Multiplication operators on the Bergman space via analytic continuation

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Abstract

In this paper, using the group-like property of local inverses of a finite Blaschke product \( \phi \), we will show that the largest \( C^* \)-algebra in the commutant of the multiplication operator \( M_\phi \) by \( \phi \) on the Bergman space is finite dimensional, and its dimension equals the number of connected components of the Riemann surface of \( \phi^{-1} \circ \phi \) over the unit disk. If the order of the Blaschke product \( \phi \) is less than or equal to eight, then every \( C^* \)-algebra contained in the commutant of \( M_\phi \) is abelian and hence the number of minimal reducing subspaces of \( M_\phi \) equals the number of connected components of the Riemann surface of \( \phi^{-1} \circ \phi \) over the unit disk.

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1. Introduction

Let \( \mathbb{D} \) be the open unit disk in \( \mathbb{C} \). Let \( dA \) denote Lebesgue area measure on the unit disk \( \mathbb{D} \), normalized so that the measure of \( \mathbb{D} \) equals 1. The Bergman space \( L^2_a \) is the Hilbert space con-
sisting of the analytic functions on $D$ that are also in the space $L^2(D, dA)$ of square integrable functions on $D$. For a bounded analytic function $\phi$ on the unit disk, the multiplication operator $M_{\phi}$ is defined on the Bergman space $L^2_a$ given by

$$M_{\phi}h = \phi h$$

for $h \in L^2_a$.

The classification of invariant subspaces or reducing subspaces of various operators acting on function spaces has proved to be one very rewarding research problem in analysis. Not only has the problem itself turned out to be important, but also the methods used to solve it are interesting. The classical Beurling theorem [6] gives a complete characterization of the invariant subspaces of the unilateral shift. Extensions of this idea have led to many important works by other investigators. On the Bergman space, the lattice of invariant subspaces of the Bergman shift $M_z$ is huge and rich [5] although a Beurling-type theorem is established in [2].

A reducing subspace $M$ for an operator $T$ on a Hilbert space $H$ is a subspace $M$ of $H$ such that $TM \subseteq M$ and $T^*M \subseteq M$. A reducing subspace $M$ of $T$ is called minimal if the only reducing subspaces contained in $M$ are $M$ and $\{0\}$. Let $\{M_{\phi}\}'$ denote the commutant of $M_{\phi}$, which is the set of bounded operators on the Bergman space commuting with $M_{\phi}$. The problem of determining the reducing subspaces of an operator is equivalent to finding the projections in the commutant of the operator. An $n$th-order Blaschke product is the function analytic on the unit disk $D$ given by

$$\phi(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a}_jz}$$

for $a_j \in D$. For an $n$th-order Blaschke product $\phi$, since $M^*_{\phi}$ belongs to the Cowen–Douglas class $B_n(D)$, [11,13], one can apply results from complex geometry to note that reducing subspaces correspond to some subspaces of a single fiber which is isomorphic to $\mathbb{C}^n$. This implies immediately that there can’t be more than $n$ pairwise orthogonal reducing subspaces of $M_{\phi}$. However, the lattice of reducing subspaces of the bounded operator $M_{\phi}$ could still be infinite.

Let $A_{\phi}$ be the von Neumann algebra, defined to be the intersection of the commutants $\{M_{\phi}\}'$ and $\{M^*_\phi\}'$. The goal of this paper is to study $A_{\phi}$ for a Blaschke product $\phi$ of finite order. This is a continuation of the investigation begun in [19,26]. In [19,26], one used the Hardy space of the bidisk to study multiplication operators on $L^2_a$ by bounded analytic functions on the unit disk $D$ and to give complete classification of the reducing subspaces of multiplication operators on $L^2_a$ by Blaschke products with order 3 or 4.

In this paper, we will take a completely different approach from the one in [19,26]. On one hand, the multiplication operators have deep connections with the analytic properties of their symbols $\phi$. On the other hand, those multiplication operators are typical subnormal operators whose minimal normal extensions have a thick spectrum and the adjoints of the multiplication operators are in the Cowen–Douglas class [11]. We will make use of two more ingredients. One is local inverses of a finite Blaschke product and their analytic continuations on a subset of the unit disk. (We follow Thomson in [28] using the term “local inverse” to denote a solution $\rho(z)$ to the implicit equation $\phi(\rho(z)) = \phi(z)$.) The germs induced by these local inverses have a group-like property by compositions of germs (for details of compositions of germs, see [29]). The group-like property was used in [10,28,27] in studying the commutant of Toeplitz operators on the Hardy space.
The other ingredient is the theory of subnormal operators [9] which, combined with properties of the Cowen–Douglas classes [11], can be used to show that unitary operators in the commutant of the multiplication operators have a nice representation. Combining the group-like property of local inverses and the nice representation of unitary operators, we will obtain a symmetric and unitary matrix representation of the action of these unitary operators acting on reproducing kernels.

Our main result in the paper is that the dimension of $A_\phi$ equals the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$. This result was obtained for Blaschke products of order 3 or 4 in [19,26] and suggests the following conjecture.

**Conjecture 1.** For a Blaschke product $\phi$ of finite order, the number of nontrivial minimal reducing subspaces of $M_\phi$ equals the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$.

Our main result implies that Conjecture 1 is equivalent to that the $C^*$-algebra $A_\phi$ is abelian. The conjecture is also equivalent to the statement whether or not the minimal reducing subspaces are orthogonal. For a Blaschke product $\phi$ with order smaller than or equal to 8, we will confirm the conjecture by showing that $A_\phi$ is abelian in the last section. In [32,19,26,18], it was shown that the $C^*$-algebra $A_\phi$ is abelian if the order of $\phi$ equals to 2, 3, 4, 5 and 6.

This paper is motivated by Richter's work on unitary equivalence of invariant subspaces of the Bergman space [20], Stephenson's work on hypergroups [22–24] and Zhu's conjecture on the number of minimal reducing subspaces of $M_\phi$ [32]. Many ideas in the paper are, however, inspired by nice works on the commutant of an analytic Toeplitz operator on the Hardy space in [10,27,28]. C. Cowen also used the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$ to describe the commutant of the multiplication operator by $\phi$ on the Hardy space in [10].

Thomson’s representation of local inverses in [28,27] is also useful in the Bergman space context. Using those local inverses as in [28], one can easily see that for an analytic and nonconstant function $f$ in the closed unit disk $\overline{\mathbb{D}}$, there exists a finite Blaschke product $\phi$ such that

\[
\{M_f\}' = \{M_\phi\}'.
\]

This shows that the multiplication operator by a finite Blaschke product will play an important role in studying the other multiplication operators on the Bergman space.

We would like to point out that the results and arguments can carry over to the weighted Bergman spaces, but do not work on the Hardy space. On the Hardy space, because the spectral measure of the minimal normal extension of the multiplication operator by $\phi$ is supported on the unit circle which is its essential spectrum. On the other hand, the spectral measure of the minimal normal extension of the multiplication operator by $\phi$ on the weighted Bergman space is supported on the unit disk, which is its spectrum. The key fact is that the spectral measure is supported on the boundary of the disk in view of the maximum principle. Although the proof doesn’t make it explicit, we believe if points on the interior of the disk are essential with respect to the spectral measure then the argument goes through.

### 2. Analytic continuation and Local inverses

First we introduce some notation. An analytic function element is a pair $(f, U)$, which consists of an open disk $U$ and an analytic function $f$ defined on this disk. A finite sequence $\mathcal{U} = \{(f_j, U_j)\}_{j=1}^n$ is a continuation sequence if

\[
\{M_f\}' = \{M_\phi\}'.
\]
\begin{itemize}
  \item $U_j \cap U_{j+1}$ is not empty for $j = 1, \ldots, m - 1$, and
  \item $f_j \equiv f_{j+1}$ on $U_j \cap U_{j+1}$, for $j = 1, \ldots, m - 1$.
\end{itemize}

Let $\gamma$ be an arc with parametrization $z(t)$, $z(t)$ being a continuous function on an interval $[a, b]$. A sequence $\{U_1, \ldots, U_m\}$ is admissible or a covering chain for $\gamma$ if each $U_j$ is an open disk, and if there exist increasing numbers $t_1, \ldots, t_m$ in $[a, b]$ such that $z(t_j) \in U_j$ for $j = 1, \ldots, m$ and

\[
  z(t) \in \begin{cases}
    U_1, & a \leq t \leq t_1, \\
    U_j \cup U_{j+1}, & t_j \leq t \leq t_{j+1}, \\
    U_m, & t_m \leq t \leq b.
  \end{cases}
\]

A continuation sequence $\mathcal{U} = \{(f_j, U_j)\}_{j=1}^m$ is an analytic continuation along the arc $\gamma$ if the sequence $U_1, \ldots, U_m$ is admissible for $\gamma$. Each of the elements $\{(f_j, U_j)\}_{j=1}^m$ is an analytic continuation of the other along the curve $\gamma$. We say that the analytic function $f_1$ on $U_1$ admits a continuation to $U_m$. A famous result on analytic continuations is the following Riemann monodromy theorem [1,17,29].

**Theorem 2.1.** Suppose $\Omega \subset \mathbb{C}$ is a simply connected open set. If an analytic element, $(f, U)$ can be analytically continued along any path inside $\Omega$, then this analytic function element can be extended to be a single-valued holomorphic function defined on the whole of $\Omega$.

Let $\phi$ be an $n$th order Blaschke product. Let

\[
  E = \mathbb{D} \setminus [\phi^{-1}(\phi(\{\beta \in \mathbb{D} : \phi'(\beta) = 0\}))].
\]

Note that $\mathbb{D} \setminus E$ is finite. For an open set $V \subset \mathbb{D}$, we define a local inverse of $\phi$ in $V$ to be a function $f$ analytic in $V$ with $f(V) \subset \mathbb{D}$ such that $\phi(f(z)) = \phi(z)$ for every $z$ in $V$. That is, $f$ is a branch of $\phi^{-1} \circ \phi$ defined in $V$.

A finite collection, $\{f_i\}$, of local inverses in $V$ is complete if for each $z$ in $V$,

\[
  \phi^{-1}(\phi(z)) \cap \mathbb{D} \subset \{f_i(z)\}
\]

and

\[
  f_i(z) \neq f_j(z)
\]

for $i \neq j$. An open set $V$ is invertible if there exists a complete collection of local inverses in $V$.

A local inverse $(f, V)$ admits an analytic continuation along the curve $\gamma$ in $E$ if there is a continuation sequence $\mathcal{U} = \{(f_j, U_j)\}_{j=1}^m$ admissible for $\gamma$ and $(f_1, U_1)$ equals $(f, V)$. A local inverse in $V \subset E$ is admissible for $\phi$ if it admits unrestricted continuation in $E$. Note that the identity function is always admissible. The set of admissible local inverses has the useful property that it is closed under composition, which can be shown as follows. Let $f$ and $g$ be admissible local inverses in open discs $V$ and $W$ centered at $a$ and $f(a)$, respectively, with $f(V) \subset W$. Let $\gamma$ be a curve in $E$ with initial point $a$. Since $f$ is admissible, it can be analytically continued along $\gamma$. There is an obvious image curve $\tilde{\gamma}$ of $\gamma$ under this analytic continuation along $\gamma$. Since $g$ is also admissible, it can be analytically continued along $\tilde{\gamma}$. By refining the covering chain of $\gamma$, if necessary, we can assume that if $\tilde{V}$ is a covering disc of $\gamma$ and $(\tilde{f}, \tilde{V})$ the corresponding
function element, then $\tilde{f}(\tilde{V})$ is contained in one of the covering discs of $\tilde{y}$. We now compose corresponding function elements in the analytic continuations along $\gamma$ and $\tilde{y}$ to obtain an analytic continuation for $(g \circ f, V)$ along $\gamma$.

Let $V$ be an invertible open disc and let $\{f_i\}$ be the family of admissible local inverses in $V$. By shrinking $V$, we can assume that each $f_i(V)$ is contained in an invertible open disc $W_i$. Let $\{g_{ij}\}_{ij}$ be the family of admissible local inverse in $W_i$. Since $g_{ij} \circ f_i$ is admissible and $g_{ij} \circ f_i \neq g_{ik} \circ f_i$ if $j \neq k$, we observe that $\{g_{ij} \circ f_i\} = \{f_j\}$, for each $i$. In particular, for each $f_i$, there exists $g_{ij}$ such that $g_{ij} \circ f_i$ is the identity function in $V$, which means $f_i^{-1} = g_{ij}$ for some $j$, and thus $f_i^{-1}$ is admissible.

For each $z \in E$, the function $\phi$ is one-to-one in some open neighborhood $D_z$ of each point $z_i$ in $\phi^{-1} \circ \phi(z) = \{z_1, \ldots, z_n\}$. Let $\phi^{-1} \circ \phi = \{\rho_k(z)\}_{k=1}^n$ be $n$ solutions $\phi(\rho(z)) = \phi(z)$. Then $\rho_j(z)$ is locally analytic and arbitrarily continuable in $E$. Assume that $\rho_1(z) = z$. Every open subset $V$ of $E$ is invertible for $\phi$. Then $\{\rho_j\}_{j=1}^n$ is the family of admissible local inverses in some invertible open disc $V \subset \mathbb{D}$. For a given point $z_0 \in V$, label those local inverses as $\{\rho_j(z)\}_{j=1}^n$ on $V$. If there is a loop $\gamma$ in $E$ at $z_0$ such that $\rho_j$ and $\rho_{j'}$ in $\{\rho_i(z)\}_{i=1}^n$ are mutually analytic continuability along $\gamma$, we can then write

$$\rho_j \sim \rho_{j'},$$

and it is easy to check that $\sim$ is an equivalence relation. Using this equivalence relation, we partition $\{\rho_i(z)\}_{i=1}^n$ into equivalence classes

$$\{G_{i_1}, G_{i_2}, \ldots, G_{i_q}\},$$

where $i_1 = 1 < i_2 < i_3 < \cdots < i_q \leq n$ for some integer $1 < q \leq n$ and $\rho_{i_k}$ is in $G_{i_k}$. Not all of the branches of $\phi^{-1} \circ \phi$ can be continued to a different branch. For example, $z$ is a single valued branch of $\phi^{-1} \circ \phi$. Then $q$ is greater than 1. Thus each element in $G_{i_k}$ extends analytically to the other element in $G_{i_k}$, but it does not extend to any element in $G_{i_l}$ if $i_k \neq i_l$. So

$$\{\rho_i(z)\}_{i=1}^n = \bigcup_{k=1}^q G_{i_k}. \quad (2.1)$$

The collection $\{G_{i_1}, G_{i_2}, \ldots, G_{i_q}\}$ does not depend on the choice of $z_0$ in $E$.

Let $k_\alpha$ denote the reproducing kernel of the Bergman space at the point $\alpha$ in $\mathbb{D}$. As in [14,28], we will use local inverses to obtain a local representation of an operator $T$ in the commutant $\{M_\phi\}'$. The proof of the following theorem is similar to the ones in [10,28].

**Theorem 2.2.** Let $\phi$ be a finite Blaschke product, $U$ be an invertible open set of $E$, and let $\{\rho_i(z)\}_{i=1}^n$ be a complete collection of local inverses on $U$. Then for each $T$ in $\{M_\phi\}'$, there are analytic functions $\{s_i(\alpha)\}_{i=1}^n$ on $U$ such that for each $h$ in the Bergman space $L^2_\alpha$,

$$Th(\alpha) = \sum_{i=1}^n s_i(\alpha) h(\rho_i(\alpha)),$$

$$T^*k_\alpha = \sum_{i=1}^n s_i(\alpha) k_{\rho_i(\alpha)}$$

for each $\alpha \in U$. Moreover, these functions $\{s_i(\alpha)\}_{i=1}^n$ admit unrestricted continuation in $E$. 
Proof. Since $T$ commutes with $M_\phi$, the adjoint $T^*$ commutes with $M_\phi^*$. Thus $T^*$ commutes with $M_\phi^*\phi(\alpha)$ for each $\alpha$ in the invertible set $U$. So the kernel of $M_\phi^*\phi(\alpha)$ is invariant for $T^*$. Note that the kernel of $M_\phi^*\phi(\alpha)$ is the finite dimensional space spanned by $\{k_{\rho_i(\alpha)}\}_{i=1}^n$. Hence for each $\alpha$ in $U$, there is a sequence $\{s_i(\alpha)\}_{i=1}^n$ of complex numbers such that

$$T^*k_\alpha = \sum_{i=1}^n s_i(\alpha)k_{\rho_i(\alpha)}.$$

Thus for each $h$ in $L^2_\alpha$, we have

$$Th(\alpha) = \langle Th, k_\alpha \rangle = \langle h, T^*k_\alpha \rangle = \left\langle h, \sum_{i=1}^n s_i(\alpha)k_{\rho_i(\alpha)} \right\rangle = \sum_{i=1}^n s_i(\alpha)\langle h, k_{\rho_i(\alpha)} \rangle = \sum_{i=1}^n s_i(\alpha)h(\rho_i(\alpha)).$$

To finish the proof we need to show that $\{s_i(z)\}_{i=1}^n$ are analytic in $U$. To do so, for each $i$, define

$$P_i(\alpha, z) = \prod_{j \neq i} (z - \rho_j(\alpha))$$

for $z$ in $\mathbb{D}$ and $\alpha$ in $U$. Thus $\{P_i(\alpha, z)\}$ is a family of functions analytic in $z$ on $\mathbb{D}$ and analytic in $\alpha$ on $U$. An easy calculation gives that for each $\alpha$ in $U$

$$\langle P_i(\alpha, .), T^*k_\alpha \rangle = \left\langle P_i(\alpha, .), \sum_{j=1}^n \overline{s_j(\alpha)}k_{\rho_j(\alpha)} \right\rangle = \sum_{j=1}^n s_j(\alpha)\langle P_i(\alpha, .), k_{\rho_j(\alpha)} \rangle = s_i(\alpha)P_i(\alpha, \rho_i(\alpha)) = s_i(\alpha)\prod_{j \neq i}(\rho_i(\alpha) - \rho_j(\alpha)).$$

Thus

$$s_i(\alpha) = \frac{\langle P_i(\alpha, .), T^*k_\alpha \rangle}{\prod_{j \neq i}(\rho_i(\alpha) - \rho_j(\alpha))}.$$
for $\alpha$ in $U$ and hence $s_i$ is analytic in $U$. Noting that $(\rho_i(z))_{j=1}^n$ admit unrestricted continuation in $E$, we conclude that the functions $(s_i(\alpha))_{j=1}^n$ admit unrestricted continuation in $E$ to complete the proof. \hfill $\Box$

3. Riemann surfaces $\phi^{-1} \circ \phi$ over $\mathbb{D}$

Let $\phi = \frac{P(z)}{Q(z)}$ be an $n$th order Blaschke product where $P(z)$ and $Q(z)$ are two coprime polynomials of degree less than or equal to $n$. In this section we will study the Riemann surface for the Blaschke product $\phi^{-1} \circ \phi$ over $\mathbb{D}$. In particular, it was shown in [26] how the number of the connected components of the Riemann surface $\phi^{-1} \circ \phi$ over $\mathbb{D}$ is related to the zeros of $\phi$ for the fourth order Blaschke product $\phi$. Let

$$f(w, z) = P(w)Q(z) - P(z)Q(w).$$

Then $f(w, z)$ is a polynomial of $w$ with degree $n$ and its coefficients are polynomials of $z$ with degree $n$. For each $z \in \mathbb{D}$, $f(w, z) = 0$ has exactly $n$ solutions in $\mathbb{D}$ counting multiplicity. An algebraic function is a function $w = g(z)$ defined for values $z$ in $\mathbb{D}$ by an equation $f(w, z) = 0$.

Recall $\mathcal{F} = \mathbb{D} \setminus E$. Then $\mathcal{F}$ is a finite set and is called the set of branch points of $\phi$, and $\phi^{-1} \circ \phi$ is an $n$-branched analytic function defined and arbitrarily continuable in $\mathbb{D} \setminus \mathcal{F}$. We denote a point of $S_\phi$ lying over $\mathbb{D} \setminus \mathcal{F}$ by $(\rho(\alpha), \alpha)$, where $\alpha$ is in $\mathbb{D} \setminus \mathcal{F}$ and $\rho$ is a branch of $\phi^{-1} \circ \phi$ defined in a neighborhood of $\alpha$.

Visualization of Riemann surfaces is complicated by the fact that they are embedded in $\mathbb{C}^2$, a four-dimensional real space. One aid to constructing and visualizing them is a method known as “cut and paste”. Here we present only details on how to construct $S_f$. For general cases, see [3,7,16]. We begin with $n$ copies of the unit disk $\mathbb{D}$, called sheets. The sheets are labeled $\mathbb{D}_1, \ldots, \mathbb{D}_n$ and stacked up over $\mathbb{D}$. Then $\{z_1, \ldots, z_m\}$ are the branch points. Suppose $\Gamma$ is a curve drawn through those branch points and a fixed point on the unit circle so that $\mathbb{D} \setminus \Gamma$ is a simply connected region. By the Riemann monodromy theorem, $n$ distinct function elements $\rho_k(z)$, $k = 1, \ldots, n$ of the algebraic equation

$$f(w, z) = 0$$

can be extended to be a single-valued holomorphic functions defined over the whole of $\mathbb{D} \setminus \Gamma$. We denote these extended functions still by $\rho_j(z)$. We may assume that $\Gamma$ consists of line segments $l_k$ to connect $z_k$ to $z_{k+1}$. The sheets $\mathbb{D}_j$ are cut open along those line segments $l_k$. Then various sheets are glued to others along opposite edges of cuts. With the point in the $k$th sheet over a value $z$ in $\mathbb{D} \setminus \Gamma$ we associate the pair of values $(\rho_k(z), z)$. In this way a one-to-one correspondence is set up between the points in $S_f$ over $\mathbb{D} \setminus \Gamma$ and the pair of points on the $n$ sheets over $\mathbb{D} \setminus \Gamma$. In order to make the correspondence continuous along the cuts exclusive of their ends, let two regions $R_1$ and $R_2$ be defined in a neighborhood of each cut $l_i$. On each of the $n$ sheets, in the region formed by $R_1$, $R_2$ and the cut $l_i$ between them exclusive of its ends, the values of the algebraic function $w = g(z)$ form again $n$ distinct holomorphic functions $\rho_k(z)$ ($k = 1, \ldots, n$), and these can be numbered so that $g_l(z) = \rho_l(z)$ in $R_1$. In the region $R_2$ the functions $g_k(z)$ are the same functions in the set $\{\rho_k(z)\}$ but possibly in a different order. We join the edge of the cut bounding $R_1$ in the $k$th sheet to the edge bounding $R_2$ in the $l$th sheet, where $l$ is so determined that $g_k(z) = \rho_l(z)$ in $R_2$. The continuous Riemann surface so formed has the property that points in the Riemann surface $S_f$ over non-branch points $\mathbb{D} \setminus \{z_1, \ldots, z_m\}$ are in one-to-one continuous
correspondence with the nonsingular points \((w, z)\) which satisfies the equation \(f(w, z) = 0\). We not only get a manifold; that is, these identifications are continuous but the Riemann surface also has an analytic structure or the match ups are analytic.

We need to use the number of connected components of the Riemann surface \(S_\phi\) in the last two sections. By the unique factorization theorem for the ring \(\mathbb{C}[z, w]\) of polynomials in \(z\) and \(w\), we can factor

\[
f(w, z) = \prod_{j=1}^{q} p_j(w, z)^{n_j},
\]

where \(p_1(w, z), \ldots, p_q(w, z)\) are irreducible polynomials. Since \(f(w, z)\) is the numerator of the rational function \(\phi(z) - \phi(w)\), if \(n_j\) is larger than 1, then the algebraic curve \(p_j(w, z) = 0\) will be contained in the critical set of \(\phi(w)\). But Bôcher’s Theorem [30] says that \(\phi\) has finitely many critical points in the unit disk \(D\). Thus we have

\[
f(w, z) = \prod_{j=1}^{q} p_j(w, z).
\]

The following theorem implies that the number of connected components equals the number of irreducible factors \(f(w, z)\). This result holds for Riemann surfaces over complex plane (cf. [7, p. 78] and [16, p. 374]).

**Theorem 3.1.** Let \(\phi(z)\) be an \(n\)th order Blaschke product and \(f(w, z) = \prod_{j=1}^{q} p_j(w, z)\). Suppose that \(p(w, z)\) is one of factors of \(f(w, z)\). Then the Riemann surface \(S_p\) is connected if and only if \(p(w, z)\) is irreducible. Hence \(q\) equals the number of connected components of the Riemann surface \(S_\phi = S_f\).

**Proof.** Let \(\{z_j\}_{j=1}^{m}\) be the branch points of \(p(w, z) = 0\) in \(\mathbb{D}\). Bôcher’s Theorem [30] says that those points \(\{z_j\}_{j=1}^{m}\) are contained in a compact subset of \(\mathbb{D}\) (it is the elementary observation that a Blachke product of order \(n\) has \(n - 1\) critical points in the unit disk (counting multiplicities)). Suppose that \(p(w, z)\) is irreducible. If the Riemann surface \(S_p\) is not connected, let \(\{\rho_k(z)\}_{k=1}^{n_p}\) be \(n_p\) distinct branches of \(p(w, z) = 0\) over \(\mathbb{D} \setminus \Gamma\). Then \(\{\rho_k(z)\}_{k=1}^{n_p}\) are also roots of the equation

\[
\phi(w) - \phi(z) = 0.
\]

Assuming that \(S_p\) is not connected, we will derive a contradiction. Suppose that one connected component of \(S_p\) is made up of the sheets corresponding to \(\{\rho_1, \ldots, \rho_{n_1}\} (n_1 < n_p)\). Let \(\sigma_s(x_1, \ldots, x_{n_1})\) be elementary symmetric functions of variables \(x_1, \ldots, x_{n_1}\) with degree \(s\):

\[
\sigma_s(x_1, \ldots, x_{n_1}) = \sum_{1 \leq j_1 < j_2 < \cdots < j_s \leq n_1} x_{j_1} x_{j_2} \cdots x_{j_s}.
\]

Since the continuation of any path in \(\mathbb{D} \setminus \mathcal{F}\) only leads to a permutation in \(\{\rho_1(z), \ldots, \rho_{n_1}(z)\}\), every \(\sigma_s(z) = \sigma_s(\rho_1(z), \ldots, \rho_{n_1}(z))\) is unchanged under such a permutation and hence is a holomorphic function well-defined on \(\mathbb{D} \setminus \{z_j\}_{j=1}^{m}\) and analytically extends on a neighborhood of the unit disk although \(\rho_j(z)\) is defined only on \(\mathbb{D} \setminus \Gamma\).
Note that $\rho_j(z)$ is in $\mathbb{D}$. Thus $\sigma_s(\rho_1(z), \ldots, \rho_{n_1}(z))$ is bounded on $\mathbb{D} \setminus \{z_j\}_{j=1}^m$. By the Riemann removable singularities theorem, $\sigma_s(\rho_1(z), \ldots, \rho_{n_1}(z))$ extends analytically on $t \mathbb{D}$ for some $t > 1$. Now we extend $\sigma_s(\rho_1(z), \ldots, \rho_{n_1}(z))$ to the complex plane $\mathbb{C}$. For each $z \in \mathbb{C} \setminus \mathbb{D}$, define

$$f_s(z) = \sigma_s\left(\frac{1}{\rho_1(\frac{1}{z})}, \ldots, \frac{1}{\rho_{n_1}(\frac{1}{z})}\right).$$

By Theorem 11.1 on page 25 [7], near an ordinary point $z = a$, each function $\rho_j(z)$ has a power series of $z - a$. By Lemma 13.1 on page 29 [7], each function $\rho_j(z)$ has a Laurent series of a fractional power of $(z - a)$ but the number of terms with negative exponents must be finite. Thus $f_s(z)$ is a meromorphic function in $\mathbb{C} \setminus \mathbb{D}$ and the point at infinity is a pole of each $f_s(z)$. Moreover, $f_s(z)$ is analytic in a neighborhood of the unit circle. So $f_s(z)$ is analytic in $t \mathbb{D} \setminus r \mathbb{D}$ for $0 < r < 1 < t$.

Next we show that on the unit circle except for one point,

$$\frac{1}{\rho_i(\frac{1}{z})} = \rho_i(z)$$

for each $i$. To do this, for each $\rho \in \{\rho_i\}_{i=1}^m$, noting that $\phi$ is analytic on a neighborhood $\mathcal{V}$ of the closure of the unit disk and $\rho$ extends analytically on the unit circle minus $\Gamma$, we have that for each $w \in \mathcal{V}$ with $|z| = 1$,

$$|\rho(z)| = 1.$$

To prove the above fact, we follow an argument in [28]. If $|\rho(z)| < 1$, as

$$\phi(\rho(z)) = \phi(z)$$

and $z$ is not a branch point of $\phi$, then $\rho(z)$ is in $\mathbb{D} \setminus \{z_j\}_{j=1}^m$. Thus $\rho^{-1}$ is a local inverse near $\rho(z)$ but its range is not contained in the unit disk $\mathbb{D}$. So $\rho^{-1}$ is not admissible and neither is $\rho$. Hence $|\rho(z)| = 1$. This means

$$\rho(z) = \frac{1}{\rho(\frac{1}{z})}$$

for $|z| = 1$.

Thus

$$\sigma_s(\rho_1(z), \ldots, \rho_{n_1}(z)) = f_s(z)$$

for $z$ on the unit circle except for one point. So

$$\sigma_s(\rho_1(z), \ldots, \rho_{n_1}(z)) = f_s(z)$$
in a neighborhood of the unit circle. Define

$$F_s(z) = \begin{cases} \sigma_s(\rho_1(z), \ldots, \rho_{n_1}(z)), & z \in \mathbb{D}, \\ f_s(z), & z \in \mathbb{C} \setminus \mathbb{D}. \end{cases}$$

Thus $F_s(z)$ is a meromorphic function in $\mathbb{C}$ and the point at infinity is a pole of each $F_s(z)$. Hence $F_s(z)$ is a rational function of $z$ and so is $\sigma_s(\rho_1(z), \ldots, \rho_{n_1}(z))$ in $\mathbb{D}$.

Now consider the polynomial

$$f_1(w, z) = w^{n_1} - \sigma_1(z)w^{n_1-1} + \cdots + (-1)^{n_1}\sigma_{n_1}(z) = \prod_{j=1}^{n_1}(w - \rho_j(z))$$

whose coefficients are rational functions of $z$. Thus

$$p(w, z) = f_1(w, z) f_2(w, z)$$

for another polynomial $f_2(w, z)$. This implies that $p(w, z)$ is reducible, which contradicts the assumption that $p(w, z)$ is irreducible. Hence the Riemann surface $S_p$ is connected.

If $p(w, z)$ is reducible, noting that every root $\rho(z)$ of $p(w, z)$ is also a root of $\phi(w) - \phi(z)$, by Bôcher’s Theorem [30], we see that $p(w, z)$ does not have multiple roots for $z \in \mathbb{D} \setminus \{z_j\}_{j=1}^{m}$, we can factor

$$p(w, z) = p_1(w, z) \cdots p_{q_1}(w, z)$$

for some irreducible polynomials $p_i(w, z)$ with degree $k_i$.

Let $\{\rho_{i_1}(z), \ldots, \rho_{k_i}(z)\}$ be roots of the equation $p_i(w, z) = 0$. Thus

$$p_i(\hat{\rho}_{ij}(z), z) = 0.$$

From the identity theorem of analytic functions, each analytic continuation $\hat{\rho}_{ij}(z)$ must still satisfy the equation

$$p_i(\hat{\rho}_{ij}(z), z) = 0,$$

and so $\hat{\rho}_{ij}(z)$ must be in $\{\rho_{i_1}\}_{j=1}^{k_i}$. Since $p_i$ is irreducible, by the above argument, we see that the continuations of the roots $\rho_{i_1}(z), \ldots, \rho_{k_i}(z)$ of $p_i(w, z)$ are always roots of $p_i(w, z)$. Hence crossing a cut permutes the set of roots, and the Riemann surface $S_{p_i}$ is connected. This gives that $S_p$ has $q_1$ connected components, one for each of factors $p_1, \ldots, p_{q_1}$. In particular, $S_f$ has $q$ connected components. This completes the proof.

4. Representation of unitary operators

In this section, we will obtain a better representation of a unitary operator in the commutant $\{M_{\phi}\}'$ on $E \setminus \Gamma$. It will be defined using the orientation $\{\{\rho_i\}_{i=1}^{n}, U\}$ of a complete collection of local inverses $\{\rho_i\}$ on $U$ for a small invertible open set $U$ of $E$. The theory of subnormal operators [9] plays also an important role in this section.
Next we need to order the set \( \{ \rho_j \}_{j=1}^n \) globally over a simply connected subset of \( E \). To do this, take an invertible small open set \( U \) of \( E \) such that the intersection of \( \rho_j(U) \) and \( \rho_k(U) \) is empty for \( j \neq k \). We can always do so by shrinking \( U \) sufficiently. In this section, we fix the small invertible open set \( U \). We label \( \{ \rho_j(z) \}_{j=1}^n \) as \( \{ \rho_1(z), \rho_2(z), \ldots, \rho_n(z) \} \) and assume \( \rho_1(z) = z \). Take a curve \( \Gamma \) through the finite set \( F \) and connecting a point on the unit circle so that \( E \setminus \Gamma \) is simply connected and disjoint from the set \( \bigcup_{j=1}^n \rho_j(U) \). Theorem 2.1 (the Riemann Monodromy theorem) gives that each of \( \{ \rho_j(z) \}_{j=1}^n \) has a unique analytic continuation on \( E \setminus \Gamma \) and hence we can view each of \( \{ \rho_j(z) \}_{j=1}^n \) as an analytic function on \( E \setminus \Gamma \) satisfying

\[
\phi(\rho_j(z)) = \phi(z),
\]

for \( z \in E \setminus \Gamma \) and \( j = 1, \ldots, n \). We retain the same labels \( \{ \rho_j(z) \}_{j=1}^n \) as \( \{ \rho_1(z), \rho_2(z), \ldots, \rho_n(z) \} \) at every point in \( E \setminus \Gamma \). The orientation is denoted by \( \{ (\rho_j(z))_{j=1}^n, U \} \). Then the composition \( \rho_k \circ \rho_i(z) \) makes sense on \( U \). As we pointed out before, \( \{ \rho_j \}_{j=1}^n \) has a group-like property under composition on \( U \), that is,

\[
\rho_k \circ \rho_l(z) = \rho_{\pi_i(k)}(z), 
\]

for some \( \pi_i(k) \in \{1, 2, \ldots, n\} \). Thus

\[
\{ \pi_i(k) \}_{k=1}^n = \{1, 2, \ldots, n\}. 
\]

For each \( \rho \in \{ \rho_i \}_{i=1}^n \), there is a unique \( \hat{\rho} \) in \( \{ \rho_i \}_{i=1}^n \) such that

\[
\hat{\rho}(\rho(z)) = \rho_1(z)
\]

on \( U \). Then the mapping \( \rho \rightarrow \hat{\rho} \) is a bijection from the finite set \( \{ \rho_i \}_{i=1}^n \) to itself. Thus \( \rho_k = \hat{\rho}_k^- \) for some \( k^- \). So for each fixed \( k \), there is a unique number \( k^- \in \{1, 2, \ldots, n\} \) such that

\[
\rho_k(z) = \hat{\rho}_k^-
\]

and hence

\[
\rho_k \circ \rho_k^-(z) = \rho_1(z);
\]

and for each fixed \( i \),

\[
\pi_i = \left( \begin{array}{cccc}
1 & 2 & \cdots & n \\
\pi_i(1) & \pi_i(2) & \cdots & \pi_i(n)
\end{array} \right)
\]

is in the permutation group \( P_n \). Now we define a mapping \( \Phi \) from the set \( \{ \rho_j \} \) of local inverses to the permutation group \( P_n \) as

\[
\Phi(i) := \pi_i.
\]
Thus for each open set $\Delta \subset U$,

$$\rho_k(\rho_i(\Delta)) = \rho_{\pi_i(k)}(\Delta). \quad (4.3)$$

Note that $\rho_k^{-1}(z)$ is also admissible in $E$ and so it is analytic locally in $E \setminus \Gamma$. But on $U$, $\rho_k^{-1}(z) = \rho_k^-(z)$ for some $k^-$. Thus

$$\rho_k^{-1}(z) = \rho_k^-(z) \quad (4.4)$$

for $z$ in $E \setminus \Gamma$. So for each subset $\Delta$ of $E \setminus \Gamma$,

$$\rho_k^{-1}(\Delta) = \rho_k^-(\Delta). \quad (4.5)$$

The proof of Theorem 2.2 gives a global form in terms of the orientation defined as above:

**Theorem 4.1.** Let $\phi$ be a finite Blaschke product. Let $U$ be a small invertible open set of $E$. Let $\{\rho_i(z)\}_{j=1}^n$ be a complete collection of local inverses on $E \setminus \Gamma$ with the orientation $\{\{\rho_j(z)\}_{j=1}^n, U\}$. Then for each $T$ in $\{M_\phi\}'$, there are analytic functions $\{s_i(\alpha)\}_{i=1}^n$ on $E \setminus \Gamma$ such that for each $h$ in the Bergman space $L^2_\alpha$,

$$Th(\alpha) = \sum_{i=1}^n s_i(\alpha)h(\rho_i(\alpha)), \quad \text{for each } \alpha \in E \setminus \Gamma.$$ 

$$T^*k_\alpha = \sum_{i=1}^n \overline{s_i(\alpha)}k_{\rho_i(\alpha)}$$

for each $\alpha$ in $E \setminus \Gamma$.

Recall that an operator $S$ on a Hilbert space $H$ is subnormal if there are a Hilbert space $K$ containing $H$ and a normal operator $N$ on $K$ such that $H$ is an invariant subspace of $N$ and $S = N|_H$. If $K$ has no proper subspace that contains $H$ and reduces $N$, we say that $N$ is a minimal normal extension of $S$. Note that $M_\phi$ is a subnormal operator. The main idea is to use the property that the spectrum of the minimal normal extension of $M_\phi$ contains each invertible set $V$ and the minimal normal extension of $M_\phi$ can be defined using the functional calculus of the minimal normal extension of the Bergman shift $M_z$ under $\phi$.

We need a few of the results on subnormal operators in [9]:

For $\phi$ in $L^\infty$, let $\tilde{M}_\phi$ denote the multiplication operator by $\phi$ on $L^2(\mathbb{D}, dA)$ given by

$$\tilde{M}_\phi g = \phi g$$

for each $g$ in $L^2(\mathbb{D}, dA)$.

- The minimal normal extension of the Bergman shift $M_z$ is the operator $\tilde{M}_z$ on $L^2(\mathbb{D}, dA)$.
- For each finite Blaschke product $\phi$, the minimal norm extension of $M_\phi$ is the operator $\tilde{M}_\phi$ on $L^2(\mathbb{D}, dA)$ (pages 435–436, Theorem VIII.2.20 in [9]).
Suppose \( S_1 \) and \( S_2 \) are two subnormal operators on a Hilbert space \( H \) and \( N_1 \) and \( N_2 \) are the minimal normal extension of \( S_1 \) and \( S_2 \) on \( K \), respectively. If \( S_1 \) and \( S_2 \) are unitarily equivalent, i.e., \( W^* S_1 W = S_2 \) for some unitary operator \( W \) on \( H \), then there is a unitary operator \( \tilde{W} \) on \( K \) such that

\[
W = \tilde{W}|_H, \\
\tilde{W}^* N_1 \tilde{W} = N_2.
\]

Thus for each unitary operator \( W \) in the commutant of \( M_\phi \), there is a unitary operator \( \tilde{W} \) on \( L^2(\mathbb{D}, dA) \) such that

\[
W = \tilde{W}|_{L^2_\alpha}, \\
\tilde{W}^* \tilde{M}_\phi \tilde{W} = \tilde{M}_\phi.
\]

The following theorem was obtained in [25]. For completeness, we will give the detailed proof of the theorem. A different proof was also given in [18].

**Theorem 4.2.** Let \( \phi \) be a finite Blaschke product. Let \( U \) be a small invertible open set of \( E \). Let \( \{\rho_i(z)\}_{j=1}^n \) be a complete collection of local inverses on \( E \setminus \Gamma \) with the orientation \( \{\{\rho_j(z)\}_{j=1}^n, U\} \). Then for each unitary operator \( W \) in \( \{M_\phi\}' \), there is a unit vector \( \{r_i\}_{i=1}^n \) in \( \mathbb{C}^n \) such that for each \( h \) in the Bergman space \( L^2_\alpha \),

\[
Wh(\alpha) = \sum_{i=1}^n r_i \rho'_i(\alpha)h(\rho_i(\alpha)), \\
W^*k_\alpha = \sum_{i=1}^n \overline{r_i} \rho'_i(\alpha)k_{\rho_i(\alpha)}
\]

for each \( \alpha \) in \( E \setminus \Gamma \).

**Proof.** Let \( W \) be a unitary operator in the commutant \( \{M_\phi\}' \). By the remark as above, there is a unitary operator \( \tilde{W} \) on \( L^2(\mathbb{D}, dA) \) such that

\[
W = \tilde{W}|_{L^2_\alpha}, \\
\tilde{W}^* \tilde{M}_\phi \tilde{W} = \tilde{M}_\phi.
\]

Since \( \tilde{M}_\phi \) is a normal operator on \( L^2(\mathbb{D}, dA) \), and \( \sigma(\tilde{M}_\phi) = \overline{\mathbb{D}} \), by the spectral theorem for normal operators [9], \( \tilde{M}_\phi \) has the following spectral decomposition:

\[
\tilde{M}_\phi = \int_{\overline{\mathbb{D}}} \lambda dE(\lambda).
\]
Note that $\tilde{M}_\phi$ equals $\phi(\tilde{M}_z)$. Thus $\tilde{M}_\phi$ commutes with $\tilde{M}_z$. So for each Lebesgue measurable subset $\Delta$ of $\overline{\mathbb{D}}$, the spectral measure

$$E(\Delta) = \int_{\Delta} dE(\lambda)$$

is a projection commuting with $\tilde{M}_z$. Since the commutant

$$\{\tilde{M}_z\}' = \{\tilde{M}_f : f \in L^\infty(\mathbb{D}, dA)\},$$

we obtain

$$E(\Delta) = \tilde{M}_{\chi_{\phi^{-1}(\Delta)}}.$$ 

Here $\chi_{\phi^{-1}(\Delta)}$ is the characteristic function of the set $\phi^{-1}(\Delta)$:

$$\chi_{\phi^{-1}(\Delta)}(z) = \begin{cases} 1, & z \in \phi^{-1}(\Delta), \\ 0, & z \notin \phi^{-1}(\Delta). \end{cases}$$

Since $\tilde{W}$ commutes with $\tilde{M}_\phi$, we have that $E(\Delta)$ commutes with $\tilde{W}$ to get

$$\tilde{W}^* \tilde{M}_{\chi_{\phi^{-1}(\Delta)}} \tilde{W} = \tilde{M}_{\chi_{\phi^{-1}(\Delta)}}.$$

Letting $\text{Supp}(g)$ denote the essential support of a function $g$ in $L^2(\mathbb{D}, dA)$, the above equality gives

$$\text{Supp}[\tilde{W}^*(\chi_{\phi^{-1}(\Delta)}g)] = \text{Supp}[\tilde{W}^* \tilde{M}_{\chi_{\phi^{-1}(\Delta)}} (g)]$$

$$= \text{Supp}[\tilde{W}^* \tilde{M}_{\chi_{\phi^{-1}(\Delta)}} \tilde{W}^* (g)]$$

$$= \text{Supp}[\tilde{M}_{\chi_{\phi^{-1}(\Delta)}} \tilde{W}^* (g)]$$

$$= \text{Supp}[\chi_{\phi^{-1}(\Delta)} \tilde{W}^* (g)]$$

$$\subset \phi^{-1}(\Delta),$$

for each Borel set $\Delta$ and each $g$ in $L^2(\mathbb{D}, dA)$.

For each $\alpha$ in $U$, let $\tilde{V}$ be the open disk with center at $\alpha$ and radius $R$ small enough such that $\tilde{V}$ is contained in $U$. Since $\tilde{V}$ is a subset of $E$, for each $\alpha$ in $\tilde{V}$, $\phi'(\alpha) \neq 0$. Note that on $\tilde{V}$

$$\phi(\rho_i(\alpha)) = \phi(\alpha).$$

Taking the derivative of both sides of the above equality, by the chain rule we have

$$\phi'(\rho_i(\alpha)) \rho_i'(\alpha) = \phi'(\alpha).$$
to get that $\rho_i'(\alpha) \neq 0$. This gives that $\rho_i$ is locally injective. Let $cl(K)$ denote the closure of a subset $K$ of the complex plane. Let $V$ be the open disk center at $\alpha$ and radius $\frac{R}{2}$. Since $V$ is a subset of $U$ and $\{cl(\rho_i(U))\}_{i=1}^n$ are disjoint, $\{cl(\rho_i(V))\}_{i=1}^n$ are disjoint. Thus

$$\phi^{-1} \circ \phi(V) = \bigcup_{i=1}^n \rho_i(V)$$

is the union of $n$ strictly separated and simply connected open sets where $\rho_1(V) = V$. Since $\{cl(\rho_i(V))\}_{i=1}^n$ are disjoint, $f$ is analytic on the closure of $\phi^{-1} \circ \phi(V)$ if

$$f(z) = \begin{cases} 1, & z \in cl(V), \\ 0, & z \in cl(\phi^{-1} \circ \phi(V)) \setminus cl(V). \end{cases}$$

By the Runge theorem [9], there exists a sequence of polynomials $\{p_k(z)\}$ of $z$ such that $p_k(z)$ uniformly converges to $f(z)$ on the closure of $\phi^{-1} \circ \phi(V)$. Therefore,

$$\lim_{k \to \infty} \|\chi_{\phi^{-1} \circ \phi(V) p_k} - \chi_{\phi^{-1} \circ \phi(V) f}\|_2 = 0. \quad (4.6)$$

This gives that for each $g$ in $L^2(\mathbb{D}, dA)$,

$$\langle \tilde{W}^* (\chi_{\phi^{-1} \circ \phi(V) p_k}), g \rangle \to \langle \tilde{W}^* (\chi_{\phi^{-1} \circ \phi(V) f}), g \rangle = \langle \tilde{W}^* (\chi_V), g \rangle. \quad (4.7)$$

The last equality follows from the fact that

$$V \subset \phi^{-1} \circ \phi(V).$$

On the other hand, since $W$ is unitary and commutes with $M_\phi$, $W^*$ also commutes with $M_\phi$. Thus Theorem 4.1 gives that there are functions $t_i(\alpha)$ and $s_i(\alpha)$ analytic on $E \setminus \Gamma$ such that

$$Wh(\alpha) = \sum_{i=1}^n t_i(\alpha)h(\rho_i(\alpha)),$$

$$W^*h(\alpha) = \sum_{i=1}^n s_i(\alpha)h(\rho_i(\alpha)) \quad (4.8)$$

for $\alpha$ in $E \setminus \Gamma$ and $h$ in $L^2_u$. It suffices to show that there is a unit vector $(r_1, r_2, \ldots, r_n)$ in $C^n$ such that

$$s_i(\alpha) = r_i \rho_i'(\alpha)$$

on $E \setminus \Gamma$.

A simple calculation gives
\[
\langle \tilde{W}^*(\chi_{\phi^{-1}}p_k) , g \rangle = \langle \tilde{W}^*(\chi_{\phi^{-1}}p_k) , g \rangle
\]
\[
= \langle \tilde{M}_{\phi^{-1}}(p_k) , g \rangle
\]
\[
= \int_{\phi^{-1}} \tilde{W}^*(p_k)(\alpha) g(\alpha) d\alpha
\]
\[
= \int \chi_{\phi^{-1}}(\sum_{i=1}^n s_i(\alpha)p_k(\rho_i(\alpha)) g(\alpha) d\alpha.
\]

The last equality follows from (4.8). Since the closure of \(\phi^{-1} \circ \phi(V)\) is a compact subset of \(E \setminus \Gamma\) and \(\{s_i(\alpha)\}_{i=1}^n\) are analytic in \(E \setminus \Gamma\), there is a constant \(M > 0\) such that
\[
\sup_{1 \leq i \leq n} \sup_{\alpha \in \phi^{-1}} |s_i(\alpha)| \leq M.
\]

Noting that \(\rho_i'(\alpha) \neq 0\) for \(\alpha\) in \(E \setminus \Gamma\) and \(p_k(z)\) uniformly converges to \(f(z)\) on the closure of \(\phi^{-1} \circ \phi(V)\), by (4.5), we see that
\[
\lim_{k \to \infty} \|\chi_{\phi^{-1}}(p_k \circ \rho_i - \chi_{\phi^{-1}} f \circ \rho_i\|_2 = 0.
\]

Therefore, we have
\[
\lim_{k \to \infty} \left\| \chi_{\phi^{-1}}(\sum_{i=1}^n s_i p_k \circ \rho_i - \chi_{\phi^{-1}} f \sum_{i=1}^n s_i f \circ \rho_i \right\|_2 = 0,
\]

to obtain
\[
\langle \tilde{W}^*(\chi_{\phi^{-1}}p_k) , g \rangle \to \left\langle \chi_{\phi^{-1}}(\sum_{i=1}^n s_i(\alpha)f(\rho_i(\alpha)), g \right\rangle. \quad (4.9)
\]

Combining (4.7) and (4.9) gives that
\[
\tilde{W}^*(\chi_{\phi^{-1}}p_k) = \chi_{\phi^{-1}}(\sum_{i=1}^n s_i(\alpha)f(\rho_i(\alpha)). \quad (4.10)
\]

Noting
\[
\text{Supp} \left[ \sum_{i=1}^n s_i(\alpha)f(\rho_i(\alpha)) \right] \subset \bigcup_{i=1}^n \text{Supp} [f \circ \rho_i]
\]
\[
\subset \bigcup_{i=1}^n \text{Supp} [\chi_{\rho_i^{-1}}(V)].
\]
\[
\subset \bigcup_{i=1}^{n} \rho_i^{-1}(V) \quad \text{by (4.5)}
\]
\[
= \phi^{-1} \circ \phi(V),
\]
we obtain
\[
\tilde{W}^*(\chi_V) = \sum_{i=1}^{n} s_i(\alpha) f(\rho_i(\alpha)).
\]

For an area-measurable set \( V \) in the unit disk, we use \( |V| \) to denote the area measure of \( V \). Since \( W \) is a unitary operator on \( L^2(\mathbb{D}, dA) \), we have
\[
|V|^2 = \| \chi_V \|^2
\]
\[
= \| W^*(\chi_V) \|^2
\]
\[
= \int_{D} \left| \sum_{i=1}^{n} s_i(\alpha) f(\rho_i(\alpha)) \right|^2 dA(\alpha)
\]
\[
= \int_{D} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i(\alpha)s_j(\alpha)f(\rho_i(\alpha))f(\rho_j(\alpha)) dA(\alpha)
\]
\[
= \int_{D} \sum_{i=1}^{n} |s_i(\alpha)|^2 |f(\rho_i(\alpha))|^2 dA(\alpha)
\]
\[
= \sum_{i=1}^{n} \int_{\rho_i^{-1}(V)} |s_i(\alpha)|^2 |f(\rho_i(\alpha))|^2 dA(\alpha)
\]
\[
= \sum_{i=1}^{n} \int_{V} \left| f(\alpha) \right|^2 \left| \frac{s_i \circ \rho_i^{-1}(\alpha)}{\rho_i \circ \rho_i^{-1}(\alpha)} \right|^2 dA(\alpha)
\]
\[
= \int_{V} \sum_{i=1}^{n} \left| \frac{s_i \circ \rho_i^{-1}(\alpha)}{\rho_i \circ \rho_i^{-1}(\alpha)} \right|^2 dA(\alpha).
\]
The fifth equality follows from
\[
\text{Supp}[f \circ \rho_i] \cap \text{Supp}[f \circ \rho_j] = \rho_i^{-1}(V) \cap \rho_j^{-1}(V)
\]
\[
= \rho_i^{-1}(V) \cap \rho_j^{-1}(V) \quad \text{by (4.5)}
\]
\[
= \emptyset.
\]
Here $\emptyset$ denotes the empty set. The sixth equality follows from the fact that $\text{Supp}[f \circ \rho_i] = \rho_i^{-1}(V)$ and the seventh equality follows from that the change of variable, $\beta = \rho_i(\alpha)$, gives

$$\int_{\rho_i^{-1}(V)} |s_i(\alpha)|^2 f(\rho_i(\alpha)) \, dA(\alpha) = \int_{V} |f(\beta)|^2 \left| \frac{s_i \circ \rho_i^{-1}(\beta)}{\rho_i' \circ \rho_i^{-1}(\beta)} \right|^2 \, dA(\beta).$$

Therefore we obtain for each $z \in U$ and for any open neighborhood $V \subset U$ of $z$, that

$$1 = \frac{1}{|V|^2} \int_{V} \left| \sum_{i=1}^{n} \frac{s_i \circ \rho_i^{-1}(\alpha)}{\rho_i' \circ \rho_i^{-1}(\alpha)} \right|^2 \, dA(\alpha).$$

Noting that $\sum_{i=1}^{n} \left| \frac{s_i \circ \rho_i^{-1}(\alpha)}{\rho_i' \circ \rho_i^{-1}(\alpha)} \right|^2$ is continuous on $U$ and letting $V$ shrink to $z$, we have

$$1 = \sum_{i=1}^{n} \left| \frac{s_i \circ \rho_i^{-1}(z)}{\rho_i' \circ \rho_i^{-1}(z)} \right|^2.$$

Applying the Laplace operator $\frac{\partial^2}{\partial z \partial \bar{z}}$ to both sides of the above equality gives

$$0 = \sum_{i=1}^{n} \left[ \frac{s_i \circ \rho_i^{-1}(z)}{\rho_i' \circ \rho_i^{-1}(z)} \right]'.
$$

Thus

$$\left[ \frac{s_i \circ \rho_i^{-1}(z)}{\rho_i' \circ \rho_i^{-1}(z)} \right]' = 0$$

on $U$. So there are constants $r_i$ such that

$$|r_1|^2 + \cdots + |r_n|^2 = 1,$$

$$\frac{s_i \circ \rho_i^{-1}(z)}{\rho_i' \circ \rho_i^{-1}(z)} = r_i$$

on $U$ for each $i$, and so

$$s_i(z) = r_i \rho_i'(z),$$

on $\rho_i^{-1}(U) = \rho_i_-(U)$ and hence on $E \setminus \Gamma$ because that both $s_i(z)$ and $\rho_i'(z)$ are analytic on $E \setminus \Gamma$. This completes the proof. \(\square\)
5. Decomposition of the Bergman space

Let \( \phi \) be a Blaschke product of order \( n \). As pointed out in the introduction, by complex geometry, one can easily see that the Bergman space can be decomposed as a direct sum of at most \( n \) nontrivial reducing subspaces of \( M_\phi \). In this section we will refine this result and show that the Bergman space can be decomposed as a direct sum of at most \( q \) nontrivial reducing subspaces of \( M_\phi \), where \( q \) is the number of connected components of the Riemann surface \( S_\phi \).

Let \( U \) be a small invertible open subset of \( E \). Let \( \{ \rho_j(z) \}_{j=1}^n \) be a complete collection of local inverses. For a unitary operator \( W \), Theorem 4.2 gives that for each \( z \) in \( U \),

\[
W^* k_\alpha = \sum_{j=1}^n \bar{r_j} \rho'_j(\alpha) k_{\rho_j(\alpha)}.
\]

For our convenience, we use \( r_\rho \) to denote \( r_\rho \). Then the above equality becomes

\[
W^* k_\alpha = \sum_{\rho \in \{ \rho_j(z) \}_{j=1}^n} \bar{r_\rho} \rho'_(\alpha) k_{\rho(\alpha)}.
\]

**Lemma 5.1.** Let \( U \) be a small invertible open subset of \( E \) and \( \{ \rho_j(z) \}_{j=1}^n \) be a complete collection of local inverses. Fix a point \( z_0 \) in \( U \). Let \( G_i \) be the set of those functions in \( \{ \rho_j(z) \}_{j=1}^n \) which are analytic extensions of \( \rho_i \) along some loop containing \( z_0 \) in \( E \). Write

\[
\{ \rho_j(z) \}_{j=1}^n = \bigcup_{k=1}^q G_{ik}.
\]

Then for each \( \alpha \) in \( U \),

\[
W^* k_\alpha = \sum_{k=1}^q \hat{r}_k \sum_{\rho \in G_{ik}} \bar{\rho}'(\alpha) k_{\rho(\alpha)},
\]

where \( \hat{r}_k = r_\rho \) for some \( \rho \) in \( G_{ik} \).

**Proof.** For each \( \alpha \) in \( U \), by Theorem 4.2, we have

\[
W^* k_\alpha = \sum_{k=1}^q \hat{r}_k \sum_{\rho \in G_{ik}} \bar{\rho}'(\alpha) k_{\rho(\alpha)}.
\]  

(5.1)

It suffices to show that for two \( \rho_0 \) and \( \hat{\rho}_0 \) in the same \( G_{ik}, r_{\rho_0} = r_{\hat{\rho}_0} \). Note that the conjugates of both sides of (5.1) are locally analytic functions of \( \alpha \) in \( E \). In fact the conjugate of the right-hand side of (5.1) is an analytic function of \( \alpha \) in the unit disk \( \mathbb{D} \). Since each of \( \{ \rho_j(z) \}_{j=1}^n \) admits unrestricted continuation in \( E \), the conjugate of the left-hand of (5.1) admits unrestricted analytic continuation in \( E \). Let \( z_0 \) be the fixed point in \( U \) and \( \gamma \) be a loop in \( E \) containing \( z_0 \).

Suppose that each \( \rho \) of the set \( \{ \rho_j(z) \}_{j=1}^n \) is extended analytically to \( \hat{\rho} \) from \( z_0 \) to \( z_0 \) along the loop \( \gamma \). In the neighborhood \( U \) of \( z_0 \), we have
Thus (5.1) and (5.2) give
\[
\sum_{k=1}^{q} \sum_{\rho \in G_{ik}} \overline{r_{\rho}}(\alpha)k_{\rho}(\alpha) = \sum_{k=1}^{q} \sum_{\rho \in G_{ik}} \overline{r_{\rho}}(\alpha)k_{\rho}(\alpha).
\]
Since the reproducing kernels are linearly independent, the above equality gives
\[
\overline{r_{\rho_0}}(\alpha) = \overline{r_{\rho_0}}(\alpha).
\]
And hence \(r_{\rho_0} = r_{\rho_0}\). This completes the proof.

**Theorem 5.2.** Let \(\phi\) be a finite Blaschke product. Then the Bergman space can be decomposed as a direct sum of at most \(q\) nontrivial minimal reducing subspaces of \(M_{\phi}\) where \(q\) is the number of connected components of the Riemann surface \(S_{\phi}\).

**Proof.** Suppose that the Bergman space is the direct sum of \(p\) nontrivial minimal reducing subspaces \(\{M_j\}_{j=1}^{p}\) of \(M_{\phi}\). That is,
\[
L_{a}^{2} = \bigoplus_{j=1}^{p} M_{j}.
\]
Let \(P_{j}\) denote the orthogonal projection from \(L_{a}^{2}\) onto \(M_{j}\). Thus \(P_{j}\) commutes with both \(M_{\phi}\) and \(M_{\phi}^{*}\). For \((\theta_1, \ldots, \theta_p)\) in \([0, 2\pi]^{p}\), let
\[
W(\theta_1, \ldots, \theta_p) = \sum_{j=1}^{p} e^{i\theta_j} P_{j}.
\]
Then \(\{W(\theta_1, \ldots, \theta_p)\}\) is a family of unitary operators in the commutant \(\{M_{\phi}\}'\). Using this family of unitary operators, we can recover \(P_{j}\) as follows:
\[
P_{j}h = \int_{[0, 2\pi]^{p}} [W(\theta_1, \ldots, \theta_p)]^{*}h d\mu_j(\theta_1, \ldots, \theta_p),
\]
for \(h\) in \(L_{a}^{2}\), where \(d\mu_j(\theta_1, \ldots, \theta_p)\) is the measure \(\frac{e^{i\theta_j}}{(2\pi)^p} d\theta_1 \cdots d\theta_p\) on \([0, 2\pi]^{p}\).

Let \(U\) be a small invertible open subset of \(E\) and \(\{\rho_{i}(z)\}_{j=1}^{n}\) be a complete collection of local inverses. Fix a point \(z_{0}\) in \(U\). Let \(G_{i}\) be the set of those local inverses in the set \(\{\rho_{i}(z)\}_{j=1}^{n}\) which are analytic extensions of \(\rho_{i}\) along some loop in \(E\) containing \(z_{0}\). Write \(\{\rho_{i}(z)\}_{j=1}^{n} = \bigcup_{k=1}^{q} G_{ik}\). By the definition, \(q\) is the number of connected components of the Riemann surface \(S_{\phi}\). To finish the proof we need only to show that
\[
p \leq q.
\]
To do this, by Lemma 5.1, we have that there are constants \( \{ \hat{r}_k(\theta_1, \ldots, \theta_p) \} \) such that

\[
\left[ W(\theta_1, \ldots, \theta_p) \right]^* k_\alpha = \sum_{k=1}^{q} \hat{r}_k(\theta_1, \ldots, \theta_p) \sum_{\rho \in G_{ik}} \rho'(\alpha) k_{\rho(\alpha)}
\]

for \( \alpha \) in \( U \). Thus

\[
P_j k_\alpha = \int_{[0,2\pi]^p} \left[ W(\theta_1, \ldots, \theta_p) \right]^* k_\alpha \, d\mu_j(\theta_1, \ldots, \theta_p)
\]

\[
= \int_{[0,2\pi]^p} \sum_{k=1}^{q} \hat{r}_k(\theta_1, \ldots, \theta_p) \sum_{\rho \in G_{ik}} \rho'(\alpha) k_{\rho(\alpha)} \, d\mu_j(\theta_1, \ldots, \theta_p)
\]

\[
= \sum_{k=1}^{q} \hat{r}_k \sum_{\rho \in G_{ik}} \rho'(\alpha) k_{\rho(\alpha)}
\]

(5.3)

where

\[
\hat{r}_k = \int_{[0,2\pi]^p} \hat{r}_k(\theta_1, \ldots, \theta_p) \, d\mu_j(\theta_1, \ldots, \theta_p).
\]

For each \( j \), let \( Z_j \) denote the zero set of the functions in \( M_j \), that is

\[
\{ z \in \mathbb{D} : f(z) = 0 \text{ for each } f \in M_j \}.
\]

Then \( Z_j \) is a countable subset of the unit disk \( \mathbb{D} \). Hence \( U \setminus \bigcup_{j=1}^{p} Z_j \) is not empty. For each \( \alpha \) in \( U \setminus \bigcup_{j=1}^{p} Z_j \), there is a function \( f_j \) in \( M_j \) such that

\[
f_j(\alpha) \neq 0.
\]

Further, we have

\[
f_j(\alpha) = \langle f_j, k_\alpha \rangle
\]

\[
= \langle P_j f_j, k_\alpha \rangle
\]

\[
= \langle f_j, P_j k_\alpha \rangle.
\]

Thus

\[
\| P_j k_\alpha \| \neq 0.
\]

(5.4)

For a fixed \( \alpha \) in \( U \setminus \bigcup_{j=1}^{p} Z_j \), let \( \Theta \) be the subspace of \( L^2_\alpha \) spanned by \( \{ P_1 k_\alpha, \ldots, P_p k_\alpha \} \). Thus (5.3) gives that \( \Theta \) is contained in the subspace of \( L^2_\alpha \) spanned by \( q \) functions

\[
\left\{ \sum_{\rho \in G_{ik}} \rho'(\alpha) k_{\rho(\alpha)} \right\}_{k=1}^{q}.
\]

So the dimension of $\Theta$ is less than or equal to $q$. On the other hand, for distinct $j$ and $l$, $P_lP_j = 0$. This gives
\[
\langle P_j k_\alpha, P_l k_\alpha \rangle = \langle P_l P_j k_\alpha, k_\alpha \rangle = 0.
\]
Thus combining the above equality with (5.4) gives that the dimension of $\Theta$ equals $p$ and so $p \leq q$. This completes the proof. $\square$

6. Matrix representation of unitary operators

For a finite Blaschke product $\phi$, it is pointed out in Section 4 that the orientation of $\{\rho_i\}_{i=1}^n$ induces elements $\{\pi_i\}_{i=1}^n$ in $P_n$. So $\{\pi_i\}_{i=1}^n$ forms a subgroup of the permutation group $P_n$. In this section, using the property that $\{\pi_i\}_{i=1}^n$ forms a group, we will obtain the following unitary matrix representation of a unitary operator in the commutant $\{M_\phi\}'$.

**Theorem 6.1.** Let $\phi$ be a finite Blaschke product with order $n$, $U$ be a small invertible open set of $E$, and $\{\rho_j(z)\}_{j=1}^n$ be a complete collection of local inverses. For $E \setminus \Gamma$ with the orientation $\{\rho_j(z)\}_{j=1}^n \cup U$. If $W$ is a unitary operator on the Bergman space which commutes with the multiplication operator $M_\phi$, then there is a unit vector $(r_1, \ldots, r_n) \in \mathbb{C}^n$ such that
\[
W^* \begin{pmatrix} \rho_1^*(\alpha)k_{\rho_1(\alpha)} \\ \rho_2^*(\alpha)k_{\rho_2(\alpha)} \\ \rho_3^*(\alpha)k_{\rho_3(\alpha)} \\ \vdots \\ \rho_n^*(\alpha)k_{\rho_n(\alpha)} \end{pmatrix} = \Gamma_W \begin{pmatrix} \rho_1^*(\alpha)k_{\rho_1(\alpha)} \\ \rho_2^*(\alpha)k_{\rho_2(\alpha)} \\ \rho_3^*(\alpha)k_{\rho_3(\alpha)} \\ \vdots \\ \rho_n^*(\alpha)k_{\rho_n(\alpha)} \end{pmatrix},
\]
for $\alpha \in U$, where the representing matrix $\Gamma_W$ is the following unitary matrix in $U_n(\mathbb{C})$:
\[
\Gamma_W = \begin{pmatrix} r_1 & r_2 & r_3 & \cdots & r_n \\ r_{\pi_1^{-1}(1)} & r_{\pi_1^{-1}(2)} & r_{\pi_1^{-1}(3)} & \cdots & r_{\pi_1^{-1}(n)} \\ r_{\pi_2^{-1}(1)} & r_{\pi_2^{-1}(2)} & r_{\pi_2^{-1}(3)} & \cdots & r_{\pi_2^{-1}(n)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ r_{\pi_n^{-1}(1)} & r_{\pi_n^{-1}(2)} & r_{\pi_n^{-1}(3)} & \cdots & r_{\pi_n^{-1}(n)} \end{pmatrix}.
\]

**Proof.** By Theorem 4.2, there is a unit vector $(r_1, r_2, \ldots, r_n) \in \mathbb{C}^n$ such that
\[
W^* k_\alpha = \sum_{i=1}^n r_i \rho_j^*(\alpha)k_{\rho_i(\alpha)}
\]
for each $\alpha$ in $E \setminus \Gamma$. Since $\bigcup_{j=1}^n \rho_j(U)$ is contained in $E \setminus \Gamma$, we have for each $l$ and $\alpha$ in $U$, that
\[ W^* \left[ \rho_l'(\alpha) k_{\rho_l(\alpha)} \right] = \rho_l'(\alpha) \sum_{i=1}^{n} \overline{r_i} \rho_i'(\alpha) k_{\rho_i(\rho_l(\alpha))} \]

\[ = \sum_{i=1}^{n} \overline{r_i} \rho_i'(\alpha) \rho_i'(\alpha) k_{\rho_i(\rho_l(\alpha))} \]

\[ = \sum_{i=1}^{n} \overline{r_i} \rho_i'(\alpha) \rho_i'(\alpha) k_{\rho_i(\rho_l(\alpha))} \]

\[ = \sum_{i=1}^{n} \overline{r_i} \rho_i'(\alpha) \rho_i'(\alpha) k_{\rho_i(\rho_l(\alpha))} \]

\[ = \overline{\rho_i'(\alpha)} k_{\rho_i(\rho_l(\alpha))} \text{ by (4.1)} \]

\[ = \sum_{i=1}^{n} \overline{r_{\pi^{-1}(i)}} \rho_i'(\alpha) k_{\rho_i(\rho_l(\alpha))}. \quad (6.3) \]

This gives the matrix representation (6.1) of \( W \). To finish the proof we need only to show that \( \Gamma_W \) is a unitary matrix.

Since \( W^* = W^{-1} \), \( W^* \) also commutes with \( M_\phi \). For a unitary operator \( W \), this follows easily from the following reason:

\[ WM_\phi = M_\phi W. \]

By multiplying both sides of the above equality by \( W^* \) and noting that

\[ W^* W = W W^* = I, \]

we have

\[ M_\phi W^* = W^* M_\phi. \]

Using Theorem 4.2 again, there is another unit vector \((s_1, s_2, \ldots, s_n)\) in \( C^n \) such that

\[ Wk_\alpha = \sum_{i=1}^{n} \overline{s_i} \rho_i'(\alpha) k_{\rho_i(\alpha)} \]  \hspace{1cm} (6.4)

for each \( \alpha \) in \( E \setminus \Gamma \). Similar to the argument estimating (6.3), we also have

\[ W \left[ \rho_l'(\alpha) k_{\rho_l(\alpha)} \right] = \sum_{i=1}^{n} \overline{s_{\pi^{-1}(i)}} \rho_i'(\alpha) k_{\rho_i(\alpha)}, \]

for \( \alpha \) in \( U \). Thus for each \( l \), we have

\[ \rho_l'(\alpha) k_{\rho_l(\alpha)} = W W^* \left[ \rho_l'(\alpha) k_{\rho_l(\alpha)} \right] \]

\[ = W \left[ W^* \left( \rho_l'(\alpha) k_{\rho_l(\alpha)} \right) \right] \]
\[
\begin{align*}
\sum_{i=1}^{n} \sum_{j=1}^{s_{\pi_i^{-1}(j)} (i)} \rho_{\pi_i^{-1}(j)}(\alpha) k_{\rho_{\pi_i^{-1}(j)}(\alpha)} &= W \left[ \sum_{i=1}^{n} \sum_{j=1}^{s_{\pi_i^{-1}(j)} (i)} \rho_{\pi_i^{-1}(j)}(\alpha) k_{\rho_{\pi_i^{-1}(j)}(\alpha)} \right] \text{ by (6.3)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{s_{\pi_i^{-1}(j)} (i)} W \left[ \rho_{\pi_i^{-1}(j)}(\alpha) k_{\rho_{\pi_i^{-1}(j)}(\alpha)} \right] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{s_{\pi_i^{-1}(j)} (i)} \sum_{k=1}^{s_{\pi_i^{-1}(j)} (i)} \rho_{\pi_i^{-1}(j)}(\alpha) k_{\rho_{\pi_i^{-1}(j)}(\alpha)} \\
&= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \sum_{k=1}^{s_{\pi_i^{-1}(j)} (i)} \rho_{\pi_i^{-1}(j)}(\alpha) k_{\rho_{\pi_i^{-1}(j)}(\alpha)} \right) \\
&= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \sum_{k=1}^{s_{\pi_i^{-1}(j)} (i)} \rho_{\pi_i^{-1}(j)}(\alpha) k_{\rho_{\pi_i^{-1}(j)}(\alpha)} \right) \\
&= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \sum_{k=1}^{s_{\pi_i^{-1}(j)} (i)} \rho_{\pi_i^{-1}(j)}(\alpha) k_{\rho_{\pi_i^{-1}(j)}(\alpha)} \right).
\end{align*}
\]

Since the reproducing kernels are linearly independent, we have
\[
\sum_{i=1}^{n} \sum_{j=1}^{s_{\pi_i^{-1}(j)} (i)} \rho_{\pi_i^{-1}(j)}(\alpha) k_{\rho_{\pi_i^{-1}(j)}(\alpha)} = 1,
\]
\[\text{and} \]
\[
\sum_{i=1}^{n} \sum_{j=1}^{s_{\pi_i^{-1}(j)} (i)} \rho_{\pi_i^{-1}(j)}(\alpha) k_{\rho_{\pi_i^{-1}(j)}(\alpha)} = 0
\]
\[\text{for } 1 \leq j, l \leq n \text{ and } j \neq l. \]

On the other hand, since both \((r_1, r_2, \ldots, r_n)\) and \((s_1, s_2, \ldots, s_n)\) are unit vectors in \(C^n\), by (4.2) we have that for each \(k\)
\[
|r_{\pi_k^{-1}(1)}|^2 + |r_{\pi_k^{-1}(2)}|^2 + \cdots + |r_{\pi_k^{-1}(n)}|^2 = |r_1|^2 + |r_2|^2 + \cdots + |r_n|^2
\]
\[= 1, \quad \text{and} \]
\[
\sum_{k=1}^{n} \left[ |s_{\pi_k^{-1}(1)}|^2 + |s_{\pi_k^{-1}(2)}|^2 + \cdots + |s_{\pi_k^{-1}(n)}|^2 \right] = \sum_{j=1}^{n} \left[ |s_{\pi_j^{-1}(1)}|^2 + |s_{\pi_j^{-1}(2)}|^2 + \cdots + |s_{\pi_j^{-1}(n)}|^2 \right]
\]
\[= \sum_{j=1}^{n} \left[ |s_1|^2 + |s_2|^2 + \cdots + |s_n|^2 \right] = n.
\]

Let \(|s_{\pi_1^{-1}(j)}|^2 + |s_{\pi_2^{-1}(j)}|^2 + \cdots + |s_{\pi_n^{-1}(j)}|^2\) be the smallest of the set
\[
\left\{ |s_{\pi_1^{-1}(k)}|^2 + |s_{\pi_2^{-1}(k)}|^2 + \cdots + |s_{\pi_n^{-1}(k)}|^2 \right\}_{k=1}^{n}.
\]

Then
\[
|s_{\pi_1^{-1}(j)}|^2 + |s_{\pi_2^{-1}(j)}|^2 + \cdots + |s_{\pi_n^{-1}(j)}|^2 \leq 1.
\]

From the Cauchy–Schwarz inequality and (6.5), one has
\[ 1 = \left| \sum_{i=1}^{n} r_{\pi_i^{-1}(i)}^{-1} s_{\pi_i^{-1}(i)} \right| \]
\[ \leq \left( \sum_{i=1}^{n} |r_{\pi_i^{-1}(i)}|^{2} \right)^{1/2} \left( \sum_{i=1}^{n} |s_{\pi_i^{-1}(i)}|^{2} \right)^{1/2} \leq 1, \]

which gives
\[ (r_{\pi_i^{-1}(1)}, r_{\pi_i^{-1}(2)}, \ldots, r_{\pi_i^{-1}(n)}) = \lambda_i (\tilde{s}_{\pi_i^{-1}(1)}, \tilde{s}_{\pi_i^{-1}(2)}, \ldots, \tilde{s}_{\pi_i^{-1}(l)}), \]

for some unimodular constant \( \lambda_i \). Thus
\[ |s_{\pi_i^{-1}(1)}|^{2} + |s_{\pi_i^{-1}(2)}|^{2} + \cdots + |s_{\pi_i^{-1}(l)}|^{2} = 1, \]

so we have
\[ \sum_{k=1, k \neq l}^{n} \left[ |s_{\pi_i^{-1}(k)}|^{2} + |s_{\pi_i^{-1}(2)}|^{2} + \cdots + |s_{\pi_i^{-1}(k)}|^{2} \right] = n - 1. \]

Repeating the above argument and by induction, we will obtain that for each \( k \)
\[ (r_{\pi_k^{-1}(1)}, r_{\pi_k^{-1}(2)}, \ldots, r_{\pi_k^{-1}(n)}) = \lambda_k (\tilde{s}_{\pi_i^{-1}(k)}, \tilde{s}_{\pi_i^{-1}(k)}, \ldots, \tilde{s}_{\pi_i^{-1}(k)}), \]

for a unimodular constant \( \lambda_k \).

By (6.6), we have
\[ (r_{\pi_i^{-1}(1)}, r_{\pi_i^{-1}(2)}, \ldots, r_{\pi_i^{-1}(n)}) \perp (\tilde{s}_{\pi_i^{-1}(j)}, \tilde{s}_{\pi_i^{-1}(j)}, \ldots, \tilde{s}_{\pi_i^{-1}(j)}) \]

for \( j \neq l \). So
\[ |r_{\pi_i^{-1}(1)}|^{2} + |r_{\pi_i^{-1}(2)}|^{2} + \cdots + |r_{\pi_i^{-1}(n)}|^{2} = 1 \quad \text{for } 1 \leq l \leq n \quad \text{and} \]
\[ (r_{\pi_i^{-1}(1)}, r_{\pi_i^{-1}(2)}, \ldots, r_{\pi_i^{-1}(n)}) \perp (r_{\pi_j^{-1}(1)}, r_{\pi_j^{-1}(2)}, \ldots, r_{\pi_j^{-1}(n)}) \]

for \( 1 \leq j \neq l \leq n \). This gives that \( \Gamma_W \) is unitary to complete the proof. \( \square \)

7. **Von Neumann algebra \( \mathcal{A}_\phi \)**

In the previous section, we have shown that every unitary \( W \) in \( \mathcal{A}_\phi \) is completely determined by \( n \times n \) matrix \( \Gamma_W \). By the Russo–Dye Theorem [21,12,31], every element in \( \mathcal{A}_\phi \) can be written as a finite linear combination of unitary operators in \( U\mathcal{A}_\phi \). Thus \( \mathcal{A}_\phi \) is finite dimensional. In this section we will show that its dimension equals the number \( q \) of connected components of the Riemann surface \( \phi^{-1} \circ \phi \) over the unit disk. Since Lemma 5.1 will show directly that the dimension of \( \mathcal{A}_\phi \) is no greater than \( q \), the main effort in this section is showing that the dimension is at least \( q \).
To that end we are going to construct \( q \) linearly independent elements in \( A_\phi \). Let \( \phi \) be a finite Blaschke product. Recall that \( \mathcal{C} \) denotes the set of the critical points of \( \phi \) in \( D \) and
\[
\mathcal{F} = \phi^{-1} \circ \phi(C),
\]
and \( E = \mathbb{D} \setminus \mathcal{F} \). Let \( z_0 \) be a point in a small invertible open set \( U_0 \) of \( E \). Let \( \{\rho_j(z)\}_{j=1}^n \) be a complete collection of local inverses. Let \( G_i \) be the set of those functions in \( \{\rho_j(z)\}_{j=1}^n \) which are analytic extensions of \( \rho_i \) along some loop in \( E \) containing \( z_0 \). Then
\[
\{\rho_j(z)\}_{j=1}^n = \bigcup_{k=1}^{q} G_{ik},
\]
where \( 1 \leq i_k \leq n \) and \( 1 \leq q \leq n \). Each element in \( G_{ik} \) extends analytically to the other elements in \( G_{ik} \), but it does not extend to any element in \( G_{il} \) if \( i_k \neq i_l \). Suppose \( \Gamma \) is the curve constructed in Section 3, drawn through these branch points and a fixed point on the unit circle so that \( \mathbb{D} \setminus \Gamma \) is a simply connected region. For each \( 1 \leq k \leq q \), define a bounded linear operator \( \mathcal{E}_k : L^2_a \rightarrow L^2_a \) by
\[
(\mathcal{E}_k f)(z) = \sum_{\rho \in G_{ik}} \rho'(z) f(\rho(z)) \tag{7.1}
\]
for \( z \in E \) and each \( f \in L^2_a \). We will prove that \( \mathcal{E}_k \) is bounded. First we show that \( \mathcal{E}_k f \) is analytic on \( \mathbb{D} \). To do so, let \( F \) be an analytic function on \( \mathbb{D} \) such that \( F' = f \). We have that
\[
g(z) = \sum_{\rho \in G_{ik}} F(\rho(z))
\]
extends to be an analytic function in \( \mathbb{D} \) (a similar argument was given in the proof of Theorem 3.1). Thus
\[
g'(z) = \sum_{\rho \in G_{ik}} \rho'(z) f(\rho(z))
\]
extends to be analytic in \( \mathbb{D} \). So the operator \( \mathcal{E}_k \) is well defined.

We observe that the label of each element in \( G_{ik} \) depends on \( \Gamma \) and a neighborhood of \( z_0 \) but the set \( G_{ik} \) does not. So we can view \( G_{ik} \) as a collection of local inverses defined locally and each element in \( G_{ik} \) may have different labels as a global function on \( E/\Gamma \). Thus the summation in (7.1) defining \( \mathcal{E}_k f(z) \) does not depend on how we choose the curve \( \Gamma \) or how we label those local inverses on a neighborhood of any point \( z_0 \) in \( E \).

First we introduce the notation \( \hat{\Gamma} \) which denotes the set
\[
\Gamma \cup \bigcup_{j=1}^{n} \{w \in E \setminus \Gamma; \ \rho_j(w) \in \Gamma\},
\]
which consists of finitely many curves on \( \mathbb{D} \). Since \( \rho \) maps \( E \) into \( E \) and is locally analytic and injective on \( E \), \( \hat{\Gamma} \) is a closed set which has area measure equal to zero.
Lemma 7.1. For each \( \rho \in \{ \rho_j \}_{j=1}^n \), \( \rho(E \setminus \Gamma) \) contains \( \mathbb{D} \setminus \hat{\Gamma} \), and hence the area measure of \( \mathbb{D} \setminus \rho(E \setminus \Gamma) \) equals zero.

Proof. Let \( z_0 \) be a point in \( \mathbb{D} \setminus \hat{\Gamma} \). Thus \( \rho_j(z_0) \) is not in \( \Gamma \) for each \( \rho_j \). Since \( \rho_j \) is locally analytic and injective in \( E \), \( \rho_j \) is an open mapping. Thus we can find a small open neighborhood \( \mathcal{V}_j \) of the point \( z_0 \) such that

\[
\rho_j(\mathcal{V}_j) \cap \Gamma = \emptyset.
\]

Let

\[
\mathcal{V} = \bigcap_{j=1}^n \mathcal{V}_j.
\]

Then \( \mathcal{V} \) is an open neighborhood of \( z_0 \) such that

\[
\bigcup_{j=1}^n \rho_j(\mathcal{V}) \subset \bigcup_{j=1}^n \rho_j(\mathcal{V}_j) \subset E \setminus \Gamma.
\]

For each \( \rho \), there is a \( \hat{\rho} \in \{ \rho_j \}_{j=1}^n \) such that

\[
\rho(\hat{\rho}(z)) = \rho_1(z) = z
\]

on an open neighborhood \( \hat{\mathcal{V}} \) of \( z_0 \) which is contained in \( \mathcal{V} \). Letting \( w_0 = \hat{\rho}(z_0) \), then \( w_0 \) is contained in \( \hat{\rho}(\mathcal{V}) \) and hence contained in \( E \setminus \Gamma \). Therefore \( z_0 \) is contained in \( \rho(E \setminus \Gamma) \). This completes the proof. \( \square \)

Since each local inverse \( \rho \) is locally injective, there is a family of pairwise disjoint open sets in \( \mathbb{D} \) on which each \( \rho \) in \( \{ \rho_j \}_{j=1}^n \) is injective on \( U_\mu \) and

\[
\left| \mathbb{D} / \bigcup_{\mu} U_\mu \right| = 0.
\]

Let \( S_k \) be the connected component of the Riemann surface \( S_\phi \) associated with \( G_{i_k} \).

Lemma 7.2. Let \( \{ U_\mu \} \) be pairwise disjoint open sets in \( \mathbb{D} \) on which each \( \rho \) in \( \{ \rho_j \}_{j=1}^n \) is injective on \( U_\mu \) and

\[
\left| \mathbb{D} / \bigcup_{\mu} U_\mu \right| = 0.
\]

Then \( \rho(U_{\mu_1}) \) is disjoint from \( \rho(U_{\mu_2}) \) if \( \mu_1 \neq \mu_2 \) and hence each \( \rho \) is injective on \( \bigcup_{\mu} U_\mu \).

Proof. As we pointed out in the proof of Theorem 3.1, each local inverse \( \rho \) has analytic continuation in \( r\mathbb{D} \setminus \mathcal{F} \) for some \( r > 1 \) and \( |\rho(z)| = 1 \) on the unit circle. So we can construct a Riemann surface \( S_r \) over \( r\mathbb{D} \setminus \mathcal{F} \) and \( S_k \) is an open region of \( S_r \). Assume \( G_{i_k} \) consists of \( n_k \) elements.
\(\tilde{\rho}_1, \ldots, \tilde{\rho}_{nk}\). For each open set \(U\) contained in the unit disk, on which \(\tilde{\rho}_j\) is injective, we define a function \(f\) on \(S_k\) by

\[f(\tilde{\rho}_j(z), z) = \tilde{\rho}_j(z)\]

for each \((\tilde{\rho}_j(z), z) \in S_k\). Then \(f\) is a holomorphic function on \(S_k\). Clearly, \(f\) extends to be a holomorphic function on \(S_r\). Let \(\omega\) be the differential 1-form on \(S_k\)

\[\omega = -\frac{i}{2} \tilde{f} df.\]

On the chart \(U_{\rho} = \{ (\rho(z), z) : z \in U \}\), it is easy to check

\[\omega = -\frac{i}{2} \rho d\rho\]

and

\[d\omega = -\frac{i}{2} d(\tilde{f} df) = d\tilde{f} \wedge df = -\frac{i}{2} (\tilde{\rho}(z) d\bar{z}) \wedge (\rho'(z) dz) = -\frac{i}{2} |\rho'(z)|^2 d\bar{z} \wedge dz.\]

Now \(\{\rho(U_\mu) : \mu, \rho \in G_{ik}\}\) forms a local chart of the Riemann surface \(S_k\) minus a set with zero surface area. Thus

\[|S_k| = \sum_{\rho \in G_{ik} \cup U_\mu} \int \frac{i}{2} d\rho \wedge d\bar{\rho} = \sum_{\rho \in G_{ik}} \sum_\mu |\rho(U_\mu)|\]

and

\[|S_k| = \int_{S_k} d\omega.\]

On the other hand, for each point \(\beta \in \mathcal{F}\) and small positive number \(\epsilon\), let \(C(\beta, \epsilon)\) be the circle with center \(\beta\) and radius \(\epsilon\) and \(\Gamma_{j,\epsilon}(\beta) = \{ (\tilde{\rho}_j(z), z) : z \in C(\beta, \epsilon) \}\). Let \(S_{k,\epsilon} = S_k \setminus \bigcup_{j=1}^{nk} \bigcup_{\beta \in \mathcal{F}} \Gamma_{j,\epsilon}(\beta)\). Clearly, \(S_{k,\epsilon}\) is an open Riemann surface with boundary

\[\partial S_{k,\epsilon} = \left[ \bigcup_{j=1}^{nk} \bigcup_{\beta \in \mathcal{F}} \Gamma_{j,\epsilon}(\beta) \right] \cup \{ |f(w, z)| = 1 : (w, z) \in S_k \}\]

in \(S_r\). By Stokes’ formula [8, Theorem 4.2], we have
\[ |S_{k,\epsilon}| = \int_{S_{k,\epsilon}} d\omega \]
\[ = \int_{\partial S_{k,\epsilon}} \omega \]
\[ = \sum_{j=1}^{n_k} \sum_{\beta \in F} \int_{F_j,\epsilon(\beta)} \omega + \int_{\{ ||f(w,z)|| = 1 : (w,z) \in S_k \}} \omega \]
\[ = -\frac{i}{2} \sum_{j=1}^{n_k} \sum_{\beta \in F} \int_{C(\beta,\epsilon)} \bar{\rho}_j d\bar{\rho}_j - \frac{i}{2} \sum_{j=1}^{n_k} \int_{|z|=1} \frac{d\bar{\rho}_j(z)}{\rho_j(z)} \]
\[ = -\frac{i}{2} \sum_{j=1}^{n_k} \sum_{\beta \in F} \int_{C(\beta,\epsilon)} \bar{\rho}_j d\bar{\rho}_j - \frac{i}{2} \int_{|z|=1} d \left[ \ln \left( \prod_{j=1}^{n_k} \bar{\rho}_j(z) \right) \right] \]

Noting that the product \( \prod_{j=1}^{n_k} \bar{\rho}_j(z) \) is a Blaschke factor of \( \frac{\phi(0)\phi}{1-\phi(0)\phi} \) with order \( n_k \), by the formula of the winding number of a closed curve, we have
\[ -\frac{i}{2} \int_{|z|=1} d \left[ \ln \left( \prod_{j=1}^{n_k} \bar{\rho}_j(z) \right) \right] = n_k \pi = n_k |D|. \]

Noting that each local inverse \( w = \rho(z) \) is a solution
\[ P(w)Q(z) - P(z)Q(w) = 0, \]
and the leading coefficient of \( w^n \) in the above polynomial is given by
\[ Q(z) - P(z)\phi(0), \]
which never vanishes on the unit disk, for each \( \beta \in F \). It follows from the Puiseux theorem ([7, Lemma 13.1 and Theorem 13.1] or [15, Theorem 8.14]) that there is a neighborhood \( D(\beta,\epsilon) \setminus \{ \beta \} \), at which \( \rho(z) \) has a power series expansion of \( (z - \beta)^{1/n} \). Thus for some positive constant \( M > 0 \),
\[ \left| \int_{C(\beta,\epsilon)} \bar{\rho}_j d\bar{\rho}_j \right| \leq M \epsilon^{1/n} \to 0, \]
and
\[ \left| \int_{D(\beta,\epsilon)\setminus\{\beta\}} d\bar{\rho}_j(z) \wedge d\bar{\rho}_j(z) \right| \leq M \epsilon^{2/n} \to 0, \]
as $\epsilon \to 0$. This gives

$$\lim_{\epsilon \to 0} |S_{k, \epsilon}| = n_k |\mathbb{D}|, \tag{7.1}$$

and

$$\lim_{\epsilon \to 0} |S_{k, \epsilon}| = |S_k|.$$

Hence

$$|S_k| = n_k |\mathbb{D}|.$$

By Lemma 7.1, we have

$$|\rho(\mathbb{D}/\Gamma)| = |\mathbb{D}|.$$

This gives

$$\sum_{\mu} |\rho(U_{\mu})| \geq |\rho(\mathbb{D}/\Gamma)| = |\mathbb{D}|.$$

Thus

$$|S_k| = \sum_{\rho \in G_i} \sum_{\mu} |\rho(U_{\mu})| \geq n_k |\mathbb{D}|.$$

But this implies

$$\sum_{\mu} |\rho(U_{\mu})| = |\mathbb{D}|.$$

Therefore

$$|\rho(U_{\mu_1}) \cap \rho(U_{\mu_2})| = 0$$

for $\mu_1 \neq \mu_2$. Since $\rho$ is an open mapping, we conclude that $\rho(U_{\mu_1})$ is disjoint from $\rho(U_{\mu_2})$. This completes the proof. \hfill \Box

The boundedness of $E_k$ follows from

$$\int_{E \setminus \Gamma} |f(\rho(z))|^2 |\rho'(z)|^2 \, dA(z) = \sum_{\mu} \int_{\rho(U_{\mu})} |f(w)|^2 \, dA(w)$$

$$= \int_{\mathbb{D}} |f(w)|^2 \, dA(w), \tag{7.2}$$
where the \( \{U_\mu\} \) are disjoint open sets in \( \mathbb{D} \) on which each \( \rho \) is injective on \( U_\mu \) and

\[
\left| \mathbb{D} / \bigcup_\mu U_\mu \right| = 0.
\]

The last equality in (7.2) follows from the above lemma.

To get \( E^*_k \), we need the following change of variable formula.

**Lemma 7.3.** For each \( \rho \in \{\rho_j\}_{j=1}^n \) and \( f \in L^2_a \),

\[
\int_{E \setminus \Gamma} \left| f(\rho(z)) \right|^2 \left| \rho'(z) \right|^2 dA(z) = \int_{\mathbb{D}} \left| f(w) \right|^2 dA(w).
\]

**Proof.** We can choose \( \{U_\mu\} \) to be disjoint open sets in \( \mathbb{D} \) such that for each \( \rho \), \( \rho \) is injective on \( U_\mu \) and

\[
\left| \mathbb{D} / \bigcup_\mu U_\mu \right| = 0.
\]

For each \( \rho \in \{\rho_j\}_{j=1}^n \) and \( f \in L^2_a \), we have

\[
\int_{E \setminus \Gamma} \left| f(\rho(z)) \right|^2 \left| \rho'(z) \right|^2 dA(z) = \int_{\bigcup_\mu U_\mu} \left| f(\rho(z)) \right|^2 \left| \rho'(z) \right|^2 dA(z)
\]

\[
= \sum_\mu \int_{\rho(U_\mu)} \left| f(w) \right|^2 dA(w)
\]

\[
= \int_{\rho(E \setminus \Gamma)} \left| f(w) \right|^2 dA(w)
\]

\[
= \int_{\mathbb{D}} \left| f(w) \right|^2 dA(w).
\]

The first equality follows from the fact that the area measure of \( \mathbb{D} \setminus E \) is zero. The second equality follows from Lemma 7.2. The last equality comes from Lemma 7.1, which states that the area measure of \( \mathbb{D} \setminus \rho(E \setminus \Gamma) \) is zero. This completes the proof. \( \square \)

Note that \( \rho_1 \) equals \( z \). Since \( \rho^{-1} \) is also in \( \{\rho_i\}_{i=1}^n \) for each \( \rho \in \{\rho_i\}_{i=1}^n \), let \( G^-_{ik} \) denote the subset of \( \{\rho_i\}_{i=1}^n \):

\[
G^-_{ik} = \{\rho: \rho^{-1} \in G_{ik}\}.
\]

We need the following lemma to find \( E^*_k \).
Lemma 7.4. For each $i_k$, there is an integer $k^-$ with $1 \leq k^- \leq q$ such that

$$G_{i_k}^- = G_{i_k^-}.$$

Proof. Assume that $z_0$ is a point in $E \setminus \Gamma$. For two elements $\rho$ and $\hat{\rho}$ in $G_{i_k}$, suppose that $\hat{\rho}$ is an analytic continuation of $\rho$ along some loop $\gamma$ in $E$ containing $z_0$. Let $\hat{\gamma}$ be the image curve of $\gamma$ under this analytic continuation along $\gamma$. We will show that $\hat{\rho}^{-1}$ is an analytic continuation of $\rho^{-1}$ along a loop at $z_0$ in $E$.

Note that $\hat{\gamma}$ is a curve connecting $\rho(z_0)$ and $\hat{\rho}(z_0)$ in $E$. Thus $\rho^{-1}$ has an analytic continuation $\hat{\rho}^{-1}$ along the curve $\hat{\gamma}$ in $E$. This gives that $\rho^{-1} \circ \rho$ has an analytic continuation $\hat{\rho}^{-1} \circ \hat{\rho}$ along the loop $\gamma$ from $z_0$ to $z_0$ in $E$, but $\rho^{-1} = z$ has only one analytic continuation $\rho^{-1}$ along any loop in $E$. Hence we have that $\hat{\rho}^{-1} \circ \rho = \rho^{-1}$ to get $\hat{\rho}^{-1} = \rho^{-1}$. This means that $\rho^{-1}$ can be analytically extended to $\hat{\rho}^{-1}$ along a curve connecting $\rho(z_0)$ and $\hat{\rho}(z_0)$ in $E$. Also we can find two curves $\gamma_1, \gamma_2$ in $E \setminus \Gamma$ such that $\rho^{-1}$ on an open neighborhood of $z_0$ is an analytic continuation of $\rho^{-1}$ on an open neighborhood of $\rho(z_0)$ along $\gamma_1$ and $\hat{\rho}^{-1}$ on an open neighborhood of $z_0$ is an analytic continuation of $\hat{\rho}^{-1}$ on an open neighborhood of $\hat{\rho}(z_0)$ along $\gamma_2$. Thus $\hat{\rho}^{-1}$ is an analytic continuation of $\rho^{-1}$ along the loop $\gamma_1 \cup \hat{\gamma} \cup (-\gamma_2)$ in $E$.

As $\{\rho_i\}_{i=1}^n$ has a group-like property under the composition defined above, there is an integer $k^-$ with $1 \leq k^- \leq q$ such that $G_{i_k}^-$ equals $G_{i_k^-}$. This completes the proof. \(\square\)

Lemma 7.5. For each integer $k$ with $1 \leq k \leq q$, there is an integer $k^-$ with $1 \leq k^- \leq q$ such that

$$E_{i_k}^* = E_{i_k^-}.$$

Proof. Polarizing the change of variable formula (7.2) gives

$$\int_{E \setminus \Gamma} f(\rho(z)) \overline{g(\rho(z))} |\rho'(z)|^2 dA(z) = \int_D f(w) g(w) dA(w)$$

for two polynomials $f, g$ of $z$. Choose a collection $\{U_\mu\}$ of disjoint open subsets such that

$$|E / [\Gamma \cup \left( \bigcup_\mu U_\mu \right)]| = 0,$$

for each $\mu$, the sets $\{\rho(U_\mu)\}_{\rho \in G_{i_k}}$ are disjoint and for each $\rho$ and $\mu$, $\rho$ is injective on $U_\mu$. Note that for each point $\rho(z) \notin \rho(\Gamma)$, by Lemma 7.4, there is a $\hat{\rho}_\mu \in G_{i_k^-}$ such that

$$\hat{\rho}_\mu(\rho(z)) = z$$

on a neighborhood $U_\mu$ of $z$. Let $w = \rho(z)$. Since $\rho$ is analytic and locally injective on $E$, the above equality gives that there is a $\tilde{\rho}_\mu \in G_{i_k}$ such that

$$\tilde{\rho}_\mu(\hat{\rho}_\mu(w)) = w$$

for $w \in \rho(U_\mu)$. Hence

$$\rho(z) = \tilde{\rho}_\mu(z)$$
for \( z \in U_\mu \) and

\[
\rho(\hat{\rho}_\mu(w)) = w
\]
on a neighborhood \( \rho(U_\mu) = \tilde{\rho}_\mu(U_\mu) \) of \( \rho(z_0) \). Let \( \chi_{\rho(U_\mu)} \) denote the characteristic function of the set \( \rho(U_\mu) \). Thus

\[
\langle \mathcal{E}_k^+ g, f \rangle = \langle g, \mathcal{E}_k f \rangle = \int_D \sum_{\rho \in G_{ik}} \sum_{\mu} \chi_{\rho(U_\mu)}(w) g(\hat{\rho}_\mu(w)) \hat{\rho}_\mu'(w) f(w) dA(w).
\]

The third equality follows from the fact that \( \hat{\Gamma} \) has area measure equal to zero and the fact that \( \sum_{\rho \in G_{ik}} f(\rho(z)) \rho'(z) \) is in the Bergman space \( L_a^2 \). The last equality follows from

\[
\hat{\rho}_\mu'(w) = \frac{1}{\rho'(\hat{\rho}_\mu(w))}.
\]

Let \( S_k \) be the connected component of the Riemann surface for \( \phi^{-1} \circ \phi \) over \( \mathbb{D} \) associated with \( G_{ik} \) and \( n_k \) the cardinality of \( G_{ik} \). Since \( S_k \) is an \( n_k \)-sheeted ramified covering of \( \mathbb{D} \), it can be pictured as \( n_k \) copies of the unit disk attached with appropriate branch points and lying over \( \mathbb{D} \). By Lemma 7.2, for each \( w \) in \( \mathbb{D} \), there are only \( n_k \) elements \( \mu_i \) such that

\[
w = \rho_{\mu_i}(z_{\mu_i})
\]
for each \( z_{\mu_i} \in U_{\mu_i}, i = 1, \ldots, n_k \). Thus, for almost all \( w \in \mathbb{D} \),

\[
\sum_{\rho \in G_{ik}} \sum_{\mu} \chi_{\rho(U_\mu)}(w) g(\hat{\rho}_\mu(w)) \hat{\rho}_\mu'(w) = \sum_{i=1}^{n_k} g(\hat{\rho}_{\mu_i}(w)) \hat{\rho}_{\mu_i}'(w).
\]
We claim that for each $w$, \( \hat{\rho}_{\mu_i}(w) \neq \hat{\rho}_{\mu_j}(w) \) if $i \neq j$. If this is not true, for some $i \neq j$,

\[
\hat{\rho}_{\mu_i}(w) = \hat{\rho}_{\mu_j}(w),
\]
as

\[
w = \rho_{\mu_i}(z_{\mu_i}) = \rho_{\mu_j}(z_{\mu_j}).
\]

Thus we have

\[
z_{\mu_i} = \hat{\rho}_{\mu_i}(\rho_{\mu_i}(z_{\mu_i}))
= \hat{\rho}_{\mu_i}(w)
= \hat{\rho}_{\mu_j}(w)
= \hat{\rho}_{\mu_j}(\rho_{\mu_j}(z_{\mu_j}))
= z_{\mu_j}.
\]

This implies that \( \rho_{\mu_i}(z_{\mu_i}) = \rho_{\mu_j}(z_{\mu_i}) \), which contradicts the fact that the intersection of \( \rho_{\mu_i}(U_{\mu_i}) \) and \( \rho_{\mu_j}(U_{\mu_i}) \) is empty. Thus the \( \rho_{\mu_j}(U_{\mu_i}) \) are disjoint.

Since \( G_{ik} \) has \( n_k \) elements and contains \( \{ \hat{\rho}_{\mu_i} \}_{i=1}^{n_k} \), we have

\[
\sum_{i=1}^{n_k} g(\hat{\rho}_{\mu_i}(w)) \hat{\rho}'_{\mu_i}(w) = \sum_{\rho \in G_{ik}} g(\rho(w)) \rho'(w)
\]
even if those \( \mu_i \) may depend on the point \( w \).

If \( g \) is a polynomial of \( z \), by the proof of Theorem 3.1, the right-hand side of the above equality is analytic in \( E \). Letting \( V \) be an open neighborhood of a branch point of \( \phi \), we have

\[
\int_{V \setminus \Gamma} |\rho'(z)|^2 dA(z) = |\rho(V \setminus \Gamma)| < \infty.
\]

This implies that \( \sum_{\rho \in G_{ik}} \rho'(z) g(\rho(z)) \) extends analytically on \( \mathbb{D} \) and is in the Bergman space \( L_a^2 \). Thus

\[
\langle \mathcal{E}_k^* g, f \rangle = \int_{\mathbb{D}} \sum_{\rho \in G_{ik}} \sum_{\mu} \chi_{\rho(U_{\mu})} g(\hat{\rho}_{\mu}(w)) \hat{\rho}'_{\mu}(w) \overline{f(w)} dA(w)
= \int_{\mathbb{D}} \left[ \sum_{\rho \in G_{ik}} g(\rho(w)) \rho'(w) \right] \overline{f(w)} dA(w)
= \langle \mathcal{E}_{k-} g, f \rangle.
\]

Since \( \sum_{\rho \in G_{ik}} g(\rho(w)) \rho'(w) \) is in the Bergman space and the polynomials are dense in the Bergman space \( L_a^2 \), we have that for any polynomial \( g \),
\[ \mathcal{E}_k^* g(z) = \sum_{\rho \in G_{ik}} \rho'(z) g(\rho(z)), \]

and hence

\[ \mathcal{E}_k^* g(z) = \mathcal{E}_k g(z) \]

for \( z \in \mathbb{D} \). By the fact that the polynomials are dense in the Bergman space \( L^2_a \), we have that \( \mathcal{E}_k^* = \mathcal{E}_k \). This completes the proof. \( \square \)

**Theorem 7.6.** Let \( \phi \) be a finite Blaschke product. The von Neumann algebra \( \mathcal{A}_\phi \) is generated by the linearly independent operators \( \mathcal{E}_1, \ldots, \mathcal{E}_q \) and hence has dimension \( q \).

**Proof.** Let \( q \) be the number of connected components of the Riemann surface \( \phi^{-1} \circ \phi \) over the unit disk. Recall that \( \mathcal{A}_\phi \) is the von Neumann algebra \( \{M\phi\}' \cap \{M_\phi^*\}' \).

To finish the proof we need show that \( \mathcal{A}_\phi \) is a finite dimensional space with dimension equal to \( q \).

By Lemma 5.1, for each unitary operator \( W \) in \( \mathcal{A}_\phi \), there are at most \( q \) distinct complex numbers \( \hat{r}_1, \ldots, \hat{r}_q \) such that for each \( \alpha \) in \( U \),

\[ W^* k_\alpha = \sum_{k=1}^{q} \hat{r}_k \sum_{\rho \in G_{ik}} \rho'(\alpha) k_\rho(\alpha) \]

\[ = \left[ \sum_{k=1}^{q} \hat{r}_k \mathcal{E}_k^* \right] k_\alpha, \]

where \( \hat{r}_k = r_\rho \) for some \( \rho \) in \( G_{ik} \). Since linear combinations of \( \{k_\alpha\}_{\alpha \in U} \) are dense in the Bergman space, we have

\[ W^* = \sum_{k=1}^{q} \hat{r}_k \mathcal{E}_k^*. \]

Thus \( \mathcal{A}_\phi \) contains at most \( q \) linearly independent unitary operators. By the Russo–Dye Theorem \([21,12,31]\), every element in \( \mathcal{A}_\phi \) can be written as a finite linear combination of unitary operators in \( \mathcal{U}_{\mathcal{A}_\phi} \). Thus \( \mathcal{A}_\phi \) is a finite dimensional space with dimension at most \( q \).

Next we show that the dimension of \( \mathcal{A}_\phi \) is at least \( q \). By (7.1), there are \( q \) bounded linear operators \( \mathcal{E}_1, \ldots, \mathcal{E}_q \) on the Bergman space. Since \( \phi(\rho(z)) = \phi(z) \), we have that

\[ M_\phi \mathcal{E}_k = \mathcal{E}_k M_\phi. \]

Thus the \( \mathcal{E}_1, \ldots, \mathcal{E}_q \) are contained in \( \{M_\phi\}' \). Now Lemma 7.5 tells us that the \( \mathcal{E}_1^*, \ldots, \mathcal{E}_q^* \) are also contained in \( \{M_\phi\}' \). This gives that the \( \mathcal{E}_1, \ldots, \mathcal{E}_q \) are contained in \( \mathcal{A}_\phi \). We claim that the \( \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_q \) are linearly independent. If this is false, there are constants \( c_1, \ldots, c_q \), not all of which are zero, such that

\[ c_1 \mathcal{E}_1 + \cdots + c_q \mathcal{E}_q = 0. \]
Thus for each $\alpha$ in $E$, we have
\[
[c_1E_1 + \cdots + c_qE_q]^*k_\alpha = 0.
\]
On the other hand, (7.1) gives
\[
[c_1E_1 + \cdots + c_qE_q]^*k_\alpha = \sum_{k=1}^{q} \tilde{c}_k \sum_{\rho \in G_k} \overline{\rho(\alpha)}k_\rho(\alpha).
\]
As in the proof of Theorem 2.2, for each $i$, define
\[
P_i(\alpha, z) = \prod_{j \neq i} (z - \rho_j(\alpha)).
\]
An easy calculation gives
\[
\langle P_i(\alpha, .), [c_1E_1 + \cdots + c_qE_q]^*k_\alpha \rangle = c_k \rho_k'(\alpha)P_{ik}(\alpha, \rho_k(\alpha)).
\]
Since $P_{ik}(\alpha, \rho_k(\alpha)) \neq 0$ and $\rho_k'(z)$ vanishes only on a countable subset of $\mathbb{D}$, we have that $c_k$ must be zero for each $k$. This is a contradiction. We conclude that the $E_1, E_2, \ldots, E_q$ are linearly independent to obtain that the dimension of $A_\phi$ is at least $q$. This completes the proof.

8. Abelian von Neumann algebra $A_\phi$

In this section, we will show that the von Neumann algebra $A_\phi$ is abelian if the order of the Blaschke product $\phi$ is smaller than or equal to 8. First we recall some concepts and notation from previous sections. For each $z \in E$, the function $\phi$ is one-to-one in some open neighborhood $D_{zi}$ of each point $z_i$ in $\phi^{-1} \circ \phi(z) = \{z_1, \ldots, z_n\}$. Let $\phi^{-1} \circ \phi = \{\rho_k(z)\}_{k=1}^{n}$ be $n$ solutions of the equation $\phi(\rho(z)) = \phi(z)$. Then $\rho_j(z)$ is locally analytic and arbitrarily continuable in $E$. Assume that $\rho_1(z) = z$. Every open subset $V$ of $E$ is invertible for $\phi$. Then $\{\rho_j\}_{j=1}^{n}$ is the family of admissible local inverses in some invertible open disc $V \subset \mathbb{D}$. For a given point $z_0 \in V$, label the local inverses as $\{\rho_i(z)\}_{i=1}^{n}$ on $V$. If there is a loop $\gamma$ in $E$ at $z_0$ such that $\rho_j$ and $\rho_j'$ in $\{\rho_i(z)\}_{i=1}^{n}$ are mutually analytically continuable along $\gamma$, we write
\[
\rho_j \sim \rho_j',
\]
and it is easy to check that $\sim$ is an equivalence relation. Let $\phi$ be a finite Blaschke product. Let $G$ be those local inverses of $\phi$ which extend analytically to only themselves in $\mathbb{D} \setminus F$. For a point $\alpha$ in the unit disk $\mathbb{D}$, the Möbius transform $\phi_\alpha(z)$ defined on $\mathbb{D}$ by
\[
\phi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}.
\]
Lemma 8.1. G is an elementary subgroup of $\text{Aut}(\mathbb{D})$ consisting of elliptic Möbius transforms and identity $\rho_1$. Moreover, there are a point $\alpha$ in $\mathbb{D}$, a unimodular constant $\lambda$ and an integer $n_G$ such that

$$G = \left\{ \frac{|\alpha|^2 - \lambda}{1 - |\alpha|^2 \lambda} \phi_{\alpha(1-\bar{\lambda})} (z) : \lambda^{n_G} = 1 \right\}.$$

Proof. Since $G$ consists of these local inverses $\rho$ which are equivalent only to themselves, each $\rho$ in $G$ has analytic continuation on $E$. Therefore each $\rho$ has an analytic continuation on the unit disk. Also $|\rho(z)| = 1$ on the unit circle and thus $\rho$ is an inner function. On the other hand, $\rho$ is locally injective. We conclude that each $\rho$ in $G$ is in $\text{Aut}(\mathbb{D})$. For two elements $\rho$ and $\tau$ in $G$, $\rho \circ \tau$ is still a local inverse in $G$, which shows that $G$ is a finite subgroup of $\text{Aut}(\mathbb{D})$. As $G$ is a finite group, letting $n_G$ be the number of elements in $G$, for each $\rho$ in $G$, we have

$$\rho^{n_G} = \rho_1(z).$$

Thus $\rho$ is elliptic and so $G$ is elementary.

According to ([4] on page 12), $G$ has an invariant point $\alpha$ in $\mathbb{D}$ and so it is a group of hyperbolic rotations about $\alpha$, that is

$$G = \left\{ \phi_{\alpha(1-\bar{\lambda})} (z) : \lambda^{n_G} = 1 \right\} = \left\{ \frac{|\alpha|^2 - \lambda}{1 - |\alpha|^2 \lambda} \phi_{\alpha(1-\bar{\lambda})} (z) : \lambda^{n_G} = 1 \right\}.$$

This completes the proof. $\square$

Remark. Clearly, the group $G$ in the above lemma is abelian.

Lemma 8.2. Suppose that $\lambda$ is the $k$th root of unity and $\alpha$ is a nonzero point in the unit disk. If $\beta_j = \frac{\alpha(1-\bar{\lambda}^j)}{1-|\alpha|^2 \bar{\lambda}^j}$, then $\{\beta_1, \beta_2, \ldots, \beta_k\}$ are distinct points in the unit disk.

Proof. Suppose that $\beta_j = \beta_l$. Then

$$\frac{\alpha(1-\bar{\lambda}^j)}{1 - |\alpha|^2 \bar{\lambda}^j} = \frac{\alpha(1-\bar{\lambda}^l)}{1 - |\alpha|^2 \bar{\lambda}^l}.$$

Since $\alpha$ does not equal zero, we have

$$\frac{(1-\bar{\lambda}^j)}{1 - |\alpha|^2 \bar{\lambda}^j} = \frac{(1-\bar{\lambda}^l)}{1 - |\alpha|^2 \bar{\lambda}^l},$$

which yields

$$\bar{\lambda}^l - \bar{\lambda}^j = |\alpha|^2 (\bar{\lambda}^l - \bar{\lambda}^j).$$

Since $\alpha$ is in the open unit disk, the above equality gives

$$\bar{\lambda}^l - \bar{\lambda}^j = 0.$$
If $1 \leq j, l \leq k$, then $j$ must equal $l$. This implies that $\{\beta_1, \beta_2, \ldots, \beta_k\}$ are distinct points. This completes the proof. \[\Box\]

Let $q$ denote the number of connected components of the Riemann surface $S_\phi$. Let $\phi$ be a Blaschke product. We say that $\phi$ is reducible if there are two other Blaschke products, $\phi_1$ and $\phi_2$, with orders larger than 1 such that $\phi = \phi_1 \circ \phi_2$.

**Lemma 8.3.** Let $\phi$ be a Blaschke product with order $n$. If $G$ has $n_G > 1$ elements, then $\phi$ is reducible and $n_G \mid n$.

**Proof.** Let $\lambda$ denote $\phi(0)$ and

$$
\psi = \phi_\lambda \circ \phi.
$$

Since for each $\rho \in G$, $\phi \circ \rho = \phi$, we have that $\psi \circ \rho = \psi$. Noting that $\psi(0) = 0$, we write

$$
\psi = z\psi_1.
$$

Since $\psi \circ \rho = \psi$, for each $\rho \in G$, $\rho$ is a factor of $\psi$. Letting $\phi_G = \prod_{\rho \in G} \rho(z)$, by Lemma 8.2, we have

$$
\psi = \left[ \prod_{\rho \in G} \rho(z) \right] \tilde{\psi}(z) = \phi_G \tilde{\psi}
$$

for a Blaschke product $\tilde{\psi}$ with order $n - n_G$, to yield

$$
\tilde{\psi} \circ \rho = \tilde{\psi}
$$

for each $\rho$ in $G$. Thus

$$
\phi = \phi_\lambda \circ \psi = \phi_\lambda \circ (\phi_G \tilde{\psi}).
$$

Repeating the above argument applied to $\tilde{\psi}$, noting that the order of $\tilde{\psi}$ is $n - n_G$, and using induction, we obtain

$$
\tilde{\psi} = \psi_2 \circ \phi_G
$$

for some Blaschke product $\psi_2$. This gives

$$
\phi = \phi_\lambda \circ (\phi_G \psi_2 \circ \phi_G) = \psi_3 \circ \phi_G,
$$

where $\psi_3 = \phi_\lambda \circ (z \psi_2)$. So $\phi$ is reducible and the order $n$ of $\phi$ equals the product of $n_G$ and the order of $\psi_3$. This completes the proof. \[\Box\]

**Corollary 8.4.** Let $\phi$ be a Blaschke product of order $n$. If $n$ is greater than or equal to 5, the number $q$ of connected components of the Riemann surface $\phi^{-1} \circ \phi$ over the unit disk does not equal $n - 1$. 
Proof. If \( q \) equals \( n - 1 \), then \( G \) has \( n - 2 \) elements. Lemma 8.2 gives that \((n - 2)|n\), which is impossible. \( \square \)

**Theorem 8.5.** Let \( \phi \) be a finite Blaschke product with order less than or equal to 8. Then \( A_\phi \) is commutative and hence, in these cases, the number of minimal reducing subspaces of \( M_\phi \) equals the number of connected components of the Riemann surface \( \phi^{-1} \circ \phi \) over the unit disk.

**Proof.** As shown in [19], the center of the algebra \( A_\phi \) contains a nontrivial projection \( P \). So if the dimension \( q \) of the algebra \( A_\phi \) is less than 5, by the classification theorem of finite dimensional von Neumann algebras [12], \( A_\phi \) is commutative. Let \( n \) be the order of the Blaschke product. So we may assume \( n \geq q \geq 5 \).

If \( n = 5 \) and \( q = 5 \), then the order of \( G \) equals 5 and \( G \) is a cyclic group. Thus \( \{E_j\}_{j=1}^n \) is commutative and hence \( A_\phi \) is commutative.

If \( n = 6 \) and \( q = 6 \), similarly we have that \( A_\phi \) is commutative. If \( q = 5 \), \( G \) has four elements and hence \( n_G \) equals 4. Lemma 8.3 implies that 4 would be a factor of 6, which is impossible.

If \( n = 7 \) and \( q = 7 \), as above, \( G \) is a cyclic group and so \( A_\phi \) is commutative. If \( q \) equals 6 or 5, then \( G \) contains either 5 or 4 or 3 elements. In any of these cases, Lemma 8.3 implies that 5, or 4 or 3 is a factor of 7. This is impossible.

If \( n = 8 \), we consider four cases \( 5 \leq q \leq 8 \).

**Case 1.** \( q = 8 \), then \( G \) is a cyclic group and so \( A_\phi \) is commutative.

**Case 2.** \( q = 7 \), then two of local inverses extend analytically each to the other in \( E \) and hence \( G \) contains 6 elements. Lemma 8.3 implies that 6 is a factor of 8, which is impossible.

**Case 3.** \( q = 6 \). In this case we will show that the center of \( A_\phi \) has dimension at least 4. So by the classification theorem of finite dimensional von Neumann algebras, \( A_\phi \) is commutative.

In this case, \( G \) contains 4 elements and is a cyclic group generated by an elliptic Möbius transform \( \rho \). Then the set of local inverses is divided into \( G_1 = \{\rho_1\} \), \( G_2 = \{\rho\} \), \( G_3 = \{\rho^2\} \), \( G_4 = \{\rho^3\} \), \( G_5 = \{\rho_5, \rho_6\} \), and \( G_6 = \{\rho_7, \rho_8\} \). Let \( E_i \) be the operator associated with \( G_i \) for \( i = 1, \ldots, 6 \). Then \( E_2 \) is a unitary operator and

\[
E_i = E_2^{i-1}
\]

for \( i = 1, \ldots, 4 \) and

\[
E_2^* = E_2^3.
\]

Moreover, \( E_2 \) commutes with \( E_i \) for \( i = 1, 2, 3, 4 \).

For two local inverses, \( \tau_1, \tau_2 \), if \( \tau_1 \) is equivalent to \( \tau_2 \), then \( \tau_1 \circ \rho \) is equivalent to \( \tau_2 \circ \rho \). We observe that both \( E_i E_2 \) and \( E_i^* \) are in \( \{E_5, E_6\} \). Thus there are permutations \( \tau \) and \( \sigma \) of \( \{5, 6\} \) such that

\[
E_i E_2 = E_\sigma(i),
\]

and

\[
E_i^* = E_\tau(i).
\]
Noting that $\sigma^2(i) = i$, we have

$$E_i E_3^3 = E_{\sigma(i)}.$$ 

Taking adjoint of both sides of the above equality gives

$$\left(E_2^*\right)^3 E_{\tau(i)} = E_{\tau(\sigma(i))}.$$ 

Since

$$\left(E_2^*\right)^3 = E_2$$ 

and $\tau$ commutes with $\sigma$, we have

$$E_2 E_i = E_{\sigma(i)}.$$ 

Thus for $i = 5, 6$,

$$E_2 E_i = E_i E_2.$$ 

This gives that $E_2$ is in the center of $A_\phi$, and so the dimension of the center is at least 4.

**Case 4.** $q = 5$. Lemma 8.3 gives that $G$ has 2 or 4 elements.

If $G$ has 4 elements, then the local inverses of $\phi$ is divided into 5 equivalent classes $G_1, \ldots, G_5$. We may assume that $G_i = \{\rho_i^{-1}(z)\}$ for $i \leq 4$, where the zero power $\rho_0(z)$ of $\rho(z)$ is $\rho_1(z)$ and $G_5 = \{\rho_5, \ldots, \rho_8\}$. Let $E_i$ be the operator associated with each $G_i$. Then $E_2$ is a unitary operator and

$$E_i = E_2^{i-1}$$ 

for $i \leq 4$ and $E_5$ is self-adjoint. Hence

$$E_i^* = E_2^{3(i-1)}.$$ 

For two local inverses $\tau_1, \tau_2$, if $\tau_1$ is equivalent to $\tau_2$, then $\tau_1 \circ \rho$ is equivalent to $\tau_2 \circ \rho$. Thus

$$E_5 E_i = E_5$$ 

for $i \leq 4$. So taking the adjoint of the above equality gives

$$E_i^* E_5 = E_5.$$ 

Therefore we have

$$E_i E_5 = E_5,$$ 

to obtain that $\{E_1, \ldots, E_5\}$ is in the center of $A_\phi$ and hence $A_\phi$ is commutative.

If $G$ has 2 elements, then the set of local inverses of $\phi$ is divided into $G_1 = \{\rho_1\}$, $G_2 = \{\rho_2\}$, $G_3 = \{\rho_3, \rho_4\}$, $G_4 = \{\rho_5, \rho_6\}$, and $G_5 = \{\rho_7, \rho_8\}$. Let $E_i$ be the operator associated with $G_i$ for
\( i \leq 5 \). We will show that \( E_2 \) is in the center of \( A_\phi \). If this is true, we obtain that the dimension of the center of \( A_\phi \) has dimension at least 2 and hence \( A_\phi \) is commutative.

Since \( E_i^* \) is in \( \{E_3, E_4, E_5\} \), we have that either every operator in \( \{E_3, E_4, E_5\} \) is self-adjoint or only one of them is self-adjoint. Now we consider two cases.

If every operator in \( \{E_3, E_4, E_5\} \) is self-adjoint, we observe that for two local inverses \( \tau_1, \tau_2 \), if \( \tau_1 \) is equivalent to \( \tau_2 \), then \( \tau_1 \circ \rho_2 \) is equivalent to \( \tau_2 \circ \rho_2 \) to get that there is a permutation \( \sigma \) of \( \{3, 4, 5\} \) such that

\[
E_i E_2 = E_{\sigma(i)}.
\]

Taking the adjoint of the above equality gives

\[
E_2 E_i = E_{\sigma(i)}.
\]

Thus \( E_2 \) is in the center of \( A_\phi \).

If only one of \( E_3, E_4, E_5 \) is self-adjoint, we may assume that \( E_3 \) is self-adjoint. For two local inverses \( \tau_1, \tau_2 \), if \( \tau_1 \) is equivalent to \( \tau_2 \), then \( \rho_2 \circ \tau_1 \circ \rho_2 \) is equivalent to \( \rho_2 \circ \tau_2 \circ \rho_2 \). Thus \( E_2 E_i E_2 \) is in \( \{E_3, E_4, E_5\} \) for \( 3 \leq i \leq 5 \). So there is a permutation \( \sigma \) of \( \{3, 4, 5\} \) such that

\[
E_2 E_i E_2 = E_{\sigma_2(i)},
\]

and

\[
E_2 E_3 E_2 = E_3.
\]

This implies that \( \sigma_2(3) = 3 \), and hence \( \sigma_2 \) is a permutation of \( \{4, 5\} \). Noting that for each \( 3 \leq i \leq 5 \),

\[
E_i^* \in \{E_3, E_4, E_5\},
\]

we see that there is the other permutation \( \sigma_1 \) of \( \{3, 4, 5\} \) with order 2 such that

\[
E_i^* = E_{\sigma_1(i)}
\]

and \( \sigma_1(3) = 3 \), which means that \( \sigma_1 \) is also a permutation of \( \{4, 5\} \). This also gives

\[
E_2 E_{\sigma_1(i)} E_2 = E_{\sigma_2(i)}^* = E_{\sigma_1(\sigma_2(i))}.
\]

Thus

\[
E_2 E_{\sigma_1(i)} E_2 = E_{\sigma_1(\sigma_2(\sigma_1^{-1}(i))))}.
\]

Since \( \sigma_1 \) and \( \sigma_2 \) are permutations of \( \{4, 5\} \), they commute with each other. Thus we have

\[
E_2 E_{i} E_2 = E_i,
\]

which implies

\[
E_2 E_{i} = E_{i} E_2,
\]

since \( E_2^2 = I \). This implies that \( E_2 \) is in the center of \( A_\phi \).
Now we will show that the number of minimal reducing subspaces of $M_\phi$ equals the number of connected components of the Riemann surface $\phi^{-1} \circ \phi$ over the unit disk. Recall that a projection $E$ in a $C^*$-algebra is called minimal if the projection $E$ does not equal 0 and the only subprojections of $E$ in the $C^*$-algebra are 0 and $E$. Then for every minimal reducing subspace $\mathcal{M}$ of $M_\phi$, we can associate a minimal projection $P_\mathcal{M}$ in the commutant $\{M_\phi\}'$ where $P_\mathcal{M}$ is the orthogonal projection from the Bergman space $L^2_a$ onto $\mathcal{M}$.

Let $\mathcal{P}$ be the set of all minimal projections in the commutant $\{M_\phi\}'$. $\mathcal{P}$ is the set of all minimal projections in the $C^*$-algebra $\{M_\phi\}' \cap \{M_\phi^*\}'$. Let $A$ be the $C^*$-algebra generated by those elements in $\mathcal{P}$. Hence $A$ equals $A_\phi$. As shown above, $A$ is a commutative $C^*$-algebra and its dimension equals $q$. Thus for any two minimal projections $P_1$ and $P_2$ in $\mathcal{P}$, $P_1$ commutes with $P_2$. Noting that both $P_1$ and $P_2$ are minimal, we have that $P_1 P_2 = 0$ to get that $P_1$ is orthogonal to $P_2$. Therefore $q$ equals the number of elements in $\mathcal{P}$. Let $\mathcal{M}_P$ denote the range of the projection $P$. Thus $\mathcal{M}_P$ is a minimal reducing subspace of $M_\phi$. So the number of minimal reducing subspaces of $M_\phi$ equals $q$. This completes the proof.

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