Collective Field Theory
for D=0 Matrix Models

Olaf Lechtenfeld †

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, D–30167 Hannover, Germany

ABSTRACT

I investigate non-perturbative aspects of zero-dimensional matrix models. Subtleties in the large-N limit of the semiclassical picture are pointed out. The tunneling of eigenvalues is seen to correspond to a chaotic sequence of recursion coefficients determining the orthogonal polynomials.

* Invited Talk at the XXVII International Symposium in Wendisch-Rietz, Germany, Sept. 7–11, 1993.
† email address: LECHTENF@ITP.UNI-HANNOVER.DE
Introduction.

Over the last four years, we have learned how to model two-dimensional euclidean quantum gravity (with topological fluctuations) by hermitean matrix models [1], in the so-called double-scaling limit [2]. The latter entails sending the size $N$ of the matrix $M$ to infinity while the couplings $g$ in the matrix potential $V$ approach a critical value $g_c$ in such a way that some combination of $N$ and $(g-g_c)$ is kept constant. In the pure gravity case, one has

$$V(M) = \frac{1}{2}M^2 + gM^4, \quad g_c = -\frac{1}{48}. \quad (1)$$

Interestingly, the critical matrix potential retains a local minimum but is unbounded from below, and analytic continuation is to be performed into a region where the partition function is ill-defined. The result is either singular or develops an imaginary part, associated with the instability towards the tunneling escape of individual matrix eigenvalues [3].

The tunneling phenomenon is most transparent in a semiclassical treatment of large-$N$ matrix models. As a first step, the saddle-point analysis was pioneered in ref. [4]. To work with a well-defined theory, I will bound the potential (1) from below by changing its ‘large-$M$’ behavior through a modification $\delta V(M) = \epsilon M^6$, with a small, positive regulator $\epsilon$. This turns out to change substantially the large-$N$ phase structure of the model [5–8]; in particular, the critical line $g_c = g_3(\epsilon)$ becomes metastable and invisible at finite $N$. Nevertheless, tunneling may now be studied directly, without the need for analytic continuation.

Instrumental to the success of the random matrix model has been the technology of orthogonal polynomials [9,10]. The saddle-point results, however, have not yet been completely understood in this framework. Only in the case of degenerate potential wells, a precise relation between orthogonal polynomial recursion coefficients $R_k$ on one side and the (multi-band) classical eigenvalue density on the other side has been established [11,7,8].
The more general connection between the two approaches involves an interpretation of eigenvalue tunneling in terms of orthogonal polynomials. This will be addressed in the third part of my talk, where I shall present a surprising resolution of a puzzle mentioned earlier [7,8,12]. The second part, following this introduction, will outline the semiclassical approach. Along the way, I will relate \( N = \infty \) saddle points to airfoils, and interpret eigenvalue tunneling in terms of the thawing of a frozen Dyson gas. A short summary shall close my talk, which reviews some research of mine conducted over the past three years and published for the most part in refs. [8,12,13].

The Semiclassical Approach.

I begin by formulating a collective field theory for the zero-dimensional hermitean one-matrix model at finite \( N \). My starting point is the partition function

\[
Z_N \propto \int d^N M e^{-N \text{tr} V(M)}
\]

for an \( N \times N \) random hermitean matrix ensemble, in a potential \( V \). Upon diagonalization \( M = \text{diag}(x_i) \) this reduces to

\[
Z_N = e^{-N^2 F_N} := \left[ \prod_{i=1}^{N} \int dx_i \right] \exp \left\{ -N \sum_i V(x_i) + \sum_{i<j} \ln(x_i - x_j)^2 \right\} ,
\]

the partition function of a two-dimensional Coulomb gas of charges restricted to a line, in an external potential \( V(x) \) at temperature \( \beta^{-1} = 1/N^2 \). I like to change variables from the matrix eigenvalues \( x_i \) to their density distribution

\[
\rho(x) := \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) .
\]

More precisely, I insert

\[
1 = \int \mathcal{D} \rho \prod_x \delta(\rho(x) - \frac{1}{N} \sum_i \delta(x - x_i))
\]

\[
= \int \int \mathcal{D} \rho \mathcal{D} \lambda \exp \left\{ i \int dx \lambda(x) \left[ \rho(x) - \frac{1}{N} \sum_i \delta(x - x_i) \right] \right\}
\]
into (3) and express the action in terms of the density,

\[ S_{N}^{0}[\rho] = N^{2} \int dx \rho(x) V(x) - \frac{1}{2} N^{2} \int \int dx dy \rho(x) f(x - y) \rho(y) + \frac{1}{2} N f(0) \]  \hspace{1cm} (6)

The self-interaction had to be regulated by replacing \( \ln z^{2} \rightarrow f(z) \) in eq. (3), choosing some suitable, \( i.e. \) symmetric and bounded, function \( f \).

Following ref. [14] I am able to perform the integration over \( x_{i} \),

\[
Z_{N} = \int \int D\rho D\lambda \ e^{-S_{N}^{0}[\rho]+i \int \lambda \rho} \left[ \prod_{i=1}^{N} \int dx_{i} \right] e^{-\frac{i}{N} \sum_{i} \lambda(x_{i})} 
\]

\[ = \int \int D\rho D\lambda \ e^{-S_{N}^{0}[\rho]+i \int \lambda \rho} \left[ \int dx \ e^{-\frac{i}{N} \lambda(x)} \right]^{N} \]  \hspace{1cm} (7)

and arrive at an \textit{exact} effective action

\[
S_{N}[\rho, \lambda] = S_{N}^{0}[\rho] - i \int dx \lambda(x) \rho(x) - N \ln \int dx \ e^{-\frac{i}{N} \lambda(x)} \]  \hspace{1cm} (8)

which is not only nonlocal in the two real fields \( \rho \) and \( \lambda \) but also non-polynomial in the latter. Interestingly, the constant mode of \( \lambda \) can be integrated out exactly to yield the constraint \( \delta(\int \rho - 1) \) that was apparent already from the definition (4). However, I shall keep those modes in the measure for the time being. In principle, another constraint arises from the positivity of \( \rho \). Perturbation theory about a strictly positive \( \hat{\rho} \), however, is insensitive to this restriction, and I will therefore ignore it in the following.

My goal is to initiate a systematic semiclassical analysis of this peculiar one-dimensional field theory. To leading order in \( \hbar \) we must determine the saddle-point configurations (\( \hat{\rho}, \hat{\lambda} \)), where the action (8) is stationary. The first variations yield

\[
0 = N^{2} V(x) - N^{2} \int dy f(x - y) \hat{\rho}(y) - i \hat{\lambda}(x) 
\]

\[ 0 = \hat{\rho}(x) - e^{-\frac{i}{N} \hat{\lambda}(x)} \int dy \ e^{-\frac{i}{N} \hat{\lambda}(y)} \]  \hspace{1cm} (9)

where \( \hat{\rho} \) comes out to be properly normalized, \( \int \hat{\rho} = 1 \). The second equation
determines \( \hat{\lambda} \) up to a constant,

\[
\hat{\lambda}(x) = iN \ln \hat{\rho}(x) + \lambda_0 \text{ ,}
\]  

(10)
a result which may be inserted into the first equation. Differentiating with respect to \( x \) to remove constants and deregulating the Coulomb repulsion I get

\[
\frac{1}{2} V'(x) + \frac{1}{2N} \frac{\hat{\rho}'}{\hat{\rho}}(x) = \int \frac{dy}{x-y} \hat{\rho}(y) \equiv \pi \mathcal{H}_x[\hat{\rho}]
\]

(11)
where \( \int \) denotes Cauchy’s principal value of the integral. The r.h.s. is known as the Hilbert transform \( \mathcal{H} \) (of \( \hat{\rho} \)) which has been thoroughly investigated [16]. Together with normalization and positivity, this equation describes the classical eigenvalue density for any finite \( N \). It is noteworthy that (11) is not homogeneous in \( \frac{1}{N} \), so its solution cannot be, either. At \( N = \infty \) the equation (11) has been widely studied and solved [17,18,4], and it was learned [5,12] that a unique solution extends to \( N < \infty \).

Unfortunately, the equation is not easily solved for finite \( N \). Even in the large-\( N \) limit some care is required, as shown by the following. For instance, should I drop the \( \hat{\rho}'/\hat{\rho} \)-term since it is down by \( \frac{1}{N} \)? A little inspection reveals that such a step is in general not consistent with the asymptotic large-\( |x| \) behavior of the equation, which demands a \( \frac{1}{x} \) fall-off for the l.h.s. due to the normalization of \( \hat{\rho} \). In fact, the formal ‘solution’

\[
\hat{\rho}_\epsilon(x) \sim \exp\left\{-N\left[V(x) - \int \ln(x-y)^2 \hat{\rho}_\epsilon(y)\right]\right\} \sim x^{2N} e^{-NV(x)}
\]

(12)
of equation (11) shows that the \( \hat{\rho}'/\hat{\rho} \) term dominates the l.h.s. of eq. (11) for \( |x| \gg 1 \) (unless \( V \sim \ln x^2 \)), so that \( \hat{\rho}_\epsilon = \mathcal{O}(e^{-N}) \) asymptotically. Note that I have attached a subscript \( \epsilon \equiv \frac{1}{N} \) to the solution to indicate that it depends on the value of \( N \).

One can read off that the effective potential seen by eigenvalues \( |x| \gg 1 \) approaches

\[*\] The first equation subsequently fixes the constant \( \lambda_0 \) for a given solution \( \hat{\rho} \).

\[\dagger\] See also ref. [15].
\( V(x) - 2 \ln |x| \). The situation is different, however, near the minimum of the potential where \( V' \) dominates the l.h.s. of eq. (11) and most of \( \hat{\rho} \) is concentrated.\(^\dagger\) When \( \hat{\rho} \) is \( \mathcal{O}(1) \) the \( \frac{\hat{\rho}'}{\hat{\rho}} \)-term may safely be neglected or treated as a \( \frac{1}{N} \) perturbation in (11), leading to

\[
\hat{\rho}_\epsilon(x) \approx -\frac{1}{\pi^2} \int \frac{dy}{x-y} \frac{1}{2} V'(y)
\]

by simply inverting the Hilbert transform in equation (11). The result is a modulation of Wigner’s semicircle distribution. When \( N \to \infty \), the crossover regions between the interior and exterior of the ‘Fermi sea’ shrink to points \( x=a, b \), and the saddle-point equation (11) turns into [4]

\[
\frac{1}{2} V'(x) = \int_a^b \frac{dy}{x-y} \hat{\rho}_0(y) \equiv \pi \mathcal{T}_x[\hat{\rho}_0] \quad \text{for } x \in [a, b]
\]

\[
\hat{\rho}_0(x) = 0 \quad \text{for } x \notin [a, b].
\]

This relation is known as the \textit{airfoil equation} and determines the vorticity \( \hat{\rho}_0 \) related to a given velocity field \( V' \) along the airfoil (\( b-a \) is the span of the wings) [17]. In our case the location of \( a \) and \( b \) is determined from the normalization of \( \hat{\rho}_0 \). Equation (14) is solved by inverting the \textit{finite} Hilbert transform \( \mathcal{T} \),

\[
\hat{\rho}_0(x) = \frac{1}{\sqrt{(b-x)(x-a)}} \left[ \frac{1}{\pi} - \frac{1}{\pi^2} \int_a^b \frac{dy}{x-y} \frac{1}{2} V'(y) \sqrt{(b-y)(y-a)} \right].
\]

Again, I have assumed a single-well potential \( V \), so that the support of \( \hat{\rho}_0 \) is a single, connected interval \([a, b]\).

It is worthwhile to give the form of the saddle-point action. Employing eq. (10)

\(^\dagger\) The case of multiple local minima will be discussed in a while.
as well as \( \ln \int e^{-\frac{i}{N} \hat{\lambda}} = -\frac{i}{N} \lambda_0 \) I find, in agreement with ref. [15], that \( \hat{\rho} = \hat{\rho}_\epsilon \)

\[
S_N[\hat{\rho}, \hat{\lambda}] = N^2 \int \hat{\rho}(x)V(x) - N^2 \int \hat{\rho}(x) \ln |x-y| \hat{\rho}(y) + N \int \hat{\rho}(x) \ln \hat{\rho}(x)
\]

\[
= \frac{N^2}{2} \int \hat{\rho}(x)(V(x) - \ln |x|) + \frac{N^2}{2} V(0) + \frac{N}{2} \int \hat{\rho}(x) \ln \hat{\rho}(x) + \frac{N}{2} \ln \hat{\rho}(0)
\]

(16)

where I dropped a (singular) term \( \frac{1}{2} N f(0) \) and made use of the saddle-point equation (11). The last integral permits an interpretation as the entropy of the distribution \( \hat{\rho} \). Taking the naive large-\( N \) limit, I obtain

\[
S_\infty[\hat{\rho}_0] = \frac{N^2}{2} \int dx \hat{\rho}_0(x) V(x) - N^2 \gamma
\]

(17)

with

\[
\gamma = \int dy \ln |x-y| \hat{\rho}_0(y) - \frac{1}{2} V(x) = \text{constant for } x \in [a, b]
\]

(18)

being the chemical potential (or Lagrange multiplier enforcing \( \int \hat{\rho}_0 = 1 \)).

At this point, I would like to drive home an essential point of my talk. The non-trivial \( N \)-dependence of the classical background \( \hat{\rho}_\epsilon \) implies that the classical limit, \( \rho \rightarrow \hat{\rho}_\epsilon \), differs from the low-temperature limit, \( N \rightarrow \infty \), because of \( \frac{1}{N} \) corrections coming from the integration measure. As a consequence, the semiclassical loop expansion will \textit{not} be identical to the topological \( \frac{1}{N} \) expansion; rather, a \textit{double} expansion arises. Beyond this, non-perturbative (in \( \frac{1}{N} \)) contributions appear already at tree level (in \( \hbar \)). It is therefore by no means clear that interchanging limits, by \textit{first} taking \( N \rightarrow \infty \) in equation (11) and \textit{then} solving equation (14) to obtain \( \hat{\rho}_0 \), provides the large-\( N \) limit, \( \lim_{\epsilon \rightarrow 0} \hat{\rho}_\epsilon \), of a proper solution to equation (11), as my notation suggests. In fact, I will now demonstrate that in general a solution \( \hat{\rho}_0 \) does not correspond to a finite-\( N \) saddle point \( \hat{\rho}_\epsilon \).

Non-perturbative effects become tangible when two or more potential wells compete for eigenvalues. For even potentials, the full complexity of the problem
appears first in the triple-well potential
\[
V(x) = \frac{1}{2}a^2 + gx^4 + \epsilon x^6 , \quad \epsilon > 0 , \tag{19}
\]
since the \(x \leftrightarrow -x\) symmetry trivializes the double-well case. I like to fix \(\epsilon\) to some small value and probe the phase diagram by decreasing the quartic coupling \(g\) along the negative axis. For \(g < g_\star = -\sqrt{3\epsilon/2}\) the potential develops three well-separated minima which become degenerate at \(g = g_\star = -\sqrt{2}\epsilon\). For sufficiently negative \(g\) the ‘Fermi sea’ must, therefore, consist of three or two disconnected oceans, called \(arcs\) or \(bands\). At \(N \to \infty\), the eigenvalue density should then be supported on one, two, or three disjoint intervals. As shown in refs. [6,7,8], the general solution of equation (14) reads
\[
\hat{\rho}_0(x) = \frac{3\epsilon}{\pi} (n^2 - x^2)^{3/2} (4a^2 - x^2)^{1/2} (4b^2 - x^2)^{1/2} (4c^2 - x^2)^{1/2} , \tag{20}
\]
describing a positive three-band density with \(0 \leq 2a < n < 2b \leq 2c\) and support on \([-2c, -2b] \cup [-2a, 2a] \cup [2b, 2c]\). Normalization imposes three conditions on \(\{n, a, b, c\}\) which leaves a one-parameter \(family\) of solutions. A convenient parameter to label these solutions is the difference \(\Delta \gamma = \gamma_o - \gamma_c\) of the chemical potentials \(\gamma\) for the outer and the central bands. The chemical potential is nothing but the integration constant appearing when integrating equation (14). Since it may take different constant values for two eigenvalues \(x_c \in [-2a, 2a]\) and \(x_o \in [2b, 2c]\) from two different bands, the difference
\[
\Delta \gamma = \int dy \ln \left| \frac{x_o - y}{x_c - y} \right| \hat{\rho}_0(y) - \frac{1}{2} \left[ V(x_o) - V(x_c) \right] \tag{21}
\]
is a genuine property of the solution \(\hat{\rho}_0\). In this situation one must replace
\[
\gamma \longrightarrow \sum_{\text{bands}} \gamma_i n_i , \quad n_i = \int dx \hat{\rho}_0(x) , \quad \sum_{\text{bands}} n_i = 1 , \tag{22}
\]
in equation (17). Extremal values of \(\Delta \gamma\) occur when the number of bands decreases, for \(b \to c\) or \(a \to 0\), and the density (20) becomes unique. However, those solutions
do not exist everywhere in the \((\epsilon, g)\) plane. For sufficiently small \(\epsilon\), the range of one-, two- and three-band solutions is given by the sequence \(g_3 < g_\alpha < g_2 < g_* < g_1 < 0\). Here, one-band densities arise for \(g > g_3\), two-band distributions occur for \(g < g_2\), and three-band solutions appear for \(g < g_1\). Hence, equation (14) admits a unique solution only for \(g > g_1\). The overlapping regions above indicate a coexistence of multiple-band densities elsewhere. Even more astonishing is the discovery that the members \(\hat{\rho}_0(x, \Delta \gamma)\) of such a family are not degenerate in free energy [8].

This contradiction in terms is resolved by noticing that infinitesimal variations within the family, which correspond to the tunneling of individual eigenvalues, are actually singular at the band edges. One might say that it requires a finite variation to move an individual eigenvalue to another well although it is only an \(O(\frac{1}{N})\) effect. Since \(\frac{\partial}{\partial \Delta \gamma} \hat{\rho}_0\) is not square-integrable the inclusion of this mode among the density fluctuations in debatable. As the chemical potential drives the tunneling of eigenvalues, one should expect the minimal action to belong to the unique family member with \(\Delta \gamma = 0\), which I call dominant. This is indeed what happens and can be checked numerically [8].

Of course, for \(N < \infty\) there is always a unique saddle-point density \(\hat{\rho}_\epsilon\), with a unique limit as \(\epsilon \to 0\), because the strict positivity of the distribution implies that \(\Delta \gamma = 0\) all along. Hence, we have

\[
\lim_{N \to \infty} \hat{\rho}_\epsilon(x) = \hat{\rho}_0(x, \Delta \gamma = 0) .
\]

With hindsight it is clear that the ‘sub-dominant’ members of a three-band family could only appear because I took the limit \(N \to \infty\) prematurely by going from equation (11) to (14). ‘Physically’ speaking, the freezing of the Dyson gas of eigenvalues at zero temperature entirely suppresses any tunneling and permits those ficticious saddle-point distributions. At finite temperature, the Dyson solid melts at the edges, and tunneling, although exponentially small, destabilizes all but the dominant solutions. The ‘entropy term’ in equation (11) plays the crucial role.

\(\star\) \(g = g_3(\epsilon)\) is the critical BPIZ line where the double scaling limit is to be taken [4].
However, I may still employ the incomplete large-$N$ saddle-point equation (14), if it is complemented by the additional global requirement $\Delta \gamma = 0$.

In this light the correct phase diagram for the potential (19) looks as follows. Again taking $\epsilon$ small but fixed, I find a new sequence $g'' < g@ < g' < 0$, where a small region of three-band dominance around the degenerate point ($g = g@$) separates two-band from one-band densities. Inspecting the shape of the potential at the transition values $g'$ and $g''$, one learns that essentially the number of degenerate absolute minima determines the number of eigenvalue bands. Interestingly, the jump from one to three well-separated bands is smoothed out by an interpolating three-band region occurring when non-degenerate wells are in some sense comparable and can compete for eigenvalues.

**Orthogonal Polynomials.**

The standard approach to matrix model calculations, and so far the only one capable of producing the topological expansion and the double-scaling limit, is the method of orthogonal polynomials [9]. Its starting observation is that the van der Monde determinant in

$$e^{-N^2 F_N} = \left[ \prod_{i=1}^{N} \int dx_i \right] \prod_{i<j} (x_i - x_j)^2 e^{-N \sum_i V(x_i)}$$

(24)

can be rewritten as

$$\prod_{i<j} |x_i - x_j| = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_N \\
x_1^2 & x_2^2 & \cdots & x_N^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1}
\end{vmatrix} = \begin{vmatrix}
P_0(x_1) & P_0(x_2) & \cdots & P_0(x_N) \\
P_1(x_1) & P_1(x_2) & \cdots & P_1(x_N) \\
P_2(x_1) & P_2(x_2) & \cdots & P_2(x_N) \\
\vdots & \vdots & \ddots & \vdots \\
P_{N-1}(x_1) & P_{N-1}(x_2) & \cdots & P_{N-1}(x_N)
\end{vmatrix},$$

(25)

with monic polynomials $P_k(x) = x^k + (\text{lower order})$. If one cleverly chooses the $P_k$
to be mutually orthogonal with respect to the measure $e^{-NV(x)}$,

$$h_k \delta_{kl} = \int dx \, e^{-NV(x)} P_k(x) P_l(x) \ ,$$  \hspace{1cm} (26)

the change of basis $x^k \rightarrow P_k(x)$ in equation (24) exactly diagonalizes the Coulomb interaction and trivializes the free energy to

$$F_N = -\frac{1}{N^2} \ln[N!h_0h_1\ldots h_{N-1}] \ .$$  \hspace{1cm} (27)

The construction of the polynomials simplifies thanks to the classic recursion relation

$$P_{k+1}(x) = x P_k(x) - R_k P_{k-1}(x) \ , \quad R_k = \frac{h_k}{h_{k-1}} \geq 0 \ ,$$  \hspace{1cm} (28)

so that it is sufficient to compute the norms $h_k$, starting with the initial condition $R_0=0$.

The Stieltjes method of constructing the orthogonal polynomials consists of iterating eq. (28) and the norm computation, (26):

$$P_k(x; R_0, \ldots, R_{k-1}) \rightarrow h_k \rightarrow R_k \rightarrow P_{k+1}(x; R_0, \ldots, R_k) \ .$$  \hspace{1cm} (29)

Unfortunately, it is quite inappropriate for numerical analysis. However, the special form $e^{-NV}$ of the measure allows for a finite recursion relation among the $R_k$ themselves, the ‘string equation’ [9]

$$\frac{k}{N} = R_k \left\{1 + 4g(R_{k-1} + R_k + R_{k+1}) + 6\epsilon(R_{k-1} + R_k + R_{k+1})^2 \\
+ 6\epsilon(R_{k-2}R_{k-1} - R_{k-1}R_{k+1} + R_{k+1}R_{k+2})\right\} \ ,$$  \hspace{1cm} (30)

displayed here for the potential (19). After solving for $R_k = R_k(R_{k-1}, \ldots, R_{k-4})$ one still needs the initial values

$$R_1 = \frac{h_1}{h_0} = \frac{\int e^{-NV} x^2}{\int e^{-NV}} \ , \quad R_2 = \frac{h_2}{h_1} = \frac{\int e^{-NV} (x^2-R_1)^2}{\int e^{-NV} x^2} \ ,$$  \hspace{1cm} (31)

besides $R_{-1} = 0 = R_0$ to start the iteration. It turns out that this procedure is numerically tractable, but $R_k$ becomes increasingly sensitive to the initial conditions for growing $k$ or $N$ [12]. Trivial but instructive is the exactly solvable
example of purely quadratic potential, i.e. \( g = \epsilon = 0 \), or \( V = \frac{1}{2} x^2 \). In this case one simply rediscovers the Hermite polynomials from \( R_k = k/N \), as appropriate for the harmonic oscillator.

Let us now investigate the large-\( N \) or planar limit, in order to connect up with the semiclassical results. Here, I have to rely on an assumption, namely a continuum approach of the \( R_k \) needs to be postulated. Writing

\[
\frac{k}{N} = \xi \in [0, 1] \quad , \quad R_k = r_{\epsilon}(\xi) \quad , \quad \epsilon = \frac{1}{N} \quad ,
\]

the simplest ansatz

\[
R_{k+1} - R_k = O\left(\frac{1}{N}\right) \quad \text{as} \quad N \to \infty
\]

implies \( r_{\epsilon}(\xi) \to r(\xi) \), a smooth positive function with \( r(0) = 0 \). Numerical studies show that this behavior indeed occurs whenever a single potential well is clearly dominant [12]. This coincides with the one-band regime of our potential. The string equation is then dramatically simplified to the algebraic relation [9]

\[
\xi = r\left\{1 + 12gr + 60\epsilon r^2\right\} \quad .
\]

The condition \( r(0) = 0 \) selects a unique branch of \( r \) which monotonically reaches \( \xi = 1 \) provided \( g > g_3(\epsilon) \), the BPIZ critical line. Finally, the free energy \( F_\infty \) is obtained by naively taking the \( N \to \infty \) limit of eq. (27),

\[
F_N = - \frac{1}{N^2} \ln N! - \frac{1}{N} \ln h_0 - \frac{1}{N} \sum_{k=1}^{N} (1 - \frac{k}{N}) \ln R_k
\]

\[
\to - \frac{1}{N^2} \ln N! + V_{\text{min}} - \int_0^1 d\xi \left(1 - \xi\right) \ln \rho(\xi) \quad ,
\]

to be compared to \( \frac{1}{N^2} S_\infty \) from equation (17).

\[\star\] excluding regions of ill-separated minima.
However, the continuity assumption (33) is clearly violated when \( g < g_{\bar{\alpha}} \), because an estimate of \( R_1 \) from equation (31) reveals that it must jump from \( O\left(\frac{1}{N}\right) \) to the square of the location of the outer potential minima, \( \approx -\frac{g_{\bar{\alpha}}}{3\epsilon} = \frac{1}{\sqrt{6\epsilon}} \), when \( g \) drops below the degenerate point, \( g = g_{\bar{\alpha}} \). Such a behavior is known from studies of double-well potentials [11,7], where an alternating sequence

\[
R_{k+2} - R_k = O\left(\frac{1}{N}\right) \quad , \quad r_{\epsilon}(\xi) \to \begin{cases} r^{(0)}(\xi) & \text{for } k \text{ even} \\ r^{(1)}(\xi) & \text{for } k \text{ odd} \end{cases} \tag{36}
\]

is observed. Under this modified assumption, the string equation (30) turns into two coupled cubic equations for \( r^{(0)} \) and \( r^{(1)} \). Their graphical solution exhibits \( r^{(0)}(\xi) \) as an increasing function starting from \( r^{(0)}(0) = 0 \), and \( r^{(1)}(\xi) \) as decreasing from \( r^{(1)}(0) = x_{\min}^2 \) [8].

Like for the saddle-point method, it is not evident which large-\( N \) assumption is to be chosen for given values of \( \epsilon \) and \( g \). It can be shown, however, that the series of \( \sqrt{R_k} \) approximates the sequence of consecutive eigenvalues \( x_k \) eventually building up to the distribution \( \hat{\rho}_{\epsilon}(x) \) in the limit \( \epsilon \to 0 \). Hence, one- and two-band regions in the phase diagram must correspond to the continuum behavior of eqs. (33) and (36), respectively.† An unsettling gap remains, however, in the three-band dominated buffer zone. How can the one-branch ansatz (33) merge with the two-branch ansatz (36) when \( r^{(1)}(0) - r^{(0)}(0) \approx x_{\min}^2 \geq \frac{1}{\sqrt{6\epsilon}} ? \) The resolution of this paradox was discovered through a numerical analysis, which uncovered a \( \xi \) interval \([\bar{\xi}, \tilde{\xi}] \) with seemingly chaotic recursion coefficients \( R_k \), interpolating between one or two branches for \( \xi < \tilde{\xi} \) and a single branch for \( \xi > \tilde{\xi}(> 1) \). The three-band region \( g'' < g < g' \) coincides with \( \tilde{\xi} < 1 \), i.e. the onset of the unpredictable behavior creeping into the relevant \( \xi \) interval \([0, 1] \). Only exactly at the degenerate point, \( g = g_{\bar{\alpha}} \), is a simple three-branch solution realized [8]. This picture has been confirmed by several groups [19,20].

† Actually, the number of branches \( r^{(i)} \) reaching the point \( \xi = 1 \) is relevant.
Conclusions.

I have reviewed some non-perturbative aspects of hermitean random matrix models, with an emphasis on the distribution of eigenvalues among several potential wells. It turned out that the classical, quasi-continuous, large-$N$ eigenvalue distribution depends non-perturbatively on $\frac{1}{N}$, so that the semiclassical loop expansion must be distinguished from the standard topological (or string loop) expansion.

For multiple-well matrix potentials, interchanging the limits $N \to \infty$ and $S_N \to$ extremum is dangerous due to the $N<\infty$ equilibration between different Dyson gas components, mediated by eigenvalue tunneling. As a consequence, the point of critical coupling becomes metastable for pure gravity when $V$ gets bounded from below.

From numerical simulations of the sequence of recursion coefficients $R_k$ for the orthogonal polynomials, I conjecture that their behavior is characterized by the critical points of the potential function $V$ itself. The consecutive equilibrium deposition of eigenvalues on the real line is suggested to be sensitive to features of $V$ at increasing values. For degenerate absolute minima one observes a quasiperiodic series of $R_k$, whereas non-degenerate minima produce chaotic behavior! The unpredictability of the coefficients reflects the competition of incommensurate potential wells for eigenvalues.

I hope to have demonstrated that the continuum limit of matrix models is more complicated than imagined originally. In view of this it would be very beneficial to understand their critical properties in the semiclassical description. An attempt to push the latter beyond the classical limit is currently in progress (see [13]).

Acknowledgements: I am grateful to the organizers for the charming and stimulating atmosphere of the Symposium, and especially K. Behrndt for his personal efforts to make it all work.
REFERENCES

1. E. Brézin and V.A. Kazakov, *Phys. Lett.* **B236** (1990), 144;
   M.R. Douglas and S.H. Shenker, *Nucl. Phys.* **B335** (1990), 635;
   D.J. Gross and A.A. Migdal, *Phys. Rev. Lett.* **64** (1990), 127,717;
   E. Brézin, M.R. Douglas, V.A. Kazakov, S.H. Shenker, *Phys. Lett.* **B237**
   (1990), 43;
   C. Crnković, P. Ginsparg and G. Moore, *Phys. Lett.* **B237** (1990), 196;
   M.R. Douglas, *Phys. Lett.* **B238** (1990), 176.

2. I.K. Kostov and A. Krzywicki, *Phys. Lett.* **B187** (1987), 149;
   I.K. Kostov and M.L. Mehta, *Phys. Lett.* **B189** (1987), 118.

3. S.H. Shenker, proceedings of the Cargèse workshop (May 1990);
   F. David, *ibid*.

4. E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, *Comm. Math. Phys.* **59**
   (1978), 35.

5. J. Jurkiewicz, *Phys. Lett.* **B245** (1990), 178.

6. G. Bhanot, G. Mandal and O. Narayan, *Phys. Lett.* **B251** (1990), 388.

7. K. Demeterfi, N. Deo, S. Jain and C.-I. Tan, *Phys. Rev.* **D42** (1990), 4105.

8. O. Lechtenfeld, R. Ray and A. Ray, *Int. J. Mod. Phys.* **A6** (1991), 4491.

9. D. Bessis, C. Itzykson and J.B. Zuber, *Adv. Appl. Math.* **1** (1980), 109.

10. T. Banks, M.R. Douglas, N. Seiberg and S.H. Shenker, *Phys. Lett.* **B238**
    (1990), 279.

11. L. Molinari, *J. Phys.* **A21** (1988), 1;
    G.M Cicuta, L. Molinari and E. Montaldi, *J. Phys.* **A23** (1990), L421;
    L. Molinari and E. Montaldi, INFN Milano preprint (1990).

12. O. Lechtenfeld, *Int. J. Mod. Phys.* **A7** (1992), 2335.

13. O. Lechtenfeld, *Int. J. Mod. Phys.* **A7** (1992), 7097.
14. D. Karabali and S. Sakita, *Int. J. Mod. Phys.* A6 (1991), 5079.

15. S. Ben-Menahem, *Nucl. Phys.* B391 (1993), 176.

16. E.C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Oxford, 2nd ed. 1948) Chap. V.

17. H. Söhngen, *Math. Z.* 45 (1939), 245.

18. F.G. Tricomi, *Quart. J. Math. Oxford Ser. (2)* 2 (1951), 199.

19. M. Sasaki and H. Suzuki, *Phys. Rev.* D43 (1991), 4015.

20. D. Senechal, *Int. J. Mod. Phys.* A7 (1992), 1491.