Explicit calculation of multi-fold contour integrals of certain ratios of Euler gamma functions. Part 1

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Abstract

In this paper, we proceed to study properties of Mellin–Barnes (MB) transforms of Usykina–Davydychev (UD) functions. In our previous papers (Allendes et al., 2013 [13], Kniehl et al., 2013 [14]), we showed that multi-fold Mellin–Barnes (MB) transforms of Usykina–Davydychev (UD) functions may be reduced to two-fold MB transforms and that the higher-order UD functions may be obtained in terms of a differential operator by applying it to a slightly modified first UD function. The result is valid in $d = 4$ dimensions, and its analog in $d = 4 - 2\varepsilon$ dimensions exits, too (Gonzalez and Kondrashuk, 2013 [6]). In Allendes et al. (2013) [13], the chain of recurrence relations for analytically regularized UD functions was obtained implicitly by comparing the left-hand side and the right-hand side of the diagrammatic relations between the diagrams with different loop orders. In turn, these diagrammatic relations were obtained using the method of loop reduction for the triangle ladder diagrams proposed in 1983 by Belokurov and Usykina. Here, we reproduce these recurrence relations by calculating explicitly, via Barnes lemmas, the contour integrals produced by the left-hand sides of the diagrammatic relations. In this a way, we explicitly calculate a family of multi-fold contour integrals of certain ratios of Euler gamma functions. We make a conjecture...
that similar results for the contour integrals are valid for a wider family of smooth functions, which includes the MB transforms of UD functions.

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1. Introduction

Off-shell triangle-ladder and box-ladder diagrams are the only families of Feynman diagrams which were calculated at any loop order, for example in $d = 4$ space–time dimensions [1–4] with all indices equal to 1 in the momentum space representation (m.s.r.) and in $d = 4 − 2\varepsilon$ space–time dimensions with indices equal to $1 − \varepsilon$ on the rungs of ladders in the m.s.r., too [5,6]. For the important case of the ladder diagrams with all indices equal to 1 in the m.s.r. in $d = 4 − 2\varepsilon$ space–time dimensions, the on-shell result for this family of diagrams is known only at the first three loops in the form of an expansion in terms of $\varepsilon$ [7,8] up to a certain power of $\varepsilon$. The off-shell result for the whole family of the ladder diagrams is unknown in $d = 4 − 2\varepsilon$ dimensions.

The momentum integrals corresponding to the family of the ladder diagrams in $d = 4$ space–time dimensions result in UD functions [2,3]. The order of the UD function is the loop order in the ladder diagram [2,3,9]. The ladder diagrams possess remarkable properties at the diagrammatic level, for example, in Refs. [10,11], it was shown that the UD functions are invariant with respect to Fourier transformations. In Refs. [9,12] it has been shown that such a property of Fourier invariance may be generalized to any three-point Green function via MB transformations.

MB transforms of UD functions were investigated in Refs. [13,14]. It has been found under some analytical regularization of Ref. [1] that the MB transform of a UD function of order $n$ is a linear combination of the MB transforms of three UD functions of order $(n − 1)$. This means that any ladder diagram of this family may be reduced via a chain of recurrence relations to the one-loop scalar massless triangle diagram, which may be expressed for any indices and in any dimensions in terms of the Appell function $F_4$ [15,16]. This chain of recurrence relations for the analytically regularized UD functions in the double-uniform limit when removing this analytical regularization, is represented as a differential operator applied a to a slightly modified first UD function [14]. It has been shown there that, if, instead of MB transforms of UD functions, we write any smooth function of the same arguments, the structure of this differential operator will remain the same in this double-uniform limit. This operator will be applied to the function of the lowest order in this chain of recurrence relations.

However, in the present paper, we show that, in the particular case when the integrand of the contour integrals on the left-hand sides of the diagrammatic relations contains MB transforms of UD functions, this chain of recurrence relations for the MB transforms of UD functions is produced by the contour integration. These contour integrals are calculated explicitly via the first and the second Barnes lemmas. Due to the observation done in the previous paragraph, we make a conjecture that similar results for the contour integrals are valid for a wider family of smooth functions written instead of MB transforms of UD functions. In our subsequent papers, we will describe this family of functions and also explain what kind of changes should be made for the contours of the integrals over complex variables for the case of other smooth functions different from certain ratios of Euler gamma functions. In this paper, we focus on the contour integration via Barnes lemmas for the case when the integrand contains MB transforms of UD functions.
The Barnes lemmas were introduced in science about a century ago. The first Barnes lemma has been proved in Ref. [17], the second Barnes lemma has been proved in Ref. [18]. They allow us to integrate a product of several Euler gamma functions in a simple manner. The Barnes lemmas will help us to demonstrate the integral relations of Refs. [13,14] by doing complex integration along contours typical for MB transformation. In Ref. [13], in order to obtain the results for the contour integrals, we simply compared the left and right parts of the diagrammatic relations.

2. Proof

The integral relation we need to prove via Barnes lemmas is Eq. (13) of Ref. [13],

$$\oint_{C} dz_2 dz_3 \, D^{(u,v)}[1 + \varepsilon_1 - z_3, 1 + \varepsilon_2 - z_2, 1 + \varepsilon_3] D^{(z_2,z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] =$$

$$J \left[ \frac{D^{(u,v-\varepsilon_2)}[1 - \varepsilon_1]}{\varepsilon_2 \varepsilon_3} + \frac{D^{(u,v)}[1 + \varepsilon_3]}{\varepsilon_1 \varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1 - \varepsilon_2]}{\varepsilon_1 \varepsilon_3} \right], \quad (1)$$

in which the parameters $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3$ are three complex variables of analytical regularization used in Ref. [1], subject to condition

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0,$$

the factor $J$ is a ratio of Euler gamma functions,

$$J = \frac{\Gamma(1 - \varepsilon_1) \Gamma(1 - \varepsilon_2) \Gamma(1 - \varepsilon_3)}{\Gamma(1 + \varepsilon_1) \Gamma(1 + \varepsilon_2) \Gamma(1 + \varepsilon_3)},$$

the function $D^{(z_2,z_3)}[v_1, v_2, v_3]$ is the MB transform of the one-loop triangle integral in momentum space $J(v_1, v_2, v_3)$ taken in Ref. [13] from Refs. [2,3,16],

$$D^{(z_2,z_3)}[v_1, v_2, v_3] = \frac{\Gamma(-z_2) \Gamma(-z_3) \Gamma(-z_2 - v_2 - v_3 + d/2) \Gamma(-z_3 - v_1 - v_3 + d/2)}{\Pi_j \Gamma(v_j)} \times \frac{\Gamma(z_2 + z_3 + v_3) \Gamma(\Sigma v_i - d/2 + z_3 + z_2)}{\Gamma(d - \Sigma v_i)}, \quad (2)$$

and, for brevity, the notation

$$D^{(u,v)}[1 + v] \equiv D^{(u,v)}[1, 1, 1 + v] \quad (3)$$

is used. The integral relation in Eq. (1) is produced by the diagrammatic relation between scalar Feynman diagrams in momentum space given in Fig. 1, which is Eq. (25) of Ref. [2]. This graphical equation was originally obtained by using the uniqueness relations from Ref. [2], and, in particular, this is why the analytic regularization indices were chosen in such a way that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$. The derivation of this diagrammatic relation is reviewed in detail in Ref. [13]. In Ref. [13], also the derivation of Eq. (1) from the diagrammatic relation of Fig. 1 may be found.

According to Eqs. (2) and (3), we write
\[ D^{(z_2,z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] = \frac{\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_2 + z_3)^2\Gamma(1 + z_2 + z_3)\Gamma(1 + z_2 + z_3 + \varepsilon_3)}{\Gamma(1 + \varepsilon_1)\Gamma(1 + \varepsilon_2)\Gamma(1 + \varepsilon_3)}, \]

The Euler gamma functions with negative signs of the integration variables of their arguments. The contribution of the corresponding residues at the points \(z_2 = \varepsilon_2\) and \(z_3 = \varepsilon_1\) in the integrand of Eq. (1), which is Eq. (4), reproduces term (5) on the right-hand side of Eq. (1). To obtain terms (6) and (7) on the right-hand side of Eq. (1), we need to use the Barnes lemmas. The first lemma has been published in 1908 in Ref. [17],

\[ \int_{C} dz \frac{\Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z)}{\Gamma(\lambda_1 + \lambda_3)\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_3)\Gamma(\lambda_2 + \lambda_4)}. \]
in which \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are complex numbers chosen in a such a way that on the right-hand side of Eq. (8) there are no singularities, while the second Barnes lemma has been published in 1910 in Ref. [18],

\[
\oint_C dz \frac{\Gamma (\lambda_1 + z) \Gamma (\lambda_2 + z) \Gamma (\lambda_3 + z) \Gamma (\lambda_4 - z) \Gamma (\lambda_5 - z)}{\Gamma (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + z)} = \\
\frac{\Gamma (\lambda_1 + \lambda_4) \Gamma (\lambda_2 + \lambda_4) \Gamma (\lambda_3 + \lambda_4) \Gamma (\lambda_1 + \lambda_5) \Gamma (\lambda_2 + \lambda_5) \Gamma (\lambda_3 + \lambda_5)}{\Gamma (\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5) \Gamma (\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5) \Gamma (\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)},
\]

(9)

in which \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \) are complex numbers chosen in a such a way that on the right-hand side of Eq. (9) there are no singularities.

The integrand of Eq. (4) may be represented as

\[
\frac{1}{z_3 - \varepsilon_1} \frac{1}{z_2 - \varepsilon_2} \Gamma (-z_2) \Gamma (-z_3) \Gamma (1 + z_2 + z_3 + \varepsilon_3) \Gamma (-u + \varepsilon_1 + z_2) \times \\
\times \Gamma (-v + \varepsilon_2 + z_3) \Gamma (1 - z_2 - z_3 + u + v) = \\
= \frac{z_2 + z_3 + \varepsilon_3}{(z_3 - \varepsilon_1)(z_2 - \varepsilon_2)} \Gamma (-z_2) \Gamma (-z_3) \Gamma (z_2 + z_3 + \varepsilon_3) \Gamma (-u + \varepsilon_1 + z_2) \times \\
\times \Gamma (-v + \varepsilon_2 + z_3) \Gamma (1 - z_2 - z_3 + u + v) = \\
= \left( \frac{1}{z_3 - \varepsilon_1} + \frac{1}{z_2 - \varepsilon_2} \right) \Gamma (-z_2) \Gamma (-z_3) \Gamma (z_2 + z_3 + \varepsilon_3) \Gamma (-u + \varepsilon_1 + z_2) \times \\
\times \Gamma (-v + \varepsilon_2 + z_3) \Gamma (1 - z_2 - z_3 + u + v),
\]

(10)

and this is a sum of two terms. We consider the second term,

\[
\oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma (-z_2) \Gamma (-u + \varepsilon_1 + z_2) \oint_C dz_3 \Gamma (-z_3) \Gamma (z_2 + z_3 + \varepsilon_3) \times \\
\Gamma (-v + \varepsilon_2 + z_3) \Gamma (1 - z_2 - z_3 + u + v),
\]

in which the integral over \( z_3 \) may be calculated via the first Barnes lemma,

\[
\oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma (-z_2) \Gamma (-u + \varepsilon_1 + z_2) \oint_C dz_3 \Gamma (-z_3) \Gamma (z_2 + z_3 + \varepsilon_3) \times \\
\Gamma (-v + \varepsilon_2 + z_3) \Gamma (1 - z_2 - z_3 + u + v) = \\
\oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma (-z_2) \Gamma (-u + \varepsilon_1 + z_2) \times \\
\times \frac{\Gamma (z_2 + \varepsilon_3) \Gamma (-v + \varepsilon_2) \Gamma (1 + \varepsilon_3 + u + v) \Gamma (1 + \varepsilon_2 + u - z_2)}{\Gamma (1 + u - \varepsilon_1)} = \\
= \frac{\Gamma (-v + \varepsilon_2) \Gamma (1 + \varepsilon_3 + u + v)}{\Gamma (1 + u - \varepsilon_1)} \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma (z_2 + \varepsilon_3) \Gamma (-z_2) \times \\
\times \Gamma (-u + \varepsilon_1 + z_2) \Gamma (1 + \varepsilon_2 + u - z_2).
\]

Now we perform the reflection of the complex variable \( z_2 \) of contour integration, \( z_2 \rightarrow -z_2 \), and apply the second Barnes lemma,
\[
\frac{\Gamma (-v + \varepsilon_2) \Gamma (1 + \varepsilon_3 + u + v)}{\Gamma (1 + u - \varepsilon_1)} \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma (z_2 + \varepsilon_3) \Gamma (-z_2) \times \\
\times \frac{\Gamma (-u + \varepsilon_1 + z_2)}{\Gamma (1 + \varepsilon_2 + u - z_2)} = \\
- \frac{\Gamma (-v + \varepsilon_2) \Gamma (1 + \varepsilon_3 + u + v)}{\Gamma (1 + u - \varepsilon_1)} \oint_C dz_2 \frac{1}{z_2 + \varepsilon_2} \Gamma (-z_2 + \varepsilon_3) \Gamma (z_2) \times \\
\times \frac{\Gamma (-u + \varepsilon_1 - z_2)}{\Gamma (1 + \varepsilon_2 + u + z_2)} = \\
- \frac{\Gamma (-v + \varepsilon_2) \Gamma (1 + \varepsilon_3 + u + v)}{\Gamma (1 + u - \varepsilon_1)} \Gamma (\varepsilon_3) \Gamma (-\varepsilon_1) \Gamma (1 + u - \varepsilon_1) \times \\
\frac{\Gamma (-u - \varepsilon_3) \Gamma (-u + \varepsilon_1) \Gamma (1 - \varepsilon_3)}{\Gamma (1 + \varepsilon_2)} \Gamma (-u) \\
= \frac{1}{\varepsilon_1 \varepsilon_3} \frac{\Gamma (1 - \varepsilon_1) \Gamma (1 + \varepsilon_3) \Gamma (1 - \varepsilon_3)}{\Gamma (1 + \varepsilon_2)} \times \\
\Gamma (-\varepsilon_3) \Gamma (-\varepsilon_1) \Gamma (-u - \varepsilon_3) \Gamma (-u + \varepsilon_1) \Gamma (1 + \varepsilon_2) \times \\
\Gamma (-u) \times \\
\frac{\Gamma (-u) \Gamma (-v) \Gamma (1 + u + v + \varepsilon_3)}{\Gamma (1 + \varepsilon_1) \Gamma (1 + \varepsilon_2) \Gamma (1 + \varepsilon_3)} \times \\
\frac{1}{\varepsilon_1 \varepsilon_3} \Gamma (1 - \varepsilon_1) \Gamma (1 + \varepsilon_3) \Gamma (1 - \varepsilon_3) \times \\
\Gamma (-u - \varepsilon_3) \Gamma (-u + \varepsilon_1) \Gamma (1 + \varepsilon_2) \times \\
\Gamma (-u) = \frac{J}{\varepsilon_1 \varepsilon_3} D^{(u, v - \varepsilon_2)}[1 - \varepsilon_1].
\]

Taking into account the factor from Eq. (4), we obtain

\[
\frac{\Gamma (-u) \Gamma (-v) \Gamma (1 + u + v + \varepsilon_3)}{\Gamma (1 + \varepsilon_1) \Gamma (1 + \varepsilon_2) \Gamma (1 + \varepsilon_3)} \times \\
\frac{1}{\varepsilon_1 \varepsilon_3} \Gamma (1 - \varepsilon_1) \Gamma (1 + \varepsilon_3) \Gamma (1 - \varepsilon_3) \times \\
\Gamma (-u - \varepsilon_3) \Gamma (-u + \varepsilon_1) \Gamma (1 + \varepsilon_2) \times \\
\Gamma (-u) = \frac{J}{\varepsilon_1 \varepsilon_3} D^{(u, v - \varepsilon_2)}[1 - \varepsilon_1].
\]

The first term in Eq. (10) analogously reproduces the term \( J / \varepsilon_2 \varepsilon_3 D^{(u, v - \varepsilon_2)}[1 - \varepsilon_1] \) on the right-hand side of Eq. (1). We need to comment that there is no double counting of residues at the points \( z_2 = \varepsilon_2 \) and \( z_3 = \varepsilon_1 \) because, after the reflection, these points become “left” poles, that is, they come from Euler gamma functions with positive signs of the integration variable in the arguments of gamma functions, while we calculate the “right” residues only, that is, the residues which come from Euler gamma functions with negative signs of the integration variable in the arguments of gamma functions.

3. Conclusion

We showed in Ref. [14] that the structure of the chain of recurrence relations for the Mellin–Barnes transforms of the analytically regularized UD functions guarantees the finiteness of the double-uniform limit when removing the analytical regularization. The limit was expressed in terms of a differential operator. This operator is the same for any smooth function written instead of the MB transforms of the UD functions and has nothing to do with the explicit form of these MB transforms. The present paper shows that the first and the second Barnes lemmas allow us to work out the contour integration only in a particular case of MB transforms of UD functions so as to produce this chain of the recurrence relations. For a wider family of smooth functions, the Barnes lemmas should be replaced with another integration trick by using more complicated contour of integration.
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References

[1] V.V. Belokurov, N.I. Ussyukina, Calculation of ladder diagrams in arbitrary order, J. Phys. A, Math. Gen. 16 (1983) 2811.
[2] N.I. Ussyukina, A.I. Davdychev, An approach to the evaluation of three- and four-point ladder diagrams, Phys. Lett. B 298 (1993) 363.
[3] N.I. Ussyukina, A.I. Davdychev, Exact results for three- and four-point ladder diagrams with an arbitrary number of rungs, Phys. Lett. B 305 (1993) 136.
[4] D.J. Broadhurst, A.I. Davdychev, Exponential suppression with four legs and an infinity of loops, Nucl. Phys. B, Proc. Suppl. 205–206 (2010) 326, arXiv:1007.0237 [hep-th].
[5] I. Gonzalez, I. Kondrashuk, Belokurov–Ussyukina loop reduction in non-integer dimension, Phys. Part. Nucl. 44 (2013) 268, arXiv:1206.4763 [hep-th].
[6] I. Gonzalez, I. Kondrashuk, Box ladders in a noninteger dimension, Theor. Math. Phys. 177 (1) (2013) 1515, Teor. Mat. Fiz. 177 (1) (2013) 276, arXiv:1210.2243 [hep-th].
[7] V.A. Smirnov, Evaluating Feynman integrals, Springer Tracts Mod. Phys. 211 (2004) 1.
[8] Z. Bern, L.J. Dixon, V.A. Smirnov, Iteration of planar amplitudes in maximally supersymmetric Yang–Mills theory at three loops and beyond, Phys. Rev. D 72 (2005) 085001, arXiv:hep-th/0505205.
[9] I. Kondrashuk, A. Vergara, Transformations of triangle ladder diagrams, J. High Energy Phys. 1003 (2010) 051, arXiv:0911.1979 [hep-th].
[10] I. Kondrashuk, A. Kotikov, Fourier transforms of UD integrals, in: B. Gustafsson, A. Vasil’ev (Eds.), Analysis and Mathematical Physics, in: Birkhäuser Book Series Trends in Mathematics, Birkhäuser, Basel, Switzerland, 2009, p. 337, arXiv:0802.3468 [hep-th].
[11] I. Kondrashuk, A. Kotikov, Triangle UD integrals in the position space, J. High Energy Phys. 0808 (2008) 106, arXiv:0803.3420 [hep-th].
[12] P. Allendes, N. Guerrero, I. Kondrashuk, E.A. Notte Cuello, New four-dimensional integrals by Mellin–Barnes transform, J. Math. Phys. 51 (2010) 052304, arXiv:0910.4805 [hep-th].
[13] P. Allendes, B.A. Kniehl, I. Kondrashuk, E.A. Notte-Cuello, M. Rojas-Medar, Solution to Bethe–Salpeter equation via Mellin–Barnes transform, Nucl. Phys. B 870 (2013) 243, arXiv:1205.6257 [hep-th].
[14] B.A. Kniehl, I. Kondrashuk, E.A. Notte-Cuello, I. Parra-Ferrada, M. Rojas-Medar, Two-fold Mellin–Barnes transforms of Ussyukina–Davydychev functions, Nucl. Phys. B 876 (2013) 322, arXiv:1304.3004 [hep-th].
[15] E.E. Boos, A.I. Davydychev, A method of evaluating massive Feynman integrals, Teor. Mat. Fiz. 89 (1991) 56, Theor. Math. Phys. 89 (1991) 1052.
[16] A.I. Davydychev, Recursive algorithm of evaluating vertex-type Feynman integrals, J. Phys. A, Math. Gen. 25 (1992) 5587.
[17] E.W. Barnes, A new development of the theory of the hypergeometric functions, Proc. Lond. Math. Soc. Second Ser. 6 (1908) 141.
[18] E.W. Barnes, A transformation of generalized hypergeometric series, Q. J. Pure Appl. Math. 41 (1910) 136.