Pairings of automorphic distributions

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Abstract

We present a pairing of automorphic distributions that applies in situations where a Lie group acts with an open orbit on a product of generalized flag varieties. The pairing gives meaning to an integral of products of automorphic distributions on these varieties. This generalizes classical integral representations or “Rankin-Selberg integrals” of \( L \)-functions, and gives new constructions and analytic continuations of automorphic \( L \)-functions.

Keywords: Automorphic forms, invariant pairings, automorphic distributions, \( L \)-functions, analytic continuation, rapid decay.

1 Introduction

Among its many and deep predictions, the Langlands conjectures attach families of \( L \)-functions to each automorphic representation. The Langlands \( L \)-functions are expected to have holomorphic continuations – or sometimes meromorphic continuations with poles only at certain specific points – and to satisfy functional equations. This has been established in relatively few cases so far, generally by one of two methods. One, the Langlands-Shahidi method, deduces the meromorphic continuation and functional equation from the analogous properties of Eisenstein series – Eisenstein series induced from the automorphic representation in question. The other is called the method of integral representations, although it is perhaps more a collection of very

\[\text{Keywords: Automorphic forms, invariant pairings, automorphic distributions, } L \text{-functions, analytic continuation, rapid decay.}\]
clever ideas than a systematic method. These integral representations express
the $L$-function as an integral of matrix coefficients (i.e., automorphic forms),
or products of matrix coefficients, of the automorphic representation; the
functional equation then follows by applying some appropriate involution,
which has the desired effect on the argument of the $L$-function.

The Langlands-Shahidi method presents the functional equation in ex-
actly the form conjectured by Langlands, with the Gamma factors predicted
by him. Its range of applicability is limited, because the group which acts on
the automorphic representation must arise as Levi component of a maximal
parabolic in some ambient group. The method of integral representations
typically does not produce the Gamma factors directly. As a result, this
method cannot always rule out all unwanted poles of the $L$-function. The
same is true, for different reasons, of the Langlands-Shahidi method.

In the papers [4, 5] we have introduced a new approach: we work not
with matrix coefficients of an automorphic representation, as in the method
of integral representations, but with its automorphic distribution. The latter,
in effect, is the datum of the embedding of the automorphic representation
into $L^2(\Gamma\backslash G)$; it does not involve the choice of particular matrix coefficients.
Our approach has the advantage of making the Gamma factors computable
and allows us to rule out all the unwanted poles. There is a cost, of course –
distributions are more difficult to work with than $C^\infty$ functions.

In this paper we develop an important analytic tool for our program,
the analogue, in the setting of automorphic distributions, of Rankin-Selberg
type integrals. Those are integral representations of $L$-functions, as integrals
of products of automorphic forms – i.e., multilinear pairings of automorphic
forms. The analytic properties of the $L$-functions are then deduced from the
integrals in question. Our main result, theorem 2.30, provides an alternative
by establishing an invariant multilinear pairing of automorphic distributions.
In our forthcoming paper [7] we shall use a particular instance of our pairings
to obtain new analytic information, such as the full analytic continuation, of
the exterior square $L$-function for $GL(2n)$. We intend to use theorem 2.30
in the future to obtain new results on other $L$-functions, results that are
inaccessible by the Rankin-Selberg and Langlands-Shahidi methods. Indeed,
for every integral representation we are aware of, there exists an analogous
distributional pairing of the type covered by theorem 2.30.
2 Statement of the main theorem

Our main result involves two reductive Lie groups, $G$ and $H$. We suppose that both are finite covers – typically but not always the trivial cover – of the groups of real points of algebraic reductive subgroups $G_\mathbb{Q} \subset GL(N_G, \mathbb{Q})$ and $H_\mathbb{Q} \subset GL(N_H, \mathbb{Q})$, respectively. By an arithmetic subgroup $\Gamma \subset G$, we mean a subgroup commensurate with the inverse image, in $G$, of the group of integral points $G_\mathbb{Z} \subset GL(N_G, \mathbb{Z})$. We use the same terminology in the case of $H$, of course. We shall suppose that $G$ is realized as a subgroup of $H$, via an inclusion

$$G \hookrightarrow H$$

that is defined over $\mathbb{Q}$; in other words, the inclusion is compatible with a $\mathbb{Q}$-homomorphism $G_\mathbb{Q} \to H_\mathbb{Q}$. It then follows that any arithmetic subgroup of $\Gamma_H \subset H$ intersects $G$ in an arithmetic subgroup $\Gamma_G \subset G$. To simplify various statements, we suppose that

$$Z_H, \text{ the center of } H, \text{ is compact.}$$

In the context of automorphic forms on $H$, that is not a restrictive hypothesis: any automorphic form on which the center of $H$ acts according to a character is completely determined by its restriction to the derived group $[H,H]$.

We consider a generalized real flag variety $Y$ for $H$ – a compact real algebraic variety with a transitive action of $H$, such that the isotropy subgroups are parabolic. If we let $P$ denote the isotropy subgroup at some base point $y_0 \in Y$, we can make the identification

$$Y \cong H/P, \quad y_0 \cong eP.$$

We do not require that the parabolic subgroup $P \subset H$ be defined over $\mathbb{Q}$. Via the embedding (2.1) $G$ acts on $Y$. In addition, we consider the datum of a connected unipotent subgroup $U \subset H$, which is normalized by $G$ and defined over $\mathbb{Q}$, such that

a) $U$ is the unipotent radical of a $\mathbb{Q}$-parabolic subgroup $P_U \subset H$,

b) $G \subset M_U$ for some Langlands decomposition $P_U = M_U \cdot U \cdot A_U \cdot U$,

c) $U \cdot G$ has an open orbit $\mathcal{O}$ in $Y$, and

d) $G$ preserves the bi-invariant measure on $U$. 

3
Since $G$ is reductive and normalizes $U$ it can be extended to a Levi component of $P_U$. Thus b) is implied by the simpler, but more restrictive condition

\[ b' \] $Z_G$, the center of $G$, is compact.

We note that the semidirect product $U \cdot G$ is defined over $\mathbb{Q}$, since both factors are.

At this point two examples may be helpful. In the first, $G$ is the group $SL^{\pm 1}(n, \mathbb{R})$ equipped with the standard $\mathbb{Q}$-structure, $H = G \times G \times G$, which contains $G$ diagonally, and $U = \{ e \}$. Thus a) and d) are vacuously satisfied, and the semisimplicity of $G$ implies b). Let $X_n$ denote the flag variety of $G = SL^{\pm 1}(n, \mathbb{R})$,

\[ X_n = \{ 0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset \cdots \subset F_n = \mathbb{R}^n \mid \dim F_k = k \} . \tag{2.5} \]

Then $G$ acts on $X_n$ with isotropy subgroups conjugate to the lower triangular subgroup. Note that $\mathbb{RP}^{n-1}$ can be regarded as a generalized flag variety for $H$. It is not difficult to see that $G$, via the diagonal embedding $G \hookrightarrow G \times G$, has a unique open orbit in $X_n \times X_n$, and that the isotropy subgroup of $G$ at any point of the open orbit is $G$-conjugate to the diagonal Cartan subgroup. The diagonal Cartan has a unique open orbit in $\mathbb{RP}^{n-1}$. It follows that $U \cdot G = G$ has a unique open orbit in $Y = X_n \times X_n \times \mathbb{RP}^{n-1}$, so (2.4c) is also satisfied.

For the second example, $G$ is again $SL^{\pm 1}(n, \mathbb{R})$ with the standard $\mathbb{Q}$-structure, $H = SL^{\pm 1}(2n, \mathbb{R}) \times G$, also with the standard $\mathbb{Q}$-structure,

\[ G \ni g \mapsto \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} , \quad g \in H = SL^{\pm 1}(2n, \mathbb{R}) \times G \tag{2.6} \]

describes the embedding of $G$ into $H$, and $Y = X_{2n} \times \mathbb{RP}^{n-1}$, which is a generalized flag variety for $H$. The abelian subgroup

\[ U = \left\{ \begin{pmatrix} c & \ast \\ \ast & e \end{pmatrix} \mid c \in M(n \times n, \mathbb{R}) \right\} \tag{2.7} \]

of $H$ is unipotent, defined over $\mathbb{Q}$, and normalized by $G$. It can also be described as the unipotent radical of the $(n, n)$ parabolic in $SL^{\pm 1}(2n, \mathbb{R})$, hence as the unipotent radical of a $\mathbb{Q}$-parabolic subgroup of $H$. Again the semisimplicity of $G$ implies b). The conjugation action of $G = SL^{\pm 1}(n, \mathbb{R})$ on $M(n \times n, \mathbb{R})$ preserves the Euclidean measure, so d) is satisfied. The
assertion c) can be reduced to the corresponding assertion about the first example: when we embed \( X_n \times X_n \) diagonally into \( X_{2n} \), and correspondingly \( X_n \times X_n \times \mathbb{RP}^{n-1} \) into \( Y \), the \( U \)-translates of \( X_n \times X_n \times \mathbb{RP}^{n-1} \) sweep out a dense open subset of \( Y \); thus, since \( G \) has an open orbit in \( X_n \times X_n \times \mathbb{RP}^{n-1} \), \( U \cdot G = G \cdot U \) does have an open orbit \( O \subset Y \).

We need to briefly recall the notion of an automorphic distribution. Let \( \Gamma \subset H \) denote an arithmetic subgroup. Then \( H \) acts unitarily on \( L^2(\Gamma \backslash H) \), via right translation. An automorphic representation consists of an irreducible unitary representation \((\pi,V)\) of \( H \), together with a \( H \)-invariant embedding

\[
(2.8) \quad j : V \rightarrow L^2(\Gamma \backslash H).
\]

If \( v \in V \) is a \( C^\infty \) vector\(^1\), then \( j(v) \) is a \( \Gamma \)-invariant \( C^\infty \) function on \( H \). That makes evaluation of \( j(v) \) at \( e \in G \) meaningful. The linear map

\[
(2.9) \quad \tau_j : V^\infty \rightarrow \mathbb{C}, \quad \tau_j(v) = j(v)(e),
\]

defined on the space of \( C^\infty \) vectors \( V^\infty \), is continuous with respect to the intrinsic topology on \( V^\infty \), and is also \( \Gamma \)-invariant. Thus \( \tau_j \) can be regarded as a \( \Gamma \)-invariant distribution vector for the dual representation \((\pi',V')\),

\[
(2.10) \quad \tau_j \in ((V')^{-\infty})^{\Gamma_H}.
\]

We refer to \( \tau_j \) as the automorphic distribution associated to the automorphic representation \((2.8)\). It completely determines \( j \), since \( V^\infty \) is dense in \( V \) and \( j(v)(h) = \langle \pi(h)v, j(v) \rangle(e) = \langle \tau_j, \pi(h)v \rangle \) for all \( v \in V^\infty, h \in H \).

To simplify the notation, we drop the subscript \( j \), and we interchange the roles of \((\pi,V)\) and its dual \((\pi',V')\); from now on,

\[
(2.11) \quad \tau \in (V^{-\infty})^{\Gamma_H}.
\]

We shall also relax the assumption that \( \tau \) corresponds to an irreducible subspace of \( L^2(\Gamma \backslash H) \), as in \((2.8)\)\((2.9)\): in the discussion that follows, \( \tau \) will denote an arbitrary \( \Gamma_H \)-invariant distribution vector for an admissible representation \((\pi,V)\) of finite length, on a reflexive Banach space\(^2\). In particular, \( \tau \) may denote the automorphic distribution arising from an Eisenstein series.

\(^1\)i.e., a vector such that \( H \ni h \mapsto \pi(h)v \) is a \( C^\infty \) \( V \)-valued function on \( H \).

\(^2\)For an expository discussion of these matters see [6, §5 and appendix].
We return to the situation of a generalized real flag variety $Y \cong H/P$, with $P \subset H$ parabolic, but not necessarily defined over $\mathbb{Q}$. To any finite dimensional complex representation

$$\mu : P \rightarrow GL(E)$$

we associate the $H$-invariant vector bundle $\mathcal{E} \rightarrow Y$ modeled on $E$. In other words, $\mathcal{E}$ is a vector bundle to which the action of $H$ on $Y$ lifts, and whose fiber at the identity coset $y_0 \cong eP$ is isomorphic to $E$ as a $P$-module. Tracing back through the definition, one obtains the description

$$C^\infty(Y, \mathcal{E}) \cong \{ f : H \overset{C^\infty}{\longrightarrow} E \mid f(hp) = \mu(p^{-1})f(h) \text{ for } h \in H, \ p \in P \}$$

of its space of $C^\infty$ sections. This isomorphism relates the action of $H$ on the space of smooth sections $C^\infty(Y, \mathcal{E})$, by means of the structure of equivariant vector bundle, to the action, via left translation, on the space of smooth $E$-valued functions on $H$.

We follow the convention of using the term “distribution” in the sense of “generalized function”: scalar valued distributions are dual to smooth measures, and thus continuous functions can be viewed as distributions. In complete analogy to (2.13), there exists a natural, $H$-invariant isomorphism

$$C^{-\infty}(Y, \mathcal{E}) \cong \{ \sigma : H \overset{C^{-\infty}}{\longrightarrow} E \mid \sigma(hp) = \mu(p^{-1})\sigma(h) \text{ for } h \in H, \ p \in P \}$$

between the space $C^{-\infty}(Y, \mathcal{E})$ of sections of $\mathcal{E}$ with distribution coefficients and the space of $E$-valued distributions on $H$ which transform in the specified manner under right translation by elements of $H$. The identity $\sigma(hp) = \mu(p^{-1})\sigma(h)$ has only symbolic meaning, of course, since distributions cannot be evaluated at points.

When one puts a Riemannian metric on $Y$ and a hermitian metric on the vector bundle $\mathcal{E}$, it makes sense to consider the space of all $L^2$ sections of $\mathcal{E} \rightarrow Y$. That is a Hilbert space, on which $H$ acts continuously, but generally not unitarily. The resulting representation of $H$ is known to be admissible, of finite length. Its spaces of $C^\infty$ and distribution vectors are naturally isomorphic to the spaces (2.13) and (2.14), respectively. From now on we keep fixed an arithmetic subgroup $\Gamma_H \subset H$ and a particular

$$\tau \in C^{-\infty}(Y, \mathcal{E})^{\Gamma_H}.$$
Then $\tau$ is an automorphic distribution in the sense of our earlier discussion. According to the Casselman embedding theorem [2] and results of Casselman-Wallach [8, §11], the space of distribution vectors for any irreducible unitary representation can be embedded into the space of distribution vectors (2.14), with appropriate choices of $Y$ and $E$. In that sense, all automorphic distributions $\tau_j$ that encode irreducible subspaces of $L^2(\Gamma H \backslash H)$ as in (2.8–2.9) can be realized as in (2.15); for details see [4, 5].

We denote the action of $H$ on sections of $E \to Y$, and also on functions and distributions on $H$, by the letter $\ell$, for left translation. In analogy to the case of automorphic forms, the automorphic distribution $\tau$ is said to be cuspidal if

$$\int_{N/(\Gamma H \cap N)} \ell(n) \tau \, dn = 0$$

for any unipotent subgroup $\{e\} \neq N \subset H$ which arises as the unipotent radical of a $\mathbb{Q}$-parabolic subgroup $P \subset H$. We also need a weaker notion. Suppose

$$H = H_1 \times H_2 \quad \text{(in the category of groups defined over $\mathbb{Q}$)}$$

can be expressed as the product of two reductive, non-abelian factors. We shall then say that $\tau$ is cuspidal with respect to the factor $H_1$ if the integral (2.16) vanishes for any unipotent subgroup $\{e\} \neq N \subset H_1$ which arises as the unipotent radical of a $\mathbb{Q}$-parabolic subgroup $P \subset H_1$. That will be a potential hypothesis in the statement of our main theorem, but under the additional assumption that

$$\text{the projection of } G \subset H \text{ into } H_j \text{ has a finite kernel,}$$

for both $j = 1, 2$.

Recall the conditions (2.4). We choose a base point $\sigma \in \mathcal{O}$ and let $(UG)_\sigma$ denote the isotropy subgroup of $U \cdot G$ at $\sigma$. The surjective map

$$p : U \cdot G \longrightarrow \mathcal{O}, \quad p(u g) = u g \cdot \sigma,$$

In those references, we consider only line bundles, rather than vector bundles $E \to Y$. For linear groups, line bundles suffice. In this paper we also consider nonlinear groups. The Cartan subgroups of a non-linear group $H$ need not be abelian, which necessitates working with representations parabolically induced from finite dimensional representations of dimension greater than one – or in geometric terms, with vector bundles rather than line bundles.
induces a \((U \cdot G)-\)equivariant identification \(O \simeq (U \cdot G)/(U \cdot G)_o\). Distribution sections of the \(H\)-equivariant vector bundle \(E \to Y\) can be restricted to the open subset \(O \subset Y\), and then pulled back from \(O \simeq (U \cdot G)/(U \cdot G)_o\) to \(U \cdot G\) via \(p\). We choose \(s \in H\) so that

\[(2.20) \quad o = sP \in H/P \cong Y, \text{ and consequently } (UG)_o = (U \cdot G) \cap sPs^{-1}.
\]

The \((U \cdot G)\)-equivariant vector bundle \(E|_O \to O \cong (U \cdot G)/(U \cdot G)_o\) is attached to the representation

\[(2.21) \quad (UG)_o \ni v \mapsto \mu(s^{-1}vs) \in GL(E),
\]

hence, in analogy to (2.14),

\[(2.22) \quad C^{-\infty}(O, E) \simeq \{ \sigma : (U \cdot G) \xrightarrow{C^{-\infty}} E \mid r(v)\sigma = \mu(s^{-1}v^{-1}s)\sigma \text{ for } v \in (UG)_o \}.
\]

Here \(r\) denotes the right translation action on possibly vector valued functions and its natural extension to distributions.

Even if the representation (2.12) is irreducible, its restriction to \(s^{-1}(UG)_o s\) may well have a trivial quotient. We suppose that is the case\(^4\), and fix

\[(2.23) \quad q : E \longrightarrow \mathbb{C} \quad ((s^{-1}(UG)_o s)-\text{equivariant projection}).
\]

Composing, from right to left, restriction of sections of \(E\) from \(Y\) to \(O\), pullback from \(O\) to \(U \cdot G\), and the projection \(q\) from \(E\) to \(\mathbb{C}\), we obtain a map \(\tilde{p}^*\) from \(C^{-\infty}(Y, E)\) to scalar valued distributions on \(U \cdot G\),

\[(2.24) \quad \tilde{p}^* : C^{-\infty}(Y, E) \longrightarrow C^{-\infty}(U \cdot G).
\]

In terms of the isomorphisms (2.14) and (2.22), is given by the explicit formula

\[(2.25) \quad (\tilde{p}^* \sigma)(ug) = q(\sigma(ugs)).
\]

\(^4\)This hypothesis is necessary to construct the distributional pairing. Geometrically it means that the restriction of \(E\) to the open orbit \(O\) has a \(U \cdot G\)-equivariantly trivial rank one quotient bundle. When we are dealing with a line bundle \(L\) rather than a vector bundle \(E\) – cf. the previous footnote – it reduces to the assumption that the restriction of \(L\) to \(O\) is \(U \cdot G\)-equivariantly trivial. In some applications, such as [7], it is vacuously satisfied.
Then \(\tilde{p}^*\) is \((U \cdot G)\)-equivariant by construction. In particular, it maps \(\Gamma_H\)-invariant distribution sections, such as \(\tau\), to scalar distributions invariant under both \(\Gamma_H \cap U\) and \(\Gamma_H \cap G\).

To continue with the hypotheses of our main theorem, we note that the two subgroups

\[
\Gamma_G = \Gamma_H \cap G \subset G \quad \text{and} \quad \Gamma_U = \Gamma_H \cap U \subset U
\]

are arithmetic. Since \(G\) normalizes \(U\), \(\Gamma_G\) normalizes \(\Gamma_U\); moreover

\[
\Gamma_U \backslash U \text{ is compact,}
\]

as is the quotient of any unipotent linear group over \(\mathbb{Q}\) modulo an arithmetic subgroup. For any character

\[
\psi : U \to \mathbb{C}^* \quad \text{such that}
\]

\[
\begin{align*}
\text{a) } & \psi \equiv 1 \text{ on } \Gamma_U, \quad \text{and} \\
\text{b) } & \psi(gug^{-1}) = \psi(u) \quad \text{for all } g \in G, \ u \in U,
\end{align*}
\]

we define the averaging operator

\[
A_\psi : C^{-\infty}(Y, \mathcal{E})^{\Gamma_H} \to C^{-\infty}(Y, \mathcal{E})^{\Gamma_U \cdot \Gamma_G},
\]

\[
A_\psi(\sigma) = \frac{1}{\text{vol}(U/\Gamma_U)} \int_{U/\Gamma_U} \psi(u) \ell(u) \sigma du.
\]

This makes sense because of (2.4d) and (2.27–2.28). We shall briefly comment on the analytic aspects of the definition in section 3.

**Theorem 2.30.** Under the hypotheses just stated, for any \(\phi \in C_c^\infty(G)\),

\[
F_{\tau, q, \psi, \phi} = \text{def} \int_G \phi(g) r(g) \tilde{p}^*(A_\psi \tau) \, dg
\]

is a left \(\Gamma_G\)-invariant smooth function on \(U \cdot G\), translating under the right action of \(U\) on \(U \cdot G\) according to the character \(\psi\), and under the left action of \(U\) according to the complex conjugate character \(\overline{\psi}\). In particular the restriction of \(F_{\tau, q, \psi, \phi}\) to \(G\) lies in \(C^\infty(\Gamma_G \backslash G)\). If \(\tau \in C^{-\infty}(Y, \mathcal{E})^{\Gamma_H}\) is cuspidal or is at least cuspidal with respect to one factor \(H_j\) in the setting (2.17–2.18), this restricted function decays rapidly along all the cusps of \(\Gamma_G \backslash G\), and the integral

\[
\int_{\Gamma_G \backslash G} F_{\tau, q, \psi, \phi}(g) \, dg
\]

is
converges absolutely. The value of the integral does not depend on the choice of \( \phi \), provided \( \phi \) satisfies the normalizing condition \( \int_G \phi(g) \, dg = 1 \). If \( \tau \) depends holomorphically on a parameter \( s \), then the value of the integral also depends holomorphically on that parameter.

Let us re-state the theorem informally, in more suggestive terms. First of all, the action of \( U \cdot G \) on its open orbit \( O \) allows us to think of \( \tilde{\rho}^*(A_{\psi} \tau) \in C^{-\infty}(U \cdot G) \) as

\[
(2.31) \quad \tilde{\rho}^*(A_{\psi} \tau)(ug) = \frac{1}{\text{vol}(\Gamma_U \backslash U)} \int_{\Gamma_U \backslash U} \psi(u_1)^{-1} q(\tau(u_1 u gs)) \, du_1.
\]

This distribution transforms by the character \( \psi \) in the \( u \)-variable, and thus has smooth dependence on \( u \in U \). It therefore makes sense to speak of its restriction to the \( G \) factor, in which it is formally automorphic under \( \Gamma_G \).

Suppose momentarily that \( \Gamma_G \) is cocompact in \( G \). Then the integral of this restriction over \( \Gamma_G \backslash G \),

\[
(2.32) \quad \frac{1}{\text{vol}(\Gamma_U \backslash U)} \int_{\Gamma_G \backslash G} \int_{\Gamma_U \backslash U} \psi(u)^{-1} q(\tau(ugs)) \, du \, dg,
\]

makes sense as the integral of a distribution against the smooth measure \( dg \) over a compact manifold. Right translating \( g \) by \( g_1 \in G \) does not change the value of (2.32), nor does integrating the resulting integral over \( g_1 \) against a smooth function \( \phi \) of compact support and total integral 1. In other words, the integral of the function

\[
(2.33) \quad F_{\tau,q,\psi,\phi}(g) = \frac{1}{\text{vol}(\Gamma_U \backslash U)} \int_G \int_{\Gamma_U \backslash U} \psi(u)^{-1} q(\tau(ugg_1 s)) \phi(g_1) \, du \, dg
\]

over \( \Gamma_G \backslash G \) – the integral in the statement of the theorem – equals (2.32), again assuming that \( \Gamma_G \) is cocompact in \( G \). In the noncompact setting (where nearly all our applications lie), we cannot directly make sense of (2.32), but rather use (2.33) to define pairing in the statement of the theorem.

\footnote{Holomorphic dependence is to be taken in the weak sense: a distribution \( \sigma_s \) depends holomorphically on the parameter \( s \) if the integral of \( \sigma_s \) against any test function depends holomorphically on \( s \). Eisenstein distributions – the analogues of Eisenstein series in the context of automorphic distributions – are typical examples of distributions depending holomorphically on a parameter.}
In applications of theorem 2.30, \( H \) typically factors as a product \( H = H_1 \times H_2 \), and correspondingly \( \tau = \tau_1 \cdot \tau_2 \) as the product of two automorphic distributions, one for each of the factors \( H_j \). One can then think of the integral of \( F_{\tau,q,\psi,\phi} \) in the theorem as defining a pairing between the two automorphic distributions. That is the reason for the title of our paper. Often one of the factors \( \tau_j \) is an Eisenstein series – not a classical Eisenstein series, of course, but its distribution version. Nonetheless one can “unfold”, just as one does in the traditional Rankin-Selberg approach. This can be carried out in one of several ways, depending on the particular application – see, for example, [5,7]. For that reason, we shall treat the unfolding not here, but in papers in which we apply theorem 2.30.

We conclude with some explicit examples of pairings that are explained in further detail in [5]. Recall that \( s \in H \) was introduced in (2.20) in order to identify the quotient \((U \cdot G)/(UG)\) with its open orbit \( \mathcal{O} \). It is a straightforward matter to make this explicit in any particular example. For instance, let us consider the open orbit of \( G = SL^{\pm 1}(2,\mathbb{R}) \) on \( \mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}P^1 \) that was considered above in the earlier examples. If \( f_1 = (1 0 \ 0 1), f_2 = (1 1 \ 0 1), \) and \( f_3 = (0 -1 \ 1 0) \), the diagonal action of \( G \) on \( H = G \times G \times G \) gives an open orbit in the flag variety \( \mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}P^1 \) with basepoint represented by \( s = (f_1, f_2, f_3) \in H \). Thus if \( \tau_1, \tau_2, \) and \( \tau_3 \) are automorphic distributions for \( G \), the integral defined in theorem 2.30 is equal to

\[
\int_{\Gamma \setminus G} \int_{G} \int_{\Gamma \setminus U} \tau_1(ghf_1) \tau_2(ghf_2) \tau_3(ghf_3) \psi(h) \phi(h) \, dh \, dg,
\]

and is related to the Rankin-Selberg \( L \)-function of \( \tau_1 \otimes \tau_2 \) when \( \tau_3 \) is an Eisenstein series distribution. As an example with a nontrivial unipotent integration, let us return to (2.6)-(2.7) and let \( \tau_1 \) be an automorphic distribution for \( SL^{\pm 1}(4,\mathbb{R}) \), and \( \tau_2 \) be an automorphic distribution for \( G = SL^{\pm 1}(2,\mathbb{R}) \). The integral defined in theorem 2.30 in this case is

\[
\frac{1}{\text{vol}(\Gamma \setminus U)} \int_{\Gamma \setminus G} \int_{G} \int_{\Gamma \setminus U} \psi(u)^{-1} \tau_1 \left( u \left( g^hf_1 \right) \right) \tau_2(ghf_3) \, du \, dh \, dg,
\]

and is related to the exterior square \( L \)-function of \( \tau_1 \) when \( \tau_2 \) is an Eisenstein series distribution.
3 Smoothing the integrand

In this section we will first prove the invariance and smoothness properties asserted in theorem 2.30, and then give a construction of $F_{\tau,q,\psi,\phi}$ as an integral of an automorphic form on $H$ in lemma 3.9. The integral is then used in the following section to prove the remaining properties asserted in the theorem.

We continue with the hypotheses and notation of the previous section, and return briefly to the definition (2.29) of the averaging operator $A_\psi$. Like the space of distributions on any $C^\infty$ manifold, $C^{-\infty}(Y,E)$ has an intrinsic topology as complete, locally convex, Hausdorff topological vector space. These are precisely the properties one needs to define the integral of a continuous, compactly supported function with values in a topological vector space. The action of $H$ on $C^{-\infty}(Y,E)$ is continuous by general principles. It follows that $u \mapsto \psi(u)\ell(u)\tau$, for $\tau \in C^{-\infty}(Y,E)^{\Gamma_U}$, can be regarded as a continuous function, defined on the compact manifold $U/\Gamma_U$, with values in the topological vector space $C^{-\infty}(Y,E)$. As such it has a well defined integral, which represents $A_\psi \tau$.

Let us first examine the smoothness and the invariance properties asserted by the theorem. For $u \in U$,

\begin{equation}
\ell(u)(A_\psi \tau) = \frac{1}{\text{vol}(U/\Gamma_U)} \int_{U/\Gamma_U} \psi(v)\ell(uv)\tau dv
\end{equation}

and therefore, in view of the $(U \cdot G)$-equivariance of $\tilde{p}^*$,

\begin{equation}
\ell(u) \tilde{p}^*(A_\psi \tau) = \psi(u) \tilde{p}^*(A_\psi \tau).
\end{equation}

In terms of the identification $U \cdot G \simeq U \times G$, the distribution $\tilde{p}^*(A_\psi \tau)$ transforms according to the character $\psi$ in the $U$-variable. Without loss of information, we can set the $U$-variable equal to the identity and – temporarily – regard $\tilde{p}^*(A_\psi \tau)$ as distribution on $G$. Convolution with a compactly supported smooth function turns any distribution on $G$ into a smooth function. We then put the $U$-variable back in and conclude:

\begin{equation}
F_{\tau,q,\psi,\phi} = \int_G \phi(g) r(g) \tilde{p}^*(A_\psi \tau) dg \quad \text{is a } C^\infty \text{ function on } U \cdot G,
\end{equation}
for every $\phi \in C^\infty(G)$. Convolution on the right commutes with left translation. Hence, for $u \in U$, (3.2) implies

$$\ell(u) F_{\tau,q,\psi,\phi} = \overline{\psi(u)} F_{\tau,q,\psi,\phi}.$$  

Now suppose $g \in G$, $u, u_1 \in U$. Then, in view of (2.28b) and (3.4),

$$r(u) F_{\tau,q,\psi,\phi}(u_1 g) = F_{\tau,q,\psi,\phi}(u_1 g u g^{-1}) = (\ell((u_1 g u g^{-1})^{-1}) F_{\tau,q,\psi,\phi})(g)$$

$$= \psi(u_1 g u g^{-1}) F_{\tau,q,\psi,\phi}(g) = \psi(u) \psi(u_1) F_{\tau,q,\psi,\phi}(g)$$

$$= \psi(u) (\ell(u_1^{-1}) F_{\tau,q,\psi,\phi}) (g) = \psi(u) F_{\tau,q,\psi,\phi}(u_1 g),$$

so $r(u) F_{\tau,q,\psi,\phi} = \psi(u) F_{\tau,q,\psi,\phi}$. Since $A_\psi \tau$ is $\Gamma G$-invariant on the left, since $\tilde{p}^* G$ is $G$-equivariant, and since right and left translation commute,

$$\ell(\gamma) F_{\tau,q,\psi,\phi} = F_{\tau,q,\psi,\phi}, \quad \text{for every } \gamma \in \Gamma G.$$  

These are the smoothness and the invariance properties of $F_{\tau,q,\psi,\phi}$, as asserted.

For the proof of rapid decay in the next section, we need to work with an auxiliary function $\Phi_{\tau,q,\psi,\phi,\phi_U}$, whose definition involves the choice of some $\phi_U \in C^\infty_c(U)$, in addition to $\tau$, $\psi$, and $\phi$. We impose the normalization conditions

$$\int_G \phi(g) \, dg = 1, \quad \int_U \phi_U(u) \, du = 1,$$

and regard $q \circ \tau$ as a scalar valued distribution on $H$, via (2.14) and (2.23). That allows us to consider

$$\Phi_{\tau,q,\psi,\phi,\phi_U} = \int_G \int_U \phi(g) \phi_U(u) \overline{\psi(u)} \, r(u) \, r(gs) \, q \circ \tau \, du \, dg,$$

as a scalar distribution on $H$. Recall that $s \in H$, as defined in (2.20), relates the base points $y_0 \in Y$ and $o \in O$.

**Lemma 3.9.** $\Phi_{\tau,q,\psi,\phi,\phi_U}$ is a $\Gamma H$-automorphic form\(^6\) on $H$ — i.e., a $\Gamma H$-invariant $C^\infty$ function of uniformly moderate growth, which transforms finitely under the action of the algebra of bi-invariant differential operators on $H$. Moreover, for all $g \in G$,

$$\frac{1}{\text{vol}(U/\Gamma U)} \int_{U/\Gamma U} \psi(u) \left( \ell(u) \Phi_{\tau,q,\psi,\phi,\phi_U} \right)(g) \, du = F_{\tau,q,\psi,\phi}(g).$$

\(^6\)a smooth automorphic form, not required to transform finitely under the right action of a maximal compact subgroup, as is often assumed.
Proof: Let $E^* \to Y$ denote the $H$-invariant vector bundle dual to $E$, and $\wedge^{\text{top}}T^*Y$ the top exterior power of the cotangent bundle of $Y$ – i.e., the line bundle whose $C^\infty$ sections are smooth measures on $Y$. We shall produce a smooth section $\omega$ of the tensor product,

$$\omega \in C^\infty(Y, E^* \otimes \wedge^{\text{top}}T^*Y),$$

such that $\Phi_{\tau,q,\psi,\phi,\phi_u}(h) = \int_Y \langle \ell(h^{-1})\tau, \omega \rangle$;

here $\langle \ell(h^{-1})\tau, \omega \rangle$ denotes the contraction between $\omega \in C^\infty(Y, E^* \otimes \wedge^{\text{top}}T^*Y)$ and $\ell(h^{-1})\tau \in C^{-\infty}(Y, E)$, resulting in a scalar valued, distribution coefficient form of top degree on the compact manifold $Y$. As such, it can be integrated over $Y$ against any smooth function, in particular the constant function 1.

We assume the existence of $\omega$ for the moment. In section 2 we mentioned that $C^{-\infty}(Y, E)$ is the space of distribution vectors $V^\infty$ for an admissible, representation $(\pi, V)$ of $H$, of finite length, on a Hilbert space. Analogously its topological dual $C^\infty(Y, E^* \otimes \wedge^{\text{top}}T^*Y)$ can be regarded as the space of $C^\infty$ vectors $(V')^\infty$ for the dual representation $(\pi', V')$. If we now let $\langle \ , \ \rangle$ denote the pairing between $V^{-\infty}$ and $(V')^\infty$, we can rewrite the equality in (3.10) as

$$\Phi_{\tau,q,\psi,\phi,\phi_u}(h) = \langle \ell(h^{-1})\tau, \omega \rangle.$$

Since $h \mapsto \ell(h^{-1})\tau$ is a smooth function on $H$, with values in the topological vector space $V^{-\infty}$, and since the pairing is linear and continuous in each variable, the description (3.11) exhibits $\Phi_{\tau,q,\psi,\phi,\phi_u}$ as a $C^\infty$ function. Moreover, any function of this type, with $\tau \in (V^{-\infty})^{\Gamma_H}$, is a $C^\infty$ automorphic form [6, (2.15)].

We continue to assume the existence of $\omega$. We shall establish the identity stated at the end of the lemma. Since it only involves the values of $\Phi_{\tau,q,\psi,\phi,\phi_u}$ on $U \cdot G$, we now regard this function as defined on $U \cdot G$, rather than on $H$ as in (3.8). Recall the definition of the averaging operator $A_\psi$ in (2.29). Left and right translation commute, hence

$$\frac{1}{\text{vol}(U/\Gamma_U)} \int_{U/\Gamma_U} \psi(u) (\ell(u) \Phi_{\tau,q,\psi,\phi,\phi_u}) \, du =$$

$$= \int_{U} \int_{G} \phi(g) \phi_U(u) \overline{\psi(u)} r(u) r(g) \tilde{p}^*(A_\psi \tau) \, dg \, du$$

$$= \int_{U} \phi_U(u) \overline{\psi(u)} r(u) F_{\tau,q,\phi,\phi} \, du$$

$$= \int_{U} \phi_U(u) |\psi(u)|^2 F_{\tau,q,\psi,\phi} \, du = F_{\tau,q,\psi,\phi}. \tag{3.12}$$
At the first step of this chain of equalities, we have used the definition (3.8) of $\Phi_{\tau,q,\psi,\phi_U}$, the definition (2.29) of $A_\psi$, as well as the characterization of the map $\tilde{p}^*$ in (2.24); at the second step we have used the definition of $F_{\tau,q,\psi,\phi}$ in the statement of the main theorem; and finally, in the third step, the transformation law (3.5) and the normalization (3.7) of $\phi_U$.

We shall construct $\omega$ first as a smooth, compactly supported, $\mathcal{E}$-valued measure on $\mathcal{O}$. Since $\mathcal{O}$ is open in $\mathcal{Y}$, we can then regard $\omega$ as an element of $C^\infty(\mathcal{Y}, \mathcal{E}^* \otimes \Lambda^{\text{top}}T^*\mathcal{Y})$. Recall the definition, above (2.19), of $(UG)_o$ as the isotropy subgroup of $U \cdot G$ at $s = s y_0 \cong s P$. Since $U \cdot G$ and $P$ are defined, respectively, over $\mathbb{Q}$ and over $\mathbb{R}$,

\begin{equation}
(UG)_o = (U \cdot G) \cap s P s^{-1}
\end{equation}

is an $\mathbb{R}$-subgroup of $H$. We define

\begin{equation}
f_0 \in C^\infty_c(U \cdot G), \quad f_0(ug) = \phi(g) \phi_U(u) \overline{\psi(u)}.
\end{equation}

Note that $U \cdot G$ is unimodular because of (2.41). Haar measure on this group can be described as the product $du \, dg$ of the Haar measures on $U$ and $G$. The normalizations (3.7) imply that $f_0 \, du \, dg$ is well-defined, independently of the scaling of $du$ and $dg$. We let $(\mu^*, E^*)$ denote the representation of $P$ dual to $(\mu, E)$, and choose $e^* \in E^*$ such that

\begin{equation}
q(e) = \langle e^*, e \rangle \quad \text{for all} \quad e \in E.
\end{equation}

Using this notation, we interpret (3.8) as an identity between distributions on $H$,

\begin{equation}
\Phi_{\tau,q,\psi,\phi_U}(h) = \int_{U \cdot G} \langle f_0(ug) e^*, \tau(ugs) \rangle \, du \, dg.
\end{equation}

The group $(UG)_o$ may not be unimodular. We pick a left invariant Haar measure $dv$; the particular normalization will turn out not to matter. Averaging first over $(UG)_o$ and integrating the result over the quotient $(U \cdot G)/(UG)_o$, we find

\begin{equation}
\Phi_{\tau,q,\psi,\phi_U}(h) = \int_{(U \cdot G)/(UG)_o} \int_{(UG)_o} \langle f_0(ugv) e^*, \tau(ugs) \rangle \, dv \frac{du \, dg}{dv};
\end{equation}

here $du \, dg/dv$ is regarded as smooth measure on $(U \cdot G)/(UG)_o$ in the obvious manner.

15
All along we have used (2.14) to regard $\tau$ as $E$-valued distribution on $H$, transforming under right translation by $p \in P$ according to $\mu(p^{-1})$. Hence

$$\langle f_0(ugv) e^*, \tau(hugv) \rangle = \langle f_0(ugv) e^*, \mu(s^{-1}v^{-1}s)\tau(hugs) \rangle = \langle f_0(ugv) \mu^*(s^{-1}vs)e^*, \tau(hugs) \rangle.$$  

The $C^\infty$ function $f : U \cdot G \to E^*$, defined by

$$(3.19) f(ug) = \int_{(UG)_0} f_0(ugv) \mu^*(s^{-1}vs)e^* dv,$$

is well defined because the support of any left translate of $f_0$ intersects $(UG)_0$ in a compact subset. As direct consequence of the construction,

$$(3.20) f(ugv) = \mu^*(s^{-1}v^{-1}s)f(ug) \quad \text{for all } v \in (UG)_0.$$  

That, in analogy to (2.22) with $C^\infty$ in place of $C^{-\infty}$, allows us to regard $f$ as a $C^\infty$ section of $E^* \to (U \cdot G)/(UG)_0 \cong O$. The fact that $f_0$ has compact support in $U \cdot G$ implies the compact support, modulo $(UG)_0$, of $f$,

$$(3.21) f \in C^\infty_c(O, E^*).$$

At this point we can reinterpret the outer integral in (3.17) as the integral of the smooth, compactly supported, $E^*$-valued measure

$$\omega = f^* \frac{du}{dv} dv \in C^\infty_c(O, E^* \otimes \Lambda^{top}T^*Y)$$

against the $E$-valued distribution section $\ell(h^{-1})\tau \in C^\infty(Y, E)$. In effect, we have verified (3.10), completing the proof of the lemma.

To construct $\omega$ in the preceding proof, we averaged $f_0$ over $(UG)_0$ on the right, and $f_0$ involved the function $\phi$ as a factor. Hence:

**Remark 3.23.** If the support of $\phi$ is kept fixed, the dependence of $\omega$ on $\phi$ is bounded in $C^k$ norm, for every $k \in \mathbb{N}$.

## 4 Rapid decay

In this section we finish the proof of theorem 2.30 using lemma 3.9 as a main tool. We shall use the results of [6], in particular theorem 4.7 of that
paper. The notation there differs from our current notation, since the two papers were written for different purposes. The role of the current ambient group $H$, or of its factors $H_j$ in (2.17), is played by $G$ in [6], and the role of the unipotent group $U$ by $N_1$. The group $G$ in [6] satisfies exactly the same hypotheses as our ambient group $H$. Since our current $G$ is reductive and defined over $\mathbb{Q}$, it can be extended to a Levi component, defined over $\mathbb{Q}$, of the parabolic subgroup $P_U \subset H$. Thus we may assume that the Langlands decomposition in (2.4b) is defined over $\mathbb{Q}$. That, in view of (2.28), makes $G$ a reductive $\mathbb{Q}$-subgroup of the group denoted by $L$ in [6, §4]. If $L_1 \subset L_2$ is an inclusion of real reductive groups defined over $\mathbb{Q}$, any Siegel set in $L_1$ is contained in the intersection of $L_1$ with an appropriately chosen Siegel set in $L_2$. Thus, for statements about rapid decay on Siegel sets of automorphic forms on some ambient group, any decay statement for $L_2$ directly implies the analogous statement for $L_1$. In our words, for our purposes it is really irrelevant whether the current $G$ is a subgroup of the $L$ in [6, §4], or all of it.

Let us restate theorem 4.7 of [6] in our current notation. We consider the automorphic form $\Phi_{\tau,q,\psi,\phi_U}$ on $H$ associated to a cuspidal automorphic distribution $\tau \in C^{-\infty}(Y, \mathcal{E})$ and $\omega \in C^\infty_c(O, \mathcal{E}^* \otimes \Lambda^{\text{top}}T^*Y)$ as in (3.10); in the proof of lemma 3.9 $\omega$ was constructed in terms of $q, \psi, \phi, \phi_U$, of course. Then for any Siegel set $\mathcal{S}_G \subset G$, any $c$ in the inverse image of $G_\mathbb{Q}$ in $G$ – cf. the discussion at the beginning of section 2 – and any $n \in \mathbb{N}$, there exists a constant $C > 0$ such that

\begin{equation}
\Phi_{\tau,q,\psi,\phi_U}(ucg) \quad \Longrightarrow \quad |\Phi_{\tau,q,\psi,\phi_U}(ucg)| \leq C\|g\|^{-n}.
\end{equation}

When $F_{\tau,q,\psi,\phi_U}(ucg)$ is defined in terms of $\Phi_{\tau,q,\psi,\phi_U}$ by the formula in lemma 3.9 the above estimate directly implies

\begin{equation}
g \in \mathcal{S}_G \quad \rightarrow \quad |F_{\tau,q,\psi,\phi_U}(ucg)| \leq C\|g\|^{-n}.
\end{equation}

A finite number of translates $c\mathcal{S}_G$ cover $\Gamma_G \backslash G$, so (4.2) implies the integrability of $F_{\tau,q,\psi,\phi_U}(ucg)$ over $\Gamma_G \backslash G$, at least when $\tau$ is cuspidal.

The preceding argument must be modified when $\tau$ is only cuspidal with respect to one of the factors in a factorization (2.17) – say with respect to $H_1$ for definiteness. In that situation [6, theorem 5.13] asserts the simultaneous rapid decay in the first factor and moderate growth in the second for $\Phi_{\tau,q,\psi,\phi_U}(h_1h_2)$, provided $h_1$ varies in a Siegel set $\mathcal{S}_1 \subset H_1$ and $h_2$ over $H_2$. Rapid decay trumps moderate growth when $\Phi_{\tau,q,\psi,\phi_U}$ is restricted to $U \cdot G$; this depends on the hypothesis (2.18), of course. Thus both (4.1) and (4.2) remain valid even in the partially cuspidal case.
Only two assertions remain to be verified to complete the proof of theorem 2.30. First of all, we need to show that the integral

\[
\int_{\Gamma \setminus G} F_{\tau,q,\psi,\phi}(g) \, dg = \int_{\Gamma \setminus G} \int_G \phi(g_1) \tilde{p}^*(A_\psi \tau)(gg_1) \, dg_1 \, dg
\]

depends only on the total integral of the smoothing function \( \phi \), but not otherwise on the particular choice of \( \phi \). Since the measure on \( \Gamma \setminus G \) is \( G \)-invariant on the right, we can right translate the argument \( g \) by any fixed \( g_2 \in G \) without changing the value of the integral (4.3). Equivalently, we can replace \( \phi \) by \( \ell(g_2) \phi \) without affecting the integral. On the infinitesimal level this means that the integral vanishes whenever \( \phi = \ell(Z) \tilde{\phi} \) is the left infinitesimal translate of some \( \phi \in C^\infty_c(G) \) by some \( Z \in \mathfrak{g} \). Thus the integral (4.3) depends only on the image of \( \phi \) in the quotient

\[
C^\infty_c(G)/\ell(\mathfrak{g})C^\infty_c(G).
\]

Since \( G \) is parallelizable, this quotient represents the top cohomology group of the de Rham complex with compact support. As such, it is naturally dual to \( H_0(G, \mathbb{C}) \), and the duality is implemented by integration over the various connected components of \( G \). If \( G \) is connected, we can conclude that

\[
\ell(\mathfrak{g})C^\infty_c(G) = \{ \phi \in C^\infty_c(G) \mid \int_G \phi(g) \, dg = 0 \},
\]

which implies the conclusion we want. The case of several connected components reduces readily to the connected case; what matters here is that replacing \( \phi \) by \( \ell(g_2) \phi \), with \( g_2 \in G \), does not affect the value of the integral.

The preceding argument involves interchanging the order of differentiation and integration. To make this legitimate, one needs to know that the absolute convergence of the integral of \( F_{\tau,q,\psi,\phi} \) over \( \Gamma \setminus G \) can be estimated – when \( \tau, \psi \), and the support of \( \phi \) are kept fixed – in terms of some \( C^k \) norm on \( \phi \). Recall the remark 3.23. Since \( F_{\tau,\psi,\phi} \) was expressed in terms of the automorphic form \( \Phi_{\tau,\psi,\phi,\phi_U} \), which involves \( \phi \) via \( \omega \) in the identity (3.11), we need to quantify the decay of \( \Phi_{\tau,\psi,\phi,\phi_U} \) in terms of some \( C^k \) norm of \( \omega \), again assuming \( \tau \) is fixed. In the notation of \([6\), theorem 5.13], \( \omega \) corresponds to the vector \( v \). The estimate in that theorem depends linearly on the \( C^k \) norm of \( v \), which provides the necessary justification.

As was remarked earlier, for us holomorphic dependence of a distribution on a complex parameter means dependence in the “weak sense” – i.e., the
integral of the distribution against any test function depends holomorphically on the parameter in question. According to (3.10), the values of the function \( \Phi_{\tau,q,\psi,\phi,\phi_U} \) can be interpreted as the result of paring the distribution section \( \tau \) of the vector bundle \( \mathcal{E} \) over the compact manifold \( Y \) against a smooth measure with values in the dual vector bundle. Thus, if \( \tau \) depends holomorphically on a complex parameter, then so does the function \( \Phi_{\tau,q,\psi,\phi,\phi_U} \), and in view of lemma 3.9 also the function \( F_{\tau,q,\psi,\phi} \). This completes the proof of theorem 2.30.

References

[1] A. Borel, *Introduction aux groupes arithmétiques*, Actualités scientifiques et industrielles 1341, Hermann, Paris, 1969.

[2] W. Casselman, *Jacquet modules for real reductive groups*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), 1980, pp. 557–563. MR 83h:22025

[3] __________, *Canonical extensions of Harish-Chandra modules to representations of \( G \)*, Canad. J. Math. 41 (1989), no. 3, 385–438. MR 90j:22013

[4] Stephen D. Miller and Wilfried Schmid, *Automorphic Distributions, L-functions, and Voronoi summation for \( GL(3) \)*, Annals of Math. 164 (2006), no. 2, 423–488. MR2247965 (2007j:11065)

[5] __________, *The Rankin-Selberg method for automorphic distributions*, Representation theory and automorphic forms, Progr. Math., vol. 255, Birkhäuser Boston, Boston, MA, 2008, pp. 111–150.

[6] __________, *On the rapid decay of cuspidal automorphic forms* (2010). preprint.

[7] __________, *The archimedean theory of the Exterior Square L-functions over \( \mathbb{Q} \)* (2010). Preprint.

[8] Nolan R. Wallach, *Real Reductive Groups I, II*, Pure and Applied Mathematics, vol. 132-I,II, Academic Press, 1988,1992.

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