Multiplicity results of fractional-Laplace system with sign-changing and singular nonlinearity

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Abstract

In this article, we study the following fractional-Laplacian system with singular nonlinearity

\[
\begin{aligned}
(P_{\lambda,\mu}) \quad 
& (-\Delta)^s u = \lambda f(x)u^{-q} + \frac{\alpha}{\alpha + \beta} b(x)u^{\alpha-1}w^\beta \quad \text{in } \Omega \\
& (-\Delta)^s w = \mu g(x)w^{-q} + \frac{\beta}{\alpha + \beta} b(x)u^\alpha w^{\alpha - 1} \quad \text{in } \Omega \\
& u, w > 0 \text{ in } \Omega, \quad u = w = 0 \text{ in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( n > 2s \), \( s \in (0, 1) \), \( 0 < q < 1 \), \( \alpha > 1 \), \( \beta > 1 \) satisfy \( 2 < \alpha + \beta < 2^*_s - 1 \) with \( 2^*_s = \frac{2n}{n - 2s} \), the pair of parameters \((\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\} \). The weight functions \( f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R} \) such that \( 0 < f, g \in L^{\frac{n}{n-2s}}(\Omega) \), and \( b : \Omega \subset \mathbb{R}^n \to \mathbb{R} \) is a sign-changing function such that \( b(x) \in L^\infty(\Omega) \). Using variational methods, we show existence and multiplicity of positive solutions of \((P_{\lambda,\mu})\) with respect to the pair of parameters \((\lambda, \mu)\).

Key words: Fractional Laplacian system, singular nonlinearity, sign-changing weight function, Variational methods.

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1 Introduction

Let $s \in (0, 1)$ and let $0 \in \Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $n > 2s$. Then we consider the following fractional system with singular nonlinearity:

$$
(P_{\lambda,\mu}) \left\{ \begin{array}{l}
(-\Delta)^s u = \lambda f(x)u^{-q} + \frac{\alpha}{\alpha+\beta} b(x) u^{\alpha-1} w^\beta \quad \text{in } \Omega \\
(-\Delta)^s w = \mu g(x)w^{-q} + \frac{\beta}{\alpha+\beta} b(x) u^\alpha w^{\beta-1} \quad \text{in } \Omega \\
u, w > 0 \quad \text{in } \Omega,
\end{array} \right.
$$

Here, $(-\Delta)^s$ is the fractional Laplacian operator defined as

$$
(-\Delta)^s u(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy \quad \text{for all } x \in \mathbb{R}^n.
$$

We assume the following assumptions on $f$ and $g$:

(a1) $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $0 < f, g \in L^{q^*} (\Omega)$, where $q^* = \frac{\alpha+\beta}{\alpha+\beta-1+q}$.

(b1) $b : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a sign-changing function such that $b^+ = \max\{f, 0\} \neq 0$ and $b(x) \in L^{\infty}(\Omega)$.

Also the pair of parameters $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $0 < q < 1$ and $\alpha > 1$, $\beta > 1$ satisfy $2 < \alpha + \beta < 2_*^s - 1$, with $2_*^s = \frac{2n}{n-2s}$, known as fractional critical Sobolev exponent.

In this work, we prove the existence of multiple non-negative solutions for a system of fractional operator with singular and sign changing nonlinearity by studying the nature of Nehari manifold with respect to the parameter $\lambda$ and $\mu$. These same result can be easily extended to $p$-fractional Laplacian operator $(-\Delta)_p^s$, defined as

$$
(-\Delta)_p^s = 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x - y|^{n+ps}} dy.
$$

This definition is consistent, up to a normalization constant depending on $n$, $s$, with linear Laplacian fractional $(-\Delta)^s$, for the case $p = 2$.

The natural space to look for solutions of the problem $(P_{\lambda,\mu})$ is the product space $W^{s,p}_0(\Omega) \times W^{s,p}_0(\Omega)$. In order to study $(P_{\lambda,\mu})$, it is important to encode the ‘boundary condition’ $u = v = 0$ in $\mathbb{R}^n \setminus \Omega$ in the weak formulation. Servadei and Valdinoci in [36] have introduced the new function spaces to study the variational functionals related to the fractional Laplacian by observing the interaction between $\Omega$ and $\mathbb{R}^n \setminus \Omega$.

For $u = v$, $\alpha = \beta$, $\alpha + \beta = r$, $\lambda = \mu$ and $f = g$, the problem $(P_{\lambda,\mu})$ reduces to the following fractional equation with singular nonlinearities

$$
(P_\lambda) \left\{ \begin{array}{l}
(-\Delta)_p^s = f(x)w^{-q} + \lambda b(x) w^r \quad \text{in } \Omega, \\
w > 0 \quad \text{in } \Omega, \\
w = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{array} \right.
$$

In [24], the author studied the existence and multiplicity of non-negative solutions to problem $(P_\lambda)$ for sign changing and singular nonlinearity. In the scalar case the problems involving
the fractional operator with singular nonlinearity have been studied by many authors, see \[40\] and references therein.

The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusions in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. For more details, one can see \[3, 19\] and reference therein. Recently the fractional elliptic equation attracts a lot of interest in nonlinear analysis such as in \[7, 36, 37, 38, 39\]. Caffarelli and Silvestre \[7\] gave a new formulation of fractional Laplacian through Dirichlet-Neumann maps. This is commonly used in the literature since it allows us to write a nonlocal problem to a local problem which allow us to use the variational methods to study the existence and uniqueness.

On the other hand, the fractional elliptic problem have been investigated by many authors, for example, \[36, 37\] for subcritical case, \[38, 39\] for critical case with polynomial type nonlinearities. Moreover, by Nehari manifold and fibering maps, the author obtained the existence of multiple solutions for fractional equations for critical \[42\] and subcritical case \[25, 26\] and reference therein. In case of square root of Laplacian, existence and multiplicity results for sublinear and superlinear type of nonlinearity with sign-changing weight functions is studied in \[41\]. In \[41\], author used the idea of Caffarelli and Silvestre \[7\], which gives a formulation of the fractional Laplacian through Dirichlet-Neumann maps. Also in case of fractional \(p\)-Laplacian, existence and multiplicity results for polynomial type nonlinearities is studied by many authors see \[25, 26, 29, 30, 34\] and reference therein. Also eigenvalue problem related to \(p\)-fractional Laplacian is studied in \[17, 33\].

For \(s = 1\), the paper by Crandall, Robinowitz and Tartar \[10\] is the starting point on semi-linear problem with singular nonlinearity. There is a large literature on singular nonlinearity see \[1, 2, 10, 11, 12, 13, 15, 16, 20, 27, 28, 31, 32, 21, 22, 23\] and reference therein. In \[9\], Chen showed the existence and multiplicity of the following problem

\[
\begin{aligned}
-\Delta w - \frac{\lambda}{|x|^2} w &= \frac{f(x)}{w^p} + \mu g(x)w^p \quad \text{in } \Omega \setminus \{0\} \\
w &> 0 \text{ in } \Omega \setminus \{0\}, \quad w = 0 \text{ in } \partial \Omega,
\end{aligned}
\]

where \(0 \in \Omega\) is a bounded smooth domain of \(\mathbb{R}^n\) with smooth boundary, \(0 < \lambda < \frac{(n-2)^2}{4}\), \(0 < q < 1 < p < \frac{n+2}{n-2}\), \(f(x) > 0\) and \(g\) is sign-changing continuous function.

To the best of our knowledge, there is no work related to system of fractional Laplacian with singular and sign-changing nonlinearity. In this work, we studied the multiplicity results for the system of fractional Laplacian equation with singular nonlinearity and sign-changing weight function with respect to the parameter \(\lambda, \mu\). This work is motivated by the work of Chen and Chen in \[2\]. But one can not directly extend all the results for fractional \(p\)-Laplacian, due to the non-local behavior of the operator and the bounded support of the test function is not preserved. Also due to the singularity of the problem, the associated functional is not differentiable in the sense of Gâteaux. The results obtained here are somehow expected but we show how the results arise out of nature of the Nehari manifold.
We will use the following notation throughout this paper: \( \| \cdot \|_q^* \), \( \| g \|_q^* \) denote the norm in \( L^{\alpha+\beta-1+q} (\Omega) \).

### 2 Preliminaries:

In this section we give some definitions and functional settings. At the end of this section, we state our main results. For this we define \( H^s(\Omega) \), the usual fractional Sobolev space \( H^s(\Omega) := \left\{ w \in L^2(\Omega); \frac{(w(x)-w(y))}{|x-y|^s} \in L^2(\Omega \times \Omega) \right\} \) endowed with the norm

\[
\| w \|_{H^s(\Omega)} = \| w \|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]  

(2.1)

To study fractional Sobolev space in details we refer \[35\].

Due to the non-localness of the operator, we define linear space as follows:

\[ X_0 = \left\{ w | w : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, w|_{\Omega} \in L^p(\Omega) \text{ and } \frac{w(x) - w(y)}{|x-y|^{2s}} \in L^2(Q); w = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \right\} \]

where \( Q = \mathbb{R}^{2n} \setminus (C\Omega \times C\Omega) \) and \( C\Omega := \mathbb{R}^n \setminus \Omega \). The space \( X_0 \) was firstly introduced by Servadei and Valdinoci \[36\]. The space \( X_0 \) endowed with the norm

\[
\| w \| = \left( \int_{Q} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{\frac{1}{2}}
\]  

(2.2)

is a Hilbert space. We notice that, the norms in (2.1) and (2.2) are not same because \( \Omega \times \Omega \) is strictly contained in \( Q \). Let \( Y = X_0 \times X_0 \) be the cartesian product of two reflexive Banach spaces, which is also reflexive Banach space with the norm

\[
\| (u, w) \| = (\| u \|_{X_0}^2 + \| w \|_{X_0}^2)^{\frac{1}{2}} = \left( \int_{Q} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy + \int_{Q} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

Now we define the space

\[
C_Y := \{ (u, w) : u, w \in C^\infty_c(\mathbb{R}^n); u = w = 0 \text{ in } \mathbb{R}^n \setminus \Omega \}.
\]

Then \( C_Y \) is a dense in the space \( Y \).

Denote \( S := \inf_{u \in X_0} \left\{ \frac{\int_{Q} |u(x) - u(y)|^2 |x-y|^{-(n+2s)} \, dx \, dy}{\int_{Q} |u|^{q|a|+\beta \delta} \, dx} \right\} \), \( \overline{S} := \inf_{u \in Y} \left\{ \frac{\| (u, w) \|}{\int_{Q} |u|^{q|a|+\beta \delta} \, dx} \right\} \) and

\[
K_{\lambda, \mu} = \lambda \int_{\Omega} f(x)(u_+)^{1-q} \, dx + \mu \int_{\Omega} g(x)(w_+)^{1-q} \, dx.
\]
Definition 2.1 A weak solution of the problem \((P_{\lambda,\mu})\) is a function \((u, w) \in Y, u, w > 0\) in \(\Omega\) such that for every \((\phi, \psi) \in Y\)
\[
\int_Q \frac{((u(x) - u(y))(\phi(x) - \phi(y)))}{|x - y|^{n+2s}} dxdy + \frac{1}{|x - y|^{n+2s}} \int_Q ((w(x) - w(y))(\psi(x) - \psi(y))) dxdy = \lambda \int_\Omega f(x)(u^{-q}w^q) dx + \mu \int_\Omega g(x)(w^{-q}w^q) dx + \frac{\alpha}{\alpha + \beta} \int_\Omega b(x)(u^\alpha w^\beta \phi(x) dx + \frac{\beta}{\alpha + \beta} \int_\Omega b(x)(u^\alpha w^\beta \psi(x) dx.
\]
In order to present the existence of positive solution of \((P_{\lambda,\mu})\), we will consider the following problem
\[
(P^+_{\lambda,\mu}) \begin{cases}
(-\Delta)^s u = \lambda f(x)u^{-q} + \frac{\alpha}{\alpha + \beta} b(x)u^\alpha - w^\beta \phi(x) & \text{in } \Omega, u, w > 0 \text{ in } \Omega, u = w = 0 \text{ in } \mathbb{R}^n \setminus \Omega,
(-\Delta)^s w = \mu g(x)w^{-q} + \frac{\beta}{\alpha + \beta} b(x)u^\alpha w^\beta - 1 \psi(x) & \text{in } \Omega
\end{cases}
\]
where \(w_+ := \max\{w, 0\}\), denote the positive part of \(w\). Then the function \((u, w) \in Y, u, w > 0\) in \(\Omega \times \Omega\) is a weak solution of the problem \((P^+_{\lambda,\mu})\) if for every \((\phi, \psi) \in Y\)
\[
\int_Q \frac{((u(x) - u(y))(\phi(x) - \phi(y)))}{|x - y|^{n+2s}} dxdy + \frac{1}{|x - y|^{n+2s}} \int_Q ((w(x) - w(y))(\psi(x) - \psi(y))) dxdy = \lambda \int_\Omega f(x)(u^{-q}w^q) dx + \mu \int_\Omega g(x)(w^{-q}w^q) dx + \frac{\alpha}{\alpha + \beta} \int_\Omega b(x)(u^\alpha w^\beta \phi(x) dx + \frac{\beta}{\alpha + \beta} \int_\Omega b(x)(u^\alpha w^\beta \psi(x) dx.
\]
We note that if \((u, w) > 0\) is a solution of \((P^+_{\lambda,\mu})\) then one can easily see that \((u, w)\) is also a solution \((P_{\lambda,\mu})\). To find the solution of \((P^+_{\lambda,\mu})\), we will use variational approach. So we define the associated functional \(J_{\lambda,\mu} : Y \to [-\infty, \infty)\) as
\[
J_{\lambda,\mu}(u, w) = \frac{1}{2} \left\| (u, w) \right\|^2 - \frac{1}{1 - q} \int_\Omega \left( f(x)u^{-q} + \mu g(x)w^{-q} \right) dx - \frac{1}{\alpha + \beta} \int_\Omega b(x)u^\alpha w^\beta dx.
\]
Here \(J_{\lambda,\mu}\) is not bounded below on \(Y\) but is bounded below on appropriate subset \(N_{\lambda,\mu}\) of \(Y\).
Therefore in order to obtain the existence results, we introduce the Nehari manifold
\[
N_{\lambda,\mu} = \left\{ (u, w) \in Y : \langle J'_{\lambda,\mu}(u, w), (u, w) \rangle = 0 \right\} = \left\{ (u, w) \in Y : \phi^*_w (1) = 0 \right\}
\]
where \(\langle , \rangle\) denotes the duality between \(Y\) and its dual space. Thus \((u, w) \in N_{\lambda,\mu}\) if and only if
\[
\left\| (u, w) \right\|^2 - \left( \lambda \int_\Omega f(x)u^{-q} dx + \mu \int_\Omega g(x)w^{-q} dx \right) - \int_\Omega b(x)u^\alpha w^\beta dx = 0 \tag{2.3}
\]
We note that \(N_{\lambda,\mu}\) contains every solution of \((P_{\lambda,\mu})\). Now as we know that the Nehari manifold is closely related to the behavior of the functions \(\phi_{u, w} : \mathbb{R}^+ \to \mathbb{R}\) defined as \(\phi_{u, w}(t) = J_{\lambda,\mu}(tu, tw)\). Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [\ref{14}]. For \((u, w) \in Y\), we have
\[
\phi_{u, w}(t) = \frac{t^2}{2} \left\| (u, w) \right\|^2 - \frac{1}{1 - q} t^{-q} K_{\lambda,\mu}(u, w) - \frac{2t^\alpha}{1 - q} \int_\Omega b(x)u^\alpha w^\beta dx,
\]
\[
\phi'_{u, w}(t) = t \left\| (u, w) \right\|^2 - t^{-q} K_{\lambda,\mu}(u, w) - t^\alpha - 1 \int_\Omega b(x)u^\alpha w^\beta dx,
\]
\[
\phi''_{u, w}(t) = \left\| (u, w) \right\|^2 + qt^{-q} K_{\lambda,\mu}(u, w) - (q + 1)t^\alpha - 2 \int_\Omega b(x)u^\alpha w^\beta dx.
\]
Then it is easy to see that \((u, w) \in \mathcal{N}_{\lambda, \mu}\) if and only if \(\phi_{u,w}'(t) = 0\) and in particular, \(u \in \mathcal{N}_{\lambda, \mu}\) if and only if \(\phi_{u,w}'(1) = 0\). Thus it is natural to split \(\mathcal{N}_{\lambda, \mu}\) into three parts corresponding to local minima, local maxima and points of inflection. For this we set

\[
\mathcal{N}_{\lambda, \mu}^+ := \{(u, w) \in \mathcal{N}_{\lambda, \mu} : \phi_{u,w}''(1) \geq 0\} = \{(tu, tw) \in Y : \phi_{u,w}'(t) = 0, \phi_{u,w}''(t) \geq 0\},
\]

\[
\mathcal{N}_{\lambda, \mu}^- := \{(u, w) \in \mathcal{N}_{\lambda, \mu} : \phi_{u,w}''(1) = 0\} = \{(tu, tw) \in Y : \phi_{u,w}'(t) = 0, \phi_{u,w}''(t) = 0\}.
\]

We also observe that if \((u, w) \in \mathcal{N}_{\lambda, \mu}\) then

\[
\phi_{u,w}''(1) = \begin{cases} 
(1 + q)\|(u, w)\|^2 - (\alpha + \beta - 1 + q) \int_{\Omega} b(x)u_+^\alpha w_+^\beta \, dx \\
(2 - \alpha - \beta)\|(u, w)\|^2 + (\alpha + \beta - 1 + q)K_{\lambda, \mu}(u, w).
\end{cases}
\]

Inspired by [9], we show that how variational methods can be used to established some existence and multiplicity results for \((P_{\lambda, \mu}^-)\). Our results are as follows:

**Theorem 2.2.** Suppose that \(\lambda \in (0, \Lambda)\), where

\[
\Lambda := \frac{(1 + q)}{(\alpha + \beta - 1 + q)} \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right)^{\frac{\alpha + \beta - 2}{1 + q}} \frac{1}{\|b\|} \left( \frac{S^{\alpha + \beta - 1 + q}}{\|a\|^{|a + \beta - 2|}} \right)^{\frac{1}{1 + q}}.
\]

Then the problem \((P_{\lambda, \mu})\) has at least two solutions

\[
(u, w) \in \mathcal{N}_{\lambda, \mu}^+, (U, W) \in \mathcal{N}_{\lambda, \mu}^- \quad \text{with} \quad \|(U, W)\| > \|(u, w)\|.
\]

### 3 Fibering map analysis

In this section, we show that \(\mathcal{N}_{\lambda, \mu}^\pm\) is nonempty and \(\mathcal{N}_{\lambda, \mu}^0 = \{(0, 0)\}\). Moreover, \(J_{\lambda, \mu}\) is bounded below and coercive. Define

\[
\Gamma := \left\{ (\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\} : 0 < \Lambda := (|\lambda||f|_{q'})^{\frac{1}{q'}} + (|\mu||g|_{q'})^{\frac{1}{q'}} < C(n, \alpha, \beta, q, S) \right\},
\]

where,

\[
C(n, \alpha, \beta, q, S) = \left( \frac{(1 + q)}{(\alpha + \beta - 1 + q)} \right)^{\frac{\alpha + \beta - 2}{1 + q}} \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right)^{\frac{1}{1 + q}} \left( \frac{1}{\|b\|_\infty} \right)^{\frac{\alpha + \beta - 2}{1 + q}} S^{\frac{2(\alpha + \beta - 1 + q)}{(n + q)(\alpha + \beta - 2)}}.
\]

**Lemma 3.1.** Let \((\lambda, \mu) \in \Gamma\). Then for each \((u, w) \in Y\) with \(K_{\lambda, \mu}(u, w) > 0\), we have the following:

(i) \(\int_{\Omega} b(x)u_+^\alpha w_+^\beta \, dx \leq 0\), then there exists a unique \(0 < t_1 < t_{\max}\) such that \((t_1 u, t_1 w) \in \mathcal{N}_{\lambda, \mu}^+\) and \(J_{\lambda, \mu}(t_1 u, t_1 w) = \inf_{t > 0} J_{\lambda, \mu}(tu, tw)\),

(ii) \(\int_{\Omega} b(x)u_+^\alpha w_+^\beta \, dx > 0\), then there exists a unique \(t_1\) and \(t_2\) with \(0 < t_1 < t_{\max} < t_2\) such that \((t_1 u, t_1 w) \in \mathcal{N}_{\lambda, \mu}^+, (t_2 u, t_2 w) \in \mathcal{N}_{\lambda, \mu}^-\) and \(J_{\lambda, \mu}(t_1 u, t_1 w) = \inf_{0 \leq t \leq t_{\max}} J_{\lambda, \mu}(tu, tw)\), \(J_{\lambda, \mu}(t_2 u, t_2 w) = \sup_{t \geq t_1} J_{\lambda, \mu}(tu, tw)\).
Proof. For $t > 0$, we define
\[
\psi_{u,w}(t) = t^{2 - \alpha - \beta} \| (u,w) \|_{\Omega}^2 - t^{-\alpha - \beta + 1 - q} K_{\lambda,\mu}(u,w) - \int_{\Omega} b(x) u^\alpha w^\beta dx.
\]
One can easily see that $\psi_{u,w}(t) \to -\infty$ as $t \to 0^+$. Now
\[
\begin{aligned}
\psi_{u,w}'(t) &= (2 - \alpha - \beta) t^{1 - \alpha - \beta} \| (u,w) \|_{\Omega}^2 + (\alpha + \beta - 1 + q) t^{-\alpha - \beta - q} K_{\lambda,\mu}(u,w), \\
\psi_{u,w}''(t) &= (2 - \alpha - \beta)(1 - \alpha - \beta) t^{-\alpha - \beta} \| (u,w) \|_{\Omega}^2 - (\alpha + \beta - 1 + q) (\alpha + \beta + q) t^{-\alpha - \beta - q - 1} K_{\lambda,\mu}(u,w).
\end{aligned}
\]
Then $\psi_{u,w}'(t) = 0$ if and only if $t = t_{\text{max}} := \frac{\| (u,w) \|^2}{(\alpha + \beta - 1 + q) K_{\lambda,\mu}(u,w)}$. Also
\[
\psi_{u,w}''(t_{\text{max}}) = (2 - \alpha - \beta)(1 - \alpha - \beta) \left[ \frac{(\alpha + \beta - 2) \| (u,w) \|^2}{(\alpha + \beta - 1 + q) K_{\lambda,\mu}(u,w)} \right]^{\frac{\alpha + \beta - 2}{\alpha + \beta}} \| (u,w) \|^2 \\
- (\alpha + \beta - 1 + q) (\alpha + \beta + q) \left[ \frac{(\alpha + \beta - 2) \| (u,w) \|^2}{(\alpha + \beta - 1 + q) K_{\lambda,\mu}(u,w)} \right]^{\frac{\alpha + \beta - 2 + q}{\alpha + \beta}} K_{\lambda,\mu}(u,w) \\
= -\| (u,w) \|^2 (\alpha + \beta - 2)(1 + q) \left[ \frac{(\alpha + \beta - 2) \| (u,w) \|^2}{(\alpha + \beta - 1 + q) K_{\lambda,\mu}(u,w)} \right]^{\frac{\alpha + \beta - 2}{\alpha + \beta}} < 0.
\]
Thus $\psi_{u,w}$ achieves its maximum at $t = t_{\text{max}}$. Now using the Hölder’s inequality and fractional Sobolev inequality, we obtain
\[
K_{\lambda,\mu}(u,w) \leq |\lambda| \int_{\Omega} |f(x)||u|^{1-q}dx + |\mu| \int_{\Omega} |g(x)||w|^{1-q}dx \\
\leq |\lambda| ||f||_q ||u||_{\alpha+\beta}^{1-q} + |\mu| ||g||_q ||w||_{\alpha+\beta}^{1-q} \\
\leq (|\lambda| ||f||_q)^{\frac{1}{1+q}} + (|\mu| ||g||_q)^{\frac{1}{1+q}} \left( \frac{\| (u,w) \|}{\sqrt{S}} \right)^{1-q} \leq \Lambda^{\frac{1+q}{2}} \left( \frac{\| (u,w) \|}{\sqrt{S}} \right)^{1-q}. \tag{3.3}
\]
\[
\int_{\Omega} b(x) u^\alpha w^\beta dx \leq \| b \|_{\infty} \left( \frac{\alpha}{\alpha + \beta} \int_{\Omega} |u|^{\alpha+\beta}dx + \frac{\beta}{\alpha + \beta} \int_{\Omega} |v|^{\alpha+\beta}dx \right) \\
\leq \| b \|_{\infty} \left( \frac{\| (u,w) \|}{\sqrt{S}} \right)^{\alpha+\beta}. \tag{3.4}
\]
Using (3.3) and (3.5) we obtain,
\[
\begin{aligned}
\psi_{u,w}(t_{\text{max}}) \\
&= \frac{(1 + q)}{(\alpha + \beta - 1 + q)} \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right) \left( \frac{\| (u,w) \|^{2+\beta}}{K_{\lambda,\mu}(u,w)} \right)^{\frac{\alpha + \beta - 2 + q}{\alpha + \beta}} - \int_{\Omega} b(x) u^\alpha w^\beta dx \\
&\geq \left[ \frac{(1 + q)}{(\alpha + \beta - 1 + q)} \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right) \left( \frac{(\sqrt{S})(1-q)}{\Lambda^{\frac{1+q}{2}}} \right)^{\frac{(\alpha + \beta - 2 + q)}{(\alpha + \beta)}} \right] - \| b \|_{\infty} \left( \frac{1}{\sqrt{S}} \right)^{\alpha+\beta} \| (u,w) \|^{\alpha+\beta} \\
&\equiv E_{\lambda,\mu}(u,w)^{\alpha+\beta}. \tag{3.6}
\end{aligned}
\]
where
\[
E_{\lambda,\mu} = \left[ \frac{(1 + q)}{(\alpha + \beta - 1 + q)} \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right) \frac{1}{\alpha + \beta} - \|b\|_\infty \left( \frac{1}{\sqrt{\lambda}} \right) ^{\alpha + \beta} \right].
\]

Then we see that \(E_{\lambda,\mu} = 0\) if and only if \(\Lambda = C(n, \alpha, \beta, q, S)\), where
\[
C(n, \alpha, \beta, q, S) = \left( \frac{(1 + q)}{(\alpha + \beta - 1 + q)} \right)^{\frac{1}{1 + q}} \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right)^{\frac{1}{1 + q}} \left( \frac{1}{\|b\|_\infty} \right) \frac{1}{\sqrt{\lambda}} - \frac{2(\alpha + \beta - 1 + q)}{S(1 + q)(\alpha + \beta - 2)}. \]

Thus for \((\lambda, \mu) \in \Gamma\), we have \(E_{\lambda,\mu} > 0\), and therefore it follows from (3.6) that \(\psi_{u,w}(t_{\max}) > 0\).

(i) If \(\int_{\Omega} b(x)u_\alpha w_\beta dx \geq 0\), then \(\psi_{u,w}(t) \to - \int_{\Omega} b(x)u_\alpha w_\beta dx < 0\) as \(t \to \infty\). Consequently, \(\psi_{u,w}(t)\) has exactly two points \(0 < t_1 < t_{\max} < t_2\) such that
\[
\psi_{u,w}(t_1) = 0 = \psi_{u,w}(t_2) \quad \text{and} \quad \psi'_{u,w}(t_1) > 0 > \psi'_{u,w}(t_2).
\]

Now we show that if \(\psi_{u,w}(t) = 0\) and \(\psi'_{u,w}(t) > 0\), then \((tu, tw) \in N^+_{\lambda,\mu}\).
\[
\psi_{u,w}(t) = 0 \Leftrightarrow \|(tu, tw)\|^2 = K_{\lambda,\mu}(tu, tw) + \int_{\Omega} b(x)(tu)_{\alpha}^\alpha (tw)_{\beta}^\beta dx
\]
\[
\Leftrightarrow (tu, tw) \in N^+_{\lambda,\mu},
\]
and therefore
\[
\psi'_{u,w}(t) > 0 \Rightarrow (2 - \alpha - \beta)t^{1-\alpha-\beta}||(u, w)||^2 - (\alpha + \beta + 1 - q)t^{-\alpha-\beta-q}K_{\lambda,\mu}(u, w) > 0
\]
\[
\Rightarrow (2 - \alpha - \beta)|||(tu, tw)||^2 + (\alpha + \beta + 1 + q) \left[ ||(tu, tw)||^2 - \int_{\Omega} b(x)(tu)_{\alpha}^\alpha (tw)_{\beta}^\beta dx \right] > 0,
\]
\[
\Rightarrow (1 + q)|||(tu, tw)||^2 - (\alpha + \beta + 1 + q) \int_{\Omega} b(x)(tu)_{\alpha}^\alpha (tw)_{\beta}^\beta dx > 0
\]
\[
\Rightarrow (tu, tw) \in N^+_{\lambda,\mu}.
\]

Similarly one can show that if \(\psi_{u,w}(t) = 0\) and \(\psi'_{u,w}(t) < 0\), then \((tu, tw) \in N^-_{\lambda,\mu}\).

Now \(\phi'_{u,w}(t) = t^{\alpha+\beta-1}\psi_{u,w}(t)\). Thus \(\phi'_{u,w}(t) < 0\) in \((0, t_1)\), \(\phi'_{u,w}(t) > 0\) in \((t_1, t_2)\) and \(\phi'_{u,w}(t) < 0\) in \((t_2, \infty)\). Hence \(J_{\lambda,\mu}(t_1u, t_1w) = \inf_{0 \leq t \leq t_{\max}} J_{\lambda,\mu}(tu, tw), J_{\lambda,\mu}(t_1w, t_2w) = \sup_{t \geq t_1} J_{\lambda,\mu}(tu, tw)\).

Moreover \((t_1u, t_1w) \in N^+_{\lambda,\mu}\) and \((t_2u, t_2w) \in N^-_{\lambda,\mu}\).

(ii) If \(\int_{\Omega} b(x)u_\alpha w_\beta dx < 0\) and \(\psi_{u,w}(t) \to - \int_{\Omega} b(x)u_\alpha w_\beta dx > 0\) as \(t \to \infty\). Consequently, \(\psi_{u,w}(t)\) has exactly one point \(0 < t_1 < t_{\max}\) such that
\[
\psi_{u,w}(t_1) = 0 \quad \text{and} \quad \psi'_{u,w}(t_1) > 0.
\]
Using \(\phi'_{u,w}(t) = t^{\alpha+\beta-1}\psi_{u,w}(t)\), we have \(\phi'_{u,w}(t) < 0\) in \((0, t_1)\), \(\phi'_{u,w}(t) > 0\) in \((t_1, \infty)\). So, \(J_{\lambda,\mu}(t_1u, t_1w) = \inf_{t \geq 0} J_{\lambda,\mu}(tu, tw)\). Hence, it follows that \((t_1u, t_1w) \in N^+_{\lambda,\mu}\).

**Corollary 3.2** Suppose that \((\lambda, \mu) \in \Gamma\), then \(N^\pm_{\lambda,\mu} = \emptyset\).

**Proof.** From (a1) and (b1), we can choose \((u, w) \in Y \setminus \{(0, 0)\}\) such that \(K_{\lambda,\mu}(u, w) > 0\) and \(\int_{\Omega} b(x)u_\alpha^\alpha w_\beta^\beta dx > 0\). By (ii) of Lemma 3.3, there exists unique \(t_1\) and \(t_2\) such that \((t_1u, t_1w) \in N^+_{\lambda,\mu}, (t_2u, t_2w) \in N^-_{\lambda,\mu}\). In conclusion, \(N^\pm_{\lambda,\mu} = \emptyset\). \(\square\)
Lemma 3.3 For \((\lambda, \mu) \in \Gamma\), we have \(N_{\lambda,\mu}^0 = \{(0,0)\} \).

Proof. We prove this by contradiction. Assume that there exists \((0,0) \neq (u, w) \in N_{\lambda,\mu}^0\). Then it follows from \((u, w) \in N_{\lambda,\mu}^0\), that

\[
(1 + q)\| (u, w) \|^2 = (\alpha + \beta - 1 + q) \int_\Omega b(x) u_+^\alpha w_+^\beta dx
\]

and consequently

\[
0 = \| (u, w) \|^2 - K_{\lambda,\mu}(u, w) - \int_\Omega b(x) u_+^\alpha w_+^\beta dx
\]

\[
= \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right) \| (u, w) \|^2 - K_{\lambda,\mu}(u, w).
\]

Therefore, as \((\lambda, \mu) \in \Gamma\) and \((u, w) \neq (0,0)\), we use similar arguments as those in (3.6) to get

\[
0 < E_{\lambda,\mu}(u, w)^{\alpha + \beta}
\]

\[
\leq (1 + q) \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \left( \frac{\| (u, w) \|^2}{K_{\lambda,\mu}(u, w)} \right)^{\frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q}} - \int_\Omega b(x) u_+^\alpha w_+^\beta dx
\]

\[
= \left( \frac{1 + q}{\alpha + \beta - 1 + q} \right) \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right)^{\frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q}} \left( \frac{\| (u, w) \|^2}{(\alpha + \beta - 1 + q) \| (u, w) \|^2} \right)^{\frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q}} - (1 + q) \| (u, w) \|^2
\]

\[
= 0,
\]

a contradiction. Hence \((u, w) = (0,0)\). That is, \(N_{\lambda,\mu}^0 = \{(0,0)\}\). \(\square\)

We note that \(\Gamma\) is also related to a gap structure in \(N_{\lambda,\mu}\):

Lemma 3.4 Suppose that \((\lambda, \mu) \in \Gamma\), then there exist a gap structure in \(N_{\lambda,\mu}\):

\[
\| (U, W) \| > A_0 > A_{\lambda,\mu} > \| (u, w) \| \text{ for all } (u, w) \in N_{\lambda,\mu}^+, (U, W) \in N_{\lambda,\mu}^-,
\]

where

\[
A_0 = \left[ \frac{(1 + q)}{\| b \|_\infty} (\sqrt{S})^{\alpha + \beta} \right]^{\frac{1}{\alpha + \beta - 2}} \text{ and } A_{\lambda,\mu} = \left[ \frac{(\alpha + \beta - 1 + q)}{(\alpha + \beta - 2)} \right]^{\frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q}} \Lambda^\frac{1}{\alpha + \beta - 2}.
\]

Proof. If \(w \in N_{\lambda,\mu}^+ \subset N_{\lambda,\mu}\), then

\[
0 < (1 + q)\| (u, w) \|^2 - (\alpha + \beta - 1 + q) \int_\Omega b(x) u_+^\alpha w_+^\beta dx
\]

\[
(\alpha + \beta - 2)\| (u, w) \|^2 < (\alpha + \beta - 1 + q) K_{\lambda,\mu}(u, w)
\]

\[
\leq (\alpha + \beta - 1 + q) \left( (\| f \|_{L^p})^{\frac{2}{1 + p}} + (\| g \|_{L^p})^{\frac{2}{1 + q}} \right)^{\frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q}} \left( \frac{\| (u, w) \|}{\sqrt{S}} \right)^{1 - q}
\]
which yields
\[
\| (u, w) \| < \left[ \frac{(\alpha + \beta - 1 + q)}{(\alpha + \beta - 2)} \left( \frac{1}{\sqrt{S}} \right)^{1/q} \right]^{1/q} \left( \| \lambda \|_{q^*} \| f \|_{q^*} \right)^{2/q} + \left( \| \mu \|_{q^*} \right)^{2/q} \equiv A_{\lambda, \mu}.
\]

If \((U, W) \in \mathcal{N}^-_{\lambda, \mu}\), then it follows from (8.5) that
\[
(1 + q)\|(U, W)\|^2 < (\alpha + \beta - 1 + q) \int_\Omega b(x)U_+^\alpha W_+^\beta\,dx \leq (\alpha + \beta - 1 + q)\| b \|_{\infty} \left( \| (U, W) \| \right)^{\alpha + \beta}
\]
which yields
\[
\| (U, W) \| > \left[ \frac{(1 + q)}{\| b \|_{\infty} (\alpha + \beta - 1 + q)} \left( \frac{1}{\sqrt{S}} \right)^{1/q} \right]^{1/q} \equiv A_0.
\]

Now we show that \(A_{\lambda, \mu} = A_0\) if and only if \(\Lambda = C(n, \alpha, \beta, q, S)\).
\[
\Lambda = C(n, \alpha, \beta, q, S) = \left( \frac{1 + q}{\| b \|_{\infty} (\alpha + \beta - 1 + q)} \right)^{\alpha + \beta} \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right)^{\frac{2}{1+q}} S^{-\frac{2(\alpha + \beta - 1 + q)}{\alpha + \beta - 2}}.
\]

\[
\Leftrightarrow A_{\lambda, \mu} = \Lambda^\frac{1}{\alpha + \beta} \left[ \frac{(\alpha + \beta - 1 + q)}{(\alpha + \beta - 2)} \left( \frac{1}{\sqrt{S}} \right)^{1/q} \right]^{1/q}
\]
\[
= \left( \frac{(1 + q)}{\| b \|_{\infty} (\alpha + \beta - 1 + q)} \right)^{\alpha + \beta - 1} \left( \frac{\alpha + \beta - 2}{\alpha + \beta - 1 + q} \right)^{\frac{2}{1+q}} S^{-\frac{2(\alpha + \beta - 1 + q)}{\alpha + \beta - 2}} \left[ \frac{(\alpha + \beta - 1 + q)}{(\alpha + \beta - 2)} \left( \frac{1}{\sqrt{S}} \right)^{1/q} \right]^{1/q}
\]
\[
\equiv A_0.
\]

Thus for all \((\lambda, \mu) \in \Gamma\), we can conclude that
\[
\| (U, W) \| > A_0 > A_{\lambda, \mu} > \| (u, w) \| \text{ for all } (u, w) \in \mathcal{N}^+_{\lambda, \mu}, (U, W) \in \mathcal{N}^-_{\lambda, \mu}.
\]

This completes the proof of the Lemma. \(\square\)

**Lemma 3.5** Suppose that \((\lambda, \mu) \in \Gamma\), then \(\mathcal{N}^-_{\lambda, \mu}\) is a closed set in \(Y\)-topology.

**Proof.** Let \(\{(U_k, W_k)\}\) be a sequence in \(\mathcal{N}^-_{\lambda, \mu}\) with \((U_k, W_k) \rightarrow (U, W)\) in \(Y\). Then we have
\[
\| (U_k, W_k) \|^2 = \lim_{k \to \infty} \| (U_k, W_k) \|^2
\]
\[
= \lim_{k \to \infty} \left[ \int_\Omega (\lambda f(x)(U_k)_+^{1-q} + \mu g(x)(W_k)_+^{1-q})\,dx + \int_\Omega b(x)(U_k)_+^\alpha (W_k)_+^\beta\,dx \right]
\]
\[
= \int_\Omega (\lambda f(x)U_+^{1-q} + \mu g(x)W_+^{1-q})\,dx + \int_\Omega b(x)U_+^\alpha W_+^\beta\,dx
\]
and
\[
(1 + q)\|(U, W)\| - (\alpha + \beta - 1 + q) \int_\Omega b(x)U_+^\alpha W_+^\beta\,dx
\]
\[
= \lim_{k \to \infty} \left[ (1 + q)\| (U_k, W_k) \|^2 - (\alpha + \beta - 1 + q) \int_\Omega b(x)(U_k)_+^\alpha (W_k)_+^\beta\,dx \right] \leq 0.
\]
i.e. \((U,W) \in \mathcal{N}^-_{\lambda,\mu} \cap \mathcal{N}^0_{\lambda,\mu}\). Since \(\{(U_k,W_k)\} \subset \mathcal{N}^-_{\lambda,\mu}\), from Lemma 3.4 we have
\[
\| (U,W) \| = \lim_{k \to \infty} \| (U_k,W_k) \| \geq A_{\lambda,\mu} > 0,
\]
that is, \((U,W) \not\in (0,0)\). It follows from Lemma 3.1 that \((U,W) \not\in \mathcal{N}^0_{\lambda,\mu}\) for any \((\lambda,\mu) \in \Gamma\). Thus \((U,W) \in \mathcal{N}^-_{\lambda,\mu}\). That is, \(\mathcal{N}^-_{\lambda,\mu}\) is a closed set in \(Y\)-topology for any \((\lambda,\mu) \in \Gamma\).

**Lemma 3.6** Let \((u,w) \in \mathcal{N}^\pm_{\lambda,\mu}\), then for any \(\Phi = (\phi,\psi) \in C_Y\), there exists a number \(\epsilon > 0\) and a continuous function \(f : B_\epsilon(0) := \{v = (v_1,v_2) \in Y : \|v\| < \epsilon\} \to \mathbb{R}^+\) such that
\[
f(v_1,v_2) > 0, f(0,0) = 1 \text{ and } f(v_1,v_2)(u + v_1 \phi, w + v_2 \psi) \in \mathcal{N}^\pm_{\lambda,\mu} \text{ for all } v \in B_\epsilon(0).
\]

**Proof.** We give the proof only for the case \((u,w) \in \mathcal{N}^+_{\lambda,\mu}\), the case \(\mathcal{N}^-_{\lambda,\mu}\) may be preceded exactly. For any \(C_Y\), we define \(F : Y \times \mathbb{R}^+ \to \mathbb{R}\) as follows:
\[
F(v,t) = t^{1+q}\| (u + v_1 \phi, w + v_2 \psi) \|^2 - t^{\alpha + \beta - 1 + q} \int_{\Omega} b(x)(u + v_1 \phi)^{\alpha}_+(w + v_2 \psi)^{\beta}_+ dx - K_{\lambda,\mu}(u + v_1 \phi, w + v_2 \psi)
\]
Since \(w \in \mathcal{N}^+_{\lambda,\mu}\), we have that
\[
F((0,0),1) = \| (u,w) \|^2 - K_{\lambda,\mu}(u,w) - \int_{\Omega} b(x)u^{\alpha}_+w^{\beta}_+ dx = 0,
\]
and
\[
\frac{\partial F}{\partial t}((0,0),1) = (1+q)\| (u,w) \|^2 - (\alpha + \beta - 1 + q) \int_{\Omega} b(x)u^{\alpha}_+w^{\beta}_+ dx > 0.
\]
Applying the implicit function Theorem at the point \((0,0,1)\), we have that there exists \(\tilde{\epsilon} > 0\) such that for \(\|v\| < \tilde{\epsilon}\), \(v \in Y\), the equation \(F(v_1,v_2), t) = 0\) has a unique continuous solution \(t = f(v_1,v_2) > 0\). It follows from \(F((0,0),1) = 0\) that \(f(0,0) = 1\) and from \(F((v_1,v_2), f(v_1,v_2)) = 0\) for \(\|v\| < \tilde{\epsilon}\), \(v \in Y\) that
\[
0 = f^{1+q}(v)\|w + v\phi\|^2 - K_{\lambda,\mu}(u + v_1 \phi, w + v_2 \psi) - f^{\alpha + \beta - 1 + q}(v) \int_{\Omega} b(x)(u + v_1 \phi)^{\alpha}_+(w + v_2 \psi)^{\beta}_+ dx
\]
\[
= \frac{\|f(v)(u + v_1 \phi, w + v_2 \psi)\|^2 - K_{\lambda,\mu}(f(v)(u + v_1 \phi), f(v)(w + v_2 \psi))}{f^{1-q}(v)}
\]
\[
- \int_{\Omega} b(x)(f(v)(u + v_1 \phi))^{\alpha}_+(f(v)(w + v_2 \psi))^{\beta}_+ dx
\]
that is,
\[
f(v_1,v_2)(u + v_1 \phi, w + v_2 \psi) \in \mathcal{N}_{\lambda,\mu}\) for all \(v \in Y, \|v\| < \tilde{\epsilon}\).
\]
Since \(\frac{\partial F}{\partial t}((0,0),1) > 0\) and
\[
\frac{\partial F}{\partial t}((v_1,v_2), f(v_1,v_2))
\]
\[
= (1+q)f^{q}(v)||(u + v_1 \phi, w + v_2 \psi)||(\alpha + \beta - 1 + q) f^{\alpha + \beta - 1 + q-1}(v) \int_{\Omega} b(x)(u + v_1 \phi)^{\alpha}_+(w + v_2 \psi)^{\beta}_+
\]
\[
= \frac{(1+q)||f(v)(u + v_1 \phi), f(v)(w + v_1 \psi)||^2}{f^{2-q}(v)} - (\alpha + \beta - 1 + q) \int_{\Omega} b(x)(f(v)(u + v_1 \phi))^{\alpha}_+(f(v)(w + v_2 \psi))^{\beta}_+ dx
\]
\[
= \frac{1}{f^{2-q}(v)} \int_{\Omega} b(x)(f(v)(u + v_1 \phi))^{\alpha}_+(f(v)(w + v_2 \psi))^{\beta}_+ dx
\]
we can take $\epsilon > 0$ possibly smaller ($\epsilon < \bar{\epsilon}$) such that for any $v = (v_1, v_2) \in Y$, $\|v\| < \epsilon$, 

$$(1+q)\|f(v)(u+v_1\phi), f(v)(w+v_2\psi)||^2 - (\alpha + \beta - 1 + q) \int_\Omega b(x)(f(v)(u+v_1\phi))_+^\alpha (f(v)(w+v_2\psi))_+^\beta dx > 0,$$

that is,

$$f(v_1, v_2)(u + v_1\phi, w + v_2\psi) \in \mathcal{N}_{\lambda, \mu}^+$$

for all $v = (v_1, v_2) \in B_\epsilon(0)$.

This completes the proof of Lemma.

□

**Lemma 3.7** $J_\lambda$ is bounded below and coercive on $\mathcal{N}_{\lambda, \mu}$.

**Proof.** For $(u, w) \in \mathcal{N}_{\lambda, \mu}$, we obtain from (3.3) that

$$J_{\lambda, \mu}(u, w) = \left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right) \| (u, w) \|^2 - \left( \frac{1}{1 - q} - \frac{1}{\alpha + \beta} \right) K_{\lambda, \mu}(u, w)$$

$$\geq \left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right) \| (u, w) \|^2 - \left( \frac{1}{1 - q} - \frac{1}{\alpha + \beta} \right) \Lambda^{1+q} \left( \frac{\| (u, w) \|}{\sqrt{S}} \right)^{1-q}.$$  

(3.7)

Now consider the function $\rho : \mathbb{R}^+ \to \mathbb{R}$ as $\rho(t) = dt^{1-q} - ct^2$, where $c, d$ are both positive constants. One can easily show that $\rho$ is convex($\rho''(t) > 0$ for all $t > 0$) with $\rho(t) \to 0$ as $t \to 0$ and $\rho(t) \to \infty$ as $t \to \infty$. $\rho$ achieves its minimum at $t_{min} = \left( \frac{d(1-q)}{2c} \right)^{1/q}$ and

$$\rho(t_{min}) = c \left( \frac{d(1-q)}{2c} \right)^{\frac{1}{1+q}} - d \left( \frac{d(1-q)}{2c} \right)^{\frac{1}{1-q}} \Lambda^{\frac{1+q}{2}} \left( \frac{1}{\sqrt{S}} \right)^{1-q}.$$  

Applying $\rho(t)$ with $c = \left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right)$, $d = \left( \frac{1}{1-q} - \frac{1}{\alpha + \beta} \right) \Lambda^{\frac{1+q}{2}} \left( \frac{1}{\sqrt{S}} \right)^{1-q}$ and $t = \| (u, w) \|, (u, w) \in \mathcal{N}_{\lambda, \mu}$, we obtain from (3.7) that

$$\lim_{\| (u, w) \| \to \infty} J_{\lambda, \mu}(u, w) \geq \lim_{t \to \infty} \rho(t) = \infty,$$

since $0 < q < 1$. That is $J_{\lambda, \mu}$ is coercive on $\mathcal{N}_{\lambda, \mu}$. Moreover it follows from (3.7) that

$$J_{\lambda, \mu}(u, w) \geq \rho(t) \geq \rho(t_{min})(\text{a constant}),$$  

(3.8)

i.e

$$J_{\lambda, \mu}(u, w) \geq -\left( \frac{1+q}{2} \right) d^{\frac{1}{1+q}} \left( \frac{1-q}{2c} \right)^{\frac{1}{1+q}} + \left( \frac{1+q}{1-q} \right) \left( \frac{\alpha + \beta - 1 + q}{2(\alpha + \beta - 2)} \right)^{\frac{q}{1+q}} \Lambda \left( \frac{1}{\sqrt{S}} \right)^{\frac{2(1-q)}{1+q}}.$$  

Thus $J_{\lambda, \mu}$ is bounded below on $\mathcal{N}_{\lambda, \mu}$. □
4 Existence of Solutions in $\mathcal{N}^\pm_{\lambda,\mu}$

Now from Lemma 3.5, $\mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$ are two closed sets in $Y$ provided $(\lambda, \mu) \in \Gamma$. Consequently, the Ekeland variational principle can be applied to the problem of finding the infimum of $J_{\lambda,\mu}$ on both $\mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$. First, consider $\{(u_k, w_k)\} \subset \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$ with the following properties:

$$J_{\lambda,\mu}(u_k, w_k) < \inf_{(u, w) \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}} J_{\lambda,\mu}(u, w) + \frac{1}{k},$$

(4.1)

$$J_{\lambda,\mu}(u, w) \geq J_{\lambda,\mu}(u_k, w_k) - \frac{1}{k} \|(u - u_k, w - w_k)\| \text{ for all } (u, w) \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}.$$

(4.2)

**Lemma 4.1** Show that the sequence $\{(u_k, w_k)\}$ is bounded in $\mathcal{N}_{\lambda,\mu}$. Moreover, there exists $0 \neq (u, w) \in Y$ such that $(u_k, w_k) \rightharpoonup (u, w)$ weakly in $Y$.

**Proof.** From equations (3.3) and (4.1), we have

$$ct^2 - dt^{1-q} = \rho(t) \leq J_{\lambda,\mu}(u, w) < \inf_{(u, w) \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}} J_{\lambda,\mu}(u, w) + \frac{1}{k} \leq C_5,$$

for sufficiently large $k$ and a suitable positive constant. Hence putting $t = \|(u_k, w_k)\|$ in the above equation, we obtain $\{(u_k, w_k)\}$ is bounded.

Let $\{(u_k, w_k)\}$ is bounded in $Y$. Then, there exists a subsequence of $\{(u_k, w_k)\}$, still denoted by $\{(u_k, w_k)\}$, such that $\|(u_k, w_k)\| \in Y$ such that $(u_k, w_k) \rightharpoonup (u, w)$ weakly in $Y$, $(u_k, w_k)(\cdot) \rightharpoonup (u, w)(\cdot)$ strongly in $(L^r(\Omega))^2$ for $1 \leq r < p^*_s$ and $u_k(\cdot) \rightharpoonup u(\cdot)$, $w_k(\cdot) \rightharpoonup w(\cdot)$ a.e. in $\Omega$.

For any $(u, w) \in \mathcal{N}^+_{\lambda,\mu}$, we have from $0 < q < 1$, $2 < \alpha + \beta < 2s^*$ that

$$J_{\lambda,\mu}(u, w) = \left(\frac{1}{2} - \frac{1}{1-q}\right) \|(u, w)\|^2 + \left(\frac{1}{1-q} - \frac{1}{\alpha + \beta}\right) \int_{\Omega} b(x)u_+^\alpha w_+^\beta dx$$

$$< \frac{1}{2} \left(\frac{1}{2} - \frac{1}{1-q}\right) \|(u, w)\|^2 + \frac{1}{2} \left(\frac{1}{1-q} - \frac{1}{\alpha + \beta}\right) \frac{1+q}{\alpha + \beta - 1+q} \|(u, w)\|^2$$

$$= \frac{1}{\alpha + \beta - 1+q} \frac{1}{\alpha + \beta - 1+q} \|(u, w)\|^2 < 0,$$

which means that $\inf_{\mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu} < 0$. Now for $(\lambda, \mu) \in \Gamma$, we know from Lemma 3.1 that $\mathcal{N}^0_{\lambda,\mu} = \{(0,0)\}$. Together, these imply that $(u_k, w_k) \in \mathcal{N}^+_{\lambda,\mu}$ for $k$ large and

$$\inf_{(u, w) \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}} J_{\lambda,\mu}(u, w) = \inf_{(u, w) \in \mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u, w) < 0.$$

Therefore, by weak lower semi-continuity of norm,

$$J_{\lambda,\mu}(u, w) \leq \liminf_{k \to \infty} J_{\lambda,\mu}(u_k, w_k) = \inf_{\mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}} J_{\lambda,\mu} < 0,$$

that is, $(u, w) \not\equiv 0$ and $(u, w) \in Y$. \(\Box\)
Lemma 4.2 Suppose \((u_k, w_k) \in \mathcal{N}^+_{\lambda, \mu}\) such that \((u_k, w_k) \rightharpoonup (u, w)\) weakly in \(Y\). Then for \((\lambda, \mu) \in \Gamma\),

\[
(1 + q) \int_\Omega (\lambda f(x)u_+^{1-q} + \mu g(x)w_+^{1-q}) dx - (\alpha + \beta - 2) \int_\Omega b(x)u_+^\alpha w_+^\beta dx > 0. \tag{4.3}
\]

Moreover, there exists a constant \(C_2 > 0\) such that

\[
(1 + q) \| (u_k, w_k) \|^2 - (\alpha + \beta - 1 + q) \int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta \geq C_2 > 0. \tag{4.4}
\]

**Proof.** For \(\{(u_k, w_k)\} \subset \mathcal{N}^+_{\lambda, \mu} (\subset \mathcal{N}_{\lambda, \mu})\), we have

\[
(1 + q)K_{\lambda, \mu}(u, w) - (\alpha + \beta - 2) \int_\Omega b(x)u_+^\alpha w_+^\beta dx
\]

\[=
\lim_{k \to \infty} \left[ (1 + q)K_{\lambda, \mu}(u_k, w_k) - (\alpha + \beta - 2) \int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta \right]
\]

\[=
\lim_{k \to \infty} \left[ (1 + q)\|(u_k, w_k)\|^2 - (\alpha + \beta - 1 + q) \int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta \right] \geq 0.
\]

Now, we can argue by a contradiction and assume that

\[
(1 + q)K_{\lambda, \mu}(u, w) - (\alpha + \beta - 2) \int_\Omega b(x)u_+^\alpha w_+^\beta dx = 0. \tag{4.5}
\]

Using \((u_k, w_k) \in \mathcal{N}_{\lambda, \mu}\), the weak lower semi continuity of norm and (4.5) we have that

\[
0 = \lim_{k \to \infty} \left[ \|(u_k, w_k)\|^2 - K_{\lambda, \mu}(u_k, w_k) - \int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta dx \right]
\]

\[\geq \|(u, w)\|^2 - K_{\lambda, \mu}(u, w)dx - \int_\Omega b(x)u_+^\alpha w_+^\beta dx
\]

\[= \left\{ \begin{array}{ll}
\|(u, w)\|^2 - \frac{\alpha + \beta - 1 + q}{\alpha + \beta - 2} \int_\Omega b(x)u_+^\alpha w_+^\beta dx \\
\|(u, w)\|^2 - \frac{\alpha + \beta - 1 + q}{\alpha + \beta - 2} K_{\lambda, \mu}(u, w).
\end{array} \right.
\]

qq Thus for any \((\lambda, \mu) \in \Gamma\) and \((u, w) \neq 0\), by similar arguments as those in (3.6) we have that

\[
0 < E_{\lambda, \mu}\|(u, w)\|^\alpha + \beta
\]

\[\leq \frac{(1 + q)}{(\alpha + \beta - 1 + q)} \frac{(\alpha + \beta - 2)}{\alpha + \beta - 1 + q} \frac{\|(u, w)\|^2}{\|K_{\lambda, \mu}(u, w)\|^{\frac{\alpha + \beta - 2 - q}{\alpha + \beta - q}} - \int_\Omega b(x)u_+^\alpha w_+^\beta dx
\]

\[= \frac{(1 + q)}{(\alpha + \beta - 1 + q)} \frac{(\alpha + \beta - 2)}{\alpha + \beta - 1 + q} \frac{\|(u, w)\|^2}{\|K_{\lambda, \mu}(u, w)\|^\frac{\alpha + \beta - 2 - q}{\alpha + \beta - q}} - \frac{(1 + q)}{(\alpha + \beta - 1 + q)} \|(u, w)\|^2
\]

\[= 0,
\]

which is clearly impossible. Now by (4.3), we have that

\[
(1 + q)K_{\lambda, \mu}(u_k, w_k) - (\alpha + \beta - 2) \int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta \geq C_2 \tag{4.6}
\]

for sufficiently large \(k\) and a suitable positive constant \(C_2\). This, together with the fact that \((u_k, w_k) \in \mathcal{N}_{\lambda, \mu}\) we obtain equation (4.4). \(\square\)
Fix \((\phi, \psi) \in C_Y\) with \(\phi, \psi \geq 0\). Then we apply Lemma 3.6 with \((u_k, w_k) \in \mathcal{N}^+_{\lambda, \mu}\) (k large enough such that \(\frac{(1-q)C_k}{k} < C_2\)), we obtain a sequence of functions \(f_k : B_{C_k}(0) \subset Y \rightarrow \mathbb{R}\) such that \(f_k(0, 0) = 1\) and \(f_k(s_1, s_2)(u_k + s_1 \phi, w_k + s_2 \psi) \in \mathcal{N}^+_{\lambda, \mu}\) for all \(s = (s_1, s_2) \in B_{C_k}(0)\). It follows from \((u_k, w_k) \in \mathcal{N}^+_{\lambda, \mu}\) and \(f_k(s_1, s_2)(u_k + s_1 \phi, w_k + s_2 \psi) \in \mathcal{N}^+_{\lambda, \mu}\) that

\[
\| (u_k, w_k) \|^2 - K_{\lambda, \mu}(u_k, w_k) - \int_{\Omega} b(x)(u_k)_+^\alpha (w_k)_+^\beta \, dx = 0
\]

and

\[
f^2_k(s_1, s_2)\left( (u_k + s_1 \phi, w_k + s_2 \psi) \right)^2 - f_k^{1-q}(s_1, s_2)K(u_k + s_1 \phi, w_k + s_2 \phi)
- f_k^{\alpha+\beta}(s_1, s_2) \int_{\Omega} b(x)(u_k + s_1 \phi)_+^\alpha (w_k + s_2 \psi)_+^\beta \, dx = 0.
\]

Choose \(0 < \rho < \epsilon_k\), and \((s_1, s_2) = (\rho v_1, \rho v_2)\) with \(\|v\| < 1\) then we find \(f_k(v_1, v_2)\) such that \(f_k(0, 0) = 1\) and \(f_k(v_1, v_2)(u_k + v_1 \phi, w_k + v_2 \psi) \in \mathcal{N}^+_{\lambda, \mu}\) for all \(v \in B_\rho(0)\).

**Lemma 4.3** For \((\lambda, \mu) \in \Gamma\) we have \(|\langle f'_k(0, 0), (v_1, v_2) \rangle|\) is finite for every \(0 \leq v = (v_1, v_2) \in C_Y\) with \(\|v\| \leq 1\).

**Proof.** From (4.7) and (4.8) we have that

\[
0 = [f_k^2(\rho v_1, \rho v_2) - 1]|| (u_k + \rho v_1 \phi, w_k + \rho v_2 \psi) \|^2 + || (u_k + \rho v_1 \phi, w_k + \rho v_2 \psi) \|^2 - || (u_k, w_k) \|^2
- [f_k^{1-q}(\rho v_1, \rho v_2) - 1] \int_{\Omega} (\lambda f(x)(u_k + \rho v_1 \phi)_+^{1-q} + \mu g(x)(w_k + \rho v_2 \psi)_+^{1-q}) \, dx
- \lambda \int_{\Omega} f(x)[((u_k + v_1 \phi)_+^{1-q} - (u_k)_+^{1-q})] \, dx - \mu \int_{\Omega} g(x)[((w_k + v_2 \psi)_+^{1-q} - (w_k)_+^{1-q})] \, dx
- [f_k^{\alpha+\beta}(\rho v_1, \rho v_2) - 1] \int_{\Omega} b(x)(u_k + \rho v_1 \phi)_+^\alpha (w_k + \rho v_2 \psi)_+^\beta \, dx
- \int_{\Omega} b(x) [((u_k + \rho v_1 \phi)_+^\alpha (w_k + \rho v_2 \psi)_+^\beta - (u_k)_+^\alpha (w_k)_+^\beta)] \, dx,
\]

since

\[
(u_k + \rho v_1 \phi)_+^{1-q}(x) - (u_k)_+^{1-q}(x) = \begin{cases} (u_k + \rho v_1 \phi)_+^{1-q}(x) - (u_k)_+^{1-q}(x) & \text{if } u_k \geq 0 \\ 0 & \text{if } u_k \leq 0, u_k + \rho v_1 \phi \leq 0 \\ (u_k + \rho v_1 \phi)_+^{1-q}(x) - (u_k)_+^{1-q}(x) & \text{if } u_k \leq 0, u_k + \rho v_1 \phi \geq 0, \end{cases}
\]

we have,

\[
\int_{\Omega} f(x)[((u_k + v_1 \phi)_+^{1-q} - (u_k)_+^{1-q})(x)] \, dx \geq 0.
\]
Similarly, one can see that
\[
\int_{\Omega} g(x) \left| \left( (w_k + v_2 \psi)^{1-q} - (w_k)^{1-q} \right)(x) \right| dx \geq 0.
\]
Now dividing by \( \rho > 0 \) and passing to the limit \( \rho \to 0 \), we derive that
\[
0 \leq \langle f_k'(0,0), (v_1, v_2) \rangle \left[ 2\|(u_k, w_k)\|^2 - (1-q)K_{\lambda,\mu}(u_k, w_k) - (\alpha + \beta) \int_{\Omega} b(x)(u_k)_+^\alpha (w_k)_+^\beta dx \right]
+ 2 \int_{\Omega} (u_k(x) - u_k(y))(v_1 \phi(x) - (v_1 \phi(y)) + (w_k(x) - w_k(y))(v_2 \psi(x) - (v_2 \psi(y))) dx dy \\
- \alpha \int_{\Omega} b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1 \phi dx - \beta \int_{\Omega} b(x)(u_k)_+^{\alpha}(w_k)_+^{\beta-1} v_2 \psi dx \\
= \langle f_k'(0,0), (v_1, v_2) \rangle \left[ (1+q)\|(u_k, w_k)\|^2 - (\alpha + \beta - 1 + q) \int_{\Omega} b(x)(u_k)_+^{\alpha}(w_k)_+^\beta dx \right]
+ 2 \int_{\Omega} (u_k(x) - u_k(y))(v_1 \phi(x) - (v_1 \phi(y)) + (w_k(x) - w_k(y))(v_2 \psi(x) - (v_2 \psi(y))) dx dy \\
- \alpha \int_{\Omega} b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1 \phi dx - \beta \int_{\Omega} b(x)(u_k)_+^{\alpha}(w_k)_+^{\beta-1} v_2 \psi dx. \tag{4.10}
\]
From (4.4) and (4.10) we know immediately that \( \langle f_k'(0,0), (v_1, v_2) \rangle \neq -\infty \). Now we show that \( \langle f_k'(0,0), (v_1, v_2) \rangle \neq +\infty \). Arguing by contradiction, we assume that \( \langle f_k'(0,0), (v_1, v_2) \rangle = +\infty \). Since
\[
|f_k(\rho v_1, \rho v_2) - 1|\|(u_k, w_k)\| + \rho f_k(\rho v_1, \rho v_2)\|(v_1 \phi, v_2 \psi)\|
\geq ||f_k(\rho v_1, \rho v_2) - 1||\|(u_k, w_k)\| + f_k(\rho v_1, \rho v_2)\|(\rho v_1 \phi, \rho v_2 \psi)\|
= ||f_k(\rho v_1, \rho v_2)(u_k + \rho v_1 \phi, w_k + \rho v_2 \psi) - (u_k, w_k)\| \tag{4.11}
\]
and
\[
f_k(\rho v_1, \rho v_2) > f_k(0,0) = 1
\]
for sufficiently large \( k \). From the definition of derivative \( f_k'(0,0), (v_1, v_2) \), applying equation (4.2) with \( (u, w) = f_k(\rho v_1, \rho v_2)(u_k + \rho v_1 \phi, w_k + \rho v_2 \psi) \in \mathcal{N}_{\lambda,\mu}^+ \), we clearly have that
\[
f_k(\rho v_1, \rho v_2) - 1 \frac{||u_k, w_k||}{k} + f_k(\rho v_1, \rho v_2) \frac{\|\rho v\phi\|}{k}
\geq \frac{1}{k} f_k(\rho v_1, \rho v_2)(u_k + \rho v_1 \phi, w_k + \rho v_2 \psi) - (u_k, w_k)
\geq J_{\lambda,\mu}(u_k, w_k) - J_{\lambda,\mu}(f_k(\rho v_1, \rho v_2)(u_k + \rho v_1 \phi, w_k + \rho v_2 \psi))
= \left( \frac{1}{2} - \frac{1}{1-q} \right) \|(u_k, w_k)\|^2 + \left( \frac{1}{1-q} - \frac{1}{\alpha + \beta} \right) f_k^\alpha(\rho v_1, \rho v_2) \|(\rho v_1 \phi, w_k + \rho v_2 \psi)\|^2 \\
+ \left( \frac{1}{1-q} - \frac{1}{\alpha + \beta} \right) \left( \int_{\Omega} b(x)(u_k)_+^{\alpha}(w_k)_+^\beta - f_k^{\alpha+\beta}(\rho v_1, \rho v_2) \int_{\Omega} b(x)(u_k + \rho v_1 \phi)_+^{\alpha}(w_k + \rho v_2 \psi)_+^\beta dx \right)
\geq \left( \frac{1+q}{1-q} \right) ||(u_k + \rho v_1 \phi, w_k + \rho v_2 \psi)\|^2 - ||(u_k, w_k)\|^2 + [f_k^\alpha(\rho v_1, \rho v_2) - 1] ||(u_k + \rho v_1 \phi, w_k + \rho v_2 \psi)\|^2 \\
- \left( \frac{1}{1-q} - \frac{1}{\alpha + \beta} \right) f_k^{\alpha+\beta}(\rho v_1, \rho v_2) \int_{\Omega} b(x) [(u_k + \rho v_1 \phi)_+^{\alpha}(w_k + \rho v_2 \psi)_+^\beta - (u_k)_+^{\alpha}(w_k)_+^\beta] dx \\
- \left( \frac{1}{1-q} - \frac{1}{\alpha + \beta} \right) [f_k^{\alpha+\beta}(\rho v_1, \rho v_2) - 1] \int_{\Omega} b(x)(u_k)_+^{\alpha}(w_k)_+^\beta dx.
\]
Dividing by \( \rho > 0 \) and passing to the limit as \( \rho \to 0 \), we can obtain that

\[
\langle f'_k(0,0), (v_1, v_2) \rangle \frac{\|(u_k, w_k)\|}{k} + \frac{\|(v_1 \phi, v_2 \psi)\|}{k} \geq \frac{1 + q}{1 - q} \langle f'_k(0,0), (v_1, v_2) \rangle - \frac{q + 1}{q} \int_{\Omega} b(x)(u_k)_+^\alpha (w_k)_+^\beta dxdy \\
- \frac{\alpha + \beta - 1 + q}{1 - q} \left[ \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1 \phi dx + \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)(u_k)_+^\alpha (w_k)_+^{\beta-1} v_2 \psi dx \right]
\]

that is,

\[
\begin{align*}
\frac{1}{k} \|(v_1 \phi, v_2 \psi)\| & \geq \frac{1 + q}{1 - q} \left[ \int_{\Omega} b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1 \phi dx + \int_{\Omega} b(x)(u_k)_+^\alpha (w_k)_+^{\beta-1} v_2 \psi dx \right] \\
& - \frac{\alpha + \beta - 1 + q}{1 - q} \left[ \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1 \phi dx + \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)(u_k)_+^\alpha (w_k)_+^{\beta-1} v_2 \psi dx \right]
\end{align*}
\]

which is impossible because \( \langle f'_k(0,0), (v_1, v_2) \rangle = +\infty \) and

\[
(1+q)\|(u_k, w_k)\|^2 - (\alpha+\beta-1+q) \int_{\Omega} b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta \frac{(1-q)}{k} \|(u_k, w_k)\| \geq C_2 \cdot \frac{(1-q)C_1}{k} > 0.
\]

In conclusion, \( \langle f'_k(0,0), (v_1, v_2) \rangle < +\infty \). Furthermore [4.14] with \( \|(u_k, w_k)\| \leq C_1 \) and two inequalities [4.10] and [4.12] also imply that

\[
\langle f'_k(0,0), (v_1, v_2) \rangle \leq C_3
\]

for \( k \) sufficiently large and a suitable constant \( C_3 \).

**Lemma 4.4** For each \( 0 \leq (\phi, \psi) \in C_Y \) and for every \( 0 \leq v = (v_1, v_2) \in Y \) with \( \|v\| \leq 1 \), we have \( \lambda f(x) u_*^\alpha v_1 \phi + \mu g(x) w_*^\alpha v_2 \psi \in L^1(\Omega) \) and

\[
\begin{align*}
\int_{Q} \frac{(u(x) - u(y))(v_1(x) - v_1(y))}{|x - y|^{n+2s}} dxdy & + \int_{Q} \frac{(w(x) - w(y))(v_2(x) - v_2(y))}{|x - y|^{n+2s}} dxdy \\
& - \int_{\Omega} (\lambda f(x) u_*^\alpha v_1 \phi + \mu g(x) w_*^\alpha v_2 \psi) dx - \int_{\Omega} b(x) u_*^{\alpha-1} v_1^2 \phi dx - \int_{\Omega} b(x) w_*^{\alpha-1} v_2^2 \psi dx \geq 0.
\end{align*}
\]
Proof. Applying (4.11) and (4.2) again, we have that
\[
\begin{align*}
[f_k(\rho v_1, \rho v_2) &- 1] \left\| (u_k, w_k) \right\| + f_k(\rho v_1, \rho v_2) \left\| \rho v \phi \right\| \\
&\geq \frac{1}{k} \left\| f_k(\rho v_1, \rho v_2)(w_k + \rho v \phi) - w_k \right\| \\
&\geq J_{\lambda, \mu}(u_k, w_k) - J_{\lambda, \mu}(f_k(\rho v_1, \rho v_2)(w_k + \rho v \phi)) \\
&= \frac{1}{2} \||u_k, w_k|| - \frac{1}{2} \left\| f_k(\rho v_1, \rho v_2)(w_k + \rho v \phi) \right\|^2 dx - \frac{1}{1 - q} \int_\Omega (\lambda f(x)(u_k)^{1-q} + g(x)(w_k)^{1-q}) dx \\
&\quad + \frac{1}{1 - q} \int_\Omega (\lambda f(x)(u_k)^{1-q} + g(x)(w_k)^{1-q}) dx \\
&\quad - \frac{1}{\alpha + \beta} \int_\Omega b(x)(u_k)^{\alpha}(w_k)^{\beta} dx + \frac{1}{\alpha + \beta} \int_\Omega b(x)(w_k)^{\alpha}(w_k + \rho v \phi)^{\beta} dx \\
&= - \frac{f_k^2(\rho v_1, \rho v_2) - 1}{2} \||u_k, w_k||^2 - \frac{f_k^2(\rho v_1, \rho v_2)(||u_k + \rho v \phi, w_k + \rho v \psi||^2 - ||u_k, w_k||^2)}{2} \\
&\quad + \frac{1}{1 - q} \int_\Omega (\lambda f(x)(w_k + \rho v \phi)^{1-q} + g(x)(w_k + \rho v \psi)^{1-q}) dx \\
&\quad + \frac{1}{1 - q} \int_\Omega a(x)((w_k + \rho v \phi)^{1-q} - (w_k)^{1-q}) dx \\
&\quad + \frac{f_k^{\alpha + \beta}(\rho v_1, \rho v_2) - 1}{\alpha + \beta} \int_\Omega b(x)(u_k + \rho v \phi)^{\alpha}(w_k + \rho v \psi)^{\beta} dx \\
&\quad + \frac{1}{\alpha + \beta} \int_\Omega b(x)((u_k + \rho v \phi)^{\alpha}(w_k + \rho v \psi)^{\beta} - (u_k)^{\alpha}(w_k)^{\beta}) dx. \\
\end{align*}
\]
Dividing by \( \rho > 0 \) and passing to the limit \( \rho \to 0^+ \), we obtain
\[
\begin{align*}
\frac{(f_k^2(0, 0), (v_1, v_2))}{k} \left\| (u_k, w_k) \right\| + \left\| (v_1 \phi, v_2 \psi) \right\| \\
&\geq - (f_k^2(0, 0), (v_1, v_2)) \left\| (u_k, w_k) \right\|^2 - K_{\lambda, \mu}(u_k, w_k) - \int_\Omega b(x)(u_k)^{\alpha}(w_k)^{\beta} dx \\
&\quad - \int_\Omega \frac{(u_k(x) - u_k(y))(\phi(x) - \phi(y)) + (w_k(x) - w_k(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}} dxdy \\
&\quad + \frac{\alpha}{\alpha + \beta} \int_\Omega b(x)(u_k)^{\alpha-1}(w_k)^{\beta} v_1 \phi dx + \frac{\beta}{\alpha + \beta} \int_\Omega b(x)(w_k)^{\alpha}(w_k + \rho v \phi)^{\beta} v_2 \psi dx \\
&\quad + \frac{1}{1 - q} \liminf_{\rho \to 0^+} \left[ \int_\Omega \frac{\lambda f(x)((u_k + \rho v \phi)^{1-q} - (w_k)^{1-q})}{\rho} dx + \int_\Omega \frac{\mu g(x)((u_k + \rho v \psi)^{1-q} - (w_k)^{1-q})}{\rho} dx \right] \\
&= - \int_\Omega \frac{(u_k(x) - u_k(y))(\phi(x) - \phi(y)) + (w_k(x) - w_k(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}} dxdy \\
&\quad + \frac{\alpha}{\alpha + \beta} \int_\Omega b(x)(u_k)^{\alpha-1}(w_k)^{\beta} v_1 \phi dx + \frac{\beta}{\alpha + \beta} \int_\Omega b(x)(w_k)^{\alpha}(w_k + \rho v \phi)^{\beta} v_2 \psi dx \\
&\quad + \frac{1}{1 - q} \liminf_{\rho \to 0^+} \left[ \int_\Omega \frac{\lambda f(x)((u_k + \rho v \phi)^{1-q} - (w_k)^{1-q})}{\rho} dx + \int_\Omega \frac{\mu g(x)((u_k + \rho v \psi)^{1-q} - (w_k)^{1-q})}{\rho} dx \right]. \\
\end{align*}
\]
is finite. Now, using (4.9), we have
\[ f(x)((w_k + \rho v_1 \phi)_{+}^{1-q} - (w_k)_{+}^{1-q}) \geq 0 \]
and similarly we have,
\[ g(x)((w_k + \rho v_2 \psi)_{+}^{1-q} - (w_k)_{+}^{1-q}) \geq 0, \text{ for all } x \in \Omega, \text{ for all } t > 0. \]

Then by the Fatou Lemma, we have that
\[
\int_{\Omega} (\lambda f(x)(u_k)_{+}^{-q}v_1 \phi + \mu g(x)(w_k)_{+}^{-q}v_2 \psi) dx \\
\leq \frac{1}{1 - q} \liminf_{\rho \to 0^+} \left[ \lambda \int_{\Omega} \frac{f(x)((u_k + \rho v_1 \phi)_{+}^{1-q} - (u_k)_{+}^{1-q})}{\rho} + \mu \int_{\Omega} \frac{g(x)((w_k + \rho v_2 \psi)_{+}^{1-q} - (w_k)_{+}^{1-q})}{\rho} dx \right] \\
\leq \|f'_k(0,0),(v_1,v_2)\|\|u_k\| + \|v_1 \phi, v_2 \psi\| \\
\leq C_1C_3\|v_1,v_2\| + \|v_1 \phi,v_2 \psi\| - \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x)(u_k)_{+}^{\alpha+1}(w_k)_{+}^{\beta}v_1 \phi dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)(w_k)_{+}^{\alpha} (u_k)_{+}^{\beta}v_2 \psi dx \\
\leq C_1C_3\|v_1,v_2\| + \|v_1 \phi,v_2 \psi\| \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)(u_k)_{+}^{\alpha+1}(w_k)_{+}^{\beta}v_1 \phi dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)(w_k)_{+}^{\alpha} (u_k)_{+}^{\beta}v_2 \psi dx \\
+ \int_{Q} \frac{(u_k(x) - u_k(y))(\phi(x) - \phi(y)) + (w_k(x) - w_k(y))(\psi(x) - \psi(y))}{|x - y|^{n + 2s}} dx dy \\
\quad - \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x)(u_k)_{+}^{\alpha+1}(w_k)_{+}^{\beta}v_1 \phi dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)(w_k)_{+}^{\alpha} (u_k)_{+}^{\beta}v_2 \psi dx \\
\leq \int_{Q} \left( \frac{(u_k(x) - u_k(y))(\phi(x) - \phi(y)) + (w_k(x) - w_k(y))(\psi(x) - \psi(y))}{|x - y|^{n + 2s}} \\
\quad + \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x)(u_k)_{+}^{\alpha+1}(w_k)_{+}^{\beta}v_1 \phi dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)(w_k)_{+}^{\alpha} (u_k)_{+}^{\beta}v_2 \psi dx \right) dx dy \\
\]
\[ \mu \int_{\Omega} g(x)w^{-q}\psi dx < \infty, \text{ for every } 0 \leq (\phi, \psi) \in C_Y \text{ which guarantees that } u_+, w_+ > 0 \text{ a.e in } \Omega. \] Putting this choice of \( v \) in (4.13), we have for every \( 0 \leq (\phi, \psi) \in C_Y \)

\[
\int_Q \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{n+2s}} dxdy + \int_Q \frac{(w(x) - w(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}} dxdy - \lambda \int_{\Omega} f(x)u^{-q}\phi dx - \mu \int_{\Omega} g(x)w^{-q}\psi dx - \frac{\alpha}{\alpha + \beta} \int_{\Omega} bu_+^{-1}w_+^\beta dxdy - \frac{\beta}{\alpha + \beta} \int_{\Omega} bu_+^\alpha w_+^{-1}\psi dx \geq 0.
\]

Hence by density argument, (4.14) holds for every \( 0 \leq (\phi, \psi) \in Y \), which completes the proof of Corollary.

**Lemma 4.6** We show that \( u > 0, w > 0 \) and \( (u, w) \in \mathcal{N}^+_{\lambda, \mu} \).

**Proof.** Using (4.14) with \( \phi = u_-, \psi = w_-, \) we obtain that

\[
0 \leq \int_Q \frac{(u(x) - u(y))(u_-(x) - u_-(y))}{|x - y|^{n+2s}} dxdy + \int_Q \frac{(w(x) - w(y))(w_-(x) - w_-(y))}{|x - y|^{n+2s}} dxdy
\]

\[
\leq -\|u_+\|^2 - \|w_+\|^2 - 2 \int_Q \frac{u_-(x)u_+(y) + w_-(x)w_+(y)}{|x - y|^{n+2s}} dxdy \leq -\|u_+\|^2 - \|w_+\|^2 \leq 0.
\]

i.e, \( u_- = w_- = 0 \text{ a.e.} \) So, \( u = u_+ > 0, w = w_+ > 0 \text{ a.e by Corollary 4.5.} \) Hence, \( u, w > 0 \) in \( \Omega \). Now using (4.14) with \( \phi = u, \psi = w, \) we obtain that

\[
\|u, w\|^2 \geq \int_{\Omega}(\lambda f(x)u_+^{1-q} + \mu g(x)w_+^{1-q})dx + \int_{\Omega} b(x)u_+^\alpha w_+^\beta dx.
\]

On the other hand, by the weak lower semi-continuity of the norm, we have that

\[
\|u, w\|^2 \leq \liminf_{k \to \infty} \|(u_k, w_k)\|^2 \leq \limsup_{k \to \infty} \|(u_k, w_k)\|^2
\]

\[
= \int_{\Omega}(\lambda f(x)u_+^{1-q} + \mu g(x)w_+^{1-q})dx + \int_{\Omega} b(x)u_+^\alpha w_+^\beta dx.
\]

Thus

\[
\|u, w\|^2 = \int_{\Omega}(\lambda f(x)u_+^{1-q} + \mu g(x)w_+^{1-q})dx + \int_{\Omega} b(x)u_+^\alpha w_+^\beta dx.
\]

Consequently, \( (u_k, w_k) \to (u, w) \) in \( Y \) and \( (u, w) \in \mathcal{N}^+_{\lambda, \mu}. \) Now from (4.3) it follows that

\[
(1 + q)\|u, w\|^2 - (\alpha + \beta - 1 + q) \int_{\Omega} b(x)u_+^\alpha w_+^\beta dx
\]

\[
= (1 + q) \int_{\Omega}(\lambda f(x)u_+^{1-q} + \mu g(x)w_+^{1-q})dx - (\alpha + \beta - 2) \int_{\Omega} b(x)u_+^\alpha w_+^\beta dx > 0,
\]

that is, \( (u, w) \in \mathcal{N}^+_{\lambda, \mu}. \) \( \square \)
Lemma 4.7 Show that \( w \) is in fact a positive weak solution of problem \((P_{\lambda,\mu})\).

Proof. Let \((u, w) = (u_1, u_2), (\phi_1, \phi_2) \in Y \) and \( \epsilon > 0 \), then we define

\[
\Psi(x) = (\Psi_1, \Psi_2) = ((u_1 + \epsilon \phi_1_+, (u_2 + \epsilon \phi_2_+))
\]

For \( i = 1, 2 \), let \( \Omega = \Omega_i \times \Gamma_i \) with

\[
\Omega_i := \{ x \in \Omega : u_i(x) + \epsilon \phi_i(x) > 0 \} \text{ and } \Gamma_i := \{ x \in \Omega : u_i(x) + \epsilon \phi_i(x) \leq 0 \}.
\]

Then \( \Psi_i|_{\Omega_i}(x) = (u_i + \epsilon \phi_i)_+ \) and \( \Psi_i|_{\Gamma_i}(x) = 0 \). Decompose

\[
Q := (\Omega_i \times \Omega^c) \cup (\Gamma_i \times \Omega^c) \cup (\Omega^c \times \Omega_i) \cup (\Gamma^c \times \Omega_i) \cup (\Omega_i \times \Omega_i) \cup (\Gamma_i \times \Gamma_i).
\]

Let \( M_i(x, y) = u_i(x, y)((u_i + \epsilon \phi_i)(x) - (u_i + \epsilon \phi_i)(y))K(x, y) \), where \( u_i(x, y) = (u_i(x) - u_i(y)) \) and \( K(x, y) = \frac{1}{|x - y|^{n+2s}} \) Then we have

1. \( \int_{\Omega_i \times \Omega^c} M_i(x, y)dxdy = \int_{\Omega^c \times \Omega_i} M_i(x, y)dxdy = 0. \)
2. \( \int_{\Gamma_i \times \Omega^c} M_i(x, y)dxdy = -\int_{\Omega^c \times \Gamma_i} u_i(x)(u_i + \epsilon \phi_i)(x)K(x, y)dxdy. \)
3. \( \int_{\Omega^c \times \Gamma_i} M_i(x, y)dxdy = -\int_{\Gamma_i \times \Omega^c} u_i(x)(u_i + \epsilon \phi_i)(x)K(x, y)dxdy. \)
4. \( \int_{\Omega_i \times \Gamma_i} M_i(x, y)dxdy = -\int_{\Gamma_i \times \Omega_i} u_i(x)(u_i + \epsilon \phi_i)(x)K(x, y)dxdy. \)
5. \( \int_{\Omega_i \times \Gamma_i} M_i(x, y)dxdy = \frac{1}{|x - y|^{n+2s}}. \)
6. \( \int_{\Omega_i \times \Omega_i} M_i(x, y)dxdy = 0. \)
7. \( \int_{\Gamma_i \times \Gamma_i} M_i(x, y)dxdy = -\int_{\Gamma_i \times \Gamma_i} u_i(x)(u_i + \epsilon \phi_i)(x) - (u_i + \epsilon \phi_i)(y))K(x, y)dxdy. \)

Now relabeling \((\psi_1, \psi_2) = (\Phi, \Psi), (u_1, u_2) = (u, w)\) and \((\phi_1, \phi_2) = (\phi, \psi)\). Then putting \((\Phi, \Psi)\) into \([11,13]\) and using \([11,15]\), we see that

\[
0 \leq \int_Q \frac{u(x, y)(\Phi(x) - \Phi(y)) + w(x, y)(\Psi(x) - \Psi(y))}{|x - y|^{n+2s}}dxdy - \int_\Omega (\lambda f(x)u_-^q \Phi + \mu g(x)w_-^q \Psi)dx
\]

\[
- \frac{\alpha}{\alpha + \beta} \int_\Omega b(x)w_+^{\alpha-1}w_+^\beta \Phi dx - \frac{\beta}{\alpha + \beta} \int_\Omega b(x)u_+^\alpha w_+^{\beta-1} \Psi dx
\]

\[
= \int_Q \frac{u(x, y)((u + \epsilon \phi)(x) - (u + \epsilon \phi)(y)) + w(x, y)((w + \epsilon \psi)(x) - (w + \epsilon \psi)(y))}{|x - y|^{n+2s}}dxdy
\]

\[
+ \int_Q \frac{u(x, y)((u + \epsilon \phi)(x) - (u + \epsilon \phi)(y)) + w(x, y)((w + \epsilon \psi)(x) - (w + \epsilon \psi)(y))}{|x - y|^{n+2s}}dxdy
\]

\[
- \int_\Omega (\lambda f(x)u_-^q (u + \epsilon \phi) + \mu g(x)w_-^q (w + \epsilon \psi)) - \int_\Omega (\lambda f(x)u_-^q (u + \epsilon \phi) + \mu g(x)w_-^q (w + \epsilon \phi))
\]

\[
- \frac{\alpha}{\alpha + \beta} \int_\Omega b(x)w_+^{\alpha-1}w_+^\beta (u + \epsilon \phi)dx - \frac{\beta}{\alpha + \beta} \int_\Omega b(x)u_+^\alpha w_+^{\beta-1} (w + \epsilon \phi)dx
\]

\[
- \frac{\alpha}{\alpha + \beta} \int_\Omega b(x)w_+^{\alpha-1}w_+^\beta (u + \epsilon \phi)dx - \frac{\beta}{\alpha + \beta} \int_\Omega b(x)u_+^\alpha w_+^{\beta-1} (w + \epsilon \phi)dx
\]

\[
\text{fractional } p\text{-Laplacian with singular nonlinearity} \]

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\[
\begin{align*}
\varepsilon \left( \int_{Q} u(x, y)(\phi(x) - \phi(y)) + w(x, y)(\psi(x) - \psi(y)) \, dx \, dy \right) & - \int_{\Omega} (\lambda f(x)u_{+}^{\alpha-q} + \mu g(x)w_{+}^{\alpha-q}) \, dx \\
- \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x)u_{+}^{\alpha-1}w_{+}^{\beta} \, dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)u_{+}^{\alpha}w_{+}^{\beta-1} \, dx + \int_{Q} |u(x) - u(y)|^{2} + |w(x) - w(y)|^{2} \, dx \, dy \\
+ \int_{Q} u(x, y)((u + \epsilon \phi)^{-}(x) - (u + \epsilon \phi)^{-}(y)) + w(x, y)((w + \epsilon \phi)^{-}(x) - (w + \epsilon \phi)^{-}(y)) \, dx \, dy \\
- \int_{\Omega} (\lambda f(x)u_{+}^{\alpha-q} + \mu g(x)w_{+}^{\alpha-q}) \, dx + \lambda \int_{\Gamma_{1}} f(x)u_{+}^{\alpha-q}(u + \epsilon \phi) \, dx + \mu \int_{\Gamma_{2}} g(x)w_{+}^{\alpha-q}(w + \epsilon \phi) \, dx \\
- \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x)u_{+}^{\alpha-1}w_{+}^{\beta} \, dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)u_{+}^{\alpha}w_{+}^{\beta-1} \, dx + 2 \int_{\Gamma_{1} \times \Gamma_{1}} \frac{|u(x) - u(y)|^{2}}{|x - y|^{n+2s}} \, dx \, dy \\
- 2 \int_{\Gamma_{2} \times \Omega} \frac{|w(x)|^{2}}{|x - y|^{n+2s}} \, dx \, dy - 2 \int_{\Gamma_{1} \times \Omega} \frac{|u(x, y)|^{2}}{|x - y|^{n+2s}} \, dx \, dy \\
- 2 \int_{\Gamma_{2} \times \Omega} \frac{|u(x) - u(y)|^{2}}{|x - y|^{n+2s}} \, dx \, dy - 2 \int_{\Gamma_{1} \times \Omega} \frac{|w(x) - w(y)|^{2}}{|x - y|^{n+2s}} \, dx \, dy \\
+ \mu \int_{\Gamma_{2}} g(x)w_{+}^{\alpha-q}(w + \epsilon \phi) \, dx - \frac{\alpha}{\alpha + \beta} \int_{\Gamma_{1}} b(x)u_{+}^{\alpha-1}w_{+}^{\beta} \, dx - \frac{\beta}{\alpha + \beta} \int_{\Gamma_{2}} b(x)u_{+}^{\alpha}w_{+}^{\beta-1} \, dx + 2 \int_{\Gamma_{2} \times \Omega} \frac{|u(x) - u(y)|^{2}}{|x - y|^{n+2s}} \, dx \, dy \\
- 2 \int_{\Gamma_{1} \times \Omega} \frac{|u(x)|^{2}}{|x - y|^{n+2s}} \, dx \, dy + \int_{\Gamma_{1} \times \Omega} \frac{|u(x)|^{2}}{|x - y|^{n+2s}} \, dx \, dy \\
+ \mu \int_{\Gamma_{1}} g(x)w_{+}^{\alpha-q}(w + \epsilon \phi) \, dx - \frac{\alpha}{\alpha + \beta} \int_{\Gamma_{1}} b(x)u_{+}^{\alpha-1}w_{+}^{\beta} \, dx - \frac{\beta}{\alpha + \beta} \int_{\Gamma_{2}} b(x)u_{+}^{\alpha}w_{+}^{\beta-1} \, dx \\
- 2 \int_{\Gamma_{2} \times \Omega} \frac{|w(x)|^{2}}{|x - y|^{n+2s}} \, dx \, dy + \int_{\Gamma_{2} \times \Omega} \frac{|w(x)|^{2}}{|x - y|^{n+2s}} \, dx \, dy \\
+ \int_{\Gamma_{1} \times \Gamma_{1}} \frac{|w(x) - w(y)|^{2}}{|x - y|^{n+2s}} \, dx \, dy + \int_{\Gamma_{1} \times \Omega} \frac{|w(x) - w(y)|^{2}}{|x - y|^{n+2s}} \, dx \, dy \\
- \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x)u_{+}^{\alpha-1}w_{+}^{\beta} \, dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)u_{+}^{\alpha}w_{+}^{\beta-1} \, dx + 2 \int_{\Gamma_{1} \times \Gamma_{1}} \frac{|u(x) - u(y)|^{2}}{|x - y|^{n+2s}} \, dx \, dy \end{align*}
\]
\[
-2 \int_{\Gamma_2 \times \Omega_2} \frac{w(x, y) w(x)}{|x - y|^{n+2s}} \, dx \, dy + \lambda \int_{\Gamma_1} f(x) u^{-q}_+ (u + \varepsilon \phi) \, dx + \mu \int_{\Gamma_2} g(x) w^{-q}_+ (w + \varepsilon \psi) \, dx \\
- \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x) u^{-1}_+ w_+^\beta (u + \varepsilon \phi) \, dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x) u^\alpha_+ w_+^{\beta - 1} (w + \varepsilon \psi) \, dx \\
- 2 \epsilon \left( \int_{\Gamma_1 \times \Omega_1} \frac{u(x) \phi(x)}{|x - y|^{n+2s}} \, dx \, dy + \int_{\Gamma_2 \times \Omega_2} \frac{w(x) \psi(x)}{|x - y|^{n+2s}} \, dx \, dy + \int_{\Gamma_1 \times \Omega_1} \frac{u(x, y) \phi(x)}{|x - y|^{n+2s}} \, dx \, dy \right) \\
+ \int_{\Gamma_2 \times \Omega_2} \frac{w(x, y) \psi(x)}{|x - y|^{n+2s}} + \int_{\Gamma_1 \times \Omega_1} \frac{u(x, y) (\phi(x) - \phi(y))}{|x - y|^{n+2s}} \, dx \, dy + \int_{\Gamma_2 \times \Gamma_2} \frac{w(x, y) (\psi(x) - \psi(y))}{|x - y|^{n+2s}} \, dx \, dy \right) \\
\leq \epsilon \left( \int_{Q} \frac{(u(x) - u(y)) (\phi(x) - \phi(y)) + (w(x) - w(y)) (\psi(x) - \psi(y))}{|x - y|^{n+2s}} \, dx \, dy \right) \\
- \int_{\Omega} (\lambda f(x) u^{-q}_+ \phi + \mu g(x) w^{-q}_+ \psi) \, dx - \frac{\alpha}{\alpha + \beta} \int_{\Omega} b(x) u^{-1}_+ w_+^\beta \phi \, dx - \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x) u^\alpha_+ w_+^{\beta - 1} \phi \, dx \right) \\
+ 2 \epsilon \left( \int_{\Gamma_1 \times \Omega_1} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right) \frac{1}{2} \left( \int_{\Gamma_1 \times \Omega_1} \frac{|\phi(x)|^2}{|x - y|^{n+2s}} \, dx \, dy \right) \frac{1}{2} \\
+ 2 \epsilon \left( \int_{\Gamma_2 \times \Omega_2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right) \frac{1}{2} \left( \int_{\Gamma_2 \times \Omega_2} \frac{|\phi(x)|^2}{|x - y|^{n+2s}} \, dx \, dy \right) \frac{1}{2} \\
+ 2 \epsilon \left( \int_{\Gamma_1 \times \Omega_1} \frac{|w(x)|^2}{|x - y|^{n+2s}} \, dx \, dy \right) \frac{1}{2} \left( \int_{\Gamma_1 \times \Omega_1} \frac{|\psi(x)|^2}{|x - y|^{n+2s}} \, dx \, dy \right) \frac{1}{2} \\
+ 2 \epsilon \left( \int_{\Gamma_2 \times \Omega_2} \frac{|w(x)|^2}{|x - y|^{n+2s}} \, dx \, dy \right) \frac{1}{2} \left( \int_{\Gamma_2 \times \Omega_2} \frac{|\psi(x)|^2}{|x - y|^{n+2s}} \, dx \, dy \right) \frac{1}{2} \\
+ \epsilon \alpha \|b\| \left( \int_{\Gamma_1} |\phi|^{\alpha + \beta} \, dx \right)^{\frac{\beta}{\alpha + \beta}} \left( \int_{\Gamma_1} (u_+)^{\alpha + \beta} \, dx \right)^{\frac{\alpha}{\alpha + \beta}} \\
+ \epsilon \beta \|b\| \left( \int_{\Gamma_2} |\phi|^{\alpha + \beta} \, dx \right)^{\frac{\beta}{\alpha + \beta}} \left( \int_{\Gamma_2} (w_+)^{\alpha + \beta} \, dx \right)^{\frac{\alpha}{\alpha + \beta}} \\
- \frac{\epsilon \alpha}{\alpha + \beta} \int_{\Gamma_1} b(x) u_+^{-1} u_+^\beta \phi \, dx - \frac{\epsilon \beta}{\alpha + \beta} \int_{\Gamma_2} b(x) u_+^\alpha w_+^{\beta - 1} \psi \, dx. 
\]

Since the measure of \( \Gamma_i = \{ x \in \Omega_1 | (u_1 + \varepsilon \phi_i)(x) \leq 0 \} \) tend to zero as \( \varepsilon \to 0 \), it follows that
\[
\int_{\Gamma_1 \times \Omega_1} \frac{|\phi_i(x)|^2}{|x - y|^{n+2s}} \, dx \, dy \to 0, \text{ as } \varepsilon \to 0, 
\]
and similarly
\[
\int_{\Gamma_1 \times \Omega_1} \frac{|\phi_i(x)|^2}{|x - y|^{n+2s}} \, dx \, dy, \int_{\Gamma_1 \times \Omega_1} \frac{|\phi_i(x) - \phi_i(y)|^2}{|x - y|^{n+2s}} \, dx \, dy, 
\]
\[
\int_{\Gamma_1} b(x)u_+^{\alpha-1}w_+^{\beta} \phi dx + \int_{\Gamma_2} b(x)u_+^\alpha w_+^{\beta-1} \psi dx, \quad \text{all are tend to } 0 \text{ as } \epsilon \to 0.
\]
Dividing by \(\epsilon\) and letting \(\epsilon \to 0\), we obtain
\[
\int_O \frac{(u(x) - u(y))(\phi(x) - \phi(y)) + (w(x) - w(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}} \, dx dy
\]
\[-\int_{\Omega} (\lambda f(x)u_+^{\alpha} + \mu g(x)w_+^{\beta}) \, dx \quad \alpha \frac{(\lambda f(x)u_+^{\alpha-1}w_+^{\beta} \phi dx) - \beta \frac{\beta}{\alpha + \beta} \int_{\Omega} b(x)u_+^{\alpha-1}w_+^{\beta} \psi dx \geq 0
\]
and since this holds equally well for \((-\phi, -\psi)\), it follows that \((u, w)\) is indeed a positive weak solution of problem \((P_{\lambda,\mu})\) and hence a positive solution of \((P_{\lambda,\mu})\).

**Lemma 4.8** There exists a minimizing sequence \(\{(U_k, W_k)\}\) in \(N_{\lambda,\mu}^-\) such that \((U_k, W_k) \to (U, W)\) strongly in \(N_{\lambda,\mu}^-\). Moreover \((U, W)\) is a positive weak solution of \((P_{\lambda,\mu})\).

**Proof.** Using the Ekeland variational principle again, we may find a minimizing sequence \(\{(U_k, W_k)\} \subset N_{\lambda,\mu}^-\) for the minimizing problem \(\inf_{N_{\lambda,\mu}^-} J_{\lambda,\mu}\) such that for \((U_k, W_k) \to (U, W)\) weakly in \(Y\) and pointwise a.e. in \(Q\). We can repeat the argument used in Lemma 4.2 to derive that when \((\lambda, \mu) \in \Gamma)\)
\[
(1 + q) \int_{\Omega} (\lambda f(x)U_+^{1-q} + \mu g(x)W_+^{1-q}) \, dx - (\alpha + \beta - 2) \int_{\Omega} b(x)U_+^{\alpha} W_+^{\beta} \, dx < 0 \quad (4.16)
\]
which yields
\[
(1 + q) \int_{\Omega} (\lambda f(x)(U_k)_+^{1-q} + \mu g(x)(W_k)_+^{1-q}) \, dx - (\alpha + \beta - 2) \int_{\Omega} b(x)(U_k)_+^{\alpha} (W_k)_+^{\beta} \, dx \leq -C_4
\]
for \(k\) sufficiently large and a suitable positive constant \(C_4\). At this point we may proceed exactly as in Lemmas 4.3, 4.4, 4.6, 4.7 and corollary 4.5, we conclude that \(U, W > 0\) is the required positive weak solution of problem \((P_{\lambda,\mu}^+)\). In particular \((U, W) \in N_{\lambda,\mu}^-\). Moreover from (4.16) it follows that
\[
(1 + q)\|(U, W)\|^2 - (\alpha + \beta - 1 + q) \int_{\Omega} b(x)U_+^{\alpha} W_+^{\beta} \, dx
\]
\[
=(1 + q) \left[ K_{\lambda,\mu}(U, W) + \int_{\Omega} b(x)U_+^{\alpha} W_+^{\beta} \, dx \right] - (\alpha + \beta - 1 + q) \int_{\Omega} b(x)U_+^{\alpha} W_+^{\beta} \, dx
\]
\[
=(1 + q)K_{\lambda,\mu}(U, W) - (\alpha + \beta - 2) \int_{\Omega} b(x)U_+^{\alpha} W_+^{\beta} \, dx < 0,
\]
that is \((U, W) \in N_{\lambda,\mu}^-\).

**Proof of the Theorem 2.2:** From Lemmas 4.7, 4.8 and 3.4 we can conclude that the problem \((P_{\lambda,\mu})\) has at least two positive weak solutions \((u, w) \in N_{\lambda,\mu}^+\), \((U, W) \in N_{\lambda,\mu}^-\) with \(\|(U, W)\| > \|(u, w)\|\) for any \((\lambda, \mu) \in \Gamma)\).
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