Existence and conditional energetic stability of solitary gravity–capillary water waves with constant vorticity

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We present an existence and stability theory for gravity–capillary solitary waves with constant vorticity on the surface of a body of water of finite depth. Exploiting a rotational version of the classical variational principle, we prove the existence of a minimizer of the wave energy \( H \) subject to the constraint \( I = 2\mu \), where \( I \) is the wave momentum and \( 0 < \mu \ll 1 \). Since \( H \) and \( I \) are both conserved quantities, a standard argument asserts the stability of the set \( D_\mu \) of minimizers: solutions starting near \( D_\mu \) remain close to \( D_\mu \) in a suitably defined energy space over their interval of existence. In the applied mathematics literature solitary water waves of the present kind are described by solutions of a Korteweg–de Vries equation (for strong surface tension) or a nonlinear Schrödinger equation (for weak surface tension). We show that the waves detected by our variational method converge (after an appropriate rescaling) to solutions of the appropriate model equation as \( \mu \downarrow 0 \).

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1. Introduction

1.1. Variational formulation of the hydrodynamic problem

1.1.1. The water-wave problem

In this paper we consider a two-dimensional perfect fluid bounded below by a flat rigid bottom \( \{ y = 0 \} \) and above by a free surface \( \{ y = d + \eta(x,t) \} \). The fluid has unit density and flows under the influence of gravity and surface tension with constant vorticity \( \omega \) so that the velocity field \((u(x, y, t), v(x, y, t))\) in the fluid domain \( \Sigma_\eta = \{ 0 < y < d + \eta(x, t) \} \) satisfies \( v_x - u_y = \omega \). We study waves that are perturbations of underlying shear flows given by \( \eta = 0 \) and \((u, v) = (\omega(d - y), 0)\)
(which may be a good description of tidal currents: see Constantin [12, ch. 2.3.2]) and are evanescent as \( x \to \pm \infty \). In terms of a generalized velocity potential \( \phi \) such that \((u, v) = (\phi_x + \omega(d-y), \phi_y)\) and stream function \( \psi \) such that \((u, v) = (\psi_y, -\psi_x)\), the governing equations are

\[
\begin{align*}
\Delta \phi &= 0, & 0 < y < d + \eta, \\
\phi_y &= 0, & y = 0, \\
\eta_t &= \phi_y - \eta_x \phi_x + \omega \eta_x, & y = d + \eta, \\
\phi_t &= -\frac{1}{2} |\nabla \psi|^2 - \omega \psi - g \eta + \beta \left[ \frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right], & y = d + \eta,
\end{align*}
\]

with \( \eta(x, t), \phi(x, y, t), \psi(x, y, t) + \frac{1}{2} \omega(d-y)^2 \to 0 \) as \( x \to \pm \infty \), where \( g \) and \( \beta \) are the acceleration due to gravity and the (positive) coefficient of surface tension, respectively (see Constantin et al. [14]).

At this point it is convenient to introduce dimensionless variables

\[
(x', y') = \frac{1}{d}(x, y), \quad t' = \left( \frac{g}{d} \right)^{1/2} t,
\]

\[
\eta'(x', t'), \quad \phi'(x', t') = \frac{1}{(gd^3)^{1/2}} \phi(x, t), \quad \psi'(x', t') = \frac{1}{(gd^3)^{1/2}} \psi(x, t)
\]

and parameters \( \omega' = \omega(d/g)^{1/2}, \beta' = \beta/gd^2 \); one obtains the equations

\[
\begin{align*}
\Delta \phi &= 0, & 0 < y < 1 + \eta, \\
\phi_y &= 0, & y = 0, \\
\eta_t &= \phi_y - \eta_x \phi_x + \omega \eta_x, & y = 1 + \eta, \\
\phi_t &= -\frac{1}{2} |\nabla \psi|^2 - \omega \psi - \eta + \beta \left[ \frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right], & y = 1 + \eta,
\end{align*}
\]

in which the primes have been dropped for notational simplicity. In particular, we seek solitary-wave solutions of (1.1)–(1.4), that is, waves of permanent form that propagate from right to left with constant (dimensionless) speed \( \nu \), so that \( \eta(x, t) = \eta(x + \nu t) \) (and of course \( \eta(x + \nu t) \to 0 \) as \( x + \nu t \to \pm \infty \)).

1.1.2. Formulation as a Hamiltonian system

We proceed by reducing the hydrodynamic problem to a pair of non-local coupled evolutionary equations for the variables \( \eta \) and \( \xi = \phi|_{y=1+\eta} \). For fixed \( \eta \) and \( \xi \), let \( \phi \) denote the unique solution to the boundary-value problem

\[
\begin{align*}
\Delta \phi &= 0, & 0 < y < 1 + \eta, \\
\phi &= \xi, & y = 1 + \eta, \\
\phi_y &= 0, & y = 0,
\end{align*}
\]

and denote the harmonic conjugate of \( \phi \) by \( \tilde{\psi} \). We define the Hilbert transform \( H(\eta) \) and Dirichlet–Neumann operator \( G(\eta) \) for this boundary-value problem by

\[
H(\eta) \xi = \tilde{\psi}|_{y=1+\eta}, \quad G(\eta) \xi = (\phi_y - \eta_x \phi_x)|_{y=1+\eta},
\]
so that $G(\eta) = -\partial_x H(\eta)$ and note that the boundary conditions (1.3), (1.4) can be written as

$$
\eta_t = G(\eta)\xi + \omega \eta \eta_x,
$$

$$
\xi_t = -\frac{1}{2(1 + \eta^2_x)}(\xi^2_x - (G(\eta)\xi)^2 - 2\eta_x \xi_x G(\eta)\xi_x)
+ \omega \eta \xi_x - \omega H(\eta)\xi - \eta + \beta \left[ \frac{\eta_x}{\sqrt{1 + \eta^2_x}} \right].
$$

Wahlén [25] observed that the above equations can be formulated as the Hamiltonian system

$$
\begin{pmatrix}
\eta_t \\
\xi_t
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-1 & \omega \partial_x^{-1}
\end{pmatrix} \begin{pmatrix}
\delta_\eta H \\
\delta_\xi H
\end{pmatrix},
$$

in which

$$
H(\eta, \xi) = \int_{-\infty}^{\infty} \left( \frac{1}{2} \xi G(\eta)\xi + \omega \xi \eta \eta_x + \frac{1}{6} \omega^2 \eta^3 + \frac{1}{2} \eta^2 + \beta(\sqrt{1 + \eta^2_x} - 1) \right) dx
$$

(note that the well-known formulation of the water-wave problem by Zakharov [26] is recovered in the irrotational case $\omega = 0$). This Hamiltonian system has the conserved quantities $H(\eta, \xi)$ (total energy) and

$$
I(\eta, \xi) = \int_{-\infty}^{\infty} (\xi \eta_x + \frac{1}{2} \omega \eta^2) dx
$$

(total horizontal momentum), which satisfies the equation

$$
\begin{pmatrix}
\eta_x \\
\xi_x
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-1 & \omega \partial_x^{-1}
\end{pmatrix} \begin{pmatrix}
\delta_\eta I \\
\delta_\xi I
\end{pmatrix};
$$

these quantities are associated with its independence of $t$ and $x$, respectively. According to (1.5) and (1.8), a solution of the form $\eta(x, t) = \eta(x + \nu t)$, $\xi(x, t) = \xi(x + \nu t)$ is characterized as a critical point of the total energy subject to the constraint of fixed momentum (cf. Benjamin [4]). It is therefore a critical point of the functional $H - \nu I$, where the speed of the wave is given by the Lagrange multiplier $\nu$. This functional depends on the single independent variable $x + \nu t$, which we now abbreviate to $x$.

A similar variational principle for waves of permanent form with a general distribution of vorticity has been used by Groves and Wahlén [16] in an existence theory for solitary waves. Groves and Wahlén interpreted their variational functional as an action functional and derived a formulation of the hydrodynamic problem as an infinite-dimensional spatial Hamiltonian system; a rich solution set is found using a centre-manifold reduction technique to convert it into a Hamiltonian system with a finite number of degrees of freedom.

In this paper we present a direct existence theory for minimizers of $H$ subject to the constraint $I = 2\mu$ for $0 < \mu < \mu_0$, where $\mu_0$ is a fixed positive constant chosen small enough for the validity of our calculations. We seek constrained minimizers in a two-step approach.
(1) Fix \( \eta \neq 0 \) and minimize \( \mathcal{H}(\eta, \cdot) \) over \( T_\mu = \{ \xi : \mathcal{I}(\eta, \xi) = 2\mu \} \). This problem (of minimizing a quadratic functional over a linear manifold) admits a unique global minimizer \( \xi_\eta \).

(2) Minimize \( J_\mu(\eta) := \mathcal{H}(\eta, \xi_\eta) \) over \( \eta \in U \setminus \{0\} \). Here \( U \) is a fixed ball centred upon the origin in a suitable function space. Because \( \xi_\eta \) minimizes \( \mathcal{H}(\eta, \cdot) \) over \( T_\mu \), there exists a Lagrange multiplier \( \nu_\eta \) such that

\[
G(\eta)\xi_\eta + \omega \eta \eta' = \nu_\eta \eta',
\]

and straightforward calculations show that

\[
\nu_\eta = \left( \frac{1}{2} \int_{-\infty}^{\infty} \eta' G(\eta)^{-1} \eta' dx \right)^{-1} \left( \mu - \frac{\omega}{4} \int_{-\infty}^{\infty} \eta^2 dx + \frac{\omega}{4} \int_{-\infty}^{\infty} (\eta^2)' G(\eta)^{-1} \eta' dx \right)
\]

so that

\[
J_\mu(\eta) = \mathcal{K}(\eta) + \frac{(\mu + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)}, \quad (1.9)
\]

where

\[
\mathcal{G}(\eta) = \frac{\omega}{4} \int_{-\infty}^{\infty} \eta^2 K(\eta) \eta dx - \frac{\omega}{4} \int_{-\infty}^{\infty} \eta^2 dx, \quad (1.10)
\]

\[
\mathcal{K}(\eta) = \int_{-\infty}^{\infty} \left( \frac{\eta^2}{2} + \beta \sqrt{1 + \eta^2} - 1 \right) dx - \frac{\omega^2}{2} \int_{-\infty}^{\infty} \frac{\eta^2}{2} K(\eta) \frac{\eta^2}{2} dx + \frac{\omega^2}{6} \int_{-\infty}^{\infty} \eta^3 dx, \quad (1.11)
\]

\[
\mathcal{L}(\eta) = \frac{1}{2} \int_{-\infty}^{\infty} \eta K(\eta) \eta dx \quad (1.12)
\]

and \( K(\eta) = -\partial_x G(\eta)^{-1} \partial_x \). This computation also shows that the dimensionless speed of a solitary wave corresponding to a constrained minimizer \( \eta \) of \( \mathcal{H} \) is

\[
\nu = \frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}.
\]

This two-step approach to the constrained minimization problem was introduced in a corresponding theory for irrotational solitary waves by Buffoni [5] who used a conformal mapping due to Babenko [1, 2] to transform \( J_\mu \) into another functional \( \tilde{J}_\mu \), depending only upon \( H(0) \), and hence simplified the necessary variational analysis. Buffoni established the existence of a (non-zero) minimizer of \( \tilde{J}_\mu \) for strong surface tension (see Buffoni [5]) and obtained partial results in this direction for weak surface tension (see Buffoni [6, 7]). A method for completing his results for weak surface tension was sketched in a short note by Groves and Wahlén [17]; in the present paper we give complete details, including non-zero vorticity in our treatment and working directly with the original physical variables. Although versions of the Babenko transformation for non-zero constant vorticity have been published (see Constantin and Varvaruca [13] and Martin [23]), finding minimizers of \( J_\mu \) over \( U \setminus \{0\} \) has the advantage of immediately yielding precise information on solutions to the original water-wave equations (1.1)–(1.4).
1.1.3. Functional-analytic framework

An appropriate functional-analytic framework for the above variational problem is introduced in §2. We work with the function spaces

\[ H^r(\mathbb{R}) = \left( \mathcal{S}(\mathbb{R}) \setminus \| \cdot \|_r \right), \quad \| \eta \|_r^2 := \int_{-\infty}^{\infty} (1 + k^2)^r |\hat{\eta}|^2 \, dk, \]

for \( r \in \mathbb{R} \) (the standard Sobolev spaces) and

\[ H^{1/2}_s(\mathbb{R}) = \left( \mathcal{S}(\mathbb{R}) \setminus \| \cdot \|_{H^{1/2}_s(\mathbb{R})} \right), \quad \| \eta \|_{H^{1/2}_s(\mathbb{R})} := \int_{-\infty}^{\infty} (1 + k^2)^{-1/2} k^2 |\hat{\eta}|^2 \, dk, \]

\[ H^{-1/2}_s(\mathbb{R}) = \left( \mathcal{S}(\mathbb{R}) \setminus \| \cdot \|_{H^{-1/2}_s(\mathbb{R})} \right), \quad \| \eta \|_{H^{-1/2}_s(\mathbb{R})} := \int_{-\infty}^{\infty} (1 + k^2)^{1/2} k^{-2} |\hat{\eta}|^2 \, dk; \]

here \( (\mathcal{S}(\mathbb{R}), \| \cdot \|) \) denotes the completion of the inner product space constructed by equipping the Schwartz class \( \mathcal{S}(\mathbb{R}) \) (or the subclass \( \mathcal{S}(\mathbb{R}) \) of Schwartz-class functions with zero mean) with the norm \( \| \cdot \| \), and \( \hat{\eta} = \mathcal{F}[\eta] \) is the Fourier transform of \( \eta \).

The mathematical analysis of \( G(\eta) \) and \( K(\eta) \) is complicated by the fact that they are defined in terms of boundary-value problems in the variable domain \( \Sigma \). Lannes [20, ch. 2 and 3] has presented a comprehensive theory for handling such boundary-value problems by transforming them into serviceable nonlinear elliptic problems in the fixed domain \( \Sigma_0 \), and here we adapt Lannes’s methods to our specific requirements. Our main results are stated in the following theorem, according to which equations (1.10)–(1.12) define analytic functionals \( G, K, L : W^{s+3/2} \to \mathbb{R} \) for \( s > 0 \). In accordance with this theorem we take \( U = B_M(0) \subseteq H^2(\mathbb{R}) \), where \( M > 0 \) is chosen small enough so that \( B_M(0) \subseteq H^2(\mathbb{R}) \) lies in \( W^{s+3/2} \) and for the validity of our calculations.

**Theorem 1.1.** Choose \( h_0 \in (0, 1) \) and define \( W = \{ \eta \in W^{1, \infty}(\mathbb{R}) : 1 + \inf \eta > h_0 \} \) and \( W^r = H^r \cap W \) for \( r \geq 0 \).

(i) The Dirichlet–Neumann operator \( G(\eta) \) is an isomorphism

\[ H^{1/2}_s(\mathbb{R}) \to H^{-1/2}_s(\mathbb{R}) \]

for each \( \eta \in W \).

(ii) The Dirichlet–Neumann operator \( G : W \to \mathcal{L}(H^{1/2}_s(\mathbb{R}), H^{-1/2}_s(\mathbb{R})) \) and Neumann–Dirichlet operator \( G^{-1} : W \to \mathcal{L}(H^{-1/2}_s(\mathbb{R}), H^{1/2}_s(\mathbb{R})) \) are analytic.

(iii) The operator \( K : W^{s+3/2} \to \mathcal{L}(H^{s+3/2}(\mathbb{R}), H^{s+1/2}(\mathbb{R})) \) is analytic for each \( s > 0 \).

1.2. Heuristics

The existence of small-amplitude solitary waves is predicted by studying the dispersion relation for the linearized version of (1.1)–(1.4). Linear waves of the form \( \eta(x, t) = \cos k(x + vt) \) exist whenever

\[ 1 + \beta k^2 - \omega v - v^2 f(k) = 0, \quad f(k) = |k| \coth |k|, \]
that is, whenever
\[ \nu = -\frac{\omega}{2f(k)} + \frac{1}{2} \left( \frac{\omega^2}{f(k)^2} + \frac{4(\beta k^2)}{f(k)} \right)^{1/2}. \]

The function \( k \mapsto \nu(k), \ k \geq 0, \) has a unique global minimum \( \nu_0 = \nu(k_0) \), and one finds that \( k_0 > 0 \) for \( \beta < \beta_c \) and \( k_0 = 0 \) (with \( \nu_0 = \nu(0) = \frac{1}{2}(-\omega + \sqrt{\omega^2 + 4}) \)) for \( \beta > \beta_c \), where
\[ \beta_c = \frac{1}{6} \left( \omega^2 + 2 - \omega \sqrt{\omega^2 + 4} \right) \]
(see figure 1). For later use let us also note that
\[ g(k) := 1 + \beta k^2 - \omega \nu_0 - \nu_0^2 f(k) \geq 0, \ k \in \mathbb{R}, \]
with equality precisely when \( k = \pm k_0 \).

Bifurcations of nonlinear solitary waves are expected whenever the linear group and phase speeds are equal, so that \( \nu'(k) = 0 \) (see Dias and Kharif [15, §3]). We therefore expect the existence of small-amplitude solitary waves with speed near \( \nu_0 \); the waves bifurcate from laminar flow when \( \beta > \beta_c \) and from a linear periodic wave train with frequency \( k_0 \nu_0(k_0) \) when \( \beta < \beta_c \). Model equations for both types of solution have been derived by Johnson [19, §§4 and 5].

**Case 1 (\( \beta > \beta_c \)).** The appropriate model equation is the Korteweg–de Vries equation
\[ -2u_T - \left( \beta - \frac{\nu_0^2}{3} \right) u_{XX} + (\omega^2 + 3)uu_X = 0, \quad (1.13) \]
in which
\[ \eta = \mu^{2/3} u(X, T) + O(\mu^{4/3}), \quad X = \mu^{1/3}(x + \nu_0 t), \quad T = 2(\omega^2 + 4)^{-1/2}\mu^{2/3} t. \]

At this level of approximation, a solution to (1.13) of the form \( u(X, T) = \phi(X + \nu_{KdV} T) \) with \( \phi(X) \to 0 \) as \( X \to \pm \infty \) corresponds to a solitary water wave with speed
\[ \nu = \nu_0 + 2(\omega^2 + 4)^{-1/2}\mu^{2/3} \nu_{KdV} = -\frac{1}{2} \omega + \frac{1}{2}(\omega^2 + 4)^{1/2} + 2(\omega^2 + 4)^{-1/2}\mu^{2/3} \nu_{KdV}. \]

The following lemma gives a variational description of the set of such solutions; the corresponding solitary waves are sketched in figure 2.
Lemma 1.2.

(i) The set of solutions to the ordinary differential equation

\[-(\beta - \frac{1}{4} \nu_0^2) \phi'' - 2 \nu_{KdV} \phi + \frac{3}{2} (\frac{1}{4} \omega^2 + 1) \phi = 0\]

satisfying \(\phi(X) \to 0\) as \(X \to \infty\) is \(D_{KdV} = \{ \phi_{KdV}(\cdot + y) : y \in \mathbb{R} \}\), where

\[\nu_{KdV} = -\frac{2(\frac{3}{16})^{2/3} (\frac{1}{4} \omega^2 + 1)^{4/3}}{(\beta - \frac{1}{4} \nu_0^2)^{1/3} (\omega^2 + 4)^{1/3}},\]

\[\phi_{KdV}(x) = -\frac{\sqrt{3}(\frac{3}{16})^{1/6} (\frac{1}{4} \omega^2 + 1)^{1/3}}{(\beta - \frac{1}{4} \nu_0^2)^{1/3} (\omega^2 + 4)^{1/3}} \text{sech}^2 \left( \frac{(\frac{3}{16})^{1/3} (\frac{1}{4} \omega^2 + 1)^{2/3}}{(\beta - \frac{1}{4} \nu_0^2)^{2/3} (\omega^2 + 4)^{1/6}} \right).\]

These functions are precisely the minimizers of the functional \(E_{KdV} : H^1(\mathbb{R}) \to \mathbb{R}\) given by

\[E_{KdV}(\phi) = \frac{1}{2} \int_{-\infty}^{\infty} ((\beta - \frac{1}{4} \nu_0^2)(\phi')^2 + (\frac{1}{4} \omega^2 + 1) \phi^3) \, dx\]

over the set \(N_{KdV} = \{ \phi \in H^1(\mathbb{R}) : \| \phi \|_3^2 = 2 \alpha_{KdV} \}\); the constant \(2 \nu_{KdV}\) is the Lagrange multiplier in this constrained variational principle and

\[c_{KdV} := \inf \{ E_{KdV}(\phi) : \phi \in N_{KdV} \} = -\frac{9(\frac{3}{16})^{1/3} (\frac{1}{4} \omega^2 + 1)^{4/3}}{(\beta - \frac{1}{4} \nu_0^2)^{1/3} (\omega^2 + 4)^{5/6}}.\]

Here the numerical value \(\alpha_{KdV} = 2 (\omega^2 + 4)^{-1/2}\) is chosen for compatibility with an estimate (see proposition 5.4) in the following water-wave theory.

(ii) Suppose that \(\{ \phi_m \} \subset N_{KdV}\) is a minimizing sequence for \(E_{KdV}\). There exists a sequence \(\{ x_m \}\) of real numbers with the property that a subsequence of \(\{ \phi_m(\cdot + x_m) \}\) converges in \(H^1(\mathbb{R})\) to an element of \(D_{KdV}\).

Case 2 \((\beta < \beta_c)\). The appropriate model equation is the cubic nonlinear Schrödinger equation

\[2i \eta_t - \frac{1}{2} g''(k_0) A_{XX} + \frac{3}{2} (\frac{1}{2} A_3 + A_4) |A|^2 A = 0,\]

in which

\[\eta = \frac{1}{2} \mu (A(X, T)e^{i k_0 (x + \nu_0 t)} + c.c.) + O(\mu^2),\]

\[X = \mu (x + \nu_0 t), \quad T = 4 k_0 (\omega + 2 \nu_0 f(k_0))^{-1} \mu^2 t\]
and $A_3, A_4$ are functions of $\beta$ and $\omega$ that are given in corollary 4.25 and proposition 4.28; the abbreviation ‘c.c.’ denotes the complex conjugate of the preceding quantity. (It is demonstrated in Appendix B that $A_3 + 2A_4$ is negative.) At this level of approximation, a solution to (1.14) of the form $A(X,T) = e^{i\nu NLS}T \phi(X)$ with $\phi(X) \to 0$ as $X \to \pm \infty$ corresponds to a solitary water wave with speed

$$\nu = \nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1}\nu^2 NLS.$$ 

The following lemma gives a variational description of the set of such solutions (see Cazenave [10, §8]); the corresponding solitary waves are sketched in figure 3.

**Lemma 1.3.**

(i) The set of complex-valued solutions to the ordinary differential equation

$$-\frac{1}{4} g''(k_0) \phi'' - 2\nu_{NLS} \phi + \frac{3}{8}(\frac{1}{2} A_3 + A_4)|\phi|^2 \phi = 0$$

satisfying $\phi(X) \to 0$ as $X \to \infty$ is

$$D_{NLS} = \{e^{i\omega} \phi_{NLS}(-y) : \omega \in [0, 2\pi), y \in \mathbb{R}\},$$

where

$$\nu_{NLS} = -\frac{9\alpha_{NLS}^2}{8g''(k_0)} \left( \frac{A_3}{2} + A_4 \right)^2,$$

$$\phi_{NLS}(x) = \alpha_{NLS} \left( -\frac{3}{g''(k_0)} \left( \frac{A_3}{2} + A_4 \right) \right)^{1/2} \text{sech} \left( \frac{3\alpha_{NLS}}{g''(k_0)} \left( \frac{A_3}{2} + A_4 \right) x \right).$$

These functions are precisely the minimizers of the functional $E_{NLS} : H^1(\mathbb{R}) \to \mathbb{R}$ given by

$$E_{NLS}(\phi) = \int_{-\infty}^{\infty} \left( \frac{1}{8} g''(k_0) |\phi'|^2 + \frac{3}{8}(\frac{1}{2} A_3 + A_4)|\phi|^4 \right) dx$$

over the set $N_{NLS} = \{\phi \in H^1(\mathbb{R}) : ||\phi||_0^2 = 2\alpha_{NLS}\}$; the constant $2\nu_{NLS}$ is the Lagrange multiplier in this constrained variational principle and

$$c_{NLS} := \inf \{E_{NLS}(\phi) : \phi \in N_{NLS} \} = -\frac{3\alpha_{NLS}^2}{4g''(k_0)} \left( \frac{A_3}{2} + A_4 \right)^2.$$

Here the numerical value $\alpha_{NLS} = \frac{1}{2}(\frac{1}{4} \nu_0 f(k_0) + \frac{1}{8}\omega)^{-1}$ is chosen for compatibility with an estimate (see proposition 5.10) in the following water-wave theory.

(ii) Suppose that $\{\phi_n\} \subset N_{NLS}$ is a minimizing sequence for $E_{NLS}$. There exists a sequence $\{x_m\}$ of real numbers with the property that a subsequence of $\{\phi_m(x + x_m)\}$ converges in $H^1(\mathbb{R})$ to an element of $D_{NLS}$. 


1.3. The main results

In this paper we establish the existence of minimizers of the functional $J_\mu$ over $U \setminus \{0\}$ and confirm that the corresponding solitary water waves are approximated by suitable scalings of the functions $\phi_{KdV}$ (for $\beta > \beta_c$) and $\phi_{NLS}$ (for $\beta < \beta_c$). The following theorem states these results more precisely.

**Theorem 1.4.**

(i) The set $B_\mu$ of minimizers of $J_\mu$ over $U \setminus \{0\}$ is non-empty.

(ii) Suppose that $\{\eta_m\}$ is a minimizing sequence for $J_\mu$ on $U \setminus \{0\}$ that satisfies

$$\sup_{m \in \mathbb{N}} \|\eta_m\|_2 < M.$$  

There exists a sequence $\{x_m\} \subset \mathbb{R}$ with the property that a subsequence of $\{\eta_m(x_m + \cdot)\}$ converges in $H^r(\mathbb{R})$, $r \in [0, 2)$, to a function $\eta \in B_\mu$.

(iii) Suppose that $\beta > \beta_c$. The set $B_\mu$ of minimizers of $J_\mu$ over $U \setminus \{0\}$ satisfies

$$\sup_{\eta \in B_\mu} \inf_{x \in \mathbb{R}} \|\phi_\eta - \phi_{KdV}(\cdot + x)\|_1 \to 0$$

as $\mu \downarrow 0$, where we write

$$\eta_1(x) = \mu^{2/3} \phi_{\eta}(\mu^{1/3} x)$$

and $\eta_1$ is obtained from $\eta$ by multiplying its Fourier transform by the characteristic function of the interval $[-\delta_0, \delta_0]$ with $\delta_0 > 0$. Furthermore, the speed $\nu_\mu$ of the corresponding solitary water waves satisfies

$$\nu_\mu = \nu_0 + 2(\omega^2 + 4)^{-1/2} \nu_{KdV} \mu^{2/3} + o(\mu^{2/3})$$

uniformly over $\eta \in B_\mu$.

(iv) Suppose that $\beta < \beta_c$. The set $B_\mu$ of minimizers of $J_\mu$ over $U \setminus \{0\}$ satisfies

$$\sup_{\eta \in B_\mu} \inf_{\omega \in [0, 2\pi], x \in \mathbb{R}} \|\phi_\eta - e^{i\omega} \phi_{NLS}(\cdot + x)\|_1 \to 0$$



Figure 3. Nonlinear Schrödinger theory predicts the existence of small-amplitude envelope solitary waves for weak surface tension.
as \( \mu \downarrow 0 \), where we write
\[
\eta_i^+(x) = \frac{1}{2} \mu \phi_n(\mu x) e^{ik_0x}
\]
and \( \eta_i^- \) is obtained from \( \eta \) by multiplying its Fourier transform by the characteristic function of the interval \([k_0 - \delta_0, k_0 + \delta_0]\) with \( \delta_0 \in (0, k_0/3) \). Furthermore, the speed \( \nu_\mu \) of the corresponding solitary water waves satisfies
\[
\nu_\mu = \nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1} \nu_{NLS} \mu^2 + o(\mu^2)
\]
uniformly over \( \eta \in B_\mu \).

The first part of theorem 1.4 is proved by reducing it to a special case of the second. We proceed by introducing the coercive penalized functional \( J_{\rho, \mu} : H^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\} \) defined by
\[
J_{\rho, \mu}(\eta) = \begin{cases} \mathcal{K}(\eta) + \frac{(\mu + \mathcal{G}(\eta))^2}{G(\eta)} + \rho(\|\eta\|_2^2), & \eta \in U \setminus \{0\}, \\ \infty, & \eta \notin U \setminus \{0\}, \end{cases}
\]
where \( \rho : [0, M^2] \to \mathbb{R} \) is a smooth increasing ‘penalization’ function that explodes to infinity as \( t \uparrow M^2 \) and vanishes for \( 0 \leq t \leq M^2 \); the number \( M \) is chosen very close to \( M \). Minimizing sequences \( \{\eta_m\} \) for \( J_{\rho, \mu} \), which clearly satisfy \( \sup_{m \in \mathbb{N}} \|\eta_m\|_2 < M \), are studied in detail in §3 with the help of the concentration-compactness principle (see Lions [21, 22]). The main difficulty here lies in discussing the consequences of ‘dichotomy’.

On the one hand the functionals \( \mathcal{G}, \mathcal{K} \) and \( \mathcal{L} \) are non-local and therefore do not act linearly when applied to the sum of two functions with disjoint supports. They are, however, ‘pseudo-local’ in the sense that
\[
\begin{align*}
\left\{ \mathcal{G} \right\}_{\mathcal{K}} (\eta^{(1)}_m + \eta^{(2)}_m) - \left\{ \mathcal{G} \right\}_{\mathcal{K}} (\eta^{(1)}_m) - \left\{ \mathcal{G} \right\}_{\mathcal{L}} (\eta^{(2)}_m) & \to 0
\end{align*}
\]
as \( m \to \infty \), where \( \{\eta^{(1)}_m\}, \{\eta^{(2)}_m\} \) have the properties that
\[
\operatorname{supp} \eta^{(1)}_m \subset [-R_m, R_m], \quad \operatorname{supp} \eta^{(2)}_m \subset \mathbb{R} \setminus (-S_m, S_m)
\]
for sequences \( \{R_m\}, \{S_m\} \) of positive real numbers with \( R_m, S_m \to \infty, R_m/S_m \to 0 \) as \( m \to \infty \) (see lemma 3.9(iii)). This result is established in § 2.2.2 by a new method that involves studying the weak formulation of the boundary-value problems defining the terms in the power-series expansion of \( K \) about \( \eta_0 \in W^{s+3/2} \). On the other hand, no \textit{a priori} estimate is available to rule out ‘dichotomy’ at this stage; proceeding iteratively, we find that minimizing sequences can theoretically have profiles with infinitely many ‘bumps’. In particular, we show that \( \{\eta_m\} \) asymptotically lies in the region unaffected by the penalization and construct a special minimizing sequence \( \{\tilde{\eta}_m\} \) for \( J_{\rho, \mu} \) that lies in a neighbourhood of the origin with radius \( O(\mu^{1/2}) \) in \( H^2(\mathbb{R}) \) and satisfies \( \|J'_\mu(\tilde{\eta}_m)\|_0 \to 0 \) as \( \mu \to \infty \). The fact that the construction is independent of the choice of \( M \) allows us to conclude that \( \{\tilde{\eta}_m\} \) is also a minimizing sequence for \( J_\mu \) over \( U \setminus \{0\} \).
The special minimizing sequence \( \{ \tilde{\eta}_m \} \) is used in §4 to establish the strict subadditivity of the infimum \( c_\mu \) of \( J_\mu \) over \( U \setminus \{ 0 \} \), that is, the inequality
\[
c_{\mu_1 + \mu_2} < c_{\mu_1} + c_{\mu_2}, \quad 0 < \mu_1, \mu_2, \mu_1 + \mu_2 < \mu_0.
\]
The strict subadditivity of \( c_\mu \) follows from the fact that the function
\[
a \mapsto a^{-q} M_{a \mu}(a \tilde{\eta}_m), \quad a \in [1, a_0],
\]
is decreasing and strictly negative for some \( q > 2 \) and \( a_0 \in (1, 2] \), where
\[
M_\mu(\eta) := J_\mu(\eta) - K_2(\eta) - \frac{(\mu + G_2(\eta))^2}{\mathcal{L}_2(\eta)}
\]
is the ‘nonlinear’ part of \( J_\mu(\eta) \) (see §4.4). We proceed by approximating \( M_\mu(\eta_m) \) with its dominant term and showing that this term has the required property.

The heuristic arguments given above suggest firstly that the spectrum of minimizers of \( J_\mu \) over \( U \setminus \{ 0 \} \) (that is, the support of their Fourier transform) is concentrated near wavenumbers \( k = \pm k_0 \), and secondly that they have the Korteweg–de Vries or nonlinear Schrödinger length-scales; the same should be true of the functions \( \tilde{\eta}_m \), which approximate minimizers. We therefore decompose \( \tilde{\eta}_m \) into the sum of a function \( \tilde{\eta}_{m,1} \), whose spectrum is compactly supported near \( k = \pm k_0 \), and a function \( \tilde{\eta}_{m,2} \), whose spectrum is bounded away from these points, and study \( \tilde{\eta}_{m,1} \) using the weighted norm
\[
\| \eta \|^2_\alpha := \int_{-\infty}^{\infty} (1 + \mu^{-4\alpha} (|k| - k_0)^4) |\hat{\eta}(k)|^2 \, dk.
\]
A careful analysis of the equation \( J'_\mu(\tilde{\eta}_m) = O(\mu^N) \) in \( L^2(\mathbb{R}) \) shows that \( \| \tilde{\eta}_{m,1} \|^2_\alpha = O(\mu) \) and \( \| \tilde{\eta}_{m,2} \|_2 = O(\mu^{2+\alpha}) \) for \( \alpha < \frac{1}{3} \) when \( \beta > \frac{\beta_c}{2} \) and for \( \alpha < 1 \) when \( \beta < \beta_c \). Using these estimates on the size of \( \tilde{\eta}_m \), we find that
\[
M_\mu(\tilde{\eta}_m) = \begin{cases} 
\int_{-\infty}^{\infty} \tilde{\eta}_{m,1}^3 \, dx + o(\mu^{5/3}), & \beta > \beta_c, \\
-\int_{-\infty}^{\infty} \tilde{\eta}_{m,1}^4 \, dx + o(\mu^3), & \beta < \beta_c.
\end{cases}
\]
That the function (1.15) is decreasing and strictly negative follows from the above estimate and the fact that \( M_\mu(\eta_m) \) is negative for any minimizing sequence \( \{ \eta_m \} \) for \( J_\mu \) over \( U \setminus \{ 0 \} \).

Knowledge of the strict subadditivity property of \( c_\mu \) (and general estimates for general minimizing sequences) reduces the proof of theorem 1.4(ii) to a straightforward application of the concentration-compactness principle (see §5.1). Parts (iii) and (iv) are derived from lemmas 1.2(ii) and 1.3(ii) by means of a scaling and contradiction argument from the estimates
\[
\| \phi_\eta \|_0^2 = 2 \left\{ \frac{\alpha_{KdV}}{\alpha_{NLS}} \right\} + o(1), \quad \left\{ \frac{E_{KdV}}{E_{NLS}} \right\} (\phi_\eta) = \left\{ \frac{c_{KdV}}{c_{NLS}} \right\} + o(1), \quad \eta \in B_\mu,
\]
which emerge as part of the proof of theorem 1.4(i) (see §5.2).
Some of the techniques used in the present paper were developed by Buffoni et al. [9] in an existence theory for three-dimensional irrotational solitary waves. While we make reference to relevant parts of that paper, many aspects of our construction differ significantly from theirs. In particular, our treatment of non-local analytic operators is more comprehensive. Their version of theorem 1.1 (see Buffoni et al. [9, lemmas 1.1 and 1.4]) is obtained using a less sophisticated ‘flattening’ transformation and shows only that the operators are analytic at the origin. Correspondingly, ‘pseudo-localness’ in the sense described above is established there only for constant-coefficient boundary-value problems (using an explicit representation of the solution by means of Green functions). Our treatment of the consequences of ‘dichotomy’ in the concentration-compactness principle (see §3) is on the other hand similar to that given by Buffoni et al. [9] and we omit proofs that are straightforward modifications of theirs; the main difference here is that negative values of the parameter $\mu$ emerge in our iterative construction of the special minimizing sequence (see the remarks below lemma 3.8).

1.4. Conditional energetic stability

Our original problem of finding minimizers of $\mathcal{H}(\eta, \xi)$ subject to the constraint $\mathcal{I}(\eta, \xi) = 2\mu$ is also solved as a corollary to theorem 1.4(ii); one follows the two-step minimization procedure described in §1.1 (see §5.1).

Theorem 1.5.

(i) The set $D_\mu$ of minimizers of $\mathcal{H}$ on the set

$$S_\mu = \{(\eta, \xi) \in U \times H^{1/2}_r(\mathbb{R}) : \mathcal{I}(\eta, \xi) = 2\mu\}$$

is non-empty.

(ii) Suppose that $\{(\eta_m, \xi_m)\} \subset S_\mu$ is a minimizing sequence for $\mathcal{H}$ with the property that $\sup_{m \in \mathbb{N}} \|\eta_m\|_2 < M$. There exists a sequence $\{x_m\} \subset \mathbb{R}$ with the property that a subsequence of $\{(\eta_m(x_m + \cdot), \xi_m(x_m + \cdot))\}$ converges in $H^r(\mathbb{R}) \times H^{1/2}_r(\mathbb{R})$, $r \in [0, 2)$, to a function in $D_\mu$.

It is a general principle that the solution set of a constrained minimization problem constitutes a stable set of solutions of the corresponding initial-value problem (see, for example, Cazenave and Lions [11]). The usual informal interpretation of the statement that a set $X$ of solutions to an initial-value problem is ‘stable’ is that a solution that begins close to a solution in $X$ remains close to a solution in $X$ at all subsequent times. Implicit in this statement is the assumption that the initial-value problem is globally well posed, that is, every pair $(\eta_0, \Phi_0)$ in an appropriately chosen set is indeed the initial datum of a unique solution $t \mapsto (\eta(t), \Phi(t))$, $t \in [0, \infty)$. At present there is no global well-posedness theory for gravity–capillary water waves with constant vorticity (although there is a large and growing body of literature concerning well-posedness issues for water-wave problems in general). Assuming the existence of solutions, we obtain the following stability result as a corollary of theorem 1.5 using the argument given by Buffoni et al. [9, theorem 5.5]. (The only property of a solution $(\eta, \xi)$ to the initial-value problem that is relevant to stability theory is that $\mathcal{H}(\eta(t), \xi(t))$ and $\mathcal{I}(\eta(t), \xi(t))$ are constant; we therefore adopt this property as the definition of a solution.)
Theorem 1.6. Suppose that \((\eta, \xi): [0, T] \rightarrow U \times H^{1/2} (\mathbb{R})\) has the properties that
\[
\mathcal{H}(\eta(t), \xi(t)) = \mathcal{H}(\eta(0), \xi(0)), \quad I(\eta(t), \xi(t)) = I(\eta(0), \xi(0)), \quad t \in [0, T],
\]
and
\[
\sup_{t \in [0, T]} \|\eta(t)\|_2 < M.
\]
Choose \(r \in [0, 2)\) and let ‘dist’ denote the distance in \(H^r (\mathbb{R}) \times H^{1/2} (\mathbb{R})\). For each \(\varepsilon > 0\) there exists \(\delta > 0\) such that
\[
\text{dist}((\eta(0), \xi(0)), D_{\mu}) < \delta \quad \Rightarrow \quad \text{dist}((\eta(t), \xi(t)), D_{\mu}) < \varepsilon
\]
for \(t \in [0, T]\).

This result is a statement of the conditional energetic stability of the set \(D_{\mu}\). Here energetic refers to the fact that the distance in the statement of stability is measured in the ‘energy space’ \(H^r (\mathbb{R}) \times H^{1/2} (\mathbb{R})\), while conditional alludes to the well-posedness issue. Note that the solution \(t \mapsto (\eta(t), \xi(t))\) may exist in a smaller space over the interval \([0, T]\), at each instant of which it remains close (in energy space) to a solution in \(D_{\mu}\). Furthermore, theorem 1.6 is a statement of the stability of the set of constrained minimizers \(D_{\mu}\); establishing the uniqueness of the constrained minimizer would imply that \(D_{\mu}\) consists of translations of a single solution, so that the statement that \(D_{\mu}\) is stable is equivalent to classical orbital stability of this unique solution (see Benjamin [3]). The phrase ‘conditional energetic stability’ was introduced by Mielke [24] in his study of the stability of irrotational solitary water waves with strong surface tension using dynamical-systems methods.

2. The functional-analytic setting

2.1. Non-local operators

The goal of this section is to introduce rigorous definitions of the Dirichlet–Neumann operator \(G(\eta)\), its inverse \(N(\eta)\) and the operator \(K(\eta) := -\partial_x (N(\eta) \partial_x)\).

2.1.1. Function spaces

Choose \(h_0 \in (0, 1)\). We consider the class
\[
W = \{\eta \in W^{1, \infty} (\mathbb{R}): 1 + \inf \eta > h_0\}
\]
of surface profiles and denote the fluid domain by
\[
\Sigma_\eta = \{(x, y) \in \mathbb{R}^2: 0 < y < 1 + \eta(x)\}, \quad \eta \in W.
\]
The observation that velocity potentials are unique only up to additive constants leads us to introduce the completion \(H^1 (\Sigma_\eta)\) of
\[
S(\Sigma_\eta) = \{\phi \in C^\infty (\Sigma_\eta): |x|^m |\partial_x^{\alpha_1} \partial_y^{\alpha_2} \phi| \text{ is bounded for all } m, \alpha_1, \alpha_2 \in \mathbb{N}_0\}
\]
with respect to the Dirichlet norm as an appropriate function space for \(\phi\). The corresponding space for the trace \(\phi|_{y=1+\eta}\) is the space \(H^{1/2} (\mathbb{R})\) defined in § 1.1.3.
Proposition 2.1. Fix \( \eta \in W \). The trace map \( \phi \mapsto \phi|_{y=1+\eta} \) defines a continuous operator \( H^1_\Sigma \to H^{1/2}(\mathbb{R}) \) with a continuous right inverse \( H^{1/2}(\mathbb{R}) \to H^1_\Sigma \).

We also use anisotropic function spaces for functions defined in the strip \( \Sigma_0 = \mathbb{R} \times (0,1) \).

Definition 2.2. Suppose that \( r \in \mathbb{R} \) and \( n \in \mathbb{N}_0 \).

(i) The Banach space \((L^\infty H^r, \| \cdot \|_{r,\infty})\) is defined by
\[
L^\infty H^r = L^\infty((0,1), H^r(\mathbb{R})), \quad \|u\|_{r,\infty} = \text{ess sup}_{y \in (0,1)} \|u(\cdot, y)\|_{H^r(\mathbb{R})}.
\]

(ii) The Banach space \((H^{r,m}, \| \cdot \|_{r,m})\) is defined by
\[
H^{r,m} = \bigcap_{j=0}^m H^j((0,1), H^{-j}(\mathbb{R})), \quad \|u\|_{r,m} = \sum_{j=0}^m \|A^{-j} \partial_y^j u\|_{L^2(\Sigma)}
\]
where \( A f = \mathcal{F}^{-1}[(1 + k^2)^{1/2} \hat{f}(k)] \).

The following propositions state some properties of these function spaces that are used in the subsequent analysis; they are deduced from results for standard Sobolev spaces (see Hörmander [18, theorem 8.3.1] for proposition 2.4).

Proposition 2.3.

(i) The space \( C_0^\infty(\Sigma) \) is dense in \( H^{r,1} \) for each \( r \in \mathbb{R} \).

(ii) For each \( r \in \mathbb{R} \) the mapping \( u \mapsto u|_{y=1} \), \( u \in C_0^\infty(\Sigma) \), extends continuously to an operator \( H^{r+1,1} \to H^{r+1/2}(\mathbb{R}) \).

(iii) The space \( H^{r+1,1} \) is continuously embedded in \( L^\infty H^{r+1/2} \) for each \( r \in \mathbb{R} \).

(iv) The space \( H^{r+1,1} \) is a Banach algebra for each \( r > 0 \).

Proposition 2.4. Suppose that \( r_0, r_1 \) and \( r_2 \) satisfy \( r_0 \leq r_1, r_0 \leq r_2, r_1 + r_2 \geq 0 \) and \( r_0 < r_1 + r_2 - \frac{1}{2} \). The product \( u_1 u_2 \) of each \( u_1 \in L^\infty H^{r_1} \) and \( u_2 \in H^{r_2,0} \) lies in \( H^{r_0,0} \) and satisfies
\[
\|u_1 u_2\|_{r_0,0} \leq c \|u_1\|_{r_1,\infty} \|u_2\|_{r_2,0}.
\]

Proposition 2.5. For each bounded linear function \( L : L^2(\mathbb{R}) \to L^\infty H^0 \) the formula \( (\eta, w) \mapsto L(\eta)w \) defines a bounded bilinear function \( L^2(\mathbb{R}) \times H^1(\Sigma) \to L^2(\Sigma) \) that satisfies the estimate
\[
\|L(\eta)w\|_0 \leq c \|L\| \|w\|_0^{1/2} \|u\|_H^{1/2} \|\eta\|_0.
\]
The assertion remains valid when \( \Sigma \) is replaced by \( \{ |x| < M \} \) or \( \{ |x| > M \} \) and the estimate holds uniformly over all values of \( M \) greater than unity.
2.1.2. The Dirichlet–Neumann operator

The Dirichlet–Neumann operator $G(\eta)$ for the boundary-value problem

$$\Delta \phi = 0, \quad 0 < y < 1 + \eta, \quad (2.1)$$
$$\phi = \xi, \quad y = 1 + \eta, \quad (2.2)$$
$$\phi_y = 0, \quad y = 0, \quad (2.3)$$

is defined formally as follows: fix $\xi = \xi(x)$, solve (2.1)–(2.3) and set

$$G(\eta)\xi = (\phi_y - \eta'\phi_x)|_{y=1+\eta}.$$  

Our rigorous definition of $G(\eta)$ is given in terms of weak solutions to (2.1)–(2.3) (see Lannes [20, proposition 2.9] for the proof of lemma 2.7).

**Definition 2.6.** Suppose that $\xi \in H^{1/2}_0(\mathbb{R})$ and $\eta \in W$. A weak solution of (2.1)–(2.3) is a function $\phi \in H^1_\ast(\Sigma_\eta)$ with $\phi|_{y=1+\eta} = \xi$ that satisfies

$$\int_{\Sigma_\eta} \nabla \phi \cdot \nabla \psi \, dx \, dy = 0$$

for all $\psi \in H^1_\ast(\Sigma_\eta)$ with $\psi|_{y=1+\eta} = 0$.

**Lemma 2.7.** For each $\xi \in H^{1/2}_0(\mathbb{R})$ and $\eta \in W$ there exists a unique weak solution $\phi$ of (2.1)–(2.3). The solution satisfies the estimate

$$\|\phi\|_{H^1_\ast(\Sigma_\eta)} \leq C\|\xi\|_{H^{1/2}_0(\mathbb{R})},$$

where $C = C(\|\eta\|_{1,\infty})$.

**Definition 2.8.** Suppose that $\eta \in W$ and $\xi \in H^{1/2}_0(\mathbb{R})$. The Dirichlet–Neumann operator is the bounded linear operator $G(\eta): H^{1/2}_\ast(\mathbb{R}) \to H^{-1/2}_\ast(\mathbb{R})$ defined by

$$\int_{-\infty}^\infty (G(\eta)\xi_1)\xi_2 \, dx = \int_{\Sigma_\eta} \nabla \phi_1 \cdot \nabla \phi_2 \, dx \, dy,$$

where $\phi_j \in H^1_\ast(\Sigma_\eta)$ is the unique weak solution of (2.1)–(2.3) with $\xi = \xi_j$, $j = 1, 2$.

2.1.3. The Neumann–Dirichlet operator

The Neumann–Dirichlet operator $N(\eta)$ for the boundary-value problem

$$\Delta \phi = 0, \quad 0 < y < 1 + \eta, \quad (2.4)$$
$$\phi_y - \eta'\phi_x = \xi, \quad y = 1 + \eta, \quad (2.5)$$
$$\phi_y = 0, \quad y = 0, \quad (2.6)$$

is defined formally as follows: fix $\xi = \xi(x)$, solve (2.4)–(2.6) and set

$$N(\eta)\xi = \phi|_{y=1+\eta}.$$  

Our rigorous definition of $N(\eta)$ is also given in terms of weak solutions; lemma 2.10 is proved in the same fashion as lemma 2.7.
Definition 2.9. Suppose that $\xi \in H_{-1/2}^s(\mathbb{R})$ and $\eta \in W$. A weak solution of (2.4)–(2.6) is a function $\phi \in H_1^s(\Sigma_{\eta})$ that satisfies
\[
\int_{\Sigma_{\eta}} \nabla \phi \cdot \nabla \psi \, dx \, dy = \int_{-\infty}^{\infty} \xi \psi|_{y=1+\eta} \, dx
\]
for all $\psi \in H_1^s(\Sigma_{\eta})$.

Lemma 2.10. For each $\xi \in H_{-1/2}^s(\mathbb{R})$ and $\eta \in W$ there exists a unique weak solution $\phi$ of (2.4)–(2.6). The solution satisfies the estimate
\[
\|\phi\|_{H_1^s(\Sigma_{\eta})} \leq C \|\xi\|_{H_{-1/2}^s(\mathbb{R})},
\]
where $C = C(\|\eta\|_{1,\infty})$.

Definition 2.11. Suppose that $\eta \in W$ and $\xi \in H_{-1/2}^s(\mathbb{R})$. The Neumann–Dirichlet operator is the bounded linear operator $N(\eta): H_{-1/2}^s(\mathbb{R}) \rightarrow H_{1/2}^s(\mathbb{R})$ defined by
\[
N(\eta)\xi = \phi|_{y=1+\eta},
\]
where $\phi \in H_1^s(\Sigma_{\eta})$ is the unique weak solution of (2.4)–(2.6).

The relationship between $G(\eta)$ and $N(\eta)$ is clarified by the following result, which follows from the definitions of these operators.

Lemma 2.12. Suppose that $\eta \in W$. The operator $G(\eta) \in \mathcal{L}(H_{1/2}^s(\mathbb{R}), H_{-1/2}^s(\mathbb{R}))$ is invertible with $G(\eta)^{-1} = N(\eta)$.

2.1.4. Analyticity of the operators

Let us begin by recalling the definition of analyticity given by Buffoni and Toland [8, definition 4.3.1] together with a precise formulation of our result in their terminology.

Definition 2.13. Let $X$ and $Y$ be Banach spaces, let $U$ be a non-empty open subset of $X$ and let $\mathcal{L}_k^s(X,Y)$ be the space of bounded $k$-linear symmetric operators $X^k \rightarrow Y$ with norm
\[
\|[m]\| := \inf\{c: \|m(f)\|_Y \leq c\|f\|_X^k \text{ for all } f \in X\}.
\]
A function $F: U \rightarrow Y$ is analytic at a point $x_0 \in U$ if there exist real numbers $\delta, r > 0$ and a sequence $\{m_k\}$, where $m_k \in \mathcal{L}_k^s(X,Y)$, $k \in \mathbb{N}_0$, with the properties that
\[
F(x) = \sum_{k=0}^{\infty} m_k(\{x - x_0\}(k)), \quad x \in B_{\delta}(x_0),
\]
and
\[
\sup_{k \geq 0} r^k \|[m_k]\| < \infty.
\]
The function is analytic if it is analytic at each point $x_0 \in U$. 
Theorem 2.14.

(i) The Dirichlet–Neumann operator \( G: W \to \mathcal{L}(H^{1/2}_{\Sigma}(\mathbb{R}), H^{-1/2}_{\Sigma}(\mathbb{R})) \) is analytic.

(ii) The Neumann–Dirichlet operator \( N: W \to \mathcal{L}(H^{-1/2}_{\Sigma}(\mathbb{R}), H^{1/2}_{\Sigma}(\mathbb{R})) \) is analytic.

To prove this theorem we study the dependence of solutions to the boundary-value problems (2.1)–(2.3) and (2.4)–(2.6) on \( \eta \) by transforming them into equivalent problems in the fixed domain \( \Sigma := \Sigma_0 \). For this purpose we define a change of variable \((x, y) = F^\delta(x, y')\) in the following way. Choose \( \delta > 0 \) and an even function \( \chi \in C_0^\infty(\mathbb{R}) \) with \( \chi(k) \in [0, 1] \) for \( k \in \mathbb{R} \), supp \( \chi \in [-2, 2] \) and \( \chi(x) \equiv 1 \) for \( |x| \leq 1 \), write

\[
\eta^\delta(x, y') = F^{-1}\left[ \chi(\delta(1 - y')k)\eta(k) \right](x)
\]

and define

\[
F^\delta(x, y') = (x, y'(1 + \eta^\delta(x, y'))) = (x, y' + f^\delta(x, y'))
\]

in which \( f^\delta(x, y') = y'\eta^\delta(x, y') \).

Lemma 2.15. Suppose that \( \eta \in W \). The mapping \( F^\delta \) is a bijection \( \Sigma \to \Sigma_\eta \) and \( \Sigma_\eta \to \Sigma_\eta \) with \( y \in C^1_b(\Sigma) \), \( y' \in C^1_b(\Sigma_\eta) \) and

\[
\inf_{(x, y') \in \Sigma} y' = \inf_{(x, y') \in \Sigma} (1 + f^\delta(x, y')) > 0
\]

for each \( \delta \in (0, \delta_{\text{max}}) \), where \( \delta_{\text{max}} = \delta_{\text{max}}(\|\eta'\|_{\infty}) \).

Proof. Writing

\[
\eta^\delta(x, y') = \int_{-\infty}^{\infty} K(s)\eta(x - \delta(1 - y')s) \, ds,
\]

where \( K = (2\pi)^{-1/2}\mathcal{F}^{-1}[\chi] \in \mathcal{S}(\mathbb{R}) \), one finds that \( \eta^\delta \in C^\infty(\Sigma) \cap C^1_b(\Sigma) \) with \( \|\eta^\delta\|_{\infty} \leq c\|\eta\|_{\infty}, \|\eta^\delta_x\|_{\infty} \leq c\|\eta'\|_{\infty}, \|\eta^\delta_y\|_{\infty} \leq c\delta\|\eta'\|_{\infty} \). It follows that \( F^\delta \in C^\infty(\Sigma) \) and \( y \in C^1_b(\Sigma) \). Furthermore, \( y(x, 0) = 0 \), \( y(x, 1) = 1 + \eta(x) \) and

\[
\partial_{y'} y = 1 + y'\eta^\delta_y + \eta^\delta
\]

\[
= 1 + y'\eta^\delta_y + \eta - \int_{y'}^{\delta} \eta^\delta
\]

\[
\geq h_0 - c\delta\|\eta'\|_{\infty}
\]

\[
\geq \frac{1}{2}h_0
\]

\[
> 0
\]

for sufficiently small \( \delta \) (depending only upon \( \|\eta'\|_{\infty}^{-1} \)), so that \( F^\delta \) is a bijection \( \Sigma \to \Sigma_\eta \) and \( \Sigma_\eta \to \Sigma_\eta \). It follows from the inverse function theorem that \( (F^\delta)^{-1} \in C^\infty(\Sigma_{\eta}) \); the estimate

\[
\det dF^\delta[x, y'] = \partial_{y'} y(x, y') \geq \frac{1}{2}h_0
\]

and the fact that \( dF^\delta \) is bounded on \( \Sigma \) imply that \( d(F^\delta)^{-1} \in C_b(\Sigma_{\eta}) \), whereby \( y' \in C^1_b(\Sigma_{\eta}) \). \( \square \)
The change of variable \((x, y) = F^{\delta}(x, y')\) transforms the boundary-value problem (2.4)–(2.6) into

\[
\nabla \cdot ((I + Q)\nabla u) = 0, \quad 0 < y < 1, \tag{2.7}
\]

\[
(I + Q)\nabla u \cdot (0, 1) = \xi, \quad y = 1, \tag{2.8}
\]

\[
(I + Q)\nabla u \cdot (0, -1) = 0, \quad y = 0, \tag{2.9}
\]

where

\[
Q = \begin{pmatrix}
  f^{\delta}_y & -f^{\delta}_x \\
  -f^{\delta}_x & \frac{-f^{\delta}_y + (f^{\delta}_x)^2}{1 + f^{\delta}_y}
\end{pmatrix}
\]

and the primes have been dropped for notational simplicity.

**Lemma 2.16.** The mapping \(W \to (L^\infty(\Sigma))^{2\times2}\) given by \(\eta \mapsto Q(\eta)\) is analytic.

It is helpful to consider the more general boundary-value problem

\[
\nabla \cdot ((I + Q)\nabla u) = \nabla \cdot G, \quad 0 < y < 1, \tag{2.10}
\]

\[
(I + Q)\nabla u \cdot (0, 1) = \xi + G \cdot (0, 1), \quad y = 1, \tag{2.11}
\]

\[
(I + Q)\nabla u \cdot (0, -1) = G \cdot (0, -1), \quad y = 0, \tag{2.12}
\]

where \(I + Q \in (L^\infty(\Sigma))^{2\times2}\) is uniformly positive definite, that is, there exists a constant \(p_0 > 0\) such that

\[
(I + Q)(x, y)\nu \cdot \nu \geq p_0|\nu|^2
\]

for all \((x, y) \in \bar{\Sigma}\) and all \(\nu \in \mathbb{R}^2\).

**Definition 2.17.** Suppose that \(\xi \in H^{-1/2}_x(\mathbb{R})\) and \(G \in (L^2(\Sigma))^2\). A weak solution of (2.10)–(2.12) is a function \(u \in H^1_\Sigma(\Sigma)\) that satisfies

\[
\int_{\Sigma} (I + Q)\nabla u \cdot \nabla w \, dx \, dy = \int_{\Sigma} G \cdot \nabla w \, dx \, dy + \int_{-\infty}^{\infty} \xi w|_{y=1} \, dx
\]

for all \(w \in H^1_\Sigma(\Sigma)\).

**Lemma 2.18.** For each \(\xi \in H^{-1/2}_x(\mathbb{R})\) and \(G \in (L^2(\Sigma))^2\) the boundary-value problem (2.10)–(2.12) has a unique weak solution \(u \in H^1_\Sigma(\Sigma)\). The solution satisfies the estimate

\[
\|u\|_{H^1_\Sigma(\Sigma)} \leq C(\|\xi\|_{H^{-1/2}_x(\mathbb{R})} + \|G\|_{L^2(\mathbb{R})}),
\]

where \(C = C(p_0^{-1})\).

Lemma 2.18 applies in particular to (2.7)–(2.9) for each fixed \(\eta \in W\) (the matrix \(I + Q\) is uniformly positive definite since it is uniformly bounded above, its determinant is unity and its upper left entry is positive). The next theorem shows that its unique weak solution depends analytically upon \(\eta\).

**Theorem 2.19.** The mapping \(W \to \mathcal{L}(H^{-1/2}_x(\mathbb{R}), H^1_\Sigma(\Sigma))\) given by \(\eta \mapsto (\xi \mapsto u)\), where \(u \in H^1_\Sigma(\Sigma)\) is the unique weak solution of (2.7)–(2.9), is analytic.
Proof. Choose \( \eta_0 \in W \) and write \( \tilde{\eta} = \eta - \eta_0 \) and

\[
Q(x, y) = \sum_{n=0}^{\infty} Q^n(x, y), \quad Q^n = \tilde{m}_n(\tilde{\eta}^{(n)}),
\]

where \( \tilde{m}_n(\tilde{\eta}^{(n)}) \in L^s_{\mathbb{R}}(W^{1, \infty}(\mathbb{R}), (L^\infty(\Sigma))^{2 \times 2}) \) satisfies

\[
\|\tilde{m}_n\| \leq C_2 r^{-n}\|\tilde{\eta}\|_{1, \infty}^n\]

(see lemma 2.16). We proceed by seeking a solution of (2.7)–(2.9) of the form

\[
u(x, y) = \sum_{n=0}^{\infty} \nu^n(x, y), \quad \nu^n = m^n_1(\{\tilde{\eta}\}^{(n)}), \tag{2.13}
\]

where \( m^n_1 \in L^s_{\mathbb{R}}(W^{1, \infty}(\mathbb{R}), H^{-1/2}_0(\Sigma)) \) is linear in \( \xi \) and satisfies

\[
\|m^n_1\| \leq C_1 B^n\|\xi\|_{H^{-1/2}_0(\mathbb{R})}
\]

for some constant \( B > 0 \).

Substituting the ansatz (2.13) into the equations, one finds that

\[
\nabla \cdot ((I + Q^0) \nabla \nu^0) = 0, \quad 0 < y < 1, \tag{2.14}
\]

\[
(I + Q^0) \nabla \nu^0 \cdot (0, 1) = \xi, \quad y = 1, \tag{2.15}
\]

\[
(I + Q^0) \nabla \nu^0 \cdot (0, -1) = 0, \quad y = 0, \tag{2.16}
\]

and

\[
\nabla \cdot ((I + Q^0) \nabla \nu^n) = \nabla \cdot G^n, \quad 0 < y < 1, \tag{2.17}
\]

\[
(I + Q^0) \nabla \nu^n \cdot (0, 1) = G^n \cdot (0, 1), \quad y = 1, \tag{2.18}
\]

\[
(I + Q^0) \nabla \nu^n \cdot (0, -1) = G^n \cdot (0, -1), \quad y = 0, \tag{2.19}
\]

for \( n \in \mathbb{N} \), where

\[
G^n = -\sum_{k=1}^{n} Q^k \nabla \nu^{n-k}.
\]

The estimate for \( m^0 \) follows directly from lemma 2.18. Proceeding inductively, suppose that the result for \( m^n \) is true for all \( k < n \). Estimating

\[
\|G^n\|_0 \leq \sum_{k=1}^{n} \|Q^k\|_\infty \|\nabla \nu^{n-k}\|_0 \tag{2.20}
\]

\[
\leq C_1 C_2 B^n\|\xi\|_{H^{-1/2}_0(\mathbb{R})}\|\tilde{\eta}\|_{1, \infty}^n \sum_{k=1}^{n} (Br)^{-k}
\]

and using lemma 2.18 again, we find that

\[
\|\nu^n\|_{H^1(\Sigma)} \leq C_1 C_2 C_3 B^n\|\xi\|_{H^{-1/2}_0(\mathbb{R})}\|\tilde{\eta}\|_{1, \infty}^n \sum_{k=1}^{\infty} (Br)^{-k}
\]

\[
\leq C_1 B^n\|\xi\|_{H^{-1/2}_0(\mathbb{R})}\|\tilde{\eta}\|_{1, \infty}^n
\]

for sufficiently large values of \( B \) (independently of \( n \)).
A straightforward supplementary argument shows that (2.13) defines a weak solution $u$ of (2.17)–(2.19).

Theorem 2.14(ii) follows from theorem 2.19, the equation $N(\eta)\xi = u|_{y=1}$ and the continuity of the trace operator $H^1_*(\Sigma) \to H^{1/2}_*(\mathbb{R})$, while theorem 2.14(i) follows from the inverse function theorem for analytic functions.

Finally, we record another useful result.

**Theorem 2.20.** For each $\eta \in W$ the norms

$$
\xi \mapsto \left( \int_{-\infty}^{\infty} \xi G(\eta)\xi \, dx \right)^{1/2}, \quad \kappa \mapsto \left( \int_{-\infty}^{\infty} \kappa N(\eta)\kappa \, dx \right)^{1/2}
$$

are equivalent to the usual norms for $H^{1/2}_*(\mathbb{R})$ and $H^{-1/2}_*(\mathbb{R})$, respectively.

**Proof.** Let $T : H^{-1/2}_*(\mathbb{R}) \to H^{1/2}_*(\mathbb{R})$ be the isometric isomorphism $\eta \mapsto \mathcal{F}^{-1}[1 + k^2]^{1/2} \hat{\eta}$, which has the property that

$$
\int_{-\infty}^{\infty} \psi \xi \, dx = \langle T\psi, \xi \rangle_{H^{1/2}_*(\mathbb{R})}, \quad \psi \in H^{-1/2}_*(\mathbb{R}), \ \xi \in H^{1/2}_*(\mathbb{R}).
$$

It follows from definition 2.8, lemma 2.12 and the calculation

$$
\langle TG(\eta)\xi, \xi \rangle_{H^{1/2}_*(\mathbb{R})} = \int_{-\infty}^{\infty} (G(\eta)\xi) \, dx = \int_{\Sigma_\eta} |\nabla \phi|^2 \, dx \, dy \geq 0,
$$

where $\phi$ is the unique weak solution of (2.1)–(2.3), that $TG(\eta)$ is a self-adjoint positive isomorphism $H^{1/2}_*(\mathbb{R}) \to H^{1/2}_*(\mathbb{R})$. The spectral theory for bounded self-adjoint operators shows that

$$
\xi \mapsto \langle TG(\eta)\xi, \xi \rangle_{H^{1/2}_*(\mathbb{R})}, \quad \xi \mapsto \langle N(\eta)T^{-1}\xi, \xi \rangle_{H^{1/2}_*(\mathbb{R})}
$$

are both equivalent to the usual norm for $H^{1/2}_*(\mathbb{R})$, so that

$$
\kappa \mapsto \langle N(\eta)\kappa, T\kappa \rangle_{H^{1/2}_*(\mathbb{R})}^{1/2}
$$

is equivalent to the usual norm for $H^{-1/2}_*(\mathbb{R})$. The assertion now follows from the first equality in the previous equation and the calculation

$$
\langle N(\eta)\kappa, T\kappa \rangle_{H^{1/2}_*(\mathbb{R})} = \int_{-\infty}^{\infty} (N(\eta)\kappa) \kappa \, dx.
$$

2.1.5. The operator $K(\eta) = -\partial_x (N(\eta)\partial_x)$

Our first result for this operator is obtained from the material presented above for $N$.

**Theorem 2.21.**

(i) The operator $K : W \to \mathcal{L}(H^{1/2}(\mathbb{R}), H^{-1/2}(\mathbb{R}))$ is analytic.
(ii) For each $\eta \in W$ the operator $K(\eta): H^{1/2}(\mathbb{R}) \to H^{-1/2}(\mathbb{R})$ is an isomorphism and the norm

$$
\zeta \mapsto \left( \int_{-\infty}^{\infty} \zeta K(\eta) \zeta \, dx \right)^{1/2}
$$

is equivalent to the usual norm for $H^{1/2}(\mathbb{R})$.

Proof. (i) This result follows from the definition of $K$ and the continuity of the operators $\partial_x: H^{1/2}(\mathbb{R}) \to H^{-1/2}(\mathbb{R})$ and $\partial_x: H^{1/2}(\mathbb{R}) \to H^{-1/2}(\mathbb{R})$.

(ii) This result is obtained by writing

$$
\int_{-\infty}^{\infty} \zeta K(\eta) \zeta \, dx = \int_{-\infty}^{\infty} \zeta' N(\eta) \zeta' \, dx 
$$

$$
\geq c \|\zeta'\|^2_{H^{-1/2}(\mathbb{R})} 
$$

$$
= c \|\zeta\|^2_{H^{1/2}},
$$

in which theorem 2.20 has been used.

In the remainder of this section we establish the following result concerning the analyticity of $K$ in higher-order Sobolev spaces, using the symbol $W'$ as an abbreviation for $W \cap H'(\mathbb{R})$.

**Theorem 2.22.** The operator $K: W^{s+3/2} \to \mathcal{L}(H^{s+3/2}(\mathbb{R}), H^{s+1/2}(\mathbb{R}))$ is analytic for each $s > 0$.

To prove theorem 2.22 it is necessary to establish additional regularity of the weak solutions $u^n$, $n \in \mathbb{N}_0$, of the boundary-value problems given by (2.14)–(2.16) and (2.17)–(2.19). We proceed by examining the general boundary-value problem (2.10)–(2.12) under additional regularity assumptions on $\zeta$ and $G$. Our result is stated in lemma 2.23, the proof of which requires an *a priori* estimate and a commutator estimate (see Lannes [20, Proposition B.10(2)] for a derivation of the latter).

**Lemma 2.23.** Suppose that $Q \in (H'^{s+1/2})^2 \times 2$ and $G \in (H^{t,1})^2$ for some $t \in (\frac{1}{2} - s, s + 1]$. The weak solution $u$ to (2.10)–(2.12) satisfies the a priori estimate

$$
\|\nabla u\|_{t,1} \leq C(\|G\|_{t,1} + \|\nabla u\|_{t,0}),
$$

where $C = C(p_0^{-1}; \|Q\|_{s+1,2})$.

Proof. Note that

$$
\|\nabla u\|_{t,1} = \|u_x\|_{t,1} + \|u_y\|_{t,1} 
$$

$$
= \|u_x\|_{t,0} + \|u_{xy}\|_{t-0} + \|u_y\|_{t,0} + \|u_{yy}\|_{t-0} 
$$

$$
\leq C(\|\nabla u\|_{t,0} + \|u_{yy}\|_{t-0})
$$

because $\|u_{xy}\|_{t-0} \leq \|u_y\|_{t,0}$, and to estimate $\|u_{yy}\|_{t-0}$ we use (2.10), which we write in the form

$$
(1 + q_{22})u_{yy} = \nabla \cdot G - \partial_x[(1 + q_{11})u_x + q_{12}u_y] - \partial_y(q_{12}u_x) - q_{22}u_y.
$$
Denoting the right-hand side of this equation by $H$, one finds that
\[
\|u_{yy}\|_{t-1,0} = \|(1 + q_{22})^{-1} H\|_{t-1,0} \\
\leq \|H\|_{t-1,0} + \|q_{22} H\|_{t-1,0} \\
\leq (1 + \|q_{22}\|_{s+1/2,\infty}) \|H\|_{t-1,0} \\
\leq C \|H\|_{t-1,0},
\]
where $\tilde{q}_{22} = -q_{22}(1 + q_{22})^{-1}$ and we have used the interpolation estimate
\[
\left\| \frac{p}{1 + p} \right\|_r \leq C_1(p_0^{-1}, \|p\|_\infty) \|p\|_r \leq C_2(p_0^{-1}, \|p\|_r)
\]
for $p \in H^r(\mathbb{R})$, $r > \frac{1}{2}$, with $1 + p(x) \geq p_0$ for all $x \in \mathbb{R}$.

It remains to estimate $\|H\|_{t-1,0}$. Observe that $\|\nabla G\|_{t-1,0} \leq \|G\|_{t,1}, \|u_{xx}\|_{t-1,0} \leq \|\nabla u\|_{t,0}$ and
\[
\|q_{ij} \nabla u_{x_{j}}\|_{t-1,0} \leq C \|Q\|_{s,1/2,\infty} \|\nabla u_{x_{j}}\|_{t-1,0} \\
\leq C \|Q\|_{s,1,1} \|\nabla u\|_{t,0}.
\]
(2.21)

The terms in $H$ involving derivatives of $Q$ are treated differently.

Suppose first that $t \leq s + \frac{1}{2}$. Combining the estimate
\[
\left\| \left\{ \frac{\partial_x}{\partial_y} \right\} q_{ij} \nabla u \right\|_{t-1,0} \leq C \left\| \left\{ \frac{\partial_x}{\partial_y} \right\} q_{ij} \right\|_{s-1/2,\infty} \|\nabla u\|_{t,0} \\
\leq C \|Q\|_{H^{s+1,2}} \|\nabla u\|_{t,0}
\]
(see proposition 2.4) and the estimate (2.21), one obtains the required result
\[
\|u_{yy}\|_{t-1,0} \leq \|H\|_{t-1,0} \leq C(\|G\|_{t,1} + \|\nabla u\|_{t,0}).
\]

For $t \in (s + \frac{1}{2}, s + 1]$ we instead estimate
\[
\left\| \left\{ \frac{\partial_x}{\partial_y} \right\} q_{ij} \nabla u \right\|_{t-1,0} \leq C \left\| \left\{ \frac{\partial_x}{\partial_y} \right\} q_{ij} \right\|_{s,0} \|\nabla u\|_{t-1-\varepsilon, \infty} \\
\leq C \|Q\|_{s+1,1} \|\nabla u\|_{t-\varepsilon, 1}
\]
with $0 < \varepsilon < \min \{\frac{1}{2}, s\}$ by proposition 2.4 to find that
\[
\|u_{yy}\|_{t-1,0} \leq C(\|G\|_{t,1} + \|\nabla u\|_{t,0} + \|\nabla u\|_{t-\varepsilon, 1}) \\
\leq C(\|G\|_{t,1} + \|\nabla u\|_{t,0} + \|u_{yy}\|_{t-1-\varepsilon, 0}).
\]

The result follows by repeating this argument a finite number of times and using the already established result for $t = s + \frac{1}{2}$.

\[\square\]

**Lemma 2.24.** Suppose that $r_0 > \frac{1}{2}$, $\Delta \in [0, 1]$ and $r \in (-\frac{1}{2}, r_0 + \Delta]$ and define $\Lambda_{\varepsilon} = \Lambda^\varepsilon \chi(\varepsilon A)$ for $\varepsilon \in [0, \varepsilon_0)$. The estimate
\[
\|\Lambda_{\varepsilon} u \|_{r, \infty} \leq c \|u\|_{r + \Delta} \|v\|_{r - \Delta}
\]
holds for each $u \in H^{r_0 + \Delta}$ and each $v \in H^{r - \Delta}$, where the constant $c$ does not depend upon $\varepsilon$.  

Lemma 2.25. Suppose that \( Q \in (H^{s+1,2})^{2\times 2} \) and \( \zeta \in H^{t+3/2}(\mathbb{R}) \), \( G \in (H^{t+1,1})^{2} \) for some \( t \in [0,s] \). The weak solution \( u \) of (2.10)-(2.12) with \( \zeta = \zeta' \) satisfies
\[
\| \nabla u \|_{t+1,1} \leq C(\| G \|_{t+1,1} + \| \zeta \|_{t+3/2}),
\]
where \( C = C(p_0^{-1}, \| Q \|_{s+1,2}) \).

Proof. Choose \( r \in (0, t+1), \varepsilon > 0 \) and note that \( A^\varepsilon_r \) is well defined as an operator on \( H^t(\Sigma) \). Writing \( w = (A^\varepsilon_r)^2 u \) in definition 2.17, we find that
\[
\int_{\Sigma} A^\varepsilon_r(P \nabla u) \cdot \nabla A^\varepsilon_r u \, dx \, dy = \int_{\Sigma} A^\varepsilon_r(\partial A^\varepsilon_r u) \cdot \nabla A^\varepsilon_r u \, dy + \int_{-\infty}^{\infty} A^\varepsilon_r \xi \partial A^\varepsilon_r u \big|_{y=1} \, dx
\]
because \( A^\varepsilon_r \) commutes with partial derivatives and is symmetric with respect to the \( L^2 \) inner product. This equation can be rewritten as
\[
\int_{\Sigma} P \nabla A^\varepsilon_r u \cdot \nabla A^\varepsilon_r u \, dx \, dy = -\int_{\Sigma} [A^\varepsilon_r, Q] \nabla u \cdot \nabla A^\varepsilon_r u \, dx \, dy + \int_{\Sigma} A^\varepsilon_r G \cdot \nabla A^\varepsilon_r u \, dx \, dy
\]
and it follows from the coercivity of \( P \) and the continuity of the trace map \( H^t(\Sigma) \to H^{t/2}(\mathbb{R}) \) that
\[
\| A^\varepsilon_r \nabla u \|_{L^2(\Sigma)} \leq C(\| [A^\varepsilon_r, Q] \nabla u \|_{L^2(\Sigma)} + \| A^\varepsilon_r G \|_{L^2(\Sigma)} + \| A^\varepsilon_r A^{1/2} \xi \|_{L^2(\mathbb{R})})
\]
\[
\leq C(\| [A^\varepsilon_r, Q] \nabla u \|_{L^2(\Sigma)} + \| G \|_{t+1,1} + \| \zeta \|_{t+3/2}).
\]

The next step is to estimate the commutator \( [A^\varepsilon_r, Q] \). For \( r \leq s + \frac{1}{2} \) we choose \( \Delta \in (0, \min(s,1)) \) and estimate
\[
\|[A^\varepsilon_r, Q] \nabla u \|_{L^2(\Sigma)} \leq C\| [A^\varepsilon_r, Q] \nabla u \|_{L^2(\Sigma)} + \| \zeta \|_{t+3/2}
\]
\[
\leq C\| [A^\varepsilon_r, Q] \nabla u \|_{L^2(\Sigma)} + \| G \|_{t+1,1} + \| \zeta \|_{t+3/2}.
\]

Using lemma 2.24 (with \( r_0 = s + \frac{1}{2} - \Delta, \Delta = \Delta' \)). For \( r \in (s + \frac{1}{2}, s+1) \), on the other hand, we choose \( \Delta \in (0, \min(s, \frac{1}{2})) \) and estimate
\[
\|[A^\varepsilon_r, Q] \nabla u \|_{L^2(\Sigma)} \leq C\| [A^\varepsilon_r, Q] \nabla u \|_{L^2(\Sigma)} + \| G \|_{t+1,1} + \| \zeta \|_{t+3/2}
\]
\[
\leq C\| [A^\varepsilon_r, Q] \nabla u \|_{L^2(\Sigma)} + \| G \|_{t+1,1} + \| \zeta \|_{t+3/2}.
\]

Using lemma 2.23.

Combining the above estimates yields
\[
\| A^\varepsilon_r \nabla u \|_{L^2(\Sigma)} \leq C(\| \nabla u \|_{t-\Delta,0} + \| G \|_{t+1,1} + \| \zeta \|_{t+3/2}),
\]
where \( \Delta \in (0, \min(s, \frac{1}{2})) \), and letting \( \varepsilon \to 0 \) and using the resulting estimate iteratively, we find that
\[
\| \nabla u \|_{t+1,0} \leq C(\| G \|_{t+1,1} + \| \zeta \|_{t+3/2} + \| u \|_{H^t(\Sigma)}),
\]
from which the result follows by lemmas 2.18 and 2.23.
The following result shows that lemma 2.25 is applicable to the boundary-value problems (2.14)–(2.16) and (2.17)–(2.19).

**Lemma 2.26.** The mapping \(W^{s+3/2} \to (H^{s+1,2})^{2 \times 2}\) given by \(\eta \mapsto Q(\eta)\) is analytic.

**Remark 2.27.** Observe that

\[
Q_x(\eta) = S_0(\eta) + R_0(\eta)L_0^2\eta'' + R_1(\eta)L_1^2\eta''',
\]

\[
Q_y(\eta) = T_0(\eta) + R_0(\eta)L_0^2\eta'' + R_1(\eta)L_2^2\eta''',
\]

where \(L_j(\cdot) = \mathcal{F}^{-1}[(i\delta)^j\chi^{(j)}((1 - y)\delta k|\mathcal{F}|)]\), \(j = 0, 1, 2\), are bounded bilinear functions \(L^2(\mathbb{R}) \to L^\infty H^0\) and

\[
S_0: \eta \mapsto \begin{pmatrix} \eta_y \delta \\ 0 - \frac{\eta_y}{1 + f_y} - \frac{0}{(1 + f_y)^2} \end{pmatrix},
\]

\[
T_0: \eta \mapsto \begin{pmatrix} 2L_1^2\eta' \\ -\frac{\eta_y}{1 + f_y} - \frac{2L_1^2\eta_y'}{1 + f_y} + \frac{2f_x\eta_y}{(1 + f_y)^2} - \frac{2(-f_y + (f_y)^2)L_2^2\eta'}{(1 + f_y)^2} \end{pmatrix},
\]

\[
R_0: \eta \mapsto \begin{pmatrix} 0 - y \\ -y \frac{2f_1^2}{1 + f_y} \end{pmatrix},
\]

\[
R_1: \eta \mapsto \begin{pmatrix} y \\ 0 - \frac{y}{1 + f_y} - \frac{y(-f_y + (f_y)^2)}{(1 + f_y)^2} \end{pmatrix}
\]

are analytic functions \(W \to (L^\infty(\tilde{\Sigma}))^{2 \times 2}\).

The regularity assertion in theorem 2.22 now follows from the next result and the continuity of the trace operator \(H^{s+1,1} \to H^{s+1/2}(\mathbb{R})\).

**Theorem 2.28.** The mapping \(W^{s+3/2} \to \mathcal{L}(H^{s+1/2}(\mathbb{R}), (H^{s+1,1})^2)\) given by \(\eta \mapsto (\zeta \mapsto \nabla u)\), where \(u \in H^1(\Sigma)\) is the unique weak solution of (2.7)–(2.9) with \(\xi = \zeta'\), is analytic.

**Proof.** Repeating the proof of theorem 2.19, replacing lemma 2.18 by lemma 2.25, lemma 2.16 by lemma 2.26 and inequality (2.20) by

\[
\|\mathcal{C}^n\|_{s+1,1} \leq \sum_{k=1}^n \|Q^k\|_{s+1,1}\|\nabla u^{n-k}\|_{s+1,1}
\]

\((H^{s+1,1} \text{ is a Banach algebra})\), we obtain the representation

\[
\nabla u(x, y) = \sum_{n=0}^\infty \nabla u^n(x, y), \quad \nabla u^n = m_2^n(\{\delta\}^{(n)}),
\]

where \(m_2^n \in \mathcal{L}_n(\mathcal{L}_{s+3/2}(\mathbb{R}), (H^{s+1,1})^2)\) is linear in \(\zeta\) and satisfies

\[
\|m_2^n\| \leq C_1 B^n\|\zeta\|_{s+3/2}
\]

for some constant \(B > 0\). \(\square\)
We conclude this section with a useful supplementary estimate for \(\|K^n(\tilde{\eta})\|\).

**Proposition 2.29.** There exists a constant \(B > 0\) such that

\[
\|K^n(\tilde{\eta})\|_0 \leq C_1 B^n (\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}'\| + k_0^2 \tilde{\eta}_0\|) (\|\zeta\|_{3/2})^{n}, \quad n \in \mathbb{N}_0.
\]

**Proof.** It suffices to establish the estimate

\[
\|\nabla u^n\|_1 \leq C_1 B^n (\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}'\| + k_0^2 \tilde{\eta}_0\|) (\|\zeta\|_{3/2})^{n}, \quad n \in \mathbb{N}_0;
\]

for \(n = 0\) this result follows from Lemma 2.25 (with \(t = 0\) and \(s = \frac{1}{2}\)).

Proceeding inductively, suppose that the estimate for \(\|\nabla u^k\|_1\) is true for all \(k < n\) and recall from the proof of Theorem 2.19 that

\[
\|Q^k\|_\infty \leq C_2 r^{-k} \|\tilde{\eta}\|_{1,\infty}^k, \quad \|G^n\|_0 \leq C_1 C_2 B^n (\|\zeta\|_{3/2})^{n} \sum_{k=1}^{n} (Br)^{-k}.
\]

Writing

\[
Q^k_x = S^k_0 + R^k_0 L^\delta_0 \tilde{\eta}_0'' + R^k_0 L^\delta_0 \tilde{\eta}_0' + R^k_1 L^\delta_0 \tilde{\eta}_0 + R^k_1 L^\delta_0 \tilde{\eta}_0'' + R^k_1 L^\delta_0 \tilde{\eta}_0''
\]

\[
= S^k_0 + \sum_{j=0}^{1} (-k_0^2 R^k_0 L^\delta_0 \tilde{\eta}_0 + R^k_1 L^\delta_0 \tilde{\eta}_0' + R^k_1 L^\delta_0 \tilde{\eta}_0' + R^k_1 L^\delta_0 (\tilde{\eta}_0'' + k_0^2 \tilde{\eta}_0)),
\]

where

\[
\|S^k_0\|_\infty \leq C_2 r^{-k} \|\tilde{\eta}\|_{1,\infty}^k, \quad \|R^k_1\|_\infty \leq C_2 r^{-k} \|\tilde{\eta}\|_{1,\infty}^k, \quad j = 0, 1
\]

(see Remark 2.27), we find that

\[
G^n_x = -\sum_{k=1}^{n} (Q^k_x \nabla u^{n-k} + Q^k \nabla u^{n-k})
\]

\[
= \sum_{k=1}^{n} \left( S^k_0 \nabla u^{n-k} + \sum_{j=0}^{1} (-k_0^2 R^k_0 L^\delta_0 \tilde{\eta}_0 + R^k_1 L^\delta_0 \tilde{\eta}_0' + R^k_1 L^\delta_0 \tilde{\eta}_0' + R^k_1 L^\delta_0 (\tilde{\eta}_0'' + k_0^2 \tilde{\eta}_0)) \nabla u^{n-k} \right.
\]

\[
+ Q^k \nabla u^{n-k})
\]

It follows that

\[
\|G^n_x\|_0 \leq \sum_{k=1}^{n} \left( (\|S^k_0\|_\infty + k_0^2 (\|R^k_0\|_\infty + \|R^k_1\|_\infty) (\|\tilde{\eta}\|_{1,\infty}) (\|\zeta\|_{3/2})^{n} \nabla u^{n-k} \right)
\]

\[
+ (\|R^k_0\|_\infty \|L^\delta_0\|_1 + \|R^k_1\|_\infty \|L^\delta_0\|_1) (\|\tilde{\eta}_0''\|_0) \nabla u^{n-k} \|_1
\]

\[
+ (\|R^k_0\|_\infty \|L^\delta_0\|_1 + \|R^k_1\|_\infty \|L^\delta_0\|_1) (\|\tilde{\eta}_0''\|_0) \nabla u^{n-k} \|_1
\]

\[
+ (Q^k \|\zeta\|_{3/2} (\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}_0'' + k_0^2 \tilde{\eta}_0\|_0) \sum_{k=1}^{n} (Br)^{-k},
\]

\[
\leq C_1 C_2 B^n (1 + 2k_0^2 r + (\|L^\delta_0\|_1 + \|L^\delta_0\|_1) (\|\tilde{\eta}_0''\|_0 + r) + 1)
\]

\[
\times (\|\zeta\|_{3/2} (\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}_0'' + k_0^2 \tilde{\eta}_0\|_0) \sum_{k=1}^{n} (Br)^{-k}.
\]
in which proposition 2.5 has been used. A similar calculation yields the same estimate for \( \|G_n\|_0 \).

Combining the estimates for \( \|G_n\|_0, \|G'\|_0 \) and \( \|G''\|_0 \) and applying lemma 2.25 (with \( t = 0 \) and \( s = \frac{1}{2} \)), one finds that

\[
\|\nabla u_n\|_1 \leq \sqrt{3}C_1C_2C_3B^n \left( 1 + 2k_0^2r + (\|L_0\| + \|L_1\|)(\|\eta''\|_0 + r) + 1 \right) \\
\times \|\zeta\|_{3/2} (\|\bar{\eta}\|_{1,\infty} + \|\bar{\eta}'' + k_0^2\bar{\eta}\|_0)^n \sum_{k=1}^n (Br)^{-k},
\]

so that

\[
\|\nabla u_n\|_1 \leq C_1B^n (\|\bar{\eta}\|_{1,\infty} + \|\bar{\eta}'' + k_0^2\bar{\eta}\|_0)^n \|\zeta\|_{3/2}
\]

for sufficiently large values of \( B \) (independently of \( n \)).

### 2.2. Variational functionals

In this section we study the functional

\[
T(\eta) = \int_{-\infty}^{\infty} f_1(\eta)K(\eta)f_2(\eta) \, dx,
\]

where \( f_1, f_2 : \mathbb{R} \to \mathbb{R} \) are polynomials with \( f_1(0) = f_2(0) = 0 \), and apply our results to the functionals \( G, K \) and \( L \).

#### 2.2.1. Analyticity of the functionals

In this section we again suppose that \( s > 0 \). The first result follows from theorem 2.21(i).

**Lemma 2.30.** Equation (2.22) defines a functional \( T : W^{s+3/2} \to \mathbb{R} \) that is analytic and satisfies \( T(0) = 0 \).

We now turn to the construction of the gradient \( T'(\eta) \) in \( L^2(\mathbb{R}) \), the main step of which is accomplished by the following lemma.

**Lemma 2.31.** Define \( \mathcal{H} : W^{s+3/2} \to L^2_s(H^{s+3/2}(\mathbb{R}),\mathbb{R}) \) by the formula

\[
\mathcal{H}(\eta)(\zeta_1, \zeta_2) = (\zeta_1, K(\eta)\zeta_2)_{0}.
\]

The gradient \( \mathcal{H}'(\eta)(\zeta_1, \zeta_2) \) in \( L^2(\mathbb{R}) \) exists for each \( \eta \in W^{s+3/2} \) and \( \zeta_1, \zeta_2 \in H^{s+3/2}(\mathbb{R}) \) and is given by the formula

\[
\mathcal{H}'(\eta)(\zeta_1, \zeta_2) = -u_{1x}u_{2x} + \frac{1 + \eta'^2}{(1 + \eta)^2} u_{1y}u_{2y} \bigg|_{y=1},
\]

where \( u_j \) is the weak solution of (2.7)–(2.9) with \( \xi = \zeta_j' \), \( j = 1, 2 \). This formula defines an analytic function \( \mathcal{H}' : W^{s+3/2} \to L^2_s(H^{s+3/2}(\mathbb{R}), H^{s+1/2}(\mathbb{R})) \).

**Proof.** It follows from the formula

\[
\mathcal{H}(\eta) = \int_{\Sigma} (I + Q(\eta)) \nabla u_1 \cdot \nabla u_2 \, dx \, dy
\]
that
\[ d\mathcal{H}[^\eta](\omega) = \int_{\Sigma} dQ[^\eta](\omega) \nabla u_1 \cdot \nabla u_2 \, dx \, dy 
+ \int_{\Sigma} (I + Q(\eta)) \nabla w_1 \cdot \nabla u_2 \, dx \, dy 
+ \int_{\Sigma} (I + Q(\eta)) \nabla u_1 \cdot \nabla w_2 \, dx \, dy, \]

(2.23)

where \( w_j = du_j(\eta)[\omega], \) \( j = 1, 2. \) Recall that
\[ \int_{\Sigma} (I + Q(\eta)) \nabla u_j \cdot \nabla v \, dx \, dy = \int_{-\infty}^{\infty} \zeta_j v[y=1] \, dx, \quad j = 1, 2, \]

for every \( v \in H^1(\Sigma) \) (see definition 2.17 with \( \xi = \zeta_j' \) and \( G = 0 \)), so that
\[ \int_{\Sigma} (dQ[^\eta](\omega) \nabla u_j \cdot \nabla v + (I + Q(\eta)) \nabla w_j \cdot \nabla v) \, dx \, dy = 0, \quad j = 1, 2, \]

(2.24)

for every \( v \in H^1(\Sigma) \). Subtracting (2.24) with \( j = 1, \) \( v = u_2 \) and \( j = 2, \) \( v = u_1 \) from (2.23) yields
\[ d\mathcal{H}[^\eta](\omega) = -\int_{\Sigma} dQ[^\eta](\omega) \nabla u_1 \cdot \nabla u_2 \, dx \, dy. \]

Finally, write \( h^\delta(x, y) = y\omega^\delta(x, y), \) where
\[ \omega^\delta(x, y) = \mathcal{F}^{-1}[\chi((\delta(y - 1)|k|)\tilde{\omega}(k)](x), \]

so that \( h^\delta = df^\delta[^\eta](\omega), \) and observe that
\[ \int_{-\infty}^{\infty} \left( -h^\delta \left( \frac{f_2^\delta y u_{1y}}{1 + f_y^\delta} \right) (u_{2x} - \frac{f_2^\delta y u_{2y}}{1 + f_y^\delta}) + \frac{h^\delta u_{1y} u_{2y}}{(1 + f_y^\delta)^2} \right) \bigg|_{y=1} \, dx 
= \frac{1}{2} \int_{\Sigma} \frac{d}{dy} \left( -h^\delta \left( \frac{f_2^\delta y u_{1y}}{1 + f_y^\delta} \right) (u_{2x} - \frac{f_2^\delta y u_{2y}}{1 + f_y^\delta}) + \frac{h^\delta u_{1y} u_{2y}}{(1 + f_y^\delta)^2} \right) \, dx \, dy 
= \int_{\Sigma} \left( -h^\delta u_{1x} u_{2x} + h^\delta u_{1x} u_{2y} + h^\delta u_{1y} u_{2x} + \frac{h^\delta u_{1y} u_{2y}}{(1 + f_y^\delta)^2} + \frac{2(f_2^\delta)^2 h^\delta u_{1y} u_{2y}}{(1 + f_y^\delta)^2} - \frac{2f_2^\delta h^\delta u_{1y} u_{2y}}{1 + f_y^\delta} \right) \, dx \, dy 
+ \int_{\Sigma} h^\delta u_{1y} \left( (1 + f_y^\delta) u_{2x} - f_x^\delta u_{2y} \right) \, dx \, dy 
+ \int_{\Sigma} h^\delta u_{2y} \left( (1 + f_y^\delta) u_{1x} - f_x^\delta u_{1y} \right) \, dx \, dy 
+ \int_{-\infty}^{\infty} \left( h^\delta f_2^\delta u_{1y} (u_{2x} - \frac{f_2^\delta y u_{2y}}{1 + f_y^\delta}) + \frac{h^\delta f_2^\delta u_{2y}}{1 + f_y^\delta} (u_{1x} - \frac{f_2^\delta y u_{1y}}{1 + f_y^\delta}) \right) \bigg|_{y=1} \, dx \]
\[ = -\int_{\Sigma} \left( dQ[\eta](\omega) \nabla u_1 \cdot \nabla u_2 + \frac{h^\delta u_{1y}}{1 + f_y^2} \nabla \cdot ((I + Q(\eta)) \nabla u_1) + \frac{h^\delta u_{2y}}{1 + f_y^2} \nabla \cdot ((I + Q(\eta)) \nabla u_2) \right) \, dx \, dy + \int_{-\infty}^{\infty} \left( \frac{h^\delta f_{2x}^\delta u_{1y}}{1 + f_y^2} u_{2x} - \frac{f_{2x}^\delta u_{2y}}{1 + f_y^2} \right) \left( u_{1x} - \frac{f_{1x}^\delta y u_{1y}}{1 + f_y^2} \right) \bigg|_{y = 1} \, dx, \]

in which the third line follows from the second by differentiating the term in braces with respect to \( y \) (note that \( h^\delta |_{y = 0} = 0 \)) and integrating by parts. One concludes that

\[ dH[\eta](\omega) = \int_{-\infty}^{\infty} \left( -u_{1x} u_{2x} + \frac{1 + (f_{2x}^\delta)^2}{1 + f_y^2} u_{1y} u_{2y} \right) h^\delta \bigg|_{y = 1} \, dx, \]

and the stated formula follows from this result and the facts that \( f^\delta |_{y = 1} = \eta \) and \( h^\delta |_{y = 1} = \omega \).

The hypotheses of the lemma imply that \( \nabla u_j \in H^{s+1,1} \) and \( \nabla u_j |_{y = 1} \in H^{s+1/2}(\mathbb{R}) \), \( j = 1, 2 \). This observation ensures that the above algebraic manipulations are valid and that \( dH[\eta] \) belongs to \( H^{s+1/2}(\mathbb{R}) \) because \( H^{s+1,1} \) and \( H^{s+1/2}(\mathbb{R}) \) are Banach algebras.

**Corollary 2.32.** The gradient \( T'(\eta) \) in \( L^2(\mathbb{R}) \) exists for each \( \eta \in W^{s+3/2} \) and is given by the formula

\[ T'(\eta) = H'(\eta)(f_1(\eta), f_2(\eta)) + f_1'(\eta)K(\eta)f_2(\eta) + f_2'(\eta)K(\eta)f_1(\eta). \]

This formula defines an analytic function \( T': W^{s+3/2} \to H^{s+1/2}(\mathbb{R}) \) that satisfies \( T'(0) = 0 \).

**Theorem 2.33.**

(i) Equations (1.10)–(1.12) define analytic functionals \( G, K, L: W^{s+3/2} \to \mathbb{R} \) that satisfy \( G(0), K(0), L(0) = 0 \).

(ii) Equation (1.9) defines an analytic functional \( J_\mu: W^{s+3/2} \setminus \{0\} \to \mathbb{R} \).

(iii) The gradients \( G'(\eta) \) and \( L'(\eta) \) in \( L^2(\mathbb{R}) \) exist for each \( \eta \in W^{s+3/2} \) and are given by the equations

\[ G'(\eta) = \frac{1}{4} \omega H'(\eta)(\eta^2, \eta) + \frac{1}{4} \omega K(\eta)\eta^2 + \frac{1}{2} \omega \eta K(\eta)\eta - \frac{1}{2} \omega \eta, \]  

\[ L'(\eta) = \frac{1}{2} H'(\eta)(\eta, \eta) + K(\eta)\eta. \]

These equations define analytic functions \( G', L': W^{s+3/2} \to H^{s+1/2}(\mathbb{R}) \) that satisfy \( G'(0) = 0 \) and \( L'(0) = 0 \).

(iv) The gradient \( K'(\eta) \) in \( L^2(\mathbb{R}) \) exists for each \( \eta \in W^2 \) and is given by

\[ K'(\eta) = \eta - \beta \left( -\frac{\eta}{\sqrt{1 + \eta^2}} \right)' \frac{\omega^2}{8} H'(\eta)(\eta^2, \eta^2) - \frac{\omega^2}{2} \eta^2 K(\eta)\eta + \frac{\omega^2}{3} \eta^2. \]

This equation defines an analytic function \( K': W^2 \to L^2(\mathbb{R}) \) that satisfies \( K'(0) = 0 \).
Theorem 2.21(ii). Turning to the estimate for $K$, observe that

$$\lim_{m \to \infty} \|T^m(\eta_1^{(1)} + \eta_2^{(2)}) - T^m(\eta_1^{(1)}) - T^m(\eta_2^{(2)})\|_0 = 0,$$

$$\lim_{m \to \infty} \|T'(\eta_1^{(1)} + \eta_2^{(2)}) - T'(\eta_1^{(1)}) - T'(\eta_2^{(2)})\|_0 = 0,$$

and

$$\lim_{m \to \infty} \langle T'(\eta_2^{(2)}), \eta_1^{(1)} \rangle = 0.$$

In particular, this result applies to $G$, $K$, and $L$.
We begin the proof of theorem 2.36 by re-examining the general boundary-value problem (2.10)–(2.12).

**Lemma 2.37.** Suppose that \{R_m\}, \{S_m\} and \{U_m\} are sequences of positive real numbers and

\[
\{Q_m\} \subseteq (L^\infty(\Sigma))^2 \times 2, \quad \{G^{(1)}_m\}, \{G^{(2)}_m\} \subseteq L^2(\Sigma), \quad \{\zeta^{(1)}_m\}, \{\zeta^{(2)}_m\} \subseteq H^{1/2}(\mathbb{R})
\]

are bounded sequences with the properties that

(i) \(S_m - U_m, U_m - R_m \to \infty\) as \(m \to \infty\);

(ii) \(\text{supp} \zeta^{(1)}_m \subseteq [-R_m, R_m]\) and \(\text{supp} \zeta^{(2)}_m \subseteq \mathbb{R} \setminus (-S_m, S_m)\);

(iii) \(\|G^{(1)}_m\|_{L^2(|x|>R_m)}, \|G^{(2)}_m\|_{L^2(|x|<S_m)} \to 0\) as \(m \to \infty\);

(iv) there exists a constant \(p_0 > 0\) such that

\[
(I + Q_m)(x, y) \nu \cdot \nu \geq p_0 |\nu|^2
\]

for all \((x, y) \in \Sigma, m \in \mathbb{N}\) and all \(\nu \in \mathbb{R}^2\).

The unique weak solutions \(u^{(j)}_m \in H^1_\nu(\Sigma)\) of the boundary-value problems

\[
\begin{align*}
\nabla \cdot ((I + Q_m)\nabla u^{(j)}_m) &= \nabla \cdot G^{(j)}_m, & 0 < y < 1, \\
(I + Q_m)\nabla u^{(j)}_m \cdot (0, 1) &= \zeta^{(j)}_m \cdot (0, 1), & y = 1, \\
(I + Q_m)\nabla u^{(j)}_m \cdot (0, -1) &= G^{(j)}_m \cdot (0, -1), & y = 0,
\end{align*}
\]

\(j = 1, 2\), satisfy the estimates

\[
\lim_{m \to \infty} \|\nabla u^{(1)}_m\|_{L^2(|x|>U_m)} = 0, \quad \lim_{m \to \infty} \|\nabla u^{(2)}_m\|_{L^2(|x|<U_m)} = 0.
\]

**Proof.** Write \(\zeta^{(2)}_m = \zeta^{(2)}_{m,+} + \zeta^{(2)}_{m,-}\), where

\[
\text{supp} \zeta^{(2)}_{m,+} \subseteq [S_m, \infty), \quad \text{supp} \zeta^{(2)}_{m,-} \subseteq (-\infty, -S_m],
\]

and let \(u^{(2)}_{m,+}, u^{(2)}_{m,-}\) be the weak solutions of the boundary-value problem (2.28)–(2.29) with \(\zeta^{(2)}_m, G^{(2)}_m\) replaced by

\[
\zeta^{(2)}_{m,+}, G^{(2)}_{m,+} := G^{(2)}_m \chi_{x>0}, \quad \zeta^{(2)}_{m,-}, G^{(2)}_{m,-} := G^{(2)}_m \chi_{x<0},
\]

respectively, so that \(u^{(2)}_m = u^{(2)}_{m,+} + u^{(2)}_{m,-}\).

Choose \(T > 0\) and take \(m\) large enough so that \(T + 1 < S_m\). Define \(\phi \in C^\infty(\mathbb{R})\) by

\[
\phi_T(x) = \begin{cases} 1, & x \leq T, \\ \chi(2(x-T)), & x > T, \end{cases}
\]

and set

\[
w_m(x, y) = \phi_T^2(x)(u^{(2)}_{m,+}(x, y) - M_T),
\]

where

\[
M_T = \int_{T \leq x \leq T+1} u^{(2)}_{m,+}(x, y) \, dx \, dy,
\]
so that supp \( u_m \subseteq (-\infty, T + 1] \times [0, 1] \) and the mean value of \( u_{m,+}^{(2)}(x,y) - M_T \) over \((T, T + 1) \times (0, 1)\) is zero. Using definition 2.17, we find that

\[
\int_\Sigma (I + Q_m) \nabla u_{m,+}^{(2)} \cdot \nabla u_m \, dx \, dy = \int_\Sigma G_{m,+}^{(2)} \cdot \nabla u_m \, dx \, dy + \int_{-\infty}^{\infty} \partial_x c_{m,+}^{(2)} w_m |_{y=1} \, dx,
\]

from which it follows that

\[
\int_\Sigma (I + Q_m) \phi_T^2 \nabla u_{m,+}^{(2)} \cdot \nabla u_{m,+}^{(2)} \, dx \, dy \leq c \left( \int_\Sigma \phi_T^2 \nabla u_{m,+}^{(2)} \, dx \, dy \right)^{1/2} \left( \int_{T \leq x \leq T+1} |u_{m,+}^{(2)} - M_T|^2 \, dx \, dy \right)^{1/2} + \left( \int_{x \leq T+1} |G_{m,+}^{(2)}|^2 \, dx \, dy \right)^{1/2} \left( \int_{T \leq x \leq T+1} |u_{m,+}^{(2)} - M_T|^2 \, dx \, dy \right)^{1/2} + \left( \int_{x \leq T} |G_{m,+}^{(2)}|^2 \, dx \, dy \right)^{1/2} \left( \int_\Sigma \phi_T^2 \nabla u_{m,+}^{(2)} \, dx \, dy \right)^{1/2},
\]

and hence that

\[
\int_\Sigma \phi_T^2 \nabla u_{m,+}^{(2)} \, dx \, dy \leq c \left( \int_{T \leq x \leq T+1} |\nabla u_{m,+}^{(2)}|^2 \, dx \, dy \right) + \int_{x \leq T+1} |G_{m,+}^{(2)}|^2 \, dx \, dy,
\]

where the Poincaré inequality

\[
\int_{T \leq x \leq T+1} |u_{m,+}^{(2)} - M_T|^2 \, dx \, dy \leq c \int_{T \leq x \leq T+1} |\nabla u_{m,+}^{(2)}|^2 \, dx \, dy
\]

has been used.

The above inequality implies that

\[
\Phi(T) \leq c_*(\Phi(T + 1) - \Phi(T) + \Psi(T + 1))
\]

for some \( c_* > 0 \), where

\[
\Phi(T) = \int_{x \leq T} |\nabla u_{m,+}^{(2)}|^2 \, dx \, dy, \quad \Psi(T) = \int_{x \leq T} |G_{m,+}^{(2)}|^2 \, dx \, dy,
\]

so that

\[
\Phi(T) \leq d_* (\Phi(T + 1) + \Psi(T + 1)),
\]

where \( d_* = c_*/(c_* + 1) \in (0, 1) \), and using this inequality recursively, one finds that

\[
\Phi(T) \leq d_*^{[r]} \Phi(T + r) + \frac{d_*}{1 - d_*} \Psi(T + r), \quad r \geq 1.
\]

In particular, this result asserts that

\[
\Phi(U_m) \leq d_*^{S_m - U_m - 1} \Phi(S_m) + \frac{d_*}{1 - d_*} \Psi(S_m)
\]
and, because
\[
\Phi(S_m) = \int_{x < S_m} |\nabla u^{(2)}_{m,+}|^2 \, dx \, dy \leq \int_{\Sigma} |\nabla u^{(2)}_{m,+}|^2 \, dx \, dy \leq \|u^{(2)}_{m,+}\|_{1/2} = O(1)
\]
and
\[
\Psi(S_m) = \int_{x < S_m} |G^{(2)}_{m,+}|^2 \, dx \, dy \leq \int_{|x| < S_m} |G^{(2)}_{m,+}|^2 \, dx \, dy = o(1)
\]
as \(m \to \infty\), we conclude that
\[
\Phi(U_m) = \int_{x < U_m} |\nabla u^{(2)}_{m,+}|^2 \, dx \, dy = o(1)
\]
as \(m \to \infty\).

A similar argument shows that
\[
\int_{x > -U_m} |\nabla u^{(2)}_{m,-}|^2 \, dx \, dy = o(1)
\]
as \(m \to \infty\), so that
\[
\int_{|x| < U_m} |\nabla u^{(2)}_{m,+}|^2 \, dx \, dy \leq \int_{|x| < U_m} |\nabla u^{(2)}_{m,+}|^2 \, dx \, dy + \int_{|x| < U_m} |\nabla u^{(2)}_{m,-}|^2 \, dx \, dy
\]
\[
= \int_{x < U_m} |\nabla u^{(2)}_{m,+}|^2 \, dx \, dy + \int_{x > -U_m} |\nabla u^{(2)}_{m,-}|^2 \, dx \, dy = o(1)
\]
as \(m \to \infty\).

The complementary estimate
\[
\int_{|x| > U_m} |\nabla u^{(1)}_{m,-}|^2 = o(1)
\]
as \(m \to \infty\) is obtained in a similar fashion.

The next step is to apply lemma 2.37 to the boundary-value problem (2.7)–(2.9).

**Lemma 2.38.** Let \(u(\eta)\) be the solution to (2.7)–(2.9) with \(\xi = \partial_x f(\eta), \eta \in U\), where \(f\) is a real polynomial. The estimates
\[
\lim_{m \to \infty} \|\nabla u(\eta^{(1)}_{m})\|_{H^1(|x| > T_m)} = 0, \quad \lim_{m \to \infty} \|\nabla u(\eta^{(2)}_{m})\|_{H^1(|x| < T_m)} = 0
\]
hold for each sequence \(\{T_m\}\) of positive real numbers with \(S_m - T_m, T_m - R_m \to \infty\) as \(m \to \infty\).

**Proof.** Choose sequences \(\{\tilde{R}_m\}, \{\tilde{S}_m\}\) of positive real numbers with \(S_m - \tilde{S}_m, \tilde{S}_m - T_m \to \infty\) and \(T_m - \tilde{R}_m, \tilde{R}_m - R_m \to \infty\) as \(m \to \infty\). The quantities \(u_{m,j}^{(j)} = u(\eta^{(j)}_{m,j})\), \(j = 1, 2\), satisfy the boundary-value problems
\[
\nabla \cdot ((I + Q^j_{m}) \nabla u_{m,j}^{(j)}) = 0, \quad 0 < y < 1,
\]
\[
(I + Q^j_{m}) \nabla u_{m,j}^{(j)} \cdot (0, 1) = f(\eta^{(j)}_{m,j})_x, \quad y = 1,
\]
\[
(I + Q^j_{m}) \nabla u_{m,j}^{(j)} \cdot (0, -1) = 0, \quad y = 0,
\]
where $Q_m^{(j)} = Q(\eta_m^{(j)})$, and lemma 2.37 asserts that

$$\lim_{m \to \infty} \| \nabla u_m^{(1)} \|_{L^2(|x|>\tilde{R}_m)} = 0, \quad \lim_{m \to \infty} \| \nabla u_m^{(2)} \|_{L^2(|x|<\tilde{S}_m)} = 0.$$

The derivatives $u_{m,x}^{(j)}$, $j = 1, 2$ are weak solutions of the boundary-value problems

$$\nabla \cdot ((I + Q_m^{(j)}) \nabla u_m^{(j)} + G_m^{(j)}) = 0 \quad \text{for } 0 < y < 1,$$

$$(I + Q_m^{(j)}) \nabla u_m^{(j)} - (0, 1) = f(\eta_m^{(j)})_{xx} + G_m^{(j)} \cdot (0, 1), \quad y = 1,$$

$$(I + Q_m^{(j)}) \nabla u_m^{(j)} - (0, -1) = G_m^{(j)} \cdot (0, -1), \quad y = 0,$$

where $G_m^{(j)} = -Q_m^{(j)} \nabla u_m^{(j)}$. Using remark 2.27 and writing $S_m^{(j)} = S_0(\eta_m^{(j)})$, $R_m^{(j)} = R_0(\eta_m^{(j)})$, $R_m^{(j)} = R_1(\eta_m^{(j)})$, one finds that

$$\| Q_m^{(j)} \nabla u_m^{(1)} \|_{L^2(|x|>\tilde{R}_m)} \leq \| S_m^{(1)} \|_{\infty} \| \nabla u_m^{(1)} \|_{L^2(|x|>\tilde{R}_m)} + c(\| R_m^{(1)} \|_{\infty} \| L_m^{j} \|_{\infty} + \| R_m^{(1)} \|_{\infty} \| L_m^{j} \|_{\infty}) \times \| (\eta_m^{(1)})'' \|_{\infty} \| \nabla u_m^{(1)} \|_{L^2(|x|>\tilde{R}_m)} \| \nabla u_m^{(2)} \|_{L^2(|x|>\tilde{R}_m)}^{1/2} \| \nabla u_m^{(1)} \|_{L^2(|x|>\tilde{R}_m)}^{1/2} = o(1) \quad (2.31)$$

as $m \to \infty$. (Lemma 2.25 asserts that $\{ \nabla u_m^{(j)} \} \subseteq H^{3/2,1}$ and hence $\{ \nabla u_m^{(j)} \} \subseteq H^1(\Sigma)$ is bounded; it follows that $\| \nabla u_m^{(1)} \|_{H^1(|x|>\tilde{R}_m)} = O(1)$ as $m \to \infty$.) A similar calculation shows that

$$\| Q_m^{(j)} \nabla u_m^{(2)} \|_{L^2(|x|<\tilde{S}_m)} = o(1) \quad \text{as } m \to \infty,$$

and lemma 2.37 yields the estimates

$$\lim_{m \to \infty} \| \nabla u_{m,x}^{(1)} \|_{L^2(|x|>T_m)} = 0, \quad \lim_{m \to \infty} \| \nabla u_{m,x}^{(2)} \|_{L^2(|x|<T_m)} = 0.$$

The calculation

$$u_m^{(j)} = -\frac{1}{1 + q_{m,22}^{(j)}} (\partial_x [(1 + q_{m,11}^{(j)}) u_{m,x} + q_{m,12}^{(j)} u_{m,y}] + \partial_y (q_{m,12}^{(j)} u_{m,x} - q_{m,22}^{(j)} u_{m,y}))$$

and estimates

$$\| q_m^{(1)} \nabla u_m^{(1)} \|_{L^2(|x|>T_m)} \leq \| q_m^{(1)} \|_{\infty} \| \nabla u_m^{(1)} \|_{L^2(|x|>T_m)} = o(1),$$

$$\left\| \begin{array}{c} \partial_x \\ \partial_y \end{array} \right\| q_m^{(1)} \nabla u_m^{(1)} \left\|_{L^2(|x|>T_m)} = o(1) \right.$$
Lemma 2.40 states another useful application of lemma 2.37 to the boundary-value problem (2.7)–(2.9); the following proposition is used in its proof.

**Proposition 2.39.** Choose \( N \in \mathbb{N} \). The estimates

\[
|Q(\eta_m^{(1)} + \eta_m^{(2)}) - Q(\eta_m^{(1)})| \leq c \text{dist}(x, [-R_m, R_m])^{-N}
\]

and

\[
|Q(\eta_m^{(1)} + \eta_m^{(2)}) - Q(\eta_m^{(1)})| \leq c \text{dist}(x, \mathbb{R} \setminus (-S_m, S_m))^{-N}
\]

hold for all \((x, y) \in \Sigma\), where \(| \cdot |\) denotes the 2 \times 2 matrix maximum norm, and remain valid when \( Q \) is replaced by \( Q_x \) or \( Q_y \).

**Proof.** Observe that

\[
\eta^\delta(x, y') = \frac{1}{1-y} \int_{\text{supp } \eta} K \left( \frac{x-s}{1-y} \right) \eta(s) \, ds,
\]

where \( K = (2\pi)^{-1/2} \delta^{-1} \mathcal{F}^{-1} |x| \in \mathcal{S}(\mathbb{R}) \). The above equation shows that \( \eta^\delta \in C^\infty(\Sigma \setminus \text{supp } \times \{1\}) \) with

\[
|\partial_x^p \partial_y^q \eta^\delta(x, y)| \leq c \text{dist}(x, \text{supp } \eta)^{-N} \| \eta \|_\infty
\]

for each \( N \in \mathbb{N} \).

Note that

\[
|(Q(\eta_1 + \eta_2) - Q(\eta_2))(x, y)| = \left| \begin{vmatrix} f_{1y} & -f_{1x} \\ -f_{3y} & 1 + f_{3y}^2 \\ -f_{2y} & 1 + f_{2y}^2 \end{vmatrix} \right|(x, y)
\]

\[
\leq c |(f_{1x}, f_{1y})|(x, y)
\]

for all \( \eta_1 \), \( \eta_2 \) and \( \eta_3 = \eta_1 + \eta_2 \in U \). It follows that

\[
|(Q(\eta_m^{(1)} + \eta_m^{(2)}) - Q(\eta_m^{(1)}))(x, y)| \leq c |(\eta_m^{(1)})^\delta(x, y), (\eta_m^{(1)})^\delta(x, y), (\eta_m^{(1)})^\delta(x, y)|
\]

\[
\leq c \text{dist}(x, [-R_m, R_m])^{-N}.
\]

The same argument yields the estimate for \( Q(\eta_m^{(1)} + \eta_m^{(2)}) - Q(\eta_m^{(1)}) \) and the corresponding results for \( Q_x \) and \( Q_y \).

**Lemma 2.40.** Let \( u(\eta) \) be the solution to (2.7)–(2.9) with \( \xi = \partial_x f(\eta), \eta \in U \), where \( f \) is a real polynomial. The estimates

\[
\lim_{m \to \infty} \| \nabla u(\eta_m^{(1)} + \eta_m^{(2)}) - \nabla u(\eta_m^{(1)}) \|_{H^1(|x|<T_m)} = 0,
\]

\[
\lim_{m \to \infty} \| \nabla u(\eta_m^{(1)} + \eta_m^{(2)}) - \nabla u(\eta_m^{(2)}) \|_{H^1(|x|>T_m)} = 0
\]

hold for each sequence \( \{T_m\} \) of positive real numbers with \( S_m - T_m, T_m - R_m \to \infty \) as \( m \to \infty \).

**Proof.** Choose sequences \( \{R_m\}, \{S_m\} \) of positive real numbers with \( S_m - \tilde{S}_m, \tilde{S}_m - T_m \to \infty \) and \( T_m - R_m, R_m - R_m \to \infty \) as \( m \to \infty \). The quantities \( w_m^{(1)} = \)
using the method given above, one finds that (see proposition 2.39), one finds that a similar argument yields

\[ (I + Q_m) \nabla w_m^{(j)} \cdot (0, 1) = f(\eta_m^{(j)}) + C_m^{(j)} \cdot (0, 1), \quad y = 1, \]

\[ (I + Q_m) \nabla w_m^{(j)} \cdot (0, -1) = C_m^{(j)} \cdot (0, -1), \quad y = 0, \]

where \( Q_m = Q(\eta_m^{(1)} + \eta_m^{(2)}) \) and

\[ C_m^{(1)} = (Q_m^{(2)} - Q_m) \nabla u_m^{(2)}, \quad C_m^{(2)} = (Q_m^{(1)} - Q_m) \nabla u_m^{(1)}. \]

Using the estimate

\[ |(Q_m^{(2)} - Q_m)(x, y)| \leq c \text{dist}(x, [-R_m, R_m])^{-N} \]

(see proposition 2.39), one finds that

\[ \|C_m^{(1)}\|_{L^2(|x| > R_m)}^2 \leq c(R_m - R_m)^{-N} \|\nabla u_m^{(2)}\|_0^2 \leq c(R_m - R_m)^{-N} \|f(\eta_m^{(2)})\|_{1/2}^2 = o(1) \]

as \( m \to \infty \) and a similar argument shows that \( \|C_m^{(2)}\|_{L^2(|x| < S_m)}^2 = o(1) \) as \( m \to \infty \). It follows from lemma 2.37 that

\[ \lim_{m \to \infty} \|u_m^{(1)}\|_{L^2(|x| > T_m)} = 0, \quad \lim_{m \to \infty} \|u_m^{(2)}\|_{L^2(|x| < T_m)} = 0. \]

The derivatives \( u_m^{(j)}, j = 1, 2 \), are weak solutions of the boundary-value problems

\[ \nabla \cdot ((I + Q_m^{(j)}) \nabla u_m^{(j)} + (Q_m^{(2)} - Q_m) \nabla u_m^{(2)}), \quad 0 < y < 1, \]

\[ (I + Q_m^{(j)}) \nabla u_m^{(j)} \cdot (0, 1) = \partial_x f(\eta_m^{(j)}) + H_m^{(j)} \cdot (0, 1), \quad y = 1, \]

\[ (I + Q_m^{(j)}) \nabla u_m^{(j)} \cdot (0, -1) = H_m^{(j)} \cdot (0, -1), \quad y = 0, \]

where

\[ H_m^{(1)} = -Q_m \nabla u_m^{(1)} + (Q_m^{(2)} - Q_m) \nabla u_m^{(2)} + (Q_m^{(2)} - Q_m) \nabla u_m^{(2)}, \]

\[ H_m^{(2)} = -Q_m \nabla u_m^{(2)} + (Q_m^{(1)} - Q_m) \nabla u_m^{(1)} + (Q_m^{(1)} - Q_m) \nabla u_m^{(1)}. \]

Treating \( \|Q_m \nabla u_m^{(1)}\|_{L^2(|x| > R_m)} \) using the method given in the proof of lemma 2.38 (see estimate (2.31)) and treating

\[ \|Q_m^{(2)} - Q_m\|_{L^2(|x| > R_m)} \]

using the method given above, one finds that \( \|H_m^{(1)}\|_{L^2(|x| > R_m)} = o(1) \) as \( m \to \infty \). A similar argument yields \( \|H_m^{(2)}\|_{L^2(|x| < S_m)} = o(1) \) as \( m \to \infty \) and it follows from lemma 2.37 that

\[ \lim_{m \to \infty} \|\nabla u_m^{(1)}\|_{L^2(|x| > T_m)} = 0, \quad \lim_{m \to \infty} \|\nabla u_m^{(2)}\|_{L^2(|x| < T_m)} = 0. \]

Finally, observe that

\[ u_m^{(1)} = \frac{1}{1 + q_m^{(1)}} \left( \partial_x [(1 + q_m^{(1)}) u_m^{(1)} + q_m^{(1)} w_m^{(1)}] + \partial_y (q_m^{(1)} w_m^{(1)}) \right) \]

\[ - q_m^{(2)} w_m^{(1)} + \nabla (Q_m^{(2)} - Q_m) \cdot \nabla u_m^{(1)} + (Q_m^{(2)} - Q_m) \Delta u_m^{(1)}. \]
The argument given in the proof of lemma 2.38 shows that
\[ \|\partial_x[(1 + q_m^{(1)}w_m^{(1)} + q_m^{(1)}w_m^{(1)} + \partial_y(q_m^{(1)}w_m^{(1)} - q_m^{(1)}w_m^{(1)}w_m^{(1)})]_{L^2(|x| > T_m)} = o(1), \]
and the method given above shows that
\[ \|\nabla(Q_m^{(2)}(2) - Q_m^{(2)}(2))\cdot \nabla u_m^{(1)}\|_{L^2(|x| > T_m)} , \| (Q_m^{(2)}(2) - Q_m^{(2)}(2))\Delta u_m^{(1)}\|_{L^2(|x| > T_m)} = o(1) \]
as \( m \to \infty \). One concludes that
\[ \lim_{m \to \infty} \| w_m^{(1)} \|_{L^2(|x| > T_m)} = 0, \]
and the complementary limit
\[ \lim_{m \to \infty} \| w_m^{(2)} \|_{L^2(|x| < T_m)} = 0 \]
is obtained in a similar fashion. \( \square \)

**Corollary 2.41.** The estimate
\[ \lim_{m \to \infty} \| \nabla u(\eta_m^{(1)} + \eta_m^{(2)}) - \nabla u(\eta_m^{(1)}) - \nabla u(\eta_m^{(2)}) \|_1 = 0 \]
holds under the hypotheses of lemmas 2.38 and 2.40.

The proof of theorem 2.36 is completed by applying the next lemma to the equation for \( T' \) given in corollary 2.32.

**Lemma 2.42.**

(i) **The estimates**
\[ \lim_{m \to \infty} \| f_1(\eta_m^{(1)} + \eta_m^{(2)})K(\eta_m^{(1)} + \eta_m^{(2)})f_2(\eta_m^{(1)} + \eta_m^{(2)}) - f_1(\eta_m^{(1)})K(\eta_m^{(1)})f_2(\eta_m^{(1)}) - f_1(\eta_m^{(2)})K(\eta_m^{(2)})f_2(\eta_m^{(2)}) \|_0 = 0 \]
and
\[ \lim_{m \to \infty} \| f_1(\eta_m^{(1)} + \eta_m^{(2)})K(\eta_m^{(1)} + \eta_m^{(2)})f_2(\eta_m^{(1)} + \eta_m^{(2)}) - f_1(\eta_m^{(1)})K(\eta_m^{(1)})f_2(\eta_m^{(1)}) - f_1(\eta_m^{(2)})K(\eta_m^{(2)})f_2(\eta_m^{(2)}) \|_{L^1(\mathbb{R})} = 0 \]
hold for all real polynomials \( f_1, f_2 \).

(ii) **The estimate**
\[ \lim_{m \to \infty} \| H'(\eta_m^{(1)} + \eta_m^{(2)})(f_1(\eta_m^{(1)} + \eta_m^{(2)}), f_2(\eta_m^{(1)} + \eta_m^{(2)})) - H'(\eta_m^{(1)})(f_1(\eta_m^{(1)}), f_2(\eta_m^{(1)})) - H'(\eta_m^{(2)})(f_1(\eta_m^{(2)}), f_2(\eta_m^{(2)})) \|_0 = 0 \]
holds for all real polynomials \( f_1, f_2 \).

(iii) **The estimate**
\[ \lim_{m \to \infty} \| H'(\eta_m^{(1)})(f_1(\eta_m^{(1)}), f_2(\eta_m^{(1)})), \eta_m^{(2)} \|_0 = 0 \]
holds for all real polynomials \( f_1, f_2 \).
Proof. (i) Observe that

\[ f_1(\eta_m^{(1)} + \eta_m^{(2)}) K(\eta_m^{(1)} + \eta_m^{(2)}), f_2(\eta_m^{(1)} + \eta_m^{(2)}) \]

\[ - f_1(\eta_m^{(1)} + \eta_m^{(2)}) K(\eta_m^{(1)} + \eta_m^{(2)}), f_2(\eta_m^{(2)}) K(\eta_m^{(2)}) f_2(\eta_m^{(2)}) \]

\[ = f_1(\eta_m^{(1)})(u_x(\eta_m^{(1)} + \eta_m^{(2)}) - u_x(\eta_m^{(1)})) + f_2(\eta_m^{(2)})(u_x(\eta_m^{(1)} + \eta_m^{(2)}) - u_x(\eta_m^{(2)})). \]

The \( L^1(\mathbb{R}) \)- and \( L^2(\mathbb{R}) \)-norms of this quantity can both be estimated by

\[ \| f_1(\eta_m^{(1)}) \|_1 \| u_x(\eta_m^{(1)} + \eta_m^{(2)}) - u_x(\eta_m^{(1)}) \|_{L^2(|x|<R_m)} \]

\[ + \| f_2(\eta_m^{(2)}) \|_1 \| u_x(\eta_m^{(1)} + \eta_m^{(2)}) - u_x(\eta_m^{(2)}) \|_{L^2(|x|>S_m)} \]

\[ \leq \| f_1(\eta_m^{(1)}) \|_1 \| \nabla u(\eta_m^{(1)} + \eta_m^{(2)}) - \nabla u(\eta_m^{(1)}) \|_{H^1(|x|<T_m)} \]

\[ O(1) \]

\[ + \| f_2(\eta_m^{(2)}) \|_1 \| \nabla u(\eta_m^{(1)} + \eta_m^{(2)}) - \nabla u(\eta_m^{(2)}) \|_{H^1(|x|>T_m)} \]

\[ O(1) \]

\[ = o(1) \]

(use the Cauchy–Schwarz inequality or the maximum norm for the polynomials).

(ii) Observe that

\[ \mathcal{H}'(\eta_m^{(1)} + \eta_m^{(2)}), f_1(\eta_m^{(1)} + \eta_m^{(2)}) \]

\[ - \mathcal{H}'(\eta_m^{(1)}), f_2(\eta_m^{(1)}) - \mathcal{H}'(\eta_m^{(2)}), f_2(\eta_m^{(2)}) \]

\[ = -u_x(\eta_m^{(1)} + \eta_m^{(2)}) v_x(\eta_m^{(1)} + \eta_m^{(2)}) + u_x(\eta_m^{(1)} + \eta_m^{(2)}) v_x(\eta_m^{(1)}) + u_x(\eta_m^{(2)}) v_x(\eta_m^{(2)}) \]

\[ + u_y(\eta_m^{(1)} + \eta_m^{(2)}) v_y(\eta_m^{(1)} + \eta_m^{(2)}) - u_y(\eta_m^{(1)} + \eta_m^{(2)}) v_y(\eta_m^{(1)}) - u_y(\eta_m^{(2)}) v_y(\eta_m^{(2)}) \]

\[ + h(\eta_m^{(1)} + \eta_m^{(2)}) u_y(\eta_m^{(1)} + \eta_m^{(2)}) v_y(\eta_m^{(1)} + \eta_m^{(2)}) \]

\[ - h(\eta_m^{(1)}), u_y(\eta_m^{(1)}) v_y(\eta_m^{(1)}) - h(\eta_m^{(2)}) u_y(\eta_m^{(2)}) v_y(\eta_m^{(2)}) \]

where

\[ h(\eta) = \frac{\eta^2 - \eta^2 - 2\eta}{(1 + \eta)^2} \]

and \( u(\eta), v(\eta) \) are the solutions to (2.7)–(2.9) with \( \xi = \partial_x f_1(\eta), \eta \in U \), and

\[ \xi = \partial_x f_2(\eta), \eta \in U, \text{ respectively.} \]

The estimates

\[ \| u_x(\eta_m^{(1)} + \eta_m^{(2)}) v_x(\eta_m^{(1)} + \eta_m^{(2)}) - (u_x(\eta_m^{(1)} + \eta_m^{(2)}))(v_x(\eta_m^{(1)} + \eta_m^{(2)})) \|_{y=1} \]

\[ \leq \| v_x(\eta_m^{(1)} + \eta_m^{(2)}) \|_{y=1} \| u_x(\eta_m^{(1)} + \eta_m^{(2)}) - u_x(\eta_m^{(1)}) \|_{y=1} \| u_x(\eta_m^{(2)}) \|_{y=1} \]

\[ = O(1) \]

\[ + \| u_x(\eta_m^{(1)} + \eta_m^{(2)}) \|_{y=1} \| v_x(\eta_m^{(1)} + \eta_m^{(2)}) - v_x(\eta_m^{(1)}) \|_{y=1} \| v_x(\eta_m^{(2)}) \|_{y=1} \]

\[ = o(1) \]

\[ = o(1) \]
and
\[
\|u_x(\eta_m^{(1)}) + u_x(\eta_m^{(2)})\|_y + v_x(\eta_m^{(1)}) + v_x(\eta_m^{(2)})
- u_x(\eta_m^{(1)})v_x(\eta_m^{(1)}) - u_x(\eta_m^{(2)})v_x(\eta_m^{(2)})|_y=1\|_0
\leq \|u_x(\eta_m^{(1)})v_x(\eta_m^{(2)})\|_y=1\|_0 + \|u_x(\eta_m^{(1)})v_x(\eta_m^{(2)})\|_y=1\|_0
\leq c\left(\|u_x(\eta_m^{(1)})\|_y=1\|_L^2(|x|<T_m)\|v_x(\eta_m^{(2)})\|_y=1\|_1\right)
= O(1)
\]
\[
+ \|u_x(\eta_m^{(1)})\|_y=1\|1\|v_x(\eta_m^{(2)})\|_y=1\|L^2(|x|<T_m)\|_1
= o(1)
\]
\[
+ \|u_x(\eta_m^{(2)})\|_y=1\|L^2(|x|<T_m)\|v_x(\eta_m^{(1)})\|_y=1\|_1
= o(1)
\]
\[
+ \|u_x(\eta_m^{(2)})\|_y=1\|1\|v_x(\eta_m^{(2)})\|_y=1\|L^2(|x|>T_m)\|_1
= O(1)
\]
\[
= o(1)
\]

imply that
\[
\|\eta_m^{(1)} + \eta_m^{(2)}\|_y=1\|1\|\eta_m^{(1)} + \eta_m^{(2)}\|_1\|1\|\eta_m^{(1)} + \eta_m^{(2)}\|_y=1\|_0 = o(1)
\]
as \(m \to \infty\); here we have used the estimate
\[
\|u_x(\eta)|_y=1\|_1 \leq c\|\nabla u\|_3/2,1 \leq c\|f_1(\eta)\|_2, \quad \eta \in U,
\]
and its counterpart for \(v\). The same argument shows that
\[
\|v(\eta_m^{(1)}) + v(\eta_m^{(2)})\|_y=1\|1\|v(\eta_m^{(1)}) + v(\eta_m^{(2)})\|_1\|1\|v(\eta_m^{(1)}) + v(\eta_m^{(2)})\|_y=1\|_0 = o(1)
\]
as \(m \to \infty\).

Because \(h(\eta_m^{(1)} + \eta_m^{(2)}) = h(\eta_m^{(1)}) + h(\eta_m^{(2)})\) and
\[
\|u_y(\eta_m^{(1)} + \eta_m^{(2)})\|_y=1\|1\|u_y(\eta_m^{(1)} + \eta_m^{(2)})\|_1\|1\|u_y(\eta_m^{(1)} + \eta_m^{(2)})\|_y=1\|_0 = o(1)
\]
as \(m \to \infty\) (see above), repeating the proof of part (i) yields the estimate
\[
\|h(\eta_m^{(1)} + \eta_m^{(2)})u_y(\eta_m^{(1)} + \eta_m^{(2)})v_y(\eta_m^{(1)} + \eta_m^{(2)})\|_y=1\|1\|h(\eta_m^{(1)} + \eta_m^{(2)})u_y(\eta_m^{(1)} + \eta_m^{(2)})v_y(\eta_m^{(1)} + \eta_m^{(2)})\|_y=1\|_0 = o(1)
\]
as \(m \to \infty\).

(iii) The methods used in part (ii) show that
\[
\|\mathcal{H}'(\eta_m^{(1)})(f_1(\eta_m^{(1)}), f_2(\eta_m^{(1)}))\|_L^2(|x|>T_m) = o(1),
\]
so that

$$
|\langle H'(\eta_m^{(1)})(f_1(\eta_m^{(1)}), f_2(\eta_m^{(1)})), \eta_m^{(2)}\rangle_0| \\
\leq \|H'(\eta_m^{(1)})(f_1(\eta_m^{(1)}), f_2(\eta_m^{(1)}))\|_{L^2(|x| > S_m)} \|\eta_m^{(2)}\|_0
$$

$$
= O(1) = o(1)

\to 0
$$
as \ m \to \infty.

3. Minimizing sequences

The goal of this section is the proof of the following theorem, the existence of the sequence advertised in which is a key ingredient in the proof that the infimum of J\_\(U\) minimizes over \(U\) is in turn used to establish the convergence (up to subsequences and translations) of any minimizing sequence for J\_\(U\) over \(U \setminus \{0\}\) that does not approach the boundary of \(U\).

**Theorem 3.1.** There exists a minimizing sequence \(\{\tilde{\eta}_m\}\) for \(J_{\mu}\) over \(U \setminus \{0\}\) with the properties that \(\|\tilde{\eta}_m\|^2 \leq c\mu\) for each \(m \in \mathbb{N}\) and \(\lim_{m \to \infty} \|J'_\mu(\tilde{\eta}_m)\|_0 = 0\).

3.1. The penalized minimization problem

We begin by studying the functional \(J_{p,\mu} : H^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}\) defined by

$$
J_{p,\mu}(\eta) = \begin{cases} 
K(\eta) + \frac{(\mu + G(\eta))^2}{\mathcal{L}(\eta)} + \rho(\|\eta\|^2_2), & \eta \in U \setminus \{0\}, \\
\infty, & \eta \notin U \setminus \{0\},
\end{cases}
$$
in which \(\rho : [0, M^2] \to \mathbb{R}\) is a smooth increasing ‘penalization’ function such that \(\rho(t) = 0\) for \(0 \leq t \leq M^2\) and \(\rho(t) \to \infty\) as \(t \uparrow M^2\). We allow negative values of the small parameter, so that \(0 < |\mu| < \mu_0\) (see the comments below lemma 3.8) and the number \(\tilde{M} \in (0, M)\) is chosen so that

$$
\tilde{M}^2 > (c^* + Dv_0 + Dv_0^{-})|\mu|;
$$
the following analysis is valid for every such choice of \(\tilde{M}\), which, in particular, may be chosen arbitrarily close to \(M\). In this inequality \(v_0\) and \(v_0^{-}\) are the speeds of linear waves with frequency \(k_0\) riding shear flows with vorticities \(\omega\) and \(-\omega\) and \(c^*, D\) are constants identified in lemmas 3.2(i) and 3.3. In §3.2 we give a detailed description of the qualitative properties of an arbitrary minimizing sequence \(\{\eta_m\}\) for \(J_{p,\mu}\); the penalization function ensures that \(\{\eta_m\}\) does not approach the boundary of the set \(U \setminus \{0\}\), in which \(J_{\mu}\) is defined.

We first give some useful \textit{a priori} estimates. Lemma 3.2(i) shows in particular that

$$
c_{p,\mu} := \inf J_{p,\mu} < 2\nu_0^0 |\mu| - c|\mu|^{-}, \quad c_{\mu} := \inf_{\eta \in U \setminus \{0\}} J_{\mu}(\eta) < 2\nu_0^0 |\mu| - c|\mu|^{-},
$$

where \(\nu_0^0\) is the speed of linear waves with frequency \(k_0\) riding a shear flow with vorticity \((\text{sgn} \mu)\omega\) (which depends only upon the sign of \(\mu\)), while lemma 3.3, whose
Lemma 3.2.

(i) There exists \( \eta^* \in U \setminus \{0\} \) with compact support and a positive constant \( c^* \) such that \( \|\eta^*_\|^2 \leq c^*|\mu|^{1/2} \), \( \rho(\|\eta^*_\|^2) = 0 \) and

\[
J_{\rho, \mu}(\eta^*_\) = J_{\mu}(\eta^*) < 2\nu^\rho_0|\mu| - c|\mu|^{r^*}, \quad r^* = \begin{cases} \frac{5}{3}, & \beta > \beta_c, \\ 3, & \beta < \beta_c. \end{cases}
\]

(ii) The inequality

\[
K_2(\eta) + \frac{(\mu + G_2(\eta))^2}{L_2(\eta)} \geq 2\nu^\rho_0|\mu|
\]

holds for each \( \eta \in H^2(\mathbb{R}) \setminus \{0\} \).

Proof. First suppose that \( \mu > 0 \). The proof of part (i) is recorded in Appendix A, while part (ii) follows from the calculation

\[
K_2(\eta) + \frac{(\mu + G_2(\eta))^2}{L_2(\eta)} = K_2(\eta) + 2\nu_0G_2(\eta) - \nu_0^2L_2(\eta) + \frac{(\mu + G_2(\eta) - \nu_0L_2(\eta))^2}{L_2(\eta)} + 2\nu_0\mu
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} g(\eta)\hat{\eta}^2 + \frac{(\mu + G_2(\eta) - \nu_0L_2(\eta))^2}{L_2(\eta)} + 2\nu_0\mu
\]

\[
\geq 2\nu_0\mu.
\]

For \( \mu < 0 \) we observe that \( J_\mu(\eta), J_{\rho, \mu}(\eta) \) and \( K_2(\eta) + (\mu + G_2(\eta))^2/L_2(\eta) \) are invariant under the transformation \( (\mu, \omega) \mapsto (-\mu, -\omega) \). \( \square \)

Lemma 3.3. Suppose that \( \gamma_1 \) and \( \gamma_2 \) belong to a bounded set of real numbers. Any critical point \( \eta \) of the functional \( \tilde{J}_\gamma : U \to \mathbb{R} \) defined by

\[
\tilde{J}_\gamma(\eta) = K(\eta) - \gamma_1G(\eta) - \gamma_2L(\eta) + \gamma_3\|\eta\|^2, \quad \gamma_3 \geq 0,
\]

satisfies the estimate

\[
\|\eta\|^2 \leq DK(\eta),
\]

where \( D \) is a positive constant that does not depend upon \( \gamma_1, \gamma_2 \) or \( \gamma_3 \).

Corollary 3.4. Any critical point \( \eta \) of \( J_{\rho, \mu} \) with \( J_{\rho, \mu}(\eta) < 2\nu^\rho_0|\mu| \) satisfies

\[
\|\eta\|^2 \leq 2\nu^\rho_0|\mu|, \quad \rho(\|\eta\|^2) = 0.
\]

Proof. Notice that any critical point \( \eta \) of \( J_{\rho, \mu} \) is also a critical point of the functional \( \tilde{J}_\gamma \), where

\[
\gamma_1 = -\frac{2(\mu + G(\eta)}{L(\eta)}, \quad \gamma_2 = \frac{(\mu + G(\eta))^2}{L(\eta)^2}, \quad \gamma_3 = 2\rho'(\|\eta\|^2).
\]
Furthermore, any function $\eta \in U$ such that

$$\frac{(\mu + G(\eta))^2}{L(\eta)} \leq 2\nu_0^2 |\mu|$$

satisfies

$$\frac{\mu^2}{L(\eta)} \leq 2\nu_0^2 |\mu| - 2\frac{\mu G(\eta)}{L(\eta)} - \frac{G(\eta)^2}{L(\eta)} \leq 2\nu_0^2 |\mu| + 2\frac{\mu |G(\eta)|}{L(\eta)} \leq c|\mu|$$

(see proposition 2.35), so that

$$|\mu| \frac{L(\eta)}{L(\eta)} \leq c.$$  \hspace{1cm} (3.1)

Observing that

$$\frac{(\mu + G(\eta))^2}{L(\eta)} \leq J_{\rho,\mu}(\eta) \leq 2\nu_0^2 |\mu|,$$

we find from proposition 2.35 and inequality (3.1) that $\gamma_1$ and $\gamma_2$ are bounded. The previous lemma shows that $\|\eta\|_2^2 \leq DK(\eta) \leq D J_{\rho,\mu}(\eta) < 2D\nu_0^2 |\mu|$, and hence $\rho(\|\eta\|_2^2) = 0$ because of the choice of $M$.

Finally, we establish some basic properties of a minimizing sequence $\{\eta_m\}$ for $J_{\rho,\mu}$. Without loss of generality we may assume that

$$\sup_{m \in \mathbb{N}} \|\eta_m\|_2 < M$$

($\|\eta_m\|_2 \to M$ would imply that $J_{\rho,\mu}(\eta_m) \to \infty$) and it follows that $\{\eta_m\}$ admits a subsequence such that $J_{\rho,\mu}(\eta_m) \to \infty$ and $\rho(\|\eta\|_2^2) = 0$ because of the choice of $M$.

**Lemma 3.5.** Every minimizing sequence $\{\eta_m\}$ for $J_{\rho,\mu}$ has the properties that

$$J_{\rho,\mu}(\eta_m) < 2\nu_0^2 |\mu| - c|\mu|^{r^*}, \quad L(\eta_m) \geq c|\mu|, \quad L_2(\eta_m) \geq c|\mu|,$$

$$M_{\rho,\mu}(\eta_m) \leq -c|\mu|^{r^*}, \quad \|\eta_m\|_{1,\infty} \geq c|\mu|^{r^*}$$

for each $m \in \mathbb{N}$, where

$$M_{\rho,\mu}(\eta) = J_{\rho,\mu}(\eta) - K_2(\eta) - \frac{(\mu + G(\eta))^2}{L_2(\eta)}.$$

**Proof.** The first and second estimates are obtained from lemma 3.2(i) and the remark leading to (3.1), while the third is a consequence of the calculation

$$c\|\eta\|_{1/2}^2 \leq \left\{ \begin{array}{l} L_2(\eta) \\ L(\eta) \end{array} \right\} \leq c\|\eta\|_{1/2}^2, \quad \eta \in U.$$  \hspace{1cm} (3.2)

Turning to the fourth estimate, observe that

$$M_{\rho,\mu}(\eta_m) \leq J_{\rho,\mu}(\eta_m) - 2\nu_0^2 |\mu| \leq -c|\mu|^{r^*}.$$
because
\[ \mathcal{K}_2(\eta) + \frac{(\mu + G_2(\eta))^2}{\mathcal{L}_2(\eta)} \geq 2\nu^0|\mu| \]
(see lemma 3.2(ii)).

Finally, it follows from the calculation
\[ \mathcal{M}_{\rho,\mu}(\eta_m) - \rho(\|\eta_m\|_2^2) \]
\[ = \mathcal{K}_{nl}(\eta_m) - \frac{\mu^2\mathcal{L}_{nl}(\eta_m)}{\mathcal{L}(\eta_m)\mathcal{L}_2(\eta_m)} - \frac{2\mu\mathcal{G}(\eta_m)\mathcal{L}_{nl}(\eta_m)}{\mathcal{L}(\eta_m)\mathcal{L}_2(\eta_m)} + \frac{2\mu\mathcal{G}_{nl}(\eta_m)}{\mathcal{L}(\eta_m)} + \frac{\mathcal{G}_2(\eta_m)\mathcal{G}_{nl}(\eta_m)}{\mathcal{L}(\eta_m)} + \frac{(\mathcal{G}(\eta_m) + \mathcal{G}_2(\eta_m))\mathcal{G}_{nl}(\eta_m)}{\mathcal{L}(\eta_m)}, \]
the inequalities
\[ |\mathcal{G}_2(\eta_m)|, |\mathcal{G}(\eta_m)| \leq c\|\eta_m\|_{1/2}, \]
\[ |\mathcal{G}_{nl}(\eta_m)|, |\mathcal{K}_{nl}(\eta_m)| \leq c\|\eta_m\|_{1,\infty}, \]
\[ |\mathcal{L}_{nl}(\eta_m)| \leq c\|\eta_m\|_{1,\infty}\|\eta_m\|_1^{1/2} \]
and (3.2) that
\[ |\mathcal{M}_{\rho,\mu}(\eta_m) - \rho(\|\eta_m\|_2^2)| \leq c(\|\eta_m\|_1). \]

The fifth estimate is obtained from this result and the fact that
\[ \mathcal{M}_{\rho,\mu}(\eta_m) - \rho(\|\eta_m\|_2^2) \leq -c|\mu|^r. \]

\[ \square \]

**Remark 3.6.** Replacing \( \mathcal{J}_{\rho,\mu}(\eta) \) by \( \mathcal{J}_{\mu}(\eta) \) and \( \mathcal{M}_{\rho,\mu}(\eta) \) by
\[ \mathcal{M}_{\mu}(\eta) := \mathcal{J}_{\mu}(\eta) - \mathcal{K}_2(\eta) - \frac{(\mu + G_2(\eta))^2}{\mathcal{L}_2(\eta)} \]
in its statement, one finds that lemma 3.5 is also valid for a minimizing sequence \( \{\eta_m\} \) for \( \mathcal{J}_{\mu} \) over \( U \setminus \{0\} \).

### 3.2. Minimizing sequences for the penalized problem

**3.2.1. Application of the concentration-compactness principle**

The next step is to perform a more detailed analysis of the behaviour of a minimizing sequence \( \{\eta_m\} \) for \( \mathcal{J}_{\mu} \) by applying the concentration-compactness principle (see Lions \[21,22\]); theorem 3.7 states this result in a form suitable for the present situation.

**Theorem 3.7.** Any sequence \( \{u_m\} \subset L^1(\mathbb{R}) \) of non-negative functions with the property that
\[ \lim_{m \to \infty} \int_{-\infty}^{\infty} u_m(x) \, dx = \ell > 0 \]
admits a subsequence for which precisely one of the following phenomena occurs.
Vanishing: for each $r > 0$ one has that
\[
\lim_{m \to \infty} \left( \sup_{\tilde{x} \in \mathbb{R}} \int_{\tilde{x}-r}^{\tilde{x}+r} u_m(x) \, dx \right) = 0.
\]

Concentration: there is a sequence $\{x_m\} \subset \mathbb{R}$ with the property that for each $\varepsilon > 0$ there exists a positive real number $R$ with
\[
\int_{-R}^{R} u_m(x + x_m) \, dx \geq \ell - \varepsilon
\]
for each $m \in \mathbb{N}$.

Dichotomy: there are sequences $\{x_m\} \subset \mathbb{R}$, $\{M^{(1)}_m\}, \{M^{(2)}_m\} \subset \mathbb{R}$ and a real number $\kappa \in (0, \ell)$ with the properties that
\[
M^{(1)}_m, M^{(2)}_m \to \infty, \quad M^{(1)}_m / M^{(2)}_m \to 0,
\]
\[
\int_{-M^{(1)}_m}^{M^{(1)}_m} u_m(x + x_m) \, dx \to \kappa, \quad \int_{-M^{(2)}_m}^{M^{(2)}_m} u_m(x + x_m) \, dx \to \kappa
\]
as $m \to \infty$. Furthermore,
\[
\lim_{m \to \infty} \left( \sup_{\tilde{x} \in \mathbb{R}} \int_{\tilde{x}-r}^{\tilde{x}+r} u_m(x) \, dx \right) \leq \kappa
\]
for each $r > 0$, and for each $\varepsilon > 0$ there is a positive real number $R$ such that
\[
\int_{-R}^{R} u_m(x + x_m) \, dx \geq \kappa - \varepsilon
\]
for each $m \in \mathbb{N}$.

Standard interpolation inequalities show that the norms $\| \cdot \|_r$ are metrically equivalent on $U$ for $r \in [0, 2)$; we therefore study the convergence properties of $\{\eta_m\}$ in $H^r(\mathbb{R})$ for $r \in [0, 2)$ by focusing on the concrete choice $r = 1$. One may assume that $\|\eta_m\|_1 \to \ell$ as $m \to \infty$, where $\ell > 0$ because $\eta_m \to 0$ in $H^r(\mathbb{R})$ for $r > \frac{3}{2}$ would imply that $J_{\rho, \mu}(\eta_m) \to \infty$. This observation suggests applying theorem 3.7 to the sequence $\{u_m\}$ defined by
\[
u_m = \eta_m^2 + \eta_m^2,
\]
so that $\|u_m\|_{L^1(\mathbb{R})} = \|\eta_m\|_1^2$. The following result deals with ‘vanishing’ and ‘concentration’ (see Buffoni et al. [9, lemmas 3.7 and 3.9]).

**Lemma 3.8.**

(i) The sequence $\{u_m\}$ does not have the ‘vanishing’ property.

(ii) Suppose $\{u_m\}$ has the ‘concentration’ property. The sequence $\{\eta_m(\cdot + x_m)\}$ admits a subsequence, abbreviated, with a slight abuse of notation, to $\{\eta_m\}$, which satisfies
\[
\lim_{m \to \infty} \|\eta_m\|_2 \leq M
\]
and converges in \( H^r(\mathbb{R}) \) for \( r \in [0,2) \) to \( \eta^{(1)} \). The function \( \eta^{(1)} \) satisfies the estimate
\[
\|\eta^{(1)}\|_2^2 \leq DK(\eta^{(1)}) < 2D\nu|\mu|,
\]
minimizes \( J_{\rho,\mu} \) and minimizes \( J_\mu \) over \( \tilde{U} \setminus \{0\} \), where
\[
\tilde{U} = \{ \eta \in H^2(\mathbb{R}) : \|\eta\|_2 < \tilde{M} \}.
\]

We now present the more involved discussion of the remaining case (‘dichotomy’), again abbreviating the subsequence of \( \{\eta_m(\cdot + x_m)\} \) identified by theorem 3.7 to \( \{\eta_m\} \). The analysis is similar to that given by Buffoni et al. [9] in their study of three-dimensional irrotational solitary waves, the main difference being that negative values of \( \mu \) are also considered, so that \( \mu \) is replaced by \( |\mu| \) in estimates (this change is necessary since the numbers \( \mu^{(1)} \) and \( \mu^{(2)} \) appearing in part (iv) of the following lemma, which are later used iteratively, may be negative). We therefore omit proofs that are straightforward modifications of those given by Buffoni et al.; note, however, that references in their paper to Appendix D (in particular theorem D.6) for ‘pseudo-local’ properties of operators should be replaced by references to §2.2.2 (in particular theorem 2.36) here.

Define sequences \( \{\eta_m^{(1)}\} \), \( \{\eta_m^{(2)}\} \) by the formulas
\[
\eta_m^{(1)}(x) = \eta_m(x) \chi\left(\frac{x}{M_m^{(1)}}\right), \quad \eta_m^{(2)}(x) = \eta_m(x) \left(1 - \chi\left(\frac{x}{M_m^{(2)}}\right)\right),
\]
so that
\[
\text{supp } \eta_m^{(1)} \subset [-2M_m^{(1)}, 2M_m^{(1)}], \quad \text{supp } \eta_m^{(2)} \subset \mathbb{R} \setminus (-M_m^{(2)}, M_m^{(2)}).
\]

**Lemma 3.9.**

(i) The sequences \( \{\eta_m\} \), \( \{\eta_m^{(1)}\} \) and \( \{\eta_m^{(2)}\} \) have the limiting behaviour
\[
\|\eta_m^{(1)}\|_2 \to \kappa, \quad \|\eta_m^{(2)}\|_2 \to \ell - \kappa, \quad \|\eta_m - \eta_m^{(1)} - \eta_m^{(2)}\|_2 \to 0
\]
as \( m \to \infty \) and satisfy the bounds
\[
\sup_{m \in \mathbb{N}} \|\eta_m^{(1)}\|_2 < M, \quad \sup_{m \in \mathbb{N}} \|\eta_m^{(2)}\|_2 < M, \quad \sup_{m \in \mathbb{N}} \|\eta_m^{(1)} + \eta_m^{(2)}\|_2 < M.
\]

(ii) The limits \( \lim_{m \to \infty} \mathcal{L}(\eta_m^{(1)}) \) and \( \lim_{m \to \infty} \mathcal{L}(\eta_m^{(2)}) \) are positive.

(iii) The functionals \( \mathcal{G}, \mathcal{K} \) and \( \mathcal{L} \) satisfy
\[
\left\{ \begin{array}{l}
(\mathcal{G}/\mathcal{K}) (\eta_m) - (\mathcal{G}/\mathcal{L}) (\eta_m^{(1)}) - (\mathcal{G}/\mathcal{K}) (\eta_m^{(2)}) \to 0,
\end{array} \right.
\]
\[
\left\| \begin{array}{l}
(\mathcal{G}/\mathcal{K}') (\eta_m) - (\mathcal{G}/\mathcal{L}') (\eta_m^{(1)}) - (\mathcal{G}/\mathcal{K}') (\eta_m^{(2)}) \to 0.
\end{array} \right.
\]
as \( m \to \infty \).
(iv) The sequences \( \{ \eta_m \} \), \( \{ \eta_m^{(1)} \} \) and \( \{ \eta_m^{(2)} \} \) satisfy
\[
\lim_{m \to \infty} \mathcal{J}_\mu(\eta_m) = \lim_{m \to \infty} \mathcal{J}_\mu^{(1)}(\eta_m^{(1)}) + \lim_{m \to \infty} \mathcal{J}_\mu^{(2)}(\eta_m^{(2)}),
\]
and the positive numbers \( \alpha^{(1)} \), \( \alpha^{(2)} \) are defined by
\[
\alpha^{(1)} = \frac{\lim_{m \to \infty} \mathcal{L}(\eta_m^{(1)})}{\lim_{m \to \infty} \mathcal{L}(\eta_m)}, \quad \alpha^{(2)} = \frac{\lim_{m \to \infty} \mathcal{L}(\eta_m^{(2)})}{\lim_{m \to \infty} \mathcal{L}(\eta_m)}.
\]

(v) The sequence \( \{ \eta_m^{(1)} \} \) converges weakly in \( H^2(\mathbb{R}) \), and strongly in \( H^r(\mathbb{R}) \) for \( r \in [0, 2) \), to a function \( \eta^{(1)} \in H^2(\mathbb{R}) \) with \( \| \eta^{(1)} \|_2^2 \leq DK(\eta^{(1)}) \) and \( \| \eta^{(1)} \|_1 \geq c|\mu|^{2r^*}. \)

(vi) The sequence \( \{ \eta_m^{(2)} \} \) is a minimizing sequence for the functional
\[
\mathcal{J}_{\rho_2, \mu^{(2)}} : H^2(\mathbb{R}) \to \mathbb{R} \cup \{ \infty \}
\]
defined by
\[
\mathcal{J}_{\rho_2, \mu^{(2)}}(\eta) = \begin{cases}
\mathcal{K}(\eta) + \frac{(\mu^{(2)} + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)} + \rho_2(\| \eta \|_2^2), & \eta \in U_2 \setminus \{ 0 \}, \\
\infty, & \eta \not\in U_2 \setminus \{ 0 \},
\end{cases}
\]
where
\[
U_2 = \{ \eta \in H^2(\mathbb{R}) : \| \eta \|_2^2 \leq M^2 - \| \eta^{(1)} \|_2^2 \}, \quad \rho_2(\| \eta \|_2^2) = \rho(\| \eta^{(1)} \|_2^2 + \| \eta \|_2^2).
\]

(vii) The sequences \( \{ \eta_m \} \) and \( \{ \eta_m^{(2)} \} \) satisfy
\[
\lim_{m \to \infty} \rho(\| \eta_m \|_2^2) = \lim_{m \to \infty} \rho_2(\| \eta_m^{(2)} \|_2^2),
\]
and
\[
\| \eta^{(1)} \|_2^2 + \lim_{m \to \infty} \| \eta_m^{(2)} \|_2^2 \leq \lim_{m \to \infty} \| \eta_m \|_2^2
\]
with equality if \( \lim_{m \to \infty} \rho(\| \eta_m \|_2^2) > 0 \).

Proof. For part (i), see Buffoni et al. [9, lemma 3.10(i) and (ii)].

Turning to part (ii), observe that \( \mathcal{L}(\eta_m^{(1)}) \to 0 \) as \( m \to \infty \) implies that \( \| \eta_m^{(1)} \|_{1/2} \to 0 \), and hence \( \| \eta_m^{(1)} \|_1 \to 0 \) as \( m \to \infty \), which contradicts part (i). The same argument shows us that \( \mathcal{L}(\eta_m^{(2)}) \not\to 0 \) as \( m \to \infty \). Because the derivative of \( \mathcal{G} \) is bounded on \( U \), we find that
\[
|\mathcal{G}(\eta_m) - \mathcal{G}(\eta_m^{(1)} + \eta_m^{(2)})| \leq c\| \eta_m - \eta_m^{(1)} - \eta_m^{(2)} \|_2 \to 0.
\]
(see part (i)), and therefore that
\[
G(\eta_m) - G(\eta_m^{(1)}) - G(\eta_m^{(2)})
\]
\[
= G(\eta_m) - G(\eta_m^{(1)} + \eta_m^{(2)}) + G(\eta_m^{(1)} + \eta_m^{(2)}) - G(\eta_m^{(1)}) - G(\eta_m^{(2)}) = o(1)
\]
as \(m \to \infty\), in which theorem 2.36 has been used. The same argument applies to \(K\) and \(L\) and establishes part (iii).

Part (iv) follows from part (iii) by a direct calculation (cf. Buffoni et al. [9, corollary 3.11]); for parts (v), (vi) and (vii) see Buffoni et al. [9, lemmas 3.12, 3.15(i) and 3.15(ii)].

3.2.2. Iteration

The next step is to apply the concentration-compactness principle to the sequence \(\{u_{2,m}\}\) given by
\[
u_{2,m} = \eta_{2,m}^2 + \eta_{2,m}^2,
\]
where \(\eta_{2,m} = \eta_{m}^{(2)}\), and repeat the above analysis. We proceed iteratively in this fashion, writing \(\{\eta_m\}\), \(\mu\) and \(U\) in iterative formulas as \(\{\eta_{1,m}\}\), \(\mu_1\) and \(U_1\), respectively. The following lemma describes the result of one step in this procedure (see Buffoni et al. [9, §3.3]).

**Lemma 3.10.** Suppose that there exist functions \(\eta^{(1)}, \ldots, \eta^{(k)} \in H^2(\mathbb{R})\) and a sequence \(\{\eta_{k+1,m}\} \subset H^2(\mathbb{R})\) with the following properties.

(i) The sequence \(\{\eta_{k+1,m}\}\) is a minimizing sequence for
\[
J_{\rho_{k+1},\mu_{k+1}} : H^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}
\]
defined by
\[
J_{\rho_{k+1},\mu_{k+1}}(\eta) = \begin{cases}
K(\eta) + \frac{(\mu_{k+1} + G(\eta))^2}{L(\eta)}, & \eta \in U_{k+1} \setminus \{0\}; \\
\infty, & \eta \notin U_{k+1} \setminus \{0\},
\end{cases}
\]
where
\[
U_{k+1} = \left\{ \eta \in H^2(\mathbb{R}) : \|\eta\|_2^2 \leq M^2 - \sum_{j=1}^{k} \|\eta^{(j)}\|_2^2 \right\}
\]
and
\[
\rho_{k+1}(\|\eta\|_2^2) = \rho\left(\sum_{j=1}^{k} \|\eta^{(j)}\|_2^2 + \|\eta\|_2^2\right),
\]
\[
\mu_{k+1} = \frac{\lim_{m \to \infty} L(\eta_{k+1,m})}{\lim_{m \to \infty} L(\eta_{m})} \left(\mu + \lim_{m \to \infty} G(\eta_{m})\right) = \lim_{m \to \infty} G(\eta_{k+1,m}).
\]

(ii) The functions \(\eta^{(1)}, \ldots, \eta^{(k)}\) satisfy
\[
0 < \|\eta^{(j)}\|_2^2 \leq DK(\eta^{(j)}) \quad j = 1, \ldots, k,
\]
and
\[ c_{p,\mu} = \sum_{j=1}^{k} J_{\mu_j}^{(i)}(\eta^{(j)}) + c_{p_{k+1},\mu_{k+1}}, \]
where
\[ \mu_j^{(1)} = \frac{\mathcal{L}(\eta^{(j)})}{\lim_{m \to \infty} \mathcal{L}(\eta_m)} \left( \mu + \lim_{m \to \infty} \mathcal{G}(\eta_m) \right) - \lim_{m \to \infty} \mathcal{G}(\eta^{(j)}), \quad j = 1, \ldots, k, \]
and \( c_{p_{k+1},\mu_{k+1}} = \inf J_{p_{k+1},\mu_{k+1}} \).

(iii) The sequences \( \{\eta_m\}, \{\eta_{k+1,m}\} \) and functions \( \eta^{(1)}, \ldots, \eta^{(k)} \) satisfy
\[
\sum_{j=1}^{k} \left\{ \frac{\mathcal{G}}{\mathcal{K}} \right\} (\eta^{(j)}) + \lim_{m \to \infty} \left\{ \frac{\mathcal{G}}{\mathcal{K}} \right\} (\eta_{k+1,m}) = \lim_{m \to \infty} \left\{ \frac{\mathcal{G}}{\mathcal{K}} \right\} (\eta_m),
\]
\[
\lim_{m \to \infty} \rho(\|\eta_m\|_2^2) = \lim_{m \to \infty} \rho_{k+1}(\|\eta_{k+1,m}\|_2^2)
\]
and
\[
\sum_{j=1}^{k} \|\eta^{(j)}\|_2^2 + \lim_{m \to \infty} \|\eta_{k+1,m}\|_2^2 \leq \lim_{m \to \infty} \|\eta_m\|_2^2
\]
with equality if \( \lim_{m \to \infty} \rho(\|\eta_m\|_2^2) > 0 \).

Precisely one of the following phenomena occurs.

(1) There exists a sequence \( \{x_{k+1,m}\} \subset \mathbb{R} \) and a subsequence of \( \{\eta_{k+1,m}(\cdot + x_{k+1,m})\} \) that satisfies
\[
\lim_{m \to \infty} \|\eta_{k+1,m}(\cdot + x_{k+1,m})\|_2^2 \leq \tilde{M}^2 - \sum_{j=1}^{k} \|\eta^{(j)}\|_2^2
\]
and converges in \( H^r(\mathbb{R}) \) for \( r \in [0, 2) \). The limiting function \( \eta^{(k+1)} \) satisfies
\[
\sum_{j=1}^{k+1} \left\{ \frac{\mathcal{G}}{\mathcal{K}} \right\} (\eta^{(j)}) = \lim_{m \to \infty} \left\{ \frac{\mathcal{G}}{\mathcal{K}} \right\} (\eta_m),
\]
\[
0 < \|\eta^{(k+1)}\|_2^2 \leq DK(\eta^{(k+1)}), \quad c_{p,\mu} = \sum_{j=1}^{k+1} J_{\mu_j}^{(1)}(\eta^{(j)})
\]
with \( \mu_{k+1}^{(1)} = \mu_{k+1} \), minimizes \( J_{p_{k+1},\mu_{k+1}} \) and minimizes \( J_{\mu_j}^{(1)} \) over \( \tilde{U}_{k+1} \setminus \{0\} \), where
\[
\tilde{U}_{k+1} = \left\{ \eta \in H^2(\mathbb{R}) : \|\eta\|_2^2 \leq \tilde{M}^2 - \sum_{j=1}^{k} ||\eta^{(j)}||_2^2 \right\}.
\]
The iteration terminates with this step.
There exist sequences \( \{ \eta_{k+1,m}^{(1)} \} \) with the following properties.

(i) The sequence \( \{ \eta_{k+1,m}^{(1)} \} \) converges in \( H^r(\mathbb{R}^2) \) for \( r \in [0,2) \) to a function \( \eta^{(k+1)} \) that satisfies the estimates

\[
0 < ||\eta^{(k+1)}||_2^2 < D K (\eta^{(k+1)}), \quad ||\eta^{(k+1)}||_2 \geq c |\mu|^{2r+1}.
\]

(ii) The sequence \( \{ \eta_{k+1,m}^{(2)} \} \) is a minimizing sequence for

\[
\mathcal{J}_{\rho_{k+2}, \mu_{k+1}^{(2)}} : H^2(\mathbb{R}) \to \mathbb{R} \cup \{ \infty \}
\]

defined by

\[
\mathcal{J}_{\rho_{k+2}, \mu_{k+1}^{(2)}} (\eta) = \begin{cases} 
K(\eta) & \text{if } \eta \in U_{k+2} \setminus \{0\}, \\
\frac{(\mu_{k+2} + G(\eta))^2}{L(\eta)} + \rho_{k+2}(||\eta||_2^2), & \eta \notin U_{k+2} \setminus \{0\}, \\
\infty, & \text{otherwise}
\end{cases}
\]

where

\[
U_{k+2} = \{ \eta \in H^2(\mathbb{R}) : ||\eta||_2^2 \leq M^2 - \sum_{j=1}^{k+1} ||\eta^{(j)}||_2^2 \}
\]

and

\[
\rho_{k+2}(||\eta||_2^2) = \rho \left( \sum_{j=1}^{k+1} ||\eta^{(j)}||_2^2 + ||\eta||_2^2 \right),
\]

\[
\mu_{k+1}^{(2)} = \lim_{m \to \infty} \frac{L(\eta_{k+1,m}^{(2)})}{L(\eta_m)} \left( \mu + \lim_{m \to \infty} G(\eta_m) - \lim_{m \to \infty} G(\eta_{k+1,m}^{(2)}) \right);
\]

furthermore

\[
c_{\rho, \mu} = \sum_{j=1}^{k+1} \mathcal{J}_{\mu_{j+1}^{(1)}} (\eta^{(j)}) + c_{\rho_{k+2}, \mu_{k+1}^{(2)}},
\]

where

\[
\mu_{k+1}^{(1)} = \mu \frac{L(\eta^{(k+1)})}{\lim_{m \to \infty} L(\eta_m)}, \quad c_{\rho_{k+2}, \mu_{k+1}^{(2)}} = \inf_{\rho_{k+2}, \mu_{k+1}^{(2)}} \mathcal{J}.
\]

(iii) The sequences \( \{ \eta_m \} \), \( \{ \eta_{k+1,m}^{(2)} \} \) and functions \( \eta^{(1)}, \ldots, \eta^{(k+1)} \) satisfy

\[
\sum_{j=1}^{k} \left\{ \mathcal{G} \mathcal{K} \right\}_{\mathcal{L}} (\eta^{(j+1)}) + \lim_{m \to \infty} \left\{ \mathcal{G} \mathcal{K} \right\}_{\mathcal{L}} (\eta_{k+1,m}^{(2)}) = \lim_{m \to \infty} \left\{ \mathcal{G} \mathcal{K} \right\}_{\mathcal{L}} (\eta_m),
\]

\[
\lim_{m \to \infty} \rho(\eta_m ||_2^2) = \lim_{m \to \infty} \rho_{k+2}(||\eta_{k+1,m}^{(2)}||_2^2)
\]

and

\[
\sum_{j=1}^{k+1} ||\eta^{(j)}||_2^2 + \lim_{m \to \infty} ||\eta_{k+1,m}^{(2)}||_2^2 \leq \lim_{m \to \infty} ||\eta_m||_2^2
\]

with equality if \( \lim_{m \to \infty} \rho(||\eta_m||_2^2) > 0 \).

The iteration continues to the next step with \( \eta_{k+2,m} = \eta_{k+1,m}^{(2)}, m \in \mathbb{N} \).
The above construction does not assume that the iteration terminates (that is ‘concentration’ occurs after a finite number of iterations). If it does not terminate, we let \( k \to \infty \) in lemma 3.10 and find that \( \| \eta^{(k)} \|_2 \to 0 \) (because
\[
\sum_{j=1}^{k} \| \eta^{(j)} \|_2^2 \leq D \sum_{j=1}^{k} K(\eta^{(j)}) \leq D \sum_{j=1}^{k} \mathcal{J}_{\mu_j}^{(1)}(\eta^{(j)}) < D c_{p,\mu} < 2 D \nu_0 |\mu|,
\]
for each \( k \in \mathbb{N} \), so that the series \( \sum_{j=1}^{\infty} \| \eta^{(j)} \|_2^2 \) converges), \( \mu_k \to 0 \) (because \( \| \eta^{(k)} \|_2^2 \geq c |\mu_k|^{2 \ast} \), \( c_{p_k,\mu_k} \to 0 \) (because \( c_{p_k,\mu_k} < 2 D |\mu| ) \) and
\[
c_{p,\mu} = \sum_{j=1}^{\infty} \mathcal{J}_{\mu_j}^{(1)}(\eta^{(j)}).
\]
For completeness we record the following corollary of lemma 3.10, which is not used in the remainder of the paper (cf. Buffoni et al. [9, corollary 3.17]).

**Corollary 3.11.** Every minimizing sequence \( \{ \eta_m \} \) for \( \mathcal{J}_{\rho,\mu} \) satisfies
\[
\lim_{m \to \infty} \| \eta_m \|_2 \leq \bar{M}.
\]

### 3.3. Construction of the special minimizing sequence

The sequence \( \{ \tilde{\eta}_m \} \) advertised in theorem 3.1 is constructed by gluing together the functions \( \eta^{(j)} \) identified in §3.2.2 with increasingly large distances between them (the index \( j \) is taken between 1 and \( k \), where \( k = \infty \) if the iteration does not terminate). The minimal distance between the functions is chosen so that the interaction between the ‘tails’ of the individual functions is negligible and \( \| \tilde{\eta}_m \|_2^2 \) is approximately \( \sum_{j=1}^{k} \| \eta^{(j)} \|_2^2 = O(\mu) \) (we return to the original physical setting in which \( \mu \) is positive). The algorithm is stated precisely in part (ii) of the following proposition (which follows immediately from part (i)); for the proof of part (i), see Buffoni et al. [9, proposition 3.20].

**Proposition 3.12.**

(i) There exists a constant \( C > 0 \) such that
\[
\left\| \sum_{j=1}^{k} \tau_{S_j} \eta^{(j)} \right\|_2^2 \leq 2 C^2 D \nu_0 \mu,
\]
where \( (\tau_X \eta^{(j)})(x) := \eta^{(j)}(x + X) \), for all choices of \( \{ S_j \}_{j=1}^{k} \). Moreover, in the case \( k = \infty \) the series converges uniformly over all such sequences.

(ii) The sequence \( \{ \tilde{\eta}_m \} \) defined by the following algorithm satisfies
\[
\| \tilde{\eta}_m \|_2^3 \leq 2 C^2 D \nu_0 \mu.
\]

(1) Choose \( R_j > 1 \) large enough so that
\[
\| \eta^{(j)} \|_{H^2(|x| > R_j)} < \frac{\mu}{2 \tau}.
\]
(2) Write $S_1 = 0$ and choose $S_j > S_{j-1} + 2R_j + 2R_{j-1}$ for $j = 2, \ldots, k$.

(3) Define

$$\tilde{\eta}_m = \sum_{j=1}^{k} \tau_{S_j+(j-1)m} \eta^{(j)}, \quad m \in \mathbb{N}.$$ 

Observe that a local translation-invariant analytic operator $\mathcal{T}: U \to \mathbb{R}$ has the property that

$$\lim_{m \to \infty} \mathcal{T}(\tilde{\eta}_m) = \sum_{j=1}^{k} \mathcal{T}(\eta^{(j)}).$$

Part (i) of the next lemma states that the functionals $G$, $K$ and $L$ behave in the same fashion (with corresponding estimates for their $L^2$-gradients); it is deduced from theorem 2.36 using the method given by Buffoni et al. [9, lemma 3.22]. Part (ii) follows from part (i) by a straightforward calculation that shows that

$$\lim_{m \to \infty} J_{\mu}(\tilde{\eta}_m) = \sum_{j=1}^{k} J_{\mu j}^{(1)}(\eta^{(j)}), \quad \lim_{m \to \infty} \left\| J_{\mu}'(\tilde{\eta}_m) - \sum_{j=1}^{k} J_{\mu j}^{(1)}(\eta^{(j)}) \right\|_0 = 0$$

(cf. Buffoni et al. [9, corollary 3.23]).

Lemma 3.13.

(i) The sequence $\{\tilde{\eta}_m\}$ and functions $\{\eta^{(i)}\}_{i=1}^{m}$ satisfy

$$\lim_{m \to \infty} \left\{ \frac{G}{K} \right\}(\tilde{\eta}_m) = \sum_{i=1}^{k} \left\{ \frac{G}{K} \right\}(\eta^{(i)}),$$

$$\lim_{m \to \infty} \left\| \left\{ \frac{G'}{K'} \right\}(\tilde{\eta}_m) - \sum_{i=1}^{k} \left\{ \frac{G'}{K'} \right\}(\eta^{(i)}) \right\|_0 = 0.$$ 

(ii) The sequence $\{\tilde{\eta}_m\}$ has the properties that

$$\lim_{m \to \infty} J_{\mu}(\tilde{\eta}_m) = c_{\rho,\mu}, \quad \lim_{m \to \infty} \| J_{\mu}'(\tilde{\eta}_m) \|_0 = 0.$$

The proof of theorem 3.1 is completed by the following proposition.

Proposition 3.14. The sequence $\{\tilde{\eta}_m\}$ is a minimizing sequence for $J_{\mu}$ over $U \setminus \{0\}$.

Proof. Let us first note that $\{\tilde{\eta}_m\}$ is a minimizing sequence for $J_{\mu}$ over $\hat{U} \setminus \{0\}$ since the existence of a minimizing sequence $\{v_m\}$ for $J_{\mu}$ over $\hat{U} \setminus \{0\}$ with $\lim_{m \to \infty} J_{\mu}(v_m) < \lim_{m \to \infty} J_{\mu}(\tilde{\eta}_m)$ would lead to the contradiction

$$\lim_{m \to \infty} J_{\rho,\mu}(v_m) = \lim_{m \to \infty} J_{\mu}(v_m) < \lim_{m \to \infty} J_{\mu}(\tilde{\eta}_m) = \lim_{m \to \infty} J_{\rho,\mu}(\tilde{\eta}_m) = c_{\rho,\mu}.$$ 

It follows from this fact and the estimate $\|\tilde{\eta}_m\|_2^2 \leq 2C^2 Dv_0 \mu$ that

$$\inf\{J_{\mu}(\eta): \|\eta\|_2 \in (0, M)\} = \inf\{J_{\mu}(\eta): \|\eta\|_2 \in (0, \sqrt{2C^2 Dv_0 \mu})\}.$$
for all $\tilde{M} \in (\sqrt{2C^2Dv_0\mu}, M)$. The right-hand side of this equation does not depend upon $\tilde{M}$; letting $\tilde{M} \to M$ on the left-hand side, one therefore finds that

$$\inf \{ J_\mu(\eta) : \|\eta\|_2 \in (0, M) \} = \inf \{ J_\mu(\eta) : \|\eta\|_2 \in (0, \sqrt{2C^2Dv_0\mu}) \} = \lim_{m \to \infty} J_\mu(\tilde{\eta}_m).$$

4. Strict subadditivity

The goal of this section is to establish that $c_\mu$ is strictly subadditive, that is,

$$c_{\mu_1 + \mu_2} < c_{\mu_1} + c_{\mu_2}, \quad 0 < |\mu_1|, |\mu_2|, \mu_1 + \mu_2 < \mu_0, \quad (4.1)$$

where negative values of the small parameter are again allowed. This fact is deduced from the facts that $c_\mu$ is an increasing strictly subhomogeneous function of $\mu > 0$, that is,

$$c_{a\mu} < ac_\mu, \quad a > 1. \quad (4.2)$$

The strict subhomogeneity property of $c_\mu$ is established by considering a 'near minimizer' of $J_\mu$ over $U \setminus \{0\}$, that is, a function in $U \setminus \{0\}$ with

$$\|\tilde{\eta}\|_2^2 \leq c_\mu, \quad J_\mu(\tilde{\eta}) < 2v_0\mu - c\mu^\ast, \quad \|J_\mu(\tilde{\eta})\|_0 \leq \mu^N,$$

and hence $L(\tilde{\eta}), L_2(\tilde{\eta}) > c_\mu$ (see the remark above (3.1) and inequality (3.2)), and identifying the dominant term in the 'nonlinear' part $M_\mu(\tilde{\eta})$ of $J_\mu(\tilde{\eta})$. In $\S\S$ 4.2 and 4.3 we show that

$$0 > M_\mu(\tilde{\eta}) = \begin{cases} c \int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, dx + o(\mu^{5/3}), & \beta > \beta_\ast, \\ -c \int_{-\infty}^{\infty} \tilde{\eta}_1^4 \, dx + o(\mu^3), & \beta < \beta_\ast, \end{cases} \quad (4.3)$$

where $\eta_1$ is obtained from $\eta \in H^2(\mathbb{R})$ by multiplying its Fourier transform by the characteristic function of the set $S = [-k_0 - \delta_0, -k_0 + \delta_0] \cup [k_0 - \delta_0, k_0 + \delta_0]$ with $\delta_0 > 0$ if $\beta > \beta_\ast$ and $\delta_0 \in (0, k_0/3)$ if $\beta < \beta_\ast$; inequality (4.2) is readily verified by approximating $M(\tilde{\eta}_m)$ by the homogeneous term identified in (4.3). The details of this procedure are given in $\S$ 4.4.

Straightforward estimates of the kind

$$\mathcal{G}_j(\tilde{\eta}_m), \mathcal{K}_j(\tilde{\eta}_m), \mathcal{L}_j(\tilde{\eta}_m) = O(\|\tilde{\eta}_m\|_2) = O(\mu^{1/2})$$

do not suffice to establish (4.3). According to the calculations presented in Appendix A, the function $\eta_\mu^\ast$, which is constructed using the Korteweg–de Vries scaling for $\beta > \beta_\ast$ and the nonlinear Schrödinger scaling for $\beta < \beta_\ast$, satisfies the estimate (4.3) (with $\tilde{\eta}$ replaced by $\eta_\mu^\ast$). The choice of $\eta_\mu^\ast$ is of course motivated by the expectation that a minimizer, and hence any near minimizer, should have the Korteweg–de Vries or nonlinear Schrödinger length-scales. Our strategy is therefore to show that $\tilde{\eta}_1$ is $O(\mu^{1/2})$ with respect to a weighted norm. To this end we consider the norm

$$\|\eta\|_\infty^2 := \int_{-\infty}^{\infty} (1 + \mu^{-4\alpha}(|k| - k_0)^4)|\hat{\eta}(k)|^2 \, dk$$
and choose \( \alpha > 0 \) as large as possible so that \( \| \tilde{\eta} \|_\alpha \) is \( O(\mu^{1/2}) \); this more detailed description of the behaviour of \( \tilde{\eta} \) allows one to obtain better estimates for \( G_j(\tilde{\eta}) \), \( K_j(\tilde{\eta}) \) and \( L_j(\tilde{\eta}) \), and thus establish (4.3) (see §§ 4.2 and 4.3 for \( \beta > \beta_c \) and \( \beta < \beta_c \), respectively).

### 4.1. Preliminaries

In this section we establish some basic facts that are used in §§ 4.2–4.4.

#### 4.1.1. Splitting of \( \eta \)

In view of the expected frequency distribution of \( \tilde{\eta} \), we split each \( \eta \in U \) into the sum of a function \( \eta_1 \) with spectrum near \( k = \pm k_0 \) and a function \( \eta_2 \) whose spectrum is bounded away from these points. To this end we write the equation

\[
J'_\mu(\eta) = K'_2(\eta) + K'_{\text{nl}}(\eta) + 2 \left( \frac{\mu + G(\eta)}{L(\eta)} \right) G'_2(\eta) + 2 \left( \frac{\mu + G(\eta)}{L(\eta)} \right) G'_{\text{nl}}(\eta) - \left( \frac{\mu + G(\eta)}{L(\eta)} \right)^2 L'_2(\eta) - \left( \frac{\mu + G(\eta)}{L(\eta)} \right)^2 L'_{\text{nl}}(\eta)
\]

in the form

\[
g(k)\tilde{\eta} = \mathcal{F} \left[ J'_\mu(\eta) - K'_{\text{nl}}(\eta) - 2 \left( \frac{\mu + G(\eta)}{L(\eta)} \right) G'_2(\eta) - 2 \left( \frac{\mu + G(\eta)}{L(\eta)} \right) G'_{\text{nl}}(\eta) + \left( \frac{\mu + G(\eta)}{L(\eta)} + \nu_0 \right) \left( \frac{\mu + G(\eta)}{L(\eta)} - \nu_0 \right) L'_2(\eta) + \left( \frac{\mu + G(\eta)}{L(\eta)} \right)^2 L'_{\text{nl}}(\eta) \right]
\]

and decompose it into two coupled equations by defining \( \eta_2 \in H^2(\mathbb{R}) \) by the formula

\[
\eta_2 = \mathcal{F}^{-1} \left[ \frac{1 - \chi_S(k)}{g(k)} \mathcal{F} \left[ J'(\eta) - K'_{\text{nl}}(\eta) \right] - 2 \left( \frac{\mu + G(\eta)}{L(\eta)} - \nu_0 \right) G'_2(\eta) - 2 \left( \frac{\mu + G(\eta)}{L(\eta)} \right) G'_{\text{nl}}(\eta) + \left( \frac{\mu + G(\eta)}{L(\eta)} + \nu_0 \right) \left( \frac{\mu + G(\eta)}{L(\eta)} - \nu_0 \right) L'_2(\eta) + \left( \frac{\mu + G(\eta)}{L(\eta)} \right)^2 L'_{\text{nl}}(\eta) \right]
\]

and \( \eta_1 \in H^2(\mathbb{R}) \) by \( \eta_1 = \eta - \eta_2 \), so that \( \eta_1 \) has support in \( S \); here we have used the fact that

\[
f \mapsto \mathcal{F}^{-1} \left[ \frac{1 - \chi_S(k)}{g(k)} \hat{f}(k) \right]
\]

is a bounded linear operator \( L^2(\mathbb{R}) \to H^2(\mathbb{R}) \).
4.1.2. Estimates for $\| \cdot \|_\alpha$

**Proposition 4.1.**

(i) The estimates $\| \eta \|_{1, \infty} \leq c \mu^{\alpha/2} \| \eta \|_\alpha$, $\| K^0 \eta \|_\infty \leq c \mu^{\alpha/2} \| \eta \|_\alpha$ hold for each $\eta \in H^2(\mathbb{R})$.

(ii) The estimates

$$\| \eta'' + k_0^2 \eta \|_0 \leq c \mu^{\alpha} \| \eta \|_\alpha, \quad k_0 \neq 0,$$

and

$$\| (K^0 \eta)^{(n)} \|_\infty \leq \mu^{\alpha/2} \| \eta \|_\alpha, \quad n \in \mathbb{N}_0,$$

hold for each $\eta \in H^2(\mathbb{R})$ with $\text{supp} \ \hat{\eta} \subseteq S$.

**Proof.**

(i) Observe that

$$\| \eta^{(j)} \|_\infty^2 \leq c \| k \| \| \hat{\eta} \|_{L^1(\mathbb{R})}, \quad j = 0, 1, \quad \quad (4.4)$$

$$\| K^0 \eta \|_\infty \leq \| (K^0 - 1) \eta \|_\infty + \| \eta \|_\infty \leq c (\| |k| \coth |k| - 1 \| \hat{\eta} \|_{L^1(\mathbb{R})} + \| \eta \|_\infty) \leq c (\| |k| \| \hat{\eta} \|_{L^1(\mathbb{R})} + \| \hat{\eta} \|_{L^1(\mathbb{R})}) \quad \quad (4.5)$$

and

$$\| |k| \hat{\eta} \|_{L^2(\mathbb{R})}^2 \leq \left( \int_{-\infty}^{\infty} \frac{k^{2j}}{1 + \mu^{-4\alpha}(k - k_0)^4} \, dk \right) \int_{-\infty}^{\infty} (1 + \mu^{-4\alpha}(k - k_0)^4) |\hat{\eta}(k)|^2 \, dk$$

$$+ \left( \int_{-\infty}^{\infty} \frac{k^{2j}}{1 + \mu^{-4\alpha}(k + k_0)^4} \, dk \right) \int_{-\infty}^{0} (1 + \mu^{-4\alpha}(k + k_0)^4) |\hat{\eta}(k)|^2 \, dk$$

$$\leq c \mu^{\alpha} \| \eta \|_\alpha^2, \quad j = 0, 1.$$

(ii) The first result follows from the calculation

$$\| \eta'' + k_0^2 \eta \|_0 = \| (k^2 - k_0^2) \hat{\eta} \|_0$$

$$\leq c \left( \int_{k_0 - \delta_0}^{k_0 + \delta_0} |k - k_0|^2 |\hat{\eta}(k)|^2 \, dk + \int_{-k_0 - \delta_0}^{-k_0 + \delta_0} |k + k_0|^2 |\hat{\eta}(k)|^2 \, dk \right)$$

$$\leq c \left( \int_{k_0 - \delta_0}^{k_0 + \delta_0} (\mu^{2\alpha} + \mu^{-2\alpha} |k - k_0|^4) |\hat{\eta}(k)|^2 \, dk \right. \right.$$  

$$+ \left. \int_{-k_0 - \delta_0}^{-k_0 + \delta_0} (\mu^{2\alpha} + \mu^{-2\alpha} |k + k_0|^4) |\hat{\eta}(k)|^2 \, dk \right)$$

$$\leq c \mu^{2\alpha} \left( \int_{k_0 - \delta_0}^{k_0 + \delta_0} (1 + \mu^{-4\alpha} |k - k_0|^4) |\hat{\eta}(k)|^2 \, dk \right.$$

$$+ \int_{-k_0 - \delta_0}^{-k_0 + \delta_0} (1 + \mu^{-4\alpha} |k + k_0|^4) |\hat{\eta}(k)|^2 \, dk \right)$$

$$= c \mu^{2\alpha} \| \eta \|_\alpha^2.$$
while the second is established by repeating the proof of the second inequality in part (i) and estimating $|k| \leq k_0 + \delta_0$. \hfill \Box

### 4.1.3. Estimates for the wave speed

The following proposition is used in particular to bound the deviation of the quantity $(\mu + G(\tilde{\eta}))/L(\tilde{\eta})$ (the speed of the corresponding travelling wave when $\tilde{\eta}$ is a minimizer of $J_\mu$ over $U \setminus \{0\}$) from the linear wave speed $\nu_0$.

**Proposition 4.2.** The function $\tilde{\eta}$ satisfies the inequalities

$$ R_1(\tilde{\eta}) \leq \frac{\mu + G(\tilde{\eta})}{L(\tilde{\eta})} - \nu_0 \leq R_2(\tilde{\eta}) $$

and

$$ R_1(\tilde{\eta}) - \tilde{M}_\mu(\tilde{\eta}) \leq \frac{\mu + G_2(\tilde{\eta})}{L_2(\tilde{\eta})} - \nu_0 \leq R_2(\tilde{\eta}) - \tilde{M}_\mu(\tilde{\eta}), $$

where

$$ R_1(\tilde{\eta}) = -\frac{\langle J'_\mu(\tilde{\eta}), \tilde{\eta} \rangle}{4\mu} + \frac{1}{4\mu} ((\mathcal{M}'(\tilde{\eta}), \tilde{\eta}) + 4\mu \tilde{M}_\mu(\tilde{\eta})), $$

$$ R_2(\tilde{\eta}) = -\frac{\langle J'_\mu(\tilde{\eta}), \tilde{\eta} \rangle}{4\mu} + \frac{1}{4\mu} ((\mathcal{M}'(\tilde{\eta}), \tilde{\eta}) + 4\mu \tilde{M}_\mu(\tilde{\eta}) - \frac{\mathcal{M}_\mu(\tilde{\eta})}{2\mu}) $$

and

$$ \tilde{M}_\mu(\tilde{\eta}) = \frac{\mu + G(\tilde{\eta})}{L(\tilde{\eta})} - \frac{\mu + G_2(\tilde{\eta})}{L_2(\tilde{\eta})}. $$

**Proof.** Taking the scalar product of the equation

$$ J'_\mu(\tilde{\eta}) = \mathcal{K}_2(\tilde{\eta}) - \left( \frac{\mu + G_2(\tilde{\eta})}{L_2(\tilde{\eta})} \right)^2 L_2(\tilde{\eta}) + 2 \left( \frac{\mu + G(\tilde{\eta})}{L(\tilde{\eta})} \right) G_2(\tilde{\eta}) + \mathcal{M}'(\tilde{\eta}) $$

with $\tilde{\eta}$ yields the identity

$$ \frac{\mu + G(\tilde{\eta})}{L(\tilde{\eta})} = \frac{\langle J'_\mu(\tilde{\eta}), \tilde{\eta} \rangle}{4\mu} + \frac{1}{2\mu} (\mathcal{K}_2(\tilde{\eta}) + (\mu + G_2(\tilde{\eta}))^2) + \frac{1}{4\mu} ((\mathcal{M}'(\tilde{\eta}), \tilde{\eta}) + 4\mu \tilde{M}_\mu(\tilde{\eta})). $$

The first inequality is derived by estimating the quantity in brackets from above and below by means of the estimate

$$ 2\nu_0 \mu \leq \mathcal{K}_2(\tilde{\eta}) + \left( \frac{\mu + G_2(\tilde{\eta})}{L_2(\tilde{\eta})} \right)^2 = J'_\mu(\tilde{\eta}) - \mathcal{M}_\mu(\tilde{\eta}) < 2\nu_0 \mu - \mathcal{M}_\mu(\tilde{\eta}) $$

and the second inequality follows directly from the first. \hfill \Box
4.1.4. Estimates for the functionals \( G, K \) and \( L \)

Turning to the functionals \( G, K \) and \( L : U \to \mathbb{R} \), denote their non-quadratic parts by \( G_{nl}, K_{nl}, L_{nl} \) and write

\[
G_{nl}(\eta) = \sum_{k=3}^{4} G_k(\eta) + G_4(\eta),
K_{nl}(\eta) = \sum_{k=3}^{4} K_k(\eta) + K_4(\eta),
L_{nl}(\eta) = \sum_{k=3}^{4} L_k(\eta) + L_4(\eta),
\]

so that

\[
G_4(\eta) = \frac{\omega}{4} \int_{-\infty}^{\infty} \eta^2 (K(\eta) - K^0 - K^1(\eta)) \eta \, dx,
(4.6)
\]

\[
K_4(\eta) = \beta \int_{-\infty}^{\infty} \left( \sqrt{1 + \eta^2} - 1 - \frac{\eta^2}{2} + \frac{\eta^4}{8} \right) \, dx - \frac{\omega^2}{2} \int_{-\infty}^{\infty} \frac{\eta^2}{2} (K(\eta) - K^0) \eta^2 \, dx,
(4.7)
\]

\[
L_4(\eta) = \frac{1}{2} \int_{-\infty}^{\infty} (K(\eta) - K^0 - K^1(\eta) - K^2(\eta)) \eta \, dx.
(4.8)
\]

We now record useful explicit formulas for the cubic and quartic parts of the functionals in terms of the Fourier-multiplier operator \( K^0 \) and give order-of-magnitude estimates for their cubic, quartic and higher-order parts.

**Proposition 4.3.** The formulas

\[
G_3(\eta) = \frac{\omega}{4} \int_{-\infty}^{\infty} \eta^2 K^0 \eta \, dx,
K_3(\eta) = \frac{\omega^2}{6} \int_{-\infty}^{\infty} \eta^3 \, dx,
L_3(\eta) = \frac{1}{2} \int_{-\infty}^{\infty} (-K^0 \eta^2 + \eta^2 \eta) \, dx
\]

and

\[
G_4(\eta) = \frac{\omega}{4} \eta^2 \eta^2 \, dx - \frac{\omega}{4} \int_{-\infty}^{\infty} \eta^2 K^0 (\eta K^0 \eta) \, dx,
K_4(\eta) = \frac{-\beta}{8} \int_{-\infty}^{\infty} \eta^4 \, dx - \frac{\omega^2}{8} \int_{-\infty}^{\infty} \eta^2 K^0 \eta^2 \, dx,
L_4(\eta) = \frac{1}{2} \int_{-\infty}^{\infty} (K^0 (\eta K^0 \eta) \eta K^0 \eta + (K^0 \eta) \eta^2 \eta) \, dx
\]

hold for each \( \eta \in U \).

**Proof.** The formulas for \( G_3, K_3, K_4 \) follow directly from (1.10) and (1.11). Equations (1.12) and (2.26) imply that

\[
L_3(\eta) = \frac{1}{2} \int_{-\infty}^{\infty} \eta K_1(\eta) \eta \, dx,
L_3^*(\eta) = \frac{1}{2} \mathcal{H}_1(\eta)(\eta, \eta) + K_1(\eta) \eta,
\]
while lemma 2.31 shows that
\[ \mathcal{H}_2'(\eta)(\zeta_1, \zeta_2) = -u_{1x}^0 u_{2x}^0 + u_{1y}^0 u_{2y}^0 |_{y=1} = -(K^0 \zeta_1')(K^0 \zeta_2) + \zeta_1' \zeta_2', \]
where \( u_j \) is the weak solution of (2.7)–(2.9) with \( \xi = \zeta_j', \ j = 1, 2 \), so that
\[ L_2'(\eta) = -\frac{1}{2}(K^0 \eta)^2 + \frac{1}{2} \eta'^2 + K_1(\eta) \eta. \tag{4.9} \]
Taking the inner product of this equation with \( \eta \), we therefore find that
\[ 3\mathcal{L}_3(\eta) = \frac{1}{2} \int_{-\infty}^{\infty} (-(K^0 \eta)^2 \eta + \eta'^2 \eta) \, dx + 2\mathcal{L}_3(\eta), \]
which yields the given formula for \( \mathcal{L}_3(\eta) \).

Similarly, (1.10) and (2.25) imply that
\[ \mathcal{G}_4(\eta) = \frac{\omega}{4} \int_{-\infty}^{\infty} \eta^2 K_1(\eta) \eta \, dx \]
and
\[ \mathcal{G}_4'(\eta) = \frac{1}{4} \omega \mathcal{H}_1'(\eta)(\eta^2, \eta) + \frac{1}{4} \omega K_1(\eta) \eta^2 + \frac{1}{2} \omega \eta K_1(\eta) \eta \]
\[ = -\frac{1}{4} \omega (K^0 \eta^2) K^0 \eta + \frac{1}{4} \omega (\eta^2)' \eta' + \frac{1}{4} \omega K_1(\eta) \eta^2 + \frac{1}{2} \omega \eta K_1(\eta) \eta. \]
The formula for \( \mathcal{G}_4(\eta) \) follows by taking the inner product of the latter equation with \( \eta \).

Finally, (1.12) and (2.26) imply that
\[ \mathcal{L}_4(\eta) = \frac{1}{2} \int_{-\infty}^{\infty} \eta K_2(\eta) \eta \, dx, \quad \mathcal{L}_4'(\eta) = \frac{1}{2} \mathcal{H}_2'(\eta)(\eta, \eta) + K_2(\eta) \eta \]
and lemma 2.31 shows that
\[ \mathcal{H}_2'(\eta)(\zeta_1, \zeta_2) = -u_{1x}^1 u_{2x}^1 - u_{2x} u_{1x}^1 + u_{1y}^1 u_{2y}^1 + u_{2y} u_{1y}^1 - 2\eta u_{1y}^0 u_{2y}^0 |_{y=1}. \]
Using (2.18), we find that
\[ u_{1y}^1 |_{y=1} = G^1 \cdot (0, 1) |_{y=1} = -(Q^1 \nabla u^0) \cdot (0, 1) |_{y=1} = \eta u_{1y}^0 + \eta' u_{2x}^0 |_{y=1} = \eta \zeta' - \eta' K^0 \zeta, \]
where \( u \) is the weak solution of (2.7)–(2.9) with \( \xi = \zeta' \), so that
\[ \mathcal{H}_2'(\eta)(\eta, \eta) = -2\eta'^2 K^0 \eta - 2K^0 \eta K_1(\eta) \eta. \]
Equating (4.9) and
\[ L_2'(\eta) = -K^0 (\eta K^0) - \frac{1}{2}(K^0 \eta)^2 - \frac{1}{2} \eta'^2 - \eta'' \eta, \]
which follows from the formula for \( \mathcal{L}_3(\eta) \), we find that
\[ K_1(\eta) \eta = -K^0 (\eta K^0) - (\eta' \eta)' \]
so that
\[ L_2'(\eta) = -\eta'^2 K^0 \eta + K^0 \eta K^0 (\eta K^0) + K^0 \eta (\eta' \eta)' + K_2(\eta) \eta. \]
The formula for \( \mathcal{L}_4(\eta) \) is obtained by taking the inner product of this expression with \( \eta \).
Proposition 4.4. The estimates

\[
\begin{align*}
\{ |G_3(\eta)| \} & \leq c \| \eta \|_2^2 (\| \eta \|_{1,\infty} + \| \eta'' + k_0^2 \eta \|_0), \\
\{ |K_3(\eta)| \} & \leq c \| \eta \|_2^2 (\| \eta \|_{1,\infty} + \| \eta'' + k_0^2 \eta \|_0), \\
\{ |L_3(\eta)| \} & \leq c \| \eta \|_2^2 (\| \eta \|_{1,\infty} + \| \eta'' + k_0^2 \eta \|_0),
\end{align*}
\]

hold for each \( \eta \in U \).

Proof. These results are obtained by estimating the right-hand sides of the formulas given in proposition 4.3 and (4.6)–(4.8) using proposition 2.29.

Proposition 4.5. The estimates

\[
\begin{align*}
\{ |G'_3(\eta)| \} & \leq c \| \eta \|_2 (\| \eta \|_{1,\infty} + \| \eta'' + k_0^2 \eta \|_0), \\
\{ |K'_3(\eta)| \} & \leq c \| \eta \|_2 (\| \eta \|_{1,\infty} + \| \eta'' + k_0^2 \eta \|_0 + \| K^0 \eta \|_{\infty}), \\
\{ |L'_3(\eta)| \} & \leq c \| \eta \|_2 (\| \eta \|_{1,\infty} + \| \eta'' + k_0^2 \eta \|_0 + \| K^0 \eta \|_{\infty}),
\end{align*}
\]

hold for each \( \eta \in U \).

Proof. We estimate the right-hand sides of the formulas

\[
\begin{align*}
G'_3(\eta) & = \frac{1}{4} \omega K^0 \eta^2 + \frac{1}{2} \omega \eta K^0 \eta, \\
K'_3(\eta) & = \frac{1}{4} \omega^2 \eta^2, \\
L'_3(\eta) & = -K^0(\eta K^0 \eta) - \frac{1}{2} (K^0 \eta)^2 - \frac{1}{2} \eta'' - \eta', \\
G'_4(\eta) & = -\frac{1}{4} \omega (K^0 \eta^2) K^0 \eta - \frac{1}{2} \omega K^0(\eta K^0 \eta^2) - \omega \eta \eta'' - \omega \eta^2 \eta' - \frac{1}{2} \omega \eta K^0(\eta K^0 \eta), \\
K'_4(\eta) & = \frac{3}{2} \beta \eta^2 \eta'' - \frac{1}{4} \omega^2 \eta^2 K^0 \eta^2, \\
L'_4(\eta) & = -2 \eta^2 K^0 \eta - 2 K^0 \eta K^1(\eta) \eta + K_2(\eta) \eta,
\end{align*}
\]

and

\[
\begin{align*}
G'_5(\eta) & = \frac{1}{4} \omega (H'(\eta) - H_1'(\eta))(\gamma^2, \eta) + \frac{1}{2} \omega (K(\eta) - K^0 K^1(\eta) \eta) \eta^2 \\
& \quad + \frac{1}{2} \omega \eta (K(\eta) - K^0 - K^1(\eta)) \eta.
\end{align*}
\]
\[ K'_4(\eta) = \beta \left( 1 - \frac{3}{2} \eta^2 - \frac{1}{(1 + \eta^2)^{3/2}} \right) \eta'' - \frac{1}{8} \omega^2 \mathcal{H}'(\eta)(\eta^2) - \frac{1}{2} \omega^2 \eta^2 (K(\eta) - K^0) \eta, \]

\[ L'_4(\eta) = \frac{1}{2} (\mathcal{H}'(\eta) - \mathcal{H}'_1(\eta) - \mathcal{H}'_2(\eta))(\eta, \eta) + (K(\eta) - K^0 - K^1(\eta) - K^2(\eta)) \eta \]

using proposition 2.29 and the estimate

\[ \| \mathcal{H}_{j+1}(\eta, \zeta) \|_0 \leq CB^j(\| \eta \|_{1,\infty} + \| \eta'' + k_0^2 \eta \|_0)^j \| \zeta \|_{3/2} \| \zeta \|_{3/2}, \quad j \in \mathbb{N}_0. \]

\[ \square \]

It is also helpful to write

\[ K'_3(\eta) = m_1(\eta, \eta), \quad G'_3(\eta) = m_2(\eta, \eta), \quad L'_3(\eta) = m_3(\eta, \eta), \]

where \( m_j \in \mathcal{L}_3^2(H^2(\mathbb{R}), L^2(\mathbb{R})), j = 1, 2, 3, \) are defined by

\[ m_1(u_1, u_2) = \frac{1}{2} \omega^2 u_1 u_2, \]

\[ m_2(u_1, u_2) = \frac{1}{2} \omega K^0(u_1 u_2) + \frac{1}{2} \omega u_1 K^0 u_2 + \frac{1}{2} \omega u_2 K^0 u_1, \]

\[ m_3(u_1, u_2) = -\frac{1}{2} K^0(u_1 K^0 u_2) - \frac{1}{2} K^0(u_2 K^0 u_1) - \frac{1}{2} K^0 u_1 K^0 u_2 - \frac{1}{2} u_1 x u_2 - \frac{1}{2} u_1 x u_2 \]

and, similarly,

\[ K_3(\eta) = n_1(\eta, \eta, \eta), \quad G_3(\eta) = n_2(\eta, \eta, \eta), \quad L_3(\eta) = n_3(\eta, \eta, \eta), \]

where \( n_j \in \mathcal{L}_3^2(H^2(\mathbb{R}), \mathbb{R}), j = 1, 2, 3, \) are defined by

\[ n_1(u_1, u_2, u_3) = \frac{1}{6} \omega^2 \int_{-\infty}^{\infty} u_1 u_2 u_3 \, dx, \]

\[ n_2(u_1, u_2, u_3) = \frac{1}{12} \omega \int_{-\infty}^{\infty} \mathcal{P}[u_1 u_2 K^0 u_3] \, dx, \]

\[ n_3(u_1, u_2, u_3) = \frac{1}{6} \int_{-\infty}^{\infty} \mathcal{P}[u'_1 u'_2 u'_3] \, dx - \frac{1}{6} \int_{-\infty}^{\infty} \mathcal{P}(K^0 u_1)(K^0 u_2) u_3 \, dx \]

and the symbol \( \mathcal{P}[\cdot] \) denotes the sum of all distinct expressions resulting from permutations of the variables appearing in its argument.

**Proposition 4.6.** The estimates

\[ \| m_j(\eta_1, u_2) \|_0 \leq c(\| \eta_1 \|_{1,\infty} + \| \eta'' + k_0^2 \eta_1 \|_0 + \| K^0 \eta_1 \|_{1,\infty}) \| u_2 \|_2, \quad j = 1, 2, 3, \]

and

\[ | n_j(\eta_1, u_2, u_3) | \leq c(\| \eta_1 \|_{1,\infty} + \| \eta'' + k_0^2 \eta_1 \|_0 + \| K^0 \eta_1 \|_{1,\infty}) \| u_2 \|_2 \| u_3 \|_2, \quad j = 1, 2, 3, \]

hold for each \( \eta \in U \) and \( u_2, u_3 \in H^2(\mathbb{R}). \)
4.1.5. Formulae for the functionals $\mathcal{M}_\mu$ and $\tilde{\mathcal{M}}_\mu$

**Lemma 4.7.** The estimates

\[
\mathcal{M}_\mu(\eta) = K_3(\eta) + 2\nu_0 G_3(\eta) - \nu_0^3 L_3(\eta) + K_4(\eta) + 2\nu_0 G_4(\eta) - \nu_0^3 L_4(\eta) \\
+ 2 \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} - \nu_0 \right) (G_3(\eta) + G_4(\eta)) \\
- \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} - \nu_0 \right) \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} + \nu_0 \right) (L_3(\eta) + L_4(\eta)) \\
+ \frac{1}{L_2(\eta)} \left( G_3(\eta) - \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} \right) L_3(\eta) \right)^2 \\
+ O(\mu^{3/2}(||\eta||_{1,\infty} + ||\eta'' + k_0^2\eta||_0)^2),
\]

\[
\langle \mathcal{M}_\mu(\eta), \eta \rangle + 4\mu \tilde{\mathcal{M}}_\mu(\eta) \\
= 3(K_3(\eta) + 2\nu_0 G_3(\eta) - \nu_0^3 L_3(\eta)) + 4(K_3(\eta) + 2\nu_0 G_3(\eta) - \nu_0^3 L_3(\eta)) \\
+ 2 \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} - \nu_0 \right) (3G_3(\eta) + 4G_4(\eta)) \\
- \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} - \nu_0 \right) \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} + \nu_0 \right) (3L_3(\eta) + 4L_4(\eta)) \\
+ \frac{4}{L_2(\eta)} \left( G_3(\eta) - \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} \right) L_3(\eta) \right)^2 \\
+ O(\mu^{3/2}(||\eta||_{1,\infty} + ||\eta'' + k_0^2\eta||_0)^2)
\]

and

\[
\tilde{\mathcal{M}}_\mu(\eta) = \mu^{-1}(G_3(\eta) + G_4(\eta)) + \mu^{-1} \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} \right) (L_3(\eta) + L_4(\eta)) \\
+ O(\mu^{1/2}(||\eta||_{1,\infty} + ||\eta'' + k_0^2\eta||_0)^2)
\]

hold for each $\eta \in U$ with $||\eta||_2 \leq c\mu^{1/2}$ and $L_2(\eta) > c\mu$.

**Proof.** Using the formulas

\[
\mathcal{M}_\mu(\eta) = K_{nl}(\eta) + \frac{(\mu + G(\eta))^2}{L(\eta)} - \frac{L_{nl}(\eta)}{L(\eta)}
\]

and

\[
\frac{1}{L(\eta)} = \frac{1}{L_2(\eta)} \left( 1 - \frac{L_{nl}(\eta)}{L(\eta)} \right),
\]

one finds that

\[
\mathcal{M}_\mu(\eta) = K_{nl}(\eta) + 2 \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} \right) G_{nl}(\eta) - \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} \right)^2 L_{nl}(\eta) \\
+ \frac{G_{nl}(\eta)}{L(\eta)} - 2 \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} \right) \frac{G_{nl}(\eta)}{L(\eta)} L_{nl}(\eta) + \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} \right)^2 L_{nl}(\eta)^2.
\]
We estimate the first line by substituting
\[
\begin{align*}
&\begin{cases}
\mathcal{G}_m(\eta) \\
\mathcal{K}_m(\eta) \\
\mathcal{L}_m(\eta)
\end{cases}
= \begin{cases}
\mathcal{G}_3(\eta) + \mathcal{G}_4(\eta) \\
\mathcal{K}_3(\eta) + \mathcal{K}_4(\eta) \\
\mathcal{L}_3(\eta) + \mathcal{L}_4(\eta)
\end{cases} + O(\mu^{3/2}(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^2)
\end{align*}
\]
(see proposition 4.4) and
\[
\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} = O(1).
\]
Writing
\[
\mathcal{G}_m(\eta) = \mathcal{G}_3(\eta) + O(\mu(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^2)
\]
(see proposition 4.4) and estimating
\[
\mathcal{G}_3(\eta) = O(\|\eta\|_\infty \|\eta\|_2^2) = O(\mu\|\eta\|_\infty)
\]
(see proposition 4.4) and estimating
\[
\mathcal{G}_3(\eta) = O(\|\eta\|_\infty \|\eta\|_2^2) = O(\mu\|\eta\|_\infty)
\]
(using the formula for \(\mathcal{G}_3(\eta)\) given in proposition 4.3) yields
\[
\mathcal{G}_m(\eta)^2 = \mathcal{G}_3(\eta)^2 + O(\mu^2(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^3)
\]
and
\[
\frac{\mathcal{L}_m(\eta)\mathcal{G}_3(\eta)^2}{\mathcal{L}_2(\eta)^2} = \frac{\mathcal{G}_3(\eta)^2}{\mathcal{L}_2(\eta)} + O(\mu^{3/2}(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^2)
\]
(recall that \(\mathcal{L}(\eta) \geq c\mathcal{L}_2(\eta)\) for \(\eta \in U\), so that
\[
\mathcal{G}_m(\eta)^2 = \frac{\mathcal{G}_3(\eta)^2}{\mathcal{L}_2(\eta)} + O(\mu^{3/2}(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^2);
\]
the remaining terms on the second line are estimated in the same fashion.

Altogether we find that
\[
\mathcal{M}_\mu(\eta) = \mathcal{K}_3(\eta) + 2\left(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)\mathcal{G}_3(\eta) - \left(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)^2\mathcal{L}_3(\eta)
\]
\[
+ \mathcal{K}_4(\eta) + 2\left(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)\mathcal{G}_4(\eta) - \left(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)^2\mathcal{L}_4(\eta)
\]
\[
+ \frac{1}{\mathcal{L}_2(\eta)}\left(\mathcal{G}_3(\eta) - \mathcal{G}_3(\eta)\left(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)^2\right)
\]
\[
+ O(\mu^{3/2}(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^2),
\]
from which the stated formula for \(\mathcal{M}_\mu(\eta)\) follows by an algebraic manipulation.

The other estimates are derived by similar calculations.

\[\square\]

4.2. The case \(\beta > \beta_c\)

We begin by estimating the wave speed.

**Proposition 4.8.** The function \(\tilde{\eta}\) satisfies the estimates
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} - v_0 \\
\frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - v_0
\end{array} \right\} &\leq c(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0 + \mu^{N-1/2}),
\end{align*}
\]
Proof. Proposition 4.4 implies that
\[
\begin{bmatrix}
|\mathcal{G}_j(\tilde{\eta})| \\
|\mathcal{K}_j(\tilde{\eta})| \\
|\mathcal{L}_j(\tilde{\eta})|
\end{bmatrix} \leq c\mu(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0), \quad j = 3, 4,
\]
and lemma 4.7 shows that
\[
|\mathcal{M}_\mu(\tilde{\eta})|, |(\mathcal{M}_\mu'(\tilde{\eta}), \tilde{\eta}) + 4\mu \mathcal{M}_\mu(\tilde{\eta})| \leq c\mu(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0),
\]
\[
|\mathcal{M}_\mu(\tilde{\eta})| \leq c(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0).
\]
The results are obtained by combining these estimates with proposition 4.2.

**Corollary 4.9.** The quantity
\[
S(\tilde{\eta}) = \mathcal{J}_\mu(\tilde{\eta}) - \mathcal{K}_\mu(\tilde{\eta}) - 2\left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} - \nu_0\right)\mathcal{G}'(\tilde{\eta}) - 2\left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})}\right)^2 \mathcal{G}''(\tilde{\eta})
\]
\[
+ \left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} + \nu_0\right)\left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} - \nu_0\right)\mathcal{L}'(\tilde{\eta}) + \left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})}\right)^2 \mathcal{L}''(\tilde{\eta})
\]
satisfies
\[
\|S(\tilde{\eta})\|_0 \leq c(\mu^{1/2}(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0 + \|\mathcal{K}^0\tilde{\eta}\|_\infty) + \mu^N).
\]

The next step is an estimate for \(\|\tilde{\eta}_1\|_\alpha\) and \(\|\tilde{\eta}_2\|_2\).

**Lemma 4.10.** The function \(\tilde{\eta}\) satisfies \(\|\tilde{\eta}_1\|_\alpha^2 \leq c\mu\) and \(\|\tilde{\eta}_2\|_2^2 \leq c\mu^{2+\alpha}\) for \(\alpha < \frac{1}{3}\).

Proof. Using the equations
\[
g(k)\tilde{\eta}_1 = \mathcal{F}[S(\tilde{\eta})], \quad \tilde{\eta}_2 = \mathcal{F}^{-1}\left[\frac{1 - \chi_S(k)}{g(k)} \mathcal{F}[S(\tilde{\eta})]\right],
\]
we find from corollary 4.9 that
\[
\|\tilde{\eta}_2\|_2 \leq c(\mu^{1/2}(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0 + \|\mathcal{K}^0\tilde{\eta}\|_\infty) + \mu^{1/2}\|\tilde{\eta}_2\|_2 + \mu^N),
\]
and therefore
\[
\|\tilde{\eta}_2\|_2 \leq c(\mu^{1/2}(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0 + \|\mathcal{K}^0\tilde{\eta}\|_\infty) + \mu^N), \tag{4.13}
\]
and
\[
\int_{-\infty}^{\infty} g(k)^2|\tilde{\eta}_1(k)|^2 dk \leq c(\mu(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0 + \|\mathcal{K}^0\tilde{\eta}\|_\infty)^2 + \mu\|\tilde{\eta}_2\|_2^2 + \mu^{2N})
\]
(see proposition 4.1). Multiplying the above inequality by \(\mu^{-4\alpha}\), using (4.13) and adding \(\|\tilde{\eta}_1\|_\alpha^2 \leq \|\tilde{\eta}\|_\alpha^2 \leq c\mu\), one finds that
\[
\|\tilde{\eta}\|_\alpha^2 \leq c(\mu^{1-4\alpha}(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0 + \|\mathcal{K}^0\tilde{\eta}\|_\infty)^2 + \mu) \leq c(\mu^{1-3\alpha}\|\tilde{\eta}\|_\alpha^2 + \mu), \tag{4.14}
\]
so that \(\|\tilde{\eta}\|_\alpha^2 \leq c\mu\) for \(\alpha < \frac{1}{3}\). The estimate for \(\tilde{\eta}_2\) follows from inequality (4.13).
It remains to identify the dominant terms in the formulas for
\[ M_\mu(\tilde{\eta}) \quad \text{and} \quad \langle M'_\mu(\tilde{\eta}), \tilde{\eta} \rangle + 4\mu \tilde{M}_\mu(\tilde{\eta}) \]
given in lemma 4.7; this task is accomplished by combining the estimates in propositions 4.11 and 4.12 and lemma 4.13.

**Proposition 4.11.** The function \( \tilde{\eta} \) satisfies the estimate
\[
\begin{align*}
\begin{cases}
G_3(\tilde{\eta}) \\
K_3(\tilde{\eta}) \\
L_3(\tilde{\eta})
\end{cases} =
\begin{cases}
G_3(\tilde{\eta}_1) \\
K_3(\tilde{\eta}_1) \\
L_3(\tilde{\eta}_1)
\end{cases} + o(\mu^{5/3}).
\end{align*}
\]

**Proof.** Using proposition 4.6, we find that
\[
\left| n_j(\tilde{\eta}_1, \tilde{\eta}_2) \right| \leq c\mu^{\alpha/2} \| \tilde{\eta}_1 \|_\alpha \left\{ \| \tilde{\eta}_1 \|_2 \| \tilde{\eta}_2 \|_2 \right\} \\leq c\mu^{2+\alpha} \\leq o(\mu^{5/3}),
\]
while
\[
\left| n_j(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3) \right| \leq c\| \tilde{\eta}_2 \|_2^3 \leq c\mu^{3+3\alpha/2} = o(\mu^{5/3});
\]
it follows that
\[
n_j(\tilde{\eta}_1 + \tilde{\eta}_2, \tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\eta}_3) - n_j(\tilde{\eta}_1, \tilde{\eta}_1, \tilde{\eta}_1) = o(\mu^{5/3})
\]
for \( j = 1, 2, 3 \). \( \square \)

**Proposition 4.12.** The function \( \tilde{\eta} \) satisfies the estimate
\[
K_3(\tilde{\eta}) + 2\nu_0 G_3(\tilde{\eta}) - \nu_0^2 L_3(\tilde{\eta}) = \frac{1}{2}(\frac{1}{4} \omega^2 + 1) \int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, dx + o(\mu^{5/3}).
\]

**Proof.** Note that
\[
G_3(\tilde{\eta}_1) = \frac{1}{4} \omega \int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, dx + \frac{1}{4} \omega \int_{-\infty}^{\infty} \tilde{\eta}_1^2 (K^0 \tilde{\eta}_1 - \tilde{\eta}_1) \, dx,
\]
\[
K_3(\tilde{\eta}_1) = \frac{1}{4} \omega^2 \int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, dx,
\]
\[
L_3(\tilde{\eta}_1) = -\frac{1}{2} \int_{-\infty}^{\infty} (K^0 \tilde{\eta}_1 - \tilde{\eta}_1)^2 \tilde{\eta}_1 \, dx
\]
\[
- \frac{1}{2} \int_{-\infty}^{\infty} (K^0 \tilde{\eta}_1 - \tilde{\eta}_1)^2 \tilde{\eta}_1 \, dx + \frac{1}{2} \int_{-\infty}^{\infty} \tilde{\eta}_1^2 \tilde{\eta}_1 \, dx
\]
(see proposition 4.3) and estimate
\[ \left| \int_{-\infty}^{\infty} \tilde{\eta}_1^2 \tilde{\eta}_1 \, dx \right| \leq \| \tilde{\eta}_1 \|_\infty \| \tilde{\eta}'_1 \|_0^2 \]
\[ \leq c \mu^{5\alpha/2} \| \tilde{\eta}_1 \|_\alpha^3 \]
\[ \leq c \mu^{3/2+5\alpha/2} \]
\[ = o(\mu^{5/3}), \]
\[ \left| \int_{-\infty}^{\infty} \tilde{\eta}_1^2 (K^0 \tilde{\eta}_1 - \tilde{\eta}_1) \, dx \right| \leq \| \tilde{\eta}_1 \|_\infty \| \tilde{\eta}_1 \|_0 \| K^0 \tilde{\eta}_1 - \tilde{\eta}_1 \|_0 \]
\[ \leq c \mu^{1/2+5\alpha/2} \| \tilde{\eta}_1 \|_\alpha^2 \]
\[ \leq c \mu^{3/2+5\alpha/2} \]
\[ = o(\mu^{5/3}), \]
\[ \left| \int_{-\infty}^{\infty} \tilde{\eta}_1 (K^0 \tilde{\eta}_1 - \tilde{\eta}_1)^2 \, dx \right| \leq \| \tilde{\eta}_1 \|_\infty \| K^0 \tilde{\eta}_1 - \tilde{\eta}_1 \|_0^2 \]
\[ \leq c \mu^{9\alpha/2} \| \tilde{\eta}_1 \|_\alpha^3 \]
\[ \leq c \mu^{3/2+9\alpha/2} \]
\[ = o(\mu^{5/3}), \]
in which the calculation
\[ \| K^0 \eta - \eta \|_0^2 = \int_{-\infty}^{\infty} (|k| \coth |k| - 1)^2 |\hat{\eta}(k)|^2 \, dk \]
\[ \leq c \int_{-\infty}^{\infty} k^4 |\hat{\eta}(k)|^2 \, dk \]
\[ = c \| \eta'' \|_0^2 \leq c \mu^{4\alpha} \| \eta \|_\alpha^2 \]
for \( \eta \in H^2(\mathbb{R}) \) has been used. One concludes that
\[ K_3(\tilde{\eta}_1) + 2\nu_0 G_3(\tilde{\eta}_1) - \nu_0^2 L_3(\tilde{\eta}_1) = \frac{1}{2} \left( \frac{1}{2} \omega^2 + \omega \nu_0 + \nu_0^2 \right) \int_{-\infty}^{\infty} \hat{\eta}_1^2 \, dx + o(\mu^{5/3}). \]

**Lemma 4.13.** The estimates
\[ M_{a^2\mu}(a\tilde{\eta}) = a^3 (K_3(\tilde{\eta}) + 2\nu_0 G_3(\tilde{\eta}) - \nu_0^2 L_3(\tilde{\eta})) + a^3 o(\mu^{5/3}), \]
\[ \langle M_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta} \rangle + 4a^2 \mu \hat{M}_{a^2\mu}(a\tilde{\eta}) = 3a^3 (K_3(\tilde{\eta}) + 2\nu_0 G_3(\tilde{\eta}) - \nu_0^2 L_3(\tilde{\eta})) + a^3 o(\mu^{5/3}) \]
hold uniformly over \( a \in [1, 2] \).

**Proof.** Using lemma 4.7, the estimates given in proposition 4.4 and
\[ \frac{\mu + G_2(\eta)}{L_2(\eta)} = O(1), \]
we find that
\[
\mathcal{M}_{a^2}(a\tilde{\eta}) = a^3 \left[ K_3(\tilde{\eta}) + 2\nu_0 G_3(\tilde{\eta}) - \nu_0^2 \mathcal{L}_3(\tilde{\eta}) + 2 \left( \frac{\mu + G_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - \nu_0 \right) G_3(\tilde{\eta}) \right. \\
- \left. \left( \frac{\mu + G_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - \nu_0 \right) \left( \frac{\mu + G_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} + \nu_0 \right) \mathcal{L}_3(\tilde{\eta}) \right] + O(a^4 \mu^{3/2} (||\tilde{\eta}||_{1, \infty} + ||\tilde{\eta}''||_0))
\]
uniformly over \(a \in [1, 2]\). The first result follows by estimating
\[
||\tilde{\eta}||_{1, \infty} + ||\tilde{\eta}''||_0 \leq c(\mu^{\alpha/2} ||\tilde{\eta}_1||_\alpha + ||\tilde{\eta}_2||_2) \leq c\mu^{1/2+\alpha/2},
\]
and \(a^4 \leq 2a^3\). The second result is derived in a similar fashion.

\[\square\]

**Corollary 4.14.** The estimates
\[
\mathcal{M}_{a^2}(a\tilde{\eta}) = \frac{1}{2} a^3 (\frac{1}{3} \omega^2 + 1) \int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, dx + a^3 o(\mu^{5/3}),
\]
\[
\langle \mathcal{M}_{a^2}(a\tilde{\eta}), a\tilde{\eta} \rangle + 4a^2 \mu \mathcal{M}_{a^2}(a\tilde{\eta}) = \frac{3}{2} a^3 (\frac{1}{3} \omega^2 + 1) \int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, dx + a^3 o(\mu^{5/3})
\]
hold uniformly over \(a \in [1, 2]\) and
\[
\int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, dx \leq -c\mu^{5/3}.
\]

**Proof.** The estimates follow from propositions 4.11 and 4.12 and lemma 4.13, while the inequality for \(\tilde{\eta}\) is a consequence of the first estimate (with \(a = 1\)) and the fact that \(\mathcal{M}_\mu(\tilde{\eta}) \leq -c\mu^{5/3}\).

\[\square\]

### 4.3. The case \(\beta < \beta_c\)

#### 4.3.1. Estimates for near minimizers

We begin with an observation that shows that the equation for \(\eta_1\) may be written as
\[
g(k)\hat{\eta}_1 = \chi_S(k)\mathcal{F}[\mathcal{S}(\eta)],
\]
where
\[
\mathcal{S}(\eta) = J'_\mu(\eta) - K'_{a\eta}(\eta) + K_3(\eta_1) - 2 \left( \frac{\mu + G(\eta)}{\mathcal{L}(\eta)} - \nu_0 \right) G'_2(\eta) \]
\[
- 2 \left( \frac{\mu + G(\eta)}{\mathcal{L}(\eta)} \right) (G'_{a\eta}(\eta) - G'_3(\eta)) \]
\[
+ \left( \frac{\mu + G(\eta)}{\mathcal{L}(\eta)} + \nu_0 \right) \left( \frac{\mu + G(\eta)}{\mathcal{L}(\eta)} - \nu_0 \right) \mathcal{L}'_2(\eta) \]
\[
+ \left( \frac{\mu + G(\eta)}{\mathcal{L}(\eta)} \right)^2 (\mathcal{L}'_{a\eta}(\eta) - \mathcal{L}'_3(\eta)).
\]
Proposition 4.15. The identity
\[ \chi_S \mathcal{F} \left[ \begin{pmatrix} G_3'(\eta_1) \\ K_3'(\eta_1) \\ L_3'(\eta_1) \end{pmatrix} \right] = 0 \]
holds for each \( \eta \in U \).

Proof. Using (4.10)–(4.12) we find that the supports of \( G_3'(\eta_1) \), \( K_3'(\eta_1) \) and \( L_3'(\eta_1) \) lie in the set \( [-2\delta_0 - 2\delta_0, 2\delta_0 - 2\delta_0] \cup [-2\delta_0, 2\delta_0] \cup [2\delta_0 - 2\delta_0, 2\delta_0 + 2\delta_0] \).

In keeping with (4.15), we write the equation for \( \eta_2 \) in the form
\[ \eta_2 + H(\eta) = \mathcal{F}^{-1} \left[ \frac{1 - \chi_S(k)}{g(k)} \mathcal{F}[\mathcal{S}(\eta)] \right], \]
where
\[ H(\eta) = \mathcal{F}^{-1} \left[ \frac{1}{g(k)} \mathcal{F} \left[ K_3'(\eta_1) + 2 \left( \frac{\mu + G(\eta)}{L(\eta)} \right) G_3'(\eta_1) - \left( \frac{\mu + G(\eta)}{L(\eta)} \right)^2 L_3'(\eta_1) \right] \right]; \]
the decomposition \( \eta = \eta_1 - H(\eta) + \eta_3 \) forms the basis of the calculations presented below. An estimate on the size of \( H(\eta) \) is obtained from (4.16) and proposition 4.6.

Proposition 4.16. The estimate
\[ \| H(\eta) \|_2 \leq c(\| \eta_1 \|_1, 1 + \| \eta'' \| + k_2^2 \eta_1 \|_0 + \| K^0 \eta_1 \|_1, \infty + \| \eta_3 \|_2) \| \eta_1 \|_2 \]
holds for each \( \eta \in U \).

The above results may be used to derive estimates for the gradients of the cubic parts of the functionals that are used in the analysis below.

Proposition 4.17. The function \( \tilde{\eta} \) satisfies the estimates
\[ \left\{ \begin{array}{l} \| G_3'(\tilde{\eta}) - G_3'(\tilde{\eta}_1) \|_0 \\ \| K_3'(\tilde{\eta}) - K_3'(\tilde{\eta}_1) \|_0 \\ \| L_3'(\tilde{\eta}) - L_3'(\tilde{\eta}_1) \|_0 \end{array} \right\} \leq c \mu^{1/2} \left( (\| \tilde{\eta}_1 \|_1, \infty + \| \tilde{\eta}'' \| + k_2^2 \tilde{\eta}_1 \|_0 + \| K^0 \tilde{\eta}_1 \|_1, \infty)^2 + \| \tilde{\eta}_3 \|_2 \right). \]

Proof. Observe that
\[ G_3'(\eta) - G_3'(\eta_1) = m_2(H(\eta), H(\eta)) + m_2(\eta_3, \eta_3) \]
\[ - 2m_2(\eta_1, H(\eta)) - 2m_2(\eta_3, H(\eta)) + 2m_2(\eta_1, \eta_3) \]
and estimate the right-hand side of this equation using propositions 4.6 and 4.16. The same method yields the results for \( K_3' \) and \( L_3' \).

Estimates for \( G_3(\tilde{\eta}), K_3(\tilde{\eta}) \) and \( L_3(\tilde{\eta}) \) are obtained in a similar fashion.

Proposition 4.18. The function \( \tilde{\eta} \) satisfies the estimates
\[ \left\{ \begin{array}{l} \| G_3(\tilde{\eta}) \| \\ \| K_3(\tilde{\eta}) \| \\ \| L_3(\tilde{\eta}) \| \end{array} \right\} \leq c(\| \tilde{\eta}_1 \|_1 + \| \tilde{\eta}'' \| + k_2^2 \tilde{\eta}_1 \|_0 + \| K^0 \tilde{\eta}_1 \|_1, \infty) \| \tilde{\eta}_3 \|_2). \]
Proof. Observe that
\[
G_3(\eta_1) = \int_{-\infty}^{\infty} F[G_3'(\eta_1)] e^{i\eta_1 k} \, dk = \frac{1}{3} \int_{-\infty}^{\infty} \chi_S(k) F[G_3'(\eta_1)] e^{i\eta_1 k} \, dk
\]
\[
= \int_{-\infty}^{\infty} \chi_S(k) F[G_3'(\eta_1)] e^{i\eta_1 k} \, dk = 0
\]
(since \(\hat{\eta}_1 = \chi_S(k)\hat{\eta}_1\)), so that
\[
G_3(\eta) = G_3(\eta) - G_3(\eta_1)
\]
\[
= -n_2(H(\eta), H(\eta), H(\eta)) + n_2(\eta_3, \eta_3, \eta_3) - 6n_2(\eta_1, H(\eta), \eta_3)
\]
\[
- 3n_2(\eta_1, \eta_1, H(\eta)) + 3n_2(\eta_1, \eta_1, \eta_3) + 3n_2(H(\eta), H(\eta), \eta_3)
\]
\[
+ 3n_2(H(\eta), H(\eta), \eta_1) + 3n_2(\eta_3, \eta_3, \eta_1) - 3n_2(\eta_3, \eta_3, H(\eta))
\]
and estimate the right-hand side of this equation using propositions 4.6 and 4.16. The same method yields the results for \(K_3\) and \(L_3\).

Estimating the right-hand sides of the inequalities
\[
\|G_n'(\tilde{\eta}) - G_n'(\tilde{\eta}_1)\| \leq \|G_n'(\tilde{\eta})\| + \|G_n'(\tilde{\eta}_1)\|,
\]
\[
\|G_n(\tilde{\eta})\| \leq |G_n(\tilde{\eta})| + \|G_n(\tilde{\eta}_1)\| + \|G_n(\tilde{\eta}_1)\|
\]
(together with the corresponding inequalities for \(K\) and \(L\)) using propositions 4.4 and 4.5, the calculation
\[
\|\eta\|_{1,\infty} + \|\eta'' + k_0^2 \eta\|_0 + \|K^0 \eta\|_{\infty}
\]
\[
\leq c(\|\eta_1\|_{1,\infty} + \|\eta''_1 + k_0^2 \eta_1\|_0 + \|K^0 \eta_1\|_{\infty} + \|H(\eta)\|_2 + \|\eta_3\|_2)
\]
\[
\leq c(\|\eta_1\|_{1,\infty} + \|\eta''_1 + k_0^2 \eta_1\|_0 + \|K^0 \eta_1\|_{1,\infty} + \|\eta_3\|_2) \quad (4.17)
\]
and propositions 4.17 and 4.18 yields the following estimates for the ‘nonlinear’ parts of the functionals.

**Lemma 4.19.** The function \(\tilde{\eta}\) satisfies the estimates
\[
\begin{align*}
\left\{ \|G_n'(\tilde{\eta}) - G_n'(\tilde{\eta}_1)\| \right\} \\
\left\{ \|K_n'(\tilde{\eta}) - K_n'(\tilde{\eta}_1)\| \right\} \\
\left\{ \|L_n'(\tilde{\eta}) - L_n'(\tilde{\eta}_1)\| \right\}
\end{align*}
\]
\[
\leq c(\mu^{1/2}(\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}'_1 + k_0^2 \tilde{\eta}_1\|_0 + \|K^0 \tilde{\eta}_1\|_{1,\infty})^2 + \mu^{1/2}\|\tilde{\eta}_3\|_2),
\]
\[
\begin{align*}
\left\{ \|G_n(\tilde{\eta})\| \right\} \\
\left\{ \|K_n(\tilde{\eta})\| \right\} \\
\left\{ \|L_n(\tilde{\eta})\| \right\}
\end{align*}
\]
\[
\leq c(\mu(\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}'_1 + k_0^2 \tilde{\eta}_1\|_0 + \|K^0 \tilde{\eta}_1\|_{1,\infty})^2 + \mu\|\tilde{\eta}_3\|_2).
\]

We now have all the ingredients necessary to estimate the wave speed and the quantity \(\|\tilde{\eta}_1\|_0\).
Proof. Combining lemma 4.7, inequality (4.17) and lemma 4.19, one finds that
\[
\left(\frac{\mu + G(\eta)}{L(\eta)} - \nu_0, \frac{\mu + G_2(\eta)}{L_2(\eta)} - \nu_0\right) \leq c\left(\|\eta_1\|_{1,\infty} + \|\eta''_1 + k_0^2 \eta_1\|_0 + \|K^0 \eta_1\|_{1,\infty}\right)^2 + \|\eta_3\|_2 + \mu^{N-1/2}).
\]

Proposition 4.20. The function $\eta$ satisfies the estimates

\[
\|\eta_1\|^2 \leq c\mu, \quad \|\eta_3\|^2 \leq c\mu^{3+2\alpha} \quad \text{and} \quad \|H(\eta)\|^2 \leq c\mu^{2+\alpha} \quad \text{for} \quad \alpha < 1.
\]

Proof. Lemma 4.19 and proposition 4.20 assert that
\[
\|S(\eta)\| \leq c\mu^{1/2}(\|\eta_1\|_{1,\infty} + \|\eta''_1 + k_0^2 \eta_1\|_0 + \|K^0 \eta_1\|_{1,\infty})^2 + \mu^{1/2} \|\eta_3\|_2 + \mu^N),
\]

whereby
\[
\|\eta_3\|_2 \leq c\mu^{1/2}(\|\eta_1\|_{1,\infty} + \|\eta''_1 + k_0^2 \eta_1\|_0 + \|K^0 \eta_1\|_{1,\infty})^2 + \mu^{1/2} \|\eta_3\|_2 + \mu^N),
\]

and therefore
\[
\|\eta_3\|_2 \leq c\mu^{1/2}(\|\eta_1\|_{1,\infty} + \|\eta''_1 + k_0^2 \eta_1\|_0 + \|K^0 \eta_1\|_{1,\infty})^2 + \mu^N)) \quad (4.18)
\]

and
\[
\int_{-\infty}^{\infty} g(k)|\eta_1|^2 \, dk \leq c\mu(\|\eta_1\|_{1,\infty} + \|\eta''_1 + k_0^2 \eta_1\|_0 + \|K^0 \eta_1\|_{1,\infty})^4 + \mu \|\eta_3\|_2^2 + \mu^{2N})
\]

Multiplying the above inequality by $\mu^{-4\alpha}$ and adding $\|\eta_1\|^2 \leq \|\eta\|^2 \leq c\mu$, one finds that
\[
\|\eta_1\|^2 \leq c\mu^{1-4\alpha}(\|\eta_1\|_{1,\infty} + \|\eta''_1 + k_0^2 \eta_1\|_0 + \|K^0 \eta_1\|_{1,\infty})^4 + \mu) \quad (4.19)
\]

where proposition 4.1 and the fact that $g(k) \geq c(|k| - k_0)^2$ for $k \in S$ have also been used.

The estimate for $\eta_1$ follows from the previous inequality using the argument given by Groves and Wahlén [17, p. 401], while those for $\eta_3$ and $H(\eta)$ are derived by estimating $\|\eta_1\|^2 \leq c\mu$ in (4.18) and proposition 4.16. 

\[\square\]
4.3.2. Estimates for the variational functional

The next step is to identify the dominant terms in the formulas for \( \mathcal{M}_\mu(\eta) \) and \( (\mathcal{M}'_\mu(\eta), \eta) + 4\mu \mathcal{M}'_\mu(\eta) \) given in lemma 4.7. We begin by examining the quantities \( \mathcal{G}_4(\eta) \), \( \mathcal{K}_4(\eta) \) and \( \mathcal{L}_4(\eta) \).

**Proposition 4.22.** The function \( \eta \) satisfies the estimates

\[
\begin{align*}
\left\{ \mathcal{G}_4(\eta) \right\} &= \left\{ \mathcal{G}_4(\eta_1) \right\} + o(\mu^3).
\end{align*}
\]

**Proof.** Write

\[
\begin{align*}
\mathcal{K}_4(\eta) &= p_1(\eta, \eta, \eta, \eta), \quad \mathcal{G}_4(\eta) = p_2(\eta, \eta, \eta, \eta), \quad \mathcal{L}_4(\eta) = p_3(\eta, \eta, \eta, \eta),
\end{align*}
\]

where \( p_j \in \mathcal{L}_4^4(H^2(\mathbb{R}), \mathbb{R}) \), \( j = 1, 2, 3 \), are defined by

\[
\begin{align*}
p_1(u_1, u_2, u_3, u_4) &= -\frac{1}{8} \int_{-\infty}^{\infty} u_1' u_2' u_3' u_4' \, dx - \frac{1}{4\pi} \omega^2 \int_{-\infty}^{\infty} \mathcal{P}[u_1 u_2 K^0(u_3 u_4)] \, dx, \\
p_2(u_1, u_2, u_3, u_4) &= \frac{1}{12} \omega \int_{-\infty}^{\infty} \mathcal{P}[u_1 u_2 u_3' u_4'] \, dx - \frac{1}{4\pi} \omega \int_{-\infty}^{\infty} \mathcal{P}[u_1 u_2 K^0(u_3 K^0 u_4)] \, dx, \\
p_3(u_1, u_2, u_3, u_4) &= \frac{1}{24} \int_{-\infty}^{\infty} \mathcal{P}[u_1 u_2 K^0 u_3 u_4'] \, dx \\
&\quad + \frac{1}{48} \int_{-\infty}^{\infty} \mathcal{P}[K^0(u_1 K^0 u_2) u_3 K^0 u_4] \, dx,
\end{align*}
\]

and estimate each term in the expansion of

\[
p_j(\eta - H(\eta) + \delta_3, \eta - H(\eta) + \delta_3, \eta - H(\eta) + \delta_3, \eta - H(\eta) + \delta_3) - p_j(\eta_1, \eta_2, \eta_3, \eta_4)
\]

for \( j = 1, 2, 3 \). Terms with zero, one or two occurrences of \( \eta_1 \) are estimated by

\[
\begin{align*}
p_j \left( \left\{ \eta_1 \right\}^{(2)} \right) \left( \left\{ H(\eta) \right\} \right) \left( \left\{ \eta_3 \right\} \right) &\leq c \left\{ \left\| \eta_1 \right\|_2 \right\}^2 \left\{ \left\| H(\eta) \right\|_2 \right\} \left\| \eta_3 \right\|_2 \leq c \mu^2 + \alpha = o(\mu^3),
\end{align*}
\]

while terms with three or four occurrences of \( \eta_1 \) are estimated by

\[
\begin{align*}
p_j \left( \left\{ \eta_1 \right\}^{(3)} \right) \left( \left\{ H(\eta) \right\} \right) &\leq c \left\{ \left\| \eta_1 \right\|^\infty \right\} \left\| \eta_3 \right\|_2 \left\| H(\eta) \right\|_2 \leq c \mu^{5/2} \left\| \eta_1 \right\|_2 \left\| \eta_3 \right\|_2 \leq c \mu^{5/2} + \alpha = o(\mu^3).
\end{align*}
\]

To identify the dominant terms in \( \mathcal{G}_4(\eta_1) \), \( \mathcal{K}_4(\eta_1) \) and \( \mathcal{L}_4(\eta_1) \) we use the following result, which shows how Fourier-multiplier operators acting upon the function \( \eta_1 \),
whose spectrum is concentrated near \( k = \pm k_0 \), may be approximated by multiplication by constants.

**Lemma 4.23.** For each \( \eta \in H^2(\mathbb{R}) \) with \( \| \eta \|_2 \leq c\mu^{1/2} \) the quantities
\[
\eta^+_1 := \mathcal{F}^{-1}[\chi_{[0,\infty)} \hat{\eta}_1], \quad \eta^-_1 := \mathcal{F}^{-1}[\chi_{(-\infty,0)} \hat{\eta}_1] = \overline{\eta}_1^+
\]
satisfy the estimates

(i) \( \eta^+_1 = \pm ik_0 \eta^+_1 + O(\mu^{1/2 + \alpha}) \),

(ii) \( K^0(\eta^+_1) = f(k_0) \eta^+_1 + O(\mu^{1/2 + \alpha}) \),

(iii) \( ((\eta^+_1)^2)' = \pm 2k_0i(\eta^+_1)^2 + O(\mu^{1 + 3\alpha/2}) \),

(iv) \( (\eta^+_1 \eta^-_1)' = O(\mu^{1 + 3\alpha/2}) \),

(v) \( K^0((\eta^+_1)^2) = f(2k_0)(\eta^+_1)^2 + O(\mu^{1 + 3\alpha/2}) \),

(vi) \( K^0(\eta^+_1 \eta^-_1) = \eta^+_1 \eta^-_1 + O(\mu^{1 + 3\alpha/2}) \).

Here the symbol \( O(\mu^\gamma) \) denotes a quantity whose Fourier transform has compact support and whose \( L^2(\mathbb{R}) \)-norm (and hence \( H^s(\mathbb{R}) \)-norm for \( s \geq 0 \)) is \( O(\mu^\gamma) \).

**Proof.** Estimates (i) and (ii) follow from the calculations
\[
\| (ik \mp ik_0) \hat{\eta}^+_1 \|^2_0 = \| (|k| - k_0) \hat{\eta}_1 \|^2_0, \quad \| (K^0 - f(k_0))(\eta^+_1)^2 \|^2_0 \leq c\| (|k| - k_0) \hat{\eta}_1 \|^2_0
\]
(because \( f(k) = f(k_0) + O(|k| - k_0) \) for \( k \in S \)) and
\[
\| (|k| - k_0) \hat{\eta}_1 \|^2_0 \leq \frac{1}{2} \int_{-\infty}^{\infty} (\mu^{2\alpha} + \mu^{-2\alpha} (|k| - k_0)^4) |\hat{\eta}_1|^2 \, dk \leq c\mu^{2\alpha} \| \eta_1 \|^2_0 \leq c\mu^{1 + 2\alpha},
\]
while (iii) and (iv) are obtained from the observations
\[
\| (\partial_x \mp 2ik_0)(\eta^+_1)^2 \|_0 = \| 2((\partial_x \mp k_0i)\eta^+_1)\eta^+_1 \|_0 \\
\leq 2\| (\partial_x \mp k_0i)\eta^+_1 \|_0 \| \eta^+_1 \|_\infty \\
\leq c\mu^{1/2 + 3\alpha/2} \| \eta^+_1 \|_\alpha \\
\leq c\mu^{1 + 3\alpha/2}
\]

and
\[
\| (\eta^+_1 \eta^-_1)' \|_0 = \| ((\partial_x - ik_0)\eta^+_1)\eta^-_1 + \eta^+_1 ((\partial_x + ik_0)\eta^-_1) \|_0 \\
\leq \| (\partial_x - ik_0)\eta^+_1 \|_0 \| \eta^-_1 \|_\infty + \| \eta^+_1 \|_\infty \| (\partial_x + ik_0)\eta^-_1 \|_0 \\
\leq c\mu^{1 + 3\alpha/2},
\]
in which proposition 4.1 has been used. Estimates (v) and (vi) are deduced from (iii) and (iv), respectively, by means of the inequalities
\[
\| (K^0 - f(2k_0))(\eta^+_1)^2 \|^2_0 \leq c\| (|k| - 2k_0)\mathcal{F}[(\eta^+_1)^2] \|^2_0 = \| (ik \mp ik_0)\mathcal{F}[(\eta^+_1)^2] \|^2_0
\]
(because \( f(k) = f(2k_0) + O(|k| - 2k_0) \) for \( k \in 2S \)) and
\[
\| (K^0 - f(0)) \eta_1^+ \eta_1^- \|_0 \leq c \| kF [\eta_1^+ \eta_1^-] \|_0^2 = \| ikF [\eta_1^+ \eta_1^-] \|_0^2
\]
(because \( f(k) = f(0) + O(|k|) \) for \( k \in [-2\delta_0, 2\delta_0] \)), and (vii) and (viii) are deduced from (iii) and (iv) in the same fashion.

**Proposition 4.24.** The function \( \tilde{\eta}_1 \) satisfies the estimates
\[
\mathcal{K}_4(\tilde{\eta}_1) = A_4^1 \int_{-\infty}^{\infty} \tilde{\eta}_1^1 dx + o(\mu^3), \quad A_4^1 = -\frac{1}{3} \beta \omega k_0^4 - \frac{1}{32} \omega^2 (f(2k_0) + 2),
\]
\[
\mathcal{G}_4(\tilde{\eta}_1) = A_4^2 \int_{-\infty}^{\infty} \tilde{\eta}_1^1 dx + o(\mu^3), \quad A_4^2 = \frac{1}{5} \omega k_0^2 - \frac{1}{12} \omega f(k_0)(f(2k_0) + 2),
\]
\[
\mathcal{L}_4(\tilde{\eta}_1) = A_4^3 \int_{-\infty}^{\infty} \tilde{\eta}_1^1 dx + o(\mu^3), \quad A_4^3 = \frac{1}{6} f(k_0)^2 (f(2k_0) + 2) - \frac{1}{2} k_0^2 f(k_0).
\]

**Proof.** Using the formulas given in lemma 4.23, we find that
\[
\int_{-\infty}^{\infty} \tilde{\eta}_1^2 \tilde{\eta}_1^2 dx = \int_{-\infty}^{\infty} ((\tilde{\eta}_1^1)^2 (\tilde{\eta}_1^1)' + (\tilde{\eta}_1^1)'(\tilde{\eta}_1^1)^2) + 4 \tilde{\eta}_1^1 \tilde{\eta}_1^- (\tilde{\eta}_1^1)'(\tilde{\eta}_1^-)' dx
\]
\[
= 2k_0^2 \int_{-\infty}^{\infty} (\tilde{\eta}_1^1)^2 (\tilde{\eta}_1^-)^2 dx + o(\mu^3),
\]
and similarly
\[
\int_{-\infty}^{\infty} K^0(\tilde{\eta}_1^1) \tilde{\eta}_1 K^0 \tilde{\eta}_1 dx = (2f(2k_0)f(k_0) + 4f(k_0))
\]
\[
\times \int_{-\infty}^{\infty} (\tilde{\eta}_1^1)^2 (\tilde{\eta}_1^-)^2 dx + o(\mu^3),
\]
\[
\int_{-\infty}^{\infty} (\tilde{\eta}_1^1)^4 dx = 6k_0^4 \int_{-\infty}^{\infty} (\tilde{\eta}_1^1)^2 (\tilde{\eta}_1^-)^2 dx + o(\mu^3),
\]
\[
\int_{-\infty}^{\infty} \tilde{\eta}_1^2 K^0(\tilde{\eta}_1^1) \tilde{\eta}_1 dx = (2f(2k_0) + 4) \int_{-\infty}^{\infty} (\tilde{\eta}_1^1)^2 (\tilde{\eta}_1^-)^2 dx + o(\mu^3),
\]
\[
\int_{-\infty}^{\infty} K^0(\tilde{\eta}_1^1) \tilde{\eta}_1 K^0 \tilde{\eta}_1 dx = (2f(2k_0)f(k_0)^2 + 4f(k_0)^2)
\]
\[
\times \int_{-\infty}^{\infty} (\tilde{\eta}_1^1)^2 (\tilde{\eta}_1^-)^2 dx + o(\mu^3),
\]
\[
\int_{-\infty}^{\infty} (K^0 \tilde{\eta}_1) \tilde{\eta}_1^2 \tilde{\eta}_1^2 dx = -6k_0^2 f(k_0) \int_{-\infty}^{\infty} (\tilde{\eta}_1^1)^2 (\tilde{\eta}_1^-)^2 dx + o(\mu^3).
\]

The result is obtained by substituting the above expressions into the explicit formulas for \( \mathcal{K}_4, \mathcal{G}_4 \) and \( \mathcal{L}_4 \) given in proposition 4.3.

**Corollary 4.25.** The function \( \tilde{\eta} \) satisfies the estimate
\[
\mathcal{K}_4(\tilde{\eta}) + 2\nu_2 \mathcal{G}_4(\tilde{\eta}) - \nu_2^2 \mathcal{L}_4(\tilde{\eta}) = A_4 \int_{-\infty}^{\infty} \tilde{\eta}_1^1 dx + o(\mu^3),
\]
where \( A_4 = A_4^1 + 2\nu_2 A_4^2 - \nu_2^2 A_4^3. \)
We now turn to the corresponding result for $\mathcal{G}_3(\tilde{\eta}), \mathcal{K}_3(\tilde{\eta})$ and $\mathcal{L}_3(\tilde{\eta})$.

**Proposition 4.26.** The function $\tilde{\eta}$ satisfies the estimate

\[
\begin{cases}
\mathcal{G}_3(\tilde{\eta}) \\
\mathcal{K}_3(\tilde{\eta}) \\
\mathcal{L}_3(\tilde{\eta})
\end{cases} = - \int_{-\infty}^{\infty} \begin{cases}
\mathcal{G}'_3(\tilde{\eta}_1) \\
\mathcal{K}'_3(\tilde{\eta}_1) \\
\mathcal{L}'_3(\tilde{\eta}_1)
\end{cases} \cdot H(\tilde{\eta}) \, dx + o(\mu^3).
\]

**Proof.** Each term in the expansion of

\[
n_2(\tilde{\eta}_1 - H(\tilde{\eta}), H(\tilde{\eta}) + \tilde{\eta}_3 - H(\tilde{\eta}) + \tilde{\eta}_3)
\]

with zero or one occurrence of $\tilde{\eta}_1$ can be estimated by

\[
\left| n_2 \left( \begin{array}{c}
\tilde{\eta}_1 \\
H(\tilde{\eta}) \\
\tilde{\eta}_3
\end{array} \right), \begin{array}{c}
\left\{ \right. \\
\left. \right. \\
\left. \right.
\end{array} \right| \leq c \left\{ \begin{array}{c}
\| \tilde{\eta}_1 \|_2 \\
\| H(\tilde{\eta}) \|_2 \\
\| \tilde{\eta}_3 \|_2
\end{array} \right\} \leq c\mu^{1/2}\mu^{2+\alpha} = o(\mu^3),
\]

while

\[
|n_2(\tilde{\eta}_1, \tilde{\eta}_1, \tilde{\eta}_3)| \leq c\| \tilde{\eta}_2 \|_2 \leq c\mu^{3/2+\alpha} = o(\mu^3)
\]

and

\[
n_2(\tilde{\eta}_1, \tilde{\eta}_1, \tilde{\eta}_1) = \mathcal{G}_3(\tilde{\eta}) = 0.
\]

It follows that

\[
\mathcal{G}_3(\tilde{\eta}) = -3n_2(\tilde{\eta}_1, \tilde{\eta}_1, H(\tilde{\eta})) + o(\mu^3)
\]

\[
= -d\mathcal{G}_3(\tilde{\eta})(H(\tilde{\eta})) + o(\mu^3)
\]

\[
= - \int_{-\infty}^{\infty} \mathcal{G}'_3(\tilde{\eta}_1)H(\tilde{\eta}) \, dx + o(\mu^3).
\]

The same argument yields the results for $\mathcal{K}_3(\tilde{\eta})$ and $\mathcal{L}_3(\tilde{\eta})$.

\[\square\]

**Proposition 4.27.** The function $\tilde{\eta}$ satisfies the estimate

\[
H(\tilde{\eta}) = \mathcal{F}^{-1} \left[ \frac{1}{g(k)} \mathcal{F}[K'_3(\tilde{\eta}_1) + 2\nu_0 G'_3(\tilde{\eta}_1) - \nu_0^2 L'_3(\tilde{\eta}_1)] \right] + o(\mu^3).
\]

**Proof.** Noting that

\[
\left| \frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\eta)} - \nu_0 \right| \leq c(\mu^{\alpha \| \tilde{\eta}_1 \|_2^2 + \| \tilde{\eta}_3 \|_2 + \mu^{N-1/2}) = O(\mu^{1+\alpha})
\]

(see corollary 4.20) and

\[
\left\{ \begin{array}{c}
\| \mathcal{G}'_3(\tilde{\eta}_1) \|_0 \\
\| K'_3(\tilde{\eta}_1) \|_0 \\
\| L'_3(\tilde{\eta}_1) \|_0
\end{array} \right\} \leq c\mu^{\alpha/2}\| \tilde{\eta}_1 \|_2 \| \tilde{\eta}_1 \|_2 = O(\mu^{1+\alpha/2})
\]

(see proposition 4.5), one finds that

\[
H(\tilde{\eta}) = \mathcal{F}^{-1} \left[ \frac{1}{g(k)} \mathcal{F}[K'_3(\tilde{\eta}_1) + 2\nu_0 G'_3(\tilde{\eta}_1) - \nu_0^2 L'_3(\tilde{\eta}_1)] \right] + O(\mu^{1+\alpha}Q(\mu^{1+\alpha/2}))
\]

(see proposition 4.5).
Combining propositions 4.26 and 4.27, one finds that
\[ \mathcal{K}_3(\bar{\eta}) + 2\nu_0 \mathcal{G}_3(\bar{\eta}) - \nu_0^2 \mathcal{L}_3(\bar{\eta}) \]
\[ = -\int_{-\infty}^{\infty} (\mathcal{K}_3'(\bar{\eta}_1) + 2\nu_0 \mathcal{G}_3'(\bar{\eta}_1) - \nu_0^2 \mathcal{L}_3'(\bar{\eta}_1)) \]
\[ \times \mathcal{F}^{-1} \left[ \frac{1}{g(k)} \mathcal{F}[\mathcal{K}_3(\bar{\eta}) + 2\nu_0 \mathcal{G}_3(\bar{\eta}) - \nu_0^2 \mathcal{L}_3(\bar{\eta})] \right] \, dx + o(\mu^3), \quad (4.20) \]
which we write as
\[ \mathcal{K}_3(\bar{\eta}) + 2\nu_0 \mathcal{G}_3(\bar{\eta}) - \nu_0^2 \mathcal{L}_3(\bar{\eta}) \]
\[ = -2 \int_{-\infty}^{\infty} M(\bar{\eta}_1^+, \bar{\eta}_1^-) \mathcal{F}^{-1} [g(k)^{-1} M(\bar{\eta}_1^+, \bar{\eta}_1^-)] \, dx \]
\[ - 4 \int_{-\infty}^{\infty} M(\bar{\eta}_1^+, \bar{\eta}_1^-) \mathcal{F}^{-1} [g(k)^{-1} M(\bar{\eta}_1^+, \bar{\eta}_1^-)] \, dx + o(\mu^3), \quad (4.21) \]
in order to determine the dominant term on its right-hand side.

**Proposition 4.28.** The function \( \bar{\eta} \) satisfies
\[ \mathcal{K}_3(\bar{\eta}) + 2\nu_0 \mathcal{G}_3(\bar{\eta}) - \nu_0^2 \mathcal{L}_3(\bar{\eta}) = A_3 \int_{-\infty}^{\infty} \bar{\eta}_1^+ \, dx + o(\mu^3), \]
where
\[ A_3 = -\frac{1}{3} g(2k_0)^{-1} (A_3^1)^2 - \frac{2}{3} g(0)^{-1} (A_3^2)^2, \]
\[ A_3^1 = \frac{1}{2} \omega \nu_0 f(2k_0) + \omega \nu_0 f(k_0) + \frac{1}{2} \omega^2 + \nu_0^2 f(2k_0) f(k_0) + \frac{1}{2} \nu_0^2 f(k_0)^2 - \frac{1}{2} k_0^2 \nu_0^2, \]
\[ A_3^2 = \frac{1}{2} \omega \nu_0 + \omega \nu_0 f(k_0) + \frac{1}{2} \omega^2 + \nu_0^2 f(k_0) + \frac{1}{2} \nu_0^2 f(k_0)^2 - \frac{1}{2} k_0^2 \nu_0^2. \]

**Proof.** Lemma 4.23 implies that
\[ M(\bar{\eta}_1^+, \bar{\eta}_1^-) = A_3^1 (\bar{\eta}_1^+)^2 + O(\mu^{1+\alpha}), \]
so that
\[ \mathcal{F}^{-1} [g(k)^{-1} M(\bar{\eta}_1^+, \bar{\eta}_1^-)] = \mathcal{F}^{-1} [g(k)^{-1} M(\bar{\eta}_1^+, \bar{\eta}_1^-)] = g(2k_0)^{-1} A_3^1 (\bar{\eta}_1^-)^2 + O(\mu^{1+\alpha}), \]
and
\[ M(\bar{\eta}_1^+, \bar{\eta}_1^-) = A_3^2 \bar{\eta}_1^+ \bar{\eta}_1^- + O(\mu^{1+\alpha}), \]
so that
\[ \mathcal{F}^{-1} [g(k)^{-1} M(\bar{\eta}_1^+, \bar{\eta}_1^-)] = g(0)^{-1} A_3^2 \bar{\eta}_1^+ \bar{\eta}_1^- + O(\mu^{1+\alpha}); \]
the result follows from these calculations and (4.21). \( \square \)

The requisite estimates for \( \mathcal{M}_\mu(\bar{\eta}) \) and \( \langle M'_\mu(\bar{\eta}), \bar{\eta} \rangle \) + 4\mu \tilde{M}_\mu(\bar{\eta}) may now be derived from corollary 4.25 and proposition 4.28.
Lemma 4.29. The estimates

\[ \mathcal{M}_{a^2}(a\hat{\eta}) = a^3(K_3(\hat{\eta}) + 2\nu_0 G_3(\hat{\eta}) - \nu_0^2 L_3(\hat{\eta})) + a^4(K_4(\hat{\eta}) + 2\nu_0 G_4(\hat{\eta}) - \nu_0^2 L_4(\hat{\eta})) + a^3 o(\mu^3), \]

\[ \langle \mathcal{M}'_{a^2}(a\hat{\eta}), a\hat{\eta} \rangle + 4a^2 \mu \mathcal{M}_{a^2}(a\hat{\eta}) \]

\[ = 3a^3(K_3(\hat{\eta}) + 2\nu_0 G_3(\hat{\eta}) - \nu_0^2 L_3(\hat{\eta})) + a^4(K_4(\hat{\eta}) + 2\nu_0 G_4(\hat{\eta}) - \nu_0^2 L_4(\hat{\eta})) + a^3 o(\mu^3) \]

hold uniformly over \( a \in [1, 2] \).

Proof. Lemma 4.7 asserts that

\[ \mathcal{M}_{a^2}(a\hat{\eta}) = a^3(K_3(\hat{\eta}) + 2\nu_0 G_3(\hat{\eta}) - \nu_0^2 L_3(\hat{\eta})) + a^4(K_4(\hat{\eta}) + 2\nu_0 G_4(\hat{\eta}) - \nu_0^2 L_4(\hat{\eta})) \]

\[ + 2 \left( \frac{\mu + G_2(\hat{\eta})}{L_2(\hat{\eta})} - \nu_0 \right) \left( a^3 G_3(\hat{\eta}) + a^4 G_4(\hat{\eta}) \right) \]

\[ - \left( \frac{\mu + G_2(\hat{\eta})}{L_2(\hat{\eta})} - \nu_0 \right) \left( a^3 L_3(\hat{\eta}) + a^4 L_4(\hat{\eta}) \right) \]

\[ + \frac{a^3}{L_2(\hat{\eta})} (G_3(\hat{\eta}) - \left( \frac{\mu + G_2(\hat{\eta})}{L_2(\hat{\eta})} \right) L_3(\hat{\eta}) \right)^2 \]

\[ + O(a^5 \nu^{3/2}(\|\hat{\eta}\|_{1, \infty} + \|\hat{\eta}'' + k_0^2 \hat{\eta}\|_0)^2) \]

uniformly over \( a \in [1, 2] \).

The first result follows by estimating

\[ \begin{align*}
\|G_3(\hat{\eta})\|_{L_3(\hat{\eta})} & = O(\nu^{3/2}), \\
\|G_4(\hat{\eta})\|_{L_4(\hat{\eta})} & = O(\nu^2),
\end{align*} \]

\[ \|\hat{\eta}\|_{1, \infty} + \|\hat{\eta}'' + k_0^2 \hat{\eta}\|_0 \leq c(\mu + \|\hat{\eta}\|_a + \|\hat{\eta}_3\|_2) \leq c\mu^{1/2 + \alpha/2} \]

(see 4.17),

\[ \left| \frac{\mu + G_2(\hat{\eta})}{L_2(\hat{\eta})} - \nu_0 \right| \leq c(\mu^{\alpha} \|\hat{\eta}\|_a^2 + \|\hat{\eta}_3\|_2 + \mu^{-1/2}) \leq c\mu^{1+\alpha} \]

and noting that

\[ G_3(\hat{\eta}) - \left( \frac{\mu + G_2(\hat{\eta})}{L_2(\hat{\eta})} \right) L_3(\hat{\eta}) = G_3(\hat{\eta}) - \nu_0 L_3(\hat{\eta}) + o(\nu^2) \]

\[ = - \int_{-\infty}^{\infty} (G_3'(\hat{\eta}_1) - \nu_0 L_3'(\hat{\eta}_1)) \]

\[ \times \mathcal{F}^{-1} \left[ \frac{1}{g(k)} \mathcal{F}(K_3'(\hat{\eta}_1) + 2\nu_0 G_3'(\hat{\eta}_1) - \nu_0^2 L_3'(\hat{\eta}_1)) \right] dx + o(\nu^3) \]
Proposition 4.4. Derivation of the strict subadditivity property

In this section we derive the strict subadditivity property (4.1). We begin by showing that $c_\mu$ is a strictly subhomogeneous increasing function of $\mu > 0$. The first of these properties is a corollary of the next proposition.

**Proposition 4.31.** There exist $a_0 \in (1, 2]$ and $q > 2$ with the property that the function $a \mapsto a^{-q} M_{a^2 \mu}(\tilde{\eta} \tilde{\eta})$, $a \in [1, a_0]$, is decreasing and strictly negative.

**Proof.** This result follows from the calculations

$$
\frac{d}{da} (a^{-5/2} M_{a^2 \mu}(\tilde{\eta} \tilde{\eta})) = a^{-7/2} (-\frac{5}{2} M_{a^2 \mu}(\tilde{\eta} \tilde{\eta}) + (M'_{a^2 \mu}(\tilde{\eta} \tilde{\eta}), a \tilde{\eta})_0 + 4 a^2 \mu M_{a^2 \mu}(\tilde{\eta} \tilde{\eta}))
$$

$$
= a^{-1/2} \left( \frac{1}{4} a^3 \left( \frac{1}{4} \omega^3 + 1 \right) \int_{-\infty}^{\infty} \tilde{\eta}_1^4 \, dx + o(\mu^{5/3}) \right)
$$

$$
\leq -c_\mu^{5/3},
$$

where $\tilde{M} = m_2 - \nu_0 m_3$ and $\gamma$ is a (possibly negative) constant. Here, the third line follows from the second by propositions 4.26 and 4.27 and the fifth from the fourth by repeating the proof of proposition 4.28.

The second result is derived in a similar fashion. \hfill \Box

**Corollary 4.30.** The estimates

$$
M_{a^2 \mu}(\tilde{\eta} \tilde{\eta}) = (a^3 A_3 + a^4 A_4) \int_{-\infty}^{\infty} \tilde{\eta}_1^4 \, dx + a^3 o(\mu^3),
$$

$$
(M'_{a^2 \mu}(\tilde{\eta} \tilde{\eta}), a \tilde{\eta}) + 4 a^2 \mu M_{a^2 \mu}(\tilde{\eta} \tilde{\eta}) = (3a^3 A_3 + 4a^4 A_4) \int_{-\infty}^{\infty} \tilde{\eta}_1^4 \, dx + a^3 o(\mu^3)
$$

hold uniformly over $a \in [1, 2]$ and

$$
\int_{-\infty}^{\infty} \tilde{\eta}_1^4 \, dx \geq c_\mu^{3}.
$$

**Proof.** The estimates follow by combining corollary 4.25, proposition 4.28 and lemma 4.29, while the inequality for $\tilde{\eta}_1$ is a consequence of the first estimate (with $a = 1$) and the fact that $M_{\mu}(\tilde{\eta}) \leq -c_\mu^{3}$. \hfill \Box
for $\beta > \beta_c$ (see corollary 4.14) and

$$\frac{d}{da}(a^{-q}M_{a^2\mu}(a\tilde{\eta})) = a^{-(q+1)}(-qM_{a^2\mu}(a\tilde{\eta}) + \{\tilde{M}_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta}\}_0 + 4a^2\mu\tilde{M}_{a^2\mu}(a\tilde{\eta}))$$

$$= a^{-(q+1)}(-q(a^3A_3 + a^4A_4 + 3a^3A_3 + 4a^4A_4) \int_{-\infty}^{\infty} \tilde{\eta}_1^4 dx + a^3 o(\mu^3))$$

$$= a^{2-q}((3 - q)A_3 + a(4 - q)A_4) \int_{-\infty}^{\infty} \tilde{\eta}_1^4 dx + o(\mu^3)$$

$$\leq -c\mu^3$$

$$< 0, \quad a \in (1, a_0), \quad q \in (2, q_0),$$

for $\beta < \beta_c$ (see corollary 4.30); here $a_0 > 1$ and $q_0 > 2$ are chosen so that $(3 - q)A_3 + a(4 - q)A_4$, which is negative for $a = 1$ and $q = 2$ (see Appendix B), is also negative for $a \in (1, a_0]$ and $q \in (2, q_0]$.

**Corollary 4.32.** The number $c_\mu$ is a strictly subhomogeneous function of $\mu > 0$.

**Proof.** The previous lemma implies that

$$M_{a\mu}(a^{1/2}\tilde{\eta}_m) \leq a^{1/2}qM_\mu(\tilde{\eta}_m) < 0, \quad a \in [1, a_0^2],$$

from which it follows that

$$c_{a\mu} \leq \mathcal{J}_{a\mu}(a^{1/2}\tilde{\eta}_m)$$

$$= \mathcal{K}_2(a^{1/2}\tilde{\eta}_m) + \frac{(a\mu + G_2(q^{1/2}\tilde{\eta}_m))^2}{L_2(a^{1/2}\tilde{\eta}_m)} + M(a^{1/2}\tilde{\eta}_m)$$

$$\leq a(\mathcal{K}_2(\tilde{\eta}_m) + \frac{(\mu + G(\tilde{\eta}_m))^2}{L(\tilde{\eta}_m)}) + a^{1/2}qM_\mu(\tilde{\eta}_m)$$

$$= a(\mathcal{K}_2(\tilde{\eta}_m) + \frac{\mu^2}{L(\tilde{\eta}_m)} + M_\mu(\tilde{\eta}_m)) + (a^{1/2}q - a)M_\mu(\tilde{\eta}_m)$$

$$\leq ac_\mu - c(a^{1/2}q - a)\mu^r, \quad a \in [1, a_0^2].$$

In the limit $n \to \infty$ the above inequality yields

$$c_{a\mu} \leq ac_\mu - c(a^{1/2}q - a)\mu^r < ac_\mu, \quad a \in (1, a_0^2].$$

For $a > a_0^2$ we choose $p \geq 2$ such that $a \in (1, a_0^{2p})$ (and hence $a^1p \in (1, a_0^2]$) and observe that

$$c_{a\mu} < a^{1/p}c_{a(p-1)/r\mu} < a^{2/p}c_{a(p-2)/r\mu} < \cdots < ac_\mu.$$  

**Lemma 4.33.** The number $c_\mu$ is an increasing function of $\mu > 0$.

**Proof.** Using proposition 4.8 for $\beta > \beta_c$ and proposition 4.20 for $\beta < \beta_c$, one finds that

$$\mu + G(\tilde{\eta}_m) = \nu_0 L(\tilde{\eta}_m) + O(\mu^{3/2}) \geq c\mu + O(\mu^{3/2})$$
so that
\[ \mu + \mathcal{G}(\tilde{\eta}_m) \geq c_* \mu \]
for some \( c_* \in (0, 1) \). Let \( d_* = 1 - c_* \) so that \( d_* \in (0, 1) \).

First suppose that \( \mu_1 \in [d_* \mu_2, \mu_2] \). Let \( \{\tilde{\eta}_m^2\} \) be the special minimizing sequence constructed in theorem 3.1 for \( \mu = \mu_2 \) and note that
\[ \mu_1 + \mathcal{G}(\tilde{\eta}_m^2) = \mu_2 + \mathcal{G}(\tilde{\eta}_m^2) - (\mu_2 - \mu_1) \geq \mu_1 - d_* \mu_2 \geq 0, \]
so that \( \mathcal{J}_{\mu_1}(\tilde{\eta}_m^2) \leq \mathcal{J}_{\mu_2}(\tilde{\eta}_m^2) \). It follows that
\[ c_{\mu_1} \leq \mathcal{J}_{\mu_1}(\tilde{\eta}_m^2) \leq \mathcal{J}_{\mu_2}(\tilde{\eta}_m^2) \to c_{\mu_2} \]
as \( n \to \infty \), that is,
\[ c_{\mu_1} \leq c_{\mu_2}. \]

For \( \mu_1 < d_* \mu_2 \) we choose \( p \geq 2 \) such that \( \mu_1 \in [d_*^q \mu_2, \mu_2] \) (and hence \( \mu_1 \in [d_*^q \mu_2^{-1}, d_*^{p-1} \mu_2] \) and obviously \( d_*^{p+1} \mu_2 \in [d_*^q \mu_2, d_*^p \mu_2] \), \( q = 0, \ldots, p - 2 \) and observe that
\[ c_{\mu_1} \leq c_{d_*^{p-1}} \mu_2 \leq c_{d_*^{p-2}} \mu_2 \leq \cdots \leq c_{\mu_2}. \]

\[ \square \]

Our final result is stated in the following theorem.

**Theorem 4.34.** The number \( c_{\mu} \) has the strict subadditivity property
\[ c_{\mu_1 + \mu_2} < c_{\mu_1} + c_{\mu_2}, \quad 0 < |\mu_1|, |\mu_2|, \mu_1 + \mu_2 < \mu_0. \]

**Proof.** Using the strict subhomogeneity of \( c(\mu) \) for \( \mu > 0 \), we find that
\[ c_{\mu_1 + \mu_2} < \frac{\mu_1 + \mu_2}{\mu_1} c_{\mu_1} = c_{\mu_1} + \frac{\mu_2}{\mu_1} c_{\mu_1} \leq c_{\mu_1} + c_{\mu_2} \]
for \( 0 < \mu_1 \leq \mu_2 \), and for \( \mu_1 < 0, \mu_2 > 0 \) with \( \mu_1 + \mu_2 > 0 \) its monotonicity for \( \mu > 0 \) shows that
\[ c_{\mu_1 + \mu_2} \leq c_{\mu_2} < c_{\mu_1} + c_{\mu_2}. \]

\[ \square \]

5. Existence theory and consequences

5.1. Minimization

The following theorem, which is proved using the results of §§3 and 4, is our final result concerning the set of minimizers of \( \mathcal{J}_\mu \) over \( U \setminus \{0\} \).

**Theorem 5.1.**

(i) The set \( B_\mu \) of minimizers of \( \mathcal{J}_\mu \) over \( U \setminus \{0\} \) is non-empty.

(ii) Suppose that \( \{\eta_m\} \) is a minimizing sequence for \( \mathcal{J}_\mu \) on \( U \setminus \{0\} \) that satisfies
\[ \sup_{m \in \mathbb{N}} \|\eta_m\|_2 < M. \]
There exists a sequence \( \{x_m\} \subset \mathbb{R} \) with the property that a subsequence of \( \{\eta_m(x_m + \cdot)\} \) converges in \( H^r(\mathbb{R}), \ r \in (0, 2) \), to a function \( \eta \in B_\mu \).
Proof. It suffices to prove part (ii), since an application of this result to the sequence \( \{ \eta_m \} \) constructed in theorem 3.1 yields part (i).

In order to establish part (ii) we choose \( M \in (\sup_{m \in \mathbb{N}} \| \eta_m \|_2, M) \), so that \( \{ \eta_m \} \) is also a minimizing sequence for the functional \( J_{\rho, \mu} \) introduced in § 3.1; the existence of a minimizing sequence \( \{ v_m \} \) for \( J_{\rho, \mu} \) with \( \lim_{m \to \infty} J_{\rho, \mu}(v_m) < \lim_{m \to \infty} J_{\rho, \mu}(\eta_m) \) would lead to the contradiction

\[
\lim_{m \to \infty} J_{\mu}(v_m) \leq \lim_{m \to \infty} J_{\rho, \mu}(v_m) < \lim_{m \to \infty} J_{\rho, \mu}(\eta_m) = \lim_{m \to \infty} J_{\mu}(\eta_m) = c_{\mu}.
\]

We may therefore study \( \{ \eta_m \} \) using the theory given in § 3.2, noting that the sequence \( \{ u_m \} \) with \( u_m = (\eta_m^{(1)})^2 + \eta_m^{(2)} \) does not have the ‘dichotomy’ property: the existence of two sequences \( \{ \eta_m^{(1)} \} \), \( \{ \eta_m^{(2)} \} \) with the features listed in lemma 3.9 is incompatible with the strict subadditivity property of \( c_{\mu} \) (see theorem 4.34).

Recall that the numbers \( \mu^{(1)}, \mu^{(2)} \) sum to \( \mu \); this fact leads to the contradiction

\[
c_{\mu} < c_{\mu^{(1)}} + c_{\mu^{(2)}} \\
\leq \lim_{m \to \infty} J_{\mu^{(1)}}(\eta_m^{(1)}) + \lim_{m \to \infty} J_{\mu^{(2)}}(\eta_m^{(2)}) \\
= \lim_{m \to \infty} J_{\mu}(\eta_m) \\
= c_{\mu}.
\]

We conclude that \( \{ u_m \} \) has the ‘concentration’ property, and hence \( \eta_m(\cdot + x_m) \to \eta^{(1)} \) as \( n \to \infty \) in \( H^{r}(\mathbb{R}) \) for every \( r \in [0, 2) \) (see lemma 3.8(ii)), whereby \( J_{\mu}(\eta) = \lim_{m \to \infty} J_{\mu}(\eta_m(\cdot + x_m)) = c_{\mu} \) so that \( \eta^{(1)} \) is a minimizer of \( J_{\mu} \) over \( U \setminus \{ 0 \} \).

The next step is to relate the above result to our original problem of finding minimizers of \( \mathcal{H}(\eta, \xi) \) subject to the constraint \( \mathcal{I}(\eta, \xi) = 2\mu \), where \( \mathcal{H} \) and \( \mathcal{I} \) are defined in (1.6) and (1.7).

**Theorem 5.2.**

(i) The set \( D_\mu \) of minimizers of \( \mathcal{H} \) on the set

\[
S_\mu = \{ (\eta, \xi) \in U \times H^{1/2}_s(\mathbb{R}) : \mathcal{I}(\eta, \xi) = 2\mu \}
\]

is non-empty.

(ii) Suppose that \( \{ (\eta_m, \xi_m) \} \subset S_\mu \) is a minimizing sequence for \( \mathcal{H} \) with the property that \( \sup_{m \in \mathbb{N}} \| \eta_m \|_2 < M \). There exists a sequence \( \{ x_m \} \subset \mathbb{R} \) with the property that a subsequence of \( \{ (\eta_m(x_m + \cdot), \xi_m(x_m + \cdot)) \} \) converges in \( H^{r}(\mathbb{R}) \times H^{1/2}_s(\mathbb{R}) \), \( r \in [0, 2) \), to a function in \( D_\mu \).

Proof. (i) We consider the minimization problem in two steps.

(1) Fix \( \eta \in U \setminus \{ 0 \} \) and minimize \( \mathcal{H}(\eta, \cdot) \) over \( T_\mu = \{ \xi \in H^{1/2}_s(\mathbb{R}) : \mathcal{I}(\eta, \xi) = 2\mu \} \); notice that \( \mathcal{H}(\eta, \cdot) \) is weakly lower semi-continuous on \( H^{1/2}_s(\mathbb{R}) \) (since \( \xi \mapsto \langle G(\eta)\xi, \xi \rangle_s^{1/2} \) is equivalent to its usual norm), while \( \mathcal{I}(\eta, \cdot) \) is weakly continuous on \( H^{1/2}_s(\mathbb{R}) \); furthermore, \( \mathcal{H}(\eta, \cdot) \) is convex and coercive. A familiar argument shows that \( \mathcal{H}(\eta, \cdot) \) has a unique minimizer \( \xi_\eta \) over \( T_\mu \).
(2) Minimize $\mathcal{H}(\eta, \xi_\eta)$ over $U \setminus \{0\}$; because $\xi_\eta$ minimizes $\mathcal{H}(\eta, \cdot)$ over $T_\mu$, there exists a Lagrange multiplier $\nu_\eta$ such that

$$G(\eta)\xi_\eta + \omega \eta \eta' = \nu_\eta \eta';$$

and a straightforward calculation shows that

$$\xi_\eta = G(\eta)^{-1}(\nu_\eta \eta' - \omega \eta'), \quad \nu_\eta = \frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}. \tag{5.1}$$

According to theorem 5.1(i), the set $B_\mu$ of minimizers of $\mathcal{J}_\mu(\eta) := \mathcal{H}(\eta, \xi_\eta)$ over $U \setminus \{0\}$ is not empty; it follows that $D_\mu$ is also not empty.

(ii) Let $\{(\eta_m, \xi_m)\} \subset U \times H^{1/2}_R$ be a minimizing sequence for $\mathcal{H}$ over $S_\mu$ with $\sup_{m \in \mathbb{N}} \|\eta_m\|_2 < M$. The inequality

$$\mathcal{H}(\eta_m, \xi_m) \leq \mathcal{H}(\eta_m, \xi_m)$$

shows that $\{(\eta_k, \xi_{\eta_k})\} \subset U \times H^{1/2}_R$ is also a minimizing sequence; it follows that $\{\eta_m\} \subset U \setminus \{0\}$ is a minimizing sequence for $\mathcal{J}_\mu$, which therefore converges (up to translations and subsequences) in $H^r_\mu(\mathbb{R})$, $r \in [0, 2)$, to a minimizer $\eta$ of $\mathcal{J}_\mu$ over $U \setminus \{0\}$.

The relations (5.1) show that $\xi_{\eta_m} \to \xi_\eta$ in $H^{1/2}_R$ and, using this result and the calculation

$$c\|\xi_m - \xi_{\eta_m}\|^2_{H^{1/2}_R} \leq \frac{1}{2}\langle G(\eta_m)(\xi_m - \xi_{\eta_m}), (\xi_m - \xi_{\eta_m}) \rangle$$

$$= 2\mathcal{H}(\eta_m, \xi_m) + 2\mathcal{H}(\eta_m, \xi_{\eta_m}) - 4\mathcal{H}(\eta_m, \frac{1}{2}(\xi_m + \xi_{\eta_m}))$$

$$\leq 2\mathcal{H}(\eta_m, \xi_m) + 2\mathcal{H}(\eta_m, \xi_{\eta_m}) - 4\mu_c$$

$$\to 2c_\mu + 2c_\mu - 4c_\mu$$

$$= 0$$

as $n \to \infty$ (recall that $\mathcal{H}(\eta_m, \xi) \geq \mathcal{H}(\eta_m, \xi_{\eta_m}) = \mathcal{J}(\eta_m) \geq c_\mu$ for all $\xi \in H^{1/2}_R$), one finds that $\xi_m \to \xi_\eta$ in $H^{1/2}_R$ as $m \to \infty$. \hfill \Box

5.2. Convergence to solitary-wave solutions of model equations

5.2.1. The case $\beta > \beta_c$

Suppose that $\eta$ is a minimizer of $\mathcal{J}$ over $U \setminus \{0\}$, write $\eta = \eta_1 + \eta_2$ according to the decomposition introduced in § 4.1 and define $\phi_\eta \in H^2_\mu(\mathbb{R})$ by the formula

$$\eta_1(x) = \mu^{2/3}\phi_\eta(\mu^{1/3}x).$$

In this section we prove that $\text{dist}(\phi_\eta, D_{KdV}) \to 0$ as $\mu \downarrow 0$, uniformly over $\eta \in B_\mu$, where $D_{KdV}$ is the set of solitary-wave solutions to the Korteweg–de Vries equation and ‘dist’ denotes the distance in $H^1_\mu(\mathbb{R})$. 
Remark 5.3. Observe that
\[
\begin{align*}
\left\{ \begin{array}{l}
K_2(\eta) \\
G_2(\eta) \\
L_2(\eta)
\end{array} \right\} = \left\{ \begin{array}{l}
K_2(\eta_1) \\
G_2(\eta_1) \\
L_2(\eta_1)
\end{array} \right\} + \left\{ \begin{array}{l}
K_2(\eta_2) \\
G_2(\eta_2) \\
L_2(\eta_2)
\end{array} \right\}
= O(\|\eta\|_2^2) = O(\mu^{2+\alpha})
\end{align*}
\]

because \( \hat{\eta}_1 \) and \( \hat{\eta}_2 \) have disjoint supports, and
\[
G_2(\eta_1) = -\frac{\mu \omega}{4} \int_{-\infty}^{\infty} \phi_\eta^2 \, dx, \quad K_2(\eta_1) = \frac{\mu}{2} \int_{-\infty}^{\infty} \phi_\eta^2 \, dx,
\]
while the estimates
\[
\int_{-\infty}^{\infty} (|k| \coth |k| - 1)|\hat{\eta}_1|^2 \, dk \leq c \int_{-\infty}^{\infty} k^2 |\hat{\eta}_1|^2 \, dk = c \|\eta\|_0^2
\]
\[
\leq c\mu^{2\alpha} \|\eta\|_\alpha^2 \leq c\mu^{1+2\alpha},
\]
\[
\int_{-\infty}^{\infty} (|k| \coth |k| - 1 - \frac{1}{3} k^2)|\hat{\eta}_1|^2 \, dk \leq c \int_{-\infty}^{\infty} k^4 |\hat{\eta}_1|^2 \, dk = c \|\eta''\|_0^2
\]
\[
\leq c\mu^{4\alpha} \|\eta\|_\alpha^2 = c\mu^{1+4\alpha}
\]
show that
\[
L_2(\eta_1) = \frac{\mu}{2} \int_{-\infty}^{\infty} \phi_\eta^2 \, dx + O(\mu^{1+2\alpha})
\]
and
\[
L_2(\eta_1) = \frac{\mu}{2} \int_{-\infty}^{\infty} \phi_\eta^2 \, dx - \frac{\beta}{3} \mu^{5/3} \int_{-\infty}^{\infty} (\phi_\eta^2) \, dx + O(\mu^{1+4\alpha}).
\]
Furthermore, corollary 4.14 implies that
\[
\mathcal{M}_\mu(\eta) = \frac{1}{2}(\frac{3}{2} \omega^2 + 1) \mu^{5/3} \int_{-\infty}^{\infty} \phi_\eta^3 \, dx + o(\mu^{5/3}).
\]

Our first result concerns the convergence of the \( L^2(\mathbb{R}) \)-norm of minimizers of \( \mathcal{J}_\mu \) over \( U \setminus \{0\} \).

Proposition 5.4. The estimate \( \|\phi_\eta\|_0^2 = 4(\omega^2 + 4)^{-1/2} + O(\mu^{2\alpha}) \) holds for each \( \eta \in B_\mu \).

Proof. It follows from
\[
\left| \frac{\mu + G_2(\eta)}{L_2(\eta)} - \nu_0 \right| \leq c\mu^{\alpha/2 + 1/2}, \quad \mathcal{L}(\eta) \leq c\mu,
\]
that
\[
\nu_0 L_2(\eta) - G_2(\eta) = \mu + O(\mu^{\alpha/2 + 3/2}),
\]
and the result is obtained by combining this estimate with
\[
\nu_0 L_2(\eta) - G_2(\eta) = \frac{1}{4} \left( 2\nu_0 + \omega \right) \mu \int_{-\infty}^{\infty} \phi_\eta^2 \, dx + O(\mu^{1+2\alpha}).
\]
The next step is to show that the Korteweg–de Vries energy \( E_{\text{KdV}}(\eta) \) corresponding to a minimizer \( \eta \) of \( J_\mu \) over \( U \setminus \{0\} \) approaches \( c_{\text{KdV}} \) in the limit \( \mu \downarrow 0 \).

**Theorem 5.5.**

(i) The number \( c_\mu \) satisfies
\[
c_\mu = 2\nu_0 \mu + c_{\text{KdV}} \mu^{5/3} + o(\mu^{5/3}).
\]

(ii) Each \( \eta \in B_\mu \) satisfies \( E_{\text{KdV}}(\phi_\eta) \to c_{\text{KdV}} \) as \( \mu \downarrow 0 \).

**Proof.** Notice that
\[
c_\mu = J_\mu(\eta) = K_2(\eta) + \frac{(\mu + G_2(\eta))^2}{L_2(\eta)} + M_\mu(\eta)
\]
\[
= 2\nu_0 \mu + K_2(\eta) + 2\nu_0 G_2(\eta) - \nu_0^2 L_2(\eta) + \left( \frac{\mu + G_2(\eta)}{\sqrt{L_2(\eta)}} - \nu_0 \sqrt{L_2(\eta)} \right)^2 + M_\mu(\eta)
\]
\[
\geq 2\nu_0 \mu + K_2(\eta) + 2\nu_0 G_2(\eta) - \nu_0^2 L_2(\eta) + M_\mu(\eta)
\]
\[
= 2\nu_0 \mu + \frac{\mu^{5/3}}{2} \int_{-\infty}^{\infty} \left( \beta - \frac{\nu_0^2}{3} \right) (\phi_\eta')^2 + \left( \frac{\omega^2}{3} + 1 \right) (\psi_\eta^3) \, dx + o(\mu^{5/3})
\]
\[
= 2\nu_0 \mu + {\mu^{5/3} E_{\text{KdV}}(\phi_\eta) + o(\mu^{5/3})},
\]
and combining this estimate with lemma A.1 yields
\[
E_{\text{KdV}}(\phi_\eta) \leq c_{\text{KdV}} + o(1).
\]

A straightforward scaling argument shows that
\[
\inf \{ E_{\text{KdV}}(\phi) : \phi \in H^1(\mathbb{R}), \| \phi \|^2_0 = 4(\omega^2 + 4)^{-1/2} a \} = a^{5/3} c_{\text{KdV}},
\]
whence
\[
E_{\text{KdV}}(\phi_\eta) \geq (1 + O(\mu^{2\alpha}))^{5/3} c_{\text{KdV}} = c_{\text{KdV}} + o(1)
\]
because \( \| \phi_\eta \|^2_0 = 4(\omega^2 + 4)^{-1/2} + O(\mu^{2\alpha}) \) (see proposition 5.4), and it follows from (5.2) that
\[
c_\mu \geq 2\nu_0 \mu + \mu^{5/3} c_{\text{KdV}} + o(\mu^{5/3}).
\]

The complementary estimate
\[
c_\mu \leq 2\nu_0 \mu + \mu^{5/3} c_{\text{KdV}} + o(\mu^{5/3})
\]
is a consequence of lemma A.1.

We now present our main convergence result.

**Theorem 5.6.** The set \( B_\mu \) of minimizers of \( J_\mu \) over \( U \setminus \{0\} \) satisfies
\[
\sup_{\eta \in B_\mu} \inf_{x \in \mathbb{R}} \| \phi_\eta - \phi_{\text{KdV}}(\cdot + x) \|_1 \to 0
\]
as \( \mu \downarrow 0 \).
Proof. Suppose that the limit is positive so that there exists $\varepsilon > 0$ and a sequence $\{\mu_m\}$ with $\mu_m \downarrow 0$ such that

$$\sup_{\eta \in C_{\mu_m}} \inf_{x \in \mathbb{R}} \|\phi_{\eta} - \phi_{KdV}(\cdot + x)\|_1 \geq \varepsilon, \quad m \in \mathbb{N},$$

and hence a further sequence $\{\eta_m\} \subset U \setminus \{0\}$ with $\eta_m \in C_{\mu_m}$ and

$$\text{dist}(\phi_{\eta_m}, D_{KdV}) = \inf_{x \in \mathbb{R}} \|\phi_{\eta_m} - \phi_{KdV}(\cdot + x)\|_1 \geq \frac{1}{2} \varepsilon, \quad m \in \mathbb{N}.$$  

On the other hand, $E_{KdV}(\phi_{\eta_m}) \to c_{KdV}$ and $\|\phi_{\eta_m}\|_0^2 \to 4(\omega^2 + 4)^{-1/2}$ as $n \to \infty$ (see proposition 5.4 and theorem 5.5(ii)); combining lemma 1.2(ii) with a straightforward scaling argument, we arrive at the contradiction of the existence of a sequence $\{x_m\} \subset \mathbb{R}$ such that a subsequence of $\{\phi_{\eta_m}(x_m + \cdot)\}$ converges in $H^1(\mathbb{R})$ to an element of $D_{KdV}$.

Remark 5.7. The previous theorem implies that $\{\|\phi_{\eta}\|_1: \eta \in B_{\mu}\}$ is bounded, so that

$$\|\hat{\eta}_1\|_{L^1(\mathbb{R})} \leq \left( \int_{-\infty}^{\infty} \frac{1}{1 + \mu^{-2/3}k^2} \, dk \right) \left( \int_{-\infty}^{\infty} (1 + \mu^{-2/3}k^2)|\hat{\eta}_1(k)|^2 \, dk \right)^{1/3}$$

$$= \mu^{2/3} \left( \int_{-\infty}^{\infty} \frac{1}{1 + \mu^{-2/3}k^2} \, dk \right) \left( \int_{-\infty}^{\infty} (1 + \mu^{-2/3}k^2)^{2/3} |\hat{\phi}_{\eta}(\frac{k}{\mu^{1/3}})|^2 \, dk \right)$$

$$= 2\pi \mu^{4/3} \|\phi_{\eta}\|_2^2$$

$$\leq c\mu^{4/3},$$

and hence $\|\eta_1\|_{1,\infty}, \|K^0\eta_1\|_\infty \leq c\mu^{2/3}$ (see (4.4) and (4.5)) and it follows from (4.13) and (4.14) that

$$\|\|\eta_1\|_{1/3}^2 \leq c\mu, \quad \|\eta_2\|_2^2 \leq \mu^{7/3}.$$  

For $\eta \in B_{\mu}$, lemma 4.10 therefore also holds with $\alpha = \frac{1}{3}$ (the result predicted in the Korteweg–de Vries scaling limit).

Our final result shows that the speed $\nu_{\mu}$ of a solitary wave corresponding to $\eta \in B_{\mu}$, which is given by the formula

$$\nu_{\mu} = \frac{\mu + G(\eta)}{L(\eta)},$$

satisfies

$$\nu_{\mu} = \nu_0 + 2(\omega^2 + 4)^{-1/2}\nu_{KdV}\mu^{2/3} + o(\mu^{2/3})$$

uniformly over $\eta \in B_{\mu}$.

Theorem 5.8. The set $B_{\mu}$ of minimizers of $J_{\mu}$ over $U \setminus \{0\}$ satisfies

$$\sup_{\eta \in B_{\mu}} \left| \frac{\mu + G(\eta)}{L(\eta)} - (\nu_0 + 2(\omega^2 + 4)^{-1/2}\nu_{KdV}\mu^{2/3}) \right| = o(\mu^{2/3}).$$
Proof. Using the identity
\[
\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} = \frac{1}{2\mu} (c_\mu - \mathcal{M}_\mu(\eta)) + \frac{1}{4\mu} (\langle \mathcal{M}'_\mu(\eta), \eta \rangle + 4\mu \mathcal{M}_\mu(\eta))
\]
(see the proof of proposition 4.2), we find that
\[
\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} = \nu_0 + \frac{1}{2} c_{\text{KdV}} \mu^{2/3} + \frac{1}{8\mu} \left( \frac{\omega^2}{3} + 1 \right) \int_{-\infty}^\infty \eta_1^3 dx + o(\mu^{2/3})
\]
\[
= \nu_0 + \frac{1}{2} c_{\text{KdV}} \mu^{2/3} + \frac{1}{8} \left( \frac{\omega^2}{3} + 1 \right) \mu^{2/3} \int_{-\infty}^\infty \phi_\nu^3 dx + o(\mu^{2/3})
\]
\[
= \nu_0 + \frac{1}{2} c_{\text{KdV}} (\phi_{\text{KdV}}) \mu^{2/3} + \frac{1}{8} \left( \frac{\omega^2}{3} + 1 \right) \mu^{2/3} \int_{-\infty}^\infty \phi_{\text{KdV}}^3 dx + o(\mu^{2/3})
\]
\[
= \nu_0 + \frac{1}{2} \mu^{2/3} \int_{-\infty}^\infty \left( \beta - \frac{\nu_0^2}{3} \right) (\phi_{\text{KdV}}')^2 + \frac{3}{2} \left( \frac{\omega^2}{3} + 1 \right) \phi_{\text{KdV}}^3 dx + o(\mu^{2/3})
\]
\[
= \nu_0 + 8(\omega^2 + 4)^{-1/2} \nu_{\text{KdV}} \mu^{2/3} + o(\mu^{2/3}),
\]
in which theorem 5.5(i), corollary 4.14 and theorem 5.6 have been used. \qed

5.2.2. The case \( \beta < \beta_c \)

Suppose that \( \eta \) is a minimizer of \( \mathcal{J}_\mu \) over \( U \setminus \{0\} \), write \( \eta = \eta_1 - H(\eta_1) + \eta_3 \) and \( \eta_1 = \eta_1^+ + \eta_1^- \) according to the decompositions introduced in § 4.3, and define \( \phi_\eta \in H^2(\mathbb{R}) \) by the formula
\[
\eta_1^+(x) = \frac{1}{2} \mu \phi_\eta(\mu x)e^{i\kappa_0 x}.
\]

In this section we prove that \( \text{dist}(\phi_\eta, D_{\text{NLS}}) \to 0 \) as \( \mu \downarrow 0 \), uniformly over \( \eta \in B_\mu \), where \( D_{\text{NLS}} \) is the set of solitary-wave solutions to the nonlinear Schrödinger equation and ‘dist’ denotes the distance in \( H^1(\mathbb{R}) \).

Remark 5.9. Note that
\[
\begin{align*}
\{ K_2(\eta) \} &= \{ K_2(\eta_1^-) \} + \{ K_2(-H(\eta) + \eta_3) \} \\
\{ \mathcal{G}_2(\eta) \} &= \{ \mathcal{G}_2(\eta_1^-) \} + \{ \mathcal{G}_2(-H(\eta) + \eta_3) \} \\
\{ \mathcal{L}_2(\eta) \} &= \{ \mathcal{L}_2(\eta_1^-) \} + \{ \mathcal{L}_2(-H(\eta) + \eta_3) \}
\end{align*}
\]

because \( \eta_1^- \) and \( \mathcal{F}[-H(\eta) + \eta_3] \) have disjoint supports.

Our first result concerns the convergence of the \( L^2(\mathbb{R}) \)-norm of minimizers of \( \mathcal{J}_\mu \) over \( U_3 \setminus \{0\} \).

Proposition 5.10. The estimate \( \| \phi_\eta \|_0^2 = (\frac{1}{4} \nu_0 f(k_0) + \frac{1}{2} \omega)^{-1} + O(\mu^\alpha) \) holds for each \( \eta \in B_\mu \).

Proof. It follows from
\[
\left| \frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} - \nu_0 \right| \leq c \mu^{1+\alpha}, \quad \mathcal{L}_2(\eta) \leq c \mu,
\]
that
\[ \nu_0L_2(\eta) - G_2(\eta) = \mu + O(\mu^{2+\alpha}). \]
(5.4)

On the other hand,
\[ \nu_0L_2(\eta) - G_2(\eta) = \nu_0L_2(\eta_1) - G_2(\eta_1) + O(\|H(\eta)\|^2_2 + \|\eta_3\|^2_2) \]
\[ = \nu_0L_2(\eta_1) - G_2(\eta_1) + O(\mu^{2+\alpha}) \]
\[ = \nu_0 \int_{-\infty}^{\infty} \eta_1^+ K_0 \eta_1^- \, dx + \frac{\omega}{2} \int_{-\infty}^{\infty} \eta_1^+ \eta_1^- \, dx + O(\mu^{2+\alpha}) \]
\[ = \left( \nu_0 f(0) + \frac{\omega}{2} \right) \int_{-\infty}^{\infty} \eta_1^+ \eta_1^- \, dx + O(\mu^{1+\alpha}) \]
\[ = \left( \frac{1}{4} \nu_0 f(0) + \frac{\omega}{8} \right) \mu \int_{-\infty}^{\infty} |\phi_\eta|^2 \, dx + O(\mu^{1+\alpha}), \]
and the result is obtained by combining this estimate with (5.4).

The next step is to show that the nonlinear Schrödinger energy \( E_{\text{NLS}}(\phi_\eta) \) corresponding to a minimizer \( \eta \) of \( J_\mu \) over \( U \setminus \{0\} \) approaches \( c_{\text{NLS}} \) in the limit \( \mu \downarrow 0 \).

**Theorem 5.11.**

(i) The number \( c_\mu \) satisfies \( c_\mu = 2\nu_0\mu + c_{\text{NLS}} \mu^3 + o(\mu^3) \).

(ii) Each \( \eta \in B_\mu \) satisfies \( E_{\text{NLS}}(\phi_\eta) \to c_{\text{NLS}} \) as \( \mu \downarrow 0 \).

**Proof.** Notice that
\[ c_\mu = J_\mu(\eta) \]
\[ = K_2(\eta) + \left( \frac{\mu + G_2(\eta)}{L_2(\eta)} \right)^2 + M_\mu(\eta) \]
\[ = 2\nu_0\mu + K_2(\eta) + 2\nu_0G_2(\eta) - \nu_0^2 L_2(\eta) + \left( \frac{\mu + G_2(\eta)}{\sqrt{L_2(\eta)}} - \nu_0 \sqrt{L_2(\eta)} \right)^2 + M_\mu(\eta) \]
\[ \geq 2\nu_0\mu + K_2(\eta) + 2\nu_0G_2(\eta) - \nu_0^2 L_2(\eta) + M_\mu(\eta), \]
(5.5)
where
\[ K_2(\eta) + 2\nu_0G_2(\eta) - \nu_0^2 L_2(\eta) \]
\[ = (K_2 + 2\nu_0G_2 - \nu_0^2 L_2)(\eta_1) + (K_2 + 2\nu_0G_2 - \nu_0^2 L_2)(-H(\eta) + \eta_3). \]
(5.6)
The second term on the right-hand side of (5.6) is estimated using the calculation
\[ (K_2 + 2\nu_0G_2 - \nu_0^2 L_2)(-H(\eta) + \eta_3) \]
\[ = (K_2 + 2\nu_0G_2 - \nu_0^2 L_2)(H(\eta)) + O(\|H(\eta)\|_2 \|\eta_3\|_2) + O(\|\eta_3\|^2_2) \]
\[ = \frac{1}{2} \int_{-\infty}^{\infty} g(k) |H(\eta)|^2 \, dk + o(\mu^3) \]
\[ = \frac{1}{2} \int_{-\infty}^{\infty} g(k)^{-1} |F[K_3(\eta_1) + 2\nu_0G_3(\eta_1) - \nu_0^2 L_3(\eta_1)]|^2 \, dk + o(\mu^3) \]
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\[-\frac{1}{2}(\mathcal{K}_3(\eta) + 2\nu_0 \mathcal{G}_3(\eta) - \nu_0^2 \mathcal{L}_3(\eta)) + o(\mu^3)\]

\[= -\frac{1}{2} A_3 \int_{-\infty}^{\infty} \eta_1^4 \mathrm{d}x + o(\mu^3)\]

\[= -\frac{3}{16} A_3 \mu^3 \int_{-\infty}^{\infty} |\phi_\eta|^4 \mathrm{d}x + o(\mu^3),\]

where we have used proposition 4.27, (4.20) and proposition 4.28. Turning to the first term on the right-hand side of (5.6), write

\[(\mathcal{K}_2 + 2\nu_0 \mathcal{G}_2 - \nu_0^2 \mathcal{L}_2)(\eta_1) = \frac{1}{2} \int_{-\infty}^{\infty} g(k) \hat{\eta}_1(k)^2 \mathrm{d}k = \int_{-\infty}^{\infty} g(k) |\tilde{\eta}_1(k)|^2 \mathrm{d}k\]

and note that

\[g(k) = \frac{1}{2} g''(k_0) (k - k_0)^2 + O(|k - k_0|^3), \quad k \in [k_0 - \delta_0, k_0 + \delta_0].\]

One finds that

\[\int_{-\infty}^{\infty} (k - k_0)^2 |\tilde{\eta}_1(k)|^2 \mathrm{d}k = \int_{-\infty}^{\infty} k^2 |\tilde{\eta}_1(k) + k_0)|^2 \mathrm{d}k\]

\[\quad = \frac{\mu^2}{4} \int_{-\infty}^{\infty} \left| \frac{\mathrm{d}}{\mathrm{d}x} \phi_\eta(\mu x) \right|^2 \mathrm{d}x = \frac{\mu^3}{4} \int_{-\infty}^{\infty} |\phi_\eta'|^2 \mathrm{d}x\]

(because \(\tilde{\eta}_1(k + k_0) = \frac{1}{2} \mu \mathcal{F}[\phi_\eta(\mu x)]\)) and

\[\int_{-\infty}^{\infty} (k - k_0)^3 |\tilde{\eta}_1(k)|^2 \mathrm{d}k \leq c \mu^{3\alpha} \|\eta_1\|_{\alpha}^2 = O(\mu^{1+3\alpha}),\]

so that

\[\int_{-\infty}^{\infty} (g(k) - \frac{1}{2} (k - k_0)^2) |\tilde{\eta}_1(k)|^2 \mathrm{d}k = o(\mu^3).\]

Altogether these calculations show that

\[(\mathcal{K}_2 + 2\nu_0 \mathcal{G}_2 - \nu_0^2 \mathcal{L}_2)(\eta_1)\]

\[= \frac{1}{8} g''(k_0) \mu^3 \int_{-\infty}^{\infty} |\phi_\eta'|^2 \mathrm{d}x - \frac{3 A_3}{16} \mu^3 \int_{-\infty}^{\infty} |\phi_\eta|^4 \mathrm{d}x + o(\mu^3).\]  (5.7)

Substituting (5.7) and

\[\mathcal{M}_\mu(\eta) = (A_3 + A_4) \int_{-\infty}^{\infty} \eta_1^4 \mathrm{d}x + o(\mu^3) = \frac{3}{8} (A_3 + A_4) \mu^3 \int_{-\infty}^{\infty} |\phi_\eta|^4 \mathrm{d}x + o(\mu^3)\]

(see corollary 4.30) into inequality (5.5) yields

\[c_\mu \geq 2\nu_0 \mu + \frac{1}{2} g''(k_0) \mu^3 \int_{-\infty}^{\infty} |\phi_\eta'|^2 \mathrm{d}x + \frac{3}{8} (A_3 + A_4) \mu^3 \int_{-\infty}^{\infty} |\phi_\eta|^4 \mathrm{d}x + o(\mu^3)\]

\[= 2\nu_0 \mu + \mu^3 \mathcal{E}_{\text{NLS}}(\phi_\eta) + o(\mu^3),\]  (5.8)

and combining this estimate with lemma A.2 yields

\[\mathcal{E}_{\text{NLS}}(\phi_\eta) \leq c_{\text{NLS}} + o(1).\]
A straightforward scaling argument shows that
\[
\inf \{ E_{\text{NLS}}(\phi) : \phi \in H^1(\mathbb{R}), \| \phi \|_0^2 = (\frac{1}{2}\nu_0 f(k_0) + \frac{1}{8} \omega)^{-1} a \} = a^3 c_{\text{NLS}},
\]
whence
\[
E_{\text{NLS}}(\phi_\eta) \geq (1 + O(\mu^\alpha))^3 c_{\text{NLS}} = c_{\text{NLS}} + o(1)
\]
because \( \| \phi_\eta \|_0^2 = (\frac{1}{2}\nu_0 f(k_0) + \frac{1}{8} \omega)^{-1} + O(\mu^\alpha) \) (see proposition 5.10), and it follows from (5.8) that
\[
c_\mu \geq 2\nu_0 \mu + \mu^3 c_{\text{NLS}} + o(\mu^3).
\]
The complementary estimate
\[
c_\mu \leq 2\nu_0 \mu + \mu^3 c_{\text{NLS}} + o(\mu^3)
\]
is a consequence of lemma A.2. \(\square\)

Our main convergence result is derived from theorem 5.11 in the same way as the corresponding result for \( \beta > \beta_c \) (see Appendix A.1).

**Theorem 5.12.** The set \( B_\mu \) of minimizers of \( J_\mu \) over \( \{0\} \) satisfies
\[
\sup_{\eta \in B_\mu} \inf_{\omega \in [0, 2\pi]} \| \phi_\eta - e^{i\omega} \phi_{\text{NLS}}(\cdot + x) \|_1 \to 0
\]
as \( \mu \downarrow 0 \).

**Remark 5.13.** The previous theorem implies that \( \{ \| \phi_\eta \|_1 : \eta \in B_\mu \} \) is bounded, so that
\[
\| \eta_1 \|_2, \infty \leq c \mu, \quad \| H(\eta_1) \|_2 \leq c \mu^3, \quad \| u_3 \|_2 \leq c \mu^5.
\]
For \( \eta \in B_\mu \), lemma 4.21 therefore also holds with \( \alpha = 1 \) (the result predicted in the nonlinear Schrödinger scaling limit).
Our final result shows that the speed \( \nu_\mu \) of a solitary wave corresponding to \( \eta \in B_\mu \), which is given by the formula
\[
\nu_\mu + \frac{\mu + G(\eta)}{\mathcal{L}(\eta)},
\]
satisfies
\[
\nu_\mu = \nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1}v_{\text{NLS}}\mu^2 + o(\mu^2)
\]
uniformly over \( \eta \in B_\mu \).

**Theorem 5.14.** The set \( B_\mu \) of minimizers of \( J_\mu \) over \( U \setminus \{0\} \) satisfies
\[
\sup_{\eta \in B_\mu} \left| \frac{\mu + G(\eta)}{\mathcal{L}(\eta)} - (\nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1}v_{\text{NLS}}\mu^2) \right| = o(\mu^2).
\]

**Proof.** Using the identity
\[
\frac{\mu + G(\eta)}{\mathcal{L}(\eta)} = \frac{1}{2\mu} (c_\mu - \mathcal{M}_\mu(\eta)) + \frac{1}{4\mu} (\langle \mathcal{M}_\mu(\eta), \eta \rangle + 4\mu \mathcal{M}_\mu(\eta))
\]
(see the proof of proposition 4.2), we find that
\[
\frac{\mu + G(\eta)}{\mathcal{L}(\eta)} = \nu_0 + \frac{1}{2}c_{\text{NLS}}\mu^2 + \frac{1}{2\mu} \left( \frac{A_3}{2} + A_4 \right) \int_{-\infty}^{\infty} \eta_1^4 \, dx + o(\mu^2)
\]
\[
= \nu_0 + \frac{1}{2}c_{\text{NLS}}\mu^2 + \frac{3}{16} \left( \frac{A_3}{2} + A_4 \right) \mu^2 \int_{-\infty}^{\infty} |\phi_\eta|^4 \, dx + o(\mu^2)
\]
\[
= \nu_0 + \frac{1}{2}c_{\text{NLS}}(\phi_{\text{NLS}})\mu^2 + \frac{3}{16} \left( \frac{A_3}{2} + A_4 \right) \mu^2 \int_{-\infty}^{\infty} |\phi_{\text{NLS}}|^4 \, dx + o(\mu^2)
\]
\[
= \nu_0 + \frac{1}{4\mu} \int_{-\infty}^{\infty} \left( \frac{1}{4} \phi_\eta''(k_0) |\phi_{\text{NLS}}|^2 + \frac{3}{2} \left( \frac{1}{2} A_3 + A_4 \right) |\phi_{\text{NLS}}|^4 \right) \, dx + o(\mu^2)
\]
\[
= 2 \left( \frac{1}{4} \nu_0 f(k_0) + \frac{3}{8} \omega \right)^{-1} v_{\text{NLS}}
\]
\[
\nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1}v_{\text{NLS}}\mu^2 + o(\mu^2),
\]
in which theorem 5.11(i), corollary 4.30 and theorem 5.12 have been used. \( \square \)

**Appendix A. Proof of lemma 3.2(i)**

**A.1. The case \( \beta > \beta_c \)**

**Lemma A.1.** Suppose that \( \mu > 0 \). There exists a continuous invertible mapping \( \mu \to \alpha(\mu) \) such that
\[
J_\mu(\eta^*) = 2\nu_0 \mu + c_{\text{KdV}} \mu^{5/3} + o(\mu^{5/3}),
\]
where
\[
\eta^*(x) = \alpha^2 \phi_{\text{KdV}}(\alpha x).
\]
Proof. Let us first note that

\[
K^0 \eta^* - \eta^* + \frac{1}{2}(\eta^*)'' = \mathcal{F}^{-1}[((|k| \coth |k| - 1 - \frac{1}{2}|k|^2) \hat{\eta}^*) = O(\alpha^{11/2}),
\]

and hence

\[
K^0 \eta^* - \eta^* = \mathcal{F}^{-1}[((|k| \coth |k| - 1) \hat{\eta}^*) = O(\alpha^{7/2}).
\]

Using these estimates and \( \| \eta^* \|_0 = O(\alpha^{3/2}) \), one finds that

\[
K_2(\eta^*) = \frac{1}{2} \alpha^3 \int_{-\infty}^{\infty} \phi^2_{\text{KdV}} \, dx + \frac{1}{2} \alpha^5 \beta \int_{-\infty}^{\infty} \phi^2_{\text{KdV}} \, dx,
\]

\[
G_2(\eta^*) = -\frac{1}{4} \alpha^3 \omega \int_{-\infty}^{\infty} \phi^2_{\text{KdV}} \, dx,
\]

\[
L_2(\eta^*) = \frac{1}{2} \int_{-\infty}^{\infty} \hat{\eta}^* K^0 \eta^* \, dx
\]

\[
= \frac{1}{2} \alpha^3 \int_{-\infty}^{\infty} \phi^2_{\text{KdV}} \, dx + \frac{1}{6} \alpha^5 \int_{-\infty}^{\infty} \phi^2_{\text{KdV}} \, dx + O(\alpha^7),
\]

and

\[
K_3(\eta^*) = \frac{1}{6} \alpha^5 \omega^2 \int_{-\infty}^{\infty} \phi^3_{\text{KdV}} \, dx,
\]

\[
G_3(\eta^*) = \frac{1}{4} \omega \int_{-\infty}^{\infty} (\eta^*)^2 K^0 \eta^* \, dx
\]

\[
= \frac{1}{4} \omega \int_{-\infty}^{\infty} (\eta^*)^3 \, dx + \frac{1}{4} \omega \int_{-\infty}^{\infty} (\eta^*)^2 (K^0 \eta^* - \eta^*) \, dx
\]

\[
= \frac{1}{4} \alpha^5 \omega \int_{-\infty}^{\infty} \phi^3_{\text{KdV}} \, dx + O(\alpha^7),
\]

\[
L_3(\eta^*) = \frac{1}{2} \int_{-\infty}^{\infty} (-K^0 \eta^*)^2 \eta^* + (\eta^*)^2 \eta^* \, dx
\]

\[
= -\frac{1}{2} \int_{-\infty}^{\infty} (\eta^*)^3 \, dx
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} (-2(K^0 \eta^* - \eta^*)(\eta^*)^2 - (K^0 \eta^* - \eta^*)(\eta^*)^2 + (\eta^*)^2 \eta^*) \, dx
\]

\[
= \frac{1}{2} \alpha^5 \int_{-\infty}^{\infty} \phi^3_{\text{KdV}} \, dx + O(\alpha^7),
\]

in which the further estimate \( \| \eta^* \|_\infty = O(\alpha^2) \) has been used (see proposition 4.3 for the formulas for \( G_3, K_3 \) and \( L_3 \)). Finally, proposition 4.4 shows that \( G_4(\eta^*), K_4(\eta^*), L_4(\eta^*) \) and \( G_\tau(\eta^*), K_\tau(\eta^*), L_\tau(\eta^*) \) are all \( O(\alpha^7) \).
The above calculations show that
\[
\mathcal{K}(\eta^*) + 2\nu_0 \mathcal{G}(\eta^*) - \nu_0^2 \mathcal{L}(\eta^*)
\]
\[
= \frac{\alpha^3}{2} \left( 1 - \omega \nu_0 - \nu_0^2 \right) \int_{-\infty}^{\infty} \phi_{KdV}^2 \, dx + \frac{1}{2} \left( \beta - \frac{\nu_0^2}{3} \right) \alpha^5 \int_{-\infty}^{\infty} \phi_{KdV}^2 \, dx
\]
\[
+ \frac{1}{2} \left( \frac{\omega^2}{3} + \omega \nu_0 + \nu_0^2 \right) \alpha^5 \int_{-\infty}^{\infty} \phi_{KdV}^2 \, dx + O(\alpha^7)
\]
\[
= \frac{\alpha^3}{2} \left( 1 - \omega \nu_0 - \nu_0^2 \right) \int_{-\infty}^{\infty} \phi_{KdV}^2 \, dx + \frac{1}{2} \left( \beta - \frac{\nu_0^2}{3} \right) \alpha^5 \int_{-\infty}^{\infty} \phi_{KdV}^2 \, dx + O(\alpha^7)
\]
\[
= \frac{\alpha^5}{2} \mathcal{E}_{KdV}(\phi_{KdV}) + O(\alpha^7).
\]

The mapping
\[
\alpha \mapsto \nu_0 \mathcal{L}(\eta^*) - \mathcal{G}(\eta^*)
\]
\[
= \frac{\alpha^3}{2} \left( \nu_0 + \frac{\omega}{4} \right) \int_{-\infty}^{\infty} \phi_{KdV}^2 \, dx + O(\alpha^5)
\]
\[
= \frac{\alpha^3}{4} \sqrt{\omega^2 + 4} \int_{-\infty}^{\infty} \phi_{KdV}^2 \, dx + O(\alpha^5)
\]
is continuous and strictly increasing and therefore has a continuous inverse \( \mu \mapsto \alpha(\mu) \); furthermore, \( \alpha(\mu) = \mu^{1/3} + o(\mu^{1/3}) \) and
\[
\mathcal{J}_\mu(\eta^*) - 2\nu_0 \mu = \mathcal{K}(\eta^*) + 2\nu_0 \mathcal{G}(\eta^*) - \nu_0^2 \mathcal{L}(\eta^*) = c_{KdV} \mu^{5/3} + o(\mu^{5/3}).
\]

A.2. The case \( \beta < \beta_c \)

**Lemma A.2.** Suppose that \( \mu > 0 \). There exists a continuous invertible mapping \( \mu \mapsto \alpha(\mu) \) such that
\[
\mathcal{J}_\mu(\eta^*) = 2\nu_0 \mu + c_{\text{NLS}} \mu^3 + o(\mu^3),
\]
where
\[
\eta^*(x) = \alpha \phi_{\text{NLS}}(\alpha x) \cos k_0 x
\]
\[
- \frac{\alpha^2}{2} g(2k_0)^{-1} A_4^1 \phi_{\text{NLS}}(\alpha x)^2 \cos 2k_0 x - \frac{\alpha^2}{2} g(0)^{-1} A_4^2 \phi_{\text{NLS}}(\alpha x)^2.
\]

**Proof.** We seek a test function \( \eta^* \) of the form
\[
\eta^*(x) = \alpha \phi_{\text{NLS}}(\alpha x) \cos k_0 x + \alpha^2 \psi(\alpha x) \cos 2k_0 x + \alpha^2 \xi(\alpha x)
\]
with \( \psi, \xi \in \mathcal{S}(\mathbb{R}) \).

Choose \( n \in \mathbb{N} \) and \( \chi \in C_0^\infty(\mathbb{R}) \). Straightforward calculations yield the formulas
\[
K^0(\chi(\alpha x)) = \chi(\alpha x) + S_1(x),
\]
where
\[
S_1(x) = \frac{1}{\alpha} \mathcal{F}^{-1} \left[ (|k| \coth |k| - 1) \chi \left( \frac{k}{\alpha} \right) \right].
\]
and
\[ K_0(\chi(ax) \cos nk_0 x) = f(nk_0)\chi(ax) \cos nk_0 x + \alpha f'(nk_0) \chi'(ax) \sin nk_0 x - \frac{1}{2} \alpha^2 f''(nk_0) \chi''(ax) \cos nk_0 x + S_2(x), \]
where
\[
S_2(x) = \frac{1}{2} \mathcal{F}^{-1} \left[ R_{nk_0}(k)(k-nk_0)^3 \left( \frac{k-nk_0}{\alpha} \right) \right] + \frac{1}{2} \mathcal{F}^{-1} \left[ R_{-nk_0}(k)(k+nk_0)^3 \left( \frac{k+nk_0}{\alpha} \right) \right]
\]
and \( R_\omega(k) = \frac{1}{2} f'''(k_\omega) \) for some \( k_\omega \) between \( k \) and \( \omega \); the remainder terms \( S_1 \) and \( S_2 \) satisfy the estimates \( \| S_1 \|_{1} = O(\alpha^{m+3/2}) \) and \( \| S_2 \|_{\infty} = O(\alpha^3), \| S_2 \|_{1} = O(\alpha^{7/2}) \). Furthermore, repeated integration by parts shows that
\[
\int_{-\infty}^{\infty} \chi(ax) \left\{ \sin \frac{\sin}{\cos}(mx) \right\} dx = O(\alpha^n)
\]
for each \( m \in \mathbb{N} \), so that
\[
\int_{-\infty}^{\infty} \chi(ax) \left\{ \sin \frac{\sin}{\cos}(m_1 x) \cdots \sin \frac{\sin}{\cos}(m_\ell x) \right\} dx = O(\alpha^n)
\]
for all \( m_1, \ldots, m_\ell \in \mathbb{N} \) with \( m_1 \pm \cdots \pm m_\ell \neq 0 \).

Estimating using the above rules, one finds that
\[
\mathcal{K}_2(\eta^*) = \frac{\alpha}{4} (1 + \beta k_0^2) \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 dx + \frac{\alpha^3}{4} \beta \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 dx + \frac{\alpha^3}{4} (1 + 4\beta k_0^2) \int_{-\infty}^{\infty} \psi^2 dx + \frac{\alpha^3}{2} \int_{-\infty}^{\infty} \xi^2 dx + O(\alpha^4),
\]
\[
\mathcal{G}_2(\eta^*) = -\frac{\alpha}{8} \omega \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 dx - \frac{\alpha^3}{8} \omega \int_{-\infty}^{\infty} \psi^2 dx - \frac{\alpha^3}{4} \omega \int_{-\infty}^{\infty} \xi^2 dx + O(\alpha^4),
\]
\[
\mathcal{L}_2(\eta^*) = \frac{\alpha}{4} f(k_0) \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 dx + \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 dx + \frac{\alpha^3}{4} \int_{-\infty}^{\infty} \psi^2 dx + \frac{\alpha^3}{2} \int_{-\infty}^{\infty} \xi^2 dx + O(\alpha^4),
\]
\[
\mathcal{K}_3(\eta^*) = \frac{\alpha^2}{8} \omega^2 \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 \psi dx + \frac{\alpha^3}{4} \omega^2 \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 \xi dx + O(\alpha^4),
\]
\[
\mathcal{G}_3(\eta^*) = \frac{\alpha^3}{8} \left( f(k_0) + \frac{f(2k_0)}{2} \right) \omega \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 \psi dx + \frac{\alpha^3}{4} \left( f(k_0) + \frac{1}{2} \right) \omega \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 \xi dx + O(\alpha^4),
\]
\[
\mathcal{L}_3(\eta^*) = \frac{\alpha^3}{4} \left( -f(k_0) f(2k_0) - \frac{f(k_0)^2}{2} + \frac{3k_0^2}{2} \right) \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 \psi dx + \frac{\alpha^3}{4} \left( -2f(k_0) - f(k_0)^2 + k_0^2 \right) \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 \xi dx + O(\alpha^4)
\]
and

\[ \mathcal{K}_4(\eta^*) = -\frac{\alpha^3}{64} (3\beta k_0^4 + \omega^2 f(2k_0) + 2) \int_{-\infty}^{\infty} \phi_{NLS}^4 \, dx + O(\alpha^4), \]

\[ \mathcal{G}_4(\eta^*) = \frac{\alpha^3}{16} \left( k_0^2 - \frac{f(k_0)(f(2k_0) + 2)}{2} \right) \omega \int_{-\infty}^{\infty} \phi_{NLS}^4 \, dx + O(\alpha^4), \]

\[ \mathcal{L}_4(\eta^*) = \frac{\alpha^3}{16} \left( f(k_0)^2 (f(2k_0) + 2) - 3k_0^2 f(k_0) \right) \int_{-\infty}^{\infty} \phi_{NLS}^4 \, dx + O(\alpha^4). \]

(see proposition 4.3 for the formulas for \( \mathcal{K}_3, \mathcal{G}_3, \mathcal{L}_3 \) and \( \mathcal{K}_4, \mathcal{G}_4, \mathcal{L}_4 \)). Finally, observe that

\[ \eta''(x) + k_0^2 \eta''(x) = \alpha^3 \phi_{NLS}(\alpha x) \cos k_0 x - 2\alpha^2 k_0 \phi_{NLS}'(\alpha x) \sin k_0 x + \alpha^4 \psi''(\alpha x) \cos 2k_0 x - 4\alpha^3 k_0 \psi'(\alpha x) \sin 2k_0 x - 3k_0^2 \alpha^2 \psi(\alpha x) \cos 2k_0 x + \alpha^4 \xi''(\alpha x), \]

so that \( \| \eta'' + k_0^2 \eta'' \|_0 = O(\alpha^{3/2}) \), and using the further estimates \( \| \eta'' \|_2 = O(\alpha^{1/2}) \) and \( \| \eta'' \|_{1,\infty} = O(\alpha) \), one finds from proposition 4.4 that \( \mathcal{K}_r(\eta^*), \mathcal{G}_r(\eta^*), \mathcal{L}_r(\eta^*) \) are all \( O(\alpha^{7/2}) \).

The above calculations show that

\[ \mathcal{K}(\eta^*) + 2\nu_0 \mathcal{G}(\eta^*) - \nu_0^2 \mathcal{L}(\eta^*) \]

\[ = \frac{\alpha^3}{8} (2\beta - \nu_0^2 f''(k_0)) \int_{-\infty}^{\infty} \phi_{NLS}^2 \, dx + \frac{\alpha^3}{4} \int_{-\infty}^{\infty} (g(2k_0) \psi^2 + A_3^1 \phi_{NLS}^2 \psi) \, dx \]

\[ + \frac{\alpha^3}{2} \int_{-\infty}^{\infty} (g(0) \xi^2 + A_3^2 \phi_{NLS}^2 \xi) \, dx + \frac{3\alpha^3}{8} A_4 \int_{-\infty}^{\infty} \phi_{NLS}^4 \, dx + O(\alpha^{7/2}) \]

\[ = \frac{\alpha^3}{8} (2\beta - \nu_0^2 f''(k_0)) \int_{-\infty}^{\infty} \phi_{NLS}^2 \, dx + \frac{\alpha^3}{4} \int_{-\infty}^{\infty} g(2k_0) \left( \psi + \frac{g(2k_0)^{-1}}{2} A_3 \phi_{NLS}^2 \right)^2 \, dx \]

\[ + \frac{\alpha^3}{4} g(0) \int_{-\infty}^{\infty} \left( \xi + \frac{g(0)^{-1}}{2} A_3 \phi_{NLS}^2 \right)^2 \, dx \]

\[ + \alpha^3 \left( \frac{3}{8} A_4 - \frac{g(2k_0)^{-1}}{16} (A_3^2)^2 - \frac{g(0)^{-1}}{8} (A_3^2)^2 \right) \int_{-\infty}^{\infty} \phi_{NLS}^4 \, dx + O(\alpha^{7/2}), \]

in which the second line follows from the first by the definitions of \( A_3^1, A_3^2, A_4 \) and the third from the second by completing the square. The choice

\[ \psi = -\frac{g(2k_0)^{-1}}{2} A_3^1 \phi_{NLS}, \quad \xi = -\frac{g(0)^{-1}}{2} A_3 \phi_{NLS} \]

therefore minimizes the value of \( \mathcal{K}(\eta^*) + 2\nu_0 \mathcal{G}(\eta^*) - \nu_0^2 \mathcal{L}(\eta^*) \) up to \( O(\alpha^{7/2}) \), whereby

\[ \mathcal{K}(\eta^*) + 2\nu_0 \mathcal{G}(\eta^*) - \nu_0^2 \mathcal{L}(\eta^*) = \alpha^3 \mathcal{E}_{NLS}(\phi_{NLS}) + O(\alpha^{7/2}) \]

\[ = c_{NLS} \alpha^3 + O(\alpha^{7/2}). \]
The mapping
\[ \alpha \mapsto \nu_0 \mathcal{L}(\eta^*) - \mathcal{G}(\eta^*) \]
\[ = \alpha \left( \frac{\nu_0}{4} f(k_0) + \frac{\omega}{8} \right) \int_{-\infty}^{\infty} \phi_{\text{NLS}}^2 \, dx + O(\alpha^2) \]
is continuous and strictly increasing and therefore has a continuous inverse \( \alpha \mapsto \alpha(\mu) \); furthermore, \( \alpha(\mu) = \mu + o(\mu) \) and
\[ \mathcal{F}_\mu(\eta^*) - 2\nu_0 \mu = \mathcal{K}(\eta^*) + 2\nu_0 \mathcal{G}(\eta^*) - \nu_0^2 \mathcal{L}(\eta^*) = \mathcal{C}_{\text{NLS}} \mu^3 + o(\mu^3). \]
\[
\]
\[ \square \]

Appendix B. The sign of \( A_3 + 2A_4 \)

The quantities \( \beta, \omega, k_0 \) and \( \nu_0 \) are related by the fact that \( g(k) \geq 0 \) with equality precisely when \( k = \pm k_0 \). It follows from the simultaneous equations \( g(k_0) = 0, \)
\( g'(k_0) = 0 \) that
\[ \beta = \frac{\nu_0^2 f'(k_0)}{2k_0}, \quad \omega = \frac{1 + \beta k_0^2 - \nu_0^2 f(k_0)}{\nu_0}, \]
and inserting these expressions for \( \beta \) and \( \omega \) into the formulas for \( A_3 \) and \( A_4 \) (see corollary 4.25 and proposition 4.28), one finds that
\[ \nu_0^8 (A_3 + 2A_4) = a_8 \nu_0^8 + a_6 \nu_0^6 + a_4 \nu_0^4 + a_2 \nu_0^2 + a_0, \]
(B1)
in which
\[ a_0 = -\frac{1}{12} h_2(k_0)^{-1}(1 + 2h_1(k_0)), \]
\[ a_2 = -\frac{1}{3} h_2(k_0)^{-1}\left( \frac{1}{2} f(2k_0) + \frac{1}{2} k_0 f'(k_0) + 2h_1(k_0)(\frac{1}{2} + \frac{1}{2} k_0 f'(k_0)) \right), \]
\[ a_4 = -\frac{1}{3} h_2(k_0)^{-1}\left( \left( \frac{1}{2} f(2k_0) + \frac{1}{2} k_0 f'(k_0) \right)^2 + 2h_1(k_0)(\frac{1}{2} + \frac{1}{2} k_0 f'(k_0))^2 \right) \]
\[ - 2(\frac{1}{12} + \frac{1}{2f(2k_0)}), \]
\[ a_6 = -\frac{2}{3} h_2(k_0)^{-1}\left( \frac{1}{2} f(k_0) f(2k_0) - \frac{3}{2} k_0^2 + \frac{1}{4} k_0 f'(k_0) f(2k_0) + \frac{1}{8} f'(k_0)^2 \right) \]
\[ \times \left( \frac{1}{2} f(2k_0) + \frac{1}{2} k_0 f'(k_0) \right) \]
\[ - \frac{4}{3} h_2(k_0)^{-1} h_1(k_0)(\frac{1}{4} k_0 f'(k_0) + \frac{1}{2} f(k_0) - \frac{1}{2} k_0^2 + \frac{1}{8} k_0^2 f'(k_0)^2) \]
\[ \times \left( \frac{1}{2} + \frac{1}{2} k_0 f'(k_0) \right) \]
\[ + 2(\frac{1}{2} k_0 f'(k_0) f(2k_0) + \frac{1}{4} k_0^2 - \frac{1}{12} k_0 f'(k_0) - \frac{1}{6} f(k_0) - \frac{1}{12} f(k_0) f(2k_0)), \]
\[ a_8 = -\frac{1}{3} h_2(k_0)^{-1}\left( \frac{1}{2} f(k_0) f(2k_0) - \frac{3}{2} k_0^2 + \frac{1}{2} f'(k_0) f(2k_0) + \frac{1}{8} f'(k_0)^2 \right) \]
\[ - \frac{2}{3} h_2(k_0)^{-1} h_1(k_0)(\frac{1}{4} k_0 f'(k_0) + \frac{1}{2} f(k_0) - \frac{1}{2} k_0^2 + \frac{1}{8} k_0^2 f'(k_0)^2) \]
\[ - 2(\frac{1}{16} k_0^3 f'(k_0) + \frac{1}{6} f(k_0)^2 (f(k_0) + 2) - \frac{1}{2} k_0^2 f(k_0) \]
\[ - 2(\frac{1}{2} k_0 f'(k_0) - f(k_0))(\frac{1}{8} k_0^2 - \frac{1}{12} f(k_0)(f(2k_0) + 2)) \]
\[ + \frac{1}{24}(\frac{1}{2} k_0 f'(k_0) - f(k_0))^2 (f(2k_0) + 2)) \]
and
\[ h_1(k_0) = \frac{-2f(2k_0) + 2f(k_0) + 3k_0 f'(k_0)}{-2 - k_0 f'(k_0) + 2f(k_0)}, \quad h_2(k_0) = \frac{2}{5} k_0 f'(k_0) + f(k_0) - f(2k_0). \]
The right-hand side of (B.1) defines a polynomial function of \( \nu_0 \) with coefficients that depend upon \( k_0 \), and the following argument shows that it is negative for all positive values of \( \nu_0 \).

First note that \( a_0, a_2 \) and \( a_4 \) are negative because

\[

h_1(k_0) = g(0)^{-1} g(2k_0)^{-1} > 0, \quad h_2(k_0) = \frac{g(2k_0)}{\nu_0^2} > 0.

\]

A lengthy calculation shows that

\[

a_8 = -\frac{k_0^3}{\sinh^3 k_0} \left( \sum_{j=0}^{\infty} \frac{a_{8,2j+1}}{(2j+1)!} k_0^{2j+1} \right)^{-1} \left( \sum_{j=0}^{\infty} \frac{b_j}{(2j)!} k_0^{2j} \right)

\]

in which explicit formulas for the coefficients \( a_{8,j} \) are computed from the above expression for \( a_8 \). Elementary estimates are used to establish that \( a_{8,j} > 0 \), so that \( a_8 \) is also negative. The argument is completed by demonstrating that \( 4a_4a_8 - a_6^2 \) is positive. For this purpose we use the calculation

\[

4a_4a_8 - a_6^2 = \frac{k_0^4}{\sinh^3 k_0} \left( \sum_{j=0}^{\infty} \frac{b_j}{(2j)!} k_0^{2j} \right)^{-1} \left( \sum_{j=0}^{\infty} \frac{c_j}{(2j)!} k_0^{2j} \right)

\]

with explicit formulas for the coefficients \( b_j \) and \( c_j \), which are also found to be positive.

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