High-energy scattering in gauge theories
and integrable spin chains

D.R. Karakhanyan, R. Kirschner

1 Yerevan Physics Institute
   Alikhanyan Br. street
   Yerevan, Armenia

2 Naturwissenschaftlich-Theoretisches Zentrum
   und Institut f"ur Theoretische Physik, Universit"at Leipzig
   Augustusplatz 10, D-04109 Leipzig, Germany

Abstract: In the leading log approximation and at large \(N_C\) the interaction of two fermionic and one gluonic reggeons is described by an integrable system corresponding to an open spin chain.

1 Reggeons in gauge theories

In the Regge limit, where the energy is large compared to the masses and momentum transfers, the perturbative expansion of the scattering amplitudes is most conveniently written in terms of the exchange of reggeons in the \(t\)-channel. The reggeons interact with the scattering near mass shell quanta and with each other via effective vertices. It is appropriate to Mellin transform the energy dependence of the amplitudes and to consider the resulting partial waves in the complex angular momentum \((j)\) plane. The reggeons correspond to poles in \(j\) moving with the transferred momentum \([1]\). The scattering amplitude of gauge-group singlet particles is represented as a convolution of two impact factors and the reggeon Green function. The impact factors describe the coupling of the in-and outgoing (in forward and backward directions, respectively) particles to the exchanged reggeons. Our discussion is restricted to the reggeon Green function describing the exchange of the interacting reggeons.

The high-energy effective action \([2]\) describes the interaction of the scattering partons and the reggeons. This action is derived from the underlying action by integrating over modes of the fields, that appear neither in the scattering nor in the exchanged quanta. On the tree level the reggeons are the modes of the original fields corresponding to the exchange in the Regge kinematics. Due to loop corrections the corresponding pole in the \(j\)-plane moves with the momentum transferred by the reggeon. Actually, the trajectory functions and the bare reggeon interactions have to be regularized to avoid infrared
divergences, which cancel however in the scattering amplitudes of gauge group singlet objects.

We disregard the case of changing number of reggeons during the exchange. The dependence on the longitudinal (time and direction of the collision axis) dimensions is trivial. At large \( N_C \) also the gauge group part of the interaction becomes trivial. The interaction is pairwise. The interaction of two reggeons at impact parameters (transverse positions) \( x_1 \) and \( x_2 \) (parametrized as complex numbers) is described by hamiltonian operators, which can be written as a sum of two terms, the first involving only the holomorphic coordinates and derivatives and the second involving only the anti-holomorphic ones. The lowest energy eigenvalues are directly related to the right-most Regge singularities generated by the (two- or multi-) reggeon exchange. The reggeons are characterized by conformal weights \((\Delta, \bar{\Delta})\) which are \((0,0)\) for gluonic reggeons and \((\frac{1}{2}, 0)\) or \((0, \frac{1}{2})\) for fermions, depending on helicity \([3, 4]\). We restrict the discussion to the holomorphic part keeping in mind the condition of single-valuedness of the reggeon Green function in the transverse plane. We denote the holomorphic part of the hamiltonian by \( H_{12}^{\Delta_1, \Delta_2} \), with \( \Delta_1, \Delta_2 \) being the holomorphic weights of the two reggeons and the lower indices being the short-hand notation for the dependence on the holomorphic coordinates \( x_1, x_2 \) and derivatives \( \partial_1, \partial_2 \), \((x_{12} = x_1 - x_2)\).

\[
H_{12}^{\Delta_1, \Delta_2} = \partial_1^{1+2\Delta_1} \ln x_{12} \partial_2^{1-2\Delta_1} + \partial_2^{1+2\Delta_2} \ln x_{12} \partial_1^{1-2\Delta_2} + \ln \partial_1 \partial_2 - 2\psi(1) = 2 \ln x_{12} + x_{12}^{1-2\Delta_1} \ln \partial_1 + x_{12}^{1-2\Delta_2} \ln \partial_2 - x_{12}^{-1+2\Delta_2} - 2\psi(1). \tag{1}
\]

The conformal (holomorphic Möbius) transformations of the wave function of the reggeon \( \Delta_1 \) at \( x_1 \) are generated by

\[
M_1^- = \partial_1, \quad M_1^0 = x_1 \partial_1 + \Delta_1, \quad M_1^+ = x_1^2 \partial_1 + 2x_1 \Delta_1. \tag{2}
\]

It is not difficult to check that the hamiltonian is symmetric, i.e. commutes with \( M_a^2 = M_1^a + M_2^a, \ a = \pm 0 \). Therefore there is a representation of this operator as a function of the Casimir operator

\[
C_{12}^{\Delta_1, \Delta_2} = -M_{12}^{0, 2} + \frac{1}{2}(M_{12}^{+}M_{12}^{-} + M_{12}^{-}M_{12}^{+}). \tag{3}
\]

We have indeed

\[
H_{12}^{\Delta_1, \Delta_2} = -\frac{1}{2}(\chi(\Delta_1 - \Delta_2)(C_{12}^{\Delta_1, \Delta_2}) + \chi(\Delta_2 - \Delta_1)(C_{12}^{\Delta_1, \Delta_2})). \tag{4}
\]

Writing the argument as \( C = m(1 - m) \) the functions \( \chi_\delta(C) \) can be expressed in terms of the logarithmic derivative of the \( \Gamma \) function

\[
\chi_\delta(C) = 2\psi(1) - \psi(m + \delta) - \psi(1 - m + \delta). \tag{5}
\]

The conformal symmetry can be established also by looking more closely at the two forms given in (1) and observing that the formal transposition of these operators \( H^T \) can be obtained from \( H \) by similarity transformations. For example, in the case of \( \Delta_1 = \frac{1}{2}, \Delta_2 = 0 \) we have

\[
(H(\frac{1}{2},0))^T = \partial_2 H^{(\frac{1}{2},0)} \partial_2^{-1} = \mathcal{P}_{12} x_{12}^{-1} H(\frac{1}{2},0) x_{12} \mathcal{P}_{12}. \tag{6}
\]

\( \mathcal{P}_{12} \) permutes the points 1 and 2. By comparison we obtain that there is an operator commuting with \( H^{(\frac{1}{2},0)} \):

\[
[A_{12}, H_{12}^{(\frac{1}{2},0)}] = 0, \quad A_{12} = \mathcal{P}_{12} x_{12} \partial_2. \tag{7}
\]
This result is directly related to conformal symmetry, because $A_{12}$ is essentially the square root of the Casimir operator: $A_{12}^2 = C^{(4,0)}_{12} - \frac{1}{4}$.

2 Three-reggeon exchange

The symmetry which has been exhibited in the representation (4) allows to obtain easily the eigenvalues and eigenfunctions of the two-reggeon system. The two-reggeon Green function can be decomposed into these eigenfunctions. For the eigenvalues we have just to substitute in (4) $C = m(1 - m)$, $m = \frac{1}{2} + n + i\nu$, $n$ integer and $\nu$ real. The contribution of multi-reggeon exchange is physically relevant not only as unitarity correction to the two-reggeon exchange. The quantum numbers in some channels cannot be transferred by just two gluons or by quark - antiquark. Much work has been done for the case of multiple exchange of gluonic reggeons. The problem has been related to integrable spin chains [5, 6]. Due to the large $N_C$ approximation one is led to a closed chain with nearest neighbour interactions. In the case of three gluonic reggeons (oderon channel) after serious efforts the leading eigenvalue has been calculated [7, 8].

We are studying the reggeon exchange with fermions included. As the first non-trivial case we have considered the quark - antiquark - gluon exchange. If the fermions have opposite helicities the holomorphic part corresponds to a one-dimensional three-body system with the conformal weights $\Delta_1 = \frac{1}{2}$, $\Delta_2 = \Delta_3 = 0$. We show that the system is integrable by constructing an additional operator, besides of the conformal generators $M_1^a + M_2^a + M_3^a$, commuting with the latter and with the hamiltonian

$$H_{123} = H^{(4,0)}_{12} + H^{(0,0)}_{23}. \tag{8}$$

For the fermions in the fundamental representation the large $N_C$ approximation results in an open chain with nearest neighbour interaction and with the fermions at the end points. Including adjoint fermions we would encounter also closed inhomogeneous chains, which is a case essential different from the one considered.

We know two ways of constructing the additional integral of motion. The first one is a simple generalization of the transposition argument presented above. Indeed we observe

$$H_{123}^T = \partial_2 \partial_3 H_{123} (\partial_2 \partial_3)^{-1} = \mathcal{P}_{123} x_{12}^{-1} x_{23}^{-1} H_{123} x_{23} x_{12} \mathcal{P}_{123}. \tag{9}$$

$\mathcal{P}_{123}$ permutes the points 1, 2, 3 into 3, 2, 1. Therefore we have the commuting operator $A_{123}$:

$$[A_{123}, H_{123}] = 0, \quad A_{123} = \mathcal{P}_{123} x_{12} x_{23} \partial_2 \partial_3. \tag{10}$$

3 Open spin chain

In the case of only gluons the multi-reggeon system is described by a closed quantum spin chain with the non-compact $sl(2)$ representations of weight $\Delta = 0$ at each site. We associate a Lax matrix to each site: $L_i(\theta) = \theta I^{(2)} + \sigma^0 M_i^0 + \sigma^+ M_i^- - \sigma^- M_i^+$. It obeys the usual Yang - Baxter - relation with the $4 \times 4$ $R$-matrix $R_{12}(\theta) = \theta I^{(4)} + \xi \mathcal{P}_{12}$. This relation generalizes to products $\prod_{i=1}^N L_i(\theta + \delta_i) = T(\theta)$. It follows that the trace $\text{tr} T(\theta)$ generates a complete set of commuting operators, which are just the ones needed for the integrable closed homogeneous chain [9, 10].
In the case of open chains the boundaries impose additional conditions on the integrable interaction \[ R_{12}(\theta_1 - \theta_2) T(\theta_1) \otimes I^{(2)} R_{12}(\theta_1 + \theta_2) I^{(2)} \otimes T(\theta_2) = \]
\[ T(\theta_2) \otimes I^{(2)} R_{12}(\theta_1 + \theta_2) I^{(2)} \otimes T(\theta_1) R_{12}(\theta_1 - \theta_2). \] (11)
and another condition related to this one by crossing. Given a matrix \( K_- \) obeying (11) when inserted for \( T \) and another matrix \( K_+ \) obeying the crossing relation one finds that \( T(\theta) = T(\theta) K_- T^{-1}(-\theta) \) obeys (11) too, provided \( T(\theta) \) obeys the ordinary Yang-Baxter relation. Furthermore, it follows that the trace \( \text{tr}(T(\theta) K_-) \) generates mutually commuting operators. Whereas the generating function \( \text{tr} T(\theta) \) for the closed chain can be visualized by drawing the closed chain and associating the Lax matrices with each site, the generating function \( \text{tr} (T(\theta) K_- T^{-1}(-\theta) K_+ \) is visualized as the closed chain built from the considered open chain together with its mirror image and with the matrices \( K_\pm \) standing for the mirror.

We obtain the integrals of motion for our three-reggeon problem for the simple choice \( K_\pm = I^{(2)} \). We have

\[ T(\theta) = L_1^{(\Delta=1/2)}(\theta + \delta_1) L_2^{(\Delta=0)}(\theta + \delta_2) L_3^{(\Delta=0)}(\theta + \delta_3). \] (12)

The trace \( \text{tr}(T(\theta) T^{-1}(-\theta)) \), e.g. with \( \delta_1 = \delta_2 = \delta_3 = \frac{1}{\beta} \), decomposes into the operators \( A_{123}^{(\Delta=1/2)} \), \( C_{123} = C_{12} + C_{23} + C_{31} \) and \( C_{23} \) with linear independent functions of \( \theta \) as coefficients. In this way we have checked that the considered three-reggeon system is properly described by an integrable open spin chain. Now the generalization to multi-reggeon systems with quark and antiquark and an arbitrary number of gluonic reggeons is straightforward.

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