CONJECTURES ABOUT CERTAIN PARABOLIC KAZHDAN–LUSZTIG POLYNOMIALS

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Abstract. Irreducibility results for parabolic induction of representations of the general linear group over a local non-archimedean field can be formulated in terms of Kazhdan–Lusztig polynomials of type $A$. Spurred by these results and some computer calculations, we conjecture that certain alternating sums of Kazhdan–Lusztig polynomials known as parabolic Kazhdan–Lusztig polynomials satisfy properties analogous to those of the ordinary ones.

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1. Introduction

Let $P_{u,w}$ be the Kazhdan–Lusztig polynomials with respect to the symmetric groups $S_r$, $r \geq 1$. Recall that $P_{u,w} = 0$ unless $u \leq w$ in the Bruhat order. Fix $m, n \geq 1$ and let $H \cong S_m \times \cdots \times S_m$ be the parabolic subgroup of $S_{mn}$ of type $(m, \ldots, m)$ ($n$ times). (In the body of the paper we will only consider the case $m = 2$, but for the introduction $m$ is arbitrary.) The normalizer of $H$ in $S_{mn}$ is $H \rtimes N$ where

$$N = \{ \tilde{w} : w \in S_n \} \text{ and } \tilde{w}(mi - j) = mw(i) - j, \ i = 1, \ldots, n, \ j = 0, \ldots, m - 1.$$ 

As a consequence of representation-theoretic results, it was proved in [LM16] that if $x, w \in S_n$ with $x \leq w$ and there exists $v \leq x$ such that $P_{v,w} = 1$ and $v$ is (213)-avoiding (i.e., there do not exist indices $1 \leq i < j < k \leq n$ such that $v(j) < v(i) < v(k)$) then

$$\sum_{u \in H} \text{sgn} \ u \ P_{\tilde{x}u, \tilde{w}}(1) = 1.$$
Conjecture 1.1. For any $x, w \in S_n$ with $x \leq w$ write
\begin{equation}
\sum_{u \in H} \text{sgn } u \ P_{xu, w} = q^{\binom{m}{2} (\ell(w) - \ell(x))} \tilde{P}_{x, w}^{(m)}.
\end{equation}
Then
1. $\tilde{P}_{x, w}^{(m)}$ is a polynomial (rather than a Laurent polynomial).
2. $\tilde{P}_{x, w}^{(m)}(0) = 1$.
3. $\tilde{P}_{x, w}^{(m)} = P_{x, w}^{(m)}$ for any simple reflection $s$ of $S_n$ such that $ws < w$.
4. $\deg \tilde{P}_{x, w}^{(m)} = m \deg P_{x, w}$. In particular, $\tilde{P}_{x, w}^{(m)} = 1$ if and only if $P_{x, w} = 1$.

Remark 1.2. (1) The left-hand side of (2) is an instance of a parabolic Kazhdan–Lusztig polynomial in the sense of Deodhar [Deo87]. They are known to have non-negative coefficients [KT02] (see also [Yun09, BY13]). Thus, the same holds for $\tilde{P}_{x, w}^{(m)}$. In particular, if (1) holds (i.e., if $\tilde{P}_{x, w}^{(m)}(1) = 1$) then the left-hand side of (2) is a priori a monomial (with coefficient 1), and the conjecture would say in this case that its degree is $\binom{m}{2} (\ell(w) - \ell(x))$.

(2) Conjecture 1.1 is trivially true for $m = 1$ (in which case $\tilde{P}_{x, w}^{(m)} = P_{x, w}$), or if $w = x$ (in which case $\tilde{P}_{x, w}^{(m)} = P_{x, w} = 1$).

(3) The summands on the left-hand side of (2) are 0 unless $x \leq w$.

(4) If $w = w_1 \oplus w_2$ (direct sum of permutations) then $x = x_1 \oplus x_2$ with $x_i \leq w_i$, $i = 1, 2$ and $\tilde{P}_{x, w}^{(m)} = \tilde{P}_{x_1, w_1}^{(m)} \tilde{P}_{x_2, w_2}^{(m)}$. Thus, in Conjecture 1.1 we may assume without loss of generality that $w$ is indecomposable (i.e., does not belong to a proper parabolic subgroup of $S_n$).

(5) The relations $\tilde{P}_{x, w}^{(m)} = \tilde{P}_{x, w}^{(m)} = \tilde{P}_{w_0, w_0, w_0, w_0}^{(m)}$ for the longest $w_0 \in S_n$ are immediate from the definition and the analogous relations for $m = 1$. It is also not difficult to see that, just as in the case $m = 1$, we have $\tilde{P}_{x, w}^{(m)} = \tilde{P}_{x, w}^{(m)}$ (and more precisely, $P_{u\bar{u}z, \bar{u}z} = P_{u\bar{u}z, \bar{u}z}$ for every $u \in H$) for any $x, w \in S_n$ and a simple reflection $s$ such that $xs < x \not< ws < w$. On the other hand, the third part of the conjecture does not seem to be a formal consequence of the analogous relation for $m = 1$.

(6) An index $i = 1, \ldots, n$ is called cancelable for $(w, x)$ if $w(i) = x(i)$ and $\#\{j < i : x(j) < x(i)\} = \#\{j < i : w(j) < w(i)\}$. It is known that in this case $P_{x, w} = P_{x^i, w^i}$ where $w^i = \Delta_{x(i)}^{-1} \circ w \circ \Delta_i, x^i = \Delta_{x(i)}^{-1} \circ x \circ \Delta_i \in S_{n-1}$ and $\Delta_j : \{1, \ldots, n - 1\} \rightarrow \{1, \ldots, n\} \setminus \{j\}$ is the monotone bijection (see [BW03, Hen07]). Clearly, if $i$ is cancelable for $(w, x)$ then $mi, mi - 1, \ldots, m(i - 1) + 1$ are cancelable for $(\bar{w}, \bar{x})$, and hence it is easy to see from the definition that $\tilde{P}_{x, w}^{(m)} = \tilde{P}_{x^i, w^i}^{(m)}$. 

\footnote{This is now known for any Coxeter group and a parabolic subgroup thereof by Libedinsky–Williamson [LW17].}
For $n = 2$ (and any $m$) Conjecture 1.1 is a special case of a result of Brenti [Bre02]. (See also [BC17].) We verified the conjecture numerically for the cases where $nm \leq 12$. In the appendix we provide all non-trivial $\tilde{P}^{(m)}_{x,w}$ in these cases. In general, already for $m = 2$, $\tilde{P}^{(m)}_{x,w}$ does not depend only on $P_{x,w}$. Nevertheless, there seems to be some correlation between $\tilde{P}^{(2)}_{x,w}$ and $(P_{x,w})^2 + P_{x,w}(q^2))/2$.

The purpose of this paper is to provide modest theoretical evidence, or a sanity check, for Conjecture 1.1 in the case $m = 2$. Namely, we prove it in the very special case that $w$ is any Coxeter element of $S_n$ (or a parabolic subgroup thereof). Note that $P_{e,w} = 1$ in this case and the conjecture predicts that $\tilde{P}^{(2)}_{x,w} = 1$ for any $x \leq w$. Following Deodhar [Deo90], the assumption on $w$ guarantees that $P_{u,\tilde{w}}$ admits a simple combinatorial formula for any $u \in S_{2n}$. (This is a special case of a result of Billey–Warrington [BW01] but the case at hand is much simpler.) Thus, the problem becomes elementary. (For an analogous result in a different setup see [Mon14].)

In principle, the method can also be used to prove Conjecture 1.1 for $m = 3$ in the case where $w$ is the right or left cyclic shift. However, we will not carry this out here. Unfortunately, for $m > 3$ the method is not applicable for any $w \neq e$ (again, by the aforementioned result of [BW01]).

In the general case, for instance if $w$ is the longest Weyl element, we are unaware of a simple combinatorial formula for the individual Kazhdan–Lusztig polynomials $P_{u,\tilde{w}}$, even for $m = 2$. Thus, Conjecture 1.1 becomes more challenging and at the moment we do not have any concrete approach to attack it beyond the cases described above. In particular, we do not have any theoretical result supporting the last part of the conjecture, which rests on thin air.

We mention that the relation (1) admits the following generalization. Suppose that $v, w \in S_n$, $P_{v,w} = 1$ and $v$ is (213)-avoiding. Then

$$\sum_{u \in HxH} \text{sgn } u P_{u,\tilde{w}}(1) = \sum_{u \in HxH \cap K} \text{sgn } u$$

for any $x \in S_{2n}$ such that $\tilde{v} \leq x \leq \tilde{w}$ ([LM16 Theorem 10.11], which uses [Lap17]). Here $K$ is the subgroup of $S_{mn}$ (isomorphic to $S_n \times \cdots \times S_n$, $m$ times) that preserves the congruence classes modulo $m$. In the case $m = 2$, sgn is constant on $HxH \cap K$ and the cardinality of $HxH \cap K$ is a power of 2 that can be easily explicited in terms of $x$ ([LM16 Lemma 10.6]). (For $m > 2$ this is no longer the case. For instance, for $m = n$ and a suitable $x$, (3) is $(-1)^{(n)}$ times the difference between the number of even and odd Latin squares of size $n \times n$.) In general, already for $m = 2$, the assumption that $v$ is (213)-avoiding is essential for (3) (in contrast to (1) if Conjecture 1.1 holds). For $m = 2$ and Coxeter elements $w$ we give in Corollary 6.2 a simple expression for $\sum_{u \in HxH} \text{sgn } u P_{u,\tilde{w}}$ for any $x \in S_{2n}$. However, at the moment we do not know how to extend it, even conjecturally, to other $w$’s such that $P_{e,w} = 1$.

It would be interesting to know whether Conjecture 1.1 admits a representation-theoretic interpretation.
Normalizers of parabolic subgroups of Coxeter groups were studied in [Lus77, How80, Bor98, BH99]. In particular, they are the semidirect product of the parabolic subgroup by a complementary subgroup, which in certain cases is a Coxeter group by itself. It is natural to ask whether Conjecture 1.1 extends to other classes of parabolic Kazhdan–Lusztig polynomials with respect to (certain) pairs of elements of the normalizer (e.g., as in the setup of [Lus03, §25.1]). Note that already for the Weyl group of type $B_2$ these parabolic Kazhdan–Lusztig polynomials may vanish, so that a straightforward generalization of Conjecture 1.1 is too naive. Nonetheless, recent results of Brenti, Mongelli and Sentinelli [BMS16] (albeit rather special) may suggest that some generalization (which is yet to be formulated) is not hopeless. Perhaps there is even a deeper relationship between parabolic Kazhdan–Lusztig polynomials pertaining to different data (including ordinary ones). (See [Sen14] for another result in this direction.) At the moment it is not clear what is the scope of such a hypothetical relationship.

Another natural and equally important question, to which I do not have an answer, is whether the geometric interpretations of the parabolic Kazhdan–Lusztig polynomials [KT02, Yun09, BY13, LW17] shed any light on Conjecture 1.1 or its possible generalizations.

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2. A result of Deodhar

2.1. For this subsection only let $G$ be an algebraic semisimple group over $\mathbb{C}$ of rank $r$, $B$ a Borel subgroup of $G$ and $T$ a maximal torus contained in $B$. We enumerate the simple roots as $\alpha_1, \ldots, \alpha_r$, the corresponding simple reflexions by $s_1, \ldots, s_r$ and the corresponding minimal non-solvable parabolic subgroups by $Q_1, \ldots, Q_r$. Let $W$ be the Weyl group, generated by $s_1, \ldots, s_r$. The group $W$ is endowed with the Bruhat order $\leq$, the length function $\ell$, and the sign character $\text{sgn} : W \to \{\pm 1\}$.

We consider words in the alphabet $s_1, s_2, \ldots, s_r$. For any word $w = s_{j_1} \ldots s_{j_l}$ we write $\pi(w) = s_{j_1} \ldots s_{j_l}$ for the corresponding element in $W$ and $\text{supp}(w) = \{\alpha_{j_1}, \ldots, \alpha_{j_l}\}$. We say that $w$ is supported in $A$ if $A \supset \text{supp}(w)$. We also write $w^r$ for the reversed word $s_{j_l} \ldots s_{j_1}$.

Let $w = s_{j_1} \ldots s_{j_l}$ be a reduced decomposition for $w \in W$ where $l = \ell(w)$. The reversed word $w^r$ is a reduced decomposition for $w^{-1}$. Note that $\text{supp}(w) = \{\alpha_i : s_i \leq w\}$ and in particular, $\text{supp}(w)$ depends only on $w$. A $w$-mask is simply a sequence of $l$ zeros and ones, i.e., an element of $\{0, 1\}^l$. For a $w$-mask $\underline{x} \in \{0, 1\}^l$ and $i = 0, \ldots, l$ we write $w^{(i)}[\underline{x}]$ for the subword of $w$ composed of the letters $s_{j_k}$ for $k = 1, \ldots, i$ with $\underline{x}_k = 1$. For $i = l$ we simply write

$$w[\underline{x}] = w^{(l)}[\underline{x}].$$

Let

$$D_w(\underline{x}) = \{i = 1, \ldots, l : \pi(w^{(i-1)}[\underline{x}]) (\alpha_{j_i}) < 0\}, \quad \varnothing_w(\underline{x}) = |D_w(\underline{x})|$$

(4)
(the defect set and defect statistics of \( \underline{x} \)) where \(| \cdot |\) denotes the cardinality of a set. We also write
\[
D^i_w(\underline{x}) = \{ i \in D_w(\underline{x}) : x_i = r \}, \quad E^i_w(\underline{x}) = \{ i \notin D_w(\underline{x}) : x_i = r \}, \quad r = 0, 1.
\]
Note that \( \ell(\underline{w}[\underline{x}]) = |E^1_w(\underline{x})| - |D^1_w(\underline{x})| \) for any \( \underline{x} \). We say that \( \underline{x} \) is full if \( x_i = 1 \) for all \( i \).

For later use we also set
\[
\text{sgn} \underline{x} = \prod_{i=1}^{l} (-1)^{x_i} \in \{ \pm 1 \}
\]
so that \( \text{sgn} \underline{w}[\underline{x}] = \text{sgn} \underline{x} \).

It is well known that
\[
\{ \pi(\underline{w}[\underline{x}]) : \underline{x} \in \{0,1\}^l \} = \{ \pi(\underline{w}[\underline{x}]) : \underline{x} \in \{0,1\}^l \text{ and } \underline{w}[\underline{x}] \text{ is reduced} \} = \{ u \in W : u \leq w \}.
\]

For any \( u \in W \) define the polynomial
\[
P^w_u = \sum_{\underline{w}[\underline{x}] \in \{0,1\}^{\ell(w)} : \pi(\underline{w}[\underline{x}]) = u} q^\phi(\underline{x}).
\]
Let
\[
\phi_w : (Q_{j_1} \times \cdots \times Q_{j_l})/B^{l-1} \rightarrow \overline{BwB}, \quad (q_1, \ldots, q_l) \mapsto q_1 \cdots q_l
\]
be (essentially) the Bott–Samelson resolution [Dem74] where \( B^{l-1} \) acts by \((q_1, \ldots, q_l) \cdot (b_1, \ldots, b_{l-1}) = (q_1b_1, b_1^{-1}q_2b_2, \ldots, b_{l-2}^{-1}q_{l-1}b_{l-1}, b_{l-1}^{-1}q_l)\).

Remark 2.1. (1) It is easy to see that \( P^w_u \) has constant term 1 if \( u \leq w \) (cf. top of p. 161 in [JW13]).

(2) It follows from the Białynicki-Birula decomposition that \( P^w_u \) is the Poincare polynomial for \( \phi^{-1}_w(BuB) \) (cf. [Deo90, Proposition 3.9], [JW13, Proposition 5.12]). In particular, since the diagram
\[
\begin{array}{ccc}
(Q_{j_1} \times \cdots \times Q_{j_l})/B^{l-1} & \xrightarrow{\phi_w} & \overline{BwB} \\
(q_1, \ldots, q_l) & \mapsto & (q_1^{-1}, \ldots, q_1^{-1})
\end{array}
\]
is commutative we have \( P^{-1}_{u^{-1}} = P^w_u \), a fact which is not immediately clear from the definition since in general \( \partial_w(\underline{x}) \neq \partial_w(\underline{x}^t) \) where \( \underline{x}^t \) denotes the reversed \( w^t \)-mask \( \underline{x}_{i+1}^t = \underline{x}_{i+1} \).

(3) In general, \( P^w_u \) heavily depends on the choice of \( w \) unless \( w \) has the property that all its reduced decompositions are obtained from one another by repeatedly interchanging adjacent commuting simple reflections, i.e., \( w \) is fully commutative.

For \( u, w \in W \) we denote by \( P_{u,w} \) the Kazhdan–Lusztig polynomial with respect to \( W \) [KL79]. In particular, \( P_{u,w} = 0 \) unless \( u \leq w \), in which case \( P_{u,w}(0) = 1 \) and all coefficients of \( P_{u,w} \) are non-negative. (This holds in fact for any Coxeter group by a recent result of
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Elias-Williamson [EW14]. We also have \( P_{w,u} = 1 \), \( \deg P_{u,w} \leq \frac{1}{2}(\ell(w) - \ell(u) - 1) \) for any \( u < w \)

\[ P_{u,w} = P_{w,u}. \]

In general, even for the symmetric group, it seems that no “elementary” manifestly positive combinatorial formula for \( P_{u,w} \) is known. However, implementable combinatorial formulas to compute \( P_{u,w} \) do exist (see [BB05] and the references therein).

The following is a consequence of the main result of [Deo90]. See [Dec94, BW01, JW13] for more details.

**Theorem 2.2.** [Dec90] Let \( w \) be a reduced decomposition for \( w \in W \). Then the following conditions are equivalent.

1. \( \deg P_{w,u} \leq \frac{1}{2}(\ell(w) - \ell(u) - 1) \) for any \( u < w \).
2. For every non-full \( w \)-mask \( x \) we have \( |E_0^w(x)| > |D_0^w(x)| \).
3. For every \( u \in W \) we have \( P_{u,w} = P_{w,u} \).
4. The Bott–Samelson resolution \( \phi_w \) is small.

In particular, under these conditions

\[ P_{u,w}(1) = \left| \{ x \in \{0, 1\}^{\ell(w)} : \pi(w[x]) = u \} \right| \]

for any \( u \in W \).

Following Lusztig [Lus93] and Fan–Green [FG97] we say that \( w \in W \) is tight if it satisfies the conditions of Theorem 2.2 (in fact, this condition is independent of the choice of \( w \)).

**3. Certain classes of permutations**

From now on we specialize to the symmetric group \( S_n \) on \( n \) letters, \( n \geq 1 \). We enumerate \( \alpha_1, \ldots, \alpha_{n-1} \) in the usual way. Thus, \( w\alpha_i > 0 \) if and only if \( w(i) < w(i+1) \). We normally write elements \( w \) of \( S_n \) as \( (w(1) \ldots w(n)) \).

Given \( x \in S_m \) and \( w \in S_n \) we say that \( w \) avoids \( x \) if there do not exist indices \( 1 \leq i_1 < \cdots < i_m \leq n \) such that

\[ \forall 1 \leq j_1 < j_2 \leq m \, w(i_{j_1}) < w(i_{j_2}) \iff x(j_1) < x(j_2). \]

Equivalently, the \( n \times n \)-matrix \( M_w \) representing \( w \) does not admit \( M_x \) as a minor.

There is a vast literature on pattern avoidance. We will only mention two remarkable closely related general facts. The first is that given \( m \), there is an algorithm, due to Guillemot–Marx, to detect whether \( w \in S_n \) is \( x \)-avoiding whose running time is linear in \( n \) [GM14]. (If \( m \) also varies then the problem is NP-complete [BBL98].) Secondly, if \( C_n(x) \) denotes the number of \( w \in S_n \) which are \( x \)-avoiding then it was shown by Marcos–Tardos that the Stanley–Wilf limit \( C(x) = \lim_{n \to \infty} C_n(x)^{1/n} \) exists and is finite [MT04], and as proved more recently by Fox, it is typically exponential in \( m \) [Fox].

We recall several classes of pattern avoiding permutations. First, consider the (321)-avoiding permutations, namely those for which there do not exist \( 1 \leq i < j < k \leq n \) such that \( w(i) > w(j) > w(k) \). It is known that \( w \) is (321)-avoiding if and only if it is fully commutative [BJS93]. The number of (321)-avoiding permutations in \( S_n \) is the Catalan
number \( C_n = \binom{2n}{n} - \binom{2n}{n-1} \) – a well-known result which goes back at least 100 years ago to MacMahon [Mac04].

We say that \( w \in S_n \) is smooth if it avoids the patterns (4231) and (3412). By a result of Lakshmibai–Sandhya, \( w \) is smooth if and only if the closure \( BwB \) of the cell \( BwB \) in \( \text{GL}_n(\mathbb{C}) \) (where \( B \) is the Borel subgroup of upper triangular matrices) is smooth [LS90]. (A generating function for the number of smooth permutations in \( S_n \) is given in [BMB07].)

It is also known that \( w \) is smooth \( \iff P_{e,w} = 1 \iff P_{u,w} = 1 \) for all \( u \leq w \) [Deo85].

The (321)-avoiding smooth permutations (i.e., the (321) and (3412)-avoiding permutations) are precisely the products of distinct simple reflections [Fan98, Wes96], i.e., the Coxeter elements of the parabolic subgroups of \( S_n \). They are characterized by the property that the Bruhat interval \( \{ x \in S_n : x \leq w \} \) is a Boolean lattice, namely the power set of \( \{ i : s_i \leq w \} \) [Ten07]. They are therefore called Boolean permutations. The number of Boolean permutations in \( S_n \) is \( F_{2n-1} \) where \( F_m \) is the Fibonacci sequence [Fan98, Wes96].

In [BW01] tight permutations were classified by Billey–Warrington in terms of pattern avoidance. Namely, \( w \) is tight if and only if it avoids the following five permutations

(8) \((321) \in S_3, \ (46718235), \ (46781235), \ (56718234), \ (56781234) \in S_8.

For the counting function of this class of permutations see [SW04].

Remark 3.1. In [Las95], Lascoux gave a simple, manifestly positive combinatorial formula for \( P_{u,w} \) in the case where \( w \) is (3412)-avoiding (a property also known as co-lexicographical). Note that a (321)-hexagon-avoiding permutation \( w \) is co-lexicographical if and only if it is a Boolean permutation, in which case \( P_{u,w} = 1 \) for all \( u \leq w \).

4. The Defect

Henceforth (except for Remark 4.6 below) we assume, in the notation of the introduction, that \( m = 2 \). Recall the group homomorphism

\[ \tilde{w} : S_n \to S_{2n} \]

given by

\[ \tilde{w}(2i) = 2w(i), \ \tilde{w}(2i-1) = 2w(i) - 1, \ i = 1, \ldots, n. \]

(Technically, \( \tilde{w} \) depends on \( n \) but the latter will be hopefully clear from the context.)

We will use Theorem 2.2 to derive a simple expression for \( P_{u,\tilde{w}} \) where \( w \) is a Boolean permutation. (Note that if \( e \neq w \in S_n \) then \( \tilde{w} \) is not co-lexicographical. Thus, Lascoux’s formula is not applicable.)

Remark 4.1. It is easy to see that \( w \) is Boolean if and only if \( \tilde{w} \) satisfies the pattern avoidance conditions of [BW01]. Thus, it follows from [ibid.] that \( \tilde{w} \) is tight. However, we will give a self-contained proof of this fact since this case is much simpler and in any case the ingredients are needed for the evaluation of \( P_{u,\tilde{w}} \).

For the rest of the paper we fix a Boolean permutation \( w \in S_n \) and a reduced decomposition \( w = s_{j_1} \ldots s_{j_l} \) for \( w \) where \( l = \ell(w) \) and \( j_1, \ldots, j_l \in \{1, \ldots, n-1\} \) are distinct. The choice of \( w \) plays little role and will often be suppressed from the notation.
For any \( x \in S_n \) let 
\[
I_x = \{ i : s_{j_i} \leq x \}.
\]
Then 
\[
x \mapsto I_x \text{ is a bijection between } \{ x \in S_n : x \leq w \} \text{ and } \mathcal{P}(\{1, \ldots, l\})
\]
where \( \mathcal{P} \) denotes the power set.

A key role is played by the following simple combinatorial objects.

**Definition 4.2.** Let \( A \) and \( B \) be subsets of \( \{1, \ldots, l\} \) with \( A \subset B \).

1. The right (resp., left) neighbor set \( \tilde{N}_A^B = w_N^B \) (resp., \( \hat{N}_A^B = w_N^B \)) of \( A \) in \( B \) with respect to \( w \) consists of the elements \( i \in B \setminus A \) for which there exists \( t > 0 \) and indices \( i_1, \ldots, i_t, \) necessarily unique, such that \( i < i_1 < \cdots < i_t \) (resp., \( i > i_1 > \cdots > i_t \)), \( \{i_1, \ldots, i_{t-1}\} \subset A, i_t \in B \setminus A \) and \( j_{ik} = j_i + k \) for \( k = 1, \ldots, t \).
2. The neighbor set of \( A \) in \( B \) with respect to \( w \) is \( N_A^B = \tilde{N}_A^B \cup \hat{N}_A^B \).
3. The neighboring function \( \nu_A^B = w_N^B : N_A^B \rightarrow B \setminus A \) is given by the rule \( i \mapsto i_t \).

Note that the sets \( \tilde{N}_A^B \) and \( \hat{N}_A^B \) are disjoint and that \( \nu_A^B \) is injective. Moreover, if \( i \in N_A^B \) then \( \nu_A^B(i) > i \) if and only if \( i \in \hat{N}_A^B \).

If \( B = \{1, \ldots, l\} \) then we suppress \( B \) from the notation. Note that 
\[
N_A^B = N_A \cap B \cup \nu_A^{-1}(B) \text{ and } \nu_A \big|_{N_A^B} = \nu_A^B.
\]
We have \( \ell(\tilde{w}) = 4l \) and a reduced decomposition for \( \tilde{w} \) is given by 
\[
\tilde{w} = s_{2j_1}s_{2j_1-1}s_{2j_1+1}s_{2j_1} \cdots s_{2j_1}s_{2j_1-1}s_{2j_1+1}s_{2j_1}.
\]
It will be convenient to write \( \tilde{w} \)-masks as elements of \( (\{0, 1\}^4)^l \). Thus, if \( \bar{x} \) is a \( \tilde{w} \)-mask then \( \bar{x}_i \in \{0, 1\}^4, i = 1, \ldots, l \) and we write \( \bar{x}_{i,k}, k = 1, 2, 3, 4 \) for the coordinates of \( \bar{x}_i \). By convention, we write for instance \( \bar{x}_i = (*, 1, *, 0) \), to mean that \( \bar{x}_{i,2} = 1 \) and \( \bar{x}_{i,4} = 0 \), without restrictions on \( \bar{x}_{i,1} \) or \( \bar{x}_{i,3} \).

For the rest of the section we fix a \( \tilde{w} \)-mask \( \bar{x} \in (\{0, 1\}^4)^l \) and let 
\[
I_f = \{ i \in \{1, \ldots, l\} : \bar{x}_i = (1, 1, 1, 1) \}.
\]
We explicate the defect set \( D_{\tilde{w}}(\bar{x}) \) of \( \bar{x} \) (see (1)).

**Lemma 4.3.** For any \( i = 1, \ldots, l \) let \( C(\bar{x}, i) \) (resp., \( \tilde{C}(\bar{x}, i) \)) be the condition 
\[
i \in \tilde{N}_{I_f} \text{ and either } \bar{x}_{\nu_{I_f}(i)} = (1, 1, 0, 1) \text{ or } \bar{x}_{\nu_{I_f}(i)} = (*, 1, *, 0)
\]
(resp., 
\[
\exists r \in \tilde{N}_{I_f} \text{ such that } \nu_{I_f}(r) = i \text{ and either } \bar{x}_r = (1, 0, 1, 1) \text{ or } \bar{x}_r = (*, *, 1, 0)).
\]
Then for all \( i = 1, \ldots, l \) we have
1. \( \pi(\tilde{w}^{(4i-4)}|\bar{x})\alpha_{2j_i} > 0 \).
2. \( \pi(\tilde{w}^{(4i-3)}|\bar{x})\alpha_{2j_i-1} < 0 \) if and only if \( \bar{x}_i = (0, *, *, *) \) and \( \tilde{C}(\bar{x}, i) \).
3. \( \pi(\tilde{w}^{(4i-2)}|\bar{x})\alpha_{2j_i+1} < 0 \) if and only if \( \bar{x}_i = (0, *, *, *) \) and \( C(\bar{x}, i) \).
(4) \( \pi(\tilde{w}^{(4i-1)}[x])\alpha_{2j} < 0 \) if and only if (exactly) one of the following conditions is satisfied

\[
\begin{align*}
\tilde{x}_i = (1, 0, 0, \ast), \\
\tilde{x}_i = (1, 0, 1, \ast) \text{ and } C(\tilde{x}, i), \\
\tilde{x}_i = (1, 1, 0, \ast) \text{ and } \tilde{C}(\tilde{x}, i).
\end{align*}
\]

Proof. (1) This is clear since \( \alpha_{2j} \notin \text{supp}(\tilde{w}^{(4i-4)}[x]) \).

(2) If \( \tilde{x}_{i,1} = 1 \) then \( \pi(\tilde{w}^{(4i-3)}[x])\alpha_{2j-1} = \pi(\tilde{w}^{(4i-3)}[x])\alpha_{2j-1} + \alpha_{2j} \) which as before is a positive root since \( \alpha_{2j} \notin \text{supp}(\tilde{w}^{(4i-4)}[x]) \).

Suppose from now on that \( \tilde{x}_{i,1} = 0 \) and let \( t \geq 0 \) be the largest index for which there exist (unique) indices \( i_t < \cdots < i_0 = i \) with \( \{i_1, \ldots, i_t\} \subset I_f \) such that \( j_{i_t} = j_i = t - 1 \) then \( \pi(\tilde{w}^{(4i-3)}[x])\alpha_{2j-1} = \alpha_{2j-1} > 0 \). Otherwise, by the maximality of \( t \), we have \( r \in \tilde{N}_{r,f} \), \( \nu_{r,f}(r) = i \) and \( \pi(\tilde{w}^{(4i-3)}[x])\alpha_{2j-1} = \pi(\tilde{w}^{(4r)}[x])\alpha_{2j-1} \). We split into cases.

(a) If \( \tilde{x}_{r,3} = 0 \) then \( \alpha_{2j+1} \notin \text{supp}(\tilde{w}^{(4r)}[x]) \) and therefore \( \pi(\tilde{w}^{(4i-3)}[x])\alpha_{2j-1} < 0 \).

(b) Assume that \( \tilde{x}_{r,3} = 1 \).

(i) If \( \tilde{x}_{r,4} = 0 \) then \( \pi(\tilde{w}^{(4r)}[x])\alpha_{2j+1} = -\pi(\tilde{w}^{(4r-2)}[x])\alpha_{2j+1} < 0 \) since \( \alpha_{2j+1} \notin \text{supp}(\tilde{w}^{(4r-2)}[x]) \).

(ii) Assume that \( \tilde{x}_{r,4} = 1 \), so that \( \pi(\tilde{w}^{(4r)}[x])\alpha_{2j+1} = \pi(\tilde{w}^{(4r-2)}[x])\alpha_{2j} \).

The latter is a positive root unless \( \tilde{x}_{r,1} = 1 \) (since otherwise \( \alpha_{2j} \notin \text{supp}(\tilde{w}^{(4r-2)}[x]) \)). If \( \tilde{x}_{r,1} = 1 \) then \( \tilde{x}_{r,2} = 0 \) (since \( r \notin I_f \)) and \( \pi(\tilde{w}^{(4i-3)}[x])\alpha_{2j-1} = -\pi(\tilde{w}^{(4r-4)}[x])\alpha_{2j} < 0 \) since \( \alpha_{2j} \notin \text{supp}(\tilde{w}^{(4r-4)}[x]) \).

(3) This is similar to the second part. We omit the details.

(4) If \( \tilde{x}_{i,1} = 0 \) then \( \pi(\tilde{w}^{(4i-1)}[x])\alpha_{2j} > 0 \) since \( \alpha_{2j} \notin \text{supp}(\tilde{w}^{(4i-1)}[x]) \).

Assume that \( \tilde{x}_{i,1} = 1 \). We split into cases.

(a) If \( \tilde{x}_{i,2} = \tilde{x}_{i,3} = 0 \) then \( \pi(\tilde{w}^{(4i-1)}[x])\alpha_{2j} = -\pi(\tilde{w}^{(4i-4)}[x])\alpha_{2j} < 0 \) since \( \alpha_{2j} \notin \text{supp}(\tilde{w}^{(4i-4)}[x]) \).

(b) For the same reason, if \( \tilde{x}_{i,2} = \tilde{x}_{i,3} = 1 \) then \( \pi(\tilde{w}^{(4i-1)}[x])\alpha_{2j} = \pi(\tilde{w}^{(4i-4)}[x])\alpha_{2j} + \alpha_{2j,1} + \alpha_{2j,1} > 0 \).

(c) If \( \tilde{x}_{i,2} = 1 \) and \( \tilde{x}_{i,3} = 0 \) then \( \pi(\tilde{w}^{(4i-1)}[x])\alpha_{2j} = \pi(\tilde{w}^{(4i-4)}[x])\alpha_{2j-1} = \pi(\tilde{w}^{(4i-3)}[x'])\alpha_{2j-1} \) where \( x'_j = x_j \) for all \( j \neq i \) and \( x'_{i,1} = 0 \). This case was considered in the second part.

(d) Similarly, the case \( \tilde{x}_{i,2} = 0 \) and \( \tilde{x}_{i,3} = 1 \) reduces to the third part.

For a subset \( A \subset \{1, \ldots, l\} \) we denote by \( A^c \) its complement in \( \{1, \ldots, l\} \).

**Corollary 4.4.** Let \( i \in (\tilde{N}_{r,f} \cup \nu_{r,f}(\tilde{N}_{r,f}))^c \).

(1) Let \( x' \in \{\{0, 1\}^4 \}^l \) be such that \( x'_j = x_j \) for all \( j \neq i \) and either \( x'_{i,1} = x_{i,1} = 0 \) or \( x'_{i,1} = x_{i,1} = 1, x'_{i,2} = x_{i,2} \) and \( x'_{i,3} = x_{i,3} \). Then \( \delta_w(x') = \delta_w(x') \).
(2) Assume that $x_{i,1} = 1$. Let $w'$ be the word obtained from $w$ by removing $s_j$, and let $x'$ be the $w'$-mask obtained from $x$ by removing $x_i$. Then

$$d_w(x) - d_{w'}(x') = \begin{cases} 1 & x_{i,2} = x_{i,3} = 0, \\ \delta_1 & x_{i,2} = 0, x_{i,3} = 1, \\ \delta_2 & x_{i,2} = 1, x_{i,3} = 0, \\ 0 & x_{i,2} = x_{i,3} = 1, \end{cases}$$

where $\delta_1$ (resp., $\delta_2$) is 1 if $C(x, i)$ (resp., $\tilde{C}(x, i)$) holds and 0 otherwise.

**Corollary 4.5.** $\tilde{w}$ is tight.

**Proof.** Assume that $I_f \neq \{1, \ldots, l\}$ i.e., $x$ is not full. We identify $\{1, \ldots, l\} \times \{1, 2, 3, 4\}$ with $\{1, \ldots, 4l\}$ by $(i, \delta) \mapsto 4(i - 1) + \delta$. For any $i \notin I_f$ let $m_i \in \{1, 2, 3, 4\}$ be the smallest index such that $x_{i,m_i} = 0$. It follows from Lemma 4.3 that $(i, m_i) \in \mathcal{E}_w^0(x)$ (see (5)). Define a map

$$h_x : D_w^0(x) \to \mathcal{E}_w^0(x)$$

according to the rule

$$h_x(i, \delta) = \begin{cases} (i, 1) & \delta = 2, \\ (\nu_l(i), 3) & \delta > 2, x_{i,1} = x_{i,3} \text{ and } x_{\nu_l(i)} = (1, 1, 0, 1), \\ (\nu_l(i), 4) & \delta > 2, x_{i,1} = x_{i,3} \text{ and } x_{\nu_l(i)} = (*, 1, *, 0), \\ (i, 3) & \delta = 4 \text{ and } x_i = (1, *, 0, 0). \end{cases}$$

By Lemma 4.3 $h_x$ is well defined since if $C(x, i)$ is satisfied then $\tilde{C}(x, \nu_l(i))$ is not satisfied. Moreover, $h_x$ is injective since $\nu_l$ is. We claim that $h_x$ is not onto. Indeed, let $i \notin I_f$ be such that $j_i$ is minimal. Then $i \notin \nu_l(N_{I_f})$, and in particular $\tilde{C}(x, i)$ is not satisfied. Thus, $(i, \delta)$ is not in the image of $h_x$ unless $x_i = (1, 0, 0, 0)$ and $\delta = 3$. Hence, $(i, m_i) \in \mathcal{E}_w^0(x)$ but $(i, m_i)$ is not in the image of $h_x$.

It follows that

$$|\mathcal{E}_w^0(x)| > |D_w^0(x)|.$$ 

Since $x$ is arbitrary (non-full), $\tilde{w}$ is tight.

**Remark 4.6.** Note that for $m = 3$, $\tilde{w}$ avoids the five permutations in (8) if and only if $w$ avoids the patterns (321), (3412), (3142), (2413). It is easy to see that these $w$'s are exactly the permutations which can be written as direct sums of (left or right) cyclic shifts. In principle, it should be possible to check Conjecture 1.1 for $m = 3$ for these permutations. We will not provide any details here. Note that for $m > 3$ $\tilde{w}$ does not avoid (56781234) unless $w = e$ so this method fails.
5. Double cosets

Let $H = H_n$ be the parabolic subgroup of $S_{2n}$ consisting of permutations which preserve each of the sets $\{2i - 1, 2i\}$, $i = 1, \ldots, n$. Thus, $H$ is an elementary abelian group of order $2^n$. Note that $H$ is normalized by $\widetilde{S}_n$. It is well-known that the double cosets $H \backslash S_{2n} / H$ are parameterized by $n \times n$-matrices with non-negative integer entries, whose sums along each row and each column are all equal to 2. By the Birkhoff–von-Neumann Theorem, these matrices are precisely the sums of two $n \times n$-permutation matrices.

We denote by $\mathcal{R}^H$ the set of bi-$H$-reduced elements in $S_{2n}$, i.e.

$$\mathcal{R}^H = \{ w \in S_{2n} : w(2i) > w(2i - 1) \text{ and } w^{-1}(2i) > w^{-1}(2i - 1) \text{ for all } i = 1, \ldots, n \}.$$  

Each $H$-double coset contains a unique element of $\mathcal{R}^H$.

Our goal in this section is to parameterize the double cosets of $H$ containing an element $\leq \bar{w}$, or equivalently, the set $\mathcal{R}^H_{\leq \bar{w}} := \{ u \in \mathcal{R}^H : u \leq \bar{w} \}$.

**Definition 5.1.** Let $\mathcal{T} = \mathcal{T}^w$ be the set of triplets $(I_e, I, I_f)$ of subsets of $\{1, \ldots, l\}$ such that $I_f \subset I_e$ and $I \subset \hat{N}_{I_f}$. We will write $\hat{I} = \hat{I} \cup \hat{I}$ (disjoint union) where $\hat{I} = I \cap \hat{N}_{I_f}$ and $\hat{I} = I \cap \hat{N}_{I_f}$.

For any $u \leq \bar{w}$ define $p(u) = p^w(i) = (I_f, \hat{I} \cup \hat{I}, I_e)$ where

$$I_e = \{ i : s_{2j_i} \leq u \}, \quad I_f = \{ i : \hat{s}_{2j_i} \leq u \},$$

$$\hat{I} = \{ i \in \hat{N}_{I_f} : \hat{s}_{2j_i}s_{2j_i+1}s_{j_i+1} \cdots \hat{s}_{j_{I_f}(i)-1}s_{2j_{I_f}(i)} \leq u \},$$

$$\hat{I} = \{ i \in \hat{N}_{I_f} : \hat{s}_{j_{I_f}(i)}s_{j_{I_f}(i)-1} \cdots \hat{s}_{j_i+1}s_{2j_i+1} \leq u \}.$$  

Clearly, $p(u) \in \mathcal{T}$ by (11). Note that if $i \in \hat{N}_{I_f}$ (resp., $i \in \hat{N}_{I_f}$) then $s_{j_i+1} \cdots s_{j_{I_f}(i)-1}$ (resp., $s_{j_{I_f}(i)-1} \cdots s_{j_i+1}$) is the cyclic shift

$$t \mapsto \begin{cases} 
    t + 1 & j_i < t < j_{I_f}(i) \\
    j_i + 1 & t = j_{I_f}(i) \\
    t & \text{otherwise}
\end{cases} \quad (\text{resp., } t \mapsto \begin{cases} 
    t - 1 & j_i + 1 < t \leq j_{I_f}(i) \\
    j_{I_f}(i) & t = j_i + 1 \\
    t & \text{otherwise}
\end{cases}).$$

Also,  

$$\text{(12) } \quad \text{if } v \leq u \leq \bar{w} \text{ and } v \in HuH \cap \mathcal{R}^H \text{ then } p(v) = p(u).$$

Indeed, for any $y \in \mathcal{R}^H$ we have $y \leq u$ if and only if $y \leq v$. Thus, $p$ is determined by its values on $\mathcal{R}^H_{\leq \bar{w}}$.

In the other direction, consider the map

$$q = q^w : \mathcal{T} \to S_{2n}$$
given by \( Q = (I_e, I_I, I_f) \mapsto \pi(\omega_Q) \) where

\[
\omega_Q = y_1 \ldots y_l, \quad y_i = \begin{cases} 
\emptyset & i \notin I_e, \\
s_{2j_i} & i \in I_f, \\
s_{2j_i} s_{2j_i+1} & i \in \tilde{I}, \\
s_{2j_i+1} s_{2j_i} & i \in I, \\
s_{2j_i} & \text{otherwise.}
\end{cases}
\]  

Clearly \( q(Q) \leq \tilde{w} \) for all \( Q \in \mathcal{T} \). We also remark that

\[
q((I_x, \emptyset, I_x)) = \bar{x} \text{ for all } x \leq w
\]  

(see \((9)\)).

**Proposition 5.2.** The map \( p \) is a bijection between \( \mathcal{R}_{\leq \tilde{w}}^H \) and \( \mathcal{T} \) whose inverse is \( q \).

The proposition will follow from Lemmas \[5.3\] and \[5.6\] below.

**Lemma 5.3.** We have \( p \circ q = \text{id}_\mathcal{T} \). In particular, \( q \) is injective. Moreover, the image of \( q \) is contained in \( \mathcal{R}_{\leq \tilde{w}}^H \).

**Proof.** Let \( Q = (I_e, I_I, I_f) \in \mathcal{T} \). We first claim that \( \omega_Q \) is a reduced word. Indeed, let

\[
\underline{x} = \begin{cases} 
(0, 0, 0, 0) & i \notin I_e, \\
(1, 1, 1, 1) & i \in I_f, \\
(1, 0, 1, 0) & i \in \tilde{I}, \\
(0, 0, 1, 1) & i \in I, \\
(0, 0, 0, 1) & \text{otherwise},
\end{cases}
\]

so that \( \omega_Q = \tilde{w}[\underline{x}] \). Then it is easy to see from Lemma \[4.3\] that \( \tilde{w}[\underline{x}] \) is reduced i.e., that \( \pi(\tilde{w}^{(i-k)|i|}) \alpha_{2j_i+k} > 0 \) whenever \( i = 1, \ldots, l \) and \( k = 1, 2, 3, 4 \) are such that \( \underline{x}_{i,k} = 1 \) where \( t_1 = t_4 = 0, t_2 = 1, t_3 = -1 \). (The condition \( C(\underline{x}, i) \) is never satisfied.)

Let us show that \( p(q(Q)) = Q \). Write \( p(q(Q)) = (I_f^\circ, \tilde{I}^\circ \cup \hat{I}^\circ, I_f^\circ) \). Since \( \omega_Q \) is reduced, it is clear from the definition and from \((9)\) that \( I_e^\circ = I_e, I_f \subset I_f^\circ, \hat{I} \subset \hat{I}^\circ \). Since the only reduced decompositions of \( \sim s_{2j_i}, \sim s_{2j_i+1} s_{2j_i+1} s_{2j_i} \) (and in particular \( s_{2j_i} \) occurs twice) we must have \( I_f = I_f^\circ \). Let \( i \in \hat{N}_f \) and suppose that \( v := s_{2j_i} s_{2j_i+1} s_{2j_i+1} \ldots s_{j_{v_I}(i)} s_{j_{v_I}(i)-1} s_{2j_{v_I}(i)} \leq \pi(\omega_Q) \). Then \( v \) is represented by a subword of \( \omega_Q \).

On the other hand, it is clear that any subword of \( \omega_Q \) supported in \( \{s_k : 2j_i \leq k \leq 2j_{v_I}(i)\} \) is a subword of \( s_{2j_i} s_{2j_i+1} s_{2j_i+1} \ldots s_{j_{v_I}(i)-1} s_{2j_{v_I}(i)} \) and the latter is a subword of \( \omega_Q \) only if \( i \in \hat{I} \). Hence, \( \hat{I} = \hat{I}^\circ \). Similarly one shows that \( \tilde{I} = \tilde{I}^\circ \).

Finally, let us show that \( q(Q) \in \mathcal{R}_H^H \). Let \( u \in H^H \cap Hq(Q)H \). Then \( u \leq q(Q) \) and by \((12)\) and the above we have \( p(u) = p(q(Q)) = Q \). It is easy to see that this is impossible unless \( u = q(Q) \). \(\square\)
Lemma 5.5. Let assertion.

For any \( i = 1, \ldots, l \) let \( \mu_{\pm}(i) \) (resp., \( \bar{r}_{\pm}(i) \)) be \( t_i \) where \( t \geq 0 \) is the largest index for which there exist (unique) indices \( i = i_0 < i_1 < \cdots < i_t \leq l \) (resp., \( i = i_0 > i_1 > \cdots > i_t > 0 \)) such that \( j_{i_0} = j_i \pm k \) for \( k = 1, \ldots, t \).

Lemma 5.5. Let \( x, x' \in \{0, 1\}^l, I_f = \{ i = 1, \ldots, l : x_i = (1, 1, 1) \} \) and \( I_e = \{ i = 1, \ldots, l : x_i \neq (0, \ast, \ast, 0), (1, 0, 0, 1) \} \).

1. Suppose that \( i \) is such that \( x_j = x_j' \) for all \( j \neq i \) and let \( \epsilon_1, \epsilon_2 \in \{0, 1\} \).
   (a) If \( x_i = (1, 0, 0, 1) \) and \( x_i' = (0, 0, 0, 0) \) then \( \pi(\bar{w}[x]) = \pi(\bar{w}[x']) \).
   (b) If \( i \notin \tilde{N}_{I_f} \cap \nu^{-1}_{I_f}(I_e) \) and either \( x_i = (1, \epsilon_1, \epsilon_2, 1), x_i' = (1, \epsilon_2, 0, 1) \) for \( \tilde{N}_{I_f} \) and \( k = j_{\tilde{I}_{\nu_I}}(i) - 1 \).
   (c) If \( i \notin \tilde{N}_{I_f} \cap \nu^{-1}_{I_f}(I_e) \) and either \( x_i = (0, \epsilon_1, 1, 0), x_i' = (0, \epsilon_2, 0, 1) \) for \( \tilde{N}_{I_f} \) and \( k = j_{\tilde{I}_{\nu_I}}(i) - 1 \).
   (d) If \( i \notin \nu_I(\tilde{N}_{I_f} \cap I_e) \) and either \( x_i = (1, \epsilon_1, 1, 0), x_i' = (1, \epsilon_2, 0, 1) \) for \( \tilde{N}_{I_f} \) and \( k = j_{\tilde{I}_{\nu_I}}(i) - 1 \).
   (e) If \( i \notin \nu_I(\tilde{N}_{I_f} \cap I_e) \) and either \( x_i = (0, \epsilon_1, 1, 0), x_i' = (0, \epsilon_2, 0, 1) \) for \( \tilde{N}_{I_f} \) and \( k = j_{\tilde{I}_{\nu_I}}(i) - 1 \).

2. Suppose that \( i \in N_{I_f} \) is such that \( x_j = x_j' \) for all \( j \neq i \). Assume that \( x_i', r = x_i, r \) for \( r = 1, 2, 3, 4, x_i', 3 = 1 - x_i, 3, x_{\nu_I(i), r} = x_{\nu_I(i), r} \) for \( r = 1, 3, 4, x_{\nu_I(i), 2} = 1 - x_{\nu_I(i), 2} \) and \( x_i, 4 = x_{i, 2}' = 0 \) where \( i_1 = \min(i, \nu_I(i)) \), \( i_2 = \max(i, \nu_I(i)) \). Then \( \pi(\bar{w}[x]) = \pi(\bar{w}[x']) \).

Proof. Part 1(a) is trivial. Part 1(b) follows from the braid relation

\[
s_{2j_1}s_{2j_1+1}s_{2j_1} = s_{2j_1+1}s_{2j_1}s_{2j_1+1}
\]

and the relation

\[
s_{2j_1}s_{2j_1+1}s_{2j_1} = \tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_k = \tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_k s_{2j_k+1}
\]

where \( k = j_{\nu_I(i)} - 1 \) if \( i \in \tilde{N}_{I_f} \) and \( k = j_{\bar{r}_{\nu_I}(i)} \) otherwise. The other parts are similar. \[\square\]
Next, we explicate, for any $\tilde{w}$-mask $\tilde{x} \in \{(0, 1)^4\}^l$, the $H$-double coset of $\pi(\tilde{w}[\tilde{x}])$, thereby finishing the proof of Proposition 5.2.

**Lemma 5.6.** For any $\tilde{x} \in \{(0, 1)^4\}^l$ let $Q_{\tilde{x}} = (I_e, \tilde{I} \cup \tilde{I}, I_f)$ where

\begin{align*}
(15a) & \quad I_e = \{i = 1, \ldots , l : x_i \neq (0, *, *, 0), (1, 0, 0, 1)\}, \\
(15b) & \quad I_f = \{i = 1, \ldots , l : x_i = (1, 1, 1, 1)\}, \\
(15c) & \quad \tilde{I} = \{i \in \tilde{N}_{I_f}^* : x_{i,1} \cdot x_{i,3} \neq \varnothing_{I_f}(i),2 \cdot \varnothing_{I_f}(i),4\}, \\
(15d) & \quad \tilde{I} = \{i \in \tilde{N}_{I_f}^* : x_{i,1} \cdot x_{i,3} \neq \varnothing_{I_f}(i),1 \cdot \varnothing_{I_f}(i),2\}.
\end{align*}

Then $\pi(\tilde{w}[\tilde{x}]) \in Hq(Q_{\tilde{x}})H$ and hence $p(\pi(\tilde{w}[\tilde{x}])) = Q_{\tilde{x}}$. In particular, the image of $q$ is $\mathcal{R}_{\tilde{w}}^H$.

**Proof.** Consider the graph $G_1$ whose vertex set consists of the $\tilde{w}$-masks and the edges connect two $\tilde{w}$-masks $\tilde{x}, \tilde{x}' \in \{(0, 1)^4\}^l$ if there exists $i = 1, \ldots , l$ such that $\tilde{x}_i = \tilde{x'}_i$ for all $j \neq i$ and one of the following conditions holds for some $\epsilon \in \{0, 1\}$ (where $Q_{\tilde{x}} = (I_e, \tilde{I} \cup \tilde{I}, I_f)$):

1. $\tilde{x}_i = (0, 0, 0, 0)$ and either $\tilde{x'}_i = (0, *, *, 0)$ or $\tilde{x'}_i = (1, 0, 0, 1)$.
2. $\tilde{x}_i = (1, 0, 0, 0)$ and $\tilde{x'}_i = (0, 0, 0, 1)$.
3. $\tilde{x}_i = (1, 0, 0, 0)$ and $\tilde{x'}_i = (1, 0, 0, 0)$ and $i \notin \tilde{N}_{I_f}^*$.
4. $\tilde{x}_i = (0, 0, 0, 1)$ and $\tilde{x'}_i = (0, 0, 0, 1)$ and $i \notin \tilde{N}_{I_f}^*$.
5. $\tilde{x}_i = (1, 0, 0, 0)$ and $\tilde{x'}_i = (1, 0, 0, 0)$ and $i \notin \nu_{I_f}^*(\tilde{N}_{I_f}^*)$.
6. $\tilde{x}_i = (0, 0, 0, 1)$ and $\tilde{x'}_i = (0, 0, 0, 1)$ and $i \notin \nu_{I_f}^*(\tilde{N}_{I_f}^*)$.
7. $\tilde{x}_i = (0, 0, 0, 1)$ and $\tilde{x'}_i = \{(1, 0, 0, 1) \quad \text{if} \quad i \in \tilde{N}_{I_f}^*\}
\{(0, 0, 1, 1) \quad \text{if} \quad i \notin \tilde{N}_{I_f}^*\}.
8. $\tilde{x}_i = (0, 0, 0, 1)$ and $\tilde{x'}_i = \{(1, 1, 0, 0) \quad \text{if} \quad i \in \nu_{I_f}^*(\tilde{N}_{I_f}^*)\}
\{(0, 0, 1, 0) \quad \text{if} \quad i \notin \nu_{I_f}^*(\tilde{N}_{I_f}^*)\}.$

It follows from the first part of Lemma 5.5 that the double coset $H\pi(\tilde{w}[\tilde{x}])H$ depends only on the $G_1$-connected component of $\tilde{x}$. It is also straightforward to check that $Q_{\tilde{x}}$ depends only on the $G_1$-connected component of $\tilde{x}$.

On the other hand, each $G_1$-connected component contains a representative $\tilde{x}$ which satisfies the following conditions for all $i$

1. If $i \notin I_e$ then $\tilde{x}_i = (0, 0, 0, 0)$.
2. If $\tilde{x}_i = (1, *, *, 0)$ then $\tilde{x}_i = (1, 1, 1, 1)$, i.e., $i \in I_f$.
3. If $\tilde{x}_i = (1, *, 1, 0)$ then $i \in \tilde{N}_{I_f}^*$.
4. If $\tilde{x}_i = (0, *, 1, 1)$ then $i \in \tilde{N}_{I_f}^*.$
(5) If $x_i = (1, 1, *, 0)$ then $i \in \nu^I_f(\widetilde{N}_{I_f}^r)$.
(6) If $x_i = (0, 1, *, 1)$ then $i \in \nu^I_f(\widetilde{N}_{I_f}^r)$.

We call such $x$ “special”. We will show that if $x$ is special then $\pi(\tilde{w}(x)) = q(Q_x)$, thereby finishing the proof.

Consider a second graph $G_2$ with the same vertex set as $G_1$, where the edges are given by the condition in the second part of Lemma 5.5 as well as the condition that there exists $i$ such that $x_j = x'_j$ for all $j \neq i$ and $x_i = (1, 0, 0, 0), x'_i = (0, 0, 0, 1)$. Thus, $\pi(\tilde{w}(x))$ depends only on the $G_2$-connected component of $x$ and once again, it is easy to verify that the same is true for $Q_x$. Note that a $G_2$-neighbor of a special $\tilde{w}$-mask is also special.

We claim that the $G_2$-connected component of a special $\tilde{w}$-mask $x$ contains one which vanishes at all coordinates $(i, 2)$ for $i \notin I_f$. We argue by induction on the number of indices $i \notin I_f$ such that $x_{i,2} = 1$. For the induction step take such $i$ with $j_i$ minimal. Since $x$ is special, by the first two conditions we have $x_{i,1} + x_{i,4} = 1$. Suppose for instance that $x_{i,1} = 0$ and $x_{i,4} = 1$. Then, by condition 0) $i \in \nu^I_f(i_1)$ for some $i_1 \in \widetilde{N}_{I_f}^r$. By minimality of $j_i$ we have $x_{i_1,2} = 0$. Also, by passing to a $G_2$-neighbor if necessary, we may assume that $x_{i_1} \neq (0, 0, 0, 1)$. Then by condition 4) we necessarily have $x_{i_1,4} = 1$ since $i_1 \notin \widetilde{N}_{I_f}^r$.

Thus, we can apply the induction hypothesis to the neighbor of $x$ in $G_2$ which differs from it precisely at the coordinates $(i, 2)$ and $(i_1, 3)$. The case $x_{i,4} = 0$ and $x_{i_1,1} = 1$ is similar.

Finally, if $x$ is special and $x_{i,2} = 0$ for all $i \notin I_f$ then $\tilde{w}(x) = \omega Q_x$ (see (13)) and hence $\pi(\tilde{w}(x)) = q(Q_x)$. The lemma follows.

**Example 5.7.** Consider the case $n = 2$ and $w = s_1$ (so that $w = s_1$ and $\tilde{w} = s_2s_1s_3s_2$). There are three $H$-double cosets. As representatives we can take the identity, $s_2$ and $\tilde{s}_1$. The corresponding triplets under $\mathfrak{p}$ are $(0, 0, 0), (\{1\}, \emptyset, \emptyset)$ and $(\{1\}, \emptyset, \{1\})$. We have

$$\pi(w(x)) \in H \iff x \in \{(0, *, *, 0), (1, 0, 0, 1)\},$$

$$\pi(w(x)) \in Hs_2H \iff x \in \{(1, *, *, 0), (0, *, *, 1), (1, 0, 1, 1), (1, 1, 0, 1)\},$$

$$\pi(w(x)) \in Hs_1 \iff x = (1, 1, 1, 1).$$

**Remark 5.8.** Consider the reduced decomposition $w^i$ for $w^{-1}$. Write $i^2 = l + 1 - i$ and similarly for sets. Then for any $A \subset B \subset \{1, \ldots, l\}$ we have $w^i \tilde{N}_A^B = (w \tilde{N}_A^B)^{i^2}$, $w^i \tilde{N}_A^B = (w \tilde{N}_A^B)^{i^2}$ and $w^i \nu_A^B(i) = \nu_A^B(i^2)$. Moreover, the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{T}^{w_i} & \xrightarrow{(I_e, I_f) \mapsto (I'_e, I'_f)} & \mathcal{T}^{w} \\
q^{w_i} \downarrow & & q^{w} \downarrow \\
\mathcal{R}_{\leq w}^{H} & \xrightarrow{w \mapsto w^{-1}} & \mathcal{R}_{\leq w}^{H} \\
\end{array}$$

**6. The main result**

Finally, we prove the main result of the paper. Recall that $w \in S_n$ is a fixed Boolean permutation with reduced decomposition $w = s_{j_1} \ldots s_{j_l}$ (with $j_1, \ldots, j_l$ distinct).
Proposition 6.1. For any $Q = (I_e, I, I_f) \in \mathcal{T}$ we have
\begin{equation}
\sum_{u \in Hq(Q)\, H} \text{sgn } u P_{u}^\tilde{w} = \sum_{\tilde{u} \in \{0,1\}^4 : p(\pi(\tilde{w}[\tilde{u}])) = Q} \text{sgn } \tilde{u} q^\tilde{u}[\tilde{u}] \\
= (-1)^{|I_e\setminus I_f|} q^{I_e[|I_f|]} N^{I_f}[I_f] (q + 1) \left| I_e \setminus (I_f \cup N^{I_f}_f) \right|.
\end{equation}

In particular,
\begin{equation}
\sum_{\tilde{u} \in \{0,1\}^4 : p(\pi(\tilde{w}[\tilde{u}])) = Q} \text{sgn } \tilde{u} = 2 \left| I_e \setminus (I_f \cup N^{I_f}_f) \right| (-1)^{|I_e\setminus I_f|}.
\end{equation}

By Theorem 2.2 and Corollary 4.5 we infer

Corollary 6.2. For any $Q = (I_e, I, I_f) \in \mathcal{T}$ we have
\begin{equation}
\sum_{u \in Hq(Q)\, H} \text{sgn } u P_{u,\tilde{w}} = (-1)^{|I_e\setminus I_f|} q^{I_e[|I_f|]} N^{I_f}[I_f] (q + 1) \left| I_e \setminus (I_f \cup N^{I_f}_f) \right|.
\end{equation}

In particular, by (14), for any $x \leq w$
\begin{equation}
\tilde{F}_{x,w}^{(2)} = \sum_{u \in H\tilde{w}} \text{sgn } u P_{u,\tilde{w}} = q^{|w[\nu(\tilde{w})]| - \ell(x)}.
\end{equation}

Proof. The first equality of (16) follows from Proposition 5.2. We prove the second one by induction on $l$. The case $l = 0$ is trivial – both sides of (16) are equal to 1. Suppose that $l > 0$ and the result is known for $l - 1$.

If $I_f = \{1, \ldots, l\}$ (so that $Q = (\{1, \ldots, l\}, \emptyset, \emptyset, \{1, \ldots, l\})$) then the only summand on the left-hand side of (16) is the one corresponding to $\tilde{x}_i = (1, 1, 1, 1)$ for all $i$ and the result is trivial.

We may therefore assume that $I_f \neq \{1, \ldots, l\}$. For convenience, denote the left-hand side of (16) by $L_Q^w$ and let
\begin{equation}
M_Q^w = \{ \tilde{x} \in \{0,1\}^4 : p(\pi(\tilde{w}[\tilde{x}])) = Q \}
\end{equation}
which is explicated in Lemma 5.4. Let $i_0$ be the element of $I^c_f$ for which $j_{i_0}$ is maximal. In particular, $\tilde{x}_{i_0} \neq (1, 1, 1, 1)$ for any $\tilde{x} \in M_Q^w$. Clearly $i_0 \notin N_{I_f}$, otherwise $j_{\nu_I(i_0)} > i_0$. Note that by (17) and Remark 5.8, the statement of Corollary 6.2 is invariant under $w \mapsto w^{-1}$ (and $w \mapsto w^\ell$). On the other hand, Corollary 6.2 is equivalent to Proposition 6.1 by Theorem 2.2. Therefore, upon inverting $w$ if necessary we may assume that
\begin{equation}
i_0 \notin \nu_{I_f}(\tilde{N}_{I_f}).
\end{equation}

In particular, we can apply Corollary 4.4.

Let $\mathbf{w}'$ be the word obtained from $\mathbf{w}$ by removing $s_{j_{i_0}}$ and let $Q' = (I'_e, I', I'_f) \in \mathcal{T}^{w'}$ where $I'_e = I_e \setminus \{i_0\}$, $I' = I \setminus (\nu_{I_f}^{-1}(\{i_0\}))$ and $I'_f = I_f$. Note that $w' N_{I_f}^{I_f} = N_{I_f}^{I_f} \setminus (\nu_{I_f}^{-1}(\{i_0\}))$. (For simplicity we suppress $\mathbf{w}$ if the notation is pertaining to it.)
To carry out the induction step we show using Lemmas 4.3 and 5.6 that

\[
L^w_Q = L^w_{Q'} \times \begin{cases} 
q & i_0 \notin I_e, \\
-(q + 1) & i_0 \in I_e \setminus \nu^{I_e}_f (N^{I_e}_f), \\
-1 & i_0 \in \nu^{I_e}_f (I), \\
-q & i_0 \in \nu^{I_e}_f (N^{I_e}_f \setminus I).
\end{cases}
\]  

(19)

We separate into cases.

(1) Assume first that \( i_0 \notin \nu^{I_e}_f (N^{I_e}_f) \) (the first two cases on the right-hand side of (19)).

In this case, in order for \( x \) to belong to \( M^w_Q \), the conditions (15c) and (15d) are independent of \( x_{i_0} \).

We claim that

\[
L^w_Q = \sum_{\bar{x} \in M^w_Q; x_0 \in \{(1,0,0,0),(1,0,0,1),(1,1,1,0)\}} \text{sgn } \bar{x} q^{\bar{\omega}(\bar{x})},
\]

i.e., that

\[
\sum_{\bar{x} \in R_{i_0}} \text{sgn } \bar{x} q^{\bar{\omega}(\bar{x})} = 0
\]

(20)

where

\[ R_{i_0} := \{ \bar{x} \in M^w_Q : x_0 \neq (1,0,0,0),(1,1,1,*), (1,1,1,*) \}. \]

We define an involution \( \iota \) on \( R_{i_0} \) by retaining \( \bar{x} \) for \( i \neq i_0 \) and changing \( x_{i_0} \) according to the rule

\[
(0,0,0,0) \leftrightarrow (0,0,1,0), \quad (0,1,0,0) \leftrightarrow (0,1,1,0), \quad (1,0,1,0) \leftrightarrow (1,0,1,1), \\
(0,0,0,1) \leftrightarrow (0,0,1,1), \quad (0,1,0,1) \leftrightarrow (0,1,1,1), \quad (1,1,0,0) \leftrightarrow (1,1,0,1).
\]

This is well defined since the condition \( x_{i_0} = (0,*,*,0) \) is invariant under the above rule. By the first part of Corollary 4.4 \( \iota \) preserves \( \bar{\omega} \). Since \( \text{sgn } \iota (\bar{x}) = -\text{sgn } \bar{x} \), the assertion (21) follows.

(a) Suppose that \( i_0 \notin I_e \). Then by (20)

\[
L^w_Q = \sum_{\bar{x} \in M^w_Q; x_0 \in \{(1,0,0,1)\}} \text{sgn } \bar{x} q^{\bar{\omega}(\bar{x})}
\]

and therefore by the second part of Corollary 4.4

\[
L^w_Q = qL^w_{Q'}.
\]

(b) Similarly, if \( i_0 \in I_e \setminus \nu^{I_e}_f (N^{I_e}_f) \) then

\[
L^w_Q = \sum_{\bar{x} \in M^w_Q; x_0 \in \{(1,0,0,0),(1,1,1,0)\}} \text{sgn } \bar{x} q^{\bar{\omega}(\bar{x})}
\]

and we get

\[
L^w_Q = -(q + 1)L^w_{Q'}.
\]
(2) Consider now the case \( i_0 \in \nu_{i_j}^L (N_{i_j}^L) \) (the last two cases on the right-hand side of (19)). In particular, \( i_0 \in I_e \setminus I_f \) so that
\[
\mathbf{x}_{i_0} \in \{(1, *, *, 0), (0, *, *, 1), (1, 0, 1, 1), (1, 1, 0, 1)\}
\]
for any \( \mathbf{x} \in M_Q^w. \)

Let \( i_1 = (\nu_{i_j}^L)^{-1}(i_0) \). By (18) \( i_1 \in N_{i_j}^L. \) We write \( L_Q^w = T_0 + T_1 \) where
\[
T_j = \sum_{\mathbf{x} \in M_Q^w; \mathbf{x}_{i_1,1} \mathbf{x}_{i_1,3} = j} \text{sgn}\ \mathbf{x} q^\omega(\mathbf{x}), \quad j = 0, 1.
\]

We first claim that
\[
(22) \quad T_j = \sum_{\mathbf{x} \in M_Q^w; \mathbf{x}_{i_1,1} \mathbf{x}_{i_1,3} = j} \text{sgn}\ \mathbf{x} q^\omega(\mathbf{x}),
\]
where
\[
R_j := \{ \mathbf{x} \in M_Q^w : \mathbf{x}_{i_1,1} \mathbf{x}_{i_1,3} = j \text{ and } \mathbf{x}_{i_0} \notin \{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 0, 1)\} \}.
\]
As before, we define \( \iota \) on \( R_j \) by keeping \( \mathbf{x}_i \) for all \( i \neq i_0 \) and changing \( \mathbf{x}_{i_0} \) according to the rule
\[
(0, 0, 0, 1) \leftrightarrow (0, 0, 1, 1), \ (0, 1, 0, 1) \leftrightarrow (0, 1, 1, 1), \ (1, 0, 1, 0) \leftrightarrow (1, 0, 1, 1).
\]
This is well defined since \( \iota \) preserves \( \mathbf{x}_{i_0,2} \cdot \mathbf{x}_{i_1,4} \). By the first part of Corollary 4.4 \( \iota \) preserves \( \partial_w \). Since \( \text{sgn} \ \iota(\mathbf{x}) = -\text{sgn} \ \mathbf{x} \), the assertion follows.

(a) Suppose that \( i_1 \notin I \). Then by (22) and (15c)
\[
T_0 = \sum_{\mathbf{x} \in M_Q^w; \mathbf{x}_{i_1,1} \mathbf{x}_{i_1,3} = 0 \text{ and } \mathbf{x}_{i_0} \in \{(1, 0, 0, 0), (1, 1, 1, 0), (1, 1, 0, 0)\}} \text{sgn}\ \mathbf{x} q^\omega(\mathbf{x})
\]
and
\[
T_1 = \sum_{\mathbf{x} \in M_Q^w; \mathbf{x}_{i_1,1} \mathbf{x}_{i_1,3} = 1 \text{ and } \mathbf{x}_{i_0} = (1, 1, 0, 1)} \text{sgn}\ \mathbf{x} q^\omega(\mathbf{x}).
\]
Thus, using the second part of Corollary 4.4 the contributions from \( \mathbf{x}_{i_0} = (1, 1, 1, 0) \) and \( \mathbf{x}_{i_0} = (1, 1, 0, 0) \) cancel and we get
\[
T_j = (-q) \times \sum_{\mathbf{x}' \in M_Q^w; \mathbf{x}'_{i_1,1} \mathbf{x}'_{i_1,3} = j} \text{sgn}\ \mathbf{x}' q^\omega(\mathbf{x}'), \quad j = 0, 1.
\]
Hence,
\[
T_0 + T_1 = (-q) \times \sum_{\mathbf{x}' \in M_Q^w} \text{sgn}\ \mathbf{x}' q^\omega(\mathbf{x}').
\]
We prove this by induction on the cardinality of $I$. We continue to assume that $w$ is a Boolean permutation, \( \{0,1\} \), and \( x_i \) is the empty set then \( x_i = 1 \). Let \( x_i' \) be such that \( x_i' = x_i \cup \{i\} \). Then \( \sum_{x_0, x_1, x_3 = 0 \text{ and } x_0 = (1,0,1) \} \). The contributions from \( x_{i_0} = (1,0,0,0) \) and \( x_{i_0} = (1,1,0,0) \) cancel and we obtain

\[
T_j = - \sum_{x' \in M_Q^w, x'_1, x'_3 = j} \text{sgn } x' q^{\bar{w}(x')}, \quad j = 0, 1
\]

and hence

\[
T_0 + T_1 = - \sum_{x' \in M_Q^w} \text{sgn } x' q^{\bar{w}(x')},
\]

Thus, we established (19) in all cases. The induction step now follows from the induction hypothesis.

\[\square\]

7. Complements

In conclusion, we relate the result of the previous section to the results of [LM16, §10]. We continue to assume that $w$ and $w$ are as in [LM16]

**Lemma 7.1.** Let $x_1, x_2 \leq w$, $I_e = I_{x_1} \cup I_{x_2}$ and $I_f = I_{x_1} \cap I_{x_2}$. Then the number of non-trivial cycles of the permutation $x_2^{-1} x_1$ is \( |I_e \setminus (I_f \cup N_{I_f}^{I_e})| \).

**Proof.** We prove this by induction on the cardinality of $I_f$. If $I_f$ is the empty set then $x_2^{-1} x_1$ is a Boolean permutation, \( \{ j : s_j \leq x_2^{-1} x_1 \} = \{ j_i : i \in I_e \} \) and

\[
N_{I_f}^{I_e} = \{ i \in I_e : \text{there exists } r \in I_e \text{ such that } j_{i_1} + 1 = j_r \}.
\]

The claim follows since any Coxeter element of the symmetric group is a single cycle.

For the induction step suppose that $I_f \neq \emptyset$. Let \( i_1 \) be the index in $I_f$ with $j_{i_1}$ minimal.

(1) Suppose first that $j_{i_1} - 1 \neq j_{i_0}$ for all $i' \in I_e$. If $j_{i_1} + 1 = j_{i_2}$ for some $i_2 \in I_e$ then we may assume upon replacing $w$ by $w^{-1}$ (and $w$ by $w^t$) if necessary that $i_2 > i_1$. Let \( x_1', x_2' \leq w \) be such that $I_{x_1'} = I_e \setminus \{i_1\}$. Then $x_2^{-1} x_1 = (x_2')^{-1} x'_1$. Letting $I'_e = I_{x_1'} \cup I_{x_2'} = I_e \setminus \{i_1\}$ and $I'_f = I_{x_1'} \cap I_{x_2'} = I_f \setminus \{i_1\}$ we have $N_{I_f}^{I_e} = N_{I_f}^{I_e}$ and therefore $I'_e \setminus (I'_f \cup N_{I_f}^{I_e}) = I_e \setminus (I_f \cup N_{I_f}^{I_e})$. Thus, the claim follows from the induction hypothesis.

(2) Otherwise, $j_{i_1} - 1 = j_{i_0}$ for some $i_0 \in I_e \setminus I_f$ (by the minimality of $i_1$). Once again, upon replacing $w$ by $w^{-1}$ (and $w$ by $w^t$) if necessary we may assume that $i_1 < i_0$. 

\[\square\]
(a) Suppose first that $i_0 \notin N_{I_f}^{I_e}$. Let $t \geq 1$ be the maximal index for which there exist indices $i_t < \cdots < i_1$ in $I_f$ such that $j_{i_t} = j_{i_0} + r$, $r = 1, \ldots, t$. By the assumption on $i_0$ and $t$, if there exists $i' < i_t$ such that $j_{i'} = j_{i_t} + 1$ then $i' > i_t$. Therefore $x_{i_2}^{-1}x_{i_1} = (x_{i_2}')^{-1}x_{i_1}'$ where $x_{i_1}', x_{i_2}' \leq w$ are such that $I_{x_{i_1}'} = I_{x_{i_2}} \setminus \{i_1, \ldots, i_t\}$. Let $I'_e = I_{x_{i_1}'} \cup I_{x_{i_2}'} = I_e \setminus \{i_1, \ldots, i_t\}$ and $I'_f = I_{x_{i_1}'} \cap I_{x_{i_2}'} = I_f \setminus \{i_1, \ldots, i_t\}$. Then $N_{I_f}^{I_e} = N_{I_f}^{I_e}$. This case therefore follows from the induction hypothesis.

(b) Suppose that $i_0 \in N_{I_f}^{I_e}$ and let $i_{t+1} < \cdots < i_1$, $t \geq 1$ be such that $j_{i_r} = j_{i_0} + r$, $r = 1, \ldots, t + 1$ with $i_1, \ldots, i_t \in I_f$ and $i_{t+1} = \nu_{i_r}^{I_e}(i_0) \in I_e \setminus I_f$. Upon interchanging $x_1$ and $x_2$ if necessary we may assume that $i_0 \in I_{x_1} \setminus I_{x_2}$. Let $u$ be the permutation

$$u(r) = \begin{cases} r + j_{i_0} & r \leq t, \\ r - t & t < r \leq j_{i_t}, \\ r & r > j_{i_t} \end{cases}$$

so that $u^{-1}s_{j_1} s_{j_2} \cdots s_{j_{t+1}} s_{j_t} \cdots s_{j_0} u = s_{j_{t+1}} s_{j_t} \cdots s_{j_0}$ and $u^{-1}s_r u = s_{r+t}$ for all $r < j_{i_0}$.

Let $w'$ be the word obtained from $w$ by removing the $t$ simple roots $s_{j_1}, \ldots, s_{j_r}$ (i.e., the indices $i_1, \ldots, i_t$) and replacing $s_r$ by $s_{r+t}$ for $r \leq j_{i_0}$. Then $u^{-1}x_{i_2}^{-1}x_{i_1} u = (x_{i_2}')^{-1}x_{i_1}'$ where $I_{x_{i_2}'} = I_{x_{i_1}'} \setminus \{i_1, \ldots, i_t\}$, $r = 1, 2$. Let $I'_e = I_{x_{i_1}'} \cup I_{x_{i_2}'} = I_e \setminus \{i_1, \ldots, i_t\}$ and $I'_f = I_{x_{i_1}'} \cap I_{x_{i_2}'} = I_f \setminus \{i_1, \ldots, i_t\}$. Then $w'N_{I_f}^{I_e} = N_{I_f}^{I_e}$ and the claim follows from the induction hypothesis.

Let $K$ be the subgroup of $S_{2n}$ (isomorphic to $S_n \times S_n$) preserving the set $\{2, 4, \ldots, 2n\}$. For any $x \in K$ let $x_{\text{odd}} \in S_n$ (resp., $x_{\text{even}} \in S_n$) be the permutation such that $x(2i - 1) = 2x_{\text{odd}}(i) - 1$ (resp., $x(2i) = 2x_{\text{even}}(i)$) for $i = 1, \ldots, n$. Thus, $x \mapsto (x_{\text{odd}}, x_{\text{even}})$ is a group isomorphism $K \simeq S_n \times S_n$. Note that $x_{\text{even}} = x_{\text{odd}}$ if and only if $x = \bar{x}_{\text{odd}}$.

We recall the following general elementary result.

**Lemma 7.2.** [LM16, Lemma 10.6] For any $x \in S_{2n}$ we have $K \cap H x H \neq \emptyset$. Let $u \in K \cap H x H$ and let $r$ be the number of non-trivial cosets of $u_{\text{odd}}^{-1} u_{\text{odd}} \in S_n$. Then $|K \cap H x H| = 2^r$. Moreover, sgn is constant on $K \cap H x H$.

Note that in general, for any $w \in S_n$, if $u \in K$ and $u_{\text{odd}}, u_{\text{even}} \leq w$ then $u \leq \bar{w}$. In the case at hand we can be more explicit.

**Lemma 7.3.** Let $u \in K$ with $u_{\text{odd}}, u_{\text{even}} \leq w$. Then

$$u = \pi(\bar{w} x_i) \text{ where } x_i = (1, \chi_{u_{\text{odd}}}(i), \chi_{u_{\text{even}}}(i), 1), \ i = 1, \ldots, l$$
and \( p(u) = (I_e, I, I_f) \) where \( I_e = I_{u_{\text{odd}}} \cup I_{u_{\text{even}}} \), \( I_f = I_{u_{\text{odd}}} \cap I_{u_{\text{even}}} \) and

\[
I = N_{I_f}^{I_e} \setminus (N_{I}^{I_{u_{\text{odd}}}} \cup N_{I}^{I_{u_{\text{even}}}})
\]

\[
= \{ i \in N_{I_f} \cap I_{u_{\text{odd}}} : \nu_{I_f}(i) \in I_{u_{\text{even}}} \} \cup \{ i \in N_{I_f} \cap I_{u_{\text{even}}} : \nu_{I_f}(i) \in I_{u_{\text{odd}}} \}.
\]

**Proof.** Since \( u \mapsto (u_{\text{odd}}, u_{\text{even}}) \) is a group isomorphism, it is enough to check (23) for the case \( n = 1 \), which is straightforward. The second part follows from Lemma 5.6. \( \square \)

**Remark 7.4.** Let \( A \) and \( B \) be subsets of \( \{1, \ldots, l\} \) with \( A \subset B \). Let \( \sim \) be the equivalence relation on \( B \setminus A \) generated by \( i \sim \nu^B_A(i) \) whenever \( i \in N^B_A \). Any equivalence class \( C \subset B \setminus A \) of \( \sim \) is of the form

\[
C = \{ i_1, \ldots, i_a \}
\]

where

1. For all \( t < a \), \( i_t \in N^B_A \) and \( \nu^B_A(i_t) = i_{t+1} \). In particular, \( j_{i_{t+1}} > j_{i_t} \).
2. \( i_a \notin N^B_A \).
3. \( i_1 \notin \nu^B_A(N^B_A) \).

Thus, each equivalence class contains a unique element outside \( N^B_A \). In particular, the number of equivalence classes of \( \sim \) is \( |B \setminus (A \cup N^B_A)| \).

**Corollary 7.5.** For any \( Q = (I_e, I, I_f) \in T \)

\[
|\{ u \in K : u_{\text{even}}, u_{\text{odd}} \leq w \text{ and } p(u) = Q \}| = 2^{\left| I_e \setminus (I_f \cup N^B_A) \right|}.
\]

Moreover,

\[
\sgn u = (-1)^{|I_e \setminus I_f|}
\]

for any \( u \in K \) such that \( u_{\text{even}}, u_{\text{odd}} \leq w \) and \( p(u) = Q \).

**Proof.** Indeed, by (10) and Lemma 7.3, the set on the left-hand side is in bijection with the set of ordered pairs \((I_1, I_2)\) of subsets of \( \{1, \ldots, l\} \) such that \( I_1 \cap I_2 = I_f, I_1 \cup I_2 = I_e \) and (24) holds. Under this bijection \( \sgn u = (-1)^{|I_1 \setminus I_2|} \) and the symmetric difference \( I_1 \Delta I_2 \) is equal to \( I_e \setminus I_f \). This implies the second part. Now, the map \((I_1, I_2) \mapsto I_1 \setminus I_2\) is a bijection between

\[
\{(I_1, I_2) : I_1, I_2 \subset \{1, \ldots, l\}, I_1 \cap I_2 = I_f \text{ and } I_1 \cup I_2 = I_e \}
\]

and \( P(I_e \setminus I_f) \). Moreover, the condition (24) holds if and only if for every equivalence class (25) of \( \sim \) as above with respect to \( A = I_f \) and \( B = I_e \) and every \( t < a \) we have \( \chi_{I_1 \setminus I_2}(i_{t+1}) = \chi_{I_1 \setminus I_2}(i_t) \) if and only if \( i_t \notin I \). Thus, by Remark 7.4, \( \chi_{I_1 \setminus I_2} \) is determined by its values on \( B \setminus (A \cup N^B_A) \), which are arbitrary. The corollary follows. \( \square \)

Combining the results of this section we obtain

**Corollary 7.6.** Let \( u \in K \). Then \( u \leq \bar{w} \) if and only if \( u_{\text{even}}, u_{\text{odd}} \leq w \). Moreover, the right-hand side of (17) is

\[
\sum_{u \in K : u \leq \bar{w} \text{ and } p(u) = Q} \sgn u = \pm |H_\mathbf{q}(Q)H \cap K|.
\]
We remark that the first part of the corollary holds in fact for any smooth $w$ \cite[Corollary 10.8]{LM16}.

**Appendix A. Numerical results**

A.1. We have calculated all the polynomials $\tilde{P}^{(m)}_{x,w}$, $x, w \in S_n$ and verified Conjecture 1.1 for $nm \leq 12$\(^2\) (Recall that Conjecture 1.1 is known for $n = 2$.) Let us call a pair $(w, x)$ in $S_n$ *reduced* if it admits no cancelable indices (see Remark 1.2(6)) and $xs < x$ (resp., $sx < x$) for any simple reflection $s$ such that $ws < w$ (resp., $sw < w$). In the following tables we list $\tilde{P}^{(m)}_{x,w}$ in the cases $nm \leq 12$ ($n, m > 1$) for all reduced pairs $(w, x)$ in $S_n$. By Conjecture 1.1 (which we checked at the cases at hand) and Remark 1.2(6), this covers all the polynomials $\tilde{P}^{(m)}_{x,w}$ without restriction on $(w, x)$. To avoid repetitions, we only list representatives for the equivalence classes of the relation $(w, x) \sim (w^{-1}, x^{-1}) \sim (w_0ww_0, w_0xw_0) \sim (w_0w^{-1}w_0, w_0x^{-1}w_0)$.

**Table 1.** $n = 4$

| $(w, x)$       | $P_{x,w}$ | $\tilde{P}^{(2)}_{x,w}$ | $\tilde{P}^{(3)}_{x,w}$ |
|----------------|-----------|-------------------------|-------------------------|
| $(3412, 1324)$ | $1 + q$   | $1 + q + q^2$           | $1 + q + q^2 + q^4$    |
| $(4231, 2143)$ |           |                        |                        |

**Table 2.** $n = 5$

| $(w, x)$       | $P_{x,w}$ | $\tilde{P}^{(2)}_{x,w}$ |
|----------------|-----------|-------------------------|
| $(35142, 13254)$, $(52431, 21543)$ | $1 + q$ | $1 + q + q^2$ |
| $(34512, 13425)$, $(45231, 24153)$ | $1 + 2q$ | $1 + 2q + 3q^2$ |
| $(45312, 14325)$ | $1 + q^2$ | $1 + q^2 + q^4$ |
| $(52341, 21354)$ | $1 + 2q + q^2$ | $1 + 2q + 4q^2 + 2q^3 + q^4$ |

Note that in the cases $n = 4, 5$ we have $\tilde{P}^{(2)}_{x,w} = (P^2_{x,w} + P_{x,w}(q^2))/2$. We split the case $n = 6$ according to two subcases.

**Table 3.** Cases for $n = 6$ where $\tilde{P}^{(2)}_{x,w} = (P^2_{x,w} + P_{x,w}(q^2))/2$

---

\(^2\)We remark that already for $m = 2$ we may have $\deg P_{x,w} > \ell(w) - \ell(x)$ even if $P_{x,w} = 1$, e.g. for $(w, x) = (35421, 13254)$. 


| $(w,x)$ | $P_{x,w}$ | $\tilde{P}^{(2)}_{x,w}$ |
|-------|--------|-----------------|
| (361452, 143265) | $1 + q$ | $1 + q + q^2$ |
| (361542, 132654) | $1 + q$ | $1 + q + q^2$ |
| (426153, 214365) | $1 + q$ | $1 + q + q^2$ |
| (562341, 251463) | $1 + q$ | $1 + q + q^2$ |
| (625431, 216543) | $1 + q$ | $1 + q + q^2$ |
| (356412, 135426) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (463152, 143265) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (465132, 143265) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (562341, 214635) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (632541, 321654) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (653421, 321654) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (364124, 132546) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (364152, 132546) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (364152, 132456) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (562341, 214635) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |

| $(w,x)$ | $P_{x,w}$ | $\tilde{P}^{(2)}_{x,w}$ |
|-------|--------|-----------------|
| (345612, 134526) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (456231, 245163) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (563421, 321654) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (634521, 321654) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (653421, 321654) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (345612, 134256) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (456231, 245163) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (563421, 321654) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (634521, 321654) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |

| $(w,x)$ | $P_{x,w}$ | $\tilde{P}^{(2)}_{x,w}$ |
|-------|--------|-----------------|
| (456231, 245163) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (563421, 321654) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (634521, 321654) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |
| (756432, 123456) | $1 + q + q^2$ | $1 + q + 2q^2 + 3q^3 + q^4$ |

Table 4. Cases for $n = 6$ where $\tilde{P}^{(2)}_{x,w} \neq (P^2_{x,w} + P_{x,w}(q^2))/2$
A.2. Implementation. For the computation, we actually wrote and executed a C program to calculate all ordinary Kazhdan–Lusztig polynomials $P_{x,w}$ for the symmetric groups $S_k$, $k \leq 12$. As far as we know this computation is already new for $k = 11$. (See [dC02] and [War11] for accounts of earlier computations, as well as the documentation of the Atlas software and other mathematical software packages.) As always, the computation proceeds with the original recursive formula of Kazhdan–Lusztig [KL79]

$$P_{x,w} = q^c P_{x,ws} + q^{1-c} P_{xs,ws} - \sum_{z : zs < z < ws} \mu(z, ws) q^{\ell(ws) - \ell(z)} P_{x,z}$$

where $\mu(x, y)$ is the coefficient of $q^{(\ell(y)-\ell(x)-1)/2}$ (the largest possible degree) in $P_{x,y}$, $s$ is a simple reflection such that $ws < w$ and $c$ is 1 if $xs < x$ and 0 otherwise. However, some special features of the symmetric group allow for a faster, if ad hoc, code. (See Remark 1.2(6) and the comments below.) For $S_{11}$ the program runs on a standard laptop (a Lenovo T470s 2017 model with 2.7 GHz CPU and 16GB RAM) in a little less than 3 hours. For $S_{12}$ the memory requirements are about 500 GB RAM. We ran it on the computer of the Faculty of Mathematics and Computer Science of the Weizmann Institute of Science (SGI, model UV-10), which has one terabyte RAM and 2.67 GHz CPU. The job was completed after almost a month of CPU time on a single core.

Let us give a few more details about the implementation. We say that a pair $(w, x)$ is fully reduced if it is reduced (see above) and $x \leq ws, sw$ for any simple reflection $s$. Recall that we only need to compute $P_{x,w}$ for fully reduced pairs. The number of fully reduced pairs for $S_{12}$, up to symmetry, is about $46 \times 10^9$. However, a posteriori, the number of distinct polynomials obtained is “only” about $4.3 \times 10^9$. This phenomenon (which had been previously observed for smaller symmetric groups) is crucial for the implementation since it makes the memory requirements feasible. An equally important feature, which once again had been noticed before for smaller symmetric groups, is that only for a small fraction of the pairs above, namely about $66.5 \times 10^6$, we have $\mu(x, w) > 0$. This fact cuts down significantly the number of summands in the recursive formula and makes the computation feasible in terms of time complexity.

We store the results as follows.

1. A “glossary” of the $\sim 4.3 \times 10^9$ different polynomials. (The coefficients of the vast majority of the polynomials are smaller than $2^{16} = 65536$. The average degree is about 10.)
2. A table with $\sim 46 \times 10^9$ entries that provides for each reduced pair the pointer to $P_{x,w}$ in the glossary.
3. An additional lookup table of size $12! \sim 0.5 \times 10^9$ (which is negligible compared to the previous one) so that in the previous table we only need to record $x$ and the pointer to $P_{x,w}$ (which can be encoded in 29 and 33 bits, respectively), but not $w$.
4. A table with $\sim 66.5 \times 10^6$ entries recording $x, w, \mu(x, w)$ for all fully reduced pairs (up to symmetry) with $\mu(x, w) > 0$.

Thus, the main table is of size $\sim 8 \times 46 \times 10^9$ bytes, or about 340 GB. This is supplemented by the glossary table which is of size $< 100$ GB, plus auxiliary tables of insignificant size.
Of course, by the nature of the recursive algorithm all these tables have to be stored in the RAM.

We mention a few additional technical aspects about the program.

(1) The outer loop is over all permutations \( w \in S_n \) in lexicographic order. Given \( w \in S_{12} \) it is possible to enumerate efficiently the pairs \((w, x)\) such that \( xs < x \) (resp., \( sx < x \)) whenever \( ws < w \) (resp., \( sw < w \)). More precisely, given such \( x < w \) we can very quickly find the next such \( x \) in lexicographic order. Moreover, one can incorporate the “non-cancelability” condition to this “advancing” procedure and then test the condition \( x \leq ws, sw \) for the remaining \( x \)'s. Thus, it is perfectly feasible to enumerate the \( \sim 46 \times 10^9 \) fully reduced pairs.

(2) On the surface, the recursive formula requires a large number of additions and multiplications in each step. However, in reality, the number of summands is usually relatively small, since the \( \mu \)-function is rarely non-zero.

(3) For each \( w \neq 1 \) the program picks the first simple root \( s \) (in the standard ordering) such that \( ws < w \) and produces the list of \( z \)'s such that \( zs < z < ws \) and \( \mu(z, ws) > 0 \). The maximal size of this list turns out to be \( \sim 100,000 \) but it is usually much much smaller. The list is then used to compute \( P_{x,w} \) (and in particular, \( \mu(x, w) \)) for all fully reduced pairs using the recursion formula and the polynomials already generated for \( w' < w \). Of course, for any given \( x \) only the \( z \)'s with \( x \leq z \) matter.

(4) Since we only keep the data for fully reduced pairs (in order to save memory) we need to find, for any given pair the fully reduced pair which “represents” it. Fortunately, this procedure is reasonably quick.

(5) The glossary table is continuously updated and stored as 1,000 binary search trees, eventually consisting of \( \sim 4.3 \times 10^6 \) internal nodes each. The data is sufficiently random so that there is no need to balance the trees. The memory overhead for maintaining the trees is inconsequential.

(6) In principle, it should be possible to parallelize the program so that it runs simultaneously on many processors. The point is that the recursive formula only requires the knowledge of \( P_{x',w'} \) with \( w' < w \), so we can compute all \( P_{x,w} \)'s with a fixed \( \ell(w) \) in parallel. For technical reasons we haven’t been able to implement this parallelization.

As a curious by-product of our computation we get

**Corollary A.1.** The values of \( \mu(x, w) \) for \( x, w \in S_{12} \) are given by

\[
\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 158, 163\}.
\]

This complements [WarII] Theorem 1.1]. The new values of \( \mu \) are 9, 10, 17, 19, 20, 21, 22.

**Table 5.** Values of \( \mu(x, w) \) and pairs attaining them for \( S_{12} \)
Complete tables listing the fully reduced pairs in $S_k$, $k \leq 12$ with $\mu > 0$ (together with their $\mu$-value) are available upon request. The size of the compressed file for $S_{12}$ is 200MB.

Finally, I would like to take this opportunity to thank Amir Gonen, the Unix system engineer of our faculty, for his technical assistance with running this heavy-duty job.

**References**

[BB05] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR 2133266 (2006d:05001)

[BBL98] Prosenjit Bose, Jonathan F. Buss, and Anna Lubiw, *Pattern matching for permutations*, Inform. Process. Lett. **65** (1998), no. 5, 277–283. MR 1620935

[BC17] Francesco Brenti and Fabrizio Caselli, *Peak algebras, paths in the Bruhat graph and Kazhdan-Lusztig polynomials*, Adv. Math. **304** (2017), 539–582. MR 3558217

[BH99] Brigitte Brink and Robert B. Howlett, *Normalizers of parabolic subgroups in Coxeter groups*, Invent. Math. **136** (1999), no. 2, 323–351. MR 1688445

[BJS93] Sara C. Billey, William Jockusch, and Richard P. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), no. 4, 345–374. MR 1241505

[BMB07] Mireille Bousquet-Mélou and Steve Butler, *Forest-like permutations*, Ann. Comb. **11** (2007), no. 3-4, 335–354. MR 2376109
[Las95] Alain Lascoux, *Polynômes de Kazhdan-Lusztig pour les variétés de Schubert vexillaires*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 6, 667–670. MR 1354702 (96g:05144)

[LM16] Erez Lapid and Alberto Mínguez, *Geometric conditions for □-irreducibility of certain representations of the general linear group over a non-archimedean local field*, 2016, arXiv:1605.08515.

[LS90] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in Sl(n)/B*, Proc. Indian Acad. Sci. Math. Sci. **100** (1990), no. 1, 45–52. MR 1051089

[Lus93] G. Lusztig, *Tight monomials in quantized enveloping algebras*, Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), Israel Math. Conf. Proc., vol. 7, Bar-Ilan Univ., Ramat Gan, 1993, pp. 117–132. MR 1261904

[Lus03] ———, *Hecke algebras with unequal parameters*, CRM Monograph Series, vol. 18, American Mathematical Society, Providence, RI, 2003. MR 1974442

[LW17] Nicolas Libedinsky and Geordie Williamson, *The anti-spherical category*, 2017, arXiv:1702.00459.

[Mac04] Percy A. MacMahon, *Combinatory analysis. Vol. I, II (bound in one volume)*, Dover Phoenix Editions, Dover Publications, Inc., Mineola, NY, 2004, Reprint of An introduction to combinatorial analysis (1920) and Combinatory analysis. Vol. I, II (1915, 1916). MR 2417935

[Mon14] Pietro Mongelli, *Kazhdan-Lusztig polynomials of Boolean elements*, J. Algebraic Combin. **39** (2014), no. 2, 497–525. MR 3159260

[MT04] Adam Marcus and Gábor Tardos, *Excluded permutation matrices and the Stanley-Wilf conjecture*, J. Combin. Theory Ser. A **107** (2004), no. 1, 153–160. MR 2063960

[Sen14] Paolo Sentinelli, *Isomorphisms of Hecke modules and parabolic Kazhdan-Lusztig polynomials*, J. Algebra **403** (2014), 1–18. MR 3166061

[SW04] Zvezdelina Stankova and Julian West, *Explicit enumeration of 321, hexagon-avoiding permutations*, Discrete Math. **280** (2004), no. 1-3, 165–189. MR 2043806

[Ten07] Bridget Eileen Tenner, *Pattern avoidance and the Bruhat order*, J. Combin. Theory Ser. A **114** (2007), no. 5, 888–905. MR 2333139

[War11] Gregory S. Warrington, *Equivalence classes for the µ-coefficient of Kazhdan-Lusztig polynomials in Sn*, Exp. Math. **20** (2011), no. 4, 457–466. MR 2859901 (2012j:05460)

[Wes96] Julian West, *Generating trees and forbidden subsequences*, Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994), vol. 157, 1996, pp. 363–374. MR 1417303

[Yun09] Zhiwei Yun, *Weights of mixed tilting sheaves and geometric Ringel duality*, Selecta Math. (N.S.) **14** (2009), no. 2, 299–320. MR 2480718

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