Gaussian Process Vector Autoregressions and Macroeconomic Uncertainty*

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Abstract. We develop a non-parametric multivariate time series model that remains agnostic on the precise relationship between a (possibly) large set of macroeconomic time series and their lagged values. The main building block of our model is a Gaussian Process prior on the functional relationship that determines the conditional mean of the model, hence the name of Gaussian Process Vector Autoregression (GP-VAR). We control for changes in the error variances by introducing a stochastic volatility specification. To facilitate computation in high dimensions and introduce convenient statistical properties tailored to match stylized facts commonly observed in macro time series, we assume that the covariance of the Gaussian Process is scaled by the latent volatility factors. We illustrate the use of the GP-VAR by analyzing the effects of macroeconomic uncertainty, with a particular emphasis on time variation and asymmetries in the transmission mechanisms. Using US data, we find that uncertainty shocks have time-varying effects, they are less persistent during recessions but their larger size in these specific periods causes more than proportional effects on real growth and employment.

JEL: C11, C14, C32, E32.

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1 Introduction

Economic relationships can change over time for a variety of reasons, including technological progress, institutional changes, major policy interventions, but also wars, terrorist attacks, stock market crashes, pandemics, etc. Standard econometric models, such as linear single and multivariate regressions assume instead stability of the parameters characterizing the conditional first and second moments of the dependent variables. When stability is formally tested, it is often rejected, see e.g. Stock and Watson (1996). This has led to the development of a variety of methods to handle structural change in econometric models.

Parameter evolution is assumed to be either observable (i.e., driven by the behavior of observable economic variables) or unobservable, and either discrete and abrupt or continuous and smooth. Examples include Threshold and Smooth Transition models (see, e.g., Tong (1990) and Teräsvirta (1994)), Markov Switching models (see, e.g., Hamilton (1989)), and double stochastic models (see, e.g., Nyblom (1989)). In all these models, a specific and fully parametrized type of parameter evolution is assumed, and then linear or non-linear filters are used for estimation in a classical context or Markov Chain Monte Carlo (MCMC) methods in a Bayesian framework. Examples of economic applications of these methods include Koop and Korobilis (2013), Aastveit, Natvik, and Sola (2017), Alessandri and Muntaz (2019), and Aastveit, et al. (2017).

Assuming a specific type of parameter evolution increases estimation efficiency but can lead to mis-specification. A more flexible alternative allows for a smooth evolution of parameters without specifying the form of parameter time variation. In a classical context, the evolution can be either deterministic (see, e.g., Robinson (1991) and Chen and Hong (2012)), or stochastic (see, e.g., Giraitis, Kapetanios, and Yates (2014, 2018) and Kapetanios, Marcellino, and Venditti (2019) for the specific case of (possibly large) vector autoregressive models (VARs)). Kernel estimators are the main tool used in this literature.

In this paper, we propose a new model that belongs to the non-parametric class and is capable of capturing, in a very flexible way, not only parameter evolution but also general non-linear relationships. Our model can be possibly applied in a large data context while keeping the flexibility and ease of use of VARs. Specifically, we combine the statistical literature on Gaussian Process (GP) regressions (see, e.g., Crawford, et al. (2019)), with that on VARs to obtain a GP-VAR model.

Borrowing ideas from the literature on Bayesian Minnesota-type VARs, the model assumes, for each variable under analysis, a different non-linear relationship with its own lags and with the

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1 A special mention is due to Primiceri (2005) who popularized the use of stochastic volatility in macroeconometrics.
lags of all the other variables (and possibly of other exogenous regressors). Gaussian Processes are used to model in a flexible but efficient way the non-linearity. They can be viewed as a non-parametric alternative to the adaptive Minnesota-type shrinkage proposed in Chan (2021).

We develop an efficient (Bayesian) estimation procedure, based on the structural form of the GP-VAR, which has the additional benefit that its complexity is linear in the number of endogenous variables and does not depend on the number of lags. Hence, estimation can be parallelized and, in addition, the conjugate structure of the model we use allows for pre-computing various matrix multiplications and Kernel operations, which further speeds up computation time. As a result, estimation is feasible also for very large models.

We showcase the GP-VAR with synthetic data generated from a highly non-linear multivariate data generating process (DGP). This DGP assumes that some equations feature parameters that exhibit structural breaks while others depend non-linearly on the lags of the endogenous variables. To assess whether the GP-VAR is also capable of recovering linear relations, one equation is a standard linear regression model. In all these cases, our approach works reasonably well in terms of detecting the nature of non-linearities.

The GP-VAR has a vast range of applicability, for both reduced form and structural analysis. This mirrors the possible applications of standard VARs but allows for much more general dynamic relationships across the variables. To illustrate its use, and to gather new insights on a topic that has recently attracted considerable attention (see, e.g., Bloom (2014)), we use the GP-VAR to investigate the effects of exogenous uncertainty shocks on US macroeconomic and financial time series.

We work with GP-VARs of different sizes, ranging from 8 up to 64 endogenous variables, the first one of them being the Jurado, Ludvigson, and Ng (2015, JLN) macroeconomic uncertainty measure, with the residuals for the associated equation interpreted as uncertainty shocks. We find that the GP-VAR model performs statistically better than a linear VAR, and it generates different responses to the uncertainty shock (when averaged over time) from the linear VAR, in particular smaller reactions of real variables, implying more limited damages to the real economy from unexpected increases in uncertainty. Interestingly, the responses of key macroeconomic and financial variables are rather robust to the model size, but for each model size we detect substantial time variation in the dynamic responses. This translates into changing effects of (similarly sized) uncertainty shocks over time. The response of uncertainty itself seems smaller in the mid 1970s, early 1980s, and during the global financial crisis. This pattern appears to be even more evident for longer-run responses. At these horizons, the response of real GDP also changes in line with that of uncertainty, with smaller responses in the aforementioned three
periods. For employment and hours worked, a similar pattern emerges. Interestingly, these are the periods where the JLN uncertainty measure peaks. A possible economic explanation for this pattern is that when uncertainty increases substantially it is likely to have policy interventions, aimed at improving economic conditions and therefore reducing uncertainty (and its persistence) and attenuating its effects on the real economy. The overall effects of the uncertainty shocks are, however, not necessarily smaller during the mentioned periods, as the larger size of the shocks could compensate the reduced effects of a same size shock.

Our proposed framework naturally allows for analyzing potential asymmetries in transmission channels. When we focus on this aspect we find that the responses to the uncertainty shock are also often asymmetric, i.e. dependent on the sign of the shock, but with no clear-cut pattern emerging across the variables or over time. When we focus on differences across shock sizes, we observe that larger shocks exert disproportionately larger effects on the economy, a finding that is difficult to recover with linear VARs and that makes sudden vast increases in uncertainty particularly dangerous.

The paper is structured as follows. Section 2 starts by providing a brief introduction to GP regressions in a time series context and then develops the GP-VAR models in subsection 2.2. This section also provides necessary details on the prior setup and posterior computation. We then analyze model performance using synthetic data in section 3. Section 4 includes our empirical work. In this section we briefly discuss the dataset, provide some in-sample model evidence and then focus on the macroeconomic implications of an uncertainty shock. The final section briefly summarizes and concludes the paper.

2 Large-dimensional Gaussian Process VARs

In this section we first provide a brief introduction to Gaussian Process (GP) regressions in Sub-section 2.1. We then develop the GP-VAR in Sub-section 2.2. Next, Sub-sections 2.3-2.5 are devoted to the development of efficient MCMC schemes to carry out posterior and structural inference. Finally, Sub-section 2.6 details how to compute (generalized) impulse response functions for the GP-VAR.

2.1 A brief introduction to Gaussian Processes

In this sub-section we briefly discuss GP regressions with a focus on time series data.\textsuperscript{2} Gaussian Process regressions are a non-parametric technique to establish a relationship between a scalar

\textsuperscript{2}For a textbook treatment, see Williams and Rasmussen (2006).
time series $y_t$ and a set of $K$ predictors $x_t$. The key advantage of this approach is that it does not rely on parametric assumptions on the precise functional relationship between $y_t$ and $x_t$. In general, a non-parametric regression is given by:

$$y_t = f(x_t) + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma^2),$$

with $f$ being some unknown regression function $f : \mathbb{R}^K \to \mathbb{R}$ and $\varepsilon_t$ denoting a Gaussian shock with zero mean and variance $\sigma^2$. Here, we do not make any assumption on the functional form of $f$. Instead of assuming, e.g., linearity for $f$ a GP regression builds on the idea that one could specify a prior over all functions $f$ which might fit the data well (or capture any other form of behavior macroeconomists might be interested in). This is achieved by assuming that $f(x_t)$ follows a Gaussian Process, or, using Bayesian terminology, we specify a Gaussian Process prior over the function space:

$$f(x_t) \sim \mathcal{GP}(\mu, K_\vartheta(x_t, x_t)),$$

with $\mu$ being a mean parameter (which we will set equal to zero without loss of generality), $K_\vartheta$ denoting a Kernel which depends on a low dimensional vector of hyperparameters $\vartheta$ and controls the behavior of the function $f$. If we condition on the data $\{y_t, x_t\}_{t=1}^T$, the GP process prior becomes a multivariate Gaussian prior:

$$f \sim \mathcal{N}(0_T, K_\vartheta),$$

with $K_\vartheta = K_\vartheta(XX')$ being a covariance matrix with typical element $\kappa_{t,\tau} = K_\vartheta(x_t, x_\tau)$, $X = (x_1, \ldots, x_T)'$ and $t \neq \tau$ referring to two values of $x$ in different points in time. This implies that in terms of full data matrices the GP regression is given by:

$$y = f + \varepsilon, \quad f \sim \mathcal{N}(0_T, K_\vartheta), \quad \varepsilon \sim \mathcal{N}(0_T, \sigma^2 I_T).$$

Up to this point we have remained silent on how to specify the Kernel (and thus the covariance structure) of the GP. In a time series context, we typically assume that $f(x_t)$ does not change much if $x_t \approx x_\tau$ (i.e., a prior view that the function is smooth). A specification which achieves this goal is the Gaussian (or squared exponential) kernel function:

$$\text{Cov}(f(x_t), f(x_\tau)) = \xi \times \exp \left( -\frac{h}{2} ||x_t - x_\tau||^2 \right),$$

5
with $\xi$ denoting a scaling parameter and $h$ the (inverse) length scale and thus $\vartheta = (\xi, h)^\prime$. Larger values of $h$ lead to a GP which displays more high frequency variation whereas lower values imply a smoothly varying mean function. The parameter $\xi$ controls the marginal variance of the function $f$. To see this, note that if $x_t = x_r$, we obtain $\text{Var}(f(x_t)) = \xi$. Williams and Rasmussen (2006) show that the use of the Gaussian kernel implies that $f(x_t)$ is mean square continuous and differentiable.

The effect of the hyperparameter $h$ on the prior and the posterior of $f$ is illustrated in Figure 1. In this figure, as an example, we model year-on-year US inflation using a GP regression. We assume that $x_t$ includes only the first lag of inflation. The left panel of the figure shows the 5th and 95th prior percentiles (with the area in between shaded in light red) as well as three random draws from the priors (in red), and the actual data (black dots). Here, we observe that the draws from the prior are very smooth and exhibit only small changes over time. When we increase $h$, the function $f$ increasingly varies over time. This is especially pronounced for the case $h = 1$ for which we find that the random draws from the prior display much more time variation and cover the range of the actual time series.

Once we have decided on a proper Kernel (such as the Gaussian kernel described above) and a suitable prior on $\sigma^2$ we can easily derive the full conditional posterior distributions $p(f|y, \sigma^2)$ and $p(\sigma^2|f, y)$, as well as the posterior predictive distribution. The full conditional of $f$ given the data and the remaining model parameters is a multivariate Gaussian distribution, while the conditional posterior of $\sigma^2$ follows an inverse Gamma distribution. The corresponding posterior moments take a well known form and are thus not discussed here. In Sub-section 2.5 we provide exact details based on our model and discuss how we sample the hyperparameters $\vartheta$.

Continuing the inflation example, when we focus on the posterior distribution (right panels of Figure 1) we find that larger values of $h$ translate into wider posterior credible intervals (the areas shaded in light red). In terms of the shape of the function we find small differences between the posterior median (solid black line) for $h = 0.01$ and for $h = 0.1$. If $h = 1$, we find larger changes in the posterior of $f$ as well. This suggests that the effect of $h$ on the posterior is rather non-linear.

Before proceeding to the discussion of our model it is worth noting that what we have discussed above is often labeled the function-space view of the GP. This is because the prior is elicited directly on $f$. Another way of analyzing GPs is based on the weight-space view. Under the weight-space view one can rewrite the GP regression as a standard regression model as
Notes: In this figure we showcase the GP regression with US inflation data. The left panels report, for different values of $h$, the 5th and 95th prior percentiles (with the area in between shaded in light red), three draws from the prior (in red), and the actual values of inflation (black dots). The right panels report the 90% posterior credible sets (shaded in light red), the posterior medians (in solid black), and actual inflation (black dots).

follows:

$$y = W_\theta \eta + \varepsilon, \quad \eta \sim \mathcal{N}(0_T, I_T),$$

with $W_\theta$ denoting the lower Cholesky factor of $K_\theta = W_\theta W_\theta'$ and $\eta$ is a Gaussian shock vector with zero mean and unit variance. This is a standard regression model with $T$ regressors, a coefficient vector $\eta$ and a Gaussian prior on $\eta$. Standard textbook formulas for the Bayesian linear regression model (see, e.g., Koop, 2003, Chapter 4) can be used to carry out posterior inference.

Yet another interpretation of the GP is regression is based on the idea that the unknown function $f(x_t)$ can be approximated with an infinite set of basis functions. If the basis functions are Gaussians with different means and variances, then we obtain the GP regression with squared exponential kernel. This follows from Mercer’s theorem, see Williams and Rasmussen (2006) for details.
2.2 The Gaussian Process VAR

The previous section has laid out the general GP regression and provided a specific example which establishes a relationship between inflation and its first lag. In this section, our aim is to generalize the GP regression to the multivariate case by developing a Gaussian Process VAR (GP-VAR).

In the following discussion, let $y_t = (y_{1t}, \ldots, y_{Mt})'$ denote an $M$-dimensional vector of macroeconomic and financial variables. Moreover, $x_t = (x_{1t}', \ldots, x_{Mt}')'$ denotes an $Mp$-dimensional vector with $x_{jt} = (y_{jt-1}', \ldots, y_{jt-p}')$ storing the "own" lags of the $j^{th}$ endogenous variable and $z_t = (z_{1t}', \ldots, z_{Mt}')$ an $(M-1)Mp$-dimensional vector of "other" lags with $z_{jt} = (y'_{j-1t}, \ldots, y'_{j-pt})'$ and $y_{jt}$ denotes the vector $y_t$ with the $j^{th}$ element excluded.

We discriminate between own and other lags of $y_t$ because we assume that lags of other endogenous variables impact a given endogenous variable differently from its own lags. The literature on Bayesian VARs (see Banbura, Giannoni, and Reichlin, 2010; Koop, 2013; Huber and Feldkircher, 2019; Chan, 2021) has captured this through shrinkage priors that treat coefficients on own and other lags differently. Specifically, priors in the Minnesota tradition often shrink parameters on other lagged variables stronger towards zero. We wish to capture this asymmetry by specifying our GP-VAR to depend on two latent processes: one driven by $x_t$ and one by $z_t$.

The structural form of the GP-VAR is then given by:

$$y_t = F(x_t) + G(z_t) + Qy_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0_M, H_t),$$  \hspace{1cm} (1)

with $F(x_t) = (f_1(x_{1t}), \ldots, f_M(x_{Mt}))'$ and $G(z_t) = (g_1(z_{1t}), \ldots, g_M(z_{Mt}))'$ and $f_j$ and $g_j$ being equation-specific functions. The function $f_j$ controls how $y_{jt}$ depends on its own lags while $g_j$ encodes the relationship between $y_{jt}$ and the lags of the other endogenous variables. The matrix $Q$ is a lower triangular matrix with zeros along its main diagonal. This matrix defines the contemporaneous relations across the elements in $y_t$. Finally, $\varepsilon_t$ is an $M$-dimensional vector of Gaussian shocks with zero mean and an $M \times M$-dimensional time-varying variance-covariance matrix $H_t = \text{diag}(\omega_{1t}, \ldots, \omega_{Mt})$. Allowing for time variation in the shocks provides additional flexibility and enables us to control for outliers in the shocks.

The model in Eq. (1) assumes that the shocks in $\varepsilon_t$ are, conditional on $Qy_t$, orthogonal and hence estimation can be carried out equation-by-equation. We will exploit this representation for simplicity and computational tractability. Defining full data matrices $Y_i = (y_{i1}', \ldots, y_{iT}')'$, $X_i = (x_{i1}', \ldots, x_{iT}')'$, $Z_i = (z_{i1}', \ldots, z_{iT}')'$ and $\varepsilon_i = (\varepsilon_{i1}', \ldots, \varepsilon_{iT}')'$ allows us to cast the $i^{th}$ equation of
the model in terms of full-data matrices:

\[ Y_i = F_i + G_i + \sum_{j=1}^{i-1} q_{ij} Y_j + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0_T, \Omega_i), \]

with \( F_i = f_i(X_i), G_i = g_i(Z_i), \Omega_i = \text{diag}(\omega_{i1}, \ldots, \omega_{iT}) \) and \( q_{ij} \) denoting the \((i, j)^{th}\) element of \( Q \). We will use this form to carry out inference about the unknown functions \( f_i \) and \( g_i \) as well as the remaining parameters and latent states of the model.

### 2.3 Function learning using conjugate Gaussian Processes

We now turn to the priors on the functions \( f_i \) and \( g_i \). For each equation-specific function \( f_i \) and \( g_i \) we specify two equation-specific GPs:

\[
\begin{align*}
    f_i(X_i) &\sim GP\left(0_T, \sqrt{\Omega_i}K_{\vartheta_i}(X_i, X_i')\sqrt{\Omega_i}\right), \\
    g_i(Z_i) &\sim GP\left(0_T, \sqrt{\Omega_i}K_{\vartheta_i}(Z_i, Z_i')\sqrt{\Omega_i}\right),
\end{align*}
\]

with \( \sqrt{\Omega_i} = \text{diag}(\sqrt{\omega_{i1}}, \ldots, \sqrt{\omega_{iT}}) \) and \( K_{\vartheta_i}(X_i, X_i') \) and \( K_{\vartheta_i}(Z_i, Z_i') \) denoting two suitable Kernels with typical elements given by:

\[
K_{\vartheta_i}(x_t, x_r) = \xi_{i1} \times \exp\left(- \frac{h_{i1}}{2} ||x_t - x_r||^2\right) \quad \text{and} \quad K_{\vartheta_i}(z_t, z_r) = \xi_{i2} \times \exp\left(- \frac{h_{i2}}{2} ||z_t - z_r||^2\right).
\]

Here, \( \vartheta_{ij} = (\xi_{ij}, h_{ij})' \) for \( i = 1, \ldots, M \) and \( j = 1, 2 \) are equation and kernel-specific hyperparameters. Notice that since the hyperparameters are allowed to differ, we essentially treat own and other lags asymmetrically through different functional approximations \( f_i \) and \( g_i \). In the following discussion we suppress the dependence of the kernels on the hyperparameters to simplify notation.

Notice that the kernel is scaled with the error variances in \( \Omega_i \). A typical diagonal element of the corresponding re-scaled Kernel is given by \( \omega_{it} \times K_{\vartheta_{i1}}(x_t, x_t) = \omega_{it} \xi_{i1} \) and \( \omega_{it} \times K_{\vartheta_{i2}}(z_t, z_t) = \omega_{it} \xi_{i2} \). Typical off-diagonal elements are given by \( \sqrt{\omega_{it}} \sqrt{\omega_{ir}} \times K_{\vartheta_{i1}}(x_t, x_r) \) and \( \sqrt{\omega_{it}} \sqrt{\omega_{ir}} \times K_{\vartheta_{i2}}(z_t, z_r) \). The interaction between the kernel and the error variances gives rise to convenient statistical and computational properties. First, note that if \( \omega_t \) is large, the corresponding prior on the unknown functions is more spread out. In macroeconomic data, \( \omega_t \) is typically large in crisis periods when the \( x_{it} \) and \( z_{it} \) are far away from their previous values. Since the diagonal elements of the kernels are determined by \( \xi_i = (\xi_{i1}, \xi_{i2})' \) the presence of \( \omega_t \) allows for larger values in the marginal prior variance and thus makes large shifts in the unknown functions more likely. Second, the interaction between \( \omega_t \) and \( \omega_{t-1} \) implies that the covariances are scaled down...
if $\omega_t \gg \omega_{t-1}$, suggesting that the informational content decreases if increases in uncertainty are substantial (i.e. $\Delta \omega_t$ is large). If $\omega_t \approx \omega_{\tau}$ and both are large, the corresponding covariance will be scaled upwards. This implies that our model learns from previous crisis episodes as well. Third, as we will show in Sub-section 2.5, interacting the kernel with the error variances leads to a conjugate Gaussian Process structure which implies that we can factor out the error volatilities and do not need to recompute several quantities during MCMC sampling. This speeds up computation enormously and allows for estimating large models.

2.4 Selecting the hyperparameters associated with the kernel

So far, we always conditioned on the hyperparameters that determine the shape of the Gaussian kernel. A simple way of specifying $\vartheta_{ij}$ is the median-heuristic approach stipulated in Chaudhuri, et al. (2017). This choice works well in a wide range of applications featuring many covariates (see, e.g., Crawford, et al., 2019). The median heuristic fixes $\xi_{i1} = \xi_{i2} = 1$ and defines the inverse of the bandwidth parameter as:

$$h_{i1} = \tilde{h}_{i1} = \text{median}_\tau \left( \frac{1}{\|x_{it} - x_{i\tau}\|_2} \right) \quad \text{and} \quad h_{i2} = \tilde{h}_{i2} = \text{median}_\tau \left( \frac{1}{\|z_{it} - z_{i\tau}\|_2} \right),$$

for $i = 1, \ldots, M$. This simple approach has the convenient property that it automatically selects a bandwidth which is consistent with the time series behavior of the elements in $y_t$. To illustrate this, suppose that $y_{it}$ is a highly persistent process (e.g., inflation or short-term interest rates). In this case, the Euclidean distance $\|x_{it} - x_{i\tau}\|_2$ for $\tau = t - 1$ will be quite small and, hence, the mean function $f_i(X_i)$ smoothly adjust. If $y_{it}$ is less persistent and displays large fluctuations (e.g., stock market or exchange rate returns), the Euclidean distance $\|x_{it} - x_{i\tau}\|_2$ will be large and, thus, $f_i(X_i)$ allows for capturing this behavior. This discussion highlights how the median-heuristic allows for flexibly discriminating between signal and noise and thus acts as a non-linear filter which purges the time series from high frequency variation, if necessary.

Given that we work with potentially large panels of time series, it is questionable that the median heuristic works for all elements in $y_t$ equally well. As a solution, we plan to use the median heuristic to set up a discrete grid for $h_{ij}$. For each element in this grid we specify a hyperprior. We use Gamma priors on all elements. For $i = 1, \ldots, M$, that is

$$\xi_{i1} \sim G \left( \frac{1}{2}, \frac{1}{2c_{\xi_1}} \right) \quad \text{and} \quad \xi_{i2} \sim G \left( \frac{1}{2}, \frac{1}{2c_{\xi_2}} \right),$$
for the linear shrinkage hyperparameters and
\[ h_{i1} \sim \mathcal{G}\left(\frac{1}{2}, \frac{1}{2c_{h1}}\right) \quad \text{and} \quad h_{i2} \sim \mathcal{G}\left(\frac{1}{2}, \frac{1}{2c_{h2}}\right), \]
for the bandwidth parameters. Here, \( c_{\xi 1}, c_{\xi 2}, c_{h1} \) and \( c_{h2} \) are scalars that define the tightness of the hyperprior. In the empirical application, we set \( c_{\xi 1} = c_{\xi 2} = c_{\xi} \) and \( c_{h1} = c_{h2} = c_{h} \). These parameters strongly influence the shape of the conditional mean and are crucial modeling choices and we set them through cross-validation. In our empirical application, we find that small values of \( c_{h} \) work reasonably well, yielding an informative prior that forces \( h_{ij} \) to zero.

Based on this set of priors we can derive the conditional posterior distribution. Since \( h_{ij} \) is placed on a grid we can pre-compute several quantities related to the kernel (such as inverses and Cholesky factors). The corresponding posterior is discrete and we can use inverse transform sampling to carry out posterior inference. Further details are provided in Sub-section 2.5.

### 2.5 Posterior computation

Posterior inference is carried out using a conceptually simple MCMC algorithm which cycles between several steps. In this section we will focus on how to sample from the posterior of \( F_i \), \( p(F_i|\bullet) \) (with \( \bullet \) denoting conditioning on everything else), and \( \theta_{i1} \). Sampling from \( p(G_i|\bullet) \) and \( p(\theta_{i2}|\bullet) \) works analogously with trivial adjustments.

As stated in Sub-section 2.1, the posterior of \( F_i \) is Gaussian for all \( i \):

\[ F_i|\bullet \sim \mathcal{N}(\bar{F}_i, \bar{V}_{F_i}), \]

with posterior moments given by:

\[
\bar{V}_{F_i} = \sqrt{\bar{\Omega}_i} \left( \mathcal{K}_{\theta_{i1}}(X_i X_i') - \mathcal{K}_{\theta_{i1}}(X_i X_i') (\mathcal{K}_{\theta_{i1}}(X_i X_i') + I_T)^{-1} \mathcal{K}_{\theta_{i1}}(X_i X_i') \right) \sqrt{\bar{\Omega}_i},
\]

\[
\bar{F}_i = \sqrt{\bar{\Omega}_i} \mathcal{K}_{\theta_{i1}}(X_i X_i') (\mathcal{K}_{\theta_{i1}}(X_i X_i') + I_T)^{-1} \sqrt{\bar{\Omega}_i} \left( Y_i - G_i - \sum_{j=1}^{i-1} q_{ij} Y_j \right).
\]

In principle, computing the inverse and Cholesky factor of \( \bar{V}_{F_i}^{-1} \) constitutes the main bottleneck when it comes to sampling \( F_i \). This is especially so if \( T \) is large. But, in common macroeconomic applications which use quarterly US data, \( T \) is moderate and thus computation is feasible. In our case, even if interest centers on using monthly data or even higher frequencies, we can exploit the convenient fact that, conditional on the hyperparameters \( \xi_{ii} \) and \( h_{ii} \), \( \left( \mathcal{K}_{\theta_{i1}}(X_i X_i') - \mathcal{K}_{\theta_{i1}}(X_i X_i') (\mathcal{K}_{\theta_{i1}}(X_i X_i') + I_T)^{-1} \mathcal{K}_{\theta_{i1}}(X_i X_i') \right) \) can be precomputed. In addi-
tion, notice that
\[
\mathbf{V}_F = \mathbf{C}_F \mathbf{C}_F' = \left( \sqrt{\Omega_i} \mathbf{B}_F \right) \left( \sqrt{\Omega_i} \mathbf{B}_F \right)',
\]
with
\[
\left( \mathbf{K}_{\theta,1}(\mathbf{X}_i, \mathbf{X}'_i) - \mathbf{K}_{\theta,1}(\mathbf{X}_i, \mathbf{X}'_i) \left( \mathbf{K}_{\theta,1}(\mathbf{X}_i, \mathbf{X}'_i) + \mathbf{I}_T \right)^{-1} \mathbf{K}_{\theta,1}(\mathbf{X}_i, \mathbf{X}'_i) \right) = \mathbf{B}_F \mathbf{B}_F',
\]
and hence the Cholesky factor \( \mathbf{C}_F \) can be pre-computed as well. This substantially speeds up computation.

Notice, however, that we conditioned on the hyperparameters. Since these are crucial for precise predictive inference, we will estimate them by constructing a grid:

\[
h_{ij} = [0.1 \bar{h}_{ij}, 2\bar{h}_{ij}] \quad \text{and} \quad \xi_{ij} = [0.04, 4], \quad \text{for} \quad i = 1, \ldots, M \quad \text{and} \quad j = 1, 2,
\]
with the intervals indicating the minimum (maximum) value supported for each hyperparameter. Within this two dimensional range, we define a discrete grid of around 1000 combinations with equally sized increments along each dimension. Notice that the grid on \( h_{ij} \) is centered around the median heuristic. For each value on this grid, we compute the corresponding kernel \( \mathbf{K}_{\theta,1}(\mathbf{X}_i, \mathbf{X}'_i) \) as well as all relevant quantities (i.e., \( \mathbf{C}_F \)). Based on these values we jointly evaluate the conditional posterior ordinate by applying Bayes theorem. The exact form of the conditional likelihood is given by:

\[
p(F_i|\theta_{i1}, \bullet) = (2\pi)^{-\frac{T}{2}} \times \det \left( \sqrt{\Omega_i} \mathbf{K}_{\theta,1}(\mathbf{X}_i, \mathbf{X}'_i) \sqrt{\Omega_i} \right)^{-\frac{1}{2}} \times \exp \left\{ -\frac{1}{2} \left( \mathbf{F}_i' \left( \sqrt{\Omega_i} \mathbf{K}_{\theta,1}(\mathbf{X}_i, \mathbf{X}'_i) \sqrt{\Omega_i} \right)^{-1} \mathbf{F}_i \right) \right\}.
\]

Note that the shape of \( \mathbf{K}_{\theta,1}(\mathbf{X}_i, \mathbf{X}'_i) \) depends on the hyperparameters \( \theta_{i1} = (\xi_{1i}, h_{1i})' \), which we want to update. For each pair of values \( \theta_{i1}^{(s)} \) on our two dimensional grid (with \( s \) denoting a specific combination), we compute the corresponding kernel \( \mathbf{K}_{\theta,1=\theta_{i1}^{(s)}}(\mathbf{X}_i, \mathbf{X}'_i) \) as well as \( \det \left( \mathbf{K}_{\theta,1=\theta_{i1}^{(s)}}(\mathbf{X}_i, \mathbf{X}'_i) \right) \) and \( \left( \mathbf{K}_{\theta,1=\theta_{i1}^{(s)}}(\mathbf{X}_i, \mathbf{X}'_i) \right)^{-1} \) prior to MCMC sampling. Hence, within our sampler evaluating the likelihood is straightforward and computationally efficient. All that remains is to multiply the likelihood with the prior. The corresponding posterior ordinates for each \( \theta_{i1}^{(s)} \) are used to compute probabilities to perform inverse transform sampling to sample from \( p(\theta_{i1}|\bullet) \).

As stated above, the full conditionals of \( \mathbf{G}_i \) and \( \theta_{i2} \) are obtained analogously. This implies
that \( p(G_i|\bullet) \) follows a \( T \)-dimensional multivariate Gaussian distribution and \( \vartheta_{i2} \) is obtained by constructing a grid and performing inverse transform sampling.

Finally, we simulate the log-volatilities using a modified variant of the algorithm proposed in Kastner and Frühwirth-Schnatter (2014). The main difference stems from the fact that the volatilities in \( \Omega_i \) also show up in the prior on \( f_i \) and \( g_i \). This calls for a minor adjustment of the original sampler by integrating out the latent processes \( F_i \) and \( G_i \) first and then sampling the log-volatilities from a standard stochastic volatility model that is normalized by dividing through the square root of the sum of the diagonal elements of the kernel matrices.

### 2.6 Generalized impulse responses in the GP-VAR

We will apply the GP-VAR to perform impulse response analysis. Since the model is highly non-linear, we need to resort to generalized impulse responses (GIRFs) originally proposed in Koop, Pesaran, and Potter (1996). To obtain the structural GIRFs in our framework we need to recover the reduced-form of Eq. (1) as follows:

\[
y_t = \tilde{Q} [F(x_t) + G(z_t)] + \eta_t, \quad \eta_t \sim N(0_M, \Sigma_t),
\]

with \( \tilde{Q} = (I_M - Q)^{-1} \) and \( \Sigma_t = \tilde{Q}H_t\tilde{Q}' \). The period-specific GIRFs for horizon \( h \), denoted by \( \delta_{ht} \), are defined as the difference of the forecast conditional on a unit structural shock \( \epsilon_{it} = 1 \) to the \( i^{th} \) variable in period \( t \) and the respective unconditional forecast (with \( \epsilon_{it} = 0 \)):

\[
\delta_{ht} = E[y_{t+h}|I_t, \epsilon_{it} = 1] - E[y_{t+h}|I_t],
\]

where \( I_t \) indicates the information available at period \( t - 1 \). The corresponding forecast is computed in a recursive manner.

Notice that \( \delta_{ht} \) is state-dependent and, due to the non-linear nature of the conditional mean function, allows for asymmetries in how \( y_t \) reacts to shocks. This gives rise to two inferential opportunities. First, one can assess how a given shock has impacted the economy in a given point in time. This allows us to investigate whether transmission mechanisms depend on the underlying state of the economy. Second, the non-linear mean function directly implies that shock transmission can be asymmetric, so that positive shocks might feed through the economy differently than negative shocks, and non-proportional, so that larger shocks can have proportionally larger effects than smaller shocks. In all our empirical work we will exploit both dimensions and focus on asymmetries in the sign, and non-proportionality in the size, of the
shock as well as explicitly consider state dependencies by computing $\delta_{ht}$ over time. Finally, we also integrate out uncertainty with respect to the state of the economy by averaging over all values of $t$.

3 Illustration using synthetic data

In this section we illustrate the computational merits of our approach and evaluate whether it successfully recovers different features of a highly non-linear DGP.

To illustrate our methods, we simulate $T = 200$ observations from a highly non-linear small-scale VAR with $M = 3$ equations. The three equations differ in terms of whether they are linear or non-linear in the parameters. Non-linearities are captured in two ways. First, we assume a break point and second we assume non-linear relations between the response variable and the lags of the other variables. In all these equations, we assume that the functions $f_j$ and $g_j$ differ to assess whether our approach is capable of discriminating between the two. The precise form of our DGP is given by:

$$y_t = F(x_t) + G(z_t) + Qy_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0_M, H_t),$$

with

$$F(x_t) = \begin{pmatrix} f_1(x_{1t}) \\ f_2(x_{2t}) \\ f_3(x_{3t}) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{p} \phi_{11,k} y_{1t-k} \\ \sum_{k=1}^{p} \phi_{22,k} y_{2t-k} \times \mathcal{I}(t \leq 100) + \phi_{22,1} y_{2t-1} \times \mathcal{I}(t > 100) \\ \frac{1}{3} \sin\left(\frac{\pi}{2} y_{3t-1} y_{3t-2}\right) + \frac{1}{6} (y_{3t-3} - 1)^2 + \frac{1}{3} y_{3t-4} + \frac{1}{3} y_{3t-5} \end{pmatrix},$$

and

$$G(z_t) = \begin{pmatrix} g_1(z_{1t}) \\ g_2(z_{2t}) \\ g_3(z_{3t}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \times \mathcal{I}(t \leq 100) + \sum_{k=1}^{p} \sum_{j \in \{1,3\}} \phi_{2j,k} y_{jt-k} \times \mathcal{I}(t > 100) \\ \frac{1}{6} \sin\left(\frac{\pi}{2} y_{1t-1} y_{2t-1}\right) + \frac{1}{12} (y_{1t-2} - 1)^2 + \frac{1}{6} y_{1t-3} + \frac{1}{6} y_{2t-5} \end{pmatrix},$$

where $\phi_{11,1} = 0.9$, $\phi_{22,1} = 0.6$, $\phi_{ij,k} \sim \mathcal{N}(0, (0.3)^2)$ for $i \neq j$ and $k = 1, \ldots, p$, and $\mathcal{I}(\bullet)$ denotes the indicator function that equals one if its argument is true and zero otherwise. The free elements in $Q$, are also simulated from a Gaussian distribution with $\eta_{ij} \sim \mathcal{N}(0, 0.1^2)$. Moreover, we introduce a stochastic volatility specification for the structural error variances $\{\omega_{jt}\}_{t=1}^3$ with each element following an independent random walk law of motion: $\log \omega_{it} = \log \omega_{it-1} + \eta_{\omega, it}$, with $\eta_{\omega, it} \sim \mathcal{N}(0, \sigma_{\omega}^2)$. For each equation, we set the initial state $\omega_{i0} = 0.1$ and the state innovation variance $\sigma_{\omega}^2 = 0.01^2$.

Figure 2 shows results obtained from simulating a single realization from the DGP. Hor-
orizontal panels refer to each equation of the DGP. The red shaded areas represent 5th and 95th posterior intervals of $F_i$, $G_i$ and $F_i + G_i$, while the solid black line denotes the actual outcome. The figure suggests that our approach is capable of detecting different functional relations between $y_t$, $x_t$ and $z_t$. Across all three equations, we find that the estimated $F_i$ tracks the actual value rather well. Considering the estimates for $G_i$ suggests that, if the actual relationship between $z_t$ and $y_t$ is non-existent, our approach accurately detects this behavior.

One key advantage of the GP-VAR is that we can remain agnostic on the form of non-linearities the conditional mean function might take. This is confirmed for synthetic data where, irrespective of the non-linear nature of the model, the estimated sum of $F_i$ and $G_i$ closely tracks the dynamics of the actual outcome. This finding holds both for the linear case (i.e. the first equation) and for highly non-linear situations (i.e. the second and third equations).

We have stressed that our approach is computationally efficient and scalable to large datasets. To investigate this claim more carefully, Figure 3 shows the time required to generate a single draw from the joint posterior for a given equation across different values of $K$ and for $T = 200$. We show the computation times for our GP-VAR and a VAR with SV. Since our approach is embarrassingly parallel the actual times for generating a draw from the joint posterior of the full system are approximately $M$ times the runtimes reported in the figure.\(^3\)

The most striking take away from the figure is that the computation time of the GP-VAR does not depend on $K$. This implies that increasing the number of lags and/or endogenous variables does not impact estimation times considerably. By contrast, the time necessary to generate a draw from the joint posterior of the VAR rises rapidly in $K$. This shows that our approach scales well in high dimensions and, in fact, is much faster than competing approaches to non-linear VAR models such as TVP-VARs or regime-switching VARs.

### 4 The macroeconomic effects of uncertainty shocks

We now turn to our analysis of the effects of uncertainty shocks using our GP-VAR. In Sub-section 4.1 we provide details on the data and in 4.2 we discuss whether the GP-VAR fits the data well and show key in-sample features. We then focus on assessing how macroeconomic uncertainty feeds through the economy. This analysis is split into several parts with Sub-section 4.3 serving as a benchmark exercise which compares our impulse responses to the ones obtained from a model similar to that used by Jurado, Ludvigson, and Ng (2015). In Sub-section 4.4 we leverage the non-linear nature of the GP-VAR and analyze how the effects of uncertainty shocks

\(^3\)Since we need to augment the $j^{th}$ equation with the contemporaneous values of the preceding $j - 1$ equations, this statement is only approximately valid.
Figure 2: Posterior distributions of $F_i$ and $G_i$ versus actual realizations $f_i$ and $g_i$ for each of the three equations in the DGP.

Notes: Results are obtained from simulating a single realization from the proposed DGP. Horizontal panels refer to each equation of the DGP. The solid black lines denote the actual outcomes of $f_i$, $g_i$, and $f_i + g_i$, the red solid lines the posterior median of $F_i$, $G_i$ and $F_i + G_i$, while the red shaded areas represent the respective 90% posterior credible sets.

change across model sizes, over time, and according to the sign or size of the shocks.

4.1 Data overview

We use the quarterly version of the dataset proposed in McCracken and Ng (2016) and consider time series that range from 1960:Q1 to 2019:Q4. We exclude the years of the Covid-19 pandemic to make comparison with the model used in Jurado, Ludvigson, and Ng (2015) possible. In
Figure 3: Computation time (in seconds) for a single draw from the joint posterior distribution.

Notes: The red dashed line refers to the computation time to obtain a draw for our GP-VAR, while the blue dashed line denotes the computation time for a single draw of a standard BVAR. Since our approach is embarrassingly parallel the actual times for generating a draw from the joint posterior of the full system is approximately $M$ times the runtimes reported in this figure. Computation times are based on a desktop machine with an AMD Ryzen 7 5800X 8-Core processor.

In the following, we consider four different specifications that differ in the number of endogenous variables used. These specifications are:

- **GP-VAR-8**: Uncertainty index (labeled as $\text{UNC}$), real GDP ($\text{RGDP}$), civilian employment ($\text{EMP}$), average weekly working hours in manufacturing ($\text{AWH}$), consumer price index ($\text{CPI}$), average hourly earnings in manufacturing ($\text{AHE}$), Fed Funds Rate ($\text{FFR}$), S&P 500 ($\text{SP500}$). This model is closely related to the specification adopted in Jurado, Ludvigson, and Ng (2015).

- **GP-VAR-16**: In addition to the variables in the VAR with $M = 8$ endogenous variables, we also include the components of GDP (such as real personal consumption and real private fixed investment), additional labour market variables (such as unemployment, initial claims, average weekly working hours and average hourly earnings across all sectors), housing starts, as well as the real M2 money stock.

- **GP-VAR-32**: On top of the variables of the VAR with $M = 16$ variables, we include important financial variables, further data on housing and data on loans.

- **GP-VAR-64**: The largest model we consider features $M = 64$ endogenous variables. The set of endogenous variables is obtained by taking the variables with $M = 32$ and including additional financial variables and data on manufacturing.
All variables are transformed to be approximately stationary and we include five lags of the endogenous variables. The precise variables included (and transformations applied to each variable) are shown in Table A.1 in the appendix. We consider these different model sizes for several reasons. First, we would like to assess how adding additional information impacts the responses of key variables to an uncertainty shock. Second, we are interested in the relationship between non-linearities and the size of the model. Third, we wish to show that a model such as the GP-VAR-64 can be estimated in a reasonable amount of time and thus illustrate the scalability of our approach.

4.2 In-sample features and model adequacy

In this section we first assess model adequacy through the widely applicable information criterion (WAIC, Watanabe, 2010, 2013). Our focus on the WAIC is predicated by the fact that it is more robust than alternative measures such as the Deviance Information Criterion (DIC, Spiegelhalter, et al., 2002) if the model features a large number of latent variables.

Let \( A \) denote a specific model. The WAIC is then given by:

\[
\text{WAIC}_A = -2 \left( \hat{\ell}_A - \hat{p}_A \right),
\]

with \( \hat{\ell}_A \) denoting an estimate of the period-wise log likelihood and \( \hat{p}_A \) the effective number of parameters. The first term measures the in-sample fit of the model while the second term penalizes model complexity.

The period-wise log likelihood is given by:

\[
\hat{\ell}_A = \sum_{t=1}^{T} \left( \frac{1}{S} \sum_{s=1}^{S} \log p_A(y_t | \theta_A^{(s)}) \right),
\]

where \( \theta_A^{(s)} \) denotes the \( s^{th} \) draw of model \( A \)'s full posterior distribution. The effective number of parameters is measured through the variance of the log likelihood across \( S \) draws from the full posterior distribution:

\[
\hat{p}_A = \sum_{t=1}^{T} \text{Var} \left( \log p_A(y_t | \{ \theta_A^{(s)} \}_{s=1}^{S}) \right).
\]

Table 1 shows relative WAIC values for different models relative to a small-scale BVAR with SV. The models differ with respect to model size and how the hyperparameters are set. In this regard, we consider three ways of sampling the hyperparameters. First, we simulate
Table 1: Widely applicable information criterion (WAIC, Watanabe, 2010) relative to a small-scale BVAR (BVAR-8) with stochastic volatility.

| Specification | relative WAIC |
|---------------|---------------|
|               | $c_\xi = c_h$ | GP-VAR-8 | GP-VAR-16 | GP-VAR-32 | GP-VAR-64 |
| Semi-automatic grid scaled by median heuristic: | | | | | |
| $h_{ij} = [0.1h_{ij}, 2h_{ij}]$ and $\xi_{ij} = [0.04, 4]$ | $10^{-3}$ | −170.02 | −187.21 | −156.53 | −129.44 |
| | $5 \times 10^{-5}$ | −125.20 | −113.23 | −71.91 | −66.03 |
| | $10^{-8}$ | 1466.38 | 1314.14 | 1269.13 | 1258.40 |
| Median heuristic and grid for linear scaling: | | | | | |
| $h_{ij} = h_{ij}$ and $\xi_{ij} = [0.04, 4]$ | $10^{-3}$ | −198.11 | −201.40 | −138.80 | −118.44 |
| | $5 \times 10^{-5}$ | −202.58 | −221.71 | −143.82 | −94.26 |
| | $10^{-8}$ | 747.38 | 767.30 | 849.97 | 901.66 |
| Median heuristic and no linear scaling: | | | | | |
| $h_{ij} = h_{ij}$ and $\xi_{ij} = 1$ for $k \in \{1, 2\}$ and $i \in \{1, \ldots, M\}$ | | | | | |
| | $10^{-3}$ | −10.91 | −138.03 | −139.56 | −54.79 |

Notes: The table shows relative WAICs. The results are benchmarked to a small-scale BVAR (BVAR-8) with stochastic volatility. To ensure comparability between models with different information sets, we evaluate the period-wise log-likelihood only for the target variables included in the VAR-8. Negative values imply that a given model improves upon the benchmark, while positive values suggest that the benchmark is preferred. The best specification within each information set (VAR-8, VAR-16, VAR-32, and VAR-64) are shaded in gray.

from the posterior of $h_{ij}$ and $\xi_{ij}$ using the approach outlined in Section 2.5. Second, we fix $h_{ij}$ using the median heuristic while sampling the linear scaling parameter $\xi_{ij}$. Finally, we consider the median heuristic and set $\xi_{ij} = 1$. We focus on the model performance for the eight target variables included in the smallest specification to compare models across sizes.

It turns out that GP-VAR is better than BVAR-SV for a variety of model specifications. The semi-automatic grid scaled by median heuristic works particularly well for the larger model. Median heuristic and grid for median scaling are instead best for the GP-VARs with 8 and 16 variables, with the latter performing best in overall terms. Median heuristic and no linear scaling is the worst performer, though the associated GP-VARs still beat the BVAR-SV for any choice of dimension. The table also includes the WAIC for different values of the hyperparameters that control the tightness of the prior on $h_{ij}$. Comparing them suggests that in larger models, introducing more shrinkage (by shrinking $h_{ij}$ to small values) leads to slightly better WAIC values. But differences are rather small and setting $c_\xi, c_h$ too small hurts model evidence substantially.

Figure 4 reports the model fit for each variable obtained from the GP-VAR, also decomposed into the estimated $F_i$ and $G_i$ functions. To enable visual comparison, we have put the scale of $F_i$ on the left axis and the scale of $G_i$ on the right axis. Simply comparing the limits of the axes reveals that $F_i$ displays much more variation for rather persistent variables such as the uncertainty indicator, the Federal Funds Rate and CPI inflation. For variables which are less persistent (such as the S&P 500) differences in variation between $F_i$ and $G_i$ become smaller.
Figure 4: Posterior summary of fitted values of the GP-VAR-8 model.

Notes: The red lines refer to the posterior median of the fit $Y_i = F_i + G_i$, the shaded red areas denote the the 84% posterior credible set of the fit, the blue lines denote the posterior median of $F_i$, the green lines the posterior median of $G_i$, and the black lines the observed series $Y_i$.

Focusing on the fit of the model we observe a good fit for all variables. This is not surprising given the flexibility of the model and its ability to capture all sort of non-linearities in the simulated data of Section 3. When we focus on the dynamics over time we observe that
the autoregressive component is relevant in most time periods while the effect of other lagged variables appears to play an important role in recessions (or turbulent times in general).

To get a feeling on how the linear shrinkage parameter evolves over time, Figure 5 plots the product of the error variance times the linear shrinkage parameter that determines the kernel of \(F\) and \(G\). A few interesting features emerge. First, less shrinkage (larger parameter values) is applied to the own lags than to the lags of the other variables. Second, more shrinkage is applied in the larger models than in the smaller models (in line with the literature on BVARs). Finally, less shrinkage is applied during recessionary times, in particular the recessions of the early ’80s and the financial crisis. This finding is, again, in line with results from the forecasting literature showing that more information is particularly useful during problematic times (see, e.g., Koop, 2013).

Finally, we investigate differences in \(h_{11}\) and \(h_{v2}\) across equations and variable types. A heatmap that shows the posterior mean of the inverse of the length scale parameters for own and other lags is in Figure 6. Recall that large values of \(h_{ij}\) \((j = 1, 2)\) imply more variation in the latent processes whereas values of \(h_{ij}\) close to zero imply less variation in \(F_i\) and \(G_i\) (or put differently, smooth paths of \(F_i\) and \(G_i\)). From the figure we learn that for employment and the S&P500, the hyperparameters are considerably higher for own lags and for both information sets. This corroborates the findings in Figure 4: the latent process that encodes the relations between the endogenous variables and their own lags soaks up most of the variation in \(y_t\). When we compare panels (a) and (b) we also find that for the small-scale GP-VAR, the scale parameter appears to be much larger (as evidenced by many red shaded cells) as in the case of the huge GP-VAR. This finding is consistent with the literature on Bayesian VARs which reports that shrinkage typically increases with the model size (see, e.g., Giannone, Lenza, and Primiceri, 2015).

4.3 Comparison with standard BVAR analysis

In this section we investigate whether our non-parametric model yields impulse responses to economic uncertainty shocks which are fundamentally different than the ones reported from a model similar to the one proposed in Jurado, Ludvigson, and Ng (2015) (JLN). In particular, while they use a classical homoskedastic VAR estimated on monthly data, we work with a quarterly Bayesian VAR with SV.

In Figure 7 we report the (average over time in the case of the GP-VAR) posterior quantiles \((16^{th}, 50^{th} and 84^{th})\) of the responses to an uncertainty shock in the JLN model and in the corresponding GP-VAR with eight endogenous variables. Uncertainty responses to its own shock
**Figure 5:** Linear shrinkage parameters of equation-specific kernels.

a) GP-VAR with 8 variables

![Grapha](image)

b) GP-VAR with 64 variables

![Graphb](image)

**Notes:** This figure reports the posterior mean of the product of the error variances \(\omega_{it}\) and the linear scaling parameters for own lags \((\xi_{i1})\) and for other lags \((\xi_{i2})\), respectively. These two quantities correspond to the diagonal elements of the re-scaled Kernels \(\omega_{it} \times K_{\theta_i1}(\pi_t, \pi_t) = \omega_{it}\xi_{i1}\) and \(\omega_{it} \times K_{\theta_i2}(\pi_t, z_t) = \omega_{it}\xi_{i2}\).
Figure 6: Inverse length scale parameters of equation-specific kernels.

a) GP-VAR with 8 variables

b) GP-VAR with 64 variables

Notes: This figure reports the posterior mean of the (inverse) length scale parameters for own lags \(h_{1i}\) and other lags \(h_{12}\).

differ between the GP-VAR and the linear model. As opposed to the BVAR, the non-parametric model produces uncertainty reactions that are smaller in magnitudes but which also peak earlier and thus display an overall decrease throughout the impulse response horizon. This has direct implications on the reactions of the other variables in the system. The responses of the growth rates of real GDP, employment and hours worked are smaller in the GP-VAR. This hints towards more limited damages to the real economy from unexpected increases in uncertainty. One thing worth emphasizing is that, as opposed to the original JLN model, our non-parametric technique does not produce evidence in favor of a real activity overshoot that is attributed to a typical wait-and-see mechanism commonly found in the literature.

Focusing on the effects on CPI inflation reveals weaker effects produced by our GP-VAR, comparable effects for hours employed and a stronger negative reaction of the Federal Funds rate up to about two-years ahead. The latter response is likely due to the fact that inflation does not increase as in JLN and therefore the deterioration in the real economy becomes the main economic driver of the policy rate. Finally, the S&P500 decreases for both JLN and GP-VAR, a bit more for the latter.

Considering the uncertainty surrounding the IRFs reveals that for most variables, the GP-VAR produces IRFs which are more precise than the ones from the linear model. This is driven by the fact that the parameter space of the GP-VAR does not increase with the number of lags. In sum, the GP-VAR produces sensible responses for all quantities under consideration with slightly weaker effects for real variables and stronger effects for financial variables (when compared to the original JLN model). While these differences are already relevant, they are based on averages over time of the GP-VAR responses, so the period by period differences could be even larger, as we will discuss later on.
4.4 The relationship between model size and economic uncertainty

Next, we assess how IRFs to uncertainty shocks change if we increase the size of the model. In principle, our flexible non-parametric approach should be capable of handling omitted variable bias by fitting more complex mean relations. Nevertheless, it is worth investigating whether more information has an effect on the estimated IRFs of the eight focus variables.

In Figure 8 we report the (averaged over time) responses to the uncertainty shock for the same 8 variables displayed in Figure 7 but obtained from GP-VARs of different dimensions (with 8, 16, 32, and 64 variables). In summary, we find some differences in terms of the point estimates but overall, the qualitative insights remain similar (with some notable exceptions). This suggests that a larger information set does not affect the results for the key variables but provides a more granular view on the reaction of the economy.

Focusing on the responses which display more pronounced changes to different information sets, we find that output and employment reactions appear to be much more persistent for larger models. Interestingly, this effect becomes more pronounced the larger the information set is.

Inflation reactions also change considerably across information sets. While we find no evidence of significant changes in inflation for small to moderately-sized models, the large model produces inflation responses which are negative and quite persistent (i.e. not approaching zero...
4.5 Asymmetries in the transmission of uncertainty shocks

The non-linear and non-parametric nature of our models allows for asymmetries in the impulse response functions. This implies that shocks propagate non-linearly through the model, giving rise to differences in the IRFs both over time but also for different shock magnitudes or signs.

4.5.1 Asymmetries with respect to the sign of the shock

The first aspect we consider relates to whether positive and negative uncertainty shocks generate different responses of the economy. In Figure 9 (a) we report the IRFs to negative and positive uncertainty shocks from the GP-VAR-8, averaged over time. The figure shows the IRFs to a positive (in orange), negative (in blue) and a negative shock multiplied by -1 (in gray).

From the figure we observe some differences. These differences mostly relate to longer-run responses. In economic terms, however, asymmetries between positive and negative shocks are rather small and both shocks tell a similar story (with differing sign). When we focus on the

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**Figure 8:** Impulse responses of target variables across different information sets.

Notes: Average generalized impulse responses (GIRFs, outlined in Sub-section 2.6) to a positive one standard deviation shock in uncertainty. Solid lines denote the posterior median, while shaded areas correspond to the 84% credible sets.

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after 16 quarters). Consistent with our finding above, this stronger reaction of inflation impacts the reaction of the Federal Funds Rate. As opposed to the small-scale model, using a larger model implies no significant reaction in the short-term interest rates over the impulse response horizon.
larger model (see Figure 9 (b)) we find that symmetry increases. Except for the uncertainty reaction and hours worked, we find only minor differences between responses. This indicates that symmetry can be a good approximation. Yet, as we will see, the picture will be different when looking at symmetry period by period and for differing shock sizes.

**Figure 9:** Shock sign asymmetries in impulse responses of target variables.

Notes: Average generalized impulse responses (GIRFs, outlined in Sub-section 2.6) to a negative (positive) one standard deviation shock in uncertainty. Solid lines denote the posterior median, while shaded areas correspond to the 84% credible sets. Here, negative $x (-1)$ denotes a negative one standard deviation shock with the respective responses being mirrored across the x-axis.
4.5.2 Asymmetries with respect to the size of the shock

Our GP-VAR also allows for analyzing how shocks of different sizes impact the economy. As opposed to a linear VAR which assumes that shocks enter linearly (and thus responses to shocks of different sizes are exactly proportional to each other) our GP-VAR is more flexible and allows for analyzing whether shocks of different magnitudes give rise to different dynamics in the IRFs. In Figure 10 we consider two shock sizes: a one standard deviation and a two standard deviation shock. To permit straightforward comparison of the shapes of the responses to differently sized shocks, we also add impulses to a two standard deviation shock which are then re-scaled to match the impact of the one standard deviation shock (the gray shaded area in the figures).

This figure gives rise to at least three observations. First, while we find little differences for the small GP-VAR, asymmetries with respect to differing shock sizes seem to increase with the information set. This is probably driven by the case that variables outside the focus set display more asymmetries and thus impact IRFs in the focus set as well.

Second, when we compare the impact magnitudes of a one standard deviation to a two standard deviation shock we find non-linear increases in (absolute) impact reactions. For some variables (e.g., uncertainty and hours worked) the impact magnitudes of a two standard error shock is more than double of the magnitudes to a one standard deviation shock. For other variables (e.g., employment, inflation and the Fed Funds rate) we find only small differences in terms of impact estimates.

Third, and turning to higher order responses we find pronounced differences not only in the size but also in the shape of the IRFs. This is especially visible for output reactions. Under a more pronounced uncertainty shock we observe that GDP growth reacts much more strongly after one year (declining by around 1.5 percentage points). Interestingly, the recovery between two and three years is slightly more pronounced) but longer-run reactions (16 quarters and more) appear to be stronger tilted to the downside. These asymmetries are also present for employment and hours worked. Apart from scaling effects, there is less evidence in favor of asymmetries in the shape of the responses for CPI inflation, average hours employed, the Fed funds rate and the S&P 500.

4.5.3 Asymmetries over time

Figure 11 reports the temporal evolution of the GIRFs to a positive uncertainty shock. To facilitate easy interpretation we focus on horizons \( h = 1, 4, 8 \) for the GP-VARs of differing dimensions. This allows us to investigate whether uncertainty transmission has changed over
Figure 10: Shock size asymmetries in impulse responses of target variables.

a) GP-VAR-8

b) GP-VAR-64

Notes: Average generalized impulse responses (GIRFs, outlined in Sub-section 2.6) to a positive two (one) standard deviation shock in uncertainty. Solid lines denote the posterior median, while shaded areas correspond to the 84% credible sets. Here, 1 sd refers to a one standard deviation shock, 2 sd indicates a two standard deviation shock, and 2 sd x (1/2) denotes a two standard deviation shock with the respective responses divided by two.
time (for some evidence using TVP-VARs, see, Mumtaz and Theodoridis (2018)).

**Figure 11:** Impulse responses of target variables over time for a fixed horizon.

![Impulse responses of target variables over time for a fixed horizon.](image)

**Notes:** Time-specific generalized impulse responses (GIRFs, outlined in Sub-section 2.6) to a positive one standard deviation shock in uncertainty. Solid lines denote the posterior median, while shaded areas correspond to the 84% credible sets. We focus on three horizons: one-quarter ($h = 1$), one-year ($h = 4$) and two-years ($h = 8$) ahead.

An interesting finding is the time-varying pattern in the response of uncertainty itself. Uncertainty reactions seem to be smaller in the mid-70s, early '80s, and during the financial crisis, a pattern that is even more evident for 1-year and 2-years-ahead responses. At these horizons, the response of real GDP growth also changes in line with that of uncertainty, with
smaller responses in the mid of these three time periods. As mentioned in the introduction, a possible reason for this pattern is that when uncertainty increases substantially it is likely to trigger policy interventions. These are aimed at improving economic conditions and therefore reducing uncertainty and attenuating its effects on the real economy. At $h = 8$, overall the response of output seems to be more negative from the mid 1980s to the end of the 1990s than in other periods. For employment and hours worked, our results suggest similar dynamics.

By contrast, the response of CPI inflation appears to be more stable over time while earnings seem to react much more strongly in the late 1970s and display little variation afterwards. For earnings, we find a particularly strong reaction in the late 1970s (especially so for $h = 4$) and stronger responses in the early 2000s (for $h = 8$). Focusing on the Federal Funds Rate, the results indicate only modest changes over time. For this variable, one exception is the 1-year-ahead response where we find more time variation around the Volcker disinflation in the late 1970s and early 1980s. When we consider stock market reactions we do not observe a systematic pattern in terms of the direction of the IRFs. But what is visible is that during recessions, posterior uncertainty increases appreciably.

For most of the variables under consideration no discernible differences in asymmetries with respect to changing the model size are visible. The main exception is the reaction of the short-term interest rate which displays much more time variation in the small-scale model as opposed to larger-sized models. This can be attributed to the fact that if we include a large information set we control for observed heterogeneity while in the case of a smaller information set (such as the one based on eight variables) the Gaussian Process picks up unobserved heterogeneity and thus generates time variation in the responses.

Up to this point the discussion has been rather informal. We have focused on specific patterns over time and considered overall movements in the IRFs. To provide more formal evidence of time variation in the responses, and to assess whether there is any evidence of time variation in the (a)symmetry and (non-)proportionality in the responses, we use two measures similar to Olivei and Tenreyro (2010). For the case of (a)symmetry the measures are:

$$D_t = \max \| \delta^+_{ht} + \delta^-_{ht} \|_1 \quad \text{and}$$

$$CD_t = \| \sum_{h=0}^{16} (\delta^+_{ht} + \delta^-_{ht}) \|_1$$

(2)

where $\delta^+_{ht}$ denotes the impulse response to a positive shock and $\delta^-_{ht}$ denotes the impulse responses to a negative shock. The measures of asymmetry represent the maximum difference and the accumulated difference in the responses, respectively. In the special case of a conditionally
linear model (conditional on $t$) both $D_t$ and $CD_t$ are equal zero, since $\delta^{-}_{ht} = -\delta^{+}_{ht}$ irrespective of $h$. The $D_t$ and $CD_t$ measures can be easily modified to assess the extent and pattern of time variation (comparing $\delta_{ht}$ with $\delta_h$, where $\delta_h$ is the average over time of $\delta_{ht}$) and of (non-)proportionality (comparing $2 \times \delta_{ht}$ with $\gamma_{ht}$, where $\gamma_{ht}$ denotes the impulse response to a 2 times larger shock than in the baseline).

In Figure 12 we report the $D_t$ and $CD_t$ measures for the various variables, computed over time and for the different GP-VAR models sizes. The benchmark we consider are the average GIRFs, so that the measures should be zero in case of no substantial departure of the time $t$ IRFs from the time average. Overall, and for each model size, there seems to emerge substantial evidence in favor of time variation. However, it is difficult to tell a clear-cut story about the precise timing of the differences. For some variables, such as uncertainty itself, average hours worked and CPI inflation, the differences with respect to the time invariant case seem to be larger on average in the first part of the sample (up to around 1985). For other variables this pattern is not so evident but there are clear periods where the differences get sizable and significant, for example for the S&P500, average earnings and employment.

Figure 13 and Figure 14 report the asymmetry measures for positive/negative uncertainty shocks and for different shock sizes for the various variables, computed over time and for the different GP-VAR models sizes. As for the previous figure, and contrary to the results averaged over time, there is substantial evidence in favor of sign asymmetry, with its importance being time-varying but no clear-cut pattern across variables or time. Yet, interestingly, for most variables the size asymmetry measures increase substantially around the financial crisis, and they also reached similar values in previous recessionary episodes. This suggests that the size of the uncertainty shocks matter and, since uncertainty shocks are larger during recessions, it implies that they can be particularly harmful in these periods.

5 Conclusions

In this paper, we have developed a flexible multivariate model that uses Gaussian Processes to model the unknown relationship between a panel of macroeconomic time series and their lagged values. Our GP-VAR is a very flexible model which remains agnostic on the precise relations between the endogenous variables and the predictors. This model can be viewed as a very flexible and general extension of the linear VAR commonly used in empirical macroeconomics. We also control for changes in the error variances by introducing a stochastic volatility specification. While a more flexible conditional mean can reduce the need of a time-varying conditional
Figure 12: Asymmetries of target variables over time.

Notes: Asymmetry measures (defined in Eq. 2) are based on time-specific generalized impulse responses (GIRFs, outlined in Sub-section 2.6) and benchmarked against the respective average GIRFs, conditional on a positive one standard deviation shock in uncertainty. Solid lines denote the posterior median, while shaded areas correspond to the 84% credible sets.
Figure 13: Shock sign asymmetry measures of target variables over time.

Notes: Asymmetry measures (defined in Eq. 2) are based on time-specific generalized impulse responses (GIRFs, outlined in Sub-section 2.6) to a negative (positive) one standard deviation shock in uncertainty. Solid lines denote the posterior median, while shaded areas correspond to the 84% credible sets.
Figure 14: Shock size asymmetry measures of target variables over time.

a) $D_t$

b) $DC_t$

Notes: Asymmetry measures (defined in Eq. 2) are based on time-specific generalized impulse responses (GIRFs, outlined in Sub-section 2.6) to a positive two (one) standard deviation shock in uncertainty. Solid lines denote the posterior median, while shaded areas correspond to the 84% credible sets.
variance, empirically we find heteroskedasticity to be relevant also for GP-VARs.

We develop efficient MCMC estimation algorithms for the GP-VAR, which are scalable to high dimensions, so much so that for large models estimation is even faster than for the corresponding BVAR-SV. Scaling the covariance of the Gaussian Process by the latent volatility factors is particularly helpful to achieve computation gains, as it permits to pre-compute several quantities before MCMC sampling. This speeds up computation enormously.

To illustrate the practical working of the GP-VAR, we first test it on simulated data from different linear and non-linear models, finding that it is capable of reproducing a variety of non-linear patterns (but also a linear behavior). Then, we re-assess the effects of uncertainty shocks by replicating and extending with the GP-VAR the analysis carried out by Jurado, Ludvigson, and Ng (2015) based on linear VARs. Overall, the empirical results we get suggest that the measurement of uncertainty and its effects with a simple linear VAR can lead to several incorrect conclusions. Not only the effects of uncertainty can be over-stated, but they can also be treated as stable over time, symmetric for positive and negative shocks, and proportional to the shock size. Instead, the GP-VAR model, which is preferred to the linear VAR by the WAIC information criteria, returns time variation in the responses, asymmetry and non-proportionality. These findings emerge for all model sizes, from 8 to 64 variables, with an intermediate 16 variable GP-VAR preferred by the information criteria. Hence, the empirical features we uncover should be also replicated by theoretical models about uncertainty and its effects, which instead at the moment typically assume stability and symmetry (see, e.g., the survey in Bloom (2014)).
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## A Data

### Table A.1: Data description.

| FRED-Mnemonic | Description | Trans. | VAR-8 | VAR-16 | VAR-32 | VAR-64 |
|---------------|-------------|--------|-------|--------|--------|--------|
| GDPCH1        | Real Gross Domestic Product | 2     | x     | x      | x      | x      |
| CHGEV (REMP)  | Civilian Employment (Thousand of Persons) | 2     | x     | x      | x      | x      |
| ANVON (AVR)   | Average Weekly Hours of Production and Nonfarm Payroll Employees: Manufacturing | 1     | x     | x      | x      | x      |
| CTDSS6000000008hX (AHR) | Real Average Hourly Earnings of Production and Nonfarm Payroll Employees: Manufacturing | 2     | x     | x      | x      | x      |
| FEDFUNDS (FFR) | Effective Federal Funds Rate (Percent) | 2     | x     | x      | x      | x      |
| S.P.500 (SFT506) | S&P’s Common Stock Price Index: Composite | 3     | x     | x      | x      | x      |
| FCCEEDC6      | Real Personal Consumption Expenditures | 2     | x     | x      | x      | x      |
| FPI           | Real private fixed investment | 2     | x     | x      | x      | x      |
| UNRATE        | Civilian Unemployment Rate (Percent) | 1     | x     | x      | x      | x      |
| CTDSS6000000008d | Average Hourly Earnings of Production and Nonfarm Payroll Employees: Goods-Producing | 2     | x     | x      | x      | x      |
| CLAIMS4000000004X | Initial Claims | 2     | x     | x      | x      | x      |
| HOUST         | Housing Starts: Total; New Privately Owned Housing Units Started | 2     | x     | x      | x      | x      |
| CTDSS6000000008d | Average Hourly Earnings of Production and Nonfarm Payroll Employees: Construction | 2     | x     | x      | x      | x      |
| MIREAL        | Real M2 Money Stock | 1     | x     | x      | x      | x      |
| OGC21        | Real Government Consumption Expenditures and Gross Investment | 2     | x     | x      | x      | x      |
| INDPRO        | IP Total Index Industrial Production Index (Index 2012=100) | 2     | x     | x      | x      | x      |
| CUMFNS        | Capacity Utilization: Manufacturing (SEC) (Percent of Capacity) | 1     | x     | x      | x      | x      |
| FAYEMS        | Emp:Nonfarm All Employee: Total nonfarm (Thousands of Persons) | 2     | x     | x      | x      | x      |
| FERMST        | New Private Housing Units Authorized by Building Permits | 2     | x     | x      | x      | x      |
| FEREFTX       | Personal Consumption Expenditures: Chain-type Price Index | 2     | x     | x      | x      | x      |
| GDCPTFX       | Gross Domestic Product: Chain-type Price Index | 2     | x     | x      | x      | x      |
| CTDSS6000000008dX | Real Average Hourly Earnings of Production and Nonfarm Payroll Employees: Construction | 2     | x     | x      | x      | x      |
| BBAA0VM       | Moody’s Seasoned Aaa Corporate Bond Yield Relative to Yield on 10-Year Treasury | 1     | x     | x      | x      | x      |
| GT30FRM       | 30-Year Treasury Constant Maturity Muni 3-Month Treasury Bill, secondary market | 1     | x     | x      | x      | x      |
| TS30FRM       | 3-Month Treasury Constant Maturity Muni Federal Funds Rate | 1     | x     | x      | x      | x      |
| AASS0FX       | Moody’s Seasoned Aaa Corporate Bond Muni Federal Funds Rate | 1     | x     | x      | x      | x      |
| BUSLOANx      | Real Commercial and Industrial Loans, All Commercial Banks | 2     | x     | x      | x      | x      |
| CONSUMERx     | Real Consumer Loans at All Commercial Banks | 2     | x     | x      | x      | x      |
| NONRESX       | Total Real Nonresidential Construction Put in Place | 2     | x     | x      | x      | x      |
| NONREVX       | Real Nonresidential Construction Put in Place | 2     | x     | x      | x      | x      |
| NONBorX       | Reserves Of Depository Institutions, Nonborrowed | 2     | x     | x      | x      | x      |
| GPDC21        | Real Gross Private Domestic Investment | 2     | x     | x      | x      | x      |
| PFNfx         | Real private fixed investment: Residential | 2     | x     | x      | x      | x      |
| PRFfx         | Real private fixed investment: Nonresidential | 2     | x     | x      | x      | x      |
| EXFOC21       | Real Exports of Goods and Services | 2     | x     | x      | x      | x      |
| IMFDC21       | Real Imports of Goods and Services | 2     | x     | x      | x      | x      |
| IPCCONDG      | IP-Consumer goods Industrial Production: Consumer Goods (Index 2012=100) | 2     | x     | x      | x      | x      |
| UNRATELTx     | Unemployment Rate for more than 27 weeks (Percent) | 1     | x     | x      | x      | x      |
| ATOTMAN       | Average Weekly Overtime Hours of Production and Nonfarm Payroll Employees: Manufacturing | 1     | x     | x      | x      | x      |
| INDHH         | Real Manufacturing’s New Orders: Durable Goods (Millions of 2012 Dollar) | 2     | x     | x      | x      | x      |
| CHIC21        | Real Private Domestic Investment: Chain-type Price Index | 2     | x     | x      | x      | x      |
| DCHS0896SQ65BEXA | Personal consumption expenditures: Goods | 2     | x     | x      | x      | x      |
| DCHS0896SQ65BSBA | Personal consumption expenditures: Services | 2     | x     | x      | x      | x      |
| MCFRES        | Consumer Price Index for All Urban Consumers: All Items Less Food & Energy | 2     | x     | x      | x      | x      |
| COFRES4       | Consumer Price Index for All Urban Consumers: Core (less food, energy, and shelter) | 2     | x     | x      | x      | x      |
| COMPRNBFR     | Nonfarm Business Sector: Real Compensation Per Hour (Index 2012=100) | 2     | x     | x      | x      | x      |
| BCH3000000008X | Business Sector: Real Compensation Per Hour (Index 2012=100) | 2     | x     | x      | x      | x      |
| TB3MS         | 3-Month Treasury Bill: Secondary Market Rate (Percent) | 1     | x     | x      | x      | x      |
| TB3MN         | 3-Month Treasury Bill: Secondary Market Rate (Percent) | 1     | x     | x      | x      | x      |
| GT30          | 30-Year Treasury Constant Maturity Rate (Percent) | 1     | x     | x      | x      | x      |
| AAA           | Moody’s Seasoned Aaa Corporate Bond Yield (Percent) | 1     | x     | x      | x      | x      |
| BBAA0X        | Moody’s Seasoned Aaa Corporate Bond Yield (Percent) | 1     | x     | x      | x      | x      |
| TS30FRM       | 3-Month Treasury Constant Maturity Muni Federal Funds Rate | 1     | x     | x      | x      | x      |
| S.P.INDX      | S&P’s Common Stock Price Index: Industrials | 2     | x     | x      | x      | x      |
| S.P.DYDX      | S&P’s Common Stock Price Index: Dividend Yield | 1     | x     | x      | x      | x      |

**Notes:** 

- **Trans.** indicates the transformation applied to each variable with (1) implying no transformation, (2) denoting year-on-year growth rates, (3) denoting quarter-on-quarter growth rates, and (4) refers to quarter-on-quarter percentage changes.