Volatility Driven Market in a Generalised Lotka Voltera Formalism

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Pacs:89.65.Gh

Abstract

The Generalized Lotka Voltera (GLV) formalism has been introduced in order to explain the power law distributions in the individual wealth \( w_i(t) \) (Pareto law) and financial markets returns (fluctuations) \( r \) as a result of the auto-catalytic (multiplicative random) character of the individual capital dynamics.

As long as the multiplicative random factor \( \lambda \) is extracted from the same probability distribution for all the individuals, the exponent of the power laws turns out to be independent on the time variations of the average \( \langle \lambda \rangle \). This explains also the stability over the past century of experimentally measured Pareto exponent.

In contrast to the scaling properties of the single time ("unconditional") probability distributions, the (auto-)correlations between observables measured at different times are not correctly reproduced by the original GLV, if the variance \( \sigma^2 \) of \( \lambda \) is time independent. In the GLV formalism the volatility \( r^2 \) auto-correlations decay exponentially while the measurements in real markets indicate a power law with a very small exponent.

We show in the present paper that by making the variance of the individual wealth changes \( \sigma^2 \) a function of the market volatility \( \langle r^2 \rangle \), one correctly reproduces the market volatility long range correlations.

Moreover, we show that this non-trivial feedback loop between the market price volatility and the variance of the investors wealth leads to non-trivial patterns in the overall market trends. If the feedback is too strong, it may even endanger the market stability.

Key words: volatility auto-correlations, power laws, econophysics, Lotka Volterra, stochastic logistic, behavioral finance
1 Introduction

In the last decade, the Microscopic Representation techniques were used in a wide range of subjects[1]. In particular, Levy Levy and Solomon [2,3](LLS) have devised a model of the financial markets in terms of a large number of virtual investors characterized each by a current wealth, portfolio structure, probability expectations and risk taking preferences (for a review see [4]). Such models allowed to uncover and study:

- Market effects of arbitrarily inhomogenous and non-rational traders behavior.
- Returns stochastic properties: autocorrelations, volatility, trading volume
- Predation, competition and symbiosis among species.
- Heavy-tailed market returns distributions related to the ratio between the capital entering the market and the increase in market stock capitalization.

In order to understand what are the crucial factors governing this complex dynamics displayed by the microscopic representation of markets, one constructed more schematic models which, while discarding some of the realistic features of LLS, still conserve the crucial dynamical features of the market.

The GLV is such a model that embodies some stylized features of the LLS model in a more generic framework. Instead of following in detail the way the market price influences each investor population and individual $i$, it was assumed that this influence can be represented through multiplying their wealth $w_i(t)$ by stochastic multiplicative factors $\lambda_i(t)$. This is naturally suggested by the LLS model simulations in which the investments of the individuals (and consequently their returns) are fractions of their wealth (as implied by the constant relative risk aversion utility functions). This is also consistent with the recent measurements by [5] of the exponent of the power law distribution of the market order volumes.

The stochastic proportionality between personal returns and personal wealth is consistent with the real data that show that the (annual) individual income distribution is proportional to the individual wealth distribution [6]. We proposed [7–11] therefore a model including the above stochastic autocatalytic properties of the capital as well as the cooperative, diffusive and competitive/predatory interactions between the investors. The GLV described below is a straightforward stochastic generalization of the Lotka-Volterra system (and of the discrete logistic equation) well known previously in population biology and social sciences.

As explained below, it automatically leads to many of the well known experimental features of the real markets. However, some of the initial GLV simplifications were too drastic. In particular the assumption that the indi-
individual returns are extracted from a probability distribution with fixed variance lead to the result that the market volatility auto-correlations decays exponentially with time. This is in stark contradiction with the measured real market properties [12–16]. In the present paper we identify the microscopic dynamical features which are responsible for "volatility clustering" effect: the fact that the variance of the individual invested wealth changes is influenced by the global market price volatility.

We show that this leads also to the emergence of a feedback loop which may in certain conditions destabilize the market.

2 Background on the simple GLV model

More than a hundred years ago, Pareto [17] discovered that the number of individuals with wealth (or incomes) with a certain value \( w \) is proportional to \( w^{-\alpha} \).

It turns out that in the conditions in which the participants in the market do not have a systematic advantage one over the other (which is in fact expected in an efficient market), realistic market dynamics of the LLS and GLV types lead always to Pareto laws.

Let us define now in more detail the GLV framework: Consider a fixed constant number of investors (N) (for the extension to a variable number of agents, see [18]). At each time step, the wealth of each investor \((i)\) is \( w_i(t) \), and the total wealth is \( W(t) = \sum_i w_i(t) \).

The time evolution of the \( w_i(t) \) is simulated by the following procedure. At each time step an investor \( i \) is chosen randomly to undergo an event that changes its invested wealth \( w_i(t) \rightarrow w_i(t+1) \). The various components in the change per unit time \( dw_i/dt \) are:

- A deterministic component related to the global status of the economy. This term is proportional to the current wealth of the investor through an arbitrary coefficient that may depend on time and on all the \( w \)’s:

\[
m(w_1, ..., w_N, t)w_i(t)
\]  
(1)

One can imagine that the coefficient \( m(w_1, ..., w_N, t) \) aggregates information on economic growth rate, taxes, social benefits, interest rates etc. and it is therefore the same for everybody. This is equivalent to the efficient market hypothesis [23].

- A purely stochastic term which takes into account the specific circumstances of each agent. The change of his wealth are still proportional to its currently
invested wealth $\eta_i(t)w_i(t)$ but the coefficient $\eta_i$ is a random number taken from a normal distribution with mean $< \eta > = 0$ and variance $< \eta_i^2 > = D_i$.

- There is a social security mechanism, or some fixed relative income that ensures that the investors do not become arbitrarily poor. This term is taken of the form $a_i \sum_j b_j w_j$. The coefficients $b_j$ represent the relative contribution of the individual $j$ to the redistributed wealth (through taxes, donations, payments) while $a_i$ represent the relative amount that the individual $i$ receives from the redistribution (through salaries, services, exchanges, pensions, social security). Without loss of generality one may assume $\sum_j b_j = 1$. In the case in which all $b_i$’s are constant: $b_i = 1/N$, the sum $w(t) = \sum b_j w_j$ reduces to the average wealth $w(t) = W(t)/N$.

Consequently, in the continuum time limit the GLV dynamics is governed by the system of $N$ coupled non-linear differential stochastic equations (in the Ito sense) with time dependent coefficients:

$$ \frac{dw_i}{dt} = [m(w_1,...,w_N,t) + \eta_i(t)]w_i(t) + a_i \sum_j b_j w_j(t) $$

Such systems are notoriously difficult to solve or even characterize qualitatively. Yet in the present case, in the limit $N \to \infty$, (and for positive, not too unequal $a_i$ and respectively $b_i$) the probability distribution of relative wealths $w_i/w$ is completely under analytic control in spite of the fact that the global wealth is very non-stationary and can have arbitrary ups and downs (corresponding to booms and crashes/recession). In particular, with the notation,

$$ a = \sum_j b_j a_j(t) $$

the relative wealth:

$$ x_i(t) = w_i(t)/w(t) $$

has been shown [19–21] to converge even in nonstationary conditions to a probability distribution that is $m(w_1,... w_N, t)$-independent:

$$ P(x_i) \sim x_i^{-1 - 2a/D_i} e^{-2a_i/(D_i x_i)}. $$

Consequently, modulo important finite $N$ corrections [18] which are outside the present scope, the dynamics (2) insures in the range $x_i > x_{min} \equiv 1/(1 + D/a)$ a power law with

$$ \alpha = 1 + 2a/D. $$
In the real measurements $\alpha$ has been found to be roughly constant around 1.5 in the last 100 years in all the western economies [18,20].

This value $\alpha \sim 3/2$ has been related to the intrinsic human biological constraints through the formula $\alpha \sim L/(L-1)$ where $L$ is the average number of dependents on the average wealth [19].

In the simulations presented here, we will use a discrete version of Eq. 2 with a particular choice of the form of the random factor and of the parameters:

$$m(w_1, ..., w_N, t) = -a - \mu w(t), \quad D_i = D, \quad a_i = a, \quad b_i = 1/N:$$

$$w_i(t + 1) = w_i(t) \cdot e^{\eta(t)} + a\left(\frac{W(t)}{N} - w_i(t)\right) - \frac{\mu}{N}w_i(t)W(t)$$ (7)

This specific choice of $m(w, t)$ is the minimal form, which embodies the 2 main relevant economic facts:

• the term $-aw_i(t)$ represents proportional taxation, while
• the term $-\frac{\mu}{N}W(t)$ models limiting global factors such as inflation (when the total numeraire wealth in the system is rising, the real value of each investors wealth is decreasing proportionally.).

The simple Eq. 7 recreates many of the observed features of stock markets, mainly the observed distribution of the wealth, the power law in the distribution of market returns and the long term rise in the total wealth of the investors, it fails to reproduce the long term correlations in the volatility of the market. We will then show how to amend this problem and how this influences the market stability.

3 Volatility correlations in the simple GLV model

The main focus of the present paper is the market volatility. Let us therefore describe first the definition, measurement and properties of this quantity in the simple GLV model Eq. 7. We will then study in the next section the modified GLV model in which the volatility determines the variance ($D$) of the random factor $\eta$.

The market return at time $t$ is defined as:

$$r(t) = \ln\left(\frac{W(t)}{W(t-1)}\right)$$ (8)

The volatility is defined as the average of the square of the returns over a
certain time period (we take it as $N$ unit time steps):

$$V = < [\ln(W(t + 1)/W(t))]^2 >_N$$  (9)

The change in $W$ between each time step is small and one can replace eq 9 by:

$$V = < [\ln(\frac{W(t) + \Delta W}{W(t)})]^2 >= < [\ln(1 + \frac{\Delta W}{W(t)})]^2 >$$

$$\sim < [\frac{\Delta W}{W(t)}]^2 >= < [(w_i(t + 1) - w_i(t))/W(t)]^2 >=$$

$$= < (x_i(t) \ast (e^{\eta_i(t)} - 1) + a(\frac{1}{N} - x_i(t)) - \frac{\mu}{N}x_i(t)W(t))^2 >=$$

$$= < x_i^2 > (D + a^2) - \frac{a^2}{N^2} \sim < x_i^2 > D$$  (10)

One sees that $V$ depends only on the distribution of the relative wealth $x_i(t)$.

As mentioned above, in GLV the probability distribution of the individual relative wealth does not change even in the presence of significant variations in the total wealth. Therefore, the classical GLV, the volatility is also a stochastic variable with a static distribution. In particular, the volatility inherits the scaling properties of the relative wealth distribution. In fact it turns out that the volatility has a Levy distribution with an index of $1 + \frac{\alpha}{2}$ (Figure 1).

The experimentally observed long term correlation of the volatility is a power law i.e: $< \frac{V(t+\tau)V(t)}{<V(t)^2>}> \propto \tau^{-\delta}$, with an exponent close to $\delta \sim 0$. By contrast, the time auto-correlation of the volatility decays exponentially in the simple GLV model. In order to reproduce the experimentally observed property we add below an auto-catalytic dependence of $D$ on $V(t)$.

4 Dynamic volatility

The volatility represents the “nervousness” of the market. It also measures the fraction of money an investor can expect to win or loose by investing in the market during a certain time interval. Therefore it is natural to assume that the variance $D$ of the random factor $\eta(t)$ in Eq. (7) is in fact a function of the volatility $V(t)$:

$$D = g(V)$$  (11)
We will further assume that \( g(V) \) can be parameterized by a power of exponent \( n \):

\[
g(V) = c_1 V^n \tag{12}
\]

In order to close the feed-back loop one has to estimate the dependence of the market volatility \( V \) on the variance \( D \) of the fluctuations of the individual wealth using equation 10. One can measure \( \langle x^2 \rangle \) as a function of \( D \) (Figure 2). This dependence turns out to be linear in the range of values used in the model (0.03-0.1). Thus in the range of values used in our simulation, one obtains that:

\[
V(D) \approx c_2 D^2 \tag{13}
\]

In fact we measured \( V(D) \) for the simple GLV model and verified that it does fit this function (Figure 3).

We are now in the position to estimate the stability of the system as a function of \( n \). By using the Eqs 11 and 13 we obtain the iterative equation describing the dynamics of the volatility:

\[
V(t + 1) = c_2 D(t + 1)^2 = c_2 c_1 V(t)^{2n} \tag{14}
\]

The condition for a stationary dynamics ("fix point") is therefore:

\[
V = cV^{2n} \Rightarrow V_{fp} = c^{\frac{1}{1-2n}} \tag{15}
\]

where \( c = c_1 c_2 \)

One can now replace Eq 14 with a continuous dynamics:

\[
\dot{V}(t + 1) - V(t) = cV(t)(V(t)^{2n-1} - \frac{1}{c})
\]

\[
\Rightarrow \dot{V} = cV(t)(V(t)^{2n-1} - \frac{1}{c}) \tag{16}
\]

We can now now estimate the stability around the steady state:

\[
\Delta \dot{V} = c(V_{fp} + \Delta V)((V_{fp} + \Delta V)^{2n-1} - \frac{1}{c})
\]

\[
\Rightarrow \Delta \dot{V} = c(2n - 1)V_{fp}^{2n-1}\Delta V
\]

\[
\Rightarrow \Delta \dot{V} = (2n - 1)\Delta V \tag{17}
\]
The fix point Eq. 15 is therefore stable for $2n - 1 < 0$, and unstable for $2n - 1 > 0$. If $n$ is exactly $n = 1/2$ then equation 14 will have a marginal steady state at any value of $V$ if $c = 1$. It will diverge for any value of $V$ if $c > 1$, and it will converge if $c < 1$ to a low enough value of $V$ where the second order dynamics will take effect. When the fix point is unstable ($n > 1/2$) the dynamics will lead to zero volatility if we start below the fix point and to infinity if we start above the fix point.

In order to check this effect we simulated the system for various $n$ values: $n=1$ (figure 4), $n = 1/3$ (figure 7,8) and $n = 1/2$ (figure 5,6)

- When $n = 1$ (i.e $2n - 1 > 0$) The values of $V$ and $D$ both diverge. The divergence mechanism is not driven by the rise in the wealth of the investors, or by better investment. It is driven purely by the rise in the variance of the market. All the feedback interplay takes place between $x$ (the normalized value of the investors wealth) and the volatility $V(t)$. The total $W(t)$ does not play an active role in this feedback loop.
- When $n = 1/3$. The values of the volatility $V$ and of $D$ both stabilize. This will be the case that will be further analyzed.
- If $n = 1/2$, and we use $c > 1$ the volatility diverges, as expected, while if $c < 1$ the values of $D$ and $V$ converge to a very low value. We will further investigate the dynamics that leads to this steady state. Interestingly enough, this value at the border between divergence and stability seems to be favored by the actual market observations. There might exists an self-organization argument that explains this fact.

We assume the diverging case does not represent realistic situations though the implied regime might be likened to some of the large fluctuations experienced in the latest years by the Nasdaq index.

However, we did not consider here, the risk-adversity of the investors might be the factor which ultimately prevents the unlimited price increase together with the volatility divergence. We also neglected adverse market effects related with the largest investors bidding against themselves.

In the converging cases ($n = 1/3$ and $n = 1/2, c < 1$) long-term volatility correlations exist (figure 8 and 9). In order to measure the power of the correlations we made a best-fit estimation to the log of the correlations with the log of $\tau$ and got for this specific set of parameters an exponent of $\delta = 0.5$. Note that this exponent is sensitive to the values of $n$ and $c$. We plan now to study in a future publication the bounds on the parameters $n$ and $c$ that can be deduced from the experimental measurements of the volatility correlations.
5 Discussion

The GLV formalism explains elegantly basic features of wealth distribution and markets dynamics. One of the main flaws of this formalism was its failure to predict the long term correlation in the markets volatility. Long term correlations imply a long term market memory. Such a memory is not included in the GLV formalism, which is based only on the current state of the market. In order for the GLV formalism to include long term correlations memory must emerge from the dynamics of the market, instead of being imposed on it externally.

We proposed a simple mechanism that can explain both the long term correlations of the volatility, and the apparent divergence of some markets. We propose that the memory is due to a positive feedback between the volatility and the nervousness of the traders. The efficient market hypothesis [23] requires that there can be no direct link between the average gain of the traders and any of the measured properties of the market. Thus long term correlations cannot be due to a positive feedback on the gains. Thus the next natural candidate for this feedback is the standard deviation of the gains.

The efficiency of the market seems to be also in contradiction with the fast rise observed in some markets. We therefore propose here that this rise is not due to the average gain of each investor, but to the standard deviation of this gain. As a result, one explains the sustained rise in the prices as the result of the auto-catalytic feedback of the standard deviation of the traders gain on itself.

We have shown using a stability analysis that a long term rise in the markets value may take place for certain values of $n$ the exponent parametrizing the dependence of the nervousness of the traders on the volatility.

If the trade volume increases sharply as a function of the arise in the volatility (very nervous traders) this will lead to a divergent rise in the market prices.

If on the contrary, the traders are very calm (a weak dependence of the standard deviation $D$ of the individual returns on the market volatility $V$), the markets average returns will be stable, and the only trace of this dependence will be the long term power-law auto-corelations of the volatility.

In a previous paper [20] the constance of the individual relative wealth Pareto distribution during the last century was explained in terms of general sociological factors (size of families...) [19]. In the present paper we showed that the long term evolution of the market may be due to psychological factors. Even very weak autocatalytic effects of the volatility on the individuals’ expectations, can determine the long term evolution of the markets.
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Figure 1 - The volatility distributions. This distribution has a power law tail over at least three orders of magnitude. The distribution was measured for a constant value of $D$ (the variance of the investors gain), and after the average wealth has stabilized due to the non linear competition factor. The exponent of the volatility power distribution is determined by the exponent of the Pareto relative wealth distribution.

Figure 2 - The average of the square of the relative individual wealth $< x_i^2 >$ is found (in the relevant parameters range) to depend linearly on the variance in the investors random gains $D = < \eta_i^2 >$. Deviations from the linear fit occur:-

- at very low values of $D$, - at large values $D \sim 1$. For such values of $D$, most of the total wealth is in the hands of the wealthiest agent: $x_{max} = 1$ and all other agents have $x_i \sim 0$. Consequently $< x_i^2 >$ saturates as it approaches its maximal possible value 1. However these parameters ranges where linearity is violated, are outside the interest of the present paper.

Figure 3 - The square root of the volatility as a function of the variance in the gain ($D$), and a linear fit. The gain variance was varied and for each variance we measured the algebraic average of the volatility once the total wealth reached equilibrium. Note that the volatility itself has a very large variance, and these results represents the average over a very long time. Thus at small time scales the average volatility may be very different than its long term average.

Figure 4 - The average wealth and volatility, when the variance in the investors gain ($D$) is linearly dependent on the volatility. These represents a very “nervous” market in which the investors are very sensitive to variations in the stock values. The dynamics in this case diverge through the positive feedback loop between the gain variance and the volatility.

Figure 5 - The average wealth and volatility, when $D$ is proportional to the square root of the volatility. This represents a marginally steady state in which the investors sensitivity to the volatility is precisely inverse to the sensitivity of the volatility on the $D$. The average volatility is stable, however very large fluctuations around the average are observed.

Figure 6 - This simulation is similar to the one presented in figure 5, with a higher value of $c$. The high value of $c$ leads to divergent total wealth and volatility.

Figure 7 - The simulated volatility with the variance proportional to the cubic root of the volatility. In this case the volatility is stable but it has long term correlations. The two drawings represent the volatility fluctuations at different time scales.

Figure 8 - Long time ($\tau$) auto-correlations of the market volatility $< V(t + \tau)V(t) > / < V(t)^2 >$. The graph was obtained from runs in which the
square standard deviation of the individual returns $D$ was proportional to $V^{1/3}$. One observes a straight line fit on the double-logarithmic plot, indicative of a power-law auto-correlations decay $\tau^{-\delta}$. 
Fig. 1.

Fig. 2.

Fig. 3.
Divergence of volatility

Time [arbitrary units]

Volatility

Fig. 4.

Divergence of total wealth

Time [arbitrary units]

Fig. 5.

Log total wealth

Log Volatility

Time steps

Log total wealth

Time steps

Fig. 6.
Snapshot over a relatively short period of the volatility.

Long term measurement of volatility.

Volatility $V = 0.5D^{1/3}$

Long term correlation with a cubic root relation between $D$ and $V$.

Fig. 7.

Fig. 8.