BIFURCATIONS AND EXACT TRAVELING WAVE SOLUTIONS OF THE ZAKHAROV-RUBENCHIK EQUATION

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Abstract. The bounded traveling wave solutions of the Zakharov-Rubenchik equation are investigated by using the method of dynamical system theorems in this paper. After suitable transformations we find that the traveling wave equations of the Zakharov-Rubenchik equation are fully determined by a second-order singular ordinary differential equation (ODE) with three real coefficients which can be arbitrary constants. We derive abundant exact bounded periodic and solitary wave solutions of the Zakharov-Rubenchik equation via studying the bifurcations and exact solutions of the derived ODE.

1. Introduction. In order to model the nonlinear interaction of high-frequency and low-frequency waves in an arbitrary medium, Zakharov [25] proposed a equation given by

\[
\begin{align*}
\frac{dB}{dt} + \triangle B &= B\rho, \\
\frac{\rho_{tt}}{\rho} - \triangle \rho &= \triangle (|B|^2).
\end{align*}
\]

Here $B$ represents the slowly varying envelope of high-frequency electric field and $\rho$ the low-frequency variation of density of ions. This equation is the so-called Zakharov equation and some generalized forms of this equation have been investigated in some literatures [1, 5, 9, 21, 22, 23, 24]. In 2007, Wang et al [24] investigated the exact and numerical solutions of a generalized Zakharov equation with certain initial conditions by using the modified Adomian decomposition method (mADM). Later on, Javidi and Golbabai [9] succeed in finding some exact and numerical

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solutions of the generalized Zakharov equation by the well-known variational iteration method. In 2009, Guo and Zhang [5] studied the existence and uniqueness of smooth solution for a generalized Zakharov equation, and proved the global existence of smooth solution in one spatial dimension at the same time. The homotopy analysis method (HAM) is applied in [1] to obtain approximations to the analytic solution of a generalized form of the Zakharov equation.

The global well-posedness of the one-dimensional Zakharov-Rubenchik equation

\[
\begin{align*}
&iB_t + \omega B_{xx} - \gamma (u - \frac{v}{2} \rho + q |B|^2) B = 0, \\
&\theta \rho_t + (u - v \rho)_x = -\gamma |B|^2_x, \\
&\theta u_t + (\beta \rho - v u)_x = \frac{\gamma}{2} v |B|^2_x,
\end{align*}
\]

has been considered in [21]. As a result, the existence and orbital stability of solitary wave solutions to equations (1) have been proved. [22] proved that the point-wise converge of the solutions \(B_\varepsilon\) of equation (1) to the solution \(B\) of the nonlinear Schrödinger equation by making the adiabatic limit \(\theta\) tend to 0. In 2005, Ponce and Saut [23] studied the well-posedness of a multi-dimensional version of system (1). The global well-posedness of one dimensional Zakharov-Rubenchik was applied [15] to study the norm growth of solutions corresponding to the Schrödinger equation term.

In this paper, we aim to investigate the bounded traveling wave solutions of equations (1). Amounts of methods have been proposed to explore the exact solutions of nonlinear PDEs which model a majority of real-world physical phenomena [4, 10, 16, 18, 29], for instance, the inverse scattering method, Backlund transformation method, Darboux transformation method, Hirota bilinear method, invariant subspace method, symmetry analysis approach, dynamical system approach and some special function etc (refer to [7, 8, 11, 19, 30, 31] and their references).

The dynamical system approach has been well applied to study the dynamics of some nonlinear partial differential equations [3, 12, 17, 26, 32] since the corresponding traveling wave systems of PDEs are dynamical systems which are always involving with some arbitrary constants such as wave speed, constant of integration and parameters of original systems. We will use dynamical system approach [6, 13, 14, 20, 27, 28] to study the bifurcation and exact bounded solutions of the assistant second-order ODE to investigate the bounded traveling wave solutions of the the one-dimensional Zakharov-Rubenchik equation.

This paper is organized as follows. In Section 2, by rescaling and using the traveling wave frame, we reduce the partial differential equations (PDEs) (1) to ODEs at first. By integrations and further calculations, we find that all traveling wave solutions of (1) are determined by a second-order ODE with three real coefficients which can be arbitrary constants. Therefore, it is necessary to study the bifurcations of this ODE which has singularity to derive all the bounded traveling wave solutions of system (1). The phase portraits in each bifurcation set are plotted. In Section 3, we show the various bounded traveling wave solutions of equations (1) and their qualitative properties via the analysis of phase portraits derived in previous section. Some exact solitary wave solutions are obtained by integration along the bounded homoclinic orbits.

2. Reduction of the one-dimensional Zakharov-Rubenchik equation. Letting \(\rho = \psi_1 + \psi_2\) and \(u = \sqrt{\beta} (\psi_1 - \psi_2)\), system (1) becomes
then integrating (8) once gives

$$\theta\psi_{1x} + (\sqrt{\beta} - \nu)\psi_{1x} = \tilde{\gamma}_2 (1 + \frac{\nu}{2\sqrt{\beta}})(|B|^2)_x,$$

$$\theta\psi_{2x} - (\sqrt{\beta} + \nu)\psi_{2x} = \tilde{\gamma}_2 (1 - \frac{\nu}{2\sqrt{\beta}})(|B|^2)_x.$$

Denote $k_1 = \gamma(\sqrt{\beta} - \frac{\nu}{2})$, $k_2 = \gamma(\sqrt{\beta} + \frac{\nu}{2})$, $k_3 = \sqrt{\beta} - \nu$, $k_4 = \sqrt{\beta} + \nu$, $k_5 = \frac{\gamma}{2} (-1 + \frac{\nu}{2\sqrt{\beta}})$ and $k_6 = \frac{\gamma}{2} (1 + \frac{\nu}{2\sqrt{\beta}})$, then system (2) can be rewritten as

$$\left\{\begin{array}{l}
iB_t = -\omega B_{xx} + k_1 B\psi_1 - k_2 B\psi_2 + \gamma q|B|^2 B, \\
\theta\psi_{1x} = -k_3 \psi_{1x} + k_5 (|B|^2)_x, \\
\theta\psi_{2x} = k_4 \psi_{2x} - k_6 (|B|^2)_x.
\end{array}\right.\quad(3)$$

To investigate the traveling wave solutions to (2) and (3), we perform the traveling wave transformations $B(x,t) = e^{-i\lambda t} e^{jf(x)} \varphi(\xi)$, $\psi_1(x,t) = \phi_1(\xi)$, $\psi_2(x,t) = \phi_2(\xi)$ and $\xi = x - ct$, where $\lambda$ and $c$ are two real numbers, $f$, $\varphi$, $\phi_1$ and $\phi_2$ are real functions of $\xi$, then system (2.2) reduces to the following nonlinear ODEs:

$$\left\{\begin{array}{l}
(\omega f'' - \nu \varphi'' + k_1 \varphi_1 - k_2 \varphi_2 - \lambda - cf')\varphi + (c - 2\omega f')i\varphi' - \omega \varphi'' + \gamma q \varphi^3 = 0, \\
- c\theta \phi_1' = -k_3 \phi_1' + k_5 \varphi^2, \\
- c\theta \phi_2' = k_4 \phi_2' - k_6 \varphi \varphi',
\end{array}\right.\quad(4)$$

where $'$ denotes the derivative with respect to $\xi$. In what follows, we will investigate the solutions of system (4) in order to study the traveling wave solutions of system (3). Integrating the second and third equation of system (4) once respectively, we have

$$\phi_1 = \frac{k_5}{k_3 - c\theta} \varphi^2 + g_1, \quad \phi_2 = \frac{k_6}{k_4 + c\theta} \varphi^2 + g_2,$$

where $g_1$ and $g_2$ are constants of integration.

Comparing the real parts and imaginary parts of the first equation of (4), respectively, yields

$$-c\varphi' = -\omega f'' \varphi - 2\omega f' \varphi'$$

and

$$\lambda \varphi + cf' \varphi = \omega f'^2 \varphi - \omega \varphi'' + k_1 \varphi_1 - k_2 \varphi_2 + \gamma q \varphi^3.$$

We see from (6) that

$$f'' \varphi = \left(\frac{c}{\omega} - 2f'\right) \varphi',$$

which is a separable first-order differential equation of the functions $\varphi$ and $f'$. For the case when $\frac{c}{\omega} - 2f' \neq 0$, separating the dependent variables $\varphi$ and $f'$ first and then integrating (8) once gives

$$-\frac{1}{2} \ln\frac{c}{|\omega|} - 2f' \varphi = \ln|\varphi| + g_3,$$

where $g_3$ is a constant of integration. It follows from (9) that

$$\varphi^2 \left(\frac{c}{2\omega} - f'\right) = g,$$

where $g = \pm \frac{1}{2} e^{-2g_3}$. Note that $g$ defined previously seems to be a nonzero constant, but it can be extended to an arbitrary real number including zero, which can be
known from the fact that $\varphi = 0$ is a solution of equation (8). Obviously, for the case when $\varphi \neq 0$, (10) can be rewritten as

$$ f' = \frac{c}{2\omega} + g\varphi^{-2}. \quad (11) $$

Substituting (5) and (11) into (7) yields

$$ \varphi'' = a\varphi + b\varphi^3 - c\omega. \quad (12) $$

where

$$ a = \frac{1}{4\omega^2} \left( -c^2 - 4\omega \left( \lambda + \sqrt{\beta} (g_2 - g_1) + \frac{\gamma}{2} (g_1 + g_2) \gamma \right) \right), \quad d = \frac{2}{\omega} \left( \frac{\beta^{3/2} (g_2 - g_1) \gamma + \frac{v^2}{2} (g_1 + g_2) \gamma}{\beta} \right) + q \right) \right) $$

and

$$ b = g_2. $$

Equation (12) is a second-order ODE, which is equivalent to the following planar dynamical system

$$ \begin{cases} \dot{\varphi} = y, \\ \dot{y} = a\varphi + b\varphi^3 + d\varphi^3. \end{cases} \quad (13) $$

By letting $d\xi = \varphi^3 d\eta$, system (13) becomes

$$ \begin{cases} \dot{\varphi} = \varphi^3 y, \\ \dot{y} = b + a\varphi^4 + d\varphi^6, \end{cases} \quad (14) $$

where $\cdot$ denotes the derivative with respect to $\eta$. It is easy to see that system (14) is a Hamiltonian system with Hamiltonian

$$ H(\varphi, y) = -\frac{b}{2} \varphi^{-2} + \frac{a}{2} \varphi^2 + \frac{d}{4} \varphi^4 - \frac{1}{2} y^2. \quad (15) $$

Clearly, (14) has the same phase portraits as (13) except the singular line $\varphi = 0$ for the case when $b \neq 0$. Let $\phi = \varphi^2$, then (14) becomes

$$ \begin{cases} \dot{\phi} = 2y\phi^2, \\ \dot{y} = b + a\phi^2 + d\phi^3. \end{cases} \quad (16) $$

Note that the function $\phi$ in system (16) is a nonnegative function, so we only need to study the bifurcations of (13) by the analysis of (16) on the right half phase plane.

However, for the case when $\frac{1}{2} - 2f' = 0$, that is

$$ f' = \frac{c}{2\omega}. \quad (17) $$

Substituting (5) and (17) into (7) leads to

$$ \varphi'' = a\varphi + d\varphi^3. \quad (18) $$

It is easy to see that equation (18) is just a special case of equation (12) with $b = 0$, so the coefficient $b$ involving in all above equations (12)-(16) is in the range $[0, +\infty)$.

3. Bifurcation of the planar dynamical system (13).

3.1. Bifurcation analysis of system (13) with $b = 0$. Let $M(\varphi_c, 0)$ be the coefficient matrix of the linearized system of (13) with $b = 0$ at the equilibrium point $(\varphi_c, 0)$ and $J(\varphi_c, 0)$ be its Jacobian determinant. Then we have

$$ J(\varphi_c, 0) = -(a + 3d\varphi_c^2). $$

By the theory of planar dynamic system, we know that for an equilibrium point
of a planar integrable systems, if $J < 0$ then the equilibrium point is a saddle point; If $J > 0$ and $\text{Trace}(M(\varphi_e), 0) = 0$ then it is a center; If $J = 0$ and Poincaré index of the equilibrium point is 0 then it is a cusp. Let $f(\varphi) = a\varphi + d\varphi^3$, we find that $(\varphi_e, 0)$ is a saddle point if $f'(\varphi_e) > 0$ and is a center if $f'(\varphi_e) < 0$, and is of cuspidal type if $f'(\varphi_e) = 0$, where $f(\varphi_e) = 0$. By careful calculations, we have the following statements.

**Case (I).** $d < 0$ & $a > 0$

Equation $f(\varphi) = 0$ has three roots and thus system (13) has three equilibrium points $(\varphi_{e\pm}, 0)$ and $(0, 0)$, where $\varphi_{e\pm} = \pm \sqrt{-\frac{a}{d}}$. Furthermore, $(\varphi_{e\pm}, 0)$ are centers since $f'(\varphi_{e\pm}) < 0$. $(0, 0)$ is a saddle point since $f'(0) > 0$. There are two homoclinic orbits connecting the saddle point and three families of periodic orbits, among which two families of periodic orbits surround the centers respectively and one family of periodic orbits enclose the two homoclinic orbits (see Fig. 1(a)).

**Case (II).** $d > 0$ & $a < 0$

System (13) has three equilibrium points $(\varphi_{e\pm}, 0)$ and $(0, 0)$ similarly. However, $(\varphi_{e\pm}, 0)$ are saddle points since $f'(\varphi_{e\pm}) > 0$ and $(0, 0)$ is a center since $f'(0) < 0$. There are two heteroclinic orbits connecting the two saddle points which construct the boundary of a family of periodic orbits surrounding the center (see Fig. 1(b)).

**Case (III).** $d \leq 0$ & $a < 0$

Equation $f(\varphi) = 0$ has only one root $\varphi = 0$. $(0, 0)$ is a center since $f'(0) < 0$. Hence, there are a family of periodic orbits surrounding the center (see Fig. 1(c)).

**Case (IV).** $d \geq 0$ & $a > 0$ or $a = 0$ & $d \neq 0$

Equation $f(\varphi) = 0$ has only one root $\varphi = 0$ and thus system (13) with $b = 0$ has unique equilibrium point $(0, 0)$ which is a saddle point for $d \geq 0$ & $a > 0$ and a cusp for $a = 0$ & $d \neq 0$. Therefore, system (13) with $b = 0$ has no bounded orbits in this case.

![Figure 1. Phase portraits of system (13) with $b = 0$. (a) $d < 0$ & $a > 0$; (b) $d > 0$ & $a < 0$; (c) $d \leq 0$ & $a < 0$.](image)

### 3.2. Bifurcation analysis of system (13) with $b > 0$

We now analyze the bifurcation of system (13) via studying system (16) by the theory of planar dynamic system. Let $f_1(\phi) = d\phi^3 + a\phi^2 + b$, then the positive roots of $f_1(\phi)$ determine the equilibrium points of system (16). By direct calculations, we firstly get the following results for system (16).
Case (I). $d > 0 \& a \geq 0$

Equation $f_1(\phi) = 0$ has no positive root, so (16) has no equilibrium point on the right-half phase plane.

Case (II). $d > 0 \& a < 0$

For the case when $0 < b < -\frac{4a^3}{27d^2}$, equation $f_1(\phi) = 0$ has two positive roots $\phi_{e_1}$ and $\phi_{e_2}$, where $\phi_{e_1} \in (-\frac{2a}{3d}, -\frac{a}{3})$ and $\phi_{e_2} = \frac{1}{12d}(\sqrt{a^2 - 2ad\phi_{e_1}} - 3d^2\phi_{e_1}^2 - a - d\phi_{e_1})$. Furthermore, $(\phi_{e_2}, 0)$ is a center and $(\phi_{e_1}, 0)$ is a saddle point since $f'_1(\phi_{e_2}) < 0$ and $f'_1(\phi_{e_1}) > 0$. System (16) has a homoclinic orbit connecting the saddle point $(\phi_{e_1}, 0)$.

For the case when $b = -\frac{4a^3}{27d^2}$, equation $f_1(\phi) = 0$ has one positive root $\phi_2$, and the corresponding equilibrium point $(\phi_2, 0)$ is a cusp. For the case when $b > -\frac{4a^3}{27d^2}$, equation $f_1(\phi) = 0$ has no positive root. Therefore, we know that system (16) has no bounded orbits on the right-half plane for the case when $b \geq -\frac{4a^3}{27d^2}$.

Case (III). $d < 0 \& a \geq 0 \text{ or } d \leq 0 \& a < 0$

Equation $f_1(\phi) = 0$ has only one positive root $\phi_e$, where $\phi_e > -\frac{a}{2}$ for $d < 0 \& a \geq 0$ and $\phi_e > 0$ for $d \leq 0 \& a < 0$. $(\phi_e, 0)$ is a center since $f'_1(\phi_e) < 0$. Therefore, system (16) has a family of periodic orbits surrounding the center on its right-half phase plane.

Recalling that $\phi = \varphi^2$, so we can get the phase portrait in each bifurcation set of system (13) via the bifurcation analysis for system (16) on the right-half phase plane. Here we just focus on the bounded orbits of system (13), so we only consider the bounded orbits of system (16). We see from the analysis above that system (16) has bounded orbits if and only if the parameters $b$ and $d$ satisfy the conditions: (A) $d > 0$, $a < 0$ & $0 < b < -\frac{4a^3}{27d^2}$; or (B) $b > 0$, $d < 0$ & $a \geq 0$ or $d \leq 0$ & $a < 0$. In what follows, we consider the bounded orbits of system (13) for which the parameters satisfy one of the two conditions.

(A) If $a$, $b$ and $d$ satisfy condition (1), that is, $d > 0$, $a < 0$ & $0 < b < -\frac{4a^3}{27d^2}$, then system (13) has four equilibrium points $(\pm \sqrt{\phi_{e_1}}, 0)$ and $(\pm \sqrt{\phi_{e_2}}, 0)$, where $(\pm \sqrt{\phi_{e_2}}, 0)$ are centers and $(\pm \sqrt{\phi_{e_1}}, 0)$ are saddle points. There are two homoclinic orbits connecting the two saddle points respectively which are boundary curves of the periodic orbits surrounding the centers $(\pm \sqrt{\phi_{e_2}}, 0)$ respectively (see Fig. 2(A)).

(B) If $a$, $b$ and $d$ satisfy condition (2), namely, $b > 0$, $d < 0$ & $a \geq 0$ or $d \leq 0$ & $a < 0$, then system (13) has two centers $(\pm \sqrt{\phi_e}, 0)$. There are two family of periodic orbits surrounding the two centers respectively (see Fig. 2(B)).

![Figure 2. Phase portraits of system (13) with $b > 0$. (A) $d > 0$, $a < 0$ & $0 < b < -\frac{4a^3}{27d^2}$; (B) $d < 0$ & $a \geq 0$ or $d \leq 0$ & $a < 0$.

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4. Bounded solutions and exact solitary wave solutions of the Zakharov-Rubenchik equation. In this section, we apply the results obtained in the previous sections to study the existence and qualitative properties of bounded solutions and explore some exact traveling wave solutions of the one-dimensional Zakharov-Rubenchik equation (1). Recalling that $B(x,t) = e^{-i\lambda t}e^{i\xi(x)}\varphi(\xi)$, $\rho(x,t) = \phi_1 + \phi_2$, and $u(x,t) = \sqrt{\beta}(\psi_1 - \psi_2) = \sqrt{\beta}(\phi_1 - \phi_2)$, where $\phi_1$ and $\phi_2$ are both determined by $\varphi(\xi)$. To be exact, $\phi_1 = \frac{k_0}{\kappa + c\varphi^2} + g_1$, and $\phi_2 = \frac{k_0}{\kappa + c\varphi^2} + g_2$. Hence, in order to obtain the bounded traveling wave solutions of the Zakharov-Rubenchik equation, we only need to explore the bounded solutions of $\varphi(\xi)$ which is determined by equation (12) or system (13). Clearly, the solutions of (12) are fully determined by the energy curve $H(\varphi,y) = h$, where $H(\varphi,y)$ is defined by (15). We will investigate the bounded solutions of (12) by integrating along the bounded orbits of system (13) in each bifurcation sets.

4.1. Bounded solutions determined by (12) with $b = 0$. All orbits of system (13) are determined by $H(\varphi,y) = h$, namely,

$$\frac{a}{2}\varphi^2 + \frac{d}{4}\varphi^4 - \frac{1}{2}y^2 = h. \quad (19)$$

It follows from Section 3.1 that system (13) has bounded orbits if and only if the parameters $a$ and $d$ satisfies: (I) $d < 0$ & $a > 0$; (II) $d > 0$ & $a < 0$; or (III) $d \leq 0$ & $a < 0$.

For case (I), that is, $d < 0$ and $a > 0$, $H(\varphi,y) = h \leq -\frac{a^2}{4d}$. $h = 0$ corresponds to the two homoclinic orbits connecting the saddle point $(0,0)$; $H(\varphi,y) = h$ with $h \in (0,-\frac{a^2}{4d})$ determines two periodic orbits surrounding the two centers and inside the homoclinic orbits, respectively. $H(\varphi,y) = h$ with $h \in (-\infty,0)$ determines a big periodic orbit which surrounds the two homoclinic orbits.

Solving for $y$ from (19) with $h = 0$ and then substituting the result in the first equation of (13), one has

$$\frac{d\varphi}{d\xi} = \pm \frac{d}{2} \sqrt{\varphi \varphi_+ \varphi_-} \ , \quad (20)$$

where $\varphi_{\pm} = \pm \sqrt{-\frac{2a}{d}}$. Integrating (20) gives

$$\varphi_{\pm}(\xi) = \pm 2 \sqrt{-\frac{2a}{d}} \frac{e^{\sqrt{2d}(\xi - \xi_0)}}{1 + e^{\sqrt{2d}(\xi - \xi_0)}} , \quad (21)$$

where $\xi_0$ is an arbitrary constant. Clearly, (21) are two solutions of (12) with $b = 0$ which has a single peak and tends to 0 as $\xi$ approaches infinity.

For arbitrary $\varphi_1 \in (-\sqrt{-\frac{2a}{d}}, \sqrt{-\frac{2a}{d}})$, $h = H(\varphi_1) \in (0,-\frac{a^2}{4d})$ and thus it follows from $H(\varphi,y) = h$ that

$$\frac{d\varphi}{d\xi} = \pm \frac{d}{2} \sqrt{(\varphi_1 \varphi^2) (\varphi_- \varphi^2)} \ , \quad (22)$$

where $\varphi_2 = \sqrt{-\frac{2a}{d} - \varphi_1^2}$. Integrating (22) once yields (refer to the formula in [2])

$$\varphi(\xi) = \pm \sqrt{\varphi_1^2 - (\varphi_1 \varphi^2) \sin^2(\sqrt{-\frac{d}{2}}\varphi_1(\xi - \xi_0), k)} \quad (23)$$
which are two families of exact periodic solutions of (12). Here “sn” is Jacobian elliptic sine function and 
\[ k = \sqrt{2(1 + \frac{a^2}{d^2})}. \]
However, for arbitrary \( \varphi_1 \in (\sqrt{-\frac{a^2}{d}}, +\infty) \), 
\[ h = H(\varphi_1) \in (-\infty, 0) \]
and thus it follows from \( H(\varphi, y) = h \) that
\[ \frac{d\varphi}{dx} = \pm \sqrt{-\frac{d}{2}} \sqrt{(\varphi^2 - \varphi_1^2)(\varphi^2 + \varphi_1^2 + \frac{2a}{d})}. \] (24)
Integrating (24) leads to
\[ \varphi(x) = \varphi_1 \text{cn}(p_1(x - \xi_0), k_1), \] (25)
where \( p_1 = \sqrt{-a - \varphi_1^2} \) and \( k_1 = \frac{\varphi_1}{\sqrt{a + \frac{a^2}{d}}}. \)

For case (II), namely, \( d > 0 \) and \( a < 0 \), the bounded solution of (12) is determined by 
\[ H(\varphi, y) = h \]
with \( h \in [\frac{a^2}{4d}, 0] \). Furthermore, \( h = -\frac{a^2}{4d} \) corresponds to the two heteroclinic orbits and 
\( h \in (-\frac{a^2}{4d}, 0) \) corresponds to a periodic orbits inside the loop enclosed by the two heteroclinic orbits. Similarly, we get from (19) with 
\[ h = -\frac{a^2}{4d} \]
and the first equation of (13) that
\[ \frac{d\varphi}{dx} = \pm \frac{1}{\sqrt{2d}}(d\varphi^2 + a). \] (26)
Integrating (26) once gives
\[ \varphi(\xi) = \pm \sqrt{-\frac{a}{d}} \tanh \sqrt{-\frac{a}{2}(\xi - \xi_0)}, \] (27)
which correspond to the two heteroclinic orbits of system (13).

For arbitrary \( \varphi_1 \in (0, +\infty) \), \( h = H(\varphi_1) \in (-\frac{a^2}{4d}, 0) \) and thus it follows from 
\( H(\varphi, y) = h \) that
\[ \frac{d\varphi}{dx} = \pm \sqrt{\frac{d}{2}} \sqrt{(\varphi^2 - \varphi_1^2)(\varphi_2^2 - \varphi_1^2)} \], (28)
where \( \varphi_2 = \sqrt{-\frac{2a}{d}} - \varphi_1^2 \). Integrating (28) once yields
\[ \varphi(x) = \varphi_1 \text{sn}(p_2(x - \xi_0), k_2), \] (29)
where \( p_2 = \sqrt{-a - \frac{d}{2}\varphi_1^2} \) and \( k_2 = \frac{\varphi_1}{\sqrt{-\frac{2a}{d} - \varphi_1^2}} \).

For case (III), that is, \( d \leq 0 \) and \( a < 0 \), the bounded solutions of (12) are periodic solutions which are determined by 
\( H(\varphi, y) = h \)
with \( h \in (-\infty, 0) \). For arbitrary \( \varphi_1 \in (0, +\infty) \), \( h = H(\varphi_1) \in (-\infty, 0) \) and thus equation (24) is derived from 
\( H(\varphi, y) = h \) for \( d < 0 \), from which we get solution (25). However, for the case when \( d = 0 \) and \( a < 0 \), we get from 
\( H(\varphi, y) = \frac{a}{2}\varphi_1^2 \) that
\[ \frac{d\varphi}{dx} = \pm \sqrt{-a} \sqrt{\varphi_1^2 - \varphi^2}. \] (30)
Solving (30) directly for \( \varphi \) gives
\[ \varphi(x) = \varphi_1 \sin(\sqrt{-a}(x - \xi_0)). \] (31)
Theorem 4.1. Let \( a = \frac{1}{4c^2} (-c^2 - 4\omega(\lambda + (\frac{\nu}{2} c_1 - c_2)\gamma), \ d = \frac{\gamma}{\omega}(\frac{\gamma}{\omega} + \nu^2 + c\theta\nu), \) and
\[
\begin{align*}
B(x,t) &= e^{-i\lambda t}e^{i\frac{\nu}{2}x+c_3}f(\xi), \\
\rho(x,t) &= \frac{\gamma(v + 2\theta)}{2((v + \theta)^2 - \beta)}\varphi^2(\xi) + c_1, \\
u(x,t) &= \frac{\gamma(2\beta - v(v + \theta))}{2((v + \theta)^2 - \beta)}\varphi^2(\xi) + c_2,
\end{align*}
\]
where \( \xi = x - ct. \) Then (31) with \( f(\xi) = \frac{\nu}{2c} \xi + c_3 \) defines some bounded traveling wave solutions of the Zakharov-Rubenchik equation.

(I) For arbitrary constants \( \lambda, c, c_1, c_2 \) and \( c_3 \) such that \( d < 0 \) and \( a > 0, \) (31) defines two bounded solitary wave solutions of the Zakharov-Rubenchik equation if \( \varphi(\xi) \) is given by (21). For arbitrary \( \varphi_1 \in (\sqrt{-\frac{\nu}{\omega}}, \sqrt{-\frac{2\nu}{\omega}}), \) (31) with (25) defines two families of periodic wave solutions. And for arbitrary \( \varphi_1 \in (\sqrt{-\frac{2\nu}{\omega}}, +\infty), \) (31) with (25) defines a family of periodic wave solutions.

(II) For arbitrary constants \( \lambda, c, c_1, c_2 \) and \( c_3 \) such that \( d > 0, a < 0, \) (31) with \( \varphi(\xi) \) given by (27) defines two bounded traveling wave solutions of the Zakharov-Rubenchik equation. Furthermore, \( \rho(x,t) \) and \( \nu(x,t) \) are solitary wave solutions, but \( B(x,t) \) acts as a kink wave solution. For arbitrary \( \varphi_1 \in (0, \sqrt{-\frac{\nu}{\omega}}), \) (31) with \( \varphi(\xi) \) given by (29) defines a family of bounded periodic traveling wave solutions.

(III) For arbitrary constants \( \lambda, c, c_1, c_2 \) and \( c_3 \) such that \( d \leq 0, a < 0, \) (31) with \( \varphi(\xi) \) given by (25) if \( d < 0 \) or (31) with \( \varphi(\xi) \) given by (31) gives a family of bounded periodic traveling wave solutions for arbitrary \( \varphi_1 \in (0, +\infty). \)

Proof. Integrating (17) once yields \( f = \frac{\nu}{2c} \xi + c_3, \) where \( c_3 \) is a constant of integration. It follows from Section 2 that for arbitrary \( g_1 \) and \( g_2, \)
\[
\begin{align*}
B(x,t) &= e^{-i\lambda t}e^{i\frac{\nu}{2}x+c_3}f(\xi), \\
\psi_1(x,t) &= \frac{\gamma(-2\sqrt{\beta} + v)}{4\sqrt{\beta}(\sqrt{\beta} - v - \theta)}\varphi^2(\xi) + g_1, \\
\psi_2(x,t) &= \frac{\gamma(2\sqrt{\beta} + v)}{4\sqrt{\beta}(\sqrt{\beta} + v + \theta)}\varphi^2(\xi) + g_2
\end{align*}
\]
solves system (2.2) if \( \varphi(\xi) \) satisfies (12) with \( b = 0. \) Let \( c_1 = g_1 + g_2 \) and \( c_2 = \sqrt{\beta}(g_1 - g_2). \) Clearly, \( c_1 \) and \( c_2 \) are also arbitrary constants since \( g_1 \) and \( g_2 \) are two arbitrary constants. In view of the transformations \( \rho = \psi_1 + \psi_2 \) and \( u = \sqrt{\beta}(\psi_1 - \psi_2), \) we see that (31) solves system (2.1) if \( \varphi(\xi) \) satisfies (12) with \( b = 0. \) It is easy to see that solutions (31) is bounded if and only if \( \varphi(\xi) \) is bounded. We know from Section 4.1 that (21), (23) or (25) defines some bounded exact solutions of (12) for \( d < 0 \) and \( a > 0; \) (27) or (29) gives some bounded exact solutions of (12) for \( d > 0 \) and \( a < 0; \) (25) gives some bounded periodic solutions of (12) for \( d < 0 \) and \( a < 0; \) (31) gives some bounded periodic solutions of (12) for \( d = 0 \) and \( a < 0. \) In view of the qualitative properties of \( \varphi(\xi) \) and (31), we prove this theorem. \( \square \)

4.2. Bounded solutions determined by (12) with \( b > 0. \) In order to study the exact solutions of (12) with \( b > 0, \) we solve \( H(\varphi, y) = h \) for \( y \) and then substitute it in the first equation of (13) yields
\[
\frac{d\varphi}{d\xi} = \pm \sqrt{-b\varphi^{-2} + a\varphi^2 + \frac{d}{2}\varphi^4 - 2h}, \quad \text{(34)}
\]
where \( h = H(\varphi_0, 0) \) is determined by the bounded orbits passing through the point \((\varphi_0, 0)\). To find the solution of (34) subject to \( \varphi(\xi_0) = \varphi_0 \), by separating the variables and then integrating it, we have

\[
\int_{\varphi_0}^{\varphi} \frac{1}{\sqrt{-b\varphi^2 + a\varphi^4 + \frac{d}{2} \varphi^4 - 2h}} \, d\varphi = \pm \int_{\xi_0}^{\xi} \, d\xi, \tag{35}
\]

Introducing \( s = \varphi^2 \) and letting \( s_0 = \varphi_0^2 \), then (35) becomes

\[
\frac{1}{2} \int_{s_0}^{s} \frac{1}{\sqrt{\frac{d}{2} s^3 + as^2 - 2hs - b}} \, ds = \pm \int_{\xi_0}^{\xi} \, d\xi. \tag{36}
\]

For the case when \( d > 0 \), \( a < 0 \) and \( 0 < b < -\frac{4a^3}{27d^2} \), the bounded solution of (12) is determined by \( H(\varphi, y) = h \), where \( h \in [H(\sqrt{\varphi_{c_2}}, 0), H(\sqrt{\varphi_{c_1}}, 0)] \). Furthermore, the corresponding Hamiltonian of the homoclinic orbit is determined by \( h_0 = H(\sqrt{\varphi_{c_1}}, 0) \). Note that

\[
d\left( \frac{d}{2} s^3 + as^2 - 2hs - b \right) = \frac{d}{2} (s - \varphi_{c_1})^2 (s - s_0), \tag{37}
\]

where \( s_0 = -2\varphi_{c_1} - \frac{2a}{d} \). Substituting (37) in (36) and then solving for \( s \) leads to

\[
s = (3\varphi_{c_1} + \frac{2a}{d}) \tanh^2 \left( \sqrt{\frac{3d}{2} \varphi_{c_1} + a (\xi - \xi_0)} \right) + s_0, \tag{38}
\]

from which we see that

\[
\varphi_{\pm} = \pm \sqrt{(3\varphi_{c_1} + \frac{2a}{d}) \tanh^2 \left( \sqrt{\frac{3d}{2} \varphi_{c_1} + a (\xi - \xi_0)} \right) + \varphi_0^2}. \tag{39}
\]

Clearly, (39) gives two exact bounded solutions of (12) with \( b > 0 \) for the case when \( d > 0 \), \( a < 0 \) and \( 0 < b < -\frac{4a^3}{27d^2} \).

For arbitrary \( s_0 \in (\varphi_{c_2}, \varphi_{c_1}) \), \( h_1 = H(\sqrt{s_0}, 0) \in (H(\sqrt{\varphi_{c_2}}, 0), H(\sqrt{\varphi_{c_1}}, 0)) \) and

\[
d\left( \frac{d}{2} s^3 + as^2 - 2hs - b \right) = \frac{d}{2} (s - s_+) (s - s_-) (s_+ - s), \tag{40}
\]

where \( s_{\pm} = -(\frac{a}{d} + \frac{1}{2} \sqrt{\Delta}) \) and \( \Delta = (s_0 + \frac{2a}{d})^2 - \frac{8b}{d s_0} \).

Substituting (40) in (36) and then solving for \( s \) leads to

\[
s = \frac{(s_+ - s_0)(s_+ - s_-)}{(s_0 - s_-) s_{\pm}^2 (p_3 (\xi - \xi_0), k_3) - (s_+ - s_-)} + s_+, \tag{41}
\]

where \( p_3 = \sqrt{\frac{d}{2} (s_+ - s_-)} \) and \( k_3 = \sqrt{\frac{s_0 - s_-}{s_+ - s_-}} \).

Therefore,

\[
\varphi_{\pm} = \pm \sqrt{(s_+ - s_0)(s_+ - s_-)/(s_0 - s_-) s_{\pm}^2 (p_3 (\xi - \xi_0), k_3) - (s_+ - s_-) + s_+}, \tag{42}
\]

are two solutions of (12) which correspond to the periodic orbits passing through \((\pm \sqrt{s_0}, 0)\), respectively. Clearly, (39) and (42) are all possible bounded solutions of (12) with \( d > 0 \), \( a < 0 \) and \( 0 < b < -\frac{4a^3}{27d^2} \).

For the case when \( d < 0 \) & \( a \geq 0 \) or \( d \leq 0 \) & \( a < 0 \), the bounded solutions of (12) are determined by \( H(\varphi, y) = h \) which are periodic solutions, where
For arbitrary constants $h \in (-\infty, H(\sqrt{\phi_e}, 0))$. For arbitrary $s_0 \in (0, \phi_e)$ and $d < 0$, $h = H(\sqrt{s_0}, 0) \in (-\infty, H(\sqrt{\phi_e}, 0))$ and
\[
\frac{d}{2} s^3 + as^2 - 2h_1 s - b = -\frac{d}{2}(s - s_0)(s - s_0)(s - s),
\]
where $s_0 < 0 < s_0 < s_+$ and $s_\pm$ defined same as what in (40). Substituting (43) in (36) and then solving for $s$ leads to
\[
s = s_+ - (s_+ - s_0)\text{sn}^2(p_4(\xi - \xi_0), k_4),
\]
where $p_4 = \sqrt{\frac{d}{2}(s_+ - s_+)}$ and $k_4 = \sqrt{\frac{s_+ - s_0}{s_+ - s_0}}$. Therefore,
\[
\varphi_\pm = \pm \sqrt{s_+ - (s_+ - s_0)\text{sn}^2(p_4(\xi - \xi_0), k_4)}
\]
are two solutions of (12) which correspond to the periodic orbits passing through $(\pm\sqrt{s_0}, 0)$, respectively.

For the case when $d = 0$, $a < 0$ and $s_0 \in (0, \phi_e)$,
\[
\frac{d}{2} s^3 + as^2 - 2h_1 s - b = -a(s - s_0)(-\frac{b}{as_0} - s).
\]
Substituting (46) in (36) and then solving for $s$ leads to
\[
s = s_0 - (\frac{b}{as_0} + s_0)\text{sn}^2(\sqrt{-a}(\xi - \xi_0)),
\]
Therefore,
\[
\varphi_\pm = \pm \sqrt{s_0 - (\frac{b}{as_0} + s_0)\text{sn}^2(\sqrt{-a}(\xi - \xi_0))}
\]
are two solutions of (12) which correspond to the periodic orbits passing through $(\pm\sqrt{s_0}, 0)$, respectively.

**Theorem 4.2.** For arbitrary constants $\lambda$, $c$, $g$, $c_1$ and $c_2$ such that $d > 0$, $a < 0$ and $0 < g^2 < -\frac{4b^2}{\lambda^2}$, (31) defines some bounded traveling wave solutions of the Zakharov-Rubenchik equation when $\varphi(\xi)$ is given by (39) or (42) and $f(\xi)$ is determined by $f' = \frac{c}{2\sigma} + g\varphi^{-2}$. The solutions are periodic wave solutions when $\varphi(\xi)$ is given by (12). For the case when $d < 0 \& a \geq 0$ or $d < 0 \& a < 0$, (31) defines some bounded periodic traveling wave solutions of the Zakharov-Rubenchik equation when $\varphi(\xi)$ is given by (45). For the case when $d = 0 \& a \geq 0$ or $d < 0 \& a \geq 0$, (31) defines two families of bounded periodic traveling wave solutions of the Zakharov-Rubenchik equation when $\varphi(\xi)$ is given by (48).

5. **Conclusion.** In this paper, we obtain traveling wave solutions of the Zakharov-Rubenchik equation with different parameter values. By using traveling wave transformations and a series of calculations, we transformed the equation to a second-order ODE which have coefficients $a$, $b$ and $d$. By using bifurcation and dynamic system theorem, all possible bounded real solutions of the involving planar dynamic systems were studied and then some exact solitary wave solutions were obtained.

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