Persistence of the steady planar normal shock structure in 3-D unsteady potential flows

Beixiang Fang¹ | Feimin Huang²,³ | Wei Xiang⁴ | Feng Xiao⁵,⁶

¹School of Mathematical Science, MOE-LSC, and SHL-MAC, Shanghai Jiao Tong University, Shanghai, China
²Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing, China
³School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China
⁴Department of Mathematics, City University of Hong Kong, Hong Kong, China
⁵Key Laboratory of Computing and Stochastic Mathematics (Ministry of Education), School of Mathematics and Statistics, Hunan Normal University, Hunan, China
⁶National Center for Mathematics and Interdisciplinary Sciences (NCMIS), Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China

Abstract
This paper concerns the dynamic stability of the steady three-dimensional (3-D) wave structure of a planar normal shock front intersecting perpendicularly to a planar solid wall for unsteady potential flows. The stability problem can be formulated as a free boundary problem of a quasi-linear hyperbolic equation of second order in a dihedral-space domain between the shock front and the solid wall. The key difficulty is brought by the edge singularity of the space domain, the intersection curve between the shock front and the solid wall. Different from the two-dimensional (2-D) case, for which the singular part of the boundary is only a point, it is a curve for the 3-D case in this paper. This difference brings new difficulties to the mathematical analysis of the stability problem. A modified partial hodograph transformation is introduced such that the extension technique developed for the 2-D case can be employed to establish the well-posed theory for the initial-boundary value problem of the linearized hyperbolic equation of
second order in a dihedral-space domain. Moreover, the extension technique is improved in this paper such that loss of regularity in the a priori estimates on the shock front does not occur. Thus, the classical nonlinear iteration scheme can be constructed to prove the existence of the solution to the stability problem, which shows the dynamic stability of the steady planar normal shock without applying the Nash–Moser iteration method.

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## 1 | INTRODUCTION

### 1.1 | Description of the problem

This paper concerns the dynamic stability of the steady three-dimensional (3-D) wave structure of a planar normal shock front intersecting perpendicularly to a planar solid wall (see Figure 1) for unsteady potential flows. As stated by Courant–Friedrichs in [30, p. 375], ‘Whether or not a flow compatible with the boundary condition occurs depends moreover on its stability’, it is important and necessary to study the stability of the normal shock structure, namely, whether or not the shock structure will basically maintain as the parameters of the flow fields are slightly perturbed. For steady flows, for which the parameters (density, velocity, pressure, and so on) do not depend on the time variable, there have been plenty of works on the existence and stability of transonic shocks, for instance, see [6, 11–14, 22, 23, 34–36, 38, 46, 47, 50, 61–63] and the references cited therein. As pointed out by von Karman in the discussion chaired by von Neumann and recorded in [60], a steady motion ‘can occur only as a limiting case’ of a physical process. Therefore, it is necessary to investigate the unsteady motions associated with the steady planar normal shocks and study their dynamic stability under unsteady perturbations. It has been established the stability of normal shocks, which are far away from physical boundaries, in [52, 53] by Majda for Euler flows, and in [54] by Majda and Thomann for potential flows. See also, for instance, [8, 55] and references therein for further studies. However, in practice, shocks often appear together with physical boundaries such as solid walls, wedges, wings, and so on. Therefore, it is important and necessary to further study the stability of shocks involving physical boundaries. In this paper, we are going to study the dynamic stability of the steady 3-D wave structure of a planar normal shock front intersecting perpendicularly to a planar solid wall (see Figure 1), namely, whether the structure will maintain, at least in a short time, under unsteady perturbations of the flow parameters. In this paper, the flows are governed by the unsteady potential flow equations, which read

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \nabla \Phi) &= 0, \\
\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + \nu(\rho) &= B_0,
\end{align*}
\]

(1.1)
where \( \nabla := (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^T \) is the gradient operator with respect to the space variables \( x := (x_1, x_2, x_3) \in \mathbb{R}^3 \) and \( t > 0 \) is the time variable. \( \rho(\rho) := \rho \gamma^{-1} - 1 \) is the specific enthalpy, \( \Phi \) the velocity potential, \( \rho \) the density, \( B_0 \) the Bernoulli constant, and \( \gamma > 1 \) the adiabatic exponent. The importance of the potential flow equations is first observed by Jacques Hadamard in [42] for the unsteady Euler equations with weak shocks. Since then, the potential flow equations have been studied by mathematicians steadily, for instance, see Bers [9], Courant–Friedrichs [30], Majda-Thomann [54], and Morawetz [57].

By the second equation of (1.1), one can express the density \( \rho \) as a function with respect to \( D\Phi := (\partial_t \Phi, \nabla \Phi) \), \( B_0 \) and \( \gamma \), that is,

\[
\rho = \mathfrak{R}(D\Phi;B_0,\gamma) := \left((\gamma - 1)(B_0 - \partial_t \Phi - \frac{1}{2}|\nabla \Phi|^2) + 1\right)^{\frac{1}{\gamma - 1}}. \tag{1.2}
\]

Replacing \( \rho \) in the first equation of (1.1) by \( \mathfrak{R}(D\Phi;B_0,\gamma) \), one deduces that \( \Phi \) satisfies a hyperbolic equation of second order:

\[
\partial_{tt} \Phi + 2 \sum_{i=1}^{3} \partial_{x_i} \Phi \partial_{tx_i} \Phi - \sum_{i,j=1}^{3} (\delta_{ij} c^2 - \partial_{x_i} \Phi \partial_{x_j} \Phi) \partial_{x_i} x_j \Phi = 0, \tag{1.3}
\]

where \( c = \sqrt{\rho \gamma^{-1}} \) is the sonic speed and

\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
\]

Let \( \Gamma_{\text{shock}} := \{(t, x) : x_1 = \lambda(t, x_2, x_3)\} \) be a smooth shock front in the flow field. Then on \( \Gamma_{\text{shock}} \), the velocity potential \( \Phi \) has to satisfy the following Rankine–Hugoniot conditions:

\[
[\Phi] = 0 \quad \text{and} \quad \partial_t \lambda [\rho] - [\rho \partial_{x_1} \Phi] + \partial_{x_2} \lambda [\rho \partial_{x_2} \Phi] + \partial_{x_3} \lambda [\rho \partial_{x_3} \Phi] = 0, \tag{1.4}
\]
where the square bracket $[m]$ stands for the jump of the quantity $m$ across the shock front $\Gamma_{\text{shock}}$; that is, assuming

$$R_{\pm} := \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 : x_1 \geq \lambda(t, x_2, x_3)\}$$

and

$$n_s := \frac{\left(\partial_1 \lambda', -1, \partial_{x_2} \lambda', \partial_{x_3} \lambda'\right)}{\sqrt{1 + |\partial_1 \lambda'|^2 + |\partial_{x_2} \lambda'|^2 + |\partial_{x_3} \lambda'|^2}}.$$  

for every $(t, x) \in \Gamma_{\text{shock}}$, there exists $\alpha > 0$ such that $(t, x) \pm \tau n_s \in R_{\pm}$ for any $\tau \in (0, \alpha)$, define

$$[m](t, x) := \lim_{\tilde{t}, \tilde{x} \to (t, x)} m(\tilde{t}, \tilde{x}) \quad \text{and} \quad [m](t, x) := \lim_{\tilde{t}, \tilde{x} \to (t, x)} m(\tilde{t}, \tilde{x}).$$

It is easy to verify that the Ranking–Hugoniot conditions are equivalent to the following free boundary conditions for $\Phi$:

$$[\Phi] = 0 \quad \text{and} \quad [\rho][\partial_1 \Phi] + [\partial_{x_1} \Phi][\rho \partial_1 \Phi] + [\partial_{x_2} \Phi][\rho \partial_{x_2} \Phi] + [\partial_{x_3} \Phi][\rho \partial_{x_3} \Phi] = 0.$$  

The steady planar normal shock structure

A steady planar normal shock solution (see Figure 1) to the potential flow equations (1.1), satisfying the Rankine–Hugoniot conditions (1.4) on the planar shock front, can be easily constructed, which is the reference state in this paper.

In Figure 1, the red rectangle stands for a steady planar normal shock front $\{x_1 = 0\}$ intersecting the solid wall $\{x_3 = 0\}$ at the edge $\{x_1 = 0\} \cap \{x_3 = 0\}$. Constants $\rho_{\pm}$ represent the density of the fluid behind and ahead of the steady planar normal shock, respectively, and $(q_{\pm}, 0, 0)$ are the constant velocities of the flow fields behind and ahead of the steady planar normal shock, respectively.

Now we give a mathematical definition to this steady planar normal shock structure. Denote by $\Gamma_0 := \{x \in \mathbb{R}^3 : x_3 = 0\}$ the flat solid wall and let $\Gamma_{\text{shock}} := \{x \in \mathbb{R}^3 : x_1 = \lambda(t, x_2, x_3) \equiv 0\}$ be the position of the steady planar normal shock. The flow field is divided by the normal shock front $\Gamma_{\text{shock}}$ into two parts $D_-$ and $D_+$, which are the regions ahead of and behind the steady shock front $\Gamma_{\text{shock}}$, respectively, that is,

$$D_\pm := \{x \in \mathbb{R}^3 : x_1 \geq \lambda(t, x_2, x_3), x_2 \in \mathbb{R}, x_3 > 0\}.$$  

The constant densities and velocities of the fluid in $D_\pm$ are given by $(\rho_\pm, (q_\pm, 0, 0))$, respectively. Then $\rho_\pm$ are determined by $q_\pm$ via (1.2), that is,

$$\rho_\pm = h((0, q_\pm, 0, 0); B_0, \gamma) = \left(\gamma - 1\right)\left(B_0 - \frac{1}{2} q_\pm^2\right) + 1.$$

Let $\Phi(t, x)$ be defined as

$$\Phi(t, x) = \begin{cases} \Phi_-(t, x) := q_- \cdot x_1 & \text{for } (t, x) \in \mathbb{R}^+ \times D_-, \\ \Phi_+(t, x) := q_+ \cdot x_1 & \text{for } (t, x) \in \mathbb{R}^+ \times D_+. \end{cases}$$
Then it is easy to see that $\Phi(t, x)$ satisfies (1.3) in the two regions $\overline{D}_-$ and $\overline{D}_+$. Moreover, it satisfies

$$\nabla \Phi(t, x) = \begin{cases} (q_-, 0, 0) & \text{for } (t, x) \in \mathbb{R}_+ \times \overline{D}_-, \\ (q_+, 0, 0) & \text{for } (t, x) \in \mathbb{R}_+ \times \overline{D}_+. \end{cases}$$

(1.10)

Thus, $\Phi(t, x)$ is a velocity potential of the flow field above the solid wall $\overline{\Gamma}_0$. Due to the Rankine–Hugoniot conditions (1.4) (or equivalently (1.7)) and the entropy condition, constants $(\rho_-, \rho_+, q_-, q_+)$ must satisfy

$$\rho_- < \rho_+, \quad \rho_- q_- = \rho_+ q_+, \quad \text{and} \quad \frac{q_-^2}{2} + \gamma(\rho_-) = \frac{q_+^2}{2} + \gamma(\rho_+).$$

(1.11)

The steady planar normal shock $\overline{\Gamma}_{\text{shock}}$ is a transonic shock: ahead of the shock front $\overline{\Gamma}_{\text{shock}}$, the uniform coming flow $(\rho_-, (q_-, 0, 0))$ is supersonic and behind the shock front $\overline{\Gamma}_{\text{shock}}$, the flow $(\rho_+, (q_+, 0, 0))$ is subsonic, that is,

$$q_-^2 > c_-^2 = \rho_-' \gamma$$

and

$$q_+^2 < c_+^2 = \rho_+' \gamma.$$ 

(1.12)

Then the triplet $(\Phi(t, x), \overline{\Gamma}_{\text{shock}}, \overline{\Gamma}_0)$ is called the steady planar normal shock structure, which will be the reference state investigated in this paper. The steady planar normal shock structure can be observed in many situations. For example, if a normal shock appears in a nozzle with flat boundary (for instance, the nozzle with rectangular cross-section), then this kind of normal shock coincides locally with the steady planar normal shock structure in Figure 1.

### 1.2 Mathematical formulation

The theme of this paper is to study the dynamic stability of the steady planar normal shock structure $(\Phi(t, x), \overline{\Gamma}_{\text{shock}}, \overline{\Gamma}_0)$, in the framework of unsteady potential flow equation (1.3). We want to know whether or not the steady planar normal shock structure persists, at least for a short time, when the uniform supersonic coming flow $(\rho_-, (q_-, 0, 0))$ is perturbed a little unsteadily and the flat solid wall $\overline{\Gamma}_0$ becomes slightly curved. Let $\mathcal{W}(x_1, x_2)$ be a smooth function. We denote by $\Gamma_0 := \{(t, x) : x_3 = \mathcal{W}(x_1, x_2)\}$ an impermeable solid boundary of the flow field. Then the whole flow field is

$$D := \{x \in \mathbb{R}^3 : x_3 > \mathcal{W}(x_1, x_2)\}.$$ 

$\Phi$ satisfies the slip boundary condition $\nabla \Phi \cdot \mathbf{n} = 0$ on $\Gamma_0$, where $\mathbf{n}$ is the unit exterior normal vector of $\Gamma_0$, that is,

$$-\partial_{x_1} \Phi \partial_{x_1} \mathcal{W} - \partial_{x_2} \Phi \partial_{x_2} \mathcal{W} + \partial_{x_3} \Phi = 0 \quad \text{on} \quad \Gamma_0.$$ 

(1.13)

Moreover, let the initial states of the fluid be also slightly perturbed such that the initial conditions for $\Phi$ are given as:

$$\Phi(0, x) = \Phi_0(x) \quad \text{and} \quad \partial_t \Phi(0, x) = \Phi_1(x),$$

(1.14)
where for $i = 0, 1$,
\[
\Phi_i(x) := \begin{cases} 
\Phi_i^+(x) & \text{for } x \in R_0^+ := \{x_1 > \lambda(0, x_2, x_3)\} \cap D, \\
\Phi_i^-(x) & \text{for } x \in R_0^- := \{x_1 < \lambda(0, x_2, x_3)\} \cap D.
\end{cases}
\] (1.15)

Here the initial position $\lambda(0, x_2, x_3)$ of the perturbed shock front $\Gamma_{\text{shock}}$ is a small perturbation of the reference shock front $\Gamma_{\text{shock}}$.

Now the dynamic stability problem (see Figure 2) can be precisely reformulated as following problem:

**Problem 1.** Suppose $\Gamma_0$ is a small perturbation of $\Gamma_0$, that is, $\mathcal{W}$ is close to zero and the initial data $(\Phi_0, \Phi_1)$ are small perturbations of $\Phi(0, x)$, that is, $\Phi_0$ is close to $\Phi(0, x)$ and $\Phi_1(x)$ is close to zero. One looks for a unique local piece-wise smooth solution $(\Phi(t, x), \lambda(t, x_2, x_3))$ to Equation (1.3) in the flow field $D = \{x \in \mathbb{R}^3 : x_3 > \mathcal{W}(x_1, x_3)\}$ such that:

(i) the shock front is given by
\[
\Gamma_{\text{shock}} := \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 : x_1 = \lambda(t, x_2, x_3)\},
\]

which divides the flow field into $D_+ := D \cap R_+$ and $D_- := D \cap R_-$, where $R_\pm$ are defined in (1.5);

(ii) $\Phi(t, x)$ is smooth up to either sides of $\Gamma_{\text{shock}}$ such that
\[
\Phi(t, x) = \begin{cases} 
\Phi^+(t, x) & \text{for } (t, x) \in D_+, \\
\Phi^-(t, x) & \text{for } (t, x) \in D_-.
\end{cases}
\]

and $\Phi^\pm(t, x)$ satisfy Equation (1.3) in $D_\pm$, respectively;

(iii) $\Phi^\pm(t, x)$ satisfy the slip boundary condition (1.13), respectively, that is,
\[
-\partial_{x_1} \Phi^\pm \partial_{x_1} \mathcal{W} - \partial_{x_2} \Phi^\pm \partial_{x_2} \mathcal{W} + \partial_{x_3} \Phi^\pm = 0 \quad \text{for } (t, x) \in \mathbb{R}^+ \times \Gamma_0,
\]
(iv) \( \Phi^\pm(t, x) \) satisfy the initial conditions (1.14)-(1.15), respectively, that is,

\[
\Phi^\pm(t, x)|_{t=0} = \Phi^\pm_0(x) \quad \text{for } x \in R^0_\pm \cap D,
\]

and

\[
\partial_t \Phi^\pm(t, x)|_{t=0} = \Phi^\pm_1(x) \quad \text{for } x \in R^0_\pm \cap D,
\]

where \( R^0_\pm \) are the ones defined in (1.15);

(v) \( (\Phi^+(t, x), \Phi^-(t, x), \lambda(t, x_2, x_3)) \) satisfy the Rankine–Hugoniot conditions in (1.4);

(vi) \( (\Phi(t, x), \lambda(t, x_2, x_3)) \) is close to the steady normal shock solution \( (\Phi, \lambda) \), that is, \( \Phi^\pm(t, x) \) is close to \( \Phi^\pm_\pm(t, x) \) in \( D_\pm \), respectively, and \( \lambda(t, x_2, x_3) \) is close to \( \lambda_\pm(t, x_2, x_3) \).

Remark 1.1. Thanks to the property of the finite speed of propagation of hyperbolic equations and the well-established mathematical theory for initial boundary value problems for hyperbolic equations with smooth boundaries (for instance, see [8]), one can assume that, without loss of generality, the perturbation only occurs near the intersection curve, where the shock front intersects the solid wall \( x_3 = \mathcal{W}(x_1, x_2) \). Therefore, this paper only solves the stability problem near the edge of the dihedral-space domain, and in a short time.

The initial boundary value problem (1.3), (1.7), and (1.13)–(1.15) is a free boundary problem in a dihedral-space domain between two surfaces, the shock front \( \Gamma_{\text{shock}} \) and the perturbed solid wall \( \Gamma_0 \). The key difficulty in the mathematical analysis of the problem comes from the singularity of the boundary of the space domain, which is not smooth along the edge of the dihedral-space domain, especially as it couples with other difficulties such as nonlinearity, free boundaries, and so on. In fact, Osher has given examples in [58, 59] showing that hyperbolic equations in cornered space domain may be ill-posed. On the other hand, for the well-posedness problem of hyperbolic equations in space domains with non-smooth boundaries, there are also positive results, for instance, see [39–41, 63]. In particular, under certain symmetry assumptions, Gazzola–Secchi [39] studied the inflow–outflow problem in a bounded cylinder. Then Yuan [63] studied the dynamic stability of normal shock in a duct with flat boundaries in two space dimensions. In both works, the symmetry assumptions play an essential role in the analysis, under which the extension techniques can be employed such that the non-smooth domain is reduced into a smooth domain. Such symmetry assumptions fail to be valid in the problem (1.3), (1.7), and (1.13)–(1.15) studied in this paper, since the solid wall \( \Gamma_0 \) is a curved surface. Hence, the methods developed in [39, 63] are not applicable. Nevertheless, the assumption that \( \Gamma_0 \) is a slightly perturbed surface from a flat one implies that there may hold some symmetry properties under certain transformation. Recently in [37], the authors develop an extension technique successfully to deal with the difficulty in a two-dimensional (2-D) cornered-space domain. However, the technique cannot be directly applied to the problem in this paper because the singular set of the boundary is no longer a single point, but a curve, which is the edge. Therefore, new methods should be developed and more careful analysis are needed to establish the well-posedness of the solutions in the dihedral-space domain.

Motivated by the extension techniques developed in [37] for 2-D case, we shall look for an appropriate transformation, under which it is possible to extend the linearized initial-boundary value problem in the dihedral-space domain into an initial-boundary value problem in the half-space domain. To make it, a modified partial hodograph transformation (see (2.5) for details),
different from the transformation employed for the 2-D case, is introduced. Then the problem in the dihedral-space domain will be extended into a problem in a half-space domain, and the unique existence of a $H^2_0$-solution (a weighted Sobolev space) can be established by employing the classical theory for initial-boundary value problems of hyperbolic equations (see [8], for instance). Similar to the 2-D case, the $H^2_0$ regularity is not sufficient to close the nonlinear iteration. Therefore, a priori estimates for higher order derivatives are required, which should be established directly in the dihedral-space domain, since the extended coefficients are of low regularity. Moreover, as the space dimension increases, the analysis needed for the a priori estimates for higher order derivatives is more complicated than the 2-D case and it should be dealt with more carefully. Finally, it is worth mentioning that a transformation (see Section 4) is introduced to reformulate the nonlinear problem (NLP), which helps to improve the extension argument develop in [37], such that the loss-of-regularity for the a priori estimates on the shock-front will not occur. Hence, instead of the Nash–Moser iteration scheme employed in [37], a classical nonlinear iteration scheme is sufficient to prove the existence of the solutions to the NLP.

Up to now, much great progress has been made in the study of weak solutions of multidimensional unsteady compressible Euler equations. For instance, see [26, 27, 44, 45, 52–56] for the study of shock waves, [1, 2, 10] for rarefaction waves, [18, 19, 28, 29] for contact discontinuities, [4, 5, 15–17, 21, 33, 48, 49] for self-similar solutions, and [3, 7, 24, 25, 31, 32, 43, 51] for the non-uniqueness of weak solutions.

The remainder of the paper is organized as follows. In Section 2, a modified partial hodograph transformation is introduced to fix the free boundary and flat the curved solid wall. Then the dynamic stability problem is reformulated as the well-posedness problem of an initial boundary value problem for a nonlinear hyperbolic equation of second order, in a dihedral-space domain with fixed boundaries. Finally, the main theorem, Theorem 2.1, is presented at the end of this section. In Section 3, we obtain the well-posedness of a general initial boundary value problem for a linear hyperbolic equation of second order in the dihedral-space domain. In Section 4, the NLP is reformulated. In Subsection 4.2, an iteration scheme is introduced to solve the reformulated NLP. Then one proves the main theorem by showing that the iteration scheme provides a sequence of functions which converges to the desired solution, and hence prove the dynamic stability of the steady planar normal shock structure.

2 PARTIAL HODOGRAPH TRANSFORMATION AND MAIN RESULT

In this section, we introduce a modified partial hodograph transformation, which is used to fix the free boundary $\Gamma_{\text{shock}}$ and straighten the perturbed solid wall $\Gamma_0$. With the aid of this transformation, the previous initial boundary value problem (1.3), (1.7), and (1.13)–(1.15) is mapped to an initial boundary value problem in a dihedral-space domain with fixed boundaries in the new coordinate system. Then Problem 1 is converted to Problem 2 and solving Problem 1 is equivalent to solve Problem 2. Finally, at the end of this section, we present our main result.

2.1 Partial hodograph transformation

Let $\Phi^{-}$ be the potential for the flow field ahead of the shock-front and $\Phi$ the one behind the shock-front. Extend $\Phi^{-}$ by solving the Equation (1.3) with the boundary condition (1.13) into the domain
ahead of the shock-front, which is at least $C^1$ across the shock-front. More precisely, first we extend $\Phi_0^-(x)$ and $\Phi_1^-(x)$ smoothly into the whole domain $\mathbb{R}^3$. Then solve the initial boundary value problem (1.3), (1.13), and (1.14), where $\Phi_i(x)$ in (1.14) is replaced by $\Phi_i^-(x)$. Obviously, such solution exists locally (this is reasonable, one can see [20] for the case of compressible Euler equations, which includes the case of potential flows) and is a solution of Problem 1 when $x_1 < \lambda(t, x_2, x_3)$. Denote by $\Phi^-(t, x)$ this smooth solution and define

$$\phi(t, x) := \Phi^-(t, x) - \Phi(t, x).$$

Then the potential equation (1.3) for $\Phi$ is reformulated as a second-order equation for $\phi$:

$$\sum_{i,j=0}^{3} a_{ij}(D\phi; D\Phi^-)\partial_{x_i x_j} \phi = \sum_{i,j=0}^{3} a_{ij}(D\phi; D\Phi^-)\partial_{x_i x_j} \Phi^-,$$

where

$$a_{00} = 1, a_{0j} = a_{j0} := \partial_{x_j} \Phi^- - \partial_{x_j} \phi = \partial_{x_j} \Phi,$$

and

$$a_{ij} = a_{ji} := -c^2 \delta_{ij} + (\partial_{x_i} \Phi^- - \partial_{x_i} \phi)(\partial_{x_j} \Phi^- - \partial_{x_j} \phi) = -c^2 \delta_{ij} + \partial_{x_i} \Phi \partial_{x_j} \Phi$$

for $i, j = 1, 2, 3$.

We introduce the following partial hodograph transformation:

$$\mathcal{P} : \begin{cases} y_0 = t \\ y_1 = \phi(t, x) \\ y_2 = x_2 + p(x) \\ y_3 = x_3 - \mathcal{W}(x_1, x_2), \end{cases}$$

where

$$p(x) = \frac{\partial_{x_2} \mathcal{W}}{1 + |\partial_{x_1} \mathcal{W}|^2 + |\partial_{x_2} \mathcal{W}|^2}(x_3 - \mathcal{W}(x_1, x_2)).$$

Here $p(x)$ is introduced to balance the perturbation on the $x_2$-direction.

**Remark 2.1.** In [37], $p(x)$ does not appear in the partial hodograph transformation. While in this paper, $p(x)$ plays an essential role, as it is used to match the perturbations on the $x_2$-direction and $x_3$-direction. As one will see from the proof of Lemma 4.1, the appearance of $p(x)$ guarantees the vanishing property of $\bar{a}_{23}$ and $\bar{a}_{32}$ on $\{y_3 = 0\}$, which is necessary to the application of the extension technique and crucial to the solvability of the linearized problem in the dihedral-space domain.

The inverse of $\mathcal{P}$ is

$$\mathcal{P}^{-1} : t = y_0, x_1 = u(y_0, y), x_2 = x_2(y_0, y), x_3 = y_3 + \mathcal{W}(u(y_0, y), x_2(y_0, y)).$$
where \((y_0, y) = (y_0, y_1, y_2, y_3)\) are the time-spatial variables in the new coordinate and \(u(y_0, y)\) is the new unknown function. Taking the partial derivatives to the equation \(y_j = \phi \circ \mathcal{P}^{-1}(y_0, y)\) with respect to \(y_j\) \((j = 0, 1, 2, 3)\), we obtain a linear system with respect to \(D_{t,x} \phi := (\partial_t \phi, \nabla \phi)\). By solving this system, one can express \(D_{t,x} \phi\) in terms of \(Du := (\partial_{y_0} u, \partial_{y_1} u, \partial_{y_2} u, \partial_{y_3} u)\),

\[
\begin{align*}
\partial_t \phi &= \frac{\partial y_0 u}{\partial y_1 u}, \\
\partial_{x_1} \phi &= -\frac{\partial x_1 p \partial y_2 u - \partial x_1 \mathcal{W} \partial y_3 u - 1}{\partial y_1 u}, \\
\partial_{x_2} \phi &= \frac{\partial x_2 p \partial x_2 \mathcal{W} \partial y_2 u + \partial x_2 \mathcal{W} \partial y_3 u - \partial y_2 u}{\partial y_1 u}, \\
\partial_{x_3} \phi &= -\frac{\partial x_3 p \partial y_2 u + \partial y_3 u}{\partial y_1 u}.
\end{align*}
\]

(2.8)

The Jacob matrix of \(\mathcal{P}\) is

\[
\begin{bmatrix}
\partial(y_0, y) \\
\partial(t, x)
\end{bmatrix} =
\begin{bmatrix}
\partial \phi & \partial_{x_1} \phi & \partial_{x_2} \phi & \partial_{x_3} \phi \\
0 & \partial_{x_1} p & \partial_{x_2} p & \partial_{x_3} p \\
0 & -\partial_{x_1} \mathcal{W} & -\partial_{x_2} \mathcal{W} & 1
\end{bmatrix} := \frac{1}{\partial_{y_1} u} J^T,
\]

where

\[
J :=
\begin{bmatrix}
\partial_{y_1} u & -\partial_{y_0} u & 0 & 0 \\
0 & \partial_{x_1} \mathcal{W} \partial y_3 u - \partial_{x_1} p \partial y_2 u + 1 & \partial_{x_1} p \partial y_1 u & -\partial_{x_1} \mathcal{W} \partial y_1 u \\
0 & (\partial_{x_2} p \partial x_2 \mathcal{W} - 1)\partial_{y_2} u + \partial y_3 u \partial_{x_2} \mathcal{W} & (\partial x_2 p + 1) \partial y_1 u & -\partial_{x_2} \mathcal{W} \partial y_1 u \\
0 & -\partial_{x_3} p \partial y_2 u - \partial y_3 u & \partial_{x_3} p \partial y_1 u & \partial_{y_1} u
\end{bmatrix}.
\]

2.2 Formulation in new coordinate

In the remaining part of this paper, time \(t\) may be denoted by \(y_0\) and vice versa. After a direct computation, we also obtain

\[
\begin{aligned}
\frac{\partial(D\phi)}{\partial(Du)} &= -\frac{1}{(\partial_{y_1} u)^2} J.
\end{aligned}
\]

Denote by \(D^2 \phi\) the Hessian matrix of \(\phi\), that is,

\[
D^2 \phi = \frac{\partial(D\phi)}{\partial(t, x)}.
\]
With the help of (2.8), by simple calculation, one has

\[
D^2 \phi = \frac{\partial (D \phi)}{\partial (Du)} \left[ \frac{\partial (y_0, y)}{\partial (t, x)} + \frac{(-\partial_{x_1 x_1} p \partial_{y_2} u + \partial_{x_1 x_1} W \partial_{y_3} u) I_{11}}{\partial_{y_1} u} \right.
\]
\[
+ \frac{(-\partial_{x_1 x_2} p \partial_{y_2} u + \partial_{x_1 x_2} W \partial_{y_3} u) I_{12}}{\partial_{y_1} u} - \frac{\partial_{x_1 x_3} p \partial_{y_2} u I_{13}}{\partial_{y_1} u}
\]
\[
+ \frac{1}{\partial_{y_1} u} ((\partial_{x_1 x_3} p \partial_{y_2} u \partial_{x_3} W + \partial_{x_3} p \partial_{y_2} u \partial_{x_1 x_2} W + \partial_{x_1 x_2} W \partial_{y_3} u) I_{21})
\]
\[
+ \frac{1}{\partial_{y_1} u} ((\partial_{x_3} p \partial_{y_2} u \partial_{x_1 x_3} W + \partial_{x_1 x_3} p \partial_{y_2} u \partial_{x_2 x_2} W + \partial_{x_2 x_2} W \partial_{y_3} u) I_{22})
\]
\[
+ \frac{\partial_{y_2} u}{\partial_{y_1} u} (-\partial_{x_1 x_3} p I_{31} - \partial_{x_2 x_3} p I_{32} - \partial_{x_3 x_3} p I_{33} + \partial_{x_3 x_3} p \partial_{x_2} W I_{23}),
\]

where \(I_{ij} := e_i^T e_j \in \mathbb{R}^{4 \times 4}\) with \(\{e_i\}_{i=0}^3\) being the canonical basis of \(\mathbb{R}^4\). Then we have

\[
\sum_{i,j=0}^3 a_{ij} \partial_{x_i x_j} \phi = \text{Tr}(A^T D^2 \phi) = -\frac{1}{(\partial_{y_1} u)^3} \sum_{i,j=0}^3 \tilde{a}_{ij} \partial_{y_i y_j} u + \sum_{i=1}^4 S_i,
\]

(2.9)

where \(A := [a_{ij}]_{4 \times 4}\) with \(a_{ij}\) being defined in (2.3)-(2.4) and \(\text{Tr}(M)\) means the trace of the square matrix \(M\). The coefficients \(\tilde{a}_{ij} = \tilde{a}_{ij}(\partial_{x_i} W, \partial_{x_j} W, Du; D\Phi^-)\) satisfy that

\[
[\tilde{a}_{ij}]_{4 \times 4} := J^T AJ = \tilde{A} = \tilde{A}^T,
\]

and

\[
S_1 = a_{11} (-\partial_{x_1 x_1} p \partial_{y_2} u + \partial_{x_1 x_1} W \partial_{y_2} u),
\]
\[
S_2 = \frac{1}{\partial_{y_1} u} (-a_{13} \partial_{x_1 x_3} p \partial_{y_2} u + a_{21} (\partial_{x_1 x_3} p \partial_{x_2} W \partial_{y_2} u + \partial_{x_3} p \partial_{x_1 x_2} W \partial_{y_2} u + \partial_{x_1 x_2} W \partial_{y_3} u)),
\]
\[
S_3 = \frac{1}{\partial_{y_1} u} (a_{22} (\partial_{x_3} \partial_{x_2} W \partial_{y_2} u + \partial_{x_2} \partial_{x_2 x_2} W \partial_{y_2} u + \partial_{x_2 x_2} W \partial_{y_3} u) + a_{23} \partial_{x_3 x_3} p \partial_{x_2} W \partial_{y_2} u),
\]
\[
S_4 = -\frac{\partial_{y_2} u}{\partial_{y_1} u} (a_{31} \partial_{x_1 x_1} p + a_{32} \partial_{x_2 x_1} p + a_{33} \partial_{x_3 x_1} p).
\]

By simple calculation, especially, one has

\[
\tilde{a}_{03} = \tilde{a}_{30} = (\partial_{y_1} u)^2 \cdot d,
\]

\[
\tilde{a}_{13} = \tilde{a}_{31} = - (\partial_{y_1} u)^2 \cdot d + \partial_{y_1} u (\partial_{x_1} W \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1) \cdot d + c^2 \partial_{x_1} W
\]
\[
- (\partial_{x_3} p \partial_{y_2} u + \partial_{y_3} u) \partial_{y_1} u (\partial_{x_3} \Phi \cdot d - c^2),
\]

(2.11)
\[
\bar{a}_{23} = \bar{a}_{32} = \frac{\partial}{\partial x_1} p(\partial_y u)^2 (\partial_x \Phi \cdot d + c^2 \partial_x W) + (\partial_x p + 1)(\partial_y u)^2 (\partial_x \Phi \cdot d + c^2 \partial_x W) \\
+ ((\partial_x p \partial_x W - 1) \partial_y u + \partial_y u \partial_x W) \partial_y u (\partial_x \Phi \cdot d + c^2 \partial_x W) \\
+ \partial_x \partial_p (\partial_y u)^2 (\partial_x \Phi \cdot d - c^2),
\] (2.12)

where \( d = \frac{\partial}{\partial x_3} \Phi - \frac{\partial}{\partial x_1} \Phi \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \Phi \frac{\partial}{\partial x_2} \).

For the other coefficients, because we do not need the properties of their trace on the boundary, they are listed in the Appendix.

From (2.2) and (2.9), we deduce that \( u \) satisfies following equation

\[
\sum_{i,j=0}^{3} \bar{a}_{ij} \frac{\partial}{\partial y_i y_j} u + \bar{a}_2 \frac{\partial}{\partial y_2} u + \bar{a}_3 \frac{\partial}{\partial y_3} u + a_{12} \frac{\partial}{\partial x_1 x_2} p(\partial_y u)^3 = - (\partial_y u)^3 \sum_{i,j=0}^{3} a_{ij} \frac{\partial}{\partial x_i x_j} \Phi^-,
\] (2.13)

where

\[
\bar{a}_2 = (\partial_y u)^2 (a_{11} \frac{\partial}{\partial x_1 x_1} + a_{13} \frac{\partial}{\partial x_1 x_3} p - a_{21} (\partial_x x_3 p \frac{\partial}{\partial x_2} W + \partial_x \frac{\partial}{\partial x_1 x_2} W)) \\
+ (\partial_y u)^2 (a_{12} \frac{\partial}{\partial x_1 x_2} p + a_{31} \frac{\partial}{\partial x_1 x_3} p + a_{32} \frac{\partial}{\partial x_2 x_3} p - a_{22} \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_2} W),
\] (2.14)

\[
\bar{a}_3 = (\partial_y u)^2 (-a_{11} \frac{\partial}{\partial x_1 x_1} W - a_{12} \frac{\partial}{\partial x_1 x_2} W - a_{21} \frac{\partial}{\partial x_1 x_2} W - a_{22} \frac{\partial}{\partial x_2 x_2} W + a_{12} \frac{\partial}{\partial x_1 x_2} p).
\] (2.15)

Assume

\[
\frac{\partial}{\partial x_2} W(x_1, 0) = 0.
\] (2.16)

Then the partial hodograph transformation \( \mathcal{P} \) maps the axis \( x_2 = 0 \) in \( (t, x) \)-coordinate to the axis \( y_2 = 0 \) in \( (y_0, y) \)-coordinate. Moreover, the perturbed solid wall \( \Gamma_0 \) and the shock front \( \Gamma_{\text{shock}} \) in \( (t, x) \)-coordinate are mapped to

\[
\Gamma_w := \{ y_0 > 0, y_1 > 0, y_3 = 0 \}
\] (2.17)

and

\[
\Gamma_s := \{ y_0 > 0, y_1 = 0, y_3 > 0 \},
\] (2.18)

respectively. Substituting the expressions of \( D_{t,x} \Phi \) and \( p(x) \) into (1.13), we find that \( u \) satisfies

\[
\frac{\partial}{\partial y_3} u = - \frac{\frac{\partial}{\partial x_1} W}{1 + \left| \frac{\partial}{\partial x_1} W \right|^2 + \left| \frac{\partial}{\partial x_2} W \right|^2} \text{ on } \Gamma_w.
\] (2.19)

Substituting (2.8) into (1.7), we obtain the Rankine–Hugoniot condition in the new coordinate variables:

\[
G(u, Du; D\Phi^-) = 0 \quad \text{on } \Gamma_s,
\] (2.20)

where

\[
G(u, Du; D\Phi^-) := [\rho][\Phi_1] + [\partial_{x_1} \Phi][\rho \partial_{x_1} \Phi] + [\partial_{x_2} \Phi][\rho \partial_{x_2} \Phi] + [\partial_{x_3} \Phi][\rho \partial_{x_3} \Phi],
\] (2.21)
where $D\Phi$ should be replaced by $D\Phi^-$ and $D_t, x \Phi$ should be replaced by $Du$ via (2.8). For the initial conditions, we assume

$$ u(y_0, y) |_{y_0} := u_0(y) \quad \text{and} \quad \partial_{y_0} u(y_0, y) |_{y_0=0} := u_1(y), $$

where $u_0$ and $u_1$ are some given functions.

For notational simplicity, one defines $Lu$ by

$$ Lu := \sum_{i,j=0}^3 \bar{a}_{ij} \partial_{y_i} \partial_{y_j} u + \bar{a}_1 \partial_{y_3} u + a_{12} \partial_{x_1} x_2 p(\partial_{y_1} u)^3 + (\partial_{y_1} u)^3 \sum_{i,j=0}^3 a_{ij} \partial_{x_i} x_j \Phi^-, $$

where the coefficients depend on $u(y_0, y)$ and its first-order derivatives, as well as $W(x_1, x_2)$ and its derivatives up to third order. Gathering (2.13), (2.19)–(2.20), and the initial conditions of $u$, we get the initial boundary value problem concerned in this paper:

$$\begin{cases}
Lu = 0 & \text{in } \Omega_T, \\
G(\partial_{x_1} W, \partial_{x_2} W, Du; D\Phi^-) = 0 & \text{on } \{y_1 = 0\}, \\
G_1 := (1 + |\partial_{x_1} W|^2 + |\partial_{x_2} W|^2)\partial_{y_3} u + \partial_{x_1} W = 0 & \text{on } \{y_3 = 0\}, \\
u(y_0, y) = u_0(y), u(y_0, y) = u_1(y) & \text{on } \{y_0 = 0\}.
\end{cases} \tag{NLP}$$

Here and after $\Omega_T := [0, T] \times \Omega$ and $\Omega := \mathbb{R}^+_{y_1} \times \mathbb{R}^{+}_{y_2} \times \mathbb{R}^{+}_{y_3}$, where $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}$ is the set of real numbers.

In the $(t, x)$-coordinate, the background state for $\phi$ is

$$ \hat{\phi}(t, x) := \Phi_-(t, x) - \Phi_+(t, x) = (q_- - q_+)x_1. $$

Then the corresponding partial hodograph transformation is

$$ y_0 = t, \quad y_1 = \hat{\phi}(t, x), \quad y_2 = x_2, \quad y_3 = x_3, \tag{2.22} $$

and its inverse transformation is

$$ t = y_0, \quad x_1 = u_b(y), \quad x_2 = y_2, \quad x_3 = y_3. \tag{2.23} $$

It is clear that

$$ x_1 = \frac{1}{q_- - q_+} \hat{\phi}(t, x) = \frac{1}{q_- - q_+} y_1. $$

Hence, we have

$$ u_b(y) = \frac{1}{q_- - q_+} y_1. \tag{2.24} $$

At the background state, that is, the state that $u = u_b$, $W(x_1, x_2) \equiv 0$, $\nabla \Phi(t, x) \equiv (q_+, 0, 0)$ and $\nabla \Phi^-(t, x) \equiv (q_-, 0, 0)$, one has

$$ a_{00} = \frac{1}{(q_- - q_+)^2} > 0, \quad a_{01} = a_{10} = \frac{q_+}{q_- - q_+} > 0, \quad a_{02} = a_{20} = 0, \tag{2.25} $$

$$ a_{11} = q_+^2 - c_+^2 < 0, \quad a_{03} = a_{30} = 0, \quad a_{13} = a_{31} = 0. \tag{2.26} $$
\[ \tilde{a}_{22} = \frac{-c_+^2}{(q_- - q_+)^2} < 0, \quad \tilde{a}_{21} = \tilde{a}_{12} = 0, \quad \tilde{a}_{23} = \tilde{a}_{32} = 0, \] (2.27)

\[ \tilde{a}_{33} = \frac{-c_+^2}{(q_- - q_+)^2} < 0. \] (2.28)

In \( y \)-coordinates, the dynamic stability problem is rewritten as the following problem:

**Problem 2.** Suppose the initial data \((u_0, u_1)\) and \( \mathcal{W} \) are small perturbations of the background state \( u_b \) and zero, respectively, and \( \nabla \Phi^- \) is close to \((q_-, 0, 0)\). Can we show the local existence and uniqueness of smooth solutions to (NLP), such that the unique solution is still close to \( u_b \)?

The remaining part of this paper is devoted to solving this problem. It is shown that one can indeed find a unique smooth solution to (NLP) near \( u_b \), if the following condition:

\[ q_- \rho_+ - q_+ \rho_- - \rho_+ > 0 \] (2.29)

holds for the constants \((\rho_-, q_-, \rho_+, q_+)\).

**Remark 2.2.** It should be noted that, as one will see from the proof of Lemma 4.1, the condition (2.29) is employed to guarantee that the steady normal shock solution satisfies the stability conditions, which are defined in (H4) below in the beginning of Section 3. However, the conditions (1.11) and (1.12) are not sufficient to yield (2.29). For example, for any \( 1 < \lambda < \frac{1 + \sqrt{5}}{2} \), choose \((q_-, \rho_-, q_+, \rho_+)\) as follows:

\[ q_- = \lambda, \quad q_+ = 1, \quad \rho_- = \left( \frac{(\gamma - 1)(\lambda^2 - 1)}{2(\lambda^{\gamma - 1} - 1)} \right)^{\frac{1}{\gamma - 1}}, \quad \text{and} \quad \rho_+ = \lambda \rho_- . \] (2.30)

Then it can be easily verified that (1.11) and (1.12) are valid, but (2.29) fails:

\[ q_- \rho_+ - q_+ \rho_- - \rho_+ = (\lambda^2 - \lambda - 1)\rho_- < 0. \] (2.31)

**Remark 2.3.** It is worth pointing out that, since the solid boundary is perturbed and no longer flat, the symmetry assumptions proposed in [39, 63] fail to be valid in this problem. Therefore, new ideas and methods must be developed to deal with the dihedral singularity, which is also completely different from the one caused by the corner singularity in [37]. These are the main new ingredients of this paper.

Now, we are ready to state our main result as following theorem:

**Theorem 2.1.** For each integer \( s_0 \geq 3 \), suppose the initial-boundary data of (NLP) satisfy the compatibility condition up to order \( s_0 + 1 \). If conditions (1.11), (1.12), (2.16) and (2.29) hold, then there exist three constants \( \eta_0 > 1, T_0 > 0 \) and \( \bar{\varepsilon} > 0 \) such that if

\[ \| u_0 - u_b \|_{H^{s_0+1}(\Omega)} + \| u_1 \|_{H^{s_0}(\Omega)} + \| \mathcal{W} \|_{W^{s_0+2,\infty}(\mathbb{R}^2)} + \| e^{-\eta t}(D\Phi^- - (q_-, 0, 0)) \|_{H^{s_0}([0,T] \times \{x_3 > \mathcal{W}(x_1, x_2)\})} \leq \varepsilon \] (2.32)
is satisfied for $0 < T \leq T_0$, $\eta \geq \eta_0$ and $\epsilon \leq \epsilon_0$, where $\| \cdot \|_{H^k}$ stands for the standard Sobolev norm. Then (NLP) admits a unique solution $u \in H^{\eta_0+1}(\Omega_T)$ satisfying

$$\| e^{-\eta t} (u - u_b) \|_{H^{\eta_0+1}(\Omega_T)} \leq C \epsilon,$$

(2.33)

where $C$ is a positive constant depending on $(q_-, q_+, \rho_-, \rho_+, T_0, \eta_0)$.

Remark 2.4. The compatibility conditions mentioned in Theorem 2.1 come from the requirement that the initial-boundary data of (NLP) should be consistent. More precisely, by initial conditions in (NLP) and the first equation of (NLP), we know that at $y_0 = 0$,

$$D^\beta u = D^\beta u_0, \quad \partial_j D^\beta u = D^\beta u_1$$

and

$$\partial^2_{y_0} D^\beta u = D^\beta \left( \frac{1}{\tilde{a}_{00}} (\tilde{f} - \sum_{(i,j) \neq (0,0)} \tilde{a}_{ij} \partial_{y_i y_j} u) \right),$$

where $D^\beta = \partial_{y_1}^{\beta_1} \partial_{y_2}^{\beta_2} \partial_{y_3}^{\beta_3}$ is the spatial derivatives and $\beta = (\beta_1, \beta_2, \beta_3)$ is the multi-index corresponds to spatial derivative and

$$\tilde{f} = (\partial_{y_1} u)^3 \sum_{i,j=0}^3 a_{ij} \partial_{x_i x_j} \Phi^- + \tilde{a}_2 \partial_{y_2} u + \tilde{a}_3 \partial_{y_3} u + a_{12} \partial_{x_1 x_2} p(\partial_{y_1} u)^3.$$

Then by induction on $k$ (that is, assume we have already known the expression of $\partial^m y_0 D^\beta u$ at $y_0 = 0$ for all $m \leq k$.) and by taking derivative $D^\beta \partial^k$ on equation (NLP)1, we will have the expression of $\partial^k y_0 D^\beta u$ at $y_0 = 0$. We omit the details for the shortness. Then we have the expression of $D^\beta u$ at $y_0 = 0$ for all $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. Let

$$u_\alpha := D^\alpha u|_{y_0=0}.$$

(2.34)

On the other hand, we have two boundary conditions in (NLP). So, for any $(k_0, k_1, k_2, k_3) \in \mathbb{N}^4$, we have

$$D^{(k_0, k_1, k_2, 0)} G = 0 \text{ on } \{y_3 = 0 \} \quad \text{and} \quad D^{(k_0, 0, k_2, k_3)} G_1 = 0 \text{ on } \{y_1 = 0 \}.$$

Let $U := (u, Du)$, then by the Faà di Bruno’s formula and the Leibniz rule, we know there exist $c_{l_1 \cdots l_{m_1}' \cdots l_{m_m}'}(U)$ and $c'_{l_1 \cdots l_{m_1}' \cdots l_{m_m}'}(U)$ such that

$$\max_{(k_0, k_1, k_2)} \sum_{m=1}^\infty \sum_{l_1 + \cdots + l_m = k_0} \sum_{l_1' + \cdots + l_{m_1}' = k_1} \sum_{l_1'' + \cdots + l_{m_m}'' = k_2} c_{l_1 \cdots l_{m_1}' \cdots l_{m_m}'}(U) \cdot (D^{(l_1, l_1', 0)} U, \ldots, D^{(l_{m_1}, l_{m_1}', l_{m_1}'', 0)} U) = 0 \text{ on } \{y_2 = 0 \}.$$
and
\[ \max(k_0, k_1, k_3) \sum_{m=1}^n \sum_{l_1+\cdots+l_m=k_0} c_{l_1}^{\prime} \cdots c_{l_m}^{\prime \prime} (U) \cdot (D^{(l_1,0,l_1')_m} U, \ldots, D^{(l_m,0,l_m')_m} U) = 0 \text{ on } \{y_1 = 0\}. \]

Here integers \( l_m, l_1', \) and \( l_2'' \) can be zero. Let \( y_0 = 0 \) and plug (2.34) into the two identities above for all integers \( k_0 + k_1 + k_2 \leq s_0 + 1 \) and \( k_0 + k_2 + k_3 \leq s_0 + 1 \). Then we can obtain the identities that the initial and boundary data must satisfy for all integers \( k_0 + k_1 + k_2 \leq s_0 + 1 \) and \( k_0 + k_2 + k_3 \leq s_0 + 1 \). These identities are called the compatibility conditions up to order \( s_0 + 1 \).

### 3 WELL-POSEDNESS OF THE LINEAR PROBLEM

In this section, we will establish the well-posedness theorem for an initial boundary value problem of a linear hyperbolic equation of second order in the dihedral-space domain. The linear theorem will be used to solve the (NLP) by introducing an iteration scheme in the next section.

In this section, we investigate the following initial boundary value problem

\[
\begin{cases}
L'(u)w = f & \text{in } \Omega_T, \\
B(u)w = g & \text{on } \Gamma_s, \\
\partial y_3 w = 0 & \text{on } \Gamma_w, \\
(w, \partial y_0 w) = (0, 0) & \text{on } \Gamma_{in} := \{y_0 = 0\},
\end{cases}
\tag{LP}
\]

where

\[
L'(u) := \sum_{i,j=0}^{3} r_{ij} \partial_{ij} + \sum_{i=0}^{3} r_i \partial_i + r,
\]

\[
B(u) := \sum_{i=0}^{3} b_i \partial_i + b,
\]

\( \Omega_T \) is the time-spatial domain defined below (NLP) in Section 2, \( \Gamma_w \) and \( \Gamma_s \) are defined by (2.17) and (2.18), respectively.

We impose following hypothesis on the coefficients of the operators \( L'(u) \) and \( B(u) \).

(\( H_1 \)) \( L'(u) \) is a hyperbolic operator of second order. \( r_{ij}, r_i \) and \( r \) are smooth functions of \( D\PhiD, Du \) and \( W(u, x_2(u, y_2, y_3)) \). Moreover, \( r_{32}, r_{31}, r_{30}, \) and \( r_2 \) vanish on the flat boundary \( \Gamma_w \). In particular, at the background solution \( u_b \), which is given in (2.24), \( r_{10} = r_{01} > 0, r_{12} = r_{21} = 0, r_{02} = r_{20} = 0, r_{33} = r_{22} < 0, r_{30} = r_{03} = r_{31} = r_{13} = r_{32} = r_{23} = 0 \) and \( r_{11} < 0 \).

(\( H_2 \)) \( b_i \) and \( b \) are smooth functions depend on \( Du \) and \( W'(u) \) and \( b_3|_{\Gamma_w} = 0 \). Furthermore, \( b = b_2 = b_3 = 0 \) at the background solution \( u_b \).

(\( H_3 \)) There exists an integer \( n_0 \geq 1 \) and \( \delta > 0 \) such that

\[
\sup_{0 \leq y_0 \leq T} \sum_{|\alpha| \leq n_0 + 3} \|D^\alpha (u - u_b)\|_{L^2(\Omega)} < \delta.
\]
At the background solution $u_b$, the following stability conditions hold for some constant $\gamma_0 > 0$:

$$|b_1| \geq \gamma_0, \quad \tilde{a}_{11} b_0 - r_{01} \geq \gamma_0, \quad \sum_{i,j=0}^{3} r^{ij} \left( \frac{r_{11} b_1}{b_1} - r_{i1} \right) \left( \frac{r_{11} b_j}{b_1} - r_{j1} \right) \geq \gamma_0.$$

Here $r^{ij}$ is the $(i, j)$th entry of the matrix $[r_{ij}]^{-1}_{4 \times 4}$, the inverse matrix of $[r_{ij}]_{4 \times 4}$.

Remark 3.1. The compatibility conditions up to order $n_0 + 3$ for (LP) can be obtained by same arguments as done in Remark 2.4.

For the linear problem (LP), we have the following theorem.

**Theorem 3.1.** Suppose assumptions $(H_1)$-$(H_4)$ are fulfilled and $\delta^k f|_{t=0} = 0$ for $k = 0, 1, 2, \ldots, n_0 + 2$ with an integer $n_0 \geq 1$. Then problem (LP) admits a smooth solution $w \in H^{n_0+3}(\Omega_T)$. Moreover, there exists $\eta_0 \geq 1$ and $T_0 > 0$ such that for all $s \leq n_0 + 3$, the following estimate

$$\sum_{|\alpha| \leq s} \eta \||e^{-\eta t} D^\alpha w||^2_{L^2(\Omega_T)} + e^{-2\eta T} \sup_{0 \leq t \leq T} ||D^\alpha w(t, \cdot)||^2_{L^2(\Omega)} + ||e^{-\eta t} D^\alpha w||^2_{L^2(\omega^c_T)} \lesssim 1$$

holds for all $\eta \geq \eta_0$ and $0 < T \leq T_0$.

We have used the symbol $\lesssim$ in (3.1). Here and after $A \lesssim B$ means that $A \leq CB$ for some positive constant $C$. Hereafter, we also use the notations:

$$\omega^c_T := (0, T) \times \omega^c, \quad \text{where} \quad \omega^c := \{y_1 = 0\} \times \mathbb{R} y_2 \times \{y_3 > 0\}.$$

To prove Theorem 3.1, one needs to prove the existence of the solutions and the energy inequality (3.1). In fact, we have following proposition:

**Proposition 3.1.** If hypothesis $(H_1)$-$(H_4)$ hold, $\delta^k f|_{t=0} = 0$ for $k = 0, 1, 2, \ldots, s - 1$, with $s \leq n_0 + 2$, then (LP) admits a smooth solution $w \in H^{n_0+3}(\Omega_T)$. Moreover, there exists $\eta_0 \geq 1$ such that for $\eta \geq \eta_0$ and $T > 0$, it holds that

$$\sum_{|\alpha| \leq s+1} \left( \eta \||e^{-\eta t} D^\alpha w||^2_{L^2(\Omega_T)} + e^{-2\eta T} ||D^\alpha w||^2_{L^2(\Omega)} + ||e^{-\eta t} D^\alpha w||_{y_1=0}^2 \right) \lesssim 1$$

holds for all $\eta \geq \eta_0$ and $0 < T \leq T_0$.

Remark 3.2. It is obvious that the estimate (3.3) is different from the inequality (3.1) in Theorem 3.1. On the right side of (3.3), the first two terms $e^{-\eta t} L'(D^\alpha w) = e^{-\eta t} (D^\alpha (L' w) - [D^\alpha, L'] w)$
and $e^{-\eta t} B(D^2 w) = e^{-\eta t} (D^2(B w) - [D^2, B]w)$ contain the commutators $[D^2, L']w$ and $[D^2, B]w$, respectively, which will be estimated carefully in the last subsection of this section.

In the coming proof of this proposition, $D^\ell v$ stands for the derivatives of function $v$ of order no higher than $\ell$ and $|D^\ell v|^p := \sum_{|\alpha| \leq \ell} |D^\alpha v|^p$ for $p = 1, 2$. In what follows, the dependence of the operators $L'$ and $B$ on $u$ is omitted. For brevity, one uses the notation $\partial_{i_1 \cdots i_{\ell}}$ to represent the partial derivative with respect to the variables $y_{i_1}, y_{i_2}, \cdots, y_{i_{\ell}}$.

Since the proof of this proposition is long, we divide it into two parts. In the first part, we will derive the existence and uniqueness of solutions to problem (LP) and illustrate how to derive the energy estimates up to second order (see Lemma 3.2). In the second part, we establish the energy estimates of higher order (see Lemma 3.5).

It should be emphasized that the estimates of derivatives higher than second order cannot be derived in the half-space domain directly, due to the fact that the regularity of the extended coefficients is not sufficient. Hence, we have to establish higher order estimates directly in the dihedral-space domain. They will be established in three steps (see the proof of Lemma 3.5). For this purpose, two multipliers are constructed to deal with the boundary terms (see Lemmas 3.3 and 3.4). In the third step (the final step), we treat the energy estimates of even order and odd order separately, because of different types of boundary conditions. The estimates in the final step of the higher order estimates still rely on the multipliers constructed in Lemmas 3.3 and 3.4. It is useful to point out the observation that both $\partial_{y_0}$ and $\partial_{y_2}$ are tangential to the boundaries $\Gamma_s$ and $\Gamma_w$. Hence, any established estimates of $w$ can be directly applied to $\partial_{y_0} w$ and $\partial_{y_2} w$, which helps us to establish the higher order estimates.

### 3.1 Well-posedness of the LP in $H^2(\Omega_T)$

In this subsection, by introducing an auxiliary problem, we show the well-posedness of the LP in $H^2(\Omega_T)$. In fact, one has the following lemma.

**Lemma 3.2 (Existence and uniqueness).** If hypothesis (H1)-(H4) hold, $\partial^k f |_{t=0} = 0$ for $k = 0, 1, 2, \ldots, s - 1$, with $s \leq n_0 + 2$, then (LP) admits a weak solution $w \in H^2(\Omega_T)$. Moreover, it satisfies

$$\sum_{|\alpha| \leq 2} \eta \| e^{-\eta t} D^\alpha w \|_{L^2(\tilde{\Omega}_T)}^2 + e^{-2\eta T} \| D^2 w(T, \cdot) \|_{L^2(\tilde{\Omega})} + \| e^{-\eta t} D^\alpha w |_{y_1=0} \|_{L^2(\omega_T)}^2 \lesssim \frac{1}{\eta} \| e^{-\eta t} f \|_{H^1(\tilde{\Omega}_T)}^2 + \sum_{|\alpha| \leq 1} \| e^{-\eta t} D^\alpha g |_{y_1=0} \|_{L^2(\omega_T)}^2$$

for $\eta$ large enough and all $T > 0$.

In this lemma, we have used the notations: $\tilde{\Omega} := \mathbb{R}^+ \times \mathbb{R} y_2 \times \mathbb{R} y_3$, $\tilde{\Omega}_T := [0, T] \times \tilde{\Omega}$, and $\omega_T := [0, T] \times \omega$, where $\omega := \{0\} \times \mathbb{R} y_2 \times \mathbb{R} y_3$.

**Proof.** The proof of Lemma 3.2 starts with the introduction of an auxiliary problem (LPE). First, one extends the coefficients of $L'(u)$, $f$, $g$ and the coefficients of $B(u)$ from $\Omega_T$ to $\tilde{\Omega}_T$ in the following way.
(i) Extend \( r_{03} = r_{30}, \ r_{13} = r_{31}, \ r_{23} = r_{32} \) and \( b_3 \) oddly with respect to \( \{y_3 = 0\} \). To be precise, we take \( r_{03} \) for example. Extend \( r_{03} \) by letting \((Er_{03})(y_0, y) := r_{03}(y_0, y)\) when \( y_3 \geq 0 \) and \((Er_{03})(y_0, y) := -r_{03}(y_0, y_1, y_2, -y_3)\) when \( y_3 < 0 \). Coefficients other than \( r_{03}, r_{13}, \) and \( r_{23} \) will be extend evenly with respect to \( \{y_3 = 0\} \) by the same manner.

(ii) Extend \( f \) and \( g \) evenly with respect to \( \{y_3 = 0\} \). For notational simplicity, one omits the \( E' \) for all extended coefficients. Then the extended problem (LPE) is defined as follows:

\[
\begin{cases}
L'(u)w = f & \text{in } \tilde{\Omega}_T, \\
E(u)w = g & \text{on } \omega_T, \\
(w, \partial_{y_0} w) = (0, 0) & \text{on } \Gamma_{in} := \{y_0 = 0\}.
\end{cases}
\]

\text{(LPE)}

\text{Remark 3.3.} Obviously, due to the regularity of the extended coefficients, (LPE) only satisfies the compatibility conditions as the one in Remark 2.4 on the wedge up to order 2. But it is enough for us to show the existence of solutions of the (LPE) in \( H^2(\tilde{\Omega}_T) \). Then the better regularity in \( \Omega_T \) of such solutions can be obtained by further argument.

By considering the well-posedness of the (LPE), and proving that the unique solution to (LPE) is the unique solution to (LP), one can show the well-posedness of the LP. Before that, let us deduce some useful inequalities. By the Sobolev embedding theorem and Assumption \((H_3)\), one has

\[
\sup_{(y_0, y) \in [0, T] \times \mathbb{R}^3} \sum_{|\alpha| \leq n_0} |D^\alpha (u - u_b)(y_0, y)| \leq C\delta.
\]

Since \( s \geq \left\lceil \frac{s+2}{2} \right\rceil \) if \( s \geq 4 \), we deduce that if \( n_0 \geq 4 \), then

\[
\sup_{(y_0, y) \in [0, T] \times \mathbb{R}^3} \sum_{|\alpha| \leq n_0+2} |D^\alpha (u - u_b)(y_0, y)| \leq C\delta. \tag{3.5}
\]

As a corollary of (3.5) and Assumption \((H_1)\), we have

\[
\sup_{(y_0, y) \in [0, T] \times \mathbb{R}^3} |D_{r_{ij}}(y_0, y)| \leq C\delta. \tag{3.6}
\]

The remainder of the proof consists of three steps. In the first two steps, one establishes the energy estimates of the solution to (LPE) up to second order. Then in the third step, we show that there exists a unique solution to (LPE) and the unique solution is indeed a solution to the LP.

\textbf{Step 1:} First-order estimate of the solution to (LPE).

Multiplying \( 2e^{-2\eta t} Qw \) on both sides of (LPE), where \( Q := \sum_{\epsilon=0}^3 \sum_{\epsilon=0}^3 Q_{\epsilon} \partial_{\epsilon} \) will be chosen properly later. Then integrate by parts over \( \tilde{\Omega}_T \) with respect to \( (y_0, y) \), we have

\[
\int_{\tilde{\Omega}_T} e^{-2\eta y_0} (QwL'w + P(w, Dw)) dy_0 dy = \int_{\tilde{\Omega}_T} e^{-2\eta y_0} H_0 |H_0(y_0, y_1, y_2, y_3) + 2\eta \int_{\tilde{\Omega}_T} e^{-2\eta y_0} H_0 dy_0 dy \tag{3.7}
\]
where
\[ H_i(Dw; Q) = 2 \sum_{j, \ell=0}^{3} r_{ij} \hat{\partial}_j w Q_{\ell} \hat{\partial}_\ell w - Q_i \sum_{j, \ell=0}^{3} r_{\ell j} \hat{\partial}_\ell w Q_j \] (i = 0, 1) \tag{3.8}

and \( P(w, Dw) \) is a quadratic polynomial in \( w \) and \( Dw \) with bounded coefficients. For later use, we also define \( H_3 \) by
\[ H_3(Dw; Q) := 2 \sum_{j, \ell=0}^{3} r_{ij} \hat{\partial}_j w Q_{\ell} \hat{\partial}_\ell w - Q_3 \sum_{j, \ell=0}^{3} r_{\ell j} \hat{\partial}_\ell w Q_j. \] \tag{3.9}

It is easy to see
\[ \| \mathcal{P} \| \leq C(|w|^2 + |Dw|^2). \]

Choosing \( Q \) appropriately as
\[ Q = \tilde{B} + \nu(\tilde{B} - \mathcal{N}) + \frac{\nu r_{01}}{B_0} \tilde{B}, \]
where \( \tilde{B} = r_{11} \sum_{j=0}^{3} b_j \hat{\partial}_j - \sum_{j=0}^{3} r_{j1} \hat{\partial}_j, \mathcal{N} = - \sum_{j=0}^{3} r_{j1} \hat{\partial}_j \) and \( \nu = \sum_{i,j=0}^{3} r_{ij} \tilde{B}_i \tilde{B}_j \), where \( B_j \) is the coefficient in \( B \) in front of \( \hat{\partial}_j \). Then by simple calculation, we obtain
\[ H_0(Dw) \geq C|Dw|^2 \quad \text{and} \quad -H_1(Dw) \geq C(|Dw|^2 + |w|^2 - |Bw|^2). \] \tag{3.10}

In view of (3.7), (3.10), and the Cauchy inequality, one has
\[
\begin{align*}
\eta \| e^{-\frac{\eta t}{2}} Dw \|_{L^2(\tilde{\Omega}_T)}^2 &+ e^{-2\eta T} \| Dw(T, \cdot) \|_{L^2(\tilde{\Omega})}^2 + \| e^{-\frac{\eta t}{2}} Dw \|_{y_1=0}^2 \\
&\leq \frac{1}{\varepsilon \eta} \| e^{-\frac{\eta t}{2}} L'(u)w \|_{L^2(\tilde{\Omega}_T)}^2 + \varepsilon \eta \| e^{-\frac{\eta t}{2}} Dw \|_{L^2(\tilde{\Omega}_T)}^2 + \| e^{-\frac{\eta t}{2}} (Bw, w) \|_{L^2(\tilde{\Omega}_T)}^2 \\
&\quad + \| (w, \hat{\partial}_t w) \|_{t=0}^2_{L^2(\tilde{\Omega}_T)}. \tag{3.11}
\end{align*}
\]

Set \( \varepsilon = \frac{1}{2C} \), then the second term on the right side is absorbed by the left-hand side term, hence we get
\[
\begin{align*}
\eta \| e^{-\frac{\eta t}{2}} Dw \|_{L^2(\tilde{\Omega}_T)}^2 &+ e^{-2\eta T} \| Dw(T, \cdot) \|_{L^2(\tilde{\Omega})}^2 + \| e^{-\frac{\eta t}{2}} Dw \|_{y_1=0}^2 \\
&\leq C \left( \frac{1}{\eta} \| e^{-\frac{\eta t}{2}} L'(u)w \|_{L^2(\tilde{\Omega}_T)}^2 + \| e^{-\frac{\eta t}{2}} Bw \|_{L^2(\tilde{\Omega}_T)}^2 + \| e^{-\frac{\eta t}{2}} w \|_{L^2(\tilde{\Omega}_T)}^2 \right) \\
&\quad + C \| (w, \hat{\partial}_t w) \|_{t=0}^2_{L^2(\tilde{\Omega}_T)}. \tag{3.12}
\end{align*}
\]

Apply (3.17) to the boundary term of \( w \) on the right-hand side of above inequality, then let \( \eta \) be properly large, so that \( \| e^{-\frac{\eta t}{2}} w \|_{L^2(\tilde{\Omega}_T)}^2 \) be absorbed by the left-hand side terms. Then we obtain
\[
\begin{align*}
\sum_{|\alpha| \leq 1} \left( \eta \| e^{-\frac{\eta t}{2}} D^\alpha w \|_{L^2(\tilde{\Omega}_T)}^2 + e^{-2\eta T} D^\alpha w(T, \cdot) \|_{L^2(\tilde{\Omega})}^2 + \| e^{-\frac{\eta t}{2}} D^\alpha w \|_{y_1=0}^2_{L^2(\tilde{\Omega}_T)} \right) \\
&\leq C \left( \frac{1}{\eta} \| e^{-\frac{\eta t}{2}} L'(u)w \|_{L^2(\tilde{\Omega}_T)}^2 + \| e^{-\frac{\eta t}{2}} Bw \|_{y_1=0}^2_{L^2(\tilde{\Omega}_T)} + \| Dw \|_{t=0}^2_{L^2(\tilde{\Omega})} \right). \tag{3.12}
\end{align*}
\]
Step 2: In this step, we establish the second-order estimate of the solution to \((\text{LPE})\).

Applying (3.12) to \(\partial_{y_0} w, \partial_{y_2} w\) and \(\partial_{y_3} w\), we obtain that

\[
\eta \|e^{-\eta t}D\partial_{y_1} w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|\partial_{y_1} w(T, \cdot)\|^2_{L^2(\Omega)} + \|e^{-\eta t}D\partial_{y_1} w|_{y_1=0}\|^2_{L^2(\omega_T)}
\]

\[
\leq \frac{1}{\eta} \|e^{-\eta t}L'\partial_{y_1} w\|^2_{L^2(\Omega_T)} + \|e^{-\eta t}B\partial_{y_1} w|_{y_1=0}\|^2_{L^2(\omega_T)} + \|D\partial_{y_1} w|_{t=0}\|^2_{L^2(\Omega)}
\]

(3.13)

holds for \(\ell' = 0, 2, 3\). By \((\text{LPE})_1\), one has

\[
\partial^2_{y_1} w = \frac{1}{r_{11}} \left( L'w - \sum_{(i,j)\neq(1,1)} r_{ij}\partial_{ij} w - \sum_{i=0}^2 r_i\partial_i w - rw \right)
\]

Hence,

\[
\eta \|e^{-\eta t}\partial_{y_1}^2 w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|\partial_{y_1}^2 w(T, \cdot)\|^2_{L^2(\Omega)} + \|e^{-\eta t}\partial_{y_1}^2 w|_{y_1=0}\|^2_{L^2(\omega_T)}
\]

\[
\leq \sum_{\ell' = 0,2,3} \eta \|e^{-\eta t}D\partial_{y_1} w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|D\partial_{y_1} w|_{t=T}\|^2_{L^2(\Omega)} + \|e^{-\eta t}D\partial_{y_1} w|_{y_1=0}\|^2_{L^2(\omega_T)}
\]

\[
+ \eta \|e^{-\eta t}L'w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|L'w\|^2_{L^2(\Omega)} + \|e^{-\eta t}L'w|_{y_1=0}\|^2_{L^2(\omega_T)}
\]

(3.15)

By (3.12), (3.13), and (3.15), we have

\[
\eta \|e^{-\eta t}\partial_{y_1}^2 w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|\partial_{y_1}^2 w(T, \cdot)\|^2_{L^2(\Omega)} + \|e^{-\eta t}\partial_{y_1}^2 w|_{y_1=0}\|^2_{L^2(\omega_T)}
\]

\[
\leq \sum_{|\alpha|\leq 1} \left( \frac{1}{\eta} \|e^{-\eta t}L'(D^2 w)\|^2_{L^2(\Omega_T)} + \|e^{-\eta t}BD^2 w\|^2_{L^2(\omega_T)} \right) + \sum_{\ell' = 0,2,3} \|D\partial_{y_1} w|_{t=0}\|^2_{L^2(\Omega)}
\]

\[
+ \eta \|e^{-\eta t}w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|w|_{t=T}\|^2_{L^2(\Omega)} + \|e^{-\eta t}w|_{y_1=0}\|^2_{L^2(\omega_T)}
\]

\[
+ \eta \|e^{-\eta t}L'w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|L'w\|^2_{L^2(\Omega)} + \|e^{-\eta t}L'w|_{y_1=0}\|^2_{L^2(\omega_T)}
\]

(3.16)

By integrating by parts with respect to \(t\) and the trace theorem, we have

\[
\eta \|e^{-\eta t}w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|w\|^2_{L^2(\Omega)} + \|e^{-\eta t}w|_{y_1=0}\|^2_{L^2(\omega_T)}
\]

\[
\leq \frac{1}{\eta} \|e^{-\eta t}\partial_t w\|^2_{L^2(\Omega_T)} + \|w|_{t=0}\|^2_{L^2(\Omega)} + \sum_{|\alpha|\leq 1} \|e^{-\eta t}D^2 w\|^2_{L^2(\Omega_T)}
\]

(3.17)

So, by (3.17) and Cauchy inequality, one has

\[
\eta \|e^{-\eta t}L'w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|L'w\|^2_{L^2(\Omega)} + \|e^{-\eta t}L'w|_{y_1=0}\|^2_{L^2(\omega_T)}
\]

\[
\leq \|L'w|_{t=0}\|^2_{L^2(\Omega)} + \frac{1}{\eta} \sum_{|\alpha|\leq 1} \|e^{-\eta t}D^2 w\|^2_{L^2(\Omega_T)} + \|e^{-\eta t}L'(D^2 w)\|^2_{L^2(\Omega_T)}
\]

\[
+ \eta \sum_{|\alpha|\leq 2} \|e^{-\eta t}D^2 w\|^2_{L^2(\Omega_T)}
\]

(3.18)
In light of (3.13), (3.15), (3.17), and (3.18), we obtain the estimate of $\delta^2_{y_1} w$, that is,

$$
\eta \| e^{-\eta T} \delta^2_{y_1} w \|^2_{L^2(\tilde{\Omega}_T)} + e^{-2\eta T} \| \delta^2_{y_1} w(T, \cdot) \|_{L^2(\tilde{\Omega})} + \| e^{-\eta T} \delta^2_{y_1} w \|_{y_1=0} \|_{L^2(\tilde{\omega}_T)} \\
\lesssim \frac{1}{\eta} \| e^{-\eta T} \partial_t w \|^2_{L^2(\tilde{\Omega}_T)} + \sum_{|\alpha| \leq 1} \| e^{-\eta T} D^\alpha w \|^2_{L^2(\tilde{\Omega}_T)} \\
+ \| D w \|_{t=0} \|_{L^2(\tilde{\Omega})} + \sum_{|\alpha| \leq 1} \left( \frac{1}{\eta} \| e^{-\eta T} L(D^\alpha w) \|^2_{L^2(\tilde{\Omega}_T)} + \| e^{-\eta T} BD^\alpha w \|^2_{L^2(\tilde{\omega}_T)} \right) \\
+ \frac{1}{\eta} \sum_{|\alpha| \leq 1} \left( \| e^{-\eta T} D^\alpha w \|^2_{L^2(\tilde{\Omega}_T)} + \| e^{-\eta T} L(D^\alpha w) \|^2_{L^2(\tilde{\Omega}_T)} + \| L w \|_{t=0} \|_{L^2(\tilde{\Omega})} \right) \\
+ \eta \sum_{|\alpha| \leq 2} \| e^{-\eta T} D^\alpha w \|^2_{L^2(\tilde{\Omega}_T)} + \sum_{|\alpha| \leq 2} \| D^\alpha w \|_{t=0} \|_{L^2(\tilde{\Omega})}.
$$

(3.19)

Adding up (3.12), (3.13) for $\ell = 0, 2, 3$ and (3.19), then set $\varepsilon$ and $\frac{1}{\eta}$ to be properly small, we have

$$
\sum_{|\alpha| \leq 2} \eta \| e^{-\eta T} D^\alpha w \|^2_{L^2(\tilde{\Omega}_T)} + e^{-2\eta T} \| D^\alpha w(T, \cdot) \|_{L^2(\tilde{\Omega})} + \| e^{-\eta T} D^\alpha w \|_{y_1=0} \|_{L^2(\tilde{\omega}_T)} \\
\lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 1} \| e^{-\eta T} L(D^\alpha w) \|^2_{L^2(\tilde{\Omega}_T)} + \| L w \|_{t=0} \|_{L^2(\tilde{\Omega})} + \| e^{-\eta T} BD^\alpha w \|^2_{L^2(\tilde{\omega}_T)} \\
+ \sum_{|\alpha| \leq 2} \| D^\alpha w \|_{t=0} \|_{L^2(\tilde{\Omega})} \\
\lesssim \sum_{|\alpha| \leq 2} \frac{1}{\eta} \left( \| e^{-\eta T} D^\alpha w \|^2_{L^2(\tilde{\Omega}_T)} + \| e^{-\eta T} D^\alpha L w \|^2_{L^2(\tilde{\Omega}_T)} \right) + \sum_{|\alpha| \leq 1} \| e^{-\eta T} D^\alpha g \|_{y_1=0} \|_{L^2(\tilde{\omega}_T)}.
$$

(3.20)

Let $\eta$ be properly large, we obtain

$$
\sum_{|\alpha| \leq 2} \eta \| e^{-\eta T} D^\alpha w \|^2_{L^2(\tilde{\Omega}_T)} + e^{-2\eta T} \| D^\alpha w(T, \cdot) \|_{L^2(\tilde{\Omega})} + \| e^{-\eta T} D^\alpha w \|_{y_1=0} \|_{L^2(\tilde{\omega}_T)} \\
\lesssim \frac{1}{\eta} \| e^{-\eta T} f \|^2_{L^2(\tilde{\Omega}_T)} + \sum_{|\alpha| \leq 1} \| e^{-\eta T} D^\alpha g \|_{y_1=0} \|_{L^2(\tilde{\omega}_T)}.
$$

(3.21)

**Step 3:** In this step, we show that the LP is well-posed by showing that the unique solution to (LPE) is indeed a solution to (LP).

Based on energy estimate (3.21), it is easy to obtain the existence of an $H^2(\tilde{\Omega}_T)$ solution $w$ of problem (LPE). In fact, the existence of (LPE) has been proved in [54, Theorem 3.3], when the coefficients and source terms belong to $H^s(\tilde{\Omega}_T)$ with $s > \left[ \frac{N+1}{2} \right] + 1$, where $N$ is the space dimension. Though the regularity of coefficients and source terms of (LPE) is not enough, we can still deduce the existence of (LPE). First, one mollifies the coefficients and the source terms by the convolution of the classical Friedrichs mollifier $\rho_\varepsilon$, then by [54, Theorem 3.3], there exists a smooth solution $w^\varepsilon$ to the regularized problem for each $\varepsilon > 0$. Thanks to our uniform $H^2(\tilde{\Omega}_T)$ estimate (3.21), $\{w^\varepsilon\}_{\varepsilon > 0}$ is strongly compact in $H^1(\tilde{\Omega}_T)$ and weakly compact in $H^2(\tilde{\Omega}_T)$. Then passing the limit by letting $\varepsilon \to 0^+$ in the regularized equation, we obtain a $H^2_\eta$-solution to the linear problem (LPE). If $f = g \equiv 0$, (3.21) implies $w \equiv 0$ in $\tilde{\Omega}_T$. This indicates that the solution to (LPE) is unique,
since \((LPE)\) is a LP. Due to our extension, it is easy to check that \(w(y_0, y_1, y_2, -y_3)\) is also a solution to \((LPE)\). By the uniqueness, we have \(w(y_0, y_1, y_2, y_3) = w(y_0, y_1, y_2, -y_3)\) for all \((y_0, y) \in \tilde{\Omega_T}\). Differentiating with respect to \(y_3\) on both sides of this equality and letting \(y_3 = 0\), one has

\[
\partial_{y_3} w|_{y_3=0} = -\partial_{y_3} w|_{y_3=0},
\]

which implies \(\partial_{y_3} w|_{y_3=0} = 0\). From (3.21) and the trace theorem, we know \(\partial_{y_3} w\) is a \(L^2\) function on \(\{y_3 = 0\}\), so above process makes sense. Therefore, we conclude that the unique solution to \((LPE)\) is indeed the unique solution to \((LP)\). This completes the proof of Lemma 3.2. □

### 3.2 Higher order estimates: Proof of (3.3)

In this subsection, we prove (3.3) by establishing higher order estimates of the solution \(w\) directly in the dihedral-space domain. The higher order estimates in the dihedral-space domain rely on the second-order estimate derived in Lemma 3.2. Moreover, in order to deal with the boundary integrals on the boundaries of the dihedral-space domain, we need two different types of multipliers. The existence of such multipliers will be given in following two lemmas. The following lemma reveals the existence of the required multipliers, which will be used to obtain higher order estimates of odd order.

**Lemma 3.3.** Let \(H_m\) be defined as in (3.8) and (3.9). For any given \(r_{ij}\) satisfying assumptions \((H_1)-(H_4)\), we can find a multiplier \(Q^d = \sum_{i=0}^3 Q_i^d \partial_i\) such that

\[
H_0(Dw; Q^d) \geq C_1 |\nabla_y w|^2 - C_2 \left( |\partial_{y_0} w|^2 + |\partial_{y_2} w|^2 + |\partial_{y_3} w|^2 \right),
\]

(3.22)

\[
-H_1(Dw; Q^d) \geq C_1 \left( |\partial_{y_0} w|^2 + |\partial_{y_2} w|^2 + |\partial_{y_3} w|^2 \right),
\]

(3.23)

where \(\nabla_y := (\partial_{y_1}, \partial_{y_2}, \partial_{y_3})\). Moreover, if \(w = 0\) on \(\{y_3 = 0\}\), then

\[
-H_3(Dw; Q^d) \geq C |\partial_{y_3} w|^2 \quad \text{on} \quad \{y_3 = 0\}.
\]

(3.24)

**Proof.** It is convenient to denote \(\partial_{y_i} w\) by \(\xi_i\) for \(i = 0, 1, 2, 3\). At the background solution, by simple calculation, one has

\[
-H_1(Dw; Q^d) = (-2r_{10}Q_{0}^d + r_{00}Q_{1}^d)\xi_0^2 + Q_{1}^d (-r_{11}\xi_1^2 + r_{22}\xi_2^2 + r_{33}\xi_3^2)
\]

\[
-2r_{11}Q_{0}^d\xi_1\xi_0 - 2r_{10}Q_{0}^d\xi_0\xi_1 + 2Q_{1}^d r_{10}Q_{0}^d\xi_0\xi_1 - 2r_{11}Q_{0}^d\xi_1\xi_0
\]

\[
\xi_0^2 + 2Q_{1}^d r_{10}Q_{0}^d\xi_0\xi_1 - 2r_{11}Q_{0}^d\xi_1\xi_0.
\]

(3.25)

Choosing \(Q_{1}^d\) such that \(-Q_{1}^d r_{11} > 0\), then (3.23) follows easily. At the background solution \(u_b\), we know \(r_{11} = -\frac{c_1^2}{(q_- - q^+)^2} < 0\). So, one just needs to let \(Q_{1}^d > 0\). For \(H_0(Dw; Q^d)\), at the background solution one has

\[
H_0(Dw; Q^d) = 2r_{10}\xi_1(Q_{0}^d\xi_0 + Q_{1}^d\xi_1 + Q_{2}^d\xi_2 + Q_{3}^d\xi_3)
\]

\[
- Q_{0}^d (r_{00}\xi_0^2 + r_{11}\xi_1^2 + r_{22}\xi_2^2 + r_{33}\xi_3^2 + 2r_{10}\xi_0\xi_1).
\]

(3.26)
If we can let the coefficient before $\xi_1^2$ be positive, then (3.23) follows immediately. In fact, it suffices to let $2r_{10}Q_1^d - Q_0^d r_{11} > 0$. Since $Q_1^d$ has been set to be positive, $r_{10} > 0$, and $r_{11} < 0$ at the background solution, it is sufficient to let $Q_0^d$ be positive. At the background solution, one has

$$-H_3(Dw; Q_1^d) = -r_{33}Q_3^d \xi_3^2.$$  (3.27)

Hence, (3.24) follows if we let $Q_3^d > 0$, since $r_{33} < 0$ at the background solution. □

For the multipliers needed in the higher estimates of even order, one has following lemma.

**Lemma 3.4.** Let $H_m$ be defined as in (3.8) and (3.9). For any given $r_{ij}$ satisfying assumptions (H1)-(H4), we can find a multiplier $Q_e = \sum_{i=0}^{3} Q_i^e$ such that

$$H_0(Dw; Q_e) \geq C_1 \left| \nabla_y w \right|^2 - C_2 \left| \partial_y^0 w \right|^2,$$  (3.28)

$$-H_1(Dw; Q_e) \geq C_1 \left| \partial_y^3 w \right|^2 - C_2 \left( \left| \partial_y^0 w \right|^2 + \left| \partial_y^1 w \right|^2 + \left| \partial_y^2 w \right|^2 \right),$$  (3.29)

where $\nabla_y := (\partial_y^1, \partial_y^2, \partial_y^3)$. Moreover, if $w = 0$ on $\{y_2 = 0\}$, then

$$-H_3(Dw; Q_e) \geq C \left| \partial_y^3 w \right|^2 \text{ on } \{y_3 = 0\}.$$  (3.30)

**Proof.** For the ease of presentation, in the proof of this lemma, denote Dw by $(\xi_0, \xi_1, \xi_2, \xi_3)$. Then at the background solution $u_b$, we have

$$-H_1(Dw; Q_e) = (2r_{10}Q_0^e + r_{00}Q_1^e)\xi_0^2 + Q_1^e(-r_{11}\xi_1^2 + r_{22}\xi_2^2 + r_{33}\xi_3^2)$$

$$-2r_{11}Q_0^e\xi_0\xi_1 - 2r_{10}Q_2^e\xi_0\xi_2 + 2r_{22}Q_3^e\xi_0\xi_3 - 2r_{11}Q_2^e\xi_1\xi_2$$

$$-2r_{11}Q_3^e\xi_1\xi_3.$$  (3.31)

Choose $Q_1^e$ such that

$$r_{33}Q_1^e > 0,$$  (3.32)

then (3.29) follows easily. We know that $r_{33} = -\frac{c_1^2}{(q_+ - q_-)^2} < 0$ at the background solution $u_b$. So, we just need to let

$$Q_1^e < 0.$$  (3.33)

Next, since at the background solution, we have

$$-H_3(Dw; Q_e) = -r_{33}Q_3^e \left| \partial_y^3 w \right|^2,$$  (3.34)

(3.30) follows if we let $Q_3^e > 0$. Finally, for (3.28), at the background solution $u_b$, we know

$$H_0(Dw; Q_e) = 2r_{10}\xi_1(Q_0^e\xi_0 + Q_1^e\xi_1 + Q_2^e\xi_2 + Q_3^e\xi_3)$$

$$- Q_0^e(r_{00}\xi_0^2 + r_{11}\xi_1^2 + r_{22}\xi_2^2 + r_{33}\xi_3^2 + 2r_{10}\xi_0\xi_1)$$
\[(2r_{10}Q_1^e - r_{11}Q_0^e)\xi_1^2 - Q_0^e (r_{22}\xi_2 + r_{33}\xi_3) + 2r_{10}\xi_1 (Q_0^e \xi_1 + Q_2^e \xi_2 + Q_3^e \xi_3) - Q_0^e (r_{00}\xi_0 + 2r_{10}\xi_0) \geq (2r_{10}Q_1^e - r_{11}Q_0^e - r_{10}\vert Q_2^e \vert - r_{10}Q_3^e)\xi_1^2 + (-Q_0^e r_{22} - r_{10}\vert Q_2^e \vert)\xi_2^2 + (-Q_0^e r_{33} - r_{10}Q_3^e)\xi_3^2 - Q_0^e r_{00}\xi_0. \]  

(3.35)

Because \( r_{11}, r_{22} \) and \( r_{33} \) are negative, we may let

\[ Q_0^e > \max \left\{-\frac{2r_{10}Q_1^e + r_{10}\vert Q_2^e \vert + r_{10}Q_3^e}{-r_{11}}, \frac{r_{10}\vert Q_2^e \vert}{-r_{22}}, \frac{r_{10}Q_3^e}{-r_{33}}\right\} > 0, \]  

(3.36)

then (3.28) follows. \( \square \)

Armed with Lemmas 3.3 and 3.4, we are able to show that (3.3) holds. Actually, we have:

**Lemma 3.5** (Higher order estimates). If hypotheses (H\(_1\))-(H\(_4\)) hold, \( \partial^k f \big|_{t=0} = 0 \) for \( k = 0, 1, 2, \ldots, s-1 \), with \( s \leq n_0 + 2 \), then there exists \( \eta_0 \geq 1 \) such that for \( \eta \geq \eta_0 \) and \( T > 0 \), the unique solution \( w \) obtained in Lemma 3.2 satisfies (3.3) for \( s \leq n_0 + 2 \).

**Proof.** The proof of this lemma consists of three steps. In the first two steps, one derives the energy inequalities up to fourth order in the dihedral-space domain. Then in the last step, we establish higher order estimates by induction.

**Step 1:** In this step, we will consider the third-order estimate. Since both \( \partial_{y_0} \) and \( \partial_{y_2} \) are tangential to both the solid wall \( \Gamma_w \) and shock front \( \Gamma_s \) and all the coefficients are smooth in the directions of \( y_0 \) and \( y_2 \). We can apply the first inequality of (3.20) to \( \partial_{y_0} w \) and \( \partial_{y_2} w \), respectively, to obtain that for \( i = 0, 2, \)

\[
\sum_{|\alpha| \leq 2} \left( \|e^{-\eta t} D^\alpha \partial_{y_i} w \|_{L^2(\tilde{\Omega}_T)}^2 + e^{-2\eta T} \| D^2 \partial_{y_i} w \|_{L^2(\tilde{\Omega})}^2 + \| e^{-\eta t} D^2 \partial_{y_i} w \big|_{y_1=0} \|_{L^2(\omega_T)}^2 \right) \leq \sum_{|\alpha| \leq 1} \left( \frac{1}{\eta} \| e^{-\eta t} L'(D^\alpha \partial_{y_i} w) \|_{L^2(\tilde{\Omega}_T)}^2 + \| e^{-\eta t} BD^\alpha \partial_{y_i} w \|_{L^2(\omega_T)}^2 \right) + \| Df \big|_{t=0} \|_{L^2(\tilde{\Omega})}^2. \]  

(3.37)

Here we use the fact that \( \sum_{|\alpha| \leq 3} \| D^\alpha w \big|_{t=0} \|_{L^2(\tilde{\Omega})}^2 \leq \sum_{j=0}^1 \| Df \big|_{t=0} \|_{L^2(\tilde{\Omega})}^2 \), which comes from the equation and the initial data. In the coming steps, the estimate we obtained in each step will be applied to \( \partial_{y_0} w \) and \( \partial_{y_2} w \) in the next step, because of the same reason as stated below Proposition 3.1. To control all other derivatives of third order, we need to estimate derivatives in the form of \( \partial_{y_1} \partial_{y_3} w \) with \( k_1 + k_3 = 3 \). Due to the limit of the regularity of the extended coefficients, we cannot obtain higher order estimate in \( \tilde{\Omega}_T \) directly. In the following steps, all estimates are restricted to the cornered time spatial domain \( \Omega_T \). We first try to obtain the first-order estimate of \( \partial_{y_1} \partial_{y_3} w \).
In fact, \( \partial_{y_1y_3} w \) satisfies

\[
\begin{cases}
L'(\partial_{y_1y_3} w) = \partial_{y_1y_3} f - [\partial_{y_1y_3}, L'] w & \text{in } \Omega_f, \\
\partial_{y_1y_3} w = 0, & \text{on } \omega_f, \\
\partial_{y_1y_3} w = \Lambda, & \text{on } \omega_f, \\
(\partial_{y_0}(\partial_{y_1y_3} w), \partial_{y_1y_3} w) = (0, 0) & \text{on } \Gamma_{in},
\end{cases}
\]  

(3.38)

where

\[
\Lambda = \frac{1}{b_1} \left( B(\partial_{y_3} w) - b_0 \partial_{y_0} w - b_2 \partial_{y_2} w - b_3 \partial_{y_3} w - b \partial_{y_3} w \right),
\]  

(3.39)

and

\[
w'_f := [0, T] \times \omega', \quad \text{where } \omega' := \{y_1 > 0\} \times \mathbb{R} y_2 \times \{y_3 = 0\}. 
\]  

(3.40)

Problem (3.38) is an initial boundary value problem in a dihedrul space-domain with two Dirichlet boundary conditions. Multiplying (3.38) by \(2e^{-2\eta t} D\partial_{y_1y_3} w\), where \(D\) is given in Lemma 3.3. Then integrating on both sides with respect to \((y_0, y)\) over \(\Omega_f\) and by using Cauchy inequality, we obtain

\[
\eta \int_{\Omega_f} e^{-2\eta t} H_0 dy_0 dy + e^{-2\eta T} \int_{\Omega} H_0 |t=T| dy - \int_{\omega'_f} e^{-2\eta t} H_1 dy_0 dy - \int_{\omega'_f} e^{-2\eta t} H_3 dy_0 dy
\]

\[
\lesssim \frac{1}{\varepsilon \eta} \| e^{-\eta t} L'(\partial_{y_1y_3} w) \|^2_{L^2(\Omega_f)} + (\varepsilon \eta + 1) \| e^{-\eta t} D\partial_{y_1y_3} w \|^2_{L^2(\Omega_f)}. 
\]  

(3.41)

By Lemma 3.3, one knows that

\[
-H_3 \geq C |\partial_{y_3}(\partial_{y_1y_3} w)|^2, 
\]  

(3.42)

\[
-H_1 \geq C_1 |\partial_{y_1}(\partial_{y_1y_3} w)|^2 - C_2 (|\partial_{y_0} \Lambda|^2 + |\partial_{y_2} \Lambda|^2 + |\partial_{y_3} \Lambda|^2), 
\]  

(3.43)

\[
H_0 \geq C_1 \| \nabla_y \partial_{y_1y_3} w \|^2 - C_2 \left| \partial_{y_0}(\partial_{y_1y_3} w) \right|^2. 
\]  

(3.44)

From (3.41)–(3.44) and letting \(\varepsilon\) be properly small, one obtains

\[
\eta \| \nabla_y \partial_{13} w \|^2_{L^2(\Omega_f)} + e^{-2\eta T} \| \nabla_y \partial_{13} w \|^2_{L^2(\Omega_f)} + \| e^{-\eta t} \partial_{113} w \|^2_{L^2(\omega'_f)} + \| e^{-\eta t} \partial_{313} w \|^2_{L^2(\omega'_f)}
\]

\[
\lesssim \frac{1}{\eta} \| e^{-\eta t} L'(\partial_{13} w) \|^2_{L^2(\Omega_f)} + \eta \| e^{-\eta t} \partial_0 \partial_{13} w \|^2_{L^2(\omega'_f)} + e^{-2\eta T} \| \partial_0 \partial_{13} w \|^2_{L^2(\omega'_f)}
\]

\[
+ \| e^{-\eta t} (\partial_{y_0} \Lambda, \partial_{y_2} \Lambda, \partial_{y_3} \Lambda) \|^2_{L^2(\omega'_f)}.
\]  

(3.45)

From (3.39), we have

\[
|\partial_{y_0} \Lambda| + |\partial_{y_2} \Lambda| + |\partial_{y_3} \Lambda|
\]

\[
\lesssim \sum_{j \neq 1} |B \partial_{3j} w| + \sum_{|\alpha| \leq 2} \left( |D^{\alpha} \partial_{y_0} w| + |D^{\alpha} w| \right)
\]

\[
+ \| b_2 \|_{L^\infty} (|\partial_{233} w| + |\partial_{233} w|) + \| b_3 \|_{L^\infty} |\partial_{333} w|. 
\]  

(3.46)
Combining (3.21), (3.45), and (3.46) and the second-order estimate, we obtain

\[ \eta \| \nabla \partial_{13} w \|^2_{L^2(\Omega_T)} + e^{-2\eta T} \| \nabla \partial_{13} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} \partial_{13} w \|^2_{L^2(\omega_T')} + \| e^{-\eta t} \partial_{313} w \|^2_{L^2(\omega_T')} \]

\[ \leq \sum_{|\alpha| \leq 1} \left( \frac{1}{\eta} \| e^{-\eta t} L' (D^\alpha \partial_0 w) \|^2_{L^2(\omega_T')} + \| e^{-\eta t} B D^\alpha \partial_0 w \|^2_{L^2(\omega_T')} \right) + \frac{1}{\eta} \sum_{|\alpha| \leq 1} \| e^{-\eta t} L' (D^\alpha w) \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} B D^\alpha w \|^2_{L^2(\omega_T')} + \frac{1}{\eta} \| e^{-\eta t} L' (\partial_{13} w) \|^2_{L^2(\Omega_T)} + \| \nabla \partial_{13} w \|^2_{L^2(\Omega_T)} \]

\[ + \| b_2 \|^2_{L^\infty(\omega_T')} \cdot \| (e^{-\eta t} \partial_{223} w, e^{-\eta t} \partial_{233} w) \|^2_{L^2(\omega_T')} + \sum_{|\alpha| \leq 1} \| e^{-\eta t} B \partial_3 \partial_1 w \|^2_{L^2(\omega_T')} \]

\[ + \| b_3 \|^2_{L^\infty(\omega_T')} \cdot \| e^{-\eta t} \partial_{333} w \|^2_{L^2(\omega_T')} + \| D f |_{t=0} \|^2_{L^2(\tilde{\Omega})}. \]

But (3.46) implies that

\[ \| e^{-\eta t} \partial_{213} w \|^2_{L^2(\omega_T')} + \| e^{-\eta t} \partial_{313} w \|^2_{L^2(\omega_T')} \]

\[ \leq \sum_{|\alpha| \leq 2} \left( \frac{1}{\eta} \| e^{-\eta t} L' (D^\alpha w) \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} B D^\alpha w \|^2_{L^2(\omega_T')} \right) + \| b_2 \|^2_{L^\infty(\omega_T')} \cdot \| (e^{-\eta t} \partial_{223} w \|^2_{L^2(\omega_T')} + \| e^{-\eta t} \partial_{233} w \|^2_{L^2(\omega_T')} \]

\[ + \| b_3 \|^2_{L^\infty(\omega_T')} \cdot \| e^{-\eta t} \partial_{333} w \|^2_{L^2(\omega_T')} + \| D f |_{t=0} \|^2_{L^2(\tilde{\Omega})}. \]

Then the sum of (3.47) and (3.48) indicates that

\[ \eta \| \nabla \partial_{13} w \|^2_{L^2(\Omega_T)} + e^{-2\eta T} \| \nabla \partial_{13} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} \nabla \partial_{13} w \|^2_{L^2(\omega_T')} + \| e^{-\eta t} \partial_{313} w \|^2_{L^2(\omega_T')} \]

\[ \leq \sum_{|\alpha| \leq 2} \left( \frac{1}{\eta} \| e^{-\eta t} L' (D^\alpha w) \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} B D^\alpha w \|^2_{L^2(\omega_T')} \right) + \| b_2 \|^2_{L^\infty(\omega_T')} \cdot \| (e^{-\eta t} \partial_{223} w \|^2_{L^2(\omega_T')} + \| e^{-\eta t} \partial_{233} w \|^2_{L^2(\omega_T')} \]

\[ + \| b_3 \|^2_{L^\infty(\omega_T')} \cdot \| e^{-\eta t} \partial_{333} w \|^2_{L^2(\omega_T')} + \| D f |_{t=0} \|^2_{L^2(\tilde{\Omega})}. \]

By (H2) and (H3), we know that \( \| b_2 \|_{L^\infty} \) and \( \| b_3 \|_{L^\infty} \) are small, provided the \( \delta \) in (H3) is set to be sufficiently small. It will be shown later that the third-order derivatives on the right-hand side of (3.49) can be absorbed by the left-hand side terms.
Armed with the second-order estimate of $\delta_{y_0} w$ and $\delta_{y_2} w$ and the estimate of $\nabla y \delta_{13} w$, one can deduce the estimate of other third-order derivatives. It is easy to see

$$
\delta_{111} w = \frac{1}{r_{11}} \left( L'(\delta_{y_1} w) - \sum_{(i,j) \neq (1,1)} r_{ij} \delta_{y_i} \delta_{y_j} w - \sum_{j=0}^3 r_j \delta_{y_j} w - r \delta_{y_3} w \right). \tag{3.50}
$$

Hence, one has

$$
|\delta_{111} w| \lesssim \left| L'(\delta_{y_1} w) \right| + \left| \nabla y \delta_{13} w \right| + \sum_{|\alpha| \leq 2} \left( |D^\alpha \delta_{y_0} w| + |D^\alpha \delta_{y_2} w| + |D^\alpha w| \right). \tag{3.51}
$$

This leads to the estimate of $\delta_{111} w$ in terms of the controlled terms on the right-hand side of (3.50). In fact, one has

$$
\eta \|e^{-\eta t} \delta_{111} w\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|\delta_{111} w\|_{L^2(\Omega)}^2 + \|e^{-\eta t} \delta_{111} w\|_{L^2(\Omega')}^2 \lesssim \sum_{|\alpha| \leq 2} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^\alpha w)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} BD^\alpha w\|_{L^2(\Omega')}^2 \right)
$$

$$
+ \|b_2\|_{L^\infty(\Omega')}^2 \cdot \left( \|e^{-\eta t} \delta_{223} w\|_{L^2(\Omega')}^2 + \|e^{-\eta t} \delta_{233} w\|_{L^2(\Omega')}^2 \right)
$$

$$
+ \|b_3\|_{L^\infty(\Omega')}^2 \cdot \|e^{-\eta t} \delta_{333} w\|_{L^2(\Omega')}^2 + \|Df|_{t=0}\|_{L^2(\Omega')}^2. \tag{3.52}
$$

For $\delta_{333} w$, we have

$$
\delta_{333} w = \frac{1}{r_{33}} \left( L'(\delta_{y_3} w) - \sum_{(i,j) \neq (3,3)} r_{ij} \delta_{y_i} \delta_{y_j} w - \sum_{j=0}^3 r_j \delta_{y_j} w - r \delta_{y_3} w \right). \tag{3.53}
$$

It is clear that $\delta_{333} w$ is the finite combination of $D^2 \delta_{y_0} w$, $D^2 \delta_{y_2} w$, $\nabla y \delta_{13} w$, $\delta_{111} w$, and $\delta_{333} w$ cover all third-order derivatives of $w$. Thus, by adding (3.37) for $i = 0, 2$, (3.49), (3.50) and (3.54) together, we obtain

$$
\sum_{|\alpha| \leq 3} \eta \|e^{-\eta t} D^\alpha w\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^\alpha w\|_{L^2(\Omega)}^2 + \|e^{-\eta t} D^\alpha w\|_{L^2(\Omega')}^2 \lesssim \sum_{|\alpha| \leq 2} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^\alpha w)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} BD^\alpha w\|_{L^2(\Omega')}^2 \right).$$

$$
+ \|b_2\|_{L^\infty(\Omega')}^2 \cdot \left( \|e^{-\eta t} \delta_{223} w\|_{L^2(\Omega')}^2 + \|e^{-\eta t} \delta_{233} w\|_{L^2(\Omega')}^2 \right)
$$

$$
+ \|b_3\|_{L^\infty(\Omega')}^2 \cdot \|e^{-\eta t} \delta_{333} w\|_{L^2(\Omega')}^2 + \|Df|_{t=0}\|_{L^2(\Omega')}^2. \tag{3.54}
$$

It is easy to see that $D^2 \delta_{y_0} w$, $D^2 \delta_{y_2} w$, $\nabla y \delta_{13} w$, $\delta_{111} w$, and $\delta_{333} w$ cover all third-order derivatives of $w$. Thus, by adding (3.37) for $i = 0, 2$, (3.49), (3.50) and (3.54) together, we obtain
\[ + \|b_2\|^2_{L^\infty(\Omega_T)} \cdot \left( \|e^{-\eta t} \partial_{223} w\|^2_{L^2(\Omega_T)} + \|e^{-\eta t} \partial_{233} w\|^2_{L^2(\Omega_T)} \right) + \|b_3\|^2_{L^\infty(\Omega_T)} \cdot \|e^{-\eta t} \partial_{333} w\|^2_{L^2(\Omega_T)} + \|Df\big|_{t=0}\|^2_{L^2(\tilde{\Omega})}. \] (3.55)

As stated before, let the \( \delta \) in \( (H_3) \) be properly small, such that the boundary integrals on \( \partial_\Gamma \) on the right-hand side of (3.55) be absorbed by the left-hand side terms. Then we conclude the third-order estimate as follows

\[
\sum_{|\alpha| \leq 3} \eta \|e^{-\eta t} D^\alpha w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|D^\alpha w\|^2_{L^2(\Omega)} + \|e^{-\eta t} D^\alpha w\|^2_{L^2(\Omega_T)} 
\lesssim \sum_{|\alpha| \leq 2} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^\alpha w)\|^2_{L^2(\Omega_T)} + \|e^{-\eta t} BD^\alpha w\|^2_{L^2(\Omega_T)} \right) + \|Df\big|_{t=0}\|^2_{L^2(\tilde{\Omega})}. \] (3.56)

**Step 2:** In this step, we will establish the fourth-order estimate in the dihedral-space domain. Applying (3.55) to functions \( \partial_{y^0} w \) and \( \partial_{y^2} w \), respectively, one obtains

\[
\sum_{i=0,2} \sum_{|\alpha| \leq 3} \eta \|e^{-\eta t} D^\alpha \partial_{y^i} w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|D^\alpha \partial_{y^i} w\|^2_{L^2(\Omega)} + \|e^{-\eta t} D^\alpha \partial_{y^i} w\|^2_{L^2(\Omega_T)} 
\lesssim \sum_{|\alpha| \leq 3} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^\alpha w)\|^2_{L^2(\Omega_T)} + \|e^{-\eta t} BD^\alpha w\|^2_{L^2(\Omega_T)} \right) + \|D^2 f\big|_{t=0}\|^2_{L^2(\Omega)}. \] (3.57)

We establish the first-order estimate of \( \partial_{113} w \) first. In fact, one notices that \( \partial_{113} w \) satisfy

\[
L'(\partial_{113} w) = \partial_{113} f - [\partial_{113}, L'] w \quad \text{in} \quad \Omega_T, \quad (3.58)
\]

\[
\partial_{113} w = 0 \quad \text{on} \quad \partial_\Gamma, \quad (3.59)
\]

\[
(\partial_{113} w, \partial_{y^0}(\partial_{113} w)) = (0, 0) \quad \text{on} \quad \Gamma_{in}. \quad (3.60)
\]

Next we need to deduce the boundary condition that \( \partial_{113} w \) satisfies on the vertical boundary \( \partial_\Gamma \). Let

\[
L_2 = 2r_{12} \partial_{12} + r_{22} \partial_{22} + 2r_{23} \partial_{32},
\]

\[
L_1 = r_{11} \partial_{11} + 2r_{13} \partial_{13} + r_{33} \partial_{33},
\]

\[
L_0 = L' - L_1 - L_2.
\]

So, we have

\[
(r_{11} \partial_{y^1} \partial_{113} + 2r_{13} \partial_{y^3} \partial_{113}) w
\]

\[
= L_1 \partial_{13} w - r_{33} \partial_{1333} w
\]

\[
= (L' - L_0 - L_2)(\partial_{13} w) - r_{33} \partial_{1333} w. \] (3.61)

For the terms on the right-hand side of above equality, only \( r_{33} \partial_{1333} w \) has not been controlled yet. Indeed, \( L'(\partial_{13} w) \) is what we need in the estimate and \( L_0(\partial_{13} w) \) and \( L_2(\partial_{13} w) \) have been controlled.
by (3.57). But by the boundary condition of \( w \) on \( \omega^c \), we note that
\[
\partial_{1333} w = \frac{1}{b_1}(B(\partial_{333} w) - b_0 \partial_y \partial_{333} w - b_2 \partial_{y_2} \partial_{333} w - b_3 \partial_{y_3} \partial_{333} w).
\]

Therefore, we deduce that
\[
|\partial_{1333} w| \lesssim |B(\partial_{333} w)| + |D^3 \partial_{y_0} w| + |D^3 \partial_{y_2} w| + \|b_3\|_{L^\infty} \cdot |\partial_{3333} w|. \tag{3.62}
\]

On the right-hand side of (3.62), the first term is what we need, the second and the third terms are controlled by (3.57). For the last term, by (H_2) and (H_3), we know that \( \|b_3\|_{L^\infty} \) is small, provided the \( \delta \) in (H_3) is appropriately small. Hence, it can be absorbed by the left-hand side of the estimate coming later, which will cover all fourth-order derivatives. On the boundary \( \omega^c \), combining (3.61) and (3.62), we obtain
\[
|\partial_{1113} w| \lesssim |L'(\partial_{13} w)| + |B(\partial_{333} w)| + |D^3 \partial_{y_0} w| + |D^3 \partial_{y_2} w| + \|b_3\|_{L^\infty} \cdot |\partial_{3333} w| + \|r_{13}\|_{L^\infty} \cdot |\partial_{1133} w|. \tag{3.63}
\]

Multiplying \( 2e^{-2\eta t} Q^e(\partial_{13} w) \), where \( Q^e \) is given in Lemma 3.4, on both sides of (3.58), integration by parts over \( \Omega_T \) and by the use of Cauchy inequality, one has
\[
2\eta \int_{\Omega_T} e^{-2\eta t} H_0(\partial_{13} w; Q^e) dy dy_0 + e^{-2\eta T} \int_{\partial_T} e^{-2\eta t} H_0(\partial_{13} w; Q^e) dy|_{t=T} \\
- \int_{\omega_T} e^{-2\eta t} H_1(\partial_{13} w; Q^e) dy dy_0 - \int_{\omega_T} e^{-2\eta t} H_3(\partial_{13} w; Q^e) dy dy_0 \\
\lesssim \frac{1}{\epsilon_1 \eta} \|e^{-\eta t} L'(\partial_{13} w)\|_{L^2(\Omega_T)}^2 + (\epsilon_1 \eta + 1) \|D\partial_{133} w\|_{L^2(\Omega_T)}^2 \\
+ \int_{\Omega} H_0(\partial_{13} w; Q^e) dy|_{t=0}. \tag{3.64}
\]

By (3.28), (3.29), and (3.30) together with the fact
\[
H_0(\partial_{13} w; Q^e) \lesssim |D\partial_{13} w|^2,
\]
we deduce that
\[
\eta \|e^{-\eta t} \nabla_y \partial_{13} w\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|\nabla_y \partial_{133} w\|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} \partial_{y_3} \partial_{133} w\|_{L^2(\omega_T)}^2 \\
+ \|e^{-\eta t} \partial_{y_3} \partial_{133} w\|_{L^2(\omega_T')}^2 \\
\lesssim \frac{1}{\epsilon_1 \eta} \|e^{-\eta t} L'(\partial_{133} w)\|_{L^2(\Omega_T)}^2 + (\epsilon_1 \eta + 1) \|D\partial_{133} w\|_{L^2(\Omega_T)}^2 \\
+ \eta \|e^{-\eta t} \partial_{y_0} \partial_{133} w\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|\partial_{y_0} \partial_{133} w\|_{L^2(\Omega_T)}^2 \\
+ \sum_{i=0}^2 \|e^{-\eta t} \partial_{y_i} \partial_{133} w\|_{L^2(\omega_T')}^2 + \|D^2 f|_{t=0}\|_{L^2(\Omega)}. \tag{3.65}
\]
Recalling (3.57) and (3.63), we obtain

\[ \eta \| e^{-\eta t} D\partial_{113} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| D\partial_{113} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} D\partial_{113} w \|^2_{L^2(\omega_T')} \]

\[ + \| e^{-\eta t} \partial_{y3} \partial_{113} w \|^2_{L^2(\omega_T')} \]

\[ \lesssim \frac{1}{\epsilon_1 \eta} \| e^{-\eta t} L'(\partial_{113} w) \|^2_{L^2(\Omega_T)} + (\epsilon_1 \eta + 1) \| D\partial_{113} w \|^2_{L^2(\Omega_T)} \]

\[ + \sum_{|\alpha| \leq 3} \left( \frac{1}{\eta} \| e^{-\eta t} L'(D^\alpha w) \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} BD^\alpha w \|^2_{L^2(\omega_T')} \right) \]

\[ + \| b_3 \|^2_{L^\infty} \cdot \| e^{-\eta t} \partial_{3333} w \|^2_{L^2(\omega_T')} + \| r_{13} \|^2_{L^\infty} \cdot \| e^{-\eta t} \partial_{1133} w \|^2_{L^2(\omega_T')} \]

\[ + \| e^{-\eta t} L'(\partial_{13} w) \|^2_{L^2(\omega_T')} + \| D^2 f \|_{t=0}^2 \].

(3.66)

With (3.57) and (3.66) in hand, we can deduce the estimate of the left derivatives of fourth order, that is, \( \partial_{1111} w, \partial_{3333} w \) and \( \partial_{1333} w \). It is clear that

\[ \partial_{1111} w = \frac{1}{r_{11}} \left( (L' - L_0 - L_2) \partial_{11} w - 2r_{13} \partial_{1113} w - r_{33} \partial_{1133} w \right). \]

(3.67)

Hence, one has

\[ \eta \| e^{-\eta t} \partial_{1111} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| \partial_{1111} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} \partial_{1111} w \|^2_{L^2(\omega_T')} \]

\[ \lesssim \sum_{|\alpha| \leq 3} \left( \eta \| e^{-\eta t} D^\alpha \partial_{y0} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| D^\alpha \partial_{y0} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} D^\alpha \partial_{y0} w \|^2_{L^2(\omega_T')} \right) \]

\[ + \sum_{|\alpha| \leq 3} \left( \eta \| e^{-\eta t} D^\alpha \partial_{y2} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| D^\alpha \partial_{y2} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} D^\alpha \partial_{y2} w \|^2_{L^2(\omega_T')} \right) \]

\[ + \eta \| e^{-\eta t} D\partial_{1133} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| D\partial_{1133} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} D\partial_{1133} w \|^2_{L^2(\omega_T')} \]

\[ + \eta \| e^{-\eta t} L'(\partial_{11} w) \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| L'(\partial_{11} w) \|^2_{L^2(\Omega)} + \| e^{-\eta t} L'(\partial_{11} w) \|^2_{L^2(\omega_T')} \].

(3.68)

For \( \partial_{3333} w \) and \( \partial_{1333} w \), it is easy to check that

\[ \partial_{1333} w = \frac{1}{r_{33}} \left( (L' - L_0 - L_2) \partial_{13} w - r_{13} \partial_{1113} w - 2r_{13} \partial_{1133} w \right), \]

\[ \partial_{3333} w = \frac{1}{r_{33}} \left( (L' - L_0 - L_2) \partial_{33} w - 2r_{13} \partial_{1333} w - r_{11} \partial_{1133} w \right). \]

Thus, both \( \partial_{3333} w \) and \( \partial_{1333} w \) can be controlled by estimated terms. In fact, we have

\[ \sum_{i=1,3} \left( \eta \| e^{-\eta t} \partial_{i333} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| \partial_{i333} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} \partial_{i333} w \|^2_{L^2(\omega_T')} \right) \]

\[ \lesssim \frac{1}{\epsilon_1 \eta} \| e^{-\eta t} L'(\partial_{113} w) \|^2_{L^2(\Omega_T)} + (\epsilon_1 \eta + 1) \| D\partial_{113} w \|^2_{L^2(\Omega)} + \| D^2 f \|_{t=0}^2 \].
\[
\sum_{|\alpha| \leq 3} \left( \frac{1}{\eta} ||e^{-\eta t} L'(D^\alpha w)||^2_{L^2(\Omega_T)} + ||e^{-\eta t} BD^\alpha w||^2_{L^2(\omega'_T)} \right) + \||e^{-\eta t} L'(\partial_{13} w)||^2_{L^2(\omega'_T)} \\
+ \|b_3\|_{L^\infty} \cdot ||e^{-\eta t} \partial_{3333} w||^2_{L^2(\omega'_T)} + \|r_{13}\|_{L^\infty} \cdot ||e^{-\eta t} \partial_{1133} w||^2_{L^2(\omega'_T)} \\
+ \eta ||e^{-\eta t} D\partial_{113} w||^2_{L^2(\Omega_T)} + e^{-2\eta T} ||D\partial_{113} w||^2_{L^2(\Omega)} + ||e^{-\eta t} D\partial_{113} w||^2_{L^2(\omega'_T)} \\
+ \sum_{i=1,3} \left( \eta \||e^{-\eta t} L'(\partial_i w)||^2_{L^2(\Omega_T)} + e^{-2\eta T} ||L'(\partial_i w)||^2_{L^2(\Omega_T)} + ||e^{-\eta t} L'(\partial_i w)||^2_{L^2(\omega'_T)} \right) \\
+ \eta ||e^{-\eta t} L'(\partial_{33} w)||^2_{L^2(\Omega_T)} + e^{-2\eta T} ||L'(\partial_{33} w)||^2_{L^2(\Omega)} + ||e^{-\eta t} L'(\partial_{33} w)||^2_{L^2(\omega'_T)}. \tag{3.69}
\]

It is not difficult to see that \(D^3 \partial y_0 w, D^3 \partial y_2 w, D \partial_{113} w, \partial_{1111} w, \partial_{1333} w,\) and \(\partial_{3333} w\) cover all derivatives of fourth order of \(w\). We add (3.57), (3.66), (3.68), and (3.69) up, let the \(\varepsilon_1\) in (3.66) and \(\delta\) be properly small and let \(\eta\) be properly large, such that the terms with smallness be absorbed by the corresponding left-hand side terms. Then we obtain

\[
\sum_{|\alpha| \leq 4} \eta ||e^{-\eta t} D^\alpha w||^2_{L^2(\Omega_T)} + e^{-2\eta T} ||D^\alpha w||^2_{L^2(\Omega)} + ||e^{-\eta t} D^\alpha w||^2_{L^2(\omega'_T)} \\
\lesssim \sum_{|\alpha| \leq 3} \left( \frac{1}{\eta} ||e^{-\eta t} L'(D^\alpha w)||^2_{L^2(\Omega_T)} + ||e^{-\eta t} BD^\alpha w||^2_{L^2(\omega'_T)} \right) + \||D^2 f|_{t=0}||^2_{L^2(\Omega)} \\
+ \eta ||e^{-\eta t} L'(\partial_{11} w)||^2_{L^2(\Omega_T)} + e^{-2\eta T} ||L'(\partial_{11} w)||^2_{L^2(\Omega)} + ||e^{-\eta t} L'(\partial_{11} w)||^2_{L^2(\omega'_T)} \\
+ \eta ||e^{-\eta t} L'(\partial_{13} w)||^2_{L^2(\Omega_T)} + e^{-2\eta T} ||L'(\partial_{13} w)||^2_{L^2(\Omega)} + ||e^{-\eta t} L'(\partial_{13} w)||^2_{L^2(\omega'_T)} \\
+ \eta ||e^{-\eta t} L'(\partial_{33} w)||^2_{L^2(\Omega_T)} + e^{-2\eta T} ||L'(\partial_{33} w)||^2_{L^2(\Omega)} + ||e^{-\eta t} L'(\partial_{33} w)||^2_{L^2(\omega'_T)}. \tag{3.70}
\]

Exploiting integration by parts to \(\int_{\Omega_T} e^{-2\eta t} \xi^2 dy dy_0\) with respect to \(t\), we can derive following inequality

\[
\eta \int_{\Omega_T} e^{-2\eta t} |\xi|^2 dy dy_0 + e^{-2\eta T} \int_{\Omega} ||\xi||^2 dy \leq \frac{1}{\eta} \int_{\Omega_T} e^{-2\eta t} |\partial_t \xi|^2 dy dy_0 + \int_{\Omega} |\xi(0)|^2 dy. \tag{3.71}
\]

Hence, we obtain

\[
\eta ||e^{-\eta t} L'(\partial_{ij} w)||^2_{L^2(\Omega_T)} + e^{-2\eta T} ||L'(\partial_{ij} w)||^2_{L^2(\Omega)} \\
\leq \frac{1}{\eta} \||e^{-\eta t} \partial_i L'(\partial_{ij} w)||^2_{L^2(\Omega_T)} + \||L'(\partial_{ij} w)|_{t=0}||^2_{L^2(\Omega)} \\
\leq \frac{1}{\eta} \left( \sum_{|\alpha| \leq 4} ||e^{-\eta t} D^\alpha w||^2_{L^2(\Omega_T)} + ||e^{-\eta t} L'(\partial_i \partial_{ij} w)||^2_{L^2(\omega'_T)} \right) + \||D^2 f|_{t=0}||^2_{L^2(\Omega)} \tag{3.72}
\]

By Gauss theorem, we also have

\[
||e^{-\eta t} L'(\partial_{ij} w)||^2_{L^2(\omega'_T)} \\
= \int_{\Omega_T} -\partial_{y_1} (e^{-2\eta T} |L'(\partial_{ij} w)|^2) dy dy_0
\]
\[
\begin{align*}
&\leq \int_{\Omega_T} e^{-2\eta t} 2|L'(\partial_{ij}w)| \cdot |\partial_{y_j} L'(\partial_{ij}w)| \, dy dy_0 \\
&\leq \int_{\Omega_T} e^{-2\eta t} \left( \frac{1}{\eta} |\partial_{y_j} L'(\partial_{ij}w)|^2 + \varepsilon \eta |L'(\partial_{ij}w)|^2 \right) \, dy dy_0 \\
&\leq \int_{\Omega_T} e^{-2\eta t} \left( \frac{1}{\eta} \left( 2|\partial_{y_j}, L'|\partial_{ij}w|^2 + 2|L'(\partial_{y_j}, \partial_{ij}w)|^2 \right) + \varepsilon \eta |L'(\partial_{ij}w)|^2 \right) \, dy dy_0 \\
&\approx \frac{1}{\eta} \sum_{|\alpha| \leq 3} \left( \|e^{-\eta t} D^2 w\|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} L'(D^2 w)\|_{L^2(\Omega_T)}^2 \right) \\
&+ \varepsilon \eta \sum_{|\alpha| \leq 4} \|e^{-\eta t} D^2 w\|_{L^2(\Omega_T)}^2. 
\end{align*}
\]

(3.73)

Substitute (3.72) and (3.73) into (3.70), let the \( \varepsilon \) in (3.73) be properly small and then let \( \eta \) be appropriately large, we conclude the fourth-order estimate as follows

\[
\sum_{|\alpha| \leq 4} \eta \|e^{-\eta t} D^2 w\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^2 w\|_{L^2(\Omega)}^2 + \|e^{-\eta t} D^2 w\|_{L^2(\Omega_T)}^2 \\
\lesssim \sum_{|\alpha| \leq 3} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^2 w)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} B D^2 w\|_{L^2(\Omega_T)}^2 \right) + \|D^2 f|_{t=0}\|_{L^2(\Omega)}. 
\]

(3.74)

**Step 3:** Higher order estimates. In this step, we will prove higher order estimates by the induction method. Assume the estimate of 2\( k \)th order has been established, that is, we have

\[
\sum_{|\alpha| \leq 2k} \eta \|e^{-\eta t} D^2 w\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^2 w\|_{L^2(\Omega)}^2 + \|e^{-\eta t} D^2 w\|_{L^2(\Omega_T)}^2 \\
\lesssim \sum_{|\alpha| \leq 2k-1} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^2 w)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} B D^2 w\|_{L^2(\Omega_T)}^2 \right) + \|D^{2k-2} f|_{t=0}\|_{L^2(\Omega)}. 
\]

(3.75)

Then one proceeds to establish the estimate of (2\( k \) + 1)th order and (2\( k \) + 2)th order on the basis of the estimate of (2\( k \))th order. In what follows, we deal with the estimate of (2\( k \))th order first. Since both \( \partial_{y_0} \) and \( \partial_{y_2} \) are tangential to the boundaries \( \Gamma_s \) and \( \Gamma_w \), the application of (3.75) to \( \partial_{y_0} w \) and \( \partial_{y_2} w \) yields

\[
\sum_{i=0,2} \sum_{|\alpha| \leq 2k} \eta \|e^{-\eta t} D^2 \partial_{y_i} w\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^2 \partial_{y_i} w\|_{L^2(\Omega)}^2 + \|e^{-\eta t} D^2 \partial_{y_i} w\|_{L^2(\Omega_T)}^2 \\
\lesssim \sum_{|\alpha| \leq 2k-2} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^2 w)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} B D^2 w\|_{L^2(\Omega_T)}^2 \right) \\
+ \|D^{2k-2} f|_{t=0}\|_{L^2(\Omega_T)}. 
\]

(3.76)
Analogous to the estimate of third order, one tries to derive the first-order estimate of $\partial^{2k-1}_{y_1} \partial^{j}_{y_3} w$. It is clear that

$$L'(\partial^{2k-1}_{y_1} \partial^{j}_{y_3} w) = \partial^{2k-1}_{y_1} \partial^{j}_{y_3} f - [\partial^{2k-1}_{y_1} \partial^{j}_{y_3}, L'] w \quad \text{in} \quad \Omega_f,$$

$$\partial^{2k-1}_{y_1} \partial^{j}_{y_3} w = 0 \quad \text{on} \quad \omega^c,$$

$$(\partial^{2k-1}_{y_1} \partial^{j}_{y_3} w, \partial^{0}_{y_0} (\partial^{2k-1}_{y_1} \partial^{j}_{y_3} w)) = (0, 0) \quad \text{on} \quad \Gamma_{in}.$$  

Next we need to deduce the boundary condition on $\omega^c$ for $\partial^{2k-1}_{y_1} \partial^{j}_{y_3} w$. It is not difficult to check that

$$\partial^{2k-1}_{y_1} \partial^{j}_{y_3} w = (L' - L_0 - L_2 - 2r_{13} \partial_{13} - r_{33} \partial_{33}) \partial^{2k-3}_{y_1} \partial^{j}_{y_3} w.$$  

It follows from (3.80) that

$$\partial^{2k-1}_{y_1} \partial^{j}_{y_3} w = \Lambda_1 \quad \text{on} \quad \Gamma_s.$$  

Moreover, from (3.80), one has

$$\left| \partial^{j}_{y_0} \Lambda_1 \right| + \left| \partial^{j}_{y_2} \Lambda_1 \right| + \left| \partial^{j}_{y_3} \Lambda_1 \right| \lesssim \sum_{|\alpha| \leq 2k} (|D^2 \partial^{j}_{y_0} w| + |D^2 \partial^{j}_{y_2} w|) + \delta |D^{2k+1} w|$$

$$+ \sum_{|\alpha| \leq 2k-1} |L'(D^2 \partial^{j}_{y_3} w)| + \left| \partial^{2k-3}_{y_1} \partial^4_{y_3} w \right|, \quad (3.82)$$

where the $\delta$ before $|D^{2k+1} w|$ comes from the smallness of $r_{13}$ due to (H$_1$) and (H$_3$). So, we have to estimate $|\partial^{2k-3}_{y_1} \partial^4_{y_3} w|$. We already know $Bw = g$ on $\omega^c$, then it is easy to verify that

$$\partial^{j}_{y_1} \partial^{2k}_{y_3} w = \frac{1}{b_1} (B \partial^{2k}_{y_3} w - (b_0 \partial^{j}_{y_0} + b_2 \partial^{j}_{y_2} + b_3 \partial^{j}_{y_3} + b) \partial^{2k}_{y_3} w). \quad (3.83)$$

Remembering that $\partial^{j}_{y_0} \partial^{2k}_{y_3} w$, $\partial^{j}_{y_2} \partial^{2k}_{y_3} w$ and $\partial^{2k}_{y_3} w$ have been controlled by (3.76) and $\|b_1\|_{L^\infty}$ is close to zero, so $\partial^{j}_{y_1} \partial^{2k}_{y_3} w$ can be regarded as known function on $\omega^c$. Furthermore, from (3.83), we have

$$\left| \partial^{j}_{y_1} \partial^{2k}_{y_3} w \right| \lesssim \sum_{|\alpha| \leq 2k} (|D^2 \partial^{j}_{y_0} w| + |D^2 w| + |D^2 \partial^{j}_{y_2} w| + |B(D^2 w)|) + \delta |D^{2k+1} w|.$$  

It is easy to check that

$$\partial^{2k-2j-1}_{y_1} \partial^{2j+2}_{y_3} w = \frac{1}{r_{33}} (L' - L_0 - L_2 - 2r_{13} \partial_{13} - r_{11} \partial_{11}) \partial^{2k-2j-1}_{y_1} \partial^{2j}_{y_3} w. \quad (3.85)$$

For the ease of presentation, let

$$\beta_j := \partial^{2k-2j+1}_{y_1} \partial^{2j}_{y_3} w, \quad (3.86)$$

$$A_j := (L' - L_0 - L_2 - 2r_{13} \partial_{13}) \partial^{2k-2j-1}_{y_1} \partial^{2j}_{y_3} w. \quad (3.87)$$
for $j = 2, 3, ..., k$. Then it is clear that
\[
|A_j| \lesssim \sum_{|\alpha| \leq 2k} (|D^2 \partial_{y_0} w| + |D^2 w| + |D^2 \partial_{y_2} w|) + \delta |D^{2k+1} w| + \sum_{|\alpha| \leq 2k-1} |L'(D^2 w)|.
\] (3.88)

From (3.85)–(3.87), we obtain
\[
\beta_{j+1} = \frac{1}{r_{33}}(A_j - r_{11} \beta_j),
\] (3.89)
which implies
\[
\beta_j = \frac{A_j - r_{33} \beta_{j+1}}{r_{11}}.
\] (3.90)

Gathering (3.84), (3.87), and (3.90), one derives a sequence $\{\beta_j\}_{j=2}^k$ that satisfies
\[
\begin{cases}
\beta_k \lesssim \sum_{|\alpha| \leq 2k} (|D^2 \partial_{y_0} w| + |D^2 w| + |B(D^2 w)| + |D^2 \partial_{y_2} w|) + \delta |D^{2k+1} w|, \\
\beta_j = \frac{A_j - r_{33} \beta_{j+1}}{r_{11}}, & j = k - 1, k - 2, ..., 3, 2, \\
|A_j| \lesssim \sum_{|\alpha| \leq 2k} (|D^2 \partial_{y_0} w| + |D^2 w| + \delta |D^{2k+1} w|) + \sum_{|\alpha| \leq 2k-1} |L'(D^2 w)|.
\end{cases}
\] (3.91)

For $j = 2, 3, ..., k$, we claim that $\beta_j$ satisfies
\[
|\beta_j| \lesssim \sum_{|\alpha| \leq 2k} (|D^2 \partial_{y_0} w| + |D^2 w| + |B(D^2 w)| + |D^2 \partial_{y_2} w|) \\
+ \delta |D^{2k+1} w| + \sum_{|\alpha| \leq 2k-1} |L'(D^2 w)|,
\] (3.92)
and hence so does $\partial^{2k-3} \partial_4 w := \beta_2$. Indeed, from (3.84) it is clear to see that $\beta_k$ satisfies (3.92). Assume $\beta_{\ell'}$ satisfies (3.92) for some $\ell' \leq k$, then by (3.88) and (3.91)$_2$, we obtain
\[
|\beta_{\ell'-1}| \lesssim |A_{\ell'-1}| + |\beta_{\ell'}| \\
\lesssim \sum_{|\alpha| \leq 2k} (|D^2 \partial_{y_0} w| + |D^2 w| + |B(D^2 w)| + |D^2 \partial_{y_2} w|) \\
+ \delta |D^{2k+1} w| + \sum_{|\alpha| \leq 2k-1} |L'(D^2 w)|,
\]
which implies $\beta_{\ell'-1}$ also satisfies (3.92). Hence, our claim holds. Therefore, one can deduce from (3.82) that
\[
\left| \partial_{y_0} \Lambda_1 \right| + \left| \partial_{y_2} \Lambda_1 \right| + \left| \partial_{y_3} \Lambda_1 \right| \\
\lesssim \sum_{|\alpha| \leq 2k} (|D^2 \partial_{y_0} w| + |D^2 w| + |B(D^2 w)| + \delta |D^{2k+1} w|) \\
+ \sum_{|\alpha| \leq 2k-1} |L'(D^2 w)|.
\] (3.93)
With the help of Lemma 3.3 and (3.93), we are able to obtain the first-order estimate of \( \partial_{y_1}^{2k-1} \partial_{y_3} w \). Multiplying \( 2e^{-2\eta t} Q^d(\partial_{y_1}^{2k-1} \partial_{y_3} w) \) on both sides of (3.77), integrating by parts over \( \Omega_T \) and then apply (3.22)–(3.24) in Lemma 3.3, one deduces that

\[
\eta \| e^{-\eta t} \nabla_y \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| \nabla_y \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\omega_T)} \\
\leq \frac{1}{\varepsilon \eta} \| e^{-\eta t} L'(\partial_{y_1}^{2k-1} \partial_{y_3} w) \|^2_{L^2(\Omega_T)} + (1 + \varepsilon \eta) \| e^{-\eta t} D \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega_T)} \\
+ \| e^{-\eta t} \partial_y \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega_T)} + \sum_{i \neq 1} \| e^{-\eta t} \partial_y \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\omega_T)}.
\] (3.94)

In light of (3.76), (3.93), and (3.94), we obtain

\[
\eta \| e^{-\eta t} D \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| D \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} D \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\omega_T)} \\
\leq \frac{1}{\varepsilon \eta} \| e^{-\eta t} L'(\partial_{y_1}^{2k-1} \partial_{y_3} w) \|^2_{L^2(\Omega_T)} + (1 + \varepsilon \eta) \| e^{-\eta t} D \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega_T)} \\
+ \sum_{|\alpha| \leq 2k} \left( \frac{1}{\eta} \| e^{-\eta t} L'(D^{\alpha} w) \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} BD^{\alpha} w \|^2_{L^2(\omega_T)} \right) + \| D^{2k-2} f | t = 0 \|^2_{L^2(\Omega_T)} \\
+ \delta \| e^{-\eta t} D^{2k+1} w \|^2_{L^2(\omega_T)} + \sum_{|\alpha| \leq 2k-1} \| e^{-\eta t} L'(D^{\alpha} w) \|^2_{L^2(\omega_T)}.
\] (3.95)

Now one turns to the estimate of derivatives other than \( D^{2k} \partial_{y_0} w, D^{2k} \partial_{y_2} w, \) and \( D \partial_{y_1}^{2k-1} \partial_{y_3} w, \) that is, the estimate of \( \partial_{y_1}^{2k+1} w \) and the estimate of derivatives in the form of \( \partial_{y_1}^{2k-j+1} \partial_{y_j} w \) with \( 3 \leq j \leq 2k + 1. \) For \( \partial_{y_1}^{2k+1} w, \) it is easy to check that

\[
\partial_{y_1}^{2k+1} w = \frac{1}{r_{11}} (L' - L_0 - L_2 - 2r_{13} \partial_{13} - r_{33} \partial_{33}) \partial_{y_1}^{2k-1} w.
\] (3.96)

Hence, \( \partial_{y_1}^{2k+1} w \) can be controlled by estimated terms. In fact, we have

\[
\eta \| e^{-\eta t} \partial_{y_1}^{2k+1} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| \partial_{y_1}^{2k+1} w \|^2_{L^2(\Omega)} + \| e^{-\eta t} \partial_{y_1}^{2k+1} w \|^2_{L^2(\omega_T)} \\
\leq \eta \| e^{-\eta t} D^{2k} \partial_{y_0} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| D^{2k} \partial_{y_0} w \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} D^{2k} \partial_{y_0} w \|^2_{L^2(\omega_T)} \\
+ \eta \| e^{-\eta t} D^{2k} \partial_{y_2} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| D^{2k} \partial_{y_2} w \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} D^{2k} \partial_{y_2} w \|^2_{L^2(\omega_T)} \\
+ \eta \| e^{-\eta t} D \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| D \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} D \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\omega_T)} \\
+ \eta \| e^{-\eta t} L'(\partial_{y_1}^{2k-1} w) \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| L'(\partial_{y_1}^{2k-1} w) \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} L'(\partial_{y_1}^{2k-1} w) \|^2_{L^2(\omega_T)} \\
\leq \frac{1}{\varepsilon \eta} \| e^{-\eta t} L'(\partial_{y_1}^{2k-1} \partial_{y_3} w) \|^2_{L^2(\Omega_T)} + (1 + \varepsilon \eta) \| e^{-\eta t} D \partial_{y_1}^{2k-1} \partial_{y_3} w \|^2_{L^2(\Omega_T)}.
\]
\( \frac{1}{\eta} \| e^{-\eta t} L'(D^2 w) \|_{L^2(\Omega_T)}^2 + \| e^{-\eta t} B D^2 w \|_{L^2(\omega_T')}^2 \) + \| D^{2k-2} f(t=0) \|_{L^2(\Omega_T)}^2

+ \delta \| e^{-\eta t} D^{2k+1} w \|_{L^2(\omega_T')}^2 + \eta \| e^{-\eta t} L'(\partial_{y_1}^{2k-1} w) \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \| L'(\partial_{y_1}^{2k-1} w) \|_{L^2(\Omega)}^2

+ \sum_{|\alpha| \leq 2k-1} \| e^{-\eta t} L'(D^3 w) \|_{L^2(\omega_T')}^2. \tag{3.97}

We remark that the last three terms in (3.97) can be estimated by same argument as (3.72) and (3.73). For all \( j = 0,1,2,\ldots,2k+1 \) we claim that

\[
\eta \| e^{-\eta t} \partial_{y_1}^{2k-j+1} \partial_{y_3}^j w \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \| \partial_{y_1}^{2k-j+1} \partial_{y_3}^j w \|_{L^2(\Omega)}^2
\]

\[
\lesssim \frac{1}{\varepsilon \eta} \| e^{-\eta t} L'(\partial_{y_1}^{2k-j+1} \partial_{y_3}^j w) \|_{L^2(\Omega_T)}^2 + (1 + \varepsilon \eta) \| e^{-\eta t} D \partial_{y_1}^{2k-1} \partial_{y_3}^j w \|_{L^2(\Omega_T)}^2
\]

\[
+ \delta \| e^{-\eta t} D^{2k+1} w \|_{L^2(\omega_T')}^2 + \sum_{|\alpha| \leq 2k-1} \eta \| e^{-\eta t} L'(D^3 w) \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \| L'(D^3 w) \|_{L^2(\Omega)}^2
\]

\[
+ \sum_{|\alpha| \leq 2k-1} \| e^{-\eta t} L'(D^3 w) \|_{L^2(\omega_T')}^2. \tag{3.98}
\]

Indeed, from (3.95) and (3.97), we know (3.98) is valid for \( j = 0,1,2 \). Suppose (3.98) holds for all \( j \leq \ell \). We proceed to show (3.98) also holds for \( j = \ell + 1 \). In fact, one has

\[
\partial_{y_1}^{2k-\ell} \partial_{y_3}^{\ell+1} w = \frac{1}{r_{33}} (L' - L_0 - L_2 - 2r_{13} \partial_{y_1}^{\ell+1} - r_{11} \partial_{y_1}^{\ell+1}) \partial_{y_1}^{2k-\ell-1} \partial_{y_3}^{\ell-1} w. \tag{3.99}
\]

Hence, we have

\[
\eta \| e^{-\eta t} \partial_{y_1}^{2k-\ell} \partial_{y_3}^{\ell+1} w \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \| \partial_{y_1}^{2k-\ell} \partial_{y_3}^{\ell+1} w \|_{L^2(\Omega)}^2
\]

\[
\lesssim \eta \| e^{-\eta t} L'(\partial_{y_1}^{2k-\ell} \partial_{y_3}^{\ell+1} w) \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \| L'(\partial_{y_1}^{2k-\ell} \partial_{y_3}^{\ell+1} w) \|_{L^2(\Omega)}^2
\]

\[
+ \| e^{-\eta t} L'(\partial_{y_1}^{2k-\ell} \partial_{y_3}^{\ell+1} w) \|_{L^2(\omega_T')}^2
\]

\[
+ \eta \| e^{-\eta t} D^{2k} \partial_{y_0} w \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \| D^{2k} \partial_{y_0} w \|_{L^2(\omega_T')}^2
\]

\[
+ \eta \| e^{-\eta t} D^{2k} \partial_{y_2} w \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \| D^{2k} \partial_{y_2} w \|_{L^2(\omega_T')}^2
\]

\[
+ \eta \| e^{-\eta t} \partial_{y_1}^{2k-\ell-2} \partial_{y_3}^{\ell-1} w \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \| \partial_{y_1}^{2k-\ell-2} \partial_{y_3}^{\ell-1} w \|_{L^2(\Omega)}^2
\]

\[
+ \| e^{-\eta t} \partial_{y_1}^{2k-\ell-2} \partial_{y_3}^{\ell-1} w \|_{L^2(\omega_T')}^2
\]

\[
+ \eta \| e^{-\eta t} \partial_{y_1}^{2k-\ell-1} \partial_{y_3}^{\ell} w \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \| \partial_{y_1}^{2k-\ell-1} \partial_{y_3}^{\ell} w \|_{L^2(\Omega)}^2
\]

\[
+ \| e^{-\eta t} \partial_{y_1}^{2k-\ell-1} \partial_{y_3}^{\ell} w \|_{L^2(\omega_T')}^2. \tag{3.100}
\]
By our induction assumption that (3.98) is valid for \( j \leq \ell' \) and (3.76), we deduce that

\[
\eta \| e^{-\eta t} \hat{\delta}^{2k-\ell} \hat{\delta}^{\ell+1} w \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| \hat{\delta}^{2k-\ell} \hat{\delta}^{\ell+1} w \|^2_{L^2(\Omega_T)} + \| e^{-\eta t} \hat{\delta}^{2k-\ell} \hat{\delta}^{\ell+1} w \|^2_{L^2(\omega_T)} \leq \frac{1}{\varepsilon \eta} \| e^{-\eta t} L'(\hat{\delta}^{2k-1} \hat{\delta} y_3 w) \|^2_{L^2(\Omega_T)} + (1 + \varepsilon \eta) \| e^{-\eta t} D^2 \hat{\delta}^{2k-1} \hat{\delta} y_3 w \|^2_{L^2(\Omega_T)} \\
+ \sum_{|\alpha| \leq 2k} \left( \frac{1}{\eta} \| e^{-\eta t} L'(D^2 w) \|^2_{L^2(\omega_T)} + \| e^{-\eta t} BD^2 w \|^2_{L^2(\omega_T)} \right) + \| D^{2k-2} f |_{t=0} \|^2_{L^2(\Omega_T)} \\
+ \sum_{|\alpha| \leq 2k} \left( \| e^{-\eta t} L'(D^2 w) \|^2_{L^2(\omega_T)} + \eta \| e^{-\eta t} L'(D^2 w) \|^2_{L^2(\Omega_T)} + e^{-2\eta t} \| L'(D^2 w) \|^2_{L^2(\Omega)} \right) \\
+ \delta \| e^{-\eta t} D^{2k+1} w \|^2_{L^2(\omega_T)},
\]

(3.101)

which implies (3.98) holds for \( j = \ell' + 1 \) and this completes the induction. Now we are able to conclude the estimate of \((2k + 1)\)th order. Since \( D^2 \hat{\delta} y_0 w, D^2 \hat{\delta} y_2 w, D\hat{\delta}^{2k-1} \hat{\delta} y_3 w, \hat{\delta} y_1 \hat{\delta} y_3 w \) and \( \hat{\delta} y_1 \hat{\delta} y_3 w(0 \leq j \leq 2k + 1) \) cover all derivatives of \((2k + 1)\)th order, the sum of (3.76) and (3.98) for \( 0 \leq j \leq 2k + 1 \) yields

\[
\sum_{|\alpha| \leq 2k+1} \left( \eta \| e^{-\eta t} D^2 w \|^2_{L^2(\omega_T)} + e^{-2\eta t} \| D^2 w \|^2_{L^2(\Omega)} + \| e^{-\eta t} D^3 w \|^2_{L^2(\omega_T)} \right) \leq \frac{1}{\varepsilon \eta} \| e^{-\eta t} L'(D^2 \hat{\delta} y_0 w) \|^2_{L^2(\Omega_T)} + (1 + \varepsilon \eta) \| e^{-\eta t} D^2 \hat{\delta}^{2k-1} \hat{\delta} y_3 w \|^2_{L^2(\Omega_T)} \\
+ \sum_{|\alpha| \leq 2k} \left( \frac{1}{\eta} \| e^{-\eta t} L'(D^2 w) \|^2_{L^2(\omega_T)} + \| e^{-\eta t} BD^2 w \|^2_{L^2(\omega_T)} \right) + \| e^{-\eta t} L'(D^2 w) \|^2_{L^2(\Omega)} \\
+ e^{-2\eta t} \| L'(D^2 w) \|^2_{L^2(\Omega)} \right) + \delta \| e^{-\eta t} D^{2k+1} w \|^2_{L^2(\omega_T)} + \| D^{2k-2} f |_{t=0} \|^2_{L^2(\Omega_T)}.
\]

(3.102)

Let \( \delta \) and \( \varepsilon \) be appropriately small and estimate the terms on the second last line of (3.102) by same arguments as (3.72) and (3.73), then let \( \eta \) be properly large, we are led to

\[
\sum_{|\alpha| \leq 2k+1} \left( \eta \| e^{-\eta t} D^3 w \|^2_{L^2(\omega_T)} + e^{-2\eta t} \| D^3 w \|^2_{L^2(\Omega)} + \| e^{-\eta t} D^3 w \|^2_{L^2(\omega_T)} \right) \leq \sum_{|\alpha| \leq 2k} \left( \frac{1}{\eta} \| e^{-\eta t} L'(D^2 w) \|^2_{L^2(\omega_T)} + \| e^{-\eta t} BD^2 w \|^2_{L^2(\omega_T)} \right) + \| D^{2k-2} f |_{t=0} \|^2_{L^2(\Omega_T)},
\]

(3.103)

which is nothing but the estimate of \((2k + 1)\)th order.

Next, we continue to derive the estimate of \((2k + 2)\)th order, on the basis of the estimate of \((2k + 1)\)th order. Apply (3.103) to \( \hat{\delta} y_0 w \) and \( \hat{\delta} y_2 w \), we have

\[
\sum_{|\alpha| \leq 2k+1} \left( \eta \| e^{-\eta t} D^2 \hat{\delta} y_0 w \|^2_{L^2(\omega_T)} + e^{-2\eta t} \| D^2 \hat{\delta} y_0 w \|^2_{L^2(\Omega)} + \| e^{-\eta t} D^3 \hat{\delta} y_0 w \|^2_{L^2(\omega_T)} \right)
\]
\begin{align*}
+ \sum_{|\alpha| \leq 2k+1} \left( \eta \| e^{-\eta t} D^{\alpha} \partial_{y_j} w \|_{L^2(\Omega_T)}^2 + e^{-2\eta t} \| D^{\alpha} \partial_{y_j} w \|_{L^2(\Omega)}^2 \right) \\
\lesssim \sum_{|\alpha| \leq 2k+1} \left( \frac{1}{\eta} \| e^{-\eta t} L'(D^{\alpha} w) \|_{L^2(\Omega_T)}^2 + \| e^{-\eta t} BD^{\alpha} w \|_{L^2(\omega_T)}^2 \right) + \| D^k f \|_{t=0}^2_{L^2(\Omega_T)}. \tag{3.104}
\end{align*}

Then we will first establish the first-order derivative of $\partial_{y_1}^{2k+1} \partial_{y_3} w$. It is clear that
\begin{align*}
L'(\partial_{y_1}^{2k} \partial_{y_3} w) &= \partial_{y_1}^{2k} \partial_{y_3} f - [\partial_{y_1}^{2k} \partial_{y_3}, L'] w \quad \text{in} \ \Omega_T, \tag{3.105} \\
\partial_{y_1}^{2k} \partial_{y_3} w &= 0 \quad \text{on} \ \omega_T^r, \tag{3.106} \\
(\partial_{y_1}^{2k} \partial_{y_3} w, \partial_{y_0} (\partial_{y_1}^{2k} \partial_{y_3} w)) &= (0,0) \quad \text{on} \ \Gamma_{in}. \tag{3.107}
\end{align*}

We have to deduce the boundary condition on $\omega^r$ for $\partial_{y_1}^{2k} \partial_{y_3} w$. By the definitions of $L', L_0$ and $L_2$, it is clear that
\begin{align*}
\partial_{y_1} (\partial_{y_1}^{2k} \partial_{y_3} w) &= (L' - L_0 - L_2 - 2r_{13} \partial_{13} - r_{33} \partial_{33}) \partial_{y_1}^{2k-1} \partial_{y_3} w. \tag{3.108}
\end{align*}

Hence, we need to determine $\partial_{y_1}^{2k-1} \partial_{y_3} w$ on $\omega^r$. From the boundary condition $Bw = g$ on $\omega^r$, we note that
\begin{align*}
\partial_{y_1} \partial_{y_3}^{2k+1} w &= (B - b_0 \partial_{y_0} - \partial_{y_2} \partial_{y_2} - b_3 \partial_{y_3}^{2k+1}) \partial_{y_1} \partial_{y_3} w.
\end{align*}

Thus, we have
\begin{align*}
\left| \partial_{y_1} \partial_{y_3}^{2k+1} w \right| &\lesssim B \partial_{y_1}^{2k+1} w + \left| D^{2k+1} \partial_{y_0} w \right| + \left| D^{2k+1} \partial_{y_2} w \right| + \delta \left| D^{2k+2} w \right|. \tag{3.109}
\end{align*}

Again by the definitions of $L', L_0$ and $L_2$, we can further deduce
\begin{align*}
\partial_{y_1}^{2k-2j+1} \partial_{y_3}^{2j+3} w &= \frac{1}{r_{33}} (L' - L_0 - L_2 - 2r_{13} - r_{11} \partial_{11}) \partial_{y_1}^{2k-2j-1} \partial_{y_3}^{2j+1} w. \tag{3.110}
\end{align*}

For $1 \leq j \leq k$, let
\begin{align*}
\alpha_j &:= \partial_{y_1}^{2k-2j+1} \partial_{y_3}^{2j+1} w, \tag{3.111} \\
B_j &:= \frac{1}{r_{33}} (L' - L_0 - L_2 - 2r_{13} \partial_{y_1}^{2k-2j-1} \partial_{y_3}^{2j+1} w. \tag{3.112}
\end{align*}

From (3.109) to (3.112), we obtain a finite sequence $\{\alpha_j\}_{j=1}^k$ satisfying
\begin{align*}
\begin{cases}
\alpha_k \lesssim B \partial_{y_1}^{2k+1} w + \left| D^{2k+1} \partial_{y_0} w \right| + \left| D^{2k+1} \partial_{y_2} w \right| + \delta \left| D^{2k+2} w \right|, \\
\alpha_{j+1} = B_j - \frac{r_{11}}{r_{33}} \alpha_j \ (j = 1, 2, \ldots, k), \\
B_j \lesssim L'(\partial_{y_1}^{2k-2j-1} \partial_{y_3}^{2j+1} w) + \left| D^{2k+1} \partial_{y_0} w \right| + \left| D^{2k+1} \partial_{y_2} w \right| + \delta \left| D^{2k+2} w \right|.
\end{cases}
\end{align*}
Analogous to the sequence \{β_j\}, by induction on \( j \), one can deduce that \( α_j \) (\( 1 ≤ j ≤ k \)) satisfies
\[
|α_j| ≤ \sum_{|α|≤2k} |L'(D^α w)| + |D^{2k+1} \partial y_0 w| + |D^{2k+1} \partial y_2 w| + δ |D^{2k+2} w| + \sum_{|α|≤2k+1} |BD^α w|,
\]
and so does \( α_1 = \partial^{2k-1} \partial^3 y_1 \). Armed with this estimate for \( \partial^{2k-1} \partial^3 y_1 \), we obtain from (3.108) that
\[
|\partial^j y_1 (\partial^{2k-1} \partial^3 y_1 w)| ≤ \sum_{|α|≤2k} |L'(D^α w)| + |D^{2k+1} \partial y_0 w| + |D^{2k+1} \partial y_2 w| + δ |D^{2k+2} w| + \sum_{|α|≤2k+1} |BD^α w|.
\]

Thanks to Lemma 3.4, we are able to derive the first-order estimate of \( \partial^{2k} \partial y_1 \partial^3 y_3 w \). Multiplying \( 2e^{-2η \tau} \) on both sides of (3.105), integration by parts over \( Ω_τ \), applying Lemma 3.4 and Cauchy inequality, one has
\[
\eta \|e^{-η \tau} \nabla y_1 \partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω_τ)}^2 + e^{-2η \tau} \|\nabla y_1 \partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω)}^2
\]
\[
+ \|e^{-η \tau} \partial^2 y_1 \partial^{2k} \partial y_3 \partial y_3 w\|_{L^2(Ω)}^2 + \|e^{-η \tau} \partial^2 y_3 \partial^{2k} \partial y_1 \partial y_3 w\|_{L^2(Ω)}^2 ≤ \frac{1}{η} \|e^{-η \tau} L' (\partial^{2k} \partial y_1 \partial^3 y_3 w)\|_{L^2(Ω_τ)}^2 + (1 + εη) \|e^{-η \tau} D\partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω_τ)}^2
\]
\[
+ \eta \|e^{-η \tau} \partial^2 y_0 \partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω_τ)}^2 + e^{-2η \tau} \|\partial^2 y_0 \partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω)}^2 + \sum_{i=0}^2 \|e^{-η \tau} \partial^i y_1 \partial^{2k} \partial y_3 \partial y_3 w\|_{L^2(Ω_τ)}^2.
\]

In view of (3.104) and (3.113), we obtain from above inequality that
\[
\eta \|e^{-η \tau} D\partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω_τ)}^2 + e^{-2η \tau} \|D\partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω)}^2
\]
\[
+ \|e^{-η \tau} D\partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω)}^2 + \|e^{-η \tau} \partial^2 y_3 \partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω)}^2 ≤ \frac{1}{η} \|e^{-η \tau} L' (\partial^{2k} \partial y_1 \partial^3 y_3 w)\|_{L^2(Ω_τ)}^2 + (1 + εη) \|e^{-η \tau} D\partial^{2k} \partial y_1 \partial^3 y_3 w\|_{L^2(Ω_τ)}^2
\]
\[
+ \sum_{|α|≤2k+1} \left( \frac{1}{η} \|e^{-η \tau} L' (D^α w)\|_{L^2(Ω_τ)}^2 + \|e^{-η \tau} BD^2 w\|_{L^2(Ω)}^2 \right) + \|D^{2k} f|_{t=0}\|_{L^2(Ω_τ)}^2 + \sum_{|α|≤2k} \|e^{-η \tau} L' (D^α w)\|_{L^2(Ω)}^2 + δ \|D^{2k+2} w\|_{L^2(Ω)}^2.
\]

Now we turn to the estimate of \( \partial^{2k+2} \partial y_1 \partial^j y_3 \) with \( 3 ≤ j ≤ 2k + 2 \). By the definitions of \( L' \), \( L_0 \) and \( L_2 \), one has
\[
\partial^{2k+2} \partial y_1 = \frac{1}{r_{11}} (L' - L_0 - L_2 - 2r_{13} \partial_{13} - r_{33} \partial_{33}) \partial^{2k} \partial y_1.
\]
So, $\partial^{2k+2} y_1$ can be estimated by controlled terms, that is,
\[
|\partial^{2k+2} y_1| \lesssim |L'(\partial^{2k} w)| + |D^{2k+1} \partial y_0 w| + |D^{2k+1} \partial^{2k} w| + |D \partial^{2k} \partial y_3 w|.
\] (3.116)

This together with (3.104) and (3.114) imply
\[
\eta \|e^{-\eta t} \partial^{2k+2} y_1 \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|\partial^{2k+2} y_1 \|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} \partial^{2k+2} y_1 \|_{L^2(\omega_T')}^2 \leq \frac{1}{\varepsilon \eta} \|e^{-\eta t} L'(\partial^{2k} w) \|_{L^2(\Omega)}^2 + (1 + \eta) \|e^{-\eta t} D^{2k} \partial y_3 w \|_{L^2(\omega_T')}^2 + \|e^{-\eta t} L'(\partial^{2k} w) \|_{L^2(\omega_T')}^2 + \sum_{|\alpha| \leq 2k+1} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^\alpha w) \|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} BD^\alpha w \|_{L^2(\omega_T')}^2 \right) + \|D^{2k} f \|_{t=0}^2 \] (3.117)

Then by simple induction argument as we use in (3.98)–(3.101), one deduces for all $3 \leq j \leq 2k + 2$ that
\[
\eta \|e^{-\eta t} \partial^{2k-j+2} y_3 w \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|\partial^{2k-j+2} y_3 w \|_{L^2(\Omega)}^2 + \|e^{-\eta t} \partial^{2k-j+2} y_3 w \|_{L^2(\omega_T')}^2 \leq \frac{1}{\varepsilon \eta} \|e^{-\eta t} L'(\partial^{2k} y_3 w) \|_{L^2(\Omega)}^2 + (1 + \eta) \|e^{-\eta t} D^{2k} \partial y_3 w \|_{L^2(\omega_T')}^2 + \sum_{|\alpha| \leq 2k+1} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^\alpha w) \|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} BD^\alpha w \|_{L^2(\omega_T')}^2 \right) + \sum_{|\alpha| \leq 2k} \|e^{-\eta t} L'(D^\alpha w) \|_{L^2(\omega_T')}^2 \] (3.118)

To this end, adding (3.104), (3.114), and (3.118) for all $3 \leq j \leq 2k + 2$ together, then let $\varepsilon$, $\delta$ be properly small and $\eta$ be appropriately large, one concludes that
\[
\sum_{|\alpha| \leq 2k+2} \left( \eta \|e^{-\eta t} D^\alpha w \|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^\alpha w \|_{L^2(\Omega)}^2 + \|e^{-\eta t} D^\alpha w \|_{L^2(\omega_T')}^2 \right) \leq \sum_{|\alpha| \leq 2k+1} \left( \frac{1}{\eta} \|e^{-\eta t} L'(D^\alpha w) \|_{L^2(\Omega_T)}^2 + \|e^{-\eta t} BD^\alpha w \|_{L^2(\omega_T')}^2 \right) + \|D^{2k} f \|_{t=0}^2 \] (3.119)

This completes the induction process from the estimate of $2k$th order to $(2k + 2)$th order and hence finishes our proof of Lemma 3.5. \qed

By combining Lemmas 3.2 and 3.5, Proposition 3.1 follows.
3.3  Proof of Theorem 3.1

Based on Proposition 3.1, we are able to prove Theorem 3.1 by estimating \( L'(D^\alpha w) \) for \( |\alpha| \leq s \leq n_0 + 3 \) carefully.

**Proof of Theorem 3.1.** It is clear that the estimate in Proposition 3.1 holds for all \( T \geq 0 \). Hence, in order to prove Theorem 3.1, we just need to estimate \( L'(D^\alpha w) \) and \( B(D^\alpha w) \). First, for \( L'(D^\alpha w) \) one has

\[
L'(D^\alpha w) = -[D^\alpha, L']w + D^\alpha (L'w) = -[D^\alpha, L']w + D^\alpha f. \tag{3.120}
\]

Then we need to estimate the commutator \([D^\alpha, L']w\). By definition, one has

\[
[D^\alpha, L']w = D^\alpha (r_{ij} \partial_{ij} w) - r_{ij} D^\alpha \partial_{ij} w + D^\alpha (r_k \partial_k w) - r_k D^\alpha \partial_k w + D^\alpha (rw) - r D^\alpha w,
\]

where we have omitted the summation for repeated indices for brevity. If \( |\alpha| = 0 \), it is clear that \([D^\alpha, L']w = 0\). We will estimate \( D^\alpha (r_{ij} \partial_{ij} w) - r_{ij} D^\alpha \partial_{ij} w \) for \( 1 \leq |\alpha| \leq 4 \) case by case, then \( D^\alpha (r_i \partial_i w) - r_i D^\alpha \partial_i w \) and \( D^\alpha (rw) - r D^\alpha w \) can be estimated in the same way. If \( |\alpha| = 1 \), then

\[
\| (D^\alpha (r_{ij} \partial_{ij} w) - r_{ij} D^\alpha \partial_{ij} w) (y_0, \cdot) \|_{L^2(\Omega)} \leq \delta \sum_{|\beta| \leq 2} \| D^\beta w (y_0, \cdot) \|_{L^2(\Omega)}. \tag{3.121}
\]

If \( |\alpha| = 2 \), one has

\[
\| (D^\alpha (r_{ij} \partial_{ij} w) - r_{ij} D^\alpha \partial_{ij} w) (y_0, \cdot) \|_{L^2(\Omega)} \leq \sum_{i,j = 0}^3 \| (D r_{ij} D^2 \partial_{ij} w) (y_0, \cdot) \|_{L^2(\Omega)} + \| (D^2 r_{ij} \partial_{ij} w) (y_0, \cdot) \|_{L^2(\Omega)} \leq \delta \sum_{|\beta| \leq 4} \| D^\beta w (y_0, \cdot) \|_{L^2(\Omega)} \sum_{|\beta| \leq 3} \| D^\beta u (y_0, \cdot) \|_{L^2(\Omega)}. \tag{3.122}
\]

If \( |\alpha| = 3 \), by the Sobolev embedding \( H^2(\Omega) \hookrightarrow L^\infty(\Omega) \), one has

\[
\| (D^\alpha (r_{ij} \partial_{ij} w) - r_{ij} D^\alpha \partial_{ij} w) (y_0, \cdot) \|_{L^2(\Omega)} \leq \sum_{i,j = 0}^3 \| D r_{ij} D^2 \partial_{ij} w (y_0, \cdot) \|_{L^2(\Omega)} + \| (D^3 r_{ij} \partial_{ij} w) (y_0, \cdot) \|_{L^2(\Omega)} + \| (D^2 r_{ij} D^2 \partial_{ij} w) (y_0, \cdot) \|_{L^2(\Omega)} \leq \delta \sum_{|\beta| \leq 4} \| D^\beta w (y_0, \cdot) \|_{L^2(\Omega)} + \delta \sum_{|\beta| \leq 4} \| D^\beta w (y_0, \cdot) \|_{L^2(\Omega)} \sum_{|\beta| \leq 3} \| D^\beta u (y_0, \cdot) \|_{L^2(\Omega)} \tag{3.123}
\]
Since in this paper \( r_{00} \) represents \( \tilde{a}_{00} \) and \( \tilde{a}_{00} = (\tilde{\varphi}, u)^2 \), by the Sobolev embedding theorem, the Hölder inequality and the Gagliardo–Nirenberg inequality, we have

\[
\| (D^2 r_{00} D\tilde{\varphi}_{00} w)(y_0, \cdot) \|_{L^2(\Omega)} 
\leq \| (D\tilde{\varphi}_{00} w)(y_0, \cdot) \|_{L^2(\Omega)} + \| (\tilde{\varphi}_{00} u D^2 \tilde{\varphi}_{00} u D\tilde{\varphi}_{00} w)(y_0, \cdot) \|_{L^2(\Omega)} 
\leq \| D\tilde{\varphi}_{00} w(y_0, \cdot) \|_{L^6(\Omega)} + \| D^2 V_y u(y_0, \cdot) \|_{L^3(\Omega)} \| D\tilde{\varphi}_{00} w(y_0, \cdot) \|_{L^6(\Omega)} 
\leq \| D^3 w(y_0, \cdot) \|_{L^2(\Omega)} + \| V^2_y D^2 u(y_0, \cdot) \|_{L^2(\Omega)} ^{\frac{1}{2}} \| D^2 u(y_0, \cdot) \|_{L^6(\Omega)} ^{\frac{1}{2}} \| D^3 w(y_0, \cdot) \|_{L^6(\Omega)} 
\leq \delta \| \nabla^2_y D^2 w(y_0, \cdot) \|_{L^2(\Omega)} \| D^2 w(y_0, \cdot) \|_{L^6(\Omega)} \| D^3 w(y_0, \cdot) \|_{L^6(\Omega)} \| D^3 w(y_0, \cdot) \|_{H^1(\Omega)} 
\leq \delta \sum_{|\beta|\leq 4} \| D\beta w(y_0, \cdot) \|_{L^2(\Omega)},
\]  

(3.124)

where in the last but one inequality, we have used the embedding \( H^1(\Omega) \hookrightarrow L^6(\Omega) \). If \((i, j) \neq (0, 0)\), then \( \partial_{ij} w \) must contain spatial derivative of \( w \). So, by the Hölder inequality, the Gagliardo–Nirenberg inequality, and the Sobolev embedding theorem, one has

\[
\| (D^2 r_{ij} D\tilde{\varphi}_{ij} w)(y_0, \cdot) \|_{L^2(\Omega)} 
\leq \| (D^2 r_{ij} w(y_0, \cdot) \|_{L^6(\Omega)} \| D\tilde{\varphi}_{ij} w(y_0, \cdot) \|_{L^3(\Omega)} 
\leq \| D^2 r_{ij} w(y_0, \cdot) \|_{H^1(\Omega)} \| \nabla_y D^2 w(y_0, \cdot) \|_{L^3(\Omega)} 
\leq \delta \| \nabla^2_y D^2 w(y_0, \cdot) \|_{L^2(\Omega)} ^{\frac{1}{2}} \| D^2 w(y_0, \cdot) \|_{L^6(\Omega)} ^{\frac{1}{2}} \| D^3 w(y_0, \cdot) \|_{L^6(\Omega)} \| D^3 w(y_0, \cdot) \|_{H^1(\Omega)} 
\leq \delta \sum_{|\beta|\leq 4} \| D\beta w(y_0, \cdot) \|_{L^2(\Omega)}.
\]  

(3.125)

Therefore, combining (3.123)-(3.125), one has for \(|\alpha| = 3\),

\[
\| (D^3 r_{ij} \tilde{\varphi}_{ij} w - r_{ij} D^3 \tilde{\varphi}_{ij} w)(y_0, \cdot) \|_{L^2(\Omega)} 
\leq \sum_{i,j=0}^3 \| Dr_{ij} D^2 \tilde{\varphi}_{ij} w(y_0, \cdot) \|_{L^2(\Omega)} + \| (D^3 r_{ij} \tilde{\varphi}_{ij} w)(y_0, \cdot) \|_{L^2(\Omega)} + \| (D^2 r_{ij} D\tilde{\varphi}_{ij} w)(y_0, \cdot) \|_{L^2(\Omega)} 
\leq \delta \sum_{|\beta|\leq 4} \| D\beta w(y_0, \cdot) \|_{L^2(\Omega)} + \delta \sum_{|\beta|\leq 4} \| D\beta w(y_0, \cdot) \|_{L^2(\Omega)} \sum_{1\leq|\beta|\leq 4} \| D\beta u(y_0, \cdot) \|_{L^2(\Omega)}.
\]  

(3.126)

If \(|\alpha| = 4\), one has

\[
\| (D^4 r_{ij} \tilde{\varphi}_{ij} w - r_{ij} D^4 \tilde{\varphi}_{ij} w)(y_0, \cdot) \|_{L^2(\Omega)} 
\leq \sum_{i,j=0}^3 \| Dr_{ij} D^3 \tilde{\varphi}_{ij} w(y_0, \cdot) \|_{L^2(\Omega)} + \| (D^4 r_{ij} \tilde{\varphi}_{ij} w)(y_0, \cdot) \|_{L^2(\Omega)} + \| (D^3 r_{ij} D\tilde{\varphi}_{ij} w)(y_0, \cdot) \|_{L^2(\Omega)}
\]
\[ + \| (D^3 r_{ij} D_s^i w) (y_0, \cdot) \|_{L^2(\Omega)} + \| (D^2 r_{ij} D_s^j w) (y_0, \cdot) \|_{L^2(\Omega)} \]
\[ \lesssim \| D r_{ij} \|_{L^\infty(\Omega)} \| D^3 w (y_0, \cdot) \|_{L^2(\Omega)} + \| D^2 r_{ij} (y_0, \cdot) \|_{L^\infty(\Omega)} \| D^4 w (y_0, \cdot) \|_{L^2(\Omega)} \]
\[ + \| D^3 r_{ij} (y_0, \cdot) \|_{L^2(\Omega)} \| D^3 w (y_0, \cdot) \|_{L^\infty(\Omega)} + \| D^2 r_{ij} (y_0, \cdot) \|_{L^\infty(\Omega)} \| D^4 w (y_0, \cdot) \|_{L^2(\Omega)} \]
\[ \lesssim \delta \sum_{|\beta| \leq 5} \| D^\beta w (y_0, \cdot) \|_{L^2(\Omega)} + \delta \sum_{|\beta| \leq 4} \| D^\beta w (y_0, \cdot) \|_{L^2(\Omega)} \sum_{|\beta| \leq 5} \| D^\beta u (y_0, \cdot) \|_{L^2(\Omega)} \]
\[ (3.127) \]

In fact, for any \(|\alpha| \leq s - 1\), we have
\[ \| (D^\alpha (r_{ij} \partial_{ij} w) - r_{ij} D^\alpha \partial_{ij} w) (y_0, \cdot) \|_{L^2(\Omega)} \]
\[ \lesssim \delta \sum_{0 \leq |\beta| \leq 4} \| D^\beta w (y_0, \cdot) \|_{L^2(\Omega)} \sum_{0 \leq |\beta| \leq |\alpha| + 1} \| D^\beta u (y_0, \cdot) \|_{L^2(\Omega)} \]
\[ + \delta \sum_{0 \leq |\beta| \leq |\alpha| + 1} \| D^\beta w (y_0, \cdot) \|_{L^2(\Omega)}. \]
\[ (3.128) \]

It is clear that \((3.121), (3.122), (3.126)\), and \((3.127)\) imply that our claim holds for \(|\alpha| \leq 4\). If \(|\alpha| \geq 5\), one has
\[
D^\alpha (r_{ij} \partial_{ij} w) - r_{ij} D^\alpha \partial_{ij} w
\]
\[ = \sum_{|\gamma| \leq |\alpha| - 1} D^\beta r_{ij} D^\gamma \partial_{ij} w \]
\[ = \sum_{1 \leq |\beta| \leq |\alpha| - 2} D^\beta r_{ij} D^\gamma \partial_{ij} w + \sum_{|\beta| > |\alpha| - 1} D^\beta r_{ij} D^\gamma \partial_{ij} w \]
\[ := Z_1 + Z_2. \]
\[ (3.129) \]

To estimate \(Z_1\), we need to calculate \(D^\beta r_{ij}\) with \(1 \leq |\beta| \leq |\alpha| - 2\) carefully. It is easy to see that \(D^\beta r_{ij}\) can be written as the sum of finitely many terms in the following form:
\[
R_{ij} \cdot \partial^{\mu_1} \mathcal{W} \cdot \partial^{\mu_2} \mathcal{W} \ldots \partial^{\mu_1} \mathcal{W} \cdot \partial^{\gamma_1} Du \cdot \partial^{\gamma_2} Du \ldots \partial^{\gamma_k} Du,
\]
\[ R_{ij} = R_{ij}(\mathcal{W}(u, x_2(u, y_2, y_3), Du, y_3 + \mathcal{W}(u, x_2(u, y_2, y_3))) \quad \text{and} \quad \mu_1 + \cdots + \mu_1 + \gamma_1 + \cdots + \gamma_k \leq |\beta| \leq |\alpha| - 2. \]
\[ \text{Since } |\alpha| \geq 4, \text{ at most one factor in above formula is not in } L^\infty(\Omega) \text{ (this means corresponding factors cannot be bounded by the derivatives of } u \text{ of order no more than } |\alpha| + 1). \]

Indeed, assume two factors are not in \(L^\infty(\Omega)\), then by the Sobolev embedding \(H^2(\Omega) \hookrightarrow L^\infty(\Omega)\), at least one of the following three cases occur.

(C1): \(\mu_{i_0} + 2 \geq |\alpha| + 2 \) and \(\mu_{j_0} + 2 \geq |\alpha| + 2\) for some \((i_0, j_0) \in \{1, 2, \ldots, l\}^2\).
(C2): \(\mu_{i_0} + 2 \geq |\alpha| + 2 \) and \(\gamma_{j_0} + 3 \geq |\alpha| + 2\) for some \(i_0 \in \{1, 2, \ldots, l\}\) and \(j_0 \in \{1, 2, \ldots, k\}\).
(C3): \(\gamma_{i_0} + 3 \geq |\alpha| + 2 \) and \(\gamma_{j_0} + 3 \geq |\alpha| + 2\) for some \((i_0, j_0) \in \{1, 2, \ldots, k\}^2\).

If case (C1) occurs, then one has
\[ |\alpha| - 2 \geq |\beta| \geq \mu_{i_0} + \mu_{j_0} \geq 2|\alpha|, \]
which is impossible for \(|\alpha| \geq 4\). Hence, (C1) does not occur. By similar arguments, one can deduce that the other two cases (C2) and (C3) do not occur either. Therefore, we have

\[
\|Z_1(y_0, \cdot)\|_{L^2} \leq \sum_{1 \leq |\beta| = |\alpha| - 2} \||D^\beta r_{ij}(y_0, \cdot)\|_{L^\infty(\Omega)} \||D^\gamma \partial_{ij} w(y_0, \cdot)\|_{L^2(\Omega)}
\]

\[
\leq \delta \sum_{|\gamma| = |\alpha| - 1} \||D^\gamma \partial_{ij} w(y_0, \cdot)\|_{L^2(\Omega)}
\]

\[
\leq \delta \sum_{|\gamma| = |\alpha| + 1} \||D^\gamma w\|_{L^2(\Omega)}.
\]  

(3.130)

For \(Z_2\), we have

\[
\|Z_2(y_0, \cdot)\|_{L^2(\Omega)} \leq \sum_{|\beta| = |\alpha| - 1} \||D^\beta r_{ij} D^\gamma \partial_{ij} w\|_{L^2(\Omega)} + \sum_{|\beta| = |\alpha|} \|(D^\beta r_{ij}\partial_{ij} w)(y_0, \cdot)\|_{L^2(\Omega)}
\]

\[
:= Z_{21} + Z_{22}.
\]  

(3.131)

For \(Z_{21}\), we first consider the case \((i, j) = (0, 0)\), since \(r_{00} = \tilde{a}_{00} = (\partial_{y_1} u)^2\), so \(r_{00}\) contains spatial derivative of \(u\). Hence, by the Hölder inequality and the Gagliardo–Nirenberg inequality, one has

\[
\|(D^\beta r_{00} D^\gamma \partial_{00} w)(y_0, \cdot)\|_{L^2(\Omega)}
\]

\[
\leq \||D^\beta (\partial_{y_1} u)^2(y_0, \cdot)\|_{L^\lambda(\Omega)} \|(D^\gamma \partial_{00} w)(y_0, \cdot)\|_{L^6(\Omega)}
\]

\[
\leq \left( \sum_{\beta_1 + \beta_2 = \beta} \||D^\beta_1 \partial_{y_1} u D^\beta_2 \partial_{y_1} u(y_0, \cdot)\|_{L^\lambda(\Omega)} \right) \||D \partial_{y_1} w(y_0, \cdot)\|_{L^6(\Omega)}.
\]  

(3.132)

By similar argument as before, it is not difficult to see that only one factor in the product \(D^\beta_1 \partial_{y_1} u D^\beta_2 \partial_{y_1} u\) is not in \(L^\infty(\Omega)\) (this means the factor cannot be bounded by the derivatives of \(u\) of order no more than \(|\beta| + 2 = |\alpha| + 1 \leq s\)). Therefore, one has

\[
\sum_{\beta_1 + \beta_2 = \beta} \||D^\beta_1 \partial_{y_1} u D^\beta_2 \partial_{y_1} u(y_0, \cdot)\|_{L^\lambda(\Omega)}
\]

\[
= 2 \||\partial_{y_1} u D^\beta \partial_{y_1} u(y_0, \cdot)\|_{L^\lambda(\Omega)} + 2 \sum_{|\beta_2| = |\beta| - 1} \||D \partial_{y_1} u D^\beta_2 \partial_{y_1} u(y_0, \cdot)\|_{L^\lambda(\Omega)}
\]

\[+ 2 \sum_{2 \leq |\beta_1| \leq |\beta| - 2} \||D^\beta_1 \partial_{y_1} u D^\beta_2 \partial_{y_1} u(y_0, \cdot)\|_{L^\lambda(\Omega)}
\]

\[
\leq (||\partial_{y_1} u(y_0, \cdot)||_{L^\infty(\Omega)} + ||D \partial_{y_1} u(y_0, \cdot)||_{L^\infty(\Omega)}) \sum_{|\beta_2| = |\beta| - 1} \||D^\beta_2 \partial_{y_1} u(y_0, \cdot)\|_{L^\lambda(\Omega)}
\]

\[+ \sum_{2 \leq |\beta_1| \leq |\beta| - 2} \||D^\beta_1 \partial_{y_1} u(y_0, \cdot)||_{L^\infty(\Omega)} \||D^\beta_2 \partial_{y_1} u(y_0, \cdot)||_{L^\lambda(\Omega)}.\]

(3.133)
Applying the Gagliardo–Nirenberg inequality in above inequality, we have
\[
\sum_{\beta_1 + \beta_2 = \beta} \| (D^{\beta_1} \partial_y u D^{\beta_2} \partial_y u)(y_0, \cdot) \|_{L^2(\Omega)} \\
\lesssim (1 + \delta) \sum_{|\beta| \geq |\gamma| \geq |\beta| - 1} \| \nabla^2_y D^\gamma u(y_0, \cdot) \|_{L^2(\Omega)}^{1/2} \| D^\gamma u(y_0, \cdot) \|_{L^3(\Omega)}^{1/2} \\
+ \delta \sum_{2 \leq |\beta_2| \leq |\beta| - 2} \| \nabla^2_y D^{\beta_2} u(y_0, \cdot) \|_{L^2(\Omega)}^{1/2} \| D^{\beta_2} u(y_0, \cdot) \|_{L^6(\Omega)}^{1/2} \\
\lesssim (1 + \delta) \sum_{|\beta| \geq |\gamma| \geq |\beta| - 1} \| \nabla^2_y D^\gamma u(y_0, \cdot) \|_{L^2(\Omega)}^{1/2} \| D^\gamma u(y_0, \cdot) \|_{H^1(\Omega)}^{1/2} \\
+ \delta \sum_{2 \leq |\beta_2| \leq |\beta| - 2} \| \nabla^2_y D^{\beta_2} u(y_0, \cdot) \|_{L^2(\Omega)}^{1/2} \| D^{\beta_2} u(y_0, \cdot) \|_{H^1(\Omega)}^{1/2} \\
\lesssim \delta. \quad (3.134)
\]

By (3.132) and (3.134), we obtain
\[
\| (D^\gamma r_{00} D \partial_{00} w)(y_0, \cdot) \|_{L^2(\Omega)} \\
\leq \delta \| D \partial_{00} w(y_0, \cdot) \|_{L^6(\Omega)} \\
\leq \delta \| D \partial_{00} w(y_0, \cdot) \|_{H^1(\Omega)} \\
\leq \delta \sum_{|\gamma| \leq 4} \| D^\gamma w(y_0, \cdot) \|_{L^2(\Omega)}. \quad (3.135)
\]

If \((i, j) \neq (0, 0)\), we apply the Gagliardo–Nirenberg inequality to \(D \partial_{i,j} w\), then by similar argument, we obtain
\[
\| (D^\gamma r_{ij} D \partial_{ij} w)(y_0, \cdot) \|_{L^2(\Omega)} \lesssim \delta \sum_{|\gamma| \leq 4} \| D^\gamma w(y_0, \cdot) \|_{L^2(\Omega)}. \quad (3.136)
\]

Combining (3.135) and (3.136), one deduces that
\[
Z_{21} \lesssim \delta \sum_{|\gamma| \leq 4} \| D^\gamma w(y_0, \cdot) \|_{L^2(\Omega)}. \quad (3.137)
\]

It is clear that
\[
Z_{22} \lesssim \delta \sum_{|\gamma| \leq 4} \| D^\gamma w(y_0, \cdot) \| \sum_{|\alpha| \leq |\alpha| + 1} \| D^\gamma u(y_0, \cdot) \|_{L^2(\Omega)}. \quad (3.138)
\]

Combining (3.129)–(3.131), (3.137), and (3.138), we obtain
\[
\| (D^\alpha (r_{ij} \partial_{ij} w) - r_{ij} D^\alpha \partial_{ij} w)(y_0, \cdot) \|_{L^2(\Omega)} \\
\lesssim \delta \sum_{0 \leq |\beta| \leq 4} \| D^\beta w(y_0, \cdot) \|_{L^2(\Omega)} \sum_{0 \leq |\beta| \leq |\alpha| + 1} \| D^\beta u(y_0, \cdot) \|_{L^2(\Omega)} \\
+ \delta \sum_{0 \leq |\beta| \leq |\alpha| + 1} \| D^\beta w(y_0, \cdot) \|_{L^2(\Omega)}. \quad (3.139)
\]
By similar arguments, it is not difficult to derive the estimate of 
\[ \|D^\alpha, L'\|_{L^2(\Omega)} \]
\[ \leq \delta \sum_{0 \leq |\beta| \leq 4} \|D^\beta w(y_0, \cdot)\|_{L^2(\Omega)} \sum_{0 \leq |\beta| \leq |x|+1} \|D^\beta u(y_0, \cdot)\|_{L^2(\Omega)} \]
\[ + \delta \sum_{0 \leq |\beta| \leq |x|+1} \|D^\beta w(y_0, \cdot)\|_{L^2(\Omega)} \]
\[ \leq \delta \sum_{0 \leq |\beta| \leq 4} \|D^\beta w(y_0, \cdot)\|_{L^2(\Omega)} \sum_{0 \leq |\beta| \leq s} \|D^\beta u(y_0, \cdot)\|_{L^2(\Omega)} \]
\[ + \delta \sum_{0 \leq |\beta| \leq s} \|D^\beta w(y_0, \cdot)\|_{L^2(\Omega)} \]
(3.140)

Then by choosing \( \eta \) be large and \( \delta \) be small, it follows from Proposition 3.1 that

\[ \sum_{|\alpha| \leq s} \eta \|e^{-\eta t}D^\alpha w\|^2_{L^2(\Omega_T)} + e^{-2\eta T} \|D^\alpha w\|^2_{L^2(\Omega)} + \|e^{-\eta t}D^\alpha w\|^2_{L^2(\omega_T')} \]
\[ \leq \frac{1}{\eta} \left( \|e^{-\eta t}u\|^2_{L^2(\omega_T')} \cdot \sum_{|\alpha| \leq 4} \sup_{0 \leq t \leq T} \|D^\alpha w(t, \cdot)\|^2_{L^2(\omega_T')} + \|e^{-\eta t}f\|^2_{L^2(\omega_T')} \right) \]
\[ + \sum_{|\alpha| \leq s-1} \|e^{-\eta t}BD^\alpha w\|^2_{L^2(\omega_T')} \]
(3.141)

Note that

\[ B(D^\alpha w) = -[D^\alpha, B]w + D^\alpha (Bw) \]
(3.142)

By similar argument to the one used for estimate of the commutator \([D^\alpha, L']\) and the trace theorem, one has

\[ \sum_{|\alpha| \leq s-1} \int_0^{y_0} e^{-2\eta t} \|BD^\alpha w(t, 0, \cdot)\|^2_{L^2(\mathbb{R}^n)} dt \]
\[ \leq \sum_{|\alpha| \leq s-1} \int_0^{y_0} e^{-2\eta t} (\delta \|D^\alpha w(t, 0, \cdot)\|^2_{L^2(\mathbb{R}^n)} + \|D^\alpha g(t, 0, \cdot)\|^2_{L^2(\mathbb{R}^n)}) dt \]
\[ \leq \sum_{|\alpha| \leq s} \delta \int_0^{y_0} e^{-2\eta t} \|D^\alpha w(t, \cdot)\|^2_{L^2(\Omega)} dt + \sum_{|\alpha| \leq s-1} \int_0^{y_0} e^{-2\eta t} \|D^\alpha g(t, 0, \cdot)\|^2_{L^2(\mathbb{R}^n)} dt \]
(3.143)

Noticing that (3.141) holds for all \( T > 0 \), so by substituting (3.143) into (3.141), letting \( s = 4 \) in (3.141) and repeating above process, then letting \( \delta, T \) and \( \frac{1}{\eta} \) be small, one can deduce that solution \( w \) of problem (LP) satisfies the estimate (3.1). This completes the proof of this theorem.
4  |  WELL-POSEDNESS OF THE NLP

4.1  |  Reformulation of the NLP

We first reformulate the NLP. Let

\[ \tilde{u} := u + y_3 \frac{\partial_{x_1} \mathcal{W}}{1 + |\partial_{x_1} \mathcal{W}|^2 + |\partial_{x_2} \mathcal{W}|^2}, \]

where \( u \) is the solution to the (NLP). Since

\[ \frac{\partial \tilde{u}}{\partial u} = 1 + y_3 \left( \frac{\partial_{x_1} \mathcal{W} + \partial_{x_1} \mathcal{W} \frac{\partial_{x_2} \mathcal{W}}{\partial u}}{1 + |\partial_{x_1} \mathcal{W}|^2 + |\partial_{x_2} \mathcal{W}|^2} \right) \left( \frac{1 - |\partial_{x_1} \mathcal{W}|^2 + |\partial_{x_2} \mathcal{W}|^2}{1 + |\partial_{x_1} \mathcal{W}|^2 + |\partial_{x_2} \mathcal{W}|^2} \right) \]

\[ - y_3 \frac{2 \partial_{x_1} \mathcal{W} \partial_{x_2} \mathcal{W}}{(1 + |\partial_{x_1} \mathcal{W}|^2 + |\partial_{x_2} \mathcal{W}|^2)^2} \left( \partial_{x_1} \mathcal{W} + \partial_{x_2} \mathcal{W} \frac{\partial_{x_2} \mathcal{W}}{\partial u} \right), \] (4.1)

one has \( \frac{\partial \tilde{u}}{\partial u} > 0 \), when \( y_3 \) and \( \| \mathcal{W} \|_{W^{2,\infty}} \) is sufficiently small. Then by the implicit function theorem, \( u \) can be expressed as a function with respect to \( \tilde{u}, y_2 \) and \( y_3 \). We assume \( u = \kappa(\tilde{u}, y_2, y_3) \) for some smooth function \( \kappa \). By the property of our background solution, that is, the nozzle wall \( \Gamma_0 \) is flat at the background solution, we have \( \tilde{u} = u_b \), if \( u = u_b \). That is to say \( \kappa(u_b, y_2, y_3) = u_b \). For notational simplicity, let

\[ N(x_1, x_2) = \left( \frac{\partial_{x_1} \mathcal{W}}{1 + |\partial_{x_1} \mathcal{W}|^2 + |\partial_{x_2} \mathcal{W}|^2} \right)(x_1, x_2). \]

Then by direct computation, one has

\[ \begin{aligned}
\frac{\partial \tilde{u}}{\partial x_1} \kappa &= \frac{1 + y_3 \partial_{x_2} N}{1 + y_3 (\partial_{x_1} N + \partial_{x_2} N)}, \\
\frac{\partial \tilde{u}}{\partial y_2} \kappa &= -\frac{y_3 \partial_{x_2} N}{1 + y_3 (\partial_{x_1} N + \partial_{x_2} N)}, \\
\frac{\partial \tilde{u}}{\partial y_3} \kappa &= -\frac{N}{1 + y_3 (\partial_{x_1} N + \partial_{x_2} N)}.
\end{aligned} \] (4.2)

It is easy to see that \( \frac{\partial \tilde{u}}{\partial x_1} \kappa \) is close to one and \( \frac{\partial \tilde{u}}{\partial y_2} \kappa \) close to zero. The second-order derivatives of \( \kappa \) with respect ot \( \tilde{u} \), \( y_2 \), \( y_3 \) is listed in the Appendix. From (2.13) to (2.15), we deduce that \( \tilde{u} \) satisfies

\[ \frac{\partial \tilde{u}}{\partial x_1} \kappa \sum_{i,j=0}^{3} \tilde{a}_{ij} \partial_{x_1} \tilde{u} \tilde{u} = F(\tilde{u}, D\tilde{u}), \] (4.3)

where

\[ F(\tilde{u}, D\tilde{u}) = \frac{\partial \tilde{u}}{\partial x_1} \kappa \tilde{a}_{ij} \partial_{x_1} \tilde{u} \tilde{u} + \sum_{j=0}^{3} \kappa \tilde{a}_{ij} \tilde{a}_{ij} \partial_{x_1} \tilde{u} + \sum_{j=0}^{3} \kappa \tilde{a}_{ij} \tilde{a}_{ij} \partial_{x_1} \tilde{u} \]
\[
\begin{align*}
+ \tilde{a}_{22}\partial_{y_2}y_2\kappa + 2\tilde{a}_{23}\partial_{y_2}y_3\kappa + \tilde{a}_{33}\partial_{y_3}y_3\kappa - (\tilde{a}_2\partial_{y_2}u + \tilde{a}_3\partial_{y_3}u) \\
- (a_{12}\partial_{x_1}x_2 p(\partial_{y_1}u)^3 + (\partial_{y_1}u)^3 \sum_{i,j=0}^{3} a_{ij}\partial_{x_i}x_j\Phi^-),
\end{align*}
\]

(4.4)

where \(\partial_{y_i}u\) can be replaced by \(\partial_{\bar{u}}\kappa \partial_{y_i}u + \partial_{y_i} \kappa \delta_{i2} + \partial_{y_3} \kappa \delta_{i3}\).

The initial conditions for \(\bar{u}\) now become

\[
\bar{u}|_{y_0=0} = u_0 + y_3N(u_0, x_2(u_0, y_2, y_3)),
\]

(4.5)

\[
\partial_{y_0}\bar{u}|_{y_0=0} = u_1 \cdot (1 + y_3\tilde{a}_{x_1}N)\left(\frac{\partial_{x_1}N}{1 + y_3\tilde{a}_{x_2}N}\right)(u_0, x_2(u_0, y_2, y_3)).
\]

(4.6)

The boundary conditions for \(\bar{u}\) are

\[
\partial_{y_3}\bar{u} = 0 \quad \text{on} \quad \omega_T',
\]

(4.7)

\[
G(\kappa(\bar{u}, y_2, y_3), D(\bar{u}, y_2, y_3)) = 0 \quad \text{on} \quad \omega_T',
\]

(4.8)

where \(\partial_{y_i}u\) should be replaced by \(\partial_{\bar{u}}\kappa \partial_{y_i}u + \partial_{y_i} \kappa \delta_{i2} + \partial_{y_3} \kappa \delta_{i3}\).

Let \(\bar{u}_j(y) := \partial_{y_j}u(y_0, y)|_{y_0=0}\), which can be derived by differentiating (4.3) with respect to \(y_0\).

Obviously, \(\bar{u}_0\) and \(\bar{u}_1\) are give by (4.5) and (4.6), respectively. Let

\[
\psi(y_0, y) := \bar{u}_0 + \bar{u}_1y_0 + \frac{\bar{u}_2}{2!}y_0^2 + \cdots + \frac{\bar{u}_{s_0}}{s_0!}y_0^{s_0}.
\]

We introduce a new unknown \(\bar{u} := \bar{u} - \psi\) and define \(\bar{u}_b := u_b - \psi\). Then \(\bar{u}\) satisfies

\[
\begin{align*}
\partial_{\bar{u}}\kappa \sum_{i,j=0}^{3} \tilde{a}_{ij}\partial_{\bar{u}}\partial_{\bar{u}}\partial_{\bar{u}}u &= F(\bar{u} + \psi, D(\bar{u} + \psi)) - \partial_{\bar{u}}\kappa \sum_{i,j=0}^{3} \tilde{a}_{ij}\partial_{\bar{u}}\partial_{\bar{u}}\psi \quad \text{in} \quad \Omega_T, \\
G(\kappa(\bar{u} + \psi, y_2, y_3), D(\kappa(\bar{u} + \psi, y_2, y_3))) &= 0 \quad \text{on} \quad \omega_T', \\
\partial_{y_3}\bar{u} &= 0 \quad \text{on} \quad \omega_T', \\
(\bar{u}, \partial_{\bar{u}}\partial_{\bar{u}}u)|_{y_0=0} &= (0, 0) \quad \text{on} \quad \Gamma_{\text{in}}.
\end{align*}
\]

(4.9)

If we can solve this problem for \(\bar{u}\), then clearly \(\bar{u} + \psi\) is the desired solution to the NLP.

\subsection*{4.2 Proof of Theorem 2.1}

In this section, we introduce a iterative scheme to deduce the existence of smooth solution to the NLP (4.9). Let \(\bar{u}_0 := 0\) and \(\bar{u}_{m+1} (m \geq 0)\) is defined as the solution to the following initial
boundary value problem

\[
\begin{aligned}
    \partial_{\nu} \kappa \sum_{i,j=0}^{3} \tilde{a}_{ij} \partial_{y_i} \partial_{y_j} \bar{u}_{m+1} &= F_m - \partial_{\nu} \kappa \partial_{y_i} \partial_{y_j} \psi \\
    B \bar{u}_{m+1} &= B \bar{u}_m - G_m & \text{on } \omega^c_T, \\
    \partial_{y_3} \bar{u}_{m+1} &= 0 & \text{on } \omega^c_T, \\
    (\bar{u}_{m+1}, \partial_{y_0} \bar{u}_{m+1})|_{y_0=0} &= (0, 0) & \text{on } \Gamma_{in},
\end{aligned}
\]  

where \( \tilde{a}_{ij} = \tilde{a}_{ij}|_{u = \varphi(\bar{u}_m + \psi, y_2, y_3)} \), \( F_m := F(\bar{u}_m + \psi, D(\bar{u}_m + \psi)) \), \( G_m := G|_{u = \varphi(\bar{u}_m + \psi, y_2, y_3)} \) and

\[
B = \partial_{\nu} \kappa \sum_{i=0}^{3} \frac{\partial G}{\partial y_i}|_{u = u_b} \cdot \partial_{y_i} + \partial_{\nu} \kappa \frac{\partial G}{\partial y_i}|_{u = u_b}
\]

and \( G = G(u, Du) \) is defined in (2.21).

Before proving the convergence of above iterative scheme, we have to verify hypothesis \((H_1)-(H_4)\). Actually we have following lemma:

**Lemma 4.1.** \((\tilde{a}_{ij})_{0 \leq i,j \leq 3}\) and \(B\) satisfy all assumptions \((H_1)-(H_4)\).

**Proof.** It is clear that \(\tilde{a}_{ij}\) are smooth functions depending on \(u\) and \(Du\). As a direct consequence of (1.13) and (2.10), \(\tilde{a}_{03}\) vanishes on \(\Gamma_w\). In view of the slip boundary condition (1.13) and (2.11), it is clear that

\[
\partial_{y_1} u(\partial_{x_1} W(\partial_{x_1} W \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1))
\]

\[
+ c^2 \partial_{y_1} u(\partial_{x_2} W(\partial_{x_2} p \partial_{y_2} \partial_{y_3} u + \partial_{x_2} \partial_{y_3} \partial_{y_2} u - \partial_{y_2} u) + \partial_{x_3} p \partial_{y_2} u + \partial_{y_3} u). \quad (4.11)
\]

But (1.13) implies

\[-\partial_{x_1} W(\Phi^- - \phi)_{x_1} - \partial_{x_2} W(\Phi^- - \phi)_{x_2} + (\Phi^- - \phi)_{x_3} = 0\]

on \(\Gamma_0\), so by the slip boundary condition of \(\Phi^-\) and the expressions of \(\phi_{x_i}\) \((i = 1, 2, 3)\) given by (2.8), it is equivalent to say that

\[
\partial_{x_1} W(\partial_{x_1} W \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1)
\]

\[
+ \partial_{x_2} W(\partial_{x_2} p \partial_{y_2} \partial_{x_1} W \partial_{y_3} u + \partial_{x_2} \partial_{y_3} \partial_{y_2} u - \partial_{y_2} u)
\]

\[
+ \partial_{x_3} p \partial_{y_2} u + \partial_{y_3} u = 0 \quad \text{on } \Gamma_w. \quad (4.12)
\]

Thus, independent of the choice of \(p(x)\), one deduces that \(\bar{a}_{13}\) vanishes on \(\Gamma_w\). Also by the slip boundary condition (1.13), we deduce that

\[
\bar{a}_{23}|_{\Gamma_w} = c^2 \partial_{y_1} u \partial_{y_1} p \partial_{x_1} W(\partial_{x_1} W \partial_{x_2} p + 1) - \partial_{x_3} p. \quad (4.13)
\]

So, requiring \(\bar{a}_{23} = 0\) on \(\Gamma_w\) is equivalent to require

\[
\partial_{x_1} W \partial_{x_1} p + \partial_{x_2} W(\partial_{x_2} p + 1) - \partial_{x_3} p = 0 \quad \text{on } \Gamma_w. \quad (4.14)
\]
With $G$ given in (2.21), by simple calculation, we have

$$G = (\rho^+ - \rho^-)(\phi_t + \nabla \phi \cdot \nabla \Phi^-) - |\nabla \phi|^2 \rho^+,$$

where $\rho^\pm = ((\gamma - 1)B_0 - \Phi^\pm - \frac{1}{2} |\nabla \Phi^\pm|^2) + 1)^{\frac{1}{\gamma - 1}}$ and $\Phi^+ := \Phi$ is the velocity potential ahead of the shock front. Replacing $D\phi$ in $G$ by $Du$ via (2.8), then differentiate $G$ with respect to $u$ and $u_{y_1}$, respectively, one can obtain the expressions of $b$ and $b_j$ ($j = 0, 1, 2, 3$).

By simple calculation, one has

$$b_3 = \frac{\partial \rho^+}{\partial (\nabla y_3 u)} (\phi_t + \nabla \phi \cdot \nabla \Phi^-) + (\rho^+ - \rho^-) \frac{\partial}{\partial (\nabla y_3 u)} (\phi_t + \nabla \phi \cdot \nabla \Phi^-)$$

$$= \mathcal{E}_1 \cdot (\phi_t + \nabla \phi \cdot \nabla \Phi^-) + (\rho^+ - \rho^-) \cdot \mathcal{E}_2,$$

where

$$\mathcal{E}_1 = \frac{(\partial x_3 p - \partial x_1 p \partial x_1 W + \partial x_3 p |\partial x_2 W|^2 - \partial x_2 W) \partial y_3 u}{(\rho^+)^{\gamma - 2} (\partial y_1 u)^2} + \frac{1 + |\partial x_1 W|^2 + |\partial x_2 W|^2) \partial y_3 u + \partial x_1 W}{(\rho^+)^{\gamma - 2} (\partial y_1 u)^2} - \frac{\partial x_1 W \partial x_1 \Phi^- + \partial x_2 W \partial x_2 \Phi^- - \partial x_3 \Phi^-}{\partial y_1 u},$$

$$\mathcal{E}_2 = \frac{\partial x_1 W \partial x_1 \Phi^- + \partial x_2 W \partial x_2 \Phi^- - \partial x_3 \Phi^-}{\partial y_1 u}.$$

Since $\Phi^-$ satisfies (1.13) on $\Gamma_w$ (equivalently on $\{y_3 = 0\}$) and $u$ satisfies (2.19) on $\{y_3 = 0\}$, in order to let $b_3 |_{y_3=0} = 0$, it suffices to require

$$\partial x_3 p(1 + |\partial x_2 W|^2) - \partial x_1 p \partial x_1 W - \partial x_2 W = 0 \quad \text{on} \quad \Gamma_w.$$

It is easy to verify that (4.14) and (4.16) are satisfied, if we let

$$p(x) := \frac{\partial x_2 W}{1 + |\partial x_1 W|^2 + |\partial x_2 W|^2}(x_3 - W).$$

With such $p(x)$, by calculating the Jacobian of $\mathcal{P}$, one can easily check that $\mathcal{P}$ is indeed invertible, when $u$ is close to $u_b$, $\partial x_3 W$ is small, and $x_3$ is close to $W(x_1, x_2)$ (this means $y_3$ is small in $(y_0, y)$-coordinate). Here we do not need the exact expression of $b_0$, $b$ and $b_j$ ($j = 0, 1, 2$). At the background solution $(u, W) = (u_b, 0)$, one has

$$b_0 := \frac{\partial G}{\partial u_{y_0}} = (q_- - q_+)^2 \left( -\frac{\rho_+ q_+}{c_+^2} (q_- - q_+) - (\rho_+ - \rho_-) \right) < 0,$$

$$b_1 := \frac{\partial G}{\partial (\nabla y_1 u)} = (q_- - q_+)^2 \left( -\frac{q_+^2 \rho_+}{c_+^2} (q_- - q_+) + \rho_+ (q_- - q_+) \right) > 0.$$
and
\[
b_2 := \frac{\partial G}{\partial (\partial y_2 u)} = 0, \quad b_3 := \frac{\partial G}{\partial (\partial y_3 u)} = 0, \quad b := \frac{\partial G}{\partial u} = 0.
\]

Moreover, with the choice of \(p(x)\), one can see that \(\tilde{a}_2\) is zero at the background solution. Hence, \(\tilde{a}_2\) is close to zero near the background solution. This allows us to put the term \(\tilde{a}_2 \partial y_2 u\) to the right side in the coming iteration scheme, so that the coefficient before \(\partial y_2 u\) be zero. Then above computations together with (2.25)-(2.28) implies (H1) and (H2) are fulfilled. We still need to verify (H4). It is clear that \(b_1\) is bounded away from zero when \(u\) is sufficiently close to \(u_b\). By simple calculation, we have

\[
\begin{align*}
\frac{\tilde{a}_{11} b_0 - \tilde{a}_{10}}{b_1} &= -\frac{q_+}{q_- - q_+} + \frac{(c_+^2 - q_+^2)(c_+^2(\rho_- - \rho_+) + q_+(q_+ - q_-)\rho_+)}{(q_+ - q_-)\rho_+ q_+^2 - c_+^2(-2q_+\rho_+ + q_- (\rho_- + \rho_+))}
\end{align*}
\]

\[
= -\frac{q_+}{q_- - q_+} + \frac{(c_+^2 - q_+^2)(c_+^2(\rho_- - \rho_+) + q_+(q_+ - q_-)\rho_+)}{(q_+^2 - c_+^2)(q_+ - q_-)\rho_+}
\]

\[
= -\frac{q_+}{q_- - q_+} - \frac{c_+^2(\rho_- - \rho_+) + q_+(q_+ - q_-)\rho_+}{(q_+ - q_-)\rho_+}
\]

\[
= \frac{1}{(q_+ - q_-)\rho_+}(-q_+\rho_+ - c_+^2(\rho_- - \rho_+) - q_+(q_+ - q_-)\rho_+ - q_+(q_+ - q_-)\rho_+ - q_+(q_+ - q_-)\rho_+ - q_+(q_+ - q_-)\rho_+ + q_+(q_+ - q_-)\rho_+ + q_+(q_+ - q_-)\rho_+ + q_+(q_+ - q_-)\rho_+ + q_+(q_+ - q_-)\rho_+) > 0.
\]

Moreover, at the background solution we have

\[
\sum_{i,j=0}^{3} \tilde{a}_{ij} (\frac{b_i}{b_1} - \tilde{a}_{i1}) (\frac{b_j}{b_1} - \tilde{a}_{j1}) = \frac{(q_- - q_+)^2(c_+^2 - q_+^2)}{c_+^2} (\frac{\tilde{a}_{11} b_0 - \tilde{a}_{10}}{b_1})^2 > 0.
\]

So, \(\gamma_0\) exists and hence (H4) is satisfied. For (H3), we can see from our proof of the convergence of the scheme that the solution \(\tilde{u} + \psi\) is still close to \(u_b\). \(\square\)

For \(s \in \mathbb{N}\), let

\[
\eta \sum_{|\alpha| \leq s} \|e^{-\eta t} D^\alpha u\|_{L^2(\Omega_T)}^2 = \|u\|_{H^s(\Omega_T)}^2
\]

and denote by \(\|u\|_{s,\eta, T}\) the usual Sobolev norm \(\|e^{-\eta t} u\|_{H^s(\Omega_T)}\). Furthermore, for simplification, one may use \(\|u\|_{y_1 = 0, s, \eta, T}\) to represent the usual Sobolev norm on the boundary \(\{y_1 = 0\}\).

**Lemma 4.2.** For any smooth function \(u\) and any \(\eta \geq 1\), we have

\[
\|e^{-\eta t} u\|_{H^s(\Omega_T)}^2 \leq \frac{1}{\eta} \|u\|_{H^s(\Omega_T)}^2,
\]

provided that \(\partial_j u = 0, j = 0, 1, \ldots, s - 1\). Here \(\|e^{-\eta t} u\|_{H^s(\Omega_T)}\) is the standard Sobolev norm.
Proof. Let

\[ A(y_0) = \int_0^{y_0} \int_\Omega e^{-2\eta t} |u|^2 \, dy \, dt \]

Then we have

\[ A(y_0) = -\frac{1}{2\eta} \int_0^{y_0} \int_\Omega (e^{-2\eta t} |u|)^2 \, dy \, dt \]

\[ = -\frac{1}{2\eta} \int_\Omega e^{-2\eta y_0} |u(y_0, \cdot)|^2 \, dy \, + \frac{1}{2\eta} \int_0^{y_0} \int_\Omega e^{-2\eta t} \partial_t u \, dy \, dt \]

\[ \leq -\frac{1}{2\eta} \int_\Omega e^{-2\eta y_0} |u(y_0, \cdot)|^2 \, dy \, + \frac{1}{\eta} \int_0^{y_0} \int_\Omega e^{-2\eta t} |\partial_t u|^2 \, dy \, dt \]

\[ = -\frac{1}{2\eta} \int_\Omega e^{-2\eta y_0} |u(y_0, \cdot)|^2 \, dy \, + \frac{1}{2} A(y_0) + \frac{1}{2\eta} \int_0^{y_0} \int_\Omega e^{-2\eta t} |\partial_t u|^2 \, dy \, dt. \]

Here for the third identity, we have used the assumption that \( u|_{t=0} = 0 \). This implies

\[ \eta A(y_0) + \int_\Omega e^{-2\eta y_0} |u(y_0, \cdot)|^2 \, dy \leq \frac{1}{\eta} \int_0^{y_0} \int_\Omega e^{-2\eta t} |\partial_t u|^2 \, dy \, dt. \] (4.22)

In particular,

\[ \eta^2 A(y_0) \leq \int_0^{y_0} \int_\Omega e^{-2\eta t} |\partial_t u|^2 \, dy \, dt \leq \int_0^{y_0} \int_\Omega e^{-2\eta t} |D u|^2 \, dy \, dt. \] (4.23)

It follows from (4.22) that

\[ \int_0^{y_0} \int_\Omega e^{-2\eta t} |D u|^2 \, dy \, dt + \eta^2 \int_0^{y_0} \int_\Omega e^{-2\eta t} |u|^2 \, dy \, dt + \int_0^{y_0} \int_\Omega e^{-2\eta t} |u|^2 \, dy \, dt \]

\[ \leq \int_0^{y_0} e^{-2\eta t} (\|u\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2) \, dt. \] (4.24)

This implies that \( \|u\|_{1,\eta,T}^2 \leq \frac{1}{\eta} \|u\|_{H^1(\Omega_T)}^2 \).

Now for \( k \in \mathbb{N} \), assume

\[ \|u\|_{k,\eta,T}^2 \leq \frac{1}{\eta} \|u\|_{H^k_{\eta}(\Omega_T)}^2. \] (4.25)

We are going to show

\[ \|u\|_{k+1,\eta,T}^2 \leq \frac{1}{\eta} \|u\|_{H^{k+1}_{\eta}(\Omega_T)}^2. \] (4.26)
Repeating the process for estimate (4.23) above \( m \) times where \( |u|^2 \) in \( A(y_0) \) is replaced by \( |D^n u|^2 \), we have

\[
\int_0^{y_0} e^{-2\eta t} \|D^n u\|^2_{L^2(\Omega)} dt \leq \eta^{-2m} \int_0^{y_0} e^{-2\eta t} \|D^{m+n} u\|^2_{L^2(\Omega)} dt
\]  
(4.27)

provided that \( \delta^l_i u |_{t=0} = 0, l = 0, 1, 2, ..., m + n - 1 \).

Note that

\[
\| u \|^2_{k+1, \eta, T} = \| u \|^2_{k, \eta, T} + \sum_{|\alpha| = k+1} \| D^\alpha (e^{-\eta t} u) \|^2_{L^2(\Omega_T)}
\]  
(4.28)

and

\[
\sum_{|\alpha| = k+1} \| D^\alpha (e^{-\eta t} u) \|^2_{L^2(\Omega_T)} = \sum_{l_1 + l_2 = k+1} \| (-\eta)^{l_1} e^{-\eta t} D^{l_2} u \|^2_{L^2(\Omega_T)}
\]
\[= \sum_{l_1 + l_2 = k+1} (\eta)^{2l_1} \| e^{-\eta t} D^{l_2} u \|^2_{L^2(\Omega_T)}.\]  
(4.29)

So, by (4.27), we have

\[
\sum_{l_1 + l_2 = k+1} (\eta)^{2l_1} \| e^{-\eta t} D^{l_2} u \|^2_{L^2(\Omega_T)} \leq \sum_{l_1 + l_2 = k+1} \| e^{-\eta t} D^{l_1+l_2} u \|^2_{L^2(\Omega_T)}
\]
\[= \sum_{|\alpha| = k+1} \| e^{-\eta t} D^{\alpha} u \|^2_{L^2(\Omega_T)}.\]  
(4.30)

From (4.25), (4.28)–(4.30), we obtain (4.26). Therefore, we derive the estimate (4.21) for any \( s \in \mathbb{N} \) by the induction method.

**Lemma 4.3** (Boundedness in the norm of high regularity). Under the assumption of Theorem 2.1, there exists a large \( \eta_* \geq 1 \) and a small \( T_* > 0 \) and small \( \varepsilon_0 > 0 \), such that for all \( \eta \geq \eta_* \) and \( T \leq T_* \), the following estimate

\[
\| \tilde{u}_m \|^2_{s, \eta, T} + \sum_{|\alpha| \leq s} \sup_{0 \leq t \leq T} \| D^\alpha \tilde{u}_m (t, \cdot) \|^2_{L^2(\Omega)} + \sum_{|\alpha| \leq s} \| e^{-\eta t} D^\alpha \tilde{u}_m \|^2_{L^2(\omega^T_\ell)} \leq \varepsilon_0^2
\]  
(4.31)

holds for all \( m \geq 0 \).

**Proof.** We prove this lemma by induction. Suppose (4.31) holds for all \( m \leq n \), we proceed to show it also holds true for \( m = n + 1 \). In view of (4.10), in order to apply Theorem 3.1 to \( \tilde{u}_{n+1} \), we need to estimate the source terms. By the definition of \( F_n \), we know that

\[
\| F_n \|^2_{s-1, \eta, T} \lesssim C' \varepsilon_0^2 \| \tilde{u}_n \|^2_{s, \eta, T},
\]  
(4.32)

Similarly, we have

\[
\| \tilde{a}^{m}_{ij} \nabla_i \psi \|^2_{s-1, \eta, T} \lesssim C'' \| \tilde{u}_n \|^2_{s, \eta, T}.
\]  
(4.33)
For the boundary term, noticing that \( G(\bar{u}_b, Du_b) = 0 \), we have
\[
\tilde{u}_n - G_n = \tilde{u}_n - (G(\bar{u}_n + \psi, D(\bar{u}_n + \psi) - G(u_b, Du_b))
\]
\[
= \tilde{u}_n - B(\bar{u}_n + \psi - u_b) + A_n^T D^2 G|_{u=\bar{u}_n+\psi-u_b} A_n
\]
\[
= B(u_b - \psi) + A_n^T D^2 G|_{u=\bar{u}_b+\psi-u_b} A_n,
\]
(4.34)
where \( A_n = (\bar{u}_n + \psi - u_b, D(\bar{u}_n + \psi - u_b)) \). Hence, we deduce that
\[
\sum_{|\alpha| \leq s} \| e^{-\eta t} D^{\alpha} (\tilde{u}_n - G_n) \|^2_{L^2(\omega_f^T)} \leq \sum_{|\alpha| \leq s} \| e^{-\eta t} D^{\alpha} (\psi - u_b) \|^2_{L^2(\omega_f^T)} + \| e^{-\eta t} \tilde{u}_n \|_{y_1=0}^2 \|_H^2(\omega_f^T).
\]
(4.35)

By Theorem 3.1 and Lemma 4.2 and in view of (4.32), (4.33), and (4.35), we deduce that
\[
\| \tilde{u}_{n+1} \|^2_{s,\eta, T} + \sup_{0 \leq t \leq T} \| D^{\alpha} \tilde{u}_{n+1}(t, \cdot) \|^2_{L^2(\Omega)} + \| e^{-\eta t} \tilde{u}_{n+1} \|_{y_1=0}^2 \leq \frac{C}{\eta} e^{4\eta T} \| \tilde{u}_n + \psi \|^2_{s,\eta, T} \cdot (\| F_n \|^2_{3,\eta, T} + \| \tilde{a}_n^i \gamma^i_\partial \psi \|^2_{3,\eta, T})
\]
\[
+ \frac{C}{\eta} e^{2\eta T} \| F_n \|^2_{s-1,\eta, T} + e^{2\eta T} \| (B \tilde{u}_n - G_n) \|_{y_1=0}^2 \|_{s-1,\eta, T}
\]
\[
\leq \frac{C}{\eta} e^{4\eta T} (\varepsilon_0^2 + \| \psi \|^2_{s,\eta, T}) \cdot (C' \varepsilon_4^4 + C'' \varepsilon_0^2)
\]
\[
+ \frac{C}{\eta} e^{2\eta T} \varepsilon_0^2 + C e^{2\eta T} (\varepsilon^2 + \varepsilon_0^4).
\]
(4.36)

Now let \( \eta_0 \) be properly large such that \( \frac{C}{\eta_0} (C' + C'' + 1) < \frac{1}{16} \). Then let \( \varepsilon_0 \) be small such that \( \varepsilon^2 < \min\left(\frac{1}{2}, \sqrt{\frac{1}{8C}}\right) \). Finally let \( T_0 \) and the \( \varepsilon \) in Theorem 2.1 be properly small such that \( e^{4\eta_0 T_0} < 2 \) and \( \| \psi \|^2_{s,\eta_0, T_0} < \varepsilon_0^2 \) and \( \varepsilon \leq \min\left(\frac{1}{16C} \varepsilon_0, \varepsilon_0\right) \). We obtain
\[
\| \tilde{u}_{n+1} \|^2_{s,\eta_0, T_0} + \sup_{0 \leq t \leq T_0} \| D^{\alpha} \tilde{u}_{n+1}(t, \cdot) \|^2_{L^2(\Omega)} + \| \tilde{a}_n^i \gamma_\partial \tilde{u}_n \|_{y_1=0}^2 \leq \frac{1}{2} \varepsilon_0^2 + 2 \times \frac{1}{8} \varepsilon_0^2 + \frac{1}{4} \varepsilon_0^2 = \varepsilon_0^2.
\]
This implies (4.31) also holds for \( m = n + 1 \). It is clear that (4.31) holds for \( m = 0 \). This completes the proof of this lemma.

It is easy to check that \( v_{m+1} := \tilde{u}_{m+1} - \tilde{u}_m \) satisfies following initial boundary value problem
\[
\begin{aligned}
\partial_\alpha \kappa \sum_{i,j=0}^{3} \tilde{a}_{ij} \partial_{ij} v_{m+1} &= F_m - F_{m-1} - \partial_\alpha \kappa (\tilde{a}_{ij}^m - \tilde{a}_{ij}^{m-1}) \partial_{ij} (\tilde{u}_m + \psi) & \text{in } \Omega_T, \\
B v_{m+1} &= B v_m - (G_m - G_{m-1}) & \text{on } \omega_T^c, \\
\partial_{y_3} v_{m+1} &= 0 & \text{on } \omega_T^c, \\
(v_{m+1}, \partial_{y_0} v_{m+1})|_{y_0=0} &= (0, 0) & \text{on } \Gamma_{in}.
\end{aligned}
\]
(4.37)

For the sequence \( \{v_m\}_{m=1}^\infty \), we have following lemma:
Lemma 4.4 (Contraction in the norm of low regularity). Under the assumption of Theorem 2.1 and suppose the $\epsilon$ in 2.1 is small enough, then there exist two constants $\eta_{\ast \ast} \geq 1$ and $T_{\ast \ast} > 0$ such that
\begin{equation}
\|v_{m+1}\|_{1, \eta_{\ast \ast}, T_{\ast \ast}}^2 + \sup_{0 \leq t \leq T_{\ast \ast}} \|D^2 v_{m+1}(t, \cdot)\|_{L^2(\Omega)}^2 + \|v_{m+1}\|_{1, \eta_{\ast \ast}, T_{\ast \ast}}^2
\leq \sigma_0 \cdot (\|v_m\|_{1, \eta_{\ast \ast}, T_{\ast \ast}}^2 + \sup_{0 \leq t \leq T_{\ast \ast}} \|D^2 v_m(t, \cdot)\|_{L^2(\Omega)}^2 + \|v_m\|_{1, \eta_{\ast \ast}, T_{\ast \ast}}^2)
\end{equation}
hold for all $m \geq 0$, where $0 < \sigma_0 < 1$ is a constant independent of $m$.

Proof. To apply Theorem 3.1, we need to estimate the source terms. In fact, we have
\begin{equation}
F_m - F_{m-1} = \partial \zeta(\int_0^1 F(u, Du)|_{u=\tilde{u}_{m-1} + \theta v_m + \psi d\theta}) \cdot v_m
+ \sum_{i=0}^3 \partial \zeta \left(\int_0^1 \partial \zeta F(u, Du)|_{u=\tilde{u}_{m-1} + \theta v_m + \psi d\theta}\right) \cdot \partial_i v_m.
\end{equation}
Hence, we deduce that
\begin{equation}
\|F_m - F_{m-1}\|_{0, \eta, T} \lesssim \epsilon_0 \|v_m\|_{1, \eta, T}.
\end{equation}
Similarly, one has
\begin{equation}
\|\partial \zeta (\tilde{a}_{ij}^m - \tilde{a}_{ij}^{m-1}) \partial_j (\tilde{u}_m + \psi)\|_{0, \eta, T} \lesssim \|v_m\|_{1, \eta, T}.
\end{equation}
For the boundary term, we have
\begin{equation}
G_m - G_{m-1} = \partial \zeta G|_{\tilde{u}=\tilde{u}_{m-1}} v_m + \sum_{i=0}^3 \partial \zeta G|_{\tilde{u}=\tilde{u}_{m-1}} \partial_i v_m
+ (v_m, Dv_m)^\top D^2 G|_{\tilde{u}=\tilde{u}_{m-1} + \theta v_m} (v_m, Dv_m),
\end{equation}
where $\theta \in (0, 1)$ and $D^2 G$ is the Hessian matrix of $G$ with respect to $(u, Du)$. Hence, one deduces that
\begin{equation}
Bv_m - (G_m - G_{m-1})
= (\partial \zeta G|_{\tilde{u}=\tilde{u}_b} - \partial \zeta G|_{\tilde{u}=\tilde{u}_{m-1}}) v_m + \sum_{i=0}^3 (\partial \zeta G|_{\tilde{u}=\tilde{u}_b} - \partial \zeta G|_{\tilde{u}=\tilde{u}_{m-1}}) \partial_j v_m
+ (v_m, Dv_m)^\top D^2 G|_{\tilde{u}=\tilde{u}_{m-1} + \theta v_m} (v_m, Dv_m).
\end{equation}
By Taylor theorem, it is clear that
\begin{equation}
(\partial \zeta G|_{\tilde{u}=\tilde{u}_b} - \partial \zeta G|_{\tilde{u}=\tilde{u}_{m-1}}) v_m
= - \left(\partial \zeta^2 G|_{\tilde{u}=\tilde{u}_b} (u_{m-1} - \tilde{u}_b) + \sum_{i=0}^3 (\partial \zeta u y_j G)|_{\tilde{u}=\tilde{u}_b} \partial y_j (\tilde{u}_{m-1} - \tilde{u}_b)\right) v_m
- \left(\mathbf{X}_{m-1}^\top (D^2 \partial \zeta G)|_{\tilde{u}=\tilde{u}_b + \theta_1 v_m} \mathbf{X}_{m-1}\right) v_m.
\end{equation}
where $\partial_1 \in (0, 1)$, $X_{m-1} = (\bar{u}_{m-1} - \bar{u}_b, D(\bar{u}_{m-1} - \bar{u}_b))$, and $D^2 \partial_u G$ is the Hessian matrix of $\partial_u G$ with respect to $(u, Du)$.

Hence, by Lemma 4.3, one has

$$\|e^{-\eta t}(\partial_u G|_{\bar{u} = \bar{u}_b} - \partial_u G|_{\bar{u} = \bar{u}_{m-1}}) v_m\|_{L^2(\alpha^c)} \leq \varepsilon_0 \|e^{-\eta t}v_m\|_{L^2(\alpha^c)}.$$  \hspace{1cm} (4.45)

Similarly one deduces that

$$\|e^{-\eta t}(\partial_{u_j} G|_{\bar{u} = \bar{u}_b} - \partial_{u_j} G|_{\bar{u} = \bar{u}_{m-1}}) \partial_j v_m\|_{L^2(\alpha^c)} \leq \varepsilon_0 \|e^{-\eta t}v_m\|_{H^1(\alpha^c)}, \quad 0 \leq i \leq 3. \hspace{1cm} (4.46)$$

It is easy to see that

$$\|e^{-\eta t}(v_m, Dv_m)^T D^2 G|_{\bar{u} = \bar{u}_{m-1} + \varepsilon v_m} (v_m, Dv_m)\|_{L^2(\alpha^c)} \leq \varepsilon_0 \|e^{-\eta t}v_m\|_{H^1(\alpha^c)}. \hspace{1cm} (4.47)$$

By (4.43) and (4.45)–(4.47), we have

$$\|e^{-\eta t} u_{m+1}\|_{L^2(\alpha^c)} \leq \varepsilon_0 \|e^{-\eta t}v_m\|_{H^1(\alpha^c)}. \hspace{1cm} (4.48)$$

Then by Theorem 3.1, one has

$$\|v_{m+1}\|_{1, \eta, T}^2 + \sum_{|\alpha| \leq 1} \sup_{0 \leq t \leq T} \|D^\alpha v_{m+1}(t, \cdot)\|_{L^2(\Omega)}^2 + \|v_{m+1}|_{y_1 = 0}\|_{1, \eta, T}^2 \leq \frac{1}{\eta} + \varepsilon_0^2 \left(\|v_m\|_{2, \eta, T}^2 + \|v_m|_{y_1 = 0}\|_{1, \eta, T}^2\right). \hspace{1cm} (4.49)$$

Above inequality holds for $\eta > \eta_*$ and $T < T_*$. From the proof of Lemma 4.3, we can further require $\varepsilon_0$ small such that $C \varepsilon_0 \leq \frac{1}{2} \varepsilon_0$, then for properly small $T_* < T_*$, we have

$$\|v_{m+1}\|_{1, \eta_*, T_*}^2 + \sum_{|\alpha| \leq 1} \sup_{0 \leq t \leq T_*} \|D^\alpha v_{m+1}(t, \cdot)\|_{L^2(\Omega)}^2 + \|e^{-\eta t}v_{m+1}|_{y_1 = 0}\|_{2, \eta_*, T_*}^2 \leq \varepsilon_0 \left(\|v_m\|_{2, \eta_*, T_*}^2 + \sum_{|\alpha| \leq 1} \sup_{0 \leq t \leq T_*} \|D^\alpha v_m(t, \cdot)\|_{L^2(\Omega)}^2 + \|v_m|_{y_1 = 0}\|_{2, \eta_*, T_*}^2\right). \hspace{1cm} (4.50)$$

Since $\varepsilon_0 < 1$, we finish the proof of this lemma by letting $\sigma_0 = \varepsilon_0$. \hspace{1cm} \[\Box\]

**Proof of Theorem 2.1.** Armed with Lemmas 4.3 and 4.4, we are able to prove Theorem 2.1. In fact, Lemma 4.4 implies that $\{\bar{u}_m\}_{m=1}^\infty$ is a Cauchy sequence in the norm of low regularity. Hence, it converges strongly such that $\bar{u}_m$ converges to some function $\bar{u}$, that is,

$$\|\bar{u}_m - \bar{u}\|_{1, \eta_*, T_*} + \sum_{|\alpha| \leq 1} \sup_{0 \leq t \leq T_*} \|D^\alpha(\bar{u}_m - \bar{u})(t, \cdot)\|_{L^2(\Omega)}^2 \hspace{1cm} (4.51)$$

$$+ \|\bar{u}_m - \bar{u}\|_{y_1 = 0}|_{1, \eta_*, T_*} \longrightarrow 0 \text{ as } m \text{ goes to infinity.}$$

Limit (4.51) also means the coefficients in the equation and boundary conditions in (4.10), $\bar{a}_{l, j}^m$, $F_m$, $G_m$ and $B\bar{u}_m$, converge to the corresponding quantities with $\bar{u}_m$ being replaced by $\bar{u}$.

On the other hand, it follows from Lemma 4.3 that $\bar{u}_m$ converges to $\bar{u}$ weakly in the norm of high regularity such that $\bar{u}$ satisfies estimate (4.31).
Hence, by passing the limit in (4.10), it is easy to see that $\bar{u} + \psi$ is the smooth solution of the (NLP) with estimate (4.31). By (4.31) and the assumption of Theorem 2.1 one has

$$
\|\bar{u} + \psi - u_b\|_{s, \eta^*, T^*} \leq \|\bar{u}\|_{s, \eta^*, T^*} + \|\psi - u_b\|_{s, \eta^*, T^*} \leq C\varepsilon_0.
$$

(4.52)

This completes the proof of Theorem 2.1. □

APPENDIX A

A.1 | Interior Coefficients

By direct computation, we can determine other coefficients.

$$\bar{a}_{00} = (\partial_{y_1} u)^2. $$

(A.1)

$$\bar{a}_{01} = \bar{a}_{10} = \partial_{y_1} u(-\partial_{y_0} u + \partial_{x_1} \Phi(\partial_{x_1} \mathcal{W} \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1))$$

$$+ \partial_{y_1} u(\partial_{x_2} \Phi(\partial_{x_3} p \partial_{y_2} u + \partial_{x_2} \mathcal{W} \partial_{y_2} u - \partial_{x_2} u) - \partial_{x_3} \Phi(\partial_{x_3} p \partial_{y_2} u + \partial_{y_2} u)).$$

(A.2)

$$\bar{a}_{02} = \bar{a}_{20} = (\partial_{y_1} u)^2(\partial_{x_1} \Phi \partial_{x_1} p + \partial_{x_1} \Phi(\partial_{x_3} p + 1) + \partial_{x_3} \Phi \partial_{x_1} p).$$

(A.3)

$$\bar{a}_{12} = \bar{a}_{21} = -\partial_{y_0} u(\partial_{x_1} \Phi \partial_{x_1} p \partial_{y_1} u + \partial_{x_1} \Phi(\partial_{x_2} p + 1) \partial_{y_1} u)$$

$$+ \partial_{y_1} u(\partial_{x_1} \mathcal{W} \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1)$$

$$\times (\partial_{y_1} p(-c^2 + |\partial_{x_1} \Phi|^2) + \partial_{x_1} \Phi \partial_{x_2} \Phi(\partial_{x_2} p + 1) + \partial_{x_1} \Phi \partial_{x_3} \Phi \partial_{x_3} p).$$

(A.4)

$$\bar{a}_{22} = \partial_{x_1} p \partial_{y_1} u((-c^2 + |\partial_{x_1} \Phi|^2) \partial_{x_1} p \partial_{y_1} u + \partial_{x_1} \Phi \partial_{x_2} \Phi(\partial_{x_2} p + 1) \partial_{y_1} u)$$

$$+ (\partial_{y_1} u)^2(\partial_{x_2} p + 1)(\partial_{x_2} \Phi \partial_{x_1} \partial_{x_1} p + (-c^2 + |\partial_{x_2} \Phi|^2) \partial_{x_2} p + 1) + \partial_{x_2} \Phi \partial_{x_3} \Phi \partial_{x_3} p)$$

$$+ \partial_{x_3} p(\partial_{y_1} u)^2(\partial_{x_3} \Phi \partial_{x_1} \partial_{x_1} p + \partial_{x_3} \Phi \partial_{x_2} \Phi(\partial_{x_2} p + 1) + (-c^2 + |\partial_{x_3} \Phi|^2) \partial_{x_1} p).$$

(A.5)

$$\bar{a}_{11} = -\partial_{y_0} u(-\partial_{y_0} u + \partial_{x_1} \Phi(\partial_{x_1} \mathcal{W} \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1)$$

$$+ \partial_{x_2} \Phi((\partial_{x_3} p \partial_{y_2} \mathcal{W} - 1) \partial_{y_3} u) + \partial_{x_2} \mathcal{W} \partial_{y_3} u)$$

$$+ (\partial_{x_3} \mathcal{W} \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1)$$

$$\times ((\partial_{x_1} \Phi \partial_{y_0} u + (-c^2 + |\partial_{x_1} \Phi|^2)(\partial_{x_1} \mathcal{W} \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1))$$

$$+ \partial_{x_3} \Phi \partial_{y_0} u(\partial_{x_1} p \partial_{y_2} u + \partial_{y_3} u) + \partial_{x_1} \Phi \partial_{x_0} \Phi(\partial_{x_1} \mathcal{W} \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1)$$

$$\times (\partial_{x_3} p \partial_{y_2} \mathcal{W} - 1) \partial_{y_3} u + \partial_{x_2} \mathcal{W} \partial_{y_3} u)$$

$$-(\partial_{x_1} p \partial_{y_3} u - \partial_{x_1} \partial_{x_1} p \partial_{y_2} u + 1)\partial_{x_1} \Phi \partial_{x_3} \Phi(\partial_{x_3} p \partial_{y_2} u + \partial_{y_3} u)$$

$$+ ((\partial_{x_3} p \partial_{y_2} \mathcal{W} - 1) \partial_{y_3} u + \partial_{x_2} \mathcal{W} \partial_{y_3} u)$$

$$\times (\partial_{x_2} \Phi \partial_{y_0} u + \partial_{x_1} \Phi \partial_{x_2} \Phi(\partial_{x_1} \mathcal{W} \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1))$$

$$+ ((-c^2 + |\partial_{x_2} \Phi|^2)((\partial_{x_3} p \partial_{y_2} \mathcal{W} - 1) \partial_{y_2} u + 1)((\partial_{x_3} p \partial_{y_2} \mathcal{W} - 1) \partial_{y_2} u + 1) + \partial_{x_2} \mathcal{W} \partial_{y_3} u)$$

$$- ((\partial_{x_3} p \partial_{y_2} \mathcal{W} - 1) \partial_{y_3} u + \partial_{x_2} \mathcal{W} \partial_{y_3} u) \partial_{x_2} \Phi \partial_{x_3} \Phi(\partial_{x_3} p \partial_{y_2} u + \partial_{y_3} u)$$

$$- (\partial_{x_3} p \partial_{y_2} u + \partial_{y_3} u)((-\partial_{x_3} \Phi \partial_{y_0} u + \partial_{x_3} \Phi \partial_{x_1} \Phi(\partial_{x_1} \mathcal{W} \partial_{y_3} u - \partial_{x_1} p \partial_{y_2} u + 1))$$
\(- (\partial_{x_3} p \partial_{y_2} u + \partial_{y_3} u) \partial_{x_3} \Phi (\Phi x_3 (p \partial_{x_2} W - 1) \partial_{y_2} u + \partial_{x_2} W \partial_{y_3} u) + (\partial_{x_3} p \partial_{y_2} u + \partial_{y_3} u)^2 (\partial_{x_3} \Phi)^2 \).

(A.6)

\[\bar{a}_{33} = \partial_{x_3} p (\partial_{y_1} u)^2 (\partial_{x_2} p + 1) \partial_{x_1} \Phi \partial_{x_3} \Phi \partial_{x_3} p + (\partial_{x_2} p + 1)(\partial_{y_1} u)^2 (\partial_{x_3} \Phi \partial_{x_3} \Phi \partial_{x_3} p + (-c^2 + |\partial_{x_3} \Phi|^2)(\partial_{x_3} p + 1) + \partial_{x_2} \Phi \partial_{x_3} \Phi \partial_{x_3} p) + \partial_{x_3} p (\partial_{y_1} u)^2 (\partial_{x_3} \Phi \partial_{x_3} \Phi \partial_{x_3} p + \partial_{x_3} \Phi \partial_{x_3} \Phi \partial_{x_3} p + 1) + (-c^2 + |\partial_{x_3} \Phi|^2) \partial_{x_3} p).\]

(A.7)

A.2 Second-Order Derivatives of \(\kappa\)

The second-order derivatives of \(\kappa\) can be computed via chain rule on the basis of the first-order derivatives.

\[\partial_{\bar{u}_i} \kappa = \partial_{x_1} (\partial_{\bar{u}} \kappa) \frac{\partial x_1}{\partial u} \partial_{\bar{u}} \kappa + \partial_{x_2} (\partial_{\bar{u}} \kappa) \frac{\partial x_2}{\partial u} \partial_{\bar{u}} \kappa\]

(A.8)

\[\kappa_{\bar{u}_i \bar{u}_j} = \partial_{x_1} (\partial_{\bar{u}} \kappa) \frac{\partial x_1}{\partial y_2} + \partial_{x_2} (\partial_{\bar{u}} \kappa) \frac{\partial x_2}{\partial y_2}\]

(A.9)

\[\kappa_{\bar{u}_i \bar{u}_j} = \partial_{x_2} N (1 + y_3 (\partial_{x_1} N) + \partial_{x_3} N) + (\partial_{x_1} N + \partial_{x_3} N)(1 + y_3 \partial_{x_2} N) \]

\[(1 + y_3 (\partial_{x_1} N + \partial_{x_2} N))^2\]

\[+ \partial_{x_1} (\partial_{\bar{u}} \kappa) \frac{\partial x_1}{\partial u} + \partial_{x_2} (\partial_{\bar{u}} \kappa) \frac{\partial x_2}{\partial u}\]

(A.10)

\[\partial_{x_{i} y_{j}} \kappa = \partial_{x_1} (\partial_{x_2} \kappa) \frac{\partial x_1}{\partial y_2} + \partial_{x_2} (\partial_{x_2} \kappa) \frac{\partial x_2}{\partial y_2}\]

(A.11)

\[\partial_{y_{i} y_{j}} \kappa = \partial_{x_1} \partial_{y_3} \kappa \frac{\partial x_1}{\partial y_3} + \partial_{x_2} \partial_{y_3} \kappa \frac{\partial x_2}{\partial y_3}\]

(A.12)

\[\partial_{y_{i} y_{j}} \kappa = \partial_{x_1} \partial_{y_3} \kappa \frac{\partial x_1}{\partial y_3} + \partial_{x_2} \partial_{y_3} \kappa \frac{\partial x_2}{\partial y_3} + \frac{N (\partial_{x_1} N + \partial_{x_2} N)}{(1 + y_3 (\partial_{x_1} N + \partial_{x_2} N))^2},\]

(A.13)

where

\[\frac{\partial x_1}{\partial u} = 1\]

(A.14)

\[\frac{\partial x_2}{\partial u} = -\frac{y_3 \partial_{x_1} N}{1 + y_3 \partial_{x_2} N}.\]

(A.15)

By calculating the inverse of \(J\), one can derive \(\frac{\partial x_1}{\partial y_i}\) \((i = 2, 3)\) and \(\frac{\partial x_2}{\partial y_i}\) \((i = 2, 3)\). For example, one has

\[\frac{\partial x_2}{\partial y_2} = \frac{1}{1 + y_3 \partial_{x_2} N}\]

(A.16)

\[\frac{\partial x_2}{\partial y_3} = -\frac{N}{1 + y_3 \partial_{x_2} N}.\]

(A.17)

And \(\partial_{\bar{u}} \kappa\), \(\partial_{x_2} \kappa\) and \(\partial_{x_3} \kappa\) are given in (4.2).
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