Hardy’s non-locality and generalized non-local theory

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Abstract

Hardy’s non-locality theorem for multiple two-level systems is explored in the context of generalized nonlocal theory. We find nonlocal but non-signaling probabilities, providing Hardy’s nonlocal argument, which are higher than those in Quantum Mechanics. Maximum probability of success of Hardy’s argument is obtained for three two-level systems in quantum as well as in a more generalized theory. Interestingly, the maximum in the nonlocal generalized theory for both the cases turns out to be same.

1 Introduction

There exist correlations between quantum systems which no local realistic theory can reproduce. This was first shown by Bell by means of an inequality, popularly known as Bell’s inequality\cite{1}. Later, Hardy\cite{2} gave a non-locality theorem which provides a manifestation of non-locality in Quantum Mechanics without using statistical inequalities involving expectation values, in contrast with Bell’s inequality. This caused much interest among physicist. The logical structure introduced by Hardy is as follows: Consider four yes-no type events $A$, $B$, $A'$ and $B'$ where $A$ and $A'$ may happen in one system and $B$ and $B'$ happen in another system which is far apart from the first. The probability of joint occurrence of and $A'$ and $B'$ (i.e., the joint probability that both the events $A'$ and $B'$ are ‘yes’) is non-zero, $B'$ always implies $A$ (i.e., if $B'$ is ‘yes’ then $A$ is also ‘yes’), $A'$ always implies $B$ (i.e., if $A'$ is ‘yes’ then $B$ is also ‘yes’), but $A$ and $B$ never occurs (i.e., the joint probability that both $A$ as well as $B$ are ‘yes’ is zero). These four statements are not compatible with local realism. The nonzero probability appearing in the argument is the measure of violation of local-realism. Any given non-maximally entangled (pure) state of two spin-$\frac{1}{2}$ particles exhibits hardy type non-locality for proper choices of observables but surprisingly no maximally entangled state of two such (spin-$\frac{1}{2}$) particles can show Hardy’s
non-locality.
Although no local-realistic theory can reproduce quantum correlations still these correlations cannot be exploited to communicate with a speed greater than that of the light in vacuum. But quantum theory is not the only nonlocal theory consistent with the relativistic causality [4]. Theories which predict nonlocal correlations and hence permit violation of Bell’s inequality but are constrained with the no signalling condition are called ‘Generalized nonlocal theory (GNLT)’. In recent years there has been an increasing interest in GNLT [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In general, quantum theory has been studied in the background of classical theory which is comparatively restrictive. The new idea is to study quantum theory from outside i.e., starting from a from general family of theories, the so called ‘Generalized nonlocal theory (GNLT)’ and to study properties common to all [8]. This might help in a better understanding of quantum nonlocality.

In this paper we study the Hardy’s nonlocality argument in the framework of GNLT. The maximum probability of success of the Hardy’s non-locality argument for two two-level quantum systems is known to be 0.09[15, 16]. We have found that if instead a more general framework of GNLT is considered, the success probability of Hardy’s argument can be increased up to 50% for two two-level systems. The Bell’s inequality is violated maximally by the maximally entangled state of two spin-1/2 particles, but surprisingly such states do not exhibit Hardy type nonlocality. In section 5 of the paper we have shown that the scenario in GNLT is quite different.

For three spin-1/2 particles, Hardy’s nonlocality theorem has been proved by Wu and Xi[17]. They have shown that almost all entangled states of three spin-1/2 particles exhibit hardy nonlocality, but their proof lack generality[18]. Later Ghosh et.al[18] proved the same for all genuinely entangled states of three spin-1/2 particles. They further have shown that in sharp contrast to bipartite cases, every maximally entangled state of three spin-1/2 particles exhibits Hardy’s nonlocality. Moreover, for these states (i.e. for maximally entangled states of three spin-1/2 particles) the maximum probability of success of Hardy’s argument is 12.5%. For a given maximally entangled state of three spin-1/2 particles, they have varied the observable settings and have found that for a particular set of observables, the probability of success of Hardy’s argument is 12.5% which is maximum as compared with the other settings. But, whether this is the maximum over all three particle entangled state or not is worth knowing. We have found that this indeed is the case, i.e., for three-qubit systems the Hardy’s nonlocality argument can run maximum upto 12.5. Then we have studied the same in the context of GNLT and find that the maximum probability is higher than that in quantum mechanics. Actually the maximum probability reaches 50% which is surprisingly same as that for two two-level system.

2 Hardy’s non-locality theorem for three qubit systems

Let $|\psi\rangle$ be a pure state of three spin-1/2 particles 1, 2 and 3 and let $(\hat{U}_j, \hat{D}_j)$ be a pair of non-commuting projection operators for the $j$-th particle (where $j = 1, 2, 3$). Let us consider the following set of conditions [3].
Choosing over all the different possibilities, that can satisfy equation (5) and keeping in mind the conditions of equation (1) will be satisfied. Returning to the question of finding the choice of the observables \( \hat{D}_j \) and \( \hat{U}_j \) for each \( j = 1, 2, 3 \). Let us first look for all the product states \( |\psi_p\rangle \equiv |\phi\rangle \otimes |\eta\rangle \otimes |\chi\rangle \) of the three qubits, each of which is orthogonal to the all the four product states \( |\phi_1\rangle \equiv |\hat{D}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1\rangle, |\phi_2\rangle \equiv |\hat{U}_1 = +1, \hat{D}_2 = +1, \hat{U}_3 = +1\rangle, |\phi_3\rangle \equiv |\hat{U}_1 = +1, \hat{U}_2 = +1, \hat{D}_3 = +1\rangle \) and \( |\phi_\eta\rangle \equiv |\hat{D}_1 = -1, \hat{D}_2 = -1, \hat{D}_3 = -1\rangle \) appeared in the first four conditions of equation (1). The motivation behind this is to identify all the genuine three-qubit entangled states \( |\psi\rangle \) for each of which all the conditions of equation (1) will be satisfied. Returning to the question of finding out the product states \( |\psi_p\rangle \), we must have here:

\[
\begin{align*}
\langle \phi | \hat{D}_1 = +1 \rangle \langle \eta | \hat{U}_2 = +1 \rangle \langle \chi | \hat{U}_3 = +1 \rangle &= 0, \\
\langle \phi | \hat{U}_1 = +1 \rangle \langle \eta | \hat{D}_2 = +1 \rangle \langle \chi | \hat{U}_3 = +1 \rangle &= 0, \\
\langle \phi | \hat{U}_1 = +1 \rangle \langle \eta | \hat{U}_2 = +1 \rangle \langle \chi | \hat{D}_3 = +1 \rangle &= 0, \\
\langle \phi | \hat{D}_1 = -1 \rangle \langle \eta | \hat{D}_2 = -1 \rangle \langle \chi | \hat{D}_3 = -1 \rangle &= 0.
\end{align*}
\]

Choosing over all the different possibilities, that can satisfy equation (5) and keeping in mind the conditions (3) as well as (4), one can show that \( |\psi_p\rangle \) can be only one of the following three product states (and nothing else):

\[
\begin{align*}
|\psi_3\rangle &= |\hat{U}_1 = -1\rangle \otimes |\hat{U}_2 = -1\rangle \otimes |\hat{D}_3 = +1\rangle, \\
|\psi_1\rangle &= |\hat{D}_1 = +1\rangle \otimes |\hat{U}_2 = -1\rangle \otimes |\hat{U}_3 = -1\rangle, \\
|\psi_2\rangle &= |\hat{U}_1 = -1\rangle \otimes |\hat{D}_2 = +1\rangle \otimes |\hat{U}_3 = -1\rangle.
\end{align*}
\]

Let \( S_1 \) be the subspace (of the eight dimensional Hilbert space \( \mathcal{H}_{123} \) of the three qubits 1, 2, 3) spanned by the product states \( |\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle \) and \( |\phi_\eta\rangle \), while \( S_2 \) be the subspace (of \( \mathcal{H}_{123} \)) spanned by the product states \( |\psi_1\rangle, |\psi_2\rangle \) and \( |\psi_3\rangle \). So \( S_1 \) must be orthogonal to \( S_2 \). It can easily be shown that the dimension of \( S_1 \) is four while that of \( S_2 \) is three. Thus the subspace
\[ S_1 \oplus S_2 \] (of \( \mathcal{H}_{123} \)) has dimension seven. Therefore, its orthogonal subspace \((S_1 \oplus S_2)^\perp\) has to be one dimensional, and the above-mentioned construction of \( \psi_0 \), \((S_1 \oplus S_2)^\perp\) must be spanned by a single entangled state \( |\psi_0\rangle \) (say) of the three qubits. So it turns out that for \( |\psi\rangle = |\psi_0\rangle \), the probability \(|(\psi|\hat{U}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1)^2\| \), appeared in the last condition of equation (1), is maximum. It is to be noted that the state \(|\hat{U}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1\rangle\) has to be of the form (as no product state can satisfy Hardy-type non-locality conditions given in equation (1)):

\[
|\hat{U}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1\rangle = a|\psi_0\rangle + b|\phi_1\rangle + c|\phi_2\rangle + d|\phi_3\rangle + e|\phi_\rangle.
\]

As \( |\psi_0\rangle \) has to be orthogonal to all the linearly independent states \(|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_\rangle, |\psi_1\rangle, |\psi_2\rangle \) and \( |\psi_3\rangle \), therefore \( |\psi_0\rangle \) has to be unique. Taking this into account, it can be shown that

\[
|\psi_0\rangle = |b_1 b_2 b_3\rangle \sqrt{\frac{1 - N}{N}} \left[ |\hat{U}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1\rangle - \frac{a_3}{b_3} |\hat{U}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = -1\rangle \right.
\]

\[
- \frac{a_2}{b_2} |\hat{U}_1 = +1, \hat{U}_2 = -1, \hat{U}_3 = +1\rangle - \frac{a_2 a_3}{b_2 b_3} \left( \frac{N}{1 - N} \right) |\hat{U}_1 = +1, \hat{U}_2 = -1, \hat{U}_3 = -1\rangle
\]

\[
- \frac{a_1}{b_1} |\hat{U}_1 = -1, \hat{U}_2 = +1, \hat{U}_3 = +1\rangle - \frac{a_1 a_3}{b_1 b_3} \left( \frac{N}{1 - N} \right) |\hat{U}_1 = -1, \hat{U}_2 = +1, \hat{U}_3 = -1\rangle
\]

\[
- \frac{a_1 a_2}{b_1 b_2} \left( \frac{N}{1 - N} \right) |\hat{U}_1 = -1, \hat{U}_2 = -1, \hat{U}_3 = +1\rangle + \frac{a_1 a_2 a_3}{b_1 b_2 b_3} \left( \frac{N}{1 - N} \right) |\hat{U}_1 = -1, \hat{U}_2 = -1, \hat{U}_3 = -1\rangle \right]
\]

(7) where

\[ N = |b_1 b_2|^2 + |b_2 b_3|^2 + |b_3 b_1|^2 - 2|b_1 b_2 b_3|^2 = |b_1 b_2 a_3|^2 + |a_1 b_2 b_3|^2 + |b_3 b_1|^2. \]

(8)

So, for given set of pairwise non-commuting observables \((\hat{U}_j, \hat{D}_j) \) \((j = 1, 2, 3)\), the maximum probability \(p(\hat{U}_1, \hat{D}_1, \hat{U}_2, \hat{D}_2, \hat{U}_3, \hat{D}_3)\) (say) with which the Hardy-type non-locality (1) will be satisfied is given by

\[
p(\hat{U}_1, \hat{D}_1, \hat{U}_2, \hat{D}_2, \hat{U}_3, \hat{D}_3) = |\langle \psi_0 | \hat{U}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1\rangle|^2 = |b_1 b_2 b_3|^2 \left( \frac{1 - N}{N} \right),
\]

(9) where

\[ |b_j|^2 = |\langle \hat{D}_j = -1 | \hat{U}_j = +1\rangle|^2 = |\langle \hat{D}_j = +1 | \hat{U}_j = -1\rangle|^2 \] for \( j = 1, 2, 3 \),

(10)

and \( N \) is given by equation (8). Any density matrix on \( \mathcal{H}_{123} \), whose support is contained in \((S_1 \oplus S_2)^\perp \oplus S_2\) and which is non-orthogonal to \(|\psi_0\rangle\) will satisfy the Hardy-type non-locality conditions given in equation (1). No other density matrix of \( \mathcal{H}_{123} \) will satisfy it.

Next we want to maximize the probability \(p(\hat{U}_1, \hat{D}_1, \hat{U}_2, \hat{D}_2, \hat{U}_3, \hat{D}_3)\) over all possible choices of the pairwise non-commuting observables \((\hat{U}_j, \hat{D}_j) \) \((j = 1, 2, 3)\). From equations (9) and (8) that \( p \) is a symmetric function of the three variables \(|b_1|^2, |b_2|^2\) and \(|b_3|^2\), where \(0 < |b_1|, |b_2|, |b_3| < 1\). So \( p \) will attain its extremum only when \(|b_1|^2 = |b_2|^2 = |b_3|^2 = k \) (say). Taking \(|b_1|^2 = |b_2|^2 = |b_3|^2 = k\) in the expression for \( p \), we find that

\[
p \equiv p(k) = \frac{k}{3 - 2k} - k^3, \ \text{with} \ 0 < k < 1.
\]

(11)
So $p(k)$ attains its maximum value when $k = 1/2$. Thus we see that the maximum value $1/8$ of $p(\hat{U}_1, \hat{D}_1, \hat{U}_2, \hat{D}_2, \hat{U}_3, \hat{D}_3)$ occurs when and only when $|b_j|^2 = |\langle \hat{D}_j = -1 | \hat{U}_j = +1 \rangle| = |\langle \hat{D}_j = +1 | \hat{U}_j = -1 \rangle|^2 = 1/2$ for $j = 1, 2, 3$. In this case, $|\psi_0\rangle$ is given by

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} |\hat{U}_1 = +1\rangle \otimes \left\{ \frac{1}{\sqrt{2}} |\hat{U}_2 = +1\rangle \otimes \frac{1}{\sqrt{2}} \left( |\hat{U}_3 = +1\rangle - e^{i(x_3 + y_3)} |\hat{U}_3 = -1\rangle \right) \right\}$$

$$- \frac{e^{i(x_2 + y_2)}}{\sqrt{2}} |\hat{U}_2 = -1\rangle \otimes \frac{1}{\sqrt{2}} \left( |\hat{U}_3 = +1\rangle + e^{i(x_3 + y_3)} |\hat{U}_3 = -1\rangle \right)$$

$$- \frac{e^{i(x_1 + y_1)}}{\sqrt{2}} |\hat{U}_1 = -1\rangle \otimes \frac{1}{\sqrt{2}} \left( |\hat{U}_3 = +1\rangle + e^{i(x_3 + y_3)} |\hat{U}_3 = -1\rangle \right)$$

$$+ \frac{e^{i(x_2 + y_2)}}{\sqrt{2}} |\hat{U}_2 = -1\rangle \otimes \frac{1}{\sqrt{2}} \left( |\hat{U}_3 = +1\rangle - e^{i(x_3 + y_3)} |\hat{U}_3 = -1\rangle \right)$$

which is nothing but a maximally entangled state of three qubits (and hence, it is locally unitarily connected to the three-qubit GHZ state for which the maximum success probability of Hardy’s argument is known to be 12.5%). Here $a_j = \langle \hat{D}_j = +1 | \hat{U}_j = +1 \rangle = - \langle \hat{D}_j = -1 | \hat{U}_j = -1 \rangle = \frac{e^{x_j}}{\sqrt{2}}$ and $b_j = \langle \hat{D}_j = -1 | \hat{U}_j = +1 \rangle = \langle \hat{D}_j = +1 | \hat{U}_j = -1 \rangle = \frac{e^{y_j}}{\sqrt{2}}$ for $j = 1, 2, 3$. Thus we conclude this section with the fact that the maximum probability of success of Hardy’s argument for three-qubit states is 12.5% and the state which respond to this maximum is the three-qubit GHZ state.

### 3 General non-signaling probabilities satisfying Hardy-type non-locality argument for two two-level systems

In this section, we will study the characters of a set of sixteen non-signaling joint probabilities $\text{Prob}(M = m, N = n)$, where $m, n \in \{+1, -1\}$, $M$ is one of the two $\{+1, -1\}$-valued random variables $A, A'$, chosen by Alice, and $N$ is one of the two $\{+1, -1\}$-valued random variables $B, B'$, chosen by Bob. We will further impose the restriction that four of these probabilities will satisfy Hardy-type non-locality conditions. In this direction, let us consider the following sixteen joint probabilities:

$$\text{Prob}(A = +1, B = +1) = p_1,$$
$$\text{Prob}(A = +1, B = -1) = p_2,$$
$$\text{Prob}(A = -1, B = +1) = p_3,$$
$$\text{Prob}(A = -1, B = -1) = p_4,$$
$$\text{Prob}(A' = +1, B = +1) = p_5,$$
$$\text{Prob}(A' = +1, B = -1) = p_6,$$
$$\text{Prob}(A' = -1, B = +1) = p_7,$$
$$\text{Prob}(A' = -1, B = -1) = p_8,$$

(13)
Prob\((A = +1, B' = +1) = p_9,\)  
Prob\((A = +1, B' = -1) = p_{10},\)  
Prob\((A = -1, B' = +1) = p_{11},\)  
Prob\((A = -1, B' = -1) = p_{12},\)

Prob\((A' = +1, B' = +1) = p_{13},\)  
Prob\((A' = +1, B' = -1) = p_{14},\)  
Prob\((A' = -1, B' = +1) = p_{15},\)  
Prob\((A' = -1, B' = -1) = p_{16}.\)

Other than being members of the interval \([0, 1],\) these sixteen probabilities must satisfy the normalization condition:

\[ p_1 + p_2 + p_3 + p_4 = 1, \quad (14) \]
\[ p_5 + p_6 + p_7 + p_8 = 1, \quad (15) \]
\[ p_9 + p_{10} + p_{11} + p_{12} = 1, \quad (16) \]
\[ p_{13} + p_{14} + p_{15} + p_{16} = 1. \quad (17) \]

Now the no-signaling constraint (i.e., the relativistic causality) implies that if Alice performs the experiment for \(A\) (or \(A'\)), the individual probabilities for the outcomes \(A = +1\) (or \(A' = +1\)) and \(A = -1\) (or \(A' = -1\)) must be independent of whether Bob chooses to perform the experiment for \(B\) or \(B'\) and similar should be the case for Bob also. So for the above-mentioned sixteen probabilities, the condition for causality to hold is given by:

\[ p_1 + p_2 = p_9 + p_{10}, \quad (18) \]
\[ p_3 + p_4 = p_{11} + p_{12}, \quad (19) \]
\[ p_5 + p_6 = p_{13} + p_{14}, \quad (20) \]
\[ p_7 + p_8 = p_{15} + p_{16}, \quad (21) \]
\[ p_1 + p_3 = p_5 + p_7, \quad (22) \]
\[ p_2 + p_4 = p_6 + p_8, \quad (23) \]
\[ p_9 + p_{11} = p_{13} + p_{15}, \quad (24) \]
\[ p_{10} + p_{12} = p_{14} + p_{16}. \quad (25) \]

We further assume that the above-mentioned probabilities respect the Hardy-type non-locality conditions\(^1\):

\[ p_1 = 0, \quad p_6 = 0, \quad p_{11} = 0, \quad p_{13} = q. \quad (26) \]

Using equation (26) into equation (18), we get

\[ p_2 \geq p_9. \quad (27) \]

\(^1\)The conditions given in equation (26) are not compatible with the notion of local-realism. To see this let us consider those local-realistic states for which values of both \(A'\) and \(B'\) are +1. The condition \(p_{13} = q\) guarantees that there exists such states. Now for these states \(p_{11}\) and \(p_6\) respectively imply \(A = +1\) and \(B = +1.\) But this contradicts the first condition \(p_1 = 0.\)
Using equation (26) into equation (22), we get
\[ p_3 \geq p_5. \]  
(28)

Using equation (26) into equation (14), we get
\[ 1 = p_2 + p_3 + p_4 \geq p_2 + p_3 \geq p_5 + p_9, \]  
(29)

using equations (27) and (28). Using equation (26) into equations (20) and (24), we get
\[ p_5 + p_9 = 2q + p_{14} + p_{15} \geq 2q. \]  
(30)

Using equations (29) and (30), we get
\[ 1 \geq p_5 + p_9 \geq 2q. \]

Thus we have
\[ q \leq \frac{1}{2}. \]  
(31)

If we now follow the argument, beginning at equation (27) and ending at equation (31), it can be easily shown that for \( q = 1/2 \), there is a unique solution for the above-mentioned sixteen joint probabilities satisfying simultaneously all the conditions (14) to (26):
\[ p_2 = p_3 = p_5 = p_8 = p_9 = p_{12} = p_{16} = \frac{1}{2} \text{ and } p_4 = p_7 = p_{10} = p_{14} = p_{15} = 0. \]  
(32)

Thus we see that above-mentioned sixteen probabilities will be non-local as well as non-signaling iff \( 0 < q \leq 1/2 \).

We know that quantum states cannot exhibit Hardys nonlocality with a probability more than 0.09 but in a more generalized nonlocal theory, the success probability of Hardys argument can be increased up to 0.5. We write below, for ready reference, the nonsignalling joint probability distribution for which the Hardys argument runs for maximum number of times:
\[ q = p_2 = p_3 = p_5 = p_6 = p_8 = p_9 = p_{12} = p_{16} = \frac{1}{2} \]
\[ p_1 = p_4 = p_7 = p_{10} = p_{11} = p_{14} = p_{15} = 0. \]  
(33)

4 General non-signaling probabilities satisfying Hardy-type non-locality argument for three two-level systems

In the framework of a general probabilistic theory, consider a physical system consisting of three subsystems shared among three far apart parties Alice, Bob and Charlie. Assume that Alice, Bob and Charlie can measure one of the two observables \( X_i, Y_i \), where \( i \) stands for the 1st (i.e., Alice), 2nd (i.e., Bob), or 3rd (i.e., Charlie) on their respective subsystems. The outcomes of each such measurements can be either up (\( U \)) or down (\( D \)). We now consider all the sixty four joint probabilities \( \text{Prob}(R_1 = j, R_2' = k, R_3'' = l) \), where \( R, R', R'' \in \{ X, Y \} \) and \( j, k, l \in \{ U, D \} \). For the sake of notational simplicity, we will denote \( X \) by 0 and \( Y \) by 1. Moreover, we will also denote \( U \) by 0 and \( D \) by 1. With this notation in mind, we can denote the above-mentioned
joint probabilities as \( \text{Prob}(i = s_1, j = s_2, k = s_3) \), where \( i, j, k \in \{0, 1\} \) and \( s_1, s_2, s_3 \in \{0, 1\} \). To make it more readable, we will denote the probability \( \text{Prob}(i = s_1, j = s_2, k = s_3) \) by \( p_{i s_1 j s_2 k s_3} \), where \( i s_1 j s_2 k s_3 \) is the binary representation of the number \( 32i + 16s_1 + 8j + 4s_2 + 2k + s_3 \). Let us now impose the normalization, non-signalling as well as Hardy’s non-locality type conditions on these sixty four joint probabilities \( p_0, p_1, \ldots, p_{63} \):

**Condition (1): (Normalization conditions):**

\[
\begin{align*}
p_0 + p_1 + p_4 + p_5 + p_{16} + p_{17} + p_{20} + p_{21} &= 1, \\
p_2 + p_3 + p_6 + p_7 + p_{18} + p_{19} + p_{22} + p_{23} &= 1, \\
p_8 + p_9 + p_{12} + p_{13} + p_{24} + p_{25} + p_{28} + p_{29} &= 1, \\
p_{10} + p_{11} + p_{14} + p_{15} + p_{26} + p_{27} + p_{30} + p_{31} &= 1, \\
p_{32} + p_{33} + p_{36} + p_{37} + p_{48} + p_{49} + p_{52} + p_{53} &= 1, \\
p_{34} + p_{35} + p_{38} + p_{39} + p_{50} + p_{51} + p_{54} + p_{55} &= 1, \\
p_{40} + p_{41} + p_{44} + p_{45} + p_{56} + p_{57} + p_{60} + p_{61} &= 1, \\
p_{42} + p_{43} + p_{46} + p_{47} + p_{58} + p_{59} + p_{62} + p_{63} &= 1.
\end{align*}
\]

**Condition (2): (Non-signalling conditions):**

\[
\begin{align*}
p_0 + p_1 &= p_2 + p_3, \\
p_4 + p_5 &= p_6 + p_7, \\
p_8 + p_9 &= p_{10} + p_{11}, \\
p_{12} + p_{13} &= p_{14} + p_{15}, \\
p_{16} + p_{17} &= p_{18} + p_{19}, \\
p_{20} + p_{21} &= p_{22} + p_{23}, \\
p_{24} + p_{25} &= p_{26} + p_{27}, \\
p_{28} + p_{29} &= p_{30} + p_{31}, \\
p_{32} + p_{33} &= p_{34} + p_{35}, \\
p_{36} + p_{37} &= p_{38} + p_{39}, \\
p_{40} + p_{41} &= p_{42} + p_{43}, \\
p_{44} + p_{45} &= p_{46} + p_{47}, \\
p_{48} + p_{49} &= p_{50} + p_{51}, \\
p_{52} + p_{53} &= p_{54} + p_{55}, \\
p_{56} + p_{57} &= p_{58} + p_{59}, \\
p_{60} + p_{61} &= p_{62} + p_{63}.
\end{align*}
\]
\[ \begin{align*}
p_0 + p_4 &= p_8 + p_{12}, \\
p_1 + p_5 &= p_9 + p_{13}, \\
p_2 + p_6 &= p_{10} + p_{14}, \\
p_3 + p_7 &= p_{11} + p_{15}, \\
p_{16} + p_{20} &= p_{24} + p_{28}, \\
p_{17} + p_{21} &= p_{25} + p_{29}, \\
p_{18} + p_{22} &= p_{26} + p_{30}, \\
p_{19} + p_{23} &= p_{27} + p_{31}, \\
p_{32} + p_{36} &= p_{40} + p_{44}, \\
p_{33} + p_{37} &= p_{41} + p_{45}, \\
p_{34} + p_{38} &= p_{42} + p_{46}, \\
p_{35} + p_{39} &= p_{43} + p_{47}, \\
p_{48} + p_{52} &= p_{56} + p_{60}, \\
p_{49} + p_{53} &= p_{57} + p_{61}, \\
p_{50} + p_{54} &= p_{58} + p_{62}, \\
p_{51} + p_{55} &= p_{59} + p_{63};
\end{align*} \]

\[ (36) \]

\[ \begin{align*}
p_0 + p_{16} &= p_{32} + p_{48}, \\
p_1 + p_{17} &= p_{33} + p_{49}, \\
p_2 + p_{18} &= p_{34} + p_{50}, \\
p_3 + p_{19} &= p_{35} + p_{51}, \\
p_4 + p_{20} &= p_{36} + p_{52}, \\
p_5 + p_{21} &= p_{37} + p_{53}, \\
p_6 + p_{22} &= p_{38} + p_{54}, \\
p_7 + p_{23} &= p_{39} + p_{55}, \\
p_8 + p_{24} &= p_{40} + p_{56}, \\
p_9 + p_{25} &= p_{41} + p_{57}, \\
p_{10} + p_{26} &= p_{42} + p_{58}, \\
p_{11} + p_{27} &= p_{43} + p_{59}, \\
p_{12} + p_{28} &= p_{44} + p_{60}, \\
p_{13} + p_{29} &= p_{45} + p_{61}, \\
p_{14} + p_{30} &= p_{46} + p_{62}, \\
p_{15} + p_{31} &= p_{47} + p_{63}.
\end{align*} \]

(37)

**Condition (3): (Hardy-type non-locality conditions):**

\[ \begin{align*}
p_{32} &= 0, \\
p_8 &= 0, \\
p_2 &= 0, \\
p_{63} &= 0, \\
p_0 &> 0.
\end{align*} \]

(38) 

Our aim is to maximize the probability \( p_0 \) subject to satisfying all the conditions given in
equations (34), (35), (36), (37), (38).

We have maximized for $p_0$ with the help of a computer programme (Mathematica) and have found the solution as

$$z_{\text{max}} \equiv p_{0 \text{max}}^\text{max} = \frac{1}{2}, x_{10} = x_{12} = x_{18} = \frac{1}{2}, \text{ and } x_i = 0 \text{ for all } i \in \{1, 2, \ldots, 19\} - \{10, 12, 18\}.$$ 

Thus we see that the maximum probability of a non-signalling Hardy-type probability distribution for three two-level systems is given by

$$p_{0 \text{max}} = \frac{1}{2},$$ 

while rest of the sixty four probabilities are given by

$$p_3 = p_{12} = p_{15} = p_{17} = p_{18} = p_{29} = p_{30} = p_{33} = p_{35} = p_{45} = p_{47} = p_{48} = p_{50} = p_{60} = p_{62} = \frac{1}{2},$$

and $p_i = 0$ for all $i \in \{1, 2, \ldots, 63\} - \{3, 12, 15, 17, 18, 29, 30, 33, 35, 45, 47, 48, 50, 60, 62\}$.

Thus the maximum probability of success of Hardy’s argument for three two-level systems too is more in GNL T than in the quantum theory.

5 Nonlocality inequality and non-signalling maximal Hardy type probabilities

Hardy’s non-locality argument is considered weaker than Bell’s inequality in quantum mechanics as every maximally entangled state of two spin-1/2 particles violates Bell’s inequality maximally but none of them satisfy Hardy type nonlocality conditions. The scenario in GNL T is quite different, however. To see this let us consider the following Bell-CHSH expression:

$$B = |E(s,t|A', B) + E(s,t|A, B') - E(s,t|A', B') - E(s,t|A, B)|$$ 

where $E(s,t|S_A, S_B) \equiv \sum_{i,j\in\{-1,1\}} ij P(i,j|S_A, S_B)$ and so on. With the help of equations (13), we can write it as:

$$E(s,t|A', B) = p_5 - p_6 - p_7 + p_8$$

$$E(s,t|A, B') = p_9 - p_{10} - p_{11} + p_{12}$$

$$E(s,t|A', B') = p_{13} - p_{14} - p_{15} + p_{16}$$

$$E(s,t|A, B) = p_1 - p_2 - p_3 + p_4$$

The probability distribution which reduces the hardy’s argument to a maximum has been appeared in equation (33). Using equation (33) in (42), we get

$$B = 4$$

But 4 is the maximum allowed value of Bell-CHSH expression by a GNL T [4] and it corresponds to that probability distribution of Hardys arguments which leads the success probability of it to the maximum in GNL T. \footnote{In all, there are four different expressions for $B$ corresponding to four different permutations of the terms in the right hand side of equation (41):}

$$B_{\alpha \beta \gamma} = |(-1)^\alpha E(s,t|A', B) + (-1)^\beta E(s,t|A, B') + (-1)^\gamma E(s,t|A', B') + (-1)^{\alpha + \beta + \gamma + 1} E(s,t|A, B)|$$
This is in sharp contrast with the quantum mechanical case where for maximum violation of Bell's inequality, no probability distribution exists for the hardy arguments.

6 Conclusion

In conclusion, we have shown here that for three two level quantum systems the maximum probability of success of Hardy's argument is 12.5% which in fact is exhibited by the maximally entangled state of three qubits namely the GHZ state. The corresponding maximum probability for two two-level quantum system is known to be 9%. It would have been interesting to search whether this trend is sustained or not i.e., whether the success probability of Hardy’s argument keeps on increasing with increase in number of qubits or not as it is known in this context that violation of Bell’s inequality increases exponentially with the increase in number of qubits which falls highly against the common acceptance [19, 20, 21, 22, 23, 24]. For this, in the appendix, we have provided a way to find out the state of $n$ two-level quantum systems which will show maximum departure from local-realism if Hardy’s logical structure is opted for this purpose. However, if instead of quantum mechanics a more general framework of GNLT is adopted, the maximum probability of success of Hardy’s argument can be enhanced for both the three two-level systems and for two two-level systems. Interestingly, the maximum success probability for both type of systems attains a common value 0.5.

Quantum non-locality has attracted much attention since its discovery because it relates quantum mechanics with special relativity. Special relativity forbids sending physical information with a speed greater than that of the light in vacuum. This is reflected in the quantum mechanical joint probabilities appearing both in the violation of Bell’s inequality as well as in the fulfillment of Hardy’s non-locality conditions, although there is no direct relevance of special theory of relativity in the postulates of non-relativistic quantum mechanics. These quantum mechanical joint probabilities are only non-local but also non-signaling. The non-local probabilities, coming out from quantum mechanical states can give rise to the maximum violation up to the amount $2\sqrt{2}$ of Bell’s inequality, whereas there are non-quantum mechanical non-local joint probabilities which give rise to the maximal possible algebraic violation (namely, 4) of Bell’s inequality, without violating the relativistic-causality [4]. Why quantum theory can not provide more than $2\sqrt{2}$ violation of the Bell’s inequality? By exploiting the theoretical structure of quantum mechanics it has been shown that a violation greater than $2\sqrt{2}$ will result in signalling in quantum mechanics [25, 26, 27]. We have seen in this paper that the no-signaling constraint

\[ P(A = +1, B = -1) = \frac{1}{2} \]
\[ P(A = +1, B' = -1) = 0 \]
\[ P(A' = -1, B = -1) = 0 \]
\[ P(A' = +1, B' = +1) = 0 \]

where $\alpha, \beta, \gamma \in 0, 1$. All the four expressions have the same algebraic maximum 4 to which different nonlocal non-quantum Hardy type conditions corresponds. For example the Hardy type probability which corresponds to the maximum of $|E(s, t|A', B) - E(s, t|A, B') + E(s, t|A', B') + E(s, t|A, B)|$ is the following:

where $\alpha, \beta, \gamma \in 0, 1$. All the four expressions have the same algebraic maximum 4 to which different nonlocal non-quantum Hardy type conditions corresponds. For example the Hardy type probability which corresponds to the maximum of $|E(s, t|A', B) - E(s, t|A, B') + E(s, t|A', B') + E(s, t|A, B)|$ is the following:
cannot restrict the maximum value of the non-zero probability appearing in the Hardy’s argument to 0.09 all by itself. In a generalized non-signalling theory this value can go up to 0.5. It will be an interesting open question to find what feature of quantum mechanics along with no-signalling condition restricts the value to 0.09

7 Acknowledgments

R.R acknowledges the support by CSIR, Government of India, New Delhi.

Appendix-I Here we try to find out the state of n qubits \((n > 3)\) which exhibits the Hardy type nonlocality with maximum probability. For the choice of two-valued two non-commuting observables \(\hat{U}_j\) and \(\hat{D}_j\) for the \(j\)-th qubit, Hardy-type non-locality argument runs as follows:

\[
\begin{align*}
\langle \hat{D}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1, \ldots, \hat{U}_n = +1 \rangle &= 0, \\
\langle \hat{U}_1 = +1, \hat{D}_2 = +1, \hat{U}_3 = +1, \ldots, \hat{U}_n = +1 \rangle &= 0, \\
\ldots &
\end{align*}
\]

(44)

\[
\begin{align*}
\langle \hat{U}_1 = +1, \hat{U}_2 = +1, \ldots, \hat{U}_{n-1} = +1, \hat{D}_n = +1 \rangle &= 0, \\
\langle \hat{D} = -1, \hat{D}_2 = -1, \hat{D}_3 = -1, \ldots, \hat{D}_n = -1 \rangle &= 0, \\
\langle \hat{U}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1, \ldots, \hat{U}_n = +1 \rangle &> 0.
\end{align*}
\]

As above, if \(S_1\) is the \((n + 1)\)-dimensional subspace of the \(n\)-qubit Hilbert space \((\mathcal{A}^2)^{\otimes n}\) linearly spanned by the \((n + 1)\) number of linearly independent product states \(|\hat{D}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1, \ldots, \hat{U}_n = +1\rangle, |\hat{U}_1 = +1, \hat{D}_2 = +1, \hat{U}_3 = +1, \ldots, \hat{U}_n = +1\rangle, \ldots, |\hat{U}_1 = +1, \hat{U}_2 = +1, \ldots, \hat{U}_{n-1} = +1, \hat{D}_n = +1\rangle, |\hat{D}_1 = -1, \hat{D}_2 = -1, \hat{D}_3 = -1, \ldots, \hat{D}_n = -1\rangle\), then all the (fully) product states, each of which is orthogonal to \(S_1\), will linearly span the \((2^n - n - 2)\)-dimensional subspace \(S_2\). So \(S_2\) is orthogonal to \(S_1\). One can also show that \(S_2\) can be linearly spanned by the following \((2^n - n - 2)\) number of linearly independent product states:

\[
\begin{align*}
|\hat{D}_1 = +1, \hat{U}_2 = -1, \hat{U}_3 = -1, \ldots, \hat{U}_n = -1\rangle, |\hat{U}_1 = -1, \hat{D}_2 = +1, \hat{U}_3 = -1, \ldots, \hat{U}_n = -1\rangle, \\
\ldots, |\hat{U}_1 = -1, \hat{U}_2 = -1, \ldots, \hat{U}_{n-1} = -1, \hat{D}_n = +1\rangle; \text{ (total no. } = n) \\
|\hat{D}_1 = +1, \hat{D}_2 = +1, \hat{U}_3 = -1, \ldots, \hat{U}_n = -1\rangle, |\hat{D}_1 = +1, \hat{U}_2 = -1, \hat{D}_3 = +1, \ldots, \hat{U}_n = -1\rangle, \\
\ldots, |\hat{U}_1 = -1, \hat{U}_2 = -1, \ldots, \hat{D}_{n-1} = +1, \hat{D}_n = +1\rangle; \text{ (total no. } = \frac{n(n-1)}{2})
\end{align*}
\]

\[
\ldots
\]
\[ |\hat{D}_1 = +1, \ldots, \hat{D}_{n-2} = +1, \hat{U}_{n-1} = -1, \hat{U}_n = -1 \rangle, \]
\[ |\hat{D}_1 = +1, \ldots, \hat{D}_{n-3} = +1, \hat{U}_{n-2} = -1, \hat{D}_{n-1} = +1, \hat{U}_n = -1 \rangle, \ldots, \]
\[ |\hat{U}_1 = -1, \hat{U}_2 = -1, \hat{D}_3 = +1, \ldots, \hat{D}_n = +1 \rangle \quad \text{(total no. } = \frac{n(n-1)}{2}). \]  

Note that the product state \[ |\hat{U}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1, \ldots, \hat{U}_n = +1 \rangle \] (which appeared in the last condition in equation (44)), is orthogonal to each of the \((2^n - n - 2)\) product states appeared in equation (45), and hence, it is orthogonal to \(S_2\) as well. So, in order that the inequality in the last condition of equation (44) is satisfied, the state \[ |\hat{U}_1 = +1, \hat{U}_2 = +1, \hat{U}_3 = +1, \ldots, \hat{U}_n = +1 \rangle \] has to have a non-zero overlap with the one-dimensional subspace \((S_1 \oplus S_2)^\perp\) of \((\mathcal{C}^2)^{\otimes n}\). Let \(|\psi_0\rangle\) be the (entangled) state spanning the subspace \((S_1 \oplus S_2)^\perp\). Thus we see that, for the given set of two-valued, pairwise non-commuting observables \((\hat{U}_j, \hat{D}_j) | j = 1, 2, \ldots, n \}\), there exits a unique state \(|\psi_0\rangle\) satisfying the Hardy’s non-locality conditions (44) with maximum probability. This \(|\psi_0\rangle\) can now be found out easily as it is orthogonal to the \((2^n - 1)\) number of linearly independent product states (described above) spanning \((S_1 \oplus S_2)\). One can then also try to maximize the probability \[ \langle \psi_0 | \hat{U}_1 = +1, \hat{U}_2 = +1, \ldots, \hat{U}_n = -1 | \psi_0 \rangle \] over all possible choices of the set \((\hat{U}_j, \hat{D}_j) | j = 1, 2, \ldots, n \}\) of observables (and thereby, over all possible choices of \(|\psi_0\rangle\)).

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