Chiral de Rham complex and the half-twisted sigma-model

Anton Kapustin

*California Institute of Technology, Pasadena, CA 91125, U.S.A.*

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Abstract

On any Calabi-Yau manifold \(X\) one can define a certain sheaf of chiral \(N = 2\) superconformal field theories, known as the chiral de Rham complex of \(X\). It depends only on the complex structure of \(X\), and its local structure is described by a simple free field theory. We show that the cohomology of this sheaf can be identified with the infinite-volume limit of the half-twisted sigma-model defined by E. Witten more than a decade ago. We also show that the correlators of the half-twisted model are independent of the Kähler moduli to all orders in worldsheet perturbation theory, and that the relation to the chiral de Rham complex can be violated only by worldsheet instantons.
1 Introduction and summary

Given a Calabi-Yau manifold $X$ equipped with a Kähler class $\omega$ and a B-field $B \in H^2(X, \mathbb{R})$, one can consider the corresponding supersymmetric sigma-model. This field theory has $N = (2,2)$ superconformal symmetry, both on the classical and quantum levels. That is, its Hilbert space carries a unitary representation of the direct sum of two copies of $N = 2$ super-Virasoro algebra, with central charge $\hat{c} = \dim_{\mathbb{C}} X$. The quantum sigma-model is a very complicated object, depending both on the complexified Kähler class $B + i\omega$ and the complex structure on $X$. In general this field theory can be described only in the limit when the volume of $X$ is large, using perturbation theory in the inverse volume.

To obtain a more manageable object, one should “truncate” the sigma-model in some way. In Ref. [1] E. Witten defined three such truncations, known as the A-model, the B-model, and the half-twisted model of $X$. The A and B-models are two-dimensional topological field theories, while the half-twisted model is a holomorphic $N = 2$ superconformal field theory with the same central charge as the original sigma-model.\(^1\) All three models are obtained in a uniform way: one starts with the operator algebra of the $N = 2$ sigma-model, shifts the holomorphic (=left-moving) and antiholomorphic (=right-moving) stress-energy tensors by derivatives of the corresponding R-currents, and considers the cohomology of the operator algebra with respect to a certain nilpotent operator (the BRST charge). The difference between the models resides in the choice of the shift of the stress-energy tensors and the choice of the BRST charge. If we distinguish the generators of the holomorphic and antiholomorphic $N = 2$ super-Virasoro with ± signs, then the shifts and the BRST operators are

\[
\begin{align*}
\text{A - model: } & T_-(z) \rightarrow T_-(z) + \frac{1}{2} \partial J_-(z), \quad T_+(\bar{z}) \rightarrow T_+(\bar{z}) + \frac{1}{2} \bar{\partial} J_+(\bar{z}), \\
& \quad Q_{\text{BRST}} = \bar{Q}_+ + Q_-,
\end{align*}
\]

\[
\begin{align*}
\text{B - model: } & T_-(z) \rightarrow T_-(z) - \frac{1}{2} \partial J_-(z), \quad T_+(\bar{z}) \rightarrow T_+(\bar{z}) + \frac{1}{2} \bar{\partial} J_+(\bar{z}), \\
& \quad Q_{\text{BRST}} = \bar{Q}_+ + \bar{Q}_-,
\end{align*}
\]

\[
\begin{align*}
\text{half - twisted model: } & T_+(\bar{z}) \rightarrow T_+(\bar{z}) + \frac{1}{2} \bar{\partial} J_+(\bar{z}), \quad Q_{\text{BRST}} = \bar{Q}_+.
\end{align*}
\]

\(^1\)Another common name for a holomorphic CFT is chiral algebra. In Ref. [2] they are called conformal vertex algebras.
Here $Q_\pm$ and $\bar{Q}_\pm$ are Fourier-components of the corresponding supercurrents; for any field $V(z, \bar{z})$ the supercommutator of $\bar{Q}_+$ with $V$ is given by

$$\{\bar{Q}_+, V\}(z, \bar{z}) = \oint \frac{d\bar{w}}{2\pi i} \bar{Q}_+(\bar{w})V(z, \bar{z}),$$

and similarly for other supercharges. Here the integral is over a small contour around $w = z$. In what follows we choose to redefine the half-twisted model slightly, by shifting $T_-(z) \rightarrow T_-(z) + \frac{1}{2} \partial J_-(z)$, as for the A-model. The operator algebra of the half-twisted model is a holomorphic CFT and contains a copy of the $N = 2$ super-Virasoro algebra. With our slight redefinition, it is a twisted $N = 2$ super-Virasoro algebra (also known as the topological $N = 2$ algebra).

Witten showed that the B-model is independent of the complexified Kähler class of $X$, while the A-model is independent of the complex structure of $X$. These topological field theories have been intensively studied, for a variety of reasons. The half-twisted model is a much richer object (its state space is always infinite-dimensional), and has not been much studied. For general $X$, essentially the only thing we know about the half-twisted model is its “Euler characteristic” defined as

$$\chi(q, \gamma) = \text{Tr}(-1)^F q^{L_0 - \frac{c}{12}} \exp(i\gamma J_-),$$

where $F$ is the total fermion number operator, and

$$L_0 = \oint \frac{dz}{2\pi i} z T_-(z), \quad J_- = \oint \frac{dz}{2\pi i} J_-(z).$$

The Euler characteristic is known to coincide with a certain two-variable elliptic genus of $X$ [3].

A few years ago mathematicians Malikov, Schechtman, and Vaintrob have defined, for any complex manifold $X$, a sheaf of holomorphic operator algebras, called the chiral de Rham complex [4]. Locally, the space of sections of this sheaf is very simple: it is the tensor product of $n$ copies of the familiar $\beta\gamma - bc$ system, where $n = \text{dim}_C X$. It was shown in Ref. [4] that if one performs arbitrary analytic reparametrizations of the $\gamma$-fields, there exist analytic transformation laws for the remaining fields which preserve the Operator Product Expansion (OPE). This surprising fact allows one to treat the constant modes of the $\gamma$-fields as local coordinates on $X$, and glue the locally-defined operator algebras into a sheaf of operator algebras on $X$. In
the case when $X$ is a Calabi-Yau, the authors of Ref. [4] showed that there is an extra structure on this sheaf: it is a sheaf of holomorphic $N = 2$ superconformal field theories, whose central charge is equal to $\dim \mathbb{C} X$. (Locally, this is obvious, because the $\beta \gamma - bc$ system has $N = 2$ superconformal symmetry. However, it is nontrivial to check that the generators of the $N = 2$ super-Virasoro algebra glue properly on double overlaps.) The cohomology of this sheaf is a certain holomorphic $N = 2$ SCFT, which is fairly complicated, in general. As explained in Ref. [5], a suitably defined Euler characteristic of this sheaf is equal to the elliptic genus of $X$. This suggests a relation with the half-twisted model of E. Witten. There are two puzzles which must be resolved to establish a precise connection. First, physicists usually do not think in terms of sheaves of CFTs: the sigma-model is a global object, from the target-space viewpoint. Second, the half-twisted model depends both on complex and Kähler structures of $X$, while the chiral de Rham complex depends only on the complex structure.

In this note we show that the cohomology of the chiral de Rham complex can be identified with the infinite-volume limit of the half-twisted sigma-model. This resolves the second puzzle. As for the first puzzle, we recall that one can compute sheaf cohomology in a number of ways. The definition used in Ref. [4] naturally leads one to the Čech resolution. On the other hand, one can also construct a “Dolbeault resolution” for the chiral de Rham complex, whose graded components are soft sheaves of holomorphic $N = 2$ SCFTs. Thus the sheaf cohomology can be computed by taking the global sections of the Dolbeault resolution and computing the cohomology of the Dolbeault differential on the resulting graded vector space. We show that this recipe for computing sheaf cohomology is the infinite-volume limit of the physical definition of the half-twisted sigma-model. This resolves the first puzzle.

Having established that the infinite-volume limit of the half-twisted model is related to the chiral de Rham complex, we then ask how this result is modified at a large but finite volume. We show that the half-twisted model does not depend on Kähler moduli to all orders in the large-volume expansion. Thus the relation between the chiral de Rham complex and the half-twisted model is perturbatively exact. We find it rather surprising that such a seemingly complicated object as the half-twisted sigma-model for an arbitrary Calabi-Yau manifold can be computed to all orders in perturbation theory by gluing together free theories ($\beta \gamma - bc$ systems). On the other hand, we expect that nonperturbative contributions from worldsheet-instanton lead to Kähler moduli dependence, much like in the A-model. Unlike the A-model,
however, the half-twisted model also depends on the complex structure of $X$.

The content of the paper is as follows. In Section 2 we describe the chiral de Rham complex and its Dolbeault resolution. In Section 3 we show that the infinite-volume limit of the half-twisted sigma-model coincides with the cohomology of the Dolbeault differential acting on the space of global sections of the Dolbeault resolution. In Section 4 we study the dependence of the half-twisted model on the Kähler moduli. In Section 5 we briefly discuss possible generalizations.

In Section 2 we make use of some basic notions of sheaf theory. For an introduction, see e.g. Chapters 1 and 2 of Ref. [6]. Readers with a sheaf-intolerance are advised to skim Section 2; the rest of the paper does not use sheaves.

## 2 Chiral de Rham complex and its Dolbeault resolution

Following Ref. [4], we first describe the chiral de Rham complex of an affine space $\mathbb{C}^n$. It is a holomorphic $N=2$ SCFT which is a tensor product of $n$ copies of the $\beta\gamma-bc$ system. That is, we have free bosonic fields $\phi^i(z), p^i(z)$ and free fermionic fields $\psi^i(z), \rho^i(z)$ with the nontrivial OPE

$$
\phi^i(z)p^j(w) \sim \frac{\delta^i_j}{z-w}, \quad \rho^i(z)\psi^j(w) \sim \frac{\delta^i_j}{z-w}.
$$

(Our convention for the fields $p^i$ differs by a sign from that of Ref. [4]. This difference in conventions accounts for some extra minus signs below.) Such OPEs follow from quantizing a quadratic action

$$
S = \frac{1}{\pi} \int d^2 z \left( p^i \partial \phi^i + \rho^i \partial \bar{\psi}^i \right).
$$

The equations of motion following from this action imply that the fields can be expanded into Laurent series:

$$
\phi^i = \sum_{n \in \mathbb{Z}} \phi^i_n z^{-n}, \quad p^i = \sum_{n \in \mathbb{Z}} p^i_n z^{-n-1},
$$

$$
\psi^i = \sum_{n \in \mathbb{Z}} \psi^i_n z^{-n}, \quad \rho^i = \sum_{n \in \mathbb{Z}} \rho^i_n z^{-n-1},
$$

From the OPEs we get the commutation relations:

$$
[\phi^i_n, p^j_{m}] = \delta^i_j \delta_{n,-m}, \quad \{\psi^i_n, \rho^j_{m}\} = \delta^i_j \delta_{n,-m},
$$
with all other supercommutators vanishing. The state space of the theory is the tensor product of the space of holomorphic functions of variables $\phi_0^i, \psi_0^i$ and the Fock space of the “oscillator” modes (the ones with $n \neq 0$). The Fock space contains a vacuum vector $|\text{vac}\rangle$ satisfying

$$\phi_n^i |\text{vac}\rangle = p_{i,n} |\text{vac}\rangle = \psi_n^i |\text{vac}\rangle = \rho_{i,n} |\text{vac}\rangle = 0, \quad \forall n > 0,$$

and is generated by acting on the vacuum vector with $n < 0$ oscillators. The remaining zero-modes, $p_{i,0}$ and $\rho_{i,0}$, act as differential operators on holomorphic functions of $\phi_0^i$ and $\psi_0^i$:

$$p_{i,0} = -\frac{\partial}{\partial \phi_0^i}, \quad \rho_{i,0} = \frac{\partial}{\partial \psi_0^i}.$$

The stress-energy tensor is defined to be

$$T_-(z) = -p_i(z)\partial \phi^i(z) - \rho_i(z)\partial \psi^i(z).$$

(Here and below we omit the symbol of normal-ordering). The supercurrents are

$$Q_-(z) = -\psi^i(z)p_i(z), \quad \bar{Q}_-(z) = \rho_i(z)\partial \phi^i(z).$$

The R-current is

$$J_-(z) = \rho_i(z)\psi^i(z).$$

Computing the OPE of these currents, one finds that they (or rather the coefficients of their Laurent expansions) form a topologically twisted $N = 2$ super-Virasoro algebra with central charge $\hat{c} = n$. The supersymmetry transformations of the fields are computed from the OPE:

$$\{ Q_-, V(w) \} = \oint \frac{dz}{2\pi i} Q_-(z)V(w), \quad \{ Q_-, V(w) \} = \oint \frac{dz}{2\pi i} \bar{Q}_-(z)V(w).$$

In this way one finds:

\begin{align}
\{ Q_-, \phi^i \} &= \psi^i, & \{ Q_-, \phi^j \} &= 0, \\
\{ Q_-, p_i \} &= 0, & \{ \bar{Q}_-, p_i \} &= -\partial \rho_i, \\
\{ Q_-, \psi^i \} &= 0, & \{ \bar{Q}_-, \psi^i \} &= \partial \phi^i, \\
\{ Q_-, \rho_i \} &= -p_i, & \{ \bar{Q}_-, \rho_i \} &= 0.
\end{align}

From the worldsheet viewpoint, $\rho_i$ and $p_i$ are 1-forms, while $\psi^i$ and $\phi^i$ are 0-forms. One can untwist the theory by subtracting $\frac{1}{2}\partial J_-$ from the stress-energy tensor; then $\rho_i$ and $\psi^i$ become worldsheet spinors of the same chirality.
Ordinarily, in free field theory one considers composite fields which are normal-ordered polynomials of the fundamental fields (which in this case are $\phi^i, p_i, \psi^i, \rho_i$) and their derivatives. In the present case, since the OPE of $\phi^i$ with itself is trivial, it makes sense to consider composite operators where arbitrary analytic functions of $\phi^i$ (and polynomial functions of other fields and derivatives of $\phi^i$) are allowed. This is explained in detail in section 3.1 of Ref. [4]. Briefly speaking, given a function $f(\phi)$, one replaces the variable $\phi$ with the corresponding Laurent series $\phi(z)$ and reexpands in powers of $z$. Although the coefficient of each power of $z$ is an infinite sum of operators on the tensor product of the bosonic Fock space and the space of holomorphic functions, one can show that it is well-defined.

To define a sheaf of holomorphic $N=2$ SCFTs, one has to invent a transformation law for the fields which is compatible with the postulated OPEs and preserves the expressions for the stress-energy tensor, supercurrents, and the R-current. Given a change of coordinates

$$\tilde{\phi}^i = g^i(\phi), \quad \phi^i = f^i(\tilde{\phi}),$$

an obvious transformation law for the fermions is

$$\tilde{\psi}^i = \frac{\partial g^i}{\partial \phi^j} \psi^j, \quad \tilde{\rho}^i = \frac{\partial f^i}{\partial \phi^j} \rho^j.$$

The correct transformation law for the bosonic field $p_i$ is less obvious:

$$\tilde{p}_i = \frac{\partial f^j}{\partial \phi^i} p_j + \frac{\partial^2 f^j}{\partial \phi^i \partial \phi^j} \frac{\partial g^k}{\partial \phi^i} \rho_k \psi^r. \quad (3)$$

In Ref. [4] this transformation law was motivated by its compatibility with the OPE (Th. 3.7 of Ref. [4]). We will see in the next section how this transformation law arises naturally in the half-twisted model.

Finally, one has to check that the $N=2$ super-Virasoro algebra is preserved under coordinate change. It turns out that this is true if and only if all Jacobians

$$\det \frac{\partial g^i}{\partial \phi^j}$$

are constant. This is possible to achieve if and only if $c_1(X) = 0$, i.e. if $X$ is a Calabi-Yau manifold.\(^2\) Thus for any Calabi-Yau $X$ one obtains a sheaf

\(^2\)Strictly speaking, this condition is equivalent to the Calabi-Yau condition only if $X$ is simply-connected. If $H^1(X) \neq 0$, then it is possible that $c_1(X) = 0$, but the canonical line bundle of $X$ is nontrivial, as a holomorphic line bundle. From the physical viewpoint, this means that the BRST cohomology does not contain the spectral flow operator.
of holomorphic $N = 2$ SCFTs. By definition, this is the chiral de Rham complex of $X$, denoted $\Omega^ch_X$. The space of sections of this sheaf over an open set $U$ will be denoted $\Omega^ch_X(U)$.

We are interested in the sheaf cohomology of the chiral de Rham complex, which, on general grounds, is a holomorphic $N = 2$ SCFT. The degree-0 part is simply the space of global sections of the chiral de Rham complex, while higher-degree cohomology can be defined using either the Čech approach or any soft resolution. In order to make a connection with the half-twisted model, we would like to consider a Dolbeault-like resolution. Locally, this is achieved by introducing bosonic variables $\phi^i$ complex-conjugate to the zero-modes of $\phi^i$ and fermionic (i.e. odd) variables $\psi^i$, and enlarging the chiral de Rham complex by allowing arbitrary smooth functions of $\phi^i$ and $\bar{\phi}^i$ and polynomial functions of the remaining fields, their $\bar{\partial}$-derivatives, and the variables $\bar{\psi}^i$. Note that while $\phi^i$ are fields (i.e. they are power series in $z$ and $\bar{z}$), $\phi^i$ are simply complex variables. Thus it is not completely obvious that the procedure is well-defined. Recall that we started with polynomial functions of the fields $\phi^i$, but then replaced them with arbitrary holomorphic functions. As explained in Ref. [4], this construction has the following generalization: instead of holomorphic function one can take an arbitrary supercommutative algebra which contains the polynomial algebra of $\phi^i$ as a subalgebra and is equipped with an action of the Lie algebra of polynomial vector fields $\frac{\partial}{\partial \phi^i}$ which satisfies the Leibniz rule. We are dealing with a special case, where the algebra in question is the algebra of smooth forms of type $(0,p)$ (for some $p$).

In this way we get a new sheaf of holomorphic $N = 2$ SCFTs graded by the $\psi^i$-degree. We will denote it $\Omega^{ch,Dol}_X$. The Dolbeault operator

$$d_{Dol} = \bar{\psi}^i \frac{\partial}{\partial \phi^i}$$

makes $\Omega^{ch,Dol}_X$ into a complex of sheaves. It is easy to see that this is a resolution of the chiral de Rham complex: this follows from the usual $\bar{\partial}$-Poincaré lemma and the fact that $d_{Dol}$ commutes with the normal-ordering of operator product. We will call this complex of sheaves the Dolbeault resolution of the chiral de Rham complex.

It is also easy to see that the Dolbeault resolution is soft. The proof is the same as for the ordinary Dolbeault resolution and makes use of the existence of partition of unity for smooth functions. Softness implies that we can
compute the sheaf cohomology of the chiral de Rham complex by considering the space of global sections of the Dolbeault resolution $\Omega_{X}^{ch,Dol}(X)$, and then computing the cohomology of $d_{Dol}$. As we will see in the next section, this global approach to computing the sheaf cohomology is much better suited for comparison with the half-twisted sigma-model.

3 The half-twisted model and its large volume limit

Now we turn to the half-twisted sigma-model whose target is a Calabi-Yau manifold. Recall that we decided to twist both the holomorphic and antiholomorphic stress-energy tensors, so the action is the same as for the A-model [1]:

$$S = \frac{1}{\pi} \int d^{2}z \left[ \frac{1}{2} (g_{i\overline{j}} + B_{i\overline{j}}) \partial \phi^{i} \overline{\partial} \phi^{\overline{j}} + \frac{1}{2} (g_{i\overline{j}} - B_{i\overline{j}}) \overline{\partial} \phi^{i} \partial \phi^{\overline{j}} + \rho_{i} \overline{D} \psi^{i} \right. $$

$$+ \left. \rho_{i} D \psi^{\overline{i}} + R^{i}_{j\overline{j}} \rho_{i} \rho_{\overline{i}} \psi^{j} \psi^{\overline{j}} \right]. \quad (4)$$

Here $\phi^{i}(z, \overline{z})$ is a coordinate representation of a map from the worldsheet $\Sigma$ to the target $X$, $\psi^{i}, \psi^{\overline{i}}$ are fermionic fields taking values in $\phi^{*}TX^{1,0}$ and $\phi^{*}TX^{0,1}$, respectively, and $\rho_{i}, \rho_{\overline{i}}$ are fermionic fields taking values in $K_{\Sigma} \otimes \phi^{*}TX^{1,0}$ and $K_{\Sigma} \otimes \phi^{*}TX^{0,1}$, respectively, where $K_{\Sigma}$ is the canonical line bundle of $\Sigma$. Further, $g_{i\overline{j}}(\phi)$ is the Kähler metric on $X$, $R^{i}_{j\overline{j}}$ is the corresponding Riemann tensor, $B_{i\overline{j}}(\phi)$ is the B-field (which is assumed here to be closed and of type $(1, 1)$), and $D$ and $\overline{D}$ are holomorphic and antiholomorphic covariant derivatives on the vector bundle $\phi^{*}TX$:

$$\overline{D} \psi^{\overline{j}} = \overline{\partial} \psi^{\overline{j}} + \Gamma^{j}_{k}(\overline{\partial} \phi^{j}) \psi^{\overline{k}}, \quad D \psi^{j} = \partial \psi^{j} + \Gamma^{j}_{k}(\partial \phi^{j}) \psi^{\overline{k}}.$$  

We are going to consider the infinite-volume limit $g_{i\overline{j}} \to \infty$. In this limit nonperturbative effects, such as worldsheet instantons, are irrelevant, and the B-field does not affect the dynamics. If we choose a purely imaginary B-field, $B = i\omega$, where $\omega$ is the Kähler form, then the first term in Eq. (4) vanishes, and the action becomes

$$S = \frac{1}{\pi} \int d^{2}z \left[ g_{i\overline{j}} \overline{\partial} \phi^{i} \partial \phi^{\overline{j}} + \rho_{i} \overline{D} \psi^{i} + \rho_{i} D \psi^{\overline{j}} + R^{i}_{j\overline{j}} \rho_{i} \rho_{\overline{i}} \psi^{j} \psi^{\overline{j}} \right]. \quad (5)$$
Alternatively, one can simply set $B = 0$ and note that the first term in Eq. (4) differs from the second one by $\frac{1}{4\pi} \int \phi^* \omega$, which vanishes for topologically trivial string worldsheets. Therefore, if one neglects the contributions of worldsheet instantons, one can replace Eq. (4) with Eq. (5).

Consider now the following action:

$$S^{(1)} = \frac{1}{\pi} \int d^2z \left[ p_i \partial \phi^i + p_i \partial \phi^j + \rho_i \partial \psi^i + \rho_i \partial \psi^\bar{j} - g^{ij} \left( p_i - \Gamma^{ij}_{k} \rho_k \psi^i \right) \left( p_j - \Gamma^{\bar{j}k}_{j} \rho_k \psi^\bar{k} \right) + R^{ij}_{\bar{k}j} \rho_i \rho_i \psi^j \psi^\bar{j} \right].$$  \hspace{1cm} (6)

The equations of motion for the bosonic fields $p_i$ and $\bar{p}_i$ are algebraic and read

$$p_i = g_i^{\bar{j}} \partial \phi^\bar{j} + \Gamma^{ij}_{k} \rho_j \psi^k, \quad \bar{p}_j = \bar{g}^{ij} \partial \phi^j + \Gamma^{\bar{k}j}_{\bar{j}} \rho_j \psi^\bar{k}. \hspace{1cm} (7)$$

Substituting the expressions for $p_i, \bar{p}_j$ back into Eq. (6), one finds the action Eq. (5), and thus Eq. (5) and Eq. (6) define equivalent theories. The action Eq. (6) is more convenient for taking the infinite-volume limit. In this limit the inverse metric $g^{\bar{j}i}$ goes to zero, so it is clear that we should keep the fields $\phi^i, p_i, \psi^i, \rho_i$ and their complex conjugates fixed. The limiting action is quadratic:

$$S^{(1)}_{\infty} = \frac{1}{\pi} \int d^2z \left[ p_i \partial \phi^i + p_i \partial \phi^\bar{j} + \rho_i \partial \psi^i + \rho_i \partial \psi^\bar{j} \right]. \hspace{1cm} (8)$$

Quantization of this action gives a free $N = 2$ SCFT. This theory is essentially a tensor product of the theory defined by the action Eq. (1) and its complex-conjugate. This statement would be exactly true if we allowed only operators which depend polynomially on $\phi^i$ and $\phi^\bar{j}$. But since the zero-modes of $\phi^i(z)$ and $\phi^\bar{j}(\bar{z})$ are local coordinates on $X$, one should allow arbitrary smooth functions of $\phi^i$ and $\phi^\bar{j}$. (The dependence on the rest of the fields and the derivatives of all the fields is polynomial.)

On the other hand, the limiting $N = 2$ SCFT is also similar to the space $\Omega^{ch,Dol}_X(X)$. The difference is that in the latter case only the zero modes of the antiholomorphic fields $\phi^\bar{j}(\bar{z}), \phi^\bar{i}(\bar{z})$ are retained. Let us show that this difference becomes irrelevant if we compare the $\bar{Q}_+$ cohomology of the theory Eq. (3) with the $d_{Dol}$ cohomology of $\Omega^{ch,Dol}_X(X)$.

The BRST transformations of the half-twisted model (before taking the
infinite-volume limit) read [1]:

\[ \delta \phi^i = 0, \quad \delta \phi^\bar{i} = \epsilon \psi^\bar{i}, \]
\[ \delta \psi^i = 0, \quad \delta \psi^\bar{i} = 0, \]
\[ \delta \rho_i = 0, \quad \delta \rho^\bar{i} = -\epsilon g_{ij} \bar{\partial} \phi^j - \epsilon \Gamma^k_{ij} \rho_k \psi^\bar{j} = -\epsilon p_i. \]

Here \( \epsilon \) is an odd parameter of the BRST transformation. The BRST variation of \( p^\bar{i} \) vanishes identically, while the BRST variation of \( p_i \) vanishes upon using the equations of motion. This can be shown using Eq. (9) and Eq. (7). The operator \( \bar{Q}^+ \) generating these transformations has the following simple form:

\[ \bar{Q}^+ = -\int d\bar{z} \psi^i p_i. \]

In the infinite-volume limit, we can expand the fields with antiholomorphic indices into antiholomorphic Laurent series:

\[ \phi^\bar{i} = \sum_{n \in \mathbb{Z}} \phi^\bar{i}_n \bar{z}^{-n}, \quad p_i = \sum_{n \in \mathbb{Z}} p_i_n \bar{z}^{-n-1}, \]
\[ \psi^\bar{i} = \sum_{n \in \mathbb{Z}} \psi^\bar{i}_n \bar{z}^{-n}, \quad \rho^\bar{i} = \sum_{n \in \mathbb{Z}} \rho^\bar{i}_n \bar{z}^{-n-1}, \]

whose coefficients are operators satisfying the commutation relations

\[ [\phi^\bar{i}_n, p^\bar{j}_m] = \delta^\bar{i}_n \delta_{n,-m}, \quad \{\psi^\bar{i}_n, \rho^\bar{j}_m\} = \delta^\bar{i}_n \delta_{n,-m}. \]

The modes with \( n \neq 0 \) (“oscillators”) are represented by operators in the Fock space in the standard manner by postulating

\[ \phi^\bar{i}_n |\text{vac}\rangle = \psi^\bar{i}_n |\text{vac}\rangle = p_i_n |\text{vac}\rangle = \rho_i_n |\text{vac}\rangle = 0, \quad \forall n > 0. \]

In terms of the coefficients of the Laurent expansion, the formula for \( \bar{Q}^+ \) reads:

\[ \bar{Q}^+ = -\sum_{n \in \mathbb{Z}} \bar{z}^n p^\bar{i}_n. \]

To compute the cohomology of \( \bar{Q}^+ \), we note that the space of states is a tensor product of the space of zero-modes and the oscillator modes (modes with \( n \neq 0 \)). The operator \( \bar{Q}^+ \) is a sum of terms, each of which acts only in one factor and does not mix modes with different \( n \). For \( n \neq 0 \), it is easy to see that the cohomology of \( \bar{Q}^+ \) in the Fock space generated by \( \phi^\bar{i}_{-n}, p^\bar{i}_{-n}, \psi^\bar{i}_{-n}, \rho^\bar{i}_{-n} \) is one-dimensional. Indeed, from the physical viewpoint the problem is identical to that of two independent supersymmetric harmonic oscillators, with
the supersymmetry generator playing the role of $\bar{Q}_+$. The cohomology of this system is well-known to be one-dimensional. From the mathematical viewpoint, we are dealing with the tensor product of a Koszul complex, corresponding to a free resolution of a point on $\mathbb{C}^n$, and the algebraic de Rham complex of $\mathbb{C}^n$.

Thus we may as well drop all the right-moving modes with $n \neq 0$. We are left with the theory where the only antiholomorphic modes are the zero-modes $\phi_0^i, p_{i,0}, \psi_0^i, \rho_{i,0}$ satisfying the canonical commutation relations

$$[\phi_0^i, p_{j,0}] = \delta_j^i, \quad \{\psi_0^i, \rho_{j,0}\} = \delta_j^i.$$ 

These are quantized in the standard manner: the states are arbitrary functions of the variables $\phi_0^i$ and $\psi_0^i$, while the canonically-conjugate variables are realized by differential operators:

$$p_{i,0} = -\frac{\partial}{\partial \phi_0^i}, \quad \rho_{i,0} = \frac{\partial}{\partial \psi_0^i}.$$ 

The operator $\bar{Q}_+$ reduces to

$$\bar{Q}_+ = \psi_0^i \frac{\partial}{\partial \phi_0^i}.$$ 

The space of states of this SCFT is exactly $\Omega_{X, \text{Dol}}^{\text{ch}}(X)$. This is obvious locally; we discuss the transformation law for going from chart to chart below. The operator $\bar{Q}_+$ acts on the space of states of this SCFT as $d_{\text{Dol}}$. Thus the $\bar{Q}_+$-cohomology of the sigma-model is isomorphic to the $d_{\text{Dol}}$-cohomology of the space $\Omega_{X, \text{Dol}}^{\text{ch}}(X)$. The latter is the same as the sheaf cohomology of the chiral de Rham complex.

Further, one can now see where the peculiar transformation law for $p_i$ comes from: while $\partial \bar{\phi}^i$ transforms homogeneously, the expression for $p_i$ also involves the term

$$\Gamma_{ik}^l \rho_j \psi^k.$$ 

The well-known transformation law for the Christoffel symbols

$$\tilde{\Gamma}_{jk}^i = \frac{\partial \bar{\phi}^m}{\partial \phi^j} \frac{\partial \phi^n}{\partial \phi^k} \Gamma_{mn}^l + \frac{\partial^2 \bar{\phi}^l}{\partial \phi^j \partial \phi^k} \frac{\partial \bar{\phi}^i}{\partial \phi^l},$$ 

then implies the transformation law Eq. (3). The fields $\rho_i$ and $\psi^i$ transform homogeneously, as expected.
Finally, let us consider the supersymmetry transformations generated by $Q_-$ and $\bar{Q}_-$. If we denote the corresponding odd parameters by $\epsilon$ and $\bar{\epsilon}$, then according to Ref. [1], they are given by

\begin{align*}
\delta \phi^i &= \epsilon \psi^i, \\
\delta \phi^\bar{i} &= -\bar{\epsilon} g^{\bar{i}i} \rho_i, \\
\delta \psi^i &= \bar{\epsilon} \partial \phi^i, \\
\delta \psi^\bar{i} &= \epsilon \Gamma^i_{\bar{j}j} g^{\bar{j}i} \rho_i \psi^\bar{k}, \\
\delta \rho_i &= -\epsilon \left( g_{\bar{i}} \partial \phi^\bar{i} + \Gamma^k_{ij} \rho_k \psi^j \right) = -\epsilon p_i, \\
\delta \rho^\bar{i} &= -\bar{\epsilon} \Gamma^i_{\bar{j}k} g^{\bar{i}k} \rho_i \rho^\bar{j}.
\end{align*}

In the infinite-volume limit they become

\begin{align*}
\delta \phi^i &= \epsilon \psi^i, \quad \delta \phi^\bar{i} = 0, \\
\delta \psi^i &= \bar{\epsilon} \partial \phi^i, \quad \delta \psi^\bar{i} = 0, \\
\delta \rho_i &= -\epsilon p_i, \quad \delta \rho^\bar{i} = 0.
\end{align*}

We see that in this limit all the antiholomorphic fields are invariant with respect to all left-moving supersymmetry transformations. On the other hand, the transformations of the fields $\phi^i, \psi^i, \rho_i$ agree with Eq. [2]. The transformation law for $p_i$ follows from the above transformations and properties of the supersymmetry algebra:

\begin{align*}
\{ Q_-, p_i \} &= -\{ Q_-, \{ Q_-, \rho_i \} \} = 0, \\
\{ \bar{Q}_-, p_i \} &= -\{ \bar{Q}_-, \{ \bar{Q}_-, \rho_i \} \} = -\partial \rho_i.
\end{align*}

This concludes the demonstration that the infinite-volume limit of the half-twisted model is isomorphic, as an $N = 2$ SCFT, to the sheaf cohomology of the chiral de Rham complex.

4 The dependence of the half-twisted model on Kähler moduli

We will show in this section that to any order in the perturbative (large-volume) expansion the half-twisted model is independent of the Kähler moduli, but may receive contributions from worldsheet instantons. Thus the correspondence with the chiral de Rham complex persists to all orders in perturbation theory, but not nonperturbatively.
To this end we have to rewrite the action of the half-twisted sigma-model as a sum of a topological term, a BRST-trivial term, and the remainder, so that all the Kähler-moduli dependence is in the first two terms. This is very similar to the argument for the A-model in Ref. [1].

The BRST transformations for the fields are displayed in Eq. (9). Let

\[ V = -\rho_i \bar{\partial}\phi^i - g^{ik}\Gamma_{jk}^i \rho_i \psi^j \rho_l. \]

It is straightforward to check that the action Eq. (4) can be written in the following way:

\[ S = \frac{1}{4\pi} \int \phi^*(\omega + iB) + \frac{1}{\pi} \int d^2z \left( \rho_i \bar{\partial}\phi^i + \{\bar{Q}_+, V\} \right). \]

The first term depends on \( \omega \) and \( B \), but it contributes only for topologically nontrivial worldsheets. The second term contains a BRST-trivial piece and a piece which depends only on the complex structure of \( X \). We conclude that the dependence on the Kähler moduli is absent to all orders in perturbation theory, if we consider only correlators of BRST-closed operators (i.e. if we work with the half-twisted sigma-model).

To go beyond perturbation theory, one has to include worldsheet instantons. As usual, the path-integral localizes on classical configurations which are BRST-invariant. In the present case, the condition is that the variation of \( \rho_i \) must vanish. This implies that the relevant classical configurations are solutions of

\[ \bar{\partial}\phi^i = 0, \]

i.e. holomorphic maps from the worldsheet to \( X \), just like in the A-model. It would be of some interest to determine the precise form of the instanton corrections to the chiral de Rham complex. This would provide a “chiral” generalization of the quantum cohomology ring.

5 Concluding remarks

The definition of the chiral de Rham complex makes sense even if \( c_1(X) \neq 0 \). However, in this case one cannot globally define the generators \( J_-(z) \) and \( Q_-(z) \) [4]. Thus one gets a sheaf of holomorphic operator algebras, but not a sheaf of \( N = 2 \) superconformal field theories. Note that although the twisted stress-energy tensor \( T_-(z) \) can still be defined, we cannot “untwist” it into
a “physical” stress-energy tensor with a nonzero central charge, because we lack $J_-(z)$. This is related to the fact that for $c_1(X) \neq 0$ the sigma-model is not conformally-invariant.

From the physical viewpoint, for $c_1(X) \neq 0$ one should consider both the $\bar{Q}_+$-cohomology of operators and the $\bar{Q}_+$-cohomology of states in the Ramond sector. While they are isomorphic vector spaces in the conformal case, this is not so in general. Rather, one can only say that the latter is a module over the former. If the theory has a mass gap, the cohomology of states is much smaller than the cohomology of operators (usually, finite-dimensional).

Another very interesting generalization is to replace the fermions $\psi^i$ and $\rho^i$ taking values in $TX$ and $T^*X$ with fermions taking values in an arbitrary holomorphic vector bundle $E$ and its dual. This generalization was considered in Refs. [4, 7], where it was found that such conformal field theories can be glued into a sheaf of conformal field theories only if the topological condition

$$ch_2(E) - ch_2(TX) = 0$$

is satisfied. Here $ch_2(E)$ is the degree-4 part of the Chern character of $E$.

From the physical viewpoint, this generalization corresponds to studying the twisted version of the heterotic sigma-model with $(0,2)$ supersymmetry. The above topological condition is the familiar condition of cancellation of worldsheet anomalies for heterotic sigma-models.

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