On the obstruction to extending a vector bundle from a submanifold

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Abstract

For the first obstruction to extending a holomorphic vector bundle from a submanifold of a complex manifold found by Griffiths we give an explicit formula in terms of the Atiyah class of the bundle.

1. Introduction

Let $Y$ be a complex manifold, $X \subset Y$ a compact submanifold, and $\iota : X \to Y$ the embedding. For a holomorphic vector bundle $E \to X$, is there a holomorphic bundle $F \to Y$ such that $E = \iota^* F$? This problem was considered by Griffiths who found a sequence of obstructions to such an extension [G, Proposition 1.1] denoted by $\omega(\alpha_\mu)$, $\mu \geq 0$. However, a proof of Proposition 1.1 is omitted and there is no actual construction of this invariants except for line bundles. In this note we give an explicit formula for the first obstruction.

Following [G] we consider the analytic space $X_1 = (X, \mathcal{O}_Y/\mathcal{I}^2)$ where $\mathcal{I} \subset \mathcal{O}_Y$ is the ideal sheaf of $X$, and interpret the embedding of $X$ into $Y$ as a composition of two morphisms $X \hookrightarrow X_1 \hookrightarrow Y$. If $E = \iota^* F$ then there is also an intermediate bundle $E_1 \to X_1$ such that $E = \iota^* E_1$ which is the natural first step in extending $E$ over $Y$. (We use the same symbol $\iota$ for both embeddings $X \to X_1$ and $X \to Y$. This should do no harm as its meaning must be obvious from the context.) The obstruction to such a bundle can be described as follows. There is a certain cohomology class $\kappa_{X/Y} \in H^1(X, \text{Hom}(\mathcal{N}_{X/Y}, \mathcal{T}_X))$ which we call a Kodaira-Spencer invariant (for reasons explained below). Denote

$$\text{At}(E) \cdot \kappa_{X/Y} = \text{tr}(\text{At}(E) \cup \kappa_{X/Y}, \mathcal{T}_X) \in H^2(X, \mathcal{N}_{X/Y}^* \otimes \text{End}(E))$$

where $\text{At}(E) \in H^1(X, \Omega^1_X \otimes \text{End}(E))$ is the Atiyah class of the bundle.

**Theorem 1** For a holomorphic vector bundle $E \to X$ there is a bundle $E_1 \to X_1$ such that $E = \iota^* E_1$ iff $\text{At}(E) \cdot \kappa_{X/Y} = 0$.

So, $\omega(\alpha_0)$ is actually a combination of the Atiyah class (which has nothing to do with the embedding) and the Kodaira-Spencer invariant (which has nothing to do with the bundle). Unfortunately, higher order obstructions probably cannot be separated in this manner into the embedding part and the bundle part.
2. Kodaira-Spencer invariant

For a sufficiently fine cover \(\{U_i\}\) of \(X\) we may choose holomorphic sections \(\Psi_i \in \Gamma(U_i, \text{Hom}(\mathcal{N}_{X/Y}, T_Y))\) such that for the natural map \(r : \text{Hom}(\mathcal{N}_{X/Y}, T_Y) \to \text{Hom}(\mathcal{N}_{X/Y}, \mathcal{N}_{X/Y})\) we have \(r \circ \Psi_i = 1\). The Kodaira-Spencer invariant \(\kappa_{X/Y} \in H^1(X, \text{Hom}(\mathcal{N}_{X/Y}, T_X))\) can be defined as the cohomology class of the cocycle

\[
\Psi_i - \Psi_j \in \Gamma(U_i \cap U_j, \text{Hom}(\mathcal{N}_{X/Y}, T_X)).
\]

It does not depend on the choice of \(\Psi_i\) so this is indeed an invariant of \(X\) as a submanifold.

If \(Y \to B\) is a deformation of \(X = X_0\) then \(\mathcal{N}_{X/Y}\) is a trivial bundle naturally isomorphic to the tangent space \(T_{B,0}\). In this case we may interpret the usual Kodaira-Spencer map \(H^0(X, \mathcal{N}_{X/Y}) \to H^1(X, T_X)\) as a cohomology class in \(H^1(X, \text{Hom}(\mathcal{N}_{X/Y}, T_X))\), and this class is equal to \(\kappa_{X/Y}\). A generalization of the Kodaira-Spencer construction to the case when \(\mathcal{N}_{X/Y}\) is not necessarily trivial seems straightforward enough but the author does not know if it actually appeared in the literature. However, this invariant does appear in \([G]\) in a different guise, it is the first obstruction to a map \(f : Y \to X\) such that \(f \circ \iota = \text{id}\) \([G, \text{Proposition 1.6}]\). In hindsight, it is not very surprising that this obstruction is related to extensions of bundles: if such a map exists then one obviously can take \(F = f^*E\).

3. Atiyah class

For sections of the bundle \(\text{End}(E)\) we use a notation which seems quite convenient but probably uncommon. We assume forth that a cover \(\{U_i\}\) of \(X\) is fixed together with trivializations of \(E\) over \(U_i\), so we have fixed transition functions \(g_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})\). Let \(U \subset X\) be an open set and \(m_{ij} : U \cap U\lambda \to \text{End}(\mathbb{C}^r)\) be some holomorphic matrices where \(r = \text{rank} \ E\). If matrices \(g_{ij}m_{ij}g_{ij}\) do not depend on \(\lambda\) (where they are defined) then we can interpret the collection \(\{m_{ij}\}\) as a section of \(\text{End}(E)\) and write \(m_{ij} \in \Gamma(U, \text{End}(E))\). (To make it a consistent notation we have to treat \(\lambda\) as a special index used for this purpose only.)

In this notation, the Atiyah class \(\text{At}(E)\) is represented \([\mathcal{A}|\mathcal{K}]\) by the cocycle

\[
g_{ij}d\lambda_{ij} - g_{ij}d\lambda_{ij} \in \Gamma(U_i \cap U_j, \Omega^1_X \otimes \text{End}(E)).
\]

(By the way, we cannot write \(g_{ij}d\lambda_{ij} \in \Gamma(U_i, \Omega^1_X \otimes \text{End}(E))\) because this term alone is not a well defined section of \(\Omega^1_X \otimes \text{End}(E)\).) If there is a bundle \(F \to Y\) such that \(E = \pi^*F\) then its Atiyah class can be introduced in the same way, but we are only interested in its restriction to \(X\) corresponding to the cocycle \(g_{ij}dG\lambda_{ij} - g_{ij}dG\lambda_{ij} \in \Gamma(U_i \cap U_j, \Omega^1_Y \otimes \text{End}(E))\) where \(G_{ij}\) are transition functions of \(F\) such that \(G_{ij} = g_{ij}\) on \(U_i \cap U_j\).

4. Vector bundles over \(X_1\)

A section of the structure sheaf of \(X_1\) may be interpreted as a pair \((f, \beta) \in \mathcal{O}_{X_1}(U)\) where \(f \in \mathcal{O}_X(U)\) and \(\beta \in \Omega^1_{Y|X}(U)\) is a form such that \(\pi^*\beta = df\). Consequently, transition functions of a vector bundle \(E_1 \to X_1\) are pairs \((g_{ij}, \Delta_{ij})\) where \(g_{ij}\) are the transition functions of \(\pi^*E_1\) and \(\Delta_{ij} \in \Gamma(U_i \cap U_j, \Omega^1_Y \otimes \text{End}(\mathbb{C}^r))\) are forms such that \(\pi^*\Delta_{ij} = dg_{ij}\) and \(\Delta_{ij}g_{ij} + g_{ij}\Delta_{jk}g_{ki} + g_{ik}\Delta_{ki} = 0\).

We say that two extensions \(E_1, E'_1 \to X_1\) of the bundle \(E \to X\) are equivalent if they are isomorphic as vector bundles over \(X_1\). (Griffiths used the same definition \([G, \text{I.4}]\). It is actually the finest equivalence in this context even if it apparently ignores the original bundle \(E\).) Assuming as before that \(g_{ij}\) are fixed, it is easy to see that two forms \(\Delta_{ij}\) and \(\Delta'_{ij}\) correspond to equivalent
Proof of the Theorem.

According to [G, Proposition 1.4], the obstructions to uniqueness of an extension of bundles if \( \Delta'_{ij} - \Delta_{ij} = B_ig_{ij} - g_{ij}B_j \) where \( B_i \in \Gamma(U_i, \Omega^1_{Y|X} \otimes \text{End}(C^r)) \) satisfy \( i^*B_i = 0 \).

The Atiyah class \( \text{At}(E_1) \in H^1(X, \Omega^1_{Y|X} \otimes \text{End}(E)) \) of such a bundle can be defined by the cocycle \( A_{ij} = g_{\lambda i}\Delta_{i\lambda} - g_{\lambda j}\Delta_{j\lambda} \) (for equivalent bundles we have \( A'_{ij} - A_{ij} = g_{\lambda i}B_ig_{i\lambda} - g_{\lambda j}B_jg_{j\lambda} \) which is, of course, a coboundary). Note that the equality \( \text{At}(E_1) = \text{At}(E'_1) \) gives another equivalence relation, in general more coarse than the natural one.

5. The proof

**Proposition 1** An extension \( E_1 \to X_1 \) of a holomorphic vector bundle \( E \to X \) exists iff \( \text{At}(E) \) is in the image of the map

\[
i^*: H^1(X, \Omega^1_{Y|X} \otimes \text{End}(E)) \to H^1(X, \Omega^1_X \otimes \text{End}(E)).
\]

**Proof.** If there is an extension then \( \text{At}(E) = i^* \text{At}(E_1) \) so this condition is necessary. Conversely, let \( D_{ij} \in \Gamma(U_i \cap U_j, \Omega^1_{Y|X} \otimes \text{End}(E)) \) be a cocycle representing a cohomology class with the image \( \text{At}(E) \). Then we have

\[
i^*D_{ij} = g_{\lambda i}dg_{i\lambda} - g_{\lambda j}dg_{j\lambda} + g_{\lambda i}C_{ij}g_{i\lambda} - g_{\lambda j}C_{ij}g_{j\lambda}
\]

for some \( C_i \in \Gamma(U_i, \Omega^1_X \otimes \text{End}(C^r)) \). The coboundary part can be lifted to \( \Omega^1_{Y|X} \) so we are free to assume that \( C_i = 0 \). Then \( (g_{ij}, \Delta_{ij}) \) with \( \Delta_{ij} = g_{\lambda i}D_{ij}g_{\lambda j} \) are transition functions of a vector bundle over \( X_1 \) with this Atiyah class. \( \square \)

**Proof of the Theorem.** From the exact sequence of sheaves

\[
0 \to N^*_X \to \Omega^1_{Y|X} \to \Omega^1_X \to 0
\]

we have a long exact sequence in cohomology

\[
H^1(X, \Omega^1_{Y|X} \otimes \text{End}(E)) \xrightarrow{i^*} H^1(X, \Omega^1_X \otimes \text{End}(E)) \xrightarrow{\alpha} H^2(X, N^*_X \otimes \text{End}(E)).
\]

Under close examination the coboundary map coincides with \( \alpha \mapsto \alpha \cdot \kappa_{X/Y} \), hence \( \text{At}(E) \cdot \kappa_{X/Y} = 0 \) iff \( \text{At}(E) \) belongs to the image of \( i^* \). \( \square \)

6. Obstructions to uniqueness

According to [Gr] Proposition 1.4], the obstructions to uniqueness of an extension of \( E \to X \) over \( X_1 \) are cohomology classes in \( H^1(X, N^*_X \otimes \text{End}(E)) \). (A proof of it is also omitted though.) It may be convenient to distinguish two different obstructions, the vector bundles \( E_1 \to X_1 \) extending the same bundle \( E \) can have different Atiyah classes \( \text{At}(E_1) \), and also there can be many bundles with the same Atiyah class among them.

We assume that the vector bundle \( E \to X \) under consideration has at least one extension and denote \( \mathcal{G}(E) = \{ E_1 \to X_1 \mid i^*E_1 = E \} \neq \emptyset \). There is a natural action of the group \( H^1(X, N^*_X \otimes \text{End}(E)) \) on this set which may be described as follows: for a given \( e \in \mathcal{G}(E) \) with transition functions \( (g_{ij}, \Delta_{ij}) \) and a given cohomology class \( \gamma \in H^1(X, N^*_X \otimes \text{End}(E)) \) we denote by \( e' = e + \gamma \in \mathcal{G}(E) \) a bundle with transition functions \( (g_{ij}, \Delta'_{ij}) \) defined by \( \Delta'_{ij} = \Delta_{ij} + g_{\lambda i}C_{ij}g_{\lambda j} \) where \( C_{ij} \in \Gamma(U_i \cap U_j, N^*_X \otimes \text{End}(E)) \) is a cocycle representing \( \gamma \).

**Proposition 2** The map

\[
H^1(X, N^*_X \otimes \text{End}(E)) \to \mathcal{G}(E), \gamma \mapsto e + \gamma
\]
is a bijection making the diagram

\[
\begin{array}{c}
\mathcal{S}(E) \xrightarrow{\text{At}} H^1(X, \Omega^1_X \otimes \text{End}(E)) \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
H^1(X, N^*_X/Y \otimes \text{End}(E)) \xrightarrow{+e} H^1(X, \Omega^1_Y \otimes \text{End}(E)) \\
- \text{At}(e)
\end{array}
\]

commutative.

A proof (using the same argument as in Proposition 1) is straightforward. As a consequence of Proposition 2 and the long exact sequence, bundles in \( \mathcal{S}(E) \) with the same Atiyah class come from global sections of the sheaf \( \Omega^1_X \otimes \text{End}(E) \).

References

A M. F. Atiyah: Complex analytic connections in fibre bundles, Trans. AMS 85 (1957), 181-207.

G P. A. Griffiths: The extension problem in Complex Analysis II; Embeddings with positive normal bundle, Am. J. Math. 88 (1966), 366-446.

K M. Kapranov: Rozansky-Witten invariants via Atiyah classes, Compositio Math. 115 (1999), 71-113.

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