Stable and metastable states in a superconducting "eight" loop in applied magnetic field

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The stable and metastable states of different configurations of a loop in the form of an eight is studied in the presence of a magnetic field. We find that for certain configurations the current is equal to zero for any value of the magnetic field leading to a magnetic field independent superconducting state. The state with fixed phase circulation becomes unstable when the momentum of the superconducting electrons reaches a critical value. At this moment the kinetic energy of the superconducting condensate becomes of the same order as the potential energy of the Cooper pairs and it leads to an instability. Numerical analysis of the time-dependent Ginzburg-Landau equations shows that the absolute value of the order parameter changes gradually at the transition from a state with one phase circulation to another although the vorticity change occurs abruptly.

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Starting with the pioneering work of Little and Parks, who studied the properties of double-connected superconductors (confined samples with two surfaces, for example rings) in applied magnetic field there has been continued theoretical and experimental interest in those systems. Changing the topology of the system is expected to have a strong influence on the superconducting properties, in particular in the presence of an external magnetic field. For instance in Ref. 2 the Möbius loop was considered and it was obtained that the phase diagram of this system differs considerably from an ordinary loop.

In the present paper we investigate a different topology, a loop having an "eight" geometry. The latter can be considered as a combination of two strongly interacting loops (see Fig.1). In comparison with two unconnected loops the direction and value of the current density in one loop of the "eight" system will depend on the direction and value of the current in the other loop. As a result, the free energy of such a system will not be merely the sum of the free energy of two unconnected loops. Moreover, we found that for certain values of the relative sizes of the two loops the current density in such a loop will be equal to zero at any value of the applied magnetic field.

In this work we study the stable and metastable states of a loop in the shape of an "eight". This geometry can be obtained by grapping two radial sides of a single loop and twisting it over 180 degrees with respect to each other. We assume that there is no physical contact at the crossing point of the "eight". We can further distinguish three configurations for this system as shown in Fig. 2. The "eight" loop was modelled as a combination of two rings with radius $R_1$ and $R_2$. We use the Ginzburg-Landau (GL) theory in order to study the structure of the superconducting state of such a system. We neglect the finite width $w$ and thickness $d$ of the loop. This is allowed when the radius of the loop is much larger than $w$ and $d << \lambda$, where $\lambda$ is the London penetration length. The last condition also allows us to neglect the screening effects (self-inductance of the system). Within these simplifications the GL equations are one-dimensional and using the gauge-invariant momentum $p$ we can write it in the following form

$$\frac{d^2 f}{ds^2} + f(1 - f^2 - p^2) = 0,$$

$$j = f^2 p.$$ (1b)

where $\psi = f(s)e^{i\phi(s)}$ is a undimensional order parameter, the momentum $p = \nabla \phi - A$ is scaled in units $\Phi_0/(2\pi \xi)$ (where $\Phi_0$ is the quantum of magnetic flux), the length of the loop is $L = 2\pi(R_1 + R_2)$ and the arc coordinate $s$ is in units of the coherence length $\xi(T)$. In these units the magnetic field is scaled in the unit $H_{c2}$ and the current density $j$ in $j_0 = e\Phi_0/8\pi^2\lambda^2\xi$.

At first we find the dependence of the momentum of the "eight" loop on the applied magnetic field. From equation $\text{div}j = 0$ it follows that the supercurrent $j = f^2 p$ is constant all over the system. Our numerical analysis of Eqs. (1a,b) shows that in the stationary state, $F$ does not depend on $s$. As a result $p = \text{const}$ along the loop and the order parameter is equal to $f = \sqrt{1 - p^2}$.

Let us first consider the circulation of the momentum.
It is that there is no oscillations in the phase diagram to very high magnetic fields. The other consequence of applied magnetic field in system (which in Fig. 2 are shown slightly separated from this property that for rings with radius $R_1$ where $\Phi = (n - (\Phi_1 + \Phi_2))$ for system $(A)$ the value for $p$ is a factor $(R_1/(R_1 + R_2))$ less as compared to the single loop case.

Using Eqs. (3,4) we can find the stable states of our systems $(A - C)$. This is determined by the global minimum of the Gibbs free energy

$$G[f] = \oint \left( \left( \frac{df}{ds} \right)^2 - f^2 + \frac{f^4}{2} + f^2p^2 \right) ds,$$

resulting in a minimum value of $p$ at given $H$. The dependence of $p(H)$ is a periodic function of $H$ with period $2/(R_1^2 - R_2^2)$, $2/R_1^2$ and $2/(R_1^2 + R_2^2)$ for systems $(A)$, $(B)$, $(C)$, respectively. Note that the maximum value of $p_{\text{max}}$ for the thermodynamically stable state of our systems $(A - C)$ is the same: $1/(R_1 + R_2)$.

**FIG. 2:** Different 1D model configurations for the "eight" loop.

$p$ for system $(A)$ of Fig. 2

$$\oint P \cdot ds = 2\pi(R_1 + R_2)p = 2\pi n - \int_1^2 A \cdot ds - \int_3^4 A \cdot ds,$$

where the points 1,2,3,4, are at the twist point of the loop which in Fig. 2 are shown slightly separated from this point for clarity. As a result we find

$$p = \frac{1}{R_1 + R_2} (n - (\Phi_1 + \Phi_2)),$$  (3)

where $\Phi_{1,2} = HR_{1,2}^2/2$ is the magnetic flux through the rings with radius $R_1$ and $R_2$, respectively (in $\Phi_0$ units), $n = \oint \nabla\phi ds/2\pi$ is an integer number defining the circulation (or vorticity) of the phase of the order parameter in the ring. From Eq. (3) follows the interesting property that for $R_1 = R_2$ the momentum and hence the current density will be equal to zero for any value of the applied magnetic field in system $(A)$. Thus this system will have no response to an applied magnetic field and consequently superconductivity should be conserved up to very high magnetic fields. The other consequence of it is that there is no oscillations in the phase diagram $H - T$ of such a system.

It is easy to show that for the systems $(B,C)$ the dependence $p(H)$ are, respectively, given by

$$p = \frac{1}{R_1 + R_2} (n - \Phi_1),$$  (4a)

$$p = \frac{1}{R_1 + R_2} (n - (\Phi_1 + \Phi_2)).$$  (4b)

In the limit $R_2 \rightarrow 0$ we obtain from Eqs. (3,4) the well known result

$$p = \frac{1}{R_1} (n - \Phi_1),$$  (5)

which is valid for a single double connected loop (see for example Ref.[6]). It is interesting to note that for system $(B)$ the value for $p$ is a factor $(R_1/(R_1 + R_2))$ less as compared to the single loop case.

The two rings in our "eight" loop are strongly coupled and as a result the Gibbs free energy of this system is not simply the sum of the energies of two uncoupled rings plus a small interaction term as for the case of two magnetically coupled rings. In Fig. 3 the dependence of the equilibrium energy of the "eight" system (case $(A)$ - curve 4) and two separate rings with different
radii (curves 1, 3) are presented. In the same figure the sum of the Gibbs free energies of the two uncoupled rings is shown (curve 2). The dependence $G(H)$ of the "eight" loop is more similar to the dependency of $G(H)$ for the separate rings then with the sum of the two energies.

Analysis shows that the state of the system for a specific value of the phase circulation $n$ may be stable (more exactly metastable) for $|p| > p_{max}$. This state becomes unstable when the second variation of the Gibbs free energy (6) becomes equal to zero. Let's consider small deviations $f_e \ll f$, $p_e \ll p$ (where the full solution are $\bar{f} = f + f_e$ and $\bar{p} = p + p_e$) from the stable solutions of our systems. The Ginzburg-Landau equations linearized with respect to the small perturbations $f$ and $p$ are given by

$$\frac{d^2 f_e}{ds^2} + f_e(1 - 3f^2 - p^2) - 2fpf_e = 0, \quad (7a)$$

$$f^2p_e + 2fpf_e = C. \quad (7b)$$

The constant $C$ in Eq. (7a) has to be equal to zero because otherwise $p_e \neq 0$ and $f_e \neq 0$ for any $f$ and $p$. Using Eq. (7b) we solve for $p_e$ and substitute it in Eq. (7a) in order to obtain the following equation

$$\frac{d^2 f_e}{ds^2} + f_e(6p^2 - 2) = 0, \quad (8)$$

which has the following solution

$$f_e(s) = A\cos(\omega s) + B\sin(\omega s), \quad (9)$$

with $\omega = \sqrt{6p^2 - 2}$. Taking into account the boundary condition $f_e(0) = f_e(L)$ we find that a state with fixed $n$ becomes unstable when the following condition is fulfilled

$$p = p_c = \frac{1}{\sqrt{3}} \sqrt{1 + \frac{1}{2(R_1 + R_2)^2}}. \quad (10)$$

Inserting $R_2 = 0$ in Eq. (10) we obtain the stability condition of a single loop as was obtained in Ref.[1] (expressed in $\Phi$, $n$ and $\xi/R_1$). In the case of large (i.e. min[$R_1, R_2$] $\gg \xi$) loops $p_c = 1/\sqrt{3}$ and the current density is equal to $j = p_c(1 - p_e^2) = j_c = 2/3\sqrt{3}$ - which is the depairing current density. Thus Eq. (10) has a simple physical meaning: when the kinetic energy of the superconducting condensate (or Cooper pair) becomes of the order of the potential energy the state becomes unstable. The finite length of the loop starts to play an essential role when $L$ is about the Cooper pair size which results in different values for $p_c$.

Using Eq. (10) it is easy to find the critical values of the magnetic field $H_{e,n}$ at which the transition from state $n$ to state $n + 1$ occurs. Consider for example system (A) (see Fig. 2), and combine Eq. (10) with Eq. (3) we obtain

$$H_{e,n} = \frac{2}{R_1 - R_2} \left[ \frac{1}{\sqrt{3}} \sqrt{1 + \frac{1}{2(R_1 + R_2)^2}} - \frac{n}{R_1 + R_2} \right], \quad (11)$$

It is interesting that for the configurations (B) and (C) the critical fields will be different but $p_c$ (and $p_{max}$) will be the same under the condition that the length of the loops are the same. Moreover for loops with $L/\xi \gg 1$ the value of the critical momentum does not depend on the size of the system. Therefore, we may conclude that condition (10) is a universal condition for one-dimensional double-connected systems of the type shown in Fig. 2. Even if the loops have a different shape, for example ellipsoid, the condition for vortex entry will be described by Eq. (10) (with the change of $(R_1 + R_2)$ by $L/2\pi$) because the order parameter in such a sample will also be uniform along the loop. Indeed, a loop of arbitrary shape with length $L$ is equivalent to a one-dimensional wire of length $L$ with periodical boundary conditions. The presence of a magnetic field leads to a nonuniform (uniform for a single circular ring) distribution of the vector potential along the wire which is compensated by the term $\nabla \phi$ in such a way that the current density and the order parameter are uniform along the wire.

We checked Eq. (10) through a numerical solution of the time-dependent one-dimensional Ginzburg-Landau
the order parameter equals zero at one point on the ring (for
from 2
of order parameter (b), the superconducting (c) and normal
\( j_s \) (d)
\( \phi_R \) increases in the point (for our parameters \( t_0 \simeq 512.98\tau \)).

\begin{equation}
- \gamma \left( \frac{\partial \psi}{\partial t} + i \varphi \psi \right) = (-i \nabla - A)^2 \psi + \psi(|\psi|^2 - 1), \quad (12a)
\end{equation}
\begin{equation}
\frac{d\varphi}{ds} - \text{Re}(\psi^* (-i \nabla - A)\psi) = 0. \quad (12b)
\end{equation}

Here time is scaled in units of \( \tau = 4\pi \sigma_n \lambda^2(T)/c^2 \), the
electrostatic potential \( \varphi \) in units of \( c\Phi_0/8\pi^2 \lambda \alpha_n \) (\( \sigma_n \)
is the normal-state conductivity), and \( \gamma \) is a relaxation constant in (12).

In Fig. 4 the dependency of \( p \) (Fig. 4(a)) and the
current density (Fig. 4(b)) on the applied magnetic field
are shown for the three systems (we took \( R_1 = 2\xi, R_2 = 1\xi \)). The magnetic field was increased linearly from \( H = 0 \) to \( 2H_{c2} \) during a time period of \( \Delta t = 2 \cdot 10^4 \tau \). The
theoretical value of \( p_c \) for such a system is \( p_c = 0.593 \).
From our numerical calculation we obtained \( p_c \simeq 0.596 \).
The difference between the numerical result and Eq. (10)
is in the range of our numerical error.

From our numerical solution of the time-dependent
Ginzburg-Landau equations we find the evolution of the
system from a state with phase circulation \( 2\pi n \) to a state
with \( 2\pi(n + 1) \). In Fig. 5 the dependence of \(|\psi|, j_s \text{ and } j_n \)
are shown for the system (A) with parameters \( R_1 = \xi, R_2 = 0.5\xi \). The magnetic field was increased
gradually to \( 2.51H_{c2} \) and then in one step to \( 2.52H_{c2} \). At
time \( t = 0 \) the "eight" loop was at a metastable state at \( H = 2.51H_{c2} \). At \( t = t_0 \simeq 513\tau \) (curve 4 in
Fig. 5) the order parameter reaches zero at a certain
point on the ring. Simultaneously the phase difference
\( \Delta \phi = \phi(+0) - \phi(-0) \) near the zero point becomes equal
to \(-\pi \). In Refs. \( 4, 7 \) the vanishing of the order parameter
in one point on a single ring was considered and it was
found that when such a solution exists the superconducting
current density is equal to zero and \( \Delta \phi = \pm \pi \) (sign
depends on the current direction). Although in our time-
dependent problem the full current is not equal to zero,
nevertheless there is a similarity between the stationary
and the non-stationary problems - in both cases when
the order parameter reaches zero there is a phase shift of
\( \Delta \phi = \pm \pi \) near the point where \( \psi = 0 \).

For \( t > t_0 \) the superconducting current (see Fig. 5(c))
changes sign near the minimum point and hence it leads
to a phase change of \( \Delta \phi = 2\pi \) (from \(-\pi \) to \(+\pi \)) according to \( 4 \) (see also Fig. 5(b)). As a result a phase circulation of \( \oint \nabla \phi ds = 2\pi \) appears in the ring (see Fig. 5(b)). Then
the order parameter increases in the point \( \psi_{\text{min}} \) and \( \Delta \phi \)
gradually decreases to 0 (see Fig. 5(b)). This is also in
qualitative agreement with the results of Refs. \( 4, 7 \), where it was found that for increasing \( \psi_{\text{min}} \) (or \( |j_n| \)) \( \Delta \phi \)
decreased to zero near the minimum point.

Because \( \nabla \phi \) is a continuous function everywhere in the
loop, the current density (superconducting \( j_s \) and normal
\( j_n \)) will also change continuously. The time evolution of
\( j_s \) and \( j_n \) are shown in Fig. 5(c) and Fig. 5(d), respectivley.
The currents are a function of the arc-coordinate during the transition process and only the sum \( j_s + j_n \)
does not depend on \( s \). For large times the normal current density decays and the order parameter becomes uniform in the system.

In conclusion, the stable and metastable states of three different configurations for the "eight" loop in a magnetic field were studied. We found that the state with fixed phase circulation becomes unstable when the value of the momentum of the superconducting electrons reaches a critical \( p_c \). Then the kinetic energy of the superconducting condensate becomes of the same order as the potential energy of the Cooper pairs resulting in an instability. Numerical analysis of the time-dependent Ginzburg-Landau equations shows that the absolute value of the order parameter changes gradually at the transition from a state with one phase circulation to another although the vorticity changes abruptly.

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