ON THE SMOOTH DEPENDENCE OF SRB MEASURES FOR PARTIALLY HYPERBOLIC SYSTEMS

ZHIYUAN ZHANG

ABSTRACT. In this paper, we study the differentiability of SRB measures for partially hyperbolic systems. We show that for any $s \geq 1$, for any integer $\ell \geq 2$, any sufficiently large $r$, any $\varphi \in C^s(T, \mathbb{R})$ such that the map $f : T^2 \rightarrow T^2, f(x, y) = (\ell x, y + \varphi(x))$ is $C^s$-stably ergodic, there exists an open neighbourhood of $f$ in $C^s(T^2, T^2)$ such that any map in this neighbourhood has a unique SRB measure with $C^{s-1}$ density, which depends on the dynamics in a $C^s$ fashion.

We also construct a $C^\infty$ mostly contracting partially hyperbolic diffeomorphism $f : T^3 \rightarrow T^3$ such that all $f'$ in a $C^2$ open neighbourhood of $f$ possess a unique SRB measure $\mu_f$ and the map $f' \mapsto \mu_f$ is strictly H"older at $f$, in particular, non-differentiable. This gives a partial answer to Dolgopyat's Question 13.3 in [12].

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1. INTRODUCTION

There is a lot of interest in understanding the ergodic aspect of partially hyperbolic systems. For conservative dynamics, one of the fundamental questions is proving ergodicity. In this direction, we have stable ergodicity conjecture which attempts to describe the generic picture of volume preserving partially hyperbolic systems. For non-conservative dynamics, one tries to describe the dynamics through studying distinguished invariant measures. A prominent role is played by SRB measures.

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DEFINITION 1. For any $C^1$ diffeomorphism $f : X \to X$ on a compact Riemannian manifold $X$, a probability measure $\mu$ on $X$ is called a SRB measure for $f$ if there exists a subset $Y(\mu) \subset X$ of positive Lebesgue measure such that for any $x \in Y(\mu)$, any continuous function $\phi$ on $X$, $\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x))$ converges to $\int \phi \, d\mu$ as $n$ tends to infinity.

A satisfactory understanding of SRB measures for generic dynamics is currently lacking, despite of having some deep results in several models, see [2, 8, 12, 21, 24] just to list a few.

For partially hyperbolic systems, the existence of SRB measures is proved for several cases: 1. mostly expanding dynamics in [2]; 2. mostly contracting dynamics in [9, 12]; 3. generically for partially hyperbolic surface endomorphisms in [24]. Known uniqueness result of SRB measures, for example in [12, 21], usually assume some form of transitivity. An even more refine question is the differentiability of SRB measures. In [12], it is shown that for partially hyperbolic, dynamically coherent, u-convergent mostly contracting $f$ on a three-dimensional manifold, there is a unique SRB measure $\nu_f$. If in addition that $f$ is also stably dynamically coherent, then $f$ is stably mostly contracting, and the SRB measure is known to exhibit Hölder dependence on the dynamics. In [12] Question 13.3, Dolgopyat asked whether or not for mostly contracting dynamics $f$, the map $f \mapsto \nu_f$ is actually smooth? We refer the readers to [11, 15] for recent advances in the study of mostly contracting dynamics.

The question of the differentiability of SRB measures had been previously studied by several authors. It has its roots in statistical physics, and has applications in averaging theory and the removability of zero Lyapunov exponents. The differentiability of SRB measures were previously known for Axiom A diffeomorphisms by [22]. For a class of rapidly mixing, partially hyperbolic systems with isometric center dynamics, the differentiability is proved by Dolgopyat in [13]. On the other hand, to the best of our knowledge, the non-differentiability of SRB measures (when the existence and uniqueness is proved) is unknown for partially hyperbolic systems, despite of having some speculations (see Problem 4 in [10]). In fact, the breakdown of the differentiability is poorly understood for multidimensional dynamics in general. For one-dimensional dynamics, Whitney-Hölder dependence is proved for a family of smooth unimodal maps in [1], with matching upper and lower bounds for the Hölder exponents. For more results on the nondifferentiability of SRB measures for one-dimensional dynamics, we refer the reader to the references in [3]. We mention that in [3], the study of the breakdown of the differentiability of SRB measures for higher dimensional dynamics was proposed as a future research direction.

One of the purpose of this paper is to prove the existence, uniqueness and differentiability of SRB measures for perturbations of a class of area-preserving endomorphisms which are special cases of those studied in [14].
We mention a recent work [16] on a similar class of systems. We note that in contrast to [12, 21], our method does not directly use any form of transitivity for the map in question. On the other hand, we give a method of constructing partially hyperbolic diffeomorphisms and endomorphisms at which the set of uGibbs states (see Definition 3 and the footnote) is not differentiable. We can also require our diffeomorphism to be mostly contracting satisfying the conditions in Theorem II [12], which is known to imply the uniqueness of SRB measure/ uGibbs state. This gives a partial answer to Question 13.3 in [12]: we have an example at which linear response breaks down, but we know no non-trivial example of mostly contracting system where linear response holds. Moreover, by Theorem I in [12], the mostly contracting diffeomorphism we construct is exponentially mixing with respect to the unique SRB measure, for Hölder observables. On the other hand, we mention that linear response can appear for slowly mixing systems, see [6].

2. Main results

Definition 2. Let $M$ be a compact Riemannian manifold. Given integers $r \geq s \geq 1$, and an open set $V \subset C^r(M, M)$. We say that $\{f_t\}_{t \in (-1, 1)}$ is a $C^s$ family in $V$ through $f_0$, if $f_t \in V$ for any $t \in (-1, 1)$, and

$$\|\{f_t\}_{t \in (-1, 1)}\|_{s,r} := \sup_{0 \leq i \leq s, 0 \leq j \leq r, (t,x) \in I \times M} \|\partial^i_0 \partial^j T f_t(x)\| < \infty$$

Given any integers $r \geq r' \geq 2$ and an open set $U \subset C^r(M, M)$. Assume that for each $f \in U$ there exists a unique SRB measure $\mu_f$. Then we say that $f \mapsto \mu_f$ is $C^\ell$ restricted to $U$, if for any $C^\ell$ family $\{f_t\}_{t \in (-1, 1)}$ in $U$ through $f$, for any $\phi \in C^r(M)$, the map $t \mapsto \int \phi d\mu_{f_t}$ is $C^{\ell'}$ at $t = 0$.

We will prove the existence, uniqueness and differentiability of SRB measures for endomorphisms close to a class of skew-products which we now define.

For any integers $r \geq 2$, $\ell \geq 2$, any $\phi \in C^r(T, \mathbb{R})$, we define a $C^\ell$ map $f : T^2 \to T^2$ by $f(x, y) = (\ell x, y + \phi(x))$, $\forall (x, y) \in T^2$. We denote by $U_{r, \ell}^{rot}$ the set of $C^\ell$ maps defined as above for all $\phi \in C^r(T, \mathbb{R})$. We say that $f$ is $C^\ell$–stably ergodic in $U_{r, \ell}^{rot}$ if all $f' \in U_{r, \ell}^{rot}$ in a $C^\ell$ open neighbourhood of $f$ are ergodic.

Theorem 1. For each $r \geq 20$, $1 \leq r' \leq \frac{r}{2} - 9$, $\ell \geq 2$, for any $f_{rot} \in U_{r, \ell}^{rot}$ that is $C^\ell$–stably ergodic in $U_{r, \ell}^{rot}$, there is a $C^r$ open neighbourhood of $f_{rot}$ in $C^r(T^2, T^2)$, denoted by $U$, such that the following is true. Any $f \in U$ admits a unique SRB measure $\mu_f$, having $C^{r' - 1}$ density, and $f \mapsto \mu_f$ is $C^\ell$ restricted to $U$.

By Theorem 3.4 in [14], we know that the set of maps in $U_{r, \ell}^{rot}$ that is $C^\ell$–stably ergodic in $U_{r, \ell}^{rot}$ form a $C^\ell$ open and dense subset of $U_{r, \ell}^{rot}$. It is obvious that our theorem does not extend to nonergodic $f_{rot}$, so in this
aspect our theorem is optimal. By Theorem 3.3 in [14], for maps in \( U_{c,r}^{rot} \), being \( C^r \)-stably ergodic in \( U_{c,r}^{rot} \) is equivalent to being infinitesimally non-integrable, defined in [14].

Our method for proving Theorem 1 is based on the work of Tsujii in his study of decay estimates. Our new input emphasis on using higher regularity and the weak perturbation theory of transfer operators in [17, 19]. We believe our method for proving the uniqueness of SRB would be of independent interest.

Our next result is on the nondifferentiability of SRB measures. As we mentioned above, the existence of SRB measure in general is already difficult. So in order to state our theorem in a more general context, we recall the following more general notion.

**Definition 3.** Let \( f : X \to X \) be a \( C^2 \) partially hyperbolic system on a compact Riemannian manifold \( X \). We denote by \( u\text{Gibbs}(f) \) the set of \( f \)-invariant Borel probability measure \( \mu \in \mathcal{M}(X) \) such that \( \mu \) has absolutely continuous conditional measures on unstable manifolds.

We will establish examples of mostly contracting partially hyperbolic systems stably having a unique SRB measure, while the SRB measures depend on the dynamics in a strictly Hölder fashion. We can even make the Hölder exponent to be arbitrarily small.

**Theorem 2.** For any \( r = 2, 3, \cdots, \infty \), for any \( \theta \in (0, 1) \), there is a \( C^r \) family \( \{f_t\}_{t \in (-1, 1)} \) in the space of \( C^r \) partially hyperbolic diffeomorphisms (resp. endomorphisms) through \( f \), and a \( C^r \) function \( \phi : X \to \mathbb{R} \) such that for any \( \{\mu_t \in u\text{Gibbs}(f_t)\}_{t \in (-1, 1)} \), the function \( t \mapsto \int \phi \mu_t \) is not \( \theta \)-Hölder at \( t = 0 \). Moreover we can choose \( f \) to satisfy Theorem II in [12], that is, \( f \) can be a stably dynamically coherent, u-convergent, mostly contracting map on \( T^3 \).

The notion u-convergent in Theorem 2 is defined in [12] for 3D partially hyperbolic systems \( f \) as follows. We say \( f \) is u-convergent if for any \( \varepsilon > 0 \), there exists an integer \( n > 0 \) such that for any two unstable manifolds of length between 1 and 2, denoted by \( V_1, V_2 \), there exists \( x_j \in V_j, j = 1, 2 \) such that \( d(f^n(x_1), f^n(x_2)) < \varepsilon \).

Our Theorem 2 give an example to Dolgopyat’s Question 13.3 in [12]. An interesting aspect of our construction is that this nondifferentiability comes with some form of stability. See Further Aspect 2.

**Further Aspect.** 1. We will later see that we can choose \( f \) in Theorem 2 so that \( \inf_{f' \in U_{c,r}^{rot}} d_{C^0}(f, f') \) can be made arbitrarily small, and to exhibit lack of transversality. Theorem 1, 2 as stated does not exclude the possible existence of a region where the SRB measures are differentiable at a

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1In some places this notion is also called SRB measure. In our paper, we reserve the term SRB measure for those with a basin of positive Lebesgue measure.
generic map, and are non-differentiable at the others (on a nonempty set). We think it is very likely that there exists a nonperturbative $C^r$ open neighbourhood of $\mathcal{U}_{\ell,r}^{\text{rot}}$ with such property. Indeed, we think some form of transversality condition would be necessary for the differentiability of SRB measures. There are other works that explore the relation between transversality and (fractional) linear response, for example [4, 5, 20].

2. The non-differentiable example we constructed is a skew product, and is stable under sufficiently localised perturbation preserving the skew product (See Corollary [13]). It would be interesting to construct an open set of diffeomorphisms where the non-differentiability of SRB measures hold.

**Plan of the paper.** We will recall Tsuboi's transversality condition in Section 3 and reduce the proof Theorem 1 to Proposition 1, which we prove in Section 4. In Section 5, we give precise conditions for the construction and verify these conditions in Subsection 5.2 and finish the proof of Theorem 2 in Subsection 5.3.

3. Transversality Property

The proof of Theorem 1 is divided into two parts using a transversality condition due to Tsuboi in [24, 25], which we now introduce.

**Definition 4.** For any $\alpha > 0$, we set

$$\mathcal{C}(\alpha) = \{(x, y) \in \mathbb{R}^2 | |y| \leq \alpha|x|\}.$$ 

More generally, for any line $L \subset \mathbb{R}^2$ containing the origin, any $\beta > 0$, we denote

$$\mathcal{C}(L, \beta) = \{(x, y) \in \mathbb{R}^2 \setminus \{0\} | \angle((x, y), L) \leq \beta\} \cup \{0\}.$$ 

Given $\ell \geq 2$, $\gamma_0 \in (\ell^{-1}, 1)$ and $\theta > 0$. Denote $\mathcal{C}_0 = \mathcal{C}(\theta)$. Then for any $f \in \mathcal{U}_{\ell,r}^{\text{rot}}$ written as $f(x, y) = (\ell x, y + \varphi(x))$ such that

$$\gamma_0(\ell - 1)\theta > \|D\varphi\|,$$

we have that $\mathcal{C}_0$ is strictly invariant under $Df$ in the sense that

$$Df_z(\mathcal{C}_0) \subset C(\gamma_0\theta)$$

for any $z \in \mathbb{T}^2$.

Here and after, for two cones $\mathcal{C}, \mathcal{C}' \subset \mathbb{R}^2$, we denote $\mathcal{C} \subset \mathcal{C}'$ if the closure of $\mathcal{C}$ is contained in the interior of $\mathcal{C}'$ except for the origin. For any cone $\mathcal{C}$, we set

$$\mathcal{C}^* = \{u \in \mathbb{R}^2 \mid \exists v \in \mathcal{C} \text{ such that } \langle u, v \rangle = 0\}.$$ 

Given any $\ell \geq 2$, $\gamma_0 \in (\ell^{-1}, 1), \theta > 0, f$ satisfying (3.2), for any $z \in \mathbb{T}^2$, any $n \geq 1$, any $w_1, w_2 \in f^{-n}(z)$, we say that $w_1 \cap w_2$ if

$$Df^n_{w_1}(\mathcal{C}_0) \cap Df^n_{w_2}(\mathcal{C}_0) = \{0\}$$
otherwise we say \( w_1 \not\mathcal{H} w_2 \). We define

\[
m(f, n) = \sup_{z \in \mathbb{T}^2} \sup_{w \in f^{-n}(z)} \ell^{-n} \# \{ \xi \in f^{-n}(z) | \xi \not\mathcal{H} w \} \leq 1,
\]

\[
m(f) = \limsup_{n \to \infty} m(f, n)^{1/\ell} \leq 1.
\]

By (3.2), it is direct to see that

\[
(3.3) \quad m(f) \leq m(f, n)^{1/\ell}, \quad \forall n \geq 1.
\]

Then we have the following easy but important consequence,

\[
The function \( f \mapsto m(f) \) is upper semicontinuous in \( C^1 \) topology.
\]

Using the exponent \( m(f) \), the proof of Theorem 1.4 in [14] splits into two parts.

**Proposition 1.** Given any integers \( r \geq 20, 1 \leq r' \leq \frac{r}{2} - 9, \ell \geq 2 \). For any \( \gamma_0 \in (\ell^{-1}, 1), \theta > 0, f \in U_{r, r}^{\text{rot}} \) satisfying (3.1) and \( m(f) < 1 \), there exists an \( C' \)-open neighbourhood of \( f \) in \( C'(\mathbb{T}^2, \mathbb{T}^2) \), denoted by \( U \), such that any \( f' \in U \) admits a unique SRB measure \( \mu_{f'} \) having \( C'^{-1} \) density, and \( f' \mapsto \mu_{f'} \) is \( C' \)-restricted to \( U \).

**Proposition 2.** For any integers \( r \geq 1, \ell \geq 2 \), any \( f \in U_{r, r}^{\text{rot}} \) that is \( C' \)-stably ergodic in \( U_{r, r}^{\text{rot}} \), there exist \( \gamma_0 \in (\ell^{-1}, 1), \theta > 0 \) satisfying (3.1) and \( m(f) < 1 \).

**Proof.** The proof is very similar to Theorem 1.4 in [25]. We denote \( f(x, y) = (\ell x, y + \varphi(x)), \forall (x, y) \in \mathbb{T}^2 \) and choose any \( \gamma_0 \in (\ell^{-1}, 1), \theta > 0 \) such that (3.1) is true. If \( m(f) = 1 \), then for any \( n \geq 1 \), there exists \( z_n \in \mathbb{T}^2 \) such that for any \( w, w' \in f^{-n}(z_n) \), \( Df_{z_n}(C_0) \cap Df_{w'}(C_0) \neq \emptyset \). Thus there exists a line in \( \mathbb{R}^2 \), denoted by \( L_n \), contained in \( C_0 \), such that \( Df_{z_n}(C_0) \subset C(L_n, C^{-\ell n}) \) for all \( w \in f^{-n}(z_n) \) and some constant \( C \) independent of \( n \). After passing to a subsequence, we can assume that \( z_n \to z, L_n \to L \). We let \( W \) be the set of \( (z', L') \in \mathbb{T}^2 \times \mathbb{P}(\mathbb{R}^2) \) such that for any \( n \geq 0 \), any \( w' \in f^{-n}(z') \), \( Df_{z_n}(C_0) \subset C(L, C^{-\ell n}) \). We easily verify that \( W \) is closed and completely invariant. Moreover, \( (z, L) \in W \). This shows that for any \( z \in \mathbb{T}^2 \) there exists \( \Psi(z) \in \mathbb{P}(\mathbb{R}^2) \) such that \( (z, \Psi(z)) \in W \). It is easy to see that the choice of \( \Psi(z) \) is unique and depends only on the first coordinate of \( z \). Let \( \psi : \mathbb{T} \to \mathbb{R} \) be a function such that \( \Psi(z) = [R(1, \psi(x))], \forall z = (x, y) \in \mathbb{T}^2 \). Then we have

\[
\ell^{-1}(\psi(x) + \varphi'(x)) = \psi(\ell x), \quad \forall x \in \mathbb{T}.
\]

Then for any two sequences \( (y_n)_{n \geq 0}, (y'_n)_{n \geq 0} \) in \( \mathbb{T} \) such that \( \ell y_{n+1} = y_n, \ell y'_{n+1} = y'_n \) and \( y_0 = y'_0 \), we have

\[
\sum_{i \geq 1} l^{-i} \varphi'(y_i) = \sum_{i \geq 1} l^{-i} \varphi'(y'_i)
\]

But this shows that \( f \) does not satisfy the infinitesimal completely non-integrability condition in Section 3.2 [14]. We then conclude the proof by Theorem 3.3 in [14]. \( \square \)
Proof of Theorem 1: Our theorem follows immediately by combining Proposition 1 and Proposition 2.

We will prove Proposition 1 in Section 4.

4. SPECTRAL GAP IN ANISOTROPIC BANACH SPACE

Our strategy for proving Proposition 1 is the following. We construct Anisotropic Sobolev spaces \( W_{\Theta,p,q} \) following Tsuji in [25]. Different from [25], we consider positive \( p, q \), which corresponds to smaller and smoother spaces. We will consider a filtration of such spaces, and establish Lasota-Yorke’s inequalities for Perron-Frobenius operator \( P \) acting on these spaces. These give us control of the essential spectrums of \( P \). Such control is ultimately due to our hypothesis that transversality strongly dominates the possible contraction in the center space. We then use a general theorem of Gouëzel-Liverani in [17] to show the differentiability result.

Throughout this section, we will need to study inequalities associated to \( f^n \) for \( f \in C^r(\mathbb{T}^2, \mathbb{T}^2) \) and for different \( n \)’s. We use \( C \) to denote positive constants which are independent of \( n \), and use \( C_n \) to denote positive constants which may depend on \( n \). Constants \( C, C_n \) are uniform in a \( C^r \) open neighbourhood of \( f \), and may vary from line to line.

4.1. Anisotropic Sobolev spaces. In this section, we will collection some basic notions from [25]. Throughout this section, we denote \( R = (-\frac{1}{4}, \frac{1}{4})^2 \) and \( Q = (-\frac{1}{3}, \frac{1}{3})^2 \).

We say \( \Theta \) is a polarisation if it is a combination \( \Theta = (C_+, C_-, \varphi_+, \varphi_-) \) of closed cones \( C_\pm \) in \( \mathbb{R}^2 \) and \( C^\infty \) functions \( \varphi_\pm : S^1 \to [0, 1] \) on the unit circle \( S^1 \subset \mathbb{R}^2 \) satisfying \( C_+ \cap C_- = \{0\} \) and

\[
\varphi_+ = \begin{cases} 
1, & \text{if } \xi \in S^1 \cap C_+, \\
0, & \text{if } \xi \in S^1 \cap C_- 
\end{cases}, \quad \varphi_- = 1 - \varphi_+
\]

For two polarisation \( \Theta = (C_+, C_-, \varphi_+, \varphi_-) \) and \( \Theta' = (C'_+, C'_-, \varphi'_+, \varphi'_-) \), we write \( \Theta < \Theta' \) if \( \mathbb{R}^2 \setminus C'_+ \subset C_- \).

For a \( C^\infty \) function \( \chi : \mathbb{R} \to [0, 1] \) satisfying \( \chi(s) = \begin{cases} 
1, & \text{for } s \leq 1 \\
0, & \text{for } s \geq 2
\end{cases} \). For a polarisation \( \Theta = (C_+, C_-, \varphi_+, \varphi_-) \), an integer \( n \geq 0 \), and \( \sigma \in \{+, -\} \), we define \( C^\infty \) function \( \psi_{\Theta,n,\sigma} : \mathbb{R}^2 \to [0, 1] \) by

\[
\psi_{\Theta,n,\sigma}(\xi) = \begin{cases} 
\varphi_{\sigma}(\xi/|\xi|) \cdot (\chi(2^{-n}|\xi|) - \chi(2^{-(n+1)}|\xi|)), & n \geq 1 \\
\chi(\xi)/2, & n = 0
\end{cases}
\]

For a function \( u \in L^2(\mathbb{R}) \), we denote the Fourier modes by

\[
F(u)(\xi) = \int e^{-2\pi iy \cdot \xi} u(y) dy, \quad \xi \in \mathbb{R}^2
\]
and define

\[ u_{\Theta,n,r}(x) = \psi_{\Theta,n,r}(D)u(x) := \int e^{2\pi i x \cdot \xi} \psi_{\Theta,n,r}(\xi) F(u)(\xi) d\xi \]

For any open set \( X \subseteq \mathbb{R}^2 \), any \( r \in (0, \infty] \), we denote by \( C^0_0(X) \) the set of compactly supported \( C' \) functions on \( X \). For any \( p \in \mathbb{R} \), for any \( u \in C^0_0(\mathbb{R}^2) \), we denote its Sobolev norm \( ||u||_{H^p} \) by

\[ ||u||_H^p = \left( \int (|\xi|^2 + 1)^p |F(u)(\xi)|^2 d\xi \right)^{\frac{1}{2}} \]

It is well-known that for \( p \in \mathbb{N} \),

\[ ||u||^2_{H^p} \sim \sum_{j=0}^{p} ||D^j u||^2_{L^2} \] \hspace{1cm} (4.1)

For an open set \( X \subseteq \mathbb{R}^2 \), we denote by \( H^p_0(X) \) the completion of \( C^0_0(X) \) with respect to \( || \cdot ||_{H^p} \).

For a polarisation \( \Theta = (C_+, C_-, \varphi_+, \varphi_-) \) and a real number \( p \), we define the semi-norms \( || \cdot ||^+_\Theta,p \) and \( || \cdot ||^-\Theta,q \) on \( C^0_0(\mathbb{R}) \) by

\[ ||u||_{\Theta,c(c)}^p = (\sum_{n \geq 0} 2^{2c(c)n} ||u_{\Theta,n,c}||^2_{L^2})^{1/2} \]

where we set \( c(+) = p \) and \( c(-) = q \).

We define the anisotropic Sobolev norm \( || \cdot ||_{\Theta,p,q} \) on \( C^0_0(\mathbb{R}) \) for real numbers \( p \) and \( q \) by

\[ ||u||_{\Theta,p,q} = \left( ||u||^{+\Theta,p}_\Theta \right)^2 + \left( ||u||^{-\Theta,q}_\Theta \right)^2 \]

For any \( p, q \in \mathbb{R} \), any polarisation \( \Theta \), we denote by \( W_{\Theta,p,q}(\mathbb{R}) \) the completion of \( C^0_0(\mathbb{R}) \) with respect to the norm \( || \cdot ||_{\Theta,p,q} \).

In the following two lemmata, we collect some basic properties of anisotropic Sobolev norms.

**Lemma 1.** For any \( 0 \leq p' < p, 0 \leq q' < q \) satisfying \( p' \geq q' \), \( p \geq q \), any polarisations \( \Theta' \subset \Theta \), we have

1. \( C^{p'}_0(\mathbb{R}) \subset H^p_0(\mathbb{R}) \subset W_{\Theta,p,q}(\mathbb{R}) \subset H^q_0(\mathbb{R}) \). If \( q \geq 2 \), then \( W_{\Theta,p,q}(\mathbb{R}) \subset C^{q-2}(\mathbb{R}) \).
2. \( W_{\Theta,p,q}(\mathbb{R}) \subset W_{\Theta',p,q}(\mathbb{R}) \).
3. We have a compact inclusion \( W_{\Theta,p,q}(\mathbb{R}) \subset W_{\Theta',p',q'}(\mathbb{R}) \).

**Proof.** The first 3 inclusions in (1) and (2) are obvious. The inclusion \( W_{\Theta,p,q}(\mathbb{R}) \subset C^{q-2}(\mathbb{R}) \) for \( q \geq 2 \) follows from \( W_{\Theta,p,q}(\mathbb{R}) \subset H^q_0(\mathbb{R}) \) and Sobolev’s embedding theorem. For (3), we refer the reader to Proposition 5.1 in [7]. \( \square \)

**Lemma 2.** Let \( r \geq 1 \) and let \( g_i : \mathbb{R}^2 \rightarrow [0, 1], 1 \leq i \leq l \), be a family of functions, \( C' \) in the interior of \( R \), and satisfy \( \sum_{i=1}^{l} g_i(x) \leq 1 \) for \( x \in R \). Let \( \Theta \) and \( \Theta' \) be
polariations such that $\Theta' < \Theta$, and let $1 \leq q \leq p \leq r$ be integers. Then for all $u \in C_0^r(R)$ we have
\[
\left( \sum_{i=1}^I \|g_iu\|_{R, p, q}^2 \right)^{\frac{1}{2}} \leq C\|u\|_{R, p, q} + C'\|u\|_{R, p-1, q-1}
\]
where $C$ does not depend on $\{g_i\}$, while $C'$ may. Further, if $\sum_{i=1}^I g_i(x) \equiv 1$ for all $x \in R$ in addition, then for all $u \in C_0^r(R)$ we have
\[
\|u\|_{R, p, q} \leq v\left( \sum_{i=1}^I \|g_iu\|_{R, p, q}^2 \right)^{\frac{1}{2}} + C'\sum_{i=1}^I \|g_iu\|_{R, p-1, q-1}
\]
where $v$ is the intersection multiplicity of the supports of the functions $g_i$ for $1 \leq i \leq I$.

Proof. This is a more general case of Lemma 2.3 in [25]. The proof follows from straightforward adaptions. The first inequality is essentially proved in Appendix C [25], the only difference being that instead of $\|g_iu\|_{L^2} \leq \|u\|_{L^2}$, we use
\[
\|g_iu\|_{H^1} \leq \|g_iDu\|_{L^2} + C(g_i)\|u\|_{H^{q-1}}
\]
The second inequality is essentially proved in Lemma 7.1 [7]. □

To exploit the expansion in the unstable direction, we consider the following situation. Let $r \geq 2, \rho \in C_0^{r-1}(R)$ be supported inside an open set $U \subset R$ and let $S : U \to S(U) \subset R$ be a $C^r$ diffeomorphism. Consider operator $L : C^{r-1}(R) \to C^{r-1}(R)$ defined by
\[
Lu(x) = \begin{cases} 
\rho(x)u(S(x)), & \forall x \in U \\
0, & \text{otherwise}
\end{cases}
\]
Assume that for polariations $\Theta = (C_\pm, \varphi_\pm), \Theta' = (C'_\pm, \varphi'_\pm)$, we have

\[(DS_\zeta)^{tr}(R^2 \setminus C_+) \subset C'_-, \quad \forall \zeta \in U\]

where $(DS_\zeta)^{tr}$ denotes the transpose of $DS_\zeta$. Put
\[
\gamma(S) = \min_{\zeta \in U} | \det DS_\zeta |
\]
\[
\Lambda(S, \Theta') = \sup \left\{ \frac{\| (DS_\zeta)^{tr}(v) \|}{\|v\|} \middle| \zeta \in U, (DS_\zeta)^{tr}(v) \notin C'_- \right\}
\]
The following is essentially contained in the proof of Lemma 2.4 in [25]. We refer the readers to the Appendix for the details

**Lemma 3.** Given integers $r \geq 7, 0 \leq q \leq p \leq \frac{r}{2} - 3$. Then the operator $L$ extends boundedly to $L : W_{\Theta, p, q}(R) \to W_{\Theta', p, q}(R)$. If in addition $q \geq 1$, then we have for $u \in W_{\Theta, p, q}(R)$ that
\[
\|Lu\|_{\Theta', q} \leq C\|p\|_{L^{\infty} \gamma(S)^{-\frac{1}{2}}\|DS\|_{q} \|u\|_{\Theta, p, q} + C'\|u\|_{\Theta, p-1, q-1}}
\]
\[
\|Lu\|_{\Theta', p} \leq C\|p\|_{L^{\infty} \gamma(S)^{-\frac{1}{2}}\Lambda(S, \Theta') \|u\|_{\Theta, p, q} + C'\|u\|_{\Theta, p-1, q-1}}
\]
here constant $C$ does not dependent on $\Theta, \Theta', S, \rho$ while $C'$ may.

For any $p, q \in \mathbb{R}$, any polarisation $\Theta$, we define a norm $\| \cdot \|_{\Theta, p, q}$ for $C^\infty(\mathbb{T}^2)$ in the following way. We construct a finite collection of translations of $R$ in $\mathbb{T}^2$, defined by $\{ R_a := \kappa_a(R) \}_{a \in A}$, where $A$ is a finite set in $\mathbb{T}^2$ and $\kappa_a : Q \rightarrow \mathbb{T}^2$ is the embedding defined by $\kappa_a(z) = z + a, \forall z \in Q$. Let $R_a = \kappa_a(R)$ and $Q_a = \kappa_a(Q)$. We assume that $\mathbb{T}^2 \subset \bigcup_{a \in A} R_a$. We choose a unit partition $\{ \rho_a \in C^\infty(\mathbb{T}^2, [0, 1]) \}_{a \in A}$ such that

$$\sum_{a \in A} \rho_a \equiv 1, \quad \text{supp}(\rho_a) \subset R_a, \forall a \in A.$$  

For each $u \in C^\infty(\mathbb{T}^2)$, we define

$$\| u \|_{\Theta, p, q} = \left( \sum_{a \in A} \| (\rho_a u) \circ \kappa_a \|_{\Theta, p, q}^2 \right)^{\frac{1}{2}}$$

and we let $W_{\Theta, p, q}(\mathbb{T}^2)$ be the completion of $C^\infty(\mathbb{T}^2)$ with respect to $\| \cdot \|_{\Theta, p, q}$.

**Remark 1.** The construction of anisotropic Banach spaces adapted to dynamically systems was originally due to Baladi and Tsujii in [11], and then used by Tsujii in [25] to study a class of suspension semi-flows. Similar ideas also appeared in [1]. In their papers, the dynamics are either uniformly hyperbolic, or have natural invariant measures, so they only studied the case where $q \leq p$ in order to be able to prove decay for rough observables. We need to consider $0 < q < p$ in order to prove our uniqueness of SRB measure.

4.2. **Transfer operators and Lasota-Yorke’s inequality.** In the rest of this section, we let $r \geq r' \geq 2$, $\ell \geq 2$ and assume that $f$ is $C^\ell$ close to $\mathcal{U}_\ell^{\text{rot}}$. It is a classical fact and easy to verify that the density $\rho$ (w.r.t. the Lebesgue measure) of any absolute continuous $f-$invariant measure $\mu$ is a fixed point of the Perron-Frobenius operator $\mathcal{P}_f : L^1(\mathbb{T}^2) \rightarrow L^1(\mathbb{T}^2)$ associated to $f$, defined by

$$\mathcal{P}_f u(z) = \sum_{w \in f^{-1}(z)} u(w) \det(Df(w))^{-1},$$

Moreover, we have for any $u, v \in L^2(\mathbb{T}^2)$ that

$$(\mathcal{P}_f u, v)_{L^2} = (u, v \circ f)_{L^2}. \quad (4.2)$$

In the following, we briefly denote $\mathcal{P} = \mathcal{P}_f$.

We define for any $n \in \mathbb{N}$, any $a, b \in A$, any $u \in C_{0}^{r-1}(R)$ that

$$P^{u}_{a,b} u(x) = \rho_a \kappa_a(x) \sum_{\kappa_b(y) \in f^{-u}(\kappa_a(x))} u(y) \det(Df^u(\kappa_b(y)))^{-1}$$
Then for any $0 \leq p, q \leq r - 1$, any polarisation $\Theta$, any $u \in C^{r-1}(\mathbb{T}^2)$, we have

$$\|P^n u\|^2_{\Theta, p,q} = \sum_{a \in A} \sum_{b \in A} (\rho_a P^n (\rho_b u)) \circ \kappa_a \|\Theta, p,q\|^2_{\Theta, p,q} \leq C \sum_{a \in A} \sum_{b \in A} (\|\rho_a P^n (\rho_b u)) \circ \kappa_a\|\Theta, p,q\|^2_{\Theta, p,q} \leq C \sum_{a \in A} \sum_{b \in A} (P^n (\rho_b u)) \circ \kappa_a\|\Theta, p,q\|^2_{\Theta, p,q},$$

(4.3)

We fix any constants $\gamma_0 \in (\ell^{-1}, 1), \theta > 0$ such that (3.2) is satisfied for $f, C_0 = C(\theta)$ and $\gamma_0$. This is true if, for example, when $f \in U_{t, r}^{\text{rot}}$ and (3.1) is satisfied. In the following, $m(f)$ is defined using cone $C_0$.

Let $\hat{\Theta}, \Theta, \Theta', \hat{\Theta}$ be polarisations denoted by

$$\hat{\Theta} = (\hat{\Theta}_+, \hat{\Theta}_-), \quad \Theta' = (\Theta_+, \Theta_-), \quad \Theta = (\Theta_+, \Theta_-), \quad \hat{\Theta} = (\hat{\Theta}_+, \hat{\Theta}_-)$$

such that

$$\Theta < \Theta' < \Theta < \hat{\Theta}$$

and

$$\Theta(\gamma_0 \theta)) \in \hat{\Theta}_- \in (\mathbb{R}^2 \backslash \hat{\Theta}_+) \in C_0^*,$$

(4.5)

$$\Theta(\gamma_0 \theta)) \in \hat{\Theta}_+ \in \hat{\Theta}_-, \quad \forall z \in T^2$$

(4.6)

Moreover, we always assume that $\mathbb{R}(0, 1)$ is contained in the interior of $\hat{\Theta}_-$. Such choice is possible since by (3.2).

$$\Theta(\gamma_0 \theta)) \in \hat{\Theta}_- \in (\mathbb{R}^2 \backslash \hat{\Theta}_+) \in C_0^*,$$

(4.7)

In the following, we fix $\hat{\Theta}, \Theta, \Theta', \hat{\Theta}$. For any $h \in (0, \log \ell)$, integer $N_0 > 0$, we let $U_{t, r}^{\text{rot}}$ be the set of $C^r$ covering maps $g : T^2 \to T^2$ of degree $\ell$ satisfying (3.2) and

1. $\|g\|_{C^r} < \ell N_0 e^{N_0 h}$,
2. $\ell N_0 e^{N_0 h} > \|Dg_{z}^{N_0}(v)\| > \ell N_0 e^{-N_0 h}, \forall z \in T^2, v \neq 0, v \in C_0$,
3. $\ell N_0 e^{N_0 h} > \|Dg_{z}^{N_0}(v)\| > e^{-N_0 h}$, $\forall z \in T^2, v \neq 0, Dg_{z}^{N_0}(v) \notin C_0$,
4. $\|Dg_{z}^{N_0}(v)\|_{\mathbb{R}^2} > \|Dg_{z}^{N_0}(v)\|_{\mathbb{R}^2} > \ell N_0 e^{-N_0 h}, \forall z \in T^2, v \neq 0, v \in \mathbb{R}^2 \backslash \hat{\Theta}_-.$

It is straightforward to check that for any $f^{\text{rot}} \in U_{t, r}^{\text{rot}}$, for any $h > 0$, there exists $N_0 = N_0(f^{\text{rot}}, h) > 0$ such that $U_{t, r}^{\text{rot}}$ contains an $C^r$ open neighbourhood of $f^{\text{rot}}$ in $C^r(T^2, T^2)$.

By (3.2), for any $f \in U_{t, r}^{\text{rot}}$, $N > N_0$, denote a local inverse branch of $f^N$ denoted by $H : U \to T^2$, i.e. $f^N H = Id_{|U}$, we have

$$D H_{z}(\mathbb{R}^2 \backslash C_0) \in (\mathbb{R}^2 \backslash C_0), \quad \forall z \in U.$$
Then for any $N > N_0$ we have
\begin{equation}
\det(Df_z^N) \geq C \ell^N e^{-Nh}, \forall z \in T^2
\end{equation}
and for all $H$ as above, we have
\begin{equation}
\|DH_z\| \leq Ce^{Nh}, \quad \det(DH_z) \geq C \ell^{-N} e^{-Nh}, \forall z \in U
\end{equation}
\begin{equation}
\|\{DH_z\}^\triangledown(v)\| \leq C \ell^{-N} e^{Nh}\|v\|, \forall z \in U, v \neq 0, (DH_z)^\triangledown(v) \notin \hat{\mathcal{C}}
\end{equation}
We have the following.

\textbf{Proposition 3.} Given integers $r \geq 13, 3 \leq q + 3 \leq p < \ell - 3$. For any \( h \in (0, \log \ell) \), integer $N_0 > 0$, any $f \in U^{h,N_0}$, any $u \in C_0^{r-1}(R)$, for any $a, b \in A$, we have
\[ \|P_{a,b}u\|_{\ell,p,q} \leq C \|u\|_{\ell,p,q} \]
If in addition that $q \geq 1$, then for any
\begin{equation}
m > m_0(f,h,p,q) := \max(e^{(2q+3)h}m(f),e^{(2p+3)h}\ell^{-(2p-1)})
\end{equation}
we have
\[ \|P_{a,b}u\|_{\ell,p,q} \leq C\tilde{m}^{n/2}\|u\|_{\ell,p,q} + C_n\|u\|_{\ell,p-1,q-1} \]

\textbf{Proof.} The proof is an easy adaptation of Lemma 2.6 in [25] using Lemma 3 instead of Lemma 2.4 in [25]. We will only give a sketched proof. The reader is referred to [25] for details.

By (4.11), we have $e^{-(2q+3)h}\tilde{m}^n > m(f)^n$. By (3.3), we can assume that $n$ is sufficiently large, so that $m(f,n) < e^{-(2q+3)h}\tilde{m}^n$. We will choose a covering of the closure of $R$ by finitely many little open cubes in $Q$ with intersection multiplicity bounded by 10, denoted by $\{D(\omega)\}_{\omega \in A}$. Take a family of $C^\infty$ functions $\{g_\omega : \mathbb{R}^2 \to [0,1]\}_{\omega \in A}$ such that $\text{supp}(g_\omega) \subset D(\omega)$ and $\sum_{\omega \in A}g_\omega(z) = 1$ for any $z \in R$.

Fix $u \in C^{r-1}(R)$, $a, b \in A$, $\omega \in A$, we denote the connected components of the preimage $f^{-n}(\kappa_a D(\omega)) \cap R_b$ by $\kappa_b(D(\omega),i), 1 \leq i \leq I(\omega)$, where $D(\omega,i) \subset R$ are open sets. By letting $D(\omega)$ to be small, we can ensure that for each $1 \leq i \leq I(\omega)$, $\kappa_a^{-1}f^n\kappa_b : D(\omega,i) \to D(\omega)$ is a $C^r$ injection; and by setting $i \notin I(\omega)$ if
\begin{equation}
\overline{Df_{w}^n(C_{0})} \cap \overline{Df_{w'}^n(C_{0})} = \{0\}, \quad \forall w \in D(\omega,i), w' \in D(\omega,j),
\end{equation}
for each $i$ there are at most $m(f,n)$ many $j$ such that $i \notin I(\omega)$.

We define functions $\{g_{\omega,i} : \mathbb{R}^2 \to [0,1]\}_{i=1}^{I(\omega)}$ by
\[ g_{\omega,i}(y) = \begin{cases} (g_\omega \rho_a \circ \kappa_a)(\kappa_a^{-1}f^n\kappa_b(y)), & y \in D(\omega,i) \\ 0, & \text{otherwise} \end{cases} \]
We claim that $g_{\omega,i}$ is $C^{r-1}$ in $R$. Indeed, it is clear that $g_{\omega,i}$ is $C^{r-1}$ in $D(\omega,i)$ and continuously extends up to the boundary. Moreover, for any $z \in \partial D(\omega,i) \cap R$, we have $\kappa_a^{-1}f^n\kappa_b(z) \in \partial D(\omega)$, for otherwise an open
neighborhood of $z$ would be mapped into $D(\omega)$, thus $z \in D(\omega, i)$, a contradiction. While $g_\omega$ vanishes on an open neighbourhood of $\partial D(\omega)$. This implies our claim, and also proves that $g_{\omega,i}$ vanish in an open neighbourhood of $\partial D(\omega, i) \cap R$. Define $g_\omega := \sum_{i=1}^{I(\omega)} g_{\omega,i}$. Since $D(\omega, i), 1 \leq i \leq I(\omega)$ are mutually disjoint, we have $0 \leq g_\omega \leq 1$. Define

$$
g(y) = \sum_{\omega \in A} g_\omega(y) = \begin{cases} \rho_n f^n \kappa_b(y), & y \in \kappa_b^{-1}(f^{-n}(R_a) \cap R_b), \\ 0, & \text{otherwise} \end{cases}$$

We can easily verify that $0 \leq g \leq 1$ and $g$ is $C'^{-1}$ in the interior of $R$.

Let $u \in C_0^{-1}(R)$. We have the following,

1. for any $1 \leq i \leq I(\omega)$, define $u_{\omega,i} := g_{\omega,i}u$. Then we have $\text{supp}(u_{\omega,i}) \subseteq D(\omega, i)$, since $g_{\omega,i}$ vanish in an open neighbourhood of $\partial D(\omega, i) \cap R$ and $u$ vanish in an open neighbourhood of $\partial R$.

2. for any $1 \leq i \leq I(\omega)$, define

$$v_{\omega,i}(x) = \begin{cases} (u_{\omega,i} \det(Df^n \circ \kappa_i(\cdot))^{-1})(\kappa_i^{-1}f^{-n}\kappa_i(x)), & x \in \kappa_i^{-1}f^n\kappa_b(D(\omega, i)) \\ 0, & \text{otherwise} \end{cases}$$

We have that $v_{\omega,i} \in C'^{-1}(R)$,

3. let $v_{\omega} := \sum_{i=1}^{I(\omega)} v_{\omega,i} = g_\omega P_{a,b}u$,

4. we have $P_{a,b}u = \sum_{\omega \in A} v_\omega$.

Denote by $S = \kappa_b^{-1}f^{-n}\kappa_a : \kappa_i^{-1}f^n\kappa_b(D(\omega, i)) \to D(\omega)$. By $f \in \mathcal{U}^{h, N_0}$ and (4.8), (4.9), (4.10), we have for any $n \geq 1$

$$\|\left(\det(Df^n)\right)^{-1}\|_{L_\infty} \leq C\ell^{-n}\epsilon^{nh},$$

$$\gamma(S) \geq C\ell^{-n}\epsilon^{-nh}, \quad \|DS\| \leq C\epsilon^{nh}, \quad \Lambda(S, \hat{\Theta}) < C\ell^{-n}\epsilon^{nh}$$

Then by Lemma [3] and our hypothesis that $p, q \in [0, \frac{1}{2} - 3)$, we have

$$\|v_{\omega,i}\|_{\Theta, p, q} \leq C_n\|u_{\omega,i}\|_{\Theta, p, q}$$

Moreover, if $q \geq 1$, then

1. $|v_{\omega,i}|_{\Theta, q} \leq C\ell^{-p}\epsilon(q + \frac{1}{2})n\|u_{\omega,i}\|_{\Theta, p, q} + C_n\|u_{\omega,i}\|_{\Theta, p, q - 1}$

2. $|v_{\omega,i}|_{\Theta, p} \leq C\ell^{-(p + \frac{1}{2})}\epsilon(p + \frac{1}{2})n\|u_{\omega,i}\|_{\Theta, p, q} + C_n\|u_{\omega,i}\|_{\Theta, p, q - 1}$

We choose polarisations $\{\Theta(\omega, i) = (C_{\omega,i,\pm}, \varphi_{\omega,i,\pm})\}_{i=1}^{I(\omega)}$ such that for all $1 \leq i \leq I(\omega)$,

$$(Df^n)^{-1}(\mathbb{R}^2 \setminus \hat{C}_+) \subseteq C_{\omega,i,-} \subseteq (\mathbb{R}^2 \setminus C_{\omega,i,+}) \subseteq \hat{C}_-, \forall x \in D(\omega, i)$$

and

$$\left(\mathbb{R}^2 \setminus C_{\omega,i,+}\right) \cap \left(\mathbb{R}^2 \setminus C_{\omega,j,+}\right) = \emptyset, \text{ if } i \not\in j$$

It is possible by $i \not\in j$, (4.5), (4.7) and (4.12).

It is clear that $\hat{\Theta} < \Theta(\omega, i)$ for all $\omega \in A, 1 \leq i \leq I(\omega)$. 

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Proof. The proof is similar to that of Lemma 2.7 in [25]. For \( k = i,j \), put 
\[ w_{k,n} = \psi_{\Theta, n,-}(D)\nu_{\omega,k} \] 
and 
\[ w'_{k,n} = \psi_{\Theta, (k),n,-}(D)\nu_{\omega,k} \] 
for \( n \geq 0 \) we have \( (w'_{i,n}, w'_{j,n}) \leq 0 \). While \( 2^{(q-1)n} \|\nu_{\omega,i}\|_{L^2} \leq \|\nu_{\omega,i}\|_{\Theta, (i), p-1,q-1} \) and \( 2^{(p-1)n} \|\nu_{\omega,j}\|_{L^2} \leq \|\nu_{\omega,j}\|_{\Theta, (j), p-1,q-1} \). We have the similar thing for \( j \). Thus by \( p \geq q + 3 \),
\[ \| (w_{i,n}, w_{j,n}) \|_{L^2} \leq (2 \cdot 2^{-(q-1+p-1)n} + 2^{-(2p-2)n}) \|\nu_{\omega,i}\|_{\Theta, (i), p-1,q-1} \|\nu_{\omega,j}\|_{\Theta, (j), p-1,q-1} \]
\[ \leq 10 \cdot 2^{-2q+1} \|\nu_{\omega,i}\|_{\Theta, (i), p-1,q-1} \|\nu_{\omega,j}\|_{\Theta, (j), p-1,q-1} \]
The lemma follows from direct computations. \( \square \)

By \( (D_{x \times y}^n)^{-1}(R^2 \setminus \mathbf{\hat{C}}) \in C_{\omega,i,-} \) and Lemma 3 we have for \( q \geq 1 \) and all \( 1 \leq i \leq T(\omega) \),
\[ \|\nu_{\omega,i}\|_{\Theta, (i), p-1,q-1} \leq C_n \|\nu_{\omega,i}\|_{\Theta, p-1,q-1} \]

Then the rest of the proof follows almost exactly that of Lemma 2.6 in [25]. By Lemma 2 for any \( u \in C_0^{\infty}(R) \), we have for any \( n \geq 1 \),
\[ \| P_{\omega,b}^{\nu, u} \|_{\Theta, p,q}^2 = \| \sum_{\omega \in A} \|\nu_{\omega}\|_{\Theta, p,q}^2 \leq C \| \sum_{\omega \in A} \|\nu_{\omega}\|_{\Theta, p,q}^2 + C_n \| \sum_{\omega \in A} \|\nu_{\omega}\|_{\Theta, p-1,q-1}^2 \]
By Lemma 3 Cauchy’s inequality and (4.14),
\[ \left( \|\nu_{\omega} \|_{\Theta, q} \right)^2 \leq C \sum \sum \ell^{-n} e^{(2q+3)n} \|\nu_{\omega,i}\|_{\Theta, p,q}^2 \|\nu_{\omega,j}\|_{\Theta, p,q}^2 + C_n \|\nu_{\omega,i}\|_{\Theta, p-1,q-1}^2 \]
\[ \leq C \tilde{n}^n \sum \|\nu_{\omega,i}\|_{\Theta, p,q}^2 + C_n \sum \|\nu_{\omega,i}\|_{\Theta, p-1,q-1}^2 \]
By Lemma 2 and the calculation of \( g \),
\[ \sum_{\omega \in A} \|\nu_{\omega,i}\|_{\Theta, p,q}^2 \leq C \| u \|_{\Theta, p,q}^2 + C_n \| u \|_{\Theta, p-1,q-1}^2 \]
\[ \sum_{\omega \in A} \|\nu_{\omega,i}\|_{\Theta, p-1,q-1}^2 \leq C_n \| u \|_{\Theta, p-1,q-1}^2 \]
By (4.16), (4.17), (4.18), we obtain
\[ \sum_{\omega \in A} \left( \|\nu_{\omega} \|_{\Theta, q} \right)^2 \leq C \tilde{n}^2 \| u \|_{\Theta, p,q}^2 + C_n \| u \|_{\Theta, p-1,q-1}^2 \]
We have
\[ \left( \|\nu_{\omega} \|_{\Theta, p} \right)^2 \leq \ell^{2n} \sum \left( \|\nu_{\omega,i} \|_{\Theta, p} \right)^2 \]
\[ \leq C \sum \ell^{-n(2p-1)} e^{(2p+3)n} \|\nu_{\omega,i}\|_{\Theta, p,q}^2 + C_n \sum \|\nu_{\omega,i}\|_{\Theta, p-1,q-1}^2 \]
Again by (4.17), (4.18), we obtain
\[
(4.20) \quad \sum_{\omega \in A} (\|v_\omega\|_{\Theta, p}^*)^2 \leq C\hat{m}^n \|u\|^2_{\Theta, p, q} + C_n \|u\|^2_{\Theta^*, p-1, q-1}
\]

Finally by Lemma 2 Lemma 3 and $\Theta < \Theta(\omega, i)$, (4.14), we have
\[
\|v_\omega\|^2_{\Theta, p-1, q-1} \leq C \sum_i \|v_{\omega, i}\|^2_{\Theta(\omega, i), p-1, q-1} \leq C \sum_i \|u_{\omega, i}\|^2_{\Theta, p, q-1}
\]
By (4.18),
\[
(4.21) \quad \sum_{\omega \in A} \|v_\omega\|^2_{\Theta, p-1, q-1} \leq C_n \|u\|^2_{\Theta^*, p-1, q-1}
\]

Then lemma follows from (4.15), (4.19), (4.20), (4.21). 

COROLLARY A. Given integers $r \geq 13, 3 \leq q < p < \frac{r}{2} - 3$. Let $h, N_0, f$ be given by Proposition 3. For any $u \in C^\infty(T^2)$, we have,
\[
\|P u\|_{\Theta, p, q} \leq C \|u\|_{\Theta, p, q}
\]
If in addition that $q \geq 1$, then for any $\tilde{m}$ in (4.11), there exists $M > 0$ such that for any $u \in C^\infty(T^2)$, any $n \in \mathbb{N}$,
\[
\|P^n u\|_{\Theta, p, q} \leq C\hat{m}^n \|u\|_{\Theta, p, q} + CM^n \|u\|_{\Theta^*, p-1, q-1}
\]

Proof. We choose an arbitrary $\bar{m} \in (m_0(f, h, p, q), \tilde{m})$ (recall (4.11)). By (4.3) and Proposition 3 we have
\[
(4.22) \quad \|P u\|^2_{\Theta, p, q} \leq C \sum_{b \in A} \|(\rho_b u) \circ \kappa_b\|^2_{\Theta, p, q} \leq C \|u\|^2_{\Theta, p, q}
\]
and for $q \geq 1$,
\[
\|P^n u\|^2_{\Theta, p, q} \leq C\bar{m}^N \sum_{b \in A} \|(\rho_b u) \circ \kappa_b\|^2_{\Theta, p, q} + C N \sum_{b \in A} \|(\rho_b u) \circ \kappa_b\|^2_{\Theta^*, p-1, q-1} \leq C\bar{m}^N \|u\|^2_{\Theta, p, q} + C N \|u\|^2_{\Theta^*, p-1, q-1}
\]
Then
\[
(4.23) \quad \|P^n u\|_{\Theta, p, q} \leq C\bar{m}^N \|u\|_{\Theta, p, q} + C N \|u\|_{\Theta^*, p-1, q-1}
\]
We fix $N$ to be a large integer so that the coefficient of $\|u\|_{\Theta, p, q}$ in (4.23) is less than $\tilde{m}\bar{m}^N$. Let $M$ be a large constant to be chosen later. We will inductively prove that for all integer $l \geq 1$,
\[
(4.24) \quad \|P^l u\|_{\Theta, p, q} \leq \tilde{m}^N \|u\|_{\Theta, p, q} + M^{Nl} \|u\|_{\Theta^*, p-1, q-1}
\]
This is true for $l = 1$ by (4.23) and by letting $M > C_N^\frac{1}{2}$. Assume that (4.24) is prove for $l$. Then by (4.22) and (4.24) we have

$$\|P^{N(l+1)}u\|_{\Theta,p,q} \leq \tilde{m}^N \|P^N u\|_{\Theta,p,q} + M^N \|P^N u\|_{\Theta,p-1,q-1}$$

$$\leq \tilde{m}^N (\tilde{m}^N \|u\|_{\Theta,p,q} + M^N \|u\|_{\Theta,p-1,q-1}) + M^N C^N \|u\|_{\Theta,p-1,q-1}$$

The last inequality follows by letting $M > 10 \max(\tilde{m}^N, \tilde{m}^{-\frac{1}{2}} C, C)$. This completes the induction. Our corollary then follows by letting $C$ in the second inequality of our corollary to be large depending on $N$. 

4.3. **Gouëzel-Liverani's perturbation lemma.** We recall an abstract result from [17 Section 8]. Let $\mathcal{B}^0 \supset \mathcal{B}^1 \supset \cdots \supset \mathcal{B}^s, s \in \mathbb{N}$, be a finite family of Banach spaces, and let $\{L_i\}_{i \in (-1,1)}$ be a family of operators acting on the above Banach spaces. Moreover, assume that

1. there exist $M > 0$, for all $t \in (-1,1)$, $\|L_i^t u\|_{\mathcal{B}^0} \leq C_0 M^r \|u\|_{\mathcal{B}^0}$,
2. there exists $\alpha \in (0, M)$, for all $t \in (-1,1)$, $\|L_i^t u\|_{\mathcal{B}^i} \leq C_0 \alpha^r \|u\|_{\mathcal{B}^i} + C_0 M^r \|f\|_{\mathcal{B}^0}$,
3. there exist operators $Q_1, \cdots, Q_{s-1}$ satisfying

$$\|Q_j\|_{\mathcal{B}^i \rightarrow \mathcal{B}^{i-1}} \leq C_1, \forall j = 1, \cdots, s-1, i = j, \cdots, s$$

4. moreover, define $\Delta_0(t) := L_i$ and $\Delta_j(t) := L_i - L_0 - \sum_{k=1}^{j-1} t^k Q_k$ for $j \geq 1$, we have

$$\|\Delta_j(t)\|_{\mathcal{B}^i \rightarrow \mathcal{B}^{i-1}} \leq C_1 |t|^{j}, \forall t \in (-1,1), j = 0, \cdots, s, i = j, \cdots, s$$

In this case, we say that $\{L_i\}_{i \in (-1,1)}$ is $(\alpha, M, C_0, C_1)$ adapted to $\{\mathcal{B}^i\}_{0 \leq i \leq s}$.

For any integer $1 \leq k \leq s$, any $t \in (-1,1)$, any $q > \alpha$ and $\delta > 0$, denote

$$V_{q,\delta} = \{z \in \mathbb{C} | |z| \geq q, d(z, \text{Sp}(L_0 : \mathcal{B}^k \rightarrow \mathcal{B}^k)) \geq \delta, \forall k = 1, \cdots, s\}$$

The following theorem in proved in [17].

**Theorem 3** (Theorem 8.1 in [17]). **Given a family of operators $\{L_i\}_{i \in (-1,1)}$ that is $(\alpha, M, C_0, C_1)$ adapted to $\{\mathcal{B}^i\}_{0 \leq i \leq s}$ and set**

$$R_s(t) = \sum_{k=0}^{s-1} \sum_{l_1 + \cdots + l_j = k} (z - L_0)^{-1} Q_{l_1} (z - L_0)^{-1} \cdots (z - L_0)^{-1} Q_{l_j} (z - L_0)^{-1}$$

then for all $q > \alpha, \delta > 0$, there exists $\eta > 0, C_2 = C_2(\alpha, M, C_0, C_1, q, \delta) > 0, \eta_0 = \eta_0(\alpha, M, C_0, C_1, q, \delta) > 0$ such that for all $z \in V_{q,\delta}$ and $t \in (-t_0, t_0)$, we have that

$$\|(z - L_0)^{-1} - R_s(t)\|_{\mathcal{B}^i \rightarrow \mathcal{B}^0} \leq C_2 |t|^{s-1+\eta}$$

Let $r \geq r' + 2 \geq 3$. Given any $C^{r+1}$ family in $C^r(T^2)$, denoted by $\{f_t\}_{t \in (-1,1)}$. For any $t \in (-1,1)$, we denote $P_t = P_{f_t}$.
By Taylor’s formula, for each \( u \in C'(\mathbb{T}^2) \), for all \( 1 \leq k \leq r' + 1 \), we have

\[
\mathcal{P}_t u = \sum_{j=0}^{k-1} \frac{1}{j!} \partial^j_t \mathcal{P}_t u |_{t=0} + \int_0^t dt_1 \cdots \int_0^{t_{k-1}} dt_k (\partial^k_t \mathcal{P}_t u)(t_k)
\]  

(4.25)

For any \( k \geq 1 \), any \( \alpha = (\alpha_1, \cdots, \alpha_k) \in \{1, 2\}^k \), we denote \(|\alpha| = k\) and define by \( \partial^\alpha \) the linear operator from \( C^\infty(\mathbb{T}^2) \) to \( C^\infty(\mathbb{T}^2) \) that

\[
\partial^\alpha u = \partial_{\alpha_k} \cdots \partial_{\alpha_1} u
\]

(4.26)

Then there exist for each \( 1 \leq k \leq r' + 1 \), functions \( J_0(k, t, x) \) which are \( C^0 \) in \( t \) and \( C^1 \) in \( x \), and for each multi-index \( \alpha, 1 \leq |\alpha| \leq k \), functions \( J_\alpha(k, t, x) \) which are \( C^{|\alpha|} \) in \( t \) and \( C^{r'-1} \) in \( x \), such that for all \( t_0 \in (-1, 1) \)

\[
\partial^k_t \mathcal{P}_t u(x)|_{t=t_0} = J_0(k, t_0, x) (\mathcal{P}_t u)(x) + \sum_{j=1}^{k} \sum_{|\alpha|=j} J_\alpha(k, t_0, x) (\partial^\alpha u)(x)|_{t=t_0}
\]

(4.27)

Moreover, for \( 1 \leq k \leq r' + 1, 1 \leq j \leq k, \alpha \in \{1, 2\}^j \), we have

\[
\sup_{t \in (-1, 1)} \| J_\alpha(k, t, \cdot) \|_{C^{r'-1}(\mathbb{T}^2)} \leq C(\| f_\alpha \|_{t \in (-1, 1)} \| r' + 1, r \})
\]

(4.28)

and

\[
\sup_{t \in (-1, 1)} \| J_0(k, t, \cdot) \|_{C^{r'-1}(\mathbb{T}^2)} \leq C(\| f_\alpha \|_{t \in (-1, 1)} \| r' + 1, r \})
\]

(4.29)

We need the following lemma.

**Lemma 5.** Let \( \Theta, \Theta' \) be two polarisations such that \( \Theta < \Theta' \), and let \( p, q \in \mathbb{N}, q \leq p \). For any \( k \geq 0 \), for any multi-index \( \alpha \in \{1, 2\}^k \) (when \( k = 0 \), we set \( \alpha = \emptyset \) and \( \partial^\alpha = 1 \)), for any \( J \in C^{p+k}(\mathbb{T}^2) \), \( J \partial^\alpha \) is a bounded operator from \( W_{\Theta, p+k,q+k}(\mathbb{T}^2) \) to \( W_{\Theta, p, q}(\mathbb{T}^2) \) with norm bounded by \( C = C(\Theta, \Theta', p, q, k, \| J \|_{C^{p+k}}) \).

**Proof.** In the following we will consider \( \Theta, \Theta', p, q \) to be fixed, so that we will not express the dependence of varies constants on them.

We prove our lemma by induction on \( k \). We denote \( \alpha = (\alpha_1, \cdots, \alpha_k) \). For any \( u \in C^\infty(\mathbb{T}^2) \), \( a \in A \), we have

\[
\partial^\alpha ((\rho_a J u) \circ \kappa_a) = \rho_a \circ \kappa_a (J \partial^\alpha u) \circ \kappa_a + (\hat{\rho} u) \circ \kappa_a + \sum_{\beta, 1 \leq |\beta| \leq k-1} (\hat{\rho}_\beta \partial^\beta u) \circ \kappa_a
\]

where \( \hat{\rho} \in C^p_0(R_a), \hat{\rho}_\beta \in C^{p+|\beta|}_0(R_a) \), \( \forall \beta, 1 \leq |\beta| \leq k-1 \). Moreover, it is direct to see that for all \( \beta, 1 \leq |\beta| \leq k-1 \),

\[
\| \hat{\rho} \|_{C^p} \leq \| \rho_a \|_{C^{p+k}} \| J \|_{C^{p+k}}, \quad \| \hat{\rho}_\beta \|_{C^{p+|\beta|}} \leq \| \rho_a \|_{C^{p+k}} \| J \|_{C^{p+k}}
\]

(4.30)

For any \( v \in C^\infty(\mathbb{R}^2) \), any \( \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2 \), we have

\[
\mathcal{F}(\partial^\alpha v)(\zeta) = (2\pi i)^k (\prod_{j=1}^{k} \zeta_{\alpha_j}) \mathcal{F}(v)(\zeta)
\]
Then for any \((n, \sigma) \in \mathbb{N} \times \{+, -\}\), for any \(\zeta \in \text{supp}(\psi_{\Theta, n, \sigma})\), we have
\[
|\mathcal{F}(\partial^n ((\rho_a J u) \circ \kappa_d)) (\zeta)| \leq C 2^{(n+1)k} |\mathcal{F}((\rho_a J u) \circ \kappa_d)(\zeta)|
\]
By Lemma 2 there exists \(C'_1 = C'_1(k, \|\hat{\rho}\|_{C^p})\) such that
\[
\|((\hat{\rho} u) \circ \kappa_d)\|_{\Theta, p, q} \leq C'_1 \|u\|_{\Theta, p, q}
\]
We have
(i) If \(k = 0\), then the boundedness of \(J\) from \(W_{\Theta, p, q}(\mathbb{T}^2)\) to \(W_{\Theta, p, q}(\mathbb{T}^2)\) follows from Lemma 2.
(ii) If \(k = 1\), then the boundedness of \(J\partial^a\) from \(W_{\Theta', p+1, q+1}(\mathbb{T}^2)\) to \(W_{\Theta, p, q}(\mathbb{T}^2)\) follows from (i), (4.30), (4.31), (4.32), and Parseval’s identity.

Otherwise, assume that our lemma is true for \(1, \cdots, k - 1\). By (4.32) for \((\hat{\rho}_\beta, \partial^\beta u)\) in place of \((\hat{\rho}, u)\) and the induction hypothesis, there exist a constant \(C'_2\) depending only on \(k - 1\) and sup \(\beta_1, \cdots, \beta_{k-1}\) \(\|\hat{\rho}_\beta\|_{C^p}\) such that
\[
\|((\hat{\rho}_\beta \partial^\beta u) \circ \kappa_d)\|_{\Theta, p, q} \leq C'_2 \|u\|_{\Theta, p, q+k+2}
\]
Then we verify our lemma for \(k\) by (4.30), (4.31), (4.32), (4.33) and Parseval’s identity. This completes the induction and thus conclude the proof. □

LEMMA 6. Let \(r, r', \ell, \gamma_0, \theta, f\) be given by Proposition 1. Then there exist \(C, M > 0, \alpha \in (0, 1),\) an open neighbourhood of \(f\) in \(C^r(\mathbb{T}^2, \mathbb{T}^2)\) denoted by \(\mathcal{U}\), and Banach spaces \(\mathcal{B}^0 \supset \mathcal{B}^1 \supset \cdots \supset \mathcal{B}^{r+1}\) satisfying that for all \(2 \leq i \leq r' + 1, \mathcal{B}^i \subset C^{i-2}(\mathbb{T}^2)\) and for all \(1 \leq i \leq r' + 1,\) the inclusion \(\mathcal{B}^i \subset \mathcal{B}^{i-1}\) is compact. Moreover, for any \(\mathcal{C}^{r+1}\) family in \(\mathcal{U}\), denoted by \(\{f_l\}_{l \in (-1, 1)}\), there exists constant \(C_1 > 0\) depending only on \(\|\{f_l\}_{l \in (-1, 1)}\|_{r'+1, r}\), such that \(\{\mathcal{P}_{f_l}\}_{l \in 1}\) is \((\alpha, M, C, C_1)\) adapted to \(\{\mathcal{B}^i\}_{i=0}^{r+1}\).

Proof. Since \(m(f) < 1\), by (3.3), there exist \(m < 1\) and a \(C^r\) open neighbourhood of \(f\) in \(C^r(\mathbb{T}^2, \mathbb{T}^2)\), denoted by \(\mathcal{U}_1\) such that \(m(f') < m\) for all \(f' \in \mathcal{U}_1\).

We choose \(\hat{\Theta} = (\hat{\zeta}_\pm, \phi_\pm), \hat{\Theta} = (\hat{\zeta}_\pm, \phi_\pm)\) such that \(\hat{C} < \hat{\zeta}\) and satisfy (4.5), (4.6) for \(f\) and \(C_0 = C(\theta),\) and \(\mathbb{R}(0, 1)\) is contained in the interior of \(\hat{C}\) except for \(\mathbb{R}(0, 1)\). Then there are an open neighbourhood of \(f\) in \(\mathcal{U}_1\), denoted by \(\mathcal{U}_2\), such that properties (4.5), (4.6) are satisfied for any \(f' \in \mathcal{U}_2\) in place of \(f\). Then fix a sequence of polarisations \(\{\Theta_k = (\zeta_{k, \pm}, \phi_{k, \pm})\}_{k=0}^{r' + 1}\) such that
\[
\Theta < \Theta_0 < \Theta_1 < \cdots < \Theta_{r'} < \Theta_{r'+1} < \Theta
\]
and define for \(0 \leq k \leq r' + 1\) that
\[
B^k = W_{\Theta_k, |\mathcal{J}| - r' - 5 + k, k}
\]
By Lemma 1 we have \(\mathcal{B}^i \subset C^{i-2}(\mathbb{T}^2)\) for all \(2 \leq i \leq r' + 1\), and the inclusion \(\mathcal{B}^i \subset \mathcal{B}^{i-1}\) is compact for all \(1 \leq i \leq r' + 1\).
On differentiability of SRB measures

For any integer $1 \leq k \leq r’ + 1$, we denote
$$
\alpha_k = \max(e^{(2k+3)hM}, e^{(r-2r'+2k-8)h} e^{-(r-2r'+2k-12)})
$$
Then we have $\alpha_k > m_0(f', h, |x| - r' - 5 + k, k)$ for all $f' \in U_2$, where $m_0$ is defined in (4.11).

By $r' \leq \frac{r}{2} - 9$, we have for any $0 \leq k \leq r' + 1, (p, q) := (|x| - r' - 5 + k, k)$ that
$$
p \geq q + 3, \quad p, q \in \left[0, r - \frac{3}{2}\right)
$$
We take an arbitrary
$$
h \in (0, \min\left(\frac{r - 2r'}{r - 2r'} - \frac{10}{6} \log \frac{r}{2r'} + 5\right))
$$
It is direct to verify that
$$
\alpha_0 := \sup_{1 \leq k \leq r'+1} \alpha_k = \max(e^{(2r'+5)hM}, e^{(r-2r'+6)h} e^{-(r-2r'-10)}) < 1
$$
We let $N_0 > 0$ be sufficiently large so that $U^{h,0}$ contains a $C'$ open neighbourhood of $f$, denoted by $U_3$. We assume that $U_3 \subset U_2$.

Take any $\alpha \in (\alpha_0^\frac{1}{r}, 1)$. We can apply Corollary A to see that there exist $C, M > 1$ such that for any $f' \in U_3$, any $u \in C^8(T^2)$, any $n \geq 1$ that
$$
\|P_{f'}u\|_{B^0} \leq M\|u\|_{B^0},
\|P_{f'}^n u\|_{B^k} \leq C\|u\|_{B^k} + CM^n\|u\|_{B^{k-1}}, \quad 1 \leq k \leq r' + 1
$$

Given any $C^{r'+1}$ family in $U_3$ denoted by $\{f_t\}_{t \in (-1, 1)}$. Let $Q_{1, t}, \cdots, Q_{r'+1, t}$ be defined by (4.27). We then let
$$
Q_j := Q_{j, 0}, \quad \forall j = 1, \cdots, r'
$$
We let $\Delta_0(t) = P_t$ for all $t \in (-1, 1)$. By (4.25), for any $1 \leq j \leq r' + 1$,
$$
\Delta_j(t) = \int_0^t dt_1 \cdots \int_0^{t_{k-1}} dt_k (\partial_j^k P_t u)(t_k) \quad \forall t \in (-1, 1)
$$
Then by Lemma 5 (4.34), (4.27), (4.28), (4.29) there exists $C_1$ depending only on $\left\|\{f_t\}_{t \in (-1, 1)}\right\|^{r'+1, r}$, such that
$$
\|Q_j\|_{B^j \rightarrow B^{j-1}} \leq C_1, \forall j = 1, \cdots, r'; i = j, \cdots, r' + 1
\|\Delta_j(t)\|_{B^j \rightarrow B^{j-1}} \leq C_1 |t|^j, \forall t \in (-1, 1), j = 0, \cdots, r' + 1, i = j, \cdots, r' + 1
$$
This concludes the proof.

Now we can prove Proposition.
Proof of Proposition 1: Let \( f \) be given by Proposition 1. We let \( B^0, \ldots, B^{r'+1} \) and \( U, C, M, \alpha \) be given by Lemma 6. Then for any \( f \in U \), \( \mathcal{P}_f \) extends to a bounded operator from \( B^i \) to \( B^i \) for all \( 0 \leq i \leq r' + 1 \).

For any \( f' \) in an open neighbourhood of \( f \) in \( C^r(\mathbb{T}^2, \mathbb{T}^2) \), we denote \( s(f, f') = d_{C^r}(f, f')^{\frac{1}{r}} \), and define \( \{ F^f_i \}_{t \in (-1,1)} \), a \( C^r \) family in \( U \) by

\[
F^f_t(x, y) = (1 - \Pi(\frac{t}{s(f, f')}) f(x, y) + \Pi(\frac{t}{s(f, f')}) f'(x, y)
\]

where \( \Pi \in C^\infty(\mathbb{R}, [0, 1]) \) such that \( \Pi(t) = \begin{cases} 0, & t < \frac{1}{2}, \\ 1, & t > \frac{1}{2} \end{cases} \), and we assume that \( d_{C^r}(f, f') \ll 1 \) so that the addition, the right hand side is interpreted as the linear interpolation between two nearby points \( f(x, y), f'(x, y) \in \mathbb{T}^2 \). Then \( \{ F^f_i \}_{t \in (-1,1)} \) constructed above satisfies that \( F^f_0 = f, F^f_{s(f, f')} = f' \), and \( \| \{ F^f_i \}_{t \in (-1,1)} \|_{r'+1} < C(f, \Pi) \). Then by Lemma 6 there exists a constant \( C_1 > 0 \) depending only on \( f \) and \( \Pi \) such that for any \( f' \) sufficiently close to \( f \) in \( C^r \), \( \{ \mathcal{P}_f^i \}_{t \in (-1,1)} \) is \( (\alpha, M, C, C_1) \) adapted to \( \{ B^i \}_{0 \leq i \leq r'+1} \).

By Lemma 6 and Hennion’s theorem in [18], for \( 1 \leq i \leq r' + 1 \), \( Sp(\mathcal{P}_f : B^i \to B^i) \cap \{ z \|z\| > \alpha \} \) contains isolated eigenvalues of finite multiplicity. By (4.12), \( 1 - \mathcal{P}_f \) is non-invertible in \( B^i \) for all \( 1 \leq i \leq r' + 1 \). Thus 1 is an eigenvalue of \( \mathcal{P}_f \) with finite multiplicity in \( B^i \). By our hypothesis that \( f \) is ergodic with respect to the Lebesgue measure on \( \mathbb{T}^2 \), we have that for all \( 1 \leq i \leq r' + 1 \), \( \text{Ker}((1 - \mathcal{P}_f) : B^i \to B^i) = \mathbb{R}u \) for function \( u \equiv 1 \in C^\infty(\mathbb{T}^2) \).

Let \( \kappa > 0 \) be a constant such that 1 is the only eigenvalue of \( \mathcal{P}_f : B^i \to B^i \) in \( B(1, \kappa) \) for all \( 1 \leq i \leq r' + 1 \). By Theorem 3 for all \( 1 \leq i \leq r' + 1 \), for all \( z \in \partial B(1, \kappa) \), we have

\[
\sup_{f', s(f, f') \leq \epsilon} \| (z - \mathcal{P}_f^i) - (z - \mathcal{P}_f) \|_{B^i \to B^i} \to 0 \quad \text{when} \ \epsilon \to 0
\]

Moreover, this convergence is uniform for all \( z \in \partial B(1, \kappa) \). By Lemma 6 the inclusion \( B^i \subset B^{-1} \) is compact for all \( 1 \leq i \leq r' + 1 \). Then using the by-now standard argument in [19], we see that there exists \( \delta > 0 \) such that for all \( f' \) such that \( d_{C^r}(f, f') < \delta \), \( \mathcal{P}_f \) has a unique simple eigenvalue in \( B(1, \kappa) \).

We now show that any \( f' \) sufficiently close to \( f \) has a unique SRB measure. We define for \( f' \) sufficiently close to \( f \) the spectral projection at 1 by \( \Pi_{f'} \). Then we have

\[
(4.35) \quad \Pi_{f'} = \frac{1}{2\pi i} \int_{|z| = \epsilon} (z - \mathcal{P}_{f'})^{-1} dz
\]

Moreover, denote \( \rho_{f'} := \Pi_{f'} 1 \in B^2 \), then it is standard to see that \( \rho_{f'} d\text{Leb} \) is \( f' \)-invariant.
For all $f'$ sufficiently close to $f$ in $C^r$, $\{\mathcal{P}_{f'}\}_{t \in (-1, 1)}$ is $(\alpha, 2M, C, C_1)$ adapted to $\{B^2, B^3\}$. Then by Theorem 3 we have for all $z, |z - 1| = \kappa$ that
\[
\| (z - \mathcal{P}_{f'})^{-1} - (z - \mathcal{P}_f)^{-1} \|_{B^1 \to B^2} \leq C_2 d_{C^r}(f, f')^{\frac{1}{2}}.
\]
By Lemma 1 and (4.35), we have
\[
\lim_{f' \to f, d_{C^r}(f', f) \to 0} \| \Pi_{f'} - \Pi_f \|_{C^0} \lesssim \lim_{f' \to f, d_{C^r}(f', f) \to 0} \| \Pi_{f'} - \Pi_f \|_{B^2} = 0.
\]
While it is clear that $f'$ sufficiently close to $f$ in $C^r(\mathbb{T}^2, \mathbb{T}^2)$, $\rho_{f'}(z) \geq \frac{1}{2}$ for all $z \in \mathbb{T}^2$. Then $\rho_{f'}d\text{Leb}$ is necessarily the unique SRB measure of $f'$. Let $\mathcal{U}$ be a sufficiently small $C^r$ open neighbourhood of $f$ satisfying all the above conditions for $f'$.

Given a $C^{r+1}$ family in $\mathcal{U}$ denoted by $\{f_t\}_{t \in (-1, 1)}$. For any $\varphi \in C^r(\mathbb{T}^2)$, any $t \in (-1, 1)$, we have
\[
\int \varphi d\mu_t = (\Pi_{f_t}(1), \varphi)_{L^2} = \frac{1}{2\pi i} \int_{|z-1|=\kappa} ((z - \mathcal{P}_{f_t})^{-1}, \varphi)_{L^2} dz
\]
Then our proposition follows from Lemma 6 and Theorem 3. \hfill \Box

5. NONDIFFERENTIABILITY OF U-GIBBS STATES

Our construction is inspired by a theorem of Halperin in the study of Anderson-Bernoulli model, stated in the Appendix of [23]. The argument in [23] is of spectral nature, and made essential use of the self-adjointness of the Schrödinger operators. Our argument is purely dynamical and focused on exploiting monotonicity and periodicity. This proof should shed some light on the study of the regularity of the density of states of 1D Schrödinger operators with strongly mixing potentials.

5.1. Markov partitions. In this section, we define for $f$ that is either a partially hyperbolic system, or an Anosov system, or a strictly expanding map, a family of submanifolds that approximate the unstable manifolds of $f$. Note that for our later purpose, we only need to ensure that for any such submanifold, its image after long iterations can be almost decomposed into submanifolds in the same class. This makes our definition much simpler than the ones used in [13].

For any compact Riemannian manifold $X$, any precompact submanifold $D \subset X$, we denote by $\text{Vol}|_D$ the normalised volume form on $D$ induced by the restriction of the Riemannian metric on $D$. The normalisation ensures that $\text{Vol}|_D(D) = 1$.

Let $f : X \to X$ be either a partially hyperbolic or an Anosov system. We denote by $E^u(x)$ the unstable subspace at $x$ of dimension $d_u$, and let $K = \{K(x)|E^u(x) \subset K(x) \subset T_xX\}_{x \in X}$ denote a continuous family of cones containing $E^u(x)$ such that the closure of $Df(K(x))$ is contained in the interior of $K(f(x))$ except for the origin.
**Definition 5.** Let $f, K$ be given as above. For any $\epsilon \in (0, 1), C_2 > 0$, we denote by $A_{\epsilon, C_2, K}(f)$ the set of submanifolds $D = \Phi((0, \epsilon)^d_0)$, where $\Phi : (0, 2\epsilon)^d_0 \to X$ is a $C^2$ immersion such that $\|\Phi^{-1}\|_{C^2}, \|\Phi\|_{C^2} < C_2, T\mathcal{D}(x) \subset K(x), \forall x \in \mathcal{D}$.

It is a standard fact that we can choose $K$ such that for all $f'$ sufficiently close to $f$ in $C^1(X, X)$, the closure of $Df'(K(x))$ is contained in the interior of $K(f'(x))$ except for the origin. Moreover, there exists a constant $C_3$ depending only on $\|f\|_{C^2}$ such that for any $D \in A_{\epsilon, C_2, K}(f)$, for any $n \geq 1$, let $\rho$ be the density of $(f^n)^* (Vol|_D)$ with respect to $Vol|_{f^n(D)}$. Then we have

$$|\log \rho(y_1) - \log \rho(y_2)| \leq C_3d_{f^n(D)}(y_1, y_2), \quad \forall y_1, y_2 \in f^n(D)$$

As a consequence, we have the following result. The proof is a standard exercise, which we omit.

**Lemma 7.** Let $C^2$ map $f : X \to X$ be either a partially hyperbolic system or an Anosov system. Then there exists a continuous family of cones $K = \{K(x)|E^u(x) \subset K(x) \subset T_xX\}_{x \in X}$, a constant $C_2 > 0$ such that the following is true. For any $x \in X$ the closure of $Df(K(x))$ is contained in $K(f(x))$ except for the origin. Moreover, for any $\kappa \in (0, 1)$ there exists $\epsilon_0 = \epsilon_0(f, \kappa)$ with the following property. For any $\epsilon \in (0, \epsilon_0)$, there exist $N_0 = N_0(\epsilon) > 0$ and a $C^2$ open neighbourhood of $f$, denoted by $U$, such that for any $f' \in U$, any $D \in A_{\epsilon, C_2, K}(f)$, any integer $N > N_0$, there exist disjoint $D_1, \cdots, D_l \in A_{\epsilon, C_2, K}(f)$, constants $c_1, \cdots, c_l > 0$ such that for all $1 \leq i \leq l$, we have $D_i \subset f^{N}(D)$, and

$$\sum_{i=1}^{l} c_i Vol|_{D_i} \leq (f^N)^* (Vol|_D) \text{ and } \sum_{i=1}^{l} c_i > 1 - \kappa$$

In the following, for any $f$ that is either a partially hyperbolic system or an Anosov system, we will always choose $K, C_2$ as in Lemma 7. We will briefly denote $A_{\epsilon, C_2, K}(f)$ by $A_{\epsilon}(f)$. When $f$ denotes a strictly expanding map, we define $A_{\epsilon}(f)$ to be the collection of balls in $X$ of radius $\epsilon$.

**5.2. Conditions for the construction.** As usual, we let $SL(2, \mathbb{R})$ denote the special linear group acting on $\mathbb{R}^2$. We have a canonical action of $SL(2, \mathbb{R})$ on $\mathbb{P}(\mathbb{R}^2)$. We use map $\psi : \mathbb{P}(\mathbb{R}^2) \to T, \psi(\mathbb{R}(\cos \pi \theta, \sin \pi \theta)) = \theta$ to identify $\mathbb{P}(\mathbb{R}^2)$ with $T$. For any $H \in SL(2, \mathbb{R})$, we denote $\hat{H} = \psi H \psi^{-1} \in Diff^0(T)$.

Let $H_0 \in SL(2, \mathbb{R})$ be a hyperbolic element with eigenvalues $e^\alpha, e^{-\alpha}$. Let $u_0, s_0 \in T$ be respectively the sink and source of $\hat{H}_0$. Then for all $H \in SL(2, \mathbb{R})$ sufficiently close to $H_0$, $H$ is still a hyperbolic element. Let $u(H), s(H) \in T$ be respectively the sink and source of $\hat{H}$. Then we can easily verify that for all $H$ sufficiently close to $H_0$ the following is true,

**(HYP):** there exists a constant $c > 0$ such that for any $\delta \in (0, \frac{1}{2})$,

$$\hat{H}^n(T \setminus B(s(H), \delta)) \subset B(u(H), c\delta^{-1}e^{-n\alpha}), \quad \forall n \geq 1$$
We let \( C_0 = \hat{H}_0 \in \text{Diff}^{\infty}(T) \). Let \( B_0 \in \text{Diff}^{\infty}(T) \) satisfy that \( B_0u_0 = s_0 \). We denote by \( \hat{C}, \hat{B} : \mathbb{R} \to \mathbb{R} \) respectively lifts of \( C_0, B_0 \). Let \( \hat{u}_0, \hat{s}_0 \in [0, 1) \) be respectively lifts of \( u_0, s_0 \). Without loss of generality, we can assume that:

(a) \( \hat{u}_0, \hat{s}_0 \) are both fixed by \( \hat{C}, \hat{B}(\hat{u}_0) = \hat{s}_0 \), (2) \( \hat{B}(\hat{u}_0) = \hat{s}_0 < \hat{u}_0 \).

Let \( M \) be a compact Riemannian manifold. Let map \( g : M \to M \) be either a \( C^{r} \) transitive Anosov diffeomorphism, or a \( C^{r+\epsilon} \) strictly expanding map. We denote by \( m \) the unique SRB measure of \( g \).

We denote by \( p_1 : M \times T \to M \), \( p_2 : M \times T \to T \) be the canonical projections. We let \( f : M \times T \to M \times T \) be a \( C^{r} \) map defined by

\[
(5.1) \quad f(z, x) = (g(z), A(z, x))
\]

where \( A : M \times T \to T \) is a \( C^{r} \) map.

We will assume that \( f \) satisfies the following,

(a) \( \sup_{z \in M} \|DA(z, \cdot)\| \) is small enough so that \( f \) is partially hyperbolic,

(b) there exists \( \hat{f} : M \times \mathbb{R} \to M \times \mathbb{R} \) such that for any \( (z, x) \in M \times \mathbb{R} \), \( \hat{p}_2 \hat{f}(z, x + 1) = \hat{p}_2 \hat{f}(z, x) + 1 \) and \( \pi \hat{f} = f \pi \), where \( \hat{p}_2 : M \times \mathbb{R} \to \mathbb{R} \), \( \pi : M \times \mathbb{R} \to M \times T \) are the canonical projections,

(c) there is an open set \( C \subset \{ z \mid A(z, \cdot) = C_0 \} \), an open set \( B \subset \{ z \mid A(z, \cdot) = B_0 \} \) and constants \( \kappa, c_0 \in (0, 1) \), such that

\[
m(C) > 1 - \kappa, \quad m(B) > c_0, \quad m(C \cup B) < 1
\]

(d) there exist \( z \in C \cap \mathbb{supp}(m) \) and an integer \( q \geq 1 \) such that \( g^q(z) = z \) and \( f^q(z, s_0) \neq (z, s_0) \).

We take an arbitrary \( \epsilon_1 \in (0, d(z, \partial C)) \) and denote map \( D : T \to T \) by

\[
D(x) = p_2f^q(z, x), \quad \forall x \in T
\]

We let \( \epsilon_2 > 0 \) be a constant such that \( D(B(s_0, \epsilon_2)) \cap B(s_0, \epsilon_2) = \emptyset \). Without loss of generality, we assume that \( \epsilon_1, \epsilon_2 \in (0, \epsilon_0(f, \frac{\epsilon}{2})) \), where \( \epsilon_0 \) is given by Lemma 7.

(e) there exists a constant \( \epsilon \in (0, \min(\epsilon_1, \epsilon_2)/10) \) such that the following is true. For any \( D \in A_\epsilon(f) \), there exist disjoint \( E_1, \cdots, E_l \in A_\epsilon(f) \), and \( d_1, \cdots, d_l > 0 \) such that for all \( 1 \leq i \leq l, \)

\[
\overline{E}_i \subset f(D) \cap (C \times T), \quad \sum_{i=1}^{l} d_i |\text{Vol}_{E_i} - f_*|\text{Vol}_{D}| \geq \sum_{i=1}^{l} d_i > 1 - \kappa
\]

Similarly, there exist disjoint \( F_1, \cdots, F_k \in A_\epsilon(f) \), and \( h_1, \cdots, h_k > 0 \), such that for all \( 1 \leq i \leq k, \)

\[
\overline{F}_i \subset f(D) \cap (B \times T), \quad \sum_{i=1}^{k} h_i |\text{Vol}_{F_i} - f_*|\text{Vol}_{D}| \geq \sum_{i=1}^{k} h_i > c_0
\]

(f) Let \( \epsilon > 0 \) be as in (e). There exists closed interval \( J_1 \subset T \setminus \{ s_0 \} \) such that there is a constant \( K \in \mathbb{N} \), such that for each \( D \in A_\epsilon(f) \), there exists \( D' \in A_\epsilon(f) \) satisfying \( D' \subset f^K(D) \cap (C \times J_1) \). We let \( f \) be
a closed interval contained in \( T \setminus \{ s_0 \} \) such that \( J_1 \subseteq J \). We denote \( \hat{f} = \pi_{R \to T}^{-1}(f) \cap [0,1) \).

(g) for any \( \varepsilon' > 0 \), any sequence \( \{ D_i \}_{i \geq 0} \subseteq A_{\varepsilon'}(f) \), any strictly increasing \( \{ K_i \}_{i \geq 0} \subseteq \mathbb{N} \), the accumulating points of \( (f^{K_n})_*(Vol|_{D_n}) \) are contained in \( uGibbs(f) \).

**Remark 2.** It is clear there exists \( \sigma > 0 \) such that properties (a), (b), (c), (d), (e) are satisfied for any \( f' : M \times T \to M \times T \) satisfying that \( d_{C'}(f, f') < \sigma \), \( p_1f' = p_1f \), and that \( f'(z, \cdot) \neq f(z, \cdot) \) is contained in a \( \sigma \)-ball.

**Remark 3.** We now explain the applicability of the above conditions. Given any \( C' \) map \( A : M \times T \to T \), we can make (a) valid via replacing \( g \) by any large power of \( g \). Condition (b) is valid for any \( A \) that is \( C^0 \) close to maps of the form \( A_0 : M \times T \to T, A_0(z, x) = x + \varphi(z) \), and for any \( g \in C'(M, M) \). For any \( \kappa, c_0 \in (0, 1), \kappa + c_0 < 1 \), we can choose \( A \) satisfying condition (c) since \( m \) has no atoms. The validity of (d) is easily satisfied. For any \( \varepsilon > 0 \), we can make (e) valid via replacing \( g \) by any large power of \( g \); this is obvious for strictly expanding \( g \); for Anosov map \( g \), this follows from (e). Lemma 7 We will verify (f) in Lemma 8

**Lemma 8.** If we have (a), (b), (c), (d), then we have (f).

**Proof.** Let \( \varepsilon \) be in (c). Let \( z \in C, q \in \mathbb{N} \) be given by (d). We denote \( C' = B(z, \varepsilon) \). By \( \varepsilon < \varepsilon_1 \), we have \( C' \subseteq C \). We denote \( J' = T \setminus B(s_0, 2\varepsilon) \). We first show that

\[
(5.2) \quad \inf_{\mu: f_*\mu = \mu, \ (p_1)_*\mu = m} \mu(C' \times J') > 0
\]

Indeed, if (5.2) was false, then there would exists a sequence \( \{ c_n > 0 \}_{n \geq 1}, \{ \mu_n, f_*\mu_n = \mu_n, (p_1)_*\mu_n = m \}_{n \geq 1} \) such that \( \lim_{n \to \infty} c_n = 0 \) and \( \mu_n(C' \times J') < c_n \) for all \( n \geq 1 \). Let \( \mu \) be an accumulating point of \( \mu_n \). It is clear that \( \mu \) is \( f \)-invariant and \( (p_1)_*\mu = m \). Moreover, we have

\[
\mu(C' \times J') \leq \liminf_{k \to \infty} \mu_k(C' \times J') \leq \liminf_{k \to \infty} c_k = 0
\]

Thus \( \mu(C' \times J') = 0 \). This implies that for \( m \) almost every \( z' \in C' \) the conditional measure \( \mu_{z'} \) on \( \{ z' \} \times T \simeq T \) is supported in \( B(s_0, 2\varepsilon) \). By \( z \in \text{supp}(m) \), we can let \( z', z'' \) be two \( m \) generic points sufficiently close to \( z \), such that \( z'' = g^a(z') \), and \( \mu_{z'a}, \mu_{z''a} \) are supported in \( B(s_0, 2\varepsilon) \). Moreover the map \( D_{z',z''} : T \to T \) defined by \( D_{z',z''}(x) = p_2f^a(z', x) \), satisfies

\[
D_{z',z''}(B(s_0, 2\varepsilon)) \cap B(s_0, 2\varepsilon) = \emptyset
\]

By the \( f \)-invariance of \( \mu \), for a generic choice of \( z', z'' \) as above, we have \( D_{z',z''}\mu_{z'} = \mu_{z''} \). This is a contradiction.

We claim that there exist arbitrarily large \( K \) such that for any \( D' \in A_{\varepsilon/2}(f) \), \( f^K(D') \cap (C' \times J') \neq \emptyset \). Indeed, if there was a sequence \( \{ D_n \}_{n \geq 1} \subseteq A_{\varepsilon/2}(f) \), a strictly increasing sequence \( \{ K_n \}_{n \geq 1} \subseteq \mathbb{N} \), such that for any \( n \geq 1 \), \( f^{K_n}(D_n) \cap (C' \times J') = \emptyset \). We let \( \mu \) be an accumulating point of \( (f^{K_n})_*(Vol|_{D_n}) \),
then $\mu(C' \times J') = 0$. By Lemma 8, $\mu \in uGibbs(f)$. Then it is clear that $(p_1)_* \mu = m$. But then $\mu(C' \times J') > 0$. Contradiction. Thus our claim is true.

We let $J'_1 = T \setminus B((s_0), \varepsilon/2)$. Then it is clear that $J' \subset J_1$ and $d(J', J'_1) \geq 3\varepsilon/2$. Take an arbitrary $D \in A_{\varepsilon/2}(f)$. We choose $D_0 \in A_{\varepsilon/2}(f)$ such that $D_0 \subset D$ and $d(D_0, \partial D) > \varepsilon/2$, where $C_2$ is in the definition of $A_{\varepsilon}(f)$. For any $K_0 > 0$, by our claim above there exists $K = K(K_0, \varepsilon) > K_0$, independent of the choice of $D, D_0$, such that $f^K(D_0) \cap (C' \times J') \neq \emptyset$. Let $(z', x')$ be a point in $f^K(D_0) \cap (C' \times J')$. Then by letting $K_0$ to be sufficiently large, we can find a neighbourhood of $(z', x')$ in $f^K(D)$, denoted by $D'$, such that $D' \in A_{\varepsilon}(f)$ and $\text{diam}(D') < \varepsilon_0$. Since $d(C' \times J', C \times J_1) > 3\varepsilon/2$, we have $D' \subset C \times J_1$. This concludes the proof. □

Remark 3 and Lemma 8 suggest a way of constructing dynamics satisfying condition (v) to (vii), as the following proposition shows.

**Proposition 4.** For any $\kappa \in (0, 1), c_0 \in (0, 1 - \kappa)$, there exists a partially hyperbolic, stably dynamically coherent, $u$-convergent, mostly contracting diffeomorphism $f$ on $T^3$ satisfying (iv) to (vi).

**Proof.** We will follow Example (a), Section 12 in [12]. Let $M = T^2$ and let $g : M \to M$ be a linear Anosov diffeomorphism. It is known that the Lebesgue measure on $M$, denoted by $m$, is the unique SRB measure for $g$. We let $C_0, B_0$ be projective actions of $SL(2, \mathbb{R})$ on $T$, satisfying the conditions in the beginning of this section. Let $C, B$ be two disjoint open sets of $M$ satisfying $m(C) > 1 - \kappa, m(B) > c_0$ and $m(C \cup B) < 1$. We let $S : M \to SL(2, \mathbb{R})$ be a $C'$ map such that $S|_C \equiv C_0, S|_B \equiv B_0$. We let $A : M \times T \to M$ be defined by $A(z, x) = \hat{S}(z)(x)$, so that (i) is satisfied. By choosing $C_0, B_0$ to be close to rotations, it is easy to choose $S$ so that (ii) is also satisfied. Since $C$ is an open set, and $m(C), m(M \setminus \overline{C \cup B}) > 0$, there exists a $g$ periodic point $z \in C \cap \text{supp}(\mu)$, and the $g$ orbit of $z$ intersects $M \setminus \overline{C \cup B}$. We can make an arbitrarily small modification on $S$ outside of $C \cup B$ so that (iii) is satisfied, and the image of $S$ generates $SL(2, \mathbb{R})$. Moreover, any such modification will not ruin (iv), (v). Now let $\varepsilon_1, \varepsilon_2$ be defined in (iv), and let $\varepsilon = \frac{\min(\varepsilon_1, \varepsilon_2)}{2}$. Let $q > 0$ be the period of $z$, i.e. $g^q(z) = z$, and define $D : T \to T$ by $D(x) = p_2 f^q(z, x)$. For integer $n \geq 1$, we define $f_n : M \times T \to M \times T$ by

$$f_n(z', x') = (g^{nq + 1}(z'), A(z', x')), \quad \forall (z', x') \in M \times T$$

and define $D_n : T \to T$ by $D_n(x) = p_2 f_n(x, z)$. It is direct to verify that $D = D_n$ for all $n \geq 1$. In particular, constant $\varepsilon_2$ is valid for all $f_n, n \geq 1$ in place of $f$.

By Remark 3 (a), (c) are satisfied when we replace $g$ by any sufficiently large power of $g$. Since the center foliation of $g$ is a $C^1$ foliation, this is known to imply stably dynamically coherence. Moreover, by the discussion in Example (a), Section 12, after replacing $g$ by any sufficiently
large power of $g$, $f$ become $u$-convergent and mostly contracting. Then $f$ satisfies Theorem II in [12], thus (b) is verified by Corollary 6.3 in [12]. By Lemma 8 we can replace $g$ by $g^{nq+1}$ for sufficiently large $n$, so that $f$ satisfies the conditions of Theorem II in [12] and (a) to (e).

5.3. **Proving nondifferentiability.** The main result of this section is the following.

**Proposition 5.** Let $r = 2, 3, \cdots, \infty$ and $f : M \times \mathbb{T} \to M \times \mathbb{T}$ be a $C^r$ map given by (5.1) satisfying (a), (b), (c), (d), (e). Then there exists a $C^r$ family \( \{f_t\}_{t \in (-1, 1)} \) of partially hyperbolic systems through $f$, a function $\phi \in C^r(M \times \mathbb{T}, \mathbb{R})$, such that for any map $t \mapsto \mu_t \in uGibbs(f_t)$, the map $t \mapsto \int \phi d\mu_t$ is not $\beta$-Hölder at $t = 0$ for any $\beta > \frac{-6 \log(1-x)}{\bar{\alpha}}$.

The following is an immediate corollary of Proposition 5 and Remark 2.

**Corollary B.** For any $f : M \times \mathbb{T} \to M \times \mathbb{T}$ given in Proposition 5 there exists $\sigma > 0$ such that for any $f^t : M \times \mathbb{T} \to M \times \mathbb{T}$ satisfying that $d_{C^r} (f, f^t) < \sigma$, $p_1 f^t = p_1 f$, and that $f^t (z, \cdot) \neq f(z, \cdot)$ is contained in a $\sigma$-ball, the same conclusion of Proposition 5 holds for $f^t$ in place of $f$.

**Proof of Proposition 5.** By (5.1)

\[
f(x, y) = (g(x), A(x, y))
\]
We let $m$ be the SRB measure of $g$, let $\varepsilon$ be given by (e), let $K \in \mathbb{N}, J_1 \Subset J \subset \mathbb{T}$ be given by (f).

We can define $f_t$ for $t \in (-1, 1)$ by

\[
f_t(z, x) = (g(z), A(z, x) + t), \quad \forall (z, x) \in M \times \mathbb{T}
\]
Let $\hat{f} : M \times \mathbb{R} \to M \times \mathbb{R}$ be given by (b). For each $t \in (-1, 1)$ we define $\hat{f}_t : M \times \mathbb{R} \to M \times \mathbb{R}$ by

\[
\hat{f}_t(z, x) = (g(z), \hat{\rho}_2 \hat{f}(z, x) + t), \quad \forall (z, x) \in M \times \mathbb{R}
\]
It is clear that for any $t \in (-1, 1)$, $\hat{\rho}_2 \hat{f}_t(z, x + 1) = \hat{\rho}_2 \hat{f}_t(z, x) + 1, \forall (z, x) \in M \times \mathbb{R}$ and $\pi \hat{f}_t = f_t \pi$. For any $z \in M$, any $x \in \mathbb{T}$, set

\[
\phi_t(z, x) = \hat{\rho}_2 \hat{f}_t(z, \hat{x}) - \hat{x}
\]
where $\hat{x}$ is any element of $\pi_{\mathbb{R} \to \mathbb{T}}^{-1}(x)$. The right hand side of the above equality is independent of different choices of $\hat{x}$. We set

\[
\phi = \phi_0.
\]

Fix any $\beta > \frac{-6 \log(1-x)}{\bar{\alpha}}$. We will construct a sequence of real numbers \( \{t_i\}_{i \in \mathbb{N}} \) converging to 0, such that for any sequence of measures \( \{\mu_i \in uGibbs(f_{t_i})\}_{i \in \mathbb{N}}, \) we have

\[
| \int \phi d\mu_i - \int \phi d(m \times \text{Leb}_\mathbb{T}) | > |t_i|^\beta
\]
It is direct to see that $|\phi_t - \phi| \equiv |t|$ for any $t \in (-1, 1)$. Thus it suffices to show that there exists a sequence $\{t_i\}_{i \in \mathbb{N}}$ converging to 0 such that for any sequence $\{\mu_i \in u\text{Gibbs}(f_{t_i})\}_{i \in \mathbb{N}}$, we have

$$|\int \phi_t d\mu_t - \int \phi d(m \times \text{Leb}_T)| > 2|t_t|^\beta$$

For any $t \in \mathbb{R}$, we let $R_t : T \to T$ be the rigid translation by $t$, i.e. $R_t(x) = x + t, \forall x \in T$. Since by our choice $C_0 = \tilde{H}_0$, for any $t$ sufficiently close to 0, $C_t := R_tC_0$ is still given by a hyperbolic element. Let $u_t, s_t$ be respectively the continuations of $u_0, s_0$. By (HYP), there exist $c, t_1 > 0$ such that for any $t \in (-t_1, t_1)$, any $\delta \in (0, \frac{1}{4})$,

$$(5.3) \quad C^n_t(T \setminus B(s_t, \delta)) \subset B(u_t, c\delta^{-1}e^{-\kappa n}) \quad \forall n \geq 1$$

We denote $\hat{C}_t = C + t, \hat{B}_t = B + t$. Then $\hat{C}_t, \hat{B}_t$ are respectively lifts of $C_t, B_t$. We let $\hat{u}_t, \hat{s}_t$ be the fixed point of $\hat{C}_t$ which are respectively the continuations of $u_0, s_0$. We have following observation.

**Lemma 9.** There exists $\gamma_1 > 0$ such that for any $t$ sufficiently close to 0, we have

$$\partial_t u_t > \gamma_1, \quad \partial_t s_t < -\gamma_1$$

**Proof.** We omit the proof for it follows from elementary computations. $\square$

We define for any $(z, x) \in M \times T$, any $n \geq 1$, any $t \in (-1, 1)$, that

$$S_n(t, z, x) = \sum_{i=0}^{n-1} \phi_t f_i(z, x)$$

We have for any $\hat{x} \in \pi_{T \to T}^{-1}(x)$ that

$$S_n(t, z, x) = \hat{\rho}_2 f_i^n(z, \hat{x}) - \hat{x}$$

By monotonicity, it is clear that

$$(5.4) \quad S_n(\delta, z, x) - S_n(-\delta, z, x) \geq 0, \forall n \geq 0, \delta \in (0, 1), (z, x) \in M \times T$$

For any $D \in \mathcal{A}_c(f)$, any integer $N_0 > 0$, real number $\delta \in (0, 1)$, we define a sequence of random variables $X = X(D, N_0, \delta) = \{X_n\}_{n \geq 1}$ on probability space $(D, \text{Vol}_D)$, defined by

$$X_n(z, x) = S_{nN_0}(\delta, z, x) - S_{nN_0}(-\delta, z, x), \quad \forall (z, x) \in D$$

In the following, for any $D \in \mathcal{A}_c(f)$, any measurable subset $E \subset D$, any random variable $F : D \to \mathbb{R}$, we will use notations $\mathbb{P}_D(E), \mathbb{E}_D(F)$ to denote respectively $\text{Vol}_D(E)$ and $\int_D F d\text{Vol}_D$.

The following lemma is the main step in the proof.

**Lemma 10.** There exists $c_2 > 0$ such that the following is true. For any sufficiently large integer $L > 1$, any $\delta \in [e^{-4L}, \frac{1}{2})$, any $D_0 \in \mathcal{A}_c(f)$, we define a random variable $Z$ on probability space $(D_0, \text{Vol}_{D_0})$ by

$$Z(z, x) = S_{2L+1+K}(\delta, z, x) - S_{2L+1+K}(-\delta, z, x), \quad \forall (z, x) \in D_0$$

then we have $Z \geq 0$ and $\mathbb{P}_{D_0}(Z \geq 1) \geq c_2(1 - \kappa)^{2L}$. 
Proof. By (1) and that \( J_1 \in \mathcal{J} \), there exists \( t_0 > 0 \) such that for any \( D_0 \in \textcolor{red}{A}_s(f) \), there is a subset of \( D_0 \), denoted by \( D_1 \), such that for all \( t \in (-t_0, t_0) \), by letting \( D_1 := f^k(D_1) \), we have \( D_1 \in \textcolor{red}{A}_s(f) \) and \( D_1 \subset C \times J \). Moreover, let \( c_1 = c_1(f, K) > 0 \) such that for any \( D_0, D_1 \in \textcolor{red}{A}_s(f) \) satisfying \( D_1 \subset f^k(D_0) \), we have \( c_1 \text{Vol} |_{D_1} \leq (f^K)_*(\text{Vol}|_{D_0}) \).

Let \( D_0 \) be given in the lemma. By (5.4), we have \( Z(z, x) \geq 0 \) for all \((z, x) \in D_0\). To simplify notations, we denote \( \rho_0 := \text{Vol}|_{D_0} \).

We will inductively construct for all \( 0 \leq k \leq 2L \), a subset of \( D_1 \) denoted by \( U_k \), such that \( U_k \subset U_{k-1} \) for \( k \geq 1 \), and the following is satisfied,

1. For each \((z, x) \in U_k\), we have
   \[
   g^{k+k}(z) \in \begin{cases} 
   C, & k \in \{0, \ldots, 2L\} \setminus \{L\}, \\
   B, & k = L
   \end{cases}
   \]

2. There exists an integer \( l_k \geq 1 \), and disjoint \( D_i \subset \textcolor{red}{A}_s(f), 1 \leq i \leq l_k \) such that \( f^{k+k}(U_k) = \bigcup_{i=1}^{l_k} D_i \).

3. There exist \( a_1, \ldots, a_{l_k} > 0 \) such that
   \[
   (f^{k+k})_*\rho_0 \geq \sum_{i=1}^{l_k} a_i \text{Vol}|_{D_i}, \quad \sum_{i=1}^{l_k} a_i \geq \begin{cases} 
   (1-k)^k c_1, & 0 \leq k \leq L - 1 \\
   (1-k)^k c_1 c_0, & k = L \\
   (1-k)^k c_1 c_0, & L < k \leq 2L
   \end{cases}
   \]

For \( k = 0 \), we let \( U_0 = D_1 \). We denote \( I_0 = 1 \) and \( D^0_1 := D^1_0 \), then (1)-(3) are clear.

Assume that we have constructed \( U_i \) for all \( i \in \{0, \ldots, k\}, k \leq 2L - 1 \), we construct \( U_{k+1} \) as follows. Let \( \{D^k_{i_1}\}_{i_1=1}^{l_k} \) be given by (3). Then by (e), for each \( 1 \leq i \leq l_k \), there exist \( k_{i,j} \in \mathbb{N} \), disjoint \( F_{i,j} \subset \textcolor{red}{A}_s(f), 1 \leq j \leq l_{k,j} \) satisfying \( F_{i,j} \subset f(\mathcal{A}_k) \), and constants \( c_{k,i,j} > 0, 1 \leq j \leq l_{k,j} \) such that

\[
(5.5) \quad \sum_{j=1}^{l_{k,j}} c_{k,i,j} \text{Vol}|_{F_{i,j}} \leq f_*(\text{Vol}|_{D^k_i})
\]

and

\[
(5.6) \quad F_{i,j} \subset \begin{cases} 
C \times \mathbb{T}, k \neq L - 1 \\
B \times \mathbb{T}, k = L - 1
\end{cases}, \quad \sum_{j=1}^{l_{k,j}} c_{k,i,j} > \begin{cases} 
1 - \kappa, k \neq L - 1 \\
0, k = L - 1
\end{cases}
\]

Then we define

\[
U_{k+1} = \bigcup_{i=1}^{l_k} \bigcup_{j=1}^{l_{k,j}} f^{-(k+k+1)}(F_{i,j})
\]

It is direct to see (1),(2) for \( k + 1 \) in place of \( k \). It remains to verify (3). By (3) for \( k \) and (5.5), we have

\[
(f^{k+k+1})_*\rho_0 \geq \sum_{i=1}^{l_k} a_i f_*(\text{Vol}|_{D^k_i}) \geq \sum_{i=1}^{l_k} a_i \sum_{j=1}^{l_{k,j}} c_{k,i,j} \text{Vol}|_{F_{i,j}}
\]
Then by (3) for \( k \) and (5.6), we deduce (3) for \( k + 1 \). This concludes the induction. In particular, by (2),(3) for \( k = 2L \), we have

\[
\mathbb{P}_{D_0}(U_{2L}) \geq (1 - \kappa)^{2L} c_0 c_1
\]

We have the following.

**Lemma 11.** For any \((z, x) \in U_{2L}, Z(z, x) \geq 1\).

**Proof.** Without loss of generality, we can assume that \( \delta > 0 \) is sufficiently small, independent of \( L \). We choose an arbitrary \( \hat{x} \in \pi_{R \to T}^{-1}(x) \). For all \( 0 \leq k \leq 2L + 1 \), we denote

\[
(z_k, x_k^+) := f_{x, \hat{x}}^{k+L}(z, \hat{x})
\]

By \((z, x) \in U_{2L} \subset D_1\), there exists \( l \in \mathbb{N} \) such that

\[
x_0^+ \in \hat{J} + l, \quad z_k \in \begin{cases} C, & \forall k \in \{0, \ldots, 2L\} \setminus \{L\} \\ B, & k = L \end{cases}
\]

Thus we have the following relations.

\[
x_L^+ = \hat{C}_L(x_0^+), \quad x_{L+1}^+ = \hat{B}_{x, \hat{x}}(x_L^+), \quad x_{L+1}^+ = \hat{C}_L(x_{L+1}^+)
\]

By \( J \in \mathbb{T} \setminus \{s_0\}, s_0 \in [0, 1), \hat{J} \subset [0, 1) \), we have either \( \hat{J} \in (s_0, s_0 + 1) \) or \( \hat{J} \in (s - 1, s) \). We will prove our lemma assuming the first case \( \hat{J} \in (s_0, s_0 + 1) \) happens. The second case is similar.

Denote

\[
\bar{a} = \bar{a}_0 + l, \quad \bar{s} = \bar{s}_0 + l,
\]

\[
\bar{a}_\pm = \bar{a}_{\pm} + l, \quad \bar{s}_\pm = \bar{s}_{\pm} + l
\]

Then by Lemma 11 we have

\[
\bar{a}_+ > \bar{a} + \gamma_1 \delta, \quad \bar{s}_+ < \bar{s} - \gamma_1 \delta
\]

\[
\bar{a}_- < \bar{a} - \gamma_1 \delta, \quad \bar{s}_- > \bar{s} + \gamma_1 \delta
\]

Then there exists \( c_3 > 0 \), such that for all sufficiently small \( \delta > 0 \), and for all \( L \),

\[
x_L^+ \in (\bar{a}_+ - c_3 e^{-L_\alpha}, \bar{a}_+ + c_3 e^{-L_\alpha}), \quad x_L^- \in (\bar{a}_- - c_3 e^{-L_\alpha}, \bar{a}_- + c_3 e^{-L_\alpha})
\]

In particular, for sufficiently large \( L \) we have

\[
x_L^+ > \bar{a} + \gamma_1 \delta, \quad x_L^- < \bar{a} - \gamma_1 \delta
\]

Then

\[
x_{L+1}^+ = \hat{B}(x_L^+) + \delta \geq \hat{B}(\bar{a}) + \delta = \bar{s} + \delta > \bar{s}_+ + \delta
\]

\[
x_{L+1}^- = \hat{B}(x_L^-) - \delta \leq \hat{B}(\bar{a}) - \delta < \bar{s} - \delta < \bar{s}_- - \delta
\]

It is easy to see that for sufficiently large \( L \),

\[
x_{L+1}^+ < \bar{s}_+ + \frac{1}{2}, \quad x_{L+1}^- > \bar{s}_- - \frac{1}{2}
\]
By (5.3), there exists $C_4 > 0$ independent of $L$ such that
\[
\begin{align*}
x_{2L+1}^+ > \bar{u} + C_4 \delta - 1 e^{-L \alpha} > \bar{u} + \gamma_1 \delta - C_4 \delta - 1 e^{-L \alpha} \\
x_{2L+1}^- < \underline{u} - 1 + C_4 \delta - 1 e^{-L \alpha} < \underline{u} - 1 - \gamma_1 \delta + C_4 \delta - 1 e^{-L \alpha}
\end{align*}
\]
As a consequence, for all sufficiently large $L$ we have
\[
Z(z, x) = x_{2L+1}^+ - x_{2L+1}^- > 1
\]
Now Lemma 10 follows from Lemma 11 and (5.7).

We have the following lower bound.

**Lemma 12.** There exists a constant $c_4 > 0$ such that for all sufficiently large integer $L \geq 1$, let $\delta = e^{-\frac{L}{4}}$, then for any $D \in A(f)$, denote $Y = [X(D, 2L + 1 + K, \delta)]$ (i.e. $Y_n = [X_n]$, \forall $n \geq 1$), we have
\[
\liminf_{n \to \infty} \frac{1}{n} E_D(Y_n) \geq c_4 (1 - \varepsilon)^{2L}
\]

**Proof.** Denote $N_0 = 2L + 1 + K$. We have
\[
X_n(z, x) = \beta_{2f_{\delta}^{nN_0}}(z, \hat{x}) - \beta_{2f_{-\delta}^{nN_0}}(z, \hat{x}), \quad \forall \hat{x} \in \tau_{\mathbb{R} \to \mathbb{T}}(x)
\]
We denote
\[
\begin{align*}
Z_n(z, x) &= \beta_{2f_{\delta}^{nN_0}}(f_{\delta}^{nN_0}(z, \hat{x})) - \beta_{2f_{-\delta}^{nN_0}}(f_{-\delta}^{nN_0}(z, \hat{x})) \\
W_n(z, x) &= \beta_{2f_{-\delta}^{nN_0}}(f_{\delta}^{nN_0}(z, \hat{x})) - \beta_{2f_{-\delta}^{nN_0}}(f_{-\delta}^{nN_0}(z, \hat{x}))
\end{align*}
\]
Then it is clear that
\[
X_{n+1} = Z_n + W_n
\]
By definition, $X_n \geq Y_n$. Then by the monotonicity and the periodicity of $f_{-\delta}$, we have
\[
W_n(z, x) \geq \beta_{2f_{-\delta}^{nN_0}}(f_{\delta}^{nN_0}(z, Y_n + \beta_{2f_{-\delta}^{nN_0}}(f_{-\delta}^{nN_0}(z, \hat{x})))) - \beta_{2f_{-\delta}^{nN_0}}(f_{-\delta}^{nN_0}(z, \hat{x})) = Y_n
\]
We have for all $(z, x) \in D$ that,
\[
Z_n(z, x) = S_{N_0}(\delta, f_{\delta}^{nN_0}(z, x)) - S_{N_0}(-\delta, f_{\delta}^{nN_0}(z, x))
\]
By Lemma 7 and by $\varepsilon < \min(\varepsilon_1, \varepsilon_2) < \varepsilon_0(\varepsilon, \frac{1}{2})$, for all $\delta$ such that $|\delta| \ll 1$, there exist $L_0 > 0$ depending only on $\varepsilon$ (in particular, independent of $D$), such that for all $L > L_0$, there exist disjoint $D_1, \cdots, D_k \in A(f)$, satisfying that $D_i \subset f_{\delta}^{nN_0}(D)$ for all $1 \leq i \leq k$, and there exist constants $d_1, \cdots, d_k > 0$ such that $\sum_{i=1}^k d_i \text{Vol}|_{D_i} \leq (f_{\delta}^{nN_0}(\text{Vol}|_D) + \sum_{i=1}^k d_i) > \frac{1}{2}$. For any $1 \leq i \leq k$, we define
\[
Z^{(i)}(z, x) = S_{N_0}(\delta, z, x) - S_{N_0}(-\delta, z, x), \quad \forall (z, x) \in D_i
\]
Then by Lemma 10 and (5.4), we have
\[E_D(Y_{n+1}) - E_D(Y_n) \geq P_D(Z_n \geq 1)\]
\[\geq \sum_{i=1}^{k} d_i P_D(Z(i) \geq 1) \geq \frac{1}{2}(1 - \kappa)^2Lc_2\]

It is direct to see that for any \(t \in (-1, 1)\), any \(\mu_t \in u\text{Gibbs}(f_t)\), we have \(\pi_*\mu_t \in u\text{Gibbs}(g)\). By the uniqueness of SRB measure for \(g\), we have \(\pi_*\mu_t = m\) for all \(\mu_t \in u\text{Gibbs}(f_t)\). Then for each \(t \in (0, 1)\), for any \(\mu_t \in u\text{Gibbs}(f_t)\) and \(\mu_{-t} \in u\text{Gibbs}(f_{-t})\), there exist a subset of \(M_0 \subset M\) with \(m(M_0) = 1\), and for each \(z \in M_0\), there exists \(x, x' \in \mathbb{T}\) such that \((z, x)\) is \(\mu_t\) generic, and \((z, x')\) is \(\mu_{-t}\) generic. Let \(E \in A_1(g)\) be such that almost every \(z \in E\) with respect to the Lebesgue measure on \(E\) belongs to \(M_0\).

We claim that: for any \(y \in \mathbb{T}\), we have
\[\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_t f_i^2(z, y) = \int \phi_t d\mu_t\]

Indeed, let \(\hat{y}, \hat{x} \in \mathbb{R}\) be respectively lifts of \(y, x \in \mathbb{T}\). Then for any \(n \geq 1\), we have for \(w = x, y\) that
\[\frac{1}{n} \sum_{i=0}^{n-1} \phi_t f_i^2(z, w) = n^{-1} S_n(t, z, w) = n^{-1}(\hat{\phi}_2 \hat{f}_i^2(z, \hat{w}) - \hat{w})\]

Since \((z, x)\) is generic for \(\mu_t\), we have
\[\lim_{n \to \infty} n^{-1}(\hat{\phi}_2 \hat{f}_i^2(z, \hat{x}) - \hat{x}) = \int \phi_t d\mu_t\]

By periodicity, we have
\[|(\hat{\phi}_2 \hat{f}_i^2(z, \hat{x}) - \hat{x}) - (\hat{\phi}_2 \hat{f}_i^2(z, \hat{y}) - \hat{y})| \leq 2(|\hat{y} - \hat{x}| + 1)\]

Then the claim follows from
\[\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_t f_i^2(z, y) = \lim_{n \to \infty} n^{-1}(\hat{\phi}_2 \hat{f}_i^2(z, \hat{y}) - \hat{y})\]
\[= \lim_{n \to \infty} n^{-1}(\hat{\phi}_2 \hat{f}_i^2(z, \hat{x}) - \hat{x}) = \int \phi_t d\mu_t\]

As a consequence, for any \(t \in (-1, 1)\), any \(\mu_t \in u\text{Gibbs}(f_t)\), for any \(D \in \mathcal{A}_k(f)\) such that \(p_1(D) \subset E\), we have
\[\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} E_D(\phi_t f_i) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int \phi_t f_i d\text{Vol}|_D = \int \phi_t d\mu_t\]

Then take an arbitrary \(D \in \mathcal{A}_k(f)\) such that \(p_1(D) \subset E\). For sufficiently large \(L\) and \(\delta = e^{-\frac{\kappa}{2}}\), by Lemma 12, our theorem then follows from
\[\int \phi_{\delta} d\mu_{\delta} - \int \phi_{-\delta} d\mu_{-\delta} = \lim_{n \to \infty} \frac{1}{n(2L + 1 + K)} E_D(X_n) \geq \lim_{n \to \infty} \frac{1}{n(2L + 1 + K)} E_D(Y_n)\]
and that
\[
\lim_{n \to \infty} \frac{1}{n(2L + 1 + K)} E_D(Y_n) > \frac{1}{(2L + 1 + K)^4(1 - \epsilon)^2L} > 4\delta^6.
\]
Here \(X = X(D, 2L + 1 + K, \delta)\) and \(Y = \lfloor X \rfloor\).

\[ \square \]

**Proof of Theorem 2** By combining Proposition 4 Proposition 5 \[ \square \]

**APPENDIX**

**Proof of Lemma 3** The proof is essentially contained in [25] Appendix B. For the convenience of the reader, we recall the proof.

As in [25] Appendix B, we let \(\Gamma = \mathbb{N} \times \{+, -\}\). We let \((c(+)c(-)) = (p, q)\) instead of \((1, 0)\) in [25], and let \((c'(+)c'(-)) = (p - 1, q - 1)\). We write \(C\) for constants that does not depend on \(S, \rho, \Theta, \Theta'\), while \(C'\) for constants that may depend on them. Let \(\mu\) be an integer such that
\[
2^{-\mu + 6}||\xi|| \leq ||(DS_x)^{tr}(\xi)|| \leq 2^{\mu - 6}||\xi||, \forall x \in U, \xi \in \mathbb{R}^2
\]
let \(\nu \leq \mu - 6\) be an integer such that
\[
2^{\nu - 1} < \Lambda(S, \Theta') \leq 2^{\nu}
\]
so that \(||(DS_x)^{tr}(\xi)|| \leq 2^\nu||\xi||, \forall x \in U, (DS_x)^{tr}(\xi) \notin C'\_\). We write as in [25] that \((m, \tau) \leftrightarrow (n, \sigma)\) if either
\[
(1) \quad (\tau, \sigma) = (+, +) \text{ and } m - \mu \leq n \leq \max(0, m + \nu + 6), \text{ or}
(2) \quad (\tau, \sigma) = \{(-, -), (+, +)\} \text{ and } m - \mu \leq n \leq m + \mu.
\]
and we write \((m, \tau) \not\leftrightarrow (n, \sigma)\) otherwise.

For \(u \in C_0^1(R)\), let \(v := Lu\). For \((n, \sigma), (m, \tau) \in \Gamma\), define
\[
v_{m, \tau}^{n, \sigma} = \psi_{n, \sigma, \tau}(D)L(u_{\Theta, m, \tau})
\]
By Parseval’s identity, we have
\[
(5.8) \quad \sum_{n, \sigma} ||v_{m, \tau}^{n, \sigma}||_{L^2}^2 \leq C ||L(u_{\Theta, m, \tau})||_{L^2}^2 \leq C\gamma(S)^{-1}||\rho||_{L^\infty}^2 ||u_{\Theta, m, \tau}||_{L^2}^2
\]
We have the following.

**LEMMA 13** (Lemma B.1 in [25]). If \((m, \tau) \not\leftrightarrow (n, \sigma)\), we have
\[
||v_{m, \tau}^{n, \sigma}||_{L^2} \leq C2^{-(r - 1)\max(m, \nu)} ||u_{\Theta, m, \tau}||_{L^2}
\]
It is clear that
\[
(5.9) \quad ||v||_{L^2} \leq C\gamma(S)^{-\frac{1}{2}}||\rho||_{L^\infty}||u||_{L^2}
\]
We claim for \(q \geq 1\) that
\[
(5.10) \quad ||v||_{H^q} \leq C\gamma(S)^{-\frac{1}{2}}||\rho||_{L^\infty}||DS^q||_{1}||u||_{1, p, \rho, q} + C'||u||_{1, p, -1, q - 1}
\]
Indeed, for any multi-index \(\alpha = (\alpha_1, \ldots, \alpha_q) \in \{1, 2\}^q\), let \(\partial^\alpha\) be as in (4.26), we can write
\[
\partial^\alpha v = \rho P_0 + P_1
\]
where we denote $S = (S_1, S_2)$ and

$$P_0 = \sum_{\beta=(\beta_1, \cdots, \beta_q)} \prod_{j=1}^q \partial_{\beta_j} S_{\beta_j} \partial^\beta u \circ S$$

$$P_1 = (\partial^\beta \rho) u \circ S + \sum_{\beta, 1 \leq |\beta| \leq q-1} \rho_\beta \partial^\beta u \circ S$$

Here $\partial^\beta \rho, \rho_\beta, \partial^\beta S_{\beta_j}$ are all $C^0$ functions.

We have

$$\|P_1\|_{L^2} \leq C\|u\|_{H^{q-1}} \leq C'\|u\|_{\Omega, p-1, q-1}$$

Moreover, we have

$$\|\rho P_0\|_{L^2} \leq C\|\rho\|_{L^\infty}^2 \|DS\|^{2q} \gamma(S)^{-1} \sup_{\beta, |\beta|=q} \|\partial^\beta u\|_{L^2}$$

By (4.1) and straightforward computation,

$$\|v\|_{H^q} \leq C \sum_{a, |a| \leq q} \|\partial^a v\|_{L^2} \leq C\|\rho\|_{L^\infty} \|DS\|^{q} \gamma(S)^{-1/2} \|u\|_{H^q} + C'\|u\|_{\Omega, p-1, q-1}$$

Then (5.10) follows from $\|u\|_{H^q} \leq \|u\|_{\Omega, p, q}$ and $p \geq q$. Then our first inequality in Lemma 3 follows from (5.10) and $\|v\|_{\Omega, p, q} \leq \|v\|_{H^q}$.

We now prove the second inequality. By definition that

$$\left(\|v\|_{\Omega, p, q}^+\right)^2 = \|\psi_{\Omega, p, q}(\partial^\beta u)\|_{L^2}^2 + \sum_{n \geq 1} 2^{2pn} \|\psi_{\Omega, p, q}(D)D\|_{L^2}^2$$

The first term on the right hand side of (5.11) is easily bounded by $C\|u\|_{\Omega, p, q}^2$ or $C\|u\|_{\Omega, p-1, q-1}^2$ if $q \geq 1$. For any $n \geq 1$, we have

$$\|\psi_{\Omega, p, q}(\partial^\beta u)\|_{L^2}^2 \leq 2\| \sum_{(m, \tau) \leftrightarrow (n, +)} v_{m, \tau}^{n, +} \|_{L^2}^2 + 2\| \sum_{(m, \tau) \leftrightarrow (n, +)} v_{m, \tau}^{n, +} \|_{L^2}^2$$

Note that $(m, \tau) \leftrightarrow (n, +)$ only if $\tau = +$ and $n \leq m + v + 6$. Thus by Cauchy’s inequality

$$\| \sum_{(m, \tau) \rightarrow (n, +)} v_{m, \tau}^{n, +} \|_{L^2}^2 \leq ( \sum_{l \geq -v + 6} 2^{-2lp}) ( \sum_{(m, +) \rightarrow (n, +)} 2^{2np} \| v_{m, +}^{n, +} \|_{L^2}^2)$$

Then summing up the above inequality weighted by $2^{2np}$ for all $n \geq 1$, and by (5.8) we obtain

$$\sum_{n \geq 1} 2^{2np} \| \sum_{(m, \tau) \rightarrow (n, +)} v_{m, \tau}^{n, +} \|_{L^2}^2 \leq C \sum_m 2^{2(v+m)p} \gamma(S)^{-1} \|\rho\|_{L^\infty}^2 \|u_{\Omega, m, +}\|_{L^2}^2$$

$$(5.13) \leq CA(S, \Omega')^2 \gamma(S)^{-1} \|\rho\|_{L^\infty}^2 \|u\|_{\Omega, p, q}^2$$
By Cauchy’s inequality and Lemma 13, we have

\[
\sum_{(n,\sigma) \in \Gamma} \| 2^{\sigma(n)} \sum_{(m,\tau) \neq (n,+)} v^{m,\tau}_{n,+} \|_{L_2}^2 
\leq C \sum_{(n,\sigma) \in \Gamma} \left( \sum_{(m,\tau)} 2^{2\sigma(n)} - 2^{\sigma(\tau)m - 2(r-1)\max(m,n)} \right) \left( \sum_{(m,\tau)} 2^{2\sigma(\tau)m} \| u_{\Theta,m,\tau} \|_{L_2}^2 \right) 
\leq C \| u \|_{\Theta,p-1,q-1}^2
\]

The last inequality follows from \( p, q \in [0, \frac{2}{2} - 3) \) and

\[
2^{2\sigma(n)} - 2^{\sigma(\tau)m - 2(r-1)\max(m,n)} \leq 2^{(2p-r+1)n - (2q+r-3)m}
\]

We conclude the proof by (5.9), (5.12), (5.13), (5.14).

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Zhiyuan Zhang
Institut de Mathématique de Jussieu—Paris Rive Gauche, Bâtiment Sophie Germain, Bureau 652
75205 PARIS CEDEX 13, FRANCE
Email address: zzzhangzhiyuan@gmail.com