Lindhard function, optical conductivity and plasmon mode of a linear triple component fermionic system

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We investigate the nature of density response of linear triple component fermions by computing the Lindhard function, dielectric function, plasmon mode and long wavelength optical conductivity of the system and compare the results with those of Weyl fermions and three dimensional free electron gas. Linear triple component fermions are the low energy quasiparticles of linear triple component semimetals, consisting of linearly dispersive and dispersionless (flat band) excitations. The presence of flat band brings about notable modifications in the response properties with respect to Weyl fermions such as induction of a new region in the particle-hole continuum, reduced plasmon energy gap, shift in absorption edge, enhanced rate of increase in energy absorption with frequency and forbidden intercone transitions in the long wavelength limit. The plasmon dispersion follows the usual $\omega \sim \omega_0 + \omega_1 q^2$ nature as observed in most of the three dimensional electronic systems.

I. INTRODUCTION

Three dimensional semimetals having linear energy spectra around the Fermi level viz. Weyl fermions and Dirac fermions have become breeding grounds for plethora of intriguing physical phenomena such as Fermi arc surface states, chiral anomaly, anomalous Hall effect etc. The quasiparticles close to the band-crossing nodes act as condensed-matter versions of Weyl and massless Dirac fermions theorized in high-energy physics. Recent studies have unveiled other classes of topological semimetals where more than two bands cross at a node and exhibit fermionic excitations with no counterpart in high energy physics. It is speculated that mirror and discrete rotational symmetries in symmorphic crystals may lead to topologically protected three-fold degenerate crossing points. First-principles calculations have shown that materials such as TaN, MoP, WC, RhSi, RhGe and ZrTe can host three-band crossings in the neighborhood of the Fermi level. In this paper, we deal with one such class of semimetals with three-band crossings, where quasiparticles around the nodes transform under pseudospin-1 representation. These are called triple-component semimetals (TCSs) and their low energy excitations are called triple component fermions (TCFs). The pseudospin degrees of freedom may emerge from specific admixtures of orbital and spin projections.

The dynamics of the TCFs are governed by the Hamiltonian $H(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \mathbf{S}$, where $\mathbf{S} = (S_x, S_y, S_z)$ denote the usual spin-1 matrices and $\mathbf{d}(\mathbf{k})$ is a vector function of $\mathbf{k}$. The band structure consists of two dispersive bands and a flat band. The TCFs can be grouped into linear, quadratic and cubic, depending on the form of $\mathbf{d}(\mathbf{k})$. For linear TCFs, the energy scales linearly with all the three components of momentum. For quadratic and cubic TCFs, the energy scales linearly with $k_x$, but as $k_x^2$ and $k_x^3$ respectively in the $k_x-k_y$ plane, where $k_\perp = \sqrt{k_x^2 + k_y^2}$. Time-reversal symmetric TCFs arise in materials with space group symmetry 199 and 214, e.g. Pd$_3$Bi$_2$S$_2$ and Ag$_2$Se$_2$Au$_2$. Material realizations of time-reversal symmetry (TRS)-breaking TCFs is still absent but are predicted to be found in magnetically ordered systems. In this work, we analyse the density and current response functions for linear TCFs in non-interacting and interacting limits.

**Linear Response:** When a system is subjected to a time and space dependent electric field (or potential), the field couples with the charge degree of freedom of electrons and drives the system into a non-equilibrium phase. The system responds by modifying its charge density and induction of charge currents. If the strength of the external field is small enough to be treated as a perturbation, the response function is obtained from the Kubo formula. For stronger fields, the system shows nonlinear response and may give rise to Floquet bands. The Kubo formula usually holds good for the typical amplitudes of electric fields used in the experiments. It is used to obtain current and density responses of the system, which are given by its conductivity and polarizability respectively. The polarizability relates the induced density fluctuation of the electron gas to the external potential, while the conductivity relates the induced current to the external field. The conductivity can be calculated from the polarizability and vice-versa.

The polarizability function in momentum-frequency space is called the Lindhard function. The imaginary part of Lindhard function is a measure of energy absorbed by the system. The electron gas allows absorption for a range of momentum and frequency, which is attributed to intraband or interband particle-hole excitations across the Fermi sea. This region in the momentum-frequency space is called the particle-hole continuum (PHC). At $T \to 0$, energy is not absorbed from the field for frequencies and momenta outside the PHC. The shape of the PHC depends on the chemical potential, band structure and overlap factor between the bands.

On inclusion of Coulomb interaction between the electrons, the Lindhard function gets renormalized by the dielectric function within Random Phase Approximation (RPA). The renormalization gives rise to plas-
mon modes\textsuperscript{53} which are perceived as collective oscillations of the interacting electrons in the uniform positively charged background of the system (jellium model). They correspond to the poles of the renormalized Lindhard function or the zeroes of the dielectric function. They may be grouped into gapped or gapless modes, which require finite and infinitesimal amounts of energy respectively to be excited at long wavelengths ($q \rightarrow 0$). The curve in momentum-frequency space along which the dielectric function vanishes is called the plasmon dispersion. Plasmons can be excited at wavelengths and frequencies given by the plasmon dispersion. They are probed using inelastic scattering experiments such as electron energy loss spectroscopy\textsuperscript{52}. The dimensionality and band structure of the system play crucial roles in determining the plasmon dispersion. It has been studied that 2D electron systems such as 2D free electron gas (FEG)\textsuperscript{53},\textsuperscript{69} graphene\textsuperscript{60–66},\textsuperscript{67} and dice lattice\textsuperscript{68} (2D pseudospin-1 system) host a gapless plasmon mode with dispersion $\sim \sqrt{q}$ at long wavelengths. In contrast, 3D FEG\textsuperscript{69}, 3D noncentrosymmetric metals\textsuperscript{69} and doped Weyl\textsuperscript{60–66} and Dirac semimetals\textsuperscript{72–74} exhibit gapped plasmon modes dispersing as $\sim \omega_0 + \omega_1 q^2$ in the long wavelength limit, where $\omega_0$ and $\omega_1$ are constants with appropriate dimensions. However, the study of response functions and plasmons in TCSs is still unexplored. We fill this gap in the research by making a comprehensive analysis of the Lindhard function, PHC, dielectric function and optical conductivity for linear isotropic TCFs and compare the results with those of Weyl fermions and 3D FEG. Furthermore, we derive approximate analytical expression of Lindhard function and plasmon dispersion in the long wavelength limit. We investigate the interplay of three bands and the effect of flat band in particular in the response functions. For the rest of the paper, ‘TCF’ and ‘Weyl fermions/semimetals’ would refer to linear isotropic TCF and isotropic type-I Weyl fermions/semimetals repectively.

This paper is organized as follows. In Sec. \textsection II, we review the low energy band structure and eigenstates of TCF. In Sec. \textsection IIIA, we obtain the Lindhard function and PHC of doped linear TCS. The calculation of dielectric function and plasmon modes of the system are shown in Sec. \textsection IIIB. A discussion on optical conductivity is presented in Sec. \textsection IIIC. Finally, the results are summarized in Sec. \textsection IV.

II. MODEL HAMILTONIAN

The Hamiltonian of TCFs around a band touching node is given by

$$H(k) = \hbar v_F \mathbf{S} \cdot \mathbf{k}. \quad (1)$$

Here, $v_F$ is the Fermi velocity and $\mathbf{S} = (S_x, S_y, S_z)$ denotes the usual spin-1 matrices. The band structure comprises of three bands viz. $E_{k+} = \hbar v_F k$ (conduction band), $E_{k-} = -\hbar v_F k$ (valence band) and $E_{k0} = 0$ (flat band). Denoting the pseudospin basis states $\{|s\rangle\}$ as $|\uparrow\rangle = (1 \ 0 \ 0)^T$, $|\downarrow\rangle = (0 \ 1 \ 0)^T$ and $|\downarrow\rangle = (0 \ 0 \ 1)^T$, where $T$ stands for transpose, the single-particle eigenstates $\{|\lambda(k)\rangle\}$ are given by

$$| + (k) \rangle = \left( \begin{array}{c} \cos^2 \theta \frac{\eta}{\sqrt{2}} e^{i\phi} \\
\sin \theta \sin \frac{\eta}{\sqrt{2}} e^{i2\phi} \\
\sin \theta \cos \frac{\eta}{\sqrt{2}} e^{i2\phi} \end{array} \right) \quad ,$$

$$| - (k) \rangle = \left( \begin{array}{c} \sin^2 \theta \frac{\eta}{\sqrt{2}} e^{-i\phi} \\
\sin \theta \sin \frac{\eta}{\sqrt{2}} e^{-i2\phi} \\
\sin \theta \cos \frac{\eta}{\sqrt{2}} e^{-i2\phi} \end{array} \right) \quad (2)$$

and

$$| 0 (k) \rangle = \left( \begin{array}{c} -\sin \frac{\eta}{\sqrt{2}} e^{-i\phi} \\
\cos \theta e^{i\phi} \\
\sin \theta e^{i\phi} \end{array} \right) \quad , \quad (3)$$

III. RESPONSE FUNCTIONS OF TCFS

A. LINDHARD FUNCTION

A brief review of the theory of linear density response for a multi-band system is presented in Appendix A. The dynamical polarization function or the Lindhard function (A16) of a non-interacting system of electrons is given by

$$\chi(q, \omega) = \sum \sum \frac{F_{\lambda,\lambda'}(k, k+q)(f_{\lambda,k} - f_{\lambda',k+q})}{h(\omega + i\eta)} + E_{\lambda,k} - E_{\lambda',k+q}, \quad (4)$$

where $g$ is the degeneracy factor, $F_{\lambda,\lambda'}(k, k+q) = |\langle \lambda(k)|\lambda'(k+q)\rangle|^2$ is the overlap between the corresponding states and $f_{\lambda,k} = [e^{\beta(E_{\lambda,k} - E_F)} + 1]^{-1}$ is the Fermi-Dirac distribution function.

For TCF, the interband and intraband overlaps between the dispersive bands is given by

$$F_{\lambda,\lambda'}(k, k+q) = \frac{1}{4} \left[ 1 + \lambda \lambda' \frac{(k+q)^2}{|k||k+q|} \right]^2, \quad \lambda, \lambda' = \pm 1 \quad (5)$$

and that between the flat and dispersive bands is

$$F_{0,\lambda}(k, k+q) = F_{\lambda,0}(k, k+q) = \frac{1}{2} \left[ 1 - \left( \frac{k}{|k||k+q|} \right)^2 \right]. \quad (6)$$

At $T \rightarrow 0$ K, the Lindhard function (4) takes the following form for $E_F > 0$ (i.e. doped TCS):

$$\chi(q, \omega) = \chi^{(+)}(q, \omega) + \chi^{(0)}(q, \omega) + \chi^{(-)}(q, \omega), \quad (7)$$

where
\[ \chi^{(+)}(q, \omega) = \lim_{\eta \to 0} \frac{g}{V} \sum_{k} \left[ \frac{F_{+,+}(k, k + q)(f_{+,k} - f_{+,k+q})}{\hbar \omega + i \eta + E_{+,k} - E_{+,k+q}} + \frac{F_{+,0}(k, k + q)f_{+,k}}{\hbar \omega + i \eta + E_{+,k} - E_{0,k+q}} - \frac{F_{0,+}(k, k + q)f_{+,k+q}}{\hbar \omega + i \eta + E_{0,k} - E_{+,k+q}} 
\right. \\
+ \left. \frac{F_{+-}(k, k + q)f_{+,k}}{\hbar \omega + i \eta + E_{+,k} - E_{-,k+q}} - \frac{F_{+,+}(k, k + q)f_{+,k+q}}{\hbar \omega + i \eta + E_{-,k} - E_{+,k+q}} \right], \]

(8)

\[ \chi^{(0)}(q, \omega) = \lim_{\eta \to 0} \frac{g}{V} \sum_{k} \left[ \frac{F_{0,+}(k, k + q)f_{0,k}}{\hbar \omega + i \eta + E_{0,k} - E_{+,k+q}} - \frac{F_{+,+}(k, k + q)f_{0,k+q}}{\hbar \omega + i \eta + E_{+,k} - E_{0,k+q}} \right] \]

(9)

and

\[ \chi^{(-)}(q, \omega) = \lim_{\eta \to 0} \frac{g}{V} \sum_{k} \left[ \frac{F_{-,+}(k, k + q)f_{-,k}}{\hbar \omega + i \eta + E_{-,k} - E_{+,k+q}} - \frac{F_{-,+}(k, k + q)f_{-,k+q}}{\hbar \omega + i \eta + E_{-,k} - E_{-,k+q}} \right]. \]

(10)

Here, we have excluded the terms which represent the intraband transitions within flat and valence bands and the interband transitions between them. This is true only for \( E_F > 0 \).

On non-dimensionalizing the quantities as \( x = k/k_F \), \( Q = q/k_F \), \( \Omega = \lim_{\eta \to 0} \hbar(\omega + i\eta)/E_F = \lim_{\eta \to 0}(\omega + i\eta)/E_F \), \( \tilde{\omega} = \hbar \omega/E_F \), and \( \tilde{\chi}^{(\lambda)}(Q, \Omega) = \chi^{(\lambda)}(q, \omega)/\chi_F \) (where \( E_F = \hbar c k_F \) and \( \chi_F = g k_F^2/(4\pi^2 \hbar c k_F) \)) and converting the summation into continuous integrals, equations (8), (9) and (10) simplify as

\[ \tilde{\chi}^{(+)}(Q, \Omega) = \int_{0}^{Q} \frac{x(\Omega + x)}{4Q} \log \left( \frac{\Omega^2 + 2\Omega x - Q^2 + 2xQ}{\Omega^2 + 2\Omega x - Q^2 - 2xQ} \right) dx \\
+ \int_{0}^{Q} \frac{(Q-x)^{2}}{16Q(x^2 + 2\Omega x + x^2)} \log \left( \frac{(Q-x)^2}{\Omega^2 + 2\Omega x - Q^2 + 2xQ} \right) dx \\
+ \int_{0}^{Q} \left[ \frac{Q^2}{2(\Omega + x)} \left[ \frac{Q^2 + 2x}{2Q^2} + \frac{(Q^2 - 2x^2)}{8Q^2 x} \log \left( \frac{(Q-x)^2}{(Q+x)^2} \right) \right] dx + (\Omega \leftrightarrow -\Omega), \right. \\
\]

(11)

\[ \tilde{\chi}^{(0)}(Q, \Omega) = \int_{0}^{Q} \frac{1}{8Q} \left[ \frac{2}{3} \left( 3Q^2 x + x^3 \right) + 2x(-2Q^2 + \Omega^2 - 2x^2) + 2Qx\Omega - \frac{(Q-x)^2(Q+x)^2}{\Omega} \log \left( \frac{x + Q - \Omega}{Q + x - \Omega} \right) \right] dx \\
+ \int_{Q}^{\Lambda} \frac{1}{8Q} \left[ \frac{2}{3} \left( 3Q^2 x + Q^3 \right) + 2Q(-2Q^2 + \Omega^2 - 2x^2) + 2Qx\Omega - \frac{(Q-x)^2(Q+x)^2}{\Omega} \log \left( \frac{x+Q}{x-Q} \right) \right] dx + (\Omega \leftrightarrow -\Omega) \\
\]

(12)
and

\[ \tilde{\chi}(-)(Q, \Omega) = \int_0^Q \frac{1}{16xQ} \left[ 2x(2Q^2 - \Omega^2 + 6Qx - 11x^2) - 2Qx(\Omega - 5x) - \frac{2}{3}(3Q^2x + x^3) - (Q - x)^2(Q + x)^2 \log \left( \frac{x + Q}{Q - x} \right) \right] \frac{(Q^2 - (\Omega - 2x)^2)^2}{\Omega - x} \log \left( \frac{2x + Q - \Omega}{2x - Q - \Omega} \right) dx 
+ \int_0^\Lambda \frac{1}{16xQ} \left[ 2Q(2Q^2 - \Omega^2 + 6Qx - 11x^2) - 2Qx(\Omega - 5x) - \frac{2}{3}(3Q^2Q + Q^3) - (Q - x)^2(Q + x)^2 \log \left( \frac{x + Q}{x - Q} \right) \right] \frac{(Q^2 - (\Omega - 2x)^2)^2}{\Omega - x} \log \left( \frac{2x + Q - \Omega}{2x - Q - \Omega} \right) dx + (\Omega + -\Omega). \] 

We restrict the limits of integration in Eqs. (12) and (13) to an ultraviolet cutoff \( \Lambda = k_c/k_F \gg 1 \). Thus, the dimensionless form of Lindhard function is

\[ \tilde{\chi}(Q, \Omega) = \tilde{\chi}^{(+)}(Q, \Omega) + \tilde{\chi}^{(0)}(Q, \Omega) + \tilde{\chi}^{(-)}(Q, \Omega). \] 

A diagram of the PHC for doped TCS \((E_F > 0)\) is shown in Fig. [1]. Like Weyl semimetals, the PHC for intraband transitions within the conduction band is bounded by \( \tilde{\omega} = Q \), \( \tilde{\omega} = 0 \) and \( \tilde{\omega} = Q - 2 \) lines, while the interband transitions between valence and conduction bands occur in the region bounded by \( \tilde{\omega} = Q \) and \( \tilde{\omega} = -Q + 2 \) lines. The flat band introduces a new region of PHC which is absent in Weyl semimetals. The PHC for interband transitions between the flat and conduction bands is above \( \tilde{\omega} = 1 \) line. So, the flat-to-conduction PHC overlaps those of intercone and intracone ones. These features were observed in dice lattice also[fig1].

![FIG. 1: Different regions of PHC for TCF. The dotted, violet and red regions indicates flat to conduction, valence to conduction and intra-conduction-band transitions respectively for \( E_F > 0 \).](image1)

![FIG. 2: Density plot of the natural logarithm of Im \( \tilde{\chi}(Q, \Omega) \) as functions of \( Q \) and \( \Omega \) for TCF.](image2)

The static Lindhard function Re \( \tilde{\chi}(Q, 0) \) as a function of \( Q \) is plotted in Fig. (3) for TCF, Weyl semimetals and 3D FEG. The function rises monotonically with \( Q \) for both TCF and Weyl fermions with the slope being higher in the former. This nature is contrary to that of FEG where the function decreases monotonically with \( Q \) with a slope discontinuity at \( \tilde{\omega} = 2 \).

**B. DIELECTRIC FUNCTION AND PLASMONS**

For TCF, the dielectric function \( \chi_{A18} \) can be written as

\[ \varepsilon(Q, \Omega) = 1 - \frac{C}{Q^2} \tilde{\chi}(Q, \Omega), \] 

where

\[ \tilde{\chi}(Q, \Omega) = \frac{1}{16xQ} \left[ 2x(2Q^2 - \Omega^2 + 6Qx - 11x^2) - 2Qx(\Omega - 5x) - \frac{2}{3}(3Q^2x + x^3) - (Q - x)^2(Q + x)^2 \log \left( \frac{x + Q}{Q - x} \right) \right] \frac{(Q^2 - (\Omega - 2x)^2)^2}{\Omega - x} \log \left( \frac{2x + Q - \Omega}{2x - Q - \Omega} \right) dx + (\Omega + -\Omega). \]
FIG. 3: Plots of \(\text{Re} [\chi(Q,0)]\) vs \(Q\) for TCF, Weyl semimetal and free electron gas (FEG). The \(\text{Re} [\chi(Q,0)]\) increases monotonically with \(Q\) for TCF and Weyl semimetals, but a decreasing function of \(Q\) for free electron gas. Also, magnitude of \(\text{Re} [\chi(Q,0)]\) for TCF is greater than that of Weyl semimetal for the same set of parameters.

FIG. 4: Density plot of the natural logarithm of loss function (27) as a function of \(q/k_F\) and \(\hbar \omega/E_F\). The plasmon mode appears as bright curve in the region where \(\text{Im} (\chi)\) vanishes. Hence, the mode is undamped. It continues to extend into the PHC where it gets damped into particle-hole excitations.

where \(C = 8g/(4\varepsilon_r \varepsilon_0 \pi^2 \hbar v_F)\). For \(v_F = 4 \times 10^5 \text{ m/s}, \varepsilon_r = 10\) and \(g = 2\), we get \(C \approx 0.347\). The undamped plasmon modes \(\tilde{\omega}_p\) for TCF can be obtained by solving the following equation for \(\Omega\) and \(Q\):

\[
1 - \frac{C}{Q^2} \text{Re} [\chi(Q,\Omega_p)] = 0. \tag{16}
\]

Since the exact solution of the Eq. (16) cannot be obtained analytically, we deduce an approximate expression of long wavelength (\(Q \ll 1\)) and low frequency (\(\tilde{\omega} \ll 1\)) plasmon mode of this system using the expansion of (14) in orders of \(Q\). The Lindhard function for small \(Q\) can

FIG. 5: Comparison of analytical solution of plasmon mode (dotted curve) for long wavelength (\(Q \ll 1\)) regime given by Eq. (25) and numerically obtained plasmon mode in the loss function plot. The agreement is good for low \(Q\) as expected.

FIG. 6: Density plot of the natural logarithm of loss function as functions of \(\varepsilon_r\) and \(\Omega\) for TCF for very small wavelengths \(Q \ll 1\). The plasmon mode (bright yellow curve) remains undamped and its frequency decreases with \(\varepsilon_r\).
be written as
\[
\tilde{\chi}(Q, \Omega) = \tilde{\chi}_{cc}(Q, \Omega) + \tilde{\chi}_{fc}(Q, \Omega) + \tilde{\chi}_{vc}(Q, \Omega),
\]
(17)
where \(\tilde{\chi}_{cc}(Q, \Omega), \tilde{\chi}_{fc}(Q, \Omega)\) and \(\tilde{\chi}_{vc}(Q, \Omega)\) are intra-conduction band, flat-to-conduction and valence-to-conduction (intercone) contributions respectively, given by
\[
\tilde{\chi}_{cc}(Q, \Omega) = \left( \frac{2}{3\Omega^2} Q^2 + \frac{2}{5\Omega^4} Q^4 + \mathcal{O}(Q^6) \right),
\]
(18)
\[
\tilde{\chi}_{fc}(Q, \Omega) = \int_0^1 \left[ \frac{-4x}{3(\Omega^2 - x^2)} Q^2 + \frac{4}{15x(\Omega^2 - x^2)} Q^4 \right] dx
\]
\[+ \int_0^\Lambda \left[ \frac{(4x)}{3(\Omega^2 - x^2)} Q^2 + \frac{4x^2(\Omega^4 - 5\Omega^2 x^2)}{15(-\Omega^2 x + x^3)^3} Q^4 \right] dx,
\]
(19)
and
\[
\tilde{\chi}_{vc}(Q, \Omega) = \left[ \int_0^1 \frac{4x^2}{-15\Omega^2 x^3 + 60x^5} + \int_0^\Lambda \frac{4x^2}{15\Omega^2 x^3 - 60x^5} \right] dx Q^4.
\]
(20)

Firstly, we obtain the plasmon energy gap \(\tilde{\omega}_p^{(0)} = \tilde{\omega}_p(Q \to 0)\) by substituting the real part of Eq. (17) up to order of \(Q^2\) in Eq. (16). The simplified form of Eq. (17) containing only the term proportional to \(Q^2\) can be written as
\[
\tilde{\chi}(Q^2, \Omega) = \frac{2}{3} \left( \frac{1}{\Omega^2} + \log \left[ \frac{1 - \Omega^2}{\Lambda^2 - \Omega^2} \right] \right) Q^2.
\]
(21)

Substituting the real part of the above expression in Eq. (16) gives
\[
1 - \frac{2}{3} C \left[ \frac{1}{(\tilde{\omega}_p^{(0)})^2} + \log \left| \frac{1}{\Lambda^2} \right| \right] + \left(-1 + \frac{1}{\Lambda^2}\right) (\tilde{\omega}_p^{(0)})^2 + \mathcal{O}((\tilde{\omega}_p^{(0)})^3) = 0.
\]
(22)

Considering \(\tilde{\omega}_p^{(0)} \ll 1\) i.e \(\hbar \tilde{\omega}_p^{(0)} \ll E_F\), we neglect the terms of the order of \((\tilde{\omega}_p^{(0)})^2\) and higher in the above equation to get the plasmon gap as
\[
\tilde{\omega}_p^{(0)} = \sqrt{\frac{\frac{2}{3} C}{1 + \frac{2}{3} C \log \Lambda^2}}.
\]
(23)

The plasmon gap depends on the cut-off \(\Lambda\). For \(\Lambda = 10\), \(\tilde{\omega}_p^{(0)} \approx 0.33\). In terms of \(E_F\), we have
\[
\omega_p^{(0)} = \frac{E_F}{\hbar} \sqrt{\frac{\frac{2}{3} C}{1 + \frac{2}{3} C \log \Lambda^2}}.
\]
(24)

So, plasmon gap is linearly proportional to \(E_F\) for large values of \(E_F\). The variation of Re \(\varepsilon(Q \to 0, \Omega)\) with \(\tilde{\omega}\) is shown in Fig. 8 for TCF, Weyl semimetals and FEG. The points marked by small circles are the plasmon energy gaps for the respective systems. The gaps show the following trend: \((\tilde{\omega}_p^{(0)})_{\text{TCF}} < (\tilde{\omega}_p^{(0)})_{\text{Weyl}} < (\tilde{\omega}_p^{(0)})_{\text{FEG}}\). Hence, for the same set of parameters, the plasmon gap of TCFs is smaller than that of (doped) Weyl semimetal.

The approximate plasmon dispersion in the long wavelength regime can be obtained by taking into account higher order terms of Eq. (17). The plasmon dispersion
upto the order of $Q^2$ is

$$\tilde{\omega}_p = \tilde{\omega}_p^{(0)} \left(1 + \frac{\xi(\tilde{\omega}_p^{(0)}) C}{2 (1 + (2C/3) \log \Lambda^2) Q^2}\right),$$

(25)

where

$$\xi(\tilde{\omega}_p^{(0)}) = \frac{4}{15} \left(\frac{3}{2(\tilde{\omega}_p^{(0)})^4} - \frac{1}{2(\tilde{\omega}_p^{(0)})^2} + \frac{3}{8}\right).$$

(26)

The plasmon mode can be traced numerically from the loss function which is defined as

$$-\text{Im} \left[ \frac{1}{\varepsilon(q, \omega)} \right] = \frac{V(q) \text{Im} [\chi]}{(1 - V(q) \text{Re} [\chi])^2 + (V(q) \text{Im} [\chi])^2}.$$  

(27)

Figure 9 shows the density plot of loss function for TCF. The plasmon mode can be spotted as the bright curve originating outside the PHC and finally merging into it. The part of the plasmon mode outside the PHC is undamped while that inside the PHC gets damped into particle-hole excitations, acquiring a finite lifetime. The zoomed version of the above plot is shown in Fig. 10, where the analytically obtained plasmon mode in Eq. (25) (labelled by dotted line) is plotted alongside the numerically obtained mode for comparison. The agreement between the two solutions holds good for low $Q$ as expected.

\section*{C. OPTICAL CONDUCTIVITY}

The optical conductivities in the non-interacting and interacting limits are related to the respective Lindhard functions as\(^{55}\)

$$\sigma(q, \omega) = \frac{i\omega e^2}{q^2} \chi(q, \omega)$$

(28)

and

$$\sigma^1(q, \omega) = \frac{i\omega e^2}{q^2} \chi^1(q, \omega)$$

(29)

respectively. The real part of optical conductivity corresponds to dissipation/absorption of energy in the medium. Using Eq. (21) in Eqs. (28) and (29), we get

$$\text{Re}[\tilde{\sigma}(Q \rightarrow 0, \Omega)] = \frac{-\tilde{\omega}}{Q^2} \text{Im} [\tilde{\chi}(Q^2, \Omega)]$$

(30)

and

$$\text{Re}[\tilde{\sigma}^1(Q \rightarrow 0, \Omega)] = \frac{-\tilde{\omega} \text{ Im} [\tilde{\chi}(Q^2, \Omega)]/Q^2}{(1 - C \text{ Re} [\tilde{\chi}(Q^2, \Omega)]/Q^2)^2 + (C \text{ Im} [\tilde{\chi}(Q^2, \Omega)]/Q^2)^2},$$

(31)

where

$$\text{Im} [\tilde{\chi}(Q^2, \Omega)] = \frac{-2}{3} \left[\frac{2\pi}{\omega} \delta(\tilde{\omega}) + \pi \Theta(\tilde{\omega}^2 - 1)\right] Q^2$$

(32)

and

$$\text{Re} [\tilde{\chi}(Q^2, \Omega)] = \frac{2}{3} \left(\frac{1}{\tilde{\omega}^2} - \pi^2 \delta(\tilde{\omega}) - \log \frac{\Lambda^2 - \tilde{\omega}^2}{1 - \tilde{\omega}^2}\right) Q^2.$$  

(33)

Here, we have defined $\text{Re}[\tilde{\sigma}(Q \rightarrow 0, \omega)] = \text{Re}[\sigma(q \rightarrow 0, \omega)]/\sigma_F$ with $\sigma_F = e^2 g k_F/(4\pi^2h)$. The variation of
Re[\hat{\sigma}(Q \to 0, \Omega)] and Re[\hat{\sigma}^\dagger(Q \to 0, \Omega)] with \omega for TCFs, Weyl semimetals and 3D FEG are plotted in Figs. [9] and [10] respectively. The zero frequency peak accounts for the intraband absorption and is evident in all the three systems. The interband absorption edges of TCF and Weyl semimetals commence at \hbar \omega = E_F and \hbar \omega = 2E_F respectively and the absorption grows linearly with frequency. For TCF, the absorption edge corresponds to the onset of flat-to-conduction absorption whereas for Weyl semimetals, it indicates valence-to-conduction (or intercone) absorption. The intercone absorption of TCF vanishes in the Q \to 0 limit since \chi_{cc}(Q, \Omega) is of the order of \Omega^4 for small Q [see Eq. (20)], which makes \hat{\sigma}_{cc}(Q, \Omega) \sim O(\Omega^2).

In the interacting limit, the zero frequency peak vanishes and new peaks emerge at frequencies corresponding to the plasmon gaps. The magnitudes of interband absorption gets suppressed for both Weyl fermions and TCF but the location of the absorption edges remain unaltered.

IV. CONCLUSION

We have explored the Lindhard function, PHC, loss function, plasmon mode and optical activity of TCF and compared the results with those of Weyl fermions and 3D free electron gas. The flat band endows the response functions with several new features which were absent in Weyl semimetals. The PHC gets extended due to transitions between flat and conduction bands which occur for frequencies above \EF/\hbar. An approximate expression for low energy plasmon dispersion has been derived within RPA using small Q expansion of the Lindhard function. The dominant contributions to the Lindhard function are of the order of \Omega^2 which represent intra-conduction band and flat-to-conduction transitions, while valence-to-conduction transitions are of the order of \Omega^4. The plasmon frequency shows the usual dependence \omega \sim \omega_0 + \omega_1 Q^2 as observed in most of the 3D electronic systems. The plasmon energy gap is proportional to \EF for \EF \gg \hbar \omega and is a decreasing function of background dielectric constant. The plasmon energy gap is reduced as compared to Weyl semimetals for the same set of parameters and no plasmon mode occurs as \EF \to 0. We obtain the analytical expression of real part of optical conductivity in the Q \to 0 limit for both noninteracting and interacting cases. Unlike Weyl semimetals, the interband optical absorption for TCF begins at \hbar \omega = E_F and the optical transitions between valence and conduction bands are forbidden in the long wavelength limit. The rate of increase in optical absorption with frequency is higher in TCFs than Weyl semimetals. On incorporating electron-electron interactions, the energy absorption gets reduced in both the systems and plasmon peaks show up at the plasmon energy gaps.

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Appendix A: Theory of linear density response

The Hamiltonian operator of an electron gas in low energy continuum model of a lattice (excluding electron-electron interactions) is given by

\[ \hat{H} = \sum_{k,\lambda} E_{\lambda k} c_{\lambda k}^\dagger c_{\lambda k}, \]  

(A1)

where \( c_{\lambda k}^\dagger \) and \( c_{\lambda k} \) are creation and annihilation operators of the single-particle states \( |\psi_{\lambda k}\rangle = |\lambda(k)\rangle|k\rangle \) with energies \( E_{\lambda k} \) and \( \lambda \) is the band index. The density operator \( \hat{\rho}(\mathbf{r}) \) is given by

\[ \hat{\rho}(\mathbf{r}) = \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}). \]  

(A2)

The field operators \( \hat{\Psi}^\dagger(\mathbf{r}) \) and \( \hat{\Psi}(\mathbf{r}) \) are generally expressed in terms of operators corresponding to momentum-spin basis \( \{|\psi_{s,k}\rangle\} \) (i.e. \( \{|s\rangle|k\rangle\} \)), which gives

\[ \hat{\rho}(\mathbf{r}) = \frac{1}{V} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \left( \sum_{k,s} c_{\mathbf{k} + \mathbf{q}}^\dagger c_{\mathbf{k}} \right). \]  

(A3)

For a three-band system, the Hamiltonian is diagonal in \( \{|\psi_{\lambda k}\rangle\} \) basis and hence it is convenient to expand \( \hat{\rho}(\mathbf{r}) \) in operators corresponding to this basis. The basis transformation equations are given by

\[ c_{\mathbf{k}} = \sum_{\lambda} \langle s|\lambda(k)\rangle c_{\lambda k}, \quad c_{\mathbf{k}}^\dagger = \sum_{\lambda} \langle s|\lambda(k)\rangle^* c_{\lambda k}^\dagger. \]  

(A4)

where \( \lambda \) is summed over \((-1, 0, 1)\). Using Eqs.(A3) and (A4) we get

\[ \hat{\rho}(\mathbf{r}) = 1/V \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \left( \sum_{\mathbf{k},\lambda_1,\lambda_2} \langle \lambda_1(k)|\lambda_2(k + \mathbf{q})\rangle c_{\lambda_1 k}^\dagger c_{\lambda_2 k + \mathbf{q}} \right). \]  

(A5)

When the system is in thermodynamic equilibrium with a reservoir at temperature T, the equilibrium electron density \( \rho(\mathbf{r}) \) given by

\[ \rho(\mathbf{r}) = \langle \hat{\rho}(\mathbf{r}) \rangle_0 = \frac{1}{Z_0} \sum_N \langle N | \hat{\rho}(\mathbf{r}) e^{-\beta \hat{H}} | N \rangle, \]  

(A6)

where \( Z_0 = \sum_N \langle N | e^{-\beta \hat{H}} | N \rangle \) is the canonical partition function, \( \beta = (k_B T)^{-1} \) and the summation runs over all the \( N \)-particle fermionic eigenstates of \( \hat{H} \). When the system is subjected to an external electric field \( \mathbf{E}_{\text{ext}}(\mathbf{r}, t) \), a perturbation of the form

\[ \hat{V}(t) = \int_{t_0} \hat{\rho}(\mathbf{r}) \phi_{\text{ext}}(\mathbf{r}, t) d\mathbf{r} \Theta(t - t_0) \]  

(A7)
gets added to the Hamiltonian \( \hat{H} \), where \( \phi_{\text{ext}}(r',t) = -e \int E_{\text{ext}}(r,t) \cdot d\mathbf{r} \) is the electric potential and \( t_0 \) is the time when the field is switched on. The new Hamiltonian is

\[
\hat{H}'(t) = \hat{H} + \hat{V}(t). \tag{A8}
\]

The time evolution of the states are now governed by \( \hat{H}'(t) \), which drives the system out of equilibrium and the electron density becomes a function of both space and time in general. Considering magnitude of the perturbation very small compared to \( \langle \hat{H} \rangle_0 \), the nonequilibrium expectation value of density up to linear order in \( \phi_{\text{ext}} \) is given by the Kubo formula as

\[
\langle \hat{\rho}(r) \rangle = \langle \hat{\rho}(r) \rangle_0 + \int dr' \int_{t_0}^\infty dt' \chi(r, r', t, t') \phi_{\text{ext}}(r', t') \tag{A9}
\]

or,

\[
\rho_{\text{ind}}(r, t) = \int dr' \int_{t_0}^\infty dt' \chi(r, r', t, t') \phi_{\text{ext}}(r', t'), \tag{A10}
\]

where \( \rho_{\text{ind}}(r, t) \equiv \langle \hat{\rho}(r) \rangle - \langle \hat{\rho}(r) \rangle_0 \) is the induced density and \( \chi(r, r', t - t') \) is the retarded density-density correlation function or polarizability given by

\[
\chi(r, r', t, t') = -i \Theta(t - t') \langle [\hat{\rho}(r, t), \hat{\rho}(r', t')] \rangle_0 / \hbar. \tag{A11}
\]

Here, \( \langle \cdots \rangle_0 \) denotes the expectation value taken with respect to the equilibrium state and \( \hat{\rho}(r, t) \) is the density operator in the interaction picture, which is defined as

\[
\hat{\rho}(r, t) = e^{i\hat{H}t/\hbar} \hat{\rho}(r) e^{-i\hat{H}t/\hbar}. \tag{A12}
\]

It can be seen that the polarizability is non-local in space and retarded in time, i.e., the response at a particular point in space at a given instant of time is correlated to the value of external field at some other point in space at any previous instant of time. Moreover, \( \langle [\hat{\rho}(r, t), \hat{\rho}(r', t')] \rangle_0 \) is always a function of \( (t - t') \) and for translationally invariant systems, it is a function of \( r - r' \). For such systems, \( \chi(r, r', t, t') \equiv \chi(r - r', t - t') \) and hence \( \rho_{\text{ind}}(r, t) \) becomes the convolution of \( \chi \) and \( \phi_{\text{ext}} \) in both time and space coordinates. By convolution theorem, we get

\[
\rho_{\text{ind}}(q, \omega) = \chi(q, \omega) \phi_{\text{ext}}(q, \omega), \tag{A13}
\]

where

\[
\chi(q, \omega) = \int d(r - r') \int_{-\infty}^{\infty} dt(t - t') \chi(r - r', t - t') e^{-i(q \cdot (r - r') - \omega(t - t'))} \tag{A14}
\]

and

\[
\phi_{\text{ext}}(q, \omega) = \int dr' \int_{-\infty}^{\infty} dt' \phi_{\text{ext}}(r', t') e^{-i(q \cdot r' - \omega t')} \tag{A15}
\]

are the Fourier transforms. On simplification, Eq. (A14) reduces to

\[
\chi(q, \omega) = \lim_{\eta \to 0} \frac{g}{\hbar(\omega + i\eta)} \sum_{k, \lambda, \lambda'} F_{\lambda, \lambda'}(k, k + q)(f_{\lambda}(k) - f_{\lambda'}(k + q)). \tag{A16}
\]

This is called the Lindhard function. In (A16), \( g \) is the degeneracy factor, \( F_{\lambda, \lambda'}(k, k + q) = |\langle \lambda(k) | \chi(k + q) | \lambda' \rangle|^2 \) is the overlap between the corresponding states and \( f_{\lambda}(k) = [e^{\beta E_{\lambda}(k)} - 1]^{-1} \) is the Fermi-Dirac distribution function.

On incorporating electron-electron interactions, the Lindhard function obtained within Random Phase Approximation (RPA) is given by

\[
\chi^{\text{RPA}}(q, \omega) = \frac{\chi(q, \omega)}{\varepsilon(q, \omega)}, \tag{A17}
\]

where superscript RPA stands for ‘interactions’, \( \chi(q, \omega) \) is the non-interacting Lindhard function given by Eq. (A16), and \( \varepsilon(q, \omega) \) is the dielectric function which has the following form:

\[
\varepsilon(q, \omega) = 1 - V(q) \chi(q, \omega). \tag{A18}
\]

Here \( V(q) = e^2/\epsilon_{\text{ext}}q^2 \) is the Fourier transform of Coulomb potential energy between electrons in SI units in a medium of background dielectric constant \( \epsilon_r \). The real space-time dielectric function \( \varepsilon(r, t) \) is the inverse Fourier transform of Eq. (A18) and acts as a response function between \( \phi_{\text{ext}} \) and \( \phi_{\text{total}} \):

\[
\phi_{\text{ext}}(r, t) = \int dr' \int_{t_0}^{\infty} dt' \varepsilon(r - r', t - t') \phi_{\text{total}}(r', t'). \tag{A19}
\]

The poles of the interacting Lindhard function in Eq. (A17) or the zeroes of the dielectric function in Eq. (A18) correspond to the collective modes of electron oscillations and are known as plasmon modes. They can be damped or undamped depending on the values of \( Q \) and \( \Omega \) of the external perturbation. The undamped plasmon modes \( \Omega \) are obtained from the zeroes of \( \text{Re} [\varepsilon(q, \omega)] \) in the region where \( \text{Im} [\chi(q, \omega)] \) vanishes.

Appendix B: Alternative derivation of real part of optical conductivity

In long wavelength limit \( (q \to 0) \), \( \text{Re} [\sigma_{xx}(\omega)] \) (excluding the zero frequency peak) can be analytically derived from Kubo formula as

\[
\text{Re} [\sigma_{xx}(\omega)] = \frac{\pi \varepsilon g^2}{(2\pi)^2 \omega} \sum_{\lambda, \lambda'} \int d^3k (f_{\lambda}(k) - f_{\lambda'}(k)) |\psi_x^{\lambda'}|^2 \delta(\Delta E_{\lambda\lambda'} - \hbar\omega), \tag{B1}
\]

where \( \Delta E_{\lambda\lambda'} = E_{\lambda}(k) - E_{\lambda'}(k) \), \( d \) is the dimensionality, \( g \) is the degeneracy and \( \varepsilon_x^{\lambda'} = \langle \psi_x^{\lambda'}(k) | \hat{\varepsilon} | \psi_{\lambda}(k) \rangle \) with
For TCF, the above expression reduces to

\[ \text{Re} [\sigma_{xx}(\omega)] = \frac{ge^2}{8\pi^2}\int d^3k \left[ (f_-(k) - f_+(k))|v_x^+|^2\delta(2hv_F k - \hbar\omega) + (f_0(k) - f_+(k))|v_x^{0+}|^2\delta(\hbar v_F k - \hbar\omega) \right]. \]  

(B2)

For TCF, \( \dot{v}_x = v_F S_x, v_x^+ = 0 \) and \( |v_x^{0+}|^2 = v_x^2(3 - \cos 2\phi + 2\cos^2 \phi \cos 2\theta)/8 \). Using these results, Eq. (B2) gives

\[ \text{Re} [\sigma_{xx}(\omega)] = \frac{ge^2\omega}{6\pi\hbar v_F} \Theta(\omega - v_F k_F), \]  

(B3)

where \( \Theta(x) \) is the usual step function. Unlike Weyl semimetals, the absorption between the linearly dispersive bands is absent in TCF. This feature is also seen in dice lattice, where it was attributed to zero (modulo 2\pi) Berry phase of the charge carriers.
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