Notes on Isolated Horizons

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Abstract

A general analysis for characterizing and classifying ‘isolated horizons’ is presented in terms of null tetrads and spin coefficients. The freely specifiable spin coefficients corresponding to isolated horizons are identified and specific symmetry classes are enumerated. For isolated horizons admitting at least one spatial isometry, a standard set of spherical coordinates are introduced and associated metric is obtained. An angular momentum is also defined.

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I. INTRODUCTION

Stationary black holes in asymptotically flat space-time obey the famous laws of black hole mechanics [1] which admit a thermodynamical interpretation [2] supported by the Hawking effect [3]. These provide an arena to study the interplay of classical gravity, quantum mechanics and thermodynamics [4]. These involve the global notions of event horizon, asymptotic flatness and stationarity. There have been two generalizations by relaxing some of the global conditions, namely the ‘trapping horizons’ of Hayward [5] and the ‘isolated horizons’ of Ashtekar et al [6].

Hayward [5] introduced the quasi-local idea of “trapping” horizons by abstracting the property that these are foliations of (suitably) marginally trapped surfaces. He gave a classification of “trapping horizons” and also defined quantities satisfying a generalized version of all the laws of black hole mechanics. The entire analysis is strictly quasi-local with no reference to any asymptotics.

Among these classes of horizons, is a special case wherein a horizon is a null hypersurface. These horizons have the property that the area of its two dimensional space-like surfaces remain constant along its null generators and the second law reduces trivially to the statement that the area is constant. Since one expects these areas to change only when there is energy flow across the horizon, these horizons suggests an intuitive idea of ‘isolation’.

With a somewhat different notion of “isolated horizons”, recent work of Ashtekar et al [6] also seeks to replace event horizons and yet get analogue of the zeroth and the first
law of black hole mechanics. In this generalization, though reference to asymptotics is retained, stationarity of usual black holes is relaxed. In particular, Ashtekar et al formulate these notions and their associated quantities such as mass, in terms of variables and actions suitable for passage to a non-perturbative quantization. This generalization permits one to deal with a situation wherein a collapse proceeds in stages punctuated by a series of ‘non activity’ across a collapsed ‘core’. Each of such stage of ‘non activity’ is modeled by an “isolated horizon”. Loss of stationarity come with a price which makes this generalization non trivial and leads to infinitely many forms of the first law \[7\]. This framework allows Ashtekar et al to compute the entropy of the so called non-rotating isolated horizons \[8\].

The latter notion of isolated horizons is more restrictive than that of trapping horizons in the sense that isolated horizons are null hyper-surfaces while trapping horizons have no such restriction. The restriction is not overly strong, the space of solutions admitting isolated horizons is infinite dimensional \[9\]. The restriction however gives some control over the space of solutions and recently Ashtekar et al \[7\] have given a very interesting interpretation of the first law(s).

In both the generalizations mentioned above, there are two part. The first is the geometric characterization of appropriate horizon. This involves both the distinguishing features of appropriate three dimensional sub-manifold of a space-time and the definitions of quantities such as surface gravity, mass, area, angular velocity, angular momentum, charge etc associated with the horizon. The second part involves showing that the quantities so defined do satisfy the laws of black hole mechanics (or a generalized version thereof).
In this work we consider a uniform treatment of general isolated horizons only. Our aim is to arrive at a characterization such that the zeroth and the first law of black hole mechanics hold. One should observe at the outset that the zeroth law is a property of a single solution of matter-Einstein field equation admitting an isolated horizon. The first law however involves comparison of certain quantities associated with several such solutions and thus is a statement about properties of the full space of such solutions. Establishing the first law is therefore expected to be more involved.

A remark about the slight difference in the approach of Ashtekar and co-workers and ours is in order. In the approach of Ashtekar et al, an action principle plays an important role, particularly for the first law and of course in the subsequent computation of the entropy. It is thus natural to view an isolated horizon as an “inner boundary” of a suitable class of space-time manifolds and transcribe characterization of isolated horizons in terms of boundary conditions on the variables entering the action formulation. On the other hand, as a first step, we wish to translate the physical idea behind isolated horizon into a geometrical characterization and try to get a handle on the space of solutions admitting such horizons. For this an action formulation is unnecessary and it is natural to view an isolated horizon as a suitable null hypersurface of a solution space-time. Since a tetrad formulation is closer to (metric) geometry, it is a natural choice for us. This has the further advantage that it can be generalized to other dimensions. If one could get the laws of black hole mechanics without the use of an action principle, then one could also treat phenomenological matter. One should also remark that since both approaches capture the same physical idea, one does not expect different results at the level of geometrical characterization. What we do seek though is a direct and systematic
arrival at a geometrical characterization of the physical notions.

The basic defining property of these horizons is that they are all null hyper-surfaces. This naturally suggests the use of null tetrad formulation. In terms of suitably adapted null tetrads and their corresponding spin coefficients, this alone immediately implies that the coefficients $\kappa = 0$ and $\rho$ is real. To this one adds the requirement that the expansion of the null geodesic generators be zero ($\rho = 0$) which incorporates the properties of being isolated (constancy of area) and potential marginal trapping. Since these are supposed to be hyper-surfaces of physical space times, it is only natural to require that Einstein equations hold and that the stress tensor satisfies a suitable energy condition. Raychoudhuri equation and the energy conditions then imply a number of consequences, one of which is that the null geodesic congruence is also shear free ($\sigma = 0$).

There is however a good deal of freedom in choosing the null tetrads, the freedom to make local Lorentz transformations. These have been conveniently classified into three types [10]. Having gotten a null direction field, the relevant freedom is reduced to only the so-called type-I and type-III transformations. Due to this freedom, one can not characterize isolated horizons in terms of the spin coefficients by themselves unless these are invariant under the local Lorentz transformation. (The particular values of the spin coefficients in the previous paragraph are indeed invariant.) One can either impose conditions on quantities that are invariant under these local Lorentz transformations or one could fix a convenient ‘gauge’ and then use the corresponding spin coefficients for characterization. In this work we use a combination of both.

The paper is naturally divided into two parts. The first part uses a gauge fixing
procedure to identify freely specifiable spin coefficients. The second part uses invariant quantities to analyze symmetry classes of the horizons and identifies suitable associated quantities.

The paper is organized as follows.

In section II we discuss the basic conditions characterizing isolated horizons and arrive at the zeroth law. We also obtain the ‘freely specifiable’ spin coefficients by a gauge fixing procedure.

In section III we discuss a symmetry classification (isometries). A characterization of “rotating horizons” is discussed and an angular momentum is identified. This section uses invariant quantities and constructs a natural set of coordinates on the null hypersurface. We assume the existence of at least one spatial symmetry for this purpose. A candidate parameterization of a ‘mass’ is also given.

Section IV contains a summary and remarks on the first law.

The notation and conventions used are those of Chandrasekhar [10] and some of these are collected in appendix A for reader’s convenience. The metric signature is (+ - - - -).

Appendix B is included to illustrate our procedures for the Kerr-Newman family.
II. CHARACTERIZATION OF ISOLATED HORIZONS

A. Basic Conditions and the Zeroth Law

Our definition of isolated horizon will turn out to be the same as the most recent one given by Ashtekar et al [7]. We will however not try to define it completely intrinsically nor view it as a boundary to be attached to an exterior asymptotically flat space-time. We will explicitly view an isolated horizon as a null hyper-surface in a four dimensional space-time which is a solution of a set of Einstein-matter field equations. The matter stress tensor is required to satisfy suitable energy condition and the solution is required to be causally well behaved. To keep the logic and the role of each of the conditions transparent, we will begin with minimal conditions, see their implications and add further conditions as needed.

(I) An isolated horizon is a null hyper-surface in a solution of four dimensional Einstein-matter field equations with matter satisfying the dominant energy condition (which implies also the weak energy and the null energy conditions). Thus, Einstein equation holds on \( \Delta \) and the matter stress tensor satisfies:

- The weak energy condition and in particular, \( T_{\mu\nu} \ell^\mu \ell^\nu \geq 0 \);

- \( T_{\mu\nu} \ell^\nu \) is causal. It is future(past) directed according as \( \ell^\mu \) is.

One can always write these equations with reference to a chosen tetrad. The null hyper-surface character of \( \Delta \), singles out one null direction and makes the use of null tetrad natural. We will thus assume that we have made a (arbitrary) choice of null tetrad, \( \ell, n, m, \bar{m} \), such that \( \ell \), at points of \( \Delta \), is along the direction of the null normal. There is of course a large class of null tetrads to choose from and this is parameterized
by the group of type-I and type-III local Lorentz transformations (see appendix A). We will refer to the type-I transformations as \((complex)\) boosts, type-III scaling as \(scaling\) and type-III rotations of \(m, \bar{m}\) as \(rotations\).

The hyper-surface orthogonality of \(\Delta\) implies that the null congruence defined by \(\ell^\mu \partial_\mu\) is a geodetic congruence \((\kappa = 0)\) and is twist-free \((\rho\) is real). The geodesics however are not affinely parameterized in general. The shear of this congruence is given in terms of \(\sigma\). Vanishing of \(\kappa\) already makes the twist and shear to be invariant under of boost transformations. Furthermore the null geodesics generate \(\Delta\) and hence \(\Delta\) has the topology of \(R \times \Sigma_2\) (We assume there are no closed causal curves). The \(R\) part can be an interval while \(\Sigma_2\) could be compact or non-compact. At present, one need not stipulate these global aspects. But later we will restrict to \(\Sigma_2\) being spherical. The topology, guarantees existence of at least one foliation, not necessarily unique, as follows.

Choose any two dimensional sub-manifold \(\Sigma_2\) of \(\Delta\) such that it is transversal to the direction field of the null normals. Fix an arbitrary null vector field along the null direction. Under diffeomorphisms generated by the null vector field, we will generate images of \(\Sigma_2\) and hence a foliation of \(\Delta\). The leaves of such a foliation will be diffeomorphic to \(\Sigma_2\). One will therefore have a natural (and arbitrary) choice of null directions tangential to the leaves. Once we make a choice of \(m, \bar{m}\), the null tetrad is uniquely completed. It then follows immediately that the null congruence defined by \(n^\mu \partial_\mu\) is also twist-free \((\mu\) is real). Since the leaves are obtained by diffeomorphism generated by \(\ell\) and the \(m, \bar{m}\) are tangential to the leaves, it follows that

\[
(\mathcal{L}_\ell m)^\mu n_\mu = (\mathcal{L}_\ell \bar{m})^\mu n_\mu = 0
\]
This in turn implies that \((\alpha + \bar{\beta} - \pi) = 0\) must hold. Note that this procedure can always be followed and hence we can always have a choice of null tetrad such that \(\mu\) is real and \(\alpha + \bar{\beta} = \pi\).

Remark: In the above we made two arbitrary choices, the vector field \(\ell\) and the initial transversal \(\Sigma_2\). We have not defined \(m, \bar{m}\) by any transport from those chosen on initial \(\Sigma_2\). There is no loss of generality though. By the available freedom of making local boosts, scaling and rotations, we can change the initial null normal by local scaling and also the initial leaf (and hence the foliation) by local boosts. These transformations are not completely general though since we want to preserve the foliation property. We will refer to these restricted transformations as residual transformation. These will get progressively further restricted. At this stage, the rotation parameter is completely free and so is the scaling parameter while the boosts parameter satisfies,

\[
\begin{align*}
Da - (\epsilon - \bar{\epsilon})a &= 0 \\
D\bar{a} + (\epsilon - \bar{\epsilon})\bar{a} &= 0 \\
\delta\bar{a} - (\bar{\alpha} - \beta)\bar{a} &= \bar{\delta}a - (\alpha - \bar{\beta})a
\end{align*}
\]

Thus from the null hyper-surface property and a procedure of choosing null tetrads we have deduced that for every given initial choice of null normal vector field and initial \(\Sigma_2\), there exist a choice of null tetrads such that the spin coefficients satisfy, \(\kappa = \rho - \bar{\rho} = \mu - \bar{\mu} = \alpha + \bar{\beta} - \pi = 0\). Further more we are free to make the residual transformations.

For the next condition one can give two different arguments. The \(\ell\) and the \(n\) congruences are orthogonal to the leaves. If the \(n\) congruence were also geodesic \((\nu = 0)\) then
depending on the expansion of the $n$-congruence, we could have the leaves as (marginally) trapped surfaces. Indeed we would like to have this property, at least in the black hole context, so at the least we require $\ell$ to be expansions-free ($\rho + \bar{\rho} = 0$). We will however leave $\nu$ unspecified for the moment.

Alternatively, this condition also shows that the induced metric on the leaves does not vary from leaf to leaf and hence the area of a leaf is a constant or equivalently, the Lie derivative of $m \wedge \bar{m}$ projected to $\Sigma_2$ is zero. Since the area is expected to change only when there is energy flow across the horizon, this captures the notion of “isolation”. In the terminology of Ashtekar et al [7], this is specification of non-expanding horizon.

(II) $\ell$ congruence is expansion-free ($\rho + \bar{\rho} = 0$).

A number of consequences follow from these conditions. The first of the energy conditions in conjunction with the Raychoudhuri equation for the expansion of $\ell$ congruence and the zero expansion condition above implies that

- shear($\ell$) = 0 ($\sigma = 0$),
- $R_{\mu\nu}^\ell \ell^\mu \ell^\nu = 0$ on $\Delta$ and by Einstein equations
- $T_{\mu\nu}^\ell \ell^\mu \ell^\nu = 0$ on $\Delta$.

Note that since the $\ell$ geodesics are not affinely parameterized, the Raychoudhuri equation has an extra term ($\epsilon + \bar{\epsilon}) \theta(\ell)$. This however does not affect the conclusion.

The $T^{\mu\nu}_{\ell}$ being causal and $T_{\mu\nu}^\ell \ell^\mu \ell^\nu = 0$ then implies that $T^{\mu\nu}_{\ell} = e \ell^\mu$, with $e$ being non-negative. Again using Einstein equation this implies that
\[ R_{\mu\nu} \ell^\nu = (8\pi e + \frac{R}{2}) \ell_\mu \] (3)

Hence we get the Ricci scalars \( \Phi_{00} = \Phi_{01} = \Phi_{10} = 0 \).

Furthermore, using definitions of the Ricci scalars we deduce,

\[ \Phi_{11} + 3\Lambda = -4\pi e \equiv -\mathcal{E} \quad (\leq 0). \] (4)

For future use we also note that the conservation of the stress tensor and the above form for it implies that,

\[ \ell \cdot \nabla \mathcal{E} \equiv D\mathcal{E} = 0; \] (5)

The Bianchi identities from appendix A imply,

\[ D(\Psi_2 - \Phi_{11} - \Lambda) = 0 \] (6)

The equation (h) from item 5 of appendix A gives,

\[ \Psi_2 + 2\Lambda = -\delta \pi - \pi(\bar{\pi} - \bar{\alpha} + \beta) + D\mu + \mu(\epsilon + \bar{\epsilon}) \] (7)

Combining the above equations we get,

\[ -K \equiv \Psi_2 - \Phi_{11} - \Lambda = -\pi \bar{\pi} - \{\delta \pi - (\bar{\alpha} - \beta)\pi\} + D\mu + \mu(\epsilon + \bar{\epsilon}) + \mathcal{E}. \] (8)

And \( D \) of the L.H.S. is zero by the Bianchi identity. The \( K \) introduced above is the (complex) curvature of \( \Sigma_2 \) as defined by [11]. Restricting now to compact leaves without boundaries one has,

\[ \int_{\Sigma_2} (K + \bar{K}) = 4\pi(1 - \text{genus}) \quad (\text{Gauss - Bonnet}) \] (9)

\[ \int_{\Sigma_2} (K - \bar{K}) = 0 \quad (\text{Penrose}) \] (10)

We note in passing that the following can be checked explicitly:
\[ \omega_+ \equiv \pi m + \bar{\pi} \bar{m} \implies d\omega_+ \sim (\delta \pi - \delta \bar{\pi}) m \wedge \bar{m} \]

\[ \omega_- \equiv i(\pi m - \bar{\pi} \bar{m}) \implies d\omega_- \sim (\delta \pi + \delta \bar{\pi}) m \wedge \bar{m} \]

(11)

Here the underlined derivatives are the compacted (rotation covariant) derivatives defined in appendix A. From this the equation (10) follows and integral of the right hand side of the second of the above equations over any leaf also vanishes.

Substituting for \( K \) in the above equations and specializing to spherical topology (genus = 0) gives,

\[ 2\pi = \int_{\Sigma_2} (\pi \bar{\pi} + \frac{\delta \pi + \delta \bar{\pi}}{2} - \mathcal{E} - D\mu) - \int_{\Sigma_2} (\epsilon + \bar{\epsilon})\mu , \]

(12)

\[ 0 = \int_{\Sigma_2} (\hat{\delta} \pi \pm \hat{\delta} \bar{\pi}) \]

(13)

We also deduce that,

\[ - (K - \bar{K}) = \Psi_2 - \bar{\Psi}_2 = -\{\delta \pi - (\bar{\alpha} - \beta)\pi\} + \{\bar{\delta} \bar{\pi} - (\alpha - \bar{\beta})\bar{\pi}\} \]

\[ = \hat{\delta} \pi - \hat{\delta} \bar{\pi} \]

(14)

Note that all these equations are manifestly invariant under rotations. Also observe that if \( \Psi_2 \) is not real, then \( \pi \) can not be set to zero.

Since \( \Psi_2 \) is invariant under boost, scaling and rotation transformation when \( \kappa = 0 \), its value has a physical meaning. Incidentally, \( \Lambda, \Phi_{11} \) are also invariant under these transformations. Thus, the complex curvature \( K \) of the leaves is an invariant.

Above we obtained imaginary part of the complex curvature in terms of derivatives of \( \pi, \bar{\pi} \) in the gauge chosen. Since \( K \) is invariant under boost transformations, the right
hand side of eqn. (14) above, must also be invariant under residual boost transformations. This implies that we must have,

\[ D\pi \equiv D\pi + (\epsilon - \bar{\epsilon})\pi = 0 \]  

(15)

Now the equations \((b + \bar{d})\) of item 5 of the Appendix A imply that,

\[ \delta(\epsilon + \bar{\epsilon}) = D\pi + (\epsilon - \bar{\epsilon})\pi = 0 \]  

(16)

Thus, \(\epsilon + \bar{\epsilon}\) is constant on each leaf but it may vary from leaf to leaf. This now limits the scaling freedom to scaling by a factor which is constant on each leaf. We may now exhaust this freedom by setting \(\epsilon + \bar{\epsilon}\) to a constant on \(\Delta\). Alternatively we may note that invariance of \(D\pi = 0\) condition under residual boost transformations implies that,

\[ D(\epsilon + \bar{\epsilon}) = 0 \]  

(17)

We have therefore already got that real part of \(\epsilon\) is a constant on \(\Delta\). This of course reduces the scaling to scaling by a constant factor.

Thus at this stage we have \(\kappa, \rho, \sigma, (\alpha + \bar{\beta} - \pi), \Phi_{00}, \Phi_{01}\) are zero and \(\mu\) is real. As a consistency of the equations with the gauge choice, we also deduced that \(\epsilon + \bar{\epsilon}\) is constant over the horizon. We have neither required \(\nu\) to be zero nor that \(\mu\) is strictly negative (positive). Physically we have already captured a geometrical property of \(\Delta\) that it is potentially foliated by marginally trapped surfaces in a physical space-time.

We already have a definition of area namely the area of a leaf with respect to the induced metric on \(\Sigma_2\) and that this area is “constant”. What could be candidate for a “surface gravity”? In the usual case of stationary black holes, it is the acceleration at the horizon of the killing vector normal to the horizon, with suitable normalization of
the stationary killing vector at infinity. Presently we don’t have any stationary killing vector. The topology of $\Delta$ however suggests that $\ell$ serves to define an evolution along $\Delta$. Thus, its acceleration, $\tilde{\kappa} \equiv (\epsilon + \bar{\epsilon})$ is a natural candidate. Indeed, as we have seen above, $\tilde{\kappa}$ is constant over $\Delta$! Identifying $\tilde{\kappa}$ with surface gravity (modulo a constant scale factor to be fixed later) we already have the zeroth law.

The rest of the logic is similar to [6]. The Bianchi identity implies,

$$(D + \tilde{\kappa})(D\mu) = 0$$

(18)

Remarks:

1. We have deduced the zeroth law using just the two conditions (non-expanding horizon), use of foliation and use of residual local Lorentz transformation. The notion of weak isolation [7] is not necessary. This is an alternative mentioned by Ashtekar et al in [4]. Although we have fixed the residual scaling freedom to a constant scaling only, we are still left with full rotation freedom and residual boost freedom.

2. All of the above works with any initial choice of $\ell$. In the process though we got the scaling to be restricted to a constant scaling only. Thus if we regard two $\ell$’s as equivalent if they differ by a constant non-zero factor, then all of the above holds for any given equivalence class of $\ell$. We could however begin with an equivalence class such that $D\mu = 0$. As long as an initial $\mu$ is non zero we can always do a local scaling transformation to the new $\mu$ to satisfy $D'\mu' = 0$. This will fix the initial $\ell$ and hence its equivalence class. Thus it is possible to choose a unique equivalence class of $\ell$ such that $D\mu = 0$. Note that this condition is preserved by the residual
transformations. At this stage we do not need to make such a choice though we will use this towards the end of this section.

3. We have obtained two constant quantities, surface gravity and area, associated with $\Delta$. We could obtain $\tilde{\kappa}$ in terms of integrals of certain expressions over a leaf via eqn. (12). Notice that in the absence of $\mu$ being constant on a leaf, Gauss-Bonnet does not give $\tilde{\kappa}$ directly in terms of the area.

\section*{B. Freely Specifiable Spin Coefficients}

As noted earlier, the zeroth law refers to a single solution while the first law refers to the class of solutions. To gain an understanding of such a class, we now proceed to identify freely specifiable spin coefficients corresponding to (non-expanding) horizons. For this of course we have to choose a suitable gauge.

Since we view $\Delta$ as a sub-manifold of a solution, we now consider excursion off-$\Delta$. We still continue with an arbitrary initial choice of the null normal vector field and initial $\Sigma_2$ (now restricting to spherical topology). On $\Delta$ we have constructed null tetrads and therefore have $n^\mu \partial_\mu$ defined. Consider geodesics specified by points on $\Delta$ and the $n^\mu \partial_\mu$. We need only infinitesimal geodesics to go infinitesimally off-$\Delta$. In this neighbourhood we construct null tetrads by parallel transporting the null tetrads from $\Delta$, $n \cdot \nabla(\text{tetrad}) = 0$. The equations of item 2 from appendix A then immediately gives, $\gamma, \nu, \tau$ to be zero off-$\Delta$ and therefore, by continuity also on $\Delta$. 

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Thus, on $\Delta$, we have six of the twelve spin coefficients to be zero, namely, $\gamma, \kappa, \nu, \rho, \tau, \sigma$. Furthermore we have real part of $\epsilon$ to be a constant, $\mu$ to be real and $\pi = \alpha + \bar{\beta}$.

Now one can always choose a gauge such that imaginary part of $\epsilon$ is zero on $\Delta$ (see the transformation equations of appendix A). This has two consequences. Firstly it reduces the rotation freedom, so far unrestricted, to rotations by parameter which is constant along the null generators. It is still local along the leaves. Secondly, the eliminant equation of appendix implies that $\alpha - \beta$ is constant along the null generators of $\Delta$. Thus both the combinations, $\alpha \pm \beta$ are now constant along the null generators and $\epsilon$ is a real constant. Now only $\alpha \pm \beta, \mu$ and $\lambda$ are non trivial functions on $\Delta$. Are all of these freely specifiable?

Not yet! The spin coefficients have still to satisfy the Einstein equations. Only the Ricci scalars enter in these equations and Weyl scalars can be thought of as derived quantities via the 18 complex equations of [10]. The eliminant equation directly give relations among the spin coefficients.

In the Appendix A, we have collected equations from [10]. These equations use the conditions derived from the non expanding horizon conditions discussed above. Since we are interested in quantities defined on $\Delta$, the equations involving $D'$ derivatives are omitted. These can be understood as specifying the derivatives off $\Delta$ in terms of quantities specified on $\Delta$.

We see immediately that the eliminant equations give no conditions on the non triv-
ial spin coefficients mentioned above. From the full set of the 18 equations of [10] one can see the following. All quantities are evaluated on ∆.

- \( \Phi_{00} \) appears in equation (a) which is identically true.

- \( \Phi_{01} \) appears in equations (c, d, e, k) together with \( \Psi_1 \). These serve to give \( \Psi_1 = 0 \) and \( D'\kappa = 0 \). This in turn implies that \( \kappa \) is zero in an infinitesimal neighbourhood of \( \Delta \).

- \( \Phi_{02} \) appears in equations (g, p). These serve to give \( D'\sigma = -\Phi_{02} \) and also

\[
D\lambda - \bar{\delta}\pi - \pi^2 + \bar{\kappa}\lambda = \Phi_{02}
\]  

(19)

This is a non trivial condition among the non vanishing coefficients and can be thought of as a differential equation for \( \lambda \) given the other quantities.

- \( \Phi_{12} \) appears in equations (i, m, o, r) together with \( \tilde{\Psi}_3 \). Use (r) to eliminate \( \Psi_3 \). The remaining equations show that \( D' (\pi - \alpha - \bar{\beta}) = 0 \) and \( \Phi_{12} \) determines only the off-\( \Delta \) derivatives. One consequence of these is that \( \pi = \alpha + \bar{\beta} \) in a neighbourhood of \( \Delta \).

- \( \Phi_{22} \) appears only in equation (n) and specifies \( D'\mu \). The equation being real, it follows that \( \mu \) is real in a neighbourhood.

- \( \Phi_{11}, \Lambda \) appear in combinations with \( \Psi_2 \) in equations (f, h, l, q). One can use (l) to determine \( \Psi_2 \) and eliminate the complex curvature of leaves, \( K \), from the remaining equations. Two of the equations then determine \( D'\epsilon \) and \( D'\rho \) while the remaining one gives,

\[
D\mu - \bar{\delta}\pi - \pi\bar{\pi} + \bar{\kappa}\mu = \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta - \delta\alpha + \bar{\delta}\beta + \Phi_{11} + 3\Lambda
\]  

(20)
Again this can be thought of as a differential equation for $\mu$ in terms of the remaining quantities. Notice that the equation (q) implies that if $\Psi_2$ is real then, $\rho$ is real in the neighbourhood.

- Lastly equation (j) just gives $\Psi_4$ in terms of $D'\lambda$.

In effect we have obtained two (differential) conditions on two of the non trivial coefficients, $\mu$ and $\lambda$ and are left with just $\alpha \pm \bar{\beta}$ as freely specifiable. Since these are constant along the null generators, these need to be given only on a leaf.

The differential equations for $\lambda, \mu$ can be ‘reduced’ further. Recall the remark about the initial choice of the null normal vector field. Generically we can choose it to be such that $D\mu = 0$. This is consistent with the differential equation due to Bianchi identity and the zeroth law. This fixes a unique equivalence class of null normals. We could consider imposing $D\lambda = 0$ condition also. If $\Phi_{02}$ is zero, as is the case for vacuum, Einstein-Maxwell and Einstein-YM systems, the condition is consistent with the differential equation. If it is non zero, then consistency with the differential equation requires $D\Phi_{02} = 0$ which in turn via Bianchi identity (e) requires $D'\Psi_0 = 0$. Constancy of $\mu$ and $\lambda$ along the generators is precisely the extra condition for isolated (as opposed to weakly isolated) horizons that is imposed by Ashtekar et al \[7\].

Without the constancy of $\mu, \lambda$ along the generators, we see that $\pi = (\alpha + \bar{\beta}), (\alpha - \bar{\beta}), \mu$ and $\lambda$ may be freely specified on a leaf. $\tilde{\kappa}$ may then be determined via the Gauss-Bonnet integral. Not all these specifications give different (non-expanding) horizons since we still have good deal of residual transformations apart from the initial choices of $\ell$ and foliation.

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With the constancy of $\mu$, $\lambda$ along generators imposed, we fix the initial equivalence class of $\ell$, determine $\lambda$ and $\bar{\kappa}\mu$ in terms of $\alpha \pm \bar{\beta}$ and $\Phi_{02}$ specified on a leaf. The residual rotation can be fixed completely by making $\alpha - \bar{\beta}$ real. We are then left with only the constant scaling and the dependence on the initial foliation (residual boosts). Notice that when $\epsilon$ is real, $\alpha - \bar{\beta}$ is invariant under boost transformations. Thus our identification of freely specifiable spin coefficients is ‘gauge invariant’. The residual transformations now just serve to demarcate equivalent isolated horizons.

Remarks

1. Apart from seeing the role of Einstein equations via the 18 equations of [10], we also determined the Weyl scalars. Of the five Weyl scalars, only $\Psi_2, \Psi_3$ are determined by freely specifiable coefficients on a leaf and Ricci scalars. Thus the (real) curvature of the leaves, and hence the metric on the leaves is also determined. The freely specifiable data consists of just one complex function, $\pi$, and one real function, $\alpha - \bar{\beta}$, on a leaf.

2. It is apparent from our analysis and also proved by Lewandowski [9] that the space of solutions admitting isolated horizons is infinite dimensional. In the above analysis we have addressed the freedom of local Lorentz transformations naturally present in a tetrad formulation. We also have the diffeomorphism invariance though. Thus two sets of data related by a diffeomorphism on a leaf must be regarded as giving the ‘same’ isolated horizon. This is of course implicit in an analysis based on an ‘initial value problem’ formulation [9]. The corresponding diffeomorphism classes must be characterized by diffeomorphism invariants. In-
integrals of scalars (e.g., invariant combinations of spin coefficients) over the leaf are just one such set of invariants. We will obtain a few of these in the next section.

To summarize: We have shown that every solution of Einstein-matter equations with matter satisfying the dominant energy condition and admitting non-expanding horizon admits foliations and a corresponding choice of null tetrads modulo constant scalings such that the free data consists of one complex function $\pi$ and one real function $\alpha - \beta$. Furthermore with $\tilde{\kappa}$ being identified as (unnormalized) surface gravity, the zeroth law holds for all such solutions.

In the next section we will consider the symmetry properties of isolated horizons and see that if leaves admit at least one isometry, there is a unique choice of foliation and the equivalence is reduced to that implied by constant scaling only.

III. CONSTANTS ASSOCIATED WITH ISOLATED HORIZONS

In this section we discuss symmetries of isolated horizons and corresponding “conserved” quantities. This discussion is carried out in terms of invariant quantities so that the conclusions are not tied to any particular gauge choice. Under certain condition we also see how this helps to fix a unique foliation.

A. Symmetries of $\Delta$

In the previous section we naturally obtained two constant quantities associated with $\Delta$, the surface gravity and the area. Noting that quantities constructed out of the spin
coefficients and their derivatives and which are invariant under the residual transformations are physical characteristics of a given isolated horizon, $\Delta$. An example is the complex curvature $K$ defined above. To look for further characteristic quantities associated with a given $\Delta$ we analyze the symmetries of these horizons.

Consider a fixed solution containing $\Delta$. This is characterized by a set of physical quantities. By definition, a symmetry of $\Delta$ is a diffeomorphism of $\Delta$ which leaves these quantities invariant. Since the induced metric on leaves is one such quantity, a symmetry must be an isometry of the leaves. However, there are further invariant quantities built from ingredients other than the induced metric on leaves, e.g. $(K - \bar{K})$. We now look for further such invariant quantities.

Since residual transformations permit change of a foliation, it is desirable to make use of quantities invariant under these. Thus in order to classify the symmetry classes and distinguish a rotating case, we look for invariant forms and vector fields on $\Delta$. Any vector field (or form) can be expressed as an expansion in terms of the appropriate tetrad basis. Depending on the transformation properties of the expansion coefficients under residual Lorentz transformations, these expansions will (or will not) preserve their form. By invariant vectors (forms) we mean form invariance under residual transformations. The general coordinate transformations of course play no role in the discussion. The use of such ‘invariant’ quantities frees us from having to keep track of the particular choices of tetrad bases. The demand of form invariance puts restrictions on the expansion coefficients which can then be taken in a convenient manner as seen below.

Noting that the tangent space of $\Delta$ is spanned by the tangent vectors $\ell, m, \bar{m}$ while
the cotangent space is spanned the \textit{cotangent} vectors \(n, m, \bar{m}\), it is easy to see that a generic invariant vector field \(X\) on \(\Delta\) is parameterized as:

\[
X \equiv \frac{1}{\tilde{k}}(\zeta + \pi C + \bar{\pi} \bar{C})\ell + Cm + \bar{C}\bar{m}
\]

(21)

where, \(C\) transforms as \(\pi\) under rotation, is invariant under boost while \(\zeta\) is real and invariant under both sets of transformations. These are our candidates for generating symmetries of \(\Delta\).

The condition that \(X\) be a Killing vector (of the metric on \(\Delta\) and \textit{not} the four dimensional space-time metric) requires, in terms of the rotation covariant derivatives (see appendix A):

\[
\delta \bar{C} = 0, \quad \delta C = 0, \quad \delta C + \bar{\delta} \bar{C} = 0.
\]

(22)

The third of the above equations can be solved identically by setting \(C \equiv i\tilde{\delta} f\) where \(f\) is an invariant function while the first two require \(f\) to satisfy \(\tilde{\delta}^2 f = 0 = \bar{\delta}^2 f\).

While looking for invariant 1-form, one should note that restricted to forms, the boost transformations have the \(\ell\) dependent terms dropped. Then there are two types of such 1-forms (real and space-like):

\[
\omega \equiv -\kappa n + \pi m + \bar{\pi} \bar{m}, \quad \omega(X) = \zeta; \quad \bar{\omega} \equiv \bar{C} m + \bar{\bar{C}} \bar{m}, \quad \bar{\omega}(X) = -(C \bar{C} + \bar{C} \bar{C})
\]

(23)

Since the 1-form \(\omega\) is invariant and built out of spin coefficients, it is a physical characteristic of \(\Delta\) and thus a symmetry generating invariant vector field must leave this invariant, \(\mathcal{L}_X \omega = 0\).
This immediately implies that $D(\zeta) = 0$. It also gives two differential equations for $\zeta$, namely,

$$\delta \zeta = -\bar{C}(K - \bar{K}), \quad \bar{\delta} \zeta = C(K - \bar{K})$$

(24)

This has several implications. Firstly, $X(\zeta) = 0$ i.e. $\zeta$ is constant along the integral curves of $X$. Secondly, taking $D$ of the equations, using the commutators of the derivatives and Bianchi identity implies that $D\bar{C} = 0$. Finally, The integrability conditions for these equations, which require that $X(K - \bar{K}) = 0$ and $\delta\bar{C} + \bar{\delta}C = 0$, are automatically true by our definition of symmetry. It follows then that the solution for $\zeta$ is unique up to an additive constant. Note that when $K$ is real, $\zeta$ must be a constant. These integrability conditions will also allow us to set up an adapted $(\theta, \phi)$ coordinate system on leaves.

One can construct an invariant vector field $Y$ (not necessarily a symmetry generator) which is orthogonal to a symmetry generator $X$ by taking $C$ going to $i\Phi C$ where $\Phi$ is a real and invariant function. We would also like to use a symmetry generator along $\ell$. Thus we define two further invariant vector fields,

$$Y \equiv \frac{1}{-\bar{K}}(\zeta - i\Phi(\pi\bar{C} - \pi C))\ell + i\Phi(C\bar{m} - \bar{C}m)$$

(25)

$$Z \equiv \frac{1}{-\bar{K}}\ell$$

(26)

Evidently, $X, Y$ are real, space-like and mutually orthogonal (for all $\Phi$). $Z$ is of course a null symmetry generator and is orthogonal to $X$ and $Y$. Observe that $\omega(X) = \zeta = \omega(Y), \omega(Z) = 1$.

We would now like to have all these vector fields to commute so that parameters of their integral curves can be taken as coordinates for $\Delta$. While commutativity of $X, Z$
is a statement about the nature of the isometry algebra, commutators involving $Y$ are just stipulations on $Y$ which in no way affect properties of $X$ or $Z$.

The commutativity of the vector fields $X, Y$ with $Z$ requires:

$$Dζ = 0, \quad DC = 0, \quad DΦ = 0$$ \hspace{1cm} (27)

Note that commutativity of $Z, X$ is already implied by the eqn. (24) while that of $Z, Y$ is a condition on $Φ$. These enable us to construct the invariant vector fields as follows. For any leaf, assume we could find a real function $f$, satisfying $\tilde{\delta}^2 f = 0$. Extend it to $∆$ by Lie dragging by $ℓ$. This implies that $f$ so constructed, satisfies the double derivative condition on all the leaves. Furthermore, these conditions themselves are invariant under residual transformations. We could similarly define $Φ, ζ$ on $∆$.

The commutativity of $X, Y$ with $X$ a Killing vector (of leaves) requires $X(Φ) = 0$ and

$$X(ζ) - Y(ζ) = 2iC\bar{C}\Phi(K - \bar{K})$$ \hspace{1cm} (28)

The above equation automatically holds due to eqn (24). The only additional information we have obtained is that the as yet arbitrary $Φ$ function is constant along integral curves of $X$. There are no conditions implied on $X$ due to the demands of commutativity.

Several consequences can be derived now.

The integral curves of $X$ - the Killing orbits - in general leave a leaf. We could however choose a foliation such that these orbits are confined to leaves. This means that
the coefficient of $\ell$ must be made zero. This can be effected by a boost transformation. This still leaves a one parameter freedom of boost residual transformation. This can be fixed by demanding that integral curves of $Y$ be similarly confined to leaves. The boost parameter effecting this is given by,

$$a + \frac{\pi}{\kappa} = \frac{i\zeta}{2\kappa\Phi C}(1 + i\Phi)$$

(29)

That the parameter $a$ satisfies the conditions of being residual transformation requires $X(\zeta) = 0$ which is already seen to be true. For the transformation parameter to be well defined, it is necessary that $\zeta$ vanishes where ever $C\Phi$ does. Since $C$ must vanish at at least one point on a leaf, $\zeta$ must also vanish at at least one point. Thus, when $K$ is real which implies that $\zeta$ is a constant, $\zeta$ in fact must be zero.

Note that this fixes the boost freedom completely. The left hand side is nothing but the transformed value of $\pi$ divided by the surface gravity. Thus in effect, this boost transformation has fixed for us a particular foliation. In this foliation we have the $X, Y$ vector fields purely tangential to the leaves. Consequently, we call the vector field $X$ as a “rotational” symmetry generator.

In this foliation $\zeta = -(\pi \tilde{C} + \pi C)$ and we see that $\zeta$ is zero iff there exist a gauge in which $\pi$ is zero. This fact will be used in defining the angular momentum in terms of $\zeta$.

Since the leaves are compact, the Killing vector field is complete. Suppose for the moment that its orbits are closed curves. Provided that the ranges of the Killing parameter is the same for all the orbits, we can adjust this range to be $2\pi$ by a constant scaling of $X$ and identify the Killing parameter to an angular coordinate $\phi$. This will genuinely make the spatial Killing vector correspond to axisymmetry. When can this be
First we need to argue that orbits of the Killing vector, $X$, are closed. Every vector field on $S^2$ will have at least one zero (or fixed points). These fixed points can be classified in elliptic, hyperbolic etc by standard linearizations [12]. The third of the eqn. (22) implies that $X$ is ‘area preserving’ and thus has zero divergence. This implies that its zeros are all elliptic and hence orbits in the vicinity are closed. Since the orbits can not intersect, all orbits in facts must be closed. The vector field $Y$ by contrast is not divergence free and thus its orbits are not closed.

For an integral curve $\gamma$ of $X$, the line integral of the invariant 1-form $\omega$ is equal to the integral of $\omega(X) = \zeta$ along $\gamma$. Since $\zeta$ is constant along such curves, its integral will be $\zeta(\gamma) \times I(\gamma)$, where $I(\gamma)$ is the range of the Killing parameter along $\gamma$. Now one can consider a family of such $\gamma$’s labeled by an infinitesimal integral curve of $Y$. Differentiate the integrals w.r.t. the parameter along $Y$. On the left hand side use Stokes theorem together with $d\omega = (K - \bar{K})m \wedge \bar{m}$ while on the right hand side use Taylor expansion. This gives the left hand side as,

$$\oint_{\gamma(\beta+\delta\beta)} \omega - \oint_{\gamma(\beta)} \omega = \int_{cyl} d\omega = \int_{cyl} (K - \bar{K})m \wedge \bar{m} \sim -2i\Phi C \bar{C}(K - \bar{K})(\beta)I(\beta)\delta\beta. \quad (30)$$

Here the surface integral is over an infinitesimal cylinder formed by the family. In the last step of course we have used the mean value theorem. The right hand side gives,

$$\{Y(\zeta)I(\beta) + \zeta(\beta)Y(I(\beta))\}\delta\beta \quad (31)$$

The equation (28) satisfied by $\zeta$ then implies that the parameter range does not vary along $Y$ and hence is a constant. We can now introduce the azimuthal angle $\phi$. Clearly
We have used the vector field $Y$ to show that azimuthal coordinate can be introduced. This is independent of the choice of $\Phi$. Can we also introduced the polar angle $\theta$? The answer is yes as seen below.

Every vector field on $S^2$ must vanish at at least one point. $X$ vanishes precisely when $C = 0$. If $\Phi$ is non singular then $Y$ also vanishes precisely where ever $X$ vanishes. One can “decompactify” the leaf by removing such a point and see that $C$ must vanish at one and only one more point. Further the integral curves of $Y$ must “begin” and “end” at these two points. This may also be seen via the Poincare-Bendixon theorem \[12\]. Further, since $X, Y$ commute, the diffeomorphisms generated by $X$ take orbits of $Y$ to orbits of $Y$. Choosing a “meridian” we could suitably adjust $\Phi$ and hence select a $Y$ so that the parameter along this curve ranges from 0 to $\pi$. Lie dragging by $X$ then defines these for other longitudes. This way we can introduce the standard spherical polar angles on the leaf (boost gauge fixed). The metric can also be written down as:

$$ds^2 = -2C\bar{C}(\Phi^2d\theta^2 + d\phi^2)$$ (32)

That such a choice of $\Phi$ is possible can be seen by construction. For the above form of the metric, one can compute the Ricci scalar in terms of $\Phi$. We also have $(K + \bar{K})$ as the curvature of the leaves. Equating these two gives a first order differential equation for $\Phi$. Its smooth behaviour on the leaf excepting the poles, fixes the constant integration and determines $\Phi$ in terms of $C$ and $(K + \bar{K})$. For the Kerr-Newman family it reproduces the precise metric components. Thus existence of an invariant Killing vector (spatial), permits us to determine both $\zeta$ and $\Phi$ and also allows us to introduce an adapted set of coordinates $\theta, \phi$ on the spherical leaves. Note that while on each of the leaf we can in-
introduce these coordinates, the choice of prime meridians on different leaves is arbitrary. These could be related by diffeomorphisms generated by $Z$.

**Remark:** Commutativity of the invariant vector fields, $X, Y, Z$, not only allows us to introduce spherical polar coordinates on leaves, it also allows us to introduce coordinates on $\Delta$ itself. With the excursion off-$\Delta$ defined via parallel transport along geodesics defined by $n^\mu \partial_\mu$, one can naturally introduce coordinates in the neighbourhood also.

On a spherical leaf, we can have either none, one or three isometries i.e. either 1) *no* such $f$, or 2) precisely *one* such $f$, or 3) precisely *three* such $f$’s. If we found two such functions (two Killing vectors), then their commutator is either zero or is also a Killing vector. On $S^2$ we can’t have two commuting Killing vectors, therefore the commutator must be non zero and a Killing vector. One can consider the commutator $X'' \equiv [X, X']$ of two Killing vectors and see that it is also an invariant Killing vector provided the $\zeta''$ satisfies the same conditions. This is possible only if $K - \bar{K} = 0$. Thus, as expected, we see that maximal symmetry is possible only if $\Psi_2$ is real. In this case of course one does not expect any rotation. The converse need not be true as shown by the “distorted” horizons [7].

**Remark:** We have fixed the boost freedom by demanding $X,Y$ be tangential to a leaf. When we have three such pairs of vector fields are all these automatically tangential to the same leaf? The answer is yes. Since three isometries implied $K$ is real, all the $\zeta, \zeta', \zeta''$ are constants and hence zero. Furthermore tangentiality of any one implies $\pi = 0$. Therefore all the three Killing vectors are tangential to the same leaf.
In the case of single isometry, we have both the possibilities namely \( K - \bar{K} \) is zero or non zero. These two can be seen intuitively as hinting that while a “rotating” body is expected to be distorted (bulged) a distorted body need not be rotating. Since maximal symmetry implies ‘no rotation’ and also that \( K \) is real, we label the two cases as:

\[ \Delta \text{ is non-rotating iff } K \text{ is real (i.e. } \Psi_2 \text{ is real) and is rotating iff } K \text{ has non zero imaginary part.} \]

We have already noted that in non-rotating case \( \zeta \) must be zero. This means that \( \pi \) can be transformed to zero. One could also see this directly from the equations of residual transformations that it is possible to choose a gauge such that \( \pi = 0 \). For the rotating case of course \( \zeta \) is non zero and \( \pi \) is also non zero in every gauge.

**B. “Conserved” quantities**

While \( \zeta \) introduced above, could be non zero, it is some function of the adapted co-ordinate \( \theta \). Its integral over the leaf is clearly a constant which is zero iff \( K \) is real. This integral is therefore a candidate for defining angular momentum. Indeed, the explicit example of Kerr space-time, discussed in appendix-B, shows that the candidate agrees with the angular momentum of the Kerr space-time. It also provides a proportionality factor. Unlike the Kerr black hole however, this angular momentum is defined in terms of quantities intrinsic to \( \Delta \). For Kerr-Newman solution also, one can find \( \zeta \) and its integral. Interestingly, one does not get the usual, total angular momentum \((J = Ma)\) of the Kerr-Newman space-time, but one gets the total angular momentum minus the contribution of the electromagnetic field in the exterior. Indeed, one can explicitly check
that the integral of $\zeta$ is precisely equal to the Komar integral \[13\] evaluated at the event horizon. This justifies the case for taking the following as the angular momentum of an isolated horizon.

Thus we define,

$$J \equiv -\frac{1}{8\pi} \int_{S^2} \zeta (im \wedge \bar{m}) \quad (33)$$

Apart from the “rotational” symmetry generator defined above, we see that $gZ$ is also a symmetry generator provided $g$ is a constant. Following identical logic as for the rotational symmetry generator, we see that analogue of $J$, is proportional to the area. Thus area is the ‘conserved’ charge associated with $Z$. Unlike $X$ whose normalization is fixed due to compactness of its orbits, $S^1$, normalization of $Z$ is not fixed. Clearly arbitrary constant linear combinations of $Z$ and the well defined “rotational” $X$ is a symmetry generator. Let us denote this as,

$$\xi_{a,b} \equiv (-aZ - bX); \quad \tilde{\zeta}_{a,b} \equiv \omega(\xi_{a,b}) = -(a + b\zeta) \quad (34)$$

We have introduced the suffixes $a, b$ to remind ourselves that the suffixed quantities depend on the arbitrary constants $a, b$.

We define, by analogy with $J$, a corresponding conserved quantity as,

$$\tilde{M}_{a,b} \equiv -\frac{1}{4\pi} \int_{S^2} \tilde{\zeta}_{a,b} = \frac{a}{4\pi} \text{Area} + 2bJ \quad (35)$$

Note that unlike $\zeta$ which is determined via a differential equation involving $(K - \bar{K})$, $\zeta_{a,b}$ is not completely determined by $(K - \bar{K})$. A Smarr-like relation still follows because of the symmetries we have and our parameterization of the general isometry, $\xi_{a,b}$ in terms
an arbitrarily introduced constants $a, b$.

Interestingly, the usual Smarr relation for the Kerr-Newman family, with $M, J$ replaced by the Komar integrals evaluated at the event horizon has exactly the same form as above with $a = \tilde{\kappa}_{KN}$ and $b = \Omega_{KN}$! The Maxwell contribution is subsumed in the Komar integrals at the horizons and there is no explicit ‘$Q\Phi$’ term.

It is thus suggestive that $M_{a,b}$ be identifiable as the mass of the isolated horizon. It is defined on the same footing as the angular momentum. For $b \neq 0$ the $\xi_{a,b}$ is a space-like isometry (analogue of stationary Killing vector) as in the case of the Kerr-Newman family. However $a, b$ are arbitrary parameters. Further, while $b$ is the analogue of the angular velocity, it is not clear why it must be non-zero when and only when $J$ is non-zero.

The notion of angular velocity is tricky though because on the one hand ‘rotation’ must be defined with respect to some observers (in the usual case asymptotic observers) on the other hand it is signaled by imaginary part of $\Psi_2$ which is a local property. At present we have not been able to resolve this issue.

Remarks:

1. The above candidate definitions are not the same as those given by Ashtekar et al [7]. There need not be any inconsistency since our definitions need not enter a first law in the usual manner. Some functions of our quantities may become the mass and angular momentum which will enter a first law.
2. Our candidate identifications are based on analogy and consistency with the Kerr-Newman family. The symmetry generators, $Z, X$, are invariant vector fields. In particular they are insensitive to the constant scaling freedom un-fixed as yet. Provided, the constants $a, b$, are also invariants, the general symmetry generator $\xi_{a,b}$ will also be an invariant vector field. If $a, b$ are proposed to be the surface gravity and the angular velocity based on analogy and dimensional grounds, these will be subject to the constant scaling freedom and one will have to face the normalization issue as done in [6,7]. If on the other hand $a, b$ (equivalently $\xi_{a,b}$) could be directly tied to the space-time under consideration, and are calculable then a first law variation may be directly obtained. In the absence of a stationary Killing vector there does not appear to be any natural choice of $\xi_{a,b}$.

Without relating the constants $a, b$ to any other intrinsic properties of $\Delta$ and/or normalizing them suitably, $M_{a,b}$ can not fully be identified as the mass of $\Delta$. Since we do not address the first law in this work, we do not pursue the identification further.

To summarize: In this section we assumed the existence of at least one spatial symmetry of $\Delta$ and deduced candidate definition of angular momentum and possible identification of mass. We also saw how this enables us to fix a unique foliation together with an adapted set of spherical polar angular coordinates. At present, we have not been able to obtain a candidate definition of angular velocity. Whether at least one spatial isometry must exist or not is also not addressed. These will be addressed elsewhere.
C. The Special Case of Spherical Symmetry

If a solution admitting isolated horizon has maximal symmetry for the leaves, we have already seen that complex curvature must in fact be real. Furthermore it must be constant due to maximal symmetry. Our gauge choice fixing the foliation then has $\pi = 0$. Equations (19) and (8) which determine $\lambda$ and $\mu$ then become,

$$\tilde{\kappa}\lambda = \Phi_{20}, \quad \tilde{\kappa}\mu = -K - \mathcal{E} \Rightarrow \tilde{\kappa}\delta\mu = -\delta\mathcal{E}$$ (36)

If $\lambda = 0$ can be shown, then equation (m) of item 5 of appendix A will give $\Psi_3 = \Phi_{21}$. Likewise $\delta\mathcal{E} = 0$ will give $\mu$ to be a constant. It appears that maximal symmetry alone does not imply either $\lambda$ or $\delta\mu$ to be equal to zero. For the Maxwell and YM matter, the form of the stress tensor together with energy condition implies $\Phi_{02} = 0$ and then $\lambda = 0$ does follow. Since the space-time itself is not guaranteed to be spherically symmetric in the neighbourhood, while $\delta\mathcal{E} = 0$ is a reasonable condition, it does not seem to be forced upon. If these two conditions however, are imposed, then the previous results on non-rotating and spherically symmetric isolated horizons are recovered. Incidentally, in this case positivity of $K$ (spherical topology) and of $\mathcal{E}$ implies that $\tilde{\kappa}\mu$ is negative. Thus for positive surface gravity, $\mu$ must be negative implying marginal trapping.

IV. SUMMARY AND CONCLUSIONS

In this work we have presented a general analysis of isolated horizons by manipulating the basic equations of null tetrad formalism. By using existence of foliations (causally well behaved solutions) and keeping track of local Lorentz transformations we have shown that non-expanding conditions (as opposed to weak isolation) are adequate
to get the zeroth law. We also used a gauge fixing procedure to identify freely speci-
fiably spin coefficients. By using invariant vector fields and 1-forms, we analyzed the
symmetries of isolated horizons and showed that if at least one spatial isometry exists,
then a unique foliation can be fixed. Further more a natural choice of spherical polar
coordinates exists for the leaves. We defined associated ‘conserved’ quantities which are
consistent with the Kerr-Newman family. In the appendix B we have illustrated our
procedures for the Kerr-Newman family.

We have not taken explicit examples of various matter sectors (except the Kerr-
Newman family). For the Einstein-Maxwell, Einstein-Yang-Mills with or without dilat-
on has been discussed in [6]. We have also not invoked any action principle. Our hope
was and still is to deduce a first law without the use of an action principle. Action
formulation(s) and its use has already been discussed in detail in [6,7].

The first law for isolated horizons is much more subtle. A very interesting formula-
tion and perspective is given by Ashtekar et al [7]. Here we will be content with some
remarks.

The usual stationary black holes are parameterized by finitely many parameters (3
for the Kerr-Newman family) and the first law also involves only a few parameters. By
contrast, the space of solutions with isolated horizons is infinite dimensional and yet a
first law is expected to involve only a few parameters. Even granting the infinitely many
forms of the first law [7, each of these still involves a few parameters only with area,
angular momentum and charges playing the role of independently variable quantities.
Why only a few quantities be expected a priori to enter a first law of mechanics for

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isolated horizons is unclear. Ashtekar et al have incorporated asymptotic flatness in an action principle and used the covariant phase space formulation to deal with the space of solutions directly. They traced the existence of first law as a necessary and sufficient condition for a Hamiltonian evolution on the covariant phase space of isolated horizons. It will be nicer to have a direct and quasi-local ‘explanation’ of the first law.

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Appendix A Summary of relevant equations for spin coefficients

In this appendix we collect together the relevant and useful formulae from [10]. The conventions are those of Chandrasekhar’s book.

1. Null tetrad basis in the tangent and cotangent spaces:

\[
E_1 = \ell^\mu \partial_\mu \equiv D, \quad E_2 = n^\mu \partial_\mu \equiv D', \quad E_3 = m^\mu \partial_\mu \equiv \delta, \quad E_4 = \bar{m}^\mu \partial_\mu \equiv \bar{\delta};
\]

\[
E^1 = n_\mu dx^\mu, \quad E^2 = \ell_\mu dx^\mu, \quad E^3 = -\bar{m}_\mu dx^\mu, \quad E^4 = -m_\mu dx^\mu
\]

2. Covariant derivatives in terms of spin coefficients:

\[
\ell_{\mu;\nu} = (\gamma + \bar{\gamma})\ell_\mu \ell_\nu + (\epsilon + \bar{\epsilon})\ell_\mu n_\nu - (\alpha + \bar{\beta})\ell_\mu m_\nu - (\bar{\alpha} + \beta)\ell_\mu \bar{m}_\nu
\]

\[
-\bar{\tau}m_\mu \ell_\nu - \bar{\kappa}m_\mu n_\nu + \bar{\sigma}m_\mu m_\nu + \bar{\rho}m_\mu \bar{m}_\nu
\]

\[
-\tau\bar{m}_\mu \ell_\nu - \kappa\bar{m}_\mu n_\nu + \rho\bar{m}_\mu m_\nu + \sigma\bar{m}_\mu \bar{m}_\nu
\]

\[
n_{\mu;\nu} = -(\epsilon + \bar{\epsilon})n_\mu n_\nu - (\gamma + \bar{\gamma})n_\mu \ell_\nu + (\alpha + \bar{\beta})n_\mu m_\nu + (\bar{\alpha} + \beta)n_\mu \bar{m}_\nu
\]

\[
+\nu m_\mu \ell_\nu + \pi m_\mu n_\nu - \lambda m_\mu m_\nu - \mu m_\mu \bar{m}_\nu
\]

\[
+\nu \bar{m}_\mu \ell_\nu + \pi \bar{m}_\mu n_\nu - \bar{\mu} \bar{m}_\mu m_\nu - \bar{\lambda} \bar{m}_\mu \bar{m}_\nu
\]

\[
m_{\mu;\nu} = \bar{\nu} \ell_\mu \ell_\nu + \bar{\pi} \ell_\mu n_\nu - \bar{\mu} \ell_\mu m_\nu - \bar{\lambda} \ell_\mu \bar{m}_\nu
\]

\[
-\tau n_\mu \ell_\nu - \kappa n_\mu n_\nu + \rho n_\mu m_\nu - \sigma n_\mu \bar{m}_\nu
\]

\[
+(\gamma - \bar{\gamma})m_\mu \ell_\nu + (\epsilon - \bar{\epsilon})m_\mu n_\nu - (\alpha - \bar{\beta})m_\mu m_\nu + (\bar{\alpha} - \beta)m_\mu \bar{m}_\nu
\]

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3. Local Lorentz transformations:

These are given using $\kappa$, $\rho$, $\sigma$, $\Psi_0$, $\Psi_1$, $\Phi_{00}$ set equal to zero. These values are invariant under both the sets of local Lorentz transformations.

**Type-I (Boosts):**

$$\ell' = \ell, \quad n' = n + \bar{a}m + a\bar{m} + a\bar{a}\ell, \quad m' = m + a\ell, \quad m' = \bar{m} + \bar{a}\ell;$$

$$\Psi_2' = \Psi_2, \quad \Psi_3' = \Psi_3 + 3\bar{a}\Psi_2, \quad \Psi_4' = \Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2;$$

$$\epsilon' = \epsilon, \quad \tau' = \tau,$$

$$\pi' = \pi + 2\bar{a}\epsilon + D\bar{a}, \quad \alpha' = \alpha + \bar{a}\epsilon, \quad \beta' = \beta + a\epsilon,$$

$$\gamma' = \gamma + a\alpha + \bar{a}(\beta + \tau) + a\bar{a}\epsilon,$$

$$\lambda' = \lambda + \bar{a}(2\alpha + \pi) + 2\bar{a}^2\epsilon + \bar{a}\delta + aD\bar{a},$$

$$\mu' = \mu + a\pi + 2\bar{a}\beta + 2a\bar{a}\epsilon + \delta\bar{a} + aD\bar{a},$$

$$\nu' = \nu + a\lambda + \bar{a}(\mu + 2\gamma) + \bar{a}(\tau + 2\beta) + a\bar{a}(\pi + 2\alpha) + 2a\bar{a}^2\epsilon$$

$$(D' + \bar{a}\delta + a\bar{a}D\bar{a});$$

**Type-III (scaling and rotations):**

$$\ell' = A^{-1}\ell, \quad n' = An, \quad m' = e^{i\theta}m, \quad \bar{m}' = e^{-i\theta}\bar{m},$$

$$\Psi_2' = \Psi_2, \quad \Psi_3' = A\Psi_3e^{-i\theta}, \quad \Psi_4' = A^2e^{-2i\theta}\Psi_4,$$

$$\mu' = A\mu, \quad \tau' = e^{i\theta}\tau, \quad \pi' = e^{-i\theta}\pi, \quad \lambda' = Ae^{-2i\theta}\lambda, \quad \nu' = A^2e^{-i\theta}\nu,$$

$$(\epsilon + \bar{\epsilon})' = A^{-1}\{(\epsilon + \bar{\epsilon}) - A^{-1}DA\}, \quad (\epsilon - \bar{\epsilon})' = A^{-1}\{(\epsilon - \bar{\epsilon}) + iD\theta\},$$

$$(\alpha + \bar{\beta})' = e^{-i\theta}\{(\alpha + \bar{\beta}) - A^{-1}\delta A\}, \quad (\alpha - \bar{\beta})' = e^{-i\theta}\{(\alpha - \bar{\beta}) + i\delta\theta}$$

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4. Type-III rotation Covariant derivatives ("compacted covariant derivatives"): 

These are defined for a quantity that transforms homogeneously under rotations.

\[ X' = e^{in\theta}X \quad , \quad n \text{ an integer} \]

\[ DX \equiv DX - n(\epsilon - \bar{\epsilon})X \quad ; \quad D'X' = e^{in\theta}DX \]

\[ \delta X \equiv \delta X + n(\bar{\alpha} - \beta)X \quad ; \quad \delta'X' = e^{i(n+1)\theta}\delta X \]

\[ \bar{\delta}X \equiv \bar{\delta}X - n(\alpha - \bar{\beta})X \quad ; \quad \bar{\delta}'X' = e^{i(n-1)\theta}\bar{\delta}X \]

The corresponding commutators are given by:

\[ [\delta, D]X = (\bar{\alpha} + \beta - \bar{\pi})DX \]

\[ [\bar{\delta}, D]X = (\alpha + \bar{\beta} - \pi)DX \]

\[ [\delta, \bar{\delta}]X = -n(K + \bar{K})X \]

5. Riemann tensor component equations:

There are in all 18 such equations together with their complex conjugates. But since we are interested in values only on the horizon, we regard the equations involving \( D' \) derivatives as specifying the same in terms of values obtained on \( \Delta \). These therefore give no conditions on \( \Delta \) itself. Three of these equations, equations (a), (b) and (k) of [10] have already been been implied by the energy conditions, Raychoudhuri equation and the zero expansion condition on the \( \ell \) congruence. These give, \( \Psi_0, \Psi_1, \Phi_{00}, \Psi_{01} \) and \( \Psi_{10} \) to be zero. This leaves us with 6 equations, 2 of the eliminant equations and 2 from the Bianchi identities. These are listed below for convenience in terms of the "compacted covariant derivatives" where relevant. The equation labels refer to the equations from Chandrasekhar’s book.
\[D\alpha - \delta \epsilon = -\alpha (\epsilon - \bar{\epsilon}) + \epsilon (\pi - \alpha - \bar{\beta}) \quad (\text{eqn. d})\]

\[D\beta - \delta \epsilon = \beta (\epsilon - \bar{\epsilon}) + \epsilon (\bar{\pi} - \bar{\alpha} - \bar{\beta}) \quad (\text{eqn. e})\]

\[D\lambda - \delta \pi = \pi^2 - \lambda (\epsilon + \bar{\epsilon}) + \Phi_{20} \quad (\text{eqn. g})\]

\[D\mu - \delta \pi = \pi \bar{\pi} - \mu (\epsilon + \bar{\epsilon}) + \Psi_2 + 2\Lambda \quad (\text{eqn. h})\]

\[\delta \alpha - \delta \beta = \alpha \bar{\alpha} + \beta \bar{\beta} - 2\alpha \beta - \epsilon (\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda \quad (\text{eqn. l})\]

\[\delta \lambda - \delta \mu = \pi (\mu - \bar{\mu}) + \mu (\alpha + \bar{\beta}) - \lambda (\bar{\alpha} + \bar{\beta}) - \Psi_3 + \Phi_{21} \quad (\text{eqn. m})\]

6. Eliminant equations:

\[\delta (\alpha + \bar{\beta} - \pi) - \bar{\delta} (\bar{\alpha} + \beta - \bar{\pi}) = -D(\mu - \bar{\mu}) - 2(\alpha \beta - \bar{\alpha} \bar{\beta}) - (\bar{\alpha} - \beta) \pi + (\alpha - \bar{\beta}) \bar{\pi} \quad (\text{eqn. b'})\]

\[D(\bar{\alpha} - \beta) + \delta (\epsilon - \bar{\epsilon}) = (2\bar{\alpha} - \bar{\pi})(\epsilon - \bar{\epsilon}) \quad (\text{eqn. f'})\]

7. Bianchi identities (terms which vanish are dropped):

\[-D\Psi_2 - D'\Phi_{00} - 2D\Lambda = 0 \quad (\text{eqn. b''})\]

\[-D(\Phi_{11} + 3\Lambda) - D'\Phi_{00} = 0 \quad (\text{eqn. i''})\]

8. Change of the metric on \(\Sigma_2\) under its diffeomorphism:

The metric on \(\Sigma_2\) is \(-(m \otimes \bar{m} + \bar{m} \otimes m)_{\mu\nu}\). We are interested in the Lie derivative of this metric with respect to a vector field on \(\Sigma_2\) projected back on to \(\Sigma_2\). For a vector field of the form, \(X = Cm + \bar{C}\bar{m}\), one gets,

\[\mathcal{L}_X(m \otimes \bar{m} + \bar{m} \otimes m) = 2\delta C(m \otimes m) + 2\delta \bar{C}(\bar{m} \otimes \bar{m}) + (\delta C + \delta \bar{C})(m \otimes \bar{m} + \bar{m} \otimes m)\]

\[\mathcal{L}_X(m \otimes \bar{m} - \bar{m} \otimes m) = (\delta C + \delta \bar{C})(m \otimes \bar{m} - \bar{m} \otimes m)\]
Appendix B Example of Kerr-Newman family

In this appendix we describe the Kerr-Newman family in terms of the spin coefficients. While these are given by [10], a slight modification is needed. Further, these are also used to illustrate our gauge fixing procedure.

Metric:

\[
ds^2 = \frac{\eta^2}{\Sigma^2} \Delta dt^2 - \left(\frac{\Sigma^2 \sin^2 \theta}{\eta^2}\right)(d\phi - \omega dt)^2 - \frac{\eta^2}{\Delta} dr^2 - \eta^2 d\theta^2,
\]

where

\[
\eta^2 \equiv r^2 + a^2 \cos^2 \theta, \quad \Sigma^2 \equiv (r^2 + a^2)^2 - a^2 \sin^2 \theta \Delta
\]

\[
\omega \equiv \frac{a(2Mr - Q^2)}{\Sigma^2}, \quad \Delta \equiv r^2 + a^2 - 2Mr + Q^2_*
\]

\[
\xi \equiv r + i a \cos \theta, \quad \bar{\xi} \equiv r - i a \cos \theta
\]

The area, surface gravity and angular velocity of the event horizon are given by \((r_+ \) is the radius of the event horizon),

\[
\text{Area} = 4\pi(r_+^2 + a^2)
\]

\[
\text{Surface gravity} = \frac{r_+ - M}{2Mr_+ - Q^2_*}
\]

\[
\text{Angular velocity} = \frac{a}{r_+^2 + a^2}
\]

The null tetrad (principle congruences) definitions (the components refer to \(t, r, \theta, \phi\) respectively):

\[
\ell^\mu \equiv \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a) \quad , \quad n^\mu \equiv \frac{1}{2\eta}(r^2 + a^2, -\Delta, 0, a) \quad ,
\]

\[
m^\mu \equiv \frac{1}{\sqrt{2\xi}}(i a \sin \theta, 0, 1, \frac{i}{\sin \theta}) \quad , \quad \bar{m}^\mu \equiv \frac{1}{\sqrt{2\xi}}(-i a \sin \theta, 0, 1, -\frac{i}{\sin \theta}) \quad .
\]

These are not well defined on the horizons \((\Delta = 0)\) since the coordinates are singular at the horizon. The \(\ell, n\) are both future directed and ‘out-going’ and ‘in-coming’
respectively. Outside the event horizon one can make a scaling by $A = 1/\Delta$, and get the $\ell$ to be well defined on the horizons. Though $n$ is ill-defined on the horizon, it does not matter since we don’t not need to use it. This scaling changes some of the spin coefficients given by Chandrasekhar. Following are these changed values, on horizons, except $\gamma$ which is not needed.

$$\kappa = \rho = \sigma = \lambda = \nu = 0, \quad \epsilon = r - M, \quad \mu = -\frac{1}{2\xi\eta^2},$$

$$\pi = \frac{iasin\theta}{\sqrt{2\xi^2}}, \quad \tau = -\frac{iasin\theta}{\sqrt{2\eta^2}}, \quad \alpha = \frac{iasin\theta}{\sqrt{2\xi^2}} - \frac{\cot\theta}{2\sqrt{2\xi}}, \quad \beta = \frac{\cot\theta}{2\sqrt{2\xi}},$$

Notice that we already have $\alpha + \bar{\beta} = \pi$, and that gauge covariant derivative of $\pi$ is zero. However $\mu$ is complex. We can make a boost transformation to make this real without disturbing the “gauge condition”. Such a transformation parameter, $b$, satisfies,

$$\mu - \bar{\mu} + b\pi - \bar{b}\bar{\pi} + 2\bar{b}\beta - 2b\bar{\beta} + \delta b - \delta\bar{b} = 0.$$

In general this gives a one parameter family of solutions. However regularity on the spherical leaves fixes this uniquely to:

$$b = -i\frac{asin\theta}{2\sqrt{2r(r^2 + a^2)}}.$$

It can be explicitly checked that this is consistent with the commutation relations.

Since $\epsilon$ is real, $\alpha - \bar{\beta}$ is unchanged. $\pi$ and $\mu$ change to,

$$\pi' = \frac{iasin\theta}{\sqrt{2}} \left[ \frac{1}{\xi^2} + \frac{r - M}{r(r^2 + a^2)} \right]$$

(39)

$$\mu' = -\frac{1}{4r(r^2 + a^2)\eta^2} \left[ 2(r^2 - a^2cos^2\theta) + \frac{a^2sin^2\theta}{r^2 + a^2} \left\{ \frac{M}{r\eta^2} + a^2sin^2\theta \right\} \right]$$

(40)

This is manifestly negative definite consistent with marginal trapping.
For subsequent computations we note that all quantities we will need will be functions only of \( \theta \) and hence *effectively* the derivatives \( \delta, \bar{\delta} \) are invariant and can be taken to be,

\[
\delta = \frac{1}{\sqrt{2\xi}} \partial_\theta, \quad \bar{\delta} = \frac{1}{\sqrt{2\xi}} \partial_\theta, \quad \text{effectively (41)}
\]

The function \( f \), that gives the rotational Killing vector, can be determined by noting that \( \bar{a} - \beta \) can be expressed as \( -\delta(\ell \ln(\sin \theta / \xi)) \). Solving \( \delta^2 f = 0 \) gives,

\[
f = -\hat{C}\cos \theta + \text{constant}, \quad \hat{C} = -(r^2 + a^2), \quad C = i\bar{\delta} f = \frac{i\hat{C}}{\sqrt{2\xi}} \sin \theta \quad (42)
\]

The constant has been fixed by demanding that the range of the Killing parameter is \( 2\pi \). This is explained below.

One can solve the equations determining \( \zeta \) and get,

\[
\zeta = \left( \frac{\hat{C} M}{a} \right) \left[ \frac{\eta^2 - 2r^2}{\eta^4} + \frac{r^2 - a^2}{(r^2 + a^2)^2} \right] + \left( \frac{\hat{C} Q^2 r}{a} \right) \left[ \frac{1}{\eta^4} - \frac{1}{(r^2 + a^2)^2} \right] \quad (43)
\]

The condition that \( \zeta \) must vanish where \( C \) vanishes, is used to fix the constant of integration.

It can be checked directly that the invariant vector field \( X \) is already tangential to leaves (has the coefficient of \( \ell \) to be zero). So no further boost transformation is needed. In terms of the coordinate basis provided by the \( t, r, \theta, \phi \), \( X \) has non vanishing components along \( t \) and \( \phi \) direction, implying that the adapted azimuthal angle does *not* coincide with the \( \phi \). This is not surprising since on the event horizon \( \phi \) is singular. One can make a coordinate transformation to the usual non-singular coordinates \( r_*, \bar{\phi}^+ \) and see explicitly that in terms of these coordinate basis, \( X \) has non vanishing component only along \( \bar{\phi}^+ \). Demanding that this be equal to 1, fixes the constant \( \hat{C} \) and
identifies $\bar{\phi}^+$ as the adapted azimuthal coordinate. This coordinate transformation of course does not affect any of the spin coefficients. It does not affect the area 2-form either.

It is straightforward to check that the integral of $\zeta$ over the sphere precisely equals the angular momentum Komar integral \cite{K3} for the Kerr-Newman solution evaluated at the horizon.

$$J \equiv -\frac{1}{8\pi} \int_{S^2} \zeta ds = -\frac{1}{8\pi} \int_{S^2} \zeta (r^2 + a^2) \sin \theta d\theta d\phi$$ (44)

For comparison we include the Komar integrals corresponding to the stationary and the axial Killing vectors on a $t = \text{constant}$ and $r = \text{constant}$ surfaces.

$$M(r) = M - Q^2 \left\{ \frac{1}{r} + \frac{a^2}{2r^3} + \frac{r^2 + a^2}{2ar^2} \left( \arctan \left( \frac{a}{r} \right) - \frac{a}{r} \right) \right\}$$

$$J(r) = Ma - Q^2 \left\{ \frac{3a}{4r} + \frac{3a^3}{4r^3} + \frac{(r^2 + a^2)^2}{4a^2r^2} \left( \arctan \left( \frac{a}{r} \right) - \frac{a}{r} \right) \right\}$$ (45)

Next, the Ricci scalar of a diagonal metric on $S^2$ is given by,

$$R = -\frac{1}{2} \left[ \partial_\theta \left( \frac{\partial_\theta g_{\phi\phi}}{\det(g)} \right) + \frac{1}{\det(g)} \partial^2_{\theta} g_{\phi\phi} \right]$$ (46)

From our procedure of introducing $\theta, \phi$ coordinates, we have $g_{\theta\theta} = -2C\bar{C}\Phi^2, g_{\phi\phi} = -2C\bar{C}$. We have found above $C$ for the Kerr-Newman family and we also have $R = K + \bar{K}$. Thus we get a differential equation for $\Phi$. This can be solved easily to get, (for both Kerr and Kerr-Newman curvatures)

$$\frac{1}{\Phi^2} = \frac{(r^2 + a^2)^2 \sin^2(\theta)}{\eta^4}$$ (47)

This reduces our standard form of the metric to the usual one for the Kerr-Newman solution.
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