Lie Group Analysis and Wavelet Analysis Method for Solution of a Stefan Problem

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Abstract
This paper deals with the solution of a one-phase Stefan melting problem that models the movement of the shoreline in a sedimentary ocean basin which is considered as a moving boundary problem with variable latent heat. The governing partial differential equations are transformed into a set of ordinary differential equations using similarity transformations via the Lie group analysis. We then propose Chebyshev wavelet analysis method for solving the resulting system. A comparison between the solutions obtained by the proposed method and those reported in the literature as well as the exact solution is made to confirm the accuracy and efficiency of the current method.

Keywords: Lie Group Analysis, Moving Boundary Problem, Stefan Problem, Similarity Transformation, Variable Latent Heat, Wavelet Analysis Method

1. Introduction
A Stefan problem involves a partial differential equation with boundary conditions in which a phase boundary can move with time. The shoreline problem is a special case of the standard melting problem in which the fixed-flux Neumann boundary condition at the origin, x = 0, is considered the latent heat term, \( \gamma s \), and s(x,t) is a linear function of position instead of being constant. Swenson et al. modeled the movement of the shoreline in a sedimentary basin with regards to sediment line subsidence, and ocean level (Figure 1). The transport and position can be given as the following diffusion equation:

\[
\frac{\partial g}{\partial t} + v \frac{\partial^2 g}{\partial x^2} + \frac{\partial h}{\partial t} = 0, \quad 0 \leq x \leq s(t)
\]  

(1)

with boundary conditions

\[
\nu \frac{\partial g}{\partial x} \big|_{x=0} = -\overline{q}(t) \quad \text{and} \quad g(s(t), t) = l(t)
\]

(2)

where \( g(x,t) \), v, h(x,t), \( \overline{q} \), and l(t) are the height of the sediment above datum, the effective fluvial diffusivity, the height of Earth surface, the prescribed sediment line flux, and the ocean level above datum, respectively.

Another associated condition for the movement of the shoreline in an offshore submarine domain is considered as:

\[
\nu \frac{\partial \overline{g}}{\partial x} \big|_{x=s(t)} = (u-s) \left[ \alpha \frac{ds}{dt} + \frac{dl}{dt} \right] - \int_{s}^{\overline{u}} \frac{\partial h}{\partial t} dx
\]

(3)

with the initial condition \( s(0) = 0 \)

(4)

where \( \alpha \) and \( u(t) \) are the position of the interface between the toe of the submarine sediment wedge and the ocean basement, respectively.

An analytical similarity solution of a Stefan problem with variable latent heat has been obtained by Voller et al. They considered a special case of the above mentioned shoreline model by supposing that the sediment line-flux, \( \overline{q} \) is fixed, the ocean level \( l \) is constant (\( l = 0 \)), and there is no subsidence of the Earth’s crust (\( h(x,t) = 0 \)) (Figure 2). With the above assumptions, the shoreline problem introduced by Equations (1)-(4), is reduced to the following equations:
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Figure 1. A Schematic cross-section of a sedimentary ocean basin\textsuperscript{13}.

\[ g(x,0) = 0, \quad v \frac{\partial g}{\partial x} \big|_{x=0} = -q, \quad g(s(t),t) = 0 \]  
\[ \text{(5)} \]

with initial and boundary conditions

\[ g(x,0) = 0, \quad v \frac{\partial g}{\partial x} \big|_{x=0} = -q, \quad g(s(t),t) = 0 \]  
\[ \text{(6)} \]

and

\[ -v \frac{\partial g}{\partial x} \big|_{x=s(t)} = s \frac{ds}{dt}, \quad s(0) = 0 \]  
\[ \text{(7)} \]

where

\[ \alpha(u-s) = \frac{\alpha \beta s}{\alpha - \beta} = \gamma s \quad (\beta < \alpha) \quad \text{and} \quad \gamma \quad \text{is a constant}. \]

The aim of this paper is to present Chebyshev wavelet analysis method for solving the shoreline problem, Equations (5)-(7), and make a comparison between the solutions generated by our method, those ones reported in reference 5, and exact solutions in reference 1. To do so, we first find the similarity transformations and employ them to convert the above partial differential problem to the corresponding ordinary differential equations.

2. Lie Group Analysis of the Shoreline Problem

It is often difficult and computationally expensive to obtain the solution of a system of partial differential equations with its relevant conditions. So it is a good idea to transfer it into corresponding ordinary differential equations system which can be solved more easily than the original system of PDEs. Hence we use scaling transformations which is a special case of Lie group transformation\textsuperscript{6} and then obtain the numerical solution of the resulting system of ODEs using Chebyshev wavelet analysis method. We introduce the scaling group of transformations as follows:

\[ G: \quad x = e^{-\varepsilon \beta_1} x^*, \quad t = e^{-\varepsilon \beta_2} t^*, \quad \psi = e^{-\varepsilon \beta_3} \psi^*, \quad s = e^{-\varepsilon \beta_4} s^*. \quad \text{(8)} \]

Here \( \varepsilon \) is the parameter of the group G and \( \beta_i \) (i = 1, …, 4) are arbitrary real numbers not all zero, simultaneously.

Equations (5)-(7) will stay fixed under the group of Transformation (8) supposing that \( \beta_1 \) 's are related to each other as

\[ \beta_2 - \beta_3 = 2 \beta_1 - \beta_3 \]
\[ \beta_1 - \beta_2 = 0 \]
\[ \beta_4 - \beta_3 = 0 \]
\[ \beta_1 - \beta_4 = 0 \]
\[ \Rightarrow \beta_2 = 2 \beta_1, \quad \beta_3 = \beta_1, \quad \beta_4 = \beta_1. \]

The characteristic equations are

\[ \frac{dx}{x} = \frac{dt}{2t} = \frac{dg}{h} = \frac{ds}{s}. \quad \text{(9)} \]

Solving system of Equations (9), we get

\[ \eta = \frac{x}{2\sqrt{t}}, g = 2\sqrt{2f(\eta)}, s = 2\sqrt{t} \zeta. \quad \text{(10)} \]

Substituting similarity Transformations (10) into Equations (5)-(7), we obtain the following transformed equations

\[ v \frac{d^2 f}{d\eta^2} + \eta \frac{df}{d\eta} - f(\eta) = 0, \quad 0 \leq \eta \leq \zeta \quad \text{(11)} \]

with

\[ v \frac{df}{d\eta} \bigg|_{\eta=0} = -q, \quad f(\eta) \bigg|_{\eta=0} = 0, \quad \text{and} \quad v \frac{df}{d\eta} \bigg|_{\eta=\zeta} = -2\gamma \zeta^2 \quad \text{(12)} \]
where $\zeta$ is a constant.

3. Chebyshev Wavelet Analysis Method

Wavelets are attained from a generating single mother wavelet by translation and dilation operations. If the dilation parameter $a$ and the translation parameter $b$ vary in a continuous manner, the continuous wavelets

$$\psi_{a,b}(t) = |a|^{\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), a, b \in R, a \neq 0.$$  

are generated. Chebyshev wavelets

$$\psi_{nm}(t) = \psi(k, n, m, t),$$

have four arguments;

$$n = 1, \ldots, 2^{k-1}.$$  

The argument $k$ is a positive integer, $m = 0, 1, \ldots, M$, is the order for Chebyshev polynomials and $t$ is the normalized time. They are defined on the interval $[0, 1]$ by

$$\begin{align*}
\phi_{n,m}(t) &= \begin{cases} 2^n p_m \left(2^n t - 2^n \pm 1\right), & \frac{n-1}{2^n} \leq t < \frac{n}{2^n} \\ 0, & \text{otherwise} \end{cases} \quad (13)
\end{align*}$$

where

$$P_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0 \\ \sqrt{\frac{2}{\pi}} T_m(t), & m \geq 1 \end{cases},$$

$$T_0(t) = 1, T_1(t) = t, T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t). \quad (14)$$

The weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$, must be dilated and translated as following:

| Table 1. The exact values, numerical values, and absolute errors at $\alpha = 1.7$ |
|---|---|---|---|---|---|
| $\beta$ | $S$ | Exact value | Numerical value | Absolute error | Our method | Absolute error |
| 0.5 | 0.1 | 0.005442 | 0.0035 | $1.6 \times 10^{-3}$ | 0.005442 | 0 |
| | 0.2 | 0.021768 | 0.0192 | $1.5 \times 10^{-3}$ | 0.021768 | 0 |
| | 0.3 | 0.048980 | 0.0475 | $4.1 \times 10^{-3}$ | 0.048980 | 0 |
| | 0.4 | 0.087075 | 0.0862 | $9.0 \times 10^{-4}$ | 0.087075 | 0 |
| | 0.5 | 0.136055 | 0.1377 | $6.4 \times 10^{-4}$ | 0.136055 | 0 |
| | 0.6 | 0.195920 | 0.1906 | $5.3 \times 10^{-4}$ | 0.195919 | $1.0 \times 10^{-6}$ |
| | 0.7 | 0.266670 | 0.2762 | $4.7 \times 10^{-4}$ | 0.26668 | $2.0 \times 10^{-6}$ |
| 1.0 | 0.1 | 0.014400 | 0.0096 | $4.8 \times 10^{-3}$ | 0.014380 | $2.0 \times 10^{-5}$ |
| | 0.2 | 0.056522 | 0.0529 | $3.6 \times 10^{-3}$ | 0.056522 | 0 |
| | 0.3 | 0.129424 | 0.1271 | $2.3 \times 10^{-3}$ | 0.129425 | $1 \times 10^{-4}$ |
| | 0.4 | 0.230087 | 0.2279 | $2.1 \times 10^{-3}$ | 0.230088 | $1.0 \times 10^{-4}$ |
| | 0.5 | 0.359500 | 0.3575 | $1.9 \times 10^{-3}$ | 0.359513 | $1.3 \times 10^{-5}$ |
| | 0.6 | 0.517700 | 0.5160 | $1.6 \times 10^{-3}$ | 0.517699 | $1.0 \times 10^{-6}$ |
| | 0.7 | 0.704640 | 0.7032 | $1.4 \times 10^{-3}$ | 0.704646 | $6.0 \times 10^{-6}$ |
| 1.5 | 0.1 | 0.066119 | 0.0429 | $2.3 \times 10^{-2}$ | 0.066188 | $6.9 \times 10^{-5}$ |
| | 0.2 | 0.264750 | 0.2445 | $2.0 \times 10^{-2}$ | 0.264753 | $3.0 \times 10^{-4}$ |
| | 0.3 | 0.595680 | 0.5841 | $1.1 \times 10^{-2}$ | 0.595695 | $1.5 \times 10^{-5}$ |
| | 0.4 | 1.058990 | 1.0495 | $9.5 \times 10^{-3}$ | 1.059015 | $2.5 \times 10^{-5}$ |
| | 0.5 | 1.654670 | 1.6528 | $1.9 \times 10^{-3}$ | 1.654711 | $4.1 \times 10^{-5}$ |
| | 0.6 | 2.382720 | 2.3813 | $1.4 \times 10^{-4}$ | 2.382739 | $1.9 \times 10^{-6}$ |
| | 0.7 | 3.243150 | 3.2423 | $8.4 \times 10^{-4}$ | 3.243234 | $8.3 \times 10^{-5}$ |
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\[ w_n(t) = w(2^k t - 2n + 1). \] (15)

A function \( f(t) \) in \( L^2[0,1] \) can be approximated by the truncated series as

\[ f(t) = \sum_{n=1}^{M} \sum_{m=0}^{2^{k-1}} c_{nm} \psi_{nm}(t) = C^T \psi(t) \] (16)

where \( C \) and \( \psi(t) \) are \( 2^{k-1}(M+1) \times 1 \) matrices given by

\[ C = \begin{bmatrix} c_{1,0} & c_{1,1} & \cdots & c_{1,M} & c_{2,0} & \cdots & c_{2,M} & \cdots & c_{2^{k-1},0} & \cdots \end{bmatrix}^T, \] (17)

\[ \psi(t) = \begin{bmatrix} \psi_{1,0}(t) & \psi_{1,1}(t) & \cdots & \psi_{1,M}(t) & \psi_{2,0}(t) & \cdots & \psi_{2,M}(t) & \cdots & \psi_{2^{k-1},0}(t) & \cdots \end{bmatrix}^T. \]

The derivative of the vector \( \psi(t) \), can be obtained

\[ \frac{d}{dt} \psi(t) = D \psi(t), \] (18)

in which \( D \) is a square operational matrix of derivative of order \( 2^{k}(M+1) \). Our description of wavelets here is a standard one which has often been used, e.g., in\(^{7-9}\) and the references therein.

4. Discretization of Problem

In this section, we propose the wavelet analysis method for solving Equations (11)-(12). To do so, let us first consider collocation points at the \( n \)th subintervals

\[ \left( \frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right) \text{ as } \]

\[ s_{nm} = \frac{1}{2}(t_m + 2n - 1), \quad n = 1, \ldots, 2^{k-1}, \quad m = 0, 1, \ldots, M \] (19)

where \( t_m \) are the extrema of \( M \)-th order Chebyshev polynomials \( T_M \) and are defined as

\[ t_m = \cos \left( \frac{m\pi}{M} \right), \quad m = 0, 1, \ldots, M. \]

We now approximate \( f(\eta) \), and its derivatives in Equations (11)-(12). Having replacing \( f(\eta) \) and its derivatives with their approximation, we collocate the Equation (11) at \( 2^{k-1}(M+1) - 2 \) suitable collocation points Equation (19) to obtain \( 2^{k-1}(M+1) - 2 \) nonlinear equations. From initial and boundary conditions (12), we also get three more equations. So we have a nonlinear system of \( 2^{k-1}(M+1) + 1 \) equations with the same number of unknowns which can be solved by Newton iteration to obtain the vector \( C \) and the parameter \( \zeta \). Having obtained the vector \( C \) and \( \zeta \), one can get temperature distribution in the domain and position of interface. For more information on the discretization, the reader is referred to\(^{7-9}\) and the references therein.

5. Numerical Results

We apply the introduced method by setting \( M = 10 \) and \( k = 1 \) to get numerical solution of the problem under consideration. We then make a comparison between our results with those obtained by Chebyshev polynomials of the second kind in\(^5\), and exact solutions given in\(^1\), to validate the appropriateness of the proposed method. In Table 1, the numerical results of time are obtained for different values of shoreline position (\( s \)) and gradient of basement (\( \beta \)) for fixed off-shore sediment (\( \alpha = 1.7 \)), sediment line flux \( q = 1 \) \( m^3/mt \), and effective fluvial diffusivity \( \nu = 2 \) \( m^2/t \). Table 1 shows that our results are in good agreement with the exact solutions and more accurate than those reported in\(^i\).

Figure 3. Dependence of shoreline position on time for different values of basement slope \( \beta \).

We repeat some of the numerical experiments conducted in\(^i\) but now using the Chebyshev wavelet analysis method. The results are displayed in Figures 3 and 4 and agree well with the corresponding Figures in Rajeev and Rai\(^5\). It is observed from Figure 3, that increasing the value of slope of basement (\( \beta \)) while keeping the value of \( \alpha \) fixed, results in decreasing movement of contact point (shoreline location) and slowness in the deposition of sediment. According to Rajeev and Rai\(^5\) this will cause...
sedimentation towards the land, enhancement of sediments deposited earlier, less movement of the shoreline towards the land, and slower sedimentation.

In Figure 4, for a fixed value of $\beta$, the effect of increasing the value of the gradient of off-shore sediment wedge is shown. As was highlighted by Rajeev and Rai more residue will be kept close to the contact point causing the progression of the shoreline to the ocean and hence the sedimentation turns out to be quick.

![Figure 4. Dependence of shoreline position on time for different values of off-shore sediment wedge $\alpha$.](image)

### 6. Conclusion

In this paper, Chebyshev wavelet analysis method based on operational matrix of derivatives was validated and then employed to seek the numerical solutions of a kind of moving-boundary problem called a Stefan problem. The numerical results indicated that the method is very accurate and gave results consistent with previous studies.

### 7. Acknowledgement

The financial support from the Universiti Sains Malaysia through RUI grant (Grant No: 1001/PMATHS/811252), is hereby acknowledged.

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