ABSTRACT

We analyze the structure of higher-order radiative corrections for processes with unstable particles. By subsequently integrating out the various scales that are induced by the presence of unstable particles we obtain a hierarchy of effective field theories. In the effective field theory framework the separation of physically different effects is achieved naturally. In particular, we automatically obtain a separation of factorizable and non-factorizable corrections to all orders in perturbation theory. At one loop this treatment is equivalent to the double-pole approximation (DPA) but generalizes to higher orders and, at least in principle, to beyond the DPA. It is known that one-loop non-factorizable corrections to invariant mass distributions are suppressed at high energy. We study the mechanism of this suppression and obtain estimates of higher-order non-factorizable corrections at high energy.
1 Introduction

An important area in the physics programme of a future electron-positron linear collider (LC) belongs to the detailed studies of the pair production of heavy unstable particles ($W$ and $Z$ bosons, top quarks, SUSY particles, etc.) [1]. One of the main goals of these measurements is the high precision determination of their parameters, primarily their masses.

In recent years there has been a growing interest in high energy photon colliders (see e.g. [2]), which in some cases offer certain advantages over the $e^+e^-$ collisions in the exploration of the unstable particles, see for example [3]. Pairs of unstable particles can also be produced in hadron collisions at the Tevatron and LHC. All these studies would allow a variety of detailed tests of the Standard Model to be performed, both in the electroweak and in the strong interaction sectors. This, in turn, requires very accurate theoretical knowledge of the production and decay mechanism and of interference effects between them. This interference is caused by the large width, $\Gamma$, of many of these objects. The widths of the $W$, $Z$ and $t$ are all of the order of 2 GeV, and many supersymmetric and other beyond-the-Standard-Model particles would also have widths in the GeV range.

Although many results of this paper are applicable to the non-relativistic case as well, we concentrate on relativistic and ultra-relativistic energy regimes. In the center-of-mass frame the physical picture of processes involving unstable particles can be viewed as a consequence of several subprocesses. First the unstable particles are produced during a short time scale $\mathcal{O}(1/E)$. Then during a large proper time, $\sim 1/\Gamma$, they separate from each other on a large distance, $\sim 1/\Gamma \cdot E/M$. Finally, they decay during a short time, $\sim 1/E$, into their decay products. Here, $E$, $M$ and $\Gamma$ are the energy, mass and width of the unstable particle respectively. If $E^2 \gg M\Gamma$, the production and decay subprocesses are well separated from each other. The so-called pole-scheme [4] is a framework which makes use of the presence of two scales in the problem by expanding the amplitudes in $\Gamma/M$. In particular, one can keep only the leading terms, neglecting all the terms suppressed by powers of $\Gamma/M$. In the case of pair production of unstable particles such an approximation is usually called the double-pole approximation (DPA). The DPA guarantees gauge invariance of the calculation and simplifies it significantly. The DPA allowed the calculation of the full $\mathcal{O}(\alpha)$ electroweak corrections to the pair production of $W$ and $Z$ bosons in $e^+e^-$ collisions [5]. It is essential for the DPA that unstable particles are close to resonance and that $E^2 \gg M\Gamma$. This is also the regime which we will specialize to throughout this paper.

Within the DPA at one loop one can readily classify the radiative effects into two types: factorizable, which act inside the separate hard subprocesses (production and decay); and non-factorizable, which interconnect various hard subprocesses. Depending on the subprocesses they interconnect, the non-factorizable corrections can be of two types: decay-decay and production-decay. The non-factorizable corrections are believed to be an important ingredient of the theoretical predictions. In particular, they affect the measurement of the mass of the unstable particle. They have been a subject of intensive study recently, see for example [6]-[21].

The properties of perturbative non-factorizable corrections depend crucially on the level of
“inclusiveness” of the distributions. The distributions can be of three distinct types. For virtual corrections only two of them are relevant: invariant mass and angular distributions. Since real gauge-boson (photon, gluon) radiation contributes to the non-factorizable corrections as well one can consider additional distributions where the non-factorizable real radiation is added and an integration over the real emission phase space is performed. At one loop in QED the non-factorizable corrections to the distributions inclusive in real photon and invariant mass are suppressed by at least $\alpha \Gamma / M$, \cite{10, 11}. In \cite{10} general arguments are given indicating that this should still be the case at higher loops. As a consequence, for a particular class of the one-loop non-factorizable corrections (initial-final state interferences) the virtual contribution is cancelled by the real emission even for the distributions exclusive in the invariant mass. In \cite{12} a first calculation of one-loop non-factorizable corrections exclusive in invariant mass but inclusive in real radiation was performed. In \cite{13, 14} the one-loop non-factorizable corrections were calculated for completely exclusive distributions. Although the latter calculations were originally performed for pair production of $W$ bosons, the results were applied also to other cases, for example to one-loop QCD non-factorizable corrections in top-quark pair production \cite{15} and to one-loop QED non-factorizable corrections to the pair production of $Z$ bosons \cite{16}.

Even though one-loop non-factorizable corrections are extensively studied by now, almost nothing is known about the higher-loop non-factorizable effects. An explicit calculation of two-loop non-factorizable corrections is well beyond present capabilities. Even a precise definition of higher-loop non-factorizable corrections is lacking at present. Higher-order non-factorizable corrections can be quite important, especially in QCD. Note that all of the one-loop results above did not assume the distributions to be inclusive in the angles. In \cite{17} it was found that the one-loop non-factorizable correction to the distribution inclusive in real photons and angles, but not in the invariant mass, is suppressed by $\alpha (M/E)^4$ at high energies, $E \gg M$. This is consistent with an earlier observation of Ref. \cite{12}, where a similar effect was observed for a simplified toy model. A similar energy dependence is also found for the one-loop QCD interconnection effects in $t \bar{t}$ production \cite{15, 17}. It is the main purpose of this paper to study the mechanism of the high energy suppression. This allows us to obtain information about the higher-order non-factorizable corrections to the distributions inclusive in angles.

An indication of the importance of higher-loop QCD corrections comes from the studies of non-perturbative QCD interconnection effects. These effects are an essential source of the systematic uncertainties in the reconstruction of the $W$-boson or top-quark parameters. Recall that not far from threshold the typical decay time, $\tau \sim 1/\Gamma \approx 0.1 \text{fm}$, is much shorter than the characteristic hadronization time, $\tau_{\text{had}} \approx 1 \text{ fm}$. Thus, the final state hadronic systems overlap between pairs of resonances ($W^+ W^-, Z^0 Z^0, t \bar{t}$, etc). Non-perturbative effects at the hadronization stage are usually modelled by a colour rearrangement between the partons produced in the two resonance decays and the subsequent parton showers \cite{18}. This topic is deeply related to the confinement physics. Thus, interconnection studies not only offer an opportunity to investigate the dynamics of unstable particles, but they also open new ways to probe confinement forces in space and time \cite{18, 19}.

So far, the experimental analysis of the non-factorizable effects have been performed only at LEP2 and mainly in the context of precise $W$ mass measurements \cite{20}. A LC will allow a
series of very accurate measurements of $W^+W^-$, $ZZ$, and $t\bar{t}$ production in a wide energy range. Especially promising for the purposes of precise mass determination and interconnection studies look high luminosity runs of a LC in the threshold regions [19, 21]. A detailed knowledge of the energy dependence of the interconnection effects in $W^+W^-$ production would allow to choose the optimal strategy for their studies at a LC.

The purpose of this paper is twofold. First, we discuss the separation between the different radiative phenomena to all orders in perturbation theory. We find the following different effects:

- There are factorizable and non-factorizable corrections.
- The non-factorizable corrections can be of the production-decay and decay-decay types.
- The non-factorizable corrections of each type can receive corrections due to two effects: interaction between the production/decay dipoles and propagation corrections.

We start with the separation of these effects in QED within the DPA by analyzing Feynman diagrams. This is an extension of the well studied one-loop QED case. Subsequently we find an effective field theory interpretation of this separation, based on the presence of a hierarchy of scales in the problem, $E^2 \gg M\Gamma$. The existence of an effective field theory allows us to generalize the separation between different effects to the more complicated (non-abelian) QCD case. Also the DPA appears naturally within the effective field theory framework. This implies – at least in principle – the possibility to study $\Gamma/M$ suppressed (beyond the DPA) effects in a consistent way.

Second, we focus on the high-energy, $E \gg M$, behavior of the non-factorizable corrections to the distributions inclusive in angles of the decay products. We study the mechanism responsible for the suppression of the non-factorizable corrections at high energy. The suppression mechanism works differently for the different effects we identified. In this paper we limit ourselves to the decay-decay interferences, leaving aside the more complicated case of the production-decay interferences. We illustrate the suppression mechanism by estimating the QED decay-decay interferences, using a high energy expansion of Feynman diagrams. Subsequently we find an effective field theory interpretation of this mechanism. Based on the effective field theory approach we develop a general framework that allows us to generalize the estimates to the non-abelian QCD case. Finally, we consider some topical applications of our results.

The paper is organized as follows. In Sect. 2 we discuss the separation of different effects in QED by analysing Feynman diagrams. We also derive the high energy estimates of the non-factorizable decay-decay interferences. In Sect. 3 we analyse the structure of the modes induced by the hierarchy of scales, $E$ and $\Gamma$. By subsequently integrating out different modes we construct an effective field theory that provides us with a separation between the various radiative effects together with the high energy estimates. In Sect. 4 we discuss some physical applications of the high energy estimates of the non-factorizable corrections. Sect. 5 gives our conclusions and outlines the possible future developments and applications of our results.
2 Estimates from high energy expansion of amplitudes

In this section we investigate higher-loop QED non-factorizable corrections in a toy model by considering the high-energy expansion of Feynman diagrams. We work within the DPA, neglecting all contributions suppressed by $\Gamma/M$. The separation of factorizable and non-factorizable corrections is achieved by using the non-factorizable currents. This procedure is an extension of the one-loop separation procedure [5, 13, 14]. We subsequently study non-factorizable corrections to the distributions inclusive in decay angles in the high energy limit, $E \gg M$.

2.1 Separation of various contributions

In the Born approximation the hierarchy of scales $E^2 \gg M\Gamma$ implies the factorization of the full matrix element into a product of matrix elements corresponding to hard (production and decay of unstable particles) and soft (propagation of unstable particles) sub-stages of the process. The factorization holds with an accuracy of at least $O(\Gamma/M)$.

Radiative corrections are due to radiation of photons/gluons with typical energy $\Omega$. If the radiation is hard, $\Omega \sim E$, the inverse propagation distance of the unstable particles, $\sim \Gamma \cdot M/E$, remains to be the only soft scale in the problem. As a result, the factorization for the process works in the same way as in the Born approximation. On the other hand, if the radiation is soft, $\Omega \sim \Gamma \cdot M/E$, there are two soft scales. The hard matrix elements still factorize but the propagation of the unstable particles and the interactions via soft particles can mix now. In general, a part of the corrections will have a form where the propagation subprocess is factorized in the same way as it is factorized in the Born approximation, and a part of the corrections will not have such a factorized form.

A correction is factorizable (non-factorizable) if the propagation sub-stage factorizes (does not factorize) with respect to the rest of the process in the same way as in the Born approximation. Corrections due to the exchange of hard particles are always factorizable whereas corrections due to the exchange of soft particles can be both factorizable and non-factorizable. In the former case the split up between factorizable and non-factorizable corrections is unambiguously defined on the basis of the energy of the exchange particles. In the latter case the split up between factorizable and non-factorizable corrections can only be done by comparing corresponding matrix elements to that in the Born approximation. One expects that factorizable and non-factorizable contributions factorize with respect to each other beyond one loop.

The toy model we will consider has a neutral, scalar, unstable particle field $\phi$ with mass $M$ and width $\Gamma$. This particle decays into massless, charged fermions, $\psi$. The charged fermions couple to a $U(1)$ gauge field, $A_\mu$. The Lagrangian of the model is

$$\mathcal{L} = \frac{1}{2}\phi(p^2 - M^2)\phi + \bar{\psi} \not{p} \psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{P} X \phi \phi + \phi \bar{\psi} \not{D} \psi. \quad (1)$$

Here $p$ is the momentum operator and $\not{p}$ is the covariant momentum operator. The first three terms in Eq. (1) describe the propagation of $\phi$, $\psi$, and $A_\mu$ fields. The $\mathcal{P}$ term describes the
production of a pair of unstable particles from some neutral source $X$, whereas the $D$ term describes their decay into the fermions. We will assume that all these terms can be treated as a perturbation. To see how factorization occurs let us consider the process

$$X \rightarrow \phi(p_1)\phi(p_2) \rightarrow \psi^+(k_1)\psi^-(k_1')\psi^-(k_2)\psi^+(k_2').$$

where $p_i = k_i + k_i'$, $i \in \{1, 2\}$ and

$$p_1^2 - M^2 \sim \Gamma M, \quad p_2^2 - M^2 \sim \Gamma M$$

We assume there are no charged particles in the initial state.

### 2.1.1 One-loop order

The complete one-loop correction consists of one-loop factorizable and one-loop non-factorizable corrections. The separation between these two contributions is known \[5\]. In order to introduce notations let us recall briefly how it works. The hard modes contribute exclusively to the factorizable corrections, thus the separation needs to be performed only for soft photons. Let us start with the non-factorizable corrections. Since charged particles appear only in the final state, as decay products there will be only decay-decay interference. The decay-decay non-factorizable correction is given by

$$\int \equiv \int \frac{d^4 l}{(2\pi)^4 l^2}.$$ 

where we used a short hand notation for the loop integral

$$\int_c = iM_0 \int (\mathcal{J}_1 \cdot \mathcal{J}_2),$$

where $\mathcal{J}_1^\mu$ and $\mathcal{J}_2^\mu$ are called the non-factorizable currents. They are given by

$$\mathcal{J}_1^\mu = \mathcal{J}_1^\mu(l, D_1) = + e \left[ \frac{k_1^\mu}{l k_1} - \frac{k_1'^\mu}{l k_1'} \right] \frac{D_1}{D_1 + 2lp_1},$$

$$\mathcal{J}_2^\mu = \mathcal{J}_2^\mu(l, D_2) = + e \left[ \frac{k_2^\mu}{-l k_2} - \frac{k_2'^\mu}{-l k_2'} \right] \frac{D_2}{D_2 - 2lp_2},$$

where

$$D_{1,2} \equiv p_{1,2}^2 - M^2 + i\Gamma M,$$

is the off-shellness of the unstable particle. By inspection of the integral Eq. (4) it can be seen that the energy of the photon relevant for non-factorizable corrections is indeed $\Omega \sim \Gamma M/E$. 

5
The contribution of a hard photon is suppressed by $O(M^2 \Gamma^2/E^2)$ with respect to the leading contribution. Thus, the corresponding contributions are beyond the DPA.

The factorizable correction is given by

$$\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{factorizable}\end{array} + \begin{array}{c}
\includegraphics[width=1cm]{factorizable}\end{array} + \ldots = M_1.
\end{align*}$$

The important property of $M_1$ is that it has exactly the same dependence on $D_{1,2}$ as the Born matrix element $M_0$. This is the crucial difference between the factorizable and non-factorizable corrections. The non-factorizable corrections contain an additional strong $D_{1,2}$ dependence. Thus, contrary to the factorizable corrections, the non-factorizable corrections distort the invariant mass distribution.

2.1.2 Two-loop order

At two-loop the separation into factorizable and non-factorizable contributions is somewhat more involved. There are four distinct contributions, two of which are non-factorizable. First of all there is a two-loop factorizable correction given by the following diagrams

$$\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{factorizable} + \begin{array}{c}
\includegraphics[width=1cm]{factorizable}\end{array} + \ldots = M_2.
\end{align*}$$

Secondly, there are contributions that can be attributed to the interference between one-loop factorizable and one-loop non-factorizable contributions. An example of such a contribution in the soft approximation is

$$\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{factorizable} + \begin{array}{c}
\includegraphics[width=1cm]{factorizable}\end{array} = iM_0 \int_{l_1, l_2} k_1^\mu D_1 \frac{k_2^\mu}{l_1 k_1} D_2 \frac{k_2}{l_2 k_2} \frac{D_2}{D_2 - 2l_2 p_2} \times \frac{k_1^\nu}{l_1 k_1} \frac{k_1^\nu}{l_1 k_1} \cdot \frac{k_1^\nu}{l_1 k_1} \cdot \frac{k_1^\nu}{l_1 k_1}.
\end{align*}$$

The two-loop corrections of this type factorize into a product of one-loop factorizable corrections times one-loop non-factorizable corrections. The same factorization holds for hard momenta flow in the factorizable loop. Combination of all contributions of this type will lead to

$$\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{factorizable} + \begin{array}{c}
\includegraphics[width=1cm]{factorizable}\end{array} = iM_1 \int_{l_1} (\mathcal{J}_1 \cdot \mathcal{J}_2).
\end{align*}$$
The first factor, $\mathcal{M}_1$, is the same as the one-loop factorizable correction. The second factor is the same as one-loop non-factorizable correction.

In addition to the two contributions mentioned above, there are also two classes of non-factorizable contributions. The first class consists of all diagrams where two photons are exchanged between the two fermion pairs. Let us list here all these diagrams. There are diagrams like

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1.png}
\end{array}
\end{align*}
\]

\[
\sim \frac{1}{2} \int_{l_1, l_2} \frac{k_1^\mu}{k_1 l_1} \frac{k_1^\nu}{k_1 l_2} \frac{k_2^\mu}{-k_2 l_1} \frac{k_2^\nu}{-k_2 l_2},
\] (12)

where the integration is over the virtual photon momenta, $l_1$ and $l_2$, with the appropriate measure, Eq. (3). We omit factors $D_{1,2}/(D_{1,2} \pm 2l_1 p_{1,2} \pm 2l_2 p_{1,2})$ in the expression above. There are three more contributions of this form corresponding to interactions: $(k_1 k'_2)(k_1 k'_2)$, $(k'_1 k_2)(k'_1 k_2)$, $(k'_1 k'_2)(k'_1 k'_2)$. Another set of diagrams contributing to this class is given by

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram2.png}
\end{array}
\end{align*}
\]

\[
\sim -\frac{1}{2} \int_{l_1, l_2} \left[ \frac{k_1^\mu}{k_1 l_1} \frac{k_1^\nu}{k_1 l_2} \frac{k_2^\mu}{-k_2 l_1} \frac{k_2^\nu}{-k_2 l_2} + \frac{k_1^\mu}{k'_1 l_1} \frac{k_1^\nu}{k'_1 l_2} \frac{k_2^\mu}{-k'_2 l_1} \frac{k_2^\nu}{-k'_2 l_2} \right].
\] (13)

There are three additional contributions of this type: $(k_1 k'_2)(k'_1 k'_2)$, $(k_1 k_2)(k'_1 k'_2)$, $(k'_1 k_2)(k'_1 k'_2)$. Finally, the last set of diagrams contributing to the first class of non-factorizable corrections is

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3.png}
\end{array}
\end{align*}
\]

\[
\sim + \int_{l_1, l_2} \frac{k_1^\mu}{k_1 l_1} \frac{k_1^\nu}{k_1 l_2} \frac{k_2^\mu}{-k_2 l_1} \frac{k_2^\nu}{-k_2 l_2}.
\] (14)

There is one more diagram of this type: $(k_1 k_2)(k'_1 k'_2)$.

Combining all these contributions we obtain a gauge invariant expression for this part of the two-loop non-factorizable correction

\[
\frac{1}{2} i \mathcal{M}_0 \int_{l_1, l_2} \left( \frac{k_1}{l_1 k_1} - \frac{k_1'}{l_1' k_1} \right)^\mu \left( \frac{k_1}{l_2 k_1} - \frac{k_1'}{l_2' k_1} \right)^\nu \left( \frac{k_2}{-l_1 k_2} - \frac{k_2'}{-l_1' k_2} \right)_\mu \left( \frac{k_2}{-l_2 k_2} - \frac{k_2'}{-l_2' k_2} \right)_\nu \frac{D_1}{D_1 + 2l_1 p_1 + 2l_2 p_1} \frac{D_2}{D_2 - 2l_1 p_2 - 2l_2 p_2}
\] (15)

Using the non-factorizable currents this can be written as

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{final_diagram.png}
\end{array}
\end{align*}
\]

\[
\sim \frac{1}{2} i \mathcal{M}_0 \int_{l_1, l_2} (\mathcal{J}_1 \cdot \mathcal{J}_2)(l_1) * (\mathcal{J}_1 \cdot \mathcal{J}_2)(l_2),
\] (16)
where the currents are the same as in the one-loop case and are given by Eq. (6). The product of two currents is defined by the following current multiplication rules

\[ J_\mu^i(l_1, D) \ast J_\nu^i(l_2, D) = J_\mu^i(l_1, D) J_\nu^i(l_2, D \pm 2l_1p_i), \]

where \( \pm = + \) for \( i = 1 \) and \( \pm = - \) for \( i = 2 \). Furthermore

\[ J_\mu^i(l_1, D) \ast J_\nu^j(l_2, D) = J_\nu^j(l_2, D) \ast J_\mu^i(l_1, D) = J_\nu^j(l_1, D) J_\mu^i(l_2, D \pm 2l_1p_i), \]

(18)

The second class of non-factorizable corrections consists of diagrams with only one photon exchange. These are diagrams of the following form

\[ \sim i M_0 \int_{l_1, l_2} (J_1 \cdot J_2) \ e^{2 \frac{l_1^2}{l_2^2}} \text{Tr} \left( \frac{1}{J_1} \frac{1}{J_2 - J_1} \right). \]

(19)

The loop integration over the fermion loop and over the non-factorizable currents does not factorize.

As we have seen, two-loop non-factorizable corrections receive contributions from two different sources: from additional interactions with the decay products and from the corrections to the propagation of the soft exchange particles. The two contributions given by Eqs. (16) and (19) respectively are separately gauge invariant and are distinguished in particular by a different effective coupling constant. Eq. (16) originates solely from additional interactions of photons with the decay products. This term contains two non-factorizable currents for each decay and is proportional to \( \alpha^2 \). Eq. (19) on the other hand receives contributions from propagator corrections as well as interactions with the decay products. This term contains only one non-factorizable current and is proportional to \( \alpha^2 N_f \), where \( N_f \) is the number of fermions in the loop.

In summary, we separated the factorizable, \( \sim \alpha_{\text{fact}} \), and non-factorizable corrections, \( \sim \alpha_{\text{nf}} \). The non-factorizable corrections are further separated into corrections due to the interaction of the photons with the (dipole of the) decay products, \( \sim \alpha_{\text{dipole}} \), and corrections due to the propagation of the soft photons, \( \sim \alpha_{\text{prop}} \). Thus, the complete two-loop corrections can be written schematically as

\[ \alpha^2 = \alpha_{\text{fact}}^2 + \alpha_{\text{fact}} \alpha_{\text{nf}} + \alpha_{\text{nf}}^2; \]

\[ \alpha_{\text{nf}}^2 = \alpha_{\text{dipole}}^2 + (\alpha_{\text{dipole}} \alpha_{\text{prop}}). \]

(20)

The parenthesis in the last term indicates that the two effects do not factorize. The split up is gauge invariant and each contribution has different physical properties.

### 2.1.3 N-loop order

The procedure described above generalizes to the separation of N-loop correction. The complete \( \alpha^N \) correction consists of interferences between factorizable and non-factorizable corrections,
factorized with respect to each other. Schematically

$$\alpha_N^N = \sum_{i=0}^{N} \alpha_{\text{fact}}^{N-i} \alpha_{\text{nf}}^{i}.$$  \hspace{1cm} (21)$$

Non-factorizable corrections can be due to the interaction of soft photons with the dipoles of the decay products or due to the correction to the propagation of the soft photons

$$\alpha_{\text{nf}}^{i} = \sum_{j=1}^{i} (\alpha_{\text{dipole}}^{j} \alpha_{\text{prop}}^{i-j}).$$  \hspace{1cm} (22)$$

A useful illustration is given by the four-loop non-factorizable diagrams shown in Fig. 1. The first diagram, Fig. 1(a), contains four dipole interactions and no propagation corrections. The second diagram, Fig. 1(b), has three dipole interactions and a first-order propagation correction. Finally, the last diagram, Fig. 1(e), has only one dipole interaction and a three loop propagator correction.

![Four-loop diagrams](image)

Figure 1: Four-loop diagrams contributing to non-factorizable corrections. The diagrams are proportional to (a) $\sim \alpha_\text{dipole}^4$, (b) $\sim \alpha_\text{dipole}^3 \alpha_\text{prop}$, (c) $\sim \alpha_\text{dipole}^2 \alpha_\text{prop}^2$, (d) $\sim \alpha_\text{dipole}^2 \alpha_\text{prop}^2$ and (e) $\sim \alpha_\text{dipole} \alpha_\text{prop}^3$.

At $N$-loop the non-factorizable correction arising solely from dipole interactions, $\sim \alpha_\text{dipole}^N$, can be written in terms of the non-factorizable currents analogously to Eqs. (17) and (18). They are given by

$$i \mathcal{M}_0 \frac{1}{N!} \left[ * \int \frac{d^4k}{(2\pi)^4 k^2} [\mathcal{J}_1 \mathcal{J}_2](k) \right]^N.$$  \hspace{1cm} (23)$$

The products of non-factorizable currents are defined according to Eqs. (17) and (18). Examination of the integral, Eq. (23), confirms that also at higher-loop level the leading contribution to the non-factorizable correction is due to soft photon exchange, $\Omega \sim \Gamma M/E$.

### 2.1.4 Charged unstable particle production

In this subsection we repeat the procedure of separating different effects for the case of charged unstable particle production. The difference to the neutral unstable particle case is that now in addition to the decay-decay interferences there will be production-decay interferences.
Let us consider a toy model with a massive, charged, scalar, unstable particle field, \( \phi \). The scalar field decays into a pair of massless fermions, \( \psi \) and \( \psi' \). One of the fermions, \( \psi \), is charged and thus couples to a \( U(1) \) gauge field, \( A_\mu \). The other fermion, \( \psi' \), is neutral. The Lagrangian of the model is given by

\[
\mathcal{L} = \phi^\dagger (p^2 - M^2) \phi + \bar{\psi} \gamma^\mu p_\mu \psi + \bar{\psi'} \gamma^\mu p_\mu \psi' - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \\
+ \left( \mathcal{P} X (\phi^\dagger \phi) + \phi^\dagger \mathcal{D}(\bar{\psi}' \psi) + \text{h.c.} \right)
\]  

(24)

The difference to Eq. (1) is that now the field \( \phi \) is charged, thus \( p \) in the kinetic term of the \( \phi \) field is the covariant derivative. Again, the \( \mathcal{P} \) and \( \mathcal{D} \) terms are responsible for the production from a neutral source \( X \) and for the decay of the unstable particles respectively. As in the neutral case, we assume that these terms can be treated perturbatively.

In order to see how the factorization picture can be generalized to the case of charged unstable particles we consider the process

\[
X \to \phi^+(p_1) \phi^-(p_2) \to \psi^+(k_1) \psi'(k'_1) \psi^-(k_2) \psi'(k'_2).
\]  

(25)

where \( p_{1,2} = k_{1,2} + k'_{1,2} \) and \( p_{1,2}^2 - M^2 \sim \Gamma M \). For simplicity we will assume that there are no charged initial state particles.

At one-loop the complete non-factorizable corrections can be written as

\[
\mathcal{M}_{\text{nf}}^{\text{virt}} = i M_0 \int \frac{d^4 l}{(2\pi)^4 l^2} \left[ J_0 \cdot J_1 + J_0 \cdot J_2 + J_1 \cdot J_2 \right].
\]  

(26)

The currents are given by

\[
J_\mu^0 = e \left[ \frac{p_1^\mu}{l p_1} - \frac{p_2^\mu}{l p_2} \right],
\]  

(27)

for photon emission from the production stage of the process and

\[
J_\mu^1 = - e \left[ \frac{p_1^\mu}{l p_1} - \frac{k_1^\mu}{l k_1} \right] \frac{D_1}{D_1 + 2 l p_1}, \quad J_\mu^2 = - e \left[ \frac{p_2^\mu}{l p_2} - \frac{k_2^\mu}{l k_2} \right] \frac{D_2}{D_2 - 2 l p_2}
\]  

(28)

for photon emission from the decay stages of the process. The main difference is that there are three interference terms now. Two terms account for photons connecting the production and the two decay sub-stages (production-decay interferences). The third term corresponds to photons connecting the two decay sub-stages (decay-decay interferences). The decay-decay interferences are similar to those in the neutral unstable particle case.

At \( N \)-loop order the full correction will be a sum of products of factorizable and non-factorizable corrections similar to Eq. (21). Non-factorizable corrections come from two sources similarly to Eq. (22). One is the interaction of the soft photons with the decay or production dipoles. This contribution can be expressed in terms of the non-factorizable currents and is given by a generalization of Eq. (23).

\[
i M_0 \frac{1}{N!} \left[ * \int \frac{d^4 l}{(2\pi)^4 l^2} \left[ J_0 \cdot J_1 + J_0 \cdot J_2 + J_1 \cdot J_2 \right] \right]^N.
\]  

(29)
The second source for $N$-th order non-factorizable corrections is photon propagation corrections to lower-order non-factorizable corrections. Finally, factorizable corrections will be corrections to the two decay sub-stages and to the production sub-stage.

In summary, in the case of production of charged unstable particles the general picture still holds. Indeed, at any order the full correction comes from several physically different effects. First of all, factorizable and non-factorizable corrections can be separated. Second, non-factorizable correction can be production-decay and decay-decay interferences. Each type of interference receives corrections from the coupling of soft photons to production and decay dipoles and from corrections to the soft photon propagation.

At this point let us comment on the possibility of a generalization of this separation procedure to QCD. We believe that in QCD the physical picture should remain the same. At any order the corrections should come from the physically different sources identified above and, thus, one should be able to separate them. It seems, however, that it will be complicated to generalize the procedure we used so far. This is because the separation given above uses manipulations of the non-factorizable currents. In QCD the gauge structure is more complicated due to the gluon self coupling. At higher order it will be hard to maintain gauge invariance of separate contributions. It would be advantageous, therefore, to find a framework, which makes use of the physical distinctions between various effects. Such a framework should be manifestly gauge invariant. In Sect. 3 we come back to this question.

2.2 Integrating over decay products

In this subsection we study the mechanism that leads to the suppression of non-factorizable corrections at high energies, $E \gg M$. The mechanism is at work for distributions integrated over the momenta of the decay products keeping their invariant masses fixed. High energy suppression arises because of the coupling of the soft photons to the decay products. Therefore we will concentrate on the non-factorizable corrections coming from the interaction with the decay dipoles, as defined in Sect. 2.1. We will not consider any further factorizable corrections nor non-factorizable propagation corrections. They are proportional to the coupling constant without any extra high energy suppression. This fact illustrates that the effects we identified in Sect. 2.1 are indeed physically different. Also, we do not discuss high energy behavior of production-decay interferences that are present in the charged case. The suppression mechanism works differently in this case and we leave this issue for the future. Thus, the estimates that will be presented in this subsection are valid for decay-decay non-factorizable interferences coming from the interaction with decay dipoles. In Sect. 2.1 this was shown to be a well defined gauge invariant subset of all corrections.

2.2.1 Neutral Unstable Particles

The first process we will consider is a neutral, unstable, scalar particle with momentum $p$ decaying into a pair of massless, charged fermions with momenta $k$ and $k'$ respectively. The
Lagrangian of the model is given in Eq. (1). We will estimate the $N$-th order virtual non-factorizable correction due to exchange of $N$-photons with momenta $l_1, \ldots, l_N$. As a first step we will integrate the decay part of the Born matrix element squared multiplied by $N$ non-factorizable currents over $k$ and $k'$, keeping $p$ fixed. According to Eq. (23) the relevant integral is

\begin{equation}
I^{\alpha_1 \ldots \alpha_N} = \int d^4k \, d^4k' \, \delta(k^2) \, \delta(k'^2) \, \delta^4(p - k - k') \times |D|^2 \times \left( \frac{k}{kl_1} - \frac{k'}{k'l_1} \right)^{\alpha_1} \cdots \left( \frac{k}{kl_N} - \frac{k'}{k'l_N} \right)^{\alpha_N}. \tag{30}
\end{equation}

This tensor corresponds to the decay part of one of the particles. $|D|^2$ is the Born matrix element squared. The indices $\alpha_1 \ldots \alpha_N$ will be contracted with the indices of the non-factorizable current from the other unstable particle.

In the center-of-mass frame at high energy the components of momenta in the formula above can be estimated as follows:

\begin{equation}
p \sim k \sim k' \sim E, \quad l_i \sim \Gamma M/E. \tag{31}
\end{equation}

The latter estimate follows from the fact that only soft photons are relevant for non-factorizable corrections. Also $(kl_i) \sim (k'l_i) \sim (pl_i) \sim \Gamma M$. Since ultra-relativistic particle decay forward $k_\mu$ and $k'_\mu$ are almost collinear to $p_\mu$ and to each other. This will lead to a suppression of the non-factorizable currents in Eq. (30). In order to make this suppression explicit, it is convenient to change variables

\begin{equation}
k^\mu = \frac{1}{2}(p^\mu + \Delta^\mu), \quad k'^\mu = \frac{1}{2}(p^\mu - \Delta^\mu), \tag{32}
\end{equation}

where $\Delta \sim M$. The phase-space integration over the final state particle momenta is given by

\begin{equation}
\int d^4k \, d^4k' \, \delta(k^2) \, \delta(k'^2) \, \delta^4(p - k - k') = \int d^4\Delta \, \delta(M^2 + \Delta^2) \, \delta(2p\Delta) \equiv \int_\Delta \tag{33}
\end{equation}

which allows to rewrite Eq. (30) as

\begin{equation}
I^{\alpha_1 \ldots \alpha_N} = \int_\Delta |D|^2 \left( \frac{k}{kl_1} - \frac{k'}{k'l_1} \right)^{\alpha_1} \cdots \left( \frac{k}{kl_N} - \frac{k'}{k'l_N} \right)^{\alpha_N}. \tag{34}
\end{equation}

In terms of the new variables, the non-factorizable currents depend on $p$ and $\Delta$. They can be expanded in $M/E$

\begin{equation}
\left( \frac{k}{kl} - \frac{k'}{k'l} \right)^\alpha = \Delta^\alpha \cdot 2 \frac{l_p}{l_p} \left( g^{\alpha'\alpha} - \frac{l_{\alpha'}p^{\alpha}}{l_p} \right) + \mathcal{O}(\Delta^3) \sim \frac{1}{\Gamma}. \tag{35}
\end{equation}

There will be only odd powers of $\Delta$ in the expansion because the exact non-factorizable current is anti-symmetric under $k \leftrightarrow k'$.

We are now ready to estimate the size of $N$-non-factorizable currents averaged over the momenta of decay products, $I^{\alpha_1 \ldots \alpha_N}$, in terms of $E$, $M$, and $\Gamma$. We want to compare this to the leading contribution of the Born matrix element squared, corresponding to $N = 0$. This is
because our goal at the end is to estimate non-factorizable corrections with respect to the Born cross section.

The estimate depends on the parity of $N$. If $N$ is odd the product of the currents contains an odd number of $\Delta_\alpha_i$. Upon integration these terms will vanish. For even $N$ Eqs. (34) and (35) lead to the following estimate:

$$I^{\alpha_1...\alpha_N} \sim \begin{cases} 0, & \text{for } N \text{ odd}, \\
\text{Born} \cdot \Gamma^{-N}, & \text{for } N \text{ even}. \end{cases}$$

(36)

In Eq. (36) we denote the leading contribution to the Born matrix element squared by “Born”. Of course, the equation above does not account for the Lorentz structure. However, the information it provides is sufficient to estimate the non-factorizable correction.

For the purpose of illustration let us make this estimates more precise by taking into account the Lorentz structure. As an example, we consider the simplest case $N = 2$:

$$I^{\alpha_1\alpha_2} \sim |D|^2 \frac{p^2}{(pl_1)(pl_2)} \left[ g^{\alpha_1\alpha_2} + \frac{(l_1l_2)}{(pl_1)(pl_2)} p^{\alpha_1} p^{\alpha_2} - \frac{1}{(pl_1)} p^{\alpha_2} - \frac{1}{(pl_2)} p^{\alpha_1} \right].$$

(37)

The first factor is the leading contribution to the Born term while the second factor is the leading contribution from the non-factorizable currents. Power counting in the second factor gives $\sim 1/\Gamma^2$, in agreement with Eq. (36).

The estimate given in Eq. (36) can be summarized in the following way: using the scaling rules given in Eq. (31) the integrals over the decay product momenta, Eqs. (34) and (35), can naively be estimated to be of the order $O((E/M \cdot 1/\Gamma)^N)$ relative to the Born term. At high energy, however, $k$ is collinear to $k'$, which introduces an extra suppression of $O(M^N/E^N)$ for even $N$. This results in the final estimate quoted above: $O(\Gamma^{-N})$ relative to the Born term.

Finally we are in a position to estimate the non-factorizable correction itself. At $N$-th order it is given by Eq. (23) and involves $N$-integrations over the momenta of virtual particles, $N$ propagators of soft exchange particles and interference of $2N$ non-factorizable currents, $N$ coming from each unstable particle. We assume that one integrates over the momenta of decay products. This allows us to use the estimates made in the previous subsection. Denoting the non-factorizable correction relative to the Born term by $\delta_{nf}$, we find for even $N$

$$\delta_{nf} \sim \left( \frac{(\Gamma M/E)^4}{(\Gamma M/E)^2} \right)^N \cdot \left( \frac{1}{\Gamma} \right)^N \cdot \left( \frac{1}{\Gamma} \right)^N \sim \left( \frac{M}{E} \right)^{2N}, \quad N \text{ even.}$$

(38)

The first factor comes from the loop integration measure and $N$ virtual particle propagators whereas the second and the third factors are due to the $2N$ non-factorizable currents. The non-factorizable correction for odd $N$

$$\delta_{nf} = 0, \quad N \text{ odd,}$$

(39)

because of the symmetry mentioned above.
2.2.2 Charged Unstable Particles

The estimates made in the previous subsection can be generalized to the case where the unstable particle is charged. We will consider a process where the unstable particle, with momentum $p^\mu$, decays into a charged and neutral fermion with momentum $k^\mu$ and $k'^\mu$ respectively. The Lagrangian of the model is given in Eq. (24). The Born matrix element squared, multiplied by $N$ non-factorizable currents and integrated over the decay product momenta now reads

$$I^{\alpha_1...\alpha_N} = \int_\Delta |\mathcal{D}|^2 \cdot \left( \frac{p}{pl_1} - \frac{k}{kl_1} \right)^{\alpha_1} \cdots \left( \frac{p}{pl_N} - \frac{k}{kl_N} \right)^{\alpha_N}. \quad (40)$$

As in the neutral case we change variables and define $\Delta^\mu$ through the relation $k = (p + \Delta)/2$ with $\Delta \sim M$. Expressing the non-factorizable currents in the new variables and expanding in $M/E$ results in

$$\left( \frac{p}{pl} - \frac{k}{kl} \right)^\alpha = -\Delta^\alpha \cdot \frac{1}{lp} \left( g^{\alpha'\alpha} - \frac{l^{\alpha'} p^\alpha}{lp} \right) + \Delta^\alpha \Delta^\beta \cdot \frac{l^{\beta'} (g^{\alpha'\alpha} - \frac{l^{\alpha'} p^\alpha}{lp})}{(lp)^2} + \mathcal{O}(\Delta^3)$$

$$\sim \frac{1}{\Gamma} \left( 1 + \frac{M}{E} + \ldots \right). \quad (41)$$

The main difference compared to the case of neutral, unstable particles is that in the charged case there are odd powers of $\Delta$ in the expansion. This is because the exact non-factorizable current is not anti-symmetric under $k \leftrightarrow k'$ anymore. As a consequence non-factorizable correction will not vanish any longer for odd $N$. Instead they will get an extra $M/E$ suppression. The reason for this additional suppression is that one of the non-factorizable currents will have to be expanded to second order in $\Delta$. For even $N$ the estimates are the same as in the neutral case. Repeating the estimates that lead to Eq. (36) one obtains

$$I^{\alpha_1...\alpha_N} \sim \begin{cases} 
\text{Born} \cdot \Gamma^{-N} \cdot M/E, & N \text{ odd}, \\
\text{Born} \cdot \Gamma^{-N}, & N \text{ even}.
\end{cases} \quad (42)$$

where “Born” again denotes the leading contribution to the Born matrix element squared.

As in the neutral case it is possible to get the Lorentz structure. Let us consider, for example, the case $N = 1$:

$$I^\alpha \sim |\mathcal{D}|^2 \cdot \frac{p^2}{(pl)^2} \left[ l^\alpha - \frac{l^2}{lp} p^\alpha \right]. \quad (43)$$

The first factor is the leading contribution to the Born term whereas the second factor is the leading contribution from the non-factorizable currents. Power counting in the second factor gives $\sim 1/\Gamma \cdot M/E$.

Using Eq. (12) we can now obtain the estimate of the decay-decay non-factorizable correction in the charged case. We obtain

$$\delta_{nf} \sim \left( \frac{M}{E} \right)^{2N}, \quad N \text{ even}, \quad (44)$$

$$\delta_{nf} \sim \left( \frac{M}{E} \right)^{2N+2}, \quad N \text{ odd}. \quad (45)$$
These estimates can be summarized as follows: the non-factorizable correction with $2N$ decay currents scales as $E^{-2N}$ if $N$ is even. Because of symmetry reasons there is an extra suppression of $1/E$ for each of the two phase-space integrations over the angles if $N$ is odd. It should be stressed again that this estimate is incomplete. The reason is that we did not consider production-decay interferences even though they are present in the charged case.

### 3 Effective Field Theory description

The estimates of non-factorizable corrections presented in the previous subsection seem to be quite technical. They show different energy scaling behavior depending on the process, particles participating in the process, type of interaction through which they interact and many other factors. Moreover, the very definition of what we mean by non-factorizable corrections rests on the separation of factorizable and non-factorizable corrections discussed in Sect. 2.1. There we discussed just one specific example (out of many) and showed how such a separation can be achieved in QED.

Despite all the different ways the non-factorizable corrections manifest themselves, all the estimates rely on one simple observation. In all processes involving the production of unstable particles close to resonance there are always two scales present: a hard scale, $\Lambda_1 \sim E$, and a soft scale, $\Lambda_2 \sim \Gamma \cdot M/E$. If $E^2 \gg M\Gamma$ there is a hierarchy between these two scales, $\Lambda_1 \gg \Lambda_2$. At the hard scale, $\Lambda_1$, the process can be described in terms of interacting stable particles. The fact that the particles are unstable is not very important and can be neglected. At the soft scale, $\Lambda_2$, the instability of the particles becomes important and cannot be neglected any longer. If one’s objective is to study effects due to the instability then the problem can be simplified by separating the two contributions and studying them separately.

This separation of scales is most naturally reflected in an effective field theory framework. In this framework one starts with identifying all the relevant degrees of freedom in the underlying theory. They are determined by the scales present in the process. These scales can come from two sources. Either they come from the underlying theory itself (masses of heavy particles, etc.) or they are introduced through kinematical constraints (CMS energy, etc.). Without kinematical scales the mode contribution is determined by the typical excitation energy. However, in the presence of kinematically induced scales the excitation energy alone is not sufficient to determine the mode contribution. Additional quantum numbers are needed to describe the mode contributions uniquely. Once the relevant modes have been identified we can integrate out the modes that do not show up as external particles.

Let us recall how the procedure of constructing an effective theory works in the simpler case without kinematical constraints. The first step is to integrate out hard momenta. This

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1In the case of heavy quark pair production close to threshold, for example, the kinematical constraints are such that particles are produced with a small velocity, $v$ (see [24, 25]). There are three modes distinguished by energy with four-momenta: ($m \sim \sim m), (\sim mv, \sim mv), (\sim mv^2, \sim mv^2)$. This is not the complete list, however. One needs an additional quantum number — three-momentum — to identify an additional mode with four-momentum ($mv, \sim mv^2$).
results in a theory where only soft degrees of freedom are dynamical. The interactions of the soft degrees of freedom are described by an effective Lagrangian

\[ \mathcal{L}_{\text{eff}} = \sum_n \frac{c_n}{\Lambda^n} O_n(\text{soft fields}). \]  

(46)

Each operator, \( O_n \), is multiplied by some power of a scale, \( \Lambda \), and a dimensionless Wilson coefficient, \( c_n \), which is a series in the coupling constant of the original theory \( c_n = \sum c_{n,i} \alpha^i \). The contributions of the hard modes are included in the Wilson coefficients. These coefficients are obtained by matching the effective theory to the original theory. Since there is a hierarchy of scales the operators \( O_n \) are local and \( \Lambda \) is a typical excitation energy of the hard modes.

Let us now investigate how this picture applies to processes with unstable particles produced close to resonance. This induces a kinematical constraint and consequently the distinction between the various modes is more subtle. First of all, modes are distinguished by their momentum, which can either be hard, \( P \sim \Lambda_1 \), or soft, \( p \sim \Lambda_2 \). By the statement \( P \sim \Lambda_1 \) we mean that at least one component of the momentum is of the order \( \Lambda_1 \) and none is much larger. However, the fact that there is a kinematical constraint means that an additional quantum number is necessary to distinguish different modes. This quantum number is off-shellness or virtuality of the particles, \( D \), defined as

\[ D \equiv P^2 - m^2, \]  

(47)

where \( P \) is the total momentum and \( m \) the mass of the particle. Indeed, if the particle is close to resonance, \( P^2 \sim m^2 \), it can happen that a particle has large momentum, \( P \sim \Lambda_1 \), but its virtuality is still small, or even zero. In this configuration observables are sensitive to small (soft) changes of momentum, \( P \rightarrow P \pm p, p \sim \Lambda_2 \). The contribution of such modes depends on two momenta: the hard on-shell momentum, \( P \), and soft momentum, \( p \). The total momentum of such a mode is \( P + p \sim \Lambda_1 \), and the virtuality is \( D \sim \Lambda_1 \Lambda_2 \).

This makes it clear that in order to account correctly for all the relevant modes we will have to distinguish between three types of momenta:

- off-shell hard momenta, \( P \sim \Lambda_1, P^2 - m^2 \sim \Lambda_1^2 \).
- on-shell hard momenta, \( P_m \sim \Lambda_1, P_m^2 = m^2 \).
- soft momenta, \( p \sim \Lambda_2 \).

In general, the virtuality can take four values: 0, \( \Lambda_2^2 \), \( \Lambda_1 \Lambda_2 \), and \( \Lambda_1^2 \). Consequently, there are the following types of modes in the problem:

- Hard modes: \( \Phi(P) \), where \( P \sim \Lambda_1, D \sim \Lambda_1^2 \).
- Resonant modes \( \Phi(P_m, p) \), where \( P_m \sim \Lambda_1, p \sim \Lambda_2, D \sim \Lambda_1 \Lambda_2 \).
- Soft modes: \( \Phi(p) \), where \( p \sim \Lambda_2, D \sim \Lambda_2^2 \).
• External modes: $\Phi(P_m)$, where $P_m \sim \Lambda_1$, $D = 0$.

All fields together with their respective energy and virtuality scales are listed in Fig. 2.

Figure 2: Relevant energy-momentum and virtuality scales with a list of corresponding fields contributing at each scale.

Let us now consider what happens if we start to integrate out fields in the case where kinematical scales are present. It is possible that fields that are integrated out and fields that are left in the Lagrangian – being distinguished by the additional quantum number – have the same typical energy. In this case the operators $O_n$ can be nonlocal. The scale $\Lambda$ is determined by the quantum number, which distinguishes the modes.

Applying this strategy to processes with unstable particles one first integrates out the hard modes, $\Phi(P)$. The resulting effective theory describes instability effects, such as propagation of unstable particles close to resonance, $\Phi(P_m, p)$, their decay into on-shell particles, $\Phi(P_m)$, and interactions between them due to soft exchanges, $\Phi(p)$. The effective Lagrangian contains nonlocal operators. The main point is that in this way we achieve a separation of factorizable and non-factorizable effects. As we will see, the factorizable corrections correspond to the hard effects encoded in the Wilson coefficients whereas all non-factorizable corrections are described by interactions of the still dynamical soft modes. Since this separation is based on a physically relevant hierarchy of scales it is possible to generalize it to more complicated cases. The usual way of separating factorizable and non-factorizable corrections is based on ad hoc manipulations of various contributions. These manipulations become increasingly tedious for more complicated cases.

As a next step one can integrate out the resonant modes, $\Phi(P_m, p)$. This results in a new effective theory that contains nonlocal interactions between external particles, $\Phi(P_m)$, coupled in the production and decay points. It also contains interactions between production and decay points due to soft exchanges, $\Phi(p)$ (non-factorizable corrections). Whereas integrating out hard modes provides a separation between factorizable and non-factorizable corrections, integrating out resonant modes provides a separation of different types of non-factorizable corrections (production-decay, decay-decay).

The last step is to integrate out the soft modes, $\Phi(p)$. This will lead to the $S$-matrix defined in terms of external particles, $\Phi(P_m)$. 

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3.1 Toy model

Let us now carry out this programme in some more detail for the toy model specified in Eq. (1). The list of relevant modes is the following:

- **Hard modes:** $\phi(P), \psi(P), A_\mu(P)$.
- **Resonant modes:** $\phi(P_M, p), \psi(P_0, p)$.
- **Soft modes:** $\psi(p), A_\mu(p)$.
- **External modes:** $\psi(P_0)$.

It should be understood that the $A_\mu(P)$ field is subject to gauge transformations at the hard scale whereas $A_\mu(p)$ is subject to gauge transformations at the soft scale. Thus, for example, the covariant derivative acting on $\psi(P)$ contains $A_\mu(P)$ and the covariant derivative acting on $\psi(P_0, p)$ contains $A_\mu(p)$.

The leading diagram for the process Eq. (2) is shown in Fig. 3(a) where the dashed line denotes a resonant field $\phi(P_M, p)$. Neglecting an overall coupling constant, this diagram is of the order $(\Lambda_1 \Lambda_2)^{-2}$.

3.2 Integrating out hard modes

The Lagrangian given in Eq. (1) describes the full theory where all modes are dynamical. The first step towards the effective theory we are aiming at is to integrate out the hard modes $\phi(P), \psi(P)$, and $A_\mu(P)$. They all have momenta $P \sim \Lambda_1$ and virtuality $D \sim \Lambda_1^2$. The resulting effective Lagrangian contains fields $\phi(P_M, p)$ ($P_M \sim \Lambda_1, P_M^2 = M^2$), $\psi(P_0, p)$ ($P_0 \sim \Lambda_1, P_0^2 = 0$), $\psi(p)$ and $A_\mu(p)$, with $p \sim \Lambda_2$, and the external field $\psi(P_0)$. The Lagrangian of the effective theory is determined through matching. Thus, we require that up to a certain order in the coupling constants and up to a certain order in $\Lambda_2/\Lambda_1$ the effective theory coincides with the underlying theory. At this point we restrict ourselves to contributions that are not suppressed by powers of $\Lambda_2/\Lambda_1$.

Let us start by integrating out $\phi(P)$. At leading order in $\alpha$, we have to consider diagrams as shown in Fig. 3. The diagram shown in Fig. 3(a) cannot contribute to the process Eq. (2) if the $\phi$-field is hard, since this would violate the kinematic constraint Eq. (3). On the other hand, the background process, Fig. 3(b), does contribute to this process, even if the $\phi$-fields are hard. Thus, integrating out $\phi(P)$ results in an operator $X(\bar{\psi}\psi)(\bar{\psi}\psi)$ in the effective Lagrangian. However, for each hard $\phi$-field we get a suppression of $\Lambda_2/\Lambda_1$. Thus, in leading order in $\Lambda_2/\Lambda_1$ there are no new operators introduced by integrating out $\phi(P)$.

Consider now $\alpha$-corrections in the leading $\Lambda_2/\Lambda_1$ approximation. As shown in Fig. 4(a), there are diagrams involving hard $\phi$-fields that are not suppressed by $\Lambda_2/\Lambda_1$. However, these diagrams do not result in new operators in the effective Lagrangian. Rather, their effect is
encoded in the (modified) coefficients of already existing operators. In fact, Fig. 3(a) results in a modification of $\mathcal{P}X\phi\phi$. Thus, $\mathcal{P}X\phi\phi \rightarrow c_{P}\mathcal{P}X\phi\phi$ where $c_{P} = 1 + \mathcal{O}(\alpha)$ is a Wilson coefficient. Of course, there are also diagrams resulting in $\alpha$ corrections to operators that are suppressed by $\Lambda_{2}/\Lambda_{1}$. An example of a diagram that gives rise to an $\alpha$ correction to the operator $X(\bar{\psi}\psi)(\bar{\psi}\psi)$ is shown in Fig. 3(b).

![Figure 3](image)

**Figure 3:** Resonant (a) and background (b) Born contributions to pair production in the underlying theory. Resonant kinematics means that $D_{1} \sim D_{2} \sim \Lambda_{1}\Lambda_{2}$.

As a next step we turn to the $U(1)$ field. In order to integrate out $A(P)$ we have to consider diagrams as shown in Fig. 4. For a hard photon, the diagram shown in Fig. 4(a) is suppressed by powers of $\Lambda_{2}/\Lambda_{1}$. This diagram only gives a leading in $\Lambda_{2}/\Lambda_{1}$ contribution for a soft photon. The diagram shown in Fig. 4(b), however, is only suppressed by a coupling constant $\alpha$, but not by powers of $\Lambda_{2}/\Lambda_{1}$. Thus, it leads to an $\alpha$ correction to the contribution of the $\phi\bar{\psi}\mathcal{D}\psi$ operator. We take this into account by multiplying this operator by a Wilson coefficient $c_{D} = 1 + \mathcal{O}(\alpha)$. Since we are dealing with a neutral field $\phi$, the diagram shown in Fig. 4(c) does not contribute.

The fields $\psi(P)$ enter through self energy insertions of $\phi(P_{M}, p)$, thus they will modify the operators bilinear in $\phi$. The corresponding operator leading in $\Lambda_{2}/\Lambda_{1}$ is $\phi P_{M}^{2}\phi = \phi M^{2}\phi$. However, at leading order in $\alpha$ this operator does not contribute. The next-to-leading order corrections in $\alpha$ result in an operator $\phi i M \Gamma \phi$. This can be seen as follows: the relative order of magnitude of a hard self energy insertion is given by $\alpha\Lambda_{1}^{4}\Lambda_{1}^{-2}(\Lambda_{1}\Lambda_{2})^{-1} = \alpha\Lambda_{1}/\Lambda_{2}$. The factor $\alpha \sim g^{2}$ comes from the coupling constant, $\Lambda_{1}^{4}$ from the phase-space integration, $\Lambda_{1}^{-2}$ from the hard fermion propagators and $(\Lambda_{1}\Lambda_{2})^{-1}$ from the additional $\phi(P_{M}, p)$ propagator. From the estimate above it is evident that the self energy insertion is not suppressed. Therefore, these
Figure 5: One-loop radiative corrections in effective theory. (a) would be counted as non-factorizable and (b) as factorizable correction in the underlying theory. If the unstable particles are charged there are diagrams in the underlying theory that contain both factorizable and non-factorizable corrections. An example is given in (c).

contributions have to be taken into account to all orders. This leads to the usual self energy resummation.

In summary, at leading order in $\Lambda_2/\Lambda_1$ the effective Lagrangian takes the form

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\phi(2P_M p + i M \Gamma)\phi + \bar{\psi}(\slashed{P}_0 + \slashed{p})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_D \phi^2 X(P_M)\phi + c_D \phi \bar{\psi} \mathcal{D}(P_M)\psi. \quad (48)$$

We should stress that the terms $\bar{\psi} \slashed{p} \psi$ and $F_{\mu\nu}F^{\mu\nu}$ in Eq. (48) are not equivalent to the ones in Eq. (1). In the underlying theory the fields $A(P)$ and $\psi(P)$ are dynamical whereas in Eq. (48) they have been integrated out.

The effective Lagrangian given in Eq. (48) is in fact completely equivalent to the double pole approximation (DPA) [4]. At Born level this can be seen as follows: The production and decay operators, $\phi \mathcal{P}(P_M)\phi$ and $\phi \bar{\psi} \mathcal{D}(P_M)\psi$, depend on the on-shell momentum $P_M$. The kinetic operator of the unstable particle depends also on the soft momentum $p$. Consequently, the Born process shown in Fig. 3(a) is described by the Lagrangian Eq. (48) as a superposition of an on-shell production subprocess, unstable particle propagation subprocess, and on-shell decay subprocess. This picture is the same as that in DPA.

The equivalence of the effective theory given by the Lagrangian Eq. (48) and DPA can be extended to $\alpha$-corrections in the leading $\Lambda_2/\Lambda_1$ approximation. In DPA the important fact is that $\alpha$-corrections are separated into a sum of factorizable and non-factorizable corrections. The diagram shown in Fig. 5(a) is a non-factorizable correction. Only soft gauge bosons give contributions to non-factorizable corrections in DPA. On the other hand the diagram Fig. 5(b) is a factorizable correction in DPA.

Let us see how this happens in our effective theory framework. There are two sources of $\alpha$-corrections. First, the Wilson coefficients contain $\alpha$-corrections. As we have seen above, these correspond to corrections obtained by integrating out hard modes. Second, there are loop corrections in the effective theory due to the exchange of soft gauge bosons. In the effective theory at leading $\Lambda_2/\Lambda_1$ order gauge bosons couple to resonant particles in the soft photon approximation. Thus, the evaluation of a diagram of the type Fig. 5(a) with a soft photon is the same in the effective theory and the DPA of the underlying theory. Furthermore, loop
integrals in the effective theory have to be carried out in dimensional regularization. A loop integral corresponding to the diagram of Fig. 5(b) with a soft photon does not have a scale and, therefore, is zero in dimensional regularization.

All of the above means that the complete factorizable corrections correspond to the contributions due to the hard modes, calculated in dimensional regularization. These contributions are included in the Wilson coefficients of the effective theory. The non-factorizable corrections are described by the dynamical degrees of freedom of the effective theory. In this way we achieve a separation of factorizable and non-factorizable effects based on the hierarchy of scales in the problem. It is clear that the effective theory description provides a separation between factorizable and non-factorizable corrections also at higher orders in $\alpha$. This is the main advantage of the effective theory approach compared to DPA.

In more complicated cases, for example when the unstable particles are charged, the separation between factorizable and non-factorizable corrections cannot be done on the basis of diagrams. There are diagrams, like the one shown in Fig. 5(c), which contain both factorizable and non-factorizable parts. The separation is still correctly reproduced within the effective theory framework. In Appendix A we show how the separation works for this example to all orders in $\Lambda_2/\Lambda_1$. In order to do that we use an expansion of loop integrals in dimensional regularization that is equivalent to effective theory calculations.

The effective Lagrangian Eq. (48) contains only local operators. On general grounds, however, one can expect nonlocal operators to be present. This is because the momentum of the fields integrated out, $P \sim \Lambda_1$, is of the same order as the momentum of the fields still present, $P_M \sim \Lambda_1$. In order to understand this mismatch let us recall that in our example hard modes do not introduce new operators at leading order in $\Lambda_2/\Lambda_1$. Thus, the resonant diagrams are described in terms of the same operators in the original and effective theory. These operators are local. As mentioned before, hard modes will introduce new operators that are suppressed by $\Lambda_2/\Lambda_1$. Consequently, nonlocal operators will appear in the effective Lagrangian only at next to leading order in $\Lambda_2/\Lambda_1$.

This brings us to the question of $\Lambda_2/\Lambda_1$ suppressed operators. As we have shown, the effective Lagrangian Eq. (48) contains leading operators in $\Lambda_2/\Lambda_1$ and is completely equivalent to DPA. The effective field theory approach should enable us – at least in principle – to construct $\Lambda_2/\Lambda_1$ suppressed operators order by order in $\Lambda_2/\Lambda_1$. This would open the possibility to go beyond DPA and calculate $\Lambda_2/\Lambda_1$ corrections in a regular (in particular, gauge invariant) way. It is not the aim of this paper to go beyond the leading in $\Lambda_2/\Lambda_1$ order. However, before proceeding, we would like to discuss this point briefly. In the construction of the effective Lagrangian we assumed the presence of two widely separated scales. One of the scales, $\Lambda_2$, was induced by the kinematical constraints that the unstable particles are close to resonance. If the unstable particles are far from resonance then there is no extra scale, and the whole effective theory approach is not applicable. This means that for different parts of the phase space one needs to perform different calculations and switch between the two as one goes from one phase-space region to the other. There is a question of accuracy and applicability of the calculation in the transition region. This question always arises when effective theories are used
for exclusive distributions. If one is interested in inclusive distributions, where one integrates over the complete phase space, then the procedure should be the following. The phase space should be divided into regions where one of the two calculations is applicable. The calculation then is to be performed in the respective regions, assuming extreme scale hierarchy everywhere in that region. Then the sum will not depend on the cutoff defining the regions at the order at which calculation is performed. It is not possible to perform one calculation, which would have the same accuracy in all the different regions of the phase space with a different scale structure. In what follows we will be interested only in contributions leading in \( \frac{\Lambda_2}{\Lambda_1} \), which corresponds to DPA. The possibility of using the effective field theory framework to go beyond DPA is interesting and very relevant, but it is still an open problem.

3.3 Integrating out resonant modes

Starting from the effective Lagrangian Eq. (48), let us now integrate out resonant modes, \( \phi(P_M, p), \psi(P_0, p) \). These modes all have momenta \( P_M \sim P_0 \sim \Lambda_1, p \sim \Lambda_2, D \sim \Lambda_1 \Lambda_2 \). This will result in another effective Lagrangian that contains soft modes, \( \psi(p) \) and \( A_\mu(p) \), with momenta \( p \sim \Lambda_2, D \sim \Lambda_2^2 \), and external modes, \( \psi(P_0) \), with momenta \( P_0 \sim \Lambda_1, D = 0 \). The importance of operators in the effective Lagrangian is determined by powers of \( \frac{\Lambda_2}{\Lambda_1} \). The operators in the new effective Lagrangian are nonlocal. The leading operator corresponding to pair production of unstable particles is a \( \psi^4 \)-operator and has the form

\[
\Lambda(x, y, z) \cdot (\bar{\psi} D \psi)(x) \cdot X \mathcal{P}(y) \cdot (\bar{\psi} D \psi)(z). \tag{49}
\]

This term describes the resonant Born diagram in the leading approximation. The factor \( \mathcal{P}(y) \) corresponds to the production of unstable particles at the point \( y \) and \( (\bar{\psi} D \psi)(x) \) corresponds to their decay at the point \( x \). As long as the unstable particles are neutral, each of these factors is gauge invariant separately. The coefficient \( \Lambda(x, y, z) \) contains derivatives with respect to \( x, y \) and \( z \), which we will denote by \( P_x, P_y \), and \( P_z \) respectively. It describes the propagation of unstable particles from the production to the decay point and has dimension \([-4]\). Only the virtuality of modes that have been integrated out can appear in the denominator. Thus, \( \Lambda \) can be estimated as

\[
\Lambda(x, y, z) \sim \frac{1}{(\Lambda_1 \Lambda_2)^2} \left( 1 + \mathcal{O}\left( \frac{\Lambda_2}{\Lambda_1} \right) \right). \tag{50}
\]

As in section 3.2 we will neglect \( \frac{\Lambda_2}{\Lambda_1} \) suppressed terms. Of course in this case we know the precise form of \( \Lambda \). From Eq. (48) we deduce that the propagator of a \( \phi(P_M, p) \)-field is given by \( (2P_M p + i \Gamma)^{-1} \). This entails

\[
\Lambda(x, y, z) = \frac{1}{2P_x p_x + i \Gamma} \cdot \frac{1}{2P_z p_z + i \Gamma}. \tag{51}
\]

When the operator \( P_x \) acts on a fermion field, \( \psi(P)(x) \), it gives its momentum \( P \); when it acts on pair of fermion fields, \( (\bar{\psi}(P_1) \psi(P_2))(x) \), it gives the sum of the momenta, \( P_1 + P_2 \). The total momentum of the fermion pair is equal to the momentum of the unstable particle. Therefore it has hard and soft components, \( (P_1 + P_2) = P + p \). Because we integrated out the unstable
particle field the effective Lagrangian does not depend on the unstable particle field itself any longer. Nevertheless, it can depend of its momentum.

Of course, in more complicated cases the coefficients in front of the nonlocal operators can be quite complicated. An explicit matching calculation is required in order to determine them. In Sect. 3.3.1 and Sect. 3.3.2 we write down the nonlocal operators explicitly and show how the nonlocal Wilson coefficients can be determined by the matching. Our main goal, however, is to reproduce the estimates of non-factorizable corrections presented in Section 2. In order to do that we will not need to know the explicit form of the nonlocal coefficients. The estimates of various quantities can be determined by the scaling properties alone. In Sect. 3.3.3 we show how the scaling properties lead to such estimates.

3.3.1 Explicit operators and matching in QED

We start by writing down an effective Lagrangian describing pair production of unstable particles. The Lagrangian will be written in terms of (nonlocal) gauge invariant operators dependent on soft fields, $\psi(p)$ and $A_\mu(p)$, and external fields $\psi(P_0)$. The soft fields transform locally under gauge transformations whereas the external fields transform globally. We will perform the matching by comparing matrix elements squared calculated in the effective theory and in the underlying theory.

The effective Lagrangian consists of kinetic terms and terms describing interactions. It has the following structure

$$L_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(p)(P_0 + \slashed{p})\psi(p) + \bar{\psi}(P_0)\slashed{P}_0\psi(P_0) + \Lambda(x, y, z) \cdot (\bar{\psi}D_{\text{int}}\psi)(x) \cdot X \cdot (\bar{\psi}P_{\text{int}}\psi)(y) \cdot (\bar{\psi}D_{\text{int}}\psi)(z),$$

(52)

where the first three terms are kinetic terms. Note that $\slashed{p}$ contains the covariant derivative whereas $p$ contains the normal derivative. Therefore, the soft fermions $\psi(p)$ do interact with the soft gauge field $A_\mu(p)$ but there is no interaction of the external field, $\psi(P_0)$, with $A_\mu(p)$. The last term in Eq. (52) describes doubly resonant production-decay processes. The field $\psi$ appearing there is an external field $\psi(P_0)$. Of course, one can easily generalize this Lagrangian to describe production of more than two unstable particles. The operators $D_{\text{int}}$ and $P_{\text{int}}$ contain operators with different powers of the gauge field $A_\mu$, describing gauge boson radiation accompanying the production/decay process. Schematically, they have the following structure

$$P_{\text{int}} = \mathcal{P}\left(1 + eF_{\mu\nu} + e^2(F_{\mu\nu})^2 + \ldots\right)$$

(53)

$$D_{\text{int}} = \mathcal{D}\left(1 + eF_{\mu\nu} + e^2(F_{\mu\nu})^2 + \ldots\right).$$

(54)

The first terms correspond to production/decay without radiation. The terms linear in $A_\mu$ correspond to production/decay accompanied by radiation of one gauge boson, and so on. If matching matrix elements would be incorrect because this does not fully reflect the fact that $\psi(P_0)$ are external fields by construction. In particular, the color traces have to be taken, because $\psi(P_0)$ is subject to global gauge transformation only.
the source $X$ and the unstable particles is neutral then there can be no radiation from
the production stage of the process

$$\mathcal{P}_{\text{int}} = \mathcal{P}.$$  \hfill (55)

Let us start with the QED case. We will write down explicitly operators corresponding to
different terms in $(\bar{\psi} \mathcal{D}_{\text{int}} \psi)(x)$. The operator corresponding to one photon radiation is

$$\left. (\bar{\psi} \mathcal{D}_{\text{int}} \psi)(X) \right|_{\sim F} = \int dxdydz \, \Sigma^{(1)}_{\mu\nu}(x - X, y - X|z - X) \left( \bar{\psi}(x) \mathcal{D} \psi(y) \right) F^{\mu\nu}(z).$$  \hfill (56)

This operator is gauge invariant because $\psi$ is an external field subject to a global gauge
transformation and $A_\mu$ is an abelian gauge field. This operator contributes only to one photon radiation. It is convenient to use a Fourier transformed function for the Wilson coefficient

$$\Sigma^{(1)}_{\mu\nu}(x - X, y - X|z - X) = \int d^3s \, e^{i s_1(x - X)} e^{i s_2(y - X)} e^{i s_3(z - X)} \Sigma^{(1)}_{\mu\nu}(s_1, s_2, s_3).$$  \hfill (57)

The matching calculation in the effective theory and in QED is

$$\left. (\bar{\psi} \mathcal{D}_{\text{int}} \psi)(X) \right|_{\sim F} \sim e^{i X(k + k'|l)} \Sigma^{(1)}_{\mu\nu}(k, k'|l) (l^\mu g^{\nu\alpha} - l^\nu g^{\mu\alpha}) = e^{-i X(k + k'|l)} \left( \frac{k}{kl} - \frac{k'}{kl} \right)^\alpha. \hfill (58)$$

Here we recognize the non-factorizable current structure that we encountered in the QED calculations of Sect. \[. The Wilson coefficient can be read off from Eq. (58) and is given by

$$\Sigma^{(1)}_{\mu\nu}(k, k'|l) = \frac{e}{2} \left( \frac{k_\mu k'_\nu}{kl} - \frac{k_\mu k_\nu}{kl} \right).$$  \hfill (59)

This result generalizes to $N$-photon radiation in a straightforward way. The operator responsible for radiation of $N$ photons is

$$\left. (\bar{\psi} \mathcal{D}_{\text{int}} \psi)(X) \right|_{\sim \mathcal{F}^N} = \int dxdyd^Nz \, \Sigma^{(N)}_{\mu_1 \ldots \mu_N \nu_1 \ldots \nu_N}(x - X, y - X|z_1 - X, \ldots, z_N - X) \times \left( \bar{\psi}(x) \mathcal{D} \psi(y) \right) F^{\mu_1 \nu_1}(z_1) \ldots F^{\mu_N \nu_N}(z_N).$$  \hfill (60)

This operator is gauge invariant and contributes solely to observable where the number of emitted photons is $N$. The matching calculation in QED is very similar to the calculations presented in Sect. \[ with a product of $N$ non-factorizable currents emerging as a result. The resulting Wilson coefficient is given by

$$\Sigma^{(N)}_{\mu_1 \nu_1 \ldots \mu_N \nu_N}(k, k'|l_1 \ldots l_N) = \frac{e^N}{2^N N!} \left( \frac{k_{\mu_1} k'_{\nu_1}}{kl_1} - \frac{k'_{\mu_1} k_{\nu_1}}{kl_1} \right) \ldots \left( \frac{k_{\mu_N} k'_{\nu_N}}{kl_N} - \frac{k'_{\mu_N} k_{\nu_N}}{kl_N} \right).$$  \hfill (61)

We can summarize the situation as follows: the propagation of soft and external fields is described by the kinetic operators in Eq. (12). The production and decay of a pair of unstable particles accompanied by radiation of an arbitrary number of photons is described by the last
term in Eq. (52) with the factors $(\bar{\psi}D_{\text{int}}\psi)(x)$ explicitly given in Eq. (60). These operators reproduce those non-factorizable corrections that are due to interaction with decay dipoles and have the non-factorizable current structure encountered in Sect. 2. As mentioned in Sect. 2 there are also non-factorizable corrections due to the coupling of soft fermions to soft gauge bosons, which induces corrections due to propagation. Since soft fermions as well as soft gauge boson fields are still dynamical in the Lagrangian of the effective theory, Eq. (52), these corrections are absolutely identical in the effective and underlying theory.

3.3.2 Explicit operators and matching in QCD

We now want to generalize the construction of the effective theory to the non-abelian case of QCD. The effective Lagrangian has a structure similar to the one encountered in the QED case, Eq. (52). However, the explicit form of the operators $(\bar{\psi}D_{\text{int}}\psi)(x)$ has to change now. Indeed, the operator responsible for one photon radiation, Eq. (56), is not gauge invariant in the non-abelian case. This is because the field tensor $F_{\mu\nu}$ is not gauge invariant if $A_{\mu}$ is a non-abelian gauge field. In fact it is impossible to construct a gauge invariant operator that contains $\bar{\psi}\psi A_{\mu}$ if $\psi$ is transforms globally and $A_{\mu}$ transforms locally under gauge transformations. This is consistent with what is observed in QCD. It merely reflects the fact that the emission of a single gluon is forbidden in QCD due to the vanishing of the color trace. The matching equation for the emission of a single gluon then is simply

$$l, \alpha, a \quad \left| \begin{array}{c} X \bullet \bigcirc \bullet \bigcirc \downarrow \downarrow \uparrow \downarrow \downarrow \bigcirc \bullet \bigcirc \downarrow \downarrow \uparrow \downarrow \downarrow \bigcirc \cdot \end{array} \right| = 0.$$  \hfill (62)

The first non-vanishing process in QCD is the one where two gluons are radiated. Accordingly, the first non-trivial gauge invariant operator in the effective theory contains two $F_{\mu\nu}$ tensors

$$(\bar{\psi}D_{\text{int}}\psi)(X) \bigg|_{F^2} = \int dxdyd^2z \Sigma^{(2)}_{\mu\nu\rho\sigma}(x - X, y - X|z_1 - X, z_2 - X) \times \left( \bar{\psi}(x)D\psi(y) \right) \text{Tr} \left( F^\mu\nu(z_1)U(z_1, z_2)F^{\sigma\rho}(z_2)U(z_2, z_1) \right).$$  \hfill (63)

Here we introduced $U(z_1, z_2)$, the path-ordered exponential defined as

$$U(x, y) = \text{P exp} \left[ -ig \int_x^y A_\mu(\omega) \, d\omega^\mu \right],$$  \hfill (64)

in order to ensure the gauge invariance of the operator. The matching calculation in the effective theory and in QCD for the two gluon process is similar to the one in QED

$$X \bullet \bigcirc \bullet \bigcirc \downarrow \downarrow \uparrow \downarrow \downarrow \bigcirc \bullet \bigcirc \downarrow \downarrow \uparrow \downarrow \downarrow \bigcirc \cdot \quad \sim \quad \text{Tr} \left( T^\alpha T^\delta \right) \Sigma^{(2)}_{\mu\nu\rho\sigma}(k, k'|l_1, l_2) \left( l_1^\mu g^{\rho\alpha} - l_1^\rho g^{\mu\alpha} \right) \left( l_2^\sigma g^{\rho\beta} - l_2^\rho g^{\sigma\beta} \right) + (1 \leftrightarrow 2)$$

25
\[ g^2 \text{Tr} \left( T^a T^b \right) \left( \frac{k}{kl_1} - \frac{k'}{k'l_1} \right)^\alpha \left( \frac{k}{kl_2} - \frac{k'}{k'l_2} \right)^\beta. \]

The solution for the Wilson coefficient can be read off from Eq. (65)

\[ \Sigma^{(2)}_{\mu \nu \sigma \rho} (k, k'|l_1, l_2) = \frac{g^2}{2^{22} \times} \left( \frac{k_\mu k'_\nu}{kl_1 k'l_1} - \frac{k'_\mu k_\nu}{kl_1 k'l_1} \right) \left( \frac{k_\sigma k'_\rho}{kl_2 k'l_2} - \frac{k'_\sigma k_\rho}{kl_2 k'l_2} \right). \]

This is similar to what we found in QED. However, there is an important difference compared to the QED case in that now the operator that is responsible for the emission of two gluons, Eq. (63), will contribute also to processes with a larger number of gluons radiated off. To illustrate this point, let us consider a process with the emission of three gluons. The diagrams of the effective theory contributing to this process are shown in Fig. 6. There will be three distinct contributions. In the first one, shown in Fig. 6(a), two gluons are emitted due to the \((\bar{\psi} \psi) F^2\) operator, Eq. (63), with one of the gluons subsequently splitting into two due to three-gluon interaction coming from the kinetic term of the gauge field, \(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\). The second is shown in Fig. 6(b) and is also due to the \((\bar{\psi} \psi) F^2\) operator. There are two sources in \((\bar{\psi} \psi) F^2\) that give rise to terms contributing to three gluon radiation. First, the non-abelian commutator terms in \( F_{\mu\nu} \) result in terms with three (and four) gauge fields. Second, the expansion of the path-ordered exponentials \( \mathcal{U}(z_1, z_2) \) creates terms with an arbitrarily large number of fields. Thus, the operator \((\bar{\psi} \psi) F^2\) given in Eq. (63) contributes to all processes with two or more gluons emitted. Finally, the third contribution to the three gluon process is shown schematically in Fig. 6(c). This contribution is due to a new \( \bar{\psi} \psi F^3 \) operator with a Wilson coefficient \( \Sigma^{(3)} \) defined as

\[ (\bar{\psi} \mathcal{D}_{\text{int}} \psi)(X) \bigg|_{F^3} = \int dx dy d^3 z \Sigma^{(3)}_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} (x - X, y - X, z_1 - X, z_2 - X, z_3 - X) \]

\[ \times \left( \bar{\psi}(x) \mathcal{D} \psi(y) \right) \text{Tr} \left( F^{\mu_1 \nu_1}(z_1) U(z_1, z_2) F^{\mu_2 \nu_2}(z_2) U(z_2, z_3) F^{\mu_3 \nu_3}(z_3) U(z_3, z_1) \right). \]

The sum of the three contributions shown in Fig. 6 should be equivalent to the full QCD calculation. This matching condition determines the Wilson coefficient \( \Sigma^{(3)} \).

This picture is the non-abelian version of what we found in QED. Non-factorizable corrections are either due to the interaction of soft gluons with the decay dipoles or due to propagation corrections of the soft gluons. In the abelian case the propagation corrections arise solely from
interaction with soft fermions. In the non-abelian case there is an additional source of propagation corrections, namely the self-couplings of gluons. What we achieved is a generalization to the non-abelian case of a gauge invariant separation between non-factorizable corrections due to interaction with decay dipole and non-factorizable corrections due to propagation. The analogy of the estimates of Sect. 2 for the non-factorizable corrections due to the interaction with decay dipoles is now simply an estimate for the Wilson coefficients \( \Sigma^{(N)} \). For example, the contribution shown in Fig. 3(c) is due to \( \mathcal{O}(g^3) \) dipole interaction. On the other hand, the contributions shown in Fig. 3(a) and (b) are due to \( \mathcal{O}(g^2) \) dipole interactions plus \( \mathcal{O}(g) \) propagation corrections. Thus, it is only within the effective theory framework that the true meaning of the estimates for non-factorizable corrections in the non-abelian case becomes clear.

The matching calculation in QCD is far more complicated than in QED. What will be important for us here is that the Wilson coefficient can be estimated as

\[
\Sigma^{(3)}_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}(k, k')|l_1, l_2, l_3) \sim g^3 \frac{1}{(\Lambda_1 \Lambda_2)^2} k_{\mu_1} k'_{\nu_1} k_{\mu_2} k'_{\nu_2} k_{\mu_3} k'_{\nu_3}. \tag{68}
\]

In the following section we concentrate on the estimates of the non-factorizable corrections due to interactions with the decay dipoles. It will be sufficient to know the estimates of the Wilson coefficients, Eq. (68), as well as the Lorentz structure and the form of the operators. The explicit form of the matching coefficients will not be important. Nevertheless, we performed an explicit matching calculation for \( \Sigma^{(3)} \) in order to convince the reader that the effective theory does exist and can be matched to QCD. We should mention that the set of operators we have chosen is sufficient only at leading order in \( \Lambda_2/\Lambda_1 \). If we were to go beyond this approximation additional operators with a more complicated \( \gamma \)-matrix structure would be needed. Details of the matching calculation are given in Appendix B.

### 3.3.3 Scaling properties and estimates

In this subsection we will generalize the estimates of the Wilson coefficients found above and then apply these in order to obtain estimates of the non-factorizable correction. Thereby we will confirm the results that have been found in Section 2 by analyzing Feynman diagrams.

Instead of working with the momenta of the fermions it will turn out to be more convenient to work with the momenta \( P_M + p \) and \( \Delta \), defined as follows:

\[
P_M + p = P_1 + P_2, \quad \Delta \equiv k - k', \quad P_M \Delta = -p \Delta, \tag{69}
\]

where \( P_M \) is the hard on-shell momentum of the resonant \( \phi \)-field that decays into the fermions with momentum \( k \) and \( k' \). \( p \) is the corresponding soft momentum, \( p \sim \Lambda_2 \). Finally, \( \Delta \) is a hard off-shell momentum, \( \Delta \sim \Lambda_1 \), however \( P_M \Delta \sim \Lambda_1 \Lambda_2 \).

The estimate found in Eq. (68) can be generalized to the Wilson coefficient \( \Sigma^{(n)} \) of the operator \( (\bar{\psi} \psi)^n \):

\[
\Sigma^{(n)}_{\mu_1 \nu_1 \ldots \mu_n \nu_n} \sim g^n \frac{1}{(\Lambda_1 \Lambda_2)^{2n}} k_{\mu_1} k'_{\nu_1} \ldots k_{\mu_3} k'_{\nu_3} \sim \frac{g^n}{\Lambda_2^{2n}}. \tag{70}
\]
Rewriting this in terms of the new momenta we find
\[
(\bar{\psi}D_{\text{int}}\psi)(P_M, p, \Delta) \bigg|_{\sim F^n} \sim \frac{(P_M \text{ or } \Delta)^{2n+m} p^n}{(\Lambda_1 \Lambda_2)^{2n+m}} (\bar{\psi}D\psi) (eF)^n
\]  
(71)

We would like to stress again here that this selection rule is valid only up to \( \Lambda_2/\Lambda_1 \) suppressed terms.

The selection rule, Eq. (71), can be understood as follows: the amplitude for radiating one soft gauge boson is proportional to 
\[
gP_M^\mu / (P_M p) \sim g/\Lambda_2, \text{ plus } \Lambda_2/\Lambda_1 \text{ suppressed contributions.}
\]
Consequently, the Feynman rules corresponding to such an interaction operator should scale as 
\[
\sim g/\Lambda_2.
\]
This means that the term linear in \( gA_\mu \) should scale as \( gA/\Lambda_2 \). As \( A_\mu \) is a soft field, this is equivalent to terms linear in \( gF_{\mu\nu} \) scaling as \( gF/\Lambda_2^2 \). These arguments can be extended to radiation of \( n \)-gauge bosons. This leads to the observation that \( (gF)^n \)-terms have to scale as \( (gF)^n/\Lambda_2^{2n} \), as stated in Eq. (70). Taking into account that only \( \Lambda_1 \Lambda_2 \) can appear in the denominator we balance the dimension of the contribution of the operators by powers of \( \Lambda_1 \) and \( \Lambda_2 \) in the numerator. Powers of \( \Lambda_1 \) correspond to hard momenta \( P \) or \( \Delta \) whereas powers of \( \Lambda_2 \) correspond to soft momenta \( p \). This leads directly to the selection rule Eq. (71).

Let us use this selection rule to study the non-factorizable correction due to the exchange of one soft \( A_\mu \) field. As mentioned above, Eq. (72), this contribution will vanish in QCD and our estimate will only be relevant for QED. The Feynman diagram corresponding to this one-loop correction is shown in Fig. 7(a). According to Eq. (71), the leading contribution to the Feynman rule for the vertex scales as
\[
(\bar{\psi}D_{\text{int}}\psi)(P_M, p, \Delta) \sim D e^{\frac{1}{\Lambda_2}}.
\]  
(72)

More explicitly, the following operators linear in \( F_{\mu\nu} \) are possible
\[
(\bar{\psi}D_{\text{int}}\psi)(P_M, p, \Delta) \bigg|_{\sim F} = \left[ \frac{P_M^\mu \Delta^\nu}{\Lambda_1 \Lambda_2} + \frac{(\Delta^\mu p^\nu + P_M^\mu p^\nu)}{\Lambda_1 \Lambda_2} \frac{P_M^2 + \Delta^2}{\Lambda_1 \Lambda_2} + \ldots \right] \cdot \frac{e}{\Lambda_1 \Lambda_2} (\bar{\psi}D\psi) F_{\mu\nu},
\]  
(73)

where in the last term we used the fact that \( P_M \Delta \) is subleading. The leading contribution to the Feynman rule is coming from the first terms in Eq. (73). With the help of Eq. (72) the one-loop correction relative to the Born term can be estimated as
\[
\delta_1 \sim \int \frac{d^4 l}{T^2} \frac{e}{\Lambda_2} \frac{e}{\Lambda_2} \sim \frac{\Lambda_2^4}{\Lambda_2^2} \frac{e}{\Lambda_2} \sim e^2,
\]  
(74)

where \( l \) is the soft momentum of the exchanged gauge boson. This shows that non-factorizable corrections to exclusive quantities are \( \mathcal{O}(\alpha) \) and are not suppressed by any additional parameters.

In order to compare to our previous results, let us recall that the estimates of non-factorizable corrections presented in Sect. 2 are valid for a special class of observables only. The estimates are valid if one integrates out part of the phase space corresponding to final state fermions, keeping the momenta and invariant masses of the unstable particles fixed. This corresponds to
the construction of operators in the effective field theory using \(P_M^\mu\) and \(p^\mu\), but not \(\Delta^\mu\). Indeed, suppose that the contribution of an operator depends on \(\Delta\). After performing a phase-space integral over the fermion decay angles this contribution will be expressed in terms of \(P_M\) and \(p\) only. Thus, in order to reproduce the estimates of Sect. 2 we will have to construct a new effective Lagrangian. This Lagrangian is the same as the previous one, but without terms containing \(\Delta\). This Lagrangian will be used to calculate observables, in which the integration over fermion decay angles is already performed. Consulting Eq. (73), the leading operator linear in \(F_{\mu\nu}\) is

\[
(\bar{\psi} D_{int} \psi)(P_M, p) \bigg|_{\sim F} = e \frac{P_M^{\mu} p^\nu P_M^2}{(\Lambda_1 \Lambda_2)^3} (\bar{\psi} D \psi) F_{\mu\nu}.
\]

Therefore, the Feynman rule corresponding to the one-gauge-boson-radiation vertex in this new Lagrangian scales as

\[
\frac{(\bar{\psi} D_{int} \psi)(P_M, p)}{(P_M, p)} \sim \mathcal{D} \frac{e P_M^2}{\Lambda_2^2 \Lambda_1^2} \frac{P_M^2}{\Lambda_1^2}.
\]

The estimate of the non-factorizable correction in this case is

\[
\delta_1 \sim \frac{\Lambda_1^4}{\Lambda_2^2} \cdot \frac{e P_M^2}{\Lambda_2^2 \Lambda_1^2} \cdot \frac{e P_M^2}{\Lambda_2^2 \Lambda_1^2} \sim e^2 \frac{M^4}{E^4}.
\]

This is an estimate similar to what we had in Sect. 3. It shows that at high energy, \(E \gg M\), non-factorizable corrections are suppressed.

Of course, it can be that the underlying theory has more symmetries. In that case one has to take them into account in the construction of the relevant operators. This is the reason why the estimate obtained in Eq. (76) does not agree with Eqs. (38) and (39). Indeed, if the unstable particle is neutral and it decays into a fermion-antifermion pair with the same flavour there is an antisymmetry under the exchange of final state particles. This leads to the statement that the one-loop non-factorizable correction is zero in this case. In fact, all odd-order-loop non-factorizable corrections are zero because of this symmetry. In the effective theory language this means that operators linear in \(F\), like the one given in Eq. (75) are forbidden. Alternatively, performing the matching one will find that the corresponding Wilson coefficients vanish. However, if there is no additional symmetry the estimate given in Eq. (76) holds.
Let us extend this analysis to two-loop corrections. The explicit form of the terms quadratic in $F_{\mu\nu}$ is

$$
(\bar{\psi}\mathcal{D}_{\text{int}}\psi)(P_M, p)\bigg|_{F^2} = e^2 \frac{1}{(\Lambda_1 \Lambda_2)^4} \left( g^{\nu\sigma} P_M^\mu P_{\rho}^\sigma P_M^2 + g^{\nu\sigma} g^{\mu\rho} P_M^4 \right) (\bar{\psi} \mathcal{D} \psi) F_{\mu\nu} F_{\sigma\rho}.
$$

(77)

The diagram corresponding to two-loop correction is shown in Fig.7(b). The Feynman rule for the vertex with the emission of two gauge bosons scales like

$$
(P_M, p) \sim \mathcal{D} \frac{e^2}{\Lambda_2^2} \cdot \frac{P_M^2}{\Lambda_1^2} \left( 1 + \frac{P_M^2}{\Lambda_1^2} \right).
$$

(78)

At high energy the leading contribution comes from the $g^{\nu\sigma} P_M^\mu P_{\rho}^\sigma P_M^2$ term and the contribution of the $g^{\nu\sigma} g^{\mu\rho} P_M^4$ term is suppressed by $M^2/E^2$. The relative two-loop non-factorizable correction can be estimated as

$$
\delta_2 \sim \int \frac{d^4 k_1 \, d^4 k_2}{k_1^2 \, k_2} \left( \frac{e^2 P_M^2}{\Lambda_2^2 \, \Lambda_1^2} \right) \left( \frac{e^2 P_M^2}{\Lambda_2^2 \, \Lambda_1^2} \right) \sim \left( \frac{P_M^4}{\Lambda_2^2 \, \Lambda_1^2} \right) \sim e^4 \frac{M^4}{E^4}.
$$

(79)

This is precisely the estimate for two-loop non-factorizable final-final corrections obtained in Sect. 4, Eq. (44).

It is possible to construct $N$th order terms in $F_{\mu\nu}$ for $\mathcal{D}_{\text{int}}$. It is clear that at high energy the leading contribution will come from operators in which as many $P_M^\mu$ as possible are contracted with $F_{\mu\nu}$. This is because the remaining $P_M^\mu$ needed for dimensional reasons will have to be contracted with each other. Since $P_M^2 \sim M^2 \ll \Lambda_2^2$, this results in a suppression at high energy. If $N$ is even $(F_{\mu\nu})^N$ has $2N$ indices to be contracted. Due to the antisymmetry of the field tensor a maximum of $N$ of these indices can be contracted with $P_M^\mu$. The remaining $P_M^\mu$ have to be contracted with each other. By looking at the selection rule, Eq. (71), it can be seen that different $m$ are possible. All of them give the same contribution if $\mu$ is contracted with $P_M^\mu$. If $\mu$ is contracted with $F_{\mu\nu}$ then some extra $P_M^\mu$ would have to be contracted with each other giving additional $M/E$ suppression. This means that the leading correction is the same as that coming from the selection rule with $m = 0$

$$
(\bar{\psi}\mathcal{D}_{\text{int}}\psi)(P_M, p)\bigg|_{F^N} \sim \frac{P_M^{2N}}{(\Lambda_1 \Lambda_2)^{2N}} (\bar{\psi} \mathcal{D} \psi)(eF)^N, \quad N \text{ even.}
$$

(80)

Note that this is in perfect agreement with the explicit result of $\Sigma(3)$ given in Appendix 3. If $N$ is odd $(F_{\mu\nu})^N$ has $2N$ indices to be contracted, out of which a maximum of $N$ can be contracted with $P_M^\mu$. The rest of the indices cannot be contracted with each other because their number is odd. Thus, at least one $p_\mu$ is necessary. Again, in the selection rule Eq. (71) different values of $m$ are possible. However, the leading contribution comes from $m = 1$. All other values of $m$ give the same result or suppressed contributions. The selection rule Eq. (74) gives for odd $N$

$$
(\bar{\psi}\mathcal{D}_{\text{int}}\psi)(P_M, p)\bigg|_{F^N} \sim \frac{P_M^{2N+1}}{(\Lambda_1 \Lambda_2)^{2N+1}} (\bar{\psi} \mathcal{D} \psi)(eF)^N, \quad N \text{ odd.}
$$

(81)
Thus, the operator $\mathcal{D}_{\text{int}}$ is of the following form

$$\langle \bar{\psi} \mathcal{D}_{\text{int}} \psi \rangle (P_M, p) =$$

$$= \langle \bar{\psi} \mathcal{D} \psi \rangle + e \frac{P^\mu_M P^\nu_M P^2_M}{(\Lambda_1 \Lambda_2)^3} \langle \bar{\psi} \mathcal{D} \psi \rangle F_{\mu\nu} + e^2 \frac{P^\mu_M P^\nu_M P^2_M}{(\Lambda_1 \Lambda_2)^4} \langle \bar{\psi} \mathcal{D} \psi \rangle F_{\mu\nu} F_{\nu}'$$

$$+ e^N \frac{(P^2_M)^{N/2} P^\mu_M \cdots P^\mu_M}{(\Lambda_1 \Lambda_2)^{2N}} T^{\sigma_1 \cdots \sigma_N} \langle \bar{\psi} \mathcal{D} \psi \rangle F_{\mu_1 \sigma_1} \cdots F_{\mu_N \sigma_N} \bigg|_{N=\text{even}}$$

$$+ e^N \frac{(P^2_M)^{(N+1)/2} P^\mu_M \cdots P^\mu_M}{(\Lambda_1 \Lambda_2)^{2N+1}} T^{\sigma_1 \cdots \sigma_{(N-1)}} \langle \bar{\psi} \mathcal{D} \psi \rangle F_{\mu_1 \sigma_1} \cdots F_{\mu_N \sigma_N} \bigg|_{N=\text{odd}} + \ldots,$$

where $T^{\sigma_1 \cdots \sigma_N}$ is a symmetric tensor of even rank constructed from $g_{\theta \sigma_j}$.

We are now ready to estimate the $N$-loop non-factorizable correction. The estimate will be different for odd and even $N$. Let us start with even $N$. The Feynman rule corresponding to the radiation of $N$ gauge bosons from the decay dipole is

$$N \left( \begin{array}{c} \hline \hline \end{array} \right) (P_M, p) \sim \mathcal{D} \frac{e^N (P^2_M)^{N/2}}{\Lambda_2^N \Lambda_1^N}, \quad N \text{ even.}$$

Correspondingly, the relative $N$-loop non-factorizable correction is

$$\delta_N \sim \int \frac{d^4k_1}{k_1^2} \ldots \frac{d^4k_N}{k_N^2} \left( \frac{e^N (P^2_M)^{N/2}}{\Lambda_2^N \Lambda_1^N} \right) \left( \frac{e^N (P^2_M)^{N/2}}{\Lambda_2^N \Lambda_1^N} \right)$$

$$\sim \left( \frac{\Lambda_1^N}{\Lambda_2^N} \right)^N \left( \frac{e^N (P^2_M)^{N/2}}{\Lambda_2^N \Lambda_1^N} \right)^2 \sim e^{2N} \left( \frac{M}{E} \right)^{2N} \quad N \text{ even.}$$

Here the first factor comes from the $N$ loop integrals and $N$ gauge boson propagators while the second factor comes from two interfering currents to radiate $N$ gauge bosons.

The case of odd $N$ is treated analogously. The Feynman rule corresponding to radiation of $N$ gauge bosons from decay is

$$N \left( \begin{array}{c} \hline \hline \end{array} \right) (P_M, p) \sim \mathcal{D} \frac{e^N (P^2_M)^{(N+1)/2}}{\Lambda_2^N \Lambda_1^{N+1}}, \quad N \text{ odd.}$$

In this case, the relative $N$-loop non-factorizable correction is

$$\delta_N \sim \int \frac{d^4k_1}{k_1^2} \ldots \frac{d^4k_N}{k_N^2} \left( \frac{e^N (P^2_M)^{(N+1)/2}}{\Lambda_2^N \Lambda_1^{N+1}} \right) \left( \frac{e^N (P^2_M)^{(N+1)/2}}{\Lambda_2^N \Lambda_1^{N+1}} \right)$$

$$\sim \left( \frac{\Lambda_1^{N+1}}{\Lambda_2^N} \right)^N \left( \frac{e^N (P^2_M)^{(N+1)/2}}{\Lambda_2^N \Lambda_1^{N+1}} \right)^2 \sim e^{2N} \left( \frac{M}{E} \right)^{2(N+1)} \quad N \text{ odd.}$$

Thus, the estimates obtained using the effective theory, Eqs. (84) and (86), agree with the estimates obtained through the analysis of higher order Feynman diagrams, Eqs. (14) and (15).
4 Applications

In this section we consider some topical applications of the techniques presented in Sect. 2 and Sect. 3 to the pair production of $W$ and $Z$ bosons and top quarks in $\gamma\gamma$ and $e^+e^-$ collisions. As above, all our considerations will be performed in the pole-scheme. We shall also confine ourselves to the case when no kinematic restrictions are imposed on the configurations of the final state particles. In general, as shown in [3], the kinematic cuts applied to the final state particles may induce significant changes in the energy behavior of the non-factorizable effects. We focus on the distributions in which the integration is performed over the momenta of the decay products keeping tracks of the invariant masses of the unstable particles.

In Sect. 2 and Sect. 3 it was shown that non-factorizable corrections due to interaction with decay dipoles are suppressed at high energy. Throughout this section we consider only this type of corrections. We do not discuss other corrections (factorizable corrections and non-factorizable corrections due to propagation corrections) for which the suppression mechanism is not applicable.

4.1 Neutral unstable particles

We begin with the basic case when only neutral (both with respect to electromagnetic and color charges) particles participate in the production process and, therefore, only interference between radiation accompanying the decays occurs. This can be exemplified by the process

$$\gamma\gamma \rightarrow Z(p_1)Z(p_2) \rightarrow 4 \text{ fermions} (k_1, k'_1; k_2, k'_2), \quad (87)$$

with $p_{1,2} = k_{1,2} + k'_{1,2}$. The non-factorizable effects appear in the first order in QED and in the second order in QCD. In the case of color singlet unstable particles the QCD interconnection is additionally suppressed by $1/N_c^2$ compared to the leading QCD contribution, see e.g. [18] ($N_c$ being the number of colors).

For the purposes of illustration it is sufficient to focus attention on the pure QED phenomena accompanying the process

$$\gamma\gamma \rightarrow Z(p_1)Z(p_2) \rightarrow e^+(k_1)e^-(k'_1)\mu^+(k_2)\mu^-(k'_2). \quad (88)$$

We present below an estimate of the energy behavior of the non-factorizable correction of order $\mathcal{O}(\alpha^N)$ induced by the exchanges of $N$ virtual photons with momenta $l_1,..,l_N$ between the decay products of the two $Z$’s.

Similarly to what we did in Sect. 2 in order to find the $N$-loop contribution we have to evaluate the integrals over the Born decay matrix element squared multiplied by the $N$ non-factorizable currents corresponding to radiation off the massless charged fermions with momenta

---

3Our approach may be generalized to the case of pair production of unstable particles in hadronic reactions (for discussion of the gluon bremsstrahlung effects accompanying the $t\bar{t}$ production at hadron colliders see for instance [24]).
where $i = 1, 2$. The new element with respect to the consideration of Sect. 2 is the appearance of the decay tensor $\Delta_{\mu\nu}$, which results from the vector nature of the decaying particles. The indices $\mu, \nu$ will be contracted with the rest of the Born matrix element squared. The indices $\alpha_1 \ldots \alpha_N$ will be contracted with the corresponding indices of the non-factorizable currents from the second unstable particle. $\Delta_{\mu\nu}$ can be decomposed into two contributions with different $P$-parity

$$\Delta_{\mu\nu} = \Delta^V_{\mu\nu} + \Delta^A_{\mu\nu}$$

The $\Delta^V_{\mu\nu}$ piece corresponds to the $(V^2 + A^2)$ contribution to the matrix element squared while $\Delta^A_{\mu\nu}$ results from the $(VA)$ interference. We shall call them $(V^2 + A^2)$ and $(VA)$ contributions respectively.

Similarly to Sect. 2 we introduce the new variables

$$k^\mu_i = \frac{1}{2} (p^\mu_i + \Delta^\mu_i), \quad k'^\mu_i = \frac{1}{2} (p'^\mu_i - \Delta^\mu_i).$$

In terms of these new variables the decay tensor components $\Delta^V_{\mu\nu}$ can be rewritten as

$$\Delta^V_{\mu\nu} = \frac{1}{2} \left[ p^\mu p^\nu - M^2 g^{\mu\nu} - \Delta^\mu \Delta^\nu \right], \quad \Delta^A_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} k^\sigma k'^\rho.$$

We immediately see from Eq. (93) that the $\Delta$-dependent part of the $(V^2 + A^2)$ component is suppressed by a factor $\sim M^2/E^2$ as compared to the leading term while the $(VA)$ component is suppressed by $O(M/E)$ factor with respect to the leading $(V^2 + A^2)$ part.

Let us perform an integration over the final state particle momenta in Eq. (89) (see Sect. 2 and, in particular, Eq. (30) and Eq. (33)) and evaluate the high-energy behavior of different components of the $N$-loop tensor $I_{\mu\nu}^{\alpha_1 \ldots \alpha_N}$

$$V(A) I_{\mu\nu}^{\alpha_1 \ldots \alpha_N} = \int \Delta \Delta^V_{\mu\nu} \left( \frac{k}{k l_1} - \frac{k'}{k' l_1} \right)^{\alpha_1} \ldots \left( \frac{k}{k l_N} - \frac{k'}{k' l_N} \right)^{\alpha_N}. $$

The result depends on whether $N$ is odd or even

$$V I_{\mu\nu}^{\alpha_1 \ldots \alpha_N} \sim \begin{cases} 0, & \text{for } N \text{ odd}, \\ \text{Born} \cdot \Gamma^{-N}, & \text{for } N \text{ even}. \end{cases} $$

$$A I_{\mu\nu}^{\alpha_1 \ldots \alpha_N} \sim \begin{cases} \text{Born} \cdot \Gamma^{-N} \cdot M/E, & \text{for } N \text{ odd}, \\ 0, & \text{for } N \text{ even}. \end{cases} $$
Here “Born” is a symbolic notation for the leading \((V^2 + A^2)\) contribution to the Born matrix element squared.

Power-counting rules established in Sect. 2 allow one to estimate the suppression factors corresponding to the different parts of the non-factorizable correction to the cross section of the process \((88)\). The results are summarized in Table 1 where the notations \((V^2 + A^2) - (V^2 + A^2)\), \((V^2 + A^2) - (VA)\), \((VA) - (VA)\) are used to specify the interference between the different contributions into the Born decay matrix element squared of each \(Z\)-boson.

| \(N\) | Born                                     |
|-------|------------------------------------------|
|       | \((V^2 + A^2) - (V^2 + A^2)\) | \((V^2 + A^2) - (VA)\) | \((VA) - (VA)\) |
| \(N\)-even | \((M/E)^{2N}\) | 0 | 0 |
| \(N\)-odd  | 0 | 0 | \((M/E)^{2N+2}\) |

Table 1: Estimates of the energy dependence of the \(\mathcal{O}(\alpha^N)\) non-factorizable correction, \(\delta_{\text{nf}}\), integrated over the momenta of the decay products for different interference contributions corresponding to the process \((88)\).

As follows from Table 1, the one-loop QED non-factorizable corrections to the cross section of the process \((88)\) for different final states may appear only as a result of the \((VA) - (VA)\) decay interference and in practice are numerically small (see Ref. \([16]\)). This term scales as \(E^{-4}\) with the CMS energy, which is in accord with the earlier observation in Ref. \([17]\). The two-loop QED interconnection effects are caused by the interference of the \(P\)-even terms \((V^2 + A^2) - (V^2 + A^2)\), and their contribution also scales as \(E^{-4}\). Symbolically, the four leading QED terms in relative non-factorizable correction, \(\delta_{\text{nf}}\), for the process \((88)\) can be presented as

\[
\delta_{\text{nf}} \sim r_A \alpha \left( \frac{M}{E} \right)^4 \left[ 1 + \alpha^2 \left( \frac{M}{E} \right)^4 + \ldots \right] + \alpha^2 \left( \frac{M}{E} \right)^4 \left[ 1 + \alpha^2 \left( \frac{M}{E} \right)^4 + \ldots \right], \tag{97}
\]

where parameter \(r_A\) is determined by the product of vector and axial couplings of the corresponding neutral current. Therefore, the actual parameter of the perturbation theory in \(\delta_{\text{nf}}\) separately for even and odd powers of \(\alpha\) is \(\alpha^2(M/E)^4\), and the series in Eq. (97) rapidly converges at \(E \gg M\).

All these observations remain valid for the decay-decay part of the QED non-factorizable corrections to the process

\[e^+e^- \to Z(p_1)Z(p_2) \to 4 \text{ fermions } (k_1, k_1'; k_2, k_2'),\tag{98}\]

The difference is that now there are also production-decay QED interferences, which were absent in the process \((88)\). Estimates for production-decay interferences are in general different from those for the decay-decay interferences.
Let us now turn to the QCD non-factorizable corrections to the processes (87) and (98) with final state quarks. Due to the group structure of QCD these can appear only at \( N \geq 2 \), see e.g. [18]. Only the \((V^2 + A^2) - (V^2 + A^2)\) decay-decay gluon interference survives at \( N = 2 \) with the energy scaling \( E^{-4} \). As we discussed in Sect. 3.3.2, in QCD the separation between higher-order non-factorizable corrections due to interaction with decay dipoles, which is of interest to us here, and non-factorizable propagation corrections is complicated due to the non-abelian structure of the gauge group. This separation is correctly understood in terms of different operators in the Lagrangian of the effective theory discussed in Sect. 3.3.2. What we estimate here are the leading non-factorizable corrections due to operators describing decay dipole interactions. Using Table 1 we can see that the four leading terms in \( \delta_{QCD}^{nf} \) can be written symbolically as

\[
\delta_{QCD}^{nf} \sim \alpha_s^2 \left( \frac{M}{E} \right)^4 \left[ 1 + \alpha_s^2 \left( \frac{M}{E} \right)^4 + \ldots \right] + r_A \alpha_s^3 \left( \frac{M}{E} \right)^8 \left[ 1 + \alpha_s^2 \left( \frac{M}{E} \right)^4 + \ldots \right].
\]

(99)

The \( N_c \) dependence of non-factorizable corrections might be of special interest. The exact formula is complicated for arbitrary order, \( N \), however, the leading \( N_c \) dependence is easy to estimate and goes as \((C_F \alpha_s)^N / N_c^2 \sim (N_c \alpha_s)^N / N_c^2\). This means that the actual parameter of the perturbation theory separately for even and odd powers of \( \alpha_s \) is \( N_c^2 \alpha_s^2 (M/E)^4 \). The same conclusion remains valid for the QCD non-factorizable corrections to the process

\[ e^+e^-[\gamma\gamma] \rightarrow W^+W^- \rightarrow 4 \text{ quarks}. \]

(100)

The main qualitative difference between the \( W^+W^- \) and \( ZZ \) interconnection results stems from the difference in the vector and axial couplings of the \( W \) and \( Z \) bosons. Note that since the \( Z \) mass and other properties are well known, \( ZZ \) production provides a unique laboratory for studying QCD interconnection phenomena [18, 19].

Formally, the application of perturbation theory to the description of non-factorizable QCD phenomena accompanying \( ZZ \) and \( W^+W^- \) production at \( E \gg M \) is restricted by the requirement that the typical space-time separation between the decay vertices does not exceed the characteristic strong interaction scale \( R_{ch} \sim 1/\mu \), where \( \mu \sim m_\pi \sim \Lambda_{QCD} \),

\[ d \sim 1/\Gamma E/M < 1/\mu, \]

(101)

see Refs. [3, 18]. In other words, remembering that the life-time of an unstable particle in the laboratory frame is

\[ t_{dec} \sim \frac{E}{M \Gamma}, \]

(102)

we can say that this condition implies that the decay happens before the formation of the first interconnecting hadrons.

At the same time, since the relevant momentum scale for the non-factorizable effects is \( k \sim 1/d \) the requirement Eq. (101) coincides with the Landau pole condition for \( \alpha_s \) associated with the \( \delta_{QCD}^{nf} \) correction. Therefore, even though formally the perturbative results at very high energies are not justified our estimates show that in the non-controllable domain the \( \delta_{QCD}^{nf} \) corrections due to interaction with products of decay are decreasing very sharply with increasing
energy, if the two outgoing hadronic systems are more and more boosted apart. Moreover, the higher is the power of $\alpha_S$ coupling to the decay dipoles the steeper is the energy fall-off. The corrections to propagation, on the other hand, are not suppressed at high energy. At large $\alpha_S$ the description of propagation becomes purely non-perturbative. It is worthwhile to mention that the non-perturbative models predict that the hadronic cross-talk between the decaying particles dampens with energy comparatively slowly $[19, 25]$. 

### 4.2 Charged unstable particles

In the case of charged, unstable particles both types of non-factorizable corrections are present: decay-decay and production-decay. We shall consider only the decay-decay part. Our analysis embraces, in particular, QED effects in the $W^+W^-$ production process and QCD effects in the production and decay of a $t\bar{t}$ pair.

Let us consider the case when the “charged” (with respect to QED or QCD interactions as appropriate) unstable particle with momentum $p^\mu$ decays into a “charged” massless particle with momentum $k^\mu$ and a “neutral” particle with momentum $k'^\mu$. We begin with vector-boson pair production accompanied by the leptonic decays. Analogously to the previous subsection we introduce the new variable $\Delta$ (see Eq. (92)) and evaluate the high-energy behavior of the $N$-loop tensors $V \, I^{\alpha_1...\alpha_N}_{\mu\nu}$

$$V \, I^{\alpha_1...\alpha_N}_{\mu\nu} = \int_\Delta \Delta^{V(A)} \cdot \left( \frac{p}{pl_1} - \frac{k}{kl_1} \right)^{\alpha_1} \ldots \left( \frac{p}{pl_N} - \frac{k}{kl_N} \right)^{\alpha_N}. \quad (104)$$

Here the contributions with different $P$-parity are determined by the Born decay matrix elements. Repeating the same procedure as in the derivation of Eqs. (95) and (96) we arrive at

$$V \, I^{\alpha_1...\alpha_N}_{\mu\nu} \sim \begin{cases} \text{Born} \cdot \Gamma^{-N} \cdot M/E, & \text{for odd } N, \\ \text{Born} \cdot \Gamma^{-N}, & \text{for even } N. \end{cases} \quad (105)$$

$$A \, I^{\alpha_1...\alpha_N}_{\mu\nu} \sim \begin{cases} \text{Born} \cdot \Gamma^{-N} \cdot M/E, & \text{for odd } N, \\ \text{Born} \cdot \Gamma^{-N} \cdot M^2/E^2, & \text{for even } N. \end{cases} \quad (106)$$

Again, “Born” is a symbolic notation for the leading $(V^2 + A^2)$ contribution to the Born matrix element squared. Recall that the main difference with respect to the case of neutral particles is caused by the presence of the odd powers of $\Delta$ in the expansion of the decay currents over $\Delta$.

The estimates for the energy scaling of the non-factorizable corrections caused by the final-final interferences are summarized in Table 2.

From Table 2 we can see immediately that the leading part of the decay-decay interference contribution to $\delta_{nf}^{QED}$ in the case of the $W^+W^-$ production scales with energy as $E^{-4}$. This was first established in Ref. [17]. Symbolically the four leading QED terms for this process can be written as

$$\delta_{nf} \sim \alpha \left( \frac{M}{E} \right)^4 \left[ 1 + \alpha^2 \left( \frac{M}{E} \right)^4 + \ldots \right] + \alpha^2 \left( \frac{M}{E} \right)^4 \left[ 1 + \alpha^2 \left( \frac{M}{E} \right)^4 + \ldots \right], \quad (107)$$
A similar argumentation remains valid for the QCD non-factorizable corrections corresponding to the $t\bar{t}$ production process

$$\gamma\gamma(e^+e^-) \to t\bar{t} \to bW^+\bar{b}W^-,$$

with $\alpha$ substituted by $\alpha_S$. Recall that in the $t\bar{t}$ case there is no color suppression in $\delta_{QCD}^{nf}$ if the pair is produced in the color singlet state and there is an additional $1/N_c^2$ suppression for the case of the octet $t\bar{t}$ state [24, 15]. Analogously to the case of color-singlet unstable particle production discussed above the perturbative results for $\delta_{QCD}^{nf}$ are fully controllable only if the restriction Eq. (101) is satisfied.

We stress again that for the charged particles case this estimates are not complete. For the evaluation of the magnitude of the total QCD interconnection effects one has to incorporate also production-decay interferences and real radiation.

### 5 Conclusions

In the near future, the direct evaluation of two-loop corrections to processes involving unstable particles is not feasible, not to mention even higher order corrections. In view of the phenomenological importance of these processes, however, it is essential to improve our understanding beyond the presently available one-loop calculations. In this paper we took a first step in this direction by analyzing higher-order radiative corrections to pair production of unstable particles with their subsequent decay.

Having a future linear collider in mind, we are particularly interested in the high-energy behavior, $E \gg M$, of the corrections. We identified various effects and studied their behavior/suppression at high energy. In particular, we considered distributions that are obtained by integrating over the angles of the decay products, but keeping their invariant mass fixed.

As a first case, we studied neutral, unstable particles decaying into fermions that interact through the exchange of abelian gauge bosons. This is the simplest case and can be analyzed by studying Feynman diagrams. By extending the well known one-loop analyses [3, 13, 14].

| $N_1, N_2$ | Born |
|------------|------|
| $N_{-even}$ | $(V^2 + A^2) - (V^2 + A^2)$ |
| $N_{odd}$  | $(M/E)^{2N}$ |

Table 2: Estimates of the energy dependence of the $O(\alpha^N)$ non-factorizable corrections, integrated over the momenta of the decay products for the decay-decay interference contributions associated with the “charged” unstable particle production.
we found that to all orders in perturbation theory the complete corrections have the following structure: there are factorizable and non-factorizable corrections and interferences between them that factorize with respect to each other, Eq. (21). The factorizable corrections are associated with a hard scale $\sim E$, whereas the non-factorizable corrections are associated with a soft scale $\sim \Gamma M/E$. Furthermore, there are two sources of non-factorizable corrections, namely interactions of the gauge bosons with the decay dipoles and propagator corrections of the gauge bosons.

Since the various corrections listed above have a different physical origin it is to be expected that their high-energy behavior is different. Indeed, we found that the corrections due to the interaction with the dipoles is — on top of the usual coupling constant suppression — additionally suppressed at high energies, Eqs. (38) and (39). The other corrections identified above do not have such an additional suppression. This analysis can be generalized to the case of charged, unstable particles and similar results are found, Eqs. (44) and (45).

The extension of such an analysis to the non-abelian case is a rather daunting task. The reason is that the self-coupling of the gauge bosons complicates the matter enormously. In particular, most diagrams will now contribute to several types of corrections, rather than just one. Therefore, it was necessary to tackle this problem in a different way. Since the separation of the full corrections into various parts relies on the presence of two different scales, $\Lambda_1 \sim E$, $\Lambda_2 \sim \Gamma M/E$ with $\Lambda_1 \gg \Lambda_2$, an effective theory approach looks very promising.

In order to pursue this we identified the relevant fields present in the underlying theory: hard, resonant, soft and external. By subsequently integrating out the various fields we obtain a hierarchy of effective field theories. The first step is to integrate out the hard modes. This leaves us with a theory where only resonant, soft and external particles are dynamical. The factorizable corrections are those that are encoded in the (higher order corrections to the) Wilson coefficients, whereas the non-factorizable corrections are the corrections due to the still dynamical degrees of freedom. What we have gained is that the generalization of this separation to higher orders in the coupling constant and to the non-abelian case is straightforward. Throughout the paper we restrict ourselves to the leading terms in $\Lambda_2/\Lambda_1$. In this approximation, the effective theory is equivalent to the DPA of the underlying theory.

The second step in constructing an effective field theory is to integrate out the resonant modes. The corresponding effective Lagrangian contains again (non-local) gauge-invariant operators, multiplied by Wilson coefficients. Some of these operators are responsible for the propagator corrections and some are responsible for the dipole interactions. Thus, in the same way as integrating out hard modes provided us with a separation of factorizable and non-factorizable interactions, integrating out resonant modes allows us to separate the two kinds of non-factorizable corrections. As a consequence of the gauge invariance of the operators, this separation is also gauge invariant.

In order to obtain estimates of the various corrections similar to those obtained by analyzing Feynman diagrams, all we have to do is to get estimates of the Wilson coefficients of the corresponding operators. These estimates can be obtained by general considerations, i.e. without explicitly knowing the Wilson coefficient. Thereby we generalize the estimates obtained in the
abelian case to the non-abelian case. Obviously, if we wanted to do a precise calculation in
the effective theory we would need the exact form of the Wilson coefficient. They have to be
obtained through matching the effective theory to the underlying theory. We have done this
matching for the operators corresponding to the emission of up to three gauge bosons, thereby
confirming the estimates obtained by general considerations.

We applied the estimates to several phenomenologically relevant processes. The actual
estimate depends on the structure of the Born term. Also, it is important to stress again that
the estimates obtained here are valid only for the non-factorizable corrections due to dipole
interactions for distributions integrated over the decay angles. Furthermore, in the effective
theory approach we considered the case of neutral, unstable particles only.

In the case of charged, unstable particles there are additional non-factorizable corrections
of the production-decay type. In the effective theory language this means that the operator
associated with $\mathcal{P}$, Eq. (53), is more complicated than given in Eq. (55). Allowing for the emis-
sion of gauge bosons from the unstable particle production process results in more complicated
operators in the effective Lagrangian. In order to obtain estimates in this case one would have
to construct these operators and obtain estimates of the corresponding Wilson coefficients.

Another possible future development is the inclusion of real gauge-boson radiation. As
mentioned in the introduction, real gauge-boson radiation contributes to the non-factorizable
corrections. In this paper, we have left this issue completely aside. However, adding real
radiation and performing an integration over the phase space could certainly be studied in the
effective theory framework.

Finally, we would like to mention the possibility to extend this analysis beyond leading order
in $\Lambda_2/\Lambda_1$. As mentioned above, if we keep only the leading terms in $\Lambda_2/\Lambda_1$ after integrating
out the hard modes we recover the DPA of the underlying theory. Since there is nothing that
prevents us to go beyond the leading terms in $\Lambda_2/\Lambda_1$, the effective theory approach opens up
the possibility to go beyond the DPA in a systematic way.

In summary, we studied the structure of higher-order corrections in the presence of unstable
particles, exploring an effective field theory approach to the problem. Even though we made
several simplifying restrictions this approach is very promising and seems to allow to look at a
whole set of problems from a different, more promising, point of view.

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A Expansion of loop integrals

In this appendix we consider an example of the separation of factorizable and non-factorizable corrections to all orders in $\Lambda_2/\Lambda_1$. We use an expansion of loop integrals in dimensional regularization by letting the loop momenta be in all relevant momentum regions [23, 27]. This expansion is equivalent to our effective field theory framework.

The diagram we consider here is shown in Fig. 5(c) and contains both factorizable and non-factorizable contributions. The scalar integral associated with this diagram is

$$I = \int \frac{dl}{l^2} \frac{1}{(l^2 - 2kl)(l^2 - 2pl + D)}, \quad (A1)$$

where $k^2 = m^2$, $p^2 = M^2$, $M \gg D \gg m$. The integral is IR and UV finite and, thus, does not require regularization. The exact result is

$$I = i\pi^2 M^2 \left[ \ln \frac{M^2}{m^2} \ln \frac{-D}{M^2 - D} - \text{Li}_2 \left( \frac{D}{M^2} \right) - \frac{1}{2} \ln^2 \frac{M^2 - D}{M^2} - \frac{\pi^2}{6} \right]. \quad (A2)$$

In order to expand the loop integral in $D/M^2$ we should assume the loop momentum to be in one of the following two regions: soft, $l \sim D/M$, and hard, $l \sim M$. Then we have to expand the integrand and perform the integrations over the full momentum space in dimensional regularization.

$$I = I_{\text{soft}} + I_{\text{hard}} = \int \frac{dl}{l^2} \frac{1}{(-2kl)(-2pl + D)} + \int \frac{dl}{l^2} \frac{1}{(l^2 - 2kl)(l^2 - 2pl)} \sum_{n=0}^{\infty} \left( \frac{-D}{l^2 - 2pl} \right)^n. \quad (A3)$$

There is only one (the leading) term in the soft momentum expansion because in higher orders the $l^2$ term in the denominator will be cancelled. The resulting integrals are scaleless and vanish in dimensional regularization. The soft integral is UV divergent, and the hard integral is IR divergent. These divergences should be regularized by dimensional regularization and they will cancel in the sum.

The soft integral is

$$I_{\text{soft}} = -\frac{i\pi^2}{M^2} \left[ \ln \frac{M}{m} \left( \frac{1}{\epsilon} - \gamma - \ln \pi + 2 \ln \frac{M^2}{-D} - \ln M^2 \right) + \ln^2 \frac{M}{m} + \frac{\pi^2}{6} \right]. \quad (A4)$$

For $n = 0$ the hard integral is given by

$$\int \frac{dl}{l^2} \frac{1}{(l^2 - 2kl)(l^2 - 2pl)} = \frac{i\pi^2}{M^2} \left[ \ln \frac{M}{m} \left( \frac{1}{\epsilon} - \gamma - \ln \pi - \ln M^2 \right) + \ln^2 \frac{M}{m} \right]. \quad (A5)$$

whereas for $n \geq 1$ the integrals are finite and given by

$$\int \frac{dl}{l^2} \frac{1}{(l^2 - 2kl)(l^2 - 2pl)} \sum_{n=0}^{\infty} \left( \frac{-D}{l^2 - 2pl} \right)^n = \frac{i\pi^2}{M^2} \left[ - \ln \left( 1 - \frac{D}{M^2} \right) \ln \frac{M^2}{m^2} - \text{Li}_2 \left( \frac{D}{M^2} \right) - \frac{1}{2} \ln^2 \left( 1 - \frac{D}{M^2} \right) \right]. \quad (A6)$$

As expected, the sum of $I_{\text{soft}}$ and $I_{\text{hard}}$ reproduce the full result.
B Matching calculation

In this appendix we give some details concerning the matching of the \((\bar{\psi}\psi)F^3\) operator defined
in Eq. (7). To do this matching, we first have to evaluate the contribution of the operator
\((\bar{\psi}\psi)F^2\), Eq. (3), to three-gluon processes. The Wilson coefficient \(\Sigma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3)}(k,k'|l_1,l_2,l_3)\)
is then determined such that the full result in the effective theory is the same as in QCD.

The first point to note is that this contribution depends on the chosen path from \(z_1\) to \(z_2\). This
path dependence has to be cancelled by the contribution of the operator \((\bar{\psi}\psi)F^3\). From this we
conclude that the precise form of the corresponding Wilson coefficient \(\Sigma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3)}\) is path dependent.

To illustrate this point we will consider the matching for two different paths. It is convenient
to split the tensor \(\Sigma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3)}\) into a color symmetric and antisymmetric piece
\[
\Sigma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3)}(k,k'|l_1,l_2,l_3) = \Sigma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3,+)}(k,k'|l_1,l_2,l_3) + \Sigma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3,-)}(k,k'|l_1,l_2,l_3)
\]

The tensor \(\Sigma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3,+)}\) will be proportional to a color factor \(1/2\) \((\text{Tr} (T^{a_1}T^{a_2}T^{a_3}) + \text{Tr} (T^{a_1}T^{a_3}T^{a_2}))\)
while the \(\Sigma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3,-)}\) terms will come with a color factor \(1/2\) \((\text{Tr} (T^{a_1}T^{a_2}T^{a_3}) - \text{Tr} (T^{a_1}T^{a_3}T^{a_2}))\).
The symmetric part is exactly the same as in the abelian theory and is given in Eq. (B1) with
\(N = 3\). The antisymmetric part is the new feature of the non-abelian case and contains all the
path dependence.

The first path we consider is the linear path. In this case
\[
\int_{z_1}^{z_2} d\omega^{\alpha_3} e^{-il_3\omega} = i \frac{z_2^{\alpha_3} - z_1^{\alpha_3}}{l_3 \cdot (z_2 - z_1)} (e^{-il_3z_2} - e^{-il_3z_1}) \tag{B3}
\]
Thus the contribution of the operator \((\bar{\psi}\psi)F^2\) to three-gluon processes given in Eq. (B1),
will contain logarithms with ratios of scalar products of momenta as arguments. Since the QCD
result does not contain such logarithms, they will have to be cancelled by the operator \((\bar{\psi}\psi)F^3\).
As a result, the corresponding Wilson coefficient \(\Sigma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3,-)}\) will contain some logarithms.
In order to facilitate the presentation of the result let us introduce the shorthand notations

\[ l_{ij...} \equiv l_i + l_j + \ldots \]  \hspace{1cm} \text{(B4)}

\[ s_{kk'}(l_i, l_j) \equiv k \cdot l_i k' \cdot l_j - k \cdot l_j k' \cdot l_i \]  \hspace{1cm} \text{(B5)}

\[ s_{kk'}(l_i, l_j, l_k) \equiv k \cdot l_i k' \cdot l_j + k \cdot l_j k' \cdot l_k + k \cdot l_j k' \cdot l_k \]  \hspace{1cm} \text{(B6)}

and the auxiliary function \( f_{aux} \)

\[ f_{aux}(k, k'; l_1, l_2, l_3) \equiv \frac{1}{4} \frac{k \cdot l_1 k \cdot l_1 k' \cdot l_2 k' \cdot l_3}{(k \cdot l_2)^2} + \frac{2}{k \cdot l_1 k \cdot l_1 k' \cdot l_2} s_{kk'}(l_1, l_2) s_{kk'}^2(l_1, l_2, l_3) - \frac{k \cdot l_2}{k \cdot l_1} \left( \frac{1}{s_{kk'}(l_1, l_2)} s_{kk'}^2(l_1, l_2, l_3) + \frac{1}{s_{kk'}(l_1, l_2)} s_{kk'}(l_1, l_2, l_3) \right) \log \left( \frac{k \cdot l_1}{k \cdot l_2} \right) \]  \hspace{1cm} \text{(B7)}

The explicit form of the matching coefficient can then be written as

\[ \Sigma^{(3;-)}_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}(k, k'|l_1, l_2, l_3) = \frac{1}{2^3} \left( k_{\mu_1 \nu_1} - k'_{\mu_1 \nu_1} \right) \left( k_{\mu_2 \nu_2} - k'_{\mu_2 \nu_2} \right) \left( k_{\mu_3 \nu_3} - k'_{\mu_3 \nu_3} \right) \Sigma^{(3;L^-)}(k, k'|l_1, l_2, l_3) \times \left( f_{aux}(k, k'; l_1, l_2, l_3) - f_{aux}(k, k'; l_1, l_3, l_2) - f_{aux}(k', k; l_1, l_2, l_3) + f_{aux}(k', k; l_1, l_3, l_2) \right) \]  \hspace{1cm} \text{(B8)}

We should mention that the form of the matching coefficient is not unique, since it only enters in a symmetric combination in any calculation. The important point is that the explicit form of the matching coefficient Eq. (B8) is compatible with the estimate given in Eq. (68).

Let us now consider a second path. In order to evaluate the expression given in Eq. (B8) we choose the following path. We start at \( z_1 \) and follow a linear path to the intermediate point \( u^{\alpha_3} = z_1^{\alpha_3} + l_3^{\alpha_3} l_3 \cdot (z_2 - z_1)/l_3 \cdot l_3 \) from where we follow again a linear path to \( z_2 \). This path has the advantage that

\[ \int_{z_1}^{z_2} \omega^{\alpha_3} e^{-i z_3 \omega} = i \frac{l_3^{\alpha_3}}{l_3 \cdot l_3} (e^{-i z_3 z_2} - e^{-i z_3 z_1}) + z_2^{\alpha_3} - z_1^{\alpha_3} - l_3^{\alpha_3} \frac{l_3 \cdot (z_2 - z_1)}{l_3 \cdot l_3} \]  \hspace{1cm} \text{(B9)}

and, therefore, Eq. (B11) will not yield logarithms upon integration. However, the presence of terms proportional to \( l_3^{\alpha_3} \) in Eq. (B3) results in a more complicated tensor structure of the Wilson coefficient. Instead of being proportional to \( k_{\mu_1} \) or \( k'_{\nu_1} \) only, the matching coefficient will also contain tensor structures with one of the fermion momenta replaced by a gluon momentum. The explicit result is

\[ \Sigma^{(3;-)}_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}(k, k'|l_1, l_2, l_3) = \frac{1}{2^3} \left( k_{\mu_1 \nu_1} - k'_{\mu_1 \nu_1} \right) \left( k_{\mu_2 \nu_2} - k'_{\mu_2 \nu_2} \right) \left( k_{\mu_3 \nu_3} - k'_{\mu_3 \nu_3} \right) \Sigma^{(3;L^-)}(k, k'|l_1, l_2, l_3) + \frac{1}{2^3} \sum_\sigma (-1)^\sigma l_{1 \mu_1} k'_{\nu_1} \left( k_{\mu_2 \nu_2} - k'_{\mu_2 \nu_2} \right) \left( k_{\mu_3 \nu_3} - k'_{\mu_3 \nu_3} \right) \Sigma^{(3;SL^-)}(k, k'|l_1, l_2, l_3) \]  \hspace{1cm} \text{(B10)}
where the sum consists of twelve terms that are obtained by taking all permutations of \{1, 2, 3\} as well as the interchange terms \(k \leftrightarrow k'\). The factor \((-1)^\sigma\) denotes the sign of the permutation of \{1, 2, 3\}. The explicit form of the scalar coefficients is

\[
\Sigma^{(3;L)}(k, k'|l_1, l_2, l_3) = \frac{1}{2 k \cdot l_2 k \cdot l_3 k' \cdot l_2 k' \cdot l_3} \left( \frac{1}{k \cdot l_1 k \cdot l_{12}} - \frac{1}{k' \cdot l_{12} k \cdot l_1} \right)
\]

\[
\Sigma^{(3;SL)}(k, k'|l_1, l_2, l_3) = \frac{1}{3 l_1 \cdot l_1 k \cdot l_3 k' \cdot l_3 l_2 \cdot l_2 k' \cdot l_3} \left( \frac{2}{k \cdot l_2 k' \cdot l_2 k' \cdot l_3} - \frac{1}{k \cdot l_{12} k' \cdot l_1 k' \cdot l_{12}} \right)
\]

Note that all terms in \(\Sigma^{(3;\ldots)}\) that are proportional to one of the gluon momenta are path dependent and will be cancelled by corresponding terms coming from the \((\bar{\psi}\psi)F^2\) operator. Therefore, as for the linear path, the explicit form of the matching coefficient is in agreement with Eq. (B8).

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