A QUANTITATIVE WEINSTOCK INEQUALITY

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Abstract. The paper is devoted to the study of a quantitative Weinstock inequality in higher dimension for the first non trivial Steklov eigenvalue of Laplace operator for convex sets. The key rule is played by a quantitative isoperimetric inequality which involves the boundary momentum, the volume and the perimeter of a convex open set of \( \mathbb{R}^n, n \geq 2 \).

Keywords: Steklov eigenvalue, isoperimetric inequality, convex sets.

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1. Introduction

Let \( \Omega \subset \mathbb{R}^n \), with \( n \geq 2 \), be a bounded connected open set with Lipschitz boundary. In this paper we consider the following Steklov eigenvalue problem for the Laplace operator

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial \Omega,
\end{cases}
\]

where \( \partial u/\partial \nu \) is the outer normal derivative to \( u \) on \( \partial \Omega \). It is well-known (see for instance [2, 14, 4]) that the spectrum is discrete then there exists a sequence of eigenvalues, \( 0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots \uparrow +\infty \) called Steklov eigenvalues of \( \Omega \). In particular, the first non trivial Steklov eigenvalue of \( \Omega \) has the following variational characterization:

\[
\sigma_1(\Omega) = \min \left\{ \frac{\int_{\Omega} |\nabla v|^2 \, dx}{\int_{\partial \Omega} v^2 \, dH^{n-1}} : v \in H^1(\Omega) \setminus \{0\}, \int_{\partial \Omega} v \, dH^{n-1} = 0 \right\},
\]

where \( H^{n-1} \) denotes the \((n-1)\)-dimensional Hausdorff measure in \( \mathbb{R}^n \). When \( \Omega = B_R \) is a ball with radius \( R \) then

\[
\sigma_1(B_R) = \frac{1}{R},
\]

moreover it has multiplicity \( n \) and the corresponding eigenfunctions are \( u_i(x) = x_{i-1} \), with \( i = 2, \ldots, n+1 \). In [17, 18] the author considers the problem of maximizing \( \sigma_1(\Omega) \) in the plane keeping...
fixed the perimeter of $\Omega$. More precisely, if $\Omega \subset \mathbb{R}^2$ is a simply connected open set, he proved the following so-called Weinstock inequality

\begin{equation}
\sigma_1(\Omega) P(\Omega) \leq \sigma_1(B_R) P(B_R),
\end{equation}

where $P(\Omega)$ denotes the Euclidean perimeter of $\Omega$. Inequality (1.4) states that, among all the planar simply connected open sets with prescribed perimeter, $\sigma_1(\Omega)$ is maximum for the disk. In [7], the authors generalize the Weinstock inequality (1.4) in any dimension in the class of convex sets. More precisely they prove that if $\Omega \subset \mathbb{R}^n$ is a bounded, convex open set, then

\begin{equation}
\sigma_1(\Omega) P(\Omega) \leq \sigma_1(B_R) P(B_R),
\end{equation}

and equality holds only if $\Omega$ is a ball. In [6] the author investigated and solved the problem of maximizing $\sigma_1(\Omega)$ keeping the volume fixed. He proved that

\begin{equation}
\sigma_1(\Omega) V(\Omega) \leq \sigma_1(B_R) V(B_R),
\end{equation}

where $V(\Omega)$ denotes the Lebesgue measure of $\Omega$. Recently in [4] a quantitative version of inequality (1.6) is proved. The aim of this paper is to prove a quantitative version of inequality (1.5). Let $\omega_n$ be the measure of the $n$-dimensional unit ball in $\mathbb{R}^n$ and let $d_H$ be the Hausdorff distance (defined in (2.5)). We consider the following asymmetry functional

\begin{equation}
A_H(\Omega) = d_H(\Omega, \Omega^*) \left( \frac{P(\Omega)}{n \omega_n} \right)^{\frac{n+1}{n-1}},
\end{equation}

where $\Omega \subset \mathbb{R}^n$ is a bounded open, convex set and $\Omega^*$ is the ball centered at the origin with $P(\Omega) = P(\Omega^*)$. We observe that $A_H(\Omega)$ is scaling invariant, hence

\[ A_H(\Omega) = A_H(F), \]

where $F$ is a convex set having the same perimeter of the unit ball, that is $P(F) = n \omega_n$. Our main result is the following:

**Theorem 1.1.** Let $n \geq 2$. There exist two constants $\delta > 0$ and $C = C(n) > 0$ such that if $\Omega$ is a bounded, convex open set of $\mathbb{R}^n$ with $\sigma_1(\Omega^*) \leq (1 + \delta) \sigma_1(\Omega)$, then

\begin{equation}
\frac{\sigma_1(\Omega^*) - \sigma_1(\Omega)}{\sigma_1(\Omega)} \geq \begin{cases} C (A_H(\Omega))^{\frac{2}{n+1}} & \text{if } n = 2 \\ C g \left( (A_H(\Omega))^2 \right) & \text{if } n = 3 \\ C (A_H(\Omega))^{\frac{2}{n+1}} & \text{if } n \geq 4 \end{cases}
\end{equation}

where $g$ is the inverse function of $f(x) = x \log \left( \frac{x}{x-1} \right)$, for $0 < x < e^{-1}$.

The key point in order to prove Theorem 1.1, is a quantitative version of a weighted isoperimetric inequality (see Theorem 3.6 for the precise statement).

2. Notation and Preliminary results

2.1. Notation and some definitions. Through the paper the unit ball centered at the origin will be denoted by $B$ and its boundary by $\mathbb{S}^{n-1}$, the ball of radius $R$ centered at the origin will be denoted by $B_R$ while the ball centered at $x$ and radius $R$ will be denoted by $B_R(x)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $E \subset \mathbb{R}^n$ be a measurable set. For the sake of completeness, we recall here the definition of the perimeter of $E$ in $\Omega$:

\[ P(E; \Omega) = \sup \left\{ \int_E \text{div}\phi \, dx : \phi \in C^\infty_c(\Omega; \mathbb{R}^n), \|\phi\|_\infty \leq 1 \right\}. \]
The perimeter of $E$ in $\mathbb{R}^n$ will be denoted by $P(E)$ and, if $P(E) < \infty$, we say that $E$ is a set of finite perimeter. If $E$ has Lipschitz boundary, then

\begin{equation}
(2.1) \quad P(E) = \mathcal{H}^{n-1}(\partial E),
\end{equation}

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

We denote by

\begin{equation}
(2.2) \quad V(E) = \int_E dx
\end{equation}

the volume of the measurable set $E \subseteq \mathbb{R}^n$, i.e. its $n$-dimensional Lebesgue measure and, if $E$ has Lipschitz boundary, by

\begin{equation}
(2.3) \quad W(E) = \int_{\partial E} |x|^2 d\mathcal{H}^{n-1}
\end{equation}

we denote the boundary momentum of $E$, where $| \cdot |$ is the euclidean norm in $\mathbb{R}^n$. We observe that $P(E), W(E)$ and $V(E)$ have the following scaling properties for $t > 0$

\begin{equation}
(2.4) \quad P(tE) = t^{n-1}P(E) \quad V(tE) = t^nV(E) \quad W(tE) = t^{n+1}W(E)
\end{equation}

Finally, we recall the definition of the Hausdorff distance between two nonempty compact sets $E, F \subset \mathbb{R}^n$ that is (see for instance [16])

\begin{equation}
(2.5) \quad d_H(E, F) = \inf \{ \epsilon > 0 : E \subset F + B_\epsilon, \ F \subset E + B_\epsilon \}
\end{equation}

Note that for convex sets $d_H(E, F) = d_H(\partial E, \partial F)$ and the following rescaling property holds

\[ d_H(tE, tF) = t d_H(E, F), \quad t > 0. \]

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, convex set and $\Omega^*$ the ball centered at the origin with $P(\Omega) = P(\Omega^*)$. We consider the following asymmetry functional related to $\Omega$

\begin{equation}
(2.6) \quad A_H(\Omega) = d_H(\Omega, \Omega^*) \left( \frac{P(\Omega)}{n\omega_n} \right)^{-\frac{1}{n-1}}
\end{equation}

**Definition 2.1.** Let $\Omega \subseteq \mathbb{R}^n$, be a bounded, open set and let $(E_j) \subset \mathbb{R}^n$ be a sequence of measurable sets and let $E \subset \mathbb{R}^n$ be a measurable set. We say that $(E_j)$ converges in measure in $\Omega$ to $E$, and we write $E_j \rightarrow E$, if $\chi_{E_j} \rightarrow \chi_E$ in $L^1(\Omega)$, or in other words if $\lim_{j \rightarrow \infty} V((E_j \Delta E) \cap \Omega) = 0$.

We recall also that the perimeter is lower semicontinuous with respect to the local convergence in measure, i.e. if the sequence of sets $(E_j)$ converges in measure in $\Omega$ to $E$, then

\[ P(E; \Omega) \leq \liminf_{j \rightarrow \infty} P(E_j; \Omega). \]

As a consequence of the Rellich-Kondrachov theorem we have this result of compactness, for a reference see for instance [1].

**Proposition 2.2.** Let $\Omega \subseteq \mathbb{R}^n$, be a bounded, open set and let $(E_j)$ be a sequence of measurable sets of $\mathbb{R}^n$ such that $\sup_j P(E_j; \Omega) < \infty$. Then, there exists a subsequence $(E_{j_k})$ converging in measure in $\Omega$ to a set $E$, such that

\[ P(E; \Omega) \leq \liminf_{k \rightarrow \infty} P(E_{j_k}; \Omega). \]

Another useful property of sets of finite perimeter is stated in the next approximation result

**Proposition 2.3.** Let $\Omega \subseteq \mathbb{R}^n$, be a bounded, open set, and let $E$ be a set of finite perimeter in $\Omega$. Then, there exists a sequence of smooth, bounded open sets $(E_j)$ converging in measure in $\Omega$ and such that $\lim_{j \rightarrow \infty} P(E_j; \Omega) = P(E; \Omega)$. 

For convex sets in particular holds the following

**Lemma 2.4.** Let \((E_j) \subseteq \mathbb{R}^n\) be a sequence of convex sets such that \(E_j \to B\) in measure, then
\[
\lim_{j \to \infty} P(E_j) = P(B).
\]

**Proof.** Since for convex sets the convergence in measure implies the Hausdorff convergence we have that \(\lim_{j \to \infty} d_H(E_j, B) = 0\) (see for instance [9]). Thus, for \(j\) big enough, there exists \(\epsilon_j\) such that
\[
(1 - \epsilon_j)E_j \subset B \subset (1 + \epsilon_j)E_j
\]
Since \(E_j\) are convex the perimeter is monotone respect to the inclusion then
\[
(1 - \epsilon_j)^{n-1}P(E_j) \leq P(B) \leq (1 + \epsilon_j)P(E_j)
\]
Letting \(j\) to infinity we have the desired result. \(\square\)

We conclude this paragraph recalling the following result (see [9]).

**Lemma 2.5.** Let \(K \subseteq \mathbb{R}^n, n \geq 2\), be a bounded, open convex set. There exists a positive constant \(C(n)\) such that
\[
\text{diam}(K) \leq C(n) \frac{P(K)^{n-1}}{V(K)^{n-2}}.
\]

2.2. **Nearly spherical sets.** In this section we give the definition of nearly spherical sets we consider in this paper and we recall some their basic properties (see for instance [5, 11, 12]).

**Definition 2.6.** Let \(n \geq 2\). An open bounded set \(E \subset \mathbb{R}^n\) is said nearly spherical parametrized by \(v\) if there exists \(v \in W^{1,\infty}(S^{n-1})\) such that
\[
\partial E = \{y \in \mathbb{R}^n: y = x(1 + v(x)), x \in S^{n-1}\}
\]
with \(||v||_{W^{1,\infty}} \leq \frac{1}{2}\).

Note also that \(||v||_L^\infty = d_H(E, B)\). The perimeter, the volume and the boundary momentum of the nearly spherical sets are given by
\[
P(E) = \int_{S^{n-1}} (1 + v(x))^{n-2} \sqrt{(1 + v(x))^2 + |Dv(x)|^2};
\]
\[
V(E) = \frac{1}{n} \int_{S^{n-1}} (1 + v(x))^n \, dH^{n-1}(x);
\]
\[
W(E) = \int_{S^{n-1}} (1 + v(x))^n \sqrt{(1 + v(x))^2 + |Dv(x)|^2}.
\]

Finally we recall two lemmas that we will use later. The first one is an interpolation result, for its proof we refer for instance to [11, 12].

**Lemma 2.7.** If \(v \in W^{1,\infty}(S^{n-1})\) and \(\int_{S^{n-1}} v \, dH^{n-1} = 0\), then
\[
||v||_{L^\infty(S^{n-1})}^{-1} \leq \begin{cases}
\pi ||Dv||_{L^2(S^{n-1})} & n = 2 \\
4||Dv||_{L^2(S^{n-1})}^2 \log (\frac{8\pi}{||Dv||_{L^\infty(S^{n-1})}}) & n = 3 \\
C(n) ||Dv||_{L^2(S^{n-1})}^2 ||Dv||_{L^\infty(S^{n-1})}^{-3} & n = 4
\end{cases}
\]

For this second lemma we refer for instance to [12].
Lemma 2.8. Let $n \geq 2$. There exists $\varepsilon_0$ such that if $E$ is a convex nearly spherical set with $V(E) = V(B)$ and $\|v\|_{W^{1,\infty}} \leq \varepsilon_0$, then

\begin{equation}
\|D_v v\|_{L^2}^2 \leq 8 \|v\|_{L^\infty}.
\end{equation}

Finally we prove the following

Lemma 2.9. Let $n \geq 2$ and let $E \subseteq \mathbb{R}^n$ be a bounded, convex nearly spherical set with $\|v\|_{W^{1,\infty}} \leq \delta$, then

\begin{equation}
d_H(E, E^*) \leq C(n)d_H(E, E^x),
\end{equation}

where $E^*$ and $E^x$ are the balls centered at the origin having respectively the same perimeter and the same volume than $E$

Proof. By properties of the Hausdorff distance we get

\begin{equation}
d_H(E, E^*) \leq d_H(E, E^x) + d_H(E^x, E^x) = d_H(E, E^x) + \left( \frac{P(E)}{n\omega_n^{1/n}} \right)^{\frac{1}{n-1}} - \left( \frac{V(E)}{\omega_n} \right)^\frac{1}{n-1} \leq d_H(E, E^x) + \left( \frac{V(E)}{\omega_n} \right)^\frac{1}{n-1} \left[ \left( \frac{P(E)}{n\omega_n^{1/n}} \right)^{\frac{1}{n-1}} - 1 \right].
\end{equation}

We stress that in the square brackets we have the isoperimetric deficit of $E$ which is scaling invariant. Let $F \subseteq \mathbb{R}^n$ be a convex nearly spherical set parametrized by $v_F$ with $\|v_F\|_{W^{1,\infty}} \leq \delta$ and $V(F) = V(B)$. Being $F$ nearly spherical and $\|v_F\|_{W^{1,\infty}} \leq \delta$, from (2.9) and Lemma 2.8 we get

\begin{equation}
\left( \frac{P(F)}{n\omega_n^{1/n} V(F)^{\frac{n-1}{n}}} \right)^{\frac{1}{n-1}} - 1 = \left( \frac{P(F)}{n\omega_n} \right)^{\frac{1}{n-1}} - 1 = \left( \frac{1}{n\omega_n} \int_{S^{n-1}} (1 + v_F(x))^{n-2} \sqrt{1 + v_F(x)^2} |D_v v_F(x)|^2 \right)^{\frac{1}{n-1}} - 1 \leq C(n)\|v_F\|_{W^{1,\infty}} \leq C(n)\|v_F\|_{L^\infty}.
\end{equation}

Then recalling that $\|v_F\|_{L^\infty} = d_H(F, B)$

\begin{equation}
\left( \frac{V(E)}{\omega_n} \right)^\frac{1}{n-1} \left[ \left( \frac{P(E)}{n\omega_n^{1/n} V(E)^{\frac{n-1}{n}}} \right)^{\frac{1}{n-1}} - 1 \right] \leq C(n)d_H(E, E^x)
\end{equation}

Substituting in (2.15), we get the claim. 

\section{An Isoperimetric Inequality}

In [6] the author proves a weighted isoperimetric inequality where the perimeter is replaced by the boundary momentum $W(E)$ defined in (2.3). More precisely he proves that if $E \subseteq \mathbb{R}^n$ is a Lipschitz set, then

\begin{equation}
\frac{W(E)}{V(E)^{\frac{n-1}{n}}} \geq \frac{W(B)}{V(B)^{\frac{n-1}{n}}} = n\omega_n^{-1/n},
\end{equation}

and the equality holds for a ball. Inequalities (3.1) implies that among sets with fixed volume, the boundary momentum and the perimeter are both minimal on balls.

In [17] in the planar case, and then in [7] in any dimension, the authors show that, for convex sets, an isoperimetric inequality for a functional involving the quantities $P(E)$, $W(E)$ and $V(E)$
holds true. More precisely, if $E \subseteq \mathbb{R}^n$ is a bounded, open convex set, they prove the following inequality
\begin{equation}
\mathcal{J}(E) = \frac{W(E)}{P(E) V(E)^{\frac{n}{n-1}}} \geq \frac{\omega_n}{\omega_{n-1}} = \omega_n^{\frac{n}{n-1}} = \mathcal{J}(B)
\end{equation}
where equality holds only on balls centered at the origin.

In the same spirit, if $F \subseteq \mathbb{R}^n$ is a bounded, open, convex set, we define the following functional
\begin{equation}
I(F) = \frac{W(F)}{V(F) P(F)^{\frac{n}{n-1}}}
\end{equation}
The following isoperimetric inequality holds

**Proposition 3.1.** Let $n \geq 2$. For every bounded, open, convex set $F$ of $\mathbb{R}^n$ it holds
\begin{equation}
I(F) \geq \frac{n}{(n \omega_n)^{\frac{n}{n-1}}} = I(B)
\end{equation}
Equality holds only for balls centered at the origin.

**Proof.** The proof follows easily by using inequalities (3.2) and (3.3) observing that
\begin{equation}
I(F) = \mathcal{J}(F) \left( \frac{P(F)}{V(F)^{\frac{1}{n-1}}} \right)^{\frac{n-2}{n-1}}
\end{equation}
\[\square\]

Our aim is to prove a quantitative version of (3.4). From now on we use the following notation
\begin{equation}
\mathcal{D}(E) = I(E) - \frac{n}{(n \omega_n)^{\frac{n}{n-1}}} = I(E) - I(B).
\end{equation}

3.1. **Stability for nearly spherical sets.** Following the Fuglede’s approach (see [11]), we first prove a quantitative version of (3.4) for nearly spherical sets defined 2.6, when $n \geq 3$.

**Theorem 3.2.** Let $n \geq 3$ and $B$ the unit ball of $\mathbb{R}^n$ centered at the origin. Then there exist three positive constants $C_1(n)$, $C_2(n)$ and $\varepsilon = \varepsilon(n)$, such that if $E \subseteq \mathbb{R}^n$ be a nearly spherical set with $P(E) = P(B)$ and $||v||_{W^{1,\infty}} \leq \varepsilon$, then
\begin{equation}
C_1(n) ||v||_{W^{1,2(\mathbb{S}^{n-1})}} \geq \mathcal{D}(E) \geq C_2(n) ||v||_{W^{1,2(\mathbb{S}^{n-1})}}^2
\end{equation}

**Proof.** We set $v = tu$, with $||u||_{W^{1,\infty}} = 1/2$, then $||v||_{W^{1,\infty}} = t ||u||_{W^{1,\infty}} = t/2$. Thus, using the expression of $P(E)$ and $W(E)$ given in (2.9) and (2.11) respectively, we get
\begin{equation}
\mathcal{D}(E) = \frac{n}{P(B)^{\frac{n}{n-1}}} \left( \int_{\mathbb{S}^{n-1}} (1 + tu(x))^n \sqrt{1 + tu(x)^2 + t^2 |D_r u(x)|^2} \, d\mathcal{H}^{n-1} \right) - 1
\end{equation}
\begin{equation}
= \frac{n}{P(B)^{\frac{n}{n-1}}} \left( \int_{\mathbb{S}^{n-1}} (1 + tu(x))^n \left( \sqrt{1 + tu(x)^2 + t^2 |D_r u(x)|^2} - 1 \right) \, d\mathcal{H}^{n-1} \right) / nV(E)
\end{equation}

Now we prove the lower bound in (3.6). We first take into account the numerator in (3.7). Let $f_k(t) = (1 + tu)^k \sqrt{(1 + tu)^2 + t^2 |D_r u|^2}$. An elementary calculation shows that
\begin{equation}
f_k(0) = 0, \quad f_k'(0) = (k + 1)u, \quad f_k''(0) = (k)(k + 1)u^2 + |D_r u|^2
\end{equation}
\begin{equation}
f_k'''(\tau) \leq 2(k + 2)(k + 1)k \left( u^3 + |u||D_r u|^2 \right)
\end{equation}
for any $\tau \in (0, t)$. Thus, since the numerator of (3.7) is given by $f_n(t) - (1 + tu)^n$, using the Lagrange expression of the rest, we can Taylor expand up to the third order we get

\[
(3.9) \quad \int_{S^{n-1}} (1 + tu(x))^n \left( \sqrt{(1 + tu(x))^2 + t^2|D_x u(x)|^2} - 1 \right) \, d\mathcal{H}^{n-1}
\geq t \int_{S^{n-1}} ud\mathcal{H}^{n-1} + nt^2 \int_{S^{n-1}} u^2 d\mathcal{H}^{n-1} + \frac{1}{2} t^2 \int_{S^{n-1}} |D_x u|^2 d\mathcal{H}^{n-1} - C(n)\varepsilon t^2 \int_{S^{n-1}} (u^2 + |D_x u|^2) \, d\mathcal{H}^{n-1}.
\]

Since $P(E) = P(B)$, we have

\[
(3.10) \quad \int_{S^{n-1}} (1 + tu(x))^{n-2} \sqrt{(1 + tu(x))^2 + t^2|D_x u(x)|^2} \, d\mathcal{H}^{n-1} = \int_{S^{n-1}} 1 \, d\mathcal{H}^{n-1}.
\]

Using (3.8) for $f_{n-2}$, we infer

\[
(3.11) \quad t \int_{S^{n-1}} ud\mathcal{H}^{n-1} \geq -\frac{n - 2}{2} t^2 \int_{S^{n-1}} u^2 d\mathcal{H}^{n-1} - \frac{t^2}{2(n - 1)} \int_{S^{n-1}} |D_x u|^2 d\mathcal{H}^{n-1} - C_1(n)\varepsilon t^2 \int_{S^{n-1}} (u^2 + |D_x u|^2) \, d\mathcal{H}^{n-1}.
\]

Then, being $n \geq 3$ by using the inequality (3.11) in (3.9), we get

\[
(3.12) \quad \int_{S^{n-1}} (1 + tu(x))^n \left( \sqrt{(1 + tu(x))^2 + t^2|D_x u(x)|^2} - 1 \right) \, d\mathcal{H}^{n-1}
\geq \left( \frac{n + 2}{2} - C_2(n)\varepsilon \right) t^2 \int_{S^{n-1}} u^2 d\mathcal{H}^{n-1} + \left( \frac{n - 2}{2(n - 1)} - C_1\varepsilon \right) t^2 \int_{S^{n-1}} |D_x u|^2 d\mathcal{H}^{n-1}.
\]

Then, choosing $\varepsilon = \frac{1}{4} \min \left\{ \frac{n + 2}{2c_2(n)}, \frac{n - 2}{2c_1(n - 1)} \right\}$, finally we obtain

\[
\mathcal{D}(E) \geq C_2(n)||tu||_{W^{1,2}(S^{n-1})}^2 \geq C_2(n)||u||_{W^{1,2}(S^{n-1})}^2
\]

which is the lower bound in (3.6). Then

\[
(3.13) \quad \frac{W(E)}{nV(E)} - 1 = \int_{S^{n-1}} (1 + v(x))^n \left( \sqrt{(1 + v(x))^2 + |D_x v(x)|^2} - 1 \right) \, d\mathcal{H}^{n-1}
\leq C(n) \int_{S^{n-1}} \left( \frac{1 + v(x)}{nV(E)} \right)^n \left( \sqrt{(1 + v(x))^2 + |D_x v(x)|^2} - 1 \right) \, d\mathcal{H}^{n-1}
\leq C(n) \int_{S^{n-1}} \left( \frac{|v(x)| + |D_x v(x)|}{nV(E)} \right)^n \, d\mathcal{H}^{n-1}
= C(n) \int_{S^{n-1}} \left( \frac{|v(x)| + |D_x v(x)|}{nV(E)} \right)^n \, d\mathcal{H}^{n-1}
\leq C(n) ||v||_{W^{1,1}(S^{n-1})}^n,
\]

where last equality follows from Hölder inequality and from the following estimate

\[
nV(E) = \int_{S^{n-1}} (1 + v(x))^n \, d\mathcal{H}^{n-1} \geq n \omega_n \left( \frac{1}{2} \right)^n
\]

□
3.2. Stability for convex sets. Before to complete the proof of the quantitative version of inequality (3.4) we need the following useful technical lemmas.

**Lemma 3.3.** Let \( n \geq 2 \), then there exists \( M > 0 \) such that if \( F \subseteq \mathbb{R}^n \) is a bounded, open, convex set with \( I(F) \leq \frac{2n}{(n\omega_n)^{\frac{n}{n-1}}} \) and \( |F| = 1 \), then \( F \subseteq Q_M \), where \( Q_M \) is the hypercube centered at the origin with edge \( M \).

*Proof.* Since the functional is scale invariant, let us suppose \( |F| = 1 \). If \( L > 1 \) and write

\[
W(F) = \int_{\partial F} |x|^2 d\mathcal{H}^{n-1} = \int_{(\partial F) \cap Q_L} |x|^2 d\mathcal{H}^{n-1} + \int_{\partial F \setminus Q_L} |x|^2 d\mathcal{H}^{n-1} \\
\geq \int_{\partial F \cap Q_L} |x|^2 d\mathcal{H}^{n-1} + L^2 P(F; C(Q_L)),
\]

where \( C(Q_L) \) is the complementary set of \( Q_L \) in \( \mathbb{R}^n \). Since \( F \) is convex, \( F \cap Q_L \) is convex as well and then the monotonicity of the perimeter implies

\[(3.14) \quad P(F) \leq P(F; C(Q_L)) + P(F; Q_L) \leq P(F; C(Q_L)) + 2L^{n-1}.\]

Suppose that \( P(F) > L^n \). Then equation (3.14) gives \( P(F; C(Q_L)) \geq L^n - 2L^{n-1} \) and thus

\[(3.15) \quad I(F) \geq \frac{\int_{\partial F \cap Q_L} |x|^2 d\mathcal{H}^{n-1} + L^2 P(F; C(Q_L))}{(P(F; C(Q_L)) + 2L^{n-1})^{\frac{1}{n-1}}} > \frac{L^{n+2} - L^{n+1}}{L^{\frac{n+1}{n}}} \]

which is not possible for \( L \) big enough since we are supposing \( I(F) < \frac{2n}{(n\omega_n)^{\frac{n}{n-1}}} \) while the last term of the above inequality diverges when \( L \to \infty \). Thus we have that there exists \( L_0 \) such that for every convex set \( F \) with \( I(F) \leq \frac{2n}{(n\omega_n)^{\frac{n}{n-1}}} \), \( P(F) < L_0^n \). Being \( |F| = 1 \) and \( P(F) \leq L_0^n \) from (2.7) we get

\[
diam(F) \leq C(n)L_0^{n(n-1)}
\]

which proves (3.14) choosing \( M = C(n)L_0^{n(n-1)} \). \( \square \)

**Lemma 3.4.** Let \( (F_j) \subseteq \mathbb{R}^n \), \( n \geq 2 \), be a sequence of convex sets such that \( I(F_j) \leq \frac{2n}{(n\omega_n)^{\frac{n}{n-1}}} \) and \( P(F_j) = P(B) \). Then there exists a convex set \( F \subseteq \mathbb{R}^n \) with \( P(F) = P(B) \) and such that, up to a subsequence

\[(3.16) \quad |F_j \Delta F| \to 0 \quad \text{and} \quad I(F) \leq \liminf I(F_j).\]

*Proof.* The existence of the limit set \( F \) comes from the proof of Lemma 3.3: since \( I(F_j) < \frac{2n}{(n\omega_n)^{\frac{n}{n-1}}} \), then there exists \( M > 0 \) such that \( F_j \subseteq Q_M \) and \( P(F_j) = P(B) \) for every \( i \in \mathbb{N} \). Thus the sequence \( \{\chi_{E_j}\}_{j \in \mathbb{N}} \) is precompact in \( BV(Q_M) \) and then there exists a subsequence and a set \( F \) such that \( |F \Delta F_j| \to 0 \) and from Lemma 2.4 we have that \( P(F) = P(B) \). Note that we can write

\[
W(F) = \sup \left\{ \int_F \text{div} \left( |x|^2 \phi(x) \right) \, dx, \quad \phi \in C_c^1(Q_M, \mathbb{R}^n), \quad ||\phi||_{\infty} \leq 1 \right\}.
\]

Then, observing that

\[
\int_F |\text{div} \left( |x|^2 \phi(x) \right)| \, dx \leq M ||\text{div} \phi||_{\infty} + M^2,
\]

we have
using the dominate convergence theorem we have that the functional

$$ F \to \int_F \text{div} \left( |x|^2 \phi(x) \right) \, dx $$

is continuous with respect to the $L^1$ convergence. Thus, since $W(F)$ is the supremum given by taking the supremum of continuous functionals, then it is lower semicontinuous and thus the inequality in (3.16).

Next result allows us to reduce the study of the stability issue for nearly spherical sets.

**Lemma 3.5.** Let $n \geq 2$. For every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $E \subseteq \mathbb{R}^n$, is a bounded, open, convex set with $P(E) = P(B)$ and $\mathcal{D}(E) < \delta_\varepsilon$, with $\mathcal{D}(E)$ defined as in (3.5), then there exists a Lipschitz function $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ such that $E$ is nearly spherical parametrized by $v$ and $\|v\|_{W^{1,\infty}} \leq \varepsilon$.

**Proof.** We first prove that $d_H(E, B) < \varepsilon$. Suppose by contradiction that there exists $\varepsilon_0 > 0$ such that for every $j \in \mathbb{N}$ exists a convex set $E_j$ with $I(E_j) - \frac{2n}{(n\omega_n)^{\frac{n}{n+1}}} < \frac{1}{j}$, $d_H(E_j, B) > \varepsilon_0$ and $P(E_j) = P(B)$. By Lemma 3.4 we have that there exists $E$ convex such that $E_j$ converges to $E$ in measure and $P(E) = P(B)$. From the semicontinuity of $W(E)$ we have that $I(E) \leq \liminf I(E_j) \leq \frac{2n}{(n\omega_n)^{\frac{n}{n+1}}}$. Since $B$ is the only minimizer of the functional $I$ we get the contradiction. Then being $E$ convex and $d_H(E, B) \leq \varepsilon$, $E$ contains the origin and so there exists a Lipschitz function $v \in L^\infty(\mathbb{S}^{n-1})$ with $\|v\|_\infty < \varepsilon$ such that

$$ \partial E = \{x(1 + v(x)), x \in \mathbb{S}^{n-1}\}. $$

Then in order to complete the proof we have to show that $\|v\|_{W^{1,\infty}}$ is small when $\mathcal{D}(E)$ is small. This is a consequence of Lemma (2.8). \[ \Box \]

Now we can prove the stability result for inequality (3.4). We first consider $n \geq 3$. The two dimensional case will be discuss separately in the next section.

**Theorem 3.6.** Let $n \geq 3$ and let $E \subseteq \mathbb{R}^n$ be a bounded, open, convex set. Then, there exist $\delta$ and $C(n) > 0$ such that if $\mathcal{D}(E) \leq \delta$ then

$$ A_H(E) \leq \begin{cases} \sqrt{\mathcal{D}(E) \log \frac{1}{\mathcal{D}(E)}} & n = 3 \\ C(n) (\mathcal{D}(E))^{\frac{2}{n+1}} & n \geq 4, \end{cases} $$

where $A_H(E)$ and $\mathcal{D}(E)$ are defined in (2.6) and (3.5) respectively.

**Proof.** Being the functional $I$ scaling invariant we can suppose that $E$ is a convex set of finite measure with $P(E) = P(B)$. Let $\varepsilon > 0$ fixed.

From Lemma 3.5 we may suppose that there exist $v \in W_{1,\infty}(\mathbb{S}^{n-1})$ with $\|v\|_{W_{1,\infty}} < \varepsilon$ such that

$$ \partial E = \{x(1 + v(x)), x \in \mathbb{S}^{n-1}\}. $$

Then if we take $\varepsilon$ small enough, by Theorem 3.2 we obtain

$$ \mathcal{D}(E) \geq C(n) \|v\|_{W^{1,2}(\mathbb{S}^{n-1})}^2. $$

Let $F = \lambda E$, with $\lambda$ such that $V(F) = V(B)$. From the isoperimetric inequality, follows that $\lambda > 1$. Since the quantity $I(E)$ is scaling invariant, one has $I(F) = I(E)$ and, from the definition of $F$, we have

$$ \partial F = \{\lambda x(1 + v(x)), x \in \mathbb{S}^{n-1}\} = \{x(1 + (\lambda - 1 + \lambda v(x))), x \in \mathbb{S}^{n-1}\}. $$

Now we proceed to the second case.
Using the definition of $\lambda$ we obtain

$$
\lambda^n - 1 = \frac{V(B)}{V(E)} - 1 = \sum_{k=1}^{n} \binom{n}{k} \int_{S_{n-1}} v^k \mathcal{H}^{k-1} \frac{1}{V(E) \sum_{0}^{n-1} \lambda^k}.
$$

and, as a consequence,

$$
\lambda - 1 = \frac{\sum_{k=1}^{n} \binom{n}{k} \int_{S_{n-1}} v^k \mathcal{H}^{k-1}}{V(E) \sum_{0}^{n-1} \lambda^k}.
$$

Let now $h(x) = \lambda - 1 + \lambda v(x)$. Note that $\|h\|_{W^{1,\infty}} < 2^n \|v\|_{W^{1,\infty}}$ and that $\lambda^n \in (1, 2)$. Moreover, using Hölder inequality, it is easy to check that

$$
||h||^2_{L^2(S^{n-1})} \leq 2^{n+2} ||v||^2_{L^2(S^{n-1})} \quad \text{and} \quad ||D_\tau h||^2_{L^2(S^{n-1})} \leq 2^{1/n} ||D_\tau v||^2_{L^2(S^{n-1})}.
$$

Thus,

$$
\mathcal{D}(F) = \mathcal{D}(E) \geq C_2(n)||h||^2_{W^1,2(S^{n-1})} \geq 2^{-n-1} C_2(n)||h||^2_{W^1,2(S^{n-1})}.
$$

Let $g = (1 + h)^n - 1$. Then, since $V(F) = V(B)$, we have $\int_{S^{n-1}} g d\mathcal{H}^{n-1} = 0$ and, from the smallness assumption on $u$, we immediately have $\frac{1}{2} |h| \leq |g| \leq 2|h|$ and $\frac{1}{2} |Dh| \leq |Dg| \leq 2|Dh|$. Now we have to distinguish the cases $n = 3$ and $n \geq 4$ because we have to apply the interpolation Lemma 2.7 to $g$. In the case $n \geq 4$, we get

$$
||h||_{L^\infty} \leq 2 ||g||_{L^\infty} \leq C(n)||D_\tau g||^2_{L^2(S^{n-1})} ||D_\tau g||^{n-3}_{L^\infty(S^{n-1})} \leq C(n)||D_\tau h||^2_{L^2(S^{n-1})} ||h||^{n-3}_{L^\infty(S^{n-1})},
$$

where in the last inequality we used (2.13). From the above chain of inequalities we deduce

$$
||h||_{L^\infty} \leq C(n)||D_\tau h||^2_{L^2(S^{n-1})}
$$

and finally recalling that $F = \lambda E$ and $V(F) = V(B)$ we get

$$
\mathcal{D}(E) \geq C_n||D_\tau h||^2_{L^2(S^{n-1})} \geq C_n||h||^{\frac{n+1}{n}}_{L^\infty} = C_n d_\mathcal{H}(F, B)^{\frac{n+1}{n}} = C_n \left( \frac{d_\mathcal{H}(E, E^t)}{V(E)^{\frac{1}{n}}} \right)^{\frac{n+1}{n}}.
$$

So we obtain the desired result (3.17) in the case $n \geq 4$ from (2.14) and the isoperimetric inequality. We proceed in an analogous way in the case $n = 3$. First we observe that, by definition of $h$, there exists a positive constant $A$ such that $\|v\|_{W^{1,1}(S^{n-1})} \leq A \|h\|_{W^{1,1}(S^{n-1})}$. Then the upper bound in (3.17) in terms of $h$, can be written as follows

$$
\mathcal{D}(E) = \mathcal{D}(F) \leq \tilde{C} \|h\|_{W^{1,1}(S^{n-1})},
$$

with $\tilde{C}$ positive constant depending on the dimension. Applying Lemma 2.7 to $g$ and using Lemma (2.8), we obtain,

$$
||h||^2_{L^\infty} \leq 4 ||g||^2_{L^\infty} \leq 16 ||D_\tau g||^2_{L^2(S^{n-1})} \log \left[ \frac{8 \varepsilon ||D_\tau g||^2_{L^\infty}}{||D_\tau g||^2_{L^2(S^{n-1})}} \right] \leq 64 ||D_\tau h||^2_{L^2(S^{n-1})} \log \left[ \frac{32 \varepsilon ||D_\tau h||^2_{L^\infty}}{||D_\tau g||^2_{L^2(S^{n-1})}} \right] \leq 64 ||D_\tau h||^2_{L^2(S^{n-1})} \log \left[ \frac{C ||h||_{L^\infty}}{||D_\tau g||_{L^2(S^{n-1})}} \right].
$$

Choosing now $||h||_{L^\infty}$ small enough, from the upper bound in (3.6), we have

$$
||h||^2_{L^\infty} \leq 64 ||Dh||^2_{L^2(S^{n-1})} \log \left[ \frac{1}{\mathcal{D}(E)} \right],
$$

where
and, as a consequence, using (3.6) and (3.23),
\[
D(E) \log \left( \frac{1}{D(E)} \right) \geq C_1(n)\|D_\gamma h\|_{L^2(S^{n-1})} \log \left( \frac{1}{D(E)} \right) \geq C\|h\|^2_{\infty} \log \left( \frac{1}{D(E)} \right) = C\|h\|^2_{\infty}.
\]

3.3. Optimality issue. In this section we will show the sharpness of inequality (3.17) and, as a consequence, we will have the sharpness for the exponent of the quantitative Weinstock inequality. We start by taking in exam the case \( n = 3 \).

**Theorem 3.7.** Let \( n = 3 \). There exists a family of convex sets \( \{E_\alpha\}_{\alpha > 0} \) such that for every \( \alpha \)
\[
D(E_\alpha) \to 0, \quad \text{when} \quad \alpha \to 0
\]
and
\[
A_H(E_\alpha) = C \sqrt{D(E_\alpha) \log \frac{1}{D(E_\alpha)}}
\]
where \( C \) is a suitable positive constant independent of \( \alpha \).

**Proof.** We follow the idea contained in [11] (Example 3.1) and we recall it here for the convinience of the reader. Let \( \alpha \in (0, \pi/2) \) and consider the following function \( \omega = \omega(\varphi) \) defined over \( S^2 \) and depending only on the spherical distance \( \varphi \), with \( \varphi \in [0, \pi] \), from a prescribed north pole \( \xi^* \in S^2 \):
\[
\omega = \omega(\varphi) = \begin{cases} 
-\sin^2 \alpha \log(\sin \alpha) + \sin \alpha (\sin \alpha - \sin \varphi) & \text{for } \sin \varphi \leq \sin \alpha \\
-\sin^2(\alpha) \log(\sin \varphi) & \text{for } \sin \varphi \geq \sin \alpha.
\end{cases}
\]

Let \( g := \omega - \bar{\omega} \), with \( \bar{\omega} \) the mean value of \( \omega \), i.e.
\[
\bar{\omega} = \int_0^{\pi/2} \omega(\varphi) \sin \varphi \, d\varphi = (1 - \log 2) \alpha^2 \sin \alpha + \mathcal{O}(\alpha^3),
\]
when \( \alpha \) goes to 0, and let
\[
R := (1 + 3g)^{1/3} = 1 + h.
\]
The \( C^1 \) function \( R = R(\varphi) \) determines in polar coordinates \((R, \varphi)\) a planar curve. We rotate this curve about the line \( \xi^* \mathbb{R} \), determining in this way the boundary of a convex and bounded set that we call \( E_\alpha \). We can observe that \( h \) and \( g \) are the same functions contained in the proof of Theorem 3.6. The set \( E_\alpha \) is indeed a nearly spherical, which has \( h \) as a representative function and with \( V(E_\alpha) = V(B) \). Therefore, taking into account the computations contained in the proof of Theorem 3.6 relative to the functions \( h \) and \( g \) and the ones contained in [11] combined with (3.6), we have
\[
\|g\|_{\infty} = \alpha^2 \log \frac{1}{\alpha} + \mathcal{O}(\alpha^2),
\]
\[
\|h\|_{\infty} \geq \frac{1}{2} \|g\|_{\infty} = \frac{1}{2} \alpha^2 \log \frac{1}{\alpha} + \mathcal{O}(\alpha^2),
\]
and
\[
\|\nabla h\|_2^2 = \|\nabla g\|_2^2 = \alpha^4 \log \left( \frac{1}{\alpha} \right) + \mathcal{O}(\alpha^4).
\]
Using (3.22), we obtain:
\[
D(E_\alpha) = O \left( \alpha^4 \log \frac{1}{\alpha} \right)
\]
Consequently
\[
D(E_\alpha) \log \left( \frac{1}{D(E_\alpha)} \right) = O \left( \alpha^2 \log \frac{1}{\alpha} \right)^2.
\]
Therefore, we have that \( D(D_\alpha) \to 0 \) as \( \alpha \) goes to 0 and, combining (3.27) with (3.29), the validity of (3.24)

We show now the sharpness of the quantitative Weinstock inequality in dimension \( n \geq 4 \).

**Theorem 3.8.** Let \( n \geq 4 \). There exists a family of convex sets \( \{P_\alpha\}_{\alpha > 0} \) such that
\[
D(P_\alpha) \to 0, \quad \text{when} \quad \alpha \to 0
\]
and
\[
\mathcal{A}_H(P_\alpha) \geq C(n) \left( D(P_\alpha) \right)^{2/(n+1)}
\]
where \( C(n) \) is a suitable positive constant.

**Proof.** In this proof we follow the construction given in [11] (Example 3.2). Let \( \alpha \in ]0, \pi/2[ \) and let \( P_\alpha \) be the convex hull of \( B \cup \{-p, p\} \), where \( p \in \mathbb{R}^n \) is given by
\[
|p| = \frac{1}{\cos \alpha}.
\]
We have that
\[
V(P_\alpha) = \omega_n + \frac{2}{n(n+1)} \omega_{n-1} \alpha^{n+1} + O(\alpha^{n+3})
\]
and
\[
P(P_\alpha) = n V(P_\alpha).
\]
We provide here the computation of the boundary momentum, that is (3.30)
\[
W(P_\alpha) = \frac{2 \omega_{n-1}}{n(n+1)} \frac{(\sin(\alpha))^{(n-1)}}{\cos(\alpha)} \left( n^2 + n + 2 \tan^2(\alpha) \right) + 2(n-1) \left[ \frac{\sqrt{\pi} \Gamma \left( \frac{n-1}{2} \right)}{2 \Gamma \left( \frac{n}{2} \right)} - \int_0^\alpha \sin^{n-2}(\theta) \, d\theta \right].
\]
As a consequence, we have
\[
(n \omega_n)^{\frac{1}{n-1}} V(P_\alpha) P(P_\alpha)^{\frac{1}{n}} D(P_\alpha) = (n \omega_n)^{\frac{1}{n-1}} \frac{2 \omega_{n-1}}{n+1} \frac{(n-2)}{n(n-1)} \alpha^{n+1} + o(\alpha^{n+3}).
\]
Since \( \mathcal{A}_H(D_\alpha) \) behaves asymptotically as \( \alpha^2 \), we have proved the desired claim. \( \square \)

4. The planar case

In this section we discuss the stability of the isoperimetric inequality (3.4) in the plane. In \( \mathbb{R}^2 \) we cannot use the same arguments used in higher dimensions, in order to obtain a stability result for (3.4). Moreover, we observe that in two dimension inequality (3.2) contained in [7] and (3.4), are proved by Weinstock in [17]. The proof is based on the representation of two dimensional convex sets via their support function. Let \( E \subset \mathbb{R}^2 \), be an open, smooth, convex set in the plane containing the origin and let \( h(\theta) \) the support function of \( E \), with \( \theta \in [0, 2\pi] \). Then Weinstock proves the following inequality (see [17] and [7] for the details)
\[
\pi W(E) - P(E) V(E) \geq \frac{P(E)}{2} \int_0^{2\pi} p^2(\theta) \, d\theta \geq 0,
\]
where for every \( \theta \in [0, 2\pi] \), \( p(x) \) is defined by
\[
h(\theta) = \frac{P(E)}{2\pi} + p(\theta).
\]
By the definition of support function it holds
\[ \int_{0}^{2\pi} h(\theta) \, d\theta = P(E). \]
Moreover, being \( E \) convex, it holds
\[ h(\theta) + h''(\theta) \geq 0. \]
Then the function \( p \) verifies
\[ \int_{0}^{2\pi} p(\theta) \, d\theta = 0, \]
and
\[ \frac{P(E)}{2\pi} + p(\theta) + p''(\theta) \geq 0. \]
We observe that
\[ \| p \|_{L^{\infty}([0, 2\pi])} = d_{H}(E, E^*), \]
where \( E^* \) is the disc centered at the origin having the same perimeter as \( E \). Let \( \theta_0 \in [0, 2\pi] \) be such that \( \| p \|_{L^{\infty}} = p(\theta_0) \). By using property (4.4) it is not difficult to prove the following result.

**Proposition 4.1.** Let \( p \) be as above then it holds
\[ p(\theta) \geq \gamma(\theta), \]
where \( \gamma(\theta) := p(\theta_0) - \frac{1}{2} \left( \frac{P(E)}{2\pi} + p(\theta_0) \right) (\theta - \theta_0)^2 \) is a parabola which vanishes at the following points
\[ \theta_{1,2} = \theta_0 \pm \sqrt{\frac{2p(\theta_0)}{\frac{P(E)}{2\pi} + p(\theta_0)}}. \]

**Proof.** By property (4.4) we obtain
\[ p(\theta) = p(\theta_0) + \int_{\theta_0}^{\theta} p'(t) \, dt = p(\theta_0) + \int_{\theta_0}^{\theta} \int_{\theta_0}^{t} p''(s) \, ds \, dt \]
\[ \geq p(\theta_0) + \int_{\theta_0}^{\theta} \int_{\theta_0}^{t} - \left( \frac{P(E)}{2\pi} + p(s) \right) \, ds \, dt \]
\[ \geq p(\theta_0) - \left( \frac{P(E)}{2\pi} + p(\theta_0) \right) \frac{(\theta - \theta_0)^2}{2}, \]
which is the claim. Then \( p \) is above the parabola \( \gamma \) which is zero at the following points
\[ \theta_{1,2} = \theta_0 \pm \sqrt{\frac{2p(\theta_0)}{\frac{P(E)}{2\pi} + p(\theta_0)}} \]
and this concludes the proof. \( \square \)

Inequality (4.1) implies Weinstock inequality but it hides also a stability result. Indeed by using the previous Proposition we get the following quantitative Weinstock inequality in the plane.

**Theorem 4.2.** Let \( E \subset \mathbb{R}^2 \), be bounded, open, convex set. Then, there exist \( \delta \) and a positive constant \( C \) such that if \( D(E) \leq \delta \) then
\[ CA_{H}(E)^{\frac{5}{2}} \leq D(E), \]
where \( A_{H}(E) \) and \( D(E) \) are defined in (2.6) and (3.5) respectively. Moreover the exponent \( \frac{5}{2} \) is sharp.
Proof. Being the functional $\mathcal{D}$ scaling invariant we can assume that $E$ is a strictly convex set of finite measure with $P(E) = P(B) = 2\pi$. From Lemma 3.5 if we take a sufficiently small $\epsilon$, there exists $\delta > 0$ such that, $\mathcal{D}(E) \leq \delta$, $E$ contains the origin, its boundary can be parametrized as above by means the support function and, by (4.5)

$$d := \|p\|_{L^\infty([0,2\pi])} \leq \epsilon$$

Under this assumptions, being in particular $|d| < \frac{\epsilon}{4}$, Proposition 4.1 gives

(4.8)

$$p(\theta) \geq d - \left(\frac{1 + d}{2}\right) (\theta - \theta_0)^2 \geq d - \frac{(\theta - \theta_0)^2}{4}$$

Denoting by $\theta_{1,2}$ the zeros of the parabola $d - \frac{(\theta - \theta_0)^2}{4}$, that is

$$\theta_{1,2} = \theta_0 \pm 2\sqrt{d},$$

by using (4.1), the isoperimetric inequality, Hölder inequality and (4.8) we get

(4.9) \quad \mathcal{D}(E) = \frac{W(E)}{P(E)V(E)} - \frac{1}{\pi} \geq \frac{\pi W(E) - P(E)V(E)}{\pi P(E)V(E)} \geq \frac{1}{2\pi^2} \int_0^{2\pi} p^2(\theta) \, d\theta

\begin{align*}
&\geq \frac{1}{2\pi^2} \int_{\theta_1}^{\theta_2} p^2(\theta) \, d\theta \\
&\geq \frac{1}{2\pi^2 (\theta_2 - \theta_1)} \left( \int_{\theta_1}^{\theta_2} p(\theta) \, d\theta \right)^2 \\
&> \frac{16}{9\pi^2} d^2.
\end{align*}

By (4.5) and (2.6), being $P(E) = 2\pi$, we get the claim.

In order to conclude the proof we have to show the sharpness of the exponent. We construct a family of strictly convex sets $E_\epsilon$, with $P(E_\epsilon) = 2\pi$ and such that

$$\mathcal{D}(E_\epsilon) \to 0 \quad \text{for} \quad \epsilon \to 0,$$

and

$$A_H(E_\epsilon) = C \epsilon^{\frac{2}{3}}.$$

Let us consider the convex set $E$ having the following support function

$$h(\theta) = 1 + p(\theta), \quad \theta \in [0, 2\pi]$$

where the function $p$ is the following

$$p(\theta) = \begin{cases} 
-b & \text{if } \theta \in [0, \alpha_1] \\
\epsilon - \frac{(\theta - \pi)^2}{4} & \text{if } \theta \in [\alpha_1, \alpha_2] \\
-b & \text{if } \theta \in [\alpha_2, 2\pi],
\end{cases}$$

Here the parameters $b$, $\alpha_1$ and $\alpha_2$ are

$$b = \frac{4}{3} \frac{\epsilon^{\frac{3}{2}}}{\pi - 2\sqrt{\epsilon}}, \quad \alpha_1 = \pi - 2\sqrt{\epsilon}, \quad \alpha_2 = \pi + 2\sqrt{\epsilon}.$$

By construction we have that

$$P(E_\epsilon) = 2\pi \quad \text{and} \quad \int_0^{2\pi} p(\theta) \, d\theta = 0.$$

Let us recall that (see for instance [17, 18, 7])

$$\begin{cases} 
V(E_\epsilon) = \frac{1}{2} \int_0^{2\pi} (h^2(\theta) + h(\theta)h''(\theta)) \, d\theta \\
W(E_\epsilon) = \int_0^{2\pi} \left( h^3(\theta) + \frac{1}{2} h^2(\theta)h''(\theta) \right) \, d\theta
\end{cases}$$
Arguing as in the proof of the Weinstock inequality a simple calculation gives

\[ \pi W(E) - P(E)V(E) = \pi \int_0^{2\pi} p^2(\theta) \left( 2 + p(\theta) + \frac{1}{2} p''(\theta) \right) d\theta = 
2\pi \int_0^{2\pi} p^2(\theta) d\theta + \pi \int_0^{2\pi} p^3(\theta) d\theta + \frac{\pi}{2} \int_0^{2\pi} p^2(\theta) p'(\theta) d\theta = C\epsilon^2 + O(\epsilon^3), \]

where \( C \) is a positive constant. This concludes the Theorem. \( \square \)

5. Proof of Theorem 1.1

The proof is a consequence of Theorems 3.6 and 4.2. In [7] the authors proved that

\[ \sigma(\Omega) \leq \frac{n V(\Omega)}{W(\Omega)}. \]

By (5.2) it holds

\[ \sigma(\Omega^*) = \left[ \frac{n \omega_n}{P(\Omega)} \right]^{1/(n-1)}, \]

then denoting by \( B \) the unit ball of \( \mathbb{R}^n \) centered at the origin, by the previous inequality and (3.3) we have

\[ \frac{\sigma(\Omega^*) - \sigma(\Omega)}{\sigma(\Omega)} = \frac{\sigma(\Omega^*)}{\sigma(\Omega)} - 1 \geq \frac{W(\Omega)}{n V(\Omega)} \left( \frac{n \omega_n}{P(\Omega)} \right)^{1/(n-1)} - 1 = \frac{(n \omega_n)^{n-1}}{n} D(\Omega) \]

Let \( \delta \) be as in Theorem 3.6, since \( \sigma_1(\Omega^*) \leq (1 + \delta) \sigma_1(\Omega) \), then \( D(E) < \delta \) and, for \( n \geq 4 \) from (3.17) in Theorem 3.6, we get

\[ \frac{\sigma(\Omega^*) - \sigma(\Omega)}{\sigma(\Omega)} \geq C(n) (A_H(E))^{n+1}. \]

If \( n = 3 \), in a similar way, we obtain that the claim follows, observing that \( f(x) = x \log \left( \frac{x}{e} \right) \) is invertible for \( 0 < x < e^{-1} \). Then, being \( D(\Omega) \) small, we can explicit it in (3.17), obtaining the thesis. The result in two dimension follows from Theorem 4.2.

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