Some features of oscillations and stability of reinforced cylindrical shells under action of movable inertial loading

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Abstract. This paper describes some features of the mathematical models for the elastic shell with movable load and for the elastic elements of changeable length. In these systems two forms of eigenoscillations - the eigen component and the accompanying one, displaced in phase to the right angle correspond to every frequency of the system. The accompanying component is caused by the mobile inertia load or by the changeable length and they are non-trivial when those factors are present. As well as for objects with time-varying length, these problems lie outside of the scope of the classical problems of mathematical physics due to that the eigenfrequencies and eigenforms will be presented as time-nondependent functions. This non-classical section of the mathematical physics is waiting for its development, new researches and generalizations. The main attention of this work is paid to the elastic systems with the mobile inertia load. The problem of oscillations and stability of thin cylindrical envelope with regular stringer set is being considered under assumption that they are regularly placed with a small step from one to another. Also, we assume that there is unsymmetrical stream of mobile inertia load. There is a wide variety of problems in area of jet engine, cooling systems, flying, and floating vehicle framework's design, which boil down to such computational schemes. The linear theory of structural-orthotropical envelopes with equilibrium equations and geometrical Donnel-Mushtary relations are being used to obtain of the discussed problem formulation. We assume that ribs of the shell are considered as Kirchhoff-Klebsh bars. The accurate solution of the problem is obtained by using of two-waves representation for motion and it has been applied for research on influence of geometry, elastic and structural parameters on the oscillation and stability properties of the shell.

1. Introduction and literature review

174 years have gone since the day of the first formulation of the problem of acting of movable loading on the elastic structure and building, after decay of Chester bridge in England in May 1847. During this time a lot of problems of dynamical impact of the movable loadings are different by their nature, behavior and influence on the elastic structures, systems, and buildings have been considered, solved and verified.

State-of-the-art of technology, increased road traffic intensity, intensification of flaw process require more accurate mechanical and mathematical models, that could reflect the essence of the phenomenon more comprehensively and precisely, so there is necessity of improvements and modifications for traditional modeling concepts or new concepts and methods of researches.

In the well-known review [10], dedicated to 100th anniversary of the problem formulation, the famous scientists in mechanical engineering Ya.G. Panovko wrote: “The problem of dynamic acting of movable load, 100th anniversary of which we have celebrated in 1947, by today did not lost their up-to-date status, the life still set new tasks and caused by this following motion of the theory to forward”.

In agile XX-XXI centuries significant increasing of masses and velocities of motion formulates a wide variety of new problems, and requires their solutions by developing new approaches in the
mechanical and mathematical modeling, renewing and improving of old methods to make allowable more comprehensive representation for quantitative and qualitative features of the kinematical and dynamical properties of the system motion.

Nowadays the keen interest to this problem arose due to the intensively usage of information technologies, that allows to research the mathematical model and to analyze their results more deeply and comprehensive. The traditional representation of mechanical systems under movable inertia loads has been changed significantly. The simple examples of those systems are bridges with moving vehicles, different kind of pipelines, thin-walled structures with additional strengthening bars, plates or envelopes loaded by moving liquid or gas.

As the problem of this class, we could treat the dynamical problems of the variable-length and time-dependent length objects, the dynamic of the objects under longitudinal motion like threads, wires, profiled rod in the rolling process, strips and chain saws, belts, cables of mining lifts and some others.

In dependency of the analytical model of inertia properties of the elastic structure and acting movable loading we could use the four variants of statement of the problem of influence of the movable loading on the elastic structures and buildings [7,11]. The most complex for applying in the practice is a fourth variant, that considers both inertia forces of the structure and inertia forces of the movable loading. Research on the qualitative and quantitative properties of the motion of such objects could be reduced to analysis of the following mathematical model

$$L(x,t, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}) \cdot w = L_1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) \cdot q(x,t),$$

where

$$q(x,t) = -\frac{q_0 + q_1}{g} \frac{\partial^2 w}{\partial t^2} - q_1 v \frac{\partial^2 w}{\partial t \partial x} - q_2 v^2 \frac{\partial^2 w}{\partial x^2}$$

with the respective boundary and initial conditions under assumption of constant motion velocity.

2. Mechanical, mathematical models and features
The main features of the mathematical models of this problem, at first, is the presence of the inertia operator $q(x,t)$ in some of its form in the differential equation. It is distinguishing detail, that the loading is dependent from the load intensity $q_1(x)$, the velocity $v$ of the loading stream, the elastic displacement $w(x,t)$.

Moreover, it is clearly, there is a strong dependency between loading intensity and the acceleration $w_{tt}(x,t)$, velocity of angular deformation $w_{tx}(x,t)$ and variation of curvature of the elastic line of the object $w_{xx}(x,t)$, that is in such systems the acting force is following after the system behavior and changing its value and direction during the deformation.

Thereby, acting force applied to the elastic object caused by movable mass is not predefined and depends on the state of the system. It is the second feature of the dynamic problem of the elastic system in the inertia force field of movable loadings. Third significant features of these problems is that the mathematical model should contain the mixed derivative of odd-order by time in one of its form that represents Coriolis acceleration of the moving inertial loading and does not allow to separate space and time variables by using the Fourier schema in the field of real functions.

An aerodynamic and hydrodynamic action of a liquid or a gas applied to the elastic object could be reduced to the same kind of the inertia operator. The velocity of the liquid stream in pipelines of the aircrafts is ranged between 50-80 m/s, and 200-250 m/s for gases and aircraft failures due to the instability of their pipelines attain 60% from the total number of failures [6,7].

The movable loading could be distributed uniformly or by some law, that could be discrete or continuously distributed with discrete inclusions with the constants or time-varying velocity.

It is known a number of applied mathematical researches, where successive approximation structure has been built.
At the beginning the rough approximation should be built, then the mechanical model and the respective mathematical model and choose the technology of analysis of the mathematical model to refine rough approximation and to get more accurate solution at the followed steps.

The quality of rough approximation has a critical sense to get more accurate solution quickly. The advantage of the rough approximate models and solutions is simplicity, transparency and evident.

2.1. Mechanical, mathematical models and research methods
As it is known, Fourier method of mathematical physics allows to get solutions of some class of partial differential equations in the explicit form \[4,5\]. Only in relatively simple cases it is possible to build up the solutions of partial differential equations as a sum of partial solutions in the form of product of the separated functions. To those equations belong the equations of oscillations of the string, a beam and some others. The direct applying of this method to the dynamic problem of elastic systems under movable inertia loading is not possible in general cases.

That is why some authors tried to use this method by its modifying and generalizing. One of the first publications was H. Steuding \[13\], where the lateral oscillations of the beam under movable distributed and concentrated loadings have been considered. The second one G.W. Housner \[12\] proved that the general solution of the partial differential equation of elastic oscillations under movable inertia loading could be obtained as a linear combination of partial solutions, those contain symmetric and antisymmetric forced forms shifted by 90 degrees in their phase. Moreover, antisymmetric forced forms occurred due to the mixed derivative of odd order by time and Coriolis’ inertia forces caused by movable loading and related through them to symmetrical forced forms. The symmetrical forced forms under non-movable loading are matched to the eigenforms of the loaded system. Two above works began the method of two-wave representation of the elastic system oscillations under movable inertia loading and its physics interpretation had been provided by O. A. Goroshko \[4-6\].

Using the method of two-wave representation of the oscillations for research of those system, that allows in some cases to obtain analytical solutions, the general solution of the differential equations could be found as a sum of two infinity series the first series is a classical part of the solution and the second one is the part of the solution caused by presence of odd-order by time mixed derivative and inertia of movable loading, that could not be discovered by using of traditional direct methods of mathematical physics. The forms of the first group are called as eigenforms and the forms of the second group known as accompanying oscillation forms of the elastic system. Accompanying oscillations could be non-trivial if the elastic system is loaded by movable inertia loading. The modes of the first group called eigenmodes, and the modes from the second one are called as accompanying modes of the elastic system oscillations. Accompanying modes are induced and non-trivial when the movable inertia loading is present.

Today more penetrating and thorough research on the dynamic problem of elastic system under movable inertia loading by the method of two-wave representation is supported by modern information technologies, that was never used before, especially in the days of H. Steuding, G. W. Housner, Ya.G. Panovko and others.

2.2. A simple case of analogue between mathematical models of the dynamic problem of elastic objects under movable loading and the static problem
As for analogues of mathematical models, it is easy to see, that the problem of lateral oscillation of the beam under uniformly distributed inertia loading in the critical mode could be reduced to the problem of solving of the differential equation \[7\]

\[ E \cdot J_{min} \cdot w''(x) = -m \cdot V^2_{cr} \cdot w''(x) , \]

with respective boundary and initial conditions, and, as it is known, could be reduced to the solving of the following differential equation
\[ E \cdot J_{\text{min}} \cdot w^{IV}(x) = -F_{\text{cr}} \cdot w^{II}(x). \]  

(4)

Analysis of these equations shows that the mathematical models of these problems are identical, that is to say some mathematical analogy exists and by using of this analogy we will get the approximate values of the critical speed of motion of the loading when the pinned beam will have buckling failure

\[ F_{\text{cr}} = \frac{\pi^2 \cdot E \cdot J_{\text{min}}}{(v \cdot l)} = m \cdot V_{\text{cr}}^2 \quad \text{or} \quad V_{\text{cr}} = \frac{\pi}{\sqrt{v l \cdot m}} \cdot \sqrt{E \cdot J_{\text{min}}}. \]  

(5)

In the formulas from (3) – (5):

- \( E \) – Young’s modulus of longitudinal elasticity of the beam material;
- \( w(x) \) - deflection of arbitrary cross-section of beam;
- \( m \) - mass of unit of beam length;
- \( J_{\text{min}} \) - the axial moment of inertia of the cross-section;
- \( F_{\text{cr}} \) is the critical value of the compression force by Euler;
- \( V_{\text{cr}} \) - the critical speed of the movable loading;
- \( v \) - coefficient of the effective length of the beam, which depends on the conditions of fixation of cross-sections.

3. Differential equation of casing radial oscillations and its solution

The problem of eigenoscillations and stability, strengthened in longitudinal direction by the regular stringer set of cylindrical shell with radius \( R \), thickness \( h \) and length \( L \), is considered. The stream of mass with constant intensity \( q_1 \) moves along stringers with a constant velocity \( V \). The stringers having the same geometry, placed with a small step between them and unsymmetrically to middle surface of the shell. Long measures theory of the shell with equations of balance and geometrical correlations in the shape of Donnel-Mushtary are used during compiling the equations of moving. The stringer set could be modelled by using of Kirchhoff – Klebsh approach. In this case the stiffness of stringers should be connected to the elasticity properties of the shell. If we will neglect tangential components of inertial forces, then strength factors which reduce to the middle surface of strengthened casing, will presented in the following form [2,9]

\[
N_1 = B(\varepsilon_1 + \mu \varepsilon_2) + \left(\frac{EF}{l}\right)\varepsilon_1 - \left(\frac{ES}{l}\right)\kappa_1, \\
N_2 = B(\varepsilon_2 + \mu \varepsilon_1), \quad N_{12} = 0,5(1 - \mu)B\varepsilon_{12}, \label{equation6}
\]

(6)

\[
M_1 = D(\kappa_1 + \mu \kappa_2) + \left(\frac{EI}{l}\right)\kappa_1 - \left(\frac{ES}{l}\right)\varepsilon_1, \\
M_2 = D(\kappa_2 + \mu \kappa_1), \quad M_{12} = \left[(1 - \mu)D + \frac{Gl}{2l}\right]\kappa_{12}. \label{equation7}
\]

(7)

In this wording our task reduces to decision the equations system of designed casing relatively the function of crook bend \( W \) and the function of effort \( \Phi \) [9]

\[
L_1(W) = \frac{1}{R} \frac{\partial^2 \Phi}{\partial x^2} - \frac{a_2 S}{h l} \left( \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} - \frac{\mu}{\partial x^2} \cdot \frac{\partial^3 \Phi}{\partial x^2} \right) + \frac{Z}{h}, \label{equation8}
\]

(8)

\[
L_2(\Phi) = -\frac{1}{R} \frac{\partial^3 W}{\partial x^2} - \frac{a_2 S}{h l} \left( \frac{\mu}{\partial x^2} \cdot \frac{\partial^3 W}{\partial x^2 \partial y^2} - \frac{\partial^3 W}{\partial x^2} \right), \label{equation9}
\]

(9)

where \( L_1 \) and \( L_2 \) have following form
\[
L_1 = \frac{D}{h} \left[ b_1 \frac{\partial^4}{\partial x^4} + 2 \cdot b_2 \frac{\partial^4}{\partial x^2 \partial y^2} + b_3 \frac{\partial^4}{\partial y^4} \right], \\
L_2 = \frac{1}{E} \left[ a_1 \frac{\partial^4}{\partial x^4} + 2 \cdot a_2 \frac{\partial^4}{\partial x^2 \partial y^2} + a_3 \frac{\partial^4}{\partial y^4} \right].
\] (10)

and

\[
a_1 = \left[ 1 + (1 - \mu^2) \cdot F/\rho l \right] \cdot a_z; \quad a_2 = 1 + \mu(1 - a_z); \quad a_3 = \frac{1}{(1 + F/\rho l)};
\]

\[
b_1 = 1 + 12 \left( 1 - \mu^2 \right) \cdot \left[ J/h^3 l - (S/h^2 l)^2 \cdot a^4 \right]; \quad b_2 = 1 + 3 \left( 1 - \mu \right) I_p / h^3 l; \quad b_3 = 1.
\]

By applying operators to the equation (8), using that inertial forces of casing, ribs and mass streams, and distributing Z along the shell, equation of the motion for functions \( W \) and \( \Phi \) could be written as below

\[
L_2 L_1 (W) = \frac{1}{R} \frac{\partial^2}{\partial x^2} \cdot L_2 (\Phi) - \frac{a_3 S}{h l} \left[ \frac{\partial^4}{\partial x^2 \partial y^2} - \mu \cdot \frac{\partial^4}{\partial x^4} \right] \cdot L_2 (\Phi) - \frac{1}{h} L_2 L_3 (W),
\] (11)

where operator \( L_3 \) has got a shape

\[
L_3 (W) = \frac{q}{g} \frac{\partial^2 W}{\partial t^2} + \frac{q_1}{g} \left( \frac{\partial^2 W}{\partial t^2} + 2 \cdot V \frac{\partial^2 W}{\partial t \partial x} + V^2 \cdot \frac{\partial^3 W}{\partial x^2} \right),
\] (12)

and \( L_2 (\Phi) \) will be used as a right part of equation (9), \( q, q_1 - \) the weight of shell and mass streams, \( F, J, S - \) the area, moment of inertia and statical moment of stringer’s cross-section area relatively middle surface of panelling, \( I_p - \) moment of inertia rib's cross-section for torsion, \( l - \) the rib's step.

4. Results of solutions

Partial solution of equation (11) will have following form

\[
W(x, y, t) = \varphi(x, y) \cdot \cos \omega t + \psi(x, y) \cdot \sin \omega t.
\] (13)

After the substitution expression (13) into equation (11) and after introducing of complex function of real variables

\[
\Phi_1(x, y) = \varphi(x, y) + i \psi(x, y),
\] (14)

will come to the partial differential equations for function \( \Phi_1(x, y) \) that does not have derivatives by time. Satisfying to the arrangement of recurring by the circular coordinate, function \( \Phi_1(x, y) \) could be presented in the following form

\[
\Phi_1(x, y) = \sum_m F_m(x) \cos \left( \frac{m \cdot y}{R} \right),
\] (15)

and after substitution (15) into equations for function \( \Phi_1(x, y) \) will come to the usual differential equation of eighth order for function \( F_m(x) \). The general solution under assumption that the roots of characteristic equation are simple, \( F_m(x) \) could be written as below [3,8]

\[
F_m(x) = \sum_{i=1}^{4} C_i \cdot e^{\lambda x},
\] (16)
where \( C_i \) — arbitrary constants, \( k_i \) - the roots of characteristic equation eighth degree with complex coefficients which have a shape [3,4]

\[
C_i k^8 + C_i k^7 + C_i k^6 + iC_i k^5 + C_i k^4 + iC_i k^3 + C_i k^2 + iC_i k + C_i = 0.
\] (17)

Using standard programs to define the roots of equation, for long or middle length shells the roots of equation (17) could be divide on two categories, that satisfy following conditions

\[
|k_1| \geq |k_2| \geq |k_3| \geq |k_4| \gg |k_5| \geq |k_6| \geq |k_7| \geq |k_8|
\] (18)

At the table there are significances of characteristic equation's (17) roots for casing with means \( L=50 \) cm, \( R=15 \) cm, \( h=0.05 \) cm, \( d_1=1 \) cm, \( d_2=0.98 \) cm, the ribs quantity in the form of tubes with external and inside diameters \( d_1 \) and \( d_2 \), \( n=15 \).

**Table 1.** Roots of the characteristic equation (17) of eighth degree with complex coefficients

| V=83 m/s | \( \omega_{k_5} = 863 \) l/s | \( k_1 \) | \( k_2 \) | \( k_3 \) | \( k_4 \) | \( k_5 \) | \( k_6 \) | \( k_7 \) | \( k_8 \) |
|---|---|---|---|---|---|---|---|---|---|
| Re \( k \) | -24.50 | 24.50 | -24.60 | 24.60 | -3.64 | 3.64 | 0 | 0 |
| Im \( k \) | -19.34 | -19.34 | 19.35 | 19.35 | 1.01 | 1.01 | -4.15 | 1.72 |

With considerations (18) according to the method Lobachevskogo-Greffe, looking for all roots of equation (17) reduce to the decision of two equations of fourth order

\[
C_0 \cdot k^4 + C_1 \cdot k^3 + C_2 \cdot k^2 + i \cdot C_3 \cdot k + C_4 = 0,
\] (19)

\[
C_4 \cdot k^4 + i \cdot C_3 \cdot k^3 + C_6 \cdot k^2 + i \cdot C_7 \cdot k + C_8 = 0,
\] (20)

After evaluation of roots of equations (19) and (20), we may set for elementary approach of roots (17), which in future we may define more precisely by one of iteration methods with any required accuracy.

After determining of the characteristic equation's (17) roots and satisfying the function (16) to the boundary conditions of pinned shell ends for \( F_m(x) \) in the form

\[
F(x) = F_{II}(x) = F_{IV}(x) = F_{VI}(x) = 0 \quad \text{for} \quad x=0.1,
\]

will have the equation

\[
\Delta = \begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
k_1^2 & k_2^2 & k_3^2 & k_4^2 & k_5^2 & k_6^2 & k_7^2 & k_8^2 \\
k_1^4 & k_2^4 & k_3^4 & k_4^4 & k_5^4 & k_6^4 & k_7^4 & k_8^4 \\
k_1^6 & k_2^6 & k_3^6 & k_4^6 & k_5^6 & k_6^6 & k_7^6 & k_8^6 \\
e^{k_1} & e^{k_2} & e^{k_3} & e^{k_4} & e^{k_5} & e^{k_6} & e^{k_7} & e^{k_8} \\
e^{k_1^2} & e^{k_2^2} & e^{k_3^2} & e^{k_4^2} & e^{k_5^2} & e^{k_6^2} & e^{k_7^2} & e^{k_8^2} \\
e^{k_1^4} & e^{k_2^4} & e^{k_3^4} & e^{k_4^4} & e^{k_5^4} & e^{k_6^4} & e^{k_7^4} & e^{k_8^4} \\
e^{k_1^6} & e^{k_2^6} & e^{k_3^6} & e^{k_4^6} & e^{k_5^6} & e^{k_6^6} & e^{k_7^6} & e^{k_8^6} \\
e^{k_1^8} & e^{k_2^8} & e^{k_3^8} & e^{k_4^8} & e^{k_5^8} & e^{k_6^8} & e^{k_7^8} & e^{k_8^8}
\end{vmatrix} = 0,
\] (21)

from equality to "0" (21), which determine the frequencies \( \omega_{nm} \) of shell oscillations.

Finally, the general solution could be written in the following form

\[
W(x,y,t) = \sum_{nm} a_{nm} \left( \text{Re}[\Phi_t(x,y)] \cdot \cos(\omega_{nm,t} + \alpha_{nm}) + \text{Im}[\Phi_t(x,y)] \cdot \sin(\omega_{nm,t} + \alpha_{nm}) \right).
\] (22)

The analysis of the shell oscillations and stability, conducted by means of given method for casings with means \( L = 50 \) cm, \( L/R = 3,4,5,6 \), \( L/h = 2500 \), \( n = 8,10,15,20 \), where \( n \) is number of stringers, for
tubes with inside diameter \(d_1=0.98\) cm and external diameter \(d_2=1\) cm as stringers, which have shown that frequencies and critical speeds have got lesser meanings with maximal number of district waves \(\text{mn}\). Besides, if a rib's number is increase, frequencies of oscillations slightly increase, but at that time critical speeds slow down.

5. Discussion of results obtained in the study

The obtained exact solution of the problem shows that the vibrations of the reinforced cylindrical shell in the field of the moving inertia loads could be presented as a superposition of two groups of oscillations. The oscillations of those groups will incorporate oscillations with the frequencies \(\omega_{\text{mn}}\), but oscillations of second group \(\varphi_{\text{mn}}(x, y)\), known as accompanying oscillations, are shifted in phase by a right angle relatively to the respective oscillations from the first group \(\psi_{\text{mn}}(x, y)\), which is known as natural or eigenoscillations.

\[
\varphi_{\text{mn}}(x, y) = \text{Re}\left[\Phi_1(x, y)\right], \quad \psi_{\text{mn}}(x, y) = \text{Im}\left[\Phi_1(x, y)\right].
\]

Note, that the accompanying vibrations are caused by a moving inertial load and are non-trivial only if the inertial loading is present. The natural forms of natural oscillations in the absence of a moving inertial loads pass into the eigenoscillations modes of the shell.

6. Conclusions

An analysis of the dependence of the main lowest oscillation frequency on the flow velocity showed that the Coriolis forces of inertia of the moving load significantly affect the fundamental vibration frequency of the shell. Already at average flow velocities, the value of the fundamental oscillation frequency, determined considering of the Coriolis inertia forces, is 20–25% lower than the value obtained without taking them into account.

An analysis of the forms of the intrinsic and accompanying vibrations of the shell shows their significant dependence on the velocity of the flows, the ratio of the moving and stationary masses of the system, and, at average speeds of movement of the load, the main form of intrinsic vibrations passes from the nodal to the nodal, the node of which moves with increasing speed. The role of the accompanying oscillations increases significantly with increasing speed and the ratio of the moving and stationary masses of the system.

If this is neglected, then a fairly distorted picture of the shell motion can be obtained not only from the quantitative, but also from the qualitative side.

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