QCD SUM RULES AND THE DETERMINATION OF LEADING TWIST NON–SINGLET OPERATOR MATRIX ELEMENTS

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Abstract
We use QCD sum rules to determine directly the leading–twist non–singlet operator matrix elements based on calculations of three–point correlator functions in configuration space. We find a different result from that obtained by integrating the structure functions' expressions obtained by Belyaev and Ioffe based on calculations of four–point correlators in momentum space. The origin of this discrepancy remains unclear.

1 Introduction
The determination of the functional dependence of the nuclear structure functions on the Bjorken scaling variable $x$ via QCD sum rules has been pioneered by Belyaev and Ioffe [1, 2]. They considered the four–point correlator

$$iT_{\mu\nu}(p,q) = \int d^4x d^4y d^4z e^{iqx} e^{ip(y-z)} \langle \{\eta(y) j_\mu(x) j_\nu(0) \overline{\eta}(z)\} \rangle_0$$

where $\eta$ is the standard three–quark current with proton quantum numbers and $j_\mu$ is the electromagnetic current (see Fig. 1). The sum rules they obtained have a limited applicability region restricted to intermediate $x$ far from the kinematical boundaries $x = 0$ and 1, and should be used for not very large $Q^2 \lesssim 10 \text{ GeV}^2$.

In this paper we propose another method to calculate the quark distributions by evaluating their moments. More precisely, we shall use QCD sum rules to determine the leading twist–two non–singlet operator matrix elements (OMEs) contributing to deep inelastic scattering from nucleon targets which, by the moments sum rules, are related to the quark distribution moments.

Figure 1: The lowest–order contribution to the four–point correlator of Equation 1.
In principle, this method should be equivalent to that of [1] since it is based on calculation of three–point correlation functions which can be produced by performing the OPE on the two electromagnetic currents in the four–point correlator approach (see Fig. 2). However, it does not have the $Q^2$ problem as it is calculated perturbatively from the perturbative dependence of the coefficient functions and the anomalous dimension of the operators in the OPE series.

It is crucial to note that, assuming the validity of the conventional OPE as was the case in [1], the calculations for higher moments do not reveal logarithmic divergences. Thus it is not necessary to use the external field method contrary to the case of the second moments where the logarithmic divergences are present [3].

2 The method

Following the standard procedure of QCD sum rules, we calculate the three–point correlation function

$$T_{ij}(p) = \int d^4(x)e^{ipx}\langle 0|T\{\eta_i(x)\hat{O}_{\beta_1...\alpha_n}\bar{\eta}_j(0)\}|0\rangle$$

(2)

where $\eta_i(x)$ is the interpolating current for a proton originally suggested in [4].

$$\eta_i(x) = (u_a^T(x)C\gamma_{\mu}u_b(x))(\gamma_5\gamma^\mu d_c(x))_i\epsilon^{abc}$$

(3)

$C$ is the charge conjugation matrix, superscript $T$ means transpose, $i$ and $j$ are spinor indices while $a$, $b$ and $c$ are colour indices. As to the operator $\hat{O}_{\beta_1...\alpha_n}$, we choose the twist–two non–singlet composite operator given by

$$\hat{O}_{\beta_1...\alpha_n} = i^n S \int d^4y\bar{\Psi}(y)\gamma_\beta D_{\alpha_1}...D_{\alpha_n}\Psi(y)$$

(4)

where $D_\alpha$ is the covariant derivative

$$D_{\alpha_1|y} = \partial_\alpha - ig(\frac{\lambda^n}{2})A_\alpha^n(y) = \partial_\alpha - ig(\frac{\lambda^n}{2})y_pG^n_{\rho\alpha}(0)$$

(5)

In fact, performing the calculations in the case of the second moment would lead to integrals of the form $\int d^4x e^{ipx}\frac{\ln x^2}{(x^2)^m}$ where $m > 2$. These integrals when Borel transformed give a divergent contribution.
in Schwinger gauge where \( y_\alpha A_\alpha^n(y) = 0 \), and \( S \) denotes the symmetrization over the indices \( \beta, \alpha_1, \ldots, \alpha_n \) with subtraction of traces in order to make the operator \( \hat{O} \) of definite spin.

The operators \( \hat{O}_{\beta\alpha_1\ldots\alpha_n} \) form an irreducible representation of the Lorentz group. Their reduced matrix elements \( A^n \) are defined by

\[
\langle N(p)|\hat{O}_{\beta\alpha_1\ldots\alpha_n}|N(p)\rangle = A_n\ p_\beta p^{\alpha_1}\ldots p^{\alpha_n} + \text{terms containing } g_{\mu\nu},
\]

and our aim is to estimate these OMEs’ \( A^n \) by the QCD sum rules method.

First, we consider the “phenomenological” representation of the three–point correlator which is expressed in terms of physical intermediate states as

\[
\langle 0|\eta|N(p)\rangle \times \text{nucleon propagator} \times \langle N(p)|\hat{O}_{\beta\alpha_1\ldots\alpha_n}|N(p)\rangle \times \text{nucleon propagator} \times \langle N(p)|\eta|0\rangle
\]

where nucleon propagator = \( i\frac{\not{p}+m}{p^2-m^2} \) and \( m \) is the mass of the nucleon \( N \). In addition to this double pole term, there are also single pole (continuum) terms suppressed relative to the former with power of the Borel parameter \( M^{-2} \) (exponentially). One can check that the combination of invariant functions which enters the coefficient at the structure \( \hat{p}\hat{p}^\beta p^{\alpha_1}\ldots p^{\alpha_n} \) or \( p^\beta p^{\alpha_1}\ldots p^{\alpha_n} \) in Equation 7 coincides (up to a numerical factor) with the combination of invariant functions at the structure \( p^\beta p^{\alpha_1}\ldots p^{\alpha_n} \) in the spin–averaged matrix element \( \langle N(p)|\hat{O}_{\beta\alpha_1\ldots\alpha_n}|N(p)\rangle \) i.e. with the reduced matrix element we are interested in, and we get:

\[
\text{Phen} = \frac{-\lambda^2}{(p^2-m^2)^2}2A_n(\hat{p}+m)p^\beta p^{\alpha_1}\ldots p^{\alpha_n} + \text{other structures},
\]

where \( \lambda \) is the constant defined by

\[
\langle 0|\eta|N(p)\rangle = \lambda v_p
\]

and \( v_p \) is the proton spinor satisfying \( (\hat{p}-m)v_p = 0 \) and the normalisation \( \sum_{\text{polarisations}} v^r v^r = 2m \).

The different tensor structures emerging from the double pole term can be used to construct a sum rule and extract \( A_n \), but in our case the sum rule from the structure \( \hat{p}\hat{p}^\beta p^{\alpha_1}\ldots p^{\alpha_n} \) is preferred. Firstly, this is because it contains the maximum number of momenta in the numerator and thus improves the convergence of the OPE series and diminishes the background contribution of excited hadronic states compared to the lowest state (proton) contribution of interest. Secondly, the structure \( \hat{p}\hat{p}^\beta p^{\alpha_1}\ldots p^{\alpha_n} \) conserves chirality: this is also a merit since for structures conserving chirality one can calculate a larger number of terms in the OPE series. Finally, for this tensor structure, several simplification tricks proved to be possible which made the calculations for the “theoretical” part manageable.

We consider now the “theoretical” part of the sum rules. In the process of the calculations we shall take into account only the operators of dimension \( d \leq 6 \) and calculate only the invariant functions at the structure \( \hat{p}\hat{p}^\beta p^{\alpha_1}\ldots p^{\alpha_n} \). The diagrams corresponding to this tensor structure are depicted in Fig. 3 where the unit operator contribution corresponds to Fig. 3a, the gluon condensate \( \langle G^\alpha_{\mu\nu}G^\alpha_{\mu\nu} \rangle \) to Fig. 3b–l and the quark condensate \( \langle \bar{\psi}\psi \rangle^2 \) to Fig. 3m. For massless quarks, it is convenient to work in configuration space and in order to calculate the gluonic condensate contribution it is far easier to work in the fixed–point gauge for the gluonic field \( z_\mu A_\mu^n(z) = 0 \).

The standard choice of the reference frame is to identify the origin \( (z = 0) \) with the coordinate of one of the currents, so we have the following two choices.
Figure 3: The “theoretical” sides of the sum rule for the three–point correlation function. The wavy lines denote the current, and the crossed circles denote the operator insertion. The solid and dashed lines denote quarks and gluons respectively.

1. The operator $\hat{O}$ is inserted at the origin (Fig. 4a): The advantage of this gauge is that, because covariant derivatives may be replaced by ordinary derivatives at the origin, the graphs where gluons originate from the operator (Fig. 3i, j, k, l) do not contribute. This simplification is, however, illusory and one should be very careful in doing the calculations in this gauge choice. In fact, because the operator contains derivatives, it is impossible to put ($z = 0$) from the very beginning since the quark loop is determined now by derivatives of the type ($\frac{\partial}{\partial z}$) $S(z, y)$ where $S(z, y)$ denotes the propagator between the two points $y$ and $z$ and is given by [6, 7, 8]:

$$S^{A,B}_{ij}(z, y) = \delta^{AB} \frac{i}{2\pi^2} \frac{(z - y)_{ij}}{(z - y)^4} + \frac{-1}{16\pi^2} \frac{ig}{2} G_{\alpha\beta} \left( \frac{\lambda^n}{2} \right)^{AB} \frac{(z - y \sigma_{\alpha\beta} + \sigma_{\alpha\beta} z - y)_{ij}}{(z - y)^2}$$

$$+ \frac{i}{2\pi^2} \frac{ig}{2} G_{\alpha\beta} \left( \frac{\lambda^n}{2} \right)^{AB} \frac{y_{\alpha} z_{\beta} (z - y)_{ij}}{(z - y)^4}$$

$$+ \frac{i}{48\pi^2} \left( \frac{ig}{2} \right)^2 G_{\alpha\beta}^m G_{\alpha\beta}^m \left( \frac{\lambda^n \lambda^m}{2} \right)^{AB} \frac{(z^2 y^2 - (z, y)^2) (z - y)_{ij}}{(z - y)^4} + \frac{-1}{12} (\bar{q} q) \delta^{AB} \delta_{ij},$$

(10)

where $A, B$ are colour indices, $i, j$ are spinor indices, $\sigma_{\alpha\beta} = \frac{i}{2} [\gamma^\alpha, \gamma^\beta]$ and $\lambda^n$ are the Gell–Mann matrices.

Since one has to differentiate first with respect to $z$ and only put $z = 0$ afterwards, we can see that the “non–translation–invariant” terms in $S(z, y)$ would give non-zero results leading to new integrals to be done.

2. The current $\bar{\eta}$ is inserted at the origin (Fig. 4b): Here the derivatives are applied at $y$ and no derivatives in the current $\bar{\eta}$ at 0 so we can put $z = 0$ from the beginning. Looking again at the expression for $S(z, y)$ we see that the graphs in Fig. 4f and h contribute nothing but those of Fig. 3i, j, k and l do contribute since the covariant derivative $D$ contains operators of the gluon field (c.f. Equation 9).

At first glance, this gives rise to some doubts as one might suspect that the gluons originating from the operator (e.g. Fig. 3k) correspond to gluons emitted from the upper propagator in the four–point correlation function formalism (Fig. 3a). If this were true...
then it would lead to inconsistencies, because in Schwinger gauge the gluon field $A_\mu$ is expressed in terms of the field strength tensor $G_{\mu\nu}$ and the standard analysis of OPE in DIS shows that this would be of higher twist effect while our operator is of leading twist! However, as the choice of gauge in Fig. 4 shows, such higher twist operators do not contribute. Thus gauge invariance requires that these operator–originated gluons have their origin in the right–hand upper propagator (Fig. 5b) and no inconsistencies persist.

We opt for the second choice of gauge: the origin at the right–hand vertex of the diagrams of Fig. 4, $y$ as the point where the operator $\hat{O}_{\beta\alpha_1...\alpha_n}$ is inserted while $x$ is the point where the nucleon is created. For this gauge choice, the calculations are easier and can be done in two steps. We do the $y$–integration first to obtain expressions of “full” propagators in the presence of the operator $\hat{O}_{\beta\alpha_1...\alpha_n}$, leaving the $x$–integration till the end.

As an example, doing the $y$–integration in Fig. 4b results in a full propagator

$$\int \frac{d^4 p}{(p^2 + i\epsilon)^n} e^{ip.x} = (-1)^n \frac{i(-1)^n 2^{2n-2} \pi^2}{\Gamma(n-1)\Gamma(n)} (x^2 - i\epsilon)^{n-2} \ln(-x^2 + i\epsilon).$$

As we are interested only in the structure $\hat{x} x^\beta x^{\alpha_1} ... x^{\alpha_n}$, the derivatives can be done giving
\[ \psi^u \otimes \psi^d = i^n \delta^{AB} \cdot i \frac{(-2)^{n-1} n! \gamma^\beta}{(x^2)^{n+1}} \left( x^{\alpha_2 \ldots \alpha_n} \gamma^{\alpha_1} + \ldots + x^{\alpha_1 \ldots \alpha_n} \gamma^{\alpha_2} \right) \]

Essentially the same techniques are used to evaluate other “full” propagators in the presence of operators even though the calculations are more complicated. After this, one can evaluate the graphs: for example, we get for the bare graph where the operator is inserted on a u–quark line the result:

\[ \frac{i^{n+1} (-2)^n n!}{\pi^b (x^2)^{n+5}} (-48 - 36n) . \]

In order to get the expression in the momentum representation, one should now perform the \( x \)-integration using the formula

\[ \int \frac{d^4x}{(x^2 - i\epsilon)^n} e^{ip.x} = \frac{i(-1)^n 2^{4-2n} \pi^2}{\Gamma(n-1)\Gamma(n)} (p^2 + i\epsilon)^{(n-2)} \ln(-p^2 - i\epsilon) \quad (12) \]

For the other graphs, the calculations were performed using the program Mathematica for symbolic calculus (we used the FeynCalc package). The results of all the graphs are listed below (the letters a,b,… correspond to those of Fig. 3; the superscripts \( u \) and \( d \) denote that the operator is acting on \( u \)– and \( d \)–quarks respectively while the subscripts \( u \) and \( d \) mean that a \( u \)– or \( d \)–quark line where the operator is not inserted emits a gluon).

\[
\begin{align*}
(a)^u &= \frac{1}{4\pi^4} \left( \frac{1}{2(n+1)} + \frac{3}{n+2} + \frac{-15}{2(n+3)} + \frac{4}{n+4} \right) p^2 \ln p^2 \\
(b\&c)^u &= \frac{g^2(G^2)}{\pi^4} \left( \frac{-1}{192(n+1)} + \frac{1}{192(n+2)} \right) \frac{1}{p^2} \\
(b\&c)^d &= \frac{g^2(G^2)}{\pi^4} \left( \frac{1}{96(n+1)} + \frac{-1}{96(n+2)} \right) \frac{1}{p^2} \\
(d)^u &= \frac{g^2(G^2)}{\pi^4} \left( \frac{-1}{432(n-1)} + \frac{1}{144n} + \frac{-1}{144(n+1)} + \frac{1}{432(n+2)} \right) \frac{1}{p^2} \\
(e)^u &= \frac{g^2(G^2)}{\pi^4} \left( \frac{-1}{192(n+1)} + \frac{-1}{48(n+2)} \right) \frac{1}{p^2} \\
(g)^u &= \frac{g^2(G^2)}{\pi^4} \left( \frac{-1}{144n} + \frac{-1}{192(n+1)} + \frac{5}{288(n+2)} \right) \frac{1}{p^2} \\
(i\&j)^u &= \frac{g^2(G^2)}{\pi^4} \left( \frac{-1}{432(n-1)} + \frac{1}{192(n+1)} + \frac{-7}{864(n+2)} \right) \frac{1}{p^2} \\
(k)^u &= 0 \\
(k)^d &= \frac{g^2(G^2)}{\pi^4} \left( \frac{-1}{192(n+1)} + \frac{1}{192(n+2)} \right) \frac{1}{p^2} \\
(l)^u &= \frac{g^2(G^2)}{\pi^4} \left( \frac{-5}{576(n+1)} + \frac{1}{72(n+2)} \right) \frac{1}{p^2} \\
(m)^u &= 0
\end{align*}
\]
where

\[ \text{Equating with Equation 8 and doing the Borel transform, we arrive at:} \]

Summing all the graphs we arrive at:

\[ \text{Theo}^u = \frac{4}{32\pi^4} \left( \frac{1}{n+1} + \frac{6}{n+2} + \frac{-15}{n+3} + \frac{8}{n+4} \right) p^2 \ln p - p^2 \]
\[ + \frac{g^2(G^2)}{32\pi^4} \left( \frac{-4}{27(n-1)} - \frac{2}{3(n+1)} + \frac{4}{27(n+2)} \right) \frac{1}{p^2} \]

\[ \text{Theo}^d = \frac{4}{32\pi^4} \left( \frac{1}{n+1} + \frac{-3}{n+3} + \frac{2}{n+4} \right) p^2 \ln p - p^2 \]
\[ + \frac{g^2(G^2)}{32\pi^4} \left( \frac{-4}{27(n-1)} - \frac{2}{3(n+1)} + \frac{28}{27(n+2)} \right) \frac{1}{p^2} \]
\[ + \frac{-4}{3} \langle \bar{q}q \rangle^2 \frac{1}{p^4} \]

Equating with Equation 8 and doing the Borel transform, we arrive at

\[ A_n^u + \frac{M^2}{m^2} C_n^u = \frac{M^6}{2\lambda_N^2} e^{m^2/M^2} \left\{ \frac{4}{n+1} + \frac{6}{n+2} + \frac{-15}{n+3} + \frac{8}{n+4} \right\} \]
\[ + b \left\{ \frac{-4}{27(n-1)} - \frac{2}{3(n+1)} + \frac{4}{27(n+2)} \right\} \]

\[ A_n^d + \frac{M^2}{m^2} C_n^d = \frac{M^6}{2\lambda_N^2} e^{m^2/M^2} \left\{ \frac{4}{n+1} + \frac{-3}{n+3} + \frac{2}{n+4} \right\} \]
\[ + b \left\{ \frac{-4}{27(n-1)} - \frac{2}{3(n+1)} + \frac{28}{27(n+2)} \right\} \]
\[ + \frac{8 a^2}{3 M^6} \]

where \( m \) is the nucleon mass,

\[ a = -(2\pi)^2 \langle 0|\bar{\psi}\psi|0 \rangle \]
\[ b = (2\pi)^2 \langle 0 | \frac{\alpha_s}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a | 0 \rangle \]

and
\[ \bar{\lambda}_N^2 = 2(2\pi)^4 \lambda^2 \].

### 3 Comparison with other results

As mentioned in the introduction, Belyaev and Ioffe [1] determined the valence up and down quark distributions in the proton by considering the four–point correlator \( T_{\mu\nu} \) (Equation [1]) where \( x = \frac{Q^2}{2p.q} \) is fixed and not too close to the boundaries \( x = 0,1 \) and \( Q^2 = -q^2 \) is assumed to be only moderately large \( (Q^2 \lesssim 10 GeV^2) \) since the logarithmic corrections \( [\alpha_S(\ln \frac{Q^2}{\Lambda_{QCD}})]^n \) were not summed. Assuming the legitimacy of using the conventional OPE for the imaginary part of the correlator, they calculated \( \text{Im} T_{\mu\nu} \) taking perturbative and non–perturbative corrections into account, performed the Borel transformation over the parameter \( p^2 \) keeping only leading powers in an expansion in \( \frac{1}{Q^2} \) and hence projecting only the leading twist contribution. They obtained the following sum rules (the effects of the continuum, the anomalous dimensions and the terms \( \propto \alpha_s \bar{q}q \) are not shown):

\[
xu_v(x, Q^2) + M^2 A^u(x, Q^2) = \frac{M^6}{2\lambda_N^2} e^{m^2/M^2} \left\{ 4x(1-x)^2(1 + 8x) \right.
+ \frac{b}{M^4} \left( -\frac{4}{27} + \frac{1}{6} - \frac{19}{12}x + \frac{97}{108}x^2 \right) \right\}
\tag{37}
\]

\[
xd_v(x, Q^2) + M^2 A^d(x, Q^2) = \frac{M^6}{2\lambda_N^2} e^{m^2/M^2} \left\{ 4x(1-x)^2(1 + 2x) \right.
+ \frac{b}{M^4} \left( -\frac{4}{27} + \frac{1}{6} - \frac{11}{12}x - \frac{7}{54}x^2 \right) \right\}
\tag{38}
\]

The standard analysis of DIS would, in principle, allow one to calculate the twist–two OMEs simply by taking moments of the structure functions

\[
A^u_n = \int_0^1 dx x^n u_v(x, Q^2) \tag{39}
\]

\[
A^d_n = \int_0^1 dx x^n d_v(x, Q^2) \tag{40}
\]

and formally one obtains

\[
A^u_n + \frac{M^2}{m^2} R^u_n = \frac{M^6}{2\lambda_N^2} e^{m^2/M^2} \left\{ 4 \left( \frac{1}{n+1} + \frac{6}{n+2} - \frac{15}{n+3} + \frac{8}{n+4} \right) \right.
+ \frac{b}{M^4} \left( -\frac{4}{27} + \frac{1}{6} - \frac{19}{12}n + \frac{97}{108}n^2 + \frac{1}{108}n + 2 \right) \right\}
\tag{41}
\]

\[
A^d_n + \frac{M^2}{m^2} R^d_n = \frac{M^6}{2\lambda_N^2} e^{m^2/M^2} \left\{ 4 \left( \frac{1}{n+1} + \frac{-3}{n+3} + \frac{2}{n+4} \right) \right.
+ \frac{b}{M^4} \left( -\frac{4}{27} + \frac{1}{6} - \frac{11}{12}n + \frac{-7}{54}n + \frac{1}{54}n^2 + \frac{8}{3}a^2 \right) \right\}
\tag{42}
\]

The term proportional to \( a^2 \) in Equation [42] corresponds to the \( \delta(1-x) \) piece which was omitted in the structure function expression in Equation [38] applicable at intermediate values of \( x \).
4 Discussion

In comparing Equations 35 and 36 with Equations 41 and 42, we see that the unit operator and $<\bar{q}q>^2$ contributions agree while there is a discrepancy in the $<G^2>$ term. In Belyaev’s opinion [9], it is not necessary that (35, 36) and (41, 42) give the same answer because the sum rules (37, 38) are used to find the quark distribution for intermediate $x$ only, and one cannot integrate them analytically over the whole region $x \in [0, 1]$ because there can be singular contributions of the OPE near the points $x = 0, 1$. However, in a formal (rather than phenomenological) sense, one can compare the two methods. Since we can identify the singularities at $x = 0, 1$ in the approach of [1], we can formally evaluate the moments and compare them with the OPE analysis. We have not been able to establish the origin of the discrepancy although, as was noted in [10], the phenomenological analysis of the OMEs [11, 12] changes only slightly using the new $<G^2>$ term expression. However the difference is still of importance and we think it essential to understand the origin of the discrepancy in the two methods. We hope this paper will stimulate further investigation of this puzzle.

Acknowledgement: I am very grateful to G.G. Ross for help in this analysis.

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