Relative Spin Polarization of Parity-Violating Asymmetries

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Parity-violating asymmetries in polarized electron scattering have been interpreted as the asymmetries between opposite helicities of incoming fermion based on the approximation of the spin polarization operator. Here exact calculations of cross sections for parity-violating asymmetries in SLAC E158 and SLD have been performed using spin projection operators. And the parity-violating factor incorporating with spin polarization and momentum has been identified and shown that its sign depends on the spin polarization of incoming particle and the relative velocity of incoming and target particles. Therefore, I suggest a new concept of relative spin polarization to interpret the parity-violating asymmetry as contributed by the antisymmetric nature of the weak interactions depending on whether the spin direction of the incoming electron is inward or outward relative to the target electron.

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I. INTRODUCTION

The parity-violating asymmetries of SLAC E158 and SLD experiments have been interpreted as the asymmetries between opposite helicities for the Standard Model [1, 2, 3, 4, 5, 6] using the chiral projection, approximated from the spin projection operator. Helicity \( h_\pm \) is defined by the momentum of a fermion and its spin orientation; if the spin orientation is in the same direction as its momentum, it is called right-handed helicity (\( h^+ \)). And helicity can be observed to be opposite depending on the reference frame and is especially indefinite in the rest frame where momentum is zero. For example, a right-handed helicity massive fermion \( u_{h^+}(+p_z,+s_z) \) in one reference frame can be observed at the same time as \( u_{h^-}(-p_z,+s_z) \) with left-handed helicity by observers in other reference frames. Chirality (L,R) is defined to indicate either of the two-component objects in a massive fermion field, and no physical measurement is available for the chirality of massive fermions and only when fermions are massless chirality becomes helicity.

After a review of spin polarization, we investigate the validity of the chiral projection as an approximation of the spin polarization operator in the practical calculation of matrix elements [7, 8, 9].

II. SPIN POLARIZATION AND CHIRAL PROJECTION

Spin polarization is defined by the direction of the spin relative to a given coordinate system, whereas helicity is relative to the momentum direction of the particle. The spin polarization vector \( s_\mu \) can be derived from the spin projection operator \( P(s) \) for an electron with spin direction at rest:

\[
P(s) = \frac{1}{4} \left( 1 + \frac{\not{p}}{m} \right) \left( 1 - \gamma^5 \not{s} \right)
\]

where the spin polarization 4-vector \( s_\mu \) is

\[
s_\mu = \left( \frac{s \cdot p}{m}, s + \frac{p(s \cdot p)}{m(E+m)} \right),
\]

satisfying that

\[
s^2 = -1 \quad \text{and} \quad s^\mu p_\mu = 0,
\]

where it is normalized to one particle per unit volume in the rest frame by the normalization factor \( c \) [13, 14, 15]. For example, the spin projection operator for an electron with the spin direction +\( s_z \) and momentum +\( p_z \) is:

\[
P(+s_z) = \frac{1}{2m} \begin{pmatrix} m & 0 & E - p_z & 0 \\ 0 & 0 & 0 & 0 \\ E + p_z & 0 & m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
= \frac{1}{4} \left( 1 + \frac{\not{p}}{m} \right) \left( 1 - \gamma^5 \not{s} \right)
\]

and the spin polarization vector is:

\[
s_\mu = \left( \frac{+p_z}{m}, 0, 0, \frac{+(p_z)p_z}{m(E+m)} \right) = \left( \frac{+p_z}{m}, 0, 0, \frac{E}{m} \right)
\]
In general, the spin projection operators of a fermion with \( \pm s_z \) can be given by:

\[
P(\pm s_z) \equiv cu_{h\pm}(\pm p_z, +s_z) \overline{u}_{h\mp}(\pm p_z, +s_z)
\]

\[
= \frac{1}{2m} \begin{pmatrix}
\begin{array}{cccc}
m & 0 & E \mp p_z & 0 \\
0 & 0 & 0 & 0 \\
0 & m & E \pm p_z & 0 \\
0 & 0 & 0 & m
\end{array}
\end{pmatrix}
\]

\[
P(-s_z) \equiv cu_{h\mp}(\pm p_z, -s_z) \overline{u}_{h\pm}(\pm p_z, -s_z)
\]

\[
= \frac{1}{2m} \begin{pmatrix}
\begin{array}{cccc}
m & 0 & E \mp p_z & 0 \\
0 & 0 & 0 & 0 \\
0 & m & E \pm p_z & 0 \\
0 & 0 & 0 & m
\end{array}
\end{pmatrix}
\]

Note that the location of the non-zero matrix elements is determined by the spin direction from the product of the two-component spinors \( \xi^\dagger \).

The approximation of the spin projection operator \( P(s) \) should be performed after the full evaluation of the matrix element, taking into account the normalization factor. For example, the trace of the matrix element calculation \( \text{tr}[P(\pm s_z)] = 1 \) vanishes for the massless approximation \( (m \rightarrow 0) \) ignoring the normalization factor \( 1/2m \):

\[
\text{tr}[P(\pm s_z)] = \frac{1}{2m} \begin{pmatrix}
\begin{array}{cccc}
m & 0 & E - p_z & 0 \\
0 & 0 & 0 & 0 \\
0 & m & E + p_z & 0 \\
0 & 0 & 0 & m
\end{array}
\end{pmatrix} \rightarrow 0.
\]

The common practice of spin polarization approximation is inaccurate compared with the accurate evaluation of the full spin projection operator. For the relativistic limit of \( m/E \rightarrow 0 \), the spin projection operators for longitudinally polarized electrons with \( s \) parallel to \( p \) can be reduced as:

\[
s^\mu = \frac{1}{m\beta}p^\mu - \frac{\sqrt{1 - \beta^2}}{\beta}g^{\mu 0}
\]

\[
\rightarrow \frac{p^\mu}{m} \quad \text{for} \quad \beta \rightarrow 1,
\]

whereas the explicit expression of the spin polarization \( P(\pm s_z) \) is now given by:

\[
P(\pm s_z) \rightarrow \frac{1}{4} \begin{pmatrix}
\begin{array}{cccc}
m & 0 & 2(E - p_z) & 0 \\
0 & m & 0 & 2(E + p_z) \\
0 & 0 & m & 0 \\
0 & 0 & 0 & m
\end{array}
\end{pmatrix}
\]

and although the approximation holds true for some specific calculations such as \( \text{tr}[P(\pm s_z)] = 1 \) it is inconsistent with the accurate evaluation of the matrix element in general. The omission of the normalization factor is also misleading because it neglects the electron mass term in the relativistic limit \( \frac{1}{2m} \):

\[
\frac{1}{2}(\gamma^1 + m) (1 - \gamma^5 \hat{s}) \rightarrow \frac{1}{2}(\gamma^1) (1 - \gamma^5 \hat{s}).
\]

whereas the contribution of the \( m \) term to the matrix element should be the same regardless of the energy \( E \) once the proper normalization is considered, since the mass factor in \( ms^\mu \) is normalized to be one by \( m^{-1} \). The simplification of the matrix element calculation in the relativistic limit can be given by:

\[
\left( \frac{1 + \gamma^5}{2} \right) \left( \frac{\gamma + m}{2m} \right) \rightarrow \left( \frac{1 + \gamma^5}{2} \right) \left( \frac{\gamma + m}{2m} \right).
\]

whereas the explicit expression of the spin polarization \( P(\pm s_z) \) is now given by:

\[
\left( \frac{1 + \gamma^5}{2} \right) \left( \frac{\gamma + m}{2m} \right) = \frac{1}{2m} \begin{pmatrix}
\begin{array}{cccc}
m & 0 & E - p_z & 0 \\
0 & m & 0 & E + p_z \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}
\end{pmatrix}
\]

This is inconsistent with the accurate evaluation of the general matrix element calculation, since the spin direction represented by a nonzero matrix element is inconsistent for massive electrons. The matrix element for a right-handed chirality relativistic electron to scatter to a left-handed chirality electron with interaction \( \gamma^\mu \) is proportional to:

\[
\overline{u}_L(p') \gamma^\mu u_R(p) = \overline{u}(p') \left( \frac{1 + \gamma_5}{2} \right) \gamma_\mu \left( \frac{1 + \gamma_5}{2} \right) u(p) \rightarrow 0
\]

whereas from the exact representation of massive fermions in terms of helicity, not of chirality, the corresponding matrix element is in general nonzero:

\[
\text{tr} \left[ \overline{u}_{h_\pm}(p') \gamma^\mu u_{h_\mp}(p) \overline{u}_{h_\pm}(p) \gamma_\nu u_{h_\mp}(p) \right] = \text{tr} \left[ \overline{u}_{h_\pm}(p') \gamma^\mu u(p) \overline{u}(p) \left( \frac{1 - \gamma^5}{2} \right) \gamma_\nu u(p) \right].
\]

Note that \( (1 \pm \gamma^5 \hat{s})/2 \) is supposed to act on \( (\gamma^1 + m)/2m \), not on \( u \) alone, and the chiral projection \( (1 \pm \gamma^5 \hat{s})/2 \) on \( u \) is inconsistent with being a proper approximation of \( (1 \pm \gamma^5 \hat{s})/2 \) for a massive fermion.

Since the chiral projection as an approximation of spin projection operators could differ from the exact calculation of spin projection operators, the matrix element should be carefully evaluated comparing with the full representation of the spin projection operators.

### III. THE PARITY-VIOLATING ASYMMETRY IN SLAC E158 AND SLD

The parity-violating asymmetry for polarized electrons scattering on an unpolarized target is given by:

\[
A_{PV} \equiv \frac{d\sigma_{h_\pm}(+p_z, +s_z) - d\sigma_{h_\pm}(-p_z, -s_z)}{d\sigma_{h_\pm}(+p_z, +s_z) + d\sigma_{h_\pm}(-p_z, -s_z)}
\]

where \( d\sigma \) denotes the differential cross section for an incoming electron of helicity \( \lambda \) on an unpolarized target. For the parity-violating asymmetry in SLAC E158
and SLD [3, 6], left- and right-handed helicity massive fermions are considered as distinguishable particles with the weak interaction structure of $\gamma^\mu(v - a\gamma^5)$, as suggested in the Standard Model [18, 14, 20], and the asymmetry is accounted for as parity violation between opposite helicities. For the electron scattering of

$$e^-(p_1, \lambda_1) + e^-(p_2, \lambda_2) \rightarrow e^-(p_3, \lambda_3) + e^-(p_4, \lambda_4), \quad (14)$$

where $(p_1, \lambda_1)$ denote the four momenta and helicities of incoming and outgoing electrons, respectively; the interaction Lagrangian for the electromagnetic and weak interactions with the $Z$ boson is given by:

$$\mathcal{L}_{\text{int}} \sim -g_\gamma e \gamma_\mu A^\mu + g_\gamma \left[ c_L e \gamma_\mu (1 - \gamma^5) e + c_R e \gamma_\mu (1 + \gamma^5) e \right] Z^\mu$$

$$= -g_\gamma e \gamma_\mu A^\mu + g_Z e \gamma_\mu (v - a\gamma^5) e Z^\mu, \quad (15)$$

where $g_\gamma = e, g_z = 4e(2\cos\theta_W \sin\theta_W)^{-1}$ and $v \equiv c_L + c_R = 1/2 - 2\sin^2\theta_W, a \equiv c_L - c_R = 1/2$. Note that $(1 \mp \gamma^5)$ in Eq. (15) are not chiral projections but chiral interaction structures, and electrons are later to be classified as left- and right-handed helicity electrons $e_\lambda$ using the spin polarization projection operator $(\mp \gamma^5) [3, 13, 14, 15, 20]$.

The total tree-level amplitude for $ee$ scattering via $\gamma$ and $Z$ exchange in the center-of-mass (CM) frame is then given by:

$$\mathcal{M} = \mathcal{M}_\gamma^d + \mathcal{M}_\gamma^u + \mathcal{M}_Z^d + \mathcal{M}_Z^u$$

$$= -\frac{g^2}{ysm^2} \pi(p_3)\gamma_\mu u(p_1) \cdot \pi(p_4)\gamma^\mu u(p_2)$$

$$+ \frac{g^2}{(1-y)s} \pi(p_4)\gamma_\mu u(p_1) \cdot \pi(p_3)\gamma^\mu u(p_2)$$

$$- \frac{g^2}{m^2} \pi(p_3)\gamma_\mu (v - a\gamma^5) u(p_1) \cdot \pi(p_4)\gamma^\mu (v - a\gamma^5) u(p_2)$$

$$+ \frac{g^2}{m^2} \pi(p_4)\gamma_\mu (v - a\gamma^5) u(p_1) \cdot \pi(p_3)\gamma^\mu (v - a\gamma^5) u(p_2)$$

where $y \equiv -(p_1 - p_3)^2/s = \sin^2(\theta_{cm}/2)$ with the CM scattering angle $\theta_{cm}$ and the CM energy $\sqrt{s} = \sqrt{(p_1 + p_2)^2/2}$.

For the explicit spin polarization calculation with the operator $(1 \mp \gamma^5)\gamma_\mu$ for inward and outward relative spin polarization corresponding to $(1 \pm \gamma^5)$, the nonzero parity-violating asymmetry arises from such terms as: $[\mathcal{M}_\gamma^d \mathcal{M}_\gamma^u]_{\lambda_1}, [\mathcal{M}_\gamma^d \mathcal{M}_Z^u]_{\lambda_1}, [\mathcal{M}_\gamma^d \mathcal{M}_Z^d]_{\lambda_1}, [\mathcal{M}_\gamma^u \mathcal{M}_Z^u]_{\lambda_1}$.

$$[\mathcal{M}_\gamma^d \mathcal{M}_\gamma^u]_{\lambda_1} = \frac{g^4}{ysm^2} \left[ \pi(p_3)\gamma_\mu (1 \mp \gamma^5) u(p_1) \cdot \pi(p_4)\gamma^\mu (v - a\gamma^5) u(p_2) \right]$$

$$\left[ \pi(p_4)\gamma_\mu u(p_2) \cdot \pi(p_3)\gamma^\mu (v - a\gamma^5) u(p_1) \right]$$

$$= \frac{g^4}{y} \beta(-E_1p_{2z} + E_2p_{1z}) va \quad (16)$$

where the incoming and target electrons are given by $p_1 = (E_1, p_{1z})$ and $p_2 = (E_2, p_{2z})$. The parity-violating asymmetry $A_{PV}$ is then given by:

$$A_{PV} = -16\beta(-E_1p_{2z} + E_2p_{1z}) va \frac{y(1-y)}{1+y^2(1-y)^4}$$

where $(-E_1p_{2z} + E_2p_{1z}) = E_1E_2(v_1 - v_2) = \sqrt{(p_1 \cdot p_2 - m^2)}$ is Lorentz invariant holding for any arbitrary frame when $v_1$ is parallel to $v_2$ and the massless approximation of $A_{PV}$ in the CM frame $p_1 \simeq (E, E)$ and $p_2 \simeq (E, -E)$ is consistent with the approximated parity-violating asymmetry of the SLAC E158 denoted as $A_{PV}^{approx}$ [20].

In SLD, the polarized differential cross section of the $e^- e^+ \rightarrow Z^0 \rightarrow f\bar{f}$ process with longitudinally polarized electrons and unpolarized positrons is

$$\frac{d\sigma^P_f}{d\Omega} = \frac{N e^2}{8 \sin^4 \theta_W \cos^4 \theta_W (s - m_{Z^0}^2 + s^2\Gamma_{Z^0}^2/m_{Z^0}^2)} \times [2(v_e^2 + a_e^2)(v_\bar{f}^2 + a_\bar{f}^2)(E_1E_2E_3E_4 + p_{1z}p_{2z}p_{3z}p_{4z})]$$

$$+ [2v_e a_e(v_\bar{f}^2 + a_\bar{f}^2)(E_1E_2E_3E_4 + p_{1z}p_{2z}p_{3z}p_{4z})]$$

$$\times [2v_e^2 + a_e^2](v_\bar{f}^2 + a_\bar{f}^2)(E_1E_2E_3E_4 + p_{1z}p_{2z}p_{3z}p_{4z})]$$

$$\times [2v_e a_{\bar{f}}(v_\bar{f}^2 + a_\bar{f}^2)(E_1E_2E_3E_4 + p_{1z}p_{2z}p_{3z}p_{4z})]$$

$$= \frac{g^4}{y} \beta(-E_1p_{2z} + E_2p_{1z}) va \quad (16)$$

Note that its massless approximation in the CM frame $p_1 \simeq (E, E)$ and $p_2 \simeq (E, -E)$ is consistent with the approximated parity-violating calculation of the SLD [6] and the total cross sections are Lorentz invariant.
IV. RELATIVE SPIN POLARIZATION FOR TWO-PARTICLE SYSTEMS

Let us investigate the exact cross section calculation under parity regarding to spin polarization and momentum of incoming and target particles. In the exact calculations, the factor that determines the signs of parity-violating terms depending on spin polarization and momentum is given by $\pm (E_1 p_{zz} - E_2 p_{1z}) = \mp E_1 E_2 (\vec{v}_1 - \vec{v}_2)$. The signs of parity-violating terms become reversed when either the spin polarization ($\pm s_z$) of the incoming particle or the relative velocity between two particles $\vec{v}_r = (\vec{v}_1 - \vec{v}_2)$ is reversed. Note that the signs of parity-violating terms can remain constant even when the helicity of incoming electron becomes opposite as its momentum is reversed ($\vec{v}_1 \rightarrow -\vec{v}_1$) under the Lorentz transformations such as in Fig. 1 since the factor $\pm (E_1 p_{zz} - E_2 p_{1z})$ is Lorentz invariant along the $z$ direction.

For a two-particle system, the spin polarization can be characterized by its direction relative to the velocity difference between the incoming and target particles $\vec{v}_r$, indicating whether the spin direction of the incoming particle points outward from or toward the target particle. Under Lorentz transformations, the differential cross sections for a right-handed helicity incoming electron can be observed as a left-handed helicity electron, but the relative spin polarization remains inward as in Fig. 1. Thus, the exact evaluation of parity-violating asymmetry $A_{PV}$ remains asymmetric in general:

$$A_{PV} = \frac{d\sigma_h^+ (p_z, +s_z, in) - d\sigma_h^- (p_z, -s_z, out)}{d\sigma_h^+ (p_z, +s_z, in) + d\sigma_h^- (p_z, -s_z, out)}$$

whereas the approximated calculation of parity-violating asymmetry, interpreted as the asymmetry between opposite helicities, vanishes $A_{PV}^{approx} = 0$.

Therefore, the parity-violating asymmetry measured in SLAC E158 and SLD should be interpreted in terms of relative spin polarization not of helicity, since the asymmetry between opposite helicities vanishes under Lorentz transformations whereas the exact calculation does not.

V. CONCLUSION

Since the approximate spin polarization $(1 \pm \gamma^5)$ may significantly differ from the exact spin polarization in evaluating the matrix element, here exact calculations of cross sections for parity-violating asymmetries in SLAC E158 and SLD have been performed using the full expression of spin projection $(1 \mp \gamma^5 \vec{r})$. And the parity-violating factor incorporating with spin polarization and momentum $\pm (E_1 p_{zz} - E_2 p_{1z}) = \mp E_1 E_2 (\vec{v}_1 - \vec{v}_2)$ has been identified and shown that its sign depends on the spin polarization of incoming particle and the relative velocity of incoming and target particles, not on the helicity of incoming particle. Therefore, I suggest a new concept of relative spin polarization to interpret the parity-violating asymmetry as contributed by the antisymmetric nature of the weak interactions depending on whether the spin direction of the incoming electron is inward or outward relative to the target electron.

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