HOLOMORPHIC ISOMETRIES INTO HOMOGENEOUS BOUNDED DOMAINS

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Abstract. We prove two rigidity theorems on holomorphic isometries into homogeneous bounded domains. The first shows that a Kähler-Ricci soliton induced by the homogeneous metric of a homogeneous bounded domain is trivial, i.e. Kähler-Einstein. In the second one we prove that a homogeneous bounded domain and the flat (definite or indefinite) complex Euclidean space are not relatives, i.e. they do not share a common Kähler submanifold (of positive dimension). Our theorems extend the results proved in [14] and [3], respectively.

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1. Introduction

The complex $n$-dimensional hyperbolic space, namely the complex unit ball $\mathbb{C}H^n \subset \mathbb{C}^n$ equipped with the hyperbolic metric $g_{\text{hyp}}$ (a multiple of the Bergman metric of constant and negative holomorphic sectional curvature) is the prototype of noncompact and nonflat Kähler manifold. Thus, many rigidity results of the Kähler geometry of the complex hyperbolic space have been the subject of study by various...
mathematicians. Of particular interest is the study of its Kähler submanifolds, or more generally, those Kähler manifolds \((M, g)\) which admits a holomorphic isometry into \((CH^n, g_{hyp})\). The first result in this direction was obtained in the celebrated work of Calabi \[2\] were it is proven (among other things) that the complex flat space does not admit a holomorphic isometry into any complex hyperbolic space. This result has been later extended by Umehara as expressed by the following striking result.

**Theorem A** \[21\] If \((M, g)\) admits a holomorphic isometry into the complex hyperbolic space then it cannot admit a holomorphic isometry into the complex flat space.

The reader is referred to \[4\] for the extension of Theorem A to the case of indefinite complex hyperbolic and flat spaces.

Another recent rigidity result on holomorphic isometries into the complex hyperbolic space is the following theorem:

**Theorem B** \[14\] If \((g, X)\) is a Kähler-Ricci soliton on a complex manifold \(M\) and \((M, g)\) admits a holomorphic isometry into the complex hyperbolic space then the soliton is trivial, i.e. \(g\) is KE.

Recall that a Kähler-Ricci soliton (KRS) \((g, X)\) (see e.g. \[14\] and \[1\] for details and references) consists of a Kähler metric \(g\) and a holomorphic vector field \(X\), called the solitonic vector field such that \(\text{Ric}_g = \lambda g + L_X g\) where \(\text{Ric}_g\) is the Ricci tensor of the metric \(g\), \(\lambda\) a constant and \(L_X g\) denotes the Lie derivative of \(g\) with respect to \(X\). Clearly KRS generalize KE metrics. Indeed any KE metric \(g\) on a complex manifold \(M\) gives rise to a trivial KRS by choosing \(X = 0\) or \(X\) Killing with respect to \(g\).

Notice that Umehara \[20\] proves that a holomorphic isometry of a KE manifold into \((CH^n, g_{hyp})\) is necessarily totally geodesic, and hence the conclusion of Theorem B is that \((M, g)\) is an open subset of a complex hyperbolic space.

Recently Cheng and Hao \[3, Theorem 3\] extend Theorem A to the case of the Bergman metric on a homogeneous bounded domain:

**Theorem C** \[3, Theorem 3\] Let \((M, g)\) be a Kähler manifold which admits a holomorphic isometry into a homogeneous bounded domain \((\Omega, g_B)\) endowed with its Bergman metric \(g_B\). Then \((M, g)\) cannot admit a holomorphic isometry into a complex Euclidian space.

The main result of the paper is the following Theorem \[14\] where it is shown that we get the same conclusion as in Theorem B and Theorem C if we replace either
the hyperbolic metric or the Bergman metric by any homogeneous metric \( g_{\Omega} \) on a homogeneous bounded domain \( \Omega \). Recall that \( g_{\Omega} \) is a homogenous metric on a bounded domain \( \Omega \subset \mathbb{C}^n \) if \( (\Omega, g_{\Omega}) \) is acted upon transitively by its group biholomorphic isometries. Notice that a bounded symmetric domain with its Bergman metric is a very special example of this instance. Moreover, there exist many homogeneous Kähler metrics on a bounded domain different form the Bergman metric which is KE by the homogeneity assumption (see Theorem 3.1 below, [12] and [15] for details).

**Theorem 1.1.** Let \((M, g)\) be a Kähler manifold which admits a holomorphic isometry into a homogeneous bounded domain \((\Omega, g_{\Omega})\). Then the following facts hold true:

(i) any KRS on \( M \) is trivial, i.e. \( g \) is KE;

(ii) \((M, g)\) cannot admit a holomorphic isometry into a complex (definite or indefinite) Euclidian space.

In [6] the authors inspired by Umehara’s work [21] have christened two Kähler manifolds \((M_i, g_i), i = 1, 2\) to be relatives if there exists a Kähler manifold \((M, g)\) and two holomorphic isometries \( \varphi_i : M \to M_i, i = 1, 2 \) (for update results on relatives Kähler manifolds the reader is referred the the survey paper [22] and reference therein). In this language we can rephrase (ii) in Theorem 1.1 by saying that a homogeneous bounded domain and the complex (definite or indefinite) flat space are not relatives (see also [17]).

It is also worth noticing that a homogeneous bounded domain admits a holomorphic isometry into the infinite dimensional complex projective space (see [13]). Unfortunately this is not of any help in the proof of (i) of Theorem 1.1 since there are examples of non trivial KRS which admits holomorphic isometries into the infinite dimensional complex projective space (see [15]).

The proof of Theorem 1.1 is based on Theorem 2.1 which is a transcendental result on holomorphic Nash algebraic functions and on Theorem 3.1 (unpublished) due to Hishi Hideyuki which gives an explicit description of the structure of a Kähler potential of the homogeneous Kähler metric \( g_{\Omega} \) on a homogeneous bounded domain.

The paper contains three other sections. Sections 2 and Section 3 are dedicated to the proofs of Theorem 2.1 and Theorem 3.1 respectively. Section 4 contains the proof of Theorem 1.1.

The authors are indebted to Hishi Hideyuki for the proof of Theorem 3.1 and for the possibility of including it in our paper.
2. A result on holomorphic Nash algebraic functions

Let \( N^m \) be the set real analytic function \( \xi : V \subset \mathbb{C}^m \rightarrow \mathbb{R} \) defined in some open neighbourhood \( V \subset \mathbb{C}^m \), such that its real analytic extension \( \tilde{\xi}(z, w) \) in a neighbourhood of the diagonal of \( V \times \text{Conj} \ V \) is a holomorphic Nash algebraic function (for background material on Nash functions, we refer the readers to [10] and [18]). We define

\[
F = \{ \xi(f_1, \ldots, f_m) | \xi \in N^m, f_j \in O_0, j = 1, \ldots, m, m > 0 \},
\]

where \( O_0 \) denotes the germ of holomorphic functions around \( 0 \in \mathbb{C} \) and we set

\[
\tilde{F} = \{ \psi \in F | \psi \text{ is of diastasis-type} \}.
\]

Here we say (see also [14]) that a real analytic function defined on a neighborhood \( U \) of a point \( p \) of a complex manifold \( M \) is of diastasis-type if in one (and hence any) coordinate system \( \{ z_1, \ldots, z_n \} \) centered at \( p \) its expansion in \( z \) and \( \bar{z} \) does not contain non constant purely holomorphic or anti-holomorphic terms (i.e. of the form \( z_j \) or \( \bar{z}_j \) with \( j > 0 \)).

The key element in the proof of Theorem 1.1 is the following Theorem 2.1. This theorem generalizes [14, Theorem 2.1]. Moreover its proof can be considered an extension of the techniques used in the proof of [3, Theorem 1].

**Theorem 2.1.** Let \( \psi_0 \in \tilde{F} \setminus R \). Then for every \( \mu_1, \ldots, \mu_\ell \in \mathbb{R} \) we have

\[
e^{\psi_0} \notin \tilde{F}^{\mu_1} \cdots \tilde{F}^{\mu_\ell} \setminus \mathbb{R},
\]

where \( \tilde{F}^{\mu_1} \cdots \tilde{F}^{\mu_\ell} = \{ \psi_1^{\mu_1} \cdots \psi_\ell^{\mu_\ell} | \psi_1, \ldots, \psi_\ell \in \tilde{F} \} \).

**Proof of Theorem 2.1.** Assume that

\[
e^{\psi_0} = \psi_1^{\mu_1} \cdots \psi_\ell^{\mu_\ell} \in \tilde{F}^{\mu_1} \cdots \tilde{F}^{\mu_\ell},
\]

with

\[
\psi_k = \xi_k \left( f_1^{(k)}, \ldots, f_m^{(k)} \right) \in \tilde{F}, \ k = 0, \ldots, \ell.
\]

Let us rename the functions involved in (3) by

\[
(\varphi_1, \ldots, \varphi_s) = \left( f_1^{(0)}, \ldots, f_m^{(0)}, f_1^{(\ell)}, \ldots, f_m^{(\ell)} \right)
\]

and let

\[
S = \{ \varphi_1, \ldots, \varphi_s \}.
\]

Let \( D \) be an open neighborhood of the origin of \( \mathbb{C} \) on which each \( \varphi_j, j = 1, \ldots, s, \) is defined. Consider the field \( \mathcal{R} \) of rational function on \( D \) and its field extension \( \mathfrak{F} = \mathcal{R}(S) \), namely, the smallest subfield of the field of the meromorphic functions on \( D \), containing rational functions and the elements of \( S \). Let \( l \) be the transcendence degree of the field extension \( \mathfrak{F}/\mathcal{R} \). If \( l = 0 \), then each element in \( S \) is
holomorphic Nash algebraic and hence $\psi_0$ is forced to be constant by \[\text{Lemma 2.2}]. Assume then that $l > 0$. Without loss of generality we can assume that $\mathcal{G} = \{\varphi_1, \ldots, \varphi_l\} \subset S$ is a maximal algebraic independent subset over $\mathbb{R}$. Then there exist minimal polynomials $P_j(z,X,Y), X = (X_1, \ldots, X_l)$, such that

$$P_j(z,\Phi(z),\varphi_j(z)) \equiv 0, \ \forall j = 1, \ldots, s,$$

where $\Phi(z) = (\varphi_1(z), \ldots, \varphi_l(z))$.

Moreover, by the definition of minimal polynomial

$$\frac{\partial P_j(z,X,Y)}{\partial Y}(z,\Phi(z),\varphi_j(z)) \neq 0, \ \forall j = 1, \ldots, s.$$

on $D$. Thus, by the algebraic version of the existence and uniqueness part of the implicit function theorem, there exist a connected open subset $U \subset D$ with $0 \in \overline{U}$ and Nash algebraic functions $\hat{\varphi}_j(z,X)$, defined in a neighborhood $U$ of $\{(z,\Phi(z)) \mid z \in U\} \subset \mathbb{C}^n \times \mathbb{C}^l$, such that

$$\varphi_j(z) = \hat{\varphi}_j(z,\Phi(z)), \ \forall j = 1, \ldots, s.$$

for any $z \in U$. Let us denote

$$\left(\hat{f}_1^{(0)}(z,X), \ldots, \hat{f}_m^{0}(z,X), \ldots, \hat{f}_1^{(l)}(z,X), \ldots, \hat{f}_m^{l}(z,X)\right) = (\hat{\varphi}_1(z,X), \ldots, \hat{\varphi}_s(z,X)),$$

(notice that, by \[\text{[1]}, f_j^{(k)}(z,\Phi(z)) = \hat{f}_j^{(k)}(z)$ for all $k = 0, \ldots, l$ and $i = 1, \ldots, m_k$).

We define

$$\hat{F}_k(z,X) := \left(\hat{f}_1^{(k)}(z,X), \ldots, \hat{f}_m^{l}(z,X)\right), k = 0, \ldots, l.$$

Consider the function

$$\Psi(z,X,w) := \tilde{\xi}_0 \left(\hat{F}_0(z,X),F_0(w)\right) - \mu_1 \log \left(\tilde{\xi}_1 \left(\hat{F}_1(z,X), F_1(w)\right)\right) - \cdots - \mu_l \log \left(\tilde{\xi}_l \left(\hat{F}_l(z,X), F_l(w)\right)\right),$$

where $\tilde{\xi}_j$ is the real analytic extension of $\xi_j$ in a neighbourhood of $(0,0) \in \mathbb{C}^{m_j} \times \text{Conj} \mathbb{C}^{m_j}$ and $F_k(w) = \left(f_1^{(k)}(w), \ldots, f_m^{l}(w)\right)$. By shrinking $U$ if necessary we can assume $\Psi(z,X,w)$ is defined on $\tilde{U} \times U$.

We claim that $\Psi(z,X,w)$ vanishes identically on this set. Recalling that $\psi_k(z,w) = \tilde{\xi}_k(F_k(z),F_k(w))$ is of diastasis-type, we see that $\xi_k(\hat{F}_k(z,X),F_k(0)) = \psi_k(0), k = 0, \ldots, l$. Since $0 \in \overline{U}$, it follows by \[\text{[3]} that $\Psi(z,X,0) \equiv 0$. Hence, in order to prove the claim, it is enough to show that $(\partial_w \Psi)(z,X,w) \equiv 0$ for all $w \in U$. Assume, by contradiction, that there exists $w_0 \in U$ such that $(\partial_w \Psi)(z,X,w_0) \neq 0$. Since $(\partial_w \Psi)(z,X,w_0)$ is Nash algebraic in $(z,X)$ there exists a holomorphic polynomial $P(z,X,t) = A_d(z,X)t^d + \cdots + A_0(z,X)$ with $A_0(z,X) \neq 0$ such that $P(z,X,(\partial_w \Psi)(z,X,w_0)) = 0$. Since, by \[\text{[3]} and \[\text{[4]} we have $\Psi(z,\Phi(z),w) \equiv 0$ we get $(\partial_w \Psi)(z,\Phi(z),w) \equiv 0$. Thus $A_0(z,\Phi(z)) \equiv 0$ which contradicts the fact
that \( \varphi_1(z), \ldots, \varphi_t(z) \) are algebraic independent over \( \Re \). Hence \((\partial_w \Psi)(z, X, w_0) \equiv 0\) and the claim is proved.

Therefore
\[
\xi_0(F_0(z, X), F_0(w)) = \left( \xi_1 \left( F_1(z, X), F_1(w) \right) \right)^{\mu_1} \cdots \left( \xi_t \left( F_t(z, X), F_t(w) \right) \right)^{\mu_t},
\]
for every \((z, X, w) \in \mathcal{U} \times \mathcal{U} \). By fixing \( w \in \mathcal{U} \) and applying [11, Lemma 2.2] we deduce that \( \xi_0 \left( F_0(z, X), F_0(w) \right) \) is constant in \((z, X)\). Thus, by evaluating at \( X = \Phi(z) \) one obtains that \( \xi_0 \left( F_0(z), F_0(w) \right) \) is constant for fixed \( w \), forcing \( \psi_0(z) = \xi_0 \left( F_0(z), F_0(z) \right) \) to be constant for all \( z \). The proof of the theorem is complete. \( \square \)

3. Kähler potential of homogeneous Kähler metrics on complex bounded domains (by Hideyuki Ishi)

Let \( \mathcal{D}_r \) be the so-called Siegel upper half plane of rank \( r \), that is, the set of complex symmetric matrices \( z \in \text{Sym}(r, \mathbb{C}) \) of size \( r \times r \) such that the imaginary part \( \Im z \) is positive definite. Note that \( \mathcal{D}_1 = \mathbb{H} \). Let \( H_r \) be the group of real lower triangular matrices with positive diagonal entries, and define
\[
\mathcal{S}_r := \left\{ b(v, T) := \begin{pmatrix} I_r & v \\ \ & T \end{pmatrix} \left| v \in \text{Sym}(r, \mathbb{R}), T \in H_r \right. \right\}
\]
which is a maximal real split solvable Lie subgroup of the real symplectic group \( \text{Sp}(r, \mathbb{R}) \). The solvable group \( \mathcal{S}_r \) acts on \( \mathcal{D}_r \) simply transitively by
\[
b(v, T) \cdot z := v + T z^T, \quad (b(v, T) \in \mathcal{S}_r, z \in \mathcal{D}_r)
\]
For a complex symmetric matrix \( w \in \text{Sym}(r, \mathbb{C}) \) and \( k = 1, \ldots, r \), we denote by \( \Delta_k(w) \) the principal minor det
\[
\begin{pmatrix}
w_{11} & \cdots & w_{k1} \\
\vdots & \ddots & \vdots \\
w_{k1} & \cdots & w_{kk}
\end{pmatrix}
\]
define
\[
\Delta_{\underline{z}}(w) := \Delta_1(w)^{s_1} \prod_{k=2}^r \left( \frac{\Delta_k(w)}{\Delta_{k-1}(w)} \right)^{s_k},
\]
which is called a generalized power function of \( w \) [8, p. 122]. We have the formulas
\[
\begin{align*}
(1) & \quad \Delta_{\underline{z}}(T w^T) = (T_{11}^{2s_1} T_{22}^{2s_2} \cdots T_{rr}^{2s_r}) \Delta_{\underline{z}}(w), \quad (T \in H_r, w \in \text{Sym}(r, \mathbb{C})), \\
(2) & \quad \underline{z} = (\alpha, \ldots, \alpha) \Rightarrow \Delta_{\underline{z}}(w) = (\det w)^{\alpha}, \\
(3) & \quad w = \text{diag}(w_{11}, w_{22}, \ldots, w_{rr}) \Rightarrow \Delta_{\underline{z}}(w) = w_{11}^{2s_1} w_{22}^{2s_2} \cdots w_{rr}^{2s_r}.
\end{align*}
\]
For \( \gamma = (\gamma_1, \ldots, \gamma_r) \in \mathbb{R}_{>0}^r \), define a positive function \( F_{\gamma} : \mathcal{D}_r \rightarrow \mathbb{R}_{>0} \) by \( F_{\gamma}(z) := \Delta_{\underline{z}}(3z)(z \in \mathcal{D}_r) \). Then it is known that \( \log F_{\gamma} \) is plurisubharmonic, which means
that \( \log F \) is a potential function of a Kähler metric \( g \) on \( \Omega \). By (1), we have

\[
\log F(b(v,T)z) = \log F(z) + \log \left(T_{11}^{-2\gamma_1} \ldots T_{rr}^{-2\gamma_r}\right) \quad (z \in \Omega, b(v,T)z),
\]

so that the metric \( g \) is invariant under the action of \( S_r \) on \( \Omega \). We see from Dorfmeister [7] that any \( S_r \)-invariant Kähler metric on \( \Omega \) is of the form \( g \). Furthermore, any homogeneous Kähler metric \( g \) on \( \Omega \) is equivalent to some \( g \), which means that there exists a biholomorphic map \( \varphi: \Omega \to \Omega \) such that \( g = \varphi^* g \). It is also known that the biholomorphic map \( \varphi \) on the Siegel upper half plane \( \Omega \) is birational. Note that \( F(z) \) is a rational function if and only if all \( \gamma_k \) are positive integers. Let us introduce a rational function \( F_k(k = 1, \ldots, r) \) defined by

\[
F_k(z) := \begin{cases} 
\Delta_1(\Im z) & (k = 1) \\
\Delta_k(\Im z)/\Delta_{k-1}(\Im z) & (k = 2, \ldots, r)
\end{cases}
\]

Then we have

\[
\log F(z) = \gamma_1 \log F_1(z) + \gamma_2 \log F_2(z) + \cdots + \gamma_r \log F_r(z) \quad (z \in \Omega).
\]

This argument can be generalized to any homogeneous Siegel domain.

**Theorem 3.1.** Let \( \Omega \) be a homogeneous Siegel domain and \( g \) a homogeneous Kähler metric on \( \Omega \). Then there exist rational functions \( F_1, \ldots, F_r \) and positive numbers \( \gamma_1, \ldots, \gamma_r \) such that \( \sum_{k=1}^r \gamma_k \log F_k(z) \) is a potential function of \( g \).

In particular, when \( g \) is the Bergman metric the coefficients \( \gamma_1, \ldots, \gamma_r \) are positive integers.

A concrete description of the rational functions \( F_k(k = 1, \ldots, r) \) can be found in Gindikin [9]. For the relation between the metric \( g \) and the coefficients \( \gamma_1, \ldots, \gamma_r \), see [5, Proof of Theorem 4].

### 4. Proof of Theorem 3.1

Given a complex manifold \( M \) endowed with a real analytic Kähler metric \( g \), Calabi introduced, in a neighborhood of a point \( p \in M \), a very special Kähler potential \( D_0^g \) for the metric \( g \), which he called diastasis (the reader is referred either to [2] or to [16] for details).

Let \( \psi \) be a real analytic Kähler potential for \( g \) centred at \( p \), by duplicating the variables \( z \) and \( \bar{z} \), the potential \( \psi \) can be complex analytically continued to a function \( \tilde{\psi} \) defined in a neighborhood \( U \times U \) of \( (p, \bar{p}) \in M \times M \), then the diastasis \( D_0^g : U \to \mathbb{R} \) is defined by

\[
D_p(z) = \tilde{\psi}(z, \bar{z}) + \tilde{\psi}(p, \bar{p}) - \tilde{\psi}(z, \bar{p}) - \tilde{\psi}(p, \bar{z}).
\]
Among all the potentials the diastasis is characterized by the fact that in every coordinate system \(\{z_1, \ldots, z_n\}\) centered at \(p\)

\[
D_p^g(z, \bar{z}) = \sum_{|j|, |k| \geq 0} a_{jk} z^j \bar{z}^k,
\]

(7)

with \(a_{j0} = a_{0j} = 0\) for all multi-indices \(j\). Clearly the diastasis \(D_p^g\) is a function of diastasis-type as defined in Section 2.

The next general proposition will be used in the proof of Theorem 1.1.

**Proposition 4.1.** Let \((M, g)\) be a real analytic Kähler manifold and let \(\{z_1, \ldots, z_n\}\) be holomorphic coordinates centered at a point \(p\), in a neighborhood \(U \subset M\) where the diastasis \(D_p^g\) is defined. Assume that

\[
e^{-D_p^g} \in \tilde{F}^{\gamma_1} \cdots \tilde{F}^{\gamma_r},
\]

(8)

for some \(\gamma_1, \ldots, \gamma_r \in \mathbb{R}\). Then any KRS \((g, X)\) on \(M\) is trivial, i.e. \(g\) is KE.

**Proof.** Let us start to write down the KRS equation

\[
\text{Ric}_g = \lambda g + L_X g
\]

(9)

in local complex coordinates \(\{z_1, \ldots, z_n\}\) for \(M\) centred at \(p\) in a neighborhood \(U\) of \(p\) where the diastasis \(D_p^g\) is defined. Since the solitonic vector field \(X\) can be assumed to be the real part of a holomorphic vector field, we can write

\[
X = \sum_{j=1}^n \left( f_j \frac{\partial}{\partial z_j} + \bar{f}_j \frac{\partial}{\partial \bar{z}_j} \right)
\]

for some holomorphic functions \(f_j, j = 1, \ldots, n,\) on \(U\).

Thus, by the definition of Lie derivative, after a straightforward computation we can write on \(U\)

\[
L_X \omega = i \frac{\partial}{\partial \bar{z}} f_X.
\]

(10)

where \(\omega\) is the Kähler form associated to \(g\) and

\[
f_X = \sum_{j=1}^n f_j \frac{\partial D_p^g}{\partial z_j} + \bar{f}_j \frac{\partial D_p^g}{\partial \bar{z}_j}.
\]

(11)

By the \(\partial \bar{\partial}\)-Lemma one deduces (cfr. [14] for details) that the local expression of (9) is given by

\[
\det \left[ \frac{\partial^2 D_p^g}{\partial z_\alpha \partial \bar{z}_\beta} \right] = e^{-\frac{1}{2} D_p^g - \frac{i}{2} f_X} + h + \bar{h},
\]

(12)

for a holomorphic function \(h\) on \(U\). Notice that the function \(\det \left[ \frac{\partial^2 D_p^g}{\partial z_\alpha \partial \bar{z}_\beta} \right]\) is finitely generated by rational functions composed with holomorphic or anti-holomorphic functions around \(p\). Moreover it is real valued, since the matrix \(\frac{\partial^2 D_p^g}{\partial z_\alpha \partial \bar{z}_\beta}\) is Hermitian.
We conclude that \( \det \left[ \frac{\partial^2 D_p}{\partial z_a \partial \bar{z}_\beta} \right] \in \mathcal{F} \). Furthermore one can check (cf. (14)) that the function \( \det \left[ \frac{\partial^2 D_p}{\partial z_a \partial \bar{z}_\beta} \right] \) is of diastasis-type, thus \( \det \left[ \frac{\partial^2 D_p}{\partial z_a \partial \bar{z}_\beta} \right] \in \tilde{\mathcal{F}} \).

From (14), one sees that

\[
\zeta := \frac{-f_X}{2} + h + \bar{h} \in \mathcal{F}.
\]

By (8), (12) and the previous observation one deduces that

\[
e^{\zeta} = \left[ e^{\frac{1}{2} D_p} \right]^\lambda \det \left[ \frac{\partial^2 D_p}{\partial z_a \partial \bar{z}_\beta} \right] \in \tilde{\mathcal{F}}_{\gamma_1} \ldots \tilde{\mathcal{F}}_{\gamma_r} \tilde{\mathcal{F}},
\]

with \( \mu_j = \frac{1}{2} \lambda \gamma_j, \ j = 1, \ldots, r \). On the one hand (14) shows that \( e^\zeta \) and hence \( \zeta \) is of diastasis-type and so, \( \zeta \in \tilde{\mathcal{F}} \). On the other hand (13) together with Theorem 2.1 force \( \zeta \) to be a constant and so \( f_X \) is the real part of a holomorphic function. Therefore, by (9) and (10) the metric \( g \) is KE.

\[\square\]

**Proof of Theorem 1.1.** Let \( \varphi : (M, g) \to (\Omega, g_\Omega) \) be a holomorphic isometry. Let \( \{z_1, \ldots, z_n\} \) be a coordinate system for \( M \) centered at \( p \), by the hereditary property of Calabi’s diastasis function (see [2] for details), we have that

\[
D^\varphi_p(z) = D^\varphi_{\varphi(p)}(\varphi(z)).
\]

Consider the realization of \( \Omega \) as a homogeneous Siegel domain as in the previous section. From the definition of Calabi’s diastasis function (6) and Theorem 3.1, we deduce that the the diastasis function for the homogeneous metric \( g_\Omega \) at given point \( v \in \Omega \) is given by:

\[
D^\varphi_p(u) = \sum_{k=1}^r \gamma_k \log \left( \frac{F_k(u, \varphi(p)) F_k(v, \varphi(p))}{F_k(u, \varphi(p)) F_k(v, \varphi(p))} \right),
\]

where \( \gamma_k \) are positive real numbers and \( F_k(u, \varphi(p)) \) are rational holomorphic functions in \( u \) and \( \varphi(p) \), \( k = 1, \ldots, r \). Therefore

\[
D^\varphi_p(\varphi(z)) = \sum_{k=1}^r \gamma_k \log \left( \frac{F_k(\varphi(z), \varphi(p)) F_k(\varphi(0), \varphi(p))}{F_k(\varphi(z), \varphi(p)) F_k(\varphi(0), \varphi(p))} \right)
\]

From (14) and (10), we get

\[
e^{D^\varphi_p(z)} = \prod_{k=1}^r \left( \frac{F_k(\varphi(z), \varphi(p)) F_k(\varphi(0), \varphi(p))}{F_k(\varphi(z), \varphi(p)) F_k(\varphi(0), \varphi(p))} \right)^\gamma_k.
\]

and hence

\[
e^{D^\varphi_p(z)} \in \tilde{\mathcal{F}}_{\gamma_1} \ldots \tilde{\mathcal{F}}_{\gamma_r}.
\]

On the other hand, from (7), we can see that each term in (10) is of diastasis-type. That is

\[
e^{D^\varphi_p(z)} \in \tilde{\mathcal{F}}_{\gamma_1} \ldots \tilde{\mathcal{F}}_{\gamma_r}.
\]
namely (8). Thus, by Proposition 4.1 we deduce that $g$ is KE, completing the proof of (i) of Theorem 1.1.

Proof of (ii). Let $(\Omega, g_\Omega)$ be a homogeneous bounded domain and let $U \subset \mathbb{C}$ be a neighbourhood of the origin and let $g_0$ the indefinite flat metric on $\mathbb{C}^n$ whose Kähler potential is given by $|z|^2 := |z_1|^2 + \cdots + |z_l|^2 - |z_{l+1}|^2 - \cdots - |z_n|^2$, $l \in \{0, 1, \ldots, n\}$. Assume by contradiction that there exists a holomorphic immersion $\varphi : U \rightarrow \Omega$ and $\eta : U \rightarrow \mathbb{C}^n$, $\eta(0) = 0$, such that

$$\varphi^*g_\Omega = \eta^*g_0.$$ 

By (14) and (16) this is equivalent to the following equation

$$\sum_{k=1}^r \gamma_k \log \left( \frac{F_k(\varphi(z), \varphi(z)) F_k(\varphi(0), \overline{\varphi(0)})}{F_k(\varphi(z), \varphi(0)) F_k(\varphi(0), \overline{\varphi(z)})} \right) = |\eta(z)|^2.$$ 

So that

$$e^{\eta(z)} = \prod_{k=1}^r \left( \frac{F_k(\varphi(z), \overline{\varphi(z)} F_k(\varphi(0), \overline{\varphi(0)})}{F_k(\varphi(z), \varphi(0)) F_k(\varphi(0), \overline{\varphi(z)})} \right)^{\gamma_k}.$$ 

By applying Proposition 4.1 and Theorem 2.1 we see that $\eta$ is forced to be constant, which contradicts the fact that $\eta$ is an immersion (unless $M$ is zero-dimensional). 

\[\square\]

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