Robust consistent a posteriori error majorants for approximate solutions of diffusion-reaction equations

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Abstract

Efficiency of the error control of numerical solutions of partial differential equations entirely depends on the two factors: accuracy of an a posteriori error majorant and the computational cost of its evaluation for some test function/vector-function plus the cost of the latter. In the paper, consistency of an a posteriori bound implies that it is the same in the order with the respective unimprovable a priori bound. Therefore, it is the basic characteristic related to the first factor. The paper is dedicated to the elliptic diffusion-reaction equations. We present a guaranteed robust a posteriori error majorant effective at any nonnegative constant reaction coefficient (r.c.). For a wide range of finite element solutions on a quasiuniform meshes the majorant is consistent. For big values of r.c. the majorant coincides with the majorant of Aubin (1972), which, as it is known, for not big r.c. (< ch\(^{-2}\)) is inconsistent and loses its sense at r.c. approaching zero. Our majorant improves also some other majorants derived for the Poisson and reaction-diffusion equations.

1 Introduction

For the successful error control of approximate solutions to the boundary value problems, the guaranteed a posteriori error majorant must be sufficiently accurate and cheap in a sense of the computational work. The first requirement can be considered as satisfied at least in part, if the majorant is consistent in respect to the order of accuracy with the a priori convergence estimate of the numerical method. Obviously, a consistent majorant is unimprovable in the order, if the a priori convergence estimate is unimprovable in the same sense. An error majorant usually depends on the approximate solution and on some other function or functions which are termed test functions. In this paper consistency assumes that it can be approved by an easily calculated test function with the use of some procedure of the linear complexity.

The term "functional a posteriori error majorants" is usually related to a posteriori error bounds possessing significant generality and some other positive properties. However, sometimes generality is attained for the price of the lost of consistency, resulting in the necessity of attracting some majorant minimization procedures over the space of admissible test functions [8, 9]. The computational cost of such procedures can exceed the cost of the numerical solution of the boundary value problem.

The majorant of Aubin [3] is one of the earliest. Let us illustrate it on a model problem

\[-\text{div}(A\text{grad }u) + \sigma u = f(x), \quad x \in \Omega \subset \mathbb{R}^m,\]
\[u|_{\Gamma_D} = \psi_D, \quad -A\nabla u \cdot \nu|_{\Gamma_N} = \psi_N,\]

(1.1)
where $\Gamma_D$, $\Gamma_N$ are not intersecting parts of the boundary $\partial \Omega = \Gamma_D \cup \Gamma_N$, $\text{mes} \Gamma_D > 0$, $\mathbf{v}$ is the internal unite normal to the boundary, $\mathbf{A}$ is a symmetric $m \times m$ matrix, and $0 < \sigma = \text{const}$. It is assumed that the matrix $\mathbf{A}$ satisfies the inequalities

$$
\mu_1 \xi \cdot \xi \leq \mathbf{A} \xi \cdot \xi \leq \mu_2 \xi \cdot \xi, \quad 0 < \mu_1, \mu_2 = \text{const},
$$

for any $x \in \Omega$ and $\xi \in \mathbb{R}^m$. Everywhere in the paper, the boundary $\partial \Omega$, the coefficients of the matrix $\mathbf{A}$, and the right hand part $f$ are assumed to be sufficiently smooth, if more specific requirements to their smoothness are absent.

The error bounds in the energy norm

$$
|\mathbf{u}| = (\|\mathbf{u}\|_{\mathbf{A}}^2 + \sigma \|\mathbf{u}\|_{L^2(\Omega)}^2)^{1/2}, \quad \|\mathbf{u}\|_{\mathbf{A}}^2 = \int_{\Omega} \nabla \mathbf{u} \cdot \mathbf{A} \nabla \mathbf{u}, \quad (1.2)
$$

are most useful in applications. For vectors $\mathbf{y} \in \mathbb{R}^m$ we introduce also the spaces $L_2(\Omega) = (L_2(\Omega))^m$, $H(\Omega, \text{div}) = \{\mathbf{y} \in L_2(\Omega) : \text{div} \mathbf{y} \in L_2(\Omega)\}$ and the norm $\|\mathbf{y}\|_{A^{-1}} = (\int_{\Omega} \mathbf{A}^{-1} \mathbf{y} \cdot \mathbf{y})^{1/2}$.

**Theorem 1.1.** Let $f \in L_2(\Omega)$, $\psi_D \in H^1(\Omega)$, $\psi_N \in L_2(\Gamma_N)$, $v$ is any function from $H^1(\Omega)$ satisfying the boundary condition on $\Gamma_D$. Then for any $\mathbf{z} \in H(\Omega, \text{div})$, satisfying on $\Gamma_N$ the boundary condition $\mathbf{z} \cdot \mathbf{v} = \psi_N$, we have

$$
|v - u|^2 \leq \|\mathbf{A} \nabla v + \mathbf{z}\|_{A^{-1}}^2 + \frac{1}{\sigma} \|f - \sigma v - \text{div} \mathbf{z}\|_{L^2(\Omega)}^2. \quad (1.3)
$$

The bound (1.3) is a particular case of the results of [1], see, e. g., Theorem 22 in Introduction and additionally Theorems 1.2, 1.4, 1.6 of ch. 10. It can be found also in [15, 17]. Obviously, this majorant becomes meaningless at $\sigma \to 0$. Let us add that in (1.3) one can use $\mathbf{z} = -\mathbf{A} \nabla w$ with any $w \in H^1(\Omega, \mathcal{L})$, satisfying the boundary condition in (1.1) on $\Gamma_N$. Here $\mathcal{L} = -\text{div}(\mathbf{A}\text{grad})$ and $H^1(\Omega, \mathcal{L}) = \{w : w \in H^1(\Omega), \mathcal{L}w \in L_2(\Omega)\}$. If $v \in C(\Omega) \cap H^1(\Omega)$ is the finite element solution, then $w = \tilde{w}(v)$ most often is obtained from $v$ by some recovery technique [13, 1, 4, 5].

Let $\hat{H}^1(\Omega) := \{v \in H^1(\Omega) : v|_{\partial \Omega} = 0\}$, for simplicity $\Gamma_D = \partial \Omega$, $\psi_D \equiv 0$, $\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the unity matrix. In [10] for the case $\sigma = 0$, it was suggested the majorant

$$
\|\nabla (v - u)\|_{L^2(\Omega)}^2 \leq (1 + \epsilon)\|\nabla v + \mathbf{z}\|_{L^2(\Omega)}^2 + c_\Omega (1 + \frac{1}{\epsilon}) \|\nabla \cdot \mathbf{z} - f\|_{L^2(\Omega)}^2, \quad \forall \epsilon > 0, \quad (1.4)
$$

where $v$ and $\mathbf{z}$ are any function and vector-function from $\hat{H}^1(\Omega)$ and $H(\Omega, \text{div})$, respectively, and $c_\Omega$ is the constant from the Friedrichs inequality.

Attempts to modify the Aubin’s majorant in such a way that it provided admissible accuracy for all $\sigma \geq 0$ were made in the papers [17, 18]. The latter suggests the majorant for all $\sigma = \text{const} \geq 0$ of the form

$$
|v - u|^2 \leq (1 + \epsilon)\|\mathbf{A} \nabla v + \mathbf{z}\|_{A^{-1}}^2 + \frac{1}{\sigma + c_\Omega (1 + \epsilon)} \|f - \sigma v - \text{div} \mathbf{z}\|_{L^2(\Omega)}^2. \quad (1.5)
$$

It was shown in [2] that the correction of arbitrary vector-function $\mathbf{z} \in H(\Omega, \text{div})$ into the vector-function $\mathbf{\tau}$, satisfying the balance/equilibrium equations, can be done by quite a rather simple techniques. In particular, it is true for the correction of the flux vector-function $\nabla u_{\text{fem}}$ into $\mathbf{\tau}(u_{\text{fem}})$. This allows to implement the a posteriori bound $|v - u| \leq \|\mathbf{A} \nabla u_{\text{fem}} + \mathbf{\tau}(u_{\text{fem}})\|_{A^{-1}}^2$ or the bound with the additional free vector-function in the right part, which we present below. For simplicity, we restrict considerations to the same homogeneous Dirichlet problem for the Poisson equation in a two-dimensional convex
domain. Let $T_k$ be the projection of the domain $\Omega$ on the axis $x_3-k$ and the equations of the left and lower parts of the boundary be $x_k = a_k(x_3-k)$, $x_3-k \in T_k$. If $\beta_k$ are arbitrary bounded functions and $\beta_1 + \beta_2 \equiv 1$, then according to (2.11)

$$\|
abla (v-u)\|_{L^2(\Omega)} \leq \|
abla v + z\|_{L^2(\Omega)} + \sum_{k=1,2} \| \int_{a_k}^{x_k} \beta_k(f - \nabla \cdot z)(\eta_k, x_3-k) \, d\eta_k \|_{L^2(\Omega)} . \tag{1.6}$$

In (1.6) on the right we have integrals from the residual and this helps to make the majorant more accurate. Besides there is an additional free function $\beta_1$ or $\beta_2$ and it’s right choice (for instance, with the use of the found approximate solution $v$) can accelerate the process of the minimization of the right part. Nevertheless, the majorant (1.6), as well as majorants (1.4), (1.5), are not consistent. Since it is practically obvious, below we discuss this matter very briefly.

The inconsistency is the most clearly visible for finite element methods of a higher smoothness. Let us turn to (1.4) in the case when $v \parallel H^3(\Omega)$, $f \in H^1(\Omega)$, and the unimprovable a priori convergence estimates $\|u - v\|_{H^k(\Omega)} \leq ch^{3-k} \|u\|_{H^k(\Omega)}$, $k = 0, 1, 2$, hold with the mesh parameter $h$ under assumption that the finite element assemblage satisfies the conditions of the generalized quasuniformity [7, 11]. For the latter conditions see, e. g., Section 3.2 in [12]. Now we see that the left part of (1.4) is estimated from above with the order $h^4$. One can set $z = -\nabla v$ making the first term in the right part equal to zero. At the same time, the second term in the right part is estimated from above only with the order $h^2$. More over since the estimates of the convergence are exact there are functions $f \in H^1(\Omega)$ for which the second term is estimated with the order $h^2$ from below. The proofs of the inconsistency of the majorants (1.5), and (1.6) are also straightforward.

If the FEM belongs to the class $C$, then we can use the so called recovered flux $z = \widetilde{z}(u_{\text{fem}})$, whose components $\widetilde{z}_k(u_{\text{fem}})$ in the simplest case are defined as functions of the same finite element space, to which belongs $u_{\text{fem}}$. Several cheap averaging procedures were developed for defining the nodal parameters of $\widetilde{z}_k$, which provide at least the same orders of accuracy for $(\partial u/\partial x_k - \widetilde{z}_k)$ and $(u - u_{\text{fem}})/\partial x_k$, see [13, 11, 4, 5]. If $f \in L^2(\Omega)$, then the order of the left part is by the multiplier $h^2$ higher again than the order of the right part.

Let $\sigma^*$ be the value from the inequality

$$\|u - v\|_{L^2(\Omega)}^2 \leq \sigma^* \|u - v\|_{A}^2 . \tag{1.7}$$

There is the multiplier $1/\sigma$ before the second norm in the right part of (1.3). In view of this, it can be shown that at $\sigma \geq \sigma^*$ the Aubin’s majorant is consistent for approximate solutions by FEM, if $\sigma^* \leq c_1 h^2$, $c_1 = \text{const}$, and some natural conditions are fulfilled, see Lemma 1 in the next section. However, at $\sigma \ll \sigma^*$ the consistency deteriorates and with $\sigma$ tending to zero the majorant becomes meaningless.

## 2 Consistent error majorant for any nonnegative reaction coefficient

We start from the presentation of a guaranteed robust error majorant valid for all $\sigma \in [0, \infty)$, which at application to the FEM solutions is consistent. We start from the presentation of a guaranteed robust error majorant valid for all $\sigma \in [0, \infty)$, which at application to the FEM solutions is consistent.

**Theorem 2.1.** Let $\Gamma_D = \partial \Omega$, the conditions of Theorem (1.7) be fulfilled, and $\sigma$ satisfy the inequality (1.7). Then

$$\|v - u\|_{L^2(\Omega)}^2 \leq \mathcal{M}(\sigma, f, v, z) = \Theta \left[ \|A \nabla v + z\|_{A^{-1}}^2 + \Theta \|f - \sigma v - \text{div} z\|_{L^2(\Omega)}^2 \right] , \tag{2.1}$$
where for $\kappa = \sigma / \sigma_s$
\[
\Theta = \left\{ \begin{array}{l}
2/(1 + \kappa), \quad \forall \sigma \in [0, \sigma_s] \\
1, \quad \forall \sigma > \sigma_s
\end{array} \right\}, \quad \theta = \left\{ \begin{array}{l}
1/\sigma_s, \quad \forall \sigma \in [0, \sigma_s] \\
1/\sigma, \quad \forall \sigma > \sigma_s
\end{array} \right\}. \quad (2.2)
\]

**Proof.** Obviously, for $\sigma \geq \sigma_s$ the majorant (2.1), (2.2) coincides with the majorant of Aubin. Consequently, it is necessary to consider only the case $\sigma < \sigma_s$. For simplicity, in the proof we set $A = I$ and $\psi_D \equiv 0$. For the solution of the problem $u$, arbitrary function $v \in H^1(\Omega)$ and vector-function $z \in H(\Omega, \text{div})$, we can write
\[
|v - u|^2 = \int_\Omega [\nabla(v - u) \cdot \nabla(v - u) + \sigma(v - u)(v - u)] =
\int_\Omega [(\nabla v + z) \cdot \nabla(v - u) - (z + \nabla u) \cdot \nabla(v - u) +
\sigma(v - u)(v - u)]. \quad (2.3)
\]
Integrating by parts the second summand in the right part and implementing the inequality
\[
a_1b_1 + a_2b_2 \leq (a_1^2 + \frac{1}{\sigma_s} a_2^2)^{1/2} (b_1^2 + \sigma_s b_2^2)^{1/2}, \quad (2.4)
\]
we find out that
\[
|v - u|^2 = \|\nabla(u - v)\|^2_{L^2(\Omega)} + \sigma\|u - v\|^2_{L^2(\Omega)} \leq
\left[ \|\nabla(u - v)\|^2_{L^2(\Omega)} + \frac{1}{\sigma_s} \|f - \sigma v + \Delta w\|^2_{L^2(\Omega)} \right]^{1/2} \times
\left[ \|\nabla(u - v)\|^2_{L^2(\Omega)} + \sigma_s\|u - v\|^2_{L^2(\Omega)} \right]^{1/2}. \quad (2.5)
\]
The use of $\beta \in (0, 1]$ and (1.7) allows us to get
\[
\|\nabla(u - v)\|^2_{L^2(\Omega)} + \sigma_s\|u - v\|^2_{L^2(\Omega)} = |u - v|^2 + (\sigma_s - \sigma)\|u - v\|^2_{L^2(\Omega)} \leq
|u - v|^2 + (\sigma_s - \sigma) \left[ \frac{2}{\sigma_s} \|\nabla(u - v)\|^2_{L^2(\Omega)} + (1 - \beta)\|u - v\|^2_{L^2(\Omega)} \right] =
1 + (\sigma_s - \sigma) \frac{2}{\sigma_s} \|\nabla(u - v)\|^2_{L^2(\Omega)} + [(1 - \beta)(\sigma_s - \sigma) + \sigma]\|u - v\|^2_{L^2(\Omega)}, \quad (2.6)
\]
The value $\beta = 2/(1 + \kappa)$ makes the relation of the multipliers before the second and first norms on the right of (2.6) equal to $\sigma$. Substituting it into (2.6) and then (2.6) into (2.5) yields
\[
|v - u|^2 \leq \frac{2}{1 + \kappa} \left[ \|\nabla(u - v)\|^2_{L^2(\Omega)} + \frac{1}{\sigma_s} \|f - \sigma v + \Delta w\|^2_{L^2(\Omega)} \right]^{1/2} |v - u|, \quad (2.7)
\]
which is equivalent to (2.1) in the case of $A = I$. \hfill \qed

As was noted above, for $\sigma \geq \sigma_s$ the majorant (2.1), (2.2) coincides with the majorant of Aubin. In the contrast to Aubin’s majorant, for all $\sigma \geq 0$ it is well defined and, more over, belongs to the class of consistent majorants when applied to the solutions by the finite element method satisfying quite natural conditions. Before formulating the respective result in Lemma 1 below, we briefly discuss these conditions.

It is assumed that the finite element space $V_h(\Omega)$, $V_h(\Omega) \subset C^0(\Omega) \cap H^1(\Omega)$, is induced by the assemblage of the finite elements, in general curvilinear, which satisfy the generalized conditions of quasiconformity with the mesh parameter $h$, see e. g. [10][12], and $V_h(\Omega) = \{ v \in V_h(\Omega) : v|_{\partial \Omega} = 0 \}$. For simplicity, we consider the FEM of the first order of accuracy,
i. e. with finite elements associated with the triangular linear and square bilinear reference elements. If \( f \in L^2(\Omega) \), boundary \( \partial \Omega \) and the coefficients of the matrix \( \mathbf{A} \) are sufficiently smooth, then the following convergence estimates can be proved:

\[
\|u - u_{\text{fem}}\|_{k,\Omega} \leq c_k h^{l-k} \|u\|_{l,\Omega}, \quad k = 0, 1, \quad l = 1, 2, \\
\|u - u_{\text{fem}}\|_{0,\Omega} \leq \min[c_c c_{k,l} h^2, \sigma^{-1}] \|f\|_{0,\Omega}, \quad \forall \sigma \geq 0,
\]

(2.8)

where \( \| \cdot \|_{k,\Omega} \) are the norms in the space \( L^2(\Omega) \) for \( k = 0 \) and in the spaces \( H^k(\Omega) \) for \( k > 0 \), whereas \( c_c, c_{k,l} = \text{const} \). If \( \sigma = 0 \), they are the well known FEM convergence estimates for regular elliptic problems, see, e. g., [14 7 10]. If \( \sigma \leq c_1 h^{-2} \), the results of [6] on elliptic projections in the space \( L^2(\Omega) \) together with the fact that at least \( u \in H^2(\Omega) \) can be used for their proof. Indeed, it is easily shown that

\[
\|u\|_{H^2(\Omega)} \leq c_o \|f\|_{L^2(\Omega)}, \quad c_o = \text{const},
\]

(2.9)

at any \( \sigma \geq 0 \), if it is true (with different constant) for \( \sigma = 0 \). The second bound (2.8) takes additionally into account the bound \( \|u - u_{\text{fem}}\|_{0,\Omega} \leq \sigma^{-1} \|f\|_{0,\Omega} \).

For the use of a posteriori majorant (2.1), (2.2), one has to bound \( c_k \). First we turn to the case \( \sigma = 0 \). By means of Nitsche trick, see e. g. [7 14], for \( e_{\text{fem}} = u - u_{\text{fem}} \) it is proved the inequality

\[
\|e_{\text{fem}}\|_{0,\Omega}^2 \leq \|e_{\text{fem}}\|_{A} \|\phi - \phi_{\text{int}}\|_{A},
\]

(2.10)

where \( \phi \) is the solution of the boundary value problem \( L\phi = e_{\text{fem}}, \phi|_{\partial \Omega} = 0 \), and, according to (2.9), \( \phi \in H^2(\Omega) \), whereas \( \phi_{\text{int}} \) is the interpolation of \( \phi \) from the finite element space \( \mathcal{V}_h(\Omega) \). Combining the approximation error bounds

\[
\|\phi - \phi_{\text{int}}\|_{k,\Omega} \leq \hat{c}_k h^{l-k} \|\phi\|_{l,\Omega}, \quad k = 0, 1, \quad l = 1, 2,
\]

(2.11)

(2.10) and (2.9) yields (1.7) with \( \sigma^{-1} \leq c_1 h^2 \) and

\[
c_k = \mu_2 c_{1,2}^2 c_o^2.
\]

(2.12)

Now we will use the notations \( e_\sigma = e_{\text{fem}} \) and \( e_0 \) for the errors of the finite element solutions of the equations \( Lu + \sigma u = f \) and \( Lu = f_1 \), respectively, with the first boundary condition \( u|_{\partial \Omega} = 0 \) and \( f_1 = f - \sigma u \). Since from the proof given above and the introduced definitions it follows that

\[
\|e_0\|_{0,\Omega}^2 \leq c_1 h^2 \|e_0\|_{A}^2,
\]

(2.13)

we come to (1.7) of the form

\[
\|e_{\text{fem}}\|_{0,\Omega}^2 \leq c_1 h^2 \|e_{\text{fem}}\|_{A}^2
\]

(2.14)

with the same, as in (2.12) and (2.13), constant \( c_1 \). Accordingly, at \( \sigma \in [0, 1/(c_1 h^2)] \) the bound (2.1) for the finite element solutions can be rewritten as

\[
|u_{\text{fem}} - u| \leq M_{\text{fem}}(\sigma, f, u_{\text{fem}}, z) =
\]

\[
\frac{2}{1 + c_1 h^2 \sigma} \left[ \|\mathbf{A} \nabla v + z\|_{A^{-1}}^2 + c_1 h^2 \|f - \sigma v - \text{div } z\|_{L^2(\Omega)}^2 \right],
\]

(2.15)

The construction of the recovered vector-function \( z = \tilde{z}(u_{\text{fem}}) \in \mathbf{H}(\Omega, \text{div}) \) can be performed with the use of the finite element fluxes \( -\mathbf{A} \nabla u_{\text{fem}} \). The convergence bounds (2.8) lead to the conclusion that at any \( \sigma \geq 0 \) the same recovery techniques can be used,
which are used for regular elliptic problems \[1,4,5,13\]. They allow to obtain such vector-functions \( z \) with components satisfying the inequalities

\[
\begin{align*}
\| \tilde{z}(\phi) \|_{L^2(\Omega)} & \leq c \| \nabla \phi \|_{L^2(\Omega)}, \quad \forall \phi \in \mathbb{V}(\Omega), \\
\| \nabla u + \tilde{z}(u_{\text{fem}}) \|_{L^2(\Omega)} & \leq \tilde{c}_1 h^{-1} \| u \|_{H^1(\Omega)}, \quad l = 1, 2, \\
\| \Delta u + \nabla \cdot \tilde{z}(u_{\text{fem}}) \|_{L^2(\Omega)} & \leq \tilde{c}_2 \| u \|_{H^2(\Omega)}.
\end{align*}
\]

(2.16)

Lemma 2.1. Let \( \Gamma_D = \partial \Omega, \psi_D \equiv 0 \) and \( f \in L^2(\Omega) \), the finite element assemble satisfy the conditions of the generalized quasiminiformity, and the convergence estimates \[2.8\] hold. Let also \( v = u_{\text{fem}} \) and for the vector-function \( z = \tilde{z}(v) \), obtained by the application of the recovery technique to the finite element fluxes \(-A \cdot \nabla v\), the inequalities \[2.16\] hold. Then for \( \sigma^{-1} \leq c_1 h^2 \) with \( c_1 \) from \[2.12\] and any \( \sigma \geq 0 \) we have

\[
M(\sigma, f, u_{\text{fem}}, \tilde{z}) \leq C h^2 \| f \|_{L^2(\Omega)}
\]

(2.17)

with the constant \( C \) independent of \( \sigma \) and \( h \).

In fact, the recovered flux is defined in such a way that at least to have the same orders of accuracy with the flux defined by the finite element solution or the same orders of accuracy in the unimprovable a priori error bounds. More over the superconvergence recovery technique (SPR) demonstrated ability to provide the superconvergent recovery on regular meshes and recovery with much improved accuracy on general meshes. The mathematical analysis approving this phenomena for some finite element methods can be found in \[19,13\]. At the same time, alongside with \[2.17\] it is not difficult to establish the consistency of the majorant \( M(\sigma, f, v, z) \) with the a priori error bounds for finite element methods of higher order of accuracy.

3 Concluding remarks

Theorem 2 and Lemma 1 are formulated for the first boundary value problem. If the natural boundary condition is posed on the part of the boundary, then the necessary changes of these results are illustrated by Theorem 1. Namely, vector-functions \( z, \tilde{z}(v) \) and functions \( w, \tilde{w}(v) \) should satisfy this boundary condition. However, in general the finite element spaces, to which these vector-functions and functions belong, do not allow to satisfy the boundary condition exactly. Therefore, they must be approximated in the corresponding trace spaces, and as a consequence the additional terms estimating influence of the approximation appear in the majorants. The technique of the estimating such additional terms is common for a posteriori bounds of different types and can be found, e. g., in \[2\], see Remark 4.5.

Results of the paper can be expanded upon more general elliptic equations of orders \( 2n, n \geq 1 \) and, in particular, to those described un Theorems 1.2, 1.4, 1.6 of ch. 10 in \[3\]. One of them is the equation \( L_n u + \sigma u = f \) with the differential operator

\[
L_n u = \sum_{|q|, |p| = n} (-1)^{|q|} D^{q} a_{q,p}(x) D^{p} u,
\]

where \( D^q v = \partial^{|q|} v / \partial x_1^{q_1} \partial x_2^{q_2} \cdots \partial x_m^{q_m} \), \( q = (q_1, q_2, \ldots, q_m) \), \( q_k \) are nonnegative whole numbers, \( |q| = q_1 + q_2 + \cdots + q_m \), \( A = \{ a_{p,q} \}_{|p|,|q| = n} \) is the matrix with the sufficiently smooth coefficients, satisfying the inequalities \( \mu \hat{1} \leq A \leq \hat{m} \hat{1} \), \( 0 \leq \mu, \hat{m} = \text{const} \), \( \forall x \in \Omega \). Here the inequality \( B \leq C \) for two nonnegative matrices \( B \) and \( C \) of the same dimension assumes that \( (C - B) \) is a nonnegative matrix.
For definiteness, we turn to the case of the first boundary condition \( \partial^k u / \partial \nu^k = 0 \), \( k = 0, 1, \ldots, (n - 1) \), \( \forall x \in \partial \Omega \), where \( \nu \) is the distance to the boundary along the normal \( \nu \), and define the norm

\[
\| v \|_A = \left( \sum_{|q|, |p| = n} \int_\Omega a_{p,q}(D^q v)D^p v dx \right)^{1/2}.
\]

Under the well known conditions, the value \( \sigma^* = \| u - v \|^2 / \| u - v \|^2, L^2(\Omega) \) for the FEM solutions \( v = u_{\text{fem}} \) is estimated from below as \( \sigma^* \geq 1/(c_n \delta h^{2n}) \). The bound of the identical to (2.1) form retains, if for the introduced differential operator \( L_n \), matrix \( A \) and number \( \sigma^* \) the norms \( \| \cdot \|_A \) and the functions \( \Theta(\kappa) \), \( \theta(\kappa) \) are correspondingly defined and the vector-function \( \nabla v \) and the function \( \text{div} z \) are replaced by \( D v = \{ D^p v \}_{p=0}^{n} \) and \( D^* z = \sum_{|q| = n} (-1)^{|q|} D^q z^{(q)} \), respectively, where \( z^{(q)} \) are components of the vector \( z \).

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