Small solutions of nonlinear Schrödinger equations near first excited states

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Abstract

Consider a nonlinear Schrödinger equation in \( \mathbb{R}^3 \) whose linear part has three or more eigenvalues satisfying some resonance conditions. Solutions which are initially small in \( H^1 \cap L^1(\mathbb{R}^3) \) and inside a neighborhood of the first excited state family are shown to converge to either a first excited state or a ground state at time infinity. An essential part of our analysis is on the linear and nonlinear estimates near nonlinear excited states, around which the linearized operators have eigenvalues with nonzero real parts and their corresponding eigenfunctions are not uniformly localized in space.

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1 Introduction

Consider the nonlinear Schrödinger equation in \( \mathbb{R}^3 \),

\[
    i \partial_t \psi = H_0 \psi + \kappa |\psi|^2 \psi, \quad \psi|_{t=0} = \psi_0, \tag{1.1}
\]

where \( H_0 = -\Delta + V \) is the linear Hamiltonian with a localized real potential \( V \), \( \kappa = \pm 1 \), and \( \psi(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C} \) is the wave function. We often drop the \( x \) dependence and write \( \psi(t) \). We assume \( \psi_0 \in H^1 \) is localized, say \( \psi_0 \in L^1 \), so that its dispersive component decays rapidly under the evolution. For any solution \( \psi(t) \in H^1(\mathbb{R}^3) \) its \( L^2 \)-norm and energy

\[
    \mathcal{E}[\psi] = \int \left( \frac{1}{2} |\nabla \psi|^2 + \frac{1}{2} V|\psi|^2 + \frac{1}{4} \kappa |\psi|^4 \right) dx \tag{1.2}
\]

are constant in \( t \). The global well-posedness for small solutions in \( H^1(\mathbb{R}^3) \) can be proven using these conserved quantities no matter what the sign of \( \kappa \) is.

We assume that \( H_0 \) has \( K + 1 \) simple eigenvalues \( e_0 < e_1 < \cdots < e_K( < 0) \) with normalized real eigenfunctions \( \phi_k \), \( k = 0, 1, \ldots, K \), where \( K \geq 2 \). They are assumed to satisfy

\[
    e_0 < 2 e_1 < 4 e_2, \tag{1.3}
\]

and some generic conditions to be specified later. Through bifurcation around zero along these eigenfunctions, one obtains \( K + 1 \) families of nonlinear bound states \( Q_{k,n} = n \phi_k + h \), \( h = O(n^3) \) and \( (h, \phi_k) = 0 \) for \( k = 0, \ldots, K \), and \( n > 0 \) sufficiently small, which solve the equation

\[
    (-\Delta + V)Q + \kappa |Q|^2 Q = EQ, \tag{1.4}
\]

for some \( E = E_{k,n} = e_k + O(n^2) \), see Lemma 2.1. They are real and decay exponentially at spatial infinity. Each of them gives an exact solution \( \psi(t, x) = Q(x)e^{-iEt} \) of (1.1). The family \( Q_{0,n} \) are called the nonlinear ground states while \( Q_{k,n}, k > 0 \), are called the \( k \)-th nonlinear excited states.

Our goal is to understand the long-time dynamics of the solutions at the presence of nonlinear bound states. The first question is the stability problem of nonlinear ground states. It is well-known that nonlinear ground states are orbitally stable in the sense that the difference

\[
    \inf_{n, \theta} \left\| \psi(t) - Q_{0,n} e^{i\theta} \right\|_{H^1(\mathbb{R}^3)} \tag{1.5}
\]

remains uniformly small for all time \( t \) if it is initially small. On the other hand, the difference is expected to approach zero locally since the majority of which is a dispersive wave that scatters to infinity. Hence one expects that it is asymptotically stable in the sense that

\[
    \left\| \psi(t) - Q_{0,n(t)} e^{i\theta(t)} \right\|_{L^2_{\text{loc}}} \to 0 \tag{1.6}
\]

as \( t \to \infty \), for a suitable choice of \( n(t) \) and \( \theta(t) \). Here \( \| \cdot \|_{L^2_{\text{loc}}} \) denotes a local \( L^2 \) norm, to be made precise in \( (1.16) \). One is also interested in how fast (1.6) converges and whether \( n(t) \) has a limit.

The second question is the asymptotic problem of the solution when \( \psi(0) \) is small but not close to ground states. It is delicate since nonlinear excited states stay there forever.
but are expected to be unstable from physical intuition. Thus, a solution may stay near an excited state for an *extremely long time* but then moves on and approaches another excited state.

We now review the literature, assuming $\psi_0$ is small in $H^1 \cap L^1$.

If $-\Delta + V$ has only one bound state, i.e., with no excited states, the asymptotic stability of ground states is proved in [24, 25], with convergence rate $t^{-3/2}$. It is then shown in [20] that all solutions with small initial data, not necessarily near ground states, will locally converge to a ground state.

Suppose $-\Delta + V$ has two bound states. the asymptotic stability of ground states is proved in [29], with a slower convergence rate $t^{-1/2}$ due to the persistence of the excited state. The problem becomes more delicate when the initial data are away from ground states. It is proved in [31] that, near excited states, there is a finite co-dimensional manifold of initial data so that the corresponding solutions locally converge to excited states. Outside of a small wedge enclosing this manifold, all solutions exit the excited state neighborhood and relax to ground states [30]. It is further showed in [32] that for all small initial data in $H^1 \cap L^1$, there are exactly three types of asymptotic profiles: vacuum, excited states or ground states. The last problem is also considered in [27].

Suppose $-\Delta + V$ has three or more bound states. The asymptotic stability of ground states is proved in [28]. In fact, it is shown that all solutions with

$$
\|\psi_0\|_{H^1 \cap L^1}^{3-\varepsilon} \leq |(\phi_0, \psi_0)| \ll 1, \quad 0 < \varepsilon \ll 1,
$$

relax to ground states. It ensures that the solution is away from excited states but allows the ground state component to be much smaller than other components.

We also mention a few related results on the asymptotic stability of ground states of nonlinear Schrödinger equations with more general nonlinearities. For small solutions, one extension is to replace the resonance condition (1.3) by weaker conditions, e.g. those by [9] and by [7]. Another extension is to assume $\psi_0 \in H^1$ without assuming $\psi_0 \in L^1$. It is first proved in [11] for $K = 0$ and dimension $N = 3$ and then extended by [18, 19] for $K = 0$ and $N = 1, 2$. It is also extended by [7] for $K \geq 1$ with (1.3) replaced by weaker conditions used by [9]. A third extension is to allow subcritical nonlinearity $\pm|\psi|^{p-1}\psi$, $p < 1 + 4/N$, see e.g. [14]. A fourth extension is to assume $K = 1$ and $e_1$ has multiplicity, see [10, 12].

The stability of large solitary waves is considered for $K = 0, 1$, by [2, 3, 4] for $N = 1$ and by [5, 6] for $N = 3$.

See [17, 22, 12] and their references for construction of stable manifolds similar to that in [31].

In this paper, our goal is to continue the study of [28] under the same assumptions, with initial data $\psi_0$ now inside a neighborhood of the first excited state $Q_{1,n}$. This is the easiest interesting case not covered in [28]. Guided by the $K = 1$ case, one expects that the solution should either converge to a first excited state (with the ground state component always negligible), or leave the excited state neighborhood after some time (which may be extremely long, say greater than $e^{-1/n}$), and then relax to a ground state.

The new difficulty of the $K > 1$ case is the existence of higher excited state components. If the solution is to converge to a first excited state with the ground state component always negligible, one can think that the ground state component is absent and the first excited state as a new ground state. Thus, in the $K > 1$ case the convergence to a first excited state is expected to be in the rate $t^{-1/2}$, much slower than $t^{-3/2}$ in the $K = 1$ case.
When the difference is of order $t^{-3/2}$, one can use centered orthogonal coordinates as in\footnote{Denote $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^d, \ d \geq 1$. For $r \in \mathbb{R}$, denote by $L^2_r$ the weighted $L^2$ spaces with $\|f\|_{L^2_r} = \| |x|^r \, f(x)\|_{L^2}$.},
\[
\psi(t) = Q_{1,n(t)} e^{i\theta(t)} + h(t), \quad h(t) = x_0(t) \phi_0 + \xi(t), \quad \xi \in \mathcal{E}_c(H_0). \tag{1.8}
\]
The equations of $\dot{h}(t)$ and $\dot{\theta}(t)$ contain linear terms in $h$. When $x_0(t)$ is negligible, these linear terms are of order $t^{-3/2}$ and hence integrable in $t$, ensuring the convergence of the parameters. However, when $K > 1$, the difference is order $t^{-1/2}$ and one cannot show the convergence of the parameters if their equations contain linear terms. To remove linear terms, one is forced to use linearized coordinates around the first excited state, to be specified later in §3.2.

We now describe a few special properties of the linearized operator around an excited state. When the function $\psi$ is close to a nonlinear bound state $Q = Q_{m,n}$ with corresponding frequency $E = E_{m,n}$, one writes $\psi = (Q(x) + h(t,x))e^{-iEt}$. The perturbation $h(t,x)$ satisfies
\[
\partial_t h = \mathcal{L} h + \text{nonlinear terms}, \tag{1.9}
\]
where the linearized operator $\mathcal{L}$ around $Q$ is given by
\[
\mathcal{L} h = -i \left\{ (H + \kappa Q^2) h + \kappa Q^2 \bar{h} \right\}, \quad H = -\Delta + V - E + \kappa Q^2. \tag{1.10}
\]
Note $HQ = 0$. Since $\mathcal{L}$ does not commute with $i$, it is not useful to consider its spectral properties. Instead one looks at its matrix version acting on $[\Re h, \Im h]$:
\[
\mathbf{L} = \begin{bmatrix} 0 & H \\ -H - 2\kappa Q^2 & 0 \end{bmatrix}. \tag{1.11}
\]
The spectral property of $\mathbf{L}$ for $m > 0$ is studied in \cite{31} and recalled in Proposition 2.4. It is a perturbation of $J(H_0 - e_m)$ with $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which has eigenvalues $\pm i(e_k - e_m)$, $k = 0, \ldots, K$. When $m > 0, k < m$ and $e_k < 2e_m$, the eigenvalues $\pm i(e_k - e_m)$ are embedded in the continuous spectrum $\pm i [\left|e_m\right|, \infty)$. These embedded eigenvalues split into a quadruple of eigenvalues of $\mathbf{L}$, $\pm \lambda_k$ and $\pm \bar{\lambda}_k$, with $\Im \lambda_k = |e_k - e_m| + O(n^2)$ and $C^{-1}n^4 < \Re \lambda_k < Cn^4$ (assuming the generic condition (1.11)). The size of their corresponding eigenvectors are roughly\footnote{\cite{6}}
\[
O_{L^2_{100}} (1) + \frac{O(n^2)}{\langle x \rangle} 1_{|x|<n^{-4}}. \tag{1.12}
\]
The second part is not localized; It is small in $L^\infty \cap L^3$, of order 1 in $L^2$, and of order $n^{6-12/p}$ in $L^p$ for $p < 2$. In particular, the projection $P_{\mathbf{L}}^c$ onto the continuous spectral subspace $\mathcal{E}_c^L$ of $\mathbf{L}$ is of order $n^{6-12/p} \gg 1$ in $L^p$ for $p < 2$, giving an extra difficulty to the usual analysis.

To overcome this difficulty, we prove decay estimates of the form (see Lemma 2.11)
\[
\| e^{i\mathbf{L} t} P_{\mathbf{L}}^c \varphi \|_{L^p} \leq C_p \left( \frac{2}{3} + \frac{2}{p} \right)^{\frac{d}{2p}} \| \varphi \|_{L^{p'}} \quad (t \geq 0), \tag{1.13}
\]
for $3 \leq p < 6$, with constant $C_p$ independent of $n$. Here $P_{\mathbf{L}}^c$ is an extended projection: It is the sum of $P_{c}^L$ and all projections onto eigenspaces whose corresponding eigenvalues have negative real parts. As shown in Remark (iii) after Lemma 2.11 these estimates with
n-independent constant are false if \( P_e^2 \) is replaced by \( P_e^L \). Also note that (1.13) is time-direction sensitive: it is true only for \( t \geq 0 \). The decay exponent above is not as good as the usual free Schrödinger evolution, but it is sufficient for us if we take \( p < 6 \) close to 6. A side benefit of extending \( P_e \) to \( P_e^2 \) is that we no longer need to track the component \((P_e^2 - P_e)\hbar\).

Our assumptions on the operator \( H_0 = -\Delta + V \) are as follows:

**Assumption A0.** \( H_0 = -\Delta + V \) acting on \( L^2(\mathbb{R}^3) \) has \( K + 1 \) simple eigenvalues \( e_0 < e_1 < \ldots < e_K < 0, K \geq 2 \), with normalized real eigenvectors \( \phi_0, \ldots, \phi_K \).

**Assumption A1.** \( V(x) \) is a real-valued function satisfying \( |\nabla^\alpha V(x)| \lesssim \langle x \rangle^{-5-s_1} \) for \( |\alpha| \leq 3 \), for some \( s_1 > 0 \). \( 0 \) is not an eigenvalue nor a resonance for \( H_0 \).

**Assumption A2.** Resonance condition. We assume that

\[
e_0 < 2e_1 < 4e_2.
\]

We further assume that, for some small \( s_0 > 0 \),

\[
\gamma_0 \equiv \inf_{0 \leq m \leq 1, |s| < s_0} \lim_{r \to 0^+} \frac{1}{r} \frac{\phi_m \phi_k^2}{-\Delta + V + e_m - e_k - e_l - s - r t} \left( f, s \right) > 0.
\]

**Assumption A3.** No-resonance condition (between eigenvalues). Let \( j_{\text{max}} = 3 \). For all \( j = 2, \ldots, j_{\text{max}} \) and for all \( k_1, \ldots, k_j, l_1, \ldots, l_j \in \{0, \ldots, K\} \), if \( e_{k_1} + \cdots + e_{k_j} = e_{l_1} + \cdots + e_{l_j} \), then there is a permutation \( \sigma \) of \( \{1, \ldots, j\} \) such that \( (l_1, \ldots, l_j) = (k_{\sigma_1}, \ldots, k_{\sigma_j}) \).

Assumption A1 ensure several estimates for linear Schrödinger evolution such as decay estimates and the \( \mathcal{W} \)-estimates for the wave operator \( W_{H_0} \). They are certainly not optimal. The main assumption in A2 is the condition \( e_{k-1} < 2e_k \). It ensures that \( H_0 + e_m - e_k - e_l \) is not invertible in \( L^2 \) for \( m < k, l \), and provides (for our cubic nonlinearity) the required resonance between eigenvalues through the continuous spectrum. Since the expression for \( \gamma_0 \) is quadratic, it is non-negative and \( \gamma_0 > 0 \) holds generically. Assumption A3 is a condition to avoid direct resonance between the eigenvalues. It is trivial if \( K = 0, 1 \). It holds true generically and is often seen in dynamical systems of ODE’s. If we relax the assumption (1.14), we may need to increase \( j_{\text{max}} \).

Fix \( r_1 > 10 \) large enough. We denote by \( L^p_{\text{loc}} \) the local \( L^p \) spaces given by the norm

\[
\|\phi\|_{L^p_{\text{loc}}(\mathbb{R}^3)} \equiv \left\{ \int_{\mathbb{R}^3} \langle x \rangle^{-pr_1} |\phi(x)|^p \, dx \right\}^{1/p}.
\]

Now we are ready to state our main theorem.

**Theorem 1.1** Assume Assumptions A0–A3 and fix \( 0 < \delta \leq \frac{1}{10} \). There are constants \( C_0, C_1 > 0 \), and small \( n_0 > 0 \) such that the following hold. If \( n = (\phi_1, \psi_0) \in (0, n_0) \) and \( \|\psi_0 - n\phi_1\|_{H^\gamma(\mathbb{R}^3)} \leq n^{1+\delta} \), then the solution \( \psi(t) \) of (1.11) with \( \psi(0) = \psi_0 \) satisfies

\[
\lim_{t \to \infty} \sup_{t} \left\| \psi(t) - Q_{m,n} e^{i\theta(t)} \right\|_{L^2_{\text{loc}}} t^{1/2} \leq C_0/n
\]

for \( m = 0 \) or \( m = 1 \), for some \( n_+ \in (C_1^{-1} n, C_1 n) \) and some \( \theta(t) \in C([0, \infty), \mathbb{R}) \).
In fact we have more detailed estimates of the solution for all time, see Propositions 4.2, 5.1, 6.3, 6.7, and 7.2. In particular, if the initial data $\psi_0$ is placed in the neighborhood of an excited state $Q_{m,n}$ with $m \geq 2$, even if $K > 2$, Propositions 4.2, 5.1, 6.3, 6.7 show that the solution will either converge to $Q_{m,n}$, for some $n_+$, or eventually exits the neighborhood, stays away from bound states for a time interval of order between $n^{-4} \log \frac{1}{n}$ and $n^{-4-2\delta}$, until it reaches the neighborhood of another bound state $Q_{m',n'}$, $m' < m$. If $m' = 0$, then Proposition 7.2 shows that $\psi(t)$ will converge to some $Q_{0,n_+}$. However, if $m' > 0$, our current analysis is not sufficient to control its evolution after this time.

We now sketch the structure of our proof and this paper.

In §2 we give the linear analysis, including the decay estimates $\langle 1.13 \rangle$.

In §3 we consider the decomposition of the solutions in different coordinates and the normal forms of their equations.

In §4 we start with the solution in a $n^{1+\delta}$-neighborhood of $Q_{1,n}$ and use linearized coordinates. We follow the evolution as long as the ground state component $z_0$ is negligible, characterized by $|z_0(t)| < n^{-3}(n^{-4-2\delta} + t)^{-1}$. If it is always negligible, we prove that the solution converges to an excited state with convergence rate $t^{-1/2}$.

In §5 we consider the case that $|z_0(t_c)| \geq n^{-3}(n^{-4-2\delta} + t_c)^{-1}$ in a first time $t_c \in [0, \infty)$, which may be 0 or extremely large, say $e^{e^{-1/n}}$. After an initial layer, we show that $|z_0(t)|$ starts to grow exponentially with exponent $Cn^4$ until it reaches the size $2n^{1+\delta}$ at time $t_0$. The time it takes, $t_0 - t_c$, is of order $n^{-4} \log \frac{2n^{1+\delta}}{|z_0(t_c)|}$. Along the way higher excited states may have size larger than $|z_0(t)|$ but can be controlled. This section is the most difficult part in the nonlinear analysis because it involves estimates not previously studied.

In §6 we study the dynamics after $t_o$ when there are at least two components of size greater than $2n^{1+\delta}$, and change to orthogonal coordinates

$$\psi = x_0\phi_0 + \cdots + x_k\phi_k + \xi, \quad \xi \in E_c(H_0). \tag{1.18}$$

Although $\xi(t_o)$ is already non-localized, we can prove “outgoing estimates” for $\xi(t_o)$, introduced in [30, 32], to capture the time-direction sensitive information of the dispersive waves. We show that, after a time of order between $n^{-4} \log \frac{1}{n}$ and $n^{-4-2\delta}$, the ground state component $x_0$ grows to order $n$ while all other components become smaller than $n^{1+\delta}$. (This is called the transition regime.)

In §7 the ground state component becomes dominant and we change to linearized coordinates around it. Again we need to keep track of out-going estimates during the coordinate change. We show that the solutions will converge to ground states with convergence rate $t^{-1/2}$. The analysis is similar to §4 but easier because it has no unstable direction. (This is called the stabilization regime.)

Analysis similar to §6 and §7 is done in [28], (and in the two-eigenvalue case near ground states in [31, 29, 30, 6, 4]). However, with weaker decay estimates like $\langle 1.13 \rangle$, we need more refined analysis. For example, since the nonlinearity is of constant order $n^3$ in the transition regime, we need to make this time interval as short as possible by taking $\delta > 0$ small. We also take $p < 6$ close to 6 to minimize our loss in estimating the $L^p$-norm of the dispersive component during this interval.

**New proof of linear decay estimates for ground states**

We end this introduction by noting that, our linear analysis, Lemmas 2.11 and 2.13 in the case $m = 0$, provide a new proof of linear estimates for the linearized operators around
ground states, which is used to prove the stability of ground states in 3D, see [5, 29, 28]. Proofs in these references either use the wave operator between \( \mathcal{L} \) and \(-i(H_0 - E)\), or use a similarity transform \( \mathcal{L} = U(-iA)U^{-1} \) for some self-adjoint perturbation \( A \) of \( H_0 - E \) and non-self-adjoint operator \( U \). Our proof here use simple perturbation argument and requires less assumptions on the potential \( V \). Moreover, this perturbation argument allows the operator \( V \) to be more general than a potential, as long as the decay and singular decay estimates for \(-\Delta + V\) hold.

## 2 Linear analysis

In this section we will study various properties of the linearized operator around a fixed bound state, in particular an excited state. The starting point is the following lemma on the existence of nonlinear bound states and their basic properties.

**Lemma 2.1 (Nonlinear bound states)** Assume Assumptions A0–A1. There exists a small \( \eta_1 > 0 \) such that for each \( k = 0, \ldots, K \) and \( n \in [0, n_k] \), there is a solution \( Q_{k,n} \in H^2 \cap W^{1,1} \) of (1.4) with \( E = E_{k,n} \) such that

\[
Q_{k,n} = n\phi_k + q(n), \quad (q, \phi_k) = 0. \tag{2.1}
\]

The pair \((q, E)\) is unique in the class \( \|q\|_{H^2} + |E - e_k| \leq n^2 \). Moreover, \( \|q\|_{H^2} \|W^{1,1} \| \leq n^3 \) and \( \|\partial_{\eta_1} q\|_{H^2} \|W^{1,1} \| \leq n^2, \|E - e_k - Cn^2\| \leq n^4 \) where \( C = \int_{\mathbb{R}} \phi_k^4 \). We also denote

\[
R_{k,n} = \frac{\partial}{\partial E_k} Q_{k,n} = \frac{\partial}{\partial q} Q_{k,n}/\frac{\partial}{\partial E_k} E_{k,n} = \frac{1}{2\alpha_1} \phi_k + O_{H^2 \cap W^{1,1}}(n).
\]

In the following we fix \( m \in \{0, \ldots, K\} \) and \( n \in [0, n_k] \). Let \( Q = Q_{m,n}, R = R_{m,n} \) and \( E = E_{m,n} \). The function \( Q \) satisfies \( HQ = 0 \) where

\[
H = H_0 - E + \kappa Q^2. \tag{2.2}
\]

The following lemma collects useful properties of \( H \).

**Lemma 2.2** Assume Assumptions A0-A1 and let \( H \) be defined as in (2.2). The operator \( H \) has \( K + 1 \) real eigenvalues \( \tilde{e}_k = e_k - e_m + O(n^2) \) with normalized eigenfunctions \( \phi_k = \phi_k + O(n^2) \). In particular, \( \tilde{e}_m = 0 \) and \( \phi_m = CQ_m \). The projection to its continuous spectral subspace is \( P^H_{\phi} f = f - \sum_k (\phi_k, f)\phi_k \). Furthermore, we have the following decay estimates

\[
\|e^{-iHT} P_{c}^H \phi\|_{L^q} \leq C|t|^{-3/2+3/q} \|\phi\|_{L^{q'}} \quad (2 \leq q \leq \infty), \tag{2.3}
\]

and singular decay estimates: for sufficiently large \( r_1 > 9/2 \), for \( 0 \leq N \leq 3 \), for \( \alpha_j \in \mathbb{C} \) with \( \text{Im} \alpha_j > 0 \), \( |\text{Re} \alpha_j + e_m| \in [a_1, a_2] \subset (0, \infty) \), \( j \leq N \),

\[
\|\langle x \rangle^{-r_1} e^{-iHT} \Pi_j^N (H - \alpha_j)^{-1} P_{c}^H \phi\|_{L^2} \leq C \langle t \rangle^{-3/2} \|\langle x \rangle^{r_1} \phi\|_{L^2}, \quad (t \geq 0). \tag{2.4}
\]

Here the constant \( C \) is independent of \( n, \phi \) and \( \alpha_j \).

Note that this lemma contains \( H = H_0 \) as a special case with \( n = 0 \). The proof of the first part is well-known by perturbation. Estimate (2.3) is by Journe-Soffer-Sogge [16]. Estimate (2.4) for \( N = 0 \) is by Jensen-Kato [15] and Rauch [21]. Estimate (2.4) for \( \alpha_1 = \cdots = \alpha_N, N \geq 1 \), was first proven by Soffer-Weinstein [26] for Klein-Gordon equations, then by Tsai-Yau [29] and Cuccagna [6] for (linearized) Schrödinger equations. The general
case is similar and a proof based on Mourre estimate is sketched below for completeness. (See [6] for a different approach).

Denote the dilation operator \( D = x \cdot p + p \cdot x \) with \( p = -i \nabla \), and the commutators
\[
\text{ad}^0_D(H) = H, \quad \text{ad}^{k+1}_D(H) = [\text{ad}^k_D(H), D], \quad k \geq 0. \tag{2.5}
\]
Fix \( g_* \in C_c^\infty(\mathbb{R}) \) with \( g_* = 1 \) on \([-1, 1]\) and \( \text{supp} g_* \subset (-2, 2) \). For each \( j \), let \( g_j(t) = g_*(t - \text{Re} z_j)/\varepsilon \). If \( \varepsilon > 0 \) is sufficiently small, \( g_j(H)\text{ad}^k_D(H)g_j(H) \) are bounded operators in \( L^2 \) for \( k \leq 3 \) and all \( j \), and the Mourre estimate holds: For some \( \theta > 0 \),
\[
g_j(H)[iH, D]g_j(H) \geq \theta g_j(H)^2, \quad \forall j. \tag{2.6}
\]
See [8]. Thus the pair \( H, D \) satisfies the assumptions of the minimal velocity estimates in [13] and Theorem 2.4 of [23], and one has
\[
\| \chi(D \leq \theta t/2)e^{-itH}g_j(H) \langle D \rangle^{-r_1}\|_{L^2 \to L^2} \leq C \langle t \rangle^{-r_1+\varepsilon_1}, \tag{2.7}
\]
where \( 0 < \varepsilon_1 \ll 1 \) and \( \chi(D \leq a) \) is the spectral projection of \( D \) associated to the interval \((-\infty, a]\). The same argument of [26] then gives [24].

2.1 Linearized operator

A perturbation solution \( \psi(x, t) \) of (1.1) of the exact solution \( Q(x)e^{-iEt} \) can be written in the form
\[
\psi(x, t) = [Q(x) + h(x, t)]e^{-iEt} \tag{2.8}
\]
for some function \( h \) which is small in a suitable sense. Then, \( h \) satisfies
\[
\partial_t h = \mathcal{L}h + \text{nonlinear terms}, \tag{2.9}
\]
where the operator \( \mathcal{L} \) is defined as
\[
\mathcal{L}h = -i\{(H_0 - E + 2\kappa Q^2)h + \kappa Q^2\bar{h}\}. \tag{2.10}
\]
The operator \( \mathcal{L} \) is linear over \( \mathbb{R} \) but not over \( \mathbb{C} \). As a result it is not useful to consider its spectral properties.

Consider the injection from scalar functions to vector functions
\[
\mathbf{j} : L^2(\mathbb{R}^3, \mathbb{C}) \to L^2(\mathbb{R}^3, \mathbb{C}^2), \quad \mathbf{j}(\varphi) = [\varphi] := \begin{bmatrix} \text{Re} \varphi \\ \text{Im} \varphi \end{bmatrix}. \tag{2.11}
\]
With respect to this injection, the operator \( \mathcal{L} \) is naturally extended to a matrix operator acting on \( L^2(\mathbb{R}^3, \mathbb{C}^2) \) with the following form
\[
\mathbf{L} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \quad \text{where} \quad \begin{cases} L_- = H - H_0 - E + \kappa Q^2, \\ L_+ = H + 2\kappa Q^2 = H_0 - E + 3\kappa Q^2. \end{cases} \tag{2.12}
\]
We will use \( \mathcal{L} = \mathbf{j}^{-1}\mathbf{L}\mathbf{j} \) for computations involving \( \mathcal{L} \).

The space \( L^2(\mathbb{R}^3, \mathbb{C}^2) \) is endowed with the natural inner product
\[
(f, g) = \int_{\mathbb{R}^3} (f \bar{g}_1 + f g_2) \, dx \tag{2.13}
\]
for \( f = [f_1, f_2] \) and \( g = [g_1, g_2] \). We will use the Pauli matrices
\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.14}
\]
2.2 Invariant subspaces

In this subsection we study the spectral subspaces of $L$. Since $L$ is a perturbation of $JH$, we first give the following lemma for comparison.

Lemma 2.3 (Invariant subspaces of $JH$) Assume Assumptions A0–A2. The space $L^2(\mathbb{R}^3, \mathbb{C}^2)$ can be decomposed as the direct sum of $JH$-invariant subspaces

$$L^2(\mathbb{R}^3, \mathbb{C}^2) = E_0^{JH} \oplus \cdots \oplus E_K^{JH} \oplus E_c^{JH}.$$  \hspace{1cm} (2.15)

For each $k \in \{0, \ldots, K\}$, the space $E_k^{JH}$ is spanned by 2 eigenvectors $[\frac{1}{k}] \bar{\phi}_k$ and $[\frac{-1}{k}] \bar{\phi}_k$ with eigenvalues $-i\bar{c}_k$ and $i\bar{c}_k$, respectively. Its corresponding orthogonal projection is $P_k^{JH} \left[ f_1 \right] = \frac{1}{k} \left( \bar{\phi}_k, f_2 \right) \bar{\phi}_k$. The subspace $E_c^{JH}$ has projection $P_c^{JH} f = \left[ P_c^{JH} f_1 \right]$.

The proof is straightforward and skipped. We next give the corresponding statements for $L$.

Proposition 2.4 (Invariant subspaces of $L$) Assume Assumptions A0–A2. Fix $m \in \{0, \ldots, K\}$ and $n \in (0, n_1]$. Let $Q = Q_m$, $R = R_m$, and $E = E_{m,n}$. The space $L^2(\mathbb{R}^3, \mathbb{C}^2)$ can be decomposed as the direct sum of $L$-invariant subspaces

$$L^2(\mathbb{R}^3, \mathbb{C}^2) = E_0^L \oplus \cdots \oplus E_K^L \oplus E_c^L.$$  \hspace{1cm} (2.16)

If $f$ and $g$ belong to different subspaces, then

$$\langle \sigma_1 f, g \rangle = 0.$$  \hspace{1cm} (2.17)

These subspaces and their corresponding projections satisfy the following.

(i) $E_m^L$ is the 0-eigenspace spanned by $[\frac{0}{Q}]$ and $[\frac{R}{0}]$, with $L \left[ \frac{0}{Q} \right] = [\frac{0}{0}]$ and $L \left[ \frac{R}{0} \right] = -[\frac{0}{Q}]$. Its projection is $P_m f = c_m \left( \sigma_1 \left[ \frac{R}{0} \right], f \right) \left[ \frac{0}{Q} \right] + c_m \left( \sigma_1 \left[ \frac{0}{Q} \right], f \right) \left[ \frac{R}{0} \right]$, $c_m = (Q, R)^{-1}$.

(ii) $E_k^L$ for $0 \leq k < m$, if such $k$ exists, is spanned by 4 eigenvectors $\Phi_k = [\frac{u_k}{-v_k}]$, $\bar{\Phi}_k$, $\Phi_3 \bar{\Phi}_k$ and $\bar{\Phi}_3 \Phi_k$, with eigenvalues $\lambda_k$, $\lambda_k$, $-\lambda_k$, and $-\lambda_k$, respectively. Here $\lambda_k = -i(e_k - e_m) + O(n^2)$, $n^4 \lesssim \Re \lambda_k \lesssim n^4$, $u_k$ and $v_k$ are complex-valued functions, $u_k = \bar{u}_k + \bar{u}_k$ and $v_k = \bar{v}_k - \bar{v}_k$, with

$$u_k^+ = \phi_k + O_{L^\infty_r}(n^2), \quad u_k^- = (H - i\lambda_k)^{-1} \phi_k + O_{L^\infty_r}(n^2)$$  \hspace{1cm} (2.18)

where $\phi_k^* = P_c^{JH} \Phi_k^* = O_{L^\infty_r}(n^2)$. Furthermore, $(u_k, v_k) = 0$ and $(u_k, v_k) = (\bar{u}_k, v_k) = 0$ for $k \neq \ell$. All $(\bar{u}_k, v_k)$, $\|u_k\|_{L^2}$ and $\|u_k\|_{L^2}$ are equal to 1 + $O(n^2)$ and $\|u_k\|_{L^2 loc} \lesssim n^2$.

The projection to $E_k^L$ is $P_k + P_k^*$ where

$$P_k f = c_k \left( \sigma_1 \bar{\Phi}_k, f \right) \Phi_k + \bar{c}_k \left( \sigma_1 \Phi_k, f \right) \bar{\Phi}_k,$$

$$P_k^* f = -c_k \left( \sigma_1 \bar{\Phi}_k, f \right) \bar{\Phi}_3 \Phi_k - \bar{c}_k \left( \sigma_1 \Phi_k, f \right) \Phi_3 \bar{\Phi}_k,$$  \hspace{1cm} (2.19)

and $c_k = (\sigma_1 \bar{\Phi}_k, \Phi_k)^{-1} = i/\int 2u_k v_k = i/2 + O(n^2)$.

(iii) $E_k^L$ for $m < k \leq K$, if such $k$ exists, is spanned by 2 eigenvectors $\Phi_k = [\frac{u_k}{-v_k}]$ and $\Phi_k$ with eigenvalues $\lambda_k$ and $\lambda_k$, respectively. Here $\Re i\lambda_k = e_k - e_m + O(n^2)$, $u_k$ and $v_k$ are real-valued, both equal to $\phi_k + O_{L^\infty_r}(n^2)$, and normalized by $(u_k, v_k) = 1$. Its projection is $P_k$, also given by (2.19), with $c_k = i/2$.
(iv) \( \mathbf{E}^L_k = \{ g : (\sigma_1 f, g) = 0, \forall f \in \mathbf{E}_k, \forall k = 0, \ldots, K \} \). Its projection is \( P^L_k f = f - \sum_{k=0}^K P_k f - \sum_{k<m} P^k f \).

Note that \( \lambda_k \) is in the first quadrant and near the imaginary axis for \( k < m \), and in the lower imaginary axis for \( k > m \). They are all perturbations of \(-i\tilde{e}_k\) of Lemma 2.3. When \( k < m \), \(-i\tilde{e}_k\) are inside the continuous spectrum \( \pm i[|E_m|, \infty) \) and their resonance make the eigenvalues split.

![Figure 1: Spectrum of \( \mathbf{L} \) around \( Q_m \), \( 0 < m < K \).](image)

**Proof.** The same proof of [31, Theorem 2.2] works in our many eigenvalue case. The only thing we need to check is the properties of \( u^+_k \) and \( u^-_k \) when \( k < m \). Fix \( k < m \). Denote by \( \Pi \) the orthogonal projection from \( L^2 \) onto \( \{ \tilde{\phi}_k, Q_m \}^\perp \), and \( B = 2\kappa Q^2_m \). We omit the subscript \( k \) below. By the defining equations \( L_m \Phi = \lambda \Phi \) and \( \Phi = [u - iv] \), \( \bar{u} \) satisfies

\[
(H^2 + HB)\bar{u} = -\bar{\lambda}^2 \bar{u}. \tag{2.20}
\]

By the same proof for the two-eigenvalue case in [31, section 2.1] (in which \( \Pi = P^H_c \)), \( \bar{u} \) can be solved in the form

\[
\bar{u} = \tilde{\phi} + h, \quad h = \Pi h = -(H^2 + \Pi HB\Pi + \tilde{\lambda}^2)^{-1}\Pi HB\tilde{\phi}. \tag{2.21}
\]

One can rewrite

\[
h = (H^2 + \tilde{\lambda}^2)^{-1}\Psi, \quad \Psi = \Pi \Psi = [1 + \Pi HB\Pi(H^2 + \tilde{\lambda}^2)^{-1}]^{-1}\Pi HB\tilde{\phi}. \tag{2.22}
\]

By resolvent estimates and a power series expansion as in [31], the function \( \Psi \) is localized and \( \|\Psi\|_{L^2} \leq Cn^2 \). Since \( v = (i\lambda)^{-1}(H + B)u \), we have \( u^\pm = \mp \frac{1}{2\zeta}(H \mp z + B)\bar{u} \) with \( z = i\bar{\lambda} = |e_k - e_m| + O(n^2) \). For \( u^+ \),

\[
u^+ = -\frac{1}{2\zeta}(H - z)\tilde{\phi} - \frac{1}{2\zeta}(H + z)^{-1}\Psi - \frac{1}{2\zeta}B\bar{u}. \tag{2.23}
\]
The first term is equal to \((1 + O(n^2))\tilde{\phi}\). Since \((H + z)^{-1}\Pi\) is order one, the remaining two terms are \(O_{3\nu, L}^\infty(n^2)\), and so is \(\phi - \tilde{\phi}\). This shows \(u^+ = \phi + O_{3\nu, L}^\infty(n^2)\). For \(u^-\),
\[
  u^- = \frac{1}{2z}(H + z)\phi + \frac{1}{2z}(H - z)^{-1}\Psi + \frac{1}{2z}B\tilde{u}.
\] (2.24)

The first term is \(O(n^2)\tilde{\phi}\). Since \((H - z)^{-1}(\Pi - P_c^H)\Psi\) are sum of eigenfunctions with \(O(n^2)\) coefficients, we get (2.18) with \(\phi_k^* = \frac{1}{2\pi}P_c^H\Psi = O_{3\nu, L}^\infty(n^2)\).

The orthogonality \((u, v) = 0\) is equivalent to \((\sigma_1, \Phi, \tilde{\Phi}) = (\sigma_1, \sigma_2, \Phi, \tilde{\Phi}) = 0\), which follow from the general fact shown in [31 §2.6] that
\[
(\sigma_1 f, g) = 0 \text{ if } Lf = \lambda f, \ Lg = \mu g, \text{ and } \lambda \neq \mu.
\] (2.25)

It also follows from (2.25) that \((u_k, v_k) = (\bar{u}_k, v_k) = 0\) for \(k \neq \ell\). That \(\|u^+\|_{L^2} = 1 + O(n^2)\) and \(\|u^\|_{L^2_{loc}} \lesssim n^2\) follow from (2.18). Note
\[
0 = (\bar{u}, \dot{v}) = (u^+ + u^{}, u^+ - u^-) = (u^+ + u^-) - (u^-, u^-) + (u^-, u^+) - (u^+, u^-).
\] (2.26)

Since the last two terms are \(O(n^2)\), we get \(\|u^-\|_{L^2} - \|u^+\|_{L^2} = O(n^2)\). Finally
\[
(\bar{u}, v) = (u^+ + u^{}, \dot{u}^+ - \dot{u}^-) = (u^+, \dot{u}^+) - (\bar{u}^-, \dot{u}^-) + (\bar{u}^-, \dot{u}^+) - (u^+, \dot{u}^-).
\] (2.27)

We have \((u^+, u^-) - (\bar{u}^+, u^-) = O(n^2)\). By (2.18) we also have (denoting \(ok = O(n^4)\))
\[
(\bar{u}^-, u^-) = ((H - \bar{z})^{-1}\bar{\phi}_k^*, (H - z)^{-1}\phi_k^*) + ok = (\bar{\phi}_k^*, (H - \bar{z})^{-2}\phi_k^*) + ok = ok
\] (2.28)

by the singular decay estimate of Lemma 2.2 with \(t = 0\). Thus \((\bar{u}_k, v_k) = 1 + O(n^2)\).

Similarly, \((\bar{u}_k, v_k) = O(n^2)\) for \(k \neq \ell\).

In the following lemma we provide more properties of \(u_k^\) -

**Lemma 2.5** Assume the same as in Proposition 2.4 and fix \(k < m\). Then

(i) \(\|\bar{u}_k\|_{L^p} \leq C_p(n^2 + n^{6-12/p})\) for \(1 \leq p \leq \infty\), in particular \(\|\bar{u}_k\|_{L^\infty} \leq Cn^2\).

(ii) \(\|e^{-iH\nu}P_c^H\bar{u}_k\|_{L^2_{\nu}} + \|e^{-iH\nu}P_c^H\bar{u}_k\|_{L^2_{\nu}} \leq Cn^2 \langle \nu \rangle^{-3/2}\) for \(s \geq 0\).

(iii) \(\|u_k\|_{H^1} \leq C\).

**Proof.** Denote \(z = i\bar{\lambda}_k\) and \(\varphi = \phi_k^*\). For (i), it suffices to check \((H - z)^{-1}\varphi\), the main part of \(u_k^\) in (2.18). Write \(H - z = -\Delta + \nu^2 + V_1\) where \(V_1 = V + \kappa Q_m^2\), \(\nu^2 = E_m + z\) with \(\text{Im } \nu > 0\). Thus \(\text{Im } \nu \sim +n^4\). By resolvent expansion,
\[
(H - z)^{-1}\varphi = (-\Delta - \nu^2)^{-1}\varphi + (-\Delta - \nu^2)^{-1}V_1(H - z)^{-1}\varphi.
\] (2.29)

Since the resolvent \((-\Delta - \nu^2)^{-1}\) has the convolution kernel \(G(x) = (4\pi|x|)^{-1}\text{exp}(i\nu|x|)\),
\[
\|(-\Delta - \nu^2)^{-1}\varphi\|_{L^p} \lesssim \|G \ast \varphi\|_{L^p} \lesssim (\|G\|_{L^p(B_1)} + \|G\|_{L^2(B_1)}) \cdot \|\varphi\|_{L^1 \cap L^2}
\] (2.30)

which is bounded by \((n^{4-12/p} + 1) \cdot n^2\). Since \(\|V_1(H - z)^{-1}\varphi\|_{L^1 \cap L^2} \lesssim \|(H - z)^{-1}\varphi\|_{L^2_{\nu}} \lesssim n^2\), we have the same bound for the second term. The above show (i).

For (ii), we only need to consider \(e^{-iH\nu}P_c^H\bar{u}_k\) since the other term follows from Lemma 2.2. By resolvent expansion \(R = (H - z)^{-1} = R_0(1 + \kappa Q_m^2 R)\) where \(R_0 = (H_0 - E_m + z)^{-1}\),
\[
P_c^H \bar{u}_k = R_0 \varphi' + O_{3\nu, L}^\infty(n^2), \quad \varphi' = P_c^H(1 + \kappa Q_m^2 R)\varphi = O_{3\nu, L}^\infty(n^2).
\] (2.31)
Thus
\[ e^{-isH_0}P_e^{H_0}u^{-} = e^{-isH_0}R_0\varphi' + O_{L^2_r}(n^2\langle s \rangle^{-3/2}). \]  

By the singular decay estimate for \( H_0 \), the first term is also of order \( O_{L^2_r}(n^2\langle s \rangle^{-3/2}) \).

To prove (iii), it suffices to prove that \( \|\nabla v\|_{L^2} = O(1) \) where \( v = (H - z)^{-1}\varphi \). It can be shown by multiplying the equation \((H - z)v = \varphi\) by \( \bar{v} \) and then integrating it on \( \mathbb{R}^3 \). □

We will need the following lemmas for scalar functions.

**Lemma 2.6** Fix \( 0 \leq k \leq K, k \neq m \). Let \( \varphi \in L^2(\mathbb{R}^3, \mathbb{C}) \) be a scalar function.

(i) \( P_k[\varphi] = \text{Re}\alpha\Phi_k, J^{-1}P_k[\varphi] = \alpha\bar{u}^+ + \alpha\bar{u}^- \), where
\[ \alpha = 2c_k(\sigma_1\bar{\Phi}_k[\varphi]) = -2c_ki[(u_k^+, \varphi) - (u_k^-, \varphi)]. \]  

(ii) \( P_k\varphi = 0 \) iff \( (\sigma_1\Phi_k, [\varphi]) = 0 \) iff \( (u_k^+, \varphi) = (u_k^-, \varphi) \).

(iii) For \( k < m \), \( P_k^*\varphi = 0 \) iff \( (\sigma_1\sigma_3\Phi_k, [\varphi]) = 0 \) iff \( (u_k^+, \varphi) = (u_k^-, \varphi) \).

**Proof.** Write \( [\varphi] = [\varphi_1^+ \varphi_2^+] \). Since \( [\varphi] \) is real, we have by (2.19) that \( P_k[\varphi] = \text{Re}\alpha\Phi_k \) with \( \alpha = 2c_k(\sigma_1\bar{\Phi}_k[\varphi]) \). Omitting the subscript \( k \), we have
\[ (\sigma_1\Phi_k, [\varphi]) = (i\bar{v}, \varphi_1) + (\bar{u}, \varphi_2) = (u^+ - u^-, -i\varphi_1) + (u^+ + u^-, \varphi_2) = -i(u^+, \varphi) + i(u^-, \varphi), \]
which gives the formula for \( \alpha \). Thus
\[ J^{-1}P_k[\varphi] = J^{-1}\text{Re}\alpha \left[ \frac{u}{-i\bar{v}} \right] = \frac{1}{2} \left\{ (\alpha u + \bar{\alpha}\bar{u}) + i(-i\alpha v + i\bar{\alpha}\bar{v}) \right\} = \alpha\bar{u}^+ + \alpha\bar{u}^-. \]  

The claim (ii) follows from (i). For (iii), since \( \sigma_3\sigma_1\sigma_3 = -\sigma_1 \), \( (\sigma_1\sigma_3\Phi_k, [\varphi]) = 0 \) is equivalent to \( 0 = (\sigma_1\Phi_k, \sigma_3[\varphi]) = (\sigma_1\Phi_k, [\varphi]) \) and hence to \( (u_k^+, \varphi) = (u_k^-, \varphi) \). □

The following lemma will be used to treat the linear term in the \( \eta \) equation.

**Lemma 2.7** (i) For \( k < m \),
\[ J\Phi_k = i\Phi_k - 2i \left[ \frac{1}{i} \right] \bar{u}_k^+. \]  

(ii) If \( f \in L^2(\mathbb{R}^3, \mathbb{C}^2) \) and \( P_kf = 0 \), then \( \|P_kJf\|_{L^2} \lesssim \|f\|_{L^2_r} \).

**Proof.** For (i), rewrite
\[ \Phi_k = \left[ \frac{u_k}{-i\bar{v}_k} \right] = \left[ \frac{1}{i} \right] \bar{u}_k^+ + \left[ \frac{1}{i} \right] \bar{u}_k^-. \]  

Applying \( J \)
\[ J\Phi_k = -i \left[ \frac{1}{i} \right] \bar{u}_k^+ + i \left[ \frac{1}{i} \right] \bar{u}_k^-. \]  

Canceling \( u_k^- \) we get (2.35).

For (ii), we have \( (\sigma_1\Phi_k, f) = (\sigma_1\Phi_k, f) = 0 \). Using \( J^* = -J \), \( J\sigma_1 = -\sigma_1J \), and (2.35),
\[ (\sigma_1\Phi_k, Jf) = -(J\sigma_1\Phi_k, f) = (\sigma_1J\Phi_k, f) = (\sigma_1(-i\Phi_k + 2i \left[ \frac{1}{i} \right] u_k^+), f) = (2i \left[ \frac{1}{i} \right] u_k^+, f). \]  

Similarly \( (\sigma_1\Phi_k, Jf) = (2i \left[ \frac{1}{i} \right] \bar{u}_k^+, f) \). This shows (ii). □

Note, in deriving (2.35) if we cancel \( u_k^+ \) instead of \( u_k^- \), we get
\[ J\Phi_k = -i\Phi_k + 2i \left[ \frac{1}{i} \right] \bar{u}_k^-. \]  

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2.3 Decay estimate

In the following two subsections we prove decay estimates for $e^{tL}$ with the constant independent of $n$. This independence is essential for our analysis of the nonlinear dynamics both inside a neighborhood of an excited and away from bound states. For example, it ensures that the time spent traveling between bound states is no longer than $O(n^{-4-2\delta})$.

This independence cannot be achieved if we restrict ourselves of $E_c$, the continuous spectral subspace, because the projection $P_c^L$ as an operator acting on $L^1$ is of order $O(n^{-6})$ due to the presence of $u_k$. Suppose $F$ is the total nonlinearity in the equation of the perturbation $h$. Our choice of parameters $a(t)$ and $\theta(t)$ makes $P_mF = 0$, but does not make $F \in E_c$. To avoid the large constant problem, we absorb the range of $P_k^c$, $k < m$, which have exponential decay, into $E_c$. The range of $P_k$ for $k < m$, which have exponential growth, is left out and will be taken care of using the evolution with correct time direction.

Define $E^c_\delta$ as the direct sum of $E^L_i$ and eigenspaces whose eigenvalues have negative real parts

$$E^c_\delta = E^L_i \oplus \text{span}_C \{ \sigma_3 \Phi_k, \sigma_3 \Phi_k : 0 \leq k < m \}. \quad (2.40)$$

Its corresponding projection is denoted as

$$P^c_\delta f = P^L_i f + \sum_{k<m} P^c_k(f) = f - P_d f, \quad P_d f = \sum_{k=0}^K P_k(f). \quad (2.41)$$

We extend the definition of $P^c_\delta$ to scalar functions by $P^c_\delta \varphi = J^{-1} P^c_\delta [\varphi]$, and similarly for $P_d$. If a scalar function $\varphi$ satisfies $[\varphi] \in E^c_\delta$, then $(\sigma_1 \Phi_k, [\varphi]) = 0$ for all $k$.

The next lemma is on the uniform bound of $H^1$-norm of $e^{tL} P^c_\delta \varphi$ for $t \geq 0$.

**Lemma 2.8** For any scalar function $\varphi \in H^1$ we have

$$\left\| e^{tL} P^c_\delta \varphi \right\|_{H^1} \leq C \| \varphi \|_{H^1}, \quad (t \geq 0), \quad (2.42)$$

where the constant $C$ is independent of $n$ and $t \geq 0$.

**Proof.** From (2.41) and (2.19), we have

$$e^{tL} P^c_\delta \varphi = e^{tL} P^L_i \varphi - \sum_{k<m} \left[ \tilde{c}_k (\sigma_1 \sigma_3 \Phi_k, \varphi) e^{-\bar{\lambda}_k t} \sigma_3 \Phi_k + c_k (\sigma_1 \sigma_3 \Phi_k, \varphi) e^{-\lambda_3 t} \sigma_3 \Phi_k \right]. \quad (2.43)$$

By Lemma 2.5 we have $\| \Phi_k \|_{H^1} = O(1)$ for all $k < m$. From this and Re $\lambda_k > 0$ for all $k < m$, we can find a constant $C > 0$ independent of $n$ such that

$$\left\| e^{tL} P^c_\delta \varphi \right\|_{H^1} \leq \left\| e^{tL} P^L_i \varphi \right\|_{H^1} + C \| \varphi \|_{H^1}. \quad (2.44)$$

Moreover, by following the proof of [31, (2.6)], we see that there exists a constant $C$ independent of $n$ such that

$$\left\| e^{tL} P^c_\delta \varphi \right\|_{H^1} \leq C \left\| P^L_i \varphi \right\|_{H^1}. \quad (2.45)$$

Again, since $\| \Phi_k \|_{H^1} = O(1)$ for all $k$, we also have $\left\| P^L_i \varphi \right\|_{H^1} \leq C \| \varphi \|_{H^1}$ for some constant $C$ which is independent of $n$. From this, (2.41), and (2.45), Lemma 2.8 follows. □
Lemma 2.9 If a scalar function $\eta$ satisfies $[\eta] \in E_c^2$, then
\[
\|\eta - P_c^H \eta\|_{L^\infty_{3\tau}} \lesssim n^2 \|\eta\|_{L^2_{loc}} + \sum_{k<m} \|\langle \tilde{u}_k \rangle, P_c^H \eta\|.
\] (2.46)

Proof. Write $\eta' = P_c^H \eta$ and
\[
\eta - \eta' = (1 - P_c^H)\eta = \sum_k (\tilde{\phi}_k, \eta) \tilde{\phi}_k.
\] (2.47)
For $k \geq m$, $|\langle \tilde{\phi}_k, \eta \rangle| \leq ok$ where $ok$ denotes $O(n^2 \|\eta\|_{L^2_{loc}})$. For $k < m$, by Lemma 2.6 (ii),
\[
(\tilde{\phi}_k, \eta) + ok = (u_k^+, \eta) = (u_k^-, \eta) = (u_k^-, \eta') + (u_k^-, \eta - \eta').
\] (2.48)
Since $\|u_k^-\|_{L^2_{loc}} \lesssim n^2$,
\[
(u_k^-, \eta - \eta') = \sum_{j=0}^K (u_k^-, \langle \tilde{\phi}_j \rangle, \eta) \tilde{\phi}_j = ok.
\] (2.49)
The above show the lemma. \hfill \Box

The following lemma provides decay estimates for $e^{-itH}u_j^-$.  

Lemma 2.10 Let $H_s$ be the self-adjoint realization of $-\Delta$ on $L^2(\mathbb{R}^3)$. Let $V$ be a localized real potential so that $H_s + V$ satisfies the decay and singular decay estimates \[2.3\] and \[2.4\]. Let $0 < n < n_0 \ll 1$, $a > 0$, and $z = a + n^4i$. Let $\varphi(t) = n^2(H_s + V - z)^{-1/2}e^{-it(H_s + V)}P_c g$ with $\|g\|_{L^1} \lesssim 1$ and $P_c = P_{H_s + V}$. Then for all $p \in (3, \infty)$, $m = \frac{1}{2} - \frac{3}{2p} \in [0, 1/2]$,
\[
\|\varphi(t)\|_{L^p} \lesssim t^{-m}(1 + t)^{-m - \min(m, 1/4)}, \quad \forall t > 0.
\] (2.50)
Above the $p$-dependent constant is uniform in $a \in [a_1, a_2] \subset (0, \infty)$ and independent of $t$ and $n$.

Proof. The case $V = 0$ is postponed to Subsection 2.4. For general $\tilde{H} = H_s + V$, by resolvent expansion and Duhamel’s formula,
\[
\varphi(t) = n^2(H_s - z)^{-1/2}e^{-it\tilde{H}}P_c g + n^2(H_s - z)^{-1} \int_0^t e^{-i(t-s)\tilde{H}}V e^{-is\tilde{H}}P_c g
\] + $n^2(H_s - z)^{-1}V(\tilde{H} - z)^{-1}e^{-it\tilde{H}}P_c g$.

Denote $\tilde{\alpha}_p(t) := t^{-m}(1 + t)^{-m - \min(m, 1/4)}$. By the estimate for $H_s$, \[2.3\], \[2.4\], and the proof for Lemma 2.5 (i), the above is bounded by
\[
\|\varphi(t)\|_{L^p} \lesssim \tilde{\alpha}_p(t) + \int_0^t \tilde{\alpha}_p(t - s) \langle s \rangle^{-3/2} ds + n^2 \langle t \rangle^{-3/2} \lesssim \tilde{\alpha}_p(t).
\] (2.51) \hfill \Box

The following is the main result of this subsection.

Lemma 2.11 (Decay estimate) For any scalar function $\varphi \in L^{9/8} \cap L^{3/2}$,
\[
\left\| e^{itP_c^\varphi} \varphi \right\|_{L^\infty_{s+t} L^2} \leq C_{\alpha_\infty}(t) \|\varphi\|_{L^{9/8} \cap L^{3/2}}, \quad (t \geq 0).
\] (2.52)

For $3 < p < 6$ and any scalar function $\varphi \in L^p$,
\[
\left\| e^{itP_c^\varphi} \varphi \right\|_{L^p} \leq C_p \alpha_p(t) \|\varphi\|_{L^p}, \quad (t \geq 0).
\] (2.53)

Above the constants are independent of $n$ and $\varphi$, and
\[
\alpha_\infty(t) := t^{-1/2} (t^{-2/3}, \quad \alpha_p(t) := t^{-\frac{3}{2} + \frac{3}{p}} \langle t \rangle^{\frac{3}{2p}}.
\] (2.54)
Remark. (i) For (2.52), we could have chosen $\varphi \in L^q \cap L^{3/2}$, $\frac{12}{11} \leq q < \frac{6}{5}$. Then $\alpha_\infty(t) = t^{-1/2} (t)^{-s}$, with $s = 3/q - 2 \in (1/2, 3/4]$ by the same proof. The exponent $q = \frac{12}{11}$ gives the optimal decay rate that Lemma 2.10 provides for $e^{-itH} P_c u_j^-$. However, when we estimate $\|x_1^3\|_{L^3}, \|\nabla^3 \varphi\|_{L^3}, \|\nabla^3 \varphi\|_{L^p}$, we prefer a larger $q$. For convenience we choose $q = 9/8$.

(ii) Suppose we keep $q = \frac{12}{11}$ with $\alpha_\infty(t) = t^{-1/2} (t)^{-s}$, and estimate $\|x_1^3\|_{L^{12/11}} \lesssim \|\nabla^3 \varphi\|_{L^3}$ (by the same reasons, we need $\frac{6}{5} < p < 6$).

(iii) These estimates are false if $P_c^x$ is replaced by $P_c$. Suppose the contrary, then they would be also true if $P_c^x$ is replaced by $P_d^x = P_c^x - P_c$. Consider the case $m = 1$ and $\varphi = \phi_0$ the $e_0$-eigenfunction of $-\Delta + V$. Then

$$\|e^{iL_p^d} \|_{L^p} \sim e^{-c n^4 t}, \quad \|\varphi\|_{L^{p'}} \sim 1.$$

(2.55)

However the former is not bounded by $C t^{-k}$ for all $t > 0$, for any $k > 0$ and $C$ independent of $n$.

Proof. Denote $\eta(t) = e^{iL_p^c} \|\varphi\|$ and $\eta' = P_c^1 e^{iH} \eta$. Lemma 2.10 implies

$$\|\eta\|_X \lesssim \|\eta'\|_X + \sum_{k < m} \|\eta_k, \eta'\|, \quad X = L^\infty + L^2.$$

(2.56)

Denote $L = JH + W_1$ with $W_1 = \left[\begin{array}{cc} 0 & 0 \\ -2\pi Q, 0 \end{array}\right]$. By Duhamel’s formula,

$$\eta'(t) = e^{iH} P_c^1 e^{iH} \|\varphi\| + \int_0^t P_c^1 e^{(t-s)H} W_1 \eta(s) ds.$$

(2.57)

By Lemma 2.10 (i),

$$J^{-1} P_c^1 \|\varphi\| = \varphi - J^{-1} Re \sum_{j=0}^K \Phi_j = \varphi - \sum_{j=0}^K (z_j \tilde{u}_j + \tilde{z}_j \tilde{u}_j)$$

(2.58)

where $z_j \in \mathbb{C}$ are bounded by $\|\varphi\|_{L^2}$ for any $q \leq 2$. Using (2.18) for $j < m$ in particular $u_j^- = (H - i\lambda_j)^{-1} \Phi_j + O_{L^\infty}(n^2), \text{Im} i\lambda_j \sim n^4$, and by Lemma 2.10 (with $p = 9, 3$) and Lemma 2.10 (with $p = \infty$),

$$\|\eta'(t)\|_X \lesssim \alpha(t) \|\varphi\|_Y + \int_0^t \langle t-s \rangle^{-3/2} n^2 \|\eta(s)\|_X ds.$$

(2.59)

where $\alpha(t) = t^{-1/2} (t)^{-2/3}$ and $Y = L^{9/8} \cap L^{3/2}$. By the same reasons,

$$\|\eta_k, \eta'\| = \|\eta_k, (H - i\lambda_k)^{-1} \eta'\| + O(n^2 \|\eta'\|_X),$$

(2.60)

and

$$\|\eta_k, (H - i\lambda_k)^{-1} \eta'\| \lesssim n^2 \|H - i\lambda_k\|_{L^2} \sim n^2 \cdot \text{RHS of (2.59)}.$$

(2.61)

Summing the estimates, we get $\|\eta(t)\|_X \lesssim \text{RHS of (2.59)}$, which implies (2.52).

The estimate (2.53) is proved similarly with $X = L^p, Y = L^{p'}$ and $\alpha(t) = \alpha_p(t) \sim \max(\tilde{\alpha}_p(t), t^{-3/2} \|\varphi\|_Y))$. \qed
2.4 Decay estimate for free evolution with resonant data

In this subsection we prove Lemma 2.10 for $H_* = -\Delta$, i.e. decay estimate for $\varphi(t) = n^2(H_* - z)^{-1} e^{-itH_*} g$ where $z = a + n^4i$, $a \sim 1$, and $g \in L^1$. The operator $(H_* - z)^{-1} e^{-itH_*}$ has symbol $(\xi^2 - z)^{-1} e^{-it\xi^2}$ and thus its Green’s function $G$ is radial and, for $r = |x|$,

$$
G(r, t) = (2\pi)^{-3} \int_0^\infty (p^2 - z)^{-1} e^{-it p^2} \int_{|\omega| = 1} e^{i p \omega} dS(\omega) \, p^2 \, dp
$$

$$
= (2\pi)^{-3} \int_0^\infty (p^2 - z)^{-1} e^{-it p^2} \frac{4\pi \sin(rp)}{rp} \, p^2 \, dp
$$

$$
= \frac{1}{4\pi^2 tr} \int_\mathbb{R} (p^2 - z)^{-1} e^{-itr} e^{i rp} \, dp.
$$

It is well known that $G(r, 0) = \frac{1}{4\pi r} e^{i\sqrt{\pi} r}$. We are not aware of an explicit formula for $G(r, t)$. Because for $3 < p \leq \infty$ we have

$$
\|\varphi(t)\|_{L^p} = \|n^2 G(t) * g\|_{L^p} \lesssim n^2 \|G(t)\|_{L^p} \|g\|_{L^1} \lesssim n^2 \|G(t)\|_{L^{3/p}, \infty} \|G(t)\|_{L^{\infty}, \infty} \|g\|_{L^1},
$$

(2.62)

estimate (2.50) follows from (2.64) of the following lemma.

**Lemma 2.12** Let $H_*$ be the self-adjoint realization of $-\Delta$ on $L^2(\mathbb{R}^3)$. Let $G(x, t)$ be the Green’s function of the operator $(H_* - z)^{-1} e^{-itH_*}$ where $z$ is the same as in Lemma 2.11. Then $G(x, t) = G(|x|, t)$ and

$$
|G(r, t)| \lesssim \begin{cases} 
\frac{r^{-1/2}}{n^4 r + r^{1/2} + (t - r)^+}, & r > 1, \frac{t}{100}, \\
t^{-3/2}, & 1 \leq r < \frac{t}{100}, \\
\min(t^{-1/2}(1 + t)^{-1}, r^{-1}), & r < 1.
\end{cases}
$$

(2.63)

In particular,

$$
\|G(\cdot, t)\|_{L^\infty_2} \lesssim t^{-1/2}(1 + t)^{-1/2}(1 + n^4 t^{1/2})^{-1}, \quad \|G(\cdot, t)\|_{L^{1, \infty}_2} \lesssim 1.
$$

(2.64)

**Proof.** We may assume $a = 1/4$. The general case follows from change of variables and is uniform for $a \in [a_1, a_2]$. Introduce a regularizing factor $e^{-\delta p^2}$ and write $(p^2 - z)^{-1}$ as a time integral (using $\Re z > 0$)

$$
G(r, t) = \lim_{\delta \to 0^+} \frac{1}{4\pi^2 r} \int_\mathbb{R} \int_0^\infty e^{-it p^2 - \delta p^2 - i(2s p^2 - z)} + i rp \, ds \, dp
$$

$$
= \lim_{\delta \to 0^+} \frac{1}{4\pi^2 r} \int_\mathbb{R} \int_0^\infty e^{isz + \frac{p^2}{4\delta}} \int_\mathbb{R} e^{-i\alpha(p - \frac{r}{2\delta})^2} \, dp \, ds, \quad \alpha = s + t - i\delta.
$$

(2.65)

Using $\int_\mathbb{R} e^{-p^2} \, dp = \sqrt{\pi}$ and

$$
\int_\mathbb{R} e^{-i\alpha(p - \beta)^2} \, dp = \int_\mathbb{R} e^{-i\alpha(p - \beta)^2} \beta \, dp = \beta \int_\mathbb{R} e^{-i\alpha p^2} \, dp = \beta (i\alpha)^{-1/2} \sqrt{\pi},
$$

(2.66)
we get

\[
G(r, t) = \lim_{\delta \to 0+} \frac{1}{4\pi^2 r} \int_0^\infty e^{isz + \frac{r^2}{4s}} (i\alpha)^{-1/2} e^{-\sqrt{\pi} r} ds = \frac{1}{8\pi^{3/2} t} \int_0^\infty e^{i^2 s + \frac{r^2}{4s}} (s + t)^{-3/2} ds = \frac{1}{8\pi^{3/2} t} \int_0^\infty e^{i\Phi_s - 3/2} ds,
\]

(2.67)

where the phase \( \Phi \) is

\[
\Phi(r, s) = sz - tz + \frac{r^2}{4s}, \quad \Phi_s = z - \frac{r^2}{4s^2}, \quad \Phi_{ss} = \frac{r^2}{2s^3}.
\]

(2.68)

Note \( z = \frac{1}{4} + n^4 i \), \( \Phi_s \) vanishes at \( s = r/(2\sqrt{z}) \sim r \), and \( \text{Re} i\Phi < 0 \) for \( s > t \).

First note

\[
|G(r, t)| \lesssim \int_t^\infty s^{-3/2} ds = C r^{-1/2},
\]

(2.69)

which is valid for all \( r > 0 \) and \( t > 0 \). We will use a stationary phase argument to get a better estimate. The main contribution should come from \( I \equiv r(1 - \mu, 1 + \mu) \) where \( 0 < \mu \leq \frac{1}{20\pi} \) will be chosen. Comparing (2.69) and (2.70) below, it is clear we do not get a better estimate unless \( \mu \) is small.

We first consider the case \( r > 1 \).

Suppose \( t \in I \). The contribution from \( s \in (t, r + \mu s) \) is bounded by

\[
|\int_t^{r+\mu r} e^{i\Phi_s - 3/2} ds| \lesssim \int_t^r s^{-3/2} ds \lesssim \mu r^{-1/2}.
\]

(2.70)

The contribution from \( (r + \mu r, \infty) \) is, with \( t_1 = r + \mu r \),

\[
\int_t^\infty e^{i\Phi_s - 3/2} ds = \int_t^{t_1} \partial_s(e^{i\Phi_s}) \frac{1}{i\Phi_s} s^{-3/2} ds = \frac{1}{i\Phi_s} e^{i\Phi_s - 3/2} \bigg|_{s=t_1} + \int_{t_1}^\infty e^{i\Phi_s} J ds,
\]

(2.71)

where

\[
J = -\frac{\partial}{\partial s} \left( \frac{1}{i\Phi_s} s^{-3/2} \right) = \frac{\Phi_{ss}}{i\Phi_s^2} s^{-3/2} + \frac{3}{2i\Phi_s s^{3/2}}.
\]

(2.72)

For \( s \geq t_1 \), we have \( |\Phi_s| \sim n^4 + (s-r)/r \) and \( |\Phi_{ss}| \lesssim s^{-1} \). Thus \( |J| \lesssim (|\Phi_s|^{-1} + |\Phi_{ss}|^{-2}) s^{-5/2} \), and the boundary term is bounded by

\[
\left| \frac{1}{i\Phi_s} e^{i\Phi_s - 3/2} \bigg|_{s=t_1} \right| \lesssim \frac{1}{|\Phi_s(t_1)|} t_1^{-3/2} \lesssim \frac{r^{-3/2}}{n^4 + \mu}.
\]

(2.73)

Decompose \( (t_1, \infty) = (t_1, 100r) \cup (100r, \infty) \). On \( (t_1, 100r) \), we have

\[
\left| \int_{t_1}^{100r} e^{i\Phi_s} J ds \right| \lesssim \int_{t_1}^{100r} \frac{r^{2-5/2}}{(n^4 r + s-r)^2} ds \lesssim \frac{r^{-1/2}}{n^4 r + t_1 - r} = \frac{r^{-3/2}}{n^4 + \mu}.
\]

(2.74)

For \( s > 100r \), we have \( |\Phi_s| \gtrsim 1 \) and

\[
\left| \int_{100r}^\infty e^{i\Phi_s} J ds \right| \lesssim \int_{100r}^\infty s^{-5/2} ds \lesssim r^{-3/2}.
\]

(2.75)
We now choose $\mu \leq \frac{1}{200}$ so that $\mu r^{-1/2} \sim r^{-3/2}$. If $r \geq 1$, we can choose $\mu = \frac{1}{200} r^{-1/2} (1 + n^8 r)^{-1/2}$ and get for $t/r \in (1 - \mu, 1 + \mu)$

$$|G(r, t)| \leq \frac{r^{-1/2}}{n^4 r^3 + r^{1/2}}. \quad (2.76)$$

If $t \in (r + \mu r, 100r)$, we can take $t_1 = t$ in the above estimates and ignore the contribution from (2.70) to get the bound for $r > 1$

$$|G(r, t)| \lesssim \frac{r^{-1/2}}{n^4 r + |t - r|}. \quad (2.77)$$

If $t > 100r$, we can replace $100r$ by $t$ in (2.75) and ignore the contribution from (2.70) and (2.74) to get (also true for $r < 1$),

$$|G(r, t)| \lesssim t^{-3/2}. \quad (2.78)$$

If $t \in (\frac{r}{100}, r - \mu r)$ and $r > 1$, the additional contribution from $s \in (t, r - \mu r)$ is estimated as in (2.71)–(2.74) with $t_1 = r - \mu r$ and $100r$ replaced by $r/100$, and bounded by (2.77), which is smaller than (2.76) for $r > 1$.

If $t \in (0, \frac{r}{100})$, we have $|\Phi_s| \sim r^2 s^{-2}$ and $|\Phi_{ss}| \sim r^2 s^{-3}$ for $s \in (t, \frac{r}{100})$. The additional contribution from $s \in (t, \frac{r}{100})$ is estimated as in (2.71)–(2.74) and bounded by

$$\left[ r^{-2} s^{-1/2} \right]_{s=t}^{r/100} + \int_{s=t}^{r/100} r^{-2} s^{-1/2} ds \lesssim r^{-3/2} \quad (2.79)$$

which is smaller than (2.76) for $r > 1$.

We now consider the case $r < 1$. Let $\alpha > 0$ be a small number to be chosen. The contribution from $s \geq \max(t, \alpha r)$ is bounded by

$$|\int_{\alpha r}^{\infty} e^{i \Phi_s} s^{-3/2} 1_{s > t} ds| \leq |\int_{\alpha r}^{\infty} s^{-3/2} ds| = C(\alpha r)^{-1/2}. \quad (2.80)$$

If $t < \alpha r$, we have $|\Phi_s|^{-1} \sim r^{-2} s^2$, $|\Phi_{ss}|/|\Phi_s| \lesssim s^{-1}$, and the contribution from $s < \alpha r$ is

$$\int_{t}^{\alpha r} e^{i \Phi_s} s^{-3/2} ds = \left[ \frac{1}{i \Phi_s} e^{i \Phi_s} s^{-3/2} \right]_{s=t}^{s=\alpha r} + \int_{t}^{\alpha r} e^{i \Phi} J ds, \quad (2.81)$$

which is bounded by

$$r^{-2} (\alpha r)^{1/2}. \quad (2.82)$$

We want $(\alpha r)^{-1/2} \sim r^{-2} (\alpha r)^{1/2}$ and we can choose $\alpha = \frac{r}{100}$, which gives $r^{-1}$ bound for $r < 1$.

In conclusion, we have proved (2.63) for all $r > 0$ and $t > 0$. $\square$

Remark. (i) Lemma 2.10 for the free case can be considered an estimate of $(f, n^2 G(t)g)$. If (2.63) cannot be improved, then Lemma 2.10 cannot be improved, even if assuming further that one of $f, g$ is in $L^2_r$ (but not both). To see it, let $g$ be the characteristic function of the unit ball. Note $|I| \sim \mu r \gg 1$ for $r \gg 1$, thus $(n^2 G) (r, t)$ has the optimal size at $r \sim t$. Since translation does not change the $L^1 \cap L^2$-norm of $f$, we can put the support of $f$ at $r \sim t$, showing the optimality of Lemma 2.10.
(ii) Although the real part of the phase, $e^{-n^4(s-t)}$, is decaying, it does not seem to improve our estimate. In the case $t \sim r \sim n^{-8}$, we have $|I| \sim \mu r \sim n^{-4}$ and the estimate (2.70) does not improve because of the factor $e^{-n^4(s-t)}$, in view of the identity
$$\int_0^{n^4} e^{-n^4s} ds = C \int_0^{n^4} ds.$$

(iii) Since $|\text{Im } \Phi_s| \sim |s-r|/r \lesssim \mu$ for $s \in I$, $e^{i\Phi}$ almost has no oscillation on $I$ if $\mu^2 r \sim \mu \cdot |I| \ll 1$. Thus, if $\mu = \varepsilon r^{-1/2}$ with $0 < \varepsilon \ll 1$, then the upper bound in (2.70) is also a lower bound. In the case $t \sim r \gg \varepsilon^{-2} n^{-8}$, we have $\mu \ll n^4$ and $\mu r^{-1/2} \gg \varepsilon^{-3/2} n^{-8} \sim n^{-8+\mu}$. Thus (2.63) is optimal in this case.

### 2.5 Singular decay estimate

We will need to identify the main part of
$$\eta(t) = \int_0^t e^{(t-s)\mathcal{L}} P_c^s e^{-i\alpha s} f(s) ds$$
(2.83)
where $\alpha \in \mathbb{C}$ with $\text{Im } \alpha > 0$ and $f(s)$ is an $L^2$-valued function of $s$ with $\dot{f}$ smaller than $f$ in a suitable sense. We will rewrite it in matrix form in order to integrate by parts. Using
$$[\varphi] = \begin{bmatrix} \text{Re } \varphi \\ \text{Im } \varphi \end{bmatrix} = \text{Re } \varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
(2.84)
and denoting $R = (\mathcal{L} + i\alpha)^{-1}$, we have
$$\eta(t) = J^{-1} P_c^s \int_0^t e^{(t-s)\mathcal{L}} \text{Re } e^{-i\alpha s} f(s) \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds$$
$$= J^{-1} P_c^s \text{Re } \left( - Re^{-iat} f(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{\mathcal{L} R} f(0) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t e^{(t-s)\mathcal{L}} \text{Re } e^{-i\alpha s} \dot{f}(s) \begin{bmatrix} 1 \\ -i \end{bmatrix} ds \right).$$
(2.85)

To estimate the last two terms, we need the following lemma.

**Lemma 2.13 (Singular decay estimate)** There is a constant $C > 0$ independent of $\alpha \in \mathbb{C}$ with $\text{Im } \alpha > 0$, $n \in [0, n_0]$, and vector function $\Psi \in L^2_v$ so that
$$\left\| J^{-1} \text{Re } e^{\mathcal{L} (\mathcal{L} + i\alpha)^{-1}} P_c^s \Psi \right\|_{L^2_{loc}} \leq C \langle t \rangle^{-3/2} \| \Psi \|_{L^2_v}, \quad (t \geq 0).$$
(2.86)

**Proof.** Denote by $\eta$ the scalar function to be estimated, $\eta(t) = J^{-1} \text{Re } e^{\mathcal{L} R} P_c^s \Psi$, and $\eta' = P_c^H \eta$. Lemma 2.49 implies
$$\| \eta \|_{L^2_{loc}} \lesssim \| \eta' \|_{L^2_{loc}} + \sum_{k \leq m} |(\tilde{u}^{-}, \eta')|.$$  
(2.87)

Denote $\mathcal{L} = JH + W_1$ with $W_1 = \begin{bmatrix} 0 & 0 \\ 0 & -2\alpha Q_{\bar{c}} \end{bmatrix}$, $R = (\mathcal{L} + i\alpha)^{-1}$ and $R_0 = (JH + i\alpha)^{-1}$. By Duhamel’s formula and resolvent expansion,
$$\eta'(t) = P_c^H J^{-1} \text{Re } \left( e^{tJH} R_0(1 + W_1 R) P_c^s \Psi + \int_0^t e^{(t-s)JH} W_1 \eta(s) ds \right).$$
(2.88)
Denote the first term on the right side by $\eta'_1(t)$. Using $P^J_c \Psi = \Psi - \sum_k P_k(\Psi)$,

$$
\eta'_1(t) = J^{-1} \Re e^{iJH} R_0 P^J_c (\Psi - \sum_{k>m} P_k(\Psi)) + \Psi_1,
$$

where $\Psi_1 = P^J_c [\sum_{k\geq m} P_k \Psi + W_1 R^J_c \Psi]$ is localized with

$$
\| \Psi_1 \|_{L^2} \lesssim n^2 \| \Psi \|_{L^2} + n^2 \| R^J_c \Psi \|_{L^2_{\text{loc}}} \lesssim n^2 \| \Psi \|_{L^2}.
$$

Note that

$$
e^{tJH} = \begin{bmatrix} \cos(tH) & \sin(tH) \\ -\sin(tH) & \cos(tH) \end{bmatrix} = \sum_{\varepsilon = \pm 1} e^{i\varepsilon tH} \frac{1}{2}(I - i \varepsilon J),
$$

and

$$(JH + i\alpha)^{-1} = (H^2 - \alpha^2)^{-1}(-JH + i\alpha),
$$

We conclude, for $R_0 = (JH + i\alpha)^{-1}$,

$$
e^{tJH} R_0 = \sum_{\varepsilon = \pm 1} e^{i\varepsilon tH} (H + \varepsilon \alpha)^{-1} \frac{-i\varepsilon}{2}(I - i \varepsilon J).
$$

By (2.94), (2.50), Lemma 2.2, and Im $\alpha > 0$,

$$
\| J^{-1} \Re e^{tJH} R_0 P^J_c (\Psi + \Psi_1) \|_{L^2_{\text{loc}}} \lesssim \langle t \rangle^{-3/2} \| \Psi \|_{L^2_t}.
$$

For $k < m$, note

$$(I + iJ) \Phi_k = 2\bar{u}^+ \left[ \frac{\varepsilon}{\Lambda} \right], \quad (I + iJ) \bar{\Phi}_k = 2\bar{u}^- \left[ \frac{\varepsilon}{\Lambda} \right].
$$

Using (2.94) and writing $P_k \Psi = a\Phi_k + b\bar{\Phi}_k$, we have

$$
\Re e^{tJH} R_0 P_k \Psi = \Re \sum_{\varepsilon = \pm 1} e^{i\varepsilon tH} (H + \varepsilon \alpha)^{-1} \frac{-i\varepsilon}{2}(I - i \varepsilon J) P_k \Psi
= \Re e^{-itH} \left\{ (H - \alpha)^{-1} i(a\bar{u}^+ + bu^-) + (H + \bar{\alpha})^{-1} i(b\bar{u}^+ + \bar{u}^-) \right\} \left[ \frac{\varepsilon}{\Lambda} \right].
$$

By (2.84),

$$
J^{-1} \Re e^{tJH} R_0 P_k \Psi = e^{-iH} \left\{ (H - \alpha)^{-1} i(a\bar{u}^+ + bu^-) + (H + \bar{\alpha})^{-1} i(b\bar{u}^+ + \bar{u}^-) \right\}.
$$

Note Im($-\bar{\alpha}$) = Im $\alpha > 0$. By Lemma 2.2 and (2.18),

$$
\| P^J_c H^{-1} \Re e^{tJH} R_0 P_c \Psi \|_{L^2_{\text{loc}}} \lesssim n^2 \langle t \rangle^{-3/2} \| \Psi \|_{L^2_t}.
$$

Thus

$$
\| \eta'(t) \|_{L^2_{\text{loc}}} \lesssim \langle t \rangle^{-3/2} \| \Psi \|_{L^2_t} + \int_0^t \langle t-s \rangle^{-3/2} n^2 \| \eta(s) \|_{L^2_{\text{loc}}} ds.
$$

On the other hand, for $j < m$, by (2.15) again,

$$
| (\bar{u}^-_j, \eta') | = (\tilde{\phi}^*_j, (H - i\bar{\lambda}_j)^{-1} \eta') + O(n^2 \| \eta' \|_{L^2_{\text{loc}}}^2).
$$

Note Im $i\bar{\lambda}_j > 0$. By Lemma 2.2 and the previous decomposition of $\eta'$,

$$
| (\tilde{\phi}^*_j, (H - i\bar{\lambda}_j)^{-1} \eta'(t)) | \lesssim n^2 \| (H - i\bar{\lambda}_j)^{-1} \eta' \|_{L^2_{\text{loc}}} \lesssim n^2. \quad \text{RHS of (2.99)}.
$$

By (2.87) and summing the estimates, we get $\| \eta(t) \|_{L^2_{\text{loc}}} \lesssim \text{RHS of (2.99)}$, which implies the lemma.

□
2.6 Upper and lower spectral projections

In this subsection we prove various estimates for the spectral projections $\Pi_{\pm}$ which are defined in \[2.4\] and corresponds to $\pm \Im z \geq |E|$ in the spectrum of $L$. In particular, Lemma 2.16 allows us to replace $P_c^2$ by $P_\pm = P_c^2 \Pi_{\pm}$ in Lemmas 2.11 and 2.13.

Decompose $L = JA + W_2 = JH + W_1$ where $A = -\Delta + |E|$, $W_2 = J(V + K^2) + W_1$, and $W_1 = \begin{bmatrix} 0 & 0 \\ -2K^2 & 0 \end{bmatrix}$. Let $R(z) = (L - z)^{-1}$, $R_0(z) = (JA - z)^{-1}$ and $R_1(z) = (JH - z)^{-1}$ be their resolvents. Note $R_0(z)$ can be decomposed as

$$R_0(z) = (JA - z)^{-1} = \begin{bmatrix} -z & A \\ -A & -z \end{bmatrix}^{-1} = (A^2 + z^2)^{-1} \begin{bmatrix} -z & -A \\ A & -z \end{bmatrix}$$

(2.102)

$R_1(z)$ has a similar formula with $A$ replaced by $H$.

Let $\Gamma_{\pm}$ be contours about the upper and lower continuous spectra $\Sigma_{\pm} = \pm |E|, +\infty)$, respectively. For an eigenvalue $\lambda$ of $L$, let $\Gamma_{\lambda}$ be a small circle centered at $\lambda$ with radius $\sim n^4$. All contours are oriented clockwise and do not intersect. Let $P_\star = \frac{1}{2\pi i} \int_{\Gamma} R(z) dz$, $* = c\pm, \lambda$, be their corresponding spectral projections. Decompose $P_c^2$ as the sum of its upper and lower half plane components:

$$P_c^2 = P_+ + P_-, \quad P_\pm = P_{c\pm} + P_{L\pm}, \quad P_{L+} = \sum_{k<m} P_{-\lambda_k}, \quad P_{L-} = \sum_{k<m} P_{-\lambda_k}.$$  \hspace{1cm} (2.103)

Also denote

$$\Pi_{\pm} = P_{\pm} + P_{R\pm}, \quad P_{R+} = \sum_{k<m} P_{\lambda_k}, \quad P_{R-} = \sum_{k<m} P_{\lambda_k}.$$  \hspace{1cm} (2.104)

Note $P_\pm = P_{c\pm} \Pi_{\pm}$.

Let

$$\delta_0 = \frac{1}{4} \min\{|e_K|, |e_k - e_{k-1}| : 1 \leq k \leq K\}, \quad \tau_0 = \frac{1}{2} e_K - e_m.$$  \hspace{1cm} (2.105)

Note $\Im \lambda_N < \tau_0 - \delta_0 < \tau_0 + \delta_0 < |E|$.

We collect a few estimates for $R_0(z)$ and $R(z)$.

**Lemma 2.14** Let $\sigma_d^0 = \{\pm i(e_k - e_m) : 0 \leq k \leq K\}$, $s > \frac{1}{2}$ and $1 < p < \infty$. We have

$$\|R_0(z)\|_{L^2_2 \rightarrow L^2_2} \leq C \langle z \rangle^{-1/2}, \quad z \notin i\mathbb{R},$$

$$\|R_1(z)\|_{L^2_2 \rightarrow L^2_2} + \|R(z)\|_{L^2_2 \rightarrow L^2_2} \leq C \langle z \rangle^{-1/2}, \quad z \notin i\mathbb{R}, \quad \text{dist}(z, \sigma_d^0) \geq \delta_0,$$

$$\|R(z)\|_{L^2_2 \rightarrow L^2_2} \leq C n^{-4}, \quad 0 < |\Re z| < \frac{1}{4} \gamma_0 n^4, \quad \text{dist}(z, \sigma_d^0) < \delta_0,$$

$$\|R_0(z)\|_{L^p \rightarrow L^p} + \|R_1(z)\|_{L^p \rightarrow L^p} + \|R(z)\|_{L^p \rightarrow L^p} \leq C_p \langle z \rangle^{-1+\varepsilon_p}, \quad |\Im z| = \tau_0.$$  \hspace{1cm} (2.106)

Above $\varepsilon_p = 0$ for $p > 1$ and $0 < \varepsilon_1 < 1$, and the constants are uniform in $n \in [0, n_0]$.

**Proof.** The first estimate is by the scalar case proved in [1] Remark 2 in Appendix A] and by (2.102). The second estimate is valid if $R(z)$ is replaced by $R_1(z) = (JH - z)^{-1}$, which is by the scalar case proved in [15 Theorem 9.2] and by (2.102) with $A$ replaced by $H$. It
is true for \( R(z) \) using the resolvent series \( R(z) = R_1(z) \sum_{k=0}^{\infty} |W_1 R_1(z)|^k \) and the fact \( W_1 \) is a small localized matrix potential. The third estimate is proved in [31, Lemma 2.5].

The last estimate for \( R_0(z) \) is by the scalar case proved in [31, Lemma 7.4] and by (2.102). It is true for \( R_1(z) \) because \( \|(H - z)^{-1}\|_{L^p \to L^p} \lesssim (z)^{-1+\varepsilon_p} \) for \( |\text{Im } z| = \tau_0 \), which follows from

\[
(H - z)^{-1} f = (H - z)^{-1} P_z f + (H - z)^{-1} \sum_{k=0}^{K} (\tilde{\phi}_k, f) \tilde{\phi}_k
\]

\[
= W^{-1}(A - z)^{-1} W P_z f + \sum_{k=0}^{K} (\tilde{e}_k - z)^{-1} (\tilde{\phi}_k, f) \tilde{\phi}_k,
\]

where \( W \) is the wave operator between \( H \) and \( A \) and \( \tilde{\phi}_k \) are normalized eigenfunctions of \( H \) with eigenvalues \( \tilde{e}_k \). Finally, the estimate for \( R(z) \) follows from the resolvent series \( R(z) = R_1(z) \sum_{k=0}^{\infty} |W_1 R_1(z)|^k \) again.

**Lemma 2.15** Let \( K_{\pm} = \Pi_{\pm}(J \mp i) \), initially defined from \( L^2 \) to \( L^2 \), \( s > 1 \). For any \( 1 \leq p \leq q < \infty \), there is a constant \( c \) so that \( \|K_{\pm} u\|_p \leq c \|u\|_q \) for any \( u \in L^2 \cap L^q \).

This is clear for the reference self-adjoint operator \( JA \), for which \( K_{\pm} = 0 \).

**Proof.** Recall \( R_0 \) is decomposed in (2.102), and \( MJ = -iM \) and \( MJ = iM \). As \( z \) approaches \( \Sigma_{\pm} = [|E|, +\infty] \), the upper continuous spectrum of \( A \), the resolvent \( (A + iz)^{-1} \) is unbounded, and we write

\[
R_0(z)J = -2iM(A - iz)^{-1}, \quad (z \sim \Sigma_{\pm}).
\]

Note right side is bounded. Similarly, as \( z \) approaches \( \Sigma_{-} = -\Sigma_{+} \), we write

\[
R_0(z)J = 2i\tilde{M}(A + iz)^{-1}, \quad (z \sim \Sigma_{-}).
\]

We now prove the bound for \( K_{+} \). The case of \( K_{-} \) is similar. Let \( \Gamma = \Gamma_{c+} \cup \Gamma_p \) and \( \Gamma_p = \cup_{k < m}(\Gamma_{\lambda_k} \cup \Gamma_{-\lambda_k}) \). By spectral projection formula and resolvent expansion,

\[
\Pi_{+} = \frac{1}{2\pi i} \int_{\Gamma} R(z)dz = \frac{1}{2\pi i} \int_{\Gamma} [1 + R_0(z)W_0 + R_0(z)W_0R(z)W_0]R_0(z)dz.
\]

By (2.108),

\[
\Pi_{+}(J - i) = -\frac{1}{\pi} \int_{\Gamma} [1 + R_0(z)W_0 + R_0(z)W_0R(z)W_0]M(A - iz)^{-1}dz = K_0 + K_1 + K_2.
\]

The above sum is well-defined as operators from \( L^2 \) to \( L^2 \).

Note that \( K_0 \) is zero since \( (A - iz)^{-1} \) is regular inside \( \Gamma \) and the rest of the integrand of \( K_0 \) does not depend on \( z \).

For \( K_1 \), the integral over \( \Gamma_{c+} \) is bounded from \( L^q \) to \( L^p \) by Lemma 7.6 of Cuccagna [C2] using Coifman-Meyer multi-linear estimates. The integral over \( \Gamma_p \) is also bounded from \( L^q \) to \( L^p \) since

\[
\int_{\Gamma_p} \|R_0(z)W_0M(A - iz)^{-1}\|_{L^q \to L^p} |dz| \leq \int_{\Gamma_p} \|R_0(z)\|_{L^p \to L^p} \|(A - iz)^{-1}\|_{L^p \to L^q} \lesssim \int_{\Gamma_p} n^{-4} \cdot 1 \lesssim 1.
\]
For $K_2$, the integrand is analytic in $z$ and has enough decay in $B(L^2 \rightarrow L^2_{-s})$ in $|z|$ by Lemma 2.14. Thus we can change the contour to $\Gamma = \mathbb{R} + \tau_0 i$. By Lemma 2.14, $\|K_2\|_{L^q \rightarrow L^p}$ is bounded by

$$
\int_{\Gamma} \|R_0(z)\|_{L^p \rightarrow L^p} \cdot \|R(z)\|_{L^q \rightarrow L^q} \cdot \|R_0(z)\|_{L^q \rightarrow L^q} \, dz | \leq C. 
$$

(2.113)

Summing the estimates we get the lemma. □

**Lemma 2.16** The projection operators $\Pi_\pm$ are bounded from $L^2_s$ to $L^2_{-s}$, $s > 1$, and from $L^p$ to $L^p$ for any $1 \leq p \leq \infty$.

**Proof.** From the definition of $K_\pm$ in Lemma 2.15 we have

$$
K_+ = \Pi_+(J - i), \quad K_- = (1 - \Pi_+ - \Pi_0)(J + i),
$$

(2.114)

where $\Pi_0 = \sum_{j \geq m} P_j$ is bounded in $L^p$. Thus

$$
\Pi_+ = \frac{i}{2} [K_+ + K_- - (1 - \Pi_0)(J + i)],
$$

(2.115)

where shows $\Pi_+ \Pi_-$ is bounded in $L^p$ for $p < \infty$ by Lemma 2.15. Similarly $\Pi_-$ and $\Pi_\pm^*$ are bounded in $L^p$ for $p < \infty$. The boundedness of $\Pi_\pm$ in $L^\infty$ follows from that of $\Pi_\pm^* \Pi_\pm$ in $L^1$ and duality.

As a corollary, Lemmas 2.11 and 2.13 hold with $P_\pm^2$ replaced by $P_\pm$ since $P_\pm = P_\pm^2 \Pi_\pm$.

### 2.7 Fermi Golden Rule

In this subsection we prove Corollary 2.20 which gives the key resonance coefficients in the normal form equations in Lemmas 3.7 and 3.8.

For any $k \neq m$, recall (2.36) that

$$
\Phi_k = [\begin{array}{cc} 1 & 1 \\ i & -i \end{array}] \bar{u}_k^+ + [\begin{array}{cc} 1 & 1 \\ i & -i \end{array}] \bar{u}_k^-.
$$

(2.116)

From (2.18), we introduce $\Phi_k^+$ and $\Phi_k^-$ which satisfy the equation $\Phi_k = \Phi_k^+ + \Phi_k^-$ where $\Phi_k^+$ is localized and

$$
\Phi_k^+ = [\begin{array}{c} \frac{1}{2} (H - \bar{\alpha}_k)^{-1} \bar{\phi}_k^* \\ -\frac{1}{2} \phi_k + O_{L^2_r}(n^2) \end{array}], \quad \Phi_k^- = [\begin{array}{c} \frac{1}{2} \phi_k + O_{L^2_r}(n^2) \\ \frac{1}{2} (H - \bar{\alpha}_k)^{-1} \bar{\phi}_k^* \end{array}].
$$

(2.117)

Note that $\phi_k^* = O_{L^2_r}(n^2)$ is defined in (2.18) and $\alpha_k = i\lambda_k = |e_k - e_m| + O(n^2)$ with $\text{Im} \alpha_k > 0$. Moreover, since $\Phi_k = \Phi_k^+ + \Phi_k^-$, from (2.19), we see that for all function $f \in L^2(\mathbb{R}^2, C^2)$

$$
P_k f = c_k(\sigma_1 \Phi_k, f) \Phi_k^+ + b_k(\sigma_1 \Phi_k, f) \Phi_k^- + c_k(\sigma_1 \Phi_k, f) \Phi_k^+ + b_k(\sigma_1 \Phi_k, f) \Phi_k^-,
$$

(2.118)

Since $\Phi_k^+$ is localized and $\Phi_k^- = O_{L^2_r}(n^2)$, it follows from Lemma 2.23 and (2.118) that for all function $f$ such with $\|f\|_{L^2_r} = O(\delta)$

$$
(P_k - P_k^{IH}) f = O(n^2 \delta) \Phi_k + O(n^2 \delta) \Phi_k^- + O(\delta) \Phi_k^+ + O(\delta) \Phi_k^- + O_{L^2_r}(n^2 \delta)
$$

(2.119)
Throughout this subsection, let $\omega$ and $\epsilon$ be two fixed numbers such that
\[
\omega \pm \text{Im} \lambda_k = O(1) \neq 0, \quad 0 < \epsilon \ll 1.
\] (2.120)
Let $\alpha = -i\omega + \epsilon$ and
\[
R = (L - \alpha)^{-1}, \quad R_0 = (JH - \alpha)^{-1}.
\] (2.121)
Note that we have
\[
R = R_0 + R_0 WR_0 + R_0 WRR_0,
\] (2.122)
where $W$ is a localized potential which is of order $\|Q\|^2$.

**Lemma 2.17** For any $k \neq m$, there exist $C > 0$ independent of $\epsilon$ and $n$ such that
\[
\|R_0 \Phi_k^\pm\|_{L^2_{\text{loc}}}^2, \quad \|R_0 \Phi_k^\pm\|_{L^2_{\text{loc}}}^2, \quad \|(R_0)^* \sigma_1 \Phi_k^\pm\|_{L^2_{\text{loc}}}^2, \quad \|(R_0)^* \sigma_1 \Phi_k^\pm\|_{L^2_{\text{loc}}}^2 \leq Cn^2.
\] (2.123)

**Proof.** We write
\[
R_0 = (JH - \alpha)^{-1} = (H^2 + \alpha^2)^{-1} \begin{bmatrix} -\alpha & -H \\ H & -\alpha \end{bmatrix}.
\] (2.124)
Then, it follows that
\[
R_0 \Phi_k^- = \begin{bmatrix} 1 \\ i \end{bmatrix} (H + i\alpha)^{-1}(H - \alpha)^{-1} \Phi_k,
\]
\[
R_0 \Phi_k^- = \begin{bmatrix} 1 \\ i \end{bmatrix} (H - i\alpha)^{-1}(H - \alpha)^{-1} \Phi_k,
\]
\[
(R_0)^* \sigma_1 \Phi_k^- = \begin{bmatrix} 1 \\ -i \end{bmatrix} (H + i\alpha)^{-1}(H - \alpha)^{-1} \Phi_k,
\]
\[
(R_0)^* \sigma_1 \Phi_k^- = \begin{bmatrix} 1 \\ -i \end{bmatrix} (H - i\alpha)^{-1}(H - \alpha)^{-1} \Phi_k.
\] (2.125)
Since $\text{Re} \alpha > 0$ and $\text{Im}(\alpha_k) > 0$ and $\phi_k \in O_{L^2}(n^2)$, our claim follows. \qed

**Lemma 2.18** There exists $C > 0$ such that for any function $f, g \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ with $f = O_{L^2}(n)$:
\[
|\langle (f, (L - \alpha)(P_c^\pm P_c^JH)(\Phi_k^-) \rangle | \leq Cn^4,
\]
\[
|\langle (f, (P_c^\pm P_c^JH)(L - \alpha)(P_c^\pm) g) | \leq Cn^4.
\] (2.126)

**Proof.** Since the proof of both estimates in (2.126) are similar, we shall only prove the first estimate of (2.126). From (2.119), we have
\[
(P_c^\pm - P_c^JH)g = \sum_{k=0}^K \{O(n^3) \Phi_k + O(n^3) \Phi_k + O(n) \Phi_k + O(n) \Phi_k + O_{L^2}(n^2)\}.
\] (2.127)
Since $L \Phi_k = \lambda_k \Phi_k$ and $\lambda_k - \alpha, \bar{\lambda_k} - \alpha$ are all non-zero order one, we get
\[
(f, P_c^\pm R(P_c^\pm - P_c^JH) g) = O(n^4) + (f, P_c^\pm R O(n) \Phi_k + O(n) \Phi_k).
\] (2.128)
By similarity, we only need to show that $|\langle f, P_c^\pm R \Phi_k^- \rangle | \leq Cn^3$. Let $\tilde{g} = [WR_0 + WRWR_0] \Phi_k^-$. By Lemma 2.17 $\|\tilde{g}\|_{L^2} \leq Cn^4$. Then, using (2.122), (2.119) and Lemma 2.17 we have
\[
|\langle f, P_c^\pm R \Phi_k^- \rangle | = |\langle (P_c^\pm)^* f, R_0 \Phi_k^- + R_0 \tilde{g} \rangle |
\]
\[
\leq |\langle (P_c^\pm)^* f, R_0 \Phi_k^- \rangle | + |\langle (R_0)^* (P_c^\pm)^* f, \tilde{g} \rangle | + Cn^3
\]
\[
\leq C \left\{ \sum_{j \neq m} |\langle \sigma_1 \Phi_j^-, R_0 \Phi_k^- \rangle | + \sum_{j \neq m} |\langle \sigma_1 \Phi_j^-, R_0 \Phi_k^- \rangle | + n^3 \right\}
\] (2.129)
\[
\leq C \left\{ \sum_{j \neq m} |\langle \sigma_1 \Phi_j^-, R_0 \Phi_k^- \rangle | + \sum_{j \neq m} |\langle \sigma_1 \Phi_j^-, R_0 \Phi_k^- \rangle | + n^3 \right\}
\]
Note that from (2.117) and (2.125), we get
\[
(\sigma_1 \Phi_j^-, R_0 \Phi_k^-) \leq C n^4, \quad (\sigma_1 \Phi_j^-, R_0 \Phi_k^-) = 0.
\]
So, from (2.129), we obtain
\[
\|(f, P_c^g R \Phi_k^-)\| \leq C n^3.
\]
This completes the proof of Lemma 2.18. \(\Box\)

Corollary 2.19 For any function \(f, g \in L^2(\mathbb{R}^2, \mathbb{C}^2)\) with \(f, g = O_{L^2}(n)\), we have
\[
(f, P_c^g (L - \alpha)^{-1} P_c^g g) = (f, P_c^{JH_0}(J(H_0 - E) - \alpha)^{-1} P_c^{JH_0}g) + O(n^4).
\]

Proof. Using (2.122) and Lemma 2.18 we have
\[
(f, P_c^g (L - \alpha)^{-1} P_c^g g) = (f, P_c^{JH}(JH - \alpha)^{-1} P_c^{JH}g) + O(n^4).
\]
Now, since that \(H - (H_0 - E) = \kappa Q^2 = O(n^2)\) and \(P_c^{JH} - P_c^{JH_0} = O_{L^2}(n^2)\), we can use the same method as in Lemma 2.18 to obtain
\[
(f, P_c^g (L - \alpha)^{-1} P_c^g g) = (f, P_c^{JH_0}(J(H_0 - E) - \alpha)^{-1} P_c^{JH_0}g) + O(n^4).
\]
This completes the proof of Corollary 2.19. \(\Box\)

Corollary 2.20 Let \(f, g \in L^2(\mathbb{R}^3, \mathbb{C})\) be a localized real functions of order \(n\) and let \(f_1 = [\frac{1}{i}] f\) and \(g_1 = [\frac{1}{i}] g\). We then have
\[
\langle [\frac{1}{i}] f, (L - \alpha)^{-1} P_c^g [\frac{1}{i}] g \rangle = -2(f, P_c^{JH_0}(H_0 - E - i\alpha)^{-1} P_c^{JH_0}g) + O(n^4),
\]
\[
\langle [\frac{1}{i}] f, (L - \alpha)^{-1} P_c^g [\frac{1}{i}] g \rangle = -2(f, P_c^{JH_0}(H_0 - E + i\alpha)^{-1} P_c^{JH_0}g) + O(n^4),
\]
\[
\langle [\frac{1}{i}] f, (L - \alpha)^{-1} P_c^g [\frac{1}{i}] g \rangle = O(n^4),
\]
\[
\langle [\frac{1}{i}] f, (L - \alpha)^{-1} P_c^g [\frac{1}{i}] g \rangle = O(n^4).
\]

Proof. By Corollary 2.19 we have
\[
\langle [\frac{1}{i}] f, (L - \alpha)^{-1} P_c^g [\frac{1}{i}] g \rangle = \langle [\frac{1}{i}] f, P_c^{JH_0}(J(H_0 - E) - \alpha)^{-1} P_c^{JH_0} [\frac{1}{i}] g \rangle + O(n^4).
\]
On the other hand,
\[
(J(H_0 - E) - \alpha)^{-1} P_c^{JH_0} [\frac{1}{i}] g = (H_0 - E + \alpha^2)^{-1} P_c^{JH_0} \begin{bmatrix} -\alpha & -(H_0 - E) \\ H_0 - E & -\alpha \end{bmatrix} [\frac{1}{i}] g = [\frac{-1}{i}] \begin{bmatrix} (H_0 - E - i\alpha)^{-1} \\ (H_0 - E - i\alpha)^{-1} \end{bmatrix} P_c^{JH_0} g.
\]
So, the first identity of our corollary follows. Similarly, we can prove all of the last three identities of the corollary. \(\Box\)

3 Equations and main terms

In our analysis we use different coordinate systems. When the solution is away from bound states, we use the orthogonal coordinates (1.18), i.e., we decompose the solution as a sum of different spectral components with respect to \(-A + V\). When the solution is near a nonlinear bound state, we use the linearized coordinates (2.17), i.e., decomposition with respect to the corresponding linearized operator instead. In subsection 3.1 we recall the equations and normal forms in orthogonal coordinates from [28]. The rest of this section is devoted to analysis in linearized coordinates. We will not use centered orthogonal coordinates (1.8), which is also mentioned in §1.
3.1 Orthogonal coordinates

Let \( t_0 \) be a fixed initial time. For \( t \geq t_0 \) we may decompose the solution with respect to \( H_0 \) as

\[
\psi(t) = \sum_{j=0}^{K} x_j(t)\phi_j + \xi, \quad \xi \in H_c(H_0), \quad \forall \ t \geq t_0. \tag{3.1}
\]

Then for \( t \geq t_0 \), as in \cite{28} Section 4 we have

\[
\begin{aligned}
i\dot{x}_j &= e_jx_j + (\phi_j, G), \quad (j = 0, \ldots, K), \\
i\partial_t \xi &= H_0 \xi + P_eH_0 G, \quad G := \kappa \psi^2 \bar{\psi}.
\end{aligned} \tag{3.2}
\]

Let

\[
G_3 := \kappa \sum_{j=0}^{K} x_j\phi_j \left( \sum_{j=0}^{K} x_j\phi_j \right) = \kappa \sum_{l,m,j=0}^{K} x_lx_m\bar{x}_j\phi_l\phi_m\phi_j. \tag{3.3}
\]

We then decompose \( \xi \) as (for details, see \cite{28} Section 4)

\[
\xi(t) = \xi^{(2)}(t) + \xi^{(3)}_{1}(t) + \xi^{(3)}_{2}(t) + \cdots + \xi^{(3)}_{5}(t), \quad \forall \ t \geq t_0, \tag{3.4}
\]

where

\[
\xi^{(2)}(t) := \sum_{l,m,j=0}^{K} x_lx_m\bar{x}_j(t)\xi_{lm}^j, \text{ with}
\]

\[
\xi_{lm}^j := -\kappa \lim_{r \to 0^+} \|H_0 - e_l - e_m + e_j - ri\|^{-1}P_eH_0 \phi_m\phi_l\phi_j, \tag{3.5}
\]

and, with \( u_j(t) = e^{ie_jt}x_j(t) \) which have less oscillation than \( x_j(t) \),

\[
\begin{aligned}
\xi^{(3)}_{1}(t) &:= e^{-iH_0(t-t_0)}\xi(t_0), \quad \xi^{(3)}_{2}(t) := -e^{-iH_0(t-t_0)}\xi^{(2)}(t_0), \\
\xi^{(3)}_{3}(t) &:= -\int_{t_0}^{t} e^{-iH_0(t-s)}P_eH_0 \sum_{l,m,j=0}^{K} e^{i(e_l-e_m+e_j)s} ds (u_lu_m\bar{u}_j)\xi_{lm}^j ds, \\
\xi^{(3)}_{4}(t) &:= \int_{t_0}^{t} e^{-iH_0(t-s)}P_eH_0^{-1}(G - G_3 - \kappa \xi^{2}\bar{\xi}) ds, \\
\xi^{(3)}_{5}(t) &:= \int_{t_0}^{t} e^{-iH_0(t-s)}P_eH_0^{-1}(\kappa \xi^{2}\bar{\xi}) ds.
\end{aligned} \tag{3.6}
\]

We recall the following two lemmas from \cite{28}:

**Lemma 3.1 (Lemma 4.1 \cite{28})** Let \( p, p' \) such that \( 4 \leq p \leq 6, \ (p)^{-1} + (p')^{-1} = 1 \). Suppose that for a fixed time \( t \geq t_0 \) and for \( 0 < n \leq n_0 \ll 1 \), we have

\[
\max_j |x_j(t)| \leq 2n, \quad \|\xi(t)\|_{L^2_{loc} \cap L^p} \leq 2n, \quad \|\xi(t)\|_{L^2} \ll 1. \tag{3.7}
\]

Then for \( u_j(t) = e^{ie_jt}x_j(t) \),

\[
\|G\|_{L^1_{loc}} \max_j |\dot{u}_j| \lesssim n^3 \quad \text{and} \quad \|G - G_3 - \kappa \xi^{2}\bar{\xi}\|_{L^1_{loc} \cap L^{p'}} \lesssim n^2 \|\xi\|_{L^2_{loc}}. \tag{3.8}
\]
There are small positive constants $n_0$ and $\varepsilon_3$ such that the following hold. Suppose $\|\psi\|_{H^1} \leq n_0$ satisfies $\|\psi - (\psi, \phi_m)\phi_m\|_{L^2} \leq \varepsilon_3|\psi, \phi_m|$.  

(i) For any $0 < n < n_0$, there exist unique $a, \theta \in \mathbb{R}$ such that
\[
\psi = [Q_{m,n} + aR_{m,n} + h]e^{i\theta},
\]
where $Q_{m,n}$ and $R_{m,n}$ are given by Lemma 2.1 $P_m h = 0$, and $|n^{-1}a| + \|h\|_{H^1} \leq \varepsilon_3n$.  

(ii) There exist unique $n(\psi) \in (0, n_0)$ and $\theta \in \mathbb{R}$ such that $a = 0$. Moreover, if $\psi$ is decomposed as in (i) with respect to another $n$, then
\[
n(\psi) = n + \frac{a}{2Cn} + O(n^3), \quad C = \kappa \int \phi_m^4.
\]

(iii) If $\psi$ is decomposed as in (i) with respect to $n_1$ and $n_2$ with $\|h_j\| \leq \rho \leq \varepsilon_3n$, $|a_j| \leq C\rho^2$, and $|n_1 - n_2| \leq n^{-1}\rho^2$, then
\[
C(n_1^2 - n_2^2) + a_1 - a_2 = O(\rho|n_1 - n_2|).
\]
It then follows that for all \( k \neq m \),

\[
\zeta_k := \mathbf{j}^{-1} \text{Re}(z_k \Phi_k) = z_k \bar{u}_k^+ + \bar{z}_k u_k^-, \quad u_k^\pm := \frac{1}{2}(\bar{u}_k \pm \bar{v}_k). \tag{3.19}
\]

Substituting (3.17) into (1.1) and using \( L \) and \( R \), we get

\[
\partial_t h - L h = F_h \equiv i^{-1}(F + \hat{\theta}(Q + aR + h)) - a\dot{Q} - \dot{a}R, \tag{3.20}
\]

where

\[
F = \kappa(Q |h_\sigma|^2 + h_\sigma^2) + \kappa|h_\sigma|^2 h_\sigma, \quad h_\sigma = aR + h. \tag{3.21}
\]

We choose \( \dot{\theta} \) and \( \dot{a} \) so that \( P_m F_h = 0 \). Thus \( F_h = (1 - P_m)i^{-1}(F + \hat{\theta}(aR + h)) \) and

\[
\begin{aligned}
\dot{a} &= (c_m Q, \text{Im}(F + \hat{\theta} h)), \\
\dot{\theta} &= F_h \equiv -[a + (c_m R, \text{Re} F)] \cdot [1 + (c_m R, R)a + (c_m R, \text{Re} h)]^{-1}.
\end{aligned} \tag{3.22}
\]

Taking \( P_c^2 \) of (3.20), we get

\[
\partial_t \eta - \mathcal{L}\eta = P_c^2 i^{-1}(F + \hat{\theta}(aR + h)). \tag{3.23}
\]

Taking 2c_k(\sigma_1 \bar{\Phi}_k, [h]) of (3.20), \( k \neq m \), we get

\[
\dot{z}_k - \lambda_k z_k = Z_k := 2c_k(\sigma_1 \bar{\Phi}_k, [F_h]). \tag{3.24}
\]

A direct computation using (2.36) shows

\[
Z_k = -2c_k \left\{ (u_k^+, F) + (u_k^-, F) + [(u_k^+, h) + (u_k^-, \mathcal{H}) + (\bar{u}_k, R)a] \hat{\theta} \right\}. \tag{3.25}
\]

Let \( \omega_k := -\text{Im} \lambda_k \) and let \( p_k(t) = z_k(t)e^{i\omega_k t} \). We have

\[
\dot{p}_k = (\text{Re} \lambda_k) p_k + e^{i\omega_k t} Z_k. \tag{3.26}
\]

Also, for any \( k \neq m \), let \( r_k := e^{-\lambda_k t} z_k \), we have,

\[
\dot{r}_k = e^{-\lambda_k t} Z_k. \tag{3.27}
\]

Note that \( r_k = p_k \) for all \( k > m \) and \( r_k = e^{-\text{Re}(\lambda_k) t} p_k \) for \( k < m \). We shall use \( r_k \) in computing the normal form for the equation of \( a \).

**Definition 3.1** Denote \( I = \{0, 1, \cdots, K\}, I^* = \{0^*, 1^*, \cdots, K^*\} \). For all \( m \in I \), let \( I_{m} = \{m+1, \cdots, K\}, I_{<m} = \{0, \cdots, m-1\}, I_{m} = I \setminus \{m\}, I^*_m = I^* \setminus \{m^*\} \) and \( \Omega_m := I_m \cup I^*_m \). For \( j \in I_m \), let

\[
\lambda_j^* = \bar{\lambda}_j, \quad \omega_j^* = -\omega_j, \quad z_j^* = \bar{z}_j, \quad r_j^* = \bar{r}_j, \quad p_j^* = \bar{p}_j, \quad u_j^* = \bar{u}_j, \quad \text{and} \quad v_j^* = \bar{v}_j. \tag{3.28}
\]

It then follows that for all \( j \in \Omega_m \), we have \( z_j(t) = e^{-i\omega_j t} p_j(t) \) and \( r_j = e^{-\lambda_j t} z_j \).
3.3 Decomposition of a

Recall \( \dot{a} = (c_m Q, \text{Im}(F + \theta h)) \). Let \( F_1 := \kappa Q (2|\zeta|^2 + \zeta^2) \), \( A^{(2)} := c_m (Q, \text{Im} F_1) \) and \( A^{(3)} := c_m (Q, \text{Im}(F - F_1 + \theta h)) \). Then, we have \( \dot{a} = A^{(2)} + A^{(3)} \). We shall impose the boundary condition of \( a \) at \( t = T \), which is in fact the condition imposed on the choice of \( E = E(T) \). Hence, we have

\[
a(t) = a(T) + \int_{T}^{t} [A^{(2)}(s) + A^{(3)}(s)] \, ds.
\] (3.29)

Recall that

\[
\zeta = \sum_{k \in I_m} \zeta_k, \quad \zeta_k = z_k \bar{u}_k + \bar{z}_k u_k.
\] (3.30)

Therefore,

\[
\text{Im} \zeta_k \zeta_l = \text{Im}[z_k z_l (\bar{u}_k \bar{u}_l + \bar{u}_k u_l) + (z_k \bar{z}_l)(\bar{u}_k u_l - \bar{u}_k u_l^*)].
\] (3.31)

Let

\[
a_{kl,1} := \kappa c_m(Q^2, \bar{u}_k \bar{u}_l + \bar{u}_k u_l), \quad a_{kl,2} := \kappa c_m(Q^2, \bar{u}_k u_l - \bar{u}_k u_l^*).
\] (3.32)

Note that \( a_{kl,1}, a_{kl,2} = O(n^2) \), \( a_{kl,1}, a_{kl,2} \) are real if both \( k, l > m \), and \( a_{kk,2} \) are purely imaginary. In particular \( a_{kk,2} = 0 \) if \( k > m \). We have

\[
A^{(2)} = \kappa c_m(Q^2, \text{Im} \sum_{k, l \in I_m} \zeta_k \zeta_l) = \text{Im} \sum_{k, l \in I_m} \{a_{kl,1} z_k \bar{z}_l + a_{kl,2} z_k \bar{z}_l\}
\] (3.33)

where

\[
b_0(t) = \sum_{k < m} b_{0k} |z_k|^2, \quad b_{0k} := \text{Im} a_{kk,2}, \quad \bar{b}_0(t) := \int_{T}^{t} b_0(s) \, ds,
\] (3.34)

\[
A^{(2)}_1 := \sum_{k, l \in I_m} a_{kl,1} z_k \bar{z}_l + \sum_{k \neq l} a_{kl,2} z_k \bar{z}_l.
\] (3.35)

Note \( |b_{0k}| \lesssim n^2 \| u_k^- \|_{L^2_{\text{loc}}} = O(n^4) \) for \( k < m \) by Lemmas 2.4 and 2.5.

We shall integrate \( A^{(2)}_1 \) by parts. Note that for all \( \lambda_k + \lambda_l = -i(\omega_k + \omega_l) + O(n^4) \) and \( \lambda_k + \bar{\lambda}_l = -i(\omega_k - \omega_l) + O(n^4) \). Therefore, \( \lambda_k + \lambda_l = O(1) \) for all \( k, l \in I_m \) and \( \lambda_k + \bar{\lambda}_l = O(1) \) for all \( k, l \in I_m \) and \( k \neq l \). We then write

\[
A^{(2)}_1 = \sum_{k, l \in I_m} a_{kl,1} e^{(\lambda_k + \lambda_l) t} r_k r_l + \sum_{k \neq l} e^{(\lambda_k + \bar{\lambda}_l) t} a_{kl,2} r_k \bar{r}_l
\]

\[
= \sum_{k, l \in I_m} \frac{a_{kl,1}}{\lambda_k + \lambda_l} \left[ \frac{d}{dt}(z_k \bar{z}_l) - e^{(\lambda_k + \lambda_l) t} \frac{d}{dt}(r_k r_l) \right]
\]
\[
+ \sum_{k \neq l} \frac{a_{kl,2}}{\lambda_k + \bar{\lambda}_l} \left[ \frac{d}{dt}(z_k \bar{z}_l) - e^{(\lambda_k + \bar{\lambda}_l) t} \frac{d}{dt}(r_k \bar{r}_l) \right].
\] (3.36)

Now, define

\[
a^{(2)}_1(t) := \text{Im} \sum_{k, l \in I_m} \frac{a_{kl,1}}{\lambda_k + \lambda_l} z_k \bar{z}_l + \text{Im} \sum_{k \neq l} \frac{a_{kl,2}}{\lambda_k + \bar{\lambda}_l} z_k \bar{z}_l
\]

\[
A_{2, rm} := \text{Im} \sum_{k, l \in I_m} \frac{a_{kl,1}}{\lambda_k + \lambda_l} e^{(\lambda_k + \lambda_l) t} \frac{d}{dt}(r_k r_l) + \text{Im} \sum_{k \neq l} \frac{a_{kl,2}}{\lambda_k + \bar{\lambda}_l} e^{(\lambda_k + \bar{\lambda}_l) t} \frac{d}{dt}(r_k \bar{r}_l).
\] (3.37)
We shall get
\[ \text{Im}(A_1^{(2)}) = \frac{d}{dt}a^{(2)}(t) - A_{2,rm}(t). \] (3.38)
So, we have \( A^{(2)} = \frac{d}{dt}a^{(2)}(t) + b_0(t) - A_{2,rm}(t) \). Therefore,
\[ a(t) = a^{(2)}(t) + b(t), \] (3.39)
where \( b(t) \) satisfies
\[ \dot{b} = b_0 + c_m(Q, \text{Im}(F - F_1 + \dot{h})) - A_{2,rm}, \quad b(T) = a(T) - a^{(2)}(T). \] (3.40)
Moreover, let \( a_{kl,3} := 2a_{kl,1}(\lambda_k + \lambda_l)^{-1} \) and \( a_{kl,4} := 2a_{kl,2}(\lambda_k + \bar{\lambda}_l)^{-1} \). Since \( a_{kl,1} \) and \( a_{kl,2} \) are of order \( n^2 \), so are \( a_{kl,3} \) and \( a_{kl,4} \). Moreover, \( a_{kl,3}, a_{kl,4} \) are purely imaginary for \( k, l \in I_m \).

Using (3.27), \( a_{kl,1} = a_{lk,1} \) and \( a_{kl,2} = -a_{lk,2} \), we obtain
\[ A_{2,rm} = \text{Im} \sum_{k,l \in I_m} a_{kl,3}z_kZ_l + \text{Im} \sum_{k \neq l} a_{kl,4}Z_kz_l. \] (3.41)

It worths noting that the benefits from using \( r_k \) instead of \( p_k \) in (3.37) is that we do not have terms of order \( zz \) for \( k \in I_m \) in (3.41). This is very essential in the normal forms.

### 3.4 Decomposition of \( \eta \)

We shall single out the main terms in \( \eta \). Recall from (3.28) that
\[ \partial_t \eta - \mathcal{L} \eta = P_{c}^2 i^{-1}(F + \dot{\theta}(aR + \zeta + \eta)). \] (3.42)

In the vector form, we have
\[ \partial_t \eta = \mathbf{L} \eta + P_{c}^2 J \dot{\theta}[\eta] + P_{c}^2 J[(F + \dot{\theta}(aR + \zeta))]. \] (3.43)

We first deal with the non-localized linear term \( J \dot{\theta}[\eta] \) using Lemma 2.13 following Buslaev-Perelman [3], also see [4, 6]. We need to revise their original statement and proof to take care of eigenvalues near the continuous spectrum.

Recall \( P_{\pm} \) are defined in subsection 2.6. Taking projection \( P_{\pm} \) of (3.43), and using
\[ P_{\pm} J + iP_{\pm} = P_{\pm}(P_{\pm} J + iP_{\pm}) = P_{\pm}[K_{\pm} - (P_{R_{\pm}} J + iP_{R_{\pm}})] = P_{\pm} K_{\pm}, \] (3.44)
we get
\[ \partial_t P_{\pm}[\eta] = \mathbf{L} P_{\pm}[\eta] + i\dot{\theta} P_{\pm}[\eta] + P_{\pm} K_{\pm} \dot{\theta}[\eta] + P_{\pm} J[(F + \dot{\theta}(aR + \zeta))]. \] (3.45)

Denote
\[ \eta_{\pm} := e^{\mp i\theta} P_{\pm}[\eta]. \] (3.46)

We have
\[ \partial_t \eta_{\pm} = \mathbf{L} \eta_{\pm} + e^{\mp i\theta} P_{\pm} \left[ K_{\pm} \dot{\theta}[\eta] + J[(F + \dot{\theta}(aR + \zeta))] \right]. \] (3.47)

\footnote{The term \( i\dot{\theta} \eta \) is not a problem in [23] in which \( \mathcal{L} \) is factorized in the form \( \mathcal{L} = U^{-1}JAU \) for some scalar self-adjoint operator \( A \). Such factorization does not exist for linearized operators near excited states. In [31], the term \( i\dot{\theta} \eta \) is removed by introducing \( \eta = P_{c}^2 e^{i\theta} \eta \) and using Strichartz estimates to control the (small) commutator term. This last method is not suitable for \( L^p \)-decay approach since the commutator term, although smaller, has the same decay rate as \( \eta \) itself. The approach of Buslaev-Perelman has the further benefit of being applicable to large soliton case.}
Recall that \( \zeta_k = (z_k \Phi_k + \bar{z}_k \Phi_k)/2 \). Note the term \( e^{\mp i\theta} P_\pm \hat{J}[\zeta] \) is not localized. However, by formula (3.35)

\[
P^2_c J \Phi_k = P^2_c \Phi'_k, \quad P^2_c J \Phi_k = P^2_c \Phi'_k, \quad \Phi'_k = \left[ -\frac{2i}{2} \right] \bar{u}_k^+ \tag{3.48}
\]

and note \( \Phi'_k \) is localized. Thus we can rewrite the linear terms in (3.47) as

\[
F_{L \pm} := e^{\mp i\theta} \left\{ K \pm [\eta] + J[a R] + \sum_{j \in I_m} (z_j \Phi'_j + \bar{z}_j \Phi'_j) \right\}, \tag{3.49}
\]

where all functions are localized, and (3.47) becomes

\[
\partial_t \eta_\pm = \mathbf{L} \eta_\pm + P_\pm \left[ e^{\mp i\theta} J[F] + F_{L \pm} \right]. \tag{3.50}
\]

In other words, for some \( t_0 \geq 0 \) and for all \( t \geq t_0 \), we have

\[
\eta_\pm(t) = e^{\mathbf{L}(t-t_0)} \eta_\pm(t_0) + \int_{t_0}^{t} e^{\mathbf{L}(t-s)} P_\pm \left\{ e^{\mp i\theta} J[F] + F_{L \pm} \right\}(s) ds. \tag{3.51}
\]

We will decompose \( \eta_\pm \) as follows. Denote

\[
\eta_{\pm, 1}^{(3)}(t) := e^{\mathbf{L}(t-t_0)} \eta_{\pm}(t_0),
\]

\[
\eta_{\pm, 4}^{(3)}(t) := \int_{t_0}^{t} e^{\mathbf{L}(t-s)} P_\pm \left\{ F_{L \pm} + e^{\mp i\theta} J[F - F_1] \right\}(s) ds. \tag{3.52}
\]

Then, we have

\[
\eta_\pm(t) = \eta_{\pm, 1}^{(3)}(t) + \eta_{\pm, 4}^{(3)}(t) + \int_{t_0}^{t} e^{\mathbf{L}(t-s)} P_\pm \left\{ e^{\mp i\theta} J[F_1] \right\}(s) ds. \tag{3.53}
\]

We shall integrate the last term in (3.53). Recall that \( F_1 = \kappa Q(2|\zeta|^2 + \zeta^2) \) is the main term in \( F \) with

\[
\zeta = \sum_{k \in I_m} \zeta_k = \sum_{k \in I_m} (z_k u_k^+ + \bar{z}_k u_k^-), \quad u_k^+ = \phi_k + O_{L^2}^t(n^2), \quad u_k^- = O_{L^2}^t(n^2). \tag{3.54}
\]

So,

\[
F_1 = \sum_{k, l \in I_m} F_{kl}[z_k z_l + 2z_k \bar{z}_l] + \sum_{k, l \in \Omega_m} \tilde{F}_{kl} z_k z_l, \quad F_{kl} = \kappa Q \phi_k \phi_l, \quad \tilde{F}_{kl} = O_{L^3}^t(n^3). \tag{3.55}
\]

In other words, we can write

\[
F_1 = \kappa \sum_{k, l \in \Omega_m} z_k z_l \tilde{\Phi}_{kl}, \tag{3.56}
\]

for some localized functions \( \Phi_{kl} \) which can be computed explicitly. In particular, \( \text{Re} \Phi_{kl} = O(n) \) and \( \text{Im} \Phi_{kl} = O(n^3) \) for all \( k, l \in \Omega_m \).

To integrate \( P_\pm e^{\mp i\theta} J[F_1] \) in \( \eta_\pm \) equation, we want to integrate terms of the form

\[
I_\pm(t) = \int_{t_0}^{t} e^{(t-s)} \mathbf{L} e^{-i\omega_3} P_\pm f(s) ds, \tag{3.57}
\]
where $\omega \in \mathbb{R}$, $f(s) \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ and $\dot{f}(s)$ decays faster than $f$. We re-write $I_\pm$ as

$$I_\pm(t) = e^{\mathbf{L} t} \int_{t_0}^t e^{-s(\mathbf{L} + i\omega)} P_\pm f(s) ds. \quad (3.58)$$

Denote $R = \lim_{\varepsilon \to 0^+} (\mathbf{L} + i\omega - \varepsilon)^{-1}$. Integration by parts gives

$$I_\pm(t) = -e^{-i\omega t} RP_\pm f(t) + e^{(t-t_0)\mathbf{L}} e^{-i\omega t_0} RP_\pm f(t_0) + \int_{t_0}^t e^{(t-s)\mathbf{L}} RP_\pm e^{-i\omega s} \dot{f}(s) ds. \quad (3.59)$$

The choice of the sign of $\varepsilon$ ensures that $e^{t\mathbf{L}} RP_\pm$ has singular decay estimate according to Lemma 2.13. We can now identify the main term of $\eta_\pm$. Since $i^{-1} F_1 = -i\kappa \sum z_k z_l \Phi_{kl}$ with summation over $k, l \in \Omega_m$,

$$J[F_1] = -\text{Re} \sum ikz_k z_l \Phi_{kl} \left[ \frac{1}{i} \right] = -\text{Re} \sum f_{kl}(t)e^{-i(\omega_k + \omega_l)t}, \quad (3.60)$$

where $f_{kl} = i\kappa p_k p_l \Phi_{kl} \left[ \frac{1}{i} \right]$. We decompose $P_\pm = \Pi_\pm P_\pm^\sharp$ since $\Pi_\pm$ does not commute with Re. Denote $R_{kl} = \lim_{\varepsilon \to 0^+} (\mathbf{L} + i(\omega_k + \omega_l) - \varepsilon)^{-1} P_\pm^\sharp$ and $\omega_{kl} = \omega_k + \omega_l$. We get

$$\int_{t_0}^t e^{(t-s)\mathbf{L}} P_\pm e^{i\theta(s)} J[F_1] ds = \eta_\pm^{(2)} + \eta_\pm^{(3)} + \eta_\pm^{3} \quad (3.61)$$

where

$$\begin{align*}
\eta_\pm^{(2)} &= e^{+i\theta(t)} \Pi_\pm \text{Re} \sum_{k, l \in \Omega_m} R_{kl} e^{-i\omega_{kl}t} f_{kl}(t) \\
\eta_\pm^{(3)} &= -e^{(t-t_0)\mathbf{L}} e^{+i\theta(t_0)} \Pi_\pm \text{Re} \sum_{k, l \in \Omega_m} R_{kl} e^{-i\omega_{kl}t_0} f_{kl}(t_0) \\
\eta_\pm^{3} &= -\int_{t_0}^t e^{(t-s)\mathbf{L}} e^{+i\theta(s)} \Pi_\pm \sum_{k, l \in \Omega_m} \left( \text{Re} R_{kl} e^{-i\omega_{kl}s} \dot{f}_{kl} + i \text{Re} R_{kl} e^{-i\omega_{kl}s} \dot{f}_{kl} \right) ds.
\end{align*} \quad (3.62)$$

Observe that

$$||\dot{f}_{kl}|| + ||\ddot{f}_{kl}||_{L^2_x} \lesssim n||\dot{\theta}||_{\mathbb{R}} + n \beta \max |p_k|, \quad \beta = \max |p_k|. \quad (3.63)$$

Now, let

$$\eta_\pm^{(3)}(t) := \sum_{j=1}^4 \eta_\pm^{(j)}(t), \quad \eta^{(j)} := e^{i\theta} \eta^{(j)}_+ + e^{-i\theta} \eta^{(j)}_-, \quad j = 2, 3. \quad (3.64)$$

Then, from (3.63) and (3.62), we obtain the decomposition of $\eta_\pm$ and $\eta$ as

$$\eta_\pm = \eta_\pm^{(2)} + \eta_\pm^{(3)}, \quad [\eta] = e^{i\theta} \eta_+ + e^{-i\theta} \eta_- = \eta^{(2)} + \eta^{(3)}. \quad (3.65)$$

We now compute the explicit form of $\eta^{(2)}$ which will be used in the computation of the key coefficients in the normal forms of $z_k$. By (3.65), (3.62), $\Pi_+ + \Pi_- = P_\pm^\sharp$, and (3.55),

$$\eta^{(2)} = e^{i\theta} \eta^{(2)}_+ + e^{-i\theta} \eta^{(2)}_- = \text{Re} \sum_{k, l \in \Omega_m} R_{kl} e^{-i(\omega_k + \omega_l)t} f_{kl}(t)$$

$$= \text{Re} \sum_{k, l \in \Omega_m} \left\{ R_{kl} z_k z_l [i] F_{kl} + 2R_{kl} z_k z_l [i] F_{kl} \right\} + \sum_{k, l \in \Omega_m} z_k z_l R_{kl} O_{L^2_x}(\eta^3). \quad (3.66)$$

Recall $F_{kl} = \kappa Q \phi_k \phi_l$. Thus the first sum contains terms of order $O(nz^2)$.
\section{Decomposition of $F$}

We now decompose $F$ into appropriate terms of the same order. We write

$$F = F_1 + F_2 + \cdots + F_5,$$  \hfill (3.67)

where

$$F_1 = \kappa Q(2|\zeta|^2 + \zeta^2),$$
$$F_2 = 2\kappa Q R b(2\zeta + \tilde{\zeta}) + 3\kappa Q R^2 b^2 + \kappa (\zeta + bR)^2 (\tilde{\zeta} + bR),$$
$$F_3 = 2\kappa Q R a^{(2)} (2\zeta + \tilde{\zeta}), \quad F_4 = 2\kappa Q [(\zeta + \tilde{\zeta}) \eta + \zeta \tilde{\eta}],$$
$$F_5 = \kappa Q [2|\eta_a|^2 + \eta_a^2] + 2\kappa Q R b(2\eta_a + \eta_a) + \kappa (aR + h)^2 (aR + \tilde{h}) - \kappa (\zeta + bR)^2 (\tilde{\zeta} + bR),$$

with $\eta_a = \eta + a^{(2)} R$. Note that $F_1$ consists of terms of order $nz^2$; $F_2$, $F_3$ and $F_4$ consist of terms no smaller than $n^2 z^3$; and $F_5$ higher order terms.

\subsection{Basic estimates and normal forms}

In this subsection, we first give some basic estimates in Lemmas 3.4, 3.5 and 3.6. We then give the normal forms of the equations of $z_k$ and $b$ in Lemmas 3.7 and 3.8.

\textbf{Lemma 3.4 (Basic Estimates)} Suppose, for a fixed time, for some $\beta \ll n \leq n_0$ and $p \geq 5$,

$$\|Q\| = n, \quad \|\eta\|_{L^2 \cap L^p} \ll 1, \quad \|\eta\|_{L^2_{\text{loc}}} \leq n,$$
$$\max_{j \neq m} |z_j| \leq \beta, \quad |a| \leq C \beta^2. \hfill (3.69)$$

For all $1 \leq r \leq 2$, denote

$$X := n\beta \|\eta\|_{L^2_{\text{loc}}} + n \|\eta\|_{L^2_{\text{loc}}}^2 + \|\eta^3\|_{L^1_{\text{loc}}},$$

$$\bar{X} := \beta^2 \|\eta\|_{L^2_{\text{loc}}}^2 + n \|\eta\|_{L^2_{\text{loc}}}^2 + \|\eta^3\|_{L^1_{\text{loc}}}, \quad Y(r,p) := n \|\eta\|_{L^p} + \|\eta^3\|_{L^r}. \hfill (3.70)$$

We have

$$\|F_5\|_{L^1_{\text{loc}}} \lesssim n\beta^4 + \bar{X}, \quad \|F_3 + F_4 + F_5\|_{L^1_{\text{loc}}} \lesssim n^2 \beta^3 + X,$$

$$\|F - F_1\|_{L^1_{\text{loc}}} \lesssim \beta^3 + X, \quad \|F\|_{L^1_{\text{loc}}} \lesssim n\beta^2 + X,$$

$$|F_0| \lesssim \beta^2 + n^{-1}X, \quad \|F - F_1\|_{L^r} \lesssim \beta^3 + n\beta \|\eta\|_{L^2_{\text{loc}}} + Y(r,p),$$

$$\|F\|_{L^r} \lesssim n\beta^2 + n\beta \|\eta\|_{L^2_{\text{loc}}} + Y(r,p). \hfill (3.71)$$

\textbf{Proof.} First note that the proof of the first five estimates of (3.71) can be found in \cite{28} Lemma 3.2]. Although \cite{28} is for $m = 0$ case, for $L^1_{\text{loc}}$ bounds the new non-localized terms for $m > 0$ are similarly estimated.

Now consider the last two $L^r$-estimates of (3.71). The only non-localized terms of $F$ are of order $(u_k z_k)^3$, $(u_{k-1} z_k)^2 \eta$, $u_{k-1} z_k \eta^2$, and $\eta^3$ for $k < m$. Since $|(u_k z_k)^3 \eta| + |u_{k-1} z_k \eta^2| \lesssim |u_{k-1} z_k|^3 + |\eta|^3$, we have

$$\|(u_k z_k)^3\|_{L^r} + \|(u_k z_k)^2 \eta\|_{L^r} + \|u_{k-1} z_k \eta^2\|_{L^r} + \|\eta^3\|_{L^r} \lesssim \beta^3 + \|\eta^3\|_{L^r}. \hfill (3.72)$$
Then for $1 \leq r \leq 2 < p/2$, the estimates of $\|F\|_{L^r}$ is the same as the estimates of $\|F\|_{L^1_{\text{loc}}}$ except the non-localized terms we just estimated and

$$
\|Qz_k u_k^\pm \eta\|_{L^r} \lesssim n^\beta \|\eta\|_{L^1_{\text{loc}}}, \quad \|Q\eta^2\|_{L^r} \lesssim n \|\eta\|_{L^p}^2.
$$

(3.73)

So, we obtain the last two estimates.

For some fixed $0 < p < 6$ which will be chosen, let us define

$$
z_L = \left( \sum_{k=0}^{m-1} |z_k|^2 \right)^{1/2}, \quad z_H = \left( \sum_{k=m+1}^{K} |z_k|^2 \right)^{1/2}.
$$

(3.74)

If $m = 0$, we set $z_L = 0$. We also denote

$$
\tilde{X} = \tilde{X}_p := n^4 z_L \|\eta\|_{L^p}^2 + n^6 z_L^2 \|\eta\|_{L^p}^2 + m \cdot n^{\frac{p(6-p)}{p}} \|\eta\|_{L^p}^3.
$$

(3.75)

Note that if $m = 0$, then $\tilde{X} = 0$. Let

$$
D = 6Kc_{\max} \gamma_0^+ / \gamma_0 = O(1)
$$

(3.76)

where $c_{\max} = \max_k 2 \int \phi_k^4$ and

$$
\gamma_0^+ = \max_{k,l,m \in I, |s| < \delta_0} \lim_{r \to \delta_0^+} \text{Im} \left( \phi_k \phi_l \phi_m, \frac{1}{H_0 + e_k - e_l - e_m - s - r} \right) P_c H_0 \phi_k \phi_l \phi_m.
$$

(3.77)

Note that $(Q_{k,n}, R_{k,n})^{-1} = 2K \int \phi_k^4 + o(1)$. We have the following lemma on the normal forms of $z_k$.

**Lemma 3.5** Assume as in the Lemma 3.4 then for all $k \neq m$, we have

$$
|Z_k| \lesssim n^2 + \tilde{X}_p + X, \quad \text{if} \quad k < m, \quad |Z_k| \lesssim n^2 + X, \quad \text{if} \quad k > m,
$$

$$
|R_k| \lesssim k^2 + \tilde{X}_p + X, \quad \text{if} \quad k < m, \quad |R_k| \lesssim k^2 + X, \quad \text{if} \quad k > m.
$$

(3.78)

Here $Z_k$ is defined in (3.25) and $R_k = R_{k,1} + R_{k,2}$ is part of $Z_k$, where

$$
R_{k,1} := -2c_k \left[ (u_k^+, F - F_1) + (u_k^-, \overline{F} - \overline{F}_1) \right],
$$

$$
R_{k,2} := -2c_k \left[ (u_k^+, h) + (u_k^-, \overline{h}) + (\bar{u}_k, R) a \right] F_\theta.
$$

(3.79)

**Proof.** Recall (3.25) that

$$
Z_k := -2c_k \left\{ (u_k^+, F) + (u_k^-, \overline{F}) + [(u_k^+, h) + (u_k^-, \overline{h}) + (\bar{u}_k, R) a] \overline{\theta} \right\}.
$$

(3.80)

For $m < k \leq K$, since $u_k^+, u_k^-$ are both real and localized, $P_k \eta = 0$, using Lemma 2.6 we have

$$
|(u_k^+, \eta) + (u_k^-, \overline{\eta})| = 2|(u_k^-, \overline{\eta})| \leq C n^2 \|\eta\|_{L^1_{\text{loc}}}.
$$

(3.81)

Therefore,

$$
|Z_k| \leq \|F\|_{L^1_{\text{loc}}} + |\overline{\theta}| |a| + |z| + n^2 \|\eta\|_{L^1_{\text{loc}}} \lesssim n^2 \beta^2 + X + [\beta^2 + n^{-1} X] (\beta + \|\eta\|_{L^1_{\text{loc}}}) \lesssim n^2 \beta^2 + X.
$$

(3.82)
Now, we consider the case when \( k < m \). We first consider the term \( 2c_k[(u_k^+, F) + (u_k^-, \overline{F})] \).
As we already see in the proof of Lemma 3.3, the only non-localized terms in \( F \) are bounded by \( |\eta^3| + \sum_{j,l,h<m} |u_j^{-1} u_i^+ u_h^+| z_L^3 \). Thus, for \( k < m \), using Hölder’s inequality and Lemma 2.5, we have
\[
\left| \left[ (u_k^+, F) - (u_k^-, \overline{F}) \right] \right| \lesssim \|F\|_{L^1_{loc}} + \left| (u_k^-) \right| + |\eta^3| + \sum_{j,l,h<m} |u_j^{-1} u_i^+ u_h^+| z_L^3 \\
\lesssim n \beta^3 + \dot{X}_p + X.
\] (3.83)
On the other hand, we have
\[
|\left( u_k^+, \eta \right) + (u_k^-, \overline{\eta})| = |(\sigma_1 \Phi, J[\eta])| \lesssim \|\eta\|_{L^2_{loc}}, \quad (k < m).
\] (3.84)
Then, it follows from Lemmas 3.4 and 2.5 that
\[
\left| \left( u_k^+, h \right) + (u_k^-, \overline{h}) + (\overline{u_k}, R)a \right|_{F_0} \lesssim \|z + n^{-1} |a| + \|\eta\|_{L^2_{loc}} \right|_{F_0} \lesssim \beta^3 + X.
\] (3.85)
This completes the proof of the estimates of \( Z_k \). By a similar way, we can obtain the estimates of \( R_k \).

**Lemma 3.6** Assume as in the Lemma 3.4, then we have
\[
|\dot{b}| \leq C[n^4 z_L^2 + n \beta^3 + nX + n^2 \beta \dot{X}].
\] (3.86)
Above \( \dot{X} = \dot{X}_p \) is defined in (3.75) and can be omitted if \( m = 0 \).

**Proof.** Recall (3.40) that
\[
\dot{b} = b_0 + c_m (Q, \text{Im}(F - F_1 + \dot{\theta} h)) - A_{2,rm}.
\] (3.87)
It follows from (3.34), (3.41) and Lemma 3.5, that
\[
|b_0| \leq C n^4 z_L^2, \quad |A_{2,rm}| \leq n^2 \beta [n \beta^2 + X + \dot{X}].
\] (3.88)
On the other hand, we have
\[
|c_m (Q, \text{Im}(F - F_1 + \dot{\theta} h))| \lesssim n \|F - F_1\|_{L^1_{loc}} + n^3 \beta^2 + |\dot{\theta}| [n^3 \beta + n \|\eta\|_{L^2_{loc}}] \lesssim n \beta^3 + nX.
\] (3.89)
So, (3.86) follows.

**Lemma 3.7** Fix \( 0 \leq m \leq K \) and \( 0 < n_1 \sim n \leq n_0 \). Let \( Q = Q_{m,n_1} \) and \( \mathcal{L} = \mathcal{L}_{m,n_1} \). Suppose \( \psi \) is decomposed as in (3.17) with respect to \( \mathcal{L} \), and for some \( 0 < \beta \ll n \)
\[
\|\eta\|_{L^2_{loc}} \lesssim \beta, \quad \|\eta\|_{L^2 \cap L_p} \ll 1, \quad \max_{k \neq m} |z_k| \leq \beta, \quad |a| \leq C \beta^2.
\] (3.90)
Then there exist functions \( q_k, Y_k \) and constants \( D_{kl} \) for \( l \neq m \) such that
\[
\dot{q}_k - \text{Re}(\lambda_k) q_k = \sum_{l > m} D_{kl} |q_l|^2 q_k + Y_k q_k + g_k, \quad \text{with} \quad |g_k - p_k| \lesssim n \beta^2, \\
|D_{kl}| \lesssim D n^2, \quad \text{Re}(D_{kl}) \leq -\gamma_0 n^2, \quad \forall k, l > m, \quad \text{and}
\]
\[
|\text{Re}(Y_k)| \lesssim n^2 \beta^2, \quad (k > m); \quad |\text{Re}(Y_k)| \lesssim n^2 \beta^2, \quad (k < m).
\] (3.91)
Recall \( \Re \lambda_k \geq n^4 \) if \( k < m \) and \( \Re \lambda_k = 0 \) if \( k > m \). Moreover, we have
\[
|g_k| \lesssim n \beta^4 + n^4 \beta z_L^2 + n^3 \beta \|\eta\|_{L^2_{loc}} + n \beta \|\eta^3\|_{L^2_{loc}} + n \beta \dot{X}_p + \dot{X}, \quad (k > m),
\]
\[
|g_k| \lesssim n^5 \beta^2 + n^4 \beta z_L^2 + n^3 \beta \|\eta\|_{L^2_{loc}} + n \beta \|\eta^3\|_{L^2_{loc}} + \dot{X}_p + \dot{X}, \quad (k < m).
\] (3.92)
Above \( \dot{X}_p \) is defined in (3.75) and can be omitted if \( m = 0 \).
Proof. In case $m = 0$, Lemma 3.4 is identical to [28, Lemma 3.4]. So, it suffices to assume $m > 0$. The main difference in case $m > 0$ is that $u_l^\pm$ are not localized and $u_l^\pm$ are complex for $l < m$. For those new terms involving $z_l$ with $l < m$, we either integrate them using integration by parts and equations of $r_l$, as in (3.30), or include them in the error terms. We sketch the proof here. Recall that for $k \in I_m$,

$$
\dot{z}_k - \lambda_k z_k = Z_k, \quad \dot{p}_k - \text{Re}(\lambda_k)p_k = e^{i\omega_k t} Z_k, \quad \dot{i}_k = e^{-\lambda_k t} Z_k,
$$

(3.93)

where $Z_k$ is defined in (3.25). For $F_j, j = 1, 2, \cdots, 5$, defined in (3.68), we let

$$
T_{k,1} := -2c_ke^{i\omega_k t}[u_k^+, F_4] + (u_k^-, F_4), \quad T_{k,2} := -2c_ke^{i\omega_k t}[(u_k^+, F_1) + (u_k^-, F_1)],
$$

$$
T_{k,3} := -2c_ke^{i\omega_k t}\{[(u_k^+, F_2 + F_3) + (u_k^-, F_2 + F_3) + [(u_k^+, \zeta) + (u_k^-, \zeta)\dot{\theta}]},
$$

$$
T_{k,4} := -2c_ke^{i\omega_k t}\{(u_k^+, F_5) + (u_k^-, F_5) + [(u_k^+, \eta) + (u_k^-, \eta) + (\bar{u}_k, R)a]\dot{\theta}\}.
$$

(3.94)

Then, we can write

$$
\dot{p}_k - \text{Re}(\lambda_k)p_k = e^{i\omega_k t} Z_k = T_{k,1} + T_{k,2} + T_{k,3} + T_{k,4}.
$$

(3.95)

The term $T_{k,1}$ contains the key terms with resonant coefficients. Recall

$$
F_4 = 2\kappa Q\sum_{j\neq m}\{(z_j \phi_j + \bar{z}_j \phi_j) \eta + z_j \phi_j \bar{\eta}\} + O(n^3 z \|\eta\|_{L_2^{\text{loc}}}^1),
$$

(3.96)

$[\eta] = \eta^{(2)} + \eta^{(3)}$, and denote $\eta_1^{(2)}$ the first sum for $\eta^{(2)}$ in (3.66), which is the main term of $\eta$. Since $\eta_1^{(2)}$ involves matrix operators $R_{kl}$, we rewrite $T_{k,1}$ in vector form and get $T_{k,1} = T_{k,1,1} +$ error term, where

$$
T_{k,1,1} := -4\kappa c_k e^{-i\omega_k t} \sum_{j\neq m}\{[[1] Q\phi_k \phi_j, \eta^{(2)}](z_j + \bar{z}_j) + ([1] Q\phi_k \phi_j, \eta^{(2)} z_j)\}.
$$

(3.97)

The error term is controlled by $n^3 \beta \|\eta\|_{L_2^{\text{loc}}}^2 + n\beta \|\eta^{(3)}\|_{L_2^{\text{loc}}}^2$. By the explicit formula of $\eta^{(2)}$ in (3.66), we see that

$$
T_{k,1,1} = \sum_{l > m} D_{kl}|p_l|^2 p_k + (\text{non-zero phase } n^2 z^3\text{-terms}) + Y_{k,1} p_k,
$$

(3.98)

where $Y_{k,1} := \sum_{l < m} D_{kl}|p_l|^2$. The first term in $T_{k,1,1}$ we will keep. The middle term we integrate using integration by parts. The last term is part of $Y_k p_k$ term. So, we get

$$
T_{k,1} = \frac{d}{dt} \tilde{T}_{k,1} + \sum_{l > m} D_{kl}|p_l|^2 p_k + Y_{k,1} p_k + g_{k,1}, \quad |\tilde{T}_{k,1}| \lesssim n^2 \beta^3.
$$

(3.99)

Moreover, $Y_{k,1}$ and $g_{k,1}$ satisfy the estimates as those of $Y_k$ and $g_k$ in the lemma. To compute $\text{Re}(D_{kl})$, we use (3.66), (3.97), and Corollary 2.20. The leading terms of $D_{kl}$ are from $([1] Q\phi_k \phi_j, \eta_1^{(2)}) \bar{z}_j$. We get

$$
\text{Re } D_{kl} = -C \text{Im}(Q\phi_k \phi_l, (H_0 - E - i[-i(\omega_l + \omega_k) + 0^+])^{-1} P_c^{H_0} Q\phi_k \phi_l) + O(n^4)
$$

$$
= -Cn^2 \text{Im}(\phi_m \phi_k \phi_l, (H_0 + s - i0^+)^{-1} P_c^{H_0} \phi_m \phi_k \phi_l) + O(n^4),
$$

(3.100)

where $C = 2\kappa^2(2 - \delta^k) \geq 2, s \in \mathbb{R}$ and $|s - (e_m - e_l - e_k)| \lesssim n^2$. In particular $\text{Re } D_{kl} = O(n^4)$ if $k < m$ or $l < m$. 

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For terms in $T_{k,2}$, we integrate as in (3.36). Since $e^{i\omega t}Z_k = T_{k,2} + R_k$ (see Lemma 3.5), we get
\[
T_{k,2} = \frac{d}{dt}(\tilde{T}_{k,2,1}) + T_{k,2,1} + O(n\beta \max_l |R_l|), \quad |\tilde{T}_{k,2,1}| \lesssim n\beta^2. \tag{3.101}
\]
The last term is moved to the error term and it can be estimated by using Lemma 3.5. The term $T_{k,2,1}$ are terms of $n^2\varepsilon^3$ and can be written as
\[
T_{k,2,1} = Y_{k,2} p_k + \text{(non-zero phase $n^2\varepsilon^3$-terms)}, \quad Y_{k,2} := \sum_{l\neq m} d_{kl} |p_l|^2. \tag{3.102}
\]
The constants $d_{kl}$ can be computed explicitly. In particular, $\text{Re}(d_{kl}) = 0$ for $k, l > m$. So, we get $|\text{Re}(Y_{k,2})| \lesssim n^4\varepsilon^2$ if $k > m$. Again, the non-zero phase terms are integrated by using integration by parts. So, we get
\[
T_{k,2} = \frac{d}{dt} \tilde{T}_{k,2} + Y_{k,2} p_k + g_{k,2}, \quad |\tilde{T}_{k,2}| \lesssim n\beta^2. \tag{3.103}
\]
Moreover, $Y_{k,2}$ and $g_{k,2}$ satisfy the estimates as those of $Y_k$ and $g_k$ in the statement of the lemma.

For terms in $T_{k,3}$, we integrate terms which are not smaller than $n^2\beta^3$. The main difficulty is from terms with $\theta$ since this term is not a polynomial expression in $z$ and $b$. The analysis is the same as in [29, 30, 28] with the use of Lemma 3.5 and (3.86). We can write
\[
T_{k,3} = \frac{d}{dt} \tilde{T}_{k,3} + Y_{k,3} p_k + g_{k,3}, \quad |\tilde{T}_{k,3}| \lesssim \beta^3. \tag{3.104}
\]
Again, $Y_{k,3}$ and $g_{k,3}$ satisfy the estimates as those of $g_k$ and $Y_k$ in the statement of the lemma. The only difference between our case here and [29, 30, 28] is the computation of $\text{Re}(Y_{k,3})$. This can be done with simple calculation and the attention that $u_l^\pm$ is complex only for $l < m$ and $\text{Im} u_l^\pm = O(n^2)$.

The term $T_{k,4}$ is part of the error term and can be estimated as $Z_k$ in Lemma 3.5:
\[
|T_{k,4}| \lesssim n\beta^4 + \bar{X}, \quad (k > m); \quad |T_{k,4}| \lesssim n\beta^4 + \bar{X} + \bar{X}_p, \quad (k < m). \tag{3.105}
\]

Now, let
\[
g_k := p_k - \tilde{T}_{k,1} - \tilde{T}_{k,2} - \tilde{T}_{k,3}, \quad Y_k := Y_{k,1} + Y_{k,2} + Y_{k,3}. \tag{3.106}
\]
Our lemma follows from (3.99), (3.103), (3.104) and (3.105). \Box

**Lemma 3.8** Assume as in Lemma 3.7 Then, there exist functions $\tilde{b}, g_b$ and numbers $B_{kl}$ for $k, l \in I_m$ such that
\[
\tilde{b} = b_0 + \sum_{k,l \in I_m} B_{kl} |z_k|^2 |z_l|^2 + g_b, \quad |b - \tilde{b}| \leq Cn\beta |\beta^2 + n \|\eta\|_{L^2_{\text{loc}}}],
\]
\[
|g_b| \leq C |n^3\beta^2 + n^5\beta z_L^2 + n^2\beta^2 z_L^2 + n\beta^5 + n^2 z_L \|\eta\|_{L^1_{\text{loc}}} + n^2 \|\eta\|_{L^2_{\text{loc}}} + n \|\eta\|_{L^2_{\text{loc}}} + n^2 \beta |\tilde{X}_p|.
\]  

(3.107)

Above $b_0$ is define in (3.34) and can be omitted if $m = 0$. Moreover, we also have $|B_{kl}| \leq Cn^2$ and $B_{kl} = -\frac{d}{dt} \text{Re} D_{kl} + O(n^4)$ where $D_{kl}$ is defined in Lemma 3.7 and $c_m = (Q_m, R_m)^{-1} = O(1) > 0$. Moreover, $\max_{kl} (|B_{kl}|/(K^{-1}g_0 n^2)) \leq \frac{L_2}{2}$.  

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Proof. Recall (3.40) that for $b_0$ defined in (3.31) and $A_{2,rm}$ defined in (3.41),
\[ b = b_0 + c_m(Q, \text{Im}(F - F_1 + \dot{\theta} h)) - A_{2,rm}. \] (3.108)

In case $m = 0$ we have $z_L = 0$, $b_0 = 0$, $\dot{X}_p = 0$, and Lemma 3.8 follows from Lemma 3.4. Therefore, we may assume $m > 0$. All of the work here is similar to that of 29 30 28, so we only give a sketch. Define
\[ b_1 := c_m(Q, \text{Im}(F - F_1)) - b_3 - b_4, \quad b_2 := c_m(Q, \text{Im} h)\dot{\theta}, \]
\[ b_3 := 2\kappa c_m \text{Im}(Q, Q[(\zeta + \zeta)\eta + \zeta \bar{\eta}]), \quad b_4 = c_m\kappa \text{Im}(Q, 2|\zeta|^2 \eta + \zeta^2 \bar{\eta}). \] (3.109)

Then, we have
\[ \dot{b} = \sum_{j=0}^4 b_j - A_{2,rm}. \] (3.110)

For terms in $b_1 + b_2$, we treat them as in 29 30 28. Note that we can write
\[ b_2 + b_2 \sim n z^3 + nbz + n^2 z^4 + b z^2 + n^{-1}b^2 z + \text{ error}. \] (3.111)

The non-zero phase terms are of the form $C_{kl}|z_k|^2|z_l|^2$ and $C_kb|z_k|^2$. Here $|C_{kl}| \leq n^2$ and $|C_k| \leq C$ for all $k,l \in I_m$. Moreover, $\text{Im} C_{kl} = \text{Im} C_k = 0$ for all $k,l \in I_{>m}$. Therefore, due to the Im-operator, the terms of the forms $C_{kl}|z_k|^2|z_l|^2$ with $C_kb|z_k|^2$ with $k,l \in I_{>m}$ are killed. If one of $k,l$ is in $I_{<m}$, we get new non-zero phase terms of the forms $C_{kl}|z_k|^2z_L^2$ with $|\text{Im} C_{kl}| \leq n^4$ and $C_L b z_L^2$ with $|\text{Im} C_L| \leq n^2$. We move these new terms into the error term. For all of the non-zero phase terms, we integrate them by using integration by parts. We get
\[ b_1 + b_2 = \frac{d}{dt} \tilde{b}_1 + \tilde{b}_2 + g_{b,1} + g_{b,2}, \quad |\tilde{b}_1| + |\tilde{b}_2| \lesssim n\beta^2, \] (3.112)

with $g_{b,1} + g_{b,2}$ satisfying the estimates as that of $g_b$ in the statement of the lemma.

For terms in $b_3$, we have $b_3 = 2\kappa c_m \text{Im}(Q^2, \zeta \eta)$. We move terms $2\kappa c_m \text{Im}(Q^2, \zeta \eta)$ to error term if $k < m$. So, we have $b_3 = b_{3,1} + b_{3,2}$ with
\[ b_{3,1} := \text{Im} \sum_{k>m} \left( \langle f_{1k}, [\eta] \rangle z_k + \langle f_{2k}, [\eta] \rangle z_k \right), \quad |b_{3,2}| \lesssim n^2 z_L \|\eta\|_{L^2_{loc}}. \] (3.113)

Here, $f_{1k}, f_{2k}$ are some explicit localized functions of order $n^2$. We need to integrate $b_{3,1}$ using equations (5.51) of $\eta_{\pm}$ as in 29. We get
\[ b_3 = \frac{d}{dt} \tilde{b}_3 + g_{b,3}, \quad |\tilde{b}_3| \leq C[n^2 \beta \|\eta\|_{L^2_{loc}} + n^3 \beta^3]. \] (3.114)

Also, $g_{b,3}$ satisfies the estimates as that of $g_b$ in the statement of the lemma.

For terms in $b_4$, as in (3.97), we write $b_4 = b_{4,1} + b_{4,er}$ with
\[ b_{4,1} = c_m\kappa \text{Im} \sum_{k,l \in I_m} \left[ \langle [\frac{1}{z_k}] Q\phi_k \phi_l, [\eta^{(2)}] z_k z_l \right] + \left[ \langle [\frac{1}{z_k}] Q\phi_k \phi_l, [\eta^{(2)}] z_k z_l \right], \]
\[ |b_{4,er}| \lesssim n^3 \beta^3 \|\eta\|_{L^2_{loc}} + n\beta^2 \|\eta^{(3)}\|_{L^2_{loc}}. \] (3.115)

From (3.66) and as in (3.99), we have
\[ b_4 = \frac{d}{dt} b_4 + \sum_{k,l \in I_{>m}} B_{kl}|z_k|^2|z_l|^2 + g_{b,4}, \quad |\tilde{b}_4| \leq Cn^2 \beta^4. \] (3.116)
The term $g_{b,4}$ satisfies the estimate as that of $g_b$ in the statement of the lemma. The computation of $B_{kl}$ is exactly the same as that of $\text{Re} D_{kl}$. The main leading terms come from

$$c_m \kappa \left( \frac{1}{i} Q \phi_k \phi_l, \eta_l^{(2)} \right) \bar{z}_k \bar{z}_l, \quad c_m \kappa \left( [\frac{1}{i}] Q \phi_k \phi_l, \eta_l^{(2)} \right) z_k z_l. \quad (3.117)$$

Using (3.66), Corollary 2.20 and by direct computation, we get

$$B_{kl} = -\frac{c_m}{2} \text{Re}(D_{kl}) + O(n^4). \quad (3.118)$$

For terms in $A_{2,rm}$, we can write

$$A_{2,rm} = n^3 z^3 + n^2 z^4 + n^2 b z^2 + \text{error}. \quad (3.119)$$

Again, zero-phase terms are of the form $c_k |z_k|^2 |z_l|^2$ and $c_{1k} b |z_k|^2$. By direct computation, it follows that $\text{Im} c_{kl} = \text{Im} c_{1k} = 0$ if $k, l \in I_{>m}$. So, those terms $c_k |z_k|^2 |z_l|^2$ and $c_{1k} b |z_k|^2$ are killed by the $\text{Im}$-operator if $k, l \in I_{>m}$. The other terms of the form $c_{kl} |z_k|^2 |z_l|^2$ and $c_{1k} b |z_k|^2$ with $k$ or $l$ in $I_{<m}$ are moved to the error term. For terms of non-zero phase, we integrate them. We get

$$A_{2,rm} = \frac{d}{dt} \bar{A}_{2,rm} + g_{b,5}, \quad |\bar{A}_{2,rm}| \lesssim n^3 \beta^3. \quad (3.120)$$

The error term $g_{b,5}$ satisfies the estimate as that of $g_b$ in the statement of the lemma. Finally, let

$$\bar{b} = b - [\bar{b}_1 + \bar{b}_2 + \bar{b}_3 - \bar{b}_4 - \bar{A}_{2,rm}], \quad g_b = \sum_{j=1}^{5} g_{b,j}. \quad (3.121)$$

Lemma 3.8 follows from (3.112), (3.114), (3.116) and (3.120).

4 Converging to an excited state

In this and the next sections, we study the dynamics when the solution is in a neighborhood of some excited states $Q_1$ at $t = 0$. We want to show that the solution either converges to an excited state, or exits the neighborhood eventually. In the first case, the ground state component is always bounded by other states. In the second case, the ground state component becomes significant after some time, denoted $t_c$ below. In this section we study the dynamics for $t < t_c$. In next section we study the dynamics for $t > t_c$ if $t_c$ is finite.

Denote $x_j(t) = (\phi_j, \psi(t))$ and $\xi(t) = P_e^{H_0} \psi(t)$. The assumption of Theorem 1.1 states that, at time $t = 0$,

$$|x_1(0)| = n, \quad \| \sum_{j \neq 1} x_j(0) \phi_j + \xi(0) \|_{H^1 \cap L^1} \leq \rho_0, \quad \rho_0 = n^{1+\delta}. \quad (4.1)$$

Denote

$$T_e := \sup_{T > 0} \left\{ T : \frac{1}{\varepsilon_3} \| \psi(t) - x_1(t) \phi_1 \|_{L^2} \leq |x_1(t)| \in ((0.9)n, (1.1)n), \quad 0 \leq t \leq T \right\}. \quad (4.2)$$

Above $\varepsilon_3 > 0$ is the small constant in Lemma 3.3 and $T_e > 0$ by (4.1). $T_e$ is the time the solution exits the neighborhood of first excited state family. Note that (4.1)–(4.2) are in terms of the orthogonal coordinates. For the majority of this section we will use linearized
coordinates which depend on the choice of $Q$, but (4.1)–(4.2) are independent of such a choice.

From Lemma 3.3 and the definition of $T_e$, for each $0 \leq T < T_e$, we can find a unique $n(T) = n(\psi(T)) \in (0, n_0)$ such that the solution $\psi(t)$ can be decomposed as

$$\psi(t) = [Q + a(t)R + \xi(t) + \eta(t)]e^{-iEt + i\theta}, \quad \forall \ 0 \leq t < T_e,$$

(4.3)

with $a(T) = 0$, where $Q = Q_{1,n(T)}$, $R = R_{1,n(T)}$ and $E = E_{1,n(T)}$. The components $\xi$ and $\eta$ are in the corresponding spectral subspaces with respect to $Q_{1,n(T)}$. Moreover we decompose

$$\xi = \sum_{j \neq 1} \xi_j, \quad \xi_j = \bar{z}_j u_j - z_j \bar{u}_j, \quad \eta = e^{i\theta} \eta_+ + e^{-i\theta} \eta_-.$$  

(4.4)

Define

$$\rho(t) := \frac{1}{n} (\Delta t + \gamma_0 t)^{-1/2}, \quad \Delta t := (n\rho_0)^{-2}, \quad \rho(0) = \rho_0,$$

(4.5)

where $\gamma_0$ is given in (1.15), and let

$$t_c := \sup_{0 < T \leq T_e} \{ t : |z_0(t)| \leq \varepsilon_4 n^{-1} \rho(t)^2, \ 0 \leq t < T \},$$

(4.6)

where $\varepsilon_4 > 0$ is a small constant to be chosen in (1.49), and $z_0$ is the coefficient of $\zeta_0$ in (4.4) with respect to $Q_{1,n(T)}$. If there does not exist any $T$ satisfying the right side of (4.6), we let $t_c = 0$.

By definition $t_c \leq T_e$ could be finite or infinite and is independent of the choice of $Q$ in (4.3). If it is finite, it is the first time that $z_0$ becomes large enough, and will not be destroyed by other components in the future. The subscript $c$ means “change” (of behavior).

The function $\rho(t)$ is an upper bound for higher bound states for $0 \leq t \leq t_c$.

If $t_c = 0$, we may skip most of this section and go directly to Lemma 4.6 and section 5.

We will bound $\eta$ in $L^p$ and $L^2_{\text{loc}}$ with fixed $p$ satisfying

$$\frac{27}{5} < p < 6, \quad \sigma = \sigma(p) = \frac{3p - 9}{2p}, \quad \frac{2}{3} < \sigma < \frac{3}{4}. \quad (4.7)$$

From now on let $0 \leq T < t_c$ and $\psi$ be decomposed as in (4.3) with respect to $Q_{1,n(T)}$. We start with the following lemma.

**Lemma 4.1 (Initial estimates)** Fix $\frac{27}{5} < p < 6$ with $\sigma(p) = \frac{3p - 9}{2p}$. We have

$$\sum_{k \neq 1} |z_k(0)|^2 \leq \frac{9}{8\rho_0}, \quad \|e^{Lt} \eta_\pm(0)\|_{L^p} \langle t \rangle^{\sigma(p)} + \|e^{Lt} \eta_\pm(0)\|_{L^2_{\text{loc}}} \langle t \rangle^{7/6} \leq C_2 \rho_0$$

(4.8)

for $t \geq 0$, for some $C_2 > 0$ uniformly in $n = n(T), 0 \leq T < T_e$.

**Proof.** Let $\psi' := e^{-i\theta(0)} \psi(0) - Q$. From (4.3) at $t = 0$, we have

$$a(0)R + \zeta(0) + \eta(0) = \psi' = e^{-i\theta(0)} \left( \sum_{j=0}^K x_j(0) \phi_j + \xi(0) \right) - Q.$$  

(4.9)

For $k \neq 1$, applying the projection $P_k$ on this equation, we get

$$|z_k(0)| \leq 2 |c_k| \left[ \|u^+_k \psi' \| + \|u^-_k \psi' \| \right] \leq (1 + o(1)) \| z_k(0) \| + n^3.$$  

(4.10)

Thus $\sum_{k \neq 1} |z_k(0)|^2 \leq \frac{9}{8\rho_0}$ by (4.1). Moreover, since $\psi'$ is localized and $\|\psi'\|_{H^1 \cap L^1} \lesssim \rho_0$, using Lemma 2.10 we get the estimates of $\eta_\pm(0)$ for $t > 1$ by Lemma 2.11 and for $0 \leq t \leq 1$ by Lemma 2.8. \( \square \)
Recall $\eta^{(3)}$ and $z_H$ are defined in (3.64) and (3.74). We now define

$$M_T := \sup_{0 \leq t \leq T} \max \left\{ \rho(t)^{-1}z_H(t), \quad 2D^{-1}\rho(t)|a(t)|, \right.$$  
$$\left[ n^{2\alpha-1} \rho(t)^{2\alpha-2\alpha} + 2C_2\rho_0(t)^{-\sigma(p)} \right]^{-1} \| \eta(t) \|_{L_p}, \quad \left[ n^{-\alpha/2} \rho^3 + n^{4/5} \rho^{7/3} + 2C_2\rho_0(t)^{-7/6} \right]^{-1} \| \eta^{(3)}(t) \|_{L_{loc}^2} \right\}. \quad (4.11)$$

Above $\alpha > 0$ is a small constant to be chosen. We can choose $\alpha = 0.01$.

Clearly $M_0 \leq 3/2$ if $n$ is sufficiently small. By continuity we have $M_T \leq 2$ for $T > 0$ sufficiently small. Our main result in this section is the following proposition, which implies $M_T \leq 3/2$ for all $T < t_c$ by a continuity argument.

**Proposition 4.2** Suppose that for some $T \in [0, t_c)$, $M_T$ is well-defined and $M_T \leq 2$. Then we have $M_T \leq 3/2$ and $n(T)/n \in (\frac{1}{4}, \frac{5}{2})$.

The proof of Proposition 4.2 is decomposed to Lemmas 4.3, 4.5

Note that $T < t_c$ and $M_T \leq 2$ imply

$$|z_0(t)| \leq \epsilon_4 n^{-1} \rho^2(t), \quad z_H(t) \leq 2\rho(t), \quad |a(t)| \leq D\rho(t)^2,$$

$$\| \eta(t) \|_{L_p} \leq 2n^{2\sigma-1}\rho(t)^{2\sigma-2\alpha} + 4C_2\rho_0(t)^{-\sigma},$$

$$\| \eta^{(3)}(t) \|_{L_{loc}^2} \leq 2n^{-\alpha/2}\rho^3 + 2n^{4/5}\rho^{7/3} + 4C_2\rho_0(t)^{-7/6}. \quad (4.12)$$

Since $[\eta] = \eta^{(2)} + \eta^{(3)}$ and $\| \eta^{(2)} \|_{L_{loc}^2} \lesssim n\rho^2$ by its definition, we get

$$\| \eta(t) \|_{L_{loc}^2} \lesssim n\rho(t)^2 + \rho_0(t)^{-7/6}. \quad (4.13)$$

It is sometimes convenient to use

$$\rho_0(t)^{-1/2} \lesssim \rho(t) \lesssim n^{-1}(t)^{-1/2}, \quad \| \eta \|_{L_p} + \| \eta(t) \|_{L_{loc}^2} \lesssim \rho. \quad (4.14)$$

**Lemma 4.3** Recall $X, \tilde{X}, F$ and $F_1$ are defined in (3.70), (3.21), and (3.68), with $\frac{27}{5} < p < 6$. Assume $M_T \leq 2$, then we have

$$\tilde{X} \lesssim n^{2}\rho^4 + \rho_0(t)^2(t)^{-7/6} + n\rho_0^2(t)^{-7/3},$$

$$X \lesssim n^{2}\rho^3 + n\rho_0(t)^{-7/6} + n\rho_0^2(t)^{-7/3}, \quad (4.15)$$

and, with $o(1)$ denoting small positive constants which go to 0 as $n + \| \psi_0 \|_{H^1} \to 0$,

$$\| F \|_{L_{p'}} \lesssim n\rho^2 + o(1)\rho_0^2(t)^{-1/4},$$

$$\| F - F_1 \|_{\frac{2}{3}L_{\frac{3}{5}} \cap L_{\frac{3}{2}}} \lesssim \rho^3 + n^{0.64}\rho^{2.54} + \rho_0^{7/4}(t)^{-5/4}. \quad (4.16)$$

**Proof.** By Hölder’s inequality for $p \geq 9/2$, and $\| \eta \|_{L_{2\cap L_p}} \ll 1$,

$$\| \eta^3 \|_{L_{loc}^1} \lesssim \| \eta \|_{L_{loc}^2} \| \eta \|_{L_{p}^{p/2}}, \quad \| \eta^3 \|_{L_1} \lesssim o(1) \| \eta \|_{L_{p}^{p/2}},$$

$$\| \eta^3 \|_{L_{p'}} \lesssim o(1) \| \eta \|_{L_{p}^{p/2}}, \quad \| \eta^3 \|_{L_{p/5\cap L_3/2}} \lesssim o(1) \| \eta \|_{L_{p}^{p/2}}. \quad (4.17)$$
From (3.70) with \( \beta = \rho \) and \( n \) replaced by \( n(T) \sim n \),

\[
\tilde{X} \lesssim \rho^2 \|\eta\|_{L^2_{loc}} + X_1, \quad X \lesssim n\rho \|\eta\|_{L^2_{loc}} + X_1, \quad X_1 = n \|\eta\|_{L^2_{loc}}^2 + \|\eta^3\|_{L^1_{loc}}.
\] (4.18)

Using (4.12), (4.13), and (4.17), one gets for \( \frac{27}{5} < p < 6 \) that

\[
X_1 \lesssim n^2 \rho^4 + \rho_0 \rho^2 \langle t \rangle^{-7/6} + n \rho_0^2 \langle t \rangle^{-7/3}.
\] (4.19)

One gets (4.15) from the above two equations.

To bound \( F = \kappa Q(2|h_\sigma|^2 + h_\sigma^2) + \kappa|h_\sigma|^2 h_\sigma \) in \( L^p \) with \( h_\sigma = aR + \zeta + \eta \), since \( \|aR\| \lesssim n^{-1} \rho^2 \), \( \|\zeta\|_{L^p} \lesssim \rho \) for \( q \geq 2 \), and \( \|\eta\|_{L^p} \leq \rho \), by (4.17) and (4.12) we get

\[
\|F\|_{L^p} \lesssim n \rho^2 + o(1) \|\eta\|_{L^p}^\frac{64}{3p} \lesssim n \rho^2 + o(1) \rho_0^2 \langle t \rangle^{-1.4510}.
\] (4.20)

Similarly, to bound \( F - F_1 \) with \( F_1 = \kappa Q(2|\zeta|^2 + \zeta^2) \), by (4.17) we have

\[
\|F - F_1\|_{L^p \cap L^\infty} \lesssim \rho^3 + n\rho \|\eta\|_{L^2_{loc}} + o(1) \|\eta\|_{L^p}^{\frac{10p}{9p-2}}.
\] (4.21)

By (4.12), \( \rho \leq n^{-1} \langle t \rangle^{-1/2} \), and \( \frac{27}{5} < p < 6 \), it is bounded by

\[
\lesssim \rho^3 + n \rho [n \rho^2 + \rho_0 \langle t \rangle^{-7/6}] + [n^{0.6471} \rho^{2.5494} + \rho_0^{1.8333} \langle t \rangle^{-1.2941}]
\lesssim \rho^3 + n^{0.64} \rho^{2.54} + \rho_0^{7/4} \langle t \rangle^{-5/4}.
\] (4.22)

Lemma 4.4 (Dispersion estimates) Assume \( M_T \leq 2 \), then for all \( 0 \leq t \leq T \), we have

\[
\|\eta(t)\|_{L^p} \lesssim \frac{3}{2} n^{2\sigma-1} \rho(t)^{2\sigma-2\alpha} + 3C_2 \rho_0(t)^{-\sigma},
\]

\[
\|\eta^3(t)\|_{L^2_{loc}} \lesssim \frac{3}{2} [n^{-\sigma} \rho^3 + n^{4/5} \rho^{7/3}] + 3C_2 \rho_0(t)^{-7/6}.
\] (4.23)

Proof. We first prove the \( L^p \)-bound. Since \( [\eta] = e^{i\theta} \eta_+ + e^{-i\theta} \eta_- \), it suffices to estimate \( \|\eta_\pm\|_{L^p} \). By (3.50) with \( t_0 = 0 \), and by Lemmas 2.11 and 2.16

\[
\|\eta_\pm\|_{L^p} \lesssim \|e^{tL} \eta_\pm(0)\|_{L^p} + \int_0^t \alpha_p(t-s) \|F_{L \pm}\|_{L^p} + \|F\|_{L^p}(s)ds.
\] (4.24)

By Lemma 4.1

\[
\|e^{tL} \eta_\pm(0)\|_{L^p} \leq C_2 \rho_0(t)^{-\sigma}.
\] (4.25)

By (3.71), Lemma 4.3 and (4.14),

\[
|\partial_t| = |F_\theta| \lesssim \rho^2 + n^{-1} X \lesssim \rho(t)^2 + \rho_0 \rho(t) \langle t \rangle^{-7/6} + \rho_0^2 \langle t \rangle^{-7/3} \lesssim \rho(t)^2.
\] (4.26)

By (3.49), (4.14), and Lemma 2.10

\[
\|F_{L \pm}\|_{L^p} \lesssim |F_\theta| (\|\eta\|_{L^p} + n^{-1}|a| + |z|) \lesssim \rho^2 \cdot \rho = \rho^3.
\] (4.27)

By Lemma 4.3 \( \|F\|_{L^p} \lesssim n \rho^2 + \rho_0^2 \langle t \rangle^{-7/5} \). Thus the integral in (4.24) is bounded by

\[
\lesssim \int_0^t \alpha_p(t-s) [n \rho^2(s) + \rho_0^2(s)^{-7/5}] ds \lesssim \rho_0^{2\alpha} n^{2\sigma-1} \rho(t)^{2\sigma-2\alpha} + \rho_0^2 \langle t \rangle^{-\sigma}.
\] (4.28)
Here we have used (4.3), $n\rho^2(s) \sim n^{-1}(\Delta t + s)^{-1}$, and $\forall 0 < \alpha < \sigma < 1$

$$
\int_0^t |t-s|^{-\sigma}(\Delta t + s)^{-1}ds \lesssim (\Delta t)^{-\alpha}(\Delta t + t)^{-\sigma + \alpha}.
$$

(4.29)

Combining (4.25) and (4.28), we get the first estimate of Lemma 4.4.

We next prove the second estimate. Recall that $\eta^{(3)}_\pm = \sum_{j=1}^4 \eta^{(3)}_{\pm,j}$, where $\eta^{(3)}_{\pm,j}$ are defined in (3.52) and (3.62) with $t_0 = 0$. By Lemmas 4.1 and 4.13 we get

$$
\left\|\eta^{(3)}_\pm,1\right\|_{L^2_{\text{loc}}} \leq C_2 \rho_0(t)^{-7/6}, \quad \left\|\eta^{(3)}_\pm,2\right\|_{L^2_{\text{loc}}} \leq C n \rho_0(t)^{-3/2}.
$$

(4.30)

For $\eta_{\pm,3}$, by Lemma 3.3, (4.14), and (4.15),

$$
\max |p_k| \lesssim n \rho^2 + \dot{X}_p + X \lesssim n \rho^2.
$$

(4.31)

By (3.33), (4.26) and the above,

$$
\|f_{kl} + \dot{\theta} f_{kl}\|_{L^2_t} \lesssim n|\dot{\theta}| \rho^2 + n \rho \max |\dot{p}_k| \lesssim n \rho^2 \rho^2 + n \rho (n \rho^2) \lesssim n^2 \rho^3.
$$

(4.32)

It follows from Lemma 2.13 that

$$
\left\|\eta^{(3)}_\pm,3\right\|_{L^2_{\text{loc}}} \leq C \int_0^t (t-s)^{-3/2} n^2 \rho^3(s)ds \leq C n \rho^3(t).
$$

(4.33)

Here we have used, for $a, b > 1$ and $S \geq 1$,

$$
\int_0^t (t-s)^{-a}(S+s)^{-b}ds \lesssim S^{1-b}(S+t)^{-a} + (S+t)^{-b},
$$

(4.34)

which is bounded by $(S+t)^{-b}$ if $a \geq b$.

For $\eta^{(3)}_\pm,4$ by Lemma 2.11 we have

$$
\left\|\eta^{(3)}_\pm,4\right\|_{L^2_{\text{loc}}} \leq C \int_0^t \alpha_\infty(t-s)\|F_{L \pm}\|_{L^9/s \cap L^{3/2}} + \|F - F_1\|_{L^9/s \cap L^{3/2}}(s)ds,
$$

(4.35)

where $\alpha_\infty(t) = t^{-1/2}(t^{-2/3})$. It follows from (4.34) that

$$
\int_0^t \alpha_\infty(t-s)\rho(s)^rds \lesssim \rho(t)^r + n^{1/3} \rho_0^{-2} \rho(t)^{7/3}, \quad r > 2.
$$

(4.36)

As for (4.27), we have $\|F_{L \pm}\|_{L^9/s \cap L^{3/2}} \lesssim \rho^3$. By Lemma 4.3 $\|F - F_1\|_{L^9/s \cap L^{3/2}} \lesssim \rho^3 + n^{0.64} \rho^{2.54} + \rho_0^{7/4} (t)^{-5/4}$. Thus

$$
\left\|\eta^{(3)}_\pm,4\right\|_{L^2_{\text{loc}}} \lesssim (\rho^3 + n^{1/3} \rho_0 \rho^{7/3}) + (n^{0.64} \rho^{2.54} + n^{0.97} \rho_0^{0.54} \rho^{7/3}) + \rho_0^{7/4} (t)^{-5/4}
$$

$$
\lesssim \rho^3 + o(1)n^{4/5} \rho^{7/3} \rho_0^{7/4} (t)^{-5/4}.
$$

(4.37)

Summing (4.30), (4.33), and (4.37), we get the bound of $\left\|\eta^{(3)}_\pm\right\|_{L^2_{\text{loc}}}$ in the lemma. □

**Lemma 4.5 (Bound states estimates)** Assume $M_T \leq 2$, then for all $0 \leq t \leq T$, we have

$$
z_H(t) \leq \frac{3}{2} \rho(t), \quad |a(t)| \leq \frac{3}{4} D \rho(t)^2, \quad |n(t) - n| \leq \frac{1}{4} n.
$$

(4.38)
Proof. For $1 < k \leq K$, from Lemma 3.7, we have a perturbation $q_k$ of $p_k$ such that

$$
\dot{q}_k = \sum_{l \neq 1} D_{kl}|q_l|^2 q_k + Y_k q_k + g_k,
$$

where

$$
|q_k - p_k| \lesssim Cn^2 \rho^2, \quad |\text{Re}(Y_k)| \leq Cn^2 z_L^2 \leq C\rho^4(t),
$$

$$
|g_k| \lesssim n^{\rho^2} + n^3 \rho \|\eta\|_{L^2_{loc}} + n\rho \|\eta^{(3)}\|_{L^2_{loc}} + \hat{X} + n\rho \hat{X}.
$$

From (3.75) and $\|\eta\|_{L^p} \leq \rho$, we have $\hat{X} \lesssim \rho^3$. Thus, from (4.12), (4.13) and Lemma 4.3, we get

$$
|g_k| \lesssim o(1)n^2 \rho^3 + n\rho \rho(t)^{-7/6} + n\rho^2(t)^{-7/3}.
$$

Since $\rho_0 = n^{1+\delta}$ and $0 < \delta < \frac{3}{2}$, it follows that

$$
\int_0^{n^{-3}T} |g_k(t)|dt \leq Cn\rho_0; \quad |g_k(t)| \leq o(1)n^2 \rho^3(t), \quad \forall t \geq n^{-3}.
$$

Now, from (4.39), we get

$$
\frac{d}{dt}|q_k| = \sum_{l \neq 1} \text{Re}(D_{kl})|q_l|^2 q_k + (\text{Re} Y_k)|q_k| + \text{Re}(\overline{q}_k / |q_k| g_k).
$$

for all $0 \leq t \leq n^{-3}$, by integrating this equation on $(0, t)$, we see that $|q_k(t) - q_k(0)| \ll \rho_0$. Using $z_H = (\sum_{k>1}|p_k|^2)^{1/2}$, $z_H(0) \leq \sqrt{9/8} \rho_0$ and $|q_k - p_k| \lesssim n^{\rho^2}$, we get

$$
z_H(t) \leq 1.1 \rho_0, \quad \forall 0 \leq t \leq n^{-3}.
$$

Now, let $f_H = (|q_2|^2 + \cdots + |q_K|^2)^{1/2}$, from (4.43) and (3.91), in particular $D_{k0}|q_0|^2 \lesssim n^2(n^{-1}\rho^2)^2 = \rho^4$, we get

$$
\dot{f}_H \leq -\frac{\gamma \rho_0 n^2}{2} f_H^3 + C[f_H \rho^4 + \sum_{k=2}^K |g_k|].
$$

By (4.12) and (4.42), we get

$$
\dot{f}_H \leq -\frac{\gamma \rho_0 n^2}{2} f_H^3 + o(1)n^2 \rho(t)^3, \quad n^{-3} \leq t \leq t_c.
$$

Let $g(t) := \frac{7}{5} \rho(t)$. We have $f_H(n^{-3}) < g(n^{-3})$ and $\dot{g} = -\frac{\gamma \rho_0 n^2}{5} g^3$, thus $\dot{f}_H(t) < \dot{g}(t)$ if $f_H(t) = g(t)$. By comparison principle,

$$
f_H(t) \leq g(t) = \frac{7}{5} \rho(t), \quad (n^{-3} \leq t \leq T),
$$

which together with (4.41) give the first estimate of the Lemma.

For the second estimate, recall that $a = a^{(2)} + b$ with $|a^{(2)}| \leq Cn^2 \rho^2(t)$. From Lemma 3.8, there is a perturbation $\tilde{b}$ such that

$$
\frac{d}{dt} \tilde{b} = b_0 + \dot{b}_0 + \sum_{1 < l, k \leq K} B_{kl}|z_l|^2 |z_k|^2 + g_b,
$$

(4.48)
where $g_b$ and $B_{kl}$ are defined in Lemma 3.8 and $b_0 = B_{00}|z_0|^4 + 2 \sum_{1<k\leq K} B_{k0}|z_0|^2|z_k|^2$. We have $|b - \tilde{b}| \leq C n^2 \rho^2$ and $|b_0| + |\tilde{b}_0| \lesssim n^4 |z_0|^2 \lesssim \varepsilon_4 n^2 \rho^4$. By Lemma 3.8 (1.12), (in particular $|z_0| \lesssim \varepsilon_4 n^{-1} \rho^2$ and this is where we choose $\varepsilon_4$), (1.13), Lemma 4.3 (1.19) and $\dot{X} \lesssim n^4 \rho^3 + \|\eta\|_L^3$,

$$|g_b| \lesssim n^3 \rho^4 + n \rho^5 + \varepsilon_4 n^2 \rho^2 \|\eta\|_{L^2} + n \rho^2 \|\eta\(3)\|_{L^2} + n X_1 + n \rho \dot{X} \lesssim \varepsilon n^2 \rho(t)^4 + \rho \|\eta\|_{L^2}.$$

(4.49)

Then, for $t \geq \Delta t = n^{-2} \rho_0^{-2}$, we have $\rho(t) \sim n^{-1} t^{-1/2}$ and

$$\int_{t}^{T} |\tilde{g}_b|(s)ds \approx \int_{t}^{T} [n^4 s^{-3/2} + s^{-7/6} - 1]ds \lesssim n^4 t^{-4/3} + t^{-7/6} \lesssim n^2 \rho(t)^2.$$  

(4.50)

For $0 \leq t \leq \Delta t$, we have $\rho(t) \sim \rho_0$ and

$$\int_{t}^{T} |\tilde{g}_b|(s)ds \leq \left( \int_{t}^{\Delta t} + \int_{\Delta t}^{\infty} \right) \|\tilde{g}_b\|(s)ds \lesssim \int_{t}^{\Delta t} n^2 \rho_0^2 (s)^{-7/6} ds + n^2 \rho_0^2 \lesssim n^2 \rho_0^2.$$  

(4.51)

Using $\int_{t}^{\infty} n^2 \rho^4 ds \lesssim \rho(t)^2$, we get have

$$\int_{t}^{T} |\tilde{b}_0 + g_b|(s)ds \leq o(1) \rho(t)^2, \; \forall \; t \in [0, T).$$  

(4.52)

Integrating (4.48) on $(t, T)$ and using $\max_{k,l}(\|B_{kl}\|)/(K^{-1} \gamma_0 n^2) \leq D_4/(4T)$, we get

$$|\tilde{b}(t)| \leq |\tilde{b}(T)| + D_4/2 \rho^2(t) + o(1) \rho^2(t) \leq |\tilde{b}(T)| + 5/9 D \rho^2(t).$$  

(4.53)

Now, since $a(T) = 0$, we get

$$|\tilde{b}(T)| = |a(T) - b(T)| + |b(T) - \tilde{b}(T)| \leq |a^{(2)}(T)| + C n^2 \rho(T)^2 \lesssim n^2 \rho(t)^2.$$  

(4.54)

Thus we have $|\tilde{b}(t)| \leq |\tilde{b}(T)| + |b(t) - \tilde{b}(T)| \leq 5/8 D \rho(t)^2$ and

$$|a(t)| \leq |a^{(2)}(t)| + |\tilde{b}(t)| + |\tilde{b}(t) - b(t)| \leq 3/4 D \rho(t)^2.$$  

(4.55)

Finally, Lemma 3.3 shows $|n(T) - n(t)| \lesssim n^{-1} |a(t)| + n^3 < n$ and the last claim of the Lemma.

The proof of Lemma 4.2 and Lemma 4.3 complete the proof of Proposition 4.2.

We now distinguish the two cases that $t_c = \infty$ and $t_c < \infty$.

Suppose $t_c = \infty$. By Lemma 3.3 (iii) we have for any $t < T < \infty$,

$$|n(t) - n(T)| \lesssim |a_{n(T)}(t)| \lesssim \rho^2(t),$$

(4.56)

which shows that $n(t)$ converges to some $n_\infty \sim n$ as $t \to \infty$. Furthermore $n(t) \sim n(0) \sim n_\infty$ and $|n(t) - n_\infty| \lesssim n^{-1} \rho^2(t)$. Together with the estimate $M_T \lesssim 3/2$ we have shown the main theorem in the case the solution converges to an excited state.

In the case $t_c < \infty$, by continuity we also have $M_{t_c} \lesssim 3/2$. we will show that the solution escapes from the first excited state family in the next section. We prepare it with the following lemma, whose proof is the same as that for $\eta_\pm(t)$ in Lemma 4.1 with the nonlinear terms set to zero for $t_c < s < t$. 

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Lemma 4.6 Suppose $t_c < \infty$. Let $\Delta t = n^{-2} \rho_{0}^{-2}$ and $\eta_{\pm}(t) = e^{i\theta(t)} P_{\pm}[\eta(t)]$ where $\eta(t)$ is as in (4.3) with respect to $Q_{1,n(t_c)}$. Then for all $t \geq t_c$, we have
\[
\| e^{L(t-t_c)} \eta_{\pm}(t_c) \|_{L^p} \leq \frac{1}{4} \Lambda_1(t), \quad \| e^{L(t-t_c)} \eta_{\pm}(t_c) \|_{L^2_{loc}} \leq \frac{1}{4} \Lambda_2(t),
\]
where for $C_2$ from Lemma 4.7 some $C_3 > 0$ and $\rho_c = \rho(t_c)$,
\[
\Lambda_1(t) = C_3[2C_2 \rho_0(t)^{-\sigma(p)} + n^{2\sigma-1} \rho_0^{-2} \rho(t)^{2\sigma-2\alpha}],
\]
\[
\Lambda_2(t) = C_3[2C_2 \rho_0(t)^{-7/6} + n \rho_c^2 (t - t_c)^{-7/6} + \rho^2(t) + n^{4/5} \rho^{7/3}(t)].
\]
Moreover, with $\sigma_2 := \min(\delta, \frac{3}{2} - \delta, \frac{2+5\delta}{15}) > 0$ and $t_c^+ := t_c + n^{-3}$,
\[
\Lambda_1(t) + \Lambda_2(t) \leq \rho_c, \quad (\forall t > t_c),
\]
\[
\Lambda_1 \leq \rho_0 \langle t \rangle^{-\sigma} + n^{1/3} \rho_c^{4/3}, \quad \Lambda_2 \leq \rho_0 \langle t \rangle^{-7/6} + n \rho_c^2, \quad (t_c < t < t_c^+),
\]
\[
\Lambda_1(t) \leq n^{1/3} \rho_c^{4/3}, \quad \Lambda_2(t) \leq n^{1+\sigma} \rho_c^2, \quad (t > t_c^+).
\]

Proof. From (3.51), we have
\[
e^{L(t-t_c)} \eta_{\pm}(t_c) = e^{L t} \eta_{\pm}(0) + \int_{0}^{t_c} e^{L(t-s)} P_{\pm} F_{L_{\pm}}(s) \eta_{\pm}(t_c) ds.
\]
We also decompose $\eta_{\pm}(t_c) = \eta_{\pm}^{(2)}(t_c) + \eta_{\pm}^{(3)}(t_c)$ with a similar formula for $e^{L(t-t_c)} \eta_{\pm}^{(3)}(t_c)$. We can bound $e^{L(t-t_c)} \eta_{\pm}^{(2)}(t_c)$ in $L^p$ and $e^{L(t-t_c)} \eta_{\pm}^{(3)}(t_c)$ in $L^2_{loc}$ using the same proof for Lemma 4.4 with the integrand set to zero for $t_c < s < t$. We also have
\[
\| e^{L(t-t_c)} \eta_{\pm}^{(2)}(t_c) \|_{L^2_{loc}} \lesssim \langle t - t_c \rangle^{-3/2} n \rho_c^2
\]
using the explicit definition of $\eta_{\pm}^{(2)}$ in (3.62) and Lemma 2.13. The above shows (4.57).

We now show (4.59). Its first part is because $\rho_0 \langle t \rangle^{-1/2} \leq \rho_c$ for all $t \geq t_c$, which follows from (4.14).

Its second part follows from $\rho(t) \sim \rho_c < \rho_0$.

For the third part with $t > t_c^+$, it suffices to show
\[
\rho_0 \langle t \rangle^{-\sigma} \lesssim n^{2\sigma-1} \rho_c^{2\sigma-2\alpha}, \quad \rho_0 \langle t \rangle^{-7/6} \lesssim n^{1+\sigma} \rho_c^2.
\]
If $t_c < \Delta t$, then $\rho \sim \rho_c \sim \rho_0$. Writing all factors as powers of $n$ using $\langle t \rangle^{-1} \leq n^3$, (4.62) is reduced to $1 + \delta + 3\sigma > 2\sigma - 1 + (2\sigma - 2\alpha)(1 + \delta)$ and $1 + \delta + 7/2 > 1 + \sigma_2 + 2(1 + \delta)$. Both are valid using $2/3 < \sigma < 3/4$, $0 < \delta < 3/2$ and $\sigma_2 < 3/2 - \delta$.

If $t_c > \Delta t$, then $\rho_c \sim n^{-1} t_c^{-1/2}$, and (4.62) is reduced to $n^{1+\delta} \langle t \rangle^{-\sigma} \lesssim n^{-1+2\alpha} t_c^{-\sigma+\alpha}$ and $n^{1+\delta} \langle t \rangle^{-7/6} \lesssim n^{-1+\sigma} t_c^{-1}$, both are correct. \(\Box\)

5 Escaping from an excited state

In this section we study the dynamics near an excited state when $t > t_c$ assuming $t_c < \infty$. We want to show that the solution will escape from the $\rho_0$-neighborhood of the excited state. Recall $\rho_0 = n^{1+\delta}$ with $0 < \delta < 3/2$. (We need $\delta \ll 1$ in next section but not here.)
Fix \( Q = Q_{1,n(t_c)} \) and decompose \( \psi(t) \) for \( t_c \leq t < T_e \) as in (4.3) and (4.4) with respect to this fixed \( Q \). At \( t = t_c \) we have Lemma 4.6 and, by definition of \( t_c \) and \( M_{t_c} \leq 3/2 \),

\[
|z_0(t_c)| \geq \varepsilon_4 n^{-1} \rho_c^2, \quad z_H(t_c) \leq \frac{3}{2} \rho_c, \quad |a(t_c)| \leq \frac{3}{4} D \rho_c^2, \quad \rho_c := \rho(t_c).
\]

Let

\[
\gamma(t) := |q_0(t)| + n^5 |\phi_0(t)|^{1/2} + \rho_c,
\]

where \( q_0(t) \) is the perturbation of \( p_0(t) \) defined in Lemma 3.7. It will be shown to be an upper bound for bound states. We have defined \( \gamma(t) \) in terms of \( |\phi_0| \) instead of \( |z_0| \) so that it is non-decreasing in \( t \) (for \( t > t_c^+ := t_c + n^{-3} \)).

Define

\[
t_0 := \sup \left\{ t \geq t_c : z_L(s) < 2n^{1+\delta}, \quad \forall s \in [t_c, t) \right\}.
\]

The time \( t_0 \) is the time that \( z_L \) becomes powerful enough in orthogonal coordinates. The subscript \( _o \) means “out” (of the neighborhood). It follows from Proposition 5.1 below that \( t_0 < T_e \) and hence the decompositions (4.3) and (4.4) are valid at least slightly beyond \( t_0 \).

Recall

\[
\frac{27}{5} < p < 6, \quad \sigma = \sigma(p) = \frac{3p - 9}{2p}, \quad \frac{2}{3} < \sigma < \frac{3}{4}.
\]

The main result of the section is the following proposition.

**Proposition 5.1** There exist constants \( C_3, D_1 > 0 \), uniform in \( n \), (with \( C_3 \) greater than that in Lemma 4.6), such that for all \( t_c \leq t \leq t_0 \), we have

\[
|q_0(t) - q_0(s)| \leq \frac{1}{10} \varepsilon_4 n^{-1} \rho_c^2, \quad (t_c \leq s \leq t \leq t_c^+ := t_c + n^{-3}),
\]

\[
\frac{|q_0(t)|}{|q_0(s)|} \in \left[ \frac{1}{2} e^{2(\Re \lambda_0)(t-s)}, e^{2(\Re \lambda_0)(t-s)} \right], \quad (t_c \leq s < t),
\]

\[
z_H(t) \leq \sqrt{\frac{6D}{\gamma_0}} \gamma(t), \quad |a(t)| \leq D_1 \gamma^2,
\]

\[
\left\| \eta(t) \right\|_{L^p} \leq n^{\sigma_1} \gamma(t)^2 + \frac{1}{2} \Lambda_1(t), \quad \sigma_1 = 4\sigma - 3 - \alpha,
\]

\[
\left\| \eta^{(3)}(t) \right\|_{L^2_{loc}} \leq C_3 n^{\frac{5}{2}} \gamma(t)^2 + C_3 \gamma(t)^3 + \frac{1}{2} \Lambda_2(t),
\]

where \( \alpha > 0 \) is so small that \(-\frac{1}{3} + 2\alpha < \sigma_1 = \frac{3(p-6)}{p} - \alpha < 0 \), and \( \Lambda_1(t) \) and \( \Lambda_2(t) \) are defined in (5.6). In particular, \( t_0 \leq T_e \) and for some constants \( c_1 \) and \( c_2 \),

\[
t_c + c_1 n^{-4} \log \frac{2\rho_0}{z_L(t_c)} \leq t_0 \leq t_c + c_2 n^{-4} \log \frac{2\rho_0}{z_L(t_c)}.
\]

The main term in the integrand of \( \eta \) is of order \( n \gamma^2 \). In the first term of its \( L^p \)-bound we lose some powers of \( n \) due to integration over a time interval of order \( n^{-4} \). On the other hand, the first term \( \gamma^3 \) of \( \left\| \eta(t) \right\|_{L^2_{loc}} \) estimate is optimal and comes from recent time terms of order \( z^{3} \) in the integrand.

---

5The term \( n^{\sigma_1} |\phi_0|^{1/2} \) is included in \( \gamma \) so that \( z_L \leq \gamma \). Explicitly: The bound of \( \left\| \eta \right\|_{L^p} \) includes \( n^{11} |\phi_0| \), see (6.32). By (6.21), the bound of \( \left\| \eta^{(3)} \right\|_{L^8_{\gamma^{3/2}}} \) and hence \( \left\| \eta^{(3)} \right\|_{L^2_{loc}} \) contains \( n^{18} z_L^m \) where \( m \to 11/6 \) as \( p \to 6 \). To bound \( z_H \) by \( \gamma \), we need \( \left\| \eta \right\|_{L^2_{loc}} \leq n \gamma^2 \) for \( n \geq 47 \) and \( \gamma = |\phi_0| + \rho_c \) is insufficient.
Proof. The lemma clearly holds true for $t = t_c$. By a continuity argument, it suffices to prove the lemma with additional weaker assumptions:

$$|q_0(t) - q_0(s)| \leq \frac{1}{2} \varepsilon_4 n^{-1} \rho_c^2, \quad (t_c \leq s \leq t \leq t_c^+),$$

$$|q_0(t)|/|q_0(s)| \in \left[ e^{\frac{1}{4}(\text{Re} \lambda_0)(t-s)}, e^{2(\text{Re} \lambda_0)(t-s)} \right], \quad (t_c^+ \leq s < t),$$

$$z_H(t) \leq 2 \sqrt{\frac{6D}{\gamma_0}} \gamma(t), \quad |a(t)| \leq 2D_1 \gamma^2,$$  \hspace{1cm} (5.7)

$$\|\eta(t)\|_{L^p} \leq 2n^\gamma \eta(t)^2 + 2\Lambda_1(t),$$

$$\left\| \eta(t) \right\|_{L^2_{\text{loc}}} \leq 2C_3 n^5 \gamma(t)^2 + 2C_3 \gamma(t)^2 + 2\Lambda_2(t).$$

At least for $t$ near $t_c$, the assumptions of Lemma 3.4 are satisfied and hence $|z_0| \leq |q_0| + |p_0 - q_0| \leq \gamma + Cn\gamma^2 = (1 + o(1)) \gamma$. Together with (5.7) and $\|\eta\| = \eta(t) + \eta(3)$, the assumptions of Lemmas 3.4 and 3.5 are valid until $t = t_o$ with $\beta = (1 + o(1)) \gamma(t)$, and

$$|z_0(t)| \leq (1 + o(1)) \gamma(t),$$

$$\|\eta(t)\|_{L^2_{\text{loc}}} \leq Cn\gamma^2(t) + \Lambda_2(t),$$  \hspace{1cm} (5.8)

$$\|\eta(t)\|_{L^2_{\text{loc}} \cap L^p} \leq \gamma(t).$$

Here we have used (4.59).

It is convenient to have an upper bound of $\gamma$ in terms of $|q_0|$. Clearly

$$\gamma^2(t) \sim |q_0|^2 + n^{10}|q_0| + \rho_c^2 \lesssim \varepsilon_4^{-1} n|q_0(t)| + \varepsilon_4^{-1} n|z_0(t_c)|.$$  \hspace{1cm} (5.9)

Since $|z_0(t_c)| \leq |q_0(t_c)| + Cn\gamma(t_c)^2 \leq |q_0(t)| + Cn\gamma(t)^2$, we get

$$\gamma(t) \lesssim \varepsilon_4^{-1} n|q_0(t)|.$$  \hspace{1cm} (5.10)

Thus we get an improved $z_0$ estimate,

$$|z_0| \leq |q_0| + Cn\gamma \leq (1 + o(1)) |q_0|.$$  \hspace{1cm} (5.11)

We can also derive from (5.7) and $|z_0(t_c)| \geq \varepsilon_4 n^{-1} \rho_c^2$ that, for any $t_c \leq s < t < t_o$,

$$|q_0(s)| \leq \frac{6}{5} |q_0(t)| e^{-\frac{1}{4}(\text{Re} \lambda_0)(t-s)}.$$  \hspace{1cm} (5.12)

We now give error estimates. For $X_1 = n \|\eta\|^2_{L^2_{\text{loc}}} + \|\eta^3\|_{L^1_{\text{loc}}}$, using (5.7), (5.8), and Hölder inequality, we have

$$X_1 \lesssim n(n^2 \gamma^4 \Lambda_2^2) + (n \gamma^2 + \Lambda_2)^4 (n^{\sigma_1} \gamma^2 + \Lambda_1)^B,$$  \hspace{1cm} (5.13)

with $A = \frac{2n-6}{p-2}$ and $B = \frac{p}{p-2}$. We claim that

$$X_1(t) \lesssim \begin{cases} n \gamma^2, & (\forall t > t_c), \\ n \rho_c^2 (t)^{-7/6} + n^{2.8} \gamma^4, & (t_c < t < t_c^+), \\ n^{2.8} \gamma^4, & (t > t_c^+). \end{cases}$$  \hspace{1cm} (5.14)
The first estimate is because $\Lambda_1 + \Lambda_2 \lesssim \rho C$. The last estimate is, using (4.59), and $A < 1.5 < B < 1.6$ with $A + B = 3$,

$$X_1(t) \lesssim n^3 \gamma^4 + (n \gamma^2)^A(n^1/3 \gamma^{4/3})^B = n^3 \gamma^4 + (n \gamma)^{2A/3} n \gamma^4 \lesssim n^2.8 \gamma^4.$$  (5.15)

When $t_c < t < t_c^+$, using $\rho \sim \rho_0$, (4.59), $\sigma_1 > -1/3$, and the previous estimate,

$$X_1(t) \lesssim n^3 \gamma^4 + n \rho^2_0 (t)^{-7/3} + (\rho_0 (t)^{-7/6} + n \gamma^2)^A(\rho_0 (t)^{-\sigma} + n^1/3 \gamma^{4/3})^B \lesssim n \rho^2_0 (t)^{-7/6} + n^2.8 \gamma^4.$$  (5.16)

For $\tilde{X}$ and $X$ defined in (3.10), we have

$$\tilde{X} \leq \gamma^2 \|\eta\|_{L^2_{loc}} + X = n^4 \gamma^4 + \gamma^2 A_2 + X(t),$$

$$X \leq n^4 \gamma^4 + n \gamma A_2 + X(t).$$  (5.17)

For $\tilde{X}_p$ defined in (3.76), we have

$$\tilde{X}_p = n^4 z_L \|\eta\|_{L^2_p} + n^6 z_L^2 \|\eta\|_{L_p} + n^6(6-p)/p \|\eta\|^3_{L_p} \lesssim n^4 z_L(n^2 \sigma \gamma^4 + \Lambda_2) + n^6 z_L(n^\sigma \gamma^2 + \Lambda_1) + n^6(6-p)/p(n^3 \sigma \gamma^4 + \Lambda_1).$$  (5.18)

Using Young’s inequality on $n^4 z_L \Lambda_2^3 + n^6 z_L^2 \Lambda_1$, and $6(6-p)/p + 3\sigma = 1 - 2\sigma > -1/2$, we get

$$\tilde{X}_p \lesssim n^3 \gamma^4 + n^5 z_L^3 + n^6(6-p)/p \Lambda_1^3.$$  (5.19)

From (3.22), (3.26), Lemmas 3.4, 3.5 and (5.7), (5.17) and (4.59), we get

$$|\dot{\theta}| \lesssim \beta^2 + n^{-1} X \lesssim \gamma^2 + n^{-1}(n^2 \gamma^3 + n \gamma A_2 + X) \lesssim \gamma^2,$$

$$|\dot{p}_k| \lesssim n^4 z_L + n \beta^2 + X \lesssim n^4 z_L + n \gamma^2 + X \lesssim n^4 z_L + n \gamma^2.$$  (5.20)

We now estimate the main terms. By Hölder inequality,

$$\|\eta^3\|_{L^{p'}} \leq \|\eta\|_{L^2}^{2(p-4)/(p-2)} \|\eta\|_{L^p}^{p-2} \|\eta^3\|_{L^{p'/2} \gamma \gamma (3/2)} \leq \|\eta\|_{L^2}^{2(p-9)/(p-2)} \|\eta\|_{L^p}^{11p/(p-2)}.\ (5.21)

Using $36/7 < p < 6$ and $-1/2 \leq \sigma = 4 \sigma - 3 - \alpha = 3 - 18/p - \alpha < 0$,

$$(n^4 \gamma^3 - \alpha \gamma^2)^{p+2} \lesssim (n^4 \gamma^3 - \alpha \gamma^2)^{p+2} \lesssim o(1) \gamma^3,$$  (5.22)

for $\alpha > 0$ sufficiently small. By Lemma 3.4 and $\|\eta\|_{L^2} \leq o(1)$, we get

$$\|F\|_{L^{p'}} \lesssim n \gamma^2 + X \|\eta\|_{L^p}^2 + \|\eta^3\|_{L^{p'/2}} \lesssim n \gamma^2 + \delta_1;$$

$$\|F - F_1\|_{L^{p'/2} \gamma \gamma (3/2)} \lesssim \gamma^3 + X \|\eta\|_{L^p}^2 + \|\eta^3\|_{L^{p'/2} \gamma \gamma (3/2)} \lesssim \gamma^3 + \delta_2.$$  (5.23)

$$\delta_2(t) = n \gamma(t) \Lambda_2(t) + n \Lambda_2^2 (t).$$

In deriving the above estimates most terms in $X_1$ are controlled by $\delta_2$ except

$$n^4 \gamma^2 A_1^B \lesssim (n^A \gamma^2 A_1 B)(n^B \gamma^2 A_1 B)^{2/(2-B)} \lesssim n^A \gamma^2 A_1 B^2/2 (2-B) + (n^B \gamma^2 A_1 B^2/2 - B) \lesssim \gamma^3 + n \Lambda_2^2.$$  (5.24)

Estimates (5.3) now follows from Lemmas 5.2 and 5.3 below.

In particular, taking $s = t_c^+$ and $t = t_\phi$, (5.5) and (5.7) together with $\Re \lambda_0 \sim n^{-4}$ and $|z_0| = (1 + o(1))|g_0|$ imply (5.6).
**Lemma 5.2 (Dispersion estimates)** For all \( t_c \leq t \leq t_o \), we have

\[
\|\eta(t)\|_{L^p} \leq \left[ n^{\sigma_1} \gamma^2 + \Lambda_1 \right](t), \quad \left\| \eta^{(3)}(t) \right\|_{L^2_{\text{loc}}} \leq \left[ C_3 n^5 \gamma^2 + C_3 \gamma^3 + \Lambda_2 \right](t). \tag{5.25}
\]

Note that \( \Lambda_2(t) \) may compete with the main terms for \( t \) near \( t_c \) but decay rapidly.

**Proof.** We first estimate \( \|\eta(t)\|_{L^p} \). It suffices to estimate \( \eta_{\pm} \) with

\[
\eta_{\pm}(t) = e^{L(t-t_c)}\eta_{\pm}(t_c) + \int_{t_c}^{t} e^{L(t-s)}P_{\pm}\left\{ F_{L\pm} + e^{\mp i\gamma J[F]} \right\} ds. \tag{5.26}
\]

By Lemma 4.11 we have

\[
\|\eta_{\pm}(t)\|_{L^p} \lesssim \left\| e^{L(t-t_c)}\eta_{\pm}(t_c) \right\|_{L^p} + \int_{t_c}^{t} \alpha_p(t-s)\left\{ \|F_{L\pm}\|_{L^p} + \|F\|_{L^p} \right\}(s)ds. \tag{5.27}
\]

By Lemma 4.6 we have \( \|e^{L(t-t_c)}\eta_{\pm}(t_c)\|_{L^p} \leq \frac{1}{4}\Lambda_1(t) \). By (3.49) and (5.20), we get

\[
\|F_{L\pm}\|_{L^p} \leq \left\| \hat{\theta} \left\|\eta\|_{L^p} + n^{-1}|a| + |z| \right\| \gamma^2 \cdot \gamma. \tag{5.28}
\]

From this, (5.23), (5.27), and \( X_1 \ll n\rho_c^2 \), we get

\[
\int_{t_c}^{t} \alpha_p(t-s)\left\{ \|F_{L\pm}\|_{L^p} + \|F\|_{L^p} \right\}(s)ds \lesssim \int_{t_c}^{t} \alpha_p(t-s)(n\gamma(s)^2 + \delta_2(s))ds. \tag{5.29}
\]

Recall \( \gamma^2 = |q_0|^2 + n^{10}|q_0| + \rho_c^2 \). By (5.7), \( \Re\lambda_0 \sim n^4 \) and \( f^t |t-s|^{-\sigma}e^{-a(t-s)}ds \lesssim a^{\sigma-1} \),

\[
\int_{t_c}^{t} \alpha_p(t-s)n|q_0|^2(s)ds \lesssim \int_{t_c}^{t} \alpha_p(t-s)n|q_0|(t)^2 e^{-\frac{t}{4}\Re\lambda_0(t-s)}ds \leq Cn^{4\sigma-3}|q_0|^2(t). \tag{5.30}
\]

The integral of \( nn^{10}|q_0| \), part of \( \delta_2 \), is bounded in the same way by \( Cn^{4(\sigma-1)+11}|q_0|(t) \).

For \( \rho_c^2 \), we have

\[
\int_{t_c}^{t} \alpha_p(t-s)n\rho_c^2ds \leq n\rho_c^2(t-t_c)^{1-\sigma} = n^{4\sigma-3-\alpha/2} \cdot \rho_c^2 n^{\alpha/2}T^{1-\sigma} \tag{5.31}
\]

where \( \alpha > 0 \) is to be chosen and \( T = n^4(t-t_c) \). Let \( A = \frac{1}{8}n^{-4}\Re\lambda_0 \) which is of order 1. If \( AT \leq 10\log\frac{1}{n} \), then \( n^{\alpha/2}T^{1-\sigma} = o(1) \) if \( n \) is sufficiently small. If \( AT \geq 10\log\frac{1}{n} \), then by (5.12)

\[
\rho_c^2T^{1-\sigma} \leq Cn|q_0(t_c)|T^{1-\sigma} \leq Cn|q_0(t)|e^{-2AT}T^{1-\sigma}. \tag{5.32}
\]

Since \( \rho_c^2T^{1-\sigma} \leq n^{10} \) and \( e^{-2AT}T^{1-\sigma} \leq C \), it is bounded by \( Cn^{11}|q_0(t)| \).

Using (4.59), the error term \( \delta_2(t) = n\gamma(t)\Lambda_2(t) + n\Lambda_2^2(t) \) is bounded by \( n^{7/3}\rho_c^2 \) when \( t > t_c^+ \) and by \( n^{7/3}\rho_c^2 + n\rho_c^2(t)^{-7/6} \) when \( t < t_c^+ \). The term \( n^{7/3}\rho_c^2 \) is smaller than the main term \( n\gamma^2 \) in (5.29) and can be absorbed, while

\[
\int_{t_c}^{t_c^+} n\rho_c^2(t)^{-7/6} dt \lesssim n\rho_c^2 \tag{5.33}
\]

which can be checked using \( \rho_c \sim \rho_0 \) for \( t_c < \Delta t \) and \( \rho_c \sim n^{-1}t_c^{-1/2} \) for \( t_c > \Delta t \).
Thus the integral in (5.24) is bounded by $n^{\sigma_1 \gamma^2}$ with $\sigma_1 = 4\sigma - 3 - \alpha$, and we have shown the first estimate of (5.25) for $\|\eta\|_{L^p}$.

Next, we estimate $\| \eta \|_{L^2_{\text{loc}}}$.

Decompose $\eta_{\pm}^{(3)} = \sum_{j=1}^{\infty} \eta_{\pm,j}^{(3)}$, where $\eta_{\pm,j}^{(3)}$ are defined explicitly in (5.52) and (5.62) with $t_0 = t_c$. From Lemmas 2.13 and 4.6, we get

$$
\| \eta_{\pm,1}^{(3)} \|_{L^2_{\text{loc}}} \leq \frac{1}{4} \Lambda_2(t), \quad \| \eta_{\pm,2}^{(3)} \|_{L^2_{\text{loc}}} \leq \frac{1}{4} C_3 n \rho_c^2 (t - t_c)^{-3/2} \leq \frac{1}{4} \Lambda_2(t). \tag{5.34}
$$

By (3.63) and (5.20), we have

$$
\langle |\hat{f}_{kl}| + |\hat{\theta} f_{kl}| \rangle_{L_{\text{loc}}} \lesssim n |\hat{\theta}| \gamma^2 + n \gamma |\hat{p}| \lesssim n (\gamma^2 \gamma^2 + n \gamma(n^2 \gamma^2)} \approx n^5 \gamma^2 + n^2 \gamma^3. \tag{5.35}
$$

By Lemma 2.13 again and $\gamma(s) \lesssim \gamma(t)$ for $s < t$, we obtain

$$
\| \eta_{\pm,3}^{(3)} \|_{L^2_{\text{loc}}} \lesssim \int_{t_c}^t \langle t - s \rangle^{-3/2} [n^5 \gamma^2 + n^2 \gamma^3](s) ds \lesssim [n^5 \gamma^2 + n^2 \gamma^3](t). \tag{5.36}
$$

Finally, $\| \eta_{\pm,4}^{(3)} \|_{L^2_{\text{loc}}}$ is bounded by $\int_{t_c}^t \alpha_\infty(t - s) I_4(s) ds$ by Lemma 2.11 with

$$
I_4 = \| F_L^\pm \|_{L^{9/8} \cap L^{3/2}} + \| F - F_1 \|_{L^{9/8} \cap L^{3/2}} \lesssim \gamma^3 + \delta_2 \tag{5.37}
$$

by (5.28) and (5.29). Using $\delta_2(t) = n \gamma(t) \Lambda_2(t) + n \Lambda_2^2(t)$ and the explicit form of $\Lambda_j$ in (4.15), we obtain

$$
\| \eta_{\pm,4}^{(3)} \|_{L^2_{\text{loc}}} \lesssim \int_{t_c}^t \alpha_\infty(t - s) [\gamma^3 + \delta_2](s) ds \lesssim \gamma^3(t) + n \rho_0^2(t)^{-3/2} \gamma(t) \approx \gamma^3(t) + o(1) \Lambda_2(t). \tag{5.38}
$$

Summing the above estimates, we get the second estimate of (5.25) for $\| \eta \|_{L^2_{\text{loc}}}$. \hfill \Box

**Lemma 5.3 (Bound states estimates)** There is a uniform in $n$ constant $D_1 > 0$ such that for all $t_c \leq t \leq t_o$, we have

$$
|q_0(t) - q_0(t_c)| \leq \frac{1}{10} \varepsilon n^{-1} \rho_c^2, \quad (t_c \leq t \leq t_c^+),
$$

$$
|g_0(t)| \leq \varepsilon \left[ e^{\frac{1}{2} (\text{Re} \lambda_0)(t-s)} + e^{\frac{4}{3} (\text{Re} \lambda_0)(t-s)} \right], \quad (t_c^+ \leq s < t), \tag{5.39}
$$

$$
z_H(t) \leq \sqrt{\frac{6D}{\gamma_0}} \gamma(t), \quad |a(t)| \leq D_1 \gamma(t)^2.
$$

**Proof.** First we estimate $g_0(t)$. From Lemma 5.7 we have

$$
\hat{q}_0(t) = (\text{Re} \lambda_0) q_0 + \bar{Y}_0 q_0 + g_0, \quad |q_0 - p_0| \lesssim n \gamma^2, \quad |\text{Re}(\bar{Y}_0)| \leq C n^2 \gamma^2 \ll n^4. \tag{5.40}
$$

Here $\bar{Y}_0 = Y_0 + \sum_{l \neq 1} D_{0l} |q_l|^2$. Moreover, from (3.92), (5.17) and (5.10), we have

$$
|g_0| \leq C [n^3 \gamma^2 + n \gamma^4 + n^3 \gamma^2 |\eta|_{L^2_{\text{loc}}} + n \gamma \| \eta \|_{L^2_{\text{loc}}} + \sqrt{\bar{X}_p + \bar{X}^2}] \leq o(1)n^4 |q_0| + \delta_3, \tag{5.41}
$$
where $\delta_3 = C(n\delta^{(6-p)/p} \Lambda_1^3 + \gamma^2 \Lambda_2 + X_1)$. If $t < t^+_c$, by (4.59)\textsubscript{2}, (5.14)\textsubscript{2} and (5.33),

$$
\delta_3(t) \lesssim n\rho_0^2(t)^{-7/6} + n\rho_c^4 + n\gamma^2\rho_c^2 + n^{2.8}\gamma^4,
$$

$$
|q_0(t) - q_0(t_c)| \leq \int_{t_c}^{t^+_c} Cn^4|q_0| + \delta_3(s) ds \leq o(1)|q_0(t_c)| + \varepsilon_4n^{-1}\rho_c^2),
$$

This shows the $q_0(t)$-estimate for $t < t^+_c$. Suppose now $t^+_c < t$. By (4.59)\textsubscript{3}, (5.14)\textsubscript{3}, and (5.10),

$$
\delta_3(t) \lesssim n\rho_0^2(t)^{-7/6} + n\rho_c^4 + n\gamma^2\rho_c^2 + n^{2.8}\gamma^4 \ll n^4|q_0|.
$$

Since $\Re \lambda_0 > 0$ is of order $n^4$, Eq. (5.40) gives

$$
0 < \frac{1}{2}(\Re \lambda_0)|q_0| \leq \frac{d}{dt}|q_0| \leq \frac{3}{2}(\Re \lambda_0)|q_0|,
$$

which implies the estimate of $|q_0(t)|$ for $t > t^+_c$.

Next, we estimate $z_H(t)$. For any $k > 1$, by Lemma 3.7, we have

$$
\frac{d}{dt}q_k = \sum_{l>1} D_{kl}|q_l|^2 q_k + Y_k q_k + g_k, \quad |q_k - p_k| \leq Cn\gamma^2.
$$

Moreover, we have

$$
|D_{kl}| \leq Dn^2, \quad |\Re(Y_k)| \leq Dn^2|z_0|^2, \quad \Re(D_{kl}) \leq -\frac{\gamma_0}{2} n^2, \quad \forall l > 1.
$$

So, we have

$$
\frac{d}{dt}(|q_k|) \leq -\frac{\gamma_0 n^2}{2} \sum_{l>1} |q_l|^2 |q_k| + 2Dn^2|q_0|^2 |q_k| + |g_k|.
$$

Let $f(t) = (\sum_{l>1} |q_l|^2)^{1/2}$. We have $f(t_c) \lesssim \rho_c$ and

$$
\dot{f}(t) \leq -\frac{\gamma_0 n^2}{2} f^3 + 2Dn^2|q_0|^2 f(t) + \sum_{k>1} |g_k|.
$$

On the other hand, from (3.92), we have

$$
|g_k| \leq C[n\gamma^4 + n^4\gamma^3 + n^3\gamma \|\eta\|_{L^2_{\text{loc}}} + n\gamma \|\eta\|_{L^3_{\text{loc}}} + n\gamma \hat{X}_p + \hat{X}] \leq o(1)n^2\gamma^3 + \delta_4,
$$

where $\delta_4 = C(n\gamma n^{6(6-p)/p} \Lambda_1^3 + n\gamma \Lambda_2 + X_1)$. If $t \leq t^+_c$, by (4.59)\textsubscript{2}, (5.14)\textsubscript{2} and (5.33),

$$
\delta_4(t) \lesssim n\rho_0^2(t)^{-7/6} + n\rho_c^4 + n^{2.8}\gamma^4
$$

$$
|f(t) - f(t_c)| \leq \int_{t_c}^{t^+_c} Cn^2\rho_c^3 + \delta_4(s) ds \leq Cn\rho_c^2 \ll \rho_c.
$$

Thus $f(t) \lesssim \rho_c$ for $t < t^+_c$. When $t^+_c < t$, since $\delta_4(t) \leq n^2\gamma^5 + n^{2+\sigma_2}\gamma^3 + n^{2.8}\gamma^4 \ll n^2\gamma^3$, for $\overline{\gamma} = (\frac{16D}{3\gamma_0})^{1/2} \gamma$,

$$
\dot{f}(t) \leq \frac{\gamma_0 n^2}{4} [\overline{\gamma}^3 - f^3], \quad (t > t^+_c).
$$

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Since $\gamma(t)$ is nondecreasing and $f(t^+_c) < \tilde{\gamma}(t^+_c)$, by comparison we get
\[
f(t) \leq \tilde{\gamma}(t), \quad \forall \, t > t^+_c. \tag{5.52}
\]
Thus $z_H(t) \leq f(t) + |f(t) - z_H(t)| \leq \tilde{\gamma}(t) + C n \gamma^2(t) < \sqrt{\frac{n^2}{\gamma(t)}}$. 

Finally, we estimate $a(t)$. By (3.39) and Lemma 3.8, $a = a^{(2)} + (b - \tilde{b}) + \tilde{b}$, where
\[
|a^{(2)}| \leq n^2 \gamma^2, \quad |b - \tilde{b}| \leq C n \gamma [\gamma^2 + n \|\eta\|_{L^2_{\text{loc}}}] \leq C n^2 \gamma^2, \tag{5.53}
\]
and
\[
\frac{d}{dt} \tilde{b} = b_0 + \sum_{k,l \neq 1} B_{kl} |z_k|^2 |z_l|^2 + g_b. \tag{5.54}
\]
Using $a(t_c) = 0$,
\[
|a(t) - 0| \leq |a^{(2)}(t)| + |b^{(2)}(t) + |b - \tilde{b}|(t)| + |b(t) - \tilde{b}(t)| \leq C n^2 \gamma^2(t) + \int_{t_c}^t |\frac{d}{dt} \tilde{b}|. \tag{5.55}
\]
From (5.34), $b_0(t) = b_0 \left|z_0(t)\right|^2$ with $b_0 = 2 \Im \kappa_0 (Q^2, \tilde{u}_0^+ u_0^-)$ and $|b_0| n^{-4} \leq C_4$ for some explicit $C_4 = O(1)$. We also have $|B_{kl}||z_k|^2 |z_l|^2 \lesssim n^2 \gamma^4$ and
\[
|g_b| \leq C [n^2 \gamma^4 + n^2 \beta^2 |z_0|^2 + n \beta^5 + n^2 |z_0| \|\eta\|_{L^2_{\text{loc}}} + n X_1 + n \gamma^2 \|\eta^{(3)}\|_{L^2_{\text{loc}}} + n \gamma \tilde{X}_p] \leq o(1)n^4 |z_0|^2 + C n^2 \gamma^4 + \delta_5, \tag{5.56}
\]
where $\delta_5 = n X_1 + (n^2 \beta L + n^2 \gamma) A_2 + n \gamma n^{6(6/p - 1)} \Lambda_3$. Thus
\[
|a(t)| \leq C n^2 \gamma^2(t) + \int_{t_c}^t (C_4 + o(1)) n^4 |q_0(s)|^2 + C n^2 \gamma^4(s) + \delta_5(s) ds. \tag{5.57}
\]
By (5.7),
\[
\int_{t_c}^t (C_4 + o(1)) n^4 |q_0(s)|^2 ds \leq \frac{6}{5} C_4 n^4 |q_0(t)|^2 \int_{t_c}^t e^{-\frac{4}{5} \Re \lambda_0 (t-s)} ds \leq \frac{24 C_4 n^4}{5 \Re \lambda_0} |q_0(t)|^2. \tag{5.58}
\]
Moreover, by the definition of $\gamma$,
\[
\int_{t_c}^t C n^2 \gamma^4(s) ds \leq \int_{t_c}^t [n^2 q_0^4 + n^{22} |q_0|^2] (s) ds + n^2 \rho_c^4 (t - t_c). \tag{5.59}
\]
The integral is bounded by $n^{-2} |q_0|^4 + n^{18} |q_0|^2 = o(1)|q_0|^2$ similarly as in (5.58), while the last term is bounded by $n^2 \rho_c^4 C n^{-4} \log |z_0(t)| = o(1) \rho_c^2$. Thus this term is $o(1) \gamma^2$. 

For the error term $\int_{t_c}^t \delta_5(s) ds$, if $t \leq t^+_c$, by (4.14) we have
\[
\delta_5(s) \leq n^2 \rho_0^2 (t)^{-7/6} + n^3 \gamma^4 + (n^2 |q_0(t_c)| + n \gamma^2) (\rho_0 (t)^{-7/6} + n \rho_c^2) + n \gamma (t_c) (\rho_0^3 (t)^{-3/2} + n \rho_c) \leq n^2 \rho_0^2 (t)^{-7/6} + o(1) n^4 \gamma^2. \tag{5.60}
\]
Thus, using (5.33), we have \( f_t^s \delta_\rho(s)ds \leq o(1)n\gamma(t_c)^2 \). If \( t > t_c^+ \), by (4.59) and (5.14), we have \( \delta_\rho(s) \leq n^{3.8}\gamma^4 + n^{2}\gamma n^{1-s}\rho_c^2 + n\gamma n\rho_c^4 = o(1)(n^2\gamma^4 + n^4\gamma^2) \), which is dominated by other terms in (5.57).

In conclusion, we have shown

\[
|a(t)| \leq D_1\gamma^2(t) \quad D_1 := \frac{5C_4n^4}{\text{Re} \lambda_0} = O(1).
\] (5.61)

This completes the proof of the Lemma 5.3.

The above finishes the proof of Proposition 5.1.

We now prove the following out-going estimate of \( \eta \) at \( t_o \).

**Lemma 5.4** For some \( C_5 > 0 \), for all \( t \geq t_o \), we have

\[
\left\| e^{(t-t_o)L} \eta(t_o) \right\|_{L^p} \leq \bar{A}_1(t) := A_1(t) + C_5n^{-2}p_0(n^{-4} + t - t_o)^{-\sigma},
\]

\[
\left\| e^{(t-t_o)L} \eta(t_o) \right\|_{L^2_{loc}} \leq \bar{A}_2(t) := A_2(t) + C_5np_0^2(t - t_o)^{-7/6}
\]

\[
+ C_5p_0^3(t - t_o)^{-1/6}n^{-4}(t - t_o + n^{-4})^{-1}
\]

\[
+ C_5n^{-3}(n^{7/3}\rho_c + \rho_c^2)(t - t_o + n^{-4})^{-7/6}.
\] (5.62)

**Proof.** For all \( t \geq t_o \), we have

\[
e^{L(t-t_0)}e^{L(t-t_c)}\eta(t_c) + \int_{t_c}^{t_o} e^{L(t-s)}F\pm F_L\eta +Je^{i\theta[F]}ds.
\] (5.63)

We first bound it in \( L^p \). By Lemma 4.6, the first term is bounded in \( L^p \) by \( A_1(t) \). The second term is bounded in \( L^p \) as in (5.29) by

\[
\leq \int_{t_c}^{t_o} \alpha_p(t-s)[\|F_L\|_{L^p'} + \|F\|_{L^p'}]ds \leq \int_{t_c}^{t_o} \alpha_p(t-s)[n\gamma^2(s) + \delta_2(s)]ds.
\] (5.64)

Note \( n\gamma^2 + \delta_2 \sim n|q_0|^2 + n^{11}|q_0| + n\rho_c^2 + \delta_2 \). By (5.7),

\[
\int_{t_c}^{t_o} \alpha_p(t-s)n|q_0|^2(s)ds \leq \frac{6}{5} \int_{t_c}^{t_o} \alpha_p(t-s)n\rho_0^2e^{-\frac{4}{5}\text{Re} \lambda_0(t_o-s)}ds.
\] (5.65)

Using

\[
\int_{t_o}^{t_o} |t-s|^{-\sigma}e^{-(t_0-s)/T}ds \leq \int_{t_o-T}^{t_o} |t-s|^{-\sigma}e^{-(t_0-s)/T}ds \leq \int_{t_o-T}^{t_o} |t-s|^{-\sigma}ds \leq T(t-t_o + T)^{-\sigma}
\] (5.66)

with \( T = 4/\text{Re} \lambda_0 \sim n^{-4} \), (5.65) is bounded by \( Cn^{-3}\rho_0^2(t - t_o + n^{-4})^{-\sigma} \).

Similarly \( \int_{t_c}^{t_o} \alpha_p(t-s)n|q_0|s)(s)ds \) is bounded by \( n^{11}p_0n^{-4}(t - t_o + n^{-4})^{-\sigma} \).

Let \( t_k \) denote the first time in \( [t_c, t_o] \) so that \( |q_0(t)| = \rho_c \). When \( t > t_k \), the integrand \( \rho_c^2 \) is dominated by \( |q_0|^2 \) and can be absorbed. By (5.7),

\[
\int_{t_c}^{t_k} \alpha_p(t-s)n\rho_c^2(s)ds \leq n\rho_c^2|t_k - t_c||t - t_k|^{-\sigma}.
\] (5.67)
Using
\[ \rho_c^2 \lesssim \varepsilon^{-4} nq_0(t_c) \lesssim \frac{6}{5} \varepsilon^{-4} n\rho_c e^{-\frac{1}{4} \Re \lambda_0(t_c-t_c)}, \]  
and \( n^4|t_k-t_o|e^{-\frac{1}{4} \Re \lambda_0(t_k-t_c)} \leq C, \) the integral in (5.67) is bounded by \( Cn^{-2}\rho_c |t-t_k|^{-\sigma}. \)

Using (4.55), the error term \( \delta_2(t) \) is bounded by \( n^{7/3}\rho_c^2 \) when \( t > t_c^+ \) and by \( n^{7/3}\rho_c^2 + n\rho_0^2(t)^{-7/6} \) when \( t < t_c^+ \). The term \( n^{7/3}\rho_c^2 \) is much smaller than the main terms and can be absorbed, while by (5.58),

\[ \int_{t_c}^{t_c^+} \alpha_p(t-s)n\rho_0^2(s)^{-7/6} \, ds \lesssim n\rho_c^2|t-t_c|^{-\sigma}. \]  

Summing the above estimates gives the first estimate of Lemma [5.3].

For the second estimate, we have \( \eta_\pm(t_o) = \eta_\pm^{(2)}(t_o) + \eta_\pm^{(3)}(t_o) \). By (3.52), (3.62) and (3.64) with \( t_0 \) replaced by \( t_c \), we have for \( \tau = t - t_o \geq 0 \)

\[ e^{L_\tau} \eta_\pm(t_o) = e^{L_\tau} \eta_\pm^{(2)}(t_o) + \sum_{j=1}^{4} e^{L_\tau} \eta_\pm^{(3)}(t_o), \]

with
\[ e^{L_\tau} \eta_\pm^{(3)}(t_c) = e^{(t-t_c)L} \eta_\pm(t_c), \quad e^{L_\tau} \eta_\pm^{(3)}(t_o) = -e^{(t-t_c)L} \eta_\pm^{(2)}(t_c), \]

\[ e^{L_\tau} \eta_\pm^{(3)}(t_o) = -\int_{t_c}^{t_o} e^{(t-s)L} e^{-i\Theta(s)} \prod_{k,l} \left( \Re R_{kl} e^{-i\Theta(s)} f_{kl} \mp i \Re R_{kl} e^{-i\Theta(s)} \hat{f}_{kl} \right)(s) ds, \]

\[ e^{L_\tau} \eta_\pm^{(3)}(t_o) = \int_{t_c}^{t_o} e^{(t-s)L} P_\pm \{ F_{L_\pm} + J e^{i\Theta[F - F_1]} \} ds. \]

From the explicit definition of \( \eta_\pm^{(2)}(t_o) \) in (3.62) and Lemma [2.13] we obtain

\[ \left\| e^{L_\tau} \eta_\pm^{(2)}(t_o) \right\| \leq Cn\rho_0^2(t-t_o)^{-3/2}. \]

By Lemma [4.6]

\[ \left\| e^{L_\tau} \eta_\pm^{(3)}(t_o) \right\|_{L^2_{loc}} \leq \frac{1}{4} \Lambda_2(t), \quad \left\| e^{L_\tau} \eta_\pm^{(3)}(t_o) \right\|_{L^1_{loc}} \leq C_3n\rho_c^2(t-t_c)^{-3/2}. \]

As in (5.36) and (5.38), we obtain

\[ \left\| e^{L_\tau} (\eta_\pm^{(3)} + \eta_\pm^{(4)})(t_o) \right\|_{L^1_{loc}} \lesssim \int_{t_c}^{t_o} \alpha_\infty(t-s)[n^5\gamma^2 + \gamma^3 + \delta_2](s) ds \lesssim I_1 + I_2, \]

where \( I_j \) are integrals over the same time interval with the following integrands

\[ (n^5|q_0|^2 + n^{15}|q_0| + |q_0|^3 + n^{15}|q_0|^{3/2}), \quad (n^{7/3}\rho_c^2 + \rho_c^3)\mathbf{1}_{[t_c,t_k]}, \quad n\rho_0^2(s)^{-7/6} \mathbf{1}_{[t_c,t_c^+}]. \]

Then

\[ I_j(t) \lesssim \int_{t_c}^{t_o} \alpha_\infty(t-s)\rho_0^3 e^{-\frac{1}{4} \Re \lambda_0(t-s)} ds \lesssim \rho_0^3 \int_{t_o-t_c}^{t_o} \langle t-s \rangle^{-7/6} ds \lesssim \rho_0^3 \langle t-t_o \rangle^{-1/6} n^{-4}(t-t_o + n^{-4})^{-1}. \]
With constant \( \varepsilon = n^{7/3} \rho_c^2 + \rho_c^3 \), using (5.68) and \( n^4(t_k - t_c)e^{-\Re \frac{1}{2} \lambda_0(t_k - t_c)} \leq C \),

\[
I_2(t) \lesssim \int_{t_c}^{t} \alpha_\infty(t - s)\varepsilon ds \leq \varepsilon(t - t_k)^{-1/6}(t_k - t_c)(t - t_c)^{-1} \\
\leq \varepsilon(t - t_k)^{-1/6}(t - t_c)^{-1}n^{-4}n^4(t_k - t_c)n\rho_c^{-1}e^{-\Re \frac{1}{2} \lambda_0(t_k - t_c)} \\
\leq \varepsilon^{-1}n^{-3}(n^{7/3} \rho_c + \rho_c^2)(t - t_k)^{-1/6}(t - t_c)^{-1}.
\]

Finally, \( I_3(t) \lesssim \int_{t_c}^{t} \alpha_\infty(t - s)n\rho_0^{-3}(s)^{-7/6} ds \leq (t - t_c)^{-7/6}n\rho_0^{-2} \). Summing the estimates we get the second part of the Lemma. \( \square \)

6 Dynamics away from bound states

In this section, we study the dynamics of the solution \( \psi(t) \) for \( t_o \leq t \leq t_i \), where \( t_o \) is the time it leaves \( 2\rho_0 \) neighborhood of first excited states, and \( t_i \) is the time it enters the \( \rho_0 \)-neighborhood of ground states, to be defined in (6.73). In this time interval we use orthogonal coordinates and decompose

\[
\psi(t) = \sum_{j=0}^{K} x_j(t) \phi_j + \xi(t), \quad \xi(t) \in E_{H_o}, \quad (t \geq t_o).
\]

We first estimate \( x_j(t_o) \) and \( \xi(t_o) \) in Lemma (6.1) for which we recall some definitions. Recall that \( \Delta t = n^{-2}\rho_0^{-2} = n^{-2(2+\delta)} \), \( 0 < \alpha \ll 1 \) is fixed and \( 0 < \delta \leq 1 \). Moreover, \( \frac{27}{5} < p < 6 \) is fixed, \( \frac{2}{3} < \sigma = \frac{3(p-3)}{2p} < \frac{3}{4} \), and \( \sigma' := \frac{3(p-2)}{2p} > \sigma \). Recall from Lemma (5.4) that \( \tilde{\Lambda}_2 = \tilde{\Lambda}_{2,1} + \tilde{\Lambda}_{2,2} \) with

\[
\tilde{\Lambda}_{2,1}(t) := \Lambda_2(t) + C_5 n\rho_0^2 \langle t - t_o \rangle^{-7/6} + C_5 n^{-3}(n^{7/3} \rho_c + \rho_c^2)(t - t_o + n^{-4})^{-7/6}, \\
\tilde{\Lambda}_{2,1}(t) := C_5 \rho_0^3 \langle t - t_o \rangle^{-1/6} n^{-4}(t - t_o + n^{-4})^{-1}.
\]

We also define

\[
\Lambda_3(t) := 3\tilde{\Lambda}_2(t) + C_6 n^3(1 + t - t_o)^{-3/2}, \quad \Lambda_4(t) := \sum_{j=1}^{3} \Lambda_{4,j}(t),
\]

where \( C_6 \) is some uniform constant defined in (6.14) and

\[
\Lambda_{4,1} := C_6 n^{-1+(4+2\delta)\alpha} (\Delta t + t)^{-\sigma + \alpha}, \quad \Lambda_{4,2} := C_6 \rho_0 (1 + t - t_o)^{-\sigma}, \\
\Lambda_{4,3} := C_6 n^{-1+\delta} (n^{-4} + t - t_o)^{-\sigma}.
\]

Note that \( \Lambda_{4,1} \) is the second term in \( \Lambda_1 \) and comes from the out-going estimate at \( t_c \); \( \Lambda_{4,3} \) is from the out-going estimate at \( t_o \) and \( \Lambda_{4,2} \) is from (6.14). Also note that

\[
\Lambda_3(t) \leq 3C_6 n^3, \quad \Lambda_4(t) \leq 2C_6 n^{\frac{5p-18}{p}+\delta} + C_6 \rho_0 (t - t_o)^{-\sigma}, \quad \frac{5p-18}{p} > \frac{5}{3}.
\]

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Lemma 6.1  At $t = t_0$ we have

\begin{equation}
(1.9)n^{1+\delta} \leq |x_0| \leq (2.1)n^{1+\delta}, \quad \left( \sum_{k \neq 1} |x_k|^2 \right)^{\frac{1}{2}} \leq 6 \sqrt{\frac{D}{\gamma_0}} \rho_0, \quad (0.9)n \leq |x_1| \leq (1.1)n. \tag{6.6}
\end{equation}

Moreover, we have for all $t \geq t_0$

\begin{equation}
\left\| e^{-i(t-t_0)H_0} \xi(t) \right\|_{L^2_{\text{loc}}} \leq \Lambda_3(t), \quad \left\| e^{-i(t-t_0)H_0} \xi(t) \right\|_{L^p} \leq \Lambda_4(t). \tag{6.7}
\end{equation}

Proof. For all $0 \leq t \leq t_0$, we have

\begin{equation}
\psi = |Q + a(t)R + \zeta + \eta|^{-iEt+it} = \sum_{j=0}^K \bar{x}_j \phi_j + \xi. \tag{6.8}
\end{equation}

Here $Q = Q_{1,n(t_0)}$. Recall $n(t_0) = n + O(n^{1+2\delta})$ by substituting (6.8) with $t = 0$ into $n = |(\phi_1, \psi_0)|$. For $j \neq 1$, taking the inner product of (6.8) at $t = t_0$ with $\phi_j$ we get

\begin{equation}
|x_j(t_0)| = O(n^3) + (1 + O(n^2))|z_j(t_0)|, \quad (j \neq 1). \tag{6.9}
\end{equation}

We also have

\begin{equation}
|x_1(t_0)| = (\phi_1, Q) + O(n^3) = n(t_0) + O(n^3) = n + O(n^{1+2\delta}). \tag{6.10}
\end{equation}

Since $|z_0(t_0)| = (1 + o(1))2\rho_0$ and $z_H(t_0) \leq \sqrt{6D/\gamma_0}(1 + o(1)))|z_0(t_0)|$, we have (6.6).

Next, we shall prove (6.7). Denote $\theta_* := iEt_0 - i\theta(t_0)$ and

\begin{equation}
x^* = e^{\theta_*} \sum_{j=0}^K \bar{x}_j(t_0) \phi_j, \quad \xi^* = e^{\theta_*} \xi(t_0), \quad \eta^* = \eta(t_0). \tag{6.11}
\end{equation}

From (6.8), we get

\begin{equation}
\xi^* = P_c H_0 \{Q + a(t_0)R + \zeta(t_0) + \eta^* - x^*\}. \tag{6.12}
\end{equation}

We write $\xi^* = \xi_1^* + \xi_2^* + \xi_3^*$ where

\begin{equation}
\xi_1^* := P_c H_0 \left\{Q + a(t_0)R + \sum_{j \neq 1} z_j(t_0) \bar{u}_j^* + \sum_{j > 1} \bar{z}_j(t_0) u_j^* - x^* \right\}, \quad \xi_2^* := P_c H_0 \bar{z}_0(t_0) u_0^*, \quad \xi_3^* := P_c H_0 \eta^*. \tag{6.13}
\end{equation}

From the explicit formulae of $Q, R, u_j^*$, we see that $\xi_1^*$ is localized and $\|\xi_1^*\| \lesssim n^3 + |a(t_0)| + \max_{j \neq 1} |z_j| n^2 \lesssim n^3$. Therefore, for all $t \geq t_0$, $\tau = t - t_0$, by Lemma 2.5 and Lemma 2.10 we have a uniform constant $C_0 > \max \{C_3, C_5\}$ such that

\begin{equation}
\left\| e^{-i\tau H_0} \xi_1^* \right\|_{L^2_{\text{loc}}} \leq \frac{1}{2} C_0 n^3 (1 + \tau)^{-3/2}, \quad \left\| e^{-i\tau H_0} \xi_1^* \right\|_{L^p} \leq \frac{1}{2} C_0 n^3 (1 + \tau)^{-\sigma'}, \tag{6.14}
\end{equation}

\begin{equation}
\left\| e^{-i\tau H_0} \xi_2^* \right\|_{L^2_{\text{loc}}} \leq \frac{1}{2} C_0 n^3 + 3(1 + \tau)^{-3/2}, \quad \left\| e^{-i\tau H_0} \xi_2^* \right\|_{L^p} \leq \frac{1}{2} C_0 \rho_0 (1 + \tau)^{-\sigma}. \tag{6.14}
\end{equation}

Here for $\tau \leq 1$ we have used $\left\| e^{-i\tau H_0} \xi_2^* \right\|_{L^p} \lesssim \left\| \xi_2^* \right\|_{H^1} \lesssim |z_0(t_0)|$. Next, we estimate $e^{-i\tau H_0} \xi_3^*$ in $L^2_{\text{loc}}$ and $L^p$. Note $|e^{-i\tau H_0} \xi_3^*| = e^{\tau [H_0-E]} |\xi_3^*|$. Recall that

\begin{equation}
L = J(H_0 - E) - W, \quad [\eta^*] = e^{i\theta(t_0)} \eta^*_+ + e^{-i\theta(t_0)} \eta^*_-, \quad \eta^*_\pm = \eta_\pm(t_0), \tag{6.15}
\end{equation}
for some localized potential $W$ of order $n^2$. By Duhamel’s principle, we have
\[ e^{\tau J(H_0 - E)[\xi_3^*]} = P_c^0 e^{\tau L_0}[\eta^*] + \int_0^\tau e^{\tau J(H_0 - E)(\tau - s)} P_c^H W e^{L_s[\eta^*]} ds. \] (6.16)

From Lemma 5.4, we get
\[ \left\| e^{\tau J(H_0 - E)[\xi_3^*]} \right\|_{L^p} \leq \sum_{\pm} \left\| e^{\tau L_0[\eta^*]} \right\|_{L^p} + C n^2 \sum_{\pm} \int_0^\tau |\tau - s|^{-\sigma'} \left\| e^{L_s[\eta^*]} \right\|_{L^2_{loc}} ds \]
\[ \leq 2\tilde{A}_1(t) + C n^2 \sum_{\pm} \int_0^\tau |\tau - s|^{-\sigma'} \tilde{A}_2(s + t_o) ds. \] (6.17)

Using the fact that
\[ \int_0^t (t - s)^{-\beta_1} (e^{-1} + s)^{-\beta_2} \leq C e^{\beta_2 - 1} (e^{-1} + t)^{-\beta_1}, \quad 0 < \beta_1 < 1 < \beta_2, \] (6.18)
we have
\[ n^2 \int_0^\tau |\tau - s|^{-\sigma'} \tilde{A}_2(s + t_o) ds \leq C n^2 \rho_0(t - t_o)^{-\sigma'} + C \rho_0(\Delta t + t)^{-\sigma'} \] (6.19)
which is $o(1) \rho_0(t - t_o)^{-\sigma}$. From this and (6.17), we get
\[ \left\| e^{\tau J(H_0 - E)[\xi_3^*]} \right\|_{L^p} \leq 2\tilde{A}_1 + o(1) \rho_0(t - t_o)^{-\sigma}. \] (6.20)

Similarly from (6.16) with $\tilde{a}_{\infty}(t) = \min\{t^{-3/2}, t^{-9/10}\}$,
\[ \left\| e^{\tau J(H_0 - E)[\xi_3^*]} \right\|_{L^2_{loc}} \leq \sum_{\pm} \left\| e^{\tau L_0[\eta^*]} \right\|_{L^2_{loc}} + C n^2 \sum_{\pm} \int_0^\tau \tilde{a}_{\infty}(\tau - s) \left\| e^{L_s[\eta^*]} \right\|_{L^2_{loc}} ds \]
\[ \leq 2\tilde{A}_2(t) + C n^2 \sum_{\pm} \int_0^\tau \tilde{a}_{\infty}(\tau - s) \tilde{A}_2(s + t_o) ds \leq 3\tilde{A}_2(t). \] (6.21)

So, (6.17) follows from (6.14), (6.20), and (6.21). This completes the proof of Lemma 6.1.

For $j \in \{0, 1, \ldots, K\}$, let $f_j := |\mu_j(t)|^2$, where $\mu_j$ is the perturbation of $x_j$ defined in Lemma 5.2. Since $\frac{d}{dt} |\mu|^2 = 2 \Re \bar{\mu} \dot{\mu}$ and $\ddot{c}_l$ are all purely imaginary, from (6.10) we have
\[ \dot{f}_j = \sum_{a, b=0}^K 2(\Re d_{ab}^j) f_a f_b f_j + 2 \Re \bar{\mu}_j g_j. \] (6.22)

Let
\[ f = \sum_{l=1}^K f_l, \quad h = \sum_{l=1}^K 2^{-l} f_l, \quad \gamma := \min\{\gamma_{0_{ab}}, \text{ for } a, b \geq 1\} > 0. \] (6.23)

Then, from (6.22), Lemma 3.2 and as in [28] (4.58), we have
\[ \frac{d}{dt} (f_0 + f)(t) \leq 2(K + 1) \max_l |\bar{\mu}_l g_l|, \quad \frac{d}{dt} (f_0 + h)(t) \geq -2(K + 1) \max_l |\bar{\mu}_l g_l|. \] (6.24)

Moreover, we have the following lemma.
Lemma 6.2 Assume as in Lemma 3.2. We have
\[ \dot{j}_0 \geq 2 \gamma f^2 f_0 + 2 \text{Re} \bar{\mu}_0 g_0, \quad \dot{j} \leq -4 \gamma f_0 f^2 + \sum_{l=1}^{K} 2 \text{Re} \bar{\mu}_l g_l. \] (6.25)

Proof. From (6.22) and Lemma 3.2 in particular (3.12), we have
\[ \dot{j}_0 - 2 \text{Re} \bar{\mu}_0 g_0 = \sum_{a,b=0}^{K} 2 \text{Re}(d^0_{a,b}) f_a f_b f_0 = \sum_{a,b=0}^{K} [2(2 - \delta^b_a) \gamma^0_{ab} - 4(2 - \delta^b_0) \gamma^0_{0b}] f_a f_b f_0. \] (6.26)

Note that \( \gamma^0_{0b} = 0 \) for any \( a \) and \( b \). Thus
\[ \dot{j}_0 - 2 \text{Re} \bar{\mu}_0 g_0 = \sum_{a,b=1}^{K} 2(2 - \delta^b_a) \gamma^0_{ab} f_a f_b f_0 \geq 2 \gamma f^2 f_0. \] (6.27)

This proves the first part of (6.25). For the second part,
\[ \dot{j} - 2 \sum_{l=1}^{K} \text{Re} \bar{\mu}_l g_l = \sum_{l=1}^{K} \sum_{a,b=0}^{K} 2[(2 - \delta^b_a) \gamma^l_{ab} - 2(2 - \delta^b_l) \gamma^l_{0b}] f_a f_b f_l. \]
\[ = \sum_{b=0}^{K} \sum_{a,l=1}^{K} 2[(2 - \delta^b_a) \gamma^l_{ab} - 2(2 - \delta^b_l) \gamma^l_{0b}] f_a f_b f_l + \sum_{l=1}^{K} \sum_{b=0}^{K} -4(2 - \delta^b_l) \gamma^l_{0b} f_0 f_b f_l. \] (6.28)

By switching \( a \) and \( l \) in the terms with factor \( \gamma^l_{ab} \), the summands in the first sum become \(-2(2 - \delta^b_l) \gamma^l_{0b} f_0 f_b f_l \leq 0 \). The summands of the second sum are also nonpositive. Keeping only terms with \( b > 0 \) in the second sum, we get
\[ \dot{j} - 2 \sum_{l=1}^{K} \text{Re} \bar{\mu}_l g_l \leq -4 \sum_{b,l=1}^{K} (2 - \delta^b_l) \gamma^0_{0b} f_0 f_b f_l \leq -4 \gamma f_0 f^2. \]

This proves the second part of (6.25). \( \Box \)

The following proposition estimates the solution in a time interval containing \([t_0, t_i]\).

Proposition 6.3 Let \( \delta_0(t) := \rho^2_0(t - t_0)^{-\frac{6}{21}} \). For all \( t \in [t_0, t_0 + \frac{6}{7} n^{-2(2+\delta)}] \), we have
\[ \frac{n}{5} \leq \max_j |x_j| \leq (\sum_{j=0}^{K} |x_j(t)|^2)^{\frac{1}{2}} \leq 2n, \]
\[ \|\xi(t)\|_{L^2_{loc}} \leq n^{3-\alpha} + \delta_0(t), \quad \|\xi(t)\|_{L^p} \leq n^{3-\alpha} |t - t_0|^\frac{6-p}{2p} + \frac{3}{2} \Lambda_4(t). \] (6.29)

Proof. Since (6.29) holds at \( t = t_0 \), we then prove it by using the continuity argument. So, we can assume the following weaker estimates: For \( t_0 \leq t \leq t_0 + \frac{6}{7} n^{-2(2+\delta)} \),
\[ \frac{n}{10} \leq \max_j |x_j| \leq (\sum_{j=0}^{K} |x_j(t)|^2)^{\frac{1}{2}} \leq 3n, \]
\[ \|\xi(t)\|_{L^2_{loc}} \leq 2|n^{3-\alpha} + \delta_0(t)| \leq n^2, \] (6.30)
\[ \|\xi(t)\|_{L^p} \leq 2n^{3-\alpha} |t - t_0|^\frac{6-p}{2p} + 3\Lambda_4(t) \leq n^{2.7} + 3\Lambda_4(t). \]
In particular \( \|\xi(t)\|_{L^2_{loc}} + \|\xi(t)\|_{L^p} \ll n \). The proof of Proposition 6.3 follows from Lemma 6.4 and Lemma 6.6 below. \( \Box \)
Lemma 6.4 For all \( t \in [t_0, t_0 + 2n^{-2(2+\delta)}] \), we have

\[
\|\xi(t)\|_{L^2_{tloc}} \leq n^{3-\alpha} + \delta_6(t), \quad \|\xi(t)\|_{L^\infty} \leq n^{3-\alpha}|t-t_0|^{\frac{6-p}{2p}} + \frac{3}{2} \Lambda_4(t). \tag{6.31}
\]

Proof. For all \( t-t_0 \leq Cn^{-2(2+\delta)} \), by (6.30), we have

\[
\|\xi(t)\|_{L^p} \lesssim n^{3-\alpha - \frac{2(2+\delta)(6-p)}{2p}} + \Lambda_4(t) \leq C\left[ n^{\frac{5+\delta}{p}(6+2\delta)} - \alpha + 3\Lambda_4(t) \right]. \tag{6.32}
\]

We have

\[
\xi(t) = e^{-iH_0(t-t_0)}\xi(t_0) + \int_{t_0}^t e^{-iH_0(t-s)}P_{\epsilon^l}^{-1}G(s)ds.
\]

So, we have

\[
\|\xi(t)\|_{L^p} \leq \Lambda_4(t) + C \int_{t_0}^t |t-s|^{\frac{3(p-2)}{2p}} \|G(s)\|_{L^{p'}_{t'}} ds.
\]

Note that \( \|G\|_{L^{p'}} \lesssim \|G_3\|_{L^{p'}} + \|G - G_3 - \kappa^2 \xi\|_{L^{p'}} + \|\kappa^2 \xi\|_{L^{p'}} \quad \text{and} \quad \|G_3\|_{L^{p'}} \lesssim n^3 \). On the other hand, from Lemma 3.1, (6.30) and (6.32), we get

\[
\|G - G_3 - \kappa^2 \xi\|_{L^1_{t} \cap L^{p'}_{t'}} \lesssim n^2 \|\xi\|_{L^2_{tloc}} \lesssim [n^{5-\alpha} + n^2 \delta_6(t)]. \tag{6.35}
\]

On the other hand, using Hölder’s inequality, we get

\[
\|\kappa^2 \xi\|_{L^{p'}} \leq \|\xi\|_{L^2_{tloc}} \frac{2(p-4)}{p-2} \|\xi\|_{L^{p'}_{t'}} \quad \text{and} \quad \|\xi\|_{L^1_{t}} \leq \|\xi\|_{L^2_{tloc}} \frac{2(p-3)}{p-2} \|\xi\|_{L^{p'}_{t'}} \tag{6.36}
\]

From this, (6.32) and since \( 0 < \delta \leq \frac{1}{10} \), we get

\[
\|\kappa^2 \xi\|_{L^{p'}} \leq \|\xi\|_{L^2_{tloc}} \frac{p+2}{p-2} \|\xi\|_{L^{p'}_{t'}} \leq o(1)[n^{5-2\alpha} + \Lambda_4(t)]^{\frac{p+2}{p-2}}. \tag{6.37}
\]

By (6.3), (6.36), and (6.37), we have

\[
\|G(s)\|_{L^{p'}} \leq C[n^3 + o(1)\bar{\delta}_2(t)], \quad \bar{\delta}_2(t) := [\rho_0(t-t_0)^{-\frac{\sigma}{p-2}}]. \tag{6.38}
\]

Therefore, using \( \sigma \frac{p+2}{p-2} > 1 \),

\[
\|\xi(t)\|_{L^p} \leq \Lambda_4(t) + C \int_{t_0}^t |t-s|^{-\frac{\sigma}{2}} n^3 + o(1)\bar{\delta}_2(s) ds
\]

\[
\leq Cn^3|t-t_0|^{\frac{6-p}{2p}} + \frac{3}{2} \Lambda_4(t). \tag{6.39}
\]

So, we have proved the estimate of \( \|\xi(t)\|_{L^p} \).

We now estimate \( \|\xi(t)\|_{L^2_{tloc}} \). By (5.5), (3.6), (6.30) and Lemma 3.1 we have

\[
\|\xi_{1}^{(3)}(t)\|_{L^2_{tloc}} \leq \Lambda_3(t), \quad \|\xi_{2}^{(3)}(t)\|_{L^2_{tloc}} \lesssim n^3(1+t-t_0)^{-3/2}. \tag{6.40}
\]

By (3.6) and the estimate of \( \max_j |\dot{u}_j| \) in Lemma 3.1 we get

\[
\|\xi_{3}^{(3)}(t)\|_{L^2_{tloc}} \lesssim \int_{t_0}^t |1+t-s|^{-3/2} n^5 ds \lesssim n^5. \tag{6.41}
\]
For $\xi^{(3)}(t)$, bounding its integrand by either $L^\infty$ or $L^p$-norm and using (6.35), we have
\[
\|\xi^{(3)}(t)\|_{L^p_{loc}} \lesssim \int_{t_0}^t \min\{|t-s|^{-\gamma}, |t-s|^{-\frac{3(p-2)}{2p}}\} \|G - G_3 - \kappa \xi^2 \xi\|_{L^1 \cap L^{p'}} ds
\]
\[
\lesssim \int_{t_0}^t \min\{|t-s|^{-\gamma}, |t-s|^{-\frac{3(p-2)}{2p}}\}[n^{5-\alpha} + n^2 \delta_6(s)] ds
\]
\[
\lesssim n^{5-\alpha} + n^2 \delta_6(t).
\]

For $\xi^{(3)}(t)$, bounding its integrand in either $L^{\frac{2p}{p-4}}$ or $L^p$, we have
\[
\|\xi^{(3)}(t)\|_{L^p_{loc}} \leq C \int_{t_0}^t \min\{|t-s|^{-\frac{6}{p}}, |t-s|^{-\frac{3(p-2)}{2p}}\} \|\xi\|_{L^p}^2 ds.
\] (6.43)

By (6.36), $\frac{p+2}{p-4} > 2$ and $2 < \frac{6p}{p+2} < p$ because $\frac{27}{8} < p < 6$,
\[
\|\xi\|_{L^p}^2 \|\xi\|^2_{L^p} \leq C \|\xi\|_{L^p} \|\xi\|^2_{L^p} \leq o(1) \|\xi\|^2_{L^p}.
\] (6.44)

Therefore, by (6.30),
\[
\|\xi^{(3)}(t)\|_{L^p_{loc}} \leq o(1) \int_{t_0}^t \min\{|t-s|^{-\frac{6}{p}}, |t-s|^{-\frac{3(p-2)}{2p}}\}[n^{5.4} + \Lambda^2_4(s)] ds
\]
\[
\leq o(1)[n^{5.4} + \Lambda_4(t)^2 + \delta_7(t)],
\] (6.45)

where
\[
\delta_7(t) := \rho_0^2(t - t_0)^{-\frac{6}{p}} + n^{-\frac{2+2\delta}{3}}(n^{-4} + t - t_0)^{-\frac{6}{p}},
\] (6.46)

and we have used $\frac{2}{3} < \sigma < 2$, (6.4), and (4.34) with $a = 6/p < b = 2\sigma - 2\alpha$, (or $b = 2\sigma$).

Collecting all of the estimates of $\xi^{(3)}$ with $j = 1, 2, 3, 4$, we have
\[
\|\xi^{(3)}(t)\|_{L^p_{loc}} \leq \Lambda_3(t) + C n^{5} + o(1)[\Lambda_4(t)^2 + \delta_7(t)].
\] (6.47)

By (6.3), we have $\Lambda_3(t) \lesssim n^3$ and $\Lambda_4(t)^2 + \delta_7(t) \leq n^3 + \delta_6(t)$. Thus
\[
\|\xi(t)\|_{L^p_{loc}} \leq \|\xi^{(2)}(t)\|_{L^2_{loc}} + \|\xi^{(3)}(t)\|_{L^2_{loc}} \leq C n^3 + o(1)\delta_6(t).
\] (6.48)

This completes the proof of the lemma.

**Lemma 6.5** For $t \in [t_0, t_0 + \frac{6}{n} n^{-2(2+\delta)}]$, the error terms $g_j(t)$ in (5.10) satisfy
\[
|g_j(t)| \leq o(1)n^{6.7+\delta} + C n^2 g(t),
\] (6.49)

where
\[
g(t) := \Lambda_3(t) + o(1)[n^{1+3\delta}(t - t_0)^{-\frac{6p}{p+2}} + \Lambda^2_4(t) + \delta_7(t)]
\] (6.50)
satisfies
\[
\int_{t_0}^\infty g(t) ds \leq o(1)n^{-\frac{2}{\delta};} \quad g(t) \leq o(1)n\rho_0^2, \quad \forall t \geq t_0 + n^{-3}.
\] (6.51)
Proof. Recall (3.11),
\[
|g_j(t)| \lesssim n^7 + n^2 \left\| \xi(3) \right\|_{L^2_{loc}} + n \left\| \xi \right\|_{L^2_{loc}} + n \left\| \xi \right\|_{L^{2(p-3)}_{loc}} \left\| \xi \right\|_{L^p}.
\] (6.52)
From (6.30) and (6.47), we get
\[
n^2 \left\| \xi(3) \right\|_{L^2_{loc}} \leq n^2 A_3 + Cn^7 + o(1)n^2[A_3^2 + \delta_7],
\]
\[
n \left\| \xi \right\|_{L^2_{loc}}^2 \leq C[n^{7-2\alpha} + n\delta_6(t)^2],
\] (6.53)
and, using \([n^2 + A_{4,1} + A_{4,3}]^{\frac{p}{p-2}} \leq o(1)n^{\frac{5+3\delta}{2}}\),
\[
\left\| \xi \right\|_{L^2_{loc}}^{\frac{2(p-3)}{p-2}} \left\| \xi \right\|_{L^p}^{\frac{p}{p-2}} \lesssim [n^{3-\alpha} + \delta_6(t)]^{\frac{2(p-3)}{p-2}} [n^{2.7} + A_4]\frac{p}{p-2}
\]
\[
\leq o(1)\left[n^{\frac{(p-3)(3-\alpha)}{p-2}} + \delta_6^{\frac{2(p-3)}{p-2}} \left\| n^{\frac{5+3\delta}{2}} + \rho_0^2 (t-t_0)^{-\frac{\rho_0^2}{p-2}} \right. \right]
\]
\[
\leq o(1)\left[n^{6.7+\delta} + \rho_0^2 (t-t_0)^{-\frac{\rho_0^2}{p-2}} \right].
\] (6.54)
Summing the estimates we get (6.49). The estimates (6.51) follow from direct checking. \(\square\)

**Lemma 6.6** For all \(t \in [t_o, t_o + \frac{6}{7}n^{-2(2+\delta)}]\), we have
\[
\frac{1}{5} n \leq \max_j |x_j(t)| \leq \left( \sum_{j=0}^{K} |x_j(t)|^2 \right)^{\frac{1}{2}} \leq 2n.
\] (6.55)

**Proof.** From the first equation of (6.24), (6.51) and \(\delta \leq \frac{1}{10}\), we get
\[
(f_0 + f)(t) \leq (f_0 + f)(t_o) + Cn \max_j \int_{t_o}^{t} |g_j(s)| ds
\]
\[
\leq (f_0 + f)(t_o) + C[o(1)n^{7.7+\delta}(t-t_o) + n^3 \int_{t_o}^{t} g(s) ds]
\]
\[
\leq (f_0 + f)(t_o) + o(1)\rho_0^2 \leq [1 + o(1)](f_0 + f)(t_o).
\] (6.56)
By (3.11), (6.30), we have \([1 - o(1)] \sum_j |x_j|^2 \leq f_o + f\). By Lemma 6.1 we get \((f_0 + f)(t_o) \leq 2n^2\). It follows from (6.56) that \((\sum_{j=0}^{K} |x_j(t)|^2)^{\frac{1}{2}} \leq 2n\).
Similarly, by integrating the second equation of (6.24), we obtain
\[
(f_0 + h)(t) \geq [1 - o(1)](f_0 + h)(t_o).
\] (6.57)
By (3.11), (6.30) and the definition of \(f_0, h\), we get
\[
(f_0 + h)(t) \leq \left[ \sum_{k=0}^{K} 2^{-k} + o(1) \right] \max_j |x_j(t)|^2.
\] (6.58)
Therefore,
\[
2 \max_j |x_j(t)|^2 \geq [1 - o(1)](f_0 + h)(t_o) \geq [1 - o(1)] \frac{1}{2} |x_1(t_o)|^2.
\] (6.59)
Hence \(\max_j |x_j(t)|^2 \geq \frac{n^2}{25}\) for all \(t \in [t_o, t_o + \frac{6}{7}n^{-2(2+\delta)}]\). \(\square\)
Proposition 6.7 There exists $t_i$ such that $t_o + \frac{\delta}{10\gamma}n^{-4}\log\frac{1}{n} < t_i \leq t_o + \frac{\gamma}{2}n^{-4-2\delta}$ and

$$\frac{n}{5} \leq |x_0(t_i)| \leq 2n, \quad (0.9)\rho_0 \leq \left(\sum_{j=1}^{K} |x_j(t_i)|^2\right)^{1/2} \leq (1.1)\rho_0. \quad (6.60)$$

Above $\gamma = \max\{1, (d^k_\alpha) - : \forall a, b, l = 0, \ldots, K\}$ and $d^k_\alpha = O(1)$ are given in 3.12.

Proof. By Lemma 6.6 we already have $|x_0| \leq 2n$. The proof is divided into four steps.

Step 1: Let $t_1 := t_o + n^{-3}$. For $t_o \leq t \leq t_1$, for any $j$, by (6.22), (6.30), (6.49), and (6.51), we get

$$|f_j(t) - f_j(t_o)| \lesssim \int_{t_o}^{t_1} [n^6 + n |g_j(s)|]ds \lesssim n^3 + n \int_{t_o}^{t_1} |n^2 g(s)|ds \leq o(1)\rho_0^2. \quad (6.61)$$

In particular, for $j = 0, 1$, we get

$$[1 - o(1)]f_j(t_o) \leq f_j(t) \leq [1 + o(1)]f_j(t_o), \quad \forall t \in [t_o, t_1]. \quad (6.62)$$

By (3.11) and the definitions of $f_j$, we get

$$[1 - o(1)]|x_j(t_o)| \leq |x_j(t)| \leq [1 + o(1)]|x_j(t_o)|, \quad \forall t \in [t_o, t_1], \quad j = 0, 1. \quad (6.63)$$

Together with (6.6), for $t \in [t_o, t_1]$, we have

$$1.8\rho_0 \leq |x_0(t)| \leq 2.2\rho_0, \quad 0.8n \leq |x_1(t)| \leq 1.2n. \quad (6.64)$$

On the other hand, for $j > 1$, from (6.61), we obtain $f_j(t) \leq f_j(t_o) + o(1)\rho_0^2$ for $t \in [t_o, t_1]$. So, by (3.11), (6.6), and the definition of $f_j$, we get

$$|x_j(t)| \leq [1 + o(1)]f_j(t)^{1/2} \leq 7\sqrt{\frac{D}{\gamma_0}}\rho_0, \quad \forall t \in [t_o, t_1], \quad \forall j > 1. \quad (6.65)$$

Step 2: Let us define

$$t_2 := \sup\{t \geq t_1 : f_0(s) < \frac{n^2}{10}, \forall s \in [t_1, t]\}. \quad (6.66)$$

By (6.64), $t_2 < t_1$. We shall prove that

$$t_1 < t_2 \leq t'_2 := t_1 + a^{-1}\log \frac{n^2}{5f_0(t_1)}, \quad a := 2\gamma\left[\frac{n^2}{50}\right]^2. \quad (6.67)$$

For all $t_1 \leq t \leq t_2$, $f_0(t) < \frac{n^2}{10}$. Note $h(t_1) \geq f_1(t_1)/2 \geq (1 + o(1))(0.8n)^2/2 \geq (0.3)n^2$. From (6.24) and Lemma 6.5 we get

$$h(t) \geq (f_0 + h)(t_1) - f_0(t) - 2(K + 1)\int_{t_1}^{t} \max_j |\mu_j||g_j(s)|ds \geq (0.3)n^2 - \frac{n^2}{10} - Cn \int_{t_1}^{t} [n^6g^1 + n^2 g(s)]ds \geq \frac{n^2}{100}. \quad (6.68)$$
By (6.25), (6.30) and (6.68), we have, for \( t \in [t_1, t_2] \),
\[
\dot{f}_0 \geq 2\gamma f^2 \rho_0 - 2|\mu_0| |g_0| \geq 2\gamma_0(2\beta)^2 f_0 - 4n|g_0| \geq 2\gamma n^2 f_0 - 4n|g_0|.
\] (6.69)
Note the coefficient of \( f_0 \) is \( a \). Thus
\[
f_0(t) \geq e^{a(t-t_1)}[f_0(t_1) - 4n \int_{t_1}^{t} e^{-a(s-t_1)} g_0(s) ds].
\] (6.70)
On the other hand, from (6.49), we have
\[
\int_{t_1}^{t} e^{-a(s-t_1)} g_0(s) ds \leq n \int_{t_1}^{t} [n^{6.7 + \delta} + n^2 g(s)] ds \leq n^{7.7 + \delta}(t - t_1) + n^2 \int_{t_1}^{t} g(s) ds \leq o(1) \rho_0^2 \leq o(1) f_0(t_1).
\] (6.71)
Therefore,
\[
f_0(t) \geq \frac{1}{2} e^{a(t-t_1)} f_0(t_1), \quad \forall t \in [t_1, t_2].
\] (6.72)
This shows \( t_2 \leq t_2' \) is finite, and \( f_0(t_2) = \frac{n^2}{10} \).

**Step 3:** Define
\[
t_i := \sup\{t \geq t_2 : f(s) > \rho_0^2, \forall s \in [t_2, t]\}.
\] (6.73)
From (6.68), we get \( t_i > t_2 \). We shall prove in Steps 3 and 4 that
\[
t_2 + \frac{\delta}{10\gamma} n^{-4} \log \frac{1}{n} \leq t_i \leq t_3 := t_2 + \frac{6}{\gamma} n^{-4-2\delta}.
\] (6.74)
By definition of \( t_i \), we get
\[
f(t) > \rho_0^2, \quad \forall t \in [t_2, t_i).
\] (6.75)
From Lemma (6.72) and (6.75), we have
\[
\frac{d}{dt}(f_0(t)) \geq 2\gamma \rho_0^4 f_0(t) - 4n|g_0|, \forall t \in [t_2, t_i).
\] (6.76)
From this and as in (6.72), we also obtain
\[
f_0(t) \geq \frac{1}{2} e^{2\gamma \rho_0^2(t-t_2)} f_0(t_2) \geq \frac{n^2}{20}, \forall t \in [t_2, t_i).
\] (6.77)
From this, (6.25), and Lemma (6.5) for \( t \in [t_1, t_i) \),
\[
\frac{d}{dt}(f(t)) \leq -4\gamma f_0(t) f(t)^2 + C n \max_{k \geq 0} |g_k| \leq -\gamma n^2 f(t)^2 + C n [n^{6.7+\delta} + n^2 g(s)].
\] (6.78)
From this and (6.75), (and \( \delta \leq \frac{1}{10} \)), we get
\[
\frac{n^2 \gamma}{6} - C n^3 \rho_0^4 g(t) < \frac{n^2 \gamma}{5} - C n [n^{6.7+\delta} + n^2 g(s)] \leq -\frac{\dot{f}}{f^2}, \forall t \in [t_2, t_i).
\] (6.79)
Note that by \((6.51), (6.68)\), Proposition \(6.3\) and \(\delta \leq \frac{1}{10}\), we have \(\forall \ t \geq t_2\)

\[
n^{-1-4\delta} \int_{t_2}^{t} g(s) ds \leq o(1)n^{-1-4\delta}n^{-2(1-\delta)/3} = o(1)n^{-\frac{5(1+2\delta)}{3}} \leq o(1)f(t_2)^{-1}. \tag{6.80}
\]

Integrating \((6.79)\) in \([t_2, t]\), we get

\[
f(t) < \left[ f(t_2)^{-1}/2 + \frac{n^{2\gamma}}{6}(t - t_2) \right]^{-1}, \quad \forall \ t \in [t_2, t]. \tag{6.81}
\]

In particular, \(\rho_0^2 < f(t) < \left[ \frac{n^{2\gamma}}{6}(t - t_2) \right]^{-1}\), which shows \(t_i \leq t_3\), and \(f(t_i) = \rho_0^2\). From this, \((6.11)\) and \((6.77)\), we get the estimates \((6.60)\). Since

\[
t_i - t_o \leq (t_i - t_2) + (t_2 - t_1) + (t_1 - t_o) \leq \frac{6}{\gamma}n^{-4-2\delta} + Cn^{-4} \log \frac{1}{n} + n^{-3} \tag{6.82}
\]

by \((6.67)\) and \((6.64)\), we get the upper bound of \(t_i - t_o\) in Prop. \(6.7\)

**Step 4:** It remains to show that \(t_i \geq t_2 + \frac{8}{107}n^{-4} \log \frac{1}{n}\). Recall \(g(t) \leq o(1)n\rho_0^2\) for all \(t \geq t_1 = t_o + n^{-3}\) from Lemma \(6.3\). By \((6.22)\) and Prop. \(6.3\)

\[
\dot{f}(t) \geq -9\gamma n^4 f(t) - Cn[n^{6.7+\delta} + n^2 g(t)] \geq -10\gamma n^4 f(t), \quad \forall \ t \in [t_1, t_i], \tag{6.83}
\]

where \(\bar{\gamma} = \max\{1, (d_{ab}^l)_- : \forall a, b, l = 0, \ldots, K\}\). This implies that

\[
t_i - t_2 \geq \frac{n^{-4} \log f(t_2)}{f(t)} \geq \frac{\delta}{10\gamma} n^{-4} \log \frac{1}{n}. \tag{6.84}
\]

For the second inequality we have used \(f(t_2) \geq h(t_2) \geq n^2/50\) by \((6.68)\). This completes the proof of Proposition \(6.7\) \(\square\)

At \(t = t_i\) the solution enters \(\rho_0\)-neighborhood of ground states and we change to linearized coordinates. For that purpose we prepare outgoing estimates at \(t = t_i\).

**Lemma 6.8** Let \(t_i\) be as in Proposition \(6.7\). For any \(t > t_i\), we have

\[
\left\| e^{-iH_0(t-t_i)} \xi(t_i) \right\|_{L^2_{\text{loc}}} \leq \left\{ \frac{1}{2} |\Lambda_{L,1}(t) + \Lambda_{L,2}(t)|, \right. \tag{6.85}
\]

\[
\left\| e^{-iH_0(t-t_i)} \xi(t_i) \right\|_{L^p} \leq \left\{ \frac{1}{2} |\Lambda_{G,1}(t) + \Lambda_{G,2}(t)|, \right. \tag{6.85}
\]

where for some constant \(C_7 \geq C_6\) and \(\sigma' = \frac{3(p-2)}{2p}\),

\[
\Lambda_{L,1}(t) := 2C_7 [n^{-1+2\delta} (t - t_o)^{-7/6} + \rho(t)^3 + n^{4/5} \rho(t)^{7/3}],
\]

\[
\Lambda_{G,1}(t) := 2C_7 [n^{-1+\delta} (t - t_o)^{-\sigma} + n^{-1+2(2+\delta)\alpha} (\Delta t + t)^{-\sigma + \alpha}],
\]

\[
\Lambda_{L,2}(t) := \frac{2n^{5/2-14+\alpha \delta}}{t - t_o} (t_i - t_o) (t - t_i)^{-1/2},
\]

\[
\Lambda_{G,2}(t) := 2C_7 n^{3} (t_i - t_o)(t-t_o)^{-\sigma'}. \tag{6.86}
\]
Proof. Decompose $e^{-i(t-t_i)H_0} \zeta(t_i) = \chi(t) + J(t)$, where

$$\chi(t) := e^{-i(t-t_o)H_0} \zeta(t_o), \quad J(t) := \int_{t_o}^{t_i} e^{-i(t-s)H_0} P_s G(s) ds.$$  \hspace{1cm} (6.87)

Denote $T = t_i - t_o$. By Lemma 6.1 and using $n^{-4} \log \frac{1}{n} \lesssim T \lesssim n^{-2} (2 + \delta)$, we have

$$\|\chi(t)\|_{L^p} \leq \Lambda_4 (t) \leq \frac{1}{2} \Lambda_{G,1}(t), \quad \|\chi(t)\|_{L'^{2}_{\text{loc}}} \leq \Lambda_3 (t) \leq \frac{1}{2} \Lambda_{L,1}(t),$$  \hspace{1cm} (6.88)

for some $C_7$. By (6.38), we have

$$\|G(s)\|_{L'^{p}} \leq C [n^3 + o(1) \delta_2 (s)] \quad \forall \ s \in [t_o, t_i], \quad \delta_2(s) = |\rho_0 (s - t_o) - \sigma|^\frac{p+2}{2}. \hspace{1cm} (6.89)$$

So, we have (using $\frac{p+2}{2} > 2$)

$$\|J(t)\|_{L^p} \leq C \int_{t_o}^{t_i} |t - s|^{-\sigma'} \|G(s)\|_{L'^p} ds \leq C \int_{t_o}^{t_i} |t - s|^{-\sigma'} [n^3 + o(1) \delta_2(s)] ds \leq C n^3 T (t - t_o)^{-\sigma'} + \rho_0^2 (t - t_o)^{-\sigma'} \leq \frac{1}{2} \Lambda_{G,2}(t).$$  \hspace{1cm} (6.90)

It remains to estimate $\|J(t)\|_{L'^{2}_{\text{loc}}}$. By (6.35) and (6.36),

$$\|G(s)\|_{L^1 \cap L'^p} \leq C n^3 + C n^2 \|\xi(s)\|_{L'^{2}_{\text{loc}}} + o(1) \|\xi(s)\|_{L'^p}.$$  \hspace{1cm} (6.91)

By (6.30) and (6.5),

$$\|G(s)\|_{L^1 \cap L'^p} \leq o(1) n^{5\frac{p-18+\mu_5}{p-2}} + \rho_0^{3/2} (s - t_o)^{-\frac{p\sigma}{p-2}}.$$  \hspace{1cm} (6.92)

Thus

$$\|J(t)\|_{L'^{2}_{\text{loc}}} \leq C \int_{t_o}^{t_i} \min \{ |t - s|^{-3/2}, (t - s)^{-\sigma'} \} \|G(s)\|_{L^1 \cap L'^p} ds \leq o(1) \int_{t_o}^{t_i} \min \{ |t - s|^{-3/2}, (t - s)^{-\sigma'} \} [n^{5\frac{p-18+\mu_5}{p-2}} + \rho_0^{3/2} (s - t_o)^{-\frac{p\sigma}{p-2}}] ds \leq o(1) n^{5\frac{p-18+\mu_5}{p-2}} T^{1/2}. \hspace{1cm} (6.93)$$

which is bounded by $\frac{1}{2} \Lambda_{L,2}(t)$. This completes the proof of the lemma. \hfill \Box

## 7 Converging to a ground state

In this section we study the solution when it is already inside a neighborhood of the ground states. It is similar to the estimates in [3, 29, 6, 28], however, it requires a proof because the dispersive component has much worse estimates. As in Section 4, for fixed $T \geq t_i$ we shall decompose $\psi(t)$ as (see 3.17)

$$\psi(t) = [Q_{0,n}(T) + a(t) R_{0,n}(T) + \zeta(t) + \eta(t)] e^{-iEt + i\theta(t)}, \quad t \in [t_i, T].$$  \hspace{1cm} (7.1)
We have \( a(T) = 0 \), and
\[
\zeta = \sum_{j=1}^{K} \zeta_j, \quad \zeta_j = \bar{z}_j u_j^- + z_j u_j^+, \quad [\eta] = \begin{bmatrix} \Re \eta \\ \Im \eta \end{bmatrix} = e^{i\theta} \eta_+ + e^{-i\theta} \eta_-.
\] (7.2)

Denote \( z_H(t) = (\sum_{j=1}^{K} |z_j(t)|^2)^{1/2} \). From Lemma 3.3 and Proposition 6.7, (7.1) is valid at least for \( T > t_i \) sufficiently close to \( t_i \). We prove in this section that this is true with suitable estimates for all \( T \geq t_i \) and, moreover, \( n(T) \) converges to some \( n_+ \sim n \) as \( T \to \infty \).

**Lemma 7.1** There exists \( C_8 > 0 \) such that if \( T > t_i \) and \( n(T)/n(t_i) \in (\frac{1}{2}, \frac{3}{2}) \), then
\[
\frac{4}{5} \rho_0 \leq z_H(t_i) \leq \frac{6}{5} \rho_0,
\] (7.3)
and, for \( t \geq t_i \),
\[
\left\| e^{L(t-t_i)} \eta_{\pm}(t_i) \right\|_{L^2_{\text{loc}}} \leq \Lambda_L(t) := \Lambda_{L,1}(t) + \Lambda_{L,2}(t) + \Lambda_{L,3},
\]
\[
\left\| e^{L(t-t_i)} \eta_{\pm}(t_i) \right\|_{L^p} \leq \Lambda_G(t) := \Lambda_{G,1}(t) + \Lambda_{G,2}(t) + \Lambda_{G,3},
\] (7.4)
where \( \Lambda_{L,1}, \Lambda_{L,2}, \Lambda_{G,1} \) and \( \Lambda_{G,2} \) are defined in Lemma 6.8. \( \Lambda_{L,3}(t) = C_8 n^3 (t-t_i)^{-3/2} \), and \( \Lambda_{G,3}(t) = C_8 n^3 (t-t_i)^{-\sigma'} \).

**Proof.** At \( t = t_i \), with \( Q = Q_{n(T)} \) and \( \Theta = E_{n(T)} t_i - \theta(t_i) \) we have
\[
Q + a(t_i) R + \zeta(t_i) + \eta(t_i) = e^{i\Theta} \left[ \sum_{j=0}^{K} x_j(t_i) \phi_j + \xi(t_i) \right].
\] (7.5)
For each \( j \geq 1 \), applying the projection \( P_j \) (see Prop. 2.4 (iii)) to (7.5), we get \( z_j(t_i) = e^{i\Theta} x_j(t_i) + O(n^3) \). By Proposition 6.7, we get (7.3). Denote
\[
\eta_1 := \sum_{j=0}^{K} e^{i\Theta} x_j(t_i) \phi_j - Q - a(t_i) - \zeta(t_i), \quad \eta_2 := e^{i\Theta} \xi(t_i).
\] (7.6)
Then \( [\eta] = P_e^{\eta_1} + P_e^{\eta_2} \). Since \( \eta_1 \) is localized and of order \( O(n^3) \), we get
\[
\left\| e^{L(t-t_i)} P_\pm \eta_1 \right\|_{L^2_{\text{loc}}} \leq C_8 n^3(t-t_i)^{-3/2}, \quad \left\| e^{L(t-t_i)} P_\pm \eta_1 \right\|_{L^p} \leq C_8 n^3(t-t_i)^{-\sigma'}.
\] (7.7)
On the other hand, we have \( L = J(H_0 - E) + W \) with \( W = O(n^2) \) which is localized. By Duhamel’s formula,
\[
e^{L(t-t_i)} P_\pm \eta_2 = P_\pm e^{J(H_0 - E)(t-t_i)} \eta_2 + \int_{t_i}^{t} P_\pm e^{L(t-s)} W e^{J(H_0 - E) s} \eta_2 ds.
\] (7.8)
Thus, using Lemma 6.8, \( \left\| e^{L(t-t_i)} P_\pm \eta_2 \right\|_{L^p} \) is bounded by
\[
\leq \left\| e^{J(H_0 - E)(t-t_i)} \eta_2 \right\|_{L^p} + C n^2 \int_{t_i}^{t} |t-s|^{-\sigma'} \left\| e^{J(H_0 - E) s} \eta_2 \right\|_{L^2_{\text{loc}}} ds
\leq \frac{1}{2} \left[ \Lambda_{G,1}(t) + \Lambda_{G,2}(t) \right] + C n^2 \int_{t_i}^{t} |t-s|^{-\sigma'} [\Lambda_{L,1} + \Lambda_{L,2}] ds.
\]

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which is bounded by $\Lambda_{G,1}(t) + \Lambda_{G,2}(t)$. Similarly, $\|e^{L(t-t_i)}P_{\pm}[\eta_2]\|_{L^2_{\text{loc}}}^2$ is bounded by

$$
\leq \left\| e^{J(H_0-E)(t-t_i)}[\eta_2] \right\|_{L^2_{\text{loc}}}^2 + Cn^2 \int_{t_i}^t \min \{ |t-s|^{-3/2}, |t-s|^{-\sigma'} \} \left\| e^{J(H_0-E)s}[\eta_2] \right\|_{L^2_{\text{loc}}}^2 \, ds
$$

$$
\leq \frac{1}{2} [\Lambda_{L,1}(t) + \Lambda_{L,2}(t)] + Cn^2 \int_{t_i}^t \min \{ |t-s|^{-3/2}, |t-s|^{-\sigma'} \} [\Lambda_{L,1} + \Lambda_{L,2}](s) \, ds,
$$

which is bounded by $\Lambda_{L,1}(t) + \Lambda_{L,2}(t)$. Summing the estimates we get the Lemma.

Denote

$$
\hat{\rho}(t) = \rho(t - t_i) = [\rho_0^{-2} + \gamma_0 n^2 (t - t_i)]^{-1/2},
$$

$$
\delta_{8}(t) = n^{-\frac{2}{5}(1-\delta)}(t - t_o)^{-6/p} + n^6(t-t_i)^{-6/p} \leq o(1)n\hat{\rho}(t)^2,
$$

and

$$
M_{T}^* := \sup_{t_i \leq t \leq T} \max_{t \leq T} \left\{ \frac{\hat{\rho}(t)^{-1}|z_H(t)|}{[\Lambda_G(t) + n^{7/9}\hat{\rho}(t)^{5/3}]^{-1} \| \eta \|_{L^p}}, \frac{\| \eta \|_{L^2_{\text{loc}}}^2}{[\Lambda_G(t) + \Lambda_G^2(t) + n^{-\alpha}\hat{\rho}(t)^3 + \delta_{8}(t)]^{-1} \| \eta \|^{(3)}_{L^2_{\text{loc}}}} \right\}.
$$

(7.9)

**Proposition 7.2** Suppose for $T \geq t_i$ we have $n(T)/n(t_i) \in \left(\frac{3}{2}, \frac{9}{2}\right)$ and $M_{T}^* \leq 3$. Then we have $M_{T}^* \leq \frac{5}{2}$ and $n(T)/n(t_i) \in \left(\frac{3}{2}, \frac{9}{2}\right)$.

This Proposition implies Theorem 14 in the case $k = 0$, see e.g. [3] [20] [6] [28].

**Proof of Proposition 7.2** The condition $M_{T}^{*} \leq 3$ means, $\forall \ t_i \leq t < T$,

$$
z_H(t) \leq 3\hat{\rho}(t), \quad |a(t)| \leq 6D\hat{\rho}(t), \quad \| \eta \|_{L^p} \leq 3[\Lambda_G(t) + n^{7/9}\hat{\rho}(t)^{5/3}],
$$

$$
\| \eta \|^{(3)}_{L^2_{\text{loc}}} \leq 3[\Lambda_G(t) + \Lambda_G^2(t) + n^{-\alpha}\hat{\rho}(t)^3 + \delta_{8}(t)].
$$

(7.10)

Let $\delta_{9}(t) := \Lambda_{L,2}(t) + \Lambda_{L,3}(t)$. Note

$$
\Lambda_{L,1}(t) + \Lambda_{G}^2(t) + \delta_{8}(t) \leq o(1)n\hat{\rho}(t)^2, \quad \sum_{j=1}^{3}[\Lambda_{L,j}(t) + \Lambda_{G,j}(t)] \leq o(1)\hat{\rho}(t).
$$

(7.11)

Thus

$$
\| \eta \|^{(3)}_{L^2_{\text{loc}}} \leq o(1)n\hat{\rho}(t)^2 + 3\delta_{9}(t), \quad \| \eta \|_{L^2_{\text{loc}}}^2 \leq Cn\hat{\rho}(t)^2 + 3\delta_{9}(t).
$$

(7.12)

We also have

$$
\| \eta \|_{L^2_{\text{loc}}} \leq o(1)\hat{\rho}(t), \quad \| \eta \|_{L^p} \leq o(1)\hat{\rho}(t), \quad t \in [t_i, T].
$$

(7.13)

Recall that $X$ and $\tilde{X}$ are defined in (3.70). From (7.11), (7.13) and

$$
\| \eta \|_{L^1_{\text{loc}}} \leq \| \eta \|_{L^2_{\text{loc}}} \| \eta \|_{L^p}^{\frac{p}{2}} \leq \| \eta \|_{L^2_{\text{loc}}} \| \eta \|_{L^p}^{\frac{p}{2}},
$$

(7.14)

we have

$$
\tilde{X} \lesssim n\hat{\rho}(t)^4 + [\rho^2 + n^2]\delta_{9}(t), \quad X \lesssim n^2\hat{\rho}(t)^3 + n\hat{\rho}\delta_{9}.
$$

(7.15)

On the other hand, from (3.71), (3.78), (7.13) and (7.15), we also obtain

$$
|\hat{\theta}| = |F_{\hat{\theta}}| \lesssim \hat{\rho}^2 + n^{-1}X \lesssim \hat{\rho}(t)^2, \quad |\hat{p}_k| = |Z_k| \lesssim n\hat{\rho}(t)^2, \quad k \geq 1.
$$

(7.16)

The proof of this proposition is divided into two lemmas.
Lemma 7.3 Suppose for some \( T > t_i \) that \( M_T^* \leq 3 \). Then, \( t \in [t_i, T] \), we have
\[
\| \eta(t) \|_{L^p} \leq \frac{5}{2} [\Lambda_G(t) + n^{7/9} \hat{\rho}(t)^{5/3}],
\]
\[
\left\| \eta^{(3)}(t) \right\|_{L^p_{loc}} \leq \frac{5}{2} [\Lambda_L(t) + \Lambda_G^2 + n^{-\alpha} \hat{\rho}(t)^3 + \delta_8(t)].
\]

Proof. Recall \( \eta = e^{-i\theta} \eta_- + e^{i\theta} \eta_+ \). By (3.55),
\[
\eta_\pm(t) = e^{L(t-t_i)} \eta_\pm(t_i) + \int_{t_i}^t e^{L(t-s)} P_\pm \{ F_{L\pm} + e^{\mp i\theta} J[F] \}(s) ds.
\]
From (3.59), (3.71), (7.16), (7.17), (5.21) and as in (4.27) and (4.16), we get
\[
\| F_{L\pm} \|_{L^p'} + \| F \|_{L^p'} \leq C n \hat{\rho}(t)^2 + o(1) \| \eta \|_{L^p}^{\frac{p+2}{p-2}}.
\]
By Lemma 7.1, \( \frac{p+2}{p-2} > 2 \) and \( \| \eta \|^2_{L^p} \leq o(1) n \rho^2 \),
\[
\| \eta_\pm \|_{L^p} \leq \Lambda_G(t) + \int_{t_i}^t (t-s)^{-\sigma'} n \hat{\rho}(s)^2 ds
\]
\[
\leq \Lambda_G(t) + C n^{-1}(\Delta t)^{-\alpha}[\Delta t + t - t_i]^{-\sigma' + \alpha} \leq \Lambda_G(t) + o(1) n^{7/9} \hat{\rho}(t)^{5/3}.
\]
Above we have used (3.59) with \( \alpha = \sigma' - 5/6 > 7/18 \). So, we get
\[
\| \eta \|_{L^p} \leq \| \eta_- \|_{L^p} + \| \eta_+ \|_{L^p} \leq 2[\Lambda_G(t) + n^{7/9} \hat{\rho}(t)^{5/3}].
\]
To estimate \( \eta^{(3)} \) in \( L^2_{loc} \), write \( \eta^{(3)} = e^{-i\theta} \eta_-(3) + e^{i\theta} \eta_+(3) \) with \( \eta^{(3)}_\pm = \sum_{j=1}^4 \eta^{(3)}_{\pm,j} \) as in (3.55) and (3.62). Using (7.11), (7.16), (7.17) and the argument of (4.30) and (4.33), we obtain
\[
\sum_{j=1}^3 \left\| \eta^{(3)}_{\pm,j}(t) \right\|_{L^2_{loc}} \leq \Lambda_L(t) + C n [\hat{\rho}(t)^3 + \hat{\rho}_0^2(t-t_i)^{-3/2}].
\]
To estimate \( \eta^{(3)}_{\pm,4} \), we write \( \eta^{(3)}_{\pm,4} = \eta^{(3)}_{\pm,4,1} + \eta^{(3)}_{\pm,4,2} \) with
\[
\eta^{(3)}_{\pm,4,1}(t) = \int_{t_i}^t e^{L(t-s)} P_\pm \{ F_{L \pm} + e^{\mp i\theta} J[F - F_1 - |\eta|^2 \eta] \}(s) ds,
\]
\[
\eta^{(3)}_{\pm,4,2}(t) = \int_{t_i}^t e^{L(t-s)} P_\pm \{ e^{\mp i\theta} J[|\kappa| \eta^2 \eta] \}(s) ds.
\]
Note that \( \{ F_{L \pm} + e^{\mp i\theta} J[F - F_1 - |\eta|^2 \eta] \} \) is localized and
\[
\| F_{L\pm} \|_{L^1 \cap L^p'} + \| F - F_1 - |\eta|^2 \eta \|_{L^1 \cap L^p'} \leq C [\hat{\rho}(t)^3 + n \Lambda_G^2(t)].
\]
So, we get
\[
\left\| \eta^{(3)}_{\pm,4,1}(t) \right\|_{L^2_{loc}} \leq C \int_{t_i}^t \min \{ |t-s|^{-3/2} |t-s|^{-\sigma'} \} \hat{\rho}(s)^3 + n \Lambda_G^2(s) ds
\]
\[
\leq C [\hat{\rho}(t)^3 + n \Lambda_G^2(t)].
\]
As in (6.44), we have
\[ \| \eta^2 \eta \|_{L^2_{\rho(t)} + L_{\rho(t)}} \leq o(1) \| \eta \|_{L^2_{\rho(t)}} \leq o(1) [\Lambda^2_G + n \frac{14}{15} \hat{\rho}(t) \frac{14}{15}]. \] (7.24)

So, it follows as in (6.45) that
\[ \| \eta^{(3)}_{\pm, 4, 2}(t) \|_{L^2_{\text{loc}}} \leq o(1) [\Lambda^2_G(t) + \hat{\rho}(t)^3 + \delta_8(t)]. \] (7.25)

Collecting (7.21), (7.23), and (7.25), we obtain the second estimate of the lemma.

**Lemma 7.4** Suppose for some $T > t_i$ that $M^*_T \leq 3$. Then, for $t \in [t_i, T]$, we have
\[ z_H(t) \leq 2\hat{\rho}(t), \quad |a(t)| \leq 4D\hat{\rho}(t)^2, \quad n(T)/n(t_i) \in \left(\frac{3}{4}, \frac{5}{4}\right), \quad \forall k \geq 1. \] (7.26)

**Proof.** From (7.14) and (7.11), we can apply Lemma 3.7 with $\beta = \hat{\rho}(t)$. So, for each $k \geq 1$, we can find $q_k$ such that
\[ \dot{q}_k = \sum_{l \geq 1} D_{kl} [q_l] [q_l^2 q_k + Y_k q_k + g_k], \quad |q_k - p_k| \leq Cn\hat{\rho}(t)^2. \] (7.27)

Moreover, $\text{Re}(Y_k) = 0$, $\text{Re} D_{kl} \leq -\frac{2n^2}{2}$ and
\[ |g_k| \leq C [n\hat{\rho}(t)^4 + n^2 \hat{\rho}(t) \| \eta \|_{L^2_{\text{loc}}} + n\hat{\rho}(t)|\eta^{(3)}|_{L^2_{\text{loc}}} + M]. \] (7.28)

So, from (7.13) and (7.16), we get
\[ |g_k| \leq o(1)n^2 \hat{\rho}(t)^3 + Cn\hat{\rho}(t)\delta_9(t). \] (7.29)

Note that
\[ \int_{t_i}^{t_i + n^{-3}} \delta_9(t) ds \leq o(1); \quad \delta_9(t) \leq o(1)n\hat{\rho}(t)^2 \quad \forall t > t_i + n^{-3}. \] (7.30)

By the argument in the proof of Lemma 4.3 we obtain
\[ |z_k(t)| \leq z_H(t) \leq 2\hat{\rho}(t), \quad \forall t \in [t_i, T]. \] (7.31)

On the other hand, from Lemma 3.8 we have
\[ \hat{b} = \sum_{k, l \geq 1} B_{kl} |z_k|^2 |z_l|^2 + g_b, \quad |b - \hat{b}| \leq Cn\hat{\rho}^2, \]
\[ |g_b| \leq C [n^3 \hat{\rho}(t) + n^2 \| \eta \|_{L^2_{\text{loc}}} + n \| \eta^{(3)} \|_{L^2_{\text{loc}}} + n\hat{\rho}^2 \| \eta^{(3)} \|_{L^2_{\text{loc}}}] . \]

Again, by using (7.13) and (7.15), we get
\[ |g_b(s)| \leq o(1)n^2 \hat{\rho}(t)^4 + Cn\delta_9(t)[n\delta_9 + \hat{\rho}(t)^2]. \] (7.32)

If $T > t > t_i + n^{-3}$, we get $\delta_9(t) \leq o(1)n\hat{\rho}(t)^2$. So,
\[ \int_{t_i + n^{-3}}^{t} |g_b(s)| ds \leq o(1) \int_{t_i + n^{-3}}^{t} n^2 \hat{\rho}(s)^4 ds \leq o(1)\hat{\rho}(t)^2. \] (7.33)
For $t_i \leq t < \min(n^{-3}, T)$, we have $|g_b(s)| \leq o(1)[n^2\rho_0^4 + n^2\rho_0\delta_0(s)]$ and, using (7.30),

$$
\int_{t_i}^t |g_b(s)|ds \leq o(1) \int_{t_i}^{t_i+n^{-3}} [n^2\rho_0^4 + n^2\rho_0\delta_0(s)]ds \leq o(1)\rho_0^2 \sim o(1)\hat{\rho}(t)^2.
$$

(7.34)

So, we get

$$
\int_{t_i}^t |g_b|(s)ds \leq o(1)\hat{\rho}(t)^2, \quad \forall \ t_i \leq t \leq T.
$$

(7.35)

From this and as in the proof of Lemma 4.5, we also get

$$
|a(t)| \leq 4D\rho(t)^2, \quad n(T)/n(t_i) \in (\frac{3}{4}, \frac{5}{4}).
$$

(7.36)

This completes the proof of Lemma 7.4. □

Acknowledgments

We thank S. Gustafson for his constant interest in this work. Part of this work was conducted while the first author visited the University of British Columbia, by the support of the 21st century COE program and the Kyoto University Foundation, and while the second author was a postdoctoral fellow at the University of British Columbia. The research of Nakanishi was partly supported by the JSPS grant no. 15740086. The research of Tsai was partly supported by the NSERC grant no. 261356-08.

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