Master equation approach to the conjugate pairing rule of Lyapunov spectra for many-particle thermostatted systems

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The master equation approach to Lyapunov spectra for many-particle systems is applied to non-equilibrium thermostatted systems to discuss the conjugate pairing rule. We consider iso-kinetic thermostatted systems with a shear flow sustained by an external restriction, in which particle interactions are expressed as a Gaussian white randomness. Positive Lyapunov exponents are calculated by using the Fokker-Planck equation to describe the tangent vector dynamics. We introduce another Fokker-Planck equation to describe the time-reversed tangent vector dynamics, which allows us to calculate the negative Lyapunov exponents. Using the Lyapunov exponents provided by these two Fokker-Planck equations we show the conjugate pairing rule is satisfied for thermostatted systems with a shear flow in the thermodynamic limit. We also give an explicit form to connect the Lyapunov exponents with the time-correlation of the interaction matrix in a thermostatted system with a color field.

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I. INTRODUCTION

The Lyapunov exponent is an essential concept to express the instability of orbits in a dynamical system. It is introduced as an exponential expansion (or contraction) rate of an infinitesimal perturbation of orbits, and its positivity implies that the system is chaotic. In general there is a Lyapunov exponent for each independent direction of the infinitesimal perturbation of the orbit, and the sorted set of such Lyapunov exponents is called the Lyapunov spectrum, and has been the subject of study in many-particle systems. For example, the existence of its thermodynamic limit \[ \lim_{N \to \infty} \frac{1}{N} \sum \lambda_i \] an effect of the rotational degrees of freedom of molecules, its stepwise structure and the Lyapunov modes \[ \lambda \] and a tracer particle effect \[ \lambda \] have been observed and discussed in the Lyapunov spectra of many-particle chaotic systems.

A characteristic of the Lyapunov spectrum that is known in Hamiltonian systems, is that the Lyapunov exponents appear as a pair, namely any positive Lyapunov exponent accompanies a negative Lyapunov exponent with its opposite sign \[ \sum \lambda_i = 0 \]. This characteristic, which is based on the symplectic structure of the Hamiltonian mechanics, is not correct in non-Hamiltonian systems, but it is interesting to know how it is modified in quasi-Hamiltonian systems such as a Hamiltonian system coupled to a thermodynamic reservoir. This problem has been considered in some thermostatted dynamics with a term to extract the heat produced in the system by external force fields, and led to the proposal of the conjugate pairing rule for thermostatted systems, which claims that the sum of any Lyapunov exponent pair is not zero but a constant regardless of the exponent numbers. \[ \sum \lambda_i = \text{constant} \]. This conjecture was confirmed by many following numerical calculations \[ \sum \lambda_i = \text{constant} \]. This also led to the discovery that some thermostatted systems contain hidden Hamiltonian structure \[ \sum \lambda_i = \text{constant} \]. The pairing rule is not only interesting as a mathematical structure of the thermostatted system but also valuable for a practical use; the conjugate pairing rule for the thermostatted system allows us to calculate non-equilibrium transport coefficients (e.g. conductivity and viscosity) from only one pair of the Lyapunov exponents, such as the largest and smallest Lyapunov exponents only \[ \sum \lambda_i = \text{constant} \].

A problem is that the necessary and the sufficient conditions for the conjugate pairing rule to hold for thermostatted systems is not clearly known. The conjugate pairing rule for the iso-kinetic thermostatted system with a color field was proved for the soft core interaction potential \[ \sum \lambda_i = \text{constant} \] and the hard core interaction potential \[ \sum \lambda_i = \text{constant} \], regardless of the number of particles. A similar discussion was done in Nosé Hamiltonian dynamics \[ \sum \lambda_i = \text{constant} \].

These works give the sufficient conditions for the conjugate pairing rule. On the other hand, it was suggested numerically that it can be violated in the presence of a magnetic field \[ \sum \lambda_i = \text{constant} \] and in inhomogeneously thermostatted systems such as a system under temperature gradient \[ \sum \lambda_i = \text{constant} \] or a system in which the peculiar momenta are thermostatted \[ \sum \lambda_i = \text{constant} \]. Another numerical work also suggested that it is not exact in the iso-energetic thermostat with a finite number of particles \[ \sum \lambda_i = \text{constant} \], although the iso-energetic thermostat should be equivalent to the iso-kinetic thermostat as the number of the particles goes to infinity, namely in the thermodynamic limit \[ \sum \lambda_i = \text{constant} \]. A special interest is the iso-kinetic thermostatted system with a shear field, which is described by the Sllod equation for the planar Coutte flow \[ \sum \lambda_i = \text{constant} \]. Early investigations supported the conjugate pairing rule for such a system \[ \sum \lambda_i = \text{constant} \]. Refs. \[ \sum \lambda_i = \text{constant} \] suggested a small deviation from the conjugate pairing rule. An analytical consideration showed that the deviation from the conjugate pairing rule should be at most fourth order in the shear rate in the case of a small shear rate in the thermodynamic limit \[ \sum \lambda_i = \text{constant} \]. However a recent numerical calculation with a more careful numbering of the Lyapunov exponents and with numerical error bars showed that within the numerical
precision the conjugate pairing rule was satisfied [3]. After all these trials, a justification of the conjugate pairing rule for the iso-kinetic thermostatted system with a shear field still remains as an open problem.

Analytical calculation of the full Lyapunov spectra for many-particle systems is still not an easy matter, and so far full Lyapunov spectra have been calculated mainly using numerical approaches. On the other hand, recently an analytical approach to the full Lyapunov spectra was proposed for many-particle systems with a random interaction [13]. This approach describes the tangent vector dynamics by a master equation, and allows the calculation of all individual positive Lyapunov exponents through the average of the magnitude of the tangent vector. It was used to explain the stepwise structure of the Lyapunov spectrum for many-particle Hamiltonian systems. However a generalization of this approach to non-equilibrium systems, especially to thermostatted systems, is not known yet.

The purpose of this paper is to generalize the master equation approach to the Lyapunov spectrum to non-equilibrium thermostatted systems, and to discuss the conjugate pairing rule of the Lyapunov spectrum for the iso-kinetic thermostatted system with a shear field. Especially we propose a method to calculate negative Lyapunov exponents using a time-reversed master equation. In this paper we concentrate on the case where the particles interact with a Gaussian white randomness. In this case the master equation that describes the tangent vector dynamics is attributed to the Fokker-Planck equation. We also restrict our consideration in the thermodynamic limit, in which the fluctuations of the friction coefficient can be neglected [26]. Under these conditions we show that the conjugate pairing rule for the thermostatted system with a shear field given by the Sllod equation is satisfied. As a special case we also discuss briefly an explicit form to connect the Lyapunov exponents with the time-correlation of the interaction matrix in a thermostatted system without a shear field.

II. ISOKINETIC THERMOSTATTED SYSTEM WITH A SHEAR FIELD AND ITS TANGENT VECTOR DYNAMICS

We consider non-equilibrium systems with an iso-kinetic thermostat. Our consideration includes the case where a shear flow is sustained by an external restriction, and for simplicity we consider a two-dimensional system consisting of $N$ particles with the same mass $m$. We introduce $\mathbf{q}^{(j)}(t) = (q^{(j)}_1(t), q^{(j)}_2(t))$, and $\mathbf{p}^{(j)}(t) = (p^{(j)}_1(t), p^{(j)}_2(t))$ as the spatial coordinate vector and the momentum vector of the $j$-particle, respectively, at time $t$ with the transpose operation $T$. (Note that all vectors in this paper are introduced as column vectors.) Equations for $\mathbf{q}^{(j)}(t)$ and $\mathbf{p}^{(j)}(t)$ are expressed as [26]

$$\frac{d\mathbf{q}^{(j)}(t)}{dt} = \frac{1}{m} \mathbf{p}^{(j)}(t) + \gamma \Xi_2 \mathbf{q}^{(j)}(t) \quad (1)$$

$$\frac{d\mathbf{p}^{(j)}(t)}{dt} = -\frac{\partial U(t)}{\partial \mathbf{q}^{(j)}(t)} - \{\gamma \Xi_2 + \alpha(t) I_2\} \mathbf{p}^{(j)}(t) \quad (2)$$

where $U(t)$ is the potential energy as a function of $\mathbf{q}^{(j)}(t)$, $j = 1, 2, \cdots, N$ and $t$ only, and we introduce $\Xi_{2k}$ as the $2k \times 2k$ matrix defined by

$$\Xi_{2k} \equiv \begin{pmatrix} 0_k & I_k \\ 0_k & 0_k \end{pmatrix} \quad (3)$$

with the $k \times k$ identical matrix $I_k$ and the $k \times k$ null matrix $0_k$. Here $\gamma$ is the shear rate as an external parameter, namely a constant gradient of the $x$ component of the local velocity in the $y$ direction, and $\alpha(t)$ is defined by

$$\alpha(t) \equiv -\frac{\sum_{j=1}^{N} \left| \mathbf{p}^{(j)}(t) \right|^2}{\sum_{j=1}^{N} \left| \mathbf{p}^{(j)}(t) \right|^2} \quad (4)$$

so that the total kinetic energy is constant in time: $d\left[ \sum_{j=1}^{N} \left| \mathbf{p}^{(j)}(t) \right|^2/(2m) \right]/dt = 0$. Eqs. (1) and (2) are called the Sllod equation for the planar Couette flow with the iso-kinetic thermostat, and gives the model for the system driven by external fields and/or a shear rate with an attached heat reservoir which removes the energy generated inside the system and maintains the temperature of the system constantly in time. As an example described by Eqs. (1) and (2), other than the system with a shear field, we may mention the color field system in which the system consists of many particles with charges of different signs and is driven by an external electric field [13, 42].

In general, the quantity $\alpha(t)$, which is interpreted as the friction coefficient, depends on the coordinates and the momenta of the particles, so is variable in time. However, it is known that the fluctuation of the quantity $\alpha(t)$ is small in a system consisting of many particles [26]. (For a justification of this point by the kinetic approach see Ref. [3], which shows that the quantity $\alpha(t)$ fluctuates with the order of $1/\sqrt{N}$ around a fixed value.) Based on this fact, in this paper we consider only the system which consists of enough particles so that the friction coefficient $\alpha(t)$ in Eq. (2) can be replaced by a fixed constant $\bar{\alpha}$.

For a convenience we represent the $4N$-dimensional phase space vector $\mathbf{\Gamma}(t)$ as a vector $(q_1^{(1)}(t), q_2^{(1)}(t), \cdots, q_x^{(N)}(t), q_y^{(1)}(t), q_y^{(2)}(t), \cdots, q_y^{(N)}(t), p_1^{(1)}(t), p_2^{(1)}(t), \cdots, p_x^{(N)}(t), p_y^{(1)}(t), p_y^{(2)}(t), \cdots, p_y^{(N)}(t))^T$. Using this notation and the assumption explained in the previous paragraph we obtain the equation

$$\frac{d\delta \mathbf{\Gamma}(t)}{dt} = \mathbf{\mathcal{L}}(t) \delta \mathbf{\Gamma}(t) \quad (5)$$

for the tangent vector $\delta \mathbf{\Gamma}(t)$. Here the matrix $\mathbf{\mathcal{L}}(t)$ is given by

$$\mathbf{\mathcal{L}}(t) \equiv \begin{pmatrix} \Phi & I_{2N}/m & \Psi \end{pmatrix} \quad (6)$$
with $2N \times 2N$ matrices $\Phi$, $\Psi$ and $R$ defined by
\[ \Phi \equiv \gamma \Xi_{2N}, \quad (7) \]
\[ \Psi \equiv -\gamma \Xi_{2N} - \bar{\alpha} I_{2N}, \quad (8) \]
\[ R(t) \equiv -\frac{\partial^2 U(t)}{\partial q(t) \partial \dot{q}(t)} \quad (9) \]
where we introduced $q(t)$ as a vector $(q_x^{(1)}(t), q_x^{(2)}(t), \ldots, q_x^{(N)}(t), q_y^{(1)}(t), q_y^{(2)}(t), \ldots, q_y^{(N)}(t))^T$.

III. RANDOM INTERACTIONS AND MASTER EQUATIONS FOR THE TANGENT VECTOR DYNAMICS

In this section we introduce a random interaction between the particles, and obtain the two kinds of master equations corresponding to the time-forward tangent vector dynamics and the time-reversed tangent vector dynamics by using Kramers-Moyal expansion technique.

A. Fokker-Planck equation for the forward dynamics of the tangent vector

We consider the case that each particle interacts with the other particles randomly enough so that the matrix $R(t) \equiv (R_{jk}(t))$ can be regarded as a Gaussian white random matrix satisfying the conditions
\[ \langle R_{\mu_1 \nu_1}(t_1) R_{\mu_2 \nu_2}(t_2) \cdots R_{\mu_{2n-1} \nu_{2n-1}}(t_{2n-1}) \rangle = 0, \quad (10) \]
\[ \langle R_{\mu_1 \nu_1}(t_1) R_{\mu_2 \nu_2}(t_2) \cdots R_{\mu_{2n} \nu_{2n}}(t_{2n}) \rangle = \sum_{P_d} D_{\mu_j \nu_j \mu_j \nu_j} D_{\mu_j \nu_j \mu_j \nu_j} \cdots D_{\mu_{2n} \nu_{2n} \mu_{2n} \nu_{2n}} \times \delta(t_{j_1} - t_{j_2}) \delta(t_{j_3} - t_{j_4}) \cdots \delta(t_{j_{2n-1}} - t_{j_{2n}}) \quad (11) \]
for any integer $n$ and a 4-th rank constant tensor $D_{jklm}$, where we take the sum over only the permutation $P_d$ : $(1, 2, \ldots, 2n) \to (j_1, j_2, \ldots, j_{2n})$, and the bracket $\langle \cdots \rangle$ means the ensemble average over random processes.

Under the randomness conditions (11) and (12) the time-evolutional equation (8) is regarded as a stochastic equation of the Langevin type, and its corresponding master equation for the probability density $\rho^{(+)}(\delta \Gamma, t)$ at time $t$ is given by
\[ -\sum_{\mu=1}^{2N} \sum_{\nu=1}^{2N} \frac{\partial}{\partial \delta q_\nu} \left( \Phi_{\mu \nu} \delta q_\nu + \frac{\delta_{\mu \nu}}{m} \delta p_\nu \right) \rho^{(+)}(\delta \Gamma, t) \]
\[ + \sum_{\mu=1}^{2N} \sum_{\nu=1}^{2N} \sum_{\mu'=1}^{2N} \sum_{\nu'=1}^{2N} \frac{1}{2} D_{\mu \nu \mu' \nu'} \delta q_\nu \delta q_\nu \]
\[ \times \frac{\partial^2}{\partial \delta p_\mu \partial \delta p_{\mu'}} \rho^{(+)}(\delta \Gamma, t) \quad (12) \]

applying the Kramers-Moyal expansion technique to the dynamics (8). Here $\delta q_j$ and $\delta p_j$ are the $j$-th components of the coordinate part $\delta \mathbf{q}$ and the momentum part $\delta \mathbf{p}$ in the tangent vector $\delta \Gamma = (\delta \mathbf{q}, \delta \mathbf{p})^T$, respectively, and $\Phi_{\mu \nu}$ and $\Psi_{\mu \nu}$ are the matrix elements of the matrices $\Phi$ and $\Psi$ defined by Eqs. (7) and (8), respectively. The derivation of Eq. (12) is given in Appendix A. Eq. (12) in the special case of $\Phi = 0_{2N}$ and $\Psi = 0_{2N}$ have already been used to discuss the stepwise structure of the Lyapunov spectrum for a many-particle Hamiltonian system [3].

B. Anti-Fokker-Planck equation for the time-reversed dynamics of the tangent vector

As shown in Ref. [31], in the case of $\gamma = 0$ and $\bar{\alpha} = 0$ the Fokker-Planck equation (12) provides the positive Lyapunov exponents as the time averaged exponential rate of the randomness average by the probability density $\rho^{(+)}(\Gamma, t)$ in the time evolution of infinitesimal perturbations of the dynamical variables. However, this method does not provide directly the negative Lyapunov exponents, because in the stochastic system the randomness average of the distance between the infinitesimal nearby trajectories should not shrink in the infinite time limit. This was not a problem in the Hamiltonian system discussed in Ref. [31], because in the Hamiltonian system the absolute values of the negative Lyapunov exponents are the same with the positive Lyapunov exponents [7]. However in the thermostatted system discussed in this paper such a simple relation of the negative and the positive Lyapunov exponents can not be expected any more. In order to overcome this problem and to provide the negative Lyapunov exponents using the master equation approach directly, we use the fact that the negative Lyapunov exponents can be regarded as the positive Lyapunov exponents for the time-reversed motion. This fact has been used in some works to calculate the negative Lyapunov exponents for chaotic systems [16,34,35].

In the iso-kinetic thermostatted system with a shear field the time-reversed motion is expressed by the "time-reversed mapping" $\mathbf{q} \rightarrow -\mathbf{q}$, $\mathbf{p} \rightarrow -\mathbf{p}$ and $\gamma \rightarrow -\gamma$. The transformation $\gamma \rightarrow -\gamma$ is justified by the fact that the direction of the shear flow changes to the opposite direction in the time-reversed motion. This justifies the time-reversal transformation $\bar{\alpha} \rightarrow -\bar{\alpha}$ for the friction coefficient by Eq. (4). The time-reversed mapping leads to the time-reversed transformations...
for the tangent vector dynamics, noting the relations (13) and (14) to connect the matrices $\Phi$ and $\Psi$ with the shear rate $\gamma$ and the friction coefficient $\bar{\alpha}$. It is important to note that the tensor $D_{\mu\nu}$ and the tangent dynamical equation (15) themselves are invariant under the transformations (13, 17).

Now we introduce the Fokker-Planck equation for the probability density $\rho^{(\gamma)}(\delta \Gamma, t)$ for the time-reversed tangent vector at time $t$ as the transformed equation of the Fokker-Planck equation (12) by the transformations (13, 17), namely

$$\frac{\partial}{\partial t} \rho^{(\gamma)}(\delta \Gamma, t) = -\sum_{\mu=1}^{2N} \sum_{v=1}^{2N} \frac{\partial}{\partial \delta q_{\mu}} \left( \Phi_{\mu\nu} \delta q_{\nu} + \frac{\delta_{\mu\nu}}{m} \delta p_{\nu} \right) \rho^{(\gamma)}(\delta \Gamma, t)$$

$$-\sum_{\mu=1}^{2N} \sum_{v=1}^{2N} \frac{\partial}{\partial \delta p_{\mu}} \Psi_{\mu\nu} \delta p_{\nu} \rho^{(\gamma)}(\delta \Gamma, t)$$

$$-\sum_{\mu=1}^{2N} \sum_{\nu=1}^{2N} \frac{1}{2} D_{\mu\nu} \delta p_{\mu} \delta q_{\nu} \frac{\partial^2}{\partial \delta p_{\mu} \partial \delta p_{\nu}} \rho^{(\gamma)}(\delta \Gamma, t).$$  

(18)

This is simply the equation with the opposite sign of the diffusion term to the forward Fokker-Planck equation (12), and is interpreted as the master equation to describe the time-evolution of the tangent vector whose initial condition is the time-reversed initial condition to the Fokker-Planck equation (12). We call this equation for the probability density $\rho^{(\gamma)}(\delta \Gamma, t)$ the "anti-Fokker-Planck equation" in this paper, and calculate the negative Lyapunov exponents as the opposites of the positive Lyapunov exponents calculated by using the Fokker-Planck equation (12). In the next section we show it actually as the special case of more general results.

IV. CONJUGATE PAIRING RULE FOR THERMOSTATTED SYSTEMS WITH A SHEAR

We have to know the time evolution of the amplitude of the tangent vector in order to calculate the Lyapunov exponents. Such a time evolution for the forward movement (the time-reversed movement) is expressed as the dynamics of the diagonal elements of the matrix $\Upsilon^{(+)}(t)$ (the matrix $\Upsilon^{(-)}(t)$) given by

$$\Upsilon^{(\pm)}(t) \equiv \langle \delta q \delta q^T \rangle_t^{(\pm)}.$$  

(19)

Here the bracket $\langle \cdots \rangle_t^{(\pm)}$ means to take the average by the probability density $\rho^{(\pm)}(\delta \Gamma, t)$, namely

$$\langle X(\delta \Gamma) \rangle_t^{(\pm)} \equiv \int d\delta \Gamma X(\delta \Gamma) \rho^{(\pm)}(\delta \Gamma, t).$$  

(20)

for any function $X(\delta \Gamma)$ of $\delta \Gamma$. In Ref. [31] the positive Lyapunov exponents were calculated by the time-averaged exponential rate of the diagonal elements of an orthogonal-transformed matrix of $\Upsilon^{(+)}(t)$. We get the negative Lyapunov exponents from the matrix $\Upsilon^{(-)}(t)$ by a similar procedure. The introduction of Lyapunov exponents by the spatial coordinate part only (or the momentum part only) of the tangent vector has been used previously by Refs. [32, 33].

We introduce the matrix $\hat{\Upsilon}^{(\pm)}(t)$ defined by

$$\hat{\Upsilon}^{(\pm)}(t) \equiv \Upsilon^{(\pm)}(t)e^{\pm \bar{\alpha} t}.$$  

(21)

As shown in Appendix E, the matrix $\hat{\Upsilon}^{(\pm)}(t)$ satisfies the differential equation

$$\frac{d^4 \hat{\Upsilon}^{(\pm)}(t)}{dt^4} - \frac{1}{2} \left[ \Omega^2 \frac{d^2 \hat{\Upsilon}^{(\pm)}(t)}{dt^2} + \frac{d^2 \hat{\Upsilon}^{(\pm)}(t)}{dt^2} (\Omega^2)^T \right]$$

$$+ \frac{1}{16} \left[ \Omega^4 \hat{\Upsilon}^{(\pm)}(t) - 2\Omega^2 \hat{\Upsilon}^{(\pm)}(t) (\Omega^2)^T \right]$$
\begin{align}
&\bar{\Upsilon}^{(\pm)}(t) (\Omega^2)^T - \frac{2}{m^2} \mathcal{D} \left\{ \frac{d\bar{\Upsilon}^{(\pm)}(t)}{dt} \right\} = 0
\end{align}

(22)

where we assumed the probability density \( \rho^{(\pm)}(\delta\Gamma, t) \) to be zero at the boundary of the tangent space. Here \( \Omega \) is the \((2N) \times (2N)\) matrix defined by

\[ \Omega \equiv \Phi - \Psi. \]

(23)

and \( \mathcal{D} \) is the linear operator to map any \((2N) \times (2N)\) matrix \( X \equiv (X_{jk}) \) to the \((2N) \times (2N)\) matrix \( \mathcal{D}\{X\} \equiv (\mathcal{D}\{X\}_{jk}) \) defined by

\[ (\mathcal{D}\{X\})_{jk} \equiv \sum_{\mu=1}^{N} \sum_{\nu=1}^{N} D_{j\mu k\nu} X_{\mu\nu}. \]

(24)

It may be noted that Eq. (22) for the matrix \( \bar{\Upsilon}^{(\pm)}(t) \) is invariant under the transformation \( \Omega \rightarrow -\Omega \).

We can choose the initial probability density \( \rho^{(\pm)}(\delta\Gamma, 0) \) arbitrarily to calculate the positive Lyapunov exponents. On the other hand in order to derive the corresponding negative Lyapunov exponents we assume the initial probability density \( \rho^{(-)}(\delta\Gamma, 0) \) for the time-reversed tangent vector to satisfy the condition

\[ d^b\bar{\Upsilon}^{(-)}(t)/dt^b|_{t=0} = d^b\bar{\Upsilon}^{(+)}(t)/dt^b|_{t=0}, \quad k = 0, 1, 2, 3 \]

at the initial time \( t = 0 \). Under this assumption it follows from Eq. (22) that

\[ \bar{\Upsilon}^{(-)}(-t)e^{-\bar{\alpha}t} = \bar{\Upsilon}^{(+)}(+t)e^{\bar{\alpha}t}, \]

(25)

because the quantities \( \bar{\Upsilon}^{(+)}(t) \) and \( \bar{\Upsilon}^{(-)}(t) \) defined by Eq. (22) satisfy the same differential equation (22) and have the same initial condition. Therefore the diagonal element \( \bar{\Upsilon}^{(\pm)}_{jj}(t) \) of any orthogonal-transformed matrix of \( \bar{\Upsilon}^{(\pm)}(t) \) must satisfy the relation

\[ \bar{\Upsilon}^{(-)}_{jj}(-t) = \bar{\Upsilon}^{(+)}_{jj}(+t)e^{2\bar{\alpha}t}. \]

(26)

This equation connects the time-forward evolution and the time-reversed evolution in the amplitudes of the spatial part of the tangent vector in the thermostatted system.

The \( j \)-th positive (or zero) Lyapunov exponent \( \lambda^{(+)}_{jj} \) and its conjugate negative (or zero) exponent \( \lambda^{(-)}_{jj} \) are given by

\[ \lambda^{(\pm)}_{jj} = \lim_{t \rightarrow \pm \infty} \frac{1}{2t} \ln \frac{\bar{\Upsilon}^{(\pm)}_{jj}(t)}{\bar{\Upsilon}^{(\pm)}_{jj}(0)}. \]

(27)

The quantity \( \bar{\Upsilon}^{(\pm)}_{jj}(t) \) satisfies the condition \( \bar{\Upsilon}^{(\pm)}_{jj}(0) = \bar{\Upsilon}^{(\pm)}_{jj}(-0) \) at the initial time, and using Eqs. (24) and (27) we obtain

\[ \lambda^{(+)}_{jj} + \lambda^{(-)}_{jj} = -\bar{\alpha}. \]

(28)

This is the conjugate pairing rule of the Lyapunov spectrum for the iso-kinetic thermostatted system with a shear field. It is clear that it is attributed to the pairing rule of the Hamiltonian system in the case of \( \alpha = 0 \).

V. CONJUGATE PAIRING RULE FOR A COLOR FIELD

For an actual calculation of the Lyapunov spectrum for an iso-kinetic thermostatted system by the master equation approach we have to know the value of tensor \( D_{j\mu k\nu} \) and to solve the differential equation (22) for the matrix \( \bar{\Upsilon}^{(\pm)}(t) \). Let’s discuss these points briefly by using a case without a shear field such as a color field system \([12,32]\), namely

\[ \gamma = 0. \]

(29)

For simplicity in this section we also assume that the tensor \( D_{j\mu k\nu} \) is expressed as the multiplication of the matrix elements of a symmetric \((2N) \times (2N)\) matrix \( \mathcal{W} \equiv (W_{jk}) \):

\[ D_{j\mu k\nu} = W_{jk}W_{\mu\nu}. \]

(30)

This assumption has already been used in Ref. [31].

As shown in Appendix C, under the assumptions (29) and (31) the equation of the matrix \( \bar{\Upsilon}^{(\pm)}(t) \) is simplified to

\[ \frac{d^2 \bar{\Upsilon}^{(\pm)}(t)}{dt^2} - \bar{\alpha}^2 \frac{d \bar{\Upsilon}^{(\pm)}(t)}{dt} - \frac{2}{m^2} \mathcal{W} \bar{\Upsilon}^{(\pm)}(t)W(t) = 0. \]

(31)

It is shown that Eq. (22) is given by taking the time-differential in both the sides of Eq. (31). Using the orthogonal matrix \( V \) diagonalizing the matrix \( \mathcal{W} \), namely

\[ (V^T \mathcal{W} V)_{jk} = \omega_j \delta_{jk} \]

(32)

with a real eigenvalue \( \omega_j \), the quantity \( \bar{\Upsilon}^{(\pm)}_{jj}(t) \) is expressed as the diagonal element of the matrix \( \bar{\Upsilon}^{(\pm)}(t) \equiv (\bar{\Upsilon}^{(\pm)}_{jk}(t)) \) defined by

\[ \bar{\Upsilon}^{(\pm)}(t) = V^T \bar{\Upsilon}^{(\pm)}(t)V. \]

(33)

Using Eq. (33) we can solve the equation for the quantity \( \bar{\Upsilon}^{(\pm)}(t) \) derived from Eq. (31), and by using Eqs. (22), (24) and (33) the Lyapunov exponents are given by

\[ \lambda^{(\pm)}_{jj} = -\frac{\bar{\alpha}}{2} \pm \frac{1}{2} \left( \lambda_j + \frac{\bar{\alpha}^2}{3\lambda_j} \right). \]

(34)

where \( \lambda_j \) is defined by

\[ \lambda_j \equiv \left[ \left( \frac{\omega_j}{m} \right)^2 + \sqrt{\left( \frac{\omega_j}{m} \right)^4 - \left( \frac{\bar{\alpha}^2}{3} \right)^2} \right]^{1/3}. \]

(35)

(See Appendix C about a derivation of Eq. (24).) It is clear that the Lyapunov exponents given by Eq. (34) satisfy the conjugate pairing rule \( \bar{\alpha} = 0 \).
Concerning the expression (24) for the Lyapunov exponent it is important to note that the tensor $D_{j\text{kin}}$ can depend on external force fields. This implies that the eigenvalue $\omega_j$ of the matrix $W$ can depend on the friction coefficient $\alpha$. If we were to assume the quantity $\omega_j$ to be independent of the friction coefficient $\alpha$, then we obtain the expression of the Lyapunov exponents as $\lambda_j^{(3)} = \pm [\omega_j/(2m)]^{2/3} - \alpha/2 + O(\alpha^2)$ from Eq. (24) in the case of $|\alpha| \leq \sqrt{2} |\omega_j/m|^{2/3}$. However this is not consistent with the numerical results in a deterministic many-hard-disk system with a color field in which the negative Lyapunov exponents rather increase as the value of the friction coefficient increases (12). This consideration suggests that we should not neglect the external force field dependence of the correlation amplitude $D_{j\text{kin}}$ at least in the color field case. The dependence of the tensor $D_{j\text{kin}}$ on the shear rate and the external force fields should be a subject for a separated paper, although the conjugate pairing rule of the Lyapunov spectrum is correct regardless of their dependence as shown in this paper.

VI. CONCLUSION AND REMARKS

In this paper we have applied the master equation approach to Lyapunov spectra to non-equilibrium iso-kinetic thermostatted systems in order to discuss a conjugate pairing rule. We considered two-dimensional many-particle system with Gaussian white random interactions between the particles. In this system the positive Lyapunov exponents are calculated by a (forward) Fokker-Planck equation for the tangent vector dynamics. We proposed a method to calculate the negative Lyapunov exponents by a time-reversed master equation, especially the anti-Fokker-Planck equation where the diffusion term has the opposite sign to the forward Fokker-Planck equation. Using the Lyapunov exponents calculated by these two Fokker-Planck equations we showed the conjugate pairing rule of the Lyapunov spectrum for iso-kinetic thermostatted systems with a shear field given by the Slrod equation in the thermodynamic limit. We also gave an concrete form to connect the Lyapunov exponents with the time-correlation of the interaction matrix in a thermostatted system without a shear field.

We discussed the conjugate pairing rule based on the iso-kinetic thermostat in the thermodynamic limit. However it is known that the iso-kinetic thermostat is formally equivalent to other thermostats such as the isoenergetic thermostat in the thermodynamic limit (28). In this sense our result should be correct in systems with such other thermostats, more explicitly as far as the friction coefficient can be regarded as a constant even in a finite number of particle systems.

In order to get the anti-Fokker-Planck equation we used the fact that the shear rate $\gamma$ changes its sign in the time reversed motion. However this time-reversed change of the sign of the shear rate to get the anti-Fokker-Planck equation may not be essential to obtain the negative Lyapunov exponents, if the Lyapunov exponents are even functions of the shear rate. We have not proved the invariance of the Lyapunov exponents under the transformation $\gamma \to -\gamma$ in this paper, but the invariance of Eq. (24) under the transformation $\Omega \to -\Omega$ implies that the Lyapunov exponents are invariant under this transformation.

We can show that all the Lyapunov exponents $\lambda_j^{(+)}$ ($\lambda_j^{(-)}$) are non-negative (non-positive) in the case of $\gamma = 0$ (See Appendix 3). This implies that in this case the number of the positive Lyapunov exponents should be equal to the number of the negative Lyapunov exponents. However we have not proved that it is also correct in the presence of a shear field: $\gamma \neq 0$. Concerning this point we should notice that a numerical calculation of the Lyapunov spectrum for the Slrod equation (1) and (2) showed that in the case of a high shear rate the number of the positive Lyapunov exponents can be less than the number of the negative Lyapunov exponents (29). Therefore it should be interesting to check whether the master equation approach to the Lyapunov spectrum can describe such a situation or not.

It should be noted that the discussion of this paper does not conclude that the conjugate pairing rule of the Lyapunov spectrum for the thermostatted system with a shear field must be satisfied rigorously in deterministic chaotic systems. To show the conjugate pairing rule in this paper we assumed the Gaussian white randomness (13) and (14) for the particle interactions, and there is no guarantee that we can justify the conjugate pairing rule by the master equation approach under a more general random interaction of particles, especially under the randomness with a long time correlation which leads to a more general master equation for the tangent vector rather than a simple Fokker-Planck equation. A generalization of the conjugate pairing rule by the master equation approach to a more general random interaction remains as one of the important future problems.

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APPENDIX A: MASTER EQUATION FOR THE TANGENT VECTOR

In this appendix we derive the Fokker-Planck equation (23) for the tangent vector space. Using the Kramers-Moyal expansion the dynamics of the probability density $\rho^{(+)}(\delta \Gamma, t)$ is given by (37)

$$
\frac{\partial \rho^{(+)}(\delta \Gamma, t)}{\partial t} = \sum_{n=1}^{\infty} \sum_{j_1=1}^{2N} \sum_{j_2=1}^{2N} \cdots \sum_{j_n=1}^{2N} (-1)^n \frac{\partial^n \Xi^{(n)}_{j_1 j_2 \cdots j_n}(\delta \Gamma, t)}{\partial \delta \Gamma_{j_1} \partial \delta \Gamma_{j_2} \cdots \partial \delta \Gamma_{j_n}} \rho^{(+)}(\delta \Gamma, t)
$$

(A.1)

where $\Xi^{(n)}_{j_1 j_2 \cdots j_n}(\delta \Gamma, t)$ is defined by...
\[\Xi^{(n)}_{j_1j_2\ldots j_n}(\delta\Gamma, t) = \lim_{s \to 0} \frac{1}{s} \langle [\delta\Gamma_{j_1}(t+s) - \delta\Gamma_{j_1}(t)] \times [\delta\Gamma_{j_2}(t+s) - \delta\Gamma_{j_2}(t)] \times \cdots \times [\delta\Gamma_{j_n}(t+s) - \delta\Gamma_{j_n}(t)] \rangle_{\delta\Gamma(t) = \delta\Gamma} \quad \text{(A.2)}\]

and \(\delta\Gamma_j(t)\) is the \(j\)-th component of the tangent vector \(\delta\Gamma(t)\).

Using Eq. (3) we obtain

\[\delta\Gamma(t+s) - \delta\Gamma(t) = \left\{ T \exp \int_t^{t+s} d\tau \mathcal{L}(\tau) \right\} - 1 \delta\Gamma(t)\]

\[= \sum_{n=1}^{\infty} \int_t^{t+s} d\tau_n \int_t^{\tau_n} d\tau_{n-1} \cdots \int_t^{\tau_2} d\tau_1 \times \mathcal{L}(\tau_n) \mathcal{L}(\tau_{n-1}) \cdots \mathcal{L}(\tau_1) \delta\Gamma(t), \quad \text{(A.3)}\]

It follows from Eqs. (3), (10), (11), (A.2) and (A.3) that

\[\Xi^{(1)}(\delta\Gamma, t) = (\Xi^{(1)}(\delta\Gamma, t), \Xi^{(1)}(\delta\Gamma, t), \cdots, \Xi^{(1)}(\delta\Gamma, t))^T\]

\[= \lim_{s \to 0} \frac{1}{s} \langle [\delta\Gamma(t+s) - \delta\Gamma(t)] \rangle_{\delta\Gamma(t) = \delta\Gamma}\]

\[= \lim_{s \to 0} \frac{1}{s} \int_t^{t+s} d\tau \left( \mathcal{L}(\tau) \right) \delta\Gamma\]

\[= \left( \Phi \delta\mathbf{q} + \frac{\delta\mathbf{p}}{m} \right) \quad \text{(A.4)}\]

and

\[\Xi^{(2)}(\delta\Gamma, t) = (\Xi^{(2)}(\delta\Gamma, t), \cdots, \Xi^{(n)}(\delta\Gamma, t))^T\]

\[= \lim_{s \to 0} \frac{1}{2s} \langle [\delta\Gamma(t+s) - \delta\Gamma(t)] \times [\delta\Gamma(t+s) - \delta\Gamma(t)]^T \rangle_{\delta\Gamma(t) = \delta\Gamma}\]

\[= \lim_{s \to 0} \frac{1}{2s} \int_t^{t+s} d\tau \int_t^{t+s} d\tau \times \mathcal{L}(\tau) \delta\Gamma \delta\Gamma^T \mathcal{L}(\tau)^T\]

\[= \left( \begin{array}{cc} 0_{2N} & 0_{2N} \\ 0_{2N} & \eta(\delta\mathbf{q}) \end{array} \right) \quad \text{(A.5)}\]

where \(\eta(\delta\mathbf{q}) \equiv (\eta_{jk}(\delta\mathbf{q}))\) is defined by

\[\eta_{jk}(\delta\mathbf{q}) \equiv \frac{1}{2} \sum_{\mu=1}^{2N} \sum_{\nu=1}^{2N} D_{jk\mu\nu} \delta q_{\mu} \delta q_{\nu}. \quad \text{(A.6)}\]

Here the only non-zero contributions come from the \(n = 1\) term of Eq. (A.3). For general \(n\), the number of delta functions from Eq. (11) must be only one less than the number of time integrals, to give a non-zero contribution. It is straightforward to show that this never happens for \(n > 1\). Concerning the terms including \(\Xi^{(n)}_{j_1j_2\ldots j_n}(\delta\Gamma, t)\), \(n = 3, 4, \cdots\) in the right-hand side of Eq. (A.1) we obtain

\[\Xi^{(n)}_{j_1j_2\ldots j_n}(\delta\Gamma, t) = 0, \quad n = 3, 4, \cdots, \quad \text{(A.7)}\]

because of the Gaussian white properties (10) and (11) of the random matrix \(R(t)\). Using Eqs. (A.1), (A.7) and (A.7) we obtain the Fokker-Planck equation (12).

**APPENDIX B: EQUATION FOR THE MATRIX \(\check{\Upsilon}(\pm)\)**

In this appendix we give details of the derivation of Eq. (23) from Eqs. (12), (18) and (19). We start this derivation by introducing the \(N \times N\) matrices \(\check{F}(\pm)(t)\) and \(\check{G}(\pm)(t)\) defined by

\[\check{F}(\pm)(t) \equiv \langle \delta q \delta p T \rangle_t^{(\pm)}, \quad \text{(B.1)}\]

\[\check{G}(\pm)(t) \equiv \langle \delta p \delta p T \rangle_t^{(\pm)}. \quad \text{(B.2)}\]

Equations (12) and (18) lead to the equations

\[\frac{d\check{\Upsilon}(\pm)(t)}{dt} = \Phi \check{\Upsilon}(\pm)(t) + \check{\Upsilon}(\pm)(t)\Phi^T + \frac{1}{m} \left[ \check{F}(\pm)(t) + \check{F}(\pm)(t)^T \right], \quad \text{(B.3)}\]

\[\frac{d\check{G}(\pm)(t)}{dt} = \Psi \check{G}(\pm)(t) + \check{G}(\pm)(t)^T \Psi^T \pm \hat{D} \left\{ \check{\Upsilon}(\pm)(t) \right\} \quad \text{(B.5)}\]

for the matrices \(\check{\Upsilon}(\pm)(t), \check{F}(\pm)(t)\) and \(\check{G}(\pm)(t)\) with the operator \(\hat{D}\) defined by Eq. (24). Here, to derive Eq. (B.5) we used the relation

\[D_{lnjk} = D_{jkln} \quad \text{(B.6)}\]

which is derived from the definition (11) of the tensor \(D_{jklm}\) and the symmetry property of the matrix \(R(t)\).

Eqs. (B.3), (B.4) and (B.5) are equivalent to

\[\frac{d\check{\Upsilon}(\pm)(t)}{dt} = \check{F}(\pm)(t) P(t)^T + P(t) \check{F}(\pm)(t)^T, \quad \text{(B.7)}\]

\[\frac{d\check{F}(\pm)(t)}{dt} = P(t) \check{G}(\pm)(t), \quad \text{(B.8)}\]
\[
\frac{d\tilde{\Theta}(\pm)(t)}{dt} = \pm \hat{D}_t \left\{ \tilde{\Theta}(\pm)(t) \right\} 
\]  
(B.9)

where \( \tilde{\Theta}(\pm)(t) \), \( \tilde{\Phi}(\pm)(t) \) and \( \tilde{G}(\pm)(t) \) are defined by
\[
\tilde{\Theta}(\pm)(t) = e^{-\Phi t} \tilde{\Theta}(\pm)(t) e^{-\Phi^T t},
\]  
(B.10)
\[
\tilde{\Phi}(\pm)(t) = e^{-\Phi t} \tilde{\Phi}(\pm)(t) e^{-\Phi^T t},
\]  
(B.11)
\[
\tilde{G}(\pm)(t) = e^{-\Phi t} \tilde{G}(\pm)(t) e^{-\Phi^T t},
\]  
(B.12)

and \( P(t) \) is the \((2N) \times (2N) \) matrix defined by
\[
P(t) = \frac{1}{m} e^{-\Omega t},
\]  
(B.13)

with the matrix \( \Omega \) defined by Eq. (23), and \( \hat{D}_t \{ \cdots \} \) is defined by
\[
\hat{D}_t \{ X \} = e^{-\Phi t} \hat{D} \left\{ e^{\Phi t} X e^{\Phi^T t} \right\} e^{-\Phi^T t}
\]  
(B.14)

for any \((2N) \times (2N) \) matrix \( X \). Here we used the relation
\[
\Phi \Psi = \Psi \Phi,
\]  
(B.15)

so that we have the equation \( \exp\{-\Phi t\} \cdot \exp\{\Psi t\} = \exp\{-\Phi - \Psi\ t\} \).

Noting that the matrix \( \tilde{G}(\pm)(t) \) is symmetric and the inverse matrix of the matrix \( P(t) \) is given by \( P(t)^{-1} = m \exp\{\Omega t\} \), we obtain
\[
2\tilde{G}(\pm)(t) = P(t)^{-1} \frac{d\tilde{\Phi}(\pm)(t)}{dt} + \frac{d\tilde{\Phi}(\pm)(t)}{dt} P(t)^{-1} \left[ P(t)^{-1} \right]^T
\]  
(B.16)

by using Eqs. (B.7), (B.8) and (B.13). Besides, using Eqs. (B.7), (B.8) and (B.13) we obtain
\[
\frac{d}{dt} \left\{ P(t)^{-1} \tilde{\Phi}(\pm)(t) \Omega^T + \Omega \tilde{\Phi}(\pm)(t) P(t)^{-1} \right\} = \Omega P(t)^{-1} \frac{d\tilde{\Phi}(\pm)(t)}{dt} \left[ P(t)^{-1} \right]^T \Omega^T
\]  
(B.17)

where we again used the relation \( \tilde{G}(\pm)(t)^T = \tilde{G}(\pm)(t) \). It follows from Eqs. (B.7), (B.8) and (B.9) that
\[
\pm 2\hat{D}_t \left\{ \tilde{\Theta}(\pm)(t) \right\} = \frac{d}{dt} P(t)^{-1} \frac{d^2\tilde{\Theta}(\pm)(t)}{dt^2} \left[ P(t)^{-1} \right]^T + \Omega P(t)^{-1} \frac{d\tilde{\Phi}(\pm)(t)}{dt} \left[ P(t)^{-1} \right]^T \Omega^T + \tilde{G}(\pm)(t) \Omega^T + \Omega \tilde{G}(\pm)(t).
\]  
(B.18)

Taking the time-differential of both the sides of Eq. (B.18), and using Eq. (B.3) we obtain
\[
\frac{d^2}{dt^2} P(t)^{-1} \frac{d^2\tilde{\Theta}(\pm)(t)}{dt^2} \left[ P(t)^{-1} \right]^T + \Omega \frac{d}{dt} P(t)^{-1} \frac{d\tilde{\Phi}(\pm)(t)}{dt} \left[ P(t)^{-1} \right]^T \Omega^T + 2 \frac{d}{dt} \hat{D}_t \left\{ \tilde{\Theta}(\pm)(t) \right\} + \tilde{G}(\pm)(t) \Omega^T + \Omega \tilde{G}(\pm)(t) = 0.
\]  
(B.19)

This is the equation for the quantity \( \tilde{\Theta}(\pm)(t) \) only.

Now we derive the equation for \( \tilde{\Phi}(\pm)(t) \) defined by Eq. (23) from Eq. (B.19) for \( \tilde{\Theta}(\pm)(t) \) defined by Eq. (B.19).

We note
\[
\Omega = 2\Phi + \bar{\alpha} I_{2N}
\]  
(B.20)
\[
= -2\Psi - \bar{\alpha} I_{2N}
\]  
(B.21)

which is derived from Eqs. (6), (8) and (23). Using Eqs. (21), (B.10) and (B.20) the matrices \( \tilde{\Phi}(\pm)(t) \) are connected with the matrix \( \tilde{\Theta}(\pm)(t) \) by
\[
\tilde{\Phi}(\pm)(t) = e^{-\Omega t/2} \tilde{\Theta}(\pm)(t) e^{-\Omega^T t/2}.
\]  
(B.22)

We introduce the multiplication \( X \otimes Y \) of \( X \) and \( Y \) which is defined by
\[
X \otimes Y = \frac{1}{2} \left( XY + (XY)^T \right)
\]  
(B.23)

for any square matrices \( X \) and \( Y \) of the same size. This multiplication is used in the relation
\[
\frac{d}{dt} e^{\pm \Omega t/2} Z(t) e^{\pm \Omega^T t/2} = e^{\pm \Omega t/2} \left[ \frac{dZ(t)}{dt} \pm \Omega \otimes Z(t) \right] e^{\pm \Omega^T t/2}
\]  
(B.24)
satisfied by any \((2N) \times (2N) \) symmetric matrix \( Z(t) \) as a function of \( t \). Noting Eqs. (B.23) and (B.24), Eq. (B.22) leads to
\[
P(t)^{-1} \frac{d\tilde{\Phi}(\pm)(t)}{dt} \left[ P(t)^{-1} \right]^T = m^2 e^{\Omega t/2} \left[ \frac{d\tilde{\Phi}(\pm)(t)}{dt} - \Omega \otimes \tilde{\Phi}(\pm)(t) \right] e^{\Omega^T t/2},
\]  
(B.25)
\[
P(t)^{-1} \frac{d^2\tilde{\Theta}(\pm)(t)}{dt^2} \left[ P(t)^{-1} \right]^T = m^2 e^{\Omega t/2} \left\{ \frac{d^2\tilde{\Theta}(\pm)(t)}{dt^2} - 2 \Omega \otimes \frac{d\tilde{\Theta}(\pm)(t)}{dt} \right\} e^{\Omega^T t/2}
\]  
(B.26)
where we used the relation $\hat{\Phi}^{(\pm)}(t)^T = \hat{\Phi}^{(\pm)}(t)$. Moreover, $\hat{D}_t$ operated on the matrix $\hat{\Phi}^{(\pm)}(t)$ is connected with $\hat{D}$ operated on the matrix $\hat{\Phi}^{(\pm)}(t)$ as

$$
\hat{D}_t \left\{ \hat{\Phi}^{(\pm)}(t) \right\} = e^{\alpha t/2} \hat{D} \left\{ \hat{\Phi}^{(\pm)}(t) \right\} e^{\alpha t/2}
$$

(B.27)

where we used Eqs. (21), (B.10) and (B.21). Inserting Eqs. (21), (B.21) and (B.27) into Eq. (B.19) and using Eq. (B.24) we obtain

$$
\frac{d^4 \hat{\Phi}^{(\pm)}(t)}{dt^4} - 2\Omega \left[ \Omega \circ \frac{d^2 \hat{\Phi}^{(\pm)}(t)}{dt^2} \right] + \Omega \left[ \Omega \circ \frac{d^2 \hat{\Phi}^{(\pm)}(t)}{dt^2} \right] t^T \\
+ \Omega \left[ \Omega \circ \left( \Omega \circ \hat{\Phi}^{(\pm)}(t) \right) \right] t^T \\
- \Omega \left[ \Omega \circ \left( \Omega \circ \hat{\Phi}^{(\pm)}(t) \right) \right] t^T \\
\pm 2m^2 \hat{D} \left\{ \frac{d^2 \hat{\Phi}^{(\pm)}(t)}{dt^2} \right\} = 0.
$$

(B.28)

Eq. (B.28) is equivalent to

$$
\frac{d^4 \hat{\Phi}^{(\pm)}(t)}{dt^4} - \frac{1}{2} \left[ \Omega^2 \frac{d^2 \hat{\Phi}^{(\pm)}(t)}{dt^2} + \frac{d^2 \hat{\Phi}^{(\pm)}(t)}{dt^2} \left( \Omega^2 \right)^T \right] \\
+ \frac{1}{16} \left[ \Omega^4 \hat{\Phi}^{(\pm)}(t) - 2\Omega^2 \hat{\Phi}^{(\pm)}(t) \left( \Omega^2 \right)^T \\
+ \hat{\Phi}^{(\pm)}(t) \left( \Omega^2 \right)^T \right] + \frac{2}{m^2} \hat{D} \left\{ \frac{d \hat{\Phi}^{(\pm)}(t)}{dt} \right\} = 0.
$$

(B.29)

By exchanging $t$ with $\pm t$ in Eq. (B.29) we obtain Eq. (22).

APPENDIX C: LYAPUNOV EXPONENTS IN THE COLOR FIELD CASE

In this appendix we consider the case with no shear field using the condition (21), and derive Eq. (22) under the assumption (B.1). We also give a derivation of Eq. (22) briefly.

Under the condition (29), the matrix $\Omega$ defined by Eq. (24) is simply an identical matrix multiplied by a constant and is given by

$$
\Omega = \alpha I_{2N},
$$

(C.1)

and the matrix $P(t)$ defined by Eq. (21) and the operator $\hat{D}_t \{ \cdots \}$ defined by Eq. (B.14) are given by

$$
P(t) = \frac{1}{m} e^{-\alpha t} I_{2N},
$$

(C.2)

$$
\hat{D}_t \{ X \} = e^{2\alpha t} \hat{D} \{ X \}
$$

(C.3)

for any $(2N) \times (2N)$ matrix $X$. Noting Eqs. (C.2) and (C.3) and the relation $\hat{\Phi}^{(\pm)}(t) = \Phi^{(\pm)}(t)$, Eqs. (B.8) and (B.9) are simply attributed into

$$
\frac{d\hat{\Phi}^{(\pm)}(t)}{dt} = \frac{1}{m} e^{-\alpha t} \hat{D} \left\{ \Phi^{(\pm)}(t) \right\}.
$$

(C.4)

$$
\frac{d\hat{G}^{(\pm)}(t)}{dt} = \pm 2m \hat{D} \left\{ \Phi^{(\pm)}(t) \right\}.
$$

(C.5)

Noting that the matrix $\hat{G}^{(\pm)}(t)$ is symmetric, Eqs. (C.4) and (C.5) lead to the differential equation

$$
\frac{d}{dt} e^{2\alpha t} \frac{d}{dt} e^{\alpha t} \frac{d}{dt} e^{-\alpha t} \frac{d}{dt} \Phi^{(\pm)}(t) = \pm \frac{2}{m^2} e^{2\alpha t} \hat{D} \left\{ \Phi^{(\pm)}(t) \right\}
$$

(C.7)

for the matrix $\Phi^{(\pm)}(t)$ only.

Now we consider the derivation of the equation for the matrix $\hat{\Phi}^{(\pm)}(t)$ (defined by Eq. (22)) from Eq. (C.7) for the matrix $\Phi^{(\pm)}(t)$. It follows from Eqs. (21) and (C.7) that

$$
\frac{d^3 \hat{\Phi}^{(\pm)}(t)}{dt^3} - \alpha^2 \frac{d^2 \hat{\Phi}^{(\pm)}(t)}{dt^2} + \frac{2}{m^2} \hat{D} \left\{ \hat{\Phi}^{(\pm)}(t) \right\} = 0.
$$

(C.8)

By exchanging $t$ with $\pm t$ in Eq. (C.8) and using Eqs. (21) and (31) we obtain Eq. (22).

Using Eqs. (21) and (31) the real function $\xi_j^{(\pm)}(t)$ of $t$ defined by

$$
\xi_j^{(\pm)}(t) = (V^T \hat{\Phi}^{(\pm)}(t) V)_{jj} = \hat{\Phi}^{(\pm)}(t)e^{\pm \alpha t}
$$

(C.9)

satisfies the equation

$$
\frac{d^3 \xi_j^{(\pm)}(t)}{dt^3} - \alpha^2 \frac{d \xi_j^{(\pm)}(t)}{dt} - \frac{2\omega^2}{m^2} \xi_j^{(\pm)}(t) = 0.
$$

(C.10)

The real solution of the linear differential equation (C.10) is expressed as

$$
\xi_j^{(\pm)}(t) = \sum_{k=1}^{\infty} \text{Re} \left\{ \psi_j^{(k)} e^{\zeta_j^{(k)} t} \right\}
$$

(C.11)

where $\psi_j^{(k)}$, $j = 1, 2, 3$ are constants determined by the initial condition, and $\xi_j^{(k)}$, $j = 1, 2, 3$ are the three solutions of the equation

$$
\zeta^3 - \alpha^2 \zeta - \frac{2\omega^2}{m^2} = 0
$$

(C.12)
for $\zeta$. Here $\text{Re}\{X\}$ means to take the real part of any imaginary number $X$. We sort the quantities $\zeta_j$, $j = 1, 2, 3$ as $\text{Re}\{\zeta_j^{(1)}\} \geq \text{Re}\{\zeta_j^{(2)}\} \geq \text{Re}\{\zeta_j^{(3)}\}$, so that using Eqs. (27), (33) and (C.9) the Lyapunov exponent $\lambda_j^{(\pm)}$ is expressed as

$$
\lambda_j^{(\pm)} = \pm \lim_{t \to +\infty} \frac{1}{2t} \ln \zeta_j^{(\pm)}(t) e^{\mp \alpha t} = -\frac{\alpha}{2} \pm \frac{1}{2} \text{Re}\{\zeta_j^{(1)}\}. \quad (C.13)
$$

It follows from Eq. (C.12) that $\zeta = \zeta_j^{(1)}$ is a real solution of Eq. (C.12) and satisfies the conditions $\zeta_j^{(1)} \geq |\alpha|$, $\lim_{\omega_j \to 0} \zeta_j^{(1)} = |\alpha|$, noting that the point $\zeta = \zeta_j^{(1)}$ is the maximum intersecting point of the graphs $y = \zeta^2 - \alpha^2 \zeta$ and $y = 2\omega_j^2/m^2$ in the $\zeta$-$y$ plane. This means that the Lyapunov exponents $\lambda_j^{(1)}$ ($\lambda_j^{(-1)}$) must be non-negative (non-positive). More concretely the quantity $\zeta_j^{(1)}$ is given by

$$
\zeta_j^{(1)} = \Lambda_j + \frac{\alpha^2}{3\Lambda_j}, \quad (C.14)
$$

with the quantity $\Lambda_j$ defined by Eq. (33). We can check that in the case of $|\alpha| \leq \sqrt{3} |\omega_j/m|^{2/3}$ the quantities $\Lambda_j$ and $\zeta_j^{(1)}$ are both real numbers, and in the case of $|\alpha| > \sqrt{3} |\omega_j/m|^{2/3}$ the quantity $\Lambda_j$ can be an imaginary number but the quantity $\zeta_j^{(1)}$ given by Eq. (C.14) is still a real number and satisfies the condition $\lim_{\omega_j \to 0} \zeta_j^{(1)} = |\alpha|$. Using Eqs. (C.13) and (C.14) we obtain Eq. (34).

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