A CHARACTERISTIC-FREE PROOF OF A BASIC RESULT
ON $\mathcal{D}$-MODULES

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Abstract. Let $k$ be a field, let $R$ be a ring of polynomials in a finite number of variables over $k$, let $\mathcal{D}$ be the ring of $k$-linear differential operators of $R$ and let $f \in R$ be a non-zero element. It is well-known that $R_f$, with its natural $\mathcal{D}$-module structure, has finite length in the category of $\mathcal{D}$-modules. We give a characteristic-free proof of this fact. To the best of our knowledge this is the first characteristic-free proof.

1. Introduction.

Throughout this paper $k$ is a field, $R = k[x_1, \ldots, x_n]$ is the ring of polynomials in a finite number of variables over $k$ and $\mathcal{D}$ is the ring of $k$-linear differential operators of $R$. The natural $\mathcal{D}$-action on $R$ induces a $\mathcal{D}$-module structure on $R_f$ for every $0 \neq f \in R$. The goal of this paper is to give a characteristic-free proof of the following well-known fact.

Theorem 1.1. $R_f$ has finite length in the category of $\mathcal{D}$-modules.

In characteristic 0 this is due to J. Bernstein [2, 3] and in characteristic $p > 0$ to R. Bøgvad [5]. In both cases proofs are based on suitable notions of holonomicity but the definitions of holonomicity in each of these two cases are completely different.

Our characteristic-free proof is made possible by V. Bavula’s wonderful paper [1] where a characteristic-free definition of holonomic modules is given. But the focus of [1] is the characteristic $p > 0$ case and this assumption is routinely made in the statements and used in the proofs.

In this paper we simplify and characteristic-freeify those of Bavula’s results that are needed for a proof of Theorem 1.1.

Finiteness properties of local cohomology modules for regular rings containing a field had originally been proven by two completely different methods in characteristic $p > 0$ [6] and in characteristic 0 [7]. In [9] we used $\mathcal{D}$-modules to give proofs of these finiteness properties that are characteristic-free modulo the fact that $R_f$, where $R = k[[x_1, \ldots, x_n]]$ is the ring of formal power series in a finite number of variables over $k$, has finite length in the

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Corollary 2.2. (a) of the notation, of Theorem 1.1 in this case. Such a definition is yet to be discovered.

Our proof of Theorem 1.1 leads to a characteristic-free proof of the finiteness properties of local cohomology modules over polynomial rings. And it suggests a way to find a similar proof in general, i.e. for all regular local rings containing a field: through a suitable characteristic-free definition of holo-

ness in the complete local case that would lead to a proof of an analogue of Theorem 1.1 in this case. Such a definition is yet to be discovered.

This paper is self-contained.

2. Preliminaries.

Let $D_{t,i} = \frac{1}{t!} \frac{\partial^t}{\partial x_i^t} : R \to R$ be the $k[x_1, \ldots, x_i, \ldots, x_n]$-linear map that sends $x_i^w$ to $\binom{w}{i} x_1^{i-w}$ ($D_{0,i}$ is the identity map). Even though $\frac{1}{t}$ is part of the notation, $D_{t,i}$ exists in all characteristics because $\binom{w}{i}$ is an integer.

The ring $R$ is in a natural way a subring of $\text{End}_k R$ (every element of $R$ corresponds to the multiplication by that element on $R$) and the following equality holds in $\text{End}_k R$.

**Proposition 2.1.** $D_{t,i} \cdot f = \sum_{s=0}^{t} D_{s,i} \cdot f \cdot D_{t-s,i}$ for every $f \in R$.

**Proof.** We have to show that for every $g \in R$

$$D_{t,i}(f \cdot g) = \sum_{s=0}^{t} D_{s,i}(f) \cdot D_{t-s,i}(g)$$

which is the well-known formula for the higher derivative of a product

$$\frac{\partial^t}{\partial x_i^t}(f \cdot g) = \sum_{s=0}^{t} \binom{t}{s} \frac{\partial^s}{\partial x_i^s}(f) \cdot \frac{\partial^{t-s}}{\partial x_i^{t-s}}(g)$$

divided by $t! \quad \Box$

**Corollary 2.2.** (a) $D_{t,i}$ commutes with $x_j$ for $j \neq i$ and with all $D_{s,j}$.

(b) $D_{t,i} x_i^w = \sum_{s=0}^{t} \binom{w}{s} x_i^s D_{t-s,i}$.

(c) $D_{t,i} \cdot D_{s,j} = \binom{s+t}{s} D_{t+s,i}$.

**Proof.** (a) and (c) are straightforward and (b) is 2.1 with $f = x_i^w. \quad \Box$

The ring $\mathcal{D}$ of $k$-linear differential operators of $R$ is the $k$-subalgebra of $\text{End}_k R$ generated by $R$ and all the $D_{t,i}s$. Corollary 2.2 implies that the products $\{x_i^{n_1} \cdots x_i^{n_t} : D_{t_1,1} \cdots D_{t_n,n}\}$ where $i_1, \ldots, i_n, t_1, \ldots, t_n$ range over all the $2n$-tuples of non-negative integers, are a $k$-basis of $\mathcal{D}$. Indeed, every element of $\mathcal{D}$ is by definition a linear combination of products of $D_{t,i}$s and $x_j$s. Using relations 2.2 (a)-(c) we can write every such product as a linear combination of products of the form $x_i^{n_1} \cdots x_i^{n_t} \cdot D_{t_1,1} \cdots D_{t_n,n}$. Thus $\mathcal{D}$ is free left $R$-module on the products $D_{t_1,1} \cdots D_{t_n,n}$ and similarly, it is a free right $R$-module on these same products.

**Corollary 2.3.** $x_i^w \cdot D_{t,i} \in \mathcal{D}x_i$ if $w > t$ and $x_i^t \cdot D_{t,i} - (1)^t \in \mathcal{D}x_i$. 
Proof. Isolating $x_i^w \cdot D_{t,i}$ from (2.2) we get

$$x_i^w \cdot D_{t,i} = D_{t,i} \cdot x_i^w - \sum_{s=1}^{w} x_i^{w-s} \cdot D_{t-s,i}$$

which implies both containments by induction on $t$. \qed

**Proposition 2.4.** Let $m \subset R$ be a $k$-rational maximal ideal of $R$ (this means that the natural map $k \hookrightarrow R/m$ is bijective). If $\delta \in \mathcal{D}$, we denote $\delta \in \mathcal{D}/\mathcal{D}m$ the image of $\delta$ under the natural surjection $\mathcal{D} \to \mathcal{D}/\mathcal{D}m$.

(i) $\mathcal{D}/\mathcal{D}m$ is the $k$-vector space with basis $\{\overline{D_{t_1,1} \cdots D_{t_n,n}}\}$ as $t_1,\ldots,t_n$ range over all non-negative integers.

(ii) Every element of $\mathcal{D}/\mathcal{D}m$ is annihilated by a power of $m$ and the socle of $\mathcal{D}/\mathcal{D}m$ is generated by $1$.

**Proof.** (i) follows from the fact that $\mathcal{D}$ is a free right $R$-module on the products $D_{t_1,1} \cdots D_{t_n,n}$ and $\mathcal{D}/\mathcal{D}m = \mathcal{D} \otimes_R (R/m)$. (ii) Since the natural map $k \hookrightarrow R/m$ is bijective, $m = (x_1-c_1,\ldots,x_n-c_n)$ where $c_1,\ldots,c_n \in K$. Viewing $x_j - c_j$ as a new $x_j$ we can assume that $m = (x_1,\ldots,x_n)$. According to (2.3) $x_j^{t_j+1}$ annihilates $\{\overline{D_{t_1,1} \cdots D_{t_n,n}}\}$, hence every element of $\mathcal{D}/\mathcal{D}m$ is annihilated by a power of $m$. Clearly $1$ belongs to the socle. It remains to show that every non-zero element $z$ can be sent to the socle by multiplication by an element of $R$. According to (i) $z$ is a $k$-linear combination of a finite number of $D_{t_1,1} \cdots D_{t_n,n}$. Let $D_{t_1,1} \cdots D_{t_n,n}$ have a maximal $t_1 + \cdots + t_n$ among all the $D_{t_1,1} \cdots D_{t_n,n}$ with non-zero coefficients in this linear combination. Hence for every other $D'_{t_1,1} \cdots D'_{t_n,n}$ with non-zero coefficient in the linear combination there is $j$ such that $t_j > t'_j$. It follows from (2.2) that $x_j^{t_j} D'_{t_j,j} \in \mathcal{D}m$. Hence $x_1^{t_1} \cdots x_n^{t_n} \overline{D_{t_1,1} \cdots D_{t_n,n}} = 0$. It similarly follows from (2.2) that $x_1^{t_1} \cdots x_n^{t_n} \overline{D'_{t_1,1} \cdots D'_{t_n,n}} = (-1)^{t_1+\cdots+t_n} 1$. Hence $(-1)^{t_1+\cdots+t_n} x_1^{t_1} \cdots x_n^{t_n} z = 1$. \qed

**Corollary 2.5.** Let $m \subset R$ be a maximal ideal such that $R/m$ is a finite separable field extension of $k$. Let $M$ be a $\mathcal{D}$-module and let $z \in M$ be a non-zero element such that its annihilator in $R$ is $m$. The set $\{D_{t_1,1} \cdots D_{t_n,n} z\}$, as $t_1,\ldots,t_n$ range over all non-negative integers, is linearly independent over $k$.

**Proof.** Replacing $M$ by the $\mathcal{D}$-submodule generated by $z$ we can assume that $M$ is generated by $z$. Let $K$ denote the algebraic closure of $k$, let $R' = K \otimes_k R = K[x_1,\ldots,x_n]$, $\mathcal{D}' = K \otimes_k \mathcal{D}$ and $M' = K \otimes_k M$. Then $\mathcal{D}'$ is the ring of $K$-linear differential operators of $R'$ and $M'$ is naturally a $\mathcal{D}'$-module. Identifying $M$ with the subset $1 \otimes_k M$ of $M'$ we conclude that it is enough to show that the set $\{D_{t_1,1} \cdots D_{t_n,n} z\} \subset M'$ is linearly independent over $K$.

Let $m_1,\ldots,m_s$ be the maximal ideals of $R'$ that lie over $m$. Since the field extension $k \hookrightarrow R/m$ is separable, $K \otimes_k R/m$ is reduced. Therefore $K \otimes_k R/m = R'/(\cap_i m_i)$. This implies that $R'z = K \otimes_k Rz \cong R'/(\cap_i m_i)$
since $Rz \cong R/m$. Now $M'$ being generated by $z$ is a surjective image of $D'/D'(\cap_i m_i)$ via the surjection is $D'/D'(\cap_i m_i) \xrightarrow{1-z} M'$.

But $D'/D'(\cap_i m_i) = D' \otimes_R R'/\cap_i m_i = D' \otimes_R (\oplus_i R'/m_i) = \oplus_i D'/D' m_i$. According to [2, 3], the socle of each $D'/D' m_i$ is generated by 1, hence so is the socle of $D'/D'(\cap_i m_i)$. This means the surjection induces a bijection on the socles and therefore it is itself a bijection. Thus $D'/D'(\cap_i m_i)$ is isomorphic to $M$ via an isomorphism that sends $D_{t_1,1} \cdots D_{t_n,n}$ to $\{D_{t_1,1} \cdots D_{t_n,n} z\}$. But the set $\{D_{t_1,1} \cdots D_{t_n,n}\}$ is linearly independent (this is a consequence of [2, 3] after a natural projection onto some $D'/D' m_i$).

**Definition 2.6.** The Bernstein filtration $k = F_0 \subset F_1 \subset F_2 \subset \ldots$ on $D$ is defined by setting $F_s$ to be the $k$-linear span of the set of products $\{x_1^{i_1} \cdots x_n^{i_n} \cdot D_{t_1,1} \cdots D_{t_n,n} | t_1 + \cdots + t_n + t_1 + \cdots + t_n \leq s\}$.

It follows from [2, 2] that $F_i \cdot F_j \subset F_{i+j}$.

### 3. Proof of Theorem 1.1

The technical heart of our proof is the following proposition.

**Proposition 3.1.** (cf. [1, 9.3]) Assume the field $k$ is separable. Let $M$ be a $D$-module and let $z \in M$ be an element such that the annihilator of $z$ in $R$ is a prime ideal of $R$. Then $\dim_k(F_i z) \geq \left(^{n+i}_{i} \right)$.

**Proof.** Let $d = \dim R/P$, let $h = n - d$, and let $K$ be the fraction field of $R/P$. Since the transcendence degree of $K$ over $k$ equals $d$ and $k$ is separable, after a possible permutation of indices we can assume that $x_{h+1}, \ldots, x_n$ are algebraically independent over $K$ and $K$ is finite and separable over the field of rational functions $K = k(x_{h+1}, \ldots, x_n)$.

Let $R' = K \otimes_R R = K[x_1, \ldots, x_h]$, let $D'$ be the ring of $K$-linear differential operators of $R'$ and let $M' = K \otimes_R M$. The ring $D'$ is a free $R'$-module on the products $D_{t_1,1} \cdots D_{t_h,h}$. Since each such product commutes with $x_j$ for $j > h$, its action on $M$ naturally extends to an action on $M'$ making $M'$ a $D'$-module. It follows from [2, 3] that the set of elements $\{D_{t_1,1} \cdots D_{t_h,h}z\} \subset M'$, as $t_1, \ldots, t_h$ run through all non-negative integers, is linearly independent over $K$. Setting $R'' = k[x_{h+1}, \ldots, x_n]$ ($K$ is the fraction field of $R''$) we conclude that the sum $\Sigma_{t_1, \ldots, t_h} R''D_{t_1,1} \cdots D_{t_h,h}z$ of $R''$-submodules of $M$ is direct, i.e. the natural surjective $R''$-module map $\oplus_{t_1, \ldots, t_h} R''D_{t_1,1} \cdots D_{t_h,h} \rightarrow \Sigma_{t_1, \ldots, t_h} R''D_{t_1,1} \cdots D_{t_h,h}z$ that sends every product $D_{t_1,1} \cdots D_{t_h,h} \in \oplus_{t_1, \ldots, t_h} R''D_{t_1,1} \cdots D_{t_h,h}$ to $D_{t_1,1} \cdots D_{t_h,h}z \in M$ is bijective. And this implies that the set $\{x_{h+1}^{i_1} \cdots x_n^{i_n} D_{t_1,1} \cdots D_{t_h,h}z\}$ of elements of $M$, as $t_1, \ldots, t_h, i_{h+1}, \ldots, i_n$ run over all non-negative integers, is linearly independent over $k$.

The elements of this set with $t_1 + \cdots t_h + i_1 + \cdots + i_n \leq i$ belong to $F_i z$. The number of these elements equals the number of monomials of degree at most $i$ in $n$ variables which is well-known to equal $\left(^{n+i}_{i} \right)$.
Definition 3.2. A $k$-filtration on a $\mathcal{D}$-module $M$ is an ascending chain of $k$-vector spaces $M_0 \subset M_1 \subset M_2 \subset \ldots$ such that $\bigcup_i M_i = M$ and $F_i M_j \subset M_{i+j}$ for all $i$ and $j$.

For example, the Bernstein filtration on $\mathcal{D}$ is a $k$-filtration.

Corollary 3.3. Let $M_0 \subset M_1 \subset M_2 \subset \ldots$ be a $k$-filtration on a $\mathcal{D}$-module $M$. There exists an integer $j$ such that $\dim_k M_i \geq \left( \frac{n+i-j}{i-j} \right)$ for all $i \geq j$.

Proof. Let $k'$ be the algebraic closure of $k$ and $R' = k' \otimes_k R$. Then the ring of $k'$-linear differential operators of $R'$ is $\mathcal{D}' = k' \otimes_k \mathcal{D}$ and $M' = k' \otimes_R M$ is in a natural way a $\mathcal{D}'$-module with $k'$-filtration $M'_0 \subset M'_1 \subset M'_2 \subset \ldots$ where $M'_i = k' \otimes_k M_i$. Since $\dim_k M_i = \dim_{k'} M'_i$, we can and do assume that $k$ is algebraically closed and in particular separable.

Let $P \subset R$ be an associated prime ideal of $M$ in $R$. This means there exists an element $z \in M$ such that the annihilator of $z$ in $R$ is $P$. Let $j$ be the smallest integer such that $z \in M_j$. Clearly $M_i \supset F_{i-j} z$, so we are done by 3.1.

The following definition of holonomicity is equivalent to but somewhat simpler than Bavula’s original definition [11 pp. 185, 198]: in particular we do not require the module $M$ to be finitely generated. But Theorem 3.5 implies that every holonomic module is finitely generated (this fact is not used in the sequel).

Definition 3.4. A $\mathcal{D}$-module $M$ is holonomic if it has a $k$-filtration with $\dim_k M_i \leq C i^n$ for all $i$ where $C$ is a constant independent of $i$.

Theorem 3.5. (cf. [11 9.6]) Every holonomic $\mathcal{D}$-module has finite length in the category of $\mathcal{D}$-modules. In fact if $M_0 \subset M_1 \subset \ldots$ is a $k$-filtration on $M$ with $\dim_k M_i \leq C i^n$, then the length of $M$ in the category of $\mathcal{D}$-modules is at most $n! C$.

Proof. Let $0 = M^0 \subset M^1 \subset \ldots M^\ell = M$ be a filtration of $M$ in the category of $\mathcal{D}$-modules. Then $(M^s/M^{s-1})_i = (M_i \cap M^s)/(M_i \cap M^{s-1})$ is a $k$-filtration on the $\mathcal{D}$-module $M^s/M^{s-1}$. Hence there is an integer $j_s$ such that $\dim_k (M^s/M^{s-1})_i \geq \left( \frac{n+i-j_s}{i-j_s} \right)$ for all $i \geq j_s$.

But $M_i = \bigoplus_{s=1}^\ell (M^s/M^{s-1})_i$, because these are vector spaces over a field $k$. Therefore $\dim_k M_i = \sum_{s=1}^\ell \dim_k (M^s/M^{s-1})_i \geq \sum_{s=1}^\ell \left( \frac{n+i-j_s}{i-j_s} \right)$ for all sufficiently big $i$. This implies $C i^n \geq \sum_{s=1}^\ell \left( \frac{n+i-j_s}{i-j_s} \right)$ for all sufficiently big $i$.

But $\left( \frac{n+i-j_s}{i-j_s} \right)$, for fixed $n$ and $j_s$, is a polynomial in $i$ of degree $n$ and top coefficient $\frac{1}{n!}$. Hence $p(i) = \sum_{s=1}^\ell \left( \frac{n+i-j_s}{i-j_s} \right)$ is a polynomial in $i$ of degree $n$ and top coefficient $\frac{\ell}{n!}$. The inequality $C i^n \geq p(i)$ holds for all sufficiently big $i$ only if $C \geq \frac{\ell}{n!}$, i.e. $\ell \leq n! C$.

If $M$ is a $\mathcal{D}$-module and $f \in R$ is a non-zero element, the module $M_f$ acquires a structure of $\mathcal{D}$-module as follows. The formula 2.1 implies

$$f \cdot D_{t,i} = D_{t,i} \cdot f - \sum_{s=1}^n D_{s,i}(f) \cdot D_{t-s,i}.$$
Replacing \( f \) by \( f^j \) in this equality and then applying it to \( \frac{m}{f^j} \in M_f \) and multiplying on the left by \( f^{-j} \) we get

\[
D_{t,i}(\frac{m}{f^j}) = f^{-j} \cdot D_{t,i}(m) - \sum_{s=0}^{n} f^{-j} \cdot D_{s,i}(f^j) \cdot D_{t-s,i}(\frac{m}{f^j})
\]

This leads to a definition of the action of \( D_{t,i} \) on \( M_f \) by induction on \( t \) the case \( t = 0 \) being trivial (since \( D_{0,i} \) is the identity map).

Modules of type \( M_f \) are not considered in [1].

**Corollary 3.6.** If \( M \) is a holonomic module and \( f \in R \), then \( M_f \) is holonomic.

**Proof.** Let \( M_0 \subset M_1 \subset \ldots \) be a \( k \)-filtration of \( M \) with \( \dim_k M_i \leq Ci^n \).

Let \( d \) be the degree of \( f \) and let \( M'_0 \subset M'_1 \subset \ldots \) be the filtration on \( M_f \) defined by \( M'_i = \left\{ \frac{m}{f^j} | m \in M_i(d+1) \right\} \). We claim this is a \( k \)-filtration, i.e. \( \cup_i M'_i = M_f \) and \( \mathcal{F}_i M'_j \subset M'_{i+j} \) for all \( i \) and \( j \).

Indeed, let \( \frac{m}{f^w} \in M_f \) be any element. Assume \( m \in M_u \). If \( u \leq w(d+1) \), then \( m \in M_{w(d+1)} \) hence \( \frac{m}{f^w} \in M'_w \). If \( u > w(d+1) \) let \( v = u - w(d+1) \). Since \( f^v \in \mathcal{F}_v \), it follows that \( f^v \cdot m \in M_{v+u} \). Since \( vd+u = (v+w)(d+1) \), we conclude that \( \frac{m}{f^w} = \frac{f^v \cdot m}{f^w} \in M'_{v+w} \). This shows that \( \cup_i M'_i = M_f \).

To prove that \( \mathcal{F}_i M'_j \subset M'_{i+j} \) all we need to show is that \( x_u M'_j \subset M'_{j+1} \) and \( D_{t,u} M'_j \subset M'_{t+j} \) for every \( u \in \{1, 2, \ldots, n\} \). Let \( \frac{m}{f^j} \in M'_j \) where \( m \in M_{j(d+1)} \).

Since \( x_u \cdot f \in \mathcal{F}_{d+1} \), it follows that \( (x_u \cdot f)m \in M'_{(j+1)(d+1)} \). Therefore \( x_u \cdot \frac{m}{f^j} = \frac{(x_u \cdot f)m}{f^{j+1}} \in M'_{j+1} \). This shows that \( x_u M'_j \subset M'_{j+1} \).

To prove that \( D_{t,u} M'_j \subset M'_{t+j} \) we use induction on \( t \) the case \( t = 0 \) being trivial since \( D_{0,u} \) is the identity map. It is enough to show that all the terms on the right side of (1), i.e. \( f^{-j} \cdot D_{t,u}(m) \) and \( f^{-j} \cdot D_{s,u}(f^j) \cdot D_{t-s,u}(\frac{m}{f^j}) \), where \( s \geq 1 \), belong to \( M'_{t+j} \).

Since \( f \in \mathcal{F}_d \), \( D_{t,u} \in \mathcal{F}_t \) and \( m \in M_{j(d+1)} \), it follows that \( f^t \cdot D_{t,u} \in \mathcal{F}_{td+t} \) and \( f^t \cdot D_{t,u}(m) \in M_{j(d+1)+td+(t+j)(d+1)} \). Thus \( f^{-j} \cdot D_{t,u}(m) = \frac{f^{t} \cdot D_{t,u}(m)}{f^{j+t+1}} \) belongs to \( M'_{t+j} \) because the top of this fraction belongs to \( M_{(t+j)(d+1)} \).

If \( s \geq 1 \), then \( D_{t-s,u}(\frac{m}{f^j}) \in M'_{s+j} \) by the induction hypothesis, i.e. there exists \( m_{t-s} \in M_{(t-s+j)(d+1)} \) such that \( D_{t-s,u}(\frac{m}{f^j}) = \frac{m_{t-s}}{f^{j+s}} \). It follows by induction on \( j \) using formula 2.3 that \( D_{s,i}(f^j) \) is divisible by \( f^{j-s} \), i.e. \( D_{s,i}(f^j) = f^{j-s} \cdot f_s \). Hence \( f^{-j} \cdot D_{s,u}(f^j) \cdot D_{t-s,u}(\frac{m}{f^j}) = \frac{f_s \cdot m_{t-s}}{f^{j+s}} \). Since the polynomial \( D_{s,i}(f^j) \) has degree \( dj - s \), the polynomial \( f_s \) has degree \( ds - s \). Hence \( f_s \cdot m_{t-s} \in M_{ds-s+(t-s+j)(d+1)} \subset M_{(t+j)(d+1)} \). The latter containment is because \( ds - s + (t - s + j)(d + 1) \leq (t + j)(d + 1) \). This shows that \( f^{-j} \cdot D_{s,u}(f^j) \cdot D_{t-s,u}(\frac{m}{f^j}) \in M'_{t+j} \) and completes the proof that \( D_{t,u} M'_j \subset M'_{t+j} \), which in turn completes the proof of our claim that \( M'_0 \subset M'_1 \subset \ldots \) is a \( k \)-filtration.
Clearly, \( \dim_k M'_i \leq \dim_k M_{i(d+1)} \leq C(i(d + 1))^n \). This implies that \( \dim_k M'_i \leq C'i^n \) where \( C' = C(d + 1)^n \).

The filtration by degree on the \( \mathcal{D} \)-module \( M = R \) (i.e. \( M_i \) consists of all the polynomials of degree at most \( i \)) shows that \( R \) is a holonomic \( \mathcal{D} \)-module. Now Theorem 1.1 follows from 3.6.

4. SOME OPEN PROBLEMS

1. Let \( 0 \to M' \to M \to M'' \to 0 \) be an exact sequence in the category of \( \mathcal{D} \)-modules. It is not hard to see that if \( M \) is holonomic, then so are \( M' \) and \( M'' \). In characteristic 0 the converse is also true, i.e. if \( M' \) and \( M'' \) are holonomic, then so is \( M \). Is this true in characteristic \( p > 0 \) as well?

2. Let \( M \) be a holonomic \( \mathcal{D} \)-module. Since \( M \) has finite length, it is finitely generated as a \( \mathcal{D} \)-module. This implies that there is a \( k \)-filtration \( M_0 \subset M_1 \subset \ldots \) on \( M \) such that \( M_0 \) is finite-dimensional over \( k \) and \( M_i = F_i M_0 \) (just take \( M_0 \) to be the \( k \)-span of a finite set of \( \mathcal{D} \)-generators of \( M \)). It is not hard to show that \( \limsup_{n \to \infty} \frac{\dim_k M_i}{i^n} \) is independent of \( M_0 \). It is well-known that \( \lim_{n \to \infty} \frac{\dim_k M_i}{i^n} \) exists in characteristic 0 and, moreover, \( n!(\lim_{n \to \infty} \frac{\dim_k M_i}{i^n}) \) is an integer in this case (called the multiplicity of \( M \)). Is \( n!(\limsup_{n \to \infty} \frac{\dim_k M_i}{i^n}) \) an integer in characteristic \( p > 0 \)? Does \( \lim_{n \to \infty} \frac{\dim_k M_i}{i^n} \) exist in characteristic \( p > 0 \)?

Since these problems are open only in characteristic \( p > 0 \), it is worth pointing out that Bavula [1] has given some striking examples of properties that hold in characteristic 0 but fail in characteristic \( p > 0 \). We briefly mention some of them.

Let a \( \mathcal{D} \)-module \( M \) be generated by a finite set \( z_1, \ldots, z_s \in M \). Let \( M_0 \) be the \( k \)-linear span of \( z_1, \ldots, z_s \) and let \( M_i = F_i M_0 \). Bavula defines the dimension of \( M \) as \( \inf\{r \in \mathbb{R} | \dim_k M_i < i^r \} \) for all sufficiently big \( i \). It is not hard to show that this definition is independent of a particular choice of a finite set of generators. In characteristic zero it coincides with the usual definition of the dimension of a finitely generated \( \mathcal{D} \)-module.

Bavula shows [1, 9.4] that \( \dim M \geq n \) for every finitely generated \( \mathcal{D} \)-module \( M \), an analog of the celebrated characteristic zero Bernstein inequality. This inequality is straightforward from 3.4.

Yet Bavula also shows that there are major differences between characteristic zero and characteristic \( p > 0 \) cases. These are

(a) in characteristic zero the set of possible values of \( \dim M \) is all integers between \( n \) and \( 2n \) while in characteristic \( p > 0 \) it is the set of all real numbers between \( n \) and \( 2n \), and

(b) in characteristic zero a finitely generated \( \mathcal{D} \)-module \( M \) is holonomic if and only if its dimension is \( n \) while in characteristic \( p > 0 \) there exist \( M \) such that \( \dim M = n \) yet \( M \) is not holonomic.
3. Perhaps the most interesting open problem is to find a characteristic-free proof of the fact that $\mathcal{R}_f$ has finite length in the category of $k$-linear $\mathcal{D}$-modules of the ring $\mathcal{R}$ of formal power series in a finite number of variables over $k$. Our proof of Theorem I.1 suggests that a suitable characteristic-free definition of holonomicity could lead to such a proof.

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