Borel hierarchies in infinite products of Polish spaces

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Abstract. Let \( H \) be a product of countably infinite number of copies of an uncountable Polish space \( X \). Let \( \Sigma_\xi \) be the class of Borel sets of additive class \( \xi \) for the product of copies of the discrete topology on \( X \) (the Polish topology on \( X \)), and let \( B = \bigcup_{\xi < \omega_1} \Sigma_\xi \). We prove in the Lévy–Solovay model that
\[
\Sigma_\xi = \Sigma_\xi \cap B
\]
for \( 1 \leq \xi < \omega_1 \).

Keywords. Borel sets of additive classes; Baire property; Lévy–Solovay model; Gandy–Harrington topology.

1. Introduction

Suppose \( X \) is a Polish space and \( N \) the set of positive integers. We consider \( H = X^N \) with two product topologies: (i) the product of copies of the Polish topology on \( X \), so that \( H \) is again a Polish space and (ii) the product of copies of the discrete topology on \( X \). Define now the Borel hierarchy in the larger topology on \( H \). To do so, we need some notation. An element of \( H \) will be denoted by \( h = (x_1, x_2, \ldots, x_n, \ldots) \) and for \( m \in N \), \( p_m(h) \) will denote the first \( m \) coordinates, that is, \( p_m(h) = (x_1, x_2, \ldots, x_m) \). For \( n \in N \) and \( A \subseteq X^n \), cyl\((A)\) will denote the cylinder set with base \( A \), that is,
\[
cyl(A) = \{ h \in H: p_n(h) \in A \}.
\]
The Borel hierarchy for the larger topology on \( H \) can now be defined as follows:
\[
\Sigma_0 = \Pi_0 = \{ \text{cyl}(A): A \subseteq X^n, \quad n \geq 1 \}
\]
and for \( \xi > 0 \),
\[
\Sigma_\xi = \left( \bigcup_{\eta < \xi} \Pi_\eta \right), \quad \Pi_\xi = \neg \Sigma_\xi.
\]
The Borel hierarchy on \( H \) with respect to the smaller topology is defined in the usual way:
\[
\Sigma_1 = \{ V: V \text{ is open in } H \text{ in the smaller topology} \}, \quad \Pi_1 = \neg \Sigma_1
\]
and, for $\xi > 1$,

$$\Sigma_\xi^* = \left( \bigcup_{\eta < \xi} \Pi_\eta \right)_\sigma; \quad \Pi_\xi^* = \neg \Sigma_\xi.$$ 

Let

$$B = \bigcup_{\xi < \omega_1} \Sigma_\xi^* = \bigcup_{\xi < \omega_1} \Pi_\xi^*.$$ 

The problem we will address in this article is whether

$$\Sigma_\xi^* = \Sigma_\xi \cap B \quad \text{for } 1 \leq \xi < \omega_1.$$ 

To tackle the problem we will use the methods of effective descriptive set theory. We therefore have to formulate the lightface version of (*) . We refer the reader to [Mo] and [L1] for definitions of lightface concepts. We take $X$ to be the recursively presentable Polish space $\omega^\omega$ hereafter.

Define

$$\Sigma_0^* = \Pi_0^* = \{ \text{cyl}(A): A \text{ is } \Delta_1^1 \text{ in } (\omega^\omega)^n, \ n \geq 1 \},$$

and, for $1 \leq \xi < \omega_1^{ck}$,

$$\Sigma_\xi^* = \bigcup_1^1 \left( \bigcup_{\eta < \xi} \Pi_\eta^* \right)$$

and

$$\Pi_\xi^* = \neg \Sigma_\xi^*,$$

where $\cup_1^1 (\bigcup_{\eta < \xi} \Pi_\eta^*)$ is a $\Delta_1^1$ union of members of $\cup_{\eta < \xi} \Pi_\eta^*$. The lightface analogue of (*) is then

$$\Sigma_\xi^* = \Delta_1^1 \cap \Sigma_\xi, \quad \text{for } 1 \leq \xi < \omega_1^{ck}.$$ 

In order to state the main result of the article, we equip $\omega^\omega$ with the Gandy–Harrington topology, that is, the topology whose base is the pointclass of $\Sigma_1^1$ sets. The key property of this topology is that it satisfies the Baire category theorem (see [L1]). Consider now the following statement of set theory:

(O) Every subset of $\omega^\omega$ has the Baire property with respect to the Gandy–Harrington topology.

The main result of the article can now be stated.

**Theorem 1.1.** Assume (O). Let $1 \leq \xi < \omega_1^{ck}$. If $A$ and $B$ are $\Sigma_1^1$ subsets of $H$ such that $A$ can be separated from $B$ by a $\Sigma_\xi$ set, then $A$ can be separated from $B$ by a $\Sigma_\xi^*$ set.

An immediate consequence is

**COROLLARY 1.2**

(O) implies (**).
The above results will be established in ZF + DC. Maitra et al [Ma] proved (*) for $\xi = 1$ in ZF + DC by a boldface argument. We will provide a lightface argument in the Appendix for (***) when $\xi = 1$. Again this will be done in ZF + DC. Barua [Ba] proved Theorem 1.1 and Corollary 1.2. His proof was by induction on $\xi$. However, he left out the proof of the base step ($\xi = 1$). We will fill in the gap in this article. The proof of Theorem 1.1 presented here parallels very closely that of Louveau [L1], whereas the proof in [Ba] relies on the more abstract developments of [L2]. In consequence, the proof given here is somewhat simpler.

The paper is organized as follows. Section 2 is devoted to definitions and notation. Section 3 contains the detailed proof of Theorem 1.1 when $\xi = 1$, while §4 sketches how the proof of Theorem 1.1 can be completed by an inductive argument. In the concluding section, we will prove (*) under appropriate hypotheses and also mention open problems.

2. Definitions, notation and preliminaries

For $n \geq 1$, the Gandy–Harrington topology on $(\omega^n)$ will be denoted by $T^n$ and the Gandy–Harrington topology on $H$ will be denoted by $T^\infty$. Following Louveau [L1], we define for each $\xi$ such that $1 \leq \xi < \omega_1^\omega$ a topology $T_\xi$ on $H$ having for its base the pointclass $\Sigma_1^\xi \cap \bigcup_{\eta<\xi} \Pi_\eta$.

Let $S$ be a second countable topology on $(\omega^n)$ (respectively, $H$). Let $A$ be a subset of $(\omega^n)$ (respectively, $H$). By the cosurrogate of $A$ we mean the largest $S$-open set $B$ such that $A \cap B$ is $T^n$-meager (respectively, $T^\infty$-meager). The surrogate of $A$ is defined to be the complement of the cosurrogate of $A$. When $S$ is the topology $T^n$, we denote the surrogate (respectively, cosurrogate) of $A$ by $\text{sur}(A)$ (respectively, $\text{cosur}(A)$). If $A \subseteq H$ and $S$ is the topology $T_\xi$, the surrogate (respectively, cosurrogate) of $A$ will be denoted by $\text{sur}_\xi(A)$ (respectively, $\text{cosur}_\xi(A)$).

**Lemma 2.1.** Let $m \geq 1$. If $A \subseteq (\omega^n)$ is $T^m$-open, then $\text{sur}^m(A)$ is the $T^m$-closure of $A$. Consequently, $\text{sur}^m(A) - A$ is $T^m$-nowhere dense. 

**Proof.** If $B$ is $\Sigma_1^m$ and $A \cap B$ is $T^m$-meager, then $A \cap B$ must be empty, because $A \cap B$ is $T^m$-open and the Baire category theorem holds for $T^m$. Consequently, $\text{cosur}^m(A)$ is the union of basic open sets of the $T^m$-topology which are disjoint with $A$. It follows that $\text{sur}^m(A)$ is the $T^m$-closure of $A$. 

**Lemma 2.2.** Assume (O). Let $m \geq 1$. If $A \subseteq (\omega^n)$, then $A \Delta \text{sur}^m(A)$ is $T^m$-meager.

**Proof.** Observe that $\omega^n$ and $(\omega^n)^m$ are recursively isomorphic, so $(\omega^n, T^1)$ and $((\omega^n)^m, T^m)$ are homeomorphic. Hence it follows from (O) that there is a $T^m$-open set $B$ such that $A \Delta B$ is $T^m$-meager. So, if $D$ is a $\Sigma_1^m$ subset of $(\omega^n)^m$, then $A \cap D$ is $T^m$-meager iff $B \cap D$ is $T^m$-meager, so that $\text{sur}^m(A) = \text{sur}^m(B)$. Since $B$ is $T^m$-open, it follows from Lemma 2.1 that $\text{sur}^m(B) - B$ is $T^m$-nowhere dense, hence $B \Delta \text{sur}^m(B)$ is $T^m$-meager. Consequently, $A \Delta \text{sur}^m(A)$ is $T^m$-meager. 

Note that the converse of Lemma 2.2 is true. Indeed, if $A \Delta \text{sur}^1(A)$ is $T^1$-meager for every $A \subseteq \omega^n$, then, as is easy to verify, $A$ has the Baire property with respect to $T^1$ for every $A \subseteq \omega^n$, that is, (O) holds.

3. The case $\xi = 1$

In this section we will prove Theorem 1.1 when $\xi = 1$. 


Following [L1], we fix a coding pair \((W, C)\) for the \(\Delta^1_1\) subsets of \(H\), that is,

(i) \(W\) is a \(\Pi^1_1\) subset of \(\omega\);
(ii) \(C\) is a \(\Pi^1_1\) subset of \(\omega \times H\);
(iii) the relations ’\(n \in W \& C(n, h)\)’ and ’\(n \in W \& \neg C(n, h)\)’ are both \(\Pi^1_1\);
(iv) for every \(\Delta^1_1\) subset \(A\) of \(H\), there is \(n \in W\) such that \(A = C_n \overset{\text{def}}{=} \{h \in H : C(n, h)\}\).

Define \(W_0\) as follows:

\[
m \in W_0 \iff m \in W \& (\exists n \geq 1)(\forall h)(\forall h')(C(n, h) \& p_n(h) 
\implies p_n(h') \implies C(n, h')).
\]

Then \(W_0\) is \(\Pi^1_1\). Indeed, \(W_0\) is just the set of codes of \(\Delta^1_1\) cylinder subsets of \(H\).

**Lemma 3.1.** If \(A\) is a \(\Sigma^1_1\) subset of \(H\), then \(cl^1_1(A)\) is \(\Pi^1_1\) and \(\Sigma^1_1\), hence \(T_2\)-open, where \(cl^1_1(A)\) is the \(T_1\)-closure of \(A\).

**Proof.** Indeed, for any \(A, cl^1_1(A)\) is \(\Pi^1_1\), because it is a countable intersection of \(\Pi^1_1\) sets. Now suppose \(A \in \Sigma^1_1\). Then

\[
h \notin cl^1_1(A) \iff (\exists n \geq 1)(\exists B)(B \text{ is a } \Sigma^1_1 \text{ subset of } (\omega^\omega)^n \& h \in cyl(B) 
\& A \cap cyl(B) = \phi)
\iff (\exists n \geq 1) (\exists B) (B \text{ is a } \Delta^1_1 \text{ subset of } (\omega^\omega)^n
\& h \in cyl(B) \& A \cap cyl(B) = \phi).
\]

To prove the previous implication \(\rightarrow\), let \(B\) be a \(\Sigma^1_1\) subset of \((\omega^\omega)^n\) such that \(h \in cyl(B)\) and \(A \cap cyl(B) = \phi\). But then \(p_n(A) \cap B = \phi\). Since \(p_n(A)\) is \(\Sigma^1_1\), it follows from Kleene’s separation theorem that there is a \(\Delta^1_1\) subset \(B'\) of \((\omega^\omega)^n\) such that \(B \subseteq B'\) and \(B' \cap p_n(A) = \phi\). Hence \(h \in cyl(B')\) and \(A \cap cyl(B') = \phi\), which establishes \(\rightarrow\).

Consequently,

\[
h \notin cl^1_1(A) \iff (\exists m)(m \in W_0 \& C(m, h) \& C_m \cap A = \phi).
\]

So \(\neg cl^1_1(A)\) is \(\Pi^1_1\). \(\square\)

**Lemma 3.2.** Assume (O). If \(A\) is a \(\Pi^1_1\) subset of \(H\), then \(A \Delta \text{sur}^1_1(A)\) is \(T^\infty\)-meager.

**Proof.** Choose subsets \(B_n\) of \((\omega^\omega)^n\), \(n \geq 1\), such that

\[
A = H - \cup_{n \geq 1} cyl(B_n).
\]

Then

\[
\text{sur}^1_1(A) - A = \text{sur}^1_1(A) \cap \cup_{n \geq 1} cyl(B_n)
\subseteq \cup_{n \geq 1} ([\text{sur}^1_1(A) \cap cyl(\text{sur}^\omega(B_n))]]
\cup [cyl(B_n) - cyl(\text{sur}^\omega(B_n))]).
\]
Now
\[ \text{cyl}(B_n) - \text{cyl} \left( \text{sur}^\emptyset(B_n) \right) = \text{cyl}(B_n) - \text{sur}^\emptyset(B_n). \]
The set on the right of the above equality is $T^\infty$-meager by virtue of Lemma 2.13 in [L2]. We will now prove that $\text{sur}_1(A) \cap \text{cyl} \left( \text{sur}^\emptyset(B_n) \right)$ is $T^\infty$-nowhere dense. Note that $\text{sur}_1(A) \cap \text{cyl} \left( \text{sur}^\emptyset(B_n) \right)$ is $T_1$-closed, hence $T^\infty$-closed. Now let $A'$ be a $\Sigma_1^1$ set contained in $\text{sur}_1(A) \cap \text{cyl} \left( \text{sur}^\emptyset(B_n) \right)$. Then
\[ \text{cyl}(p_n(A')) \subseteq \text{cyl} \left( \text{sur}^\emptyset(B_n) \right). \]
Hence
\[ A \cap \text{cyl}(p_n(A')) \subseteq \text{cyl} \left( \text{sur}^\emptyset(B_n) \right) - \text{cyl}(B_n) \]
\[ = \text{cyl} \left( \text{sur}^\emptyset(B_n) - B_n \right). \]
Consequently, by virtue of Lemma 2.2 and Lemma 2.13 in [L2], $A \cap \text{cyl}(p_n(A'))$ is $T^\infty$-meager. Since $\text{cyl}(p_n(A'))$ is $T_1$-open, it follows that $\text{cyl}(p_n(A')) \subseteq \text{cosur}_1(A)$. Hence $A'$ is empty because $A'$ is also contained in $\text{sur}_1(A)$. Thus $\text{sur}_1(A) \cap \text{cyl} \left( \text{sur}^\emptyset(B_n) \right)$ is $T^\infty$-nowhere dense. It follow from (1) that $\text{sur}_1(A) - A$ is $T^\infty$-meager. Since $A - \text{sur}_1(A)$ is easily seen to be $T^\infty$-meager, we are done. \hfill \Box

Lemma 3.3. If $A$ and $B$ are $\Sigma_1^1$ subsets of $H$ such that $A$ can be separated from $B$ by a $\Sigma_1^1$ set, then $A \cap \text{cl}_1(B) = \emptyset$.

Proof. Suppose $D$ is a $\Pi_1^1$ subset of $H$ such that $A \cap D = \emptyset$ and $B \subseteq D$. Hence, by Lemma 3.2, $B - \text{sur}_1(D)$ is $T^\infty$-meager. But $B - \text{sur}_1(D)$ is $T^\infty$-open, so $B \subseteq \text{sur}_1(D)$.

Since $\text{sur}_1(D)$ is $T_1$-closed, $\text{cl}_1(B) \subseteq \text{sur}_1(D)$. Now $A \cap \text{sur}_1(D)$ is $T^\infty$-meager, so $A \cap \text{cl}_1(B)$ is $T^\infty$-meager. By Lemma 3.1, $A \cap \text{cl}_1(B)$ is $\Sigma_1^1$, hence $A \cap \text{cl}_1(B)$ must be empty. \hfill \Box

Lemma 3.4. If $A$ and $B$ are $\Sigma_1^1$ subsets of $H$ such that $A \cap \text{cl}_1(B) = \emptyset$, then $A$ can be separated from $B$ by a $\Sigma_1^1$ set.

Proof. Define
\[ P(h, n) \leftrightarrow h \notin A \lor (n \in W_0 \land C(n, h) \land C_n \cap B = \emptyset). \]
Then $P$ is $\Pi_1^1$ and $(\forall h)(\exists n)P(h, n)$. By Kreisel’s selection theorem [Mo], there is a $\Delta_1^0$-recursive function $f: H \rightarrow \omega$ such that $(\forall h)P(h, f(h))$. Let
\[ D = \{ n \in \omega: n \in W_0 \land C_n \cap B = \emptyset \}. \]
Then $D$ is $\Pi_1^1$ and $f(A) \subseteq D$. Since $f(A)$ is $\Sigma_1^1$, there is a $\Delta_1^1$ set $E \subseteq \omega$ such that $f(A) \subseteq E \subseteq D$. Let
\[ R(h, n) \leftrightarrow n \in E \land C(n, h), \]
then $R$ is $\Delta_1^1$, because if
\[ R'(h, n) \leftrightarrow n \in E \land \neg C(n, h), \]
them both $R$ and $R'$ are $\Pi_1^1$, $R \cap R' = \emptyset$ and $R \cup R' = H \times E$. Set
\[ G_n = \{ h: R(h, n) \}, \quad n \in \omega. \]
Then $\bigcup_{n \geq 0} G_n$ is a $\Sigma_1^1$ set which separates $A$ from $B$. \hfill \Box

Lemmas 3.2, 3.3 and 3.4 establish Theorem 1.1 for $\xi = 1$.
4. Proof of Theorem 1.1

The proof of Theorem 1.1 is by induction on $\xi$. So we fix $\xi > 1$ and assume Theorem 1.1 is true for all $\eta < \xi$. Lemmas 3.1–3.4 can be formulated and proved at level $\xi$, thereby completing the proof of Theorem 1.1 at level $\xi$. We omit the proofs because they are exactly like the proofs of Lemmas 7, 8, 9 and Theorem B in [L1].

We observe that the inductive hypothesis that Theorem 1.1 hold at all levels $\eta < \xi$ is by itself not sufficiently strong to prove the analogue of Lemma 3.2 at level $\xi$ and hence the theorem itself at that level. For this we need that analogues of Lemma 3.2 hold at all levels $\eta < \xi$. It is at this point in the proof that assumption (O) is needed to ensure that Lemma 3.2 hold at level $\xi = 1$, the higher levels of Lemma 3.2 then being proved by inducting up from the base level.

5. Concluding remarks

For $\alpha \in \omega^\omega$, we now consider the following statement of set theory:

\((\alpha)\) Every subset of $\omega^\omega$ has the Baire property with respect to the topology whose base is the pointclass of $\Sigma^1_1(\alpha)$ sets.

It is straightforward to relativize Theorem 1.1 to $\alpha$ under the assumption that \((\alpha)\) holds. The next result is provable in $ZF + DC$.

**Theorem 5.1.** Let $X$ be an uncountable Polish space and let $H = X^N$. Then, for $1 \leq \xi < \omega_1$, $\Sigma^\xi = \Sigma^\xi \cap B$.

Under the assumption that there is an inaccessible cardinal, Solovay [S] proved that $ZF + DC$ holds in the Lévy–Solovay model. Furthermore, it was observed by Louveau (p.43 of [L2]) that the statement $(\forall \alpha)((\alpha))$ holds as well in the model.

Whether Theorem 5.1 is provable in ZFC remains an open problem. Indeed, we do not have an answer to the problem even when $\xi = 2$.

It is not difficult to prove that the axiom of determinacy implies $(\forall \alpha)((\alpha))$ so that Theorem 5.1 is provable in $ZF + AD$ (see [Mo]). On the other hand, the axiom of choice implies $\neg(O)$ in ZF.

Appendix

We will now prove Theorem 1.1 for $\xi = 1$ without assuming (O). In view of Lemma 3.4, it will suffice to prove that $A \cap cl(B) = \emptyset$. Define $P(h, n) \leftrightarrow (n \geq 1) \& (\exists h')(p_n(h)h' \in B)$, where $p_n(h)h'$ is the catenation of $p_n(h)$ and $h'$. Note that $P$ is $\Sigma^1_1$. Let $h \in \tilde{B} \leftrightarrow (\forall n \geq 1)P(h, n)$, so that $\tilde{B}$ is the closure of $B$ in the product of discrete topologies on $H$. Consequently, $\tilde{B} \subseteq H - A$. Define $Q(h, n) \leftrightarrow (n \geq 1) \& (\neg P(h, n) \vee h \notin A)$. Then $Q$ is clearly $\Pi^1_1$ and $(\exists h)Q(h, n)$. So there is a $\Delta^1_1$-recursive function $f : H \to \omega$ such that $(\forall h)Q(h, f(h))$. Let
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\[ S(h, n) \leftrightarrow (n \geq 1) \land (f(h) \neq n \lor h \notin A). \]

Claim.

(i) \( S \) is \( \Pi^1_1 \),

(ii) \( (\forall h)(\forall n \geq 1)(P(h, n) \rightarrow S(h, n)) \),

(iii) \( h \notin A \leftrightarrow (\forall n \geq 1)S(h, n) \).

To see (ii), assume \( P(h, n) \). Then we must have \( h \in A \rightarrow f(h) \neq n \). Hence \( S(h, n) \).

For (iii), suppose \( h \notin A \). Clearly, then \( (\forall n \geq 1)S(h, n) \). Suppose now that \( h \in A \). Then there is \( n \) such that \( f(h) = n \), hence \( \neg S(h, n) \). (iii) now follows.

Now turn each \( S_n \) into a cylinder set as follows. Define

\[ R(h, n) \leftrightarrow (\forall h')(S(p_n(h)h', n)), \]

so \( R \) is \( \Pi^1_1 \). Note that \( P_n \) and \( R_n \) are cylinder sets, that is,

\[ P(h, n) \land p_n(h) = p_n(h') \rightarrow P(h', n) \]

and

\[ R(h, n) \land p_n(h) = p_n(h') \rightarrow R(h', n). \]

Claim. \( (\forall h)(\forall n)(P(h, n) \rightarrow R(h, n)) \).

So suppose \( P(h, n) \). Then, for every \( h' \), \( P(p_n(h)h', n) \), hence \( S(p_n(h)h', n) \), so \( R(h, n) \).

To complete the proof, let \( h \in A \). Then there is \( n \geq 1 \) such that \( \neg S(h, n) \), hence \( \neg R(h, n) \). Now \( \neg R_n \) is \( \Sigma^1_1 \) and \( \Pi_0 \) because \( R_n \) is a cylinder set. Moreover, \( \neg R_n \cap B = \emptyset \) because \( \neg R_n \subseteq \neg P_n \) and \( \neg P_n \cap B = \emptyset \). Hence \( \neg R_n \) is a \( T_1 \)-open set containing \( h \) and disjoint from \( B \). So \( h \notin c_1(B) \).

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