ELM-Based Distributed Cooperative Learning Over Networks

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Abstract—This paper investigates distributed cooperative learning algorithms for data processing in a network setting. Specifically, the extreme learning machine (ELM) is introduced to train a set of data distributed across several components, and each component runs a program on a subset of the entire data. In this scheme, there is no requirement for a fusion center in the network due to e.g., practical limitations, security, or privacy reasons. We first reformulate the centralized ELM training problem into a separable form among nodes with consensus constraints. Then, we solve the equivalent problem using distributed optimization tools. A new distributed cooperative learning algorithm based on ELM, called DC-ELM, is proposed. The architecture of this algorithm differs from that of some existing parallel/distributed ELMs based on MapReduce or cloud computing. We also present an online version of the proposed algorithm that can learn data sequentially in a one-by-one or chunk-by-chunk mode. The novel algorithm is well suited for potential applications such as artificial intelligence, computational biology, finance, wireless sensor networks, and so on, involving datasets that are often extremely large, high-dimensional and located on distributed data sources. We show simulation results on both synthetic and real-world data sets.

Index Terms—Extreme learning machine (ELM), consensus, distributed optimization, distributed cooperative learning.

I. INTRODUCTION

The topic of learning in parallel and distributed environment has received increasing attention in recent years [1]–[7]. Research in this area has focused on aspect that the learning behavior is geographically or logically scattered among a network of learners, in which all these learners perform their tasks cooperatively. In particular, this paper is concerned with distributed learning problems that arise when training data are gathered from a set of nodes connected in a network, but their communication to a fusion center is costly or even unavailable. Three basic reasons can be identified: 1) Increasingly big data applications in machine learning often require to deal with large-scale data sets or data sets available at different locations. But these applications are too difficult to be implemented by centralized methods; 2) As the diversity of data sources and formats increase, it is more required to build such distributed learning systems that have some good properties such as scalability and robustness; 3) For some practical cases (e.g., knowledge discovery from clinical records), the learners need a learning system that does not ask them to share their data for privacy and security reasons.

For centralized learning, on the other hand, extreme learning machine (ELM) [8]–[10] has attracted significant attention in the last decade, due to its fast convergence and high efficiency in a variety of application areas such as medical diagnosis, image processing, and system modeling, just to name a few [11]–[14]. Classic ELM aims to get better generalization performance by reaching not only the smallest norm of the output weights but also the smallest training error in a single location. The outputs of a new data set is solely based on the so-called output weights $\beta$, calculated by $\beta = H^\dagger T$, where $H^\dagger$ denotes the Moore-Penrose generalized inverse of the hidden layer output matrix $H$ [15]. To train an ELM, the key point is to solve the matrix $H^\dagger$. The orthogonal projection method is often efficiently used in two cases: $H^\dagger = (H^TH)^{-1}H^T$ when $H^TH$ is nonsingular, or $H^\dagger = H^\dagger (HH^T)^{-1}$ when $HH^T$ is nonsingular [10]. However, when the number of samples grows large, the great memory consumption of matrix inversion and multiplication in a single machine is unbearable. Although the ELM of centralized version is very simple to implement, it will be considered not suitable for dealing with extremely large data sets.

Parallel computing [16] technique is one of the most popular approaches to large-scale learning. Several recent works in the literature have addressed the parallel/distributed ELM based on MapReduce architecture (see, for instances [17], [18]). This method performs optimally when the algorithm is parallel and can be decomposed into a large number of independent computations. The MapReduce-based method can be viewed in a master-slave mode. The slaves specify the calculation of matrices $P = H^TH$ and $Q = H^TT$ in terms of a map and a reduce function. The master calculates the final output weights by $\beta = P^{-1}Q$. A secure and practical mechanism for outsourcing ELM in cloud computing has been presented in [19], to reduce the training time while assuring the confidentiality of the input and the output. The cloud server is mainly responsible for calculating the Moore-Penrose generalized inverse $H^\dagger$ which is the heaviest operation computationally. Heeswijk et al. [20] propose a GPU-accelerated ELM ensemble for large scale regression applications. The proposed method first trains multiple ELMs independently through an efficient leave-one-out (LOO) cross-validation on the whole training sets, and then ensembles these ELMs through weighted voting. The LOO method accomplishes both model training and model structure selection for each component ELM. In addition, the LOO error is employed to calibrate the ensemble weights $\beta$. However, these parallel designs can be implemented in a networked architecture only if a fusion center is available to get desired $\beta$ from all intermediate stages.
The objective of this paper is to develop a fully distributed ELM algorithm, which relies on in-network data processing across a network. Although incremental solvers are possible, they make communication inefficient for requiring plenty of information transmissions. In addition, incremental algorithms generally require the nodes of the network to construct a Hamiltonian cycle, i.e., a closed path going through all nodes once. Finding such a cycle is known to be an NP-hard problem and particularly traversing the Hamiltonian cycle in a node-by-node sequential path causes a significant delay to build the final computation.

The novel approach proposed in the present paper trains ELMs in a fully distributed manner, without the need for a centralized fusion center or a master coordinator. To achieve this goal, we first introduce the communication model well suited for distributed applications, and then recast the centralized ELM training problem into an equivalent separable convex optimization problem with consensus constraints imposed on the output weights [21]. Motivated by recent progresses in distributed optimization [22], [23], we develop the so-called distributed cooperative ELM (DC-ELM) based solely on message exchanges among neighboring nodes, which is provably convergent to the centralized ELM. Compared to existing alternatives, the proposed DC-ELM exhibits the following distinct features.

- **Fully distributed.** The DC-ELM is a fully distributed scheme which works on a network without any central coordinator. All the nodes connected in the network have equal status. Each node only needs to train its own target and exchange desired information with its immediate neighbors. However, the training result obtained at each node is equivalent to the centralized one obtained by using all the training sets over the network.

- **Scalability.** The DC-ELM algorithm here relies on in-network data processing and one-hop neighborhood information exchanges. This makes the communication overhead per node affordable, even when the network size increases.

- **Robustness.** If the fusion center fails, a parallel ELM with fusion center design will fail altogether. In contrast, if a single or a few nodes fail for some reasons, the DC-ELM will still operate as long as the network remains connected.

- **Privacy preservation.** The design of communication topology is recognized to be essential for DC-ELM training. Learners in the network use their local data sources independently, and pass information to their neighbors. This avoids sharing original or sensitive data with others, and guarantees input data privacy.

The remainder of this paper is organized as follows. Section II briefs the basic ELM and distributed optimization. Section III presents the novel distributed cooperative extreme learning machine. Section IV provides simulation results for the performance of the proposed scheme. We summarize our findings and conclude in Section V.

II. BASIC ELM AND DISTRIBUTED OPTIMIZATION

This section briefs the basic ELM algorithm and the background of distributed optimization problem.

A. Basic ELM

The ELM, proposed in [8] and [9], can be considered as a learning system like feedforward neural networks. Its merits have been well documented in various applications [11]–[14]. This subsection gives a brief description of the basic ELM. For more details, we refer the reader to the recent review paper [24].

The ELM was developed for generalized single-hidden layer feedforward networks (SLFNs), which performs well in both regression and classification applications [9], [10]. The output function of ELM for generalized SLFNs is given by

$$ f_L(x) = \sum_{l=1}^{L} \beta_l g(h_l(x)) $$

where $x \in \mathbb{R}^D$ is the input vector, $\beta_l \in \mathbb{R}^M$ is the output weight vector, $h_l(x)$ denotes the output function $g(w_l, b_l, x)$ of the $l$th hidden layer neuron with respect to $x$. For $N$ distinct labeled samples $\{(x_n, t_n)\}_{n=1}^{N}$, SLFNs with $L$ hidden neurons can approximate these samples with a zero error. It means that there exist $(w_l, b_l)$ and $\beta_l$ such that

$$ \sum_{l=1}^{L} \beta_l g(w_l, b_l, x_n) = t_n, \quad n = 1, \ldots, N. $$

The above equations can be written in matrix form as

$$ H\beta = T $$

where

$$ H = \begin{bmatrix} h_1(x_1) & \cdots & h_L(x_1) \\ \vdots & \ddots & \vdots \\ h_1(x_N) & \cdots & h_L(x_N) \end{bmatrix}_{N \times L}, $$

$$ T = \begin{bmatrix} t_{11} & \cdots & t_{1M} \\ \vdots & \ddots & \vdots \\ t_{N1} & \cdots & t_{NM} \end{bmatrix}_{N \times M}, $$

and

$$ h(x_n) = [h_1(x_n), \ldots, h_L(x_n)] $$

with $h_l(x_n) = g(w_l, b_l, x_n)$ being the output of the $l$th hidden layer neuron for $x_n$. The nonlinear mapping function $g$ can be any nonlinear piecewise continuous function, such as sigmoid function and Gaussian function [10].

Since feedforward networks try to reach the smallest training error, the smaller the norms of weights are, the better generalization performance the networks tend to have. In general, the ELM minimizes both the norm of the output weights and the training error, i.e.,

$$ \min \| \beta \|^2 \quad \text{and} \quad \| H\beta - T \|^2 $$
where \(\|\cdot\|\) denotes the Frobenius norm. In the program of a centralized ELM, the output weight matrix \(\beta^*\) is calculated by \(\beta^* = H^T T\), where \(H^T\) is the Moore-Penrose Inverse of \(H\) \[15\]. Then the output function of ELM can be modeled as \(f(x) = h(x)\beta^* = h(x)H^T T\).

According to the ridge regression theory, the diagonal of a symmetric matrix can be incremented by a positive biasing constant \(\frac{1}{L}\) to gain the better stability and generalization performance. Thus, we can obtain
\[
\beta^* = \begin{cases} 
\left( \frac{L}{2} + H^T H \right)^{-1} H^T T, & L \leq N \\
H^T \left( \frac{L}{2} + HH^T \right)^{-1} T, & N \leq L
\end{cases}
\]
where \(I_L\) denotes the identity matrix of dimension \(L\).

Towards the same goal with \[2\], a constrained-optimization-based ELM can be formulated as
\[
\begin{aligned}
\min & \quad \frac{1}{2} \|\beta\|^2 + C \sum_{i=1}^N \|\xi_i\|^2 \\
\text{s.t.} & \quad h(x_n)\beta = i_n - \xi_n^T, \quad n = 1, \ldots, N.
\end{aligned}
\]
By substitute the constraints of \[3\] into its objective function, we also have
\[
\begin{aligned}
\min & \quad \frac{1}{2} \|\beta\|^2 + C \|H\beta - T\|^2 \\
\end{aligned}
\]
where \(C > 0\) can be viewed as a tunable parameter which provides a tradeoff between the regularized item and the training errors.

### B. Distributed Optimization

Since the seminal work \[25\], parallel/distributed computing has been a major research topic for decades in the area of system and control theory. The field of distributed optimization has been investigated either as a specific case of distributed computing, or as a specific function optimization technique \[25\]–\[31\]. The goal of this problem is to optimize the aggregate sum of convex functions known by a couple of agents only, i.e., each agent \(i\) associates a scalar cost function \(f_i : \mathbb{R}^D \rightarrow \mathbb{R}\) and a nonempty local constraint set \(\chi_i\), with \(\chi_i \subseteq \mathbb{R}^D\), and all the agents cooperatively optimize the overall function \(\sum f_i\) in a distributed manner. Here, the word ‘distributed’ means that the problem is not available at a single location but is distributed among different agents. Each agent only uses information acquired by itself and other information from its immediate neighbors. The problem minimizing a global function of a sum of \(n\) components is formulated as follows:
\[
\begin{aligned}
\min & \quad f(x) = \sum_{i=1}^n f_i(x) \\
\text{s.t.} & \quad x \in \chi = \bigcap_{i=1}^n \chi_i
\end{aligned}
\]

where each function \(f_i\) is convex and each set \(\chi_i\) is convex and compact. For the unconstrained case, \(\chi = \mathbb{R}^D\).

This problem arises in many practical applications, such as parameter estimation, resource allocation, and machine learning in wireless sensor networks. Since many researchers focus attention on this topic in recent years, a lot of works on this problem have been proposed. Up to now, a class of subgradient-based methods have been reported in the literature, e.g., see \[25\], \[27\]–\[31\]. These methods can be classified into two main groups: incremental methods and methods based on consensus.

1) Incremental Methods: Incremental method \[27\], \[41\] is popular for solving the problem \[6\], which goes through a cycle of \(n\) ordered nodes. The algorithm includes the following iterations, which is initiated by setting an arbitrary estimate \(x_0\):
\[
\begin{aligned}
z_{0,k+1} &= z_{n,k} = x_k, \\
z_{i,k+1} &= P_\chi(z_{i-1,k+1} - \alpha g_{i,k+1}), \quad i = 1, \ldots, n
\end{aligned}
\]
where \(\alpha\) is a small positive step size, \(k\) is the iteration index, \(x_k\) is the estimate at the end of the cycle at iteration \(k\), \(z_{i,k+1}\) is the intermediate estimate updated at node \(i\) at iteration \(k+1\), and \(P_\chi\) denotes projection on the set \(\chi\), and \(g_{i,k+1} \in \partial f_i(z_{i-1,k+1})\).

The operation of this method is very simple. An estimate of the optimizer is passed node by node along the cycle. Each node updates the estimate by adjusting the previous value with the subgradient of its own function, and then passes the update to the next node. The communication network must be unchanged and has a Hamiltonian cycle. However, the Hamiltonian cycle may not exist in a network and finding such a cycle is an NP-complete problem.

2) Consensus Methods: Another approach is based on consensus techniques \[21\]. The basic idea of this method is to distribute the entire optimization task among agents connected in a network using a consensus mechanism. In this context, each agent maintains an estimate, and locally runs a subgradient step to update the estimate. Simultaneously, each agent communicates with its neighbors in order to achieve an agreement on the estimate of the optimizer \[29\], \[42\], \[43\]. Each agent \(i\) generates its new estimate according to the following iterations:
\[
\begin{aligned}
z_{i,k} &= \sum_{j \in \mathcal{N}_i} a_{ij} x_{j,k}, \\
x_{i,k+1} &= P_\chi(z_{i,k} - \alpha_k g_{i,k})
\end{aligned}
\]

starting with some initial state \(x_{i,0} \in \chi\), where \(a_{ij}\) are nonnegative weights, \(\mathcal{N}_i\) denotes the set of neighbors of agent \(i\), \(\alpha_k > 0\) is a stepsize, \(g_{i,k}\) denotes the subgradient of \(f_i\) at \(z_{i,k}\). Each agent \(i\) updates its estimate \(x_{i,k+1}\) by combining the estimates acquired from its neighbors and the local subgradient step.

### III. DISTRIBUTED COOPERATIVE ELM (DC-ELM)

This section first models the communication of the decentralized system and reformulates the centralized ELM into an equivalent distributed form. For guaranteeing the performance, we then present a discrete-time distributed convex optimization algorithm to solve the problem.

A. Communication Model

We model the communication network by an undirected, connected, static \(V\)-node graph \(G(V, E)\), with vertices \(V = \...\)
Fig. 1. Network example of $V$ cooperative nodes. The nodes are represented by white circles, and the connectivity among nodes is denoted by a line joining them. The network is always assumed to be connected so that no nodes are isolated from the rest. At every node $i \in \mathcal{V} = \{1, \ldots, V\}$, a labeled training set $S_i = \{(x_{n,i}, t_{n,i})\}_{n=1}^{N_i}$ of size $N_i$ is available. Each node is associated with an individual ELM. The goal is to cooperatively learn a global ELM by local computation and communication.

$\{1, \ldots, V\}$, and edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ representing available communication links among nodes. Node $i \in \mathcal{V}$ can communicate with node $j \in \mathcal{V}$ if the edge $(i, j)$ is included in $\mathcal{E}$, or, in other words, $j$ belongs to the neighborhood set of $i$, denoted as $\mathcal{N}_i$. The adjacency matrix of $\mathcal{G}$ is denoted by $A = [a_{ij}] \in \mathbb{R}^{V \times V}$, where $a_{ii} = 0$, $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The degree matrix of $\mathcal{G}$ is $D = \text{diag}(d_1, \ldots, d_V) \in \mathbb{R}^{V \times V}$, where $d_i = \sum_{j=1}^{V} a_{ij}$ for $i = 1, \ldots, V$, and the Laplacian matrix of $\mathcal{G}$ is defined as $L = D - A$. The Laplacian is a positive semidefinite matrix, hence its eigenvalues can be sorted and represented in the following manner $0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_V(L)$. Furthermore, a graph is connected if and only if $\lambda_2(L) > 0$ (see [43] for instance). In the literature, $\lambda_2(L)$ is referred to as the algebraic connectivity of the network [45].

B. Problem Formulation

In a classical setting of ELM, the data set is provided to a single, centralized learning node. Instead, with reference to Fig. 1 we suppose that for each node $i \in \mathcal{V}$, a labeled training set $S_i = \{(x_{n,i}, t_{n,i})\}_{n=1}^{N_i}$ of size $N_i$ is available, where $x_{n,i} \in \mathbb{R}^D$ is an input data vector and $t_{n,i} \in \mathbb{R}^M$ denotes its corresponding label. The total number of training samples across the network is $N = \sum_{i=1}^{V} N_i$.

If all $\{S_i\}_{i=1}^{V}$ were centrally prepared at a fusion center, then the global variable $\beta^*$ describing the centralized output function $f^*(x) = h(x)\beta^*$ could be obtained by solving the following problem

$$\beta^* = \arg \min \frac{1}{2} \|\beta\|^2 + \frac{C}{2} \sum_{i=1}^{V} \sum_{n=1}^{N_i} \|\xi_{n,i}\|^2$$

s.t. $h(x_{n,i})\beta = t_{n,i}^T - \xi_{n,i}^T$

$\forall i \in \mathcal{V}, n = 1, \ldots, N_i$.

where $\xi_{n,i} \in \mathbb{R}^M$ is the training error of the output neurons with respect to the training sample $x_{n,i}$ in the $i$th computing node.

We decouple the problem (7) by assigning copies of the common variable $\beta$ to each node and then adding equivalent constraints to force them to agree. Let $\beta_i$ denote the copy held by node $i$. We constrain all copies to be equal across neighboring nodes, and rewrite (7) as

$$\min \frac{1}{2} \sum_{i=1}^{V} \|\beta_i\|^2 + \frac{VC}{2} \sum_{i=1}^{V} \sum_{n=1}^{N_i} \|\xi_{n,i}\|^2$$

s.t. $h(x_{n,i})\beta_i = t_{n,i}^T - \xi_{n,i}^T$

$\beta_i = \beta_j, \quad \forall i, j \in \mathcal{V}, \quad \forall j \in \mathcal{N}_i, \quad n = 1, \ldots, N_i$.

Lemma 1: If the graph $\mathcal{G}$ is connected and $\{\beta_i\}_{i=1}^{V}$ denotes a feasible solution of (8), the problem (7) and (8) are equivalent.

Proof: See Appendix A.

By substituting the constraints of (8) into its objective function, we obtain the following equivalent optimization problem

$$\min \sum_{i=1}^{V} \left( \frac{1}{2} \|\beta_i\|^2 + \frac{VC}{2} \|H_i \beta_i - T_i\|^2 \right)$$

s.t. $\beta_i = \beta_j, \quad \forall i \in \mathcal{V}, \quad \forall j \in \mathcal{N}_i$

where the hidden layer output matrix $H_i \in \mathbb{R}^{N_i \times L}$ is calculated based on the local training set $S_i = \{(x_{n,i}, t_{n,i})\}_{n=1}^{N_i}$. The training data target matrix $T_i \in \mathbb{R}^{N_i \times L}$ is also obtained at node $i$. Notice that each matrix $H_i$ is a block of the overall centralized hidden layer output matrix $H$ that contains $N_i$ rows of $H$.

It is possible to solve the problem (9) in distributed way because each node $i$ can optimize its own local objective, and impose the local consensus constraint $\beta_i = \beta_j$ by exchanging messages with any node $j$ in its neighborhood. The global consensus can be achieved under the assumption that the communication network is connected.

C. Distributed Optimization Model

To investigate the problem (9), we intend to develop a discrete-time version of distributed optimization model derived from (22). Consider the following problem

$$z^* = \arg \min_z \left( \frac{1}{V} \sum_{i=1}^{V} u_i(z) \right)$$

where $u_i : \mathbb{R}^L \to \mathbb{R}$ represents the local utility function of node $i \in \mathcal{V}$, observed by this node only, and all nodes wish to cooperatively decide an optimal consensus value $z^*$ of the global objective $u(z)$.

Assumption 1: Suppose that each function $u_i(z)$ is twice continuously differentiable and strongly convex.

Proposition 1: With Assumption 1 there exists a unique $z^* \in \mathbb{R}^L$ such that the gradient $\nabla u(z^*) = 0$ and the problem (10) is solved.

Proof: See Theorem 6 in [46]}

Let $z_i(k)$ represent the estimate of $z^*$ completing each iteration $k$ for node $i$ in its local memory, and $z_i(0)$ represent the initial value. With this setup, the goal may be stated as

$$\lim_{k \to \infty} z_i(k) = z^*, \quad \forall i \in \mathcal{V}.$$
To design an algorithm that ensures (11), consider the following condition
\[
\sum_{i \in \mathcal{V}} \nabla u_i(z_i(k)) = 0, \quad k = 1, 2, \ldots
\] (12)
which says that the \( z_i(k) \)'s evolve in a way that the sum of the gradients \( \nabla u_i \)'s, evaluated at the \( z_i(k) \)'s, is always conserved at zero. Moreover, consider condition
\[
\lim_{k \to \infty} z_i(k) = z^c, \quad \forall i \in \mathcal{V}, \text{ for some } z^c \in \mathbb{R}^L
\] (13)
which says that the \( z_i(k) \)'s gradually dissipate their differences and asymptotically achieve some arbitrary consensus \( z^c \in \mathbb{R}^L \).

Note that if (12) is met, then \( \lim_{k \to \infty} \sum_{i \in \mathcal{V}} \nabla u_i(z_i(k)) = \lim_{k \to \infty} 0 = 0 \). If, in addition, (13) is met, then due to the continuity of every \( \nabla u_i \)'s, \( \lim_{k \to \infty} \sum_{i \in \mathcal{V}} \nabla u_i(z_i(k)) = \sum_{i \in \mathcal{V}} \lim_{k \to \infty} \nabla u_i(z_i(k)) = \sum_{i \in \mathcal{V}} \sum_{k=1}^{\infty} \nabla u_i(z^c) = \nabla u(z^c). \)

From above results, we get \( \sum_{k \in \mathcal{V}} \nabla u_i(z_i(k)) = 0, \) and asymptotically achieve some arbitrary consensus \( z^c \). Therefore, to design an algorithm that solves (10), it is sufficient to make the conditions (12) and (13) satisfied, where \( z^c \) is implicitly obtained.

If \( z_i(0) \) satisfy \( \sum_{i \in \mathcal{V}} \nabla u_i(z_i(0)) = 0 \), and \( z_i(k + 1) \) are related to \( z_i(k) \) through
\[
\sum_{i \in \mathcal{V}} \nabla u_i(z_i(k + 1)) = \sum_{i \in \mathcal{V}} \nabla u_i(z_i(k)), \quad k = 0, 1, \ldots
\] (14)
then (12) holds. To satisfy \( \sum_{i \in \mathcal{V}} \nabla u_i(z_i(0)) = 0 \), it suffices that each node \( i \in \mathcal{V} \) computes \( z^*_i \) on its own and sets
\[
z_i(0) = z^*_i, \quad \forall i \in \mathcal{V}
\] (15)
since \( \nabla u_i(z^*_i) = 0 \). To satisfy (14), we consider the iterations described as follows
\[
z_i(k + 1) = \left( \nabla u_i(z_i(k)) \right)^{-1} \left( \nabla u_i(z_i(k)) - \sum_{j \in \mathcal{N}_i} a_{ij} (z_j(k) - z_i(k)) \right)
\] (16)
where \( 0 < \gamma < \frac{1}{d_{\max}} \) (see, Lemma 3 in [17]) and \( d_{\max} \) is the maximum degree of the graph \( \mathcal{G} \). The operator \( \left( \nabla u_i \right)^{-1} \) in (16) denotes the inverse operation of \( \nabla u_i \). It means that the following equation holds.
\[
\nabla u_i(z_i(k + 1)) - \nabla u_i(z_i(k)) = \sum_{j \in \mathcal{N}_i} a_{ij} (z_j(k) - z_i(k)).
\] (17)

Based on the above analysis, we can get a solution of optimization problem in (10) by using an algorithm presented in (15) and (16). Now, we give a theorem that guarantees the convergence of the algorithm.

**Theorem 1**: Under Assumption 11 if the graph \( \mathcal{G} \) is connected, and the positive parameter \( \gamma \) is chosen sufficiently small, then the algorithm in (16) asymptotically solves the problem (10) for the initial condition (15).

**Proof**: See Appendix [18]
Algorithm 1: DC-ELM

1 **Initialization:** Set the number of hidden neurons $L$ and the tunable parameter $C$. For all $i \in \mathcal{V}$, chose the type of activation function $g(w, b, x)$, set the same random weights and bias $\{w_i, b_i\}_{i=1}^{L}$ for each network node. Set $k = 0$ and the maximum number of iterations $k_{\text{max}}$.

2 Calculate $H_i$, $T_i$ using local training data set $S_i$.

3 $P_i \leftarrow H_i^T H_i$, $Q_i \leftarrow H_i^T T_i$.

4 $\Omega_i \leftarrow \left(\frac{I_C}{C} + P_i\right)^{-1}$.

5 $\beta_i = \Omega_i Q_i$.

6 repeat

7 $\beta_i \leftarrow \beta_i + \frac{1}{\sqrt{\nu_i}} \sum_{j \in \mathcal{N}_i} a_{ij}(\beta_j - \beta_i)$

8 Send $\beta_i$ to $\mathcal{N}_i$, and receive $\beta_j$, $j \in \mathcal{N}_i$.

9 $k \leftarrow k + 1$

10 until $k > k_{\text{max}}$.

Proof: See Appendix C.

The DC-ELM iterations (20) and (21) are summarized as Algorithm 1. All nodes have available $V_C$. Every node $i$ computes its local matrices $H_i$, $T_i$, and also computes the $L \times L$ matrix $\Omega_i$, which remains unchanged during the whole process of iterations. Then node $i$ renews its own estimate $\beta_i$ with all its one-hop neighbors $j \in \mathcal{N}_i$.

### E. Online DC-ELM

Traditional ELM used for batch learning assumes that all the training data are ready before execution. However, in many practical applications data may arrive sequentially instead of being available from the start, and may be learned one-by-one or chunk-by-chunk. Under such conditions, a number of data instances will be added or removed at each location. In the following we describe an online version of the DC-ELM algorithm.

Each node $i$ can get the hidden layer $H_i$ and the training data target matrix $T_i$ from the initial data set $S_i = \{(x_{n,i}, t_{n,i})\}_{n=1}^{N_i}$. Also, it can get $\Delta H_i$ and $\Delta T_i$ from the newly arrived training data set $S_i = \{(x_{n,i}, t_{n,i})\}_{n=1}^{N_i}$.

For convenience of explanation, we define $H_i \triangleq [H_i^T, \Delta H_i]^T$, $T_i \triangleq [T_i^T, \Delta T_i]^T$, $H_i^+ \triangleq [H_i^+, \Delta H_i]^T$, and $T_i^+ \triangleq [T_i^+, \Delta T_i]^T$, where $H_i^+$ denotes the enlarged hidden layer output matrix with newly arrived training data set $\Delta S_i$ added to the original data set $S_i$; $H_i^+$ denotes the reduced hidden layer output matrix with old expired training data set $\Delta S_i$ removed from the original data set $S_i$. Corresponding actions can be made to get $T_i^+$ and $T_i^-$. When the training data increases or decreases, the row size of matrices $H_i$ and $T_i$ is also enlarged or reduced.

At each node $i$, when some new data have been added and meanwhile some old data have been removed, the local training can be expressed in term of augmented reduced hidden layer output matrix $H_i$ and corresponding label matrix $T_i$, where we denote $H_i \triangleq [H_i^T, \Delta H_i]^T$, $T_i \triangleq [T_i^T, \Delta T_i]^T$. In very large-scale distributed learning, the data set may still be large at a component node. We could easily have $N_i \gg L$ for training a local ELM. Since the size of $H_i^T H_i$ is always much smaller than that of $H_i H_i^T$, we get the optimal output weights $\beta_i^*$ by

$$\beta_i^* = \left(\frac{I_C}{C} + H_i^T H_i\right)^{-1} H_i^T T_i. \tag{22}$$

Here, the matrix $\frac{I_C}{C} + H_i^T H_i$ is totally dense and even too large to handle. According to the matrix multiplication operator, we have

$$H_i^T H_i = H_i^+ T_i^+ + \Delta H_i^+ \Delta H_i^T,$$

$$H_i^T T_i = H_i^+ T_i^+ + \Delta H_i^+ \Delta T_i^T.$$

Thus $H_i^T H_i$ and $H_i^T T_i$ can be incrementally computed.

For all $i$, $H_i^+ T_i^+$ can get the hidden layer output matrix with all its one-hop neighbors $j \in \mathcal{N}_i$. With all its one-hop neighbors $j \in \mathcal{N}_i$, and receive $\beta_j$, $j \in \mathcal{N}_i$.

Then, by (22), we also have

$$\beta_i^* = \left(\frac{I_C}{C} + P_i\right)^{-1} Q_i. \tag{25}$$

Updating (25) involves the inversion of the $L \times L$ matrix $\frac{I_C}{C} + P_i$, that may be computationally demanding for sufficiently large $L$. Fortunately, the following lemma, easily deduced from the Sherman-Morrison-Woodbury formula [48], can be invoked to obtain a low-rank update of an inverse of a matrix.

**Lemma 2:** If matrices $A$ and $I + VA^{-1}U^T$ are invertible, then $A + U^T V$ is invertible and

$$(A + U^T V)^{-1} = A^{-1} - A^{-1} U (I + VA^{-1}U^T)^{-1} VA^{-1}. \tag{26}$$

Since $\Delta N_i$ and $\Delta N_j$ are significant smaller than $N_i$, the matrix inversion $(\frac{I_C}{C} + H_i^T H_i)^{-1}$ can be computed chunk by chunk. If the expired training data set $\Delta S_i$ has been removed, the matrix inversion changes to

$$\Omega_i \leftarrow \left(\frac{I_C}{C} + P_i\right)^{-1}$$

$$= \left(\frac{I_C}{C} + P_i - \Delta H_i^+ \Delta H_i\right)^{-1}$$

$$= \left(\frac{I_C}{C} + P_i\right)^{-1} + \left(\frac{I_C}{C} + P_i\right)^{-1} \Delta H_i^+ \Delta N_i$$

$$- \Delta H_i \left(\frac{I_C}{C} + P_i\right)^{-1} \Delta H_i^+ \left(\frac{I_C}{C} + P_i\right)^{-1}$$

$$= \Omega_i + \Omega_i \Delta H_i^+ \Delta N_i - \Delta H_i \Omega_i \Delta H_i^+ \Delta H_i \Omega_i. \tag{26}$$
Algorithm 2: Online DC-ELM

1. Initialization: Set the number of hidden neurons $L$ and the tunable parameter $C$. For all $i \in V$, chose the type of activation function $g(w, b, x)$, set the same random weights and bias $\{w_i, b_i\}_{i=1}^L$ for each network node. Set $k = 0$ and the maximum number of iterations $k_{\text{max}}$.

2. Calculate $H_i, T_i$ using local training data set $S_i$

3. $P_i \leftarrow H_i^T H_j, Q_i \leftarrow H_i^T T_i$

4. $\Omega_i \leftarrow \left( \frac{1}{C} I + P_i \right)^{-1}$

5. if $\Delta S_i = \{(x_{n,i}, t_{n,i})\}_{n=1}^{\Delta N_i}$ removed then

6. $\Omega_i \leftarrow \Omega_i + \Omega_i H_i^T (I_{\Delta N_i} - \Delta H_i \Omega_i \Delta H_i^T)^{-1} \Delta H_i \Omega_i$

7. $Q_i \leftarrow Q_i - \Delta H_i^T \Delta T_i$

8. end

9. if $\boxed{\Delta S_i = \{(x_{n,i}, t_{n,i})\}_{n=1}^{\Delta N_i}}$ added then

10. $\Omega_i \leftarrow \Omega_i - \Omega_i H_i^T (I_{\Delta N_i} + \Delta H_i \Omega_i \Delta H_i^T)^{-1} \Delta H_i \Omega_i$

11. $Q_i \leftarrow Q_i + \Delta H_i^T \Delta T_i$

12. end

13. Set $\beta_i = \Omega_i Q_i$, and $k = 0$

14. repeat

15. $\beta_i \leftarrow \beta_i + \frac{1}{\sqrt{C}} \Omega_i \sum_{j \in \mathcal{N}_i} a_{ij} (\beta_j - \beta_i)$

16. Send $\beta_i$ to $N_i$, and receive $\beta_j, j \in \mathcal{N}_i$

17. $k \leftarrow k + 1$

18. until $k > k_{\text{max}}$

As can be seen from the right-hand side of the last equality in (26), the dimensions of matrices $I_{\Delta N_i} - \Delta H_i \Omega_i \Delta H_i^T$ become $\Delta N_i$, which may be smaller than $L$, and $\Omega_i$ can be called from the node’s memory stored before. Likewise, if the newly arrived training data set $\Delta S_i$ also has been added in, then the matrix inversion becomes

$$\tilde{\Omega}_i \triangleq \left( \frac{1}{C} I + P_i \right)^{-1}$$

$$= \left( \frac{1}{C} I + P_i \right)^{-1} - \frac{1}{C} \Delta H_i^T (I_{\Delta N_i} - \Delta H_i \Omega_i \Delta H_i^T)^{-1} \Delta H_i \Omega_i$$

$$+ \Delta H_i \left( \frac{1}{C} I + P_i \right)^{-1} \Delta H_i^T \left( \frac{1}{C} I + P_i \right)^{-1}$$

$$= \Omega_i - \Omega_i \Delta H_i^T (I_{\Delta N_i} + \Delta H_i \Omega_i \Delta H_i^T)^{-1} \Delta H_i \Omega_i.$$

One can also recognize that the inversion of matrix $I_{\Delta N_i} + \Delta H_i \Omega_i \Delta H_i^T$ enjoys significant computational savings through the last equality in (27), since $\Delta N_i$ is always smaller than $L$. After completion of the aforementioned updates, we finally obtain

$$\tilde{\beta}_i = \tilde{\Omega}_i \tilde{Q}_i$$

where $\tilde{Q}_i$ is updated using (24).

The online DC-ELM is summarized as Algorithm 2. Note that the only difference between the DC-ELM and this Online DC-ELM is that the latter one adds steps 4-7 to the DC-ELM algorithm when local training sets decrease, or adds steps 8-11 when local training sets increase.

IV. SIMULATIONS

This section provides experiments on both synthetic and real-world data sets to examine the feasibility of the proposed DC-ELM algorithm.

A. Test Case 1: Approximation of ‘SinC’ function with noise

The algorithms for the ELM are tested here on a synthetic data set. In this example, both the centralized ELM and the DC-ELM algorithms are used to approximate the ‘SinC’ function, a popular choice to illustrate ELMs for regression in the literature.

$$y(x) = \begin{cases} \sin(x), & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Consider a connected undirected network with $V = 4$ nodes and maximum degree $d_{\text{max}} = 2$ depicted in Fig. 2. Network communications take place according to its corresponding adjacency matrix $A$. Each node $i$ in the network acquires a local labeled training set $S_i = \{(x_{n,i}, y_{n,i})\}_{n=1}^{N_i}$. Each local training set consists of $N_i = 1,250$ labeled examples, which are randomly generated, where $x_{n,i}$’s are uniformly randomly distributed on the interval (-10, 10). Thus the network’s global training set $S = \{(x_n, y_n)\}_{n=1}^{N} = \bigcup_{i=1}^{V} S_i$ consists of $N = \sum_{i=1}^{V} N_i = 5,000$ training examples. Likewise, a testing set $S_t = \{(x_n, y_n)\}_{n=1}^{N_t}$ with $N_t = 5,000$ examples is randomly created to evaluate the generalization performance of the algorithms. To make the experiment more practical, we add noise uniformly distributed in [-0.2, 0.2] to all the training samples, while testing samples remain noise-free.

For both the centralized ELM and the DC-ELM, the feature mapping $h(x) = [g(w_1, b_1, x), \ldots, g(w_L, b_L, x)]$ is known to all users. The choice for the activation function is the Sigmoid function, which can be written as

$$g(w, b, x) = \frac{1}{1 + \exp(- (w \cdot x + b))}$$

where $\{(w_i, b_i)\}_{i=1}^{L}$ are randomly generated based on uniform distribution in advance. Moreover, all the nodes in the network share the same set $\{(w_i, b_i)\}_{i=1}^{L}$ for DC-ELM training.

For the training and testing experiences, the user-specified parameters are $(L, C)$ for the centralized ELM while one more parameter $\gamma$ for the DC-ELM. We set $C$ chosen from the range $\{2^2, 2^4, \ldots, 2^{12}, 2^{14}\}$. Each problem has been conducted for 50 trials. Simulation results including the mean square error...
As can be seen from this figure, the centralized ELM can achieve a good generalization performance as long as the number of hidden neurons \( L \) is large enough. Moreover, the results in [9] show that the performance of (centralized) ELM with Sigmoid function is not sensitive to \( L \). There are \( L=100 \) hidden layer neurons assigned in the simulation for both the centralized ELM and the DC-ELM per each network node. In other words, \( L \) need not be specified; instead, we only need to specify one parameter \( C \). In the same setting of \( C \) and \( L \), we chose appropriate parameter \( \gamma \), satisfying \( 0 < \gamma < \frac{1}{\gamma_{\text{max}}} \) for the DC-ELM. The parameter \( \gamma \) is set at \( \gamma = \frac{1}{10} \) and \( \gamma = \frac{1}{2} \) to illustrate the convergence performance in the DC-ELM tests. For comparison, the empirical risk of the centralized ELM in [7] is defined as

\[
R_c \triangleq \frac{1}{N_t} \sum_{n=1}^{N_t} \frac{1}{2} |\hat{y}_n - \tilde{y}_n| \tag{31}
\]

where \( \hat{y}_n \) is the predicted label for \( \tilde{x}_n \). The average empirical risk of the DC-ELM, as a function of iteration index, is defined as

\[
R_d(k) \triangleq \frac{1}{V N_t} \sum_{i=1}^{V} \sum_{n=1}^{N_t} \frac{1}{2} |\hat{y}_{in}(k) - \tilde{y}_{in}(k)| \tag{32}
\]

where \( \hat{y}_{in}(k) \) is the label prediction at iteration \( k \) and node \( i \) for \( \tilde{x}_n \), \( n = 1, \ldots, N_t \).

Fig. 3 shows the prediction performance of the centralized ELM and the DC-ELM after 100 iterations. The centralized and distributed empirical risks are also included for comparison. Fig. 4(a) shows the global training set, along with the prediction results found by the centralized ELM and the DC-ELM at the four nodes with \( C = 2^{28} \) and \( \gamma = \frac{1}{17} \). It depicts that the centralized ELM gets poor performance and the DC-ELM diverges when parameters \( (L, C, \gamma) \) are not appropriately selected. In Fig. 4(b) and Fig. 4(c), the risk of the DC-ELM algorithm is reduced as the number of iterations increases. In particular, Fig. 4(c) shows the true and the approximated function of the centralized ELM algorithm and prediction results of the DC-ELM after 100 iteration with \( C = 2^{8} \) and \( \gamma = \frac{1}{2} \). For the centralized ELM, the performance of the algorithms satisfies the ELM universal approximation capability theorems for sigmoid functions. The DC-ELM at all nodes converge to the equivalent centralized ELM.

B. Test Case 2: MNIST Dataset

We next illustrate the performance of the proposed approach on a handwritten character recognition problem, using the MNIST images dataset [9]. The MNIST database contains \( 28 \times 28 \) pixel-images of handwritten digits 0 to 9; each pixel takes on an integer value from 0 (black) to 255 (white). We consider a binary problem of classifying digit 3 versus digit 6 (see Fig. 5 for sample images) using the DC-ELM algorithm.
For this experiment each image is vectorized to a $748 \times 1$ vector. We use 10,000 training samples (5,000 per digit) with the corresponding labels, and a set of 1,800 test samples (900 per digit). Both the training samples and the test samples are from the MNIST database without preprocessing.

For simulation, we consider two randomly generated networks with $V=25$, algebraic connectivity 0.9234, maximum degree 11, and average degree per node 7.44 (see Fig. 6(a)); and $V=100$, algebraic connectivity 0.3023, maximum degree 22, and average degree per node 11.02 (see Fig. 6(b)). We divide the entire training set into $V$ subsets with equal size, i.e., there are $N_i=400$ training samples for each node $i$ in the network with $V=25$ nodes, and there are $N_i=100$ training samples for each node $i$ in the network with $V=100$ nodes. We set $\gamma = 0.076$ for the network with 25 nodes and $\gamma = 0.038$ for the network with 100 nodes. In both cases, the number of hidden neurons $L$ at each node is set to be 25, and tunable parameter $C = 2^{-2}$.

The figures in this section correspond to one run of the DC-ELM for the network with noiseless communication links. Fig. 4 shows the evolution of the test error for the network after 3,000 iterations with 25 nodes and 100 nodes, respectively. Fig. 7(a) shows the evolution of the test error with an equivalent centralized ELM testing accuracy 0.8989, and Fig. 7(b) shows the evolution of the test error with an equivalent centralized ELM testing accuracy 0.9200.

V. CONCLUSIONS

In this age of data explosion, distributed processing is effective to deal with large and/or distributed data from industrial, commercial, and other resources. In this paper, we have considered the problem of cooperative learning in a network setting based on ELM. The idea is motivated by consensus theory and recent progresses in distributed multi-agent optimization. The key skill is to establish a distributed architecture for training ELMs on large data sets maintained over decentralized locations. By decomposing the centralized ELM problem into a couple of subproblems with consensus constraints, we reformulated the problem as an equivalent separable distributed optimization problem, which can be solved by local communication and individually processing on only a subset of the training data. It is proved that the proposed distributed learning algorithm converges to the equivalent centralized solution. The novel algorithm is well suited for applications including distributed machine learning, large-scale data mining, and knowledge discovery on sensitive or private data sets.

In order to construct an overall learning machine, the pro-
posed DC-ELM algorithm accesses data distributed among the network nodes and runs iteratively by local communication. At every iteration, the estimates of the local ELM output weights are exchanged between neighbors. To guarantee the convergence of the proposed algorithm, we have investigated a discrete-time version of distributed optimization model. An online variant of the DC-ELM algorithm has been introduced for scenarios when new data have been added or outdated data have been removed. The DC-ELM algorithm was applied to both regression and classification problems. The performance of the DC-ELM algorithm is very close to that of the centralized ELM algorithm.

This paper provides a preliminary attempt to establish distributed ELM framework. Future research will focus on appropriate modification of the proposed algorithm for the case when the ELM feature mapping $h(x)$ is unknown. In this case, it seems possible to improve the results of this paper using kernel method, since we can define a kernel matrix as $K = H^T H$. The corresponding kernel function $k(x_i, x_j)$ is given in [10]. In addition, more future studies can be conducted on different network topologies (such as time-varying digraph), reduction of the amount of information exchanging, and more practical applications.

**APPENDIX A**

**PROOF OF LEMMA 1**

As any feasible solution of (33) satisfies $\beta_1 = \ldots = \beta_V = \beta$, problem (33) becomes

$$
\begin{align*}
\min & \quad V \left( \frac{1}{2} \beta_1^2 + \frac{C}{2} \sum_{i=1}^{V} \sum_{n=1}^{N_i} \| \xi_{n,i} \|^2 \right) \\
\text{s.t.} & \quad h(x_{n,i}) \beta = \mathbf{t}_{n,i}^T - \mathbf{t}_{n,i}^T \\
& \quad \forall i \in V, \ n = 1, \ldots, N_i
\end{align*}
$$

(34)

which is equivalent to (2), since the constant $V$ can be dropped form the objective function in (33), and $\beta$ is a feasible solution of (7).

**APPENDIX B**

**PROOF OF THEOREM 1**

Before going to the details of the proof, we first give an assumption on the convexity of objective functions and two properties that will be used later.

**Assumption 2:** With Assumption 1, there exists a constant $\theta_i > 0$ for each $i \in V$ such that the following equivalent conditions hold

$$
\begin{align*}
& \quad u_i(y) - u_i(z) - \nabla u_i(z)^T (y - z) \geq \theta_i \frac{1}{2} \| y - z \|^2, \\
& \quad \nabla^2 u_i(z) \geq \theta_i \mathbf{I}_L, \forall y, z \in \mathbb{R}^L
\end{align*}
$$

(35)

where $\nabla^2 u_i$ denotes the symmetric Hessian of $u_i$ and $\geq$ denotes matrix inequality (i.e., $\mathbf{A} \geq \mathbf{B}$ means $\mathbf{A} - \mathbf{B}$ is a positive semidefinite matrix).

**Lemma 3 (Rayleigh Inequality, [50]):** Consider a nonsingular symmetric matrix $\mathbf{A} \in \mathbb{R}^{L \times L}$, and let $\lambda_{\min}$ and $\lambda_{\max}$ be, respectively, the minimum and maximum eigenvalues of $\mathbf{A}$. Under these conditions, for any $z \in \mathbb{R}^L$, the following inequality holds:

$$
\lambda_{\min} \| z \|^2 \leq z^T A z \leq \lambda_{\max} \| z \|^2.
$$

(36)

**Proposition 2 (Mean-Value Theorem, [32]):** For each $i \in V$, the twice-differentiable function $u_i$, it holds that

$$
\nabla u_i(y) = \nabla u_i(z) + \left( \int_0^1 \nabla^2 u_i \left( z + t(y - z) \right) dt \right) (y - z).
$$

(37)

**Proposition 3:** Since graph $G$ is undirected (i.e., $a_{ij} = a_{ji}$), We deduce that $\sum_{j \in N_i} a_{ij} \xi_j (k) - z_i (k) = 0$. By applying it to (17), we have

$$
\sum_{i \in V} \nabla u_i (z_i(k) + 1) = \sum_{i \in V} \nabla u_i (z_i(k)).
$$

(38)

Next we rewrite the distributed optimization model (16) and (15) in a matrix form as

$$
\nabla u(z(k+1)) - \nabla u(z(k)) = - \gamma (L \otimes I_L) z(k).
$$

(39)

$z(0) = z^0$

(40)

which is a legitimate Lyapunov function $V : \mathbb{R}^{2LV} \rightarrow \mathbb{R}$, defined as

$$
V(z) = \sum_{i \in V} \left( u_i(z^*) - u_i(z_i(k)) - \nabla u_i(z_i(k))^T (z^* - z_i(k)) \right). \quad (41)
$$

According to (35), it is easy to verify that

$$
V(z) \geq \sum_{i \in V} \frac{\theta_i}{2} \| z^* - z_i(k) \|^2, \quad \forall z \in \mathbb{R}^{2LV}
$$

(42)

which implies $V(z) > 0 \forall z \neq z^*$ and $V(z) \rightarrow \infty$ as $\| z \| \rightarrow \infty$. Thus, $V$ in (41) is a legitimate Lyapunov function candidate, which may be used to ensure (11).

In the following, we further verify the difference of function $V$ at $z(k+1)$ and $z(k)$. It follows from (41) that

$$
\begin{align*}
V(z(k+1)) & - V(z(k)) \\
& = - \sum_{i \in V} \left( u_i(z_i(k+1)) - u_i(z_i(k)) + \nabla u_i(z_i(k+1)) \right. \\
& \quad - \nabla u_i(z_i(k)))^T (z^* - \nabla u_i(z_i(k+1))^T z_i(k+1) + \nabla u_i(z_i(k))^T z_i(k)) \\
& = - \sum_{i \in V} \left( u_i(z_i(k+1)) - u_i(z_i(k))) \right. \\
& \quad - \left. \nabla u_i(z_i(k))^T (z_i(k+1) - z_i(k)) \right) \\
& \quad - \nabla u_i(z_i(k+1)) - \nabla u_i(z_i(k))^T z_i(k+1) \\
& \quad + \sum_{i \in V} \nabla u_i(z_i(k+1)) - \nabla u_i(z_i(k))^T z_i(k+1).
\end{align*}
$$

(43)
Due to (34) and (38), we obtain
\[
V(z(k+1)) - V(z(k)) \leq \left( \nabla u(z(k+1)) - \nabla u(z(k)) \right)^T z(k+1). \tag{44}
\]

Assumption 2 implies that \( \nabla^2 u_i \) is positive definite and invertible. Referring to Proposition 2, we write \( z(k+1) \) as
\[
z(k+1) = z(k) + \left( \int_0^1 \nabla^2 u \left( z(k) + t(z(k+1) - z(k)) \right) \right)^{-1} \left( \nabla u(z(k+1)) - \nabla u(z(k)) \right) dt \tag{45}
\]
where \( \nabla^2 u \left( z(k) + t(z(k+1) - z(k)) \right) = \nabla^2 u \left( z_k + t(z_{k+1} - z_k) \right) \). By (35), we can get
\[
\nabla^2 u \left( z_k + t(z_{k+1} - z_k) \right) \geq \psi \Omega_{LV}, \tag{46}
\]

Substituting (45) into (44), following Rayleigh Inequality, we obtain
\[
V(z(k+1)) - V(z(k)) = \left( \nabla u(z(k+1)) - \nabla u(z(k)) \right)^T z(k) + \left( \int_0^1 \nabla^2 u \left( z(k) + t(z_{k+1} - z_k) \right) \right)^{-1} \left( \nabla u(z(k+1)) - \nabla u(z(k)) \right) dt \]
\[
\leq \left( - \gamma (\Omega \otimes I_L) z(k) \right)^T z(k) + \frac{1}{\gamma} \left( \nabla u(z(k+1)) - \nabla u(z(k)) \right) \leq 0.
\]

Let \( \eta = \sup \{ \eta : \eta \mathcal{L}^2 \leq 2 \mathcal{L} \} \). Then, it follows that
\[
V(z(k+1)) - V(z(k)) \leq - \gamma z(k)^T (\mathcal{L} \otimes I_L) z(k) + \frac{2\gamma^2}{\eta^2} \nabla^2(t(k)) - \frac{2}{(i,j) \in \mathcal{E}} a_{ij} \right) \leq 0.
\]

If \( \gamma \) can be chosen such that \( 0 < \gamma < \frac{1}{2\eta^2} \), \( \{ V(z(k)) \} \) is nonnegative and non-increasing. The proof is completed.

APPENDIX C
PROOF OF THEOREM 2

For convenience of analysis, we let
\[
B(k) = \begin{bmatrix} \beta_1(k) \\ \vdots \\ \beta_V(k) \end{bmatrix}, \quad B^* = \begin{bmatrix} \beta_1^* \\ \vdots \\ \beta_V^* \end{bmatrix},
\]
\[
\Omega = \text{blockdiag} \{ \Omega_1, \ldots, \Omega_V \}
\]
where \( \Omega_i = \left( \frac{1}{\tau_i} + H_i^T H_i \right)^{-1} \), \( \beta_i^* = \left( \frac{1}{\tau_i} + H_i^T H_i \right)^{-1} H_i^T T_i \).

The equations (20) and (21) can be written in matrix form as
\[
B(k+1) = \left( I_{LV} - \frac{\gamma}{V} \Omega (\mathcal{L} \otimes I_L) \right) B(k) \tag{48}
\]
\[
B(0) = B^* \tag{49}
\]

where \( \mathcal{L} \in \mathbb{R}^{V \times V} \) is the Laplacian matrix and \( \otimes \) denotes the Kronecker product. Since \( G \) is an undirected graph, \( I_V^2 \mathcal{L} = 0 \), and \( \mathcal{L} I_V = 0 \), where \( I_V \) (0) denotes a vector of all ones (zeros) of size \( V \).

Let \( W = I_{LV} - \Omega (\mathcal{L} \otimes I_L) \), it is easy to verify that
\[
W^T W = \begin{bmatrix} I_{LV} - \frac{\gamma}{V} \Omega (\mathcal{L} \otimes I_L) \end{bmatrix} I_{LV}
= I_{LV} - \frac{\gamma}{V} \Omega (\mathcal{L} \otimes I_L) (I_V \otimes I_L)
= I_{LV} - \frac{\gamma}{V} \Omega 0_V \otimes I_L
= I_{LV},
\]

and also
\[
1_{LV}^T W = 1_{LV}^T \left( I_{LV} - \frac{\gamma}{V} \Omega (\mathcal{L} \otimes I_L) \right) 1_{LV}
= 1_{LV}^T \frac{\gamma}{V} \Omega 1_V \otimes I_L
= 1_{LV} - \frac{\gamma}{V} \Omega (I_V \otimes 1_L) (\mathcal{L} \otimes I_L)
= 1_{LV} - \frac{\gamma}{V} \Omega 0_V \otimes 1_L
= 1_{LV}^T. \tag{51}
\]

It is immediate to check that the matrix \( W \) is doubly stochastic. Due to that \( \Omega \) is positive definite and \( \mathcal{L} \otimes I_L \) is positive semidefinite, then \( \lambda (\Omega (\mathcal{L} \otimes I_L)) \geq 0 \), we have
\[
\lambda (W) = \lambda \left( I_{LV} - \frac{\gamma}{V} \Omega (\mathcal{L} \otimes I_L) \right)
= 1 - \frac{\gamma}{V} \lambda (\Omega (\mathcal{L} \otimes I_L))
\leq 1. \tag{52}
\]

The essential spectral radius \( \rho_{\text{ess}}(W) \) of the stochastic matrix \( W \) is defined as the second largest eigenvalue in modulus of the matrix \( W \), i.e. if we consider the ordered eigenvalues in modulus \( 1 = | \lambda_1(W) | \geq | \lambda_2(W) | \geq \cdots \geq | \lambda_V(W) | \), then \( \rho_{\text{ess}}(W) = | \lambda_2(W) | \). It is well-known that for stochastic matrices \( W \) yielding consensus, the convergence is exponential and its rate is given by the essential spectral radius of \( W \).
