Exponents of 2-multiarrangements and
freeness of 3-arrangements

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Abstract

We give the upper bound of differences of exponents for balanced 2-multiarrangements in terms of the cardinality of hyperplanes. Also, we give a shift isomorphism of 2-multiarrangements like Coxeter arrangements when the difference of exponents is maximum. As an application, a sufficient numerical and combinatorial condition for 3-arrangements to be free is given.

0 Introduction

Let $V$ be an $\ell$-dimensional vector space over a field $\mathbb{K}$ of characteristic zero, $S = \mathbb{K}[x_1, \ldots, x_\ell]$ the coordinate ring and $\text{Der}(S) = \bigoplus_{i=1}^{\ell} S \cdot \partial_{x_i}$, the module of $S$-regular derivations. A hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes in $V$. In this article $\mathcal{A}$ is assumed to consist of linear hyperplanes unless otherwise specified. Such an arrangement is called central. For each $H \in \mathcal{A}$ let us fix a linear form $\alpha_H \in V^*$ such that $\ker \alpha_H = H$. A function $m : \mathcal{A} \to \mathbb{Z}_{>0}$ is called a multiplicity and a pair $(\mathcal{A}, m)$ is a multiarrangement. Then we can define the logarithmic derivation module $D(\mathcal{A}, m)$ by

$$D(\mathcal{A}, m) := \{ \theta \in \text{Der}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \ (\forall H \in \mathcal{A}) \}.$$ 

$D(\mathcal{A}, m)$ is a reflexive module in general. When $D(\mathcal{A}, m)$ is a free module of rank $\ell$, we say that $(\mathcal{A}, m)$ is free and for the homogeneous basis $\theta_1, \ldots, \theta_\ell$, we define

$$\exp(\mathcal{A}, m) := (\deg \theta_1, \ldots, \deg \theta_\ell),$$

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where \( \deg \theta := \deg \theta(\alpha) \) for some \( \alpha \in V^* \) such that \( \theta(\alpha) \neq 0 \). When \( m \equiv 1 \) a multiarrangement \((\mathcal{A}, 1)\) is the same as an arrangement, which is sometimes called a \textbf{simple arrangement} and \( D(\mathcal{A}, 1) =: D(\mathcal{A}) \). An \( \ell \)-arrangement is that in \( V \simeq K^\ell \).

The freeness of an arrangement \( \mathcal{A} \) has been studied by a lot of mathematicians for a long time. Actually it is very difficult to determine whether a given arrangement is free or not. For example, whether the freeness of simple arrangements depends only on the combinatorics of arrangements or not has been unsolved for a long time, which is called the \textbf{Terao conjecture} and still open. Recently, new freeness criterions were found by Yoshinaga in [14] and [15] in terms of restricted multiarrangements. Hence it has become important to study the freeness of multiarrangements. In particular, by [15], to solve the Terao conjecture of 3-arrangements, to determine exponents of 2-multiarrangements is essential.

Since 2-multiarrangements are free, we can always define the exponents. However, contrary to the simple arrangement case, the behavior of \( \exp(\mathcal{A}, m) \) is complicated. One of the approaches to understand it is \[3\] in which a multiplicity lattice is introduced and studied. The aim of this article is the further analysis of the theory of multiplicity lattices, introduction of a generalized Euler derivations called \((\mathcal{A}, m)\)-Euler derivations, and apply them to the freeness problem of 3-arrangements as desired. Let us explain these in details below.

For a 2-multiarrangement \((\mathcal{A}, m)\) with \( \exp(\mathcal{A}, m) = (d_1, d_2) \), let us define
\[
\Delta(m) := |d_1 - d_2|.
\]

We say that a multiplicity \( m \) is \textbf{balanced} if
\[
m(K) \leq \sum_{H \in \mathcal{A} \setminus \{K\}} m(H) \quad (\forall K \in \mathcal{A}).
\]

In [3] the structure of multiplicity lattices was considered and studied by using \( \Delta \). The theory constructed there will be used to prove results in this article. See [3], Lemma 1.5, Theorem 1.6 and section one for details.

Note that, if \( m \) is not balanced, then \( \exp(\mathcal{A}, m) \) can be easily computed, see Proposition 4.2. Hence for the Terao conjecture of 3-arrangements, we have to study the exponents of balanced 2-multiarrangements. Since \( d_1 + d_2 = |m| \) when \( \exp(\mathcal{A}, m) = (d_1, d_2) \), to know exponents is equivalent to know \( \Delta(m) \). Then it is a natural question to ask for a 2-multiarrangement \((\mathcal{A}, m)\), is there any upper bound of \( \Delta(m) \) when \( m \) is balanced? Experimental computations imply that \( |\mathcal{A}| - 2 \) might be the upper bound. In fact, \( \Delta(m) = h - 2 \) when \( m \equiv 1 \). The first main result in this article is to prove that it is in fact the strict upper bound.
Theorem 0.1
Let $\mathcal{A}$ be a 2-arrangement with $|\mathcal{A}| = h > 2$. If $m : \mathcal{A} \to \mathbb{Z}_{>0}$ is balanced, then $\Delta(m) \leq h - 2$.

For the proof, we use results in [3] and the affine connection $\nabla$. Then it is an interesting problem to ask whether there are some special properties if $m$ is balanced and $\Delta(m) = |\mathcal{A}| - 2$. When $m \equiv 1$, this condition is satisfied. Let us agree that the lower degree basis $\theta$ for $D(\mathcal{A}, m)$ is the homogeneous derivation $\theta$ such that $\{\theta, \varphi\}$ is an $S$-basis for $D(\mathcal{A}, m)$ and that $\deg \theta \leq \deg \varphi$. Then the lower degree basis for $D(\mathcal{A})$ is the Euler derivation, which is apparently special. The answer is interesting.

Theorem 0.2
Let $\mathcal{A}$ be an arrangement in $\mathbb{K}^2$ with $|\mathcal{A}| = h > 2$ and $m_0 : \mathcal{A} \to \mathbb{Z}_{>0}$ be a balanced multiplicity such that $\Delta(m_0) = h - 2$. Assume that one of the following two holds:

1. $|\mathcal{A}| = h = 3$ and $m_0 - 1$ is balanced, or
2. $|\mathcal{A}| = h \geq 4$.

Then the lower degree basis $\theta_0$ for $D(\mathcal{A}, m_0)$ gives rise to an isomorphism

$$\Phi_0 : D(\mathcal{A}, m) \to D(\mathcal{A}, m_0 + m - 1)$$

defined by

$$\Phi_0(\theta) := \nabla_\theta \theta_0$$

with

$$m : \mathcal{A} \to \{+1, 0\}.$$ 

The isomorphism $\Phi_0$ is first introduced in [13], generalized in [6] and [2] all for Coxeter multiarrangements. In these papers, the invariant theory of Coxeter groups and the existence of the primitive derivation played important roles. On the contrary, Theorem 0.2 do not need them and the same statement can be true for all 2-arrangements.

As an application of Theorem 0.1 a freeness condition for 3-arrangement is also given. It is well-known that when $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$ the characteristic polynomial $\chi(\mathcal{A}, t)$ splits into

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - d_i).$$

(0.1)
The formula (0.1) is the famous Terao’s factorization theorem proved in [10]. However, the converse does not hold. Theorem 0.1 combined with the result in [15] gives the converse for some cases. In other words, if a 3-arrangement has a splitting characteristic polynomial with certain exponents, then it is free as follows.

**Theorem 0.3**
Let $\mathcal{A}$ be an affine 2-arrangement, $c\mathcal{A}$ its coning with the infinite hyperplane $H_0 \in c\mathcal{A}$. Put $|\mathcal{A}| = k$ and $\chi(\mathcal{A}, t) = (t^2 - kt + c_2)$. Also, let $(\mathcal{A}, m)$ be the Ziegler restriction (see Definition 1.3) of $c\mathcal{A}$ onto $H_0$ with $|\mathcal{A}| = h > 2$.

If $(\mathcal{A}, m)$ is balanced and $\chi(\mathcal{A}, t) = (t - d)(t - d - h + 2)$ or $\chi(\mathcal{A}, t) = (t - d)(t - d - h + 3)$ for some integer $d$, then $c\mathcal{A}$ is free.

Note that every central 3-arrangement can be obtained as the coning of a certain affine 2-arrangement. Hence Theorem 0.3 says that only the combinatorics determines the freeness of some 3-arrangements. More explicitly, if we define the category of 3-arrangements $PB_3$ which consists of $\mathcal{A}$ such that every Ziegler restriction is balanced, and there exists $H_0 \in \mathcal{A}$ such that $\chi(\mathcal{A}, t) = (t - 1)(t - d)(t - d')$ for some $d, d' \in \mathbb{Z}$ with $|d - d'| \geq h - 3$, where $h = |\mathcal{A} \cap H_0|$. Then we have the following:

**Corollary 0.4**
The Terao conjecture (1.1) is true in $PB_3$.

Since the Terao conjecture is true for non-balanced 3-arrangements (see Proposition 1.2 or [12]), Corollary 0.4 is the first step for the Terao conjecture of balanced 3-arrangements.

The organization of this article is as follows. In section one we introduce some notions and results which will be used in this article. In section two we prove Theorem 0.1. In section three we prove Theorem 0.2. In section four we prove Theorem 0.3 and Corollary 0.4. Also we give several applications of Theorem 0.1 and examples of free 3-arrangements.

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1 Preliminaries

In this section let us summarize results and definitions which will be used in this article. For a general reference, see [8]. We use the notation in the introduction. For an affine $\ell$-arrangement $\mathcal{A}$, the coning $c\mathcal{A}$ of $\mathcal{A}$ is an $(\ell + 1)$-arrangement obtained by replacing $\{\alpha = k\} \in \mathcal{A}$ ($\alpha \in V^*$, $k \in \mathbb{K}$)
by \(\{\alpha = k z_\infty\} \in c A\) combined with the infinite hyperplane \(H_0 = \{z_\infty = 0\} \in c A\). For a simple arrangement \(A\) define the intersection lattice \(L(A)\) by
\[
L(A) := \{\bigcap_{H \in B} H \mid B \subset A\}
\]
This is a partially ordered set with the reverse inclusion order and the unique minimum element \(V \in L(A)\). The Möbius function \(\mu\) on \(L(A)\) is defined by \(\mu(V) = 1\) and by
\[
\mu(X) := -\sum_{X \subset Y} \mu(Y) \ (X \in L(A) \setminus \{V\}).
\]
A characteristic polynomial \(\chi(A, t)\) is defined by
\[
\chi(A, t) := \sum_{X \in L(A)} \mu(X) t^{\dim X}
\]
The following is one of the most important problems among the arrangement theory.

**Conjecture 1.1 (Terao)**
The freeness of a simple arrangement \(A\) depends only on its intersection lattice \(L(A)\).

For a multiarrangement \((A, m)\), put
\[
|m| := \sum_{H \in A} m(H).
\]
The following is the most fundamental result in the free arrangement theory. For the proof, see [8] and [16].

**Theorem 1.2 (Saito’s criterion)**
Let \(\theta_1, \ldots, \theta_\ell\) be homogeneous derivations in \(D(A, m)\). Then \(A\) is free with basis \(\\{\theta_1, \ldots, \theta_\ell\}\) if and only if \(\\{\theta_1, \ldots, \theta_\ell\}\) is \(S\)-independent and \(\sum_{i=1}^\ell \deg \theta_i = |m|\).

To use Yoshinaga’s freeness criterion, we often use the Ziegler restriction.

**Definition 1.3 (Ziegler restriction)**
Let \(A\) be a simple arrangement and fix \(H_0 \in A\). A Ziegler restriction \((A'', m_0)\) of \(A\) with respect to \(H_0\) is defined by
\[
A'' := \{H \cap H_0 \mid H \in A \setminus \{H_0\}\},
m_0(K) := |\{H \in A \setminus \{H_0\} \mid H \cap H_0 = K\}| \ (K \in A'').
\]
Then for $D_0(A) := \{ \theta \in D(A) \mid \theta(\alpha_{H_0}) = 0 \}$, the restriction map
\[ \pi : D_0(A) \to D(A'', m_0) \]
is defined by taking a residue of $\alpha_{H_0}$. See [10] for details.

Next let us introduce the shift isomorphism, which will be generalized in Theorem 0.2 for 2-arrangements. In this paragraph we assume that $\mathbb{K} = \mathbb{R}$. Let $A$ be a Coxeter arrangement with the Coxeter group $W$. Put $R := S^W = \mathbb{R}[P_1, \ldots, P_\ell]$ with homogeneous basic invariants $P_1, \ldots, P_\ell$ by Chevalley’s theorem. Let $F$ be a quotient field of $S$. We may assume that $\deg P_i = h$ and $\deg P_i (i \neq \ell)$, where $h$ is the Coxeter number of $W$. Let $D \in \text{Der}(R) := \oplus_{i=1}^\ell R \cdot \partial P_i$ be the invariant derivation of degree $-h + 1$, called the primitive derivation. Then for the Euler derivation $\theta_E$ and the affine connection $\nabla$ defined by
\[ \nabla_\theta(\sum_{i=1}^\ell f_i \partial x_i) := \sum_{i=1}^\ell \theta(f_i) \partial x_i \]
for $\theta \in \text{Der}(F) := \text{Der}(S) \otimes_S F$, the following shift isomorphism holds.

**Theorem 1.4 ([6], Theorem 2)**

For $m : A \to \{ +1, 0 \}$ and $k \in \mathbb{Z}_{\geq 0}$, the $S$-morphism
\[ \Phi : D(A, m) \to D(A, 2k + m) \]
defined by
\[ \Phi(\theta) := \nabla_\theta \nabla_D^{-k} \theta_E \]
is an isomorphism.

For the most generalized version of shift isomorphisms, see Theorem 0.7 in [2]. In the rest of this section assume that $A$ is a 2-arrangement in $\mathbb{K}^2$. Let us recall results in [3]. For a multiarrangement $(A, m)$ with $\exp(A, m) = (d_1, d_2)$, recall that
\[ \Delta(m) = |d_1 - d_2|. \]
Then the multiplicity lattice $\Lambda$ and the subset $\Lambda'$ is defined by
\[ \Lambda : = \{ m : A \to \mathbb{Z}_{\geq 0} \}, \]
\[ \Lambda' : = \{ m \in \Lambda \mid \Delta(m) \neq 0 \}. \]
Then $\Delta$ is a function
\[ \Delta : \Lambda \to \mathbb{Z}_{\geq 0}. \]
Also, define
\[ \Lambda_K : = \{ m \in \Lambda \mid m(K) > \sum_{H \in \mathcal{A} \setminus \{K\}} m(H) \} \quad (K \in \mathcal{A}), \]
\[ \Lambda_0 : = \Lambda' \setminus \cup_{H \in \mathcal{A}} \Lambda_H. \]

Note that \( \Lambda_0 \) is denoted by \( \Lambda'_\phi \) in \([3]\). Put
\[ d(m, m') := \sum_{H \in \mathcal{A}} |m(H) - m'(H)| \quad (m, m' \in \Lambda). \]

Then the following structure theorems hold for \( \Lambda_0 \).

**Lemma 1.5** ([3], Lemma 4.2)
For \( m_1, m_2 \in \Lambda \) such that \( d(m_1, m_2) = 2 \), \( m_1(H) = m_2(H) \) for \( H \in \mathcal{A} \setminus \{H_0\} \) and \( m_1(H_0) = m_2(H_0) + 1 \), it holds that \( |\Delta(m_1) - \Delta(m_2)| = 1 \).

**Theorem 1.6** ([3], Theorem 3.2)
Let \( C \subset \Lambda_0 \) be a maximal connected component of \( \Lambda_0 \). Then there exists the unique point \( m \in C \), called the peak point of \( C \), such that
\[ \Delta(m) \geq \Delta(\mu) \quad (\forall \mu \in C). \]
Moreover,
\[ C = \{ \mu \in \Lambda' \mid d(m, \mu) < \Delta(m) \} \]
and for \( \mu \in C \),
\[ \Delta(\mu) = \Delta(m) - d(m, \mu). \]

The maximal connected component of \( \Lambda_0 \) in Theorem 1.6 is just called a (finite) component, and \( \Lambda_K \) an infinite component. Also, the following independency property holds.

**Proposition 1.7**
Let \( m_1, m_2 \in \Lambda \) such that \( d(m_1, m_2) = 2 \), \( \Delta(m_i) = 1 \) \((i = 1, 2)\), \( m_1(H) = m_2(H) \) for \( H \in \mathcal{A} \setminus \{H_1, H_2\} \), \( m_1(H_1) = m_2(H_1) + 1 \) and \( m_1(H_2) + 1 = m_2(H_2) \). Assume that for two multiplicities
\[ \mu(H) : = \max\{m_1(H), m_2(H)\} \quad (H \in \mathcal{A}), \]
\[ \mu'(H) : = \min\{m_1(H), m_2(H)\} \quad (H \in \mathcal{A}), \]
it holds that \( \Delta(\mu) = \Delta(\mu') = 0 \). Then the lower degree bases \( \theta_i \) of \( D(\mathcal{A}, m_i) \) \((i = 1, 2)\) are \( S \)-independent.
Proof. This is the special case of Lemma 4.17 in [3]. □

Remark 1.8

We always start a multiarrangement \((A, m)\) such that \(m : A \rightarrow \mathbb{Z}_{\geq 0}\). However, in the arguments in the rest of this article, it often happens that a new multiplicity \(m'\) attains zero at some hyperplane \(H \in A\). However, as we have seen in the above, the theory in [3] is constructed in the multiplicity lattice \(\Lambda = \{m : A \rightarrow \mathbb{Z}_{\geq 0}\}\). Hence there are no problems.

2 Proof of Theorem 0.1

In this section let us prove Theorem 0.1.

Proof of Theorem 0.1 Assume that \(\Delta(m) > h - 2\). We may assume that \(\{x_jx_2 = 0\} \subset A\), which may not be orthogonal. Then there exist derivations \(\partial_{x_1}, \partial_{x_2}\) of degree zero such that \(\langle \partial_{x_1}, \partial_{x_2}\rangle = \text{Der}(S)\) and that \(\partial_{x_i}(x_j) = \delta_{ij}\). Let \(C\) be a connected component of the multiplicity lattice of \(A\) such that \(m \in C\). Since \(\Delta\) attains the maximum value at the peak point of \(C\), we may assume that \(m\) is the peak point of \(C\). Let \(\theta\) be the lower degree basis for \(D(A, m)\) with \(\deg \theta = d\). Then \(\exp(A, m) = (d, d + \Delta(m))\). If we put \(n_i := m(x_i = 0) (i = 1, 2)\), \(n_H := m(H) (H \in A)\) and

\[
\begin{align*}
\theta(x_j) &= x_j^{n_j} f_j (j = 1, 2), \\
\theta(\alpha_H) &= \alpha_H^{n_H} f_H (H \in A \setminus \{x_1x_2 = 0\}),
\end{align*}
\]

then, for \(\{i, j\} = \{1, 2\}\),

\[
\begin{align*}
\nabla_{\partial_{x_i}} \theta(x_j) &= x_j^{n_j} \partial_{x_i}(f_j), \\
\nabla_{\partial_{x_i}} \theta(x_i) &= x_i^{n_i-1}(n_i f_i + x_i \partial_{x_i}(f_i)), \\
\nabla_{\partial_{x_i}} \theta(\alpha_H) &= \alpha_H^{n_H-1}(n_H \partial_{\alpha_H}(\alpha_H f_H + \alpha_H \partial_{\alpha_H}(f_H)) (H \in A \setminus \{x_1x_2 = 0\}).
\end{align*}
\]

Hence the derivation \(\nabla_{\partial_{x_i}} \theta\) is belonging to \(D(A, m - m_i)\), where

\[
m_i(H) = \begin{cases} 0 & \text{if } H = \{x_j = 0\}, \\ 1 & \text{if } H \neq \{x_j = 0\}
\end{cases}
\]

with \(\{i, j\} = \{1, 2\}\). Since \(|m_i| = |A| - 1 = h - 1\) and \(\Delta(m) > h - 2\), it holds that \(m - m_i \in C\) or on the boundary of \(C\) for \(i = 1, 2\). Since \(\exp(A, m - m_i) = (d, d + \Delta(m) - h + 1)\) by Theorem 1.6 \(d \leq d + \Delta(m) - h + 1\) and \(\deg \nabla_{\partial_{x_i}} \theta = d - 1 (i = 1, 2)\), it holds that

\[
\nabla_{\partial_{x_i}} \theta = 0 (i = 1, 2).
\]

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Since $\text{char}(K) = 0$, it holds that $\theta = a\partial_{x_1} + b\partial_{x_2}$ with $a, b \in K$, which is a contradiction. \hfill \Box

**Remark 2.1**

If $\text{char}(K) = p > 0$, then the statement in Theorem 0.1 does not hold. For example, assume that $p = 2$ and consider a balanced multiarrangement $(A, m)$ defined by

$$x_1^4x_2^4(x_1 + x_2)^4 = 0.$$  

Then Theorem 1.3 shows that $x_1^4\partial_{x_1} + x_2^4\partial_{x_2}$ form a basis for $D(A, m)$. Hence $\Delta(m) = 4 > |A| - 2 = 1$.

On the other hand, the proof of Theorem 0.1 says that, if $\text{char}(K) = p > 0$, $m \in \Lambda_0$ is a peak point and $\Delta(m) > h - 2$, then the degree of the lower degree basis for $D(A, m)$ can be divided by $p$. See also [7].

### 3 Proof of Theorem 0.2

In this section we prove Theorem 0.2.

**Proof of Theorem 0.2**  The assumptions and Theorem 0.1 imply that $m_0$ is the peak point of some finite component $C \subset \Lambda_0$. We may assume that $\{x_1x_2 = 0\} \subset A$ and take $\partial_{x_1}, \partial_{x_2}$ in the same way as those in the proof of Theorem 0.1. By the argument in [13], $\Phi_0(\theta) \in D(A, m_0 + m - 1)$ for $\theta \in D(A, m)$. Put $d := \deg \theta_0$. Then

$$|m_0 + m - 1| = (2d + h - 2) + |m| - h = 2d + |m| - 2.$$  

On the other hand, for a basis $\{\theta_1, \theta_2\}$ for $D(A, m)$,

$$\deg \nabla_{\partial_{x_1}} \theta_0 + \deg \nabla_{\partial_{x_2}} \theta_0 = (|m| - 2) + 2d.$$  

Noting that all multiplicities on $A$ are free, by Theorem 1.2 and arguments in [13], it suffices to show that $\nabla_{\partial_{x_1}} \theta_0$ and $\nabla_{\partial_{x_2}} \theta_0$ are $S$-independent. Define two multiplicities $m_1$ and $m_2$ by

$$m_i(H) = \begin{cases} 0 & \text{if } H \neq \{x_j = 0\}, \\ 1 & \text{if } H = \{x_j = 0\}, \end{cases}$$  

where $\{i, j\} = \{1, 2\}$. Then $\nabla_{\partial_{x_i}} \theta_0 \in D(A, m_0 + m_i - 1)$ by the same arguments as in the proof of Theorem 0.1. Since $d(m_0, m_0 + m_i - 1) = h - 1$, it holds that $\exp(A, m_0 + m_i - 1) = (\deg \theta_0 - 1, \deg \theta_0)$ by Lemma 1.5 and Theorem 1.6. Note that if $\nabla_{\partial_{x_i}} \theta_0 \neq 0$, then it is of degree $\deg \theta_0 - 1$ and is the lower degree basis for $D(A, m_0 + m_i - 1)$. Let us check that
\n\n∇_\partial_{x_i} \theta_0 \neq 0. We may assume that \( i = 1 \) and suppose that \( \nabla_{\partial_{x_1}} \theta_0 = 0 \). Then 
\[ \theta_0 = ax_2^i \partial_{x_1} + bx_2^i \partial_{x_2} \quad (a, b \in \mathbb{K}), \]
which is only tangent to one of following three arrangements of hyperplanes:

1. \( \{(bx_1 - ax_2)x_2 = 0\} \) if \( a \neq 0, \ b \neq 0 \).

2. \( \{x_1x_2 = 0\} \) if \( a = 0, \ b \neq 0 \).

3. \( \{x_2 = 0\} \) if \( b = 0 \).

Any case contradicts \( h > 2 \). Hence \( \nabla_{\partial_{x_i}} \theta_0 \neq 0 \) \((i = 1, 2)\) and both derivations are the lower degree basis of degree \( d - 1 \) for \( D(A, m_0 + m_i - 1) \).

Note that \( \Delta(m_0 + m_i - 1) = 1 \) and \( \Delta(m_0 + m_1 + m_2 - 1) = 0 \). Hence, by Proposition 1.7 it suffices to show that \( \Delta(m_0 - 1) = 0 \).

Assume that \( \Delta(m_0 - 1) \neq 0 \). Then Lemma 1.5 shows that \( \Delta(m_0 - 1) = 2 \).

Then the result in [3], or the addition theorem in [5] says that 
\[ x_1 \nabla_{\partial_{x_1}} \theta_0 = x_1 \theta'_{x_1}, \nabla_{\partial_{x_2}} \theta_0 = x_2 \theta'_{x_2} \]
(up to scalars) for the lower degree basis \( \theta' \) for \( D(A, m_0 - 1) \). Hence it holds

\[ \theta_0 = ax_1^i x_2^{d-i} \partial_{x_1} + bx_1^i x_2^{d-i} \partial_{x_2} = x_1^i x_2^{d-i} (a \partial_{x_1} + b \partial_{x_2}) \]

with \( 1 \leq i \leq d - 1 \). Apparently \( \theta_0 \) is tangent to \( x_1 = 0 \) and \( x_2 = 0 \) with multiplicity \( i \) and \( d - i \) respectively, can be tangent to \( bx_1 - ax_2 = 0 \) and tangent to no other hyperplanes. When \( |A| = 3 \) and \( m_0 - 1 \in \Lambda_0 \), the statement is true by [11] and Proposition 1.7. If \( |A| \geq 4 \) then \( \theta_0 \) cannot be in \( D(A, m_0) \). Hence \( \Delta(m_0 - 1) = 0 \) and Proposition 1.7 completes the proof. \( \square \)

Theorem 0.2 says that, if \( \Delta(m) = h - 2 \), then \( D(A) \simeq D(A, m) \) as \( S \)-modules. Since \( \theta_0 \) in Theorem 0.2 for \( D(A) \) is the Euler derivation \( \theta_E \), we introduce the following definition.

\[ \Delta(m_0) = h > 0. \]

Hence \( f \) is a monomial. The same argument to \( g \) shows that \( \theta_0 \) is of the form

\[ \theta_0 = ax_1^i x_2^{d-i} \partial_{x_1} + bx_1^i x_2^{d-i} \partial_{x_2} = x_1^i x_2^{d-i} (a \partial_{x_1} + b \partial_{x_2}) \]

with \( 1 \leq i \leq d - 1 \). Apparently \( \theta_0 \) is tangent to \( x_1 = 0 \) and \( x_2 = 0 \) with multiplicity \( i \) and \( d - i \) respectively, can be tangent to \( bx_1 - ax_2 = 0 \) and tangent to no other hyperplanes. When \( |A| = 3 \) and \( m_0 - 1 \in \Lambda_0 \), the statement is true by [11] and Proposition 1.7. If \( |A| \geq 4 \) then \( \theta_0 \) cannot be in \( D(A, m_0) \). Hence \( \Delta(m_0 - 1) = 0 \) and Proposition 1.7 completes the proof. \( \square \)
Definition 3.1
The derivation \( \theta_0 \) in Theorem 0.2 is called to be the \((A, m)\)-Euler derivation.

Obviously the Euler derivation \( \theta_E \) is \((A, 1)\)-Euler for all 2-arrangements.

Let us see the other examples below.

Example 3.2
Let \((A, m)\) be an \(A_2\)-type multiarrangement. By [11] we know that \( \Delta(m) = 1 \) if \(|m| \) is odd and \( m \) is balanced. Since \(|A| = 3\), Theorem 0.2 shows that every lower degree basis for \( D(A, m) \) such that \(|m| \equiv 1 \text{ (mod 2)}\) and \( m, m - 1 \in \Lambda_0 \) is \((A, m)\)-Euler.

Historically a lot of \((A, 2k + 1)\)-Euler derivations have been constructed by using the invariant theory for Coxeter arrangements \( A \), see [13] and [6]. In these papers to prove the independency is an important part. By using Theorem 0.2 we can give an another proof when \( \ell = 2 \).

Corollary 3.3
Let \( A \) be a Coxeter arrangement in \( \mathbb{R}^2 \) corresponding to the Coxeter group \( W \) and put \( R := S^W = \mathbb{R}[P_1, P_2] \) the invariant ring with basic invariants. Define \( D := \partial_{P_2} \in \text{Der}(R) \) the primitive derivation. Then the derivation \( E_k := \nabla_{\partial_{P_2}}^k \theta_E \) is \((A, 2k + 1)\)-Euler for \( k \in \mathbb{Z} \).

Proof. Assume that \( k \geq 0 \). Then \( \deg E_k = hk + 1 \) with \( h = |A| \). Also, \(|(A, 2k + 1)| = h(2k + 1) \) and a constant multiplicity is balanced. Since \( E_k \in D(A, 2k + 1) \) by [4], Theorem 0.2 completes the proof.

Assume that \( k < 0 \). Note that the same theory in [3] holds true for \(-\Lambda := \{m : A \rightarrow \mathbb{Z}_{<0}\}\). Then, noting that \( \nabla_{\partial_{\omega}}(\omega \wedge d\alpha) = \nabla_{\partial_{\omega}}(\omega \wedge d\alpha) \) for a rational differential form \( \omega \) and \( \alpha \in V^* \), the same argument as the above completes the proof. \( \square \)

The proof of Theorem 0.2 implies the following.

Proposition 3.4
Let \((A, m_0)\) be a 2-multiarrangement with \(|A| = h \geq 4\), \( m_0 \in \Lambda_0 \) and \( \Delta(m_0) = h - 2 \). Then \( m_0 - 1 \) is balanced.

Proof. Assume not. Then it is obvious that \( \nabla_{\partial_{\omega}} \theta_0 \) and \( \nabla_{\partial_{\omega}} \theta_0 \) in the proof of Theorem 0.2 are \( S \)-dependent since they are in the same infinite component and that \( \Delta(m_0 - 1) = 2 \). \( \square \)
4 Freeness condition for 3-arrangements

Before the proof of Theorem 0.3 let us recall one of Yoshinaga’s freeness criterions.

Theorem 4.1 ([15], Theorem 3.2)
Let $\mathcal{A}$ be a central 3-arrangement with the infinite hyperplane $H_0 \in \mathcal{A}$. Let $(\mathcal{A} \cap H_0, m_0)$ be the Ziegler restriction onto $H_0$ with $\exp(\mathcal{A} \cap H_0, m_0) = (d_1, d_2)$. Also put
$$\chi(\mathcal{A}, t) = (t-1)(t^2 - c_1 t + c_2).$$
Then, for the Ziegler restriction map
$$\pi : D_0(\mathcal{A}) \to D(\mathcal{A} \cap H_0, m_0),$$
it holds that
$$\dim \operatorname{coker} \pi = c_2 - d_1 d_2 \geq 0.$$  
In particular, $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, d_1, d_2)$ if and only if $c_2 = d_1 d_2$.

The following is an immediate consequence of Theorem 4.1 and well-known, see [12] for example. Here we give a proof.

Proposition 4.2
The Terao conjecture is true for a 3-arrangement $\mathcal{A}$ such that its Ziegler restriction is not balanced.

Proof. Fix an infinite hyperplane $H_0 \in \mathcal{A}$ and put $(\mathcal{A}'', m)$ the Ziegler restriction of $\mathcal{A}$ onto $H_0$ such that $m(K) > \sum_{H \in \mathcal{A}' \setminus \{K\}} m(H)$ for some $K \in \mathcal{A}'$. We may assume that $\alpha_K = x_1$. Then
$$\prod_{H \in \mathcal{A}' \setminus \{K\}} \alpha_H^{m(H)} \partial x_2$$
is the lower degree basis for $D(\mathcal{A}'', m)$. Hence $\exp(\mathcal{A}, m) = (m(K), |m| - m(K))$ and Theorem 4.1 says that $\mathcal{A}$ is free if $\chi(\mathcal{A}, t) = (t-1)(t-m(K))(t-|m| + m(K))$, which is combinatorial. \qed

Now let us consider the balanced cases. Let us recall a notation. Let $\overline{\mathcal{A}}$ be an affine 2-arrangement, $c\overline{\mathcal{A}}$ its coning with the infinite hyperplane $H_0 \in c\overline{\mathcal{A}}$. Put $|\overline{\mathcal{A}}| = k$ and $\chi(\overline{\mathcal{A}}, t) = (t^2 - kt + c_2)$. Also, let $(\mathcal{A}, m)$ be the Ziegler restriction of $c\overline{\mathcal{A}}$ onto $H_0$ with $|\mathcal{A}| = h > 2$. We say that an affine simple 2-arrangement $\overline{\mathcal{A}}$ is balanced if $m(K) \leq \sum_{H \in \mathcal{A}' \setminus \{K\}} m(H)$ for any $K \in \mathcal{A}$. Now we have prepared for the proof of Theorem 0.3.
Proof of Theorem 0.3. First assume that $c_2 = d \times (d + h - 2)$ and $k = 2d + h - 2$. Put $\exp(\mathcal{A}, m) = (d_1, d_2)$. Note that $d_1 + d_2 = 2d + h - 2 = k$. Since $m$ is balanced, Theorem 0.1 implies that

$$d_1d_2 \geq d(d + h - 2).$$

Also, Theorem 4.1 implies that

$$c_2 = d(d + h - 2) \geq d_1d_2.$$

If $\exp(\mathcal{A}, m) \neq (d, d+h-2)$, then $d < \min\{d_1, d_2\}$ and $\max\{d_1, d_2\} < d+h-2$ by Theorem 0.1. Hence

$$c_2 = d(d + h - 2) \geq d_1d_2 > d(d + h - 2),$$

which is a contradiction. Hence $\exp(\mathcal{A}, m) = (d, d+h-2)$, and Theorem 4.1 completes the proof.

Second assume that $c_2 = d \times (d + h - 3)$ and $k = 2d + h - 3 = d_1 + d_2$. In this case, $k - (h - 2)$ is an odd number. Hence $\Delta(m) = |d_1 - d_2| \leq h - 3$. By the same arguments as the above,

$$c_2 = d(d + h - 3) \geq d_1d_2 \geq d(d + h - 3),$$

which, combined with $h = |\mathcal{A}| > 2$, completes the proof. □

Proof of Corollary 0.4. Immediate by Theorem 0.3. □

Theorem 0.1 has a lot of applications on the characteristic polynomials, freeness and chambers of 3-arrangements as follows.

Theorem 4.3
In the above notation, assume that $\overline{\mathcal{A}}$ is balanced and $\chi(\overline{\mathcal{A}}, t) = (t-a)(t-b)$ with $a \leq b$.

1. If $k = a + b = 2d + h - 2$ for some integer $d$, then $d \leq a \leq b \leq d + h - 2$.

2. If $k = a + b = 2d + h - 3$ for some integer $d$, then $d \leq a \leq b \leq d + h - 3$.

Proof. Since the proof is the same, we only prove (1). By Theorem 4.1 it holds that

$$ab \geq d_1d_2$$

with $\exp(\mathcal{A}, m) = (d_1, d_2)$. Note that $d_1 + d_2 = a + b$. By Theorem 0.1

$|d_1 - d_2| \leq h - 2$. Hence, if $b - a > h - 2$, then $ab - d_1d_2 < 0$, which is a contradiction. □
Theorem 4.4
In the above notation, assume that $\mathcal{A}$ is balanced and $\mathbb{K} = \mathbb{R}$. Let $\text{ch}(\mathcal{A})$ be the set of connected components of $\mathbb{R}^2 \setminus \bigcup_{H \in \mathcal{A}} H$.

(1) If $k = 2d + h - 2$ for some integer $d$, then $c_2 \geq d(d + h - 2)$ and $|\text{ch}(\mathcal{A})| \geq 1 + k + d(d + h - 2)$.

In particular, the equation holds only if $c\mathcal{A}$ is free.

(2) If $k = 2d + h - 3$ for some integer $d$, then $c_2 \geq d(d + h - 3)$ and $|\text{ch}(\mathcal{A})| \geq 1 + k + d(d + h - 3)$.

In particular, the equation holds only if $c\mathcal{A}$ is free.

Proof. The same as that of Theorem 4.3.

These results say that, if $\mathcal{A}$ is balanced, then the characteristic polynomial $\chi(\mathcal{A}, t)$ is irreducible, or splits with a restricted splitting type seen in the above. Also, the freeness of these arrangements are determined by the intersection lattice, or more explicitly, by the characteristic polynomial.

The following can be proved by using results in [11]. We give an another proof here.

Corollary 4.5
Let $c\mathcal{A}$ be a $3$-arrangement with the infinite hyperplane $H_0 \in c\mathcal{A}$ and $(\mathcal{A}, m)$ the Ziegler restriction of $c\mathcal{A}$ onto $H_0$. Assume that $|\mathcal{A}| = 3$. Then the freeness of $c\mathcal{A}$ depends only on $L(\mathcal{A})$. In particular, so is that of the deformation of the Coxeter arrangement of type $A_2$.

Proof. If $(\mathcal{A}, m)$ is not balanced, then Proposition 4.2 completes the proof. Assume that $(\mathcal{A}, m)$ is a balanced Coxeter multiarrangement of type $A_2$. Since $|\mathcal{A}| - 2 = 1$, Theorem 4.3 completes the proof.

Example 4.6
Let $\mathcal{A}$ be the Coxeter arrangement of type $B_2$ and $\overline{\mathcal{A}}$ its deformation as in [1]. The freeness of such deformations have not yet classified, nor have the exponents of the multiarrangements $(\mathcal{A}, m)$. Some of the freeness of $c\overline{\mathcal{A}}$ was classified in Propositions 2.3 and 2.4 in [1]. They were proved by using the addition theorem. If we use Theorem 0.3, the explicit formula of the Poincaré
polynomial ([1], Lemma 2.1) is enough to show the freeness. In other words, Theorem 4.3 says that if $\mathcal{A}$ is balanced and $c\mathcal{A}$ splits, then it is of the form

$$
\chi(\mathcal{A}, t) = (t - d)^2,
$$

$$
\chi(\mathcal{A}, t) = (t - d)(t - d + 1) \text{ or }
$$

$$
\chi(\mathcal{A}, t) = (t - d)(t - d + 2).
$$

Then Theorem 0.3 says that $c\mathcal{A}$ is free if $\chi(\mathcal{A}, t)$ splits into the form of the second and third types in the above.

More generally, when $|\mathcal{A}| = 4$ and not necessarily of type $B_2$, the following holds.

**Corollary 4.7**

Let $\mathcal{A}$ be a central 3-arrangement such that $|\mathcal{A} \cap H_0| = 4$ for some $H_0 \in \mathcal{A}$. If $|\mathcal{A}|$ is even, then the freeness of $\mathcal{A}$ depends only on $L(\mathcal{A})$.

**Proof.** If the deconing $d\mathcal{A} := (\mathcal{A} \setminus \{H_0\})|_{a_{H_0}=1}$ (a converse operation of the coning) is not balanced, then Proposition 4.2 completes the proof. Assume that $d\mathcal{A}$ is balanced. In this assumption, the splitting type of the characteristic polynomial is always

$$
\chi(\mathcal{A}, t) = (t - 1)(t - d)(t - d - 1)
$$

by Theorem 4.3. Hence Theorem 0.3 completes the proof. □

**References**

[1] T. Abe, The stability of the family of $B_2$-type arrangements. Comm. in Algebra 37 (2009), no. 4, 1193–1215.

[2] T. Abe, A generalized logarithmic module and duality of Coxeter multiarrangements. [arXiv:0807.2552v1].

[3] T. Abe and Y. Numata, Exponents of 2-multiarrangements and multiplicity lattices. [arXiv:0706.0009].

[4] T. Abe and H. Terao, A primitive derivation and logarithmic differential forms of Coxeter arrangements. Math. Z. 264 (2010), no. 4, 813–828.

[5] T. Abe, H. Terao and M. Wakefield, The Euler multiplicity and addition-deletion theorems for multiarrangements. J. London Math. Soc. 77 (2008), no. 2, 335–348.
[6] T. Abe and M. Yoshinaga, Coxeter multiarrangements with quasi-
constant multiplicities. *J. Algebra* **322** (2009), no. 8, 2839–2847.

[7] Y. Numata, An algorithm to construct a basis for the module of
logarithmic vector fields. [arXiv:0707.0004](https://arxiv.org/abs/0707.0004).

[8] P. Orlik and H. Terao, *Arrangements of hyperplanes*. Grundlehren
der Mathematischen Wissenschaften, **300**. Springer-Verlag, Berlin, 1992.

[9] H. Terao, Arrangements of hyperplanes and their freeness I, II. *J.
Fac. Sci. Univ. Tokyo* **27** (1980), 293–320.

[10] H. Terao, Generalized exponents of a free arrangement of hyper-
planes and Shephard-Todd-Brieskorn formula. *Invent. Math.* **63**
(1981), 159–179.

[11] A. Wakamiko, On the Exponents of 2-Multiarrangements. *Tokyo J.
Math.* **30** (2007), no. 1, 99–116.

[12] M. Wakefield and S. Yuzvinsky, Derivations of an effective divisor on
the complex projective line. *Trans. Amer. Math. Soc.* **359** (2007),
no. 9, 4389–4403.

[13] M. Yoshinaga, The primitive derivation and freeness of multi-
Coxeter arrangements. *Proc. Japan Acad. Ser. A* **78** (2002), no.
7, 116–119.

[14] M. Yoshinaga, Characterization of a free arrangement and conjecture
of Edelman and Reiner. *Invent. Math.* **157** (2004), no. 2, 449–
454.

[15] M. Yoshinaga, On the freeness of 3-arrangements. *Bull. London
Math. Soc.* **37** (2005), no. 1, 126–134.

[16] G. M. Ziegler, Multiarrangements of hyperplanes and their freeness.
in *Singularities* (Iowa City, IA, 1986), 345–359, Contemp. Math.,
**90**, Amer. Math. Soc., Providence, RI, 1989.