Conditional Independence Beyond Domain Separability
Discussion of Engelke and Hitz (2020)

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1 A general definition of conditional independence

We congratulate Engelke and Hitz on a thought-provoking paper on graphical models for extremes. A key contribution of the paper is the introduction of a novel definition of conditional independence for a multivariate Pareto distribution. Here, we outline a proposal for independence and conditional independence of general random variables whose support is a general set $\Omega$ in $\mathbb{R}^d$. Our proposal includes the authors’ definition of conditional independence, and the analogous definition of independence as special cases. By making our proposal independent of the context of extreme value theory, we highlight the importance of the authors’ contribution beyond this particular context.

Definition 1: Suppose that $Y = (Y_A, Y_B, Y_C)$ is a random vector with support $\Omega$ in $\mathbb{R}^d$, where $A, B, C$ are disjoint sets whose union is $\{1, \ldots, d\}$. Let $U \times V$ denote the Cartesian product of $U$ and $V$.

(a) We say $Y_A$ is conditionally outer independent of $Y_C$ given $Y_B$ if there exists a random vector $W = (W_A, W_B, W_C)$ with support $L_A \times L_B \times L_C$ in $\mathbb{R}^d$ such that (i) $\Omega \subset L_A \times L_B \times L_C$; (ii) $(W | W \in \Omega) \overset{d}{=} Y$; (iii) $W_A \perp W_C | W_B$. In this case, we write $Y_A \perp_{o} Y_C | Y_B$.

If $B = \emptyset$, we say $Y_A$ is outer independent of $Y_C$, denoted as $Y_A \perp_{o} Y_C$.

(b) We say $Y_A$ is conditionally inner independent of $Y_C$ given $Y_B$ if for any $S_A \times S_B \times S_C \subset \Omega$ such that $S_k$ is a measurable subset of $\mathbb{R}^\text{dim}(Y_k)$, $k \in \{A, B, C\}$ and $P(Y \in S_A \times S_B \times S_C) > 0$, we have $Y_A \perp_{i} Y_C | (Y_B, Y \in S_A \times S_B \times S_C)$. In this case, we write $Y_A \perp_{i} Y_C | Y_B$.

If $B = \emptyset$, we say $Y_A$ is inner independent of $Y_C$, denoted as $Y_A \perp_{i} Y_C$.

Proposition 1: Suppose $\Omega = [0, \infty)^d \setminus [0, 1]^d$ as in EH’s case. Then

\[ Y_A \perp_{o} Y_C | Y_B \iff Y_A \perp_{i} Y_C | Y_B \iff Y_A \perp_{e} Y_C | Y_B. \]

where $\perp_{e}$ denotes the notion of conditional independence introduced by Engelke and Hitz (2020). In particular, if $B = \emptyset$, then

\[ Y_A \perp_{o} Y_C \iff Y_A \perp_{i} Y_C \iff Y_A \perp_{e} Y_C. \]

Remark 1: We do not place any distributional assumptions on $Y$ in Proposition 1.

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Engelke and Hitz showed that if $Y$ is multivariate Pareto and admits a positive and continuous density, then $Y_A \perp \perp Y_C$. This does not rule out the possibility of $Y_A \perp \perp Y_C$ for general Pareto distributions. For example, consider two independent standard Pareto distributions $X_1$ and $X_2$. Following eqn. (6) in Engelke and Hitz (2020), all the probability mass of $Y$ lies on $(1, \infty) \times \{0\}$ and $\{0\} \times (1, \infty)$ so that it does not admit a density with respect to Lebesgue measure. Nevertheless $Y_1 \perp \perp Y_2$. This observation is generalized in Proposition 2.

**Proposition 2:** Suppose $\Omega = \left([0, \infty)^{|A|} \setminus [0, 1]^{|A|}\right) \times \{0\} \times \{0\} \cup \left([0, \infty)^{|B|} \times [0, 1]^{|B|}\right) \times \{0\} \cup \left([0, \infty)^{|C|} \times \{0\} \times \{0\} \times \{0\}\right)$, where $|A|$ is the cardinality of set $A$. Then

$$
Y_A \perp \perp o Y_C \Leftrightarrow Y_A \perp \perp Y_C \Leftrightarrow Y_A \perp \perp e Y_C \Leftrightarrow Y_A \perp \perp e Y_C.
$$

In particular, if $B = \emptyset$, then

$$
Y_A \perp \perp o Y_C \Leftrightarrow Y_A \perp \perp Y_C \Leftrightarrow Y_A \perp \perp e Y_C.
$$

# Proofs of propositions

## 2.1 Proof of Proposition 1 in the case where $B = \emptyset$

**Proof.** We need to show

$$
Y_A \perp \perp o Y_C \Rightarrow Y_A \perp \perp Y_C \Rightarrow Y_A \perp \perp e Y_C \Rightarrow Y_A \perp \perp o Y_C.
$$

The first two claims are straightforward. We now prove the third claim in (1).

Let $H^1_k \subset L^1_k = [0, 1]^{|k|}$, $H^2_k \subset L^2_k = [0, \infty)^{|k|} \setminus [0, 1]^{|k|}$, $H_k = H^1_k \cup H^2_k$, $L_k = L^1_k \cup L^2_k$, $k \in \{A, C\}$. Obviously, $H^1_k \cap H^2_k = \emptyset$ and $L^1_k \cap L^2_k = \emptyset$. Besides, $\Omega = [0, \infty)^d \setminus [0, 1]^d \subset L_A \times L_C$.

We let

$$
P(W_A \in H^1_A, W_C \in H^1_C) = \frac{P(W_A \in H^1_A, W_C \in L^2_C)P(W_A \in L^2_A, W_C \in H^1_C)}{P(W_A \in L^2_A, W_C \in L^2_C)},
$$

and

$$
P(W_A \in H^1_A, W_C \in H^1_C) = \lambda P(Y_A \in H^1_A, Y_C \in H^1_C), \quad (i, j) \neq (1, 1),$$

where $\lambda$ is a normalizing constant. Simple calculation yields

$$
\lambda = \frac{1}{1 + \frac{P(Y_A \in L^1_A, Y_C \in L^2_C)P(Y_A \in L^2_A, Y_C \in L^1_C)}{P(Y_A \in L^2_A, Y_C \in L^2_C)}}.
$$

It is easy to see that $(W \mid W \in \Omega)^d = Y$. We then prove

$$
W_A \perp \perp W_C.
$$

If $k \in A$ and $\{W_A \in H_A\} \cap \{W_A : W_k > 1\} = \emptyset$, then

$$
P(W_A \in H_A, W_C \in H_C \mid W_k > 1) = P(W_A \in H_A \mid W_k > 1)P(W_C \in H_C \mid W_k > 1) = 0.
$$

If $k \in A$ and $\{W_A \in H_A\} \cap \{W_A : W_k > 1\} \neq \emptyset$, then $\{W_A \in H_A\} \cap \{W_A : W_k > 1\} = \{W_A \in H^2_A, W_k > 1\}$ and
\[ P(W_A \in H_A, W_C \in H_C, W_k > 1) = \lambda P(Y_A \in H_A, Y_C \in H_C, Y_k > 1) = \lambda P(Y_C \in H_C \mid Y_A \in H_A, Y_k > 1) P(Y_A \in H_A, Y_k > 1) = P(Y_C \in H_C \mid Y_k > 1) P(W_A \in H_A, W_k > 1) = P(W_C \in H_C \mid Y_k > 1). \]

So we have
\[
P(W_C \in H_C \mid W_A \in H_A, W_k > 1) = P(Y_C \in H_C \mid Y_k > 1) = \frac{P(Y_C \in H_C, Y_k > 1)}{P(Y_k > 1)} = \frac{P(W_C \in H_C, W_k > 1)/\lambda}{P(W_k > 1)/\lambda} = \frac{P(W_C \in H_C, W_k > 1)}{P(W_k > 1)} = P(W_C \in H_C \mid W_k > 1).
\]

Thus, from (3) and (4), we have for any \( k \in A \), \( W_A \parallel W_C \mid W_k > 1 \). Similarly, we have for any \( k \in C \), \( W_A \parallel W_C \mid W_k > 1 \). In summary,
\[
\forall k \in \{1, \ldots, d\} : W_A \parallel W_C \mid W_k > 1.
\]

We then show that for any \( H_A^1, H_A^2, H_A^3, H_C^1, H_C^2 \), we have
\[
\frac{P(W_A \in H_A^1, W_C \in H_C^1)}{P(W_A \in H_A^2, W_C \in H_C^1)} = \frac{P(W_A \in H_A^1, W_C \in H_C^2)}{P(W_A \in H_A^2, W_C \in H_C^2)}.
\]

To prove (6), note that
\[
\frac{P(W_A \in H_A^1, W_C \in H_C^1)}{P(W_A \in H_A^2, W_C \in H_C^1)} = \frac{P(W_A \in H_A^1 \mid W_C \in H_C^2)}{P(W_A \in H_A^2 \mid W_C \in H_C^2)} P(W_A \in H_A^1, W_C \in H_C^1) = \frac{P(W_A \in H_A^1 \mid W_C \in H_C^1)}{P(W_A \in H_A^2 \mid W_C \in H_C^1)} P(W_A \in H_A^1, W_C \in H_C^1) = P(W_A \in H_A^1 \mid W_C \in L_C^1) P(W_A \in H_A^1, W_C \in H_C^1) \quad \text{due to (5)}
\]
\[
\frac{P(W_A \in H_A^1)}{P(W_A \in H_A^2)} = \frac{P(W_A \in H_A^1 \mid W_C \in H_C^1)}{P(W_A \in H_A^2 \mid W_C \in H_C^1)} = \frac{P(W_A \in H_A^1 \mid W_C \in H_C^1)}{P(W_A \in H_A^2 \mid W_C \in H_C^1)} P(W_A \in H_A^1, W_C \in H_C^1) = P(W_A \in H_A^1 \mid W_C \in H_C^1) P(W_A \in H_A^1, W_C \in H_C^1) \quad \text{due to (5)}
\]
\[
\frac{P(W_C \in L_C^1 \mid W_A \in L_A^1)}{P(W_C \in L_C^2 \mid W_A \in L_A^1)} = \frac{P(W_C \in L_C^1)}{P(W_C \in L_C^2)} = \frac{P(W_C \in L_C^1 \mid W_A \in L_A^1)}{P(W_C \in L_C^2 \mid W_A \in L_A^1)} P(W_C \in L_C^1) = P(W_C \in L_C^1 \mid W_A \in L_A^1) P(W_C \in L_C^1) \quad \text{due to (2)}.
\]

We then show that
\[
P(W_C \in L_C^2 \mid W_A \in L_A^1) = P(W_C \in L_C^2 \mid W_A \in L_A^1).
\]
Similarly we have

\[ \frac{P(W_A \in L_A^1, W_C \in L_C^1)}{P(W_A \in L_A^2, W_C \in L_C^1)} = \frac{P(W_A \in L_A^1, W_C \in L_C^2)}{P(W_A \in L_A^2, W_C \in L_C^2)} \]

Then

\[
\text{LHS of (7)} = \frac{P(W_A \in L_A^2, W_C \in L_C^2)}{P(W_A \in L_A^2)} = \frac{P(W_A \in L_A^2, W_C \in L_C^2)}{P(W_A \in L_A^1, W_C \in L_C^2) + P(W_A \in L_A^2, W_C \in L_C^1)} = \frac{P(W_A \in L_A^1, W_C \in L_C^2)}{P(W_A \in L_A^1)} = P(W_C \in L_C^2 | W_A \in L_A^1) = \text{RHS of (7)}. \]

Similarly we have

\[ P(W_C \in L_C^1 | W_A \in L_A^2) = P(W_C \in L_C^1 | W_A \in L_A^1). \]

We then show that

\[ P(W_C \in H_C^2 | W_A \in L_A^2) = P(W_C \in H_C^2 | W_A \in L_A^1). \]

To see this, due to (6),

\[ \frac{P(W_A \in L_A^1, W_C \in H_C^1)}{P(W_A \in L_A^2, W_C \in H_C^1)} = \frac{P(W_A \in L_A^1, W_C \in L_C^2)}{P(W_A \in L_A^2, W_C \in L_C^2)}. \]

Furthermore, due to (7),

\[ \frac{P(W_C \in H_C^1 | W_A \in L_A^1)}{P(W_C \in H_C^1 | W_A \in L_A^2)} = \frac{P(W_C \in L_C^2 | W_A \in L_A^1)}{P(W_C \in L_C^2 | W_A \in L_A^2)} = 1. \]

Due to (5) and (6), it follows directly that

\[ P(W_C \in H_C^2 | W_A \in L_A^1) = P(W_C \in H_C^2 | W_A \in L_A^2) = P(W_C \in H_C^2 | W_A \in H_A^2) = P(W_C \in H_C^2). \]  

(8)

Similarly we have

\[ P(W_C \in H_C^1 | W_A \in L_A^1) = P(W_C \in H_C^1 | W_A \in H_A^2) = P(W_C \in H_C^1). \]  

(9)

We still need to show

\[ P(W_C \in H_C^2 | W_A \in H_A^1) = P(W_C \in H_C^2 | W_A \in L_A^1), \]  

(10)

\[ P(W_C \in H_C^1 | W_A \in H_A^1) = P(W_C \in H_C^1 | W_A \in L_A^1). \]  

(11)

To show (10), note that due to (5) and (6),
\[
P(W_C \in H_C^2 \mid W_A \in H_A^1) \\
= \frac{P(W_C \in H_C^2, W_A \in H_A^1)}{P(W_A \in H_A^1)} \\
= \frac{\lambda P(Y_C \in H_C^2, Y_C \in L_C^2, Y_A \in H_A^1)}{P(W_A \in H_A^1) + P(W_A \in H_A^1, W_C \in L_C^1)} \\
= \frac{\lambda P(Y_C \in H_C^2 \mid Y_C \in L_C^2) P(Y_A \in H_A^1, Y_C \in L_C^2)}{P(W_A \in H_A^1, W_C \in L_C^1) + P(W_A \in H_A^1, W_C \in L_C^2)} \\
= \frac{P(W_A \in H_A^1, W_C \in L_C^2) P(W_A \in H_A^1, W_C \in L_C^1)}{P(W_A \in L_A^2, W_C \in L_C^1) + 1}
\]

does not depend on \(H_A^1\).

To show (11), note that due to (2) and (6),
\[
P(W_C \in H_C^1 \mid W_A \in H_A^1) \\
= \frac{P(W_A \in H_A^1, W_C \in H_C^1)}{P(W_A \in H_A^1, W_C \in L_C^2)} \\
= \frac{P(W_A \in H_A^1, W_C \in L_C^1) + P(W_A \in H_A^1, W_C \in L_C^2)}{P(W_A \in H_A^1, W_C \in L_C^1) + P(W_A \in H_A^1, W_C \in L_C^2)} \\
= \frac{P(W_A \in H_A^1, W_C \in L_C^2) P(W_A \in H_A^1, W_C \in L_C^1)}{P(W_A \in L_A^2, W_C \in L_C^1) + 1}
\]

does not depend on \(H_A^1\).

From (8), (9), (10) and (11), we have
\[
P(W_C \in H_C \mid W_A \in H_A) = P(W_C \in H_C).
\]

Hence \(W_A \perp \perp W_C\).

\[\square\]

2.2 Proof of Proposition 1 for a general \(B\)

Proof. We need to show
\[
Y_A \perp \perp o Y_C \mid Y_B \Rightarrow Y_A \perp \perp e Y_C \mid Y_B \Rightarrow Y_A \perp \perp o Y_C \mid Y_B 
\]

(12)

The first two claims are straightforward. We now prove the third claim in (12).
Let $H_k^1 \subseteq L_k^1 = [0, 1]^{k_1}, H_k^2 \subseteq L_k^2 = [0, \infty)^{k_1} \setminus [0, 1]^{k_1}, H_k = H_k^1 \cup H_k^2, L_k = L_k^1 \cup L_k^2, k \in \{A, B, C\}$. Obviously, $H_k^1 \cap H_k^2 = \emptyset$ and $L_k^1 \cap L_k^2 = \emptyset$. Besides, $\Omega = [0, \infty)^d \setminus [0, 1]^d \subseteq A \times B \times C$.

We let

$$P(W_A \in H_A^1, W_B \in H_B^1, W_C \in H_C^1) = \frac{P(W_A \in H_A^1, W_B \in H_B^1, W_C \in H_C^1)P(W_A \in L_A^2, W_B \in H_B^1, W_C \in H_C^1)}{P(W_A \in L_A^2, W_B \in H_B^1, W_C \in L_C^2)},$$

(13)

and for $(i, j, k) \neq (1, 1, 1)$,

$$P(W_A \in H_A^1, W_B \in H_B^2, W_C \in H_C^k) = \lambda P(Y_A \in H_A^1, Y_B \in H_B^2, Y_C \in H_C^k),$$

where $\lambda$ is a normalizing constant. Simple calculation yields

$$\lambda = \frac{1}{1 + \frac{P(Y_A \in L_A^1, Y_B \in L_B^1, Y_C \in L_C^2)P(Y_A \in L_A^2, Y_B \in L_B^2, Y_C \in L_C^2)}{P(Y_A \in L_A^2, Y_B \in L_B^2, Y_C \in L_C^2)}}.$$

It is easy to see that $(W | W \in \Omega) \overset{d}{=} Y$. We then prove

$$W_A \parallel W_C | W_B.$$

If $k \in A$ and $\{W_A \in H_A\} \cap \{W_A : W_k > 1\} = \emptyset$, then

$$P(W_A \in H_A, W_C \in H_C | W_B \in H_B, W_k > 1) = P(W_A \in H_A | W_B \in H_B, W_k > 1) \times P(W_C \in H_C | W_B \in H_B, W_k > 1) = 0.$$  (14)

If $k \in A$ and $\{W_A \in H_A\} \cap \{W_A : W_k > 1\} \neq \emptyset$, then $\{W_A \in H_A\} \cap \{W_A : W_k > 1\} = \{W_A \in H_A^2, W_k > 1\}$. Note that

$$P(W_A \in H_A, W_B \in H_B, W_C \in H_C, W_k > 1) = \lambda P(Y_A \in H_A, Y_B \in H_B, Y_C \in H_C, Y_k > 1) = \lambda P(Y_C \in H_C | Y_A \in H_A, Y_B \in H_B, Y_k > 1)P(Y_A \in H_A, Y_B \in H_B, Y_k > 1) = P(Y_C \in H_C | Y_B \in H_B, Y_k > 1)P(W_A \in H_A, W_B \in H_B, W_k > 1)$$

So we have

$$P(W_C \in H_C | W_A \in H_A, W_B \in H_B, W_k > 1) = \frac{P(Y_C \in H_C | Y_B \in H_B, Y_k > 1)}{P(Y_B \in H_B, Y_k > 1)} = \frac{P(W_C \in H_C, W_B \in H_B, W_k > 1) / \lambda}{P(W_B \in H_B, W_k > 1) / \lambda} = \frac{P(W_C \in H_C, W_B \in H_B, W_k > 1)}{P(W_B \in H_B, W_k > 1)} = P(W_C \in H_C | W_B \in H_B, W_k > 1).$$

(15)

Thus, from (14) and (15), we have for any $k \in A$, $W_A \parallel W_C | W_B, W_k > 1$. Similarly, we have for any $k \in C$, $W_A \parallel W_C | W_B, W_k > 1$. 

6
If \( k \in B \) and \( \{ W_B \in H_B \} \cap \{ W_B : W_k > 1 \} = \emptyset \), then according to the definition in Shorack (2017, §7.4),

\[
P(W_C \in H_C \mid W_A \in H_A, W_B \in H_B, W_k > 1) = P(W_C \in H_C).
\]  
(16)

If \( k \in B \) and \( \{ W_B \in H_B \} \cap \{ W_B : W_k > 1 \} \neq \emptyset \), then \( \{ W_B \in H_B \} \cap \{ W_B : W_k > 1 \} = \{ W_B \in H_B^2, W_k > 1 \} \). Besides, similar to the proof of (15), we have

\[
P(W_C \in H_C \mid W_A \in H_A, W_B \in H_B, W_k > 1) = P(W_C \in H_C \mid W_B \in H_B, W_k > 1).
\]  
(17)

Thus, from (16) and (17), we have for any \( k \in B \), \( W_A \perp W_C \mid W_B, W_k > 1 \).
In summary,

\[
\forall k \in \{1, \ldots, d\} : W_A \perp W_C \mid W_B, W_k > 1.
\]  
(18)

From (18), obviously, we have

\[
P(W_C \in H_C \mid W_A \in H_A, W_B \in H_B^2) = P(W_C \in H_C \mid W_B \in H_B^2).
\]  
(19)

We then show that for any \( H_A^1, H_B^1, H_C^1, H_A^2, H_C^2 \), we have

\[
\frac{P(W_A \in H_A^1, W_C \in H_C^2 \mid W_B \in H_B^1)}{P(W_A \in H_A^1, W_C \in H_C^2 \mid W_B \in H_B^1)} = \frac{P(W_A \in H_A^1, W_C \in H_C^1 \mid W_B \in H_B^1)}{P(W_A \in H_A^1, W_C \in H_C^1 \mid W_B \in H_B^1)}
\]  
(20)

To prove (20), note that

\[
\begin{align*}
P(W_A \in H_A^1, W_C \in H_C^2 \mid W_B \in H_B^1) & = \frac{P(W_A \in H_A^1, W_C \in H_C^2, W_B \in H_B^1)}{P(W_A \in H_A^1, W_C \in H_C^2, W_B \in H_B^1)} P(W_A \in H_A^1, W_C \in H_C^2 \mid W_B \in H_B) \\
& = \frac{P(W_A \in H_A^1 \mid W_C \in H_C^2, W_B \in H_B^1)}{P(W_A \in H_A^1 \mid W_C \in H_C^2, W_B \in H_B^1)} P(W_A \in H_A^1, W_C \in H_C^2, W_B \in H_B^1) \\
& = \frac{P(W_A \in H_A^1 \mid W_C \in H_C^2, W_B \in H_B^1)}{P(W_A \in H_A^1 \mid W_C \in H_C^2, W_B \in H_B)} P(W_A \in H_A^1, W_C \in H_C^2, W_B \in H_B) \\
& = \frac{P(W_A \in H_A^1, W_C \in H_C^2 \mid W_B \in H_B^1) P(W_A \in H_A^1, W_C \in L_C \mid W_B \in H_B)}{P(W_A \in H_A^1, W_C \in H_C^2, W_B \in H_B^1)} \\
& = \frac{P(W_A \in H_A^1, W_C \in L_C \mid W_B \in H_B)}{P(W_A \in H_A^1, W_C \in H_C^2, W_B \in H_B^1)} \\& \text{due to (18)}
\end{align*}
\]

We then show that

\[
P(W_C \in L_C^2 \mid W_A \in L_A^2, W_B \in H_B^1) = P(W_C \in L_C^2 \mid W_A \in L_A^1, W_B \in H_B^1).
\]  
(21)

To prove (21), note that due to (20),

\[
\begin{align*}
P(W_A \in L_A^1, W_C \in L_C^2 \mid W_B \in H_B^1) & = \frac{P(W_A \in L_A^1, W_C \in L_C^2, W_B \in H_B^1)}{P(W_A \in L_A^1, W_C \in L_C^2, W_B \in H_B^1)} P(W_A \in L_A^1, W_C \in L_C^2 \mid W_B \in H_B) \\
& = \frac{P(W_A \in L_A^1, W_C \in L_C^2, W_B \in H_B^1)}{P(W_A \in L_A^1, W_C \in L_C^2, W_B \in H_B)} P(W_A \in L_A^1, W_C \in L_C^2, W_B \in H_B^1) \\
& = \frac{P(W_A \in L_A^1, W_C \in L_C^2 \mid W_B \in H_B^1) P(W_A \in L_A^1, W_C \in L_C^2, W_B \in H_B)}{P(W_A \in L_A^1, W_C \in L_C^2, W_B \in H_B^1) + P(W_A \in L_A^1, W_C \in L_C^2, W_B \in H_B)} \\
& \text{LHS of (21)}
\end{align*}
\]
Similarly we have

\[ P(W_C \in L^1_C \mid W_A \in L^2_A, W_B \in H^1_B) = P(W_C \in L^1_C \mid W_A \in L^1_A, W_B \in H^1_B) . \]

We then show that

\[ P(W_C \in H^2_C \mid W_A \in L^2_A, W_B \in H^1_B) = P(W_C \in H^2_C \mid W_A \in L^1_A, W_B \in H^1_B) . \]

To see this, due to (20),

\[ \frac{P(W_A \in L^1_A, W_C \in H^1_C \mid W_B \in H^1_B)}{P(W_A \in L^1_A, W_C \in H^1_C \mid W_B \in H^1_B)} = \frac{P(W_A \in L^1_A, W_C \in L^2_C \mid W_B \in H^1_B)}{P(W_A \in L^1_A, W_C \in L^2_C \mid W_B \in H^1_B)} . \]

Furthermore, due to (21),

\[ \frac{P(W_C \in H^1_C \mid W_A \in L^1_A, W_B \in H^1_B)}{P(W_C \in H^1_C \mid W_A \in L^2_A, W_B \in H^1_B)} = \frac{P(W_C \in L^2_C \mid W_A \in L^1_A, W_B \in H^1_B)}{P(W_C \in L^2_C \mid W_A \in L^2_A, W_B \in H^1_B)} = 1 . \]

Due to (18) and (20), it follows directly that

\[ P(W_C \in H^1_C \mid W_A \in L^1_A, W_B \in H^1_B) = P(W_C \in H^2_C \mid W_A \in L^2_A, W_B \in H^1_B) = P(W_C \in H^2_C \mid W_A \in H^2_A, W_B \in H^1_B) = P(W_C \in H^2_C \mid W_B \in H^1_B) . \] (22)

Similarly we have

\[ P(W_C \in H^1_C \mid W_A \in L^1_A, W_B \in H^1_B) = P(W_C \in H^1_C \mid W_A \in H^2_A, W_B \in H^1_B) = P(W_C \in H^1_C \mid W_B \in H^1_B) . \] (23)

We still need to show

\[ P(W_C \in H^2_C \mid W_A \in H^1_A, W_B \in H^1_B) = P(W_C \in H^2_C \mid W_A \in L^1_A, W_B \in H^1_B) , \] (24)

\[ P(W_C \in H^1_C \mid W_A \in H^1_A, W_B \in H^1_B) = P(W_C \in H^1_C \mid W_A \in L^1_A, W_B \in H^1_B) . \] (25)

To show (24), note that due to (18) and (20),

\[ \frac{P(W_C \in H^2_C \mid W_A \in H^1_A, W_B \in H^1_B)}{P(W_A \in H^1_A, W_B \in H^1_B)} = \frac{\lambda P(Y_C \in H^2_C, Y_C \in L^2_C, Y_A \in H^1_A, Y_B \in H^1_B)}{P(W_A \in H^1_A, W_B \in H^1_B, W_C \in L^2_C) + P(W_A \in H^1_A, W_B \in H^1_B, W_C \in L^2_C)} \]
\[= \frac{\lambda P(Y_C \in H_2^C \mid Y_B \in H_1^B, Y_C \in L_2^C) P(Y_A \in H_1^A, Y_B \in H_1^B, Y_C \in L_2^C)}{P(W_A \in L_2^A, W_B \in H_1^B, W_C \in L_2^C) + P(W_A \in H_1^A, W_B \in H_1^B, W_C \in L_2^C)}
\]

\[= \frac{P(Y_C \in H_2^C \mid Y_B \in H_1^B, Y_C \in L_2^C) P(W_A \in L_2^A, W_B \in H_1^B, W_C \in L_2^C)}{P(W_A \in L_2^A, W_B \in H_1^B, W_C \in L_2^C) + P(W_A \in H_1^A, W_B \in H_1^B, W_C \in L_2^C)} + 1\]

does not depend on \(H_1^A\).

To show (25), note that due to (13) and (20),

\[P(W_C \in H_2^C \mid W_A \in H_1^A, W_B \in H_1^B) = \frac{P(W_A \in H_1^A, W_B \in H_1^B, W_C \in H_1^C)}{P(W_A \in H_1^A, W_B \in H_1^B, W_C \in L_2^C) + P(W_A \in H_1^A, W_B \in H_1^B, W_C \in H_1^C)}\]

\[= \frac{P(W_A \in H_1^A, W_B \in H_1^B, W_C \in L_2^C) P(W_A \in L_2^A, W_B \in H_1^B, W_C \in L_2^C)}{P(W_A \in L_2^A, W_B \in H_1^B, W_C \in L_2^C) + P(W_A \in H_1^A, W_B \in H_1^B, W_C \in L_2^C)} + 1\]

does not depend on \(H_1^A\).

From (22), (23), (24) and (25), we have

\[P(W_C \in H_2 \mid W_A \in H_2, W_B \in H_2) = P(W_C \in H_2 \mid W_B \in H_2).\]

Besides, because of (19), we can obtain that

\[P(W_C \in H_2 \mid W_A \in H_2, W_B \in H_2) = P(W_C \in H_2 \mid W_B \in H_2).\]

We have hence completed the proof.

2.3 Proof of Proposition 2 in the case where \(B = \emptyset\)

Proof. Use the same notation as before, denote \([0, 1]^{[k]}\) as \(L_k^1\), \([0, \infty)^{[k]} \setminus [0, 1]^{[k]}\) as \(L_k^2\) for \(k \in \{A, C\}\).

Besides, denote \([0]^{[k]}\) as \(O_k\).

We need to show

the support of \(Y = (Y_A, Y_C)\) is \(\Omega = (L_2^2 \times 0_C) \cup (0_A \times L_2^C) \Rightarrow Y_{A\perp o} Y_C\),

(26)

the support of \(Y = (Y_A, Y_C)\) is \(\Omega = (L_2^2 \times 0_C) \cup (0_A \times L_2^C) \Rightarrow Y_{A\perp o} Y_C\),

(27)

the support of \(Y = (Y_A, Y_C)\) is \(\Omega = (L_2^2 \times 0_C) \cup (0_A \times L_2^C) \Rightarrow Y_{A\perp o} Y_C\).

(28)

Under (26), (27) and (28), we can obtain that if the support of \(Y = (Y_A, Y_C)\) is \(\Omega = (L_2^2 \times 0_C) \cup (0_A \times L_2^C)\), then

\[Y_{A\perp o} Y_C \Leftrightarrow Y_{A\perp o} Y_C \Leftrightarrow Y_{A\perp o} Y_C.\]
First, we show (26). Since the support of $Y = (Y_A, Y_C)$ is $\Omega = (L_A^2 \times 0) \cup (0_A \times L_C^2)$, then $S_A \times S_C \subset (L_A^2 \times 0)$ or $S_A \times S_C \subset (0_A \times L_C^2)$. When $S_A \times S_C \subset (L_A^2 \times 0)$, since $Y_C = 0$ regardless of the value of $Y_A$, then $Y_A, Y_C \mid (Y \in S_A \times S_C)$. When $S_A \times S_C \subset (0_A \times L_C^2)$, since $Y_A = 0_A$ regardless of the value of $Y_C$, then $Y_A, Y_C \mid (Y \in S_A \times S_C)$. Thus, based on the definition of $Y_A, Y_C$, we have $Y_A, Y_C$.

Second, we show (27). Since the support of $Y = (Y_A, Y_C)$ is $\Omega = (L_A^2 \times 0) \cup (0_A \times L_C^2)$, then when $Y_k > 1$ for some $k \in A$, we have $Y \in (L_A^2 \times 0)$. Since $Y_C = 0$ regardless of the value of $Y_A$, then $Y_A, Y_C \mid Y_k > 1$. When $Y_k > 1$ for some $k \in C$, we have $Y \in (0_A \times L_C^2)$. Since $Y_A = 0_A$ regardless of the value of $Y_C$, then $Y_A, Y_C \mid Y_k > 1$. Thus, based on the definition of $Y_A, Y_C$, we have $Y_A, Y_C$.

Third, we show (28). Since the support of $Y = (Y_A, Y_C)$ is $\Omega = (L_A^2 \times 0) \cup (0_A \times L_C^2)$, then

$$P(Y_A \in L_A^2, Y_C = 0_A \text{ or } Y_A = 0_A, Y_C \in L_C^2) = P(Y_A \in L_A^2, Y_C = 0) + P(Y_A = 0_A, Y_C \in L_C^2) = 1.$$

Suppose $P(Y_A \in L_A^2, Y_C = 0) = \alpha$ $0 < \alpha < 1$, then $P(Y_A = 0_A, Y_C \in L_C^2) = 1 - \alpha$. Now we introduce two new random vectors $W_A$ and $W_C$, assume that they have the distributions as follows:

$$F_{W_A}(w_A) = \begin{cases} (1 - p_A)P(U_A \leq w_A), & w_A \in L_A^2, \\ (1 - p_A) + p_AP(Y_A \leq w_A \mid Y_A \in L_A^2, Y_C = 0), & w_A \in L_A^2, \end{cases}$$

and

$$F_{W_C}(w_C) = \begin{cases} (1 - p_C)P(U_C \leq w_C), & w_C \in L_C^1, \\ (1 - p_C) + p_CP(Y_C \leq w_C \mid Y_A = 0_A, Y_C \in L_C^2), & w_C \in L_C^2, \end{cases}$$

where $p_A \in (0, 1)$, $p_C \in (0, 1)$, $U_A$ is a random vector which follows the uniform distribution in $L_A^2$, and $U_C$ is a random vector which follows the uniform distribution in $L_C^2$.

To make $W_A, W_C$, let $F(w_A, w_C) = F_{W_A}(w_A)F_{W_C}(w_C)$. Thus, if $w_A \in L_A^2, w_C \in L_C^1$,

$$F(w_A, w_C) = (1 - p_A)(1 - p_C)P(U_A \leq w_A)P(U_C \leq w_C).$$

If $w_A \in L_A^1, w_C \in L_C^2$,

$$F(w_A, w_C) = (1 - p_A)P(U_A \leq w_A)\{(1 - p_C) + p_CP(Y_C \leq w_C \mid Y_A = 0_A, Y_C \in L_C^2)\}.$$

If $w_A \in L_A^2, w_C \in L_C^1$,

$$F(w_A, w_C) = \{(1 - p_A) + p_AP(Y_A \leq w_A \mid Y_A \in L_A^2, Y_C = 0)\}(1 - p_C)P(U_C \leq w_C).$$

If $w_A \in L_A^2, w_C \in L_C^2$, then

$$F(w_A, w_C) = \{(1 - p_A) + p_AP(Y_A \leq w_A \mid Y_A \in L_A^2, Y_C = 0)\} \times \{(1 - p_C) + p_CP(Y_C \leq w_C \mid Y_A = 0_A, Y_C \in L_C^2)\}.$$

If the density function of $F(w_A, w_C)$ exists, then

$$f(w_A, w_C) = \begin{cases} (1 - p_A)(1 - p_C), & w_A \in L_A^1, w_C \in L_C^1, \\ \frac{(1 - p_A)PC(Y_A = 0_A, Y_C \leq w_C)}{\partial w_C}, & w_A \in L_A^1, w_C \in L_C^2, \\ \frac{p_A(1 - p_C)P(Y_A \leq w_A, Y_C = 0)}{\partial w_A}, & w_A \in L_A^2, w_C \in L_C^1, \\ \frac{p_A(1 - p_C)P(Y_A \leq w_A, Y_C = 0)}{\partial w_A}, & w_A \in L_A^2, w_C \in L_C^2. \end{cases}$$
Since the support of \( Y = (Y_A, Y_C) \) is \( \Omega = (L_A^2 \times 0_C) \cup (0_A \times L_C^2) \), then \( \partial P(Y_A = 0_A, Y_C \leq w_C) / \partial w_C \neq 0 \) for \( w_C \in L_C^2 \) and \( \partial P(Y_A \leq w_A, Y_C = 0_C) / \partial w_A \neq 0 \) for \( w_A \in L_A^2 \). Thus, from \( f(w_1, w_2) \), we have the support of \( W = (W_A, W_C) \) is \([0, \infty)^{\alpha A + |C|} \), which contains \( \Omega \). Based on the definition of conditional density in Shorack (2017, §7.4), we have that the conditional density at \((w_A, 0_C), (w_A \in L_A^2)\) and \((0_A, w_C), (w_C \in L_C^2)\) given that all the points are on the curve of \( \Omega \) are

\[
\int_{L_A^2} f(w_A, 0_C) dw_A + \int_{L_C^2} f(0_A, w_C) dw_C
\]

\[
= \frac{p_A(1-p_C)}{p_A(1-p_C)} P(Y_A \in L_A^2, Y_C = 0_C) + \frac{(1-p_A)p_C}{1-\alpha} P(Y_A = 0_A, Y_C \in L_C^2)
\]

\[
= \frac{p_A(1-p_C)}{\alpha} \frac{\partial P(Y_A \leq w_A, Y_C = 0_C)}{\partial w_A}
\]

and

\[
\int_{L_A^2} f(w_A, 0_C) dw_A + \int_{L_C^2} f(0_A, w_C) dw_C
\]

\[
= \frac{(1-p_A)p_C}{1-\alpha} \frac{\partial P(Y_A = 0_A, Y_C \leq w_C)}{\partial w_C}
\]

\[
= (1-\alpha) \{ p_A(1-p_C) + (1-p_A)p_C \}.
\]

Then we can get the value of \( P(W_A \leq w_A, W_C \leq w_C \mid W_A \in L_A^2, W_C = 0_C \text{ or } W_A = 0_A, W_C \in L_C^2) \). When \( w_A \in L_A^2, w_C \in L_C^1 \),

\[
P(W_A \leq w_A, W_C \leq w_C \mid W_A \in L_A^2, W_C = 0_C \text{ or } W_A = 0_A, W_C \in L_C^2) = 0 = P(Y_A \leq w_A, Y_C \leq w_C).
\]

When \( w_A \in L_A^1, w_C \in L_C^2 \),

\[
P(W_A \leq w_A, W_C \leq w_C \mid W_A \in L_A^2, W_C = 0_C \text{ or } W_A = 0_A, W_C \in L_C^2)
\]

\[
= \frac{\int_{[0,1]} f(0_A, t) dt}{\int_{L_A^2} f(w_A, 0_C) dw_A + \int_{L_C^2} f(0_A, w_C) dw_C}
\]

\[
= \frac{\int_{L_A^2} f(w_A, 0_C) dw_A + \int_{L_C^2} f(0_A, w_C) dw_C}{(1-p_A)p_C P(Y_A = 0_A, Y_C \leq w_C)}
\]

\[
= \frac{(1-\alpha) \{ p_A(1-p_C) + (1-p_A)p_C \}}{(1-\alpha) \{ p_A(1-p_C) + (1-p_A)p_C \}},
\]

where the last equation is due to the support of \( Y = (Y_A, Y_C) \) being \( \Omega = (L_A^2 \times 0_C) \cup (0_A \times L_C^2) \). When \( w_A \in L_A^2, w_C \in L_C^1 \),

\[
P(W_A \leq w_A, W_C \leq w_C \mid W_A \in L_A^2, W_C = 0_C \text{ or } W_A = 0_A, W_C \in L_C^2)
\]

\[
= \frac{- \int_{[0,1]} f(t, 0_C) dt}{\int_{L_A^2} f(w_A, 0_C) dw_A + \int_{L_C^2} f(0_A, w_C) dw_C}
\]

\[
= \frac{p_A(1-p_C) P(Y_A \leq w_A, Y_C = 0_C)}{(1-\alpha) \{ p_A(1-p_C) + (1-p_A)p_C \}}
\]

\[
= \frac{p_A(1-p_C) P(Y_A \leq w_A, Y_C \leq w_C)}{(1-\alpha) \{ p_A(1-p_C) + (1-p_A)p_C \}}.
\]

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When \( w_A \in L_A^2, w_C \in L_C^2 \), due to \( P(Y_A \leq w_A, Y_C \leq w_C) = P(Y_A \leq w_A, Y_C = 0_C) + P(Y_A = 0_A, Y_C \leq w_C) \),

\[
P(W_A \leq w_A, W_C \leq w_C \mid W_A \in L_A^2, W_C = 0_C \text{ or } W_A = 0_A, W_C \in L_C^2) = 
\frac{\int_{[0, w_A] \setminus [0, 1]} f(t, 0)dt + \int_{[0, 1]} f(0, t)dt}{\int_{[0, w_A] \setminus [0, 1]} f(t, 0)dt + \int_{[0, 1]} f(0, t)dt}.
\]

In summary,

\[
P(W_A \leq w_A, W_C \leq w_C \mid W_A \in L_A^2, W_C = 0_C \text{ or } W_A = 0_A, W_C \in L_C^2) =
\begin{cases}
0, & \text{if } w_A \in L_A^1, w_C \in L_C^1, \\
\frac{(1-p_A)p_C P(Y_A \leq w_A, Y_C = 0_C)}{(1-\alpha)(p_A(1-p_C) + (1-p_A)p_C)} & \text{if } w_A \in L_A^2, w_C \in L_C^2, \\
\frac{\alpha P_A(1-p_C)/(1-\alpha)(p_A(1-p_C) + (1-p_A)p_C)}{(1-\alpha)(p_A(1-p_C) + (1-p_A)p_C)} & \text{if } w_A \in L_A^2, w_C \in L_C^1, \\
\frac{(1-\alpha)P_A(1-p_C)P(Y_A \leq w_A, Y_C = 0_C)}{(1-\alpha)(p_A(1-p_C) + (1-p_A)p_C)} + \frac{(1-p_A)p_C P(Y_A \leq w_A, Y_C \leq w_C)}{(1-\alpha)(p_A(1-p_C) + (1-p_A)p_C)} & \text{if } w_A \in L_A^2, w_C \in L_C^2.
\end{cases}
\]

From (29), \( P(W_A \leq w_A, W_C \leq w_C \mid W_A \in L_A^2, W_C = 0_C \text{ or } W_A = 0_A, W_C \in L_C^2) = P(Y_A \leq w_A, Y_C \leq w_C) \), if and only if

\[
\begin{cases}
(1-p_A)p_C \in (1-\alpha)(p_A(1-p_C) + (1-p_A)p_C) \quad & 1, \\
\alpha P_A(1-p_C)/(1-\alpha)(p_A(1-p_C) + (1-p_A)p_C) = 1, \\
(1-\alpha)P_A(1-p_C)P(Y_A \leq w_A, Y_C = 0_C) + (1-p_A)p_C P(Y_A \leq w_A, Y_C \leq w_C) = 0.
\end{cases}
\]

or equivalently,

\[
\alpha = \frac{p_A(1-p_C)}{p_A(1-p_C) + (1-p_A)p_C}.
\]

We have hence shown that under condition (30), \( \mathbf{W} \mid \mathbf{W} \in \Omega \) \( \overset{d}{=} \mathbf{Y} \) if the density function of \( F(w_A, w_C) \) exists.

If the density function of \( F(w_A, w_C) \) does not exist, from the definition of conditional probability in Shorack (2017, §7.4), we know that (29) still satisfies the condition for conditional probability. Thus, we can define \( P(W_A \leq w_A, W_C \leq w_C \mid W_A \in L_A^2, W_C = 0_C \text{ or } W_A = 0_A, W_C \in L_C^2) \) as that in (29). Under condition (30), we also have \( P(W_A \leq w_A, W_C \leq w_C \mid W_A \in L_A^2, W_C = 0_C \text{ or } W_A = 0_A, W_C \in L_C^2) = P(Y_A \leq w_A, Y_C \leq w_C) \). Then \( \mathbf{W} \mid \mathbf{W} \in \Omega \) \( \overset{d}{=} \mathbf{Y} \).

In summary, based on the definition of \( Y_A, Y_C \), we have \( Y_A \perp \! \! \! \perp Y_C \).

\( \square \)

### 2.4 Proof of Proposition 2 for a general \( B \)

**Proof.** We need to show

the support of \( \mathbf{Y} = (Y_A, Y_B, Y_C) \) is \( \Omega = (L_A^2 \times 0_B \times 0_C) \cup (0_A \times L_B^2 \times 0_C) \cup (0_A \times 0_B \times L_C^2) \Rightarrow Y_A, Y_C \).
the support of $\mathbf{Y} = (\mathbf{Y}_A, \mathbf{Y}_B, \mathbf{Y}_C)$ is $\Omega = (L_2^A \times 0_B \times 0_C) \cup (0_A \times L_2^B \times 0_C) \cup (0_A \times 0_B \times L_2^C) \Rightarrow \mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid \mathbf{Y}_B$, \hspace{1cm} (32)

the support of $\mathbf{Y} = (\mathbf{Y}_A, \mathbf{Y}_B, \mathbf{Y}_C)$ is $\Omega = (L_2^A \times 0_B \times 0_C) \cup (0_A \times L_2^B \times 0_C) \cup (0_A \times 0_B \times L_2^C) \Rightarrow \mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid \mathbf{Y}_B$. \hspace{1cm} (33)

Under (31), (32) and (33), we can have if the support of $\mathbf{Y} = (\mathbf{Y}_A, \mathbf{Y}_B, \mathbf{Y}_C)$ is $\Omega = (L_2^A \times 0_B \times 0_C) \cup (0_A \times L_2^B \times 0_C) \cup (0_A \times 0_B \times L_2^C)$, then

$$Y_A \sqsubseteq \mathbf{Y}_C \mid \mathbf{Y}_B \iff Y_A \sqsubseteq \mathbf{Y}_C \mid \mathbf{Y}_B \iff Y_A \sqsubseteq \mathbf{Y}_C \mid \mathbf{Y}_B.$$  

First, we show (31). Since the support of $\mathbf{Y} = (\mathbf{Y}_A, \mathbf{Y}_B, \mathbf{Y}_C)$ is $\Omega = (L_2^A \times 0_B \times 0_C) \cup (0_A \times L_2^B \times 0_C) \cup (0_A \times 0_B \times L_2^C)$, then $S_A \times S_B \times S_C \subseteq (L_2^A \times 0_B \times 0_C), (0_A \times L_2^B \times 0_C) \text{ or } (0_A \times 0_B \times L_2^C)$. When $S_A \times S_B \times S_C \subseteq (L_2^A \times 0_B \times 0_C), \mathbf{Y}_B = 0_B$, then $\mathbf{Y}_C = 0_C$ regardless of the value of $\mathbf{Y}_A$, then $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid (\mathbf{Y}_B, \mathbf{Y} \in S_A \times S_B \times S_C)$. When $S_A \times S_B \times S_C \subseteq (0_A \times L_2^B \times 0_C), \mathbf{Y}_B \in L_2^B$. Given $\mathbf{Y}_B = \mathbf{y}_B$, since $\mathbf{Y}_A = 0_A$ and $\mathbf{Y}_C = 0_C$, then $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid (\mathbf{Y}_B, \mathbf{y}_B \in S_A \times S_B \times S_C)$. When $S_A \times S_B \times S_C \subseteq (0_A \times 0_B \times L_2^C), \mathbf{Y}_B = 0_B$, then $\mathbf{Y}_C = 0_C$ regardless of the value of $\mathbf{Y}_C$, then $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid (\mathbf{Y}_B, \mathbf{y}_B \in S_A \times S_B \times S_C)$. Thus, based on the definition of $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid \mathbf{Y}_B$, we have $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid \mathbf{Y}_B$.

Second, we show (32). Since the support of $\mathbf{Y} = (\mathbf{Y}_A, \mathbf{Y}_B, \mathbf{Y}_C)$ is $\Omega = (L_2^A \times 0_B \times 0_C) \cup (0_A \times L_2^B \times 0_C) \cup (0_A \times 0_B \times L_2^C)$, then when $y_k = 1$ for some $k \in A$, we have $\mathbf{Y} \in (L_2^A \times 0_B \times 0_C)$ and $\mathbf{Y}_B = 0_B$. Given $\mathbf{Y}_B = 0_B$ and $y_k > 1$, then $\mathbf{Y}_C = 0_C$ regardless of the value of $\mathbf{Y}_A$, then $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid (\mathbf{Y}_B, y_k > 1)$. When $y_k > 1$ for some $k \in B$, we have $\mathbf{Y} \in (0_A \times L_2^B \times 0_C)$ and $\mathbf{Y}_B \in L_2^B$. Given $\mathbf{Y}_B = \mathbf{y}_B$ and $y_k > 1$, since $\mathbf{Y}_A = 0_A$ and $\mathbf{Y}_C = 0_C$, then $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid (\mathbf{Y}_B, y_k > 1)$. When $y_k > 1$ for some $k \in C$, we have $\mathbf{Y} \in (0_A \times 0_B \times L_2^C)$ and $\mathbf{Y}_B = 0_B$. Given $\mathbf{Y}_B = 0_B$ and $y_k > 1$, since $\mathbf{Y}_A = 0_A$ regardless of the value of $\mathbf{Y}_C$, then $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid (\mathbf{Y}_B, y_k > 1)$. Thus, based on the definition of $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid \mathbf{Y}_B$, we have $\mathbf{Y}_A \sqsubseteq \mathbf{Y}_C \mid \mathbf{Y}_B$.

Third, we show (33). Since the support of $\mathbf{Y} = (\mathbf{Y}_A, \mathbf{Y}_B, \mathbf{Y}_C)$ is $\Omega = (L_2^A \times 0_B \times 0_C) \cup (0_A \times L_2^B \times 0_C) \cup (0_A \times 0_B \times L_2^C)$, then

$$P(Y_A \in L_2^A, \mathbf{Y}_B = 0_B, \mathbf{Y}_C = 0_C \text{ or } Y_A = 0_A, \mathbf{Y}_B \in L_2^B, \mathbf{Y}_C = 0_C \text{ or } Y_A = 0_A, \mathbf{Y}_B = 0_B, \mathbf{Y}_C \in L_2^C) = P(Y_A \in L_2^A, \mathbf{Y}_B = 0_B, \mathbf{Y}_C = 0_C) + P(Y_A = 0_A, \mathbf{Y}_B \in L_2^B, \mathbf{Y}_C = 0_C)$$

$$+ P(Y_A = 0_A, \mathbf{Y}_B = 0_B, \mathbf{Y}_C \in L_2^C) = 1.$$  

Suppose that $P(Y_A \in L_2^A, \mathbf{Y}_B = 0_B, \mathbf{Y}_C = 0_C) = \alpha_1$ and $P(Y_A = 0_A, \mathbf{Y}_B \in L_2^B, \mathbf{Y}_C = 0_C) = \alpha_2$, then $P(Y_A = 0_A, \mathbf{Y}_B = 0_B, \mathbf{Y}_C \in L_2^C) = 1 - \alpha_1 - \alpha_2$, where $\alpha_1$ and $\alpha_2$ satisfy the conditions of $\alpha_1 > 0, \alpha_2 > 0$ and $0 < \alpha_1 + \alpha_2 < 1$. Now we introduce three new random vectors $\mathbf{W}_A, \mathbf{W}_B$ and $\mathbf{W}_C$, assume that they have the distributions as follows:

$$F_{\mathbf{W}_A}(\mathbf{w}_A) = \begin{cases} 
(1 - p_A)P(U_A \leq \mathbf{w}_A), & \mathbf{w}_A \in L_2^A, \\
(1 - p_A) + p_AP(Y_A \leq \mathbf{w}_A) \mid Y_A \in L_2^A, \mathbf{Y}_B = 0_B, \mathbf{Y}_C = 0_C), & \mathbf{w}_A \in L_2^A,
\end{cases}$$

$$F_{\mathbf{W}_B}(\mathbf{w}_B) = \begin{cases} 
(1 - p_B)P(U_B \leq \mathbf{w}_B), & \mathbf{w}_B \in L_2^B, \\
(1 - p_B) + p_BP(Y_B \leq \mathbf{w}_B) \mid Y_A = 0_A, \mathbf{Y}_B \in L_2^B, \mathbf{Y}_C = 0_C), & \mathbf{w}_B \in L_2^B,
\end{cases}$$

and

$$F_{\mathbf{W}_C}(\mathbf{w}_C) = \begin{cases} 
(1 - p_C)P(U_C \leq \mathbf{w}_C), & \mathbf{w}_C \in L_2^C, \\
(1 - p_C) + p_CP(Y_C \leq \mathbf{w}_C) \mid Y_A = 0_A, \mathbf{Y}_B = 0_B, \mathbf{Y}_C \in L_2^C), & \mathbf{w}_C \in L_2^C,
\end{cases}$$
where \( p_A \in (0, 1), p_B \in (0, 1) \), \( p_C \in (0, 1) \), and \( U_A, U_B, U_C \) are random vectors which follow the uniform distributions in \( L_A^1, L_B^1 \) and \( L_C^1 \), respectively.

To make \( W_A, W_B, W_C \) independent, let \( F(w_A, w_B, w_C) = F_{W_A}(w_A)F_{W_B}(w_B)F_{W_C}(w_C) \). Furthermore, we have \( W_A \perp W_C \mid W_B \). Besides, if \( w_A \in L_A^1, w_B \in L_B^1, w_C \in L_C^1 \),

\[
F(w_A, w_B, w_C) = (1 - p_A)(1 - p_B)(1 - p_C)P(U_A \leq w_A)P(U_B \leq w_B)P(U_C \leq w_C).
\]

If \( w_A \in L_A^1, w_B \in L_B^1, w_C \in L_C^1 \),

\[
F(w_A, w_B, w_C) = (1 - p_A)(1 - p_B)P(U_A \leq w_A)P(U_B \leq w_B) \times \{(1 - p_C) + p_CP(Y_C \leq w_C \mid Y_A = 0_A, Y_B = 0_B, Y_C \in L_C^2)\}.
\]

If \( w_A \in L_A^1, w_B \in L_B^2, w_C \in L_C^1 \),

\[
F(w_A, w_B, w_C) = (1 - p_A)(1 - p_C)P(U_A \leq w_A)P(U_C \leq w_C) \times \{(1 - p_B) + p_BP(Y_B \leq w_B \mid Y_A = 0_A, Y_B \in L_B^2, Y_C = 0_C)\}.
\]

If \( w_A \in L_A^1, w_B \in L_B^2, w_C \in L_C^1 \),

\[
F(w_A, w_B, w_C) = (1 - p_A)P(U_A \leq w_A)\{(1 - p_B) + p_BP(Y_B \leq w_B \mid Y_A = 0_A, Y_B \in L_B^2, Y_C = 0_C)\} \times \{(1 - p_C) + p_CP(Y_C \leq w_C \mid Y_A = 0_A, Y_B = 0_B, Y_C \in L_C^2)\}.
\]

If \( w_A \in L_A^2, w_B \in L_B^1, w_C \in L_C^1 \),

\[
F(w_A, w_B, w_C) = (1 - p_B)P(U_B \leq w_B)\{(1 - p_A) + p_AP(Y_A \leq w_A \mid Y_A \in L_A^2, Y_B = 0_B, Y_C = 0_C)\} \times \{(1 - p_C) + p_CP(Y_C \leq w_C \mid Y_A = 0_A, Y_B = 0_B, Y_C \in L_C^2)\}.
\]

If \( w_A \in L_A^2, w_B \in L_B^2, w_C \in L_C^1 \),

\[
F(w_A, w_B, w_C) = (1 - p_C)P(U_C \leq w_C)\{(1 - p_A) + p_AP(Y_A \leq w_A \mid Y_A \in L_A^2, Y_B = 0_B, Y_C = 0_C)\} \times \{(1 - p_B) + p_BP(Y_B \leq w_B \mid Y_A = 0_A, Y_B \in L_B^2, Y_C = 0_C)\}.
\]

If \( w_A \in L_A^2, w_B \in L_B^2, w_C \in L_C^2 \),

\[
F(w_A, w_B, w_C) = \{(1 - p_A) + p_AP(Y_A \leq w_A \mid Y_A \in L_A^2, Y_B = 0_B, Y_C = 0_C)\} \times \{(1 - p_B) + p_BP(Y_B \leq w_B \mid Y_A = 0_A, Y_B \in L_B^2, Y_C = 0_C)\} \times \{(1 - p_C) + p_CP(Y_C \leq w_C \mid Y_A = 0_A, Y_B = 0_B, Y_C \in L_C^2)\}.
\]
If the density function of \( F(w_A, w_B, w_C) \) exists, then by calculation
\[
f(w_A, w_B, w_C) = \frac{(1 - p_A)(1 - p_B)(1 - pc)}{p_A(1 - p_B)(1 - pc)} \frac{\partial P(Y_A = 0_A, Y_B = 0_B, Y_C = w_C)}{\partial w_C}.
\]
Since the support of \( Y = (Y_A, Y_B, Y_C) \) is \( \Omega = (L_A^2 \times 0_B \times 0_C) \cup (0_A \times L_B^2 \times 0_C) \cup (0_A \times 0_B \times L_C^2) \), then \( \partial P(Y_A = 0_A, Y_B = 0_B, Y_C = w_C) / \partial w_C \neq 0 \) for \( w_C \in L_C^2 \), \( \partial P(Y_A = 0_A, Y_B = 0_B, Y_C = 0_C) / \partial w_B \neq 0 \) for \( w_B \in L_B^2 \) and \( \partial P(Y_A = 0_A, Y_B = 0_B, Y_C = 0_C) / \partial w_A \neq 0 \) for \( w_A \in L_A^2 \). Thus, from \( f(w_A, w_B, w_C) \), we can obtain that the support of \( W = (w_A, w_B, w_C) \) is \( [0, \infty)^{|A|+|B|+|C|} \), which contains \( \Omega \). Based on the definition of conditional density in Shorack (2017, §7.4), if \( w_A \in L_A^2, w_B \in L_B^2 \) and \( w_C \in L_C^2 \), then we have that the conditional densities at \( (w_A, 0_B, 0_C) \), \( (0_A, w_B, 0_C) \) and \( (0_A, 0_B, w_C) \), given that all the points are on the curve of \( \Omega \) are
\[
f(w_A, 0_B, 0_C) = \int_{L_A^2} f(w_A, 0_B, 0_C) dw_A + \int_{L_B^2} f(0_A, w_B, 0_C) dw_B + \int_{L_C^2} f(0_A, 0_B, w_C) dw_C
\]
\[
= \frac{p_A(1 - p_B)(1 - pc)}{\alpha_1(p_A(1 - p_B)(1 - pc) + (1 - p_A)p_B(1 - pc) + (1 - p_A)(1 - p_B)pc)},
\]
and
\[
f(0_A, w_B, 0_C) = \int_{L_A^2} f(0_A, w_B, 0_C) dw_A + \int_{L_B^2} f(0_A, 0_B, w_C) dw_B + \int_{L_C^2} f(0_A, 0_B, w_C) dw_C
\]
\[
= \frac{(1 - p_A)p_B(1 - pc)}{\alpha_2(p_A(1 - p_B)(1 - pc) + (1 - p_A)p_B(1 - pc) + (1 - p_A)(1 - p_B)pc)},
\]
and
\[
f(0_A, 0_B, w_C) = \int_{L_A^2} f(0_A, 0_B, w_C) dw_A + \int_{L_B^2} f(0_A, 0_B, w_C) dw_B + \int_{L_C^2} f(0_A, 0_B, w_C) dw_C
\]
\[
= \frac{(1 - p_A)(1 - p_B)(1 - pc)}{\alpha_1(1 - \alpha_2)(1 - p_A)(1 - p_B)(1 - pc) + (1 - p_A)p_B(1 - pc) + (1 - p_A)(1 - p_B)pc)}.
\]
Then we can get the value of \( P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W_A \in L_A^2, W_B = 0_B, W_C = 0_C \) or \( W_A = 0_A, W_B \in L_B^2, W_C = 0_C \) or \( W_A = 0_A, W_B = 0_B, W_C \in L_C^2 \) = \( P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega) \). When \( w_A \in L_A^2, w_B \in L_B^2, w_C \in L_C^2 \),
\[
P(w_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega) = 0 = P(Y_A \leq w_A, Y_B \leq w_B, Y_C \leq w_C).
\]
When \( w_A \in L_A^1, w_B \in L_B^1, w_C \in L_C^2 \),

\[
P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega) = \int_{0 \leq t \leq 1} f(0_A, 0_B, t) dt
\]

\[
= \int_{L_A^1} f(w_A, 0_B, 0_C) dw_A + \int_{L_B^1} f(0_A, w_B, 0_C) dw_B + \int_{L_C^2} f(0_A, 0_B, w_C) dw_C
\]

\[
= \frac{1 - p_A)(1 - p_B)P(\{ Y_A = 0_A, Y_B = 0_B, Y_C = w_C \cap (1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}
\]

\[
= \frac{(1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}}{(1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}}.
\]

When \( w_A \in L_A^1, w_B \in L_B^1, w_C \in L_C^2 \),

\[
P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega) = \int_{0 \leq t \leq 1} f(0_A, t, 0_C) dt
\]

\[
= \int_{L_A^1} f(w_A, 0_B, 0_C) dw_A + \int_{L_B^1} f(0_A, w_B, 0_C) dw_B + \int_{L_C^2} f(0_A, 0_B, w_C) dw_C
\]

\[
= \frac{1 - p_A)(1 - p_B)P(\{ Y_A = 0_A, Y_B = 0_B, Y_C = 0_C \cap (1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}
\]

\[
= \frac{(1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}}{(1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}}.
\]

When \( w_A \in L_A^1, w_B \in L_B^1, w_C \in L_C^1 \),

\[
P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega) = \int_{0 \leq t \leq 1} f(t, 0_B, 0_C) dt
\]

\[
= \int_{L_A^1} f(w_A, 0_B, 0_C) dw_A + \int_{L_B^1} f(0_A, w_B, 0_C) dw_B + \int_{L_C^2} f(0_A, 0_B, w_C) dw_C
\]

\[
= \frac{p_A(1 - p_B)(1 - p_C)P(\{ Y_A = w_A, Y_B = 0_B, Y_C = 0_C \cap (1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}
\]

\[
= \frac{p_A(1 - p_B)(1 - p_C)P(\{ Y_A = w_A, Y_B = 0_B, Y_C = 0_C \cap (1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}}{(1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}}.
\]

When \( w_A \in L_A^2, w_B \in L_B^1, w_C \in L_C^2 \), due to \( P(Y_A \leq w_A, Y_B \leq w_B, Y_C \leq w_C) = P(Y_A = 0_A, Y_B = w_B, Y_C = 0_C) + P(Y_A = 0_A, Y_B = 0_B, Y_C = w_C) \),

\[
P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega) = \int_{0 \leq t \leq 1} f(t, 0_B, 0_C) dt
\]

\[
= \int_{L_A^2} f(w_A, 0_B, 0_C) dw_A + \int_{L_B^2} f(0_A, w_B, 0_C) dw_B + \int_{L_C^2} f(0_A, 0_B, w_C) dw_C
\]

\[
= \frac{p_A(1 - p_B)(1 - p_C)P(\{ Y_A = w_A, Y_B = 0_B, Y_C = 0_C \cap (1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}
\]

\[
= \frac{p_A(1 - p_B)(1 - p_C)P(\{ Y_A = w_A, Y_B = 0_B, Y_C = 0_C \cap (1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}}{(1 - \alpha_1 - \alpha_2)\{p_A(1 - p_B)(1 - p_C) + (1 - p_A)p_B(1 - p_C) + (1 - p_A)(1 - p_B)p_C\}}.
\]

When \( w_A \in L_A^2, w_B \in L_B^1, w_C \in L_C^2 \), due to \( P(Y_A \leq w_A, Y_B \leq w_B, Y_C \leq w_C) = P(Y_A = 0_A, Y_B = w_B, Y_C = 0_C) + P(Y_A = 0_A, Y_B = 0_B, Y_C = w_C) \),
\[w_A, Y_B = 0_B, Y_C = 0_C + P(Y_A = 0_A, Y_B = 0_B, Y_C = 0_C)\]

\[P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega)\]

\[
\int_{0_A,0_B,0_C} f(t,0_B,0_C)dt + \int_{0_C,0_B,0_C} f(0_A,0_B,t)dt
\]

\[
\int_{L^2} f(w_A,0_B,0_C)dw_A + \int_{L^2} f(0_A,w_B,0_C)dw_B + \int_{L^2} f(0_A,0_B,w_C)dw_C
\]

\[
\{p_A(1-p_B)(1-p_C)(1-\alpha_1 - \alpha_2) - (1-p_A)(1-p_B)p_C\alpha_1\}P(Y_A \leq w_A, Y_B = 0_B, Y_C = 0_C)
\]

\[
\alpha_1(1-\alpha_1-\alpha_2)\{p_A(1-p_B)(1-p_C) + (1-p_A)p_B(1-p_C) + (1-p_A)(1-p_B)p_C\}
\]

\[
(1-p_A)(1-p_B)p_CP(Y_A \leq w_A, Y_B \leq w_B, Y_C \leq w_C)
\]

\[
\frac{\alpha_1\alpha_2\{p_A(1-p_B)(1-p_C) + (1-p_A)p_B(1-p_C) + (1-p_A)(1-p_B)p_C\}}{(1-p_A)p_B(1-p_C) + (1-p_A)p_B(1-p_C) + (1-p_A)(1-p_B)p_C}.
\]

When \(w_A \in L^2, w_B \in L^2, w_C \in L^2\), due to \(P(Y_A \leq w_A, Y_B \leq w_B, Y_C \leq w_C) = P(Y_A \leq w_A, Y_B = 0_B, Y_C = 0_C) + P(Y_A = 0_A, Y_B \leq w_B, Y_C = 0_C) + P(Y_A = 0_A, Y_B = 0_B, Y_C \leq w_C)\),

\[P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega)\]

\[
\int_{0_A,0_B,0_C} f(t,0_B,0_C)dt + \int_{0_B,0_B,0_C} f(0_A,t,0_C)dt
\]

\[
\int_{L^2} f(w_A,0_B,0_C)dw_A + \int_{L^2} f(0_A,w_B,0_C)dw_B + \int_{L^2} f(0_A,0_B,w_C)dw_C
\]

\[
\{p_A(1-p_B)(1-p_C)(1-\alpha_1 - \alpha_2) - (1-p_A)(1-p_B)p_C\alpha_1\}P(Y_A \leq w_A, Y_B = 0_B, Y_C = 0_C)
\]

\[
\frac{\alpha_1(1-\alpha_1-\alpha_2)}{(1-p_A)(1-p_B)(1-p_C) + (1-p_A)p_B(1-p_C) + (1-p_A)(1-p_B)p_C}.
\]

By comparison, we can find that \(P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega) = P(Y_A \leq w_A, Y_B \leq w_B, Y_C \leq w_C)\) if and only if

\[
\begin{align*}
\alpha_1 &= \frac{p_A(1-p_B)(1-p_C)}{p_A(1-p_B)(1-p_C) + (1-p_A)p_B(1-p_C) + (1-p_A)(1-p_B)p_C}, \\
\alpha_2 &= \frac{p_A(1-p_B)(1-p_C)}{p_A(1-p_B)(1-p_C) + (1-p_A)p_B(1-p_C) + (1-p_A)(1-p_B)p_C}.
\end{align*}
\]

We have hence shown that under condition (34), \((W \mid W \in \Omega) \overset{d}{=} Y\) if the density function of \(F(w_A, w_B, w_C)\) exists.
If the density function of $F(w_A, w_B, w_C)$ does not exist, from the definition of conditional probability in Shorack (2017, §7.4), we know that the above formulas for $P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega)$ still satisfy the condition for conditional probability. Thus, we still can define $P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega)$ in the same way as before. Under condition (34), we also have $P(W_A \leq w_A, W_B \leq w_B, W_C \leq w_C \mid W \in \Omega) = P(Y_A \leq w_A, Y_B \leq w_B, Y_C \leq w_C)$. Then $(W \mid W \in \Omega) \overset{d}{=} Y$.

In summary, according to the definition of $Y_A \perp \perp Y_C \mid Y_B$, we have $Y_A \perp \perp Y_C \mid Y_B$. □

References

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