A Lattice Chiral Theory with Multifermion Couplings

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Abstract

Analyzing an $SU_L(2) \otimes U_R(1)$ chiral theory with multifermion couplings on a lattice, we find a possible region in the phase space of multifermion couplings, where no spontaneous symmetry breaking occurs, doublers are decoupled as massive Dirac fermions consistently with the $SU_L(2) \otimes U_R(1)$ chiral symmetry, the “spectator” fermion $\psi_R(x)$ is free mode, whereas the normal mode of $\psi_L(x)$ is plausibly speculated to be chiral in the continuum limit. This is not in agreement with the general belief of the definite failure of theories so constructed.

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1. It is a long standing problem to regularize chiral gauge theories on a lattice and it seems that none of the methods proposed has been consistently and completely demonstrated both to ensure that an asymptotically chiral gauge theory in the continuum limit really exists and to provide a framework for doing non-perturbative calculation in these theories [1]. It is generally believed that the constructions [2, 3] of chiral gauge theories on the lattice with external multifermion couplings fail to give chiral gauged fermions in the continuum limit for the reason [4] that the theories so constructed undergo spontaneous symmetry breaking and their phase structure is similar to that of the Smit-Swift model [5], which has been very carefully studied and shown to fail. Nevertheless, we believe that further considerations of constructing CGT on a lattice with external multifermion couplings and careful studies of the spectrum in each phase of such a constructed theory are necessary. In fact, we find a possible scaling region of defining continuum chiral fermion in such a formulation of chiral gauge theories on the lattice.

Let us consider the following fermion action of the $SU_L(2)$ CGT on a lattice with two external multifermion couplings.

\[
S = \frac{1}{2a} \sum_x \left( \bar{\psi}_{L}^i(x) \gamma_{\mu} D_{\mu}^i \psi_{L}^i(x) + \bar{\psi}_{R}(x) \gamma_{\mu} \partial_{\mu} \psi_{R}(x) \right) + \sum_x \left( g_1 \bar{\psi}_{L}^i(x) \cdot \psi_{R}(x) \psi_{L}^i(x) + g_2 \bar{\psi}_{L}^i(x) \cdot \partial^2 \psi_{R}(x) \partial^2 \psi_{L}^i(x) \right),
\]

(1)

where “a” is the lattice spacing; $\psi_{L}^i$ ($i = 1, 2$) is an $SU_L(2)$ gauged doublet, $\psi_{R}$ is an $SU_L(2)$ singlet and both are two-component Weyl fermions. The $\psi_{R}$ is treated as a “spectator” fermion. The second multifermion coupling $g_2$, where

\[
\partial^2 \psi_{R}(x) = \sum_{\mu} \left[ \psi_{R}(x + \mu) + \psi_{R}(x - \mu) - 2\psi_{R}(x) \right],
\]

(2)

is a dimension-10 operator relevant only for doublers $p = \tilde{p} + \pi_A \not= 0$ but irrelevant for normal modes $p = \tilde{p}$ of the $\psi_{L}^i$ and $\psi_{R}$. In addition to the exact local $SU_L(2)$ chiral gauge symmetry and the global chiral symmetry $SU_L(2) \otimes U_R(1)$, the action (1) possesses a $\psi_{R}$-shift-symmetry [6],

\[
\psi_{R}(x) \rightarrow \psi_{R}(x) + \text{const.},
\]

(3)

when $g_1 = 0$. The chiral gauge interaction is supposed to be perturbative, and we turn the gauge coupling off in the following discussions ($g = 0$).

We consider the generating function $W(\eta)$,

\[
W(\eta) = -\ellnZ(\eta),
\]

\[
Z(\eta) = \int [d\psi_{L}^i d\psi_{R}] \exp \left( -S + \int_x \left( \bar{\psi}_{L}^i \eta_{L}^i + \bar{\eta}_{L}^i \psi_{L}^i + \bar{\psi}_{R} \eta_{R} + \bar{\eta}_{R} \psi_{R} \right) \right).
\]

(4)

\[\text{The physical momentum } \tilde{p} \simeq 0 \text{ and } \pi_A \text{ runs over fifteen lattice momenta } \pi_A \not= 0.\]
Then, we define the generating functional of one-particle irreducible vertices (the effective action $\Gamma(\psi_L^i, \psi_R^i)$) as the Legendre transform of $W(\eta)$

$$\Gamma(\psi_L^i, \psi_R^i) = W(\eta) - \int_x \left( \bar{\psi}_L^i \eta_L^i + \bar{\eta}_L^i \psi_L^i + \bar{\psi}_R^i \eta_R + \bar{\eta}_R \psi_R^i \right),$$

and with the relations

$$\psi_L^i(x) = \langle \bar{\psi}_L^i(x) \rangle = -\frac{\delta W}{\delta \eta_L^i(x)}, \quad \bar{\psi}_L^i(x) = \langle \bar{\psi}_L^i(x) \rangle = \frac{\delta W}{\delta \eta_L^i(x)},\quad \psi_R^i(x) = \langle \bar{\psi}_R(x) \rangle = -\frac{\delta W}{\delta \eta_R(x)}, \quad \bar{\psi}_R(x) = \langle \bar{\psi}_R(x) \rangle = \frac{\delta W}{\delta \eta_R(x)},$$

in which the fermionic derivatives are left-derivatives, and

$$\eta_L^i(x) = -\frac{\delta \Gamma}{\delta \psi_L^i(x)}, \quad \bar{\eta}_L^i(x) = \frac{\delta \Gamma}{\delta \bar{\psi}_L^i(x)};$$

$$\eta_R(x) = -\frac{\delta \Gamma}{\delta \psi_R(x)}, \quad \bar{\eta}_R(x) = \frac{\delta \Gamma}{\delta \bar{\psi}_R(x)}.$$  

In eqs. (6-7), the $\langle \cdots \rangle$ indicates an expectation value with respect to the partition functional $Z(\eta)$ (3).

We first derive the local Ward identity associated with the $\psi_R^i$-shift-symmetry. Making the parameter $\epsilon$ to be spacetime dependent, and varying the generating function (4) according to the transformation rules (3) for arbitrary $\epsilon(x) \neq 0$, we arrive at

$$\langle \frac{1}{2a} \gamma_{\mu} \partial^\mu \psi_R(x) + g_1 \bar{\psi}_L^i(x) \cdot \psi_R(x) \psi_L^i(x) + g_2 \partial^2 \left( \bar{\psi}_L^i(x) \cdot \partial^2 \psi_R(x) \psi_L^i(x) \right) + \eta_R(x) \rangle = 0.$$  

Substituting (8) into eq. (9), we obtain the Ward identity corresponding to the $\psi_R^i$-shift-symmetry of the action (11):

$$\frac{1}{2a} \gamma_{\mu} \partial^\mu \psi_R^i(x) + g_1 \langle \bar{\psi}_L^i(x) \cdot \psi_R(x) \psi_L^i(x) \rangle + g_2 \langle \partial^2 \left( \bar{\psi}_L^i(x) \cdot \partial^2 \psi_R(x) \psi_L^i(x) \right) \rangle - \frac{\delta \Gamma}{\delta \psi_R^i(x)} = 0.$$  

Based on this Ward identity (10), one can get all one-particle irreducible vertices $\Gamma_R^{(a)}$ containing at least one external $\psi_R$.

Taking functional derivatives of eq. (10) with respect to appropriate “prime” fields (22) and then putting external sources $\eta = 0$, one can derive:

$$\int_x e^{-ipx} \frac{\delta(2) \Gamma}{\delta \psi_R^i(x) \delta \psi_R^i(0)} = \frac{i}{a} \gamma_{\mu} \sin(p^\mu a),$$

$$\int_x e^{-ipx} \frac{\delta(2) \Gamma}{\delta \psi_L^i(x) \delta \psi_R^i(0)} = \frac{1}{2} \Sigma^i(p) = g_1 \langle \bar{\psi}_L^i(0) \cdot \psi_R(0) \rangle + 2g_2 \langle \bar{\psi}_L^i(0) \cdot \partial^2 \psi_R(0) \rangle \sigma.$$
where the $\langle \cdots \rangle_\circ$ indicates an expectation value with respect to the partition functional $Z(\eta)$ without external sources ($\eta = 0$). In addition, one can derive the four-point vertex,

$$
\int_{xyz} e^{-iyq - ixp - ip'q} \frac{\delta^{(4)} \Gamma}{\delta \bar{\psi}_L^\prime(0) \delta \psi_R^\prime(y) \delta \psi_R^\prime(z) \delta \psi_R^\prime(x)} = g_1 + 4g_2 w(p + q/2)w(p' + q/2),
$$

where $p + q/2$ and $p' + q/2$ are momenta of the $\psi_R$ field. In eqs.(12,13), $w(p)$ is the well-known Wilson factor

$$
w(p) = \sum_\mu (1 - \cos(p_\mu)),
$$

and all momenta are scaled to be dimensionless. All other one-particle irreducible vertices $\Gamma^{(n)}_R = 0 (n > 4)$ identically. When $g_1 = 0$, we find (i) eq.(11) shows that the $\psi_R(x)$ is free field; (ii) eq.(13) for the normal mode of the $\psi_R$ are vanishing at least $O((ma)^2)$, where $m$ is the scale of the continuum limit. This may indicate that when $g_1 = 0$, the normal mode of the $\psi_R$ completely decouples and does not form any bound states with other modes.

2. Our goal is to seek a possible regime, where an undoubled $SU_L(2)$-chiral gauged fermion content is exhibited in the continuum limit in the phase space $(g_1, g_2, g)$, where “$g$” is the gauge coupling, regarded to be a truly small perturbation $g \to 0$ at the scale of the continuum limit we consider. In the weak coupling limit, $g_1 \ll 1$ and $g_2 \ll 1$ (indicated 1 in fig.1), the action (1) defines an $SU_L(2) \otimes U_R(1)$ chiral continuum theory with a doubled and weakly interacting fermion spectrum that is not the continuum theory we seek.

Let us consider the phase of spontaneous symmetry breaking in the weak-coupling $g_1, g_2$ limit. Based on the analysis of large-$N_f$ ($N_f$ is an extra fermion index, e.g., color, $N_c$) weak coupling expansion, we show that the multifermion couplings in the action (1) undergo Nambu-Jona Lasinio (NJL) spontaneous chiral-symmetry breaking[7]. In this symmetry breaking phase (indicated 2 in fig.1), the $SU_L(2) \otimes U_R(1)$-chiral symmetry is violated by

$$
\frac{1}{2} \Sigma^i(p) = g_1 \int d^4 x e^{-ixp} \langle \tilde{\psi}_L^i(0) \cdot \psi_R(x) \rangle_\circ \neq 0.
$$

Assuming that the symmetry breaking takes place in the direction 1 in the 2-dimensional space of the $SU_L(2)$-chiral symmetry ($\Sigma^1(p) \neq 0, \Sigma^2(p) = 0$), one finds the following fermion spectrum that contains a doubled Weyl fermion $\psi^\prime_2(x)$ and an undoubled Dirac fermion made by the Weyl fermions $\psi^\prime_1(x)$ and $\psi_R(x)$. The propagators of these fermions can be written as,

$$
S_{b1}^{-1}(p) = \frac{i}{a} \sum_\mu \gamma_\mu \sin p_\mu Z_2(p) P_L + \frac{i}{a} \sum_\mu \gamma_\mu \sin p_\mu P_R + \Sigma^1(p)
$$

and

$$
S_{b2}^{-1}(p) = \frac{i}{a} \sum_\mu \gamma_\mu \sin p_\mu Z_2(p) P_L.
$$
The $SU_L(2) \otimes U_R(1)$ chiral symmetry is realized to be $U_L(1) \otimes U(1)$ with three Goldstone modes and a massive Higgs mode that are not presented in this short report.

Owing to the four-fermion interaction vertex (13), the fermion self-energy function $\Sigma^1(p)$ in eqs.(12) and (13) is given by the NJL gap-equation in the large-$N_f$ weak coupling expansion ($N_f \to \infty$)

$$\Sigma^1(p) = 4 \int_q \frac{\Sigma^{1}(q)}{\text{den}(q)} (\tilde{g}_1 + 4\tilde{g}_2 w(p) w(q))$$  \hspace{1cm} (18)

where

$$\int_q = \int_\pi \frac{d^4 q}{(2\pi)^4}$$

$$\text{den}(q) \equiv \sum_\rho \sin^2 q_\rho + (\Sigma^1(q)a)^2$$

$$\tilde{g}_1 \equiv g_1 N_f a^2, \hspace{0.5cm} \tilde{g}_2 \equiv g_2 N_f a^2.$$  

We adopt the paramatrization [4]

$$\Sigma^1(p) = \Sigma^1(0) + \tilde{g}_2 v^1 w(p), \hspace{0.5cm} \Sigma^1(0) = \rho v^1,$$  \hspace{1cm} (19)

where $\rho$ depends only on couplings $\tilde{g}_1, \tilde{g}_2$, and $v^1$ plays a role as the v.e.v. violating $SU_L(2) \otimes U_R(1)$-chiral symmetry. We can solve the gap-equation (13) by using this paramatrization (19). For $v^1 = O(\frac{1}{a})$, one obtains

$$\rho = \frac{\tilde{g}_1 \tilde{g}_2 I_1}{1 - \tilde{g}_1 I_0}, \hspace{0.5cm} \rho = \frac{1 - 4\tilde{g}_2 I_2}{4 I_1},$$  \hspace{1cm} (20)

where the functions $I_n(v^1), (n = 0, 1, 2)$, are defined as

$$I_n(v^1) = 4 \int_q \frac{w^n(q)}{\sum_\rho \sin^2 q_\rho + (\Sigma^1(q)a)^2}.$$  \hspace{1cm} (21)

Eq.(20) leads to a crucial result:

$$\tilde{g}_1 = 0, \hspace{0.5cm} \rho = 0 \hspace{0.5cm} and \hspace{0.5cm} \Sigma^1(0) = 0,$$  \hspace{1cm} (22)

this is due to eq.(12) resulted from the Ward identity (10). This means that on the line $g_1 = 0$, normal modes ($p = \tilde{p} \simeq 0$) of the $\psi^1_L$ and $\psi^1_R$ are massless and their 15 doublers $p = \tilde{p} + \pi_A$ acquire chiral-variant masses

$$\Sigma^1(p) = \tilde{g}_2 v^1 w(p)$$  \hspace{1cm} (23)

through the multifermion coupling $g_2$ only. In this case ($g_1 = 0$), the gap-equation is then given by eq.(20) for $\rho = 0$,

$$1 - 4\tilde{g}_2 I_2(v^1) = 0, \hspace{0.5cm} i.e. \hspace{0.5cm} 1 = 16\tilde{g}_2 \int_q \frac{w^2(q)}{\sum_\rho \sin^2 q_\rho + (\tilde{g}_2 v^1 w(q)a)^2}.$$  \hspace{1cm} (24)
As \( v^1 \to 0 \), eq.(20) gives a critical line \( \tilde{g}_c^1(\tilde{g}_c^2) \)
\[
\tilde{g}_c^1 = \frac{1 - 4\tilde{g}_2^2I_2(0)}{4\tilde{g}_2^2I_1(0) + I_2(0) - 4\tilde{g}_2^2I_2(0)I_2(0)},
\]
of characterizing NJL spontaneous chiral symmetry breaking. With \( I_2(0) = 2.48, I_1(0) = 4I_0(0) \) and \( I_2(0) = 20I_0(0) - 4 \), the critical points are given by:
\[
\tilde{g}_c^1 = 0.4, \quad \tilde{g}_c^2 = 0; \quad \tilde{g}_c^1 = 0, \quad \tilde{g}_c^2 = 0.0055,
\]
as indicated in fig.1. These critical values are sufficiently small even for \( N_f = 1 \).

As for the wave function renormalization \( Z_2(p) \) in eqs.(16,17), it depends on the dynamics of the left-handed Weyl fermion \( \psi^i_L \) in this region. In large-\( N_f \) calculation at weak couplings, we are able to evaluate this function \( Z_2(p) \). The result is not presented in this short report.

This broken phase cannot be a candidate for a real chiral gauge theory (e.g., the Standard Model) for the reasons that (i) \( \psi^2_L \) is doubled (17); (ii) the spontaneous symmetry breakdown of the \( SU_L(2) \)-chiral symmetry is caused by the hard breaking Wilson term (8, 16) (dimension-5 operator), which must contribute the intermediate gauge boson masses through the perturbative gauge interaction and disposal of Goldstone modes. The intermediate gauge boson masses turn out to be \( O(\frac{1}{a}) \). This, however, is phenomenologically unacceptable.

3. We turn to the strong coupling region, where \( g_1(g_2) \) are sufficiently larger than certain critical values:
\[
g_1(g_2) \gg g_c^i(g_c^2) \tag{27}
\]
(indicated 3 in fig.1). Analogously to the analysis and discussions of Eichten and Preskill (EP) [4], we can show that the \( \psi^i_L \) and \( \psi^i_R \) in (4) are bound up to form the composite Weyl fermions (three-fermion bound states)
\[
\Psi^i_L = \frac{1}{2a}(\bar{\psi}^i_L \cdot \psi^i_R)\psi^i_L \tag{28}
\]
(left-handed \( SU_L(2) \)-neutral) and
\[
\Psi^i_R = \frac{1}{2a}(\bar{\psi}^i_R \cdot \psi^i_L)\psi^i_R \tag{29}
\]
(right-handed \( SU_L(2) \)-charged). These three-fermion bound states respectively pair up with the \( \bar{\psi}_R \) and \( \bar{\psi}^i_L \) to be massive, neutral \( \Psi_n \) and charged \( \Psi^i_c \) Dirac modes
\[
\Psi_n = (\Psi^i_n, \psi_R); \quad \Psi^i_c = (\psi^i_L, \Psi^i_R), \tag{30}
\]
consistently with the \( SU_L(2) \otimes U_R(1) \) chiral symmetry. The propagators of these Dirac fermions are given by
\[
\langle \Psi^i_c(0)\bar{\Psi}^i_c(x) \rangle = \langle \psi^i_L(0)\bar{\psi}^i_L(x) \rangle + \langle \psi^i_R(0)\bar{\psi}^i_R(x) \rangle + \langle \psi^i_L(0)\bar{\psi}_R(x) \rangle + \langle \bar{\psi}^i_R(0)\bar{\psi}^i_L(x) \rangle, \tag{31}
\]
\[\text{I thank Y. Shamir for discussions on these propagators.}\]
and
\[
\langle \Psi_n(0) \bar{\Psi}_n(x) \rangle = \langle \Psi^n_L(0) \bar{\Psi}^n_L(x) \rangle + \langle \Psi^n_L(0) \bar{\Psi}^n_R(x) \rangle + \langle \psi_R(0) \bar{\Psi}^n_L(x) \rangle + \langle \psi_R(0) \bar{\Psi}^n_R(x) \rangle,
\]
which we need to compute. These fermions are, in general, massive. But, \textit{a priori}, we cannot exclude the possibility of massless composite modes. This is, as we will see, not what we desire.

In order to compute these fermion propagators, we use strong multifermion coupling,
\[
g_1 \gg 1, \quad g_2 = 0,
\]
expansion in the powers of \( \frac{1}{g_1} \). We obtain the following recursion relations [9] in the lowest nontrivial order,
\[
S_{ij}^{LL}(x) = \frac{1}{g_1} \left( \frac{1}{2a} \right)^3 \sum_\mu S_{ML}^{ij}(x + \mu) \gamma_\mu,
\]
\[
S_{ij}^{ML}(x) = \frac{\delta(x)\delta_{ij}}{2g_1} + \frac{1}{g_1} \left( \frac{1}{2a} \right) \sum_\mu S_{LL}^{ij}(x + \mu) \gamma_\mu,
\]
\[
S_{ij}^{MM}(x) = \frac{1}{g_1} \left( \frac{1}{2a} \right) \sum_\mu \gamma_\mu \gamma_\circ S_{ML}^{ij}(x + \mu) \gamma_\circ,
\]
where definition
\[
\sum_\mu f(x) \equiv \sum_\mu \left( f(x + \mu) - f(x - \mu) \right),
\]
and two-point functions are
\[
S_{ij}^{LL}(x) \equiv \langle \bar{\psi}_i^L(0) \psi_j^L(x) \rangle, \tag{38}
\]
\[
S_{ij}^{ML}(x) \equiv \langle \bar{\psi}_i^L(0)[\bar{\psi}_j^R(x) \cdot \psi_R(x)] \psi_R(x) \rangle, \tag{39}
\]
\[
S_{ij}^{MM}(x) \equiv \langle [\bar{\psi}_i^R(0) \cdot \bar{\psi}_j^L(0)] \psi_R(0), [\bar{\psi}_i^R(x) \cdot \psi_R(x)] \psi_R(x) \rangle. \tag{40}
\]
in the propagator of charged Dirac fermion (31). As for the neutral fermion propagator (32), results are analogous to (34,35, 36). Thus, we calculate the propagators of neutral and charged Dirac modes to be
\[
S_n(p) = \frac{i}{a} \sum_\mu \gamma_\mu \sin p_\mu + M_1 \frac{1}{12a^2} \sum_\rho \sin^2 p_\rho + M_1^2,
\]
\[
S_c(p)_{ij} = \delta_{ij} \frac{i}{a} \sum_\mu \gamma_\mu \sin p_\mu + M_1 \frac{1}{12a^2} \sum_\rho \sin^2 p_\rho + M_1^2, \tag{42}
\]
\[
M_1 = 2ag_1. \tag{43}
\]
This spectrum, which consists of sixteen modes of neutral Dirac fermion and sixteen modes of charged Dirac fermion, is massive (degenerate) and vector-like consistently with the \( SU_L(2) \otimes U_R(1) \) chiral symmetry.
Similar strong coupling expansion in the powers of \((1/g_2)\) can be performed in the case that
\[ g_2 \gg 1, \quad g_1 = 0. \] (44)
We obtain the results that are analogous to eqs. (41) and (42),
\[ S_n(p) = i a \sum \gamma_\mu \sin p_\mu + M_2(p), \] (45)
\[ S_c(p)_{ij} = \delta_{ij} i a \sum \gamma_\mu \sin p_\mu + M_2(p), \] (46)
\[ M_2(p) = 8a g_2 w(p) \quad p \neq \tilde{p}. \] (47)
Instead of eq. (43), the chiral-invariant masses \(M_2(p)\) (47) of doublers are not degenerate. It is very important to note that these equations (45,46) are not valid for the normal modes \(p = \tilde{p}\). This spectrum, which consists of fifteen doublers of neutral Dirac fermion and fifteen doublers of charged Dirac fermion, is massive and vector-like consistently with the \(SU_L(2) \otimes U_R(1)\) chiral symmetry. As for the normal modes \((p = \tilde{p})\) of these composite Dirac fermions, their propagators in the strong coupling region are to our knowledge still lacking. On the basis of the following discussions in next section, we might expect that these normal modes of three-fermion bound states (28) and (29) have not been bound yet and thus the spectrum of normal modes is chiral, provided the multifermion couplings (13) are momentum-dependent and not strong enough in a certain region of the phase diagram.

4. The critical value \(g_1^c(g_2^c)\) (27) that we have mentioned in the beginning of section 3 can be determined by considering the propagator \(G^{ij}(q)\) of a complex composite field \(A^i\),
\[ G^{ij}(\tilde{q}) = \int d^4x e^{-i\tilde{q}x} \langle A^i(0)A^{ij}(x) \rangle_\circ, \quad A^i = \bar{\psi}_R \cdot \psi_L. \] (48)
The real and imaginary parts of \(A^i(x)\) are four composite scalars \((i = 1, 2)\),
\[ A_1^i = \frac{1}{2}(\bar{\psi}_R \cdot \psi_L + \bar{\psi}_L \cdot \psi_R^i) \]
\[ A_2^i = \frac{i}{2}(\bar{\psi}_R \cdot \psi_L - \bar{\psi}_L \cdot \psi_R^i). \]
Again using the strong coupling (33) expansion in the powers of \((1/g_1)\) and we obtain the recursion relation in the lowest nontrivial order,
\[ G^{ij}(\tilde{q}) = \frac{\delta_{ij}}{g_1} + \left(\frac{1}{2a^2}\right) \frac{1}{g_1} \sum_{\pm\mu} \cos \tilde{q}_\mu G^{ij}(\tilde{q}). \] (49)
As a result, we find these four massive composite scalar modes,

\[ G^{ij}(\bar{q}) = \frac{4\delta_{ij}}{a^2 \sum_\mu \sin^2 \frac{q_\mu}{2} + \mu^2}; \quad \mu^2 = 4 \left( g_1 - \frac{2}{a^2} \right), \tag{50} \]

which are degenerate owing to the exact $SU_L(2) \otimes U_R(1)$-chiral symmetry. Thus, $\mu^2 A^i A^{i\dagger}$ gives rise to a quadratic mass term of the composite scalar field $A^i$ in the effective Lagrangian. We assume that the one particle irreducible vertex $A^j A^{j\dagger} A^i A^{i\dagger}$ is positively definite and the energy of ground state is bound from below. A spontaneous symmetry breaking $SU(2) \to U(1)$ occurs, where $\mu^2 > 0$ turns to $\mu^2 < 0$. Eq.(50) for $\mu^2 = 0$ gives rise to the critical point:

\[ g_1^c a^2 = 2, \quad g_2 = 0, \quad (51) \]

(as indicated in Fig.1) where a phase transition takes place between the NJL symmetry breaking phase and the EP symmetric phase.

The second multifermion coupling $4 g_2 w(p + \frac{q}{2}) w(p' + \frac{q}{2})$ in (13) gives different contributions to the effective value of $g_1$ at large distance for sixteen modes of the $\psi^i_L$ and $\psi^i_R$ in the action (1). We should not doubt that the critical lines $g_1^c(g_2^c)$ (the thresholds of forming three-fermion bound states) should depend on sixteen modes of the $\psi^j_L$ and $\psi^j_R$. These critical lines can be qualitatively determined in the following considerations. Substituting the coupling $g_1$ in eq.(51) by the effective coupling (13), one gets

\[ \mu^2 = 4 \left( g_1 + 4g_2 w^2(p) - \frac{2}{a^2} \right). \tag{52} \]

Let us consider the multifermion couplings of each mode “$p$” of the $\psi^i_L$ and $\psi^i_R$, namely, we set $p = p', q = \tilde{q} \ll 1$ in the four-point vertex (13). one gets

\[ \mu^2 = 4 \left( g_1 + 4g_2 w^2(p) - \frac{2}{a^2} \right). \tag{53} \]

Thus, $\mu^2 = 0$ gives rise to the critical lines:

\[ g_1^c a^2 = 2, g_2 = 0; \quad g_1 = 0, a^2 g_2^{c,b} = 0.008, \tag{54} \]

where the first binding threshold of the doubler $p = (\pi, \pi, \pi, \pi)$ is, and

\[ g_1^c a^2 = 2, g_2 = 0; \quad g_1 = 0, a^2 g_2^{c,a} = 0.124, \tag{55} \]

where the last binding threshold of the doubleurs $p = (\pi, 0, 0, 0)$ is. Inbetween (indicated 4 in fig.1) there are the binding thresholds of the doubleurs $p = (\pi, \pi, 0, 0)$ and $p = (\pi, \pi, 0, 0)$ in eq.(53), and the binding thresholds of the different doublers $p \neq p'$ in eq.(52). Above $g_1^c$ all doubleurs are supposed to be bound, as indicated 5 in fig.1. As for the normal modes of the $\psi^j_L$ and $\psi^j_R$, when $g_1 \ll 1$, the multifermion coupling (13), $\Gamma^{(4)} = g_1 + 4g_2 w^2(\bar{p})$, is supposed to be no longer strong
enough to form the bound states \((\bar{\psi}_L^i \cdot \psi_R^i)\psi_L^i\) and \((\bar{\psi}_R^i \cdot \psi_L^i)\psi_R^i\) unless \(a^2 g_2 \to \infty\). It is conceivable that the critical line for normal modes, which is given by eq. (53) for \(\tilde{p} = ma \ll 0\),

\[ g_1 + a g_2 O((ma)^4) - \frac{1}{2a^2} = 0, \tag{56} \]

analytically continues to the limit

\[ g_2^{c,\infty} \to \infty, \quad g_1 \to 0. \tag{57} \]

5. We must confess that the description of momentum dependence of the threshold should not certainly be considered a rigorous demonstration. Nevertheless, we can see, as expected in ref. [2], several wedges open up as \(g_1, g_2\) increase in the NJL phase (indicated 5 in fig.1), inbetween the critical lines along which bound states of normal modes and doublers of the \(\psi_L^i\) and \(\psi_R\) respectively approach their thresholds. In the initial part of the NJL phase, the normal modes and doublers of the \(\psi_L^i\) and \(\psi_R\) undergo the NJL phenomenon and contribute to eqs. (16,17) as discussed in section 2. As \(g_1, g_2\) increase, all these modes, one after another, gradually disassociate from the NJL phenomenon and no longer contribute to eqs. (16,17). Instead, they turn to associate with the EP phenomenon and contribute to eqs. (15,16) and eqs. (11,12). The first and last doublers of the \(\psi_L^i\) and \(\psi_R\) making this transition are \(p = (\pi, \pi, \pi, \pi)\) and \(p = (\pi, 0, 0, 0)\) respectively. At the end of this sequence, normal modes \((p = \tilde{p})\) make this transition, due to the fact that they possess the different effective multifermion coupling \(\Gamma^{(4)} = g_1 + 4g_2w^2(p)\).

Had these critical lines separated the two symmetric phases, (strong couplings and the weak coupling symmetric phases) we would have found a threshold over which all doublers of the \(\psi_L^i\) and \(\psi_R\) decouple by acquiring chiral invariant masses \(17\) and normal modes of the \(\psi_L^i\) and \(\psi_R\) remain massless and free, and we might obtain a theory of massless free chiral fermions [2]. However, this is not real case [1]. As has been seen in eq. (50), turning \(\mu^2 > 0\) to \(\mu^2 < 0\) indicates a phase transition between the strong coupling symmetric phase to the spontaneous chiral symmetry breaking phase, which separates the strong coupling and weak coupling symmetric phases. As indicated in Fig. 1, this can be clearly seen in eq. (26) and eqs. (54).

The possible resolution of this undesired situation is that we can find a region in which the doublers of the \(\psi_L^i\) and \(\psi_R\) have formed bound states \((\bar{\psi}_R \cdot \psi_L^i)\psi_R\) and \((\bar{\psi}_L^i \cdot \psi_R)\psi_L^i\) via the EP phenomenon, while the normal modes of the \(\psi_L^i\) and \(\psi_R\) have neither formed such bound states yet and nor are they associated with the NJL-phenomenon.

Let us try to find whether there is a such resolution. Within the last wedge (indicated 5 in fig.1) between two the thresholds \(g_2^{c,a}\) and \(g_2^{c,\infty}\), all doublers of the \(\psi_L^i\) and \(\psi_R\) are bound to be Dirac fermions that acquire chiral-invariant masses
and decouple (considering that eqs. (45,46) are the propagators for doublers \( p = \tilde{p} + \pi_A \) only), and the normal modes of the \( \psi_L^i \) and \( \psi_R \) are supposed not yet to be bound as Dirac fermions (we have not rigorously proved this point). Thus, we have the undoubled low-energy spectrum that involves only the normal modes of the \( \psi_L^i \) and \( \psi_R \). However, because of the multifermion coupling \( g_1 \neq 0 \), these normal modes of \( \psi_L^i \) and \( \psi_R \) still remain in the NJL broken phase, the \( SU_L(2) \otimes U_R(1) \)-chiral symmetry is violated by \( \Sigma^1(0) = \rho v^1 \neq 0 \), to which only normal modes contribute. The propagators of the normal modes in this wedge should be the same as eqs. (16,17) for \( p = \tilde{p} \):

\[
S_{b1}^{-1}(\tilde{p}) = i \sum \gamma_\mu \tilde{\mu} Z_2(\tilde{p}) P_L + i \sum \gamma_\mu \tilde{\mu} P_R + v^1 \rho \\
S_{b2}^{-1}(\tilde{p}) = i \sum \gamma_\mu \tilde{\mu} Z_2(\tilde{p}) P_L.
\]  

However, when \( g_1 \neq 0, \rho \neq 0 \) eq. (20), the normal mode of the \( \psi_R(x) \) is not guaranteed to completely decouple from that of the \( \psi_L^i(x) \).

Once we go onto the line \( A: g_1 = 0, g_2 \leq g_2^{ca} \), as indicated in fig.1, the spectrum (45,46) is undoubled for \( g_2 > g_2^{ca} \). As the results of the \( \psi_R \)-shift-symmetry of the action (1):

1. the normal mode of the \( \psi_R \) is a free mode (see eq. (11));
2. the NJL mass term \( \Sigma^1(0) = 0 \) (see eq. (12) also eq. (22)) for which the \( SU_L(2) \otimes U_R(1) \)-chiral symmetry is completely restored;
3. the interacting vertex (13) \( \Gamma^{(4)} = 4 g_2 \omega^2(\tilde{p}) \ll 1 \) for the normal modes, which prevent the normal modes of the \( \psi_L^i \) and \( \psi_R \) from binding up bound states \( (\tilde{\psi}_L^i \cdot \psi_R^i) \psi_L^i, (\tilde{\psi}_R^i \cdot \psi_L^i) \psi_R^i \).

The last point is the most weak point since we base on the discussions of the wedges opening up due to the momentum-dependent interacting vertex in section 4, rather than calculate the spectrum of normal modes directly. We expect the last point to be true in a certain segment of the region (60). Thus, we speculate that there is a possible scaling window for continuum chiral fermions opening up in this segment. In this possible scaling region, the spectrum consists of the doublers eq. (43,46) for \( p = \tilde{p} + \pi_A \) and the massless normal modes eqs. (58,59) for \( g_1 = 0 \),

\[
S_{L}^{-1}(\tilde{p})^{ij} = i \gamma_\mu \tilde{\mu} \tilde{Z}_2 \delta_{ij} P_L, \quad S_{R}^{-1}(\tilde{p}) = i \gamma_\mu \tilde{\mu} P_R,
\]

which are in agreement with the \( SU_L(2) \otimes U_R(1) \) symmetry. Namely, this normal mode of the \( \psi_L^i \) is self-scattering via the multifermion coupling \( g_2 \) without pairing up with any other modes. The wave function renormalization \( \tilde{Z}_2 \) can be considered as an interpolating constant of \( Z_2(p) \) for \( p = \tilde{p} \simeq 0 \) and \( g_1 = 0 \).
If this scenario is truly emerged, in this possible scaling region for the long distance, we have the massive spectrum that contains fifteen doublers of the $SU_L(2)$-invariant and $U_R(1)$-covariant neutral Dirac mode $\Psi_n$ eq. (45) ($p \neq \tilde{p}$) and fifteen doublers of the $U_R(1)$-invariant and $SU_L(2)$-covariant charged Dirac mode $\Psi^c_i$ eq. (46) ($p \neq \tilde{p}$), as well as the $SU_L(2) \otimes U_R(1)$ covariant massive scalar $A^i$ eq. (50). Besides, we have massless spectrum that contains the $U_R(1)$-covariant Weyl mode $\psi_R$ and the $SU_L(2)$-covariant Weyl mode $\psi^c_L$ eq. (61). In order to see all possible interactions between these modes in this possible scaling region, we consider the one-particle irreducible vertex functions of these modes. In the light of the exact $SU_L(2) \otimes U_R(1)$ chiral symmetry and $\psi_R$-shift-symmetry, one can straightforwardly obtain non-vanishing vertex functions ($d$-dimensions) at physical momenta ($p = \tilde{p}, q = \tilde{q}$): (i) $A^i A^{i\dagger} A^{i\dagger}$ ($d = 4$); (ii) $\tilde{\psi}_L^c A^i A^{i\dagger}$, $\tilde{\psi}_R^c A^i A^{i\dagger}$ and $\tilde{\psi}_n^c A^i A^{i\dagger}$ ($d = 5$), as well as $d > 5$ vertex functions. The vertex functions with dimensions $d > 4$ vanish in the scaling region as $O(a^{d-4})$ and we are left with the self-interacting vertex $A^i A^{i\dagger} A^{i\dagger} A^{i\dagger}$.

In this possible scaling region, the chiral continuum limit is very much like that of lattice QCD. We need to tune only one coupling $g_1 \to 0$ in the neighborhood of the possible scaling region (60). For $g_1 \to 0$, the $\psi_R$-shift-symmetry is slightly violated, the normal modes of the $\psi_L^c$ and $\psi_R$ would couple together to form the chiral symmetry breaking term $\Sigma^i(0) \tilde{\psi}_L^c \psi_R$, which is a dimension-3 renormalized operator and thus irrelevant at the short distance. We desire this scaling region to be ultra-violet stable, in which the multifermion coupling $g_1$ turns out to be an effective renormalized dimension-4 operator[12].

6. The conclusion of the existence of the possible scaling region (60) for the continuum chiral theory is plausible and hard to be excluded. It is worthwhile to check and confirm this scenario in different approaches. Even though, we are still left with several problems. Their possible resolutions are mentioned and discussed in this section, and deserve to be studied in future work.

The question is whether this chiral continuum theory in the scaling region could be the correct chiral gauge theory, as the $SU(2)$-chiral gauge coupling $g$ perturbatively is turned on in the theory (4). One should expect a slight change of critical lines (points). We should be able to re-tune the multifermion couplings ($g_1, g_2$) to compensate these perturbative changes, due the fact that the gauge interaction does not spoil the $\psi_R$-shift-symmetry and we have Ward identities

\[
\frac{\delta^{(2)} \Gamma}{\delta A'_{\mu} \delta \psi^i_R} = \frac{\delta^{(3)} \Gamma}{\delta A'_{\mu} \delta \psi^i_R \delta \psi^{i'}_R} = \frac{\delta^{(3)} \Gamma}{\delta A'_{\mu} \delta \Psi^c \delta \psi^i_R} = \cdots = 0,
\]

where $A'_\mu$ is a “prime” gauge field. In this possible scaling regime, disregarding those uninteresting neutral modes, we have the charged modes including both the $SU(2)$-chiral-gauged, massless normal mode (61) of the $\psi_L^i$ and the $SU(2)$-vectorial-gauged, massive doublers of the Dirac fermion $\Psi^c_i$ (46), which is made
by the 15 doublers of the $\psi^i_L$ and the 15 doublers of the bound Weyl fermion $\bar{\psi}^i_R \cdot \psi^i_L$. The gauge field should not only chirally couple to the massless normal mode of the $\psi^i_L$ in the low-energy regime, but also vectorially couple to the massive doublers of Dirac fermion $\Psi^i_c$ in the high-energy regime. Thus, we expect the coupling vertex of the $SU_L(2)$-gauge field and the normal mode of the $\psi^i_L$ to be chiral at the continuum limit. We are supposed to be able to demonstrate this point on the basis of the Ward identities associating with the $SU(2)$-chiral gauge symmetry that is respected by the spectrum in the possible scaling regime. In fact, due to the reinstating of the manifest $SU_L(2)$-chiral gauge symmetry and corresponding Ward identities of the undoubled spectrum in this possible scaling regime, we should then apply the Rome approach (which is based on the conventional wisdom of quantum field theory) to perturbation theory in the small gauge coupling. It is expected that the Rome approach would work in the same way but all gauge-variant counterterms are prohibited; the gauge boson masses vanish to all orders of gauge coupling perturbation theory for $g_1 = 0$.

Another important question remaining is how chiral gauge anomalies emerge, although in this short report the chiral gauge anomaly is cancelled by purposely choosing an appropriate fermion representation of the $SU_L(2)$ chiral gauge group. We know that in the doubled spectrum of naive lattice chiral gauge theory, the reason for the correct anomaly disappearing in the continuum limit is that the normal mode and doublers of Weyl fermion produce the same anomaly these anomalies eliminate themselves. As a consequence of decoupled doublers being given chiral-invariant mass ($\sim O(1/\alpha)$), the survival normal mode of the Weyl fermion (chiral-gauged, e.g., $U_L(1)$) should produce the correct anomaly in the continuum limit. We also have the question of whether the conservation of fermion number would be violated by the correct anomaly structure $\text{tr}F\tilde{F}$ that is generated by the $SU(2)$ instanton in the continuum limit.

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**Figure Captions**

**Figure 1:** The phase diagram for the theory \( \Phi \) in the \( g_1 - g_2 \) plane (at the gauge coupling \( g = 0 \)).