Riemann hypothesis is not correct

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Abstract. This paper use Nevanlinna’s Second Main Theorem of the value distribution theory, we got an important conclusion by Riemann hypothesis. this conclusion contradicts the Theorem 8.12 in Titchmarsh’s book ”Theory of the Riemann Zeta-functions”, therefore we prove that Riemann hypothesis is incorrect.

Keyword. Nevanlinna’s Second Main Theorems, Riemann zeta function

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First, we give some notations, definitions and theorems in the theory of value distribution, its contents see the references [1] and [2].

We write

\[
\log^+ x = \begin{cases}
\log x & 1 \leq x \\
0 & 0 \leq x < 1
\end{cases}
\]

It is easy to see that \( \log x \leq \log^+ x \).

Let \( f(z) \) is a non-constant meromorphic function in the circle \( |z| < R, 0 < R \leq \infty \). \( n(r, f) \) represents the number of poles of \( f(z) \) on the circle \( |z| \leq r \ (0 < r < R) \), the multiplicity of poles is included. \( n(0, f) \) represents the order of pole of \( f(z) \) in the origin. For arbitrary complex number \( a \neq \infty \), \( n(r, \frac{1}{f-a}) \) represents the number of zeros of \( f(z) - a \) in the circle \( |z| \leq r \ (0 < r < R) \), the multiplicity of zeros is included. \( n(0, \frac{1}{f-a}) \) represents the order of zero of \( f(z) - a \) in the origin.
We write

\[ m(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(re^{i\varphi})| \, d\varphi \]

\[ N(r, f) = \int_{0}^{r} \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r \]

and \( T(r, f) = m(r, f) + N(r, f) \).

\( T(r, f) \) is called the characteristic function of \( f(z) \).

**Lemma 1.** If \( f(z) \) is a analytical function in the circle \(|z| < R \) (\( 0 < R \leq \infty \)), we have

\[ T(r, f) \leq \log^+ M(r, f) \leq \frac{\rho + r}{\rho - r} T(\rho, f)(0 < r < \rho < R) \]

where \( M(r, f) = \max_{|z|=r} |f(z)| \)

The lemma 1 follows from the References [1], page 57.

**Lemma 2.** Let \( f(z) \) is a non-constant meromorphic function in the circle \(|z| < R \) (\( 0 < R \leq \infty \)). \( a_\lambda (\lambda = 1, 2, ..., h) \) and \( b_\mu (\mu = 1, 2, ..., k) \) are the zeros and poles of \( f(z) \) in the circle \(|z| < \rho \) (\( 0 < \rho < R \)) respectively, each zero or pole repeated according to their multiplicity, and \( z = 0 \) is neither zero nor pole of the function \( f(z) \), then, in the circle \(|z| < \rho \), we have the following formula

\[ \log |f(0)| = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(\rho e^{i\varphi})| \, d\varphi - \sum_{\lambda=1}^{h} \log \frac{\rho}{|a_\lambda|} + \sum_{\mu=1}^{k} \log \frac{\rho}{|b_\mu|} \]

this formula is called Jensen formula.

The lemma 2 follows from the References [1], page 48.

**Lemma 3.** Let \( f(z) \) is the meromorphic function in the circle \(|z| \leq R\), and

\[ f(0) \neq 0, \infty, 1, \quad f'(0) \neq 0 \]

when \( 0 < r < R \), we have
\[ T(r, f) < 2 \left\{ N(R, \frac{1}{f}) + N(R, f) + N(R, \frac{1}{f-1}) \right\} \]

\[ + 4 \log^+ |f(0)| + 2 \log^+ \frac{1}{R|f'(0)|} + 24 \log \frac{R}{R-r} + 2328 \]

This is a form of Nevanlinna’s Second Main Theorem.

The lemma 3 follows from the References [1], the theorem 3.1 of the page 75.

Now, we make some preparations.

**LEMMA 4.** if \( f(x) \) is a function of the nonnegative degressive, we have

\[
\lim_{N \to \infty} \left( \sum_{n=a}^{N} f(n) - \int_{a}^{N} f(x) \, dx \right) = \alpha
\]

where \( 0 \leq \alpha \leq f(a) \). in addition, if \( x \to \infty \), \( f(x) \to 0 \), we have

\[
\left| \sum_{a \leq n \leq \xi} f(n) - \int_{a}^{\xi} f(\nu) \, d\nu - \alpha \right| \leq f(\xi - 1), \quad (\xi \geq a + 1)
\]

The lemma 4 follows from the References [3], the theorem 2 of the page 91.

Let \( s = \sigma + it \) is the complex number, when \( \sigma > 1 \), Riemann Zeta function is

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

When \( \sigma > 1 \), we have

\[
\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log n}
\]

where \( \Lambda(n) \) is Mangoldt function.
LEMMA 5. If $t$ is any real number, we have

1. \[0.0426 \leq | \log \zeta(4 + it) | \leq 0.0824\]
2. \[| \zeta(4 + it) - 1 | \geq 0.0426\]
3. \[0.917 \leq | \zeta(4 + it) | \leq 1.0824\]
4. \[| \zeta'(4 + it) | \geq 0.012\]

PROOF.

1. \[| \log \zeta(4 + it) | \leq \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^4 \log n} \leq \sum_{n=2}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} - 1 \leq 0.0824\]

2. \[| \log \zeta(4 + it) | \geq \frac{1}{2^4} - \sum_{n=3}^{\infty} \frac{1}{n^4} = 1 + \frac{2}{2^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{9}{8} - \frac{\pi^4}{90} \geq 0.0426\]

2. \[| \zeta(4 + it) - 1 | = \left| \sum_{n=2}^{\infty} \frac{1}{n^{4+it}} \right| \geq \frac{1}{2^4} - \sum_{n=3}^{\infty} \frac{1}{n^4} = 1 + \frac{2}{2^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{9}{8} - \frac{\pi^4}{90} \geq 0.0426\]

3. \[| \zeta(4 + it) | = \left| \sum_{n=1}^{\infty} \frac{1}{n^{4+it}} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \leq 1.0824\]

3. \[| \zeta(4 + it) | = \left| \sum_{n=1}^{\infty} \frac{1}{n^{4+it}} \right| \leq 1 - \sum_{n=2}^{\infty} \frac{1}{n^4} = 2 - \sum_{n=1}^{\infty} \frac{1}{n^4} = 2 - \frac{\pi^4}{90} \geq 0.917\]
(4) \[
| \zeta'(4+it) | = \left| \sum_{n=2}^{\infty} \frac{\log n}{n^{4+it}} \right| \geq \frac{\log 2}{2^4} - \sum_{n=3}^{\infty} \frac{\log n}{n^{4}}
\]

by Lemma 4, we have
\[
\sum_{n=3}^{\infty} \frac{\log n}{n^{4}} = \int_{3}^{\infty} \frac{\log x}{x^{4}} \, dx + \alpha
\]
where \(0 \leq \alpha \leq \frac{\log 3}{3^4}\)

\[
\int_{3}^{\infty} \frac{\log x}{x^{4}} \, dx = -\frac{1}{3} \int_{3}^{\infty} \log x \, dx^{-3} = \frac{\log 3}{3^4} + \frac{1}{3} \int_{3}^{\infty} x^{-4} \, dx
\]

\[
= \frac{\log 3}{3^4} - \frac{1}{3^2} \int_{3}^{\infty} x^{-3} \, dx = \frac{\log 3}{3^4} + \frac{1}{3^5}
\]

therefore
\[
\sum_{n=3}^{\infty} \frac{\log n}{n^{4}} \leq \frac{\log 3}{3^4} + \frac{1}{3^5} + \frac{\log 3}{3^4}
\]

\[
| \zeta'(4+it) | \geq \frac{\log 2}{2^4} - 2 \frac{\log 3}{3^4} - \frac{1}{3^5} \geq 0.012
\]

This completes the proof of Lemma 5.

Let \(\delta = \frac{1}{100}\), \(c_1, c_2, \ldots\), is the positive constant.

**LEMMA 6.** When \(\sigma \geq \frac{1}{2}, \ |t| \geq 2\), we have
\[
| \zeta(\sigma + it) | \leq c_1 |t|^\frac{1}{2}
\]

The lemma 6 follows from the References [4], the theorem 2 of the page 140.

**LEMMA 7.** If \(f(z)\) is the analytic function in the circle \(|z - z_0| \leq R\), \(0 < r < R\), in the circle \(|z - z_0| \leq r\), we have
\[ |f(z) - f(z_0)| \leq \frac{2r}{R - r} (A(R) - Ref(z_0)) \]

where \( A(R) = \max_{|z-z_0| \leq R} Re f(z) \).

The lemma 7 follows from the References [4], the theorem 2 of the page 61.

Now, we assume that Riemann hypothesis is correct, and abbreviation as RH. In other words, when \( \sigma > \frac{1}{2} \), the function \( \zeta(\sigma + it) \) has no zeros. The function \( \log \zeta(\sigma + it) \) is a multi-valued analytic function in the region \( \sigma > \frac{1}{2}, t \geq 1 \). we choose the principal branch of the function \( \log \zeta(\sigma + it) \), therefore, if \( \zeta(\sigma + it) = 1 \), then \( \log \zeta(\sigma + it) = 0 \).

**Lemma 8.** If RH is correct, when \( \delta = \frac{1}{100} \), \( \sigma \geq \frac{1}{2} + 2\delta \), \( t \geq 16 \), we have

\[ |\log \zeta(\sigma + it)| \leq c_2 \log t + c_3 \]

**Proof.** In Lemma 7, we choose \( f(z) = \log \zeta(z + 4 + it) \), \( z_0 = 0 \), \( R = \frac{7}{2} - \delta \), \( r = \frac{7}{2} - 2\delta \), \( t \geq 16 \). Because \( \log \zeta(z + 4 + it) \) is the analytic function in the circle \( |z| \leq R \), by Lemma 7, in the circle \( |z| \leq r \), we have

\[ |\log \zeta(z + 4 + it) - \log \zeta(4 + it)| \leq \frac{7}{\delta} (A(R) - Re \log \zeta(4 + it)) \]

therefore

\[ |\log \zeta(z + 4 + it)| \leq \frac{7}{\delta} (A(R) + |\log \zeta(4 + it)| + |\log \zeta(4 + it)|) \]

by Lemma 6, we have

\[ A(R) = \max_{|z-z_0| \leq R} \log |\zeta(z + 4 + it)| \leq \frac{1}{2} \log t + \log c_1 \]

by Lemma 5, we have

\[ |\log \zeta(z + 4 + it)| \leq c_2 \log t + c_3 \]

therefore, when \( \sigma \geq \frac{1}{2} + 2\delta \), we have

\[ |\log \zeta(\sigma + it)| \leq c_2 \log t + c_3 \]
This completes the proof of Lemma 8.

**Lemma 9.** If RH is correct, when $\delta = \frac{1}{100}$, $t \geq 16$, $\rho = \frac{7}{2} - 2\delta$, in the circle $|z| \leq \rho$, we have

$$N\left( \rho, \frac{1}{\zeta(z + 4 + it)} - 1 \right) \leq \log \log t + c_4$$

**Proof.** In Lemma 2, we choose $f(z) = \log \zeta(z + 4 + it)$, $R = \frac{7}{2} - \delta$, $\rho = \frac{7}{2} - \delta$, $a_{\lambda}$ ($\lambda = 1, 2, ..., h$) are the zeros of the function $\log \zeta(z + 4 + it)$ in the circle $|z| < \rho$, each zero repeated according to their multiplicity. Because the function $\log \zeta(z + 4 + it)$ has no poles in the circle $|z| < \rho$, and $\log \zeta(4 + it)$ is not equal to zero, we have

$$\log |\log \zeta(4 + it)| = \frac{1}{2\pi} \int_0^{2\pi} \log |\log \zeta(4 + it + \rho e^{i\varphi})| \, d\varphi - \sum_{\lambda=1}^h \log \frac{\rho}{|a_{\lambda}|}$$

by Lemma 5 and Lemma 8, we have

$$\sum_{\lambda=1}^h \log \frac{\rho}{|a_{\lambda}|} \leq \log \log t + c_4$$

because $z = 0$ is neither zero nor pole of the function $\log \zeta(z + 4 + it)$, if $r_0$ is a sufficiently small positive number, we have

$$\sum_{\lambda=1}^h \log \frac{\rho}{|a_{\lambda}|} = \int_{r_0}^{\rho} \left( \log \frac{\rho}{t} \right) \left( \frac{d n(t, \frac{1}{f})}{f} \right) \left. \right|_{r_0}^{\rho}$$

$$+ \int_{r_0}^{\rho} \frac{n(t, \frac{1}{f})}{t} \, dt = \int_0^{\rho} \frac{n(t, \frac{1}{f})}{t} \, dt = N\left( \rho, \frac{1}{f} \right)$$

$$= N\left( \rho, \frac{1}{\log \zeta(z + 4 + it)} \right) \geq N\left( \rho, \frac{1}{\zeta(z + 4 + it) - 1} \right)$$

This completes the proof of Lemma 9.
THEOREM. If RH is correct, when $\sigma \geq \frac{1}{2} + 4\delta$, $\delta = \frac{1}{100}$, $t \geq 16$, we have

$$|\zeta(\sigma + it)| \leq c_8 \ (\log t)^{c_6}$$

proof. In Lemma 3, we choose $f(z) = \zeta(z + 4 + it)$, $t \geq 16$, $R = \frac{7}{2} - 2\delta$, $r = \frac{7}{2} - 3\delta$. by Lemma 5, we have $f(0) = \zeta(4 + it) \neq 0, \infty, 1,$ and $|f'(0)| = |\zeta'(4 + it)| \geq 0.012$, $|f(0)| = |\zeta(4 + it)| \leq 1.0824$. because $\zeta(z + 4 + it)$ is the analytic function, and it have neither zeros nor poles in the circle $|z| \leq R$, we have

$$N(R, \frac{1}{f}) = 0 , \quad N(R, f) = 0$$

therefore, by Lemma 9, we have

$$T(r, \zeta(z + 4 + it)) \leq 2\log \log t + c_5$$

In Lemma 1, we choose $R = \frac{7}{2} - 2\delta$, $\rho = \frac{7}{2} - 3\delta$, $r = \frac{7}{2} - 4\delta$. by the maximal principle, in the circle $|z| \leq r$, we have

$$\log^+ |\zeta(z + 4 + it)| \leq c_6 \log \log t + c_7$$

therefore, when $\sigma \geq \frac{1}{2} + 4\delta$, we have

$$\log^+ |\zeta(\sigma + it)| \leq c_6 \log \log t + c_7$$

$$\log |\zeta(\sigma + it)| \leq c_6 \log \log t + c_7$$

$$|\zeta(\sigma + it)| \leq c_8 \ (\log t)^{c_6}$$

This completes the proof of Theorem.

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