NON-COMMUTATIVE GROUPOIDS OBTAINED FROM THE FAILURE OF 3-UNIQUENESS IN STABLE THEORIES

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Abstract. We construct a possibly non-commutative groupoid from the failure of 3-uniqueness of a strong type. The commutative groupoid constructed by John Goodrick and Alexei Kolesnikov in [1] lives in the center of the groupoid.

A certain automorphism group approximated by the vertex groups of the non-commutative groupoids is suggested as a “fundamental group” of the strong type.

1. Introduction

In singular homology theory, one of the differences between the fundamental group and the first homology group $H_1$ is that the former is not necessarily commutative while the latter is. In the earlier papers [1], [2], [3] by Goodrick, Kolesnikov (and the first author), an analogue of homotopy/homology theory is developed in the context of model theory but where the “fundamental group” introduced is always commutative. In this paper, by taking an approach closer to the original idea of homotopy theory, we suggest how to construct a different fundamental group in a non-commutative manner. More precisely, from a symmetric witness to the failure of 3-uniqueness in a stable theory, we construct a new groupoid $\mathcal{F}$ whose “vertex groups” $\text{Mor}_\mathcal{F}(a, a)$ need not be abelian. In fact, we will show that $\text{Mor}_\mathcal{G}(a, a) \leq Z(\text{Mor}_\mathcal{F}(a, a))$, where $\mathcal{G}$ is the commutative groupoid constructed in [1] and [2]. We may call $\mathcal{F}$ a non-commutative groupoid constructed from the symmetric witness. But unlike the groupoid $\mathcal{G}$, this new groupoid $\mathcal{F}$ is definable only in certain cases (e.g. under $\omega$-categoricity); in general, it is merely invariant over some set.

We work in a complete stable theory $T$ with a fixed monster model $\mathcal{M} = \mathcal{M}^eq$. Unless said otherwise, tuples are from $\mathcal{M}$ and sets $A, B, \ldots$ are small subsets of $\mathcal{M}$; and there is an independence notion among

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sets, defined by nonforking. For tuples \( a_0, a_1, \ldots \), we write \( a_{01} \) to denote \( a_0 a_1 \) and so on. Throughout this paper we also fix an algebraically closed set \( A \) and a complete type \( p \) of possible infinite arity over \( A \).

For a tuple \( c, \overline{c} \) denotes \( acl(cA) \). If \( \{ a, b, c \} \) is an \( A \)-independent set of realizations of \( p \), then we let

\[ \tilde{ab} := dcl(\overline{abc}) \cap ab. \]

Due to stationarity, this set only depends on \( a \) and \( b \). The rest notational convention we take is standard. For example, \( a \equiv_B b \) means \( tp(a/B) = tp(b/B) \); and \( Aut(C/B) \) is the group of elementary maps from \( C \) onto \( C \) fixing \( B \) pointwise. In addition, \( Aut(tp(f/B)) \) means \( Aut(Y/B) \) where \( Y \) is the solution set of \( tp(f/B) \).

Now we recall definitions of notions which we will use throughout.

A groupoid is a category where every morphism is invertible. Hence in a groupoid, for each object \( a \), \( \text{Mor}(a, a) \) forms a group called a vertex group. If all the vertex groups are abelian we call the groupoid abelian or commutative. We say a groupoid is connected (finitary, resp.) if for any two objects \( a, b \), \( \text{Mor}(a, b) \) is non-empty (finite, resp.). If a groupoid is connected then each vertex group is isomorphic.

Originally, 3-uniqueness is defined functorially in \([5],[2]\), but as we will not use amalgamation notion the following equivalent definition would suffice in this note.

**Definition 1.1.** \([2]\) We say the fixed complete type \( p \) has 3-uniqueness over \( A \) if whenever \( \{ a_0, a_1, a_2 \} \) is an \( A \)-independent set of realizations of \( p \), and for \( 0 \leq i < j \leq 2 \), \( \sigma_{ij} \in Aut(\overline{a_{ij}/a_i a_j}) \), then \( \sigma_{01} \cup \sigma_{02} \cup \sigma_{12} \) is also an elementary map.

**Fact 1.2.** \([5],[2]\) Let \( a, b, c \models p \) be independent over \( A \). Then \( p \) has 3-uniqueness over \( A \) iff \( \tilde{ab} = dcl(\overline{ab}) \).

We now recall a certain automorphism group which plays a role of the (abelianized) fundamental group of \( p \) in the homotopy theory of model theory introduced in \([1],[3]\).

**Definition 1.3.** \([2],[3]\) Let \( \{ a, b, c \} \) be \( A \)-independent set of realizations of \( p \). We let

\[ \Gamma_2(p) := Aut(\tilde{ab}/\overline{ab}). \]

Since the homology groups of \( p \) will not be dealt with, we do not recall the definition of those, but only point out the following proved in \([3]\).

**Fact 1.4.** The group \( \Gamma_2(p) \) is profinite abelian and isomorphic to the type’s 2nd homology group \( H_2(p) \).
There indeed is a mismatch in numbering. The group $\Gamma_2(p)$ corresponds to the fundamental group $\pi_1$ (or its abelianization), and so should do $H_2(p)$ to the first homology group in algebraic topology. An higher dimensional version of Fact 1.4 is proved in [4].

Our goal in this paper is to introduce a possibly non-commutative “fundamental group” $\Pi_2$ of $p$, in which $\Gamma_2(p)$ places in the center. In section 2, we give a motivational example. Namely the model of a connected groupoid having a given vertex group $G$. It turns out that $\Pi_2(p) = G$ and $\Gamma_2(p) = Z(G)$ if we take $p$ as the 1-type of any object.

In section 3, we develop a general theory for constructing our desired non-commutative connected finitary groupoid from a symmetric witness to the non-3-uniqueness of $p$.

In section 4, we show that $\Pi_2(p)$, a certain automorphism group partially approximated by the vertex groups of non-commutative groupoids constructed from the failure of 3-uniqueness of $p$, is a normal subgroup of $\text{Aut}(\tilde{a}b/\tilde{a})$ where $a, b \models p$ are $A$-independent, and $\Gamma_2(p)$ is central in $\Pi_2(p)$.

2. Finitary groupoid examples

Let $G$ be an arbitrary finite group. Now let $T_G$ be the complete stable theory of the connected finitary groupoid $(O, M, ., \text{init, ter})$ with the standard setting. Namely the sorts $O, M$ represent the infinite sets of all objects and morphisms, respectively; $.$ is the composition map between morphisms; and $\text{init, ter} : M \to O$ are maps indicating initial and terminal objects, respectively, of a morphism. Moreover $G_a := \text{Mor}(a, a)$ is isomorphic to $G$ for any $a \in O$. Now due to weak elimination of imaginaries, $\emptyset = \text{acl}^o(\emptyset)$, and we let the $p(x)$ be the unique 1-type over $\emptyset$ of any object.

Remark 2.1. [1, 4.2] Fix $a \in O$, and for $u(\neq a) \in O$, choose $g_u \in G_u$. Then the following map $\sigma$ is a structure automorphism of the groupoid:

1. $\sigma$ is the identity map on $O$, and on $G_a$;
2. for $u(\neq a) \in O$, we have $\sigma(f) = g_u.f$ or $= f.g_u^{-1}$, if $f \in \text{Mor}(a, u)$, or $\in \text{Mor}(u, a)$, respectively; and
3. for $u, v(\neq a) \in O$ and $f \in \text{Mor}(u, v)$, we have $\sigma(f) = g_v.f.g_u^{-1}$.

We fix distinct $a, b \in O$ and a morphism $f_0 \in \text{Mor}(a, b)$. Due again to weak elimination of imaginaries it follows that $ab, ab$, and $\text{Mor}(a, b)G_aG_b$ are all interdefinable, and

$$\Gamma_2(p) = \text{Aut}(\tilde{a}b/\tilde{a}, \tilde{b}) = \text{Aut}(\text{Mor}(a, b)/aG_aG_b).$$
Hence indeed (note that $\text{Mor}(a, b) \subseteq \text{dcl}(f_0 G_a)$)
\[
\Gamma_2(p) = \text{Aut}(X/aG_a b G_b).
\]
where $X$ is the finite solution set of $\text{tp}(f_0/aG_a b G_b)$.

Now for $f \in X$ there is unique $x \in G_a$ such that $f = f_0.x$, and we claim that this $x$ must be in $Z(G_a)$.

**Claim 2.2.** For $x \in G_a$, we have $g := f_0.x \in X$ iff $x \in Z(G_a)$.

**Proof.** ($\Rightarrow$) Since $g \in X$, $f_0 \equiv_{G_a} g$. Then for any $y \in G_a$, we have
\[
f_0.y.f_0^{-1}(\in G_b) = g.y.g^{-1} = f_0.x.y.x^{-1}.f_0^{-1}.
\]
Hence $x \in Z(G_a)$.

($\Leftarrow$) There is $z \in G_b$ such that $f_0 = z.g$. Now since $x \in Z(G_a)$, for any $y \in G_b$ we have
\[
g^{-1}.y.g.x^{-1} = f_0^{-1}.z.y.z^{-1}.f_0.x^{-1} = x^{-1}.f_0^{-1}.z.y.z^{-1}.f_0 = g^{-1}.z.y.z^{-1}.g.x^{-1}.
\]
Hence $y = z.y.z^{-1}$, i.e. $z \in Z(G_b)$. Hence then by Remark 2.1 there is an automorphism fixing $aG_a b G_b$ pointwise while sending $f_0$ to $g$. Hence $g \in X$.  

**Claim 2.3.** $\Gamma_2(p) = Z(G)$.

**Proof.** The proof will be similar to that of Proposition 2.22 in [2]. Note firstly that due to Claim 2.2, $Z(G_a)$ acts on $X$ as an obvious manner. This action is clearly regular. Secondly $\text{Aut}(X/aG_a b G_b)$ also regularly acts on $X$. Moreover since each $\sigma \in \text{Aut}(X/aG_a b G_b)$ fixes $G_a$ pointwise, it clearly follows that the two actions commute. Hence they are the same group. 

Due to Remark 2.1 we have $f \equiv_{aG_a b} f_0$ for any $f \in \text{Mor}(a, b)$, i.e., $\text{Mor}(a, b)$ is the solution set of $\text{tp}(f_0/aG_a b)$ or $\text{tp}(f_0/ab)$. Moreover for $f \in \text{Mor}(a, b)$, it follows
\[
\text{dcl}(f \bar{a}) = \text{dcl}(f_0 \bar{a}) = \text{dcl}(\text{Mor}(a, b), \bar{a}) = \text{dcl}(\text{Mor}(a, b), ab) = ab.
\]
Hence,
\[
\text{Aut}(ab/\bar{a}) = \text{Aut}(\text{Mor}(a, b)/\bar{a}) = \text{Aut}(\text{tp}(f_0/ab)).
\]
We further claim the following.

**Claim 2.4.** $G$ is isomorphic to $\text{Aut}(\text{Mor}(a, b)/\bar{a}) = \text{Aut}(\text{Mor}(a, b)/G_a)$. Hence $\Gamma_2(p) = \text{Aut}(\text{Mor}(a, b)/G_a G_b) = Z(\text{Aut}(ab/\bar{a}))$.

**Proof.** We know $G$ and $G_b$ are isomorphic. Now given $\sigma \in G_b$ we assign an automorphism $\sigma \in \text{Aut}(\text{Mor}(a, b)/\bar{a})$ sending $f(\in \text{Mor}(a, b)) \mapsto \sigma.f$. This mapping is well-defined, since if $g \in \text{Mor}(a, b)$ so that $g = \text{dcl}(f \bar{a}) = \text{dcl}(f_0 \bar{a}) = \text{dcl}(\text{Mor}(a, b), \bar{a}) = \text{dcl}(\text{Mor}(a, b), ab) = ab.$

Hence,
\[
\text{Aut}(ab/\bar{a}) = \text{Aut}(\text{Mor}(a, b)/\bar{a}) = \text{Aut}(\text{tp}(f_0/ab)).
\]

f.µ for some µ ∈ G_a, then σ(g) = σ(f).µ = σ.f.µ = σ.g. Now this correspondence is clearly 1-1 and onto. It is obvious that the correspondence is a group isomorphism.

In the following section we try to search this phenomenon in the general stable theory context. Namely given the abelian groupoid built from a symmetric witness introduced in [1], we construct an extended groupoid possibly non-abelian but the abelian groupoid places in the center of the new groupoid. In the case of above T_G, as we seen the morphism group of the abelian groupoid is Z(G), but in the extended one the morphism group is equal to G.

3. The non-commutative groupoid F

We recall the notion of symmetric witnesses introduced in [1] or [3].

Definition 3.1. A (full) symmetric witness to non-3-uniqueness (over the algebraically closed set A) is a tuple (a_0, a_1, a_2, f'_01, f'_12, f'_02, θ(x, y, z)) such that a_0, a_1, a_2 and f_01, f_12, f_02 are finite tuples, \{a_0, a_1, a_2\} is independent over A, θ(x, y, z) is a formula over A, and:

1. a_ij ⊆ f_ij ∈ \overline{a_i} \setminus dcl(\overline{a_i}, \overline{a_j});
2. a_0 f_01 ⊆ A a_12 f_12 ⊆ A a_02 f_02;
3. f_01 is the unique realization of θ(x, f_12, f_02), and so are f_12, f_02 of θ(f_01, y, f_02), θ(f_01, f_12, z), respectively; and
4. each tp(f_ij/\overline{a_i} \overline{a_j}) is isolated by tp(f_ij/\overline{a_ij} A).

The following (proved in [1]) is the key technical point saying that we have “enough” symmetric witnesses:

Proposition 3.2. If (a'_0, a'_1, a'_2) is the beginning of a Morley sequence of finite tuples over A and f' is a finite tuple in \overline{a'_0} \setminus dcl(\overline{a'_0}, \overline{a'_1}), then there is some full symmetric witness (a_0, a_1, a_2, f, g, h, θ) such that f' ∈ dcl(f.A) and a'_i ∈ dcl(a_i A) ⊆ \overline{a'_i} for i = 0, 1, 2.

Hence if the complete type p does not have 3-uniqueness over A, then there is a symmetric witness (a_0, a_1, a_2, ...) over A such that a_i ∈ \overline{c_i} for some A-independent realizations c_0, c_1, c_2 of p.

From now on for notational simplicity, we suppress A to ∅ (by naming the set). We fix some more notations that we will refer to throughout the rest. Fix a symmetric witness W = (b_0, b_1, b_2, f'_01, f'_12, f'_02, θ) to the failure of 3-uniqueness over ∅ = acl(∅). We put p(x) = tp(b_0). The following facts are shown in [1],[2] (see also [3],[4]).

Definition 3.3. By a generic abelian groupoid in p, we mean an ∅-type-definable connected finitary groupoid G such that
(1) \( \text{Ob}(\mathcal{G}) = p(\mathcal{M}) \), \( \text{Mor}(\mathcal{G}) \) is \( \emptyset \)-type-definable, and maps

\[ \text{init, ter : Mor}(\mathcal{G}) \rightarrow \text{Ob}(\mathcal{G}) \]

indicating initial and terminal objects of a morphism respectively are \( \emptyset \)-definable, and so is the composition map between morphisms;

(2) for \( f \in \text{Mor}_G(b_0, b_1) \),

\[ \text{Mor}_G(b_0, b_1) = \{ g \mid g \equiv_{b_0} f \} = \{ g \mid g \equiv_{b_0}^{-1} b_1 f \} \]; and

(3) for each \( a \models p \), the vertex group \( \text{Mor}_G(a, a) \) is finite abelian.

**Fact 3.4.** From the witness \( W \), we can construct an \( \emptyset \)-type-definable generic abelian groupoid \( \mathcal{G} \) in \( p \) such that

(1) there exists \( b_{01} \)-definable bijection

\[ \pi_{01} : \{ f' \mid f' \equiv_{b_{01}} f_{01} \} = \{ f' \mid f' \equiv_{b_0}^{-1} f'_{01} \} \rightarrow \text{Mor}_G(b_0, b_1), \text{ and} \]

(2) \( \theta \) represent the composition, i.e., for \( f_{ij} : \pi_{ij}(f'_{ij}) \in \text{Mor}_G(b_i, b_j) \), we have \( f_{12}f_{01} = f_{02} \).

**Remark 3.5.** We also fix such \( \mathcal{G} \), the groupoid obtained from the witness \( W \). For \( a, b \models p \), for convenience, \( X_{ab} \) denotes \( \text{Mor}_G(a, b) \), and \( X_a = X_{aa} \) denotes the vertex group \( \text{Mor}_G(a, a) \). Hence if \( a, b \) are independent then \( X_{ab} \) is the solution set of \( \text{tp}(f/ab) \) (so of \( \text{tp}(f/ab) \)) where \( f \in X_{ab} \).

As is known, \( \eta : X_a \rightarrow X_b \) sending \( \sigma \) to \( f \circ \sigma \circ f^{-1} \) for some (any) \( f \in X_{ab} \) is a canonical group isomorphism. Hence there lives a finite abelian group \( G = (\bigcup \{ X_a \mid a \in \text{Ob}(\mathcal{G}) \}) / \sim \) (where \( \sigma \sim \sigma' \) if \( \eta(\sigma) = \sigma' \)) in \( \text{acl}(\emptyset) \), which is canonically isomorphic to each group \( X_a \). We call \( G \) the **binding group of the groupoid** \( \mathcal{G} \). (Hence in the example in section 2, if the finite group \( G \) there is abelian, then it lives in \( \text{acl}(\emptyset) \) as the binding group, so if we name \( \text{acl}(\emptyset) \), then for any \( a \in O \), \( \text{dcl}(a) = \text{acl}(a) = \text{acl}(aG_a) \). This need not hold if \( G \) there is not abelian.)

The group \( (G, \cdot) \) naturally acts on the set \( \text{Mor}(\mathcal{G}) \). For \( f \in X_{ab} \) and \( \sigma \in G \), the \( (\emptyset\)-definable) left action \( \sigma \cdot f \) is given by the composition \( \sigma \cdot f \), where \( \sigma \) is the unique element in \( \sigma \cap X_b \); the right action \( f \cdot \sigma \) is given similarly. But the two actions are equal. Namely, for all \( f \in X_{ab} \), \( g \in X_{bc} \), for all \( \sigma, \tau \in G \) we have \( \sigma \cdot f = f \cdot \sigma ; \) \( (g \cdot f) \cdot \sigma = g \cdot (f \cdot \sigma) \); and \( f \cdot (\sigma \cdot \tau) = (f \cdot \sigma) \cdot \tau \). Clearly this action on \( X_{ab} \) is regular, and \( |G| = |X_{ab}| \).

We fix more notations. Given independent \( a, b \models p \) and \( f_{ab} \in X_{ab} \) (so \( b_{01}f_{01} \equiv abf_{ab} \)), we write \( G_{ab} \) to denote \( \text{Aut}(\text{tp}(f_{ab}/ab)) = \).
Let $X$ and $f$ too, where $b_{01}f_{b_{01}} \equiv abf_{ab}$, since $\text{dcl}(f_{ab}) = \text{dcl}(f_{ab}') \ni a, b$.

**Fact 3.6.** Let $a, b \models p$ be independent.

1. If we let $(f_0, g_0) \sim (f_1, g_1) \in X_{ab}^2$ when there is $\sigma \in G$ such that $g_j = \sigma f_j$ ($j = 0, 1$), then $\sim$ is an equivalence relation on $X_{ab}^2$ and the map $[\cdot] : X_{ab}^2/\sim \to G$ sending $[(f_j, g_j)]$ to $\sigma$ is the unique bijection such that for $f, g, h \in X_{ab}$,

$$[(f, g)].[(g, h)] = [(f, h)].$$

2. For any $f, g \in X_{ab}$ and $\sigma \in G$, we have $\text{dcl}(f) = \text{dcl}(g)$ and $\text{tp}(f, \sigma f) = \text{tp}(g, \sigma g)$.

3. There exists the canonical isomorphism $\rho_{ab} : G \to G_{ab}$ sending $\mu \in G$ to $\mu \in G_{ab}$ such that $\mu.f = \mu(f)$ for some (any) $f \in X_{ab}$.

In other words $G$ again uniformly and canonically binds all the groups of the form $G_{ab}$ with independent $a, b \models p$.

**Proof.** (1) This easily follows from the regularity of the action of $G$ on $X_{ab}$.

(2) comes from that $G \subseteq \text{acl}(\emptyset)$.

(3) Here the abelianity of $G$ is used. It needs to show that $\mu.f = \mu(f)$ implies $\mu.g = \mu(g)$, for any $f, g \in X_{ab}$ and $\mu \in G_{ab}$. Now there is $\sigma \in G_{ab}$ such that $g = \sigma(f)$. Then from (2), or directly, $\sigma(\mu.f) = \mu.\sigma(f) = \mu.g$. Thus $\mu(g) = \mu \circ \sigma(f) = \sigma \circ \mu(f) = \sigma(\mu.f) = \mu.g$. The rest can easily be checked. \hfill $\Box$

Now we are ready to extend the construction method given in [1] to find another groupoid $\mathcal{F}$ (which in general is not $\emptyset$-type-definable but $\emptyset$-invariant) from the fixed symmetric witness $W$. The class $\text{Ob}(\mathcal{F})$ of objects will be the same as $\text{Ob}(\mathcal{G})$. But $\mathcal{F}$ need not be abelian as the vertex group $\text{Mor}_\mathcal{F}(b_i, b_i)$ will be isomorphic to $\text{Aut}(Y_{01}/\overline{b_0})$, where $Y_{01}$ is the possibly infinite set

$$Y_{01} = Y_{01} := \{ f \in \text{dcl}(f_{01}, \overline{b_0}) | f \equiv_{b_{01}} f_{01} \text{ and } \text{dcl}(f \overline{b_0}) = \text{dcl}(f_{01} \overline{b_0}) \}.$$

Note that $\text{dcl}(f_{01}, \overline{b_0}) = \text{dcl}(f_{01} b_1 \overline{b_0})$ since $b_1 \in \text{dcl}(f_{01})$. Moreover $Y_{01}$ and $Y_{01}'$, the set defined the same way as $Y_{01}$ but substituting $f'_{01}$ for $f_{01}$, are interdefinable. Furthermore, we shall see that $\text{Mor}_\mathcal{G}(b_i, b_i) \leq Z(\text{Mor}_\mathcal{F}(b_i, b_i))$ (Claim 3.9). We will call $\mathcal{F}$ a non-commutative groupoid constructed from the symmetric witness $W$.

**Remark 3.7.** The set $Y_{01}$ defined above depends only on $b_0$ and $b_1$ and not on the choice of $f_{01} \in X_{b_{01}}$.

**Proof.** Due to Facts 3.4(2) and 3.6, even if we replace $f_{01}$ by any $g \in X_{b_{01}}$, we obtain the same $Y_{01}$. \hfill $\Box$
**Lemma 3.8.** A set $C = \{c_i\}_i$ of realizations of $p$ with $b_0 \perp C$, and $g_i \in X_{b_0 c_i}$ are given. Then for $\sigma \in X_{b_0}$, there is an automorphism $\mu = \mu_\sigma$ of $M$ fixing each $c_i$ and $b_0$ pointwise and $\mu(g_i) = g_i \sigma$. Similarly, if $D = \{d_i\}_i(\perp b_0)$ is a set of realizations of $p$ and $h_i \in X_{d_0 b_0}$, then there is an automorphism $\tau$ fixing $d_i$ and $b_0$ such that $\tau(h_i) = \sigma h_i$.

**Proof.** Take $d \models p$ independent from $b_0 C$; and take $h \in X_{b_0 d}$. For each $i$, there is $h_i \in X_{d_0 i}$ such that $g_i = h_i h$. Now by stationarity we have $g_0 \equiv X_{b_0 C} g_0 \sigma$ witnessed by some automorphism $\mu$ sending $g_0$ to $g_0 \sigma$ and fixing $b_0, C d$ pointwise. Then $\mu(g_i) = \mu(h_i) = \mu(h)$ since $h_i \in C d$. Now there is unique $\tau \in X_{b_0}$ such that $\mu(h) = h \tau$. Thus $\mu(g_0) = g_0 \sigma = h_0 \sigma$. Hence $\sigma = \tau$. Similarly there is $\tau_i \in X_{b_0}$ such that $\mu(g_i) \equiv g_i \tau_i$, and then $\mu(g_i) = g_i \tau_i = h_i \sigma$. Hence $\tau_i = \sigma$, so $\mu(g_i) \equiv g_i \sigma$ as desired. The second clause can be proved similarly. \[\square\]

Now consider $F_{b_01} = F_{01} := \text{Aut}(Y_{01}/b_0)$.

**Claim 3.9.**

1. $X_{b_01} \subseteq Y_{01} \subseteq b_0$.

2. The action of $F_{01}$ on $Y_{01}$ (obviously by $\sigma(g)$ for $\sigma \in F_{01}$ and $g \in Y_{01}$) is regular (so $|F_{01}| = |Y_{01}|$ but can be infinite). Hence given $\mu \in G_{01} := G_{b_01}$, there is its unique extension in $F_{01}$ (we may identify those two). Thus $Y_{01}$ is $b_01$-invariant.

3. If we let $(f_0, g_0) \sim (f_1, g_1) \in Y_{01}^2$ when there is (unique) $\sigma \in F_{01}$ such that $g_j = \sigma(f_j)$ ($j = 0, 1$), then $\sim$ is an equivalence relation on $Y_{01}^2$ and the map $[\cdot] : Y_{01}^2 \sim F_{01}$ sending $[(f_j, g_j)]$ to $\sigma$ is the unique bijection such that for $f, g, h \in Y_{01}^2$,

$$[(g, h)] \circ [(f, g)] = [(f, h)].$$

4. $G_{01} \leq Z(F_{01})$. Hence for $f, k \in Y_{01}$ and $\sigma \in G_{01}$, we have $f, \sigma(f) \equiv_{b_0} k, \sigma(k)$; and for any $\mu \in F_{01}$, $b' = \mu(b_1)$ and $b_1$ are interdefinable.

5. Suppose that $\tau \in F_{01}$ and $f, g \in X_{b_01}$, so for unique $e \in X_{b_0}$ and $\sigma \in G_{01}$, we have $g = \sigma(f) = f e$. Then $f, \tau(f) \equiv_{b_0} g, \tau(g)$; $\tau(g) = \tau(f, e) = \tau(f).e$; and $\sigma(f, \tau(f)) = (f, e, \tau(f).e)$.

**Proof.** (1) is clear.

(2) comes from the fact that for any $g_0, g_1 \in Y_{01}$, they are interdefinable over $b_0$, and $Y_{01} \subseteq dcl(g_i b_0) = dcl(f_{01} b_0)$. Hence from (1), it follows $G_{01}$ is a subgroup of $F_{01}$. The rest clearly follows.

(3) comes from (2), particularly the regularity of the action.

(4) Suppose $\sigma \in G_{01}, \tau \in F_{01}$ are given. Let $g = \sigma(f_{01}) = f_{01} \sigma_0$ for some $\sigma_0 \in X_{b_0}$, and let $h = \tau(f_{01})$. Then $\tau(g) = \tau(f_{01} \sigma_0)$, and since $\tau$ fixes $b_0, = \tau(f_{01}) \sigma_0 = h \sigma_0$. Now by Lemma 3.8, there is
an automorphism fixing \( b_0 \) and sending \((f_0, h)\) to \((f_0, \sigma_0, h, \sigma_0)\), so \((f_0, h) \equiv_{b_0} (f_0, \sigma_0, h, \sigma_0) = (g, \tau(g))\). But since \( h \in \text{dcl}(f_0, b_0) \) and \( g = \sigma(f_0) \), we must have that \( \sigma(h) = \tau(g) \), so \( \sigma \circ \tau(f_0) = \tau \circ \sigma(f_0) \).

Then due to regularity, we conclude \( \sigma \in Z(F_{01}) \).

Hence if \( k = \mu(f) \) for some \( \mu \in F_{01} \), then \( \mu(f, \sigma(f)) = (k, \sigma(k)) \), in particular \( f, \sigma(f) \equiv_{b_0} k, \sigma(k) \) \( \text{(*)} \). Now if an automorphism \( \sigma' \) fixes \( b_0 \) then clearly we can assume \( \sigma' \in G_{01} \). Then by \( \text{(*)} \), \( \sigma' \) fixes \( b \) too. Similarly it follows \( b_1 \in \text{dcl}(b_0) \). Then due to stationarity it too follows \( \text{dcl}(b_1) = \text{dcl}(b') \).

(5) Due to \( \text{(4)} \), \( \sigma(f, \tau(f)) = (g, \tau(g)) \). Hence \( f, \tau(f) \equiv_{b_0} g, \tau(g) \). Now since \( f \) fixes \( b_0 \supseteq X_{ab} \), particularly it fixes \( e \). Hence \( \tau(f, e) = \tau(f).\tau(e) = \tau(f).e \), and the last one follows too.

But for \( \tau \in F_{01} \) and \( f, g \in Y_{01} \), in general \( \text{tp}(f, \tau(f)) \neq \text{tp}(g, \tau(g)) \) (in contrast to Claim \( \text{3.9(4),(5)} \) and Fact \( \text{3.6(2),(3)} \)). Neither needs \( G_{01} \) be equal to \( Z(F_{01}) \) (see Example \( \text{3.18} \)).

For the rest of the paper we fix independent \( a, b \models p \) and \( f_{ab} \in X_{ab} \). We define \( Y_{ab} \) just like \( Y_{01} \) but with \( b_0, f_0 \) being replaced by \( ab, f_{ab} \).

**Lemma 3.10.** Let \( c \models p \) and \( c \normalfont{\perp} ab \). Let \( g \in X_{ca} \). Then for \( f \in Y_{ab} \), it follows \( h = f.g \in Y_{cb} \). Moreover for \( h_0 = f_{ab}g \), we have

\[
h_0f_{ab} \equiv_{\mathcal{F}} hf \quad \text{and} \quad f_{ab}f_{\mathcal{F}} \equiv_{\mathcal{F}} h_0h_{\mathcal{F}}.
\]

**Proof.** Note that \( h_0 \in X_{cb} \). By stationarity, there is a \( \mathcal{F} \)-automorphism \( \mu \) such that \( \mu(f_{ab}) = f \). Then \( \mu(h_0) = \mu(f_{ab}.g) = \mu(f_{ab}).\mu(g) = f.g = h \in \mathcal{F}h \). We want to see that \( h, h_0 \) are interdefinable over \( \mathcal{F} \). Suppose not say there is \( h' \equiv_{\mathcal{F}h} h \) and \( h' \neq h \). Then again by stationarity there is a \( \mathcal{F} \)-automorphism \( \tau \) such that \( \tau(h_0)h = h_0h' \). Then for \( f = h.g^{-1} \) and \( f' := h'.g^{-1} \), we have \( f \neq f' \) but \( \tau(f_{ab}, f) = \tau(h_0,g^{-1},h.g^{-1}) = (h_0.g^{-1},h'.g^{-1}) = (f_{ab}, f') \), a contradiction. Similarly one can show that \( h_0 \in \text{dcl}(\mathcal{F}h) \). Hence \( h \in Y_{cb} \).

Now \( \mu \) witnesses \( h_0f_{ab} \equiv_{\mathcal{F}} hf \). To show \( f_{ab}f_{\mathcal{F}} \equiv_{\mathcal{F}} h_0h_{\mathcal{F}} \), choose \( d(\models p) \perp abc \). Now for \( k_0 \in X_{db} \), by our proof there is \( k \in Y_{ab} \) such that \( f = k.(k_0^{-1}.f_{ab}) \). Then \( h = k.k_0^{-1}.(f_{ab}.g) = k.k_0^{-1}.h_0 \). Now by stationarity, \( f_{ab}f_{\mathcal{F}} \equiv_{\mathcal{F}} h_0h_{\mathcal{F}} \). Since \( k, k_0 \in bd \), as desired \( f_{ab}f_{\mathcal{F}} \equiv_{bd} h_0h_{\mathcal{F}} \). \( \square \)

Now we start to construct the new groupoid mentioned. Our first approximation of \( \text{Mor}_F(a,b) \) is \( Y_{ab} \). Beware that \( Y_{ab} \supseteq X_{ab} \) need not be definable nor type-definable. It is just an \( ab \)-invariant set. So our groupoid \( F \) will only be invariant, and it will be definable only under additional hypotheses (e.g. \( \omega \)-categoricity).
We recall the binding group $G$ acting on $G$ as described in Remark 3.5. The action need not be a structure automorphism, since for $\sigma \in G$, in general $\text{id}_a \not= \sigma \cdot \text{id}_a \in X_a$, but it is so for $f \in X_{ab}$ (or more generally as in Lemma 3.8 above). As pointed out in Fact 3.6(3), there is the group isomorphism $\rho_{ab} : G \to G_{ab}$ such that $\rho_{ab}(\sigma)(f) = \sigma \cdot f$ for any $f \in X_{ab}$. We write $\sigma_{ab}$ for $\rho_{ab}(\sigma)$. But when there is no chance of confusion, we use $\sigma$ for both $\sigma \in G$ and $\sigma_{ab} \in G_{ab}$. Moreover, $\sigma_a$ denotes the unique element in $\sigma \cap X_a$. Hence for $f \in X_{ab}$, $\sigma(f) = \sigma \cdot f = \sigma_a \cdot f = f \cdot \sigma_a$

**Remark 3.11.** For $\sigma \in G$, and $f \in X_{ab}$ and $g \in X_{cd}$ with $cd \equiv ab$, since $G \subseteq \text{acl}(\emptyset)$ we have $f, \sigma \cdot f \equiv g, \sigma \cdot g$.

Now let $F_{ab} := \text{Aut}(Y_{ab}/\overline{a})$. Then as in Claim 3.9(2), $G_{ab} \leq F_{ab}$. As just said for any $cd \equiv ab$, there is the canonical isomorphism between $\rho_{cd} \circ \rho_{ab}^{-1} : G_{ab} \to G_{cd}$. We somehow try to find the canonically extended isomorphism between $F_{ab}$ and $F_{cd}$ as well. We do this as follows. Fix an enumeration of $Y_{ab} = \{g_i\}_i \cup \{g_j'\}_j$ such that $X_{ab} = \{g_i\}_i$, (and the rest construction depends on this). Let $Y_{cd} = \{h_i\}_i \cup \{h_j'\}_j$ such that $X_{cd} = \{h_i\}_i$, and $\langle g_i \rangle \cap \langle g_j' \rangle \equiv \langle h_i \rangle \cap \langle h_j' \rangle$. Then due to regularity of the action, for each $i$ or $j$ there is unique $\mu^{ab}_i$ or $\mu^{cd}_j \in F_{ab}$ such that $\mu_i(g_0) = g_i$ or $\mu_j(g_0) = g_j'$. Similarly we have $\mu^{cd}_i$ or $\mu^{cd}_j \in F_{cd}$.

**Claim 3.12.** The correspondence $\mu^{ab}_i \mapsto \mu^{cd}_i$ or $\mu^{cd}_j \mapsto \mu^{cd}_j$ is a well-defined isomorphism from $F_{ab}$ to $F_{cd}$ extending $\rho_{cd} \circ \rho_{ab}^{-1}$.

**Proof.** Assume $\{k_i\}_i \cup \{k_j'\}_j$ is another arrangement of $Y_{cd}$ such that $\langle k_i \rangle \cap \langle k_j' \rangle \equiv \langle h_i \rangle \cap \langle h_j' \rangle$. Then $k_0 = \sigma(h_0)$ for some $\sigma \in G_{cd}$. Thus by Claim 3.9 we have $\sigma(h_0), \mu_i^cd(h_0) = (k_0, \mu_i^cd(k_0))$ and so $k_0, \mu_i^cd(h_0) \equiv k_0, \mu_i^cd(k_0)$. Then due to interdefinability, we must have $\mu_i^cd(k_0) = k_i$. Similarly $\mu_j^cd(k_0) = k_j'$. Hence the map is well-defined. It easily follows that the map in fact is an isomorphism. Moreover due to 3.11 we see that it extends $\rho_{cd} \circ \rho_{ab}^{-1}$.

Hence now we fix an extended binding group $F \geq G$ isomorphic to $F_{01}$. (Contrary to $G \subseteq \text{acl}(\emptyset)$, $F$ need not live in $\text{acl}(\emptyset)$.) Then there is a canonical isomorphism $\rho_{cd} : F \to F_{cd}$ extending $\rho_{cd}$ in such a way that $\rho_{cd} \circ (\rho_{cd})^{-1} = \text{the correspondence defined above}$. Now for $\mu \in F$, we use $\mu_{cd}$ or simply $\mu$ to denote $\rho_{cd}(\mu)$. Note that a mapping $\mu \cdot f := \mu_{cd}(f)$ is clearly a regular action of $F$ on $Y_{cd}$ extending that of $G$ on $X_{cd}$.

**Claim 3.13.** If $cd \not\subseteq a$, then for $f \in X_{cd}$, $g \in X_{ac}$, we have $\mu(f \cdot g) = (\mu \cdot f) \cdot g$. 
Proof. This follows from Lemma 3.10.

Assume now \( c(\models p) \downarrow ab \), and \( g \in Y_{ab}, h \in Y_{bc} \) are given. We want to define a composition \( h.g \in Y_{ac} \) extending that for \( \mathcal{G} \). Note now \( g = \tau_0(g_0) \) and \( h = \sigma_0(h_0) \) for some \( \tau_0, \sigma_0 \in F \) and \( g_0 \in X_{ab}, h_0 \in X_{bc} \). We define \( h.g := (\sigma_0 \circ \tau_0)(h_0,g_0) = \sigma_0 \circ \tau_0(h_0,g_0) \).

Claim 3.14. The composition map is well-defined, invariant under any \((A\text{-})\)automorphism of \( \mathcal{M} \), and extends that of \( \text{Mor}(\mathcal{G}) \). For any \( f \in Y_{ac} \), there is unique \( h' \in Y_{bc} \) \((g' \in Y_{ab}, \text{resp.})\) such that \( f = h'.g \) \((f = h.g, \text{resp.})\).

Proof. Let \( g = \tau_1(g_1) \) and \( h = \sigma_1(h_1) \) for some \( \tau_1, \sigma_1 \in F \) and \( g_1 \in X_{ab}, h_1 \in X_{bc} \). Then since \( \sigma_0^{-1} \circ \sigma_1(h_1) = h_0 \) and \( \tau_0^{-1} \circ \tau_1(g_1) = g_0 \), due to uniqueness we have that both \( \sigma_0^{-1} \circ \sigma_1, \tau_0^{-1} \circ \tau_1 \) are in \( G \) so in the center of \( F \). Now due to Lemma 3.10,

\[
\begin{align*}
\sigma_0 \circ \tau_0(h_0,g_0) &= \sigma_0 \circ \tau_0 \circ \sigma_0^{-1} \circ \sigma_0(h_0,g_0) \\
&= \sigma_0 \circ \tau_0 \circ \sigma_0^{-1}(\sigma_0(h_0),g_0) \\
&= \sigma_0 \circ \tau_0 \circ \sigma_0^{-1}(\sigma_1(h_1),g_0) \\
&= \sigma_0 \circ \tau_0 \circ \sigma_0^{-1}(\sigma_1^{-1}(\tau_0(h_0),g_0)) \\
&= \sigma_0 \circ \tau_0(\sigma_1^{-1}(\tau_0(h_0),g_0)) \\
&= \sigma_0 \circ \tau_0(\sigma_1^{-1}(\tau_0(g_0))) \\
&= \sigma_0 \circ \tau_0(\sigma_1^{-1}(\tau_0(g_0))) \\
&= \sigma_0 \circ \tau_0(\tau_1(h_1,g_1)).
\end{align*}
\]

Automorphism invariance clearly follows from the same property for \( \text{Mor}(\mathcal{G}) \) and the choice of the isomorphism \( \rho_{ab}^F \). Moreover by taking \( \tau_0 = \sigma_0 = \text{id} \), we see that the composition clearly extends that for \( \mathcal{G} \). Lastly \( f = \tau(f_1) \) for some \( f_1 \in X_{ac} \). Now there is \( h'_1 \in X_{bc} \) such that \( f_1 = h'_1.g_1 \). Put \( h' = \tau \circ \tau_1^{-1}(h'_1) \). Then by the definition, \( f = (\tau \circ \tau_1^{-1}) \circ \tau_1(h'_1,g_1) = h'.g \). For any \( h'' \neq h' \in Y_{bc} \) it easily follows that \( f \neq h''.g \). Hence \( h' \) is unique such element.

The rest of the construction of \( \mathcal{F} \) will be similar to that of \( \mathcal{G} \) in [11]. \( \text{Ob}(\mathcal{F}) \) will be the same as \( \text{Ob}(\mathcal{G}) = p(\mathcal{M}) \). Now for arbitrary \( c, d \models p \), an \( n \)-step directed path from \( c \) to \( d \) is a sequence \((c_0, g_1, c_1, g_2, \ldots, c_n)\) such that \( c = c_0, d = c_n, c_{i-1}c_i \equiv ab \) and \( g_i \in Y_{c_{i-1}c_i} \). Let \( D^n(c,d) \) be the set of all \( n \)-step directed paths. For \( q = (c_0, g_1, c_1, g_2, \ldots, c_n) \in D^n(c,d) \) and \( r = (d_0, h_1, d_1, h_2, \ldots, d_m) \in D^m(c,d) \) we say they are equivalent (write \( r \sim s \)) if for some \( c^* (\models p) \downarrow qr \) and \( g^* \in Y_{c^*,d^*} \), we have \( g^*_n = h^*_m \in Y_{c^*,d^*} \) where \( g^*_0 = h^*_0 = g^* \) and \( g^*_i+1 = g^*_{i+1} \) \((i = 0, \ldots, n-1)\) and \( h^*_{j+1} = h^*_{j+1} \) \((j = 0, \ldots, m-1)\). Due to stationarity the relation is independent from the choices of \( c^* \) and \( g^* \), and is an equivalence relation. Similarly to Lemma 2.12, one can easily see using Claim 3.13 that for any \( q \in D^n(c,d) \), there is \( r \in D^2(c,d) \) such that \( q \sim r \). Then \( D^2(c,d)/\sim \) will be our \( \text{Mor}_\mathcal{F}(c,d) \), and composition will be concatenation of paths. The identity morphism in \( \text{Mor}_\mathcal{F}(c,c) \) can be
defined just like in [I 2.15]. Now our groupoid $\mathcal{F}$ is clearly connected, and it extends $\mathcal{G}$ (see Proposition 3.17). An argument similar to that in [I 2.14] implies there is a canonical $ab$-invariant 1-1 correspondence between $Y_{ab}$ and $\text{Mor}_\mathcal{F}(a, b)$. Indeed the same argument shows that for any $c, d \models p$ (not necessarily independent), there too exists a canonical injection from $X_{cd}$ to $\text{Mor}_\mathcal{F}(c, d)$.

Now for $f \in Y_{ab}$ (or $\in X_{cd}$, resp.), in the rest we let $f$ denote the corresponding element in $\text{Mor}_\mathcal{F}(a, b)$ (or $\text{Mor}_\mathcal{F}(c, d)$, resp.). Then similarly to $Y_{ab}$, it follows

$$\text{Mor}_\mathcal{F}(a, b) = \{x \in \text{dcl}(\overline{a}) | x \equiv_{\mathcal{F}} f \text{ and } \text{dcl}(x) = \text{dcl}(\overline{f}) \} \subseteq \overline{ab}.$$ 

But $\mathcal{F}$ need not be definable nor type-definable nor hyperdefinable. It is just an invariant groupoid.

As pointed out in 3.9, $Y_{ab}$ is $ab$-invariant. Now if it is type-definable then as it is a bounded union of definable sets, by compactness it indeed is definable and a finite set. (This happens when $T$ is $\omega$-categorical.) For this case let us add a bit more explanations that are not explicitly mentioned in [I]. By compactness now, $\sim$ turns out to be definable: Note that $D^2(p) := \bigcup\{D^2(c, d) | c, d \models p\}$ is $\emptyset$-type-definable. Then there clearly is an $\emptyset$-definable equivalence relation $E$ on $D^2(p)$ each of whose class is of the form $D^2(c, d)$. In each $E$-class, there are exactly $|Y_{ab}|$-many $\sim$-classes. Hence $\sim$ is $\emptyset$-definable relatively on $D^2(p)$ as well. Hence $[r] \in \text{Mor}_\mathcal{F}(c, d)$ is an imaginary element and the maps $[r] \mapsto c$ or $d$ (the first and last components of $r$) are $\emptyset$-definable init, ter maps, respectively. Similarly the composition map of morphisms is $\emptyset$-definable. Therefore $\mathcal{F}$ is a (relatively) $\emptyset$-definable groupoid.

We return to the general context of the invariant $\mathcal{F}$. For notational simplicity, use $Y_{cd}$ to denote $\text{Mor}_\mathcal{F}(c, d)$, and use $Y_c$ for $Y_{cc}$. We state some observations regarding $\mathcal{F}$.

**Remark 3.15.**

(1) Note that for $\sigma \in F_{ab} \text{ and } f \in Y_{ab}$, we have $\sigma(f) \in Y_{ab}$ and both $f, \sigma(f) \in Y_{ab}$. However depending on context, $\sigma(f)$ may be in $Y_{ab}$, or $Y_{\sigma(b)}$. For $f, g \in Y_{ab}$, clearly $fg \equiv gg$. Hence in this sense obviously $\sigma(f) \in Y_{ab}$. But since $\sigma(f)$ is in $Y_{\sigma(b)}$ too, we can say $\sigma(f) \in Y_{\sigma(b)}$, as well. To remove this confusion, one may put a prefix $ab$ to any $f \in Y_{ab}$. But instead, in the rest we only regard $\sigma(f) \in Y_{ab}$.

(2) We know that $Y_{ab} \subseteq \overline{ab}$. For any $c \models p$, we too have $Y_{bc} \subseteq \overline{bc}$.

We can assume $a \not\models bc$. Now let $f \in Y_{bc}$. Suppose that $f \not\in \overline{bc}$, and let $\{f_i\}$ be a set of infinitely many conjugates of $f$ over $\overline{bc}$. Now due to stationarity, we can then assume that all $f_i$’s have
the same type over \(\sigma \cup ab\). Hence given \(x \in Y_{ab}\), all \(f_i.x \in Y_{ac}\) have the same type over \(ac\), contradicting \(f_i.x \in \overline{ac}\).

We get now the following results for \(\mathcal{F}\) similarly to those of \(\mathcal{G}\).

**Proposition 3.16.** The group \(F_{ab}\) is isomorphic to \(Y_{ab}\). In fact for any \(\sigma \in Y_{ab}\), there is \(\sigma_b \in F_{ab}\) such that for any \(f \in Y_{ab}\), \(\sigma_b(f) = \sigma.f\). Hence \(F_{ab} = \{\sigma_b \mid \sigma \in Y_{ab}\}\).

**Proof.** The proof will be similar to that of Claim 2.4. Define a map \(\eta : Y_{ab} \to F_{ab}\) such that for \(\sigma \in Y_{ab}\) and any \(f \in Y_{ab}\), we let \(\eta(\sigma)(f) = g\) where \(g = \sigma.f\). Hence due to that \(ff = gg\), we have \(\eta(\sigma)(f) = g\) too.

Then \(\eta\) is a well-defined, since if \(h \in Y_{ab}\), then there is \(x \in Y_{a} \subseteq \overline{a}\) such that \(h = f.x\), and thus \(\eta(\sigma)(h) = \eta(\sigma)(f).x = g.x = \sigma.f.x = \sigma_h\).

Moreover clearly \(\eta\) is 1-1 and onto since any \(\mu \in F_{ab}\) is determined by \((f_{ab}, \mu(f_{ab}))\). It is obvious \(\eta\) is in fact a group isomorphism. Now we take \(\sigma_b = \eta(\sigma)\). \(\square\)

**Proposition 3.17.** For \(c \models p \bot ab\) and \(f \in Y_{ab}, g \in Y_{bc}, h \in Y_{ac}\), we have \(h = g.f\) (the composition map is defined before Claim 3.14) iff \(h = g.f\). Moreover, \(\mathcal{F}\) extends the composition of \(\mathcal{G}\).

**Proof.** Since the composition relation defined in 3.14 is invariant relation, we can find an \(\emptyset\)-invariant relation \(\theta(x, y, z)\) such that for any \(a'b'c' \equiv abc\) and \(f' \in Y_{a'y}, g' \in Y_{b'c'}, h' \in Y_{a'c'},\) we have \(h' = g'.f'\) iff \(\theta(a'b'f', b'c'g', a'c'h')\) holds. Then the rest proof of the proposition will be exactly the same as that of [2, 2.19], hence we omit it.

We now step by step show that \(\mathcal{F}\) extends the composition of \(\mathcal{G}\). Let \(c \models p\) be given such that \(ac \bot b\). Let \(y \in X_{ab}, z \in X_{bc}\). Choose \(a' \models p \bot abc\), and \(x \in X_{a'a}\). Then \(z.y.x \in X_{a'c}\), and by the definitions of the concatenating composition and the injection from \(X_{ac}\) to \(Y_{ac}\), we have \(z.y = z.y\) in \(\mathcal{F}\).

Now more generally let \(d \models p\) be given such that \(acd \bot b\). Let \(s \in X_{ac}, t \in X_{ct}\). Choose \(u \in X_{ab}\). Then there are \(v \in X_{bc}\) and \(w \in X_{bd}\) such that \(s = v.u \in X_{bc}\), and \(w.v^{-1} = t\) (in \(\mathcal{G}\)). Then by the previous argument, in \(\mathcal{F}\), we have \(t.s = w.v^{-1}.v.u = w.v^{-1}w.u = w.u = w.u = t.s\). \(\square\)

We now give an example where \(X_a\) is a proper subgroup of \(Z(Y_{ab})\).

**Example 3.18.** Consider the same example \((O, M, ., \text{init}, \text{ter})\) as in section 2, but where the binding group \(G\) is abelian. Namely it is a connected finitary abelian groupoid. We add one more sort \(I\) and an equivalence relation \(E\) on \(I\) such that each \(E\)-class has 2 elements, and there also is a projection function \(\pi_E : I \to O\) (all are in the language)
so that \( O = I/E \). We let \( N \) be the resulting extended structure. Now choose \( c \neq d \in O \), \( f \in \text{Mor}(c,d) \), and let \( \{c_0, c_1\} = \pi^{-1}(c) \), and \( \{d_0, d_1\} = \pi^{-1}(d) \). Now clearly \((c_0c_1c, d_0d_1d, \ldots, f, \ldots, (x,y,z))\) where \( (x,y,z) \) is a formula indicating the composition \( z = y.x \), can be considered as a symmetric witness. Let \( G \) be the abelian groupoid obtained from the symmetric witness. Then clearly \( X_{c_0c_1c} \) is isomorphic to \( G \). Let \( F \) be the groupoid as we constructed above from \( G \). Then since an automorphism of \( N \) can swap \( d_0, d_1 \) while fixing \( c_0c_1cd \), we have that \( G \) is central in \( F := Y_{c_0c_1c} \), while \( G \) has only two conjugates in \( F \). Then it easily follows that \( F \) is abelian too, so \( G \leq Z(F) = F \).

4. Approximation of the non-commutative groups

In this last section we discuss a possible limit of the vertex groups of the non-commutative groupoids we have constructed in previous section. As before, we keep suppressing \( A = \text{acl}(A) \) to \( \emptyset \). Fix a complete type \( q \) (of possibly infinite arity) over \( \emptyset \). Choose independent \( u, v, w \models q \). Recall that

\[
\Gamma_2(q) := \text{Aut}(\overline{uv}/\overline{w}, \overline{v}),
\]

where \( \overline{uv} := \overline{uv} \cap \text{dcl}(\overline{uw}, \overline{w}) \).

The following fact is simply a restatement of Proposition 3.2 and Fact 3.4.

**Fact 4.1.** Let a finite tuple \( f \in \overline{uv} \setminus \text{dcl}(\overline{w}, \overline{v}) \) be given. Then there are a generic abelian groupoid \( G' \) in \( q' \in S(\emptyset) \) of finite arity; and independent \( u', v' \models q' \) with \( f' \in \text{Mor}_{\overline{G'}}(u', v') \) such that \( f \in \text{dcl}(f') \), and \( u' \subseteq \overline{u}, v' \subseteq \overline{v} \).

We let

\[
I = I_q := \{ f \in \overline{u}_f, \overline{v}_f : G_f \text{ is a generic abelian groupoid in } \text{tp}(u_f) = \text{tp}(v_f) \text{ such that } f \in \text{Mor}_{G_f}(u_f, v_f) \text{ with independent finite tuples } u_f(\subseteq \overline{u}), v_f(\subseteq \overline{v}) \}.
\]

On the other hand we let,

\[
J = J_q := \{ f \in \overline{u}_f, \overline{v}_f : F_f \text{ is a non-commutative groupoid obtained from } \text{the generic abelian groupoid } G_f \text{ in } \text{tp}(u_f) = \text{tp}(v_f) \text{ such that } f \in \text{Mor}_{F_f}(u_f, v_f) \text{ with independent finite tuples } u_f(\subseteq \overline{u}), v_f(\subseteq \overline{v}) \}.
\]

For \( f \in I_q \), we write \( G_f \) to denote \( G_{u_fv_f} \) as in section 3. Now by Fact 4.1 \((I, \leq_I) \) with letting \( f \leq_I f' \) iff \( f \in \text{dcl}(f') \), \( \text{init}(f) \subseteq \text{dcl}(\text{init}(f')) \), and \( \text{ter}(f) \in \text{dcl}(\text{ter}(f')) \) is a direct system.

Now for \( f \leq_I f' \) \( f' \in I \), any \( \sigma' \in G_{f'} \) fixes \( u_fv_f \) pointwise. Hence \( \sigma'(f) \in X_{u_fv_f} \), and we write \( (\sigma' \upharpoonright f) \) to denote the unique \( \sigma \in G_f \) such that \( \sigma(f) = \sigma'(f) \); and \( \chi'_{f} : G_{f'} \rightarrow G_f \) to denote the group
homomorphism sending $\sigma'$ to $(\sigma' \upharpoonright f)$. Due to stationarity it indeed is an epimorphism. Clearly $\chi_f^f = \text{id}$. Then

$$S_I := \{\{G_f \: | \: f \in I\}, \{\chi_f^f \: | \: f \leq_I f' \in I\}\}$$

forms a directed system of finite abelian groups. As pointed out in \[2,\text{Theorem 2.25}\], the inverse limit of $S_I$ is isomorphic to $\Gamma_2(q)$, so it is a profinite abelian group.

However for $\{F_f \: | \: f \in J\}$ where $F_f := F_{uv^f}$ as in section 3, it is not clear how to give an order relation and transition maps to make this a directed system of groups. There are a couple of obstacles to do this. For example, in general given an elementary map $\sigma$ of $M$, and a tuple $cd$, even if $cd$ and $\sigma(cd)$ are interdefinable, $c$ and $\sigma(c)$ need not be so, and vice versa. But we can consider partial transition maps among $F_f$'s and their limit as follows. We let

$$\Pi_2(q) := \{\sigma \in \text{Aut}(\tilde{uv}/\tilde{u}) : \text{for any } f \in J, \text{ dcl}(f\tilde{u}) = \text{dcl}(\sigma(f)\tilde{u})\}.$$

For $f \in J$ and $\sigma \in \Pi_2(q)$, we similarly write $(\sigma \upharpoonright f)$ to denote the unique $\sigma' \in F_f$ such that $\sigma(f) = \sigma'(f)$. We let

$$\Pi_f := \{(\sigma \upharpoonright f) : \sigma \in \Pi_2(q)\}.$$

**Proposition 4.2.** The following hold.

1. $\Pi_2(q) := \{\sigma \in \text{Aut}(\tilde{uv}/\tilde{u}) : \text{for any } f \in I, \text{ dcl}(f\tilde{u}) = \text{dcl}(\sigma(f)\tilde{u})\}$.
2. For $f \in J$, we have $G_f \leq \Pi_f \leq F_f$. Now

$$S_J := \{\{\Pi_f \: | \: f \in J\}, \{\chi_f^f \: | \: f \leq_J f' \in J\}\}$$

forms a directed system of groups, where $\leq_J$ and $\chi_f^f$ are similarly defined as in $S_I$. Moreover $\Pi_2(q)$ is the inverse limit of $S_J$.
3. $\Gamma_2(q) \leq Z(\Pi_2(q))$.
4. Both $\Gamma_2(q)$ and $\Pi_2(q)$ are normal subgroups of $\text{Aut}(\tilde{uv}/\tilde{u})$.

**Proof.** (1) Clear (see Claim 3.9(4)).

(2) That $\Pi_f \leq F_f$ is clear by definition, and that $G_f \leq \Pi_f$ is also clear since $\Gamma_2(q) \leq \Pi_2(q)$. Now since every automorphism in $\Pi_f$ is the restriction of that in $\Pi_2(q)$, it follows that $S_J$ forms a directed system of groups with the transition maps $\chi_f^f$. The rest proof of (2) is standard. Let $\Pi$ be the inverse limit of $S_J$. We define a homomorphism $\varphi : \Pi_2(q) \rightarrow \Pi$ by sending $\sigma \in \Pi_2(q)$ to the element in $\Pi$ represented by the function $f \in J \mapsto (\sigma \upharpoonright f) \in \Pi_f$. This embedding is obviously one-to-one and due to compactness it is surjective too.

(3) comes from (2) and that $G_f \leq Z(\Pi_f)$. 
Let $\sigma \in \Gamma_2(q)$. Then any of its conjugates in $\text{Aut}(\overline{uv}/\overline{u})$ fixes $\overline{v}$ pointwise. Hence $\Gamma_2(q) \leq \text{Aut}(\overline{uv}/\overline{u})$.

Now let $\sigma \in \Pi_2(q)$ and $\mu \in \text{Aut}(\overline{uv}/\overline{u})$. Then for any $f \in I$, we have $g := \mu(f) \in I$ too, and $g$ and $\sigma(g)$ are interdefinable over $\overline{u}$. Hence so are $f$ and $\mu^{-1} \circ \sigma(g) = \mu^{-1} \circ \sigma \circ \mu(f)$ over $\overline{u}$. Therefore $\mu^{-1} \circ \sigma \circ \mu \in \Pi_2(q)$ and (4) is proved. \qed

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