Positivity constraints on initial spin observables in inclusive reactions

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Abstract

For any inclusive reaction of the type $A_1(\text{spin1}/2) + A_2(\text{spin1}/2) \rightarrow B + X$, we derive new positivity constraints on spin observables and study their implications for theoretical models in view, in particular, of accounting for future data from the polarized $pp$ collider at BNL-RHIC. We find that the single transverse spin asymmetry $A_N$, in the central region for several processes, for example jet production, direct photon production, lepton-pair production, is expected to be such that $|A_N| \lesssim 1/2$, rather than the usual bound $|A_N| \leq 1$.

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Let us consider an inclusive reaction of the type

\[ A_1(\text{spin}1/2) + A_2(\text{spin}1/2) \rightarrow B + X , \quad (1) \]

where the spins of both initial spin1/2 particles can be in any possible directions and no polarization is observed in the final state. The observables of this reaction, which are the spin-dependent differential cross sections with respect to the momentum of \( B \), can be expressed in terms of the discontinuities (with respect to the invariant mass squared of \( X \)) of the amplitudes for the forward three-body scattering

\[ A_1 + A_2 + \overline{B} \rightarrow A_1 + A_2 + \overline{B} , \quad (2) \]

as given by the generalized optical theorem. We assume parity conservation, so the complete knowledge of this reaction requires the determination of eight real functions, which is the number of independent spin observables [1]. In order to define these observables, we recall the standard notation used in Ref. [2] \( (A_1A_2|BX) \), by which the spin directions of \( A_1, A_2, B \) and \( X \) are specified in one of the three possible directions \( L, N, S \). Since the final spins are not observed, we have in fact \( (A_1A_2|00) \) and \( L, N, S \) are unit vectors, in the center-of-mass system, along the incident momentum, along the normal to the scattering plane which contains \( A_1, A_2 \) and \( B \), and along \( N \times L \), respectively. In addition to the unpolarized cross section \( \sigma_0 = (00|00) \), there are seven spin dependent observables, two single transverse spin asymmetries

\[ A_{1N} = (N0|00) \quad \text{and} \quad A_{2N} = (0N|00) , \quad (3) \]

and five double-spin asymmetries

\[ A_{LL} = (LL|00) , \quad A_{SS} = (SS|00) , \quad A_{NN} = (NN|00) , \quad (4) \]

\[ A_{LS} = (LS|00) \quad \text{and} \quad A_{SL} = (SL|00) . \]

The state of polarization of the two spin1/2 particles \( A_1 \) and \( A_2 \) is characterized by the \( 2 \times 2 \) density matrices \( \rho_1 \) and \( \rho_2 \) defined as

\[ \rho_i = 1/2(\mathbb{1}_2 + e_i \cdot \sigma) \quad i = 1,2 , \quad (5) \]
where $e_1$ and $e_2$ are the polarization unit vectors of $A_1$ and $A_2$, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ stands for the three $2 \times 2$ Pauli matrices and $\mathbb{I}_2$ is the $2 \times 2$ unit matrix. The state of polarization of the incoming system in the reaction (1) is described by the $4 \times 4$ density matrix $\rho$, which is the direct product $\rho = \rho_1 \otimes \rho_2$.

The spin-dependent cross section corresponding to (1) is

$$\sigma(e_1, e_2) = \text{Tr}(M \rho),$$

where $M$ denotes the $4 \times 4$ cross section matrix which we shall parametrize in the following way

$$M = \sigma_0[1 + A_{1N}e_{1z} \otimes \mathbb{I}_2 + A_{2N}e_{2z} \otimes \mathbb{I}_2 + A_{NN}e_{1z} \otimes e_{2z} + A_{LL}e_{1x} \otimes e_{2x} + A_{SS}e_{1y} \otimes e_{2y} + A_{LS}e_{1x} \otimes e_{2y} + A_{SL}e_{1y} \otimes e_{2x}].$$

Here $\mathbb{I}_4$ is the $4 \times 4$ unit matrix and $\sigma_0$ stands for the spin-averaged cross section. This expression is fully justified, since we have explicitly

$$\sigma(e_1, e_2) = \sigma_0 [1 + A_{1N}e_{1z} + A_{2N}e_{2z} + A_{NN}e_{1z}e_{2z} + A_{LL}e_{1x}e_{2x} + A_{SS}e_{1y}e_{2y} + A_{LS}e_{1x}e_{2y} + A_{SL}e_{1y}e_{2x}].$$

The crucial point is that $M$ is a Hermitian and positive matrix and in order to derive the positivity conditions one should write the explicit expression of $M$ as given by Eq. (7). Then one observes that by permuting two rows and two columns, it reduces to the simple form

$$\begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix},$$

where $M_{\pm}$ are $2 \times 2$ Hermitian matrices which must be positive, leading to the following two strongest constraints

$$(1 \pm A_{NN})^2 \geq (A_{1N} \pm A_{2N})^2 + (A_{LL} \pm A_{SS})^2 + (A_{LS} \pm A_{SL})^2.$$
As special cases of Eq. (9), we have the six weaker constraints

\[ 1 \pm A_{NN} \geq |A_{1N} \pm A_{2N}|, \quad (10) \]

\[ 1 \pm A_{NN} \geq |A_{LL} \pm A_{SS}|, \quad (11) \]

and

\[ 1 \pm A_{NN} \geq |A_{LS} \pm A_{SL}|. \quad (12) \]

These constraints are very general \(^\S\) and must hold in any kinematical region and for many different situations such as electron–proton scattering, electron–positron scattering or quark–quark scattering, but we now turn to a specific case, which is of direct relevance to the spin programme at the BNL-RHIC polarized \(pp\) collider [7]. Now let us consider a proton–proton collision and let us call \(y\) the rapidity of the outgoing particle \(B\). In this case since the initial particles are identical, we have \(A_{1N}(y) = A_{2N}(-y)\) and \(A_{LS}(y) = A_{SL}(-y)\) \(^\P\). In this case Eq. (9), which becomes two constraints among five independent spin observables, reads

\[
(1 \pm A_{NN}(y))^2 \geq (A_{1N}(y) \pm A_{1N}(-y))^2 + (A_{LL}(y) \pm A_{SS}(y))^2 + (A_{LS}(y) \pm A_{LS}(-y))^2.
\]

(13)

This implies in particular, for \(y = 0\),

\[ 1 + A_{NN}(0) \geq 2|A_N(0)|, \quad (14) \]

and

\[ 1 + A_{NN}(0) \geq 2|A_{SL}(0)|, \quad (15) \]

\(^\S\)Let us consider three spin asymmetries whose values lie between -1 and +1. For a simultaneous measurement of these three spin observables, the allowed region in a three-dimensional plot is a cube of volume \(2^3 = 8\). However it can be shown that inequalities like Eqs. (10,11,12) reduce strongly the allowed region, to a three- dimensional polygon of volume 8/3. I thank J.M. Richard for this interesting observation.

\(^\P\)I thank J.C. Collins for drawing my attention to this point and J.M. Virey for a clarifying discussion.
so that, the allowed range of $A_N$ and $A_{SL}$ is strongly reduced, if $A_{NN}$ turns out to be large and negative. Conversely if $A_{NN} \approx 1$, these constraints are useless. Note that, in the kinematical region accessible to the $pp$ polarized collider, a calculation of $A_{NN}$ for direct photon production and jet production has been performed [8]; it was found that $|A_{NN}|$ is of the order of 1 or 2%. Similarly, based on Ref. [9], this double transverse spin asymmetry for lepton pair production was estimated to be a few per-cent [10]. The direct consequence of these estimates is that $|A_N|$ and $|A_{SL}|$, for these processes\textsuperscript{\textdagger}, are essentially bounded by 1/2. In addition, from Eq. (11), there are two other non-trivial constraints: $1 \geq |A_{LL} \pm A_{SS}|$.

Single transverse spin asymmetries in inclusive reactions at high energies are now considered to be directly related to the transverse momentum of the fundamental partons involved in the process. This new viewpoint, which has been advocated to explain the existing data in semi-inclusive deep inelastic scattering [12, 13], will have to be more firmly established also by means of future data from BNL-RHIC. On the theoretical side several possible leading-twist QCD mechanisms [14, 15] have been proposed to generate these asymmetries in leptoproduction [16, 17], but also in $pp$ collisions. We believe that these new positivity constraints on spin observables for a wide class of reactions will be of interest for model builders as well as for future measurements.

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\textsuperscript{\textdagger}It is amusing to recall that, using a phenomenological approach for lepton-pair production, bounds on $|A_N|$ larger than 50% were obtained in Ref. [11], but at that time it was not known that $A_{NN}$ is small.
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