CASSON INVARIANT OF KNOTS ASSOCIATED WITH DIVIDES

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Abstract. I present a formula for the Casson invariant of knots associated with divides. The formula is written in terms of Arnold’s invariants of pieces of the divide. Various corollaries are discussed.

Introduction

A divide \( P \) is the image of a generic relative immersion of a 1-dimensional compact (not necessarily connected) manifold into the standard unit disc \( D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \) in the plane \( \mathbb{R}^2 \). The relativity of the immersion implies that the boundary of the manifold is mapped into the boundary of \( D \). The genericity condition means that a divide has only transversal double points as singularities and is transversal to \( \partial D \) at its boundary points. Divides are considered modulo ambient isotopy in \( D \).

With every divide \( P \) one can associate a link \( L(P) \subset S^3 \) using the following construction. Consider a standard projection \( \pi \) of \( S^3 = \{(x, y, u, v) \in \mathbb{R}^4 \mid x^2 + y^2 + u^2 + v^2 = 1\} \) onto \( D \), i.e. \( \pi(x, y, u, v) = (x, y) \). Preimage of a point \( p \in D \) under \( \pi \) is either a circle, if \( p \) belongs to the interior of \( D \), or a single point. Now \( L(P) \) consists of all points \( (x, y, u, v) \in \pi^{-1}(P) \subset S^3 \) such that either \( (x, y) \in \partial P \subset \partial D \) or \( (u, v) \) is tangent to \( P \) at \( (x, y) \). It is easy to see that for any \( p \in P \) its preimage \( \pi^{-1}(p) \cap L(P) \) in \( L(P) \) consists of either 1, 4 or 2 points, depending on whether \( p \) is a boundary, double or generic point of \( P \). \( L(P) \) is indeed a link with \( 2c + i \) components, where \( c \) and \( i \) are the numbers of closed and non-closed components of \( P \), respectively. Ambiently isotopic divides obviously give rise to ambiently isotopic links.

Divides and associated links were originally considered by N. A’Campo [1] and are closely related to the real morsifications of isolated complex plane curve singularities (see also [2, 3]).

In this paper I present a formula for the Casson invariant of the knot associated with a given divide with no closed and only one non-closed components. Such divides are called \( I \)-divides in this paper. The Casson invariant of a link \( L \) can be defined as \( \frac{1}{2} \Delta'_L(1) \), where \( \Delta_L(t) \) is the Alexander polynomial of \( L \) and \( \Delta'_L(1) \) is the value of its second derivative at 1. It is also a unique Vassiliev invariant of degree 2 that takes values 0 on the unknot and 1 on a trefoil. Since the Casson invariant of a link \( L \) is equal to the sum of its values on the components of \( L \), only the case of knots is interesting.

Amazingly enough, the formula is written in terms of Arnold’s invariants \( J^\pm \) and \( \text{St} \) of pieces of the divide (see section 1.2 or [6] for definitions).

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In [7] S. Chmutov constructed a second order $J^\pm$-type invariant of long curves, called $J^\pm_2$, and proved that its value on an I-divide is equal to the Casson invariant of the associated knot. Since the invariant was defined by an actuality table only, the computation of its values on a curve with even as few as 2 double points was rather involved. The original definition also did not provide a mean to compute changes of $J^\pm_2$ under self-tangency perestroikas. My formula fills these gaps (see Lemmas 3.2.A, 3.2.C and section 3.3).

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1. Main ingredients and definitions

1.1. Diagrams of divide links. Let $P \subset D$ be a divide. In this section I present a way to draw a usual diagram with over- and under-crossings of the link $L(P)$. The following construction is due to M. Hirasawa [10] with some minor modifications (see also [1]).

One first perturbs $P$ slightly without changing its ambient isotopy type so that the following two conditions are met:

- at every double point the two branches of $P$ are parallel to the main diagonals $y = \pm x$ of the plane;
- $P$ has only finitely many points where the tangent vector is parallel to the $y$-axis, and at every such a point the projection of $P$ onto the $x$-axis has either a local minimum or a local maximum.

The next step is to double the divide, i.e. to draw two parallel strands along the divide and to join them at the end points (see Figure 1.a). The resulting closed curves have a natural orientation according to the “keeping right” rule. One puts over- and under-crossings near double points according to the rule depicted in Figure 1.b. And finally at every point of $P$ with the tangent vector parallel to the
y-axis the strand pointing downwards should make a “jump through infinity”. That means to go far right (left) above (below) the rest of the diagram and to return back below (above) it, according to whether the projection of $P$ onto the $x$-axis has a local maximum or minimum at that point (see Figure 2.c,d). Denote the diagram obtained by $D(P)$. For example, the divide depicted in Figure 2.a gives rise to a (right) trefoil (see Figure 2).

1.1.A. Theorem (Hirasawa [10]). The diagram $D(P)$ represents the link $L(P)$ in $S^3$.

1.2. Arnold’s invariants of plane curves. Let $C$ be a generic plane curve, i.e. a ($C^1$-smooth) immersion of the circle into the plane that has only transversal double points as singularities. For any such a curve one can define its Whitney index or simply index as the total rotation number of tangent vector to the curve. This number is, clearly, the degree of the map that associates a direction of the tangent vector to every point of the circle. The index of a curve $C$ is denoted by $w(C)$. It is easy to see that the index does not change under a regular homotopy of a curve that is a $C^1$-smooth homotopy in the class of $C^1$-immersions. Moreover, two plane curves are regular homotopic if and only if their Whitney indices are the same.

In a generic regular homotopy connecting two generic curves only a finite number of non-generic ones can appear. Each of these curves differs from a generic one either in exactly one point of triple transversal self-intersection or in exactly one point of self-tangency. In a point of self-tangency the velocity vectors of the tangent branches can have either the same directions or the opposite ones. In the first case the self-tangency is said to be direct and in the second inverse (see Figure 3). The type of the self-tangency does not change under reversing of orientation.
Hence there are three types of singular curves that may appear in a generic regular homotopy. Passages through these curves correspond to three \textit{perestroikas} of generic curves.

Consider the triple point perestroika more carefully. Just before and just after the passage through a singular curve with a triple point, there is a small triangle close to the place of perestroika, which is formed by three branches of the curve. This triangle is said to be \textit{vanishing}. The orientation of the curve defines a cyclic order of sides of the triangle. This is the order in which one meets the sides while traveling along the curve. This cyclic order gives the orientation of the triangle and, therefore, the orientation of its sides. Denote by $q$ the number of sides of the vanishing triangle for which the orientation obtained coincides with the orientation of the curve ($q$, obviously, takes value between 0 and 3).

Define a \textit{sign} of the vanishing triangle as $(-1)^q$. The sign does not change under reversing of orientation of the curve. One can easily check that before and after the perestroika the sings of the vanishing triangles are different.

1.2.A. Definitions (Arnold [6]). 1. A triple point perestroika is said to be \textit{positive} if the newborn vanishing triangle is positive.

2. A self-tangency perestroika is said to be \textit{positive} if it increases (by 2) the number of self-intersection points of the curve.

The following theorem provides a definition of invariants of generic (plane) curves.

1.2.B. Theorem (Arnold [6]). \textit{There exist three integers} $\text{St}(C)$, $J^+(C)$, and $J^-(C)$ \textit{assigned to an arbitrary generic plane curve} $C$ \textit{that are uniquely defined by the following properties.}

(i) $\text{St}$, $J^+$ and $J^-$ are invariant under a regular homotopy in the class of generic curves.
(ii) $\text{St}$ does not change under self-tangency perestroikas and increases by 1 under a positive triple point perestroika.
(iii) $J^+$ does not change under triple point and inverse self-tangency perestroikas and increases by 2 under a positive direct self-tangency perestroika.
(iv) $J^-$ does not change under triple point and direct self-tangency perestroikas and decreases by 2 under a positive inverse self-tangency perestroika.
(v) On the standard curves $K_\omega$, shown in Figure 4, $St$, $J^+$, and $J^-$ take the following values:

- $St(K_0) = 0$, $St(K_{\omega+1}) = \omega$ ($\omega = 0, 1, 2, \ldots$);
- $J^+(K_0) = 0$, $J^+(K_{\omega+1}) = -2\omega$ ($\omega = 0, 1, 2, \ldots$);
- $J^-(K_0) = -1$, $J^-(K_{\omega+1}) = -3\omega$ ($\omega = 0, 1, 2, \ldots$).

Remark. The normalization of $St$ and $J^\pm$, which is fixed by the last property, makes them additive with respect to the connected sum of curves. It is easy to see that the invariants are independent of the orientation of a curve.

1.3. Formulas for Arnold’s invariants. It is very complicated to use the definition given above to compute Arnold’s invariants for a given curve with sufficiently many double points. The two theorems stated below provide explicit formulas, which make those computations easy and straightforward.

Consider a generic plane curve $C$. For every region $r$ of $C$ one defines its index with respect to $C$ as the total rotation number of the radius vector that connects an arbitrary interior point of $r$ to a point traveling along $C$. This number is clearly independent of the choice of the point in $r$ and is denoted by $\text{ind}_C(r)$. An index $\text{ind}_C(e)$ of an edge $e$ of $C$ is defined as a half-sum of the indices of the two regions adjacent to $e$. An index $\text{ind}_C(v)$ of a double point $v$ of $C$ is a quarter-sum of the indices of the four regions adjacent to $v$.

1.3.A. Theorem (Viro [13]). Let $C$ be a generic plane curve and $\tilde{C}$ be a family of embedded circles obtained as a result of smoothing of the curve $C$ at each double point with respect to the orientation (see Figure 4). Then

$$J^+(C) = 1 - \sum_{r \in \mathcal{R}_C} \text{ind}_C^2(r) \chi(r) + n, \quad (1.1)$$

$$J^-(C) = 1 - \sum_{r \in \mathcal{R}_C} \text{ind}_C^2(r) \chi(r), \quad (1.2)$$

where $\mathcal{R}_C$ is the set of all regions of $\tilde{C}$, $\chi$ is the Euler characteristic, and $n$ is the number of double points of the curve $C$.

Fix now a base point $f$ on a generic plane curve $C$ that is not a double point. One can enumerate all edges by numbers from 1 to $2n$ (where $n$ is again the number of double points of $C$) following the orientation and assigning 1 to the edge with the point $f$.

Consider an arbitrary double point $v$ of $C$. There are two edges pointing to $v$. Let them have numbers $i$ and $j$ such that the tangent vector to the edge $i$ and the tangent vector to the edge $j$ give a positive orientation of the plane. One defines a sign $s(v)$ of $v$ to be $\text{sign}(i - j)$.

Figure 5. The curve smoothing at a self-intersection point
1.3.B. Theorem (12). Let $C$ be a generic plane curve. Then
\[
\text{St}(C) = \sum_{v \in \mathcal{V}_C} \text{ind}_C(v)s(v) + \text{ind}_C^2(f) - \frac{1}{4},
\]
where $\mathcal{V}_C$ is the set of all double points of $C$.

Remark. If the point $f$ is chosen on an exterior edge (that is an edge that bounds the exterior region), then $\text{ind}_C(f) = \pm \frac{1}{2}$ and one gets
\[
\text{St}(C) = \sum_{v \in \mathcal{V}_C} \text{ind}_C(v)s(v). \tag{1.4}
\]

Remark. One can find another formulas for Arnold’s invariants in [8, 11, 12].

2. Formula for the Casson invariant

2.1. The main results. Let $P \subset D$ be an I-divide. Choose an arbitrary orientation on $P$. For any double point $v$ of $P$ denote by $O_v$ and $I_v$ closed and non-closed curves obtained by smoothing $P$ at $v$ with respect to the orientation (see Figure 6). The end points of $P$ split the boundary $\partial D$ into two arcs. Complete $P$ with one of those arcs, so that the orientation of $P$ induces a counter-clockwise orientation on the arc. Call the resulting closed curve $\overline{P}$. Finally, for a given plane curve $C$ denote $1 - J^-(C)$ by $\tilde{J}(C)$.

Remark. The closure $\overline{P}$ of $P$ clearly depends on the orientation of $P$ (see Figure 6).

Remark. $1 - J^-(C)$ can be expressed with a very simple formula $1 - J^-(C) = \sum_{r \in \mathcal{R}_C} \text{ind}_C^2(r) \chi(r)$ and therefore deserves a separate notation.

2.1.A. Theorem. Let $P$ be an I-divide (i.e. a divide with only one component that is non-closed). Then the Casson invariant $v_2$ of the knot $L(P)$ is given by
\[
v_2(L(P)) = \sum_{v \in \mathcal{V}_P} (\tilde{J}(O_v) + \frac{1}{4} \#(O_v \cap I_v)) + \frac{J^+(\overline{P}) + 2 \text{St}(\overline{P})}{4}, \tag{2.1}
\]
where $\mathcal{V}_P$ is the set of all double points of $P$ and $\#(O_v \cap I_v)$ is the number of intersection points of $O_v$ with $I_v$.

Remark. $J^+(\overline{P})$ and $\text{St}(\overline{P})$ obviously depend on the orientation of $P$. But the sum $J^+(\overline{P}) + 2 \text{St}(\overline{P})$ does not, since all the other ingredients of (2.1) are independent of the orientation. This fact can also be verified directly.

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1Notation rationale: letters $O$ and $I$ were chosen because they conveniently represent the topological type of the corresponding curves.
2.2. Special cases. It is well known [5] that for a tree-like generic plane curve $C$ the sum $J^+(C) + 2\text{St}(C)$ is always 0. Here a curve $C$ is called tree-like if the smoothing at every double point of $C$ produces two disjoint curves. One can define a tree-like divide in a similar fashion. It follows from the definition that $\#(O_v \cap I_v) = 0$ for every double point $v$ of a tree-like divide $P$. Hence the formula (2.1) can be simplified.

2.2.A. Let $P$ be a tree-like divide. Then

$$v_2(L(P)) = \sum_{v \in \mathcal{V}_P} \tilde{J}(O_v). \quad (2.2)$$

A slalom divide (see [4] for definitions) is a special case of tree-like divides. In this case the numbers $\tilde{J}(O_v)$ can be calculated directly.

2.2.B. Let $P$ be a slalom divide. Then

$$v_2(L(P)) = \sum_{v \in \mathcal{V}_P} (1 + n(O_v)), \quad (2.3)$$

where $n(O_v)$ is the number of the double points of $O_v$. Since this number is always non-negative, the Casson invariant of a slalom knot is always strictly positive.

Remark. Formula 2.1 admits an easy rewrite in terms of Gauss diagrams of divides (see [11] for definitions and examples of the technique). In particular, the term $\sum_{v \in \mathcal{V}_P} \frac{1}{4}(O_v \cap I_v)$ is half the number of generic intersections of chords in the Gauss diagram of a divide $P$.

3. Proof of the Theorem 2.1.A

3.1. Normalization. Denote the right-hand side of (2.1) by $\mathcal{X}(P)$. It is easy to check that the desired equality $v_2(L(P)) = \mathcal{X}(P)$ holds true for the standard divides $D_0, D_1, \ldots$. Here a standard divide $D_k$ looks like a standard curve $K_{k+1}$ cut at the external edge and appropriately immersed into the unit disc. Indeed, $v_2(L(D_0)) = 0 = \mathcal{X}(D_0)$ and $v_2(L(D_1)) = 1 = \mathcal{X}(D_1)$, since $L(D_1)$ is a trefoil knot (see Figure 2). Furthermore, $L(D_n)$ is the connected sum of $n$ copies of $L(D_1)$ for $n > 1$ and, hence, $v_2(L(D_n)) = v_2(\#nL(D_1)) = nv_2(L(D_1)) = n = n\tilde{J}(K_1) = \sum_{v \in \mathcal{V}_{D_n}} \tilde{J}(O_v) = \mathcal{X}(D_n)$.

3.2. Invariance under perestroikas. Since one can transform any divide into a standard one with a finite sequence of Arnold’s perestroikas, it is enough to check that $v_2(L(P))$ and $\mathcal{X}(P)$ change in the same way under the perestroikas. It is proved in Lemmas which follow.

3.2.A. LEMMA. Let $P'$ be the result of a positive inverse self-tangency perestroika of $P$. Assign $\Delta \mathcal{X} = \mathcal{X}(P') - \mathcal{X}(P)$ and $\Delta v_2 = v_2(L(P')) - v_2(L(P))$. Then $\Delta \mathcal{X} = \Delta v_2$.

PROOF. Recall that divides under consideration are additionally equipped with an orientation. Without a loss of generality, I assume that at every double point the two branches of $P$ are oriented upwards. One can always perturb $P$ slightly without changing its isotopy type in order to satisfy this condition, as it is shown in Figure 3. I also assume that the inverse self-tangency perestroika is happening
“horizontally”, i.e. the two new double points are created side by side like depicted in Figure 8. Denote these two points by \(a\) and \(b\).

The common part of \(P\) and \(P'\) consists of 3 non-closed curves. Two of them contain initial and final points of \(P\) (or \(P'\)) and I denote them by \(\alpha\) and \(\beta\), respectively. The third curve is denoted by \(\gamma\). Without a loss of generality, one may assume that \(\alpha\), \(\beta\), and \(\gamma\) are placed as in Figure 8. They may, of course, intersect each other and have multiple self-intersections.

Observe now that there are two crossing points of \(L(P')\) close to \(a\) and \(b\) such that changing over- and under-crossings at them allows one to pull 4 strands of \(L(P')\) placed between \(a\) and \(b\) away from each other and to obtain \(L(P)\). Those points are marked with small circles in Figure 8 and are denoted by \(\varepsilon_a\) and \(\varepsilon_b\), respectively.

More precisely, let \(K_a\) be the result of over- and under-crossings change of \(L(P')\) at \(\varepsilon_a\) and let \(K_b\) be the result of such a change of \(K_a\) at \(\varepsilon_b\). Then \(K_b\) and \(L(P)\) are ambiently isotopic and, hence, \(v_2(L(P)) = v_2(K_b)\). It is well known how the value of \(v_2\) behaves under these crossing changes. Namely, the following skein relation holds true:

\[
v_2 (L(P')) - v_2 (K_a) = \text{lk}(L_a', L_a''),
\]

where \(\text{lk}\) is the linking number of a two-component link.

Let now \(L_a\) be the result of smoothing of \(L(P')\) at \(\varepsilon_a\) with respect to the orientation and let \(L_b\) be the result of smoothing of \(K_a\) at \(\varepsilon_b\). Both \(L_a\) and \(L_b\) are links with two components \(L'_a, L''_a\) and \(L'_b, L''_b\), respectively. Then \([1]\) implies that \(v_2(L(P')) - v_2(K_a) = \text{lk}(L_a', L''_a)\) and \(v_2(K_a) - v_2(K_b) = \text{lk}(L'_a, L''_a)\).

Thus one concludes that \(\Delta v_2 = \text{lk}(L_a', L''_a) + \text{lk}(L'_b, L''_b)\). Let us compute the first linking number. Denote the diagrams of \(L'_a\) and \(L''_a\) by \(D'_a\) and \(D''_a\), respectively. Then the linking number in question is half the sum of signs of crossings from \(D'_a \cap D''_a\). Here the sign of a crossing is +1 or −1 if it looks like the first or second summand in \([1]\), respectively. In Figure 8 \(D''_a\) is depicted with a dotted line.
\[ \alpha \beta \beta \alpha \alpha \gamma \gamma \alpha \beta \gamma \beta \gamma \gamma \]

**Figure 9.** Contribution of crossing points to \(2\text{lk}(L'_a, L''_a)\)

\[ \alpha \beta \gamma \gamma \alpha \beta \gamma \beta \gamma \gamma \]

**Figure 10.** Contribution of “jumps through infinity” to \(2\text{lk}(L'_a, L''_a)\)

\(D'_a\) and \(D''_a\) may intersect each other either near a double point of the divide or where a branch of \(D(P')\) jumps through infinity. Self-intersections of \(\alpha\) and \(\beta\) do not contribute to twice the linking number, since the corresponding pieces of \(D(P)\) belong entirely to \(D''_a\) and \(D'_a\), respectively. Contribution of other double points is shown in Figure 9. It is 2 for every intersection point of \(\beta\) with \(\alpha\) and \(\gamma\) and every self-intersection of \(\gamma\). The contribution is 0 for every intersection of \(\alpha\) with \(\gamma\). Points \(a\) and \(b\) both contribute 2. Hence the part of \(lk(L'_a, L''_a)\) coming from crossings located near self-intersections of \(P'\) is \(#(\alpha \cap \beta) + #(\beta \cap \gamma) + n(\gamma) + 2\).

Let \(p\) be a point of local minimum or maximum on \(P'\) with respect to the projection onto the \(x\)-axis. Denote by \(F_p\) the corresponding piece of \(D(P')\) that makes a jump through infinity. One may assume that all these pieces are pairwise disjoint and intersect \(P'\) transversely at finitely many points. Intersections of \(F_p\) with branches of \(D(P')\) corresponding to \(\alpha\) and \(\beta\) do not contribute to twice the linking number (see the two leftmost pictures in Figure 10). Contribution from intersections with \(\gamma\) is either 2 if \(F_p\) belongs to \(D'_a\) \((D''_a)\) and the intersected branch of \(\gamma\) is oriented upwards (downwards) or \(-2\) otherwise (see Figure 10). Moreover, one should add 2 if \(p \in \gamma\) and is a local maximum. All in all, the contribution of \(F_p\) to twice the linking number is summarized in the following table.
where $\bar{\gamma}$ is the natural closure of $\gamma$ and $\mathcal{C}$ and $\bar{\mathcal{C}}$ denote the sets of all local minima and maxima on the curve $C$, respectively (with an appropriate orientation, if necessary). Recall that $\text{ind}_\gamma(p)$ is half-integer for $p \in \gamma$. Hence the contribution of $F_p$ is always even, as it should be.

It is obvious that $\#(\tilde{\gamma}) - \#(\bar{\gamma}) = 0$, since $\bar{\gamma}$ is closed. The difference between $\gamma$ and $\bar{\gamma}$ is a local maximum near the point $b$. Therefore $\#(\tilde{\gamma}) - \#(\bar{\gamma}) = -1$. Finally

$$\text{lk}(L_a', L_a'') = \sum_{p \in \tilde{\gamma} \cup \bar{\gamma}} \text{ind}_\gamma(p) - \sum_{p \in \beta \cup \bar{\gamma}} \text{ind}_\gamma(p) + \sum_{p \in \tilde{\gamma} \cup \bar{\gamma}} \text{ind}_\gamma(p) - \sum_{p \in \beta \cup \bar{\gamma}} \text{ind}_\gamma(p)$$

$$+ \#(\alpha \cap \beta) + \#(\beta \cap \gamma) + n(\gamma) + 3/2.$$  

Computation of $\text{lk}(L_a', L_a'')$ is almost the same. One should only exchange the roles of $\alpha$ and $\beta$ everywhere and to take into account that $a$ does not contribute to the linking number anymore. The contribution of $b$ remains 2. Therefore

$$\Delta v_2 = 2 \sum_{p \in \tilde{\gamma} \cup \bar{\gamma}} \text{ind}_\gamma(p) - 2 \sum_{p \in \tilde{\gamma} \cup \bar{\gamma}} \text{ind}_\gamma(p)$$

$$+ 2\#(\alpha \cap \beta) + \#(\alpha \cap \gamma) + \#(\beta \cap \gamma) + 2n(\gamma) + 2. \quad (3.2)$$

It is simpler to compute $\Delta \mathcal{X}$. First of all, $J^+(\bar{P}) = J^+(\bar{P})$ and $\text{St}(\bar{P}) = \text{St}(\bar{P})$, since neither $J^+$ nor $\text{St}$ changes under an inverse self-tangency perestroika. Let $v$ be a double point of $P$. If $v$ is a self-intersection point of either $\alpha$, $\beta$ or $\gamma$, then neither $\bar{J}(O_v)$ nor $\#(O_v \cap I_a)$ changes. If $v \in \alpha \cap \gamma$ or $v \in \beta \cap \gamma$, then $\bar{J}(O_v)$ does not change, but $\#(O_v \cap I_a)$ increases by 2 under the perestroika. Finally, if $v \in \alpha \cap \beta$, then $\#(O_v \cap I_a)$ does not change, but $O_v$ experiences a positive inverse self-tangency perestroika and, hence, $\bar{J}(O_v)$ increases by 2. Moreover, $\#(O_v \cap I_a) = \#(O_v \cap I_b) = \#(\alpha \cap \gamma) + \#(\beta \cap \gamma)$. Combining these facts together one can conclude that

$$\Delta \mathcal{X} = 2\#(\alpha \cap \beta) + \#(\alpha \cap \gamma) + \#(\beta \cap \gamma) + \bar{J}(O_a) + \bar{J}(O_b). \quad (3.3)$$

Subtracting (3.3) from (3.2) one gets

$$\Delta v_2 - \Delta \mathcal{X} = 2 \sum_{p \in \tilde{\gamma} \cup \bar{\gamma}} \text{ind}_\gamma(p) - 2 \sum_{p \in \tilde{\gamma} \cup \bar{\gamma}} \text{ind}_\gamma(p) + 2n(\gamma) + 2 - \bar{J}(O_a) - \bar{J}(O_b).$$
Figure 11. Positive direct self-tangency perestroika of a divide

It easily follows from Lemma 3.2.B below that this expression is always zero. Indeed, \( \check{J}(O_a) = \check{J}(\gamma) + 1 - (\text{ind}_\gamma(a) - 1/2) \) and \( \check{J}(O_b) = \check{J}(\gamma) + \text{ind}_\gamma(b) + 1/2 \) with \( \text{ind}_\gamma(a) = \text{ind}_\gamma(b) \), where \( \check{J}(\gamma) \) denotes

\[
\sum_{p \in \gamma \cup \gamma} \text{ind}_\gamma(p) - \sum_{p \in \gamma \cup \gamma} \text{ind}_\gamma(p) + n(\gamma).
\]

\[\Box\]

3.2.B. Lemma. Let \( C \) be an oriented closed plane curve such that

- at every double point the two branches of \( C \) are parallel to the main diagonals \( y = \pm x \) of the plane;
- \( C \) has only finitely many points where the tangent vector is parallel to the \( y \)-axis, and at every such a point the projection of \( C \) onto the \( x \)-axis has either a local minimum or a local maximum.

Then \( \hat{J}(C) = \#(\mathcal{X}) + \#(\mathcal{X}') + \sum_{p \in \gamma \cup \gamma} \text{ind}_\gamma(p) - \sum_{p \in \gamma \cup \gamma} \text{ind}_\gamma(p) \).

The proof is elementary and is left to the reader. Hint: smoothing at a double point of \( C \) with respect to the orientation does not change the expression above.

3.2.C. Lemma. Let \( P' \) be the result of a positive direct self-tangency perestroika of \( P \). Assign \( \Delta \mathcal{X} = \mathcal{X}(P') - \mathcal{X}(P) \) and \( \Delta v_2 = v_2(L(P')) - v_2(L(P)) \). Then \( \Delta \mathcal{X} = \Delta v_2 \).

The proof is similar to the one of Lemma 3.2.A. The corresponding picture of a divide and its link close to the place of perestroika is shown in Figure 11. In this case

\[
\Delta v_2 = 2 \sum_{p \in \gamma \cup \gamma} \text{ind}_\gamma(p) - 2 \sum_{p \in \gamma \cup \gamma} \text{ind}_\gamma(p) + \#(\alpha \cap \gamma) + \#(\beta \cap \gamma) + 2n(\gamma) + 1
\]

\[= \#(\alpha \cap \gamma) + \#(\beta \cap \gamma) + \check{J}(O_a) + \check{J}(O_b) + 1 = \Delta \mathcal{X} \tag{3.4}\]

Turn attention to triple point perestroikas now. It follows from the definition that if a divide \( P \) experiences a triple point perestroika, then the ambient isotopy type of the corresponding link does not change.

3.2.D. Lemma. Let \( P' \) be the result of a positive triple point perestroika of \( P \). Assign \( \Delta \mathcal{X} = \mathcal{X}(P') - \mathcal{X}(P) \). Then \( \Delta \mathcal{X} = 0 \).
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Figure 12. Different types of triple points

Figure 13. Positive triple point perestroika of type $A_+$

Proof. One can distinguish two kinds of triple points, depending on whether
the three branches of a curve (or a divide) at the point are directed into the same
half-plane or not. A triple point is said to be of type $B$ in the former case and
of type $A$ in the latter one (see Figure 12). Accordingly, there are two kinds of
triple point perestroikas. It is easy to check that a triple point perestroika of type
$B$ is equivalent to a sequence of a type $A$ triple point perestroika and several self-
tangency perestroikas. It is therefore enough to consider only type $A$ perestroikas
in the proof.

Triple points of type $A$ can further be classified as being of type $A_+$ or $A_-$,
depending of whether the cyclic order of the branches defines a positive or negative
orientation of the plane (see Figure 12). Without a loss of generality, I will restrict
my consideration to perestroikas of type $A_+$ only. The corresponding picture of
divides before and after a perestroika is shown in Figure 13.

It is now straightforward to see that $\text{St}(P') = \text{St}(P) + 1$, $J(O_{a'}) = J(O_a) - 2$,
$J(O_{b'}) = J(O_b)$, $J(O_{c'}) = J(O_c)$, and $\#(O_v \cap I_v') = \#(O_v \cap I_v) + 2$ for $v \in \{a, b, c\}$. Hence $\Delta x' = 1/2 - 2 + 6/4 = 0$. 

3.3. Applications to Chmutov’s $J_2^\pm$ invariant. $J_2^\pm$ is a second order $J^\pm$-type
invariant of long curves (or I-divides). Chmutov [8] defined it by explicitly specifying
its actuality table (i.e. values of the invariant on all the chord diagrams with two
chords) and its values on standard divides with at most one self-tangency point.
He also proved that $v_2(L(P)) = J_2^\pm(P)$ for any I-divide $P$. The definition of $J_2^\pm$
did not allow one to integrate this invariant, i.e. to compute its values on chord
diagrams with one chord. Since these values are nothing more than changes of $J_2^\pm$
under self-tangency perestroikas, formulas (3.3) and (3.4) provide an answer to this
question.
3.3.A. Corollary. Let $P$ be an I-divide and $P'$ be the result of a positive inverse self-tangency perestroika of $P$. Let $\alpha$, $\beta$, $a$, and $b$ be as in the proof of Lemma 3.2.A (see Figure 3). Then
\[
J^\pm_2(P') - J^\pm_2(P) = 2\tilde{J}(O_a) + 2\text{ind}_{O_a}(b) + \#(O_a \cap I_a) + 2\#(\alpha \cap \beta) - 1 \\
= 2\tilde{J}(O_b) - 2\text{ind}_{O_b}(a) + \#(O_b \cap I_b) + 2\#(\alpha \cap \beta) + 1.
\] (3.5)

The proof follows from (3.3) and the facts that \#($O_a \cap I_a$) = \#($O_b \cap I_b$) = \#($\alpha \cap \gamma$) + \#($\beta \cap \gamma$), $\tilde{J}(O_a) = \tilde{J}(O_b) - 2\text{ind}_{O_a}(b) + 1$ and $\text{ind}_{O_a}(b) = \text{ind}_{O_b}(a)$. 

3.3.B. Corollary. Let $P$ be an I-divide and $P'$ be the result of a positive direct self-tangency perestroika of $P$. Let $a$ and $b$ be as in the proof of Lemma 3.2.C (see Figure 7). Then
\[
J^\pm_2(P') - J^\pm_2(P) = 2\tilde{J}(O_a) + \#(O_a \cap I_a) = 2\tilde{J}(O_b) + \#(O_b \cap I_b).
\] (3.6)

The proof follows from (3.4) and the facts that \#($O_a \cap I_a$) = \#($O_b \cap I_b$) = \#($\alpha \cap \gamma$) + \#($\beta \cap \gamma$) + 1 and $\tilde{J}(O_a) = \tilde{J}(O_b)$. 

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