BOUNDED TOPOLOGIES ON BANACH SPACES AND SOME OF THEIR USES IN ECONOMIC THEORY: A REVIEW

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Abstract. Known results are reviewed about the bounded and the convex bounded variants, $b\mathcal{T}$ and $cb\mathcal{T}$, of a topology $\mathcal{T}$ on a real Banach space. The focus is on the cases of $\mathcal{T} = w(P^*, P)$ and of $\mathcal{T} = m(P^*, P)$, which are the weak* and the Mackey topologies on a dual Banach space $P^*$. The convex bounded Mackey topology, $cbm(P^*, P)$, is known to be identical to $m(P^*, P)$. As for $bm(P^*, P)$, it is conjectured to be strictly stronger than $m(P^*, P)$ or, equivalently, not to be a vector topology (except when $P$ is reflexive). Some uses of the bounded Mackey and the bounded weak* topologies in economic theory and its applications are pointed to. Also reviewed are the bounded weak and the compact weak topologies, $bw(Y, Y^*)$ and $kw(Y, Y^*)$, on a general Banach space $Y$, as well as their convex variants ($cbw$ and $ckw$).

1. Introduction

Nonmetric topologies on the norm-dual, $P^*$, of a real Banach space $(P)$ can become much more manageable when restricted to bounded sets. For example, given a convex subset of $P^*$, or a real-valued concave function on $P^*$, the bounded weak* topology, $bw^* := bw(P^*, P)$, can serve to show that the set in question is weakly* closed, or that the function is weakly* upper semicontinuous. In economic theory, such uses of the Krein-Smulian Theorem are made in [9, Proposition 1.1, Theorems 4.4 and 4.7], [22, Proposition 1 and Example 5], [24, Lemma 4.1] and [25, Section 6.2]. In applications of economic equilibrium models, this can be an indispensable tool for verifying that the production sets that describe the technologies are weakly* closed, and that the profit and cost functions are weakly* semicontinuous (which is needed for equilibria to exist, and for the dual pairs of programmes to have no duality gaps): see [20, Lemma 17.1], [21, Lemma 6.1] and [25, Lemmas 6.2.3–6.2.5].

When $P$ is $L^1(T, \sigma)$, the space of integrable real-valued functions on a set $T$ that carries a sigma-finite measure $\sigma$—and so $P^*$ is the space of essentially bounded functions $L^\infty(T)$—another useful “bounded” topology on $L^\infty$ is the bounded Mackey topology, $bm(L^\infty, L^1)$. This is because a concave real-valued function, $F$, is continuous for the “plain” Mackey topology, $m(L^\infty, L^1)$, if (and only if) it is $bm(L^\infty, L^1)$-continuous, i.e.,
m \((L^\infty, L^1)^\)-continuous on bounded sets—or, equivalently, if (and only if) \(F\) is continuous along bounded sequences (in \(L^\infty\)) that converge in measure (on subsets of \(T\) of finite measure). ¹ Thus the reduction to bounded sets provides direct access to the methods of integral calculus, which can greatly simplify verification of Mackey continuity [22, Example 5]. And, in economic equilibrium analysis, Mackey continuity of a concave utility or production function \((F)\) is essential for representing the price system by a density, as is done in [1] and [23]. In addition, the use of convergence in measure furnishes economic interpretations of Mackey continuity [22, Sections 4 and 5]. (Thus it also makes clear the restrictiveness of this condition and of the resulting density form of the price system, which excludes the singularities that alone can represent capital charges when these are extremely concentrated in time or space. The alternative is not to exclude the “intractable” singular functionals but to re-represent them [37].)

In the case that the concave function \(F\) is defined and finite on the whole space \(L^\infty\), the equivalence of \(m \((L^\infty, L^1)^\)\)-continuity to \(bm \((L^\infty, L^1)^\)\)-continuity can be shown by using the Fenchel-Legendre conjugacy. This is a result of Delbaen and Owari [9, Proposition 1.2], who also extend it to the case of a general dual Orlicz space instead of \(L^\infty\) [9, Theorem 4.5] and apply it in the mathematics of finance [9, Theorem 4.8]. Their argument shows first that if \(F\) is \(bm \((L^\infty, L^1)^\)\)-continuous then the superlevel sets of its concave conjugate (a function on \(L^1\)) are uniformly integrable, and then applies the Dunford-Pettis Compactness Criterion and the Moreau-Rockafellar Theorem (on the conjugacy between continuity and sup-compactness); the first step is made also in [26, Theorem 5.2 (i) and (iv)].

The case of a nondecreasing concave \(F\) that is defined only on the nonnegative cone \(L^\infty_+\) (and does not have a finite concave extension to \(L^\infty\)) requires a different method: it relies on Mackey continuity of the lattice operations in \(L^\infty\), as well as on the monotonicity of \(F\) [22, Proposition 3 and Example 4].

For a finite-valued concave \(F\) defined on the whole space, the equivalence of Mackey continuity to bounded Mackey continuity extends to the case of a general dual Banach space, \(P^*\), as the domain of \(F\): see Delbaen and Orihuela [8, Theorem 8]. (Equivalence to sequential Mackey continuity follows when \(P\) is strongly weakly compactly generated [8, Corollary 11].) A fortiori, those linear functionals (on \(P^*\)) that are continuous for the bounded Mackey topology, \(bm^* := bm \((P^*, P)^\)\), are actually continuous for \(m^* := m \((P^*, P)^\)\), i.e., belong to \(P\). It follows that the convex bounded Mackey topology,

¹Continuity along such sequences is also known as the Lebesgue property (of \(F\)): see, e.g., [26, Definition 1.2]. It is equivalent to \(bm \((L^\infty, L^1)^\)\)-continuity because the topology of convergence in measure (on sets of finite measure), \(T_\sigma\), is both metrizable (on \(L^\infty\)) and equal to \(m \((L^\infty, L^1)^\)\) on bounded subsets of \(L^\infty\): see, e.g., [1, Example 8.47 (3)] and [16, pp. 222–223]. The metric in [1, Example 8.47 (3)] is for the case of \(\sigma (T) < +\infty\), but \(T_\sigma\) is metrizable also when \(\sigma\) is sigma-finite. Also, on \(L^\infty\) globally, \(T_\sigma\) is weaker than \(m \((L^\infty, L^1)^\)\).
cbm \((P^*, P)\), is identical to the “plain” Mackey topology\(^3\) cbm\(^*\) = m\(^*\) for every \(P\)\(^3\). Implicit in [22, Proposition 1], this result is shown more simply by applying Grothendieck’s Completeness Theorem; it is quoted here from [38] as Proposition 2.2. It does not follow that bm\(^*\) equals m\(^*\) because bm\(^*\) is not known to be a vector topology and, indeed, it is conjectured not to be one (unless \(P\) is reflexive): see [38] or Conjecture 2.3 here.

As for the convex bounded variant of the weak* topology, cbw* is the same as bw* (since the latter is locally convex by the Banach-Dieudonné Theorem), and so it is strictly stronger than the “plain” w* (unless \(P\) is finite-dimensional).

Every Banach space, \(Y\), whether dual or not, carries also the bounded weak topology and its convex variant, bw and cbw (which differ from each other unless \(Y\) is reflexive, in which case bw\(^*\) = bw = cbw). Studied in [13] and [35], bw and cbw are briefly discussed at the end of Section 2. The space \((Y)\) carries also the compact weak topology and its convex variant, kw and ckw. Introduced in [14], this concept produces a new topology (or two) if and only if \(Y\) contains the sequence space \(l^1\) (if it does not, then kw = bw and so ckw = cbw too): see Section 4. These topologies are used in studying function spaces and linear operations: see [11] and [27, Chapter 4] for such uses of bw and cbw, and [14] and [15] for those of kw and ckw.

2. The bounded and convex bounded topologies

The weakest and the strongest of those locally convex topologies on a dual Banach space \(P^*\) which yield \(P\) as the continuous dual are denoted by w \((P^*, P)\) and m \((P^*, P)\), abbreviated to w* and m*. Known as the weak and the Mackey topologies, on \(P^*\) for its pairing with \(P\), the two can be called the weak* and the Mackey topologies (since the other Mackey topology on \(P^*\), m \((P^*, P^{**})\), is identical to the norm topology). The bounded weak* topology on \(P^*\) is denoted by bw \((P^*, P)\), abbreviated to bw*. It can be defined by stipulating that a subset of \(P^*\) is bw*-closed if and only if its intersection with every closed ball in \(P^*\) is w*-closed (or, equivalently, w*-compact). In other words, bw* is the strongest topology that is equal to the weak* topology on every bounded subset (of \(P^*\)). Directly from its definition, bw* is stronger than w* (and is strictly so unless \(P\) is finite-dimensional). The Banach-Dieudonné Theorem identifies bw* as the topology of uniform convergence on norm-compact subsets of \(P\): see, e.g., [16] p. 159: Theorem 2], [18, 18D: Corollary (b)] or [33 IV.6.3: Corollary 2]. It follows that: (i) bw* is locally...

\(^2\)For \(P = L^1\) only, that cbm \((L^\infty, L^1)\) = m \((L^\infty, L^1)\) has been shown earlier by methods specific to this space, in [6] III.1.6 and III.1.9] and in [29] Theorem 5).

\(^3\)It also follows that bm*-continuity upgrades to m*-continuity not only for linear functionals but also for general linear maps, i.e., every bm*-continuous linear map of \(P^*\), into any topological vector space, is m*-continuous (on \(P^*\)). This is because, for a linear map of a space with topologies of the forms b\(\mathcal{T}\) and cb\(\mathcal{T}\), its b\(\mathcal{T}\)-continuity implies cb\(\mathcal{T}\)-continuity [6 I.1.7], and because cbm* = m*. 
convex, and (ii) bw* is weaker than m*. So, since every convex m*-closed set is w*-closed, it follows that every convex bw*-closed set is w*-closed; this the Krein-Smulian Theorem, for which see, e.g., [10, V.5.7], [18, 18E: Corollary 2] or [33, IV.6.4]. Also, given that bw* is locally convex by Part (i), Part (ii)—that bw* ⊆ m*—can be restated as: the bw*-continuous dual of P* equals P. This equality requires the norm (of P) to be complete. Indeed, it is a special case of Grothendieck’s Completeness Theorem; for this case see, e.g., [10, V.5.6], [18, 19E: Corollary 1] or [33, IV.6.2: Corollary 2].

By its definition, bw* is a case of the general concept of “bounding” a locally convex topology T, on a space Y with a norm ||·||, to produce the strongest topology that is equal to T on every norm-bounded subset (of Y). It is assumed that: (i) T is weaker than the norm topology, and (ii) the closed unit ball of Y is T-closed; such a T is said to be compatible with the norm (of Y), and (Y, ||·||, T) is then called a Saks space [6, p. 6]. The resulting bounded T-topology, denoted here by bT, is stronger than T (and weaker than the norm). Put in other words, a subset of Y is bT-closed if and only if its intersection with every closed ball of Y is T-closed. By [5, Theorem 5]—as is noted also in [11, 2.7], [12, p. 410] and [13, p. 72]—bT is always semi-linear (i.e., both vector addition and scalar multiplication are separately continuous in either variable), but generally it need not be linear (although bw* is). A map of Y (into a topological space) is bT-continuous if and only if its restrictions to bounded sets are T-continuous [12, Theorem 1 (b)].

When (Y, ||·||, T) is a Saks space, the norm-compatible topology T can be “mixed” with the norm to produce the strongest vector topology that is equal to T on every norm-bounded subset (of Y). Remarkably, this is also the strongest locally convex topology that is equal to T on every bounded set: this is shown in [6, I.1.4 and I.1.5 (iii)] and [36, 2.2.2], and is stated also in [11, 1.39 and 1.40]—where the resulting topology is denoted by γ (||·||, T), or by γ (B, T) with B for the bounded sets. Here, this convex bounded T-topology is denoted by cbT; it is stronger than T and weaker than bT [4]. (The three are, however, sequentially equivalent (i.e., have the same convergent sequences) when Y = P* is a dual Banach space and T is stronger than w* (e.g., when T is m* or w* itself).  

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4To deduce that bw* is weaker than m*, recall that m(P*, P) is the topology of uniform convergence on all weakly compact subsets of P. See, e.g., [1, Section 5.18] or [33, IV.3.2: Corollary 1], where the compacts are required to be convex and circled as well, but here “convex” can be omitted because P is a Banach space and one can apply Krein’s Theorem—for which see, e.g., [1, Theorem 6.35, named Krein-Smulian], [11, V.6.4], [18, 19E] or [33, IV.11.4]. That “circled” can be omitted is obvious [33, I.5.2].

5To prove that the bw* of P* is P from the standard formulation of Grothendieck’s Theorem (Theorem 4.1 here), apply it to P* as E—with w* as Σ and the bounded subsets of P* as Σ, and hence with P as E' and the norm topology of P as the Σ-topology—to conclude that a linear functional on P* is w*-continuous if it is so on bounded sets, i.e., if it is bw*-continuous.

6And so T is compatible with the norm of Y if T is weaker than the norm topology but stronger than w(Y, Y*), or at least stronger than w(Y, P) when Y = P*.

7Obviously, cbT = bT if and only if bT is locally convex (or, equivalently, is a vector topology).
This is because, unlike a general uncountable net, a $\mathcal{T}$-convergent sequence, being $w^*$-convergent, is bounded by the Banach-Steinhaus Theorem—and so it is $b\mathcal{T}$-convergent. A linear map of $Y$ (into a topological vector space) is $cb\mathcal{T}$-continuous if (and only if) it is $b\mathcal{T}$-continuous, i.e., if (and only if) its restrictions to bounded sets are $\mathcal{T}$-continuous [6, I.1.7]. As a case of this, a linear functional on $Y$ is $b\mathcal{T}$-continuous if (and only if) it is $b\mathcal{T}$-continuous (but, to avoid misapplying this, recall that $b\mathcal{T}$ need not be a vector topology).

For $Y = P^*$ with $\mathcal{T} = w^* := w(P^*, P)$, where $P$ is a (real) Banach space, the bounded weak* topology is itself locally convex, and so it is identical to its convex variant: $cbw^* = bw^*$. The case of the convex bounded Mackey topology is different: $cbm^* = m^*$ (on the whole space $P^*$). As is set out next, this follows from Grothendieck’sCompleteness Theorem [33, IV.6.2], which is quoted for easy reference. \[\text{Theorem 2.1 (Grothendieck). Let } \mathcal{T} \text{ be a locally convex topology on a real vector space } E. \text{ Then additionally } \mathcal{S} \text{ is a saturated family of } \mathcal{T}\text{-bounded sets covering } E, \text{ the } \mathcal{T}\text{-dual of } E \text{ is complete under the } \mathcal{S}\text{-topology (the topology of uniform convergence on every } S \in \mathcal{S}) \text{ if and only if every linear functional (on } E) \text{ that is } \mathcal{T}\text{-continuous on each } S \in \mathcal{S} \text{ is actually } \mathcal{T}\text{-continuous on the whole space } E \text{ (i.e., is in the } \mathcal{T}\text{-dual of } E). \]

\[\text{Proposition 2.2 (} [38] \text{). Let } P \text{ be a real Banach space, and } P^* \text{ its norm-dual. Then } cbm(P^*, P) = m(P^*, P). \]

\[\text{Proof. Apply Theorem 2.1 to } P^* \text{ as } E \text{—with } m^* \text{ as } \mathcal{T} \text{ and the bounded subsets of } P^* \text{ as } \mathcal{S}, \text{ and hence with } P \text{ as the } \mathcal{T}\text{-dual and the norm topology of } P \text{ as the } \mathcal{S}\text{-topology—to conclude that a linear functional on } P^* \text{ is } m^*\text{-continuous if it is so on bounded sets (i.e., if it is } bm^*\text{-continuous). } A \text{ fortiori, it is } m^*\text{-continuous (i.e., is in } P) \text{ if it is } cbm^*\text{-continuous. In other words, } cbm^* \text{ yields the same dual space as } m^* \text{ (viz., } P). \text{ This proves that } cbm^* = m^* \text{ (since } cbm^* \text{ is both locally convex and stronger than } m^*). \]

As for $bm^*$, it is conjectured to be different from $m^*$. \[\text{Conjecture 2.3 (} [38] \text{). For } P = L^1[0, 1] \text{ at least, and possibly for every nonreflexive Banach space } P, \text{ the topology } bm(P^*, P) \text{ is strictly stronger than } m(P^*, P) \text{—or, equivalently, } bm(P^*, P) \text{ is not linear.} \]

This conjecture is based on what it takes to establish that a $bm^*$-continuous $\mathbb{R}$-valued function $F$, on a nonreflexive space $P^*$, is $m^*$-continuous: the Delbaen-Orihuela-Owari results of [8, Theorem 8] and [9, Proposition 1.2 and Theorem 4.5] require $F$ to be

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8 Grothendieck’s Completeness Theorem is also stated in [10], p. 73: Corollary 1.

9 A family, $\mathcal{S}$, of subsets of a locally convex space is called saturated [33], p. 81] if: (i) all subsets of every member of $\mathcal{S}$ belong to $\mathcal{S}$, (ii) all scalar multiples of every member of $\mathcal{S}$ belong to $\mathcal{S}$, and (iii) for each finite $\mathfrak{F} \subseteq \mathcal{S}$, the closed convex circled hull of the union of $\mathfrak{F}$ belongs to $\mathcal{S}$.

10 Alternatively, Cooper’s special case of Grothendieck’s Theorem [6, I.1.17 (ii)] can be applied—to $P^*$ as his $E$, with $m^*$ as $\tau$ and the bounded subsets of $P^*$ as $\mathcal{B}$, and hence with $P^{**}$ as $E^*_\mathcal{B}$ and $cbm^*$ as his $\gamma = \gamma(\mathcal{B}, \tau)$—to conclude that the $cbm^*$-dual equals the $m^*$-dual (so $cbm^* = m^*$).
concave (or convex), and it is hard to imagine (even when \( P^* = L^\infty [0, 1] \)) how the convexity assumption might be disposed of entirely—as would be necessary for \( bm^* \) to equal \( m^* \).

**Comments** (completeness of \( m(P^*, P) \) and lattice properties of \( m(L^\infty, L^1) \)):

- As is observed in, e.g., [17, pp. 97–98] and [34, 1.1], \( m(P^*, P) \) is complete. This is a different application of Grothendieck’s Theorem—one that swaps the spaces’ roles and works in the “other direction” to prove completeness (of \( m^* \), on \( P^* \)), rather than using completeness (of the norm on \( P \)) as here (to prove that \( cbm^* = m^* \)). And although the argument of [17] and [34] does use the norm-completeness of \( P \), this is needed only in its first step, which uses Krein’s Theorem [18, 19E] rather than Grothendieck’s.

- In [34, 2.1] it is also shown that \( m(P^*, P) \) is (completely) metrizable on bounded sets if and only if \( P \) is strongly weakly compactly generated (SWCG).

- For \( P = L^1 \), it follows from the Dunford-Pettis Compactness Criterion that \( m(L^\infty, L^1) \) is a Lebesgue topology, i.e., it is (i) locally solid (that is, it makes \( L^\infty \) a topological vector lattice), and (ii) order-continuous: see, respectively, [4] p. 535] or [1] Theorem 9.36] or [2] Chapter 6: Exercise 4], and [2] 9.1, equivalence of (i) and (ii)] or [3] 3.12, equivalence of (1) and (2)]. In [29] Theorems 4 and 5], \( m(L^\infty, L^1) \) is shown to be the strongest Lebesgue topology on \( L^\infty \), by an argument which also shows that \( m(L^\infty, L^1) = cbm(L^\infty, L^1) \). In [29] Theorem 6], the Lebesgue property of \( m(L^\infty, L^1) \) is used to show that this topology is also complete—but, like the equality \( cbm^* = m^* \), this too is actually true of \( m^* \) for every Banach space \( P \) (whether ordered or not).

- The \( m(L^\infty, L^1) \)-continuity of lattice operations has various uses in economic theory, such as those in [19], [22, Example 4] and [23, Proof of Theorem 15], in addition to those mentioned in [1] p. 361].

**Comments** (on \( bm(Y, Y^*) \), \( bm(P^*, P) \) and \( bw(Y, Y^*) \)):

- As for \( bm(Y, Y^*) \), where \( Y \) is any Banach space, it is of course the norm topology of \( Y \), since \( m(Y, Y^*) \) is.

- When \( P \) is reflexive, setting \( Y = P^* \) above (with \( Y^* = P^{**} = P \)) shows that \( bm(P^*, P) = m(P^*, P) \), and that it then is the norm topology of \( P^* \).

- In no space can \( bm^* \) be both linear and different from \( m^* \). This is in contrast to the case of \( bw^* \) and \( w^* \) (but this is not strange because the “bounding” strengthens the topology, and in the cases of \( w^* \) and \( m^* \) it starts at the opposite extremes, in

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11Note also that the sufficient condition of [6] I.4.2] for \( bT \) to equal \( cbT \) does not apply to \( T = m^* \) (since it means that \( T \) is the weak* topology on the dual of a Fréchet space [6, I.4.1 and I.2.A]).

12In detail: the mixed topology \( \gamma = \gamma(\|\cdot\|_\infty, T_\sigma) \)—where \( T_\sigma \) is the topology of convergence in measure (on sets of finite measure)—is shown in [29] Theorems 2 and 4] to yield \( L^1(\sigma) \) as its dual, and to be the strongest Lebesgue topology on \( L^\infty \) (which is of interest in itself). It is then deduced [29] Theorem 5] that \( m(L^\infty, L^1) = \gamma \). But \( \gamma = cbm(L^\infty, L^1) \) because \( T_\sigma \) is equal to \( m(L^\infty, L^1) \) on bounded sets, as is seen from the Dunford-Pettis Criterion [16] pp. 222–223].
strength, of the range of the locally convex topologies on $P^*$ for its pairing with $P$).

- “Bounding” the weak topology (when it is not $w^*$) fails to produce a linear one. That is, the bounded weak topology, $bw(Y, Y^*)$ or $bw$ for brevity, on a Banach space $Y$ is not locally convex (except, of course, when $Y$ is reflexive, in which case $Y^*$ is also the unique norm-predual of $Y$, and $bw = bw^*$): see [13, 3.7] or [27, 4.2.8]. It follows—as a case of [6, I.1.4 and I.1.5 (iii)] that is noted also in [11, 2.5] and [13, p. 72]—that $bw$ is not even linear (unless $Y$ is reflexive). In other words, $bw$ is strictly stronger than $cbw$, the convex bounded weak topology. And it is not $bw(Y, Y^*)$ but $cbw(Y, Y^*)$ that equals the restriction (to $Y$) of $bw(Y^{**}, Y^*)$, the bounded weak* topology of the second norm-dual $Y^{**}$ (when it differs from $Y$): see [13, 2.6]. For $Y = c_0$, the space of real sequences converging to zero, an example of a $bw$-closed set that is not $cbw$-closed is given in [35, 4.8]; it is reproduced in [7, p. 48, II.5(2)(b)] and [11, 2.1]. In [13, 3.1], this example is generalized to any separable nonreflexive $Y$ that is sequentially reflexive or, equivalently by [30], does not contain an isomorphic copy of $l^1$ (the space of summable sequences).

3. A summary of comparisons of $\mathcal{T}$, $cb\mathcal{T}$ and $b\mathcal{T}$ for $\mathcal{T} = w^*$, $m^*$, $w$

Except for the one which is only conjectured, the following strict inclusions and equalities hold (for the topologies as families of open/closed sets):

- $w^* \subsetneq cbw^* = bw^*$. The equality holds by the Banach-Dieudonné Theorem; the inclusion is strict for all infinite-dimensional Banach spaces [7, p. 48, II.5(2)(a)].
- $m^* = cbm^*$ (Proposition 2.2). Is $m^* \subsetneq bm^*$? (Conjecture 2.3, for nonreflexive spaces.)
- $w \subsetneq cbw \subsetneq bw$. That the second inclusion is strict (unless the space is reflexive and so $w = w^*$) is shown in [13, 2.6].

4. The compact weak and convex compact weak topologies

Replacing the bounded sets in the definition of $bw$ by weak compacts produces the compact weak topology, $kw(Y, Y^*)$ or $kw$ for brevity, on a Banach space $Y$ (paired with its norm-dual $Y^*$). Introduced in [14], $kw(Y, Y^*)$ is, then, defined as the strongest topology that is equal to $w(Y, Y^*)$ on every $w(Y, Y^*)$-compact subset [14, 2.3 (b)]. In other words, a subset of $Y$ is $kw$-closed if and only if its intersection with every $w$-compact set is $w$-closed or, equivalently, $w$-compact [14, 2.1]. An equivalent characterization is that $kw$-closed sets are the same as sequentially $w$-closed sets [14, 2.2 (a) and (b)]; this follows from the Eberlein-Smulian Theorem, for which see, e.g., [1, Theorem 6.34], [2, 19.4], [3, 2.15] or [10, V.6.1]. (So $kw$ is always weaker than the norm topology.) Another equivalent definition of $kw(Y, Y^*)$ is as the strongest topology having the same convergent sequences as $w(Y, Y^*)$; the equivalence can be shown by using [14, 2.2 and 2.3 (b)].

The “compacting” of the weak topology fails, however, to produce a linear one, except when it results in either $bw^*$ or the norm topology $m(Y, Y^*)$. As is noted in [14, p.
kw \((Y,Y^*)\) is always semi-linear (like every \(bT\)) by [5, Theorem 5]. But if \(kw\) is linear then it is even locally convex, and it is so if and only if \(Y\) is either (i) a reflexive space (in which case \(kw = bw = bw^*\)) or (ii) an infinite-dimensional Schur space, i.e., a Banach space in which weakly convergent sequences are norm-convergent (in which case \(kw = m(Y,Y^*)\), but \(bw\) is not linear because \(Y\) is then nonreflexive): see [14, 2.9 and 2.5]. In the other cases, \(kw(Y,Y^*)\) is therefore different from the convex compact weak topology. Denoted by \(ckw(Y,Y^*)\) or \(ckw\) for brevity, this is defined as the strongest locally convex topology that is equal to \(w(Y,Y^*)\) on \(w\)-compact sets (which is the same as the strongest locally convex topology that is weaker than \(kw\)).

Furthermore, if \(Y\) is reflexive then, a fortiori, it is sequentially reflexive (i.e., \(m(Y^*,Y^*)\)-convergent sequences, in \(Y^*\), are the same as the norm-convergent a.k.a. \(m(Y^*,Y^{**})\)-convergent ones) or, equivalently by [30], \(Y\) does not contain an isomorphic copy of \(l^1\) (i.e., no subspace of \(Y\) is linearly homeomorphic to \(l^1\)). By contrast, if \(Y\) is an infinite-dimensional Schur space then it is not sequentially reflexive (i.e., contains \(l^1\)): see [32, p. 2411, consequence II]. This dichotomy corresponds exactly to equality or inequality of \(kw\) and \(bw\), i.e., \(cbw = ckw\) if and only if \(bw = kw\), which is the case if and only if \(Y\) is sequentially reflexive (i.e., does not contain \(l^1\)):

In sum, there are four—mutually exclusive and collectively exhaustive—cases of strict inclusions and equalities (for the topologies as families of open/closed sets):

1. If \(Y\) is reflexive then \(cbw = bw = kw = ckw\) (and all four are equal to \(bw^*\)).
2. If \(Y\) is not reflexive but is sequentially reflexive (or, equivalently, does not contain \(l^1\) as an isomorphic copy) then \(ckw = cbw \subset bw = kw\).
3. If \(Y\) is an infinite-dimensional Schur space (and hence contains \(l^1\), i.e., is not sequentially reflexive) then \(cbw \subset bw \subset kw = ckw\) (since \(kw\) is then equal to the norm topology \(m(Y,Y^*)\)).
4. If \(Y\) is not a Schur space but contains \(l^1\) (i.e., is not sequentially reflexive) then \(cbw \subset ckw \subset bw \subset kw\). That is, \(ckw\) and \(bw\) are two topologies that both lie strictly between \(cbw\) and \(kw\) but are different from each other (\(ckw\) is locally convex, \(bw\) is not even linear)—and so all four topologies are different. In this case \(ckw\) is a new topology, i.e., it is different from all the others (\(w\), \(cbw\), \(bw\), \(kw\), and \(m(Y,Y^*)\)).

In this context it is worth noting that if a Banach space, \(P\), contains any infinite-dimensional Schur space, then it contains also the specific Schur space \(l^1\) and, furthermore, so does \(P^*\); it obviously follows that if \(P\) contains \(l^1\) then so does \(P^*\) [28, Corollaries 9 and 10]. In addition, as is shown in [31, Theorem 3] and noted also in [28, p. 371], if \(P\) contains \(l^1\) then \(P^*\) is not a Schur space—and so, for every \(n \geq 1\), the \(n\)-th norm-dual of \(P\) contains \(l^1\) but is not a Schur space.

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13 An infinite-dimensional Banach space cannot be both reflexive and a Schur space; this can be seen from [30] and [32, p. 2411, consequence II].

14 Both \(kw\) and \(ckw\) can also be defined as, respectively, the strongest topology and the strongest locally convex topology with the same compacts as \(w(Y,Y^*)\).
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