Evolution of correlated multiplexity through stability maximization

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Investigating relation between various structural patterns found in real-world networks and stability of underlying systems is crucial to understand importance and evolutionary origin of such patterns. We evolve multiplex networks, comprising of anti-symmetric couplings in one layer, depicting predator-prey relation, and symmetric couplings in the other, depicting mutualistic (or competitive) relation, based on stability maximization through the largest eigenvalue. We find that the correlated multiplexity emerges as evolution progresses. The evolved values of the correlated multiplexity exhibit a dependence on the inter-link coupling strength. Furthermore, the inter-layer coupling strength governs the evolution of disassortativity property in the individual layers. We provide analytical understanding to these findings by considering star like networks in both the layers. The model and tools used here are useful for understanding the principles governing the stability as well as importance of such patterns in the underlying networks of real-world systems.

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Introduction: There are many complex systems which comprise of interacting units having different types of coupling behavior rather than a fixed type. Incorporation of multiple types of coupling behaviour gives rise to multiplex networks [1]. Over the past decades, studies of structural and dynamical properties of networks with fixed type of coupling behavior have enhanced our understanding about various collective behaviours of the underlying system unraveling its complexity [2]. However, in order to mimic real world systems in a better manner, the multiplex network becomes a more preferred framework, as it provides understanding to various dynamical or structural features of underlying real world systems which are beyond the limit of single network framework incorporating only one type of the coupling behaviour [3, 4]. Further, the multiplex network framework has been proposed to represent ecological systems in a better manner [4], where activator and inhibitor species exhibit distinct patterns through intra-layer diffusion and inter-layer interactions [4].

Furthermore, ecological systems having large size are known to be less stable which is quantified by the largest real part of eigenvalues of the corresponding Jacobian matrices [7]. Therefore, it becomes crucial to have characterization of various existing features which bring upon stability to a system regardless of its size and the complexity. The largest real part of eigenvalues ($R_{\text{max}}$) has been used in measuring fitness of evolution of various structural properties such as degree distribution and clustering coefficients [5]. However, these studies were restricted to single networks. In this Letter, we evolve networks having multiple types of coupling behaviours using genetic algorithm so that the evolved network is more stable in terms of $R_{\text{max}}$. Inhibitory and excitatory features define the pairwise coupling, which are termed as mutualistic, predator-prey or competitive. These features are known to affect the dynamical evolution as well as stationary states of the ecological and biological systems [7, 8, 10]. We show that the correlated multiplexity is an emerged feature when a system having different types of coupling behaviour is evolved through stability maximization. This implicates that such patterns in real-world systems are required for maintaining their stability. Interestingly, inter-layer coupling strength governs the value of correlated multiplexity, further highlighting importance of multilayer framework for understanding systems having inhibitory and excitatory coupling behaviour.

Theoretical framework: In order to generate the multiplex networks having different types of interaction behaviour, and to investigate evolution of such multiplex networks with the stability maximization, we first construct the behaviour matrices defining interaction behaviour of the pairs of nodes in a given system. A pair of nodes $(i,j)$ in the behaviour matrix are said to have (I) predator-prey relation if $b_{ij} = +1$ (or -1) and $b_{ji} = -1$ (or +1), (II) mutulistic relation if both $b_{ij}$ and $b_{ji}$ are $+1$, (III) competitive relation if both $b_{ij}$ and $b_{ji}$ are $-1$. The behaviour matrix is defined initially and is fixed throughout the generations of the evolution.

Next, we generate $P$ number of Erdös-Rényi (ER) random networks of size $(N)$, which is termed as the initial population of networks for further evolution with the genetic algorithm (GA). Elements in the corresponding adjacency matrices $[a_{jk}]$ of the ER random networks take value 1 and 0 depending upon whether there exists a connection between $j^{th}$ and $k^{th}$ nodes or not. For each of
the network (say $i$) in the initial population, a multiplex network ($M^i$) is constructed using the behaviour matrix as follows (Fig. 1). We pick up all the pairs of interacting nodes from the Erdős–Rényi (ER) random network and look for their corresponding interactions in the behaviour matrix. If the interaction is of predator-prey type, then we connect the same pair of nodes in the predator-prey layer of the multiplex network. If the interaction is of mutualistic (competitive) type, we connect the same pair of nodes in the mutualistic (competitive) layer of the multiplex network. This leads to a multiplex network having two layers, in which one layer is having anti-symmetric couplings and another layer is having the symmetric couplings. The adjacency matrix of the first layer is denoted by $[p_{ij}]$ and that of the second layer is denoted by $[q_{ij}]$ (Fig. 1).

Further, we incorporate fluctuations in the coupling strength $[11]$ by introducing a uniform random number $X$ which ranges between 0 and 1. The weighted entries ($c_{ij}$ and $d_{ij}$) of the resulting matrices $C$ and $D$ are given as

\[ c_{ij} = \begin{cases} b_{ij}X & \text{if } p_{ij} = 1 \\ 0 & \text{if } p_{ij} = 0. \end{cases} \]

\[ d_{ij} = \begin{cases} b_{ij}X & \text{if } q_{ij} = 1 \\ 0 & \text{if } q_{ij} = 0. \end{cases} \]

Now, the multiplex network ($M^i$) can be defined as

\[ M^i = \begin{bmatrix} C^i & I \\ I & E_y D^i \end{bmatrix} \]  

where $i$ denotes the index of the network population, $C^i$ represents the nodes in the first (predator-prey) layer of the $i^{th}$ network population and $D^i$ stands for the nodes in the second (mutualistic or competitive) layer. $I$ is a $N \times N$ identity matrix and $E_y$ stands for the relative coupling strength of the mutualistic (or competitive) layer with respect to the predator-prey layer.

Fitness of the multiplex network ($M^i$) is defined by the largest real part of the eigenvalues ($R_{\text{max}}$). We evaluate eigenvalues of all the multiplex networks in the $P$ initial population. A network having a lower $R_{\text{max}}$ value is considered to be more fitter than the networks having higher $R_{\text{max}}$ values $[7]$. We identify $P/2$ number of multiplex networks from the initial $P$ population which are more fitter than the rest of $P/2$ population and select the ER random networks corresponding to these $P/2$ fitter multiplex networks. These networks form the first half of the population (parent networks) for the next generation of GA. The second half of the population comprises of the child networks which have been generated using the fitter networks as parents. This second half of the population is generated as follows. After randomly selecting a pair of the fitter ER random networks (first half of the population) as a pair of the parent networks for creating one child network, we divide the adjacency matrices of the selected parents into blocks of dimension $N_B \times N_B$ such that these blocks cover the entire matrix without any overlap. The adjacency matrix of the child network is then generated by creating blocks of the same dimension in the child matrix. A block in the child adjacency matrix is then filled by selecting a block at the same position of the parent matrices with equal probability. This matrix is denoted as the child matrix. The next child matrix is generated by selecting a different pair of the fitter parent networks. In the similar fashion, we create $P/2$ number of child networks, which form the second half of the population for the next generation of GA. To the end, we have $P$ number of networks in the next generation, which, using the fixed behaviour matrix, creates the $P$ multiplex networks as described above (Fig. 1). Various structural properties and $R_{\text{max}}$ of the multiplex networks are recorded for each time step during the evolution.

We quantify the degree-degree correlations of a network by considering the Pearson (degree-degree) correlation coefficient given as $[12],$

\[ r = \frac{[M^{-1} \sum_{i=1}^{M} j_i k_i] - [M^{-1} \sum_{i=1}^{M} \frac{1}{2} (j_i + k_i)]^2}{[M^{-1} \sum_{i=1}^{M} \frac{1}{2} (j_i^2 + k_i^2)] - [M^{-1} \sum_{i=1}^{M} \frac{1}{2} (j_i + k_i)]^2} \]  

(2)

where $j_i$, $k_i$ are the degrees of nodes at both the ends of the $i^{th}$ connection and $M$ represents the total connections in the network.

We calculate the correlation between the degrees of the mirror nodes in the pair of the layers as the evolution progresses, which is termed as correlated multiplexity ($L_{\text{corr}}$) and is given as $[12],$

\[ L_{\text{corr}} = \frac{\sum_{i} (k_i^1 - \langle k^1 \rangle) (k_i^2 - \langle k^2 \rangle)}{\sum_{i} (k_i^1 - \langle k^1 \rangle)^2 \sum_{i} (k_i^2 - \langle k^2 \rangle)^2}^{1/2}, \]  

(3)

where $k_i^1$ and $k_i^2$ are the degrees of $i^{th}$ node in the first and
the second layers, respectively. The terms $\langle k^1 \rangle$ and $\langle k^2 \rangle$ denote the average degrees of the first and the second layers, respectively. $L_{\text{corr}}$ is the average over the $L_{\text{corr}}$ values of the population used in the GA.

Results: We evolve the multiplex networks by minimization of $R_{\text{max}}$ through GA as described in Fig. 1. The $R_{\text{max}}$ value of the multiplex networks is either contributed by both the predator-prey as well as the mutualistic layer or by one the layers depending upon the relative coupling strength parameter $E_y$ and the ratio of connections belonging to each layer. For the equal number of connections in both the layers and $E_y = 1$, $R_{\text{max}}$ is contributed by mutualistic (or competitive) layer due to presence of the anti-symmetricity in the predator-prey coupling behaviour [14]. With an increase in the number of generations of GA, the connections tend to shift to the layer yielding lower values of $R_{\text{max}}$. As a result large number of the connections shift to the predator-prey layer having anti-symmetric couplings making this configuration having minimum $R_{\text{max}}$ values. To avoid this trivial structure, we mask the contribution of mutualistic (or competitive) layer towards $R_{\text{max}}$ by introducing a coupling strength parameter $E_y$ (Eq. 1). A lower value of $E_y$ will effectively reduce the contribution of the mutualistic (or competitive) layer towards $R_{\text{max}}$. Since $R_{\text{max}}$ for the ER random network scales with its average connectivity [15], a decrease in the $R_{\text{max}}$ values of the mutualistic (or competitive) layer arises due to decrease in the total number of connections in that layer. This results in an increase in the number of connections in the predator-prey layer, since the average degree remains fixed in the multiplex networks throughout the evolution. After selection of $E_y$ parameter resulting to non-trivial solutions, the process of optimization towards minimization of $R_{\text{max}}$ of the multiplex networks takes place through shuffling of connections between both the layers. There arises a point where the values of average degree for the predator-prey ($d_p$) and mutualistic (or competitive) ($d_q$) layers saturate and no significant changes are observed with an increase in the number of generations (Figs. 2 (a), (b) and (c)).

After adjustment in $E_y$ values such that both the layers contribute significantly, we investigate the structural properties of the individual layers of the evolved multiplex networks. Fig. 2 (f) depicts that the nodes in the predator-prey layer exhibit negative degree-degree correlations calculated by Eq. 2. This indicates the presence or abundance of star-like structures in that layer. Owing to the enhanced stability of networks having the star-like structure [16], it is not surprising that the $R_{\text{max}}$ of the evolved networks are low as well GA, which minimizes $R_{\text{max}}$ during the evolution, leads to the dis-assortative networks. Note that the mutualistic layer shows a high degree of disassortativity which was not observed in the case of evolution of isolated networks having mutualistic interaction (Fig. 2 (c)). What stands interesting is that with an increase in the generations of GA, the Pearson degree-degree correlation coefficient ($L_{\text{corr}}$), evaluated for the degrees of the mirror nodes of the duplex networks (Eq. 3), increases abruptly after certain initial generations and attains maximum value (Fig. 2 (d)). The emerged values of $L_{\text{corr}}$ turn out be a measure of correlated multiplexity and indicate importance of such features in ecological systems for their stability [17].

Furthermore, the structural parameters in the evolved multiplex networks are affected by inter-layer coupling strength ($D_x$). With an increase in the $D_x$, we observe that there is a linear increase in the emerged value of the correlated multiplexity (Fig. 3 (a)). Moreover, for lower values of $D_x$, the mutualistic layer of the evolved multiplex networks does not manifest any significant change in the disassortativity as compared to that of the initial networks population. Whereas, for the larger values of $D_x$, the mutualistic layer of the evolved networks exhibit a higher disassortativity (Fig. 3 (b)). However, the values of disassortativity are not significantly affected by $D_x$ in the predator-prey layer of the evolved multiplex networks (Fig. 3 (c)).

In order to understand the emergence of correlated multiplexity in mirror nodes and how correlated multiplexity leads to low $R_{\text{max}}$, we consider a multiplex network with individual layer being represented by a star network constructed as follows. The first interaction layer comprises of all the anti-symmetric couplings representing predator-prey relation while all the couplings in the second layer are of mutualistic type. For the calculation of $R_{\text{max}}$ we consider two cases. In the first case, we create an inter-link between the hubs of both the layers and then create inter-links between all other lower degree nodes which leads to a structure having a positive degree-degree correlation among the mirror nodes. To all the inter-links, we assign inter-connection weight as $D_x$. 

![Fig. 2: Average value of (a) $d_p$, (b) $d_q$, (c) $R_{\text{max}}$, (d) $L_{\text{corr}}$, (e) $r_{\text{in}}$ and (f) $r_{\text{out}}$ as evolution progresses. The average of is taken for 1000 networks in the population. Size of the ER networks and consequently of the individual layer in the corresponding multiplex networks remain $N = 100$. Average degree of the ER random networks here is 16.](attachment:image.jpg)
The weighted adjacency matrix is then given by,

\[ X = \begin{bmatrix} S_p & D_x L \\ D_x L' & E_y S_m \end{bmatrix} \]

where \( S_p, S_m \) and \( L \) stands for the star-like networks with the anti-symmetric couplings representing predator-prey behaviour, star-like networks with the symmetric couplings representing mutualistic behavior and \( N \times N \) identity, respectively. The matrix \( F = X^2 - D^2 I \) is of rank four, therefore, it has only four non-zero complex eigenvalues. The non-zero eigenvalues of \( F \) are calculated by the following reduced characteristic polynomial.

\[
\begin{bmatrix} \lambda - \frac{(N-1)(E_y^2 - 1) + Z^{1/2}}{2} \\ \lambda - \frac{(N-1)(E_y^2 - 1) - Z^{1/2}}{2} \end{bmatrix}^2 = 0 \quad (4)
\]

where \( Z = \{(N-1)(E_y^2 - 1))^2 - 4(N-1)(D_x^2(1 - E_y^2) - (N-1)E_y^2)\}

And the eigenvalues of \( X \) is given by,

\[ e = f(\lambda) = (\lambda + x^2)^{1/2} \]

where \( f(\lambda) \) is any function defined for the eigenvalues of \( F \) such that \( P^{-1}FP = V \), where columns of \( P \) are eigenvectors of matrix \( F \) and \( P^{-1}f(F)P = f(V) \).

In the second case, the hub of the first layer is connected to a lower degree node in the second layer, and the hub of the second layer is connected through an interconnection to a node having lower degree in the first layer. This leads to the negative degree-degree correlation in the mirror nodes. The weighted adjacency matrix is then given by

\[ Y = \begin{bmatrix} S_p & D_x L' \\ D_x L' & E_y S_m \end{bmatrix} \]

where \( L' \) stands for the connection matrix from the hub to the lower degree node and the vice versa. Again, the rank of this matrix \( Y \) is four and therefore its characteristic polynomial is reducible to,

\[(\mu^2 - \alpha)(\mu^2 - \beta) = 0 \quad (5)\]

where \( \alpha = \{(2D_x^2 + (N-1)E_y^2 - N + 1) + (N^2E_y^4 + 2N^2E_y^2 + N^2 - 2NE_y^4 - 2NE_y^4 - 4NE_y^2 - 2N^2D_x^2E_y^2 - 4D_x^2 + E_y^4 + 2E_y^2 + 1)^{1/2}\}/2 \) and \( \beta = \{(2D_x^2 + (N-1)E_y^2 - N + 1) - (N^2E_y^4 + 2N^2E_y^2 + N^2 - 2NE_y^4 - 4NE_y^2 - 2N^2 + 4D_x^2E_y^2 - 4D_x^2 + E_y^4 + 2E_y^2 + 1)^{1/2}\}/2 \)

We investigate the behaviour of \( R_{\text{max}} \) as a function of inter-link coupling strength \( D_x \), calculated using Eq. 4 and Eq. 5 and find that with an increase in \( D_x \), \( R_{\text{max}} \) increases for the Y type of matrices (Fig. 4 (a)). On the contrary, \( D_x \) does not display any significant impact on \( R_{\text{max}} \) of the X type matrices. Though for simplicity of the calculation, in the above results, we consider only two inter-links for Y type of matrices, whereas matrices of X type have \( N \) inter-links. However, to establish our results, we numerically calculate the values of \( R_{\text{max}} \) with an increase in the number of inter-links \( m \) connecting lower degree nodes of both the layers of the E type matrices.

Fig. 4 (b) demonstrates that an increase in the number of inter-links in the E type matrices does not have a significant impact on \( R_{\text{max}} \) values, as increase in the number of inter-links between lower degree nodes in E type matrices does not provide larger radius to Gerschgorin circles and thus is not sufficient to increase the upper bound for the largest eigenvalue of the underlying matrices. This renders the \( R_{\text{max}} \) values of E type matrices with two inter-links to be comparable with those of the D type matrices. On summary, there is a large regime of \( D_x \) where networks with inter-links connecting similar degree nodes of both the layers have more stable configuration as compared to the networks having inter-links connecting dissimilar degree nodes, thus leading to an emergence of the correlated multiplexity in course of evolution through stability maximization.

The above results on the correlated multiplexity are
based on the networks in which one of the layers is of predator-prey type and other is of the mutualistic type. However, the results presented so far remain largely unaffected for networks having layer of predator-prey type and the other of the competitive type due to the square of the coupling strength term ($E^2_{ij}$) in Eq. 4 and Eq. 5. This renders the eigenvalues of the multiplex networks with predator-prey and mutualistic layers to be same as the multiplex networks with predator-prey and competitive layers. This further indicates the emergence of correlated multiplexity in both the cases where predator-prey layer is multiplexed with the mutualistic layer or the competitive layer. Furthermore, there can be a more realistic situation with all the three behaviours, namely predator-prey, mutualistic and competitive which are co-existing in a system. This will lead to a multiplex network having three layers corresponding to the predator-prey, mutualistic and competitive. For this case also, correlated multiplexity between the competitive and mutualistic layers will emerge due to the pair-wise emergence of correlated multiplexity between predator-prey layer with the mutualistic and the competitive layers.

**Conclusion:** To sum up, using multiplex networks model having inhibitory and excitatory couplings, we demonstrate that there is an emergence of the correlated multiplexity as stability of the networks is maximized in terms of $R_{\text{max}}$. Further, the individual layer of the multiplex network exhibits the disassortative interaction pattern and disassortativity in the predator-prey layer brings upon a less $R_{\text{max}}$ values to the evolved networks [16]. Moreover, we find that the inter-layer coupling strength controls the values of emergent disassortativity in the mutualistic layer, as well as values of the correlated multiplexity across the layers. This is interesting as isolated networks with only mutualistic couplings do not show any significant evolution by the stability maximization [18]. It further indicates that the multiplex framework is helpful in explaining the existence of various structural properties in the individual layers (such as disassortativity) as well those of the multiplex network (such as correlated multiplexity). Furthermore, expressions for the eigenvalues of a multiplex network having star-like configuration in each layer, which are inter-linked in different ways, explain the reason behind the evolution by GA.

Earlier works had proposed models which incorporate the coupling fluctuations occurring due to the environmental perturbations [11]. However, these studies were limited to the single layer networks. The multiplex networks model proposed in our work, i.e. diffusion among the species through inter-links having anti-symmetric couplings and intra-links comprising of symmetric couplings, may refine our understanding about real world systems to explain their dynamics and functionality [14, 19].

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