**Δ-Filtrations and Projective Resolutions for the Auslander–Dlab–Ringel Algebra**

**TERESA CONDE**

**Abstract.** The ADR algebra $R_A$ of an Artin algebra $A$ is a right ultra strongly quasihereditary algebra (RUSQ algebra). In this paper we study the $Δ$-filtrations of modules over RUSQ algebras and determine the projective covers of a certain class of $R_A$-modules. As an application, we give a counterexample to a claim by Auslander–Platzeck–Todorov, concerning projective resolutions over the ADR algebra.

1. Introduction

It is natural to ask whether there exist “Schur algebras” for arbitrary Artin algebras. That is, given an Artin algebra $A$, we would like to have an $A$-module whose endomorphism algebra is quasihereditary, so that it has finite global dimension and a highest weight theory. Such modules do exist. A suitable candidate was introduced by Auslander in [1]. He showed that the endomorphism algebra of

$$G = \bigoplus_{i=1}^{\text{LL}(A)} A/\text{Rad}^i A$$

has finite global dimension (here LL$(A)$ denotes the Loewy length of $A$). Subsequently, Dlab and Ringel proved that this endomorphism algebra is actually a quasihereditary algebra ([2]). For practical purposes one considers the basic version of $\text{End}_A(G)^{op}$ instead. We denote this ‘Schur-like’ endomorphism algebra by $R_A$ and call it the Auslander–Dlab–Ringel algebra (ADR algebra) of $A$. The original algebra $A$ is then Morita equivalent to $\xi R_A \xi$ for an idempotent $\xi$ in $R_A$, and this is also analogous to the situation of symmetric groups and Schur algebras.

It was recently proved in [3] that the ADR algebra has a particularly neat quasihereditary structure. The ADR algebra is not only right strongly quasihereditary in the sense of Ringel ([22]); $R_A$ is actually a right ultra strongly quasihereditary algebra (RUSQ algebra) as defined in [3] (see also [2], Chapter 2). The ADR algebra is not the only strongly quasihereditary algebra arising from a module theoretical context. Other examples of strongly quasihereditary algebras include: the Auslander algebras, associated to algebras of finite type; the endomorphism algebras

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constructed by Iyama, used in his famous proof of the finiteness of the representation dimension of Artin algebras ([16]); certain cluster-tilted algebras studied by Geiß–Leclerc–Schröer ([15], [14]), Buan–Iyama–Reiten–Scott ([4]) and Iyama–Reiten ([17]). The cluster-tilted algebras in [4], [15], [14] and [17] are actually RUSQ up to dualisation, as implicitly proved in [15] (see also [7]).

This paper complements the investigation on RUSQ algebras and on ADR algebras initiated in [8]. We start by studying the \(\Delta\)-filtrations of modules over RUSQ algebras. In Section 3 we show that the RUSQ algebras satisfy the following key property: every submodule of a direct sum of standard modules is still a direct sum of standard modules. This has several consequences. In particular, it gives rise to special (uniquely determined) filtrations of \(\Delta\)-good modules over RUSQ algebras, called \(\Delta\)-semisimple filtrations. These can be described recursively as follows.

Given a \(\Delta\)-good module \(N\), let \(\delta_1(N)\) be the largest submodule of \(N\) which is a direct sum of standard modules, and for \(i \geq 1\) define \(\delta_{i+1}(N)\) as the module satisfying the identity \(\delta_{i+1}(N)/\delta_i(N) = \delta_1(N/\delta_i(N))\).

The ideas in Section 3 are then applied to the ADR algebra. As a main contribution of Section 4, we prove the following (which corresponds to Lemma 4.3 and Theorem 4.4).

**Theorem.** Let \(M\) be in \(\text{mod} A\). Then \(N = \text{Hom}_A(G, M)\) lies in \(F(\Delta)\) and the socle series of \(M\) determines the \(\Delta\)-semisimple filtration of \(N\). Formally,

\[
\delta_m(N) = \text{Hom}_A(G, \text{Soc}_m M),
\]

for all \(m\). The factors of the \(\Delta\)-semisimple filtration of \(N\) only depend on the factors of the socle series of \(M\) and on the Loewy length of the projective indecomposable \(A\)-modules.

Next, we describe the right minimal add-\(G\)-approximations of rigid modules in \(\text{mod} A\), or equivalently, the projective covers of the \(R_A\)-modules \(\text{Hom}_A(G, M)\), with \(M\) rigid. Recall that a module is said to be rigid if its radical series coincides with its socle series. We prove the following theorem in Section 5.

**Theorem.** Let \(M\) be a rigid module in \(\text{mod} A\), with Loewy length \(m\). Then the projective cover of \(M\) in \(\text{mod}(A/\text{Rad}^m A)\) is a right minimal add-\(G\)-approximation of \(M\).

This simple yet useful result, combined with the conclusions in Section 4, is then used to provide a counterexample to a claim by Auslander, Platzeck and Todorov in [2, §7], about the projective resolutions of modules over the ADR algebra, for which no proof was given. To be precise, we show the following.

**Proposition.** ADR algebras need not satisfy the descending Loewy length condition on projective resolutions.

2. Preliminaries

In this section we introduce the language of preradicals and give some background on quasihereditary algebras, RUSQ algebras and the ADR algebra.

Throughout this paper the letters \(B\) and \(A\) shall denote arbitrary Artin algebras over some commutative artinian ring \(C\). All the modules will be finitely generated left modules. The notation \(\text{mod} B\) will be used for the category of (finitely generated) \(B\)-modules. Given a class of \(B\)-modules \(\Theta\), let add \(\Theta\) be the full subcategory
of mod $B$ consisting of all modules isomorphic to a summand of a direct sum of modules in $\Theta$. The additive closure of a single module $M$ is denoted by $\text{add } M$.

2.1. Preradicals. Preradicals generalise the classic notions of radical and socle of a module. The results and definitions stated in this section are elementary and most of the proofs may be found in [3, Chapter 2] and [23, Chapter VI] (see also [7, Section 1.3]).

2.1.1. Definition and first properties.

Definition 2.1. A preradical $\tau$ in mod $B$ is a subfunctor of the identity functor $1_{\text{mod } B}$, i.e., $\tau$ assigns to each module $M$ a submodule $\tau(M)$, such that each morphism $f : M \to N$ induces a morphism $\tau(f) : \tau(M) \to \tau(N)$ given by restriction.

A submodule $N$ of a $B$-module $M$ is called a characteristic submodule of $M$ if $f(N) \subseteq N$, for every $f$ in $\text{End}_B(M)$. By definition, it is clear that the module $\tau(M)$ is a characteristic submodule of $M$, for every preradical $\tau$ and for every module $M$. It is also evident that every preradical is an additive functor which preserves monics.

To each preradical $\tau$ we may associate the functor $1/\tau : \text{mod } B \to \text{mod } B$, which maps $M$ to $M/\tau(M)$. Note that the functor $1/\tau$ preserves epics.

Example 2.2. For any class $\Theta$ of $B$-modules, the operators defined by

\[
\text{Tr}(\Theta, M) := \sum_{f : f \in \text{Hom}_B(U, M), U \in \Theta} \text{Im } f,
\]

\[
\text{Rej}(M, \Theta) := \bigcap_{f : f \in \text{Hom}_B(M, U), U \in \Theta} \text{Ker } f,
\]

for $M$ in $\text{mod } B$, are preradicals in $\text{mod } B$. The module $\text{Tr}(\Theta, M)$, called the trace of $\Theta$ in $M$, is the largest submodule of $M$ generated by $\Theta$. Symmetrically, $\text{Rej}(M, \Theta)$, the reject of $\Theta$ in $M$, is the submodule $N$ of $M$ such that $M/N$ is the largest factor module of $M$ cogenerated by $\Theta$. If $\varepsilon$ is a complete set of simple $B$-modules, then $\text{Tr}(\varepsilon, -) = \text{Soc}(-)$ and $\text{Rej}(-, \varepsilon) = \text{Rad}(-)$.

The statements below are immediate consequences of the definition of preradical.

Lemma 2.3. Let $\tau$ be a preradical in $\text{mod } B$. Let $N$ and $M$ be $B$-modules, with $N \subseteq M$, and let $(M_i)_{i \in I}$ be a finite family of $B$-modules. The following hold:

1. $\tau(N) \subseteq N \cap \tau(M)$;
2. $\tau(M + N)/N \subseteq \tau(M)/N$;
3. $\tau \left( \bigoplus_{i \in I} M_i \right) = \bigoplus_{i \in I} \tau(M_i)$.

2.1.2. Hereditary and cohereditary preradicals. We shall now look at preradicals which satisfy specific properties.

Definition 2.4. A preradical $\tau$ is called idempotent if $\tau \circ \tau = \tau$. Symmetrically, we say that $\tau$ is a radical if $\tau \circ (1/\tau) = 0$.

Note that $\text{Tr}(\Theta, -)$ is an idempotent preradical for every class of $B$-modules $\Theta$. Similarly, the functor $\text{Rej}(-, \Theta)$ is a radical.
**Definition 2.5.** A preradical $\tau$ is **hereditary** if $\tau(N) = N \cap \tau(M)$, for all $M$ and $N$ in $\text{mod } B$ such that $N \subseteq M$. Dually, a preradical $\tau$ is said to be **cohereditary** if $(\tau(M) + N)/N = \tau(M/N)$ for $N \subseteq M$, $M$ and $N$ in $\text{mod } B$.

**Example 2.6.** The functors $\text{Soc}(-)$ and $\text{Rad}(-)$ are the typical examples of a hereditary preradical and of a cohereditary preradical, respectively.

**Lemma 2.7** ([23, Chapter VI, §1]). Let $\tau$ be a preradical in $\text{mod } B$. The following statements are equivalent:

1. $\tau$ is hereditary;
2. $\tau$ is a left exact functor;
3. the functor $1/\tau$ preserves monics.

Moreover, any hereditary preradical is idempotent.

**Remark 2.8.** There is a result ‘dual’ to Lemma 2.7 for cohereditary preradicals.

It is possible to construct hereditary (and cohereditary) preradicals out of special classes of modules.

**Definition 2.9.** A class $\bullet$ of modules in $\text{mod } B$ is **hereditary** if every submodule of a module in $\text{add } \bullet$ is generated by $\bullet$.

**Lemma 2.10.** If $\bullet$ is a hereditary class of modules then $\text{Tr}(\bullet, -)$ is a hereditary preradical in $\text{mod } B$.

**Proof.** Consider $N \subseteq M$, with $M$ and $N$ in $\text{mod } B$. The module $\text{Tr}(\bullet, M)$ is generated by some module $M'$ which is a (finite) direct sum of modules in $\bullet$. Consider the pullback square

$$
\begin{array}{ccc}
M'' & \longrightarrow & N \cap \text{Tr}(\bullet, M) \\
\downarrow & & \downarrow \\
M' & \longrightarrow & \text{Tr}(\bullet, M)
\end{array}
$$

As $\bullet$ is a hereditary class, $M''$ is generated by $\bullet$. Hence $N \cap \text{Tr}(\bullet, M)$ is generated by $\bullet$ as well. Since $\text{Tr}(\bullet, N)$ is the largest submodule of $N$ generated by $\bullet$, we must have $N \cap \text{Tr}(\bullet, M) \subseteq \text{Tr}(\bullet, N)$.

\[ \square \]

### 2.2. Quasihereditary algebras, RUSQ algebras and the ADR algebra.

We now introduce some notation and state basic results about RUSQ algebras and ADR algebras.

**2.2.1. Quasihereditary algebras.** Given an Artin algebra $B$, we may label the isomorphism classes of simple $B$-modules by the elements of a finite poset $(\Phi, \subseteq)$. Denote the simple $B$-modules by $L_i$, $i \in \Phi$, and use the notation $P_i$ (resp. $Q_i$) for the projective cover (resp. injective hull) of $L_i$.

Let $\Delta(i)$ be the **standard module** with label $i \in \Phi$, that is

$$
\Delta(i) = P_i/\text{Tr} \left( \bigoplus_{j \not\leq i} P_j, P_i \right).
$$
The module \( \Delta(i) \) is the largest quotient of \( P_i \) whose composition factors are all of the form \( L_j \), with \( j \subseteq i \). Dually, denote the costandard modules by \( \nabla(i) \). The module

\[
\nabla(i) = \text{Rej} \left( Q_i, \bigoplus_{j : j \not\subseteq i} Q_j \right)
\]
is the largest submodule of \( Q_i \) with all composition factors of the form \( L_j \), with \( j \subseteq i \).

Let \( \mathcal{F}(\Delta) \) be the category of all \( B \)-modules which have a \( \Delta \)-filtration, that is, a filtration whose factors are standard modules. The category \( \mathcal{F}(\nabla) \) is defined dually. We call \( M \in \mathcal{F}(\Delta) \) a \( \Delta \)-good module.

The notation \([M : L]\) will be used for the multiplicity of a simple module \( L \) in the composition series of \( M \). In a similar manner, \((M : \Delta(i))\) shall denote the multiplicity of \( \Delta(i) \) in a \( \Delta \)-filtration of a module \( M \) in \( \mathcal{F}(\Delta) \) (this is well defined).

**Definition 2.11.** We say that \((B, \Phi, \subseteq)\) is quasihereditary if the following hold for every \( i \in \Phi \):

1. \([\Delta(i) : L_i] = 1\);
2. \( P_i \in \mathcal{F}(\Delta) \);
3. \((P_i : \Delta(i)) = 1\), and \((P_i : \Delta(j)) \neq 0 \Rightarrow j \supseteq i\).

Let \((B, \Phi, \subseteq)\) be quasihereditary. It was proved by Ringel in [21] (see also [13]) that there is a unique indecomposable \( B \)-module \( T(i) \) (up to isomorphism) for every \( i \) which has both a \( \Delta \)- and a \( \nabla \)-filtration, with one composition factor labelled by \( i \), and all the other composition factors labelled by \( j, j \subseteq i \). The standard module \( \Delta(i) \) can be embedded in \( T(i) \) – the corresponding monomorphism is a left minimal \( \mathcal{F}(\nabla) \)-approximation of \( \Delta(i) \) and \( T(i)/\Delta(i) \) lies in \( \mathcal{F}(\Delta) \).

**2.2.2. Strongly and ultra strongly quasihereditary algebras.** Following Ringel ([22]), a quasihereditary algebra \((B, \Phi, \subseteq)\) is said to be right strongly quasihereditary if \( \text{Rad} \Delta(i) \in \mathcal{F}(\Delta) \) for all \( i \in \Phi \). This property holds if and only if the category \( \mathcal{F}(\Delta) \) is closed under submodules (see [10], [12, Lemma 4.1*] and [22, Appendix]).

Let \((B, \Phi, \subseteq)\) be an arbitrary quasihereditary algebra, as before. Additionally, suppose that \( B \) satisfies the following two conditions:

- **(A1):** \( \text{Rad} \Delta(i) \in \mathcal{F}(\Delta) \) for all \( i \in \Phi \) (i.e. \( B \) is right strongly quasihereditary);
- **(A2):** \( Q_i \in \mathcal{F}(\Delta) \) for all \( i \in \Phi \) such that \( \text{Rad} \Delta(i) = 0 \).

We call these algebras right ultra strongly quasihereditary algebras (RUSQ algebras, for short).

**Remark 2.12.** It was proved in [7, §2.5.1] that the definition of RUSQ algebra given in [8] is equivalent to the one above.

Let \((B, \Phi, \subseteq)\) be a RUSQ algebra. It is always possible to label the elements in \( \Phi \) as

\[
\Phi = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq l_i\},
\]

for certain \( n, l_i \in \mathbb{Z}_{\geq 0} \), so that \([\Delta(k, l) : L_{i,j}] \neq 0\) implies that \( k = i \) and \( j \geq l \) (see [8, §5]). We shall always assume that the elements in \( \Phi \) are labelled in such a way.

The following proposition summarises some properties of the RUSQ algebras.

**Proposition 2.13 ([8, §5]).** Let \((B, \Phi, \subseteq)\) be a RUSQ algebra. The following hold:
(1) $\mathcal{F}(\Delta)$ is closed under submodules;
(2) $\text{Rad} \Delta (i, j) = \Delta (i, j + 1)$ for $j < l_i$, and $\Delta (i, l_i) = L_{i,l_i}$;
(3) each $\Delta (i, j)$ is uniserial and has composition factors $L_{i,j}, \ldots, L_{i,l_i}$, ordered from the top to the socle;
(4) $Q_{i,l_i} = T (i, 1)$;
(5) for $M \in \mathcal{F}(\Delta)$, the number of standard modules appearing in a $\Delta$-filtration of $M$ is given by $\sum_{i=1}^{n} [M : L_{i,l_i}]$;
(6) a module $M$ belongs to $\mathcal{F}(\Delta)$ if and only if $\text{Soc} M$ is a (finite) direct sum of modules of type $L_{i,l_i}$.

2.2.3. The ADR algebra. Fix an Artin algebra $A$. Given a module $M$ in $\text{mod } A$, we shall denote its Loewy length by $LL(M)$. Let $A$ have Loewy length $L$ (as a left module). We want to study the basic version of the endomorphism algebra of $\bigoplus_{j=1}^{l_i} A / \text{Rad}^j A$.

For this, let $\{P_1, \ldots, P_n\}$ be a complete set of projective indecomposable $A$-modules and let $l_i$ be the Loewy length of $P_i$. Define

$$G = G_A := \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{l_i} P_i / \text{Rad}^j P_i.$$ 

The Auslander–Dlab–Ringel algebra of $A$ (ADR algebra of $A$) is defined as $R = R_A := \text{End}_A (G)^{op}$.

The projective indecomposable $R$-modules are given by

$$P_{i,j} := \text{Hom}_A (G, P_i / \text{Rad}^j P_i),$$

for $1 \leq i \leq n$, $1 \leq j \leq l_i$.

Denote the simple quotient of $P_{i,j}$ by $L_{i,j}$ and define

$$(2) \Lambda := \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq l_i\};$$

so that $\Lambda$ labels the simple $R$-modules. Define a partial order, $\preceq$, on $\Lambda$ by

$$(3) (i, j) \preceq (k, l) \Leftrightarrow j > l.$$ 

According to [8, §4], $(R, \Lambda, \preceq)$ is a RUSQ algebra and the labelling $\Lambda$ is compatible with $(1)$.

**Theorem 2.14** ([8, §3, §4]). The ADR algebra $(R, \Lambda, \preceq)$ is a RUSQ algebra. For every $(i, j)$ in $\Lambda$, we have $\Delta (i, j) \cong \text{Rad}^{j-1} P_{i,1}$ and there are short exact sequences

$$(4) 0 \longrightarrow \text{Hom}_A (G, \text{Rad} P_i / \text{Rad}^j P_i) \longrightarrow P_{i,j} \longrightarrow \Delta (i, j) \longrightarrow 0.$$

3. $\Delta$-semisimple modules and $\Delta$-semisimple filtrations

For a quasihereditary algebra $(B, \Phi, \sqsubseteq)$, we say that a $B$-module is $\Delta$-semisimple if it is a direct sum of standard modules. Every module $M$ in $\mathcal{F}(\Delta)$ has some submodule $N$ such that:

**$\text{(B)}$:** $N$ is $\Delta$-semisimple and $M/N$ is in $\mathcal{F}(\Delta)$.

Given a module $M$ in $\mathcal{F}(\Delta)$, we may consider the submodules of $M$ which are maximal with respect to property [B]. The module $M$ may have more than one such submodule (see Example 2.20 in [18]). However, according to [18], the submodules of $M$ which are maximal with respect to [B] are unique up to isomorphism.
Suppose now that $B$ is a RUSQ algebra. The $\Delta$-semisimple modules over RUSQ algebras are particularly well behaved. As we will see in Corollary 3.3 the property of being $\Delta$-semisimple is closed under submodules in this case. Furthermore, every module $M$ in $\mathcal{F}(\Delta)$ has exactly one submodule $D_M$ which is maximal with respect to property $[B]$. The module $D_M$ is actually the unique maximal $\Delta$-semisimple submodule of $M$ (with respect to inclusion). Moreover, $D_M$ will be obtained by applying a certain hereditary preradical (as in Definition 2.5) to the module $M$. Since $M/D_M$ still lies in $\mathcal{F}(\Delta)$, we may proceed iteratively and define the $\Delta$-semisimple filtration (which will be unique) and the $\Delta$-semisimple length of any module in $\mathcal{F}(\Delta)$.

### 3.1. $\Delta$-semisimple modules

We are interested in submodules of $\Delta$-good modules which are maximal with respect to property $[B]$. As a consequence of Theorem 2.17 in [13], these are unique up to isomorphism.

**Theorem 3.1** ([18, Theorem 2.17]). Let $(B, \Phi, \sqsubseteq)$ be a quasihereditary algebra, and let $M$ be in $\mathcal{F}(\Delta)$. Any two submodules of $M$ which are maximal with respect to property $[B]$ are isomorphic.

Note that $\Delta$-semisimple modules are in general well behaved with respect to quotients in the following way: every $\Delta$-good factor module of a $\Delta$-semisimple module is still $\Delta$-semisimple. This assertion follows from the fact that $\mathcal{F}(\Delta)$ is closed under taking kernels of epimorphisms ([12, Lemma 1.5]) and from Theorem 3.2 in [18].

We wish to study the $\Delta$-semisimple modules over a RUSQ algebra $(B, \Phi, \sqsubseteq)$. We shall assume that the set $\Phi$ is as described in (1). In this subsection we prove some key properties of the $\Delta$-semisimple modules over RUSQ algebras. Namely, we show that the property of being $\Delta$-semisimple is closed under taking submodules.

**Lemma 3.2.** Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. Let $M$ be in mod $B$ and suppose that there is a short exact sequence

$$
0 \longrightarrow \Delta(k, l) \xrightarrow{f} M \longrightarrow \Delta(i, j) \longrightarrow 0,
$$

with $(k, l), (i, j) \in \Phi$. If $\text{Soc}M \neq \text{Soc}\Delta(k, l)$, then (5) splits.

**Proof.** Note that $\text{Soc}(\text{Im}f) = \text{Soc}\Delta(k, l) = L_{k, l}k$ (see Proposition 2.13 part 3). As $\text{Soc}M \neq \text{Soc}\Delta(k, l)$, there is some nonzero submodule $M'$ of $M$ such that $\text{Im}f \cap M' = 0$. Let $g : \text{Im}f \oplus M' \longrightarrow Q_{k, l}k$ be the morphism which embeds $\text{Im}f$ in $Q_{k, l}k$ and maps $M'$ to zero. By the injectivity of $Q_{k, l}k$, $g$ extends to a map $g' : M \longrightarrow Q_{k, l}k$. Note that $\text{Im}f \cap \text{Ker}g' = 0$ as $g'|_{\text{Im}f} = g|_{\text{Im}f}$ is an injective map. Thus $\text{Im}f \oplus \text{Ker}g'$ is a submodule of $M$.

By part 4 of Proposition 2.13 $Q_{k, l}k$ is in $\mathcal{F}(\Delta)$. Since $\mathcal{F}(\Delta)$ is closed under taking submodules, then both $\text{Im}g'$ and $\text{Ker}g'$ lie in $\mathcal{F}(\Delta)$. Denote by $\Delta[N]$ the number of standard modules appearing in a $\Delta$-filtration of $N \in \mathcal{F}(\Delta)$. As $\Delta[M] = 2$, $\Delta[\text{Im}g'] > 0$ and $\Delta[\text{Ker}g'] > 0$, it follows that $\Delta[\text{Im}g'] = \Delta[\text{Ker}g'] = 1$. So either $\text{Im}g' \cong \Delta(k, l)$ and $\text{Ker}g' \cong \Delta(i, j)$, or $\text{Im}g' \cong \Delta(i, j)$ and $\text{Ker}g' \cong \Delta(k, l)$.

If $\text{Ker}g' \cong \Delta(i, j)$, then the submodule $\text{Im}f \oplus \text{Ker}g'$ of $M$ must coincide with $M$ (as both modules have the same Jordan–Hölder length). In this case the monic $f$ splits.
If $\text{Im } g' \cong \Delta(i, j)$, then $(k, l) \subseteq (i, j)$ as
\[
\Delta(k, l) \cong \text{Im } f \subseteq \text{Im } g \subseteq \text{Im } g'.
\]
But then part (b) of Lemma 1.3 in [12] implies that (5) is a split exact sequence. □

We now use the previous result to give a characterisation of the $\Delta$-semisimple modules over a RUSQ algebra.

**Corollary 3.3.** Let $(B, \Phi, \subseteq)$ be a RUSQ algebra and let $M$ be in $\mathcal{F}(\Delta)$. Then $M$ is $\Delta$-semisimple if and only if the number of simple summands of $\text{Soc } M$ coincides with the number of factors in a $\Delta$-filtration of $M$. Moreover, any submodule of a $\Delta$-semisimple module is still $\Delta$-semisimple.

**Proof.** Let $M$ be in $\mathcal{F}(\Delta)$. Denote by $P(M)$ the following assertion: “the number of simple summands of $\text{Soc } M$ coincides with the number of factors in a $\Delta$-filtration of $M$”. By parts 5 and 6 of Proposition 2.13, $P(M)$ is true if and only if the composition factors of $M$ are exactly the summands of its socle. From this equivalence, it is easy to see that the truth of $P(M)$ implies the truth of $P(N)$ for $N \subseteq M$. Let $N \in \mathcal{F}(\Delta)$ be a submodule of $M$ such that $M/N \in \mathcal{F}(\Delta)$. Using Proposition 2.13, we also conclude that the truth of $P(M)$ implies the truth of $P(M/N)$.

If $M$ is a $\Delta$-semisimple module then $P(M)$ is clearly true. Suppose now that $P(M)$ holds for $M \in \mathcal{F}(\Delta)$. We wish to show that $M$ is $\Delta$-semisimple. We prove this by induction on the number $z$ of factors in a $\Delta$-filtration of $M$. If $z = 1$ the result is obvious. Suppose now that $z \geq 2$. Let $M_1$ be a submodule $M$ satisfying $M/M_1 \in \mathcal{F}(\Delta)$ and $M_1 \cong \Delta(i, j)$ for some $(i, j) \in \Phi$. By the remark in the first paragraph, $P(M/M_1)$ holds. Using induction, we conclude that $M/M_1$ is $\Delta$-semisimple. Therefore $M/M_1 = \bigoplus_{i=1}^{z-1} N_i/M_1$, where each $N_i/M_1$ is isomorphic to a standard module. Applying again the observations in the first paragraph, we deduce that assertion $P(N_i)$ must hold, so, by induction, each $N_i$ is a $\Delta$-semisimple module. Using that both $M_1$ and $N_i/M_1$ are standard modules, we conclude that $N_i = M_1 \oplus M_{i+1}$, where $M_{i+1}$ is a submodule of $M$ isomorphic to a standard module. Note that $\bigoplus_{i=1}^{z-1} M_1$ is a submodule of $M$ which has the same Jordan–Hölder length as $M$. This implies that $M = \bigoplus_{i=1}^{z-1} M_i$, which proves that $M$ is $\Delta$-semisimple. We have just shown that $M$ is a $\Delta$-semisimple module if and only if assertion $P(M)$ holds.

Let now $N$ be a submodule of a $\Delta$-semisimple module $M$. Then $P(M)$ is true, which implies that $P(N)$ holds. Therefore $N$ is a $\Delta$-semisimple module. □

In the next subsection we are going to show that every $\Delta$-good module over a RUSQ algebra has a unique maximal $\Delta$-semisimple submodule. First, we check that arbitrary quasihereditary algebras do not possess this property.

**Example 3.4.** Consider the quiver

\[
Q = \begin{array}{c}
0 \\
\delta \\
1 \\
\alpha \\
\beta \\
2
\end{array}
\]

\[
\begin{array}{c}
\varepsilon \\
\gamma \\
\gamma_0
\end{array}
\]

\[
\begin{array}{c}
\delta \\
\gamma \\
\gamma_0
\end{array}
\]

\[
\begin{array}{c}
\varepsilon \\
\gamma \\
\gamma_0
\end{array}
\]

\[
\begin{array}{c}
\delta \\
\gamma \\
\gamma_0
\end{array}
\]

\[
\begin{array}{c}
\varepsilon \\
\gamma \\
\gamma_0
\end{array}
\]
and the bound quiver algebra $B = KQ/I$, where $I$ is the ideal generated by the elements $\varepsilon \beta - \delta \gamma_0$ and $\gamma_0 \gamma_1$. The algebra $B$ is quasihereditary with respect to the labelling poset $0 < 1 < 2 < 3$. The modules

$$\begin{array}{c}
\begin{array}{c}
0, \\
\varepsilon, \\
0
\end{array} \\
\begin{array}{c}
1, \\
1, \\
1
\end{array} \\
\begin{array}{c}
\alpha, \\
\alpha, \\
0
\end{array}
\end{array}$$

are the corresponding standard $B$-modules. The projective cover $P_2$ of the simple module with label 2 has the following structure

$$\begin{array}{c}
\begin{array}{c}
2, \\
3, \\
1
\end{array} \\
\begin{array}{c}
\varepsilon, \\
0, \\
1
\end{array} \\
\begin{array}{c}
\alpha, \\
\alpha, \\
0
\end{array}
\end{array}$$

The modules $\Delta (1) \oplus \Delta (2)$ and $\Delta (3) \oplus \Delta (0)$ are both maximal $\Delta$-semisimple submodules of $P_2$. The quotient of $P_2$ by each of these submodules does not belong to $\mathcal{F} (\Delta)$, i.e. none of these submodules of $P_2$ satisfies property (B).

3.2. The preradical $\delta$ and $\Delta$-semisimple filtrations. Let $(B, \Phi, \subseteq)$ be an arbitrary quasihereditary algebra. As pointed out in the previous subsection, the submodules of a module $M$ in $\mathcal{F} (\Delta)$ which are maximal with respect to property (B) are all isomorphic, but they are not necessarily unique. We have also seen that a module $M$ in $\mathcal{F} (\Delta)$ may have more than one maximal $\Delta$-semisimple submodule with respect to inclusion (Example 3.4). We shall prove that both these maximal submodules are unique and actually coincide when the underlying algebra is a RUSQ algebra. For this, we use the general theory of preradicals introduced in Subsection 2.1.

**Lemma 3.5.** Let $(B, \Phi, \subseteq)$ be a RUSQ algebra. The corresponding set $\Delta$ of standard $B$-modules is a hereditary class in $\text{mod} B$. In particular, $\text{Tr} (\Delta, -)$ is a hereditary preradical in $\text{mod} B$.

**Proof.** Let $N$ be a submodule of a module in add $\Delta$, so $N$ is contained in some $\Delta$-semisimple module $M$. By Corollary 3.3, $N$ is $\Delta$-semisimple, so it is trivially generated by $\Delta$. Hence the set $\Delta$ is hereditary. Lemma 2.11 implies that $\text{Tr} (\Delta, -)$ is a hereditary preradical in $\text{mod} B$. □

**Remark 3.6.** The preradical $\text{Tr} (\Delta, -)$ is not usually hereditary for an arbitrary quasihereditary algebra $B$ (not even if $B$ is right strongly quasihereditary).

From now onwards we shall denote the functor $\text{Tr} (\Delta, -)$ by $\delta$.

**Definition 3.7.** For a RUSQ algebra $(B, \Phi, \subseteq)$, let $\delta$ be the hereditary preradical $\text{Tr} (\Delta, -)$ in $\text{mod} B$.

Next, we give a description of the submodule $\delta (M)$ of a module $M \in \mathcal{F} (\Delta)$.
Let \( (B, \Phi, \subseteq) \) be a RUSQ algebra, and let \( M \in \mathcal{F}(\Delta) \). Then \( \delta(M) \) is the largest \( \Delta \)-semisimple submodule of \( M \). Furthermore, \( M/\delta(M) \) lies in \( \mathcal{F}(\Delta) \). In particular, \( \delta(M) \) is the largest submodule of \( M \) satisfying property \([B]\).

Proof. By the definition of \( \text{Tr}(\Delta, -) \), there is an epic \( f \) from a \( \Delta \)-semisimple module \( M' \) to \( \delta(M) \). Note that both \( \delta(M) \) and \( \text{Ker} f \) are in \( \mathcal{F}(\Delta) \), since this category is closed under submodules. As a consequence, \( f \) must be a split epic. Hence \( \delta(M) \) is \( \Delta \)-semisimple. By the definition of \( \text{Tr}(\Delta, -) \) it is clear that every \( \Delta \)-semisimple submodule of \( M \) must be contained in \( \delta(M) \). This shows that \( \delta(M) \) is the largest \( \Delta \)-semisimple submodule of \( M \in \mathcal{F}(\Delta) \).

To conclude this proof it is enough to show that \( M/\delta(M) \) lies in \( \mathcal{F}(\Delta) \). We start by proving that this holds for the injective modules \( Q_{i,l} \) (recall Proposition \([2,13]\)). Note that \( \Delta(i,1) \subseteq \delta(Q_{i,l}) \), as \( \Delta(i,1) \) is a submodule of \( T(i,1) \). Since \( Q_{i,l} \) has simple socle \( L_{i,l} \), then \( \delta(Q_{i,l}) \) has to be isomorphic to some standard module \( \Delta(i,j) \). But then we must have \( \Delta(i,1) = \delta(Q_{i,l}) \), and consequently \( Q_{i,l}/\delta(Q_{i,l}) = T(i,1)/\Delta(i,1) \) is in \( \mathcal{F}(\Delta) \). Let now \( Q \) be a finite direct sum of injective modules of type \( Q_{i,l} \). The module \( Q/\delta(Q) \) still lies in \( \mathcal{F}(\Delta) \) because preradicals preserve finite direct sums (see part 3 of Lemma \([2,3]\)). Consider now \( M \in \mathcal{F}(\Delta) \). By Proposition \([2,13]\) the injective hull \( q_0 : M \rightarrow Q_0(M) \) of \( M \in \mathcal{F}(\Delta) \) is such that \( Q_0(M) \) is a direct sum of injectives of type \( Q_{i,l} \). By part 3 of Lemma \([2,7]\) \( q_0 \) gives rise to a monic \( M/\delta(M) \rightarrow Q_0(M)/\delta(Q_0(M)) \), and by our previous observation \( Q_0(M)/\delta(Q_0(M)) \) lies in \( \mathcal{F}(\Delta) \). As \( \mathcal{F}(\Delta) \) is closed under submodules, the module \( M/\delta(M) \) belongs to \( \mathcal{F}(\Delta) \). \( \square \)

Example 3.9. Note that for an arbitrary quasihereditary algebra the modules \( \delta(M), M \in \mathcal{F}(\Delta) \), are not usually \( \Delta \)-semisimple (not even \( \Delta \)-good). Indeed, for the algebra in Example \([3,4]\) we have \( \delta(P_2) = \text{Tr}(\Delta, P_2) = \text{Rad} P_2 \), which is not \( \Delta \)-semisimple.

3.2.1. Filtrations arising from preradicals. Our next goal is to define \( \Delta \)-semisimple filtration and \( \Delta \)-semisimple length for modules in \( \mathcal{F}(\Delta) \), over some RUSQ algebra \( B \). For this, some elementary results about preradicals are needed.

Let \( \tau \) and \( \upsilon \) be preradicals (over an arbitrary Artin algebra \( B \)). Write \( \tau \leq \upsilon \) if \( \tau \) is a subfunctor of \( \upsilon \). The functor \( \tau \circ \upsilon \) is a preradical, and \( \tau \circ \upsilon \leq \upsilon \). For \( M \) in \( \text{mod} B \) define \( \tau \circ \upsilon(M) \) as the submodule of \( M \) containing \( \upsilon(M) \), satisfying
\[
\tau(M/\upsilon(M)) = \tau \circ \upsilon(M)/\upsilon(M).
\]
The operator \( \tau \circ \upsilon \) is still a preradical. By construction, \( \upsilon \leq \tau \circ \upsilon \).

By the characterisation of hereditary radicals given in Lemma \([2,7]\) it follows that \( \tau \circ \upsilon \) is hereditary if both \( \tau \) and \( \upsilon \) are hereditary. We also have that \( \tau \circ \upsilon \) is hereditary, whenever \( \tau \) and \( \upsilon \) are both hereditary – the functor \( 1/(\tau \circ \upsilon) \) is naturally isomorphic to \( (1/\tau) \circ (1/\upsilon) \).

Similarly to the composition of preradicals, the operation \( \circ \) is associative. Given a preradical \( \tau \), let \( \tau^0 \) be the identity functor in \( \text{mod} B \) and let \( \tau_0 \) be the zero preradical. For \( m \in \mathbb{Z}_{\geq 0} \), define \( \tau^m := \tau \circ \tau^{m-1} \) and \( \tau_m := \tau \circ \tau_{m-1} \). We summarise the properties of these preradicals.

Lemma 3.10. Let \( \tau \) be a preradical in \( \text{mod} B \).

1. For every \( m \geq 1 \), \( \tau^m \leq \tau^{m-1} \) and \( \tau_{m-1} \leq \tau_m \).
2. For every \( M \) in \( \text{mod} B \) there is \( m \geq 0 \) such that \( \tau^m(M) = \tau^{m+1}(M) \).
3. For every \( M \) in \( \text{mod} B \) there is \( m \geq 0 \) such that \( \tau_m(M) = \tau_{m+1}(M) \).
The preradicals \( \tau_m \) (and \( \tau^m \)), \( m \in \mathbb{Z}_{\geq 0} \), give rise to special filtrations.

**Lemma 3.11** ([2] §1.3.3). Let \( \tau \) be a preradical. Suppose that \( \tau(M) \neq 0 \) for every nonzero \( B \)-module \( M \). Given \( M \) in mod \( B \), there is a unique integer \( l^{(\tau,\bullet)}(M) = n \geq 0 \) such that \( \tau_n(M) = M \), and \( \tau_{m-1}(M) \subset \tau_m(M) \) for every \( m \) satisfying \( 1 \leq m \leq n \). Moreover, for \( m \leq l^{(\tau,\bullet)}(M) \), we have

\[
l^{(\tau,\bullet)}(M/\tau_m(M)) = n - m.
\]

**Lemma 3.12** ([2] §1.3.3). Let \( \tau \) be a hereditary preradical. Then \( \tau_m \) is also a hereditary preradical, and \( \tau_m \circ \tau_{m'} = \tau_{\min(m,m')} \) for every \( m, m' \geq 0 \). Furthermore, if \( \tau(M) \neq 0 \) for every \( M \neq 0 \), the following hold for \( N \) and \( M \) in mod \( B \):

1. if \( N \subset M \) then \( l^{(\tau,\bullet)}(N) \leq l^{(\tau,\bullet)}(M) \);
2. if \( n \leq l^{(\tau,\bullet)}(M) \), then \( \tau_n(M) \) is the largest submodule \( N \) of \( M \) such that \( l^{(\tau,\bullet)}(N) = n \).

3.2.2. \( \Delta \)-semisimple filtrations and \( \Delta \)-semisimple length. Suppose once again that \((B, \Phi, \subset)\) is a RUSQ algebra, and consider the hereditary preradical \( \delta \). Note that \( \delta(M) \neq 0 \) for every nonzero module \( M \) in mod \( B \) as

\[
\text{Soc } M \subset \text{Tr}(\Delta, M) = \delta(M).
\]

In fact, we have \( \text{Soc } M = \text{Soc } \delta(M) \). We may construct the preradicals \( \delta_m \) in mod \( B \) defined recursively in (3.2.1). Then Lemmas 3.10, 3.11 and 3.12 hold for the preradicals \( \delta_m \). In particular, \( \delta_m \) is a hereditary preradical for every \( m \in \mathbb{Z}_{\geq 0} \).

**Lemma 3.13.** Let \((B, \Phi, \subset)\) be a RUSQ algebra. If \( M \) is in \( \mathcal{F}(\Delta) \) then so is \( M/\delta_m(M) \), for any \( m \geq 0 \).

**Proof.** By Proposition 3.8 the claim holds for \( m = 1 \). Suppose \( m \geq 2 \). Then

\[
M/\delta_m(M) \cong (M/\delta(M))/ (\delta_{m-1} \bullet \delta(M)/\delta(M)) = (M/\delta(M))/ (\delta_{m-1}(M/\delta(M))),
\]

so by induction \( M/\delta_m(M) \) belongs to \( \mathcal{F}(\Delta) \).

Given a module \( M \) in \( \mathcal{F}(\Delta) \), we may consider the filtration

(6)

\[
0 \subset \delta(M) \subset \cdots \subset \delta_m(M) = M,
\]

where \( m = l^{(\delta,\bullet)}(M) \) is as defined in Lemma 3.11. The factors of this filtration are \( \Delta \)-semisimple: by Lemma 3.13 and Proposition 3.8 the modules \( \delta_i(M)/\delta_{i-1}(M) = \delta(M/\delta_{i-1}(M)) \) are \( \Delta \)-semisimple. We call (6) the \( \Delta \)-semisimple filtration of \( M \in \mathcal{F}(\Delta) \).

**Definition 3.14.** The \( \Delta \)-semisimple length of a module \( M \) in \( \mathcal{F}(\Delta) \), denoted by \( \Delta,\text{ssl} M \), is the length of the \( \Delta \)-semisimple filtration of \( M \), i.e. it is given by the number \( l^{(\delta,\bullet)}(M) \) (as in Lemma 3.11).

4. \( \Delta \)-semisimple filtrations of modules over the ADR algebra

The ADR algebra of an Artin algebra \( A \), \( R = (R_A, \Lambda, \leq) \), is our prototype of a RUSQ algebra. We now prove some results specific to the \( \Delta \)-semisimple filtrations of \( \Delta \)-good modules over the ADR algebra. Throughout this section the underlying quasifereditary algebra will be \((R, \Lambda, \leq)\), where the poset \((\Lambda, \leq)\) is as defined in [2] and [3]. For the proof of the next results note that the left exact functor \( \text{Hom}_A(G, -) \) is fully faithful since \( G \) is a generator of mod \( A \) (see [1] §8–§10)].
Lemma 4.1. Let $M_1$ and $M_2$ be in $\text{mod } A$, with $M_1 \subseteq M_2$. There is a canonical embedding

$$\text{Hom}_A(G, M_2) / \text{Hom}_A(G, M_1) \hookrightarrow \text{Hom}_A(G, M_2/M_1)$$

and the induced morphisms

$$\text{Hom}_R(P_{i,l}, \text{Hom}_A(G, M_2) / \text{Hom}_A(G, M_1)) \hookrightarrow \text{Hom}_R(P_{i,l}, \text{Hom}_A(G, M_2/M_1)),$$

$$\text{Hom}_R(\text{Hom}_A(G, M_2/M_1), Q_{i,l}) \hookrightarrow \text{Hom}_R(\text{Hom}_A(G, M_2) / \text{Hom}_A(G, M_1), Q_{i,l})$$

are isomorphisms.

Proof. The functor $\text{Hom}_A(G, -)$ is left exact. Thus, it maps the canonical epic $\pi : M_2 \rightarrow M_2/M_1$ to the morphism $\pi_*$, which factors as

$$\text{Hom}_A(G, M_2) \xrightarrow{\pi_*} \text{Hom}_A(G, M_2/M_1) \xrightarrow{\iota} \text{Hom}_A(G, M_2/M_1) / \text{Hom}_A(G, M_1)$$

Consider the monic $\iota_*$ obtained by applying the functor $\text{Hom}_R(P_{i,l}, -)$ to $\iota$. Let $f_*$ be in $\text{Hom}_R(P_{i,l}, \text{Hom}_A(G, M_2/M_1))$. Then $f_* = \text{Hom}_A(G, f)$, for a map $f : P_{i,l} \rightarrow M_2/M_1$ in $\text{mod } A$. Since $P_{i,l}$ is projective, $f = \pi \circ t$ for some $t : P_{i,l} \rightarrow M_2$. So $f_* = \pi_* \circ t_* = \iota \circ \pi \circ t_* = \iota_* (\pi \circ t_*)$, where $t_* = \text{Hom}_A(G, t)$. This shows that $\iota_*$ is surjective, hence it is an isomorphism. The proof that $\iota^*$ is an isomorphism is analogous. \qed

Let $M_1$ and $M_2$ be in $\text{mod } A$, with $M_1 \subseteq M_2$. We shall regard the canonical embedding in Lemma 4.1

$$\text{Hom}_A(G, M_2) / \text{Hom}_A(G, M_1) \hookrightarrow \text{Hom}_A(G, M_2/M_1)$$

as an inclusion of $R$-modules. According to Lemma 3.6 in [3], $\text{Hom}_A(G, M)$ lies in $\mathcal{F}(\Delta)$ for every $M$ in $\text{mod } A$. Since the category $\mathcal{F}(\Delta)$ is closed under submodules then both $\text{Hom}_A(G, M_2) / \text{Hom}_A(G, M_1)$ and $\text{Hom}_A(G, M_2/M_1)$ are $\Delta$-good modules. Lemma 4.1 is hinting at a close relation between the $\Delta$-filtrations of the modules $\text{Hom}_A(G, M_2) / \text{Hom}_A(G, M_1)$ and $\text{Hom}_A(G, M_2/M_1)$. We spell out this idea below.

Corollary 4.2. Let $M_1$ and $M_2$ be in $\text{mod } A$, with $M_1 \subseteq M_2$. Write $M = \text{Hom}_A(G, M_2/M_1)$ and $M' = \text{Hom}_A(G, M_2) / \text{Hom}_A(G, M_1)$. All the composition factors of $M$ of type $L_{i,l}$ appear as composition factors of its submodule $M'$. In particular, $M$ and $M'$ have the same number of composition factors of type $L_{i,l}$. Moreover, $M$ and $M'$ lie in $\mathcal{F}(\Delta)$, $\text{Soc } M = \text{Soc } M'$, and the modules $M$ and $M'$ are filtered by the same number of standard modules.

Proof. By Lemma 4.1 all the composition factors of $M$ isomorphic to $L_{i,l}$ appear as composition factors of its submodule $M'$. As $M$ lies in $\mathcal{F}(\Delta)$ then, by Proposition 2.13 $\text{Soc } M$ is a direct sum of simples of type $L_{i,l}$. Thus $\text{Soc } M' = \text{Soc } M$. Part
Lemma 4.3. Let \((7)\). Consider now an arbitrary morphism \(f\). Note that \(f\) is semisimple filtration of \(\text{Hom}_A(G, M)\) in \(\mathcal{F}(\Delta)\).

\[\text{Hom}_A(G, \text{Soc}_j M) = \text{Tr} \left( \bigoplus_{(k,l), l \leq j} P_{k,l}, \text{Hom}_A(G, M) \right).\]

Moreover, if \(\text{Soc}_j M/\text{Soc}_{j-1} M = \bigoplus_{\theta \in \Theta} L_{x_\theta}\), then

\[\text{Hom}_A(G, \text{Soc}_j M)/\text{Hom}_A(G, \text{Soc}_{j-1} M) = \bigoplus_{\theta \in \Theta} \Delta(x_\theta, j).\]

Proof. By [1, Proposition 10.2] (see also [8, Lemma 3.3]), \(\text{Hom}_A(G, \text{Soc}_j M)\) is generated by projectives \(P_{k,l}\) satisfying \(l \leq j\). This proves one of the inclusions in (7). Consider now an arbitrary morphism \(f_* : P_{k,l} \to \text{Hom}_A(G, M)\), with \(l \leq j\). Note that \(f_* = \text{Hom}_A(G, f)\) for a certain map \(f : P_k/\text{Rad}^i P_k \to M\). Clearly, \(\text{Im} f \subseteq \text{Soc}_j M\). But then

\[\text{Im} f_* \subseteq \text{Hom}_A(G, \text{Im} f) \subseteq \text{Hom}_A(G, \text{Soc}_j M).\]

As \(f_*\) was chosen arbitrarily, the other inclusion follows. This proves identity (7).

To prove the second claim in the statement of the lemma, set

\[M' := \text{Hom}_A(G, \text{Soc}_j M)/\text{Hom}_A(G, \text{Soc}_{j-1} M),\]

and assume that \(\text{Soc}_j M/\text{Soc}_{j-1} M\) is isomorphic to \(\bigoplus_{\theta \in \Theta} L_{x_\theta}\). Recall that \(\Delta(i, 1)\) is isomorphic to \(\text{Hom}_A(G, L_i)\) (see Theorem 2.14). Lemma 4.1 and Corollary 4.2 imply that \(M'\) is contained in

\[\text{Hom}_A(G, \text{Soc}_j M/\text{Soc}_{j-1} M) = \bigoplus_{\theta \in \Theta} \text{Hom}_A(G, L_{x_\theta}) = \bigoplus_{\theta \in \Theta} \Delta(x_\theta, 1)\]

and that these modules have the same socle. By Corollary 3.3, \(M'\) is \(\Delta\)-semisimple. Finally, by the identity (7) (applied to \(j\) and \(j - 1\)), the module \(M'\) must be generated by projectives of type \(P_{i,j}\). This proves the second assertion of the lemma.

Lemma 4.3 and Theorem 4.4 are very useful to compute examples. For the proof of the next result, recall the characterisation of the preradical \(\delta\) in Subsection 3.2, namely Proposition 3.8 and Lemma 3.13.

Theorem 4.4. Let \(M\) be in mod \(A\). The socle series of \(M\) induces the \(\Delta\)-semisimple filtration of \(\text{Hom}_A(G, M)\). Formally,

\[\delta_m(\text{Hom}_A(G, M)) = \text{Hom}_A(G, \text{Soc}_m M),\]

for all \(m \in \mathbb{Z}_{\geq 0}\). In particular, \(\Delta, \text{ssl}(\text{Hom}_A(G, M)) = \text{LL}(M)\).

Proof. For \(m\) satisfying \(1 \leq m \leq \text{LL}(M)\) we prove the claim by induction on \(m\), starting with \(m = 1\). Note that \(\text{Hom}_A(G, \text{Soc} M)\) is a direct sum of standard modules of type \(\text{Hom}_A(G, L_i) = \Delta(i, 1)\), so \(\text{Hom}_A(G, \text{Soc} M) \subseteq \delta(\text{Hom}_A(G, M))\).

Since the functor \(\text{Hom}_A(G, -)\) preserves injective hulls (see [8, Lemma 4.4]), the modules \(\text{Hom}_A(G, \text{Soc} M)\) and \(\text{Hom}_A(G, M)\) have the same socle. Hence the previous inclusion must be an equality.
Suppose now that $2 \leq m \leq \text{LL}(M)$, and set
\[ Z_1 := \text{Hom}_A (G, M) / \text{Hom}_A (G, \text{Soc} m^{-1} M) \]
\[ Z_2 := \text{Hom}_A (G, \text{Soc} m M) / \text{Hom}_A (G, \text{Soc} m^{-1} M). \]
Since $\text{Hom}_A (G, -)$ preserves injective hulls, the modules $\text{Hom}_A (G, M / \text{Soc} m^{-1} M)$ and $\text{Hom}_A (G, \text{Soc} m M / \text{Soc} m^{-1} M)$ have the same socle. But then, by Corollary 4.2, $Z_1$ and $Z_2$ have the same socle. Moreover, $Z_1$ belongs to $\mathcal{F} (\Delta)$. By Lemma 4.3, $Z_2$ must be contained in $\delta (Z_1)$. So both $\delta (Z_1)$ and $Z_2$ are $\Delta$-semisimple modules with the same socle. By Corollary 4.2, $Z_1 / Z_2$ must be in $\mathcal{F} (\Delta)$. Since $\mathcal{F} (\Delta)$ is closed under submodules, then $\delta (Z_1)$ is in $\mathcal{F} (\Delta)$. We must have $\delta (Z_1) / Z_2 = 0$, otherwise this factor module would have some composition factor of type $L_{	ext{LL}}$. By induction, we may suppose that $\delta_{m-1} (\text{Hom}_A (G, M)) = \text{Hom}_A (G, \text{Soc} m^{-1} M)$. Then, the identity $Z_2 = \delta (Z_1)$ translates to
\[ \text{Hom}_A (G, \text{Soc} m M) / \delta_{m-1} (\text{Hom}_A (G, M)) = \delta_m (\text{Hom}_A (G, M)) / \delta_{m-1} (\text{Hom}_A (G, M)). \]
This implies that $\delta_m (\text{Hom}_A (G, M)) = \text{Hom}_A (G, \text{Soc} m M), 1 \leq m \leq \text{LL}(M)$. The same identity holds trivially for $m = 0$ and for $m \geq \text{LL}(M)$. \hfill \Box

5. Projective covers of modules over the ADR algebra

We would like to determine the projective covers of modules over the ADR algebra $R_A$ of $A$. For a module $M$ in mod $A$, the projective cover $p_*$ of $\text{Hom}_A (G, M)$ in mod $R_A$ is the image of an epic $p_*$ with domain in add $G$, through the functor $\text{Hom}_A (G, -)$. The morphism $p_*$ is a special kind of map: it is the right minimal $G$-approximation of $M$ in mod $A$.

The problem of finding approximations is hard in general. However, as we shall see in Theorem 5.1, it is very easy to compute right add $G$-approximations of rigid modules. Recall that a module is rigid if its radical series coincides with its socle series.

Theorem 5.1 (or rather consequences of this result – Corollary 5.2 and Proposition 5.3) will be very useful when dealing with examples. In Subsection 5.2, we will use Corollary 5.2 and Proposition 5.3 to give a counterexample to a claim by Auslander, Platzeck and Todorov (2) about the projective resolutions of modules over the ADR algebra.

5.1. Approximations of rigid modules. Let $\mathcal{X}$ be a class of $A$-modules. We recall the definition of right $\mathcal{X}$-approximation and of right minimal morphism. A morphism $f : X \rightarrow M$ in mod $A$, with $X$ in $\mathcal{X}$, is said to be a right $\mathcal{X}$-approximation of $M$ if $\text{Hom}_A (X', f)$ is an epic for all $X'$ in $\mathcal{X}$. A map $f : M \rightarrow N$ in mod $A$ is called a right minimal morphism if every endomorphism $g : M \rightarrow M$ satisfying $f = f \circ g$ is an automorphism.

The right add $G$-approximations of a module $M$ in mod $A$ are in bijection with epics in mod $R_A$,
\[ \text{Hom}_A (G, X) \longrightarrow \text{Hom}_A (G, M), \]
where $X \in \text{add } G$. This bijection restricts to a one-to-one correspondence between right minimal add $G$-approximations in mod $A$ and projective covers in mod $R_A$. Since $G$ is a generator, the functor $\text{Hom}_A (G, -)$ is particularly well behaved: it
is fully faithful and it is such that the projective cover of a module $M$ in $\mod A$ factors through its add $G$-approximation. The latter statement implies that every right add $G$-approximation is an epimorphism.

**Theorem 5.1.** Let $M$ be a rigid module in $\mod A$ such that $\text{LL}(M) = m$. The projective cover of $M$ in $\mod(A/\text{Rad}^m A)$ is a right minimal add $G$-approximation of $M$.

**Proof.** Let $M$ be a rigid module with Loewy length $m$. Consider the projective cover of $M$ as an $(A/\text{Rad}^m A)$-module,

$$\varepsilon : P_0(M) \longrightarrow M.$$ 

We want to prove that $\varepsilon$ is a right minimal add $G$-approximation. By definition, $\varepsilon$ is a right minimal morphism, so it is enough to prove that every map $f : P_i/\text{Rad}^j P_i \longrightarrow M$, with $(i, j) \in \Lambda$, factors through $\varepsilon$. Note that this holds for $j \geq m$, as $\varepsilon$ is an epic in $\mod(A/\text{Rad}^j A)$ and $P_i/\text{Rad}^j P_i$ is a projective $(A/\text{Rad}^j A)$-module. So suppose that $j < m$. Then

$$\text{Im} f \subseteq \text{Soc}_j M = \text{Rad}^{m-j} M,$$

using that $M$ is rigid. Observe that both $\text{Rad}^{m-j} M$ and $\text{Rad}^{m-j}(P_0(M))$ are annihilated by $\text{Rad}^j A$, i.e. they lie in $\mod(A/\text{Rad}^j A)$. Now note that the functor $\text{Rad}^{m-j}(-)$ preserves epics. This can be seen directly, or can be deduced by looking at Example 2.6 and Remark 2.8, recalling that the composition of cohereditary preradicals is still a cohereditary preradical. Therefore we have the diagram

$$
\begin{array}{ccc}
P_i/\text{Rad}^j P_i & \xrightarrow{f} & M \\
\downarrow & & \downarrow \\
\text{Rad}^{m-j}(P_0(M)) & \xrightarrow{\text{Rad}^{m-j}M} & \text{Rad}^{m-j} M \\
& \exists t & \\
& \text{Im} f & \left\uparrow_{\iota_M} \\
& & \\
& & \\
\end{array}
$$

where $t$ exists because $P_i/\text{Rad}^j P_i$ is a projective in $\mod(A/\text{Rad}^j A)$. Thus

$$f = \iota_M \circ (\text{Rad}^{m-j} \varepsilon) \circ t = \varepsilon \circ \iota_{P_0(M)} \circ t,$$

where $\iota_{P_0(M)}$ denotes the inclusion of $\text{Rad}^{m-j}(P_0(M))$ in $P_0(M)$. \hfill $\Box$

As an immediate consequence of Theorem 5.1 we get the following result.

**Corollary 5.2.** Let $M$ be a rigid module in $\mod A$ with $\text{LL}(M) = m$. Suppose that $\varepsilon$ is the projective cover of $M$ in $\mod(A/\text{Rad}^m A)$. Then $\text{Hom}_A(G, \varepsilon)$ is the projective cover of $\text{Hom}_A(G, M)$ in $\mod R_A$.

The simple modules over the ADR algebra $R_A$ are “linked to each other” in a neat way. When all projective indecomposable modules are rigid then the ‘glueing’ of the simple modules (and of the standard modules) is even nicer.

**Proposition 5.3.** Let $(i, j)$ and $(k, l)$ be in $\Lambda$. Then $\text{Ext}^1_{R_A}(L_{i,j}, L_{k,l}) \neq 0$ implies that either $(k, l) = (i, j + 1)$ or $l \leq j - 1$. If the $A$-module $P_i/\text{Rad}^j P_i$ is rigid then $\text{Ext}^1_{R_A}(L_{i,j}, L_{k,l}) \neq 0$ implies that either $(k, l) = (i, j + 1)$ or $l = j - 1$. 

---

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In particular, the latter statement holds when all the projective indecomposable $A$-modules are rigid.

**Proof.** Observe that $\text{Ext}^1_{K^A}(L_{i,j}, L_{k,l}) \neq 0$ if and only if the simple module $L_{k,l}$ is a summand of $\text{Rad} \, P_{i,j} / \text{Rad}^2 \, P_{i,j}$. The short exact sequence (4) in the statement of Theorem 2.14 gives rise to the exact sequence

$$
0 \longrightarrow \text{Hom}_A(G, \text{Rad} \, P_i / \text{Rad}^2 \, P_i) \longrightarrow \text{Rad} \, P_{i,j} \longrightarrow \text{Rad} \, \Delta \, (i,j) \longrightarrow 0
$$

where $\text{Rad} \, \Delta \, (i,j) = \Delta \, (i,j + 1)$. If $L_{k,l}$ is a summand of the top of $\text{Rad} \, P_{i,j}$, then either $(k,l) = (i,j + 1)$ or $L_{k,l}$ is a summand of the top of $\text{Hom}_A(G, \text{Rad} \, P_i / \text{Rad}^2 \, P_i)$. In the latter case, we must have $l \leq j - 1$ by [1, Proposition 10.2] (see also [8, Lemma 3.3]).

If $P_i / \text{Rad}^2 \, P_i$ is rigid, then $\text{Rad} \, P_i / \text{Rad}^2 \, P_i$ is also rigid. In this case, Corollary 5.2 implies that the summands of the top of $\text{Hom}_A(G, \text{Rad} \, P_i / \text{Rad}^2 \, P_i)$ are of type $L_{k,j - 1}$.

The next example shows that the rigidity condition in the statement of Theorem 5.1 cannot be omitted.

**Example 5.4.** Consider the quiver

$$
Q = \begin{array}{ccc}
1 & \alpha & 2 \\
\beta & 3 & \gamma \\
\varepsilon & 4 & \eta \\
5 & 6
\end{array}
$$

and the path algebra $A = KQ$. Let $M$ be the $A$-module $P_1 / L_6$, that is,

$$
M := \begin{array}{ccc}
1 & \text{L} & \text{R} \\
2 & 3 & 4 \\
5
\end{array}
$$

Observe that $\text{LL} \, (M) = \text{LL} \, (A) = 3$, and that $M$ is not a rigid module.

Consider the epic $\pi : P_1 \longrightarrow M$ and note that the simple module $L_4$ can be embedded in $M$. It is not difficult to check that the epic

$$
[\pi \ 1_{L_4}] : P_1 \oplus L_4 \longrightarrow M
$$

is a right minimal add $G$-approximation of $M$. This map is not a projective cover of $M$, so the rigidity condition in the statement of Theorem 5.1 is necessary.

Using the approximation (8), one easily sees that the $R_A$-module $\text{Hom}_A(G, M)$ can be represented as

$$(1,3) \quad (4,1) \quad (5,1)
$$

$$(2,1) \quad (3,2) \quad (4,2)
$$
5.2. An application. Motivated in part by the theory of quasihereditary algebras, Auslander, Platzeck and Todorov studied in [2] the homological properties of idempotent ideals. In this paper the authors defined a new class of algebras – the Artin algebras satisfying the descending Loewy length condition – and proved, in Theorem 7.3, [2], that every such algebra is quasihereditary.

Definition 5.5 ([2, §7]). An Artin algebra $B$ satisfies the descending Loewy length condition (DLL condition, for short) if for every $M$ in mod $B$, a minimal projective resolution
\[ \cdots \to P_i(M) \to \cdots \to P_0(M) \to \varepsilon M \to 0 \]
satisfies $\text{LL}(P_{i+1}(M)) < \text{LL}(P_i(M))$, for all $i \geq 1$ such that $P_i(M) \neq 0$.

In [2] the authors claim that the Artin algebras of global dimension 2, the ADR algebras $R_A$, and the $l$-hereditary algebras (introduced in [20]) all satisfy the DLL condition. The main purpose of Theorem 7.3 in [2] was thus to give a unified proof of results in [11], [9] and [5], already established in the literature.

It is not difficult to check that Artin algebras of global dimension 2 and that $l$-hereditary algebras do satisfy the DLL condition. Unfortunately, it is not true that the ADR algebra $R_A$ satisfies the DLL condition for every choice of $A$. As we shall see, the Loewy length of the projectives in a projective resolution in mod $R_A$ may increase by an arbitrarily large number.

In order to see this, consider the following example: define $A := KQ/I$, where $K$ is a field, $Q$ is the quiver
\[ Q = \begin{array}{c}
\varepsilon \\
1
\end{array} \xrightarrow{\alpha_1} \begin{array}{c}
2 \\
\beta_1
\end{array} \xrightarrow{\alpha_2} \begin{array}{c}
3 \\
\beta_2
\end{array} \xrightarrow{\varepsilon} 1 \]
and $I$ is the admissible ideal
\[ I = \langle \alpha_2 \alpha_1, \beta_1 \beta_2, \beta_2 \alpha_2 - \alpha_1 \beta_1, \varepsilon \beta_1, \alpha_1 \varepsilon, \varepsilon^n \rangle, \]
with $n > 1$ fixed.

We may represent the projective indecomposable $A$-modules as
\[
\begin{array}{c}
1 \\
\vdots \\
1 \\
1 \\
1
\end{array}
\begin{array}{c}
2 \\
2 \\
2
\end{array}
\begin{array}{c}
3 \\
3 \\
3
\end{array}
\begin{array}{c}
1 \\
1 \\
1
\end{array}
\begin{array}{c}
1 \\
1 \\
1
\end{array}
\begin{array}{c}
2 \\
2 \\
2
\end{array}
\begin{array}{c}
3 \\
3 \\
3
\end{array}
\]

Note that the $A$-module $L_3$ is in the socle of $P_3$. Thus, using the labelling in [2], the $R_A$-module $P_{3,3}$ contains a copy of $\Delta(3,1)$. The module $\Delta(3,1)$ has socle $L_{3,3}$, so we may consider the corresponding quotient module $M := P_{3,3}/L_{3,3}$.

Proposition 5.6. Let $A$ be the algebra introduced previously and consider the corresponding ADR algebra $R_A$. Let $M$ be the $R_A$-module defined above. The DLL condition fails for the $R_A$-module $M$ when $n \geq 5$. Indeed, we have $\text{LL}(P_1(M)) \leq 6$ and $\text{LL}(P_2(M)) \geq 1 + n$, so $\text{LL}(P_2(M)) \geq \text{LL}(P_1(M))$ for $n \geq 5$. 

Proof. Using that \(\LL(P_3) = 3\), together with Theorem 2.14, we conclude that \(\text{Rad} P_{3,3} = \text{Hom}_A (G, \text{Rad} P_3)\). Since \(\text{Rad} P_3\) is rigid, Corollary 5.2 implies that the minimal projective presentation of \(L_{3,3}\) is of the form

\[
P_{2,2} \longrightarrow P_{3,3} \longrightarrow L_{3,3} \longrightarrow 0.
\]

We claim that \(\LL(P_{3,3}) \leq 6\) and \(\LL(P_{2,2}) \geq 1 + n \geq 5\). Note that this will imply the statement in the proposition, as \(P_{i+1} (M) = P_i (L_{3,3})\).

We start by showing that \(\LL(P_{2,2}) \geq 1 + n\). To see this, note that the \(A\)-module \(L_1\) is in the socle of \(P_2 / \text{Rad}^2 P_2\). Thus, the \(R_A\)-module \(P_{2,2}\) contains a copy of \(\Delta (1, 1)\). Note that \(\LL(\Delta (1, 1)) = n\) (see Proposition 2.13), so \(\LL(P_{2,2}) \geq 1 + n\) (actually this is an equality).

Using that \(P_{3,3} = \text{Hom}_A (G, P_3)\), we deduce through a routine computation that \(P_{3,3}\) has dimension 6. Consequently, \(P_{3,3}\) has Jordan–Hölder length 6 and \(\LL(P_{3,3}) \leq 6\).

We have just shown that the module \(P_{3,3}\) has Loewy length at most 6. One can actually prove that \(\LL(P_{3,3}) = 5\) and, as a consequence, refine the statement of Proposition 5.6 for \(n \geq 4\).

In order to emphasise the usefulness of the results in Subsection 5.1, we compute the exact Loewy length of \(P_{3,3}\).

**Lemma 5.7.** The module \(P_{3,3}\) has Loewy length equal to 5.

**Proof.** Observe that

\[
\LL(P_{3,3}) = 1 + \LL(\text{Rad} P_{3,3}) = 1 + \LL(\text{Hom}_A (G, \text{Rad} P_3)).
\]

Define \(N_1 := \text{Hom}_A (G, \text{Rad} P_3)\). We claim that \(\LL(N_1) = 4\).

By Lemma 4.3 and Theorem 4.4, the module \(N_1\) has a \(\Delta\)-semisimple filtration

\[
0 \subset \text{Hom}_A (G, L_3) \subset N_1,
\]

with factors \(\Delta (3, 1)\) and \(\Delta (2, 2)\). In particular, \(N_1\) has socle \(L_{3,3}\). Consider now the pullback diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \Delta (3, 1) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & N_1
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
Note that $P_2$ is rigid. Using Proposition 5.3 (and the structure of $\Delta(3, 1)$), we conclude that $\text{Rad} N_2$ must have top $L_{3,2}$. Thus, $N_2$ has the following structure:

$$
N_2 = \begin{pmatrix}
(2, 3) \\
\uparrow \\
(3, 2) \\
\uparrow \\
(3, 3)
\end{pmatrix}
$$

Recall that Lemma 2.7 applies to $\text{Soc} (\cdot)$ and $\text{Soc} (\cdot)^2$. The image of the monic $\Delta(3, 1) \to N_1$ through the functor $1/\text{Soc}(\cdot)$ gives rise to the inclusion $L_{3,2} \subseteq \text{Soc}_2 N_1/\text{Soc}_1 N_2$. By applying $1/\text{Soc}(\cdot)$ to the monics $\Delta(3, 1) \to N_1$ and $N_2 \to N_1$ we deduce that

$$
L_{3,1} \oplus L_{2,3} \subseteq \text{Soc}_3 N_1/\text{Soc}_2 N_1.
$$

Now observe that $\text{Rad} P_3$ is a rigid module. Corollary 5.2 implies that $\text{Top} N_1 = L_{2,2}$. Since $N_1$ has exactly 5 composition factors (and we have looked at them all), we conclude that $\text{LL}(N_1) = 4$. This proves the result. □

Remark 5.8. In [19], Lin and Xi extended Dlab and Ringel’s result in [9] to endomorphism algebras of semilocal modules. The authors noticed that this class of algebras (which contains the ADR algebra) does not generally satisfy the DLL condition (see Example 3 in [19]).

Remark 5.9. Although the DLL condition does not hold for the ADR algebra $R_A$ in general, $R_A$ satisfies a property similar to the DLL condition. The following was implicitly proved in [4], within the proof of Proposition 10.2.

Let $M$ be in $\text{mod} A$ with $\text{LL}(M) = m$, and let $\varepsilon : X \to M$ be the right minimal add $G_A/\text{Rad} m_A$-approximation of $M$. Then $\varepsilon$ is the right minimal add $G$-approximation of $M$ and $\text{LL}(\text{Ker} \varepsilon) < m = \text{LL}(X)$.

As a consequence, the projective resolutions in $\text{mod} R_A$ come from exact sequences in $\text{mod} A$ whose Loewy length decreases strictly. To be precise, for every $N$ in $\text{mod} R_A$ there is an exact sequence of $A$-modules

$$
0 \to X_i \to \cdots \to X_2 \to X_1 \to X_0,
$$

with $X_i$ in add $G$ satisfying $\text{LL}(X_{i+1}) < \text{LL}(X_i)$ for all $i \geq 1$, such that

$$
0 \to \text{Hom}_A (G, X_i) \to \cdots \to \text{Hom}_A (G, X_0) \xrightarrow{\varepsilon} N \to 0
$$

is a minimal projective resolution for $N$ (see §3.3.2 in [7] for details).

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Institute of Algebra and Number Theory, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

E-mail address: tconde@mathematik.uni-stuttgart.de