ON BEILINSON’S HODGE AND TATE CONJECTURES
FOR OPEN COMPLETE INTERSECTIONS

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Contents

§0 Introduction
§1 Jacobian rings for open complete intersections
§2 Hodge theoretic implication of generalized Jacobian rings
§3 Beilinson’s Hodge conjecture
§4 Beilinson’s Tate conjecture
§5 Implication on injectivity of Chern class map for $K_1$ of surfaces
References

§0. Introduction.

In his lectures in [G1], M. Green gives a lucid explanation how fruitful the infinitesimal method in Hodge theory is in various aspects of algebraic geometry. A significant idea is to use Koszul cohomology for Hodge-theoretic computations. The idea originates from Griffiths work [Gri] where the Poincaré residue representation of the cohomology of a hypersurface played a crucial role in proving the infinitesimal Torelli theorem for hypersurfaces. Since then many important applications of the idea have been made in different geometric problems such as the generic Torelli problem and the Noether-Lefschetz theorem and the study of algebraic cycles (see [G1, Lectures 7 and 8]).

In this paper we introduce Jacobian rings of open complete intersections and to apply it to the Beilinson’s Hodge and Tate conjectures. Here, by “open complete intersection” we mean a pair $(X, Z = \bigcup_{1 \leq j \leq s} Z_j)$ where $X$ is a smooth complete intersection in $\mathbb{P}^n$ and $Z_j \subset X$ is a smooth hypersurface section such that $Z$ is a simple normal crossing divisor on $X$. Our Jacobian rings give an algebraic description of the infinitesimal part of mixed Hodge structure on the cohomology $H^m(X \setminus Z, \mathbb{Q})$ with $m = \dim(X)$. It is a natural generalization of the Poincaré residue representation of the cohomology of a hypersurface in [Gri].

The Beilinson’s Hodge and Tate conjectures (cf. [J1, Conjecture 8.5 and 8.6]) concern the surjectivity of the Chern class maps for open varieties. To be more precise we let $U$ be a smooth variety over $k$.

(0-1) (Hodge version) When $k = \mathbb{C}$ the conjecture predicts the surjectivity of the Chern class map from the higher Chow group to the Betti cohomology (cf. [Bl] and [Sch])

$$ch^{i,j}_{B,U} : CH^j(U, 2j - i) \otimes \mathbb{Q} \to (2\pi \sqrt{-1})^j W_2j H^i_B(U(\mathbb{C}), \mathbb{Q}) \cap F^j H^i_B(U(\mathbb{C}), \mathbb{C})$$

where $W_\ast$ (resp. $F^\ast$) denotes the weight (resp. Hodge) filtration of the mixed Hodge structure on the Betti cohomology group.

(0-2) (Tate version) When $k$ is a finite extension of $\mathbb{Q}$ the conjecture predicts the surjectivity of the Chern class map from the higher Chow group to the etale cohomology

$$ch^{i,j}_{et,U} : CH^j(U, 2j - i) \otimes \mathbb{Q}_\ell \to H^i_{et}(U \times_k \overline{k}, \mathbb{Q}_\ell(j)_{Gal(\overline{k}/k)})$$

where $\overline{k}$ is an algebraic closure of $k$. 

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Conjecture (0-1). (A. Beilinson [Bei]) The above maps are surjective in case $i = j$.

The following are some remarks on the conjecture.

(i) In case that $U$ is proper and $i = 2j$, the surjectivity of the above maps is equivalent to the Hodge and Tate conjectures for algebraic cycles of codimension $j$ on $U$. Thus Conj. (2-1) is analogous to the Hodge and Tate conjectures for algebraic cycles on a projective smooth variety.

(ii) Let $X$ be a projective smooth curve and $U \subset X$ be a non-empty open subset. Then the surjectivity of $\text{ch}^{1,1}_{B,U}$ (resp. $\text{ch}^{1,1}_{et,U}$) for follows from the Abel’s theorem. Indeed it is equivalent to the injectivity (modulo torsion) of the Abel-Jacobi map (resp. $\ell$-adic Abel-Jacobi map)

$$CH_0(X)_{\text{deg}=0} \to \text{Jac}(X)$$

restricted on the subspace of $CH_0(X)_{\text{deg}=0}$ generated by cycles supported on $X \setminus U$. Here $CH_0(X)_{\text{deg}=0}$ is the group of zero-cycles on $X$ of degree zero modulo rational equivalence and $\text{Jac}(X)$ is the Jacobian variety of $X$.

(iii) When $X$ is a projective smooth surface and $U \subset X$ is the complement of a simple normal crossing divisor $Z \subset X$, then the surjectivity of $\text{ch}^{2,2}_{B,U}$ or $\text{ch}^{2,2}_{et,U}$ has an implication on the injectivity of the Chern class map for $CH^2(X, 1)$, which can be viewed as an analogue of the Abel’s theorem for $K_1$ of surfaces. The detail will be discussed in §5.

(iv) As a naive generalization of the Beilinson’s conjecture, one may ask if the Chern class maps in (0-1) and (0-2) are surjective for any $i, j \geq 0$. Jannsen ([J1, 9.11]) has shown that the map in (0-1) in case $i = 1, j = 2q - 1 \geq 3$ is not surjective in general by using a theorem of Mumford [Mu], which implies the Abel-Jacobi for cycles of codimension $\geq 2$ is not injective even modulo torsion.

In order to state the main result on the Beilinson conjectures we fix a field $k$ of characteristic zero and a non-singular quasi-projective variety $S$ over $k$. We also fix integers $d_1, \ldots, d_r, e_1, \ldots, e_s \geq 1$. Assume that we are given schemes over $S$

$$\mathbb{P}^n_S \leftrightarrow X \leftrightarrow Z = \bigcup_{1 \leq j \leq s} Z_j$$

whose fibers are open complete intersections. We assume that the fibers of $X/S$ are smooth complete intersection of multi-degree $(d_1, \ldots, d_r)$ in $\mathbb{P}^n$ and that those of $Z_j \subset X$ are smooth hypersurface section of degree $e_j$. Let $f : X \to S$ be the natural morphism and write $U = X \setminus Z$. Let $U_x$ denote the fiber of $U = X \setminus Z$ over $x \in S$. In §2 we will introduce an invariant $c_S(X, Z)$ that measures the “generality” of the family (0-1), or how many independent parameters $S$ contains (cf. Rem. (2-2)).

Theorem (0-1). Assume $\sum_{1 \leq i \leq r} d_i \geq n + 1 + c_S(X, Z)$. Let $m = n - r + 1$.

1. Assume $k = \mathbb{C}$. There exists $E \subset S(\mathbb{C})$ which is the union of countable many proper analytic subset of $S(\mathbb{C})$ such that $\text{ch}^{m,m}_{B,U_x}$ is surjective for all $x \in E \setminus E$.

2. Assume that $k$ is a finite extension of $\mathbb{Q}$ and $S(k) \neq \emptyset$. Let $\pi : S \to \mathbb{P}^N_k$ be a dominant quasi-finite morphism. There exist a subset $H \subset \mathbb{P}^N_k(k)$ such that:

   (i) $\text{ch}^{m,m}_{et,U_x}$ is surjective for any closed point $x \in S$ such that $\pi(x) \in H$.

   (ii) Let $\Sigma$ be any finite set of primes of $k$ and let $k_v$ be the completion of $k$ at $v \in \Sigma$. Then the image of $H$ in $\prod_{v \in \Sigma} \mathbb{P}^N_k(k_v)$ is dense.

Indeed the target spaces of the maps $\text{ch}^{m,m}_{B,U_x}$ and $\text{ch}^{m,m}_{et,U_x}$ are non-zero if $s \geq n - r + 1$ (recall that $s$ is the number of the irreducible components of $Z$) and we give explicit elements in $CH^m(U_x, m)$ whose images span them.

Now we explain how the paper is organized. In §1 we state the fundamental results on the generalized Jacobian rings, the duality theorem and the symmetrizer lemma. The proof is given in another paper [AS1]. It is based on the basic techniques to compute Koszul cohomology developed by M. Green ([G2] and [G3]). In §2 we give a Hodge theoretic implication of the results in §1 which plays a crucial role in the proof of Th. (0-1). Th. (0-1)(1) is proven in §3 and Th. (0-1)(2) is proven in §4 by using the results in
\[ \text{§2. In §5 we explain an implication of the Beilinson’s conjectures on the injectivity of Chern class maps for } K_1 \text{ of surfaces.} \]

\[ \text{§1. Jacobian rings for open complete intersections.} \]

The purpose of this section is to introduce Jacobian rings for open complete intersections and state their fundamental properties. Throughout the whole paper, we fix integers \( r, s \geq 0 \) with \( r + s \geq 1 \), \( n \geq 2 \) and \( d_1, \ldots, d_r, e_1, \ldots, e_s \geq 1 \). We put

\[ d = \sum_{i=1}^{r} d_i, \quad e = \sum_{j=1}^{s} e_j, \quad \delta_{\text{min}} = \min_{1 \leq i \leq s} \{d_i, e_j\}, \quad d_{\text{max}} = \max_{1 \leq i \leq r} \{d_i\}, \quad e_{\text{max}} = \max_{1 \leq j \leq s} \{e_j\}. \]

We also fix a field \( k \) of characteristic zero. Let \( P = k[X_0, \ldots, X_n] \) be the polynomial ring over \( k \) in \( n + 1 \) variables. Denote by \( P^d \subset P \) the subspace of the homogeneous polynomials of degree \( d \). Let \( A \) be a polynomial ring over \( P \) with indeterminants \( \mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_s \). We use the multi-index notation

\[ \mu^a = \mu_1^{a_1} \cdots \mu_r^{a_r} \quad \text{and} \quad \lambda^b = \lambda_1^{b_1} \cdots \lambda_s^{b_s} \quad \text{for} \quad a = (a_1, \ldots, a_r) \in \mathbb{Z}_{\geq 0}^r, \quad b = (b_1, \ldots, b_s) \in \mathbb{Z}_{\geq 0}^s. \]

For \( q \in \mathbb{Z} \) and \( \ell \in \mathbb{Z} \), we write

\[ A_q(\ell) = \bigoplus_{a+b=q} P^{ad+be+\ell} \cdot \mu^a \lambda^b \quad (a = \sum_{i=1}^{r} a_i, \quad b = \sum_{j=1}^{s} b_j, \quad ad = \sum_{i=1}^{r} a_i d_i, \quad be = \sum_{j=1}^{s} b_j e_j) \]

By convention \( A_q(\ell) = 0 \) if \( q < 0 \).

**Definition(1-1).** For \( \mathcal{F} = (F_1, \ldots, F_r), \mathcal{G} = (G_1, \ldots, G_s) \) with \( F_i \in P^{d_i}, \quad G_j \in P^{e_j} \), we define the Jacobian ideal \( J(\mathcal{F}, \mathcal{G}) \) to be the ideal of \( A \) generated by

\[ \sum_{1 \leq i \leq r} \frac{\partial F_i}{\partial X_k} \mu_i + \sum_{1 \leq j \leq s} \frac{\partial G_j}{\partial X_k} \lambda_j, \quad \text{for} \quad F_i, G_j \lambda_j \quad (1 \leq i \leq r, 1 \leq j \leq s, 0 \leq k \leq n). \]

The quotient ring \( B = B(\mathcal{F}, \mathcal{G}) = A/J(\mathcal{F}, \mathcal{G}) \) is called the **Jacobian ring of** \( (\mathcal{F}, \mathcal{G}) \). We denote

\[ B_q(\ell) = B_q(\ell)(\mathcal{F}, \mathcal{G}) = A_q(\ell)/A_q(\ell) \cap J(\mathcal{F}, \mathcal{G}). \]

**Definition(1-2).** Suppose \( n \geq r + 1 \). Let \( \mathbb{P}^n = \text{Proj} \ P \) be the projective space over \( k \). Let \( X \subset \mathbb{P}^n \) be defined by \( F_1 = \cdots = F_r = 0 \) and let \( Z_j \subset X \) be defined by \( G_j = F_1 = \cdots = F_r = 0 \) for \( 1 \leq j \leq s \). We also call \( B(\mathcal{F}, \mathcal{G}) \) the Jacobian ring of the pair \( (X, Z = \cup_{1 \leq j \leq s} Z_j) \) and denote \( B(\mathcal{F}, \mathcal{G}) = B(X, Z) \) and \( J(\mathcal{F}, \mathcal{G}) = J(X, Z) \).

In what follows we fix \( \mathcal{F} \) and \( \mathcal{G} \) as Def.(1-1) and assume the condition

\[ (1-1) \quad F_i = 0 \quad (1 \leq i \leq r) \quad \text{and} \quad G_j = 0 \quad (1 \leq j \leq s) \quad \text{intersect transversally in} \ \mathbb{P}^n. \]

We mention three main theorems. The first main theorem concerns with the geometric meaning of Jacobian rings.

**Theorem(I).** Suppose \( n \geq r + 1 \). Let \( X \) and \( Z \) be as Definition (1-2).

1. For integers \( 0 \leq q \leq n - r \) and \( \ell \geq 0 \) there is a natural isomorphism

\[ \phi_{X,Z}^q : B_q(d + e - n - 1 + \ell) \xrightarrow{\cong} H^q(X, \Omega_{X}^{n-r-q}(\log Z)(\ell))_{\text{prim}}. \]
Here $\Omega^p_X(\log Z)$ is the sheaf of algebraic differential $q$-forms on $X$ with logarithmic poles along $Z$ and ‘prim’ means the primitive part:

$$H^q(X, \Omega^p_X(\log Z)(\ell))_{\text{prim}} = \begin{cases} 
\text{Coker}(H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) \to H^q(X, \Omega^p_X)) & \text{if } q = p \text{ and } s = \ell = 0, \\
H^q(X, \Omega^p_X(\log Z)(\ell)) & \text{otherwise}.
\end{cases}$$

(2) There is a natural map

$$\psi(X,Z) : B_1(0) \to H^1(X, T_X(-\log Z)) \to H^1(X, T_X(-\log Z))$$

which is an isomorphism if $\dim(X) \geq 2$. Here $T_X(-\log Z)$ is the $O_X$-dual of $\Omega^1_X(\log Z)$ and the group on the right hand side is defined in Def.(1-3) below. The following map

$$H^1(X, T_X(-\log Z)) \otimes H^q(X, \Omega^p_X(\log Z)) \to H^{q+1}(X, \Omega^p_X-1(\log Z)).$$

induced by the cup-product and the contraction $T_X(-\log Z) \otimes \Omega^p_X(\log Z) \to \Omega^p_X(1)(\log Z)$ is compatible through $\psi(X,Z)$ with the ring multiplication up to scalar.

Roughly speaking, the generalized Jacobian rings describe the infinitesimal part of the Hodge structures of open variety $X \setminus Z$, and the cup-product with Kodaira-Spencer class coincides with the ring multiplication up to non-zero scalar. This result was originally invented by P. Griffiths in case of hypersurfaces and generalized to complete intersections by Konno [K]. Our result is a further generalization.

**Definition(1-3).** Let the assumption be as in Th.(I). We define $H^1(X, T_X(-\log Z))_{\text{alg}}$ to be the kernel of the composite map

$$H^1(X, T_X(-\log Z)) \to H^1(X, T_X) \to H^2(X, O_X),$$

where the second map is induced by the cup product with the class $c_1(O_X(1)) \in H^1(X, \Omega^1_X)$ and the contraction $T_X \otimes \Omega^1_X \to O_X$. It can be seen that

$$\dim_k(H^1(X, T_X(-\log Z))/H^1(X, T_X(-\log Z))_{\text{alg}}) = \begin{cases} 1 \text{ if } X \text{ is a K3 surface}, \\
0 \text{ otherwise}.
\end{cases}$$

The second main theorem is the duality theorem for the generalized Jacobian rings.

**Theorem(II).** (1) There is a natural map (called the trace map)

$$\tau : B_{n-r}(2(d - n - 1) + e) \to k.$$

Let

$$h_p(\ell) : B_p(d - n - 1 + \ell) \to B_{n-r-p}(d + e - n - 1 - \ell)^*$$

be the map induced by the following pairing induced by the multiplication

$$B_p(d - n - 1 + \ell) \otimes B_{n-r-p}(d + e - n - 1 - \ell) \to B_{n-r}(2(d - n - 1) + e) \to k.$$ 

When $r > n$ we define $h_p(\ell)$ to be the zero map by convention.

(2) The map $h_p(\ell)$ is an isomorphism in either of the following cases.

(i) $s \geq 1$ and $p < n - r$ and $\ell < e_{\text{max}}$.

(ii) $s \geq 1$ and $0 \leq \ell \leq e_{\text{max}}$ and $r + s \leq n$.

(iii) $s = \ell = 0$ and either $n - r \geq 1$ or $n - r = p = 0$.

(3) The map $h_{n-r}(\ell)$ is injective if $s \geq 1$ and $\ell < e_{\text{max}}$.

We have the following auxiliary result on the duality.
Theorem (II'). Assume $n - r \geq 1$ and consider the composite map

$$
\eta(X, z) : H^0(X, \Omega_X^{n-r}(\log Z)) \xrightarrow{(q_X^*z)^{-1}} B_0(d + e - n - 1) \xrightarrow{h_{n-r}(0)^*} B_{n-r}(d - n - 1)^*
$$

where the second map is the dual of $h_{n-r}(0)$. Then $\eta(X, z)$ is surjective and we have (cf. Def. (1-4) below)

$$\text{Ker}(\eta(X, z)) = \wedge^X_{G_1, \ldots, G_s}.$$

Definition (1-4). Let $G_1, \ldots, G_s$ be as in Def. (1-1) and let $Y_j \subset \mathbb{P}^n$ be the smooth hypersurface defined by $G_j = 0$. Let $X \subset \mathbb{P}^n$ be a smooth projective variety such that $Y_j$ (1 \leq j \leq s) and $X$ intersect transversally. Put $Z_j = X \cap Y_j$. Take an integer $q$ with $0 \leq q \leq s - 1$. For integers $1 \leq j_1 < \cdots < j_q + 1 \leq s$, let

$$\omega_X(j_1, \ldots, j_q+1) \in H^0(X, \Omega_X^q(\log Z)) \quad (Z = \sum_{1 \leq j \leq s} Z_j)$$

be the restriction of

$$\sum_{q=1}^{q+1} (-1)^{q-1} (j_1) \wedge G_{j_1} \wedge \cdots \wedge \frac{dG_{j_q}}{G_{j_q}} \wedge \cdots \wedge \frac{dG_{j_{q+1}}}{G_{j_{q+1}}} \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(\log Y))$$

where $Y = \sum_{1 \leq j \leq s} Y_j \subset \mathbb{P}^n$. We let

$$\wedge^q_X(G_1, \ldots, G_s) \subset H^0(X, \Omega_X^q(\log Z))$$

be the subspace generated by $\omega_X(j_1, \ldots, j_q+1)$. For $1 \leq j_1 < \cdots < j_q \leq s - 1$ we have

$$e_s \cdot \omega_X(j_1, \ldots, j_q, s) = \frac{dG_{j_1}}{g_{j_1}} \wedge \cdots \wedge \frac{dG_{j_q}}{g_{j_q}} \wedge g_j = (G_j^* / G_j^*) \in \Gamma(U, \mathcal{O}_U) \quad (U = X \setminus Z)$$

and $\omega_X(j_1, \ldots, j_q, s)$ with $1 \leq j_1 < \cdots < j_q \leq s - 1$ form a basis of $\wedge^q_X(G_1, \ldots, G_s)$.

Our last main theorem is the generalization of Donagi’s symmetrizer lemma [Do] (see also [DG], [Na] and [N]) to the case of complete open intersections at higher degrees.

Theorem (III). Assume $s \geq 1$. Let $V \subset B_1(0)$ is a subspace of codimension $c \geq 0$. Then the Koszul complex

$$B_p(\ell) \otimes \wedge^{q+1} V \rightarrow B_{p+1}(\ell) \otimes \wedge^q V \rightarrow B_{p+2}(\ell) \otimes \wedge^{q-1} V$$

is exact if one of the following conditions is satisfied.

(i) $p \geq 0$, $q = 0$ and $\delta_{\text{min}} + \ell \geq c$.
(ii) $p \geq 0$, $q = 1$ and $\delta_{\text{min}} + \ell \geq 1 + c$ and $\delta_{\text{min}}(p + 1) + \ell \geq d_{\text{max}} + c$.
(iii) $p \geq 0$, $\delta_{\text{min}}(r + c) + \ell \geq d + q + c$, $d + e_{\text{max}} - n - 1 > \ell \geq d - n - 1$ and either $r + s \leq n + 2$ or $p \leq n - r - \lfloor q/2 \rfloor$, where $\lfloor s \rfloor$ denotes the Gaussian symbol.

§2. Hodge theoretic implication of generalized Jacobian rings.

Let the assumption be as in §1. We fix a non-singular affine algebraic variety $S$ over $k$ and the following schemes over $S$:

$$\begin{align*}
(2-1) \quad & \mathbb{P}_k^n \leftrightarrow X \leftrightarrow Z = \bigcup_{1 \leq j \leq s} Z_j
\end{align*}$$
whose fibers are as in Def.(1.2). Let \( f : \mathcal{X} \to S \) be the natural morphism and write \( U = \mathcal{X} \setminus Z \). For integers \( p, q \) we introduce the following sheaf on \( S_{\text{ar}} \)

\[
H^{p,q}(U/S) = R^q f_* \Omega^p_{\mathcal{X}/S}(\log Z),
\]

where \( \Omega^p_{\mathcal{X}/S}(\log Z) = p \Omega^1_{\mathcal{X}/S}(\log Z) \) with \( \Omega^1_{\mathcal{X}/S}(\log Z) \), the sheaf of relative differentials on \( \mathcal{X} \) over \( S \) with logarithmic poles along \( Z \). In case \( s \geq 1 \) the Lefschetz theory implies \( H^{p,q}(U/S) = 0 \) if \( p + q \neq n - r \). In case \( s = 0 \) it implies \( H^{p,q}(\mathcal{X}/S)_{\text{prim}} = 0 \) if \( p + q \neq n - r \) where “prim” denotes the primitive part (cf. Th.(I)(1)). The results in §1 implies that under an appropriate numerical condition on \( d_i \) and \( e_j \) we can control the cohomology of the following Koszul complex

\[
\Omega^{-1}_{S} \otimes H^{a+1, b-1}(U/S) \xrightarrow{\nabla} \Omega^1_S \otimes H^{a,b}(U/S).
\]

Here \( \nabla \) is induced by the Kodaira-Spencer map

\[
(2) \quad \kappa_{(\mathcal{X}, Z)} : \Theta_S \to R^1 f_* T_{\mathcal{X}/S}(- \log Z),
\]

with \( \Theta_S = \mathcal{H}\text{om}_{S}(\Omega^1_S, \mathcal{O}_{S}) \) and \( T_{\mathcal{X}/S}(- \log Z) = \mathcal{H}\text{om}_{\mathcal{X}}(\Omega^1_{\mathcal{X}/S}(\log Z), \mathcal{O}_{\mathcal{X}}) \), and the map

\[
R^1 f_* T_{\mathcal{X}/S}(- \log Z) \otimes R^{b-1} f_* \Omega^a_{\mathcal{X}/S}(\log Z) \to R^b f_* \Omega^a_{\mathcal{X}/S}(\log Z)
\]

induced by the cup product and \( T_{\mathcal{X}/S}(- \log Z) \otimes \Omega^a_{\mathcal{X}/S}(\log Z) \to \Omega^a_{\mathcal{X}/S}(\log Z) \), the contraction. For the application to the Beilinsons conjectures the kernel of the following map plays a crucial role

\[
\nabla^{p,q} : H^{p,q}(U/S) \xrightarrow{\nabla} \Omega^1_S \otimes \mathcal{O} H^{p-1,q+1}(U/S).
\]

In case \( s = 0 \) we let \( \nabla^{p,q} \) denote the primitive part of the above map

\[
H^{p,q}(\mathcal{X}/S)_{\text{prim}} \xrightarrow{\nabla} \Omega^1_S \otimes \mathcal{O} H^{p-1,q+1}(\mathcal{X}/S)_{\text{prim}}.
\]

The key result is the following. The notations will be introduced later (cf. Def.(2-1) and (2-2)).

**Theorem(2-1).** Assume \( p + q = m = n - r \geq 1 \).

1. Assuming \( 1 \leq p \leq m - 1 \) and \( \delta_{\min}(p-1) + d \geq n + 1 + c_S(\mathcal{X}, Z) \), we have \( \text{Ker}(\nabla^{p,q}) = 0 \).

2. Assume \( \delta_{\min}(n-r-1) + d \geq n + 1 + c_S(\mathcal{X}, Z) \). Then \( \text{Ker}(\nabla^{m,0}) \) is generated as \( \mathcal{O}_S \)-module by \( \omega_{U/S}(\sigma) \) with \( \sigma \in J \). In particular \( \text{Ker}(\nabla^{m,0}) = 0 \) if \( s \leq n - r \).

For \( x \in S \) let \( U_x \subset X_x \supset Z_x \) denote the fibers of \( U \subset \mathcal{X} \supset Z \).

**Definition(2-1).** Let \( G_j \in H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(e_j)) \) be a non-zero element defining \( Z_i \subset \mathcal{X} \).

1. For \( 1 \leq j \leq s - 1 \) put

\[
g_j = (G_j^x, G_j^s) \in \Gamma(U, \mathcal{O}_U) = CH^1(U, 1)
\]

where \( CH^1(*, 1) \) is the Bloch’s higher Chow group (cf. [Bl]).

2. Define the index set \( J = \{ \sigma = (j_1, \ldots, j_m) \mid 1 \leq j_1 < \cdots < j_m \leq s - 1 \} \) where \( m = n - r \) is the relative dimension of \( \mathcal{X}/S \). For \( \sigma = (j_1, \ldots, j_m) \in J \) let

\[
c_{U/S}(\sigma) = (g_{j_1}, \ldots, g_{j_m}) \in CH^m(U, m)
\]

be defined by the product \( CH^1(U, 1) \otimes \cdots \otimes CH^1(U, 1) \to CH^m(U, m) \). For \( x \in S \) let \( c_{U_x}(\sigma) \in CH^m(U_x, m) \) be the restriction of \( c_{U/S}(\sigma) \).

3. For \( \sigma = (j_1, \ldots, j_m) \in J \) let

\[
\omega_{U/S}(\sigma) = \frac{d g_{j_1}}{g_{j_1}} \wedge \cdots \wedge \frac{d g_{j_m}}{g_{j_m}} \in \Gamma(S, f_* \Omega^m_{\mathcal{X}/S}(\log Z))
\]

where \( \Omega^m_{\mathcal{X}/S}(\log Z) = p \Omega^1_{\mathcal{X}/S}(\log Z) \) with \( \Omega^1_{\mathcal{X}/S}(\log Z) \), the sheaf of relative differentials on \( \mathcal{X} \) over \( S \) with logarithmic poles along \( Z \).
Lemma (2-1). \( \omega_{U/S}(\sigma) \) with \( \sigma \in J \) are linearly independent over \( \mathcal{O}_S \).

Proof. For \( \tau = (j_1, \ldots, j_m) \in J \) write \( Z_\tau = Z_{j_1} \cap \cdots \cap Z_{j_m} \). By taking Poincaré residues along \( Z_{j_1}, \ldots, Z_{j_m} \), we get the \( \mathcal{O}_S \)-linear map
\[
\text{Res}_\tau : \Gamma(S, f_* \Omega_X^m(\log Z)) \to \Gamma(Z_\tau, \mathcal{O}_{Z_\tau})
\]
and we have
\[
\text{Res}_\tau(\omega_{U/S}(\sigma)) = \begin{cases} 1 & \text{if } \sigma = \tau, \\ 0 & \text{if } \sigma \neq \tau. \end{cases}
\]
This proves Lem. (2-1). \( \square \)

Definition (2-2). For \( x \in S \) let
\[
\kappa_x^{\log} : T_xS \to H^1(X_x, T_{X_x}(\log Z_x)) \quad (\text{resp. } \psi_{(X_x, Z_x)} : B_1(0) \to H^1(X_x, T_{X_x}(-\log Z_x)))
\]
be the Kodaira-Spencer map (resp. the map in Th. (II) (2) for \( (X_x, Z_x) \)). We define
\[
c_S(X, Z) = \max_{x \in S} \{ \dim_k(\text{Im}(\psi_{(X_x, Z_x)}))/\text{Im}(\psi_{(X_x, Z_x)}) \cap \text{Im}(\kappa_x^{\log}) \}.
\]

Remark (2-1). If \( n - r \geq 2 \) and \( X_x \) is not a K3 surface, \( \psi_{(X_x, Z_x)} \) is surjective so that
\[
c_S(X, Z) = \max_{x \in S} \{ \dim_k(Coker(\kappa_x^{\log})) \otimes_{\mathcal{O}_S} k(x) \}.
\]

Remark (2-2). We may use the following more intuitive invariant than \( c_S(X, Z) \). Let \( P^d \subset P \) be as in \( \S 1 \). The dual projective space
\[
\mathbb{P}(P^d) = \mathbb{P}_k^N(n, d) \quad (N(n, d) = \binom{n+d}{d} - 1)
\]
parametrizes hypersurfaces \( Y \subset P^n \) of degree \( d \) defined over \( k \). Let
\[
B \subset \bigcup_{1 \leq j \leq r} \mathbb{P}_k^N(n, d_j) \times \bigcup_{1 \leq j \leq s} \mathbb{P}_k^N(n, c_j)
\]
be the Zariski open subset parametrizing such \( (Y_\nu)_{1 \leq \nu \leq r+s} \) that \( Y_1 + \cdots + Y_{r+s} \) is a simple normal crossing divisor on \( \mathbb{P}_k^n \). We consider the family
\[
X_B \leftarrow Z_B = \bigcup_{1 \leq j \leq s} Z_{B,j} \quad \text{over } B
\]
whose fibers are \( X \leftarrow Z = \bigcup_{1 \leq j \leq s} Z_j \) with \( X = Y_1 \cap \cdots \cap Y_r \) and \( Z_j = X \cap Y_{r+j} \). Let \( T \subset B \) be a non-singular locally closed subvariety of codimension \( c \geq 0 \) and let \( S \to T \) be a dominant map. Assume that the family (2-1) is the pullback of \( (X_B, Z_B)/B \) via \( S \to B \). Then we have \( c_S(X, Z) \leq c \) and the statement of Th. (2-1) holds with \( c_S(X, Z) \) replaced by \( c \).

Now we prove Th. (2-1). We only show the second assertion and leave the first to the readers. The fact \( \omega_{U/S}(\sigma) \in \Gamma(S, \text{Ker}(\nabla^{m,0})) \) follows from the fact that \( \omega_{U/S}(\sigma) \) lies in the image of
\[
H^0(X, \Omega_{X/k}^m(\log Z)) \to \Gamma(S, f_* \Omega_X^m(\log Z)),
\]
where \( \Omega_{X/k}^m(\log Z) \) is the sheaf of differential forms of \( X \) over \( k \) with logarithmic poles along \( Z \). Fix \( 0 \in S \) and let \( X \subseteq Z \) be the fibers of the family (2-1). Let \( \Sigma \subset H^0(X, \Omega_X^m(\log Z)) \) be the subspace generated by \( \omega_{U/S}(\sigma)(0) \) with \( \sigma \in J \). It suffices to show the injectivity of
\[
H^0(X, \Omega_X^m(\log Z))/\Sigma \to \Omega_{X,0}^1 \otimes H^1(X, \Omega_X^{m-1}(\log Z))
\]
that is induced by \( \nabla^{m,0} \). By Th. (II) and Th. (II') in \( \S 1 \) this is reduced to show the surjectivity of
\[
V \otimes B_{n-r-1}(d - n - 1) \to B_{n-r}(d - n - 1)
\]
where \( B_1(0) \supset V := \psi_{(X,Z)}^{-1}(\text{Im}(T_0(S))) \xrightarrow{\kappa_x^{\log}} H^1(X, T_X(-\log Z))) \) (cf. Def. (2-2)). By definition \( V \) is of codimension \( c \) in \( B_1(0) \). Hence the desired assertion follows from Th. (III)(i) in \( \S 1 \). \( \square \)
In this section we assume that $k = \mathbb{C}$. Let $S_{an}$ be the analytic site on $S(\mathbb{C})$. For a coherent sheaf $\mathcal{F}$ on $S_{zar}$ let $\mathcal{F}^{an}$ be the associated analytic sheaf on $S_{an}$. We introduce local systems on $S_{an}$

$$H^q_{\mathbb{Q}}(U/S)(p) = R^q g_* \mathbb{Q}(p) \quad \text{and} \quad H^q_{\mathbb{C}}(U/S) = R^q g_* \mathbb{C},$$

where $g : U \to S$ is the natural morphism. Let $H^q_{\mathbb{Q}}(U/S)$ be the sheaf of holomorphic sections of $H^q_{\mathbb{C}}(U/S)$ and let $F^p H^q_{\mathbb{C}}(U/S) \subset H^q_{\mathbb{C}}(U/S)$ be the holomorphic subbundle given by the Hodge filtration on the cohomology of fibers of $U/S$. We have the analytic Gauss-Manin connection

$$\nabla : H^q_{\mathbb{C}}(U/S) \to \Omega^1_{S_{an}} \otimes H^q_{\mathbb{Q}}(U/S)$$

that satisfies $\nabla(F^p H^q_{\mathbb{C}}(U/S)) \subset \Omega^1_{S_{an}} \otimes F^{p-1} H^q_{\mathbb{C}}(U/S)$. The induced map

$$F^p H^p_{\mathbb{C}}(U/S) / F^{p+1} \to \Omega^1_{S_{an}} \otimes F^{p-1} H^p_{\mathbb{C}}(U/S) / F^p$$

is identified with $(\nabla^p \Gamma)^{an}$ via the identification $F^p H^p_{\mathbb{C}}(U/S) / F^{p+1} = H^{p,q}(U/S)^{an}$. Therefore Th.(2-1) implies the following.

**Theorem (3-1).** Assume $d \geq n + 1 + c_S(X, Z)$.

1. If $s \geq 1$, $F^1 H^m_{\mathbb{C}}(U/S) \subset H^m_{\mathbb{C}}(U/S)$ is generated over $\mathbb{C}$ by

$$\delta_{U/S} : \frac{[d g_{j_1}]}{g_{j_1}} \cup \ldots \cup \frac{[d g_{j_m}]}{g_{j_m}} \in \Gamma(S_{an}, H^m_{\mathbb{C}}(U/S)(m))$$

with $\sigma = (j_1, \ldots, j_m) \in J$. Here $\frac{[d g]}{g}$ is the cohomology class of $\frac{d g}{g}$.

2. If $s = 0$, $F^1 H^m_{\mathbb{C}}(X/S) \subset H^m_{\mathbb{C}}(X/S)_{\text{prim}} = 0$, where $H^m_{\mathbb{C}}(X/S)_{\text{prim}}$ is the primitive part of $H^m_{\mathbb{C}}(X/S)$.

**Theorem (3-2).** Assume $\delta_{\text{min}}(n - r - 1) + d \geq n + 1 + c_S(X, Z)$. There exists $E \subset S(\mathbb{C})$ such that:

(i) $E$ is the union of countable many closed analytic subset of codimension $\geq 1$.

(ii) For $\forall x \in S(\mathbb{C}) - E$,

$$H^m_B(U_x, Q(m)) \cap F^m H^m_B(U_x, \mathbb{C}) = \bigoplus_{\sigma \in J} Q \cdot c_{B,U_x}(\sigma).$$

**Proof.** Fix a base point $0 \in S$ and let $U \subset X \supset Z$ be the fibers of $U \subset X \supset Z$ over 0. Let $0 \in \Delta \subset S(\mathbb{C})$ be an open disk. Identifying $H^m_B(U, Q(m))$ with $\Gamma(\Delta, H^m_{\mathbb{C}}(U/S)(m))$, we set

$$\Delta_{\gamma} = \{ x \in \Delta | \gamma(x) \in F^m H^m_B(U_x, \mathbb{C}) \}$$

for $\gamma \in H^m_B(U, Q(m))$. Let $\tilde{\gamma} \in \Gamma(\Delta, H^m_{\mathbb{Q}}(U/S)/F^m H^m_{\mathbb{Q}}(U/S))$ be the image of $\gamma \in \Gamma(\Delta, H^m_{\mathbb{C}}(U/S)(m)) \subset \Gamma(\Delta, H^m_{\mathbb{Q}}(U/S))$. Then $\Delta_{\gamma} \subset \Delta$ is the zero locus of $\tilde{\gamma}$ and hence it is an analytic subset. Write $H^m_B(U, Q(m)) = \{ \gamma_i \}_{i \in I}$ as a set. Note that $I$ is a countable set. Setting $A = \{ i \in I | \Delta_{\gamma_i} = \Delta \}$ and $B = \{ i \in I | \Delta_{\gamma_i} \not\subset \Delta \}$, we have $I = A \cup B$ and $A \cap B = \emptyset$. We put $E_\Delta = \bigcup_{\gamma \in B} \Delta_{\gamma}$. It suffices to show Th.(3-2)(ii) holds for $\forall x \in \Delta - E_\Delta$.

By definition, for $\forall x \in \Delta - E_\Delta$ we have as a set

$$H^m_B(U_x, Q(m)) \cap F^m H^m_B(U_x, \mathbb{C}) = \{ \gamma_i(x) \}_{i \in A}$$

that implies

$$H^m_B(U_x, Q(m)) \cap F^m H^m_B(U_x, \mathbb{C}) \simeq \Gamma(\Delta, H^m_{\mathbb{Q}}(U/S)(m) \cap F^m H^m_{\mathbb{Q}}(U/S))$$

$$(*)$$

$$\cap \Gamma(\Delta, \text{Ker}(F^m H^m_{\mathbb{Q}}(U/S) \to \bigoplus_{\sigma \in J} S(\mathbb{C}) \otimes \mathbb{C})).$$

Note that $c_{B,U_x}(\sigma) = \omega_{U/S}(\sigma)(x)$ under the natural identification of $F^m H^m_B(U_x, \mathbb{C})$ with the fiber of $H^{m,0}(U/S)$ over $x$. Hence the desired assertion follows from Th.(2-1) and $(*)$ and Lem.(2-1).
§4. Beilinson’s Tate conjecture.

In this section we show Th.(0-1)(2) in the introduction. It will follows from Th.(4-3) below by using the theory of Hilbert set (cf. [La]). Let the assumption be as in the beginning of §2. Write \( m = n - r \).

**Theorem (4-1).** Assume \( d \geq n + 1 + c_S(X, \mathcal{Z}) \). Let \( \eta \) be a geometric generic point of \( S \) and let \( U_\eta \subset X_\eta \) be the fibers of \( U \subset X \) over \( \eta \).

1. Assuming \( s \geq 1 \), we have
   \[
   H^m_{et}(U_\eta, \mathbb{Q}_\ell(m))^{\pi_1(S, \eta)} = \bigoplus_{\sigma \in J} \mathbb{Q}_\ell \cdot ch^m_{et, U_\eta}(c_{U_\eta}(\sigma)).
   \]

2. \( H^m_{et}(X_\eta, \mathbb{Q}_\ell(m))^{\pi_1(S, \eta)} = 0 \).

**Remark (4-1).** Let \( \tilde{S} \to S \) be etale and \( X = \tilde{X} \) be the base change. Then \( c_S(X, \mathcal{Z}) = c_{\tilde{S}}(\tilde{X}, \tilde{\mathcal{Z}}) \) Hence Th.(4-1) holds if one replaces \( \pi_1(S, \eta) \) by any open subgroup of finite index.

**Theorem (4-2).** Assume \( d \geq n + 1 + c_S(X, \mathcal{Z}) \). Let \( U \subset X \) be the fibers of \( U \subset X \) over a fixed base point \( 0 \in S(C) \).

1. Assuming \( s \geq 1 \), we have
   \[
   H^m_B(U(C), \mathbb{Q}(m))^{\pi_1(S, 0)} = \bigoplus_{\sigma \in J} \mathbb{Q} \cdot ch^m_{B, U}(c_U(\sigma)).
   \]

2. \( H^m_B(X(C), \mathbb{Q}(m))^{\pi_1(S, 0)} = 0 \).

First we deduce Th.(4-1) from Th.(4-2). By the Lefschetz principle we may assume that \( k \) is a subfield of \( \mathbb{C} \) finitely generated over \( \mathbb{Q} \). We fix an embedding \( k(\eta) \to \mathbb{C} \) and let \( 0 \in S(C) \) be the corresponding \( \mathbb{C} \)-valued point of \( S \). Write \( S_C = S \otimes_k \mathbb{C} \) and put \( U_C = X_S \operatorname{Spec}(\mathbb{C}) \) via \( \operatorname{Spec}(\mathbb{C}) \to S \). We have the comparison isomorphisms ([SGA4, XVI Th.4.1])

\[
H^m_B(U_C, \mathbb{Q}) \otimes \mathbb{Q}_\ell \xrightarrow{\cong} H^m_{et}(U(C), \mathbb{Q}_\ell) \xrightarrow{\cong} H^m_{et}(U_\eta, \mathbb{Q}_\ell)
\]

that are equivariant with respect to the maps of topological and algebraic fundamental groups:

\[
\pi_1^{top}(S_C, 0) \to \pi_1^{alg}(S_C, 0) \to \pi_1^{alg}(S, \eta).
\]

The desired assertion follows at once from this. \( \square \)

First we show Th.(4-2)(2). Recall the notation in Th.(3-1). Write \( H = H^m_B(X(C), \mathbb{Q})^{\pi_1(S, 0)} \). We have the natural isomorphism

\[
H \otimes \mathbb{C} \xrightarrow{\cong} \Gamma(S_{an}, H^m_C(X/S)_{prim}).
\]

By [D] \( H \) is a sub-Hodge structure of \( H^m_B(X(C), \mathbb{Q}) \) so that we have the Hodge decomposition

\[
H \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q} \text{ with } H^{p,q} \subset \Gamma(S_{an}, F^p H^m_C(X/S) \cap H^m_C(X/S)_{prim}).
\]

By Th.(3-1) this implies \( H^{p,q} = 0 \) for \( p \geq 1 \) which implies \( H = 0 \) by the Hodge symmetry. \( \square \)

In order to show Th.(4-2)(1) we need the following

**Lemma (4-1).** Let the assumption be as in Th.(4-2). Then \( H := H^m_B(U(C), \mathbb{Q})^{\pi_1(S, 0)} \) is a submixed Hodge structure of \( H^m_B(U(C), \mathbb{Q}) \) and \( F^1 H_C = H_C \) where \( H_C = H \otimes \mathbb{C} \).
Proof. The fact that $H$ is a submixed Hodge structure follows from the theory of mixed Hodge modules [SaM]. We show the second assertion. We recall that the graded subquotients of the weight filtration of $H^m_B(U(C, Q))$ is given by

$$Gr^{m+p}_m H^m(U, Q) = \bigoplus_{1 \leq j_1 < \cdots < j_p \leq s} H^{m-p}(Z_{j_1} \cap \cdots \cap Z_{j_p}, \mathbb{Q}(-p))_{prim}. \quad (0 \leq p \leq m)$$

Therefore it suffices to note that $H^m_B(X(C), Q)_{\pi_1(S, 0)} = 0$ as we have just shown. □

Finally we complete the proof of Th. (4-2)(1). The linear independence of $\text{ch}^m_{B, U}(c_U(\sigma))$ with $\sigma \in J$ follows from Lem. (2-1). We show that they span $H^m_B(U(C), Q(m))_{\pi_1(S, 0)}$. Recall the notation in Th. (3-1). By Lem. (4-1) we have

$$H^m_B(U(C), Q(m))_{\pi_1(S, 0)} \simeq \Gamma(S_{an}, H^m_B(U/S)(m)) \subset \Gamma(S_{an}, H^m_B(U/S) \cap F^1 H^m_B(U/S)).$$

Hence the assertion follows from Th. (3-1)(1) by noting $\text{ch}^m_{B, U}(c_U(\sigma)) = \delta_{U/S}(\sigma)(0)$. □

Theorem (4-3). Let the assumption be as in Th. (0-1)(2). Then there exists an irreducible variety $\widetilde{S}$ over $k$ with a finite etale  
covering $\phi : \widetilde{S} \to S$ for which the following holds: Let $\pi = \pi \circ \phi$ and let $H \subset \mathbb{P}^N(k)$ be the subset of such points that $\pi^{-1}(y)$ is irreducible. For $\forall \pi(x) \in S$ such that $\pi(x) \in H$ we have

$$H^m_{et}(U_{\pi}, Q_{\ell}(m))^{\text{Gal}(k(\pi)/k(x))} = \bigoplus_{\sigma \in J} Q_{\ell} \cdot \text{ch}^m_{et, U_{\pi}}(c_{U_{\pi}}(\sigma)).$$

Proof. (cf. [T] and [BE]) By choosing a $k$-rational point 0 of $S$, we get the decomposition

$$\pi_1(S, \overline{\eta}) = \pi_1(S, \overline{\eta})^{geo} \times \text{Gal}(\overline{k}/k),$$

where $\pi_1(S, \overline{\eta})^{geo}$ classifies the finite etale coverings of $S$ that completely decompose over 0. Let

$$\Gamma = \text{Im}(\pi_1(S, \overline{\eta})^{geo} \to \text{GL}_{\mathbb{Q}_{\ell}}(H^m_{et}(U_{\overline{\eta}}, Q_{\ell}(m)))).$$

From the fact that $\Gamma$ contains an $\ell$-adic Lie group as a subgroup of finite index, we have the following fact (cf. [T]): There exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that a continuous homomorphism of pro-finite group $G \to \Gamma'$ is surjective if and only if $G \to \Gamma \to \Gamma'/\Gamma'$ is surjective as a map of sets. Let $\phi : \widetilde{S} \to S$ be a finite etale covering that corresponds to the inverse image of $\Gamma'$ in $\pi_1(S, \overline{\eta})^{geo}$ and let $H$ be defined as in Th. (4-3). Fix $x \in S$ with $\pi(x) \in H$ and let $\overline{\pi}$ be a geometric point of $x$. By choosing a “path” $\overline{x} \to \overline{\pi}$, we get the map $\text{Gal}(k(\overline{x})/k(x)) = \pi_1(x, \overline{x}) \to \pi_1(S, \overline{\eta})$. By the definition of $H$ the composite of $\iota$ with $\pi_1(S, \overline{\eta}) \to \Gamma \to \Gamma'/\Gamma'$ is surjective so that $\text{Gal}(k(\overline{x})/k(x))$ surjects onto $\Gamma$ by the above fact. This implies the isomorphism

$$H^m_{et}(U_{\overline{x}}, Q_{\ell}(m))^{\pi_1(x, \overline{x})} \iso H^m_{et}(U_{\overline{\eta}}, Q_{\ell}(m))^{\pi_1(S, \overline{\eta})}.$$

Now Th. (4-3) follows from Th. (4-1). □

Now Th. (0-1)(2) is a consequence of Th. (4-3) and the following result (cf. [La]):

Lemma (4-2). Let $k$ be a number field and let $V$ be an irreducible variety over $k$. Let $\pi : V \to \mathbb{P}^N(k)$ be an etale morphism and let $H \subset \mathbb{P}^N(k)$ be the subset of such points $x$ that $\pi^{-1}(x)$ is irreducible. Let $\Sigma$ be any finite set of primes of $k$ and let $k_v$ be the completion of $k$ at $v$. Then the image of $H$ in $\prod_{v \in \Sigma} \mathbb{P}^N(k_v)$ is dense.
§5. Implication on injectivity of Chern class maps for $K_1$ of surfaces.

Let $X$ be a projective smooth surface over a field $k$. Let $U \subset X$ be the complement of a simple normal crossing divisor $Z \subset X$. In this section we discuss an implication of the surjectivity of $\text{ch}_{B,U}^{2,2}$ and $\text{ch}_{et,U}^{2,2}$ on the Chern class maps for $CH^2(X,1)$. Recall that $CH^2(X,1)$ is by definition the cohomology of the following complex

$$K_2(k(X)) \xrightarrow{\partial_{\text{tame}}} \bigoplus_{C \subset X} k(C)^* \xrightarrow{\partial_{\text{div}}} \bigoplus_{x \in X} \mathbb{Z},$$

where the sum on the middle term ranges over all irreducible curves on $X$ and that on the right hand side over all closed points of $X$. The map $\partial_{\text{tame}}$ is the so-called tame symbol and $\partial_{\text{div}}$ is the sum of divisors of rational functions on curves. Thus an element of $\text{Ker}(\partial_{\text{div}})$ is given by a finite sum $\sum_i (C_i, f_i)$, where $f_i$ is a non-zero rational function on an irreducible curve $C_i \subset X$ such that $\sum_i \text{div}(f_i) = 0$ on $X$. We recall that $K_2(k(X))$ is generated as an abelian groups by symbols $\{f,g\}$ for non-zero rational functions $f,g$ on $X$ and that

$$\partial_{\text{tame}}(\{f,g\}) = ((f)_0, g) + ((f)_\infty, 1/g) + ((g)_0, 1/f) + ((g)_\infty, f),$$

where $(f)_0$ (resp. $(f)_\infty$) is the zero (resp. pole) divisor of $f$. An important tool to study $CH^2(X,1)$ is the Chern class map: In case $k = \mathbb{C}$ it is given by

$$ch_{D,X}^{2,1} : CH^2(X,1) \to H^3_D(X,\mathbb{Z}(2)),$$

where the group on the right hand side is the Deligne cohomology group (cf. [EV] and [J2]). Assuming the first Betti number $b_1(X) = 0$, we have the following explicit description of $ch_{D,X}^{2,1}$. Take $\alpha = \sum_i (C_i, f_i) \in \text{Ker}(\partial_{\text{div}})$. Under the isomorphism (cf. [EV, 2.10])

$$H^3_D(X,\mathbb{Z}(2)) \simeq \frac{H^2(X,\mathbb{C})}{H^2(X,\mathbb{Z}(2)) + F^2H^2(X,\mathbb{C})} \simeq \frac{F^1H^2(X,\mathbb{C})^*}{H_2(X,\mathbb{Z})},$$

$ch_{D,X}^{2,1}(\alpha)$ is identified with a linear function on complex valued $C^\infty$-forms $\omega$ and we have

$$ch_{D,X}^{2,1}(\alpha)(\omega) = \frac{1}{2\pi i} \sum_i \int_{C_i - \gamma_i} \log(f_i)\omega + \int_\Gamma \omega,$$

where $\gamma_i := f_i^{-1}(\gamma_0)$ with $\gamma_0$, a path on $\mathbb{P}^1_k$ connecting 0 with $\infty$ and $\Gamma$ is a real piecewise smooth 2-chain on $X$ such that $\partial\Gamma = \bigcup \gamma_i$ which exists due to the assumption $\alpha \in \text{Ker}(\partial_{\text{div}})$ and $b_1(X) = 0$.

In case $k$ is a finite extension of $\mathbb{Q}$ we have the Chern class map

$$ch_{cont,X}^{2,1} : CH^2(X,1) \otimes \mathbb{Z}_\ell \to H^3_{cont}(X,\mathbb{Z}_\ell(2)),$$

where $H^i_{cont}$ denotes the continuous etale cohomology of $X$ (cf. [J3]).

Now let $Z \subset X \supset U$ be as in the begining of this section and write $Z = \bigcup_{1 \leq i \leq r} Z_i$ with $Z_i$, smooth irreducible curves intersecting transversally with each other. We consider the following subgroup of $\text{Ker}(\partial_{\text{div}})$

$$CH^1(Z,1) = \text{Ker}(\bigoplus_{1 \leq i \leq r} k(Z_i)^* \xrightarrow{\partial_{\text{div}}} \bigoplus_{x \in Z} \mathbb{Z}).$$

By the localization theory for higher Chow group we have the exact sequence

$$CH^2(U,2) \xrightarrow{\partial_{\text{div}}} CH^1(Z,1) \to CH^2(X,1)$$

where the first map coincides up to sign with the composite of the natural map $CH^2(U,2) \to K_2(k(X))$ and $\partial_{\text{tame}}$. 
Theorem (5-1). (1) Assume \( k = \mathbb{C} \) and that there exists a subspace \( \Delta \subset CH^2(U, 2) \otimes \mathbb{Q} \) such that the restriction of \( ch^{2,1}_{D,U} \) on \( \Delta \) is surjective. Let \( \alpha \in CH^1(Z, 1) \) and assume \( ch^{2,1}_{D,X} (\alpha) = 0 \). Then \( \alpha \in \partial_Z (\Delta) \) in \( CH^1(Z, 1) \otimes \mathbb{Q} \). In particular \( \alpha = 0 \) in \( CH^2(X, 1) \otimes \mathbb{Q} \).

(2) Assume that \( k \) is a finite extension of \( \mathbb{Q} \). The analogous fact holds for \( ch^{2,2}_{ct,U} \) and \( ch^{2,1}_{ct,X} \).

Remark (5-1). The main results in §3 and §4 imply that in case \( X \) is a generic hypersurface of degree \( d \geq 4 \) and \( Z \) is the union of generic hypersurface sections on \( X \), there exists \( \Delta \) satisfying the assumption of Th. (5-1) with such explicit generators as given in Def. (2-1)/(2).

The following result is a direct consequence of Th. (5-1) and Th. (0-1). Let \( Z \subset X \) be as in the introduction and let \( Z_x \subset X_x \) be its fibers over \( x \in S \).

Corollary (5-2). Assume \( \sum_{1 \leq i \leq r} d_i \geq n + 1 + c_S(X, Z) \) and \( n - r = 2 \).

(1) Assume \( k = \mathbb{C} \). There exists \( E \subset S(\mathbb{C}) \) which is the union of countable many proper analytic subset of \( S(\mathbb{C}) \) such that \( ch^{2,1}_{D,Z} \) restricted on the image of \( CH^1(Z, 1) \) is injective modulo torsion for \( \forall x \in S(\mathbb{C}) \setminus E \).

(2) Assume that \( k \) is a finite extension of \( \mathbb{Q} \) and \( S(k) \neq \emptyset \). Let \( \pi : S \to \mathbb{P}^N_k \) be a dominant quasi-finite morphism. There exist a subset \( H \subset \mathbb{P}^N_k \) such that:

(i) \( ch^{2,1}_{D,X} \) restricted on the image of \( CH^1(Z, 1) \) is injective modulo torsion for any closed point \( x \in S \) such that \( \pi (x) \in H \).

(ii) Let \( \Sigma \) be any finite set of primes of \( k \) and let \( k_v \) be the completion of \( k \) at \( v \in \Sigma \). Then the image of \( H \) in \( \prod_{v \in \Sigma} \mathbb{P}^N_k (k_v) \) is dense.

Remark (5-2). Let the notation be as in Cor. (5-2)/1. In the forthcoming paper [AS2] it is shown that there exist \( x \in S(\mathbb{C}) \setminus E \) such that the image of \( CH^1(Z, 1) \) in \( CH^2(X, 1) \) is non-torsion. Thus Cor. (5-2) has indeed non-trivial implication on the injectivity of the Chern class map.

Proof of Th. (5-1). The idea of the following proof is taken from [J1, 9.8]. We only treat the first assertion. The second is proven by the same way. We have the commutative diagram (cf. [Bl] and [J2, 3.3 and 1.15]).

Here the horizontal sequences are the localization sequences for higher Chow groups and Deligne cohomology groups and they are exact. The vertical maps are Chern class maps. By a simple diagram chasing it suffices to show that \( ch^{1,1}_{D,Z} \) is injective and \( ch^{2,2}_{D,U}(\Delta) + \text{Im}(\iota) \) spans \( H^2_D(U, \mathbb{Q}(2)) \). In order to show the first assertion we note the commutative diagram

\[
0 \to \bigoplus_{1 \leq i \leq r} CH^1(Z_i, 1) \to CH^1(Z, 1) \to \bigoplus_{x \in W} \mathbb{Z} \quad \simeq \quad ch^{0,0}_{D,Z} \\
0 \to \bigoplus_{1 \leq i \leq r} H^1_D(Z_i, \mathbb{Z}(1)) \to H^1_D(Z, \mathbb{Z}(1)) \to \bigoplus_{x \in W} H^0_D(x, \mathbb{Z}(0)) \\
0 \to \bigoplus_{1 \leq i \neq j \leq r} Z_i \cap Z_j 
\]

where \( W = \bigcap_{1 \leq i \neq j \leq r} Z_i \cap Z_j \). The horizontal sequences come from the Mayer-Vietoris spectral sequence and they are exact. Thus the desired assertion follows from the fact that \( ch^{1,1}_{D,Z} \) is an isomorphism (cf. [J2, 3.2]). To show the second assertion we recall the exact sequence (cf. [EV, 2.10])

\[
0 \to H^1(U, \mathbb{C})/H^1(U, \mathbb{Z}(2)) \to H^2_D(U, \mathbb{Z}(2)) \to H^2(U, \mathbb{Z}(2)) \cap F^2H^2(U, \mathbb{C}) \to 0
\]
and the same sequence with $U$ replaced by $X$. We have $\chi^{2,2}_{B,U} = \beta \cdot \chi^{2,2}_{D,U}$ and the commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}^* \otimes CH^1(U,1) & \to & CH^1(U,1) \otimes CH^1(U,1) \\
\downarrow \log \otimes \chi^{1,1}_{B,U} & & \downarrow \text{product} \\
\mathbb{C}/\mathbb{Z}(1) \otimes H^1(U,\mathbb{Z}(1)) & \to & CH^2(U,2) \\
\downarrow \simeq & & \downarrow \chi^{2,2}_{D,U} \\
H^1(U,\mathbb{C})/H^1(U,\mathbb{Z}(2)) & \simeq & H^2_{D}(U,\mathbb{Z}(2))
\end{array}
$$

Therefore the desired assertion follows from the surjectivity of

$$
CH^1(U,1) \xrightarrow{\chi^{1,1}_{B,U}} H^1(U,\mathbb{Z}(1))/H^1(X,\mathbb{Z}(1))
$$

which is easily seen. □

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