Structural Resolution: a Framework for Coinductive Proof Search and Proof Construction in Horn Clause Logic

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Logic programming (LP) is a programming language based on first-order Horn clause logic that uses SLD-resolution as a semi-decision procedure. Finite SLD-computations are inductively sound and complete with respect to least Herbrand models of logic programs. Dually, the corecursive approach to SLD-resolution views infinite SLD-computations as successively approximating infinite terms contained in programs’ greatest complete Herbrand models. State-of-the-art algorithms implementing corecursion in LP are based on loop detection. However, such algorithms support inference of logical entailment only for rational terms, and they do not account for the important property of productivity in infinite SLD-computations. Loop detection thus lags behind coinductive methods in interactive theorem proving (ITP) and term-rewriting systems (TRS).

Structural resolution is a newly proposed alternative to SLD-resolution that makes it possible to define and semi-decide a notion of productivity appropriate to LP. In this paper, we prove soundness of structural resolution relative to Herbrand model semantics for productive inductive, coinductive, and mixed inductive-coinductive logic programs.

We introduce two algorithms that support coinductive proof search for infinite productive terms. One algorithm combines the method of loop detection with productive structural resolution, thus guaranteeing productivity of coinductive proofs for infinite rational terms. The other allows to make lazy sound observations of fragments of infinite irrational productive terms. This puts coinductive methods in LP on par with productivity-based observational approaches to coinduction in ITP and TRS.

Categories and Subject Descriptors: F.3.2 [Semantics of Programming Languages]: Operational Semantics; F.4.1 [Mathematical Logic]: Logic and Constraint Programming

Additional Key Words and Phrases: Logic programming, resolution, induction, coinduction, infinite derivations, Herbrand models.

ACM Reference Format:
Ekaterina Komendantskaya, et al., 2015. Structural Resolution ACM Trans. Comput. Logic V, N, Article A (January YYYY), 25 pages.
DOI:http://dx.doi.org/10.1145/0000000.0000000

1. INTRODUCTION
1.1. A Symmetry of Inductive and Coinductive Methods
Logic Programming (LP) is a programming language based on Horn clause logic. If \( P \) is a logic program and \( r \) is a (first-order) term, then LP provides a mechanism for automatically inferring whether or not \( P \) logically entails \( r \). The traditional (inductive) approach to LP is based on least fixed point semantics [Kowalski 1974; Lloyd 1987] of logic programs, and defines, for every such program \( P \), the least Herbrand model for \( P \), i.e., the set of all (finite) ground terms inductively entailed by \( P \).
Example 1.1. The program $P_1$ defines the set of natural numbers:

0. $\text{nat}(0) \leftarrow$
1. $\text{nat}(s(X)) \leftarrow \text{nat}(X)$

The least Herbrand model for $P_1$ comprises the terms $\text{nat}(0)$, $\text{nat}(s(0))$, $\text{nat}(s(s(0)))$, ...

The clauses of $P_1$ can be viewed as inference rules $\frac{\text{nat}(0)}{\text{nat}(0)}$ and $\frac{\text{nat}(x)}{\text{nat}(s(x))}$, and the least Herbrand model can be seen as the set obtained by the forward closure of these rules. Some approaches to LP and first-order sequent calculi are based on this inductive view [Baelde 2008, Brotherston and Simpson 2011] of programs, which is entirely standard [Sangiorgi 2012]. A similar view underlies inductive type definitions in interactive theorem proving (ITP) [Agda Development Team 2015, Coq Development Team 2015]. For example, $P_1$ also corresponds to the following Coq definition of an inductive type:

\begin{verbatim}
Inductive nat : Type :=
| 0 : nat
| S : nat -> nat.
\end{verbatim}

In addition to viewing logic programs inductively, we can also view them coinductively. The greatest complete Herbrand model for a program $P$ takes the backward closure of the rules derived from $P$'s clauses, thereby producing the largest set of finite and infinite ground terms coinductively entailed by $P$. For example, the greatest complete Herbrand model for $P_1$ is the set containing all of the finite terms in the least Herbrand model for $P_1$, together with the term $\text{nat}(s(s(\ldots)))$ representing the first limit ordinal. The coinductive view of logic programs corresponds to coinductive type definitions in ITP.

As it turns out, some logic programs have no natural inductive semantics and should instead be interpreted coinductively:

Example 1.2. The program $P_2$ defining streams of natural numbers comprises the clauses of $P_1$ and the following additional one:

2. $\text{nats}(s\text{cons}(X, Y)) \leftarrow \text{nat}(X), \text{nats}(Y)$

No terms defined by $\text{nats}$ are contained in the least Herbrand model for $P_2$, but $P_2$'s greatest complete Herbrand model contains infinite terms representing infinite streams of natural numbers, like e.g. the infinite term $t = \text{nats}(s\text{cons}(0, s\text{cons}(0, \ldots)))$ representing the infinite stream of zeros.

The program $P_2$ corresponds to the following Coq definition of a coinductive type:

\begin{verbatim}
CoInductive nats : Type :=
| SCons : nat -> nats -> nats.
\end{verbatim}

The formal relation between logic programs, Herbrand models, and types is analysed in [Heintze and Jaffar 1992].

1.2. Preconditions for an Operational Semantics?

The (least and greatest complete) Herbrand models for programs, as defined by (forward and backward rule closure, respectively, of) their clauses, provide one important way to understand logic programs. But an equally important way is via their computational behaviours. Rather than using Herbrand models to give meaning to “inductive” and “coinductive” logic programs, we can also use the operational properties of SLD-resolution to assign programs semantics that take into account the computational behaviours that deliver those models. Ideally, we would like to do this in such a way that the symmetry between the Herbrand model interpretations of inductive and coinduc-
active programs as the sets of terms (i.e., the types) they define is preserved by these computational interpretations.

The transition from types to computations is natural in ITP, where recursive functions consume inputs of inductive types and, dually, corecursive functions produce outputs of coinductive types. Since systems such as Coq and Agda require recursive functions to be terminating in order to be sound, and since SLD-resolution similarly requires a logic program’s derivations to be terminating in order for them to be sound with respect to that program’s least Herbrand model, we might dually expect a logic program’s non-terminating derivations to compute terms in its greatest complete Herbrand model. However, non-termination does not play a role for coinduction dual to that played by termination for induction. In particular, the fact that a logic program admits non-terminating SLD-derivations does not, on its own, guarantee that the program’s computations completely capture its greatest complete Herbrand model:

**Example 1.3.** The following “bad” program gives rise to an infinite SLD-derivation:

0. \texttt{bad}(f(X)) \leftarrow \texttt{bad}(f(X))

Although this program does not compute any infinite terms, the infinite term \texttt{bad}(f(f(...))) is in its greatest complete Herbrand model.

It is important to note that the “badness” of this program is unrelated to the fact that LP is untyped. The following corecursive function is equally “bad”, and will be rejected by Coq:

\[
\text{CoInductive } \text{Stream } A : \text{Type} := \\
\text{SCons} : A \to \text{Stream } A \to \text{Stream } A.
\]

\[
\text{CoFixpoint bad } (f : A \to A) (x : A) : \text{Stream } A := \text{bad } f (f x).
\]

The problem here actually lies in the fact that both the LP and the ITP versions of the above “bad” program fail to satisfy the important property of productivity. The productivity requirement on corecursive programs for systems such as Coq and Agda reflects the fact that an infinite computation can only be consistent with its intended coinductive semantics if it is \textit{globally productive}, i.e., if it actually produces an infinite object in the limit. But in order to give an operational meaning to “in the limit” — which is not itself a computationally tractable concept — productivity is usually interpreted in terms of finite observability. Specifically, a function can be (finitely) observed to be globally productive if each part of its infinite output can be generated in finite time. We call this kind of productivity \textit{observational productivity}. A similar notion of an observationally productive infinite computation has also been given for stream productivity in term rewriting systems (TRS) \cite{Endrullis et al. 2010, Endrullis et al. 2015}. Moreover, a variety of syntactic guardedness checks have been developed to semi-decide observational productivity in ITP in practice \cite{Coquand 1994, Giménez 1998, Bertot and Komendantskaya 2008}. However, prior to \cite{Komendantskaya et al. 2017}, LP did not have any notion of an observationally productive program, and therefore did not have a corresponding operational semantics based on any such notion.

### 1.3. Symmetry Broken

It is well-known that termination captures the least Herbrand model semantics of (inductive) logic programs computationally: the terminating and successful SLD-derivations for any program \( P \) give a decision procedure for membership in the least Herbrand model for \( P \). For example, after a finite number of SLD-derivation steps we can conclude that \texttt{nat}(x) is in the least Herbrand model for program \( P_1 \) if \( x = 0 \). Termination of SLD-derivations thus serves as a computational precondition for deciding logical entailment.

But for programs, like \( P_2 \), that admit non-terminating derivations, SLD-resolution gives only a \textit{semi}-decision procedure for logical entailment. Indeed, if an SLD-derivation for a program and a query terminates with success, then we definitely know that the program logically entails the term being queried, and thus that this term is in the greatest complete Herbrand model for the program.
But if an SLD-derivation for the program and query does not terminate, then we can infer nothing. It is therefore natural to ask:

**Question:** Is it possible to capture the greatest complete Herbrand model semantics for potentially non-terminating logic programs computationally? If so, how?

That is, can we restore the symmetry between terminating and potentially non-terminating logic programs so that the correspondence between a terminating program’s Herbrand semantics and its computational behaviour also holds for non-terminating programs?

In one attempt to match the greatest complete Herbrand semantics for potentially non-terminating programs, an operational counterpart — called *computations at infinity* — was introduced in the 1980s [Lloyd 1987; van Emden and Abdallah 1985]. The operational semantics of a potentially non-terminating logic program $P$ was then taken to be the set of all infinite ground terms computable by $P$ at infinity. For example, the infinite ground term $t$ in Example [1.2] is computable by $P_2$ at infinity starting with the query $? \leftarrow \text{nat}(x)$. Although computations at infinity do better capture the computational behaviour of non-terminating logic programs, they are still only sound, and not complete, with respect to those programs’ greatest complete Herbrand models. For example, the infinite term $\text{bad}(f(f(...)))$ is in the greatest complete Herbrand model for the “bad” program of Example [1.3] as noted there, but is not computable at infinity by that program.

Interestingly, computations at infinity capture the same intuition about globally productive infinite SLD-derivations that underlies the productivity requirement for corecursive functions in ITP [Coquand 1994; Giménez 1998; Bertot and Komendantskaya 2008] and productive streams in TRS [Endrullis et al. 2010; Endrullis et al. 2015]. That is, they insist that each infinite SLD-derivation actually produces an (infinite) term. This observation leads us to adapt the terminology of [Lloyd 1987; van Emden and Abdallah 1985] and say that a logic program $P$ is SLD-*productive* if every infinite SLD-derivation for $P$ computes an infinite term at infinity. SLD-productivity captures the difference in computational behavior between programs, like $P_2$, that actually do compute terms at infinity, from “bad” programs, like that of Example [1.3] that do not. While computations at infinity are not complete with respect to greatest complete Herbrand models for non-SLD-productive logic programs, for SLD-productive programs they are. For example, the SLD-productive program below is similar to our non-SLD-productive “bad” program and its greatest complete Herbrand model is computed in the same way:

$$0. \text{good}(f(X)) \leftarrow \text{good}(X)$$

But because this program is SLD-productive — and, therefore, “good” — the infinite term $\text{good}(f(f(...)))$ corresponding to the problematic term above is not only in its greatest complete Herbrand model, but is also computable at infinity.

In light of the above, we concentrate on productive logic programs, shifting our focus away from greatest complete Herbrand models and toward computations at infinity, to give such programs a more computationally relevant semantics. But a big challenge still remains: even for productive programs, the notion of computations at infinity does not by itself give rise to implementations. Specifically, although SLD-productivity captures the important requirement that infinite computations actually produce output, it does not give a corresponding notion of finite observability, as ITP and TRS stream productivity approaches productivity do. We therefore refine our question above to ask:

**Question (refined):** Can we formulate a computational semantics for LP that redefines productivity in terms of finite observability, as is done elsewhere in the study of programming languages, and that does this in such a way that it both yields implementations and ensures soundness and completeness.
with respect to computations at infinity (rather than greatest complete Herbrand models)? If so, how?

Thirty years after the initial investigations into coinductive computations, coinductive logic programming, implemented as CoLP, was introduced [Gupta et al. 2007; Simon et al. 2007]. CoLP provides practical methods for terminating infinite SLD-derivations. CoLP’s coinductive proof search is based on a loop detection mechanism that requires the programmer to supply annotations marking every predicate as either inductive or coinductive. For coinductive predicates, CoLP observes finite fragments of SLD-derivations, checks them for unifying subgoals, and terminates when loops determined by such subgoals are found. A similar loop detection method is employed for type class inference in the Glasgow Haskell Compiler (GHC) [L¨amml and Jones 2005], and CoLP itself is used for type class inference in Featherweight Java [Ancona and Lagorio 2011].

Example 1.4. If \texttt{nats} is marked as coinductive in \texttt{P}, then the query \texttt{? :- nats(X)} gives rise to an SLD-derivation with a sequence of subgoals \texttt{nats(X)} \leftarrow \texttt{X = scons(0,Y)} \texttt{nats(Y)} \leftarrow \ldots. Observing that \texttt{nats(X)} and \texttt{nats(Y)} unify and thus comprise a loop, CoLP concludes that \texttt{nats(X)} has been proved and returns the answer \texttt{X = scons(0,X)} in the form of a “circular” term indicating that \texttt{P} logically entails the term \(t\) in Example 1.2.

CoLP is sound, but incomplete, relative to greatest complete Herbrand models [Gupta et al. 2007; Simon et al. 2007]. But, perhaps surprisingly, it is neither sound nor complete relative to computations at infinity. CoLP is not sound because our “bad” program from Example 1.3 computes no infinite terms at infinity for the query \texttt{? :- bad(X)}, whereas CoLP notices a loop and reports success (assuming the predicate bad is marked as coinductive). CoLP is not complete because not all terms computable at infinity by all programs can be inferred by CoLP. In fact, CoLP’s loop detection mechanism can only terminate if the term computable at infinity is a rational term [Courcelle 1983; Jaffar and Stuckey 1986]. Rational terms are terms that can be represented as trees that have a finite number of distinct subtrees, and can therefore be expressed in a closed finite form computed by circular unification. The “circular” term \(X = scons(0,X)\) in Example 1.4 is so expressed. For irrational terms, CoLP simply does not terminate.

Example 1.5. The program \texttt{P3} defines addition on the Peano numbers, together with the stream of Fibonacci numbers:

\begin{align*}
0. & \texttt{add(0,Y,Z) :-} \\
1. & \texttt{add(s(X),Y,Z) :- add(X,Y,Z)} \\
2. & \texttt{fibs(X,Z) :- add(X,Y,Z),fibs(X,Y,Z).}
\end{align*}

From a coinductive perspective, \texttt{P3} is semantically and computationally meaningful. It computes the infinite term \(t^* = \texttt{fibs(0,s(0),\ldots,cons(s(0),cons(s(0),\ldots)))}\), and thus the stream of Fibonacci numbers (in the third argument to \texttt{fibs}). The term \(t^*\) is both computable at infinity by \texttt{P3} and contained in \texttt{P3}'s greatest complete Herbrand model. Nevertheless, when CoLP processes the sequence \texttt{fibs(0,s(0),\ldots,cons(s(0),\ldots))}, \texttt{fibs(s(0),s(0),\ldots)}), \texttt{fibs(s(0),s(0),\ldots)}), \ldots of subgoals for the program \texttt{P3} and query \texttt{\ldots}, it cannot unify any two of them, and thus does not terminate.

The upshot is that CoLP cannot faithfully capture the operational meaning of computations at infinity.

1.4. Structural Resolution for Productivity

It has been strongly argued in [Komendantskaya et al. 2014; Komendantskaya et al. 2017; Fu and Komendantskaya 2016] that the recently discovered structural resolution can help to combine the intuitive notion of computations at infinity and the coinductive reasoning à la CoLP. We explain the main idea behind structural resolution by means of an example.
Example 1.6. The coinductive program \( P_4 \) has the single clause

\[
0. \text{from}(X, \text{scons}(X, Y)) \leftarrow \text{from}(s(X), Y)
\]

Given the query \( ? \leftarrow \text{from}(0, X) \), and writing \([\ldots]\) as an abbreviation for the stream constructor \( \text{scons} \) here, we have that the infinite term \( t' = \text{from}(0, [0, s(0), s(s(0)), \ldots]) \) is computable at infinity by \( P_4 \) and is also contained in the greatest Herbrand model for \( P_4 \). By the same argument as in Examples 1.4 and 1.5, coinductive reasoning on this query cannot be handled by the loop detection mechanism of CoLP because the term \( t' \) is irrational.

Structural resolution allows us to separate the infinite derivation steps that compute \( t' \) at infinity into term rewriting and unification steps as shown below, with term rewriting steps shown vertically and unification steps shown horizontally. This separation makes it easy to see that \( P_4 \) is finitely observable, in the sense that all of its derivations by term rewriting alone terminate.

\[
\begin{align*}
\{X \leftarrow [0, X']\} & \quad \{X' \leftarrow [s(0), X'']\} & \quad \{X'' \leftarrow [s(s(0)), X''']\} \\
\text{from}(0, X) & \quad \text{from}(0, [0, X']) & \quad \text{from}(0, [0, s(0), X'']) \\
& \quad \text{from}(s(0), X') & \quad \text{from}(s(0), [s(0), X'']) \\
& \quad \text{from}(s(a(0)), X'')
\end{align*}
\]

It is intuitively pleasing to represent sequences of term rewriting reductions as trees. We call these rewriting trees to mark their resemblance to TRS \cite{Terese2003}. Full SLD-derivation steps can be represented by transitions between rewriting trees. These transitions are determined by most general unifiers of rewriting tree leaves with program clauses; see Section 2 below, as well as \cite{Johann2015}, for more detail. In \cite{Fu2015, Johann2015} this method of separating SLD-derivations into rewriting steps and unification-driven steps is called structural resolution, or S-resolution for short.

With S-resolution in hand, we can define a logic program to be (observationally) productive if it is finitely observable, i.e., if all of its rewriting trees are finite. Our “good” program above is again productive, whereas the “bad” one is not — but now “productive” means productive in this new observational sense. Productivity in S-resolution corresponds to termination in TRS, see \cite{Fu2015}. In addition, \cite{Komendantskaya2017} presents an algorithm and an implementation that semi-decides observational productivity.

One question remains: if termination is an effective pre-condition for semi-deciding inductive soundness in LP, can observational productivity in S-resolution become a similarly effective pre-condition for reasoning about computations at infinity?

To answer this question, we present a complete study of inductive and coinductive properties of S-resolution, and establish the following results:

— S-resolution is inductively sound and complete relative to the least Herbrand model semantics;
— S-resolution is coinductively sound relative to the greatest Herbrand model semantics;
— Infinite observationally productive computations by S-resolution are sound and complete relative to SLD-computations at infinity;

The above results prove that indeed the notion of observational productivity simplifies reasoning about global productivity (given by SLD-computations at infinity).

However, thanks to separation of rewriting and substitution steps, S-resolution can be soundly used to lazily observe finite fragments of infinite irrational terms. We introduce an algorithm for such sound observations, we attach an implementation to this paper: https://github.com/coalp

1.5. Paper overview

The paper proceeds as follows. In Section 2 we introduce background definitions concerning LP, including least and greatest complete Herbrand model semantics and operational semantics of SLD- and S-resolution given by reduction systems. In Section 2.4 we prove the soundness, and show
the incompleteness, of S-resolution reductions with respect to least Herbrand models. In Section 3 we regain completeness of proof search by introducing rewriting trees and rewriting tree transitions (which we call S-derivations), and proving the soundness and completeness of successful S-derivations with respect to least Herbrand models. This completes the discussion of inductive properties of proof-search by S-resolution, and lays the foundation for developing a new coinductive operational semantics for LP via S-resolution in Sections 4 and 5. In Section 4 we define S-computations at infinity and show that they are sound and complete relative to SLD-computations at infinity. We reconstruct the standard soundness result for computations at infinity relative to greatest complete Herbrand models, but now for S-computations at infinity.

In Section 5 we conclude and discuss related work.

2. PRELIMINARIES

In this section we introduce Horn clauses, and recall the declarative (big-step) semantics of logic programs given by least and greatest complete Herbrand models. We also introduce structural resolution by means of an operational (small-step) semantics. To enable the analysis of coinductive semantics and infinite terms, we adopt the standard view of terms as 

Courcelle 1983, Jaffar and Stuckey 1986, Lloyd 1987.

2.1. First-Order Signatures, Terms, Clauses

We write \( \mathbb{N}^* \) for the set of all finite words over the set \( \mathbb{N} \) of natural numbers. The length of \( w \in \mathbb{N}^* \) is denoted \( |w| \). The empty word \( \varepsilon \) has length 0; we identify \( i \in \mathbb{N} \) and the word \( i \) of length 1. Letters from the end of the alphabet denote words of any length, and letters from the middle of the alphabet denote words of length 1. The concatenation of \( w \) and \( u \) is denoted \( wu \); \( v \) is a prefix of \( w \) if there exists a \( u \) such that \( w = vu \), and a proper prefix of \( w \) if \( u \neq \varepsilon \).

A set \( L \subseteq \mathbb{N}^* \) is a (finitely branching) tree language provided: i) for all \( w \in \mathbb{N}^* \) and all \( i, j \in \mathbb{N} \), if \( wj \in L \) then \( w \in L \) and, for all \( i < j \), \( wi \in L \); and ii) for all \( w \in L \), the set of all \( i \in \mathbb{N} \) such that \( wi \in L \) is finite. A non-empty tree language always contains \( \varepsilon \), which we call its root. The depth of a tree language \( L \) is the maximum length of a word in \( L \). A tree language is finite if it is a finite subset of \( \mathbb{N}^* \), and infinite otherwise. A word \( w, v \in L \) is also called a node of \( L \). If \( w = w_0w_1...w_l \) then \( w_0w_1...w_k \) for \( k < l \) is an ancestor of \( w \). The node \( w \) is the parent of \( i, v \), and nodes \( w_i \) for \( i \in \mathbb{N} \) are children of \( w \). A branch of a tree language \( L \) is a subset \( L' \) of \( L \) such that, for all \( w, v \in L' \), \( w \) is an ancestor of \( v \) or \( v \) is an ancestor of \( w \). If \( L \) is a tree language and \( w \) is a node of \( L \), the subtree of \( L \) at \( w \) is \( L[w = \{ v \mid vw \in L \}] \).

A signature \( \Sigma \) is a non-empty set of function symbols, each with an associated arity. The arity of \( f \in \Sigma \) is denoted \( \text{arity}(f) \). To define terms over \( \Sigma \), we assume a countably infinite set \( \text{Var} \) of variables disjoint from \( \Sigma \), each with arity 0. Capital letters from the end of the alphabet denote variables in \( \text{Var} \). If \( L \) is a non-empty tree language and \( \Sigma \) is a signature, then a term over \( \Sigma \) is a function \( t: L \rightarrow \Sigma \cup \text{Var} \) such that, for all \( w \in L \), \( \text{arity}(t(w)) = |\{i \mid wi \in L\}| \). Terms are finite or infinite according as their domains are finite or infinite. A term \( t \) has a depth \( \text{depth}(t) = \max\{|w| \mid w \in L\} \). The subtree \( \text{subterm}(t, w) \) of \( t \) at node \( w \) is given by \( t'(v) = t(wv) \) for each \( v \in L \).

The set of finite (infinite) terms over a signature \( \Sigma \) is denoted by \( \text{Term}^f(\Sigma) \) (\( \text{Term}^i(\Sigma) \)). The set of all (i.e., finite and infinite) terms over \( \Sigma \) is denoted by \( \text{Term}(\Sigma) \). Terms with no occurrences of variables are ground. We write \( \text{GTerm}(\Sigma) \) (\( \text{GTerm}^f(\Sigma) \), \( \text{GTerm}^i(\Sigma) \)) for the set of finite (infinite, all) ground terms over \( \Sigma \).

A substitution over \( \Sigma \) is a total function \( \sigma: \text{Var} \rightarrow \text{Term}(\Sigma) \). Substitutions are extended from variables to terms homomorphically: if \( t \in \text{Term}(\Sigma) \) and \( \sigma \in \text{Subst}(\Sigma) \), then the application \( \sigma(t) \) is \( (\sigma(t))(w) = t(w) \) if \( t(w) \notin \text{Var} \), and \( (\sigma(t))(w) = (\sigma(\chi))(\varepsilon) \) if \( w = uv \), \( t(u) = \chi \), and \( \chi \in \text{Var} \).

We say that \( \sigma \) is a grounding substitution for \( t \) if \( \sigma(t) \in \text{GTerm}(\Sigma) \), and is just a grounding substitution if its codomain is \( \text{GTerm}(\Sigma) \). We write \( \text{id} \) for the identity substitution. The set of all substitutions over a signature \( \Sigma \) is \( \text{Subst}(\Sigma) \) and the set of all substitutions over \( \Sigma \) with only finite terms in their codomains is \( \text{Subst}^f(\Sigma) \). Composition of substitutions is denoted by juxtaposition. Composition is associative, so we write \( \sigma_3\sigma_2\sigma_1 \) rather than \( (\sigma_3\sigma_2)\sigma_1 \) or \( \sigma_3(\sigma_2\sigma_1) \).
A substitution \( \sigma \in \text{Subst}(\Sigma) \) is a unifier for \( t, u \in \text{Term}(\Sigma) \) if \( \sigma(t) = \sigma(u) \), and is a matcher for \( t \) against \( u \) if \( \sigma(t) = u \) for some \( \sigma \in \text{Subst}^0(\Sigma) \); note that if \( t, u \in \text{Term}(\Sigma) \), i.e., if \( t \) and \( u \) are finite terms, then the codomain of \( \sigma \) can be taken, without loss of generality, to involve only finite terms. A substitution \( \sigma_1 \in \text{Subst}(\Sigma) \) is more general than a substitution \( \sigma_2 \in \text{Subst}(\Sigma) \) if there exists a substitution \( \sigma \in \text{Subst}^0(\Sigma) \) such that \( \sigma \sigma_1(X) = \sigma_2(X) \) for every \( X \in \text{Var} \). A substitution \( \sigma \in \text{Subst}(\Sigma) \) is a most general unifier (mgu) for \( t \) and \( u \), denoted \( t \sim_\sigma u \), if it is a unifier for \( t \) and \( u \) and is more general than any other such unifier. A most general matcher (mgm) \( \sigma \) for \( t \) against \( u \), denoted \( t \sim_\sigma u \), is defined analogously. Both mgus and mgms are unique up to variable renaming if they exist. Unification is reflexive, symmetric, and transitive, but matching is reflexive and transitive only. Mgus and mgms are computable by Robinson’s seminal unification algorithm [Robinson 1965].

In many unification algorithms, the occurs check condition is imposed, so that substitution bindings of the form \( X \mapsto t(X) \), where \( t(X) \) is a term containing \( X \), are disallowed. In this case, mgus and mgms can always be taken to be idempotent, i.e., such that the sets of variables appearing in their domains and codomains are disjoint. The occurs check is critical for termination of unification algorithms, and this is, in turn, crucial for the soundness of classical SLD-resolution; see below.

In logic programming, a clause \( C \) over \( \Sigma \) is a pair \( (A, \{B_0, \ldots, B_n\}) \), where \( A \in \text{Term}(\Sigma) \) and \( \{B_0, \ldots, B_n\} \) is a list of terms in \( \text{Term}(\Sigma) \); such a clause is usually written \( A \leftarrow B_0, \ldots, B_n \). Note that the list of terms can be the empty list \([\ ]\). We will identify the singleton list \([t]\) with the term \( t \) when convenient. The head \( A \) of \( \Sigma \) is denoted head\((C)\) and the body \( B_0, \ldots, B_n \) of \( C \) is denoted body\((C)\). A goal clause \( G \) over \( \Sigma \) is a clause ? \leftarrow B_0, \ldots, B_n \) over \( \Sigma \cup \{?\} \), where ? is a special symbol not in \( \Sigma \cup \text{Var} \). We abuse terminology and consider a goal clause over \( \Sigma \) to be a clause over \( \Sigma \). The set of all clauses over \( \Sigma \) is denoted Clause\((\Sigma)\). A logic program over \( \Sigma \) is a total function from a set \( \{0, 1, \ldots, n\} \subseteq \mathbb{N} \) to the set of non-goal clauses over \( \Sigma \). The clause \( P(i) \) is called the \( i \)-th clause of \( P \). If a clause \( C \) is \( P(i) \) for some \( i \), we write \( C \in P \). The set of all logic programs over \( \Sigma \) is denoted LP\((\Sigma)\).

The predicate of a clause \( C \) is the top symbol of the term head\((C)\). The predicates of a program are the predicates of its clauses. We assume that all logic programs are written within first-order Horn logic, with proper syntactic checks implied on the predicate position. If this assumption is made, the algorithm of SLD-resolution as well as other alternative algorithms we consider in the following sections do not introduce any syntactic inconsistencies to the operational semantics.

The arity of \( P \in \text{LP}(\Sigma) \) is the number of clauses in \( P \), i.e., is \( |\text{dom}(P)| \), and is denoted arity\((P)\). The arity of \( C \in \text{Clause}(\Sigma) \) is \( |\text{body}(C)| \), and is similarly denoted arity\((C)\).

We extend substitutions from variables to clauses and programs homomorphically. We omit these standard definitions. The variables of a clause \( C \) can be renamed with “fresh” variables to get an \( \alpha\)-equivalent clause that is interchangeable with \( C \). We assume variables have been renamed when convenient. This is standard and helps avoid circular (non-terminating) unification and matching.

2.2. Big-step Inductive and Coinductive Semantics for LP

We recall the least and greatest complete Herbrand model constructions for LP [Lloyd 1987]. We express the definitions in the form of a big-step semantics for LP, thereby exposing duality of inductive and coinductive semantics for LP in the style of [Sangiorgi 2012]. We start by giving inductive interpretations to logic programs.

**Definition 2.1.** Let \( P \in \text{LP}(\Sigma) \). The big-step rule for \( P \) is given by

\[
\frac{P \models \sigma(B_1), \ldots, P \models \sigma(B_n) \quad P \models \sigma(A)}{P \models \sigma(A)}
\]

where \( A \leftarrow B_1, \ldots, B_n \) is a clause in \( P \) and \( \sigma \in \text{Subst}^0(\Sigma) \) is a grounding substitution.

Following standard terminology (see, e.g., [Sangiorgi 2012]), we say that an inference rule is applied forward if it is applied from top to bottom, and that it is applied backward if it is applied
from bottom to top. If a set of terms is closed under forward (backward) application of an inference rule, we say that it is closed forward (resp., closed backward) under that rule. If the \(\ell^b\) clause of \(P \in \text{LP}(\Sigma)\) is involved in an application of the big-step rule for \(P\), then we may say that we have applied the big-step rule for \(P\).

Definition 2.2. The least Herbrand model for \(P \in \text{LP}(\Sigma)\) is the smallest set \(M_P \subseteq \text{GTerm}(\Sigma)\) that is closed forward under the big-step rule for \(P\).

Example 2.3. The least Herbrand model for \(P_1\) is \(\{\text{nat}(0), \text{nat}(s(0)), \text{nat}(s(s(0))), \ldots\}\).

The requirement that \(M_P \subseteq \text{GTerm}(\Sigma)\) entails that only ground substitutions \(\sigma \in \text{Subst}(\Sigma)\) are used in the forward applications of the big-step rule involved in the construction of \(M_P\). Next we give coinductive interpretations to logic programs. For this we do not impose any finiteness requirement on the codomain terms of \(\sigma\).

Definition 2.4. The greatest complete Herbrand model for \(P \in \text{LP}(\Sigma)\) is the largest set \(M_P^{\omega} \subseteq \text{GTerm}^{\omega}(\Sigma)\) that is closed backward under the big-step rule for \(P\).

Example 2.5. The greatest complete Herbrand model for \(P_1\) is \(\{\text{nat}(0), \text{nat}(s(0)), \text{nat}(s(s(0))), \ldots\} \cup \{\text{nat}(s(s(\ldots)))\}\). Indeed, there is an infinite inference for \(\text{nat}(s(s(\ldots)))\) obtained by repeatedly applying the big-step rule for \(P_1(1)\) backward.

Definitions 2.2 and 2.4 could alternatively be given in terms of least and greatest fixed point operators, as in, e.g., [Lloyd 1987]. To ensure that \(\text{GTerm}(\Sigma)\) and \(\text{GTerm}^{\omega}(\Sigma)\) are non-empty, and thus that the least and greatest Herbrand model constructions are as intended, it is standard in the literature to assume that \(\Sigma\) contains at least one function symbol of arity 0. We will make this assumption throughout the remainder of this paper.

2.3. Small-step Semantics for LP

Following [Fu and Komendantskaya 2015], we distinguish the following three kinds of reduction for LP.

Definition 2.6. If \(P \in \text{LP}(\Sigma)\) and \(t_1, \ldots, t_n \in \text{Term}(\Sigma)\), then

- SLD-resolution reduction: \([t_1, \ldots, t_i, \ldots, t_n] \leadsto_P [\sigma(t_1), \ldots, \sigma(t_i-1), \sigma(B_0), \ldots, \sigma(B_m), \sigma(t_{i+1}), \ldots, \sigma(t_n)]\) if \(A \leftarrow B_0, \ldots, B_m \in P\) and \(t_i \sim_{\sigma} A\).

- rewriting reduction: \([t_1, \ldots, t_i, \ldots, t_n] \rightarrow_P [t_1, \ldots, t_{i-1}, \sigma(B_0), \ldots, \sigma(B_m), t_{i+1}, \ldots, t_n]\) if \(A \leftarrow B_0, \ldots, B_m \in P\) and \(A \sim_{\sigma} t_i\).

- substitution reduction: \([t_1, \ldots, t_i, \ldots, t_n] \rightarrow_P [\sigma(t_1), \ldots, \sigma(t_i), \ldots, \sigma(t_n)]\) if \(A \leftarrow B_0, \ldots, B_m \in P\) and \(t_i \sim_{\sigma} A\).

We assume, as is standard in LP, that all variables are renamed apart when terms are matched or unified against the program clauses.

If \(r\) is any reduction relation, we will abuse terminology and call any (possibly empty) sequence of \(r\)-reduction steps an \(r\)-reduction. When there exists no list \(L\) of terms such that \([t_1, \ldots, t_i, \ldots, t_n] \rightarrow_P L\) we say that \([t_1, \ldots, t_n]\) is in \(\rightarrow\)-normal form with respect to \(P\). We write \([t_1, \ldots, t_n] \rightarrow^* P\) to indicate the reduction of \([t_1, \ldots, t_n]\) to its \(\rightarrow\)-normal form with respect to \(P\) if this normal form exists, and to indicate an infinite reduction of \([t_1, \ldots, t_n]\) with respect to \(P\) otherwise. We write \(\rightarrow^n\) to denote rewriting by at most \(n\) steps of \(\rightarrow\), where \(n\) is a natural number. We will use similar notations for \(\sim\) and \(\leadsto\) as required. Throughout this paper we may omit explicit mention of \(P\) and/or suppress \(P\) as a subscript on reductions when it is clear from context.

We are now in a position to define the structural resolution reduction, also called the S-resolution reduction for short. We have:

Definition 2.7. For \(P \in \text{LP}(\Sigma)\), the structural resolution reduction with respect to \(P\) is \(\leadsto^1_P \circ \rightarrow^\omega_P\).
A similar notion of observational productivity, in terms of strong normalisation of term rewriting, has recently been introduced for co-patterns in functional programming [Basold and Hansen 2015].

Program $P$ nevertheless fails to be observationally productive because there exist no inductively successful reductions. For SLD-resolution reductions this agrees with standard logic programming terminology.

If we regard the term $t$ as a “query”, then we may regard the composition $\sigma_n \circ \ldots \circ \sigma_1$ of the substitutions $\sigma_1, \ldots, \sigma_n \in \text{Subst}(\Sigma)$ involved in the steps of an inductively successful SLD-resolution reduction for $t$ as an “answer” to this query, and we may think of the reduction as computing this answer. Such a composition for an initial sequence of SLD-resolution reductions in a possibly non-terminating SLD-resolution reduction for $t$ can similarly be regarded as computing a partial answer to that query. We use this terminology for rewriting and S-resolution reductions as well.

Example 2.8. The following are SLD-resolution, rewriting, and S-resolution reductions, respectively, with respect to $F_1$:

$\text{nat}(X) \rightarrow \text{nat}(X', \text{nat}(Y)) \rightarrow \text{nat}(Y) \rightarrow \ldots$

$\text{nat}(X)$

$\text{nat}(X) \rightarrow^\mu [\text{nat}(X), \text{nat}(Y)] \rightarrow^\mu [\text{nat}(s\text{cons}(X', Y))] \rightarrow^\mu [\text{nat}(X'), \text{nat}(Y)] \rightarrow^\mu [\text{nat}(s\text{cons}(X', Y))] \rightarrow^\mu \ldots$

In the S-resolution reduction above, $[\text{nat}(X)] \rightarrow^\mu [\text{nat}(X)]$ in 0 steps, since $\text{nat}(X)$ is already in $\rightarrow$-normal form. The initial sequences of the SLD-resolution and S-resolution reductions each compute the partial answer $\{X \mapsto s\text{cons}(0, s\text{cons}(X', Y))\}$ to the query $\text{nat}(X)$.

The observation that, even for coinductive program like $P_2$, $\rightarrow^\mu$ reductions are finite and thus can serve as measures of finite observation, has led to the following definition of observational productivity in LP, first introduced in [Komendantskaya et al. 2014].

Definition 2.9. A program $P \in \text{LP}(\Sigma)$ is observationally productive if $\rightarrow^p$ is strongly normalising, i.e., if every rewriting reduction with respect to $P$ is finite.

Example 2.10. The programs $P_1, P_2, P_3, P_4$, and $P_5$ are all observationally productive, as is the program $P_6$ defined in Example 2.11 below. By contrast, the “bad” program of Example 2.13 and the program $P_6$ defined in Example 2.11 below are not.

A similar notion of observational productivity, in terms of strong normalisation of term rewriting, has recently been introduced for copatterns in functional programming [Basold and Hansen 2015].

A general analysis of observational productivity for LP is rather subtle. Indeed, there are programs $P$ and queries $t$ for which there are inductively successful SLD-resolution reductions, but for which $P$ nevertheless fails to be observationally productive because there exist no inductively successful S-resolution reductions.

Example 2.11. Consider the graph connectivity program $\text{Sterling and Shapiro 1986}$ $P_0$ given by:

0. $\text{conn}(X, Y) \leftarrow \text{conn}(X, Z), \text{conn}(Z, Y)$
1. $\text{conn}(a, b) \leftarrow$
2. $\text{conn}(b, c) \leftarrow$

Although there exist inductively successful SLD-resolution reductions for $\text{conn}(X, Y)$ with respect to $P_0$, there are no such inductively successful S-resolution reductions. Indeed, the only S-resolution reductions for $\text{conn}(X, Y)$ with respect to $P_0$ are infinite rewriting reductions that, with each rewriting reduction, accumulate an additional term involving $\text{conn}$. A representative example of such an $S$-
resolution reduction is

\[ \text{conn}(X,Y) \rightarrow \text{conn}(X',X''), \text{conn}(X',Y) \rightarrow \text{conn}(X,X''), \text{conn}(X',Y) \rightarrow \ldots \]

Thus, \( P_6 \) is not observationally productive.

With this in mind, we first turn our attention to analysing the inductive properties of S-resolution reductions.

### 2.4. Inductive Properties of S-Resolution Reductions

In this section, we discuss whether, and under which conditions, S-resolution reductions are inductively sound and complete. First we recall that SLD-resolution is inductively sound and complete \cite{Lloyd1987}. The standard results of inductive soundness and completeness for SLD-resolution \cite{Lloyd1987} can be summarised as:

**Theorem 2.12.** Let \( P \in \text{LP}(\Sigma) \) and \( t \in \text{Term}(\Sigma) \).

\( \iff \) (Inductive soundness of SLD-resolution reductions) If \( t \sim^{P_n}_P \) for some \( n \) and computes answer \( \theta \), then there exists a term \( t' \in \text{GTerm}(\Sigma) \) such that \( t' \in M_P \) and \( t' \) is an instance of \( \theta(t) \).

\( \iff \) (Inductive completeness of SLD-resolution reductions) If \( t \in M_P \), then there exists a term \( t' \in \text{Term}(\Sigma) \) that yields an SLD-resolution reduction \( t' \sim^{P_n}_P \) that computes answer \( \theta \in \text{Subst}(\Sigma) \) such that \( t \) is an instance of \( \theta(t') \).

We now show that, in contrast to SLD-resolution reductions, S-resolution reductions are inductively sound but incomplete. We first establish inductive soundness.

**Theorem 2.13.** (Inductive soundness of S-resolution reductions) If \( t \sim^{P_n}_P \) for some \( n \) and computes answer \( \theta \), then there exists a term \( t' \in \text{Term}(\Sigma) \) such that \( t' \in M_P \) and \( t' \) is an instance of \( \theta(t) \).

**Proof.** The proof is by induction on \( n \) in \( \sim^{P_n}_P \). It is a simple adaptation of the soundness proof for SLD-resolution reductions given in, e.g., \cite{Lloyd1987}.

To show that S-resolution reductions are not inductively complete, it suffices to provide one example of a program \( P \) and a term \( t \) such that \( P \models \theta(t) \) but no inductively successful S-resolution reduction exists for \( t \). We will in fact give two such examples, each of which is representative of a different way in which S-resolution reductions fail to be inductively complete.

**Example 2.14.** Consider \( P_6 \) and the S-resolution reduction shown in Example 2.11. The instantiation \( \text{conn}(a,c) \) of \( \text{conn}(X,Y) \) is in the least Herbrand model of \( P_6 \), but there are no finite S-resolution reductions, and therefore no inductively successful S-resolution reductions, for \( P_6 \) and the query \( \text{conn}(X,Y) \). This shows that programs that are not observationally productive need not be inductively complete.

In light of Example 2.14 it is tempting to try to prove the inductive completeness of S-resolution reductions for observationally productive logic programs only. However, this would not solve the problem, as the following example confirms:

**Example 2.15.** Consider the program \( P_7 \) given by:

0. \( p(c) \leftarrow \)
1. \( p(X) \leftarrow q(X) \)

We have that \( P_7 \models p(c) \) for the instantiation \( p(c) \) of \( p(X) \), but there is no inductively successful S-resolution reduction for \( P_7 \) and \( p(X) \).

Program \( P_7 \) is an example of overlapping program, i.e., a program containing clauses whose heads unify. We could show that, for programs that are both observationally productive and non-overlapping, S-resolution reductions are inductively complete. However, restricting attention to non-
overlapping programs would seriously affect generality of our results, and would have the effect of making S-resolution even less suited for inductive proof search than SLD-resolution. We prefer instead to refine S-resolution so that it is inductively complete for all programs. The question is whether or not such refinement is possible.

An intuitive answer to this question comes from reconsidering Example 2.15. There, the interleaving of $\rightarrow^\mu$ and $\rightarrow^1$ has the effect of restricting the search space. Indeed, once the rewriting portion of the only possible S-resolution reduction on $p(X)$ is performed, the new subgoal $q(X)$ prevents us from revisiting the initial goal $p(X)$ and unifying it with the clause $P_7(0)$, as would be needed for an inductively successful S-resolution reduction. This is how we lose inductive completeness of the proof search.

One simple remedy would be to redefine S-resolution reductions to be $(\rightarrow^1 \circ \rightarrow^n)$-reductions, where $n$ ranges over all non-negative integers. This would indeed restore inductive completeness of S-resolution for overlapping programs. But it would at the same time destroy our notion of observational productivity, which depends crucially on $\rightarrow^\mu$. An alternative solution would keep our definitions of S-resolution reductions and observational productivity intact, but also find a way to keep track of all of the unification opportunities arising in the proof search. This is exactly the route we take here.

Kowalski [Kowalski 1974] famously observed that Logic Programming = Logic + Control. For Kowalski, the logic component was given by SLD-resolution reductions, and the control component by an algorithm coding the choice of search strategy. As it turns out, SLD-resolution reductions are sound and complete irrespective of the control component. What we would like to do in this paper is revise the very logic of LP by replacing SLD-resolution reductions with S-resolution reductions. As it turns out, defining a notion of S-resolution that is both inductively complete and capable of capturing observational productivity requires imposing an appropriate notion of control on this logic. In the next section we therefore define S-resolution in terms of rewriting trees. Rewriting trees allow us to neatly integrate precisely the control on S-resolution reductions needed to achieve both of these aims for the underlying logic of S-resolution reductions. We thus arrive at our own variant of Kowalski’s formula, namely Structural Logic Programming = Logic + Control — but now the logic is given by S-resolution reductions and the control component is captured by rewriting trees. The remainder of the paper is devoted to developing the above formula into a formal theory.

3. INDUCTIVE SOUNDNESS AND COMPLETENESS OF STRUCTURAL RESOLUTION

To ensure that S-resolution reductions are inductively complete, we need to impose more control on the rewriting reductions involved in them. To do this, we first note that the rewriting reduction in Example 2.15 can be represented as the tree

\[
p(X) \\
\quad \mid \\
q(X)
\]

Now, we would also like to reflect within this tree the fact that $p(X)$ to unifies with the head of clause $P_7(0)$ and, more generally, to reflect the fact that any term can, in principle, unify with the head of any clause in the program. We can record these possible unifications in tree form, as follows:

\[
p(X) \\
\quad \mid \\
P_7(0) \mid q(X) \\
\quad \mid \\
P_7(0) \mid P_7(1)
\]

We can now follow-up each of these possibilities and in this way extend our proof search. To do this formally, we distinguish two kinds of nodes: and-nodes, which capture terms coming from clause bodies, and or-nodes, which capture the idea that every term can, in principle, match several clause heads. We also introduce or-node variables to signify the possibility of unifying a term with
the head of a clause when the matching of that term against that clause head fails. This careful tracking of possibilities allows us to construct the inductively successful S-resolution reduction for $p(X)$ and program $P_i$ shown in Figure 1. The figure depicts two rewriting trees, each modelling all possible rewriting reductions for the given query (represented as a goal clause) with respect to $P_i$. Rewriting trees have alternating levels of or-nodes and and-nodes, as well as or-node variables ($X_1$, $X_2$, and $X_3$ in the figure) ranging over rewriting trees. By unifying $p(X)$ with $P_i(0)$ we replace the or-node variable $X_1$ in the first rewriting tree with a new rewriting tree (in this case consisting of just the single node $p(c) \leftarrow$) to transition to the second rewriting tree shown. When a node contains a clause, such as $p(c) \leftarrow$, that has an empty body, it is equivalent to an empty subgoal. Thus, the underlined subtree of the second rewriting tree in Figure 1 represents the inductively successful S-resolution reduction $P_i \models p(c) \rightarrow [\]$.

3.1. Modeling $\rightarrow^\mu$ by Rewriting Trees

We now proceed to define the construction formally. For this, we first observe that a clause $C$ over a signature $\Sigma$ that is of the form $A \leftarrow B_0, \ldots, B_r$ can be naturally represented as the total function (also called $C$) from the finite tree language $L = \{\varepsilon, 0, \ldots, n\}$ of depth 1 to Term($\Sigma$) such that $C(\varepsilon) = A$ and $C(i) = B_i$ for $i = dom(C) \setminus \{\varepsilon\}$. With this representation of clauses in hand, we can formalise our notion of a rewriting tree.

**Definition 3.1.** Let $V_R$ be a countably infinite set of variables disjoint from $Var$. If $P \in LP(\Sigma)$, $C \in \text{Clause}(\Sigma)$, and $\sigma \in \text{Subst}(\Sigma)$ is idempotent, then the tree $\text{rew}(P, C, \sigma)$ is the function $T : \text{dom}(T) \rightarrow \text{Term}(\Sigma) \cup \text{Clause}(\Sigma) \cup V_R$, where $\text{dom}(T) \neq \emptyset$ is a tree language defined simultaneously with $\text{rew}(P, C, \sigma)$, such that:

1. $T(\varepsilon) = \sigma(C)$ and, for all $i \in \text{dom}(C) \setminus \{\varepsilon\}$, $T(i) = \sigma(C(i))$.
2. For $w \in \text{dom}(T)$ with $|w|$ even and $|w| > 0$, $T(w) \in \text{Clause}(\Sigma) \cup V_R$. Moreover,
   (a) if $T(w) \in V_R$, then $\{j \mid wj \in \text{dom}(T)\} = \emptyset$.
   (b) if $T(w) = B \in \text{Clause}(\Sigma)$, then there exists a clause $P(i)$ and a $\theta \in \text{Subst}(\Sigma)$ such that $\text{head}(P(i)) \leftarrow_\theta \text{head}(B)$. Moreover, for every $j \in \text{dom}(P(i)) \setminus \{\varepsilon\}$, $wj \in \text{dom}(T)$ and $T(wj) = \sigma(\theta(P(i)(j)))$.
3. For $w \in \text{dom}(T)$ with $|w|$ odd, $T(w) \in \text{Term}(\Sigma)$. Moreover, for every $i \in \text{dom}(P)$, we have
   (a) $wi \in \text{dom}(T)$.
   (b) $T(wi) = \left\{ \begin{array}{ll} \sigma(\theta(P(i))) & \text{if } \text{head}(P(i)) \leftarrow_\theta T(w) \\ \text{a fresh } X \in V_R & \text{otherwise} \end{array} \right.$
4. No other words are in $\text{dom}(T)$.

$T(w)$ is an or-node if $|w|$ is even, and an and-node if $|w|$ is odd.

We assume, as is standard in LP, that all variables are renamed apart when terms are matched or unified against the program clauses.
If $P \in \mathbf{LP}(\Sigma)$, then $T$ is a rewriting tree for $P$ if it is either the empty tree or $\text{rew}(P, C, \sigma)$ for some $C$ and $\sigma$. Since mgms are unique up to variable renaming, $\text{rew}(P, C, \sigma)$ is as well. A rewriting tree for a program $P$ is finite or infinite according as its domain is finite or infinite. We write $\text{Rew}(P)$ ($\text{Rew}^\omega(P)$, $\text{Rew}^\infty(P)$) for the set of all finite (infinite, all) rewriting trees for $P$.

This style of tree definition mimics the classical style of defining terms as maps from a tree language to a given domain [Lloyd 1987; Courcelle 1983]. As with tree representations of terms, _arity constraints_ are imposed on rewriting trees. The arity constraints in items 2b and 3a specify that the arity of an and-node is the number of clauses in the program and the arity of an or-node is the number of terms in its clause body. The arity constraint in item 2a specifies that or-node variables must have arity 0. Or-node variables indicate where in a rewriting tree substitution can take place.

Example 3.2. The rewriting trees $\text{rew}(P_3, ?, \leftarrow \text{fibs}(0, s(0), X), \text{id})$, $\text{rew}(P_3, ?, \leftarrow \text{fibs}(0, s(0), \text{cons}(0, S)), \text{id})$, and $\text{rew}(P_3, ?, \leftarrow \text{fibs}(0, s(0), \text{cons}(0, S)), \{Z \mapsto s(0))\}$ are shown in Figure 2. Note the or-node variables and the arities. An or-node can have arity 0, 1, or 2 according as its clause body contains 0, 1, or 2 terms, and every and-node has arity 3 because $P_3$ has three clauses.

Although perhaps mysterious at first, the third parameter $\sigma$ in Definition 3.1 for $T = \text{rew}(P, C, \sigma)$ is necessary account for variables occurring in $T$ not affected by mgms computed during $T$’s construction. It plays a crucial role in ensuring that applying a substitution to a rewriting tree again yields a rewriting tree [Johann et al. 2015]. We have:

**Definition 3.3.** Let $P \in \mathbf{LP}(\Sigma)$, $C \in \text{Clause}(\Sigma)$, $\sigma, \sigma' \in \text{Subst}(\Sigma)$ idempotent, and $T = \text{rew}(P, C, \sigma)$. Then the rewriting tree $\sigma'(T)$ is defined as follows:

1. for every $w \in \text{dom}(T)$ such that $T(w)$ is an and-node or non-variable or-node, $(\sigma'(T))(w) = \sigma'(T(w))$.
2. for every $wi \in \text{dom}(T)$ such that $T(wi) \in V_R$, if $\text{head}(P(i)) \prec_\theta \sigma'(T)(v)$, then $(\sigma'(T))(wi) = \text{rew}(P, \theta(P(i)), \sigma'\sigma)(v)$. (Note that $v = \varepsilon$ is possible.) If no mgm of $\text{head}(P(i))$ against $\sigma'(T)(w)$ exists, then $(\sigma'(T))(wi) = T(wi)$.

Both conditions in the above definition are critical for ensuring that $\sigma'(T)$ satisfies Definition 3.1. We then have the following substitution theorem for rewriting trees. It is proved in [Johann et al. 2015]. The operation of substitution on rewriting trees is also introduced in the same way in [Bonchi and Zanasi 2015].

**Theorem 3.4.** Let $P \in \mathbf{LP}(\Sigma)$, $C \in \text{Clause}(\Sigma)$, and $\theta, \sigma \in \text{Subst}(\Sigma)$. Then $\theta(\text{rew}(P, C, \sigma)) = \text{rew}(P, C, \theta\sigma)$.

We can now formally establish the relation between rewriting reductions and rewriting trees. We first have the following proposition, which is an immediate consequence of Definitions 2.9 and 3.1.

**Proposition 3.5.** $P \in \mathbf{LP}(\Sigma)$ is observationally productive iff, for every term $t \in \text{Term}(\Sigma)$ and every substitution $\sigma \in \text{Subst}(\Sigma)$, $\text{rew}(P, ?, \leftarrow t, \sigma)$ is finite.

We can further establish a correspondence between certain subtrees of rewriting trees and inductively successful S-resolution reductions.

**Definition 3.6.** A tree $T'$ is a rewriting subtree of a rewriting tree $T$ if $\text{dom}(T') \subseteq \text{dom}(T)$ and the following properties hold:

1. $T'(\varepsilon) = T(\varepsilon)$.
2. If $w \in \text{dom}(T')$ with $|w|$ even, then $T'(w) = T(w)$, $wi \in \text{dom}(T')$ for every $wi \in \text{dom}(T)$, and $T'(wi) = T(wi)$.
3. If $w \in \text{dom}(T')$ with $|w|$ odd, then $T'(w) = T(w)$, there exists a unique $i$ with $wi \in \text{dom}(T)$ such that $wi \in \text{dom}(T')$, and $T'(wi) = T(wi)$ for this $i$. 

A.14
Rewriting subtrees can be either finite or infinite. Note that the and-nodes in item 2 grow children by universal quantification, whereas the or-nodes in item 3 grow them by existential quantification.

**Definition 3.7.** If \( T \in \text{Rew}^{0}(P) \), then an or-node of \( T \) is an **inductive success node** if it is a non-variable leaf node of \( T \). If \( T' \) is a finite rewriting subtree of \( T \) all of whose leaf nodes are inductive success nodes of \( T \), then \( T' \) is an **inductive success subtree** of \( T \). If \( T \) contains an inductive success subtree then we call \( T \) an **inductive success tree**.

The following proposition is immediate from Definitions 3.1 and 3.7.

**Proposition 3.8.** If \( P \in \text{LP}(\Sigma) \) and \( t \in \text{Term}(\Sigma) \), then \( P \vdash t \rightarrow^{n} [\_] \) for some \( n \) iff \( \text{rew}(P, ? \leftarrow t, \sigma) \) is an inductive success tree.

With these preliminary results in hand we can now begin to show that rewriting trees impose on S-resolution reductions precisely the control required to prove their soundness and completeness with respect to least Herbrand models. We first observe that:

**Theorem 3.9.** Let \( P \in \text{LP}(\Sigma) \) and \( t \in \text{Term}(\Sigma) \).

- If \( \text{rew}(P, ? \leftarrow t, \sigma) \) is an inductive success tree for some \( \sigma \in \text{Subst}(\Sigma) \) then, for every instance \( t' \in \text{GTerm}(\Sigma) \) of \( \sigma(t), t' \in \text{M}_{P} \).

- If \( t \in \text{M}_{P} \), then there exists a grounding substitution \( \theta \in \text{Subst}(\Sigma) \) such that \( \text{rew}(P, ? \leftarrow t, \theta) \) is an inductive success tree.

**Proof.** The proof is by induction on the depth of rewriting trees. \( \square \)

**Example 3.10.** The term \( \text{conn}(a, c) \) is in \( \text{M}_{P} \). The tree \( \text{rew}(P_{0}, ? \leftarrow \text{conn}(a, c), \text{id}) \) is not an inductive success tree, as Figure 3 shows. However, \( \text{rew}(P_{0}, ? \leftarrow \text{conn}(a, c), \theta) \), for \( \theta = \{Z \mapsto b\} \), is indeed an inductive success tree. This accords with Theorem 3.9.

### 3.2. Modeling \( \leftarrow \) by Transitions Between Rewriting Trees

Next we define transitions between rewriting trees. Such transitions are defined by the familiar notion of a resolvent, and assume a suitable algorithm for renaming “free” clause variables apart [Johann et al. 2015]. Let \( P \in \text{LP}(\Sigma) \) and \( t \in \text{Term}(\Sigma) \). If \( \text{head}(P(i)) \sim_{\theta} t \), then \( \theta \) is called the
rew (P₀, ? ← conn (a, c), id) is not an inductive success tree. However, the right tree rew (P₀, ? ← conn (a, c), {Z ← b}) is. The inductive success subtree of the right tree is underlined.

**Definition 3.11.** Let \( T = \text{rew} (P, C, \sigma) \in \text{Rew}^ω (P) \). If \( X = T(w) \in V_R \), then the rewriting tree \( T_X \) is defined as follows: If the external resolvent \( \theta \) for \( P(i) \) and \( T(w) \) is null, then \( T_X = \text{rew} (P, C, \theta \sigma) \).

If \( X \in V_R \), we denote the computation of \( T_X \) from \( T \in \text{Rew}^ω (Σ) \) by \( T \rightarrow T_X \). The operation \( T \rightarrow T_X \) is a tree transition for \( P \) and \( C \); specifically, we call the tree transition \( T \rightarrow T_X \) the tree transition for \( T \) with respect to \( X \). A tree transition for \( P \in \text{LP}(Σ) \) is a tree transition for \( P \) and some \( C \in \text{Clause}(Σ) \). If \( T \rightarrow T_X \) is a tree transition and if \( X = T(w) \), then we say that both the node \( T(w) \) and the branch of \( T \) that this node lies on are expanded in this transition. A (finite or infinite) sequence \( T_0 = \text{rew} (P, ?, t, id) \rightarrow T_1 \rightarrow T_2 \rightarrow \ldots \) of tree transitions for \( P \) is a structural tree resolution derivation, or simply an S-derivation for short, for \( P \) and \( t \). An S-derivation for \( P \) and \( t \) is said to be an S-refutation, or an inductive proof, for \( t \) with respect to \( P \), if it is of the form \( T_0 \rightarrow T_1 \rightarrow \ldots \rightarrow T_n \) for some \( n \), where \( T_n \) is an inductive success tree. Figure 2 shows an initial fragment of an infinite S-derivation for the program \( P_3 \) and \( \text{fib}s(0, s(0), X) \). The derivations shown in Figures 1 and 3 are inductive proofs for \( P_1 \) and \( p(c) \), and for \( P_0 \) and \( \text{conn}(a, c) \), respectively. Note that the final trees of Figures 1 and 3 show nodes corresponding to (finite) inductively successful S-reductions for \( P_1 \) and \( p(c) \), and for \( P_0 \) and \( \text{conn}(a, c) \), respectively, underlined.

If each \( \theta_i \) is the external resolvent associated with the tree transition \( T_{i-1} \rightarrow T_i \) in an S-derivation \( T_0 = \text{rew} (P, ?, t, id) \rightarrow T_1 \rightarrow \ldots \rightarrow T_n \), then \( \theta_1, \ldots, \theta_n \) is the sequence of resolvents associated with that S-derivation. In this case, each tree \( T_i \) in the S-derivation is given by \( \text{rew} (P, ?, t, \theta_1, \ldots, \theta_i) \). Note how the third parameter composes the mgsu.

**Example 3.12.** The S-derivation in Figure 2 starts with \( \text{rew} (P_3, ?, f \text{ib}s(0, s(0), X), id) \). Its second tree can be seen as \( \text{rew} (P_3, ?, f \text{ib}s(0, s(0), X), \theta_1) \), where \( \theta_1 = \{ X \mapsto \text{cons}(0, S) \} \), and its third tree as \( \text{rew} (P_3, ?, f \text{ib}s(0, s(0), X), \theta_2, \theta_1) \), where \( \theta_2 = \{ Z \mapsto s(0) \} \). Here, \( \theta_1 \) and \( \theta_2 \) are the resolvents for the tree transitions for the first and the second trees with respect to \( X_3 \) and \( X_4 \), respectively.

We have just formally rendered the formula *Structural Logic Programming = S-Resolution Reductions + Control*: we embedded proof search choices and or-node variable substitutions into S-resolution reductions via rewriting trees, thus obtaining the notion of an S-derivation and the inductive proof methodology we call *structural resolution*, or *S-resolution* for short. It now remains to exploit the inductive and coinductive properties of our new theory of S-resolution.

3.3. Inductive Soundness and Completeness of S-Resolution

Before exploiting the coinductive properties of S-resolution we investigate its inductive properties. Some S-derivations for a program \( P \) and a term \( t \) may be S-refutations and some not, but termination of one S-derivation in other than an inductive success tree does not mean no S-refutation exists for
P and t. This reflects the facts that inductive success is an existential property, and that entailment for Horn clauses is only semi-decidable. In this section we present our inductive soundness and completeness results for S-resolution. We note that these do not require logic programs to be either observationally productive or non-overlapping.

Example 3.13. An S-derivation for the program $P_6$ and $\text{conn}(a, c)$ is shown in Figure 3. The program $P_6$ is not observationally productive. An inductive success subtree of the derivation’s final tree is indicated by underlining. It contains the inductive success nodes labelled $\text{conn}(a, b) \leftarrow$ and $\text{conn}(b, c) \leftarrow$. Since its final tree is an inductive success tree, this S-derivation is an S-refutation for $P_6$ and $\text{conn}(a, c)$.

Example 3.14. An S-refutation for the overlapping program $P_7$ and $p(c)$ is shown in Figure 1. An inductive success subtree of the derivation’s final tree is indicated by underlining.

Inductive soundness and completeness of S-resolution are simple corollaries of Theorem 3.9.

**Theorem 3.15.** Let $P \in \mathbf{LP}(\Sigma)$ and $t \in \mathbf{Term}(\Sigma)$.

— (Inductive soundness of S-resolution) If there is an S-refutation for $P$ and $t$ that computes answer $\theta$, and $t'$ is a ground instance of $\theta(t)$, then $t' \in \mathbf{GTerm}(\Sigma)$.

— (Inductive completeness of S-resolution) If $t \in M_P$, then there exists a term $t' \in \mathbf{Term}(\Sigma)$ that yields an S-refutation for $P$ and $t'$ that computes answer $\theta \in \mathbf{Subst}(\Sigma)$ such that $t$ is an instance of $\theta(t')$.

We also have the following corollary of Theorem 3.9.

**Corollary 3.16.** Let $P \in \mathbf{LP}(\Sigma)$ and $t \in \mathbf{Term}(\Sigma)$. If there is an S-refutation $T_0 = \text{rew}(P, ? \leftarrow t, id) \rightarrow T_1 \rightarrow \ldots \rightarrow T_n$ with associated external resolvents $\sigma_1, \ldots, \sigma_n$ then, for all grounding substitutions $\theta \in \mathbf{Subst}(\Sigma)$ for $\sigma_1 \ldots \sigma_n(\theta)$, $\theta \sigma_1 \ldots \sigma_n \in M_P$.

For an S-refutation $\text{rew}(P, ? \leftarrow t, id) \rightarrow T_1 \rightarrow \ldots \rightarrow T_n$ with associated external resolvents $\sigma_1, \ldots, \sigma_n$, the rewriting tree $T_n = \text{rew}(P, ? \leftarrow t, \sigma_n, \ldots, \sigma_1)$ can be regarded as a proof witness constructed for the query $t$.

The correspondence between the soundness and completeness of S-refutations and the classical theorems of LP captures the (existential) property of inductive success in S-resolution reductions. Our results do not, however, mention failure, which is a universal (and thus more computationally expensive) property to establish. Theorems 3.9 and 3.15 also show that rewriting trees can distinguish derivations proving logical entailment existentially --- i.e., for some (ground) instances only --- from those proving it universally --- i.e., for all (ground) instances. Indeed, Theorems 3.9 and Theorem 3.15 show that proof search by unification has existential properties.

Example 3.17. Since $\text{rew}(P_7, ? \leftarrow \text{nat}(X), id)$ is not an inductive success tree, $P_7$ does not logically entail the universally quantified formula $\forall X. \text{nat}(X)$. Similarly, since $\text{rew}(P_6, ? \leftarrow \text{conn}(X, Y), id)$ is not an inductive success tree, $P_6$ does not logically entail $\forall X, Y. \text{conn}(X, Y)$. On the other hand, if we added a clause $\text{conn}(X, X) \leftarrow$ to $P_6$, then, for resulting program $P_6'$, $\text{rew}(P_6', ? \leftarrow \text{conn}(X, X), id)$ would be an inductive success tree, and we would be able to infer that $P_6'$ does indeed logically entail $\forall X. \text{conn}(X, X)$.

Throughout this section, finiteness of inductive success subtrees (and thus of their corresponding rewriting reductions and S-derivations) has served as a precondition for our inductive soundness and completeness results. In the next section we restore the broken symmetry by defining coinductive proof methods that require observational productivity of S-derivations as a precondition of coinductive soundness.

4. COINDUCTIVE SOUNDNESS OF S-RESOLUTION

In this section, we show that S-resolution can capture not just inductive declarative and operational semantics of LP, but coinductive semantics as well. We start by defining greatest complete Herbrand
models of logic programs, following [Lloyd 1987] closely, then proceed by defining a notion of S-computations at infinity, and conclude with a soundness theorem relating the two. We take time to compare the computational properties of SLD-computations at infinity and S-computations at infinity, and prove that the latter extends the former. Since this section develops the theory of S-resolution for coinductive LP, observational productivity is a necessary precondition for establishing its results.

A first attempt to give an operational semantics corresponding to greatest complete Herbrand models of logic programs was captured by the notion of a computation at infinity for SLD-resolution [van Emden and Abdallah 1985; Lloyd 1987]. Computations at infinity are usually given relative to an ultrametric on terms, constructed as follows:

\[ \text{Definition 4.1.} \quad \text{A truncation for a signature } \Sigma \text{ is a mapping } \gamma : \mathbb{N} \times \text{Term}^\omega(\Sigma) \rightarrow \text{Term}(\Sigma \cup \diamond), \]

where \( \diamond \) is a new nullary symbol not in \( \Sigma \), and, for all \( t \in \text{Term}^\omega(\Sigma) \) and \( n \in \mathbb{N} \), the following conditions hold:

\[ \begin{align*}
\text{— } & \text{dom}(\gamma(n,t)) = \{ m \in \text{dom}(t) \mid |m| \leq n \}, \\
\text{— } & \gamma(n,t) = t(m) \text{ if } |m| < n, \text{ and} \\
\text{— } & \gamma(n,t) = \diamond \text{ if } |m| = n.
\end{align*} \]

For \( t, s \in \text{Term}^\omega(\Sigma) \), we define \( \gamma(s,t) = \min\{n \mid \gamma(n,s) \neq \gamma(n,t)\} \), so that \( \gamma(s,t) \) is the least depth at which \( t \) and \( s \) differ. If we further define \( d(s,t) = 0 \) if \( s = t \) and \( d(s,t) = 2^{-\gamma(s,t)} \) otherwise, then \( (\text{Term}^\omega(\Sigma), d) \) is an ultrametric space.

The definition of SLD-computable at infinity relative to a given ultrametric is taken directly from [Lloyd 1987]:

\[ \text{Definition 4.2.} \quad \text{An SLD-resolution reduction is fair if either it is finite, or it is infinite and, for } \text{every atom } B \text{ appearing in some goal in the SLD-derivation, (a further instantiated version of) } B \text{ is chosen within a finite number of steps. The term } t \in \text{GTerm}^\omega(\Sigma) \text{ is SLD-computable at infinity with respect to a program } P \in \text{LP}(\Sigma) \text{ if there exist a } t' \in \text{Term}(\Sigma) \text{ and an infinite fair SLD-resolution reduction } C_0 = t', G_1, G_2, \ldots, G_k, \ldots \text{ with mgus } \theta_1, \theta_2, \ldots, \theta_k \ldots \text{ such that } d(t, \theta_1 \ldots \theta_k(t')) \rightarrow 0 \text{ as } k \rightarrow \infty. \text{ If such a } t' \text{ exists, we say that } t \text{ is SLD-computable at infinity by } t'. \]

The fairness requirement ensures that infinite SLD-resolution reductions that infinitely resolve against some subgoals while completely ignoring others do not satisfy the definition of SLD-computable at infinity. For example, \( \text{from}(0, [0, \lambda(x). \lambda(y). x], \lambda(x). [0, \lambda(x). x], \ldots]) \) is not SLD-computable at infinity by \( P_3 \) because no computation that infinitely resolves with subgoals involving only \( \text{from} \) is fair.

In this section we see that SLD-Computations at Infinity = Global Productivity + Control. Here, “global productivity” (as opposed to observational productivity) requires that each fair infinite SLD-resolution reduction for a program computes an infinite term at infinity. The “control” component determines the proof search strategy for SLD-computations at infinity to be constrained by fairness. We will see other variations on Kowalski’s formula below.

Letting \( P \in \text{LP}(\Sigma) \) and defining \( C_P = \{ t \in \text{GTerm}^\omega(\Sigma) \mid t \text{ is SLD-computable at infinity with respect to } P \text{ by some } t' \in \text{Term}(\Sigma) \} \), we have that \( C_P \subseteq M_P^\omega \) (van Emden and Abdallah 1985; Lloyd 1987).

4.1. S-Computations at Infinity

We can define a notion of computation at infinity for S-resolution to serve as an analogue of Definition 4.2 for SLD-resolution. As a method of “control” appropriate to S-resolution, we introduce light typing for signatures, similar to that in [Gupta et al. 2007; Simon et al. 2007]. We introduce two types — namely, inductive and coinductive — together with, for any signature \( \Sigma \), a typing function \( \text{Ty} : \Sigma \rightarrow \tau \) for \( \Sigma \) that marks each symbol in \( \Sigma \) as one or the other. We adopt the convention that

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any symbol not explicitly marked as coinductive is taken to be marked as inductive by default. We note that in SLD-computations at infinity all symbols are implicitly marked as coinductive.

We extend the typing as inductive or coinductive from symbols to terms and to nodes of rewriting trees. A term $t \in \text{Term}(\Sigma)$ is inductive or coinductive according as $t(\varepsilon)$ is. If $P \in \text{LP}(\Sigma)$ and $T \in \text{Rew}^P(P)$, then an and-node $T(w)$ is coinductive if $T(w)(\varepsilon)$ is coinductive, and is inductive otherwise; an or-node in $T(w)$ is inductive or coinductive according as its parent node is. A variable or-node $T(w) = X$ is open if there exists a tree transition $T \rightarrow T_X$, and is closed otherwise. A variable or-node is coinductively open if it is open and coinductive. If $T'$ is a rewriting subtree of $T$, then $T'$ is coinductively open if it contains coinductively open nodes, and is inductively closed if all of its open nodes are coinductive.

S-computations at infinity focus on observationally productive programs and rely on properties of lightly typed rewriting trees. We have:

**Definition 4.3.** Let $P \in \text{LP}(\Sigma)$ be observationally productive, let $Ty$ be a typing function for $\Sigma$, and let $t \in \text{GTerm}^\omega(\Sigma)$. We say that $t' \in \text{Term}(\Sigma)$ finitely approximates $t$ with respect to $P$ and $Ty$, or is a finite approximation of $t$ with respect to $P$ and $Ty$, if the following hold:

1. There is an infinite S-derivation $T_0 = \text{rew}(P, t \leftarrow \varepsilon, \text{id}) \rightarrow T_1 \rightarrow \ldots T_k \rightarrow \ldots$ with associated resolvents $\theta_1, \theta_2, \ldots \theta_k$ such that $d(t, \theta_1, \ldots, \theta_k(t')) \rightarrow 0$ as $k \rightarrow \infty$.
2. This derivation contains infinitely many trees $T_{i_1}, T_{i_2}, \ldots$ with an infinite sequence of corresponding rewriting subtrees $T'_{i_1}, T'_{i_2}, \ldots$ such that
   i. each $T'_{i_j}$ is inductively closed and coinductively open
   ii. each coinductive variable node is open and, for each such node $T'_{i_j}(w)$ in each $T'_{i_j}$, there exists $m > j$ such that $T'_{i_m}(w)$ is coinductively open for some $v$.

Then $t$ is S-computable at infinity with respect to $P$ and $Ty$ if there is a $t' \in \text{Term}(\Sigma)$ such that $t'$ finitely approximates $t$ with respect to $P$ and $Ty$. We define $S^\infty_P = \{ t \in \text{GTerm}^\omega(\Sigma) | t$ is S-computable at infinity with respect to $P$ and $Ty \}$.

Here we see that S-Computations at Infinity = Global Productivity of S-Derivations + Control. The first condition in Definition 4.3 ensures “global productivity” and the second is concerned with “control”. But Definition 4.3’s requirement that programs are observationally productive is also used to control S-derivations via observations. We will see below that, as the “control” component becomes increasingly sophisticated, it can capture richer cases of coinductive entailment than ever before.

**Example 4.4.** Consider $P_3$ and let $Ty$ be the type function marking (only) the predicate fibs as coinductive. If $t' = \text{fibs}(0, s(0). X)$, then $t'$ finitely approximates, with respect to $P_3$ and $Ty$, the infinite ground term $t^*$ from Example 1.5 representing the stream of Fibonacci numbers. Thus $t^*$ is S-computable at infinity with respect to $P_3$ and $Ty$. Figure 2 shows an initial fragment of the S-derivation witnessing this. The infinite term $t^*$ is also SLD-computable at infinity with respect to $P_3$.

Each of the “control” requirements i, ii, and iii in Definition 4.3 is crucial to the correct formulation of a notion of a finite approximation for S-resolution, and thus to the notion of S-computability at infinity. For Condition i, we note that some S-derivations expand inductive nodes infinitely, which can block the expansion of coinductive nodes. We do not want such S-derivations to be valid finite approximations. For example, we want $\text{nats}(s\text{cons}(0, s\text{cons}(0, \ldots )))$ to be S-computable at infinity with respect to $P_2$ if nats is marked coinductive and nat is marked inductive, but we do not want $\text{nats}(s\text{cons}(s(\ldots )). Y))$ to be so computable. Condition i ensures that only S-derivations that infinitely expand only coinductive nodes are valid finite approximations.

For Condition ii, we note that some S-derivations may have unsuccessful inductive subderivations. We do not want these to be valid finite approximations. For example, $P_3$ admits such deriva-
tions. Condition ii ensures that only S-derivations with successful inductive subderivations are valid finite approximations.

For Condition iii, we note that even within one rewriting subtree there may be several choices of coinductive nodes to expand in a S-derivation. We want all such nodes to be infinitely expanded in a valid finite approximation. For example, if \( P_k \) comprises the clauses of \( P_2 \) and \( P_3 \) with \( \text{fib} \) and \( \text{nats} \) marked coinductive, together with \( \text{fibnats}(0,X) \leftarrow \text{fib}(0,a(0),X),\text{nats}(Y) \), then S-derivations that infinitely expand \( \text{fib} \) but only finitely expand \( \text{nats} \) compute at infinity terms of the form \( \text{fibnats}(\text{cons}(0,(\text{cons}(a(0),...)),\text{scons}(t_1,t_2))) \) for some finite terms \( t_1 \) and \( t_2 \). Since these computations do not expose the coinductive nature of \( \text{nats} \), we do not want these to be valid finite approximations. But we do want S-derivations that compute terms of the form \( \text{fibnats}(\text{cons}(0,(\text{cons}(a(0),...)),\text{scons}(\ldots,\text{cons}(\ldots))) \) to be valid finite approximations. Condition iii ensures that only S-derivations infinitely expanding all coinductive nodes are valid finite approximations.

### 4.2. Soundness of S-Computations at Infinity

We now investigate the relationship between SLD- and S-computations at infinity. The next two examples show that, for a given \( P \in \text{LP}(\Sigma) \) and a typing function \( Ty \) for \( \Sigma \), \( C_P \subseteq S^Ty_P \) needs not hold.

**Example 4.5.** To see that \( C_P \subseteq S^Ty_P \) needs not hold, we first note that the infinite term \( t = nats(s(s(...))) \) is SLD-computable at infinity with respect to \( P_1 \) by \( nats(X) \), and is thus in \( C_{P_1} \). But if \( Ty \) marks \( nats \) as inductive, then \( t \notin S^Ty_P \). Similarly, in the mixed inductive-coinductive setting we have that \( t' = nats(\text{scons}(s(s(...)),\text{scons}(0,\text{scons}(a(0),...))) \) is SLD-computable at infinity with respect to \( P_2 \) by \( nats(X) \), and is thus in \( C_{P_2} \). But if \( Ty \) is the typing function that marks only \( \text{nats} \) as coinductive then, since \( \text{nats} \) is (implicitly) marked as inductive, \( t' \notin S^Ty_P \).

Although for any specific typing function \( Ty \) we need not have \( C_P \subseteq S^Ty_P \), considering all typing functions simultaneously recovers a connection between SLD-computability at infinity and S-computability at infinity.

**Definition 4.6.** If \( P \in \text{LP}(\Sigma) \) is observationally productive, then \( \bar{S}_P = \bigcup\{ S^Ty_P | Ty \) is a typing function for \( \Sigma \} \).

The rest of this section formalises the relationship between \( C_P, \bar{S}_P \), and \( M^0_P \).

**Proposition 4.7.** Let \( P \in \text{LP}(\Sigma) \) be observationally productive. The infinite term \( t \in \text{Term}^\omega(\Sigma) \) is SLD-computable at infinity by \( t' \in \text{Term}(\Sigma) \) with respect to \( P \) if and only if there exists a typing function \( Ty \) for \( \Sigma \) such that \( t \) is S-computable at infinity by \( t' \) with respect to \( P \) and \( Ty \).

**Proof.** We must show that, for any \( t \in \text{GTerm}^\omega(\Sigma) \), if \( t \) is SLD-computable at infinity by \( t' \) with respect to \( P \), then there is a typing function \( Ty \) for \( \Sigma \) such that \( t \) is S-computable at infinity by \( t' \) with respect to \( P \) and \( Ty \). Since \( t \) is SLD-computable at infinity, there exist a \( t' \in \text{Term}(\Sigma) \) and an infinite fair SLD-resolution reduction \( D \) of the form \( G_0 = t' \rightarrow G_1 \rightarrow G_2 \rightarrow \ldots \rightarrow G_k \rightarrow \ldots \) with mgu \( \theta_1, \theta_2, \ldots, \theta_k \ldots \) such that \( d(t, \theta_k \ldots \theta_1(t')) \rightarrow 0 \) as \( k \rightarrow \infty \).

To show that \( t \) is in \( \bar{S}_P \), consider \( t' \), let \( Ty \) be the typing function marking all symbols in \( \Sigma \) as coinductive. We construct an infinite S-derivation \( D' \) by first observing that each SLD-resolution reduction step in \( D \) proceeds either by matching or by unification. If \( G_i, G_{i+1}, \ldots \) is the sequence of lists in \( D \) out of which SLD-resolution reductions steps proceed by unification, then let \( D' \) be the infinite S-derivation \( T_0 = \text{rew}(P, t' \rightarrow t, id) \rightarrow T_1 \rightarrow \ldots \rightarrow T_j \rightarrow \ldots \), where \( T_j = \text{rew}(P, t' \rightarrow t', \theta_{i_j} \ldots \theta_{i_k}) \). We claim that \( t \) is S-computable at infinity with respect to \( P \) and \( Ty \) via the infinite S-derivation \( D' \).

The first condition of Definition 4.3 is satisfied because \( d(t, \theta_k \ldots \theta_1(t')) \rightarrow 0 \) as \( k \rightarrow \infty \) by the properties of \( D \), and thus \( d(t, \theta_j \ldots \theta_1(t')) \rightarrow 0 \) as \( j \rightarrow \infty \) by construction of \( D' \). To see that the
second condition of Definition 4.8 is satisfied, recall that D is fair and infinite. Since D is infinite and Ty does not permit inductive typing, D* contains (inductively closed and) coinductively open rewriting trees infinitely often. As a result, D* satisfies i. Since Ty does not permit inductive typing, D* satisfies ii trivially. And D* satisfies iii because D is both fair and infinite.

In the opposite direction, suppose D* = T0, T1, ... is an infinite S-derivation that computes t at infinity. We need to show that there exists a corresponding SLD-derivation that is fair and non-failing. It is easy to construct D by following exactly the same resolvents as in D*. We only need to show that such D is fair and non-failing. By definition, T0, T1, ... should contain coinductive subtrees T0, T1, ... in which every open coinductive node is resolved against infinitely often. This means that corresponding derivation D will be fair with respect to coinductively typed subgoals. If the subtrees T0, T1, ... do not involve inductive subgoals, then we have that D is fair. (Because D* is non-terminating and non-failing, D using the same resolvents will be nonfailing, too.) Suppose T0, T1, ... contained inductive subgoals. By definition of D*, every S-derivation step that resolved against a coinductive node is followed by a number of S-derivation steps that successfully close all of the inductive subgoals in the corresponding rewriting subtrees. But that means that, in the corresponding derivation D, these inductive subgoals will be chosen infinitely often and will not fail. This completes the proof.

We have the following immediate corollaries:

**Corollary 4.8.** If P ∈ LP(Σ) is observationally productive, then Cp = ŠP.

**Corollary 4.9.** (Coinductive soundness of S-computations at infinity) If P ∈ LP(Σ) is observationally productive, then ŠP ⊆ M₀ₚ.

**Proof.** Using Corollary 4.8 and the fact that Cp ⊆ M₀ₚ.

Corollary 4.9 shows that, for observationally productive programs, S-computations at infinity are sound with respect to greatest complete Herbrand models. The corresponding completeness result — namely, that M₀ₚ ⊆ ŠP — does not hold, even if P is observationally productive. The problem arises when P does not admit any infinite S-resolution reductions. For example, if P₉ is the program with the single clause anySuccessor(s(X)) ←, then anySuccessor(s(s(...))) ∈ M₀ₚ. But P₉ admits no infinite S-derivations, so no (infinite) terms are S-computable at infinity with respect to P₉ and Ty for any typing function Ty. A similar problem arises when P fails the occurs check. For example, if P₁₀ comprises the single clause p(X, f(X)) ← p(X, X), with p marked as coinductive, then p(f(...), f(...)) is in M₀ₚ but is not S-computable at infinity with respect to P₁₀ and Ty. This case is subtly different from the first one, since P₁₀ defines the pair of infinite terms X = f(X) only if unification without the occurs check is permitted.

Corollary 4.9 ensures that (finite) coinductive terms logically entail the infinite terms they finitely approximate. But there may, in general, be programs for which coinductive terms also logically entail other finite terms.

**Example 4.10.** Consider the program P₁₂ comprising the clause of P₉ and the clause

1. p(Y) ← from(0, X)

and suppose Ty types only from as coinductive. Although from(0, X) finitely approximates an infinite term with respect to P₁₁ and Ty, no infinite instance of p(Y) is S-computable at infinity with respect to P₁₁ and Ty. Nevertheless, p(0) and other instances of p(Y) are logically entailed by P₁₁ and thus in M₀ₚ.

The following definition takes such situations into account:

ACM Transactions on Computational Logic, Vol. V, No. N, Article A, Publication date: January YYYY.
Definition 4.11. Let \( P \in \text{LP}(\Sigma) \) be observationally productive, let \( Ty \) be a typing function for \( \Sigma \), and let \( t \in \text{Term}(\Sigma) \). Then \( t \) is implied at infinity with respect to \( P \) and \( Ty \) if there exist terms \( t_1, \ldots, t_n \in \text{GTerm}^o(\Sigma) \), each of which is \( S \)-computable at infinity with respect to \( P \) and \( Ty \), and there exists a sequence of rewriting reductions \( t \to \ldots \to [t'_1, \ldots, t'_n] \) such that, for each \( i \), \( \theta(t'_i) = t_i \) for some \( \theta \in \text{Subst}^o(\Sigma) \). We define \( SI^\omega_P = \{ t \in \text{GTerm}^o(\Sigma) \mid t \text{ is } S \text{-computable at infinity or } S \text{-implied at infinity with respect to } P \text{ and } Ty \} \).

Example 4.12. Consider once again the term \( p(\gamma) \) from Example 4.10. Although \( p(\gamma) \) is not computable at infinity with respect to \( P_{12} \) and \( Ty \) as in Example 4.10, it is indeed implied at infinity with respect to \( P_{12} \) and \( Ty \).

Defining \( \widetilde{SI}_P = \bigcup \{ SI^\omega_P \mid Ty \text{ is a typing function for } \Sigma \} \) gives the following corollary of Corollary 4.9.

Corollary 4.13. If \( P \in \text{LP}(\Sigma) \) is observationally productive, then \( \widetilde{SI}_P \subseteq M^\omega_P \).

5. CONCLUSIONS, RELATED WORK, AND FUTURE WORK

This paper gives a first complete formal account of the declarative and operational semantics of structural (i.e., \( S \)-) resolution. We started with characterisation of \( S \)-resolution in terms of big-step and small-step operational semantics, and then showed that a rewriting tree representation of this operational semantics is inductively sound and complete, as well as coinductively sound. Since observational productivity is one of the most striking features of \( S \)-resolution, much of this paper’s discussion is centered around the subject of productivity in its many guises: SLD-computations at infinity, \( S \)-computations at infinity, \( S \)-derivations with loop detection for rational terms (“observations of a coinductive proof”) and sound observations of infinite derivations for irrational terms (“sound observations”). We have shown how an approach to productivity based on \( S \)-resolution makes it possible to formalise the distinction between global and observational productivity. This puts \( \text{LP} \) (and the broader family of resolution-based methods) on par with coinductive methods in ITP and TRS. We have also shown that our new notion of observational productivity supports the formulation of a new coinductive proof principle based on loop detection; moreover that proof principle is sound relative to \( S \)-computations at infinity and SLD-computations at infinity from the 1980s [Lloyd 1987; van Emden and Abdallah 1985].

The webpage https://github.com/coalp contains implementation of sound observations of \( S \)-derivations and the Coq code supporting the proof-theoretic analysis of the loop detection method and its possible extensions.

The research reported herein continues the tradition of study of infinite-term models of Horn clause logic [Jaffar and Stuckey 1986; Lloyd 1987; van Emden and Abdallah 1985; Jaume 2000]. In particular, we have given a full characterisation of \( S \)-resolution relative to the least and greatest fixed point semantics of \( \text{LP} \), as is standard in the classical \( \text{LP} \) literature. Moreover, we have connected the classical work on least and greatest complete Herbrand models of \( \text{LP} \) to the more modern coalgebraic notation [Sangiorgi 2012] in Section 2. Our definitions of term trees and rewriting trees relate to the line of research into infinite (term-) trees [Courcelle 1983; Jaffar and Stuckey 1986; Johann et al. 2015].

\( S \)-resolution arose from coalgebraic studies of \( \text{LP} \) [Komendantskaya et al. 2014; Komendantskaya and Power 2011a; Komendantskaya and Power 2011b], and these were subsequently developed into a bialgebraic semantics [Bonchi and Zanasi 2013; Bonchi and Zanasi 2015]. However, the bialgebraic development takes the coalgebraic semantics of \( \text{LP} \) in a direction different from our productivity-based analysis of \( S \)-resolution. Investigating possible connections between observational productivity of logic programs and their bialgebraic semantics offers an interesting avenue for future work.

Another related area of research is the study of coinduction in first order calculi other than Horn clause logic [Baeld and Nadathur 2012], including fixed-point linear logics.
(e.g. MuLJ) [Baelde 2008] and coinductive sequent calculi [Brotherston and Simpson 2011]. One important methodological difference between MuLJ (implemented as Bedwyr) [Baelde 2008] and S-resolution is that Bedwyr begins with a strong calculus for (co)induction and explores its implementations, while S-resolution begins with LP’s computational structure and constructs such a calculus directly from it. Notably, Bedwyr requires cycle/invariant detection, accomplished via heuristics that are incomplete but practically useful. S-resolution may in the future provide further automation for systems like Bedwyr.

The definition of observationally productive logic programs given in this paper closely resembles the definitions of productive and guarded corecursive functions in ITP — particularly in Coq [Bertot and Komendantskaya 2008] and Agda, as illustrated in Introduction.

Further analysis of the relationship between that coinductive proof principle and the one developed here would require the imposition of a type-theoretic interpretation on S-resolution. A type-theoretic view of S-resolution for inductive programs is given in [Fu and Komendantskaya 2013, Fu and Komendantskaya 2016]. A preliminary investigation of how coinductive hypothesis formation for Horn clauses can be interpreted type-theoretically is given in [Fu et al. 2016].

Productivity has also become a well-established topic of research within TRS community; see, e.g., [Endrullis et al. 2010, Endrullis et al. 2015]. The definition of productivity for TRS relates to observational productivity defined in this paper, and reflects the intuition of finite observability of fragments of computations. However, because S-resolution productivity is defined via termination of rewriting reductions, it also strongly connects to the termination literature for TRS [Terese 2003]. Our definition in Section 2 of S-resolution in terms of reduction systems makes the connection between S-resolution and TRS explicit (see also [Fu and Komendantskaya 2016]), and thus encourages cross-pollination between research in S-resolution and TRS.

The fact that productivity of S-resolution depends crucially on termination of rewriting reductions makes this work relevant to co-patterns [Abel et al. 2013]. In particular, [Basold and Hansen 2015] considers a notion of productivity for co-patterns based on strong normalisation of term-rewriting. This is similar to our notion of observational productivity for logic programs. Further investigation of applications of S-resolution in the context of co-patterns is under way.

Observationally productive S-derivations may be seen as an example of clocked corecursion [Atkey and McBride 2013], where finite rewriting trees give the measures of observation in a corecursive computation. Formal investigation of this relation is a future work.

Overall, we see the work presented here as laying a new foundation for automated coinductive inference well beyond LP. In particular, we expect our new methods to allow us to extend type inference algorithms for a variety of programming languages [Ancona and Lagorio 2011, Lämmel and Jones 2005, Abel et al. 2013] to accommodate richer forms of conduction. We are currently exploring this enticing new research direction.

6. ACKNOWLEDGMENTS

We thank the following colleagues for discussions that encouraged and inspired this work: Andreas Abel, Davide Ancona, Henning Basold, Peng Fu, Gopal Gupta, Helle Hansen, Martin Hofmann and Tom Schrijvers. We particularly thank Vladimir Komendantskij and František Farka, who at different times implemented prototypes of CoAlgebraic Logic Programming (CoALP) and S-Resolution: their input has been invaluable for shaping this work.

REFERENCES

Andreas Abel, Brigitte Pientka, David Thibodeau, and Anton Setzer. 2013. Copatterns: programming infinite structures by observations. In POPL’13, 27–38.

Agda Development Team. 2015. AGDA Reference Manual. (2015). http://appserv.cs.chalmers.se/users/ulfn/wiki/agda.php.

Davide Ancona and Giovanni Lagorio. 2011. Idealized coinductive type systems for imperative object-oriented programs. RAIRO - Theory of Information and Applications 45, 1 (2011), 3–33.

Robert Atkey and Conor McBride. 2013. Productive coprogramming with guarded recursion. In ICFP’13. ACM, 197–208.
A. LIST OF EXAMPLES OF LOGIC PROGRAMS USED ACROSS ALL SECTIONS

| Program reference | Program clauses | Program meaning suggested by Herbrand models |
|-------------------|----------------|---------------------------------------------|
| $P_1$             | $0.\text{nat}(0) \leftarrow$<br>$1.\text{nat}(\text{s}(X)) \leftarrow \text{nat}(X)$ | The set of all natural numbers |
| $P_2$             | $P_1$ and<br>$2.\text{nats}(	ext{scons}(X, Y)) \leftarrow \text{nat}(X)\cdot\text{nats}(Y)$ | The set of natural numbers union the set of streams of natural numbers |
| $P_3$             | $0.\text{add}(0, Y, Y) \leftarrow$<br>$1.\text{add}(	ext{s}(X), Y, \text{s}(Z)) \leftarrow \text{add}(X, Y, Z)$<br>$2.\text{fibs}(X, Y, \text{cons}(X, S)) \leftarrow \text{add}(X, Y, Z)\cdot\text{fibs}(Y, Z, S)$ | The set of terms satisfying the relation of addition and terms denoting infinite streams of Fibonacci numbers |
| $P_4$             | $0.\text{from}(X, \text{scons}(X, Y)) \leftarrow \text{from}(	ext{s}(X), Y)$ | The set containing one term representing the infinite stream $0 :: \text{s}(0) :: \text{s}(\text{s}(0)) :: \ldots$ |
| $P_5$             | $0.\text{from}(X, \text{scons}(X, Y)) \leftarrow \text{from}(X, Y), \text{error}(0)$ | The empty set |
| $P_6$             | $0.\text{conn}(X, Y) \leftarrow \text{conn}(X, Z)\cdot\text{conn}(Z, Y)$<br>$1.\text{conn}(a, b) \leftarrow$<br>$2.\text{conn}(b, c) \leftarrow$ | The set $\{\text{conn}(a, b), \text{conn}(b, c), \text{conn}(a, a), \text{conn}(a, b), \text{conn}(b, c), \text{conn}(b, b), \text{conn}(c, c)\}$ |
| $P_7$             | $0.\text{p}(c) \leftarrow$<br>$1.\text{p}(X) \leftarrow \text{q}(X)$ | The set $\{\text{p}(c), \text{q}(c)\}$ |
| $P_8$             | $P_2, P_3$ and<br>$\text{fibnats}(X, Y) \leftarrow \text{fibs}(0, \text{s}(0), X)\cdot\text{nats}(Y)$ | The union of sets for $P_2$ and $P_3$ plus all terms where predicate $\text{fibnats}$ has all terms given in models for $P_2$ in the first argument and all terms given in models for $P_3$ in the second argument |
### $P_9$

| Rule | Description |
|------|-------------|
| anySuccessor(s(X)) ← | The set \{anySuccessor(s(0)), anySuccessor(s(s(0)), ...}\} |

### $P_{10}$

| Rule | Description |
|------|-------------|
| p(X,f(X)) ← p(X,X) | The set \{p(f(f(\ldots)), f(f(\ldots)))\}. |

### $P_{11}$

| Rule | Description |
|------|-------------|
| $P_4$ and p(Y) ← from(0,X) | The set as for $P_4$ union the set of all terms $p(t)$, where $t$ is a term from the Herbrand base of that program. |

### $P_{12}$

| Rule | Description |
|------|-------------|
| zeros(scons(0,X)) ← zeros(X) | The set contains one term – denoting the stream of zeros. |