Half-linear differential equations of fourth order: oscillation criteria of solutions

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Abstract

In this paper, we are concerned with the oscillation of solutions to a class of fourth-order delay differential equations with \( p \)-Laplacian like operators

\[
(r(t)|x'''(t)|^{p_1-2} x'''(t))' + q(t)|x(\tau(t))|^{p_2-2} x(\tau(t)) = 0
\]

under the condition

\[
\int_{t_0}^{\infty} \frac{1}{r^{1/p_1-1}(s)} \, ds = \infty.
\]

Also, we establish new criteria for the oscillatory behavior of fourth-order differential equations with middle term

\[
(r(t)|x'''(t)|^{p_1-2} x''(t))' + \sigma(t)|x'''(t)|^{p_1-2} x'''(t) + q(t)|x(\tau(t))|^{p_2-2} x(\tau(t)) = 0
\]

under the condition

\[
\int_{t_0}^{\infty} \left[ \frac{1}{r(s)} \exp\left(-\int_{t_0}^{s} \frac{\sigma(\eta)}{r(\eta)} \, d\eta\right) \right]^{1/p_1-1} \, ds = \infty.
\]

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Throughout this paper, we assume that $p_i > 1$, $i = 1, 2$, are real numbers and $j \geq 1$, $r, \sigma, q \in C([t_0, \infty), [0, \infty))$, $r(t) > 0$, $q(t) > 0$, $r'(t) + \sigma(t) \geq 0$, $\tau(t) \in C([t_0, \infty), \mathbb{R})$, $\tau(t) \leq t$, $\lim_{t \to \infty} \tau(t) = \infty$.

**Definition 1.1** A nontrivial solution $x$ of (1) and (3) is termed oscillatory or nonoscillatory according to whether it does or does not have infinitely many zeros.

**Definition 1.2** Equations (1) and (3) are called oscillatory if all their solutions are oscillatory.

Half-linear delay differential equations arise in a variety of phenomena including mixing liquids, economics problems, biology, medicine, physics, engineering and automatic control problems, as well as vibrational motion in flight, and human self-balancing, see [1–6]. In particular, differential equations with $p$-Laplacian like operators, as the classical half-linear or Emden–Fowler differential equations, have numerous applications in the study of non-Newtonian fluid theory, porous medium problems, chemotaxis models, etc.; see [7–10]. We can also refer to [11–13] for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms.

In what follows, we state some background details that motivate the analysis of (1) and (3). In recent years, numerous significant results for the oscillation of delay differential equations have been shown in [14–26].

Chiu and Li [4] considered the oscillatory behavior of a class of scalar advanced and delayed differential equations with piecewise constant generalized arguments, which extended the theory of functional differential equations. The authors in [27–34] studied the asymptotic properties of different orders of some differential equations. For more details on this theory, we refer the reader to the papers [35–43].

In 2014, Li et al. [44] presented some open problems for the study of qualitative properties of solutions to differential equations, and the authors used the Riccati technique to find oscillation conditions for the studied equations.

Zhang et al. [45] investigated a higher-order half-linear/Emden–Fowler delay equation with $p$-Laplacian like operators

$$\left(r(t)|x^{(k-1)}(t)|^{p-1}x^{(k-1)}(t)\right)' + \sigma(t)|x^{(k-1)}(t)|^{p-2}x^{(k-1)}(t) + q(t)f(x(\tau(t))) = 0.$$ 

In particular, the authors in [46] used the integral average technique and obtained several oscillation criteria of the delay equation

$$\left(r(t)|x^{(k-1)}(t)|^{p-2}x^{(k-1)}(t)\right)' + q(t)g(x(\tau(t))) = 0,$$

where $\kappa$ is even and under the condition

$$\int_{t_0}^{\infty} \frac{1}{r^{1/(p-1)}(s)} \, ds = \infty.$$ 

The motivation for this article is to continue the previous works [23, 31].

On the basis of the above discussion, we will establish criteria for the oscillation of (1) and (3) by Riccati and comparison techniques under (2) and (4). Finally, two examples are presented to show the significance of the conclusions.
2 Auxiliary results

To establish oscillation criteria for (1) and (3), we give the following lemmas in this section.

**Lemma 2.1** ([33]) Let \( h \in C^n([t_0, \infty), (0, \infty)) \). Suppose that \( h^{(n)}(t) \) is of a fixed sign on \([t_0, \infty), h^{(n)}(t) \) not identically zero and that there exists \( t_1 \geq t_0 \) such that, for all \( t \geq t_1 \),

\[
h^{(n-1)}(t)h^{(n)}(t) \leq 0.
\]

If we have \( \lim_{t \to \infty} h(t) \neq 0 \), then there exists \( t_\lambda \geq t_0 \) such that

\[
h(t) \geq \lambda \frac{\lambda^{n-1}}{(n-1)!} \frac{|h^{(n-1)}(t)|}{t^{n-1}}
\]

for every \( \lambda \in (0,1) \) and \( t \geq t_\lambda \).

**Lemma 2.2** ([32]) If the function \( x \) satisfies \( x^{(i)}(t) > 0 \), \( i = 0, 1, \ldots, n \), and \( x^{(n+1)}(t) < 0 \), then

\[
\frac{x(t)}{t^n/n!} \geq \frac{x'(t)}{t^{n-1}/(n-1)!}.
\]

**Lemma 2.3** ([34]) Let \( V > 0 \). Then

\[
Uu - V u^{(\kappa+1)/\kappa} \leq \frac{\kappa^{\kappa}}{(\kappa+1)^{\kappa+1}} U^{\kappa+1} V^{-\kappa}.
\]

**Lemma 2.4** Let (2) hold. If \( x \) is an eventually positive solution of (1), then \( x' > 0 \) and \( x'' > 0 \).

**Proof** The proof is obvious and therefore is omitted. \( \square \)

**Lemma 2.5** If

\[
\int_{t_0}^\infty \left( M p^{2-p_1} \beta(s) \nu(s) r^2(s)(\beta'(s))^{p_1} \frac{r(s)\beta'(s)s^{p_1-1}}{p_1^{p_1-1} \mu^{p_1-1} s^{p_1-1}} \right) ds = \infty
\]

for some \( \mu \in (0,1) \), then \( x'' < 0 \).

**Proof** Let \( x''(t) > 0 \). From Lemmas 2.2 and 2.1, we find

\[
\frac{x(\tau(t))}{x(t)} \geq \frac{\tau^3(t)}{t^3}
\]

and

\[
x'(t) \geq \frac{\mu}{2} t^2 x''(t).
\]

Let

\[
\zeta(t) := \beta(t) \frac{r(t)x''(t)^{p_1-1}}{x^{p_1-1}(t)} > 0.
\]


From (7), (8), and (9), we find
\[ \zeta'(t) \leq \frac{\beta'(t)}{\beta(t)} \phi(t) - \beta(t)q(t)\frac{\tau^{3(p_1-1)}(t)}{\varepsilon t^{p_1-1}} x^{p_2-1}(t) \]
\[ - \frac{(p_1 - 1)\mu}{2} \frac{t^2}{\beta^{1/p_1-1}(t)^{p_1-1}(t)} \zeta^{1+(1/p_1-1)}(t). \]
(10)

Since \( x'(t) > 0 \), there exist \( t_2 \geq t_1 \) and a constant \( M > 0 \) such that \( x(t) > M \) for all \( t \geq t_2 \).

Using inequality (5) with \( U = \beta'/\beta, V = \kappa \mu t^2/(2\gamma^{1/\kappa}(t) \beta^{1/\kappa}(t)) \) and \( u = \zeta \), we get
\[ \zeta'(t) \leq -MP^{p_2-1} \beta(t)q(t)\frac{\tau^{3(p_2-1)}(t)}{t^{3(p_2-1)}} + \frac{2^{p_2-1}}{p_1} \frac{1}{\mu p_1-1} \frac{\beta^{1/p_1-1}(t)^{p_1-1}}{2^{p_1-1}} \beta^{1/p_1-1}(t). \]

This implies that
\[ \int_{t_1}^{t} \left( MP^{p_2-1} \beta(s)q(s)\frac{\tau^{3(p_2-1)}(s)}{s^{3(p_2-1)}} - \frac{2^{p_2-1}}{p_1} \frac{1}{\mu p_1-1} \frac{\beta^{1/p_1-1}(s)^{p_1-1}}{2^{p_1-1}} \beta^{1/p_1-1}(s) \right) ds \leq \zeta(t_1), \]
which contradicts (6). The proof is complete.

For convenience, we denote
\[ R(t) := \int_{t}^{\infty} \left( \frac{1}{r(s)} \int_{t}^{\infty} q(s) \, ds \right) \frac{1}{t^{(p_2-1)}} \, d\eta, \]
\[ \bar{R}(t) := \mu_2^{(p_2-1)/(p_1-1)} \int_{t}^{\infty} \left( \frac{1}{r(s)} \int_{t}^{\infty} q(s) \left( \frac{\tau(s)}{s} \right)^{p_2-1} \, ds \right) \frac{1}{t^{(p_2-1)}} \, d\eta, \]
\[ \vartheta_{t_0}(t) := \exp \left( \int_{t_0}^{t} \frac{\sigma(s)}{r(s)} \, ds \right), \]
and
\[ \hat{R}(t) := \mu_2^{(p_2-1)/(p_1-1)} \int_{t}^{\infty} \left( \frac{1}{r(s)\vartheta_{t_0}(t)} \int_{t_0}^{\infty} \vartheta_{t_0}(s) \left( \frac{\tau(s)}{s} \right)^{p_2-1} \, ds \right) \frac{1}{t^{(p_2-1)}} \, d\eta, \]
where \( \mu_2 \in (0, 1). \)

We shall establish oscillation conditions for (3) by converting into the form (1). It is not difficult to see that
\[ \frac{1}{\vartheta_{t_0}(t)} \frac{d}{dt} \left( \mu(t)r(t)(x''(t))^{p_1-1} \right) = \frac{1}{\vartheta_{t_0}(t)} \left[ \vartheta_{t_0}(t) \left( \frac{\vartheta_{t_0}(t)}{\vartheta_{t_0}(t)^{p_1-1}} \right) + \vartheta_{t_0}(t)^{p_1-1} \right] \]
\[ - \frac{\vartheta_{t_0}(t)^{p_1-1}}{\vartheta_{t_0}(t)^{p_1-1}} \vartheta_{t_0}(t)r(t)(x''(t))^{p_1-1}, \]
which with (3) gives
\[ \left( \vartheta_{t_0}(t)r(t)(x''(t))^{p_1-1} \right) + \vartheta_{t_0}(t)^{p_1-1} = 0. \]
3 Main results

In this section, we establish oscillation criteria for (1) and (3) by the Riccati transformation and comparison technique.

**Theorem 3.1** If the equation

\[
\eta'(t) + \frac{\lambda p_2^{-1}}{6p_1^{-1}} q(t) \tau^{\frac{3(p_2-1)}{p_1}}(t) r^{\frac{(p_2-1)/(p_1-1)}{(p_2-1)/p_1-1}}(\tau(t)) \eta^{\frac{(p_2-1)/p_1-1}{(p_2-1)/p_1-1}}(\tau(t)) = 0 \quad (11)
\]

is oscillatory, then (1) is oscillatory.

**Proof** Let (1) have a nonoscillatory solution in \([t_0, \infty)\). Then there exists \(t_1 \geq t_0\) such that \(x(t) > 0\) and \(x(\tau(t)) > 0\) for \(t \geq t_1\). Let

\[
\eta(t) := r(t) \left(x''(t)\right)^{p_2-1} > 0 \quad \text{[from Lemma 2.4]},
\]

which with (1) gives

\[
\eta'(t) + q(t)x^{p_2-1}(\tau(t)) = 0. \quad (12)
\]

Since \(x\) is positive and increasing, we see \(\lim_{t \to \infty} x(t) \neq 0\). So, using Lemma 2.1, we find

\[
x^{p_2-1}(\tau(t)) \geq \frac{\lambda p_2^{-1}}{6p_1^{-1}} \tau^{\frac{3(p_2-1)}{p_1}}(t) \left(x''(\tau(t))\right)^{p_2^{-1}} \quad (13)
\]

for all \(\lambda \in (0,1)\). By (12) and (13), we see that

\[
\eta'(t) + \frac{\lambda p_2^{-1}}{6p_1^{-1}} q(t) \tau^{\frac{3(p_2-1)}{p_1}}(t) \left(x''(\tau(t))\right)^{p_2^{-1}} \leq 0.
\]

So, \(\eta\) is a positive solution of the inequality

\[
\eta'(t) + \frac{\lambda p_2^{-1}}{6p_1^{-1}} q(t) \tau^{\frac{3(p_2-1)}{p_1}}(t) r^{\frac{(p_2-1)/(p_1-1)}{(p_2-1)/p_1-1}}(\tau(t)) \eta^{\frac{(p_2-1)/p_1-1}{(p_2-1)/p_1-1}}(\tau(t)) \leq 0.
\]

By using [40, Theorem 1], we find that (11) also has a positive solution, which is a contradiction. The proof is complete. \(\square\)

**Corollary 3.2** Let \(p_2 = p_1\) and (2) hold. If

\[
\lim_{t \to \infty} \int_{t}^{\infty} \frac{\lambda p_2^{-1}}{6p_1^{-1}} q(s) r^{\frac{(p_2-1)/(p_1-1)}{(p_2-1)/p_1-1}}(\tau(s)) \, ds > \frac{1}{e}, \quad (14)
\]

then (1) is oscillatory.

**Theorem 3.3** Let \(p_2 \geq p_1\) and (6) hold for some \(\mu \in (0,1)\). If

\[
u''(t) + \lambda^{p_2-p_1} \tilde{R}(t) \nu(t) = 0 \quad (15)
\]

is oscillatory, then (1) is oscillatory.
Proof Assume to the contrary that (1) has a nonoscillatory solution in \([t_0, \infty)\). Without loss of generality, we only need to be concerned with positive solutions of equation (1). Then there exists \(t_1 \geq t_0\) such that \(x(t) > 0\) and \(x(\tau_i(t)) > 0\) for \(t \geq t_1\). From Lemmas 2.2 and 2.4, we have that

\[
x'(t) > 0, \quad x''(t) < 0 \quad \text{and} \quad x'''(t) > 0
\]

for \(t \geq t_2\), where \(t_2\) is sufficiently large. Now, integrating (1) from \(t\) to \(l\), we have

\[
r(l)(x'''(l))^{p_1-1} = r(t)(x'''(t))^{p_1-1} - \int_t^l q(s)x^{p_2-1}(\tau(s)) \, ds.
\]

Using Lemma 3 in [34] with (16), we get

\[
\frac{x(\tau(t))}{x(t)} \geq \lambda \frac{\tau(t)}{t},
\]

which with (17) gives

\[
r(l)(x'''(l))^{p_1-1} - r(t)(x'''(t))^{p_1-1} + \lambda^{p_2-1} x^{p_1-1}(t) \int_t^l q(s) \left( \frac{\tau(s)}{s} \right)^{p_2-1} x^{p_1-1}(s) \, ds \leq 0.
\]

It follows, by \(x' > 0\), that

\[
r(l)(x'''(l))^{p_1-1} - r(t)(x'''(t))^{p_1-1} + \lambda^{p_2-1} x^{p_1-1}(t) \int_t^\infty q(s) \left( \frac{\tau(s)}{s} \right)^{p_2-1} x^{p_1-1}(s) \, ds \leq 0.
\]

Taking \(l \to \infty\), we have

\[
-x''(t) \geq \lambda^{(p_2-1)/(p_1-1)} x^{(p_2-1)/(p_1-1)}(t) \int_t^\infty q(s) \left( \frac{\tau(s)}{s} \right)^{p_2-1} x^{(p_1-1)}(s) \, ds \leq 0,
\]

that is,

\[
x''(t) \geq \lambda^{(p_2-1)/(p_1-1)} x^{(p_2-1)/(p_1-1)}(t) \left( \int_t^\infty q(s) \left( \frac{\tau(s)}{s} \right)^{p_2-1} x^{(p_1-1)}(s) \, ds \right)^{1/(p_1-1)}.
\]

Integrating the above inequality from \(t\) to \(\infty\), we obtain

\[
-x'(t) \geq \lambda^{(p_2-1)/(p_1-1)} x^{(p_2-1)/(p_1-1)}(t) \int_t^\infty \left( \frac{1}{r(\eta)} \int_\eta^\infty q(s) \left( \frac{\tau(s)}{s} \right)^{p_2-1} x^{(p_1-1)}(s) \, ds \right)^{1/(p_1-1)} \, d\eta,
\]

hence

\[
x''(t) \leq -\tilde{R}(t)x^{(p_2-1)/(p_1-1)}(t).
\]

Letting

\[
\phi(t) = \frac{x'(t)}{x(t)},
\]

and

\[
r(t) = r(t) + \lambda^{p_2-1},
\]

we have

\[
x''(t) \leq -\tilde{R}(t)x^{(p_2-1)/(p_1-1)}(t).
\]

Taking \(l \to \infty\), we have

\[
x''(t) \leq -\tilde{R}(t)x^{(p_2-1)/(p_1-1)}(t).
\]
then $\phi(t) > 0$ for $t \geq t_1$ and

$$\phi'(t) = \frac{x''(t)}{x(t)} - \left(\frac{x'(t)}{x(t)}\right)^2.$$ 

By using (19) and the definition of $\phi(t)$, we see that

$$\phi'(t) \leq -\tilde{R}(t) \frac{x^{(p_2-1)/p_1-1}(t)}{x(t)} - \phi^2(t).$$  (20)

Since $x'(t) > 0$, there exists a constant $M > 0$ such that $x(t) \geq M$ for all $t \geq t_2$. Then (20) becomes

$$\phi'(t) + \phi^2(t) + M^{p_2-1/p_1} \tilde{R}(t) \leq 0.$$  (21)

From [39], we obtain that (15) is nonoscillatory if and only if there exists $t_3 > \max\{t_1, t_2\}$ such that (21) holds, which is a contradiction. Theorem is proved. □

**Theorem 3.4** Let $p_2 \geq p_1$, $\tau'_i(t) > 1$ and (6) hold for some $\mu \in (0,1)$. If

$$\left(\frac{1}{\tau'(t)} u'(t)\right)' + M^{p_2-1/p_1-2} R(t) u(t) = 0$$  (22)

is oscillatory, then (1) is oscillatory.

**Proof** From the proof of Theorem 3.3, we find that (17) holds. So, it follows from $\tau'_i(t) \geq 0$ and $x'(t) \geq 0$ that

$$r(t)x''(l)^{p_1-1} - r(t)x''(t)^{p_1-1} + x^{p_2-1}(\tau(t)) \int_t^l q(s) \, ds \leq 0.$$  (23)

Thus, (16) becomes

$$x''(t) \leq -R(t)x^{(p_2-1)/(p_1-1)}(\tau(t)).$$  (24)

Letting

$$\delta(t) = \frac{x'(t)}{x(\tau(t))},$$  (25)

then $\delta(t) > 0$ for $t \geq t_1$, and

$$\delta'(t) = \frac{x''(t)}{x(\tau(t))} - \frac{x'(t)}{x^2(\tau(t))} x'(\tau(t)) \tau'(t)$$

$$\leq \frac{x''(t)}{x(\tau(t))} - \tau'(t) \left(\frac{x'(t)}{x(\tau(t))}\right)^2.$$  

From (24) and (25), we find that

$$\delta'(t) + M^{p_2-1/p_1-2} R(t) + \tau'(t) \delta^2(t) \leq 0.$$  (26)
From [39], we find that (22) is nonoscillatory if and only if there exists $t_3 > \max\{t_1, t_2\}$ such that (26) holds, which is a contradiction. Theorem is proved. □

**Corollary 3.5** Let $p_2 = p_1$ and (6) hold. If

$$\lim_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left( H(t, s) \tilde{R}(s) - \frac{1}{4} h^2(t, s) \right) ds = \infty$$

or

$$\lim_{t \to \infty} \inf_{t} \int_{t}^{\infty} \tilde{R}(s) ds > \frac{1}{4},$$

(27)

then (1) is oscillatory.

**Corollary 3.6** Let $p_2 = p_1$ and (6) hold. If $\varepsilon \in (0, 1/4]$ such that

$$t^2 \tilde{R}(s) \geq \varepsilon$$

and

$$\lim_{t \to \infty} \sup_{t} \left( t^{-1} \int_{t_0}^{t} s^{-2} \tilde{R}(s) ds + t^{1-2} \int_{t}^{\infty} \tilde{R}(s) ds \right) > 1,$$

where $\tilde{\varepsilon} = \frac{1}{2}(1 - \sqrt{1 - 4\varepsilon})$, then (1) is oscillatory.

**Corollary 3.7** Let $p_1 = p_2$ and (4) hold. If

$$\lim_{t \to \infty} \int_{t(t)}^{t} \lambda^{p_2^{-1}} \partial_{t_0}(s)q(s)\tau_{j}^{3(p_2-1)}(s) \frac{\partial_{t_0}(s)q(s)\tau_{j}^{3(p_2-1)}(s)}{\partial r} \frac{r(s)(\beta'(s))^{p_1}}{\mu^{p_1-1}(s)\beta', \beta'}(s) ds > \frac{1}{e},$$

(28)

then (3) is oscillatory.

**Corollary 3.8** Let $p_1 = p_2$, (4), and

$$\int_{t_0}^{t} \left( M^{p_2-p_1} \beta(s) \partial_{t_0}(s)q(s) \frac{\tau_{3e}(s)}{s^{3e}} - \frac{2^{p_2-1}}{p_1^{p_1-1}} \frac{r(s)(\beta'(s))^{p_1}}{\mu^{p_1-1}(s)\beta', \beta'}(s) \right) ds = \infty,$$

hold for some $\mu \in (0, 1)$. If

$$\lim_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left( H(t, s)\tilde{R}(s) - \frac{1}{4} h^2(t, s) \right) ds = \infty$$

or

$$\lim_{t \to \infty} \inf_{t} \int_{t}^{\infty} \tilde{R}(s) ds > \frac{1}{4},$$

then (3) is oscillatory.
Corollary 3.9 Let \( p_1 = p_2 \) and (28) hold. If \( \epsilon \in (0, 1/4] \) such that

\[
t^{1/2} \tilde{R}(s) \geq \epsilon
\]

and

\[
\limsup_{t \to \infty} \left( t^{-1} \int_0^t s^{2-\epsilon} \tilde{R}(s) \, ds + t^{1-2\epsilon} \int_t^\infty s^{-\epsilon} \tilde{R}(s) \, ds \right) > 1,
\]

where \( \tilde{\epsilon} \) is defined as in Corollary 3.6, then (3) is oscillatory.

4 Examples and discussion

Two examples are presented to show the applications of our results. The first example is given to demonstrate Corollaries 3.2 and 3.5.

Example 4.1 For \( t \geq 1 \), consider the equation

\[
\left( t^3 (x'''(t))^3 \right)' + \frac{q_0}{t^5} x^3(\gamma t) = 0,
\]

we see that \( p_1 = p_2 = 4, r(t) = t^3, \tau(t) = \gamma t \) and \( q(t) = q_0/t^7, \gamma \in (0, 1] \) and \( q_0 > 0 \). So, we obtain

\[
\tilde{R}(t) = \lambda \left( \frac{q_0}{6} \right)^{1/3} \frac{1}{\gamma^{1/2} t^{2}}.
\]

By Corollary 3.2 and Corollary 3.5, equation (29) is oscillatory if

\[
q_0 > \frac{6^3}{e(\ln t)^3} \gamma^6 t^7,
\]

\[
q_0 > \left( \frac{3^4}{2} \right) \frac{1}{\gamma^7},
\]

and

\[
q_0 > 6 \left( \frac{1}{4^7} \right)^{3},
\]

respectively. Thus, equation (29) is oscillatory if

\[
q_0 > \max \left\{ \left( \frac{3^4}{2} \right) \frac{1}{\gamma^9}, 6 \left( \frac{1}{4^7} \right)^{3} \right\} = \left( \frac{3^4}{2} \right) \frac{1}{\gamma^9}.
\]

(30)

Now, we give the second example to demonstrate Corollary 3.8.

Example 4.2 Consider the equation

\[
\left( t^3 (x'''(t))^3 \right)' + \left( x'''(t) \right)^3 + \frac{q_0}{t^5} x^3(t/2) = 0, \quad t \geq 1, q_0 > 0.
\]

(31)
Let $p_1 = p_2 = 4$, $r(t) = t^3$, $\sigma(t) = 1$, $\tau(t) = t/2$, and $q(t) = q_0/t^{5}$. Thus, it is easy to verify that

\[
\int_{t_0}^{\infty} \left[ \frac{1}{r(s)} \exp \left( -\int_{t_0}^{s} \frac{\sigma(\eta)}{r(\eta)} \, d\eta \right) \right]^{1/p_1-1} \, ds
\]

\[
= \int_{t_0}^{\infty} \left[ \frac{1}{s^3} \exp \left( -\int_{t_0}^{s} \frac{1}{s^3} \, dx \right) \right]^{1/3} \, ds = \infty.
\]

Using Corollary 3.8, equation (31) is oscillatory.

5 Conclusion
The oscillation conditions of the fourth-order differential equations with $p$-Laplacian like operators are obtained in this study. In order to improve and simplify prior results in the literature, we expanded the results in [23, 31] to fourth-order equations and used the Riccati transformation and comparison techniques. It is interesting to extend our results to even-order damped differential equations with $p$-Laplacian like operators

\[
(r(t)(x^{(k-1)}(t))^{p-1})' + \sigma(t)|x''(t)|^{p_1-2}x''(t) + q(t)f(x(\tau(t))) = 0
\]

under the condition

\[
\int_{t_0}^{\infty} \left[ \frac{1}{r(s)} \exp \left( -\int_{t_0}^{s} \frac{\sigma(\eta)}{r(\eta)} \, d\eta \right) \right]^{1/p_1-1} \, ds < \infty.
\]

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The authors declare that they have no competing interests.

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The authors declare that they have read and approved the final manuscript.

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References

1. Hale, J.K.: Partial neutral functional differential equations. Rev. Roum. Math. Pures Appl. 39, 339–344 (1994)
2. MacDonald, N.: Biological Delay Systems: Linear Stability Theory. Cambridge Studies in Mathematical Biology, vol. 8. Cambridge University Press, Cambridge (1989)
3. Bohner, M., Hassan, T.S., Li, T.: Fite-Hille-Willmer-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments. Indag. Math. 29(2), 548–560 (2018)
4. Chiu, K.-S., Li, T.: Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments. Math. Nachr. 292(10), 2153–2164 (2019)
5. Agarwal, R.P., Bazighifan, O., Ragusa, M.A.: Nonlinear neutral delay differential equations of fourth-order: oscillation of solutions. Entropy 23, 129 (2021)
6. Tang, S., Li, T., Thandapani, E.: Oscillation of higher-order half-linear neutral differential equations. Demonstr. Math. 1, 101–109 (2013)
7. Bohner, M., Li, T.: Oscillation of second-order p-Laplace dynamic equations with a nonpositive neutral coefficient. Appl. Math. Lett. 37, 72–76 (2014)
8. Bohner, M., Li, T.: Kamenev-type criteria for nonlinear damped dynamic equations. Sci. China Math. 58(7), 1445–1452 (2015)
9. Dzurina, J., Grace, S.R., Jadlovská, I., Li, T.: Oscillation criteria for even-order Emden–Fowler neutral differential equations with a sublinear neutral term. Math. Nachr. 293(5), 910–922 (2020)
10. Li, T., Pintus, N., Viglialoro, G.: Properties of solutions to porous medium problems with different sources and boundary conditions. Z. Angew. Math. Phys. 70(3), Article ID 86 (2019)
11. Frassu, S., Viglialoro, G.: Boundedness for a fully parabolic Keller-Segel model with sublinear segregation and superlinear aggregation. Acta Appl. Math. 171(1), 19 (2021)
12. Frassu, S., van der Mei, C., Viglialoro, G.: Boundedness in a nonlinear attraction-repulsion Keller-Segel system with production and consumption. J. Math. Anal. Appl. 504(2), 125428 (2021)
13. Li, T., Viglialoro, G.: Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime. Differ. Integral Equ. 34(5), 315–336 (2021)
14. Li, T., Rogovchenko, Yu.V.: Oscillation of second-order neutral differential equations. Math. Nachr. 288(10), 1150–1162 (2015)
15. Li, T., Rogovchenko, Yu.V.: Oscillation criteria for even-order neutral differential equations. Appl. Math. Lett. 61, 35–41 (2016)
16. Li, T., Rogovchenko, Yu.V.: Oscillation criteria for second-order superlinear Emden–Fowler neutral differential equations. Monatshefte Math. 184(3), 489–500 (2017)
17. Li, T., Rogovchenko, Yu.V.: On asymptotic behavior of solutions to higher-order sublinear Emden–Fowler delay differential equations. Appl. Math. Lett. 67, 53–59 (2017)
18. Li, T., Rogovchenko, Yu.V.: On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations. Appl. Math. Lett. 105, Article ID 106293 (2020)
19. Agarwal, R.P., Zhang, C., Li, T.: Some remarks on oscillation of second order neutral differential equations. Appl. Math. Comput. 274, 178–181 (2016)
20. Li, T., Zhang, C., Thandapani, E.: Asymptotic behavior of fourth-order neutral dynamic equations with noncanonical operators. Taiwan. J. Math. 18(4), 1003–1019 (2014)
21. Zhang, C., Agarwal, R.P., Bohner, M., Li, T.: Oscillation of fourth-order delay dynamic equations. Sci. China Math. 58(1), 143–160 (2015)
22. Dzurina, J., Jadlovská, I.: A note on oscillation of second-order delay differential equations. Appl. Math. Lett. 69, 126–132 (2017)
23. Bohner, M., Grace, S.R., Jadlovská, I.: Sharp oscillation criteria for second-order neutral delay differential equations. Math. Methods Appl. Sci. 43, 100401–10055 (2020)
24. Baculíková, B.: Oscillation of second-order nonlinear noncanonical differential equations with deviating argument. Appl. Math. Lett. 91, 68–75 (2019)
25. Baculíková, B., Dzurina, J., Graef, J.R.: On the oscillation of higher-order delay differential equations. Math. Slovaca 187, 387–400 (2012)
26. Grace, S., Agarwal, R., Graef, J.: Oscillation theorems for fourth-order functional differential equations. J. Appl. Math. Comput. 30, 75–88 (2009)
27. Kiguradze, I.T., Chanturiya, T.A.: Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. Kluwer Academic, Dordrecht (1993)
28. Zhang, C., Li, T., Saker, S.: Oscillation of fourth-order delay differential equations. J. Math. Sci. 201, 296–308 (2014)
29. Baculíková, B., Dzurina, J., Graef, J.R.: On the oscillation of higher-order delay differential equations. Math. Slovaca 187, 387–400 (2012)
30. Zhang, C., Li, T., Sun, B., Thandapani, E.: On the oscillation of higher-order half-linear delay differential equations. Appl. Math. Lett. 24, 1618–1621 (2011)
31. Elabbasy, E.M., Thandapani, E., Moaaz, O., Bazighifan, O.: Oscillation of solutions to fourth-order delay differential equations with middle term. Open J. Math. Sci. 3, 191–197 (2019)
32. Chatzarakis, G.E., Grace, S.R., Jadlovská, I., Li, T., Tunc, E.: Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients: Complexity 2019, Article ID 5691758 (2019)
33. Agarwal, R., Grace, S., O’Regan, D.: Oscillation Theory for Difference and Functional Differential Equations. Kluwer Academic, Dordrecht (2000)
34. Bazighifan, O., Alzabut, A., Almarni, B., Marin, M.: An oscillation criterion of nonlinear differential equations with advanced term. Symmetry 13, 843 (2021)
35. Bazighifan, O., Dassios, I.: Riccati technique and asymptotic behavior of fourth-order advanced differential equations. Mathematics 8, 590 (2020)
36. Chatzarakis, G.E., Li, T.: Oscillations of differential equations generated by several deviating arguments. Adv. Differ. Equ. 2017, 292 (2017)
37. Chatzarakis, G.E., Li, T.: Oscillation criteria for delay and advanced differential equations with nonmonotone arguments. Complexity 2018, Article ID 8237634 (2018)
38. Bazighifan, O., Postolache, M.: Improved conditions for oscillation of functional nonlinear differential equations. Mathematics 8, 552 (2020)
39. Agarwal, R., Shieh, S.L., Yeh, C.C.: Oscillation criteria for second order retarded differential equations. Math. Comput. Model. 26, 1–11 (1997)
40. Philos, C.: On the existence of nonoscillatory solutions tending to zero at $\infty$ for differential equations with positive delay. Arch. Math. (Basel) 36, 168–178 (1981)
41. Hille, E.: Non-oscillation theorems. Trans. Am. Math. Soc. 64, 234–253 (1948)
42. Moaaz, O., Kumam, P., Bazighifan, O.: On the oscillatory behavior of a class of fourth-order nonlinear differential equation. Symmetry 12, 524 (2020)
43. Zhang, Q., Yan, J.: Oscillation behavior of even order neutral differential equations with variable coefficients. Appl. Math. Lett. 19, 1202–1206 (2006)
44. Li, T., Baculikova, B., Dzurina, J., Zhang, C.: Oscillation of fourth order neutral differential equations with $p$-Laplacian like operators. Bound. Value Probl. 56, 41–58 (2014)
45. Zhang, C., Agarwal, R.P., Li, T.: Oscillation and asymptotic behavior of higher-order delay differential equations with $p$-Laplacian like operators. J. Math. Anal. Appl., 409 (2014)
46. Park, C., Moaaz, O., Bazighifan, O.: Oscillation results for higher order differential equations. Axioms 9, 1–10 (2020)