LOCAL CURVATURE ESTIMATES FOR THE RICCI-HARMONIC FLOW

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ABSTRACT. In this paper we give an explicit bound of $\Delta_{g(t)} u(t)$ and the local curvature estimates for the Ricci-harmonic flow

$$\partial_t g(t) = -2 \text{Ric}_{g(t)} + 4 \nabla_{g(t)} u(t) \otimes \nabla_{g(t)} u(t), \quad \partial_t u(t) = \Delta_{g(t)} u(t)$$

under the condition that the Ricci curvature is bounded along the flow. In the second part these local curvature estimates are extended to a class of generalized Ricci flow, introduced by the author [42], whose stable points give Ricci-flat metrics on a complete manifold, and which is very close to the $(K,N)$-super Ricci flow recently defined by Xiangdong Li and Songzi Li [35]. Next we propose a conjecture for Einstein’s scalar field equations motivated by a result in the first part and the bounded $L^2$-curvature conjecture recently solved by Klainerman, Rodnianski and Szeftel [24]. In the last two parts of this paper, we discuss two notions of “Riemann curvature tensor” in the sense of Wylie-Yeroshkin [23,24,67,68], respectively, and Li [44], whose “Ricci curvature” both give the standard Bäcklund-Ricci curvature [1], and the forward and backward uniqueness for the Ricci-harmonic flow.

CONTENTS

1. Introduction 2
2. Gradient and local curvature estimates 13
  2.1. The boundedness of $\Delta_{g(t)} u(t)$ 15
  2.2. Local curvature estimates 20
3. Results for a generalized Ricci flow 25
  3.1. Long time existence 26
  3.2. Bounded scalar curvature 30
4. Bounded $L^2$-curvature conjecture for the Einstein scalar field equations 39
  4.1. Initial value problem 40
  4.2. Bounded $L^2$-curvature conjecture for Einstein’s equations 40
  4.3. Bounded $L^2$-curvature conjecture for the Einstein scalar field equations 41
5. Sm and Wylie-Yeroshkin Riemann curvature 41
  5.1. Integral inequalities for scalar and Ricci curvatures 43
  5.2. Killing vector fields with constant length 47
  5.3. Remark on $Rm^L$ and $Rm^{Wy}$ 48
6. Uniqueness for the Ricci-harmonic flow 53
  6.1. Forward uniqueness 53
  6.2. Backward uniqueness 60
Appendix A. Evolution equations of the Ricci-harmonic flow 66

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1. INTRODUCTION

In this paper we continue the study (see [42, 43]) of Ricci-harmonic flow
\begin{align}
\partial_t g(t) &= -2 \text{Ric}_{g(t)} + 4 \nabla_{g(t)} u(t) \otimes \nabla_{g(t)} u(t), \\
\partial_t u(t) &= \Delta_{g(t)} u(t)
\end{align}
on a complete $n$-dimensional manifold $M$, where $g(t)$ is a family of smooth Riemannian metrics on $M$ and $u(t)$ is a family of smooth functions on $M$. The short time existence established by List [46, 47] and later extended by Müller [49, 51], says that, given an initial data $(g_0, u_0)$ with $g_0$ and $u_0$ being, respectively, a smooth Riemannian metric and a smooth function on $M$, the system (1.1) exists over a maximal time interval $[0, T_{\text{max}})$, where $T_{\text{max}}$ is a finite number or infinity.

In the following we consider the Ricci-harmonic flow on $M \times [0, T]$ or on $M \times [0, T)$, depending on different situations, with $T \in (0, T_{\text{max}})$. We brief our main results below.

Convention: From now on we always omit the time variable $t$ and write $\Box, \Delta, \nabla, u, R_m, \text{Ric}, R, dV_g, | \cdot |$ for $\Box_{g(t)} := \partial_t - \Delta_{g(t)} - \nabla_{g(t)} u(t), R_{m,g(t)}, \text{Ric}_{g(t)}, R_{g(t)}, dV_{g(t)}, | \cdot |_{g(t)}$ in the concrete computations, respectively. We write $P \lesssim Q$ or $Q \gtrsim P$ for two quantities $P$ and $Q$, if $P \leq C Q$ for some uniform constant $C$ depending only on $g_0, u_0$, and the dimension $n$. The uniform constants $C$ may vary from lines to lines.

(O) Some known results. It is known that the Ricci-harmonic flow shares the many properties with the Ricci flow. Here we list some results both for the Ricci-harmonic flow and the Ricci flow, see Table 1. Besides these results, there are other works on the Ricci-harmonic flow including gradient estimates, eigenvalues, entropies, functionals, and solitons, etc., see [8, 13, 18, 19, 20, 21, 43, 44, 50].

In particular, we mention a result on the gradient estimates of $u(t)$. Under the curvature condition
\begin{align}
\sup_{M \times [0, T]} |\text{Ric}_{g(t)}|_{g(t)} \leq K,
\end{align}
Theorem B.2 shows that
\begin{align}
|\nabla_{g(t)} u(t)|_{g(t)}^2 \lesssim K
\end{align}
on $M \times [0, T]$, where $\lesssim$ depends only on $n$. Moreover, under a stronger condition
\begin{align}
\sup_{M \times [0, T]} |R_m|_{g(t)} \leq K,
\end{align}

\footnote{Given a flow on $M \times [0, T]$ or $M \times [0, T)$, a uniform constant $C$ in this paper means a positive constant depending only on the initial data of the flow, $M$, and $T$. Of course, when $T$ varies, $C$ varies too.}
TABLE 1. Ricci-harmonic flow (RHF) via Ricci flow (RH)

| Known properties for $T_{\text{max}} < \infty$ | RF | RHF |
|-----------------------------------------------|----|-----|
| $\lim_{t \to T_{\text{max}}} \max_M |\text{Rm}_{g(t)}|^2_{g(t)} = \infty$ | $[22]$ | $[46]$ |
| $\lim_{t \to T_{\text{max}}} \max_M |\text{Ric}_{g(t)}|^2_{g(t)} = \infty$ | $[54]$ | $[13]$ |
| **either** $\lim_{t \to T_{\text{max}}} \max_M R_g(t) = \infty$ or $\max_M R_g(t) \lesssim 1$ and $\lim_{t \to T_{\text{max}}} \frac{|W_{g(t)}|}{R_{g(t)}} = \infty$ | $[7]$ | $[43]$ |
| $T < \infty$, $n = 4$, $|R_{g(t)}|_{g(t)} \lesssim 1 \Rightarrow \int_M |\text{Ric}_{g(t)}|^2_{g(t)} dV_{g(t)} \lesssim 1$ and $\int_M |\text{Rm}_{g(t)}|^2_{g(t)} dV_{g(t)} \lesssim 1$ | $[4, 55]$ | $[43]$ |
| **Conjecture** : $\lim_{t \to T_{\text{max}}} \max_M R_g(t) = \infty$ | see $[7]$ | $[44]$ |
| **Pseudo-locality theorem** | $[52]$ | $[18]$ |

1 For the general Ricci-harmonic flow, this result was proved by Müller [49].
2 Recently, a new and elementary proof was given by Kotschwer, Munteanu, and Wang [32].
3 Here $W_{g(t)}$ is the Weyl tensor of $g(t)$, and for RF we let $u(t) \equiv 0$. According to the evolution equation for $R_{g(t)}$ (see (A.8) – (A.9)) and the maximum principle, we can assume that $R_{g(t)} > 0$ for all $t$.
4 This conjecture is due to Hamilton and was verified for the Kähler-Ricci flow by Zhang [72] and the Type-I Ricci flow by Enders, Müller and Topping [17].

Theorem B.2 also gives

\begin{equation}
|\nabla^2_{g(t)} u(t)|^2_{g(t)} \lesssim K
\end{equation}

on $M \times [0, T]$, where $\lesssim$ depends only on $n$.

However, Cheng and Zhu [13] proved that under the condition (1.2), the Riemannian curvature remains bounded and hence the Hessian $\nabla^2_{g(t)} u(t)$ is bounded along the flow.

(A) Gradient estimates under the condition (1.2). In this section, we assume that $M$ is closed. It is clear from (A.8) that the gradient of $u(t)$ along the flow (1.1) is

\(2\)Their bound is implicit, however, following the argument of [32], we can give an explicit bound.
bounded\(^3\) without any curvature condition, in terms of an initial data. According to Lemma [A.1] integrating (A.8) over \(M \times [0, T]\), one can prove that the total space-time \(L^2\)-norm of \(\nabla^2 g(t)u(t)\) is bounded\(^4\), i.e.,
\[
\int_0^T \int_M |\nabla^2 g(t)u(t)|^2_{g(t)} \, dV_t \, dt \lesssim e^T
\]
for any \(T < T_{\text{max}}\), without any curvature conditions such like (1.2) or (1.4). For fixed \(T \in (0, T_{\text{max}})\), one can get an upper bound, without any curvature conditions, for \(\nabla^2 g(t)u(t)\) and then \(\Delta g(t)u(t)\) on \(M \times [0, T]\), however this bound may depend on time \(T\) and even tends towards to infinity when \(T \to T_{\text{max}}\).

The first result in this paper is to obtain the time-independent (i.e., depends on \(T_{\text{max}}\)) bound of \(\Delta g(t)u(t)\) under the curvature condition (1.2). We have mentioned that this time-independent bound has been implicitly obtained in [13], and our contribution is to obtain an explicit bound of \(\Delta g(t)u(t)\). Under the condition (1.2), one can prove (see Proposition [2.6])
\[
(1.6) \int_M |\Delta g(t)u(t)|^2 dV_{g(t)} \leq C(1 + K)e^{C(1 + K)t} \leq C(1 + K)e^{C(1 + K)T_{\text{max}}}, \quad t \in [0, T],
\]
for some uniform constant \(C\). This \(L^2\)-norm bound together with the non-collapsing result (see Corollary [2.5]) implies that
\[
|\Delta g(t)u(t)| \leq \frac{C(1 + K)}{(1 + T)^{n/2}} \exp \left[ C \left( 1 + T + KT + e^{C\sqrt{T}} \right) \right]
\]
over \(M \times [0, T]\), where \(C\) is a uniform constant.

**Theorem 1.1.** (See also Theorem 2.6) If the Ricci-harmonic flow (1.1) satisfies the curvature condition (1.2), then \(\Delta g(t)u(t)\) is bounded and an explicit bound is given by (1.7).

**(B) Results for a generalized Ricci flow.** The author [43] introduced a class of generalized Ricci flow (For motivation see Section 3), called \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow:
\[
(1.8) \quad \partial_t g(t) = -2\text{Ric} g(t) + 2\alpha_1 \nabla g(t)u(t) \otimes \nabla g(t)u(t),
(1.9) \quad \partial_t u(t) = \Delta g(t)u(t) + \beta_1 |\nabla g(t)u(t)|^2_{g(t)} + \beta_2 u(t).
\]
Here \(\alpha_1, \beta_1, \beta_2\) are given constants. Under a technical condition “regular” (for definition, see subsection 3.1) the system (1.8) – (1.9) has the following estimate:
\[
(1.10) \quad |\nabla g(t)u(t)|_{g(t)} \leq C
\]
for some uniform constant \(C\) depending only on \(\alpha_1, \beta_1, \beta_2, n\). Thus, roughly speaking, the regular condition on constants \(\alpha_1, \beta_1, \beta_2\) guarantees the boundedness of \(\nabla g(t)u(t)\).

\(^3\)When \(M\) is complete and noncompact, the same result is still true for List’s solution of the Ricci-harmonic flow; see Theorem [A.3] and Theorem [B.1].

\(^4\)A general estimate is obtained in [A.25], where a generalized Ricci flow is considered.
The flow (1.8) – (1.9) is connected with the \((K, N)\)-super Ricci flow introduced in [35], along which the W-entropy is constant. For more detail, see Section 3.

**Theorem 1.2. (See also Theorem 3.2)** Let \((g(t), u(t))_{t \in [0, T]}\) be a solution to the regular \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow (1.8) – (1.9) on a closed \(n\)-dimensional manifold \(M\) with \(T \leq \infty\) and the initial data \((g_0, u_0)\). Assume that \(S_{g(t)} + C \geq C_0 > 0\) along the flow for some uniform constants \(C, C_0 > 0\). Then

\[
\sin_{g(t)}(t) \leq C_1 + C_2 \max_{M \times [0, T]} \frac{|W_{g(s)}(s)| + |\nabla^2_{g(s)} u(s)|^2_{g(s)}|}{S_{g(s)} + C}
\]

where

\[
\sin_{g(t)}(t) = \frac{\Ric_{g(t)} - \alpha_1 \nabla_{g(t)} u(t) \otimes \nabla_{g(t)} u(t)}{S_{g(t)}},
\]

\[
S_{g(t)} = \text{tr}_{g(t)} \sin_{g(t)}(t) = R_{g(t)} - \alpha_1 |\nabla_{g(t)} u(t)|^2_{g(t)},
\]

\[
\sin_{g(t)} := \sin_{g(t)} - \frac{S_{g(t)}}{n} g(t)
\]

and \(W_{g(t)}\) is the Weyl tensor field of \(g(t)\).

Here the technical assumption \(S_{g(t)} + C \geq C_0 > 0\) is necessary in the theorem, since, due to the undermined signs of \(\alpha_1, \beta_1, \beta_2\), we cannot in general deduce any bounds for \(S_{g(t)}\) from the evolution equations of (1.8) – (1.9) (see, for example, the evolution equation (3.11)). In the simplest case, \(\alpha_1 \geq 0\) and \(\beta_1 = \beta_2 = 0\) (i.e., Ricci-harmonic flow), we have a lower bound from Lemma A.1.

This result is obtained by applying Hamilton-Cao’s method (see for example [7, 33]) on (1.8) – (1.9). More precisely, we consider the quantity

\[
f_1 := \frac{|\sin_{g(t)}(t) + \frac{C}{n} g(t)|^2_{g(t)}}{(S_{g(t)} + C)^2},
\]

and deduce the following evolution inequality

\[
\square f \leq 2 \langle \nabla f, \nabla \ln (S + C) \rangle + 4(S + C) f \left( -f + \frac{2}{n-2} f^{1/2} + C + \frac{|W|^2 + |\nabla^2 u|^2}{S + C} \right).
\]

A direct consequence of the maximum principle applied on (1.13) yields (1.11). In conclusion, we can derive the long-time existence of (1.8) – (1.9).

**Corollary 1.3. (See also Corollary 3.3)** Let \((g(t), u(t))_{t \in [0, T]}\) be a solution to the regular \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow (1.8) – (1.9) on a closed \(n\)-dimensional manifold \(M\) with \(T \leq \infty\) and the initial data \((g_0, u_0)\). Then only one of the followings cases occurs:

(a) \(T = \infty\);

(b) \(T < \infty\) and \(|\Ric_{g(t)}(t)|_{g(t)} \leq C\) for some uniform constant \(C\);

(c) \(T < \infty\) and \(|\Ric_{g(t)}(t)|_{g(t)} \to \infty\) as \(t \to T\). In this case, there are only two subcases:

(c1) \(|R_{g(t)}(t)|_{g(t)} \to \infty\),

(c2) \(|R_{g(t)}(t)|_{g(t)} \leq C\) for some uniform constant \(C\) and there exist uniform constants \(C_1, C_2 > 0\) such that \(S_{g(t)} + C_1 \geq C_2 > 0\) and

\[
\frac{|W_{g(t)}(t)|_{g(t)} + |\nabla^2_{g(t)} u(t)|^2_{g(t)}}{S_{g(t)} + C_1} \to \infty.
\]
as \( t \to T \).

This is a general picture of the long time existence for regular \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flows, containing results in \([7, 43]\). Since the signs of \(\alpha_1, \beta_1, \beta_2\) are not determined, we can not discard the case (b) which is of course true for Ricci-harmonic flow and Ricci flow.

In the case of dimension \( n = 4 \), we can consider another quantity

\[
(1.14) \quad f_2 := \frac{|\text{Sic}_{g(t)}|^2}{S_{g(t)} + C}.
\]

Since the sign of \(\alpha_1\) is undermined, some term in the evolution inequality (see \((3.37)\)) for \(f_2\) will contain the \(L^4\)-norm of \((g(t), u(t)))\). Though we can prove

\[
(1.15) \quad \left| \nabla^2_{g(t)} u(t) \right|_{L^1_{[0,T]}L^2(M)} \leq C e^{CT} \leq C e^{CT_{\max}}
\]

for any \( T \in (0, T_{\max}) \), it is not clear whether we can get its (spatial) \(L^4\)-norm\(^5\).

**Corollary 1.4. (See also Corollary 3.6)** Let \((g(t), u(t)))_{t \in [0,T]}\) be a solution to the regular \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow \((1.8) - (1.9)\) on a closed 4-manifold \(M\) with \( T \leq \infty \) and the initial data \((g_0, u_0)\). Assume that \(S_{g(t)} + C \geq C_0 > 0\) and \(S^2_{g(t)} \leq C_1 < \infty \) along the flow for some uniform constants \(C, C_0, C_1 > 0\). Then

\[
\int_0^T \int_M |\text{Sic}_{g(t)}|^2_{g(t)} dV_{g(t)} dt \leq C'(1 + s) e^{C' s} +
\]

\[
(1.16) \quad C'[1 - \text{sgn}(\alpha_1, 0)]e^{C' s} \int_0^s \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt,
\]

\[
\int_0^T \int_M |\text{Rm}_{g(t)}|^2_{g(t)} dV_{g(t)} dt \leq C'(1 + s) e^{C' s} +
\]

\[
(1.17) \quad C'[1 - \text{sgn}(\alpha_1, 0)]e^{C' s} \int_0^s \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt,
\]

for all \( s \in [0, T] \), where \(C' = C'(g_0, u_0, \alpha_1, \beta_1, \beta_2, C, C_0, C_1, A_1, \chi(M))\) is a uniform constant. Here \( |\nabla^2_{g(t)} u(t)|^2_{g(t)} \leq A_1\) holds along the flow (by the regularity) for some uniform constant \(A_1 > 0\) (which depends only on \(g_0, u_0\) and \(\alpha_0, \beta_1, \beta_2\)).

In the corollary, the shorthand notion \(\text{sgn}(\alpha_1, 0)\) is defined to be 1 if \(\alpha_1 \geq 0\), and 0 otherwise. When \(\alpha_1\) is nonnegative, the inequalities \((1.16)\) and \((1.17)\) give us uniform \(L^1_{[0,T]}L^2(M)\)-norms to \(\text{Sic}_{g(t)}\) and \(\text{Rm}_{g(t)}\). Moreover, using an equality (above \((3.43)\)) for \(\text{Vol}_t\) and \(|\nabla^2_{g(t)} u(t)|^2_{g(t)} \leq A_1\), the integral of \(\text{Sic}_{g(t)}\) can be replaced by the integral of \(\text{Ric}_{g(t)}\). Thus, in the case that \(\alpha_1 \geq 0\), **Corollary 1.4** gives uniform \(L^1_{[0,T]}L^2(M)\)-norms to \(\text{Ric}_{g(t)}\) and \(\text{Rm}_{g(t)}\).

To derive \(L^1_{[0,T]}L^P(M)\)-norms for \(\text{Sic}_{g(t)}\) and \(\text{Rm}_{g(t)}\), for simplicity, introduce the basic assumption \(BA\), original defined in \([4, 55]\) (see also \([43]\)), for a solution \((g(t), u(t)))_{t \in [0,T]}\) to the regular \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow:

\(^5\)Under the curvature condition \((1.4)\), we can always get a bound for the \(L^4\)-norm of \(\nabla^2_{g(t)} u(t)\) by Theorem \([43, 1.2]\).
Theorem B.1. Actually, by \( \Lambda \), one has the estimate
\[
\text{Theorem 1.5. (See also Theorem 3.7)} \quad \text{Suppose that for any } s 
\text{on } n \text{ and } p, \text{such that \( 1 \leq s \leq 1 \) along the flow;}
\text{the last condition is obtained from the regularity of the flow and the third condition implies } \int_0^t |\nabla g(t)u(t)|^2 dt \leq A_1 \text{ along the flow.}
\]

Theorem 1.5. (See also Theorem 3.7) Suppose that \( (g(t), u(t))_{t \in [0,T]} \) satisfies \( BA \). Then
\[
\int_0^s \int_M |\text{Sic}_g(t)|^4_{g(t)} dV_{g(t)} dt \leq \bar{C}(1+s) e^{\bar{C}s} + \bar{C}[1 - \text{sgn}(\alpha_1, 0)] e^{\bar{C}s} \int_0^s \int_M |\nabla^2 g(t)u(t)|^4 dV_{g(t)} dt,
\]
(1.18)
\[
\int_s^T \int_M |\text{Sic}_g(t)|^2_{g(t)} dV_{g(t)} dt \leq \left[ (T-s)e^T \text{Vol}_0 \right]^{1/p} e^{\bar{C}T} \left[ \bar{C}(1+T) \right]^{2/p},
\]
(1.19)
for any \( s \in [0, T) \). Here \( \bar{C} \) is a uniform constant which depends only on \( g_0, u_0, \alpha_1, \beta_1, \beta_2, A_1, \chi(M) \).

(C) A conjecture for the Einstein scalar field equations. This conjecture is based on the following, where \( M \) is a complete manifold.

Theorem 1.6. (See also Theorem 2.7) Let \( (g(t), u(t))_{t \in [0,T]} \) be a solution to the Ricci-harmonic flow. Suppose there exist constants \( \rho, K, L, P > 0 \) and \( x_0 \in M \) such that \( B_{g_0}(x_0, \rho/\sqrt{K}) \) is compactly contained on \( M \) and the following conditions hold on \( B_{g_0}(x_0, \rho/\sqrt{K}) \times [0, T] \). For any \( p \geq 3 \), there is a constant \( C \), depending only on \( p \) and \( n \), such that
\[
\int_{B_{g_0}(x_0, \rho/2\sqrt{K})} |\text{Rm}_{g(t)}|^p_{g(t)} dV_{g(t)} \leq C \Lambda_1 e^{\Lambda_2 T} \int_{B_{g_0}(x_0, \rho/\sqrt{K})} |\text{Rm}_{g_0}|^p_{g_0} dV_{g_0}
\]
(1.20)
\[
+ CKP \left( 1 + \rho^{-2p} \right) e^{\Lambda_2 T} \text{Vol}_{g(t)} \left( B_{g_0}(x_0, \rho/\sqrt{K}) \right).
\]
Here \( \Lambda_1 := 1 + K \) and \( \Lambda_2 := K + L^2 + P^2(1 + K^{-1}) \).

This theorem extends \( [32] \) to the Ricci-harmonic flow. Along the argument in \( [32] \) we consider the quantity
\[
\frac{d}{dt} \int_M |\text{Rm}|^p_{g(t)} dV_t
\]

\[\text{The second assumption } |\nabla g(t)u(t)|_{g(t)} \leq L \text{ can not be derived by Theorem 6.1. Actually, by Theorem 6.1 one has the estimate}
\]
\[
|\nabla g(t)u(t)|_{g(t)} \leq \rho^2 C_u \left( \frac{K}{\rho^2 + 1} \right) = C_u \left( K + \frac{p^2}{T} \right)
\]
which depends on time \( t \).
where \( \phi \) is a Lipschitz function with support in \( B_{g(0)}(x_0, \rho/\sqrt{K}) \). For the Ricci-harmonic flow, extra integrals involving derivatives of \( u(t) \) appear in the computation, however, these integrals can be treated with the help of the last assumptions in (1.20).

By Hölder’s inequality we can get a similar upper bound for \( p = 2 \). From Theorem 1.6 we can get an upper bound for the \( L^2 \)-norm of \( Rm_g(t) \). Motivated by the inequality (1.21), we in this section impose a conjecture for the Einstein scalar field equations, which is analogous to the corresponding conjecture for the Einstein vacuum equations proved by Klainerman, Rodnianski, and Szeftel [26, 56-59, 60].

Consider Einstein’s scalar field equation or Einstein Klein-Gordon system

\[
R_{\alpha \beta} - \frac{1}{2}R g_{\alpha \beta} = T_{\alpha \beta}, \quad T_{\alpha \beta} = 2 \partial_\alpha u \partial_\beta u - \frac{1}{2} |D u|^2 g,
\]

where \( u \) is a smooth function on a four dimensional Lorentzian space-time \((M, g)\), \( R_{\alpha \beta}, R \), and \( D \) denote, respectively, the Ricci curvature tensor, scalar curvature, and the Levi-Civita connection of \( g \). In this case, the Einstein equation (1.15) can be written as

\[
R_{\alpha \beta} - 2 \partial_\alpha u \partial_\beta u = 0.
\]

As discussed in [53], we should impose a matter equation

\[
\Delta u = 0
\]

for \( u \), where \( \Delta := D^\alpha D_\alpha \). Hence we should consider a system of PDEs

\[
R_{\alpha \beta} - 2 \partial_\alpha u \partial_\beta u = 0, \quad \Delta u = 0.
\]

An initial data set \((\Sigma, g, k, u_0, u_1)\) for (1.23) consists of a three dimensional manifold \( \Sigma \), a Riemannian metric \( g \), a symmetric 2-tensor \( k \), together with two functions \( u_0 \) and \( u_1 \) on \( \Sigma \), all assumed to be smooth, verifying the constraint equations,

\[
\nabla^i k_{ij} - \nabla^j k_{ij} = u_1 \nabla^i u_0,
\]

\[
R - |k|^2 + (\text{tr} k)^2 = u_1^2 + |\nabla u_0|^2,
\]

where \( \nabla \) is the Levi-Civita connection of \( g \).

Given an initial data set \((\Sigma, g, k, u_0, u_1)\), the Cauchy problem consists in finding a four-dimensional Lorentzian manifold \((M, g)\) and a smooth function \( u \) on \( M \) satisfying (1.22), and also an embedding \( i : \Sigma \to M \) such that

\[
i^g g = g, \quad i^g u = u_0, \quad i^g K = k, \quad i^g (Nu) = u_1,
\]

where \( N \) is the future-directed unit normal to \( i(\Sigma) \) and \( K \) is the second fundamental form of \( i(\Sigma) \). The local existence and uniqueness result for globally hyperbolic developments can be found in, for example, [53], Theorem 14.2. For stability and instability for Einstein’s scalar field equation, we refer to [14, 15, 33, 34, 62, 63, 64, 65].

For Einstein’s equations (i.e., \( u = 0 \) in (1.22), and the corresponding initial data set is denoted by \((\Sigma, g, k)\)), Klainerman [25] proposed the following conjecture:
The Einstein vacuum equations admit local Cauchy developments for initial data sets \((\Sigma, g, k)\) with locally finite \(L^2\)-curvature and locally finite \(L^2\)-norm of the first covariant derivatives of \(k\).

This conjecture was recently solved by Klainerman, Rodnianski and Szeftel [26]. Motivated by Theorem 1.6 and Klainerman’s conjecture, we propose the following

**Conjecture 1.7.** (See also Conjecture 4.2) The Einstein scalar field equations admit local Cauchy developments for initial data sets \((\Sigma, g, k, u_0, u_1)\) with locally finite \(L^2\)-curvature, locally finite \(L^2\)-norm of the first covariant derivatives of \(k\), locally finite \(L^2\)-norm of the covariant derivatives (up to second order) of \(u_0\), and locally finite \(L^2\)-norm of the covariant derivatives (up to first order) of \(u_1\).

**D) Some notions on Riemann curvatures of Bakry-Émery Ricci curvature.** We now compare our curvature \(S_m\) with \(\alpha_1 = 2\) (see (3.10)) with a notion of curvature introduced recently by Wylie and Yeroshkin [68]. Let \((M, g)\) be a Riemannian manifold with a smooth function \(u\). Wylie and Yeroshkin introduced the following weighted connection

\[
(1.27) \quad \nabla^u_X Y := \nabla_X Y - (Yu)X - (Xu)Y.
\]

By Proposition 3.3 in [68], we have

\[
(1.28) \quad R^u_{i j k l} = R_{i j k l} + \nabla_j \nabla_k u g_{i l} - \nabla_i \nabla_k u g_{j l} + \nabla_j \nabla_l u g_{i k} - \nabla_i \nabla_l u g_{j k},
\]

where \(R^u_{i j k l}\) is the induced Riemann curvature tensor associated to the connection \(\nabla^u\). The Ricci curvature associated to \(\nabla^u\) is defined by

\[
(1.29) \quad R^u_{j k} := g^{i e} R^u_{i j k e} = R_{j k} + (n - 1) \nabla_j \nabla_k u + (n - 1) \nabla_k \nabla_j u.
\]

Here the last formula also follows from Proposition 3.3 in [68]. Recall from (3.10) that (with \(\alpha_1 = 2\))

\[
(1.30) \quad S_{i j k l} = R_{i j k l} - \nabla_i \nabla_k u g_{j l} - \nabla_i \nabla_l u g_{j k}.
\]

From now on, we are given a smooth function \(u\) on \(M\) and write

\[
(1.31) \quad R^L_{i j k l} := S_{i j k l}, \quad R^{WY}_{i j k l} := R^u_{i j k l}, \quad R^L_{j k} := g^{i e} R^L_{i j k e}, \quad R^{WY}_{j k} := R^u_{j k}, \quad R^{L WY}_{j k} := R^{WY}_{j k}, \quad R^{L WY}_{j k} := g^{j k} R^{L WY}_{j k}.
\]

From (1.29) and (1.30), we have

\[
\text{Ric}^L = \text{Ric} - 2du \otimes du, \quad \text{Ric}^{WY} = \text{Ric} + (n - 1) du \otimes du + (n - 1) \nabla^2 u.
\]

Consider another Ricci curvature of \(R^{WY}_{i j k l}\):

\[
\hat{R}^{WY}_{j k} := g^{i e} R^{WY}_{i j k e} = R_{j k} + \left(\Delta u + |\nabla u|^2\right) g_{j k} - \nabla_j \nabla_k u - \nabla_k \nabla_j u.
\]

There are some relations between these two notions on “Riemann curvature tensors, e.g.,

\[
\text{Ric}^{L WY} - \text{Ric}^{WY} = \left(\Delta u + |\nabla u|^2\right) g - n \left(\nabla^2 u + du \otimes du\right).
\]
We note that $\text{Ric}^L$ and $\text{Ric}^{\text{WY}}$ are actually the Ricci curvatures in the sense of Bakey-Émery [1]. We here use our notions to keep the paper smoothly.

We now have four different types of Ricci curvatures, $\text{Ric}$, $\text{Ric}^L$, $\text{Ric}^{\text{WY}}$, and $\hat{\text{Ric}}^{\text{WY}}$, and three different types of scalar curvatures, $\text{R}$, $\text{R}^L$, and $\text{R}^{\text{WY}}$. In order to compare those quantities, we introduce a notation $\mathcal{P} \leq_{\text{1, } \mu} \mathcal{Q}$, which is an integral inequality with respect to the measure $\mu$.

Given two scalar quantities $\mathcal{P}$, $\mathcal{Q}$ on $(M, g)$, and a measure $\mu$, we write $\mathcal{P} \leq_{\text{1, } \mu} \mathcal{Q}$ if the following inequality holds. When $d\mu$ is the volume form $dV$, we simply write (1.34) as $\mathcal{P} \leq_{\text{1}} \mathcal{Q}$. When $d\mu$ is the measured volume form $c/\mu$, we write (1.34) as $\mathcal{P} \leq_{\text{1, } f} \mathcal{Q}$. Similarly, we can define $\mathcal{P} \leq_{\mu, \text{Q}}$.

**Proposition 1.8.** (See also Proposition 5.4) For any measure $\mu$ on $M$ and smooth function $u$ on $M$, we have

\[ R^L \leq_{\text{1, } \mu} R, \quad R \leq_{\text{1}} R^{\text{WY}}, \quad R =_{\text{1, } \mu} R^{\text{WY}}. \]

This proposition shows that $R^L \leq_{\text{1}} R \leq_{\text{1}} R^{\text{WY}}$ and $R^L =_{\text{1, } \mu} R^{\text{WY}}$. Thus, in the sense of integrals, $R^L$ is weaker and $R^{\text{WY}}$ is stronger than $R$, respectively.

Next we consider the similar question on Ricci curvatures. Let $(M, g)$ be a closed Riemannian manifold with a smooth function $u$, and $\mu$ be a given measure on $M$. Given two Ricci curvatures $\text{Ric}^\bullet$, $\text{Ric}^\diamond \in \mathfrak{R}_{\text{4}} := \{ \text{Ric}, \text{Ric}^L, \text{Ric}^{\text{WY}}, \hat{\text{Ric}}^{\text{WY}} \}$, we say

\[ \text{Ric}^\bullet \leq_{\text{1, } \mu} \text{Ric}^\diamond \]

if $\text{Ric}^\bullet(X, X) \leq_{\text{1, } \mu} \text{Ric}^\diamond(X, X)$ holds for all vector fields $X \in \mathfrak{X}(M)$. Similarly we can define $\text{Ric}^\bullet \leq_{\text{1}} \text{Ric}^\diamond$ and $\text{Ric}^\bullet \leq_{\text{1, } f} \text{Ric}^\diamond$. We say

\[ \text{Ric}^\bullet \leq_{\text{1K}, \mu} \text{Ric}^\diamond \]

if $\text{Ric}^\bullet(X, X) \leq_{\text{1, } \mu} \text{Ric}^\diamond(X, X)$ holds for all Killing vector fields $X \in \mathfrak{X}_K(M)$, where $\mathfrak{X}_K(M)$ is the space of all Killing vector fields on $M$. Similarly we can define $\text{Ric}^\bullet \leq_{\text{1K}} \text{Ric}^\diamond$ and $\text{Ric}^\bullet \leq_{\text{1K, } f} \text{Ric}^\diamond$.

Consider the subset $\mathfrak{X}_\text{KC}(M)$ of $\mathfrak{X}_K(M)$, which consists of Killing vector fields on $M$ with constant norm. we say

\[ \text{Ric}^\bullet \leq_{\text{1KC}, \mu} \text{Ric}^\diamond \]

if $\text{Ric}^\bullet(X, X) \leq_{\text{1, } \mu} \text{Ric}^\diamond(X, X)$ holds for all $X \in \mathfrak{X}_\text{KC}(M)$. Similarly we can define $\text{Ric}^\bullet \leq_{\text{1KC}} \text{Ric}^\diamond$ and $\text{Ric}^\bullet \leq_{\text{1KC, } f} \text{Ric}^\diamond$. We then obtain the following two results.

**Theorem 1.9.** (see also Theorem 5.6) Let $(M, g)$ be a closed Riemannian manifold with a smooth function $u$ and $\mu$ be a given measure on $M$. Then we have

(i) $\text{Ric}^L \leq_{\text{1, } \mu} \text{Ric}$.

(ii) $\text{Ric} \leq_{\text{1KC}} \text{Ric}^{\text{WY}}$.

(iii) $\text{Ric} \leq_{\text{1KC}} \hat{\text{Ric}}^{\text{WY}}$. 

A consequence of Theorem 1.9 indicates

\[(1.39) \quad \text{Ric}^L \leq \text{IKC} \text{Ric} \leq \text{IKC} \text{Ric}^WY \quad \text{and} \quad \text{Ric}^L \leq \text{IKC} \text{Ric} \leq \text{IKC} \text{Ric}^WY .\]

**Theorem 1.10.** *(See also Theorem 5.8)* Let \((M, g)\) be a closed Riemannian manifold with a smooth function \(u\) and \(\mu\) be a given measure on \(M\). Then we have

(i) \(\text{Ric} \leq \text{IKC}_f \text{Ric}^WY\), and

(ii) \(\text{Ric} \leq \text{IKC}_f \text{Ric}^WY\).

where \(f := u - u_{\min} + c_0\) and \(c_0 \geq 1/e\).

For a given odd-dimensional sphere, we can always find a Riemannian metric \(g\) and a Killing vector field \(X\) of constant length with respect to \(g\).

**Proposition 1.11.** *(See also Proposition 5.9)* On each of 28 homotopical seven-dimensional spheres \(M\), there exist a Riemannian metric \(g\) and a nonzero vector field \(X\), such that

- \(\text{Ric}^L(X, X) \leq \text{Ric}(X, X) \leq \text{Ric}^WY(X, X)\) and \(\text{Ric}^L(X, X) \leq \text{Ric}(X, X) \leq \text{Ric}^WY(X, X)\) hold.

- for any smooth function \(u\) on \(M\), \(\text{Ric}^L(X, X) \leq f \cdot \text{Ric}(X, X) \leq f \cdot \text{Ric}^WY(X, X)\) and \(\text{Ric}^L(X, X) \leq \text{Ric}(X, X) \leq f \cdot \text{Ric}^WY(X, X)\) hold, where \(f := u - u_{\min} + c_0\) with \(c_0 \geq 1/e\).

We say that a Riemannian metric \(g\) on \(M\) is of cohomogeneity 1 if some compact Lie group \(G\) acts smoothly and isometrically on \(M\) and the space of orbits \(M/G\) with respect to this action is one-dimensional.

**Proposition 1.12.** *(See also Proposition 5.10)* Let \(n \geq 2\) and \(e > 0\). On the sphere \(S^{2n-1}\), there are a (real-analytic) Riemannian metric \(g_e\) of cohomogeneity 1, with the property that all section curvatures of \(g_e\) differ from 1 at most by \(e\), and a (real-analytic) nonzero vector field \(X_e\), such that

- \(\text{Ric}_{g_e}^L(X_e, X_e) \leq \text{Ric}_{g_e}(X_e, X_e) \leq \text{Ric}_{g_e}^WY(X_e, X_e)\) and \(\text{Ric}_{g_e}^L(X_e, X_e) \leq \text{Ric}_{g_e}(X_e, X_e) \leq \text{Ric}_{g_e}^WY(X_e, X_e)\) hold.

- for any smooth function \(u\) on \(M\), \(\text{Ric}_{g_e}^L(X_e, X_e) \leq f \cdot \text{Ric}_{g_e}(X_e, X_e) \leq f \cdot \text{Ric}_{g_e}^WY(X_e, X_e)\) and \(\text{Ric}_{g_e}^L(X_e, X_e) \leq \text{Ric}_{g_e}(X_e, X_e) \leq f \cdot \text{Ric}_{g_e}^WY(X_e, X_e)\) hold, where \(f := u - u_{\min} + c_0\) with \(c_0 \geq 1/e\).

Berestovskii and Nikonorov [2] observed that (see Remark 5.11) that if the \(S^1\)-action obtain by Tuschmann [61] is free, then we can find for \(e > 0\) a Killing vector field \(X_e\) of unit length with respect to \(g_e\).

The nonnegativity of \(R^L_{ijk\ell}\) was used in [66] to prove the compactness for gradient shrinking Ricci harmonic solitons. There is no useful relation between \(Rm^L\) and \(Rm^WY\). More precisely, we can find (see Example 5.12) a Riemannian manifold \((M, g)\) so that \(Rm^L(X, Y, Y, X) < Rm^WY(X, Y, Y, X)\) for some triple \((X, Y, u)\) of smooth vector fields \(X, Y\) and smooth function \(u\), and \(Rm^L(X, Y, Y, X) > Rm^WY(X, Y, Y, X)\) for another such triple \((X', Y', u')\).

**(E) Uniqueness for the Ricci-harmonic flow.** The uniqueness problem for the Ricci flow was proved by Hamilton [22] in the compact setting, Chen-Zhu [12]
for forward uniqueness of complete solutions, and Kotschwar \([28, 29]\) for forward and backward uniqueness of complete solutions.

It is a natural problem to prove the forward and backward uniqueness for the Ricci-harmonic flow. Actually, the forward uniqueness of the Ricci-harmonic flow when the underlying manifold is compact, was proved by List \([46]\). In this paper, we use the strategy of Kotschwar to prove the uniqueness for complete solutions of the Ricci-harmonic flow. List \([46]\) has proved that (see Theorem 1.3) if \((M, g)\) is a complete and non-compact Riemannian manifold satisfying

\[
|\text{Rm}_g|_g + |u| + |\nabla_g u|_g + |\nabla^2_g u|_g \lesssim 1
\]

then a local time existence holds for the Ricci-harmonic flow with the initial data \((g, u)\), and moreover

\[
|\text{Rm}_{g(t)}|_{g(t)} + |u(t)| + |\nabla_{g(t)} u(t)|_{g(t)} + |\nabla^2_{g(t)} u(t)|_{g(t)} \lesssim 1.
\]

Hence one may expect that the uniqueness of the Ricci-harmonic flow holds in the class \(\{(g, u) : \text{Rm}_g, \nabla_g u, \nabla^2_g u \text{ bounded}\}\). Surprisingly we prove that the uniqueness of the Ricci-harmonic flow holds in a larger class \(\{(g, u) : \text{Rm}_g \text{ bounded}\}\). The reason is that if the Riemann curvature is bounded along the flow \((1.1)\) then all derivatives of \(u(t)\) is still bounded by Theorem 5.2. To state the results, consider the following curvature condition

\[
(1.40) \sup_{M \times [0, T]} \left( |\text{Rm}_{g(t)}|_{g(t)} + |\text{Rm}_{g(t)}|_{g(t)} \right) \leq K,
\]

where \(K\) is some uniform constant.

**Theorem 1.13.** (See also Theorem 5.1) Suppose that \((g(t), u(t))\) and \((\tilde{g}(t), \tilde{u}(t))\) are two smooth complete solutions of \((1.1)\) satisfying \((1.40)\). If \((g(0), u(0)) = (\tilde{g}(0), \tilde{u}(0))\), then \((g(t), u(t)) = (\tilde{g}(t), \tilde{u}(t))\) for each \(t \in [0, T]\).

The basic idea on proving Theorem 1.13 follows from the approach of Kotschwar \([29]\) who considered the quantity

\[
\mathcal{E}(t) := \int_M \left[ t^{-1} |g(t) - \tilde{g}(t)|^2_{g(t)} + t^{-\beta} \left| \Gamma_{g(t)} - \Gamma_{\tilde{g}(t)} \right|^2_{g(t)} + \left| \text{Rm}_{g(t)} - \text{Rm}_{\tilde{g}(t)} \right|^2_{g(t)} \right] e^{-\eta} dV_{g(t)}
\]

for the Ricci flow, where \(\beta \in (0, 1)\) and \(\eta\) is a cutoff function (so that the integral is well-defined as \(t\) tends to zero). In our setting, the corresponding quantity for the Ricci-harmonic flow takes the form

\[
\mathcal{E}(t) := \int_M \left[ t^{-1} |g(t) - \tilde{g}(t)|^2_{g(t)} + t^{-\beta} \left| \Gamma_{g(t)} - \Gamma_{\tilde{g}(t)} \right|^2_{g(t)} + \left| \text{Rm}_{g(t)} - \text{Rm}_{\tilde{g}(t)} \right|^2_{g(t)} \right] e^{-\eta} dV_{g(t)}
\]

It can be shown

\[
(1.43) \quad \mathcal{E}'(t) \leq N \mathcal{E}(t)
\]

on \([0, T_0]\), for some \(T_0 \ll 1\) and \(N > 0\). From \(1.43\), together with the initial data \(\mathcal{E}(0) = 0\), we get \(\mathcal{E}(t) \equiv 0\) on \([0, T_0]\) and then on \([0, T]\).
Theorem 1.14. (See also Theorem 6.6) Suppose that \((g(t), u(t))\) are two smooth complete solutions of (1.1) satisfying (1.40). If \((g(T), u(T)) = (\tilde{g}(T), \tilde{u}(T))\), then \((g(t), u(t)) \equiv (\tilde{g}(t), \tilde{u}(t))\) for each \(t \in [0, T]\).

To prove Theorem 1.14 we use an idea of Kotschwar [28] and set
\[
T := Rm - \bar{Rm}, \quad U := \bar{\nabla}Rm - \bar{\nabla}\bar{Rm},
\]
\[
h := g - \bar{g}, \quad A := \nabla - \bar{\nabla}, \quad B := \nabla A,
\]
\[
y := \nabla^2 u - \bar{\nabla}^2 \bar{u}, \quad z := \nabla^3 u - \bar{\nabla}^3 \bar{u},
\]
\[
v := u - \bar{u}, \quad w := \nabla u - \bar{\nabla}\bar{u}, \quad x := \nabla w,
\]
\[
X := T \oplus U \oplus y \oplus z, \quad Y := h \oplus A \oplus v \oplus w \oplus x.
\]

As the same argument of [28], we can prove
\[
\left|\Box_{\tilde{g}(t)} X^2_{\tilde{g}(t)}\right| \lesssim |X^2_{\tilde{g}(t)} + |Y^2_{\tilde{g}(t)}| + |\nabla X^2_{\tilde{g}(t)}|
\]
(on any \([\delta, T]\), where \(\delta \in (0, T)\). A result of Kotschwar (see Theorem 6.6 below) implies \(X = Y = 0\) on \([\delta, T]\), and then on \([0, T]\).

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2. Gradient and local curvature estimates

In this section we assume that \((g(t), u(t))_{[0, T]}\) is a solution of (1.1) on a closed \(n\)-dimensional manifold \(M\) and use the convention in Section 1. From the equation (A.8) in Lemma A.1 we see that \(\nabla u^2\) is uniformly bounded, i.e.,
\[
|\nabla u| \leq L
\]
on \(M \times [0, T]\) for some uniform constant \(L\) depending only on the initial data \((g_0, u_0) = (g(0), u(0))\). We also notice from (A.9) that
\[
R - 2\nabla u^2 \geq -1 \implies R \geq -1.
\]
Moreover, integrating over the space-time \(M \times [0, T]\), we get
\[
\frac{d}{dt} \int_M |\nabla u|^2 dV_t = \int_M \partial_t |\nabla u|^2 dV_t + \int_M |\nabla u|^2 \partial_t dV_t
\]
\[
= \int_M \left[-2|\nabla^2 u|^2 - 4|\nabla u|^4 - (R - 2|\nabla u|^2)\right] dV_t
\]
and then
\[
\frac{d}{dt} \int_M |\nabla u|^2 dV_t + 2 \int_M |\nabla^2 u|^2 dV_t = -4 \int_M |\nabla u|^4 dV_t - \int_M (R - 2|\nabla u|^2) dV_t.
\]
Denoting $\text{Vol}_t$ the volume of $(M, g(t))$, we have
\begin{equation}
\int_0^t \int_M |\nabla^2 u|^2 dV_t dt \leq L^2 \text{Vol}_0 + C \int_0^t \text{Vol}_s ds
\end{equation}
from (2.1) and (2.2), where $C$ is a uniform constant. Using (A.10) and (2.2), we have
\[
\partial_t \text{Vol}_t = \int_M \partial_t dV_t = \int_M \left( -R + 2|\nabla u|^2 \right) dV_t \leq C \text{Vol}_t;
\]
consequently,
\begin{equation}
\text{Vol}_t \leq e^{Ct} \text{Vol}_0.
\end{equation}
Plugging (2.5) into (2.4) we conclude that
\begin{equation}
\int_0^t \int_M |\nabla^2 u|^2 dV_t dt \leq \left( L^2 + e^{Ct} \right) \text{Vol}_0 \leq C(1 + L^2)e^{Ct} \lesssim e^{Ct}.
\end{equation}
According to (A.5) we obtain
\begin{equation}
\Box \Delta u = -4|\nabla u|^2 \Delta u + 2R_{ij}\nabla^i \nabla^j u - 4\nabla_i u \nabla_j u \nabla^i \nabla^j u.
\end{equation}
In particular, the square of $\Delta u$ satisfies
\[
\partial_t |\Delta u|^2 = 2\Delta u \partial_t \Delta u = \Delta |\Delta u|^2 - 2|\nabla \Delta u|^2 - 8|\nabla u|^2 |\Delta u|^2 + 4 \left( R_{ij}\nabla^i \nabla^j u \right) \Delta u \\
- 8 \left( \nabla_i u \nabla_j u \nabla^i \nabla^j u \right) \Delta u \\
\leq \Delta |\Delta u|^2 - 2|\nabla \Delta u|^2 - 8|\nabla u|^2 |\Delta u|^2 + 4 \left( |\text{Ric}| + 2|\nabla u|^2 \right) |\nabla^2 u||\Delta u| \\
\leq \Delta |\Delta u|^2 + 2 \left( |\text{Ric}| + 2|\nabla u|^2 \right) |\Delta u|^2 + 2 \left( |\text{Ric}| + 2|\nabla u|^2 \right) |\nabla^2 u|^2.
\]
Taking integrations on both sides yields
\begin{equation}
\frac{d}{dt} \int_M |\Delta u|^2 dV_t = \int_M \partial_t |\Delta u|^2 dV_t + \int_M |\Delta u|^2 \left( -R + 2|\nabla u|^2 \right) dV_t \\
\leq \int_M \left( 2|\text{Ric}| - R + 6|\nabla u|^2 \right) |\Delta u|^2 dV_t + \int_M \left( 2|\text{Ric}| + 4|\nabla u|^2 \right) |\nabla^2 u|^2 dV_t.
\end{equation}
When the Ricci curvature is uniformly bounded, together with (2.6), we can prove that the $L^2$-norm of $\Delta u$ is finite.

**Proposition 2.1.** If the curvature condition (2.2) holds, then
\begin{equation}
\int_M |\Delta u|^2 dV_t \leq C(1 + K)e^{C(1 + K)T}
\end{equation}
for some uniform constant $C > 0$.

**Proof.** Compute, using (2.6) and (2.2),
\[
\frac{d}{dt} \int_M |\Delta u|^2 dV_t \leq \left( 2K + C + 6L^2 \right) \int_M |\Delta u|^2 dV_t + \left( 2K + 4L^2 \right) \int_M |\nabla^2 u|^2 dV_t.
\]
Therefore
\[
\frac{d}{dt} \left[ e^{-\left( 2K + C + 6L^2 \right) t} \int_M |\Delta u|^2 dV_t \right] \leq \left( 2K + 4L^2 \right) e^{-\left( 2K + C + 6L^2 \right) t} \int_M |\nabla^2 u|^2 dV_t.
\]
From (2.4), we obtain
\[
\int_M |\Delta u|^2 dV_t \leq Ce^{(2K+C+6L^2)t + (2K + 4L^2)} \int_0^t \int_M |\nabla^2 u|^2 dt
\leq Ce^{(2K+C+6L^2)t + C(2K + 4L^2)(1 + L^2)} e^{Ct}
\]
which implies (2.9). \hfill \square

2.1. The boundedness of $\Delta_{g(t)} u(t)$. According to [13], Chen and Zhu proved an analog of Sesum’s theorem for the Ricci-harmonic flow. As a consequence, we see that $\Delta_{g(t)} u(t)$ is uniformly bounded. Our contribution in this paper is to give an explicit bound for $\Delta_{g(t)} u(t)$.

We first review a non-collapsing theorem for the Ricci-harmonic flow. Suppose that $M$ is a closed manifold. For any Riemannian metric $g$, any smooth functions $u, f$, and any positive number $\tau$, define (see [46])
\[
W(g, u, f, \tau) := \int_M \left[ \tau \left( S_g + |\nabla_g f|^2_g \right) + f - n \right] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g
\]
with $S_g := R_g - 2|\nabla_g u|^2_g$, and
\[
\mu(g, u, \tau) := \inf \left\{ W(g, u, f, \tau) : f \in C^\infty(M) \text{ and } \int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g = 1 \right\}.
\]
Observe that
\[
\mu(\tau g, u, \tau) = \mu(g, u, 1), \quad \tau > 0.
\]

**Proposition 2.2.** If $(g(t), u(t), \tau(t))_{t \in [0,T]}$ solves
\[
\begin{align*}
\partial_t g(t) &= -2\text{Ric}_{g(t)} + 4\nabla_{g(t)} u(t) \otimes \nabla_{g(t)} u(t), \\
\partial_t u(t) &= \Delta_{g(t)} u(t), \\
\frac{d}{dt} \tau(t) &= -1,
\end{align*}
\]
then $\mu(g(t), u(t), \tau(t))$ is monotone nondecreasing in time $t$.

**Proof.** See [46, 47, 49, 51]. \hfill \square

In the definition (2.10), introduce the function
\[
w := \left[ \frac{e^{-f}}{(4\pi\tau)^{n/2}} \right]^{1/2}
\]
so that we can rewrite the functional $W$ as
\[
W(g, u, f, \tau) = \int_M \left[ \tau \left( w^2 S_g + 4|\nabla_g w|^2_g \right) - \left( 2 \ln w + \frac{n}{2} \ln(4\pi\tau) + n \right) w^2 \right] dV_g
\]
\[
= \tau \int_M S_g w^2 dV_g - \left[ \frac{n}{2} \ln(4\pi\tau) + n \right] \int_M w^2 dV_g
- 2 \int_M w^2 \ln w dV_g - 2\tau |\nabla_g w|^2_g dV_g.
\]
The last term in (2.15) can be handed by the logarithmic Sobolev inequality (see [10], Lemma 17.1): for any \( a > 0 \) there exists a constant \( C(a, g) \) such that if \( \varphi > 0 \) satisfies \( \int_M \varphi^2 dV_g = 1 \), then

\[
\int_M \varphi^2 \ln \varphi dV_g - a \int_M |\nabla_g \varphi|_g^2 dV_g \leq C(a, g),
\]

where

\[
C(a, g) = a \text{Vol}(M, g)^{-2/n} + \frac{n^2}{4ae^2C_s(M, g)}
\]

and \( C_s(M, g) \) denotes the \( L^2 \)-Sobolev constant.

**Lemma 2.3.** Let \( M \) be a closed \( n \)-dimensional manifold. For any Riemannian metric \( g \), any smooth function \( u \), and any \( \tau > 0 \), we have

\[
\mu(g, u, \tau) \geq \tau S_{g, \text{min}} - 2C(2\tau, g) - \frac{n}{2} \ln(4\pi\tau) - n,
\]

(2.18)

\[
\mu(g, u, \tau) \leq \tau S_{g, \text{avg}} + \text{Vol}(M, g) - \frac{n}{2} \ln(4\pi\tau) - n.
\]

(2.19)

Here \( S_{g, \text{min}} := \min_M S_g \) and \( S_{g, \text{avg}} \) denotes the average of \( S_g \) over \( (M, g) \).

**Proof.** Taking \( f = \ln \text{Vol}(M, g) - \frac{2}{4\pi} \ln(4\pi\tau) \) in (2.11) gives the first inequality. The second estimate follows from (2.15) and (2.16). \( \square \)

**Lemma 2.4.** For every \( n \geq 2 \), \( \rho \in (0, \infty) \), and \( D > 0 \), there exists \( c = c(n, \rho, D) > 0 \) such that if \( (M, g) \) is a closed \( n \)-dimensional Riemannian manifold, \( u \) is a smooth function on \( M \), and if for some \( r \in (0, \rho] \) and \( A < \infty \) we have \( \mu(g, u, r^2) > -A \), then for any \( p \in M \) with \( \text{Ric}_g \geq -Dr^{-2} \) on \( B_g(p, r) \) and \( R_g \leq Dr^{-2} \) on \( B_g(p, r) \), we have

\[
\text{Vol}_g(B_g(p, r)) \geq kr^n
\]

(2.20)

where \( k := ce^{-A} \).

**Proof.** Since \( S_g = R_g - 2|\nabla_g u|^2 \leq R_g \), the proof is almost the same as the proof of Proposition 5.37 in [11]. \( \square \)

Actually, the constant \( c \) in Lemma 2.4 can be explicitly determined. We can take the constant \( c \) in such a way that it depends only on \( n \) and \( C \). From the proof of Proposition 5.37 in [11], we have

\[
\mu(g, u, r^2) \leq \ln \frac{\text{Vol}_g(B_g(p, r))}{r^n} + C'(n, r) + \frac{1}{e}(4\pi)^{-n/2}c_x(n, r)
\]

where

\[
C'(n, r) := 36(4\pi)^{-n/2}c_x(n, r) + D, \quad C_x(n, r) := \frac{n}{2} \ln(4\pi) + \ln C(n, r)
\]

and

\[
C(n, r) = \int_0^r \frac{\sinh(\sqrt{K't})}{\sqrt{K'}} dt / \int_0^{r/2} \frac{\sinh(\sqrt{K't})}{\sqrt{K'}} dt, \quad K' := \frac{D}{(n-1)r^2}.
\]
The last quantity $C(n, r)$ can be bounded as
\[
C(n, r) = \int_0^r \left( e^{\sqrt{\kappa}t} - e^{-\sqrt{\kappa}t} \right) dt \int_0^{r/2} \left( e^{\sqrt{\kappa}t} - e^{-\sqrt{\kappa}t} \right) dt
\]
\[
= \frac{e^{\sqrt{\kappa}r} + e^{-\sqrt{\kappa}r} |r|}{e^{\sqrt{\kappa}r} + e^{-\sqrt{\kappa}r} |r|/2} = \frac{e^{\sqrt{\kappa}r} + e^{-\sqrt{\kappa}r} - 2}{e^{\sqrt{\kappa}r/2} + e^{-\sqrt{\kappa}r/2} - 2}
\]
\[
= \frac{(e^{\sqrt{\kappa}r/2} - e^{-\sqrt{\kappa}r/2})^2}{(e^{\sqrt{\kappa}r/4} - e^{-\sqrt{\kappa}r/4})^2} = \frac{(e^{\sqrt{\kappa}r/2} - 1)^2}{e^{\sqrt{\kappa}r/2(\sqrt{\kappa}r/2 - 1)^2}}
\]
\[
= \frac{(e^{\sqrt{\kappa}r/2} + 1)^2}{e^{\sqrt{\kappa}r/2}} = e^{\sqrt{\kappa}r/2} + 2 + e^{-\sqrt{\kappa}r/2} \leq 3 + e^{\sqrt{\kappa}r/2}.
\]
Hence
\[
C(n, r) \leq 3 + e^{\sqrt{D}/(n-1)}
\]
and the constant $c$ in Lemma 2.4 can be taken to be
\[
c = C(n) \exp \left( -C(n) \exp \left( (n) \sqrt{D} \right) \right),
\]
where $C(n)$ is a uniform constant depending only on $n$.

From the rescaling
\[
|\text{Ric}_{\tilde{g}}|_{\tilde{g}} = r^{-2} |\text{Ric}_g|_g, \quad B_{r^2 g}(p, r) = B_g(p, 1),
\]
we can conclude from Lemma 2.4 that

Corollary 2.5. For every $n \geq 2$ and $D > 0$, there exists $c = c(n, D) > 0$ such that if $(M, g)$ is a closed $n$-dimensional Riemannian manifold, $u$ is a smooth function on $M$, and if for some $A < \infty$ we have $\mu(g, u, 1) > -A$, then for any $p \in M$ with $|\text{Ric}_g|_g \leq D$ on $B_g(p, 1)$, we have
\[
\text{Vol}_g(B_g(p, 1)) \geq \kappa
\]
where $\kappa = ce^{-A}$. Moreover, $c = c(n, D)$ can be taken to be given in (2.22) for some constant $C(n)$ depending only on $n$.

Proof. Write $\tilde{g} := r^2 g$ for any given $r > 0$. Then the conditions $\mu(g, u, 1) > -A$ and $|\text{Ric}_g|_g \leq C$ on $B_g(p, 1)$ become
\[
\mu(\tilde{g}, u, r^2) = \mu(g, u, 1) > -A, \quad |\text{Ric}_{\tilde{g}}|_{\tilde{g}} = \frac{1}{r^2} |\text{Ric}_g|_g \leq \frac{C}{r^2} \text{ on } B_g(p, r).
\]
We obtain from Lemma 2.4 that $\kappa r^2 \leq \text{Vol}_g(B_g(p, r)) = r^2 \text{Vol}_g(B_g(p, 1))$. \qed

To prove the boundedness of $\Delta_{g(t)} u(t)$, we first verify that $\mu(g(t), u(t), 1)$ is always bounded from below by some uniform constant. Set, for each real number $\tau$,
\[
C^\infty_\tau(M) := \left\{ f \in C^\infty(M) : \int_M e^{-f} \frac{1}{(4\pi \tau)^{n/2}} dV_g = 1 \right\}.
\]
The mapping
\[
C^\infty_\tau(M) \ni f \mapsto \tilde{f} := f + \frac{\nu}{2} \ln \frac{\tau_2}{\tau_1} \in C^\infty_{\tau_1}(M)
\]
is one-to-one and onto. Choose $\tilde{f} \in C^0(M)$ so that $\mu(g, u, \tau_1) = \mathcal{W}(g, u, \tilde{f}, \tau_1)$ and define $f := \tilde{f} - \frac{n}{2} \ln \frac{\tau_2}{\tau_1} \in C^0(M)$. Hence, for $\tau_1, \tau_2 > 0$,
\[
\mu(g, u, \tau_2) \leq \mathcal{W}(g, u, f, \tau_2)
\]
\[
= \int_M \left[ \tau_2 \left( S_g + |\nabla g f|^2_g + \tilde{f} - n \right) - \frac{n}{2} \ln \frac{\tau_2}{\tau_1} \right] e^{-f} (4\pi\tau_1)^{n/2} dV_g
\]
\[
= \int_M \left[ \tau_1 \left( S_g + |\nabla g f|^2_g + \tilde{f} - n \right) - \frac{n}{2} \ln \frac{\tau_2}{\tau_1} \right] e^{-f} (4\pi\tau_1)^{n/2} dV_g
\]
\[
- \frac{n}{2} \ln \frac{\tau_2}{\tau_1} + (\tau_2 - \tau_1) \int_M \left( S_g + |\nabla g f|^2_g \right) e^{-f} (4\pi\tau_1)^{n/2} dV_g
\]
\[
= \mu(g, u, \tau_1) - \frac{n}{2} \ln \frac{\tau_2}{\tau_1} + (\tau_2 - \tau_1) \int_M \left( S_g + |\nabla g f|^2_g \right) e^{-f} (4\pi\tau_1)^{n/2} dV_g
\]
When $\tau_1 \geq \tau_2 > 0$, one has
\[
\mu(g, u, \tau_2) \leq \mu(g, u, \tau_1) - \frac{n}{2} \ln \frac{\tau_2}{\tau_1} + (\tau_2 - \tau_1) S_{g, \min}.
\]
In particular, if $0 < \tau(t) \leq 1$, then the inequality (2.25) implies
\[
\mu(g(t), u(t), \tau(t)) \leq \mu(g(t), u(t), 1) - \frac{n}{2} \ln \tau(t) + |\tau(t) - 1| S_{g(t), \min}.
\]
By the monotonicity of the Ricci-harmonic flow, Proposition 2.2, we obtain from (2.26) that
\[
\mu(g(t), u(t), 1) \geq \mu(g(0), u(0), \tau(0)) + \frac{n}{2} \ln |\tau(0) - t| + |1 + t - \tau(0)| S_{g(0), \min}
\]
when $1 + t - \tau(0) \geq 0$ and $\tau(0) - t > 0$. In particular, together with Lemma 2.4, and (2.4),
\[
\mu(g(t), \mu(t), 1) \geq \frac{n}{2} \ln \frac{\tau(0) - t}{4\pi\tau(0)} + (1 + t) S_{g(0), \min} - \frac{2C(2\tau(0), g(0)) - n}{2}
\]
whenever $\tau(0) - 1 \leq t < \tau(0)$.

**Theorem 2.6.** There exists a uniform constant $C$ depending only on $n, g(0)$, and $u(0)$ such that the following statement is true: If $|\text{Ric}_{g(t)}|_{g(t)} \leq K$ on $M \times [0, T]$, then
\[
|\Delta_{g(t)} u(t)|_{g(t)} \leq \frac{C(1 + K)}{(1 + T)^{n/2}} \exp \left[ C \left( 1 + T + 1 + KT + e^{C\sqrt{T}} \right) \right]
\]
over any geodesic ball $B_{g(t)}(p, \sqrt{1 + T})$. In particular, the estimate (2.28) holds on $M \times [0, T]$.

**Proof.** Let
\[
\bar{t} := \frac{t}{T + 1}, \quad \bar{T} := \frac{T}{T + 1}, \quad \bar{g}(\bar{t}) := \frac{1}{T + 1} g((T + 1) \bar{t}), \quad \bar{u}(\bar{t}) := u((T + 1) \bar{t})
\]
Then $(\bar{g}(\bar{t}), \bar{u}(\bar{t}))_{\bar{t} \in [0, \bar{T}]}$ is a solution of the Ricci flow with $\bar{T} \in (0, 1)$.

In this case we choose $\bar{t}(0) := (1 + \bar{T})/2$ so that $\bar{t}(0) - \bar{t} \geq 2(1 + \bar{T})/2 - \bar{T} = (1 - \bar{T})/2 > 0$.
and $1 + \bar{t} - \bar{t}(0) \geq 1 - (1 + \bar{T})/2 = (1 - \bar{T})/2 > 0$. Therefore the estimate \(2.27\) applied to the rescaling Ricci-harmonic flow holds for all $t \in [0, \bar{T}]$, i.e.,

$$
\mu(\tilde{g}(\bar{t}), \bar{u}(\bar{t}), 1) \geq \frac{n}{2} \ln \frac{1}{1 + \bar{T}} - \frac{(1 + \bar{T})|S_{\tilde{g}(0),\min}|}{2} - 2C(1 + \bar{T}, \tilde{g}(0)) - \ln 4\pi - n.
$$

Since the $L^2$-Sobolev constant $C_s(M, g)$ is invariant under scaling the metric, it follows from \(2.22\) and \(A.9\) that

$$
\mu(\tilde{g}(\bar{t}), \bar{u}(\bar{t}), 1) \geq \frac{n}{2} \ln \frac{1 + \bar{T}}{1 + \bar{T}} - \left(1 + \frac{T}{1 + \bar{T}}\right)(T + 1)|S_{\tilde{g}(0),\min}| - \ln 4\pi - n
$$

$$
- 2 \left[\frac{1 + 2T}{1 + T} \left[(1 + T)^{-n/2}\text{Vol}(M, g(0))\right]^{-2/n} + \frac{n^2}{4e^{2(1 + 2T)C_s(M, g(0))}}\right]
$$

$$
= -\frac{n}{2} \ln(1 + 2T) - (1 + 2T) \left|S_{\tilde{g}(0),\min}\right| + \frac{2}{\text{Vol}(M, g(0))^{n/2}}
$$

$$
- \ln 4\pi - n - \frac{n^2(1 + T)}{2e^{2(1 + 2T)C_s(M, g(0))}}
$$

Consequently,

$$\mu(\tilde{g}(\bar{t}), \bar{u}(\bar{t}), 1) \geq -C(1 + 2T)$$

for some uniform constant $C$ depending only on $g(0)$ and $u(0)$. Because $|\tilde{\text{Ric}}_{\tilde{g}(\bar{t})}|_{\tilde{g}(\bar{t})} = |\text{Ric}g(t)|_{g(t)}(1 + T) \leq \text{K}/(1 + T)$ on $B_{\tilde{g}(t)}(p, 1) = B_{\tilde{g}(t)}(p, T/1 + T)$, we have

$$
\text{Vol}_{\tilde{g}(t)}\left(B_{\tilde{g}(t)}(p, T/1 + T)\right) \geq C(1 + T)^{n/2} \exp \left[-C\left(1 + 2T + e^{C\sqrt{K}/(1 + T)}\right)\right]
$$

\(2.30\)

We now can prove the estimate \(2.28\). Suppose otherwise that

$$|\Delta_g u(t)| \geq C\left((1 + K)T + 1 + 2T + e^{C\sqrt{K}}\right)$$

over some geodesic ball $B_{\tilde{g}(t)}(p, T/1 + T)$ and for some time $t \in [0, T]$. On the other hand, from \(2.22\) and \(2.30\), we get

$$C(1 + K)e^{C(1 + K)T} \geq \int_{\text{Vol}_{\tilde{g}(t)}(B_{\tilde{g}(t)}(p, T/1 + T))} |\Delta_{\tilde{g}(t)} u(t)|^2 dV_{\tilde{g}(t)}$$

$$\geq 2C(1 + K)\exp \left[2C\left((1 + K)T + 1 + 2T + e^{C\sqrt{K}}\right)\right] \cdot \text{Vol}_{\tilde{g}(t)}\left(B_{\tilde{g}(t)}(p, T/1 + T)\right)
$$

$$\geq 2C(1 + K)e^{C(1 + K)T} \exp \left[C\left(1 + 2T + e^{C\sqrt{K}}\right)\right]\geq 2C(1 + K)e^{C(1 + K)T}.$$
2.2. Local curvature estimates. In this subsection we assume that
\[ |\text{Ric}| \leq K, \quad |\nabla u| \leq L, \quad |\nabla^2 u| \leq P \]
over an open subset \( \Omega \) in \( M \) and \( \phi \) is a Lipschitz function with support in \( \Omega \).

From Lemma A.1 we can deduce that
\[
\Box |\text{Ric}|^2 = -2|\nabla \text{Ric}|^2 + 4R_{pijq} R^{pij} R^{qij} - 8R_{pijq} R^{pij} \nabla^p u \nabla^q u
\]
\[ + 8 \Delta u R^{ij} \nabla_i \nabla_j u - 8R^{ij} \nabla_i \nabla_j u \nabla^k u \nabla_k \]  
(2.31)
\[ \leq -\frac{1}{2} |\nabla u|^2 |\Delta u| + \text{CKL}^2 |\nabla u|^2 + \frac{\text{CK}^2 L^2}{2} \]
(2.32)
\[ \leq -\frac{1}{2} |\nabla u|^2 |\Delta u| + \text{CK} |\nabla^2 u|^2 + \text{CK} |\nabla u|^2 |\Delta u| - \frac{\text{CK}^2 L^2}{2}, \]
by the fact at \( |\Delta u| \leq \sqrt{\text{P}} |\nabla^2 u|. \) Similarly, from (A.6), we have
\[ |\nabla \text{Rm}|^2 \leq -\frac{1}{2} |\nabla u|^2 |\Delta u| + \text{C} |\nabla u|^2 |\Delta u| - \frac{\text{C} L^2 |\nabla u|^2}{2}. \]
(2.33)
Moreover, we can prove that, see Lemma A.2
\[ \partial_t |\text{Rm}|^2 = \nabla^2 \text{Ric} * \text{Rm} + \text{Ric} * \text{Rm} * \text{Rm}
\]
(2.34)
\[ + \text{Rm} * \nabla^2 u * \nabla^2 u + \text{Rm} * \text{Rm} * \nabla u * \nabla u. \]
As in (2.33), we consider the quantity
\[ \frac{d}{dt} \int_M |\text{Rm}|^p \phi \phi^p dV \]
which can be rewritten as, using (2.34),
\[ \frac{d}{dt} \int_M |\text{Rm}|^p \phi \phi^p dV = \int_M (\partial_t |\text{Rm}|^p) \phi \phi^p dV + \int_M |\text{Rm}|^p \phi \phi^p \left( -R + \frac{1}{2} |\nabla u|^2 \right) dV 
\]
\[ = \frac{p}{2} \int_M |\text{Rm}|^{p-2} \left[ \nabla^2 \text{Ric} * \text{Rm} + \text{Ric} * \text{Rm} * \text{Rm} + \text{Rm} * \nabla^2 u * \nabla^2 u \right] dV 
\]
\[ + \text{Rm} * \text{Rm} * \nabla u * \nabla u \phi \phi^p dV
\]
\[ \leq C \int_M |\text{Rm}|^{p-2} \left( \nabla^2 \text{Ric} * \text{Rm} \right) \phi \phi^p dV + \text{CK} \int_M |\text{Rm}|^{p-1} \phi \phi^p dV
\]
\[ + \text{CP}^2 \int_M |\text{Rm}|^{p-1} \phi \phi^p dV + \text{CL}^2 \int_M |\text{Rm}|^{p-2} \phi \phi^p dV
\]
\[ + C \int_M |\text{Rm}|^{p-2} \left( \text{Rm} * \nabla u * \nabla^3 u \right) \phi \phi^p dV. \]
From (2.5), (2.6), and (2.7) in (2.32), we know that
\[ C \int_M |\text{Rm}|^{p-2} \left( \nabla^2 \text{Ric} * \text{Rm} \right) \phi \phi^p dV \leq \frac{1}{K} \int_M |\nabla \text{Ric}|^2 |\text{Rm}|^{p-1} \phi \phi^p dV 
\]
\[ + \text{CK} \int_M |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi \phi^p dV + \text{CK} \int_M |\text{Rm}|^{p-1} |\nabla \phi|^2 \phi \phi^{p-2} dV. \]
Combining all terms yields
\[
\frac{d}{dt} \int_M |\text{Rm}|^p \phi^{2p} dV_t \leq \frac{1}{K} \int_M (\nabla \text{Ric})^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t
\]
(2.35)
\[\quad + CK \int_M |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t + CK \int_M |\text{Rm}|^{p-1} |\nabla \phi|^2 \phi^{2p-2} dV_t \]
\[\quad + C(K + L^2) \int_M |\text{Rm}|^{p-1} \phi^{2p} dV_t + CP^2 \int_M |\text{Rm}|^{p-1} \phi^{2p} dV_t \]
In (2.35) the first two terms are “bad terms”, since these contain derivatives of curvature. As in (3.2) we set
\[
B_1 := \frac{1}{K} \int_M (\nabla \text{Ric})^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t, \quad B_2 := \int_M |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t.
\]
We also introduce
\[
A_1 := \int_M |\text{Rm}|^p \phi^{2p} dV_t, \quad A_2 := \int_M |\text{Rm}|^{p-1} \phi^{2p} dV_t,
\]
\[
A_3 := \int_M |\text{Rm}|^{p-1} |\nabla \phi|^2 \phi^{2p-2} dV_t, \quad A_4 := \int_M |\text{Rm}|^{p-1} |\nabla \phi|^2 \phi^{2p-2} dV_t.
\]
Then the estimate (2.35) can be rewritten as
(2.36)
\[\frac{d}{dt} \int_M |\text{Rm}|^p \phi^{2p} dV_t \leq B_1 + CK B_2 + CK A_4 + C(K + L^2) A_1 + CP^2 A_2.
\]
Using (2.35) yields
\[
B_1 \leq \int_M \left[ \frac{1}{2K} (\Delta - \partial_t) |\text{Ric}|^2 + C(L^2 + K) |\text{Rm}| + C(P^2 + KL^2) \right] |\text{Rm}|^{p-1} \phi^{2p} dV_t
\]
\[= \frac{1}{2K} \int_M (\Delta - \partial_t) |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + C(L^2 + K) A_1 + C(P^2 + KL^2) A_2
\]
\[= \frac{1}{2K} \int_M (\Delta |\text{Ric}|^2) |\text{Rm}|^{p-1} \phi^{2p} dV_t + C(L^2 + K) A_1 + C(P^2 + KL^2) A_2
\]
\[- \frac{1}{2K} \int_M |\partial_t (|\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t) - |\text{Ric}|^2 (\partial_t |\text{Rm}|^{p-1}) \phi^{2p} dV_t
\]
\[\quad - |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} (-R + 2 |\nabla u|^2) dV_t \]
\[= -\frac{1}{2K} \left[ \int_M \langle \nabla |\text{Ric}|^2, \nabla |\text{Rm}|^{p-1} \rangle \phi^{2p} dV_t + \int_M \langle \nabla |\text{Ric}|^2, \nabla \phi^{2p} \rangle |\text{Rm}|^{p-1} dV_t \right]
\]
\[\quad - \frac{1}{2K} \frac{d}{dt} \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + C(L^2 + K) A_1 + C(P^2 + KL^2) A_2
\]
\[\quad + \frac{1}{2K} \int_M |\text{Ric}|^2 (\partial_t |\text{Rm}|^{p-1}) \phi^{2p} dV_t + CK L^2 A_2 + CK A_1
\]
\[\leq -\frac{1}{2K} \left[ \int_M \langle \nabla |\text{Ric}|^2, \nabla |\text{Rm}|^{p-1} \rangle \phi^{2p} dV_t + \int_M \langle \nabla |\text{Ric}|^2, \nabla \phi^{2p} \rangle |\text{Rm}|^{p-1} dV_t \right]
\]
\[- \frac{1}{2K} \frac{d}{dt} \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{1}{2K} \int_M |\text{Ric}|^2 (\partial_t |\text{Rm}|^{p-1}) \phi^{2p} dV_t
\]
\[\quad + C(L^2 + K) A_1 + C(P^2 + KL^2) A_2.
\]
From (2.10) and (2.11) in [32], one has

\begin{align}
(2.37) \quad -\frac{1}{2K} \int_M \left\langle \nabla |\text{Ric}|^2, \nabla |\text{Rm}|^{p-1} \right\rangle \phi^{2p} dV_t & \leq \frac{1}{10} B_1 + CKB_2, \\
(2.38) \quad -\frac{1}{2K} \int_M \left\langle \nabla |\text{Ric}|^2, \nabla \phi^{2p} \right\rangle |\text{Rm}|^{p-1} dV_t & \leq \frac{1}{10} B_1 + CKA_4.
\end{align}

According to (2.34), we have

\begin{align*}
\frac{1}{2K} \int_M |\text{Ric}|^2 \left( \partial_t |\text{Rm}|^{p-1} \right) \phi^{2p} dV_t &= \frac{p-1}{4K} \int_M |\text{Ric}|^2 \left( \left| |\text{Rm}|^{p-3} \partial_t |\text{Rm}|^2 \right| \phi^{2p} dV_t \\
&= \frac{C}{K} \int_M \left| \text{Ric} \right|^2 |\text{Rm}|^{p-3} \phi^{2p} \left[ \nabla^2 \text{Ric} \ast |\text{Rm}| + \text{Ric} \ast |\text{Rm}| \right] dV_t \\
&\quad + |\text{Rm}| \ast \nabla^2 u \ast |\text{Rm}| + \text{Rm} \ast \nabla u \ast \nabla u \right] dV_t \\
&\leq \frac{C}{K} \int_M \left| \text{Ric} \right|^2 |\text{Rm}|^{p-3} \phi^{2p} \left( \nabla^2 \text{Ric} \ast |\text{Rm}| \right) dV_t + CKA_1 + C(P^2 + KL^2)A_2
\end{align*}

From the proof of (2.13) – (2.15) in [32], we can deduce that

\begin{align*}
\frac{C}{K} \int_M \left| \text{Ric} \right|^2 |\text{Rm}|^{p-3} \phi^{2p} \left( \nabla^2 \text{Ric} \ast |\text{Rm}| \right) dV_t &\leq \frac{1}{5} B_1 + CKB_2 + CKA_4.
\end{align*}

In summary, we arrive at

\begin{align}
(2.39) \quad \frac{1}{2K} \int_M |\text{Ric}|^2 \left( \partial_t |\text{Rm}|^{p-1} \right) \phi^{2p} dV_t &\leq \frac{1}{5} B_1 + CKB_2 + CKA_1 + C(P^2 + KL^2)A_2 + CKA_4.
\end{align}

Plugging (2.37), (2.38), and (2.39) into the inequality for $B_1$, we get

\begin{align}
B_1 &\leq CKB_2 + C(K + L^2)A_1 + CKA_4 \\
&\quad + C(P^2 + KL^2)A_2 - \frac{1}{2K} \frac{d}{dt} \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t.
\end{align}

To deal with the term $B_2$, we use the evolution equation (2.33) and then obtain

\begin{align*}
B_2 &\leq \int_M \left[ \frac{1}{2} \left( \Delta - \partial_t \right) |\text{Rm}|^2 + C |\text{Rm}|^3 + C(L^2 + P^2) |\text{Rm}|^2 \right] |\text{Rm}|^{p-3} \phi^{2p} dV_t \\
&= \frac{1}{2} \int_M \left( \Delta |\text{Rm}|^2 \right) |\text{Rm}|^{p-3} \phi^{2p} dV_t + CA_1 + C(L^2 + P^2)A_2 \\
&\quad - \frac{1}{2} \int_M \left( \partial_t |\text{Rm}|^2 \right) |\text{Rm}|^{p-3} \phi^{2p} dV_t \\
&\leq C \int_M |\nabla \text{Rm}| |\nabla \phi| |\text{Rm}|^{p-2} \phi^{2p-1} dV_t + CA_1 + C(L^2 + P^2)A_2 \\
&\quad - \frac{1}{2} \int_M \left( \partial_t |\text{Rm}|^2 \right) |\text{Rm}|^{p-3} \phi^{2p} dV_t \\
&\leq \frac{1}{2} B_2 + CA_4 + CA_1 + C(L^2 + P^2)A_2 - \frac{1}{2} \frac{d}{dt} \int_M \left( \partial_t |\text{Rm}|^2 \right) |\text{Rm}|^{p-3} \phi^{2p} dV_t.
\end{align*}

As the proof of (2.18) – (2.19) in [32], we have

\begin{align*}
-\frac{1}{2} \int_M \left( \partial_t |\text{Rm}|^2 \right) |\text{Rm}|^{p-3} \phi^{2p} dV_t = -\frac{1}{2} \int_M \left[ \partial_t \left( \left| \text{Rm} \right|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t \right) \\
&\quad - \frac{1}{2} \int_M \left( \partial_t |\text{Rm}|^2 \right) |\text{Rm}|^{p-3} \phi^{2p} dV_t \right]
\end{align*}
\[- |Rm|^2 \left( \partial_t |Rm|^{|p-3|} \right) \phi^{2p} dV_t - |Rm|^{|p-1|} \phi^{2p} \partial_t dV \]
\[
= - \frac{1}{2} \frac{d}{dt} \int_M |Rm|^{p-1} \phi^{2p} dV_t + \frac{p-3}{4} \int_M |Rm|^{p-3} \left( \partial_t |Rm|^2 \right) \phi^{2p} dV_t
- \frac{1}{2} \int_M R |Rm|^{p-1} \phi^{2p} dV_t + \int_M |Rm|^{p-1} \nabla u^2 \phi^{2p} dV_t
\]

and therefore
\[- \frac{1}{2} \int_M \left( \partial_t |Rm|^2 \right) |Rm|^{p-3} \phi^{2p} dV_t \leq - \frac{1}{p-1} \frac{d}{dt} \int_M |Rm|^{p-1} \phi^{2p} dV_t + CA_1 + CL^2 A_2.\]

In summary,
\[
(2.41) \quad B_2 \leq - \frac{1}{p-1} \frac{d}{dt} \int_M |Rm|^{p-1} \phi^{2p} dV_t + CA_1 + CA_4 + C(L^2 + P^2) A_2.
\]

From (2.36), (2.40), and (2.41), we finally obtain
\[
\frac{d}{dt} \left[ A_1 + CKA_2 + \frac{1}{2K} \int_M \left| \text{Ric} \right|^2 |Rm|^{p-1} \phi^{2p} dV_t \right] \leq C(K + L^2) A_1
+ CKA_4 + C(KL^2 + KP^2 + P^2 + KL) A_2.
\]

**Theorem 2.7.** Let \((g(t), u(t))_{t \in [0,T]}\) be a solution to the Ricci-harmonic flow. Suppose there exist constants \(\rho, K, L, P > 0\) and \(x_0 \in M\) such that \(B_{g(0)}(x_0, \rho / \sqrt{K})\) is compactly contained on \(M\) and
\[
|\text{Ric}_{g(t)}|_{g(t)} \leq K, \quad |\nabla g(t) u(t)|_{g(t)} \leq L, \quad |\nabla^2 g(t) u(t)|_{g(t)} \leq P
\]
on \(B_{g(0)}(x_0, \rho / \sqrt{K}) \times [0,T].\) For any \(p \geq 3,\) there is a constant \(C,\) depending only on \(n\) and \(p,\) such that
\[
\int_{B_{g(0)}(x_0, \rho / 2\sqrt{K})} |\text{Rm}_{g(t)}|^p g(t)^{1/2} dV_{g(t)} \leq CA_1 e^{C \Lambda_2 T} \int_{B_{g(0)}(x_0, \rho / \sqrt{K})} |\text{Rm}_{g(0)}|^p g(0)^{1/2} dV_{g(0)}
+ CK^p \left( 1 + \rho^{-2p} \right) e^{C \Lambda_2 T} \text{Vol}_{g(t)} \left( B_{g(0)}(x_0, \rho / \sqrt{K}) \right).
\]

Here \(\Lambda_1 := 1 + K\) and \(\Lambda_2 := K + L^2 + P^2(1 + K^{-1}).\)

**Proof.** Choose \(\Omega := B_{g(0)}(x_0, \rho / \sqrt{K})\) and
\[
\phi := \left( \frac{\rho / \sqrt{K} - d_{g(0)}(x_0, \cdot)}{\rho / \sqrt{K}} \right)^{+}.
\]

Then \(e^{-2Kt} g(0) \leq g(t) \leq e^{2Kt} g(0)\) and \(|\nabla g(t) \phi|_{g(t)} \leq e^{Kt} |\nabla g(0) \phi|_{g(0)} \leq \sqrt{K} e^{Kt} / \rho\) for any \(t \in [0, T].\) Let
\[
U := \int_M |\text{Rm}|^p \phi^{2p} dV_t + CK \int_M |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{1}{2K} \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t.
\]

Then
\[
U' \leq C(K + L^2) U + CKA_4 + C(KL^2 + KP^2 + P^2 + KL) \frac{U}{K}.
\]
For $A_4$, we can estimate it as follows:

$$
A_4 = \int_M |\text{Rm}|^{p-1} \nabla \phi_2 \phi^{2p-2} dV \leq \int_{B_{g(0)}(x_0, \varphi / \sqrt{K})} |\text{Rm}|^{p-1} \phi^{2p-2} K \rho^{2-2e^{2KT}} dV
$$

$$
\leq \int_{B_{g(0)}(x_0, \varphi / \sqrt{K})} \left( \frac{|\text{Rm}|^{p-1} \phi^{2p-2}}{p} \right)^{\frac{1}{p-1}} \left( K \rho^{2-2e^{2KT}} \right)^{\frac{1}{p-1}} dV
$$

$$
\leq A_1 + K^p e^{2KP} p \rho^{-2p} \text{Vol}_{g(t)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right)
$$

$$
\leq U + CK^p \rho^{-2p} e^{2KP} \text{Vol}_{g(t)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right).
$$

Hence

$$
U' \leq C \left[ K + L^2 + L + p^2 \left( 1 + \frac{1}{K} \right) \right] U
$$

$$
+ CK^p+1 \rho^{-2p} e^{2KP} \text{Vol}_{g(t)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right).
$$

Since, for each $\tau \in [0, T]$,

$$
\text{Vol}_{g(t)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right) \leq e^{KT} \text{Vol}_{g(\tau)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right),
$$

as argued in (2.27) of [32], we deduce that

$$
(2.42) \quad U(\tau) \leq e^{CA_2T} \left[ U(0) + C \rho^{-2p} K^p \text{Vol}_{g(t)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right],
$$

According to the Young inequality

$$
\int_M |\text{Rm}_{g(0)}|^{p-1} \phi^{2p} dV_{g(0)} = \int_M \left| |\text{Rm}|^{p-1} \phi^{2p} \right| dV_{g(0)}
$$

$$
\leq \frac{p-1}{p} \int_M |\text{Rm}_{g(0)}|^{p} \phi^{2p} dV_{g(0)} + \frac{1}{p} \int_M \phi^{2p} dV_{g(0)},
$$

we obtain

$$
U(0) \leq C(1 + K) \int_M \left| |\text{Rm}_{g(0)}|^{p} \phi^{2p} \right| dV_{g(0)} + C \text{Vol}_{g(0)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right)
$$

$$
\leq C(1 + K) \int_M \left| |\text{Rm}_{g(0)}|^{2p} \phi^{2p} \right| dV_{g(0)}
$$

$$
+ CKe^{CKT} \text{Vol}_{g(\tau)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right)
$$

$$
\leq C(1 + K) \int_{B_{g(0)}(x_0, \varphi / \sqrt{K})} |\text{Rm}_{g(0)}|^{2p} dV_{g(0)}
$$

$$
+ CKe^{CKT} \text{Vol}_{g(\tau)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right),
$$

and $\phi \geq 1/2$ on $B_{g(0)}(x_0, \rho / 2\sqrt{K})$, we complete the proof. \[\square\]

The same method can be applied to the regular Ricci flow (see Section 3), since all computations only involve the evolution equations for the metrics, which take the same forms in the Ricci-harmonic flow.
3. Results for a Generalized Ricci Flow

In this section we extend the main estimates in \(33\) to a generalized Ricci flow introduced in \(32\), where I proved that for any complete \(n\)-manifold \(M\), the following two conditions are equivalent:

(i) there exists a Ricci-flat Riemannian metric on \(M\);
(ii) there exists real numbers \(a, \beta\), a smooth function \(u\) on \(M\), and a Riemannian metric \(g\) on \(M\) such that

\[
0 = -R_{ij} + \alpha \nabla_i \nabla_j u, \quad 0 = \Delta_g u + \beta |\nabla_g u|^2_g.
\]  

The main ingredient in the proof is Chen’s result \(9\) which says that any complete noncompact steady gradient Ricci soliton has nonnegative scalar curvature.

Observe that the first equation in \(31\) is actual the vanishing of the \(\infty\)-Bakry-Émery Ricci tensor. Indeed, the \(N\)-Bakry-Émery Ricci tensor is defined by

\[
\text{Ric}_{g,N,f} := \text{Ric}_g + \nabla^2 f - \frac{df \otimes df}{N - n}
\]

for \(N\) finite, and

\[
\text{Ric}_{g,\infty,f} := \text{Ric}_g + \nabla^2 f
\]

for \(N\) infinite. Thus the first equation in \(31\) is equivalent to \(\text{Ric}_{g,\infty, -au} = 0\).

Motivated by the above equivalence conditions I introduced the following generalized Ricci flow \(32\):

\[
\partial_t g(t) = -2 \text{Ric}_{g(t)} + 2\alpha_1 \nabla g(t) u(t) \otimes \nabla g(t) u(t) + 2\alpha_2 \nabla^2 g(t) u(t),
\]

\[
\partial_t u(t) = \Delta_g u(t) + \beta_1 |\nabla g(t) u(t)|^2_g(t) + \beta_2 u(t).
\]

Here \(\alpha_1, \alpha_2, \beta_1, \beta_2\) are given constants. This system is called \((\alpha_1, \alpha_2, \beta_1, \beta_2)\)-Ricci flow. In particular, when \((\alpha_1, \alpha_2, \beta_1, \beta_2) = (2, 0, 0, 0)\), we get the Ricci-harmonic flow \(11\). In view of \(32\), we see that \(34\) can be written as

\[
\partial_t g(t) = -2 \text{Ric}_{g(t), N, -a u(t)},
\]

where \(N = n + \alpha_2^2 / \alpha_1\) if \(\alpha_1 \neq 0\), and \(N = \infty\) if \(\alpha_1 = 0\).

According to Proposition 2.12 in \(42\), we know that an \((\alpha_1, \alpha_2, \beta_1, \beta_2)\)-Ricci flow is equivalent to the \((\alpha_1, 0, \beta_1 - \alpha_2, \beta_2)\)-flow. By this reduction, in the following we main study the \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow:

\[
\partial_t g(t) = -2 \text{Ric}_{g(t)} + 2\alpha_1 \nabla g(t) u(t) \otimes \nabla g(t) u(t),
\]

\[
\partial_t u(t) = \Delta_g u(t) + \beta_1 |\nabla g(t) u(t)|^2_g(t) + \beta_2 u(t).
\]

Here \(\alpha_1, \beta_1, \beta_2\) are given constants. Recall the notion \(\square_{g(t)} = \partial_t - \Delta_g(t)\) and introduce as in \(43\).

\[
\text{Sic}_{g(t)} := \text{Ric}_{g(t)} - \alpha_1 \nabla g(t) u(t) \otimes \nabla g(t) u(t),
\]

\[
S_{g(t)} := \text{tr}_g(\text{Sic}_{g(t)}) = R_g(t) - \alpha_1 |\nabla g(t) u(t)|^2_g(t).
\]

Another interesting flow involving \((g(t), u(t))\) is the so-called super-Ricci flow introduced by X. D. Li and S. Z. Li \(35\) and in terms of our notions \(32\) and \(33\).
can be written as, according to whether or not \( N \) is infinity
\[
\frac{1}{2} \partial_t g(t) + \text{Ric}_{g(t),N,u(t)} \geq Kg(t)
\]
which is called a \((K,N)\)-super Ricci flow, where \( N \in \mathbb{R} \), and, respectively,
\[
\frac{1}{2} \partial_t g(t) + \text{Ric}_{g(t),\infty,u(t)} \geq Kg(t)
\]
which is called a \( K\)-super Perelman Ricci flow. Under the super-Ricci flow, the authors studied Harnack inequalities \([35,37,39]\), W-entropy formulas \([35,45,38,39,40]\), \((K,N)\)-Ricci solitons \([38]\), etc. For example, they proved that if \((g(t),u(t))_{t \in [0,T]}\) satisfies
\[
\frac{1}{2} \partial_t g(t) + \text{Ric}_{g(t),N,u(t)} = 0, \quad \partial_t u(t) = \frac{1}{2} \text{tr}_{g(t)} \left( \partial_t g(t) \right),
\]
then the \( W \)-entropy \( W_N(f(t)) := \frac{d}{dt} \left[ t \mathcal{H}_N(f(t)) \right] \) with
\[
\mathcal{H}_N(f(t)) := - \int_M f(t) \ln f(t) dV_{g(t)} - \frac{N}{2} \left[ 1 + \ln(4\pi t) \right]
\]
is constant along the flow, where \( f(t) \) is the fundamental solution to the heat equation \( \partial_t f(t) = \Delta_{g(t)} f(t) - \langle \nabla_{g(t)} u(t), \nabla_{g(t)} f(t) \rangle_{g(t)} \). However, we can not apply this result to our flow \([3.3]\) or
\[
\partial_t g(t) = -2 \text{Ric}_{g(t),N,-\alpha_2 u(t)},
\]
since the second equation \([3.3]\) may not satisfy the constraint equation \( \partial_t u(t) = \frac{1}{2} \text{tr}_{g(t)} \partial_t g(t) \).

In \([43]\) we also introduced a “Riemann curvature” type for RHF
\[S_{ijkl} := R_{ijkl} - \frac{\alpha_1}{2} \left( g_{ij} \nabla u \nabla k u + g_{kl} \nabla j u \nabla i u \right)\]
so that \( S_{ij} = g^{kl} S_{klij} = g^{kl} S_{ijkl} = R_{ij} - \alpha_1 \nabla i u \nabla j u \). A related construction of “Riemann curvature” type for \([52]\) is given in \([68]\). A detailed discussion of these two notions of “Riemann curvature” will be given in Section 5.

According to Lemma \([44]\) we have

**Lemma 3.1.** Under the flow \([3.6] \) to \([3.7] \),
\[
\Box S = 2 \text{Sic}^2 \left[ 2 \alpha_1 |\Delta u|^2 - 2 \alpha_1 \beta_2 |\nabla u|^2 - 4 \alpha_1 \beta_1 |\nabla u|^2 |\nabla i u \nabla j u| \right]
\]
\[= 2 \left[ 2 \text{Sic}^2 \left( 2 \alpha_1 |\Delta u|^2 - 2 \alpha_1 \beta_2 |\nabla u|^2 - 4 \alpha_1 \beta_1 |\nabla u|^2 |\nabla i u \nabla j u| \right) \right]
\]
\[
\Box S_{ij} = 2 S_{kij} S^{kl} - 2 S_{kkl} S_{ijkl} + 2 \alpha_1 u \nabla i u \nabla j u \]
\[
- 2 \alpha_1 \beta_2 \nabla i u \nabla j u - 2 \alpha_1 \beta_1 \nabla k u (\nabla j u \nabla i u + \nabla i u \nabla j u) \nabla k u.
\]

3.1. **Long time existence.** Given an initial data \((g_0, u_0)\), define \( c_0 := |\nabla g_0 u_0|^2 \)\_\(g_0\). We always assume that \( c_0 \) is a positive number. According to Corollary \(2.10\) and Definition \(2.11\) in \([42]\), we say the \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow is regular, if \( \alpha_1, \beta_1, \beta_2 \) satisfy one of the following conditions:

(i) \( \beta_2 \leq 0 \) and \( \alpha_1 \geq \beta_1^2 \);

(ii) \( \beta_2 > 0 \) and \( c_0^{-1} \beta_2 + \beta_1^2 \geq \alpha_1 > \beta_1^2 \).
Then Corollary 2.10 in [42] tells us that

\begin{equation}
|\nabla u|^2 \lesssim 1
\end{equation}

along the \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow equations (3.6) – (3.7), where \(\lesssim\) depends only on \(\alpha_1, \beta_1, \beta_2\) and \(c_0\).

**Theorem 3.2.** Let \((g(t), u(t))_{t \in [0, T)}\) be a solution to the regular \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow on a closed \(n\)-dimensional manifold \(M\) with \(T \leq \infty\) and the initial data \((g_0, u_0)\). Assume that \(S_{g(t)} + C \geq C_0 > 0\) along the flow for some uniform constants \(C, C_0 > 0\). Then

\begin{equation}
\frac{\text{Sin}_{g(t)}(g(t))}{S_{g(t)} + C} \leq C_1 + C_2 \max_{M \times [0, t]} \sqrt{\frac{|W_{g(s)}|_{g(s)} + |\nabla^2 g(s) u(s)|^2_{g(s)}|}{S_{g(s)} + C}}
\end{equation}

where \(\text{Sin}_{g(t)} := \text{Sic}_{g(t)} - \frac{\sqrt{g(t)} g(t)}{n}\) is the trace-free part of \(\text{Sic}_{g(t)}\) and \(W_{g(t)}\) is the Weyl tensor field of \(g(t)\).

Here the assumption \(S_{g(t)} + C > 0\) is necessary in the theorem, since, due to the undermined sign of \(\alpha_1, \beta_1, \beta_2\), we can not in general deduce any bounds for \(S_{g(t)}\) from the evolution equation (3.11). In the simplest case, \(\alpha_1 \geq 0\) and \(\beta_1 = \beta_2 = 0\) (i.e., Ricci-harmonic flow), we have a lower bound from (3.11).

**Proof.** As in [43], consider the quantity

\begin{equation}
f := \frac{|\text{Sin}_{g(t)}(g(t))|^2}{S_{g(t)} + C} = \frac{|\text{Sic}_{g(t)} + \frac{\sqrt{g(t)} g(t)}{n}|}{S_{g(t)} + C} - \frac{1}{n} (S_{g(t)} + C)^{2-\gamma}, \quad \gamma > 0
\end{equation}

and set

\begin{equation}
\text{Sic}'_{g(t)} := \text{Sic}_{g(t)} + \frac{C}{n} g(t), \quad S'_{g(t)} := S_{g(t)} + C.
\end{equation}

From the identity (3.21) in [11], we have

\[
\Box \frac{|\text{Sic}'|^2}{(S')^\gamma} = \frac{1}{(S')^\gamma} \Box |\text{Sic}'|^2 - \gamma \frac{|\text{Sic}'|^2}{(S')^{\gamma+1}} \Box S - \gamma (\gamma + 1) \frac{|\text{Sic}'|^2}{(S')^{\gamma+2}} \nabla S'^2 + \frac{2 \gamma}{(S')^{\gamma+1}} \left\langle \nabla |\text{Sic}'|^2, \nabla S' \right\rangle.
\]

Using (3.12), we get

\begin{align}
\Box |\text{Sic}|^2 &= -2 |\nabla \text{Sic}|^2 + 4 Sm(\text{Sic}, \text{Sic}) + 2 \left\langle \text{Sic}, 2 \alpha_1 \Delta u \nabla u - 2 \alpha_1 \beta_2 \nabla u \otimes \nabla u \right\rangle \\
&- 4 \alpha_1 \beta_1 \left\langle \text{Sic}, \nabla u \otimes \nabla |\nabla u|^2 \right\rangle
\end{align}

(3.17)
where Sm(Sic, Sic) = $S_{kiem}S^{ijkl}g_{kl}$. As in [43], we can prove

$$\Box|\text{Sic}'|^2 = \Box|\text{Sic}|^2 + \frac{2C}{n} \Box S$$

$$= -2|\nabla \text{Sic}'|^2 + 4\text{Sm}(\text{Sic}, \text{Sic}) + \frac{4C}{n}|\text{Sic}|^2$$

$$+ 4\alpha_1 \left( \text{Sic}', \Delta u \nabla^2 u - \beta_2 \nabla u \otimes \nabla u \right) - 4\alpha_1 \beta_1 \left( \text{Sic}', \nabla u \otimes \nabla |\nabla u|^2 \right)$$

$$= -2|\nabla \text{Sic}'|^2 + 4\text{Sm}(\text{Sic}', \text{Sic}') - \frac{4C}{n}|\text{Sic}'|^2 + \frac{4C^2}{n^2} S'$$

$$+ 4\alpha_1 \left( \text{Sic}', \Delta u \nabla^2 u - \beta_2 \nabla u \otimes \nabla u \right) - 4\alpha_1 \beta_1 \left( \text{Sic}', \nabla u \otimes \nabla |\nabla u|^2 \right)$$

and

$$\Box \frac{|\text{Sic}'|^2}{(S')^\gamma} = -\frac{2}{(S')^\gamma} |\nabla \text{Sic}'|^2 - \frac{2\gamma|\text{Sic}'|^4}{(S')^{\gamma+1}} + \frac{4}{(S')^\gamma}\text{Sm}(\text{Sic}', \text{Sic}')$$

$$- \gamma(\gamma + 1) \frac{|\text{Sic}'|^2 |\nabla S'|^2}{(S')^{\gamma+3}} + \frac{2\gamma}{(S')^{\gamma+1}} \left( |\nabla |\text{Sic}'|^2, \nabla S' \right) + \frac{4C^2}{n^2} \frac{S'}{(S')^\gamma}$$

$$- \frac{2C}{n} \frac{(1 - \gamma)}{(S')^{\gamma+1}} |\text{Sic}'|^2 |\nabla S'|^2 + \frac{4\alpha_1}{(S')^\gamma} \langle \text{Sic}', \Xi \rangle - \frac{2\alpha_1 \gamma}{(S')^{\gamma+1}} \Box \Xi$$

where $tr\Xi$ is the trace of $\Xi$ with respect to $g(t)$, and

$$(3.18) \quad \Xi := \Delta u \nabla^2 u - \beta_2 \nabla u \otimes \nabla u - \beta_1 \nabla u \otimes \nabla |\nabla u|^2.$$ From the identities

$$\left\langle \nabla \frac{|\text{Sic}'|^2}{(S')^\gamma}, \nabla S' \right\rangle = \frac{1}{(S')^\gamma} \left\langle \nabla |\text{Sic}'|^2, \nabla S' \right\rangle - \frac{\gamma}{(S')^{\gamma+1}} |\nabla S'|^2 |\text{Sic}'|^2,$$

$$|\nabla \text{Sic}'|^2 = |Z'|^2 - |\text{Sic}'|^2 |\nabla S'|^2 + S' \langle \nabla |\text{Sic}'|^2, \nabla S' \rangle,$$

where $Z'$ is a 3-tensor with components $Z'_{ijk} = S'_{ij}S'_{jk} - S'_{ik}S'_{jk}$, we have

$$\Box \frac{|\text{Sic}'|^2}{(S')^\gamma} = \frac{2(\gamma - 1)}{S'} \left\langle \nabla \frac{|\text{Sic}'|^2}{(S')^\gamma}, \nabla S' \right\rangle - \frac{2}{(S')^{\gamma+2}} |Z'|^2 - \frac{2\gamma|\text{Sic}'|^4}{(S')^{\gamma+1}}$$

$$- \frac{(2 - \gamma)(\gamma - 1)}{(S')^{\gamma+1}} |\text{Sic}'|^2 |\nabla S'|^2 + \frac{4}{(S')^\gamma}\text{Sm}(\text{Sic}', \text{Sic}') + \frac{4C^2}{n^2} \frac{S'}{(S')^\gamma}$$

$$- \frac{2C}{n} \frac{(1 - \gamma)}{(S')^{\gamma+1}} |\text{Sic}'|^2 |\nabla S'|^2 + \frac{4\alpha_1}{(S')^\gamma} \langle \text{Sic}', \Xi \rangle - \frac{2\alpha_1 \gamma}{(S')^{\gamma+1}} \Box \Xi.$$ The identity

$$\Box (S')^{2-\gamma} = (2 - \gamma)(S')^{1-\gamma} \Box S' - (2 - \gamma)(1 - \gamma)(S')^{-\gamma} |\nabla S'|^2$$

implies

$$\Box f = 2(\gamma - 1) \langle \nabla f, \nabla \ln S' \rangle - \frac{2}{(S')^{\gamma+2}} |Z'|^2 - (2 - \gamma)(\gamma - 1) |\nabla \ln S'|^2 f$$

$$+ \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3.$$
where
\[
D_1 := -\frac{2(2-\gamma)}{n}(S')^{1-\gamma}|\text{Sin}'|^2 + \frac{4}{(S')^\gamma} \text{Sm}(\text{Sin}', \text{Sin}') - \frac{2\gamma|\text{Sin}'|^4}{(S')^{1+\gamma}},
\]
\[
= \frac{2}{(S')^{\gamma+1}} \left[ (2-\gamma)|\text{Sin}'|^2|\text{Sin}|^2 - 2 \left(|\text{Sin}'|^4 - S'Sm(\text{Sin}', \text{Sin}') \right) \right],
\]
\[
= \frac{2}{(S')^{\gamma+1}} \left[ -\gamma(S')^{2\gamma}f^2 + \left( \frac{2n-4}{n(n-1)} - \frac{\gamma}{n} \right) (S')^{\gamma+2}f - \frac{4(S')^{4} \text{Sin}^3}{n-2 (S')^{3}} \right.
\]
\[+ 2(S')^{3} \text{W} \left( \frac{\text{Sin}}{S'}, \frac{\text{Sin}}{S'} \right) + \frac{2\alpha_1}{n-2} \left( (S')^{2} \text{Sin}' - \frac{n}{2} S' \text{Sin}'^2, \nabla u \otimes \nabla u \right)
\]
\[- \left. \frac{2}{n-1} \left( \frac{C}{n} + \frac{\alpha |\nabla u|^2}{n-1} \right) (S')^{3} - (S')^{\gamma+1}f \right] \right],
\]
\[
D_2 := 4C \left[ n(S')^2 - \frac{(1-\gamma)S' + \frac{C}{(S')^{\gamma+1}} |\text{Sin}'|^2 - 2 - \gamma \text{C} - 2S'}{2n (S')^{\gamma-1}} \right],
\]
\[
= \frac{4C}{n^2} \left[ \frac{C}{S'} + \frac{S'}{(S')^{\gamma}} + \frac{1}{(S')^{\gamma-2}} + nf \left( 1 - \frac{C}{2S'} \right) \right],
\]
\[
D_3 := \frac{4\alpha_1}{(S')^{\gamma}} (\text{Sin}', \Xi) - \frac{2\alpha_1 \text{tr} \Xi}{(S')^{\gamma+1}} \left( \gamma|\text{Sin}'|^2 + \frac{2-\gamma}{n} |S'|^2 \right)
\]
and $\text{Sin}^3 = \text{Sin}_i \text{Sin}_j \text{Sin}^k$ and $(\text{Sin}'^2)_{ij} = S'_i S'_j$. Some detailed computations can be found in [43].

In particular, for $\gamma = 2$, we have from [43] that
\[
(3.20) \quad \Box f = 2(\nabla f, \nabla \ln S') - 2 \left( \frac{\text{Sin}}{S'} \right)^2 + D_1 + D_2 + D_3,
\]
where
\[
D_1 = 4S' \left[ -f^2 - \frac{f}{n(n-1)} - \frac{2 \text{Sin}^3}{S'} + \frac{1}{S'} \text{W} \left( \frac{\text{Sin}}{S'}, \frac{\text{Sin}}{S'} \right) \right.
\]
\[- \left. \frac{1}{S'} \left( \frac{C}{n} + \frac{\alpha |\nabla u|^2}{n-1} \right) \left( \frac{1}{n} - \frac{f}{n-1} \right) + \frac{1}{S'} \left( \frac{\text{Sin}^2}{S'}, \nabla u \otimes \nabla u \right) \right]
\]
\[
+ \left. \frac{1}{S'} \left( \frac{\alpha |\nabla u|^2}{2n(n-2)} \right) \right],
\]
\[
D_2 = \frac{4C}{n^2} \left[ \frac{C}{S'} + \frac{S'}{(S')^{3}} + nf \left( 1 - \frac{C}{S'} \right) \right],
\]
\[
D_3 = \frac{4\alpha_1}{S'} \left[ (\text{Sin}', \Xi) - \text{ftr} \Xi \right].
\]

Since the flow is regular, we have a uniform upper bound for $|\nabla u|$, together with $S' \geq C_0 > 0$, and then
\[
D_1 \leq 4S' \left[ -f^2 - \frac{f}{n(n-1)} + \frac{2}{n-2} f^{3/2} + \tilde{C} \frac{|\text{W}|}{S'} f + \tilde{C} + \tilde{C} f \right],
\]
\[
D_2 \leq 4S' (\tilde{C} + \tilde{C} f),
\]
\[
D_3 \leq 4\tilde{C} S' (f + f^{1/2}) \frac{|\nabla^2 u|^2}{S'}
\]
for some uniform constant $\tilde{C}$ depending only on $n, C_0, C, \alpha_1, \beta_1, \beta_2, \gamma_0,$ and $u_0$. Without loss of generality, we may assume that $f \geq 1$. In this case we have

$$\Box f \leq 2 \langle \nabla f, \nabla \ln S' \rangle + 4S' f \left( -\frac{2}{n-2} f^{1/2} + \tilde{C} + \frac{|W| + |\nabla^2 u|^2}{S'} \right).$$

Now the maximum principle yields the desired estimate. $\square$

As immediate consequence, we have the following

**Corollary 3.3.** Let $(g(t), u(t))_{t \in [0,T)}$ be a solution to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow on a closed $n$-dimensional manifold $M$ with $T \leq 0$ and the initial data $(g_0, u_0)$. Then only one of the following cases occurs:

- (a) $T = \infty$;
- (b) $T < \infty$ and $|\text{Ric}_{g(t)}|_{g(t)} \lesssim 1$;
- (c) $T < \infty$ and $\text{Ric}_{g(t)}|_{g(t)} \to \infty$ as $t \to T$. In this case, there are only two subcases:
  - (c1) $R_{g(t)}|_{g(t)} \to \infty$;
  - (c2) $|R_{g(t)}|_{g(t)} \lesssim 1$ and there exist some uniform constants $C_1, C_2 > 0$ such that
    $$\frac{|W_{g(t)}|_{g(t)} + |\nabla^2 u(t)|_{g(t)}^2}{S_{g(t)} + C_1} \to \infty$$
    as $t \to T$.

This is a general property of the long time existence for a regular Ricci flow, generalizing results in [7, 43]. Since the signs of $\alpha_1, \beta_1, \beta_2$ are not determined, we can not discard the case (b) which is true for Ricci-harmonic flow and Ricci flow.

### 3.2. Bounded scalar curvature.

We assume that $(g(t), u(t))_{t \in [0,T)}$ is a solution to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow on a closed 4-dimensional manifold $M$ with $T \leq \infty$ and the initial data $(g_0, u_0)$, and also assume that $S_{g(t)} + C \geq C_0 > 0$ along the flow for some uniform constants $C, C_0 > 0$. According to (3.13) one has

$$|\nabla_{g(t)} u(t)|_{g(t)}^2 \leq A_1$$

along the flow.

In the proof of Theorem 3.2 we actually proved the following identity

$$\Box \frac{|\text{Sic}|^2}{S + C} = -2 \frac{|Z|^2}{(S + C)^2} - 2 \frac{|\text{Sic}|^4}{(S + C)^2} + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S + C} \left[ \text{tr} \Xi |\text{Sic}|^2 - 2(S + C) \langle \text{Sic}, \Xi \rangle \right],$$

where $Z$ is the 3-tensor with components

$$Z_{i\dot{k}} = S' \nabla_i \text{S}_{\dot{k}} - \text{S}_{\dot{k}} \nabla_i S' = (S + C) \nabla_i \text{S}_{\dot{k}} - \text{S}_{\dot{k}} \nabla_i S,$$

and $\Xi$ is given in (3.18). Observe that the last term in (3.21) can be written as

$$\text{tr} \Xi |\text{Sic}|^2 - 2(S + C) \langle \text{Sic}, \Xi \rangle$$

$$= \left[ (S + C) \left| \Delta u \frac{\text{Sic}}{\sqrt{S + C}} - \sqrt{S + C} \nabla^2 u \right|^2 - (S + C)^2 |\nabla^2 u|^2 \right]$$
From the evolution equation
\[
\partial_t (S + C) \left( \nabla u \frac{\text{Sic}}{\sqrt{S + C}} - \frac{\text{Sic}}{\sqrt{S + C}} \nabla u \right)^2 - (S + C)^2 \|\nabla u\|^2
\]
and then

\[
2\beta_1 (S + C) \left[ \left( \nabla^2 u, \nabla u \otimes \nabla u \right) \frac{\text{Sic}}{S + C} - 2 \left( \text{Sic}, \frac{1}{2} \nabla u \otimes \nabla |\nabla u|^2 \right) \right].
\]

To further analysis, we need to know an estimate for $|\nabla^2 u|^2$. Recall that

\[
\Box |\nabla u|^2 = 2\beta_2 |\nabla u|^2 - 2|\nabla^2 u|^2 - 2\alpha_1 |\nabla u|^4 + 4\beta_1 \langle \nabla u \otimes \nabla u, \nabla^2 u \rangle.
\]

Given any $\epsilon > 0$, we have from (3.24) that

\[
\Box |\nabla u|^2 \leq 2\beta_2 |\nabla u|^2 - 2|\nabla^2 u|^2 - 2\alpha_1 |\nabla u|^4 + 4\beta_1 \left( \epsilon |\nabla^2 u|^2 + \frac{1}{4\epsilon} |\nabla u|^4 \right)
\]

\[
= 2\beta_2 |\nabla u|^2 - 2 \left( 1 - 2\epsilon |\beta_1| \right) |\nabla^2 u|^2 - 2 \left( \alpha_1 - \frac{|\beta_1|}{2\epsilon} \right) |\nabla u|^4.
\]

From the evolution equation $\partial_t dV_t = -S dV_t$, we arrive at

\[
\frac{d}{dt} \int_M |\nabla u|^2 dV_t = \int_M \partial_t |\nabla u|^2 dV_t + \int_M |\nabla u|^2 \partial_t dV_t
\]

\[
= \int_M \Box |\nabla u|^2 dV_t - \int_M S |\nabla u|^2 dV_t
\]

\[
\leq 2\beta_2 \int_M |\nabla u|^2 dV_t - 2 \left( 1 - 2\epsilon |\beta_1| \right) \int_M |\nabla^2 u|^2 dV_t
\]

\[
- \int_M S |\nabla u|^2 dV_t - 2 \left( \alpha_1 - \frac{|\beta_1|}{2\epsilon} \right) \int_M |\nabla u|^4 dV_t
\]

and then

\[
\frac{d}{dt} \int_M |\nabla u|^2 dV_t \leq -2 \left( 1 - 2\epsilon |\beta_1| \right) \int_M |\nabla^2 u|^2 dV_t
\]

\[
+ (2|\beta_2| + C) \int_M |\nabla u|^2 - 2 \left( \alpha_1 - \frac{|\beta_1|}{2\epsilon} \right) \int_M |\nabla u|^4 dV_t,
\]

because of $S + C \geq C_0 > 0$.

1. $\beta_1 = 0$. In this case, the inequality becomes

\[
\frac{d}{dt} \int_M |\nabla u|^2 dV_t \leq -2 \int_M |\nabla^2 u|^2 dV_t + (2|\beta_2| + C) \int_M |\nabla u|^2 dV_t - 2\alpha_1 \int_M |\nabla u|^4 dV_t.
\]

When $\alpha_1 \geq 0$, we furthermore have

\[
\frac{d}{dt} \int_M |\nabla u|^2 dV_t \leq -2 \int_M |\nabla^2 u|^2 dV_t + (2|\beta_2| + C) \int_M |\nabla u|^2 dV_t
\]

or in this form

\[
\frac{d}{dt} \left[ e^{-2(2|\beta_2| + C)t} \int_M |\nabla u|^2 dV_t \right] \leq -2e^{-2(2|\beta_2| + C)t} \int_M |\nabla^2 u|^2 dV_t.
\]

Integrating over the interval $[0, t]$ and using (3.21), we obtain

\[
2 \int_0^t \int_M |\nabla^2 u|^2 dV_t dt + \int_M |\nabla u|^2 dV_t \leq e^{(2|\beta_2| + C)t} A_1 \text{Vol}_0
\]
where $\text{Vol}_0$ is the volume of the initial metric $g_0$. When $\alpha_1 < 0$, we similar have
\[
\frac{d}{dt} \int_M |\nabla u|^2 dV_t \leq -2 \int_M |\nabla^2 u|^2 dV_t + (2|\beta_2| + 2|\alpha_1| A_1 + C) \int_M |\nabla u|^2 dV_t.
\]
Replacing $|\beta_2|$ by $|\beta_2| + |\alpha_1| A_1$ in (3.26), the case that $\alpha_1$ is negative implies
\[(3.27) \quad 2 \int_0^t \int_M |\nabla^2 u|^2 dV_t + \int_M |\nabla u|^2 dV_t \leq e^{(2|\beta_2| + 2|\alpha_1| A_1 + C)t} A_1 \text{Vol}_0.
\]

(2) $\beta_1 \neq 0$. In this case we choose $\epsilon = 1/4|\beta_1|$ in (3.26) and obtain
\[
\frac{d}{dt} \int_M |\nabla u|^2 dV_t \leq - \int_M |\nabla^2 u|^2 dV_t + (2|\beta_2| + C) \int_M |\nabla u|^2 dV_t - 2(\alpha_1 - 2\beta_1^2) \int_M |\nabla u|^4 dV_t.
\]

A similar argument used to obtain equations (3.26) and (3.27) we get
\[(3.28) \quad \int_0^t \int_M |\nabla^2 u|^2 dV_t + \int_M |\nabla u|^2 dV_t \leq e^{(2|\beta_2| + 2|\alpha_1 - 2\beta_1^2| A_1 + C)t} A_1 \text{Vol}_0.
\]

Finally, from (3.27) and (3.28), we have
\[(3.29) \quad \int_0^t A_2(t) dt \leq e^{(2|\beta_2| + 2|\alpha_1 - 2\beta_1^2| A_1 + C)t} A_1 \text{Vol}_0.
\]

where
\[(3.30) \quad A_2(t) := \int_M |\nabla^2 u|^2 dV_t.
\]

Introduce
\[(3.31) \quad \Lambda := \frac{1}{(S + C)^2} \left[ \text{tr} \Xi |\text{Sic}|^2 - 2(S + C) \langle \text{Sic}, \Xi \rangle \right], \quad f := \frac{|\text{Sic}|^2}{S + C}
\]
and rewrite (3.22) as
\[(3.32) \quad \Box f = -2 \frac{|Z|^2}{(S + C)^3} - 2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S + C} - 2\alpha_1 \Lambda.
\]

To determine a lower bound for $\alpha_1 \Lambda$ we consider the following cases.

(i) $\alpha_1 \geq 0$. In this case we shall also find a lower bound for $\Lambda$.

(ia) When $\beta_2 \leq 0$, we have
\[
\Lambda \geq -|\nabla^2 u|^2 - |\beta_2||\nabla u|^2 - 2|\beta_1| \left| \frac{|\nabla^2 u||\nabla u|^2}{S + C} f - 2|\beta_1| \left( f + \frac{|\nabla u|^4}{S + C} \right) \right|
\]
\[
\geq -|\nabla^2 u|^2 - |\beta_2||\nabla u|^2 - 2|\beta_1| \left| \frac{|\nabla^2 u||\nabla u|^2}{C_0} f - 2|\beta_1| \left( f + \frac{|\nabla u|^4}{C_0} \right) \right|
\]

(ib) When $\beta_2 > 0$, we get the same estimate in (ia), where, in the case, the term $-|\beta_2||\nabla u|^2$ is now replaced by
\[
- \frac{\beta_2}{S + C} \left( \frac{\text{Sic}}{\sqrt{S + C}} - \frac{\sqrt{S + C}}{|\nabla u|} \nabla u \otimes \nabla u \right)^2
\]
which is bounded below by
\[
- \frac{2\beta_2}{S + C} \left( f|\nabla u|^2 + \frac{S + C}{|\nabla u|^2} |\nabla u|^4 \right) \geq -2\beta_2 |\nabla u|^2 \left( 1 + \frac{f}{C_0} \right).
\]
From (ia) – (ib), we obtain for any $\beta_2$

$$
\Lambda \geq -|\nabla^2 u|^2 - 2|\beta_2||\nabla u|^2 \left(1 + \frac{f}{C_0}\right)
$$

(3.33)

$$
-2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{C_0}f - 2|\beta_1| \left(f + \frac{|\nabla^2 u||\nabla u|^4}{C_0}\right).
$$

(ii) $\alpha_1 < 0$. In this case we shall find an upper bound for $\Lambda$.

(iia) When $\beta_2 > 0$, we have

$$
\Lambda \leq \frac{1}{(S + C)^2} \left[(S + C)\left|\frac{\Delta u}{\sqrt{S + C}} - \sqrt{S + C} \nabla^2 u\right|^2 + \beta_2(S + C)|\nabla u|^2\right]
$$

$$
+ 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S + C}f + 2|\beta_1| \left(f + \frac{|\nabla u|^4|\nabla^2 u|^2}{S + C}\right)
$$

$$
+ 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S + C}f + 2|\beta_1| \left(f + \frac{|\nabla u|^4|\nabla^2 u|^2}{S + C}\right)
$$

$$
\leq \frac{2}{S + C} \left[(\Delta u)^2f + (S + C)|\nabla^2 u|^2\right] + \beta_2|\nabla u|^2
$$

$$
+ 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S + C}f + 2|\beta_1| \left(f + \frac{|\nabla u|^4|\nabla^2 u|^2}{S + C}\right)
$$

$$
\leq 2|\nabla^2 u|^2 \left(1 + \frac{4f}{C_0}\right) + \beta_2|\nabla u|^2
$$

$$
+ 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S + C}f + 2|\beta_1| \left(f + \frac{|\nabla u|^4|\nabla^2 u|^2}{S + C}\right)
$$

(iib) When $\beta_2 \leq 0$, we also have

$$
\Lambda \leq 2|\nabla^2 u|^2 \left(1 + \frac{4f}{C_0}\right) - 2|\beta_2||\nabla u|^2 \left(1 + \frac{f}{C_0}\right)
$$

$$
+ 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S + C}f + 2|\beta_1| \left(f + \frac{|\nabla u|^4|\nabla^2 u|^2}{S + C}\right).
$$

From (iia) – (iib), we obtain for any $\beta_2$

$$
\Lambda \leq 2|\nabla^2 u|^2 \left(1 + \frac{4f}{C_0}\right) + 2|\beta_2||\nabla u|^2 \left(1 + \frac{f}{C_0}\right)
$$

$$
+ 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{C_0}f + 2|\beta_1| \left(f + \frac{|\nabla u|^4|\nabla^2 u|^2}{C_0}\right).
$$

(3.34)
Lemma 3.4. If $\alpha_1 \geq 0$, one has
\[
\frac{d}{dt} \int_M f \, dV_t \leq \int_M \left[ -2f^2 + 4 \frac{Sm(S_i, S_i)}{S + C} - fS \right] \, dV_t
\]
(3.35)
\[
+ \int_M 4\alpha_1 \left[ |\beta_1| + \frac{A_1(|\beta_2| + |\beta_1||\nabla^2 u|)}{C_0} \right] f \, dV_t
\]
\[
+ 2\alpha_1 \left( 1 + \frac{2A_1^2|\beta_1|}{C_0} \right) A_2 + 4\alpha_1 A_1|\beta_2| \text{Vol}_t.
\]
If $\alpha_1 < 0$, one has
\[
\frac{d}{dt} \int_M f \, dV_t \leq \int_M \left[ -2f^2 + 4 \frac{Sm(S_i, S_i)}{S + C} - fS \right] \, dV_t
\]
(3.36)
\[
- \int_M 4\alpha_1 \left[ |\beta_1| + \frac{A_1(|\beta_2| + |\beta_1||\nabla^2 u|) + 4|\nabla^2 u|^2}{C_0} \right] f \, dV_t
\]
\[
- 2\alpha_1 \left( 2 + \frac{2A_1^2|\beta_1|}{C_0} \right) A_2 - 4\alpha_1 A_1|\beta_2| \text{Vol}_t.
\]
Here $\text{Vol}_t$ denotes the volume of $g(t)$.

This follows immediately from (3.32) – (3.34). According to (3.35) and (3.36) we have
\[
\frac{d}{dt} \int_M f \, dV_t \leq \int_M \left[ -2f^2 + 4 \frac{Sm(S_i, S_i)}{S + C} - fS \right] \, dV_t + \int_M 4\alpha_1 \left[ |\beta_1| + \frac{A_1(|\beta_2| + |\beta_1||\nabla^2 u|) + 4(1 - \text{sgn}(\alpha_1, 0))|\nabla^2 u|^2}{C_0} \right] f \, dV_t
\]
(3.37)
\[
+ 4|\alpha_1| \left( 1 + \frac{A_1^2|\beta_1|}{C_0} \right) A_2 + 4|\alpha_1|A_1|\beta_2| \text{Vol}_t,
\]
where $\text{sgn}(\alpha_1, 0) = 1$ if $\alpha_1 \geq 0$, and otherwise 0. For $\alpha_1 \geq 0$, the above estimates shall imply integrals bounds for $|S_i|, |Sm|$ as in [43]. On the other hand, the case $\alpha_1 < 0$ will prevent us to obtain the previous-type of estimates, since we have no control on $|\nabla^2 u|^2$ even in the integral sense. Hence in the case that $\alpha_1$ is negative, we will impose another condition\footnote{For the Ricci-harmonic flow or the $(\alpha, 0, 0, 0)$-Ricci flow, the estimate (3.38) is always true provided that the curvature condition (1.4) holds.}.

\[
||\nabla^2 g(t)u(t)||^2_{g(t)} \leq \tilde{A}_1
\]
(3.38)
along the flow for some uniform constant $\tilde{A}_1$. Note that the condition (3.38) is stronger than (3.29). A weakened condition is
\[
||\nabla^2 g(t)u(t)||_{L^4(M, g(t))} \leq \tilde{A}_1
\]
(3.39)
along the flow for some uniform constant $\tilde{A}_1$.\footnote{For the Ricci-harmonic flow or the $(\alpha, 0, 0, 0)$-Ricci flow, the estimate (3.38) is always true provided that the curvature condition (1.4) holds.}
In the case of dimension 4, we have the following Gauss-Bonnet-Chern formula

\[ 32\pi^2 \chi(M) = \int_M \left[ |Rm|^2 - 4|\text{Ric}|^2 + R^2 \right] dV_f \]

for any Riemannian metric \( g \) on \( M \), where \( \chi(M) \) is the Euler characteristic number of \( M \). Applying the formula (3.40) to \( g(t) \) and noting that (see Lemma 3.1 in [43])

\[ |Rm|^2 - 4|\text{Ric}|^2 + R^2 = |Sm|^2 - 4|\text{Sic}|^2 + S^2 - \frac{13}{2}a_1^2|\nabla u|^4 - 9a_1\text{Sic}(\nabla u, \nabla u) + 2a_1S|\nabla u|^2 \]

we have (see (3.12) in [43])

\[ \int_M \left[ |Sm|^2 - 4|\text{Sic}|^2 + S^2 \right] dV_f = 32\pi^2 \chi(M) + \frac{13}{2}a_1^2 \int_M |\nabla u|^4 dV_f \]

\[ + 9a_1 \int_M \text{Sic}(\nabla u, \nabla u) dV_f - 2a_1 \int_M S|\nabla u|^2 dV_f. \]

Moreover we also have (see (3.15) in [43])

\[ \int_M \left[ -2f^2 + \frac{4Sm(\text{Sic}, \text{Sic})}{S+C} - fS \right] dV_f \leq \int_M \left( -f^2 + 36Cf + 574S^2 \right) dV_f \]

\[ + 8 \left[ 32\pi^2 \chi(M) + 13a_1^2A_1^2\text{Vol}_0e^{(2|\beta_2|+2|a_1-2\beta_2|}|A_1+C|}\right] \]

where we used 3.28 to control the integral

\[ \int_M |\nabla u|^4 dV_f. \]

Plugging (3.42) into (3.37) and using (2.6) yields

\[ \frac{d}{dt} \int_M f dV_f \leq \int_M \left( -\frac{1}{2}f^2 + C_1f + C_2S^2 \right) dV_f \]

\[ + C_3A_2 + C_4 \left[ 1 - \text{sgn}(\alpha_1, 0) \right] \int_M |\nabla^2 u|^4 dV_f + C_5e^{C_6t} + C_7. \]

where

\[ C_1 = 36C + 4|\alpha_1||\beta_1| + \frac{4|\alpha_1||A_1||\beta_2|}{C_0} = C_1(C, C_0, \alpha_1, \beta_1, \beta_2, A_1), \]

\[ C_2 = 574, \]

\[ C_3 = \frac{16|\alpha_1|^2A_1^2|\beta_1|^2}{C_0^2} + 4|\alpha_1| \left( 1 + \frac{A_1^2|\beta_1|}{C_0} \right) = C_3(C_0, \alpha_1, \beta_1, A_1), \]

\[ C_4 = \frac{256|\alpha_1|^2}{C_0^2} = C_4(C_0, \alpha_1), \]

\[ C_5 = (104a_1^2A_1^2 + 4|\alpha_1||A_1||\beta_2|)\text{Vol}_0 = C_5(\alpha_1, \beta_2, A_1, \text{Vol}_0), \]

\[ C_6 = 2|\beta_2| + 2|\alpha_1 - 2\beta_2||A_1| + C = C_6(C, \alpha_1, \beta_1, \beta_2, A_1), \]

\[ C_7 = 256\pi^2 \chi(M). \]

Note that \( C_1, C_6 \) are linear functions of \( A_1 \) and \( C_3, C_5 \) are quadratic functions of \( A_1 \). Furthermore, \( C_2, C_7 \) are constants depending only on the topological quantities of \( M \). Finally, \( C_4 \) depends only on \( \alpha_1 \), and the term containing \( C_4 \) in (3.43) vanishes provided that \( \alpha_1 \geq 0 \).
Theorem 3.5. Let \((g(t), u(t))\) be a solution to the regular \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow on a closed 4-manifold \(M\) with \(T \leq \infty\) and the initial data \((g_0, u_0)\). Assume that \(S_{g(t)} + C \geq C_0 > 0\) along the flow for some uniform constants \(C, C_0 > 0\). Then

\[
\int_M \frac{|S g(s)|^2}{S g(s)} dV_{g(s)} + \int_0^s \int_M \frac{|S g(t)|^2}{S g(t)} dV_{g(t)} dt
\]

\[
\leq C'(1 + s) e^{C's} + C'e^{C's} \int_0^s S^2 dV_t dt
\]

\[
+ C'[1 - \text{sgn}(\alpha_1, 0)] e^{C's} \int_0^s |\nabla^2 u|^4 dV_t dt,
\]

\[
\int_M |S g(s)|^2 dV_{g(s)} \leq C'(1 + s) e^{C's} + C'e^{C's} \int_0^s S^2 dV_t dt
\]

\[
+ C'[1 - \text{sgn}(\alpha_1, 0)] e^{C's} \int_0^s |\nabla^2 u|^4 dV_t dt,
\]

\[
\int_0^s \int_M |S g(t)|^2 dV_{g(t)} dt \leq C'(1 + s) e^{C's} + C'e^{C's} \int_0^s S^2 dV_t dt
\]

\[
+ C'[1 - \text{sgn}(\alpha_1, 0)] e^{C's} \int_0^s |\nabla^2 u|^4 dV_t dt,
\]

for all \(s \in [0, T]\), where \(C' = C'(g_0, u_0, \alpha_1, \beta_1, \beta_2, C, C_0, A_1, \chi(M))\) is a uniform constant. Here \(|\nabla g(t)| u(t)|^2_{g(t)} \leq A_1\) holds along the flow (by the regularity) for some uniform constant \(A_1 > 0\) (which depends only on \(g_0, u_0\) and \(\alpha_0, \beta_1, \beta_2\)).

Proof. Integrating (3.13) over \([0, s]\) we obtain

\[
e^{-C_1 t} \int_M f dV_t + \frac{1}{2} e^{-C_1 s} \int_0^s \int_M f^2 dV_t dt
\]

\[
\leq \int_0^s \left[ C_3 A e^{-C_1 t} + C_7 e^{-C_1 t} + C_5 e^{(C_6 - C_1) t} \right] dt + C_2 \int_0^s e^{-C_1 t} \int_M S^2 dV_t dt
\]

\[
+ C_4 \int_0^s e^{-C_1 t} \left[ 1 - \text{sgn}(\alpha_1, 0) \right] \int_M |\nabla^2 u|^4 dV_t dt + \int_M f dV_t \bigg|_{t=0}.
\]

Using (3.29), the above inequality becomes

\[
\int_M f dV_s + \int_0^s \int_M f^2 dV_t dt \leq 2 e^{C_1 s} \left( \int_M f dV_t \bigg|_{t=0} \right)
\]

\[
+ 2 C_3 e^{(C_1 + C_6) s} + \frac{2 C_7}{C_1} e^{C_1 s} + 2 C_5 \cdot \left\{ e^{(C_1 + C_6) s}/|C_1 - C_6|, \quad C_1 = C_6, \right\}
\]

\[
+ 2 C_3 e^{C_1 s} \int_0^s \int_M S^2 dV_t dt + 2 C_5 [1 - \text{sgn}(\alpha_1, 0)] e^{C_1 s} \int_0^s \int_M e^{-C_1 t} |\nabla^2 u|^4 dV_t dt.
\]

\[
\leq C_8 (1 + s) e^{(C_1 + C_6) s} + 2 C_3 e^{C_1 s} \int_0^s \int_M S^2 dV_t dt
\]
Corollary 3.3

where for some uniform constant \( C \) which depends only on \( g_0, u_0, \alpha_1, \beta_1, \beta_2, C, C_0, \) and \( A_1, \chi(M) \).

The second and third estimates follows from (see (3.22) and below in [43])

\[
|\text{Sic}| \leq 2 \left| \frac{\text{Sic}}{S + C} \right| + C \frac{\chi(t)}{2}, \quad |\text{Sic}| \leq 8 \left| \frac{\text{Sic}}{(S + C)^2} \right| + C^2,
\]

and \( \text{Vol}_t \leq e^{Ct} \text{Vol}_0 \).

The last estimate follows from (3.41), as the argument in [43].

Corresponding to case (c2) in Corollary 3.3 we obtain

**Corollary 3.6.** Let \( (g(t), u(t))_{t \in [0,T]} \) be a solution to the regular \((a_1, 0, \beta_1, \beta_2)\)-Ricci flow on a closed 4-manifold \( M \) with \( T \leq \infty \) and the initial data \((g_0, u_0)_0\). Assume that \( S_{g(t)} + C \geq C_0 > 0 \) and \( S^2_{g(t)} \leq C_1 < \infty \) along the flow for some uniform constants \( C, C_0, C_1 > 0 \). Then

\[
\int_0^s \int_M |\text{Sic}_{g(t)}|^2_{g(t)} dV_{g(t)} dt \leq C'(1 + s)e^{C_s}
\]

\[
\int_0^s \int_M |\text{Sm}_{g(t)}|^2_{g(t)} dV_{g(t)} dt \leq C'(1 + s)e^{C_s}
\]

for all \( s \in [0, T] \), where \( C' = C'(g_0, u_0, a_1, \beta_1, \beta_2, C, C_0, C_1, A_1, \chi(M)) \) is a uniform constant. Here \( |\nabla g(t)|^2_{g(t)} \leq A_1 \) holds along the flow (by the regularity) for some uniform constant \( A_1 > 0 \) which depends only on \( g_0, u_0, a_0, \beta_1, \beta_2 \).

To get the \( L^1_{[0, T]} L^p(M) \)-estimate of \( \text{Sic}_{g(t)} \), introduce the basic assumption \( \text{BA} \) for a solution \((g(t), u(t))_{t \in [0,T]}\) to the regular \((a_1, 0, \beta_1, \beta_2)\)-Ricci flow:

(a) \( M \) is a closed 4-manifold;
(b) \( T < \infty \);
(c) \( -1 \leq S_{g(t)} \leq 1 \) along the flow;
(d) \( |\nabla g(t)|^2_{g(t)} \leq A_1 \) along the flow.

The last condition is obtained from the regularity of the flow and the third condition implies \( S_{g(t)} + C \geq C_0 > 0 \), where \( C = 2 \) and \( C_0 = 1 \).

Under \( \text{BA} \), given \( \epsilon > 0 \), one has (see the proof of Theorem 3.4 in [43])

\[
\int_M \left[ -2f^2 + 4 \frac{\text{Sm(Sic, Sic)}}{S + 2} - fS \right] dV_t \leq \int_M \left[ - \left( 2 - \frac{4}{\epsilon^2} \right) f^2 + (12\epsilon^2 + 1) f + \epsilon^2 \left( |\text{Sm}|^2 - 4 |\text{Sic}|^2 + S^2 \right) \right] dV_t
\]

where \( f = |\text{Sic}|^2/(S + 2) \). In this case, \( C_0 = 1 \), so that \( \int_M \left[ -2f^2 + 4 \frac{\text{Sm(Sic, Sic)}}{S + 2} - fS \right] dV_t \)

\[
\frac{d}{dt} \int_M f dV_t \leq \int_M \left[ -2f^2 + 4 \frac{\text{Sm(Sic, Sic)}}{S + 2} - fS \right] dV_t
\]
\[ + \int_M 4|\alpha_1| \left[ |\beta_1| + A_1(|\beta_2| + |\beta_1| |\nabla^2 u|) + 4(1 - \text{sgn}(\alpha_1, 0))|\nabla^2 u|^2 \right] f dV_t \\
+ 4|\alpha_1|(1 + A_1^2 |\beta_1|) + 4|\alpha_1| |A_1| |\beta_2| e^2 Vol_0. \]

\[ \leq \int_M \left[ - \left( 2 - \frac{12}{e^2} \right) f^2 + \left( 12e^2 + 1 + 4|\alpha_1| |\beta_1| + 4|\alpha_1| |\beta_1| \right) f \right] dV_t \]

\[ + e^2 \int_M \left( |S_m|^2 - 4|\text{Sic}^2 + S^2| \right) dV_t + 4|\alpha_1|(1 + A_1^2 |\beta_1|)A_2 + 4|\alpha_1| |A_1| |\beta_2| e^2 Vol_0 \]

\[ + e^2 \int_M \left[ (\alpha_1 \beta_1)^2 A_1^2 |\nabla^2 u|^2 + 256|\alpha_1|^2 (1 - \text{sgn}(\alpha_1, 0)) |\nabla^2 u|^4 \right] dV_t. \]

On the other hand, the identity (3.41) implies (see (3.13) in [43])

\[ \int_M \left[ |S_m|^2 - 4|\text{Sic}^2 + S^2| \right] dV_t \leq 32\pi^5 \chi(M) + 13a_1^2 \int_M |\nabla u|^4 dV_t + \frac{243}{26} \int_M f dV_t. \]

Therefore, taking \( e^2 = 12 \) and using (3.28),

\[ \frac{d}{dt} \int_M f dV_t \leq \int_M \left( -f^2 + \tilde{C}_1 f \right) dV_t + \tilde{C}_2 A_2 + \tilde{C}_3 + \tilde{C}_4 e\tilde{c}s \]

\[ + \tilde{C}_6 [1 - \text{sgn}(\alpha_1, 0)] \int_M |\nabla^2 u|^4 dV_t, \]

where

\[ \tilde{C}_1 = 145 + 4|\alpha_1 \beta_1| + 4|\alpha_1 \beta_2| A_1 + \frac{1458}{13} = \tilde{C}_1(\alpha_1, \beta_1, \beta_2, A_1), \]

\[ \tilde{C}_2 = 4|\alpha_1|(1 + A_1^2 |\beta_1|) + 12(\alpha_1 \beta_1)^2 A_1^2 = \tilde{C}_2(\alpha_1, \beta_1, A_1), \]

\[ \tilde{C}_3 = 385\pi^5 \chi(M), \]

\[ \tilde{C}_4 = \left( 156a_1^2 A_1 + 4|\alpha_1| |A_1| |\beta_2| \right) Vol_0 = \tilde{C}_4(\alpha_1, \beta_2, A_1, Vol_0), \]

\[ \tilde{C}_5 = 2|\beta_2| + 2|\alpha_1 - 2\beta_1| A_1 + 2 = \tilde{C}_5(\alpha_1, \beta_1, \beta_2, A_1), \]

\[ \tilde{C}_6 = 3072|\alpha_1|^2 = \tilde{C}_6(\alpha_1). \]

**Theorem 3.7.** Suppose that \((g(t), u(t))_{t \in [0, T]}\) satisfies BA. Then

\[ \int_M |\text{Sic} g(s) G(s)|^2 dV_{g(s)} \leq \tilde{C}(1 + s) e^{\tilde{c}s} \]

\[ \int_M |\text{Sic} g(s) G(s)|^2 dV_{g(s)} \leq \tilde{C}(1 + s) e^{\tilde{c}s} \]

\[ + \tilde{C}[1 - \text{sgn}(\alpha_1, 0)] e^{\tilde{c}s} \int_0^s \int_M |\nabla^2 u(t)|^4 dV_{g(t)} dt, \]

\[ \int_0^s \int_M |\text{Sic} g(t) G(t)|^4 dV_{g(t)} dt \leq \tilde{C}(1 + s) e^{\tilde{c}s} \]

\[ + \tilde{C}[1 - \text{sgn}(\alpha_1, 0)] e^{\tilde{c}s} \int_0^s \int_M |\nabla^2 u(t)|^4 dV_{g(t)} dt, \]

\[ \int_s^T \int_M |\text{Sic} g(t) G(t)|^p dV_{g(t)} dt \leq \left( (T - s) e^T Vol_0 \right)^{\frac{1 + p}{p}} e^{\tilde{c}T} \tilde{C}(1 + T), \]

\[ + \tilde{C}[1 - \text{sgn}(\alpha_1, 0)] \int_0^T \int_M |\nabla^2 u(t)|^4 dV_{g(t)} dt \]

\[ \right)^{\frac{1}{4}}. \]
for any $s \in [0, T)$ and $0 < p < 4$. Here $\tilde{C}$ is a uniform constant which depends only on $g_0, u_0, \alpha_1, \beta_1, \beta_2, A_1, \chi(M)$.

Proof. From (3.51) and (3.29), one has

$$e^{-\tilde{C}_1 s} \int_M f dV_s + e^{-\tilde{C}_4 s} \int_0^s \int_M f^2 dV_t dt \leq \int_M f dV_s \bigg|_{t=0}$$

$$+ \tilde{C}_2 e^{\tilde{C}_5 s} + \tilde{C}_3 \left( 1 - e^{-\tilde{C}_1 s} \right) + \tilde{C}_4 \cdot \left\{ e^{(\tilde{C}_1 + \tilde{C}_5) s} \right\}$$

and hence

$$\int_M f dV_s + \int_0^s \int_M f^2 dV_t dt \leq \tilde{C}_7 (1 + s) e^{(\tilde{C}_1 + \tilde{C}_5) s}$$

for some uniform constant $\tilde{C}_7$ which depends only on $g_0, u_0, \alpha_1, \beta_1, \beta_2, A_1, \chi(M)$.

Because $S \leq 1$, we have $\frac{1}{2} |\text{Sic}| \leq f \leq |\text{Sic}|$. The above estimate immediately implies (3.52) and (3.54). The second estimate follows from (3.41) as the argument in [43] (see the proof of Theorem 3.5). The last estimate follows from $\text{Vol}_t \leq e^t \text{Vol}_0$ and the Hölder inequality. □

According to our definition of $\text{Sic}, \text{Sm}$ in (3.10), we see that the boundedness of $\text{Sic}, \text{Sm}$ is equivalent to the boundedness of $\text{Sic}, \text{Rm}$ for any regular Ricci flow.

**Corollary 3.8.** Suppose that $(g(t), u(t))_{t \in [0, T)}$ satisfies BA. Then

$$\int_M |\text{Ric}_{g(t)}|^2_{g(t)} dV_{g(t)} \leq \tilde{C} (1 + s) e^{\tilde{C} s}$$

(3.56)

$$+ \tilde{C} [1 - \text{sgn}(\alpha_1, 0)] e^{\tilde{C} s} \int_0^s \int_M |\nabla g(t)| u(t)^4 dV_{g(t)} dt,$$

$$\int_M |\text{Rm}_{g(t)}|^2_{g(t)} dV_{g(t)} \leq \tilde{C} (1 + s) e^{\tilde{C} s}$$

(3.57)

$$+ \tilde{C} [1 - \text{sgn}(\alpha_1, 0)] e^{\tilde{C} s} \int_0^s \int_M |\nabla g(t)| u(t)^4 dV_{g(t)} dt,$$

for any $s \in [0, T)$. Here $\tilde{C}$ is a uniform constant which depends only on $g_0, u_0, \alpha_1, \beta_1, \beta_2, A_1, \chi(M)$.

4. **Bounded $L^2$-Curvature Conjecture for the Einstein Scalar Field Equations**

From Theorem 2.7 we can get an upper bound for the $L^2$-norm of $\text{Rm}_{g(t)}$. Motivated by this estimate, we in this section impose a conjecture for the Einstein scalar field equations, which is analogous to the corresponding conjecture for the Einstein vacuum equations proved by Klainerman, Rodnianski, and Szefetel [26] [56] [57] [58] [59] [60].
4.1. **Initial value problem.** In this section we recall some basic results for Einstein scalar field equations from [53]. Consider Einstein’s equation

\[ R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = T_{\alpha\beta} \]

where \( R_{\alpha\beta} \) and \( R \) denote, respectively, the Ricci curvature tensor and scalar curvature of a four dimensional Lorentzian space-time \((M, g)\). If the energy-momentum tensor \( T_{\alpha\beta} \) is chosen as

\[ T_{\alpha\beta} = 2\partial_\alpha u \partial_\beta u - \frac{1}{2} |Du|^2 g, \]

where \( D \) is the Levi-Civita connection of \( g \) and \( u \) is a smooth function on \( M \). In this case, the Einstein equation (4.1) can be written as

\[ R_{\alpha\beta} - 2\partial_\alpha u \partial_\beta u = 0. \]

As discussed in [53], we should impose a matter equation

\[ \Delta u = 0 \]

for \( u \), where \( \Delta := D^a D_a \). Hence we should consider a system of PDEs

\[ R_{\alpha\beta} - 2\partial_\alpha u \partial_\beta u = 0, \quad \Delta u = 0, \]

which is called the Einstein scalar field equation or the Einstein-Klein-Gordon equation.

An initial data set \((\Sigma, g, k, u_0, u_1)\) for (4.3) consists of a three dimensional manifold \( \Sigma \), a Riemannian metric \( g \), a symmetric 2-tensor \( k \), together with two functions \( u_0 \) and \( u_1 \) on \( \Sigma \), all assumed to be smooth, verifying the constraint equations,

\[ \nabla^i k_{ij} - \nabla_i \text{tr} k = u_1 \nabla_i u_0, \]

\[ R - |k|^2 + (\text{tr} k)^2 = u_1^2 + |\nabla u_0|^2, \]

where \( \nabla \) is the Levi-Civita connection of \( g \).

Given an initial data set \((\Sigma, g, k, u_0, u_1)\), the Cauchy problem consists in finding a four-dimensional Lorentzian manifold \((M, g)\) and a smooth function \( u \) on \( M \) satisfying (4.3), and also an embedding \( i: \Sigma \to M \) such that

\[ i^*g = g, \quad i^*u = u_0, \quad i^*K = k, \quad i^*(N u) = u_1, \]

where \( N \) is the future-directed unit normal to \( i(\Sigma) \) and \( K \) is the second fundamental form of \( i(\Sigma) \).

The local existence and uniqueness result for globally hyperbolic developments can be found in [53], Theorem 14.2. For stability and instability for Einstein’s scalar field equation, we refer to [14, 15, 33, 34, 62, 63, 64, 65].

4.2. **Bounded \( L^2 \)-curvature conjecture for Einstein’s equations.** For Einstein’s equations (i.e., \( u = 0 \) in (4.3)), and the initial data is denoted by \((\Sigma, g, k)\), Klainerman [25] proposed the following conjecture:

*The Einstein vacuum equations admits local Cauchy developments for initial data sets \((\Sigma, g, k)\) with locally finite \( L^2 \)-curvature and locally finite \( L^2 \)-norm of the first covariant derivatives of \( k \).*
This conjecture was recently solved by Klainerman, Rodnianski and Szefetel [26]. To give a precise result, we assume that the space-time \((M, g)\) to be foliated by the level surfaces \(\Sigma_t = t^{-1}(t)\) of a time function \(t\). Let \(T\) denote the unit normal to \(\Sigma_t\), and let \(k\) the second fundamental form of \(\Sigma_t\), i.e., \(k_{ij} := -g(D_i T, e_j)\), where \((e_i)_{1 \leq i \leq 3}\) denote an arbitrary frame on \(\Sigma_t\). We also assume that the \(\Sigma_t\)-foliation is maximal, i.e., we have

\[
\text{tr}_g k = 0
\]

where \(g = g(t)\) is the induced metric on \(\Sigma_t\).

**Theorem 4.1. (Klainerman-Rodnianski-Szefetel, 2015)** Let \((M, g)\) an asymptotically flat solution to the Einstein vacuum equations together with a maximal foliation by space-like hyper-surfaces \(\Sigma_t\) defined as level hyper-surfaces of a time function \(t\). Assume that the initial slice \((\Sigma_0, g, k)\) is such that the Ricci curvature \(\text{Ric} \in L^2(\Sigma)\), \(\nabla k \in L^2(\Sigma)\), and \(\Sigma\) has a strictly positive volume radius on scales \(\leq 1\), i.e.,

\[
|\text{vol}_g(B_g(p, r))| > 0.
\]

Then there exists a time

\[
T := T \left( ||\text{Ric}||_{L^2(\Sigma)}, ||\nabla k||_{L^2(\Sigma)}, r_{\text{vol}}(\Sigma, 1) \right) > 0
\]

and a constant

\[
C := C \left( ||\text{Ric}||_{L^2(\Sigma)}, ||\nabla k||_{L^2(\Sigma)}, r_{\text{vol}}(\Sigma, 1) \right) > 0
\]

such that the following control

\[
||\text{Rm}||_{L^\infty(\Sigma)} \leq C, \quad ||\nabla k||_{L^\infty(\Sigma)} \leq C, \quad \inf_{t \in [0, T]} r_{\text{vol}}(\Sigma, 1) \geq \frac{1}{C},
\]

holds on \(t \in [0, T]\).

### 4.3. Bounded \(L^2\)-curvature conjecture for the Einstein scalar field equations.

Motivated by Theorem 2.7 and Theorem 4.1, we propose the following

**Conjecture 4.2.** The Einstein scalar field equations admit local Cauchy developments for initial data sets \((\Sigma, g, k, u_0, u_1)\) with locally finite \(L^2\)-curvature, locally finite \(L^2\)-norm of the first covariant derivatives of \(k\), locally finite \(L^2\)-norm of the covariant derivatives (up to second order) of \(u_0\), and locally finite \(L^2\)-norm of the covariant derivatives (up to first order) of \(u_1\).

An interesting question related to Theorem 4.1 is

**Question 4.3.** Can we extend Theorem 4.1 to the Einstein scalar field equations?

### 5. Sm and Wylie-Yeroshkin Riemann curvature

In this section we compare our curvature \(Sm\) with \(a_1 = 2\) (3.10) with a notion of curvature introduced recently by Wylie and Yeroshkin [68]. For other notions of sectional curvatures, see [23, 24, 67].

Let \((M, g)\) be a Riemannian manifold of dimension \(n\) with a smooth function \(u\). Wylie and Yeroshkin introduced the following weighted connection

\[
\nabla^u_X Y := \nabla_X Y - (Yu)X - (Xu)Y.
\]
By Proposition 3.3 in [68], we have
\begin{equation}
R^u_{ijk} = R_{ijkt} + \nabla_j \nabla_k g^t_{il} - \nabla_l \nabla_k g^t_{ij} + \nabla_j \nabla_k g^t_{il} - \nabla_l \nabla_k g^t_{ij},
\end{equation}
where \(R^u_{ijk} := (Rm^u(\partial_i, \partial_j)\partial_k, \partial_l)\) and \(Rm^u\) is the induced Riemann curvature tensor associated to the connection \(\nabla^u\). The Ricci curvature associated to \(\nabla^u\) is defined by
\begin{equation}
R^u_{jk} := g^{i\ell} R^u_{ijkt} = R_{ijk} + (n - 1) \nabla_j \nabla_k u + (n - 1) \nabla_j \nabla_k u.
\end{equation}
Here the last formula also follows from Proposition 3.3 in [68].

Recall from (3.11) that (with \(a_1 = 2\))
\begin{equation}
S_{ijk} = R_{ijk} - \nabla_i \nabla_k g^t_{jl} - \nabla_l \nabla_k g^t_{ij}.
\end{equation}
From now on, we are given a smooth function \(u\) on \(M\) and write
\begin{equation}
R^L_{ijk} := S_{ijk}, \quad R^WY_{ijk} := R^u_{ijk},
\end{equation}
\begin{equation}
R^L_{jk} := g^{i\ell} R^L_{ijkt}, \quad R^WY_{jk} := R^u_{jk} = g^{i\ell} R^WY_{ijkt},
\end{equation}
\begin{equation}
R^L := g^{i\ell} R^L_{ijkt}, \quad R^WY := g^{i\ell} R^WY_{ijkt}.
\end{equation}
From (5.3) and (5.4), we have
\begin{equation}
\text{Ric}^L = \text{Ric} - 2du \otimes du, \quad \text{Ric}^WY = \text{Ric} + (n - 1)du \otimes du + (n - 1)\nabla^2 u.
\end{equation}

**Remark 5.1.** We note that \(\text{Ric}^L\) and \(\text{Ric}^WY\) are actually the Ricci curvatures in the sense of Bakey-Émery [1]. We here use our notions to keep the paper smoothly.

There is another type of Ricci curvature given by
\begin{equation}
\text{Ric}^WY := g^{i\ell} R^WY_{ijkt} = R_{jk} + (\Delta u + |\nabla u|^2) g_{jk} - \nabla_j \nabla_k u - \nabla_j u \nabla_k u.
\end{equation}

**Lemma 5.2.** (Basic identities for \(R^L_{ijk}\) and \(R^WY_{ijk}\)) We have
\begin{equation}
R^WY_{ijk} - R^L_{ijk} = \nabla_j \nabla_k g^t_{il} + \nabla_k \nabla_j g^t_{il} + \nabla_l \nabla_j u \nabla_k u - \nabla_l \nabla_k u \nabla_j u,
\end{equation}
\begin{equation}
R^WY_{ijk} - R^WY_{jik} = \nabla_i \nabla_j \nabla_k g^t_{il} + \nabla_i \nabla_k g^t_{jl} - \nabla_i \nabla_j g^t_{kl} - \nabla_k \nabla_j g^t_{il} - \nabla_j \nabla_k g^t_{il} - \nabla_l \nabla_j g^t_{ik} - \nabla_j \nabla_i g^t_{lk} - \nabla_k \nabla_i g^t_{lj} - \nabla_l \nabla_i g^t_{jk},
\end{equation}
\begin{equation}
R^WY_{ijk} - R^WY_{kij} = \nabla_j \nabla_k g^t_{il} - \nabla_i \nabla_k g^t_{jl} - \nabla_i \nabla_j g^t_{kl} - \nabla_k \nabla_j g^t_{il} - \nabla_j \nabla_k g^t_{il} - \nabla_l \nabla_j g^t_{ik} - \nabla_j \nabla_i g^t_{lk} - \nabla_k \nabla_i g^t_{lj} - \nabla_l \nabla_i g^t_{jk},
\end{equation}
\begin{equation}
R^L_{ijk} - R^L_{ikj} = \nabla_j \nabla_k g^t_{il} - \nabla_i \nabla_k g^t_{jl} - \nabla_i \nabla_j g^t_{kl},
\end{equation}
\begin{equation}
R^L_{ijk} - R^L_{ikj} = \nabla_j \nabla_k g^t_{il} - \nabla_i \nabla_k g^t_{jl} - \nabla_i \nabla_j g^t_{kl},
\end{equation}
\begin{equation}
R^L_{ikj} - R^L_{kij} = \nabla_k \nabla_i g^t_{lj} - \nabla_i \nabla_j g^t_{kl},
\end{equation}
\begin{equation}
R^L_{ikj} - R^L_{kij} = \nabla_k \nabla_i g^t_{lj} - \nabla_i \nabla_j g^t_{kl},
\end{equation}
\begin{equation}
\frac{1}{2} (R^WY_{ijk} - R^WY_{jik}) = (R^L_{ijk} - R^L_{ikj}) + \nabla_j \nabla_k g^t_{il} - \nabla_l \nabla_j g^t_{ik},
\end{equation}
\begin{equation}
\text{Ric}^WY - \text{Ric}^W = \left(\Delta u + |\nabla u|^2\right) g - n \left(\nabla^2 u + du \otimes du\right).
5.1. Integral inequalities for scalar and Ricci curvatures. We now have four different types of Ricci curvatures, \( \text{Ric}, \text{Ric}^L, \text{Ric}^{\text{WY}}, \) and \( \text{Ric}^{\text{\hat{W}Y}} \), and three different types of scalar curvatures, \( \text{R}, \text{R}^L, \) and \( \text{R}^{\text{WY}} \). In order to compare those quantities, we introduce a notation \( \mathcal{P} \leq_{1,\mu} \mathcal{Q} \), which is an integral inequality with respect to the measure \( \mu \).

**Definition 5.3.** Given two scalar quantities \( \mathcal{P}, \mathcal{Q} \) on \( (M, g) \), and a measure \( \mu \), we write \( \mathcal{P} \leq_{1,\mu} \mathcal{Q} \) if the following inequality holds:

\[
\int_M \mathcal{P} \, d\mu \leq \int_M \mathcal{Q} \, d\mu
\]

(5.16)

When \( d\mu \) is the volume form \( dV \), we simply write (5.16) as \( \mathcal{P} \leq \mathcal{Q} \). When \( d\mu \) is the measured volume form \( e^f \, dV \), we write (5.16) as \( \mathcal{P} \leq_{1,f} \mathcal{Q} \). Similarly, we can define \( \mathcal{P} \leq_{1,\mu} \mathcal{Q} \).

**Proposition 5.4.** For any measure \( \mu \) on \( M \) and smooth function \( u \) on \( M \), we have

\[
R^L \leq_{1,\mu} R, \quad R \leq_{1,\mu} R^{\text{WY}}, \quad R^{\text{L}} =_{1,\mu} R^{\text{WY}}.
\]

Proof. It follows from the definitions

\[
R^L = R - 2|\nabla u|^2, \quad R^{\text{WY}} = R + (n-1)(|\nabla u|^2 + \Delta u)
\]

and the fact that integral of \( (\Delta u + |\nabla u|^2) \) with respect to \( e^f \, dV \) is zero. \( \square \)

This proposition shows that \( R^L \leq_{1,\mu} R \leq_{1,\mu} R^{\text{WY}} \) and \( R^L \leq_{1,\mu} R =_{1,\mu} R^{\text{WY}} \). Thus, in the sense of integrals, \( R^L \) is weaker and \( R^{\text{WY}} \) is stronger than \( R \), respectively.

Next we consider the similar question on Ricci curvatures.

**Definition 5.5.** Let \( (M, g) \) be a closed Riemannian manifold with a smooth function \( u \), and \( \mu \) be a given measure on \( M \). Given two Ricci curvatures \( \text{Ric}^\bullet, \text{Ric}^{\circ} \in \mathcal{R}ic_4 := \{ \text{Ric}, \text{Ric}^L, \text{Ric}^{\text{WY}}, \text{Ric}^{\text{\hat{W}Y}} \} \), we say

\[
\text{Ric}^\bullet \leq_{1,\mu} \text{Ric}^{\circ}
\]

(5.18)

if \( \text{Ric}^\bullet (X, X) \leq_{1,\mu} \text{Ric}^{\circ} (X, X) \) in the sense of Definition 5.3 for all vector fields \( X \in \mathfrak{X}(M) \). Similarly we can define \( \text{Ric}^\bullet \leq_{1,\mu} \text{Ric}^{\circ} \) and \( \text{Ric}^\bullet \leq_{1,f} \text{Ric}^{\circ} \).

We say

\[
\text{Ric}^\bullet \leq_{1K,\mu} \text{Ric}^{\circ}
\]

(5.19)

if \( \text{Ric}^\bullet (X, X) \leq_{1,\mu} \text{Ric}^{\circ} (X, X) \) in the sense of Definition 5.3 for all Killing vector fields \( X \in \mathfrak{X}_K(M) \), where \( \mathfrak{X}_K(M) \) is the space of all Killing vector fields on \( M \). Similarly we can define \( \text{Ric}^\bullet \leq_{1K} \text{Ric}^{\circ} \) and \( \text{Ric}^\bullet \leq_{1K,f} \text{Ric}^{\circ} \).

Consider the subset \( \mathfrak{X}_{K\mathcal{C}}(M) \) of \( \mathfrak{X}_K(M) \), which consists of Killing vector fields on \( M \) with constant norm. we say

\[
\text{Ric}^\bullet \leq_{1K\mathcal{C},\mu} \text{Ric}^{\circ}
\]

(5.20)

if \( \text{Ric}^\bullet (X, X) \leq_{1,\mu} \text{Ric}^{\circ} (X, X) \) in the sense of Definition 5.3 for all \( X \in \mathfrak{X}_{K\mathcal{C}}(M) \). Similarly we can define \( \text{Ric}^\bullet \leq_{1K\mathcal{C}} \text{Ric}^{\circ} \) and \( \text{Ric}^\bullet \leq_{1K\mathcal{C},f} \text{Ric}^{\circ} \).

**Theorem 5.6.** Let \( (M, g) \) be a closed Riemannian manifold with a smooth function \( u \) and \( \mu \) be a given measure on \( M \). Then we have
(i) $\text{Ric}^L \leq_{\text{L,J}} \text{Ric}$.  
(ii) $\text{Ric} \leq_{\text{IKC}} \text{Ric}^{\text{WY}}$.  
(iii) $\text{Ric} \leq_{\text{IKC}} \hat{\text{Ric}}^{\text{WY}}$.

Proof. From (5.5), we have $\text{Ric}^L(X, X) = \text{Ric}(X, X) - 2\langle X, \nabla u \rangle^2$ and then the first part follows. To prove the last result, we compute

$$
\int_M \left[ \text{Ric}^{\text{WY}}(X, X) - \text{Ric}(X, X) \right] dV = (n - 1) \int_M \left[ \langle X, \nabla u \rangle^2 + X^i X^j \nabla_i \nabla_j u \right] dV.
$$

Then we suffice to verify that the last integral is nonnegative for any Killing vector field $X$. According to (5.23), we can prove (ii) and (iii) immediately. Indeed, for (iii),

$$
\int_M \left[ \hat{\text{Ric}}^{\text{WY}}(X, X) - \text{Ric}(X, X) \right] dV = \frac{3}{2} \int_M u \Delta |X|^2 dV + \int_M |X|^2 |\nabla u|^2 - \langle X, \nabla u \rangle^2] dV
\geq \frac{3}{2} \int_M u \Delta |X|^2 dV
$$

for any Killing vector field $X$. □

To prove Lemma 5.7, we first recall Yano’s formula (see [69, 70, 71] and, [45] for related topics)

$$
\int_M |\mathcal{L}_X g|^2 dV = \int_M [||\nabla X|^2 + |\text{div}(X)|^2 - \text{Ric}(X, X)] dV, \quad X \in \mathfrak{X}(M),
$$

gives a necessary and sufficient condition to $X$ being Killing or not. Namely,

$$
X \in \mathfrak{X}_K(M) \iff \Delta X + \nabla \text{div} X + \text{Ric}(X) = 0 = \text{div} X.
$$

**Lemma 5.7.** For any vector field $X$ and any smooth function $u$, we have

$$
\int_M X^i X^j \nabla_i \nabla_j u dV = \int_M u \langle X, \Delta X + \nabla \text{div} X + \text{Ric}(X) \rangle dV
\quad + \int_M \frac{1}{2} u |\mathcal{L}_X g|^2 - \Delta |X|^2] dV - \int_M \langle X, \nabla u \rangle \text{div} X dV.
$$

In particular, when $X \in \mathfrak{X}_K(M)$, we get

$$
\int_M X^i X^j \nabla_i \nabla_j u dV = -\frac{1}{2} \int_M u \Delta |X|^2 dV = -\frac{1}{2} \int_M |X|^2 \Delta u dV.
$$

Proof. We start from the computation:

$$
\int_M X^i X^j \nabla_i \nabla_j u dV = -\int_M \nabla_i u \nabla_j (X^i X^j) dV
= -\int_M \nabla_i u \left( X^i \text{div} X + X^i \nabla_j X^j \right) dV
= -\int_M \langle X, \nabla u \rangle \text{div} X dV + \int_M u \nabla_j (X^j X^i) dV.
$$
The last integral, denoted by \( I_u \), as a function of \( u \), is equal to

\[
I_u = \int_M u(\nabla_j X^i \nabla_i X^j) dV + \int_M u X^i \nabla_j X^j dV
\]

\[
= \int_M u(\nabla_j X^i \nabla_i X^j) dV + \int_M u X^i \left( \nabla_i \text{div} X + R_{ij} X^j \right) dV
\]

\[
= \int_M u \left[ \nabla_j X^i \nabla_i X^j + \text{Ric}(X, X) \right] dV - \int_M \text{div} X (\langle X, \nabla u \rangle + u \text{div} X) dV
\]

\[
= \int_M u \left[ \nabla_j X^i \nabla_i X^j + \text{Ric}(X, X) - |\text{div}(X)|^2 \right] dV - \int_M (X, \nabla u) \text{div} X dV.
\]

Using the following identities:

\[
\frac{1}{2} |\mathcal{L}_X g|^2 = |\nabla X|^2 + \nabla^i X^j \nabla_i X^j,
\]

\[
\int_M u |\text{div}(X)|^2 dV = \int_M (u \text{div} X) \nabla_i X^i dV
\]

\[
= - \int_M X^i (\nabla_i u \text{div} X + u \nabla_i \text{div} X) dV
\]

\[
= - \int_M (X, \nabla u) \text{div} X dV - \int_M u (X, \nabla \text{div} X) dV,
\]

\[
\int_M u |\nabla X|^2 dV = \int_M u \nabla^i X_j \nabla_i X^j dV = - \int_M X^i \left( \nabla_i u \nabla^i X_j + u \Delta X_j \right) dV
\]

\[
= - \int_M u (X, \Delta X) dV - \frac{1}{2} \int_M \nabla_i u \nabla_i |X|^2 dV,
\]

we obtain

\[
(5.25) \quad I_u = \int_M u \left[ (X, \Delta X + \nabla \text{div} X + \text{Ric}(X)) dV + \frac{1}{2} |\mathcal{L}_X g|^2 - \frac{1}{2} \Delta |X|^2 \right] dV.
\]

Then (5.23) follows from (5.25).

A consequence of Theorem 5.6 indicates

\[
(5.26) \quad \text{Ric}^L \leq_{IKC} \text{Ric} \leq_{IKC} \text{Ric}^{\text{WY}} \quad \text{and} \quad \text{Ric}^L \leq_{IKC} \text{Ric} \leq_{IKC} \hat{\text{Ric}}^{\text{WY}}.
\]

To prove an analogous result in Theorem 5.6 between \( \text{Ric} \) and \( \text{Ric}^{\text{WY}}, \hat{\text{Ric}}^{\text{WY}} \) along constant Killing vector fields, for some measure \( \mu \) other than the volume form, we first do the following computation. For the measure \( e^f dV \), where \( f \equiv f(u) \) is a smooth function depends only on \( u \), we have

\[
(5.27) \quad \int_M \left[ \text{Ric}^{\text{WY}}(X, X) - \text{Ric}(X, X) \right] e^f dV = (n - 1) \int_M \left[ \langle X, \nabla u \rangle^2 + X^i X^j \nabla_i \nabla_j u \right] e^f dV.
\]
As above, the last integral is equal to
\[ \int_M X^i X^j \nabla_i \nabla_j \tilde{u} e^\tilde{f} dV = - \int_M \nabla_i \nabla_j (e^f X^i X^j) dV \]
\[ = - \int_M \nabla_i \mu (e^f X^i X^j + X^i \nabla X^j + X^j \nabla X^i) e^f dV \]
\[ = - \int_M (X, \nabla u) \nabla X e^f dV - \int_M f' \langle \nabla u, X \rangle^2 e^f dV \]
\[ + \int_M u e^f \nabla_j (X^i \nabla_i X^j) dV + \int_M u f' e^f \nabla_j u X^i \nabla_i X^j \]
\[ = - \int_M (X, \nabla u) \nabla X e^f dV - \int_M f' \langle \nabla u, X \rangle^2 e^f dV \]
\[ + \int_M u e^f \nabla_j (X^i \nabla_i X^j) dV + \int_M u f' e^f \nabla_j u X^i \nabla_i X^j \]
\[ = - \int_M (X, \nabla u) \nabla X e^f dV - \int_M u f' e^f \nabla_j u X^i \nabla_i X^j \]
\[ + I_{ue^f} - I_{u^2 f e^f} - \int_M u \nabla_j \left( u f' e^f \right) X^i \nabla_i X^j dV \]
(5.28)
where we used the notion $I_u$ that is explicitly given in (5.25). Next we require
\[ u f' e^f = 1 \implies \text{we can take } f(u) = \ln u. \]
However, in the above function $f(u)$, we should assume that $u$ is strictly positive. It motivates us to consider the modified function associated with $u$.

Given a smooth function $u$ on the closed Riemannian manifold $(M, g)$, set
\[ \bar{u} := u - u_{\min} + c_0 \geq c_0 \geq c_0 e^{-1}, \quad \bar{f} := \ln \bar{u}, \]
where $u_{\min} := \min_M u$ and $c_0 \geq e^{-1}$ is the unique constant such that $c_0 \ln c_0 = 1$. Replacing $(u, f)$ by $(\bar{u}, \bar{f})$ in (5.27) (since $\nabla \bar{u} = \nabla u$), we immediately obtain
\[ \int_M \left[ W(Y) - \text{Ric}(X, X) \right] e^f dV = (n - 1) \int_M \left[ (X, \nabla \bar{u})^2 + X^i \nabla_i \bar{u} \bar{u} \right] e^\bar{f} dV \]
where
\[ \int_M X^i X^j \nabla_i \nabla_j \bar{u} e^\bar{f} dV = - \int_M (X, \nabla \bar{u}) \nabla X e^f dV - \int_M \bar{f} \langle \nabla \bar{u}, X \rangle^2 e^f dV \]
\[ + \int_M u e^f \nabla_j (\bar{u} f' e^f) X^i \nabla_i X^j dV \]
\[ - \int_M (X, \nabla \bar{u}) \nabla X e^f dV - \int_M \frac{1}{\bar{u} \ln \bar{u}} (X, \nabla \bar{u})^2 e^\bar{f} dV \]
\[ + I_{ue^f} - I_{u^2 f e^f}. \]
When $X \in \mathcal{X}_{KC}(M)$, we can simplify the above integral into
\[ \int_M X^i X^j \nabla_i \nabla_j \bar{u} e^\bar{f} dV = - \int_M \frac{1}{\bar{u} \ln \bar{u}} (X, \nabla \bar{u})^2 e^\bar{f} dV \]
and hence
\[ \int_M \left[ W(Y) - \text{Ric}(X, X) \right] e^\bar{f} dV = (n - 1) \int_M \left( 1 - \frac{1}{\bar{u} \ln \bar{u}} \right) (X, \nabla \bar{u})^2 e^\bar{f} dV. \]
Since the function $t \ln t, \ t \geq e^{-1}$, is increasing, it follows that $\bar{u} \ln \bar{u} \geq c_0 \ln c_0 = 1$. Therefore, we obtain the first part of the following theorem.

**Theorem 5.8.** Let $(M, g)$ be a closed Riemannian manifold with a smooth function $u$ and $\mu$ be a given measure on $M$. Then we have
(i) $\text{Ric} \leq_{\text{IKC}, f} \text{Ric}^{\text{WY}}$, and
(ii) $\text{Ric} \leq_{\text{IKC}, f} \text{Ric}^{\text{WY}}$.

where $f := u - u_{\min} + c_0$ and $c_0 \geq 1/e$.

Proof. The proof of the second part is precisely the same as that of (iii) in Theorem 5.6.

\section{Killing vector fields with constant length.}

In Theorem 5.6 and Theorem 5.8, we proved integral inequalities along Killing vector fields with constant length. In this subsection we review some existence results on such vector fields.

Eisenhart \cite{16} proved that a unit vector field $X$ on a (connected) complete Riemannian manifold $(M, g)$ is the unit Killing vector field if and only if the angles between $X$ and tangent vectors to each geodesic in $(M, g)$ are constant along this geodesic. As earlier, Bianchi \cite{5} proved that A Killing vector field $X$ on a complete Riemannian manifold $(M, g)$ has constant length if and only if every integral curve of $X$ is a geodesic in $(M, g)$. For a proof, we refer to \cite{2}.

There are two necessary conditions for the existence of Killing vector fields of constant length on a given Riemannian manifold $(M, g)$, one is $\chi(M) = 0$ by Hopf’s theorem while another one, by Bott’s theorem \cite{6}, is that all the Pontrjagin numbers of the oriented cover of $M$ are zero.

The existence of Killing vector fields with constant length on a complete Riemannian manifold $(M, g)$ is connected with Clifford-Wolf translations or Clifford translations, which is an isometry $s^{CW}$ on $(M, g)$ such that $d(x, s^{CW}(x)) \equiv$ constant for all $x \in M$.

\begin{itemize}
  \item If a one-parameter isometry group generated by a Killing vector field $X$ consists of Clifford-Wolf translations, then $X$ had constant length.
  \item If $X$ is a Killing vector field of constant length on a compact Riemannian manifold $(M, g)$, then the isometries $\gamma(t)$, generated by $X$, are Clifford-Wolf translations for sufficiently small $|t|$.
\end{itemize}

The first fact is obvious by definition. The compactness condition in the second fact can be generalized to the condition that $(M, g)$ has the injectivity radius, bounded from below by some positive constant. A proof can be found in \cite{2}.

\textbf{Proposition 5.9.} On each of 28 homotopical seven-dimensional spheres $M$, there exist a Riemannian metric $g$ and a nonzero vector field $X$, such that

\begin{itemize}
  \item $\text{Ric}^L(X, X) \leq_1 \text{Ric}(X, X) \leq_1 \text{Ric}^{\text{WY}}(X, X)$ and $\text{Ric}^L(X, X) \leq_1 \text{Ric}(X, X) \leq_1 \text{Ric}^{\text{WY}}(X, X)$ hold.
  \item for any smooth function $u$ on $M$, $\text{Ric}^L(X, X) \leq_{1, f} \text{Ric}(X, X) \leq_{1, f} \text{Ric}^{\text{WY}}(X, X)$ and $\text{Ric}^L(X, X) \leq \text{Ric}(X, X) \leq_{1, f} \text{Ric}^{\text{WY}}(X, X)$ hold, where $\tilde{f} := u - u_{\min} + c_0$ with $c_0 \geq 1/e$.
\end{itemize}

Proof. According to \cite{2} (see Corollary 11) and \cite{48}, we can take $X$ to be a Killing vector field of constant length with respect to $g$. Then the results follow from Theorem 5.6 and Theorem 5.8.

We say that a Riemannian metric $g$ on $M$ is of cohomogeneity 1 if some compact Lie group $G$ acts smoothly and isometrically on $M$ and the space of orbits $M/G$ with respect to this action is one-dimensional.
Proposition 5.10. Let \( n \geq 2 \) and \( \epsilon > 0 \). On the sphere \( S^{2n-1} \), there are (real-analytic) Riemannian metric \( g_e \), of cohomogeneity 1, with the property that all section curvatures of \( g_e \) differ from 1 at most by \( \epsilon \), and (real-analytic) nonzero vector field \( X_e \), such that

- \( \text{Ric}_{g_e}^L(X_e, X_e) \leq 1 \text{ Ric}_{g_e}^W(X_e, X_e) \leq 1 \text{ Ric}_{g_e}^W(Y, Y) \) and \( \text{Ric}_{g_e}^L(X_e, X_e) \leq 1 \text{ Ric}_{g_e}^W(Y, Y) \) hold.
- for any smooth function \( u \) on \( M \), \( \text{Ric}_{g_e}^L(X_e, X_e) \leq f \text{ Ric}_{g_e}^W(X_e, X_e) \) and \( \text{Ric}_{g_e}^L(X_e, X_e) \leq f \text{ Ric}_{g_e}^W(X_e, X_e) \) hold, where \( f := u - u_{\text{min}} + c_0 \) with \( c_0 \geq 1/\epsilon \).

Proof. According to [2] (see Theorem 21), we can take \( X \) to be a Killing vector field \( X \) of unit length with respect to \( g \). Then the results follow from Theorem 5.5 and Theorem 5.8. \( \square \)

Remark 5.11. Fix \( n \geq 2 \) and \( D > 0 \). Let \( \mathcal{G}_{n,D} \) denote the class of simply-connected \( n \)-dimensional Riemannian manifolds \( (M, g) \) with the sectional curvature \( |\text{Sec}_g| \leq 1 \) and with \( \text{diam}(M, g) \leq D \). The class \( \mathcal{G}_{n,\sqrt{5}} \) contains a subclass \( \mathcal{W} \mathcal{G}_{n,\sqrt{5}} \), which consists of all simply-connected \( n \)-dimensional Riemannian manifold \( (M, g) \) with the sectional curvature \( 0 < \delta < \text{Sec}_g \leq 1 \).

Tuschmann [61] proved that there is a positive number \( v := v(n, D) \) with the following property: if \( (M, g) \in \mathcal{G}_{n,D} \) satisfies \( \text{vol}(M, g) < v \), then

(i) there is a smooth locally free action of the \( S^1 \)-action on \( M \), and

(ii) for every \( \epsilon > 0 \) there exists a \( S^1 \)-invariant metric \( g_\epsilon \) on \( M \) such that

\[ e^{-\epsilon} g \leq g_\epsilon < e^\epsilon g, \quad |\nabla g - \nabla g_\epsilon| < \epsilon, \quad |\nabla g_\epsilon\text{Rm}_{g_\epsilon}| < C(n, i, \epsilon). \]

If the \( S^1 \)-action in (i) is free, Berestovskii and Nikonov [2] observed that we can find a Killing vector field \( X_e \) of unit length with respect to \( g_e \). Consequently, in this case,

- \( \text{Ric}_{g_e}^L(X_e, X_e) \leq 1 \text{ Ric}_{g_e}^W(X_e, X_e) \leq 1 \text{ Ric}_{g_e}^W(Y, Y) \) and \( \text{Ric}_{g_e}^L(X_e, X_e) \leq 1 \text{ Ric}_{g_e}^W(Y, Y) \) hold.
- for any smooth function \( u \) on \( M \), \( \text{Ric}_{g_e}^L(X_e, X_e) \leq f \text{ Ric}_{g_e}^W(X_e, X_e) \) and \( \text{Ric}_{g_e}^L(X_e, X_e) \leq f \text{ Ric}_{g_e}^W(X_e, X_e) \) hold, where \( f := u - u_{\text{min}} + c_0 \) with \( c_0 \geq 1/\epsilon \).

5.3. Remark on \( \text{Rm}^L \) and \( \text{Rm}^W \). The nonnegativity of \( R_{ij}^L \) was used in [66] to prove the compactness for gradient shrinking Ricci harmonic solitons.

There is no useful relation between \( \text{Rm}^L \) and \( \text{Rm}^W \). More precisely, we can find a Riemannian manifold \( (M, g) \) so that \( \text{Rm}^L(X, Y, Y, X) < \text{Rm}^W(X, Y, Y, X) \) for some triple \( (X, Y, u) \) of smooth vector fields \( X, Y \) and smooth function \( u \), and \( \text{Rm}^L(X, Y, Y, X) > \text{Rm}^W(X, Y, Y, X) \) for another such triple \( (X', Y', u') \).

Example 5.12. Consider the Euclidean space \( (\mathbb{R}^n, g_{\mathbb{R}^n}) \) with the flat Riemannian metric \( g_{\mathbb{R}^n} \). Consider a smooth function \( u(x) = \varphi(r) \) with \( r = |x|^2 \), where \( \varphi(r) \) is a smooth function of variable \( r \). Let \( T = x^i \partial / \partial x^i \) denote the position vector field on \( \mathbb{R}^n \). Then

\[ \nabla_i u = 2T_i \varphi', \quad \nabla_i \nabla_j u = 4T_i T_j \varphi'' + 2\delta_{ij} \varphi'. \]
According to (5.5), we have
\[ R_{ijk}^L = -4\phi'' T_i T_k \delta_{j\ell} - 4\phi'^2 T_i T_j \delta_{k\ell}, \]
\[ R_{ijk}^W = \left( 4\phi'' T_j T_k + 2\phi' \delta_{jk} \right) \delta_{i\ell} - \left( 4\phi'' T_i T_k + 2\phi' \delta_{ik} \right) \delta_{j\ell} + 4\phi'^2 \delta_{i\ell} T_j T_k - 4\phi'^2 \delta_{j\ell} T_i T_k, \]
so that
\[ R_{m}^I(X, Y, Y, X) = -8\phi'^2 \langle X, T \rangle \langle Y, T \rangle \langle X, Y \rangle, \]
\[ R_{m}^W(X, Y, Y, X) = 4(\phi'' + \phi'^2) \left( \langle X \rangle^2 \langle Y \rangle^2 - \langle X, T \rangle \langle Y, T \rangle \langle X, Y \rangle \right) + 2\phi' \langle X \rangle^2 \langle Y \rangle^2 - \langle X, Y \rangle^2 \]
For any vector fields \( X, Y \). Choosing \( X = T \) and \( Y \) with the property that \( \langle Y, T \rangle = 0 \) yields
\[ R_{m}^I(T, Y, Y, T) = 0, \quad R_{m}^W(T, Y, Y, T) = 2\phi' \langle T \rangle^2 \langle Y \rangle^2. \]
When \( \phi(r) = r \), we have \( R_{m}^I(T, Y, Y, T) \leq R_{m}^W(T, Y, Y, T) \). On the other hand, for \( \phi(r) = -r \), we have \( R_{m}^I(T, Y, Y, T) \geq R_{m}^W(T, Y, Y, T) \).

There is also no useful pointwise relation between \( R_{m} \) and \( R_{m}^I \). By definition,
\[ R_{m}^I(X, Y, Y, X) = R_{m}(X, Y, Y, X) - 2 \langle X, \nabla u \rangle \langle Y, \nabla u \rangle \langle X, Y \rangle \]
for any vector fields \( X, Y \in \mathfrak{X}(M) \).

**Remark 5.13.** (1) For \( n \geq 2 \), take \( X, Y \in \mathfrak{X}(M) \) so that \( \langle X, Y \rangle = 0 \) at a point. Then, at this point, we get \( R_{m}^I(X, Y, Y, X) = R_{m}(X, Y, Y, X) \).

(2) When \( X = \nabla u \), we have
\[ R_{m}^I(\nabla u, Y, Y, \nabla u) = R_{m}(\nabla u, Y, Y, \nabla u) - 2 \langle \nabla u \rangle^2 \langle Y, \nabla u \rangle^2 \leq R_{m}(\nabla u, Y, Y, \nabla u). \]

(3) For \( n \geq 3 \), take \( X, Y \in \mathfrak{X}(M) \) so that \( \langle X, \nabla u \rangle, \langle Y, \nabla u \rangle, \langle X, Y \rangle < 0 \) at a point. Then, at this point, \( R_{m}^I(X, Y, Y, X) > R_{m}(X, Y, Y, X) \).

(4) In general, for any vector fields \( X, Y \in \mathfrak{X}(M) \), we have
\[ R_{m}^I(X, X, X, X) = -2 \langle X, \nabla u \rangle^2 \langle X \rangle^2 \leq 0 = R_{m}(X, X, X, X) \]
and the equality holds at a point if and only if \( \langle X, \nabla u \rangle = 0 \) or \( X = 0 \) at this point.

To give a relation between \( R_{m}^I(X, Y, Y, X) \) and \( R_{m}(X, Y, Y, X) \) in the sense of IKC, we let
\[ J(X, Y) := \int_M \langle X, \nabla u \rangle \langle Y, \nabla u \rangle \langle X, Y \rangle dV, \quad X, Y \in \mathfrak{X}(M). \]

It is clear that \( J(X, Y) = J(Y, X) \). Compute
\[ \int_M \langle X, \nabla u \rangle \langle Y, \nabla u \rangle \langle X, Y \rangle dV = \int_X \nabla_i u \left[ X^l \langle Y, \nabla u \rangle \langle X, Y \rangle \right] dV \]
\[ = -\int_M \left[ \text{div}(X) \langle Y, \nabla u \rangle \langle X, Y \rangle + X^l \nabla_i \left( \langle Y, \nabla u \rangle \langle X, Y \rangle \right) \right] dV \]
\[ = -\int_M \left[ \text{div}(X) \langle Y, \nabla u \rangle \langle X, Y \rangle + X^l \nabla_i (Y^j \nabla_j u X^k Y_k) \right] dV \]

...
Similarly,
\[
\int_Y \nabla_j \nabla_j u X^k Y_k + \int_Y \nabla_j \nabla_j X^k Y_k + \int_Y \nabla_j \nabla_j X^k Y_k \right) dV
\]

\[
= - \int_M u \text{div}(X) \langle Y, \nabla u \rangle \langle X, Y \rangle dV - \int_M u X^i \nabla_j \nabla_j X^k Y_k dV
\]

\[
- \int_M u X^i \nabla_j \nabla_j u X^k Y_k dV - \int_M u X^i \nabla_j \nabla_j X^k Y_k dV - \int_M u X^i \nabla_j \nabla_j X^k Y_k dV
\]

\[
= - \int_M u \text{div}(X) \langle Y, \nabla u \rangle \langle X, Y \rangle dV - J_1 - J_2 - J_3 - J_4.
\]

Using the definition of Lie derivative that \((\mathcal{L}_Y g)_{ik} = \nabla_i Y_k + \nabla_k Y_i\), we have

\[
J_4 = \int_M u X^i \nabla_j \nabla_j u X^k Y_k dV = - \int_M u \nabla_j (u X^i \nabla_j X^k Y_k) dV
\]

\[
= - \int_M u \left[ \nabla_j X^i \nabla_j X^k Y_k + u \nabla_j X^i \nabla_j X^k Y_k + u X^i \nabla_j \nabla_j X^k Y_k \right] dV
\]

\[
= - J_4 - \int_M u^2 \left[ \text{div}(Y) X^i \nabla_j X^k Y_k + X^i \nabla_j X^k \nabla_j Y_k + X^i \nabla_j X^k \nabla_j Y_k \right] dV
\]

\[
= - J_4 - \int_M u^2 \left[ \text{div}(Y) X^i \nabla_j X^k Y_k + X^i \nabla_j X^k \nabla_j Y_k + X^i \nabla_j X^k \nabla_j Y_k \right] dV;\]

hence

\[
(5.31)
\]

\[
J_4 = - \frac{1}{2} \int_M u^2 \left[ \text{div}(Y) X^i \nabla_j X^k Y_k + X^i \nabla_j X^k \nabla_j Y_k + X^i \nabla_j X^k \nabla_j Y_k \right] dV.\]

In particular

\[
(5.32)
\]

\[
J_4 = - \frac{1}{2} \int_M u^2 \langle X^i \nabla_j X^k \nabla_j Y_k \rangle dV = - \frac{1}{2} \int_M u^2 \langle X^i \nabla X^k \nabla_j Y_k \rangle dV, \quad Y \in \mathfrak{X}_K(M).
\]

Similarly,

\[
J_3 = \int_M u X^i \nabla_j \nabla_j u Y^k Y_k dV = - \int_M u \nabla_j (u X^i \nabla_j Y^k Y_k) dV
\]

\[
= - \int_M u \left[ \nabla_j X^i \nabla_j Y^k Y_k + u \nabla_j X^i \nabla_j Y^k Y_k + u X^i \nabla_j \nabla_j Y^k Y_k \right] dV
\]

\[
= - J_3 - \int_M u^2 \left[ \text{div}(Y) X^i \nabla_j Y^k Y_k + X^i \nabla_j Y^k \nabla_j Y_k + X^i \nabla_j Y^k \nabla_j Y_k \right] dV,
\]
Hence
\[ J_3 = -\frac{1}{2} \int_M u^2 \left[ \text{div}(Y) X^i Y^k \nabla_i X_k + X^i Y^k \nabla_j \nabla_i X_k \right. \]
\[ + \left. \nabla_j X^i \nabla_i X_k Y^k + \left\langle \nabla_X X, \nabla_Y Y \right\rangle \right] dV. \]  
(5.33)

From (5.32) and (5.33), we obtain
\[ J(X, Y) = -\int_M u \nabla_j u \nabla_X Y^j (X, Y) dV - \int_M u X^i Y^j \nabla_i u \nabla_j (X, Y) dV \]
\[ + \frac{1}{2} \int_M u^2 \left[ X^i Y^j \nabla_i \nabla_j X_k + \nabla_Y X^i \nabla_i X_k Y^k + \left\langle \nabla_X X, \nabla_Y Y \right\rangle \right] dV \]
\[ + \frac{1}{2} \int_M u^2 \left( X^i Y^k \nabla_k \nabla_i Y_j \right) dV \]  
(5.34)

According to the following identities
\[ X^i Y^k Y^j \nabla_i X_k = Y^k Y^j \nabla_i (X^i \nabla_i X_k) - Y^k Y^j \nabla_i X_k \]
\[ = Y^k Y^j \nabla_i X_k - \nabla_Y X^i \nabla_i X_k Y^k \]
\[ = \left\langle Y, \nabla_Y \nabla_X X \right\rangle - \left\langle Y, \nabla_Y X^i \nabla_i X_k \right\rangle, \]
\[ X^i Y^k X^j \nabla_i \nabla_j Y_j = Y^k X^j \nabla_i X^i \nabla_i \nabla_j Y_j - Y^k X^j \nabla_i \nabla_i X^i \nabla_j Y_j \]
\[ = Y^k X^j \nabla_i X^i \nabla_i \nabla_j Y_j - \nabla_Y X^i \nabla_i Y_j X^k \]
\[ = \left\langle X, \nabla_Y \nabla_X Y \right\rangle - \left\langle X, \nabla_Y X \right\rangle, \]
we find that
\[ \left\langle \nabla_X X, \nabla_Y Y \right\rangle + \left\langle \nabla_Y \nabla_X X, Y \right\rangle + X^i Y^k \left( Y^j \nabla_j X_k + X^i \nabla_i \nabla_j Y_j \right) \]
\[ = \left\langle \nabla_X X, \nabla_Y Y \right\rangle + \left( Y, \nabla_Y \nabla_X X \right) + \left\langle X, \nabla_Y \nabla_X Y \right\rangle \]
\[ + \left( Y, \nabla_Y \nabla_X X \right) - \left\langle X, \nabla_Y \nabla_X X \right\rangle - \left\langle Y, \nabla_Y \nabla_X X \right\rangle \]
\[ = \left\langle X, \nabla_Y \nabla_X Y \right\rangle - \left\langle \nabla_Y \nabla_X X \right\rangle + \left\langle \nabla_X X, \nabla_Y Y \right\rangle + \left\langle Y, \nabla_Y \nabla_X X \right\rangle \]
\[ = \text{Rm}(Y, X, Y, X) + \left( \nabla_X \nabla_Y X - \nabla_Y X \nabla_X \right) + \left( \nabla_X X, \nabla_Y Y \right) + \left( Y, \nabla_Y \nabla_X X \right). \]

Plugging it into (5.34) and using \( \nabla_X X = 0 \) for any \( X \in \mathcal{X}_{KC}(M) \) yields
\[ J(X, Y) = -\int_M u (X, Y) \left[ \left\langle \nabla_X X, \nabla_Y Y \right\rangle + X^i Y^j \left( \nabla_i \nabla_j Y_j \right) \right] dV \]
\[ + \frac{1}{2} \int_M u^2 \left[ -\text{Rm}(X, Y, Y, X) - \left\langle X, \nabla_Y \nabla_X Y \right\rangle \right] dV, \; X, Y \in \mathcal{X}_{KC}(M). \]
(5.35)
Since \( J \) is symmetric, we can rewrite (5.33) in a symmetric form. Changing \( X \) and \( Y \) in (5.33) we have

\[
J(X, Y) = - \int_M u(X, Y) [(\nabla u, \nabla Y X) + X^{ij} Y^i \nabla_j u] dV
\]

\[
+ \frac{1}{2} \int_M u^2 \left[ -Rm(X, Y, Y, X) - \langle Y, \nabla_{\nabla Y X} X \rangle \right] dV, \quad X, Y \in \mathcal{X}_K(M).
\]

Combining it with (5.35) we arrive at

\[
J(X, Y) = - \int_M u(X, Y) \left[ \frac{1}{2} (\nabla u, \nabla X Y + \nabla Y X) + X^{ij} Y^i \nabla_j u \right] dV
\]

\[
+ \frac{1}{2} \int_M u^2 \Lambda dV, \quad X, Y \in \mathcal{X}_K(M)
\]

(5.36)

where

\[
\Lambda := -Rm(X, Y, Y, X) - \frac{1}{2} \langle X, \nabla_{\nabla X Y} Y \rangle - \frac{1}{2} \langle Y, \nabla_{\nabla Y X} X \rangle.
\]

The following obvious identities

\[-Rm(X, Y, Y, X) - \langle X, \nabla_{\nabla X Y} Y \rangle = -Rm(X, Y, Y, X) + \langle Y, \nabla_{\nabla X Y} X \rangle - \nabla_X Y(X, Y),\]

\[-Rm(X, Y, Y, X) - \langle Y, \nabla_{\nabla Y X} X \rangle = -Rm(X, Y, Y, X) + \langle X, \nabla_{\nabla Y X} Y \rangle - \nabla_Y X(Y, X),\]

imply

\[
\Lambda = -Rm(X, Y, Y, X) - \frac{1}{4} \langle X, Y \rangle (\nabla_X Y + \nabla_Y X)
\]

\[
+ \frac{1}{4} \left[ \langle Y, \nabla_{[X,Y]} X \rangle + \langle X, \nabla_{[Y,X]} Y \rangle \right].
\]

(5.38)

In summary, we obtain

\[
\int_M Rm^1(X, Y, Y, X) dV = \int_M u(X, Y) \left[ (\nabla u, \nabla_X Y + \nabla_Y X) + 2X^{ij} Y^i \nabla_j u \right] dV
\]

\[
+ \frac{1}{2} \int_M u^2 \left[ \langle X, \nabla_{\nabla X Y} Y \rangle + \langle Y, \nabla_{\nabla Y X} X \rangle \right] dV
\]

\[
+ \int_M (1 + u^2) \text{Rm}(X, Y, Y, X) dV, \quad X, Y \in \mathcal{X}_K(M).
\]

(5.39)

or

\[
\int_M Rm^1(X, Y, Y, X) dV = \int_M u(X, Y) \left[ (\nabla u, \nabla_X Y + \nabla_Y X) + 2X^{ij} Y^i \nabla_j u \right] dV
\]

\[
+ \frac{1}{4} \int_M u^2 \langle X, Y \rangle (\nabla_X Y + \nabla_Y X) dV
\]

\[
- \frac{1}{4} \int_M u^2 \left[ \langle Y, \nabla_{[X,Y]} X \rangle + \langle X, \nabla_{[Y,X]} Y \rangle \right] dV
\]

\[
+ \int_M (1 + u^2) \text{Rm}(X, Y, Y, X) dV, \quad X, Y \in \mathcal{X}_K(M).
\]

(5.40)
6. Uniqueness for the Ricci-harmonic flow

In the section, we prove the forward and backward uniqueness of solutions for the Ricci-harmonic flow. Suppose that \((M, g_0)\) is a complete Riemannian manifold of dimension \(n\) and \(u_0\) is a smooth function on \(M\). Consider the Ricci-harmonic flow

\[
\partial_t g(t) = -2\text{Ric}_{g(t)} + 4\nabla_{g(t)} u(t) \otimes \nabla_{g(t)} u(t), \quad \partial_t u(t) = \Delta_{g(t)} u(t),
\]

on \(M \times [0, T]\).

6.1. Forward uniqueness. We now use the idea in [29] to prove the following

**Theorem 6.1. (Forward uniqueness of the Ricci-harmonic flow)** Suppose that \((g(t), u(t))\) and \((\tilde{g}(t), \tilde{u}(t))\) are two smooth complete solutions of (6.1) with

\[
\sup_{M \times [0, T]} \left( |\text{Rm}_{g(t)}|_{g(t)} + |\text{Rm}_{\tilde{g}(t)}|_{\tilde{g}(t)} \right) \leq K,
\]

for some uniform constant \(K\). If \((g(0), u(0)) = (\tilde{g}(0), \tilde{u}(0)) = (g_0, u_0)\), then \(g(t) \equiv \tilde{g}(t)\) for each \(t \in [0, T]\).

We now write

\[
g := g(t), \quad \text{Rm} := \text{Rm}_{g(t)}, \quad \tau := \Gamma_{g(t)}, \quad u := u(t), \quad \nabla := \nabla_{g(t)}, \quad dV := dV_{g(t)},
\]

and

\[
\tilde{g} := \tilde{g}(t), \quad \tilde{\text{Rm}} := \text{Rm}_{\tilde{g}(t)}, \quad \tilde{\tau} := \Gamma_{\tilde{g}(t)}, \quad \tilde{u} := \tilde{u}(t), \quad \tilde{\nabla} := \nabla_{\tilde{g}(t)}, \quad d\tilde{V} := dV_{\tilde{g}(t)}.
\]

We further fix a norm \(| \cdot | := | \cdot |_{g(t)}\). The \((p, q)\)-tensor fields \(T = (T_{ij} \cdots v_{jk})\) are smooth sections of the \((p, q)\)-tensor bundle \(T^p_q(M)\) over \(M\). For \((p, q) = (0, 0)\), we write \(T^0_0(M)\) as \(C^\infty(M)\). Introduce

\[
(h,A,T,v,w,y) \in T^1_2(M) \oplus T^2_2(M) \oplus T^3_2(M) \oplus T^0_2(M) \oplus T^1_0(M) \oplus T^2_0(M)
\]

according to the following definitions:

\[
\begin{align*}
(6.3) \quad h & \equiv h(t) := g - \tilde{g}, \quad h_{ij} := g_{ij} - \tilde{g}_{ij}, \\
(6.4) \quad A & \equiv A(t) := \nabla - \tilde{\nabla}, \quad A^k_{ij} = \Gamma^k_{ij} - \tilde{\Gamma}_{ij}^k, \\
(6.5) \quad T & \equiv T(t) := \text{Rm} - \tilde{\text{Rm}}, \quad T_{ijk}^f = \tilde{T}_{ijk}^f - \tilde{\tilde{T}}_{ijk}^f, \\
(6.6) \quad v & \equiv v(t) := u - \tilde{u}, \\
(6.7) \quad w & \equiv w(t) := \nabla u - \tilde{\nabla} \tilde{u}, \quad w_i = \nabla_i v, \\
(6.8) \quad y & \equiv y(t) := \nabla^2 u - \tilde{\nabla}^2 \tilde{u}, \quad y_{ij} = \nabla_i \nabla_j u - \tilde{\nabla}_i \tilde{\nabla}_j \tilde{u}.
\end{align*}
\]

From the definitions, we have

\[
y_{ij} = \partial^2_{ij} u - \tilde{\partial}^2_{ij} \tilde{u} - \left( \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij} \right) \partial_k \tilde{u} + \Gamma^k_{ij} (\partial_k \tilde{u} - \partial_k u)
\]

\[
= \left[ \left( \partial^2_{ij} u - \Gamma^k_{ij} \partial_k u \right) - \left( \partial^2_{ij} \tilde{u} - \Gamma^k_{ij} \partial_k \tilde{u} \right) \right] - A^k_{ij} \partial_k \tilde{u}
\]

\[
= \nabla_i w_j - A^k_{ij} \partial_k \tilde{u} = \frac{1}{2} (\nabla_i w_j + \nabla_j w_i) - A^k_{ij} \partial_k \tilde{u}
\]

so that

\[
y = \nabla w + A \ast \tilde{\nabla} \tilde{u}.
\]
Further, we can conclude
\[
\nabla_k \bar{g}^{ij} = \bar{g}^{ij} A_{k\ell} + \overset{\cdots}{g}^{ij} A_{k\ell}^* \quad \nabla \bar{g}^{-1} = \bar{g}^{-1} \ast A.
\]

Further,
\[
\partial_t h_{ij} = -2 T_{ij}^\ell + 4 \left( \partial_i \tilde{u} \partial_j \tilde{u} - \partial_i \tilde{u} \partial_j \tilde{u} \right)
\]
so that
\[
\partial_t h_{ij} = -2 T_{ij}^\ell + w_i w_j + w_i \partial_j \tilde{u} + w_j \partial_i \tilde{u} = -2 T_{ij}^\ell + w \ast w + w \ast \tilde{u}.
\]

The evolution of \( \partial_t A_{ij}^k \) can be derived similarly as that in \([29]\). Indeed,
\[
\partial_t A_{ij}^k = \bar{g}^{mk} \left( \nabla_i \tilde{R}_{jm} + \nabla_j \tilde{R}_{im} - \tilde{g}_m \tilde{R}_{ij} \right) - \bar{g}^{mk} \left( \nabla_i R_{jm} + \nabla_j R_{im} - \nabla_m R_{ij} \right)
\]
\[
+ 4 \left( \bar{g}^{mk} \nabla_m u \nabla_j \tilde{u} - \bar{g}^{mk} \partial_m u \tilde{v} \tilde{v} \right)
\]
\[
= \bar{g}^{-1} \ast h \ast \tilde{u} \nabla \tilde{R}_{ij} + A \ast \tilde{R}_{ij} + 1 \ast \nabla T
\]
\[
+ 4 \left[ \left( \bar{g}^{mk} - \bar{g}^{mk} \right) \tilde{v} \tilde{v} \nabla_i \tilde{u} \tilde{v} \tilde{v} + \bar{g}^{mk} \left( \nabla_m u \nabla_j \tilde{u} - \nabla_m u \tilde{v} \tilde{v} \right) \right]
\]
where the third line comes from the computation in \([29]\). In the last line, writing
\[
\nabla_m u \nabla_j \tilde{u} - \nabla_m \tilde{u} \tilde{v} \tilde{v} = \left( \nabla_m u - \nabla_m \tilde{u} \right) \tilde{v} \tilde{v} \nabla_j \tilde{u} + \nabla_m \left( \nabla_i \nabla_j \tilde{u} \right) \tilde{v} \tilde{v} \nabla_j \tilde{u}
\]
we can conclude
\[
\partial_t A = \bar{g}^{-1} \ast h \ast \tilde{u} \nabla \tilde{R}_{ij} + A \ast \tilde{R}_{ij} + 1 \ast \nabla T
\]
\[
+ \bar{g}^{-1} \ast h \ast \tilde{u} \ast \tilde{u} \ast \tilde{u} + w \ast \tilde{u} \ast \tilde{u} + \nabla \tilde{u} \ast y
\]
(6.13)

The same argument gives
\[
\partial_t T_{ij}^\ell = \nabla_a \left( \bar{g}^{\ell \ell} \nabla_b \tilde{R}_{ij}^\ell - \bar{g}^{\ell \ell} \nabla_b \tilde{R}_{ij}^\ell \right) + \bar{g}^{-1} \ast A \ast \tilde{R}_{ij} + \bar{g}^{-1} \ast h \ast \tilde{R}_{ij} + \tilde{R}_{ij} + T \ast \tilde{R}_{ij}
\]
\[
+ \nabla \nabla \tilde{u} \ast \tilde{u} \ast \tilde{u} + w \ast \tilde{u} \ast \tilde{u} + \nabla \tilde{u} \ast y
\]
\[
= \left( \bar{g}^{\ell \ell} - \bar{g}^{\ell \ell} \right) \tilde{v} \tilde{v} \nabla_i \tilde{m} \tilde{v} \tilde{v} + \bar{g}^{\ell \ell} \left( \nabla_i \nabla_m u \nabla_k \tilde{v} \tilde{v} \nabla_j \tilde{v} \tilde{v} - \tilde{v} \tilde{v} \nabla_i \nabla_m u \tilde{v} \tilde{v} \nabla_j \tilde{v} \tilde{v} \right)
\]
\[
+ \nabla \nabla \tilde{u} \left( \nabla_k \nabla_j \tilde{u} - \nabla_k \tilde{v} \tilde{v} \right) \ast h \ast \tilde{u} \ast \tilde{u} + y \ast \tilde{u} \ast \tilde{u} + y \ast \nabla \tilde{u} \ast y
\]
\[
\text{it follows that}
\]
\[
\partial_t T_{ij}^\ell = \nabla_a \left( \bar{g}^{\ell \ell} \nabla_b \tilde{R}_{ij}^\ell - \bar{g}^{\ell \ell} \nabla_b \tilde{R}_{ij}^\ell \right) + \bar{g}^{-1} \ast A \ast \tilde{R}_{ij} + \bar{g}^{-1} \ast h \ast \tilde{R}_{ij} + \tilde{R}_{ij} + T \ast \tilde{R}_{ij}
\]
\[
+ \nabla \nabla \tilde{u} \left( \nabla_k \nabla_j \tilde{u} - \nabla_k \tilde{v} \tilde{v} \right)
\]
(6.14)
Finally, we compute the evolution equation of $v$. Because
\[
\partial_t v = \Delta u - \tilde{\Delta} \tilde{u} = \Delta (u - \tilde{u}) + \left( g^{ij} \nabla_i \nabla_j - \tilde{g}^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \right) \tilde{u}
\]
we get
\[
(6.15) \quad \partial_t v = \Delta v + \tilde{g}^{-1} * h * \tilde{\nabla}^2 \tilde{u} + A * \tilde{\nabla} \tilde{u}.
\]
From (6.12) - (6.15), we arrive at
\[
(6.16) \quad |\partial_t h| \lesssim |T| + |w|^2 + |w||\tilde{\nabla} \tilde{u}|,
\]
\[
|\partial_t A| \lesssim |\tilde{g}^{-1}|h||\tilde{\nabla} Rm| + |A||Rm| + |T|
\]
\[
+ |\tilde{g}^{-1}|h||\tilde{\nabla}^2 \tilde{u}| + |w||\tilde{\nabla}^2 \tilde{u}| + |\nabla u||y|,
\]
\[
|\partial_t T - \Delta T - \mathrm{div} S| \lesssim |\tilde{g}^{-1}||h||\tilde{\nabla} Rm| + |\tilde{g}^{-1}|h||Rm|^2 + |T|(|Rm| + |\tilde{Rm}|)
\]
\[
(6.17) \quad + |\tilde{g}^{-1}||h||\tilde{\nabla}^2 \tilde{u}|^2 + |y|(|\nabla^2 u| + |\tilde{\nabla}^2 \tilde{u}|),
\]
\[
|\partial_t v - \Delta v| \lesssim |\tilde{g}^{-1}|h||\tilde{\nabla}^2 \tilde{u}| + |A||\tilde{\nabla} \tilde{u}|
\]
(6.19)

Here the tensor $S = (S_{ijk}^\ell)$ is defined as
\[
S_{ijk}^\ell := g^{ab} \nabla_b \tilde{K}_{ijk}^\ell - \tilde{g}^{ab} \tilde{\nabla}_b \tilde{K}_{ijk}^\ell = \tilde{g}^{-1} * h * \tilde{\nabla} Rm + A * \tilde{Rm}
\]
and satisfies
\[
(6.20) \quad |S| \lesssim |\tilde{g}^{-1}||h||\tilde{\nabla} Rm| + |A||\tilde{Rm}|.
\]

To further study, we need a version of Bernstein-Bando-Shi (BBS) estimate on higher derivatives of $Rm$ and $u$. Under the curvature condition (6.2), according to Theorem B.2, we have
\[
(6.21) \quad |\nabla u| \lesssim 1, \quad |\tilde{\nabla} \tilde{u}| \lesssim 1,
\]
and
\[
(6.22) \quad |\nabla Rm| + |\nabla^2 u| + |\tilde{\nabla} Rm| + |\tilde{\nabla}^2 \tilde{u}| \lesssim 1.
\]

**Proposition 6.2.** Assume the curvature condition (6.2). We first prove
\[
(6.23) \quad |h(t)| \leq K_0 t, \quad |A(t)| \leq K_0^{1/2}, \quad |v(t)| \leq K_0 t,
\]
on $M \times [0, T]$ for some uniform constant $K_0 = K_0(n, K, T, g_0, u_0)$. 

**Proof.** Since $g(0) = \tilde{g}(0) = g_0$, the inequality (6.23) is trivial for $t = 0$. In the following we may without loss of generality assume that $t \in (0, T]$. Given a space-time point $(p, t) \in M \times (0, T]$, the first can be proved by using (6.21)
\[
|h(p, t)| \lesssim |h(p, t)|_{g_0} \lesssim \int_0^t |\partial_s h(p, s)|_{g_0} ds \lesssim \int_0^t |\partial_s h(p, s)| ds
\]
\[
\lesssim \int_0^t \left( |T| + |w|^2 + |w||\tilde{\nabla} \tilde{u}| \right) (p, s) ds \lesssim \int_0^t ds \lesssim t.
\]
For second one, using (6.22),
\[
|A(p, t)| \lesssim |A(p, t)|_{g_0} \lesssim \int_0^t |\partial_s A(p, s)|_{g_0} ds \lesssim \int_0^t |\partial_s A(p, s)| ds \\
\lesssim \int_0^t \left[ |\nabla \text{Rm}| + |\nabla \text{Rm}| + |\nabla u| |\nabla^2 u| + |\nabla \bar{u}| |\nabla^2 \bar{u}| \right] ds \\
\lesssim \int_0^t ds \lesssim t \lesssim t^{1/2}.
\]

The last one follows from $|\partial_t v| \lesssim |\nabla^2 u| + |\nabla^2 \bar{u}|$.

Notice that the above proposition gives an explicit bound for $\nabla^2 u$ and hence for $\Delta u$, provided the condition (6.2) holds. However, Theorem 1.1 or Theorem 2.6 gives an explicit bound for $\Delta u$ under a weaker condition (1.2).

To prove the uniqueness, we need the following

**Lemma 6.3.** Consider a smooth family $(g(t))_{t \in [0, T]}$ of complete metrics on $M$ with $g(t) \geq \gamma^{-1} g$ for some uniform constant $\gamma$, where $g := g(0)$. Choose any given point $x_0 \in M$ and set $r(x) := d_g(x, x_0)$. Then the following statement is true: For any given constants $L_1, L_2 > 0$, there exist a constant $T' := T'(n, \gamma, L_1, L_2, T) > 0$ and a function $\eta : M \times [0, T'] \to \mathbb{R}$ smooth in $t$, Lipschitz on $M \times \{t\}$, and satisfying

\[
\partial_t \eta \geq L_1 |\eta|_{g(t)}^2, \quad e^{-\eta} \leq e^{-L_2 t^2}
\]
on $M \times [0, t]$ for all $t \in (0, T']$.

**Proof.** See [29]. Actually, we can pick $T' = \min\{T, 1/4(\gamma L_1 L_2)^{1/2}\}$.

Under the condition (6.2), Lemma 6.3 implies that for any given (sufficiently small and independent of $T$) constant $B > 0$ there exist a constant $T'' = T'(n, K, B, T) > 0$ and such a function $\eta : M \times [0, T'] \to \mathbb{R}$ satisfy

\[
e^{-\eta} \leq e^{-Bt^2/T}, \quad \partial_t \eta \geq B|\nabla \eta|^2
\]
on $M \times [0, t]$ for all $t \in (0, T']$; hence $\partial_t \eta, |\nabla \eta|^2$ are $e^{-\eta} dV$-integrable for all $t \in (0, T']$. For detail, see [29].

There are three claims about the integrability of $\mathcal{E}(t)$ defined below. The proof is similar to that in [29], except for some extra terms.

(a) $t^{-1}|h|^2, t^{-\beta}|A|^2, |T|^2, |v|^2$ and $|w|^2$ are uniformly bounded. It has been proved in Proposition 6.2.

(b) $\partial_t |h|^2, \partial_t |A|^2, \partial_t |T|^2, \partial_t |v|^2$, and $\partial_t |w|^2$ are uniformly bounded on $M \times [0, T]$ and consequently are $e^{-\eta} dV$-integrable for all $t \in (0, T')$. Compute

\[
|\partial_t |h|^2| = |\partial_t (g^{-1} * g^{-1} * h * h)| \\
= |g^{-1} * g^{-1} * g^{-1} * \partial_t g * h * h + g^{-1} * g^{-1} * h * \partial_t h| \\
\lesssim |\text{Ric}| |h|^2 + |h| |\partial_t h| \lesssim |h|^2 + |h| \left( |T| + |w|^2 + |w| \right) \lesssim 1,
\]
by (6.16) and (6.22). For \( \partial_t |A|^2 \) one has

\[
|\partial_t |A|^2| = |\partial_t (g^{-1} * g^{-1} * g * A * A) = g^{-1} * g^{-1} * g * \partial_t g * A * A + g^{-1} * g^{-1} * g * A * \partial_t A \\
\leq |\text{Ric}| |A|^2 + |A| |\partial_t A| \\
\leq |A|^2 + |A| \left( |\nabla \text{Rm}| + |\nabla \text{Rm}| + |\nabla u||\nabla^2 u| + |\nabla \theta||\nabla^2 \theta| \right) \lesssim 1,
\]

by (6.21), (6.22), (6.23), and the proof of Proposition 6.2. Similarly, we can deal with \( \partial_t |S|^2 \), \( \partial_t |v|^2 \), and \( \partial_t |w|^2 \). For example,

\[
|\partial_t |v|^2| \lesssim |v| \left( |\nabla^2 u| + |\nabla^2 \theta| \right) \lesssim |v| \lesssim 1
\]

using again Proposition 6.2.

Fixed \( \beta \in (0, 1) \), introduce the quantity

\[
(6.26) \quad \mathcal{E}(t) := \int_M \left( t^{-1}|h|^2 + t^{-\beta}|A|^2 + |T|^2 + |v|^2 + |w|^2 \right) e^{-\eta}dV.
\]

Here the function \( \eta \) is determined by (6.25).

(c) \( \mathcal{E}(t) \) is differentiable on \([0, T']\) and \( \lim_{t \to 0} \mathcal{E}(t) = 0 \). This follows from Proposition 6.2 and Lebesgue’s dominated convergence theorem.

The above claims (a)–(c) allow us frequently to take time derivatives inside the integrals.

**Proposition 6.4.** Assume that the curvature condition (6.2) is satisfied. There exist uniform constants \( N = N(n, K, T, g_0, u_0) > 0 \) and \( T_0 = T_0(n, K, T, g_0, u_0, \beta) \in (0, T] \) such that

\[
(6.27) \quad \mathcal{E}'(t) \leq N \mathcal{E}(t)
\]

for \( t \in [0, T_0] \). Hence \( \mathcal{E}(t) \equiv 0 \) for all \( t \in [0, T_0] \).

**Proof.** As in [29], for any \( t \in (0, T) \) and \( \alpha \in (0, 1) \), define

\[
(6.28) \quad \mathcal{G}(t) := \int_M |T|^2 e^{-\eta}dV, \quad \mathcal{H}(t) := \int_M t^{-1}|h|^2 e^{-\eta}dV,
\]

\[
(6.29) \quad \mathcal{I}(t) := \int_M t^{-\beta}|A|^2 e^{-\eta}dV, \quad \mathcal{J}(t) := \int_M |\nabla T|^2 e^{-\eta}dV,
\]

\[
(6.30) \quad \mathcal{V}(t) := \int_M |v|^2 e^{-\eta}dV, \quad \mathcal{D}(t) := \int_M |v|^2 e^{-\eta}dV, \quad \mathcal{B}(t) := \int_M |w|^2 e^{-\eta}dV.
\]

Then

\[
(6.31) \quad \mathcal{E}(t) = \mathcal{G}(t) + \mathcal{H}(t) + \mathcal{I}(t) + \mathcal{V}(t) + \mathcal{B}(t).
\]

We denote by \( C \) any constant depending only on \( n \), and by \( N \) any constant depending on \( n, K, T, g_0, u_0, \beta \).

Since \( g(t) \) are all uniformly equivalent to \( g = g(0) \), we can replace the norm \( |\cdot| \) in (6.26) by \( |\cdot|_g \). Hence we may regard the norm \( |\cdot| \) in (6.26) is independent of time.
(1) **Estimate for $G'$.** Start with

$$ G' = \int_M \left[ \partial_t |T|^2 - |T|^2 \partial_t \eta + \left( -R + 2|\nabla u|^2 \right) |T|^2 \right] e^{-\eta} dV $$

$$ \leq NG + \int_M \left[ 2(\partial_t T, T) - |T|^2 \partial_t \eta \right] e^{-\eta} dV. $$

Using (6.18), (6.20), (6.2), (6.29), and (6.31), we have

$$ G' \leq NG + \int_M \left[ 2(\Delta T + \text{div} S, T) - |T|^2 \partial_t \eta \right] e^{-\eta} dV $$

$$ \leq NG + \int_M \left[ \left( 2(\Delta T + \text{div} S, T) - |T|^2 \partial_t \eta \right) + Nt^{-1/2} |A||T| + Nh||T| + N|T|^2 + N|y||T| \right] e^{-\eta} dV $$

The same argument in [29] implies

$$ \int_M \left[ 2(\Delta T + \text{div} S, T) - |T|^2 \partial_t \eta \right] e^{-\eta} dV \leq -\mathcal{J} + N\mathcal{H} + Nt^\beta \mathcal{I} $$

by choosing an appropriate constant $B$ in (6.25). Therefore

$$ (6.32) \quad G' \leq NG + N\mathcal{H} + \left( N + t^{\beta-1} \right) \mathcal{I} - \mathcal{J} + \frac{1}{4} \mathcal{D}. $$

(2) **Estimate for $H'$.** Since $\partial_t \eta \geq 0$, we get

$$ H' = -t^{-1} \mathcal{H} + t^{-1} \int_M \left[ 2(\partial_t h, h) - |h|^2 \partial_t \eta + |h|^2 \left( -R + 2|\nabla u|^2 \right) \right] e^{-\eta} dV $$

$$ \leq (N - t^{-1}) \mathcal{H} + Ct^{-1} \int_M |h| \left( |T|^2 + |w|^2 + |\nabla u| \right) e^{-\eta} dV $$

$$ \leq (N - t^{-1}) \mathcal{H} + Ct^{-1} \int_M |h||T| e^{-\eta} dV + Nt^{-1} \int_M |h||w| e^{-\eta} dV $$

$$ \leq \left( N - \frac{1}{2} t^{-1} \right) \mathcal{H} + C\mathcal{G} + N \int_M |w|^2 e^{-\eta} dV. $$

Thus

$$ (6.33) \quad H' \leq \left( N - \frac{1}{2} t^{-1} \right) \mathcal{H} + C\mathcal{G} + NB. $$
(3) Estimate for $I'$. As in the estimation of $H'$, we have

$$I' \leq (N - \beta t^{-1})I + Ct^{-\beta} \int_M |A| \left( |g^{-1}||\nabla \bar{Rm}||h| + |A||\bar{Rm}| + |\nabla T| + |\bar{g}^{-1}||\bar{\nabla}^2 \bar{u}||h| + |w|\bar{\nabla}^2 \bar{u} + |\nabla u||y| \right) e^{-\eta} dV$$

$$\leq (N - \beta t^{-1})I + \int_M \left( Ct^{-\beta}|\nabla T||A| + N t^{-\frac{\beta}{2}}|h||A| + N t^{-\beta}|h||A| + N t^{-\beta}|A||y| \right) e^{-\eta} dV$$

$$\leq (N - \beta t^{-1})I + \left( Ct^{-\beta}I + \mathcal{J} \right) + \left( N \mathcal{H} + Ct^{-\beta}I \right) + \left( N \mathcal{H} + tI \right) + \left( Ct^{-\beta}I + NB \right) + \left( N t^{-\beta}I + \frac{1}{4}D \right).$$

Consequently

$$\tag{6.34} I' \leq N \mathcal{H} + \left( N - \beta t^{-1} + N t^{-\beta} \right) I + \mathcal{J} + NB + \frac{1}{4}D.$$

(4) Estimate for $V'$. This estimate is similar to (1),

$$V' \leq NV + \int_M \left[ 2(\partial_t v, v) - |v|^2 \partial_\eta \right] e^{-\eta} dV$$

$$\leq NV + \int_M \left[ 2(\Delta v, v) - |v|^2 \partial_\eta + C|v| \left( |\bar{g}^{-1}||\bar{\nabla}^2 \bar{u}||h| + |A||\bar{\nabla} \bar{u}| \right) \right] e^{-\eta} dV$$

$$\leq NV + \int_M \left[ 2(\Delta v, v) - |v|^2 \partial_\eta \right] e^{-\eta} dV + N \int_M \left( |v||h| + |v||A| \right) e^{-\eta} dV$$

$$\leq NV + t\mathcal{H} + t^{\beta}I + \int_M \left[ -2|\nabla v|^2 + 2|v||\nabla \eta||\nabla v| - |v|^2 \partial_\eta \right] e^{-\eta} dV.$$

Thus, by choosing an appropriate constant $B$ in $W$,

$$\tag{6.35} V' \leq NV + t\mathcal{H} + t^{\beta}I - B.$$

(5) Estimate for $B'$. Because the evolution equation $A.4$ is linear in $\nabla, u$, we conclude that

$$\tag{6.36} \partial_t w = \Delta w + Rm \ast w$$

and hence

$$B' \leq NB + \int_M \left[ 2(\partial_t w, w) - |w|^2 \partial_\eta \right] e^{-\eta} dV$$

$$\tag{6.37} \leq NB + \int_M \left[ 2(\Delta w, w) - |w|^2 \partial_\eta \right] e^{-\eta} dV$$

$$\leq NB - \int_M |\nabla w|^2 e^{-\eta} dV$$

by choosing an appropriate constant $B$ in $A.25$. From $A.28 - A.27$, we arrive at

$$\tag{6.38} E' \leq NE - t^{-1} \left( \beta - t^{\beta} - N t^{-1} \right) I + \frac{1}{2}D - \int_M |\nabla w|^2 e^{-\eta} dV.$$
On the other hand, the equation (6.9) yields
\[
\frac{1}{2}D = \frac{1}{2} \int_M |g|^2 e^{-\eta} dV \leq \int_M \left[ |\nabla w|^2 + N|A|^2 |\nabla \tilde{u}|^2 \right] e^{-\eta} dV
\]
\[
\leq \int_M |\nabla w|^2 e^{-\eta} dV + Nt^\beta \mathcal{I}.
\]
Therefore, the inequality (6.38) can be rewritten as
\[
(6.39) \quad \mathcal{E}' \leq N\mathcal{E} - t^{-1} \left( \beta - t^\beta - Nt^{1-\beta} \right) \mathcal{I}.
\]
Now we can choose appropriate constants $T_0$ and $N$ such that the term $\beta - t^\beta - Nt^{1-\beta}$ is nonnegative for any $t \in [0, T_0]$. In this case, the inequality (6.43) gives us the desired estimate $\mathcal{E}'(t) \leq N\mathcal{E}(t)$ on $[0, T_0]$. □

The proof of Theorem 6.5. Now the proof immediately follows from the above Proposition 6.4.

6.2. Backward uniqueness. In this subsection we use the main idea in [28] to prove the backward uniqueness of the Ricci-harmonic flow. Recall

**Theorem 6.5. (Kotschwar [28])** Consider a smooth family of complete Riemannian metrics $(g(t))_{t \in [0, T]}$ on a smooth n-dimensional manifold $M$, satisfying the evolution equation
\[
(6.40) \quad \partial_t g(t) = b(t),
\]
and a symmetric, positive-definite family of $(2,0)$-tensor fields $\Lambda(t), t \in [0, T]$, with
\[
(6.41) \quad \Box := \partial_t - \Delta_{g(t),g(t)}, \quad \Delta_{\Lambda,g(t)} := \text{tr}_{\Lambda(t)} \nabla^2_{g(t)}.
\]
Let $\mathcal{X}, \mathcal{Y}$ be finite direct sums of the $(p,q)$-bundle $T^p_q(M)$ over $M$, and $X, Y \in C^\infty(\mathcal{X} \times [0, T]), Y \in C^\infty(\mathcal{Y} \times [0, T])$. Suppose the following assumptions hold:

1. there exist positive constants $P, Q, a_1, a_2$ such that
   \[
   |b(t)|^2_{g(t)} + |\nabla g(t)b(t)|^2_{g(t)} \leq P,
   |\partial_t \Lambda(t)|^2_{g(t)} + |\nabla g(t)\Lambda(t)|^2_{g(t)} \leq Q,
   a_1g^{-1}(t) \leq \Lambda(t) \leq a_2g^{-1}(t),
   \]
2. there exists a nonnegative constant $K$ such that $\text{Ric}_{g(t)} \geq -Kg(t),$
3. there exist positive constants $a, A$ and some point $x_0 \in M$ such that
   \[
   (6.42) \quad |X(x,t)|^2_{g(t)} + |\nabla g(t)X(x,t)|^2_{g(t)} + |Y(x,t)|^2_{g(t)} \leq Ae^{\text{ad}_{g(t)}^A(x_0.x)},
   \]
4. $X$ and $Y$ satisfy the inequality
   \[
   (6.43) \quad |\Box X|_{g(t)}^2 \leq C \left( |X|_{g(t)}^2 + |\nabla g(t)X|_{g(t)}^2 + |Y|_{g(t)}^2 \right),
   \]
   \[
   (6.44) \quad |\partial_t Y|_{g(t)}^2 \leq C \left( |X|_{g(t)}^2 + |\nabla g(t)Y|_{g(t)}^2 + |Y|_{g(t)}^2 \right)
   \]
   for some positive constant $C$.

If $X(T) = Y(T) = 0$, then $X = Y \equiv 0$ on $M \times [0, T]$.

**Theorem 6.6. (Backward uniqueness of the Ricci-harmonic flow)** Suppose that $(g(t), u(t))$ and $(\tilde{g}(t), \tilde{u}(t))$ are two smooth complete solutions of (6.1) satisfying (6.2) for some uniform constant $K$. If $(g(T), u(T)) = (\tilde{g}(T), \tilde{u}(T))$, then $(g(t), u(t)) \equiv (\tilde{g}(t), \tilde{u}(t))$ for each $t \in [0, T]$. 
Recall notions in (6.3) – (6.8),

\[ h := g - \tilde{g}, \quad h_{ij} = g_{ij} - \tilde{g}_{ij}, \]
\[ A := \nabla - \tilde{\nabla}, \quad A_{ij}^{k} = \Gamma_{ij}^{k} - \tilde{\Gamma}_{ij}^{k}. \]
\[ T := Rm - \tilde{Rm}, \quad T_{ikj}^{\ell} = R_{ikj}^{\ell} - \tilde{R}_{ikj}^{\ell}, \quad v := \nabla u - \tilde{\nabla} \tilde{u}, \]
\[ w_{i} := \nabla_{i} v, \quad y := \nabla^{2} u - \tilde{\nabla}^{2} \tilde{u}, \quad y_{ij} = \nabla_{i} \nabla_{j} u - \nabla_{i} \tilde{\nabla} \tilde{u}. \]

Define new tensor fields

(6.45) \[ B := \nabla A, \]
(6.46) \[ U := \nabla Rm - \tilde{\nabla} \tilde{Rm}, \]
(6.47) \[ x := \nabla w, \]
(6.48) \[ z := \nabla^{3} u - \tilde{\nabla}^{3} \tilde{u}. \]

Consider direct sums of tensor fields

\[ X := (T \oplus U) \oplus (y \oplus z), \quad Y := (h \oplus A \oplus B) \oplus (v \oplus w \oplus x). \]

Using Lemma [X.2] we obtain

\[ \partial_{t} \Gamma = g^{-1} * \nabla Rm + g^{-1} * \nabla u * \nabla^{2} u, \]
\[ \partial_{t} Rm = \Delta Rm + g^{-1} * Rm * Rm + g^{-1} * Rm * \nabla u * \nabla u + g^{-1} * \nabla^{2} u * \nabla^{2} u, \]
\[ \partial_{t} \nabla Rm = \Delta \nabla Rm + g^{-1} * Rm * \nabla Rm + g^{-1} * \nabla Rm * \nabla u * \nabla u + g^{-1} * \nabla^{2} u * \nabla^{2} u, \]
\[ + g^{-1} * Rm * \nabla u * \nabla^{2} u + g^{-1} * \nabla^{2} u * \nabla^{3} u. \]

We also recall

\[ \tilde{g}^{ij} - g^{ij} = \tilde{g}^{ia} \tilde{g}^{jb} h_{ab} \quad \text{or} \quad \tilde{g}^{-1} - g^{-1} = \tilde{g}^{-1} * g^{-1} * h \]

and

(6.49) \[ \nabla \nabla h_{ab} = A_{ca}^{p} \tilde{g}_{pb} + A_{cb}^{p} \tilde{g}_{ap} \quad \text{or} \quad \nabla h = A * \tilde{g}. \]

The last identity follows from

\[ \nabla \nabla h_{ab} = \partial_{c} h_{ab} - \Gamma_{ca}^{p} h_{pb} - \Gamma_{cb}^{p} h_{ap} \]
\[ = \partial_{c} \tilde{g}_{ab} - \partial_{c} \tilde{g}_{ab} - \Gamma_{ca}^{p} \tilde{g}_{pb} + \Gamma_{cb}^{p} \tilde{g}_{ap} - \Gamma_{ca}^{p} \tilde{g}_{ap} + \Gamma_{cb}^{p} \tilde{g}_{ap} \]
\[ = \nabla \nabla \tilde{g}_{ab} - \partial_{c} \tilde{g}_{ab} + (A_{ca}^{p} + \tilde{A}_{ca}^{p}) \tilde{g}_{pb} + (A_{cb}^{p} + \tilde{A}_{cb}^{p}) \tilde{g}_{ap} \]
\[ = \nabla \nabla h_{ab} - \tilde{\nabla} \nabla \tilde{g}_{ab} + A_{ca}^{p} \tilde{g}_{pb} + A_{cb}^{p} \tilde{g}_{ap}. \]
Lemma 6.7. One has
\begin{align}
\partial_t h_{ij} &= -2 T_{ij}^e \\
&= 4 \left(w_i w_j + w_i \nabla_j \tilde{u} + w_j \nabla_i \tilde{u}\right), \quad (6.50)
\end{align}
\begin{align}
\partial_t A_{ij}^k &= -g^{mk} \left( U_{ipjm}^p + U_{ijpm}^p - U_{mpij}^p \right) \\
&+ g^{mk} g^{ma} h_{ab} \left( \nabla_i \bar{R}_{jm} + \nabla_j \bar{R}_{im} - \bar{R}_{mij} \right) + 4 g^{mk} \nabla_m u y_{ij} \\
&+ 4 g^{mk} w_m \nabla_i \nabla_j \tilde{u} - 4 g^{mk} g^{lb} h_{ab} \nabla_m \tilde{u} \nabla_i \nabla_j \tilde{u}, \quad (6.51)
\end{align}
\begin{align}
\partial_t B &= \left[ \nabla U + h * A * \tilde{g}^{-1} * \nabla \bar{R} m + A * \tilde{g}^{-1} * \nabla \bar{R} m \\
&+ h * \tilde{g}^{-1} * \nabla^2 \bar{R} m + A * U + A * \nabla \bar{R} m \right] + \tilde{g}^{-1} * h * \nabla \tilde{u} * \nabla^2 \bar{u} \事情
\end{align}
\begin{align}
\Box T &= \left[ h * \tilde{g}^{-1} * \nabla^2 \bar{R} m + A * \nabla \bar{R} m + T * \bar{R} m + T * T + A * A * \bar{R} m \\
&+ B * \bar{R} m + h * \tilde{g}^{-1} * \bar{R} m + \bar{R} m \right] + h * \tilde{g}^{-1} * \bar{R} m * \nabla \tilde{u} * \nabla \bar{u} \事情
\end{align}
\begin{align}
\Box U &= \left[ h * \tilde{g}^{-1} * \nabla^3 \bar{R} m + A * \nabla^2 \bar{R} m + A * A * \nabla \bar{R} m + B * \nabla \bar{R} m \\
&+ h * \tilde{g}^{-1} * \bar{R} m * \nabla \bar{R} m + U * \bar{R} m + T * \nabla \bar{R} m + T * U \right] \\
&+ h * \tilde{g}^{-1} * \nabla \bar{R} m * \nabla \tilde{u} * \nabla \bar{u} + U * w * w + w * \nabla \bar{R} m \事情
\end{align}
\begin{align}
 h * \tilde{g}^{-1} * Rm * \nabla \tilde{u} * \nabla \bar{u} + U * Rm * w * \nabla \tilde{u} + U * \nabla \tilde{u} * \nabla \bar{u} \\
+ h * \tilde{g}^{-1} * Rm * \nabla \tilde{u} * \nabla^2 \bar{u} + T * \nabla \tilde{u} * y + Rm * \nabla \tilde{u} * y \\
+ T * w * \nabla^2 \bar{u} + Rm * w * \nabla^2 \bar{u} + T * \nabla \tilde{u} * \nabla^2 \bar{u} + T * w * y \\
+ \bar{R} m * w * y + h * \tilde{g}^{-1} * \nabla^2 \bar{u} * \nabla^3 \tilde{u} + y * z + y * \nabla^3 \tilde{u} + z * \nabla^2 \bar{u}. \事情
\end{align}

Proof. For $h_{ij}$, one has
\begin{align}
\partial_t h_{ij} &= \partial_t g_{ij} - \partial_t \tilde{g}_{ij} = -2 R_{ij} + 4 \nabla_i u \nabla_j u + 2 \tilde{R}_{ij} - 4 \nabla_i \tilde{u} \nabla_j \tilde{u} \\
&= -2 T_{ij}^e + 4 \left(w_i + \nabla_i \tilde{u}\right) \left(w_j + \nabla_j \tilde{u}\right) - 4 \nabla_i \tilde{u} \nabla_j \tilde{u}
\end{align}
which implies (6.50).
In (6.51), the terms enclosed in the bracket were derived in [28]. The remaining terms are
\begin{align}
4 g^{mk} \nabla_m u \nabla_i \nabla_j u - 4 g^{mk} \nabla_m \tilde{u} \nabla_i \nabla_j \tilde{u}
\end{align}
which, using $y_{ij} = \nabla_i \nabla_j u - \nabla_i \nabla_j \bar{u}$, $w_m = \nabla_m y - \nabla_m \bar{u}$, and $\tilde{g}^{mk} - g^{mk} = \tilde{g}^{ma} \delta^{kb} h_{ab}$, is equal to

$$\tilde{g}^{mk} \nabla_m u \left( y_{ij} + \nabla_i \nabla_j \bar{u} \right) - \tilde{g}^{mk} \nabla_m \bar{u} \nabla_i \nabla_j \bar{u}$$

$$= \tilde{g}^{mk} \nabla_m u y_{ij} + \left( \tilde{g}^{mk} \nabla_m u - \tilde{g}^{mk} \nabla_m \bar{u} \right) \nabla_i \nabla_j \bar{u}$$

$$= \tilde{g}^{mk} \nabla_m u y_{ij} + \left[ \tilde{g}^{mk} \left( \nabla_m u - \nabla_m \bar{u} \right) \right] \nabla_i \nabla_j \bar{u}$$

$$= \tilde{g}^{mk} \nabla_m u y_{ij} + \tilde{g}^{mk} w_m \nabla_i \nabla_j \bar{u} - \tilde{g}^{ma} \delta^{kb} h_{ab} \nabla_m \bar{u} \nabla_i \nabla_j \bar{u}.$$  

In particular, (6.51) implies

$$\partial_t A = \left( g^{-1} * U + \tilde{g}^{-1} * \tilde{g}^{-1} * h * \nabla \tilde{Rm} \right).$$

According to the relation $\partial_t B = \partial_t \nabla A = \nabla \partial_t A + \partial_t \Gamma * A$, we obtain (6.52) where the bracket follows from (28) and the remaining terms are

$$g^{-1} * \nabla u * \nabla^2 u * A + g^{-1} * \nabla^2 u * y + g^{-1} * \nabla u * \nabla y + g^{-1} * \nabla w * \nabla^2 \bar{u}$$

$$+ \tilde{g}^{-1} * w * \nabla \nabla^2 \bar{u} + g^{-1} * \nabla \tilde{g}^{-1} * h * \nabla \bar{u} * \nabla^2 \bar{u} + g^{-1} * \tilde{g}^{-1} * \nabla h * \nabla \bar{u} * \nabla^2 \bar{u}$$

$$+ g^{-1} * \tilde{g}^{-1} * h * \nabla \nabla \bar{u} * \nabla^2 \bar{u} + g^{-1} * \tilde{g}^{-1} * h * \nabla \bar{u} * \nabla \nabla^2 \bar{u}.$$  

Applying the formula

(6.55)

$$\nabla W = \nabla \bar{W} + A * W$$

for any tensor field $W$, $\nabla h = \tilde{g} * A$, and $\nabla \tilde{g}^{-1} = \tilde{g}^{-1} * A$, which follows from

$$\nabla g_{ij} = \tilde{g}^{ia} \delta^{jb} \nabla_{h_{ab}} = \tilde{g}^{ia} \delta^{jb} \left( A^p_{ka} \delta^{pb} + A^p_{kb} \delta^{ap} \right) = \tilde{g}^{ia} A^j_{ka} + \tilde{g}^{ib} A^j_{kb},$$

we complete the proof of (6.52).

To prove the last two identities, recall from (28) that

(6.56)

$$\nabla^2 \bar{W} = \nabla^2 \bar{W} + A * \nabla \bar{W} + B * W + A * A * W$$

for any tensor field $W$. In particular,

$$\tilde{\Delta} \tilde{Rm} = \tilde{g}^{-1} * h * \nabla^2 \tilde{Rm} + \Delta \tilde{Rm} + A * \nabla \tilde{Rm} + B * \tilde{Rm} + A * \tilde{Rm},$$

$$\tilde{\Delta} \tilde{\nabla} \tilde{Rm} = \tilde{g}^{-1} * h * \nabla \nabla \tilde{Rm} + \Delta \tilde{\nabla} \tilde{Rm} + A * \tilde{\nabla} \tilde{Rm} + B * \tilde{\nabla} \tilde{Rm} + A * A * \tilde{\nabla} \tilde{Rm}.$$  

For (6.53), we have

$$\square T = (\partial_t - \Delta) \left( \text{Rm} - \tilde{\text{Rm}} \right)$$

$$= g^{-1} * \text{Rm} * \text{Rm} + g^{-1} * \text{Rm} * \nabla u * \nabla u + g^{-1} * \nabla^2 u * \nabla^2 u - \partial_t \tilde{\text{Rm}} + \Delta \tilde{\text{Rm}}$$

$$= g^{-1} * \text{Rm} * \text{Rm} + g^{-1} * \text{Rm} * \nabla u * \nabla u + g^{-1} * \nabla^2 u * \nabla^2 u$$

$$- \left[ \tilde{\Delta} \tilde{\text{Rm}} + g^{-1} * \tilde{\text{Rm}} * \tilde{\text{Rm}} + \tilde{\Delta} \tilde{\nabla} \tilde{\text{Rm}} + \tilde{\nabla} \tilde{\nabla} \tilde{\text{Rm}} + B * \tilde{\text{Rm}} + A * A * \tilde{\text{Rm}} \right]$$

$$+ \left[ \text{Rm} - h * g^{-1} * \nabla^2 \text{Rm} + A * \nabla \text{Rm} + B * \text{Rm} + A * A * \text{Rm} \right]$$

$$= g^{-1} * \left( T + \text{Rm} \right) * \left( T + \text{Rm} \right) - g^{-1} * \text{Rm} * \text{Rm}$$

$$+ g^{-1} * \left( \text{Rm} + \nabla \tilde{u} \right) * \left( \text{Rm} + \nabla \tilde{u} \right) - g^{-1} * \text{Rm} * \nabla \tilde{u} * \nabla \tilde{u}$$

$$+ g^{-1} * \left( \nabla^2 \tilde{u} \right) * \left( \nabla^2 \tilde{u} \right) - g^{-1} * \nabla^2 \tilde{u} * \nabla^2 \tilde{u}.$$
Simplifying terms gives (6.53). Similarly,
\[\square U = h \ast \tilde{g}^{-1} \ast \tilde{\nabla}^3 Rm + A \ast \tilde{\nabla}^2 Rm + B \ast \tilde{\nabla} Rm + A \ast A \ast \tilde{\nabla} Rm\]
+ \[\left[\tilde{g}^{-1} \ast \left( T + \tilde{R}m \right) \ast (U + \tilde{\nabla} Rm) - \tilde{g}^{-1} \ast \tilde{R}m \ast \tilde{\nabla} Rm \right]\]
+ \[\left[\tilde{g}^{-1} \ast \left( U + \tilde{\nabla} Rm \right) \ast \left( w + \tilde{\nabla} \tilde{u} \right) \ast (w + \tilde{\nabla} \tilde{u}) - \tilde{g}^{-1} \ast \tilde{\nabla} \tilde{R}m \ast \tilde{\nabla} \tilde{u} \ast \tilde{\nabla} \tilde{u} \right]\]
+ \[\left[\tilde{g}^{-1} \ast \left( T + \tilde{R}m \right) \ast \left( w + \tilde{\nabla} \tilde{u} \right) \ast \left( y + \tilde{\nabla}^2 \tilde{u} \right) - \tilde{g}^{-1} \ast \tilde{R}m \ast \tilde{\nabla} \tilde{u} \ast \tilde{\nabla}^2 \tilde{u} \right]\]
+ \[\left[\tilde{g}^{-1} \ast \left( y + \tilde{\nabla}^2 \tilde{u} \right) \ast \left( z + \tilde{\nabla}^3 \tilde{u} \right) - \tilde{g}^{-1} \ast \tilde{\nabla}^2 \tilde{u} \ast \tilde{\nabla}^3 \tilde{u} \right].\]

Simplifying terms gives (5.54).

According to Theorem [5.2] the condition \(|Rm|_{g(t)} + |\tilde{R}m|_{g(t)} \leq K\) implies that, for all \(m \geq 0\), there exist constants \(C_m = C_m(\delta, K, n, T) > 0\) such that
\[|\nabla^m Rm| + |\nabla^{m+1} Rm| + |\tilde{\nabla}^m Rm|_{g(t)} + |\tilde{\nabla}^{m+1} \tilde{u}|_{g(t)} \leq C_m\]
on \(M \times [0, T]\). We also have
\[\frac{1}{\gamma} g(t) \leq \tilde{g}(t) \leq \gamma g(t)\]
on \(M \times [0, T]\), for some positive constant \(\gamma = \gamma(K, T)\). Hence \(\nabla^m Rm, \nabla^{m+1} u,\nabla^m Rm, \nabla^{m+1} \tilde{u}, m \geq 0\), and \(n^{-1}\) are uniformly bounded with respect to \(g(t)\) on \([0, T]\) so that we can replace the norm \(|\cdot|_{g(t)}\) by \(|\cdot| := |\cdot|_{g(t)}\).

Lemma 6.8. \(h, A, B, T, U\) are uniformly bounded with respect to \(g(t)\) on \([0, T]\). Moreover, \(v, w, x, y, z\) are also uniformly bounded with respect to \(g(t)\) on \([0, T]\).

**Proof.** The first part was proved in [28] in the exact manner. The second part follows immediately from Lemma [5.1] \(\square\)

Lemma 6.9. Using the above lemma, one has
\[|\partial_t h|^2 \lesssim |T|^2 + |w|^2,\]
\[|\partial_t A|^2 \lesssim |U|^2 + |h|^2 + |y|^2 + |w|^2,\]
\[|\partial_t B|^2 \lesssim |\nabla U|^2 + |h|^2 + |A|^2 + |U|^2 + |y|^2 + |\nabla y|^2 + |x|^2 + |w|^2,\]
\[|\partial_t v|^2 \lesssim |y|^2 + |h|^2,\]
\[|\partial_t w|^2 \lesssim |z|^2 + |A|^2 + |h|^2 + |v|^2,\]
\[|\partial_t x|^2 \lesssim |\nabla z|^2 + |B|^2 + |A|^2 + |h|^2 + |v|^2 + |w|^2,\]
\[|\nabla y|^2 \lesssim |h|^2 + |A|^2 + |B|^2 + |T|^2 + |w|^2 + |y|^2,\]
\[|\nabla z|^2 \lesssim |h|^2 + |A|^2 + |B|^2 + |T|^2 + |z|^2 + |U|^2 + |y|^2 + |w|^2.\]

**Proof.** The first four inequality follows from Lemma [6.9] For the next three inequalities, we verify only the inequality for \(|\partial_t v|^2\). By definition,
\[\partial_t v = \Delta u - \Delta \tilde{u} = g^{ij} \tilde{\nabla}_i \nabla_j u - g^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \tilde{u} = g^{ij} \left( \nabla_i \nabla_j u - \tilde{\nabla}_i \tilde{\nabla}_j \tilde{u} \right) + (g^{ij} - \tilde{g}^{ij}) \tilde{\nabla}_i \tilde{\nabla}_j \tilde{u} = g^{ij} y_{ij} - \tilde{g}^{ia} g^{jb} h_{ab} \tilde{\nabla}_i \tilde{\nabla}_j \tilde{u}\]
so that $\partial_t v = g^{-1} * y + \tilde{g}^{-1} * h * \nabla^2 \tilde{u}$. Similarly
\[
\partial_t w = g^{-1} * \nabla y + \tilde{g}^{-1} * A * h * \nabla^2 \tilde{u} + \tilde{g}^{-1} * A * \nabla^2 \tilde{u} \\
+ \tilde{g}^{-1} * h * (\nabla^3 \tilde{u} + A * \nabla^2 \tilde{u})
\]
with $\nabla y = \nabla (\nabla^2 u - \nabla^2 \tilde{u}) = z + A * \nabla^2 \tilde{u}$. For the final two inequalities we only verify the inequality for $|\square y|^2$. From the identity
\[
\tilde{\Delta} \nabla^2 \tilde{u} = \tilde{g}^{-1} \nabla^2 \nabla^2 \tilde{u}
= \Delta \nabla^2 \tilde{u} + h * \tilde{g}^{-1} * \nabla^4 \tilde{u} + A * \tilde{g}^3 \tilde{u} + B * \nabla^2 \tilde{u} + A * A * \nabla^2 \tilde{u},
\]
and
\[
\partial_t \nabla^2 u = \Delta \nabla^2 u + Rm * \nabla^2 u + g^{-1} * \nabla u * \nabla u * \nabla^2 u,
\]
we obtain
\[
\square y = \left[ Rm * \nabla^2 u + g^{-1} * \nabla u * \nabla u * \nabla^2 u \right] - \partial_t \nabla^2 \tilde{u} + A \nabla^2 \tilde{u}
= h * \tilde{g}^{-1} * \nabla^4 \tilde{u} + A * \tilde{g}^3 \tilde{u} + B * \nabla^2 \tilde{u} + A * A * \nabla^2 \tilde{u}
+ (T + Rm) * \nabla^2 u - Rm * \nabla^2 \tilde{u}
+ g^{-1} * (w + \nabla \tilde{u}) * (w + \nabla \tilde{u}) * (y + \nabla^2 \tilde{u}) - g^{-1} * \nabla \tilde{u} * \nabla \tilde{u} * \nabla^2 \tilde{u}
+ w * w * y + \tilde{\nabla} \tilde{u} * \tilde{\nabla} \tilde{u} + y + w * \nabla^2 \tilde{u} + w * \tilde{\nabla} \tilde{u} * \nabla^2 \tilde{u}
+ h * \tilde{g}^{-1} * \nabla \tilde{u} * \nabla \tilde{u} * \nabla^2 \tilde{u}.
\]
For $|\square z|^2$, we need the evolution equation
\[
\partial_t \nabla^3 u = \Delta \nabla^3 u + g^{-1} * Rm * \nabla^3 u + g^{-1} * \nabla Rm * \nabla^2 u \\
+ g^{-1} * \nabla u * \nabla^2 u * \nabla^2 u + g^{-1} * \nabla u * \nabla u * \nabla^3 u.
\]
and the identity
\[
\tilde{\Delta} \nabla^3 \tilde{u} = \Delta \nabla^3 u + h * \tilde{g}^{-2} * \tilde{\nabla}^5 u + A * \tilde{g}^4 u + B * \tilde{g}^3 u + A * A * \tilde{g}^3 u.
\]
Thus we prove the results. □

*The proof of Theorem 6.6.* The above two lemmas, Lemma 6.8 and Lemma 6.9 imply
\[
|\square T|^2 \lesssim |h|^2 + |A|^2 + |T|^2 + |B|^2 + |w|^2 + |y|^2, \\
|\square U|^2 \lesssim |h|^2 + |A|^2 + |B|^2 + |U|^2 + |T|^2 + |w|^2 + |y|^2 + |z|^2.
\]
Together Lemma 6.9 we have
\[
|\square X|^2_{g(t)} \lesssim |X|^2_{g(t)} + |Y|^2_{g(t)}, \\
|\partial_t Y|^2_{g(t)} \lesssim |X|^2_{g(t)} + |Y|^2_{g(t)} + |\nabla X|^2_{g(t)}.
\]
To apply Theorem 6.5 we need to check the boundedness of $X$, $\nabla X$ and $Y$ on $[0, T]$, which are however, followed from Lemma 6.1. Therefore, $X = Y \equiv 0$ on $[0, T]$. Thus, $(g(t), u(t)) = (\tilde{g}(t), \tilde{u}(t)) \equiv 0$ on $[0, T]$. 

\[\text{Theorem 6.5}\]
APPENDIX A. EVOLUTION EQUATIONS OF THE RICCI-HARMONIC FLOW

We review some basic evolution equations of the Ricci-harmonic flow. Consider a Ricci-harmonic flow

\[ \partial_t g(t) = -2 \text{Ric}_g(t) + 4 du(t) \otimes du(t), \quad \partial_t u(t) = \Delta_{g(t)} u(t) \]

on a smooth manifold \( M \). As before, we follow the convention in Section [I].

**Lemma A.1.** Under the flow \( (A.1) \), we have

\[
\square R_{ij} = -2R_{ik}R_{kj} + 2R_{pqj}R^{pq}_{ij} - 4R_{pqj} \nabla^p u \nabla^q u + 2 \Delta u \nabla_i \nabla_j u, \\
\square \partial_t u = -R_{ij} \nabla^i u, \\
\square \nabla_i \nabla_j u = 2R_{pqj} \nabla^p u_\nabla^q u - R_{ij} \nabla_j \nabla^p u - R_{ip} \nabla_i \nabla^p u - 4|\nabla u|^2 \nabla_i \nabla_j u, \\
\square R_{ijkl} = 2 \left( B_{ijkt} - B_{ijtk} - B_{ijlk} + B_{ijkl} \right) \\
\quad - \left( R_{jk}R_{pjkt} + R_{kp}R_{ijkt} + R_{kjt}R_{ijl} \right) \\
\quad + 4 \left( \nabla_i \nabla^p \nabla_j \nabla_k u - \nabla_i \nabla_k \nabla_j \nabla^p u \right),
\]

where \( B_{ijkt} := -g^{pr}g^{qs}R_{ipj}R_{ksq}, \) and

\[
\square R = 2|\text{Ric}|^2 + 4|\Delta u|^2 - 4|\nabla^2 u|^2 - 8\text{Ric}(\nabla u, \nabla u), \\
\square |\nabla u|^2 = -2|\nabla^2 u|^2 - 4|\nabla u|^4, \\
\square \left( R - 2|\nabla u|^2 \right) = 2 |\text{Ric} - 2\nabla u \otimes \nabla u|^2 + 4|\Delta u|^2, \\
\partial_t dV = - \left( R - 2|\nabla u|^2 \right) dV.
\]

**Proof.** See for example [42, 46, 47, 49, 51]. Note that in our notation for \( R_{ijk} \) defined by

\[
R_{ijk} = \partial_i \Gamma_{jk}^\ell - \partial_j \Gamma_{ik}^\ell + \Gamma_{ik}^p \Gamma_{jp}^\ell - \Gamma_{jk}^p \Gamma_{ip}^\ell,
\]

the tensors \( R_{ijk} \) is the minus of those used in [46]. \( \square \)

**Lemma A.2.** As \((1,3)\)-tensor, we have

\[
\partial_t \text{Rm} = g^{-1} \nabla^2 \text{Ric} + g^{-1} \text{Ric} \ast \text{Rm} + g^{-1} \nabla^2 u \ast \nabla^2 u + g^{-1} \text{Rm} \ast \nabla u \ast \nabla u,
\]

\[ (A.11) \]
\[ \partial_t R_{i j k}^\ell = \left[ \Delta R_{i j k}^\ell + g^{p q} \left( R_{i p}^\ell R_{q j k}^\ell - 2 R_{i j p k}^\ell R_{q r}^\ell + 2 R_{p j i}^\ell R_{q k}^\ell \right) - g^{p q} \left( R_{i p k}^\ell R_{q j r}^\ell \right) \right] \]

(A.12)

\[ \partial_t \nabla_a R_{i j k}^\ell = \left[ \Delta \nabla_a R_{i j k}^\ell + g^{p q} \nabla_a \left( R_{i j p}^\ell R_{q k}^\ell - 2 R_{i p j}^\ell R_{q k}^\ell + 2 R_{i j}^\ell R_{p k}^\ell \right) \right] - g^{p q} \left( R_{i p}^\ell \nabla_a R_{q j k}^\ell - R_{i j p}^\ell \nabla_a R_{q k}^\ell - R_{i p j}^\ell \nabla_a R_{q k}^\ell \right) + g^{p q} R_{p q} R_{a i j k}^\ell \]

\[ - 4 g^{p q} \nabla_a R_{i j p}^\ell \nabla_a \nabla_q u - 4 g^{p q} R_{i j p}^\ell \nabla_a \nabla_q u - 4 g^{p q} R_{i j p}^\ell \nabla_a \nabla_q u - 4 g^{p q} R_{i j p}^\ell \nabla_a \nabla_q u \]

\[ + 4 g^{p q} \nabla_a \nabla_p u \nabla_q k \nabla_i u + 4 g^{p q} \nabla_i \nabla_p u \nabla_a \nabla_k \nabla_j u \]

\[ - 4 g^{p q} \nabla_a \nabla_k u \nabla_i \nabla_p u - 4 g^{p q} \nabla_i \nabla_k u \nabla_p u \]

\[ + 4 g^{p q} R_{i j p}^\ell \nabla_a \nabla_b u \nabla_k u - 4 g^{p q} \nabla_i \nabla_a \nabla_k u \nabla_j u \]

\[ - 4 g^{p q} \nabla_a \nabla_i u \nabla_k u \nabla_j u - 4 g^{p q} \nabla_i \nabla_k u \nabla_j u \]

(A.13)

We also need the existence result for the Ricci-harmonic flow.

**Theorem A.3. (List, 2005)** Let \((M, g_0)\) be a smooth complete \(n\)-dimensional Riemannian manifold with bounded curvature \(|\text{Rm}_{g_0}| \leq K_0\). Consider a smooth function \(u_0\) on \(M\) satisfying

\[ |u_0|^2_{g_0} + |\nabla_{g_0} u_0|^2_{g_0} \leq C_0, \quad |\nabla^2_{g_0} u_0|^2_{g_0} \leq C_1. \]

Here \(K_0, C_0, C_1\) are some positive constants. Then there exists a positive constant \(T := T(n, K_0, C_0)\) such that the initial value problem (A.1) with \((g(0), u(0)) = (g_0, u_0)\) has a smooth solution \((\hat{g}(t), u(t))\) on \(M \times [0, T]\). Moreover, the solution satisfies

\[ \frac{1}{C_{g_0}} \leq g(t) \leq C_{g_0}, \quad t \in [0, T], \]

for some constant \(C := C(n, K_0, C_0, C_1)\), and on \(M \times [0, T]\) there is a bound

\[ |\text{Rm}_{g(t)}|_{g(t)}^2 + |u(t)|^2_g + |du(t)|^2_{g(t)} + |\nabla^2_{g(t)} u(t)|_{g(t)}^2 \leq C' \]

(A.16) for another constant \(C' := C'(n, K_0, C_0, C_1).\)

**Proof.** See [42, 43, 47, 49, 51].

**APPENDIX B. SOME ESTIMATES OF THE RICCI-HARMONIC FLOW**

In this section we assume that \((g(t), u(t))\) is a solution to (A.1) on \([0, T]\) where \(M\) is a complete \(n\)-dimensional smooth manifold. Consider a geodesic ball \(B_{g(T)}(x_0, R)\) centered at a fixed point \(x_0 \in M\) with radius \(R > 0\).

**Theorem B.1. (Interior estimates)** Under the above hypotheses, we have

(i) If the following estimate

\[ \sup_{B_{g(T)}(x_0, R)} |\text{Ric}_{g(t)}|_{g(t)} \leq \frac{C}{R^2} \]

(B.1)
Proof. See [46, 47]. \qed

**Theorem B.2.** Suppose that \((g(t), u(t))\) is a solution to (A.1) on \(M \times [0, T]\), where \(M\) is a complete \(n\)-dimensional smooth manifold and \(T \in (0, \infty)\).

(i) If the following estimate

\[(B.6) \quad \sup_{M \times [0, T]} \| \text{Ric}_{g(t)} \|_{g(t)} \leq K\]

holds for some positive constant \(K\), then

\[(B.7) \quad \| \nabla_{g(t)} u(t) \|_{g(t)}^2 \leq 2KC_n\]

on \(M \times [0, T]\), for some positive constant \(C_n\) depending only on \(n\).

(ii) If the following estimate

\[(B.8) \quad \sup_{M \times [0, T]} \| \text{Rm}_{g(t)} \|_{g(t)} \leq K\]

holds for some positive constant \(K\), then

\[(B.9) \quad \| \nabla_{g(t)}^m \text{Rm}_{g(t)} \|_{g(t)}^2 + \| \nabla_{g(t)}^{m+2} u(t) \|_{g(t)}^2 \leq C_{n,m}(4K)^{1 + \frac{m}{2}}, \quad m \geq 0,\]

on \(M \times [0, T]\), for some positive constant \(C_{n,m}\) depending only on \(n\) and \(m\).

(iii) If \((g(t), u(t))\) is constructed in Theorem A.3 with the initial data \((g_0, u_0)\) satisfying the condition (A.14), then

\[(B.10) \quad \| \nabla_{g(t)}^m \text{Rm}_{g(t)} \|_{g(t)}^2 + \| \nabla_{g(t)}^{m+2} u(t) \|_{g(t)}^2 \leq C'_{n,m}, \quad m \geq 0,\]

on \(M \times [0, T]\), for some positive constant \(C'_{n,m}\) depending only on \(n\), \(m\), \(K_0, C_0, C_1\).
Proof. Given a space-time point \((x_0, t) \in M \times (0, T]\) and consider the geodesic ball \(B_{g(t)}(x_0, \sqrt{t}/2)\). Since

\[
\sup_{B_{g(t)}(x_0, \sqrt{t})} (\sqrt{t})^2 |\text{Ric}_{g(t)}|_{g(t)} \leq (\sqrt{t})^2 K
\]

by \((\text{B.1})\) and \((\text{B.6})\), it follows that

\[
\sup_{B_{g(t)}(x_0, \sqrt{t}/2)} |\nabla g(t)u(t)|^2_{g(t)} \leq (\sqrt{t})^2 K C_n \left( \frac{1}{(\sqrt{t})^2} + \frac{1}{t} \right) = 2KC_n.
\]

In particular, \(|\nabla g(t)u(t)|^2_{g(t)}(x_0) \leq 2KC_n\).

For (ii), consider the same geodesic ball \(B_{g(t)}(x_0, \sqrt{t}/2)\) and apply \((\text{B.4})\). The part follows from \textbf{Theorem A.3} and the second one.

APPENDIX C. EVOLUTION EQUATIONS OF THE RICCI-HARMONIC FLOW

Consider the \((\alpha_1, 0, \beta_1, \beta_2)\)-Ricci flow:

\begin{align}
\partial_t g(t) &= -2\text{Ric}_{g(t)} + 2\alpha_1 \nabla g(t)u(t) \otimes \nabla g(t)u(t), \\
\partial_t u(t) &= \Delta g(t)u(t) + \beta_1 |\nabla g(t)u(t)|^2_{g(t)} + \beta_2 u(t)
\end{align}

on a smooth manifold \(M\), where \(\alpha_1, \beta_1, \beta_2\) are given constants.

**Lemma C.1.** Under the flow \((\text{C.1}) - (\text{C.2})\), we have

\[
\square R_{ij} = -2R_{jk} R^k_{\ j} + 2R_{pigraph} R^{pq} - 2\alpha_1 R_{pigraph} \nabla^{\ p} u \nabla^{\ q} u
\]

\[
\square R_{ijkl} = 2(B_{ijkl} - B_{ijlk} - B_{jkil} + B_{jilk}) - (R_{\ k}^{\ \ n} R_{ipnq} + R_{\ i}^{\ \ n} R_{pknq}) + 2\alpha_1 (\nabla_i \nabla_j u \nabla_k u - \nabla_i \nabla_k u \nabla_j u),
\]

\[
\square |\nabla u|^2 = 2|\text{Ric}|^2 + 2\alpha_1 |\nabla u|^2 - 2\alpha_1 |\nabla^2 u|^2 - 4\alpha_1 (\text{Ric} \cdot \nabla u \otimes \nabla u),
\]

\[
\square \nabla_i \nabla_j u = 2R_{pigraph} \nabla^{\ p} \nabla^{\ q} u + \beta_2 \nabla_i \nabla_j u - R_{\ ip} \nabla^{\ p} \nabla_j u - R_{\ jp} \nabla^{\ p} \nabla_i u
\]

\[
+ 2\beta_1 \nabla_i \nabla^k u \nabla_k \nabla_j u + 2\beta_1 R_{pigraph} \nabla^p \nabla^q u \nabla^{\ p} \nabla^{\ q} u,
\]

\[
\square (\nabla_i u \nabla_j u) = -\nabla^k u (R_{jk} \nabla_i u + R_{ik} \nabla_j u) - 2\nabla_i \nabla^k u \nabla_j \nabla_k u + 2\nabla_i \nabla_j u \nabla_k u
\]

\[
+ 2\beta_1 \nabla^k u (\nabla_i u \nabla_j \nabla_k u + \nabla_j u \nabla_i \nabla_k u),
\]

where \(B_{ijkl} := \sqrt{g^{\ mn}} R_{ijam} R_{klns} \).

**Proof.** See \([\text{B.2}]\). \(\square\)

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