ANALOGICAL PROPORTIONS IN MONOUNARY ALGEBRAS

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ABSTRACT. This paper studies analogical proportions in monounary algebras consisting only of a universe and a single unary function. We show that the analogical proportion relation is characterized in the infinite monounary algebra formed by the natural numbers together with the successor function via difference proportions.

1. INTRODUCTION

Analogical proportions are expressions of the form “a is to b what c is to d” written \( a : b :: c : d \) at the core of analogical reasoning which itself is at the core of artificial general intelligence (e.g. Boden, 1998; Gentner, 1983; Gust et al., 2008; Hesse, 1966; Hofstadter, 2001; Hofstadter & Sander, 2013; Krieger, 2003; Pólya, 1954; Prade & Richard, 2021; Winston, 1980).

The purpose of this paper is to study Antić’s (2022) abstract algebraic framework of analogical proportions — recently introduced in the general setting of universal algebra — in monounary algebras containing only a single unary function.

The motivation for studying proportions in that specific context is that monounary algebras are simple enough to provide a convenient context to analyze interesting concepts like congruences in combination with proportions, and rich enough to yield interesting novel insights.

The rest of the paper is structured as follows.

In §3, we repeat that every monounary algebra satisfies — as an instance of the general framework — for all elements of the domain (cf. Theorem 4):

- p-symmetry: \( a : b :: c : d \iff c : d :: a : b \),
- inner p-symmetry: \( a : b :: c : d \iff b : a :: d : c \),
- inner p-reflexivity: \( a : a :: c : c \),
- p-reflexivity: \( a : b :: a : b \),
- p-determinism: \( a : a :: a : d \iff d = a \).

To the contrary, we will provide counterexamples showing that there are monounary algebras and elements such that:

- central permutation: \( a : b :: c : d \iff a : c :: b : d \),
- strong inner p-reflexivity: \( a : a :: c : d \Rightarrow d = c \),
- strong p-reflexivity: \( a : b :: a : d \Rightarrow d = a \),
- p-commutativity: \( a : b \not\equiv b : a \),
- p-transitivity: \( a : b :: c : d \) and \( c : d :: e : f \Rightarrow a : b :: e : f \),
- inner p-transitivity: \( a : b :: c : d \) and \( b : e :: d : f \Rightarrow a : e :: c : f \).
This is in line with what we have observed in the general context of arbitrary algebras in Theorem 28 in Antić (2022).

Moreover, in §3 we shall prove that analogical proportions and congruences are in general not compatible in the following sense:

In each of the following cases, there is a monounary algebra \( \mathcal{A} = (A, \cdot) \), and elements \( a, b, c, d \in A \) such that:

- \( a : b :: c : d \) whereas \( a/\theta : b/\theta \nmid c/\theta : d/\theta \) (Theorem 7).
- \( a/\theta : b/\theta :: c/\theta : d/\theta \) whereas \( a : b \nmid c : d \) (Theorem 8).
- \( a : b \nmid a/\theta : b/\theta \) (Theorem 9).
- \( a/\theta b \) and \( c/\theta d \) whereas \( a : b \nmid c : d \) (Theorem 10).

In §4 on the other hand, we shall prove a positive result showing that in the monounary algebra 

\[
\mathcal{A} = (\mathbb{N}, S) \text{ consisting of the natural numbers together with the unary successor function, we obtain the Difference Proportion Theorem saying that within the abstract framework we have}
\]

\[
a : b :: c : d \text{ holds in } (\mathbb{N}, S) \iff a - b = c - d.
\]

This is more surprising than it might appear, as the abstract framework is not explicitly geared towards proportions in monounary algebras.

In a broader sense, this paper is a further step towards a mathematical theory of analogical proportions and analogical reasoning in general.

2. Analogical proportions in monounary algebras

In this section, we interpret the abstract algebraic framework of analogical proportions in Antić (2022) within monounary algebras of the form \( \mathcal{A} = (A, \cdot) \), where \( A \) is a universe and \( \cdot : A \to A \) is a unary function.

We shall now recall the abstract algebraic framework of analogical proportions in Antić (2022) where we restrict ourselves to monounary algebras of the form \( \mathcal{A} = (A, \cdot) \), where \( A \) is a set and \( \cdot : A \to A \) is a unary function on \( A \) (we can imagine \( \cdot \) to be a generalized “successor” function). Terms in \( \mathcal{A} \) have the form \( S^kz \), for \( k \geq 0 \).

In what follows, let \( \mathcal{A} = (A, S_A) \) and \( \mathcal{B} = (B, S_B) \) be two monounary algebras — we will always omit the subscripts from notation and simply write \( \cdot \) instead of \( S_A \) and \( S_B \).

We define the analogical proportion entailment relation in two steps:

1. Define the set of justifications of an arrow \( a \to b \) in \( \mathcal{A} \) by

\[
\uparrow_{\mathcal{A}} (a \to b) := \left\{ S^kz \to S^\ell z \mid a \to b = S^k o \to S^\ell o, \text{ for some } o \in A \right\},
\]

extended to an arrow proportion \( a \to b :: c \to d \) in \( (\mathcal{A}, \mathcal{B}) \) by

\[
\uparrow_{(\mathcal{A}, \mathcal{B})} (a \to b :: c \to d) := \uparrow_{\mathcal{A}} (a \to b) \cap \uparrow_{\mathcal{B}} (c \to d).
\]

A justification is trivial in \( (\mathcal{A}, \mathcal{B}) \) iff it justifies every arrow proportion in \( \mathcal{A} \), and we say that \( J \) is a trivial set of justifications in \( (\mathcal{A}, \mathcal{B}) \) iff every justification in \( J \) is trivial.

Now we say that \( a \to b :: c \to d \) holds in \( (\mathcal{A}, \mathcal{B}) \) — in symbols,

\[
(\mathcal{A}, \mathcal{B}) \models a \to b :: c \to d
\]

iff

---

\(^1\)This observation was the motivation for introducing proportional congruences in Antić (2023b).

\(^2\)We use here the updated notation of Antić (2023a) where we write \( \uparrow \) instead of \( \text{Jus} \) for sets of justifications; see Antić (2023a) for motivation.

\(^3\)Read as “\( a \) transforms into \( b \) as \( c \) transforms into \( d \)”.\n
(a) either $\Uparrow A (a \to b) \cup \Uparrow B (c \to d)$ consists only of trivial justifications, in which case there is neither a non-trivial relation from $a$ to $b$ in $A$ nor from $c$ to $d$ in $B$; or

(b) $\Uparrow (A, B) (a \to b : c \to d)$ is maximal with respect to subset inclusion among the sets $\Uparrow (A, B) (a \to b : c \to d')$, $d' \in B$, containing at least one non-trivial justification, that is, for any element $d' \in B$

$$\emptyset \subseteq \Uparrow (A, B) (a \to b : c \to d) \subseteq \Uparrow (A, B) (a \to b : c \to d')$$

implies

$$\emptyset \subseteq \Uparrow (A, B) (a \to b : c \to d') \subseteq \Uparrow (A, B) (a \to b : c \to d).$$

We abbreviate the above definition by simply saying that $\Uparrow (A, B) (a \to b : c \to d)$ is $d$-maximal.

(2) Finally, the analogical proportion entailment relation is most succinctly defined by

$$a : b :: c : d \iff a \to b : c \to d \quad \text{and} \quad b \to a : d \to c$$

with

$$c \to d : a \to b \quad \text{and} \quad d \to c : b \to a.$$ 

This means that in order to prove $(A, B) \models a : b :: c : d$, we need to check the first two relations in the first line with respect to $|$ in $(A, B)$, and the last two relations in the same line in $(B, A)$.

We will always write $\mathcal{A}$ instead of $(\mathcal{A}, \mathcal{A})$ and we often omit the explicit reference to the underlying algebras.

Let $\mathbb{N} := \{n, n+1, n+2, \ldots\}$. Given some justification $S^k \to S^\ell \in \Uparrow (A \to b)$, $k, \ell \geq 0$, we can depict the two cases $k \leq \ell$ and $\ell \leq k$ as follows:

$$\begin{align*}
\text{\ include &} \quad \text{\includegraphics[width=0.5\textwidth]{example1}} \\
\text{\ include &} \quad \text{\includegraphics[width=0.5\textwidth]{example2}}
\end{align*}$$

This is an abstract version of the situation in $(\mathbb{N}, S)$ where, for example, $S^2 \to S^2 \in \Uparrow (1 \to 2)$ and $S^2 \to S^2 \in \Uparrow (2 \to 1)$ both have the following pictorial representation:

$$\begin{align*}
2 = S^2, & \quad S \quad 2 = S^2 \\
1 = S^0, & \quad S \quad 1 = S^0 \\
0 & \quad 0
\end{align*}$$

Example 1. Consider the monounary algebra

\footnote{We ignore trivial justifications and write “$\emptyset \subseteq \ldots$” instead of “[trivial justifications] $\subseteq \ldots$” et cetera.}
We expect $a : b :: c : d$ to fail as it has no non-trivial justification. In fact,

$$\uparrow (a \to b) \cup \uparrow (c \to d) = \{ z \to S^\ell z \mid \ell \geq 1 \} \neq \emptyset$$

and

$$\uparrow (a \to b : c \to d) = \emptyset$$

show

$$a : b \not:: c : d.$$  

**Example 2.** Consider the monounary algebra

The relation between 1 and 2 is a “loop”, whereas the relation between 3 and 4 is not as there is no edge from 4 back to 3; instead, there is a loop at 4. We therefore expect the following relations:

(1) $1 : 2 \not:: 3 : 4$.

(2) $1 : 2 :: 4 : 4$.

To prove the first item, we shall show

(3) $2 \to 1 : 4 \to 3$.

For this, compute

$$\uparrow (2 \to 1) = \{ S^k z \to S^\ell z \mid 2 \to 1 = S^k a \to S^\ell a, \text{ for some } a, k, \ell \in \mathbb{N} \}$$

$$= \{ S^k z \to S^\ell z \mid (k \text{ is even and } \ell \text{ is odd}) \text{ or } (k \text{ is odd and } \ell \text{ is even}) \}$$

and

$$\uparrow (4 \to 3) = \{ S^k z \to z \mid k \geq 1 \} \subseteq \{ S^k z \to S^\ell z \mid k, \ell \in \mathbb{N} \} = \uparrow (4 \to 4).$$

We thus have

$$\uparrow (2 \to 1 : 4 \to 4) = \uparrow (2 \to 1) \cap \uparrow (4 \to 4) = \uparrow (2 \to 1)$$
whereas
\[ \uparrow (2 \to 1 : 4 \to 3) = \uparrow (2 \to 1) \cap \uparrow (4 \to 3) = \{ S^k z \to z \mid k \text{ is odd} \} \subseteq \uparrow (2 \to 1 : 4 \to 4). \]

This shows (3) and therefore (1).

To prove (2), we need to show the following relations:

(4) \[ 1 \to 2 : 4 \to 4, \quad 2 \to 1 : 4 \to 4, \quad 4 \to 4 : 1 \to 2, \quad 4 \to 4 : 2 \to 1. \]

The first two relations are immediate consequences of the following observation:
\[ \emptyset \subsetneq \uparrow (a \to b : 4 \to 4) = \uparrow (a \to b), \quad \text{for all } a, b \in \{1, 2\}. \]

Now
\[ \uparrow (4 \to 4 : a \to b) = \uparrow (a \to b), \quad \text{for all } a, b \in \{1, 2\}, \]
shows that
\[ \uparrow (4 \to 4 : 1 \to 2) = \uparrow (1 \to 2) \quad \text{and} \quad \uparrow (4 \to 4 : 2 \to 1) = \uparrow (2 \to 1) \]
are both non-empty and maximal with respect to the last argument. This shows (4) and thus (2).

Define the monounary algebra \((\{0, 1\}, S)\) by \(S 0 := 1\) and \(S 1 := 0\). We can imagine 0 and 1 to be boolean truth values and \(S\) to be negation. On the other hand, in \((\mathbb{N}, S)\) let \(S a := a + 1\) be the unary successor function on the natural numbers \(\mathbb{N} = \{0, 1, 2, \ldots\}\) starting at 0. We can depict the algebras as:

![Diagram of monounary algebras]

The next result shows how we can capture evenness and oddness via analogical proportions between the two algebras.
Theorem 3.

\((\mathbb{N}, S), ([0, 1], S) \models a : b :: c : d \iff (c = d \text{ and } a - b \equiv 0 \mod 2) \text{ or } (c \neq d \text{ and } a - b \equiv 1 \mod 2)\).

Proof. We have

\[\uparrow_{(\mathbb{N}, S)} (a \rightarrow b) = \begin{cases} S^k z \rightarrow S^{k+b-a} z \quad 0 \leq k \leq a & \text{if } a \leq b \\ S^k z \rightarrow S^{k+b-a} z \quad k \text{ and } k + b - a \text{ are both even or odd} & \text{if } a \leq b \text{ and } c = d, \\ S^k z \rightarrow S^{k+b-a} z \quad k \text{ even and } k + b - a \text{ odd or vice versa} & \text{if } a \leq b \text{ and } c \neq d, \\ S^k z \rightarrow S^{k+b-a} z \quad a - b \leq k \leq a & \text{if } a > b \text{ and } c = d, \\ S^k z \rightarrow S^{k+b-a} z \quad k \text{ and } k + b - a \text{ are both even or odd} & \text{if } a > b \text{ and } c \neq d, \\ S^k z \rightarrow S^{k+b-a} z \quad k \text{ even and } k + b - a \text{ odd or vice versa} & \text{if } a > b \text{ and } c \neq d. \end{cases}\]

Moreover, we have

\[\uparrow_{([0, 1], S)} (c \rightarrow d) = \begin{cases} \{S^k z \rightarrow S^\ell z \mid (k, \ell \text{ even}) \text{ or } (k, \ell \text{ odd})\} & \text{if } c = d, \\ \{S^k z \rightarrow S^\ell z \mid (k \text{ even and } \ell \text{ odd}) \text{ or } (k \text{ odd and } \ell \text{ even})\} & \text{if } c \neq d. \end{cases}\]

Hence,

\[\uparrow_{(\mathbb{N}, S), ([0, 1], S)}(a \rightarrow b : c \rightarrow d) = \uparrow_{(\mathbb{N}, S)} (a \rightarrow b) \cap \uparrow_{([0, 1], S)} (c \rightarrow d)\]

From elementary algebra we know that

\[
\text{even + even = even and even + odd = odd and odd + odd = even.}
\]

Therefore,

\[
k \text{ is even } \Rightarrow [k + b - a \text{ is even } \iff b - a \text{ is even}],
\]

\[
k \text{ is odd } \Rightarrow [k + b - a \text{ is odd } \iff b - a \text{ is even}],
\]

\[
k \text{ is even } \Rightarrow [k + b - a \text{ is odd } \iff b - a \text{ is odd}],
\]

\[
k \text{ is odd } \Rightarrow [k + b - a \text{ is even } \iff b - a \text{ is odd}].
\]

Hence,

\[\uparrow_{(\mathbb{N}, S), ([0, 1], S)}(a \rightarrow b : c \rightarrow d) = \begin{cases} \uparrow_{([0, 1], S)} (a \rightarrow b) & \text{if } (c = d \text{ and } b - a \text{ is even)} \text{ or } (c \neq d \text{ and } b - a \text{ is odd}) \\ \emptyset & \text{otherwise}. \end{cases}\]

This proves

\[(\mathbb{N}, S), ([0, 1], S) \models a \rightarrow b :: c \rightarrow d \iff (c = d \text{ and } b - a \text{ is even}) \text{ or } (c \neq d \text{ and } b - a \text{ is odd}).\]

Since \(b - a\) is even iff \(a - b\) is and the same holds for oddness, analogous arguments prove the remaining arrow proportions and thus the theorem.

\(\square\)
3. Proportional axioms

In the tradition of the ancient Greeks, Lepage (2003) (cf. Miclet et al., 2008, pp. 796-797) introduced (in the linguistic context) a set of proportional axioms as a guideline for formal models of analogical proportions — his list has since been extended by a number of authors (e.g. Miclet & Prade, 2009; Prade & Richard, 2013; Barbot, Miclet, & Prade, 2019; Lim, Prade, & Richard, 2021; Antić, 2022):

\[
\begin{align*}
&a: b :: c: d &\iff & c: d :: a: b & (\text{p-symmetry}), \\
&a: b :: a: b & & & (\text{p-reflexivity}), \\
&a: b :: c: d &\iff & b: a :: d: c & (\text{inner p-symmetry}), \\
&a: b :: c: d &\iff & a: c :: b: d & (\text{central permutation}), \\
&a: a :: a: d &\iff & d = a & (\text{p-determinism}), \\
&a: a :: c: d &\implies & d = c & (\text{strong inner p-reflexivity}), \\
&a: a :: c: c & & & (\text{inner p-reflexivity}), \\
&a: b :: a: d &\implies & d = b & (\text{strong p-reflexivity}), \\
\end{align*}
\]

\[
\begin{align*}
&a: b :: c: d &\quad c: d :: e: f &\implies a: b :: e: f & (\text{p-transitivity}), \\
\end{align*}
\]

\[
\begin{align*}
&a: b :: c: d &\quad b: e :: d: f &\implies a: e :: c: f & (\text{inner p-transitivity}). \\
\end{align*}
\]

We have the following analysis of the proportional axioms in monounary algebras:

**Theorem 4.** The analogical proportion relation, restricted to monounary algebras, satisfies

- p-symmetry,
- inner p-symmetry,
- p-reflexivity,
- p-determinism,

and, in general, does not satisfy

- central permutation,
- strong inner p-reflexivity,
- strong p-reflexivity,
- p-commutativity,
- p-transitivity,
- inner p-transitivity.

**Proof.** We have the following proofs:

- The proof of p-symmetry, inner p-symmetry, p-reflexivity, and p-determinism is the same as the corresponding proofs of Theorem 28 in Antić (2022).
- The disproof of central permutation is the same as in Theorem 28 in Antić (2022): it fails, for example, in the monounary algebra given by (we omit the loops $So := o$, for $o \in \{b,c,d\}$, in the figure)

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Lepage (2003) uses different names for the axioms — we have decided to remain consistent with the nomenclature in Antić (2022, §4.2).
The disproof of strong inner p-reflexivity is the same as in Theorem 28 in Antić (2022): it fails, for example, in the monounary algebra

\[
\begin{array}{cc}
S & d \\
a & S \\
c & S
\end{array}
\]

Strong p-reflexivity fails, for example, in the monounary algebra

\[
\begin{array}{ccc}
S & S & S \\
a & b & d
\end{array}
\]

p-Transitivity fails, for example, in the monounary algebra

\[
\begin{array}{cccc}
S & S & S & S \\
a & b & c & d \\
e & f & e & f'
\end{array}
\]

We first prove the relations

\[(5) \quad a : b :: c : d \quad \text{and} \quad c : d :: e : f.\]

Since

\[\uparrow (a \to b) = \{s^kz \to z \mid k \geq 1\} = \uparrow (c \to d) = \uparrow (e \to f),\]

we have

\[a \to b : c \to d \quad \text{and} \quad c \to d : a \to b,\]
\[c \to d : e \to f \quad \text{and} \quad e \to f : c \to d.\]

Moreover, since

\[\uparrow (b \to a) = \{z \to S^kz \mid k \geq 1\} = \uparrow (d \to c) = \uparrow (f \to e),\]

we have

\[b \to a : d \to c \quad \text{and} \quad d \to c : b \to a,\]
\[d \to c : f \to e \quad \text{and} \quad f \to e : d \to c,\]

which thus proves \((5)\).
On the other hand,
\[ \emptyset \neq (a \to b : e \to f) = \{ S^k z \to z \mid k \geq 1 \} \]
\[ \supseteq \{ S^k z \to z \mid k \geq 1 \} \cup \{ S^k z \to S z \mid k \geq 2 \} \]
\[ = (a \to b : e \to f') \]
shows
\[ a \to b \not\vdash e \to f \]
and thus
\[ a : b \not\vdash e : f. \]

- The disproof of inner p-transitivity is the same as in Theorem 28 in Antić (2022): it fails, for example, in the monounary algebra given by (we omit the loops \( S o := o \), for \( o \in \{ b, e, c, d, f \} \), in the figure)

\[ \begin{array}{cccc}
  & e & b & d & f \\
  S & a & c
\end{array} \]

**Remark 5.** Since p-transitivity fails in general, the relation :: is in general not an equivalence relation.

**Problem 6.** Characterize those monounary algebras which satisfy p-transitivity.

4. Congruences

This section is a collection of results relating congruences to analogical proportions in monounary algebras. Recall that an equivalence relation \( \theta \) on \( \mathbb{A} = (A, S) \) is a congruence iff
\[ a \theta b \implies S a \theta S b, \text{ for all } a, b \in A. \]
The factor algebra obtained from \( \mathbb{A} \) with respect to \( \theta \) is given by
\[ \mathbb{A}/\theta := (A/\theta, S/\theta), \]
where
\[ A/\theta := \{ a/\theta \mid a \in A \} \]
contains the congruence classes
\[ a/\theta := \{ b \in A \mid a \theta b \} \]
with respect to \( \theta \), and \( S/\theta : A/\theta \to A/\theta \) is defined by
\[ S/\theta(a/\theta) := S a/\theta. \]

**Theorem 7.** There is a monounary algebra \( \mathbb{A} = (A, S) \), a congruence \( \theta \) on \( \mathbb{A} \), and elements \( a, b, c, d \in A \) such that
\[ \mathbb{A} \models a : b :: c : d \text{ whereas } \mathbb{A}/\theta \not\models a/\theta : b/\theta :: c/\theta : d/\theta. \]

**Proof.** Define the monounary algebra \( \mathbb{A} \) by
The identities
\[ \uparrow_{\mathbf{A}}(a \rightarrow b) \cup \uparrow_{\mathbf{A}}(c \rightarrow d) = \emptyset \quad \text{and} \quad \uparrow_{\mathbf{A}}(b \rightarrow a) \cup \uparrow_{\mathbf{A}}(d \rightarrow c) = \emptyset \]

imply \( \mathbf{A} \models a \cdot b :: c :: d \).

Now define the congruence \( \theta = \{\{a, a'\}, \{b, b'\}, \{c\}, \{d\}\} \) yielding the factor algebra \( \mathbf{A}/\theta \) given by

\[ \{b, b'\} = b/\theta \quad \{d\} = d/\theta \]

\[ \{a, a'\} = a/\theta \quad \{c\} = c/\theta. \]

Now
\[ \uparrow_{\mathbf{A}/\theta}(a/\theta \rightarrow b/\theta) \cup \uparrow_{\mathbf{A}/\theta}(c/\theta \rightarrow d/\theta) = \{z \rightarrow S z, \ldots\} \neq \emptyset \]

and
\[ \uparrow_{\mathbf{A}/\theta}(a/\theta \rightarrow b/\theta \cdot c/\theta \rightarrow d/\theta) = \emptyset \]

imply
\[ \mathbf{A}/\theta \models a/\theta \cdot b/\theta :: c/\theta \cdot d/\theta. \]

\[ \square \]

**Theorem 8.** There is a monounary algebra \( \mathbf{A} = (A, S) \), a congruence \( \theta \) on \( \mathbf{A} \), and elements \( a, b, c, d \in A \) such that
\[ \mathbf{A}/\theta \models a/\theta :: b/\theta \cdot c/\theta : d/\theta \quad \text{whereas} \quad \mathbf{A} \not\models a : b :: c : d. \]

**Proof.** Define the monounary algebra \( \mathbf{A} \) by
Since
\[ \uparrow_{\mathcal{A}} (a \rightarrow b) \cup \uparrow_{\mathcal{A}} (c \rightarrow d) \neq \emptyset \quad \text{and} \quad \uparrow_{\mathcal{A}} (a \rightarrow b \cdot c \rightarrow d) = \emptyset, \]
we have
\[ \mathcal{A} \not| a \rightarrow b : c \rightarrow d \quad \text{and therefore} \quad \mathcal{A} \not| a : b :: c : d. \]

Now define the congruence \( \theta := \{(a, a'), (b, b'), (c), (d)\} \) yielding the factor algebra \( \mathcal{A}/\theta \) given by

We clearly have
\[ \mathcal{A}/\theta \models a/\theta : b/\theta :: c/\theta : d/\theta. \]

\[ \square \]

**Theorem 9.** There is a monounary algebra \( \mathcal{A} = (A, S) \), a congruence \( \theta \) on \( \mathcal{A} \), and elements \( a, b \in A \) such that
\[ (\mathcal{A}, \mathcal{A}/\theta) \not| a : b :: a/\theta : b/\theta. \]

**Proof.** Define the monounary algebra \( \mathcal{A} \) by

Now define the congruence \( \theta := \{(a, a'), (b, b')\} \) yielding the factor algebra \( \mathcal{A}/\theta \) given by
\[
\{a, a'\} = a/\theta \xrightarrow{S} \{b, b'\} = b/\theta.
\]

Since

\[\uparrow_\mathbb{N} (a \to b) \cup \uparrow_{\mathbb{N}/\theta} (a/\theta \to b/\theta) = \{z \to S z, \ldots\} \neq \emptyset\]

whereas

\[\uparrow_{\mathbb{N}} (a \to b : a/\theta \to b/\theta) = \emptyset,\]

we have

\[\mathbb{N}, \mathbb{N}/\theta \nmid a \to b : a/\theta \to b/\theta \quad \text{and therefore} \quad \mathbb{N}, \mathbb{N}/\theta \nmid a : b :: a/\theta : b/\theta.\]

\[\square\]

**Theorem 10.** There is a monounary algebra \(\mathbb{N} = (A, S)\), a congruence \(\theta\) on \(\mathbb{N}\), and elements \(a, b, c, d \in A\) such that

\[a \theta b \quad \text{and} \quad c \theta d \quad \text{whereas} \quad \mathbb{N} \nmid a : b :: c : d.\]

**Proof.** Define the monounary algebra \(\mathbb{N}\) by

\[
\begin{array}{ccc}
S & S & S \\
\downarrow & \downarrow & \downarrow \\
a & b & c \\
\text{S} & \text{S} & \text{S} \\
d
\end{array}
\]

and define the congruence \(\theta := \{(a, b), (c, d)\}\). We then have

\[a \theta b \quad \text{and} \quad c \theta d \quad \text{whereas} \quad \mathbb{N} \nmid a : b :: c : d.\]

\[\square\]

5. **Difference proportion theorem**

Arithmetic or difference proportions are characterized as

\[a : b :: c : d \iff a - b = c - d\]

and have been considered already by the ancient Greeks\(^6\). In this section, we are going to see that difference proportions naturally occur in the prototypical infinite monounary algebra given by the natural numbers \(\mathbb{N} = \{0, 1, 2, \ldots\}\) together with the unary successor function \(S : \mathbb{N} \to \mathbb{N}\) defined by \(Sa := a + 1\). This is more surprising than it might appear, as the abstract framework is not geared towards proportions in monounary algebras — the forthcoming theorem is thus further conceptual evidence of the robustness of the underlying framework:

**Theorem 11** (Difference Proportion Theorem). For any natural numbers \(a, b, c, d \in \mathbb{N}\), we have

\[(\mathbb{N}, S) \vDash a : b :: c : d \iff a - b = c - d \quad (\text{difference proportion}).\]

\(^6\)https://en.wikipedia.org/wiki/Proportion_(mathematics)
In addition to the positive part of Theorem 4, we have the following remaining proofs:

**Theorem 12.** All the proportional axioms hold in every proportional polymorphism of the axioms in Antić (2022, §4.3) within the natural numbers with successor: $\mathbb{N} = \langle \mathbb{N}, S \rangle$.

By definition, $\mathbb{N}$ is injective in $\mathbb{N}$, every justification $S^k z \rightarrow S^l z$ of $a \rightarrow b : c \rightarrow d$ in $\langle \mathbb{N}, S \rangle$ is a characteristic justification by Uniqueness Lemma 23 in Antić (2022). By definition, $S^k z \rightarrow S^l z$ is a justification of $a \rightarrow b : c \rightarrow d$ in $\langle \mathbb{N}, S \rangle$ iff, for some $o_1, o_2 \in \mathbb{N}$,

$$a = S^k o_1 \quad \text{and} \quad b = S^l o_1 \quad \text{and} \quad c = S^k o_2 \quad \text{and} \quad d = S^l o_2,$$

which is equivalent to

$$a = k + o_1 \quad \text{and} \quad b = \ell + o_1 \quad \text{and} \quad c = k + o_2 \quad \text{and} \quad d = \ell + o_2.$$

This holds iff $a - b = c - d$ as desired.

As a direct consequence of the Difference Proportion Theorem, we have the following analysis of the axioms in Antić (2022, §4.3) within the natural numbers with successor:

**Theorem 12.** All the proportional axioms hold in $\langle \mathbb{N}, S \rangle$ except for p-commutativity.

**Proof.** In addition to the positive part of Theorem 4, we have the following remaining proofs:

- p-Commutativity fails since $a - b \neq b - a$ whenever $a \neq b$.
- Central permutation follows from $a - b = c - d \Rightarrow a - c = b - d$.
- Strong inner p-reflexivity follows from $a - a = c - d \Rightarrow d = c$.
- Strong p-reflexivity follows from $a - b = a - d \Rightarrow d = b$.
- p-Transitivity follows from

$$a - b = c - d \quad \text{and} \quad c - d = e - f \quad \Rightarrow \quad a - b = e - f.$$

- Inner p-transitivity follows from

$$\begin{align*}
\frac{a - b = c - d \quad b - e = d - f}{a - b + b - e = c - d + d - f} \quad \Rightarrow \quad a - e = c - f.
\end{align*}$$

- Central p-transitivity is a direct consequence of transitivity. Explicitly, we have

$$a - b = b - c \quad \text{and} \quad b - c = c - d \quad \Rightarrow \quad a - b = c - d.$$

We call an “extern” unary function $f : A \rightarrow A$ a proportional polymorphism (or p-polymorphism) (Antić, 2023b) on the monounary algebra $\mathfrak{A} = \langle A, S \rangle$ iff, for all $a, b, c, d \in A$,

$$\mathfrak{A} \models fa : fb :: fc : fd.$$

**Fact 13.** The successor function $S$ is a p-polymorphism on $\langle \mathbb{N}, S \rangle$.

**Proof.** We have the following derivation:

$$\begin{align*}
(\mathbb{N}, S) & \models a : b :: c : d \\
\frac{a - b = c - d}{Sa - Sb = Sc - Sd} \quad \Rightarrow \quad \mathbb{N}, S) \models Sa : Sb :: Sc : Sd.
\end{align*}$$

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7 The monotonicity axiom is irrelevant here as we are interested in monounary algebras only.
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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

DATA AVAILABILITY STATEMENT

The manuscript has no data associated.

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