N=2 Supergravity Lagrangians with Vector-Tensor Multiplets

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Abstract

We discuss the coupling of vector-tensor multiplets to $N = 2$ supergravity.
1 Introduction and summary

Theories with extended supersymmetry have provided a rich testing ground for the study of many phenomena in field theory and string theory [1]. This paper deals with $N = 2$ supergravity in four-dimensional spacetimes, whose coupling to a number of matter multiplets has been worked out in considerable detail. The most well-known matter multiplets are the vector multiplet and the hypermultiplet. Off-shell a vector multiplet comprises $8 + 8$ bosonic and fermionic degrees of freedom [2]. The off-shell representation of hypermultiplets [3] is more subtle. To remain off-shell one can either choose a formulation with (off-shell) central charges based on $8 + 8$ degrees of freedom and ensuing constraints, or one must accept a description based on an infinite number of degrees of freedom. For the latter description, harmonic superspace provides a natural setting [4]. The tensor multiplet [5], which is dual to a massless hypermultiplet, also comprises $8 + 8$ off-shell degrees of freedom. All three multiplets describe $4 + 4$ physical degrees of freedom.

There are two other matter multiplets describing the same number of physical states. First there is the vector-tensor multiplet [6, 7], which is dual to a vector multiplet. Secondly, there exists a double-tensor multiplet, which contains two tensor gauge fields and which is dual to a hypermultiplet. The off-shell representation of both these multiplets requires the presence of off-shell central charges and their superconformal formulation requires the presence of background fields. All multiplets appear naturally in the context of four-dimensional $N = 2$ supersymmetric compactifications of the three ten-dimensional supergravity theories, and may therefore play a role in the effective low-energy actions associated with appropriate string compactifications. For instance, the vector-tensor multiplet can be associated with the supermultiplet of vertex operators of four-dimensional heterotic $N = 2$ supersymmetric string vacua, which contains the operators of the axion-dilaton complex together with an extra vector gauge field. Similarly, the tensor and the double-tensor multiplet would appear in the context of type-IIA and type-IIB string compactifications.

At the level of the four-dimensional effective actions, the latter multiplets are usually converted into vector multiplets and hypermultiplets, which, at least in string-perturbation theory, yields an equivalent description. We should stress that this conversion rests on a purely on-shell equivalence. The question whether certain off-shell configurations are preferred by string theory has a long history (for a recent discussion, see [8]). At any rate, not every system of vector multiplets or hypermultiplets can be converted back into vector-tensor, tensor or double-tensor multiplets, so there are certain restrictions (see, e.g. [9]). Furthermore, recent experience in dual systems, for instance in the context of three spacetime dimensions [10], has taught us that the answer to these questions involves nonperturbative issues. While in [7] the vector-tensor multiplet was introduced, motivated by heterotic string perturbation theory, it meanwhile turned out that vector-tensor multiplets have a different role to play and emerge in heterotic compactifications at the nonperturbative level. This phenomenon was initially described in the context of six-dimensional heterotic string compactifications, where it turned out that certain singularities in the effective action were associated with noncritical strings becoming tensionless [11]. In six-dimensions this is related to the presence of tensor multiplets. In four dimensions, vector-tensor multiplets play a similar role [12]. Couplings of the vector-tensor multiplet appear in two varieties. One type of coupling could be of six-dimensional origin [13]. The origin of the second coupling is less clear. For recent discussions on the couplings of six-dimensional tensor multiplets, we refer to [14].

In two previous publications [15, 16] we have developed the coupling of the vector-tensor multiplet in the context of the superconformal multiplet calculus. So far we restricted ourselves to rigid supersymmetry and we constructed all possible couplings to a general (background) configuration of vector multiplets that are invariant under rigid scale and chiral U(1) transformations. The requirement of scale and chiral invariance forces the scalar fields of the vector multiplet to
act as compensators. In the context of rigid supersymmetry this feature does not represent a restriction: at the end one can always freeze the vector multiplets to constants, thereby causing a breaking of scale invariance. In the case of local supersymmetry it is more subtle to freeze the vector multiplets. The reason for insisting on rigid scale and chiral invariance, is that, in this form, one can rather straightforwardly incorporate the coupling to supergravity by employing the superconformal multiplet calculus [17, 18]. In this paper, we report on the results of the extension to local supersymmetry. We give a comprehensive treatment of matter couplings to $N = 2$ supergravity. For the vector-tensor multiplet we make use of a formulation based on a finite number of off-shell degrees of freedom, which employs off-shell central charges. These degrees of freedom are described by one vector, one two-rank tensor gauge field, two real scalar fields, one of them auxiliary, and a doublet of Majorana spinors. In this formulation the gauge field associated with the central charge is known to exhibit rather peculiar couplings [19, 17]. Recently another specific example of such a coupling was studied in [20].

In [16] we established the existence of two different vector-tensor multiplets. Their difference is encoded in the Chern-Simons couplings between the vector and tensor gauge fields, whose form is constrained by supersymmetry. One version, first discussed in [15], is characterized by the fact that the vector and tensor gauge field of the vector-tensor multiplet exhibit a direct Chern-Simons coupling. This leads to unavoidable nonlinearities (in terms of the vector-tensor multiplet fields) of the action and transformation rules. This theory is formulated with at least one abelian vector multiplet, which provides the gauge field for the central-charge transformations. When freezing this vector multiplet to a constant we obtain a vector-tensor multiplet with a self-interaction. It takes the form (after a suitable rescaling of the fields)

$$\mathcal{L} \propto \frac{1}{2} \phi (\partial_{\mu} \phi)^2 + \frac{1}{4} \phi (\partial_{\mu} V_{\nu} - \partial_{\nu} V_{\mu})^2 + \frac{3}{4} \phi^{-1} (\partial_{[\mu} B_{\nu \rho]} - V_{[\mu} \partial_{\nu} V_{\rho]} )^2$$

$$+ \frac{1}{2} \phi \bar{\lambda}^i \slashed{\partial} \lambda_i - 2 \phi (\phi^{(5)})^2 - \frac{1}{4} i (\bar{\epsilon}^{ij} \bar{\lambda}_i \sigma^{\mu \nu} \lambda_j - \text{h.c.}) (\partial_{\mu} V_{\nu} - \partial_{\nu} V_{\mu})$$

$$- \frac{1}{24} \phi^{-1} \bar{\lambda}^i \gamma^\mu \gamma^\nu \gamma^\rho \lambda_i (\partial_{[\mu} B_{\nu \rho]} - V_{[\mu} \partial_{\nu} V_{\rho]} ) + \frac{3}{16} \phi^{-1} (\bar{\lambda}^i \gamma_{\mu} \lambda_i )^2$$

$$+ \frac{1}{32} \phi^{-1} (\bar{\epsilon}^{ij} \bar{\lambda}_i \sigma_{\mu \nu} \lambda_j)^2 + \text{h.c.},$$

where we have included an auxiliary field, $\phi^{(5)}$. We will comment on its role in due course. The second version of the vector-tensor multiplet, which requires more than one abelian vector multiplet, avoids the direct Chern-Simons coupling between the vector and tensor field of the vector-tensor multiplet, but there are nonvanishing Chern-Simons couplings with the additional vector multiplets. In this case the action remains quadratic in terms of the vector-tensor multiplet fields.

Recently a number of papers appeared dealing with the superspace formulation of vector-tensor multiplets [21, 22, 23, 24, 25]. Most of this work concerns the linear version of the vector-tensor multiplet with its corresponding Chern-Simons couplings, which can be obtained by dimensional reduction from six dimensions [14]. Unfortunately, even in the framework of harmonic superspace, it turns out that it is not possible to avoid an explicit central charge with corresponding constraints [23]. On the other hand, the complexity of our results clearly demonstrates the need for a suitable superspace formulation. For rigid supersymmetry, the self-interaction (1.1) has been derived recently in harmonic superspace [23, 24].

This paper is organized as follows. In section 2 we present a survey of the superconformal multiplet calculus and establish our notation. In section 3 we introduce the vector-tensor multiplet and discuss its superconformal transformation rules. Section 4 contains the derivation of the locally supersymmetric actions for vector-tensor multiplets. In section 5 we discuss their dual version in terms of vector multiplets. A number of useful formulæ has been collected in an appendix.
2 Superconformal Multiplet Calculus

Off-shell formulations of supergravity theories can be described in a form that is gauge equivalent to a superconformal theory. In four spacetime dimensions this enables a relatively concise organization of the field content. It also allows the systematic construction of supersymmetric Lagrangians via techniques known collectively as multiplet calculus. In this section we review the organization of the field content. It also allows the systematic construction of supersymmetric theories to a superconformal theory. In four spacetime dimensions this enables a relatively concise formulation of supergravity theories can be described in a form that is gauge equivalent to a superconformal theory. In four spacetime dimensions this enables a relatively concise organization of the field content. It also allows the systematic construction of supersymmetric Lagrangians via techniques known collectively as multiplet calculus. In this section we review the organization of the field content. It also allows the systematic construction of supersymmetric theories to a superconformal theory. In four spacetime dimensions this enables a relatively concise organization of the field content. It also allows the systematic construction of supersymmetric Lagrangians via techniques known collectively as multiplet calculus. In this section we review the organization of the field content. It also allows the systematic construction of supersymmetric theories to a superconformal theory. In four spacetime dimensions this enables a relatively concise organization of the field content.
Given the $S$-supersymmetry variations one may compute the special conformal boosts from the commutator

$$[\delta_S(\eta_1), \delta_S(\eta_2)] = \delta_K(\Lambda_K^n), \quad \text{with } \Lambda_K^n = \tilde{\eta}_2\gamma^n\eta_1^i + \text{h.c.} \quad (2.4)$$

Poincaré supergravity theories are obtained by coupling the Weyl multiplet to additional superconformal multiplets containing Yang-Mills and matter fields. The resulting superconformal theory then becomes gauge equivalent to a theory of Poincaré supergravity. This is conveniently exploited by imposing gauge conditions on certain components of the extra superconformal multiplets. Subsequently one can eliminate the auxiliary superconformal fields. The additional multiplets are necessary to provide compensating fields and to overcome a deficit in degrees of freedom between the Weyl multiplet and the minimal field representation of Poincaré supergravity. For instance, the graviphoton, represented by an abelian vector field in the Poincaré supergravity, is provided by an $N = 2$ superconformal vector multiplet.

In the following subsections we briefly describe the Weyl multiplet, vector multiplets, hypermultiplets and linear multiplets.

### 2.1 The Weyl multiplet

We already specified the fields belonging to the Weyl multiplet. The Weyl and chiral weights and the fermion chiralities of the Weyl-multiplet fields, the composite connections, and also those of the supersymmetry transformation parameters, are shown in Table 2.1. The Weyl and chiral weights

$$\phi(x) \rightarrow \exp[w\Lambda_D(x)+ic\Lambda_{U(1)}(x)]\phi(x). \quad (2.5)$$

Here we summarize the transformation rules for the independent fields under $Q$- and $S$-supersymmetry and under $K$-transformations,

$$\begin{align*}
\delta\varepsilon^{\mu}_{\mu} & = \bar{\epsilon}^{\mu}\gamma^{\mu}\psi_{\mu} + \text{h.c.}, \\
\delta\psi_{\mu} & = 2\mathcal{D}_{\mu}\epsilon^{\mu} - \frac{1}{2}\sigma\cdot T^{ij}\gamma_{\mu}\epsilon_{j} - \eta_{\mu}\eta_{j}^{i}, \\
\delta b_{\mu} & = \frac{1}{2}\bar{\epsilon}_{\mu}\phi_{\mu} - \frac{3}{4}\bar{\epsilon}_{\mu}\chi_{\mu} - \bar{\psi}_{\mu}\psi_{\mu} + \text{h.c.} + \Lambda_{K}^{a}\epsilon^{a}_{\mu}, \\
\delta\Lambda_{\mu} & = \frac{1}{2}i\bar{\epsilon}_{\mu}\phi_{\mu} + \frac{3}{4}i\bar{\epsilon}_{\mu}\chi_{\mu} + \bar{\psi}_{\mu}\psi_{\mu} + \text{h.c.}, \\
\delta\chi_{\mu} & = 2\bar{\epsilon}_{\mu}\phi_{\mu} - 3\bar{\epsilon}_{\mu}\chi_{\mu}^{i} + 2\tilde{\eta}_{\mu}\psi_{\mu} + \text{h.c. ; traceless}, \\
\delta T_{ab} & = 8\bar{\epsilon}[\hat{R}_{ab}(Q)]^i, \\
\delta D & = \epsilon^{i}\mathcal{D}_{i} + \text{h.c.},
\end{align*} \quad (2.6)$$

where $\mathcal{D}_{\mu}$ are derivatives covariant with respect to Lorentz, dilatational, $U(1)$ and $SU(2)$ transformations, and $D_{\mu}$ are derivatives covariant with respect to all superconformal transformations. Both $\mathcal{D}_{\mu}$ and $D_{\mu}$ are covariant with respect to the additional gauge transformations associated with possible gauge fields of the matter multiplets. The quantities $\hat{R}_{ab}(Q), \hat{R}_{ab}(U(1))$ and $\hat{R}_{ab}(SU(2))_{i}$ are supercovariant curvatures related to $Q$-supersymmetry, $U(1)$ and $SU(2)$ transformations. Their precise definitions are given in the appendix. The gauge fields for Lorentz, special conformal, and $S$-supersymmetry transformations are denoted $\omega_{\mu_{\nu}}, \phi_{\mu}^{i},$ and $f_{\mu}^{a},$ respectively. These are composite objects, which depend in a complicated way on the independent
fields (see the appendix). Under supersymmetry and special conformal boosts they transform as follows,
\[
\delta \omega^{ab}_{\mu} = -\frac{\epsilon^i}{4} \sigma^{ab}_{\mu} \phi_{\mu i} - \frac{1}{2} \epsilon^i T^{ab}_{ij} \psi_{\mu j} + \frac{3}{2} \epsilon^i \gamma_{\mu} \sigma^{ab} \chi_i
\]
\[
+ \epsilon^i \gamma_{\mu} R^{ab}(Q)_{ij} - \frac{i}{\sqrt{2}} \sigma^{ab} \psi_{\mu i} + h.c. + 2 \Lambda^{[a}_{\mu} \epsilon^{b]}_i ,
\]
\[
\delta \phi^{i}_{\mu} = -2 f^{i}_{\mu ij} \epsilon^j - \frac{1}{4} \hat{D} T^{ij} \cdot \sigma \gamma_{\mu} \epsilon^j + \frac{3}{2} (\bar{\chi}_j \gamma^{a} \epsilon^j) \gamma_a \psi_{\mu i} - (\bar{\chi}_j \gamma^a \psi_{\mu i}) \gamma_a \epsilon^j
\]
\[
+ \frac{1}{2} \hat{R}(SU(2))_{ij} \cdot \sigma \gamma_{\mu} \epsilon^j + i \hat{R}(U(1))_{i} \cdot \sigma \gamma_{\mu} \epsilon^j + 2 D_{\mu} \epsilon^j + \Lambda^{a}_{\mu} \gamma_a \psi_{\mu i} ,
\]
\[
\delta f^{a}_{\mu} = -\frac{\epsilon}{4} \psi^{i}_{\mu} D_{\mu} T^{ij} - \frac{3}{2} e^{a}_{\mu} \epsilon^i \hat{D} \chi_i - \frac{3}{2} \bar{\epsilon} \gamma^a \psi_{\mu i} D
\]
\[
+ \epsilon^i \gamma_{\mu} D_{\mu} \hat{R}^{ab}(Q)_{ij} + \frac{1}{2} i \gamma^a \psi_{\mu i} + h.c. + D_{\mu} \Lambda^{a}_{\mu} .
\]
(2.7)

### 2.2 The vector multiplet

The \(N = 2\) vector multiplet transforms in the adjoint representation of a given gauge group. For each value of the group index \(I\), there are \(8 + 8\) component degrees of freedom off-shell, including a complex scalar \(X^I\), a doublet of chiral fermions \(\Omega^I\), a vector gauge field \(W^I_{\mu}\), and a real \(SU(2)\) triplet of scalars \(Y_{ij}^I\). The Weyl and chiral weights and the fermion chirality of the vector-multiplet component fields are listed in table 2.2. Under \(Q\)- and \(S\)-supersymmetry these transform as follows,
\[
\delta X^I = \epsilon^i \Omega^I_i ,
\]
\[
\delta \Omega^I_i = 2 \hat{D} X^I \epsilon_i + \epsilon_{ij} \sigma \cdot F^{I} - \epsilon^j + Y^I_{ij} \epsilon^j - 2 g f_{JK} X^J \bar{X}^K \epsilon_{ij} \epsilon^j + 2 X^I \eta_i ,
\]
\[
\delta W^I_{\mu} = \epsilon_{ij} \epsilon_{\mu j} \Omega^I_i + 2 \epsilon_{ij} \bar{\epsilon} X^I \psi^j_{\mu} + h.c. ,
\]
\[
\delta Y^I_{ij} = 2 \epsilon_{ij} \hat{D} \Omega^I_{ij} + 2 \epsilon_{ik} \epsilon_{jl} \epsilon^{(k} \Omega^I_{lj)} - 4 g f_{JK} \epsilon_{ij} \bar{X}^J \Omega^K - \epsilon_{ijk} \bar{X}^I \Omega^K ,
\]
(2.8)

where \(f_{JK} x^I\) are the structure constants of the group, \([t_I, t_J] = f_{IJ} t_K\), and \(g\) is a coupling constant. The field strengths \(F_{\mu
u}\) are defined by
\[
F^{I}_{\mu\nu} = 2 \partial_{\mu} W^I_{\nu j} - g f_{JK} W^J_{\mu} W^K_{\nu} - (\epsilon_{ij} \bar{\epsilon}^i \gamma_{\mu j} \Omega^{Ij} + \epsilon_{ij} \bar{\epsilon}^i X^I \psi^j_{\mu} + \frac{1}{2} \epsilon_{ij} \bar{\epsilon}^i X^I T^{ij}_{\mu\nu} + h.c.) .
\]
(2.9)

They satisfy the Bianchi identity
\[
D^{b} \left( F^{+I}_{ab} - F^{-I}_{ab} + \frac{1}{4} X^I T^{ij}_{ab} \epsilon^{ij} - \frac{1}{4} \bar{X}^I T^{ij}_{ab} \epsilon^{ij} \right) = \frac{3}{4} \left( \bar{\epsilon}^i \gamma_a \Omega^{Ij} \epsilon_{ij} - \bar{\epsilon}^i \gamma_a \Omega^{Ij} \epsilon^{ij} \right) .
\]
(2.10)

\(^3\)The real triplet \(Y^I_{ij}\) satisfies \(Y^I_{ij} = Y^I_{ji}\) and \(Y_{ij}^I \epsilon_{ij} = \epsilon_{jk} \bar{Y}_{ij}^J \epsilon^{jk}\).
Under supersymmetry they transform as follows,

$$\delta F^I_{ab} = -2\varepsilon^{ij}\bar{e}_i\gamma_a D_{bj}\Omega^I_j - 2\varepsilon^{ij}\bar{\eta}_i\sigma_{ab}\Omega^I_j + \text{h.c.}. \tag{2.11}$$

The transformation rules (2.8) satisfy the commutator relation (2.1), including a field-dependent gauge transformation on the right-hand side, which acts with the following parameter

$$\theta^I = 4\varepsilon^{ij}\bar{e}_i\epsilon_{1j}X^I + \text{h.c.}. \tag{2.12}$$

The covariant quantities of the vector multiplet constitute a so-called reduced chiral multiplet. A general chiral multiplet contains 16 + 16 off-shell degrees of freedom and an arbitrary Weyl weight factor $w$ (corresponding to the Weyl weight of its lowest component). The covariant quantities of the vector multiplet may be obtained from a chiral multiplet with $w = 1$ by the application of a set of reducibility conditions, one of which is the Bianchi identity.

### 2.3 The hypermultiplet

A finite field configuration describing off-shell hypermultiplets must have a nontrivial central charge. This charge acts on a basic unit underlying $r$ hypermultiplets, which consists of $r$ quaternions $A^i_\alpha$ and $2r$ chiral fermions $\zeta^\alpha$. The Weyl and chiral weights and fermion chirality of these fields are listed in table 2.2.

The index $\alpha$ runs from 1 to $2r$. As the basic unit contains twice as many fermionic as bosonic components, it is necessary to assume the presence of an infinite number of them. These multiple “copies”, which will be distinguished by appending successive “$z$” indices to the fields, will be organized in a linear chain, such that the central charge maps each one of them into the next one. For instance, on $A^i_\alpha$ it acts as $\delta A^i_\alpha = zA^i_\alpha(z)$, where $z$ is the transformation parameter. Successive applications of the central charge thus generate an infinite sequence,

$$A^i_\alpha \rightarrow A^i_\alpha(z) \rightarrow A^i_\alpha(zz) \rightarrow \text{etcetera}, \tag{2.13}$$

and similarly on the fermionic fields. The supersymmetry transformation rules for the basic fields are summarized as follows,

$$\delta A^i_\alpha = 2\varepsilon^{ij}\bar{\eta}_i\zeta_j, \tag{2.14}$$

$$\delta \zeta^\alpha = \bar{\delta} A^i_\alpha\epsilon^i + 2X^0 A^i_\alpha\bar{\epsilon}^i\epsilon_j + 2gX^\alpha_\beta A^i_\beta\bar{\epsilon}^i\epsilon_j + A^i_\alpha\eta^j,$$

where $X^0$ is the scalar component of a background vector multiplet which supplies the gauge field for the central charge, and $X^\alpha_\beta$ is the scalar component of a Lie-algebra valued vector multiplet.

Central charge transformations commute with the supersymmetry transformations when acting on the hypermultiplet fields. It follows that the supersymmetry transformation for e.g. $A^i_\alpha(z)$ is

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4Our notation is such that the $2 \times 2r$ matrix $A^i_\alpha$, with complex conjugate $A^i_\alpha$, satisfies the constraint $A^i_\alpha = \varepsilon^{ij}\rho_{\alpha\beta}A_j^\beta$ where, under certain conditions $[\underline{2}]$, $\rho_{\alpha\beta}$ can be brought in block-diagonal form, $\rho = \text{diag}(i\sigma_2, i\sigma_2, \ldots)$.

5Solving this constraint reduces $A^i_\alpha$ to a sequence of $r$ quaternions $(q_1, \ldots, q_r)$, where each quaternion is represented by the $2 \times 2$ matrix $q_a = q^{(0)}_a + iq^{(1)}_a\sigma_1 + iq^{(2)}_a\sigma_2 + iq^{(3)}_a\sigma_3$.

6A hierarchy such as (2.13) arises naturally when starting from a five-dimensional supersymmetric theory with one compactified coordinate, but this interpretation is not essential.

7Our conventions are such that $X^\alpha_\beta = X^I(t_I)^\alpha_\beta$ and $\bar{X}^\beta_\alpha = \bar{X}^I(t_I)^\beta_\alpha$, where $t_I$ are the generators of the Lie algebra. Consistency requires that $(t_I)^\alpha_\beta = -\rho_{\alpha\gamma}(t_I)^\gamma_\beta\rho^{\beta\delta}$. A nontrivial action requires the existence of an hermitean tensor (which is not necessarily positive-definite, as one of the hypermultiplets may act as a compensator), which restricts the gauge group to a subgroup of (a noncompact version of) $\text{USp}(2r)$.

8Later when we discuss the vector-tensor multiplet we will see that $[\delta_z(z), \delta_q(\epsilon)]$ closes into the tensor and vector gauge transformations that are associated to the vector-tensor multiplet. The hypermultiplet is inert under the latter transformations.
obtained from (2.14) by placing a “z” index onto the rule for $A_4^\alpha$. Similarly, the transformation rules for all fields higher in the hierarchy can be obtained from those corresponding to lower-lying fields by appending successive “z” indices. In order to close the supersymmetry algebra an infinite number of constraints must be imposed. The fields $(A_4^\alpha, \zeta^\alpha, A_i^{\alpha(z)})$ are not affected by the constraints. As a result they constitute the fundamental $8r + 8r$ degrees of freedom contained in the $r$ hypermultiplets. The constraints, which relate higher-$z$ elements of the central charge hierarchy to the fundamental degrees of freedom, are described by the following two relationships,

\[ \zeta^{\alpha(z)} = -\frac{1}{2X^0} \left[ \rho^{\alpha\beta} \partial \zeta^\beta + \Omega^0 \bar{\partial} A_i^\alpha + g\Omega^i_{\hat{\alpha} \beta} A_i^{\hat{\alpha}} + 2gX^\alpha A_4^\beta + \frac{1}{8\sigma} T_{ij} \varepsilon^{ij} \zeta^\alpha - \frac{3}{2} \varepsilon^{ij} \chi_i A_i^\alpha \right], \]

\[ A_i^{\alpha(zz)} = -\frac{1}{4|X^0|^2} \left[ (D^a D_a + \frac{3}{2} D) A_i^\alpha + \varepsilon_{ik} Y^0 j_k A_j^{\alpha(z)} + 2\left( \eta^{\alpha \beta} \Omega^0_{ij} \zeta^\gamma - \varepsilon_{ij} \Omega^0_{\hat{\alpha} \beta} \zeta^{\hat{\alpha}} \right) \right. \\
+ 2g(\bar{X}^0 X^\alpha \beta + X^0 \bar{X}^\beta) A_i^{\beta(z)} + 2g(\rho^{\alpha\beta} \Omega^0_{ij} \zeta^\gamma - \varepsilon_{ij} \Omega^0_{\hat{\alpha} \beta} \zeta^{\hat{\alpha}}) \\
+ \left. g\varepsilon_{ik} Y^j k\zeta^\alpha A_i^\beta + 2g^2 \{ \bar{X}, X \}^{\alpha \beta} A_i^\beta \right]. \quad (2.15) \]

All other constraints are obtained from these by application of the central charge.

An important observation is that the constraints (2.15) are algebraic relationships. For instance, the equation for $\zeta^{\alpha(z)}$ involves $\zeta^\beta(z)$ on the right hand side, through the covariant derivative $D_\mu \zeta^\beta = \partial_\mu \zeta^\beta - W^0_{\mu} \zeta^{(z)} + \cdots$. Taking the complex conjugate of the equation for $\zeta^{\alpha(z)}$, we obtain the analogous equation for $\zeta^{\beta(z)}$, which we may then substitute back. Similar manipulations may be done to the equation for $A_i^{\alpha(zz)}$. In this manner we can restructure the constraints into the following form,

\[ \zeta^{\alpha(z)} = -\frac{1}{4} X^0 \left( |X^0|^2 + \frac{1}{4} W^0 W^0 \right)^{-1} \left( \rho^{\alpha\beta} \zeta^\beta + \cdots \right), \]

\[ A_i^{\alpha(zz)} = -\frac{1}{4} \left( |X^0|^2 + \frac{1}{4} W^0 W^0 \right)^{-1} \left( \partial^2 A_i^\alpha + \cdots \right). \quad (2.16) \]

The above infinite-dimensional hierarchical structure of basic units $(A_4^\alpha, \zeta^\alpha)$ endowed with an infinite sequence of constraints leaving precisely $8 + 8$ degrees of freedom, was worked out in [18]. However, we should recall that this approach does not enable one to derive the most general couplings of hypermultiplets. These can be obtained in the harmonic-superspace formulation, which avoids the presence of an off-shell central charge at the expense of an infinite number of unconstrained fields. For the vector-tensor multiplet there seems no way to avoid the central charge [23]. Therefore we use the same method as outlined above for the construction of Lagrangians for interacting vector-tensor multiplets. This is described in the next section.

2.4 The linear multiplet

A linear multiplet contains three scalar fields transforming as an SU(2) triplet. The defining condition is that, under supersymmetry, these scalars transform into a doublet spinor. Furthermore it contains a Lorentz vector, subject to a constraint. The linear multiplet can transform in a real representation of some gauge group, as well as under a central charge. For this reason the supersymmetry transformations contain the Lie-algebra valued components of a vector multiplet associated with this gauge group. This is exactly the same as for the hypermultiplets, but here we do not introduce extra indices to indicate the matrix-valued character. The terms associated with the gauge group carry a coupling constant $g$. The central-charge transformations are simply incorporated into the generic gauge group and will not be indicated explicitly. The Weyl
and chiral weights and the fermion chirality of the component fields of the linear multiplet are listed in table 2.2.

The transformation rules for the component fields of the linear multiplet are as follows

\[
\begin{align*}
\delta L_{ij} &= 2\epsilon_{i}^{\phantom{i}j} \varphi_{j} + 2\epsilon_{ik}^{\phantom{ik}j} \epsilon^{(k} \varphi^{l)}, \\
\delta \varphi^{i} &= \mathcal{D} L^{ij} \epsilon_{j} + \mathcal{E} L^{ij} \epsilon_{j} - G \epsilon_{j} + 2g L^{ij} \epsilon_{jk} \epsilon^{k} + 2L^{ij} \eta_{j}, \\
\delta G &= -2\epsilon_{i}^{\phantom{i}j} \mathcal{D} \epsilon_{j} - \epsilon_{i}^{\phantom{i}j} \left(6\epsilon_{ji} L^{ij} + \frac{1}{2} \epsilon^{ij} \epsilon^{kl} \sigma \cdot T_{jkl} \varphi_{l}\right) + 2g \mathcal{X} \left(\epsilon^{ij} \epsilon_{i} \varphi_{j} - \text{h.c.}\right) - 2g \epsilon_{i}^{ij} \eta^{j} \epsilon_{j} + 2\eta_{i} \varphi_{j}, \\
\delta E_{a} &= 2\epsilon_{ij}^{\phantom{ij}c} \sigma_{ab} D_{b}^{a} \epsilon^{j} + \frac{1}{4} \epsilon_{i}^{\phantom{i}j} \gamma_{a} \epsilon_{j}^{ij} \gamma_{a}^{j} L^{ij} - \frac{1}{2} \sigma \cdot T_{j} \epsilon^{j} \varphi_{j}, \\
&\quad+ 2\mathcal{X} \epsilon^{c} \gamma_{a} \epsilon_{i}^{ij} \varphi_{i} + g \epsilon^{c} \gamma_{a} \epsilon_{j}^{ij} \gamma_{a} \varphi_{j} \epsilon_{i} + \text{h.c.}, \\
&\quad(2.17)
\end{align*}
\]

where \( L_{ij} = \epsilon_{ik}^{\phantom{ik}j} L^{kl} \) and

\[
2D_{a} E_{a} = g \left(\frac{1}{2} Y^{ij} L_{ij} - 2X G - 2\tilde{\Omega} \varphi_{i}\right) - 3\epsilon^{ij} \chi^{j} \epsilon_{ij} + \text{h.c.}. \quad (2.18)
\]

For \( g = 0 \) the above constraint can be solved and \( E_{a} \) can be written as the (supercovariant) field strength of a two-rank tensor gauge field \( E_{\mu \nu} \). The solution takes the form

\[
E_{a} = \frac{1}{2} \eta_{a}^{\phantom{a}c} \epsilon_{\mu \nu} \rho \sigma D_{\nu} E_{\rho \sigma}. \quad (2.19)
\]

The resulting multiplet is known as the \( N = 2 \) tensor multiplet.

### 2.5 Multiplet calculus

The identification of the various rules for multiplying multiplets is a central aspect of the multiplet calculus. This has been explicitly described in previous papers [17, 18]. There are product rules that define how to construct multiplets from products of certain other multiplets. For some of the multiplets one can find density formulae, which yield a superconformally invariant action upon integration over spacetime. In the context of our work here the most relevant density formula is the one involving an abelian vector and a linear multiplet. The linear multiplet can transform under a central charge, in which case the vector multiplet must be the one that supplies the gauge field for the central charge transformations. Apart from this the linear multiplet must be neutral under the gauge group. The density formula reads,

\[
es^{-1} \mathcal{L} = X^{0} G - \left(\frac{1}{2} Y_{\mu}^{ij} + \frac{1}{2} \eta_{\mu}^{ij} \sigma^{\mu \nu} \psi_{\nu}^{ij}\right) L_{ij} + \varphi^{i} \left(\Omega^{i} + X^{0} \gamma^{i} \psi_{\mu i}\right) \]

\[
- \frac{1}{2} W_{a}^{0} \left(2 \varphi^{a b} \sigma^{c d} \psi_{b c}^{i} \epsilon_{ij} - \frac{1}{2} \epsilon^{a b c d} \varphi^{i} \epsilon_{b c} \psi_{a d}^{i} L_{ij} \epsilon^{j k}\right) + \text{h.c.}, \quad (2.20)
\]

where the vector-multiplet fields carry a superscript “0” to indicate that they belong to an abelian vector multiplet, possibly associated with central-charge transformations.
3 The vector-tensor multiplets

3.1 Central charges and Chern-Simons terms

From off-shell counting it follows immediately that the vector-tensor multiplet must be subject to a central charge when it is based on a finite number of off-shell components. Just as in [15, 16] we use the same strategy as presented for hypermultiplets in the previous section. The basic unit of the vector-tensor multiplet consists of a scalar field $\phi$, a vector gauge field $V_\mu$, a tensor gauge field $B_{\mu\nu}$ and a doublet of spinors $\lambda_i$. This unit consists of seven bosonic and eight fermionic components. To close the supersymmetry algebra off shell, we must assume the existence of an infinite hierarchy of these units, again distinguished by appending successive indices “$z$”. The central charge then raises the number of “$z$” indices, such as, for instance, in $\delta_z \phi = z\phi^{(z)}$. Successive applications thus generate a sequence of terms,

$$\phi \rightarrow \phi^{(z)} \rightarrow \phi^{(zz)} \rightarrow \text{etcetera},$$

(3.1)

and similarly on all other fields. It will turn out that $\phi^{(z)}$ corresponds to an auxiliary field. All other objects in the hierarchy, $\phi^{(zz)}$, $V_{\mu}^{(z)}$, $V_{\mu}^{(zz)}$, etcetera, are dependent, and will be given by particular combinations of the independent fields. Hence we end up with precisely $8 + 8$ degrees of freedom.

In order to couple the vector-tensor multiplet to supergravity we employ the superconformal multiplet calculus. When the supersymmetry is local then also the central-charge transformations must be local. Therefore we must couple the vector-tensor multiplets to at least one vector multiplet, whose gauge field couples to the central charge. However, for reasons that have been described in [16], it is advisable to couple the vector-tensor multiplet to a more general background of vector multiplets, so we consider $n$ vector multiplets. One of these provides the gauge field for the central charge, which we denote by $W_\mu^0$. This must be an abelian gauge field. The remaining $n - 1$ vector multiplets supply additional background gauge fields $W_\mu^A$, which need not be abelian. The index $A$ is taken to run from 2 to $n$, for reasons we explain shortly. Also, since $W_\mu^0$ is the gauge field for the central charge, the associated transformation parameter $\theta^0$ is identified with the central charge parameter $z$ introduced above, i.e., $z \equiv \theta^0$. The vector gauge transformations act as follows on the background gauge fields,

$$\delta W_\mu^0 = \partial_\mu z, \quad \delta W_\mu^A = \partial_\mu \theta^A + f_{BC}^A \theta^B W_\mu^C.$$  

(3.2)

In addition to the central charge, the vector-tensor multiplet has its own gauge transformations associated with the tensor $B_{\mu\nu}$ and the vector $V_\mu$. We reserved the index 1 for the vector field $V_\mu$ of the vector-tensor multiplet. (The reason for this choice is based on the dual description of our theory, where the vector-tensor multiplet is replaced with a vector multiplet, so that the dual theory involves $n + 1$ vector multiplets.) In the interacting theory, the tensor field $B_{\mu\nu}$ necessarily couples to Chern-Simons forms. This coupling is evidenced by the transformation behavior of the tensor. To illustrate this, if we ignore the central charge (other than its contribution to $W_\mu^0$), then the vector field of the vector-tensor multiplet would transform as

$$\delta V_\mu = \partial_\mu \theta^1,$$

(3.3)

and the tensor field would transform as

$$\delta B_{\mu\nu} = 2\partial_\mu \Lambda_{\nu} + \eta_{IJ} \theta^I \partial_\mu W_{\nu}^I,$$

(3.4)

where $\theta^I$ and $\Lambda_\mu$ are the parameters of the transformations gauged by $W_{\mu}^I$ and $B_{\mu\nu}$ respectively, and the index $I$ is summed from 0 to $n$. As mentioned above, in this context $W_{\mu}^1$ is identified
with $V_\mu$. Closure of the combined vector and tensor gauge transformations requires that $\eta_{IJ}$ be a constant tensor invariant under the gauge group. There is an ambiguity in the structure of $\eta_{IJ}$, which derives from the possibility of performing field redefinitions. Without loss of generality, $\eta_{IJ}$ can be modified by absorbing a term proportional to $W^I_\mu W^J_\nu$ times some group-invariant antisymmetric tensor into the definition of the tensor field $B_{\mu\nu}$. Without loss of generality, we thus remove all components of $\eta_{IJ}$ except for $\eta_{11}, \eta_{1A}$ and $\eta_{AB}$, and also we render $\eta_{AB}$ symmetric. Also note that, since $\eta_{1A}$ is invariant under the gauge group, it follows that $\eta_{1A} W^A_\mu$ is an abelian gauge field.

The situation is actually more complicated, since $V_\mu$ and $B_{\mu\nu}$ are also subject to the central-charge transformation. As described above, under this transformation these fields transform into complicated expressions, denoted $V^{(z)}_\mu$ and $B^{(z)}_{\mu\nu}$, respectively, which involve other fields of the theory. Accordingly, we deform the transformation rule (3.3) to

$$\delta V_\mu = \partial_\mu \theta^1 + z V^{(z)}_\mu,$$  \hspace{1cm} (3.5)

and, at the same time, (3.4) to

$$\delta B_{\mu\nu} = 2 \partial_{[\mu} \Lambda_{\nu]} + \eta_{11} \theta^1 \partial_{[\mu} V_{\nu]} + \eta_{1A} \theta^1 \partial_{[\mu} W^A_{\nu]} + \eta_{AB} \theta^A \partial_{[\mu} W^B_{\nu]} + z B^{(z)}_{\mu\nu}. \hspace{1cm} (3.6)$$

All $\theta^0$-dependent terms, including any such Chern-Simons contributions, are now contained in $V^{(z)}_\mu$ and $B^{(z)}_{\mu\nu}$, which are determined by closure of the full algebra, including supersymmetry. The deformed transformation rules must still lead to a closed gauge algebra. In particular one finds that

$$[\delta_z, \delta_{\text{vector}}(\theta^1)] = \delta_{\text{tensor}} \left( \frac{1}{2} z \eta_{11} \theta^1 V^{(z)}_\mu \right). \hspace{1cm} (3.7)$$

This implies that $V^{(z)}_\mu$ and the combination $B^{(z)}_{\mu\nu} + \eta_{11} V_{[\mu} V^{(z)}_{\nu]}$ both transform covariantly under the central charge, but are invariant under all other gauge symmetries. However, under local supersymmetry, they do not transform covariantly, as we will see below (cf. 3.14). The resulting gauge algebra now consists of the standard gauge algebra for the vector fields augmented by a tensor gauge transformation. Observe that we have neither specified $V^{(z)}_\mu$ nor $B^{(z)}_{\mu\nu}$, which are determined by supersymmetry and will be discussed in the next section. As it turns out these terms give rise to additional Chern-Simons terms involving $W^0_\mu$ that depend on the scalar fields. The presence of these terms is a direct result of the deformation of the standard algebra of tensor and vector gauge transformations.

Before giving specific results on the local supersymmetry transformations, we discuss a crucial feature of our results. It turns out [13, 16] that the coefficients $\eta_{IJ}$ that encode the Chern-Simons terms cannot all be zero, as otherwise the supersymmetry variations turn singular and supersymmetric completions in the action will vanish. In fact, one can show that there are just two inequivalent representations of the vector-tensor multiplet. One is the case where $\eta_{11} = 0$. In this case there is no Chern-Simons coupling between the tensor and the vector fields of the vector-tensor multiplet. The choice $\eta_{11} = 0$ removes the conspicuous self-interaction between the vector-tensor multiplet fields and in fact the supersymmetry transformations become linear in these fields (but not in the background fields) and the action quadratic. However, in this case not all the $\eta_{1A}$ Chern-Simons coefficients can vanish simultaneously. Therefore we are dealing with at least three abelian gauge fields, namely, $W^0_\mu, \eta_{1A} W^A_\mu$ and $V_\mu$. In the case of rigid supersymmetry, one can freeze some or all of the vector multiplets to a constant, but this will not alter the structure of the couplings.

This first class seems to coincide with the theories one obtains by reducing (1,0) tensor multiplets in six spacetime dimensions to four dimensions. The tensor multiplet comprises a scalar, a self-dual tensor gauge field and a symplectic Majorana spinor. The self-dual tensor field decomposes in four dimensions into the vector and tensor gauge fields of the vector-tensor
multiplet. To have also a vector field that couples to the central charge presumably requires the dimensional reduction of a theory of tensor multiplets coupled to supergravity. A recent study of various Chern-Simons terms in six dimensions was carried out in [14].

The second, inequivalent class of couplings is characterized by the fact that \( \eta_{11} \neq 0 \). In that case, it turns out that one can absorb certain terms of the background multiplets into the definition of the vector-tensor fields such that all the coefficients \( \eta_{1A} \) vanish [15]. In this case we have at least two abelian vector fields, namely \( W_\mu^0 \) and \( V_\mu \).

Hence in practical situations the Chern-Simons coefficients can be restricted to satisfy either \( \eta_{11} = 0 \) or \( \eta_{1A} = 0 \). In the following we will not pay much attention to this fact, but simply evaluate the transformation rules and the action for general values of the coefficients \( \eta_{11}, \eta_{1A}, \eta_{AB} \).

### 3.2 The vector-tensor transformation rules

In [13, 16] the transformation rules for the vector-tensor multiplet have been determined by imposing the supersymmetry algebra iteratively on the multiplet component fields. In this procedure, the supersymmetry transformation rules for vector multiplets remain unchanged. Therefore, the algebra represented by the vector-tensor multiplet in the presence of a vector multiplet background is fixed up to gauge transformations which pertain exclusively to the vector-tensor multiplet. The most relevant commutator in this algebra involves two supersymmetry transformations and was given in (2.1). In the present situation we have that

\[
\begin{align*}
\delta_{\text{gauge}} &= \delta_z \left( 4\varepsilon^{ij}\bar{\epsilon}_2 \epsilon_{1j} X^0 + \text{h.c.} \right) + \delta_{\theta A} \left( 4\varepsilon^{ij}\bar{\epsilon}_2 \epsilon_{1j} X^A + \text{h.c.} \right) \\
&+ \delta_{\text{vector}} \left( \theta^1(\epsilon_1, \epsilon_2) \right) + \delta_{\text{tensor}} \left( \Lambda_\mu(\epsilon_1, \epsilon_2) \right). 
\end{align*}
\] (3.8)

The field \( X^0 \) is the complex scalar of the vector multiplet associated with the central charge. The field-dependent parameters \( \theta^1(\epsilon_1, \epsilon_2) \) and \( \Lambda_\mu(\epsilon_1, \epsilon_2) \) are found by imposing the \( Q \)-supersymmetry commutator on the vector-tensor multiplet. They will be specified in due course.

In this paper we repeat the derivation of the transformation rules, but now in the context of local supersymmetry. This means that we follow the same procedure, but now in a background of conformal supergravity combined with vector multiplets. Because the transformation rules for the superconformal fields are also completely known, the supersymmetry algebra is determined up to the gauge and central-charge transformations associated with the vector-tensor multiplet itself. The procedure followed in [15, 16] is tailor-made for an extension to local supersymmetry. First of all, we already insisted on rigid scale and chiral invariance. Because of that, the scalar fields of the vector multiplets will play the role of compensating fields to balance possible differences in scaling weights of the various terms. Secondly, one of the vector multiplets was required to realize the central charge in a local fashion. In the context of the superconformal multiplet calculus, local dilations, chiral and central-charge transformations are necessary prerequisites for the coupling to supergravity.

As was already discussed in [15], there remains some flexibility in the assignment of the scaling and chiral weights for the vector-tensor multiplet. By exploiting the scalar fields of the vector multiplets we may arbitrarily adjust the weights for each of the vector-tensor components by suitably absorbing functions of \( X \) and \( X^A \). In this way we choose the weights for the vector-tensor components to be as shown in table 3.1. The bosonic vector-tensor fields must all have chiral weight \( c = 0 \) since they are all real. To avoid a conflict between scale transformations and vector-tensor gauge transformations we adjusted \( V_\mu \) and \( B_{\mu\nu} \) to be also neutral under scale transformations. Note that there remains a freedom to absorb additional combinations of the

\[ ^8 \text{Henceforth we will suppress the superscript on } X^0 \text{ and define } X \equiv X^0 \text{ to simplify the formulae.} \]
| field | $\phi$ | $V_\mu$ | $B_{\mu\nu}$ | $\lambda_i$ | $\phi^{(k)}$ |
|-------|------|------|----------|---------|---------|
| $w$   | 0    | 0    | 0        | $\frac{1}{2}$ | 0       |
| $c$   | 0    | 0    | 0        | $\frac{1}{2}$ | 0       |
| $\gamma_5$ | | | | + | |

Table 3.1: Scaling and chiral weights ($w$ and $c$, respectively) and fermion chirality ($\gamma_5$) of the vector-tensor component fields.

background fields into the definition of $\phi$ and $\lambda_i$. Furthermore, the fields $V_\mu$ and $B_{\mu\nu}$ can be redefined by appropriate additive terms. Needless to say, it is important to separate relevant terms in the transformation rules from those that can be absorbed into such field redefinitions. In deriving our results this aspect has received proper attention.

In order to define the vector-tensor multiplet as a superconformal multiplet, we must also choose the assignments under the special $S$-supersymmetry transformations (which in turn determine the behaviour under special conformal boosts $K$). We have assumed that the scalar $\phi$ is $S$- and $K$-invariant, which leads to consistent results. While this is a natural assignment for the lowest-dimensional component of a supermultiplet, we found no rigorous arguments to rule out other assignments. The choice we made is the simplest one and, as it turns out, implies that all the vector-tensor fields remain $S$- and $K$-invariant. The latter follows from the commutator of $Q$- with $S$-supersymmetry, and subsequently, by using the $[S, S]$ commutation relation, which yields a $K$-transformation.

The transformation rules coincide with the ones found in [15, 16] apart from the presence of certain covariantizations. As before we suppress nonabelian terms for the sake of clarity; they are not important for the rest of this paper. We are not aware of arguments that would prevent us from switching on the nonabelian interactions. Furthermore we introduce the following notation for homogeneous, holomorphic functions of zero degree that occur frequently in our equations,

$$g = i\eta_{A} \frac{X^A}{X}, \quad b = -\frac{1}{4}i\eta_{AB} \frac{X^A X^B}{X^2}.$$  \hspace{1cm} \text{(3.9)}

For arbitrary Chern-Simons coefficients $\eta_{IJ}$, the transformation rules under $Q$-supersymmetry are (we emphasize that in the remainder of this section and in section 4, the index $I$ does not take the value $I = 1$),

$$\delta \phi = \bar{\epsilon}^i \lambda_i + \bar{\epsilon}_i \lambda^i,$$

$$\delta V_\mu = i\epsilon^{ij} \epsilon_i \gamma_\mu \left( 2X \lambda_j + \phi \Omega^0_j \right) - iW^{0}_\mu \epsilon^i \lambda_i + 2i\phi X \epsilon^{ij} \bar{\epsilon}_i \psi_{\mu j} + \text{h.c.},$$

$$\delta B_{\mu\nu} = -2\epsilon^{ij} \sigma_{\mu\nu|} |X|^{2} \left( 4\eta_{11} \phi - 2 \text{Re} g \right) \lambda_i - 2\epsilon^{ij} \sigma_{\mu\nu} X \left( 2\eta_{11} \phi^2 \Omega^0_i + \phi \bar{X} \partial_f \bar{\Omega}^i - 4i \text{Re} \left[ \partial_f (X b) \right] \Omega^i \right) - 2\epsilon^{ij} \gamma_{[\mu} \psi_{\nu i]} \bar{X} \left( 2\eta_{11} \phi^2 X + \bar{X} \partial_f \bar{X} X^f - 4i \text{Re} \left[ \partial_f (X b) \right] X^f \right) + i\epsilon^{ij} \bar{\epsilon}_{\mu} \gamma_\nu \left( \eta_{11} \left( 2X \lambda_j + \phi \Omega^0_j \right) - i\eta_{A} \Omega^A_j \right) + 2i\epsilon^{ij} \bar{\epsilon}_{\mu} \gamma_{\nu i} \left( \eta_{11} \phi - g \right) + \epsilon^{ij} \bar{\epsilon}_{\mu} \psi_{\nu i} \left( \eta_{11} \phi^2 - \phi g - 4ib \right)$$

$$+ 2\epsilon^{ij} \bar{\epsilon}_{\mu} \psi_{\nu i} W^{0}_{\mu} \left( \eta_{11} \phi^2 + \phi \Omega^0_j + \eta_{11} \phi^2 \Omega^0_j - i\eta_{A} \phi \Omega^A_j - 4i \partial_f (X b) \Omega^f \right) + 2\epsilon^{ij} \bar{\epsilon}_{\mu} \psi_{\nu i} W^{0}_{\mu} \left( \eta_{11} \phi^2 - \phi g - 4ib \right)$$
\[ +\varepsilon^{ij}\dot{\epsilon}_{ij}\gamma^a_{\mu}W^{A}_{\mu}A_{a}^{\mu} + 2\varepsilon^{ij}\varepsilon_{ij}\psi_{\mu}W^{A}_{\mu}\eta_{AB}X^{B} \]

\[ -i\eta_{11}W_{(\mu}^{\nu)}\varepsilon^{\mu}\lambda_{i} + \text{h.c.}, \]

\[ \delta \lambda_{i} = \left( \mathcal{D}_{\mu} - i\dot{\psi}^{(z)}_{\mu} \right) \varepsilon^{\mu} - \frac{i}{2X}\varepsilon^{\mu}\sigma_{\mu} \cdot \left( \mathcal{F}^{-}(V) - i\phi\mathcal{F}^{-0} \right) \varepsilon^{\mu} + 2\varepsilon^{ij}\dot{X}_{\phi}^{(z)}\dot{\epsilon}^{i} \]

\[ -\frac{1}{X}(\dot{\epsilon}^{i}\lambda_{j})\Omega_{i}^{0} - \frac{1}{X}(\dot{\epsilon}^{i}\Omega_{0}^{0})\lambda_{i} \]

\[ -\frac{1}{2X(2\eta_{11}\phi - \text{Re}g)}\varepsilon^{i} \left[ 2\eta_{11}\phi\dot{Y}_{ij}^{0} + \phi\dot{X}_{ij}\bar{g}Y_{ij}^{f} - 4i\text{Re}\partial_{j}(Xb)Y_{ij}^{f} \right] \]

\[ -2\eta_{11} \left( X\dot{\lambda}_{i}\lambda_{j} - \dot{X}\varepsilon_{i}^{\mu}\varepsilon_{j}^{\nu}\dot{\lambda}^{\nu} \right) \]

\[ + X \left( X\partial_{j}\bar{g}\Omega_{ij}^{f} - \dot{X}\varepsilon_{i}^{\mu}\varepsilon_{j}^{\nu}\partial_{j}\bar{g}\Omega^{k}\lambda^{l} \right) \]

\[ + i \left( \partial_{j}\partial_{j}(Xb)\Omega_{ij}^{f} + \varepsilon_{i}^{\mu}\varepsilon_{j}^{\nu}\partial_{j}\partial_{j}(Xb)\Omega^{k}\Omega^{l} \right) \]  \quad (3.10)

Except from the explicit gravitino fields in the variations of \( V_{\mu} \) and \( B_{\mu\nu} \), all extra covariantizations are implicitly contained in covariant derivatives and field strengths.

Let us now first define a number of quantities that appear in \( (3.10) \) or are related to them. The supercovariant field strengths for the vector-tensor multiplet gauge fields are equal to

\[ \mathcal{F}_{\mu\nu}(V) = 2\partial_{[\mu}V_{\nu]} - 2W_{\mu}^{(z)}(z) + \frac{1}{4}\varepsilon_{\mu}} \left[ \mathcal{X}(\dot{\epsilon}^{\mu})_{\lambda} - \text{h.c.} \right] \]

\[ H^{\mu} = \frac{1}{2}i\varepsilon^{\mu\nu\lambda\sigma} \left[ \partial_{\nu}B_{\lambda\sigma} - \eta_{11}V_{\nu}\partial_{\lambda}V_{\sigma} - \eta_{11}V_{\nu}\partial_{\sigma}W_{\lambda}^{A} \right] \]

\[ - \eta_{AB}W_{\mu}^{A}\partial_{\lambda}W_{\sigma}^{A} - W_{\nu}^{0} \left( \mathcal{F}_{\nu}^{(s)} + \eta_{11}V_{\sigma}^{(z)} \right) \]

\[ - \left[ \psi_{\mu}^{(s)} \gamma^{\mu\nu} \left( 2X\lambda_{j} + \phi\Omega_{0}^{j} \right) \right] \lambda_{i} \]

\[ + X \left( 2\eta_{11}\phi\Omega_{ij}^{f} - 4i\text{Re}\partial_{j}(Xb)\Omega_{ij}^{f} \right) + \text{h.c.} \]

\[ + \left[ \psi_{\mu}^{(s)} \gamma^{\mu\nu} \right] \left[ X \left( 2\eta_{11}\phi^{2}X + \phi\dot{X}X^{f} \partial_{j}\bar{g} - 4iX^{f} \right) \text{Re}\partial_{j}(Xb) \right] + \text{h.c.} \]. \quad (3.11)

The Bianchi identities corresponding to the field strengths \( (3.11) \) are straightforward to determine and read,

\[ D_{\mu} \left( \mathcal{F}_{\mu\nu}^{(s)}(V) + \frac{1}{4}\varepsilon_{\mu\nu}(\mathcal{X}^{(s)} + \mathcal{X}) \right) \]

\[ = -V_{\mu}^{(z)} \left[ \mathcal{F}_{\mu\nu}^{(s)} - \frac{1}{4}(\mathcal{X}^{(s)} + \mathcal{X}) \right] - \frac{3}{4}i\varepsilon_{\mu\nu}(2X\lambda^{j} + \phi\Omega_{0}^{j}) + \text{h.c.} , \]

\[ \quad D_{\mu}H^{\mu} = -\frac{1}{4i}\left[ \eta_{11}\mathcal{F}_{\mu\nu}(V) + \eta_{11}A_{\mu}^{(s)}(V) + \eta_{AB}\mathcal{F}_{\mu\nu}^{A} + 2\mathcal{F}_{\mu\nu}^{B}(z) \right] \]

\[ - \frac{1}{16} \left[ T_{ij}^{(s)} \left( 2\eta_{11}\phi X + \eta_{11}A_{\mu}^{(s)}(V) + i\phi X^{A} + 2\eta_{AB}A_{\mu}^{A}X^{B} \right) \right] \]

\[ + 2X\mathcal{F}_{\mu\nu}^{0} + \mathcal{X}^{0}(\eta_{11}\phi^{2} - \phi g - 4ib) \]  \quad (3.12)

Observe that the Bianchi identity for \( H^{\mu} \) is not linear in the vector-tensor fields. On the right-hand side there are nonlinear terms that are either of second-order (the term proportional to \( \eta_{11} \)) or of zeroth-order (the term proportional to \( \eta_{AB} \)) in the vector-tensor fields. Furthermore the quantity \( \mathcal{B}_{\mu\nu}^{(z)} \) does not depend homogeneously on the vector-tensor fields either as will become clear soon. Hence, generically the vector-tensor multiplet is realized in a nonlinear fashion, as we have already pointed out in the previous subsection.
Furthermore, the following quantities appear in the above formulae, which are the supercovariant part of the z-transformed vector and tensor fields,

\[
\hat{V}_a^{(z)} = \frac{-1}{2|X|^2(2\eta_{11} \phi - \text{Re} g)} \left\{ H_a - \left[ iXD_a \tilde{X}^I \left( 2\eta_{11} \phi^2 \delta_I^0 + \phi \tilde{X} \partial_I \tilde{g} - 4i\text{Re}[\partial_I(Xb)] \right) + \text{h.c.} \right] \right\}
\]

+ fermion terms,

\[
\hat{B}_{ab}^{(z)} = -\frac{i}{2} \text{Im} \ g \mathcal{F}_{ab}(V) + \frac{i}{2}(2\eta_{11} \phi - \text{Re} g) \mathcal{F}_{ab}(V) - \frac{i}{2}\phi(\eta_{11} \phi - \text{Re} g) \mathcal{F}_{ab}^0
\]

\[
+ \frac{i}{2} \text{Im}(X \partial_I g) \mathcal{F}_{ab}^0 + 4 \text{Im} \left[ \partial_I(Xb) \mathcal{F}_{ab}^0 \right] + \text{fermion terms}. \quad (3.13)
\]

The caret indicates that these expressions are fully covariant with respect to all local symmetries; they do not coincide with the image of \(V_\mu\) and \(B_{\mu
u}\) under the central charge, \(V_\mu^{(z)}\) and \(B_{\mu
u}^{(z)}\). The latter are given by

\[
V_\mu^{(z)} = e_\mu^a \hat{V}_a^{(z)} + \frac{i}{2} \left( i\tilde{\psi}\lambda_i + \text{h.c.} \right),
\]

\[
B_{\mu
u}^{(z)} = e_\mu^a e_\nu^b \hat{B}_{ab}^{(z)} - \eta_{11} \bar{V}_\mu V_\nu^{(z)}
\]

\[
+ \frac{i}{2} \left[ X^{ij} (\tilde{\psi} \psi_{ij} + \bar{T}_{\mu\nu ij}) (\eta_{11} \phi^2 - \phi g - 4ib) + 2X^{ij} \tilde{\psi} \psi_{[ij]} \eta_{\mu\nu} \lambda_j (2\eta_{11} \phi - g)
\]

\[
+ \epsilon^{ij} \psi_{[ij]} \eta_{\mu\nu} \left( \eta_{11} \phi^2 \Omega_j^0 - i\eta_{1A} \phi \Omega_j^A - 4i \partial_I(Xb) \Omega_j^I \right) + \text{h.c.} \right]. \quad (3.14)
\]

There are of course similar expressions for \(\lambda_i^{(z)}\) and \(\phi^{(zz)}\), which are of less direct relevance. Because the fields \(\phi\) and \(\lambda_i\) are themselves covariant, the action of the central charge will yield covariant expressions.

The results for the central charge transformations are determined from the commutator,

\[
[\delta_Q(\epsilon), \delta_z(z)] = \delta_{\text{vector}} \left( iz \epsilon^i \lambda_i + \text{h.c.} \right) + \delta_{\text{tensor}} \left( \Lambda_\mu(\epsilon, z) \right), \quad (3.15)
\]

where

\[
\Lambda_\mu(\epsilon, z) = \frac{i}{2} \epsilon^i \epsilon_{\mu} \gamma_{ij} \left( 2X(2\eta_{11} \phi - g) \lambda_j + \eta_{11} \phi^2 \Omega_j^0 - i\eta_{1A} \phi \Omega_j^A - 4i \partial_I(Xb) \Omega_j^I \right)
\]

\[
+ \epsilon^{ij} \epsilon_{\mu} \psi_{ij} X \left( \eta_{11} \phi^2 - \phi g - 4ib \right) + \frac{i}{2} \epsilon^i \eta_{11} V_\mu \epsilon^i \lambda_i + \text{h.c.}, \quad (3.16)
\]

which implies that the supersymmetry transformations of \(\phi^{(z)}\), \(\lambda_i^{(z)}\), are just the \(z\)-transformed versions of \(\delta_Q \phi, \delta_Q \lambda_i\) as given in (3.10). Hence, with the exception of \(\phi^{(z)}\) all the \(z\)-transformed fields are subject to constraints. By acting on these constraints with central-charge transformations, one recovers an infinite hierarchy of constraints. These relate the components of the higher multiplets \(\left( V_\mu^{(z)}, B_{\mu\nu}^{(z)}, \lambda_i^{(z)}, \phi^{(zz)} \right)\), et cetera to the lower ones, in such a way as to retain precisely 8 + 8 independent degrees of freedom.

At this point we specify the expressions for the vector and tensor gauge transformations in the commutator (3.8),

\[
\theta^I(\epsilon_1, \epsilon_2) = 4i\phi X \epsilon^{ij} \epsilon_1 \epsilon_2 \epsilon_{j1} + \text{h.c.},
\]

\[
\Lambda_\mu(\epsilon_1, \epsilon_2) = 2\epsilon_2^i \gamma_{\mu} \epsilon_1 \left( 2\eta_{11} \phi^2 X + \phi \tilde{X} X^I \partial_I \tilde{g} - 4iX^I \text{Re}[\partial_I(Xb)] \right)
\]

\[
+ 2i \epsilon^{ij} \epsilon_2 \epsilon_1 X \left( V_\mu(\eta_{11} \phi - g) - iW_\mu^0(\eta_{11} \phi^2 - \phi g - 4ib) \right)
\]

\[
+ 2e^{ij} \epsilon_2 \epsilon_1 W_\mu^A \eta_{AB} X^B + \text{h.c.}. \quad (3.17)
\]
We close this section with a number of supersymmetry variations of various quantities defined above. The supercovariant field strengths transform as follows:

$$\delta F_{ab}(V) = -2i\varepsilon^{ij}\hat{e}_{i}\gamma_{[a}D_{b]}(2X\lambda_{j} + \phi\Omega_{j}^{0}) - 2\varepsilon^{ij}\hat{e}_{i}\gamma_{[a}\Omega_{j}^{0}\dot{V}_{b]}^{(z)} - i\varepsilon^{ij}\lambda_{i}F_{ab}^{0}$$

$$\delta H^{a} = 4i\varepsilon^{i\sigma^{ab}}D_{b}[\lambda_{i}X^{2}(2\eta_{11}\phi - Re g)\lambda_{i}]$$

The variation of the covariant fields $\dot{V}_{a}^{(z)}$ and $\dot{B}_{ab}^{(z)}$ equals

$$\delta \dot{V}_{a}^{(z)} = i\varepsilon^{ij}\hat{e}_{i}\gamma_{a}(2X\lambda_{j} + \phi\Omega_{j}^{0})^{(z)} + i\varepsilon^{iD_{a}\lambda_{i} - \frac{i}{2}\varepsilon^{ij}\gamma_{a}\sigma \cdot T^{ij}\lambda_{j} - \frac{i}{2}\varepsilon^{ij}\gamma_{a}\lambda_{i} + h.c.$$  

$$\delta \dot{B}_{ab}^{(z)} = -4i\varepsilon^{i\sigma^{ab}}X^{2}(2\eta_{11}\phi - Re g)\lambda_{i}^{(z)}$$

The same structure is repeated as one goes higher up in the central-charge hierarchy. It was already observed in [15] that the transformations of the higher-z fields involve objects both at the next and at the preceding level. The transformations of the basic vector-tensor fields as given in (3.10) are special in this respect. They involve only the next level as there is no lower level. The consistency of this is ensured by the gauge transformations of the fields $V_{\mu}$ and $B_{\mu\nu}$, which allows for a truncation of the central charge hierarchy from below.

4 Invariant actions involving vector-tensor multiplets

In this section we present the construction of invariant actions for the vector-tensor multiplet, using the multiplet calculus described in section 2. We start by constructing a general linear
4.1 The linear multiplet

It is possible to form products of vector-tensor multiplets, using the background vector multiplets judiciously, so as to form $N = 2$ linear multiplets. One starts by constructing the lowest component $L_{ij}$ of the linear multiplet in terms of vector-tensor fields as well as the background fields, which must have weights $w = 2$ and $c = 0$ and transform into a spinor doublet under $Q$-supersymmetry. We also note that $L_{ij}$ must transform as a real vector under chiral SU(2) transformations. The only vector-tensor component which transforms under SU(2) is the fermion $\lambda_i$. For the vector multiplets, only the fermions $\Omega_i^I$ and the auxiliary fields $Y_{ij}^I$ transform non-trivially under SU(2). Therefore, the most general possible linear multiplet must be based on an $L_{ij}$ of the following form

$$L_{ij} = XA \bar{\lambda}_i \lambda_j + XA \epsilon_{ik} \epsilon_{jl} \bar{\lambda}^k \lambda^l + XB_I \bar{\lambda}_i \Omega_{ij}^I + XB_j \epsilon_{ik} \epsilon_{jl} \bar{\lambda}^k \Omega_{ij}^l$$

$$+ C_{IJ} \bar{\Omega}_i^I \Omega_{ij}^J + \bar{C}_{IJ} \epsilon_{ik} \epsilon_{jl} \bar{\Omega}_{ik} \Omega_{jl}^J + G_I Y_{ij}^I,$$  \hspace{1cm} (4.1)

where $A$, $B_I$, $C_{IJ}$ and $G_I$ are functions of $\phi$, $X^I$ and $\bar{X}^I$. In this section the index $I$ does not take the value $I = 1$. In order that $L_{ij}$ has weights $w = 2$ and $c = 0$, the functions $A$ and $G_I$ must have weights $w = c = 0$, while $B_I$ and $C_{IJ}$ have weights $w = -c = -1$. Obviously, the reality condition on $L_{ij}$ requires that $G_I$ be real. As before, we suppress the superscript zeroes of the central-charge vector multiplet for the sake of clarity. We also expect the linear multiplet to transform only under the central charge and not under the gauge transformations associated with the other vector multiplets, but this is not important for most of the construction.

Requiring that $L_{ij}$ transforms into a spinor doublet as indicated in (2.17), puts stringent requirements on each of the functions $A(\phi, X^I, \bar{X}^I)$, $B_I(\phi, X^I, \bar{X}^I)$, $C_{IJ}(\phi, X^I, \bar{X}^I)$ and $G_I(\phi, X^I, \bar{X}^I)$, which take the form of coupled first-order, linear differential equations. These equations are exactly the same as in the rigid case, which were given in [16]. We will not repeat them here but immediately present their solution, which is a linear combination of three distinct solutions, each with an independent physical interpretation. The most interesting of these is given as follows,

$$[A]_1 = \eta_{11}(\phi + i\zeta) - \frac{1}{2}g,$$

$$[B_I]_1 = -\frac{1}{2}(\phi + i\zeta) \partial_I g - 2i \partial_I b,$$

$$[C_{IJ}]_1 = -\frac{1}{2}i(\phi + i\zeta) \partial_I \partial_J(Xb),$$

$$[G_I]_1 = \text{Re}\left\{\frac{1}{3} \eta_{11}(\phi + i\zeta)^3 - \frac{1}{2}i\zeta(\phi + i\zeta)g|\delta_I^0 + \frac{1}{2}(\phi + i\zeta)X\partial_I(g\phi + 4ib)\right\},$$  \hspace{1cm} (4.2)

where

$$\zeta(\phi, X^I, \bar{X}^I) = \frac{\text{Im}(\phi\phi + 4ib)}{2\eta_{11}\phi - \text{Re}g}.$$  \hspace{1cm} (4.3)

In terms of the action, which will be discussed shortly, this solution provides the couplings which involve the vector-tensor fields. The remaining two solutions, which we discuss presently, give rise either to a total divergence or to interactions which involve only the background fields. The latter of these correspond to previously known results. The second solution takes the form,

$$[A]_2 = i\eta_{11}\zeta' - i\alpha,$$
where $\gamma = \frac{1}{4}i\alpha A X^A/X$ is a holomorphic homogeneous function of the background scalars $X^A$ and $X^0$; $\alpha$ and $\alpha A$ are arbitrary real parameters. Furthermore

$$\zeta'(\phi, X^I, \bar{X}^I) = \frac{2\alpha\phi + 4\Re \gamma}{2\eta_{11}\phi - \Re g}.$$  \hfill (4.5)

Note that this solution could be concisely included into the first solution by redefining $g \to g + 2i\alpha$ and $b \to b + \gamma$. In fact, this second solution indicates that the functions $g$ and $b$ are actually defined modulo these shifts. In terms of the action, this ambiguity is analogous to the shift of the theta angle in an ordinary Yang-Mills theory.

The third and final solution is given by

$$[\mathcal{A}]_3 = 0, \quad [\mathcal{B}_I]_3 = 0, \quad [\mathcal{C}_{IJ}]_3 = -\frac{i}{2}i\partial_I\partial_J(f(X)/X), \quad [\mathcal{G}_I]_3 = -\frac{i}{2}\Im \partial_I(f(X)/X).$$  \hfill (4.6)

Where $f(X)$ is a holomorphic function of $X^0$ and $X^A$, of degree 2. In terms of the action, this solution corresponds to interactions amongst the background vector multiplets alone. Since the possible vector multiplet self-couplings have been fully classified, this solution does not provide us with new information. The function $f(X)$ provides the well-known holomorphic prepotential for describing the background self-interactions.

All solutions have in common that they are homogeneous functions of $X^I$ and $\bar{X}^I$: $\mathcal{A}$ and $\mathcal{G}_I$ are of degree 0 and $\mathcal{B}_I$ and $\mathcal{C}_{IJ}$ are of degree $-1$. This is a result of the fact that the field $\phi$ has $w = 0$. Furthermore we note the identities,

$$X^I \mathcal{B}_I = X^I \mathcal{C}_{IJ} = 0,$$  \hfill (4.7)

which ensure that $L_{ij}$ is invariant under $S$-supersymmetry, in accord with (2.14).

Now that we have determined the scalar triplet $L_{ij}$, in terms of the specific functions $\mathcal{A}(\phi, X^I, \bar{X}^I)$, $\mathcal{B}_I(\phi, X^I, \bar{X}^I)$, $\mathcal{C}_{IJ}(\phi, X^I, \bar{X}^I)$, and $\mathcal{G}_I(\phi, X^I, \bar{X}^I)$ given above, we can generate the remaining components of the linear multiplet, $\varphi_i$, $G$, and $E_{\mu}$ by varying (4.1) with respect to supersymmetry. Given the complexity of the transformation rule for $\lambda_i$ found in (3.14), it is clear that a fair amount of work is involved in carrying out this process. However, since we are only interested in the bosonic part of the action, we are only interested in the bosonic part of $E_a$ and $G$, viz. (2.21).

The higher components of the linear multiplet are then given by

$$\varphi^i = -\bar{X}(D\phi + i\bar{V}(\phi))(A\lambda^i + \frac{1}{2}\bar{B}_I\Omega^{ii}) + \mathcal{G}_I\bar{D}\Omega^{ii}$$
$$-\frac{i}{2}e^{ij}\sigma \cdot (F(V) - i\phi F^0)(A\lambda_j + \frac{1}{2}\bar{B}_I\Omega^{ij})$$
$$+\frac{i}{2}e^{ij}\sigma \cdot F(X\bar{B}_I\lambda_j + 2\mathcal{C}_{IJ}\Omega_j^I)$$
$$-\bar{D}\bar{X}^i(\bar{X}\bar{B}_I\lambda^i + 2\mathcal{C}_{IJ}\Omega^{ij})$$
$$-|X|^2\phi(2A\lambda_j + B_I\Omega_j^I)$$
$$+\frac{1}{2}Y^{ij}(i\partial_\phi \mathcal{G}_I)\lambda_j + (\partial_j \mathcal{G}_I)\Omega_j^I + 3 \text{ fermion terms}.$$
\[ G = \bar{X} \hat{A} (D_a \phi + i \hat{V}^{(z)}_a) (D^a \phi + i \hat{V}^{(z)}_a) \\
+ 2 \bar{X} \hat{I} D_a \bar{X}^I (D^a \phi + i \hat{V}^{(z)}_a) \\
+ 4 \hat{C}_{IJ} D_a \bar{X}^I D^a \bar{X}^J - 2 \hat{G}_I D_a D^a \bar{X}^I \\
+ \frac{1}{4X} (F(V)^- - i \phi F^0^-)_{ab} \left( A(F(V)^- - i \phi F^0^-) + 2i X \hat{B}_I F^{I-} \right)_{ab} \\
- \hat{C}_{IJ} F_{ab}^{I-} F^{I-} - \frac{1}{4}(\partial_1 \hat{G}_I + X^{-1} P_1 \partial_\phi \hat{G}_I) Y^{ij} Y^{-ij} \\
- \frac{1}{2} \hat{G}_I F^{I+} T_{ij}^{ab} \varepsilon_{ij} + \text{fermion terms}, \]

\[ E_a = \text{Re} \left( -4|X|^2 \phi^{(z)} (\bar{A} (D_a \phi + i \hat{V}^{(z)}_a) + \hat{B}_I D_a X^I) -2i (D^b \phi + i \hat{V}^{(z)}_b) (A (F(V)^- - i \phi F^0^-) + i X \hat{B}_I F^{I-}) -2D^b X^I (i \hat{B}_I (F(V)^{ab} - i \phi F^{0a}_b) - 4 \hat{C}_{IJ} F_{ab}^{I-}) -2\hat{G}_I D^b (F^{I-} - \frac{1}{4} \bar{X}^I T_{ij}^{ab} \varepsilon_{ij}) \right) + \text{fermion terms}. \]

Here we used the notation

\[ P_I = -\frac{1}{2} \phi \delta_I \bar{g} + i \frac{\text{Im} \left( \phi X \partial_I g + 4i \partial_I (X \phi) \right)}{2(2\eta_{11} - \text{Re} g)}. \]

The appearance of terms containing \( T_{ij}^{ab} \) may seem strange because this field does not appear in the transformation rules for \( \lambda_i \) and \( \Omega_i \). However, this field appears in the variation of \( \partial \Omega_i \) and in the Bianchi identities for \( F_{ab}^I \), which have to be used to obtain \( G \) and \( E_a \). Having derived the complete linear multiplet we can construct the action.

### 4.2 The action

Now we want to use the linear multiplet components derived above in the action formula (2.20). Since this linear multiplet transforms under the central charge we need to use the central-charge vector multiplet in the action formula, as explained in section 2. This yields an action that is both invariant under local supersymmetry and local gauge transformations. Carrying out this calculation we note the following term in Langrange density,

\[ \mathcal{L} = 4e \bar{X} \hat{C}_{IJ} D^a X^I D_a X^J - 2e \hat{G}_I X D_a D^a \bar{X}^I \cdots, \]

which we rewrite by splitting off a total derivative. This leads to derivatives of the function \( \hat{G}_I \), which we rewrite using its explicit form (or the differential equations of which it is a solution). After this manipulation, the bosonic terms of the full action read,

\[ e^{-1} \mathcal{L} = -2 \hat{G}_I X \bar{X}^I \left( \frac{1}{6} \mathcal{R} - D \right) + |X|^2 A (\partial_\mu \phi - i \hat{V}^{(z)}_{\mu}) \left( \bar{A} \left( F(V)^{\mu -} - i \phi F^{0a}_\mu \right) + \partial_\mu \bar{F}^{I-}_{ab} + W_\mu \left( \partial_\nu \phi - i \hat{V}^{(z)}_\nu \right) \right) + i X \hat{B}_I \left( F(V)^{- \mu a} + i \hat{V}^{(z)}_{\mu \nu} \right) W_\mu \left( \partial_\nu \phi - i \hat{V}^{(z)}_\nu \right) + \hat{C}_{IJ} F^{I-} \left( X \bar{F}^{I-}_{ab} + 4 W_\mu D_\mu X \bar{X}^I \right) \]
The nonlinear vector-tensor multiplet: to give the solutions for the two inequivalent representations described in section 3.2. To go into the details of this. Consequently, the corresponding transformation rules contain significant nonlinearities. As was shown in \[16\], in this case it is possible to remove the parameter \(\eta_{11}\) and therefore the Chern-Simons coupling, \(V\). Without loss of generality, we then define \(\eta_{11} = 1\) and \(\eta_{1A} = 0\). In this case the functions \(A(\phi, X^I, \bar{X}^I)\), \(B_I(\phi, X^I, \bar{X}^I)\), \(C_{IJ}(\phi, X^I, \bar{X}^I)\), and \(G_I(\phi, X^I, \bar{X}^I)\) which define the linear multiplet and, more importantly, the vector-tensor Lagrangian \(\mathcal{L}_{12}\) are given by the following expressions

\[
\mathcal{A} = \phi + i\phi^{-1}(b + \bar{b}), \\
\mathcal{B}_I = -2i\partial_I b, \\
\mathcal{C}_{IJ} = -\frac{1}{2}i(\phi + i\phi^{-1}(b + \bar{b})\partial_I \partial_J (Xb) - \frac{1}{8}i\partial_I \partial_J (X^{-1}f)), \\
\mathcal{G}_I = \text{Re}\left(\frac{1}{4}\phi^3 \delta^0_0 + 2i\phi X\partial_I b - 2\phi^{-1}(b + \bar{b}) \partial_I (Xb)\right) - \frac{1}{2}\text{Im} \partial_I (X^{-1}f). \\
\]  

For the sake of clarity, we have absorbed the parameters \(\alpha\) and \(\alpha_A\) into the functions \(b\) and \(g\) in the manner described immediately after equation (4.3). Substituting these functions in the Lagrangian \((\mathcal{L}_{12})\), it is easy to see that the action contains, besides the total derivative and terms that depend only on the background vector multiplet fields, a cubic part and a linear part in vector-tensor fields. This is the immediate generalization to a background with more than one vector multiplet of the Lagrangian described in \[15\]. The linear vector–tensor multiplet: As described previously, if \(\eta_{11} = 0\), implying the absence of the \(V\) Chern-Simons coupling, we obtain a vector-tensor multiplet which is distinct from the nonlinear case just discussed. In this case, it is not possible to perform a field redefinition to remove all of the \(\eta_{1A}\) parameters and the supersymmetry transformation rules are linear in terms of the vector-tensor component fields. The functions \(A(\phi, X^I, \bar{X}^I)\), \(B_I(\phi, X^I, \bar{X}^I)\), \(C_{IJ}(\phi, X^I, \bar{X}^I)\), and \(G_I(\phi, X^I, \bar{X}^I)\) which
define the linear multiplet and, more importantly, the vector-tensor Lagrangian (4.11) are now given by the following expressions

\[ A = -\frac{1}{2} g, \]
\[ B_I = -\frac{1}{g + \bar{g}} \left( \phi \bar{g} \partial_I g + 2i(g + \bar{g}) \partial_I (b + \bar{b}) \right), \]
\[ C_{IJ} = -\frac{1}{g + \bar{g}} \left( i \phi \bar{g} + 2(b + \bar{b}) \right) \partial_I \partial_J (Xb) - \frac{1}{8} i \partial_I \partial_J (X^{-1} f), \]
\[ G_I = \frac{1}{g + \bar{g}} \Re \left\{ \phi \bar{g} X \partial_I (\phi g + 4ib) - 2i(b + \bar{b}) \partial_I [X(\phi g + 4ib)] \right\}. \]  

(4.13)

As above, for the sake of clarity we have absorbed the parameters \( \alpha \) and \( \alpha_A \) into the functions \( b \) and \( g \) in the manner described immediately after equation (4.5). Substituting these functions into the Lagrangian (4.11), one obtains a Lagrangian that contains, besides the total derivative terms and a part that depends exclusively on the background mentioned above, a quadratic part and a linear part in vector-tensor fields.

5 Dual versions of vector-tensor actions

As we already mentioned in the introduction, a vector-tensor multiplet is classically equivalent to a vector multiplet. The theory which we have presented, involving one vector-tensor multiplet and \( n \) vector multiplets is classically equivalent to a theory involving \( n + 1 \) vector multiplets. Since these latter theories are well understood, it is of interest to determine what subset of vector multiplet theories are classically equivalent to vector-tensor theories. Furthermore, low-energy effective string Lagrangians with \( N = 2 \) supersymmetry are usually described in terms of vector multiplets, such that by going to the vector multiplet language one can more easily verify which string theories are described by the vector-tensor multiplets we constructed above. A significant restriction along these lines has to do with the Kähler spaces on which the scalar fields of the theory may live. In the case of \( N = 2 \) vector multiplets these consist of “special Kähler” spaces, and the associated geometry is known as special geometry. For the case of effective Lagrangians corresponding to heterotic \( N = 2 \) supersymmetric string compactifications, this space must contain, at least at weak string coupling, an SU(1,1)/U(1) coset factor parametrized in terms of the complex scalar corresponding to the axion/dilaton complex. According to a well-known theorem \([27]\) this uniquely specifies the special Kähler space.

Perhaps not too surprisingly, the observations made in \([16]\) are not altered by going to local supersymmetry. Thus we will find that the vector-tensor multiplets we have been studying in the present article, fail to exhibit the SU(1,1)/U(1) factor, at least if one insists that it is the vector-tensor scalar and tensor field (the latter after a duality transformation, to be discussed below) that parametrize this subspace. Therefore it is impossible to associate this scalar and the tensor field with the (perturbative) heterotic dilaton-axion complex. However, they do play a natural role in the description of the non-perturbative heterotic string effects we alluded to in the introduction.

One goes about constructing the dual vector multiplet formulation, in the usual manner, by introducing a Lagrange multiplier field \( a \), which, upon integration, enforces the Bianchi identity on the field strength \( H_\mu \). The relevant term to add to the Lagrangian is therefore

\[ e^{-1} \mathcal{L}(a) = a D_\mu H^\mu \]
\[ + \frac{1}{16} i a \left[ \eta_{11} \tilde{F}_{\mu\nu}(V) F^{\mu\nu}(V) + \eta_{1A} \tilde{F}_{\mu\nu}(V) F^{\mu\nuA} + \eta_{AB} \tilde{F}_{\mu\nu} A^{\mu\nuB} + 2 \tilde{F}_{\mu\nu} \tilde{G}^{\mu\nu}(a) \right] \]
\[ + \frac{1}{16} i a \left[ T_{ij}^{\mu\nu} \left( 2 \eta_{11} \phi X F_{\mu\nu}(V) + \eta_A (X^2 A F_{\mu\nu}(V) + i \phi X F_{\mu\nu}^A) + 2 \eta_{AB} X^A F_{\mu\nu}^B \right) \right] \]
\[ +2X \dot{B}^{(z)}_{\mu\nu} + X F_{\mu
u}^0 (\eta_{11} \phi^2 - \phi g - 4i\bar{b}) - \text{h.c.} \]. \tag{5.1} \\

Note that we dropped the explicit fermionic terms, as we will do in the remainder of this section. Including the Lagrange multiplier term, we treat \( H_{\mu} \) as unconstrained and integrate it out in the action, thereby trading the single on-shell degree of freedom represented by \( B_{\mu\nu} \) for the real scalar \( a \). Doing this, we obtain a dual theory involving only vector multiplets. To perform these operations, it is instructive to note that all occurrences of \( H_{\mu} \) in \([4.11]\) and \([5.1]\) are most conveniently written in terms of \( V_{\mu}^{(a)} \), which can be done using \([3.13]\). Because we are suppressing the fermions in what follows, we will henceforth drop the caret on \( V_{\mu}^{(a)} \). All such terms can then be collected, and written as follows,

\[ \mathcal{L}(V_{\mu}^{(a)}) = \frac{1}{4} e (2\eta_{11} \phi - \text{Re} g) \left( W^0_{\mu\nu} W^0_{\mu\nu} - (W^0_\lambda W^0_{\mu\lambda} + 4|X|^2 g^{\mu\nu}) \left( V_{\mu}^{(a)} V_{\nu}^{(a)} - 2V_{\mu}^{(a)} \partial_\nu (a - \zeta) \right) , \right. \]
\[ \left. \text{where } \zeta \text{ was defined in } [4.3]. \right. \tag{5.2} \]

It is interesting how the terms involving \( V_{\mu}^{(a)} \) factorize into the form given in \([5.2]\). The equation of motion for \( H_{\mu} \) is conveniently written in terms of \( V_{\mu}^{(a)} \), which follows immediately from \([5.2]\). It is given by the following simple expression,

\[ V_{\mu}^{(a)} = \partial_\mu (a - \zeta). \tag{5.3} \]

We also impose the equations of motion for the auxiliary fields, \( \phi^{(a)} = Y_{ij} I = 0 \) (up to fermionic terms). After substituting these solutions, we manipulate the result into the familiar form for the bosonic Lagrangian involving vector multiplets,

\[ e^{-1} \mathcal{L} = \frac{1}{2} i (F_I X^I - X^I \dot{F}_I) \left( -\frac{1}{6} R + \mathcal{D} \right) + \frac{1}{2} i (\mathcal{D}_\mu F_I \mathcal{D}^\mu X^I - \mathcal{D}_\mu X^I \mathcal{D}^\mu \dot{F}_I) \]
\[ - \frac{1}{8} i F_{IJK} F^{+I} \mathcal{F}^{J+\mu
u} \mathcal{F}^{+\mu
u J} \frac{1}{16} i (F_I - X^I \dot{F}_I) T_{\mu
u}^I \varepsilon_{ij} \]
\[ + \frac{1}{128} i (F_I - X^I \dot{F}_I) X^I \left( T_{\mu
u ij} \varepsilon^{ij} \right)^2 + \text{h.c.}, \tag{5.4} \]

characterized by a holomorphic function \( F(X^0, X^1, X^A) \), which is homogeneous of degree two. Here the field strengths are equal to \( F_{\mu\nu} = 2\partial_{[\mu} W_{\nu]} - g_{IJK} W^I_{\mu} W^K_{\nu} \). In \([5.4]\), a subscript \( I \) denotes differentiation with respect to \( X^I \). The natural bosons in the dual theory are found to be

\[ X^1 = X^0 (a - \zeta + i\phi) , \]
\[ W^I_{\mu} = V_{\mu} + (a - \zeta) W^0_{\mu} , \tag{5.5} \]

and one can check that these transform as components of a common vector multiplet. For the general case, the dual theory obtained in this manner is described by the following holomorphic prepotential,

\[ F(X^0, X^1, X^A) = -\frac{1}{X^0} \left( \frac{1}{2} \eta_{11} X^1 X^1 X^1 + \frac{1}{2} \eta_{1A} X^1 X^1 X^A + \eta_{AB} X^1 X^A X^B \right) \]
\[ -\alpha X^1 X^1 + \alpha_A X^1 X^A + f(X^0, X^A) . \tag{5.6} \]

The quadratic terms proportional to \( \alpha \) and \( \alpha_A \) (defined in section 4.1) give rise to total derivatives since their coefficients are real. The term involving the function \( f(X^0, X^A) \) represents the self-interactions of the background vector multiplets. The first three terms in \([5.6]\) encode the couplings of the erstwhile vector-tensor fields, \( \phi \) and \( a \), and it is these which we are most interested in. As mentioned above, it is relevant to investigate whether the Kähler space described by this prepotential function can contain an \( \text{SU}(1,1)/\text{U}(1) \) factor parametrized by the field \( X^1/X^0 \).
According to the theorem of [27], this requires that \(X^1/X^0\) appears linearly in the prepotential. This is obviously not the case for (5.6), as we have quadratic and cubic terms which cannot be removed by absorbing some of the other fields into the would-be dilaton field \(X^1/X^0\). As discussed earlier in this paper, the best one can do is to remove either \(\eta_{11}\) or \(\eta_{1A}\). There exists an obstruction to removing both of these. We recall that these parameters are related to the Chern-Simons couplings of the tensor field in the dual formulation. The obstruction to removing the unwanted terms in the prepotential derives from the inability to formulate an interacting off-shell vector-tensor theory without any such Chern-Simons couplings.

In the present supergravity context it is important to note that the duality transformation we just described, does not interfere with the fields of the Weyl multiplet. This can be seen by nothing that (5.2), (5.3) and (5.5) are completely identical to the relations found in [16] in the rigid supersymmetric case. This implies that the Weyl multiplet is not involved in the duality transformation and can be kept off-shell. The vector multiplets are not realized off-shell after the duality transformation, but the auxiliary fields \(Y_{ij}\) can be reinstated afterwards. In this respect it is instructive to compare our results to the analysis performed in [9]. Here the most general vector-multiplet theories admitting a (reverse) dualization into an antisymmetric tensor theory, were considered. They were found to precisely comprise the cases described here, plus the \(\eta_{11} = 0, \eta_{1A} = 0\) case which is relevant for weakly coupled heterotic strings. However, in this last case the dualization into an antisymmetric tensor theory can no longer be carried out with the Weyl multiplet as a spectator. In particular, one is forced to first eliminate the U(1) chiral gauge field \(A_{\mu}\), which in the Poincaré theory plays the role of an auxiliary field.

Irrespective of these considerations, we note that the results we obtained in this article are a concise description of two very different situations. As described in detail in section 3, depending on whether the parameter \(\eta_{11}\) is vanishing or not, indicating the absence or presence, respectively, of a \(V \wedge dV\) Chern-Simons coupling to the tensor field, the theory takes on very distinct characters. It is instructive then, to summarize our results independently for each of these two cases.

For the nonlinear vector-tensor multiplet, we obtain a dual description involving only vector multiplets, characterized by the following holomorphic prepotential,

\[
F = -\frac{X^1}{X^0} \left( \frac{\eta_{11}X^1 X^1 + \eta_{AB}X^A X^B}{4} \right) - \alpha X^1 X^1 + \alpha_A X^1 X^A + f(X^0, X^A). \tag{5.7}
\]

As already mentioned above, the quadratic terms proportional to \(\alpha\) and \(\alpha_A\) represent total derivatives, and the last term involves the background self-interactions. Notice that in this case the prepotential is cubic in \(X^1\). No higher-dimensional tensor theory is known that gives rise to this coupling.

For the linear vector-tensor multiplet the dual description in terms of only vector multiplets is characterized by the following prepotential,

\[
F = -\frac{X^1}{X^0} \left( \frac{\eta_{1A}X^1 X^A + \eta_{AB}X^A X^B}{2} \right) - \alpha X^1 X^1 + \alpha_A X^1 X^A + f(X^0, X^A). \tag{5.8}
\]

Again, as discussed above, the quadratic terms involving \(\alpha\) and \(\alpha_A\) represent total derivatives, while the last term involves the background self-interactions. Notice that in this case the prepotential has a term quadratic in \(X^1\), which cannot be suppressed. Such a term also arises from the reduction of six-dimensional tensor multiplets to four dimensions. In that case, the presence of the quadratic term is inevitable, because it originates from the kinetic term of the tensor field [13]. Observe that we have at least three abelian vector fields coupling to the vector-tensor multiplet, namely \(W^0_\mu, W^1_\mu\) and \(\eta_{1A}W^A_\mu\).
The work presented in this paper represents an exhaustive analysis of the $N = 2$ vector-tensor multiplet coupled to supergravity and a number of background vector multiplets. One of these vector multiplets provides the gauge field that couples to the central charge. Although we considered only a single vector-tensor multiplet, our methods can be straightforwardly applied to theories where several of these multiplets are present. We have presented the complete and general superconformal transformation rules in this context, and have shown that these actually include two distinct cases, one of which is nonlinear in the vector-tensor components, and the other of which is linear. The difference between these two cases is encoded in the coefficients of the Chern-Simons couplings, denoted by $\eta_{IJ}$. Furthermore we have constructed a supersymmetric action for this system, and exhibited its bosonic part. The dual descriptions in terms of vector multiplets have been obtained, and the respective prepotential functions exhibited.

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A Conventions and definitions

Throughout the article we use $\mu, \nu, \cdots = 0, 1, 2, 3$ to denote curved indices, and $a, b, \cdots = 0, 1, 2, 3$ for local Lorentz indices. Our (anti)symmetrizations are always with weight one, so e.g.

$$[ab] = \frac{1}{2}(ab - ba), \quad (A.1)$$

We take

$$\gamma_{ab} = \eta_{ab} + 2\sigma_{ab}, \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \quad (A.2)$$

where $\eta_{ab}$ is of signature $(- + + +)$. The complete antisymmetric tensor satisfies

$$\varepsilon^{abcd} = e^{-1}\varepsilon^{\mu\nu\lambda\sigma} e_{\mu}^a e_{\nu}^b e_{\lambda}^c e_{\sigma}^d, \quad \varepsilon^{0123} = i, \quad (A.3)$$

which implies

$$\sigma_{ab} = -\frac{1}{2}\varepsilon_{abcd}\sigma^{cd}\gamma_5. \quad (A.4)$$

The dual of an antisymmetric tensor field $F_{ab}$ is given by

$$\tilde{F}_{ab} = \frac{1}{2}\varepsilon_{abcd}F^{cd}. \quad (A.5)$$

and the (anti)selfdual part of $F_{ab}$ reads

$$F_{ab}^{\pm} = \frac{1}{2}(F_{ab} \pm \tilde{F}_{ab}). \quad (A.6)$$

Note that under hermitian conjugation (h.c.) selfdual becomes antiselfdual and vice-versa. Any SU(2) index $i$ or any quaternionic index $\alpha$ changes position under h.c., for instance

$$(T_{ab}^{ij})^* = T_{ab}^{ij}, \quad (A_{\alpha}^i)^* = A_{\alpha}^i. \quad (A.7)$$

The superconformal algebra consists of general coordinate, local Lorentz, dilatation, special conformal, chiral U(1) and SU(2), and $Q$- and $S$-supersymmetry transformations. When vector and/or vector-tensor multiplets are present additional gauge symmetries must be included. A covariant general coordinate transformation is defined as follows

$$\delta^{(cov)}(\xi) = \delta_{\text{gct}}(\xi) + \sum_A \delta_A(-\xi^\mu h_\mu(A)), \quad (A.8)$$

where the sum is over all superconformal (except the g.c.t) and additional gauge transformations, each with parameter $-\xi^\mu h_\mu(A)$, where $h_\mu(A)$ is the gauge field associated with $\delta_A$. The superconformal gauge fields are normalized as in [18].

$$h_\mu^{ab}(M) = \omega_\mu^{ab}, \quad h_\mu^i(U(1)) = A_\mu^i, \quad h_\mu^i(SU(2)) = -\frac{1}{2}V_\mu^i, \quad (A.9)$$

$$h_\mu^a(Q) = \frac{1}{a}\psi_\mu^a, \quad h_\mu^i(S) = \frac{1}{2}\phi_\mu^i,$$

and

$$h_\mu(\text{gauge}) = W_\mu^I, V_\mu, B_{\mu\nu}. \quad (A.10)$$

The symbol $D_\mu$ denotes a fully covariant derivative and is defined as

$$D_\mu = \partial_\mu - \sum_A \delta_A(h_\mu(A)). \quad (A.11)$$
We use $D_{\mu}$ to denote a covariant derivative with respect to $M$, $D$, $U(1)$, $SU(2)$ and gauge transformations.

The composite gauge fields $\omega_{\mu}^{ab}$, $\phi_{\mu}^{i}$ and $f_{\mu}^{a}$ contained in the Weyl multiplet, are given by

$$
\omega_{\mu}^{ab} = -2\epsilon^{\nu[a\partial_{\mu}e_{\nu}]} - e^{\nu[a\partial_{\mu}e_{\nu}]}e_{\mu}e_{\nu}e_{\mu} - 2\epsilon_{\mu}^{[a\partial_{\nu}e_{\mu}]} e_{\nu} - \frac{1}{4} (\bar{\psi}^{i}_{\mu} \gamma_{[a} \psi^{b]}_{\mu} + \bar{\psi}^{i}_{\mu} \gamma_{[a} \psi^{b]}_{\mu} + h.c.) ,
$$

$$
\phi_{\mu}^{i} = (\sigma_{\rho\sigma} \gamma_{\mu} - \frac{1}{4} \sigma_{\rho\sigma} \sigma^{\rho\sigma}) (D_{\mu} \psi^{i})_{\rho} - \frac{1}{4} \sigma_{\rho\sigma} T_{ij} \gamma_{\rho} \psi_{ij} + \frac{1}{2} \sigma_{\rho\sigma} \chi^{i} ,
$$

$$
f_{\mu}^{a} = \frac{1}{6} \mathcal{R} - D - \frac{1}{12} \epsilon^{\nu_{\mu\rho\sigma}} \bar{\psi}^{i}_{\mu} \gamma_{\nu} D_{\rho} \psi_{\sigma} - \frac{1}{12} \bar{\psi}^{i}_{\mu} \gamma_{\nu} T^{\mu\nu} - \frac{1}{4} \bar{\psi}^{i}_{\mu} \gamma^{i} \chi_{i} + h.c.) .
$$

The following supercovariant curvatures appear in the main text,

$$
\hat{R}_{\mu\nu}(Q)^{i} = 2D_{[\mu} \psi_{\nu]}^{i} - \gamma_{[\mu} \phi_{\nu]}^{i} - \frac{1}{4} \sigma \cdot T^{ij} \gamma_{[\mu} \psi_{\nu]}^{j} ,
$$

$$
\hat{R}_{\mu\nu}(U(1)) = 2\partial_{[\mu} A_{\nu]} - i \left( e^{[\mu} \bar{\psi}^{i}_{\rho} \phi_{\nu]}^{i} + \frac{1}{2} \bar{\psi}^{i}_{[\mu} \gamma_{\nu]} \chi_{i} - h.c. \right) ,
$$

$$
\hat{R}_{\mu\nu}(SU(2))^{i} = 2\partial_{[\mu} \psi_{\nu]}^{i} + \psi_{[\mu} \gamma^{i} \psi_{\nu]}^{i}
+ \left( 2\bar{\psi}_{[\mu} \phi_{\nu]}^{i} - 3\bar{\psi}_{[\mu} \gamma_{\nu]} x_{i} - (h.c.; traceless) \right) .
$$

In actual computations one may benefit from using the following relationships

$$
\gamma^{\mu} (\hat{R}_{\mu\nu}(Q)^{i} + \sigma_{\mu} \chi^{i}) = 0 ,
$$

$$
2D_{[\mu} e_{\nu]}^{a} - \bar{\psi}_{[\mu} \gamma^{a} \psi_{\nu]}^{a} = 0 .
$$

and

$$
\begin{align*}
\sigma_{ab} &= -\frac{1}{2} \epsilon_{abcd} \sigma^{cd} \gamma_{5} , \\
\sigma_{ab} \sigma_{ab} &= -3 , \\
\gamma^{c} \sigma_{ab} \gamma_{c} &= 0 , \\
[\gamma^{c}, \sigma_{ab}] &= 2\delta_{[a}^{b]} \gamma_{5} , \\
[\sigma_{ab}, \sigma_{cd}] &= -4 \delta_{[a}^{c} \delta_{b]}^{d} , \\
\sigma_{ab}, \sigma_{cd} &= -\delta_{[a}^{c} \delta_{b]}^{d} + \frac{1}{2} \epsilon_{ab}^{c} \epsilon_{cd}^{a} \gamma_{5} .
\end{align*}
$$

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