Comparison of various higher order shear deformation theories for static and modal analysis of composite beam

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\textbf{Abstract:} This study deals with comparison of various higher order shear deformation theories including exponential shear deformation beam theory (ESDBT), trigonometric shear deformation beam theory (TSDBT) and hyperbolic shear deformation beam theory (HSDBT) for static and modal analysis of composite beam. Static analysis of generally laminated composite beam with simply supported edges and uniformly distributed load has been performed and solution has been obtained to the governing differential equations derived using Hamilton’s principal. The results are obtained by developing MATLAB codes for various theories. These results for test beam are compared with third order theory (TOT). The effect of span to thickness ratio on the accuracy is studied. The exponential shear deformation beam theory (ESDBT) yields more accurate results than other theories for both static and modal analysis of the beam.

1 \textbf{Introduction}

The analysis of composite beams is, of course, significantly more difficult than that of metallic counterparts. Many investigations have been devoted to the static and dynamic analyses of those structures and various theories and models have been proposed for the modeling of laminated composite beams. The classical laminated beam theory, based on Euler–Bernoulli hypothesis, is inaccurate for reasonably thick laminated beams and/or for highly anisotropic composite beams. The inaccuracy is due to neglecting the transverse shear and normal strains in the laminate. Transverse shear deformation is a major issue in the analysis of composite beams. A lot of work has been done to estimate it correctly and efficiently for design purposes. In order to account for the effect of low ratio of transverse shear modulus to the in-plane modulus, the first-order shear deformation theory (FSDT) in which the transverse shear strain is assumed to be constant in the beam depth direction has been developed with the assumption that the displacement in the beam thickness direction does not restrict cross-section to remain perpendicular to the deformed centroidal line. For design of composite and sandwich beams, accurate knowledge of deflection and stresses under static and dynamic loads, natural frequencies and buckling loads are required.

N.J.Pagano [1] studied the limitations of classical laminated plate theory by obtaining exact solutions for composite laminates under symmetrical bending and comparing solutions of several specific boundary value problems in this theory to the corresponding theory of elasticity solutions. Reddy J.N, Wang C.M and Lee K.H [2] developed the exact relationships between the deflections, slopes or
rotations, shear forces and bending moments. The developed relationships were capable of obtaining solutions of third-order beam theory from any known Euler-Bernouli or Timoshenko beam theory solutions of beams with any set of boundary conditions and transverse loads. Chandrashekhara [3] did the modeling of laminated beams by a systematic reduction of the constitutive relations of the three-dimensional anisotropic body. The basic equations of the beam theory here are those of the parabolic shear deformation theory. Soldatos and Elishakoff [4] developed a transverse shear and normal deformable orthotropic beam theory. Cho M and Parmerter R R [5] developed an efficient higher-order plate theory for laminated composites. The theory for symmetric laminated composites is obtained by superposing a cubic varying displacement field on a zig-zag linearly varying displacement. Karama et al. [6] developed a new multi-layer laminated composite structure model to predict the mechanical behavior of multi-layered laminated composite structures. It was observed that this new model is more precise than older ones as compared to the results by the finite element method (ABAQUS). Tahani [7] analyzed of laminated composite beams using layerwise displacement theories developed two laminated beam theories for beams with general lamination within the displacement field of a layerwise theory. In the first theory, an existing layerwise laminated plate theory is adapted to laminated beams. Kapuria, Dumir[8] Assessed the zigzag theory for static loading, buckling, free and forced response of composite and sandwich beams. Li, Hu [9] carried out the dynamic stiffness analysis of laminated composite beams using trigonometric shear deformation theory. Li, Zhen [10] Compared various shear deformation theories for free vibration of composite beams with general layups using spectral finite element method and natural frequencies were computed using witttrick Williams algorithm.

2 Modeling of beam with various higher order theories
Consider a composite beam (Figure 1) of width $b$, thickness $h$ and length $a$, made of $L$ perfectly bonded orthotropic layers with longitudinal axis $x$. It is subjected to transverse load on the bottom and the top surfaces, with no variation along the width $b$. The axis along the width is $y$.

![Figure 1- Geometry of the composite beam.](image)

For a beam of uniform cross-section, the mid-plane of the beam is chosen as the $xy$-plane. For the analysis of non-uniform beams using the finite element method, the $xy$-plane would not be the mid-plane at every cross-section, but a mid-plane for most of the length of the beam. Hence for the ease of formulation of the finite element method based on this theory, let the planes $z = z_0$ and $z = z_L$ be the bottom and the top surfaces of the beam. For a uniform beam, $z_0 = -h/2$ and $z_L = h/2$. The $z$-coordinate of the bottom surface of the $k^{th}$ layer (numbered from the bottom) is denoted as $z_{k-1}$ and its material symmetry direction 1 is at an angle $\theta_k$ to the $x$-axis. The reference plane $z = 0$ either passes through or is the bottom surface of the $k^{th}$ layer.

For a beam with a small width, the assumptions for mathematical simplification of 1D model used in the present theory are: assume plane state of stress($\sigma_y = \tau_{yz} = \tau_{xy} = 0$), neglect transverse normal
stress \((\sigma_z \approx 0)\) and assume the axial and transverse displacements \(u, w\) are independent of \(y\). The strain-displacement relations for the directions \(x, z\) are
\[
\varepsilon_x = u_x, \quad \varepsilon_z = w_x, \quad \gamma_{zx} = u_x + w_x, \quad (1)
\]

Where subscript comma denotes differentiation with respect to subscripted variable. With these assumptions, the general 3D linear constitutive equations for the stresses \(\sigma_x, \tau_{zx}\) reduce to
\[
\sigma_x = \tilde{Q}_{11}\varepsilon_x; \quad \tau_{zx} = \tilde{Q}_{55}\gamma_{zx}; \quad (2)
\]

Where \(\tilde{Q}_{11}, \tilde{Q}_{55}\) is expressed in terms of Young’s moduli \(Y_i\), shear moduli \(G_{ij}\), Poisson’s ratio \(\nu_{ij}\).

The deflection is approximated as
\[
w(x, z) = w_0(x); \quad (3)
\]

The general axial displacement for the beam has the following expression
\[
u = u_0(x) - zw_{0,x}(x) + Z(z)\phi_0(x) \quad (4)
\]

With \(Z(z)\) having the following expression for various higher order theories is

For TSDBT: \(Z(z) = (h/\pi)\sin(\pi z/h) \quad (5a)\)

For HSDBT: \(Z(z) = h\sinh \left( \frac{z}{h} \right) - z\cosh \left( \frac{1}{h} \right) \quad (5b)\)

For ESDBT: \(Z(z) = ze^{-2(z/h)^2} \quad (5c)\)

While for TOT the axial displacement has a variation with \(Z(z)\) having the following expression
\[
Z(z) = (z - 4z^3/3h^2) \quad (6)
\]

3 Governing equations

The equilibrium equations of the beam and boundary conditions are derived from variational principle.
\[
\int_V (\sigma_{ij}\delta\varepsilon_{ij})dV + \int_\Gamma (T_{ij}^n\delta u_i)d\Gamma = 0 \quad (7)
\]

\(V\) and \(\Gamma\) are the volume and surface area of the beam. Stress vector
\[
T_i^n = \sigma_{ij}n_j.
\]

Using the notation \(<...> = \sum_{k=1}^{L} \int_{z_{k-1}}^{z_k} (...) bdz\), the beam stress resultants \(N_x, M_x, P_x\) are defined by
\[
[N_x, M_x, P_x] = <\sigma_x[1, z, Z(z)]>, \quad Q_x = <Z_x(z)\tau_{zx}>, \quad F_z = b(p_z^1 + p_z^2)
\]

\[
I_{11}\ddot{u}_0 - I_{12}\ddot{w}_{0,x} + I_{13}\ddot{\phi}_x - N_{x,x} = 0, \quad (8a)
\]

\[-I_{21}\ddot{u}_0 + I_{22}\ddot{w}_{0,xx} - I_{11}\ddot{w}_0 - I_{23}\ddot{\phi}_{x,x} + M_{x,xx} + F_z = 0, \quad (8b)
\]

\[I_{31}\ddot{u}_0 - I_{32}\ddot{w}_{0,xx} + I_{33}\ddot{\phi}_x - P_{x,xx} + Q_x = 0, \quad (8c)
\]
the resultants defined by eq. (8) can be related to \(u_0, w_0, \psi_0\) by

\[
\begin{bmatrix}
N_x \\
M_x \\
P_x
\end{bmatrix} = \begin{bmatrix}
A_{11} & B_{11} & E_{11} \\
B_{11} & D_{11} & F_{11} \\
F_{11} & F_{11} & H_{11}
\end{bmatrix} \begin{bmatrix}
u_{0,x} \\
-w_{0,xx} \\
\phi_{0,x}
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
A_{11} \\
B_{11} \\
E_{11}
\end{bmatrix} \begin{bmatrix}
F_0^2 \\
F_0^2 \\
F_0^2
\end{bmatrix}
\]

\[Q_x = [F_{55}]_0\phi_0\] (9)

Where the elements of the beam stiffness matrix \(A\) and inertia \(I\) are defined in terms of material constants by

\[
\begin{bmatrix}
A_{11} & B_{11} & E_{11} \\
D_{11} & F_{11} \\
H_{11}
\end{bmatrix} = \begin{bmatrix}
\bar{Q}_{11}[1, z, z(z)] >, \\
\bar{Q}_{11}[z^2 zZ(z)] >, \\
\bar{Q}_{11}[Z(z)]^2 >
\end{bmatrix}
\]

\[F_{55} = \bar{Q}_{55}[Z_x(z)]^2 >\] (10a)

\[
\begin{bmatrix}
I_{11} & I_{12} & I_{13} \\
I_{22} & I_{23} \\
I_{33}
\end{bmatrix} = \begin{bmatrix}
\bar{Q}[1, z, Z(z)] >, \\
\bar{Q}[z^2 zZ(z)] >, \\
\bar{Q}[Z(z)]^2 >
\end{bmatrix}
\]

\[I_{22} = \bar{Q} >\] (10b)

The equations of motion in terms of displacement \(u_0, w_0, \psi_0\) are obtained from expression of stress resultants obtained in eq.(9) as follows where \(\vec{U}\) is displacement vector \([u_0, w_0, \psi_0]^T\) and \(\vec{P}\) is force vector. \(L\) and \(\bar{L}\) are linear differential operator matrices and \(\bar{L}^n\) is non linear differential operator matrix.

\[
\ddot{\vec{L}}\vec{U} + \bar{L}\vec{U} + \bar{L}^n\vec{U} = \vec{P}
\] (11)

To assess the accuracy of higher order theories studied here Fourier series solution with simply supported end conditions of the beam is obtained with

\[
(w_0, N_x, M_x, P_x, P_z) = \sum_{n=1}^{\infty} (w_0, N_x, M_x, P_x, P_z)_n \sin n\pi x
\]

\[
(u_0, \phi_0, Q_x) = \sum_{n=1}^{\infty} (u_0, \phi_0, Q_x)_n \cos n\pi x
\]

Substituting these into the equation of motion eq.(11) neglecting non-linear terms we get following Fourier component stiffness matrix for static analysis with \(\tilde{n} = n\pi/a\). Substituting these in eqs.(11) yields for \(n^{th}\) Fourier component,

\[
\bar{M}\ddot{\vec{U}}^{\tilde{n}} + \bar{C}\dot{\vec{U}}^{\tilde{n}} + \bar{K}\vec{U}^{\tilde{n}} = \vec{P}^{\tilde{n}}
\] (12)

The inertia and stiffness matrices \(M, K\) are symmetric and have following values for various theories

\[
K_{11} = \tilde{n}^2 A_{11}, K_{12} = -\tilde{n}^2 B_{11}, K_{13} = \tilde{n}^2 E_{11}, K_{22} = \tilde{n}^4 D_{11}, K_{23} = -\tilde{n}^3 F_{11}, K_{33} = \tilde{n}^2 H_{11} + F_{55}
\]

\[
M_{11} = I_{11}, M_{12} = -\tilde{n} I_{12}, M_{13} = I_{13}, M_{22} = \tilde{n}^2 I_{22} + \tilde{l}_{22}, M_{23} = -\tilde{n} I_{23}, M_{33} = I_{33}
\]
For static analysis the time derivative terms are equated to zero to reduce the equation for obtaining
displacement by the following equation
\[ K\ddot{U}^n = \ddot{p}^n \]
With solution of the form
\[ \ddot{U}^n = K^{-1}\ddot{p}^n \]  
(13)

Whereas for modal analysis the damping matrix and load vector are equated to zero to reduce the
eq.(12) to the following eigen value problem
Equation (12) yields for undamped vibrations at \( \omega_n^2 \), let \( \ddot{U}^n = \ddot{U}_0^n \cos \omega_n t \) the following generalized
eigen-value problem for \( \omega_n^2 \):
\[ K\ddot{U}_0^n = \omega_n^2 M\ddot{U}_0^n \]  
(14)

Where \( \ddot{U}_0^n \) are trough the thickness mode shapes for the given spatial mode \( n \).

4 Results and discussion

The beam studied is a simply supported composite beam of material consisting of four plies of equal
thickness 0.25h, it has symmetric lay-up [0/90/90/0]. The material properties are: [(Y1, Y2, Y3,G12,G23,G31), v_{12}, v_{13}, v_{23}]= [(181, 10.3, 10.3, 7.17, 2.87, 7.17) GPa, 0.25, 0.25, 0.33]. The density of material studied is 1578 kg/m^3. Static response is obtained for a uniformly distributed load \( p_0 = -1 \) (acting downward) on the top of the beam. Based on convergence studies, all results for the
given uniformly distributed load loads are reported for 23 odd terms in the Fourier series. The results
are non-dimensionalised with S=a/h and \( Y_0 = 10.3 \) GPa and \( \rho_0 = 1578 \) Kg/m^3 for given composite as follows
\[ \ddot{w}_0 = 100w_0Y_0/hS^4p_0, \]
\[ \ddot{\sigma}_x = \sigma_x/S^2p_0, \]
\[ \ddot{\omega}_n = \omega_n aS(\rho_0/Y_0)^{1/2} \]

The deviation in terms of percentage error in results of the given beam under uniformly distributed
load obtained in present work through MATLAB code, are listed in Table 1 for deflection \( \ddot{w}_0 = \ddot{w}(0.5a,0) \), Table 2 contain results of stresses \( \ddot{\sigma}_x (0.5a,0.5h) \) for top and \( \ddot{\sigma}_x (0.5a,-0.5h) \) for bottom
whereas Table 1 also contain results obtained for first natural frequency i.e. \( \ddot{\omega}_1 \) with S = 5 (thick),
S = 10 (moderately thick), S = 20 (thin) and S = 100 (very thin) for ESDBT, TSDBT, HSDBT and
TOT along with exact results. Also the percentage errors in \( \ddot{w}_0, \ddot{\omega}_1 \) and \( \ddot{\sigma}_x \) at top and bottom are plotted with span to thickness ratio S in figures 2 to 5. Following observations can be made from these
results. For all cases, the error in \( \ddot{w}_0 \) is least among all for ESDBT, which for S=10 is 1.55%.
Compared to TOT, this error in \( \ddot{w}_0 \) for ESDBT is almost one order less for the given beam. The error
in the maximum \( \ddot{\sigma}_x \) at the bottom and top is small for ESDBT. The maximum magnitude of % errors
in \( \ddot{\sigma}_x \) at the bottom correspond to S=5(thick), and are 2.45, 3.53, 4.88 and 4.80 for ESDBT, TSDBT,
HSDBT and TOT respectively for uniformly distributed load. Also the maximum magnitude of % errors in \( \sigma_x \) at the top surface correspond to S=5(thick), and are 3.44, 4.51, 5.85 and 5.70 for ESDBT, TSDBT,
HSDBT and TOT respectively for uniformly distributed load. And in same order the maximum % error in first natural frequency \( \ddot{\omega}_1 \) are -1.54, -1.97, -2.58 and -2.52. The errors show that ESDBT is most accurate followed by TSDBT, TOT and HSDBT respectively.
Table 1: % errors in non-dimensional central deflection $\bar{w}_0$ and first natural frequency $\bar{\omega}_1$.

| S   | Exact values [8] | %error using ESDBT | %error using TSDBT | %error using TOT | %error using HSDBT | Exact values [8] | %error using ESDBT | %error using TSDBT | %error using TOT | %error using HSDBT |
|-----|------------------|---------------------|--------------------|------------------|-------------------|------------------|------------------|--------------------|------------------|-------------------|
| 5   | -2.67            | 2.63                | 3.49               | 4.55             | 4.66              | 6.81             | -1.54            | -1.97              | -2.52            | -2.58             |
| 10  | -1.43            | 1.56                | 2.02               | 2.57             | 2.62              | 9.34             | -0.84            | -1.08              | -1.37            | -1.39             |
| 20  | -1.12            | 0.53                | 0.68               | 0.86             | 0.88              | 10.64            | -0.28            | -0.36              | -0.46            | -0.47             |
| 100 | -1.01            | 0.04                | 0.04               | 0.04             | 0.04              | 11.19            | -0.01            | -0.01              | -0.01            | -0.02             |

Table 2: % errors in axial stresses $\sigma_x$ for top and bottom.

| S   | Exact values [8] | %error using ESDBT | %error using TSDBT | %error using TOT | %error using HSDBT | Exact values [8] | %error using ESDBT | %error using TSDBT | %error using TOT | %error using HSDBT |
|-----|------------------|---------------------|--------------------|------------------|-------------------|------------------|------------------|--------------------|------------------|-------------------|
| 5   | -1.07            | 3.44                | 4.51               | 5.70             | 5.85              | 1.06             | 2.45             | 3.53               | 4.80             | 4.88             |
| 10  | -0.91            | 1.04                | 1.40               | 1.74             | 1.75              | 0.90             | 0.74             | 1.10               | 1.43             | 1.45             |
| 20  | -0.86            | 0.26                | 0.36               | 0.49             | 0.42              | 0.86             | 0.18             | 0.28               | 0.44             | 0.34             |
| 100 | -0.85            | 0.03                | 0.07               | 0.04             | 0.08              | 0.85             | 0.06             | 0.07               | 0.04             | 0.08             |

Figure 2: % error in $\bar{w}_0$ with span to thickness ratio ($S$).

Figure 3: % error in first natural frequency $\bar{\omega}_1$. 

plot for % error in nondimensionalized $w_0$ with $S$
plot for % error in natural frequency $\omega_1$ with $S$
5 Conclusion
In general, the ESDBT yields best results among all the new three theories studied here while TSDBT being lesser accurate but better than TOT and HSDBT which give least accurate results and close to each other. For static and modal analysis the results obtained for moderately thick beams with $S=10$ HSDBT and TOT yield poor results for the test beam. The error becomes very small (<1%) for thin beams with $S = 20$ for both static and modal analysis. Any theory can be used for span to thickness ratio $S \geq 20$. Moreover, the results obtained from the developed MATLAB code are very close to the standard results \cite{8} and therefore, the code can be treated as standard.

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