1D Aging

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We derive exact expressions for a number of aging functions that are scaling limits of non-equilibrium correlations, \( R(t_w, t_w + t) \) as \( t_w \to \infty, t/w \to \theta \), in the 1D homogenous \( q \)-state Potts model for all \( q \) with \( T = 0 \) dynamics following a quench from \( T = \infty \). One such quantity is \( \langle \sigma_n(t_w) \sigma_n(t_w + t) \rangle \) when \( n/\sqrt{t_w} \to z \). Exact, closed-form expressions are also obtained when one or more interludes of \( T = \infty \) dynamics occur. Our derivations express the scaling limit via coalescing Brownian paths and a “Brownian space-time spanning tree,” which also yields other aging functions, such as the persistence probability of no spin flip at \( 0 \) between \( t \) and \( t + w \).

In a typical aging experiment, a system is rapidly quenched from high to low \( T \). After a time \( t_w \) following the quench, an external parameter (e.g., temperature or external field) is changed. The response \( R(t_w, t_w + t) \) of the system (e.g., decay of thermoremanent magnetization) is then measured at time \( t_w + t \). Aging can also be observed without a sudden parameter change; e.g., in the out-of-phase component \( \chi'' \) of the ac susceptibility \( \chi \). In either case, the essence of aging is that as \( t_w \to \infty, t \to \infty \), the response depends only on the ratio \( t/t_w \):

\[
\lim_{t \to \infty, t_w \to \infty \atop t/w \to \theta} R(t_w, t_w + t) = R(\theta). \tag{1}
\]

(Other scaling forms are discussed in \( \square \).) Equilibrium responses are time-translation-invariant, so aging is a non-equilibrium, history-dependent phenomenon.

There already exist a few exact results for related quantities measuring coarsening \( \square \) or persistence \( \square \). Exact results for persistence exponents (and fraction of persistent spins) in 1D Potts models appear in \( \square \), and results on coarsening quantities (such as domain size distributions) appear in \( \square \). There are fewer exact results for aging quantities. An exception is the well understood Ising chain, for which two-time correlations have been derived \( \square \) (see also \( \square \)). However, the methods used seem specialized to the Ising case. For general Potts models, exact results have been obtained only for \( F_q(t_w, t_w + t) \), the probability of no spin flip at the origin (in 1D) between \( t_w \) and \( t_w + t \). This was analyzed in \( \square \) for a semi-infinite chain and, for \( q = \infty \), in \( \square \) on the full 1D lattice. In the following sections we present our general method and compute exact results for a variety of aging quantities in the continuum scaling limit, including as special cases rederivations of the results of \( \square \).

Preliminaries. Consider the homogeneous \( q \)-state ferromagnetic Potts model on the 1D integer lattice, where the Potts spin variables \( \sigma_n \), for \( -\infty < n < \infty \), can take the values \( 1, \ldots, q \). We study \( T = 0 \) dynamics following a quench from \( T = \infty \) — i.e., in the initial \( \sigma(0) \), each site independently takes a random value uniformly from \( 1, \ldots, q \). For \( q = \infty \), each site takes its own unique
value.) The standardly used (as in [14]) continuous time $T = 0$ dynamics, that of the 1D voter model (see, e.g., [15]), is given by independent Poisson “clock” processes at each site $n$, all of rate one, indicating when a flip at $n$ is considered. When the clock at $n$ rings, $\sigma_n$ takes the value of one neighbor, chosen by a fair coin toss (regardless of whether it previously agreed with either or both neighbors). There are other natural $T = 0$ dynamics, but we defer their analysis to a later paper.

In studying this evolution, it is convenient to use a well-known mapping to a 1D reaction-diffusion system of “kinks” [13 21]. A kink corresponds to a site $n + 1/2$ in the dual lattice where $\sigma_n \neq \sigma_{n+1}$. The initial configuration is a random arrangement of kinks (with density $(q-1)/q$) that subsequently execute 1D random walks. For the Ising model (where $q = 2$), the walks are purely annihilating, while for $q = \infty$ they are purely coalescing; for other $q$ both annihilation and coalescence occur.

At time $t_w$, there will be a characteristic distribution, depending on $q$, of walkers (i.e., kinks). One aging quantity is the persistence probability $F_q(t_w, t_w + t)$, mentioned earlier, of no flip at the origin between $t_w$ and $t_w + t$. A related quantity is $G_q(0, 0; t_w, t_w + t) = \langle \delta_{\sigma_0(t_w), \sigma_0(t_w + t)} \rangle$, the probability $P$ that $\sigma_0(t_w + t) = \sigma_0(t)$ (regardless of intervening flips). More generally, $G_q(m, n; s, s') = P(\sigma_n(s') = \sigma_m(s))$, and the spin-spin correlation $C_q = [q/(q-1)](G_q - 1/q)$, which is $(\overline{\sigma}_m(s) \cdot \overline{\sigma}_n(s))$ in the “tetrahedral” representation of Potts spins. Other aging quantities will be discussed later.

### FIG. 1. An eight-site lattice for the $q = \infty$ model. At $t = 0$ all spins have distinct values, labeled a-h. Both forward in $t$ coalescing walks representing domain boundary motion (solid lines), and backward in $t$ coalescing walks representing “ancestry” (dashed lines) are shown; horizontal segments are at times of Poisson clock rings. This diagram can be used for any $q$; e.g., for $q = 2$, abdh might all be +1 and cef -1.

Coalescing Brownian paths and spanning trees. The coalescing random walks of kinks correspond exactly to the motion of the boundaries between clusters of like spins for the $q = \infty$ model. Whether $q = \infty$ or not, to determine $\sigma_m(t_w)$, it is natural to trace backward in time to see successively from which neighbor the spin value came as various clocks rang. This leads to a dual process [13] of coalescing random walks on the original lattice such that all sites $n$ whose (backward in time) walkers have coalesced and are located at $t$ at time zero have $\sigma_n(t_w) = \sigma(0)$ (see Fig. 1).

Thus, for $q = \infty$, the equal-time $G_\infty(m, n; t_w, t_w) = P(\sigma_m(t_w) = \sigma_n(t_w))$ is just the probability that two independent (backward in time) random walks starting at $m$ and $n$ meet within time $t_w$. But the dual process also works for unequal times, so $G_\infty(m, n; t, t + t) = \delta(t)$ equals the probability that two walkers, one starting at $m$ and the other starting $t$ units of time “earlier” at $n$ meet and hence coalesce between times $0$ and $t_w$.

Not surprisingly, in the scaling limit, the walkers (both forward and backward) become particles doing Brownian motion. Concretely, when $t_w \to \infty$, one rescales the lattice by $a = 1/\sqrt{t_w}$ and time by $a^2$, so that backward walkers starting from sites 0 and $\sqrt{t_w}z$ become Brownian particles, one from 0 and the other starting $\theta = \lim t/t_w$ units of time “earlier” from $z$. This will be used below.

More surprisingly, a scaling limit is valid not just for a few walkers, but simultaneously for walkers starting from every lattice site at $t = 0$ [14]. The limit essentially has Brownian particles starting from every point on the continuous line at $t = 0$, but for any $t > 0$, coalescing has reduced them to a discrete set.

An extended limit, useful for understanding aging of persistence quantities, includes all starting times and simultaneously the (backward in time) dual particles with all their starting times. The collection of all forward (resp., backward) space-time paths forms a spanning tree of continuum space-time in the sense of [24].

Spin-spin correlation. As in the previous section, we express $G_\infty(0, \sqrt{t_w}z; t_w, t_w + t)$ in the scaling limit via coalescing dual Brownian paths. The limit of $G_\infty$, denoted $g(z, \theta) = g_\infty(z, \theta)$, is the probability that the backward (i.e., dual) Brownian paths starting at the space-time points $(z, 1 + \theta)$ and $(0, 1)$ coalesce before time 0. Denote the location at time $t - s$ of the backward Brownian path starting at a generic $(x, t)$ by $\tilde{B}_x(s)$, $s \geq 0$. Conditioning on the value $x$ of $\tilde{B}_{z,1+\theta}(\theta)$, we have

$$g(z, \theta) = \int_{-\infty}^{\infty} dx \, e^{-\left(x-z\right)^2/2\theta^2} g(x),$$

(2)

with $g(x) = P(A_x)$, where $A_x$ is the event that $\tilde{B}_{0,1}$ and $\tilde{B}_{x,1}$ coalesce at some $s \in [0, 1]$.

Note from [24] that $g(z, \theta)$ satisfies the heat equation, $\partial g/\partial \theta = (1/2) \partial^2 g/\partial z^2$ (see, e.g., [11] for a corresponding result in the Ising case), with $g(z, 0) = P(A_z)$. Since $A_z$ is the event that $\tilde{B}_{0,1}(s) - \tilde{B}_{x,1}(s) = 0$ for some $s \in [0, 1]$, and the difference of two independent (before coalescing) Brownian motions of rate 1 is a Brownian motion of rate 2, we can rewrite $P(A_z)$ as $P(B_z(s) = 0$ for some $s \in [0, 1]) = 1 - P(B_z \neq 0$ during $[0, 1])$,
where $B_x(s)$ is a Brownian motion starting at $x$ at rate $2$. By a standard argument using the Reflection Principle and the symmetry in $z$, the latter probability equals $P(B_x(1) > 0) - P(B_{-x}(1) > 0)$, and this equals

$$P(0 < B_0(1) < |x|) = 2P(0 < B_0(1) < |x|) = \frac{\phi(|x|/\sqrt{2})}{\sqrt{2/\pi} \int_0^\infty e^{-t^2/2}}.$$  

Substituting in (3) and rewriting again, we find that $1 - g(z, \theta)$ equals $(2\theta)^{-1/2} |h(z) + h(-z)|$, where $h(z) = \int_{-\infty}^\infty dx \ e^{-x^2/2\theta} \phi((x + z)/\sqrt{2})$. After further analysis,

$$g(z, \theta) = \psi(|z|/\sqrt{2 + \theta}, \sqrt{2/\theta}),$$

where $\psi(a, b) = \sqrt{2/\pi} \int_a^\infty e^{-t^2/2} \phi(bt)$, and finally

$$g(z, \theta) = \frac{2}{\pi} \int_0^{\sqrt{2/\theta}} dt \ e^{-x^2(1+2t)/(2(2+\theta))}.$$

Eqns. (3)-(4) simplify in particular cases, e.g.

$$g(0, \theta) = \frac{1}{2} \arctan \sqrt{2/\theta},$$

$$g(z, 0) = 1 - \phi(|z|/\sqrt{2}),$$

$$g(z, 2) = \frac{1}{2} \left[ 1 - \phi^2(|z|/2) \right].$$

$g(0, \theta)$ gives the scaling limit probability that $\sigma_m(t_w + t) = \sigma_m(t_w)$, regardless of flips during $(t_w, t_w + t)$. $g(z, 0)$ is the scaling limit equal-time two-point correlation function and its exact formula in (3) is implicit or explicit in earlier work on inter-particle distributions [1][2][3] and [4].

The $q < \infty$ case of $g_q(z, \theta)$ is simply related to the $q = \infty$ case just discussed. Clearly, $\sigma_\infty(t_w + t) = \sigma_\infty(t_w)$ (in the scaling limit) if the backward Brownian paths starting at the space-time points $(0, 1)$ and $(z, 1 + \theta)$ coalesce before time $0$. If not, there is still a $1/q$ probability that those paths end at time $0$ on sites with the same spin value. Hence $g_q(z, \theta) = g(z, \theta) + \frac{1}{q} (1 - g(z, \theta))$ and so $c_q(z, \theta)$ is

$$\lim_{t/w \to \infty, t/w \to \infty} C_q(0, m; t_w, t_w + t) = g(z, \theta),$$

which in particular does not depend on $q$. Thus our exact results [4] and [3] reproduce, as a special case, the known Ising result for $c_2(z, \theta)$ (see [5]).

$T = \infty$ interludes. We now modify the dynamics by inserting an interval of duration $\Delta$ with $T = \infty$ dynamics. I.e., when the clock at $n$ runs during such an interlude (which it still does at rate one), $\sigma_n$ chooses a value uniformly at random from \{1, $\ldots$, $q$\} (including the previous value); for $q = \infty$, the new value is chosen to be distinct from all other sites at that time. The entire interlude is inserted at a time $t_l$ (of the unmodified dynamics), which may be either in $(0, t_w)$ or $(t_w, t_w + t)$.

Using the backward random walks of the (unmodified) $q = \infty$ model, to analyze the aging quantities $G_q^\infty = C_q^\infty$ for the modified dynamics, we consider walkers starting from $(m, t_w)$ and $(n, t_w + t)$ and the events $A$ that they coalesce at a time $\tau > 0$ and $B$, that for $\tau < t_l$, they pass through the $T = \infty$ interlude with no clock ring. Then $G_q^\infty = P(A \cap B)$. Furthermore, by the nature of the $T = \infty$ dynamics for finite $q$, it is clear that $G_q^\infty = G_q^\infty + \frac{1}{q}$ and $G_q^\infty = G_q^\infty$ for all $q$.

Next note that the probability that a walker passes through the interlude with no clock ring is $e^{-\Delta}$. For $t_l$ in $(t_w, t_w + t)$, $P(A)$ is just the unmodified $G_q^\infty$, so $C_q^\infty = e^{-\Delta} C_q^\infty = e^{-\Delta} G_q^\infty = e^{-\Delta} g$ in the scaling limit; this is independent of the location of $t_l$ within $(t_w, t_w + t)$.

For $t_l < t_w$, we partition $A$ into $A_1$, where $t_l > \tau$ and there are two independent walkers during the interlude (so $P(B) = e^{-2\Delta}$) and the remainder $A_2$. $A_2$ is the event that coalescence occurs at $\tau > t_l$, so $P(A_2) = G_q^\infty(m, n; t_w - t_l, t_w + t - t_l)$ and $C_q^\infty = P(A_2) = e^{-2\Delta} (P(A) - P(A_2))$. Taking the limit with $\Delta$ fixed, $(t_w - t_l)/t_l \to \rho > 0$, $t/l \to \theta > 0$ and $(n - m)/t_w \to z$, the spin-spin correlation with one $\Delta$-interlude is (see Fig. 2):

$$c_q^\infty(z, \theta, \rho) = g(z, \theta/\rho) + e^{-2\Delta} [g(z, \theta) - g(z, \theta/\rho)].$$

![FIG. 2. One-interlude $c_q^\infty(z)$ curves with $\theta = 1, \Delta = \frac{1}{2}$: solid ($c_q^\infty(z, 1) = g(z, 1)$) for $t_l = 0$, dashed for $t_l = \frac{1}{2} t_w$, dot-dashed for $t_l = t_w - \frac{1}{2}$, and dotted for $t_l$ in $(t_w, t_w + t)$.](image-url)

Similar arguments for multiple $\Delta_j$-interludes at times $(1 - \rho_j) t_w$ ($0 < \rho_1 < \ldots < \rho_t < 1$) and $\Delta_j$-interludes at times $(1 + \rho_j) t_w$ ($0 < \rho_k < \theta$) yield for $c_q^\infty$:

$$e^{-\Delta} \sum_{j=0}^{t} \exp(-2 \sum_{i=0}^{j} \Delta_i) [g(z, \theta/\rho_{j+1}) - g(z, \theta/\rho_j)].$$

where $\Delta = \sum_{k=0}^{\frac{t}{2}} \Delta_k$, $\Delta_0 = 0$, $\rho_0 = 0$ and $\rho_{t+1} = 1$.

**Persistence.** Let $\tilde{N} = \tilde{N}(t_w, t_w + t)$ be the number of distinct backward walkers remaining at time zero from all those starting at the origin during $(t_w, t_w + t)$. When $\tilde{N} = k$ and $q < \infty$, the probability of no flips at 0 in this time interval is $(1/q)^{k-1}$ and so the persistence probability $P_{\tilde{Q}}(t_w, t_w + t) = (1/q)^{\tilde{N}-1}$ (for $q = \infty$, $F_{\tilde{Q}} = P(\tilde{N} = 1)$). In the scaling limit the distribution of $\tilde{N}$ converges to that of $N(1, 1 + \theta)$, the number of distinct particles, in
the dual Brownian spanning tree, surviving at time zero from all particles starting at the origin at all times during
\((1, 1 + \theta)\). Writing \(h_k(\theta)\) for \(P(N(1, 1 + \theta) = k)\), we see that \(F_q\) converges to the aging function
\[
f_q(\theta) = \begin{cases} 
\sum_{k=1}^{\infty} h_k(\theta)(1/q)^{k-1} & \text{if } q < \infty \\
h_1(\theta) = P(N(1, 1 + \theta) = 1) & \text{if } q = \infty 
\end{cases}.
\]

The persistence function \(f(\theta)\) (and hence \(h_1(\theta)\)) can be evaluated exactly, thus rederiving a result of \([3]\) by quite different methods, as follows. Let \((-X, Y)\) denote the (random) spatial interval with the same \((q = \infty)\) spin value at time 1 as the origin. The event \(A_{x,y}\) that \(X > x\) and \(Y > y\) means that the backward Brownian paths starting at \((-x,1)\) and \((y,1)\) coalesce before time 0. Proceeding as in our analysis of \([2]\), we see that \(P(A_{x,y}) = 1 - \phi((x+y)/\sqrt{2})\). The probability density of \((X,Y)\) is then \(\mu(x,y) = 1/\pi (x+y)e^{-(x+y)^2/2}\) for \(x,y > 0\). Now, given \((X,Y) = (x,y)\), \(f_\infty(\theta)\) is the probability that the forward Brownian paths starting at \((-x,1)\) and \((y,1)\) do not touch the origin during \((1,1+\theta)\), and thus
\[
f_\infty(\theta) = \int_0^\infty \int_0^\infty dx dy \mu(x,y) \phi\left(\frac{x}{\sqrt{\theta}}\right) \phi\left(\frac{y}{\sqrt{\theta}}\right).
\]

After further analysis of the same kind used to derive \([3]-[4]\), one finds \(f_\infty(\theta) = \frac{1}{2} \arcsin(1/(1+\theta))\) as in \([3]\).

This formula is consistent with the \(q = \infty\) persistence exponent value of one \([14,26,27]\) (for \(t_w\) fixed and \(t \to \infty\)) since \(f_\infty(\theta)\) is asymptotic to \(1/\theta\).

Discussion. We presented a powerful and very general approach, based on coalescing random walks and Brownian paths run forward and backward in time, to nonequilibrium dynamics in 1D. It yields exact, closed-form expressions in the scaling limit for a variety of aging (and persistence) quantities including the spin-spin correlation \(\langle \sigma_x(t_w) \cdot \sigma_y(t_w + t) \rangle\) for the \(q\)-state Potts model for all \(q\), following a quench from \(T = \infty\) to \(T = 0\). This type of approach, based on an exact analysis of the space-time scaling limit for the entire dynamical process, should yield exact expressions for aging functions in a wide variety of 1D systems.

We also presented an exact, closed-form expression for the spin-spin correlation when the system undergoes a sequence of \(T = \infty\) interludes. Perhaps surprisingly, we find that the effect of such interludes is independent of their timing provided they occur during the interval \((t_w, t_w + t)\). We believe our methods may work also for \(T < \infty\) interludes, which represent a common experimental situation \([1]\), and for other aging quantities of interest; these analyses will be deferred to a later paper.

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