On Harrell-Stubbe Type Inequalities for the Discrete Spectrum of a Self-Adjoint Operator

Mark S. Ashbaugh† 1, Lotfi Hermi‡ 2

† Department of Mathematics
University of Missouri
Columbia, MO 65211-4100, USA
Email: mark@math.missouri.edu
‡ Department of Mathematics
University of Arizona
Tucson, AZ 85721, USA
Email: hermi@math.arizona.edu

Abstract

We produce a new proof and extend results by Harrell and Stubbe for the discrete spectrum of a self-adjoint operator. An abstract approach—based on commutator algebra, the Rayleigh-Ritz principle, and an “optimal” usage of the Cauchy-Schwarz inequality—is used to produce “parameter-free”, “projection-free” versions of their theorems. We also analyze the strength of the various inequalities that ensue. The results contain

1Partially supported by National Science Foundation (USA) grant DMS-9870156.
2Corresponding author. Part of this paper appeared in L.H.’s 1999 Ph.D. Dissertation.

Copyright © 2007 by xxx
classical bounds for the eigenvalues. Extensions of a variety of inequalities à la Harrell-Stubbe are illustrated for both geometric and physical problems.

Key Words: eigenvalues of the Laplacian; Dirichlet eigenvalue problem for domains in Euclidean space; eigenvalues of elliptic operators; Payne–Pólya–Weinberger inequality; Hile–Protter inequality; H. C. Yang inequality; Harrell-Stubbe inequalities; universal eigenvalue estimates; reverse Chebyshev inequality.

I. Introduction

In this paper, we continue our work started in [9]. A semibounded operator modeled after the Dirichlet Laplacian on a bounded domain \( \Omega \subset \mathbb{R}^n \) (or a Schrödinger operator with magnetic potential) is given. We provide universal bounds for its eigenvalues. These are estimates for the eigenvalues that do not involve domain dependencies [38] (see also [3], [2]). This is a problem related to a classical result of Payne, Pólya, and Weinberger [36], [37] (abbreviated as PPW) for the eigenvalues \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \) (multiplicities included) of the fixed membrane problem

\[
-\Delta u = \lambda u \quad \text{in} \quad \Omega, \\
\quad u = 0 \quad \text{on} \quad \partial \Omega. 
\]

We provide, based on the Rayleigh-Ritz principle and trial functions, alternative proofs and extensions of recent results which were obtained by Harrell and Stubbe [22]. Our main divergence from their method is the use of the “optimal” Cauchy-Schwarz inequality exploited in [9] (see also [3], [2], [8], and [41]) and the fact that we employ the Rayleigh-Ritz inequality and not algebraic identities (see also [29] for yet another alternative).
We also consider the inequalities
\[ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{2p}{n} \sum_{i=1}^{m} \lambda_i (\lambda_{m+1} - \lambda_i)^{p-1} \quad \text{for} \quad p \geq 2 \] (2)
(see ineq. (14) in Theorem 9, p. 1805 of [22]) and
\[ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{4}{n} \sum_{i=1}^{m} \lambda_i (\lambda_{m+1} - \lambda_i)^{p-1} \quad \text{for} \quad p \leq 2 \] (3)
(see ineq. (11) in Theorem 5, p. 1801 of [22]), which stem from two different considerations in Harrell and Stubbe’s work.

The classical Hile-Protter [24] and H. C. Yang [41] inequalities appear as special cases of (3) for \( p = 0 \) and 2 respectively. In this paper, we will in fact show that (3) improves monotonically for \( 0 \leq p \leq 2 \) (Theorem 14). The classical PPW inequality
\[ \lambda_{m+1} - \lambda_m \leq \frac{4}{n} \frac{\sum_{i=1}^{m} \lambda_i}{m}, \] (4)
is obviously weaker than the \( p = 1 \) case of (3). In fact, it is weaker than the \( p = 0 \) case of (3) (see [24]), which is also easy to see.

In the literature, the case \( p = 1 \) is referred to as the “Yang 2” bound (see [3], [2]). It is, of course, explicitly given by
\[ \lambda_{m+1} \leq \left( 1 + \frac{4}{n} \right) \frac{\sum_{i=1}^{m} \lambda_i}{m}, \] (5)
The case \( p = 0 \) (herein referred to as HP) reads, explicitly,
\[ \frac{mn}{4} \leq \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_{m+1} - \lambda_i}. \] (6)

The general framework for this paper provides extensions à la Harrell-Stubbe for various geometric and physical problems. In fact Harrell-Stubbe type inequalities are valid for all the situations for which H. C. Yang-style improvements have been proved in [9] and illustrated in [10].

In this paper, an analysis of (2) is also provided. It is proved that the case \( p = 2 \) (i.e., the H. C. Yang inequality, also referred to as “Yang 1”; see [2], [3]) is the strongest for \( p \geq 2 \) (Theorem 16).
II. General Framework

Our setting is that of [9]. We provide an “algebraized” version of the membrane problem described in the introduction. Such a scheme follows a line of thought first adopted by Harrell and Davies (see [18], [34]) in 1988. This abstraction has the advantage of unifying many results for gaps of eigenvalues of subdomains of Riemannian manifolds and a variety of geometric and physical situations. This point of view was advocated by Harrell and Michel [20], [21], [34], Harrell and Stubbe [22], Hook [26], Levitin and Parnovski [29], and Ashbaugh and Hermi [9]. This point of view provides improvements à la H. C. Yang of results in [14], [20], [21], [26], [30], [31], [32], and [42] as described in [9] and [10]. See also [16] and [17] where further applications and generalizations to new settings are presented.

A complex Hilbert space \( \mathcal{H} \) with inner product \( \langle \cdot, \cdot \rangle \) is given. \( \langle \cdot, \cdot \rangle \) is taken to be linear in its first argument, conjugate linear in its second. We let \( A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H} \) be a self-adjoint operator defined on a dense domain \( \mathcal{D} \) which is semibounded below and has a discrete spectrum \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \). Let \( \{B_k : A(\mathcal{D}) \to \mathcal{H}\}_{k=1}^{N} \) be a collection of symmetric operators which leave \( \mathcal{D} \) invariant, and let \( \{u_i\}_{i=1}^{\infty} \) be the normalized eigenvectors of \( A \), \( u_i \) corresponding to \( \lambda_i \). This family of eigenvectors is further assumed to be an orthonormal basis for \( \mathcal{H} \). The commutator of two operators, \([A, B]\), is defined by \([A, B] = AB - BA\), and \( \|u\| = \sqrt{\langle u, u \rangle} \).

As in [9], we define

\[
\rho_i = \sum_{k=1}^{N} \langle [A, B_k]u_i, B_ku_i \rangle \quad (7)
\]

and

\[
\Lambda_i = \sum_{k=1}^{N} \| [A, B_k]u_i \|^2. \quad (8)
\]

In [9], we have shown that the classical inequalities of PPW, HP, and H. C. Yang follow from the same general set-up and the following theorem.
Theorem 1. The eigenvalues $\lambda_i$ of the operator $A$ satisfy the inequalities

$$\sum_{i=1}^{m} \rho_i \leq \frac{\sum_{i=1}^{m} \Lambda_i}{\lambda_{m+1} - \lambda_m},$$

(9)

$$\sum_{i=1}^{m} \rho_i \leq \sum_{i=1}^{m} \frac{\Lambda_i}{\lambda_{m+1} - \lambda_i},$$

(10)

and

$$\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \rho_i \leq \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) \Lambda_i.$$

(11)

These give abstract versions of the PPW, HP, and Yang inequalities, respectively, and even at this level, (11) is stronger than (9) and (10), and (10) is stronger than (9).

III. Extending The Work of Harrell and Stubbe

Based solely on the “traditional” tools (the Rayleigh-Ritz principle, simple trial functions, the Cauchy-Schwarz inequality, . . .), in this section we provide alternative proofs and generalizations of the results of Harrell and Stubbe [22]. In their work, they wanted to understand the nature of Yang’s inequalities [41]. Our proofs tie in with the abstract commutator approach used by various authors [20], [21], [22], in their work on geometric bounds for eigenvalues of elliptic operators. The proofs provide further insight into the extensions in [22] (explaining, for example, what terms are being dropped in arriving at their inequalities). A separate section (Section V) is dedicated to comparing the bounds obtained from the approach given in this section to those of the works of Hile-Protter and H. C. Yang. Another section (Section VI) provides illustrations of various extensions of known bounds for geometric and physical problems; for more in this direction see [10].

Theorem 2. Let the function $g(\lambda)$ be nonnegative and nondecreasing on the eigenvalues $\{\lambda_i\}_{i=1}^{m}$ of $A$. Then the eigenvalues $\{\lambda_i\}_{i=1}^{m+1}$ of $A$
satisfy the inequality
\[
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 g(\lambda_i) \rho_i \leq \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) g(\lambda_i) \Lambda_i.
\] (12)

Note. It is enough that \( g \) be a nonnegative and nondecreasing function defined on \((0, \lambda_{m+1})\), as will typically be the case in applications.

Proof. As in [9], we start with the Rayleigh-Ritz inequality
\[
\lambda_{m+1} \leq \frac{\langle A\phi, \phi \rangle}{\langle \phi, \phi \rangle}
\] (13)
and the test function
\[
\phi_i = Bu_i - \sum_{j=1}^{m} a_{ij} u_j,
\] (14)
where \( B \) is one of the \( B_k \)'s, \( k = 1, \ldots, N \). The orthogonality condition
\[
\langle \phi, u_j \rangle = 0
\]
for \( j = 1, 2, \ldots, m \) makes \( a_{ij} = \langle Bu_i, u_j \rangle \). The symmetry of \( B \) makes \( a_{ji} = a_{ij} \). As in [9], (13) reduces to
\[
\lambda_{m+1} - \lambda_i \leq \frac{\langle [A, B]u_i, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle}.
\] (15)

The calculations in [9] yield
\[
\langle [A, B]u_i, \phi_i \rangle = \langle [A, B]u_i, Bu_i \rangle - \sum_{j=1}^{m} (\lambda_j - \lambda_i) |a_{ij}|^2.
\] (16)

By (15), \( \langle [A, B]u_i, \phi_i \rangle \geq 0 \). Thus, by the “optimal” Cauchy-Schwarz inequality (see Lemma 3.1 of [9]),
\[
\frac{\langle [A, B]u_i, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \leq \frac{\| [A, B]u_i \|^2 - \sum_{j=1}^{m} (\lambda_j - \lambda_i)^2 |a_{ij}|^2}{\langle [A, B]u_i, \phi_i \rangle}.
\] (17)
HARRELL-STUBBE INEQUALITIES

We then obtain

\[(\lambda_{m+1} - \lambda_i) \left( \langle [A, B] u_i, B u_i \rangle - \sum_{j=1}^{m} (\lambda_j - \lambda_i) |a_{ij}|^2 \right) \]

\[\leq \| [A, B] u_i \|^2 - \sum_{j=1}^{m} (\lambda_j - \lambda_i)^2 |a_{ij}|^2, \]

(18)

or, upon combining the sums involving \(|a_{ij}|^2|.

\[(\lambda_{m+1} - \lambda_i) \langle [A, B] u_i, B u_i \rangle \]

\[\leq \| [A, B] u_i \|^2 - \sum_{j=1}^{m} (\lambda_i - \lambda_j)(\lambda_{m+1} - \lambda_j)|a_{ij}|^2, \]

(19)

Since \(B\) is one of the \(B_k\)'s, \(a_{ij} \equiv a_{ij}^k\). Let

\[A_{ij} \equiv \sum_{k=1}^{N} |a_{ij}^k|^2. \quad (20)\]

Hence \(A_{ji} = A_{ij} \geq 0\). Replacing \(B\) by \(B_k\) in (19), summing over \(k\) for \(1 \leq k \leq N\), and incorporating the definitions of \(\rho_i, \Lambda_i, \) and \(A_{ij}\), we obtain

\[(\lambda_{m+1} - \lambda_i) \rho_i \leq \Lambda_i - \sum_{j=1}^{m} (\lambda_{m+1} - \lambda_j)(\lambda_i - \lambda_j)A_{ij}. \]

(21)

Multiplying both sides by

\[(\lambda_{m+1} - \lambda_i) g(\lambda_i) \]

\((g \geq 0\) is needed here to preserve the sense of our inequality, and of course \(i < m + 1\) is assumed) and summing over \(i, 1 \leq i \leq m\), gives

\[\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 g(\lambda_i) \rho_i \leq \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)g(\lambda_i)\Lambda_i \]

\[\quad - \sum_{i,j=1}^{m} (\lambda_{m+1} - \lambda_i)(\lambda_{m+1} - \lambda_j)A_{ij}(\lambda_i - \lambda_j)g(\lambda_i). \]

(22)
If $g$ were constant the double sum in $i$ and $j$ here would vanish due to antisymmetry (recall that $A_{ij}$ is symmetric), allowing us to conclude that the theorem holds in this case. Indeed, it is this case which motivated our choice of multiplier for (21). This is how Yang’s main inequality (the $p = 2$ case of (3)) was proved in [3] and [2]. For more general $g$, we can use the notions of similarly (resp., oppositely) ordered (see [15], pp. 43, 261-262) to impose a sign on the double sum; in particular, it transpires that the double sum is nonnegative if \( \{\lambda_i\}_{i=1}^m \) and \( \{g(\lambda_i)\}_{i=1}^m \) are similarly ordered, or, what amounts to the same thing here, if \( \{g(\lambda_i)\}_{i=1}^m \) is a nondecreasing sequence. To see this, we rewrite the double sum with $i$ and $j$ interchanged and average the two expressions, giving

\[
\sum_{i=1}^m (\lambda_{m+1} - \lambda_i)^2 g(\lambda_i) \rho_i \leq \sum_{i=1}^m (\lambda_{m+1} - \lambda_i) g(\lambda_i) A_i - \frac{1}{2} \sum_{i,j=1}^m (\lambda_{m+1} - \lambda_i) (\lambda_{m+1} - \lambda_j) A_{ij} \times (\lambda_i - \lambda_j) (g(\lambda_i) - g(\lambda_j)).
\]  

(23)

The factor \( (\lambda_i - \lambda_j) (g(\lambda_i) - g(\lambda_j)) \) and the nonnegativity of the rest, shows that the double sum will be nonnegative whenever \( \{g(\lambda_i)\}_{i=1}^m \) is nondecreasing, and, since the double sum is preceded by a minus sign, its contribution to the right-hand side of the inequality is nonpositive, yielding the desired conclusion, i.e., inequality (12). □

Remarks. 1. If one assumes that $g$ is nondecreasing and $C^1$ (or just differentiable) on the positive half-axis, then by the mean value theorem

\[
g(\lambda_i) - g(\lambda_j) = g'(\xi_{ij}) (\lambda_i - \lambda_j)
\]

for some $\xi_{ij} > 0$ where $g'(\xi_{ij}) \geq 0$. Therefore the double sum giving the second term on the right-hand side of (23) is nonnegative, and since it is subtracted, the statement of the theorem follows.

2. If we make the replacement $g(\lambda) = h(\lambda_{m+1} - \lambda)$, then the hypotheses on $h$ would be that $h$ is nonnegative and nonincreasing on the sequence \( \{\lambda_{m+1} - \lambda_i\}_{i=1}^m \), or, perhaps a little more naturally,
that $h$ is nonnegative and nonincreasing on $(0, \lambda_{m+1})$. The inequality in Theorem 2, when written in terms of $h$, becomes

$$
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_{i})^{2} h(\lambda_{m+1} - \lambda_{i}) \rho_{i} \leq \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_{i}) h(\lambda_{m+1} - \lambda_{i}) \Lambda_{i}.
$$

(24)

3. Setting $f(\lambda) = (\lambda_{m+1} - \lambda)^{2} g(\lambda)$, or, equivalently, in the notation of Remark 2, $f(\lambda) = (\lambda_{m+1} - \lambda)^{2} h(\lambda_{m+1} - \lambda)$, for $0 < \lambda \leq \lambda_{m+1}$, (12) can be written as

$$
\sum_{i=1}^{m} f(\lambda_{i}) \rho_{i} \leq \sum_{i=1}^{m} \frac{f(\lambda_{i})}{\lambda_{m+1} - \lambda_{i}} \Lambda_{i}.
$$

(25)

This is the statement of Theorem 5 in [22] (when one specializes to their setting, which leads to $\rho_{i} = N, \Lambda_{i} = 4\lambda_{i}$, as in our Corollary 4 below). The condition that the function $f(\lambda)(\lambda_{m+1} - \lambda)^{-2}$ (in their case) be nondecreasing is equivalent to the statement of Theorem 2 which seems more natural to the problem. The H. C. Yang type inequality (11) obtains when $f(\lambda) = (\lambda_{m+1} - \lambda)^{2}$, i.e., when $g(\lambda) \equiv 1$ (or, equivalently, when $h(\lambda) \equiv 1$). As noted earlier, the second term on the right-hand side of (23) is identically zero in this case.

**Corollary 3.** Let $p \leq 2$. Then

$$
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_{i})^{2} \rho_{i} \leq \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_{i})^{p-1} \Lambda_{i}.
$$

(26)

**Proof.** We make the choice $g(\lambda) = (\lambda_{m+1} - \lambda)^{p-2}$ (or equivalently $h(\lambda) = \lambda^{p-2}$ if applying Remark 2) for $\lambda \geq 0$ in Theorem 2. □

**Remark.** We note that (11) and (10) are particular instances of this corollary (for $p = 2$ and $p = 0$, respectively) while (9) is a weaker result obtained from (10) by replacing $\lambda_{m+1} - \lambda_{i}$ by $\lambda_{m+1} - \lambda_{m}$.

**Corollary 4.** Suppose $A = -\sum_{k=1}^{N} T_{k}^{2}$ where the $T_{k}$’s are skew-symmetric with domains $\mathcal{D}(T_{k})$ such that $\mathcal{D} = \mathcal{D}(A) \subset \mathcal{D}(T_{k})$ and $T_{k}(\mathcal{D}) \subset \mathcal{D}(T_{k})$ and suppose that $[T_{k}, B_{k}]u = \delta_{kk}u$. Then $\rho_{i} = N, \Lambda_{i} = \ldots$
4\lambda_i, and for \( p \leq 2 \)

\[
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{4}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} \lambda_i.
\] (27)

**Proof.** The details of the calculations of \( \rho_i \) and \( \Lambda_i \) in this case are provided by Lemma 2.3 and Corollary 2.4 of [9]. \( \square \)

**Remark.** Inequality (3) is a particular case of this corollary with \( A = -\Delta, T_k = \frac{\partial}{\partial x_k}, B_k = x_k \), and \( N = n \), the spatial dimension.

For a symmetric operator \( C \) and \( \alpha \in \mathbb{R} \), \( C \geq \alpha \) if \( \langle Cu, u \rangle \geq \alpha \langle u, u \rangle \) for all vectors \( u \in \mathcal{D}(C) \). Moreover, \( A \geq B \) for symmetric operators \( A \) and \( B \) if \( \mathcal{D}(B) \subset \mathcal{D}(A) \) and \( A - B \geq 0 \) on \( \mathcal{D}(B) \).

**Corollary 5.** Suppose there exist \( \gamma, \beta \) such that

\[
0 < \gamma \leq [B_k, [A, B_k]]
\] (28)

and

\[
-\sum_{k=1}^{N} [A, B_k]^2 \leq \beta A.
\] (29)

Then, for \( p \leq 2 \),

\[
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{2\beta}{\gamma N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} \lambda_i.
\] (30)

**Proof.** As observed in [9] (see Theorem 2.5) the conditions of this corollary yield immediately \( \rho_i \geq \frac{1}{2} \gamma N \) and \( \Lambda_i \leq \beta \lambda_i \). Substituting these inequalities into Corollary 3, we obtain the desired result. \( \square \)

We now deal with a second set of inequalities treated by Harrell-Stubbe in [22]. We adopt their definition: A real function \( f(x) \) is said to satisfy condition (H1) if there exists a function \( r(x) \) such that

\[
(H1) \quad \frac{f(x) - f(y)}{x - y} \geq \frac{r(x) + r(y)}{2}.
\]

As an example, a function whose derivative \( f' \) is concave satisfies this condition.
**Lemma 6.** Suppose a $C^1$ function $f$ is such that its derivative $f'$ is concave. Then $f$ satisfies (H1) with $r(x) = f'(x)$.

**Proof.** For each $\xi$ between $x$ and $y$, $\exists \mu \in [0, 1]$ such that $\xi = \mu x + (1 - \mu) y$. Without loss of generality, we can assume $x \geq y$. Since $f'$ is concave

$$f'(\xi) \geq \mu f'(x) + (1 - \mu) f'(y).$$

Integrating over $\xi$ from $y$ to $x$ yields (on the right we integrate in $\mu$ from 0 to 1 noting that $d\xi = (x - y)d\mu$)

$$f(x) - f(y) \geq \frac{1}{2}(x - y)(f'(x) + f'(y)),$$

and the lemma is immediate. \(\square\)

**Theorem 7.** Let $f(x)$ be an (H1) function for some $r(x)$. Then, we have

$$\sum_{i=1}^{m} f(\lambda_i)p_i \leq -\frac{1}{2} \sum_{i=1}^{m} r(\lambda_i)\Lambda_i + R,$$

where

$$R = \sum_{k=1}^{N} \sum_{i=1}^{m} \sum_{j=m+1}^{\infty} |\langle [A, B_k]u_i, u_j \rangle|^2 \left( \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i} + \frac{1}{2} r(\lambda_i) \right).$$

**Remark.** We recall that, since $\{u_i\}_{i=1}^{\infty}$ is a basis for $\mathcal{H}$, one may write

$$[A, B_k]u_i = \sum_{j=1}^{\infty} \langle [A, B_k]u_i, u_j \rangle u_j.$$  \hspace{1cm} (33)

Furthermore, we have

$$\| [A, B_k]u_i \|^2 = \sum_{j=1}^{\infty} |\langle [A, B_k]u_i, u_j \rangle|^2.$$  \hspace{1cm} (34)

This makes

$$\sum_{j=m+1}^{\infty} |\langle [A, B_k]u_i, u_j \rangle|^2 < \infty$$  \hspace{1cm} (35)

for each $i = 1, \cdots, m$. Thus the expression for $R$ given above is well-defined.
Proof. With the substitution \( f(\lambda) = (\lambda_{m+1} - \lambda)^2 g(\lambda) \) and, \textit{a priori}, no conditions on the function \( g(\lambda) \), calculations down to (23) can be carried out as above.

Recalling the definitions of \( \Lambda_i \) in (8) and that of \( A_{ij} \) in (20), we rewrite (23) in the form

\[
\sum_{i=1}^{m} f(\lambda_i) \rho_i \leq \sum_{k=1}^{N} \sum_{i=1}^{m} \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i} \|[A, B_k]u_i\|^2
- \frac{1}{2} \sum_{k=1}^{N} \sum_{i,j=1}^{m} (\lambda_{m+1} - \lambda_i)(\lambda_{m+1} - \lambda_j)(\lambda_i - \lambda_j) |\langle B_k u_i, u_j \rangle|^2 \times \left( \frac{f(\lambda_i)}{(\lambda_{m+1} - \lambda_i)^2} - \frac{f(\lambda_j)}{(\lambda_{m+1} - \lambda_j)^2} \right).
\]

(36)

The gap formula \( \langle [A, B_k]u_i, u_j \rangle = (\lambda_j - \lambda_i) \langle B_k u_i, u_j \rangle \) gives

\[
\sum_{i=1}^{m} f(\lambda_i) \rho_i \leq \sum_{k=1}^{N} \sum_{i=1}^{m} \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i} \|[A, B_k]u_i\|^2
- \frac{1}{2} \sum_{k=1}^{N} \sum_{i,j=1}^{m} (\lambda_{m+1} - \lambda_i)(\lambda_{m+1} - \lambda_j) \frac{|\langle [A, B_k]u_i, u_j \rangle|^2}{\lambda_i - \lambda_j} \times \left( \frac{f(\lambda_i)}{(\lambda_{m+1} - \lambda_i)^2} - \frac{f(\lambda_j)}{(\lambda_{m+1} - \lambda_j)^2} \right).
\]

(37)

(If \( \lambda_i \) ever equals \( \lambda_j \) here, one should interpret the term(s) in which this occurs as 0 by using the gap formula in reverse.) As noted in [22], the second term on the right-hand side can be reduced to

\[
- \frac{1}{2} \sum_{k=1}^{N} \sum_{i,j=1}^{m} |\langle [A, B_k]u_i, u_j \rangle|^2 \left( \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} + \frac{f(\lambda_j)}{\lambda_{m+1} - \lambda_j} + \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i} \right).
\]

(38)
Symmetry of this expression in $i$ and $j$ reduces (37) to
\[
\sum_{i=1}^{m} f(\lambda_i) \rho_i \leq \sum_{k=1}^{N} \sum_{i=1}^{m} \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i} \|[A, B_k]u_i\|^2 \\
- \frac{1}{2} \sum_{k=1}^{N} \sum_{i,j=1}^{m} \left|\langle [A, B_k]u_i, u_j \rangle \right|^2 \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \\
- \sum_{k=1}^{N} \sum_{i,j=1}^{m} \left|\langle [A, B_k]u_i, u_j \rangle \right|^2 \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i},
\] (39)
i.e.,
\[
\sum_{i=1}^{m} f(\lambda_i) \rho_i \leq \sum_{k=1}^{N} \sum_{i=1}^{m} \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i} \left(\|[A, B_k]u_i\|^2 - \sum_{j=1}^{m} \left|\langle [A, B_k]u_i, u_j \rangle \right|^2\right) \\
- \frac{1}{2} \sum_{k=1}^{N} \sum_{i,j=1}^{m} \left|\langle [A, B_k]u_i, u_j \rangle \right|^2 \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}.
\] (40)
Since $f$ satisfies condition (H1), this reduces to
\[
\sum_{i=1}^{m} f(\lambda_i) \rho_i \leq \sum_{k=1}^{N} \sum_{i=1}^{m} \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i} \left(\|[A, B_k]u_i\|^2 - \sum_{j=1}^{m} \left|\langle [A, B_k]u_i, u_j \rangle \right|^2\right) \\
- \frac{1}{4} \sum_{k=1}^{N} \sum_{i,j=1}^{m} \left|\langle [A, B_k]u_i, u_j \rangle \right|^2 \left(r(\lambda_i) + r(\lambda_j)\right).
\] (41)
Symmetry in $i$ and $j$ reduces the second term of the right-hand side to
\[
-\frac{1}{2} \sum_{k=1}^{N} \sum_{i,j=1}^{m} r(\lambda_i) \left|\langle [A, B_k]u_i, u_j \rangle \right|^2.
\] (42)
This, with identity (34), gives
\[
\sum_{i=1}^{m} f(\lambda_i) \rho_i \leq \sum_{k=1}^{N} \sum_{i=1}^{m} \sum_{j=m+1}^{\infty} \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i} \left|\langle [A, B_k]u_i, u_j \rangle \right|^2 \\
- \frac{1}{2} \sum_{k=1}^{N} \sum_{i,j=1}^{m} r(\lambda_i) \left|\langle [A, B_k]u_i, u_j \rangle \right|^2.
\] (43)
Noting that (34) gives
\[
\sum_{j=1}^{m} |\langle [A, B_k]u_i, u_j \rangle|^2 = \| [A, B_k]u_i \|^2 - \sum_{j=m+1}^{\infty} |\langle [A, B_k]u_i, u_j \rangle|^2, \tag{44}
\]
we obtain
\[
\sum_{i=1}^{m} f(\lambda_i) \rho_i \leq \sum_{k=1}^{N} \sum_{i=1}^{m} \sum_{j=m+1}^{\infty} \frac{f(\lambda_i)}{\lambda_{m+1} - \lambda_i} |\langle [A, B_k]u_i, u_j \rangle|^2 \\
+ \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{m} \sum_{j=m+1}^{\infty} r(\lambda_i) |\langle [A, B_k]u_i, u_j \rangle|^2 \\
- \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{m} r(\lambda_i) \| [A, B_k]u_i \|^2, \tag{45}
\]
which, upon incorporating the definitions of \( \Lambda_i \) and \( \mathcal{R} \), is the statement of the theorem. \( \square \)

**Corollary 8.** Let \( p \geq 2 \), then
\[
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \rho_i \leq \frac{p}{2} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} \Lambda_i. \tag{46}
\]

**Proof.** For \( p \geq 2 \), \( \lambda \leq \lambda_{m+1} \), \( f(\lambda) = (\lambda_{m+1} - \lambda)^p \), is such that
\[
f'(\lambda) = -p(\lambda_{m+1} - \lambda)^{p-1}
\]
is concave, and
\[
\frac{f(\lambda)}{\lambda_{m+1} - \lambda} + \frac{1}{2} r(\lambda) = \left(1 - \frac{p}{2}\right) (\lambda_{m+1} - \lambda)^{p-1} \leq 0.
\]
Thus \( \mathcal{R} \leq 0 \) and inequality (31) completes the proof. \( \square \)

Using the same function \( f(\lambda) \) as in the proof of Corollary 8 and the skew-symmetric operators of Corollary 4 yields an analog of Corollary 4 for the case \( p \geq 2 \):

**Corollary 9.** Suppose \( A = -\sum_{k=1}^{N} T_k^2 \) where the \( T_k \)'s are skew-symmetric with the same conditions as those of Corollary 4. Then
for $p \geq 2$

$$\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{2p}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} \lambda_i.$$  \hfill (47)

Similarly, one obtains a $p \geq 2$ analog of Corollary 5:

**Corollary 10.** Let $p \geq 2$, and suppose there exist $\gamma, \beta$ such that the conditions of Corollary 5 are satisfied, then

$$\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{p\beta}{\gamma N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} \lambda_i.$$  \hfill (48)

Corollaries 9 and 10 follow from the facts about $\rho_i$ and $\Lambda_i$ given in conjunction with Corollaries 4 and 5, respectively.

**IV. The Case of a Schrödinger-like Operator**

In this section, we consider an operator $H = A + V$ defined on $\mathcal{D} \subset \mathcal{H}$, where $A$ and $V$ are self-adjoint operators, $A = -\sum_{k=1}^{N} T_k^2$, and the $T_k$’s are skew-symmetric with domains $T_k(\mathcal{D})$ satisfying $\mathcal{D} \equiv \mathcal{D}(A) \subset \mathcal{D}(T_k)$ and $T_k(\mathcal{D}) \subset \mathcal{D}(T_k)$. This operator is modeled on the Schrödinger operator. We assume that the spectrum of $H$ is discrete consisting of eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$, and we let $\{u_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of eigenvectors corresponding to $\{\lambda_i\}_{i=1}^{\infty}$. We further take a family of symmetric operators $\{B_k : H(\mathcal{D}) \to \mathcal{H}\}_{k=1}^{N}$ which leave $\mathcal{D}$ invariant, such that $[T_k, B_k]u_i = \delta_{ik}u_i$. As in Section II, the quantities $\rho_i$ and $\Lambda_i$ are given by

$$\rho_i = \sum_{k=1}^{N} \langle [H, B_k]u_i, B_ku_i \rangle,$$

and

$$\Lambda_i = \sum_{k=1}^{N} ||[H, B_k]u_i||^2.$$  

In obvious notation, we have $\rho_i = \rho_i^A + \rho_i^V$, corresponding to the decomposition $H = A + V$. The following theorem generalizes Theorem 4.1 of [9].
Theorem 11. Suppose \([V, B_k] = 0\) for \(1 \leq k \leq N\). Then \(\rho_i = N\), \(\Lambda_i = 4(\lambda_i - \langle Vu, u_i \rangle)\). Moreover, for \(p \leq 2\)

\[
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{4}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1}(\lambda_i - \langle Vu, u_i \rangle)
\]

and for \(p \geq 2\)

\[
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{2p}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1}(\lambda_i - \langle Vu, u_i \rangle).
\]

Proof. For the details of the calculations of \(\rho_i\) and \(\Lambda_i\), see [9]. The rest follows via our previous considerations. \(\square\)

Corollary 12. Suppose \(V \geq M > 0\). Then, the inequalities in the previous theorem reduce to

\[
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{4}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1}(\lambda_i - M) \text{ for } p \leq 2
\]

and

\[
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{2p}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1}(\lambda_i - M) \text{ for } p \geq 2.
\]

V. Comparing the Bounds

This section deals with the different bounds for \(\lambda_{m+1}\) arising from the Harrell and Stubbe considerations [22] and their extensions as detailed above. We will assume that the operators \(A\) and \(B_k, 1 \leq k \leq N\), satisfy the conditions (28) and (29) of Corollary 5, namely

\[
\gamma \leq [B_k, [A, B_k]]
\]

and

\[
-\sum_{k=1}^{N} [A, B_k]^2 \leq \beta A
\]

for some \(\beta, \gamma > 0\).
A. Case $p \leq 2$

We first treat the case $p \leq 2$, namely inequality (30) (or (3) in the Introduction). We assume $m \geq 2$. For $m = 1$ all bounds reduce to

$$\lambda_2 \leq \left(1 + \frac{2\beta}{\gamma N}\right) \lambda_1.$$  

We set

$$f_p(\sigma) = \frac{1}{m} \sum_{i=1}^{m} (\sigma - \lambda_i)^p - \frac{2\beta}{\gamma N} \frac{1}{m} \sum_{i=1}^{m} (\sigma - \lambda_i)^{p-1} \lambda_i,$$

for $\sigma \geq \lambda_m$. The unique zero of $f_p(\sigma)$ larger than $\lambda_m$ is denoted by $\sigma_p$ (the existence and uniqueness of $\sigma_p$ are addressed in Proposition 13 below). It can be thought of as a function of the moments $S_\ell$, for $\ell = 1, 2, \cdots$, in the eigenvalues,

$$S_\ell = \frac{1}{m} \sum_{i=1}^{m} \lambda_i^\ell. \quad (53)$$

This point of view becomes clear upon expansion in infinite series in $\sigma$ (unless $p = 1$ or 2). We first establish the following.

**Proposition 13.** For $0 \leq p \leq 2$, there exists a unique root $\sigma_p$ of $f_p(\sigma) = 0$ larger than $\lambda_m$.

**Remark.** The $p = 2$ case of this proposition is treated in detail in [9] (see also [3]). This proposition also holds when $m = 1$, and then $\sigma_p = (1 + \frac{2\beta}{\gamma N}) \lambda_1$ (for all $p$).

**Proof.** Existence. For $2 \geq p > 1$, $f_p(\lambda_m) \leq 0$ by Corollary 5 with $m$ replaced by $m - 1$. This also holds, but as a strict inequality, for $p = 1$, since the inequality follows from the $p = 1$ case of Corollary 5 (with $m$ replaced by $m - 1$), but $f_p(\lambda_m)$ has one extra strictly negative term coming from the $i = m$ term in the second sum in the definition of $f_p$. Finally, for $1 > p \geq 0$,

$$\lim_{\sigma \rightarrow \lambda_m^+} f_p(\sigma) = -\infty.$$  

Moreover, for $p > 0$

$$\lim_{\sigma \rightarrow \infty} f_p(\sigma) = \infty,$$
while for \( p = 0 \)

\[
\lim_{\sigma \to \infty} f_0(\sigma) = 1.
\]

Hence, by continuity, the existence of a zero larger than \( \lambda_m \) of \( f_p(\sigma) = 0 \) (where \( 0 \leq p \leq 2 \)) is guaranteed. In fact we can say more.

**Uniqueness.** First, observe that

\[
f_0(\sigma) = 1 - \frac{2\beta}{\gamma N} \frac{1}{m} \sum_{i=1}^{m} \frac{\lambda_i}{\sigma - \lambda_i}.
\]

Hence \( f_0'(\sigma) > 0 \) and \( f_0(\sigma) \) is monotonically increasing from \( -\infty \) to 1. This establishes the uniqueness of \( \sigma_0 \) (this is the Hile-Protter bound for \( \lambda_{m+1} \) derived in [24] (see also [3], [2], [9], [20], [21], [22], [34], [39]). For \( 0 < p \leq 1 \),

\[
f_p'(\sigma) = \frac{1}{m} \left( p \sum_{i=1}^{m} (\sigma - \lambda_i)^{p-1} - (p-1) \frac{2\beta}{\gamma N} \sum_{i=1}^{m} (\sigma - \lambda_i)^{p-2} \lambda_i \right).
\]

(The second term is identically 0 if \( p = 1 \).) Since \( p - 1 \leq 0 \) for \( 0 < p \leq 1 \), \( f_p'(\sigma) > 0 \) for \( \sigma > \lambda_m \) and the uniqueness of \( \sigma_p \) is established in this case. Note that this handles the \( p = 1 \) case since, as already observed, \( f_1(\lambda_m) < 0 \) (strict inequality). This case can also be treated via explicit and elementary calculation (as can the \( p = 2 \) case).

For \( 1 < p < 2 \), we note that \( f_p'(\sigma) \) is not clearly of one sign as before, since \( f_p'(\sigma) \to -\infty \) as \( \sigma \to \lambda_m^+ \), while \( f_p'(\sigma) \to \infty \) as \( \sigma \to \infty \). We therefore have recourse to a convexity argument. Differentiating, it becomes clear that in this case \( f_p''(\sigma) > 0 \) for \( \sigma > \lambda_m \). Hence \( f_p'(\sigma) \) is strictly increasing from \( -\infty \) (value at \( \lambda_m \) for \( 1 < p < 2 \)) to \( \infty \) (value at \( \infty \)). One can then find a unique \( \xi_p > \lambda_m \) for which \( f_p'(\xi_p) = 0 \). Moreover, \( f_p'(\sigma) < 0 \) for \( \lambda_m < \sigma < \xi_p \) and \( f_p'(\sigma) > 0 \) for \( \sigma > \xi_p \). The uniqueness of \( \sigma_p \) is therefore ascertained with \( \lambda_m < \xi_p < \sigma_p \). Finally, the case of \( p = 2 \) follows easily in much the same way as for \( 1 < p < 2 \) using now the fact that \( f_2(\sigma) \) is a quadratic in \( \sigma \) with second order term \( \sigma^2 \). Thus \( f_p \) is again concave up and the result follows.

□

Since \( f_p(\lambda_{m+1}) \leq 0 \) (viz., (30)), it obtains that \( \lambda_{m+1} \leq \sigma_p \) for any \( 0 \leq p \leq 2 \). Moreover, since \( f_p((1 + \frac{2\beta}{\gamma N})\lambda_m) > 0 \), we have the inequalities
\[ \lambda_m \leq \lambda_{m+1} \leq \sigma_p < \left(1 + \frac{2\beta}{\gamma N}\right) \lambda_m. \quad (54) \]

We are now ready to prove the statement announced in [9]: “\( \sigma_p \) improves with \( p \), for \( p \leq 2 \)” This is contained in the following theorem.

**Theorem 14.**

\[ \lambda_{m+1} \leq \sigma_{p_2} \leq \sigma_{p_1} \quad \text{if} \quad 0 \leq p_1 \leq p_2 \leq 2. \quad (55) \]

**Proof.** This is done in several reductions. We observe that the statement \( \sigma_{p_2} \leq \sigma_{p_1} \) is equivalent to showing that \( f_{p_1}(\sigma_{p_2}) \leq 0 \) (since \( f_{p_1}(\sigma) \) is below the \( \sigma \)-axis on \( (\lambda_m, \sigma_{p_1}) \)). That is

\[
\sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_1} \leq \frac{2\beta}{\gamma N} \sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_1-1} \lambda_i.
\]

Since

\[
\frac{2\beta}{\gamma N} = \frac{\sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_2}}{\sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_2-1} \lambda_i},
\]

the statement of the theorem is then equivalent to

\[
\frac{\sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_1}}{\sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_1-1} \lambda_i} \leq \frac{\sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_2}}{\sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_2-1} \lambda_i}.
\]

Or

\[
\sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_1} \sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_2-1} \lambda_i \leq \sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_2} \sum_{i=1}^{m} (\sigma_{p_2} - \lambda_i)^{p_1-1} \lambda_i.
\]

We now use the following version of the “Chebyshev Inequality” (see, for example, p. 43 of [15]).

**Lemma 15 (Weighted Reverse Chebyshev Inequality).** Let \( \{a_i\}_{i=1}^{m} \) and \( \{b_i\}_{i=1}^{m} \) be two oppositely ordered real sequences, and let \( \{w_i\}_{i=1}^{m} \) be a sequence of nonnegative weights. Then the following
inequality holds
\[
\sum_{i=1}^{m} w_i \sum_{i=1}^{m} w_i a_i b_i \leq \sum_{i=1}^{m} w_i a_i \sum_{i=1}^{m} w_i b_i. \tag{58}
\]

Inequality (57) is then a corollary to this lemma with \( w_i = (\sigma_{p_2} - \lambda_i)^{p_1} \), \( a_i = \frac{\lambda_i}{\sigma_{p_2} - \lambda_i} \), and \( b_i = (\sigma_{p_2} - \lambda_i)^{p_2-p_1} \). The sequence \( \{a_i\} \) is increasing, while the sequence \( \{b_i\} \) is decreasing by virtue of the fact that \( p_2 \geq p_1 \). Hence the result of the theorem. \( \square \)

**Remarks.**

1. This theorem contains the statement announced by H. C. Yang [41] that his inequality \((p = 2; \text{ also referred to as } \text{“Yang 1”})\) implies an “averaged” version of this inequality \((p = 1; \text{ also referred to as } \text{“Yang 2”})\) which in turn implies the Hile-Protter result \((p = 0)\). In [2], this statement is summarized in the implication that (for each \( m = 1, 2, \ldots \))

\[
\text{Yang 1 } \implies \text{ Yang 2 } \implies \text{ Hile-Protter}.
\]

2. A proof of this result is not given in [41]. Proofs are given in [3] and [9] (see also [2]). Our proof here is basically that of [9], but gives a more general result. This theorem shows that of the class of Harrell-Stubbe-type inequalities with \( p \leq 2 \), the optimum obtains when \( p = 2 \) (H. C. Yang).

3. The PPW inequality is of course weaker than the HP inequality. Thus the HP inequality provides a tighter bound for \( \sigma_p \) than the bound

\[
\sigma_p \leq \lambda_m + \frac{2\beta}{\gamma N} S_1, \tag{59}
\]

which is the PPW inequality in this setting. Also, it is perhaps worth noting that the HP inequality provides a tighter bound for \( \sigma_p \) than that given in (54), viz.,

\[
\sigma_p < \left(1 + \frac{2\beta}{\gamma N}\right) \lambda_m
\]

(which is itself a simple consequence of the PPW inequality).
4. In terms of the “moments”, \( S_\ell = \frac{1}{m} \sum_{i=1}^{m} \lambda_i^\ell \), Yang 2 reads,

\[
\sigma_1 = \left(1 + \frac{2\beta}{\gamma N}\right) S_1,
\]

(60)

while Yang 1 translates as

\[
\sigma_2 = \left(1 + \frac{\beta}{\gamma N}\right) S_1 + \left\{ \left(1 + \frac{\beta}{\gamma N}\right)^2 S_1^2 - \left(1 + \frac{2\beta}{\gamma N}\right) S_2 \right\}^{1/2}.
\]

(61)

B. Case \( p \geq 2 \)

We now turn our attention to Harrell and Stubbe’s second extension of H. C. Yang’s result, namely Corollary 10 (where \( p \geq 2 \)). We will show that bounds for \( \lambda_{m+1} \) provided by (48) (or (2) in the Introduction) obtained when \( p \geq 2 \) are weaker than those for \( p = 2 \). In fact they get worse monotonically with increasing \( p \).

In this case, the function \( f_p(\sigma) \) takes a slightly altered form, which we denote by \( \tilde{f}_p(\sigma) \):

\[
\tilde{f}_p(\sigma) = \frac{1}{m} \sum_{i=1}^{m} (\sigma - \lambda_i)^p - \frac{p\beta}{\gamma N} \frac{1}{m} \sum_{i=1}^{m} (\sigma - \lambda_i)^{p-1} \lambda_i.
\]

(62)

We will denote this function by \( \tilde{f}_{p,m}(\sigma) \) in case the explicit dependence of \( \tilde{f}_p(\sigma) \) on \( m \geq 1 \) is required. The existence of a root of \( \tilde{f}_p(\sigma) = 0 \) greater than or equal to \( \lambda_m \) is guaranteed. This is because of \( \tilde{f}_{p,m}(\lambda_m) = \frac{m-1}{m} \tilde{f}_{p,m-1}(\lambda_m) \leq 0 \) (viz., (48)) and \( \lim_{\sigma \to 0} \tilde{f}_p(\sigma) = \infty \), and because \( \tilde{f}_p(\sigma) \) is continuous on \((\lambda_m, \infty)\).

We handle this case somewhat differently from how we handled the case for \( 0 \leq p \leq 2 \). In that case we established the existence of a unique root of \( f_p(\sigma) \) greater than \( \lambda_m \). In the present case we do not establish uniqueness (although it may well obtain) but rather define (existence follows from our comments above) a root \( \tilde{\sigma}_p \) of \( \tilde{f}_p(\sigma) \) which is greater than or equal to \( \lambda_m \) and serves our purposes. For \( p \geq 2 \) we define \( \tilde{\sigma}_p \) via

\[
\tilde{\sigma}_p = \sup\{s \geq \lambda_m | \tilde{f}_p(\sigma) \leq 0 \text{ for } \lambda_m \leq \sigma \leq s\}.
\]

(63)
By continuity, \( \tilde{\sigma}_p \) is in fact realized as a maximum over the set given on the right, and we have \( \tilde{f}_p(\sigma) \leq 0 \) for \( \lambda_m \leq \sigma \leq \tilde{\sigma}_p \). Indeed, it must also be true that \( \tilde{f}_p(\tilde{\sigma}_p) = 0 \), i.e., that \( \tilde{\sigma}_p \) is a root of \( \tilde{f}_p(\sigma) \) which is greater than or equal to \( \lambda_m \). Note, too, that from the fact that \( \tilde{f}_2(\sigma) \) is a quadratic with leading term \( \sigma^2 \) and that \( \tilde{f}_2(\sigma) \leq 0 \) on \([\lambda_m, \lambda_{m+1}]\), it is clear that \( \tilde{\sigma}_2 \) as defined above is identical to \( \sigma_2 \) as defined previously (for the case of \( 0 \leq p \leq 2 \)), which is also just the explicit upper bound for \( \lambda_{m+1} \) coming from Yang’s first (or main) inequality, that is, the expression given on the right-hand side of (61).

As remarked earlier in a similar context, \( \tilde{\sigma}_p \) can be thought of as a function of the moments in the first \( m \) eigenvalues providing an upper bound for \( \lambda_{m+1} \). In this case, \( \tilde{f}_p(\sigma) \) takes the form

\[
\tilde{f}_p(\sigma) = \sigma^p + \sum_{k=1}^{N_p} (-1)^k \binom{p}{k} \left( 1 + \frac{\beta k}{\gamma N} \right) S_k \sigma^{p-k}
\]

where \( S_k \) is defined by (53) and \( N_p = p \) if \( p \) is an integer and \( N_p = \infty \) if \( p \) is not an integer. By convention, the binomial coefficient \( \binom{p}{k} \) denotes

\[
\frac{p(p-1)(p-2) \cdots (p-k+1)}{k!}
\]
even when \( p \) is not an integer.

To proceed, we need to know that \( \tilde{\sigma}_p \) as defined above for \( p > 2 \) really does provide a bound for \( \lambda_{m+1} \) (for this it is not enough to know Corollary 10, i.e., ineq. (48), since our definition of \( \tilde{\sigma}_p \) does not preclude the possibility that \( \tilde{f}(\sigma) = 0 \) has further roots beyond \( \tilde{\sigma}_p \), or further places where \( \tilde{f}_p(\sigma) < 0 \), and that \( \lambda_{m+1} \) is then somewhere to the right of \( \tilde{\sigma}_p \) as defined above).

There are (at least) three ways we could think to proceed at this point.

1. Use the \( p = 2 \) case and a technique of Aizenman and Lieb [1] (see also [27], [28]) to show that \([\lambda_m, \tilde{\sigma}_2] \subset [\lambda_m, \tilde{\sigma}_p] \) for \( p > 2 \) and hence that \( \lambda_{m+1} \leq \tilde{\sigma}_2 \leq \tilde{\sigma}_p \) for \( p \geq 2 \), by our definition of \( \tilde{\sigma}_p \).

2. Specialize to the case of ineq. (2) from our Introduction, i.e., to the case of the Laplacian (and certain generalizations), where Harrell and Stubbe [22] have already provided results implying that...
\[ \lambda_{m+1} \leq \tilde{\sigma}_p \] (and indeed that \( \tilde{f}_p(\sigma) \leq 0 \) on \([\lambda_m, \lambda_{m+1}]\)). For these results we refer to ineq. (14) in Theorem 9 on p. 1805 and, in particular, the conditions that go with it. Specifically, their results show that (for \( \tilde{f}_p(\sigma) \) as in (62) but with \( \beta p/\gamma N \) replaced by \( 2p/n \)) \( \tilde{\sigma}_p \geq \lambda_m + (p/2)(\lambda_{m+1} - \lambda_m) \) and since \( \lambda_m + (p/2)(\lambda_{m+1} - \lambda_m) > \lambda_{m+1} \) for \( p > 2 \), the desired result follows. One can consult [22], [10] for generalizations of \(-\Delta\) to which the Harrell-Stubbe results are already known to apply.

(3) Extend the approach and methods of Harrell and Stubbe [22] so that we know that, for the operators considered here and for \( \tilde{f}_p(\sigma) \) as defined by (62), \( \tilde{f}_p(\sigma) \leq 0 \) for all \( \sigma \in [\lambda_m, \tilde{\sigma}_2] \) and \( p \geq 2 \). Thus, under this approach one would begin by seeking a version of Harrell and Stubbe’s Theorem 9 (p. 1805 of their paper) that applies in our general setting.

In what follows we will follow (1) since it gives the most self-contained approach from our chosen point of view. One could also build on (2), which puts one farther along with the problem at the start, but, as mentioned above, leads to a more restricted result. Finally, (3) would probably also work, and lead to results analogous to and as general as those of (1) (even, perhaps, to results which are a bit stronger), but as we have not worked through the details of this we leave it aside.

**Theorem 16.** Suppose \( p \geq 2 \). Then \( \tilde{f}_p(\sigma) \leq 0 \) for all \( \sigma \in [\lambda_m, \tilde{\sigma}_2] \) and hence, since \( \lambda_{m+1} \leq \tilde{\sigma}_2, \lambda_{m+1} \leq \tilde{\sigma}_p \). Moreover \( \tilde{\sigma}_p \geq \tilde{\sigma}_2 (= \sigma_2) \).

**Proof.** We know that

\[
 m\tilde{f}_2(\sigma) = \sum_{i=1}^{m}(\sigma - \lambda_i)^2 - \frac{2\beta}{\gamma N} \sum_{i=1}^{m}(\sigma - \lambda_i) \lambda_i \leq 0 \quad (64)
\]

for all \( \sigma \in [\lambda_m, \tilde{\sigma}_2] \), and, in particular, for all \( \sigma \in [\lambda_m, \lambda_{m+1}] \). Because this holds for all values of \( m \) (and specifically for 1, 2, \ldots, \( m \) replacing \( m \) above), if we introduce the notation

\[
(\lambda - t)_+ = \begin{cases} 
\lambda - t & \text{if } t \leq \lambda, \\
0 & \text{if } t > \lambda,
\end{cases} \quad (65)
\]

\[ \lambda_{m+1} \leq \tilde{\sigma}_p \] (and indeed that \( \tilde{f}_p(\sigma) \leq 0 \) on \([\lambda_m, \lambda_{m+1}]\)). For these results we refer to ineq. (14) in Theorem 9 on p. 1805 and, in particular, the conditions that go with it. Specifically, their results show that (for \( \tilde{f}_p(\sigma) \) as in (62) but with \( \beta p/\gamma N \) replaced by \( 2p/n \)) \( \tilde{\sigma}_p \geq \lambda_m + (p/2)(\lambda_{m+1} - \lambda_m) \) and since \( \lambda_m + (p/2)(\lambda_{m+1} - \lambda_m) > \lambda_{m+1} \) for \( p > 2 \), the desired result follows. One can consult [22], [10] for generalizations of \(-\Delta\) to which the Harrell-Stubbe results are already known to apply.

(3) Extend the approach and methods of Harrell and Stubbe [22] so that we know that, for the operators considered here and for \( \tilde{f}_p(\sigma) \) as defined by (62), \( \tilde{f}_p(\sigma) \leq 0 \) for all \( \sigma \in [\lambda_m, \tilde{\sigma}_2] \) and \( p \geq 2 \). Thus, under this approach one would begin by seeking a version of Harrell and Stubbe’s Theorem 9 (p. 1805 of their paper) that applies in our general setting.

In what follows we will follow (1) since it gives the most self-contained approach from our chosen point of view. One could also build on (2), which puts one farther along with the problem at the start, but, as mentioned above, leads to a more restricted result. Finally, (3) would probably also work, and lead to results analogous to and as general as those of (1) (even, perhaps, to results which are a bit stronger), but as we have not worked through the details of this we leave it aside.

**Theorem 16.** Suppose \( p \geq 2 \). Then \( \tilde{f}_p(\sigma) \leq 0 \) for all \( \sigma \in [\lambda_m, \tilde{\sigma}_2] \) and hence, since \( \lambda_{m+1} \leq \tilde{\sigma}_2, \lambda_{m+1} \leq \tilde{\sigma}_p \). Moreover \( \tilde{\sigma}_p \geq \tilde{\sigma}_2 (= \sigma_2) \).

**Proof.** We know that

\[
 m\tilde{f}_2(\sigma) = \sum_{i=1}^{m}(\sigma - \lambda_i)^2 - \frac{2\beta}{\gamma N} \sum_{i=1}^{m}(\sigma - \lambda_i) \lambda_i \leq 0 \quad (64)
\]

for all \( \sigma \in [\lambda_m, \tilde{\sigma}_2] \), and, in particular, for all \( \sigma \in [\lambda_m, \lambda_{m+1}] \). Because this holds for all values of \( m \) (and specifically for 1, 2, \ldots, \( m \) replacing \( m \) above), if we introduce the notation

\[
(\lambda - t)_+ = \begin{cases} 
\lambda - t & \text{if } t \leq \lambda, \\
0 & \text{if } t > \lambda,
\end{cases} \quad (65)
\]
then (64) extends to all $\sigma \leq \bar{\sigma}_p$ as
\[
\sum_{i=1}^{m} (\sigma - \lambda_i)^2_+ - \frac{2\beta}{\gamma N} \sum_{i=1}^{m} (\sigma - \lambda_i)_+ \lambda_i \leq 0. \tag{66}
\]

Note that each time $\sigma$ passes below a $\lambda_i$ another term drops away (on both sides), leaving us with a variant of ineq. (64) where the only change is that $m$ is less. And finally, when $\sigma$ crosses $\lambda_1$ we are left with the trivial inequality $0 \leq 0$.

We now rewrite (66) as
\[
\sum_{i=1}^{m} (\sigma - \lambda_i - r)^2_+ - \frac{2\beta}{\gamma N} \sum_{i=1}^{m} (\sigma - \lambda_i - r)_+ \lambda_i \leq 0, \tag{67}
\]
which holds for all $r \geq 0$ if $\sigma \leq \bar{\sigma}_2$. In particular we consider $\sigma \in (\lambda_1, \bar{\sigma}_2]$. If we integrate this inequality against $r^{p-3}$ for $0 < r < \infty$ we can use the beta function integral
\[
B(s, t) = \int_{0}^{1} u^{s-1} (1 - u)^{t-1} \, du \text{ for } s, t > 0 = \frac{\Gamma(s) \Gamma(t)}{\Gamma(s+t)} \tag{68}
\]
to evaluate the integrals (this is the “trick” of Aizenman and Lieb [1]). We have (for $\alpha > -1, p > 2$)
\[
\int_{0}^{\infty} (\sigma - \lambda_i - r)^{\alpha} r^{p-3} \, dr = 0 \quad \text{if } \sigma - \lambda_i \leq 0.
\]
This integral reduces to
\[
\int_{0}^{\sigma - \lambda_i} ((\sigma - \lambda_i) - r)^{\alpha} r^{p-3} \, dr \quad \text{if } \sigma - \lambda_i > 0. \tag{69}
\]
Changing variables via $r = (\sigma - \lambda_i)u$ for $\sigma - \lambda_i$ positive, we arrive at
\[
\int_{0}^{\infty} (\sigma - \lambda_i - r)^{\alpha} r^{p-3} \, dr = (\sigma - \lambda_i)^{\alpha+p-2} B(p-2, \alpha + 1) \tag{70}
\]
and hence ineq. (67) becomes (for $p > 2$)
\[
B(p-2, 3) \sum_{i=1}^{m} (\sigma - \lambda_i)^{p-1}_+ - \frac{2\beta}{\gamma N} B(p-2, 2) \sum_{i=1}^{m} (\sigma - \lambda_i)^{p-1}_+ \lambda_i \leq 0,
\tag{71}
\]
or, since
\[
\frac{B(p - 2, 2)}{B(p - 2, 3)} = \frac{\Gamma(p - 2)\Gamma(2)}{\Gamma(p)} \frac{\Gamma(p + 1)}{\Gamma(p - 2)\Gamma(3)}
\]
and \(\Gamma(p + 1) = p\Gamma(p), \Gamma(1) = 1\),
\[
\sum_{i=1}^{m} (\sigma - \lambda_i)^{p} - \frac{p\beta}{\gamma N} \sum_{i=1}^{m} (\sigma - \lambda_i)^{p-1} \lambda_i \leq 0,
\]
for all \(\sigma \in (\lambda_1, \tilde{\sigma}_2]\).

Thus the inequality \(\tilde{f}_p(\sigma) \leq 0\) holds for all \(\sigma \in [\lambda_m, \tilde{\sigma}_2]\) (and similarly when \(m\) is replaced by any positive integer if \(\tilde{\sigma}_2\) is understood as \(\tilde{\sigma}_{2,m}\); the root \(\tilde{\sigma}_2\) when \(\tilde{f}_2(\sigma) = \tilde{f}_{2,m}(\sigma)\); in particular, we have \(\tilde{f}_{2} \leq 0\) for \(\sigma \in [\lambda_k, \lambda_{k+1}]\) since \([\lambda_k, \lambda_{k+1}] \subset [\lambda_k, \tilde{\sigma}_{2,k}]\). The definition of \(\tilde{\sigma}_{p,m}\) now implies that \(\lambda_{m+1} \leq \tilde{\sigma}_{2,m} \leq \tilde{\sigma}_{p,m}\) (and, in fact, that \(\tilde{f}_{p,m}(\sigma) \leq 0\) on \([\lambda_m, \tilde{\sigma}_{p,m}] \supset [\lambda_m, \tilde{\sigma}_{2,m}]\)), or dropping again the \(m\) subscript on \(\tilde{\sigma}_{p}\), \(\lambda_{m+1} \leq \tilde{\sigma}_{2} \leq \tilde{\sigma}_{p}\) for \(p \geq 2\), which is the final conclusion we wished to draw.

Thus the \(p = 2\) bound for \(\lambda_{m+1}\) equals or surpasses all the bounds \(\tilde{\sigma}_p\) for \(p \geq 2\) coming from ineq. (48) via our definition of the \(\tilde{\sigma}_p\)'s. This is certainly enough, from one point of view, to dismiss the inequality (48) for all \(p \geq 2\) from further consideration but we cannot resist drawing one final conclusion from the Aizenman-Lieb technique.

**Theorem 17.** Suppose \(q \geq p \geq 2\). Then
\[
\lambda_{m+1} \leq \tilde{\sigma}_2 \leq \tilde{\sigma}_p \leq \tilde{\sigma}_q.
\]

**Proof.** One proceeds from ineq. (72) much as we did from ineq. (66) above, first putting it in the form
\[
\sum_{i=1}^{m} (\sigma - \lambda_i - r)^{p} - \frac{p\beta}{\gamma N} \sum_{i=1}^{m} (\sigma - \lambda_i - r)^{p-1} \lambda_i \leq 0
\]
which we know to hold for all \(\sigma \leq \tilde{\sigma}_p\) and \(r \geq 0\). One then integrates in \(r\) much as before, except that this time one multiplies by \(r^{q-1}\) (for \(q > p\)) before integrating from 0 to \(\infty\). This leads to
\[
\sum_{i=1}^{m} (\sigma - \lambda_i)^{q} - \frac{p\beta}{\gamma N} \frac{B(q-p,p)}{B(q-p,p+1)} \sum_{i=1}^{m} (\sigma - \lambda_i)^{q-1} \lambda_i \leq 0
\]
for all $\sigma \leq \tilde{\sigma}_p$, which we can extend to all $\sigma \leq \tilde{\sigma}_q$ by how we defined the $\tilde{\sigma}_p$’s. Noting that

$$\frac{B(q - p, p)}{B(q - p, p + 1)} = \frac{\Gamma(p)\Gamma(q + 1)}{\Gamma(p + 1)\Gamma(q)} = \frac{q}{p}$$

we see that we have arrived at

$$\sum_{i=1}^{m} (\sigma - \lambda_i)^q - \frac{q\beta}{\gamma N} \sum_{i=1}^{m} (\sigma - \lambda_i)^{q-1}\lambda_i \leq 0, \quad (76)$$

which is what we sought, since we have that $\tilde{f}_q(\sigma) \leq 0$ for $\sigma \in [\lambda_m, \tilde{\sigma}_p]$ and, extending via the definition of $\tilde{\sigma}_q$, for all $\sigma \in [\lambda_m, \tilde{\sigma}_q]$.

Thus we have $\tilde{\sigma}_2 \leq \tilde{\sigma}_p \leq \tilde{\sigma}_q$ for $q \geq p$ and since we know that $\lambda_{m+1} \leq \tilde{\sigma}_2$ this completes the proof of the theorem. □

A small remark here is that there is a nice identity $\tilde{f}_p'(\sigma) = p\tilde{f}_{p-1}(\sigma)$, showing that zeros of $\tilde{f}_{p-1}$ are critical points of $\tilde{f}_p$. While this allows one to start analyzing the behavior of $\tilde{f}_p$ based upon that of $\tilde{f}_{p-1}$ we were not able to build a general approach along these lines. And, at best, even if successful this approach would only allow comparisons of $\tilde{\sigma}_p$’s for values of $p$ differing by an integer.

We end this section by mentioning that the (standard) Reverse Chebyshev Inequality implies (via an argument similar to that used in our proof of Theorem 14 above) that $g_m(\tilde{\sigma}_p) \geq \frac{m\gamma N}{p\beta}$ for

$$g_m(\sigma) = \sum_{i=1}^{m} \frac{\lambda_i}{\sigma - \lambda_i}. $$

This holds for any choice of $\tilde{\sigma}_p$ for $p \geq 2$. Thus, we have the upper estimate

$$\tilde{\sigma}_p \leq \lambda_m + \frac{p\beta}{\gamma N}S_1$$

by replacing the quantities $\tilde{\sigma}_p - \lambda_i$ by the smallest, i.e., $\tilde{\sigma}_p - \lambda_m$ in the expression of $g_m(\tilde{\sigma}_p)$ (note that we already have $\tilde{\sigma}_p \geq \lambda_m$). This bound is in the spirit of the PPW bound (4) (cf. also (59)) except for a $p$ in place of a 2 on the right-hand side.

It is not clear at this stage whether the Harrell-Stubbe inequality is stronger than that of Hile-Protter (the $p = 0$ case of (30) in its
generalized form, or (6) originally) for all $p > 2$ or not. It surely is, by continuity, for some range of $p$'s just larger than 2.

VI. Applications to Physical and Geometric Problems

In this section, we illustrate some applications of the abstract formulation described earlier. Physical and geometric problems are considered. Our results improve earlier bounds for various eigenvalue problems by Harrell and Michel [20], for eigenvalues of domains in $\mathbb{S}^2$ and $\mathbb{H}^2$, as well as other bounds by Hook [26]. The general strategy, as explained in [10] (see also [3], [18], [19], [20], [21], [26], [29], [30], [31], [32], [41], [42]), is to write the operator $A$ in the form $A = -\sum_{k=1}^{N} T_k^2 + V$, where the $T_k$'s are skew-symmetric. The auxiliary symmetric operators $B_k$ are chosen such that $[T_\ell, B_k] = \delta_{\ell k}$ and $[V, B_k] = 0$. The “potential” $V$ is either zero or appropriately bounded below. Sometimes it is more appropriate to reduce to a situation like that in Corollary 5 of Section III. Once this is done, a family of new inequalities of the Harrell-Stubbe-type is obtained for the eigenvalue problem at hand. We illustrate this via several examples.

A. Classical PPW, HP, and Yang Inequalities for the Fixed Membrane

For the classical “fixed membrane” problem described in Section I, $A = -\Delta, T_j = \frac{\partial}{\partial x_j}$ and $B_j = x_j$, for $1 \leq j \leq n$ are the appropriate choices. We have

$$A = -\sum_{j=1}^{n} T_j^2,$$

and

$$[T_\ell, B_k] = \delta_{\ell k}.$$
Under the Dirichlet boundary conditions of the problem, the $T_j$’s are skew-symmetric with respect to the inner product

$$\langle u, v \rangle = \int_{\Omega} u v \, dx.$$  

The classical inequalities of PPW, HP, and Yang then follow straightforwardly via the results presented in Sections III and IV, as do their Harrell-Stubbe-style generalizations.

**B. The Inhomogeneous Membrane Problem**

This is of course a generalization of the fixed membrane problem in the previous section. In this case, the density $q(x)$ of the membrane is not uniform over $\Omega \subset \mathbb{R}^n$. The eigenvalue model for this problem is given by

$$-\Delta u = \lambda q(x) u \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial \Omega.$$

(77)

We assume $0 < q_{\min} \leq q(x) \leq q_{\max} < \infty$. The operator $A$ takes the form $A = -\frac{\Delta}{q(x)}$. It is symmetric with respect to the inner product $\langle u, v \rangle_q = \int_{\Omega} u(x)v(x)q(x)dx$. The real eigenfunctions $\{u_i\}_{i=1}^{\infty}$ satisfy

$$\int_{\Omega} u_i(x)u_j(x)q(x)dx = \delta_{ij},$$

and the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ are given by $\lambda_i = \int_{\Omega} |\nabla u_i|^2 dx$. With $B_k = x_k$, for $1 \leq k \leq n$, one is led to $\rho_i \geq \frac{n}{q_{\max}}$ and $\Lambda_i \leq \frac{4\lambda_i}{q_{\min}}$. Hence, we have the following extension of a result of Ashbaugh [3].

**Theorem 18.** The eigenvalues of the inhomogeneous membrane problem with density function $0 < q_{\min} \leq q(x) \leq q_{\max} < \infty$ satisfy the inequalities

$$\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p} \leq \frac{4}{n} \frac{q_{\max}}{q_{\min}} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} \lambda_i \text{ for } p \leq 2 \quad (78)$$
and
\[ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{2p}{n} \frac{q_{\text{max}}}{q_{\text{min}}} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} \lambda_i \text{ for } p \geq 2. \] (79)

**Remark.** Ashbaugh’s result (see Section 4 of [3]) is a refinement and strengthening of a result first proved by Cheng [14] in the context of a minimal hypersurface \( \Omega \) in \( \mathbb{R}^{n+1} \). See [3] as well as [6], [7], and [10], for further references and/or discussion.

### C. Domains in \( \mathbb{S}^2 \) and \( \mathbb{H}^2 \)

Our generalized approach can be used to improve some inequalities relating the eigenvalues of the Laplace-Beltrami operator on a bounded domain \( \Omega \) in \( \mathbb{S}^2 \) or \( \mathbb{H}^2 \) (with Dirichlet boundary conditions). Consider the stereographic projections of \( \mathbb{S}^n \) to \( \mathbb{R}^n \), for \( n \geq 2 \), via projection from the south pole of \( \mathbb{S}^n \). Then the metric is given by ([13], p. 58)
\[ ds^2 = p(x)^2 |dx|^2, \]
where \( p(x) = \frac{2}{1 + |x|^2} \),

where \( | \cdot | \) denotes the Euclidean norm. Hence,
\[ g_{ij} = p^2 \delta_{ij}, \quad \mathcal{G} = (g_{ij}) = p^2 I, \quad \mathcal{G}^{-1} = (g^{ij}) = \frac{1}{p^2} I, \quad \sqrt{\mathcal{G}} = p^n, \]

where \( g = \det \mathcal{G} \) and
\[ \Delta = \frac{1}{p^n} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( p^{n-2} \frac{\partial}{\partial x_i} \right). \]

A Euclidean disk of radius \( r \) centered at the origin in \( \mathbb{R}^n \) corresponds to a geodesic disk of radius \( \alpha \) in \( \mathbb{S}^n \) centered at the north pole, where \( r \) and \( \alpha \) are related by \( r = \tan \frac{\alpha}{2} \).

For \( n = 2 \), the Laplace-Beltrami operator takes the form
\[ \Delta_{\mathbb{S}^2} = \frac{1}{p^2} \Delta_{\mathbb{R}^2}. \]

An eigenvalue of the problem
\[ -\Delta_{\mathbb{S}^2} u = \lambda u \quad \text{in} \quad \Omega \subset \mathbb{S}^2, \]
(for Ω a bounded domain) with Dirichlet boundary conditions is also an eigenvalue of the inhomogeneous membrane problem
\[ \Delta_{\mathbb{R}^2} u = \lambda p^2 u \]
also with Dirichlet boundary conditions. This is then an inhomogeneous membrane problem with \( q(x) = p(x)^2 \). It is obvious that \( q_{\text{max}} \leq 4 \). Moreover,
\[ q_{\text{min}} = \frac{4}{\left(1 + |x|_{\text{max}}^2\right)^2}. \]

We also have \( |x|_{\text{max}} = \tan \frac{\Theta}{2} \) by virtue of the correspondence between geodesic and Euclidean disks, where \( \Theta \) is the outer radius of \( \Omega \), i.e., the geodesic radius of the circumscribing circle (without loss of generality, we can assume that this circle is centered on the north pole). We have
\[ q_{\text{min}} = \frac{4}{\left(1 + \tan^2 \frac{\Theta}{2}\right)^2} = 4 \cos^4 \frac{\Theta}{2} = \left(1 + \cos \Theta\right)^2. \]
Therefore,
\[ \frac{q_{\text{max}}}{q_{\text{min}}} \leq \frac{4}{\left(1 + \cos \Theta\right)^2}. \]
The following is then an extension—à la Harrell-Stubbe—of earlier works by Harrell-Michel [20], Harrell [19], Cheng [14], and Ashbaugh [3].

**Theorem 19.** The eigenvalues of the Laplace-Beltrami operator on a bounded domain \( \Omega \subset \mathbb{S}^2 \) with Dirichlet boundary conditions satisfy the following inequalities
\[
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{8}{\left(1 + \cos \Theta\right)^2} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} \lambda_i \text{ for } p \leq 2
\]
(80)
and
\[
\sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^p \leq \frac{4p}{(1 + \cos \Theta)^2} \sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^{p-1} \lambda_i \text{ for } p \geq 2.
\]

(81)

where \(0 < \Theta < \pi\) designates the outer-radius of \(\Omega\), i.e., the radius of the circumscribing geodesic circle.

**Remark.** It is to be noted that H. C. Yang [41] and Ashbaugh [3] produced universal (i.e., domain independent) inequalities for \(\Omega \subset \mathbb{S}^n\) (see part B of Section 5 of [3]). Following the same arguments one can produce the following (see Section D below for more discussion and the essence of the proof of this theorem).

**Theorem 20.** The eigenvalues of the Laplace-Beltrami operator on a bounded domain \(\Omega \subset \mathbb{S}^n\) with Dirichlet boundary conditions satisfy the following inequalities
\[
\sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^p \leq \frac{1}{n} \sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^{p-1}(4\lambda_i + n^2) \text{ for } p \leq 2
\]

(82)

and
\[
\sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^p \leq \frac{p}{2n} \sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^{p-1}(4\lambda_i + n^2) \text{ for } p \geq 2.
\]

(83)

To consider bounds for the eigenvalues of the Laplace-Beltrami operator on a bounded domain \(\Omega \subset \mathbb{H}^2\), we might consider the problem using any of several models for \(\mathbb{H}^2\). We restrict ourselves to the half-plane model for illustrative purposes. We refer the reader to [10], [19], [20] for more discussion. Here once again the problem can be thought of as an inhomogeneous membrane problem (a point of view advocated by Bandle in [12]) since the Laplace-Beltrami operator is given by

\[
\Delta_{\mathbb{H}^2} = y^2 \Delta_{\mathbb{R}^2}.
\]

The density function is given by \(q(x) = 1/y^2\) for \(x = (x, y) \in \mathbb{H}^2\). Our extension then reads.
\textbf{Theorem 21.} The eigenvalues of the Laplace-Beltrami operator on a bounded domain $\Omega \subset \mathbb{H}^2$ satisfy the following inequalities

\[ \sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^p \leq 2 \sup_{\Omega} y^2 \sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^{p-1}, \quad \text{for } p \leq 2 \]  

and

\[ \sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^p \leq p \sup_{\Omega} y^2 \sum_{i=1}^{m}(\lambda_{m+1} - \lambda_i)^{p-1} \quad \text{for } p \geq 2. \]  

\section{D. Eigenvalues of Homogeneous and Minimally Immersed Submanifolds}

Let $M^n$ be an $n$-dimensional compact manifold (without boundary) of finite volume $V$. Consider the problem of estimating the eigenvalues of the Laplace-Beltrami operator on $M^n$. The earliest bounds for this problem were found by Cheng [14] in 1975. He considered the problem of estimating these eigenvalues when $M^n$ is immersed in the Euclidean space $\mathbb{R}^N$. Very shortly thereafter, Maeda [33] considered the analogous problem for domains in the sphere $\mathbb{S}^N$ (cf. Subsection C above), and for minimally immersed submanifolds of $\mathbb{S}^N$. Also, P. C. Yang and S.-T. Yau [42] dealt with this problem in the case of a minimally immersed submanifold of the sphere $\mathbb{S}^N$. The results of Maeda and of Yang and Yau (as corrected by Leung [31]) are essentially that

\[ \lambda_{m+1} - \lambda_m \leq n + \frac{2}{n(m+1)} \left( \sqrt{\Lambda^2 + n^2 \Lambda(m+1)} + \Lambda \right), \]

where $\Lambda = \sum_{i=1}^{m} \lambda_i$. (We note that $\lambda_0 = 0$ is the first eigenvalue for this problem since $M^n$ is compact.) Beyond that Leung [31], following the approach of Hile and Protter [24], produced an HP-type formula in the spirit of Maeda and Yang and Yau.

In 1995 Harrell and Michel [21] (see also [34], [20]) showed, via a general trace inequality, that one can produce simpler and “natural” inequalities which avoid introducing square root terms such as that found in the bound above. Finally, H. C. Yang [41] produced, in the same spirit, the strongest version of all bounds to date. His 1991
preprint only gradually became known to researchers in the field. A revised preprint was circulated in 1995, but neither version was ever published.

For further background on the history and context of the methods discussed above one can consult [3] (see also [10]).

Bringing in ideas from Yang [41] and Harrell-Stubbe [22], as developed in this paper, we arrive at the Harrell-Stubbe-type bounds contained in the following theorem.

**Theorem 22.** Let $M^n$ be an $n$-dimensional minimally immersed submanifold of $\mathbb{S}^N \subset \mathbb{R}^{N+1}$, then the eigenvalues of the Laplacian $-\Delta_M$, $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, satisfy the following inequalities

\[
\sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{1}{n} \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^{p-1}(4\lambda_i + n^2) \quad \text{for } p \leq 2 \tag{86}
\]

and

\[
\sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{p}{2n} \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^{p-1}(4\lambda_i + n^2) \quad \text{for } p \geq 2 \tag{87}
\]

**Proof.** The minimality of the immersion in $\mathbb{S}^N$ is guaranteed by the condition that the coordinate functions of the immersion are eigenfunctions of the Laplace-Beltrami operator on $M^n$ with eigenvalue $n$. The auxiliary operators are given by the coordinate functions in this case. Moreover, $\rho_i = n$ and $\Lambda_i \leq n^2 + 4\lambda_i$. Feeding this data into ineqs. (26) and (46) (see Corollaries 3 and 8) yields the desired results. □

Li [32] dealt with the eigenvalue problem for a compact homogeneous space. The key to his result and all subsequent improvements by Harrell and Michel [20], [21] (see also [34], [10]) is the following lemma.

**Lemma 23.** (Li [32]) Let $M^n$ be a compact homogeneous manifold of finite volume $V$ and let $\{\phi_{1,\alpha}\}_{\alpha=1}^{k}$ be a real orthonormal basis for the $k$-dimensional eigenspace of the first non-zero eigenvalue $\lambda_1$. 


Then
\[ \sum_{\alpha=1}^{k} \phi_{1,\alpha}^2 = \frac{k}{V} \quad \text{and} \quad \sum_{\alpha=1}^{k} |\nabla \phi_{1,\alpha}|^2 \leq \frac{\lambda_1 k}{V}. \]

Using this lemma, Li was able to prove that
\[ \lambda_{m+1} - \lambda_m \leq \lambda_1 + \frac{2}{m+1} \left( \sqrt{\Lambda^2 + (m+1)\Lambda \lambda_1 + \Lambda} \right). \]

This is of course an inequality in the spirit of Maeda, Yang-Yau, and Leung (cf. also [3]). We have the following improvement (and “natural extension” of the classical inequalities of PPW, HP, and H. C. Yang).

**Theorem 24.** Let \( M^n \) be a compact homogeneous manifold of finite volume \( V \) and let \( \{\phi_{1,\alpha}\}_{\alpha=1}^{k} \) be an orthonormal basis for the \( k \)-dimensional eigenspace of the first non-zero eigenvalue \( \lambda_1 \). Then, its eigenvalues satisfy the following
\[ \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^{p-1}(4\lambda_i + \lambda_1) \quad \text{for} \quad p \leq 2 \quad (88) \]
and
\[ \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{p}{2} \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^{p-1}(4\lambda_i + \lambda_1) \quad \text{for} \quad p \geq 2. \quad (89) \]

**Proof.** The choices we make are \( B_j = \phi_{1,j} \) for \( j = 1, \ldots, k \) (the eigenfunctions of Lemma 23) and \( T_j = [-\Delta, B_j] \). Then, \( \rho_i = \frac{\lambda_1 k}{V} \) and
\[ \Lambda_i \leq \frac{\lambda_1 k}{V} (4\lambda_i + \lambda_1). \]
Hence the desired results follow via Corollaries 3 and 8 (ineqs. (26) and (46)). \( \square \)

### E. Second Order Elliptic Operators

In [26], Hook considered a general, second order, elliptic partial differential equation with constant coefficients of the form
\[ \mathcal{A} u \equiv - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} = \lambda u \quad (90) \]
HARRELL-STUBBE INEQUALITIES

on a bounded domain \( \Omega \subset \mathbb{R}^n \) with Dirichlet boundary conditions. With the assumption that \( A = [a_{ij}] \) is a symmetric positive definite real matrix, he was able to produce HP-type bounds for the eigenvalues of this problem. In [10] we succeeded in producing H. C. Yang-type bounds for this problem thus strengthening Hook’s results. The essential ingredient is to rewrite the problem in the form

\[-e^w \cdot x \text{div}(A e^w \cdot x \text{grad } u) = \lambda u,\]  

where \( w \in \mathbb{R}^n \) is a constant vector given by \( w = A^{-1}b \), and \( b = [b_i] \) appearing in equation (90). The matrix \( A \) is diagonalized according to \( A = U^{-1}KU \), with \( U \) a real orthogonal matrix. The standard basis \( e_1, e_2, \ldots, e_n \) is then transformed according to \( v_j = U^{-1}e_j \) to produce a new orthonormal basis \( v_1, v_2, \ldots, v_n \). The operators \( T_j \) are given by

\[ T_j u = (v_j, \sqrt{A} \text{grad } u) - \frac{1}{2}(v_j, \sqrt{A}w) u, \]  

where \((\cdot, \cdot)\) denotes the usual dot product in \( \mathbb{R}^n \), \( \sqrt{A} \) denotes the positive definite square root of \( A \), and \( j = 1, 2, \ldots, n \). The operators \( T_j \) are skew-symmetric with respect to the inner product

\[ \langle u, v \rangle = \int_\Omega u \overline{v} e^{-w \cdot x} dx. \]  

Also, our original operator \( A \) satisfies

\[ \langle A u, u \rangle = \sum_{j=1}^{n} \langle T_j u, T_j u \rangle + \frac{1}{4} \langle Aw, w \rangle \langle u, u \rangle \]  

in the given inner product. The auxiliary operators \( B_k \) are chosen to be of the form \( Bu = \phi(x)u \), a multiplication by a real-valued function \( \phi \) of the coordinates. The commutation conditions \([T_j, B_k] = \delta_{jk}\) are equivalent to

\[ (\sqrt{A} v_j, \text{grad } \phi_k) = \delta_{jk}. \]  

The vectors \( \{v_1, v_2, \ldots, v_n\} \) form a basis for \( \mathbb{R}^n \) and the same is the case for \( \{\sqrt{A} v_1, \sqrt{A} v_2, \ldots, \sqrt{A} v_n\} \) since \( \sqrt{A} \) is invertible. We form the matrix \( C \) with columns given by the elements of \( \sqrt{A} v_1, \sqrt{A} v_2, \ldots, \sqrt{A} v_n \). \( C \) is then invertible. We let \( F = [f_{jk}] \) be its inverse.
Condition (95) is equivalent to

\[ \Phi C = I, \]

where \( \Phi \) is the matrix with rows given by \( \text{grad} \phi_1, \text{grad} \phi_2, \ldots, \text{grad} \phi_n \), and \( I \) is the identity matrix. Hence, \( F = C^{-1} = \Phi \) and

\[ \frac{\partial \phi_j}{\partial x_k} = f_{jk}. \]

The functions

\[ \phi_j = \sum_{j=1}^{n} f_{jk} x_k \]

satisfy the conditions we seek.

**Theorem 25.** With \( M = \sqrt{A} \), the eigenvalues of problem (90) satisfy the inequalities

\[
\sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{4}{n} \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} (\lambda_i - \frac{1}{4} \|M^{-1}b\|^2) \text{ for } p \leq 2
\]

(96)

and

\[
\sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{2p}{n} \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} (\lambda_i - \frac{1}{4} \|M^{-1}b\|^2) \text{ for } p \geq 2.
\]

(97)

**Remarks.**

1. The use of the orthonormal vectors \( v_j \) in our definition of the operators \( T_j \) in (92) is not necessary either for the skew-symmetry of the \( T_j \)'s in the inner product \( \langle \cdot, \cdot \rangle \) nor for the identity (94); for both of these it is enough that the vectors \( \{v_j\}_{j=1}^{n} \) form an orthonormal basis.

2. A more direct approach to this result, since here we deal only with the case of constant coefficients, is simply to transform away the symmetric matrix \( A \) via a linear change of variables (in fact, the change from \( x_k \) to \( \phi_j \) given above), arriving at a transformed problem on a new bounded domain \( \Omega \) in the variables \( \phi_j \) where the
second-order part of the operator is just the Laplacian (in the variables $\phi_j$, with the $\phi_j$’s viewed as Euclidean variables) and with a first-order part involving a new $b$-vector, $\tilde{b} = M^{-1}b$. The first-order term can then be entirely eliminated via the change of dependent variable, $v = e^{-(b,\phi)/2} u$ (here $\phi$ denotes the vector having the $\phi_j$’s as components), producing an eigenvalue problem $-\Delta v = \mu v$ for the Laplacian on a bounded domain, still with homogeneous Dirichlet boundary conditions, and with the only modification being that the eigenvalue parameter $\lambda$ becomes $\mu = \lambda - \|\tilde{b}\|^2/4$. Finally, since we already know the inequalities (3) and (2) for the Laplacian, we obtain the results of the theorem simply by replacing all $\lambda$’s in those inequalities by $\mu$’s, where $\mu_i = \lambda_i - \|M^{-1}b\|^2/4$.

F. Sturm-Liouville Problem

Let $I = (a, b) \subset \mathbb{R}$. Hook [26] considered the following Sturm-Liouville problem on $I$

$$\mathcal{A} u = -(p u')' + qu = \lambda u, \quad u(a) = u(b) = 0,$$

where $p(x) > 0$ and $q(x)$ are real-valued functions on $I$. The differential operator $\mathcal{A}$ is symmetric with respect to the inner product $\langle u, v \rangle = \int_a^b u \overline{v} \, dx$. One is able to prove that (see [10], [26])

$$\langle \mathcal{A} u, u \rangle = \langle T u, T u \rangle + \langle Q u, u \rangle,$$

(99)

where $T u = \frac{1}{2} \left( \sqrt{p} \, u' + (\sqrt{p} \, u')' \right)$, and $Q u = Q(x) u$ for

$$Q(x) = q(x) - \frac{1}{16} \frac{p'(x)^2}{p(x)} + \frac{1}{4} \frac{p''(x)}{p(x)}.$$

$T$ is skew-symmetric. The symmetric operator $B$ is chosen to be of the form $B u = \phi(x) u$ with $\phi$ real-valued. The commutation condition $[T, B] = 1$ yields the form

$$\phi(x) = \int \frac{dx}{\sqrt{p(x)}}.$$

The following theorem is then immediate.
Theorem 26. Suppose $Q(x) \geq M$ for $x \in (a, b)$. Then, the eigenvalues of problem (98) satisfy the inequalities

\[ \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^p \leq 4 \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} (\lambda_i - M) \text{ for } p \leq 2 \]  (100)

and

\[ \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^p \leq 2p \sum_{i=0}^{m} (\lambda_{m+1} - \lambda_i)^{p-1} (\lambda_i - M) \text{ for } p \geq 2. \]  (101)

Remark. Hook [26] considered two more problems for which he produced HP-type bounds. The first is a diagonal $n$-dimensional version of the Sturm-Liouville problem. The second is a Schrödinger operator with magnetic potential. For both problems H. C. Yang-type inequalities were produced in [10] and as such extensions à la Harrell-Stubbe are valid as well.

References

1. Aizenman, M.; Lieb, E. H. *On semi-classical bounds for eigenvalues of Schrödinger operators*, Phys. Lett. *66A* (1978), 427–429.
2. Ashbaugh, M. S. *Isoperimetric and universal inequalities for eigenvalues*, in *Spectral Theory and Geometry*, E. B. Davies and Yu. Safarov, editors, London Math. Soc. Lecture Note Series, vol. *273*, Cambridge University Press, 1999, pp. 95–139.
3. Ashbaugh, M. S. *The universal eigenvalue bounds of Payne-Pólya-Weinberger, Hile-Protter, and H. C. Yang*, Proc. Indian Acad. Sci. (Math. Sci.), *112* (2002), 3–30.
4. Ashbaugh, M. S.; Benguria, R. D. *Proof of the Payne-Pólya-Weinberger conjecture*, Bull. Amer. Math. Soc. *25* (1991), 19–29.
5. Ashbaugh, M. S.; Benguria, R. D. *A sharp bound for the ratio
of the first two eigenvalues of Dirichlet Laplacians and extensions, Ann. Math. 135 (1992), 601–628.

6. Ashbaugh, M. S.; Benguria, R. D. More bounds on eigenvalue ratios for Dirichlet Laplacians in n dimensions, SIAM J. Math. Anal. 24 (1993), 1622–1651.

7. Ashbaugh, M. S.; Benguria, R. D. Isoperimetric inequalities for eigenvalue ratios, in Partial Differential Equations of Elliptic Type, Cortona, 1992, A. Alvino, E. Fabes, and G. Talenti, editors, Symposia Mathematica, vol. 35, Cambridge University Press, 1994, pp. 1–36.

8. Ashbaugh, M. S.; Benguria, R. D. Bounds for ratios of the first, second, and third membrane eigenvalues, in Nonlinear Problems in Applied Mathematics: In Honor of Ivar Stakgold on his Seventieth Birthday, T. S. Angell, L. Pamela Cook, R. E. Kleinman, and W. E. Olmstead, editors, Society for Industrial and Applied Mathematics, Philadelphia, 1996, pp. 30–42.

9. Ashbaugh, M. S.; Hermi, L. A unified approach to universal inequalities for eigenvalues of elliptic operators, Pacific J. Math. 217 (2004), 201–219.

10. Ashbaugh, M. S.; Hermi, L. On Yang-type bounds for eigenvalues with applications to physical and geometric problems, preprint.

11. Ashbaugh, M. S.; Hermi, L. Universal inequalities for higher-order elliptic operators, preprint.

12. Bandle, C. Isoperimetric inequality for some eigenvalues of an inhomogeneous, free membrane, SIAM J. Appl. Math. 22 (1972), 142–147.

13. Chavel, I. Riemannian Geometry: A Modern Introduction, Cambridge University Press, Cambridge, 1993.

14. Cheng, S.-Y. Eigenfunctions and eigenvalues of Laplacian, in Differential Geometry, S. S. Chern and R. Osserman, editors, Proc. Symp. Pure Math. 27, part 2, American Mathematical Society, Providence, Rhode Island, 1975, pp. 185–193.

15. Hardy, G. H.; Littlewood, J. E.; Pólya, G. Inequalities, second edition, Cambridge University Press, 1952.
16. Harrell II, E. M. *Commutators, eigenvalue gaps, and mean curvature in the theory of Schrödinger operators*, Comm. Part. Diff. Eqs. **32** (2007), 401–413.

17. Harrell II, E. M.; El Soufi, A.; Ilias, S. *Universal inequalities for the eigenvalues of Laplace and Schrödinger operators on submanifolds*, preprint.

18. Harrell II, E. M. *General bounds for the eigenvalues of Schrödinger operators*, in *Maximum Principles and Eigenvalue Problems in Partial Differential Equations*, P. W. Schaefer, editor, Pitman Research Notes in Mathematics Series, vol. **175**, Longman Scientific and Technical, Harlow, Essex, United Kingdom, 1988, pp. 146–166. See, in particular, the Appendix, pp. 161–163.

19. Harrell II, E. M. *Some geometric bounds on eigenvalue gaps*, Commun. Part. Diff. Eqs. **18** (1993), 179–198.

20. Harrell II, E. M.; Michel, P. L. *Commutator bounds for eigenvalues, with applications to spectral geometry*, Commun. Part. Diff. Eqs. **19** (1994), 2037–2055. *Erratum*, Commun. Part. Diff. Eqs. **20** (1995), 1453.

21. Harrell II, E. M.; Michel, P. L. *Commutator bounds for eigenvalues of some differential operators*, in *Evolution Equations*, G. Ferreyra, G. Goldstein, and F. Neubrander, editors, Lecture Notes in Pure and Appl. Math. Series, vol. **168**, Marcel Dekker, New York, 1995, pp. 235–244.

22. Harrell II, E. M.; Stubbe, J. *On trace identities and universal eigenvalue estimates for some partial differential operators*, Trans. Amer. Math. Soc. **349** (1997), 1797–1809.

23. Hersch, J.; Rota, G.-C., editors. *George Pólya: Collected Papers, vol. III: Analysis*, MIT Press, Cambridge, Massachusetts, 1984.

24. Hile, G. N.; Protter, M. H. *Inequalities for eigenvalues of the Laplacian*, Indiana Univ. Math. J. **29** (1980), 523–538.

25. Hile, G. N.; Yeh, R. Z. *Inequalities for eigenvalues of the biharmonic operator*, Pac. J. Math. **112** (1984), 115–133.

26. Hook, S. M. *Domain-independent upper bounds for eigenvalues of elliptic operators*, Trans. Amer. Math. Soc. **318** (1990), 615–
27. Hundertmark, D.; Simon, B. Lieb-Thirring inequalities for Jacobi matrices, J. Approx. Theo. 118 (2002), 106–130.
28. Laptev, A. Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces, J. Funct. Anal. 151 (1997), 531–545.
29. Levitin M.; Parnovski, L. Commutators, spectral trace identities, and universal estimates for eigenvalues, J. Funct. Anal. 192 (2002), 425–445.
30. Lee, J. M. The gaps in the spectrum of the Laplace-Beltrami operator, Houston J. Math. 17 (1991), 1–24.
31. Leung, P. F. On the consecutive eigenvalues of the Laplacian of a compact minimal submanifold in a sphere, J. Austral. Math. Soc. Ser. A 50 (1991), 409–416.
32. Li, P. Eigenvalue estimates on homogeneous manifolds, Comment. Math. Helv. 55 (1980), 347–363.
33. Maeda, M. On the eigenvalues of Laplacian, Sci. Rep. Yokohama Nat. Univ. Sect. I 24 (1977), 29–33.
34. Michel, P. L. Eigenvalue Gaps for Self-Adjoint Operators, Ph.D. Thesis, Georgia Institute of Technology, 1994.
35. Mitrinović, D. S. Analytic Inequalities, Grundlehren der mathematischen Wissenschaften, Berlin-Heidelberg-New York, Springer Verlag, 1970, p. 42.
36. Payne, L. E.; Pólya, G.; Weinberger, H. F. Sur le quotient de deux fréquences propres consécutives, Comptes Rendus Acad. Sci. Paris 241 (1955), 917–919 (reprinted as pp. 410–412 of [23] with comments by J. Hersch on p. 518).
37. Payne, L. E.; Pólya, G.; Weinberger, H. F. On the ratio of consecutive eigenvalues, J. Math. and Phys. 35 (1956), 289–298 (reprinted as pp. 420–429 of [23] with comments by J. Hersch on p. 521).
38. Protter, M. H. Universal inequalities for eigenvalues, in Maximum Principles and Eigenvalue Problems in Partial Differential Equations, P. W. Schaefer, editor, Pitman Research Notes in Mathematics Series, vol. 175, Longman Scientific and Technical, Harlow, Essex, United Kingdom, 1988,
pp. 111–120.

39. Protter, M. H. *Upper bounds for eigenvalues of elliptic operators*, in *Partial Differential Equations and Applications*, P. Marcellini, G. Talenti, and E. Vesentini, editors, Marcel Dekker, New York, 1996, pp. 271–277.

40. Thompson, C. J. *On the ratio of consecutive eigenvalues in n-dimensions*, Stud. Appl. Math. 48 (1969), 281–283.

41. Yang, H. C. *Estimates of the difference between consecutive eigenvalues*, 1995 preprint (revision of International Centre for Theoretical Physics preprint IC/91/60, Trieste, Italy, April 1991).

42. Yang, P. C.; Yau, S.-T. *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 7 (1980), 55–63.

Received 28 December 2007
Revised xxx 200x