Fluctuating Trajectories and Switching Rates of a Synthetic Two-Level System

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The Kerr Parametric Oscillator (KPO) is a nonlinear resonator system that is often described as a synthetic two-level system. We reveal that activated switches between these levels follow a fluctuating and curving trajectory in phase space. Such fluctuations make it hard to establish a precise count, or even a useful definition, of the “lifetime” of the levels. Addressing this issue, we compare several methods to overcome this systemic noise, and estimate a lifetime for the levels. Moreover, we establish that a peak in the Allan variance of fluctuations can also be used to determine the levels’ lifetime. Our work provides a basis for characterizing KPO networks for simulated annealing.

Two-level systems (TLSs) are fundamental building blocks of quantum information. For a long time, the study of TLSs was confined to large ensembles [1, 2]. Later it became possible to engineer individual TLS, e.g., by introducing atomic defects into crystal lattices [3–5], by manipulating individual electrons in quantum dots [5–7], by defining artificial atoms with superconducting circuits [8, 9], or by addressing individual charge states in ions [10, 11]. Such highly controllable TLSs serve as carriers of quantum information – qubits – which play a key role in quantum computers [12, 13] and quantum sensing [14].

More recently, synthetic TLSs generated in driven nonlinear resonators have caught the attention of the physics community [8, 15]. A particularly prominent example is the Kerr Parametric Oscillator (KPO, also known as parametron) [16–27] whose potential energy is pumped close to twice its resonance frequency, \( f_p = 2f_0 \approx 2f_0 \). If the modulation strength \( \lambda \) exceeds a threshold \( \lambda_{th} \), the device responds with oscillations locked to \( f_d \) within a certain frequency range. This well-known “period doubling” of the response relative to the pump gives rise to two stable “phase states” with the same amplitude but separated by a phase difference of \( \pi \). The phase states can be used to encode the two polarization states (up/down) of a classical spin. This analogy has given rise to the idea of using networks of coupled KPOs to build noisy intermediate-scale quantum (NISQ) machines [28, 29]. These machines can simulate the dynamics of mathematical problems that overwhelm traditional computers, such as the ground state of an Ising Hamiltonian [30–37], or of other complex systems that can be mapped onto the same framework, such as the travelling salesman problem [38], the number partitioning problem [39], and the MAX-CUT problem [40–42].

An important quantity for many application of TLSs is their lifetime \( \tau \) [43]. It is the typical time spent on a level before the interaction with a fluctuating environment induces a (seemingly) spontaneous “jump” from one state to the other. For a TLS with non-degenerate energy lev-
els in the quantum regime, $\tau$ can be estimated unambiguously by repeated strong measurements of the decay from the excited level to the ground level. Unfortunately, the discrete (projective) nature of the measurement rules out the observation of the transition path the system follows during an individual decay. Such fluctuating quantum trajectories can only be observed in special cases under weak measurements [44, 45], hinting at added complexity beyond telegraph noise [46, 47].

The situation is markedly different for a KPO. Here, the levels are formed by coherent bosonic states and are not separated by an energy gap but by a phase gap [24]. This allows for situations with negligible back-action where the fluctuations during a single switch can in principle be tracked precisely. Nevertheless, there exist very few studies of the fascinating physics unfolding during individual switches [48–51], and none in the regime of strong fluctuations. In the latter, the fluctuations overwhelm the regime where switches can be assigned to narrow channels in phase space [50, 51]. Experimental understanding of switching is important, because the activated switching between phase states forms the basis for an entire class of proposed noisy intermediate-scale quantum simulators [52]. To calibrate the speed with which such machines converge on a solution, it is important to clarify how the phase state lifetime $\tau$ in a KPO can be defined and measured.

In this letter, we study a classical micromechanical KPO and investigate its switching behavior. We show that individual switches between phase states follow a complex random-walk process in phase space. The trajectories, which we can measure precisely for individual switching events, are curving through phase space and exhibit significant fluctuations. This makes it difficult to decide how a switching event can be defined for such a device, and how $\tau$ can be measured at all. To answer this question, we apply several methods previously used to characterize the rates of charge and parity state switching in Cooper pair boxes and superconducting qubits [53, 54]. Furthermore, we propose a new method to calculate the switching rate that is based on the Allan variance of the resonator displacement [55, 56]. In the final part of the paper, we compare all methods and find good agreement between several (but not all) of them, allowing us to draw a conclusion as to how the switching rate $\Gamma$, and thus $\tau$, can be reliably estimated in a our system.

Our KPO consists of a micro-electromechanical resonator (MEMS) in a setup schematically shown in Fig. 1(a). The resonator is a doubly-clamped beam, with the length of 200 $\mu$m, width 3 $\mu$m, and 60 $\mu$m in thickness with a lumped mass of 25.4 ng made from highly-doped single crystal silicon and fabricated in a wafer-scale encapsulation process [57]. Electrodes on both sides separated from the conducting beam with a gap $\approx 1\mu$m enable capacitive driving and sensitive detection of oscillations in the presence of a bias voltage, $V_{bias} = 10$ V [58]. We use a Zurich Instruments HF2LI lock-in amplifier to apply the driving voltage $V_{in}$ and to read out the resonator displacement $x \propto V_{out} = u\cos(\omega t) - v\sin(\omega t)$ with quadrature amplitudes $u$ and $v$. For convenience, we drop the proportionality factor between $x$ and $V_{out}$ and identify in the following $x \equiv V_{out}$ [59].

Our mechanical resonator can be described by the nonlinear equation of motion (in units of the measured electrical signal)

$$\ddot{x} + \omega_0^2 [1 - \lambda \cos(2\omega_d t)] x + \alpha x^3 + \gamma \dot{x} = \xi. \quad (1)$$

Here, dots indicate time derivatives, $\omega_0/2\pi = f_0 = 439.56$ kHz is the resonance frequency, $\alpha = 1.47 \times 10^{18}$ V$^{-2}$ s$^{-2}$ the coefficient of the Duffing non-linearity, $\gamma = \omega_0/Q$ the damping rate, and $Q = 3580$ the quality factor of the resonator. The potential energy term ($\propto x$) is pumped with the parametric modulation depth $\lambda = 2V_{in}/(V_{th}Q)$, where $V_{th} = 320$ mV is the voltage threshold for parametric oscillations at $f_d = f_0$ (demodulation frequency). The potential modulation arises because the electrostatic force due to $V_{in}$ pulls the beam closer towards one electrode. The force is

![FIG. 2. Phase space representation of states and switching. (a) $u$ and $v$ quadratures of a single phase state switch composed of 2170 data points measured with a 15 ps integration time at 14391 samples per second. Bright dots and dark lines correspond respectively to raw data and to a 10-point moving average that allows to reduce the influence of detection noise. A dashed line indicates the threshold between the phase states. $\Delta = 0$, $V_{th} = 0.4$ V, and $\sigma_{std} = 0.6$ V. (b) Phase space representation of the data in (a). White squares indicate the attractor points measured in the absence of noise, and a dashed line indicates the threshold between the phase states. (c) Probability density of the KPO steady state calculated with a numerical evolution of a Fokker-Planck description of the system [56]. Dark blue indicates zero probability of measuring the KPO at a position in phase space, bright yellow indicates a high probability (scale not normalized).]
nonlinear, i.e., it grows stronger for small beam-electrode distances, which corresponds to a change in the overall spring constant that the beam experiences. As a consequence, the drive generates small frequency variations \( \delta f_0 \propto V_{in} \). The drive \( \xi \) in Eq. 1 represents a fluctuating thermal bath (see SM for details).

Figure 1(b) shows the \( v \)-quadrature response of the resonator during two sweeps of \( f_d \) from positive to negative detuning \( \Delta \equiv f_d - f_0 \). Close to \( \Delta = 50 \) Hz, the response jumps from \( v = 0 \) to \( v = \pm 50 \) µV, marking a bifurcation point of the underlying nonlinear system. At the bifurcation, the resonator experiences a spontaneous symmetry breaking, also known as a period-doubling bifurcation or a discrete time-translation breaking \([60, 61]\). At this point, the resonator jumps to a positive or negative response with equal probability. The two responses belong to stable attractors (1 and 2) with opposite phases, i.e., \( v_1 = -v_2 \) and \( u_1 = -u_2 \).

To study switching between the phase states of our KPO, we apply white electrical noise \( \xi \) characterized by a standard deviation \( \sigma_{std} \) (over a bandwidth of \( 30 \) MHz) that causes the state of the resonator to fluctuate around its initial solution. If the fluctuations are large enough, they will occasionally carry the resonator across the threshold in the middle between the phase states. The resonator is then captured by the opposite attractor, corresponding to a switch of the synthetic TLS. Several such processes can be observed in Fig. 1(c). For this observation, it would appear natural to attribute a lifetime to the inverse switching rate, \( \tau = 1/\Gamma \). However, below we demonstrate that this simple method can underestimate \( \tau \) by overcounting switching events.

For a deeper understanding of the system’s transient behaviour during switching events, we perform measurements with a high temporal resolution. In Fig. 2(a)-(b), we display a narrow time segment before, during, and after a single switch. We find many data points in the unstable zone between the two phase states. A 10-point moving average filter helps to visualize the trajectory of the system during the transition. The total switching time is roughly 10 ms, much longer than the lock-in integration time of 15 µs and the moving-average filter time of 700 µs. The measurement error of each data point is 3.7 µV, in agreement with the measured point-to-point fluctuations, but significantly smaller than the \( \sim 10 \) µV fluctuations visible on the 5 ms scale.

Our observation depicted in Fig. 2 demonstrates that activated switches between the phase states are not deterministic, but include prominent random elements. For instance, in the phase-space representation of the switch in Fig. 2(b) we can clearly see that the system performs a winding path close to the origin. In our device, the fluctuations generally have a slight preference for counterclockwise rotations around the phase states and clockwise ones around the origin. This can be explained by the combination of the drive and the nonlinearity, which lead to an effective detuning of the fluctuations from the lock-in amplifier clock \([62]\). In the corresponding Fokker-Planck steady-state calculation presented in Fig. 2(c), we therefore find a broad region with a significant probability density between the phase states, in contrast to the narrow channel reported previously \([50]\).

These visualizations of the fluctuating trajectories expose a fundamental problem in estimating the lifetime \( \tau \): since transitions follow no straight lines, they can cross any point in phase space multiple times during a single switching event. An example of this can be observed in Fig. 2(b), where the averaged (dark) trajectory crosses the dotted threshold line from bottom to top, describes a clockwise winding that traverses back across the threshold, and finally crosses the line a third time before completing the switch. A simple counting algorithm would in this case register three crossing events during a single switch. In general, any counting method based on a simple threshold (such as a line) will therefore overestimate the switching number \( N_{switch} \) during the full time \( T \), and therefore also \( \Gamma = N_{switch}/T \).

The problem of overestimating the switching count can be reduced by defining multiple thresholds that have to be crossed in a particular order to constitute an event. In Fig. 3(a), we demonstrate this with the example of two circles in phase space. The count is increased by one each time a circular threshold is left and the opposing circle is entered. This method allows to ignore small fluctuations, but it requires a subjective measure that impacts the estimated \( \Gamma \), in our case the radii of the circular thresholds. Calibrating the measured switching rate \( \Gamma \) as a function of the radius allows to reduce this degree of arbitrariness (see SM), but it cannot be removed entirely.

To avoid overcounting and subjective dependencies, it is desirable to extract \( \Gamma \) from a method that does not require thresholds at all. Interestingly, the lifetime of superconducting qubits can be determined via their charge-parity power spectral density (PSD) \([53, 63]\). Assuming that the switching is dominated by telegraph noise, the PSD of \( v \) of our KPO can be fitted to a Lorentzian function,

\[
PSD(f) = \frac{2F^2\tau}{4 + (2\pi f\tau)^2}.
\]

where the lifetime \( \tau \) corresponds to the characteristic time scale between level switching events, and \( F \) is the measurement fidelity \([63]\). In Fig. 3(b), we fit the measured displacement power spectral density with Eq. 2 yielding a value for \( \Gamma = 1/\tau \) which is comparable with the rate estimated using the previous two methods. The method can also be applied after a Fourier transform by fitting the sliding average autocorrelation with the function \( AC(t) = Ae^{-2\Delta t} \) under the assumption of stationarity and ergodicity (not shown).

Crucially, the autocorrelation is intimately related to the Allan variance. Originally invented to characterize the fidelity of clocks, the Allan variance measures the frequency fluctuations of a resonator as a function of integration time \( \tau_A \). As we are interested in the time \( \tau \) over which the typical fluctuations of \( u \) (or \( v \)) of our KPO are
FIG. 3. Methods used to estimate the phase state lifetime. All plots show the same 15 min data set, an extract of which is shown in Fig. 1(c). Data was recorded at 899 samples per second with an integration time $\tau = 143 \mu$s and with $\Delta = 0$, $V_{in} = 0.4 V$, and $\sigma_{std} = 1 V$. (a) Phase space representation of the two phase states and of switching between them, cf. Fig. 2. White squares indicate the attractors measured in the absence of noise, and the dotted line and circles indicate different threshold methods outlined in the text. The estimated activation rates are $\Gamma_{\text{line}} \approx 13 H \pm 0.1 H$ and $\Gamma_{\text{cir}} \approx 4.3 H \pm 0.07 H$, with standard deviations calculated assuming Poisson statistics of the jumps. (b) PSD analysis of the fluctuations in terms of a telegraph noise model, cf. Eq. (2), yielding a fit result $\Gamma_{\text{psd}} \approx 3.6 H \pm 0.01 H$. Bright and dark lines correspond to the measured data and to the fit, respectively. (c) Allan variance of the measured fluctuations (bright), with a maximum at $\Gamma_{\text{Allan}} \approx 1 / \tau = 4.0 H \pm 0.1 H$, where the precision is limited by the separation of points in $\tau$. A dark line is the function expected (with arbitrary vertical scaling) for pure telegraph noise with a mean switching rate of 4 Hz, see Eq. (5).

maximal, we apply the Allan variance formalism [56, 64] to the measured values,

$$\sigma_{\text{Allan}}^2(\tau_A) = \frac{1}{2\tau_A} \langle (a_{n,2} - 2a_{n,1} + a_{n,0})^2 \rangle_n,$$

In this notation,

$$a_{n,s} = \sum_{m=0}^{n+\tau_A/dt} v(m)$$

are integrals over the measured $v$ values (or $u$ values) and $\langle \cdots \rangle_n$ denotes the mean over $n$, running from $n = 1$ to $n = N - \tau_A/dt$, where $N$ is the total number of data points and $dt$ is the sampling time. Assuming that the signal is dominated by telegraph-like switching with lifetime $\tau$ and amplitude $B$, we obtain [55]:

$$\sigma_{\text{Allan}}^2(\tau_A) = -B^2 \frac{4\tau_A/\tau \pm e^{-4\tau_A/\tau} - 4e^{-2\tau_A/\tau} + 3}{4\tau_A^2/\tau^2}.$$
noise, and the definition of the lifetime is well established [66]. In contrast, in our synthetic TLS, the switching statistics can be much richer and its observation are not necessarily limited by quantum backaction [45]. The present work highlights this richness in a classical synthetic TLS coupled to a noisy bath. The noise induces random transitions between different oscillation states that are separated by a phase gap. In this general setting, the observed telegraph noise is replaced by a complex random walk in phase space that reflects a synthetic potential landscape [67]. In spite of this complexity, we have shown that the notion of a “level” lifetime can be recovered when utilizing an appropriate metric for the switching rate $\Gamma$. This allows us to strengthen the analogy between synthetic and natural TLS, which is of importance for the rapidly growing field of NISQ simulators [28, 29], where statistical switching between synthetic levels plays a fundamental role as a randomizing source of “simulated annealing” [30–32, 38, 40, 41, 68].

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Supplementary Material for “Fluctuating Trajectories and Switching Rates of a Synthetic Two-Level System”

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S1. PROBABILITY DENSITY

In this section, we briefly detail our analysis steps yielding the probability density shown in Fig. 2(c) in the main text. First, we find the deterministic equations of motion for the “slow” quadrature amplitudes $u$ and $v$. To this end, we use the so-called van der Pol transformation and lowest-order Krylov–Bogoliubov averaging method [1–3], to replace the full time-dependent equation of motion [cf. Eq. (1) in the main text] by time-independent averaged equations of motion. The white noise term $\xi$ is transformed in a similar way using the method described in [4, 5] which yields

$$\dot{u} = -\frac{\gamma u}{2} - \left( \frac{3\alpha}{8\omega_d} X^2 + \frac{\omega_0^2 - \omega_d^2}{2\omega_d} + \frac{\lambda \omega_d^2}{4\omega_d^2} \right) v + \frac{1}{\sqrt{2\omega_0}} \Xi_u,$$

$$\dot{v} = -\frac{\gamma v}{2} + \left( \frac{3\alpha}{8\omega_d} X^2 + \frac{\omega_0^2 - \omega_d^2}{2\omega_d} - \frac{\lambda \omega_d^2}{4\omega_d^2} \right) u + \frac{1}{\sqrt{2\omega_0}} \Xi_v,$$  \hspace{1cm} (S1)

where $\omega_d = 2\pi f_d$ is the angular demodulation frequency and $X^2 = u^2 + v^2$. $\Xi_u$ and $\Xi_v$ are new independent white noise processes of strength $\sigma$. Equations (S1) provide a good description of the system when $\lambda, \gamma/\omega_0, (\alpha/\omega_d^2)x^2$, and $\alpha \sigma^2/\omega_0^2$ are small.

We use the stochastic differential equations (S1) to derive the corresponding Fokker-Planck equation [6, 7] that describes the time evolution of the probability density $p(u,v,t)$:

$$\partial_t p(u,v,t) = -\partial_u \left\{ \frac{1}{2\omega_d} \left[ \gamma \omega_d u + \left( \frac{3}{4} X^2 + \left( \frac{\omega_0^2 - \omega_d^2}{2\omega_d} + \frac{\lambda \omega_d^2}{2\omega_d^2} \right) v \right) p \right] \right. $n$ 

$$- \partial_v \left[ \frac{1}{2\omega_d} \left( \gamma \omega_d v + \left( -\frac{3}{4} X^2 - \left( \frac{\omega_0^2 - \omega_d^2}{2\omega_d} + \frac{\lambda \omega_d^2}{2\omega_d^2} \right) u \right) p \right] + \frac{1}{2} \left( \frac{\sigma}{\sqrt{2\omega_d}} \right)^2 (\partial_{u}^2 + \partial_{v}^2) p. \hspace{1cm} (S2)$$

We then solve this partial differential equation (PDE) numerically for the steady state ($\dot{p} = 0$) which is reached after long times and plot the outcome for the experimental parameters in Fig. 2(c).

S2. ALLAN VARIANCE FOR A SYSTEM DOMINATED BY TELEGRAPH NOISE

In this section, we provide a short derivation of the Allan variance of a system subject to telegraph noise [cf. Eq. (5) in the main text]. The Allan variance can be calculated via the autocorrelation function. For a random telegraph signal $x$ which switches between $B$ and $-B$ the autocorrelation function is given by [8]

$$\langle x(t)x(t+\tau_\lambda) \rangle = B^2 e^{-2\tau_\lambda/\tau}, \hspace{1cm} (S3)$$
where $1/\tau$ is the mean rate of transitions. The Allan variance is obtained by the following expectation value for the cumulative amplitudes $y(t) = \int_0^t x(t')dt'$

$$\sigma_A^2(\tau_A) = \frac{1}{2\tau_A^2} \langle (y(t + 2\tau_A) - 2y(t + \tau_A) + y(t))^2 \rangle$$  \hspace{1cm} (S4)

plug-in definition of $y$

$$\equiv \frac{1}{2\tau_A^2} \left\langle \left( \int_0^{t+2\tau} x(t')dt' - 2 \int_0^{t+\tau} x(t')dt' + \int_0^t x(t')dt' \right)^2 \right\rangle$$

cancel terms in parenthesis

$$\equiv \frac{1}{2\tau_A^2} \left\langle \left( - \int_0^{t+\tau} x(t')dt' + \int_0^{t+2\tau} x(t')dt' \right)^2 \right\rangle$$

expand the square

$$\equiv \frac{1}{2\tau_A^2} \left\langle 2 \int_0^{t+\tau} \int_0^t x(t')x(t'')dt'dt'' - 2 \int_0^{t+\tau} \int_0^t x(t' + \tau_A)x(t'')dt'dt'' \right\rangle$$

apply Eq. (S3)

$$-B^2 \frac{4\tau_A}{\tau} e^{-4\tau_A/\tau} - 4e^{-2(\tau'' + \tau_A)/\tau} dt''dt'$$

evaluate the integral

$$-B^2 \frac{4\tau_A^2}{\tau^2} e^{-4\tau_A/\tau} - 4e^{-2\tau_A/\tau} + 3 \frac{4\tau_A^2}{\tau^2}.$$  

We can now look for the maximum of the obtained expression by setting the first derivative equal to 0, i.e. $d\sigma_A^2/d\tau_A \equiv 0$. Expanding the numerator in $\tau_A$ around $\tau$ up to forth order, we find that the maximum is located at $\tau_A \approx 0.946\tau \approx \tau$.

S3. INVESTIGATION OF CIRCLE THRESHOLD RADIUS

![FIG. S1. Parametron switching rate dependence on the radius of the threshold circle.](image-url)

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