On the Grace-Danielsson inequality for tetrahedra

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Abstract

The difference between the (squared) sides of the Grace-Danielsson inequality for tetrahedra will be represented as a sum of two nonnegative terms. This gives another proof of the inequality. Examining the denominator allows us to characterize the infinite triangular prism as a degenerate tetrahedron. We give conditions for equality (for a zero gap) as well, and some examples are included.

Keywords: tetrahedron, Grace-Danielsson inequality, inradius, circumradius

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1. Introduction

It is a classic result of 18th century mathematics (Chapple, Euler) that

\[ d^2 = R(R - 2r) \]  

holds for the distance \( d \) between the circumcenter and the incenter of a triangle with circumradius \( R \) and inradius \( r \). Although a wish to generalize it to tetrahedra failed (cf. Gergonne, Durrande, 19th century), an inequality

\[ d^2 \leq (R + r)(R - 3r) \]  

is still valid for all tetrahedra with circumradius \( R \), inradius \( r \) and distance \( d \) between them (see Grace [1], Milne [3] for further references. Milne and others cite Danielsson [2] which is not readily available). The ideas of Milne, using quantum information theory can now be found in his recently published dissertation [9] as well.

Now, it is a natural question, how big is the difference between the right and left hand sides. Our aim is to become a representation, from which inequality [2] evidently follows. To this we first rewrite the inequality as

\[ R^2 - d^2 - 3r^2 \geq 2rR, \]

and square both sides to avoid irrationality, getting thus a purely algebraic form.

When we met this inequality, we tried to prove it by means of semidefinite programming, more concretely, by sum of squares programming. However, this attempt failed due to perpetual “Out of memory” problems. This is why we
were forced to have a closer look at this representation problem and to write our own routines in Maple and Matlab (the latter for quickening and/or checking purposes). Thus the result obtained also serves as an illustration for the positive answer for Hilbert’s 17th problem ².

As regards variables used, we prefer to choose the coordinates of the three vertices (forming the so-called basic face) and the tangent point of the insphere lying on the basic face, as well as the inradius $r$ – and calculate the coordinates of the remaining vertex, the circumradius $R$ and the distance $d$. With this choice we get the wanted rational representation. Our method can be hence considered – apart from the use of programming languages – elementary.

Using coordinates, as is known, goes hand in hand with long calculations, however the relevant formulas here can be well managed by means of the symbolic programming language Maple. The details will be given in the proof of Theorem 1, followed by some special cases (concerning the choice of the tangent point), illustrative examples, and by investigating the degeneracy also in two dimensions (for a triangle instead of for a tetrahedron).

2. The main theorem

At first we formulate the theorem, giving a two-term representation for the gap, where the quantities on the right hand side will be explained in the course of the proof.

Theorem 1. For a tetrahedron with inradius $r$, circumradius $R$, and distance $d$ between the incenter and circumcenter we have

$$
(R^2 - d^2 - 3r^2)^2 - (2rR)^2 = r^4 \frac{(u_1r^2 + v_1)^2 + (u_2r^2 + v_2)^2}{a_0(A - Br^2)}
$$

with polynomials $u_1, v_1, u_2, v_2, A, B, a_0$, where $A, B, a_0$ are positive, and all these quantities depend only on one face, called the basic triangle.

Proof. First we describe our method in a more detailed form. Let $x, y, z$ be the vertices of the basic triangle, considered lying on the horizontal plane. Let $c$ be an interior point of it, and $r > 0$ be a given number. We will find the fourth vertex $w$ such that the insphere of the tetrahedron $\{x, y, z, w\}$ has radius $r$ and touches the basic triangle at its inner point $c$.

To this aim we need to draw the three tangent planes and then to find their common point $w$. Since a too large value $r$ contradicts the requirements, it is essential to know the largest possible, the critical value of the inradius. In case of the critical situation all the three edges are parallel, forming a semi-infinite triangular prism. (Example 3 illustrates this behavior, while the planar version is discussed in Lemma 3.)

Denote by $x = (x_1, x_2, 0)$, $y = (y_1, y_2, 0)$, $z = (z_1, z_2, 0)$ the vertices of the basic triangle (oriented counterclockwise), by $(c_1, c_2, 0)$ the interior point chosen, and let $r > 0$ be a given number. (Note that we write $(x_1, x_2, x_3)$ for coordinates, instead of $(x, y, z)$, used in some cases.)
As a first step, we determine the three tangent points \( X, Y, Z \) on the insphere. Here \( X, Y, Z \) are opposite to vertices \( x, y, z \), resp. Calculate then the intersection point \( w \) of the tangent planes spanned by the triangles \( \{ x, y, Z \}, \{ y, z, X \} \) and \( \{ z, x, Y \} \). Although the first two coordinates \( w_1, w_2 \) of \( w \) are complicated, the third – and most relevant – can be handled well. With the quantities

\[
\begin{align*}
a_0 &= 2 \text{ area } \Delta xyz = x_1y_2 + y_1z_2 + z_1x_2 - y_1x_2 - z_1y_2 - x_1z_2, \\
a_x &= 2 \text{ area } \Delta cyz = c_1y_2 + y_1z_2 + z_1c_2 - y_1c_2 - z_1y_2 - c_1z_2, \\
a_y &= 2 \text{ area } \Delta czx = x_1c_2 + c_1z_2 + z_1x_2 - c_1x_2 - z_1c_2 - x_1z_2, \\
a_z &= 2 \text{ area } \Delta yc x = x_1y_2 + y_1c_2 + c_1x_2 - y_1x_2 - c_1y_2 - x_1c_2,
\end{align*}
\]

and

\[
A = a_xa_ya_z, \quad B = \|x\|^2a_x + \|y\|^2a_y + \|z\|^2a_z - \|c\|^2a_0
\]

we get the formula

\[
w_3 = \frac{2rA}{A - Br^2}, \quad (4)
\]

Here the quantity \( A \) is – as a product of three triangle areas – obviously positive, while the same property for \( B \) will be proved in Lemma 1 below. Then it is seen that \( w_3 > 0 \) for \( r \) sufficiently small, and that its critical value is

\[
r_{crit} = \sqrt{A/B},
\]

where the tetrahedron becomes a prism. Notice that by means of this critical value the third coordinate of \( w \) can be rewritten as

\[
w_3 = \frac{2r r_{crit}^2}{r_{crit}^2 - r^2}, \quad (5)
\]

Now we determine the circumcenter \( o \). For this case – in contrast with vertex \( w \) – the first and second coordinates are relatively simple, while the third one is difficult (but not needed here). We have

\[
\begin{align*}
2a_0 o_1 &= \|x\|^2(y_2 - z_2) + \|y\|^2(z_2 - x_2) + \|z\|^2(x_2 - y_2), \\
2a_0 o_2 &= \|x\|^2(z_1 - y_1) + \|y\|^2(x_1 - z_1) + \|z\|^2(y_1 - x_1).
\end{align*}
\]

Observe that the orthogonal projection \((o_1, o_2, 0)\) of the circumcenter \((o_1, o_2, o_3)\) of the tetrahedron coincides with the circumcenter of the basic triangle, due to elementary considerations, hence \(o_1, o_2\) do not depend on \(w\).

Now we are in the position to calculate the circumradius \( R = \|o - x\| \) and the distance \( d = \|o - c\| \) of the circumcenter and the incenter. In possession of these, a quite circumstantial manipulation in Maple is needed to get a more concise form for the gap, resulting in the two term-representation of the theorem. Of the remaining four variables, \( u_1 \) and \( u_2 \) have a fairly simple form:

\[
u_1 = 4a_0(c_1 - o_1), \quad u_2 = 4a_0(c_2 - o_2), \quad (6)
\]
while \(v_1\) and \(v_2\) are polynomials of degree five with \(108-108\) terms, but they can be rewritten into a something shorter sum-of-products form, see Appendix. The subtask of determining \(v_1\) and \(v_2\) knowing the \(u_i\)-s leads to the following.

Assume that \(\alpha = u_1^2 + u_2^2\), \(\beta\), \(\gamma\) are known and \(v_1\), \(v_2\) are asked to satisfy

\[
\alpha r^4 + \beta r^2 + \gamma = (u_1 r^2 + v_1)^2 + (u_2 r^2 + v_2)^2,
\]

then the solution is given by

\[
v_1 = \frac{u_1 \beta \pm u_2 \text{dis}}{2\alpha}, \quad v_2 = \frac{u_2 \beta \mp u_1 \text{dis}}{2\alpha}, \quad \text{dis} = \sqrt{4 \alpha \gamma - \beta^2}.
\]

Since the discriminant \(\text{dis}\) (having originally as much as 370900 terms!) proves to be a complete square, we arrive at the representation (3), expressing the gap as a rational function of the variables. The theorem is proved. \(\Box\)

Since the right hand side in (3) is non-negative, we have thus another proof for the Grace-Danielsson inequality. It remains to prove the positivity of \(B\).

**Lemma 1.** The polynomial \(B\) in the denominator of (3) is positive, further, \(B\) is independent of translation.

**Proof.** Let

\[
t_x = \frac{a_x}{a_0}, \quad t_y = \frac{a_y}{a_0}, \quad t_z = \frac{a_z}{a_0},
\]

then these positive numbers sum up to one and we have to show

\[
\|c\|^2 \leq \|x\|^2 t_x + \|y\|^2 t_y + \|z\|^2 t_z.
\]

The well known barycentric formula

\[
c = \frac{a_x x + a_y y + a_z z}{a_x + a_y + a_z} = t_x x + t_y y + t_z z
\]

implies in our case

\[
\|c\|^2 < \left( t_x \|x\| + t_y \|y\| + t_z \|z\| \right)^2,
\]

whence the Cauchy-Schwarz inequality applied for vectors

\[
(\sqrt{t_x}, \sqrt{t_y}, \sqrt{t_z}), \quad (\|x\| \sqrt{t_x}, \|y\| \sqrt{t_y}, \|z\| \sqrt{t_z})
\]

yields the positivity of \(B\). To prove the second assertion, translate now the vertices by \(h\) to get

\[
B(h) = \|x - h\|^2 a_x + \|y - h\|^2 a_y + \|z - h\|^2 a_x - \|c - h\|^2 a_0
\]

\[
= \|x\|^2 a_x + \|y\|^2 a_y + \|z\|^2 a_z - \|c\|^2 a_0
\]

\[
- 2h^T (a_x x + a_y y + a_z z - a_0 c)
\]

\[
+ \|h\|^2 (a_x + a_y + a_z - a_0).
\]
The coefficient of \( \|h\|^2 \) is obviously zero by additivity of the area, while the linear term disappears because of the basic property of barycentric coordinates. Thus \( B(h) \equiv B(0) = B \), which was to be shown. □

Now we give a numerical example using a Heronian tetrahedron, for which the essential quantities all are rational (cf. [5]), justifying thus the use of Maple.

**Example 1.** Let the vertices of the basic triangle, and the tangent point of the insphere be \( x = (0,0,0) \), \( y = (154,0,0) \), \( z = (55,132,0) \), \( c = (90,48,0) \), and choose \( r = 10 \). Then the fourth vertex and the circumcenter become

\[
w = \left( \frac{215490}{2309}, \frac{339416}{6927}, \frac{49280}{2309} \right), \quad o = \left( \frac{77}{8}, \frac{363}{8}, -\frac{15818598389}{93098880} \right).
\]

Further we have

\[
R^2 = \frac{319462309835987155321}{8667401457254400}, \quad d^2 = \frac{282073185661355308921}{8667401457254400}, \quad f = \frac{198873308525}{145467},
\]

\[
a_0 = 20328, \quad a_x = 3696, \quad a_y = 9240, \quad a_z = 7392, \quad A = 252444487680, \quad B = 158802336, \quad u_1 = 1057056, \quad v_1 = -7868399616, \quad u_2 = 213444, \quad v_2 = -2363251968.
\]

**Question.** Since the tetrahedron, a 3-dimensional simplex, has a two-term gap given by the right hand side of (3), one can put the question: how many terms (if any) can represent the gap for a simplex in \( n > 3 \) dimensions?

See to this John Baez’s blog [4] citing Greg Egan, for the concrete form

\[d^2 \leq (R + (n - 2)r)(R - nr)\]

of the inequality in \( n \) dimensions – or the equivalent, “Pythagorean” form

\[d^2 + (n - 1)r^2 \leq (R - r)^2. \tag{7}\]

Back to \( n = 3 \), the next example shows that equality in (2) (or in (7)) can occur for non-regular tetrahedra, in contrast with Euler’s inequality \( R \geq 2r \), where equality is valid only for regular triangles.

**Example 2.** Let the vertices of the basic triangle, the inner point chosen, and the inradius be

\[x = (-1,0,0), \quad y = (1,0,0), \quad z = (0,\sqrt{3},0), \quad c = (0,\frac{1}{\sqrt{3}},0), \quad r = \frac{1}{2}.
\]

From these data the method gives the fourth vertex \( w \), the circumcenter \( o \),

\[
w = \left( 0, \frac{1}{\sqrt{3}}, \frac{4}{3} \right), \quad o = \left( 0, \frac{1}{\sqrt{3}}, \frac{11}{6} \right),
\]

and the further parameters

\[
a_x = a_y = a_z = \frac{2\sqrt{3}}{3}, \quad A = \frac{8\sqrt{3}}{9}, \quad B = \frac{8\sqrt{3}}{3}, \quad R = \frac{13}{6}, \quad d = \frac{4}{3}.
\]

Therefore (7) turns into equality thanks to the Pythagorean identity \( 3^2 + 4^2 = 5^2 \). This result also follows by Lemma 2 below.
3. Some special cases and examples

The next lemma describes the gap with disappearing \( u_i \)-s, and \( v_i \)-s, resp.

**Lemma 2.** (a) For a basic triangle touched by the insphere at its circumcenter \( c_1 = o_1, c_2 = o_2 \) the \( u_i \)-s vanish and the gap (3) is given by

\[
\frac{r^2}{64 a_0^3 (A - Br^2)} \|x - y\|^4 \|y - z\|^4 \|z - x\|^4 (g_1^2 + g_2^2) \tag{8}
\]

with

\[
g_1 = (x_1^2 + 3x_2^2)(z_2 - y_2) + (y_1^2 + 3y_2^2)(x_2 - z_2) + (z_1^2 + 3z_2^2)(y_2 - z_2) + 2x_1x_2(z_1 - y_1) + 2y_1y_2(x_1 - z_1) + 2z_1z_2(y_1 - x_1),
\]

\[
g_2 = (x_2^2 + 3x_1^2)(y_1 - z_1) + (y_2^2 + 3y_1^2)(z_1 - x_1) + (z_2^2 + 3z_1^2)(x_1 - y_1) + 2x_1x_2(y_2 - z_2) + 2y_1y_2(z_2 - x_2) + 2z_1z_2(x_2 - y_2).
\]

(b) For a basic triangle touched by the insphere at its incenter, the \( v_i \)-s vanish and the gap is given by

\[
\frac{16 a_0 r^6}{A - Br^2} ((c_1 - o_1)^2 + (c_2 - o_2)^2). \tag{9}
\]

(c) Furthermore, if the basic triangle is regular with \( c_1 = o_1, c_2 = o_2 \), then the gap is 0.

**Proof.**

(a) The statement follows by (8). Factorization by Maple gives

\[
v_i = \frac{1}{8 a_0^3} \|x - y\|^2 \|y - z\|^2 \|z - x\|^2 g_i, \quad i = 1, 2,
\]

with the third degree polynomials \( g_1, g_2 \) above.

(b) If we calculate the incenter (by using barycentric coordinates), it appears that \( v_1 = v_2 = 0 \) holds, and the result follows from (9).

(c) In an equilateral triangle circumcenter and incenter are coincident at the center of rotational symmetry, so if the basic triangle is equilateral, and the touching point is the center of symmetry, the gap is given either by case (b), equation (9), i.e. is 0 \( (c_i = o_i) \) or by case (a), equation (8). Both equations having to give the same result, it implies that the term \( (g_1^2 + g_2^2) \) in (8) has to be 0 because no other terms of this equation can cancel. Therefore the polynomials \( g_1 \) and \( g_2 \) of (3) are nil. This can be checked using Maple: to force the basic triangle to be regular, we further substitute

\[
z_1 = \frac{1}{2}(x_1 + y_1 + \sqrt{3}(y_2 - x_2)), \quad z_2 = \frac{1}{2}(x_2 + y_2 + \sqrt{3}(x_1 - y_1))
\]

in \( g_1, g_2 \), to get \( g_1 = g_2 = 0 \). □
Figure 1. The circumcenter is inside the tetrahedron

Remark 1. It was the referee’s idea to give a Mapleless proof for the zero gap property (c). Also, he provided a proof (essentially part (c2) below), where he obtained (11) below by using (5), as a consequence of (4).

We added (c1) to get a self-contained proof for (11), and (c3) to draw the attention to cases different from that shown in Figure 1.

Theorem 2. If a tetrahedron has a face which is an equilateral triangle and an insphere which touches this face at its center of rotational symmetry, then the gap $G = R^2 - d^2 - 3r^2 - 2rR$ of the Grace-Danielsson inequality is always zero.

Proof. We derive two relations, involving $(w_3, r)$ and $(w_3, R)$, resp. Like in the proof of Theorem 1, we use lower case letters for vertices, and capitals for the tangent points (e.g. $Z$ is the tangent point of the insphere on the face opposite to $z$). Denote by $l$ the edge length of the basic equilateral triangle, i.e. let $l = \|x - y\| = \|y - z\| = \|z - x\|$, then we have $\|(x + y)/2 - c\| = l\sqrt{3}/6$ and $\|c - z\| = l\sqrt{3}/3$, where $c = (0, 0, 0)$ is the origin.

(c1) Let inc = $(0, 0, r)$ be the center of the inscribed sphere, then $\|w - Z\|$ can be determined from the rectangular triangle $\Delta(w, Z, \text{inc})$ using Pythagoras’ theorem, cf. Figure 1:

$$\|w - Z\|^2 = \|w - \text{inc}\|^2 - \|z - \text{inc}\|^2 = (w_3 - r)^2 - r^2 = w_3(w_3 - 2r),$$

while the similarity of this triangle to $\Delta(w, \frac{w_3 + w}{2}, c)$ implies

$$\frac{\sqrt{w_3(w_3 - 2r)}}{r} = \frac{w_3}{l\sqrt{3}/6}. $$
This immediately gives
\[ w_3 - 2r = \frac{12r^2}{l^2} w_3, \] (10)
which implies
\[ w_3 = \frac{2l^2r}{l^2 - 12r^2}. \] (11)

(c2) Calculating the circumradius by Pythagoras’ theorem applied to the rectangular triangle $\Delta(o, c, z)$ gives (see Figure 1):
\[ R^2 = \|o - c\|^2 + \|c - z\|^2 = (w_3 - R)^2 + \frac{l^2}{3} \]
with
\[ w_3 (2R - w_3) = \frac{l^2}{3}. \] (12)
Therefore, by virtue of (10) and (12) it follows that
\[ G = (R - r)^2 - d^2 - 4r^2 = (R - r)^2 - (R + r - w_3)^2 \]
\[ = (w_3 - 2r) (2R - w_3) - 4r^2 = 4r^2 - 4r^2 = 0. \]

(c3) Note finally, that the order of points $w, o, inc, c$ is not necessarily that given in Figure 1, hence the relationship between $w_3, R, r$ and $d$ varies, as well. The precise formula for the distance $d$ is
\[ d = \begin{cases} R + r - w_3, & \text{if } 0 < r < r_{reg}, \\ w_3 - R - r, & \text{if } r_{reg} < r < r_{crit} \end{cases} \]
where $r_{reg} = \frac{l}{2\sqrt{6}}$ is the inradius of the regular tetrahedron (in which case $w_3 = R + r$ and $d = 0$ hold), hence $|d| = |R + r - w_3|$, and the unified formula $d^2 = (R + r - w_3)^2$ is valid. □

In what follows, we examine the analogous planar problem for triangles. Our aim is to obtain a formula for the critical value of the inradius.

Problem. Given the line segment $I = [0, 1]$ with an interior point $p$, $0 < p < 1$, find the supremum $r_{crit}$ of positive numbers $r$, for which $r$ is the inradius of some triangle with one side equal to $I$. First we illustrate the situation.

Remark 2. Figure 1 below shows a small incircle, resulting in triangle $\Delta ABU$, a critical circle (giving two parallel straight lines $l_1$ and $l_2$ instead of a triangle), and a (too) big circle, for which the tangent lines intersect at $V$, on the other (lower) side of the horizontal axis. The big circle is then an ex-circle for triangle $\Delta AVB$. The data for this plot are
\[ p = 0.4, \ r_{small} = 0.2, \ r_{crit} = \sqrt{0.24} \approx 0.49, \ r_{big} = 0.8. \]
Note that the center $K = (0.4, 0.49)$ of the critical circle is quite close to – but not identical with – the vertex $U = (0.36, 0.48)$ of the small right triangle.
**Lemma 3.** For the above planar problem we have \( r_{\text{crit}} = \sqrt{p(1-p)} \).

**Proof.** Triangle \( \Delta ABK \) with \( A = (0,0), B = (1,0), K = (p, r_{\text{crit}}) \) is a right triangle. To this draw the tangent line to the critical circle, parallel to \( AB \). Then \( K \) is the centre of the rhomb bordered by the lines \( l_1, l_2 \) and the two horizontal tangent lines, hence \( \angle BKA \) is a right angle indeed.

Using now the well known property: "the altitude to the hypotenuse is the geometric mean of the two segments of the hypotenuse" of rectangular triangles, the statement follows. \( \square \)

After this evasion we go back to three dimensions. In the next example we calculate the critical inradius, however, in contrast with the two dimensional case, we can do it only for special data.

**Example 3.** Let the vertices of the basic triangle be

\[
x = (-\sqrt{2}, -1, 0), \quad y = (\sqrt{2}, -1, 0), \quad z = (0, 1, 0),
\]

and let the origo be the given interior point. We show that \( r_{\text{crit}} = 1/\sqrt{2} \). Take for this the sphere \( S \) of radius \( r = 1/\sqrt{2} \) centered at \( (0, 0, 1/\sqrt{2}) \), and determine the tangent points \( X, Y, Z \) of the three non-horizontal faces. They are

\[
X = \left( \frac{2\sqrt{2} \cdot 2}{5}, \frac{2\sqrt{2}}{5} \right), \quad Y = \left( -\frac{2\sqrt{2} \cdot 2}{5}, \frac{2\sqrt{2}}{5} \right), \quad Z = \left( 0, -\frac{2}{3}, \frac{2\sqrt{2}}{3} \right).
\]
The pairwise intersections of the tangent planes give the rays
\[
\left(\sqrt{2}, b, 2\sqrt{2}(1 + b)\right), \quad \left(-\sqrt{2}, b, 2\sqrt{2}(1 + b)\right), \quad \left(0, b, 2\sqrt{2}(b - 1)\right)
\]
with a free parameter \(b\). Since they share the common direction \((0, 1, 2\sqrt{2})\), the result follows.

According to Maple, the gap in (3) for these vertices \((x, y, z)\) is \(r^2(1 - 2r^2)\), showing another evidence for equality \(r_{\text{crit}} = 1/\sqrt{2}\). However the quickest way is to show that the points \(X, Y, Z\) of tangency with the centre \(K = (0, 0, 1/\sqrt{2})\) of the sphere are coplanar (cf. [7]), i.e.

\[
\begin{vmatrix}
2\sqrt{2} & 2 & 2\sqrt{2} & 1 \\
-2\sqrt{2} & 2 & -2\sqrt{2} & 1 \\
0 & -2\sqrt{2} & 2\sqrt{2} & 1 \\
0 & 0 & -1/\sqrt{2} & 1
\end{vmatrix} = 0.
\]

Finally we mention Pech’s method [8] proving Euler’s inequality \(R \geq 2r\) for triangles (a consequence of (1)), to show another idea making use of a computer. He writes down the known equations
\[
r - \frac{2K}{a + b + c} = 0, \quad R - \frac{abc}{4K} = 0, \quad R - 2r - k = 0,
\]
as well as Heron’s formula
\[
16K^2 - (a + b + c)(a + b - c)(a - b + c)(-a + b + c) = 0,
\]
including \(r, R\), the area \(K\), the three sides \(a, b, c\) of a triangle, and a slack variable \(k\). Using the CoCoA (short for Computations in Commutative Algebra) system he finds that \(R \geq 2r\) holds iff
\[
a^3 - a^2b - ab^2 + b^3 - a^2c - b^2c - ac^2 - bc^2 + c^3 \geq 0,
\]
which is easily shown by observing that this polynomial equals
\[
\frac{1}{2} [(a + b - c)(a - b)^2 + (b + c - a)(b - c)^2 + (c + a - b)(c - a)^2].
\]

Note that Pech’s method is much simpler than ours, thanks to its coordinate-free approach, however, it applies to the planar case \(n = 2\), and – on the other hand –, it does not concern the distance \(d\). For \(n = 3\) it would be a challenge to express \(d\) by help of lengths only – without using coordinates.

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Appendix A. Appendix

The polynomials $v_1$ and $v_2$ have the following, fairly symmetrical form:

\[
v_1 = c_1 (c_1^2 + c_2^2) a_0 - 2c_1^2 a_0 a_1 \\
+ y_1 z_1(y_2 - z_2)(c_1^2 - x_1^2 - (c_2 - x_2)^2) \\
+ z_1 x_1(z_2 - x_2)(c_1^2 - y_1^2 - (c_2 - y_2)^2) \\
+ x_1 y_1(x_2 - y_2)(c_1^2 - z_1^2 - (c_2 - z_2)^2) \\
+ c_1^2(x_2^2 - y_2^2) + y_2^2(x_2 - z_2) + z_2^2(y_2 - x_2) \\
+ c_1^2(x_1^2 - z_2^2) + y_1^2(x_2 - z_2) + z_1^2(y_2 - x_2) \\
+ x_1^2(y_2 - z_2)(c_1(y_1 + z_1) + c_2(y_2 + z_2) + z_2 y_2) \\
+ y_1^2(z_2 - x_2)(c_1(z_1 + x_1) + c_2(z_2 + x_2) - z_2 x_2) \\
+ z_1^2(x_2 - y_2)(c_1(x_1 + y_1) + c_2(x_2 + y_2) - x_2 y_2) \\
+ 2c_1c_2(x_1 x_2(y_2 - z_2) + y_1 y_2(z_2 - x_2) + z_1 z_2(x_2 - y_2)) \\
+ 2c_1c_2(x_1(z_2^2 - y_2^2) + y_1(x_2^2 - z_2^2) + z_1(y_2^2 - x_2^2)) \\
+ c_1(x_1 x_2(z_2^2 - y_2^2) + y_1 y_2(x_2^2 - z_2^2) + z_1 z_2(y_2^2 - x_2^2)) \\
+ 3c_1 x_1 y_2 z_2(y_2 - z_2) + x_2 y_1 z_2(z_2 - x_2) + x_2 y_2 z_1(x_2 - y_2),
\]

and

\[
v_2 = c_2 (c_1^2 + c_2^2) a_0 - 2c_2^2 a_0 a_2 \\
= y_2 z_2(y_1 - z_1)((c_1 - x_1)^2 - c_2^2 + x_2^2) \\
= x_2 z_2(z_1 - x_1)((c_1 - y_1)^2 - c_2^2 + y_2^2) \\
= x_2 y_2(z_1 - y_1)((c_1 - z_1)^2 - c_2^2 + z_2^2) \\
= c_1^2(x_1^2(y_1 - z_1) + y_1^2(z_1 - x_1) + z_1^2(x_1 - y_1)) \\
= c_2^2(x_1^2(y_1 - z_1) + y_1^2(z_1 - x_1) + z_1^2(x_1 - y_1)) \\
= x_2^2(z_1 - y_1)(c_1(y_1 + z_1) + c_2(y_2 + z_2) - y_1 z_1) \\
= y_2^2(x_1 - z_1)(c_1(z_1 + x_1) + c_2(z_2 + x_2) - z_1 x_1) \\
= z_2^2(y_1 - x_1)(c_1(x_1 + y_1) + c_2(x_2 + y_2) - x_1 y_1) \\
= 2c_1c_2(x_1 x_2(z_1 - y_1) + y_1 y_2(x_1 - z_1) + z_1 z_2(y_1 - x_1)) \\
= 2c_1c_2(x_1^2(y_1^2 - z_1^2) + y_1^2(z_1^2 - x_1^2) + z_1^2(x_1^2 - y_1^2)) \\
= c_1(x_1 x_2(y_1^2 - z_1^2) + y_1 y_2(z_1^2 - x_1^2) + z_1^2(x_1^2 - y_1^2)) \\
= 3c_2(x_1^2(y_2 z_1 - y_1 z_2) + y_1^2(x_1 z_2 - x_2 z_1) + z_1^2(x_2 y_1 - x_1 y_2)).
\]

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