Free oscillations of a beam with installation (1st part)

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Abstract. The transverse oscillations of a beam carrying a discrete mass in a span are studied. The mathematical model of oscillations is presented as a boundary value problem from the main partial differential equation of the fourth order hyperbolic type in the spatial coordinate, of the second order in time, the boundary conditions and the conditions for the sections’ conjugation. The technical theory of the rods bending oscillations, based on Bernoulli’s hypothesis, is used. The spectral problem of determining the eigenvalues and eigenmodes of oscillations (the Sturm-Liouville problem), necessary for analyzing the problems of forced oscillations, is considered. It is argued that the solution by analytical methods is impractical due to the large volume of transformations and cumbersome calculations. Methods of separation of variables and finite differences are used. An algorithm for solving the problem, implemented in the Matlab software environment in the form of high-precision graphic-analytical calculations, is developed. The conclusions for practical applications of the results are made.

1. Introduction
Beams oscillations have currently an extensive bibliography, since they are the widespread elements of buildings and structures and are used in installation in almost all the economy sectors. In recent years, the widespread use of robots, automated factories and workshops has led to the fact that beams experience non-traditional, non-classical technogenic external influences of a dynamic and kinematic nature. To date, transverse and longitudinal oscillations of beams and, in general, rods have an extensive bibliography [1-6]. But the scientific and technological revolution of recent decades as well as the requirements of the digital economy in recent years require more and more accurate calculations of structures, including beams.

Seismic and technogenic loads are special and very dangerous for beams, due to their random nature, which is very much tied to uncertainties at the earthquakes’ occurrence place and the multiparametric intensity of their flow. Studies show that the dynamic behavior of structures depends significantly on the beams and actions’ parameters, which form a large area in the multidimensional parameters space. Due to the limited volume of the given publication, in this case there is no way to move on to such problems. But the authors intend, having laid the foundation for this development, to continue the research, devoting it to the identified problems.

This work is devoted to the free oscillations of the beams carrying installation that is a source of dynamic loads. Its value lies in the fact that in such a spectral problem such beam parameters, which are necessary for the study of forced oscillations, such as frequencies and modes, will be obtained.

[Note: The content of the document is a summary and does not include the full text. Additional details can be found in the original document.]
2. Mathematical model of oscillations

The meaningful statement of the problem is based on the design diagram of the beam, carrying a discrete mass and shown in Figure 1. The beam is acted upon in the transverse direction by static forces from its own distributed mass and discrete mass.

In the transition to a mathematical model of free oscillations, we will use the technical theory of bending rods oscillations. Then the main equation has the form of a homogeneous differential equation of hyperbolic type in partial derivatives [2].

\[ bu''''(x, t) + m\ddot{u}(x, t) + em\dddot{u}(x, t) = 0, \quad b = EJ, \quad x \in (0, l), \quad t > -\infty. \]  

(1)

**Figure 1.** Design scheme of the beam

**Figure 2.** Sections conjugation

Here \( u(x, t) \) - is the desired function of beam deflections during motion, \( b = EJ \) - is the bending stiffness of the beam, \( E \) is an elastic modulus of the beam material, \( J \) is an axial moment of the cross section inertia, \( m \) - is the linear beam mass, \( \varepsilon \) - is a specific coefficient of linear viscous internal friction. The dashes in the superscript denote the deflection function differentiation along the coordinate \( x \), dots over the symbols denote time differentiation \( t \). Static beam deflections will not be considered; they are calculated using the trivial methods of structural mechanics. In this case, it will be assumed that the oscillations occur around the equilibrium position.

Note that the presence of a discrete mass \( M \) will lead to the breakup of the second and third derivatives of the function \( u(x, t) \) at the point of its location, as a result of which it is necessary to include the conditions for conjugation of sections to the left and right of it in the mathematical model of oscillations. Let us consider the situation near the mass \( M \) (Figure 2). It should be borne in mind that \( M \) is a material point with zero dimensions, so the left and right edges of the element will be considered drawn through one point.

The main equation (1) is joined by the boundary conditions corresponding to Figure 1 and the conditions for conjugating the sections to the left and right of the discrete mass. The boundary conditions for the hinged ends have the form:

\[ u(0, t) = 0, \quad u'(0, t) = 0, \quad u(l, t) = 0, \quad u''(l, t) = 0. \quad t > -\infty. \]  

(2)

Some conditions for sections conjugation are kinematic and are that the function \( u(x, t) \) and its first and second derivatives should be smooth at the points where the discrete masses are located \( x_k \):

\[ u(l/2 - 0, t) = u(l/2 + 0, t), \quad u'(l/2 - 0, t) = u'(l/2 + 0, t). \]  

(3)

The need to fulfill the conditions (3) is obvious from the continuity and structure considerations. Other conditions are dynamic and arise due to the presence of force factors at the adjacent sections’ conjugation (Figure 2): bending moments on the left and right \( M_l, M_r \), lateral forces \( Q_l, Q_r \), d’Alembert force of inertia \( D \). The cross-sections on the left and right at this point rotate the same way. Figure 2
also implies that $M_l = M_r$. Equality of bending moments in the sections on the left and on the right corresponds to the function smoothness with respect to the second derivatives

$$bu''(l/2-0, t) = bu''(l/2+0, t).$$ (4)

Transverse forces $Q_l$, $Q_r$ and d’Alembert force of inertia $D$ act in a vertical direction and create a translational movement of the installation. The d’Alembert force of inertia is

$$D = Q_l - Q_r$$ (5)

and is expressed by the time derivative

$$D = M_\ddot{u}(l/2, t).$$

From the above-said it follows that the equation (5) should be included in the conjugation conditions, but at the same time there is no such need for (3) and (4), since the smoothness and continuity of the function are ensured by them. Let us turn to expressions for the shear forces in terms of the deflection function and get

$$M_\ddot{u}(l/2, t) = b[u''(l/2 - 0, t) - u''(l/2 + 0, t)].$$ (6)

As it is possible to see, the smoothness of the function $u(x, t)$ on the third spatial derivative is not provided, and this should be taken into account in the forthcoming calculations. From the point of view of structural mechanics, this means that the transverse forces have jumps by the value of the d’Alembert force.

Equation (1), boundary conditions (2) and conjugation condition (6) form a mathematical model that makes it possible to study free, forced deterministic and stochastic oscillations of a beam carrying installation with a source of impact.

3. Free oscillations

The problem of free oscillations is mandatory before considering forced oscillations, since without such its results as natural frequencies and forms, a consistent consideration of forced oscillations is impossible. Therefore, we move on to setting and solving the problem.

Note that free oscillations are described by homogeneous basic equations and additional conditions and do not require initial conditions in time. We use the variables separation method [7] and represent the deflection function of the longitudinal axis of the beam in the form of a product

$$u(x, t) = X(x)e^{\lambda t},$$ (7)

where $X(x)$ – is the desired eigenform function, $\lambda = -\epsilon + j\omega$ – is a characteristic indicator, $\epsilon, \omega$ – are damping coefficient and angular frequency of free oscillations, $j$ – is an imaginary unit. This representation is based on the fact that it should describe the process of damped oscillations. Coefficient $\epsilon$ will be determined by bibliographic sources, frequencies $\omega$ arbitrary numbers in the class of real numbers $\mathbb{R}$. Substituting (7) in (1), (2), (6), we can reduce the result by a common factor and obtain a system of homogeneous linear algebraic equations

$$B(\lambda)X = 0,$$ (8)

where $B(\lambda)$ - is a square matrix of coefficients that contain the desired frequency $\omega$. Their determination by analytical methods is associated with cumbersome calculations and the need to solve a system of transcendental equations. A more versatile and simpler way to identify a spectral pair $(\omega, X)$ consists in using the numerical methods [8] and computer calculating systems. Let us apply the finite difference method (FDM) and software package Matlab. FDM requires a transition from the domain of a continuous argument $0 \leq x \leq l$ to the domain of a discrete argument (mesh)
\[ L_h = \{ x_i = (i - 1)h, \quad i = 1, 2, \ldots, n \} \]
with the step \( h = l / (n - 1) \). After applying the method of variables separation, we get
\[
b X''''(x) + AX(x) = 0, \quad A = m^2(\lambda + \varepsilon). \tag{9}
\]
\[
X''(0) = 0, \quad b X'''(0) - c X(0) = 0, \quad X''(l) = 0, \quad b X'''(l) + c X(l) = 0. \tag{10}
\]
\[
b X''(x_k - 0) - X''(x_k + 0) - M_{i} \lambda^2 X_i = 0, \quad i = 1, 2. \tag{11}
\]

Hereinafter \( x_i = \ell/2 \) is the coordinate of the beam sections’ conjugation point. It is necessary to note that the conjugation conditions (3), (4) are not indicated because of the smoothness of the function itself \( X(x) \) and its first and second derivatives on the boundary of the sections.

We will perform the necessary procedures FDM by replacing the derivatives with their finite-difference analogs with accuracy \( O(h^2) \) and instead of the main equation we obtain (1)
\[
X_{i-2} - 4X_{i-1} + \mu X_i - 4X_{i+1} + X_{i+2} = 0, \quad \mu = 6 + \Delta h^2/\beta. \tag{12}
\]
We carry out the similar actions for the boundary and conjugation conditions.

Left end:
\[
X_i = 0, \quad 2X_i - 5X_2 + 4X_3 - X_4 = 0, \tag{13}
\]
Right end:
\[
- X_{n-3} + 4X_{n-2} - 5X_{n-1} + 2X_n = 0, \quad X_n = 0. \tag{14}
\]
Conjugation condition at the point \( x_i \):
\[
3X_{k-4} - 14X_{k-3} + 24X_{k-2} - 18X_{k-1} + \eta X_k - 18X_{k+1} + 24X_{k+2} - 14X_{k+3} - 14X_{k+4} = 0, \quad \eta = 10 - m \lambda^2/\beta. \tag{15}
\]

The mathematical model of free oscillations is now represented by a system of algebraic equations (12) - (16), which we write in a matrix-vector form
\[
A(\lambda)Y = 0. \tag{16}
\]
where \( A(\lambda) \) is a square coefficient matrix of the \( n \) th order, \( Y \) is a vector of discrete arguments replacing an eigenfunction \( X(x) \). It should be borne in mind that instead of exact values \( X \) approximate values \( Y = \{y_1, y_2, \ldots, y_n\} \), \( y_i = X(x_i) \) will be determined, \( i \) is the finite difference mesh node number. The coefficient matrix has the form
\[
A(\lambda) = \begin{pmatrix}
1 & -4 & \mu & -4 & 1 & \cdots & \\
2 & 4 & -1 & \cdots & & \cdots & \\
1 & \mu & -4 & 1 & \cdots & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
1 & -4 & \mu & -4 & 1 & \cdots & \\
3 & -14 & 24 & -18 & \eta & -18 & 24 & -14 & 3 & \cdots & \\
1 & \mu & -4 & 1 & \cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
1 & -4 & \mu & -4 & 1 & \cdots & \\
-1 & 4 & -5 & 2 & \cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\end{pmatrix}.
\]

Zero elements are not written here. Elements of the main diagonal \( \eta \) contain eigenvalues, which can be determined from the condition for the nonzero solutions existence to the system of equations (16), which is in the fact that
\[
\det A(\lambda) = 0. \tag{17}
\]

This equation is a high-order algebraic equation, and for the large \( n \) values its compilation and solution [9] presents significant difficulties. For the cases like this spectral problem, the problem of finding the eigenvalues and eigenvectors can be significantly simplified. There are two reasons for this:

\[\text{(18)}\]
1. There is no need to solve the complete eigenvalue problem. The practical need in the overwhelming majority of cases is to use only a few of the smallest eigenvalues and corresponding vectors for the beams, i.e., to solve the partial eigenvalue problem.

2. The used numerical finite difference method in conjunction with the Matlab computer complex and the ability to build a high-precision graph of the characteristic equation with subsequent implementation on a computer screen makes it possible to set the eigenvalues in very simple ways. This method of solving the problem of eigenvalues and vectors was used by the authors in a number of problems and shows a high degree of accuracy in solving the problems of rods stability and oscillations [3].

The applied algorithm consists of the following sequential actions:

1. The square matrix is compiled according to the numerical method $A(\omega)$ n order, the elements of the main diagonal of which contain natural frequency.

2. The function graph $\det A(\omega)$ is built and displayed on the monitor screen.

3. By the points $\det A(\omega) = 0$ the eigenvalues are read from the screen $\omega_k$, where $k = 1, 2 \ldots$ in the required quantities.

4. At the zero points of the determinant, the rank of the matrix $A$ is decreased to the value $R = n - 1$. Therefore, one of the nonzero components of the vector is necessary to take an arbitrary number like $Y_k = 1$.

5. The column containing $Y_k$, is transferred with a change of signs to opposite to the right side of equation (16) and it becomes inhomogeneous.

6. To make a matrix $A$ square again, any of its lines is deleted and thus the equation (16) becomes square of the $n-1$ order.

$$AY = d. \quad (17)$$

7. The vector $Y$ of the $n-1$ dimensions is determined from (17). It is supplemented by the newly appointed $Y_k = 1$ and the desired order $n$ eigenvector is obtained.

Example 1. Initial data: The beam consists of two steel I-beams No.30, $l = 8$ m, $E = 210$ GPa, $J = 14160$ cm$^4$, $m = 73$ kg/m, $M = 4000$ kg, $\varepsilon = 0,25$ s$^{-1}$, $n = 1001$, spatial step $h = l/(n-1)$, frequency step $h_\omega = 0,25$ s$^{-1}$.

The result of the solution is displayed on the monitor screen (Fig. 3) as a graph $\omega - \det$, where the dot marks the first and second natural frequencies. The values of two elements of the natural frequencies’ spectrum are determined by reading from the monitor screen

$$\omega_k = \{25.51 \quad 393.69\} \text{ s}^{-1}.$$

These numbers are highly accurate, since the monitor screen makes it possible to increase the picture size and view the fragments and neighborhoods of points in a very enlarged form.

The use of regulatory documents [10, 11] shows that the first frequency of free oscillations corresponds to the highest dynamic factor under seismic impacts on the beam $\beta = 2.5$ (Fig. 4) for the structures on soils of all categories. From the point of view of this beam reliability, it is advisable to reduce it by changing the oscillatory system parameters. A qualified solution to such a problem should be based on methods of dynamic oscillation damping, methods of dynamic mechanical systems’ optimal design. The arsenal of techniques and special means for such cases is very extensive and the choice depends significantly on the specific circumstances.

Oscillations in and around the second natural frequency are high-frequency with 63 revolutions per second and a dynamic coefficient equal to one. They are not dangerous for the beam, since they do not fall into the dangerous zone for seismic oscillations. The third natural frequency was not determined due to its very significant value, which practically does not occur for installation.
1. I, II category soils, 2. I, II category soils.

Figure 3. Free oscillations frequencies

Figure 4. The dynamic coefficient dependence on oscillation frequencies

The next task is to find the eigenvectors $Y_k (k = 1, 2, 3... n)$ matrices $A$. Those that correspond to the frequencies of free oscillations found in the example are of interest. They can be determined from the system of equations (16) with known eigenvalues $\omega_k$. In this case, the determinant of the matrix $A$ becomes zero, which means that its rank decreases and is equal to $R = n-1$. Hence it follows that the components of the vector $Y$ can be calculated only up to a factor. In this case, one of the nonzero components of the vector $Y$ can be taken equal to an arbitrary number and excluded from the required unknowns. Then the system of equations (16) turns from homogeneous into inhomogeneous, and the remaining components of the vector $Y$ are determined from it.

The resulting eigenforms, normalized to unity and corresponding to eigenvalues, are shown in the graphs in Figure 5.

Figure 5. Free oscillation shapes

The first eigenform almost coincides with the half-wave of the sinusoid. Similar half-waves are contained in the second form. Moreover, they correspond to the boundary conditions and conjugation conditions: the ends are hinged supported, therefore, have zero deviations; the heavy discrete mass $M$ remains stationary during oscillations in the second form. Based on these features, it can be assumed that the results obtained for determining the frequencies and eigenforms are reliable.

4. Summary
1. Mathematical models to study free oscillations of a continuous-discrete beam structure have been created.
2. A high-precision graphical-analytical method for solving the spectral problem for beams has been proposed.
3. An effective numerical method for determining natural frequencies and forms of free oscillations has been developed.
4. It has been proposed to remove the first natural frequency from the large dynamic coefficients zone according to Building Codes.

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