Fixed Point and Bregman Iterative Methods for Matrix Rank Minimization

Donald Goldfarb
Columbia University
Joint with Shiqian Ma and Lifeng Chen

Compressive Sensing Workshop
Duke University
25-26 February 2009
Affinely Constrained Matrix Rank Minimization (ACMRM) problem

\[
\begin{align*}
\min \quad & \text{rank}(X) \\
\text{s.t.} \quad & \mathcal{A}(X) = b,
\end{align*}
\]

where \( X \in \mathbb{R}^{m \times n} \), \( \mathcal{A} : \mathbb{R}^{m \times n} \to \mathbb{R}^p \), \( b \in \mathbb{R}^p \).

Special case: Matrix Completion (MC) problem

\[
\begin{align*}
\min \quad & \text{rank}(X) \\
\text{s.t.} \quad & X_{ij} = M_{ij}, (i, j) \in \Omega
\end{align*}
\]
Analogy to Compressed Sensing

- If $x$ is square and diagonal, ACMRM becomes CS problem
  \[
  \min \|x\|_0 \\
  \text{s.t. } Ax = b,
  \]
  where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $\|x\|_0 \equiv \text{card}\{x_i \neq 0\}$

- Basis Pursuit (BP):
  \[
  \min \|x\|_1 \\
  \text{s.t. } Ax = b.
  \]

**Theorem** (Candès and Tao 2006, Rudelson and Vershynin 2005) When $A$ is Gaussian random and partial Fourier, with high probability, BP gives the optimal solution of the CS problem for $b$ of a size of $m = O(k \log(n/k))$ and $O(k \log(n)^4)$, respectively.
NNM for Affinely Constrained MRM

Nuclear Norm Minimization (NNM):

\[
\begin{align*}
\text{min} & \quad \|X\|_* \\
\text{s.t.} & \quad A(X) = b,
\end{align*}
\]

where \(\|X\|_* = \sum_i \sigma_i\) and \(\sigma_i = i\text{th singular value of matrix } X\).
NNM for Affinely Constrained MRM

Nuclear Norm Minimization (NNM):

\[
\begin{align*}
\min \ & \|X\|_* \\
n\text{s.t.} \ & A(X) = b,
\end{align*}
\]

where \(\|X\|_* = \sum_i \sigma_i\) and \(\sigma_i = i\text{th singular value of matrix } X\).

**Theorem** (Recht, Fazel and Parrilo, 2007)

Rewrite \(A(X) = b\) as \(A \text{ vec}(X) = b\). If the entries of \(A \in \mathbb{R}^{p \times mn}\) are suitably random, e.g., i.i.d. Gaussian, then with very high probability, \(m \times n\) matrices of rank \(r\) can be recovered by solving the NNM problem whenever

\[p \geq Cr(m + n) \log(mn),\]

where \(C\) is a positive constant.
**Theorem** (Candès and Recht, 2008)

Let \( M \in \mathbb{R}^{n_1 \times n_2} \) have rank \( r \) with SVD \( M = \sum_{k=1}^{r} \sigma_k u_k v_k^\top \), where the families \( \{u_k\}_{1 \leq k \leq r} \) and \( \{v_k\}_{1 \leq k \leq r} \) are selected uniformly at random among all families of \( r \) orthonormal vectors. Let \( n = \max(n_1, n_2) \). Then \( \exists C, c \) s.t. if

\[
|\Omega| \equiv p \geq C n^{5/4} r \log n,
\]

the minimizer of the problem NNM is unique and equal to \( M \) with probability at least \( 1 - cn^{-3} \). In addition, if \( r \leq n^{1/5} \), then the recovery is exact with probability at least \( 1 - cn^{-3} \) provided that

\[
p \geq C n^{6/5} r \log n.
\]
Dual Problem of NNM:

\[
\begin{align*}
\text{max} & \quad b^\top z \\
\text{s.t.} & \quad \|A^*(z)\|_2 \leq 1.
\end{align*}
\]

SDP formulation of NNM:

\[
\begin{align*}
\min_{X,W_1,W_2} & \quad \frac{1}{2}(\text{Tr}(W_1) + \text{Tr}(W_2)) \\
\text{s.t.} & \quad \begin{bmatrix}
W_1 & X \\
X^\top & W_2
\end{bmatrix} \succeq 0 \\
A(X) & = b.
\end{align*}
\]

SDP formulation of Dual of NNM:

\[
\begin{align*}
\max_z & \quad b^\top z \\
\text{s.t.} & \quad \begin{bmatrix}
I_m & A^*(z) \\
A^*(z)^\top & I_n
\end{bmatrix} \succeq 0.
\end{align*}
\]
Optimality Conditions for Unconstrained NNM Problem

- Unconstrained Nuclear Norm Minimization (UNNM):

\[
\min \mu \|X\|_* + \frac{1}{2} \|A(X) - b\|_2^2.
\]

- Optimality condition:

\[
\begin{align*}
0 &\in \mu \partial \|X^*\|_* + A^*(A(X^*) - b) \\
\partial \|X\|_* &= \{UV^T + W : U^T W = 0, WV = 0, \|W\|_2 \leq 1\}.
\end{align*}
\]
Optimality Conditions for Unconstrained NNM Problem

- **Unconstrained Nuclear Norm Minimization (UNNM):**

\[
\min \mu \|X\|_* + \frac{1}{2} \|A(X) - b\|_2^2.
\]

- **Optimality condition:**

\[
0 \in \mu \partial \|X^*\|_* + A^*(A(X^*) - b).
\]

\[
\partial \|X\|_* = \{UV^T + W : U^T W = 0, WV = 0, \|W\|_2 \leq 1\}.
\]

**Theorem:** Let \( X \in \mathbb{R}^{m \times n} \) have SVD \( X = U\Sigma V^T \). Then \( X \) is optimal for UNNM iff \( \exists \) a matrix \( W \in \mathbb{R}^{m \times n} \) s.t.

\[
\mu(UV^T + W) + A^*(A(X) - b) = 0,
\]

\[
U^T W = 0, WV = 0, \|W\|_2 \leq 1.
\]
\[ 0 \in \mu \partial \| X^* \|_* + A^*(A(X^*) - b), \]

Let

\[ Y^* = X^* - \tau A^*(A(X^*) - b), \]

then the optimality condition reduces to

\[ 0 \in \tau \mu \partial \| X^* \|_* + X^* - Y^*, \]

i.e., \( X^* \) is the optimal solution to

\[
\min_{X \in \mathbb{R}^{m \times n}} \tau \mu \| X \|_* + \frac{1}{2} \| X - Y^* \|_F^2
\]
Nonnegative Vector Shrinkage Operator. Assume $x \in \mathbb{R}^n_+$. $\forall \nu > 0$, $s_\nu(x) := \bar{x}$, with $\bar{x}_i = \begin{cases} x_i - \nu, & \text{if } x_i - \nu > 0 \\ 0, & \text{o.w.} \end{cases}$

Matrix Shrinkage Operator. Assume $X \in \mathbb{R}^{m \times n}$ and the SVD of $X$ is $X = U \text{Diag}(\sigma)V^\top$, $U \in \mathbb{R}^{m \times r}$, $\sigma \in \mathbb{R}^r_+$, $V \in \mathbb{R}^{n \times r}$. $\forall \nu > 0$, $S_\nu(X) := U \text{Diag}(\bar{\sigma})V^\top$, with $\bar{\sigma} = s_\nu(\sigma)$. 
**Theorem:** Given $Y \in \mathbb{R}^{m \times n}$, rank($Y$) = $t$ and SVD $Y = U_Y \text{Diag}(\gamma) V_Y^\top$, where $U_Y \in \mathbb{R}^{m \times t}$, $\gamma \in \mathbb{R}^t$, $V_Y \in \mathbb{R}^{n \times t}$, and a scalar $\nu > 0$,

$$X := S_\nu(Y) = U_Y \text{Diag}(s_\nu(\gamma)) V_Y^\top$$

is an optimal solution of the problem

$$\min_{X \in \mathbb{R}^{m \times n}} f(X) := \nu \|X\|_* + \frac{1}{2} \|X - Y\|_F^2.$$
Fixed Point Method for UNNM

Fixed Point Iterative Scheme

\[
\begin{align*}
Y^k &= X^k - \tau A^*(A(X^k) - b) \\
X^{k+1} &= S_{\tau \mu}(Y^k).
\end{align*}
\]

**Lemma:** Matrix shrinkage operator is non-expansive. i.e.,

\[\|S_{\nu}(Y_1) - S_{\nu}(Y_2)\|_F \leq \|Y_1 - Y_2\|_F.\]

**Theorem:** The sequence \(\{X^k\}\) generated by the fixed point iterations converges to some \(X^* \in \mathcal{X}^*\) (the optimal set of UNNM).
Initialize: Given $X_0$, $\bar{\mu} > 0$. Select $\mu_1 > \mu_2 > \cdots > \mu_L = \bar{\mu} > 0$. Set $X = X_0$.

For $\mu = \mu_1, \mu_2, \ldots, \mu_L$, do

1. While NOT converged, do
   1. Select $\tau > 0$
   2. Compute $Y = X - \tau A^*(A(X) - b)$, and SVD of $Y$, $Y = U \text{Diag}(\sigma) V^\top$
   3. Compute $X = U \text{Diag}(s_{\tau \mu}(\sigma)) V^\top$

2. End while

End for
Bregman Iterative Method

- $\ell_1$-regularized problem

$$\min_x J(x) + \frac{1}{2} \|Ax - b\|_2^2, \text{ where } J(x) = \mu \|x\|_1.$$  

- Bregman distance:

$$D_p^J(u, v) := J(u) - J(v) - \langle p, u - v \rangle, \text{ where } p \in \partial J(v).$$

- Bregman iterative regularization procedure

$$x^{k+1} \leftarrow \min_x D_p^J(x, x^k) + \frac{1}{2} \|Ax - b\|_2^2$$
Bregman Iterative Scheme

Optimality condition: \( 0 \in \partial J(x^{k+1}) - p^k + A^\top (Ax^{k+1} - b) \), thus

\[
p^{k+1} := p^k - A^\top (Ax^{k+1} - b).
\]

So the Bregman iterative scheme is

\[
\begin{align*}
x^{k+1} &\leftarrow \min_x D^p_j(x, x^k) + \frac{1}{2} \|Ax - b\|^2_2 \\
p^{k+1} &\leftarrow p^k - A^\top (Ax^{k+1} - b).
\end{align*}
\]

or equivalently,

\[
\begin{align*}
b^{k+1} &\leftarrow b + (b^k - Ax^k) \\
x^{k+1} &\leftarrow \min_x J(x) + \frac{1}{2} \|Ax - b^{k+1}\|^2_2.
\end{align*}
\]
Bregman Iterative Method

- $b^0 \leftarrow 0$, $X^0 \leftarrow 0$,
- for $k = 0, 1, \ldots$ do
  - $b^{k+1} \leftarrow b + (b^k - A(X^k))$,
  - $X^{k+1} \leftarrow \arg \min_X \mu \|X\|_* + \frac{1}{2} \|A(X) - b^{k+1}\|_2^2$. 

Donald Goldfarb

Fixed Point and Bregman Iterative Methods for Matrix Rank Minimization
Approximate SVD Technique

Monte-Carlo approximate SVD (Drineas et.al.2006)

- Input: \( A \in \mathbb{R}^{m \times n}, 1 \leq k \leq c \leq n \).
- Output: \( U_k \in \mathbb{R}^{m \times k} \) and \( \Sigma_k \).
  - For \( j = 1 \) to \( c \),
    - Randomly choose a column \( A^{(i)} \) of \( A \)
    - Set \( C^{(j)} = A^{(i)} / \sqrt{c/n} \).
  - Compute SVD of \( C^\top C : \sum_{j=1}^{c} \sigma_j^2 y^j y^j^\top \).
  - Compute \( u^j = C y^j / \sigma_j \) for \( j = 1, \ldots, k \).
  - Return \( U_k \), where \( U_k^{(j)} = u^j \), and \( \Sigma_k = \text{diag}(\sigma_j, j = 1, \ldots, k) \).
Approximate SVD Technique

Monte-Carlo approximate SVD (Drineas et.al.2006)

- **Input:** $A \in \mathbb{R}^{m \times n}$, $1 \leq k \leq c \leq n$.
- **Output:** $U_k \in \mathbb{R}^{m \times k}$ and $\Sigma_k$.
  - For $j = 1$ to $c$,
    - Randomly choose a column $A^{(i)}$ of $A$
    - Set $C^{(j)} = A^{(i)}/\sqrt{c/n}$.
  - Compute SVD of $C^\top C$: $\sum_{j=1}^c \sigma_j^2 y_j y_j^\top$.
  - Compute $u_j = C y_j / \sigma_j$ for $j = 1, \ldots, k$.
  - Return $U_k$, where $U_k^{(j)} = u_j$, and $\Sigma_k = \text{diag}(\sigma_j, j = 1, \cdots, k)$.

**Theorem:** With high probability, the following estimate holds for both $\xi = 2$ and $\xi = F$:

$$\|A - A_{k_s}\|_\xi^2 \leq \min_{D: \text{rank}(D) \leq k_s} \|A - D\|_\xi^2 + \text{poly}(k_s, 1/c_s)\|A\|_F^2,$$

where $A_k = U_k \Sigma_k V_k^\top$, $V_k = A^\top U_k \Sigma_k^{-1}$. 
Numerical Tests: Stopping Rules and Solvers

\[ \|U_k V_k^T + g^k / \mu\|_2 - 1 < gtol, \]
\[ \frac{\|X^{k+1} - X^k\|_F}{\max\{1, \|X^k\|_F\}} < xtol, \]

- FPC1. Exact SVD, stopping rule: (2).
- FPC2. Exact SVD, stopping rule: (1) and (2).
- FPC3. Exact SVD with debiasing, stopping rule: (2).
- FPCA. Approximate SVD, stopping rule: (2).
- Bregman. Bregman iterative method using FPC2 to solve the subproblems.
Numerical Tests Randomly Created MC Problems

- Generation: generate matrices $M_L \in \mathbb{R}^{m \times r}$ and $M_R \in \mathbb{R}^{n \times r}$ with i.i.d. Gaussian entries; set $M = M_L M_R^\top$.

- Sample a subset $\Omega$ of $p$ entries of $M$ uniformly at random.

Measures:

- $rel.\ err. := \frac{\|X_{opt} - M\|_F}{\|M\|_F}$; Claim recovery if $rel.\ err. < 1e - 3$.

- $SR = p/(mn)$ (sampling ratio)

- $FR = r(m + n - r)/p$ (Note if $FR > 1$, it is not possible to recover the matrix)

- $NS = \text{the number of problems successfully solved}$
Comparisons on small problems (m=n=40, p=800, SR=0.5)

| r   | FR   | Solver | NS | avg. secs. | avg. rel.err. |
|-----|------|--------|----|------------|---------------|
| 1   | 0.0988 | FPC1   | 50 | 1.81       | 1.67e-9       |
|     |       | FPC2   | 50 | 3.61       | 1.32e-9       |
|     |       | FPC3   | 50 | 16.81      | 1.06e-9       |
|     |       | SDPT3  | 50 | 1.81       | 6.30e-10      |
| 2   | 0.1950 | FPC1   | 42 | 3.05       | 1.01e-6       |
|     |       | FPC2   | 42 | 17.97      | 1.01e-6       |
|     |       | FPC3   | 49 | 16.86      | 1.26e-5       |
|     |       | SDPT3  | 44 | 1.90       | 1.50e-9       |
| 3   | 0.2888 | FPC1   | 35 | 5.50       | 9.72e-9       |
|     |       | FPC2   | 35 | 20.33      | 2.17e-9       |
|     |       | FPC3   | 42 | 16.87      | 3.58e-5       |
|     |       | SDPT3  | 37 | 1.95       | 2.66e-9       |
| 4   | 0.3800 | FPC1   | 22 | 9.08       | 7.91e-5       |
|     |       | FPC2   | 22 | 18.43      | 7.91e-5       |
|     |       | FPC3   | 29 | 16.95      | 3.83e-5       |
|     |       | SDPT3  | 29 | 2.09       | 1.18e-8       |
| 5   | 0.4688 | FPC1   | 1  | 10.41      | 2.10e-8       |
|     |       | FPC2   | 1  | 17.88      | 2.70e-9       |
|     |       | FPC3   | 5  | 16.70      | 1.78e-4       |
|     |       | SDPT3  | 8  | 2.26       | 1.83e-7       |
| 6   | 0.5550 | FPC1   | 0  | —          | —             |
|     |       | FPC2   | 0  | —          | —             |
|     |       | FPC3   | 0  | —          | —             |
|     |       | SDPT3  | 1  | 2.87       | 6.58e-7       |
Comparison between FPC and Bregman (m=n=40, p=800, SR = 0.5)

| Problem | FPC2 | Bregman |
|---------|------|---------|
|         | max. rel.err | max. rel.err |
| r       | NIM (NS) |         |         |
| 1       | 0.0988  | 32 (50) | 2.22e-9 | 1.87e-15 |
| 2       | 0.1950  | 29 (42) | 5.01e-9 | 2.96e-15 |
| 3       | 0.2888  | 24 (35) | 2.77e-9 | 2.93e-15 |
| 4       | 0.3800  | 10 (22) | 5.51e-9 | 3.11e-15 |
Comparison of FPCA and SDPT3  
\((m=n=40, p=800, SR=0.5)\)

| Problems | FPCA | SDPT3 |
|----------|------|-------|
|          | FR   | NS    | avg. sec. | avg. rel.err | NS | avg. secs. | avg. rel.err |
| 1        | 0.0988 | 50    | 4.24      | 6.60e-7      | 50 | 1.84      | 6.30e-1      |
| 2        | 0.1950 | 50    | 4.35      | 1.08e-6      | 44 | 1.93      | 1.50e-9      |
| 3        | 0.2888 | 50    | 4.83      | 1.83e-6      | 37 | 1.99      | 2.66e-9      |
| 4        | 0.3800 | 50    | 4.92      | 2.56e-6      | 29 | 2.12      | 1.18e-8      |
| 5        | 0.4688 | 50    | 5.06      | 3.38e-6      | 8  | 2.30      | 1.83e-7      |
| 6        | 0.5550 | 50    | 5.48      | 3.72e-6      | 1  | 2.89      | 6.58e-7      |
| 7        | 0.6388 | 50    | 5.79      | 4.78e-6      | 0  | —         | —            |
| 8        | 0.7200 | 50    | 6.03      | 8.57e-6      | 0  | —         | —            |
| 9        | 0.7987 | 49    | 6.75      | 1.27e-5      | 0  | —         | —            |
| 10       | 0.8750 | 32    | 8.71      | 7.49e-5      | 0  | —         | —            |
| 11       | 0.9487 | 0     | —         | —            | 0  | —         | —            |

\[ FR = r(m + n - r)/p \]
Medium sized matrices: \( (m=n=100,p=2000,SR=0.2) \)

| Problems | FPCA | SDPT3 |
|----------|------|-------|
|          |      |       |
| \( r \) | \( r(m + n - r)/p \) |       |
| 1        | 0.0995 | 50 7.94 6.11e-6 | 47 15.10 1.55e-9 |
| 2        | 0.1980 | 50 8.17 6.51e-6 | 31 16.02 7.95e-9 |
| 3        | 0.2955 | 50 9.09 7.36e-6 | 13 19.23 1.05e-4 |
| 4        | 0.3920 | 50 9.33 1.09e-5 | 0  —  — |
| 5        | 0.4875 | 49 9.91 2.99e-5 | 0  —  — |
| 6        | 0.5820 | 47 10.81 3.99e-5 | 0  —  — |
| 7        | 0.6755 | 44 12.63 8.87e-5 | 0  —  — |
| 8        | 0.7680 | 31 16.30 1.24e-4 | 0  —  — |
| 9        | 0.8595 | 2 17.88 6.19e-4 | 0  —  — |
| 10       | 0.9500 | 0  —  — | 0  —  — |
Medium sized matrices: \((m=n=100, p=3000, SR=0.3)\)

| Problems | FPCA | SDPT3 |
|----------|------|-------|
|          | FR   | NS avg. secs. | avg. rel.err | NS avg. secs. | avg. rel.err |
| r        | FR   | avg. secs. | avg. rel.err | avg. secs. | avg. rel.err |
| 1        | 0.0663 | 50 | 8.39 | 1.83e-6 | 50 | 36.68 | 2.01e-10 |
| 2        | 0.1320 | 50 | 8.53 | 1.86e-6 | 50 | 36.50 | 1.13e-9 |
| 3        | 0.1970 | 50 | 9.30 | 2.11e-6 | 46 | 38.50 | 1.28e-5 |
| 4        | 0.2613 | 50 | 9.72 | 2.88e-6 | 42 | 41.28 | 4.60e-6 |
| 5        | 0.3250 | 50 | 9.87 | 3.60e-6 | 32 | 43.92 | 7.82e-8 |
| 6        | 0.3880 | 50 | 9.96 | 3.93e-6 | 17 | 49.60 | 3.44e-7 |
| 7        | 0.4503 | 50 | 10.19 | 4.27e-6 | 3 | 59.18 | 1.43e-4 |
| 8        | 0.5120 | 50 | 10.65 | 4.38e-6 | 0 | — | — |
| 9        | 0.5730 | 50 | 11.74 | 5.01e-6 | 0 | — | — |
| 10       | 0.6333 | 50 | 11.76 | 6.30e-6 | 0 | — | — |
| 11       | 0.6930 | 50 | 12.08 | 8.29e-6 | 0 | — | — |
| 12       | 0.7520 | 50 | 13.67 | 2.64e-5 | 0 | — | — |
| 13       | 0.8103 | 48 | 16.00 | 2.95e-5 | 0 | — | — |
| 14       | 0.8680 | 40 | 20.51 | 1.35e-4 | 0 | — | — |
| 15       | 0.9250 | 0 | — | — | 0 | — | — |
| 16       | 0.9813 | 0 | — | — | 0 | — | — |

\[ FR = r(m + n - r) / p \]

Donald Goldfarb

Fixed Point and Bregman Iterative Methods for Matrix Rank Minimization
Large matrices: \( (m=n=1000, p=2e+5, SR=0.2) \)

| Problems | FPCA |
| --- | --- |
| \( r \) | FR | NS | avg. secs. | avg. rel.err |
| 50 | 0.4875 | 10 | 1500.7 | 2.73e-6 |
| 51 | 0.4970 | 10 | 1510.2 | 2.75e-6 |
| 52 | 0.5065 | 10 | 1515.0 | 2.80e-6 |
| 53 | 0.5160 | 10 | 1520.6 | 2.79e-6 |
| 54 | 0.5254 | 10 | 1535.9 | 2.77e-6 |
| 55 | 0.5349 | 10 | 1543.6 | 2.80e-6 |
| 56 | 0.5443 | 10 | 1556.3 | 2.78e-6 |
| 57 | 0.5538 | 10 | 1567.3 | 2.74e-6 |
| 58 | 0.5632 | 10 | 1586.4 | 2.69e-6 |
| 59 | 0.5726 | 10 | 1576.1 | 2.66e-6 |
| 60 | 0.5820 | 10 | 1602.0 | 2.55e-6 |
Hold out 2 ratings for each user.

Mean Absolute Error (MAE)

\[ MAE = \frac{1}{2N} \sum_{i=1}^{N} |\hat{r}_{i1} - r_{i1}| + |\hat{r}_{i2} - r_{i2}|. \]

Normalized Mean Absolute Error (NMAE)

\[ NMAE = \frac{MAE}{r_{max} - r_{min}}. \]
Numerical Results

**Table:** Numerical results of FPC1 for Jester joke data set

| num.user | num.samp | samp.ratio | rank | $\sigma_{\text{max}}$ | $\sigma_{\text{min}}$ | NMAE | Time |
|----------|----------|-------------|------|-----------------------|-----------------------|------|------|
| 100      | 7172     | 0.7172      | 79   | 285.6520              | 3.4916e-004           | 0.1727 | 34.3 |
| 1000     | 71152    | 0.7115      | 100  | 786.3651              | 38.4326               | 0.1667 | 304.8125 |
| 2000     | 140691   | 0.7035      | 100  | 1.1242e+003           | 65.0607               | 0.1582 | 661.6563 |

**Table:** Numerical results of FPCA for Jester joke data set

| num.user | num.samp | samp.ratio | $\epsilon_{k_s}$ | $c_s$ | rank | $\sigma_{\text{max}}$ | $\sigma_{\text{min}}$ | NMAE | Time |
|----------|----------|-------------|------------------|------|------|-----------------------|-----------------------|------|------|
| 100      | 7172     | 0.7172      | 1e-2             | 25   | 20   | 295.1449              | 32.6798               | 0.1627 | 26.7344 |
| 1000     | 71152    | 0.7115      | 1e-2             | 100  | 85   | 859.2710              | 48.0393               | 0.2008 | 808.5156 |
| 1000     | 71152    | 0.7115      | 1e-4             | 100  | 90   | 859.4588              | 44.6220               | 0.2101 | 778.5625 |
| 2000     | 140691   | 0.7035      | 1e-4             | 200  | 100  | 1.1518e+003           | 63.5244               | 0.1564 | 1.1345e+003 |