A NOTE ON THE EXISTENCE OF PROPER WEAKLY CYCLIC Z SYMMETRIC MANIFOLDS

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Abstract. The object of the present note is to discuss about the defining condition of weakly cyclic Ricci symmetric manifolds [15] and weakly cyclic Z-symmetric manifolds [2] and the existence of such notion by proper examples.

1. Introduction

During the study of Einstein like manifolds, by the decomposition of covariant derivative of Ricci tensor, Gray [6] introduced two new classes of semi-Riemannian manifold, namely, manifolds of cyclic parallel Ricci tensor (or class A) and manifolds of Codazzi type Ricci tensor (or class B) lying between the class of Ricci symmetric manifolds and the class of manifolds with constant scalar curvature. Again generalizing the notion of class A, weakly Ricci symmetric manifolds by Tamássy and Binh [18] and generalized pseudo Ricci symmetric manifolds by Chaki [1], in 2006 Shaikh and Jana [15] introduced the notion of weakly cyclic Ricci symmetric manifolds and studied its geometric properties. In 2015 De et al. [2] studied weakly cyclic Z-symmetric manifolds by considering the generalized Z-tensor of Mantica and Molinari [8] and claimed their existence by an example. These two notions was latter studied by many authors (see, [3], [7], [9], [10]). The present note deals with the reduced form of the defining condition of weakly cyclic Ricci symmetric and weakly cyclic Z-symmetric manifolds along with their existence. The most important fact that the claim about the existence of weakly cyclic Ricci symmetric manifolds in [15] and the existence of weakly cyclic Z-symmetric manifolds in [2] are not true. The present note provides the existence of weakly cyclic Ricci symmetric and weakly cyclic Z-symmetric manifolds by means of three interesting examples.

2. Weakly cyclic Z-symmetric manifolds

Let $M$ be an $n$-dimensional ($n \geq 3$) connected smooth semi-Riemannian manifold equipped with a semi-Riemannian metric $g$. Let $\nabla$, $R$, $S$ and $r$ be respectively the Levi-Civita connection, Riemann-Christoffel curvature tensor, Ricci tensor and scalar curvature of $M$. Recently in [8],
Mantica and Molinari defined a generalized $Z$-tensor as a symmetric (0, 2) tensor of the form $Z = S + \phi g$, where $\phi$ is an arbitrary scalar function.

In 2015, De et al. [2] have studied weakly cyclic $Z$-symmetric manifolds. A Riemannian manifold $M$ is said to be weakly cyclic $Z$-symmetric [2] if

\[(\nabla_X Z)(X_1, X_2) + (\nabla_{X_1} Z)(X, X_2) + (\nabla_{X_2} Z)(X_1, X) = A(X) Z(X_1, X_2) + B(X_1) Z(X, X_2) + D(X_2) Z(X_1, X),\]

for three 1-forms $A$, $B$ and $D$ on $M$, called the associated 1-forms. Such an $n$-dimensional manifold is denoted by $(WCZS)_n$ and it is called proper if it is not weakly $Z$-symmetric [8]. If $\phi \equiv 0$, then (2.1) reduces to

\[(\nabla_X S)(X_1, X_2) + (\nabla_{X_1} S)(X, X_2) + (\nabla_{X_2} S)(X_1, X) = E(X) S(X_1, X_2) + E(X_1) S(X, X_2) + E(X_2) S(X_1, X),\]

and the manifold reduces to a weakly cyclic Ricci symmetric manifold (briefly, $(WCRS)_n$) introduced by Shaikh and Jana [15] in 2006. A $(WCRS)_n$ is said to be proper if it is not weakly Ricci symmetric. We mention that Shaikh and his coauthors [9], [10] studied the weakly cyclic Ricci symmetric manifolds with some additional conditions. It may be noted that cyclic pseudo Ricci symmetric manifolds are also studied by Shaikh and Hui [12], [13], [14].

In the AMS Mathematical Review of [9] [MR2640817], the reviewer Dacko has remarked that the defining condition (2.2) of a $(WCRS)_n$ reduces to

\[(\nabla_X S)(X_1, X_2) + (\nabla_{X_1} S)(X, X_2) + (\nabla_{X_2} S)(X_1, X) = E(X) S(X_1, X_2) + E(X_1) S(X, X_2) + E(X_2) S(X_1, X),\]

where $E = \frac{1}{3}(A + B + D)$. Recently, Shaikh et al. [11] showed the following:

**Theorem A:** [11] Let $T$ be a $(0, k)$-tensor, $k \geq 2$, on a semi-Riemannian manifold $M$, symmetric with respect to $p$-number ($2p \leq k$) of indices with other $p$-number of indices taken together (including non-identical arrangement). If the manifold satisfies

\[(\nabla_X T)(X_1, X_2, \cdots, X_k) = \pi(X) T(X_1, X_2, \cdots, X_k) \]

\[+ \pi_1(X_1) T(X, X_2, \cdots, X_k) + \cdots + \pi_k(X_k) T(X_1, X_2, \cdots, X),\]

then each corresponding 1-form $\pi_i$ among those $p$ indices can be replaced by the same in pair.

Since $(\nabla_X Z)(X_1, X_2) + (\nabla_{X_1} Z)(X, X_2) + (\nabla_{X_2} Z)(X_1, X)$ is symmetric in $X_1, X_2$ and $X_3$, motivating from the AMS review of [9] and in view of Theorem A, we get the following result:
Proposition 2.1. A semi-Riemannian manifold \((M, g)\) satisfying \((2.1)\) also satisfies

\[
(\nabla_X Z)(X_1, X_2) + (\nabla_{X_1} Z)(X, X_2) + (\nabla_{X_2} Z)(X_1, X) = E(X) Z(X_1, X_2) + E(X_1) Z(X, X_2) + E(X_2) Z(X_1, X),
\]

where \(E = \frac{A + B + D}{3}\).

Remark 2.1. We note that there may exist some \((WCZS)_n\) (resp., \((WCRS)_n\)) whose associated 1-forms \(A, B\) and \(D\) are different, i.e., the associated 1-forms of the structure \((WCZS)_n\) (resp., \((WCRS)_n)\) may not unique. We also note that the metric given in Example 2 (resp., Example 1) is a \((WCZS)_4\) (resp., \((WCRS)_4\)) with unique associated 1-forms. Still there is a natural problem to find out a proper \((WCZS)_n\) (resp., \((WCRS)_n)\) with non-unique associated 1-forms, i.e., a \((WCZS)_n\) (resp., \((WCRS)_n)\) with \(A \neq B \neq D\). Although we present an improper example of \((WCRS)_n\) with \(A \neq B \neq D\).

Remark 2.2. Very recently, Kim [7] studied some geometric properties of a \((WCZS)_n\). All the results of this paper are established by considering \(B - D \neq 0\) and hence these results are standing on the problem of the existence of \((WCZS)_n\) with \(A \neq B \neq D\).

Remark 2.3. The Theorem 8.1 of [2] is based on the sufficient condition for a quasi Einstein manifold to be Ricci semisymmetric. Thus it is an easy consequence of Theorem 1 of [4]. Again the Theorem 8.2 of [2] is an easy consequence of Lemma 5.6 of [16] and Theorem 8.1.

3. Proper examples of \((WCRS)_n\) and \((WCZS)_n\)

The metric in Example 9.1 of [2] (hence Example 1 of [7]) is the metric given in Example 3 of [15]. We note that this is not a \((WCZS)_4\) for \(\phi = \frac{1}{(x^4)^{1/4}}\) due to unaccented of \(Z_{14,1} = -\frac{8}{9(x^4)^{1/3}}\). If we take \(\phi = \frac{2}{9(x^4)^{1/3}}\), then the Example 9.1 of [2] is a \((WZS)_4\) and hence improperly a \((WCZS)_4\).

It may be mentioned that the examples given in [15] also do not satisfy the defining condition of weakly cyclic Ricci symmetric manifold. We now present two proper examples of \((WCRS)_4\) and \((WCZS)_4\).

Example 1: Consider the Gödel type metric ([5], [17]) in cylindrical coordinate system

\[
ds^2 = (dt + h(r) \, d\phi)^2 - (f(r))^2 \, d\phi^2 - dr^2 - dz^2,
\]

where \(h(r) = \frac{r^2}{8}\) and \(f(r) = \frac{r}{8}\sqrt{8 + r^2}\). The non-zero (upto symmetry) components of \(R\) and \(S\) are given by

\[
R_{1212} = -\frac{r^2}{64}, \quad R_{1313} = R_{1323} = R_{2323} = -\frac{1}{r^2 + 8},
\]

\[
S_{11} = -\frac{2}{r^2 + 8}, \quad S_{12} = S_{22} = -\frac{r^2 + 16}{8(r^2 + 8)}, \quad S_{33} = \frac{8}{(r^2 + 8)^2}.
\]
Again the non-zero (upto symmetry) components of $\nabla R$ and $\nabla S$ are given by
\[
\frac{1}{2} R_{1213,2} = R_{123,2} = -R_{1223,1} = \frac{r^3}{64 (r^2 + 8)}, \quad R_{1313,3} = R_{1323,3} = R_{2323,3} = \frac{4r}{(r^2 + 8)^2},
\]
\[
S_{11,3} = 6S_{13,1} = \frac{6r}{(r^2 + 8)^2}, \quad S_{12,3} = S_{22,3} = \frac{r (r^2 + 24)}{4 (r^2 + 8)^2},
\]
\[
S_{13,2} = S_{23,1} = S_{23,2} = \frac{r}{8 (r^2 + 8)}, \quad S_{33,3} = -\frac{32r}{(r^2 + 8)^3}.
\]
It is easy to check that if we consider $A = B = D = \{0, 0, -\frac{4r}{8 + r^2}, 0\}$, then (2.2) holds and hence the manifold is a $(WCRS)_4$.

**Example 2:** Let us consider the Lorentzian manifold $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^1 > 0\}$ endowed with the metric
\[
ds^2 = (x^1)^2 (dx^1)^2 + (x^2)^2 (dx^2)^2 + (x^3)^2 (dx^3)^2 - (x^4)^{-2} (dx^4)^2.
\]
Then the non-zero (upto symmetry) components of $R$, $S$, $\nabla R$ and $\nabla S$ are given by
\[
R_{1212} = R_{1313} = -R_{2323} = 1, \quad \frac{1}{3} R_{1414} = -R_{2424} = -R_{3434} = \frac{1}{(x^1)^4},
\]
\[
S_{11} = -S_{22} = -S_{33} = \frac{1}{(x^1)^2}, \quad S_{14} = -\frac{1}{(x^1)^6},
\]
\[
-R_{1212,1} = -2R_{1223,3} = -R_{1313,1} = 2R_{1323,2} = R_{2323,1} = \frac{4}{x^1},
\]
\[
-\frac{1}{3} R_{1414,1} = R_{1424,2} = R_{1434,3} = R_{2424,1} = R_{3434,1} = \frac{4}{(x^1)^5},
\]
\[
-S_{11,1} = 2S_{12,2} = 2S_{13,3} = S_{22,1} = S_{33,1} = \frac{4}{(x^1)^3}, \quad S_{44,1} = \frac{4}{(x^1)^7}.
\]
Let us now consider $\phi = \frac{a}{(x^1)^4} - \frac{1}{(x^1)^4}$, $a$ is a constant, and then the non-zero (upto symmetry) components of $Z = S + \phi g$ and $\nabla Z$ are given by
\[
Z_{11} = \frac{a}{(x^1)^4}, \quad Z_{22} = Z_{33} = -\frac{2(x^1)^2 - a}{(x^1)^4}, \quad Z_{44} = -\frac{a}{(x^1)^8},
\]
\[
Z_{11,1} = -\frac{6a}{(x^1)^5}, \quad Z_{12,2} = Z_{13,3} = \frac{2}{(x^1)^3}, \quad Z_{22,1} = Z_{33,1} = \frac{2(4(x^1)^2 - 3a)}{(x^1)^5}, \quad Z_{44,1} = \frac{6a}{(x^1)^9}.
\]
If we consider $A = B = D = \{-\frac{a}{x}, 0, 0, 0\}$, then (2.1) holds and hence the manifold is a $(WCZS)_4$.

**Remark 3.1.** Very recently De et al. studied $(WCZS)_4$ spacetimes (connected four dimensional semi-Riemannian manifold with Lorentzian metric). They claimed that such spacetimes are quasi-Einstein (Proposition 2.1). In our Example 2, $M$ is a $(WCZS)_4$ spacetime but it is not quasi-Einstein. To prove Proposition 2.1 of [3], the authors considered the vector field corresponding to $F = B - D$ as unit timelike but it is not true for any $(WCZS)_4$ spacetime, e.g., $F \equiv 0$ in
the above example. Hence from Example 2 we can conclude that the Proposition 2.1 of \[3\] is not always true.

Note: In Example 1 (resp., Example 2), the metric is proper (WCRS)$_4$ (resp., (WCZS)$_4$) but the associated 1-forms $A, B$ and $D$ are all equal and hence unique. We now present an example of improper (WCRS)$_4$ with non-unique associated 1-forms.

Example 3: Let $M$ be a non-empty open connected subset of $\mathbb{R}^4$ endowed with the semi-Riemannian metric

\[ (3.3) \quad ds^2 = e^{x^1} \left[ (x^3)^2 + x^3 + 1 \right] (dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (x^1)^2(dx^4)^2. \]

Then the non-zero (upto symmetry) components of $R$, $S$, $\nabla R$ and $\nabla S$ are given by

\[ R_{1313} = -e^{x^1}, \quad R_{1313,1} = -e^{x^1}, \quad S_{11} = e^{x^1}, \quad S_{11,1} = e^{x^1}. \]

Obviously the metric is recurrent as well as Ricci recurrent and hence it is improperly a (WCRS)$_4$. One can easily check that it satisfies the this metric satisfies the weakly cyclic Ricci symmetry condition (2.2) for $A = \{\alpha, 0, 0, 0\}$, $B = \{\beta, 0, 0, 0\}$ and $D = \{3 - \alpha - \beta, 0, 0, 0\}$, where $\alpha$ and $\beta$ are any non-vanishing scalars.

References

[1] Chaki, M.C., On pseudo Ricci symmetric manifolds, Bulg. J. Phys., 15 (1988), 526–531.
[2] De, U.C., Mantica, C.A. and Suh, J., On weakly cyclic $Z$ symmetric manifolds, Acta. Math. Hungarica, 146(1) (2015), 153–167.
[3] De, U.C., Mantica, C.A., Molinari, L.G. and Suh, Y.J., On weakly cyclic $Z$ symmetric spacetimes, Acta Math. Hungarica, 149(2) (2016), 1–16.
[4] De, U.C., Sengupta, J. and Saha, D., Conformally flat quasi Einstein spaces, Kyungpook Math. J., 46 (2006), 417–423.
[5] Deszcz, R., Hotloś, M., Jelowicki, J., Kundu, H. and Shaikh, A.A., Curvature properties of Gödel metric, Int J. Geom. Method Mod. Phy., 11 (2014), 20 pages.
[6] Gray, A., Einstein-like manifolds which are not Einstein, Geom. Dedicata, 7 (1978), 259–280.
[7] Kim, J., Notes on weakly cyclic $Z$-symmetric manifolds, Bull. Korean Math. Soc., 55(1) (2018), 227–237.
[8] Mantica, C.A. and Molinari, L.G., Weakly $Z$ symmetric manifolds, Acta Math. Hungar., 135 (2012), 8096.
[9] Shaikh, A.A., Choi, J. and Jana S.K., On conformally flat weakly cyclic Ricci symmetric manifolds, Tensor(N.S.), 70(3) (2008), 241-254.
[10] Shaikh, A.A., Das, L.S. and Jana, S.K., Weakly cyclic Ricci symmetric manifolds admitting semi-symmetric metric connection, Acta Mathematica Academiae Paedagogicae Nyregyhziensis, 27(2) (2011), 307-323.
[11] Shaikh, A.A., Deszcz, R., Hotlós, M., Jelovicki, J. and Kundu, H., On pseudosymmetric manifolds, Publ. Math. Debrecen, 86 (3-4) (2015), 433–456, arXiv:1405.2181v1 [math.DG] 9 May 2014.
[12] Shaikh, A.A. and Hui, S.K, On pseudo cyclic Ricci symmetric manifolds, Asian-European J. Math., 2(02) (2009), 227–237.
[13] Shaikh, A.A. and Hui, S.K., *On pseudo cyclic Ricci symmetric spacetimes*, Adv. Stud. Contemp. Math., **20**(3) (2010), 425–432.

[14] Shaikh, A.A. and Hui, S.K., *On pseudo cyclic Ricci symmetric manifolds admitting semi-symmetric metric connection*, SCIENTIA, Series A: Mathematical Science, **19-20** (2010), 73–80.

[15] Shaikh, A.A. and Jana S.K., *On weakly cyclic Ricci symmetric manifolds*, Ann. Pol. Math., **89**(3) (2006), 139–146.

[16] Shaikh, A.A. and Kundu, H., *On equivalency of various geometric structures*, J. Geom., **105**(2014), 139–165, DOI: 10.1007/s00022-013-0200-4, arXiv:1301.7214v3 [math.DG], 31 Jul 2013.

[17] Shaikh, A. A. and Kundu, H., *On curvature properties of Som-Raychaudhuri spacetime*, J. Geom. **108**(2) (2016), 501–515.

[18] Tamássy, L. and Binh, T. Q., *On weak symmetries of Einstein and Sasakian manifolds*, Tensor (N. S.), **53** (1993), 140–148.

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