OPTIMAL PREVENTIVE “MAINTENANCE-FIRST OR -LAST” POLICIES WITH GENERALIZED IMPERFECT MAINTENANCE MODELS

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(Communicated by Ryan Loxton)

Abstract. This paper presents modified preventive maintenance policies for an operating system that works at random processing times and is imperfectly maintained. The system may suffer from one of the two types of failures based on a time-dependent imperfect maintenance mechanism: type-I (minor) failure, which can be rectified by minimal repair, and type-II (catastrophic) failure, which can be removed by corrective maintenance. When the system needs to be maintained, two policies “preventive maintenance-first (PMF) and preventive maintenance-last (PML)” may be applied. In each maintenance interval, before any type-II failure occurs, the system is maintained at a planned time \( T \) or at the completion of a working time, whichever occurs first and last, which are called PMF and PML, respectively. After any maintenance activity, the system improves but its failure characteristic is also altered. At the \( N \)-th maintenance, the system is replaced rather than maintained. For each policy, the optimal preventive maintenance schedule \( (T, N)∗ \) that minimizes the mean cost rate function is derived analytically and determined numerically in terms of its existence and uniqueness. The proposed models provide a general framework for analyzing the maintenance policies in reliability theory.

1. Introduction. Most systems deteriorate with age and use, and eventually, fail in random environment. Maintenance actions are performed to keep the system in a working condition, reducing maintenance cost and the risk of failures during the operation. Maintenance procedures carried out can be either corrective maintenance (CM) or preventive maintenance (PM) [15]. CM is an unplanned action that is carried out after a failure and may require more time and higher cost; hence, we need to perform planned PM to reduce maintenance cost and prevent sudden failures.

2020 Mathematics Subject Classification. Primary: 60K05; Secondary: 90B25, 62N05.
Key words and phrases. Preventive maintenance, minimal repair, replacement, random working time, optimization, reliability.

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Therefore, determining an optimal PM policy to improve the reliability and physical performance of a complex system is a major issue in reliability engineering.

Pham and Wang [18] suggest that maintenance can be classified according to the degree to which the operating conditions of an item are restored by maintenance. They proposed three accepted categories: perfect, minimal, and imperfect. Perfect maintenance is assumed to return the system to its initial condition (i.e., “as good as new”) immediately after maintenance. A minimal repair restores the system to its functioning condition just prior to the failure (i.e., “as bad as old”), and it does not affect the failure rate [1]. However, it is more realistic that maintenances would not make a system as good as new, but make it younger, which is known as imperfect maintenance. The most significant among the imperfect maintenance models include probability model [4, 5], virtual age model [11, 12], and improvement factor method [14, 23]. These imperfect maintenance models have been recently applied to cumulative repair-cost limit replacement policies [7, 8] and used systems [10, 25]. This study investigates the preventive maintenance first (PMF) and preventive maintenance last (PML) policies for systems that have an improved failure characteristic between maintenances and whose failure rate increases with frequency after preventive maintenances are performed on the systems.

In the meanwhile, time-dependent probability model proposed by Block et al. [4] is also adopted to model another imperfect maintenance action in this study. System failures always occur stochastically and can be roughly classified into two types: the first type is the catastrophic failure, in which a system fails suddenly and completely and must be replaced with a new one immediately; and the second type is the minor failure, in which a system fails gradually over time with performance deterioration. The minor failure is rectified by a minimal repair in practical situations. The above imperfect maintenance action is examined using the time-dependent probability mechanism in this study.

Periodic PM, which specifies that a system should be maintained at the fixed time instants or the multiples of the base period $T(kT, k = 1, 2, \ldots)$, is one of the commonly used PM policies. However, for an industrial system, if a job to be processed by the system has a variable working cycle, it would be less interruptive to the job processing if the maintenance is performed after the job is completed [22]. The problem of scheduling maintenances on an operating system with random processing times has received much attention in the literature [15]. Maintenance policies on systems with random working times have recently been studied in the literature that investigated replacement policies [9, 16, 6] and inspection policies [26]. In these studies, the system is maintained preventively at a planned time $T$ or at the completion of a random working time in each maintenance interval.

In addition, most existing maintenance policies assume that systems are maintained or replaced before a failure after a predefined threshold that can be calculated based on the system’s age, operating time, usage number, etc., whichever occurs first. This kind of policy can be categorized as preventive maintenance-first (PMF) policy. In practice, PMF policy often results in too frequent or unnecessary maintenance activities. Sometimes, system managers may want to operate the system as long as possible before a failure. For this type of situations, Chen et al. [9] proposed the notion of “whichever occurs last”. Such kind of maintenance policy has been applied in some cumulative damage models [27] and is called preventive maintenance-last (PML) policy.
This study investigates both PML and PMF policies for an operating system with random working time. Systems are subject to two types of failures (catastrophic and minor) based on the time-dependent probability mechanism in each maintenance interval. Each minor failure is rectified by a minimal repair, and each catastrophic failure is removed by corrective maintenance. The system is maintained preventively before a catastrophic failure at a planned time $T$ or at the completion of a random working time, whichever occurs first and last, which are the proposed PMF and PML, respectively. After the maintenance (PM or CM), the failure rate reduces to zero and then increases with the number of maintenance operations carried out. At the $N$-th maintenance, the system is replaced rather than maintained. This study develops an optimal preventive replacement schedule $(T^*, N^*)$ of PMF and PML policies to minimize the mean cost rate functions.

The main differences between this study and the existing literature are as follows.

- A new methodology for periodic preventive maintenance on systems with random working time is developed to deal with optimization problems of maintenance first or last.
- An improvement factor method is adopted with which the system undergoing imperfect maintenance has an improved life distribution after maintenances and its failure rate increases with frequency, no matter imperfect maintenances are conducted at a planned time, or at the completion of a working time, or immediately after any catastrophic failure.
- In most of the existing studies on random working time, the system is either replaced or minimally repaired at failure. It is more generalized and practical to consider the time-dependent failure characteristic between replacement and minimal repair.

The remainder of this paper is organized as follows. Section 2 describes the imperfect maintenance model for an operating system. Section 3 presents preventive maintenance policies, develops the mean cost rate functions, and obtains the optimal PM schedule. In Section 4, a computational example is given to illustrate the comparisons between the PMF and PML models. Finally, Section 5 concludes the paper with a summary.

2. System description. We consider a preventive maintenance system with random working times. In such a system, minimal repair, maintenance, and replacement are carried out according to the following scheme.

1. The failure time $X$ of the system has a general distribution $F(t) \equiv P(X \leq t)$ and probability density function $f(t) \equiv \partial F(t)/\partial t$. Then, the failure rate $r(t) \equiv f(t)/F(t)$ is assumed to increase to $r(\infty) \equiv \lim_{t \to \infty} r(t)$, where $\Phi(t) \equiv 1 - \Phi(t)$ for any function $\Phi(t)$. The cumulative hazard $\Lambda(t) \equiv \int_0^t r(u)du = -\ln F(t)$ is the mean number of failures that occur in $[0, t]$.

2. Suppose that a job has a working time $Y$ with a general distribution $G(t) \equiv P(Y \leq t)$ and $Y$ is independent of $X$. Preventive maintenance (PM) is planned to be performed at a fixed time interval $T$ or at the completion of $Y$ in each PM interval.

3. In the $i$-th PM interval, if a system’s failure occurs at time $t$, the system experiences one of the two types of failures: type-I failure (minor) that occurs with probability $q_i(t)$ and is corrected by a minimal repair, and type-II failure (catastrophic) that occurs with probability $p_i(t)(= 1 - q_i(t))$ and requires
a corrective maintenance (CM). It is often costlier to continue performing maintenance operation on a system than replacing it after a certain number of maintenances repairs. Hence, the system is replaced at the \( N \)-th maintenance.

4. We assume maintenance is a normal repair action in which the system improves but its failure characteristic is also altered after maintenance. After each maintenance (either PM or CM), the age of the system is restored to zero and the system’s failure rate increases with the number of maintenances. In other words, if the system has a failure rate \( r_i(t) \) in the \( i \)-th PM interval, then \( r_i(t) \) increases with \( i \) (i.e., \( r_i(t) \leq r_{i+1}(t) \)) and \( r_i(0) = r_{i+1}(0) \) for all \( i = 1, 2, \ldots \), where \( r_1(t) \equiv r(t) \).

5. Let \( Z_i \) be the waiting time until the first type-II failure in the \( i \)-th PM interval. From Brown and Proschan [5], the survival function of \( Z_i \) is directly obtained

\[
\bar{F}_{Z_i}(t) \equiv P(Z_i > t) = \exp \left( - \int_0^t p_i(u)r_i(u)du \right),
\]

where the cumulative hazard \( \Lambda_i(t) \equiv \int_0^t r_i(u)du \) is the mean number of failures that occur in \([0, t] \). Then, the mean number of type-I failures (minimal repairs) in the \( i \)-th PM interval can be derived as \( \Lambda_{q_i}(t) \equiv \int_0^t q_i(u)r_i(u)du \) (see [2]).

Moreover, the following assumptions have been made. A replacement (or renewal) cycle is defined as the time interval between the installation of the system and the first replacement or between two consecutive replacements. All failures are instantly detected and maintained. Repairs, maintenances, and replacements are completed instantaneously. After a replacement, the system becomes brand new and resets to time 0.

3. **Preventive maintenance.** Various costs for the maintenance model are defined as follows. Let \( C_O \) and \( C_R \) be the costs of preventive maintenance and replacement, respectively. We assume that \( C_R > C_O > 0 \). We also assume that the corrective maintenance (or replacement) activity is unplanned, hence denote that \( C_B \) is the additional cost due to unplanned maintenance (or replacement). Then, the costs of corrective maintenance and replacement are \( C_O + C_B \) and \( C_R + C_B \), respectively. The minimal repair cost in the \( i \)-th PM interval is defined as \( c_i \).

3.1. **Preventive maintenance-first (PMF).** The procedure of the PMF policy is as follows. Suppose that a PM is performed before any type-II failure occurs at a planned time \( T \) or at the completion of a working time \( Y \), whichever occurs first. The CM is performed immediately after any type-II failure in each PM interval. The system is replaced at the \( N \)-th maintenance.

The corresponding probability of each maintenance situation in a renewal cycle can be derived as below. In the \( i \)-th PM interval, the probability that the system is maintained at age \( T \) is

\[
P(Z_i > T, Y > T) = \bar{F}_{Z_i}(T)\bar{G}(T),
\]

the probability that it is maintained at \( Y \) is

\[
P(Y \leq T, Y \leq Z_i) = \int_0^T \bar{F}_{Z_i}(t) d\bar{G}(t),
\]
and the probability that it is maintained immediately after the first type-II failure is

\[ P(Z_i \leq T, Z_i \leq Y) = \int_0^T \tilde{G}(t) dF_{Z_i}(t), \quad (4) \]

where equation (2), (3), and (4) sum to 1. Hence, the mean time of the renewal cycles is

\[ \sum_{i=1}^N \left[ T\bar{F}_{Z_i}(T)\tilde{G}(T) + \int_0^T t\bar{F}_{Z_i}(t) dG(t) + \int_0^T t\tilde{G}(t) dF_{Z_i}(t) \right] \]

\[ = \sum_{i=1}^N \int_0^T \bar{F}_{Z_i}(t)\tilde{G}(t) dt. \quad (5) \]

Next, the total mean number of type-I failures before maintenance in the \( i \)-th PM interval is

\[ \Lambda_{q_i}(T)\bar{F}_{Z_i}(T)\tilde{G}(T) + \int_0^T \Lambda_{q_i}(t)\bar{F}_{Z_i}(t) dG(t) + \int_0^T \Lambda_{q_i}(t)\tilde{G}(t) dF_{Z_i}(t) \]

\[ = \int_0^T \bar{F}_{Z_i}(t)\tilde{G}(t) q_i(t) r_i(t) dt. \quad (6) \]

Then, using equation (2), (3), (4), and (6), the mean cost of the renewal cycles can be derived as follows,

\[ \sum_{i=1}^{N-1} \left[ C_O \bar{F}_{Z_i}(T)\tilde{G}(T) + \int_0^T \bar{F}_{Z_i}(t) dG(t) \right] + (C_O + C_B) \int_0^T \tilde{G}(t) dF_{Z_i}(t) \]

\[ + c_i \int_0^T \bar{F}_{Z_i}(t)\tilde{G}(t) q_i(t) r_i(t) dt + C_R \int_0^T \bar{F}_{Z_i}(T)\tilde{G}(T) dG(t) \]

\[ + (C_R + C_B) \int_0^T \tilde{G}(t) dF_{Z_i}(t) + c_N \int_0^T \bar{F}_{Z_i}(t)\tilde{G}(t) q_N(t) r_N(t) dt \]

\[ = (N-1)C_O + C_R + C_B \sum_{i=1}^N \int_0^T \tilde{G}(t) dF_{Z_i}(t) + \sum_{i=1}^N c_i \int_0^T \bar{F}_{Z_i}(t)\tilde{G}(t) q_i(t) r_i(t) dt. \quad (7) \]

Consequently, using equation (5) and (7), the mean cost rate of the renewal cycles for PMF policy is

\[ J_F(T, N) \equiv \frac{(N-1)C_O + C_R + C_B \sum_{i=1}^N \int_0^T \tilde{G}(t) dF_{Z_i}(t) + \sum_{i=1}^N c_i \int_0^T \bar{F}_{Z_i}(t)\tilde{G}(t) q_i(t) r_i(t) dt}{\sum_{i=1}^N \int_0^T \bar{F}_{Z_i}(t)\tilde{G}(t) dt}. \quad (8) \]

The main objective of this formulation is to determine the optimal PMF policy \((T_F, N_F)\)$^*$ (i.e., \(T_F^*\) and \(N_F^*\)) that minimizes the mean cost rate \(J_F(T, N)\). If \(J_F(T, N)\) is jointly convex in \((T, N)\), then there exists an optimal PMF policy \((T_F, N_F)^*\). First, we derive an optimal age \(T_F^*\) that minimizes \(J_F(T, N)\) with respect to \(T\) for a given \(N\). Differentiating \(J_F(T, N)\) with respect to \(T\) and setting it to zero, we see that \(\partial J_F(T, N)/\partial T = 0\) if and only if

\[ Q_F(T; N) = (N-1)C_O + C_R, \quad (9) \]
where
\[ Q_F(T; N) \equiv \varphi_F(T; N) \sum_{i=1}^{N} \int_{0}^{T} \tilde{F}_{Z_i}(t) \tilde{G}(t) dt \]
\[ - \left[ C_B \sum_{i=1}^{N} \int_{0}^{T} \tilde{G}(t) dF_{Z_i}(t) + \sum_{i=1}^{N} c_i \int_{0}^{T} \tilde{F}_{Z_i}(t) \tilde{G}(t) q_i(t) r_i(t) dt \right]. \quad (10) \]
\[ \varphi_F(T; N) \equiv \frac{C_B \sum_{i=1}^{N} f_{Z_i}(T) + \sum_{i=1}^{N} c_i \tilde{F}_{Z_i}(T) q_i(T) r_i(T)}{\sum_{i=1}^{N} \tilde{F}_{Z_i}(T)}. \quad (11) \]

Furthermore, let \( \theta_F \equiv \frac{C_B \sum_{i=1}^{N} \int_{0}^{\infty} \tilde{G}(t) dF_{Z_i}(t) + \sum_{i=1}^{N} c_i \int_{0}^{\infty} \tilde{F}_{Z_i}(t) \tilde{G}(t) q_i(t) r_i(t) dt}{\sum_{i=1}^{N} \int_{0}^{\infty} \tilde{F}_{Z_i}(t) \tilde{G}(t) dt}. \) We can summarize the structure properties of optimal policy \( T^*_F \) that minimizes \( J_F(T; N) \) with respect to \( T \) for a given \( N \) as follows.

**Theorem 1.** Suppose that \( \varphi_F(T; N) \) is continuous and increasing in \( T \). If \( \lim_{T \to \infty} \varphi_F(T; N) > \theta_F \), then there exists a finite and unique \( T^*_F \) which satisfies equation (9) and the optimal mean cost rate is
\[ J_F(T^*_F, N) = \varphi_F(T^*_F; N); \quad (12) \]
otherwise, the optimal policy is \( T^*_F \to \infty \).

**Proof.** For a given \( N \), taking the first order derivative of \( J_F(T, N) \) with respect to \( T \) and setting it to zero implies equation (9). If \( \varphi_F(T; N) \) is increasing at \( T \), it causes that \( Q_F(T; N) \) is also increasing at \( T \). Further, \( Q_F(0; N) = 0 \) and
\[ Q_F(\infty; N) \equiv \lim_{T \to \infty} Q_F(T; N) = \varphi_F(\infty; T) \sum_{i=1}^{N} \int_{0}^{\infty} \tilde{F}_{Z_i}(t) \tilde{G}(t) dt \]
\[ - \left[ C_B \sum_{i=1}^{N} \int_{0}^{\infty} \tilde{G}(t) dF_{Z_i}(t) + \sum_{i=1}^{N} c_i \int_{0}^{\infty} \tilde{F}_{Z_i}(t) \tilde{G}(t) q_i(t) r_i(t) dt \right], \]
where \( \varphi_F(\infty; N) \equiv \lim_{T \to \infty} \varphi_F(T; N) \). If \( \varphi_F(\infty; T) > \theta_F \), then \( Q_F(\infty; N) > (N - 1)C_O + C_R \). Thus, since \( Q_F(0; N) < (N - 1)C_O + C_R \) and the increasing property of \( Q_F(T; N) \), there exists a finite and unique \( T^*_F (0 < T^*_F < \infty) \) satisfies equation (9), which minimizes \( J_F(T; N) \) in equation (8).

Function \( \varphi_F(T; N) \) can be regarded as the expected marginal cost of this policy at age \( T \) for a given \( N \). The concept of marginal cost has been studied and used to minimize mean cost rate problem for age replacement policy, for details, see e.g. [3], [19], and [20]. The expected marginal cost of this policy at age \( T \) is expressed as a linear combination of some of its components such as replacement and repair costs. Optimal policy \( T^*_F \) that minimizes \( J_F(T; N) \) must satisfy \( J_F(T^*_F, N) = \varphi_F(T^*_F; N) \). To compute \( T^*_F \), draw functions \( J_F(T, N) \) and \( \varphi_F(T; N) \) on the same graph and locate the intersection point with high precision. This can be easily done in practice. If \( \varphi_F(T; N) \) is continuous and strictly increasing, then there exists a unique intersection point. In Section 2, we have assumed that \( r_i(t) \) increases with \( t \) for all \( i = 1, 2, \cdots \); then, the marginal cost \( \varphi_F(T; N) \) also increases.
with \( T \) and hence, we can confirm the existence and uniqueness of the optimal solution.

Next, we derive an optimal number \( N^*_F \) that minimizes \( J_F(T, N) \) for a given \( T \). We see that inequalities \( J_F(T, N + 1) \geq J_F(T, N) \) and \( J_F(T, N) < J_F(T, N - 1) \) hold if and only if

\[
L_F(N; T) \geq C_R \text{ and } L_F(N - 1; T) < C_R,
\]

where

\[
L_F(N; T) = \begin{cases} 
\zeta_F(N; T) \sum_{i=1}^{N} \int_0^T \bar{F}_{Z_i}(t) \bar{G}(t) dt \\
-(N-1)C_O - C_B \sum_{i=1}^{N} \int_0^T \bar{G}(t) dF_{Z_i}(t) \\
- \sum_{i=1}^{N} c_i \int_0^T \bar{F}_{Z_i}(t) \bar{G}(t) q_i(t) r_i(t) dt, \\
\text{for } n = 1, 2, \ldots, \\
0, \text{ for } n = 0,
\end{cases}
\]

\[
\zeta_F(N; T) = \begin{cases} 
C_O + C_B \int_0^T G(t) dF_{Z_{N+1}}(t) + c_{N+1} \int_0^T \bar{F}_{Z_{N+1}}(t) \bar{G}(t) q_{N+1}(t) r_{N+1}(t) dt, \\
\text{for } n = 0, \\
0, \text{ for } n = 0,
\end{cases}
\]

Optimal policy \( N^*_F \) that minimizes the mean cost rate \( J_F(T, N) \) for a given \( T \) can be summarized as follows.

**Theorem 2.** If \( \zeta_F(N; T) \) is increasing in \( N \) and \( \lim_{N \to \infty} \zeta_F(N; T) = \infty \), then there exists a finite and unique \( N^*_F \) that satisfies \( L_F(N^*_F; T) \geq C_R \) and \( L_F(N^*_F - 1; T) < C_R \) for \( N^*_F = 1, 2, \ldots. \)

**Proof.** For a fixed \( T \), the inequalities \( J_F(T, N + 1) \geq J_F(T, N) \) and \( J_F(T, N) < J_F(T, N - 1) \) imply inequation (13). If \( \zeta_F(N; T) \) is increasing in \( N \), then \( L_F(N; T) \) is also increasing in \( N \) since

\[
L_F(N + 1; T) - L_F(N; T) = [\zeta_F(N + 1; T) - \zeta_F(N; T)] \sum_{i=1}^{N+1} \int_0^T \bar{F}_{Z_i}(t) \bar{G}(t) dt > 0.
\]

Further, \( L_F(\infty; T) = \lim_{N \to \infty} L_F(N; T) = \infty \) since \( \lim_{N \to \infty} \zeta_F(N; T) = \infty. \) Then there exists a unique \( N^*_F \) satisfies inequation (13), which minimizes \( J_F(T, N) \) in equation (8).

Function \( \zeta_F(N; T) \) can be interpreted as the expected marginal cost of this policy at number \( N \) of the working times for a given \( T \). Then, the inequation (13) is equivalent to

\[
\zeta_F(N; T) \geq J_F(T, N) \text{ and } \zeta_F(N - 1; T) < J_F(T, N - 1).
\]

Optimum \( N \) (i.e., \( N^*_F; \min_{N} J_F(T, N) = J_F(T, N^*_F) \)) must satisfy the equivalent condition, which indicates that it is worth increasing the prefixed number of working times \( N \) if \( \zeta_F(N; T) < J_F(T, N) \).

In particular, some special cases of this model can be demonstrated as follows.

- **Case F1.** \( q_i(t) = 0. \) This case considers that the system is maintained at \( T, Y, \) or at any failure in each PM interval, whichever occurs first. The system
is replaced at the $N$-th maintenance. If we set $q_i(t) = 0$ in equation (8), the mean cost rate becomes

$$J_{F_i}(T, N) = \frac{(N - 1)C_O + C_R + C_B \sum_{i=1}^{N} \int_0^T \bar{G}(t) \int_0^T G(t) dt}{\sum_{i=1}^{N} \int_0^T \bar{F}_i(t) G(t) dt},$$

where $\bar{F}_i(t) = \exp(-\Lambda_i(t))$.

- **Case F2.** $q_i(t) = 0$ and $N = 1$. This case is considered by Chen et al. [9], in which the system is replaced at time $T$, $Y$, or at any failure, whichever occurs first. If we set $q_i(t) = 0$ and $N = 1$ in equation (8), then the mean cost rate is

$$J_{F_2}(T, 1) = \frac{C_R + C_B \int_0^T \bar{G}(t) \int_0^T G(t) dt}{\int_0^T F(t) G(t) dt}.$$

- **Case F3.** $q_i(t) = 1$. This case considers that the system is maintained at $T$ or $Y$, and undergoes minimal repair at any failure in each PM interval, whichever occurs first. The system is replaced at the $N$-th maintenance. If we set $q_i(t) = 1$ in equation (8), the mean cost rate becomes

$$J_{F_3}(T, N) = \frac{(N - 1)C_O + C_R + \sum_{i=1}^{N} \int_0^T \bar{G}(t) \int_0^T G(t) dt}{\sum_{i=1}^{N} \int_0^T \bar{F}_i(t) G(t) dt}.$$

- **Case F4.** $q_i(t) = 1$ and $N = 1$. This case is considered by Sugiura et al. [22], in which the system is replaced at time $T$ or $Y$, and undergoes minimal repair at any failure. If we set $q_i(t) = 1$ and $N = 1$ in equation (8), then the mean cost rate is

$$J_{F_4}(T, 1) = \frac{C_R + c_i \int_0^T \bar{G}(t) \int_0^T G(t) dt}{\int_0^T G(t) dt}.$$

- **Case F5.** $Y \to \infty$. This case is considered by Sheu and Liou [21], in which the system is maintained at time $T$ or after any type-II failure, and undergoes minimal repair at any type-I failure in each PM interval. The system is replaced at the $N$-th maintenance. If we set $Y \to \infty$ in equation (8), then the mean cost rate is

$$J_{F_5}(T, N) = \frac{(N - 1)C_O + C_R + C_B \sum_{i=1}^{N} \int_0^T \bar{F}_i(t) \int_0^T G(t) dt + \sum_{i=1}^{N} c_i \int_0^T \bar{F}_i(t) q_i(t) \int_0^T \bar{G}(t) \int_0^T G(t) dt}{\sum_{i=1}^{N} \int_0^T \bar{F}_i(t) \int_0^T G(t) dt}.$$

- **Case F6.** $Y \to \infty$ and $q_i(t) = 0$. This case is considered by Nguyen and Murthy [17], in which the system is maintained at time $T$ or at any failure in each PM interval, and replaced at the $N$-th maintenance.

- **Case F7.** $Y \to \infty$ and $q_i(t) = 1$. This case is considered by Nakagawa [13], in which the system is maintained at time $T$ and undergoes minimal repair at any failure in each PM interval, and replaced at the $N$-th maintenance.

3.2. **Preventive maintenance-last (PML).** Under a PML, a PM is performed before any type-II failure occurs at age $T$ or at the completion of working time $Y$, whichever occurs last. CM is also performed immediately after any type-II failure occurs in each PM interval. The system is replaced at the $N$-th maintenance.

In the $i$-th PM interval, the probability that the system is maintained at age $T$ is

$$P(Z_i > T, Y \leq T) = \bar{F}_i(T) G(T),$$

where
the probability that it is maintained at number \( N \) is
\[
P(Y > T, Y \leq Z_i) = \int_T^\infty \tilde{F}_{Z_i}(t)dG(t),
\]
and the probability that it is maintained immediately after the first type-II failure is
\[
P(Z_i \leq T) + P(Z_i > T, Z_i \leq Y) = F_{Z_i}(T) + \int_T^\infty \tilde{G}(t)dF_{Z_i}(t),
\]
where equation (21), (22), and (23) sum to 1. Hence, the mean time of renewal cycles is
\[
\sum_{i=1}^N \left[ \int_0^T \tilde{F}_{Z_i}(t)dt + \int_T^\infty \tilde{F}_{Z_i}(t)\tilde{G}(t)dt \right].
\]
(24)

Next, the total mean number of type-I failures before maintenance in the \( i \)-th PM interval is
\[
\Lambda_q(T)\tilde{F}_{Z_i}(T)G(T) + \int_T^\infty \Lambda_q(t)\tilde{F}_{Z_i}(t)dG(t) + \int_0^T \Lambda_q(t)dF_{Z_i}(t)
\]
(25)
\[+ \int_T^\infty \Lambda_q(t)\tilde{G}(t)dF_{Z_i}(t) = \int_0^T \tilde{F}_{Z_i}(t)q_i(t)r_i(t)dt + \int_T^\infty \tilde{F}_{Z_i}(t)\tilde{G}(t)q_i(t)r_i(t)dt.
\]
Then, using equation (21), (22), (23), and (25), the mean cost of renewal cycles can be derived as follows,
\[
\sum_{i=1}^{N-1} \{ C_O[\tilde{F}_{Z_i}(T)G(T) + \int_T^\infty \tilde{F}_{Z_i}(t)dG(t)] + (C_O + C_B)[F_{Z_i}(T) + \int_T^\infty \tilde{G}(t)dF_{Z_i}(t)]
\]
\[+ c_i \left[ \int_0^T \tilde{F}_{Z_i}(t)q_i(t)r_i(t)dt + \int_T^\infty \tilde{F}_{Z_i}(t)\tilde{G}(t)q_i(t)r_i(t)dt \right] \}
\[C_R[\tilde{F}_{Z_N}(T)G(T) + \int_T^\infty \tilde{F}_{Z_N}(t)dG(t)] + (C_R + C_B)[F_{Z_N}(T) + \int_T^\infty \tilde{G}(t)dF_{Z_N}(t)]
\]
\[+ c_N \left[ \int_0^T \tilde{F}_{Z_N}(t)q_N(t)r_N(t)dt + \int_T^\infty \tilde{F}_{Z_N}(t)\tilde{G}(t)q_N(t)r_N(t)dt \right]
\]
\[= (N - 1)C_O + C_R + C_B \sum_{i=1}^N \left[ F_{Z_i}(T) + \int_T^\infty \tilde{G}(t)dF_{Z_i}(t) \right]
\]
\[+ \sum_{i=1}^N c_i \left[ \int_0^T \tilde{F}_{Z_i}(t)q_i(t)r_i(t)dt + \int_T^\infty \tilde{F}_{Z_i}(t)\tilde{G}(t)q_i(t)r_i(t)dt \right].
\]
(26)

Consequently, using equation (24) and (26), the mean cost rate of renewal cycles for the PML policy is
\[
J_L(T, N) = \frac{(N - 1)C_O + C_R + C_B \sum_{i=1}^N \left[ F_{Z_i}(T) + \int_T^\infty \tilde{G}(t)dF_{Z_i}(t) \right]
\]
\[+ \sum_{i=1}^N c_i \left[ \int_0^T \tilde{F}_{Z_i}(t)q_i(t)r_i(t)dt + \int_T^\infty \tilde{F}_{Z_i}(t)\tilde{G}(t)q_i(t)r_i(t)dt \right]}{\sum_{i=1}^N \left[ \int_0^T \tilde{F}_{Z_i}(t)dt + \int_T^\infty \tilde{F}_{Z_i}(t)\tilde{G}(t)dt \right]}.
\]
(27)

If \( J_L(T, N) \) is jointly convex in \((T, N)\), then there exists an optimal PML policy \((T_L, N_L)^* \) (i.e., \( T_L^* \) and \( N_L^* \)) that minimizes the mean cost rate \( J_L(T, N) \). First,
differentiating $J_L(T, N)$ in equation (27) with respect to $T$ for a given $N$ and setting it equal to zero, we see that $\partial J_L(T, N)/\partial T = 0$ if and only if

$$Q_L(T; N) = (N - 1)C_O + C_R,$$

where

$$Q_L(T; N) = \varphi_L(T; N) \sum_{i=1}^{N} \left[ \int_{0}^{T} \bar{F}_Z(t)dt + \int_{T}^{\infty} F_Z(t)\bar{G}(t)dt \right] - \left\{ C_B \sum_{i=1}^{N} \left[ F_Z(T) + \int_{T}^{\infty} \bar{G}(t)dF_Z(t) \right] + \int_{0}^{T} F_Z(t)q_i(t)r_i(t)dt + \int_{T}^{\infty} F_Z(t)G(t)q_i(t)r_i(t)dt \right\}. \quad (29)$$

$$\varphi_L(T; N) = \frac{C_B \sum_{i=1}^{N} F_Z(T) + \sum_{i=1}^{N} c_i F_Z(T)q_i(T)r_i(T)}{\sum_{i=1}^{N} F_Z(T)} = \frac{\sum_{i=1}^{N} \left( C_B p_i(T) + c_i q(T) \right) F_Z(T)r_i(T)}{\sum_{i=1}^{N} F_Z(T)} \quad (30)$$

Using the method similar to Theorem 1, there exists a finite and unique $T_L^*$ which satisfies equation (28) and the optimal mean cost rate is

$$J_L(T_L^*, N) = \varphi_L(T_L^*; N). \quad (31)$$

Next, for a given $T$, we see that inequalities $J_L(T, N + 1) \geq J_L(T, N)$ and $J_L(T, N) < J_L(T, N - 1)$ hold if and only if

$$L_L(N; T) \geq C_R \text{ and } L_L(N - 1; T) < C_R, \quad (32)$$

where

$$L_L(N; T) \equiv \begin{cases} \zeta_L(N; T) \sum_{i=1}^{N} \left[ \int_{0}^{T} \bar{F}_Z(t)dt + \int_{T}^{\infty} F_Z(t)\bar{G}(t)dt \right] - (N - 1)C_O - C_B \sum_{i=1}^{N} \left[ F_Z(T) + \int_{T}^{\infty} \bar{G}(t)dF_Z(t) \right] - \sum_{i=1}^{N} c_i \left[ \int_{0}^{T} \bar{F}_Z(t)q_i(t)r_i(t)dt + \int_{T}^{\infty} F_Z(t)\bar{G}(t)q_i(t)r_i(t)dt \right], & \text{for } n = 1, 2, \ldots, \\ 0, & \text{for } n = 0, \end{cases} \quad (33)$$

$$\zeta_L(N; T) \equiv C_O + C_B \left[ F_{ZN+1}(T) + \int_{T}^{\infty} \bar{G}(t)dF_{ZN+1}(t) \right] + c_{N+1} \left[ \int_{0}^{T} F_{ZN+1}(t)q_{N+1}(t)r_{N+1}(t)dt + \int_{T}^{\infty} F_{ZN+1}(t)\bar{G}(t)q_{N+1}(t)r_{N+1}(t)dt \right] \quad (34)$$

$$\int_{0}^{T} \bar{F}_Z(t)dt + \int_{T}^{\infty} F_Z(t)\bar{G}(t)dt$$

If $J_L(T, N)$ is jointly convex in $(T, N)$, then the optimal PML policy $(T, N)^*$ must satisfy equation (28) and in equation (32). In general, an explicit analytical expression can be obtained by using the method similar to that used in Subsection 3.1.

In particular, some special cases of this model can be demonstrated as follows.
functions of a renewal cycle for PMF and PML policies are
\( J \) (9) and (28) for a given larger than those of \( J \) J J

3.3. Comparison of two policies. In Subsections 3.1 and 3.2, the mean cost rate functions of a renewal cycle for PMF and PML policies are \( J_F(T, N) \) and \( J_L(T, N) \), respectively. Both numerator and denominator of \( J_L(T, N) \) in equation (27) are larger than those of \( J_F(T, N) \) in equation (8). We compare the above two policies for a given \( N \). There exist both finite and unique \( T_F \) and \( T_L \) which satisfy equation (9) and (28) for a given \( N \), respectively. First, compare the left-hand sides of equation (9) and (28).

Denote that
\[ Q(T) \equiv Q_L(T; N) - Q_F(T; N) \]
\[ = \varphi(T; N) \sum_{i=1}^{N} \left[ \int_{0}^{T} \bar{F}_Z_i(t)G(t)dt + \int_{T}^{\infty} \bar{F}_Z_i(t)G(t)dt \right] \\
- C_B \sum_{i=1}^{N} \left[ \int_{0}^{T} G(t)dF_Z_i(t) + \int_{T}^{\infty} \ddot{G}(t)dF_Z_i(t) \right] \]
where \( \phi(T; N) \equiv \phi_F(T; N) = \phi_L(T; N) \).

Let \( \theta \equiv C_B \sum_{i=1}^{N} \int_0^T G(t) dF_Z_i(t) + \sum_{i=1}^{N} c_i \int_0^\infty F_{Z_i}(t) G(t) q_i(t) r_i(t) dt \). Clearly,

\[
Q(0) \equiv \lim_{T \to 0} Q(T) = -C_B \int_0^\infty G(t) dF_Z_i(t) - \sum_{i=1}^{N} c_i \int_0^\infty F_{Z_i}(t) G(t) q_i(t) r_i(t) dt < 0, \\
Q(\infty) \equiv \lim_{T \to \infty} Q(T) > 0 \text{ if } \phi(\infty; N) \equiv \lim_{T \to \infty} \phi(T; N) > \theta,
\]

\[
Q'(T) \equiv \phi'(T; N) \sum_{i=1}^{N} \left[ \int_0^T F_{Z_i}(t) G(t) dt + \int_T^\infty F_{Z_i}(t) G(t) dt \right] > 0 \text{ where } Q'(T) \equiv \partial Q(T)/\partial T \text{ and } \phi'(T; N) \equiv \partial \phi(T; N)/\partial T. \text{ Thus, there exists a finite and unique } T^*_A \text{ which satisfies } Q(T) = 0.
\]

Second, denote that

\[
H(T^*_A) \equiv \phi(T^*_A; N) \sum_{i=1}^{N} \int_0^{T_A} F_{Z_i}(t) G(t) dt - \left[ C_B \sum_{i=1}^{N} \int_0^{T_A} G(t) dF_{Z_i}(t) + \sum_{i=1}^{N} c_i \int_0^{T_A} F_{Z_i}(t) G(t) q_i(t) r_i(t) dt \right]. (40)
\]

Thus, we conclude that if \( H(T^*_A) \geq (N-1)C_O + C_R \), then \( T^*_F \leq T^*_L \) and hence, from equation (12) and (31), PMF is better than PML (i.e., \( J_F(T^*_F, N) \leq J_L(T^*_L, N) \)). Conversely, if \( H(T^*_A) < (N-1)C_O + C_R \), then \( T^*_F > T^*_L \) and PML is better than PMF.

3.4. Algorithms of two policies. In practice, the repair cost is a random variable and the decision to repair or replace a failed system may depend on the estimated repair cost. Here, we consider a repair limit replacement policy in the \( i \)-th PM interval, where, in a failure state, one replaces the system or repairs it depending on the random cost \( C_i \). A system undergoes a corrective replacement if \( C_i > \delta_i(t)c_\infty \) and is minimally repaired if \( C_i \leq \delta_i(t)c_\infty \), where \( c_\infty \) is the constant cost and \( \delta_i(t) \) (\( 0 \leq \delta_i(t) \leq 1 \)) can be interpreted as a fraction of the constant cost \( c_\infty \). We consider the following parametric form of the repair cost limit function \( \delta_i(t) = \delta_i e^{-\lambda t} \) with \( 0 \leq \delta_i \leq 1 \) and \( \lambda \geq 0 \). Suppose that the random repair cost \( C_i \) has a distribution \( K(u) \), and density function \( k(u) \), with mean \( \mu \) and standard deviation \( \sigma \).

If a system fails at age \( t \) in the \( i \)-th PM interval, it is either replaced with probability \( p_i(t) = 1 - \int_0^{\delta_i(t)c_\infty} k(u) du \) or undergoes minimal repair with probability \( q_i(t) = \int_0^{\delta_i(t)c_\infty} k(u) du \). It is noteworthy that we have \( \delta_i(t) = \delta_i \) if \( \lambda \to 0 \), so \( q_i(t) \) reduces to \( q_i \) (the constant minimal repair probability).

According to the results above, we present two algorithms that can be used to numerically compute the optimal PM policies.

4. Numerical example. In this section, we refer to the example case considered by Zhao and Nakagawa [25] and provide a hypothetical numerical example to illustrate the application of the proposed PM policies. Suppose that a computer or communication system deteriorates with its age, and fails according to the Weibull distribution \( F(t) \equiv P(X \leq t) \) with scale parameter \( \alpha \) and shape parameter \( \beta \), i.e., \( F(t) = 1 - \exp[(-\alpha t)^\beta] \). Then, the failure rate is \( r_i(t) = \beta \alpha \alpha^2 t^{\beta-1} \) in the \( i \)-th PM
interval. Further, suppose that the system successively executes computer jobs, and each job has a variable working cycle $Y_j (j = 1, 2, \cdots)$ with an exponential distribution $G(t) = 1 - \exp(-\theta t)$.

According to the algorithms presented in Subsection 3.4, we make $q_i = q$ to simplify the computation. Parameter $q$ (minimal repair probability) is varied to determine its influence on the optimal preventive maintenance policy and the related minimum cost rate which are reported in Tables 1 and 2.

From the numerical results shown in Tables 1 and 2, we make the following observations:

- As predicted by our model, a finite and unique optimal policy exists for each model.
- The minimum cost rates diminish and the optimal preventive maintenance schedules increase when the minimal repair probability increases. This is intuitive, as the lower probability of corrective maintenance leads to a lower mean cost and an extended preventive maintenance schedule.
- The PML is more advantageous than PMF in terms of the minimum cost rates (i.e., $J_L (T_L^*, N_L^*) < J_F (T_F^*, N_F^*)$).

---

**Algorithm 1** Find optimal PMF policy

**Input:** $C_O, C_R, C_B, c_\infty, K(\cdot), F(\cdot), G(\cdot), r(\cdot), \delta_i, \lambda$.

Step 1. Compute $\tilde{F}_Z(t)$ as defined by equation (1).

Step 2. Set $N = 1$ and $J_F(T_F^*(0), 0) = \infty$.

Step 3. Find the solution $T_F^*(N)$ that satisfies equation (9).

(i.e., $Q_F(T_F^*(N); N) = (N - 1)C_O + C_R$)

Step 4. Compute $J_F(T_F^*(N), N)$ as defined by equation (8).

Step 5. If $J_F(T_F^*(N), N) \geq J_F(T_F^*(N - 1), N - 1)$, then $N_F^* = N - 1, T_F^* = T_F^*(N - 1)$, $J_F(T_F^*, N_F^*) = J_F(T_F^*(N - 1), N - 1)$ and go to Output;

otherwise, set $N = N + 1$ and go to Step 3.

**Output:** $T_F^*, N_F^*, J_F(T_F^*, N_F^*)$.

Stop.

---

**Algorithm 2** Find optimal PML policy

**Input:** $C_O, C_R, C_B, c_\infty, K(\cdot), F(\cdot), G(\cdot), r(\cdot), \delta_i, \lambda$.

Step 1. Compute $\tilde{F}_Z(t)$ as defined by equation (1).

Step 2. Set $N = 1$ and $J_L(T_L^*(0), 0) = \infty$.

Step 3. Find the solution $T_L^*(N)$ that satisfies equation (28).

(i.e., $Q_L(T_L^*(N); N) = (N - 1)C_O + C_R$)

Step 4. Compute $J_L(T_L^*(N), N)$ as defined by equation (27).

Step 5. If $J_L(T_L^*(N), N) \geq J_L(T_L^*(N - 1), N - 1)$, then $N_L^* = N - 1, T_L^* = T_L^*(N - 1)$, $J_L(T_L^*, N_L^*) = J_L(T_L^*(N - 1), N - 1)$ and go to Output;

otherwise, set $N = N + 1$ and go to Step 3.

**Output:** $T_L^*, N_L^*, J_L(T_L^*, N_L^*)$.

Stop.
Table 1. Optimal PMF policies and minimum cost rates of the imperfect maintenance models for varied failure rates.  
\[ C_O = 500, C_R = 1500, C_B = 500, c_\infty = 1000, C \sim N(100, 25^2), G(t) = 1 - e^{-t}, \beta = 2 \]

| \( \delta \) | \( q \) | \( N_F^* \) | \( T_F^* \) | \( J_F(T_F^*, N_F^*) \) | \( N_F^* \) | \( T_F^* \) | \( J_F(T_F^*, N_F^*) \) |
|---|---|---|---|---|---|---|---|
| 1 | 0.9 | 10 | 3.6330 | 823.6007 | 6 | 3.6504 | 895.7270 |
| 9/11 | 0.8 | 10 | 3.1563 | 887.5352 | 6 | 3.2182 | 967.3210 |
| 7/11 | 0.7 | 9 | 2.7194 | 974.0102 | 5 | 2.8597 | 1062.4599 |
| 5/11 | 0.6 | 9 | 2.2434 | 1088.4253 | 5 | 2.3676 | 1188.1660 |
| 4/11 | 0.5 | 9 | 2.0239 | 1158.3947 | 5 | 2.1378 | 1264.9318 |
| 3/11 | 0.4 | 9 | 1.8197 | 1238.3947 | 5 | 1.9228 | 1352.3516 |
| 2/11 | 0.3 | 8 | 1.6632 | 1328.3542 | 5 | 1.7239 | 1451.5804 |
| 1/11 | 0.1 | 8 | 1.4887 | 1430.3649 | 5 | 1.5420 | 1563.8858 |

Table 2. Optimal PML policies and minimum cost rates of the imperfect maintenance models for varied failure rates.  
\[ C_O = 500, C_R = 1500, C_B = 500, c_\infty = 1000, C \sim N(100, 25^2), G(t) = 1 - e^{-t}, \beta = 2 \]

| \( \delta \) | \( q \) | \( N_L^* \) | \( T_L^* \) | \( J_L(T_L^*, N_L^*) \) | \( N_L^* \) | \( T_L^* \) | \( J_L(T_L^*, N_L^*) \) |
|---|---|---|---|---|---|---|---|
| 1 | 0.9 | 6 | 2.4933 | 522.8141 | 4 | 2.4939 | 568.2837 |
| 9/11 | 0.8 | 6 | 2.2631 | 590.6646 | 4 | 2.2725 | 642.1931 |
| 7/11 | 0.7 | 6 | 2.0081 | 682.0260 | 4 | 2.0243 | 741.7077 |
| 5/11 | 0.6 | 6 | 1.7438 | 802.4905 | 4 | 1.7642 | 872.8862 |
| 4/11 | 0.5 | 6 | 1.6133 | 875.6308 | 4 | 1.6347 | 952.5069 |
| 3/11 | 0.4 | 6 | 1.4863 | 958.5882 | 4 | 1.5081 | 1042.7916 |
| 2/11 | 0.3 | 6 | 1.3641 | 1052.3871 | 4 | 1.3857 | 1144.8479 |
| 1/11 | 0.1 | 6 | 1.2477 | 1158.1623 | 4 | 1.2689 | 1259.9027 |

5. Conclusions. In this paper, we have studied the optimal preventive maintenance-first and preventive maintenance-last policies for a system with imperfect maintenance. To determine the optimal preventive maintenance time and threshold number, the mean cost rate for each model is derived by incorporating costs related to maintenances and replacements. The existence and uniqueness of each optimal policy that minimizes the cost rate are justified analytically and computed numerically. The models provide a more general framework for analyzing the maintenance policies for a repairable system.

Future research directions may focus on the system with the following factors: (i) sequential PM, in which a system is maintained at sequential intervals over several
unequal time lengths; (ii) the probability of random failure that depends on the failure number or the system age, which is more general than a constant probability of random failure; and (iii) the system replacement caused by cumulative damage exceeding a specified level.

Acknowledgments. The authors would like to thank the referees for their insightful comments and suggestions, which greatly enhanced the clarity of the article. All of the suggestions were incorporated directly in the text. This research was supported by the Ministry of Science and Technology of Taiwan, ROC, under Grant No. MOST104-2410-H-147-004 and MOST104-2410-H-147-005.

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Received April 2016; revised March 2018.

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