The Dynamical Problem for a Non Self-adjoint Hamiltonian

Fabio Bagarello and Miloslav Znojil

Abstract. After a compact overview of the standard mathematical presentations of the formalism of quantum mechanics using the language of C*-algebras and/or the language of Hilbert spaces we turn attention to the possible use of the language of Krein spaces. In the context of the so-called three-Hilbert-space scenario involving the so-called PT-symmetric or quasi-Hermitian quantum models a few recent results are reviewed from this point of view, with particular focus on the quantum dynamics in the Schrödinger and Heisenberg representations.

Mathematics Subject Classification (2000). Primary 47B50; Secondary 81Q65 47N50 81Q12 47B36 46C20.

Keywords. Metrics in Hilbert spaces, hermitizations of a Hamiltonian.

1. Introduction

In the analysis of the dynamics of a closed quantum system $S$ a special role is played by the energy $H$, which is typically the self-adjoint operator defined by the sum of the kinetic energy of $S$ and of the potential energy giving rise to the conservative forces acting on $S$. The most common approaches in the description of $S$ are the following:

1. The algebraic description (AD): in this approach the observables of $S$ are elements of a C*-algebra $\mathfrak{A}$ (which coincides with $B(H)$ for some Hilbert space $H$). This means, first of all, that $\mathfrak{A}$ is a vector space over $\mathbb{C}$ with a multiplication law such that $\forall A, B \in \mathfrak{A}$, $AB \in \mathfrak{A}$. Also, two such elements can be summed up and the following properties hold: $\forall A, B, C \in \mathfrak{A}$ and $\forall \alpha, \beta \in \mathbb{C}$ we have

$$A(BC) = (AB)C, \quad A(B + C) = AB + AC, \quad (\alpha A)(\beta B) = \alpha \beta (AB).$$

An involution is a map $*: \mathfrak{A} \to \mathfrak{A}$ such that

$$A^{**} = A, \quad (AB)^* = B^* A^*, \quad (\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$$
A *-algebra $\mathfrak{A}$ is an algebra with an involution *. $\mathfrak{A}$ is a normed algebra if there exists a map, the norm of the algebra, $\| \cdot \|: \mathfrak{A} \to \mathbb{R}_+$, such that:

$$
\|A\| \geq 0, \quad \|A\| = 0 \iff A = 0, \quad \|\alpha A\| = |\alpha| \|A\|,
\|A + B\| \leq \|A\| + \|B\|, \quad \|AB\| \leq \|A\| \|B\|.
$$

If $\mathfrak{A}$ is complete wrt $\|\cdot\|$, then it is called a Banach algebra, or a Banach *-algebra if $\|A^*\| = \|A\|$. If further $\|A^* A\| = \|A\|^2$ holds for all $A \in \mathfrak{A}$, then $\mathfrak{A}$ is a $C^*$-algebra.

The states are linear, positive and normalized functionals on $\mathfrak{A}$, which look like $\rho(A) = \text{tr}(\hat{\rho} A)$, where $\mathfrak{A} = B(\mathcal{H})$, $\hat{\rho}$ is a trace-class operator and $\text{tr}$ is the trace on $\mathcal{H}$. This means in particular that

$$
\rho(\alpha_1 A + \alpha_2 B) = \alpha_1 \rho(A) + \alpha_2 \rho(B)
$$

and that, if $\mathfrak{A}$ has the identity $\mathbb{I}$,

$$
\rho(A^* A) \geq 0; \quad \rho(\mathbb{I}) = 1.
$$

An immediate consequence of these assumptions, and in particular of the positivity of $\rho$, is that $\rho$ is also continuous, i.e., that $|\rho(A)| \leq \|A\|$ for all $A \in \mathfrak{A}$.

The dynamics in the Heisenberg representation for the closed quantum system $\mathcal{S}$ is given by the map

$$
\mathfrak{A} \ni A \to \alpha^t(A) = U_t A U_t^\dagger \in \mathfrak{A}, \quad \forall t
$$

which defines a 1-parameter group of *-automorphisms of $\mathfrak{A}$ satisfying the following conditions

$$
\alpha^t(\lambda A) = \lambda \alpha^t(A), \quad \alpha^t(A + B) = \alpha^t(A) + \alpha^t(B),
\alpha^t(AB) = \alpha^t(A) \alpha^t(B), \quad \|\alpha^t(A)\| = \|A\|, \quad \text{and} \quad \alpha^{t+s} = \alpha^t \alpha^s.
$$

In the Schrödinger representation the time evolution is the dual of the one above, i.e., it is the map between states defined by $\hat{\rho} \to \hat{\rho}_t = \alpha^t \hat{\rho}$.

2. The Hilbert space description (HSD): this is much simpler, at a first sight. We work in some fixed Hilbert space $\mathcal{H}$, somehow related to the system we are willing to describe, and we proceed as follows:

- each observable $A$ of the physical system corresponds to a self-adjoint operator $\hat{A}$ in $\mathcal{H}$;
- the pure states of the physical system correspond to normalized vectors of $\mathcal{H}$;
- the expectation values of $A$ correspond to the following mean values: $\langle \psi, \hat{A} \psi \rangle = \rho_\psi(\hat{A}) = \text{tr}(P_\psi \hat{A})$, where we have also introduced a projector operator $P_\psi$ on $\psi$ and $\text{tr}$ is the trace on $\mathcal{H}$;
- the states which are not pure, i.e., the mixed states, correspond to convex linear combinations $\hat{\rho} = \sum_j w_j \rho_{\psi_j}$, with $\sum_j w_j = 1$ and $w_j \geq 0$ for all $j$;
- the dynamics (in the Schrödinger representation) is given by a unitary operator $U_t := e^{iHt/\hbar}$, where $H$ is the self-adjoint energy operator, as follows: $\hat{\rho} \to \hat{\rho}_t = U_t^\dagger \hat{\rho} U_t$. In the Heisenberg representation the states do not evolve in time while