Rastall gravity extension of the standard ΛCDM model: Theoretical features and observational constraints

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We present a detailed investigation of the Rastall gravity extension of the standard—six-parameter base—ΛCDM model. We review the model for two simultaneous modifications of different nature in the Friedmann equation due to the Rastall gravity: the new contributions of the material (actual) sources (considered as effective source) and the altered evolution of the material sources. We discuss the role/behavior of these modifications with regard to some low redshift tensions, including the $H_0$ tension, prevailing within the standard ΛCDM. We constrain the model at the level of linear perturbations, and obtain the first constraints through a robust and accurate analysis using the latest full Planck cosmic microwave background (CMB) data, with and without including baryon acoustic oscillations (BAO) data. We find that the Rastall parameter $\epsilon$ (null for general relativity) is consistent with zero at 68% CL (with a tendency towards positive values, $-0.0001 < \epsilon < 0.0007$ (CMB+BAO) at 68% CL), which in turn implies no significant statistical evidence for deviation from general relativity, and also a precision of $O(10^{-4})$ for the coefficient $-1/2$ of the term $g_{\mu\nu}R$ in the Einstein field equations of general relativity (guaranteeing the local energy-momentum conservation). We explore the consequences led by the Rastall gravity on the cosmological parameters in the light of the observational analyses. It turns out that the effective source, with a present-day density parameter $\Omega_{\phi0} = -0.0010 \pm 0.0013$ (CMB+BAO, 68% CL), dynamically screens the usual vacuum energy at high redshift, but this mechanism barely works due to the opposition by the altered evolution of cold dark matter. Consequently, two simultaneous modifications of different nature in the Friedmann equation by the Rastall gravity act against each other, and do not help to considerably relax the so called low redshift tensions. Our results may offer a guide for the research community that studies the Rastall gravity in various aspects of gravitation and cosmology.

I. INTRODUCTION

The most successful description of the dynamics and the large-scale structure of the Universe via excellent agreement with a wide variety of the currently available data [1–5], in the literature so far, is known to be presented by the standard (base) Lambda cold dark matter (ΛCDM) model that relies on the inflationary paradigm [6–9]. Despite its great success, in addition to the notoriously challenging theoretical issues related to Λ (or vacuum energy) [10–13], it has recently begun to suffer from some persistent tensions of various degrees of significance between some existing data sets (see, e.g., [14–18] for further reading). Such tensions have received immense attention as these could be signaling new physics beyond the well established fundamental theories of physics underpinning the standard ΛCDM model. For example, the value of the Hubble constant $H_0$ predicted by the cosmic microwave background (CMB) Planck data [21–25] within the framework of the standard ΛCDM model is in serious disagreement with direct model-independent local measurements [19–22]. This tension becomes even more compelling as it worsens (relieves only partially) when the Λ is replaced by the simplest minimally coupled single-field quintessence (phantom or quintom models), see [23–26] and [27] for further references. Surprisingly, it has been reported that the $H_0$ tension—as well as a number of other persistent low-redshift tensions—may be alleviated by a dynamical dark energy whose density can assume negative values or vanish at high redshift [27–40]. The fact that the full CMB Planck data favor positive spatial curvature—which imitates a negative energy density source with an equation of state parameter equal to $-1/3$—on top of the standard ΛCDM model in contrast to the inflationary paradigm might be signaling a need for such dark energy sources [5] (see also [41–43]).

Dark energy that assumes negative energy values at large redshift came to the agenda when it turned out that, within the standard ΛCDM model, the Ly-α forest measurement of the baryon acoustic oscillations (BAO) by the BOSS collaboration prefers a smaller value of the dust density parameter compared to the value preferred by the CMB data [28]. They then reported a clear detection of dark energy consistent with $\Lambda > 0$ for $z < 1$, but with a preference for negative energy density values for $z > 1.6$, and argued that this Ly-α data from $z \approx 2.34$ can be described by a non-monotonic evolution of $H(z)$—i.e., of the total energy density of the Universe within general relativity (GR)—, which is difficult...
to achieve in any model with non-negative dark energy density [29]. The Planck collaboration excludes the Ly-α data from their default BAO compilation as it persistently remains in large tension with the standard ΛCDM model [5]. They argue, in line with [29], that it is difficult to construct well-motivated extensions to the standard ΛCDM model that can resolve the tension with the Ly-α data, and suggest further work to assess whether this is a statistical fluctuation caused by small systematic errors, or is a signature of new physics. Of course, an actual (physical) dark energy source with a negative density would be physically ill, which might be pointing to the necessity of considering modified theories of gravity (see [44–50] for reviews on DE and modified theories of gravity), from which an effective dark energy source with desired features could be defined. In line with all these, a study [51, 52] for observational analyses when the Brans-Dicke theory or its extensions are considered. There is a large range of examples, related to these three possibilities, that exist in various models in which Λ relaxes from a large initial value via an adjustment mechanism [57, 58]. (ii) In models in which Λ itself spontaneously switches sign [58–60]. (iii) In cosmological models based on Gauss-Bonnet gravity [60], (iv) in braneworld models [61, 62]. (v) In loop quantum cosmology [63, 64], (vi) in higher-dimensional cosmologies that accommodate dynamical reduction of the internal space [65, 68], (vii) in generalisations of the form of the matter Lagrangian in a non-linear way [69, 71], (viii) in some constructions within the unimodular gravity violating the local energy-momentum conservation law [11].

In this paper, we carry out a detailed theoretical and observational investigation of the Rastall gravity [72, 73], which presents a simple mathematical generalization of GR leading to physically rich features that could be related to the cosmological points discussed above. Although it presents a simple generalization of GR (derived from Einstein-Hilbert action) at the level of the field equations, there is no consensus on that it could be derived from an action of a well established fundamental theory, but some attempts in this direction have been made. It was shown in [74] that its field equations can be derived from a variational principle in a Weyl-Cartan theory of gravity, in which the metricity condition for the connection is dropped and the torsion is allowed. Some Lagrangian formulations have been proposed within the framework of $f(R, T)$ [75, 76] and $f(R, \mathcal{L}_m)$ gravities [77]. There are also criticisms claiming that the Rastall gravity is devoid of any difference from GR, and corresponds to a trivial re-arrangement of the matter sector in GR [74, 78].

The Rastall gravity, when considered at the level of the field equations, is indeed a simple mathematical generalization of the standard Einstein field equations of GR adding the term $\epsilon g_{\mu\nu} R$ to the field equations with an arbitrary coefficient: $R_{\mu\nu} + \alpha g_{\mu\nu} R = T_{\mu\nu}$ with $\alpha$ being a real constant. The particular case $\alpha = \frac{1}{2}$ leading to the Einstein tensor $G_{\mu\nu}$ of GR is unique as it, through the twice contracted Bianchi identities, yields $\nabla^\mu G_{\mu\nu} = 0$, and therefore guarantees the local conservation of the energy-momentum tensor (EMT) of the total matter content, i.e., $\nabla^\mu T_{\mu\nu} = 0$. Therefore, any deviation from $\alpha = \frac{1}{2}$ (GR) will lead to two simultaneous modifications of different nature in the standard Einstein field equations: (i) The new term $\epsilon g_{\mu\nu} R$ (with $\epsilon = \alpha - \frac{1}{2}$) of the form of the usual vacuum energy of quantum field theory ($T_{\mu\nu} = g_{\mu\nu} \Lambda$) appears in the spacetime geometry side of the Einstein field equations of GR. (ii) The evolution of the energy density of an actual material source gets altered from its usual one in GR, in a certain way differing from the non-conservation of the EMT described by $\nabla^\mu T_{\mu\nu} = -\epsilon \nabla_\nu R$. These two simultaneous features tempted us to carefully study the extension of the standard ΛCDM model replacing the gravity theory from GR to Rastall gravity due to the following reasons: The new term $\epsilon g_{\mu\nu} R$ could dynamically screen usual vacuum energy at high redshift for a certain range of $\epsilon$ as suggested, e.g., in [29]. In addition, this extension could also modify inverse proportionality of the dust (e.g., the CDM) energy-momentum to the comoving volume scale factor. It is not clear whether these two simultaneous modifications in the Friedmann equation will support or act against each other, and then whether these together could address the low redshift tensions.

The Rastall gravity has been attracting a lot of attention by the communities in the field of gravitation and cosmology in the recent years. See, for instance, [79] for black hole solutions, [80] for gravitational collapse, [81] for thermodynamic analysis, and [82] for some cosmological applications. It has been suggested in [83] that, when the Renyi entropy of non-extensive systems is attributed to the horizon of flat Friedman RobertsonWalker Universe in Rastall gravity, the late time acceleration can be generated from the non-conservation of dust. It was argued in [84] that Rastall theory provides a proper platform for generalizing the unimodular gravity (the trace-free Einstein gravity) [85, 87], wherein the
usual vacuum energy does not gravitate but the cosmological constant arises as an integration constant. The authors of Ref. [83] impose the Rastall gravity contributions from the non-conservation of dust upon dark energy which makes it clustering. Then, it is suggested in [83,89] that this model resembles the standard ΛCDM model at the background level (with an ε not strictly constrained, provided that the dark energy yields an equation of state parameter very close to minus unity), while in [90] that the evolution of the growth index displays a significant deviation from that in the standard ΛCDM model. One of the main motivations behind such attempts is that observational signatures of non-conservation in the dark sector is expected in the non-linear regime on intermediate or small scales, and is not inconsistent with the currently available cosmological data (see [91] for details.). In fact, just upon the proposal of the Rastall gravity, the Rastall parameter has been quite tightly constrained from local physics, relying basically on the non-conservation property of the model, to be |ε| \lesssim 10^{-15}, which suggests that the Rastall gravity deviates from GR only negligibly and thereby it is rather unattractive. Using realistic equations of state for the neutron star interior, an astrophysical constraint is placed on the Rastall gravity suggests that it is well consistent with GR as |ε| \lesssim 10^{-2} [92]. In contrast, the study [93] on much larger scales, using 118 galaxy-galaxy strong gravitational lensing systems, reports the constraint ε = −0.163 ± 0.001 (68% CL) excluding GR.

Here, we study the robust and accurate observational constraints (at the level of linear perturbations of the background using the latest high precision cosmological data) on the Rastall gravity extension of the standard ΛCDM model. Such constraints would not only be important to see whether the Rastall gravity is a good candidate for studying the cosmological tensions discussed above or not, but also, as being robust and accurate, may provide a guide for the research community that studies the Rastall gravity in various aspects of gravitation and cosmology. In Section II, we construct the Rastall gravity extension of the base ΛCDM model at the background level. In Section III, we carry out a preliminary investigation of the model which provides a guide to its working and parameters. In Section IV, we derive the linear perturbation equations. In Section V, we present the observational constraints on the model parameters using the latest full Planck CMB data, with and without including BAO data, in comparison to the standard ΛCDM model while we present a statistical comparison of the fit via Bayesian evidence in Section VI. We conclude the findings of our study in Section VII.

II. RASTALL GRAVITY EXTENSION OF THE BASE ΛCDM MODEL

The Rastall gravity offers a simple generalization of the standard Einstein field equations of GR by relaxing the contribution of the term g_{μν}R to the field equations and leads to the following modified Einstein field equations:

\[ R_{μν} - \left( \frac{1}{2} + \epsilon \right) g_{μν}R = \kappa T_{μν}, \]  

where \( \kappa \) is Newton’s constant scaled by a factor of 8\pi and we henceforth set \( \epsilon = 1 \) and units are used such that \( c = 1 \), \( R_{μν} \) is the Ricci tensor, \( R \) is the curvature scalar, \( g_{μν} \) is the metric tensor, and \( T_{μν} \) is the EMT described the material content. The real constant \( \epsilon \) is the Rastall parameter that measures the deviation from GR (\( \epsilon = 0 \)).

This modification in the spacetime geometry side (l.h.s.) of the Einstein field equations of GR corresponds to two simultaneous modifications of different nature in the material content side (r.h.s.): (i) Firstly, this modification is mathematically equivalent to adding, in a certain way, new contributions of the actual material sources to the right hand side of the standard Einstein field equations, which then can be interpreted as an effective source accompanying to the actual material sources considered in the model. For, we can rewrite \[1\] in a mathematically equivalent way as follows:

\[ R_{μν} - \frac{1}{2} g_{μν}R = T_{μν} + \hat{T}_{μν}, \]

where

\[ \hat{T}_{μν} = -\frac{1}{1 + 4\epsilon} g_{μν}T \]

is the EMT describing the effective source that arises from the actual material source. Here, we have made use of the relation \( T \equiv g^{μν}T_{μν} = -(1 + 4\epsilon)R \) between the trace of the EMT of the actual material source and the curvature scalar, obtained by contracting \[1\] with the inverse metric tensor \( g^{μν} \). (ii) Secondly, this modification leads, in general, to a violation of the local conservation of the EMT of an actual material source (therefore, that of the effective source as well), as its divergence is not necessarily null, viz.,

\[ \nabla^{μ}T_{μν} = -\epsilon \nabla_ν R = \frac{ε}{1 + 4ε} \nabla_ν T. \]

It implies that the evolution of the energy density of an actual material source is, in general, modified compared to its usual evolution in general relativistic models. The reason is that only the covariant derivative of the Einstein tensor \( (G_{μν} \equiv R_{μν} - \frac{1}{2}g_{μν}R) \) part of the Rastall gravity is guaranteed to be null \( \nabla^μG_{μν} = 0 \) through the twice contracted Bianchi identities.

In this work, we study the Rastall gravity extension of the standard (the six-parameter base) ΛCDM model parameterized by only one additional degree of freedom, the Rastall parameter \( \epsilon \), while keeping all the constituents (e.g., the physical ingredients of the Universe, the laws of the local physics) of the standard model as usual. Accordingly, we consider the spatially maximally symmetric and flat Friedmann-Robertson-Walker (FRW) spacetime

\[ ds^2 = -dt^2 + a^2(t) d\bar{x}^2, \]
where the scale factor $a(t)$ is function of proper time $t$ only. For describing the standard material content of the Universe, as usual, we consider the perfect fluid EMTs:

$$T^{\mu\nu}_i = (\rho_i + p_i)u^\mu u^\nu + p_i g^{\mu\nu},$$

where the index $i$ runs over the different actual sources described by the EoS of the form $p_i/\rho_i = w_i = \text{const.}$ (with $\rho_i$ and $p_i$ being the energy density and the pressure of the $i^{th}$ fluid, respectively), $u^\mu$ is the four-velocity of the medium satisfying $u^\mu u^\mu = -1$ and $\nabla_{\nu}u^\mu u^\nu = 0$.

We proceed with writing the modified Friedmann equations in a proper manner, namely, in a manner clearly identifying and handling the two simultaneous modifications of different nature in the material content side (r.h.s.) of the Einstein field equations of GR due to the Rastall gravity. We first note that the EMT describing the effective source is always of the form that of the usual vacuum energy-like effective source accompanying to the actual sources and are not independent functions but are fully determined by the actual pressure of the usual vacuum energy-like effective source.

The Einstein field equations of the model under consideration can explicitly be written as a set of two linearly independent differential equations with the unknown functions $H$ and $\rho_i$ as follows:

$$3H^2 = \sum_i \rho_i + \rho_X,$$  \hspace{1cm} (9)

$$-3H^2 - 2\dot{H} = \sum_i w_i \rho_i + \rho_X,$$  \hspace{1cm} (10)

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and the overdot denotes the derivative with respect to the cosmic time $t$. Note that $\rho_X = \sum_i \hat{\rho}_i$ and $\rho_X = \sum_i \hat{\rho}_i$ (satisfying $\rho_X = -\rho_P$) stand for the total energy density and pressure of the usual vacuum energy-like effective source accompanying to the actual sources and are not independent functions but are fully determined by the actual sources through the relation given in (9). The dynamics of the actual sources—and hence the dynamics of the effective source as well—can directly be obtained from the continuity equation (4), which explicitly is

$$\sum_i [\hat{\rho}_i + 3H(1 + w_i)\rho_i] = \frac{\epsilon}{1 + 4\epsilon} \sum_i \hat{\rho}_i (1 - 3w_i).$$  \hspace{1cm} (11)

It is reasonable to suppose that the fluids (actual sources) on cosmological scales are minimally interacting, i.e., interacting only gravitationally, which implies the separation of (11) for each type of fluid. In this case, we find the following redshift ($z = -1 + 1/a$) dependency for the background evolution of the energy density of the $i^{th}$ fluid (actual source):

$$\rho_i = \rho_i(0) (1 + z)^3(w_i + 1)[1 + \frac{1 - 3w_i}{1 + 3w_i + w_{vac}}].$$  \hspace{1cm} (12)

Here and henceforth a subscript 0 denotes the present-day ($z = 0$) value of any quantity. We see that, except the cases $w_i = -1$ and $\frac{1}{3}$, the redshift dependence of the energy density of an actual source is modified with respect to its standard dependence $\rho_i = \rho_i(0) (1 + z)^3(1 + w_i)$. Thus, we have the usual relations $\rho_{vac} = \rho_{vac0} = \text{const.}$, for the usual vacuum energy ($w_{vac} = -1$) and $\rho_r = \rho_r(0) (1 + z)\frac{4}{3}$ for radiation ($w_r = \frac{1}{3}$), while a modified redshift dependence $\rho_m = \rho_m(0) (1 + z)^{3+\frac{4}{3+\epsilon}}$ for dust ($w_m = 0$).

Next, considering (5) and (12), it turns out that despite the fact that the effective source resembles the usual vacuum energy with an EoS parameter $\epsilon_{\rho_X} = w_X = -1$, its energy density is not a constant, but

$$\rho_X = \frac{\epsilon}{1 + 4\epsilon} \sum_i (3w_i - 1)\rho_i(0) (1 + z)^3(1 + \frac{1 - 3w_i}{1 + 3w_i + w_{vac}}).$$

This obviously results from the non-conservation of the EMT of the actual material sources, see (4). Consequently, we have

$$\rho_X = \hat{\rho}_{vac} + \hat{\rho}_i + \hat{\rho}_m,$$  \hspace{1cm} (13)

where

$$\hat{\rho}_{vac} = -\frac{4\epsilon}{1 + 4\epsilon} \rho_{vac0}, \quad \hat{\rho}_i = 0 \quad \text{and} \quad \hat{\rho}_m = -\frac{\epsilon}{1 + 4\epsilon} \rho_{m0}(1 + z)^{3+\frac{4}{3+\epsilon}}.$$  \hspace{1cm} (14)

Note that $\hat{\rho}_i = 0$ due to the fact that radiation is traceless ($T = 0$). Therefore, it does not contribute to the effective source, i.e., to $\rho_X$, as like it has been preserving its usual evolution $\rho_i \propto (1 + z)^4$. The usual vacuum energy ($w_{vac} = -1$) of the QFT, on the other hand, still contributes to the Friedmann equation like a cosmological constant, albeit with a rescaled value as $\rho_{\Lambda} = \rho_{vac} + \rho_{vac0} = \frac{1}{1 + 4\epsilon} \rho_{vac0}$. According to this, for a positive vacuum energy, $\rho_{vac} > 0$, to contribute to the Friedmann equation like a positive cosmological constant, $\rho_{\Lambda} > 0$, there is a condition $\epsilon > -\frac{1}{4}$. In what follows, we stick to this condition considering the fact that the recent observations provide a clear detection of dark energy consistent with $\rho_{\Lambda} > 0$ in the vicinity of present-day Universe, viz., for $z < 1$ (see, for instance, (29)). The further condition $\epsilon > 0$ leads to $\rho_{\Lambda} > 0$ (viz., $\rho_{vac} < 0$ and $\rho_m < 0$). Namely, under this condition, we have $\rho_X(z = 0) = -\frac{\epsilon}{1 + 4\epsilon} (4\rho_{vac} + \rho_{m0}) < 0$, and $\rho_X$ continuously growing in larger negative values in the past. Thus, in this case ($\epsilon > 0$), the effective source $\rho_X$ dynamically screens the vacuum energy $\rho_{vac}$ in the finite past, and in particular, the complete screening (viz., when $\rho_{vac} + \rho_X = 0$) takes place at the redshift:

$$z_* = -\left(\frac{4\rho_{vac0}}{\rho_{m0}}\right)^{\frac{1+\epsilon}{3+\epsilon}} - 1.$$  \hspace{1cm} (16)
This situation achieved for $\epsilon > 0$ is of particular interest and tempting for its further theoretical and observational investigation, as it has recently been reported in [27–40] that a number of persistent low-redshift tensions, including the $H_0$ tension, may be alleviated by a dynamical dark energy that assumes negative energy density values (or in cosmological models wherein the cosmological constant is dynamically screened) at finite redshift.

Finally, the modified Friedmann equation [9] for the Rastall gravity extension of the standard $\Lambda$CDM (Rastall-$\Lambda$CDM) reads

$$3H^2 = \rho_{\text{vac}0} + \rho_{m0} (1 + z)^{3+\frac{2}{\epsilon}} + \rho_0 (1 + z)^4 + \rho_X, \quad (17)$$

where

$$\rho_X = -\frac{\epsilon}{1+4\epsilon} \left[ 4\rho_{\text{vac}0} + \rho_{m0} (1 + z)^{3+\frac{2}{\epsilon}} \right]. \quad (18)$$

This can be rewritten in terms of density parameters, $\Omega_0 = \frac{\rho_0}{3H_0^2}$, as follows:

$$\frac{H^2}{H_0^2} = \Omega_{\text{vac}0} + \Omega_{m0} (1 + z)^{3+\frac{2}{\epsilon}} + \Omega_0 (1 + z)^4$$
$$+ \Omega_{X0} \frac{4\rho_{\text{vac}0} + \rho_{m0} (1 + z)^{3+\frac{2}{\epsilon}}}{4\rho_{\text{vac}0} + \rho_{m0}}, \quad (19)$$

where

$$\Omega_{X0} = -\frac{\epsilon}{1+4\epsilon} (4\Omega_{\text{vac}0} + \Omega_{m0}), \quad (20)$$

and the consistency relation $\Omega_{\text{vac}0} + \Omega_{m0} + \Omega_0 + \Omega_{X0} = 1$ is satisfied.

III. A PRELIMINARY INVESTIGATION

In this section, we present a preliminary investigation for a demonstration of how the Rastall gravity extension of the standard $\Lambda$CDM model works, and a guide to the values of the parameters of the model. To do so, we first derive some useful parameters that we shall use to discuss some of the features/limitations of the model.

First of all, since the usual radiation evolution is not affected from the Rastall gravity extension, we can safely use the relevant standard equations. The photon energy density today $\rho_0$ is then still well constrained, relying on a simple relation: $\rho_\gamma = \frac{\pi^2}{15} T_0^4 \Omega_{\text{CMB}}$ with the CMB monopole temperature $\Omega_{\text{CMB}} [94]$, which is very precisely measured to be $T_{\text{CMB}} = 2.7255 \pm 0.0006$ K [95]. We suppose, in line with standard particle physics, three neutrino species ($N_{\text{eff}} = 3.046$) with minimum allowed mass $\sum m_\nu = 0.06$ eV. Then, the radiation density parameter can be given in standard way: $\Omega_\gamma = \Omega_\text{CMB} + \Omega_\text{e} = 4.18343 \times 10^{-5} h^{-2}$, where $h$ is the dimensionless reduced Hubble constant parametrizing the Hubble constant via $H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1} [94]$. Using a reasonable value for the Hubble constant, for instance, $H_0 = 68 \text{ km s}^{-1} \text{ Mpc}^{-1}$, we find that the density parameter of the radiation today is negligible, viz., $\Omega_\gamma = 9.0472 \times 10^{-5}$. Neglecting this small contribution of radiation today, we have $\Omega_{\text{vac}0} + \Omega_{m0} + \Omega_{X0} = 1$. Using this relation with $\Omega_{X0} = -\frac{\epsilon}{1+4\epsilon} (4\Omega_{\text{vac}0} + \Omega_{m0})$, we can write the present-day density parameter of the effective source in terms of $\Omega_{m0}$ and $\epsilon$ as follows:

$$\Omega_{X0} = -4\epsilon + 3\epsilon \Omega_{m0}. \quad (21)$$

From this, we read off $\frac{\rho_{\text{vac}0}}{\rho_{m0}} = -\epsilon \frac{4}{\Omega_{m0}} - 3$, while we have $\frac{\rho_{\text{vac}0}}{\rho_{m0}} \approx -\epsilon \frac{4}{\Omega_{m0}}$ for $z > 0$, say, in the early Universe, see (12) and (15). Using the relation (21) along with (16), the redshift at which the vacuum energy is completely screened by the effective source, reads as

$$z_* = \frac{1}{\epsilon} \left( \frac{1}{\Omega_{m0}} + \frac{4}{\Omega_{m0}} - \frac{1}{\epsilon} - 3 \right)^{\frac{1+3\epsilon}{1+6\epsilon}} - 1. \quad (22)$$

Since the evolution of radiation remains unaltered in the Rastall gravity, we basically do not expect any modification in the standard history of the Universe throughout the radiation epoch. Yet, the modified evolution of dust would have consequences on the transition from radiation to dust domination. The radiation-matter(dust) transition is one of the most important epochs in the history of the Universe, as it alters the growth rate of density perturbations: during the radiation epoch, which yields $H^2(z) \propto (1 + z)^4$, perturbations will be small because $H(z)$ is nearly frozen but once matter domination commences as $H(z)$ flattens to yield $H^2(z) \propto (1 + z)^3$ during the dust epoch, perturbations on all length scales are able to grow by gravitational instability and therefore it sets the maximum of the matter power spectrum. The modified matter-radiation equality ($\rho_r = \rho_m$) redshift reads

$$z_{eq} = \left( \frac{\Omega_{m0}}{\Omega_\gamma} \right)^{\frac{1+3\epsilon}{1+6\epsilon}} - 1, \quad (23)$$

where, for a given value of the ratio of the density parameters—of course, we suppose $\frac{\Omega_{m0}}{\Omega_\gamma} > 1$—positive (negative) $\epsilon$ values shift $z_{eq}$ to larger (smaller) values. This in turn shifts the turnover in the matter power spectrum via a highly sensitive parameter to the modifications to GR, namely, the wavenumber of a mode that enters the horizon at the radiation-matter transition:

$$k_{eq} = \frac{H_{eq}}{1 + z_{eq}} = H_0 \sqrt{\frac{2 + 7\epsilon}{1 + 4\epsilon} \Omega_{m0} \left( \frac{\Omega_{m0}}{\Omega_\gamma} \right)^{\frac{1+3\epsilon}{1+6\epsilon}}}, \quad (24)$$

where we have ignored $\Omega_{\text{vac}0}$ since its contribution is safely negligible.

Note that the condition $\epsilon > -\frac{1}{4}$ that we introduced for $\rho_\Lambda > 0$ in the previous section, ensures the real positive values of $k_{eq}$ and $H^2(z) \propto (1 + z)^3 \frac{2}{\epsilon} \epsilon$ during the dust era to be flatter than $H^2(z) \propto (1 + z)^4$ during the radiation era. These two parameters, $H_{eq}$ and $k_{eq}$, are not expected to deviate much from the ones obtained within the standard $\Lambda$CDM model. Therefore, these are very useful to give an opinion whether a cosmological model
is well behaved at high redshifts, for instance, with regard to the CMB data relevant to \( z \approx 1100 \).

We can now make use of the parameters derived here for a preliminary investigation of the model: The values of these parameters may be utilized for making estimations on \( \epsilon \) by manipulating the late time dynamics of the Universe in our model, for instance, to better describe the existing model independent \( H_0 \) data as well as the BAO data from \( z \lesssim 2.4 \). We, of course, must also check the price paid for this manipulation from the chosen \( \epsilon \) value in the dynamics of the earlier Universe, for instance, in the cosmological parameters physically related/sensitive to the presence/amount of radiation.

We proceed with a reasonable set of values: \( \Omega_{m0} = 0.31 \) and \( H_0 = 68 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \), acceptable with regard to both the recent Planck predictions and the model independent the tip of the red giant branch (TRGB) \( H_0 \) value, along with the choice of \( \epsilon = 0.06 \). We find, for the present-day Universe, a considerable amount of negative contribution of the effective source, \( \Omega_{X0} = -0.184 \) [see (21)], which is then compensated by the increased value of the usual vacuum energy, \( \Omega_{\rho_{vac}} = 0.874 \), so that to yield a cosmological constant-like contribution, \( \Omega_{\Lambda0} = 0.69 \), as in the standard \( \Lambda \text{CDM} \) (\( \epsilon = 0 \)). On the other hand, this enhanced amount of usual vacuum is completely screened by the effective source at \( z_e = 2.4 \) [see (22) and Figure 1], which is pretty close to the values suggested in (21), for relaxing a number of persistent low-redshift tensions, including the \( H_0 \) tension, that arise within the standard \( \Lambda \text{CDM} \) model. It is worth noting that, as may be seen in the same figure, for \( z_e > 2.4 \), the dust energy density assumes values larger than the total energy density of the Universe. Yet, the ratio of the energy density of the effective source to that of the dust is \( \frac{\rho_X}{\rho_{vac}} = -0.59 \) today but it settles in a value pretty close to zero, \( \frac{\rho_X}{\rho_{vac}} \approx -0.048 \), at high redshifts (\( z \gg 0 \)). This implies the impact of the effective source on the dynamics of the Universe diminishes (yet not completely) with the increasing redshift. However, the price (due to the modification in the EMT conservation) we paid for this tempting result (say, the screening of the vacuum energy) is that the dust energy density grows considerably faster than it does in the usual GR, \( \rho_m \propto (1 + z)^3 \) (which is also tracked by the effective source at high redshifts), whereas the radiation energy density always grows as usual, \( \rho_r \propto (1 + z)^4 \). This leads to unrealistic values for the key parameters relevant to the early Universe, namely, \( z_{eq} = 15303 \) and \( k_{eq} = 0.04470 \, \text{Mpc}^{-1} \), which are extremely different than the values \( z_{eq} = 3391 \) and \( k_{eq} = 0.01045 \, \text{Mpc}^{-1} \) obtained in the case of the standard \( \Lambda \text{CDM} \) model (\( \epsilon = 0 \)). This situation signals that the Rastall-\( \Lambda \text{CDM} \) model with \( \epsilon = 0.06 \), which is tempting as it leads to \( z_e = 2.4 \) in line with (29, 30, 38), is not well behaved at large redshifts. Thus, it is conceivable that the high redshift cosmological data would not allow such large positive values of \( \epsilon \).

Next, we proceed to have a closer look at the dynamics of the Universe by focusing on a narrow redshift range \( 0 \lesssim z \lesssim 3 \), within which we can, in a relatively straightforward way, compare the \( H(z) \) function of the Rastall-\( \Lambda \text{CDM} \) model with the model independent \( H_0 \) measurements, e.g., \( H_0 = 69.8 \pm 0.8 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \) from the TRGB \( H_0 \) (22), and the latest high precision BAO data: \( H(z = 0.57) = 97.9 \pm 3.4 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \) (96) and \( H(z = 2.34) = 222.4 \pm 5.0 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \) from the latest BAO data (28). To do so, along with these data points, we plot the function \( H(z)/(1 + z) \) versus \( z \) in Figure 2 considering four different pair of values of the parameters \( \epsilon \) and \( H_0 \) as mentioned in the legend of this figure, wherein we keep \( \Omega_{m0} = 0.31 \) in all the cases (two of which correspond to the cases in Figure 1). First, we notice that the Rastall-\( \Lambda \text{CDM} \) model (green curve, \( \epsilon = 0.06 \), \( H_0 = 68 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \), which leads to \( z_e = 2.4 \)) does better than the GR-\( \Lambda \text{CDM} \) (red curve, \( \epsilon = 0 \), \( H_0 = 68 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \) to
describe the lower-redshift BAO data \((z = 0.57)\), but the faster growth of \(H(z)/(1 + z)\) at high redshifts leads to an increased tension of the Rastall-ACDM model with the high-redshift BAO Ly-\(\alpha\) data \((z = 2.34)\) compared to the standard GR-ACDM model. This seems to suggest that negative values of \(\epsilon\) can help better representation of the high-redshift BAO Ly-\(\alpha\) data \((z = 2.34)\) in the Rastall-ACDM model, which of course implies compromise from the feature of screening usual vacuum energy by the effective source at a finite redshift. For, we see that the Rastall-ACDM model (blue solid curve, \(\epsilon = -0.06, H_0 = 68 \text{ km s}^{-1}\text{Mpc}^{-1}\)) better represents the high-redshift BAO Ly-\(\alpha\) data \((z = 2.34)\) due to slow growth of \(H(z)/(1 + z)\) at higher redshifts. But in this case, this model worsens in representing the lower-redshift BAO data \((z = 0.57)\) compared to the GR-ACDM model. However, surprisingly, if we use a larger \(H_0\) value as well, for instance, \(H_0 = 70 \text{ km s}^{-1}\text{Mpc}^{-1}\) very close to the model independent TRGB \(H_0\) measurement, it turns out that the Rastall-ACDM model (blue dashed curve, \(\epsilon = -0.06, H_0 = 70 \text{ km s}^{-1}\text{Mpc}^{-1}\)) reconciles with all the three data points simultaneously. This makes negative \(\epsilon\) values promising with regard to addressing the so called \(H_0\) tension on top of a good description of the both BAO data (the standard \(\Lambda\)CDM model has known to be suffering from). However, most likely, it would lead to an inconsistency with the CMB data, as the values of \(\epsilon\) leading to a significant improvement in this direction would give rise to unacceptable amount of shifts in the values of \(z_{eq}\) and \(k_{eq}\). Indeed, \(\epsilon = -0.06\), that we have used just to develop an opinion, leads to the unacceptable values \(z_{eq} = 790\) and \(k_{eq} \sim 0.025 \text{ Mpc}^{-1}\), which obviously signals spoiling of a successful description of the early Universe.

The lesson we learned in this section may be summarized as follows: Through this preliminary investigation, it is not possible to reach to a decisive conclusion whether the \(H_0\) and/or BAO data show tendency of \(\epsilon\) deviating from zero (GR) in a certain direction. Moreover, a significant improvement with regard to \(H_0\) and/or BAO data would most likely lead to spoiling of a successful description of the early Universe, which signals that CMB data would keep \(\epsilon\) values close to zero. Therefore, we expect only an insignificant deviation from the standard \(\Lambda\)CDM model when it is extended from GR via the Rastall gravity. A conclusive answer, of course, cannot be given unless we rigorously confront/constrain the model with the observational data.

IV. LINEAR PERTURBATIONS

In this section, we derive the general form of the equations which describe small cosmological perturbations within the Rastall gravity extension of the standard \(\Lambda\)CDM model. We consider the perturbed RW metric, \(g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}\), where \(g_{\mu\nu}^{(0)}\) indicates background of the spatially flat RW metric \((\text{5})\) with a small fluctuation, \(h_{\mu\nu}\), and we choose the synchronous gauge \((h_{\mu0} = 0)\). The line element has the form

\[
ds^2 = a(\eta)^2 \left[ -d\eta^2 + (\delta_{jk} + h_{jk}) dx^j dx^k \right],
\]

(25)

where \(x^j\), with \(j = (1, 2, 3)\), are the spatial components in Cartesian coordinates and \(\eta\) is the conformal time. The comoving coordinates are related to the proper time \(t\) and positions \(r\) by \(dx^i = d\eta = \frac{dr}{a(\eta)}, dx = \frac{dr}{a(\eta)}\). We introduce the perturbations as follows:

\[
\rho_i = \rho_i^{(0)} + \delta \rho_i, \quad p_i = p_i^{(0)} + \delta p_i, \quad u_i = u_i^{(0)} + \delta u_i,
\]

(26)

where the superscript \((0)\) indicates the background functions and \(\delta \rho_i, \delta u_i\) and \(\delta p_i\) are the perturbed quantities in energy density, four-velocity and pressure, respectively. We also introduce the following definitions:

\[
\delta_i \equiv \frac{\delta \rho_i}{\rho_i}, \quad c_{s,i}^2 = \frac{\delta p_i}{\delta \rho_i}, \quad \theta \equiv \delta_b \delta u_i, \quad h \equiv \frac{h k^i_k}{a},
\]

(27)

where \(c_{s,i}^2\) is an adiabatic sound speed squared.

Within the formalism, the continuity equation reads

\[
\sum_i \left[ \delta_i \left( \frac{1 + 3(1 + w_i)}{1 + 4 \epsilon} \right) + 3 \mathcal{H} (c_{s,i}^2 - w_i) \delta_i \right. + \left. (1 + w_i) \left( \theta_i + \frac{h^i_k}{2} \right) \right] = 0,
\]

(28)

from the perturbations of the actual EMT conservation equation [see (4)] for \(\nu = 0\). Here the prime denotes derivative with respect to \(\eta\) and \(\mathcal{H} = \frac{\dot{a}}{a}\). For \(\nu = i\), Euler equation reads:

\[
\sum_i \left[ \theta'_i + \frac{1 - 3w_i}{1 + 3c(1 + w_i)} \mathcal{H} \theta_i - \frac{k^j \delta_i}{(1 + 4 \epsilon)(1 + w_i)} \left[ \epsilon + (1 + \epsilon)c_{s,i}^2 \right] \right] = 0.
\]

(29)

The relativistic species remains unaltered in Rastall gravity both at the background and perturbative levels. Thus, the Boltzmann hierarchy for the relativistic relics follows the standard procedures as described in (97) (see also [98]). We suppose that the evolution of baryons \((w_b = c_{s,b}^2 = 0)\) is not altered by the Rastall gravity but that of the cold dark matter \((w_{\text{cdm}} = c_{s,\text{cdm}}^2 = 0)\), for which we modify the procedures given in (97) [98] accordingly [99]. And, the first order continuity and Euler equations from (28) and (29) read

\[
\delta'_{\text{cdm}} + \frac{1 + 3 \epsilon}{1 + 4 \epsilon} \mathcal{H} \theta_{\text{cdm}} + \frac{h'}{2} = 0,
\]

(30)

\[
\theta'_{\text{cdm}} + \frac{1}{1 + 3 \epsilon} \mathcal{H} \theta_{\text{cdm}} - \frac{\epsilon}{1 + 4 \epsilon} k^2 \delta_{\text{cdm}} = 0.
\]

(31)
V. OBSERVATIONAL CONSTRAINTS

Considering the background and perturbation dynamics presented above, in what follows, we explore the full parameter space of the Rastall-$\Lambda$CDM model—namely, the Rastall gravity extension of the six-parameter base $\Lambda$CDM based on the GR via the Rastall parameter $\epsilon$, and, for comparison, that of the standard $\Lambda$CDM (GR-$\Lambda$CDM) model. The baseline seven free parameters set of the Rastall-$\Lambda$CDM model is, therefore:

$$\mathcal{P} = \{\omega_b, \omega_{\text{cdm}}, \theta_s, A_s, n_s, \tau_{\text{reio}}, \epsilon\},$$

where the first six parameters are the baseline parameters of the standard $\Lambda$CDM model, namely: $\omega_b$ and $\omega_{\text{cdm}}$ are respectively the dimensionless densities of baryons and cold dark matter; $\theta_s$ is the ratio of the sound horizon to the angular diameter distance at decoupling; $A_s$ and $n_s$ are respectively the amplitude and spectral index of the primordial curvature perturbations, and $\tau_{\text{reio}}$ is the optical depth to reionization. The uniform priors used for the model parameters are $\omega_b \in [0.018, 0.024]$, $\omega_{\text{cdm}} \in [0.10, 0.14]$, $100 \theta_s \in [1.03, 1.05]$, $\ln(10^{10} A_s) \in [3.0, 3.18]$, $n_s \in [0.9, 1.1]$, $\tau_{\text{reio}} \in [0.04, 0.125]$, and $\epsilon \in [-0.04, 0.04]$.

In order to constrain the models, we use the latest Planck CMB and BAO data: We use the recently released full Planck-2018 [5] CMB temperature and polarization data which comprise the low-\ell temperature and polarization likelihood at $\ell \leq 29$, temperature (TT) at $\ell \geq 30$, polarization (EE) power spectra, and cross correlation of temperature and polarization (TE). The Planck-2018 CMB lensing power spectrum likelihood [104] is also included. Along with the Planck CMB data, we consider the measurements of BAO provided by the distribution of galaxies in galaxy-redshift surveys. We use BAO distance measurements probed by (i) Six Degree Field Galaxy Survey (6dFGS) at effective redshift $z_{\text{eff}} = 0.106$ [104], (ii) the Main Galaxy Sample of Data Release 7 of Sloan Digital Sky Survey (SDSS-MGS) at effective redshift $z_{\text{eff}} = 0.15$ [104], (iii) the LOWZ and CMASS galaxy samples of Data Release 11 (DR11) of the Baryon Oscillation Spectroscopic Survey (BOSS) LOWZ and BOSS-CMASS at effective redshifts $z_{\text{eff}} = 0.32$ and $z_{\text{eff}} = 0.57$, respectively [86], (iv) correlation of Lyman-\alpha forest absorption and quasars at $z_{\text{eff}} = 2.35$ obtained in SDSS DR14 [107]. Also, we use the measurement obtained in [108], where the BAO scale is measured at $z_{\text{eff}} = 2.34$.

We have implemented the model in publicly available CLASS [98] code, and used the Multinest [109] algorithm in the parameter inference Monte Python [110] code with uniform priors on the model parameters to obtain correlated Monte Carlo Markov Chain samples and Bayesian evidence. Further, we have used the GetDist Python package to analyze the samples. We obtain the observational constraints on all the Rastall-$\Lambda$CDM model parameters by using first only the CMB data and then the combined CMB+BAO data. For comparison purpose, we also show the constraints on the GR-$\Lambda$CDM model parameters. The CMB data set alone is known to well constrain the six baseline parameters of the GR-$\Lambda$CDM model. The Rastall-$\Lambda$CDM model under consideration carries an additional parameter, namely $\epsilon$. Therefore, we also combine the BAO data with CMB in order to obtain possibly tighter constraints on the Rastall-$\Lambda$CDM model parameters, and also to break any possible degeneracy of the new parameter $\epsilon$ with the other baseline parameters.

Table I displays the constraints, at 68% and 95% confidence levels (CLs), on the baseline seven free parameters and some derived parameters of the Rastall-$\Lambda$CDM model.

| Parameter | CMB | CMB + BAO |
|-----------|-----|-----------|
| $10^5\omega_b$ | $2.246 \pm 0.016 + 0.031$ | $2.239 \pm 0.014 + 0.028$ |
| $\omega_{\text{cdm}}$ | $0.135 \pm 0.0228 + 0.0055$ | $0.133 \pm 0.012 + 0.025$ |
| $10^{10} A_s$ | $1.0418 \pm 0.0034 + 0.0006$ | $1.0420 \pm 0.003 + 0.0005$ |
| $10^{-10} A_\ell$ | $0.057 \pm 0.006 + 0.0014$ | $0.059 \pm 0.007 + 0.0014$ |
| $\epsilon$ | $-0.0000 \pm 0.0004 + 0.0000$ | $0.0000 \pm 0.0004 + 0.0000$ |
| $\Omega_{\text{m0}}$ | $0.347 \pm 0.009 + 0.019$ | $0.349 \pm 0.008 + 0.019$ |
| $\Omega_{\text{de}}$ | $0.316 \pm 0.007 + 0.014$ | $0.306 \pm 0.006 + 0.011$ |
| $\Omega_{\text{x0}}$ | $0.650 \pm 0.029 + 0.052$ | $0.650 \pm 0.029 + 0.052$ |
| $\Omega_\text{x0}$ | $0.0000 \pm 0.0000 + 0.0000$ | $0.0000 \pm 0.0000 + 0.0000$ |
| $H_0$ | $65.10 \pm 1.80 \pm 1.50$ | $68.31 \pm 1.76 \pm 1.95$ |
| $s_8$ | $0.792 \pm 0.063 + 0.012$ | $0.812 \pm 0.063 + 0.012$ |

TABLE I. Constraints (68% and 95% CLs) on the free and confidence levels (CLs), on the baseline seven free parameters and some derived parameters of the GR-$\Lambda$CDM model.
FIG. 3. One-dimensional marginalized distributions of the free and some derived parameters of the Rastall-ΛCDM and GR-ΛCDM models.

model and, for comparison, on those of the GR-ΛCDM model, both from the CMB and combined CMB+BAO data sets. For these constraints presented for both models in the same table, Figure 3 shows the one-dimensional marginalized distributions and Figure 4 shows the two-dimensional (68% and 95% CLs) marginalized distributions of the derived parameters with regard to the baseline free parameters. From all these, we immediately notice that, as it is the case for the GR-ΛCDM model as well, the CMB+BAO data set puts tight constraints on the parameters of the Rastall-ΛCDM model, when compared to the constraints put by CMB data set alone. On the other hand, in contrast to the GR-ΛCDM model, when the BAO data set is not included, we notice larger error bounds (loose constraints) on some of the Rastall-ΛCDM model parameters, for instance, the error bounds of $\omega_{cdm}$ and the derived parameters are larger in Rastall-ΛCDM model compared to the GR-ΛCDM model, as a consequence of the deviation of the Rastall parameter $\epsilon$ from zero. It turns out that the parameter of our main concern, the Rastall parameter $\epsilon$ measuring the deviation from GR, is constrained as

\[
\epsilon = -0.0010 \pm 0.0008 \pm 0.0015 \quad \text{from CMB},
\]

\[
\epsilon = 0.0003 \pm 0.0004 \pm 0.0008 \quad \text{from CMB+BAO},
\]

at 68% and 95% CLs. The constraints from the CMB data as well as the combined CMB+BAO data set suggest that, in line with our conclusion reached via a preliminary investigation in Section III, the Rastall parameter $\epsilon$ is well consistent with zero at 95% CL, which in turn implies that there is no significant statistical evidence for deviation from GR via Rastall gravity. We note however that, as may be seen from the probability regions and mean values of $\epsilon$, the Rastall parameter prefers negative values in the case of CMB data while positive values when BAO data set is included (CMB+BAO data). Further, although they are mostly minor, the Rastall gravity extension of the standard ΛCDM has some consequences on the cosmological parameters, which deserve further discussion that could be informative about the features of the Rastall gravity and/or whether it is a promising modified gravity or not.

In Figure 5 we present the two-dimensional (68% and 95% CLs) marginalized distributions that show how the six of the baseline parameters of the GR-ΛCDM model are affected by the Rastall gravity extension, i.e., $\epsilon$. We observe that $\epsilon$ is negatively correlated with $\omega_{cdm}$ and $\omega_b$ for both CMB data and the combined CMB+BAO data sets. Other four parameters, $\theta_s$, $A_s$, $n_s$ and $\tau_{reio}$ seem
to have minor positive correlations with $\epsilon$ in the case of CMB data but no correlation in the case of CMB+BAO data. Accordingly, in Table I and Figure 3 one may notice the shifts in the mean values and one dimensional probability distributions of different parameters. Also, see Figure 4 for the consequences of the Rastall gravity extension, i.e., $\epsilon$, via the two-dimensional marginalized distributions of the derived parameters with regard to the baseline free parameters. The last column of the same figure is of particular interest as it displays the two-dimensional marginalized distributions of the derived parameters with regard to the constraints on the Rastall parameter $\epsilon$. We notice that smaller values of $\epsilon$ lead to larger values of the present-day density parameter of matter $\Omega_m$ because of the negative correlation between these two parameters. It is in line with (12) which suggests that matter energy density dilutes less efficiently with time in a Universe with negative values of $\epsilon$, and thereby leading to larger matter density parameter in the present-day Universe. Accordingly, in the case of CMB data, where $\epsilon$ has higher probability to lie in the negative range, we see higher values of $\Omega_m$ in Rastall-$\Lambda$CDM when compared to the GR-$\Lambda$CDM. Namely, the CMB data set predicts $\Omega_m = 0.347^{+0.024}_{-0.027}$ for the Rastall-$\Lambda$CDM model, while $\Omega_m = 0.316^{+0.007}_{-0.007}$ for the GR-$\Lambda$CDM. On the other hand, the combined CMB+BAO data set prefers larger probability region of $\epsilon$ in the positive range, and thereby it predicts smaller values of $\Omega_m$ in the Rastall-$\Lambda$CDM when compared to the GR-$\Lambda$CDM. For, the combined CMB+BAO data set predicts...
We notice a positive correlation between the present-day density parameter of the usual vacuum energy and the Rastall parameter at both the data sets. The CMB (CMB+BAO) data set favors smaller (larger) values of the density parameter of the usual vacuum energy density, viz., $\Omega_{\text{vac}} = 0.650^{+0.029}_{-0.026}$ ($\Omega_{\text{vac}} = 0.697^{+0.010}_{-0.010}$) for the Rastall-ΛCDM while $\Omega_{\text{vac}} = 0.684^{+0.007}_{-0.007}$ ($\Omega_{\text{vac}} = 0.692^{+0.006}_{-0.006}$) for the GR-ΛCDM. This reduced (enhanced) amount of the usual vacuum energy density parameter is however compensated just slightly by that of the effective source (which behaves like a cosmological constant at $z \sim 0$) as the CMB (CMB+BAO) data set favors its positive (negative) values owing to the almost perfect negative correlation between its present-day density parameter $\Omega_X$ and the Rastall parameter $\epsilon$. Namely, the constraint on total present-day density parameter of those of the usual vacuum energy and the effective source reads $\Omega_{\text{vac}} + \Omega_X = 0.653^{+0.027}_{-0.024}$ (68% CL) from the CMB data set; and $\Omega_{\text{vac}} + \Omega_X = 0.696^{+0.009}_{-0.009}$ (68% CL) from the combined CMB+BAO data set. Figure 6 displays the two-dimensional posterior distributions of $\{\Omega_m, \Omega_{\text{vac}}\}$ colour coded by $\Omega_X$ with CMB+BAO data. The line across the contours is given by $\Omega_m + \Omega_{\text{vac}} = 1$.

$\Omega_m = 0.304^{+0.009}_{-0.009}$ for the Rastall-ΛCDM model, while $\Omega_m = 0.308^{+0.006}_{-0.006}$ for the GR-ΛCDM model, see Table I. We notice a positive correlation between the present-day density parameter of the usual vacuum energy and the Rastall parameter for both the data sets. The CMB (CMB+BAO) data set favors smaller (larger) values of the density parameter of the usual vacuum energy density, viz., $\Omega_{\text{vac}} = 0.650^{+0.029}_{-0.026}$ ($\Omega_{\text{vac}} = 0.697^{+0.010}_{-0.010}$) for the Rastall-ΛCDM while $\Omega_{\text{vac}} = 0.684^{+0.007}_{-0.007}$ ($\Omega_{\text{vac}} = 0.692^{+0.006}_{-0.006}$) for the GR-ΛCDM. This reduced (enhanced) amount of the usual vacuum energy density parameter is however compensated just slightly by that of the effective source (which behaves like a cosmological constant at $z \sim 0$) as the CMB (CMB+BAO) data set favors its positive (negative) values owing to the almost perfect negative correlation between its present-day density parameter $\Omega_X$ and the Rastall parameter $\epsilon$. Namely, the constraint on total present-day density parameter of those of the usual vacuum energy and the effective source reads $\Omega_{\text{vac}} + \Omega_X = 0.653^{+0.027}_{-0.024}$ (68% CL) from the CMB data set; and $\Omega_{\text{vac}} + \Omega_X = 0.696^{+0.009}_{-0.009}$ (68% CL) from the combined CMB+BAO data set. Figure 6 displays the two-dimensional posterior distributions of $\{\Omega_m, \Omega_{\text{vac}}\}$ colour coded by $\Omega_X$, at 68% and 95% CLs for the combined CMB+BAO data set. Here, we notice that the posterior distribution sample points of $\Omega_X$ (and the corresponding contours for 68% and 95% CLs) pretty much cluster/lie around a line that deviates from the GR-ΛCDM line $\Omega_m + \Omega_{\text{vac}} = 1$, due to the presence of $\Omega_X$.

We note that while the CMB data set by alone favors positive energy density values for the effective source $\rho_X$ accompanying to the actual energy sources due to the Rastall gravity extension, the CMB+BAO data set including BAO data (relatively low-redshift data compared to the CMB data set) favors negative energy density values for it, namely, $\Omega_{X_0} = 0.0030^{+0.0023}_{-0.0023}$ (68% CL) from the CMB data set, and $\Omega_{X_0} = -0.0010^{+0.0013}_{-0.0013}$ (68% CL) from the combined CMB+BAO data set. This shows that, when the BAO data set is included, the effective source $\rho_X$ indeed screens the usual vacuum energy at finite redshift and that the Rastall gravity may be counted among the cosmological models that were suggested for alleviating a number of persistent low-redshift tensions, including the $H_0$ tension, by a dynamical dark energy that assumes negative energy density values (or by a mechanism dynamically screening the cosmological constant) at finite redshift. In Figure 7, in order to visualize the screening mechanism, we show the evolution of the total energy density of the effective source plus the usual vacuum energy (scaled to the present-day critical energy density of the Universe), $\rho_X + \rho_{\text{vac}}$, versus the redshift with probability regions up to 95% CL (the darker the more probable), using the 	exttt{figivenx} python package [112]. We see that the effective source completely screens the usual vacuum energy, $\rho_X + \rho_{\text{vac}} = 0$, at a red-shift $z_*=11.68$ (68% CL) and $z_*>13.65$ (95% CL). However, these are too large compared to the $z_*$ values suggested in, for instance, Refs. [29,30,38]. This might be signaling that this mechanism does not work efficiently enough in the Rastall-ΛCDM model under consideration. For instance, we indeed observe almost a perfect positive correlation between the Hubble constant $H_0$ and $\epsilon$, which implies that larger values of $\epsilon$ would correspond to larger values of $H_0$. We see in Table I that, in comparison to the GR-ΛCDM model, the combined CMB+BAO data set favors slightly larger mean value for $H_0$ in the Rastall-ΛCDM model, which seems to be an improvement for a better agreement with, for instance, the model independent $H_0$ values measured from the distance ladder measurements (e.g., $H_0 = 69.8 \pm 0.8$ from a recent calibration of the TRGB applied to Type Ia supernovae [22]), see Figures 8 and 9. However, a more careful look reveals that this improvement is not robust. The combined CMB+BAO data set predicts $H_0 = 68.31^{+0.76}_{-1.50}$ km s$^{-1}$ Mpc$^{-1}$ for the Rastall-ΛCDM model, while $H_0 = 67.92^{+0.43}_{-0.82}$ km s$^{-1}$ Mpc$^{-1}$ for the GR-ΛCDM model. We note that, in contrast to the GR-ΛCDM model, $H_0$ in the case of the Rastall-ΛCDM model, even at 68% CL, agrees with the TRGB $H_0$ value. This may be found promising, but, this is because of the large widening in the one-dimensional...
The horizontal blue band is for the model independent TRGB $H_0$ measurement $H_0 = 69.8 \pm 0.8 \text{km s}^{-1}\text{Mpc}^{-1}$ [22].

![Figure 8](image)

**FIG. 8.** Two-dimensional (68% and 95% CLs) marginalized distributions of $H_0$ versus $\Omega_0$ for the Rastall-$\Lambda$CDM model. The horizontal blue band is for the model independent TRGB $H_0$ measurement $H_0 = 69.8 \pm 0.8 \text{km s}^{-1}\text{Mpc}^{-1}$ [22].

![Figure 9](image)

**FIG. 9.** $H(z)/(1+z)$ vs $z$ with 68% and 95% error regions in case of CMB+BAO data. Here, the red curve stands for the GR-ΛCDM model corresponding to the mean values of the parameters. The three error bars stand for $H_0 = 69.8 \pm 0.8 \text{km s}^{-1}\text{Mpc}^{-1}$ from the TRGB $H_0$ [22], $H(z = 0.57) = 97.9 \pm 3.4 \text{km s}^{-1}\text{Mpc}^{-1}$ [96], and $H(z = 2.34) = 222.4 \pm 5.0 \text{km s}^{-1}\text{Mpc}^{-1}$ from the latest BAO data [28].

In case of CMB data, we notice larger error regions of all the six baseline parameters (except $\theta_s$, $A$, and $\tau_{reio}$) and other derived parameters in Rastall-$\Lambda$CDM model when compared to the GR-ΛCDM model. Indeed, the additional parameter $\epsilon$ in Rastall-$\Lambda$CDM model penalizes the statistical fit of this model to the data when compared to the GR-ΛCDM model. However, inclusion of BAO data, that is, CMB+BAO data put tight constraints on all the model parameters, and thereby reduce the error regions of the parameters considerably in both the models. In the following section, we calculate Bayesian evidences of the two models, and thereby do a comparison of the statistical fit.
VI. BAYESIAN EVIDENCE OF THE FIT

For comparing statistical fit of the models under consideration in this work to the observational data, we use Bayesian evidence. In this regard Bayes’ theorem reads

\[
P(\Theta|D,M) = \frac{\mathcal{L}(D|\Theta,M)\pi(\Theta|M)}{\mathcal{E}(D|M)},
\]

(32)

for a given model \(M\) with the set of parameters \(\Theta\) and the cosmological data \(D\). Here, \(P(\Theta|D,M)\) is the posterior distribution of the parameters \(\Theta\); \(\mathcal{L}(D|\Theta,M)\) is the likelihood function; \(\pi(\Theta|M)\) is the prior probability of the model parameters, and \(\mathcal{E}(D|M)\) is the Bayesian evidence calculated as

\[
\mathcal{E}(D|M) = \int_M \mathcal{L}(D|\Theta,M)\pi(\Theta|M)d\Theta.
\]

(33)

Further, we compute the ratio of the posterior probabilities for a model \(M_a\) with respect to a reference model \(M_b\) as

\[
P(M_b|D)\pi(M_a)|M_b|D) = B_{ab} P(M_a) / P(M_b),
\]

(34)

where \(B_{ab}\) is the Bayes’ factor, evaluated as

\[
B_{ab} = \frac{\mathcal{E}_a}{\mathcal{E}_b}.
\]

(35)

The Jeffreys’ scale \([1,3]\) is used to interpret the Bayes’ factor by calculating \(\ln B_{ab}\). The value of \(\ln B_{ab}\) lying in the range \([0,1]\) implies the strength of the evidence to be weak or inconclusive, while a definite or positive evidence is implied by the values in the range \([1,3]\). Further, the strength of the evidence is strong for \(\ln B_{ab}\) lying in \([3,5]\), and is the strongest for \(\ln B_{ab}\) greater than 5.

Table I displays the Bayesian evidence of Rastall-\(\Lambda\)CDM in comparison with the GR-\(\Lambda\)CDM model in the case of CMB and CMB+BAO data, where \(\mathcal{E}_{\text{Rastall}}\) and \(\mathcal{E}_{\text{GR}}\) respectively stand for the Bayesian evidences of the Rastall-\(\Lambda\)CDM and GR-\(\Lambda\)CDM models. We observe a definite evidence (\(\ln \mathcal{E}_{\text{Rastall,GR}} \in [1,3]\)) in the case of CMB data, whereas weak evidence (\(\ln \mathcal{E}_{\text{Rastall,GR}} \in [0,1]\)) is observed in case of the CMB+BAO data. Thus, the GR-\(\Lambda\)CDM model finds a better fit to the CMB data in comparison to the Rastall-\(\Lambda\)CDM model, but the weak evidence in case of the combined CMB+BAO data, suggests that both the models fit equally well to the CMB+BAO data, as expected.

VII. CONCLUSIONS

The Rastall gravity \([1]\) provides a simple generalization of the standard Einstein field equations of GR by relaxing the contribution of the term \(g_{\mu\nu}R\). This modification in the spacetime geometry side of the standard Einstein field equations corresponds to two simultaneous modifications of different nature in the material content side: (i) It adds new contributions of the actual material sources to the right hand side of the standard Einstein field equations in the form that of the usual vacuum energy of quantum field theory, which we interpreted as an effective source accompanying to the actual material sources considered in the model. (ii) It leads, in general, to a violation of the local conservation of the energy momentum tensor of an actual material source—and therefore, that of the effective source as well—, which implies that the evolution of the energy density of an actual material source is, in general, modified compared to its usual evolution in general relativistic models.

We have constructed the extension of the standard \(\Lambda\)CDM model (GR-\(\Lambda\)CDM) by switching the gravity theory from GR to the Rastall gravity (Rastall-\(\Lambda\)CDM), see Section IV. We then have reviewed it—via a preliminary investigation of its features for a demonstration of how it works, and a guide to the values of its parameters—in a proper manner, namely, in a manner clearly identifying and handling the two simultaneous modifications of different nature in the material content side of the Einstein field equations of GR, see Sections IV and V. It has then turned out that it is not possible to reach a decisive conclusion only through a preliminary investigation, for instance, whether the \(H_0\) and/or BAO data show tendency of \(\epsilon\) deviating from zero (GR) in a certain direction. Further, we also have learned that a significant improvement with regard to \(H_0\) and/or BAO data would most likely lead to spoiling of a successful description of the early Universe, which signals that CMB data would keep \(\epsilon\) values close to zero. These inspections have led us to expect only an insignificant deviation from the standard \(\Lambda\)CDM model when it is extended from GR via the Rastall gravity, and persuaded us that a conclusive answer cannot be given unless the model is rigorously confronted/constrained with the observational data.

Considering the background and perturbation dynamics (Section IV), we have explored the full parameter space of the Rastall-\(\Lambda\)CDM model—viz., the Rastall gravity extension of the six-parameter base GR-\(\Lambda\)CDM described by the additional parameter \(\epsilon\)—using the latest CMB data set as well as the latest combined CMB+BAO data set, see Section V. Also, for comparison, we have presented the corresponding constraints/results on the GR-\(\Lambda\)CDM model. It turned out that, as it is the case for the GR-\(\Lambda\)CDM model as well, the CMB+BAO data set puts tight constraints on the parameters of the Rastall-\(\Lambda\)CDM model and that, in contrast to the case for the GR-\(\Lambda\)CDM model, the CMB data set by alone puts loose constraints (larger error bounds) on some of the Rastall-

| \(\ln \mathcal{E}_{\text{Rastall}}\) | \(\ln \mathcal{E}_{\text{BAO}}\) | \(\ln \mathcal{E}_{\text{CMB}}\) | \(\ln \mathcal{E}_{\text{CMB+BAO}}\) |
|-----------------|-----------------|-----------------|-----------------|
| \(-1407.02 \pm 0.21\) | \(-1413.67 \pm 0.21\) | \(-1404.56 \pm 0.19\) | \(-1413.30 \pm 0.19\) |
| \(-2.46 \pm 0.28\) | \(-0.37 \pm 0.28\) |
ACDM model parameters—particularly, on the dimensionless density of cold dark matter \(\omega_{\text{cdm}}\) and all of the derived parameters—as a consequence of a wider range of deviation of the Rastall parameter \(\epsilon\) from zero. Yet, both analyses suggest that, in line with our conclusion reached via a preliminary investigation in Section [11] the Rastall parameter \(\epsilon\) is well consistent with zero at 95% CL, which in turn implies that there is no significant statistical evidence for deviation from GR via Rastall gravity. We note however that, as may be seen from the probability regions and mean values of \(\epsilon\), the Rastall parameter prefers negative values in the case of CMB data while positive values when BAO data set is included (CMB+BAO data). Despite the fact that they are basically minor within the allowed small range of the Rastall parameter from the data, we have explored the consequences of/tendencies led by the Rastall gravity on the cosmological parameters in the light of the observational analyses. Our results can be a guide for the research community that studies the Rastall gravity in various aspects of gravitation and cosmology, where, in general, as we have found in this work that the Rastall parameter cannot be out of range \(-0.0001 < \epsilon < 0.0007\) at 68% CL. Being that range an observational boundary imposed from high precision full CMB data set along with the BAO data set, it, in principle, must be obeyed as a new bound in any qualitative study within this modified theory of gravity. Finally—in support of our conclusions here—comparing statistical fit of these two models to the observational data by using Bayesian evidence, the GR-ΛCDM model finds a better fit to the CMB data in comparison to the Rastall-ΛCDM model, but the weak evidence in case of the combined CMB+BAO data, suggests that both the models fit equally well to the CMB+BAO data.

It also is worth mentioning as one of our conclusions that, if we assume that the standard physical ingredients of the Universe considered here are the true physical ingredients of the actual Universe, our finding that the term \(g_{\mu\nu}R\) contributes to the Einstein field equations [1] with a coefficient in the range \((-0.5001, -0.4993)\) from the combined CMB+BAO data at 68% CL, i.e., a coefficient equal to \(-1/2\) with a precision of \(O(10^{-4})\), can be taken as another new demonstration of the power of general relativity (which guarantees the local conservation of the total energy momentum tensor relying on the twice contracted Bianchi identity).

On the other hand, the Rastall gravity, in fact, possesses interesting features that could be of interest in the context of cosmology, for instance, to address some of the tensions prevailing within the standard ACDM model based on GR. One particular example may be that, for positive values of \(\epsilon\), the effective source arising due to the Rastall gravity assumes negative energy density values and screens usual vacuum energy in line with Refs. [27, 40], which suggest such a scenario for alleviating a number of persistent low-redshift tensions, including the so called Hubble constant \(H_0\) tension (deficiency). Indeed, our observational analyses show an almost perfect positive correlation between \(H_0\) and \(\epsilon\), which implies that larger values of \(\epsilon\) would correspond to larger values of \(H_0\). And, as the combined CMB+BAO data set favors slightly positive values of \(\epsilon\), this feature of the model works in the right direction and leads to predictions of larger \(H_0\) values (compared to the GR-ACDM model) consistent with, for instance, the model independent TRGB \(H_0\) measurements. However, a more careful look revealed that this improvement is not robust as it arises from the large widening in the one-dimensional marginalized probability distribution of \(H_0\), viz., largely increased errors, while a minor shift to the larger mean value of \(H_0\). We remind that the effective source comes along with a modification in the energy density redshift dependence of the actual matter source (viz., CDM) due to the EMT non-conservation feature of the Rastall gravity. Such a modification would obviously be more and more effective on the dynamics of the Universe with the increasing redshift. Therefore, in case of the combined CMB+BAO data, it is conceivable that the high redshift data (viz., the CMB data relevant to \(z \sim 1100\)), in particular, tend to keep the redshift dependence of the actual matter source very close to its usual \((1+z)^3\) dependence, i.e., \(\epsilon\) values almost equal to zero, and then does not allow the Rastall gravity to successfully realize a scenario wherein the usual vacuum energy is dynamically screened by the effective source. The lesson we learned from this is that, as we have seen the first signs in this direction in our preliminary investigations of the Rastall-ACDM model, the two simultaneous modifications of different nature in the material content side of the standard Einstein field equations, arising from the relaxation of the contribution of the term \(g_{\mu\nu}R\) on the spacetime geometry side, act against each other. And, through a further modification of the Rastall gravity, it could probably be possible to reach a new modified theory of gravity which is relaxed from such a dichotomy between the two (or more) simultaneous modifications of different nature in the material content side of the standard Einstein field equations arising from a modification in the spacetime geometry side. We find this result important as such situations may exist in some other similar type of modified gravity theories.

**ACKNOWLEDGMENTS**

Ö.A. acknowledges the support by the Turkish Academy of Sciences in scheme of the Outstanding Young Scientist Award (TÜBA-GEBIP). N.K. acknowledges the post-doctoral research support from the Istanbul Technical University (ITU). S.K. and S.S. gratefully acknowledge the support from SERB-DST project No. EMR/2016/000258. R.C.N. would like to thank the Brazilian agency FAPESP for financial support under Project No. 2018/18036-5.
[1] A.G. Riess et al. [Supernova Search Team], Observational evidence from supernovae for an accelerating Universe and a cosmological constant, Astron. J. 116, 1009 (1998). astro-ph/9805201
[2] P.A.R. Ade et al. [Planck Collaboration], Planck 2015 results. XIII. Cosmological parameters, Astron. Astrophys. 594, A13 (2016). astro-ph/1502.01589
[3] S. Alam et al. [BOSS Collaboration], The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological analysis of the DR12 galaxy sample, Mon. Not. Roy. Astron. Soc. 470, 2617 (2017). 1607.03155
[4] T.M.C. Abbott et al. [DES Collaboration], Dark Energy Survey year 1 results: Cosmological constraints from galaxy clustering and weak lensing, Phys. Rev. D 98, 043526 (2018). 1708.01530
[5] N. Aghanim et al. [Planck Collaboration], Planck 2018 results. VI. Cosmological Parameters, 1807.06209
[6] A.A. Starobinsky, A New Type of Isotropic Cosmological Models Without Singularity, Phys. Lett. B 91, 99 (1980). [Adv. Ser. Astrophys. Cosmol. 3, 130 (1987)].
[7] A.H. Guth, The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems, Phys. Rev. D 23, 347 (1981). [Adv. Ser. Astrophys. Cosmol. 3, 139 (1987)].
[8] A.D. Linde, A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems, Phys. Lett. B 108, 389 (1982). [Adv. Ser. Astrophys. Cosmol. 3, 149 (1987)].
[9] A. Albrecht, P.J. Steinhardt, Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking, Phys. Rev. Lett. 48, 1220 (1982), [Adv. Ser. Astrophys. Cosmol. 3, 158 (1987)].
[10] S. Weinberg, The cosmological constant problem, Rev. Mod. Phys. 61, 1 (1989).
[11] V. Sahni, A.A. Starobinsky, The Case for a positive cosmological Lambda term, Int. J. Mod. Phys. D 9, 373 (2000). astro-ph/9904398
[12] P.J.E. Peebles, B. Ratra, The Cosmological constant and dark energy, Rev. Mod. Phys. 75, 559 (2003). astro-ph/0207347
[13] T. Padmanabhan, Cosmological constant-the weight of the vacuum, Phys. Rept. 380, 235 (2003). hep-th/0212290
[14] J.S. Bullock, M. Boylan-Kolchin, Small-Scale Challenges to the ΛCDM Paradigm, Ann. Rev. Astron. Astrophys. 55, 343 (2017). 1707.04256
[15] W.L. Freedman, Cosmology at a Crossroads, Nat. Astron. 1, 0121 (2017). 1706.02739
[16] E. Di Valentino, Crack in the cosmological paradigm, Nat. Astron. 1, 569 (2017). 1709.04046
[17] G.B. Zhao et al., Dynamical dark energy in light of the latest observations, Nature (London) 1, 627 (2017). 1701.08165
[18] M. Raveri, W. Hu, Concordance and discordance in cosmology, Phys. Rev. D 99, 043506 (2019). 1806.04649
[19] A.G. Riess et al., A 2.4% Determination of the Local Value of the Hubble Constant, Astrophys. J. 826, 56 (2016). 1604.01424
[20] A.G. Riess et al., Milky Way Cepheid Standards for Measuring Cosmic Distances and Application to Gaia DR2: Implications for the Hubble Constant, Astrophys. J. 861, 126 (2018). 1804.10655
[21] A.G. Riess, S. Casertano, W. Yuan, L.M. Macri, D. Scolnic, Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond ΛCDM, Astrophys. J. 876, 85 (2019). 1903.07603
[22] W.L. Freedman et al., The Carnegie-Chicago Hubble Program. VIII. An Independent Determination of the Hubble Constant Based on the Tip of the Red Giant Branch, Astrophys. J. 882, 34 (2019). 1907.05022
[23] S. Vagnozzi et al., Constraints on the sum of the neutrino masses in dynamical dark energy models with w(z) > -1 are tighter than those obtained in ΛCDM, Phys. Rev. D 98, 083501 (2018). astro-ph/1801.08553
[24] E.O. Colgán, H. Yavartanoo, Testing the Swampland: H0 tension, Phys. Lett. B 797, 134907 (2019). 1905.02555
[25] E. Di Valentino, R.Z. Ferreira, L. Visinelli, U. Danielsson, Late time transitions in the quintessence field and the H0 tension, Phys. Dark Univ. 26, 100385 (2019). 1906.11255
[26] E. Di Valentino, A. Melchiorri, J. Silk, Cosmological constraints in extended parameter space from the Planck 2018 Legacy release, JCAP 01, 013 (2020). 1908.01391
[27] L. Visinelli, S. Vagnozzi, U. Danielsson, Revisiting a negative cosmological constant from low-redshift data, Symmetry 11, 1035 (2019). 1907.07953
[28] T. Delubac et al. (BOSS Collaboration), Baryon acoustic oscillations in the Lyo forest of BOSS DR11 quasars, Astron. Astrophys. 574, A59 (2015). 1404.1801
[29] E. Aubourg et al., Cosmological implications of baryon acoustic oscillation measurements, Phys. Rev. D 92, 123516 (2015). 1411.1074
[30] V. Sahni, A. Shahlooo, A.A. Starobinsky, Model independent evidence for dark energy evolution from Baryon Acoustic Oscillations, Astrophys. J. 793, L40 (2014). 1408.2205
[31] E. Mortsell, S. Dhawan, Does the Hubble constant tension call for new physics?, JCAP 09, 025 (2018). 1801.07260
[32] V. Poulin, K.K. Boddy, S. Bird, M. Kamionkowski, Implications of an extended dark energy cosmology with massive neutrinos for cosmological tensions, Phys. Rev. D 97, 123504 (2018). 1803.02474
[33] S. Capozziello, Ruchika, A.A. Sen, Model-independent constraints on dark energy evolution from low-redshift observations, Mon. Not. Roy. Astron. Soc. 484, 4484 (2019). 1806.03943
[34] Y. Wang, L. Pogosian, G.B. Zhao, A. Zucca, Evolution of Dark Energy Reconstructed from the Latest Observations, Astrophys. J. 869, L8 (2018). 1807.03772
[35] K. Dutta, Ruchika, A. Roy, A.A. Sen, M.M. Sheikh-Jabbari, Beyond ΛCDM with low and high redshift data: implications for dark energy, Gen. Rel. Grav. 52, 15 (2020). 1808.06623
[36] A. Banihashemi, N. Khosravi, A.H. Shirazi, Ups and Downs in Dark Energy: phase transition in dark
sector as a proposal to lessen cosmological tensions, 1808.02472

37. A. Banilashemi, N. Khosravi, A.H. Shirazi, Ginzburg-Landau Theory of Dark Energy: A Framework to Study Both Temporal and Spatial Cosmological Tensions Simultaneously, Phys. Rev. D 99, 083509 (2019). 1810.11007

38. O. Akarsu, J.D. Barrow, L.A. Escamilla, J.A. Vazquez, Graduated dark energy: Observational hints of a spontaneous sign switch in the cosmological constant, Phys. Rev. D 101, 063528 (2020). 1912.08751

39. G. Ye, Y. Piao, Is the Hubble tension a hint of AdS around recombination?, Phys. Rev. D 101, 083507 (2020). 2001.02451

40. A. Perez, D. Sadarsky, E. Wilson-Ewing, Resolving the $H_0$ tension with diffusion, 2001.07536

41. S. Kumar, Consistency of the nonflat ΛCDM model with the new result from BOSS, Phys. Rev. D 92, 103512 (2015). 1507.04684

42. E. Di Valentino, A. Melchiorri, J. Silk, Planck evidence for a closed Universe and a possible crisis for cosmology, Nat. Astron. 4, 196 (2019). 1911.02087

43. E. Di Valentino, A. Melchiorri, J. Silk, Cosmic Discordance: Planck and luminosity distance data exclude LCDM, 2003.04935

44. E.J. Copeland, M. Sami, S. Tsujikawa, Dynamics of dark energy, Int. J. Mod. Phys. D 15, 1753 (2006). hep-th/0603057

45. R.H. Caldwell, M. Kamionkowski, The Physics of Cosmic Acceleration, Ann. Rev. Nucl. Part. Sci. 59, 397 (2009). 0903.0866

46. T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis, Modified gravity and cosmology, Phys. Rept. 513, 1 (2012). 1106.2176

47. A. De Felice, S. Tsujikawa, $f(R)$ Theories, Living Rev. Rel. 13, 3 (2010). 1002.4928

48. S. Capozziello, M. De Laurentis, Extended Theories of Gravity, Phys. Rep. 509, 167 (2011). 1108.6266

49. S. Nojiri, S.D. Odintsov, V.K. Okkonomov, Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution, Phys. Rep. 692, 1 (2017). 1705.11098

50. S. Nojiri, S.D. Odintsov, Unified cosmic history in modified gravity: From $F(R)$ theory to Lorentz non-invariant models, Phys. Rep. 505, 59 (2011). 1011.0544

51. B. Boisseau, G. Esposito-Farese, D. Polarski, A.A. Starobinsky, Reconstruction of a Scalar-Tensor Theory of Gravity in an Accelerating Universe, Phys. Rev. Lett. 85, 2236 (2000). gr-qc/0001066

52. V. Sahni, A. Starobinsky, Reconstructing Dark Energy, Int. J. Mod. Phys. D 15, 2105 (2006). astro-ph/0610020

53. C. Umlită, M. Ballardini, F. Finelli, D. Paoletti, CMB and BAO constraints for an induced gravity dark energy model with a quartic potential, JCAP 08, 017 (2015). 1507.00718

54. M. Ballardini, F. Finelli, C. Umlită, D. Paoletti, Cosmological constraints on induced gravity dark energy models, JCAP 05, 067 (2016). 1601.03387

55. Ö. Akarsu, N. Katurc, N. Özdemir, J.A. Vázquez, Anisotropic massive Brans-Dicke gravity extension of the standard ΛCDM model, Eur. Phys. J. C 80, 32 (2020). 1903.06679

56. M. Rossi, M. Ballardini, M. Braglia, F. Finelli, D. Paoletti, A.A. Starobinsky, C. Umlită, Cosmological constraints on post-Newtonian parameters in effectively massless scalar-tensor theories of gravity, Phys. Rev. D 100, 103524 (2019). 1906.10218

57. A.D. Dolgov, Field model with a dynamic cancellation of the cosmological constant, JETP Lett. 41, 345 (1985), [Pisma Zh. Eksp. Teor. Fiz. 41, 280 (1985)].

58. F. Bauer, J. Sola, H. Stefancic, Dynamically avoiding fine-tuning the cosmological constant: the “Relaxed Universe”, JCAP 12, 029 (2010). 1006.3944

59. S.A. Franchino-Viñas, S. Mignemi, Asymptotic freedom for $Λ^2$ QFT in Snyder-de Sitter space, [1911.08921]

60. S.Y. Zhou, E.J. Copeland, P.M. Saffin, Cosmological constraints on $f(G)$ dark energy models, JCAP 07, 009 (2009). 0903.4610

61. V. Sahni, Y. Shtanov, Braneworld models of dark energy, JCAP 11, 014 (2003). astro-ph/0202346

62. P. Brax, C. van de Bruck, Cosmology and braneworlds: a review, Class. Quant. Grav. 20, R201 (2003). hep-th/0303095

63. A. Ashtekar, T. Pawlowski, P. Singh, Quantum nature of the big bang: Improved dynamics, Phys. Rev. D 74, 084003 (2006). gr-qc/0607039

64. A. Ashtekar, P. Singh, Loop quantum cosmology: a status report, Class. Quant. Grav. 28, 213001 (2011). 1108.0893

65. A. Chodos, S.L. Detweiler, Where has the fifth-dimension gone?, Phys. Rev. D 21, 2167 (1980).

66. T. Dereli, R.W. Tucker, Dynamical Reduction of Internal Dimensions in the Early Universe, Phys. Lett. B 125, 133 (1983).

67. O. Akarsu, T. Dereli, Late time acceleration of the 3-space in a higher dimensional steady state Universe in dilaton gravity, JCAP 02, 050 (2013). 1210.8106

68. J.G. Russo, P.K. Townsend, Late-time cosmic acceleration from compactification, Class. Quant. Gravit. 36, 095008 (2019). 1811.03660

69. O. Akarsu, N. Katurc, S. Kumar, Cosmic acceleration in a dust only Universe via energy-momentum powered gravity, Phys. Rev. D 97, 024011 (2018). 1709.02367

70. C.V.R. Board, J.D. Barrow, Cosmological models in energy-momentum-squared gravity, Phys. Rev. D 96, 123517 (2017). 1709.09501

71. O. Akarsu, J.D. Barrow, C.V.R. Board, N.M. Uzun, J.A. Vázquez, Screening Λ in a new modified gravity model, Eur. Phys. J. C 79, 846 (2019). 1903.11519

72. P. Rastall, Generalization of the Einstein theory, Phys. Rev. D 6, 3357 (1972).

73. P. Rastall, A theory of gravity, Can. J. Phys. 54, 66 (1976).

74. L.L. Smalley, Variational Principle for a Prototype Rastall Theory of Gravitation, II Nuovo Cimento 80, 42 (1984).

75. H. Shabani, A. Hadi Ziaie, A connection between Rastall-type and $f(R,T)$ gravities, EPL 129 20004 (2020). [Europhys. Lett. 129 20004 (2020)]. 2003.02064

76. W.A.G. De Moraes, A.F. Santos, Lagrangian formalism for Rastall theory of gravity and Godel-type Universe, Gen. Rel. Grav. 51, 167 (2019). 1912.06471

77. L. Lindblom, W.A. Hiscock, Criticism of some non-conservative gravitational theories, J. Phys. A: Math. Gen. 15, 1827 (1982).

78. M. Visser, Rastall gravity is equivalent to Einstein grav-
C.E.M. Batista, M.H. Daouda, J.C. Fabris, O.F. Piattella, D.C. Rodrigues, Rastall cosmology and the ACDM model, Phys. Rev. D 85, 084008 (2012). 1112.4143

H. Moradpour, A. Bonilla, E.M.C. Abreu, J.A. Neto, Accelerated cosmo in a nonextensive setup, Phys. Rev. D 96, 123504 (2017). 1711.08338

M. Daouda, J.C. Fabris, A.M. Oliveira, F. Smirnov, H.E.S. Velten, Non-conservative unimodular type gravity, Int. J. Mod. Phys. D 28, 1950175 (2019). 1802.01413

W.G. Ururu, A Unimodular Theory of Canonical Quantum Gravity, Phys. Rev. D 40, 1048 (1989).

G.F.R. Ellis, H. van Elst, J. Murugan, J. P. Uzan, On the trace-free Einstein equations as a viable alternative to general relativity, Class. Quant. Grav. 28, 225007 (2011). 1008.1196

C.E.M. Batista, J.C. Fabris, O.F. Piattella, A.M. Velasquez-Toribio, Observational constraints on Rastall’s cosmology, Eur. Phys. J. C 73, 2425 (2013). 1208.6527

J.C. Fabris, O.F. Piattella, D.C. Rodrigues, M.H. Daouda, Rastalls cosmology and its observational constraints, AIP Conf. Proc. 1647, 50 (2015). 1405.5669

W. Khylee, J. Dutta, Linear growth index of matter perturbations in Rastall gravity, Phys. Lett. B 797, 134796 (2019). 1907.09221

C. van de Bruck, C.C. Thomas, Dark energy, the swamps and the equivalence principle, Phys. Rev. D 100, 023515 (2019). 1904.07082

A.M. Oliveira, H.E.S. Velten, J.C. Fabris, L. Casarini, Neutron stars in rastall gravity, Phys. Rev. D 92, 044020 (2015). 1506.00567

R. Li, J. Wang, Z. Xu, X. Guo, Constraining the rastall parameters in static space-times with galaxy-scale strong gravitational lensing, Mon. Not. Roy. Astron. Soc. 486, 2407 (2019). 1903.08790

S. Dodelson, Modern Cosmology, Acad. Press New York U.S.A. (2003).

D.J. Fixsen, The Temperature of the Cosmic Microwave Background, Astrophys. J. 707, 916 (2009). 0911.1955

L. Anderson et al. [BOSS Collaboration], The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: baryon acoustic oscillations in the Data Releases 10 and 11 Galaxy samples, Mon. Not. Roy. Astron. Soc. 441, 24 (2014). 1312.3877

C.P. Ma, E. Bertschinger, Cosmological Perturbation Theory in the Synchronous and Conformal Newtonian Gauges, Astrophys. J. 455, 7 (1995). astro-ph/9506072

D. Bias, J. Lesgourgues, T. Tram, The Cosmic Linear Anisotropy Solving System (CLASS). Part II: Approximation schemes, JCAP 07, 034 (2011). 1104.2933

As we mentioned above, if baryons evolve following the modification rules imposed by Rastall gravity on GR, then the full Boltzmann equations for the baryons should also be modified. Also, universal couplings generally require screening mechanisms to protect baryonic interactions in high density environments, where the local gravity measurements are tightly constrained. Also, non-standard dynamics of the ordinary particles is tightly constrained from the BBN prevision. However, the relevant investigation of these various factors on physics of baryons is beyond the scope of the current paper and deserves to be investigated in detail in some future work. Thus, in the present paper, the baryons are assumed to be minimally coupled to gravity.

J. Khoury, Theories of Dark Energy with Screening Mechanisms, 1011.5909

V. Vikram, J. Sakstein, C. Davis, A. Neil, Astrophysical tests of modified gravity: Stellar and gaseous rotation curves in dwarf galaxies, Phys. Rev. D 97, 104055 (2018). 1711.08338

J. Ellis, S. Kalara, K.A. Olive, C. Wetterich, Density-dependent couplings and astrophysical bounds on light scalar particles, Phys. Lett. B 228, 264 (1989).

K. Haginara et al. [Particle Data Group], Review of Particle Properties, Phys. Rev. D. 66, 010001 (2002).

N. Aghanim et al. [Planck Collaboration], Planck 2018 results. VIII. Gravitational lensing, 1807.06210

F. Beutler et al., The 6dF Galaxy Survey: baryon acoustic oscillations and the local Hubble constant, Mon. Not. Roy. Astron. Soc. 416, 3017 (2011). 1106.3366

A.J. Ross et al., The clustering of the SDSS DR7 main Galaxy sample-I: A 4 per cent distance measure at z = 0.15, Mon. Not. Roy. Astron. Soc. 449, 835 (2015). 1409.3212

M. Blomqvist et al., Baryon acoustic oscillations from the cross-correlation of Ly-α absorption and quasars in eBOSS DR14, Astron. Astrophys. 629, A86 (2019). 1904.05334

V. de Sainte Agathe et al., Baryon acoustic oscillations at z = 2.34 from the correlations of Ly-α absorption in eBOSS DR14, Astron. Astrophys. 629, A85 (2019). 1904.05400

F. Feroz, M.P. Hobson, M. Bridges, MultiNest: an efficient and robust Bayesian inference tool for cosmology and particle physics, Mon. Not. Roy. Astron. Soc. 398, 1601 (2009). 0809.3347

B. Audren et al., Conservative constraints on early cosmology with Monte Python, JCAP 02, 001 (2013). 1210.7183

S. Kumar, R.C. Nunes, S.K. Yadav, Dark sector interaction: a remedy of the tensions between CMB and LSS data, Eur. Phys. J. C 79, 576 (2019). 1903.04865

W. Handley, fiveness: A Python package for functional posterior plotting, J. Open Source Softw. 3, 849 (2018). 1908.01711

R.E. Kass, A.E. Raftery, Bayes Factors, J. Am. Statist. Assoc. 90, 773 (1995).