method can be described abstractly. Such an approach both clarifies the structure of the proof and prepares the ground for possible generalisations to, e.g., the call-by-value \( l \)-calculus or calculi with additional features as pattern matching or control operators. We want to show that a certain strategy \( \rightarrow \) for the \( l \)-calculus provides a unitary and invariant cost model, i.e. that the number of \( \rightarrow \) steps is a measure polynomially related to the number of transitions on Turing
We are looking for an appropriate strategy $\leadsto_X$ within the LSC which is invariant with respect to both $\leadsto$ and Turing machines. Then we need two theorems, which together form the main result of the paper:

1. **High-Level Implementation**: $\leadsto$ terminates iff $\leadsto_X$ terminates. Moreover, $\leadsto$ is implemented by $\leadsto_X$ with only a polynomial overhead. Namely, $t \leadsto_X^k u$ iff $t \leadsto^h u_\downarrow$ with $k$ polynomial in $h$ (our actual bound will be quadratic);

2. **Low-Level Implementation**: $\leadsto_X$ is implemented on Turing machines with an overhead in time which is polynomial in both $k$ and the size of $t$.

The high-level part relies on the following notion.

**Definition 4.1.** Let $\leadsto$ be a deterministic strategy on $l$-terms and $\leadsto_X$ a strategy of the LSC. The pair $(\leadsto, \leadsto_X)$ is a **high-level implementation system** if whenever $t$ is an $l$-term and $\rho : t \leadsto_X^* u_\downarrow$ then:

1. **Normal Form**: if $u$ is an $\leadsto_X$-normal form then $u_\downarrow$ is an $\leadsto$-normal form.
2. **Projection**: $\rho|_u : t \leadsto^* u_\downarrow$ and $|\rho| = |\rho|_{dB}$.
3. **Trace**: the number $|u|_1$ of ES in $u$ is exactly the number $|\rho|_{dB}$ of $dB$-steps in $\rho$.
4. **Syntactic Bound**: the length of a sequence of substitution steps from $u$ is bounded by $|u|_1$.

Concretely, the high-level implementation system at work in the paper will take as $\leadsto$ the LO strategy of the $l$-calculus and as $\leadsto_X$ a variant of the linear LO strategy for the LSC. A variant is required because, as we will explain, the linear LO strategy of the LSC does not satisfy the syntactic bound property.

The normal form and projection properties address the qualitative part of the high-level implementation theorem, i.e. the part about termination. The normal form property guarantees that $\leadsto_X$ does not stop prematurely, so that when $\leadsto_X$ terminates $\leadsto$ cannot keep going. The projection property guarantees that termination of $\leadsto$ implies termination of $\leadsto_X$. The two properties actually state a stronger fact: $\leadsto$ steps can be identified with the $dB$-steps of the $\leadsto_X$ strategy.

The trace and syntactic bound properties are instead used for the quantitative part of the theorem, i.e. to provide the polynomial bound. The two properties together provide a bound on the number of $l$-$\Sigma$-steps in a $\leadsto_X$ derivation with respect to the number of $dB$-steps, that—by the identification of $\beta$ and $dB$ redexes—is exactly the length of the associated $\leadsto$ derivation.

The high-level part can now be proved abstractly.

**Theorem 4.2 (High-Level Implementation).** Let $t$ be an ordinary $l$-term and $(\leadsto, \leadsto_X)$ a high-level implementation system. Then:

1. $t$ is $\leadsto$-normalising iff it is $\leadsto_X$-normalising.
2. If $\rho : t \leadsto_X^* u_\downarrow$ then $\rho|_u : t \leadsto^* u_\downarrow$ and $|\rho| = O(|\rho|_{dB}^2)$.

**Proof.** 1. $\Rightarrow$ Suppose that $t$ is $\leadsto_X$-normalisable and let $\rho : t \leadsto_X^* u_\downarrow$ a derivation to $\leadsto_X$-normal form. By the projection property there is a derivation $t \leadsto^* u_\downarrow$. By the normal form property $u_\downarrow$ is a $\leadsto$-normal form.

$\Leftarrow$ Suppose that $t$ is $\leadsto$-normalisable and let $\tau : t \leadsto^k u$ be the derivation to $\leadsto$-normal form (unique by determinism of $\leadsto$). Assume, by contradiction, that $t$ is not $\leadsto_X$-normalisable. Then there is a family of $\leadsto_X$-derivations $\rho_i : t \leadsto_X^* u_i$ with $i \in \mathbb{N}$, each one extending the previous one. By the syntactic bound property, $\leadsto_X$ can make only a finite number of $l$-$\Sigma$ steps (more generally, $\leadsto_X$ is strongly normalising in the LSC). Then the sequence $\{|\rho_i|_{dB}\}_{i \in \mathbb{N}}$ is non-decreasing and unbounded. By the projection property, the family $\{\rho_i\}_{i \in \mathbb{N}}$ unfolds to a family of $\leadsto$-derivations $\{\rho_i\}_{i \in \mathbb{N}}$ of unbounded length (in particular greater than $k$), absurd.

2. By the projection property, it follows that $\rho|_u : t \leadsto^* u_\downarrow$. Moreover, to show $|\rho| = O(|\rho|_{dB}^2)$ it is enough to show $|\rho| = O(|\rho|_{dB}^2)$. Now, $\rho$ has the shape:

$$t = r_1 \rightarrow_{\rho_1} p_1 \rightarrow_{\rho_2} r_2 \rightarrow_{\rho_3} p_2 \rightarrow_{\rho_4} \ldots r_k \rightarrow_{\rho_k} p_k \rightarrow_{\rho_{k+1}} u.$$ 

By the syntactic bound property, we obtain $b_i \leq |\rho_i|_1$. By the trace property we obtain $|\rho_i|_1 = \sum_{j=1}^i a_j$, and so $b_i \leq \sum_{j=1}^i a_j$. Then:
By the subterm property, implementing one step takes time polynomial (if not linear) in \(|u|\).

Proof. By the subterm property, implementing one step takes time polynomial (if not linear) in \(|u|\). An immediate consequence of the subterm property is the no size explosion property, i.e. that \(|u| \leq (k + 1) \cdot |t|\). By the selection property selecting the next redex takes time polynomial in \(|u|\), that by the no size explosion property is polynomial in \(k\) and \(|t|\). The composition of polynomials is again a polynomial, and so selecting the redex takes time polynomial in \(k\) and \(|t|\). Hence, the reduction can be implemented in polynomial time.

In [ADL12], we proved that head reduction and linear head reduction form a high-level implementation system and that linear head reduction is mechanisable, even if we did not use such a terminology, nor were we aware of the presented abstract scheme. In order to extend such a result to normal forms we need to replace head reduction with a normalising strategy (i.e. a strategy reaching the \(\beta\)-normal form, if any).

One candidate for \(\leadsto\) is the LO strategy \(\leadsto_{LO}\). Such a choice is natural, as \(\leadsto_{LO}\) is normalising, it produces standard derivations, and it is an iteration of head reduction. What is left to do, then, is to find a strategy \(\leadsto_X\) for ES, which is both mechanisable and a high-level implementation of \(\leadsto_{LO}\). Unfortunately, the linear LO strategy, here noted \(\leadsto_{LO}\) and first defined in [ABKL14], is mechanisable but the pair \((\leadsto_{LO}, \leadsto_{LO})\) is not a high-level implementation system.

In general, mechanisable strategies are not hard to find. As we will show in Sect. 6, the whole class of standard derivations for ES has the subterm property. In particular, the linear strategy \(\leadsto_{LO}\) — which is standard — enjoys all the other properties but for the syntactic bound property.

Such a problem will be solved by LO useful derivations, to be introduced in Sect. 5 that will be shown to be both mechanisable and a high-level implementation of \(\leadsto_{LO}\). Useful derivations avoid those substitution steps that only explicit the normal form without contributing to explicit \(\beta/\beta\)-redexes (that, by the projection property, can be identified). LO useful derivations will have all the nice properties of LO derivations and moreover will stop on shared, minimal representations of normal forms, solving the problem with linear LO derivations.

Let us point out that our analysis would be vacuous without evidence that useful normal forms are a reasonable representation of \(l\)-terms. In other words, we must be sure that ES do not hide (too much of) the inherent difficulty of reducing \(\lambda\)-terms under the carpet of sharing. In [ADL12], we solved this issue by providing an efficient algorithm for checking the equality of any two LSC terms — thus in particular of useful normal forms — without computing their unfoldings (that otherwise would reintroduce an exponential blow-up). Some further discussion can be found in Sect. 10.
5 Useful Derivations

In this section we define a constrained, optimised notion of reduction, that will be the key to the High-Level Implementation Theorem. The idea is that an optimised step takes place only if it somehow contributes to explicit a β/dB-redex. Let an **applicative context** be defined by $A := S(L)$, where $S$ and $L$ are a shallow and a substitution context, respectively (note that applicative contexts are not made out of applications only; for instance $\ell x.(\langle \rangle)[y\leftarrow u]r$ is an applicative context). Then:

**Definition 5.1** (Useful/Useless Steps and Derivations). A **useful step** is either a dB-step or a 1s-step $S(x) \rightarrow_{1s} S(r)$ (in compact form) s.t. $r

1. either contains a β-redex,
2. or is an abstraction and $S$ is an applicative context.

A **useless step** is a 1s-step that is not useful. A **useful derivation** (resp. **useless derivation**) is a derivation whose steps are useful (resp. useless).

Let us give some examples. The steps

\[(tx)[x\leftarrow(l y)u] \rightarrow_{1s} (t((l y)u))[x\leftarrow(l y)u];\]
\[(xt)[x\leftarrow l.y] \rightarrow_{1s} ((l y)t)[x\leftarrow l.y];\]

are useful because they move or create a β/dB-redex (first and second case of the definition, respectively) while

\[(lx.y)[y\leftarrow z z] \rightarrow_{1s} (lx.(z z))[y\leftarrow z z]\]

is useless. However, useful steps are subtler, for instance

\[(tx)[x\leftarrow z z][z\leftarrow l.y] \rightarrow_{1s} (t(z z))[x\leftarrow z z][z\leftarrow l.y]\]

is useful also if it does not move or create β/dB-redexes, because it does so up to relative unfolding, i.e. $(z z)\langle\rangle[\cdot][z\leftarrow y] = (l y)y y$ that is a β/dB-redex.

Note that useful steps concern future creations of β-redexes and yet their definition circumvent the explicit use of residuals, relying on relative unfoldings only.

**Leftmost-Outermost Useful Derivations.** The notion of small-step evaluation that we will use to implement LO β-reduction is the one of LO useful derivation. We need some preliminary definitions.

Let $R$ be a redex. Its **position** is defined as follows:

1. If $R$ is a dB-redex $S(L(lx.t)u) \rightarrow_{dB} S(L(t[x\leftarrow u]))$ then its position is given by the context $S$ surrounding the changing expression; β-redexes are treated as dB-redexes.
2. If $R$ is a 1s-redex, expressed in compact form $S(x) \rightarrow_{1s} S(u)$, then its position is the context $S$ surrounding the variable occurrence to substitute.

The left-to-right outside-in order on redexes is expressed as an order on positions, i.e. contexts. Let us warn the reader about a possible source of confusion. The left-to-right outside-in order in the next definition is sometimes simply called left-to-right (or simply left) order. The former terminology is used when terms are seen as trees (where the left-to-right and the outside-in orders are disjoint), while the latter terminology is used when terms are seen as strings (where the left-to-right is a total order). While the study of standardisation for the LSC [ABKL14] uses the string approach (and thus only talks about the left-to-right order and the leftmost redex), here some of the proofs require a delicate analysis of the relative positions of redexes and so we prefer the more informative tree approach and define the order formally.

**Definition 5.2.** The following definitions are given with respect to general (not necessarily shallow) contexts, even if apart from the next section we will use them only for shallow contexts.

1. The **outside-in order**:
1. Root: ⟨·⟩ ≺₀ C for every context C.$ \neq$ ⟨·⟩;
2. Contextual closure: If C ≺₀ E then E⟨C⟩ ≺₀ E⟨D⟩ for any context E.

Note that ≺₀ can be seen as the prefix relation ≺ₚ on contexts.

3. The left-to-right order: C ≺ₕ D is defined by:
   1. Application: If C ≺ₚ t and D ≺ₚ u then Cu ≺ₜ uD;
   2. Substitution: If C ≺ₚ t and D ≺ₚ u then C[x:=u] ≺ₜ t[x:=D];
   3. Contextual closure: If C ≺ₜ D then E⟨C⟩ ≺ₜ E⟨D⟩ for any context E.

4. The left-to-right outside-in order: C ≺ₜ D if C ≺ₜ D or C ≺ₜ D.

The following are a few examples. For every context C, it holds that ⟨·⟩ $\neq_L C$. Moreover,

\[
(lx.(·))t ≺ₜ (lx.(·)[y:=u])r t;
(r⟨·⟩)t ≺ₜ (rt)(·);
\]

\[
t[x:=·]u ≺ₜ t[x:=r][·].
\]

The next lemma guarantees that we defined a total order.

**Lemma 5.3** (Totality of ≺ₜ). If C ≺ₚ t and D ≺ₚ t then either C ≺ₜ D or D ≺ₜ C or C = D.

**Proof.** By induction on t. In general we can avoid to analyse the cases where C or D is empty, because ⟨·⟩ ≺₀ E (and so ⟨·⟩ ≺ₜ E) for any context E and there is no context E $\neq$ ⟨·⟩ s.t. E ≺ₜ ⟨·⟩. Cases:

1. **Variable.** Both C and D are the empty context, and so C = D.
2. **Abstraction lx. u.** It follows from the i.h.
3. **Application ur.** If both contexts have their hole in u or both in r we conclude by the i.h.
   Otherwise C has its hole in, say, u and D in r, and then C ≺ₜ D, i.e. C ≺ₜ D.
4. **Substitution u[x:=r].** Exactly as the previous case.

The orders above can be extended from contexts to redexes, in the expected way, e.g. for ≺ₜ given two redexes R and Q of positions S and P we write R ≺ₜ Q if S ≺ₜ P. Now, we can define the notions of derivations we are interested in.

**Definition 5.4** (Leftmost-Outermost (Useful) Redex). Let t be a term and R a redex of t. R is the **leftmost-outermost** (resp. **leftmost-outermost useful**, LOU for short) redex of t if R ≺ₜ Q for every other redex (resp. useful redex) Q of t. We write t $\rightarrow_{LO} u$ (resp. t $\rightarrow_{LOU} u$) if a step reduces the LO (resp. LOU) redex.

We need to ensure that LOU derivations are mechanisable and form a high-level implementation system when paired with LO derivations. In particular, we will show:

1. the **subterm** and **trace properties**, by first showing that they hold for every standard derivation, in Sect. 4 and then showing that LOU derivations are standard, in Sect. 5.

2. the **normal form** and **projection properties**, by a careful study of unfoldings and LO/LOU derivations, in Sect. 5.

3. the **syntactic bound property**, passing through the abstract notion of nested derivation, in Sect. 6.

4. the **selection property**, by exhibiting a polynomial algorithm to test whether a redex is useful or not, in Sect. 7.

**6 Standard Derivations**

We need to show that LOU derivations have the subterm property. It could be done directly. However, we will proceed in an abstract way, by first showing that the subterm property is a property of standard derivations for the LSC, and then showing (in Sect. 7) that LOU derivations are standard. The detour has the purpose of shedding a new light on the notion of standard.
derivation, a classic concept in rewriting theory. For the sake of readability, we use the concept of residual without formally defining it (see [ABKL14] for details).

**Definition 6.1 (Standard Derivation).** A derivation $\rho : R_1 ; \ldots ; R_n$ is **standard** if $R_i$ is not the residual of a redex $Q \prec_{LO} R_j$ for every $i \in \{2, \ldots, n\}$ and $j < i$.

The same definition where terms are ordinary $l$-terms gives the ordinary notion of standard derivation.

Note that any single reduction step is standard. Then, notice that standard derivations select redexes in a left-to-right and outside-in way, but they are not necessarily LO. For instance, the derivation

$$(((lx.z)y)[y\leftarrow z] \rightarrow_{1s} ((lx.z)y)[y\leftarrow z] \rightarrow_{1s} ((lx.z)y)[y\leftarrow z]$$

is standard even if the LO redex (i.e. the dB-redex on $x$) is not reduced. The extension of the derivation with $(((lx.z)y)[y\leftarrow z] \rightarrow_{dB} z[x\leftarrow z][y\leftarrow z]$ is not standard. Last, note that the position of a $1s$-step is given by the substituted occurrence and not by the ES, that is $(xy)[x\leftarrow u][y\leftarrow t] \rightarrow_{1s} (xt)[x\leftarrow u][y\leftarrow t] \rightarrow_{1s} (ut)[x\leftarrow u][y\leftarrow t]$ is not standard.

In [ABKL14] it is showed that in the full LSC standard derivations are complete, i.e. that whenever $t \rightarrow^* u$ there is a standard derivation from $t$ to $u$. The shallow fragment does not enjoy such a standardisation theorem, as the residuals of a shallow redex need not be shallow. This fact however does not clash with the technical treatment in this paper. The shallow restriction is indeed compatible with standardisation in the sense that:

1. **The linear LO strategy is shallow:** if the initial term is a $l$-term then every derivation reduced by the linear LO strategy is shallow (every non-shallow redex $R$ is contained in a substitution, and every substitution is involved in an outer redex $Q$);

2. **$\prec_{LO}$-ordered shallow derivations are standard:** any strategy picking shallow redexes in a left-to-right and outside-in fashion does produce standard derivations (it follows from the easy fact that a shallow redex $R$ cannot turn a non-shallow redex $Q$ s.t. $Q \prec_{LO} R$ into a shallow redex). Moreover, the only redex swaps we will consider (Lemma 7.1) will produce shallow residuals.

We are now going to show a fundamental property of standard derivations. The subterm property states that at any point of a derivation $\rho : t \rightarrow^* u$ only sub-terms of the initial term $t$ are duplicated. It immediately implies that any rewriting step can be implemented in time polynomial in the size $|t|$ of $t$. A first consequence is the fact that $|u|$ is linear in the size of the starting term and the number of steps, that we call the no size explosion property.

These properties are based on a technical lemma relying on the notions of box context and box subterm, where a box is the argument of an application or the content of an explicit substitution, corresponding to explicit boxes for promotions in the proof nets representation of $l$-terms with ES.

**Definition 6.2 (Box Context, Box Subterm).** Let $t$ be a term. **Box contexts** (that are not necessarily shallow) are defined by the following grammar, where $C$ is an generic context:

$$B ::= t(\cdot) \mid t[x\leftarrow(\cdot)] \mid C[B].$$

A **box subterm** of $t$ is a term $u$ s.t. $t = B(u)$ for some box context $B$.

We are now ready for the lemma stating the fundamental invariant of standard derivations.

**Lemma 6.3 (Standard Derivations Preserve Boxes on Their Right).** Let $\rho : t_0 \rightarrow^k t_k \rightarrow t_{k+1}$ be a standard derivation and let $S$ be the position of the last contracted redex, $k \geq 0$, and $B \prec_{p} t_{k+1}$ be a box context s.t. $S \prec_{LO} B$. Then the box subterm $u$ identified by $B$ (i.e. s.t. $t_{k+1} = B(u)$) is a box subterm of $t_0$.

**Proof.** By induction on $k$. If $k = 0$ the statement trivially holds. Otherwise, let us call $R_k$ the step $t_k \rightarrow t_{k+1}$ and consider the last contracted redex $R_{k-1} : t_{k-1} \rightarrow t_k$ of position $P_{k-1}$. By i.h. the statement holds wrt box contexts $B \prec_{p} t_k$ s.t. $P_{k-1} \prec_{LO} B$. The proof analyses the position $P_k$ of $R_k$ with respect to the position $P_{k-1}$ of $R_{k-1}$, often distinguishing between the left-to-right $\prec_L$ and the outside-in $\prec_O$ suborders of $\prec_{LO}$. Cases:
1. \( P_k \) is equal to \( P_{k-1} \). Clearly if \( P_{k-1} = P_k \prec_L B \) then the statement holds because of the i.h. (reduction clearly does not affect boxes on the left of the hole of the position). We need to check the box contexts s.t. \( P_k \prec_O B \). Note that \( R_{k-1} \) cannot be a \( \rightarrow_{db} \) redex, because if \( t_{k-1} = P_{k-1}(L(l x r)p) \rightarrow_{db} P_{k-1}(L(r[x\leftarrow p])) = t_k \) then \( P_{k-1} = P_k \) is not the position of any redex in \( t_k \). Hence, \( R_{k-1} \) is a \( \rightarrow_1 \) step and there are two cases. If it substitutes a:

1. Variable, i.e. the sequence of steps \( R_{k-1}; R_k \) is

   \[
   t_{k-1} = P_{k-1}(x) \rightarrow_1 P_{k-1}(y) \rightarrow_1 P_{k-1}(r) = t_{k+1}
   \]

   Then all box subterms of \( r \) come from box subterms that also appear on the left of \( P_{k-1} \) in \( t_k \) and so they are box subterms of \( t_0 \) by i.h.

2. dB-redex, i.e. the sequence of steps \( R_{k-1}; R_k \) is

   \[
   t_{k-1} = P_{k-1}(x) \rightarrow_{db} P_{k-1}(L(l y r)p) \rightarrow_{db} P_{k-1}(L(r[y\leftarrow p])) = t_{k+1}
   \]

   Then all box subterms of \( L(r[y\leftarrow p]) \) come from box subterms of \( L(l y r)p \) and so they are box subterms of \( t_0 \) by i.h.

3. \( P_{k-1} \) is internal to \( P_k \), i.e. \( P_k \prec_O P_{k-1} \) and \( P_k \neq P_{k-1} \). This case is only possible if \( R_k \) has been created upwards by \( R_{k-1} \), otherwise the derivation would not be \( \prec_{LO} \)-standard. There are only two possible cases of creations upwards:

1. dB creates dB, i.e. \( R_{k-1} \) is

   \[
   t_{k-1} = P_{k-1}(L(l y L' l(z x r)p)) \rightarrow_{db} P_{k-1}(L(L' l(z x r)[y \leftarrow p])) = t_k
   \]

   and \( P_{k-1} \) is applicative, that is \( P_{k-1} = P_k(L''(\cdot) s) \), so that \( R_k \) is

   \[
   t_k = P_k(L''(L(L' l(z x r)[y \leftarrow p])) s) \rightarrow_{db} P_k(L''(L(L' r[z s])(y \leftarrow p))) = t_{k+1}
   \]

   The box subterms of \( L'' \) and \( r \) (included) are box subterms of \( t_k \) whose box context \( B \) is \( P_{k-1} \prec_L B \) and so they are box subterms of \( t_0 \) by i.h.. The other box subterms of \( L''(L(L' r[z s])(y \leftarrow p)) \) are instead box subterms of \( L(l y L' l(z x r)p) \) and so they are box subterms of \( t_0 \) by i.h.

2. \( 1 \) creates \( dB \), i.e. \( R_{k-1} \) is

   \[
   t_{k-1} = P_{k-1}(x) \rightarrow_{db} P_{k-1}(L(l y r)) = t_k
   \]

   and \( P_{k-1} \) is applicative, i.e. \( P_{k-1} = P_k(L' \cdot) p \) so that \( R_k \) is

   \[
   P_k(L'(L(l y r)p)) \rightarrow_{db} P_k(L'(L(r[y \leftarrow p])))
   \]

   The box subterms of \( L' \) and \( p \) (included) are box subterms of the ending term of \( R_{k-1} \) whose box context \( B \) is \( P_{k-1} \prec_L B \) and so they are box subterms of \( t_0 \) by i.h.. The other box subterms of \( L'(L(r[y \leftarrow p])) \) are also box subterms of \( L(l y L' l(z x r)p) \) and so they are box subterms of \( t_0 \) by i.h.

3. \( P_k \) is internal to \( P_{k-1} \), i.e. \( P_{k-1} \prec_O P_k \) and \( P_k \neq P_{k-1} \). Cases of \( R_{k-1} \):

1. dB-step, i.e. \( R_{k-1} \) is

   \[
   t_{k-1} = P_{k-1}(L(l x r)p) \rightarrow_{db} P_{k-1}(L(r[x \leftarrow p])) = t_k
   \]

   Then the hole of \( P_k \) is necessarily inside \( r \). Box subterms identified by a box context \( B \) s.t. \( P_k \prec_L B \) in \( t_{k+1} \) are also box subterms of \( t_k \), and so we conclude applying the i.h.. For box subterms identified by a \( B \) of \( t_{k+1} \) s.t. \( P_k \prec_O B \) we have to analyse \( R_k \). Suppose that \( R_k \) is a:

   a. dB-step. Note that in a root dB-step (i.e. at top-level) all the box subterms of the reduct are box subterms of the redex. In this case the redex is contained in \( r \) and so by i.h. all such box subterms are box subterms of \( t_0 \).
• 1a-step, i.e. $R_k$ has the form $t_k = P_k(x) \rightarrow_{1a} P_k(s) = t_{k+1}$. In $t_k$ $s$ is identified by a box context $B$ s.t. $P_k \prec_1 B$. From $P_{k-1} \prec_1 B$ we obtain $P_{k-1} \prec_1 B$ and so all box subterms of $s$ are box subterms of $t_0$ by i.h.

2. 1a-step: $R_{k-1}$ is $t_{k-1} = P_{k-1}(x) \rightarrow dB P_{k-1}(s) = t_k$. It is analogous to the dB-case: $R_k$ takes place inside $s$, whose box subterms are box subterms of $t_0$, by i.h. If $R_k$ is a dB-reduct then it only rearranges constructors in $s$ without changing box subterms, otherwise it substitutes something coming from a substitution that is on the left of $P_{k-1}$ and so whose box subterms are box subterms of $t_0$ by i.h.

3. $P_k$ is on the left of $P_{k-1}$, i.e. $P_{k-1} \prec_1 P_k$. So for $B$ s.t. $P_k \prec_1 B$ we conclude immediately using the i.h., because there is a box context $B'$ in $t_k$ s.t. $P_{k-1} \prec_1 B'$ and identifying the same box subterm of $B$. For $B$ s.t. $P_k \prec_1 B$ we reason as in case [2]

From the invariant, one easily obtains the subterm property, that in turn implies the no size explosion and the trace properties.

**Corollary 6.4.** Let $\rho : t \rightarrow^k u$ be a standard derivation.

1. Subterm: every $\rightarrow_{1a}$-step in $\rho$ duplicates a subterm of $t$.

2. No Size Explosion: $|u| \leq (k+1) \cdot |t|$.

3. Trace: if $t$ is an ordinary $l$-term then $|u| = |\rho| dB$.

The subterm property of standard derivations is specific to small-step evaluation at a distance, and it is the crucial half of the notion of mechanisable strategy. It allows to see the standardisation theorem as the unveiling of a very abstract machine, hidden inside the calculus itself.

**Proof.** 1. By induction on $k$. If $k = 0$ the statement is evidently true. Otherwise, by i.h. for every $\rho : t \rightarrow^{k-1} r$ all its $\rightarrow_{1a}$-steps duplicated subterms of $t$. If the next step is a dB-step we conclude, otherwise by Lemma 6.3 it duplicates a subterm of $u$ which is a box subterm, and so a subterm of $t$.

2. It follows immediately from the subterm property.

3. If $t$ is an ordinary $l$-term then by the subterm property only ordinary $l$-terms are duplicated. In particular, no explicit substitution constructor is ever duplicated by 1a-steps in $\rho$: if $r \rightarrow_{1a} p$ is a step of $\rho$ then $|r| = |p| dB$. Every dB-step, instead, introduces a substitution, i.e. $|u| = |\rho| dB$.

Let us conclude the section with a further invariant of standard derivations. It is not needed for the invariance result, but sheds some light on the shallow subsystem under. Let a term be **shallow** if its substitutions do not contain substitutions. The invariant is that if the initial term is a $l$-term then standard shallow derivations involve only shallow terms. This fact is the only point of this section relying on the assumption that reduction is shallow (the standard hypothesis is also necessary, consider $(lx)(ly)(z) \rightarrow dB (lx)(y[yz]) \rightarrow dB x[y[yz]]$).

**Lemma 6.5 (Shallow Invariant).** Let $t$ be a $l$-term and $\rho : t \rightarrow^k u$ be a standard derivation. Then $u$ is a shallow term.

**Proof.** By induction on $k$. If $k = 0$ the statement is evidently true. Otherwise, by i.h. every explicit substitution in $r$, where $\rho : t \rightarrow^{k-1} r$, contains a $l$-term. If the next step is a:

1. **1a-step** By the subterm property the step duplicates a term without substitutions, and — since reduction is shallow — it does not put the duplicated term in a substitution. Therefore, for every substitution of $u$ there is a substitution of $r$ with the same content. We then conclude applying the i.h.

2. **dB-step** It is easily seen that the argument of the dB-step is on the right of the previous step, so that by Lemma 6.3 it contains a (box) subterm of $t$. Then, the substitution created by the dB-step contains a subterm of $t$, that is an ordinary $l$-term by hypothesis. The step does not affect any other substitution, because reduction is shallow, and so we conclude.
7 The Subterm and Trace Properties, via Standard Derivations

While LO derivations are evidently standard, a priori LOU derivations may not be standard, if the reduction of a useful redex \( R \) could turn a useless redex \( Q \prec_{LO} R \) into a useful redex. Luckily, this is not possible, i.e. uselessness is stable by reduction of \( \prec_{LO} \)-majorants, as proved by the next lemma.

Lemma 7.1 (Useless Persistence). Let \( R : t \to_{1\Delta} u \) be a useless redex and \( Q : t \to r \) be a useful redex s.t. \( R \prec_{LO} Q \). The unique residual \( R' \) of \( R \) after \( Q \) is shallow and useless.

**Proof.** Let \( R : P\langle S(x)[x \to r]\rangle \to_{1\Delta} P\langle S(r)[x \to r]\rangle \). Uselessness of \( R \) implies that \( r \mid_P \) is a normal \( l \)-term and if \( r \mid_P \) is an abstraction then \( P\langle S(x)[x \to r]\rangle \) is not an applicative context. Note that 1\( l \)-steps cannot change the useless nature of \( R \), because 1) useful/useless redexes are defined via unfolding and 2) 1\( l \)-steps cannot change the applicative nature of a context (indeed that could happen if the hole is in a substitution and it is substituted on an applied variable occurrence, but here it is impossible because contexts are shallow). So, in the following we suppose that \( Q \) is a dB-redex.

By induction on \( P \), the external context of \( R \). Cases:

1. **Empty context (\( \cdot \)).** Consider \( Q \), that necessarily takes the form
   \[ Q : S\langle x\rangle[x \to r] \to P\langle x\rangle[x \to r] \]
   i.e. reduction takes place in the context \( S \). The only way in which \( R' \) can be useful is if \( Q \) turned the non-applicative context \( S \) into an applicative context \( P \), assuming that \( r \mid_P \) is an abstraction. The useful redex \( Q \) can change the nature of \( S \) only if \( S = T\langle L\langle ly,L'\rangle p\rangle \) and \( T \) is applicative, so that \( Q \) is
   \[ T\langle L\langle ly,L'\langle x\rangle p\rangle \rangle[x \to r] \to dB T\langle L\langle L'\langle x\rangle \rangle\rangle[x \to r] \]
   with \( P = T\langle L\langle L'\langle y \to p \rangle \rangle \rangle \) applicative context. But then \( Q \prec_{LO} R \), against hypothesis. Absurd.

2. **Inductive cases:**
   1. **Abstraction**, i.e. \( P = ly.T \). Both redexes \( R \) and \( Q \) take place under the outermost abstraction, so we conclude using the i.h.
   2. **Left of an application**, i.e. \( P = Ts \). Note that \( Q \) cannot be the eventual root dB-redex (i.e. if \( T \) is of the form \( L\langle ly,V \rangle \) then \( Q \) is not the dB-redex involving \( ly \) and \( s \)), because this would contradict \( R \prec_{LO} Q \). If the redex \( Q \) takes place in \( T\langle S\langle x\rangle[x \to r]\rangle \) then we use the i.h. Otherwise \( Q \) takes place in \( s \), the two redexes are disjoint, and commute. Evidently, the residual \( R' \) of \( R \) after \( Q \) is still shallow and useless.
   3. **Right of an application**, i.e. \( P = sT \). For the useless part it is similar to the previous case. Now, \( R' \) is shallow, because \( R \) is shallow and \( Q \) cannot put its residual into an explicit substitution, because—as in the previous case—\( Q \) cannot be the eventual root dB-redex.
   4. **Substitution**, i.e. \( P = T[y \to s] \). Both redexes \( R \) and \( Q \) take place under the outermost explicit substitution \( [y \to s] \), so we conclude using the i.h. \( \square \)

Using the lemma above and a technical property of standard derivations (the *enclave axiom*, see [ABKL14]) we obtain:

Proposition 7.2 (LOU-Derivations Are Standard). Let \( \rho \) be a LOU derivation. Then \( \rho \) is a standard derivation.

**Proof.** Suppose not. Then \( \rho \) writes as \( \tau; R; \sigma \) where \( \tau \) is the maximum standard prefix of \( \rho \) (that is necessarily non-empty, because a single step is always standard). Now, let \( \tau \) be \( R_1; \ldots ; R_k \) and \( R_i : t_i \to t_{i+1} \) with \( i \in \{1, \ldots , k\} \). If \( \tau; R \) is not standard there is a term \( t_i \) and a redex \( Q \) of \( t_i \) s.t.

1. \( R \) is a residual of \( Q \) after \( R_1; \ldots ; R_k \);
2. $Q \prec_{LO} R_i$.

By induction on $l = k - i + 1$ (i.e. on the length of the sequence $R_i; \ldots; R_k$) we prove that $R$ is useless. This fact will give us a contradiction, because by hypothesis $R$ is useful.

If $l = 1$ by Lemma 7.1 the unique residual $Q_{i+1}$ of $Q$ after $R_k$ is useless. But this residual, being unique, is necessarily equal to $R$. We conclude. If $l > 0$ consider $R_{i+1}$ and the unique residual $Q_{i+1}$ of $Q$ after $R_i$, which is useless by Lemma 7.1. Both $Q_{i+1}$ and $R_{i+1}$ are redexes of $t_{i+1}$. Two cases:

1. $R_{i+1} \prec_{LO} Q_{i+1}$. The order $\prec_{LO}$ satisfies Melliès’ axiomatics for standardisation, in particular the axiom Enclave (see [ABKL14], Sect. 9). We recall such an axiom, that will give us a contradiction. The axiom assumes two redexes $A$ and $B$ in a term $t$ s.t. $B \prec_{LO} A$ and so $B$ has a unique residual $B'$ after the reduction of $A$, and it has two parts:
   1. Creation: if $A$ creates a redex $C$ then $B' \prec_{LO} C$;
   2. Nesting: If $A \prec_{LO} C$ and $C'$ is a residual of $C$ after $A$ then $B' \prec_{LO} C'$

Now, we have two cases:

1. $R_{i+1}$ has been created by $R_i$. Then by the creation part of the enclave axiom applied to $Q \prec_{LO} R_i$ we obtain $Q_{i+1} \prec_{LO} R_{i+1}$, absurd.
2. $R_{i+1}$ is a residual after $R_i$ of a redex $R_{i+1}^{-1}$ of $t_i$. Then by the nesting part of the enclave axiom applied to $Q \prec_{LO} R_i$ we obtain $Q_{i+1} \prec_{LO} R_{i+1}$, absurd.
3. $Q_{i+1} \prec_{LO} R_{i+1}$. Then we can apply the i.h. and conclude that $R$ is useless.

We conclude applying Corollary 6.4.

Corollary 7.3 (Subterm and Trace). LOU derivations have the subterm and the trace properties, and only involve shallow terms.

8 The Normal Form and Projection Properties

For the normal form property it is necessary to show that the position of a redex in an unfolded term $\downarrow t$ can be traced back to the position of a useful redex in the original term $t$.

Given that unfoldings turn explicit into implicit substitutions, we need a few preliminary lemmas about the stability by substitutions of the notions of applicative contexts, outside-in order, and position. Remember that all contexts are implicitly assumed to be shallow.

Lemma 8.1 (Applicative Contexts Are Stable by Substitution). Let $A$ be an applicative context. Then $A\{x\leftarrow t\}$ is an applicative context.

Proof. Straightforward induction on $A$.

Lemma 8.2 ($\prec_{O}$ Is Stable by Substitution). If $S \prec_{O} P$ then $S\{x\leftarrow t\} \prec_{O} P\{x\leftarrow t\}$.

Proof. By induction on $S$.

The definition of useful redex has a case about applicative contexts. The normal form property asks us to prove that the projection of a useful normal form is a $\beta$ normal form. Thus, we are forced to study how applicative contexts interact with substitution and unfoldings. This is why the following two lemmas have particularly technical statements and proofs.

Lemma 8.3 (Positions and Substitution). Let $t$ and $u$ be $l$-terms and $t\{y\leftarrow u\} = S\{x\}$. Then

1. there exists $P$ s.t. $t = P\{z\}$ with $z \in \{x, y\}$ and $P\{y\leftarrow u\} \prec_{O} S$.
2. if in addition $S$ is applicative then either
   1. $P$ is applicative and $P\{y\leftarrow u\} = S$, or
   2. $z = y$ and there is an applicative context $T$ s.t. $u = T\{x\}$.

Remark 8.4. In Point 1 asking that $P\{y\leftarrow u\} = S$ is equivalent to:

1. $z\{y\leftarrow u\} = x$, because both $t\{y\leftarrow u\} = P\{z\}{y\leftarrow u} = P\{y\leftarrow u\}{z\{y\leftarrow u\}}$ and $t\{y\leftarrow u\} = S\{x\}$;
2. $z = x$ or $u = x$, because it is equivalent to $z\{y\leftarrow u\} = x$. 

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Proof. By induction on $t$. Cases:

- **Variable.** Sub-cases:
  1. $t = x$.
     - then $S = \langle \rangle$ and it is enough to take $P := \langle \rangle$, for which $\langle \rangle \{y \mapsto u\} = \langle \rangle = S$.
  2. $S = \langle \rangle$ is not applicative, so there is nothing to prove.

- **$t = y$.**
  1. then $S \prec p \cdot u$ and $P := \langle \rangle$ verifies the statement (because $\langle \rangle \prec_\mathcal{O} T$ for any context $T$).
  2. If $S$ is applicative then we are in case 2 and the applicative context $T$ for $u$ is $S$ itself.

- **$t = z$.** Impossible because then $t \{y \mapsto u\} = z \neq S(x)$.

- **Abstraction.** It follows immediately from the i.h.

- **Application $t = ur$.** Then $t \{y \mapsto u\} = r \{y \mapsto u\} p \{y \mapsto u\}$. Two cases:
  1. **The hole of $S$ is in the left subterm $r \{y \mapsto u\}$.** Then $S = Tp \{y \mapsto u\}$ and so $r \{y \mapsto u\} = T(x)$.
     1. By i.h. we obtain $V$ s.t. $r = V(z)$ with $z \in \{x, y\}$ and $V \{y \mapsto u\} \prec_\mathcal{O} T$. Then $P := Vp$ is s.t. $t = P(z)$ and $P \{y \mapsto u\} = V \{y \mapsto u\} p \{y \mapsto u\} \prec_\mathcal{O} T p \{y \mapsto u\} = S$.
     2. If $S$ is applicative there are two sub-cases:
        1. **$T$ is applicative.** Then we conclude using the i.h.
        2. **$T$ is not applicative.** Then $T = \langle \rangle$ and the context $V$ given by the i.h. is empty, i.e. $P = \langle \rangle p$ is applicative and verifies $P \{y \mapsto u\} = \langle \rangle p \{y \mapsto u\} = S$, i.e. this is an instance of case 1.
  2. **The hole of $S$ is in the right subterm $p \{y \mapsto u\}$.** Then $S = r \{y \mapsto u\} T$ and so $p \{y \mapsto u\} = T(x)$.
     1. Analogous to the previous case.
     2. If $S$ is applicative then $T$ is applicative and we use the i.h.

\[
\square
\]

The next lemma states the technical relationship between positions and unfolding. It will be used in both the proofs of the normal form property (Proposition 8.7) and of the key lemma for the projection property (Lemma 8.13).

**Lemma 8.5 (Positions and Unfolding).** Let $t$ be a LSC term and $t_\downarrow = S(x)$, where $S$ does not capture $x$. Then

1. there exists a shallow context $P$ s.t. $t = P(y)$ (possibly with $y = x$) and $P_\downarrow \prec_\mathcal{O} S$;
2. if in addition $S$ is applicative then either
   1. $P$ is applicative and $P_\downarrow = S$, or
   2. there is an applicative context $A$ s.t. $y_\downarrow A = A(x)$.

**Remark 8.6.** In Point 1 $P_\downarrow = S$ is equivalent to $y_\downarrow A = x$, because $t_\downarrow = P_\downarrow (y_\downarrow A)$ (by Lemma 8.13) and $t_\downarrow = S(x)$ (by hypothesis).

**Proof.** We prove both points together, by induction on $t$. Cases:

- **Variable.**
  1. If $t = x$ then $t_\downarrow = x$, $S = \langle \rangle$ and it is enough to take $P := \langle \rangle$, for which $P_\downarrow = \langle \rangle \prec_\mathcal{O} \langle \rangle = S$.
  2. $S$ cannot be applicative.

- **Abstraction.** It follows immediately by the i.h., since $t_\downarrow$ is an abstraction.

- **Application $t = ur$.** By definition $t_\downarrow = u_\downarrow r_\downarrow$ and $t_\downarrow = S(x)$ implies $S = Tr_\downarrow$ or $S = u[T]$.
  1. For both cases of $S$ the i.h. gives $P'$ satisfying the statement wrt $T(x)$. We conclude taking $P := P' r_\downarrow$ or $P := u[P']$, depending on the case under consideration.
  2. Suppose that $S$ is applicative. If $T$ is applicative then it follows from the i.h. Otherwise $S = \langle \rangle r_\downarrow$, and that can happen only if $u_\downarrow = x$, i.e. if $u = L(y)$ (possibly with $y = x$) and $y_\downarrow A = x$. Then take $P := Lr$, which is applicative and verifies $t = P(y)$ and $P_\downarrow = S$. We conclude because $y_\downarrow A = y_\downarrow Lr = y_\downarrow A = x$, and so we are in case 1.

- **Substitution $t = u[y \mapsto r]$.** By definition $t_\downarrow = u_\downarrow \{y \mapsto r_\downarrow\}$.
Proof. By induction on \( \beta \).

Proposition 8.7

1. By Lemma \( \text{S.3} \) there exists \( T \) s.t. \( u \downarrow = T\{z\} \) with \( z \in \{y,x\} \), and \( T\{y-r\} \prec_O S \). By i.h. applied to \( u \), there also exists a context \( V \) s.t. \( u = V(x') \) (possibly with \( x' = z \)) and \( V \downarrow \prec_O T \). Take \( P := V[y-r] \). By definition \( P \downarrow = V\{y-r\} \). By Lemma \( \text{S.2} \) applied to \( V \downarrow \prec_O T \) we obtain \( \forall \{y-r\} \prec_O T\{y-r\} \), implying \( P \downarrow \prec_O T\{y-r\} \prec_O S \), that is \( P \downarrow \prec_O S \).

2. If \( S \) is applicative then by Lemma \( \text{S.3} \) one of the following cases applies:

   1. \( T \) is applicative and \( T\{y-r\} = S \). The i.h. gives us one of the two following cases:

      1. \( V \) is applicative and \( V \downarrow = T \). Then \( P \) (defined as \( V[y-r] \)) is applicative and \( P \downarrow = V\{y-r\} = V\{y-r\} = T\{y-r\} = S \). So we are in case 1.

      2. There is an applicative context \( B \) s.t. \( z' \downarrow_B = B(z) \). We have \( t = P(z') \) so we want to show that there is an applicative context \( A \) s.t. \( z' \downarrow_A = A(x) \), i.e. that we are in case 2. Note that \( z' \downarrow_B = z' \downarrow_V = z' \downarrow_V[y-r] = B(z)\{y-r\} \). Two further sub-cases:

         1. \( z = x \). Then \( z' \downarrow_B = P\{y-r\} = B\{y-r\}(x) \). We conclude taking \( A := B\{y-r\} \), that is applicative by Lemma \( \text{S.4} \).

         2. \( z = y \). Note that by \( V \downarrow = T \) and Remark \( \text{S.4} \) it follows that \( r \downarrow = x \). Then \( z' \downarrow_B = P\{y\} \{y-x\} = B\{y\}\{y-x\}(x) \). We conclude taking \( A := B\{y-x\} \) that is applicative by Lemma \( \text{S.4} \).

   2. \( z = y \) and there is an applicative context \( B \) s.t. \( r \downarrow = B(x) \). Then \( t = P(y) = V(y\{y-r\} \), and \( y \downarrow_B = y \downarrow_V \{y-r\} = y \downarrow_V \{y-r\} = r \downarrow = B(x) \). So taking \( A := B \) we are in case 2.

We now have the tools to prove the normal form property.

Proposition 8.7 (Normal Form). Let \( t \) be a LSC term in useful normal form. Then \( t \downarrow \) is a \( \beta \)-normal form.

Proof. By induction on \( t \). Cases:

1. \( \text{Variable} \ t = x \). Obvious.

2. \( \text{Abstraction} \ t = lx.u \). By i.h. \( u \downarrow \) is a normal form. We conclude, since by definition \( t \downarrow = lx.u \downarrow \).

3. \( \text{Application} \ t = ur \). By hypothesis \( u \) cannot be an abstraction (otherwise \( t \) would not be an useful normal form). By definition \( t \downarrow = u[r] \downarrow \) and by i.h. \( u \downarrow \) and \( r \downarrow \) are normal forms. We only need to show that \( u \downarrow \) cannot be an abstraction. Suppose it is. Then \( u \) has the form \( L(x) \) with \( L \) a substitution context acting on \( x \) and (i.e. \( L = L(L''[x-r]) \)) s.t. \( x \downarrow_L \) is an abstraction. Note that \( x \) occurs in an applicative context (\( Lx \)), and so the action of \( L \) on \( x \) is an useful step, against the hypothesis that \( t \) is an useful normal form.

4. \( \text{Substitution} \ t = u[x-r] \). By i.h. \( u \downarrow \) is a normal form. Then \( t \downarrow = u \downarrow \{x-r\} \downarrow \) is not a normal form only if there is \( S \) s.t. \( u \downarrow = S(x) \), and one of the two following cases holds:

   1. \( r \downarrow \) has a \( \beta \)-redex. Then by Lemma \( \text{S.5} \) applied to \( u \) there exists a context \( P \) s.t. \( u = P(y) \) (possibly with \( y = x \)) and \( P \downarrow \prec_O S \). From \( u \downarrow = P \downarrow B \downarrow \) (by Lemma \( \text{S.13} \) and \( P \downarrow \prec_O S \)) we obtain that there exists \( T \) s.t. \( y \downarrow_P = T(x) \). So,\n
      1. \( t = P(y)[x-r] \).

      2. the context \( P[x-r] \) can act on \( y \) (also in the case that \( y = x \)), and

      3. \( y \downarrow_P[x-r] = y \downarrow_P\{x-r\} = T(x)\{x-r\} = T\{x-r\}(x) \) contains a \( \beta \)-redex.

   Then \( t \) has a useful redex, against hypothesis, absurd.

   4. \( r \downarrow \) is an abstraction and \( S \) is applicative. Then by Lemma \( \text{S.3} \) applied to \( u \) there exists a context \( P \) s.t. \( u = P(y) \) (possibly with \( y = x \)), \( P \downarrow \prec_O S \), and either

      1. \( P \) is applicative and \( P \downarrow = S \). By Remark \( \text{S.0} \) this is equivalent to \( y \downarrow_P = x \). Note that:

         1. \( t = P(y)[x-r] \).

         2. \( P[x-r] \) is applicative,

         3. \( y \downarrow_P[x-r] = y \downarrow_P\{x-r\} = x\{x-r\} = r \downarrow \) is an abstraction.
Then $t$ has a useful redex, against hypothesis, absurd.

4. There exists an applicative context $A$ s.t. $y\downarrow_{P} = A(x)$.
   1. $t = P(y)[x\rightarrow r]$.
   2. the context $P[x\rightarrow r]$ can act on $y$ (also in the case that $y = x$), and
   3. $y_{P[x\rightarrow r]} = y_{P[x\rightarrow r]} = A(x)[x\rightarrow r] = A(r)\downarrow$ is a $\beta$-redex, because $r\downarrow$ is an abstraction.

Then $t$ has a useful redex, against hypothesis, absurd. 

The next lemma shows that useful reductions match their intended semantics, in the sense that every useful redex contributes somehow to a $\beta$-redex. It is not needed for the invariance result.

**Lemma 8.8** (Inverse Normal Form). Let $t$ be a LSC term s.t. $t\downarrow$ is a $\beta$-normal form. Then $t$ is a useful normal form.

**Proof.** By contraposition. Suppose that $t \rightarrow u$ by reducing a useful redex. A formal proof would be by induction on the useful step, however there are high-level observations that allow to avoid a formal proof. Note that since shallow dB-redexes cannot be erased by unfolding (because they take place in a shallow context) they project on $\beta$-redexes. Thus, we can assume that the step is a $\lambda$-step. Now, since applicative contexts unfold to applicative contexts, it is evident by the definition of useful $\lambda$-redex that a useful redex unfolds to a $\beta$-redex (that, as for dB-redexes, cannot be erased by the unfolding because its position is shallow). In any case $t\downarrow$ is not $\beta$-normal.

For the projection property, we first need to show that the LO order is stable by unfolding. As for positions, we first show that the LO order is stable by substitution.

**Lemma 8.9** ($\prec_{LO}$ and Substitution). Let $t$ be a l-term, $S \prec_{p} t$ and $P \prec_{p} t$. If $S\{x\rightarrow u\} \prec_{LO} P\{x\rightarrow u\}$ then $S \prec_{LO} P$.

**Proof.** By induction on $t$. Cases:

1. **Variable.** If $t$ is a variable then both $S$ and $P$ are the empty context, and so are the contexts $S\{x\rightarrow u\}$ and $P\{x\rightarrow u\}$. Hence $S\{x\rightarrow u\} \not\prec_{LO} P\{x\rightarrow u\}$ and the statement trivially holds.

2. **Abstraction $lx.r$.** It follows from the i.h.

3. **Application $rp$.** If $S\{x\rightarrow u\}$ is empty then $S$ is empty and $P\{x\rightarrow u\}$ is non-empty, that implies $P$ non-empty. Then $S \prec_{LO} P$. If $S\{x\rightarrow u\}$ is non-empty then so are $S$, $P\{x\rightarrow u\}$, and $P$. We have $t\{x\rightarrow u\} = r\{x\rightarrow u\}p\{x\rightarrow u\}$. Cases:
   1. $S\{x\rightarrow u\}$ and $P\{x\rightarrow u\}$ both have their holes in $r\{x\rightarrow u\}$, i.e. $S\{x\rightarrow u\} = T\{x\rightarrow u\}p\{x\rightarrow u\}$ and $P\{x\rightarrow u\} = V\{x\rightarrow u\}p\{x\rightarrow u\}$ for some contexts $T$ and $V$. Since $S\{x\rightarrow u\} \prec_{LO} P\{x\rightarrow u\}$ implies $T\{x\rightarrow u\} \prec_{LO} V\{x\rightarrow u\}$, by i.h. we obtain $T \prec_{LO} V$ and so $Tp \prec_{LO} Vp$, i.e. $S \prec_{LO} P$.
   2. $S\{x\rightarrow u\}$ and $P\{x\rightarrow u\}$ both have their holes in $p\{x\rightarrow u\}$. Analogous to the previous case.
   3. $S\{x\rightarrow u\}$ has its hole in $r\{x\rightarrow u\}$ and $P\{x\rightarrow u\}$ in $p\{x\rightarrow u\}$, i.e. $S\{x\rightarrow u\} = T\{x\rightarrow u\}p\{x\rightarrow u\}$ and $P\{x\rightarrow u\} = r\{x\rightarrow u\}V\{x\rightarrow u\}$ for some contexts $T$ and $V$. Then $S = Tp$ and $P = Vr$, therefore $S \prec_{LO} P$.

**Lemma 8.10** ($\prec_{LO}$ and Unfolding). Let $t$ be a LSC term, $S \prec_{p} t$ and $P \prec_{p} t$. If $S\downarrow \prec_{LO} P\downarrow$ then $S \prec_{LO} P$.

**Proof.** By induction on $t$. Note that $P\downarrow$ cannot be the empty context, because there is no context $S$ s.t. $S \prec_{LO} \langle \cdot \rangle$. Moreover, if $S\downarrow$ is the empty context then $S$ is the empty context and the statement is immediately verified, because $\langle \cdot \rangle \prec_{LO} P$ for all contexts $P$. Thus we can always exclude the cases where $S\downarrow$ or $P\downarrow$ is empty. Cases of $t$:

1. **Variable $x$.** We have $x\downarrow = x$ and a variable does not admit two different contexts, so there is nothing to prove.

2. **Abstraction $lx.u$.** Then $(lx.u)\downarrow = lx.u\downarrow$. Then $S = lx.T$ and $P = lx.V$, with $T \prec_{LO} V$, and we conclude using the i.h. and the closure by contexts of $\prec_{LO}$.
3. **Application ur.** Then \((ur)\downarrow = ujr\downarrow\). If \(S\downarrow\) and \(P\downarrow\) both have their hole in \(u\downarrow\) or both in \(r\downarrow\) then we conclude using the i.h. and the closure by contexts of \(\prec_\text{LO}\). Otherwise \(S\downarrow \prec_r u\downarrow\) and \(P\downarrow \prec_r r\downarrow\). Then necessarily \(S \prec_r u\downarrow\) and \(P \prec_r r\downarrow\), hence \(S \prec_\text{LO} P\).

4. **Substitution u[x→r].** Then \(t_j = u[x→r]\downarrow = u_j[x→r]\downarrow\). Since contexts are shallow, \(S = T[x→r]\) and \(P = V[x→r]\) for some contexts \(T \prec_r u\downarrow\) and \(V \prec_r u\downarrow\). Then \(S_j = T_j[x→r]\downarrow\) and \(P_j = V_j[x→r]\downarrow\) and the hypothesis becomes \(T_j[x→r] \prec_\text{LO} V_j[x→r]\downarrow\). Lemma 8.10 gives \(T\downarrow \prec_\text{LO} V\downarrow\). The i.h. gives \(T \prec_\text{LO} V\), that implies \(T[x→r] \prec_\text{LO} V[x→r]\), i.e. \(S \prec_\text{LO} P\).

We also need the two following straightforward properties.

**Lemma 8.11.** Let \(t\) and \(u\) be \(\ell\)-terms. If \(t \mapsto_\beta r\) then \(t[x→u] \mapsto_\beta u[x→u]\).

**Proof.** By induction on \(S\).

We now dispose of all the ingredients for the proof of the key lemma on which the projection theorem relies on. We use \(\mapsto_\beta\) for \(\beta\)-reduction at top level.

**Lemma 8.13 (LOU dB-Step Projects on \(→_\text{LO}\)).** Let \(t\) be a LSC term and \(L\) be a substitution context. Then \(t\downarrow_{S(L)} = L(t)\downarrow_{S}\).

**Proof.** By induction on \(S\).

The first point is an ordinary projection of reductions. The second one is instead involved, as it requires to prove that if \(R_j\) is not LO then \(R\) is not LOU, i.e. to be able to somehow trace LO redexes back through unfoldings. The proof is by induction on \(S\), that by hypothesis is the position of the LOU redex. The difficult case—not surprisingly—is when \(S = P[x→p]\), and where the two lemmas relating the unfolding with positions (Lemma 8.3) and the order (Lemma 8.10) are applied. The proof also uses the normal form property, when the position \(S\) is on the argument \(p\) of an application \(rpc\). Since \(R\) is LOU, \(r\) is useful-normal. To prove that \(R_j\) is the LO \(\beta\) redex in \((rp)\downarrow = r[p]\downarrow\) we use the fact that \(r_j\) is normal.

**Proof.** By induction on \(S\). **Notation:** if \(L = [x_1\mapsto r_1]\ldots[x_n\mapsto r_n]\) we denote with \(\hat{L}\) the list of implicit substitutions \(\{x_1\mapsto r_1\}\ldots\{x_n\mapsto r_n\}\). Cases:

1. **Empty context \(S = \langle\rangle\).** Then \(t = t' = L(\ell x.p)\), \(t_j = t'_j\downarrow_{S'}\), \(u = u'\), \(u_j = u'_j\downarrow_{S'}\), and

\[
\begin{align*}
t = L(\ell x.p) & \xrightarrow{\alpha} L[r[x→p]] = u \\
t_j = (\ell x.\hat{L})(p_j) & \xrightarrow{\beta} r_j[\hat{L}[x→p]] = r_j[x→p]\hat{L} = u_j
\end{align*}
\]

where the first equality in the South-East corner is given by the fact that \(x\) does not occur in \(L\) and the variables on which \(\hat{L}\) substitutes do not occur in \(p\), as it is easily seen by looking at the starting term. Thus the implicit substitutions \(\hat{L}\) and \(\{x→p\}\) commute. The redex \(R\) is LOU and \(R_j\) is the LO \(\beta\) redex in \(t_j\).

2. **Abstraction \(S = lx.p\).** It follows immediately by the i.h.

3. **Left of an application \(S = Pr\).** By Lemma 8.13 we know that \(t_j = P_j[t'_j[p]]r_j\). Using the i.h. we derive the following diagram:
4. **Right of an application**

Suppose that the redex $R$ reduced in the top side of the diagram is the LOU redex in $t$, and so in $P(t')$. The i.h. also gives that the redex $R_\downarrow$ reduced in the bottom side is LO in $P_\downarrow(t'_4\lvert p)$. Suppose it is not LO in $t_\downarrow$. This is only possible if $P_\downarrow(t'_4\lvert p)$ is an abstraction so that $t_\downarrow$ is a $\beta$-redex. Note that $P(t')$ is not of the form $L(\langle x.s\rangle)$, otherwise the step $P(t') \rightarrow_{\text{ab}} P(u') \rightarrow_{\text{ab}}$ would not be the LO step in $t$. Then $P(t')$ has the form $L(x)$ (because a term of the form $L(ps)$ would not unfold into an abstraction), which is not a $\alpha$-normal, absurd.

5. **Substitution** $S = P[x\leftarrow p]$. By i.h. $P_\downarrow(t'_4\lvert p) \rightarrow_{\beta} P_\downarrow(u'_4\lvert p)$ then:

\[
\begin{align*}
t_\downarrow &= P_\downarrow(t'_4\lvert p)\{x\leftarrow p\_4\lvert p\} = \text{Lemma 3.13} \\
P_\downarrow(x\leftarrow p\_4\lvert p') \rightarrow_{\beta} P_\downarrow(u\_4\lvert p) = \text{Lemma 3.11} \\
P_\downarrow(u\_4\lvert p)\{x\leftarrow p\_4\lvert p\} &= \text{Lemma 3.12} \\
\end{align*}
\]

Suppose that the redex $R$ reduced in $t$ is LO but the redex $R_\downarrow$ reduced in $t_\downarrow$ is not. The i.h. also gives that the redex $P_\downarrow(t'_4\lvert p) \rightarrow_{\beta} P_\downarrow(u'_4\lvert p)$ is LO. Consequently, the LO redex of $t^\downarrow$ has been created by the substitution $\{x\leftarrow p\_4\lvert p\}$, i.e. $P_\downarrow(t'_4\lvert p) = T(x)$ with $T_\leftarrow_{\text{LO}} P_\downarrow$ and $p_\downarrow$ has a $\beta$-redex or $p_\downarrow$ is an abstraction and $T$ is applicative. Before treating the two cases separately, we deal with some common facts. By Lemma 3.3 there is a context $V$ s.t. $P(t') = V(y)$ and $V_\downarrow \leftarrow_{\text{LO}} T$. By $T_\leftarrow_{\text{LO}} P_\downarrow$ we obtain $V_\downarrow \leftarrow_{\text{LO}} P_\downarrow$, and by Lemma 3.10 $V \leftarrow_{\text{LO}} P$. Then $V[x\leftarrow p] \leftarrow_{\text{LO}} P[x\leftarrow p] = S$. To conclude we need to show that $V[x\leftarrow p]$ is the position of an useful redex, that being on the left of $S$ would give us a contradiction. Cases:

1. $p_\downarrow$ has a $\beta$-redex. From $P_\downarrow(t'_4\lvert p) = T(x)$, $P(t') = V(y)$ and $V(y)_\downarrow = \text{Lemma 3.12} V[y\leftarrow x]$, it follows that $V[y\leftarrow x_\downarrow] = T(x)$, and from $V_\downarrow \leftarrow_{\text{LO}} T$ we obtain $x \in \text{fv}(y\leftarrow x)$. So, $p_\downarrow$ is a subterm of $y\leftarrow x_\downarrow = y\leftarrow x_\downarrow$. Summing up:
   1. $t = V(y)[x\leftarrow p]$.
   2. $y\leftarrow x_\downarrow$ contains a $\beta$-redex.

Then $V[x\leftarrow p]$ is the position of a useful redex in $t$ on the left of $R$, absurd.

2. $p_\downarrow$ is an abstraction and $T$ is applicative. By Lemma 3.3 there are two sub-cases:
   1. $V$ is applicative and $V_\downarrow = T$. By Remark 3.6 we obtain $y\leftarrow x = x$, and so:
      1. $t = V(y)[x\leftarrow p]$.
      2. $V[x\leftarrow p]$ is an applicative context.
      3. $y\leftarrow x = y\leftarrow x_\downarrow = x[x\leftarrow p] = p_\downarrow$ is an abstraction.

Then $t$ has a useful redex on the left of $R$, absurd.

4. there exists an applicative context $A$ s.t. $y\leftarrow x = A(x)$. Then:
1. \( t = V(y)[x=p] \),
2. \( y_i[x=p] = y_j[x=p] = A(x)[x=p] = A[p] \) is a \( \beta \)-redex.

Then \( t \) has a useful redex on the left of \( R \), absurd.

Projection of derivations now follows as an easy induction:

**Theorem 8.14** (Projection). Let \( t \) be a LSC term and \( \rho : t \rightarrow^*_\text{LOU} u \). Then there is a LO \( \beta \)-derivation \( \rho \downarrow_1 \rightarrow^*_\beta u \downarrow_1 \) s.t. \( |\rho\downarrow_1| = |\rho|_{\text{Rb}} \).

**Proof.** By induction on the length \( k \) of \( \rho \). If \( k = 0 \) the statement is trivially true. If \( k > 0 \) then \( \rho \) has the form \( t \rightarrow^k p \rightarrow u \) and the prefix \( t \rightarrow^k p \) by i.h. verifies the statement. Cases of \( p \rightarrow u \):

1. \( p \rightarrow_{12} u \): then \( p\downarrow_1 = u\downarrow_1 \) and there is nothing to prove.
2. \( p \rightarrow_{38} u \): by Lemma 8.13 such a step unfolds to exactly one LO \( \beta \)-step \( p\downarrow_1 \rightarrow^\beta u\downarrow_1 \) that together with the i.h. proves the statement.

\[ \square \]

### 9 The Syntactic Bound Property, via Nested Derivations

In this section we show that LOU derivations have the syntactic bound property. Instead of proving this fact directly, we introduce an abstract property, the notion of nested derivation and then prove that 1) nested derivations ensure the syntactic bound property, and 2) LOU derivations are nested. Such an approach helps to understand both LOU derivations and the syntactic bound property.

**Definition 9.1** (Nested Derivation). Two 1s-steps \( t \rightarrow_{12} u \rightarrow_{12} r \) are **nested** if the second one substitutes on the subterm substituted by the first one, i.e. if exist \( S \) and \( P \) s.t. the two steps have the compact form \( S(x) \rightarrow_{12} S(P(y)) \rightarrow_{12} S(P(u)) \). A derivation is nested if any two consecutive substitution steps are nested.

For instance, the first of the following two sequences of steps is nested while the second is not:

\[
\begin{align*}
(xy)[x\leftarrow yt][y\leftarrow u] &\rightarrow_{12} ((yt)y)[x\leftarrow yt][y\leftarrow u] \\
&\rightarrow_{12} ((yt)y)[x\leftarrow yt][y\leftarrow u]; \\
(xy)[x\leftarrow yt][y\leftarrow u] &\rightarrow_{12} ((yt)y)[x\leftarrow yt][y\leftarrow u] \\
&\rightarrow_{12} ((yt)y)[x\leftarrow yt][y\leftarrow u].
\end{align*}
\]

The idea is that nested derivations ensure the syntactic bound property because no substitution can be used twice in a nested sequence \( u \rightarrow^k_{12} r \), and so \( k \) is necessarily bounded by \( |u|_i \).

**Lemma 9.2** (Nested + Subterm = Syntactic Bound). Let \( t \) be a l-term, \( \rho : t \rightarrow^n u \rightarrow^k_{12} r \) be a derivation having the subterm property and whose suffix \( u \rightarrow^k_{12} r \) is nested. Then \( k \leq |u|_i \).

**Proof.** Let \( u = u_0 \rightarrow_{12} u_1 \rightarrow_{12} \ldots \rightarrow_{12} u_k = r \) be the nested suffix of \( \rho \) and \( u_i \rightarrow_{12} u_{i+1} \) one of its steps, for \( i \in \{0, \ldots, k-2\} \). Let us use \( S_i \) for the external context of the step, i.e. the context s.t. \( u_i = S_i(P(x)[x\leftarrow p]) \rightarrow_{12} S_i(P(p)[x\leftarrow p]) = u_{i+1} \). The following nested step \( u_{i+1} \rightarrow_{12} u_{i+2} \) substitutes on the substituted occurrence of \( p \). By the subterm property, \( p \) is a subterm of \( t \) and so it has no explicit substitution. Then the explicit substitution acting in \( u_{i+1} \rightarrow_{12} u_{i+2} \) is on the right of \( [x\leftarrow p] \), i.e. the external context \( S_{i+1} \) is a prefix of \( S_i \), in symbols \( S_{i+1} \prec_{Q} S_i \). Since the derivation \( u_0 \rightarrow_{12} u_1 \rightarrow_{12} \ldots \rightarrow_{12} u_k \) is nested we obtain a sequence \( S_k \prec_{Q} S_{k-1} \prec_{Q} \ldots \prec_{Q} S_0 \) of contexts of \( u \). In particular, every \( S_i \) corresponds to a different explicit substitution in \( u \), and so \( k \leq |u|_i \). \( \square \)
We are left to show that our small-step implementation of \( \beta \rightarrow \) — LOU derivations — indeed are nested derivations with the subterm property. We already know that they have the subterm property (Corollary 7.3), so we only need to show that they are nested.

We need a technical lemma. A context \( S \) is LO if given a fresh variable \( x \) the position \( P \) of any redex in \( S(x) \) is s.t. \( S \sim_{LO} P \).

**Lemma 9.3.** Let \( t \) be a \( l \)-term in normal form and \( S \) a LO context (possibly containing ES). If \( t \downarrow_S \) has a \( \beta \)-redex then there is a context \( P \) s.t. \( t = P(x) \), \( S \) acts on \( x \), and the induced step \( S(P(x)) \rightarrow_{1a} S(P(u)) \) is LOU.

**Proof.** By induction on \( t \). Cases:

1. **Variable \( t = x \).** Then \( S \) acts on \( x \) otherwise \( t \downarrow_S \) would not have a \( \beta \)-redex, i.e. \( S \) has the form \( T(V[x\rightarrow u]) \). Let \( P := \langle \cdot \rangle \) and consider the induced step \( S(x) \rightarrow_{1a} S(u) \). To prove that it is useful we have to analyse \( u \). Now, \( t \downarrow_S = x \downarrow_S = x \downarrow_{T(V[x\rightarrow u])} = u \downarrow_T \) and so \( u \downarrow_T \) has a \( \beta \)-redex, that is, the step is useful. Since \( P := \langle \cdot \rangle \) and \( S \) is LO, it follows that the step is also LO.

2. **Abstraction \( t = \lambda y.r \).** Note that \( t \downarrow_S = (\lambda y.r) \downarrow_S = r \downarrow_S([y\leftarrow \cdot]) \). Then \( r \downarrow_S([y\leftarrow \cdot]) \) has a \( \beta \)-redex and we can apply the i.h., and obtain a context \( P' \) satisfying the statement wrt \( r \). To conclude it is enough to take \( P := \langle \cdot \rangle P' \).

3. **Application \( t = rp \).** Note that the hypothesis on \( t \) forbids \( r \) to be an abstraction. There are three sub-cases, depending on where is the LO \( \beta \)-redex in \( t \downarrow_S = r \downarrow_S P \downarrow_S \):  
   1. **It is the outermost application of \( t \), i.e. \( r \downarrow_S \) is an abstraction.** Since \( r \) is not an abstraction, it can only be a variable \( x \) on which \( S \) acts by substituting \( u \) (i.e. \( S = T(V[x\rightarrow u]) \)). As in the variable case, we obtain \( x \downarrow_S = u \downarrow_T \), i.e. \( u \downarrow_T \) is an abstraction. Then let \( P := \langle \cdot \rangle P' \). The induced step \( S(P(x)) \rightarrow_{1a} S(P(u)) \) is useful because \( P := xp \) is applicative, and it is LO because both \( S \) and \( P \) are LO.
   2. **It is in the left subterm \( r \downarrow_S \).** It is enough to apply the i.h. as in the abstraction case (obtaining a context \( P' \) satisfying the statement wrt \( r \)), and then take \( P := P'P \).
   3. **It is in the right subterm \( P \downarrow_S \).** Analogous to the previous case. \( \square \)

We conclude with:

**Proposition 9.4.** LOU derivations are nested, and so they have the syntactic bound property.

**Proof.** We prove the following implication: if the reduction step \( S(x) \rightarrow_{1a} S(u) \) is LOU, and the LOU redex in \( S(u) \) is a \( 1a \)-redex then its position \( P \) occurs in \( u \), i.e. \( S \sim_{LO} P \) or \( S = P \), i.e. the two steps are nested. The syntactic bound property then follows from Proposition 7.3 (LOU derivations are standard), Corollary 6.4 (standard derivations have the subterm property), and Lemma 9.2 (nested plus the subterm property implies the syntactic bound property). Two cases, depending on why the reduction step \( S(x) \rightarrow_{1a} S(u) \) is useful:

1. **\( S \) is applicative and \( u \downarrow_S \) is an abstraction.** Two sub-cases:
   1. **\( u \) is an abstraction.** Then the LOU redex in \( S(u) \) is the multiplicative redex having \( u \) as abstraction, and there is nothing to prove (because the LOU redex is not a substitution redex).
   2. **\( u \) is not an abstraction.** Then it must a variable \( z \) (because it is a \( \lambda \)-term), and \( z \downarrow_S \) is an abstraction. But then \( S(u) \) is simply \( S(z) \) and the given occurrence of \( z \) marks another useful substitution redex, that is the LOU redex because \( S \) already was the position of the LOU redex at the preceding step.
2. **\( u \downarrow_S \) is not an abstraction or \( S \) is not applicative, but \( u \downarrow_S \) contains a \( \beta \)-redex.** Two sub-cases:
   1. **\( u \) contains a dB-redex itself.** Then consider the position \( T \) of the LO dB-redex \( R \) in \( u \). Two sub cases:
      1. **\( R \) is also the LOU redex in \( S(t) \).** Then there is nothing to prove, because the LOU redex is not a substitution redex.
2. \( R \) is not the LOU redex in \( S(t) \). Then there is a LOU substitution redex \( Q \) of position \( V \prec_{\text{LO}} S(T) \). Since
\begin{enumerate}
\item \( u\subscript{\prec_{\text{LO}}} \) is not an abstraction or \( S \) is not applicative,
\item the previous step was the \( \text{LOU} \) step,
\item useless steps do not become useful by reducing useful redexes on their right Lemma 7.1
\end{enumerate}
we necessarily have \( S \prec_{\text{LO}} V \) and we conclude.

4. \( u \) does not contain a redex. Remember that \( u\subscript{\prec_{\text{LO}}} \) does contain a \( \beta \)-redex, so we can apply Lemma 9.3 and obtain that there exists \( T \) s.t. \( u = T(z) \) and \( S(T(z)) \) identifies the LOU redex.

\[ \square \]

At this point, we proved all the abstract properties implying the high-level implementation theorem.

10 The Selection Property, or Computing Functions in Compact Form

This section proves the selection property for LOU derivations, which is the missing half of the proof that they are mechanisable, i.e. that they enjoy the low-level implementation theorem. The proof consists in providing a polynomial algorithm for testing the usefulness of a substitution step. The subtlety is that the test has to check whether a term in the form \( t\subscript{\prec_{\text{LO}}} \) contains a \( \beta \)-redex, or whether it is an abstraction, without explicitly computing \( t\subscript{\prec_{\text{LO}}} \) (which, of course, takes exponential time in the worst case). If one does not prove that this can be done in time polynomial in (the size of) \( t \) and \( S \), then firing each reduction step can cause an exponential blowup!

Our algorithm consists in the simultaneous computation of 4 correlated functions on terms in compact form, two of which will provide the answer to our problem. We need some abstract preliminaries about computing functions in compact form.

A function \( f \) from \( n \)-uples of \( \lambda \)-terms to a set \( A \) is said to have \textit{arity} \( n \), and we write \( f : n \rightarrow A \) in this case. The function \( f \) is said to be:
\begin{itemize}
\item \textit{Efficiently computable} if there is a polynomial time algorithm \( \mathcal{A} \) such that for every \( n \)-uple of \( \lambda \)-terms \((t_1, \ldots, t_n)\), the result of \( \mathcal{A}(t_1, \ldots, t_n) \) is precisely \( f(t_1, \ldots, t_n) \).
\item \textit{Efficiently computable in compact form} if there is a polynomial time algorithm \( \mathcal{A} \) such that for every \( n \)-uple of LSC terms \((t_1, \ldots, t_n)\), the result of \( \mathcal{A}(t_1, \ldots, t_n) \) is precisely \( f(t_1\subscript{\prec_{\text{LO}}}, \ldots, t_n\subscript{\prec_{\text{LO}}}) \).
\item \textit{Efficiently computable in compact form relatively to a context} if there is a polynomial time algorithm \( \mathcal{A} \) such that for every \( n \)-uple of pairs of LSC terms and contexts \(((t_1, S_1), \ldots, (t_n, S_n))\), the result of \( \mathcal{A}((t_1, S_1), \ldots, (t_n, S_n)) \) is precisely \( f(t_1\subscript{\prec_{\text{LO}}}, \ldots, t_n\subscript{\prec_{\text{LO}}}, S_1, \ldots, S_n) \).
\end{itemize}

An example of function is \textit{alpha} : \( 2 \rightarrow \mathbb{B} \), which given two \( \lambda \)-terms \( t \) and \( u \), returns \textit{true} if \( t \) and \( u \) are \( \alpha \)-equivalent and \textit{false} otherwise. In \textbf{ADL12}, \textit{alpha} is shown to be efficiently computable in compact form, via a dynamic programming algorithm \( \mathcal{B}_{\alpha} \) taking in input two LSC terms and computing, for every pair of their subterms, whether the (unfoldings) are \( \alpha \)-equivalent or not. Proceeding bottom-up, as usual in dynamic programming, allows to avoid the costly task of computing unfoldings explicitly, which takes exponential time in the worst-case. More details about \( \mathcal{B}_{\alpha} \) can be found in \textbf{ADL12}.

Each one of the functions of our interest take values in one of the following sets:
\[
\mathcal{V\text{ARS}} = \text{the set of finite sets of variables} \\
\mathbb{B} = \{ \text{true}, \text{false} \} \\
\mathbb{T} = \{ \text{var}(x) \mid x \text{ is a variable} \} \cup \{ \text{lam}, \text{app} \}
\]

Elements of \( \mathbb{T} \) represent the \textit{nature} of a term. The functions are:
\begin{itemize}
\item \textit{nature} : \( 1 \rightarrow \mathbb{T} \), which returns the nature of the input term;
\item \textit{redex} : \( 1 \rightarrow \mathbb{B} \), which returns \textit{true} if the input term contains a redex and \textit{false} otherwise;
\end{itemize}
be V use of a special notation: given two sets of variables V, W and on terms is A(g) defined in Figure 2. The algorithm computing × apvars redex Note that they all have arity 1 and that showing correctness First of all, we need to convince ourselves about the correctness of the proposed algorithms: do they really compute the function g? Actually, the way the algorithms are defined, namely by primitive recursion on the input terms, helps very much here: a simple induction suffices to prove the following:

**Proposition 10.1.** The algorithms A_g, B_g, C_g are all correct, namely for every λ-term t, for every

\[ A_g(x) = (\text{var}(x), \text{false}, \emptyset, \{x\}) \]

\[ A_g(lx.t) = (\text{lamb}, b, V_t - \{x\}, W_t - \{x\}) \]

where \( A_g(t) = (n_t, b, V_t, W_t) \);

\[ A_g(tu) = (\text{app}, b_t \lor b_u \lor (n_t = \text{lamb}), V_t \cup V_u \cup \{x | n_t = \text{var}(x)\}, W_t \cup W_u) \]

where \( A_g(t) = (n_t, b, V_t, W_t) \) and \( A_g(u) = (n_u, b_u, V_u, W_u) \);

\[ B_g(x) = (\text{var}(x), \text{false}, \emptyset, \{x\}) \]

\[ B_g(lx.t) = (\text{lamb}, b, V_t - \{x\}, W_t - \{x\}) \]

where \( B_g(t) = (n_t, b, V_t, W_t) \);

\[ B_g(tu) = (\text{app}, b_t \lor b_u \lor (n_t = \text{lamb}), V_t \cup V_u \cup \{x | n_t = \text{var}(x)\}, W_t \cup W_u) \]

where \( B_g(t) = (n_t, b, V_t, W_t) \) and \( B_g(u) = (n_u, b_u, V_u, W_u) \); and:

- \( n_t = \text{var}(x) \Rightarrow n = n_u; \quad n_t = \text{var}(y) \Rightarrow n = \text{var}(y) \);
- \( n_t = \text{lamb} \Rightarrow n = \text{lamb} \); \( n_t = \text{app} \Rightarrow n = \text{app} \);
- \( b = b_t \lor (b_u \land x \in W_t) \lor ((n_u = \text{lamb}) \land (x \in V_u)) \);
- \( V = (V_t - \{x\}) \cup V_u \downarrow x, W_t \cup \{y | n_u = \text{var}(y) \land x \in V_t\} \);
- \( W = (W_t - \{x\}) \cup W_u \downarrow x, W_t \)

\[ B_g(t[x\leftrightarrow u]) = (n, b, V, W) \]

where \( B_g(t) = (n_t, b, V_t, W_t) \) and \( B_g(u) = (n_u, b_u, V_u, W_u) \).

---

Figure 1: Computing g in explicit form.

Figure 2: Computing g in implicit form.

- apvars \( : 1 \rightarrow \mathcal{VARS} \), which returns the set of variables occurring in applicative position in the input term;
- freevars \( : 1 \rightarrow \mathcal{VARS} \), which returns the set of free variables occurring in the input term.

Note that they all have arity 1 and that showing **redex** and **nature** to be **efficiently computable in compact form relatively to a context** is precisely what is required to prove the efficiency of useful reduction.

The four functions above can all be proved to be efficiently computable (in the three meanings). It is convenient to do so by giving an algorithm computing the product function nature × redex × apvars × freevars \( : 1 \rightarrow \mathcal{VARS} \times \mathcal{VARS} \times \mathcal{VARS} \) (which we call g) compositionally, on the structure of the input term, because the four function are correlated (for example, tu has a redex, i.e. redex(tu) = true, if t is an abstraction, i.e. if nature(t) = lamb). The algorithm computing g on terms is A_g and is defined in Figure 1.

The interesting case in the algorithms for the two compact cases is the one for ES, that makes use of a special notation: given two sets of variables V, W and a variable x, V \downarrow x, W is defined to be V if x \in W and the empty set \emptyset otherwise. The algorithm B_g computing g on LSC terms is defined in Figure 2. The algorithm computing g on pairs in the form \((t, S)\) (where t is a LSC term and S is a shallow context) is defined in Figure 3.

First of all, we need to convince ourselves about the *correctness* of the proposed algorithms: do they really compute the function g? Actually, the way the algorithms are defined, namely by primitive recursion on the input terms, helps very much here: a simple induction suffices to prove the following:

**Proposition 10.1.** The algorithms A_g, B_g, C_g are all correct, namely for every λ-term t, for every
bounded time, e.g., the number of recursive calls triggered by $A$.

Proof. The three algorithms are defined by primitive recursion. More specifically:

- Any call $A_g(t)$ triggers at most $|t|$ calls to $A_g$;
- Any call $B_g(t)$ triggers at most $|t|$ calls to $B_g$;
- Any call $C_g(t, S)$ triggers at most $|t| + |S|$ calls to $C$ and at most $|S|$ calls to $B$.

Now, the amount of work involved in any single call (not counting the, possibly recursive, calls) is itself polynomial, simply because the tuples produced in output are made of objects whose size is itself bounded by the length of the involved terms and contexts.

The three algorithms are defined by primitive recursion. More specifically:

- The equation $A_g(t) = g(t)$ can be proved by induction on the structure of $t$. Some interesting cases:
  - If $t = ur$, then we know that:
    $$A_g(ur) = (\text{app}, b_r \lor b_r \lor (n_u = \text{lam}), V_r \cup V_r \cup \{x \mid n_u = \text{var}(x)\}, W_r \cup W_r)$$
    where $A_g(u) = (n_u, b_u, V_u, W_u)$ and $A_g(r) = (n_r, b_r, V_r, W_r)$;

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- If $t = ur$, then we know that:
  $$A_g(ur) = (\text{app}, b_r \lor b_r \lor (n_u = \text{lam}), V_r \cup V_r \cup \{x \mid n_u = \text{var}(x)\}, W_r \cup W_r)$$
  where $A_g(u) = (n_u, b_u, V_u, W_u)$ and $A_g(r) = (n_r, b_r, V_r, W_r)$;

Now, first of all observe that redex($t$) = true if and only if there is a redex in $u$ or a redex in $r$ or if $u$ is a lambda-abstraction. Moreover, the variables occurring in applicative position in $t$ are those occurring in applicative position in either $u$ or in $r$ or $x$, if $u$ is $x$ itself. Similarly, the variables occurring free in $t$ are simply those occurring free in either $u$ or in $r$. The thesis then descends easily from the inductive hypothesis.

- The equation $B_g(u) = g(u\downarrow)$ can be proved by induction on the structure of $u$, using the correctness of $A$.
- The equation $C_g(u, S) = g(u\downarrow_S)$ can be proved by induction on the structure of $u$, using the correctness of $A$.

This concludes the proof.

The way the algorithms above have been defined also helps while proving that they work in bounded time, e.g., the number of recursive calls triggered by $A_g(t)$ is linear in $|t|$ and each of them takes polynomial time. As a consequence, we can also easily bound the complexity of the three algorithms at hand.

**Proposition 10.2.** The algorithms $A_g, B_g, C_g$ all work in polynomial time. Thus LOU derivations are mechanisable.

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Now, the amount of work involved in any single call (not counting the, possibly recursive, calls) is itself polynomial, simply because the tuples produced in output are made of objects whose size is itself bounded by the length of the involved terms and contexts.

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- Any call $B_g(t)$ triggers at most $|t|$ calls to $B_g$;
- Any call $C_g(t, S)$ triggers at most $|t| + |S|$ calls to $B$ and at most $|S|$ calls to $C$.

Now, the amount of work involved in any single call (not counting the, possibly recursive, calls) is itself polynomial, simply because the tuples produced in output are made of objects whose size is itself bounded by the length of the involved terms and contexts.
11 Summing Up

The various ingredients from the previous sections can be combined together so as to obtain the following result:

**Theorem 11.1 (Invariance).** There is an algorithm which takes in input a $l$-term $t$ and which, in time polynomial in $\#_{\rightarrow_{\text{LOS}}}(t)$ and $|t|$, outputs an LSC term $u$ such that $u \rightarrow_{\text{LOS}}$ is the normal form of $t$.

As we have already mentioned, the algorithm witnessing the invariance of $l$-calculus does not produce in output a $l$-term, but a compact representation in the form of a term with ES. Theorem 11.1 together with the fact that equality of terms can be checked efficiently in compact form entail the following formulation of invariance, akin in spirit to, e.g., Statman’s Theorem [Sta79]:

**Corollary 11.2.** There is an algorithm which takes in input two $l$-terms $t$ and $u$ and checks whether $t$ and $u$ have the same normal form in time polynomial in $\#_{\rightarrow_{\text{LOS}}}(t)$, $\#_{\rightarrow_{\text{LOS}}}(u)$, $|t|$, and $|u|$.

If one instantiates Corollary 11.2 to the case in which $u$ is a normal form, one obtains that checking whether the normal form of any term $t$ is equal to $u$ can be done in time polynomial in $\#_{\rightarrow_{\text{LOS}}}(t)$, $|t|$, and $|u|$. This is particularly relevant when the size of $u$ is constant, e.g., when the $l$-calculus computes decision problems and the relevant results are truth values.

Please observe that whenever one (or both) of the involved terms are not normalisable, the algorithms above (correctly) diverge.

12 Discussion

Here we further discuss invariance and some potential optimisations, that, however, are outside the scope of this work (which only deals with asymptotical bounds and is thus foundational in spirit).

**Mechanisability vs Efficiency.** Let us stress that the study of invariance is about *mechanisability* rather than *efficiency*. One is not looking for the smartest or shortest evaluation strategy. But rather, for one that does not hide the complexity of its implementation in the cleverness of its definition, as it is the case for Lévy’s optimal evaluation. Indeed, an optimal derivation can be even shorter then the shortest sequential strategy, but — as shown by Asperti and Mairson [AM98] — its definition hides hyper-exponential computations, so that optimal derivations do not provide an invariant cost model. The leftmost-outermost strategy, is a sort of maximally unshared normalising strategy, where redexes are duplicated whenever possible (and unneeded redexes are never reduced), somehow dually with respect to optimal derivations. It is exactly this inefficiency that induces the subterm property, the key point for its mechanisability. It is important to not confuse two different levels of sharing: our LOU derivations share *subterms*, but not *computations*, while Lévy’s optimal derivations do the opposite. By sharing computations, they collapse the complexity of many steps into a single one, making the number of steps an unreliable measure.

**Call-by-Value and Call-by-Need.** Call-by-name evaluation is in many cases less efficient than call-by-value or call-by-need evaluation. Since we follow the call-by-name policy, the same kind of inefficiency shows up here. However, as already said, invariance is not about absolute efficiency: call-by-name and call-by-value are incomparable — sometimes one can even be exponentially faster than the other, sometimes the other way around — but this fact does not forbid both to be invariant, i.e. reasonably mechanisable.

We did not prove call-by-value/need invariance. Nonetheless, we strove to provide an abstract view of both the problem and of the architecture of our solution, having already in mind the adaptation to call-by-value/need $l$-calculi. Recently, the first author and Sacerdoti Coen show [ASCT14]
that (in the much simpler weak case) these policies provide an improved high-level implementation theorem, where evaluation in the LSC has a linear overhead, rather than quadratic.

**Usefulness.** Another source of inefficiency is the fact that at each reduction step we need to check whether the LO redex is useful before firing it, and this potentially amounts to doing a global analysis of the term. One could imagine decorating terms with additional tags in such a way that the check for usefulness becomes local and updating tags is not too costly, so that useful reduction may be implemented more efficiently. In particular, building on the already established relationships between the LSC and abstract machines [ABM14], we expect to be able to design an abstract machine implementing LOU evaluation and testing for usefulness in time linear in the size of the starting term.

### 13 Conclusions

This work can be seen as the last tale in the long quest for an invariant cost model for the \( \lambda \)-calculus. In the last ten years, the authors have been involved in various works in which parsimonious time cost models have been shown to be invariant for more and more general notions of reduction, progressively relaxing the conditions on the use of sharing [DLM08, DLM12, ADL12]. None of the results in the literature, however, concerns reduction to normal form as instead we do here.

By means of explicit substitutions — our tool for sharing — we provided the first full answer to a long-standing open problem: we proved that the \( l \)-calculus is indeed a reasonable machine, by showing that the length of the leftmost-outermost derivation to normal form is an invariant cost model.

The solution required the development of a whole new toolbox: an abstract deconstruction of the problem, a detailed study of unfoldings, a theory of useful derivations, and a general view of functions efficiently computable in compact form. Along the way, we showed that standard derivations for explicit substitutions enjoy the crucial subterm property. Essentially, it ensures that standard derivations are mechanisable, unveiling a very abstract notion of machine hidden deep inside the \( l \)-calculus itself, and also a surprising perspective on the standardisation theorem, a classic result apparently unrelated to the complexity of evaluation.

Among the downfalls of our results, one can of course mention that proving systems to characterise time complexity classes equal or larger than \( \text{P} \) can now be done merely by deriving bounds on the number of leftmost-outermost reduction steps to normal form. This could be useful, e.g., in the context of light logics [GRDR07, CDLRDR08, BT09]. The kind of bounds we obtain here are however more general than those obtained in implicit computational complexity (since we deal with a universal model of computation).

While there is room for finer analyses (e.g., studying call-by-value or call-by-need evaluation), we consider the understanding of time invariance essentially achieved. However, the study of complexity measures for \( l \)-terms is far from being over. Indeed, the study of space complexity for functional programs has only made its very first steps [Sch07, GMRDR08, DLS10], and not much is known about invariant space cost models.

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