A representation of angular momentum ($SU(2)$) algebra

Wu-Sheng Dai$^{1,2}$ *and Mi Xie$^3$ †

1 School of Science, Tianjin University, Tianjin 300072, P. R. China
2 LuiHui Center for Applied Mathematics, Nankai University & Tianjin University,
Tianjin 300072, P. R. China
3 Department of Physics, Tianjin Normal University, Tianjin 300074, P. R. China

Abstract

This paper seeks to construct a representation of the algebra of angular momentum ($SU(2)$ algebra) in terms of the operator relations corresponding to Gentile statistics in which one quantum state can be occupied by $n$ particles. First, we present an operator realization of Gentile statistics. Then, we propose a representation of angular momenta. The result shows that there exist certain underlying connections between the operator realization of Gentile statistics and the angular momentum ($SU(2)$) algebra.

PACS codes: 03.65.Fd, 75.10.Jm

Keywords: representation of angular momentum, operator realization, intermediate statistics

1. Introduction

It is known that there exist some interesting connections between the algebra of angular momentum operators and the algebra of boson operators. We begin by recalling two kinds of representations of angular momentum operators: the Holstein-Primakoff [1] and the Schwinger [2] representations. These two representations are very successful in describing magnetism in various quantum systems [3, 4, 5].

The main idea of the Schwinger representation is to map the angular momentum operators onto boson operators $a_1$ and $a_2$ according to

$$
\begin{align*}
J_+ &= a_1^\dagger a_2, \\
J_- &= a_2^\dagger a_1, \\
J_z &= \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2),
\end{align*}
$$

(1)

where $a_1$ and $a_2$ are two independent boson operators satisfying $[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 1$, and any other pair of operators commute. The Holstein-Primakoff representation is

$$
\begin{align*}
J_+ &= \frac{1}{2}\sqrt{2j+N}a, \\
J_- &= \frac{1}{2}\sqrt{2j-N}a^\dagger, \\
J_z &= j-N, \quad (N = 0, 1, \cdots 2j),
\end{align*}
$$

(2)

*Email:daiwusheng@tju.edu.cn
†Email:xiemi@mail.tjnu.edu.cn
where \( j \) is the magnitude of the angular momentum. But, obviously, the representation is not faithful because for \( N > 2j \) it leads to unphysical values of angular momentum. In other words, the statistics corresponding to the creation and annihilation operators \( a \) and \( a^\dagger \), though they satisfy the boson commutation relations, is not the real Bose-Einstein statistics, since the occupation number \( N \) in Bose-Einstein statistics can take any integer. This means that, in fact, the Holstein-Primakoff transformation is not really a bosonic realization of the algebra of angular momentum, but a realization corresponding to a certain kind of intermediate statistics between Bose-Einstein and Fermi-Dirac statistics.

By comparison of the Schwinger and the Holstein-Primakoff representations, we see that if we want to obtain a bosonic realization of the algebra of angular momentum (\( SU(2) \) algebra), we need to use \( two \) independent boson operators \( a_1 \) and \( a_2 \); otherwise, if we want to give a realization with only \( one \) operator \( a \), we need a kind of statistics beyond the Bose-Einstein and the Fermi-Dirac cases. This result implies that we can establish a kind of representation for the angular momentum with a single set of creation and annihilation operators. Eventually, we find that the angular momentum can be represented in terms of operators corresponding to a kind of fractional statistics —— Gentile statistics [6].

As a generalization of Bose-Einstein and Fermi-Dirac statistics, fractional statistics has been discussed for many years. As is well known, the wave function will change a phase factor when two identical particles exchange. The phase factor can be \(+1\) or \(−1\) related to bosons or fermions, respectively. When this result is generalized to an arbitrary phase factor \( e^{i\theta} \), the concept of anyon is obtained [7, 8]. The corresponding statistics is fractional statistics. Another way leading to fractional statistics is based on counting the number of many-body quantum states, i.e., generalizing the Pauli exclusion principle [6, 9, 10]. Such an idea can be used to deal with the interacting many-body problem [11]. The most direct generalization is to allow more than one particles occupying one state. Based on this idea, Gentile constructed a kind of fractional statistics [6]. Bose-Einstein or Fermi-Dirac statistics becomes its limit case when the maximum occupation number of one state equals to \( \infty \) or \( 1 \), respectively; Moreover, when the maximum occupation number is very large but not infinity, Gentile statistics can achieve a kind of statistics which is very different from the Bose-Einstein case [12].

Different kinds of statistics correspond to different relations of creation and annihilation operators: bosons correspond to commutativity and fermions to anti-commutativity. To construct a representation for angular momentum, we, first, need to establish a set of operator relations corresponding to Gentile statistics, i.e., the maximum occupation number is \( n \ (1 < n < \infty) \). Based on such operator relations, we can establish a representation for angular momenta with only one set of creation and annihilation operators. Furthermore, we will discuss the relation between the maximum occupation number \( n \) in Gentile statistics and the magnitude of the angular momentum \( j \). The result shows that there exist certain underlying connections between Gentile statistics and the angular momentum (or, in other words, \( SU(2) \) algebra).

In the following, we will (1) establish a set of operator relations based on the idea given by Gentile that a quantum state can be occupied by \( n \) particles in Section 2, and then (2) present a kind of realization of the algebra of angular momentum by use of these creation and annihilation operators in Section 3.

2. Operator realization of Gentile statistics

We know that nature realizes only two kinds of particles: one obeys Bose-Einstein statistics and the other obeys Fermi-Dirac statistics. In other words, the only kinds of particles which appear in nature are either bosons, for which symmetric wave functions are required, or fermions, which
have only antisymmetric functions. The maximum occupation numbers for any state in the
two kinds of statistics, due to the Pauli exclusion principle are \( \infty \) and 1, respectively. However,
Gentile suggested a scheme of intermediate statistics \[6\], in which the maximum occupation
number, denoted by \( n \), can be chosen arbitrarily between \( \infty \) and 1. In the following, we will
construct a set of relations of creation, annihilation and number operators of the particles which
obey the intermediate statistics. The commutativity and the anti-commutativity will be the
two limit cases of such operator relations.

Let \( |\nu\rangle_n \) express the state which contains \( \nu \) particles, where subscript \( n \) represents that no
more than \( n \) particles can be accommodated in the state. The \( |\nu\rangle_n \) is the eigenstate of number
operator \( N \):

\[
N|\nu\rangle_n = \nu|\nu\rangle_n.
\]

Let \( a^\dagger \) be the creation operator and \( b \) the annihilation operator (the reason why the creation
and the annihilation operators are not hermitian conjugate will be explained later). Since \( n \) is
the maximum occupation number, the creation and annihilation operators \( a^\dagger \) and \( b \) satisfy the
following conditions:

\[
\begin{aligned}
\{ &a^\dagger|n\rangle_n = 0 \text{ or } (a^\dagger)^{n+1}|0\rangle_n = 0, \\
&b|0\rangle_n = 0 \text{ or } b^{n+1}|n\rangle_n = 0.
\end{aligned}
\]

(4)

The relations between creation and annihilation operators for fermions and bosons are

\[
\begin{aligned}
\{ &aa^\dagger - a^\dagger a = 1 \text{ for bosons}, \\
&aa^\dagger + a^\dagger a = 1 \text{ for fermions}.
\end{aligned}
\]

(5)

The relations that we want to find must return to the cases of bosons and fermions when \( n = \infty \)
and \( n = 1 \), so a direct way is to assume the commutation relation takes the form

\[
ba^\dagger = f(n)a^\dagger b = 1,
\]

(6)

and \( f(n) \) satisfies

\[
f(1) = -1, \quad f(\infty) = 1.
\]

(7)

For being consistent with these conditions, we choose

\[
f(n) = e^{i\frac{2\pi n}{n+1}}.
\]

(8)

Thus, the basic commutation relation is

\[
ba^\dagger - e^{i\frac{2\pi n}{n+1}}a^\dagger b = 1.
\]

(9)

To coincide with the restrictions of Eqs. (4) and (9), the results of \( a^\dagger \) and \( b \) acting on the state
vectors can be chosen in the following form:

\[
\begin{aligned}
\{ &a^\dagger|\nu\rangle_n = \sqrt{\frac{1-e^{i2\pi\nu/(n+1)}}{1-e^{i2\pi/(n+1)}}} |\nu+1\rangle_n = \sqrt{\langle \nu+1 \rangle_n} |\nu+1\rangle_n, \\
&b|\nu\rangle_n = \sqrt{\frac{1-e^{i2\pi\nu/(n+1)}}{1-e^{i2\pi/(n+1)}}} |\nu-1\rangle_n = \sqrt{\langle \nu \rangle_n} |\nu-1\rangle_n,
\end{aligned}
\]

(10)

where the following notation is introduced:

\[
\langle \nu \rangle_n \equiv \frac{1 - e^{i2\pi \nu/(n+1)}}{1 - e^{i2\pi/(n+1)}} = \sum_{j=0}^{\nu-1} e^{i2\pi j/(n+1)}.
\]

(11)
According to Eq. (10), all the states can be generated from the ground state \( |0\rangle_n \) by successive operations of \( a^\dagger \):

\[
|\nu\rangle_n = \frac{(a^\dagger)^\nu}{\sqrt{\prod_{j=1}^{\nu} \langle j \rangle_n}} |0\rangle_n.
\]

(12)

It is easy to verify that \( n\langle \nu | \nu' \rangle_n = \delta_{\nu\nu'} \).

The commutation relations between number operator and \( a^\dagger \) and \( b \) can be directly derived:

\[
[N, a^\dagger] = a^\dagger, \quad [N, b] = -b.
\]

(13)

Similar to the cases of bosons and fermions, the number operator can be expressed by the creation and annihilation operators \( a^\dagger, a, b^\dagger \) and \( b \). It should be emphasized that the expression of \( N \) is not unique. In fact, in the case of fermion, \( b^\dagger f b_f \) and \((b^\dagger f b_f)^2\), etc., all can be used as the number operator, where \( b^\dagger_f \) and \( b_f \) are the creation and annihilation operators of fermions. The number operator can be expressed as

\[
N = \frac{n + 1}{2\pi} \arccos \left[ \frac{1}{2} (ab^\dagger + ba^\dagger - a^\dagger b - b^\dagger a) \right] = \frac{n + 1}{4} - \frac{n + 1}{2\pi} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+1}(n!)^2(2n + 1)} (ab^\dagger + ba^\dagger - a^\dagger b - b^\dagger a)^{2n+1}.
\]

(14)

\( (ab^\dagger + ba^\dagger - a^\dagger b - b^\dagger a)^{2n+1} \)

The reason why creation and annihilation operators are not hermitian conjugate of each other can be realized from Eq. (10). Brief calculation gives

\[
a^\dagger b|\nu\rangle_n = \langle \nu\rangle_n |\nu\rangle_n.
\]

(16)

\( \langle \nu\rangle_n \), the eigenvalue of \( a^\dagger b \), is not a real number in general, so \( a^\dagger b \) is not an hermitian operator. It means that the creation operator can not be the hermitian conjugate of the annihilation operator. Only when \( n = \infty \) or \( n = 1 \), \( \langle \nu\rangle_n \) becomes real, so the operator \( a^\dagger b \) will be hermitian. In such cases, we have \( (a^\dagger b)^\dagger = a^\dagger b \) and hence \( a^\dagger = b^\dagger \). This is just the case of bosons or fermions.

For considering the conjugate operators of \( a^\dagger \) and \( b \), taking the hermitian conjugate of Eq. (9), we have

\[
ab^\dagger - e^{-i\frac{2\pi}{n+1}} b^\dagger a = 1.
\]

(17)

Furthermore, a set of relations for \( a \) and \( b^\dagger \) analogous to Eq. (10) can be obtained easily replacing \( \langle \nu\rangle_n \) and \( \langle \nu + 1\rangle_n \) by their complex conjugate. The results show that \( b^\dagger \) and \( a \) can play roles similar to creation and annihilation operators \( a^\dagger \) and \( b \).

According to the results of the operators acting on the state vectors, the following operator relations among \( a^\dagger, b, b^\dagger \) and \( a \) can be obtained:

\[
a = b^*, \quad a^\dagger a = b^\dagger b, \quad a a^\dagger = b b^\dagger, \quad a a^\dagger a + a^\dagger a = 2 \cos \frac{\pi}{n+1} a a^\dagger a, \quad b b^\dagger + b^\dagger b = 2 \cos \frac{\pi}{n+1} b b^\dagger b, \text{ etc.}
\]

(18)

and

\[
a^\dagger a|\nu\rangle_n = b^\dagger b|\nu\rangle_n = |\langle \nu\rangle_n| |\nu\rangle_n
\]

\[
a a^\dagger |\nu\rangle_n = b b^\dagger |\nu\rangle_n = |\langle \nu + 1\rangle_n| |\nu\rangle_n.
\]

(19)
3. Representation of algebra of angular momentum

So far we have obtained an operator realization of Gentile statistics in which one quantum state can be occupied by \( n \) particles. Based on this result, we now consider the realization of angular momenta.

Straightforwardly, one can check that for the case of \( n = 1 \) (the fermion case) we have

\[
\begin{align*}
J_+ &= a^\dagger, \\
J_- &= a, \\
J_z &= N - \frac{n}{2},
\end{align*}
\]

and

\[
\begin{align*}
[J_+, J_-] &= 2J_z, \\
[J_z, J_\pm] &= \pm J_\pm.
\end{align*}
\]

Obviously, the relations among the operators \( J_+, J_- \) and \( J_z \) are the same as those of angular momentum \( J = J_x i + J_y j + J_z k \) (where \( J_\pm = J_x \pm iJ_y \)). It should be emphasized that, this result only holds for \( n = 1 \). In the case of \( n = 1 \), the maximum occupation number is 1 and hence there are only two possible states: \(|0\rangle_{n=1} \) and \(|1\rangle_{n=1} \). This means that corresponding to the case \( n = 1 \), the magnitude of the angular momentum \( j \) must be \( \frac{1}{2} \) since for \( j = \frac{1}{2} \) there are also only two states: \(|+\frac{1}{2}\rangle \) and \(|-\frac{1}{2}\rangle \).

Similarly, we can also construct a realization for angular momenta \( j = 1 \) in terms of the creation and annihilation operators corresponding to \( n = 2 \) since they both have three states: \(|+1\rangle, |0\rangle, |−1\rangle \) and \(|0\rangle_{n=2}, |1\rangle_{n=2}, |2\rangle_{n=2} \). It can be verified in a straightforward fashion that in this case

\[
\begin{align*}
J_+ &= \sqrt{2}a^\dagger, \\
J_- &= \sqrt{2}a, \\
J_z &= N - \frac{n}{2},
\end{align*}
\]

and \( J_\pm, J_z \) satisfy the relations Eq. (21).

These results imply an underlying relation between the magnitude of the angular momentum \( j \) and the maximum occupation number \( n \): To realize the angular momentum \( J \), one needs to use the creation and annihilation operators which correspond to \( n = 2j \). Then, we wish, therefore, to construct realizations for the other angular momenta.

For \( j = \frac{3}{2} \), we can chose

\[
\begin{align*}
J_+ &= \lambda_1 a^\dagger + \lambda_2 b^\dagger, \\
J_- &= \lambda_1 a + \lambda_2 b, \\
J_z &= N - \frac{n}{2},
\end{align*}
\]

where

\[
\begin{align*}
\lambda_1 &= \frac{1}{2\sqrt{2}}(2 + 2^{3/4})(1 + i), \\
\lambda_2 &= \frac{1}{2^{1/4}} - 1.
\end{align*}
\]

These relations hold for \( n = 3 \).

For \( j = 2 \) and \( n = 4 \):
\[
\begin{aligned}
J_+ &= \lambda_1^* a^\dagger + \lambda_2^* b^\dagger, \\
J_- &= \lambda_1 a + \lambda_2 b, \\
J_z &= N - \frac{n}{2},
\end{aligned}
\]  
(25)

where

\[
\lambda_1 = \sqrt{\frac{2}{5}} \left( 5\sqrt{5} - 5 + \sqrt{182}\sqrt{5} - 370 - i \frac{\sqrt{2}}{4} \sqrt{5 + 3\sqrt{5} + \sqrt{62 + 158}} \right),
\]

\[
\lambda_2 = \sqrt{5} - \sqrt{\frac{22}{\sqrt{5}}} - 5.
\]  
(26)

In the above cases, the operators \( J_+ \) and \( J_- \) can be expressed as a linear combination of the creation and annihilation operators. However, for \( j \geq \frac{5}{2} \) and \( n \geq 5 \), we need to use high order terms of the creation and annihilation operators such as \( a^\dagger a^\dagger \). For instance, in the case of \( j = \frac{5}{2} \) and \( n = 5 \)

\[
\begin{aligned}
J_+ &= \lambda_1^* a^\dagger + \lambda_2^* b^\dagger + \lambda_3^* a^\dagger a^\dagger, \\
J_- &= \lambda_1 a + \lambda_2 b + \lambda_3 a^\dagger a, \\
J_z &= N - \frac{n}{2},
\end{aligned}
\]  
(27)

where

\[
\begin{aligned}
\lambda_1 &= \sqrt{\frac{9}{2} - \frac{1}{\sqrt{3}}} - i 3^{1/4}, \\
\lambda_2 &= 0, \\
\lambda_3 &= i \sqrt{\frac{1}{\sqrt{3}} - \frac{1}{2}}.
\end{aligned}
\]  
(28)

Also, for other values of \( j \) and \( n \), we can obtain the realizations for other angular momentum operators.

In terms of creation and annihilation operators corresponding to Gentile statistics, we have constructed a kind of realization for angular momentum (\( SU(2) \)) algebra. It is important to notice that such a realization scheme is not unique.

4. Discussion

At the beginning of this paper, we have compared two kinds of representations (the Holstein-Primakoff and the Schwinger representations) of the algebra of angular momentum, and pointed out that if one wants to realize the algebra of angular momentum by means of only one set of creation and annihilation operators, he must necessarily make a departure from Bose-Einstein statistics and make use of a kind of intermediate statistics just like what has been done in the Holstein-Primakoff representation. One aim of this paper is to provide an operator realization of algebra of angular momentum. To achieve this, we begin by first showing that we can provide an operator realization of Gentile statistics (a kind of immediate statistics in which one quantum state can be occupied by \( n \) particles). Based on such operator relations for the creation and annihilation operators, we construct a representation of the algebra of angular momentum (\( SU(2) \) algebra).
Different \( n \) (the maximum occupation number) corresponds to different kinds of statistics. The two special cases are \( n = 1 \) and \( \infty \), which are Bose-Einstein statistics and Fermi-Dirac statistics. Moreover, for the other values of \( n \), e.g. \( n = 2, 3, ... \), we have various kinds of intermediate statistics. In Section 3 we find that the angular momentum with magnitude \( j = \frac{1}{2} \) can be expressed by the creation and annihilation operators corresponding to \( n = 1 \), and moreover, the case \( j = \frac{1}{2} \) corresponds to \( n = 2 \), the case \( j = \frac{3}{2} \) corresponds to \( n = 3 \) and so on. The parameter \( n \) is physically meaningful, it labels various kinds of statistics. The result implies that there exist some underlying connections between the angular momentum (\( SU(2) \)) algebra and the operator realization of Gentile statistics.

The representation for angular momentum introduced above is based on the operator realization of Gentile statistics. Clearly, the particles which obey Gentile statistics must not be real particles. In other words, this representation, just as the case in the Holstein-Primakoff representation, is related to a kind of imaginary particles. It can be expected that, this result can be used to deal with some complex interaction systems.

We would like to thank Dr. Yong Liu for sending us some important references, and we are very indebted to Dr. G. Zeitrauman for his encouragement. This work is supported in part by LiuHui fund.

References

[1] T. Holstein and H. Primakoff, Phys. Rev. 58 (1940) 1098.
[2] J. Schwinger, Quantum Theory of Angular Momentum, ed. L. Biedenharn, Academic Press, New York, 1965.
[3] C. Timm, S. M. Girvin and P. Henelius, Phys. Rev. B 58 (1998) 1464.
[4] C. Timm and P. J. Jensen, Phys. Rev. B 62 (2000) 5634.
[5] C. Kittel, Quantum theory of solids, John Wiley, New York, 1987.
[6] G. Gentile, Nuovo. Cim. 17 (1940) 493; A. Khare, it Fractional Statistics and Quantum Theory, World Scientific, Singapore, 1997.
[7] F. Wilczek, Phys. Rev. Lett. 48 (1982) 1144.
[8] F. Wilczek, Phys. Rev. Lett. 49 (1982) 957.
[9] F. D. M. Haldane, Phys. Rev. Lett. 67 (1991) 937.
[10] Y.-S. Wu, Phys. Rev. Lett. 73 (1994) 922.
[11] M. D. Johnson and G. S. Canright, Phys. Rev. B 41 (1990) 6870.
[12] W.-S. Dai and M. Xie, to appear in Ann. Phys. (N. Y.); preprint cond-mat/0310066.