Derived invariants from topological Hochschild homology

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Abstract

We consider derived invariants of varieties in positive characteristic arising from topological Hochschild homology. Using theory developed by Ekedahl and Illusie–Raynaud in their study of the slope spectral sequence, we examine the behavior under derived equivalences of various \( p \)-adic quantities related to Hodge–Witt and crystalline cohomology groups, including slope numbers, domino numbers, and Hodge–Witt numbers. As a consequence, we obtain restrictions on the Hodge numbers of derived equivalent varieties, partially extending results of Popa–Schell to positive characteristic.

Key Words. Derived equivalence, Hodge numbers, the de Rham–Witt complex, dominoes.

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1 Introduction

In this paper we study derived invariants of varieties in positive characteristic.

• Let \( X \) and \( Y \) be smooth proper \( k \)-schemes for some field \( k \). We say that \( X \) and \( Y \) are \textbf{Fourier–Mukai equivalent}, or \textbf{FM-equivalent}, if there is a complex \( P \in \mathcal{D}^b(X \times_k Y) \) such that the induced functor \( \Phi_P: \mathcal{D}^b(X) \to \mathcal{D}^b(Y) \) is an equivalence, where \( \mathcal{D}^b(–) \) denotes the dg category of bounded complexes of coherent sheaves. This is equivalent to asking for \( \mathcal{D}^b(X) \) and \( \mathcal{D}^b(Y) \) to be equivalent as \( k \)-linear dg categories. When \( X \) and \( Y \) are smooth and projective, they are FM equivalent if and only if there is a \( k \)-linear triangulated equivalence \( \mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \) by Orlov’s theorem (see [Huy06, Theorem 5.14]).

• Let \( h \) be a numerical or categorical invariant of smooth proper \( k \)-schemes. We say that \( h \) is a \textbf{derived invariant} if whenever there is a Fourier–Mukai equivalence \( \mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \) we have \( h(X) = h(Y) \).

The Hochschild homology groups \( \text{HH}_*(X/k) \) of a smooth proper variety \( X \), which are finite dimensional vector spaces over \( k \), are derived invariants. In characteristic 0, and in characteristic \( p \) for \( p \gg d = \dim X \), the Hochschild–Kostant–Rosenberg isomorphism [HKR62] relates the Hochschild homology groups to Hodge cohomology groups \( H^*(X, \Omega^*_X) \). We briefly review this story in Section 2. This relationship has been extensively studied, and plays a key role in our understanding of derived categories of varieties, especially over the complex numbers.

Suppose now that \( k \) is a perfect field of characteristic \( p > 0 \). To a smooth proper variety \( X \) over \( k \) we may associate via topological methods certain \( p \)-adic analogs of Hochschild homology: the topological Hochschild homology groups \( \text{THH}_*(X) \), as well as the related groups \( \text{TR}_*(X) \) and \( \text{TP}_*(X) \). We recall this theory in Section 3. These are modules over \( W = W(k) \) (the ring of Witt vectors of \( k \)) which are equipped with certain extra semilinear structures, and whose construction moreover depends only on \( \mathcal{D}^b(X) \). Furthermore, by a result of Hesselholt [Hes96], the \( \text{TR}_*(X) \) may be computed in terms of the Hodge–Witt cohomology groups...
H^*(W\Omega^*_X)$, similar to the classical relationship between Hochschild homology and Hodge cohomology. Our goal in this paper is to study these objects as derived invariants of the variety $X$.

Our key technical results are obtained in Section 4, where we analyze the spectral sequence connecting Hodge–Witt cohomology groups to $\text{TR}_d(X)$, and study the extent to which the extra structures of $F, V, d$ are preserved. We also recall the theory of coherent $R$-modules introduced by Illusie and Raynaud [IR83] in their study of the slope spectral sequence, and explain how our noncommutative structures fit into this framework.

For the remainder of the paper we study consequences of this theory. In Section 5, we observe the following fact, which extends results of Bragg–Lieblich [BL18]. Given a smooth proper $d$-dimensional $k$-scheme, following Artin–Mazur [AM77], we let $\Phi^d_X$ be the functor on augmented Artin local $k$-algebras defined by $\Phi^d_X(A) = \ker (H^d(X \times \text{Spec}k \text{ Spec} A, G_m) \to H^d(X, G_m))$.

**Proposition 1.1.** Let $X$ and $Y$ be Calabi–Yau $d$-folds over a perfect field $k$ of positive characteristic. If $X$ and $Y$ are $\text{FM}$-equivalent, then $\Phi^d_X \cong \Phi^d_Y$. In particular, the heights of $X$ and $Y$ are equal.

When $d = 2$, one calls $\Phi^2_X$ the formal Brauer group of $X$. Proposition 1.1 implies in particular that the height of a K3 surface is a derived invariant, which was already known. We then study derived invariants of surfaces in more detail. We find another proof that the Artin invariant is a derived invariant of supersingular K3 surfaces, and recover a result of Tirabassi on Enriques surfaces.

In Sections 5.2, 5.3, and 5.4, we introduce various numerical $p$-adic invariants. Specializing to the case of varieties of dimension $d \leq 3$, we prove the following (for definitions, see the body of the paper).

**Theorem 1.2.** Let $k$ be a perfect field of positive characteristic $p$, let $W = W(k)$ be the ring of $p$-typical Witt vectors over $k$, and let $K = W[p^{-1}]$ be the fraction field of $W$. The following are derived invariants of smooth proper varieties of dimension $d \leq 3$ over $k$:

1. the slopes of Frobenius with multiplicity acting on the rational Hodge–Witt cohomology groups $H^j(W\Omega^*_X) \otimes_W K$ and rational crystalline cohomology groups $H^j(X/K)$,
2. the domino numbers $T^{i,j}$,
3. the Hodge–Witt numbers $h^{i,j}_W$,
4. the Zeta function $\zeta(X)$ (if $X$ is defined over a finite field), and
5. the Betti numbers $b_i = \dim_K H^i(X/K)$.

Part (4) was previously proved by Honigs in [Hon18], whose methods also suffice to prove (1) and (5). The proofs of statements (2) and (3), however, crucially rely on the topological derived invariants.

In Section 5.5 we consider the question of whether the Hodge numbers $h^{i,j} = \dim_K H^j(X, \Omega^i_X)$ are derived invariants. For context, we note that a conjecture of Orlov [Orl05] states that the rational Chow motive $h_X$ of a variety $X$ is a derived invariant. In characteristic 0, the Hodge numbers are determined by the rational Chow motive, and so Orlov’s conjecture implies that the Hodge numbers are derived invariant. This consequence has been verified by Popa–Schnell [PS11] for varieties of dimension $d \leq 3$. However, as discussed in Section 5.5, their proof breaks in several ways in positive characteristic. In characteristic $p$, the Hodge numbers are related (in a somewhat subtle way) to Hodge–Witt cohomology groups. Using this relationship and Theorem 1.2, we prove the following.

**Theorem 1.3.** Suppose that $X$ and $Y$ are $\text{FM}$-equivalent smooth proper varieties of dimension $d$ over a field $k$ of positive characteristic.

1. If $d \leq 2$, then $h^{i,j}(X) = h^{i,j}(Y)$ for all $i, j$. 
(2) If \( d \leq 3 \), then \( \chi(\Omega_X^i) = \chi(\Omega_X^i) \) for all \( i \).

We remark that this result uses topological Hochschild homology constructions in a key way. Even in the case of surfaces, we do not know a direct proof using only Hochschild homology.

Under a mild additional assumption, we are able to strengthen this result for \( d = 3 \). We say that a smooth proper variety \( X \) over a perfect field \( k \) of positive characteristic is Mazur–Ogus if the Hodge–de Rham spectral sequence for \( X \) degenerates at \( E_1 \) and the crystalline cohomology groups of \( X \) are torsion-free. The class of Mazur–Ogus schemes includes smooth complete intersections, abelian varieties, and K3 surfaces.

**Theorem 1.4.** Suppose that \( X \) and \( Y \) are FM-equivalent smooth proper varieties of dimension \( d = 3 \) over a perfect field \( k \) of positive characteristic \( p \geq 3 \). If \( X \) is Mazur–Ogus, then so is \( Y \), and \( h^{i,j}(X) = h^{i,j}(Y) \) for all \( i, j \).

Finally, in Section 6, we compute \( \text{TR} \) and \( \text{TP} \) for twisted K3 surfaces and discuss how to recover the fine structure of the Mukai lattice from \( \text{TP} \).

**Conventions.** We will use many spectral sequences in this paper. They all converge for smooth and proper schemes or dg categories, so we will say nothing more about convergence in our discussion.

If \( X \) is a smooth proper scheme over \( k \), we will write \( H^*(X/W) \) for the crystalline cohomology groups of \( X \) relative to \( W = W(k) \) and \( H^*(W\Omega^*_X) = H^*(X, W\Omega^*_X) \) for the Hodge–Witt cohomology groups of \( X \), as defined in \([\text{Ill79}]\).

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## 2 Hochschild homology

Let \( k \) be a commutative ring. For any \( k \)-linear dg category \( \mathcal{C} \), the Hochschild homology of \( \mathcal{C} \) over \( k \) is an object \( \text{HH}(\mathcal{C}/k) \in D(k) \) which is equipped with an action of the circle \( S^1 \). The homology groups \( H_i(\text{HH}(\mathcal{C}/k)) = H_i(\mathcal{C}/k) \) are the Hochschild homology groups of \( \mathcal{C} \) over \( k \). For a scheme \( X/k \), we let \( \text{HH}(X/k) = \text{HH}(\text{Perf}(X)/k) \), where \( \text{Perf}(X) \) is the \( k \)-linear dg category of perfect complexes on \( X \). This is a noncommutative invariant of \( k \)-schemes meaning in particular that if \( \text{Perf}(X) \simeq \text{Perf}(Y) \), then \( \text{HH}(X/k) \simeq \text{HH}(Y/k) \) as complexes with \( S^1 \)-action. For some details on Hochschild homology and the constructions below from a classical perspective, see \([\text{Lod98}]\); for details on Hochschild homology from a modern perspective, see \([\text{BMS19}]\).

While the Hochschild homology of a smooth proper \( k \)-scheme \( X \) is a derived invariant, one often computes it via the following spectral sequence, which is not.

**Definition 2.1.** The Hochschild–Kostant–Rosenberg spectral sequence

\[
E_2^{s,t} = H^t(X, \Omega_X^s) \Rightarrow \text{HH}_{s-t}(X/k), \tag{1}
\]

is the descent, or local-to-global, spectral sequence for Hochschild homology.\(^1\) Here, \( \Omega_X^s = \Omega_{X/k}^s \) are the sheaves of de Rham forms relative to \( k \).

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\(^1\)With this indexing the differentials \( d_r \) have bidegree \((r-1, r)\); this convention has the advantage that the \( E_2 \) page of (1) agrees with the \( E_1 \) page of the Hodge–de Rham spectral sequence (4).
The HKR spectral sequence is known to degenerate for smooth schemes in characteristic zero, or more generally when \( \dim(X) \) is invertible in \( k \). It also degenerates in characteristic \( p \) when \( \dim(X) \leq p \) by [AV17]. In general, when \( \dim(X) > p \), the HKR spectral sequence does not degenerate; for examples, see [ABM]. If (1) degenerates, then there exist non-canonical isomorphisms

\[
\text{HH}^i(X/k) \cong \bigoplus_j \text{H}^{j-i}(X, \Omega^j_X)
\]

of \( k \)-vector spaces for each \( i \). The above discussion implies the following well known result.

**Theorem 2.2.** Let \( X \) and \( Y \) be FM-equivalent smooth proper schemes of dimension \( d \) over a field \( k \) of characteristic \( p \). If \( p = 0 \) or \( p \geq d \), there exist isomorphisms

\[
\bigoplus_j \text{H}^{j-i}(X, \Omega^j_X) \cong \bigoplus_j \text{H}^{j-i}(Y, \Omega^j_Y).
\]

In particular,

\[
\sum_j h^{j,j-i}(X) = \sum_j h^{j,j-i}(Y)
\]

for all \( i \).

From Hochschild homology, one constructs several other noncommutative invariants, namely the cyclic homology \( \text{HC}(C/k) = \text{HH}(C/k)_{hS^1} \) obtained using the \( S^1 \)-homotopy orbits, the negative cyclic homology \( \text{HC}^{-}(C/k) = \text{HH}(C/k)^{hS^1} \) obtained using the \( S^1 \)-homotopy fixed points, and the periodic cyclic homology \( \text{HP}(C/k) = \text{HH}(C/k)^{tS^1} \) obtained using the \( S^1 \)-Tate construction. See [Lod98] background. For a \( k \)-scheme \( X \), each of these theories is computed by two spectral sequences: a noncommutative spectral sequence and a de Rham spectral sequence. We review the theory for \( \text{HP}(X/k) \); the other cases are similar.

**Definition 2.3.** By definition of the Tate construction (see for example [NS18]), there is a Tate spectral sequence

\[
E_2^{s,t} = \widehat{\text{H}}^s(CP^\infty, \text{HH}_t(X/k)) \Rightarrow \text{HP}_{-s-t}(X/k) \quad (2)
\]

computing \( \text{HP}(X/k) \), with differentials \( d_r \) of bidegree \( (r, r-1) \), where \( \widehat{\text{H}}^s(CP^\infty, -) \) is a 2-periodic version of the cohomology of \( CP^\infty \). When computing \( \text{HP}_*(X/k) \) via a mixed complex as in [Lod98], this is the spectral sequence arising from the filtration by columns. This is often called the noncommutative Hodge–de Rham spectral sequence.

**Definition 2.4.** Let \( X \) be a smooth and proper \( k \)-scheme. There is a de Rham–HP spectral sequence

\[
E_2^{s,t} = H_{\text{dr}}^{s-t}(X/k) \Rightarrow \text{HP}_{-s-t}(X/k), \quad (3)
\]

with differentials \( d_r \) of bidegree \( (r, 1-r) \), where \( R\Gamma_{\text{dr}}(X/k) \) is the de Rham cohomology of \( X/k \) and \( H_{\text{dr}}^{s-t}(X/k) = H^{s-t}(R\Gamma_{\text{dr}}(X/k)) \). This spectral sequence was constructed in [BMS19] in the \( p \)-adically complete situation and in [Ant18] in general. In characteristic zero, it can easily be extracted from [TV11].

Finally, for a smooth proper \( k \)-scheme, we have the Hodge–de Rham spectral sequence

\[
E_1^{s,t} = H^s(X, \Omega^t_{X/k}) \Rightarrow H_{\text{dr}}^{s+t}(X/k) \quad (4)
\]

which has differentials \( d_r \) of bidegree \( (r, 1-r) \). We summarize our situation in Figure 1.
The Tate spectral sequence (2) itself, meaning the collection of pages and differentials, is a derived invariant. However, there is no reason for the de Rham–HP spectral sequence (3) to be derived invariant, although the objects it computes are derived invariants.

**Remark 2.5.** Playing these spectral sequences off of each other can be profitable. For example, if \( k \) is a field and if \( X/k \) is smooth and proper and if the HKR spectral sequence (1) degenerates (for example if \( \dim(X) \leq p \)), then the degeneration of the Tate spectral sequence (2) implies degeneration of the de Rham–HP spectral sequence (3) and the Hodge–de Rham spectral sequence (4). Similarly, if the HKR and Hodge–de Rham spectral sequences degenerate, then the Tate spectral sequence degenerates if and only if the de Rham–HP spectral sequence degenerates.

If \( k \) is a perfect field, the Tate spectral sequence (2) computing HP degenerates when \( \mathcal{E} \) is smooth and proper over \( k \), \( \mathrm{HH}_i(\mathcal{E}/k) = 0 \) for \( i \notin [-p, p] \), and \( \mathcal{E} \) lifts to \( W_2(k) \) by work of Kaledin [Kal08, Kal17] (see also Mathew’s paper [Mat17]). Using this fact, we prove a theorem which implies Hodge–de Rham degeneration in many cases.

**Theorem 2.6.** Let \( k \) be a perfect field of positive characteristic \( p \). Let \( X, Y \) be smooth proper schemes such that \( D^b(X) \simeq D^b(Y) \) and \( \dim(X) = \dim(Y) \leq p \). If \( X \) lifts to \( W_2(k) \), then the Hodge–de Rham spectral sequence degenerates for \( Y \).

**Proof.** The HKR spectral sequence (1) degenerates for both \( X \) and \( Y \) by [AV17]. Since \( X \) lifts to \( W_2(k) \), the Tate spectral sequence (2) degenerates for \( X \). This tells us the total dimension of \( \mathrm{HP}(X/k) \simeq \mathrm{HP}(Y/k) \) and hence implies that the Tate spectral sequence (2) degenerates for \( Y \) as well. The existence of the convergent de Rham–HP spectral sequence (3) now implies that the Hodge–de Rham spectral sequence (4) degenerates for \( Y \) by counting dimensions.

The theorem is some evidence for a positive answer to the following question.

**Question 2.7** (Lieblich). Let \( X \) and \( Y \) be FM-equivalent smooth proper varieties over a perfect field \( k \) of positive characteristic \( p \). If \( X \) lifts to characteristic 0 (or lifts to \( W_2(k) \), etc.), does \( Y \) also lift?

### 3 Topological Hochschild homology

In this section, we introduce the main tools of this paper, which are topological analogs of the invariants of the previous section. Here, topological means that one works relative to the sphere spectrum S, which is the initial commutative ring (spectrum) in homotopy theory. To any stable \( \infty \)-category or dg category \( \mathcal{E} \), one can associate a spectrum \( \mathrm{THH}(\mathcal{E}) = \mathrm{HH}(\mathcal{E}/S) \) with \( S^1 \)-action. This is again a noncommutative invariant and there are various analogs of the spectral sequences of the previous section. We will especially be interested in the topological periodic cyclic homology

\[
\mathrm{TP}(\mathcal{E}) = \mathrm{THH}(\mathcal{E})^{|S^1|}.
\]
Topological Hochschild homology THH(ℂ) is equipped with an even richer structure than simply an \(S^1\)-action: it is a cyclotomic spectrum, a notion introduced by Bökstedt–Hsiang–Madsen [BHM93] to study algebraic \(K\)-theory and recently recast by Nikolaus and Scholze in [NS18]. We use the following definition, which is a slight alteration of the main definition of [NS18].

**Definition 3.1.** A \(p\)-typical cyclotomic spectrum is a spectrum \(X\) with an \(S^1\)-action together with an \(S^1\)-equivariant map \(X \to X^{\text{nc}_p}\), called the cyclotomic Frobenius, where \(X^{\text{nc}_p}\) is equipped with the \(S^1\)-action coming from the isomorphism \(S^1 \cong S^1/C_p\). We let \(\text{CycSp}_p\) denote the stable \(\infty\)-category of \(p\)-typical cyclotomic spectra. If \(k\) is a commutative ring, then \(\text{THH}(k)\) is a commutative algebra object of \(\text{CycSp}_p\) and we let \(\text{CycSp}_{\text{THH}(k)}\) denote the stable \(\infty\)-category of \(\text{THH}(k)\)-modules in \(p\)-typical cyclotomic spectra.

The exact nature of a cyclotomic spectrum will not concern us much, except in the extraction of homotopy objects with respect to a natural \(t\)-structure on cyclotomic spectra studied in [AN18].

**Definition 3.2.** A \(p\)-typical Cartier module is an abelian group \(M\) equipped with endomorphisms \(F\) and \(V\) such that \(FV = p\) on \(M\). A Dieudonné module over a perfect field \(k\) is a \(W = W(k)\)-module \(M\) equipped with endomorphisms \(F\) and \(V\) satisfying \(FV = VF = p\) and which are compatible with the Witt vector Frobenius \(\sigma\) in the sense that

\[
F(\sigma(a)m) = \sigma(a)F(m) \quad \text{and} \quad V(\sigma(a)m) = aV(m)
\]

for \(a \in W\) and \(m \in M\). Note that we do not require that \(M\) is finitely generated or torsion free.

A Cartier or Dieudonné module \(M\) is derived \(V\)-complete if the natural map \(M \to R\lim n M/V^n\) is an equivalence, where \(M/V^n\) is the cofiber of \(V^n: M \to M\) in the derived category of abelian groups. Let \(\widehat{\text{Cart}}_p\) denote the abelian category of derived \(V\)-complete Cartier modules and let \(\widehat{\text{Dieu}}_k\) denote the abelian category of derived \(V\)-complete Dieudonné modules over \(k\).

The full subcategory of \(\text{CycSp}_p\) of \(p\)-typical cyclotomic spectra \(X\) such that \(\pi_iX = 0\) for \(i < 0\) defines the connective part of a \(t\)-structure on \(\text{CycSp}_p\) and similarly for \(\text{CycSp}_{\text{THH}(k)}\). The main theorem of [AN18] identifies the heart.

**Theorem 3.3 ([AN18]).** Let \(k\) be a perfect field of positive characteristic \(p\). There are equivalences of abelian categories \(\text{CycSp}_p^\otimes \simeq \widehat{\text{Cart}}_p\) and \(\text{CycSp}_{\text{THH}(k)}^\otimes \simeq \widehat{\text{Dieu}}_k\).

To a \(p\)-typical cyclotomic spectrum \(X\), we can associate a new spectrum \(\text{TR}(X)\) with \(S^1\)-action and with natural endomorphisms \(F\) and \(V\) making the homotopy groups of \(\text{TR}(X)\) into Cartier modules. The construction of \(\text{TR}(X)\) was introduced by Hesselholt in [Hes96]. It is proven in [AN18] that \(\pi_i\text{TR}(X) \cong \pi_i^{\text{nc}(X)}\) as Cartier modules under the equivalence of Theorem 3.3, where \(\pi_i^{\text{nc}}(X)\) denotes the \(i\)th homotopy object of \(X\) with respect to the \(t\)-structure on cyclotomic spectra. In the case of a scheme \(X\), we let \(\text{TR}(X) = \text{TR}(\text{THH}(X))\). If \(k\) is a perfect field of positive characteristic \(p\), then for any cyclotomic spectrum \(X \in \text{CycSp}_{\text{THH}(k)}\), the homotopy groups \(\pi_iX = \pi_i\text{TR}(X)\) are equipped with differentials

\[
\text{TR}_0(X) \xrightarrow{d} \text{TR}_{i+1}(X)
\]

coming from the \(S^1\)-action making \(\text{TR}_*(X)\) into an \(R\)-module, where \(R\) is the Cartier–Dieudonné–Raynaud ring; see Section 4.

**Example 3.4.** When \(k\) is a perfect field of positive characteristic \(p\), \(\text{THH}(k)\) is in the heart \(\text{CycSp}_{\text{THH}(k)}^\otimes\) and corresponds to the ring of Witt vectors \(W(k)\) with its Witt vector Frobenius and Vershiebung maps. In this language, the result is due to [AN18, Theorem 2], but the underlying computation that \(\pi_1\text{THH}(k) \cong W(k)\) is due to Hesselholt–Madsen [HM97, Theorem 5.5]. The fact should be compared to the fundamental computation of Bökstedt which says that \(\pi_1\text{THH}(k) \cong k[b]\) where \(|b| = 2\) (see [Bök85] for the case of \(k = \mathbb{F}_p\) and [HM97, Corollary 5.5] for the general case).
Let $A$ be a smooth commutative $k$-algebra where $k$ is a perfect field of positive characteristic $p$. The homotopy groups of $\text{THH}(A)$ were computed in [Hes96], where it is shown that $\pi_* \text{THH}(A) \cong \Omega_{A/k} \otimes_k k[b]$, where $|b| = 2$ and $\Omega_{A/k}^i$ lives in homological degree $i$. This is analogous to the HKR isomorphism for Hochschild homology, but more complicated thanks to Bökstedt's class $b$.

However, when working with $\text{TR}$ instead, Hesselholt proved in [Hes96] an exact de Rham–Witt analog of the HKR isomorphism: there is a graded isomorphism $\text{TR}^*(A) \cong W\Omega^*_{A/k}$ compatible with the $F$, $V$, and $d$ operations, where $W\Omega^*_{A/k}$ is the de Rham–Witt complex of $A$ as studied in [Ill79].

Now, we can define the topological or crystalline analogs of the four spectral sequences from Section 2.

**Definition 3.5.** Let $X$ be a smooth proper scheme over a perfect field $k$ of positive characteristic $p$ and let $\mathcal{C}$ be a smooth proper dg category over $k$.

(a) Using Hesselholt’s local calculation, the descent spectral sequence for $\text{TR}$ is

$$E_2^{s,t} = H^t(X, W\Omega_X^s) \Rightarrow \text{TR}_{s-t}(X).$$

With this indexing, the differentials $d_r$ have bidegree $(r-1, r)$. This is the topological analog of (1).

(b) There is a Tate spectral sequence

$$E_2^{s,t} = \hat{H}^s(\mathbb{C}P^\infty, \text{TR}_t(\mathbb{C})) \Rightarrow \text{TP}_{t-s}(\mathbb{C})$$

computing $\text{TP}(\mathbb{C})$, with differentials $d_r$ of bidegree $(r, r-1)$. This is the TP analog of (2). It was constructed in [AN18, Corollary 10]. See Figure 3. By analogy with (2), this spectral sequence could be called the “non-commutative slope spectral sequence”.

(c) There is a crystalline–TP spectral sequence

$$E_2^{s,t} = H^{s,t}_{\text{cryst}}(X/W) \Rightarrow \text{TP}_{s-t}(X),$$

with differentials $d_r$ of bidegree $(r, 1-r)$. This is the topological analog of (3) and is due to [BMS19].

(d) The Hodge–de Rham spectral sequence (4) is replaced with the slope spectral sequence

$$E_1^{s,t} = H^t(X, W\Omega_X^s) \Rightarrow H^{s+t}_{\text{cryst}}(X/W)$$

of [Ill79]; it has differentials $d_r$ of bidegree $(r, 1-r)$.

Figure 2 gives the topological analog of Figure 1.

Figure 2: Four spectral sequences associated to a smooth proper scheme $X$ over a perfect field $k$ of positive characteristic.

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2We will use $W\Omega^*_{A/k}$ to denote the graded abelian group underlying the complex $W\Omega^*_{A/k}$. 
Remark 3.6. As in the case of Hochschild homology, the Tate spectral sequence is a noncommutative invariant, but \textit{a priori} the other three spectral sequences are not.

Remark 3.7. The $W$-modules appearing in Figure 2 are equipped with various extra structures, and the interactions of these structures with the four spectral sequences will be critical in our analysis. Specifically,

1. The groups $H^*(W\Omega^*_X)$ and $TR_*(X)$ are Dieudonné modules, and the descent spectral sequence (5) takes place in the abelian category of derived $V$-complete Dieudonné modules. In particular, the differentials on each page commute with $F$ and $V$.

2. The groups $H^*(X/K) = H^*(X/W) \otimes_W K$ and $TP_*(X) \otimes_W K$ are $F$-isocrystals (see Definition 5.9), and up to certain Tate twists, the crystalline–TP spectral sequence (7) is compatible with these $F$-isocrystal structures.

3. The differentials in the slope (8) and Tate (6) spectral sequences do \textit{not} commute with $F$ and $V$. Rather, they satisfy the relations in Figure 4.

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\cdots & TR_{t+1}(\mathcal{E}) & 0 & TR_{t+1}(\mathcal{E}) & 0 & TR_{t+1}(\mathcal{E}) & \cdots \\
\cdots & TR_t(\mathcal{E}) & 0 & TR_t(\mathcal{E}) & 0 & TR_t(\mathcal{E}) & \cdots \\
\cdots & TR_{t-1}(\mathcal{E}) & 0 & TR_{t-1}(\mathcal{E}) & 0 & TR_{t-1}(\mathcal{E}) & \cdots \\
\vdots & \vdots & \vdots \\
\end{array}
\]

Figure 3: A portion of the $E_2$-page of the Tate spectral sequence (6) computing TP. The Tate spectral sequence is 2-periodic in the columns and for $\mathcal{E}$ smooth and proper over a perfect field of characteristic $p$ it is bounded in the rows by [AN18, Corollary 5].

We will need the following proposition.

Proposition 3.8. If $X$ is smooth and proper over a perfect field of positive characteristic $p$, then the four spectral sequences of Definition 3.5 degenerate rationally.

Proof. For the slope spectral sequence, this is due to Illusie–Raynaud [IR83]. For the crystalline–TP spectral sequence, it is proved by Elmanto in [Elm18] using an argument of Scholze. The other two cases follow by counting dimensions.

Remark 3.9. Suppose that $\mathcal{E}$ is a smooth proper dg category over a perfect field $k$ and suppose that $\mathcal{E}$ lifts to a smooth proper dg category $\mathcal{E}$ over $W$. Then, $TP(\mathcal{E})$ is equivalent to $HP(\mathcal{E}/W)$ by an unpublished argument of Scholze. In this way, one might view $TP$ as a noncommutative version of crystalline cohomology. See also forthcoming work of Petrov and Vologodsky.
4 Compatibility of the descent and slope spectral sequences

For the remainder of the paper, we use the following notation.

**Notation 4.1.** Unless otherwise stated, $k$ denotes a perfect field $k$ of characteristic $p > 0$. We write $W = W(k)$ for the ring of Witt vectors of $k$ and $K = W[p^{-1}]$ for the field of fractions of $W$. We let $\sigma$ denote the Frobenius on $W$.

Let $X$ be a smooth proper scheme over $k$. The Hodge–Witt cohomology groups $H^i(W\Omega^\bullet_X)$ and the homotopy groups $\text{TR}^i(X)$ are modules over $W$ and are equipped with $\sigma$-linear operators $F, V$, which satisfy the relations $FV = VF = p$. That is, they are Dieudonné modules in the sense of Definition 3.2. The differentials $H^i(W\Omega^\bullet_X) \to H^i(W\Omega^{i+1}_X)$ and $\text{TR}^i(X) \to \text{TR}^{i+1}(X)$ do not commute with $F$ and $V$, but instead satisfy the relations of Figure 4. Following Illusie–Raynaud [IR83], we formalize these properties using the Cartier–Dieudonné–Raynaud algebra relative to $k$, which is the graded ring

$$R = R^0 \oplus R^1$$

generated by $W$ and by operations $F$ and $V$ in degree 0 and by $d$ in degree 1, subject to the relations of Figure 4.

$$FV = VF = p$$
$$Fa = \sigma(a)F \text{ for all } a \in W$$
$$V\sigma(a) = aV \text{ for all } a \in W$$
$$da = ad \text{ for all } a \in W$$
$$d^2 = 0$$
$$FdV = d$$

Figure 4: The relations in the Raynaud ring.

Note that $VF = p = FV$ and $FdV = d$ imply the standard relations $Vd = pdV$ and $dF = pFd$. In particular, $R^0 = W_0[F, V]$ is the usual Raynaud algebra. Note also that a (left) module over $R^0$ is the same thing as a Dieudonné module in the sense of Definition 3.2. We will use the terms interchangeably. A graded left $R$-module is a complex $M^\bullet = [\cdots \to M^i \to M^{i+1} \to \cdots]$ of $R^0$-modules whose differential $d$ satisfies $FdV = d$. We will refer to such an object simply as an $R$-module.

**Remark 4.2.** A similar structure is studied by Bhatt–Lurie–Mathew in [BLM18]. They introduce the notion of a Dieudonné complex. A saturated Dieudonné complex in the sense of [BLM18, Definition 2.2.1] is naturally an $R$-module for the Raynaud algebra relative to $\mathbb{F}_p$ by [BLM18, Proposition 2.2.4].

If $M$ is an $R$-module, we denote by $M[n]$ the graded module with degree shifted by $n$ so that $M[n]^i = M^{i-n}$. Note that this notation is not consistent with that of [Eke86, Eke84], although we will use the sign conventions from op. cit.

For each $j$, the differentials in the de Rham–Witt complex make the complex

$$H^j(W\Omega^\bullet_X) \overset{\text{def}}{=} [0 \to H^j(W\Omega^0_X) \to H^j(W\Omega^1_X) \to H^j(W\Omega^2_X) \to \cdots]$$

(10)
into an $R$-module, where $H^i(W_{\Omega^s_X})$ is placed in degree $i$ (see [Ill79]). Similarly, by [AN18, Section 6.2], the $S^1$-action on $TR(X)$ gives rise to a complex

$$TR_*(X) \overset{\text{def}}{=} \{ \cdots \to TR_{t-1}(X) \overset{d}{\to} TR_t(X) \overset{d}{\to} TR_{t+1}(X) \overset{d}{\to} \cdots \}$$

which is an $R$-module, where $TR_t(X)$ is placed in degree $t$.

The $E_1$-page of the slope spectral sequence (8) is the same as the $E_2$-page of the descent spectral sequence (5), and is depicted in Figure 5.

---

Let $F^*TR(X) = R^\Gamma(X, \tau \geq \star TR(O_X))$ be the filtration giving rise to the descent spectral sequence. The differentials in the slope and descent spectral sequences are compatible in the following sense.

**Lemma 4.3.** Let $X$ be a smooth proper variety over a perfect field $k$ of positive characteristic. The de Rham–Witt differential $H^t(W_{\Omega^s_X}) \to H^t(W_{\Omega^{s+1}}_X)$ is compatible with the descent spectral sequence (5); specifically,

(i) there is a self-map $E_{s,t} \to E_{s+1,t}$ of degree $(1,0)$ of the descent spectral sequence which on the $E_2$-page is the de Rham–Witt differential;

(ii) for each $r \geq 2$, the resulting complexes

$$E_{r,t} = \{ \cdots \to E_{r-1,t} \overset{d}{\to} E_{r,t} \overset{d}{\to} E_{r+1,t} \overset{d}{\to} \cdots \}$$

are naturally $R$-modules;

---

3In [Eke84] this complex is denoted by $R^j\Gamma(X, W_{\Omega^s_X})$ (see page 190). We will avoid this notation due to its potential for confusion with the $j$-th hypercohomology of $W_{\Omega^s_X}$.
(iii) for each $t \in \mathbb{Z}$, the differential $d_r$ in the descent spectral sequence is a map of $R$-modules $E_r^{\bullet,t} \to E_r^{\bullet,t+r}[r-1]$.

(iv) the filtration on $\text{TR}_*(X)$ coming from the spectral sequence may be interpreted as a filtration by $R$-modules

$$0 = F^\bullet_{\dim(X)+1} \subseteq F^\bullet_{\dim(X)} \subseteq \cdots \subseteq F^\bullet_{\bullet} \subseteq F^\bullet_0 = \text{TR}_*(X),$$

where

$$F^\bullet_t = \text{im} \left( \pi_t F^{i+t} \text{TR}(X) \to \text{TR}_i(X) \right)$$

with isomorphisms

$$F^\bullet_t / F^\bullet_{t+1} \cong E^\bullet_{\infty}[t].$$

Note that for each $i$, the filtration on $\text{TR}_i(X)$ induced by the $F^\bullet_t$ is a shift of the usual filtration coming from the descent spectral sequence.

**Proof.** Consider the sheaf $\text{TR}(\mathcal{O}_X)$ of spectra with $S^1$-action on the Zariski site of $X$ and the filtration $F^* \text{TR}(\mathcal{O}_X) = \tau_{\geq *} \text{TR}(\mathcal{O}_X)$ arising from the Postnikov tower in sheaves of spectra with $S^1$-action. The graded piece $gr^i \text{TR}(\mathcal{O}_X)$ is equivalent to $W^{i}X[t]$. Taking global sections, we obtain a filtration $F^* \text{TR}(X) = F^* R\Gamma(X, \text{TR}(\mathcal{O}_X))$ with graded pieces $gr^i \text{TR}(X) = R\Gamma(X, W^iX)[t]$. The descent spectral sequence is by definition the spectral sequence of this filtration (with a commonly-used reindexing so that it begins with the $E_2$-page). Now, consider the $S^1$-action on $\text{TR}(\mathcal{O}_X)$. This induces a map $\text{TR}(\mathcal{O}_X) \to \text{TR}(\mathcal{O}_X)[-1]$ which automatically respects the filtration since it is just the canonical Postnikov filtration. In particular, this means that $\text{TR}(X) \to \text{TR}(X)[-1]$ induces a filtered map

$$F^* \text{TR}(X) \to F^{*+1} \text{TR}(X)[-1].$$

It follows that there is a map of spectral sequences associated to the two filtrations. But the spectral sequence coming from $F^{*+1} \text{TR}(X)[-1]$ is just a re-grading of spectral sequence coming from $F^* \text{TR}(X)$. In particular, we can view the differential as a self-map of the spectral sequence (5) of degree $(1,0)$. On the graded pieces of the filtrations, we get maps $R\Gamma(X, W^iX)[t] \to R\Gamma(X, W^{i+1}X)[t]$. Hesselholt checks in [Hes96, Theorem C] that this map is induced by the de Rham–Witt differential $W^iX \to W^{i+1}X$. In particular, on the $E_2$-page, we see the de Rham–Witt differential. This proves part (i).

We have already remarked that the spectral sequence (5) takes places in the abelian category of Dieudonné modules; in other words, the differentials in the descent spectral sequence commute with $F$ and $V$. But, by part (i), the differentials in the descent spectral sequence also commute with the de Rham–Witt differential $d$ in the sense that for all $t$ and $r \geq 2$ the diagram

$$\begin{array}{ccc}
E^r_{*,t} & \to & E^r_{*,t+1} \\
\downarrow & & \downarrow \\
E^{r+r-1,*+r} & \to & E^{r+r,*+r}
\end{array}$$

commutes. Now, parts (ii) and (iii) follow by induction where the base case is the $R$-module (10).

Finally, to prove part (iv), it is enough to note that the filtered map $d: F^* \text{TR}(X) \to F^{*+1} \text{TR}(X)[-1]$ implies that the inclusion $F^\bullet \subseteq \text{TR}_*$ is compatible with the differential. Since it is also compatible with $F$ and $V$ and since $\text{TR}_*(X)$ is an $R$-module, it follows that $F^\bullet$ is an $R$-submodule. This completes the proof. \qed
4. Compatibility of the descent and slope spectral sequences

To investigate the fine structure of the Hodge–Witt cohomology groups, Illusie and Raynaud [IR83] introduced a certain subcategory of the category of $R$-modules, which we briefly review. Forgetting the $F$ and $V$ operations, an $R$-module gives rise to a complex of $W(k)$-modules with cohomology groups $H^i(M)$. This complex comes equipped with a canonical decreasing $(V + dV)$-filtration given by $\text{Fil}^n M^i = V^n M^i + dV^n M^{i-1} \subseteq M^i$. We say that $M$ is complete if each $M^i$ is complete and separated for the $(V + dV)$-topology. We say that a complete $R$-module $M$ is profinite if $M^i/\text{Fil}^n M^i$ has finite length as a $W(k)$-module for each $i$ and $n$. An $R$-module $M$ is coherent if it is bounded (i.e., $M^i = 0$ for $|i|$ sufficiently large), profinite, and $H^i(M)$ is a finitely generated $W$-module for all $i$ (see [IR83, Théorème I.3.8] and [IR83, Définition I.3.9]).

Illusie and Raynaud showed in [IR83, Théorème II.2.2] that the Hodge–Witt complex $H^j(W\Omega^*_X)$ is coherent for each $j$.\(^4\) In the derived setting, we consider the $R$-module $TR_n(X)$ (11).

**Proposition 4.4.** If $X$ is a smooth proper $k$-scheme, then $TR_n(X)$ is a coherent $R$-module.

**Proof.** As recorded in [Eke84, page 191], the category of coherent $R$-modules is closed under kernels, cokernels, and extensions in the category of all $R$-modules. As remarked above, the Hodge–Witt complex $H^i(W\Omega^*_X)$ is coherent for each $j$. It follows from the compatibility of Lemma 4.3 that the $R$-modules $E^r_{i,j}$ appearing as the rows of the pages in the descent spectral sequence for $X$ are coherent for each $r \geq 2$. The descent spectral sequence degenerates at some finite stage for degree reasons, so the $E^\infty_{i,j}$ are also coherent. But $TR_n(X)$ admits a finite filtration by $R$-submodules whose successive quotients are isomorphic to the $E^\infty_{i,j}$ by Lemma 4.3. The category of coherent $R$-modules is closed under extensions, so it follows inductively that each piece of the filtration is coherent. In particular, $TR_n(X)$ is coherent, as desired. \(\square\)

Using results of Illusie-Raynaud and Ekedahl, we obtain the following consequences for the structure of the $R$-module $TR_n(X)$.

**Proposition 4.5.** If $X$ is a smooth proper $k$-scheme of dimension $d$ over a perfect field of positive characteristic, then for each $i \geq d - 2$,

1. $TR_i(X)$ is finitely generated over $W$, and
2. the differential $d : TR_{i-1}(X) \to TR_i(X)$ vanishes.

**Proof.** As explained in [Ill83, Section 3.1], the domino numbers $T^{i,j}(X)$ of $X$ are zero if $j \leq 1$ or $i \geq d - 1$, and therefore $H^i(W\Omega^*_X)$ is finitely generated over $W$ if $j \leq 1$ or $i = d$ (we review the definition of the domino numbers in Section 5.3). The descent spectral sequence then gives (1). Claim (2) then follows from Proposition 4.4 and Lemma 4.6. \(\square\)

**Lemma 4.6.** Let $M^*$ be a coherent $R$-module. For each $i$, the following are equivalent.

1. $M^i$ is finitely generated over $W$.
2. The differentials $M^{i-1} \to M^i$ and $M^i \to M^{i+1}$ vanish.

**Proof.** That (ii) implies (i) follows from the definition of coherence. For the converse, see [IR83, Corollaire II.3.8, II.3.9] for the case of the de Rham–Witt complex and [IR83, II.3.1(f)] for a general coherent $R$-module. \(\square\)

\(^4\)Note that this implies that while some $H^j(W\Omega^*_X)$ might be non-finitely generated as a $W(k)$-module, the terms appearing in the $E_2$-page of the slope spectral sequence (8) are all finitely generated $W(k)$-modules.
5 Derived invariants of varieties

Let $X$ be a smooth proper variety over $k$ of dimension $d$. By construction, the Dieudonné modules $\text{TR}_*(X)$ together with their differentials are derived invariants of $X$. Our goal in the following sections is to relate their structure to that of the crystalline and Hodge–Witt cohomology groups of $X$. Our main tool is the descent spectral sequence (5), which relates the $\text{TR}_*(X)$ to the Hodge–Witt cohomology groups $H^*(W\Omega^*_X)$ of $X$.

We begin with the observation that the descent spectral sequence induces natural isomorphisms

$$\text{TR}_{-d}(X) \cong H^d(W\Omega_X) \quad \text{and} \quad \text{TR}_d(X) \cong H^0(W\Omega^d_X).$$

It follows that the Dieudonné-modules $H^d(W\Omega_X)$ and $H^0(W\Omega^d_X)$ are derived invariants. We record a few easy consequences of this. Recall that if $X$ is Calabi–Yau, then $H^d(W\Omega_X)$ is the Dieudonné module associated to the Artin–Mazur formal group $\Phi^d(X, G_m)$ of $X$ (see [AM77]).

**Theorem 5.1.** If $X$ is Calabi–Yau, then the formal group $\Phi^d(X, G_m)$ is a derived invariant, as is the height of $X$.

In the case of K3 surfaces over an algebraically closed field, this is well known, being an easy consequence of the derived invariance of the rational Mukai crystal $H(X/K)$ introduced in section 2.2 of [LO15].

**Remark 5.2.** In fact, the formal group $\Phi^d(X, G_m)$ is also a twisted derived invariant. Indeed, if $\alpha \in \text{Br}(X)$, then $\text{TR}_{-d}(X, \alpha) \cong H^d(W\Omega_X)$ as well.

**Example 5.3.** Theorem 5.1 is especially interesting in light of Yobuko’s result [Yob19] which says that any finite-height Calabi–Yau variety lifts to $W_2(k)$. We see that if $X$ and $Y$ are derived equivalent Calabi–Yau varieties and if $X$ has finite height, then so does $Y$ and they both lift to $W_2(k)$. This gives some cases in which Question 2.7 has a positive answer.

The relationship between the remaining Hodge–Witt cohomology groups and the $\text{TR}_*(X)$ is subtle in general. There are several difficulties here: first, there may be non-trivial differentials in the descent spectral sequence, which need to be understood to relate the Hodge–Witt cohomology groups to the $E_\infty$-page. Second, the $E_\infty$-page only determines the graded pieces of the induced filtration on $\text{TR}_*(X)$. Unlike in the case of Hochschild homology, this is not enough to determine $\text{TR}_*(X)$ up to isomorphism, as there are non-trivial extensions in the category of $R$-modules. Finally, as in the case of Hochschild homology, to compute the $E_\infty$-page from $\text{TR}_*(X)$ one needs the additional data of the filtration on $\text{TR}_*(X)$, which is not a derived invariant.

In the remainder of this section we will attempt to overcome these obstacles in various situations. We begin in Section 5.1 by analyzing the case of surfaces in detail. Here, the situation is sufficiently restricted to allow us to (almost) prove that the entire first page of the slope spectral sequence is a derived invariant.

We then discuss the groups $\text{TR}_*(X)$ up to isogeny, that is, after inverting $p$. This is the classical theory of slopes. A major simplification occurs here, in that all of the relevant spectral sequences degenerate after inverting $p$ by Proposition 3.8. The resulting analysis follows the well established patterns involved in extracting derived invariants from Hochschild homology (see also Remark 5.18).

We then study the torsion in the Hodge–Witt cohomology, which contains information lost upon inverting $p$. We focus on the dominoes associated to differentials on the first page of the slope spectral sequence. These encode in an elegant way the infinitely generated $p$-torsion in the Hodge–Witt cohomology groups, which is present in many interesting (and geometrically well behaved) examples, such as certain K3 surfaces and abelian varieties. Here we introduce new non-classical derived invariants, defined using the differentials in the Tate spectral sequence (6). We then discuss Hodge–Witt numbers, which combine the information from the slopes of isocrystals and the domino numbers.
We finally apply these results to study the derived invariance of Hodge numbers in positive characteristic. We note that in order to obtain information on the Hodge numbers from Hodge–Witt cohomology, it is necessary to remember the infinite $p$-torsion. The information from the slopes of the isocrystals, while perhaps more elementary, does not suffice.

We are able to obtain the strongest results for surfaces and threefolds. For future use we will record a visualization of the information contained in the slope and descent spectral sequences in these cases. We record the following lemma.

**Lemma 5.4.** Let $X$ be a smooth proper $k$-scheme of dimension $d$.

1. If $d \leq 2$, then the descent spectral sequence for $X$ degenerates at $E_2$.

2. If $d = 3$, then the only possibly nonzero differentials on the $E_2$ page of the descent spectral sequence for $X$ are as pictured in Figure 7.

**Proof.** Note that, in general, the descent spectral sequence degenerates at $E_{d+1}$ for degree reasons. This gives the result for $d = 1$. If $d = 2$ the only possible differentials in the $E_2$-page are $H^0(W\Omega_X^1) \to H^2(W\Omega_X^1)$ and $H^0(W\Omega_X^1) \to H^2(W\Omega_X^2)$. By Proposition 3.8, the differentials vanish after inverting $p$. Since $H^2(W\Omega_X^2)$ is torsion-free, the latter differential is zero. The former is zero by functoriality and the fact that $H^0(W\Omega_X^1) \cong \text{TR}_0(k)$. By the same reasoning, when $d = 3$ we conclude that the differentials $H^0(W\Omega_X^1) \to H^2(W\Omega_X^1)$ and $H^1(W\Omega_X^1) \to H^0(W\Omega_X^2)$ vanish.

Suppose now that $X$ is a smooth proper surface over $k$. The first page of the slope spectral sequence for $X$ is given in Figure 6.

![Figure 6: The $E_1$-page of the slope spectral sequence and the $E_2$-page of the descent spectral sequence for TR of a smooth proper surface $X$ over $k$. The horizontal arrow is the only possibly non-zero differential on the first page of the slope spectral sequence. All differentials on all pages of the descent spectral sequence vanish (see Lemma 5.4). The red box (the small $1 \times 1$ box in the upper left) indicates the source of the only possibly non-zero domino, the blue box (the $1 \times 2$ box) indicates the possible locations of nilpotent torsion, and the black box (the stair-step shaped box) indicates the possible locations of semi-simple torsion (see Remark 5.23).](image-url)

If $X$ is a smooth proper threefold over $k$, then the first page of the slope spectral sequence is given in Figure 7.
5.1 Derived invariants of surfaces

Suppose that $X$ is a surface. By Proposition \ref{prop:derived_invariants}, the $R$-module $\text{TR} \bullet(X)$ has only one possibly nonzero differential, and so looks like

$$\text{TR} -2(X) \xrightarrow{d} \text{TR} -1(X) \xrightarrow{0} \text{TR} 0(X) \xrightarrow{0} \text{TR} 1(X) \xrightarrow{0} \text{TR} 2(X).$$

Furthermore, $\text{TR} i(X)$ is finitely generated for $i \geq 0$. By Lemma \ref{lem:descent_spectral_sequence}, the descent spectral sequence degenerates at $E_2$, so by Lemma \ref{lem:filtration} we have a filtration

$$0 = F^3 \subset F^2 \subset F^1 \subset F^0 = \text{TR} \bullet(X)$$

by coherent sub-$R$-modules such that $F^i/F^{i+1} \cong \text{H}^i(\text{W}^\bullet \Omega^\bullet X)[i]$. This filtration yields short exact sequences

$$0 \rightarrow F^2 \rightarrow \text{TR} \bullet(X) \rightarrow \frac{\text{TR} \bullet}{F^2} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \frac{F^1}{F^2} \rightarrow \frac{\text{TR} \bullet(X)}{F^2} \rightarrow \frac{\text{TR} \bullet(X)}{F^1(X)} \rightarrow 0$$
and hence we have commuting diagrams

$$
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
H^2(W\mathcal{O}_X) & d & H^2(W\Omega^1_X) & 0 \to H^2(W\Omega^2_X) \\
\downarrow & & & \\
TR_{-2}(X) & d & TR_{-1}(X) & 0 \to TR_0(X) \to 0 \to TR_1(X) \to 0 \to TR_2(X) \\
\downarrow & & & \\
H^1(W\mathcal{O}_X) & 0 \to (F^0_s/F^1_s)_0 & 0 \to TR_1(X) & 0 \to TR_2(X) \\
\downarrow & & & \\
0 & 0 & & 
\end{array}
$$

(12)

and

$$
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
H^1(W\mathcal{O}_X) & 0 \to H^1(W\Omega^1_X) & 0 \to H^1(W\Omega^2_X) \\
\downarrow & & & \\
H^1(W\mathcal{O}_X) & 0 \to (F^0_s/F^1_s)_0 & 0 \to TR_1(X) & 0 \to TR_2(X) \\
\downarrow & & & \\
H^0(W\mathcal{O}_X) & 0 \to H^0(W\mathcal{O}_X) & 0 \to H^0(W\Omega^1_X) \\
\downarrow & & & \\
0 & 0 & & 
\end{array}
$$

(13)

where the columns are exact sequences of \( R \)-modules and the rows are \( R \)-modules.

**Theorem 5.5.** Let \( X \) and \( Y \) be smooth proper surfaces over \( k \). If \( X \) and \( Y \) are FM-equivalent, then for

\[ (i, j) \in \{(0, 0), (1, 0), (2, 0), (1, 1), (2, 1), (0, 2), (2, 2)\} \]

there exists an isomorphism

\[ H^j(W\Omega^i_X) \cong H^j(W\Omega^i_Y) \]

of Dieudonné-modules. Furthermore, there exist isomorphisms

\[ H^1(W\mathcal{O}_X) \otimes K \cong H^1(W\mathcal{O}_Y) \otimes K \]
\[ H^2(W\mathcal{O}_X) \otimes K \cong H^2(W\mathcal{O}_Y) \otimes K \]
\[ H^2(W\Omega^1_X) \otimes [p^\infty] \cong H^2(W\Omega^1_Y) \otimes [p^\infty] \]

of \( R^0 \)-modules and a commutative diagram

$$
\begin{array}{ccc}
H^2(W\mathcal{O}_X) & d & H^2(W\Omega^1_X)[p^\infty] \\
\downarrow & & \downarrow \\
H^2(W\mathcal{O}_Y) & d & H^2(W\Omega^1_Y)[p^\infty].
\end{array}
$$
Proof. By diagram (13), we have an isomorphism \( TR_2(X) \cong H^0(W\Omega_X^2) \), and by diagram (12), we have an isomorphism \( H^2(W\Omega_X) \cong TR_{-2}(X) \). This produces the desired isomorphisms for \((i, j) = (2, 0), (0, 2)\). By Corollary 5.13, the isogeny class of \( H^0(W\Omega_X^1) \) is a derived invariant. But \( H^0(W\Omega_X^1) \) is torsion free of slope zero, so in fact \( H^0(W\Omega_X^1) \cong H^0(W\Omega_Y^1) \). We consider the short exact sequence

\[
0 \to H^1(W\Omega_X^2) \to TR_1(X) \to H^0(W\Omega_X^1) \to 0
\]

whose terms are finitely generated \( R^0 \)-modules, and similarly for \( Y \). As \( H^0(W\Omega_X^1) \) is torsion free of slope zero, this sequence splits, and we conclude that \( H^1(W\Omega_X^2) \cong H^1(W\Omega_Y^2) \). We conclude the result for \((i, j) = (0, 0), (1, 1), (2, 2)\) from the derived invariance of \( TR_0(X) \). Consider next the short exact sequence

\[
0 \to H^2(W\Omega_X^1) \to TR_{-1}(X) \to H^1(W\Omega_X) \to 0
\]

coming from diagram (12). As \( H^1(W\Omega_X) \) is torsion free, the map \( H^2(W\Omega_X^1) \to TR_{-1}(X) \) induces an isomorphism on torsion.

Remark 5.6. Theorem 5.5 almost shows that the entire first page of the slope spectral sequence is derived invariant.

As a corollary, we recover the following result.

Corollary 5.7 ([Bra18, Corollary 3.4.3]). Suppose that \( X \) and \( Y \) are FM-equivalent K3 surfaces over \( k \). If \( X \) is supersingular, then so is \( Y \), and \( \sigma_0(X) = \sigma_0(Y) \).

Proof. The image of the differential

\[
d : H^2(W\Omega_X) \to H^2(W\Omega_X^1)
\]

is \( p \)-torsion. Moreover, the Artin invariant of \( X \) is equal to the dimension of the \( k \)-vector space \( \ker d \). This follows for instance from the descriptions given in [Ill79, Section II.7.2]. The result follows from Theorem 5.5.

We also find another proof of the following result of Tirabassi [Tir18] on derived equivalences of Enriques surfaces. Recall from [BM76] that an Enriques surface \( X \) in characteristic 2 is either classical, singular, or supersingular, depending on whether the Picard scheme \( \text{Pic}_X/k \) is \( \mathbb{Z}/2, \mu_2, \text{or } \alpha_2 \), respectively. We call this the type of the Enriques surface.

Corollary 5.8 (Tirabassi). If \( X \) is an Enriques surface over an algebraically closed field of characteristic 2, then the type of \( X \) is a derived invariant.

Proof. The \( E_1 \) pages of the slope spectral sequences of Enriques surfaces in characteristic 2 are recorded in [Ill79, Proposition II.7.3.6]. In particular, we see by Theorem 5.5 that in this case the first page of the slope spectral sequence is a derived invariant, and this is more than enough to recover the type of \( X \).

5.2 Slopes and isogeny invariants

In this section we investigate derived invariants arising from topological Hochschild homology after inverting \( p \).

Definition 5.9. An \( F \)-isocrystal is a finite dimensional \( K \)-vector space \( V \) equipped with a \( \sigma \)-linear map \( \Phi : V \to V \). A morphism of \( F \)-isocrystals is a map of vector spaces commuting with the respective semilinear maps.
By fundamental results of Dieudonné and Manin, it is known that when \(k\) is algebraically closed, the category of \(F\)-isocrystals is abelian semisimple, and its simple objects are in bijection with rational numbers \(\lambda \in \mathbb{Q}\) (see for example [Dem72]). The slopes of an \(F\)-isocrystal \((M, \Phi)\) are the collection (with multiplicities) of the rational numbers appearing in the decomposition of \(M \otimes_K K^{un}\) into simple objects. Given a subset \(S \subset \mathbb{Q}\), we write \(M_S\) for the sub-isocrystal of \(M\) whose slopes are those in the subset \(S\).

For a smooth proper \(k\)-scheme \(X\), we let \(\Gamma\) denote the crystalline cohomology of \(X\) over \(W\) and we let \(\Gamma(X/K) = \Gamma(X/W) \otimes_W K\). Each rational crystalline cohomology group \(H^i(X/K)\) of \(X\) comes with an endomorphism \(\Phi\) induced by the absolute Frobenius of \(X\), and the pair \((H^i(X/K), \Phi)\) is an \(F\)-isocrystal. Given a rational number \(\lambda\), we define the slope number of the \(i\)-th crystalline cohomology of \(X\) by

\[
h_{\text{cryst,}\lambda}^i(X) = \dim_K H^i(X/K)_{[\lambda]}.
\]

There are two facts which allow us to get some control on the slope numbers. The first is that Poincaré duality implies the existence of a perfect pairing

\[
H^i(X/K) \otimes_K H^{2d-i}(X/K) \rightarrow H^{2d}(X/K) \cong K(-d)
\]

of isocrystals, where \(K(-d)\) is the 1-dimensional isocrystal of slope \(d\). Thus, for each \(\lambda \in \mathbb{Q}\) we have a perfect pairing

\[
H^i(X/K)_{[\lambda]} \otimes_K H^{2d-i}(X/K)_{[d-\lambda]} \rightarrow K(-d).
\]

This implies

\[
h_{\text{cryst,}\lambda}^i = h_{\text{cryst,}d-\lambda}^{2d-i}.
\]

Suppose that \(X\) is projective. The hard Lefschetz theorem in crystalline cohomology (see [KM74]) implies that if \(u = c_1(L) \in H^2(X/K)\) is the rational crystalline Chern class of an ample line bundle, then cupping with powers of \(u\) gives isomorphisms

\[
u^i \colon H^{d-i}(X/K) \cong H^{d+i}(X/K).
\]

Since \(u\) generates a 1-dimensional subspace of \(H^2(X/K)\) which is closed under Frobenius and has pure slope 1, it follows that \(u^i\) induces isomorphisms

\[
u^i \colon H^{d-i}(X/K)_{[\lambda]} \cong H^{d+i}(X/K)_{[i+\lambda]}(i).
\]

This implies

\[
h_{\text{cryst,}\lambda}^{d-i} = h_{\text{cryst,}d+i}^{d+i},
\]

or equivalently

\[
h_{\text{cryst,}\lambda}^i = h_{\text{cryst,}i-\lambda}^{d+i}.
\]

In fact, by [Suh12, Corollary 2.2.4] the relation (16) still holds only under the assumption that \(X\) is smooth and proper.

The Frobenius endomorphism on rational crystalline cohomology comes from an endomorphism of complexes \(\Gamma(X/W)\). In particular, there is a Frobenius-fixed \(W\)-lattice \(H^i(X/W)\)/tors inside \(H^i(X/K)\). This implies that the slopes \(\lambda\) appearing in crystalline cohomology are all non-negative. Equation (15) implies that the slopes are additionally bounded above by \(d\) and finally (16) implies that the slopes of \(H^i(X/K)\) are bounded above by \(i\).

The Hodge–Witt cohomology groups \(H^i(W\Omega^i_X)\) also come with a \(\sigma\)-linear operator, denoted by \(F\). Again, the pair \((H^i(W\Omega^i_X) \otimes K, F)\) is an \(F\)-isocrystal, and given a rational number \(\lambda \geq 0\) we write

\[
h_{\text{dR,}\lambda}^{ij}(X) = \dim_K H^j(X, W\Omega^i_X)_{[\lambda]}.
\]
By [III79, Corollaire II.3.5], we have a canonical isomorphism
\[ (H^{j-i}(W\Omega^i_X), p^i F) \cong H^j(X/K)_{(i, i+1)} \] (18)
of $F$-isocrystals. In particular, $h^{i,j}_{\text{dR}, \lambda}$ is non-zero only if $\lambda \in [0, 1)$ in which case
\[ h^{i,j}_{\text{dR}, \lambda} = h^{i+j}_{\text{crys}, i+\lambda}. \]

It follows from (16) that
\[ h^{i,j}_{\text{dR}, \lambda} = h^{i+j}_{\text{crys}, i+\lambda} = h^{d-i-j}_{\text{crys}, d-j+\lambda} = h^{d-j,d-i}_{\text{dR}, \lambda}, \] (19)
which we will use below. This last equality is a crystalline analog of Hodge symmetry, which we do not have access to in de Rham cohomology.

We now discuss these invariants under FM-equivalence. We record the following result, which is entirely analogous to Theorem 2.2 (note however that we need no restrictions on $p$).

**Theorem 5.10.** If $X$ and $Y$ are FM-equivalent smooth proper $k$-schemes, then there are isomorphisms
\[ \bigoplus_j H^{j-i}(W\Omega^j_X) \otimes W K \cong \bigoplus_j H^{j-i}(W\Omega^j_Y) \otimes W K \]
of $F$-isocrystals. In particular, we have
\[ \sum_j h^{i,j-1}_{\text{dR}, \lambda}(X) = \sum_j h^{i,j-1}_{\text{dR}, \lambda}(Y) \]
for each $i$ and $\lambda$.

**Proof.** By Proposition 3.8, the descent spectral sequence (5) degenerates after tensoring with $K$. We therefore obtain a (non-canonical) decomposition
\[ \text{TR}_i(X) \otimes W K \cong \bigoplus_j H^{j-i}(W\Omega^j_X) \otimes W K \] (20)
of $F$-isocrystals, and similarly for $Y$. The $F$-crystals TR$_i$ are derived invariants, so we get the claimed isomorphism, and the equality of slope numbers follows. \qed

We now define slope numbers for TR by letting
\[ h^{\text{TR}}_{n, \lambda} = \dim_K(\text{TR}_n(X) \otimes W K)_{[\lambda]}. \]
The proof of Theorem 5.10 expressed the fact that
\[ h^{\text{TR}}_{n, \lambda} = \sum_{i-j=n} h^{i,j}_{\text{dR}, \lambda}. \]

We recall the following result of Popa–Schnell (extended to positive characteristic in Theorem A.1 of [Hon18] by Achter, Casalaina-Martin, Honigs, and Vial).

**Theorem 5.11.** If $X$ and $Y$ are FM-equivalent smooth proper varieties over an arbitrary field $k$, then $(\text{Pic}^0_X)_{\text{red}}$ is isogenous to $(\text{Pic}^0_Y)_{\text{red}}$. 
5.2 Slopes and isogeny invariants

Remark 5.12. The theorem is stated in [Hon18] only for smooth projective varieties. But, the only place projectivity is used in the proof is to guarantee the existence of a FM-equivalence, which we assume to exist.

Equivalently, the isogeny class of the Albanese variety is a derived invariant. This has the following immediate consequence.

Corollary 5.13. If $X$ and $Y$ are FM-equivalent smooth proper varieties over our perfect field $k$ of positive characteristic, then the $F$-crystals $H^1(X/W)$ and $H^1(Y/W)$ are isogenous and there are equalities $h^1_{\text{crys},\lambda}(X) = h^1_{\text{crys},\lambda}(Y)$ of slope numbers for all $\lambda \in [0,1]$.

Remark 5.14. In any characteristic, the tangent space to $\text{Pic}^0_X$ at the origin is naturally identified with $H^1(X, \mathcal{O}_X)$. If the characteristic of $k$ is zero, then $\text{Pic}^0_X$ is automatically reduced, so Theorem 5.10 implies that the Hodge number $h^{0,1}$ is a derived invariant in characteristic 0. Similarly, in characteristic 0, the Hodge number $h^{1,0}$ is determined by the dimension of the Albanese of $X$, and hence is also a derived invariant.

In positive characteristic, the isogeny class of the Albanese does not in general determine the Hodge numbers $h^{0,1}$ and $h^{1,0}$, and therefore Theorem 5.10 does not imply the invariance of $h^{0,1}$ or $h^{1,0}$ under derived equivalence.

In dimension $\leq 3$, combining Theorem 5.10 and Corollary 5.13 with the constraints (15) and (16) give us complete control of the isocrystals $H^i(X/K)$ and $H^i(W\Omega^\vee_X) \otimes W K$.

Theorem 5.15. Suppose that $X$ and $Y$ are FM-equivalent smooth proper $k$-schemes of dimension $\leq 3$. For each $i,j$, there exist isomorphisms

$$H^i(W\Omega^\vee_X) \otimes W K \cong H^i(W\Omega^\vee_Y) \otimes W K \quad \text{and} \quad H^i(X/K) \cong H^i(Y/K)$$

of $F$-isocrystals. In particular, we have

$$h^{i,j}_{\text{dR},\lambda}(X) = h^{i,j}_{\text{dR},\lambda}(Y) \quad \text{and} \quad h^i_{\text{crys},\lambda}(X) = h^i_{\text{crys},\lambda}(Y)$$

for all $i,j,\lambda$.

Proof. We prove that the derived invariance of the $F$-isocrystal $H^1(X/K)$ given by Corollary 5.13 and the derived invariance of TR is enough to get the derived invariance of each $h^{i,j}_{\text{dR},\lambda}$. This is enough to prove the result for Hodge–Witt cohomology and the statement for crystalline cohomology follows from the degeneration of the slope spectral sequence. We fix $\lambda \in [0,1)$. We know to begin that $h^{i,j}_{\text{dR},\lambda}$ is a derived invariant for

$$(i,j) \in \{(0,0), (3,3), (0,1), (1,0), (3,2), (2,3), (0,3), (3,0)\}$$

from the derived invariance of $H^0(X/K)$, $H^6(X/K)$, $H^1(X/K)$, $H^5(X/K)$ (by Poincaré duality), TR$\ldots_3(X)$, and TR$\ldots_3(X)$, respectively. Now,

$$h^1_{2,\Lambda} = h^{3,0}_{\text{dR},\Lambda} + h^{3,1}_{\text{dR},\Lambda},$$
$$h^1_{1,\Lambda} = h^{2,0}_{\text{dR},\Lambda} + h^{2,1}_{\text{dR},\Lambda} + h^{1,0}_{\text{dR},\Lambda},$$
$$h^0_{0,\Lambda} = h^{3,1}_{\text{dR},\Lambda} + h^{2,2}_{\text{dR},\Lambda} + h^{1,1}_{\text{dR},\Lambda} + h^{0,0}_{\text{dR},\Lambda},$$
$$h^1_{-1,\Lambda} = h^{2,0}_{\text{dR},\Lambda} + h^{1,2}_{\text{dR},\Lambda} + h^{0,1}_{\text{dR},\Lambda},$$
$$h^3_{-2,\Lambda} = h^{1,3}_{\text{dR},\Lambda} + h^{0,2}_{\text{dR},\Lambda}.$$
But,

\[
\begin{align*}
\hat{h}^{3,1}_{\text{dR,} \lambda} &= \hat{h}^{2,0}_{\text{dR,} \lambda}, \\
\hat{h}^{2,2}_{\text{dR,} \lambda} &= \hat{h}^{1,1}_{\text{dR,} \lambda}; \\
\hat{h}^{1,3}_{\text{dR,} \lambda} &= \hat{h}^{0,2}_{\text{dR,} \lambda}
\end{align*}
\]

by (19). When combining this with the derived invariance of the Hodge–Witt slope numbers already established above and the derived invariance of the TR slope numbers, we conclude that each de Rham–Witt slope number is a derived invariant, as desired.

We obtain another proof of the following result of Honigs [Hon18].

**Corollary 5.16.** Suppose that \(X\) and \(Y\) are smooth proper schemes over a finite field \(\mathbb{F}_q\) of dimension \(\leq 3\). If \(X\) and \(Y\) are FM-equivalent, then \(\zeta(X) = \zeta(Y)\).

**Proof.** The eigenvalues of Frobenius acting on the \(\ell\)-adic cohomology of \(X_{\mathbb{F}_q}\) are determined by the slopes of the crystalline cohomology of \(X_{\mathbb{F}_q}\). These slopes are derived invariant by Theorem 5.15.

Recall that the Betti numbers of \(X\) are defined to be \(b_n(X) = \dim_k H^n_{\text{dR}}(X/k)\) for \(X\) smooth and proper over \(k\). In general, we have only an inequality \(b_n(X) \leq \dim_k H^n_{\text{dR}}(X/k)\), with equality if and only if \(H^n(X/W)\) and \(H^{n+1}(X/W)\) are torsion-free.

**Corollary 5.17.** If \(X\) and \(Y\) are FM-equivalent smooth proper \(k\)-schemes of dimension \(\leq 3\), then \(b_n(X) = b_n(Y)\) for each \(n\).

**Remark 5.18.** Given a Fourier–Mukai equivalence \(\Phi_P : \mathcal{D}^{b}(X) \to \mathcal{D}^{b}(Y)\), the crystalline Mukai vector \(v(P)\) of \(P\) gives rise to a correspondence \(H^*(X/K) \to H^*(Y/K)\). The usual formalism shows that this correspondence is an isomorphism of \(K\)-vector spaces, and the results of Section 5.2 can be proven by an analysis of the Künneth components of \(v(P)\). Thus, our use of topological constructions, while more intrinsic, is not strictly necessary to access the information contained in the Hodge–Witt cohomology groups up to isogeny. However, the topological constructions appear to be necessary in order to control the torsion in the Hodge–Witt cohomology groups. As mentioned above, remembering this information is crucial in order to access the Hodge numbers.

### 5.3 Domino numbers

To control the infinitely generated \(p\)-torsion in the Hodge–Witt cohomology groups, Illusie and Raynaud introduce in [IR83] certain structures called dominoes and domino numbers. We review their definition and prove the domino numbers are derived invariants in low dimensions. If \(M\) is an \(R\)-module, we set

\[
\begin{align*}
V^{-\infty}Z^iM &= \{ x \in M^i | dV^n(x) = 0 \text{ for all } n \geq 0 \} \quad \text{and} \\
F^{\infty}B^iM &= \{ x \in M^i | x \in F^n d(M^{i-1}) \text{ for some } n \geq 0 \}.
\end{align*}
\]

**Definition 5.19.** A coherent \(R\)-module \(M\) is a domino if there exists an integer \(i\) such that \(M^n = 0\) for \(n \neq i, i+1\), \(V^{-\infty}Z^i = 0\), and \(F^{\infty}B^{i+1} = M^{i+1}\).

For a further explication of this definition, we refer the reader to [IR83, Définition 2.16] and the surrounding material as well as [Ill83, Section 2.5]. Given any \(R\)-module \(M\), each differential \(M^i \to M^{i+1}\) of \(M\) factors as

\[
\begin{array}{ccccc}
M^i & \xrightarrow{d} & M^{i+1} \\
\downarrow & & \downarrow \\
M^i/V^{-\infty}Z^i & \xrightarrow{=} & F^{\infty}B^{i+1}
\end{array}
\]

(21)
and, if $M$ is coherent, then the $R$-module
\[ \text{Dom}^i(M) = [M^i/V^{-\infty}Z^i \to F^\infty B^{i+1}] \]
is a domino. We sometimes refer to this as the domino associated to the differential $M^i \to M^{i+1}$ of $M$.

**Definition 5.20.** If $D$ is a domino supported in degrees $i$ and $i+1$, then
\[ T(D) = \dim_k(D^i/V^i) \]
is finite. We refer to it as the dimension of the domino $D$. If $M$ is a coherent $R$-module, we let $T^i(M) = T(\text{Dom}^i(M))$.

Illusie and Raynaud define in [IR83, Section 1.2.D] certain simple one dimensional dominoes $U_{\sigma}$, depending on an integer $\sigma$. By [IR83, Proposition 1.2.15], every domino is a finite iterated extension of the $U_{\sigma}$. We will use the notation
\[ \text{Dom}^{i,j}(X) \overset{\text{def}}{=} \text{Dom}^i(\text{H}^j(W\Omega^\bullet_X)), \]
(23)
\[ T^{i,j}(X) \overset{\text{def}}{=} T^i(\text{H}^j(W\Omega^\bullet_X)). \]
(24)

**Example 5.21.** If $X$ is a K3 surface over a perfect field, then the differential
\[ d : H^2(W\Omega_X) \to H^2(W\Omega^1_X) \]
is non-zero if and only if $X$ is supersingular, in which case it is a domino of dimension 1, isomorphic to $U_{\sigma_0}$ where $\sigma_0$ is the Artin invariant of $X$ (see [Ill79, III.7.2]).

We recall that Ekedahl showed in [Eke84, Theorem IV.3.5] that $\text{Dom}^{i,j}(X)$ and $\text{Dom}^{d-i-2,d-j+2}(X)$ are naturally dual (in a certain sense); as a consequence we have the equality
\[ T^{i,j} = T^{d-i-2,d-j+2} \]
(26)
of Domino numbers [Eke84, Corollary IV.3.5.1].

**Remark 5.22.** The domino associated to an $R$-module supported in two degrees whose differential is zero is zero. Thus, we have $T^{i,j} = 0$ if $i \geq d$ or $j > d$. By (26), this implies the vanishing of various other domino numbers which are not obviously zero. For instance, if $X$ is a surface, then we see that the only possible nontrivial domino number of $X$ is $T^{0,2}$, the dimension of the domino associated to the differential (25) in the slope spectral sequence.

**Remark 5.23.** Ekedahl studies in [Eke84] a certain canonical filtration of a coherent $R$-module $M$, one piece of which is composed of the dominoes defined above (see also [Ill83]). From this filtration Ekedahl shows how to partition the torsion of $M$ according to its behavior under $V$ and $F$: semisimple torsion, nilpotent torsion, and dominoes. For surfaces and threefolds, the possible degrees in which each of these types of torsion may appear are indicated in Figures 6 and 7. We will study only the domino torsion in this document, although the semisimple and nilpotent torsion are undoubtedly interesting as well.

Let $X$ be a smooth and proper $k$-scheme. We now consider the complex $\text{TR}_\bullet(X)$, which by Proposition 4.4 is a coherent $R$-module. We define
\[ \text{Dom}^{\text{cyc}}_i(X) \overset{\text{def}}{=} \text{Dom}^i(\text{TR}_\bullet(X)), \]
(27)
\[ T^{\text{cyc}}_i(X) \overset{\text{def}}{=} T^i(\text{TR}_\bullet(X)). \]
(28)

By construction, the $T^{\text{cyc}}_i$ are derived invariants of $X$, and we refer to them as the derived domino numbers of $X$. We will use the spectral sequence (5) to relate them to the usual domino numbers $T^{i,j}$ of $X$. We note the following lemma.
Lemma 5.24. If $0 \to L \to M \to N \to 0$ is an exact sequence of coherent $R$-modules, then for each $i$ we have $T^i(M) = T^i(L) + T^i(N)$.

Proof. See [MR15, Lemma 2.5].

Proposition 5.25. If $X$ is a smooth proper $k$-scheme of dimension $d$, then $T^{0,d}(X) = T^{\text{cy}}_{-d}(X)$. In particular, $T^{0,d}(X)$ is a derived invariant.

Proof. We interpret the rows of the pages of the descent spectral sequence for $X$ as $R$-modules by Lemma 4.3. We have an exact sequence

$$H^{d-2}(W\Omega^*_X)[-1] \xrightarrow{d_2} H^d(W\Omega^*_X) \to E^*_d \to 0.$$  

The image of $d_2$ is a coherent $R$-submodule $\text{im}(d_2) \subseteq H^d(W\Omega^*_X)$, and $\text{Dom}^0(\text{im}(d_2)) = 0$ for degree reasons. Hence, $T^{0,d}(X) = T^0(H^d(W\Omega^*_X)) = T^0(E^*_d)$. The first two terms of $E^*_d$ do not see any further differentials, and so

$$T^{0,d}(X) = T^0(E^*_d) = T^0(E^\infty_d).$$  

Consider the filtration $F^*_i$ of $\text{TR}_s(X)$ from Lemma 4.3. It follows inductively from Lemma 5.24 and the isomorphisms of Lemma 4.3(iv) that $T^{-d}(\text{TR}_s(X)/F^*_i) = 0$ for all $i$. Combined with (29), we obtain

$$T^{0,d}(X) = T^0(E^*_d) = T^{-d}(F^*_d) = T^{-d}(\text{TR}_s(X)) = T^{\text{cy}}_{-d}(X),$$

as desired.

Definition 5.26. Let $X$ be a smooth proper $k$-scheme. We say that the descent spectral sequence for $X$ is degenerate at the level of dominoes if for each $i, j$ we have $T^{i,j}(X) = T^r(E^*_r)$ for all $r \geq 2$, where $E^*_r$ denotes the coherent $R$-module from Lemma 4.3 arising in the descent spectral sequence.

In low dimensions, this condition is automatic.

Lemma 5.27. Let $X$ be a smooth proper $k$-scheme. If $X$ has dimension $d \leq 3$, then the descent spectral sequence for $X$ is degenerate at the level of dominoes.

Proof. If $d \leq 2$, the descent spectral sequence is degenerate. Suppose $d = 3$. The only possibly nonzero dominoes of $X$ are depicted in Figure 7. The result follows immediately from Lemma 5.24.

Proposition 5.28. Let $X$ be a smooth proper $k$-scheme. If the descent spectral sequence for $X$ is degenerate at the level of dominoes, then for each $i$ we have

$$T^{\text{cy}}_i(X) = \sum_{j \geq 0} T^{i+j,j}(X).$$

Proof. We have

$$T^i(F^i_d/F^{i+1}_d) = T^i(E^*_d[j]) = T^{i+j}(E^*_\infty) = T^{i+j,j}(X).$$

The result follows from Lemma 5.24.

Theorem 5.29. If $X$ and $Y$ are $\text{FM}$-equivalent smooth proper $k$-schemes of dimension $\leq 3$, then for all $i, j$ we have $T^{i,j}(X) = T^{i,j}(Y)$.

Proof. If $X$ and $Y$ are surfaces, the result follows from Proposition 5.25. The only possibly nonzero domino numbers of a threefold are $T^{0,2}, T^{0,3}, T^{1,2}$, and $T^{1,3}$. By Proposition 5.25 $T^{0,3} = T^{\text{cy}}_{-3}$ is a derived invariant. By duality (26), $T^{1,2} = T^{0,3}$ is also derived invariant. By Lemma 5.27, the descent spectral sequence of a threefold is degenerate at the level of dominoes, so by Proposition 5.28 we have that $T^{\text{cy}}_{-2} = T^{0,2} + T^{1,3}$ is derived invariant. But, by duality again, $T^{0,2} = T^{1,3}$, and hence both terms are themselves derived invariant.
5.4 Hodge–Witt numbers

We recall certain $p$-adic invariants introduced by Ekedahl in Section IV of [Eke86]. We refer the reader also to Crew’s article [Cre85] and Illusie’s article [Ill83]. Let $X$ be a smooth and proper $k$-scheme. We define the Hodge–Newton numbers of $X$ by

$$m_{i,j} = \sum_{\lambda \in [i,i+1)} (i + 1 - \lambda) h_{\text{crys},\lambda}^{i+j} + \sum_{\lambda \in [i-1,i)} (\lambda - i + 1) h_{\text{crys},\lambda}^{i+j}. \quad (30)$$

One can show that the $m_{i,j}$ are in fact non-negative integers, and by [Eke86, Lemma VI.3.1] they satisfy the relations

$$m_{i,j} = m_{j,i}, \quad (31)$$
$$m_{i,j} = m_{d-i,d-j}. \quad (32)$$

The Hodge–Witt numbers of $X$ are defined as

$$h_{W}^{i,j} = m_{i,j} + T^{i,j} - 2T^{i-1,j+1} + T^{i-2,j+2}. \quad (33)$$

By [Eke86, Proposition VI.3.2, VI.3.3] these satisfy

$$h_{W}^{i,j} = h_{W}^{d-i,d-j}, \quad (34)$$

and, if $X$ has dimension $d \leq 3$, one has

$$h_{W}^{i,j} = h_{W}^{i,j}. \quad (35)$$

By [Eke86, Theorem IV.3.2, IV.3.3], the Hodge–Witt numbers are related to the Hodge numbers $h_{i,j} = h^{i,j}(X,\Omega^{i}_X)$ by the inequalities

$$h_{W}^{i,j} \leq h^{i,j} \quad (36)$$

and by Crew’s formula, which states that

$$\sum_{j} (-1)^{j} h_{W}^{i,j} = \sum_{j} (-1)^{j} h^{i,j} = \chi(\Omega^{i}_X). \quad (37)$$

Finally, we have

$$b_{n} = \sum_{i+j=n} h_{W}^{i,j} \quad (38)$$

for each $n$.

As an immediate consequence of Theorem 5.15 and Theorem 5.29, we have the following result.

**Theorem 5.30.** If $X$ and $Y$ are FM-equivalent smooth proper $k$-schemes of dimension $\leq 3$, then for all $i, j$ we have $h_{W}^{i,j}(X) = h_{W}^{i,j}(Y)$.

5.5 Hodge numbers

Let us turn to the question of whether the Hodge numbers $h^{i,j}$ are derived invariants of smooth proper $k$-schemes. The answer to this is known to be yes up to dimension $3$ in characteristic $0$ by [PS11]. For curves (in any characteristic) it is an easy consequence of the Hochschild–Kostant–Rosenberg (HKR) theorem. For surfaces in characteristic $0$, it follows from HKR together with Hodge symmetry and Serre duality; for threefolds in characteristic $0$ it follows with the addition input of the theorem of Popa and Schnell (Theorem 5.11).
These arguments fail in several places in positive characteristic. First, the HKR isomorphism is only known to hold in general if \( d \leq p \). Second, Hodge symmetry fails in general already for surfaces. Finally, in positive characteristic the isogeny class of the Albanese does not determine the Hodge numbers \( h^{0,1} \) or \( h^{1,0} \) (see Remark 5.14).

In the case of surfaces, we are able to overcome these difficulties with the additional input of our results on Hodge–Witt numbers from Section 5.4, which in turn rely on the results on domino numbers of Section 5.3. Despite its elementary statement, we do not know a direct proof of Theorem 5.31 avoiding topological Hochschild homology machinery.

**Theorem 5.31.** Suppose that \( X \) and \( Y \) are smooth proper surfaces over an arbitrary field \( k \). If \( X \) and \( Y \) are FM-equivalent, then \( h^{i,j}(X) = h^{i,j}(Y) \) for all \( i, j \).

**Proof.** As described in Theorem 2.2, the HKR isomorphism implies that \( h^{2,0}, h^{1,1}, \) and \( h^{0,2} \) are derived invariants, as are the sums \( h^{0,1} + h^{1,2} \) and \( h^{1,0} + h^{2,1} \). If \( k \) has characteristic 0, Serre duality and Hodge symmetry give the result. Suppose that \( k \) has positive characteristic. We can assume \( k \) is perfect since passage to the perfection does not change the Hodge numbers. Theorem 5.30 and (37) gives that \( \chi(\Omega_X^i) \) is a derived invariant for each \( i \). It follows that \( h^{0,1} \) and \( h^{1,0} \) are derived invariants. By the HKR isomorphism, \( h^{1,2} \) and \( h^{2,1} \) are derived invariants. \( \Box \)

**Remark 5.32.** For the positive characteristic case of the preceding theorem, we may instead argue as follows. As observed in the beginning of Section 5.1, the cohomology group \( H^2(W\mathcal{O}_X)_\mathbb{R} \), together with its \( R^{\mathbb{R}} \)-module structure (that is, with its action of \( F \) and \( V \)), is derived invariant. The length of the \( V \)-torsion of \( H^2(W\mathcal{O}_X) \) is equal to the dimension of the tangent space at the origin of \( \text{Pic}_X^0 \) minus the dimension of the tangent space at the origin of \( (\text{Pic}_X^0)_{\text{red}} \) (see for instance [Ill79, Remarque II.6.4]). By Theorem 5.11, the latter is a derived invariant as well. It follows that the dimension of the tangent space of \( \text{Pic}_X^0 \) at the origin, which is equal to the Hodge number \( h^{0,1} \), is a derived invariant. We then conclude by Serre duality and HKR, as before.

We next consider threefolds in positive characteristic. To ensure that the HKR spectral sequence degenerates, we might restrict our attention to characteristic \( p \geq 3 \).\(^5\) Even with this restriction, the failure of Hodge symmetry and of Popa–Schnell to determine \( h^{0,1} \) means that we do not have enough control to prove derived invariance of Hodge numbers of threefolds in general (see however Theorem 5.37). We record the following consequence of Theorem 5.30.

**Theorem 5.33.** Suppose that \( X \) and \( Y \) are smooth proper schemes of dimension 3 over \( k \), a perfect field of positive characteristic \( p \). If \( X \) and \( Y \) are FM-equivalent, then \( \chi(\Omega_X^i) = \chi(\Omega_Y^i) \) for each \( 0 \leq i \leq 3 \).

**Proof.** This follows immediately from Theorem 5.30 and (37). \( \Box \)

**Remark 5.34.** Suppose that \( p \geq 3 \), so that the spectral sequence (1) associated to a threefold \( X \) degenerates. The HKR isomorphism then gives that certain sums of Hodge numbers of \( X \) are derived invariants, as described in Theorem 2.2. Combined with Serre duality and the obvious Hodge number \( h^{0,0} = 1 \), we obtain 13 linearly independent relations which are preserved by derived equivalences on the 16 total Hodge numbers.

It is not hard to check that the relations in the conclusion of Theorem 5.33 are not in the span of these relations. Precisely, the result of Theorem 5.33 gives exactly one new linear relation on Hodge numbers that is preserved under derived equivalence.

\(^5\)The examples of [ABM] of varieties with non-degenerate HKR spectral sequence \( 2p \)-dimensional. We do not know an example of a threefold in characteristic 2 with non-degenerate HKR spectral sequence.
5.6 Mazur–Ogus and Hodge–Witt varieties

In this section, we prove the derived invariance of certain conditions on de Rham and Hodge–Witt cohomology. We continue Notation 4.1, so that $k$ is a perfect field of positive characteristic $p$.

Following Joshi [Jos14, 2.31], we make the following definition.

**Definition 5.35.** Let $X$ be a smooth proper $k$-scheme. We say that $X$ is **Mazur–Ogus** if

(a) the Hodge–de Rham spectral sequence for $X$ degenerates at $E_1$, and

(b) the crystalline cohomology groups of $X$ are torsion free.

We view this as a rather mild set of assumptions, which still allow for a lot of interesting behaviors in the Hodge–Witt cohomology of $X$. For instance, K3 surfaces, abelian varieties, and complete intersections in projective space are Mazur–Ogus.

**Lemma 5.36.** If $X$ is a smooth proper $k$-scheme, then the following conditions are equivalent:

1. $X$ is Mazur–Ogus;
2. $b_n(X) = \sum_{i+j=n} h^{i,j}(X)$ for all $n$;
3. $h^{i,j}(X) = h^{i,j}_W(X)$ for all $i, j$.

**Proof.** In general, we have inequalities

$$b_n(X) \leq \dim H^n_{dR}(X/k) \leq \sum_{i+j=n} h^{i,j}(X)$$

The first of these is an equality if and only if $H^n_{dR}(X/W)$ and $H^{n+1}(X/W)$ are torsion free, and the second is an equality if and only if the Hodge–de Rham spectral sequence in degree $n$ degenerates at $E_1$. This shows (1) $\iff$ (2). Using (36) and (38) we deduce (2) $\iff$ (3). $\square$

**Theorem 5.37.** Let $X$ be a smooth proper $k$-scheme of dimension $\leq 3$. If $X$ is Mazur–Ogus and if $Y$ is a smooth proper $k$-scheme such that $D^b(X) \cong D^b(Y)$, then $Y$ is Mazur–Ogus and we have $h^{i,j}(X) = h^{i,j}(Y)$ for all $i, j$.

**Proof.** Using Lemma 5.36, Theorem 5.30, and (36) we have

$$h^{i,j}(X) = h^{i,j}_W(X) = h^{i,j}_W(Y) \leq h^{i,j}(Y)$$

for each $i, j$. Using the assumption that $p \geq 3$, we have by Theorem 2.2 that

$$\sum_j h^{j,-i}(X) = \sum_j h^{j,-i}(Y)$$

for each $i$. We conclude that $h^{i,j}(X) = h^{i,j}(Y)$ for all $i, j$ and hence $h^{i,j}_W(Y) = h^{i,j}(Y)$ for all $i$ and $j$. By Lemma 5.36 we conclude that $Y$ is Mazur–Ogus. $\square$

**Definition 5.38.** Following [IR83, Section IV.4], we say that a smooth proper $k$-scheme is **Hodge–Witt** if $H^i(X, W\Omega^j_X)$ is finitely generated as a $W$-module for all $i$ and $j$. We say that $X$ is **derived Hodge–Witt** if $TR_i(X)$ is finitely generated as a $W$-module for all $i$. 
Hodge–Witt is implied by ordinarity, but is weaker than it. For example, a K3 surface is Hodge–Witt if and only if it is non-supersingular, whereas it is ordinary if and only if the associated formal group has height 1 (see the [Ill79, II.7.2]).

**Proposition 5.39.** Let $X$ be a smooth proper $k$-scheme. The following are equivalent.

1. $X$ is Hodge–Witt.
2. The slope spectral sequence for $X$ degenerates at $E_1$.
3. $T^{i,j}(X) = 0$ for all $i,j$.

**Proof.** By [IR83, IV.4.6.2], $X$ is Hodge–Witt if and only if the slope spectral sequence for $X$ degenerates at $E_1$, and so we have (1) $\iff$ (2). We have (1) $\iff$ (3) by for instance [Ill83, 3.1.4].

**Theorem 5.40.** Let $X$ be a smooth proper $k$-scheme. If $X$ is of dimension $\leq 3$, then $X$ is Hodge–Witt if and only if it is derived Hodge–Witt.

**Proof.** By [Ill83, 3.1.4], we have that for $j = 0,1$ the Hodge–Witt cohomology groups $H^i(W\Omega^j_X)$ are finitely generated for each $i$. The differentials in the descent spectral sequence have vertical degree 2 (with our conventions), and hence $X$ is Hodge–Witt if and only if the terms $E^2_{i,j}$ appearing on the $E_\infty$-page of the descent spectral sequence are finitely generated for all $i,j$.

Furthermore, each of the $TR_i(X)$ admits a filtration whose successive quotients are given by terms appearing on the $E_\infty$ page of the descent spectral sequence for $X$. In particular, we see that all of the $TR_i(X)$ are finitely generated $W$-modules if and only if $E^2_{i,j}$ is finitely generated for all $i,j$.

**Corollary 5.41.** Let $X$ and $Y$ be smooth proper threefolds over $k$. If $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ and if $X$ is Hodge–Witt, then so is $Y$.

In particular, the equivalent conditions recorded in Proposition 5.39 are all derived invariants in dimension $d \leq 3$.

**Example 5.42.** Joshi shows in [Jos07, Corollary 6.2] that if $X$ is an $F$-split threefold, then it is Hodge–Witt. Thus, Corollary 5.41 applies to $F$-split threefolds. This has recently been extended to quasi-$F$-split threefolds by Nakajima [Nak19, Corollary 1.8] and in particular applies then to all finite height Calabi–Yau threefolds by [Yob19].

**Remark 5.43.** Using the equivalent conditions of Proposition 5.39, one can give a less direct proof of Theorem 5.40 using Proposition 5.28.

## 6 Twisted K3 surfaces

In this section we will study some examples with interesting behavior in the slope and descent spectral sequences. Specifically, we will completely compute $TR$ and $TP$ for twisted K3 surfaces. This will give a different perspective on the twisted K3 crystals defined and studied in [BL18]. For the remainder of this section, we let $k$ be an algebraically closed field of positive characteristic $p$.

While we have only discussed derived invariants of varieties so far, much of our discussion carries over unchanged to twisted varieties. For instance, $TR$ and $TP$ are defined for an abstract dg category. Given $\alpha \in \text{Br}(X)$ a Brauer class on a smooth proper variety $X$, we let $TR_\alpha(X,\alpha)$ and $TP_\alpha(X,\alpha)$ denote their application to the natural enhancement of the bounded derived category of $\alpha$-twisted coherent sheaves on $X$. There is a descent spectral sequence

$$E^{s,t}_2 = H^s(X, W\Omega^t_X) \Rightarrow TR_{s-t}(X,\alpha)$$

(39)
We conclude that there exist (non canonical) isomorphisms which computes the $\text{TR}_*(X, \alpha)$ in terms of the Hodge–Witt cohomology groups of the underlying variety $X$. We remark that the Hodge–Witt cohomology groups of a K3 surface are determined very explicitly in [III79, II.7.2]. We will see that, although the objects appearing on the $E_2$ page of (39) are the same as those on the $E_2$ page of the descent spectral sequence for $X$, the differentials may be different. We let $d^r_2$ denote the differentials in the $\alpha$-twisted descent spectral sequence.

Let $X$ be a K3 surface over $k$. We begin by computing the topological periodic cyclic homology groups $\text{TP}_*(X)$ of $X$ in terms of crystalline cohomology. Since the crystalline cohomology groups of $X$ are all torsion-free, the crystalline–TP spectral sequence (7) degenerates; the corresponding filtration splits noncanonically and gives an isomorphism of $W$-modules

$$\text{TP}_{2i}(X) \cong H^0(X/W) \oplus H^2(X/W) \oplus H^4(X/W)$$

for each $i$. In general, the $K$-vector spaces obtained by tensoring $\text{TP}_i$ with $W$ admit a natural Frobenius operator coming from the cyclotomic Frobenius and are so endowed with a structure of $F$-isocrystal. However, after inverting $p$ this isomorphism does not carry the Frobenius on the left hand side to the natural Frobenius operator $\Phi$ on crystalline cohomology. Rather, consider the Mukai crystal

$$H(X/W) = H^0(X/W)(-1) \oplus H^2(X/W) \oplus H^4(X/W)(1)$$

as introduced in [LO15]. The above can then be upgraded to isomorphisms $\text{TP}_{2i}(X) \cong H(X/K)(i + 1)$ of $F$-isocrystals for each $i$, where $H(X/K) = H(X/W) \otimes_W K$. In fact, for any (possibly twisted) surface, the Frobenius is defined integrally on $\text{TP}_i(X, \alpha)$ for $i \leq -2$. The filtration on $\text{TP}$ can be split even integrally, and so we obtain an isomorphism

$$\text{TP}_{2i}(X) \cong H(X/W)(i + 1)$$

of $F$-crystals for each $i \leq -1$.

### 6.1 Finite height

Let $(X, \alpha)$ be a twisted K3 surface over $k$, and suppose that $X$ has finite height. By the computation of the Hodge–Witt cohomology groups of $X$ in [III79, Section II.7.2], we see that the slope spectral sequence for $X$ and the descent spectral sequences for $X$ and $(X, \alpha)$ are all degenerate for degree reasons. Hence, both $\text{TR}_0(X)$ and $\text{TR}_0(X, \alpha)$ admit a filtration by $R^0$-submodules with graded pieces $H^0(W\Omega_X^1)$, $H^2(W\Omega_X^1)$, and $H^2(W\Omega_X^2)$. As these groups are all torsion-free, this filtration certainly splits at the level of $W$-modules. In fact, by computing the appropriate Ext groups, one can show that it even splits at the level of $R^0$-modules. We conclude that there exist (non canonical) isomorphisms

$$\text{TR}_i(X) \cong \text{TR}_i(X, \alpha) \cong \begin{cases} H^2(W\Omega_X^1) & \text{if } i = -2, \\ H^0(W\Omega_X^1) \oplus H^2(W\Omega_X^2) & \text{if } i = 0, \\ H^0(W\Omega_X^2) & \text{if } i = 2 \end{cases} (41)$$

of $R^0$-modules, and $\text{TR}_i(X) = \text{TR}_i(X, \alpha) = 0$ otherwise.

We next compute TP. Because $\text{TR}$ is concentrated in even degrees, the Tate spectral sequences for $X$ and $(X, \alpha)$ degenerate and TP is also concentrated in even degrees. We have a filtration on $\text{TP}_{2i}(X, \alpha)$ with graded pieces $H^0(W\Omega_X^2)$, $\text{TR}_0(X, \alpha)$, and $H^2(W\Omega_X)$. Moreover, for $i \leq -1$, keeping track of the appropriate

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6By an argument of Scholze, the filtration from the crystalline–TP spectral sequence can be canonically split rationally by using Adams operations; see [Elm18].

7To see this, one must use the Nygaard filtration.
Tate twists this yields a filtration by $F$-crystals,\(^\text{8}\) whose graded pieces are $H^0(W\Omega^2_X)(i-1)$, $\text{TR}_0(X,\alpha)(i)$, and $H^2(W\Omega_X)(i+1)$.

In particular, this determines the $\text{TP}_{2i}(X,\alpha)$ as $F$-isocrystals. To determine their structure as $F$-crystals, one needs some additional input. This can be done by comparison with the B-field constructions of [BL18], from which one can compute the Hodge polygon of $\text{TP}_{2i}(X,\alpha)$. Using Katz’s Newton–Hodge decomposition [Kat79, Theorem 1.6.1], one then deduces that the filtration on $\text{TP}_{2i}(X,\alpha)$ in fact splits canonically, and so we have a canonical isomorphism

$$\text{TP}_{2i}(X,\alpha) = H^2(W\Omega_X)(i+1) \oplus \text{TR}_0(X,\alpha)(i) \oplus H^0(W\Omega^2_X)(i-1)$$

of $F$-crystals for each $i \leq -1$. In particular, by (41), we see that $\text{TP}_{2i}(X,\alpha)$ as $F$-crystals for each $i \leq -1$, although not canonically.

### 6.2 Supersingular

We now consider a twisted K3 surface $(X,\alpha)$ where $X$ is supersingular. We will examine the descent spectral sequence for $(X,\alpha)$. In particular, we will see that it is not degenerate.

We begin with a few general facts. For any smooth $k$-scheme $X$, there is a natural map of étale sheaves

$$\mathbb{G}_m \xrightarrow{\text{dlog}} W\Omega^1_X$$

given on sections by $f \mapsto \frac{df}{f}$. There is an induced map on cohomology

$$\text{dlog} : H^2(X, \mathbb{G}_m) \to H^2(W\Omega^1_X).$$

#### Lemma 6.1

Let $X$ be a smooth proper $k$-scheme and let $\alpha \in H^2(X, \mathbb{G}_m)$.

(a) The differential

$$d^2_\alpha : H^0(W\mathcal{O}_X) \to H^2(W\Omega^1_X)$$

appearing in the $E_2$-page of the $\alpha$-twisted descent spectral sequence (39) sends the canonical generator $1 \in W = H^0(W\Omega_X)$ to $\text{dlog}(\alpha)$.

(b) If $X$ is a surface, then all other differentials in the twisted descent spectral sequence are zero.

**Proof.** Let $\text{K}^{\text{et}}(X,\alpha)$ denote the $\alpha$-twisted étale $K$-theory of $X$. There is a descent spectral sequence

$$E^{s,t}_2 = H^{s,t}_\text{et}(X, \text{K}_s(\mathcal{O}_X)) \Rightarrow \text{K}^{\text{et}}_{s-t}(X,\alpha).$$

We also have natural isomorphisms $\mathbb{Z} \cong K_0(\mathcal{O}_X)$ and $\mathbb{G}_m \cong K_1(\mathcal{O}_X)$. The $d^2$-differential $H^0_\text{et}(X,\mathbb{Z}) \to H^2_\text{et}(X,\mathbb{G}_m)$ sends $1$ to $\alpha$ by [Ant11, Theorem 8.5]. Now, the map $K_1(\mathcal{O}_X) \cong \mathbb{G}_m \to \text{TR}_1(\mathcal{O}_X) \cong W\Omega^1$ is given by the $\text{dlog}$ map; see [GH99, Lemma 4.2.3].\(^\text{9}\) Thus, part (a) follows from the compatibility between the descent spectral sequences for $\text{K}^{\text{et}}(\alpha)$ and $\text{TR}(X,\alpha)$ using the trace map $\text{K}^{\text{et}}(\alpha) \to \text{TR}(X,\alpha)$ and especially the commutative diagram

$$\begin{array}{ccc}
H^0_\text{et}(X,\mathbb{Z}) & \xrightarrow{d^2_\alpha} & H^2(X,\mathbb{G}_m) \\
\downarrow & & \downarrow \text{dlog} \\
H^0(W\mathcal{O}_X) & \xrightarrow{d^2} & H^2(W\Omega^1_X).
\end{array}$$

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\(^\text{8}\)By [III79, II.7.2a] the Hodge–Witt cohomology groups of $X$ are finitely generated and torsion free.

\(^\text{9}\)Note that in Geisser–Hesselholt, the map is given by $-\text{dlog}$, but this sign depends on a choice of the HKR isomorphism $\text{TR}_1(\mathcal{O}_X) \cong W\Omega^1_X$ which amounts to a choice of an orientation on the circle.
For part (b), note that all of the differentials are torsion but that $H^2(W\Omega^2_X)$ is torsion-free. Thus, $d_2^2 : H^0(W\Omega_X) \to H^2(W\Omega^2_X)$ is the only possible non-zero differential for a surface.

We conclude from the above that the descent spectral sequence for a twisted surface $(X, \alpha)$ is degenerate at $E_2$ if and only if $\text{dlog}(\alpha) = 0$, and is always degenerate at $E_3$ for degree reasons.

Let us now return to the situation where $X$ is a supersingular K3 surface. We record the following result.

**Lemma 6.2.** If $X$ is a supersingular K3 surface over an algebraically closed field $k$ of positive characteristic $p$, then the sequence

$$0 \to \text{Br}(X) \xrightarrow{\text{dlog}} H^2(W\Omega^1_X) \xrightarrow{1-F} H^2(W\Omega^1_X) \to 0$$

is exact, where the left arrow is the map on cohomology induced by (42).

**Proof.** As $H^1(W\Omega^1_X)$ is finitely generated, its endomorphism $1-F$ is surjective by [Ill79, Lemme II.5.3]. By flat duality, $H^4(X, \mathbb{Z}_p(1)) = 0$. By [Ill79, Théorème II.5.5] we therefore obtain a short exact sequence

$$0 \to H^3(X, \mathbb{Z}_p(1)) \to H^2(W\Omega^1_X) \xrightarrow{1-F} H^2(W\Omega^1_X) \to 0$$

where as usual we put

$$H^3(X, \mathbb{Z}_p(1)) \overset{\text{def}}{=} \lim\limits_{\leftarrow} H^3(X, \mu_{p^n}).$$

As a consequence of flat duality, Artin showed that this inverse system is constant, and hence the natural map

$$H^3(X, \mathbb{Z}_p(1)) \iso H^3(X, \mu_p)$$

is an isomorphism. Furthermore, the boundary map induced by the Kummer sequence gives an isomorphism

$$\text{Br}(X) \iso H^3(X, \mu_p). \quad (44)$$

For these facts, see the proof of [Art74, Theorem 4.3] on page 559. We thus find an isomorphism $\text{Br}(X) \iso H^3(X, \mu_p)$. Using the definitions of the maps involved, one checks that the resulting map $\text{Br}(X) \to H^2(W\Omega^1_X)$ is the map on cohomology induced by (42).

We remark that the isomorphism (44) implies $\text{Br}(X)$ is $p$-torsion; in fact, as recorded in [Art74], there is an abstract isomorphism of groups $\text{Br}(X) \iso k$. Lemma 6.2 implies in particular that $\text{dlog}$ is injective. The Brauer group of a supersingular K3 surface is non-trivial, so combined with Lemma 6.1 we have produced examples of twisted surfaces with non-degenerate descent spectral sequence. We record the $E_2$ and $E_3$ pages in the following figure.

![Figure 8: The $E_2$ and $E_3 = E_\infty$ pages of the descent spectral sequence for $(X, \alpha)$. The horizontal arrows are the maps induced by the differentials appearing on the $E_1$ page of slope spectral sequence for $X$. The term $\text{ord}(\alpha)W$ is the kernel of the non-zero differential, which is generated by $\text{ord}(\alpha) \in W$, where $\text{ord}(\alpha) = \text{ord}(\text{dlog}(\alpha))$ is the order of $\alpha$.](image-url)
As in the finite height case, we therefore have noncanonical isomorphisms
\[ \text{TR}_0(X) \cong \text{TR}_0(X, \alpha) \cong H^0(W\Omega_X) \oplus H^1(W\Omega^1_X) \oplus H^2(W\Omega^2_X), \]
(45)
of \( R^0 \)-modules, where we use ord(\( \alpha \)) to give an isomorphism between \( W \cong H^0(W\Omega_X) \) and ord(\( \alpha \))\( W \). We have a commuting diagram
\[
\begin{array}{ccc}
H^2(W\Omega_X) & \xrightarrow{d} & H^2(W\Omega^1_X) \\
\text{TR}_{-2}(X, \alpha) & \xrightarrow{d} & \text{TR}_{-1}(X, \alpha) \\
\text{K}(X, \alpha) & \xrightarrow{0} & \text{TR}_0(X, \alpha)
\end{array}
\]
where the vertical arrows are induced by the descent spectral sequence, and the right lower differential vanishes by Lemma 4.6. Finally, we have that \( \text{TR}_1(X, \alpha) = \text{TR}_2(X, \alpha) = 0 \).

The differential \( d \) is surjective, and we let \( \text{K}(X, \alpha) \) denote its kernel, so that we have a short exact sequence
\[ 0 \rightarrow \text{K}(X, \alpha) \rightarrow H^2(W\Omega_X) \xrightarrow{d} H^2(W\Omega^1_X) \xrightarrow{dlog(\alpha)} 0. \]
We know that \( K(X) = K(X, 0) \) is a \( k \)-vector space of dimension \( \sigma_0(X) \). Recall from [BL18, Corollary 3.4.23] that the Artin invariant of a twisted supersingular K3 surface is given by \( \sigma_0(X, \alpha) = \sigma_0(X) + 1 \) if \( \alpha \neq 0 \) and \( \sigma_0(X, \alpha) = \sigma_0(X) \) otherwise. We conclude that \( K(X, \alpha) \) is a \( k \)-vector space of dimension \( \sigma_0(X, \alpha) \). In particular, this shows that the Artin invariant \( \sigma_0(X, \alpha) \) is a derived invariant of \( (X, \alpha) \), which gives another proof of [Bra18, Corollary 3.4.3].

The topological periodic cyclic homology \( \text{TP}(X, \alpha) \) is computed by the Tate spectral sequence (6), whose \( E_2 \) and \( E_3 \) pages are pictured in Figures 9 and 10.

\[ \cdots \xrightarrow{\alpha} \text{TR}_0(X, \alpha) \xrightarrow{0} \text{TR}_0(X, \alpha) \xrightarrow{0} \text{TR}_0(X, \alpha) \cdots \]
\[ \cdots \xrightarrow{dlog(\alpha)} \xrightarrow{dlog(\alpha)} \xrightarrow{dlog(\alpha)} \cdots \]
\[ \cdots \xrightarrow{dlog(\alpha)} \xrightarrow{dlog(\alpha)} \xrightarrow{dlog(\alpha)} \cdots \]
\[ \cdots \xrightarrow{dlog(\alpha)} \xrightarrow{dlog(\alpha)} \xrightarrow{dlog(\alpha)} \cdots \]

Figure 9: A portion of the \( E_2 \)-page of the Tate spectral sequence for \((X, \alpha)\).

\[ \cdots \xrightarrow{0} \xrightarrow{0} \xrightarrow{0} \xrightarrow{0} \xrightarrow{0} \xrightarrow{0} \cdots \]
\[ \cdots \xrightarrow{0} \xrightarrow{0} \xrightarrow{0} \xrightarrow{0} \xrightarrow{0} \xrightarrow{0} \cdots \]
\[ \cdots \xrightarrow{K(X, \alpha)} \xrightarrow{K(X, \alpha)} \xrightarrow{K(X, \alpha)} \cdots \]

Figure 10: A portion of the \( E_3 = E_\infty \)-page of the Tate spectral sequence for \((X, \alpha)\).

We therefore find short exact sequences
\[ 0 \rightarrow \text{TR}_0(X, \alpha) \rightarrow \text{TP}_1(X, \alpha) \rightarrow K(X, \alpha) \rightarrow 0 \]
(46)
of $W$-modules for all even $i$, and $TP_i(X, \alpha) = TP_i(X) = 0$ for $i$ odd. In particular, keeping track of the respective Frobenius actions as in the previous section, we find for each $i \leq -1$ a short exact sequence
\[ 0 \to \text{TR}_0(X, \alpha)(-i) \to TP_{2i}(X, \alpha) \to K(X, \alpha) \to 0 \quad (47) \]
of $W$-modules, where the left hand arrow is a map of $F$-crystals.

If $\alpha = 0$, then using (40) and (45) one checks that the inclusion $\text{TR}_0(X)(-1) \to TP_{-2}(X)$ above is isomorphic to the inclusion of the Tate module of $\tilde{H}(X/W)$. The cokernel of this inclusion is the same as the cokernel of the inclusion of the Tate module of $H^2(X/W)$. Thus, $K(X)$ is naturally identified with the characteristic subspace associated to $X$ by Ogus [Ogu79].

In general, one can show that for any $i \leq -1$ there is an isomorphism
\[ TP_{2i}(X, \alpha) \cong \tilde{H}(X/W, B)(i) \]
of $F$-crystals, where $\tilde{H}(X/W, B)$ is the twisted K3 crystal attached to $(X, \alpha)$ in [Bra18]. As in the classical setting, the construction of $\tilde{H}(X/W, B)$ depends on a non-canonical choice of a $B$-field lift of $\alpha$, although the isomorphism class of the resulting crystal is independent of this choice. Furthermore, under this isomorphism, the inclusion $\text{TR}_0(X, \alpha)(-1) \to TP_{-2}(X, \alpha)$ is identified with the inclusion of the Tate module of $H(X/W, B)$, so that $K(X, \alpha)$ is identified with the characteristic subspace associated to $(X, \alpha)$ defined in [Bra18]. In particular, as the dimension of $K(X, \alpha)$ is determined by the $F$-crystal structure on $TP_{2i}(X, \alpha)$, we see that if $\alpha \neq 0$ then $TP_{2i}(X)$ is not isomorphic to $TP_{2i}(X, \alpha)$ as an $F$-crystal for any $i \leq -1$, in contrast to the finite height case. We remark that the derived Torelli theorem of [Bra18] states that $\mathcal{D}^b(X, \alpha)$ is determined by the $F$-crystal $\tilde{H}(X/W, B)$ together with its Mukai pairing.

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