Abstract

Let $A$ and $B$ be two central simple algebras of a prime degree $n$ over a field $F$ generating the same subgroup in the Brauer group $\text{Br}(F)$. We show that the Chow motive of a Severi-Brauer variety $\text{SB}(A)$ is a direct summand of the motive of a generalized Severi-Brauer variety $\text{SB}_d(B)$ if and only if $[A] = \pm d[B]$ in $\text{Br}(F)$. The proof uses methods of Schubert calculus and combinatorial properties of Young tableaux, e.g., the Robinson-Schensted correspondence.

Keywords: Severi-Brauer variety, Chow motive, Robinson-Schensted correspondence, Grassmannian

1 Introduction

Let $X$ be a twisted flag $G$-variety for a linear algebraic group $G$. The main result of paper [CPSZ] says that under certain restrictions the Chow motive of $X$ can be expressed in terms of motives of “minimal” flag varieties, i.e., those which correspond to maximal parabolic subgroups of $G$. A natural question arises: is it possible to decompose the motive of such a “minimal” flag?

A particular case of such a decomposition was already provided in [CPSZ]. More precisely, a “minimal flag” $X$ was a generalized Severi-Brauer variety of ideals of reduced dimension 2 in a division algebra $B$ of degree 5 and the decomposition was (see [CPSZ Theorem 2.5])

$$ \mathcal{M}(\text{SB}_2(B)) \simeq \mathcal{M}(\text{SB}(A)) \oplus \mathcal{M}(\text{SB}(A))(2), $$
where $[B] = 2[A]$ in the Brauer group.

In the present paper we provide an affirmative answer on this question for any adjoint group $G$ of inner type $A_n$ of a prime rank $n$. Namely, we show that the motive of a generalized Severi-Brauer variety always contains (as a direct summand) the motive of a Severi-Brauer variety.

1.1 Theorem. Let $A$ and $B$ be two central simple algebras of a prime degree $n$ over a field $F$ generating the same subgroup in the Brauer group $\text{Br}(F)$. Then the motive of a Severi-Brauer variety $\text{SB}(A)$ is a direct summand of the motive of a generalized Severi-Brauer variety $\text{SB}_d(B)$ if and only if

$$[A] = \pm d[B] \text{ in } \text{Br}(F).$$

1.2 Remark. Theorem 1.1 may also be considered as a generalization of the result by N. Karpenko (see [Ka00, Criterion 7.1]) which says that the motives of Severi-Brauer varieties $\text{SB}(A)$ and $\text{SB}(B)$ of central simple algebras $A$ and $B$ are isomorphic if and only if $[A] = \pm [B]$ in $\text{Br}(F)$.

1.3 Remark. We expect that Theorem 1.1 holds in the case when $d$ and $n$ are coprime (see Section 7). Observe that if $d$ and $n$ are not coprime, then the theorem fails. It can be already seen on the level of generating functions. Namely, consider the Grassmann variety $G_2(4)$ of 2-planes in a 4-dimensional affine space ($n = 4$ and $d = 2$). Then for the generating functions we have

$$P(G_2(4), t) = (t^2 + 1)(t^2 + t + 1) \text{ and } P(\mathbb{P}^3, t) = t^3 + t^2 + t + 1.$$  

Obviously $P(\mathbb{P}^3, t)$ doesn’t divide $P(G_2(4), t)$. Indeed, it can be shown that $P(\mathbb{P}^{n-1}, t)$ divides $P(G_d(n), t)$ if and only if $d$ and $n$ are coprime.

1.4 Remark. For motives with $\mathbb{Z}/n\mathbb{Z}$-coefficients there is the following decomposition (see [CPSZ, Proposition 2.4])

$$\mathcal{M}(\text{SB}_d(B)) \simeq \bigoplus_i \mathcal{M}(\text{SB}(B))(i)^{a_i},$$

where the integers $a_i$ are coefficients of the quotient of Poincaré polynomials $\frac{P(G_d(n), t)}{P(\mathbb{P}^{n-1}, t)} = \sum_i a_i t^i$ (see 2.8). Note that in this case the motives of Severi-Brauer varieties corresponding to different classes of algebras generating the same subgroup in the Brauer group are isomorphic (see [Ka00, Section 7]), i.e., $\mathcal{M}(\text{SB}(A)) \simeq \mathcal{M}(\text{SB}(B))$. 
The proofs are based on Rost Nilpotence Theorem for projective homogeneous varieties proved by V. Chernousov, S. Gille and A. Merkurjev in [CGM05]. Briefly speaking, this result reduces the problem of decomposing the motive of a variety $X$ over $F$ into the question about algebraic cycles in the Chow ring $\text{CH}(X_s \times X_s)$ over the separable closure $F_s$. Namely, the motive of $\text{SB}(A)$ is a direct summand of the motive of $\text{SB}_d(B)$ if there exist two cycles $f$ and $g$ in $\text{CH}(\mathbb{P}^{n-1} \times G_d(n))$ such that both cycles belong to the image of the restriction map $\text{CH}(\text{SB}(A) \times \text{SB}_d(B)) \to \text{CH}(\mathbb{P}^{n-1} \times G_d(n))$, and the correspondence product $g^t \circ f$ is the identity.

We define $f$ and $g$ to be the Schur functions of total Chern classes of certain bundles on $\mathbb{P}^{n-1} \times G_d(n)$. Using the language and properties of Schur functions we show that the identity $g^t \circ f = \text{id}$ is a direct consequence of the Robinson-Schensted correspondence, one of classical combinatorial facts about Young tableaux.

The paper is organized as follows. In section 2 we remind several definitions and notation used in the proofs. These include Chow motives, rational cycles, and generalized Severi-Brauer varieties. In section 3 we describe the subgroup of rational cycles of the Chow group of the product of two generalized Severi-Brauer varieties. Indeed, we provide an explicit set of generators for this subgroup modulo $n$ in terms of Schur functions. In section 4 we use this description for proving some known results on motives of Severi-Brauer varieties. Section 5 is devoted to the proof of the main theorem. In section 6 we prove the crucial congruence used in the proof of the main theorem. In the last section we discuss the case of $G_2(n)$, where $n$ is an odd integer (not necessarily prime).

2 Preliminaries

In the present section we remind definition of the category of Chow motives over a field $F$ following [Ma68] and [Ka01]. We recall the notion of a rational cycle and state the Rost Nilpotence Theorem for idempotents following [CGM05]. We recall several auxiliary facts concerning generalized Severi-Brauer varieties following [KMRT98].

2.1 (Chow motives). Let $F$ be a field and $\text{Var}_F$ be the category of smooth projective varieties over $F$. We define the category $\text{Cor}_F$ of correspondences over $F$. Its objects are non-singular projective varieties over $F$. For morphisms, called correspondences, we set $\text{Mor}(X, Y) := \text{CH}^{\dim X}(X \times Y)$. For
two correspondences \( \alpha \in \text{CH}(X \times Y) \) and \( \beta \in \text{CH}(Y \times Z) \) we define the composition \( \beta \circ \alpha \in \text{CH}(X \times Z) \)

\[
\beta \circ \alpha = \text{pr}_{13*}(\text{pr}_{12}^*(\alpha) \cdot \text{pr}_{23}^*(\beta)),
\]

where \( \text{pr}_{ij} \) denotes the projection on product of the \( i \)-th and \( j \)-th factors of \( X \times Y \times Z \) respectively and \( \text{pr}_{ij*}, \text{pr}_{ij}^* \) denote the induced push-forwards and pull-backs for Chow groups. The composition \( \circ \) induces a ring structure on the abelian group \( \text{CH}^\dim X(X \times X) \). The unit element of this ring is the class of diagonal cycle \( \Delta_X \).

The pseudo-abelian completion of \( \text{Cor}_F \) is called the category of \textit{Chow motives} and is denoted by \( \mathcal{M}_F \). The objects of \( \mathcal{M}_F \) are pairs \( (X, p) \), where \( X \) is a non-singular projective variety and \( p \) is a projector, that is, \( p \circ p = p \). The motive \( (X, \Delta_X) \) will be denoted by \( \mathcal{M}(X, \Delta_X) \).

By construction \( \mathcal{M}_F \) is a self-dual tensor additive category, where the duality is given by the transposition of cycles \( \alpha \mapsto \alpha^t \) and the tensor product is given by the usual fiber product \( (X, p) \otimes (Y, q) = (X \times Y, p \times q) \). Moreover, the contravariant Chow functor \( \text{CH} : \text{Var}_F \to \text{Z-Ab} \) (to the category of \textit{Z-graded} abelian groups) factors through \( \mathcal{M}_F \), i.e., one has the commutative diagram of functors

\[
\begin{array}{ccc}
\text{Var}_F & \xrightarrow{\text{CH}} & \text{Z-Ab} \\
\Gamma & \downarrow & \downarrow R \\
\mathcal{M}_F & & \\
\end{array}
\]

where \( \Gamma : f \mapsto \Gamma_f \) is the (contravariant) graph functor and \( R : \mathcal{M}_F \to \text{Z-Ab} \) is the (covariant) realization functor given by \( R : (X, p) \mapsto \text{im}(p^*) \), where \( p^* \) is the composition

\[
p^* : \text{CH}(X) \xrightarrow{\text{pr}_1^*} \text{CH}(X \times X) \xrightarrow{p} \text{CH}(X \times X) \xrightarrow{\text{pr}_2^*} \text{CH}(X).
\]

Consider the morphism \( (\text{id}, e) : \mathbb{P}^1 \times \{\text{pt}\} \to \mathbb{P}^1 \times \mathbb{P}^1 \). The image of the induced push-forward \( (\text{id}, e)_* \) doesn’t depend on the choice of a point \( e : \{\text{pt}\} \to \mathbb{P}^1 \) and defines the projector in \( \text{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1) \) denoted by \( p_1 \). The motive \( L = (\mathbb{P}^1, p_1) \) is called \textit{Lefschetz motive}. For a motive \( M \) and a nonnegative integer \( i \) we denote by \( M(i) = M \otimes L^\otimes i \) its \textit{twist}. Observe that

\[
\text{Mor}((X, p)(i), (Y, q)(j)) = q \circ \text{CH}^\dim X+i-j(X \times Y) \circ p.
\]
2.2 (Product of cellular varieties). Let $G$ be a split linear algebraic group over a field $F$. Let $X$ be a projective $G$-homogeneous variety, i.e., $X = G/P$, where $P$ is a parabolic subgroup of $G$. The abelian group structure of $\text{CH}(X)$, as well as its ring structure, is well-known. Namely, $X$ has a cellular filtration and the generators of Chow groups of the bases of this filtration correspond to the free additive generators of $\text{CH}(X)$ (see [Ka01]). Note that the product of two projective homogeneous varieties $X \times Y$ has a cellular filtration as well, and $\text{CH}^*(X \times Y) \cong \text{CH}^*(X) \otimes \text{CH}^*(Y)$ as graded rings. The correspondence product of two cycles $\alpha = f_\alpha \times g_\alpha \in \text{CH}(X \times Y)$ and $\beta = f_\beta \times g_\beta \in \text{CH}(Y \times X)$ is given by
\[(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \deg(g_\alpha \cdot f_\beta) (f_\alpha \times g_\beta),\] (3)
where $\deg : \text{CH}(Y) \to \text{CH}(\{\text{pt}\}) = \mathbb{Z}$ is the degree map.

2.3 (Rational cycles). Let $X$ be a projective variety of dimension $n$ over $F$. Let $F_s$ denote the separable closure of $F$. Consider the scalar extension $X_s = X \times_F F_s$. We say a cycle $J \in \text{CH}(X_s)$ is rational if it lies in the image of the pull-back homomorphism $\text{CH}(X) \to \text{CH}(X_s)$. For instance, there is an obvious rational cycle $\Delta_{X_s}$ on $\text{CH}^n(X_s \times X_s)$ that is given by the diagonal class.

Let $E$ be a vector bundle over $X$. Then the total Chern class $c(E_s)$ of the pull-back induced by the scalar extension $F_s/F$ is rational. Let $L/F$ be a finite separable field extension which splits $X$, i.e., there is an induced isomorphism $\text{CH}(X_L) \cong \text{CH}(X_s)$. Then the cycle $\deg(L/F) \cdot J, J \in \text{CH}(X_s)$, is rational. Observe that all linear combinations, intersections and correspondence products of rational cycles are rational.

2.4 (Rost nilpotence). We will use the following fact (see [CGM05, Cor. 8.3]) that follows from the Rost Nilpotence Theorem. Let $X$ be a twisted flag $G$-variety for a semisimple group $G$ of inner type over $F$. Let $p_s$ be a non-trivial rational projector in $\text{CH}^n(X_s \times X_s)$, i.e., $p_s \circ p_s = p_s$. Then there exists a non-trivial projector $p$ on $\text{CH}^n(X \times X)$ such that $p \times_F F_s = p_s$. Hence, existence of a non-trivial rational projector $p_s$ on $\text{CH}^n(X_s \times X_s)$ gives rise to the decomposition of the Chow motive of $X$
\[\mathcal{M}(X) \cong (X, p) \oplus (X, \Delta_X - p)\] (4)

2.5 (Tautological and quotient bundles). Let $A$ be a central simple algebra of degree $n$ over $F$. Consider a generalized Severi-Brauer variety
SB_d(A) of ideals of reduced dimension d of A. Over the separable closure it
becomes isomorphic to the Grassmannian G_d(n) of d-dimensional planes in a
n-dimensional affine space.

There is a tautological vector bundle over SB_d(A) of rank dn denoted by
τ_d^A and given by the ideals of A of reduced dimension d considered as vector
spaces over F. Over the separable closure F_s this bundle becomes isomorphic
to Hom(E_n, τ_d) = τ_d ⊕ n_d, where E_n denotes the trivial bundle of rank n and τ_d
the tautological bundle over the Grassmannian G_d(n).

The universal quotient bundle over SB_d(A), denoted by κ_d^A, is the quotient
E_A/τ_d^A of a trivial bundle E_A of rank n^2 modulo the tautological bundle τ_d^A.
Clearly, κ_d^A is of rank (n - d)n. Over F_s this bundle becomes isomorphic to
Hom(E_n, κ_d) = κ_d ⊕ n_d, where κ_d is the universal quotient bundle over G_d(n).

We will extensively use the following fact

2.6 Lemma. Let A be a central simple algebra over F, r a positive integer
and B a division algebra which represents the class of the r-th tensor power
of A in the Brauer group. Let X = SB(A) × SB_d(B^op). Then the bundle
T_s = pr_1^*(τ_r^A) ⊗ pr_2^*(τ_d) over X is a pull-back of some bundle T over X. As
a consequence, any Chern class of T_s is a rational cycle.

Proof. According to [Pa94, 10.2] the image of the restriction map on K_0
K_0(SB(A) × SB_d(B^op)) → K_0(ℙ^n-1 × G_d(n))
is generated by classes of bundles ind(A^r ⊗ B^j) · [pr_1^*(τ_r^A) ⊗ pr_2^*(τ_d)]. □

2.7 Remark. Observe that the similar fact holds if one replaces the tensor
power by an exterior (lambda) power. This is due to the fact that [Λ^r A] =
[A^⊗ r] in Br(F) (see [KMRT98, 10.A.]).

2.8 (Poincaré polynomial). By [Fu97] the Poincaré polynomial of a Chow
group of a Grassmannian G_d(n) is given by the Gaussian polynomial

P(G_d(n), t) = \binom{n}{d} (t) = \frac{(1 - t^n)(1 - t^{n-1}) \ldots (1 - t^{n-d+1})}{(1-t)(1-t^2) \ldots (1-t^d)}.

Observe that for a projective space, i.e., for d = 1, the respective polynomial
takes the most simple form

P(ℙ^n-1, t) = \frac{1 - t^n}{1-t}.

Observe also that if n is a prime integer, then the polynomial P(ℙ^n-1, t)
always divides P(G_d(n), t).
3 The subgroup of rational cycles

The goal of the present section is to provide an explicit set of generators for the image of the restriction map \( CH(X) \to CH(X_s) \) modulo \( n \), where \( X \) is a product of a Severi-Brauer variety by a generalized Severi-Brauer variety corresponding to division algebras of a prime degree \( n \).

3.1 (Grassmann bundle structure). Let \( A \) be a division algebra of degree \( n \) over \( F \), \( r \) a positive integer and \( B \) a division algebra which represents the class of the \( r \)-th tensor power of \( A \) in \( Br(F) \). According to [IK00, Proposition 4.3] the product \( SB(A) \times SB_d((A^\otimes r)^{op}) \) can be identified with the Grassmann bundle \( G_d(V) \) over \( SB(A) \), where \( V = (\tau_1^A)^{\otimes r} \) is a locally free sheaf of (right) \( A^\otimes r \)-modules. By Morita equivalence we may replace \( A^\otimes r \) by \( B \) and, hence, obtain that the product \( X = SB(A) \times SB_d(B^{op}) \) is the Grassmann bundle \( G_d(W) \) over \( SB(A) \) for a locally free sheaf of (right) \( B \)-modules \( W \). The tautological bundle \( T \) over \( G_d(W) \) is the bundle

\[
T = pr_1^*(W) \otimes_B pr_2^*(\tau_B^{op}).
\]

Let \( Q = pr_1^*(W)/T \) denote the universal quotient bundle over \( G_d(W) \). Over the separable closure it can be identified with

\[
Q_s = pr_1^*(\tau_1^{\otimes r}) \otimes pr_2^*(\kappa_d)
\]

where \( \tau_1 \) and \( \kappa_d \) are the tautological and quotient bundles over \( \mathbb{P}^{n-1} \) and \( G_d(n) \) respectively. We shall write this bundle simply as \( \tau_1^{\otimes r} \otimes \kappa_d \) meaning the respective pull-backs.

3.2 (Grassmann bundle theorem). According to Grassmann bundle theorem the Chow ring \( CH(X) \) is a free \( CH(SB(A)) \)-module with the basis \( \Delta_\lambda(c(Q)) \), where \( \lambda \) runs through the set of all partitions \( \lambda = (\lambda_1, \ldots, \lambda_d) \) with \( n - d \geq \lambda_1 \geq \ldots \geq \lambda_d \geq 0 \), \( c(Q) \) is the total Chern class of the quotient bundle \( Q \) and \( \Delta_\lambda \) is the Schur function. In other words, for any \( k \) we have the decomposition

\[
CH^k(X) \cong \bigoplus_\lambda \Delta_\lambda(c(Q)) \cdot pr_1^*(CH^{k-|\lambda|}(SB(A)))
\]
3.3. Observe that the decomposition (3) is compatible with the scalar extension \( F_s/F \). Hence, any rational cycle \( \alpha \in \text{CH}^k(X_s) \) can be represented uniquely as the sum of cycles

\[
\alpha = \sum_{\lambda} \Delta_{\lambda}(c(Q_s)) \cdot \alpha_{\lambda},
\]

where \( \alpha_{\lambda} \in \text{CH}^{k-|\lambda|}(\mathbb{P}^{n-1}) \) is rational. If \( n \) is a prime integer, then all rational cycles of positive codimensions in \( \text{CH}(\mathbb{P}^{n-1}) \) are divisible by \( n \) (see [Ka95, Corollary 4]). Hence, considering (6) modulo \( n \) we obtain

3.4 Lemma. If \( n \) is a prime integer, the cycles \( \Delta_{\lambda}(c(Q_s)) \), where \( \lambda \) runs through the set of all partitions, generate the subgroup of rational cycles of \( \text{CH}(X_s) \) modulo \( n \). In particular case \( d = 1 \) the basis of the subgroup of rational cycles of \( \text{CH}^k(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) \) modulo \( n \) consist of Chern classes \( c_k(\tau_1^\otimes r \otimes \kappa_1) \), where \( k = 0 \ldots n - 1 \).

4 Applications to Severi-Brauer varieties

The following two lemmas were proven by N. Karpenko (see [Ka96, Theorem 2.2.1] and [Ka00, Criterion 7.1]). In the present section we provide short proofs of these results restricting to algebras of prime degrees.

4.1 Lemma. The motive of a Severi-Brauer variety of a division algebra \( A \) of a prime degree \( n \) is indecomposable.

Proof. Consider the product \( X = SB(A) \times SB(A) \). It can be identified with the product \( SB(A) \times SB(B^{op}) \), where \( [B] = [A^{n-1}] \). Apply Lemma 3.3 to the case \( d = 1 \) and \( r = n - 1 \). We obtain that in codimension \( k = n - 1 \) there is only one basis element of the subgroup of rational cycles modulo \( n \)

\[
\Delta_{n-1} = c_{n-1}(\tau_1^{\otimes n-1} \otimes \kappa_1) = \sum_{i=0}^{n-1} (1 - n)^i \cdot H^i \times H^{n-1-i} \in \text{CH}^{n-1}(X_s)
\]

which is congruent modulo \( n \) to the diagonal cycle \( \Delta = \sum_{i=0}^{n-1} H^i \times H^{n-1-i} \).

If the motive of \( SB(A) \) splits, then there must exist a rational projector \( p \in \text{CH}^{n-1}(X_s) \) such that \( \Delta \pm p \) and \( p \) are non-trivial. By composition rule (4) if \( p \) is a projector, then

\[
p = \sum_{i} \pm H^i \times H^{n-1-i},
\]

where the index \( i \) runs through a subset of \( \{0 \ldots n - 1\} \). From the other hand, since \( p \) is rational, it must be a multiple of \( \Delta \) modulo \( n \). So the only possibility for \( p \) is to coincide either with \( \pm \Delta \) or with \( 0 \), contradiction. \( \square \)
4.2 Lemma. Let $A$ and $B$ be two division algebras of a prime degree $n$ generating the same subgroup in the Brauer group. Then the motives of $SB(A)$ and $SB(B^{op})$ are isomorphic iff $A = B$ or $B^{op}$.

Proof. Take $r$ such that $[B] = [A^r]$ in $\text{Br}(F)$. and apply Lemma 3.1 for $X = SB(A) \times SB(B^{op})$. We obtain that the subgroup of rational cycles of codimension $k = n - 1$ is generated by the cycle

$$\Delta_{n-1} = c_{n-1}(\tau_1^r \otimes \kappa_1) = \sum_{i=0}^{n-1} (-r)^i \cdot H^i \times H^{n-1-i} \in \text{CH}^{n-1}(X_s).$$

According to composition rule (3) and Rost nilpotence theorem (see 2.4) any motivic isomorphisms between $SB(A)$ and $SB(B^{op})$ is given by a lifting of a rational cycle of the kind $\sum_{i=0}^{n-1} \pm H^i \times H^{n-1-i} \in \text{CH}^{n-1}(X_s)$ and vice versa.

To finish the proof observe that a rational cycle of this kind is a multiple of $\Delta_{n-1}$ modulo $n$ iff $r \equiv \pm 1 \mod n$. \hfill \qed

5 Generalized Severi-Brauer varieties

In the present section we prove the main theorem of the paper which is formulated as follows

5.1 Theorem. Let $A$ and $B$ be two division algebras of a prime degree $n$ generating the same subgroup in the Brauer group. Take an integer $r$ such that $[B] = [A^r]$. Then the motive of a Severi-Brauer variety $SB(A)$ is a direct summand of the motive of a generalized Severi-Brauer variety $SB_d(B)$ if and only if

$$d \cdot r \equiv \pm 1 \mod n.$$

The cases $d = 1$ and $d = n - 1$ were considered in Lemma 4.2. From now on we assume $1 < d < n - 1$. Moreover, by duality we may assume $d \leq \lfloor \frac{n}{2} \rfloor$.

Proof ($\Rightarrow$) Consider the product $X = SB(A) \times SB_d(B^{op})$. According to Theorem 4.4 the subgroup of rational cycles of $\text{CH}^k(X_s)$ is generated (modulo prime $n$) by the cycles

$$\Delta_\lambda = \Delta_\lambda(c(\tau_1^r \otimes \kappa_d)).$$
where \( \lambda \) runs through all partitions with \(|\lambda| = k\). By [Fu98, Example A.9.1] we have

\[
\Delta_\lambda = \sum_{\mu \subseteq \lambda} d_{\lambda,\bar{\mu}} \cdot c_1(\tau^k) \cdot \Delta_\mu(c(\kappa_d)) = \\
= \sum_{i=0}^{k} (-r)^{k-i} \cdot H^{k-i} \times \left( \sum_{\mu \subseteq \lambda, |\mu| = i} d_{\lambda,\bar{\mu}} \omega_{\mu} \right),
\]

(7)

where \( \bar{\lambda} \) denotes the conjugate partition for \( \lambda \), i.e., obtained by interchanging rows and columns in the respective Young diagram, \( H \) is the class of a hyperplane section of \( \mathbb{P}^{n-1} \), \( \omega_{\mu} \) denotes the additive generator of \( CH_{|\mu|}(\mathbb{G}_d(n)) \) corresponding to a partition \( \mu \) and the coefficients \( d_{\lambda,\bar{\mu}} \) are the binomial determinants

\[
d_{\lambda,\bar{\mu}} = \left| \begin{pmatrix} n-i \\ \bar{\mu}_j + n - d - j \end{pmatrix} \right|_{1 \leq i,j \leq n-d}
\]

Let \( k = N \), where \( N = d(n-d) \) is the dimension of \( \mathbb{G}_d(n) \). In this codimension there is only one partition \( \lambda \) with \(|\lambda| = N\), namely, the maximal one \( \lambda = (n-d, \ldots, n-d) \). Let \( g \) denote the cycle (7) corresponding to this maximal partition, i.e.,

\[
g = \sum_{m=0}^{n-1} (-r)^{m} \cdot H^{m} \times \left( \sum_{|\mu| = N-m} d_{\lambda,\bar{\mu}} \omega_{\mu} \right),
\]

(8)

where \( \bar{\lambda} = (d, d, \ldots, d) \) and the coefficients \( d_{\lambda,\bar{\mu}} \) are given by

\[
d_{\lambda,\bar{\mu}} = \left| \begin{pmatrix} n-i \\ \bar{\mu}_j + n - d - j \end{pmatrix} \right|_{1 \leq i,j \leq n-d}
\]

From now on we denote the coefficient \( d_{\lambda,\bar{\mu}} \) by \( d_{\mu'} \), where \( \mu' \) is the dual partition \( (n-d - \mu_d, \ldots, n-d - \mu_1) \). Observe that \(|\mu'| = N - |\mu|\).

For an integer \( m \) denote by \( g^{(m)} \) the summand of (8) for the chosen index \( m \) and by \( d_{\mu'}^{(m)} \) the respective coefficients, i.e.,

\[
g^{(m)} = (-r)^{m} \cdot H^{m} \times \left( \sum_{|\mu'| = m} d_{\mu'}^{(m)} \omega_{\mu} \right)
\]

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Consider the summands \( g^{(0)} \) and \( g^{(1)} \). Since the Chow group \( \text{CH}(\mathbb{G}_d(n)) \) has only one additive generator in the last two codimensions \( N \) and \( N - 1 \) denoted by \( \omega_N \) and \( \omega_{N-1} \) respectively, we obtain that

\[
g^{(0)} = 1 \times \omega_N \quad \text{and} \quad g^{(1)} = -rd \cdot H \times \omega_{N-1}
\]  

(9)

Now we are ready to finish the \((\Rightarrow)\) part of the theorem.

Assume that the motive of \( \text{SB}_d(B^\text{op}) \) contains the motive of \( \text{SB}(A) \) as a direct summand. Then there must exist two rational cycles \( \alpha \in \text{CH}^{n-1}(\mathbb{P}^{n-1} \times \mathbb{G}_d(n)) \) and \( \beta \in \text{CH}^N(\mathbb{G}_d(n) \times \mathbb{P}^{n-1}) \) such that \( \beta \circ \alpha = \text{id} \). According to the composition rule \( (3) \) this can happen only if the coefficients before the monomials \( \omega_N \times 1 \) and \( \omega_{N-1} \times H \) of the cycle \( \beta \) are equal to \( \pm 1 \). But all rational cycles in codimension \( N \) are generated (modulo \( n \)) by the transposed cycle \( g^t \) which has coefficients 1 and \( -rd \) before the respective monomials (see \( (9) \)). This can only be possible if \( rd \equiv \pm 1 \mod n \).

**Proof** \((\Leftarrow)\) Assume that the congruence \( rd \equiv \pm 1 \mod n \) holds. We want to produce two rational cycles

\[
\alpha \in \text{CH}^{n-1}(\mathbb{P}^{n-1} \times \mathbb{G}_d(n)) \quad \text{and} \quad \beta \in \text{CH}^N(\mathbb{G}_d(n) \times \mathbb{P}^{n-1})
\]

such that \( \beta \circ \alpha = \text{id} \). According to the composition rule \( (3) \) this can happen only if the coefficients before the monomials \( \omega_N \times 1 \) and \( \omega_{N-1} \times H \) of the cycle \( \beta \) are equal to \( \pm 1 \). But all rational cycles in codimension \( N \) are generated (modulo \( n \)) by the transposed cycle \( g^t \) which has coefficients 1 and \( -rd \) before the respective monomials (see \( (9) \)). This can only be possible if \( rd \equiv \pm 1 \mod n \).

Assume \( rd \equiv 1 \mod n \). Consider the bundle \( \kappa_1 \otimes \Lambda^d(\tau_d) \) of rank \( n - 1 \) on the product \( \mathbb{P}^{n-1} \times \mathbb{G}_d(n) \) and define the cycle \( f \) as

\[
f = c_{n-1}(\kappa_1 \otimes \Lambda^d(\tau_d)) = \sum_{m=0}^{n-1} c_{n-1-m}(\kappa_1) c_1(\tau_d)^m =
\]

\[
= \sum_{m=0}^{n-1} (-1)^m \cdot H^{n-1-m} \times \omega_1^m = \sum_{m=0}^{n-1} (-1)^m \cdot H^{n-1-m} \times \left( \sum_{|\rho|=m} c_{\rho}^{(m)} \omega_\rho \right)
\]

Observe that \( f \) is a rational cycle, since \( [A] = [B^\otimes d] \). If \( rd \equiv -1 \mod n \), then we take the bundle \( \Lambda^{n-d}(\kappa_d) \) instead of \( \Lambda^d(\tau_d) \) and obtain the same formulae but without the coefficient \((-1)^m\).

The coefficients \( c_{\rho}^{(m)} \) appearing in the presentation of \( \omega_1^m \) in terms of additive generators \( \omega_\rho \) of \( \text{CH}^m(\mathbb{G}_d(n)) \) have the following nice property

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5.2 Lemma. For any partition $\rho$ with $m = |\rho| < n$ the coefficient $c^{(m)}_{\rho}$ is coprime with $n$.

Proof. According to [Fu98, Example 14.7.11.(ii)] we have

$$c^{(m)}_{\rho} = \deg(a_0, \ldots, a_{d-1}) = \frac{m!}{a_0! a_1! \ldots a_{d-1}!} \prod_{i>j}(a_i - a_j), \quad (10)$$

where the set $a = (a_0, a_1, \ldots, a_{d-1})$ is defined by $a_i = \rho_{d-i} + i, \ i = 0, \ldots, d-1$. Observe that the set $a$ has the property $0 \leq a_0 < \ldots < a_{d-1} \leq n-1$ and corresponds to a Schubert variety of dimension $m$ which is dual to the Schubert variety of codimension $m$ corresponding to the partition $\rho$ (see [Fu98, 14.7]). □

5.3 Remark. The constructed cycle $f \in \text{CH}^{n-1}(\mathbb{P}^{n-1} \times G_d(n))$ has the following geometric interpretation. Consider the Plücker embedding $SB_d(B) \rightarrow SB(\Lambda^d(B))$. Its graph defines a correspondence $\Gamma \in \text{CH}(SB(\Lambda^d(B)) \times SB_d(B))$. It is known that the motive of $SB(\Lambda^d(B))$ splits as a direct sum of twisted motives of $SB(A)$ (see [Ka96, Corollary 1.3.2]). Let $i : \mathcal{M}(SB(A)) \rightarrow \mathcal{M}(SB(\Lambda^d(B)))$ be the respective splitting. Then over the separable closure the composition $\Gamma \circ i$ coincides with the cycle $f$ corresponding to the case $rd \equiv -1 \mod n$. After replacing $B$ by $B^{op}$ the respective composition $\Gamma \circ i$ will give the cycle $f$ corresponding to the case $rd \equiv 1 \mod n$.

Consider the transposed cycle $g^t$ introduced in the first part of the proof

$$g^t = \sum_{m=0}^{n-1} (-1)^m \cdot \left( \sum_{|\mu'|=m} (r^m d^{(m)}_{\mu'}) \cdot \omega_{\mu'} \right) \times H^m$$

Consider the $m$-th summand of the composition $g^t \circ f$

$$(g^t \circ f)^{(m)} = \left( r^m \sum_{|\mu'|=m} c^{(m)}_{\mu'} \cdot d^{(m)}_{\mu'} \right) \cdot H^{n-1-m} \times H^m$$

Note that if $rd \equiv -1 \mod n$, then the coefficient $(-1)^m$ will appear on the right hand side.

Assume that the following formulae holds

$$\sum_{|\mu'|=m} c^{(m)}_{\mu'} \cdot d^{(m)}_{\mu'} \equiv d^m \mod n. \quad (11)$$
Then the coefficient of $(g^t \circ f)^{(m)}$ is congruent to 1 modulo $n$. We claim that it is possible to modify the cycles $g^t$ and $f$ by adding cycles divisible by $n$ in such a way that the coefficient of $(g^t \circ f)^{(m)}$ becomes equal to 1 for each $m$.

First, we modify the cycle $f$. For each $m$, $0 \leq m \leq n - 1$, we do the following procedure. Consider the $m$-th summand $f(m) = (-1)^m \sum_{|\mu'|=m} c_{\mu'} \cdot H^{n-1-m} \times \omega_{\mu'}$

and its coefficients $c_{\mu'}^{(m)}$. For $m = 0$ and 1 there is only one additive generator of $\text{CH}^m(\mathbb{G}_d(n))$ ($\omega_0$ and $\omega_1$) and the respective coefficients are $c\mu'(0) = c\mu'(1) = 1$. So we set $\alpha\mu'(0) = f(0)$ and $\alpha\mu'(1) = f(1)$. For $1 < m \leq n - 1$ the number of generators of $\text{CH}^m(\mathbb{G}_d(n))$ is greater than 1 and all the coefficients $c_{\mu'}^{(m)}$ are coprime with $n$, in particular, they are all non-zero. In this case we can modify each $c_{\mu'}^{(m)}$ modulo $n$ by adding cycles of the kind $a \cdot n\omega_{\mu'}$, $a \in \mathbb{Z}$, to the cycle $f(m)$ in such a way that the greatest common divisor of resulting coefficients, denoted by $\alpha_{\mu'}^{(m)}$, becomes equal to 1. As a result, we obtain a new cycle $\alpha^{(m)}$ having the coefficients $\alpha_{\mu'}$ instead of $c_{\mu'}$.

5.4 Definition. Define a cycle $\alpha$ as $\alpha = \sum_{m=0}^{n-1} \alpha^{(m)}$. By construction of $\alpha^{(m)}$ we have

- $\alpha^{(0)} = \alpha^{(1)} = 1$;
- $\alpha$ is rational (congruent modulo $n$ to $f$);
- all the coefficients $\alpha_{\mu'}$ are coprime with $n$;
- for each $m$ the g.c.d of coefficients $\alpha_{\mu'}^{(m)}$ is 1;
- for each $m$ the coefficient of $(g^t \circ \alpha)^{(m)}$ is congruent to 1 modulo $n$.

Next we modify the second cycle $g^t$. For each $m$ we apply the following obvious observation.

5.5 Lemma. Let $a_1, \ldots, a_l$ be a finite set of integers with g.c.d. = 1. Assume $\sum_i a_ib_i \equiv 1 \mod n$ for some integers $b_i$. Then there exist integers $b'_i \equiv b_i \mod n$ such that $\sum_i a_ib'_i = 1$. 

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to the congruence
\[ \sum_{|\mu'|=m} c^{(m)}_{\mu'} (r^m d^{(m)}_{\mu'}) \equiv 1 \mod n, \]
where \(a_i = \alpha^{(m)}_{\mu'}\) are coefficients of the cycle \(\alpha^{(m)}\) and \(b_i = r^m d^{(m)}_{\mu'}\) are coefficients of the cycle \((g^t)^{(m)}\). As a result, we obtain a new cycle \(\beta^{(m)}\) having coefficients \(b'_i\) instead of \(b_i\).

5.6 Definition. Define a cycle \(\beta\) as \(\beta = \sum_{m=0}^{n-1} \beta^{(m)}\). By construction of \(\beta^{(m)}\) we have

- \(\beta\) is rational (congruent modulo \(n\) to \(g^t\))
- for each \(m\) the coefficient of \((\beta \circ \alpha)^{(m)}\) is equal to 1.

To finish the proof observe that the constructed cycles \(\alpha\) and \(\beta\) are rational and the composition \(\beta \circ \alpha\) is the diagonal cycle. This implies that the composition \(\alpha \circ \beta\) is a rational projector which gives rise to the decomposition of motives with integral coefficients (see (11))

\[ \mathcal{M}(SB_d(B^{op})) \simeq \mathcal{M}(SB(A)) \oplus H \]
for some motive \(H\).

6 The proof of (11)

In the present section we prove the following

6.1 Lemma. \[ \sum_{|\mu'|=m} c^{(m)}_{\mu'} \cdot d^{(m)}_{\mu'} \equiv d^m \mod n, \]

Proof. First, we express the coefficients \(c_{\mu'}\) and \(d_{\mu'}\) in terms of binomial determinants. According to [Fu98, Example 14.7.11] we have

\[ c_{\mu'} = \deg(\omega_1^m \cdot \omega_{\mu}) = \frac{m! \cdot b_0! \cdot b_1! \ldots b_{d-1}!}{a_0! \cdot a_1! \ldots a_{d-1}!} \cdot \left| \begin{array}{c} (a_i) \\ (b_j) \end{array} \right|_{0 \leq i, j \leq d-1} \]  (12)

where \(a_i = n-d+i-\mu_i+1\) and \(b_j = j\) for \(i, j = 0 \ldots d-1\). Observe that the sets \(a = (a_0, \ldots, a_{d-1})\) and \(b = (b_0, \ldots, b_{d-1})\) with \(0 \leq a_0 < \ldots < a_{d-1} \leq n-1\) and
0 ≤ b_0 < ... < b_{d-1} ≤ n - 1 correspond to the classes of Schubert varieties \( \omega_\mu \) and \( \omega_N = \{ pt \} \) of dimensions \( \sum_{i=0}^{d-1} (a_i - i) = m \) and \( \sum_{j=0}^{d-1} (b_j - j) = 0 \) respectively.

From another hand side, we have

\[
d_{\mu'} = \left| \binom{n-i}{\mu_j + n - d - j} \right|_{1 \leq i,j \leq n-d} = \left| \binom{a_i}{b_j} \right|_{0 \leq i,j \leq n-d-1}
\]

where \( \tilde{a}_i = d + i \) and \( \tilde{b}_j = \tilde{\mu}_{n-d-j} + j \) for \( i, j = 0 \ldots n - d - 1 \). Here the sets \( \tilde{a} = (\tilde{a}_0, \ldots, \tilde{a}_{n-d-1}) \) and \( \tilde{b} = (\tilde{b}_0, \ldots, \tilde{b}_{n-d-1}) \) with \( 0 \leq \tilde{a}_0 < \ldots < \tilde{a}_{n-d-1} \leq n-1 \) and \( 0 \leq \tilde{b}_0 < \ldots < \tilde{b}_{n-d-1} \leq n - 1 \) correspond to the classes of Schubert varieties on the dual Grassmannian \( \mathbb{G}_{n-d}(n) \) of dimensions \( N \) and \( N - m \) respectively. Observe that the Schubert variety corresponding to the set \( \tilde{b} \) is dual to the Schubert variety corresponding to the partition \( \mu \). Hence, we have

\[
\omega_1^m = \frac{m! \cdot \tilde{b}_0! \tilde{b}_1! \ldots \tilde{b}_{n-d-1}!}{\tilde{a}_0! \tilde{a}_1! \ldots \tilde{a}_{n-d-1}!} \cdot d_{\mu'} \cdot \omega_\tilde{b}
\]

And by duality we obtain

\[
m! \cdot \tilde{b}_0! \tilde{b}_1! \ldots \tilde{b}_{n-d-1}! \cdot d_{\mu'} = \frac{m! \cdot b_0! b_1! \ldots b_{d-1}!}{a_0! a_1! \ldots a_{d-1}!} \cdot \left| \binom{a_i}{b_j} \right|_{0 \leq i,j \leq d-1}
\]

Since the integers \( a_0, \ldots, a_{d-1} \) form the complement of \( \tilde{b}_0, \ldots, \tilde{b}_{n-d-1} \) in the set of integers from 0 to \( n - 1 \) (see [Fu98, Example 14.7.5]), we obtain

\[
d_{\mu'} = \left| \binom{a_i}{b_j} \right|_{0 \leq i,j \leq d-1}
\] (13)

According to [Fu98, Example 14.7.11] we express the binomial determinant appearing in (12) and (13) in terms of Vandermonde determinant \( D_a = \prod_{i<j} (a_j - a_i) \) and get

\[
c_{\mu'} = \frac{m! \cdot D_a}{a_0! a_1! \ldots a_{d-1}!}, \quad d_{\mu'} = \frac{D_a}{0! 1! \ldots (d-1)!}.
\] (14)

Then the formulae (14) we want to prove turns into

\[
\frac{m!}{0! 1! \ldots (d-1)!} \cdot \sum_a \frac{D_a^2}{a_0! a_1! \ldots a_{d-1}!} \equiv d^m \mod n,
\] (15)

where the sum is taken over all sets of integers \( a = (a_0, \ldots, a_{d-1}) \) such that \( 0 \leq a_0 < \ldots < a_{d-1} \leq n - 1 \) and \( \sum_{i=0}^{d-1} a_i = m + \frac{d(d-1)}{2} \).
The case $d = 2$. In this case (15) follows from the following elementary fact

$$
\frac{m!}{2} \cdot \sum_{0 \leq x_1, x_2, x_1 + x_2 = m+1} \frac{(x_1 - x_2)^2}{x_1!x_2!} = 2^m
$$

(16)

To prove it consider the following chain of identities

$$
\frac{m! \cdot (x_1 - x_2)^2}{2 \cdot x_1!x_2!} = \frac{(m + 1)!}{x_1!(m + 1 - x_1)!} \cdot \left( \frac{m + 1}{2} - \frac{2x_1(m + 1 - x_1)}{m + 1} \right) =
$$

$$
= \frac{m + 1}{2} \left( \frac{m + 1}{x_1} \right) - 2m \left( \frac{m - 1}{x_1 - 1} \right)
$$

Then taking the sum we obtain the desired equality

$$
\sum_{x_1=0}^{m+1} \frac{m + 1}{2} \left( \frac{m + 1}{x_1} \right) - 2m \left( \frac{m - 1}{x_1 - 1} \right) = \frac{m + 1}{2} \cdot 2^{m+1} - 2m \cdot 2^{m-1} = 2^m
$$

Observe that for $m < n - 1$ the left hand side of (15) coincides with the left hand side of (16), hence, we obtain the equality in (15) (not just a congruence modulo $n$). For $m = n - 1$ the left hand side of (15) is equal to $2^{n-1} - n$.

The general case. It turns out that the identity (16) is a particular case of the following combinatorial identity known as Robinson-Schensted correspondence (see [Fu97, 4.3.(5)])

$$
\sum_{|\xi|=m} d_\xi(d) \cdot f^\xi = d^m,
$$

(17)

where the sum is taken over all partitions $\xi = (\xi_1 \geq \xi_2 \geq \ldots \geq \xi_d \geq 0)$ with $|\xi| = m$, $d_\xi(d)$ denote the number of Young tableaux on the shape $\xi$ whose entries are taken from the set $(1, \ldots, d)$ and $f^\xi$ denote the number of standard tableaux on the shape $\xi$.

By using Hook length formulae (see [Fu97, 4.3, Exercise 9]) we obtain

$$
f^\xi = \frac{m! \cdot D_l}{l_0! \ldots l_{d-1}!},
$$

(18)
where \( l = (l_0, \ldots, l_{d-1}) \) is a strictly increasing set of non-negative integers defined from the partition \( \xi = (\xi_1, \ldots, \xi_d) \) by \( l_{d-i} = \xi_i + d - i \) and \( D_l \) is the Vandermonde determinant for \( l \). By definition we have \( \sum_{i=0}^{d-1} (l_i - i) = m \).

By [GV85, Corollary 13] we have

\[
\binom{l_i j}{0 \leq i,j \leq d-1} = \frac{D_l}{0! \cdots (d-1)!}
\]

(19)

Observe that if the set \( l \) is bounded by \( n-1 \), i.e., \( l_{d-1} \leq n-1 \), the expressions (18) and (19) coincide with the expressions (14) defining the coefficients \( c_{\mu'} \) and \( d_{\mu'} \) respectively (take \( \xi = \mu' \) and \( a = l \)).

Assume that \( l_{d-1} \geq n \). Then \( l_{d-2} \leq n-1 \), i.e., \( l_{d-1} \) is the only element of \( l \) which exceeds \( n-1 \). Indeed, if this is not the case then we have a sequence of inequalities

\[
n - 1 + \frac{d(d-1)}{2} \geq m + \frac{d(d-1)}{2} = 0 + 1 + \ldots + l_{d-3} + l_{d-2} + l_{d-1} \geq 0 + 1 + \ldots + (d-3) + n + (n+1) = \frac{(d-3)(d-2)}{2} + 2n + 1
\]

which can be rewritten as

\[
d \geq \frac{n+5}{2}.
\]

But we have assumed from the beginning that \( d \leq \left[ \frac{n}{2} \right] \), contradiction.

Moreover, by the similar arguments one can check that \( l_{d-1} < 2n \).

Now consider the product \( x = d_\xi(d) \cdot f^\xi \), when \( \xi \) is a partition for which the respective \( l_{d-1} \geq n \). Since \( n \) is prime, the denominator of \( x \) is divisible by \( n \) but not by \( n^2 \). Since \( x \) is an integer, the numerator of \( x \) must be divisible by \( n \) as well. From this we conclude that \( D_l \) is divisible by \( n \). But the numerator of \( x \) is the product of the square of \( D_l \) by something, hence, it is divisible by \( n^2 \). So we obtain that \( x \) must be divisible by \( n \). This means that modulo \( n \) the left hand side of (17) is congruent to the left hand side of (15).

\hfill \Box

7 Grassmannian \( \mathbb{G}(2, n) \)

The goal of the present is to extend Theorem 1.1 to the case of algebras of an arbitrary odd degree \( n \geq 5 \) and \( d = 2 \).
7.1 Theorem. Let $A$ and $B$ be two central simple algebras of an odd degree $n$ over a field $F$ generating the same subgroup in the Brauer group $\text{Br}(F)$. Then the motive of a Severi-Brauer variety $\text{SB}(A)$ is a direct summand of the motive of a generalized Severi-Brauer variety $\text{SB}_2(B)$ if and only if

$$[A] = \pm 2[B] \text{ in } \text{Br}(F).$$

Proof. Consider the cycles $g$ and $f$ defined in Section 5. Clearly, $g$ and $f$ are rational, since Lemma 2.6 holds for any algebras.

$(\Rightarrow)$ Repeating the arguments of 3.3 for an odd integer $n$ and $d = 2$ one obtains (in view of [Ka95, Corollary 4]) that the subgroup of rational cycles of $\text{CH}(X_s)$ modulo $n$ is generated in codimension $N = \dim \mathbb{G}_d(n)$ by the cycles

$$\frac{n}{(n,N-|\lambda|)} \cdot H^{N-|\lambda|} \times 1 \cdot \Delta_\lambda(c(Q_s)),$$

for all partitions $\lambda$ with $N - (n - 1) \leq |\lambda| \leq N$. Observe that in the case $|\lambda| = N$ one obtains precisely the cycle $g$.

Since the coefficient $\frac{n}{(n,N-|\lambda|)}$ is divisible by $n$ when $|\lambda| = N - 1$, we may exclude the cycles with $|\lambda| = N - 1$ from the set of generators. The latter implies that the last two non-trivial monomials (see (9)) of any rational cycle in $\text{CH}^N(X_s)$ come only from the cycle $g$. And we finish the proof as in Section 5.

$(\Leftarrow)$ We claim that it is still possible to modify cycles $f$ and $g'$ modulo $n$ in such a way that the obtained rational cycles $\alpha$ and $\beta$ will satisfy the property $\beta \circ \alpha = \text{id}$, hence, providing the desired motivic decomposition for $\mathcal{M}(\text{SB}_2(B))$.

The following lemma allows us to take $\alpha = f$ (see Definition 5.4).

7.2 Lemma. Fix a codimension $m$, $0 \leq m \leq n - 1$. For any partition $\rho$ with $|\rho| = m$ let $c^{(m)}_\rho$ be the coefficient appearing in Lemma 5.2. Then the greatest common divisor of all the coefficients $c^{(m)}_\rho$ is 1.

Proof. Since $d = 2$, for any codimension $m$ which is less than $n - 1$, there is a partition $\rho = (m,0)$ with $c^{(m)}_\rho = 1$. In the last codimension $m = n - 1$ consider two partitions $\rho = (1,n - 2)$ and $\rho' = (2,n - 3)$. By (10) the respective coefficients $c^{(n-1)}_\rho$ and $c^{(n-1)}_{\rho'}$ are equal to $\deg(1,n - 1) = n - 2$ and $\deg(2,n - 2) = \frac{(n-1)(n-4)}{2}$. Since $n$ is odd, $n - 2$ and $\frac{(n-1)(n-4)}{2}$ are coprime. \qed
Choose $\beta$ as in the Section 5. To finish the proof we have to prove the congruence (11). But this was done already in Section 3 where we treated the case $d = 2$. Namely, we proved that for $m < n − 1$ there is an exact equality (not just a congruence modulo $n$) and for $m = n − 1$ the left hand side of (11) is, indeed, equal to $2^{n−1} − n$. This finishes the proof of Theorem 7.1.

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