ON A COMBINATORIAL PROBLEM OF ASMUS SCHMIDT

WADIM ZUDILIN† (Cologne & Moscow)

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Abstract. For any integer \( r \geq 2 \), define a sequence of numbers \( \{c_k^{(r)}\}_{k=0,1,...} \), independent of the parameter \( n \), by

\[
\sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^r = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} c_k^{(r)}, \quad n = 0, 1, 2, \ldots.
\]

We prove that all the numbers \( c_k^{(r)} \) are integers.

1. Stating the problem

The following curious problem was stated by A. L. Schmidt in [Sc1] in 1992.

Problem 1. For any integer \( r \geq 2 \), define a sequence of numbers \( \{c_k^{(r)}\}_{k=0,1,...} \), independent of the parameter \( n \), by

\[
\sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^r = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} c_k^{(r)}, \quad n = 0, 1, 2, \ldots.
\]

Is it then true that all the numbers \( c_k^{(r)} \) are integers?

An affirmative answer for \( r = 2 \) was given in 1992 (but published a little bit later), independently, by Schmidt himself [Sc2] and by V. Strehl [St]. They both proved the following explicit expression:

\[
c_n^{(2)} = \sum_{j=0}^{n} \binom{n}{j}^3 = \sum_{j} \binom{n}{j}^2 \binom{2j}{n}, \quad n = 0, 1, 2, \ldots,
\]

which was observed experimentally by W. Deuber, W. Thumser and B. Voigt. In fact, Strehl used in [St] the corresponding identity as a model for demonstrating various

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proof techniques of binomial identities. He also proved an explicit expression for the sequence $c_n^{(3)}$, thus answering affirmatively to Problem 1 in the case $r = 3$. But for this case Strehl had only one proof based on Zeilberger’s algorithm of creative telescoping. Problem 1 was restated in [GKP] (the last Research Problem on p. 256) with indication (on p. 549) that H.S. Wolf had shown the desired integrality of $c_n^{(r)}$ for any $r$ but only for any $n \leq 9$.

We recall that the first non-trivial case $r = 2$ is deeply related to the famous Apéry numbers $\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$, the denominators of rational approximations to $\zeta(3)$. These numbers satisfy a 2nd-order polynomial recursion discovered by R. Apéry in 1978, while an analogous recursion (also 2nd-order and polynomial) for the numbers (2) was indicated by J. Franel already in 1894.

The aim of this paper is to give an answer in the affirmative to Problem 1 (Theorem 1) by deriving explicit expressions for the numbers $c_n^{(r)}$, and also to prove a stronger result (Theorem 2) conjectured in [St, Section 4.2].

**Theorem 1.** The answer to Problem 1 is affirmative. In particular, we have the explicit expressions

$$c_n^{(4)} = \sum_j \binom{2j}{j} \binom{n}{j} \sum_k \binom{k+j}{k-j} \binom{j}{n-k} \binom{k}{j} \binom{2j}{k-j},$$

$$c_n^{(5)} = \sum_j \binom{2j}{j} \binom{n}{j} \sum_k \binom{k+j}{k-j}^2 \binom{2j}{n-k} \binom{2j}{k-j},$$

and in general for $s = 1, 2, \ldots$

$$c_n^{(2s)} = \sum_j \binom{2j}{j}^{2s-1} \binom{n}{j} \sum_{k_1} \binom{j}{n-k_1} \binom{k_1}{j} \binom{k_1+j}{k_1-j} \sum_{k_2} \binom{2j}{k_1-k_2} \binom{k_2+j}{k_2-j} \cdots$$

$$\times \sum_{k_{s-1}} \binom{2j}{k_{s-2}-k_{s-1}} \binom{k_{s-1}+j}{k_{s-1}-j} \binom{2j}{k_{s-1}-j},$$

$$c_n^{(2s+1)} = \sum_j \binom{2j}{j}^{2s} \binom{n}{j} \sum_{k_1} \binom{2j}{n-k_1} \binom{k_1+j}{k_1-j} \sum_{k_2} \binom{2j}{k_1-k_2} \binom{k_2+j}{k_2-j} \cdots$$

$$\times \sum_{k_{s-1}} \binom{2j}{k_{s-2}-k_{s-1}} \binom{k_{s-1}+j}{k_{s-1}-j} \binom{2j}{k_{s-1}-j},$$

where $n = 0, 1, 2, \ldots$.

**2. Very-well-poised preliminaries**

The right-hand side of (1) defines the so-called Legendre transform of the sequence $\{c_k^{(r)}\}_{k=0,1,\ldots}$. In general, if

$$a_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} c_k = \sum_{k=0}^n \binom{2k}{k} \binom{n+k}{n-k} c_k,$$
then by the well-known relation for inverse Legendre pairs one has
\[
\binom{2n}{n} c_n = \sum_{k} (-1)^{n-k} d_{n,k} a_k,
\]
where
\[
d_{n,k} = \binom{2n}{n-k} - \binom{2n}{n-k-1} = \frac{2k+1}{n+k+1} \binom{2n}{n-k}.
\]
Therefore, putting
\[
t_{n,j}^{(r)} = \sum_{k=j}^{n} (-1)^{n-k} d_{n,k} \binom{k+j}{k-j}^r,
\]
we obtain
\[
\binom{2n}{n} c_{n}^{(r)} = \sum_{j=0}^{n} \binom{2j}{j}^r t_{n,j}^{(r)}.
\]
The case \(r = 1\) of Problem 1 is trivial (that is why it is not included in the statement of the problem), while the cases \(r = 2\) and \(r = 3\) are treated in [Sc2], [St] using the fact that \(t_{n,j}^{(2)}\) and \(t_{n,j}^{(3)}\) have a closed form. Namely, it is easy to show by Zeilberger’s algorithm of creative telescoping [PWZ] that the latter sequences, indexed by either \(n\) or \(j\), satisfy simple 1st-order polynomial recursions. Unfortunately, this argument does not exist for \(r \geq 4\).

V. Strehl observed in [St, Section 4.2] that the desired integrality would be a consequence of the divisibility of the product \((\binom{2j}{j})^r \cdot t_{n,j}^{(r)}\) by \(\binom{2n}{n}\) for all \(j\), \(0 \leq j \leq n\). He conjectured a much stronger property, which we are now able to prove.

**Theorem 2.** The numbers \((\binom{2n}{n})^{-1} (\binom{2j}{j}) t_{n,j}^{(r)}\) are integers.

Our general strategy of proving Theorem 2 (and hence Theorem 1) is as follows: rewrite (5) in a hypergeometric form and apply suitable summation and transformation formulae (Propositions 1 and 2 below).

Changing \(l\) to \(n - k\) in (5) we obtain
\[
t_{n,j}^{(r)} = \sum_{l=0}^{n} (-1)^{n-2l} \frac{2n-2l+1}{2n-l+1} \binom{2n}{l} \binom{n-l+j}{n-l-j}^r,
\]
where the series on the right terminates. It is convenient to write all such terminating sums simply as \(\sum_{l}\), which is, in fact, a standard convention (see, e.g., [PWZ]). The ratio of the two consecutive terms in the latter sum is equal to
\[
\frac{-(2n+1) + l}{1 + l} \cdot \frac{-\frac{1}{2} (2n-1) + l}{-\frac{1}{2} (2n+1) + l} \cdot \frac{(-n-j) + l}{(-n+j) + l}^r,
\]

hence
\[
t_{n,j}^{(r)} = \binom{n+j}{n-j}^r \cdot {}_{r+2}F_{r+1} \left( \begin{array}{c} -(2n+1), -\frac{1}{2} (2n-1), -(n-j), \ldots, -(n-j) \\ -\frac{1}{2} (2n+1), -(n+j), \ldots, -(n+j) \end{array} \right),
\]
is a very-well-poised hypergeometric series. (We will omit the argument \(z = 1\) in further discussions.)

The following two classical results—Dougall’s summation of a \(\pFq{5}{4}\)-series (proved in 1907) and Whipple’s transformation of a \(\pFq{7}{6}\)-series (proved in 1926)—will be required to treat the cases \(r = 3, 4, 5\) of Theorems 1 and 2.
Proposition 1 [Ba, Section 4.3]. We have

\[ _5F_4 \left( \begin{array}{c} a, 1 + \frac{1}{2}a, c, d, -m \\ \frac{1}{2}a, 1 + a - c, 1 + a - d, 1 + a + m \end{array} \right) = \frac{(1 + a)_m (1 + a - c - d)_m}{(1 + a - c)_m (1 + a - d)_m} \] (7)

and

\[ _7F_6 \left( \begin{array}{c} a, 1 + \frac{1}{2}a, b, c, d, e, -m \\ \frac{1}{2}a, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a + m \end{array} \right) = \frac{(1 + a)_m (1 + a - d - e)_m}{(1 + a - d)_m (1 + a - e)_m} \cdot 4F_3 \left( \begin{array}{c} 1 + a - b - c, d, e, -m \\ 1 + a - b, 1 + a - c, d + e - a - m \end{array} \right) \] (8)

where \( m \) is a non-negative integer.

Application of (7) gives (without creative telescoping)

\[ t_{n,j}^{(3)} = \left( \frac{n + j}{n - j} \right)^3 \frac{(-2n)_n (-2n + 2(n - j))}{(2n)_n (2n - j)} = \frac{(2n)!}{(3j - n)! (n - j)!^3}, \]

which is exactly the expression obtained in [St, Section 4.2]. Therefore, from (6) we have the explicit expression

\[ c_n^{(3)} = \left( \frac{2n}{n} \right)^{-1} \sum_j \left( \frac{2j}{j} \right)^3 \frac{(2n)!}{(3j - n)! (n - j)!^3} = \sum_j \left( \frac{2j}{j} \right)^2 \left( \begin{array}{c} 2j \\ n - j \end{array} \right) \left( \begin{array}{c} n \\ j \end{array} \right)^2. \]

For the case \( r = 5 \), we are able to apply the transformation (8):

\[ t_{n,j}^{(5)} = \left( \frac{n + j}{n - j} \right)^5 \frac{(-2n)_n (-2n + 2(n - j))}{(2n)_n (2n - j)} \times 4F_3 \left( \begin{array}{c} -2j, -(n - j), -(n - j), -(n - j) \\ -(n + j), -(n + j), 3j - n + 1 \end{array} \right) \]

\[ = \left( \frac{n + j}{n - j} \right)^2 \frac{(2n)!}{(3j - n)! (n - j)!^3} \sum_l \frac{(-2j)_l (-n - j)_l^3}{l! (-n + j)_l^3 (3j - n + 1)_l} \]

\[ = \frac{(2n)!}{(2j)! (n - j)!^2} \sum_l \frac{(n - l + j)^2}{n - l} \left( \begin{array}{c} 2j \\ l \end{array} \right) \left( \begin{array}{c} 2j \\ n - l - j \end{array} \right) \]

\[ = \frac{2n)!}{(2j)! (n - j)!^2} \sum_k \left( \begin{array}{c} k + j \\ k - j \end{array} \right) \left( \begin{array}{c} 2j \\ n - k \end{array} \right) \left( \begin{array}{c} 2j \\ k - j \end{array} \right), \]

hence

\[ \left( \frac{2n}{n} \right)^{-1} \left( \frac{2j}{j} \right)^2 t_{n,j}^{(5)} = \sum_k \left( \begin{array}{c} k + j \\ k - j \end{array} \right) \left( \begin{array}{c} 2j \\ n - k \end{array} \right) \left( \begin{array}{c} 2j \\ k - j \end{array} \right) \]

are integers and from (6) we derive formula (4).
To proceed in the case $r = 4$, we apply the version of formula (8) with $b = (1 + a)/2$ (so that the series on the left reduces to a $6F_5(1)$-very-well-poised series):

$$t_{n,j}^{(4)} = \binom{n + j}{n - j}^4 \frac{(-2n)_{n-j}(-2n + 2(n - j))_{n-j}}{(-2n + (n - j))^2_{n-j}} \times 4F_3\left( -j, -(n - j), -(n - j), -(n - j) \mid -n, -(n + j), 3j + n + 1 \right)$$

$$= \binom{n + j}{n - j} \frac{(2n)!}{(3j - n)! (n - j)!} \sum_l \frac{(-j)_l (-n - j)_l}{l! (-n - j)_l (3j - n + 1)_l}$$

$$= \frac{(2n)! j!}{n! (n - j)! (2j)!} \sum_l \binom{n - l + j}{n - l} \binom{j}{l} \binom{n - l - j}{n - l} \binom{2j}{n - l - j}$$

from which, again, $\frac{(2n)!}{n!} \binom{2j}{j} t_{n,j}^{(4)} \in \mathbb{Z}$ and we arrive at formula (3).

3. Andrews’s multiple transformation

It seems that ‘classical’ hypergeometric identities can cover only the cases\(^1\) $r = 2, 3, 4, 5$ of Theorems 1 and 2. In order to prove the theorems in full generality, we will require a multiple generalization of Whipple’s transformation (8). The required generalization is given by G.E. Andrews in [An, Theorem 4]. After making the passage $q \to 1$ in Andrews’s theorem, we arrive at the following result.

**Proposition 2.** For $s \geq 1$ and $m$ a non-negative integer,

$$2s + 3F_{2s+2} \left( \begin{array}{ccccccc}
\frac{a_1 + \frac{1}{2}a_2}{b_1}, & b_1, & c_1, & b_2, & c_2, & \ldots \\
\frac{a_2}{b_2}, & 1 + a - b_1, 1 + a - c_1, 1 + a - b_2, 1 + a - c_2, & \ldots & \frac{b_s, c_s, -m}{1 + a - b_s, 1 + a - c_s, 1 + a + m} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right)$$

$$= \frac{(1 + a)_m (1 + a - b_s - c_s)_m}{(1 + a - b_s)_m (1 + a - c_s)_m} \sum_{l_1 \geq 0} \frac{(1 + a - b_1 - c_1)_l (b_2)_l (c_2)_l}{l_1! (1 + a - b_1)_l (1 + a - c_1)_l}$$

$$\times \sum_{l_2 \geq 0} \frac{(1 + a - b_2 - c_2)_l (b_3)_l (c_3)_l}{l_2! (1 + a - b_2)_l (1 + a - c_2)_l} \ldots$$

$$\times \sum_{l_{s-1} \geq 0} \frac{(1 + a - b_{s-1} - c_{s-1})_{l_{s-1}} (b_s)_l \ldots (c_s)_{l_1 + \ldots + l_{s-1}}}{l_{s-1}! (1 + a - b_{s-1})_{l_1 + \ldots + l_{s-1}} (1 + a - c_{s-1})_{l_1 + \ldots + l_{s-1}}}$$

$$\times \frac{(-m)_{l_1 + \ldots + l_{s-1}}}{(b_s + c_s - a - m)_{l_1 + \ldots + l_{s-1}}}.$$

**Proof of Theorem 2.** As in Section 2, we will distinguish the cases corresponding to the parity of $r$.

\(^1\)This is not really true since Andrews’s ‘non-classical’ identity below is a consequence of very classical Whipple’s transformation and the Pfaff-Saalschütz formula.
If \( r = 2s + 1 \), then setting \( a = -(2n + 1) \) and \( b_1 = c_1 = \cdots = b_s = a_s = -m = -(n - j) \) in Proposition 2 we obtain

\[
\binom{n+j}{n-j}^{2s-2} \frac{(2n)!}{(3j-n)!(n-j)!^2} \sum_{l_1} \left( \binom{2j}{l_2} \frac{(-n-j)_{l_1 + l_2}}{(-n+j)_{l_1 + l_2}} \right)^2 \]

\[
\times \sum_{l_2} \left( \binom{2j}{l_2} \frac{(-n-j)_{l_1 + l_2}}{(-n+j)_{l_1 + l_2}} \right)^2 \cdots
\]

\[
\times \sum_{l_{s-1}} \left( \binom{2j}{l_{s-1}} \frac{(-n-j)_{l_1 + \cdots + l_{s-1}}}{(-n+j)_{l_1 + \cdots + l_{s-1}}} \right)^2
\]

\[
\times \frac{(-1)^{l_1 + \cdots + l_{s-1} } (-n-j)_{l_1 + \cdots + l_{s-1}}}{(3j-n+1)_{l_1 + \cdots + l_{s-1}}}
\]

\[
= \frac{(2n)!}{(2j)! (n-j)!^2} \sum_{l_1} \left( \binom{j}{l_1} \frac{(-n-j)_{l_1}}{(-n)_{l_1}} \frac{(-n-j)_{l_1}}{(-n+j)_{l_1}} \right) \sum_{l_2} \left( \binom{2j}{l_2} \frac{n-l_1 + l_2}{n-l_1 - j} \right)^2 \cdots
\]

\[
\times \sum_{l_{s-1}} \left( \binom{2j}{l_{s-1}} \frac{n-l_1 - \cdots - l_{s-1} + j}{n_l - j} \right)^2 \left( \binom{2j}{n_l - j} \right).
\]

If \( r = 2s \), we apply Proposition 2 with the choice \( a = -(2n+1) \), \( b_1 = (a+1)/2 = -n \) and \( c_1 = b_2 = \cdots = b_s = a_s = -m = -(n-j) \):

\[
\binom{n+j}{n-j}^{2s-2} \frac{(2n)!}{(3j-n)!(n-j)!^3} \sum_{l_1} \left( \binom{j}{l_2} \frac{(-n-j)_{l_1 + l_2}}{(-n+j)_{l_1 + l_2}} \right)^2 \cdots
\]

\[
\times \sum_{l_2} \left( \binom{2j}{l_2} \frac{(-n-j)_{l_1 + l_2}}{(-n+j)_{l_1 + l_2}} \right)^2 \cdots
\]

\[
\times \sum_{l_{s-1}} \left( \binom{2j}{l_{s-1}} \frac{(-n-j)_{l_1 + \cdots + l_{s-1}}}{(-n+j)_{l_1 + \cdots + l_{s-1}}} \right)^2
\]

\[
\times \frac{(-1)^{l_1 + \cdots + l_{s-1} } (-n-j)_{l_1 + \cdots + l_{s-1}}}{(3j-n+1)_{l_1 + \cdots + l_{s-1}}}
\]

\[
= \frac{(2n)!}{n!(n-j)! (2j)!} \sum_{l_1} \left( \binom{j}{l_1} \frac{n-l_1}{n-l_1 + j} \right) \left( \binom{n-l_1}{j} \right) \sum_{l_2} \left( \binom{2j}{l_2} \frac{n-l_1 - l_2 + j}{n-l_1 - j} \right)^2 \cdots
\]

\[
\times \sum_{l_{s-1}} \left( \binom{2j}{l_{s-1}} \frac{n-l_1 - \cdots - l_{s-1} + j}{n_l - j} \right)^2 \left( \binom{2j}{n_l - j} \right).
\]

In both cases, the desired integrality

\[
\binom{2n}{n}^{-1} \binom{2j}{j} t_{n,j}^{(r)} \in \mathbb{Z}, \quad j = 0, 1, \ldots, n,
\]
clearly holds, and Theorem 2 follows.

Theorem 1 is an immediate consequence of Theorem 2.

We would like to conclude the paper by the following $q$-question.

**Problem 2.** Find and solve an appropriate $q$-analogue of Problem 1.

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Moscow Lomonosov State University
Department of Mechanics and Mathematics
Vorobiovy Gory, GSP-2, Moscow 119992 RUSSIA
URL: [http://wain.mi.ras.ru/index.html](http://wain.mi.ras.ru/index.html)
E-mail address: wadim@ips.ras.ru