Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence

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Abstract

The spectral curve is the key ingredient in the modern theory of classical integrable systems. We develop a construction of the “quantum spectral curve” and argue that it takes the analogous structural and unifying role on the quantum level also. In the simplest but essential case the “quantum spectral curve” is given by the formula $\text{det}(L(z) - \partial_z)$ [Talalaev04] (hep-th/0404153).

As an easy application of our constructions we obtain the following: quite a universal recipe to define quantum commuting hamiltonians from classical ones, in particular an explicit description of a maximal commutative subalgebra in $U(\mathfrak{gl}_n[t]/t^N$ and in $U(\mathfrak{gl}_n[t^{-1}] t^N$; the relation (isomorphism) of the constructed commutative subalgebra with the center on the critical level of $U(\mathfrak{gl}_n)$ by the AKS-type arguments; an explicit formula for the center generators and conjecture on $W$-algebra generators; a recipe to obtain the $q$-deformation of these results; a simple and explicit construction of the Langlands correspondence and new points of view on its higher-dimensional generalization and relation to ”D-connections”; a relation between the “quantum spectral curve” and the Knizhnik-Zamolodchikov equation; new generalizations of the KZ-equation; a conjecture on rationality of the solutions of the KZ-equation for special values of level.

In simplest cases we observe coincidence of the “quantum spectral curve” and the so-called Baxter equation, our results provide a general construction of the Baxter equation. Connection with the KZ-equation offers a new powerful way to construct the Baxter’s $Q$-operator. Generalizing the known observations on the connection between the Baxter equation and the Bethe ansatz we formulate a conjecture relating the spectrum of the underlying integrable model and the properties of the “quantum spectral curve”. Our results are deeply related with the Sklyanin’s approach to separation of variables.

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1 Introduction and main results

The spectral curve is the key ingredient in the modern theory of classical integrable systems. In this paper we argue that the construction from [Talalaev04-N1] of “quantum characteristic polynomial” (QCP) takes this structural and unifying role on the quantum level. The generalizations and applications were discussed in [ChervovTalalaev04-N2], [MukhinTarasovVarchenko05-N3].

One usually defines a spectral curve of a classical integrable model as

\[ \det(L(z) - \lambda) = 0 \]

where \( L(z) \) is the corresponding Lax operator. The moduli of such a curve define conserved quantities, the dynamics is linearized on the Jacobian (possibly generalized) of the spectral curve etc. The “quantum characteristic polynomial” (QCP) in the simplest case can be informally written as

"det" \( (L(z) - \partial_z) \)

There are another names for QCP - ”universal G-oper”, ”quantum spectral curve” or ”universal Baxter equation”. We show in this paper that principal questions concerning quantum integrable systems are intrinsically encoded in QCP.

We show the relevance of QCP to the following questions:

1. Quantization problem
   QCP provides a universal recipe to construct quantum integrable hamiltonian systems from classical ones. From the mathematical point of view it gives a way to construct commutative subalgebras and central elements in various associative algebras like universal enveloping algebras and quantum groups.

2. Solution related ...
   QCP provides a natural generalization of the Baxter equation and hence gives a way to find the spectrum of quantum integrable models. With its help we relate the Baxter type equations to the Knizhnik-Zamolodchikov type equations (see also [ChervovTalalaev04-N2]), giving a new approach to the construction of the Baxter Q-operator which is essential for finding the spectrum of quantum commuting hamiltonians and other problems. The explanation of these constructions comes from Sklyanin’s ideas on separation of variables which are deeply related to the ”quantum characteristic polynomial”.

3. Langlands correspondence
   QCP with the AKS construction provides an elementary demonstration of the local geometric Langlands correspondence over \( \mathbb{C} \) as a quite particular case of the general picture. We shall show (see also [ChervovTalalaev06-1-N4, ChervovTalalaev06-2-N5]) that it provides a simple and explicit construction for the global geometric Langlands correspondence.

We also remark on relations with the \( D \)-dimensional Langlands correspondence (\( D \)-connections), Drinfeld-Sokolov reduction and matrix models. In subsequent papers we apply the theorem above to quantize the Hitchin system.
Let us briefly formulate the main constructions of the paper (see the main text for the notations and clarification).

1.1 Main hero

Let us present here the construction of the "quantum characteristic polynomial" in the simplest but illustrative case. More general constructions are described below.

Consider an associative algebra $A_q$ and $Mat_n \otimes A_q$-valued function $L(z)$ (in application $L(z)$ will be the so-called Lax operator of some integrable system).

**Definition 1** The "quantum characteristic polynomial" QCP of $L(z)$ is an element of $A_q \otimes Diff[z]$, where $Diff[z]$ is the associative algebra of differential operators in variable $z$, defined by the following expression:

$$\text{"det"}(L(z) - \partial_z) \overset{\text{def}}{=} Tr A_n(L_1(z) - \partial_z) \ldots (L_n(z) - \partial_z) = \sum_{k=0}^{n} C_n^k(-1)^{n-k} QI_k(z) \partial_z^{n-k} \quad (1)$$

where $L_i(z) \in Mat_n \otimes A_q((z))$ is obtained via the inclusion $Mat_n \hookrightarrow Mat_n \otimes A_q$ as the $i$-th component. $A_n$ is an antisymmetrization operator in $\mathbb{C}^{n \otimes n}$, the trace is taken over $Mat_n \otimes n$.

**Remark 1** If one puts $\lambda$ instead of $\partial_z$ in the formulas above and assumes $A$ to be a commutative algebra then one obtains the usual characteristic polynomial of $L(z)$ i.e. $\text{det}(L(z) - \lambda)$.

**Remark 2** Actually the construction above gives a correct definition only in the rational case. Proper modifications for other cases are described below.

1.2 Commutative subalgebra in $U(\mathfrak{gl}_n[t])/t^N$

**Theorem** Consider the standard Lax operators for $U(\mathfrak{gl}_n[t])/t^N$ with $N$ finite or infinite:

$$L(z) = \sum_{i=0}^{N-1} \Phi_i z^{-(i+1)} \quad (2)$$

Then the expressions $QI_k(z)$ given by Talalaev’s quantum characteristic polynomial:

$$\text{"det"}(L(z) - \partial_z) \overset{\text{def}}{=} Tr A_n(L_1(z) - \partial_z) \ldots (L_n(z) - \partial_z) = \sum_{k=0}^{n} C_n^k(-1)^{n-k} QI_k(z) \partial_z^{n-k} \quad (3)$$

commute with each other

$$[QI_k(u), QI_m(v)] = 0$$
Remark 3 Actually slightly more general theorem is proved below (see theorem 2).

Remark 4 One can see from the formula above (see [Talalaev04-N1]) or from the results of [ChervovRybnikovTalalaev04-N6] that the quadratic and cubic generators of this commutative subalgebra can be described as $TrL^2(z), TrL^3(z)$ but this is not true for the higher order generators: $TrL^4(z)$ does not commute with $TrL^2(u)$.

1.3 The center of $U_{\text{crit}}(\hat{\mathfrak{gl}}_n)$ and W-algebras

Theorem Let $U_{\text{crit}}(\hat{\mathfrak{gl}}_n)$ be the universal enveloping algebra of $\hat{\mathfrak{gl}}_n$ at the critical level, denote by $I$ the isomorphism of vector spaces $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t \mathfrak{gl}_n^{\text{op}}[t]) \to U_{\text{crit}}(\hat{\mathfrak{gl}}_n)$ given by $I(a \otimes b) = aS(b)$, where $S$ is the standard antipode. Consider the standard Lax operators for $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t \mathfrak{gl}_n^{\text{op}}[t])$

$$L^{\text{full}}(z) = \sum_{i=-\infty}^{+\infty} \Phi_i z^{-(i+1)}$$

- the expressions $QI_k(z) \in U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t \mathfrak{gl}_n^{\text{op}}[t])$ given by Talalaev’s quantum characteristic polynomial:

\[ "\text{det}\"(L^{\text{full}}(z) - \partial_z) \overset{\text{def}}{=} TrA_n(L_1^{\text{full}}(z) - \partial_z)\ldots(L_n^{\text{full}}(z) - \partial_z) = \sum_{k=0}^{n} C_n^k (-1)^{n-k} QI_k(z)\partial_z^{n-k} \]

commute with each other

$$[QI_k(u), QI_m(v)] = 0$$

- The commutative subalgebra in $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t \mathfrak{gl}_n^{\text{op}}[t])$ generated by $QI_k(z)$ goes under the map $I$ to the center of $U_{\text{crit}}(\hat{\mathfrak{gl}}_n)$, moreover this defines an isomorphism of commutative algebras.

Symbolically this can be written as follows:

\[ :"\text{det}\"(L_{\text{full}}(z) - \partial_z) : \text{generates the center at the critical level} \]

Here $:\ldots:$ stands for the normal ordering which should be understood as the map $I$. This agrees with the standard prescription for the center.

Conjecture

\[ :"\text{det}\"(L_{\text{full}}(z) - \partial_z) : \text{generates the W-algebra out of the critical level} \]
1.4 Langlands correspondence

Let us explain how our construction produces the local geometric Langlands correspondence over \( \mathbb{C} \) at the critical level (the global case will be treated in subsequent publications [ChervovTalalaev06-1-N4], [ChervovTalalaev06-2-N5] in order to keep the volume of present).

The local geometric Langlands correspondence over \( \mathbb{C} \) at the critical level aims to construct a bijection between representations of \( U_{\text{crit}}(\widehat{\mathfrak{gl}}_n) \) (automorphic side) and connections on \( \text{Spec} \mathbb{C}((z)) \) (Galois side). We approach this problem as follows: representations are usually defined by fixing values of Casimir elements; it was proved in [GWHMFF-N7] that the center \( Z(U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)) \) is huge enough (see also subsection above) and for any character \( \chi : Z(U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)) \to \mathbb{C} \) there exists an irreducible representation \( V_\chi \) of \( U_{\text{crit}}(\widehat{\mathfrak{gl}}_n) \) on which the center acts via \( \chi \). Now according to the theorem of the previous subsection one can obtain the character

\[
\chi_M : M_q \to \mathbb{C} \quad \text{where} \quad M_q \subset U(\mathfrak{gl}_n[t^{-1}] \oplus t \mathfrak{gl}_n^{\text{op}}[t])
\]

is the commutative subalgebra defined with the help of "det" \( (L^{\text{full}}(z) - \partial_z) \). Now one obtains a connection on \( \text{Spec} \mathbb{C}((z)) \) in the following way: one considers the scalar differential operator of order \( n \)

\[
\chi_M["\text{det}"(L^{\text{full}}(z) - \partial_z)] = \sum_{k=0}^{n} C_n^k(-1)^{n-k} \chi_M[QI_k(z)] \partial_z^{n-k}
\]

which defines a connection on the trivial rank \( n \) bundle \( (GL(n)\text{-oper}) \) over \( \text{Spec} \mathbb{C}((z)) \). Schematically the desired correspondence looks like:

\[
V_\chi \in \text{Rep}(U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)) \rightleftharpoons \text{connection defined from } \chi_M["\text{det}"(L^{\text{full}}(z) - \partial_z)]
\]

1.5 Separation of variables

**Conjecture.** Consider a Poisson algebra \( A \), its center \( Z \) and its maximal Poisson commutative subalgebra \( M \) given by the spectral invariants of the Lax operator (this means that there exists a \( \text{Mat}_n \otimes A \)-valued function\(^4\) \( L(z) \) such that: \( \forall z, \lambda \ det(L(z) - \lambda) \) belongs to \( M \) and moreover \( M \) is generated by these values). Denote also by \( A_q \) the deformation quantization of \( A \), \( Z_q - \) its center, \( L_q(z) \) - the corresponding\(^5\) Lax operator for \( A_q \).

**Then** \( \exists z_i, \lambda_i \) in \( A \) and \( \hat{z}_i, \hat{\lambda}_i \) in \( A_q \) called separated coordinates

(more precisely these variables belong to a finite algebraic extension of the field of fractions of \( A \) and \( A_q \), this means that in general one cannot express these variables in terms of rational functions of elements of \( A, A_q \) but needs to use algebraic functions i.e. to solve algebraic equations)

such that:

\[
\{z_i, \lambda_j\} = \delta_{ij} f(z_i, \lambda_j) \quad \{\lambda_i, \lambda_j\} = \{z_i, z_j\} = 0
\]

\(^4\)rational for simplicity  
\(^5\)Typically \( L_q(z) = L(z) \)
• \([\hat{z}_i, \hat{\lambda}_j] = \delta_{ij} f_q(\hat{z}_i, \hat{\lambda}_j)\)  \([\hat{\lambda}_i, \hat{\lambda}_j] = [\hat{z}_i, \hat{z}_j] = 0\)

• \(z_i, \lambda_i\) and \(Z\) are algebraically independent and algebraically generate \(A\)
  (analogously for \(\hat{z}_i, \hat{\lambda}_i, Z_q\) and \(A_q\))

• \(\forall i \ det (L (z_i) - \lambda_i) = 0\)  \(\forall i \ "det" (L_q (\hat{z}_i) - \hat{\lambda}_i) = 0\)

• Let \(\hat{z}, \hat{\lambda}\) be auxiliary variables (not elements of \(A_q\)) which satisfy the commutation relation \([\hat{z}, \hat{\lambda}] = f_q(\hat{z}, \hat{\lambda})\) (the same as for quantum separated variables), then
  \("det" (L_q(\hat{z}) - \hat{\lambda})\) generates the commutative subalgebra in \(A_q\) which quantizes subalgebra \(M\).

Except for the last item these ideas are due to Sklyanin. The last item assumes that the formula \("det" (L(z) - \partial_z)\) is the simplest example of the general construction given by \("det" (L(\hat{z}) - \hat{\lambda})\).

**Remark 5** This conjecture is mainly the belief that separation of variables can be found for any reasonable integrable system. If the parameter \(z\) lives on some algebraic curve (not just \(\mathbb{CP}^1\)) then \(z, \lambda\) should be considered as generators of the ring of differential operators on that curve minus some finite set of points.

• Despite the fact that the change of variables to separated ones is given by highly complicated algebraic functions it is expected that the generators of \(M, M_q\) can be expressed via separated variables by relatively simple rational expressions analogous to those proposed in [BabelonTalon02-N8, EnriquezRubtsov01-N9].

• Classical separated variables can be found by the Sklyanin’s ”magic recipe”: One should take the properly normalized Backer-Akhiezer function \(\Phi(z)\), i.e. the solution of \((L(z) - \lambda) \Phi(z) = 0\) and consider the poles \(z_i\) of this function together with the values of \(\lambda\) at these poles.

1.6 Universal Baxter equation, \(Q\)-operator, \(G\)-opers, Bethe ansatz

The Baxter equation (Baxter \(T - Q\) relation) and the Baxter’s \(Q\)-operator are considered now as the most powerful tools to find the spectrum of integrable models. The known constructions are mostly restricted to the cases of models related to \(GL(2), GL(3)\).

Here we propose a general point of view on the Baxter equation and the \(Q\)-operator and formulate a conjecture, which relates the Baxter equation to the spectrum of an integrable model. We explain that \(G\)-opers considered in the Langlands correspondence are closely related to this subject.

Let \(A_q, M_q, \hat{z}, \hat{\lambda}\) be as in the previous subsection. Consider \((\pi, V)\) - a representation of the algebra \(A_q\). Let \(\hat{z}, \hat{\lambda}\) be realized as operators on the space \(\mathbb{C}((z))\), where \(\hat{z}\) acts as multiplication by \(z\), and \(\hat{\lambda}\) acts as some differential (difference) operator \(\hat{\lambda}\).

**Construction of the Baxter equation.** The differential (difference) equation

\[ \pi("det" (L_q(z) - \hat{\lambda})) Q(z) = 0 \]
for an $\text{End}(V)$-valued function $Q(z)$ is called the Baxter equation and the solution $Q(z)$ - the Baxter’s $Q$-operator.

One can see that for the $SU(2)$-Gaudin, $SU(2)$-XXX, XXZ models this equation coincides with a degeneration of the original Baxter equation.

Main Conjecture Consider a character $\chi : M_q \rightarrow \mathbb{C}$ of the commutative algebra $M_q$, consider the differential (difference) equation $\chi(\text{"det"}(L_q(z) - \tilde{\lambda}))q(z) = 0$, then this equation has trivial monodromy ("$q$-monodromy" i.e. Birkhoff’s connection matrix [Birkhoff-N10]) iff there exist: a unitary representation $(\pi, V)$ for $A_q$ and a joint eigenvector $v$ for $M_q$ such that $\pi(m)v = \chi(m)v \quad \forall m \in M_q$.

Historic remark. The conjecture above was anticipated in the literature. The origin of such claims goes back to R. Baxter, M. Gaudin, further E. Sklyanin has related this to the separation of variables. Recently some close results were obtained by E. Frenkel, E. Mukhin, V. Tarasov and A. Varchenko [FMTV-N11]. Our main source of knowledge and inspiration on the subject is [Frenkel95-N12] and the conjecture in this form is motivated by this paper.

Corrections. For the models related not to $GL(n)$ but to a semisimple group $G$ the monodromy disappear only after passing to the Langlands dual group $G^L$. If irregular singularities appear in $\chi(\text{"det"}(L(z) - \tilde{\lambda}))$ one should require the vanishing of the Stokes matrices.

Simplification for compact groups. In the case of the $U(n)$-Gaudin and $U(n)$-XXX models the conjecture can be simplified requesting instead of absence of monodromy the rationality of all functions $q(z)$ satisfying the equation above. In these cases the conjecture should essentially follow (at least in one direction) from the results of [FMTV-N11] (see also [ChervovTalalaev04-N2]).

Relation to G-opers. For the case $A_q = U(\mathfrak{gl}_n[t])$ the operator $\chi(\text{"det"}(L(z) - \tilde{\lambda}))$ is precisely the G-oper which was considered in the Langlands correspondence.

Relation to Bethe equations. It is known that the condition of absence of monodromy can be efficiently written down as the Bethe equations in the case of the Gaudin and XXX models. In general an analogous procedure is not known. However for example in the case of regular connections the necessary condition for absence of monodromy is the very trivial condition for the residues of the connection to have integer spectrum. Hopefully the Bethe equations for the spectrum is the consequence of this similar local condition and by some unknown reasons the absence of local monodromies guarantees the absence of the global one.

"Bethe roots" are zeros of function $q(z)$. It is known that the Bethe equations are written on some auxiliary variables $w_i$. It can be seen in examples that $w_i$ are the zeroes of the function $q(z)$ solving the equation $\chi(\text{"det"}(L(z) - \tilde{\lambda}))q(z) = 0$. It is tempting to think that it is a general phenomenon.

Explanations. The explanation of the conjecture is obvious from the point of view of
separation of variables (see main text), another explanation is related to an alternative construction of the Langlands correspondence and relates it to the Bohr-Sommerfeld conditions (see ChervovTalalaev06-2-N5).

1.7 Knizhnik-Zamolodchikov and Baxter equations

Let us preserve the notations from the subsection above.

- **Observation** Let \( A_q = U(\mathfrak{gl}_n \oplus \ldots \oplus \mathfrak{gl}_n), L^G(z) \) be the standard Lax operator for the Gaudin model, let \( (\pi, V) \) be a representation of \( A_q \), then the equation on a \( V \otimes \mathbb{C}^n \)-valued function \( \Psi(z) \) of the type \( (\partial_z - \pi(L^G(z)))\Psi(z) = 0 \) coincides precisely with one component of the standard system of rational KZ-equations.

- **Notations** Let us take \( A_q, L(z) \) as in the previous subsection and \( (\pi, V) \) - a representation of \( A_q \), let us call the equation on an \( A_q \otimes \mathbb{C}^n \)-valued function \( \Psi(z) \) of the type \( (\lambda - L_q(z))\Psi(z) = 0 \) the **universal generalized KZ-equation**.

- **KZ rationality conjecture** The standard KZ-equation \( (\partial_z - \pi(L^G(z)))\Psi(z) = 0 \) has only rational solutions.

- **Generalized KZ rationality conjecture** Consider \( (\pi, V) \) - a unitary representation of \( A_q \), consider the equation \( (\lambda - \pi(L_q(z)))\Psi_\pi(z) = 0 \) for a \( V \otimes \mathbb{C}^n \)-valued function \( \Psi_\pi(z) \). Then this equation has trivial monodromy ("\( q \)-monodromy" i.e. Birkhoff’s connection matrix [Birkhoff-N10]).

**Corrections.** For the models related not to \( GL(n) \) but to a semisimple group \( G \) the monodromy disappears only after passing to the Langlands dual group \( G^L \). If irregular singularities appear in \( (\lambda - \pi(L_q(z))) \) one should require the vanishing of the Stokes matrices.

- **Baxter from KZ** The main message of ChervovTalalaev04-N2 was the following: one is able to construct solutions of the Baxter equation from solutions of the KZ type equation. This goes as follows: let \( \Psi(z) \) be a solution of the equation \( (\lambda - L_q(z))\Psi(z) = 0 \). Then any component \( \Psi_i(z) \) of the vector \( \Psi(z) \) provides a solution of the universal Baxter equation: \( \det(L(z) - \lambda))\Psi_i(z) = 0 \). \(^6\)

\(^6\)Moreover, in ChervovTalalaev06-03-N13 we show that there is a gauge transformation of the KZ-type connection \( (\lambda - L_q(z)) \) to the Drinfeld-Sokolov form corresponding to the differential (difference) operator "\( \det(L(z) - \lambda)) \).
1.8 Acknowledgements

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2 Commutative family in $U(\mathfrak{gl}_n[t])/t^N$

In this section we show how to define a commutative subalgebra in the algebra

$U(\mathfrak{gl}_n[t])/t^N = 0)$ for $N = 1, 2, \ldots, \infty$. Moreover we show that it can be given exactly by the same explicit formula as in [Talalaev04-N1]. The motivation for the construction is quite straightforward: the construction from [Talalaev04-N1] produces the quantum commutative subalgebra in $U(\mathfrak{gl}_n \oplus \mathfrak{gl}_n \oplus \ldots \oplus \mathfrak{gl}_n)$ which quantizes the classical Gaudin hamiltonians. On the other hand $U(\mathfrak{gl}_n \oplus \mathfrak{gl}_n \oplus \ldots \oplus \mathfrak{gl}_n) = U(\mathfrak{gl}_n[t]/(P_N(t) = 0))$ where $P_N(t)$ is a polynomial of degree $N$ with different roots. Passing to the limit $P_N(t) \rightarrow t^N$ one can expect to obtain a commutative subalgebra in $U(\mathfrak{gl}_n[t])/t^N = 0)$ of the same size. Though passing to the limit is quite delicate operation it is quite natural that there will be no problems in this situation because of the general experience of the theory of integrable systems and results of [Chernyakov03-N14, MussoPetreraRagnisco04-N15], where similar constructions have been realized in the classical case. We give a direct, self-contained proof not appealing to other considerations.

2.1 Main notation "det" $(L(z) - \partial_z)$

In sections 2,3,4 we use the following notation. Consider an associative algebra $A$, let $L(z)$ be an arbitrary element in $Mat_n \otimes A((z))$ (usually called the Lax operator), then we denote by "det" $(L(z) - \partial_z)$ and call it the "quantum characteristic polynomial" of $L(z)$ the following expression

"det" $(L(z) - \partial_z) \overset{def}{=} Tr A_n(L_1(z) - \partial_z) \ldots (L_n(z) - \partial_z)$

We denote by $QI_k(z)$ the coefficients of this differential operator after choosing an order: $\partial_z^i$ on the right

$\sum_{k=0}^{n} C_n^k (-1)^{n-k} QI_k(z) \partial_z^{n-k} \overset{def}{=} Tr A_n(L_1(z) - \partial_z) \ldots (L_n(z) - \partial_z)$ (9)

In these formulas $L_i(z)$ is an element of $Mat_n \otimes A((z))$ obtained via the inclusion $Mat_n \hookrightarrow Mat_n \otimes^n$ as the $i$-th component. The element $A_n$ is an antisymmetrization operator in $\mathbb{C}^{n \otimes n}$. 
2.2 Quantization of the Gaudin model

Let us recall the Talalaev’s result [Talalaev04-N1] on quantization of the Gaudin model [Gaudin-N16]. The Lax operator is a rational function with distinct poles:

\[ L^G(z) = K + \sum_{i=1}^{N} \frac{\Phi_i^G}{z - z_i} \]

where \( K \) is an arbitrary constant matrix, \( \Phi_i^G \in \text{Mat}_n \otimes \bigoplus \mathfrak{gl}_n \subset \text{Mat}_n \otimes U(\mathfrak{gl}_n) \otimes N \) is defined by the formula:

\[ \Phi_i^G = \sum_{kl} E_{kl} \otimes e^{(i)}_{kl} \]  

where \( E_{kl} \) form the standard basis in \( \text{Mat}_n \) and \( e^{(i)}_{kl} \) form a basis in the \( i \)-th copy of \( \mathfrak{gl}_n \). The quantum commutative family is constructed with the help of the quantum characteristic polynomial

\[ "\text{det}"(L^G(z) - \partial_z) = \text{Tr}A_n(L^G_1(z) - \partial_z)\ldots(L^G_n(z) - \partial_z) = \sum_{k=0}^{n} C^k_n (-1)^{n-k} QI_k(z) \partial_z^{n-k} \]  

Theorem [Talalaev04-N1] the coefficients \( QI_k(z) \) commute

\[ [QI_k(z), QI_m(u)] = 0 \]

and quantize the classical Gaudin hamiltonians.

Remark 6 Actually in [Talalaev04-N1] the case \( K = 0 \) was considered. The case \( K \neq 0 \) is similar and can be found in the appendix to the present paper.

2.3 Commutative family in \( U(\mathfrak{gl}_n[t])/t^N \)

Family of structures

Let us recall some standard facts about \( r \)-matrix and bihamiltonian structures (see [HarnadHurtubrise02-N17, FalquiMusso03-N18, Reyman-Semenov87-N19]). Actually these papers discuss only the Poisson structures but there is really no difference in our situation between classical and quantum cases.

For a polynomial \( P_N(t) \) of degree \( N \) the vector space \( U(\mathfrak{gl}_n[t])/(P_N(t) = 0) \) can be identified with the space \( U(\mathfrak{gl}_n[t])/t^N = 0 \). Hence consideration of different \( P_N(t) \) gives different algebraic structures on the same vector space. Let us recall how to write these algebraic structures with the help of \( r \)-matrices. Consider the standard Lax operator for \( U(\mathfrak{gl}_n[t])/t^N = 0 \) and the standard Lax operator for \( U(\mathfrak{gl}_n[t])/t^N = 0 \) with a constant term \( K \):

\[ L(z) = \sum_{i=0}^{N-1} \Phi_i z^{-(i+1)} \]

\[ L_K(z) = K + \sum_{i=0}^{N-1} \Phi_i z^{-(i+1)} \]  

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Here $K$ is an arbitrary constant matrix, $z$ - a formal parameter (not the same as $t$ !),

$$
\Phi_i \in \left( Mat_n \otimes U(\mathfrak{gl}_n[t])/\langle t^N = 0 \rangle \right) \cong \left( Mat_n[ U(\mathfrak{gl}_n[t])/\langle t^N = 0 \rangle] \right)
$$

are given by

$$
\Phi_i = \sum_{kl} E_{kl} \otimes e_{kl} t^i \iff (\Phi_i)_{kl} = e_{kl} t^i
$$

(13)

where $E_{kl} \in Mat_n$, $e_{kl} t^i \in U(\mathfrak{gl}_n[t])$. Both $E_{kl}$ and $e_{kl}$ are the matrices with the only $(k,l)$-th nontrivial matrix element equal to 1. We consider them as elements of different algebras: the first is an element of the associative algebra $Mat_n$, the second - of the universal enveloping algebra $U(\mathfrak{gl}_n[t])$.

Let us consider the standard $r$-matrix notations for the commutators in presence of an arbitrary polynomial $f(z)$ of degree less or equal then $N$ :

$$
\left[L(z), L(u)\right] = \left[\frac{P}{z - u}, L(z) \frac{f(u)}{u^N} + L(u) \frac{f(z)}{z^N}\right]
$$

(14)

Recall the following well-known simple fact:

**Proposition 1**

- For $f(z) = z^N$ the algebraic structure defined by the relation (14) on the vector space $U(\mathfrak{gl}_n[t])/\langle t^N = 0 \rangle$ coincides with the standard algebraic structure on $U(\mathfrak{gl}_n[t])/\langle t^N = 0 \rangle$ and the isomorphism is identical: $\Phi_i \to \Phi_i$.

- For the generic polynomial with distinct roots $f(z) = \prod_{i=1}^{N} (z - z_i)$ the algebra defined on the vector space $U(\mathfrak{gl}_n[t])/\langle t^N = 0 \rangle$ is isomorphic to $U(\mathfrak{gl}_n \oplus \mathfrak{gl}_n \oplus ... \oplus \mathfrak{gl}_n)$ and the isomorphism $(\Phi_0, ..., \Phi_{N-1}) \to (\Phi^G_0, ..., \Phi^G_{N-1})$ is given by the decomposition to simple fractions of:

$$
L(z) z^N / f(z) = \sum_i \frac{\Phi^G_i}{z - z_i}
$$

**Remark 7**  For an arbitrary polynomial $f(z)$ with possibly multiple roots the algebra defined on the vector space $U(\mathfrak{gl}_n[t])/\langle t^N = 0 \rangle$ is isomorphic to $U(\mathfrak{gl}_n[t])/\langle f(t) = 0 \rangle$ and the isomorphism is defined by the decomposition to simple fractions with possibly higher order poles of $L(z) z^N / f(z)$ in an obvious way.

**Remark 8**  In the classical case the formula above defines a pencil of compatible Poisson structures parameterized by $f(z)$. 

12
Commutative family

**Theorem 1**  *The Talalaev’s formula*

\[ "\text{det}"(L_K(z) - \partial_z) \]  
(15)

defines a commutative subalgebra in \(U(\mathfrak{gl}_n[t])/(t^N = 0)\). The subalgebra is maximal for \(K\) with distinct eigenvalues.

**Proof**  Let us start with the case \(K = 0\). According to the result of [Talalaev04-N1] the coefficients of "\(\text{det}\"(L^G(z) - \partial_z)\" generate a commutative subalgebra in \(U(\mathfrak{gl}_n \oplus \mathfrak{gl}_n \oplus \ldots \oplus \mathfrak{gl}_n)\) where \(L^G(z)\) is the Gaudin Lax operator. In our situation \(L(z)z^N/f(z)\) is of the Gaudin type if \(f(z)\) has distinct roots (see Proposition 1) and hence the expression "\(\text{det}"(L(z)z^N/f(z) - \partial_z)\" gives a commutative subalgebra in \(U(\mathfrak{gl}_n[t])/(f(t) = 0)\) for such \(f(z)\). We have to analyze the properties of the limit \(f(t) \to t^N\) to prove the result.

The fact that the constructed family stays commutative and does not degenerate in the limit is evident in a sense due to the following arguments:

1. The considered family of algebras is a family of universal enveloping algebras for the family of Lie algebras constructed on the same vector space by relations with different structure constants. This means that one has the same Poincaré-Birkhoff-Witt basis for the whole family of algebras.

2. The structure constants depend on coefficients of the function \(f(z)\) polynomially.

3. The quantities \(Q_i(z)\) being expressed in the common PBW basis have coefficients which depend rationally on coefficients of \(f(z)\).

4. At generic points and at the point \(f(z) = z^N\) the expressions \(Q_i(z)\) are well defined (they are not defined only at \(f(z) = 0\)).

5. The commutators \([Q_i(z), Q_j(u)]\) are rational functions on coefficients of \(f(z)\) on a such open set where \(Q_i(z)\) are well defined.

The proposition now follows from the fact that the quantities \(Q_i(z)\) are well defined at the point \(f(z) = z^N\) and the fact that there is only the zero rational function which vanishes on an open set.

Let us remark that the size of the commutative algebra remains the same after passing to the limit \(f(z) \to z^N\). Consider the classical limit which is the projection to the associated graded algebra in this case. The generators \(Q_i(z)\) map to the coefficients of the classical characteristic polynomial \(I_i(z)\). For all \(f(z)\) the Poisson commutative subalgebra generated by the coefficients of the expansion of \(I_i(z)\) at poles remains of the same dimension which is well known. This implies the same fact on the quantum level.

The case \(K \neq 0\) goes through the same lines because the quantization of the Gaudin model with the free term is also valid (see appendices).

**Remark 9**  One also has to note that the obtained subalgebra is maximal for \(K\) with distinct eigenvalues due to maximality of the Bethe subalgebra [NazarovOlshansky95-N20].
2.4 Commutative family in $U(\mathfrak{gl}_n[t])$

**Proposition 2** The commutative subalgebra in $U(\mathfrak{gl}_n[t])$ is given by the same formula as in the case of truncated algebra

$$"\det"(L(z) - \partial_z)$$

where $L(z) = \sum_{i=0}^{\infty} \Phi_i z^{-(i+1)}$.

This is a straightforward corollary of Theorem 1. The proof follows from the consideration of the natural projections $U(\mathfrak{gl}_n[t]) \to U(\mathfrak{gl}_n[t]/t^n)$.

Supposing that an element in $U(\mathfrak{gl}_n[t])$ is nonzero one can always find such $n$ that the projection to $U(\mathfrak{gl}_n[t]/t^n)$ is also nonzero. From this one sees that commutativity in $U(\mathfrak{gl}_n[t])$ is a corollary of commutativity in $U(\mathfrak{gl}_n[t]/t^n)$.

2.5 Historic remarks

**Remark 10** The classical analogue of the formula $"\det"(L(z) - \partial_z)$ is just the characteristic polynomial $\det(L(z) - \lambda)$. It follows easily from the $r$-matrix technique that it provides classical commuting expressions in $S(\mathfrak{gl}_n[t])/t^N$. The commuting quantities obtained in the way described above give rise to the analogue of the Gaudin integrable system well-known for a long time [ReymanSemenovFrenkel79-N21, AdlerMoerbeke80-N22].

**Remark 11** The central elements in $S(\mathfrak{gl}_n[t]/(t^N)$ (i.e. $U(\mathfrak{gl}_n[t]/(t^N)$) have been described in [RaisTauvel92-N23] (respectively [Molev97-N24]). Our subalgebra can be easily extended to a maximal subalgebra (see main text). Molev gave an explicit formula for the generators of center as $\tilde{\det}\tilde{L}(z)$ where his $\tilde{\det}$ and $\tilde{L}(z)$ are slightly different from the our’s, it would be interesting to clarify the relation.

**Remark 12** The considered commutative subalgebra in $U(\mathfrak{gl}_n[t])$ is in a sense a degeneration of the commutative subalgebra in Yangian $Y(\mathfrak{gl}_n[t])$ (called the Bethe subalgebra in [NazarovOlshansky95-N20]) which seems to be discovered by the Faddeev’s school (see ex. [Sklyanin95SV-N25] formula 3.16) but we are unable to find the precise reference. In the mathematical literature it has been analyzed first in [NazarovOlshansky95-N20], where all the proofs are provided and the generalization to semisimple Lie algebras found. The work [Talalaev04-N1] is heavily based on this paper.

2.6 The most general formulation

**Theorem 2** Consider an associative algebra $A$ and a slightly more general Lax operator $L(z) \in A \otimes \text{Mat}_n((z))$:

$$L(z) = \sum_{i=-M}^{L} \Phi_i z^{-(i+1)}$$

(17)
where \(M \geq 0, L > 0\) (one of \(M, L\) can be infinite, for the case \(M = L = \infty\) see proposition below)
such that this operator satisfy the commutation relation:

\[
\left[ \frac{1}{L(z)}, \frac{2}{L(u)} \right] = \left[ \frac{P}{z - u}, \frac{1}{L(z)} + \frac{2}{L(u)} \right]
\]

(18)

Then the expressions \(QI_k(z)\) given by:

\[
"det"(L(z) - \partial_z) = TrA_n(L_1(z) - \partial_z) \ldots (L_n(z) - \partial_z) = \sum_{k=0}^{n} C_n^k (-1)^{n-k} QI_k(z) \partial_z^{n-k}
\]

(19)

commute with each other

\[
[QI_k(u), QI_m(v)] = 0
\]

Proof This follows from the fact that the commutation relation \(18\) implies that the algebra generated by \(\Phi_{i,kl}\) is a factor algebra of \(U(\mathfrak{gl}_n[t]/t^M \oplus t \mathfrak{gl}_n^{op}[t]/t^L)\). The proof of the result for the algebra \(U(\mathfrak{gl}_n[t]/t^M \oplus t \mathfrak{gl}_n^{op}[t]/t^L)\) follows from the proof of the Theorem \(1\) for the case \(N = M + L\). The only change in the proof is that one needs to take the limit \(f(z) \to z^{M+1}\) (not \(f(z) \to z^N\)).

The case \(M = \infty\) xor \(N = \infty\) can be obtained by the same arguments as in section 2.4.

For the case \(M = \infty\) and \(N = \infty\) the theorem is not directly applicable because it seems to be not true that commutation relation \(18\) implies that \(A\) is a factor algebra of \(U(\mathfrak{gl}_n[t] \oplus t \mathfrak{gl}_n^{op}[t])\). All other arguments work without any change □

So we obtain the following:

**Proposition 3** If \(M = \infty, N = \infty\) and the algebra generated by \(\Phi_{i,kl}\) is a factor algebra of \(U(\mathfrak{gl}_n[t] \oplus t \mathfrak{gl}_n^{op}[t])\), then the proposition above is also true.

This proposition for the case \(A = U(\mathfrak{gl}_n[t] \oplus t \mathfrak{gl}_n^{op}[t])\) will be essential in the next sections.

### 2.7 Remark on the GL-invariance

It is quite clear that:

**Proposition 4** The commutative subalgebra obtained by \("det"(L(z) - \partial_z)\) is invariant with respect to the action of \(\mathfrak{gl}_n \in \mathfrak{gl}_n[t, t^{-1}]\).

**Remark 13** This is analogous to the fact that the Gaudin’s hamiltonians are invariant with respect to the diagonal action of \(GL_n\).

Let us remark that instead of introducing \(K\) into the Lax operator to produce a maximal commutative subalgebra one can add to our subalgebra the generators of a maximal commutative subalgebra constructed from \(\mathfrak{gl}_n\), for example of the Gelfand-Zeitlin subalgebra (see [FalquiMusso04-N26](#) for the Gaudin case).
2.8 Quantization of the "argument translation" method

For the case $gl_n = gl_n[t]/t$ the Lax operator is simply $L_{MF}(z) = K + \Phi/z$. The classical commutative subalgebra in $S(gl_n)$ obtained from coefficients of $det(L_{MF}(z) - \lambda)$ was called Mishenko-Fomenko subalgebra in [Vinberg90-N27]. It originates from Manakov’s paper [Manakov76-N28] and was deeply developed by Mishenko, Fomenko and their school (see [Mishenko-Fomenko78-N29] and subsequent papers) under the name of the "argument translation method" (see also [AdamsHarnadHurtubise96-N30]).

From the results above it is clear that:

**Proposition 5** The Talalaev’s formula $\text{"det"}(L_{MF}(z) - \partial_z)$ defines a commutative subalgebra in $U(gl_n)$ that quantizes the MF-subalgebra.

**Remark 14** Relation with the other constructions. The formula above gives an explicit recipe for quantization. By [Tarasov03-N31] the quantization is unique in this situation and hence coincides with the previous results [Tarasov00-N32, NazarovOlshansky95-N20]. It was conjectured in [Vinberg90-N27] and proved in [Tarasov00-N32] that the symmetrization map $S(gl_n) \to U(gl_n)$ produces a commutative subalgebra in $U(gl_n)$ from the MF-subalgebra in $S(gl_n))$. The low degree hamiltonians can be constructed as $TrL_{MF}(z)^k$ for $k < 6$, but $TrL_{MF}(z)^6$ does not commute with $TrL_{MF}(z)^3$ ([ChervovRybnikovTalalaev04-N6]).

**Remark 15** Limit to Gelfand-Zeitlin. It was shown in [Vinberg90-N27] that the Gelfand-Zeitlin subalgebra in $U(gl_n)$ can be obtained as a limit of the MF-subalgebras. It would be very interesting to understand the behaviour of our construction in this limit and the relation with deep results of GerasimovKharchevLebedev02-N33 on the quantum Gelfand-Zeitlin integrable system.

2.9 Coordinate change in QCP

Consider the Lie algebra $tgl_n[t]/t^{N+1}$ and the Lax operator:

$$L(z) = \sum_{i=1}^{N} \Phi_i/z^{i+1}$$  \hspace{1cm} (20)

where $\Phi_i$ are given by the formula 13.

In our study of Hitchin system the following proposition will be used:

**Theorem 3** The Talalaev’s formula

$$\text{"det"}(L(z) - \partial_z)$$ \hspace{1cm} (21)

defines a commutative subalgebra in $U(tgl_n[t])/(t^{N+1})$
Remark 16  The proof is given in Appendix B. Let us only make a comment that this theorem does not follow directly from theorem 2 since this Lax operator does not satisfy the basic commutation relation:

\[ [\hat{L}(z), \hat{L}(u)] \neq \frac{P}{z-u}, \hat{L}(z) + \hat{L}(u) \]  

Actually the following more general fact was proved:

**Proposition 6**  The following invariant formula

\[ "\text{det}\"(L(z)dz - d^{\text{deRham}}) \]

defines the same commutative subalgebra for any change of variables \( z = f(z') \).

We see that the quantum Lax operator can be naturally interpreted as a connection, this is prompting for higher-dimensional generalizations (see below).

3  The center of \( U(\widehat{\mathfrak{gl}_n}) \) at the critical level

Let us explain the coincidence of the commutative subalgebras obtained above with the ones constructed from the center of \( U_{\text{crit}}(\widehat{\mathfrak{gl}_n}) \) \cite{GWHMFF-N7}. Some related considerations for the Gaudin model can be found in \cite{EnriquezRubtsov95-N34, SemenovTianShansky97-N35}.

3.1  Commutative subalgebras from the center

The idea to obtain a commutative subalgebra from the center is due to Adler, Kostant and Symes \cite{AKS7980-N36} and Lebedev-Manin \cite{LebedevManin79-N37}. It was deeply developed by Reyman and Semenov-Tian-Shansky. Let us recall the relevant material on AKS constructions following \cite{SemenovTianShansky97-N35}.

Let \( \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \) be a finite dimensional Lie algebra which is a direct sum of two its Lie subalgebras. Let one introduce a new Lie algebra structure on \( \mathfrak{g} \): \([ (h_1, f_1), (h_2, f_2) ] = ([h_1, h_2], -[f_1, f_2]) \) where \((h_i, f_i) \in \mathfrak{g}, h_i \in \mathfrak{g}_+, f_i \in \mathfrak{g}_- \). Denote \( \mathfrak{g} \) with the new bracket by \( \mathfrak{g}_r \).

One has two isomorphisms of linear spaces

\[ \phi_{cl} : S(\mathfrak{g}) \to S(\mathfrak{g}_+) \otimes S(\mathfrak{g}_-^\text{op}) \simeq S(\mathfrak{g}_r) \]
\[ \phi : U(\mathfrak{g}) \to U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-^\text{op}) \simeq U(\mathfrak{g}_r) \]

the inverse maps are given by:

\[ \phi_{cl}^{-1}(a \otimes b) = as(b), \quad \phi^{-1}(a \otimes b) = as(b) \]  

where \( s \) is the canonical antipode.

**Lemma 1**  The center of the Poisson algebra \( Z_{cl} \subset S(\mathfrak{g}) \) maps to a commutative subalgebra in \( S(\mathfrak{g}_+) \otimes S(\mathfrak{g}_-^\text{op}) \) via \( \phi_{cl} \) where \( \mathfrak{g}_-^\text{op} \) is \( \mathfrak{g}_- \) with inverted bracket.
Lemma 2 The center $Z \subset U(\mathfrak{g})$ maps to a commutative subalgebra in $U(\mathfrak{g}_+) \otimes U(\mathfrak{g}^{op})$ via $\phi$.

For convenience we propose a proof of these classical facts in appendix C.

Remark 17 To use this result in infinite dimensional case one has to consider an appropriate completion, for the case of $U_{crit}(\hat{\mathfrak{gl}}_n)$ it is sufficient to consider a bigrading $deg(gt^k) = (k, 0)$, $deg(gt^{-k}) = (0, k)$ for $k \geq 0$. The only thing to prove is that the central elements are elements of the considered completion $U_{crit}(\hat{\mathfrak{gl}}_n)$. This is indeed the case due to the arguments of the classical limit. In what follows we omit the dot for the completed algebras $U_{crit}(\hat{\mathfrak{gl}}_n)$, $U(\mathfrak{g}_r)$, corresponding tensor products etc.

There is another canonical isomorphism of linear spaces

$$\varsigma : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}_+) \oplus U(\mathfrak{g})\mathfrak{g}_-$$

Let $\varphi$ be the projection onto the first summand $U(\mathfrak{g}_+)$. 

Lemma 3 The center $Z \subset U(\mathfrak{g})$ maps to a commutative subalgebra in $U(\mathfrak{g}_+)$ via $\varphi$.

Proof Let $c_1, c_2 \in Z$.

$$[c_1 - \varphi(c_1), c_2 - \varphi(c_2)] = [\varphi(c_1), \varphi(c_2)]$$

r.h.s. $\in U(\mathfrak{g}_+)$; l.h.s. $\in U(\mathfrak{g})\mathfrak{g}_-$; hence both are zeroes $\square$

For applying this to current algebras let us firstly recall the following

Proposition 7 Consider $\mathfrak{g} = \mathfrak{gl}_n[t, t^{-1}] = \mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_n[t]$

$$L_{full}(z) = \sum_{i=-\infty,\infty} \Phi_i z^{-i-1}$$

then the Lie algebra structure $g_r$ can be described by the commutation relation

$$[\frac{1}{L_{full}(z)}, \frac{2}{L_{full}(u)}] = [\frac{P}{z - u}, \frac{1}{L_{full}(z)} + \frac{2}{L_{full}(u)}]$$ (24)

3.2 $Z(U_{crit}(\hat{\mathfrak{gl}}_n))$ and the commutative subalgebra in $U(t\mathfrak{gl}_n[t])$

Let us denote by $U_{crit}(\hat{\mathfrak{gl}}_n)$ the algebra $U(\hat{\mathfrak{gl}}_n)$ at the critical level (for details see [GWHMFF-N7] where it was proved that $U_{crit}(\hat{\mathfrak{gl}}_n)$ has big center).

It also follows from this paper that the AKS construction works in this situation, i.e. the natural projection map $\varphi : U_{crit}(\hat{\mathfrak{gl}}_n)) \rightarrow U(t\mathfrak{gl}_n[t])$, restricted to $Z(U_{crit}(\hat{\mathfrak{gl}}_n))$ produces a huge commutative subalgebra in $U(t\mathfrak{gl}_n[t])$. 

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Theorem 4 The commutative subalgebra in $U(t\mathfrak{gl}_n[t])$ defined in the present paper with the help of "det" $(L(z) - \partial_z)$ coincides with the subalgebra obtained from the center of $U_{crit}(\mathfrak{gl}_n)$ by the natural projection $\varphi: U_{crit}(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n[t])$.

Proof The proof is based on [Rybnikov06-N38] where it was proved that the centralizer in $U(t\mathfrak{gl}_n[t])$ of the quadratic Gaudin’s hamiltonian $H_2$ is the commutative subalgebra obtained from the center of $U(\mathfrak{gl}_n)$ at the critical level. It is quite surprising that such a huge subalgebra is defined only by one element, similar results are known for Mishenko-Fomenko subalgebra [Tarasov03-N31], [Rybnikov05-N39], Calogero hamiltonians [OOS94-N40]. Since the commutative subalgebra given by "det" $(L(z) - \partial_z)$ commutes with $H_2$ it is a subset of the algebra obtained from the center. The maximality property provides that they coincide. Instead of appealing to maximality one can say that the classical versions of both subalgebras coincide and hence they have the same size $\square$

Remark 18 The same proof works in the case of the projection to $U(\mathfrak{gl}_n[t])$. One has just to take into account that both subalgebras are invariant with respect to the global $GL(n)$ action (see subsection 2.7).

Remark 19 The AKS construction can be modified to incorporate the Lax operator with the free term $K$ (see [AdamsHarnadHurtubise96-N30]). The subalgebra defined with the help of this modified construction should coincide with the subalgebra obtained from "det" $(L_K(z) - \partial_z)$.

3.3 Explicit description of $Z(U_{crit}(\mathfrak{gl}_n))$ and W-algebras.

Let us introduce the following notation $L_{full}(z) = \sum_{i=-\infty}^{\infty} \Phi_i z^{-i-1}$.

Theorem 5 The center of $U_{crit}(\mathfrak{gl}_n)$ is isomorphic as a commutative algebra to the commutative subalgebra in $U(\mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_{op}^n[t])$ defined by the Talalaev’s formula "det" $(L_{full}(z) - \partial_z)$. The isomorphism is given by the map

$I: U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_{op}^n[t]) \to U_{crit}(\mathfrak{gl}_n)$,
$I: h_1 \otimes h_2 \to h_1 s(h_2)$  (25)

where $s$ is the standard antipode: $s(h) = -h$ on generators.

Symbolically this can be written as follows:

:"det" $(L_{full}(z) - \partial_z)$: generates the center at the critical level  (26)

Here : $\ldots :$ stands for the normal ordering which should be understood as the map $I$. This construction agrees with the standard prescription for the center.

Remark 20 For the quadratic elements this was known (Sugavara’s formula).

Conjecture 1

:"det" $(L_{full}(z) - \partial_z)$: generates the W algebra out of the critical level  (27)
Proof of theorem 5 The proof is based on the same strategy as in [Rybnikov06-N38], namely we first prove that the algebra generated by the coefficients of the “quantum characteristic polynomial” (QCP) of $L_{full}(z)$ coincides with the centralizer of the quadratic coefficients and further use the Sugavara formula for the quadratic central elements to prove that its images in $U(gl_n[t^{-1}] \otimes U(tg_n[t])$ coincide with the quadratic coefficients of QCP. To prove the first statement we use a special limit of the commutative family.

For the reader’s convenience let us recall once more the structure relations

\[
\begin{align*}
[\hat{L}^{1}_{full}(z), \hat{L}^{2}_{full}(u)] &= [\hat{L}^{1}_{full}(z), P\delta(z - u)] \quad \text{in } gl_n[t, t^{-1}] \\
[\hat{L}^{1}_{full}(z), \hat{2}_{full}(u)] &= \left[ \frac{P}{z-u}, \hat{L}^{1}_{full}(z) + \hat{2}_{full}(u) \right] \quad \text{in } gl_n[t^{-1}] \oplus tgl^{'op}_n[t]
\end{align*}
\]

The Lie algebra $gl_n[t^{-1}] \oplus tgl^{'op}_n[t]$ here is the r-matrix Lie algebra $g_r$ for $g = gl_n[t, t^{-1}]$ defined by the decomposition $gl_n[t, t^{-1}] = gl_n[t^{-1}] \oplus tgl^{'op}_n[t]$. From the commutation relations above and standard r-matrix computations one obtains that $TrL^m_{full}(z)$ are central in $Sgl_n[t, t^{-1}]$ and $TrL^m_{full}(z)$ generate a Poisson commutative subalgebra in $S(gl_n[t^{-1}] \oplus tgl^{'op}_n[t])$.

Let us consider a family of automorphisms of the algebra $U(gl_n[t^{-1}] \otimes U(tg_n[t])$ defined in terms of the Lax operator as follows: let $K$ be a generic diagonal $n \times n$ matrix, the Lax operator

\[ L^h_{full}(z) = L_{full}(z) + hK \]

satisfy the same r-matrix relation as $L_{full}(z)$, hence this transformation provides a family of automorphisms depending on a parameter $h$. One has also a family of commutative subalgebras $M^h$ in $U(gl_n[t^{-1}] \otimes U(tg_n[t])$ defined by ”det”($L^h_{full}(z) - \partial_z$). $M^h$ centralizes the quadratic hamiltonian $QI_2(L^h_{full}(z))$. The k-th generator $QI_k(z, h)$ has the following scalar leading term in $h$

\[ QI_k(z, h) = h^k TrA_1K_1K_2\ldots K_k + O(h^{k-1}) \]

We prefer to slightly change the basis of generators

\[ QI_k(z, h) \mapsto \widehat{QI}_k(z, h) = (QI_k(z, h) - h^k TrA_1K_1K_2\ldots K_k)h^{-k+1} \]

and consider the limit $h \to \infty$

\[ \widehat{QI}_k(z, h) \to Tr(L_{full}(z)K^{k-1}) \]

In this limit the considered expressions generate the Cartan subalgebra $\mathfrak h = \mathfrak h_- \otimes \mathfrak h_+ = U(h[t^{-1}]) \otimes U(th[t])$ Let us demonstrate that this subalgebra coincides with the centralizer of

\[ H^\infty_2(z) = \lim_{h \to \infty} \widehat{QI}_2(z, h) = \sum_{i=-\infty,\infty} Tr(\Phi_iK)z^{-i-1} \]
Remark 21  All over this subsection the completion subject to the bigrading of Remark 17 is meant.

It is obvious that $\mathcal{F} \subset C(H_2^\infty(z))$. Let the diagonal elements of $K$ be $(k_1, \ldots, k_n)$. We denote by $h_i \in \mathcal{F}$ the sum

$$h_i = \sum_{s=1}^{n} (\Phi_i)_{ss} k_s$$

Then $H_2^\infty(z) = \sum_{i=-\infty, \infty} h_i z^{-i-1}$. Let us firstly note that the considered centralizer must commute with $h_1$ and $h_{-1}$. Let us consider an infinite series $\sum_{i=-\infty}^{\infty} x_i y_i$ such that $x_i \in U(g[t^{-1}])$, $y_i \in U(t^g[t])$ which is an element of the considered completion, i.e. there is a finite number of terms of each bigrading. The operators $[h_1, \ast]$ and $[h_{-1}, \ast]$ are homogeneous of bidegrees $(0, 1)$ and $(1, 0)$ respectively, hence the question can be restricted to the original algebra (not completed) and the answer is given by

$$C(h_1) = U(gl_n[t^{-1}]) \otimes \mathcal{F}_+ \quad C(h_{-1}) = \mathcal{F} \otimes U(t^g[t])$$

Their intersection in a completed sense is exactly $\mathcal{F}$.

We have obtained that at the generic point $\hbar$ the commutative subalgebra $M^\hbar$ belongs to the centralizer of $QI_2$ and in the limit $\hbar \to \infty$ generate the centralizer. By the general principle $M^\hbar$ coincides with the centralizer also at the generic point. The only thing to do to finish the proof is to recall the Sugawara formula for the quadratic central elements of $U_{crit}(gl_n)$

$$c_2(z) =: Tr(L^2_{full}(z)) :$$

which projects to $QI_2(z)$ up to a central element in $U(gl_n[t^{-1}]) \otimes U(t^g[t])$ □

4  Geometric Langlands correspondence

Langlands program is one of the most important themes in modern mathematics. Let us explain how the material of the present paper is related to this program. The Langlands correspondence has been transferred to $\mathbb{C}$-schemes in [BeilinsonDrinfeld-N41], has attracted much attention in [L-N42] (see [Frenkel05-N43] for up-to-date introduction). Recently a connection with the S-duality in gauge theory ([SDual-N44]) has been discussed [Witten05-N45] (see also [LP-N47] for the other relations with physics). The survey [Frenkel95-N12] is close to the setup of the present work, it was the source of inspiration for us for years.

4.1  Local Langlands correspondence over $\mathbb{C}$ (critical level)

The aim of the Langlands correspondence is to relate two type of objects: a representation of the Galois group of a 1-dimensional scheme and a representation of the group $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adels and $G$ is a reductive group. Let us restrict to the case

\footnote{A fundamental manuscript deriving the Langlands duality from the physical arguments appeared recently [WittenKapustin06-N46]}
Let $G = GL_n$. Consider the local version of the Langlands correspondence over $\mathbb{C}$ which means that the scheme is just the formal punctured disc $Spec\mathbb{C}((z))$.

**Local Langlands correspondence** over $\mathbb{C}$ states the following bijection

$$RepsGL_n(\mathbb{C}((z))) \cong \text{Morphisms} \left( \text{Galois}_{Spec\mathbb{C}((z))} \to GL_n(\mathbb{C}) \right)$$

Using the common wisdom one substitutes representations of the Lie group by representations of the corresponding Lie algebra and representations of the Galois group by connections on $Spec\mathbb{C}((z))$. One obtains the following **reformulation**

$$RepsU(\widehat{\mathfrak{gl}}_n) \cong \text{Classes of connections on } Spec\mathbb{C}((z))$$

Naively one expects to parameterize representations of $U(\widehat{\mathfrak{gl}}_n)$ by values of central elements in these representations, it is not true in general but according to [GWHMFF-N7] there exists a class of representations of $U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)$ parameterized by values of Casimirs on them. This means that for $\chi : Z(U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)) \to \mathbb{C}$ one is able to construct $V_\chi$ which is a representation of $U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)$ on which the center acts via this character. Hence by **restricting to the critical level** and using [GWHMFF-N7] we have

$$\text{FF}: RepsU_{\text{crit}}(\widehat{\mathfrak{gl}}_n) \cong \left( \chi : Z(U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)) \to \mathbb{C} \right)$$

According to theorem 5 the center of $U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)$ is isomorphic (as a commutative algebra) to the commutative subalgebra $M_{\text{full}} \subset U(\mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_n[t])$ defined with the help of $"det"(L_{\text{full}}(z) - \partial_z)$. This provides the following correspondence

$$\left( \chi : Z(U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)) \to \mathbb{C} \right) \cong \left( \chi : M_{\text{full}}^f \to \mathbb{C} \right)$$

To obtain a connection on $Spec\mathbb{C}((z))$ we have to apply the character to the ”quantum characteristic polynomial”

$$\chi("det"(L_{\text{full}}(z) - \partial_z)) = \sum_{k=0}^{n} C_k^k (-1)^{n-k} \chi[QI_k(z)] \partial_z^{n-k}$$

**Summary**

We have the following realization of the correspondence:

- Representation $V_\chi$ of $U_{\text{crit}}(\widehat{\mathfrak{gl}}_n)$
- connection on $Spec\mathbb{C}((z))$ defined by the scalar differential operator:

$$\chi("det"(L_{\text{full}}(z) - \partial_z)) = \sum_{k=0}^{n} C_k^k (-1)^{n-k} \chi[QI_k(z)] \partial_z^{n-k}$$
Remark 22  Agreement with filtration. One can see that the construction above respects the natural filtration. The algebra \( U(\mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_n^{\text{op}}[t]) \) has the natural projections to \( U(\mathfrak{gl}_n[t^{-1}]/t^k \oplus t\mathfrak{gl}_n^{\text{op}}[t]/t^l) \). Let \( M_{K,L} \subset U(\mathfrak{gl}_n[t^{-1}]/t^k \oplus t\mathfrak{gl}_n^{\text{op}}[t]/t^l) \) be the image of such a projection. Consider a character \( \chi \) of \( Z(U_{\text{crit}}(\hat{\mathfrak{gl}}_n)) \) which is the preimage of some character \( \chi_{M_{K,L}} \), then the corresponding \( G \)-oper can be given by \( \chi_{M_{K,L}}(\det(L_{K,L}(z) - \partial_z)) \), where \( L_{K,L}(z) = \sum_{i=-K,\ldots,L} \Phi_i/z^{i+1} \).

Global case of the correspondence
In the case of the global Langlands correspondence over \( \mathbb{C} \) it was advocated in [BeilinsonDrinfeld-N41] that representations of the unramified Galois group (i.e. the fundamental group) are parameterized by Hitchin’s D-modules on the moduli stack of vector bundles. Moreover it was proved that Hitchin’s D-modules comes from central elements \( Z(U_{\text{crit}}(\hat{\mathfrak{gl}}_n)) \). We will show in the subsequent publications ([ChervovTalalaev06-1-N4], [ChervovTalalaev06-2-N5]) that one can apply the construction developed in the present paper to obtain the global Langlands correspondence for algebraic curves using the methods developed in [ChervovTalalaev03-1-N48], [ChervovTalalaev03-2-N49].

4.2  Higher-dimensional Langlands correspondence (speculation)
The generalization of the Langlands correspondence to \( d \)-dimensional schemes is extremely complicated question, a little is known about it. The abelian case - the higher dimensional class field theory has been developed by [ParshinKato78-N50] (see [HLF-N51] for survey). There is only one paper known to us which tries to deal with the nonabelian case: [Kapranov95-N52]. The paper [GinzburgKapranovVasserot95-N53] deals with the case of surfaces, but the main conjecture presented therein is more likely the ”deformation” of D1-geometric Langlands correspondence then truly D2-conjecture, in the same way as the Beauville-Mukai integrable system is a ”deformation” of the Hitchin system.

One can try to formulate the Langlands correspondence as a correspondence between representations of \( GL(A) \) where \( A \) is the ring of \( d \)-dimensional Parshin-Beilinson adel of a scheme \( S \) ([Adels-N54]) and ”\( d \)-representations” of the Galois group of \( S \) ([Kapranov95-N52]). In the abelian case this formulation is different from Parshin-Kato point of view, it is dual to it (see last section in [Kapranov95-N52]). Possibly the most general formulation should state a relation between ”\( r \)-representations” of \( GL(A) \) and ”\( d - r + 1 \)-representations” of the Galois group. The obvious difficulties with such a formulations are the following: the notion of a ”\( d \)-representation” of a group is not well-known at present (at least to the authors, see however [GanterKapranov06-N55]). One can try to substitute the notion of a \( d \)-representation by the notion of a flat \( d \)-connection, which is also not well-known at the moment.

The notion of an abelian 2-connection is nevertheless relatively well understood due to the ideas of Brylinsky and Hitchin. A way to deal with this case was proposed by P. Severa (see [BresslerChervov02-N56]) who argued that the Courant algebroid should be treated as an infinitesimal analogue of a gerbe. We will try to argue below that this gives useful hints for understanding the multi-dimensional Langlands correspondence. Possibly the very
The complicated recent theory of [BreenMessing01-N57] is relevant to the non-abelian Langlands correspondence, another hope is related to the remarkable constructions developed in [Akhmedov05-N58] which possibly will lead to the right understanding of the theories of non-abelian $d$-connections and ”gauge theories of nonabelian $d$-forms” [Baez-N59], but it is far from being clear at the moment. Moreover the left-hand-side of the correspondence is also not so clear because a little is known about the representation theory of groups over $d$-dimensional local fields (see however [BravermanGaitsgoryKazhdan0305-N60]). As one can see in the one-dimensional case it was quite important to centrally extend the group $GL((z))$. In $d$-dimensions the natural cocycle which corresponds to the relevant extension is of degree $d+1$, so we cannot stay in the realm of groups to make this extension. Possibly the right thing to do is to make an extension in the category of $A_\infty$-algebras, but after doing this the relevant notion of a representation is not clear. (As an alternative way to $A_\infty$-algebras one can consider the language of grouplike monoidal categories which works in the abelian case [Osipov06-N61]). Moreover in the 1-dimensional geometric Langlands correspondence the critical level was important, it seems nobody has no idea what is an analog of this phenomenon in $d$-dimensions.

Possibly some ways round of the difficulties above exist: by the analogy with the gerbe case one can hope that to each $d+1$ connection one can attribute a $d$ connection on the loop space, so iterating this construction one can return back to the case of usual connections but on some hugely infinite-dimensional space. Instead of doing a 1-dimensional extension in the sense of $A_\infty$-algebras one should make an infinite-dimensional central extension, something like the Mickelson-Faddeev central extension.

Nevertheless we express the dream that an analogue of the Talalaev’s formula "$\det"(L(z) - \partial_z) also exists in $d$-dimensions and should be of the following type. The $d$-dim Lax operator is

$$L_d(z_1, \ldots, z_d) = \sum_{i_1, \ldots, i_d} \Phi_{i_1, \ldots, i_d} z_1^{-(i_1+1)} \ldots z_d^{-(i_d+1)}$$

By the same arguments as above we see that $\det(L_d(z) - \lambda)$ defines the center of the Poisson algebra $S(gl_n((t_1, \ldots, t_d)))$. One can hope that there exists a "critical level" such that the algebra (or $A_\infty$-algebra, or whatever) $U_{crit}(gl_n((t_1, \ldots, t_d)))$ posses the center of the same size as the center of $S(gl_n((t_1, \ldots, t_d)))$, those characters parameterize not all but at least some reasonable class of representations of $U_{crit}(gl_n((t_1, \ldots, t_d)))$

The dream One can hope that there exists a formula of the type

$$\tilde{\det}(L(z) - (\lambda)) := \sum_i QI_i(z)(\lambda)^i$$

such that the expressions $QI_i(z)$ define central elements in $Z(U_{crit}(gl_n((t_1, \ldots, t_d))))$ and each character of the center $\chi : Z(U_{crit}(gl_n((t_1, \ldots, t_d)))) \rightarrow \mathbb{C}$ produces a $d$-G-oper and hence a $d$-connection on $Spec(\mathbb{C}((z_1, \ldots, z_d)))$ in the following simple way: $\sum_i \chi(QI_i(z))(\lambda)^i$

This would be a higher-dimensional analogue of the local Langlands correspondence.

Remark 23 $L(z_1, \ldots, z_d)$ should be thought of as a $d$-form.
Remark 24  In the abelian case it seems that the right hand side of the correspondence is quite clear: one should consider the expression $\chi(d + L(z))$ as a connection on the Courant algebroid. The question is what should be on the LHS? In the 1-dimensional case there is a reformulation of the Langlands correspondence: on the LHS instead of representations of the adelic group one considers sheaves on the Jacobian. In 2-dimensions there should be something analogous and it should be very simple at least for 2-forms which have only regular singularities (the case of the tame ramification).

Remark 25  From the construction in $d = 1$ one can guess that $\hat{\lambda}$ is somehow analogous to the de-Rham differential $d$. The condition of flatness of the connection is trivial since any $d$-form on a $d$-scheme is closed. More precisely one should speak not about $C((z_1, ..., z_d))$ but about $d$-dimensional local fields in the sense of Parshin-Beilinson.

5  Separation of variables

Let us give some comments why the natural quantization of the characteristic polynomial $\det(L(z) - \lambda)$ is "$\det"(L(z) - \partial_z)". The most mysterious part is the appearance of $\partial_z$ instead of $\lambda$. In fact this is conceptually based on the idea of separation variables.

5.1 Separation of variables for classical systems

Consider a Poisson algebra $A$ and its maximal Poisson commutative subalgebra $M$ constructed with the help of the Lax operator with spectral parameter (this means that there exists a $Mat_n \otimes A$-valued function $L(z)$ such that: $\forall z, \lambda$: $\det(L(z) - \lambda)$ belongs to $M$ and all the elements of $M$ can be expressed as functions of $\det(L(z) - \lambda)$ for different $z, \lambda$).

Claim (Sklyanin)  (Classical separation of variables) Assume we have a Poisson algebra $A$, its center $Z$, its maximal Poisson commutative subalgebra $M$ given by the Lax operator $L(z)$, then there exists such elements $z_i, \lambda_i$ in $A$ (called separated coordinates) that

- $\{z_i, \lambda_j\} = \delta_{ij} f(z_i, \lambda_j)$  $\{\lambda_i, \lambda_j\} = \{z_i, z_j\} = 0$
- $z_i, \lambda_i$ and $Z$ are algebraically independent and algebraically generate $A$
- $\forall i \ det(L(z_i) - \lambda_i) = 0$.

Remark 26  In fact the variables $z_i, \lambda_i$ belong not to $A$ but to some algebraic extension of the field of fractions of $A$, this means that $z_i, \lambda_i$ are algebraic functions (not polynomial).

Remark 27  Sklyanin gave the "magic recipe" to find separated variables:
One should take the properly normalized Backer-Akhiezer function $\Phi(z, \lambda)$, i.e. the solution of $(L(z) - \lambda)\Phi(z, \lambda) = 0$ and consider the poles $(z_i, \lambda_i)$ of this function on the corresponding spectral curve. The obtained set of functions $z_i, \lambda_i$ is such that their number is precisely what one needs and the Poisson brackets between them are separated: $\{z_i, \lambda_j\} = \delta_{ij} f(z_i, z_j)$. The non-algorithmic problem of the method is to find the proper normalization of the BA-function but it can be done for a large class of models (see surveys [Sklyanin92QISM-N63], [Sklyanin95SV-N25]).
Remark 28  This point of view is usually adopted in modern texts on separation of variables, it is related in an obvious way with the classical definition [LandauLivshitsV1-N64].

Remark 29  Let us emphasize that the action-angle variables are not the same as separated ones in general. According to the textbook point of view the action-angle variables give an example of separated variables, but not vice versa in general. Modern practice of integrable systems coins the following crucial difference between these two sets of variables: we always hope that there are separated variables such that they can be expressed via algebraic functions of the original variables, meanwhile in general there is no hope to do this for the action-angle variables: even in the simplest example - the harmonic oscillator \( H = p^2 + q^2 \) - the angle variable \( \phi = \arctg(p/q) \) is not an algebraic function.

Remark 30  There exists an alternative (but deeply related) geometric recipe to find separated coordinates using bihamiltonian structures. It is being developed by Italian group (F. Magri, G. Falqui, M. Pedroni, et.al.) [BiHSV-N65]. The recipe characterizes separated coordinates as Poisson-Nijenhuis coordinates. At the moment the bihamiltonian method have not been generalized to the quantum level. It would be a marvelous advance to find a quantum analogue of the Lenard-Magri chain on the quantum level, this means that starting from the Casimirs on the quantum level, one should be able to reproduce all the Lenard-Magri chains of the quantum commuting hamiltonians. One should find a recipe to obtain the formula \( "\text{det}"(L(z) - \partial_z) \) from the Casimirs defined by the Molev’s formula \([\text{Molev97-N24}]\) \( \tilde{\text{det}}(\tilde{L}(z)) \), such that the classical limit reproduces the classical Lenard-Magri chain (see [FalquiMusso03-N18]).

5.2 Quantum separation of variables and QCP

Let us comment on quantum separation of variables and its relation to the quantum characteristic polynomial along the Sklyanin’s ideas.

Let \( A, M, Z, L(z) \) be as in the previous subsection. Denote by \( A_q \) the deformation quantization of the algebra \( A \), \( Z_q \) its center, \( L_q(z) \) - the quantum Lax operator.

Conjecture 2 (Quantum separation of variables) Assume that the bracket in separated variables is \( \{z_i, \lambda_j\} = \delta_{ij} \). Then there exist elements \( \hat{z}_i, \hat{\lambda}_i \) (called separated coordinates) in \( \hat{A}_q \) - an algebraic extension of the field of fractions of \( A_q \), such that

- \( "\text{det}"(L_q(z) - \partial_z/\hbar) \) generates a commutative subalgebra in \( A_q \) which quantizes the subalgebra \( M \)
- \( [\hat{z}_i, \hat{\lambda}_j] = \hbar \delta_{ij} \)
- \( \hat{z}_i, \hat{\lambda}_i \) and \( Z_q \) are algebraically independent and generate \( \hat{A}_q \)
- \( \hat{z}_i, \hat{\lambda}_i \) satisfy the quantum characteristic equation

\[
\sum_{k=0}^{n} QI_k(\hat{z}_i)C_n^k(-1)^{n-k}(\hat{\lambda}_i)^{n-k} = 0
\]
On definition of \( L_q(z) \) from \( L(z) \). In numerous examples the matrix elements of \( L(z) \) are generators of the algebra \( A \) and the algebra \( A_q \) can be described as an algebra with the same set of generators, so typically \( L_q(z) = L(z) \). We usually write \( L(z) \) instead of \( L_q(z) \).

Field of fractions. The construction of the field of fractions for the noncommutative algebra \( A_q \) hopefully can be done in the spirit of [GelfandKirillov66-N66]. For example in the case of an associative algebra (not necessarily the universal enveloping algebra) of less than exponential growth the remark of [KirillovKontsevichMolev83-N67] (section 1.4) shows that the Ore condition is satisfied. The way round these problems is to work in the same manner as in [EnriquezRubtsov01-N9].

Remark 31 Despite the expected high complexity of the expressions \( \hat{z}_i, \hat{\lambda}_i \) in terms of the original generators in the case of the Gaudin model it is known that the generators of \( M \) can be expressed via \( \hat{z}_i, \hat{\lambda}_i \) by relatively simple, rational expressions proposed in [BabelonTalon02-N8, EnriquezRubtsov01-N9]. It is natural to believe that it is a general phenomena.

Example In the context of the present paper one should take \( A = S(\mathfrak{gl}_n[t]/t^N), L(z) = L_q(z) = \sum_i \Phi_i z^{-(i+1)}, A_q = U(\mathfrak{gl}_n[t]/t^N), M_q \) is a commutative subalgebra generated by ”\( \text{det}(L(z) - \partial_z) \).”

5.3 Generalizations of the definition of QCP

For the case when the Poisson bracket between the separated variables is more complicated - \( \{z_i, \lambda_j\} = \delta_{ij} f(z_i, \lambda_j) \), our approach should be modified. One should quantize this Poisson bracket, define \( \hat{z}_i, \hat{\lambda}_i \) and define an analog of the QCP by:

\[
\text{”det”}(L(\hat{z}) - \hat{\lambda}) = \sum_{k=0}^n QI_k(\hat{z})C_n^k(-1)^{n-k}(\hat{\lambda})^{n-k} \tag{35}
\]

here \( \hat{z}, \hat{\lambda} \) are auxiliary variables (not elements of \( A_q \)) which satisfy the same commutation relation as quantum separated variables \( \hat{z}_i, \hat{\lambda}_j \).

Let us describe several examples of the generalized QCP. In all cases we believe that it will satisfy the expected properties:

- ”\( \text{det”}(L(\hat{z}) - \hat{\lambda}) \) generates a commutative subalgebra in \( A_q \) which quantizes the subalgebra defined by \( \text{det}(L(z) - \lambda) \).

- Separated variables \( \hat{z}_i, \hat{\lambda}_i \) satisfy the quantum characteristic equation

\[
\sum_{k=0}^n QI_k(\hat{z}_i)C_n^k(-1)^{n-k}(\hat{\lambda}_i)^{n-k} = 0
\]

27
5.3.1 QCP for the Yangian (XXX-model)

Consider the case of the Yangian $\mathcal{Y}(\mathfrak{g}_n)$. It is the symmetry algebra of XXX-spin chain, Toda, DST and Thirring models. It is well known that the Poisson bracket between separated variables $z$ and $\lambda$ is given by $\{z, \lambda\} = \lambda$.

The QCP for $\mathcal{Y}(\mathfrak{g}_n)$ should be defined by the formula (see ChervovTalalaev04-N2: introduction, remark 3):

\[
\text{QCP}^{\text{Yang}} = \text{Tr} A_n(e^{-\hbar \partial_u T_1(z)} - 1)(e^{-\hbar \partial_u T_2(u)} - 1) \ldots (e^{-\hbar \partial_u T_n(u)} - 1)
\]

(36)

It was proved in Talalaev04-N1, ChervovTalalaev04-N2 that it produces a commutative subalgebra in Yangian. It is just the Bethe subalgebra considered in Sklyanin95SV-N25 formula 3.16 and in NazarovOlshansky95-N20.

5.3.2 QCP for $U_q(\widehat{\mathfrak{g}_n})$ and elliptic algebras (XXZ,XYZ-models)

Let us express some preliminary ideas on the generalization of the QCP to quantum affine and elliptic algebras.

For the quantum affine algebra $U_q(\widehat{\mathfrak{g}_n})$ the QCP should have the form:

\[
\text{QCP}^{U_q} = \text{Tr} A_n^{q}(L_1^+(\hat{z}) - \hat{\lambda})(L_2^+(\hat{z}) - \hat{\lambda}) \ldots (L_n^+(\hat{z}) - \hat{\lambda})
\]

(37)

where the variables $\hat{z}, \hat{\lambda}$ should satisfy a relation of the type: $\hat{z}\hat{\lambda} = q\hat{\lambda}\hat{z}$, $L_i^+(z)$ is the positive current for $U_q(\widehat{\mathfrak{g}_n})$. $A_n^{q}$ should be a version of the $q$-antisymmetrizer considered in ArnaudonCrampeDoikouFrappatRagoucy05-N68 Appendix C:

\[
A_n^{q} = 1/n! \prod_{1 \leq a < b \leq n} R_{ab}^{\text{trig}}(q^{a-b})
\]

(38)

This formula generalizes the standard (Cherednik’s) formula for the standard antisymmetrizer in terms of the rational $R$-matrix.

We hope that there is an analogous formula in the elliptic case:

\[
\text{QCP}^{\text{ell}} = \text{Tr} A_n^{\text{ell}}(L_1^+(\hat{z}) - \hat{\lambda})(L_2^+(\hat{z}) - \hat{\lambda}) \ldots (L_n^+(\hat{z}) - \hat{\lambda})
\]

(39)

where $A_n^{\text{ell}}$ is an appropriate antisymmetrizer, defined by some analog of the formula above with the elliptic $R$-matrix. Possibly the facts and methods from BelavinJimbo02-N69 are relevant to the definition of $A_n^{\text{ell}}$. The quantum relation between $\hat{z}, \hat{\lambda}$ can be obtained from the known Baxter equation. We hope that with appropriate modifications the QCP can be defined for different types of elliptic Lax operators: Sklyanin’s like (see e.g. Zabrodin99-N70 formula for $T(\lambda)$ before formula 5.2), $GL(n)$-analogs of Sklyanin’s type Laxes ChernyakovLevinOlshanetskyZotov06-N71 or based on elliptic algebras JimboKonnoOdakeShiraishi97-N72.
5.3.3 QCP for the trigonometric and elliptic Gaudin models

XXZ and XYZ models can be degenerated to the trigonometric and elliptic Gaudin models. It was proved in [SklyaninTakebe98-N73] that the Poisson bracket in terms of the separated variables in these cases is given by \( \{ z, \lambda \} = 1 \). So it is natural to guess that the QCP for these models should be defined in a similar way:

\[
\text{QCP}^{\text{trig,ell}}_{\text{Gaudin}} = \text{Tr} A_n^{\text{trig,ell}} \left( L_1^+ (z) - \partial_z \right) \left( L_2^+ (z) - \partial_z \right) \ldots \left( L_n^+ (z) - \partial_z \right)
\]

where \( A_n^{\text{trig,ell}} \) are the analogs of the antisymmetrizer discussed in the previous subsection.

5.4 Universal Baxter equation, G-opers and Bethe ansatz

The Baxter equation (Baxter \( T - Q \) relation) and the Baxter \( Q \)-operator are considered now as the most powerful tools to find the spectrum of integrable models. The equation and the \( Q \)-operator were introduced in the context of XYZ model in [Baxter72-N74]. Nowadays studying the Baxter equation and its applications is an active field of research (see [QOp-N75] and survey [Sklyanin00QOp-N76]). However the known constructions are mostly restricted to the cases of models related to \( GL(2), GL(3) \).

Here we propose a general point of view on the Baxter equation and \( Q \)-operator; formulate the conjecture, which relates them to the spectrum of an integrable model; connect it with the Knizhnik-Zamolodchikov equation and propose a new way to construct \( Q \)-operator. We also explain that G-opers considered in the Langlands correspondence are closely related to these considerations.

Let \( A_q, M_q, L(z), \hat{z}, \hat{\lambda} \) be as in the previous subsection, i.e. \( A_q \) - an associative algebra (algebra of quantum observables), \( M_q \subset A_q \) - a maximal commutative subalgebra (sub-algebra of quantum commuting hamiltonians) constructed with the help of the QCP "\( \text{det} \)" for the Lax operator \( L(z) \). Consider \( (\pi, V) \) - a representation of the algebra \( A_q \). Let the quantum separated variables \( \hat{z}, \hat{\lambda} \) be realized as operators on the space \( \mathbb{C}((z)) \), where \( \hat{z} \) acts as the multiplication by the function \( z \), and \( \hat{\lambda} \) acts as some differential (difference) operator \( \hat{\lambda} \).

**Construction of Baxter equation** The differential (difference) equation

\[
\pi(\text{"det"}(L(\hat{z}) - \hat{\lambda}))Q(z) = 0
\]

for an \( \text{End}(V) \)-valued function \( Q(z) \) is called the Baxter equation.

One can see that for the \( SU(2) \)-Gaudin, \( SU(2) \)-XXX, XXZ models this coincides with the degeneration of the original Baxter equation.

Let us mention that the basic properties of the \( Q \)-operator:

\[
[Q(z), Q(u)] = 0, \quad [Q(z), m] = 0, \forall m \in M_q
\]

are almost automatic in our approach.

**Main Conjecture** Consider a character \( \chi : M_q \to \mathbb{C} \) of the commutative algebra \( M_q \), consider the scalar differential (difference) equation \( \chi(\text{"det"}(L(\hat{z}) - \hat{\lambda}))q(z) = 0 \), then this equation has trivial monodromy ("\( q \)-monodromy" i.e. Birkhoff’s connection matrix
iff there exist a unitary representation \((\pi, V)\) for \(A_q\) and a joint eigenvector \(v\) for all commuting hamiltonians \(m \in M_q\) such that \(\pi(m)v = \chi(m)v\).

**Corrections.** For the models related not to \(GL(n)\) but to a semisimple group \(G\) the monodromy disappear only after passing to the Langlands dual group \(G^L\). If irregular singularities appear in \(\chi(\text{"det"}(L(z) - \tilde{\lambda}))\) one should require the vanishing of the Stokes matrices.

**Simplification for compact groups.** In the case of the \(U(n)\)-Gaudin and \(U(n)\)-XXX models (or in the other words \(GL(n)\)-models with finite-dimensional representations) the conjecture can be simplified requesting instead of absence of monodromy the rationality of all functions \(q(z)\) satisfying the equation above. In these cases the conjecture should essentially follow (at least in one direction) from the results of [FMTV-N11] (see also [ChervovTalalaev04-N2]).

The results from [Zabrodin99-N70] hint that for the \(U(n)\)-XXZ and \(U(n)\)-XYZ there should be some trigonometric and "elliptic polynomial" solutions of the Baxter equation. Zabrodin presented a solution of the Baxter equation for the \(SL(2)\)-XYZ-model in the form of elliptic generalization of the hypergeometric series, the generalizing classical series representation of the Jacobi polynomials. The general theory of elliptic analogs of hypergeometric functions developed in [Spiridonov05-N77] should be relevant here.

**The case of noncompact groups.** We are quite sure that the conjecture above is true (at least for generic \(\chi\)) for the case of compact groups (or their \(q\)-analogs) but possibly it should be corrected for the case of noncompact ones, requiring additional analytical properties for \(Q(z)\). It seems that the situation can be clarified considering as an example the closed Toda chain, which is connected with the representation of the noncompact form of the Yangian for \(sl(2)\). It is interesting to compare our approach with the classical results due to Gutzwiller, Sklyanin, Pasquier, Gaudin (see [KharchevLebedev99-N78]).

**Relation to G-opers.** For the case \(A_q = U(\mathfrak{gl}_n[t])\) the differential operator \(\chi(\text{"det"}(L(z) - \tilde{\lambda}))\) is precisely the \(G\)-oper which was considered in the Langlands correspondence.

**Relation to Bethe equations.** It is known that the condition of absence of monodromy can be efficiently written down as the Bethe equations in the case of the Gaudin and XXX models. In general an analogous procedure is not known. However for example in the case of regular connections the necessary condition for absence of monodromy is the very trivial condition for the residues of the connection to have integer spectrum. Hopefully the Bethe equations for the spectrum is the consequence of the similar local condition and by some unknown reasons the absence of local monodromies guarantees the absence of the global one.

"Bethe roots" are zeros of the function \(q(z)\). Let us give an example for the Gaudin model related to \(SU(2)\), taken from [Frenkel95-N12] section 5.4. Consider the Hilbert space.
$V_1 \otimes \ldots \otimes V_N$, where $V_i$ are finite-dimensional irreps of $sl(2)$ of weights $\lambda_i$. Considering a character $\chi$ we see that the equation $\chi("det"(L(z) - \partial_z))$ takes the form:

$$
\chi("det"(L(z) - \partial_z)) = \partial^2_z - \sum_i \chi(C_i)/(z - z_i)^2 - \chi(H_i)/(z - z_i) = \\
= \partial^2_z - \sum_i (\lambda_i(\lambda_i + 2))/4(z - z_i)^2 - (\mu_i)/(z - z_i)
$$

(42)

where $C_i$ are Casimirs of the $i$-th copy of $sl(2)$ and $H_i$ are Gaudin hamiltonians.

One can see that the Bethe equations:

$$
\sum_i \lambda_i/(w_j - z_i) - \sum_s 2/(w_j - w_s) = 0
$$

(43)

are equivalent to the condition that the function $q(z) = \prod_i^N (z - z_i)^{-\lambda_i/2} \prod_j (z - w_j)$ satisfies the equation:

$$
(\partial^2_z - \sum_i (\lambda_i(\lambda_i + 2))/4(z - z_i)^2 - (\mu_i)/(z - z_i))q(z) = 0
$$

(44)

One sees that the Bethe roots $w_i$ arise as zeros of the function $q(z)$ which seems to be a general phenomenon.

**Explanation by separation of variables.** The best way to explain the origin of this conjecture is from the point of view of the separation of variables. Consider the quantum separated variables $\tilde{z}_i, \tilde{\lambda}_i$, then representations of $A_q$ can be realized in the space of functions of $z_i$ where $\tilde{\lambda}_i, \tilde{z}_i$ act as operators $\tilde{z}_i, \tilde{\lambda}_i$ and Casimir elements act by some constants. From the basic identity $"det"(L(\tilde{z}_i) - \tilde{\lambda}_i) = 0$ we see that joint eigen-vectors for $m \in M_q$ can be found in a factorized form:

$$
\Psi(z_1, \ldots, z_n) = \Psi^{sv}(z_1) \ldots \Psi^{sv}(z_n)
$$

where $\Psi^{sv}(z)$ satisfies the equation $\chi("det"(L(\tilde{z}) - \tilde{\lambda}))\Psi(z) = 0$. By the standard quantum-mechanical intuition we see that this solution belongs to the physical Hilbert space iff it is single valued, i.e. have trivial monodromy.

Let us formulate the important problem which should be of interest not only for experts in mathematical physics but also in pure representation theory.

**Problem** To construct a universal Baxter $Q$-operator directly in the algebra $A_q$, i.e. to construct an $A_q$-valued function $Q(z)$ which satisfies the ”universal Baxter equation”:

$$
("det"(L(\tilde{z}) - \tilde{\lambda}))Q(z) = 0
$$

**Approach** The most important profit of our approach shown in [ChervovTalalaev04-N2] (see also sections below) is the following: $Q(z)$ can be constructed from $\Psi(z)$ - the fundamental solution of the equation

$$
(\tilde{\lambda} - L_q(\tilde{z}))\Psi(z) = 0
$$
which is the universal analogue of the KZ-equation. In our paper the case of the Gaudin and XXX models was considered but we believe that it is a general phenomenon.

**Remark 32** In [DoreyTateo98-N80] amusing observations were made: in the simplest case they relate the regularized characteristic polynomial of Schrödinger operator $H = (\partial_x^2 + x^4)$ (called spectral determinant and defined as $D(E) = C_0 \prod_i (1 + E/E_i)$, where $C_0$ is some normalizing constant, $E_i$ - eigenvalues of $H$) to the Baxter $T - Q$ relation for some quantum integrable model. They observe the coincidence with the relation from [Voros92-N81]:

$$D(E\epsilon^{-1})D(E)D(\epsilon E) = D(E\epsilon^{-1}) + D(E) + D(\epsilon E) + 2$$

where $\epsilon = \exp(2\pi i/3)$, with the relation for some quantity related to the ground state energy in the theory of perturbed parafermions. Moreover they found similar relations between spectral determinants of $H = (\partial_x^2 + x^4)$ and $Q$-operator’s vacuum eigen-values in the conformal field theories. It is tempting to ask a question is there any analogous relations for the Gaudin model? This is possibly related to the bispectrality [DualIS-N82] which often occurs in integrable systems, this means that a joint eigen-function of the commuting hamiltonians $H_x \Psi(x, \lambda) = h(\lambda) \Psi(x, \lambda)$ often satisfy some dual set of equations $H_\lambda^D \Psi(x, \lambda) = h^D(x) \Psi(x, \lambda)$.

### 6 KZ equation

#### 6.1 KZ equation via the Lax operator and rational solutions

Let us emphasize the connection of the material above with the KZ-equation which is of somewhat different (but related) nature then the connection of KZ-equation and Gaudin model known before (see [KZ-Gaudin-N83]).

Denote by $e_{kl}$ the standard basis in $\mathfrak{gl}_n$. Recall that the Lax operator for the Gaudin model (see formula 10) for the case $K = 0$ is given by:

\[
L^G(z) = \sum_{i=1\ldots N} \Phi_i^G \frac{E_{kl} \otimes e^{(i)}_{kl}}{z - z_i}
\] (45)

Recall the definition of the KZ equation [KnizhnikZamolodchikov84-N84] for $\mathfrak{gl}_n$. Let $z_i \in \mathbb{C}$, $(\pi_i, V_i)$ be arbitrary representations of $\mathfrak{gl}_n$. The KZ-equations is a system of ordinary differential equations for a $V_0 \otimes \ldots \otimes V_N$-valued function $\Psi(z_0, \ldots, z_N)$ given by:

\[
(\partial_{z_j} + \sum_{i \neq j} \sum_{kl} \frac{\pi_j(e_{kl}) \otimes \pi_i(e_{lk})}{z_j - z_i})\Psi(z_0, \ldots, z_N) = 0
\] (46)

Let $V_0 = \mathbb{C}^n$ be the antifundamental representation i.e. $\pi_0(e_{kl}) = -E_{lk} \in Mat_n$.

**Observation** The expression

\[
(\pi_1 \otimes \ldots \otimes \pi_N)(\partial_{z} - L^G(z))
\]
coincides precisely with one of the KZ-operators
\[ \partial_{z_0} + \sum_{i \neq j} \sum_{kl} \pi_0(e_{kl}) \otimes \pi_i(e_{lk}) \frac{z_0 - z_i}{z_0 - z_i} \]
under the agreement \( z = z_0 \).
Let us introduce a degenerated version of the KZ-equation which is natural from the point of view of the present paper. Consider \( L(z) = \sum_{i=0,\ldots,N} \Phi_i/z^{i+1} \) the Lax operator for \( U(\mathfrak{gl}_n[t]/t^N) \) and an arbitrary finite-dimensional representation \( \pi \) of \( U(\mathfrak{gl}_n[t]/t^N) \) in a vector space \( V \).

**Definition 2** Let us call the degenerated KZ-equation for a \( V \)-valued function \( \Psi(z) \) the following one:
\[
(\partial_z - \pi L(z)) \Psi(z) = 0 \tag{47}
\]

In view of the connection with Bethe ansatz theory and the Baxter equation (see next subsection) the following conjecture is important:

**Conjecture 3** For the case of finite dimensional representations the KZ-equation \[46\] and its degeneration \[47\] has only rational solutions.

**Remark 33** We have tested the conjecture above in some simple examples with the help of ”Mathematica” and obtained the rational solutions.

**Remark 34** We can conjecture even that all the solutions of the equations
\[
(\partial_z - k\pi L(z)) \Psi(z) = 0 \quad \text{and} \quad (\partial_z - k\pi L^G(z)) \Psi(z) = 0
\]
are rational functions for \( k \) integer.

**Remark 35** Our conjecture can be considered as a special limit of the Kohno-Drinfeld theorem, which says that the monodromy representation of the KZ-equation is equivalent to the representation of the braiding group defined by the \( R \)-matrix of the corresponding quantum group. This quantum \( R \)-matrix becomes an identity matrix for our case. The problem is that the proof of this theorem works only for generic values of \( k \), the integer values cannot be treated in such a simple manner.

**Remark 36** Another support for the conjecture comes from the Matsuo-Cherednik correspondence [MatsuoCherednik92-N85], which relates solutions of the modified KZ-equation to solutions of the Calogero model, by another hand it is well-known that solutions of the Calogero model for integer values of \( k \) are polynomials. The obstacle to apply directly this result to our problem is that when the relevant parameter \( \lambda \) tends to zero the modified KZ equations degenerate not to the usual KZ-equations (as one can naively expect), but to slightly different equations ([Cherednik98-N86], [FelderVeselov01-N87]). We are indebted to A. Veselov for discussion on this point.
Remark 37  Also we were informed by A. Veselov about his common unpublished result with G. Felder which gives explicit formulas for the rational KZ solutions for the $GL_n$ case if one considers representations in $\mathbb{C}^n$.

Remark 38  For the case of the modified KZ-equation of the form:

$$(\partial_z - k\pi L^G_K(z))\Psi(z) = 0$$

where $L^G_K(z)$ is the Gaudin Lax operator with a constant term $K$ one should expect the appearance of non-polynomial solutions which nevertheless have no monodromy.

Remark 39  For the KZ-equation related to semisimple groups one should expect mild monodromy which disappears after passing to the Langlands dual group.

Remark 40  Interesting polynomial solutions of the KZ and q-KZ equations were found in [Pasquier05-N88, DiFrancescoZinnJustin05-N89] but their situation is different from ours - the value $1/k$ is integer, there is no rational fundamental solution. Their solution is somehow related to the conjecture [RazumovStroganov01-N90] on the ground-state of the $O(1)$ loop model and to the quantum Hall effect. We are indebted to P. Zinn-Justin for discussions on this point.

6.2 Generalized KZ-equations

The ideas presented above force us to introduce generalized versions of the KZ-equation, related to models other than the Gaudin model.

Let us take $A_q$, $L(z)$, $\hat{\lambda}$, $\hat{z}$ as in previous subsections, i.e. $A_q$ - an associative algebra (algebra of quantum observables), $\hat{\lambda}$, $\hat{z}$ - auxiliary variables with the same commutation relation as for separated variables, $\hat{z}_i, \hat{\lambda}_i$ - quantum separated variables. Let the operators $\hat{z}, \hat{\lambda}$ provide a representation of $\hat{\lambda}, \hat{z}$ in the space $\mathbb{C}(\mathbb{C}[z])$.

As an example, alternative to the Gaudin model, one can consider $A_q$ - the Yangian $Y(gl_n)$, then $\hat{z}$ is an operator of multiplication by $z$, $\hat{\lambda} = exp(\partial_z)$. Another example is the quantum affine algebra $A_q = U_q(gl_n)$, here $\hat{z}$ is an operator of multiplication by $exp(z)$, $\hat{\lambda} = exp(ln(q)\partial_z)$. Other examples can be defined from the elliptic algebra, elliptic Gaudin model, the versions of the models above related to semisimple groups.

Notations  Let $(\pi, V)$ be a representation of $A_q$, let us call the equation on an $A_q \otimes \mathbb{C}^n$-valued function $\Psi(z)$ of the type $(\hat{\lambda} - L(\hat{z}))\Psi(z) = 0$ the universal generalized KZ-equation.

Let us call the equation on a $V \otimes \mathbb{C}^n$-valued function $\Psi_\pi(z)$ of the type:

$$(\hat{\lambda} - \pi(L(\hat{z})))\Psi_\pi(z) = 0$$

the generalized KZ-equation.

Conjecture 4  (Generalized KZ rationality conjecture)  Let $(\pi, V)$ be a unitary representation of $A_q$, consider the differential (difference) equation $(\hat{\lambda} - \pi(L(\hat{z})))\Psi_\pi(z) = 0$ for a $V \otimes \mathbb{C}^n$-valued function $\Psi_\pi(z)$. Then this equation has trivial monodromy ("q-monodromy" i.e. Birkhoff’s connection matrix [Birkhoff-N10]).

Corrections.  For the models related not to $GL(n)$ but to a semisimple group $G$ the
monodromy disappears only after passing to the Langlands dual group $G^L$. If irregular singularities appear in $\chi(\text{"det"}(L(z) - \tilde{\lambda}))$ one should require the vanishing of the Stokes matrices.

Relations to $q$-KZs. The $q$-analogs of the KZ equations were defined in [FrenkelReshetikhin92-N91], [Smirnov92-N92], it would be interesting to find relations.

6.3 Baxter equation and Q-operator from KZ equation

Let us preserve the notations of the previous sections. As it was argued above the equation $\pi(\text{"det"}(L(z) - \tilde{\lambda}))Q(z) = 0$ should be considered as the generalized Baxter equation, on the other hand the equation ($(L(\tilde{z}) - \tilde{\lambda}))\Psi(z) = 0$ can be considered as the generalized KZ-equation.

The main message of [ChervovTalalaev04-N2] was the following: one is able to construct solutions of the Baxter equation from solutions of the KZ-equation. This goes as follows: let $\Psi(z)$ be a solution of the equation $(\tilde{\lambda} - L_q(z))\Psi(z) = 0$. Then any component $\Psi_i(z)$ of the vector $\Psi(z)$ provides a solution of the universal Baxter equation: $\text{"det"}(L(z) - \tilde{\lambda}))\Psi_i(z) = 0$.\(^8\)

The construction above allows to deduce (at least in one direction) the main conjecture on the Baxter equation from the KZ rationality conjecture [ChervovTalalaev04-N2]. Also it gives a hope to apply profound theory of the KZ equation and Cherednik algebras to solve the Baxter equation.

7 Other questions

7.1 Generalization to semisimple Lie algebras

The construction of a commutative family for a semisimple Lie algebra $g$ presumably can be obtained precisely by the same formula:

$$\text{"det"}(L^\theta(z) - \partial_z)$$

where $L^\theta(z)$ is defined as follows: consider the quadratic Casimir element $C \in U(g)$, define

$$\Phi^\theta = 1/2\pi_1(\Delta(C) - C \otimes 1 - 1 \otimes C) \in \text{Mat}_n \otimes U(g)$$

where $\Delta(C)$ is the standard coproduct for $U(g)$, $\pi_1$ is an arbitrary $n$-dimensional representation for $g$. Define $L^\theta(z)$ for the algebra $U(g[t]/t^N)$ by the formulas $L^\theta(z) = \sum_i \Phi^\theta t_i z^{-(i+1)}$.

It is likely that this construction works only for ADE Lie algebras, for the others one should possibly introduce $\Phi^\theta$ which incorporates $g$ and the Langlands dual $g^L$. This construction possibly gives not always maximal subalgebras but extendable to maximal by some expression like $\text{Pfaffian}(L^\theta(z))$.

\(^8\)Moreover, in [ChervovTalalaev06-03-N13] we show that there is a gauge transformation of the KZ-type connection $(\tilde{\lambda} - L_q(z))$ to the Drinfeld-Sokolov form corresponding to the differential (difference) operator $\text{"det"}(L(z) - \tilde{\lambda}))$.  

35
All the conjectures presented above should also be true in this more general situation, with the only difference that speaking about monodromy of the KZ-equation and $G$-opers one expects that it became trivial only after passing to the Langlands dual group.

7.2 Generalization to quantum groups

As we already discussed it is natural to believe that the construction of the QCP can be generalized to the case of the Yangians, quantum affine algebras and elliptic algebras. As we already stated we also hope the other results presented here (the separation of variables, the construction of the Baxter equation and its relation to the spectrum, relation between KZ and Baxter equations) should be true in this more general setting.

Let us also remark that hopefully our approach produces not only the center at the critical level but also the deformed $W$-algebras out of the critical level. Following ReshetikhinSemenov90-N93, SemenovTianShansky97-N35 one should introduce the properly defined full current $L_{full} = L_+(z)L_-^{-1}(z)$, then we hope that

$$: QCP(L(z)) := \det(L_{full} (\hat{z}) - \hat{\lambda}) : \quad (50)$$

defines the center at the critical level and the deformed $W$-algebras out of the critical level for the centrally extended Yangian double KhoroshkinTolstoy94-N94, Khoroshkin96-N95, Iohara96-N96, quantum affine algebras $U_q(\hat{g})$ and possibly for elliptic algebras also. In the case of quantum affine algebras $U_q(\hat{g})$ this deformed $W$-algebras should coincide with the ones introduced in qWalg-N97. The center of quantum affine algebras was described in completely different way in ReshetikhinSemenov90-N93, DingEtingof94-N98 it would be interesting to clarify the connection.

The $: ... :$ should be understood in the sense of the AKS map $I : A_q^+ \otimes A_q^- \rightarrow A_q$ given by $I(a \otimes b) = as(b)$, see SemenovTianShansky97-N35.

It would be also interesting to explore an analogue of the approach above in the case of Lie superalgebras and their $q$-deformations.

7.3 Application to other integrable systems

There are several systems related to the Gaudin model: the Calogero-Moser system, tops, the Neumann model, generalized bending flows. Our approach should be applicable for them also.

**Calogero-Moser system** Rational and trigonometric Calogero-Moser system with or without spin can be obtained by reduction from the system whose Lax operator satisfies precisely the commutation relation of the Gaudin model:

$$[\hat{L}(z), \hat{L}(u)] = \left[ \frac{P}{z-u}, \hat{L}(z) + \hat{L}(u) \right] \quad (51)$$
In the case of the trigonometric model the Lax operator is given (see [ChervovTalalaev03-1-N48], [ChervovTalalaev06-1-N4] for details) by:

\[ L(z) = \Phi^L/(z - a) + \Phi^R/(z - b) + \Phi^{\text{orbit}}/(z - c) \]  (52)

where the non-reduced phase space is \( T^*GL(n) \times \text{Orbit} \), \( \Phi^L, \Phi^R \) are the elements related to the left and right invariant vector fields on \( GL(n) \), \( \Phi^{\text{orbit}} \) is the element of the coadjoint orbit. The hamiltonian reduction is proceeded with respect to the action of \( GL(n) \) by conjugation on \( T^*GL(n) \) and the natural action on the coadjoint orbit. The system obtained by the classical reduction is the trigonometric Calogero-Moser system.

Since it is generally believed that the reduction and the quantization commute it is expected that by elaborating the reduction of the Gaudin-type model and using our results related to it one can obtain:

- construction of quantum commuting hamiltonians
- construction of the Baxter equation and \( Q \)-operator

**Classical and Quantum tops**  In [Reyman93-N99] one can find the description of multidimensional tops and the discussion of their quantization with the help of the center of the affine algebra at the critical level. Our results give a direct way to quantize the Lax operators described there and approach the problem of construction of the \( Q \)-operator and Baxter equation.

**Neumann system.**  Neumann system is a special case of the \( SL(2, \mathbb{R}) \)-Gaudin model related to special unitary representations of \( SL(2, \mathbb{R}) \). In beautiful papers [BabelonTalon92-N100], [BellonTalon04-N101] the separation of variables and construction of the Baxter equation were found for this model.

Let us advertise that our approach based on the relation between the KZ-equation and the Baxter equation can allow one to find explicit solutions for the Baxter equation using the solutions of the KZ-equation for \( SL(2, \mathbb{R}) \).

**Bending flows**  Bending flows is an amusingly simple and nice integrable system discovered in [KapovichMillson96-N102] (see also [BallesterosRagnisco98-N103], [FlaschkaMillson01-N104]). The generalization to the case of \( SL(n) \) was proposed in [FalquiMusso04-N26]. The peculiarity of the proposal is to introduce not only one Lax operator but the set of Laxes: \( L_a(z) = \sum_{i=1,...,a-1} \Phi_i + z\Phi_a \). Classical commuting hamiltonians were defined as \( \det(L_a(z) - \lambda) \).

We propose that quantum commuting bending flows can be defined in the same way as above: "\( \det^q(L_a(z)/(z(z-1)) - \partial_z) \). It follows from the theorems above that for each \( a = 1, ..., n \) the quantum hamiltonians defined in this way commute, but one needs to prove the commutativity for different \( a \), we hope to elaborate this in future.
7.4 Hamilton-Cayley identity and ”Quantum eigenvalues”

In the subsequent publication [ChervovTalalaev06-03-N13] we will describe the generalization of Hamilton-Cayley identity for the quantum Lax operator with spectral parameter of the Gaudin and related models (i.e. for $\mathfrak{gl}_n[t]$ Lie algebra). In this generalization the quantum characteristic polynomial plays the role of the standard characteristic polynomial in the classical Hamilton-Caley identity. For the Lax operators without spectral parameters the question has been deeply explored in [HC-N103]. Some physical applications of classical Hamilton-Caley identity can be found in [BerensteinUrrutia93-N106]. Let us mention that in [GelfandKrobLascouxLeclercRetakhThibon94-N107] section 8.6 page 96 the generalization of the Hamilton-Cayley identity was found for the matrix with the coefficients in arbitrary noncommutative algebra, but this generalization is of somewhat different nature - the coefficients of their identity are given not by scalars, but by the diagonal matrices with (generically) pairwise distinct entries (see for example formula 166). Nevertheless it is quite possible that there exists a connection between these constructions.

Another related question is to find a factorization of the quantum characteristic polynomial i.e. to define ”quantum eigenvalues”. This question seems to be deeply related to quantum separated variables. In [GurevichSaponov01-N108] another version of the characteristic polynomial for the $GL(2)$ Lax operator was factorized explicitly, hopefully similar ideas related to the notion of quantum bundles should work in our case.

Also the paper [MolevRetakh03-N109] hints that there is possibly non-trivial intersection between our approach and the theory created in [GelfandQDet-N110], so hopefully their general theory can be applied to our particular cases. Let us also mention that the questions dealt in [Kirillov00-N111], [Rozhkovskaya-N112] seems to be very close to the ones here.

7.5 Matrix models

It was pointed out to us by A. Alexandrov that in [AlexandrovMironovMorozov04-N113] it appeared the ”quasi-classical” curve of the type $y^2 = \sum_i H_i x^i$, where $H_i$ are operators. It is related to a completely different field - the genus zero part of the partition function of matrix models was expressed with its help. So it is natural to speculate that all genus partition function of the model is related to the solution of the equation given by the ”quantum spectral curve”: $(\partial^2 y - \sum_i H_i x^i) \psi(x) = 0$. It always happens in matrix models and Gromov-Witten theory that all genus partition function is in a sense a quantization of the genus zero part.

7.6 Relation with the Drinfeld-Sokolov reduction

The result of Drinfeld-Sokolov is that the hamiltonian reduction $\hat{\mathfrak{g}}_n^* /// n_+$ produces naturally the space of differential operators. This plays a crucial role in the construction of the Langlands correspondence. The quantum characteristic polynomial ”det$(L(z) - \partial_z)$
A Gaudin model with constant term

The crucial point in the quantization for the standard Gaudin model (see [Talalaev04-N1]) is the construction of the commutative family in $Y(\mathfrak{gl}_n)$ due to [Molev97-N24]. Actually the generators

$$\tau_k(u, h) = Tr A_n T_1(u, h) T_2(u - h, h) \ldots T_k(u - h(k - 1), h) C_{k+1} \ldots C_n \quad k = 1, \ldots n$$

for a constant matrix $C$ commute (here the matrix $C$ does not depend on the spectral parameter $u$). The Gaudin model without constant term was quantized with the choice $C = 1$. To add a constant term one has to consider the matrix $C$ of the form

$$C = 1 - hK$$

Composing now the Yangian-type universal $G$-oper of the form

$$Q_{\text{char}}(u, h) = Tr A_n (e^{-h\partial_z T_1(u, h) - (1 - hK)}) \ldots (e^{-h\partial_z T_n(u, h) - (1 - hK)})$$

$$= \sum_{j=0}^{n} \tau_j(u - h, h)(-1)^{n-j} C_j e^{-j h\partial_z}$$

where the generators $\tau_j$ correspond to the choice $C = 1 - hK$ one obtains

$$Q_{\text{char}}(u, h) = h^n q_{\text{char}}(L^K(u) - \partial_z) + O(h^{n+1})$$

where

$$L^K(u) = K + \sum_i \frac{\Phi_i}{u - z_i}$$

B Lax operator with trivial residue

One of the natural geometrical conditions for the Lax operator is the trivial residue at a pole which correspond to the holomorphity condition for the Lax operator on a singular curve or to the moment map condition subject to the hamiltonian reduction. Despite the Lax operator of the form

$$\tilde{L}(z) = \sum_{i=1}^{N} \frac{\Phi_i}{z^{i+1}}$$

does not satisfy the $r$-matrix linear commutation relation with $r(z) = P z^{-1}$ one can quantize the corresponding system by the same formula as for the complete Lax operator
L(z) given by the formula 
\[ L(z) = \sum_{i=0}^{N} \Phi_i / z^{i+1} \]
Let us introduce the positive Lax operator 
\[ L^+(z) = \sum_{i=1}^{N} \Phi_i z^{i-1} \] (56)
$L^+(z)$ satisfy the following relation 
\[ [L^+(z), \hat{L}^+(u)] = -[\frac{P}{z-u}, L^+(z) + \hat{L}^+(u)] \] (57)
By theorem 2 the quantum characteristic polynomial constructed with the help of $L^+(z)$ 
\[ qchar(-L^+(z) - \partial_z) = TrA_n(-L_1^+(z) - \partial_z) \ldots (-L_n^+(z) - \partial_z) \] (58)
has commutative coefficients.
On the other hand the transformation 
\[ L^+(z) \mapsto -\frac{1}{z^2}L^+(\frac{1}{z}) = \tilde{L}(z) \]
maps the space of positive Lax operators to the space of operators of the form $\tilde{L}(z)$.
Introducing the variable $u = z^{-1}$ we obtain 
\[ qchar(-L^+(z) - \partial_z) = (-1)^n TrA_n(u^2(\tilde{L}_1(u) - \partial_u) \ldots u^2(\tilde{L}_n(u) - \partial_u) \] (59)
Let us introduce the partial characteristic polynomials 
\[ qchar_k(\tilde{L}(u) - \partial_u) = TrA_n(\tilde{L}_1(u) - \partial_u) \ldots (\tilde{L}_k(u) - \partial_u) \] (60)
and recall the combinatorial result from \cite{ChervovTalalaev04-N2} (Theorem 1) that the coefficients of these expressions coincide with the coefficients of the complete characteristic polynomial up to constant scalars:
if one introduces $QI_i$ as coefficients of 
\[ qchar(\tilde{L}(u) - \partial_u) = \sum_{i=0}^{n} C_n^i (-1)^{n-i} QI_i(u) \partial_u^{n-i} \]
then 
\[ qchar_k(\tilde{L}(u) - \partial_u) = \sum_{i=0}^{k} C_k^i (-1)^{k-i} QI_i(u) \partial_u^{k-i} \]
Now putting all expressions $u^2$ into the formula (59) to the left one obtains 
\[ qchar(-L^+(z) - \partial_z) = \sum_{k=0}^{n} f_k(u) qchar_k(\tilde{L}(u) - \partial_u) \]
\[ = \sum_{j=0}^{n} \left( \sum_{i=j}^{n} QI_{i-j}(u) C_i^j (-1)^j f_i(u) \right) \partial_u^j \] (61)
for some polynomials $f_k(u)$. By construction this is a differential operator with commuting coefficients. To finish the proof one needs to verify that the linear transformation from this coefficients to $Q I_i(u)$ is not degenerate over the field of meromorphic functions. It is true because the matrix is triangular with $f_n(u)$ on the diagonal. This function is equal $(-1)^n u^{2n}$ due to the higher order term count and hence is not identically zero.

C AKS lemmas

Let $g = g_+ \oplus g_-$ be a finite dimensional Lie algebra which is a direct sum of two its Lie subalgebras. One has an isomorphism of linear spaces related to some normal ordering

$$\phi : U(g) \rightarrow U(g_+) \otimes U(g_-)$$

Lemma 1 The center $Z \subset U(g)$ maps to a commutative subalgebra in $U(g_+) \otimes U(g_-)$ via $\phi$ where $g^{op}$ is $g_-$ with inverted bracket.

Proof Let us denote the commutator in $U(g_+) \otimes U(g_-)$ by $[*,*]_R$. Let $c_1, c_2$ be two central elements in $U(g)$ taken in the form

$$c_i = \sum_j x_j^{(i)} y_j^{(i)} \quad x_j^{(i)} \in U(g_+), \ y_j^{(i)} \in U(g_-)$$

$$[\phi(c_1), \phi(c_2)]_R = \sum_j x_j^{(1)} y_j^{(1)} \sum_k x_k^{(2)} y_k^{(2)} = \sum_{j,k} [x_j^{(1)}, x_k^{(2)}]_R y_j^{(1)} y_k^{(2)} + x_j^{(1)} x_k^{(2)} [y_j^{(1)}, y_k^{(2)}]_R$$

Due to the definition of the algebraic structure

$$[x_j^{(1)}, x_k^{(2)}]_R = [x_j^{(1)} x_k^{(2)}] = [y_j^{(1)}, y_k^{(2)}] = -[y_j^{(1)} x_k^{(2)}]$$

$$[\phi(c_1), \phi(c_2)]_R = \sum_k [c_1, x_k^{(2)}] y_k^{(2)} - \sum_j x_j^{(1)} [y_j^{(1)}, c_2]$$

which is zero due to the centrality of $c_1, c_2$ □

Remark 41 The result is valid in the classical case if one consider the Poisson algebra $S(g)$ and the following isomorphisms of linear spaces

$$\phi_{cl} : S(g) \rightarrow S(g_+) \otimes S(g_-)$$

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