LONG-TIME ESTIMATES FOR HEAT FLOWS ON ALE MANIFOLDS

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Abstract. We consider the heat equation associated to Schrödinger operators acting on vector bundles on asymptotically locally Euclidean (ALE) manifolds. Novel $L^p - L^q$ decay estimates are established, allowing the Schrödinger operator to have a non-trivial $L^2$-kernel. We also prove new decay estimates for spatial derivatives of arbitrary order, in a general geometric setting. Our main motivation is the application to stability of non-linear geometric equations, primarily Ricci flow, which will be presented in a companion paper. The arguments in this paper use that many geometric Schrödinger operators can be written as the square of Dirac type operators. By a remarkable result of Wang, this is even true for the Lichnerowicz Laplacian, under the assumption of a parallel spinor. Our analysis is based on a novel combination of the Fredholm theory for Dirac type operators on ALE manifolds and recent advances in the study of the heat kernel on non-compact manifolds.

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1. Introduction

Given a Schrödinger operator $\Delta_V = \nabla^* \nabla + R$ on a manifold $M^n$, acting on sections of a vector bundle $V$, we have a natural evolution problem

$$\partial_t u + \Delta_V u = 0, \quad u|_{t=0} = u_0$$

for the associated heat equation. Assuming that the potential $R$ is bounded (or decays at infinity), we are interested in the evolution operator

$$e^{-t\Delta_V} : L^p \to L^q,$$  \hspace{1cm} (1)

mapping the initial data $u_0$ to the solution $u(t, \cdot)$, which we will, by abuse of notation, also call the heat kernel. One would like to understand the following questions:

(i) What are sharp decay estimates in time for the heat kernel (1)?
(ii) Does each spatial derivative of the heat kernel decay faster?
(iii) What if $\Delta_V$ has a non-trivial $L^2$-kernel (null space)?
(iv) What if $V$ is a non-trivial vector bundle?

In the case that $M$ is compact or $M = \mathbb{R}^n$, the answers to all these questions are classical by now. There is a vast literature on these questions on general non-compact manifolds (see the surveys [Cou03, Gri99, SC10]), but all work so far has only considered a subset of these questions, see [CZ07, CDS20, Dev14, Dev18, She13] for some important recent work. The main purpose of this paper is to present new methods, which simultaneously treat all questions (i-iv) on a large class of non-compact manifolds, known as asymptotically locally Euclidean (ALE) manifolds. In particular, we prove decay estimates for arbitrary order spatial derivatives, using the Fredholm theory for Dirac type operators on ALE manifolds. This extends recent results (see e.g. [CD03, Dev18, MO20]), where decay of the first order spatial derivative of the heat kernel on functions is proven.

Our main motivation for this work is an application to Ricci flow. The simplest non-trivial non-compact Ricci-flat manifolds are ALE manifolds. In a companion paper, we are interested in proving their (non-linear) $L^p$-stability under Ricci flow. The linearized Ricci flow equation can, after a choice of gauge, be put in the form (1), where $\Delta_V$ is the Lichnerowicz Laplacian. Since the moduli space of Ricci-flat ALE manifolds is non-trivial in general, the Lichnerowicz Laplacian will typically have a non-trivial $L^2$-kernel, motivating (iii). In order to prove stability under Ricci flow, we need sharp estimates for the associated heat kernel, also for the derivatives, motivating (i) and (ii). The Lichnerowicz Laplacian acts on symmetric 2-tensors, which in general is a highly non-trivial vector bundle, also at infinity, motivating (iv). The key observation we use is due to Wang in [Wan91], where he proves that the Lichnerowicz Laplacian can, provided the manifold carries a parallel spinor, be written as the square of a Dirac type operator.

Our main results in the present paper will, in particular, apply to any square of a Dirac type operator, with suitable asymptotic structure at infinity. In addition to the Lichnerowicz Laplacian, the results apply to many geometric operators, including the Laplace-Beltrami, classical and twisted Dirac, Hodge-Laplace and Rarita-Schwinger operators. In each case, one gets a slightly different decay result. This is closely related to growth and decay rates of harmonic sections at infinity. We investigate this systematically throughout the paper.

1.1. Geometric setup. Before we explain the main results of this paper, let us introduce the geometric setting we are working in. We define

$$\mathbb{R}^n_{>1} := \mathbb{R}^n \setminus B_1 = (1, \infty) \times S^{n-1},$$

where $B_1$ is the ball of radius 1 around the origin. We will from now on assume that $n \geq 3$. Let $\Gamma$ be a finite subgroup of $SO(n)$, which acts freely on $S^{n-1}$. The quotient $S^{n-1}/\Gamma$ is a smooth
compact manifold with induced metric $g_{S^{n-1}/\Gamma}$. We define the locally Euclidean cone
\[ \mathbb{R}^n_{\infty}/\Gamma := (1, \infty) \times (S^{n-1}/\Gamma), \]
with the flat metric $\hat{g} = dr^2 + r^2 g_{S^{n-1}/\Gamma}$ and Levi-Civita connection $\hat{\nabla}$. Given a tensor field $h$, the notation
\[ h \in O_{\infty}(r^\alpha) \]
means that for each $k \in \mathbb{N}_0$, there is a constant $C_k > 0$, such that
\[ |\hat{\nabla}^k h|_{\hat{g}} \leq C_k r^{\alpha-k} \]
as $r \to \infty$ in $\mathbb{R}^n_{\infty}/\Gamma$. In this paper, we consider manifolds which are asymptotic to a locally Euclidean cone in the following sense:

**Definition 1.1** (ALE and AE manifolds). A complete Riemannian manifold $(M^n, g)$ is called asymptotically locally Euclidean, with one end of order $\tau > 0$, if there is a compact subset $K \subset M$ and a diffeomorphism $\phi : M_{\infty} := M \setminus K \to \mathbb{R}^n_{\infty}/\Gamma$ such that
\[ \phi^* g - \hat{g} \in O_{\infty}(r^{-\tau}). \]
The diffeomorphism $\phi$ will also be called “coordinate system at infinity”. If $\Gamma = \{1\}$, we call $(M^n, g)$ asymptotically Euclidean.

This definition naturally extends to ALE and AE manifolds with a finite number of ends. In this paper, we allow multiple ends unless stated otherwise. AE manifolds can be easily constructed by blowing up a compact Riemannian manifold $(M, g)$ at a finite number of points. Similarly, ALE manifolds can be constructed by blowing up a compact Riemannian orbifold at its (finitely many) orbifold points. We are particularly interested in (but do not restrict our analysis to) Ricci-flat manifolds, which are harder to construct, but do exist:

**Example 1.2** (Ricci-flat ALE manifolds). The simplest example of a Ricci-flat ALE manifold (different from $\mathbb{R}^4$) is the Eguchi-Hanson manifold. Let $\alpha_1, \alpha_2, \alpha_3$ be the standard left-invariant one-forms on $S^3$. For each $\epsilon > 0$, define the Eguchi-Hanson metric
\[ g_{eh, \epsilon} := \frac{r^2}{(r^4 + \epsilon^4)^{\frac{1}{2}}} \left( dr \otimes dr + r^2 \alpha_1 \otimes \alpha_1 + (r^4 + \epsilon^4)^{\frac{1}{2}} (\alpha_2 \otimes \alpha_2 + \alpha_3 \otimes \alpha_3) \right), \]
for $r > 0$. After we quotient by $\mathbb{Z}_2$, we can smoothly glue in an $S^2$ at $r = 0$ to get the (complete) Eguchi-Hanson manifold $(TS^2, g_{eh, \epsilon})$, which is ALE with $\Gamma = \mathbb{Z}_2$ and hyperkähler, hence Ricci-flat. This is an example of Kronheimer’s classification of hyperkähler ALE manifolds [Kro89]: Each 4-dimensional hyperkähler ALE manifold is diffeomorphic to a minimal resolution of $(\mathbb{R}^4 \setminus \{0\})/\Gamma$, where $\Gamma \subset SU(2)$ be a discrete subgroup acting freely on $S^3$.

Let throughout the paper $V \to M$ be a real or complex vector bundle, equipped with a positive definite symmetric or Hermitian inner product $\langle \cdot, \cdot \rangle$ and a compatible connection $\nabla$. We will consider Schrödinger operators of the form
\[ \nabla^* \nabla + \mathcal{R}, \]
where $\mathcal{R}$ is a smooth symmetric endomorphism field of $V$. We will assume that $\nabla^* \nabla + \mathcal{R}$ is asymptotic to a standard Laplacian at infinity, so that we can apply the elliptic theory. In order to do this, we need the restriction of $V$ to $M_{\infty}$ to be a quotient by $\Gamma$ of a trivial bundle $\mathbb{R}^n_{\infty} \times K^m$. In other words, we want $V$ to satisfy the same $\Gamma$-equivariance at infinity, as the manifold does:
**Assumption 1.3** (The vector bundle at infinity). We assume that there is a representation
\[ \Gamma \to \text{End}(\mathbb{K}^m), \]
respecting the standard Riemannian or Hermitian inner product, and a bundle isomorphism \( \Phi \),
such that the following diagram commutes:
\[ \begin{array}{ccc}
V_\infty & \overset{\Phi}{\longrightarrow} & (\mathbb{R}_{>1}^n \times \mathbb{K}^m)/\Gamma \\
\downarrow & & \downarrow \\
M_\infty & \overset{\phi}{\longrightarrow} & \mathbb{R}_{>1}^n/\Gamma \\
\end{array} \]

**Remark 1.4.** The representation (3) is in most geometric examples simply induced by the action
of \( \Gamma \) on \( \mathbb{R}_{>1}^n \), such examples include the tensor bundle and the spinor bundle. Note, however,
that the action of \( \Gamma \) is indeed non-trivial in these examples.

1.2. Main results.

1.2.1. *Almost Euclidean heat kernel estimates.* We have the following assumptions on our
Schrödinger operator \( \Delta^\ast \nabla^\ast \nabla + \mathcal{R} \):

**Definition 1.5.** Let \( \Phi \) be a trivialization as in Assumption 1.3. A Schrödinger operator
\[ \Delta_V := \nabla^\ast \nabla + \mathcal{R}, \]
with a symmetric endomorphism field \( \mathcal{R} \), is said to be *asymptotic to a Euclidean Laplacian* if
\[ \Phi^\ast \nabla - \tilde{\nabla} \in \mathcal{O}_\infty (r^{-1-\tau}), \]
\[ \Phi^\ast \mathcal{R} \in \mathcal{O}_\infty (r^{-2-\tau}), \]
where \( \tilde{\nabla} \) is the connection \( (\mathbb{R}_{>1}^n \times \mathbb{K}^m)/\Gamma \), induced by the trivial connection.

**Remark 1.6.** For example the connection \( \nabla \) could be the Levi-Civita connection on the tensor
or spinor bundles and the potential \( \mathcal{R} \) could be expressed in terms of the curvature of
\( M \). In these cases, (4) and (5) typically follow from (2).

Under these assumptions, \( \Delta_V \) is self-adjoint on \( L^2(M, V) \) and \( \ker_{L^2}(\Delta_V) \) is finite dimensional
(see Lemma 3.1 for the latter statement). For fixed \( 1 \leq p \leq q \leq \infty \), the natural heat kernel estimate one would like to have is
\[ \|e^{-t\Delta_V}u\|_{L^q} \leq Ct^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}\|u\|_{L^p}. \]
for \( u \in L^p \) and \( t > 0 \), where \( C = C(p, q) \). For the Euclidean Laplacian \( \Delta \) on \( \mathbb{R}^n \), this estimate
holds for any \( 1 \leq p \leq q \leq \infty \) and any \( u \in L^p \). However, if the Schrödinger operator has an
\( L^2 \)-kernel (null space), then (6) cannot hold true, since elements in the kernel are *stationary*
solutions to the heat equation. One therefore wants to prove decay on elements \( L^2 \)-orthogonal
to the kernel:

**Definition 1.7.** Let \( \Delta_V = \nabla^\ast \nabla + \mathcal{R} \) be a Schrödinger operator which is asymptotic to a
Euclidean Laplacian, in the sense of Definition 1.5.

(i) The heat kernel is said to satisfy **Euclidean heat kernel estimates** if for each
\( 1 \leq p \leq q \leq \infty \) the estimate (6) holds, for each \( u \in L^p \).

(ii) Assume that \( \ker_{L^2}(\Delta_V) \subset \mathcal{O}_\infty (r^{-n}) \). The heat kernel is said to satisfy **almost Euclidean heat kernel estimates** if for each \( 1 < p \leq q < \infty \), the estimate (6) holds for each \( u \in L^p \),
which is \( L^2 \)-orthogonal to \( \ker_{L^2}(\Delta_V) \).
The condition $\ker_{L^2}(\Delta_V) \subset O_\infty (r^{-n})$ implies that $\ker_{L^2}(\Delta_V) \subset L^q$, for all $q \in (1, \infty]$. Thus, the orthogonal projection $\Pi : L^p \to \ker_{L^2}(\Delta_V)$ is well-defined for all $p \in [1, \infty)$. Our first main result is the following:

**Theorem 1.8 (Heat kernel estimate).** Let $\Delta_V = \nabla^* \nabla + R$ be a Schrödinger operator which is asymptotic to a Euclidean Laplacian, in the sense of Definition 1.5. Assume that $(\Delta_V u, u)_{L^2} \geq 0$, for all $u \in C^\infty_c$, and that $\ker_{L^2}(\Delta_V) \subset O_\infty (r^{-n})$. Then $e^{-t\Delta_V}$ satisfies almost Euclidean heat kernel estimates.

For a Schrödinger operator which is asymptotic to a Euclidean Laplacian, one in general has $\ker_{L^2}(\Delta_V) \subset O_\infty (r^{2-n})$ (see e.g. [Pac13]). The motivation for assuming that elements in the $L^2$-kernel of $\Delta_V$ decay as $r^{-n}$, comes from the fact that harmonic differential forms on ALE manifolds typically have this decay, see Proposition 4.3. This may be applied in many geometric situations, which we present below.

**Remark 1.9.** Our heat kernel estimate is nicely complemented by a result of Devyver [Dev14, Thm. 1.2.1], where he proves that Euclidean estimates hold if the $L^2$-kernel is trivial. Note also that, in case $n \geq 9$, then Theorem 1.8 follows by a very general result of Devyver in [Dev18, Thm. 1.7]. Let us also mention another interesting approach, using microlocal analysis, to obtain pointwise decay for the heat kernel (not derivatives) on functions by Sher in [She13, Thm. 2], based on the resolvent estimates in [GH08].

**Remark 1.10.** Let us finally note that if the heat kernel $k_t(x,y)$ of $e^{-t\Delta_V}$ satisfies Gaussian bounds, i.e. there exist constants $C_1, C_2 > 0$ such that

$$k_t(x,y) \leq C_1 t^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{C_2 t}},$$

then $e^{-t\Delta_V}$ satisfies Euclidean heat kernel estimates.

1.2.2. Derivative estimates. Let us now turn to the derivative estimates for the heat kernel. Our method is based on the elliptic theory for Dirac type operators on ALE manifolds, c.f. Section 2. We will prove derivative estimates for Schrödinger operator, which are squares of Dirac type operators of the following type:

**Definition 1.11.** Let $\Phi$ be a trivialization as in Assumption 1.3. Consider a first order formally self-adjoint differential operator

$$\mathcal{D}_V := A(\nabla) + B,$$

acting on sections of $V$, where $A$ and $B$ are smooth homomorphism fields. Then $\mathcal{D}_V$ is called a **Dirac type operator** if $\Phi^* \mathcal{D}_V^2$ is a Schrödinger operator and called **asymptotic to a Euclidean Dirac operator** if

$$\Phi^* A - A_0 \in O_\infty (r^{-\tau}),$$

$$\Phi^* B \in O_\infty (r^{-1-\tau}),$$

where $A_0$ is a constant homomorphism such that

$$\nabla^* \nabla = \left( A_0(\nabla) \right)^2$$

on $(\mathbb{R}^n_{>1} \times K^m) / \Gamma$.

Two important examples are the Hodge de Rham operator and the classical Dirac operator on ALE manifolds. As already mentioned, even the Lichnerowicz Laplacian is the square of a twisted Dirac operator, under the assumption of a parallel spinor.
Remark 1.12. Note that the assumptions (7), (8) and (9) imply that $\Delta_V := D^2_V$ is a Schrödinger operator, which is asymptotic to a Euclidean Laplacian, in the sense of Definition 1.5. The non-negativity in this case is automatic:

$$(\Delta_V u, u)_{L^2} = (D_V u, D_V u)_{L^2} \geq 0$$

for all $u \in C^\infty_c$. Moreover, if $\ker_L(\Delta_V) \subset O^\infty_{\infty}(r^{-n})$, then $\ker_L(\Delta_V) = \ker_L(D_V)$.

Motivated by the Euclidean case, one would ideally like to get derivative estimates

$$\|\nabla^k e^{-t\Delta_V} u\|_{L^p} \leq Ct^{-\frac{k}{2}} \|u\|_{L^p}$$

for $1 \leq p \leq \infty$, and all $u \in L^p$, which is $L^2$-orthogonal to $\ker_L(\Delta_V)$. However, on non-flat manifolds the story becomes very delicate. We will show below that for Schrödinger operators, which are squares of Dirac type operators as above, we in fact obtain (10) for a large range of $k$ and $p$. Interestingly, there seems to be a certain threshold, beyond which we only get the following weaker estimate:

$$\|\nabla^k \circ e^{-t\Delta_V} u\|_{L^p} \leq Ct^{-\frac{n}{2}+\frac{1}{p}+\epsilon} \|u\|_{L^p}$$

for all $t \geq t_0$, where $l \in \mathbb{N}_0$ is a fixed nonnegative integer and $\epsilon > 0$ and $t_0 > 0$ can be made arbitrarily small and $C = C(k, p, t_0, \epsilon)$. In order to state our main result for the derivative of the heat kernel, let us make the following definition:

**Definition 1.13.** We say that $e^{-t\Delta_V}$ satisfies derivative estimates of degree $l \in \mathbb{N}_0 \cup \{\infty\}$ if for each $k \in \mathbb{N}_0$, we have the estimate (10) for all $p \in (1, \infty)$, if $k \leq l$,

$$p \in \left(1, \frac{n}{k-l}\right), \quad \text{if } k \geq l+1,$$

and the estimate (11) for all

$$p \in (1, \infty) \setminus \left(1, \frac{n}{k-l}\right), \quad \text{if } k \geq l+1,$$

and all $u \in L^p$, which are $L^2$-orthogonal to $\ker_L(\Delta_V)$. Note that the decay rates match nicely at $p = \frac{n}{k-l}$.

The degree $l$ corresponds, in a certain sense, to the growth rate of the slowest growing harmonic section. (In fact, we will be able to allow slower growing harmonic sections, but we then require that the covariant derivative of the section to an appropriate order vanishes.) See the next subsection for the precise assumptions about this. There might, in general, exist bounded harmonic sections on the ALE manifold. Without further assumptions, one therefore would hope for derivative estimates of degree $l = 0$, which is our next main result:

**Theorem 1.14** (Derivative estimates). Let $\Delta_V = D^2_V$, where $D_V$ is a formally self-adjoint Dirac type operator, which is asymptotic to a Euclidean Dirac operator in the sense of Definition 1.11. Assume that $\ker_L(\Delta_V) \subset O^\infty_{\infty}(r^{-n})$. Then $e^{-t\Delta_V}$ satisfies derivative estimates of degree $0$.

To the best of our knowledge, this is the first result about the long-time behavior of arbitrarily high derivatives under the heat flow on non-compact manifolds. So far, derivatives of the heat flow have been only studied for Schrödinger operators on functions and, in that case, only the first derivative, c.f. the discussion in Subsection 1.2.4 below. The following proposition provides a simple way of improving the degree $l$ in Theorem 1.14:
Proposition 1.15. For all $0 \leq m \leq l$, assume that $\mathcal{D}_m$ are formally self adjoint Dirac type operators on $V \otimes T^*M^\otimes m$, which are asymptotic to a Euclidean Dirac operator, in the sense of Definition 1.11. Assume also that

$$\nabla \circ (\mathcal{D}_m - 1)^2 = (\mathcal{D}_m)^2 \circ \nabla, \quad \nabla \circ \nabla^* \leq C \cdot (\mathcal{D}_m)^2, \quad \nabla^* \circ \nabla \leq C \cdot (\mathcal{D}_{m-1})^2$$

holds for $1 \leq m \leq l$. If

(i) $\ker L_2(\mathcal{D}_l) \subset \mathcal{O}_\infty(r^{-n})$,
(ii) $\ker L_2(\mathcal{D}_l) = \{0\}$,

then $e^{-t\Delta_V}$ satisfies derivative estimates of degree $l$. Here, "strong" is explained in Definition 1.16 below.

Definition 1.16. We say that $e^{-t\Delta_V}$ satisfies strong derivative estimates of degree $l \in \mathbb{N}_0$ if the estimates in Definition 1.13 hold and, additionally, in case $k \geq l + 1$ and $p \in (1, \infty) \setminus \left(1, \frac{n}{k-1}\right]$, we may set $\epsilon = 0$ in the estimate (11).

Example 1.17. The Hodge de Rham operator $d + d^*$ on $\mathbb{R}^n$ satisfies the assumptions of Proposition 1.15 for every $l \in \mathbb{N}$.

We give further applications of Theorem 1.8, Theorem 1.14 and Proposition 1.15 to various concrete geometric differential operators below.

1.2.3. Improved derivative estimates. Theorem 1.14 can be improved by putting assumptions on the harmonic sections. In order to explain the method, let us first recall that elements in $\ker (\Delta_V) = \ker (\mathcal{D}_V^2)$ have an expansion near infinity, in the integer growth rates

$$\ldots, r^{-n}, r^{1-n}, r^{2-n}, 1, r, r^2, \ldots,$$

coming from the asymptotic rates of harmonic functions near infinity on $\mathbb{R}^n_{>1}$. The idea of the improved derivative estimates can be loosely formulated as follows:

If we can control the behavior of harmonic sections up to growth rate $l - 1$, then derivative estimates hold of degree $l$.

More precisely, we would like to have is the following implication:

$$\forall k \leq l : \quad \mathcal{D}_V^k u = 0, \quad u = o(r^k) \implies \nabla^k u = 0. \quad (16)$$

Under this assumption, the main theorem below says that (weak) derivative estimates of degree $l$ hold. Note that the stronger implication

$$\forall k \leq l : \quad \mathcal{D}_V^k u = 0, \quad u = o(r^k) \implies u = 0, \quad (17)$$

clearly implies (16).

Remark 1.18. Let us give an example, which shows that the implication (17) really is less general than (16): We choose $M = \mathbb{R}^n$ and $\mathcal{D}_V = d + d^*$ on differential forms. Any linear function $u$ on $\mathbb{R}^n$ solves $\mathcal{D}_V^k u = \Delta u = 0$. Therefore (16) does not hold for $l = 1$. However, $\nabla^2 u = 0$, so (17) is true for $l = 1$. In fact, it is even true for any $l \in \mathbb{N}$: One can show that any function $u$ satisfying $(d + d^*)^k u = 0$ and $u \in o(r^k)$ is a polynomial of degree $\leq k - 1$, c.f. the computations in [BGM71]. Therefore, $\nabla^k u$ as well.

In many applications, we are only interested in the decay for the heat kernel of a part of the vector bundle $V$. For example, we think of forms of a certain degree or as the spinors of positive or negative chirality. We will include this in the main theorem of this section as follows:
Theorem 1.19 (Improved derivative estimates). Let $\mathcal{D}_V$ be a formally self-adjoint Dirac type operator, which satisfies the assumptions of Theorem 1.14 and let $E \subset V$ be a parallel subbundle. Assume additionally that
\[ \mathcal{D}_V^2 u \in C^\infty(E) \] (18)
for all $u \in C^\infty(E)$ and that there exists an $l \in \mathbb{N}$ such that the implication
\[ \forall k \leq l : \quad \mathcal{D}_V^k u = 0, \quad u = o \left( r^k \right) \quad \implies \quad \nabla^k u = 0 \] (19)
holds for all $u \in C^\infty(E)$. Then, $e^{-t(\mathcal{D}_V)^2}|_E$ satisfies weak derivative estimates of degree $l$. Here, “weak” means that we loose an arbitrarily small $\epsilon > 0$ in decay at certain exceptional values of $p$, see Definition 1.20 below.

As already mentioned, it is useful to remember that (17) implies (16). In other words, if there are no harmonic sections of growth less than $l$, then weak derivative estimates hold of degree $l$.

Definition 1.20. We say that $e^{-t\Delta_V}$ satisfies weak derivative estimates of degree $l \in \mathbb{N}$ if the estimates in Definition 1.13 are satisfied for all $p$ and $k$, except if
\[ p \in \left\{ \frac{n}{k}, \frac{n}{k-1}, ..., n \right\}, \quad \text{if } k \leq l, \]
\[ p \in \left\{ \frac{n}{k}, \frac{n}{k-1}, ..., \frac{n}{k-l+1} \right\}, \quad \text{if } k \geq l + 1. \] (20)
in which case we instead have the following slightly weaker estimate: For each $\epsilon > 0$ and each $t_0 > 0$, there exists a constant $C = C(k, p, \epsilon, t_0)$, such that
\[ \| \nabla^k \circ e^{-t\Delta_V} u \|_{L^p} \leq Ct^{-\frac{k}{2} + \epsilon} \| u \|_{L^p} \]
holds for every $u$, which is $L^2$-orthogonal to $\ker_{L^2}(\Delta_V)$ and for every $t \geq t_0$.

In [CCH06], the authors studied the Riesz transform on AE manifolds. Due to boundedness of the Riesz transform they show that
\[ \| \nabla \circ e^{-t\Delta} \|_{p \to p} \leq Ct^{-\frac{1}{2}} \]
for $p \in (1, n)$. On the other hand, if the manifold has more than one end, this estimate does not hold for $p \geq n$ due to unboundedness of the Riesz transform (c.f also [ACDH04, Thm. 1.3]). It is also explained in [CCH06] that the problem is caused by nonconstant bounded harmonic functions on the manifold. Our Theorem 1.19 is a significant generalization of these results.

Example 1.21. The condition (16) is satisfied for the trivial line bundle $E = \Lambda^0 \mathbb{R}^n \cong \mathbb{R}$ as a subbundle of $V = \Lambda \mathbb{R}^n$ with the Hodge de Rham operator $d + d^*$. 

1.2.4. A discussion about the threshold in the derivative estimates. A long-standing problem in harmonic analysis is the question for which $p$ the Riesz transform
\[ \nabla \circ \Delta^{-\frac{1}{2}} : L^p(M) \to L^p(T^*M). \] (21)
is a bounded operator. This problem has been studied in the context of derivative estimates, (see e.g. the papers [Dev18, CDS20]), but also as an independent problem on asymptotically Euclidean [CCH06] and asymptotically conical manifolds [GH08, GH09]. In fact, it is straightforward to see that (21) implies the optimal derivative estimate
\[ \| \nabla \circ e^{-t\Delta} \|_{p \to p} \leq \| \Delta \circ e^{-t\Delta} \|_{p \to p} \leq \| \Delta^\perp \circ e^{-t\Delta} \|_{p \to p} \leq Ct^{-\frac{1}{2}}. \] (22)
However, the converse implication (22) $\Rightarrow$ (21) was established in [ACDH04, Thm. 1.3] under geometric conditions which do hold for ALE manifolds.
The latter result suggests a strong link between derivative estimates of arbitrary order and elliptic estimates. As the proofs of our results build on optimal elliptic estimates and invertibility of elliptic operators, we believe that the decay rates we established are optimal (possibly up to removing the arbitrarily small factor $\epsilon > 0$).

**Remark 1.22.** Our main results yield a good interpretation of heat kernel estimates and derivative estimates of degree of $L^p$. For simplicity, assume that $u$ is rotationally symmetric at infinity. Then $u \in L^p$ means that $u = o(r^{-\frac{n}{p}})$, i.e. $L^p$ corresponds to the spatial decay rate $r^{-\frac{n}{p}}$. Similarly, $L^q$ corresponds to the spatial decay rate $r^{-\frac{n}{q}}$. On the other hand the norm of the map

$$e^{-t\Delta V} : L^p \to L^q$$

has the temporal decay rate $t^{-\frac{n}{p}(\frac{2}{p} - \frac{1}{q})} = t^{\frac{2}{p}(\frac{n}{p} - (\frac{n}{q})})$. Heuristically, one gets this temporal decay rate by taking the difference of the two spatial decay rates $-\frac{n}{p}$ and $-\frac{n}{q}$, corresponding to $L^p$ and $L^q$, and multiplying by a factor $1/2$, which comes from parabolic rescaling $(t, r) \mapsto (\alpha^2 t, \alpha r)$. Moreover, if $\nabla^k u \in L^p$ (i.e. $\nabla^k u = o(r^{-\frac{n}{p}})$), one heuristically has $u = o(r^{-\frac{n}{p}})$ due to elliptic estimates for weighted Sobolev spaces. If $k \in \mathbb{N}$ is so small that $k - \frac{n}{p} < l$, then the norm of the map

$$\nabla^k \circ e^{-t\Delta V} : L^p \to L^p$$

has temporal decay rate $t^{-\frac{n}{p}(\frac{2}{p} - \frac{1}{q})} = t^{\frac{2}{p}(k - \frac{n}{p} - 1)}$. In this case, one gets this temporal rate by taking the difference of the spatial decay (and potentially growth) rates $o(r^{-\frac{n}{p}})$ and $o(r^{k - \frac{n}{p}})$ and again multiplying by the factor $1/2$. If $k \in \mathbb{N}$ is so large that $k - \frac{n}{p} \geq l$, then the norm of the map

$$\nabla^k \circ e^{-t\Delta V} : L^p \to L^p$$

has (up to an arbitrary small $\epsilon > 0$) temporal decay rate $t^{-\frac{n}{p} + \frac{2}{p} - \frac{1}{q} - \frac{1}{q}} = t^{\frac{2}{p}(k - l - \frac{n}{p})}$. This can be heuristically explained as follows: There is a critical growth rate $o(r^{l})$, $l \in \mathbb{N}_0$ (which is, in view of Theorem 1.14, the growth rate of a harmonic section whose behavior we can not control) which acts as a barrier. In this case, the temporal decay rate is obtained by taking the difference of the spatial decay rate $r^{-\frac{n}{q}}$ and the critical spatial growth rate $r^{l}$ (which is smaller than $r^{k - \frac{n}{p}}$) and again multiplying by the factor $1/2$.

1.2.5. Application to geometric operators. The first prominent example to which we apply Theorem 1.8 and Theorem 1.14 is the Hodge Laplacian:

**Corollary 1.23.** Let $(M, g)$ be an ALE manifold and $\Delta_H = (d + d^*)^2$ be the Hodge Laplacian on the exterior algebra $\Lambda M$. Suppose that $\mathcal{H}_1(M) := \ker_{L^2}(\Delta_H|_{T^* M}) = \{0\}$. Then, $e^{-t\Delta_H}$, satisfies almost Euclidean heat kernel estimates and derivative estimates of degree $0$.

This result requires in addition a careful analysis of the decay of harmonic forms at infinity. These decay properties in turn have some interesting consequences for the $L^p$-cohomology of these spaces. We will discuss this below.

Using this result and various bundle identifications for special holonomy metrics, we can study further operators under an additional geometric condition.

**Corollary 1.24.** Let $(M, g)$ be an ALE spin manifold which carries a parallel spinor. Then, almost Euclidean heat kernel estimates and derivative estimates of degree $0$ are satisfied by the following three operators:

(i) $e^{-t(\mathcal{D}_{T^* M})^2}$, where $\mathcal{D}_{T^* M}$ is the twisted Dirac operator on vector-spinors.

(ii) $e^{-tQ^2}$, where $Q$ is the Rarita-Schwinger operator on $3/2$-spinors.

(iii) $e^{-t\Delta_L}$, where $\Delta_L$ is the Lichnerowicz Laplacian on symmetric $2$-tensors.
In addition, we also prove estimates for the linearized Ricci curvature and the linearized de Turck vector field along $e^{-t\Delta}$. These will be relevant for applications in Ricci flow.

Using Proposition 1.15 and Theorem 1.19, we obtain better estimates for restrictions of the above operators to subbundles:

**Corollary 1.25.** Let $(M, g)$ be an ALE manifold with $\mathcal{H}_1(M) = \{0\}$ and $\Delta$ be its Laplace-Beltrami operator. Then $e^{-t\Delta}$ satisfies Euclidean heat kernel estimates and strong derivative estimates of degree 1. Moreover, if $(M, g)$ is not AE, then $e^{-t\Delta}$ satisfies derivative estimates of degree 2.

**Corollary 1.26.** Let $(M, g)$ be an ALE manifold and $\Delta_{H^1}$ be the Hodge Laplacian on $T^*M$.

(i) If $\mathcal{H}_1(M) = \{0\}$, $e^{-t\Delta_{H^1}}$ satisfies Euclidean heat kernel estimates and strong derivative estimates of degree 0.

(ii) If $\text{Ric} \geq 0$, $e^{-t\Delta_{H^1}}$ satisfies weak derivative estimates of degree 1.

(iii) If $(M, g)$ carries a parallel spinor, $e^{-t\Delta_{H^1}}$ satisfies derivative estimates of degree 1.

Note that the assumptions in (iii) imply the assumptions in (ii) which in turn imply the assumptions in (i).

**Corollary 1.27.** Let $(M, g)$ be an ALE spin manifold with nonnegative scalar curvature and let $D$ be its Dirac operator acting on spinors. Then, $e^{-tD^2}$ satisfies Euclidean heat kernel estimates and weak derivative estimates of degree 1. Moreover, if $(M, g)$ carries a parallel spinor, $e^{-t\Delta_{H^1}}$ satisfies derivative estimates of degree 1.

### 1.3. Structure of the paper.

The paper is structured as follows: In Section 2, we develop elliptic estimates for Dirac type operators on ALE manifolds. In particular, we develop scale broken estimates for Dirac type operators, thereby extending the results of [Bar86] for the Laplacian. These play a pivotal role for developing derivative estimates under heat flows.

Section 3 is the technical core of the paper and splits into three subsections. In Subsection 3.1, we prove Theorem 1.8. In Subsection 3.2, we prove a general decay result for first order differential operators, which admit good commutation properties, composed with heat flows. Subsections 3.3 and 3.4 are devoted to the proofs of Theorem 1.14, Proposition 1.15 and Theorem 1.19 and combine the results of the previous subsections with elliptic estimates and isomorphism properties of Dirac type operators on weighted Sobolev spaces.

The main result of Section 4 is Proposition 4.3, which proves an improved decay for harmonic $k$-forms on ALE manifolds, especially for $k \neq 1, n - 1$. The result follows from elliptic theory, combined with a careful analysis of closed and coclosed forms on Euclidean space, both carried out in Subsection 4.1. Consequences for the cohomology will be discussed in Subsection 4.2. In the final two Sections 5 and 6, we are going to apply all these results to natural geometric differential operators and prove the results in Subsection 1.2.5.

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### 2. Elliptic estimates for Dirac type operators

In this section, we establish elliptic estimates and Fredholm operators of Dirac type operators $D_V$ which are asymptotic to a Euclidean Dirac operator, in the sense of 1.11.
On the locally Euclidean cone. Let \( \mathbb{R}_+^n := \mathbb{R}^n \setminus \{0\} \). Let us consider the vector bundle 
\[
(R^n_+ \times K^m)/\Gamma \to \mathbb{R}_+^n/\Gamma,
\]
over the locally Euclidean cone, to which our ALE manifold converges as \( r := |x| \to \infty \), see Definition 1.1. By Definition 1.11, we know that \( D_V \) is asymptotic to a Dirac type operator
\[
D_\infty := A_0(\nabla),
\]
which squares to the Schrödinger operator
\[
\Delta_\infty := \nabla^* \nabla = (D_\infty)^2,
\]
where \( \nabla \) is the Levi-Civita connection of the flat metric. We will work with the following weighted Sobolev spaces:

**Definition 2.1.** For any \( k \in \mathbb{N}_0 \), \( p \in (1, \infty) \) and any \( \delta \in \mathbb{R} \), the weighted Sobolev space \( W^{k,p}_\delta (\mathbb{R}_+^n/\Gamma) \) is the space of \( (\mathbb{R}_+^n \times K^m)/\Gamma \)-valued sections \( u \in W^{k,p}_{\text{loc}} (\mathbb{R}_+^n/\Gamma) \) such that the norms
\[
\|u\|_{k,p,\delta} := \sum_{j=0}^{k} \left( \int_{\mathbb{R}_+^n} |(r\nabla)^j u|^p r^{-\delta p-n} dr \right)^{1/p}
\]
are finite. We also use the notation \( L^p_\delta := W^{0,p}_\delta \).

Note that sections in these spaces are allowed to have a “blow-up” at \( r = 0 \), at least for some choices of \( \delta \). The operator \( D_\infty \) is an isomorphism between certain weighted Sobolev spaces. In order to formulate this, we need to define certain **exceptional values**:

**Definition 2.2.** We define the sets
\[
\mathcal{E}_1 := \{ k \in \mathbb{Z} \mid k \neq -1, -2, \ldots, 2-n \}, \\
\mathcal{E}_2 := \mathcal{E}_1 \cup \{ 2-n \}.
\]

The main proposition is the following:

**Proposition 2.3 (The isomorphism).** Assume that \( k \in \mathbb{N}_0 \), \( p \in (1, \infty) \) and that \( \delta \in \mathbb{R} \setminus \mathcal{E}_1 \). Then
\[
D_\infty : W^{k+1,p}_\delta \to W^{k,p}_{\delta - 1}
\]
is an isomorphism and there is a constant \( C = C(n,k,p,\delta) \), such that
\[
\|u\|_{k+1,p,\delta} \leq C \|D_\infty u\|_{k,p,\delta-1}.
\]

In particular, for each \( \delta \in \mathbb{R} \setminus \mathcal{E}_1 \), the operator \( D_\infty \) is a Fredholm operator with trivial kernel and cokernel. This proposition relies on the following fact:

**Lemma 2.4.** Assume that \( k \in \mathbb{N}_0 \), \( p \in (1, \infty) \) and that \( \delta \in \mathbb{R} \setminus \mathcal{E}_2 \). Then
\[
\Delta_\infty : W^{k+2,p}_\delta \to W^{k,p}_{\delta - 2}
\]
is an isomorphism and there is a constant \( C = C(n,k,p,\delta) \), such that
\[
\|u\|_{k+2,p,\delta} \leq C \|\Delta_\infty u\|_{k,p,\delta-2}.
\]

**Proof.** We lift the sections with the projection map
\[
(R^n_+ \times K^m) \to (\mathbb{R}_+^n \times K^m)/\Gamma
\]
to Euclidean space \( \mathbb{R}^n_+ \). Lifting \( \Delta_\infty \) to \( R^n_+ \times K^m \), we get \( \Delta_{\mathbb{R}^n_+} \) acting on each \( K \)-component. The statement now follows from [Bar86, Thm. 1.7].

The second lemma we need is the following:
Lemma 2.5. Assume that
\[ D_\infty u = 0, \]
for \( u \in L^p_\delta \), for some \( \delta \in \mathbb{R} \). Then
\[ u = 0. \]

Proof. Note that
\[ \Delta_\infty u = D_\infty^2 u = 0. \]
As in the proof of the previous lemma, we first lift the section \( u \) to \( \mathbb{R}^n \times K^m \) to a section (vector-valued function) \( \hat{u} \in L^p_\delta (\mathbb{R}^n) \). We conclude that
\[ \Delta_\infty \hat{u} = 0, \]
where \( \Delta_\infty \) acts on each component of \( \hat{u} \). Local elliptic regularity implies that \( \hat{u} \in C^\infty (\mathbb{R}^n) \). Fix an eigensection \( \phi \) of \( \Delta_{S^{n-1}} \), with eigenvalue \( \lambda \), and define the smooth function
\[ r \mapsto (\hat{u}(r, \cdot), \phi)_{L^2(S^{n-1})}. \]
By writing the Laplace operator in polar coordinates, we note that
\[ \delta \varphi \mbox{ an eigensection} \]
This is a contradiction to the assumption that \( \hat{u} \leq S \) on \( S^{n-1} \). Hence, \( \hat{u} \) is injective, it follows that \( \Delta_\infty r \mathcal{E}_{\delta} = 0 \), unless \( (\hat{u}(r, \cdot), \phi)_{L^2(S^{n-1})} = 0 \). Since \( \phi \) was an arbitrary eigenfunction on the \( S^{n-1} \), it follows that \( \hat{u} = 0 \) and therefore \( u = 0. \)

We now prove Proposition 2.3:

Proof of Proposition 2.3. By Lemma 2.4, we know that
\[ \Delta^{-1}_\infty : W^{k-1,p}_\delta \to W^{k+1,p}_\delta \]
is bijective, for all \( \delta \in \mathbb{R} \setminus \mathcal{E}_2 \). By Lemma 2.5, we know that
\[ \Delta^{-1}_\infty \circ D_\infty : W^{k,p}_\delta \to W^{k+1,p}_\delta \]
is an injective bounded map for all \( \delta \in \mathbb{R} \setminus \mathcal{E}_2 \). Moreover, since
\[ \Delta^{-1}_\infty \circ D_\infty \circ D_\infty = \text{id}, \]
and \( D_\infty \) is injective, it follows that \( \Delta^{-1}_\infty \circ D_\infty \) are also surjective, and we have found the bounded bijective inverse for \( \delta \in \mathbb{R} \setminus \mathcal{E}_2 \). If \( \delta \in (\mathbb{R}(\mathcal{E}_1) \setminus (\mathbb{R}(\mathcal{E}_2)) = \mathcal{E}_2 \setminus \mathcal{E}_1 = \{2 - n\} \), we use that
\[ \Delta^{-1}_\infty : W^{k+1,p}_\delta \to W^{k-1,p}_\delta \]
is bijective and hence
\[ D_\infty \circ \Delta_\infty^{-1} : W^{k,-1,p}_{\delta} \rightarrow W^{k,p}_{\delta} \]
is an injective bounded map satisfying
\[ D_\infty \circ D_\infty \circ \Delta_\infty^{-1} = \text{id}. \]
Therefore, we also found the bijective inverse for \( \delta = 2 - n \).

**On ALE manifolds.** We now fix a Dirac type operator \( D_V \), which is asymptotic to Euclidean Dirac operator, in the sense of Definition 1.11. The goal here is to prove that \( D_V \) is a Fredholm operator between suitable weighted Sobolev spaces. Fix a point \( p \in M \). Define the distance function
\[ \rho(x) := \sqrt{1 + d(x,p)^2}, \]
where \( d \) is the Riemannian distance. By enlarging the compact subset \( K \) in Definition 1.1, if necessary, we may assume that \( \rho|_{M_\infty} \) is smooth and \( M_\infty = \rho^{-1}(R_0, \infty) \), for some large \( R_0 > 0 \).

**Definition 2.6.** For any \( k \in \mathbb{N}_0, p \in [1, \infty) \) and any \( \delta \in \mathbb{R} \), the weighted Sobolev space \( W^{k,p}_{\delta}(M) \) is the space of \( V \)-valued sections \( u \in W_{loc}^{k,p}(M) \) such that
\[ \| u \|_{k,p,\delta} := \left( \sum_{j=0}^{k} \left( \int_M \rho(\nabla)^j u)^p \rho^{-\delta - n} dx \right)^{1/p} \]
is finite. We also use the notation \( L^p_M := W^{0,p}_{\delta} \).

The main difference to the weighted Sobolev spaces on \( \mathbb{R}^n / \Gamma \), introduced above, is that the sections are not allowed to “blow up” at some interior point. This comes from the fact that \( \rho \geq 1 \) everywhere on \( M \). For \( k \leq l \) and \( \delta_1 \leq \delta_2 \), it is immediate from the definition that
\[ \| u \|_{k,p,\delta_1} \leq \| u \|_{l,p,\delta_2} \]
and we therefore have the inclusions
\[ W^{l,p}_{\delta_1}(M) \subset W^{k,p}_{\delta_2}(M). \]
For \( \delta_1 < \delta_2 \) and \( q \in (p, \infty) \), an application of the Hölder inequality yields
\[ \| u \|_{k,p,\delta_2} \leq \| 1 \|_{0,q,\delta_2-\delta_1} \| u \|_{k,q,\delta_1} \leq C \| u \|_{k,q,\delta_1}, \]
\[ \text{where } r \in (p, \infty) \text{ was chosen so that } \frac{1}{p} = \frac{1}{q} + \frac{1}{r}. \]
Hence if \( \delta_1 < \delta_2 \), the inclusion
\[ W^{k,\delta}_{\delta_1}(M) \subset W^{k,\delta}_{\delta_2}(M) \]
holds for any \( q \in (p, \infty) \). We will use both inequalities frequently throughout the paper. The following is the main elliptic estimate:

**Proposition 2.7.** Let \( k, m \in \mathbb{N} \), such that \( m \leq k \) and \( \delta, \ldots, \delta - (m - 1) \in \mathbb{R} \setminus \mathcal{L}_1 \) and let \( p \in (1, \infty) \). There are constants \( C > 0 \) and \( R > 0 \) (depending on \( n, \delta \) and \( p \)), such that
\[ \| u \|_{k,p,\delta} \leq C \| D_{\delta}^p u \|_{k-m,p,\delta-m} + C \| u \|_{L^p(B_R)} \]
for all \( u \in W^{k,p}_{\delta}(M) \).

**Proof.** In this proof, we simplify the notation by identifying \( D_V \) with \( \Phi_* D_V \) and \( u \) with \( \Phi_* u \), etc, where \( \Phi \) is the vector bundle isomorphism from Assumption 1.3. We follow the strategy in [Bar86, Proof of Thm. 1.10]. Let us start with the case when \( k = 1 \). For each \( R \geq R_0 \), let \( B_R \subset \mathbb{R}^n / \Gamma \) denote the ball of radius \( R \). Let \( \chi_R \in C_{c}^\infty(B_R) \) such that \( \chi_R = 1 \) on \( B_{\frac{R}{2}} \). Now define
\[ u_\infty := (1 - \chi_R) u. \]
By Proposition 2.3, we have
\[ \|u\|_{k,p,\delta} \leq C \|u\|_{k,p,\delta} \leq C \|D_{V}u\|_{k-1,p,\delta-1} \]
\[ \leq C \|D_{V}u\|_{k-1,p,\delta-1} + C \|(D_{V} - D_{\infty})u\|_{k-1,p,\delta-1}. \]

Since \( D_{V} \) is asymptotic to \( D_{\infty} \) as in Definition 1.11, we get the estimate
\[ \|(D_{V} - D_{\infty})u\|_{k-1,p,\delta-1} \leq \|(A - A_{0})(du)\|_{k-1,p,\delta-1} + \|B_{0}u\|_{k-1,p,\delta-1} \]
\[ \leq C \sup_{r \leq \varepsilon_{-1}} \|r(r\nabla)^{\delta}(A - A_{0})(x)\|_{\infty} \|u\|_{k,p,\delta} \]
\[ + C \sup_{r \geq \varepsilon_{-1}} \|r(r\nabla)^{\delta}B_{0}(x)\|_{\infty} \|u\|_{k,p,\delta}. \]

Fixing \( R \) large enough, we can therefore make the factors involving \( A - A_{0} \) and \( B_{0} \) small, hence we have
\[ \|u\|_{k,p,\delta} \leq C \|D_{V}u\|_{k-1,p,\delta-1}, \]
for all \( u \). By the elliptic theory on bounded domains (applied to \( B_{R} \)), we know that
\[ \|D_{V}u\|_{k-1,p,\delta-1} \leq \|(1 - \chi_{R})D_{V}u\|_{k-1,p,\delta-1} + \|A(d\chi_{R} \otimes u)\|_{k-1,p,\delta-1} \]
\[ \leq C \|(1 - \chi_{R})D_{V}u\|_{k-1,p,\delta-1} + C \|A(d\chi_{R} \otimes u)\|_{k-1,p,\delta-1} \]
\[ \leq C \|D_{V}u\|_{k-1,p,\delta-1} + C \|u\|_{L^{p}(B_{R})}. \]

Using again the elliptic theory on the bounded domain \( B_{R} \), we conclude that
\[ \|u - u_{\infty}\|_{k,p,\delta} = \|\chi_{R}u\|_{k,p,\delta} \]
\[ \leq C \|D_{V}u\|_{k-1,p,\delta-1} + C \|\chi_{R}u\|_{k-1,p,\delta-1} \]
\[ \leq C \|D_{V}u\|_{k-1,p,\delta-1} + C \|\chi_{R}u\|_{k-1,p,\delta-1} \]
\[ \leq C \|D_{V}u\|_{k-1,p,\delta-1} + C \|u\|_{L^{p}(B_{R})}. \]

The statement when \( m = 1 \) follows. Iterating this, checking that the conditions on \( \delta \) suffice, the proposition follows. \( \square \)

We get the following important gradient estimate:

**Corollary 2.8.** Let \( k \in \mathbb{N} \), \( p \in (1, \infty) \) and assume that \( k - \frac{n}{p} \notin \mathbb{N}_{0} \). Then
\[ \|\nabla^{k}u\|_{L^{p}(M)} \leq C \left( \|D_{V}^{k}u\|_{L^{p}(M)} + \|u\|_{L^{p}(B_{R})} \right) \]
for all \( u \in W^{k,p}(M) \).

**Proof.** Let \( \delta := k - \frac{n}{p} \).

For this choice of \( \delta \), we have
\[ \|\nabla^{k}u\|_{L^{p}(M)} \leq \|u\|_{k,p,\delta}, \quad \|u\|_{L^{p}(M)} = \|u\|_{0,p,\delta-k}. \]

For any section \( u \). Due to the assumption, note that
\[ \delta = k - \frac{n}{p}, \ldots, \delta - (k - 1) = 1 - \frac{n}{p} \in \mathbb{R}\setminus[1], \]
since \( 1 - \frac{n}{p} > 1 - n \). Thus, Proposition 2.3 implies the statement. \( \square \)

A direct consequence, which turns out to be useful, is:
Corollary 2.9. Let $k \in \mathbb{N}_0$, $p \in (1, \infty)$ and assume that $k - \frac{n}{p} \notin \mathbb{N}_0$. Then for any $q \in [p, \infty]$, we have
\[
\| \nabla^k u \|_{L^p(M)} \leq C \left( \| \mathcal{D}_V^k u \|_{L^p(M)} + \| u \|_{L^q(M)} \right)
\]
for all $u \in W^{k,p}_\delta(M)$.

The above estimate has restrictions on $\delta$. Let us therefore mention also the standard Sobolev estimate, which holds for any $\delta$:

Proposition 2.10. Let $k, m \in \mathbb{N}$, such that $m \leq k$, $\delta \in \mathbb{R}$ and $p \in (1, \infty)$. There is a constant $C > 0$ (depending on $n, \delta$ and $p$), such that
\[
\| u \|_{k,p,\delta} \leq C \| \mathcal{D}_V^k u \|_{k-m,p,\delta-m} + C \| u \|_{k-m,p,\delta}
\]
for all $u \in W^{k,p}_\delta(M)$.

Proof. This is proven by standard methods, using local elliptic regularity theory together with scaling techniques (c.f. [Bar86, Prop. 1.6]). \qed

From this, we conclude:

Corollary 2.11. Let $k \in \mathbb{N}_0$ and $p \in (1, \infty)$. Then for any $q \in [p, \infty)$ satisfying $\frac{1}{p} - \frac{1}{q} \in [0, \frac{n}{m})$, we have
\[
\| \nabla^k u \|_{L^p(M)} \leq C \left( \| \mathcal{D}_V^k u \|_{L^p(M)} + \| u \|_{L^q(M)} \right)
\]
for all $u \in W^{k,p}_\delta(M)$.

Proof. Let
\[
\delta := k - \frac{n}{p}.
\]
Due to the assumptions, we have $q \geq p$ and $\delta > -\frac{n}{q}$. Due to Proposition 2.10 and the Hölder inequality, we then get
\[
\| \nabla^k u \|_{L^p(M)} \leq \| u \|_{k,p,\delta} \leq C \left( \| \mathcal{D}_V^k u \|_{0,p,\delta-k} + \| u \|_{0,p,\delta} \right)
\]
\[
\leq C \left( \| \mathcal{D}_V^k u \|_{0,p,\delta-k} + \| u \|_{0,q,\delta} \right) \leq C \left( \| \mathcal{D}_V^k u \|_{L^p(M)} + \| u \|_{L^q(M)} \right),
\]
which finishes the proof. \qed

3. Long time estimates for heat flows of Schrödinger operators

In this section, we prove our main general results on the heat kernel and its derivatives: Theorem 1.8, Theorem 1.14, Proposition 1.15 and Theorem 1.19.

3.1. Heat kernel estimates. We work now under the assumptions of Theorem 1.8, which is the result we prove in this subsection. In particular, we have assumed that
\[
\ker (\Delta_V) \subset \mathcal{O}_\infty \left( p^{-n} \right) \subset L^p(M)
\]
for each $p \in (1, \infty)$. The key observation for the $L^2$-kernel, which is crucial for our analysis, is the following:

Lemma 3.1. The $L^2$-projection
\[
\Pi : L^p(M) \rightarrow \ker_{L^2}(\Delta_V) \subset L^q(M)
\]
is a finite range operator, in particular it is bounded, for all $p \in [1, \infty)$ and $q \in (1, \infty)$. 
Proof. That the $L^2$-kernel is finite dimensional follows from standard Fredholm theory, for example, by a slight generalization of [Bar86, Prop. 2.2]. Let $e_1, \ldots, e_m$ be an $L^2$-orthonormal basis of $\ker_{L^2}(\Delta_V)$. The $L^2$-projection onto the kernel is given by

$$\Pi : u \mapsto \sum_{i=1}^{m} \langle u, e_i \rangle_{L^2} e_i.$$ 

We get the estimate

$$\|\Pi(u)\|_{L^q} \leq \sum_{i=1}^{m} |\langle u, e_i \rangle_{L^2}| \|e_i\|_{L^q} \leq \sum_{i=1}^{m} \|u\|_{L^p} \|e_i\|_{L^p'} \|e_i\|_{L^q},$$

which is bounded since $p' = \frac{p}{p-1} > 1$. □

We will work with the operator $\Delta_V + \alpha \Pi$, which is self-adjoint on $L^2$ and has trivial kernel, for each $\alpha > 0$. The main heat kernel estimate for our analysis is the following:

**Theorem 3.2** (Heat kernel estimate). There is an $\alpha_0 > 0$, such that for each $\alpha \leq \alpha_0$ and for each $1 < p \leq q < \infty$, we have

$$\left\| e^{-t(\Delta_V + \alpha \Pi)} \right\|_{p \to q} \leq Ct^{-\frac{n}{2} \left(1\frac{1}{p} - \frac{1}{q} \right)},$$

for some $C = C(p,q)$.

Our main decay result on the heat kernel would follow as a simple corollary:

**Proof of Theorem 1.8, assuming Theorem 3.2.** We only need to note that $e^{-t(\Delta_V + \alpha \Pi)} u = e^{-t \Delta_V} u$ for all $u \in L^p$, which are $L^2$-orthogonal to $\ker_{L^2}(\Delta_V)$, i.e. those $u$ with $\Pi(u) = 0$. □

3.1.1. The resolvent estimate. The main ingredient needed in order to prove Theorem 3.2 is the following resolvent estimate:

**Proposition 3.3** (Resolvent estimate). For each $1 < p \leq q < \infty$, there is an $m \in \mathbb{N}_0$, such that

$$\left\| (\Delta_V + \alpha \Pi + \lambda)^{-1} \right\|_{p \to q} \leq C\lambda^{-\frac{n}{2} \left(1\frac{1}{p} - \frac{1}{q} \right)},$$

for all $\lambda > 0$.

As in [Dev14], let us write

$$\Delta_V = \nabla^* \nabla + \mathcal{R}_+ - \mathcal{R}_-,$$

where $\mathcal{R}_+$ and $\mathcal{R}_-$ are non-negative endomorphisms of $V$. We define

$$\mathcal{H} := \nabla^* \nabla + \mathcal{R}_+,$$

for any $\lambda > 0$ and write

$$\Delta_V + \alpha \Pi + \lambda = \mathcal{H} + \lambda + \alpha \Pi - \mathcal{R}_- = (\mathcal{H} + \lambda)(1 - (\mathcal{H} + \lambda)^{-1}(\mathcal{R}_- - \alpha \Pi)),

which implies that

$$(\Delta_V + \alpha \Pi + \lambda)^{-1} = (1 - T_\lambda)^{-1}(\mathcal{H} + \lambda)^{-1}.$$ (25)

The resolvent estimate, Proposition 3.3, can be divided into two steps, resolvent estimates for $\mathcal{H}$ and proving that $(1 - T_\lambda)^{-1}$ is bounded. The point is that $\mathcal{H}$ is a positive perturbation of $\nabla^* \nabla$, and resolvent estimates for such operators are classical. We have the following estimate in our situation:
Lemma 3.4 (The resolvent estimate for $\mathcal{H}$). For all $1 \leq p \leq q \leq \infty$, we have
\[
\|((\mathcal{H} + \lambda)^{-1}\|_{p \to q} \leq C\lambda^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}.
\]

Proof. This is immediate from the following computation, where we use [Dev14, Cor. 2.1.1]:
\[
\|((\mathcal{H} + \lambda)^{-1}\|_{p \to q} = \left\| \int_{0}^{\infty} e^{-t(\mathcal{H} + \lambda)} dt \right\|_{p \to q}
\leq \int_{0}^{\infty} \left\| e^{-t\mathcal{H}} \right\|_{p \to q} dt
\leq C \int_{0}^{\infty} e^{-t\lambda} \frac{1}{S_n^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}} dt
\leq C\lambda^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}.
\]

Given the resolvent estimate for $\mathcal{H}$, it suffices to prove the boundedness in $L^p$ for the operator $(1 - T_\lambda)^{-1}$:

Lemma 3.5. The $(1 - T_\lambda)^{-1} : L^p \to L^p$ is bounded for all $p \in (1, \infty)$, independently of $\lambda$.

The proof of this lemma is significantly more involved, and relies in a crucial way on the assumed decay of $\mathcal{R}_-$ and mapping properties of $\Pi$ (coming from the decay assumption on elements in the kernel of $\Delta_V$). We postpone the proof of the lemma to later in this section and instead show how one deduces Proposition 3.3 and Theorem 3.2:

Proof of Proposition 3.3. The proof follows by combining Lemma 3.4 and Lemma 3.5.

In order to use the resolvent estimate to prove Theorem 3.2, we need to prove a basic $L^2$-estimate for the heat operator:

Lemma 3.6. For each $m \in \mathbb{N}_0$:
\[
\left\| (1 + t(\Delta_V + \alpha \Pi))^m e^{-t(\Delta_V + \alpha \Pi)} \right\|_{L^2 \to L^2} \leq C(m).
\]

Proof. Given initial data in $C_0^\infty(M)$, standard theory implies that $e^{-t\Delta_V}u_0 \in C^\infty(M)$ for all $t > 0$. Another standard argument (c.f. for example the proofs of [DK20, Lem. 3.3 & 3.4]) then implies that for each $k \in \mathbb{N}_0$, we have
\[
e^{-t\Delta_V}u_0 \in H^k(M),
\]
for $t$ small enough, where $H^k(M)$ are the usual $L^2$-based (non-weighted) Sobolev spaces on $M$. Moreover, elliptic theory implies that the projection map
\[
\Pi : L^2 \to H^k
\]
is continuous for all $k \in \mathbb{N}_0$. For a $u_0 \in C^\infty_c$, let $u := e^{-t(\Delta_V + \alpha \Pi)}u_0$ and let
\[
\mathcal{E}^{2m}(u_0, t) := \sum_{j=0}^{2m} \frac{t^j}{j!} ((\Delta_V + \alpha \Pi)^j u_0)_{L^2}.
\]
Note that because
\[(1 + t(\Delta V + \alpha \Pi))^m = \sum_{j=0}^{m} \binom{m}{j} t^m (\Delta V + \alpha \Pi)^j,\]
and \(u \in H^k\) for all \(k \in \mathbb{N}_0\) and \(t > 0\), there is a constant \(C = C(m)\) such that
\[
\| (1 + t(\Delta V + \alpha \Pi))^m u \|^2 \leq C E^{2m}(u_0, t).
\]
Using that \(\partial_t u = -(\Delta V + \alpha \Pi) u\) and \(u \in H^k\) for any \(k\) and \(t > 0\) again, we compute that
\[
\frac{d}{dt} E^{2m}(u_0, t) = \sum_{j=1}^{2m} \frac{i^j}{(j-1)!} \langle (\Delta V + \alpha \Pi)^j u, u \rangle_{L^2} - 2 \sum_{j=0}^{2m} \frac{t^j}{j!} \langle (\Delta V + \alpha \Pi)^{j+1} u, u \rangle_{L^2}
\]
\[
= \sum_{j=0}^{2m-1} \frac{t^j}{j!} \langle (\Delta V + \alpha \Pi)^{j+1} u, u \rangle_{L^2} - 2 \sum_{j=0}^{2m} \frac{t^j}{j!} \langle (\Delta V + \alpha \Pi)^{j+1} u, u \rangle_{L^2}
\]
\[
\leq 0,
\]
where we have used the assumption in Theorem 1.8, saying that \(\Delta V \geq 0\). Integrating, we get
\[
E^{2m}(u_0, t) \leq E^{2m}(u_0, 0) = \|u_0\|^2_{L^2}
\]
for all \(t > 0\) which finishes the proof of the lemma. \(\square\)

Using this, we may now apply the resolvent estimate, Proposition 3.3, to obtain the following lemma:

**Lemma 3.7.** For all \(p \in (1, \infty)\), we have
\[
\left\| e^{-t(\Delta V + \alpha \Pi)} \right\|_{p \to p} \leq C.
\]
Moreover, for all \(1 < p \leq 2 < q < \infty\), we have
\[
\left\| e^{-t(\Delta V + \alpha \Pi)} \right\|_{p \to q} \leq C t^{-\frac{q}{2}}(\frac{p}{q} - \frac{1}{2}).
\]

**Proof.** By Proposition 3.3, with \(t := \frac{1}{4}\), and Lemma 3.6, we have the estimate
\[
\left\| e^{-t(\Delta V + \alpha \Pi)} \right\|_{p \to 2} \leq \left\| e^{-t(\Delta V + \alpha \Pi)} \right\|_{2 \to 2} \left\| (1 + t(\Delta V + \alpha \Pi))^{-m} \right\|_{2 \to 2}
\]
\[
\leq \left\| (1 + t(\Delta V + \alpha \Pi))^{-m} e^{-t(\Delta V + \alpha \Pi)} \right\|_{2 \to 2} \left\| (1 + t(\Delta V + \alpha \Pi))^{-m} \right\|_{p \to 2}
\]
\[
\leq C t^{-\frac{q}{2}}(\frac{p}{q} - \frac{1}{2}),
\]
for all \(p \in (1, 2]\). In particular, the first assertion holds for \(p = 2\). [Dev18, Prop. 2.6] implies the first assertion for \(p \in (1, 2)\) and duality implies the same assertion for \(p \in (2, \infty)\). By duality, we also conclude that
\[
\left\| e^{-t(\Delta V + \alpha \Pi)} \right\|_{2 \to q} \leq C t^{-\frac{q}{2}}(\frac{p}{q} - \frac{1}{2}),
\]
for all \(q \in [2, \infty)\). We get
\[
\left\| e^{-t(\Delta V + \alpha \Pi)} \right\|_{p \to q} \leq \left\| e^{-\frac{q}{2}(\Delta V + \alpha \Pi)} \right\|_{2 \to q} \left\| e^{-\frac{q}{2}(\Delta V + \alpha \Pi)} \right\|_{p \to 2}
\]
\[
\leq C t^{-\frac{q}{2}}(\frac{p}{q} - \frac{1}{2}) t^{-\frac{q}{2}}(\frac{p}{q} - \frac{1}{2})
\]
\[
= C t^{-\frac{q}{2}}(\frac{p}{q} - \frac{1}{2})
\]
as claimed. \(\square\)
The proof of Theorem 3.2 now follows easily:

**Proof of Theorem 3.2.** The idea is to interpolate the two estimates in Lemma 3.7. Let us assume that

\[ 1 < p \leq q \leq 2 \]

and choose \( \theta \in (0, 1) \) such that

\[ \frac{1}{q} = \frac{1 - \theta}{p} + \frac{\theta}{2}. \]

This is equivalent to

\[ \frac{1}{p} - \frac{1}{q} = \theta \left( \frac{1}{p} - \frac{1}{2} \right). \]

Let us denote \( u := e^{-t(\Delta_V + \alpha \Pi)} u_0 \). Interpolation and Lemma 3.7 gives

\[
\| u \|_{L^q} \leq \| u \|_{L^p}^{1 - \theta} \| u \|_{L^2}^\theta \leq \left\| e^{-t(\Delta_V + \alpha \Pi)} \right\|_{p \to p}^{1 - \theta} \| u_0 \|_{L^p}^{1 - \theta} \left\| e^{-t(\Delta_V + \alpha \Pi)} \right\|_{p \to 2}^\theta \| u_0 \|_{L^p}^\theta \leq C t^{-\frac{\theta}{2}} \| u_0 \|_{L^p} \;\]

Duality implies the case when

\[ 2 \leq p \leq q < \infty, \]

which finishes the proof. Concatenating these estimates gives the case

\[ 1 < p \leq 2 \leq q < \infty, \]

which finishes the proof of Theorem 3.2. \( \square \)

3.1.2. **The proof of Lemma 3.5.** We now return to the proof of Lemma 3.5, which we left out so far. We write

\[ \mathcal{H}_\lambda := \mathcal{H} + \lambda. \]

Recall that the statement of Lemma 3.5 is that

\[ (1 - T_\lambda)^{-1} : L^p \to L^p, \]

is bounded, for all \( p \in (2, \infty) \), where

\[ T_\lambda = \mathcal{H}^{-1}_\lambda (\mathcal{R}_- + \alpha \Pi). \]

We start by proving the following lemma:

**Lemma 3.8.** For each \( p, q \in (1, \infty) \), there is an \( N = N(p, q) \in \mathbb{N} \), such that

\[ T_\lambda^N : L^p \to L^q \]

is bounded, with the bound independent of \( \lambda \geq 0 \). Moreover, for each \( p \in (1, \infty) \),

\[ T_\lambda : L^p \to L^p \]

is bounded, with the bound independent of \( \lambda \geq 0 \).

**Proof.** We are going to show that

\[ T_\lambda : L^p \to L^q \]

is bounded, with the bound independent of \( \lambda \geq 0 \), if

\[ \frac{\tau}{n} < \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}. \]
where \( \tau > 0 \) is the constant from Definition 1.5. This statement would clearly imply both assertions in the lemma. By Lemma 3.4, we know that the map
\[
\mathcal{H}_\lambda^{-1} : L^s \to L^q
\]
is bounded, with the bound independent of \( \lambda \geq 0 \), if
\[
\frac{1}{s} := \frac{2}{n} + \frac{1}{q},
\]
It therefore remains to show that
\[
\mathcal{R}_- - \alpha \Pi : L^p \to L^s,
\]
for any \( p, s \in (1, \infty) \) such that
\[
-\tau \frac{n}{p} < \frac{1}{s} - \frac{1}{p} \leq \frac{2}{n},
\]
or equivalently
\[
0 \leq \frac{1}{s} - \frac{1}{p} < \frac{2 + \tau}{n}.
\]
The part \(-\alpha \Pi\) of the map (26) is bounded, by Lemma 3.1. We check the boundedness of \( \mathcal{R}_- \) by applying the Hölder inequality
\[
\|\mathcal{R}_- u\|_{L^s} \leq \|\mathcal{R}_-\|_{L^p, s} \|u\|_{L^p},
\]
which holds for all
\[
\frac{1}{s} - \frac{1}{p} = \frac{1}{\mu}
\]
By Definition 1.5, we know that \( \mathcal{R}_- \in \mathcal{O}(r^{-2-\tau}) \), which implies that
\[
\mathcal{R}_- \in L^\tau, \quad \forall \mu \in \left(\frac{n}{2 + \tau}, \infty\right),
\]
i.e. precisely in the range
\[
\frac{1}{\mu} = \frac{1}{s} - \frac{1}{p} \in \left[0, \frac{2 + \tau}{n}\right).
\]
This completes the proof.

We write (formally, for the moment)
\[
(1 - T_{\lambda})^{-1} = \sum_{m=0}^{\infty} T_{\lambda}^m.
\]
By Lemma 3.8, any finite sum
\[
\sum_{m=0}^{N} T_{\lambda}^m : L^p \to L^p
\]
is bounded. We therefore need estimates of \( T_{\lambda}^m : L^p \to L^p \), for large \( m \). Let us introduce the Banach space \( H_{0,\lambda}^1 \), which is the closure of \( C_c^\infty \) under the norm
\[
\|u\|_{H_{0,\lambda}^1}^2 := \|\nabla u\|_{L^2}^2 + \langle \mathcal{R}_+ u, u \rangle_{L^2} + \lambda \|u\|_{L^2}^2,
\]
for any \( \lambda > 0 \).

**Remark 3.9.** Note that
\[
\mathcal{H}_\lambda : H_{0,\lambda}^1 \to L^2
\]
is an isometry.
The idea (following Devyver [Dev14]) is to divide the operator $T^m_\lambda$ into parts:

$$T^m_\lambda : L^p \xrightarrow{\text{(i)}} H^1_{0,\lambda} \xrightarrow{\text{(ii)}} H^1_{0,\lambda} \xrightarrow{\text{(iii)}} L^p$$

for all $m > N_1 + N_2$ and $p \in (2, \infty)$, and estimate maps (i), (ii) and (iii) separately. The map $\iota$ is bounded independently of $\lambda \geq 0$, by the Sobolev embedding theorem. The integers $N_1, N_2$, which are supposed to be independent of $m$, are yet to be chosen.

**Remark 3.10.** Lemma 3.8 implies that there is an $N_2 \in \mathbb{N}$, such that (iii) is bounded.

We continue with the map (i):

**Lemma 3.11.** For each $p \in (1, \infty)$, there is an $N_1 \in \mathbb{N}$, such that the map

$$T^{N_1}_\lambda : L^p \rightarrow H^1_{0,\lambda}$$

is bounded.

**Proof.** Recall that $T_\lambda = \mathcal{H}_\lambda^{-1} (\mathcal{R}_- - \alpha \Pi)$. By [Dev14, Prop. 2.1.3] and Remark 3.9, the maps

$$\mathcal{H}_\lambda^{-\frac{1}{2}} : L^{\frac{2}{1+\alpha_0}} \rightarrow L^2,$$

$$\mathcal{H}_\lambda^{\frac{1}{2}} : L^2 \rightarrow H^1_{0,\lambda},$$

are bounded, independently of $\lambda \geq 0$. Moreover, since $\mathcal{R}_- \in L^\infty$ and Lemma 3.1, we have $\mathcal{R}_- - \alpha \Pi : L^{\frac{2}{1+\alpha_0}} \rightarrow L^{\frac{2}{1+\alpha_0}}$ is bounded, for each $p \in (1, \infty)$. We conclude that $\mathcal{H}_\lambda^{-1} (\mathcal{R}_- - \alpha \Pi) : L^{\frac{2}{1+\alpha_0}} \rightarrow H^1_{0,\lambda}$ is bounded, with a bound independent of $\lambda \geq 0$. By Lemma 3.8 implies that there is an $N \in \mathbb{N}$, such that

$$T^N_\lambda : L^p \rightarrow L^{\frac{2}{1+\alpha_0}}$$

is bounded, with a bound independent of $\lambda \geq 0$. We conclude the lemma with $N_1 : N + 1$. □

Finally, we consider map (ii):

**Lemma 3.12.** There is a constant $\epsilon \in (0, 1)$ such that

$$\|T^k_\lambda\|_{H^1_{0,\lambda} \rightarrow H^1_{0,\lambda}} \leq (1 - \epsilon)^k,$$

for all $k \in \mathbb{N}$ and for all $\lambda \geq 0$.

**Proof.** Defining, using Remark 3.9,

$$\mathcal{A}_\lambda := \mathcal{H}_\lambda^{-\frac{1}{2}} (\mathcal{R}_- - \alpha \Pi) \mathcal{H}_\lambda^{\frac{1}{2}},$$

we therefore have

$$T_\lambda = \mathcal{H}_\lambda^{-\frac{1}{2}} \mathcal{A}_\lambda \mathcal{H}_\lambda^{\frac{1}{2}},$$

and consequently

$$\|T_\lambda\|_{H^1_{0,\lambda} \rightarrow H^1_{0,\lambda}} = \|\mathcal{A}_\lambda\|_{2 \rightarrow 2}.$$

We claim that $\mathcal{A}_\lambda : L^2 \rightarrow L^2$ is a compact operator with eigenvalues in $(-1 + \epsilon, 1 - \epsilon)$ if $\alpha \in (0, \alpha_0)$ for some $\alpha_0$, sufficiently small. This would imply that

$$\|T^k_\lambda\|_{H^1_{0,\lambda} \rightarrow H^1_{0,\lambda}} \leq \|\mathcal{A}_\lambda\|_{2 \rightarrow 2}^k \leq (1 - \epsilon)^k.$$
as claimed.

**Step 1: Compactness of** $A_\lambda$. By [Dev14, Lem. 2.2.2], we know that the term $\mathcal{H}_\lambda^\frac{1}{2} R_- \mathcal{H}_\lambda^\frac{1}{2}$ is a compact operator. By [Dev14, Prop. 2.1.3], the operators

$$\mathcal{H}_\lambda^\frac{1}{2} : L^2 \to L^\frac{\infty}{2},$$

$$\mathcal{H}_\lambda^\frac{1}{2} : L^\frac{\infty}{2} \to L^2,$$

are bounded, independent of $\lambda$. Moreover, Lemma 3.1 implies that the projection

$$\Pi : L^\frac{\infty}{2} \to L^\frac{\infty}{2}$$

is compact, so therefore the composition is compact as well. This proves that $A_\lambda$ is a compact operator.

**Step 2: The eigenvalues are contained in** $(-\infty, 1]$. By the non-negativity of $\Delta V$, we know that

$$\left\| \mathcal{H}_\lambda^\frac{1}{2} u \right\|_{L^2}^2 = \langle \mathcal{H}_\lambda u, u \rangle_{L^2} \geq \langle \mathcal{H} u, u \rangle_{L^2}$$

$$= \langle (\Delta V + \alpha \Pi) u, u \rangle_{L^2} + \langle (R_- - \alpha \Pi) u, u \rangle_{L^2}$$

$$\geq \langle (R_- - \alpha \Pi) u, u \rangle_{L^2}$$

for all $u \in C_c^\infty$. This implies that for all $v := \mathcal{H}_\lambda^\frac{1}{2} u \in \mathcal{H}_\lambda^\frac{1}{2} (C_c^\infty)$, we have

$$\left\| v \right\|_{L^2}^2 = \left\| \mathcal{H}_\lambda^\frac{1}{2} u \right\|_{L^2}^2 \geq \langle (R_- - \alpha \Pi) u, u \rangle_{L^2} = \langle (\Delta V + \alpha \Pi) \mathcal{H}_\lambda^\frac{1}{2} v, \mathcal{H}_\lambda^\frac{1}{2} v \rangle_{L^2} = \langle A_\lambda v, v \rangle_{L^2}.$$ 

Since $C_c^\infty(M)$ is dense in $H^1_{0, \lambda}$ by construction and

$$\mathcal{H}_\lambda^\frac{1}{2} : H^1_{0, \lambda} \to L^2$$

is an isometry, it follows that $\mathcal{H}_\lambda^\frac{1}{2} (C_c^\infty)$ is dense in $L^2$. We have therefore proven that

$$\langle A_\lambda v, v \rangle_{L^2} \leq \left\| v \right\|_{L^2}^2$$

for all $v \in L^2(M)$, from which it follows that all eigenvalues of $A_\lambda$ are in $(-\infty, 1]$.

**Step 3: We now show that each eigenvalue of** $A_\lambda$ **is less than** $1 - \epsilon$ **for an** $\epsilon > 0$, **which is independent of** $\lambda \geq 0$. For this, we first prove that 1 is not an eigenvalue of $\mathcal{A} := A_0$. Assume, to reach a contradiction, that $\mathcal{A} v = v$ for some $v \in L^2$. Then $u := \mathcal{H}_\lambda^{-\frac{1}{2}} v \in H^1_0$ satisfies (weakly) the equation

$$(\Delta V + \alpha \Pi) u = \mathcal{H}_\lambda^{-\frac{1}{2}} (\operatorname{id} - \mathcal{A}) \mathcal{H}_\lambda^{-\frac{1}{2}} u = \mathcal{H}_\lambda^{-\frac{1}{2}} (\operatorname{id} - \mathcal{A}) v = 0.$$ 

In other words, we have

$$\Delta V u = -\alpha \Pi(u). \tag{28}$$

Since $u \in H^1_0 \hookrightarrow L^\frac{\infty}{2}$ by the Sobolev inequality, we know that $\Pi(u) \in C^\infty$. By local elliptic regularity, we conclude that $u \in C^\infty(M)$ so that (28) is satisfied classically. Using again that $u \in L^\frac{\infty}{2}$, we note that for all $w \in \ker_{L^2}(\Delta V)$, we have

$$\langle \Delta V u, w \rangle_{L^2} = \langle v, \Delta V u \rangle_{L^2} = 0,$$

i.e. that $\Delta V u \perp \ker_{L^2}(\Delta V)$. But equation (28) implies that $\Delta V u \in \ker_{L^2}(\Delta V)$, hence $\Delta V u = 0$. Equation (28) therefore implies that

$$0 = \Delta V u = -\alpha \Pi(u).$$
By this and Lemma 3.8, we conclude that 
\[ m > N \]
for all \( v \in L^2 \). This is the claim, when \( \lambda = 0 \). Using the bound
\[ \left\| H^{\frac{1}{2}} v \right\|_{2 \to 2} \leq \left\| H^{\frac{1}{2}} \right\|_{H^1_0 \to L^2} \left\| \epsilon \right\|_{H^1_0 \to H^1_0} \left\| H^{\frac{1}{2}} \right\|_{L^2 \to H^1_0} \leq 1, \]
we conclude that
\[ \langle A_\lambda v, v \rangle_{L^2} = \langle (R - \alpha \Pi) H^{\frac{1}{2}} v, H^{\frac{1}{2}} v \rangle_{L^2} \]
\[ = \langle A H^{\frac{1}{2}} H^{\frac{1}{2}} v, H^{\frac{1}{2}} H^{\frac{1}{2}} v \rangle_{L^2} \]
\[ \leq (1 - \epsilon) \left\| H^{\frac{1}{2}} H^{\frac{1}{2}} v \right\|_{L^2}^2 \]
\[ \leq (1 - \epsilon) \left\| v \right\|_{L^2}^2, \]
which means that the eigenvalues of \( A_\lambda \) are also bounded by \( 1 - \epsilon \), as claimed.

**Step 4:** We show that we can choose \( \alpha_0 \) small enough to ensure that all eigenvalues are greater than \( -(1 - \epsilon) \). Since \( R_- \) is a non-negative endomorphism and \( \Pi = \Pi^2 \) is self-adjoint, we get
\[ \langle A_\lambda v, v \rangle_{L^2} = \langle H^{\frac{1}{2}} (R_- - \alpha \Pi) H^{\frac{1}{2}} v, v \rangle_{L^2} \]
\[ = \langle R_- H^{\frac{1}{2}} v, H^{\frac{1}{2}} v \rangle_{L^2} - \alpha \langle \Pi H^{\frac{1}{2}} v, H^{\frac{1}{2}} v \rangle_{L^2} \]
\[ \geq -\alpha \left\| \Pi H^{\frac{1}{2}} v \right\|_{L^2}^2 \]
\[ \geq -\alpha C \left\| \Pi \right\|_{L^2 \to L^2}^2 \left\| v \right\|_{L^2}^2, \]
where we used boundedness of the maps \( H^{\frac{1}{2}} : L^2 \to H^1_{0, \lambda} \subset L^{\frac{2}{1-\alpha}} \) and \( \Pi : L^{\frac{2}{1-\alpha}} \to L^2 \) in the last inequality. We may now choose
\[ \alpha_0 := (1 - \epsilon) C^{-1} \left\| \Pi \right\|_{L^2 \to L^2}^2 \]
and conclude that
\[ \langle A_\lambda v, v \rangle_{L^2} \geq -(1 - \epsilon) \left\| v \right\|_{L^2}^2, \]
independently of \( \lambda \geq 0 \) and \( \alpha \leq \alpha_0 \), as claimed. This completes the proof of the lemma. \( \square \)

We may finally prove Lemma 3.5:

**Proof of Lemma 3.5.** Since we intend to write
\[ (1 - T_\lambda)^{-1} = \sum_{m=0}^{\infty} T_\lambda^m \]
we need estimates on \( T_\lambda^m \). By the decomposition (27) and Remark 3.10, Lemma 3.11 and Lemma 3.12, we have an estimate of the form
\[ \left\| T_\lambda^m \right\|_{p \to p} \leq C(1 - \epsilon)^m, \]
for all \( m > N_1 + N_2 \), where \( N_1, N_2 \in \mathbb{N} \) are fixed and \( \epsilon \in (0, 1) \), with \( C > 0 \) independent of \( m \). By this and Lemma 3.8, we conclude that
\[ \sum_{m=0}^{\infty} \left\| T_\lambda^m \right\|_{p \to p} \leq \sum_{m=0}^{N_1 + N_2} \left\| T_\lambda \right\|_{p \to p}^m + C \sum_{m=N_1 + N_2 + 1}^{\infty} (1 - \epsilon)^m < \infty. \]
This implies that

\[(1 - T_\lambda)^{-1} = \sum_{m=0}^{\infty} T_\lambda^m : L^p \to L^p\]

is bounded for all \(p \in (1, \infty)\), independently of \(\lambda \geq 0\). This finishes the proof. \(\square\)

### 3.2. Commuting operators

In this section, we lay the groundwork for our derivative estimates. We assume that we have a second vector bundle \(W \to M\), with the same assumptions as for \(V\). The following theorem is the main result here:

**Theorem 3.13.** Consider two Schrödinger operators

\[\Delta_V := \nabla^* \nabla + R,\]
\[\Delta_W := \overline{\nabla^* \nabla + R},\]

on \(V\) and \(W\), respectively, which are both assumed to satisfy the assumptions of Theorem 1.8. Let \(P : C^\infty(M,V) \to C^\infty(M,W)\) be a first-order differential operator such that

\[P \circ \Delta_V = \Delta_W \circ P\]

and such that there exist two constants \(C_1, C_2 > 0\) satisfying

\[P^* \circ P \leq C_1 \cdot \Delta_V, \quad P \circ P^* \leq C_2 \cdot \Delta_W.\]  \hspace{1cm} (29)

Then for all \(1 < p \leq q < \infty\), and all \(k \in \mathbb{N}_0\), there are constants \(C = C(n,k,p,q)\) such that

\[\|P e^{-t\Delta_V} u\|_{p \to q} \leq C t^{-\frac{n}{2}} \|u\|_{L^2},\]
\[\|P^* e^{-t\Delta_W} u\|_{p \to q} \leq C t^{-\frac{n}{2}} \|u\|_{L^2}.\]

**Remark 3.14.** The theorem applies directly to the case when \(\Delta_V = P^* P\) and \(\Delta_W = PP^*\), in particular when \(\Delta_V = (\mathcal{D}_V)^2\). However, we need this general formulation in later applications.

**Remark 3.15.** Note that because

\[P^* \Delta_W = (\Delta_W P)^* = (P \Delta_V)^* = \Delta_V P^*,\]

both the assumptions as well as the assertion are remain the same if we simultaneously interchange \(P\) and \(P^*\) as well as \(\Delta_V\) and \(\Delta_W\).

**The proof of Theorem 3.13.** In this subsection we work under the assumptions in Theorem 3.13. The lemmas below generalize the results [Dav92, CS08, Dev18] and the techniques in this subsection are heavily inspired by these papers. Before we start with proving estimates, let us note the following remark on the geometry of ALE manifolds:

**Remark 3.16 (Volume of balls).** Note that, for each ALE manifold, there is a constant \(C > 0\) such that

\[\frac{1}{C} r^n \leq \text{vol}(B(x,r)) \leq Cr^n,\]

for all \(x \in M\) and all \(r \geq 0\), where \(B(x,r)\) is the ball of radius \(r\) centered at \(x\). This is simply due to the Euclidean volume growth at infinity.

The the first step is to prove an \(L^2\)-estimate with a weight function \(\phi = e^{\alpha \psi}\), which we will later choose carefully.

**Lemma 3.17.** Let \(\psi : M \to \mathbb{R}\) be a smooth bounded function with \(|d\psi| \leq 1\), \(\alpha \in \mathbb{R}\) and \(\phi = e^{\alpha \psi} : M \to \mathbb{R}\). Then, there exists a constant \(C > 0\) such that

\[\|\phi P e^{-t\Delta_V} u\|_{L^2} \leq Ct^{-\frac{n}{4}} e^{C \alpha^2 t} \|u\|_{L^2}^2,\]

for all \(u \in L^2(M,V)\).
Combining the estimates (31 - 33), we get
\[
(\phi \nabla^* \nabla v, \phi v)_{L^2} = (\nabla v, \nabla (\phi \cdot \phi v))_{L^2}
\]
\[
= (\nabla v, \phi \nabla (\phi v))_{L^2} + (\nabla v, [\nabla, \phi] \phi v)_{L^2}
\]
\[
= (\phi \nabla v, \nabla (\phi v))_{L^2} + (\phi \nabla v, [\nabla, \phi] \phi v)_{L^2}
\]
\[
= (\nabla (\phi v), \nabla (\phi v))_{L^2} - ([\nabla, \phi] v, \nabla (\phi v))_{L^2}
\]
\[
+ (\nabla (\phi v), [\nabla, \phi] v)_{L^2} - ([\nabla, \phi] v, [\nabla, \phi] v)_{L^2}
\]
\[
= (\nabla^* \nabla (\phi v), \phi v)_{L^2} - \alpha^2 ||d\psi|| \phi v||_{L^2}^2,
\]
from which it follows that
\[
(\phi \Delta V v, \phi v)_{L^2} = (\Delta V (\phi v), \phi v)_{L^2} - \alpha^2 ||d\psi|| \phi v||_{L^2}^2.
\]
(30)

This identity will be used several times throughout the proof. For any $v \in H^k$, $k \geq 2$, integration by parts yields
\[
\frac{d}{dt} ||\phi u||_{L^2}^2 = 2 \text{Re}(\phi \partial_t u, \phi u)_{L^2}
\]
\[
= -2 \text{Re}(\phi \Delta V v, \phi u)_{L^2}
\]
\[
= -2 \text{Re}(\Delta V \phi u, \phi u)_{L^2} + 2\alpha^2 ||d\psi|| \phi u||_{L^2}^2
\]
\[
\leq -\frac{2}{C_1} ||P \phi u||_{L^2}^2 + 2\alpha^2 ||\phi u||_{L^2}^2.
\]
(31)

We want to combine this with
\[
\frac{1}{C_1} ||\phi Pu||_{L^2}^2 \leq \frac{2}{C_2} ||[P, \phi] u||_{L^2}^2 + \frac{2}{C_2} ||P \phi u||_{L^2}^2.
\]
(32)

By definition of the principal symbol, we have
\[
[P, \phi] u = \sigma_P(\nabla \phi) u = \alpha \sigma \sigma_P(\nabla \psi) u.
\]

Because $|\nabla \psi| \leq 1$, we can estimate
\[
||[P, \phi] u||_{L^2} \leq \alpha ||\sigma \sigma_P||_{L^\infty} ||\phi u||_{L^2} \leq C\alpha ||\phi u||_{L^2}.
\]
(33)

Combining the estimates (31 - 33), we get
\[
\frac{d}{dt} ||\phi u||_{L^2}^2 \leq -\frac{1}{C_1} ||\phi Pu||_{L^2}^2 + C\alpha^2 ||\phi u||_{L^2}^2.
\]
(34)

Using that
\[
\Delta W \geq \frac{1}{C_2} PP^* \geq 0,
\]
and equation (30), note that
\[
\frac{d}{dt} ||\phi Pu||_{L^2}^2 = 2 \text{Re}(\phi P \partial_t u, \phi Pu)_{L^2}
\]
\[
= -2 \text{Re}(\phi \Delta V u, \phi Pu)_{L^2}
\]
\[
= -2 \text{Re}(\phi \Delta W u, \phi Pu)_{L^2}
\]
\[
= -2 \text{Re}(\Delta W \phi Pu, \phi Pu)_{L^2} + 2\alpha^2 ||d\psi|| \phi Pu||_{L^2}^2
\]
\[
\leq 2\alpha^2 ||\phi Pu||_{L^2}^2.
\]
(35)
Defining the energy
\[ E(u) := \|\phi u\|_{L^2}^2 + \frac{1}{C_1} \|\phi Pu\|_{L^2}^2, \]
and combining the estimates (34) and (35), we get
\[ \frac{d}{dt} E(u) \leq C\alpha^2 E(u), \]
which completes the proof. \(\square\)

We observe the following consequence:

**Lemma 3.19.** The theorem holds for \(p \in (1, 2]\) and \(q \in [2, \infty] \).

**Proof.** Note that due to (29), \(\ker_{L^2}(\Delta_V) \subset \ker_{L^2}(P)\) and \(\ker_{L^2}(\Delta_W) \subset \ker_{L^2}(P^*)\). Therefore,
\[ P \circ e^{-t\Delta_V} = P \circ e^{-t(\Delta_V + \alpha\Pi_V)}, \quad P^* \circ e^{-t\Delta_W} = P^* \circ e^{-t(\Delta_W + \alpha\Pi_W)}. \]
By duality, this implies
\[ e^{-t\Delta_V} \circ P^* = e^{-t(\Delta_V + \alpha\Pi_V)} \circ P^*, \quad e^{-t\Delta_W} \circ P = e^{-t(\Delta_W + \alpha\Pi_W)} \circ P. \]
We conclude that
\[ P \circ e^{-t\Delta_V} = P \circ e^{-\frac{t}{4}\Delta_V} \circ e^{-\frac{t}{4}\Delta_V} \]
\[ = e^{-\frac{t}{4}\Delta_V} \circ P \circ e^{-\frac{t}{4}\Delta_V} \}
\[ = e^{-\frac{t}{4}(\Delta_W + \alpha\Pi_W)} \circ P \circ e^{-\frac{t}{4}(\Delta_V + \alpha\Pi_V)} \]
\[ = e^{-\frac{t}{4}(\Delta_W + \alpha\Pi_W)} \circ P \circ e^{-\frac{t}{4}(\Delta_V + \alpha\Pi_V)} \circ e^{-\frac{t}{4}(\Delta_V + \alpha\Pi_V)} \]
\[ = e^{-\frac{t}{4}(\Delta_W + \alpha\Pi_W)} \circ P \circ e^{-\frac{t}{4}\Delta_V} \circ e^{-\frac{t}{4}(\Delta_V + \alpha\Pi_V)}. \]
Let now \(p \in (1, 2]\) and \(q \in [2, \infty] \). By the assumptions in Theorem 3.13, we may estimate \(e^{-\frac{t}{4}(\Delta_W + \alpha\Pi_W)}\) and \(e^{-\frac{t}{4}(\Delta_V + \alpha\Pi_V)}\) and by Lemma 3.17, with \(\alpha = 0\), we may estimate \(P \circ e^{-\frac{t}{4}}\) to get:
\[ \left\| P \circ e^{-t\Delta_V} \right\|_{p \to q} \leq \left\| e^{-\frac{t}{4}(\Delta_W + \alpha\Pi_W)} \right\|_{L^2, L^q} \left\| P \circ e^{-\frac{t}{4}\Delta_V} \right\|_{L^2, L^2} \left\| e^{-\frac{t}{4}(\Delta_V + \alpha\Pi_V)} \right\|_{L^p, L^2} \]
\[ \leq Ct^{-\frac{1}{2}\left(\frac{1}{p} + \frac{1}{4}\right)} \cdot t^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}} \]
\[ = Ct^{-\frac{1}{2}(\frac{1}{p} + \frac{1}{4}) - \frac{1}{2}} \]
as claimed. \(\square\)

Another consequence of Lemma 3.17 is the following:

**Lemma 3.20** (The localized \(L^2 - L^2\) estimate). Let \(A, B \subset M\) be measurable subsets of \(M\) and \(\chi_A, \chi_B\) be the characteristic function of \(A, B\), respectively. Let \(v \in C^\infty(V, M)\) and suppose that \(\chi_{A^c} \in L^2(V, M)\). Then there is a constant \(C > 0\), such that
\[ \left\| \chi_B P e^{-t\Delta_V} \chi_A \cdot v \right\|_{L^2} \leq Ct^{-1/2} e^{-\frac{d(A, B)}{4}} \left\| \chi_A \cdot v \right\|_{L^2}. \]

**Proof.** A similar proof has been done for the scalar heat equation in [Dav92, Thm. 2]. Choose a bounded smooth function \(\psi\) such that \(\psi|_A = 0\), \(\psi|_B = \frac{d(A, B)}{2}\) and \(|\nabla \psi| \leq 1\) and let \(\phi = e^{\alpha \psi}\). Lemma 3.17 implies that for any \(f \in L^2(W)\) we have
\[ \left\langle \chi_B P e^{-t\Delta_V} \chi_A \cdot v, \phi^{-1} \chi_B f \right\rangle_{L^2} \leq \left\| \phi P e^{-t\Delta_V} \chi_A \cdot v \right\|_{L^2} \left\| \phi^{-1} \chi_B f \right\|_{L^2} \]
\[ \leq Ct^{-\frac{1}{2}} e^{C\alpha^2 t} \left\| \phi \chi_A \cdot v \right\|_{L^2} e^{-\frac{d(A, B)}{4}} \left\| f \right\|_{L^2} \]
If we set \( \alpha = \beta \frac{d(A, B)}{t} \) and \( \beta > 0 \) such that \( C \beta^2 - \frac{\beta}{2} < 0 \), we obtain
\[
\langle \chi_B v, f \rangle_{L^2} \leq Ct^{-\frac{1}{2}} e^{-\frac{d(A, B)^2}{4t}} \| \Phi_{\gamma, \nu} \|_{L^2} \| f \|_{L^2}.
\]
Because \( f \in L^2(W) \) was arbitrary, this finishes the proof. \( \square \)

In order to generalize the localized estimate to general \( p \) and \( q \), the following lemma in complex analysis plays a crucial role:

**Lemma 3.21.** Let \( F \) be an analytic function on \( \mathbb{C}_+ := \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \} \). Assume that, for given \( C_1, C_2, \gamma, \nu > 0 \), such that
\[
|F(z)| \leq D_1, \quad \forall z \in \mathbb{C}_+,
\]
\[
|F(t)| \leq D_1 e^{-\frac{t}{2}}, \quad \forall t \in \mathbb{R}_+,
\]
\[
|F(z)| \leq D_2 \left( \frac{\text{Re}(z)}{C} \right)^{-\frac{\nu}{2}}, \quad \forall z \in \mathbb{C}_a,
\]
where
\[
\mathbb{C}_a := \{ z \in \mathbb{C} \setminus \{0\} \mid \text{Re}(\frac{\gamma}{z}) \geq 1 \} = \{ z \in \mathbb{C} \setminus \{0\} \mid \left| z - \frac{\gamma}{2} \right| \leq \frac{\gamma}{2} \}.
\]
Then
\[
|F(z)| \leq e D_2 \left( \frac{2 \gamma}{|z|} \right)^\nu e^{-\text{Re}(\frac{\gamma}{z})}, \quad \forall z \in \mathbb{C}_a.
\]

**Proof.** See [CS08, Prop. 2.3]. \( \square \)

**Lemma 3.22** (The localized \( L^2 - L^p \) estimate). Let \( A, B, \chi_A, \chi_B \) and \( v \) be as in Lemma 3.20. Then for \( p \in [2, \infty) \), there is a constant \( C > 0 \), such that
\[
\| \chi_B P e^{-t \Delta} \chi_A v \|_{L^p} \leq Ct^{-\frac{\nu}{2}} e^{-\frac{d(A, B)^2}{4t}} \| \chi_A v \|_{L^2}.
\]
Moreover, for all \( p \in (1, 2] \), we have
\[
\| \chi_B P e^{-t \Delta} \chi_A v \|_{L^p} \leq C t^{-\frac{\nu}{2}} e^{-\frac{d(A, B)^2}{4t}} \| \chi_A v \|_{L^2}.
\]
Similar proofs for different situations are available in [CS08, p. 536] and [Dev18, p. 43-45].

**Proof.** Note that for large \( t \), the estimate essentially follows from Lemma 3.19. We therefore first focus on small \( t > 0 \). Fix \( f_1 \in C_c^\infty(V) \), with \( \text{supp}(f_1) \subset A \), and \( f_2 \in C_c^\infty(W) \), with \( \text{supp}(f_2) \subset B \). By the Spectral Theorem, note that the function
\[
F : \mathbb{C}_+ \rightarrow \mathbb{C},
\]
\[
z \mapsto \langle f_1, P e^{-z \Delta} f_2 \rangle_{L^2} = \langle P^* f_1, e^{-z \Delta} f_2 \rangle_{L^2},
\]
is analytic on \( \mathbb{C}_+ := \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \} \). We would like to apply Lemma 3.21 to the function \( F \). The Spectral Theorem implies that
\[
|F(z)| = |\langle f_1, P e^{-z \Delta} f_2 \rangle_{L^2}|
\]
\[
= |\langle P^* f_1, e^{-z \Delta} f_2 \rangle_{L^2}|
\]
\[
\leq \| P^* f_1 \|_{L^2} \| e^{-z \Delta} f_2 \|_{L^2}
\]
\[
\leq \| P^* f_1 \|_{L^2} \sup_{\lambda \geq 0} |e^{-\lambda \Delta}| \| f_2 \|_{L^2}
\]
\[
\leq \| P^* f_1 \|_{L^2} \| f_2 \|_{L^2}
\]
for all \( z \in \mathbb{C}_+ \). We continue with the bound on the positive real axis. Let us define
\[
\gamma := \frac{d(A, B)^2}{2C_0},
\]
where \( C_0 \) is the constant from Lemma 3.20. Using that \( x^{\frac{1}{2}} e^{-x} \) is uniformly bounded for all \( x \in \mathbb{R} \), note that Lemma 3.20 yields
\[
|F(t)| = |(f_1, Pe^{-t\Delta_V} f_2)|_2
\]
\[
\leq \|f_1\|_2 \|P e^{-t\Delta_V} e^{-t\Delta_V} f_2\|_2
\]
\[
\leq Ct^{-\frac{\nu}{2}} e^{-\frac{d(A, B)^2}{2C_0}} \|f_1\|_2 \|e^{-t\Delta_V} f_2\|_2
\]
Choosing
\[
D_1 := \max \left( \|P^* f_1\|_2, C_1 \gamma^{-\frac{1}{2}} \|f_1\|_2 \|f_2\|_2 \right),
\]
we have so far verified (36) and (37). By the Spectral Theorem, we have \( e^{-z \Delta_V} = e^{-t \Delta_V} \circ e^{-is \Delta_V} \), where \( z = t + is \), and
\[
\|e^{-is \Delta_V}\|_{2 \rightarrow 2} \leq 1, \quad \forall s \in \mathbb{R}.
\]
(39)
Let \( p' \) denote the Hölder conjugate of \( p \). Since \( p \geq 2 \), we may apply Lemma 3.19 and (39) to obtain
\[
|F(z)| \leq \|f_1\|_{L^{p'}} \|P \circ e^{-t \Delta_V} \circ e^{-is \Delta_V} f_2\|_{L^p}
\]
\[
\leq Ct^{-\frac{\nu}{2}} \left( \frac{1}{p'} \right)^{1/2} \|f_1\|_{L^{p'}} \|e^{-is \Delta_V} f_2\|_2
\]
\[
\leq C\gamma^{-\frac{\nu}{2}} \left( \frac{1}{p'} \right)^{1/2} \|f_1\|_{L^{p'}} \|f_2\|_2,
\]
where
\[
\nu := n \left( \frac{1}{2} - \frac{1}{p} \right) + 1.
\]
Note that \( \nu \geq 1 \), since \( p \geq 2 \). This verifies the bound (38) with
\[
D_2 := C\gamma^{-\frac{\nu}{2}} \|f_1\|_{L^{p'}} \|f_2\|_2.
\]
Now, Lemma 3.21 together with the fact that \( x^{\frac{1}{2}} e^{-x} \) is uniformly bounded on \( \mathbb{R} \), imply
\[
|F(t)| \leq \epsilon D_2 \left( \frac{2\gamma}{\epsilon} \right)^{\nu} e^{-\frac{\nu}{2}}
\]
\[
= Ct^{-\frac{\nu}{2}} \left( \frac{\gamma}{\epsilon} \right)^{\frac{\nu}{2}} e^{-\frac{\nu}{2}} \|f_1\|_{L^{p'}} \|f_2\|_2
\]
\[
\leq Ct^{-\frac{\nu}{2}} \|f_1\|_{L^{p'}} \|f_2\|_2
\]
\[
= Ct^{-\frac{\nu}{2}} \left( \frac{1}{p'} - \frac{1}{p} \right) e^{-\frac{d(A, B)^2}{2C_0}} \|f_1\|_{L^{p'}} \|f_2\|_2
\]
for all $t \leq \gamma = \frac{(A,B)^2}{2C_0}$. To sum up, this estimate and Lemma 3.19 imply that there are two constants $C_1, C_2 > 0$, such that

$$|F(t)| \leq \begin{cases} C_1 t^{-\frac{(p-1)}{2(p-1)}} e^{-\frac{(A,B)^2}{2C_0}} \|f_1\|_{L^{p'}} \|f_2\|_{L^p}, & 0 < t \leq \frac{(A,B)^2}{2C_0}, \\ C_2 t^{-\frac{(p-1)}{2(p-1)}} \|f_1\|_{L^{p'}} \|f_2\|_{L^p}, & 0 < t. \end{cases}$$

Choosing $C_3 := \max(C_1, C_2 e^{\frac{1}{2}t})$, we get that

$$C_2 \leq C_3 e^{-\frac{1}{2}t} \leq C_3 e^{-\frac{(A,B)^2}{2C_0}},$$

for all $t \geq \frac{(A,B)^2}{2C_0}$. Inserting this in the above estimate, we conclude that

$$|F(t)| \leq C_3 t^{-\frac{(p-1)}{2(p-1)}} e^{-\frac{(A,B)^2}{2C_0}} \|f_1\|_{L^{p'}} \|f_2\|_{L^p},$$

for all $t > 0$. In other words,

$$\|(f_1, P e^{-t\Delta v} f_2)_{L^2}\| \leq C_3 t^{-\frac{(p-1)}{2(p-1)}} e^{-\frac{(A,B)^2}{2C_0}} \|f_1\|_{L^{p'}} \|f_2\|_{L^p},$$

for all $t > 0$ and all smooth $f_1$, with $\text{supp}(f_1) \subset A$, and smooth $f_2$, with $\text{supp}(f_2) \subset B$. By density in $L^{p'}$ and $L^2$, we get that

$$\| (f, \chi_B P e^{-t\Delta v} \chi_A v)_{L^2} \| = \| (\chi_B f, P e^{-t\Delta v} \chi_A v)_{L^2} \| \leq C_3 t^{-\frac{(p-1)}{2(p-1)}} e^{-\frac{(A,B)^2}{2C_0}} \|f\|_{L^{p'}} \|\chi_A v\|_{L^p},$$

for all $t > 0$ and all $f \in L^2 \cap L^{p'}$. This proves the first assertion in the lemma. The second estimate follows by duality. \qed

The proof of the following lemma is similar to [CS08, Cor. 4.14]:

**Lemma 3.23.** Theorem 3.13 holds with

$$p = q \in (1, \infty).$$

**Proof.** It suffices to prove the first inequality for $p \in (1, 2]$, the case $p \in [2, \infty)$ follows from duality. Fix $t > 0$ and let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence such that $B \left( x_k, \frac{\sqrt{t}}{2} \right)$ are disjoint and $\cup_{k \in \mathbb{N}} B(x_k, \sqrt{t}) = M$. Define the disjoint subsets

$$B_k := B(x_k, \sqrt{t}) \setminus \cup_{l=1}^{k-1} B(x_l, \sqrt{t})$$

and note that $\cup_{k \in \mathbb{N}} B_k = M$. It follows that

$$v = \sum_{k=1}^{\infty} \chi_k v,$$

where $\chi_k$ be the characteristic function of the set $B_k$. We have

$$\|P \circ e^{-t\Delta v} v\|_{L^p} = \left( \sum_k \left\| \chi_k P \circ e^{-t\Delta v} v \right\|_{L^p}^p \right)^{\frac{1}{p}} = \left( \sum_k \left( \sum_t \left\| \chi_k P \circ e^{-t\Delta v} \chi_tv \right\|_{L^p}^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}.$$
By the Hölder inequality, Remark 3.16 and Lemma 3.22, we note that
\[
\left\| \chi_k P \circ e^{-t\Delta V} \chi_l v \right\|_{L^p} \leq \left\| \chi_k \right\|_{L^\infty} \left\| \chi_k P \circ e^{-t\Delta V} \chi_l v \right\|_{L^2} \\
\leq \text{vol}(B_k)^{\frac{2-p}{2p}} \left\| \chi_k P \circ e^{-t\Delta V} \chi_l v \right\|_{L^2} \\
\leq \mathcal{C} t^{-\frac{q}{2}} e^{-\frac{d(B_k, B_l)^2}{B_k}} \left\| \chi_l v \right\|_{L^p}.
\]
Inserting this in the above estimate, we get
\[
\left\| P \circ e^{-t\Delta V} v \right\|_{L^p} \leq \mathcal{C} t^{-\frac{q}{2}} \left( \sum_k \left( \sum_l \left( e^{-\frac{d(B_k, B_l)^2}{B_k}} \left\| \chi_l v \right\|_{L^p} \right) \right)^p \right)^{\frac{1}{p}}.
\]
(40)

In order to continue the estimate, we need a certain inequality for sequences $c_{kl}$ and $a_l$. Consider
\[
T(a)_k := \sum_l |c_{kl}a_l|
\]
as a sequence in $k$. For each $p \in [1, \infty]$, we claim that
\[
\|T(a)\|_{L^p(Z)} = \max \left\{ \sum_k \sum_l |c_{kl}||a_l|, \sup_k \sum_l |c_{kl}| \right\} \|a\|_{L^p(Z)}.
\]
(41)

By the Riesz-Thorin interpolation theorem, it suffices to prove this for $p = 1$ and $p = \infty$. We start with $p = 1$:
\[
\|T(a)\|_{L^1(Z)} = \sum_k \sum_l |c_{kl}a_l| = \sum_k \left( \sum_l |c_{kl}| \right) |a_l| \\
\leq \sup_k \sum_l |c_{kl}| \sum_l |a_l| \leq \sup_k \sum_l |c_{kl}| \|a\|_{L^1(Z)}
\]
and continue with $p = \infty$:
\[
\|T(a)\|_{L^\infty(Z)} = \sup_k \sum_l |c_{kl}a_l| \leq \sup_k \sum_l |c_{kl}| \|a\|_{L^\infty(Z)}.
\]

This proves (41). Applying (41) with
\[
c_{kl} := e^{-\frac{d(B_k, B_l)^2}{B_k}}, \\
a_l := \left\| \chi_l v \right\|_{L^p},
\]
the estimate (40) now implies that
\[
\left\| P \circ e^{-t\Delta V} v \right\|_{L^p} \leq \mathcal{C} t^{-\frac{q}{2}} \sup_k \sum_l e^{-\frac{d(B_k, B_l)^2}{B_k}} \left( \sum_m \left\| \chi_m v \right\|_{L^p} \right)^p \\
= \mathcal{C} t^{-\frac{q}{2}} \sup_k \sum_l e^{-\frac{d(B_k, B_l)^2}{B_k}} \left\| v \right\|_{L^p}.
\]
(42)

It remains to estimate
\[
\sup_k \sum_l e^{-\frac{d(B_k, B_l)^2}{B_k}} = \sup_k \sum_{N=0}^{\infty} \sum_k e^{-\frac{d(B_k, B_l)^2}{B_k}}
\]
and similarly, following lemma is merely a corollary of Theorem 3.13: We estimate \( D \) where

\[
\sum_{N=0}^{\infty} \sum_{k \leq 2N \sqrt{t}} e^{-\frac{k^2}{t}} \leq \sup_{l} \sum_{N=0}^{\infty} \sum_{k \leq 2N \sqrt{t}} e^{-\frac{k^2}{t}}.
\]

We estimate \( A_{l}(N) \) using Remark 3.16: Consider

\[
U := \bigcup_{k \in A_{l}(N)} B(x_{l}, \sqrt{t}/2) \subset B(x_{l}, (2N + 2)\sqrt{t})
\]

Since \( B(x_{l}, \sqrt{t}/2) \cap B(x_{j}, \sqrt{t}/2) = \emptyset \) for all \( l \neq j \) and by Remark 3.16, we get

\[
\sharp A_{l}(N) \frac{1}{C} \left( \frac{\sqrt{t}}{2} \right)^{n} \leq \text{vol}(U) \leq C(2N + 2)^{n}(\sqrt{t})^{n}
\]

We conclude that

\[
\sharp \{ k \mid d(B_{k}, B_{l}) \leq 2N \sqrt{t} \} \leq \sharp A_{l}(N) \leq CN^{n}
\]

Inserting the estimates into (42), we conclude that

\[
\| P \circ e^{-t\Delta_{V}} \|_{L^{p}} = Ct^{-\frac{n}{2}} \sup_{l} \sum_{k} e^{-\frac{Nk^{2}}{tN}} \| v \|_{L^{p}} \leq Ct^{-\frac{n}{2}} \sum_{N=0}^{\infty} N^n e^{-\frac{N^2}{2t}} \leq Ct^{-\frac{n}{2}},
\]

which finishes the proof of the lemma. \( \square \)

**Proof of Theorem 3.13.** As in the proof of Lemma 3.19, we write

\[ P \circ e^{-t\Delta_{V}} = P \circ e^{-t\Delta_{V} \circ e^{t\Delta_{w}}} = e^{-t\Delta_{V}} \circ P \circ e^{-t\Delta_{w}} = e^{-t\Delta_{V} + a\Pi_{w}} \circ P \circ e^{-t\Delta_{w}} \]

and similarly,\[ P^* \circ e^{-t\Delta_{w}} = e^{-t\Delta_{w} + a\Pi_{w}} \circ P^* \circ e^{-t\Delta_{w}} \cdot t^{-\frac{n}{2}} = Ct^{-\frac{n}{2} + \frac{n}{2} - \frac{n}{2} - \frac{n}{2}}.\]

Let now \( 1 < p \leq q < \infty \). By the assumptions of Theorem 3.13 and by Lemma 3.23, we have

\[
\| P \circ e^{-t\Delta_{V}} \|_{p \rightarrow q} \leq \| e^{-t\Delta_{V} + a\Pi_{w}} \|_{p \rightarrow q} \leq Ct^{-\frac{n}{2} + \frac{n}{2} - \frac{n}{2} - \frac{n}{2}}.
\]

The estimate

\[
\| P^* \circ e^{-t\Delta_{w}} \|_{p \rightarrow q} \leq Ct^{-\frac{n}{2} + \frac{n}{2} - \frac{n}{2} - \frac{n}{2}}
\]

is shown completely analogously. \( \square \)

**3.3. Derivative estimates.** Our goal here is to prove our first main derivative estimate, Theorem 1.14. This is where we begin to combine the Fredholm theory for the Dirac type operators and Theorem 3.13. We work with Schrödinger operators, which are squares of Dirac type operators

\[ \Delta_{V} = (D_{V})^{2}, \]

where \( D_{V} \) is asymptotic to a Euclidean Dirac operator, in the sense of Definition 1.11. The following lemma is merely a corollary of Theorem 3.13:

**Lemma 3.24.** For each \( p \in (1, \infty) \) and \( k \in \mathbb{N} \), we have

\[
\| D_{V}^{k} e^{-tD_{V}^{2}} \|_{p \rightarrow p} \leq Ct^{-\frac{n}{2}}
\]

for some \( C = C(k, p) \).
Proof. Because $\mathcal{D}_V$ commutes with $e^{-t\mathcal{D}_V}$, we have
\[
\mathcal{D}_V^k \circ e^{-t\mathcal{D}_V} = \mathcal{D}_V \circ e^{-t\mathcal{D}_V} \circ \cdots \circ \mathcal{D}_V \circ e^{-t\mathcal{D}_V} = \left(\mathcal{D}_V \circ e^{-t\mathcal{D}_V}\right)^k.
\]
Applying Theorem 3.13 to $\Delta V = \Delta W = \mathcal{D}_V^2$ and $P = \mathcal{D}_V$ thus implies
\[
\left\|\mathcal{D}_V^k \circ e^{-t\mathcal{D}_V}\right\|_{p \to p} \leq \left\|\mathcal{D}_V \circ e^{-t\mathcal{D}_V}\right\|^k_{p \to p} \leq Ct^{-\frac{k}{2}},
\]
which finishes the proof of the lemma.

We finally are ready to prove the first main derivative estimate:

Proof of Theorem 1.14. For a given $u_0 \in L^p$, which is $L^2$-orthogonal to $\ker_{L^2}(\mathcal{D}_V^k)$, let us write $u = e^{-t\mathcal{D}_V} u_0$. By Remark 1.12, we may apply Theorem 3.2, which implies that almost Euclidean heat kernel estimates hold, i.e. the case $k = 0$ is thus proven. We turn to the case $k \geq 1$. Let first $p \in (1, \frac{2}{k})$, which is only a non-empty set if $k \leq n$. Note that $k - \frac{2}{q} \in (1-n, 0)$, so the assumptions in Corollary 2.9 are satisfied. Corollary 2.9, Lemma 3.24 and Theorem 1.8 yield
\[
\left\|\nabla_k u\right\|_{L^p} \leq C \left(\left\|\mathcal{D}_V^k u\right\|_{L^p} + \left\|u\right\|_{L^q}\right) \leq Ct^{-\frac{k}{2}} \left\|u_0\right\|_{L^p} + Ct^{-\frac{k}{2}}(\frac{1}{t^{k-1}}) \left\|u_0\right\|_{L^q},
\]
for any $q \in [p, \infty)$. Now, since $p < \frac{2}{k}$, there is a $q \in [p, \infty)$, large enough so that
\[
\frac{n}{2} \left(\frac{1}{p} - \frac{1}{q}\right) = \frac{k}{2}.
\]
Inserting this $q$ in the above estimate, we conclude that
\[
\left\|\nabla_k u\right\|_{L^p} = Ct^{-\frac{k}{2}} \left\|u_0\right\|_{L^p},
\]
as claimed. We now let $p \in (1, \infty) \setminus (1, \frac{2}{k})$ and fix an $\epsilon > 0$. Choose $q \in (p, \infty)$ so large that $\frac{1}{2q} < \epsilon$. Since $\frac{1}{p} - \frac{1}{q} \in [0, \frac{2}{k})$, Corollary 2.11 and Lemma 3.24 yield
\[
\left\|\nabla_k u\right\|_{L^p} \leq C \left(\left\|\mathcal{D}_V^k u\right\|_{L^p} + \left\|u\right\|_{L^q}\right) \leq C \left(t^{-\frac{k}{2}} + t^{-\frac{k}{2}}(\frac{1}{t^{k-1}})\right) \left\|u_0\right\|_{L^p} \leq Ct^{-\frac{k}{2} + \epsilon} \left\|u_0\right\|_{L^p}
\]
for $t \geq t_0 > 0$, which finishes the proof of the theorem.

We use Theorem 1.14 to prove our first improved derivative estimates:

Proof of Proposition 1.15. By (15), we have
\[
(D_m)^2 \geq \nabla^* \nabla \geq 0
\]
for $0 \leq m \leq l - 1$. Thus, these $D_m$ have a trivial $L^2$-kernel and satisfy the assumptions of Theorem 1.8. By (15), we may therefore apply Theorem 3.13 with
\[
P = \nabla, \quad \Delta V = (D_{m-1})^2, \quad \Delta W = (D_m)^2
\]
to get
\[
\left\|\nabla \circ e^{-t(D_m)^2}\right\|_{p \to p} \leq Ct^{-\frac{1}{2}}
\]
for all $0 \leq m \leq l - 1$ and $p \in (1, \infty)$. Again, by (15), we get
\[
\nabla^k \circ e^{-t(D_0)^2} = \nabla \circ e^{-\frac{t}{k}(D_{k-1})^2} \circ \cdots \circ \nabla \circ e^{-\frac{t}{k}(D_0)^2},
\]
which implies that
\[
\left\|\nabla^k \circ e^{-t(D_0)^2}\right\|_{p \to p} \leq \prod_{m=0}^{k-1} \left\|\nabla \circ e^{-\frac{t}{k}(D_m)^2}\right\|_{p \to p} \leq Ct^{-\frac{k}{2}}
\]
for \( k \leq l \). Similarly, for \( k \geq l + 1 \), we get

\[
\left\| \nabla^k \circ e^{-t(D_0)^2} \right\|_{p \to p} \leq \left\| \nabla^{k-1} \circ e^{-\frac{t}{(D_1)^2}} \right\|_{p \to p} \prod_{m=0}^{l-1} \left\| \nabla \circ e^{-\frac{t}{(D_m)^2}} \right\|_{p \to p} \\
\leq C t^{-\frac{k}{2}} \left\| \nabla^{k-1} \circ e^{-\frac{t}{(D_1)^2}} \right\|_{p \to p}.
\]

We conclude that \( e^{-t(D_0)^2} \) satisfies (strong) derivative estimates of degree \( l \) if \( e^{-t(D_1)^2} \) satisfies (strong) derivative estimates of degree 0. Since the latter conditions follow from Theorem 1.14 and Remark 1.9, respectively, this finishes the proof of the theorem. \( \square \)

3.4. Improved derivative estimates. We now turn to the proof of Theorem 1.19. We will need the following consequence of Proposition 2.7:

**Lemma 3.25.** Let \( D_V \) be a self-adjoint Dirac type operator acting on sections of a vector bundle \( V \), which is asymptotic to a Euclidean Dirac operator in the sense of Definition 1.11. Let \( E \subset V \) be a parallel subbundle of \( V \). Let \( k \in \mathbb{N}_0 \) and \( p \in (1, \infty) \) and assume that

\[
\delta_1, \ldots, \delta - (k - 1) \in \mathbb{R}\setminus\mathcal{E}_1.
\]

The operator

\[
D_V|_E : W^{k,p}_{\delta}(E) \to W^{0,p}_{\delta-k}(V),
\]

is a semi-Fredholm operator, i.e. has finite dimensional kernel and closed range. In particular, there is a closed subspace \( X_\delta \subset W^{k,p}_{\delta}(E) \), such that

\[
W^{k,p}_{\delta}(E) = \ker (D_V|_E) \oplus X_\delta
\]

and a \( C > 0 \), such that

\[
\|u\|_{k,p,\delta} \leq C \|D_V u\|_{0,p,\delta-k}
\]

for all \( u \in X_\delta \).

**Proof.** The proof is standard, given Proposition 2.7. We first show that the kernel is finite dimensional. Let \( (u_i)_i \subset \ker (D_V|_E) \subset W^{k,p}_{\delta}(V) \) be a bounded sequence. Proposition 2.7 implies that

\[
\|u_i\|_{k,p,\delta} \leq C \|u_i\|_{L^p(B_R)}.
\]

By Rellich’s lemma, the embedding \( W^{k,p}_{\delta}(V) \hookrightarrow L^p(B_R) \) is compact, hence there is a converging subsequence \( u_{i_j} \to u \) in \( L^p(B_R) \). The estimate (46) implies that \( u_{i_j} \to u \) in \( W^{k,p}_{\delta}(V) \) as well. In other words, any bounded sequence has a convergent subsequence. It follows that the unit ball in \( \ker (D_V|_E) \) is a compact subset of \( W^{k,p}_{\delta}(V) \), implying that it is finite dimensional. We now show that the range is closed, by first proving estimate (45). Since \( \ker (D_V|_E) \subset W^{k,p}_{\delta}(V) \) is finite dimensional, there is a closed subspace \( X_\delta \subset W^{k,p}_{\delta}(V) \), such that (44) holds. Assume now that there is no constant \( C > 0 \), such that (45) holds. Then there is a sequence \( (u_i)_i \subset X_\delta \subset W^{k,p}_{\delta}(E) \), such that

\[
\|u_i\|_{k,p,\delta} = 1, \quad \|D_V u_i\|_{0,p,\delta-k} \to 0.
\]

Again, Rellich’s lemma and Proposition 2.7 implies in this case the existence of a \( u_i \), converging in \( L^p(B_R) \). Inserting this in combination with (47) into Proposition 2.7 implies that \( u_i \) converges in \( X_\delta \). Since \( \|u_i\|_{k,p,\delta} = 1 \), the limit is non-zero, i.e. \( u_i \to u \neq 0 \) in \( X_\delta \subset W^{k,p}_{\delta}(E) \). Continuity of \( D_V \) implies that \( D_V u_i \to D_V u \neq 0 \), which contradicts \( \|D_V u_i\|_{0,p,\delta-k} \to 0 \). This proves (45). If now \( D_V u_i \to f \), for \( (u_i)_i \subset X_\delta \), then (45) implies that \( (u_i)_i \to u \) in \( X_\delta \). Continuity of \( D_V \) implies that \( D_V u_i \to D_V u = f \), showing that the range is closed. \( \square \)
The following lemma allows us to apply eventually apply Lemma 3.25 to prove the improved derivative estimates:

**Lemma 3.26.** Assume the same as in Theorem 1.19. For any \( k \in \mathbb{N}_0 \) and \( p \in (1, \infty) \), such that
\[
k - \frac{n}{p} \notin \mathbb{N}_0, \quad k - \frac{n}{p} \leq l,
\]
we have
\[
\left\| \nabla^k u \right\|_{L^p} \leq C \left\| D_{\nabla}^k u \right\|_{L^p},
\]
for all \( u \in W_{k - \frac{n}{p}}^{k,p}(E) \).

**Remark 3.27.** Note that the assumptions on \( k \) and \( p \) are equivalent to (12), (13) and (20).

**Proof.** The proof is based on applying Lemma 3.25 with a fixed \( \delta := k - \frac{n}{p} \). Since \( k - \frac{n}{p} \notin \mathbb{N}_0 \)
and since \( 1 - \frac{n}{p} > 1 - n \), it is clear that \( \delta = k - \frac{n}{p}, \ldots, \delta - (k - 1) = 1 - \frac{n}{p} \notin \mathcal{E}_1 \). Lemma 3.25 therefore implies that
\[
D_{\nabla}^\delta |_{E} : W_{k - \frac{n}{p}}^{k,p}(E) \rightarrow W_{0,p}^{0,p}(V) = L^p(V)
\]
is a semi-Fredholm operator. Assume that \( u \in \ker (D_{\nabla}^\delta |_{E}) \subset W_{k - \frac{n}{p}}^{k,p}(E) \). By Hölder’s inequality on \( B_R \), it follows that \( u \in L^\beta(B_R) \), for all \( q \in [p, \infty) \). Therefore Proposition 2.7 implies that \( u \in W_{k - \frac{n}{p}}^{k,q}(E) \), for all \( q \in [p, \infty) \). By [Bar86, Thm. 1.2], it follows that \( u \in o(r^\delta) \). If \( k \leq l \), then \( u \in o(r^\delta) \subset o(r^k) \), so by the assumption in Theorem 1.19, we conclude that \( \nabla^k u = 0 \). If instead \( k \geq l \), then we use that \( u \in o(r^\delta) \subset o(r^k) \), which by the assumption in Theorem 1.19 implies that \( \nabla^l u = 0 \) and hence \( \nabla^k u = 0 \). To sum up, we have shown that
\[
\nabla^k u = 0
\]
for all \( u \in \ker (D_{\nabla}^\delta |_{E}) \). This implies that
\[
\nabla^k u = \nabla^k \text{proj}_{X_\delta}(u)
\]
for all \( u \in W_{k - \frac{n}{p}}^{k,p}(E) \), where \( \text{proj}_{X_\delta} \) is the projection onto \( X_\delta \), given by the split (44). On the other hand, by construction of \( X_\delta \), we have
\[
D_{\nabla}^\delta u = D_{\nabla}^\delta \text{proj}_{X_\delta}(u),
\]
for all \( u \in W_{k - \frac{n}{p}}^{k,p}(E) \). Applying the estimate (45), we get
\[
\left\| \nabla^k u \right\|_{L^p} = \left\| \nabla^k \text{proj}_{X_\delta}(u) \right\|_{L^p} \leq \left\| \text{proj}_{X_\delta}(u) \right\|_{k,p,k-\frac{n}{p}} \leq \left\| D_{\nabla}^\delta \text{proj}_{X_\delta}(u) \right\|_{L^p} = \left\| D_{\nabla}^\delta u \right\|_{L^p},
\]
for all \( u \in W_{k - \frac{n}{p}}^{k,p}(E) \). \( \square \)

The previous lemma takes care of the case \( k - \frac{n}{p} < l \). We now provide a corresponding estimate when \( k - \frac{n}{p} \geq l \):

**Lemma 3.28.** Assume the same as in Theorem 1.19. For any \( k \in \mathbb{N}_0 \) and \( p \in (1, \infty) \), such that
\[
k - \frac{n}{p} \geq l,
\]
any \( q \in (p, \infty) \) and an arbitrary
\[
\eta \in \left( l - \frac{n}{q}, l \right) \cap (l - 1, l),
\]
we have
\[
\left\| \nabla^k u \right\|_{L^p(M)} \leq C \left( \left\| D_{\nabla}^\delta u \right\|_{L^p(M)} + \left\| D_{\nabla}^\delta u \right\|_{L^p(M)} \right)
\]
for all \( u \in W_{k - \frac{n}{p}}^{k,p}(E) \).
Remark 3.29. Note that the conditions on \( p \) and \( k \) are equivalent to (14).

When applying this lemma to prove Theorem 1.19, we will assume that \( q \) is very large, so we think of \( \eta \) as being very close to \( l \).

Proof. Note that \( \eta_i, \ldots, \eta_i - (l - 1) \notin \mathcal{E}_1 \). We consider
\[
\mathcal{D}_V^e|E : W_{\eta_i}^{l,p}(E) \to W_{\eta_i}^{0,p}(V).
\]
By Lemma 3.25 and arguing as in the proof of Lemma 3.26, we know that
\[
\ker(\mathcal{D}_V^e|E) \subset o(\rho^l).
\]
By the assumption in Theorem 1.19, we conclude that
\[
\nabla^l u = 0,
\]
for all \( u \in \ker(\mathcal{D}_V^e|E) \). This implies that
\[
\nabla^l u = \nabla^l \text{proj}_{X_\eta}(u), \quad \mathcal{D}_V^e u = \mathcal{D}_V^e \text{proj}_{X_\eta}(u)
\]
for all \( u \in W_{\eta_i}^{l,p}(E) \), where \( \text{proj}_{X_\eta} \) is the projection onto \( X_\eta \), given by
\[
W_{\eta_i}^{l,p}(E) = \ker(\mathcal{D}_V^e) \oplus X_\eta.
\]
Since \( k \geq l + 1 \), we also get
\[
\nabla^k u = \nabla^k \text{proj}_{X_\eta}(u), \quad \mathcal{D}_V^e u = \mathcal{D}_V^e \text{proj}_{X_\eta}(u)
\]
for all \( u \in W_{\eta_i}^{k,p}(E) \subset W_{\eta_i}^{l,p}(E) \). Using this, and applying Proposition 2.10 with \( \delta := k - \frac{n}{p} \), we get
\[
\|\nabla^k u\|_{L^p} = \|\nabla^k \text{proj}_{X_\eta}(u)\|_{L^p} \leq \|\text{proj}_{X_\eta}(u)\|_{k,p,\delta}
\]
\[
\leq C \left( \|\mathcal{D}_V^e \text{proj}_{X_\eta}(u)\|_{0,p,\delta-k} + \|\text{proj}_{X_\eta}(u)\|_{0,p,\delta} \right)
\]
\[
\leq C \left( \|\mathcal{D}_V^e u\|_{0,p,\delta-k} + \|\text{proj}_{X_\eta}(u)\|_{0,p,\delta} \right).
\]

It remains to estimate the second term on the right hand side. For this, we note that
\[
\delta > \eta, \quad \eta - l > -\frac{n}{q}.
\]
By Proposition 2.7, we get the estimate
\[
\|\text{proj}_{X_\eta}(u)\|_{0,p,\delta} \leq C \|\text{proj}_{X_\eta}(u)\|_{l,\eta} \leq C \|\mathcal{D}_V^e \text{proj}_{X_\eta}(u)\|_{0,p,\eta-l}
\]
\[
\leq C \|\mathcal{D}_V^e \text{proj}_{X_\eta}(u)\|_{0,q,-\frac{n}{q}} = C \|\mathcal{D}_V^e u\|_{L^q},
\]
for all \( u \in W_{\eta_i}^{k,p}(E) \). This finishes the proof. \( \square \)

We are finally in shape to apply these estimates to prove Theorem 1.19:

Proof of Theorem 1.19. Assume that \( u_0 \in L^p(E) \) is \( L^2 \)-orthogonal to \( \ker_{L^2} (\mathcal{D}_V^e) \), and denote \( u = e^{-t\mathcal{D}_V^e} u_0 \). Since \( \mathcal{D}_V^e \) maps sections in \( E \) to sections in \( E \), it follows that \( u \) is a section in \( E \).

By Theorem 1.8, we know that
\[
u \in L^p(E) = W_{\eta_i}^{0,p}(E) \subset W_{\eta_{i-1}}^{0,p}(E),
\]
and by Lemma 3.24, we know that
\[
\mathcal{D}_V^e u \in L^p(E) = W_{\eta_i}^{0,p}(E).
\]
Proposition 2.10 implies therefore that
\[ u \in W^{k,p}_{k-n,p}(E), \]
which allows us to apply the previous lemmas in what comes. For any \( k \in \mathbb{N}_0 \) and \( p \in (1,\infty) \), such that
\[ k - \frac{n}{p} \not\in \mathbb{N}_0, \quad k - \frac{n}{p} < l, \]
Lemma 3.26 and Lemma 3.24 implies that
\[ \|\nabla^k u\|_{L^p} \leq C \|\nabla^m u\|_{L^p}^{\frac{1}{p}} \|u\|_{L^p}^{\frac{n}{p}} \leq C (\|\nabla^m u\|_{L^p} + \|\nabla^l u\|_{L^p})^{\frac{1}{p}} \|u_0\|_{L^p}^{\frac{n}{p}} \]
\[ \leq C (t^{-\frac{n}{p}} + t^{-\frac{n}{2} + \frac{l}{2}})^{\frac{1}{p}} \|u_0\|_{L^p} \]
\[ \leq C (t^{-\frac{n}{p}} + t^{-\frac{n}{2} + \frac{l}{2}})^{\frac{1}{p}} \|u_0\|_{L^p} \]
\[ = Ct^{-\frac{n}{p}} \|u_0\|_{L^p}, \]
for \( t \geq t_0 \), as claimed in the theorem. Now assume that
\[ k - \frac{n}{p} \in \mathbb{N}_0, \quad k - \frac{n}{p} < l. \]
This allows us to choose an \( m \in \mathbb{N}_0 \), such that
\[ m - \frac{n}{p} = l. \]
For a given \( \epsilon > 0 \), let \( q \in (p,\infty) \) be large enough such that \( \frac{n}{2q} \leq \frac{2n}{p} \). We now combine the Gagliardo-Nirenberg interpolation inequality with Theorem 1.8, Lemma 3.28 and Lemma 3.24 to get
\[ \|\nabla^k u\|_{L^p} \leq C \|\nabla^m u\|_{L^p}^{\frac{1}{p}} \|u\|_{L^p}^{\frac{n}{p}} \leq C (\|\nabla^m u\|_{L^p} + \|\nabla^l u\|_{L^p})^{\frac{1}{p}} \|u_0\|_{L^p}^{\frac{n}{p}} \]
\[ \leq C (t^{-\frac{n}{p}} + t^{-\frac{n}{2} + \frac{l}{2}})^{\frac{1}{p}} \|u_0\|_{L^p} \]
\[ \leq C (t^{-\frac{n}{p}} + t^{-\frac{n}{2} + \frac{l}{2}})^{\frac{1}{p}} \|u_0\|_{L^p} \]
\[ = Ct^{-\frac{n}{p}} \|u_0\|_{L^p}, \]
for \( t \geq t_0 \), which completes the proof. \( \square \)

For future applications, we note the following corollary:

**Corollary 3.30.** Suppose that we have a parallel subbundle \( E \subset V \) which is invariant under \( D^2_V \)
and assume that \( \Delta_E := D^2_V|_E \) is of the form
\[ \Delta_E = \nabla^* \nabla + R, \]
with a nonnegative symmetric endomorphism \( R \in C^\infty(\text{End}(E)) \). Then, if \( (M,g) \) has only one end, \( e^{-\Delta_E} \) satisfies Euclidean heat kernel estimates and weak derivative estimates of degree 1.
Proof. Let \( u \in C^\infty(E) \) with \( u = o(r) \) be such that \( Dv u = 0 \). Then \( \Delta_E u = 0 \). Because there are no exceptional values between 0 and 1, \( u \) is bounded. Because

\[
\Delta |u|^2 = -|\nabla u|^2 - (\mathcal{R} u, u) \leq 0
\]

and \((M, g)\) has only one end, the maximum principle implies equality on the right hand side. Therefore, \( \nabla u \equiv 0 \). Because this implies \( \ker_{L^2}(\Delta_E) = \{0\} \), the first assertion follows from Remark 1.9. The second assertion follows from Theorem 1.19 above. \( \square \)

4. Harmonic forms on ALE manifolds

4.1. Decay of harmonic forms. On the model cone. We start by analyzing the decay of harmonic functions on \( \mathbb{R}^n_+ \).

Lemma 4.1. Let \( f \) be a harmonic function on \( \mathbb{R}^n_+ \), which decays at infinity, i.e. \( f \to 0 \) as \( r \to \infty \). Then,

\[
f(x) = A r^{-n} + \langle B, x \rangle r^{-n} + g(x),
\]

where \( A \in \mathbb{R}, B \in \mathbb{R}^n \) and \( g \in \mathcal{O}_\infty(r^{-n}) \).

Proof. By [ABR01], c.f. also [Che20, Lem. 3.1], we have

\[
f(x) = f(r, \theta) = \sum_{i=0}^{\infty} C_i \cdot \varphi_i(\theta) \cdot r^{2-n-i},
\]

where \( C_i \in \mathbb{R}, (r, \theta) \) are polar coordinates and \( \varphi_i \in C^\infty(S^{n-1}) \) is a normalized eigenfunction on the sphere to the \( i \)-th eigenvalue. Clearly, \( \varphi_0 \) is constant. The eigenfunction \( C_1 \varphi_1 \) is the restriction of a linear function \( x \to \langle B, x \rangle \) to \( S^{n-1} \), see [BGM71, p. 159–161]. Therefore,

\[
f(x) = A r^{-n} + \langle B, x \rangle r^{-n} + g(x)
\]

where the remainder term

\[
g(x) = g(r, \theta) = \sum_{i=2}^{\infty} C_i \cdot \varphi_i(\theta) \cdot r^{2-n-i}
\]

is again a harmonic function. Standard arguments from elliptic theory imply that this series converges in all derivatives. Therefore, \( g \in \mathcal{O}_\infty(r^{-n}) \), which finishes the proof of the lemma. \( \square \)

We continue by analyzing the decay of harmonic forms on \( \mathbb{R}^n_+ \):

Lemma 4.2. Let \( \omega \) be differential form of degree \( k \) on \( \mathbb{R}^n_+ \), which satisfies \( d \omega = 0 \) and \( d^* \omega = 0 \) and decays at infinity, i.e. \( |\omega| \to 0 \) as \( r \to \infty \).

- If \( k = 0, n \), we have \( \omega = 0 \).
- If \( k = 1 \), we have \( \omega = \lambda \cdot d\Phi + \mathcal{O}_\infty(r^{-n}) \), where \( \Phi = r^{2-n} \) is the fundamental solution.
- If \( k = n - 1 \), we have \( \omega = \lambda \cdot d^* (\Phi \cdot d\vololate) + \mathcal{O}_\infty(r^{-n}) \).
- If \( 2 \leq k \leq n - 2 \), we have \( \omega = \mathcal{O}_\infty(r^{-n}) \).

Proof. In the case \( k = 0 \) or \( k = n \), the statement is trivial. Therefore, we may assume from now on that \( 1 \leq k \leq n - 1 \). We write \( \omega \) in the standard coordinates as

\[
\omega = \sum_{i_1, i_2, \ldots, i_k=1}^{n} \omega_{i_1 \ldots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k},
\]

where

\[
\omega_{i_1 \ldots i_k} = \omega(\partial_{i_1}, \ldots, \partial_{i_k})
\]
are totally anti-symmetric in all indices. Since $\Delta = dd^* + d^*d$, we have $\omega \in \ker \Delta$. This implies that each component function $\omega_{i_1 \ldots i_k}$ is a also harmonic function on $\mathbb{R}^n \setminus \{0\}$. By Lemma 4.1, we get

$$\omega_{i_1 \ldots i_k} = \omega_{i_1 \ldots i_k}^{(0)} r^{2-n} + \sum_{j=1}^{n} \omega_{j,i_1 \ldots i_k}^{(1)} x_j r^{n-1} + \mathcal{O}_\infty(r^{-n}),$$

where for each fixed $j$, the coefficients $\omega_{i_1 \ldots i_k}^{(0)}$ and $\omega_{j,i_1 \ldots i_k}^{(1)}$, are totally anti-symmetric in $i_1, \ldots, i_k$.

By the local expression for $0 = \partial^r \omega$, we get

$$0 = (d^r \omega)_{i_2 \ldots i_k}(x)$$

$$= - \sum_{l=1}^{n} \partial_l \omega_{i_2 \ldots i_k}(x)$$

$$= - \sum_{l=1}^{n} \omega_{i_2 \ldots i_k}^{(0)} x_l (2-n) r^{n-1} - \sum_{j,l=1}^{n} \omega_{j,l,i_2 \ldots i_k}^{(1)} (\delta_{jl} - n \frac{x_j x_l}{r^2}) r^{n-1} + \mathcal{O}_\infty(r^{-n-1}),$$

for all $x \in \mathbb{R}^n$. Due to the different fall-off behavior, the two terms involving $\omega_{i_2 \ldots i_k}^{(0)}$ and $\omega_{j,l,i_2 \ldots i_k}^{(1)}$ terms have to vanish separately, i.e.

$$\sum_{l=1}^{n} \omega_{i_2 \ldots i_k}^{(0)} x_l = 0, \quad \sum_{j,l=1}^{n} \omega_{j,l,i_2 \ldots i_k}^{(1)} (\delta_{jl} - n \frac{x_j x_l}{r^2}) = 0,$$

for all $x \in \mathbb{R}^n$. By inserting the unit vectors $x = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^n$ into the first equation, one successively concludes that

$$\omega_{i_2 \ldots i_k}^{(0)} = 0.$$

For the next term, fix $i_2, \ldots, i_k$ and define the quadratic form

$$C_{jl} := \omega_{j,l,i_2 \ldots i_k}^{(1)}.$$

The second equation is equivalent to

$$0 = \sum_{j,l=1}^{n} \omega_{j,l,i_2 \ldots i_k}^{(1)} (\delta_{jl} - n \frac{x_j x_l}{r^2}) r^{n-1} - \sum_{j=1}^{n} \omega_{j,i_2 \ldots i_k}^{(1)} x_j r^{n-1} + \mathcal{O}_\infty(r^{-n-1}),$$

for all $x \in \mathbb{R}^n$, where we have used that $r^2 = (x,x)$. This implies that the trace-free part of the symmetric part of $C$ vanishes. In other words, we have proven that

$$\omega_{j,l,i_2 \ldots i_k}^{(1)} + \omega_{l,j,i_2 \ldots i_k}^{(1)} = \delta_{jl} \eta_{i_2 \ldots i_k},$$

for all $j, l$ and coefficients $\eta_{i_2 \ldots i_k}$ which are totally antisymmetric in $i_2, \ldots, i_k$.

We first treat the case $k = 1$. In this case, the equation (49) reads

$$\omega_{j,l}^{(1)} + \omega_{l,j}^{(1)} = 2 \lambda \delta_{jl},$$

for some constant $\lambda \in \mathbb{R}$. We now use that $d\omega = 0$ and conclude that

$$0 = \partial_k \omega_l - \partial_l \omega_k$$

$$= \sum_{j=1}^{n} \omega_{j,1}^{(1)} (\delta_{jk} - n \frac{x_j x_k}{r^2}) r^{n-1} - \sum_{j=1}^{n} \omega_{j,k}^{(1)} (\delta_{jl} - n \frac{x_j x_l}{r^2}) r^{n-1} + \mathcal{O}(r^{-n-1}).$$

Inserting $x$, such that $x_l = \delta_{jl}$, we get

$$0 = \omega_{k,1}^{(1)} + (n-1) \omega_{l,k}^{(1)}.$$
Combining this with (50), we conclude that
\[ \omega_{k,l}^{(1)} = \lambda \delta_{kl}. \]
Inserting this is the expansion of harmonic functions, one checks that
\[ \omega = \lambda \cdot d\Phi + \mathcal{O}_\infty(r^{-n}) \]
if \( k = 1 \). The statement for \( k = n - 1 \) follows by Hodge duality.

We now turn to the case when \( 2 \leq k \leq n - 2 \). Since \( k \geq 2 \), we may insert \( j = l = i_2 \) into (49) and conclude that
\[ \eta_{i_2 \ldots i_k} = \delta_{i_2 i_2} \eta_{i_2 \ldots i_k} = 2 \omega_{i_2,i_2 \ldots i_k}^{(1)} = 0 \]
and therefore,
\[ \omega_{j,i_2 \ldots i_k}^{(1)} + \omega_{l,i_2 \ldots i_k}^{(1)} = 0, \]
which means that
\[ \omega_{j,i_1 \ldots i_k}^{(1)} \]
is totally anti-symmetric in all indices, including \( j \). In order to show that in fact \( \omega_{j,i_1 \ldots i_k}^{(1)} \) vanishes, we use our assumption \( d\omega = 0 \). This is locally given by
\[ 0 = (d\omega)_{i_1 \ldots i_{k+1}} = \frac{1}{k+1} \sum_{l=1}^{k+1} (-1)^{l+1} \partial_l \omega_{i_1 \ldots i_{k+1}}. \]
Inserting the expansion for \( r \to 0 \), we again conclude that term separately must vanish, which implies that
\[ 0 = \sum_{l=1}^{k+1} (-1)^{l+1} \partial_l \left( \sum_{j=1}^{n} \omega_{j,i_1 \ldots i_{k+1}}^{(1)} x_j r^{-n} \right) \]
\[ = \sum_{j=1}^{n} \sum_{l=1}^{k+1} (-1)^{l+1} \omega_{j,i_1 \ldots i_{k+1}}^{(1)} \left( \delta_{jl} - n \frac{x_j x_l}{r^2} \right) r^{-n}. \]
We evaluate this expression for a certain choice of \( x \). Since \( k \leq n - 2 \), there is an \( m \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{k+1}\} \).
Evaluating (51) at the \( m \)th unit vector, i.e. at \( x_i = \delta_{im} \), note that all \( x_{ij} = 0 \) and hence
\[ 0 = \sum_{j=1}^{n} \sum_{l=1}^{k+1} (-1)^{l+1} \omega_{j,i_1 \ldots i_{k+1}}^{(1)} \left( \delta_{jl} - n \frac{x_j x_l}{r^2} \right) \]
\[ = \sum_{l=1}^{k+1} (-1)^{l+1} \omega_{i_1,i_2 \ldots i_{k+1}}^{(1)} \]
\[ = (k+1) \omega_{i_1,i_2 \ldots i_{k+1}}^{(1)}, \]
where we have used that
\[ \omega_{i_1,i_2 \ldots i_{k+1}}^{(1)} \]
is totally anti-symmetric in all indices, proven above. We conclude that
\[ \omega_{i_1,i_2 \ldots i_{k+1}}^{(1)} = 0, \]
for any \( i_1, \ldots, i_{k+1} \). Inserting this in the asymptotic expansion above, we conclude that
\[ \omega = \mathcal{O}_\infty(r^{-n}) \]
as claimed, if \( 2 \leq k \leq n - 2 \). \[ \square \]
On ALE manifolds. Let $\Delta_H$ the Hodge-Laplace operator.

Proposition 4.3. Let $(M^n, g)$ be an ALE manifold. Let $\omega$ be harmonic differential form on $M$, i.e.

$$\Delta_H \omega = 0,$$

and suppose that $u \in L^p$ for some $p \in (1, \infty)$. Then, $d\omega$ and $d^*\omega$ are both vanishing and $\omega = O_\infty(r^{1-n})$. Moreover, if $\omega$ is not a one-form or an $n-1$-form, then $\omega = O_\infty(r^{-n})$.

Proof. The Hodge-Laplace operator can be written as

$$\Delta_H = dd^* + d^*d = (d + d^*)^2,$$

where

$$d^*\eta = (-1)^{kn+n+1} d \eta,$$

for any differential form $\eta$ of order $k$, and the Hodge-star operator is defined through

$$\eta_1 \wedge \eta_2 = g(\eta_1, \eta_2)\text{Vol}_g.$$

It is therefore clear that the operator $D := d + d^*$ is asymptotic to the Dirac type operator

$$D_{\mathbb{R}^n} := d + d^*_{\mathbb{R}^n}$$

in the sense of Definition 1.11, at rate $\tau$. We may therefore apply Proposition 2.10, with

$$D^l = (d + d^*)^l, \quad l \geq 2$$

and conclude that $\omega \in W^{l, p}_\mathbb{R}$, for all $l \in \mathbb{N}$. Let now $\phi : M_\infty \to \mathbb{R}^n_+ / \Gamma$ be a chart at infinity to with respect to which $M$ is ALE of order $\tau$. Let

$$\pi : \mathbb{R}^n \to \mathbb{R}^n_+ / \Gamma$$

be the quotient map. We extend $(\phi^{-1} \circ \pi)^* \omega$ smoothly to a differential form $\tilde{\omega}$ on $\mathbb{R}^n_+$ such that

$$\tilde{\omega} \equiv 0 \text{ on } \mathbb{R}^n_\neq, \quad \tilde{\omega} \equiv \phi_\ast \omega \text{ on } \mathbb{R}^n_+.$$

Noting that

$$0 = \Delta_H \tilde{\omega} = (d(d^* - d^*_\mathbb{R}) + (d^* - d^*_{\mathbb{R}^n}) d) \tilde{\omega} + \Delta_{\mathbb{R}^n} \tilde{\omega},$$

(52) on $\mathbb{R}^n_+$, we first conclude that

$$\Delta_{\mathbb{R}^n} \tilde{\omega} \in W^{l-2, p}_{\mathbb{R}^n}((\mathbb{R}^n_+)).$$

(53)

Shrinking $\tau > 0$ if necessary, we can make sure that $-\frac{n}{p} - \tau \notin \mathbb{R} \setminus \mathbb{E}_2$. Lemma 2.4 implies therefore that there is a $k$-form $\alpha \in W^{l,p}_{\mathbb{R}^n}((\mathbb{R}^n_+))$, such that

$$\Delta_{\mathbb{R}^n} \alpha = \Delta_{\mathbb{R}^n} \tilde{\omega},$$

on $\mathbb{R}^n_+$. Note that $\alpha$ may a priori depend on the weight, but is independent of $l$, because $W^{l, p}_{\mathbb{R}^n} \subset W^{l+1, p}_{\mathbb{R}^n}$ for all $l \in \mathbb{N}$. Thus, $\alpha \in W^{l,p}_{\mathbb{R}^n}$ for all $l \in \mathbb{N}$. Recall that $\omega \in W^{l,p}_{\mathbb{R}^n}$, so that $\tilde{\omega} \in W^{l,p}_{\mathbb{R}^n}$ for all $l \in \mathbb{N}$. Therefore, Sobolev embedding (c.f. [Bar86, Thm. 1.2]) implies that

$$\alpha \in O_\infty(r^{-\frac{n}{p} - \tau}), \quad \tilde{\omega} \in O_\infty(r^{-\frac{n}{p}}).$$

We thus have a harmonic form $\beta := \omega - \alpha$ on $\mathbb{R}^n_+$, which decays pointwise at infinity. Each component of $\beta$ with respect to the standard basis is a harmonic function, i.e.

$$\Delta_{\mathbb{R}^n} \beta_{i_1, \ldots, i_k} = 0.$$
for each \(i_1, \ldots, i_k \in \{1, \ldots, n\}\). Because each of these components decay at infinity, we have \(\beta_{i_1 \ldots i_k} \in \mathcal{O}_\infty(r^{2-n})\) and therefore also \(\beta \in \mathcal{O}_\infty(r^{2-n})\). This implies that after cutting of the interior part, we get

\[
\chi(\frac{1}{2}, \infty)(r) \cdot \beta \in W^{l,p}_\eta(\mathbb{R}^n_+)
\]

for each \(\eta > 2 - n\). Combining with the fact that

\[
\alpha \in W^{l,p}_\eta(\mathbb{R}^n_+) - \tau
\]

and \(\tilde{\omega} \equiv 0\) on \(\mathbb{R}^n_+\), we conclude that

\[
\tilde{\omega} = \alpha + \beta \in W^{l,p}_\eta(\mathbb{R}^n_+),
\]

for every \(l \in \mathbb{N}\) and \(\delta \in (2-n,0)\) satisfying \(\delta \geq -\frac{2}{p} - \tau\), i.e. we have “gained” an order \(\tau\) of decay.

Starting the same procedure from \((52)\) again and writing \(\tilde{\omega}\) as sum \(\tilde{\omega} = \alpha + \beta\) (with potentially different \(\alpha, \beta\)), we conclude

\[
\tilde{\omega} = \alpha + \beta \in W^{l,p}_\delta(\mathbb{R}^n_+),
\]

for every \(l \in \mathbb{N}\) and \(\delta \in (2-n,0)\) satisfying \(\delta \geq -\frac{2}{p} - 2\tau\). After iterating this procedure a finite number of times, we obtain the following assertion: For each \(\delta > 2 - n\), we may write \(\tilde{\omega}\) as a sum \(\tilde{\omega} = \alpha + \beta\) with \(\alpha \in W^{l,p}_\delta(\mathbb{R}^n_+)\) and \(\beta\) being harmonic and decaying pointwise at infinity. Therefore, by [Bar86, Thm. 1.2] again,

\[
\alpha \in \mathcal{O}_\infty(r^{\delta - \tau}), \quad \beta \in \mathcal{O}_\infty(r^{2-n}).
\]

If \(\delta > 2 - n\) is chosen so small that \(\delta - \tau < 2 - n\), we conclude \(\tilde{\omega} \in \mathcal{O}_\infty(r^{2-n})\) so that

\[
\omega \in \mathcal{O}_\infty(r^{2-n})
\]
as well. In particular,

\[
d\omega \in \mathcal{O}_\infty(r^{1-n}), \quad \star d\omega \in \mathcal{O}_\infty(r^{1-n}).
\]

Recalling that \(\Delta_H \omega = 0\), we integrate by parts on a large ball \(B_R\) of radius \(R\) on the manifold \(M\):

\[
0 = \int_{B_R} g(\Delta_H \omega, \omega) d\text{Vol} = \int_{B_R} g(d^* \omega, \omega) d\text{Vol} + \int_{\partial B_R} g(d^* d\omega, \omega) d\text{Vol}_{\partial B_R}
\]

\[
= \int_{B_R} |d^* \omega|^2 g(\omega) d\text{Vol} + \int_{\partial B_R} g(d^* \omega, \nu) d\text{Vol}_{\partial B_R}
\]

\[
+ \int_{B_R} |d\omega|^2 g(\omega) d\text{Vol} - \int_{\partial B_R} g(\omega, \nu d\omega) d\text{Vol}_{\partial B_R}.
\]

Because \(n \geq 3\), we have \(2(1-n) < -n\). Consequently, \(d\omega \in L^2(M, \Lambda^{k+1}M)\) and \(d^* \omega \in L^2(M, \Lambda^{k-1}M)\). For the boundary terms, we get

\[
\int_{\partial B_R} g(d^* \omega, \nu) d\text{Vol}_{\partial B_R} - \int_{\partial B_R} g(\omega, \nu d\omega) d\text{Vol}_{\partial B_R} = \mathcal{O}(R^{n-1+(1-n)+(2-n)}) = \mathcal{O}(R^{2-n}).
\]

Therefore, letting \(R \to \infty\) yields

\[
0 = \|d\omega\|_{L^2} + \|d^* \omega\|_{L^2},
\]

so that \(d\omega = 0\) and \(d^* \omega = 0\). The equations \(d^* \omega = 0\) and \(\omega = \alpha + \beta\) now imply

\[
d_{2n}^* \beta = (d_{2n}^* - d^*) \omega - d_{2n}^* \alpha = \mathcal{O}_\infty(r^{\delta-1-\tau}).
\]

for each \(\delta > 2 - n\). By Lemma 4.1, the harmonic functions have the expansion

\[
\tilde{\beta}_{i_1 \ldots i_k} = \tilde{\beta}^{(0)}_{i_1 \ldots i_k} r^{2-n} + \sum_{j=1}^n \tilde{\beta}^{(1)}_{j,i_1 \ldots i_k} x_j r^{-n} + \mathcal{O}_\infty(r^{-n}).
\]

(54)
Similarly as in the proof of Lemma 4.2, we compute, c.f. (48),

\[
(d^n_{R^n} \beta)_{i_1 \ldots i_k} = -(2 - n) \sum_{l=1}^{n} \beta^{(0)}_{i_1 \ldots i_k} x_l r^{-n} + O_\infty(r^{-n}) \in O_\infty(r^{\delta - 1 - \tau})
\]

If \( \delta \) was chosen so small that \( \delta - 1 - \tau < 1 - n \), we conclude as in Lemma 4.2 that \( \beta^{(0)}_{i_1 \ldots i_k} = 0 \). Therefore, \( \beta = O_\infty(r^{1-n}) \) and

\[
\beta = O_\infty(r^{1-n})
\]

is immediate. We conclude further that

\[
\tilde{\omega} = \alpha + \beta \in W^{\delta,p}_\delta(\mathbb{R}^n)
\]

for each \( l \in \mathbb{N} \) and \( \delta_1 \in (1 - n, 2 - n) \) satisfying \( \delta_1 \geq \delta - \tau \). Repeating the above procedure starting with (52) again a finite number of times, we obtain the following: For each \( \delta > 1 - n \), we may write \( \tilde{\omega} \) as a sum \( \tilde{\omega} = \alpha + \beta \) with \( \alpha \in W^{\delta_1-p}_{\delta - \tau}(\mathbb{R}^n) \) and \( \beta \) being harmonic and decaying faster than \( r^{\delta - n} \). Therefore, by [Bar86, Thm. 1.2],

\[
\alpha \in O_\infty(r^{\delta - \tau}), \quad \beta \in O_\infty(r^{1 - n}).
\]

If \( \delta > 2 - n \) is chosen so small that \( \delta - \tau < 1 - n \), we conclude \( \tilde{\omega} \in O_\infty(r^{1-n}) \) and hence,

\[
\omega \in O_\infty(r^{1-n}),
\]

as \( r \to \infty \). If \( \omega \) is a one-form or an \( n - 1 \)-form, the proof is finished. Otherwise, we continue as follows: The equations \( d\omega = 0 \) and \( d^*\omega = 0 \) imply

\[
d\beta = d\alpha = O_\infty(r^{\delta - 1 - \tau}),
\]

\[
d^n_{R^n} \beta = (d^n_{R^n} - d^*)\omega - d^n_{R^n} \alpha = O_\infty(r^{\delta - 1 - \tau}).
\]

Since we can choose \( \delta \) so small that \( \delta - \tau - 1 < -n \), we may conclude from expansion (54), using (48) and (51) in the proof of Lemma 4.2 that

\[
\beta^{(0)}_{i_1 \ldots i_k} = \beta^{(1)}_{j_1 \ldots j_k} = 0.
\]

Hence,

\[
\beta = O_\infty(r^{-n}).
\]

Repeating the above procedure starting with (52) again, we can for each \( \delta > -n \) write \( \tilde{\omega} = \alpha + \beta \) with \( \alpha \in W^{\delta_1-p}_{\delta - \tau}(\mathbb{R}^n) \) and \( \beta \) being harmonic and decaying faster than \( r^{1-n} \). By [Bar86, Thm. 1.2],

\[
\alpha \in O_\infty(r^{\delta - \tau}), \quad \beta \in O_\infty(r^{-n})
\]

Choose \( \delta > -n \) so small that \( \delta - \tau \leq -n \). Then we conclude

\[
\tilde{\omega} = O_\infty(r^{-n}).
\]

Thus, \( \omega \in O_\infty(r^{-n}) \) as \( r \to \infty \), which finishes the proof of the proposition. \( \square \)

4.2. \( L^p \)-cohomology. In this section, we discuss some consequences of Proposition 4.3 for the reduced \( L^p \)-cohomology which extend and complement the results in [CCH06, Dev14]. For \( 1 \leq k \leq n - 1 \) we introduce the notations

\[
im_{L^p}(d_{k-1}) = \overline{d(\mathcal{C}_c^\infty(\Lambda^{k-1}M)_{L^p})}, \quad \nim_{L^p}(d_{k+1}^*) = d^*(\mathcal{C}_c^\infty(\Lambda^{k+1}M)_{L^p})
\]

and

\[
\ker_{L^p}(d_k) = \{ \omega \in L^p(\Lambda^k M) \mid d\omega = 0 \}, \quad \ker_{L^p}(d_{k+1}) = \{ \omega \in L^p(\Lambda^k M) \mid d^*\omega = 0 \},
\]

\[
\ker_{L^p}(\Delta_{H_k}) = \{ \omega \in L^p(\Lambda^k M) \mid \Delta_{H_k}\omega = 0 \}.
\]
We have the inclusions
\[
\text{im}_{L^p}(d_{k-1}) \subset \ker_{L^p}(d_k), \quad \text{im}_{L^p}(d_{k+1}^*) \subset \ker_{L^p}(d_k),
\]
\[
\text{im}_{L^p}(d_{k-1}) \cap \text{im}_{L^p}(d_{k+1}^*) \subset \ker_{L^p}(d_k) \cap \ker_{L^p}(d_k^*) = \ker_{L^p}({\Delta}_H),
\]
where the inequality is due to Proposition 4.3. Now the k'th reduced $L^p$-cohomology is defined as
\[
H^k_{\bar{p}}(M) := \frac{\ker_{L^p}(d_{k-1})}{\text{im}_{L^p}(d_k)}.
\]
In the following, we will compare $H^k_{\bar{p}}(M)$ with $H_k(M) = \ker_{L^p}({\Delta}_H).

**Lemma 4.4.** Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,
\[
\text{im}_{L^p}(d_{k-1}) \cap \ker_{L^p}({\Delta}_H) \cap \ker_{L^p}({\Delta}_H) = \{0\}, \quad (55)
\]
\[
\text{im}_{L^p}(d_{k+1}^*) \cap \ker_{L^p}({\Delta}_H) \cap \ker_{L^p}({\Delta}_H) = \{0\}. \quad (56)
\]
In particular, if $\ker_{L^p}({\Delta}_H) \subset \ker_{L^p}({\Delta}_H)$, we have
\[
\text{im}_{L^p}(d_{k-1}) \cap \text{im}_{L^p}(d_{k+1}^*) = \text{im}_{L^p}(d_{k-1}) \cap \ker_{L^p}({\Delta}_H) = \text{im}_{L^p}(d_{k+1}^*) \cap \ker_{L^p}({\Delta}_H) = \{0\}.
\]

**Proof.** Let $\omega \in \ker_{L^p}({\Delta}_H) \cap \ker_{L^p}({\Delta}_H)$ and suppose that there exists $\alpha_i \in C^\infty_c({\mathbb{A}}^{k-1}M)$ such that $d\alpha_i \to \omega$ in $L^p$. By integration by parts and Proposition 4.3, $(d\alpha_i, \omega)_{L^2} = (\alpha_i, d^*\omega)_{L^2} = 0$ for all $i$ and therefore,
\[
\|\omega\|_{L^2}^2 = (\omega - d\alpha_i, \omega)_{L^2} \leq \|\omega - d\alpha_i\|_{L^p} \|\omega\|_{L^q} \to 0.
\]
Thus, $\omega = 0$, which proves (55). The proof of (56) is completely analogous. \hfill \square

For the proof of the next lemma, we recall that for a subspace $V$ of a Banach space $X$, its annihilator is defined by
\[
\text{Ann}(V) = \{x^* \in X^* \mid x^*(v) = 0 \text{ for all } v \in V\} \subset X^*.
\]
If $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, we clearly have
\[
\text{Ann}(\text{im}_{L^p}(d_{k-1})) = \ker_{L^p}(d_k^*), \quad \text{Ann}(\text{im}_{L^p}(d_{k+1}^*)) = \ker_{L^p}(d_k).
\]

**Lemma 4.5.** Let $p \in (1, \infty)$, if $2 \leq k \leq n - 2$ and $p \in \left(\frac{n}{n-1}, n\right)$, if $k \in \{1, n - 1\}$. Then we have
\[
L^p({\mathbb{A}}^{k}M) = \text{im}_{L^p}(d_{k-1}) \oplus \text{im}_{L^p}(d_{k+1}^*) \oplus \ker_{L^p}({\Delta}_H). \quad (57)
\]

**Proof.** By Proposition 4.3, we have $\ker_{L^p}({\Delta}_H) = \ker_{L^p}({\Delta}_H)$ by the assumptions on $p$ and $k$. By Lemma 4.4, the sum on the right hand side of (57) is indeed direct. To finish the proof, it suffices to show that the annihilator of the direct sum vanishes. We have
\[
\text{Ann}(\text{im}_{L^p}(d_{k-1}) \oplus \text{im}_{L^p}(d_{k+1}^*) \oplus \ker_{L^p}({\Delta}_H)) = \ker_{L^p}(d_k^*) \cap \ker_{L^p}(d_k) \cap \text{Ann}(\ker_{L^p}({\Delta}_H))
\]
\[
= \ker_{L^p}({\Delta}_H) \cap \text{Ann}(\ker_{L^p}({\Delta}_H))
\]
However, the right hand side is zero as $\ker_{L^p}({\Delta}_H) = \ker_{L^p}({\Delta}_H)$. This finishes the proof of the lemma. \hfill \square

**Corollary 4.6.** Let $p \in (1, \infty)$, if $2 \leq k \leq n - 2$ and $p \in \left(\frac{n}{n-1}, n\right)$, if $k \in \{1, n - 1\}$. Then,
\[
H^k_{\bar{p}}(M) \cong H_k(M).
\]
Lemma 4.8. Let

$$\ker_{L^p}(d_k) \cap \text{im}_{L^p}(d_{k+1}^* \cap \ker_{L^p}(d_k) \cap \ker_{L^p}(d_k^* \cap \ker_{L^p}(\Delta H_k),$$

but since \( \ker_{L^p}(\Delta H_k) = \ker_{L^p}(\Delta H_k) \), we have \( \text{im}_{L^p}(d_{k+1}^* \cap \ker_{L^p}(\Delta H_k) = \{0\} \) by Lemma 4.4. Therefore, \( \ker_{L^p}(d_k) \cap \text{im}_{L^p}(d_{k+1}^*) = \{0\} \) as well. Intersecting (57) with \( \ker_{L^p}(d_k) \) yields

$$\ker_{L^p}(d_k) = \text{im}_{L^p}(d_{k-1}) \oplus \ker_{L^p}(\Delta H_k).$$

Because

$$\mathcal{H}_k(M) = \ker_{L^p}(\Delta H_k) = \ker_{L^p}(\Delta H_k),$$

the statement of the corollary is immediate. \( \square \)

Let us now continue with the cases left out in the case \( k = 1 \) and \( k = n-1 \). By Hodge duality, we may restrict to the case \( k = 1 \). To analyse \( H^2_0(M) \) for \( p \in (1, \infty) \setminus \left(\frac{n}{n-1}, n\right) \), we want to understand \( \ker(\Delta H) \) more in detail. For this purpose, we define

$$\ker_s(\Delta H_k) = \{ \omega \in \ker(\Delta H_k) \mid \omega \in \mathcal{O}_\infty(p^k) \}.$$

From the proof of Proposition 4.3, we obtain

$$\ker_{L^p}(\Delta H_k) = \ker_{1-n}(\Delta H_k), \text{ if } p \in \left(\frac{n}{n-1}, \infty\right),$$

$$\ker_{L^p}(\Delta H_k) = \ker_{n}(\Delta H_k), \text{ if } p \in \left(1, \frac{n}{n-1}\right].$$

(58)

In the following we consider the cases of small and large \( p \) separately. Let us start with large \( p \).

Lemma 4.7. Let \( p \in [n, \infty) \). Then we have

$$L^p(\Lambda^1 M) = (\text{im}_{L^p}(d_0) + \text{im}_{L^p}(d_2^*)) \oplus \ker_{-n}(\Delta H_k).$$

(59)

Proof. Let \( q \in \left(1, \frac{n}{n-1}\right] \) be such that \( 1 = \frac{1}{p} + \frac{1}{q} \). Then, \( \ker_{L^q}(\Delta H_k) = \ker_{L^q}(\Delta H_k) = \ker_{L^q}(\Delta H_k) \cap \ker_{L^q}(\Delta H_k) \) and from Lemma 4.4, we get that

$$\ker_{L^q}(\Delta H_k) \cap \ker_{L^q}(\Delta H_k) \cap \ker_{L^q}(\Delta H_k) = \{0\},$$

so that the sum on the right hand side of (59) is indeed direct. The annihilator of the sum is given by

$$\text{Ann}(\text{im}_{L^p}(d_0) + \text{im}_{L^p}(d_2^*) \oplus \ker_{-n}(\Delta H_k)) = \ker_{L^q}(d_1) \cap \ker_{L^q}(d_2^*) \cap \text{Ann}(\ker_{-n}(\Delta H_k))$$

$$= \ker_{L^q}(\Delta H_k) \cap \text{Ann}(\ker_{-n}(\Delta H_k))$$

$$= \ker_{L^q}(\Delta H_k) \cap \text{Ann}(\ker_{-n}(\Delta H_k)) = \{0\}.$$

The latter implies that we have equality in (59), which proves the lemma. \( \square \)

Lemma 4.8. Let \( p \in [n, \infty) \). Then we have subspaces \( X_i, Y_i, Z \subset L^p(\Lambda^1 M), i = 1, 2 \), such that

$$L^p(\Lambda^1 M) = X_2 \oplus X_1 \oplus Z \oplus Y_1 \oplus Y_2 \oplus \ker_{-n}(\Delta H_k),$$

(60)

$$\text{im}_{L^p}(d_0) = X_2 \oplus X_1 \oplus Z,$$

(61)

$$\text{im}_{L^p}(d_2^*) = Z \oplus Y_1 \oplus Y_2,$$

(62)

$$\ker_{L^p}(d_1) = X_2 \oplus X_1 \oplus \ker_{-n}(\Delta H_k),$$

(63)

$$\ker_{L^p}(d_2^*) = X_1 \oplus \ker_{-n}(\Delta H_k),$$

(64)

$$\ker_{1-n}(\Delta H_k) = X_1 \oplus Y_1 \oplus Y_2 \oplus \ker_{-n}(\Delta H_k).$$

(65)
In particular, we have Corollary 4.9. Let $\Delta := \text{im}_{L^p}(d_0) \cap \text{im}_{L^p}(d_0^*) \subset \ker_{L^p}(\Delta_{H_1})$. Because $\ker_{L^p}(\Delta_{H_1})$ is finite-dimensional, we may take a complement $X_1$ of $\Delta$ in $\text{im}_{L^p}(d_0) \cap \ker_{L^p}(\Delta_{H_1})$ and a complement $X_2$ of $Z \oplus X_1$ in $\text{im}_{L^p}(d_0)$. Similarly, we may take a complement $Y_1$ of $Z$ in $\text{im}_{L^p}(d_0^*) \cap \ker_{L^p}(\Delta_{H_1})$ and a complement $Y_2$ of $Z \oplus Y_1$ in $\text{im}_{L^p}(d_0^*)$. Then, (60) follows from Lemma 4.7 and (61),(62) follow from construction. Now we are going to show (63). Note that

$$X_2 \subset \text{im}_{L^p}(d_0) \subset \ker_{L^p}(d_1), \quad X_1 \oplus Z \oplus Y_1 \oplus \ker_{-n}(\Delta_{H_1}) \subset \ker_{L^p}(\Delta_{H_1}) \subset \ker_{L^p}(d_1).$$

Because $Y_2 \subset \text{im}_{L^p}(d_0^*) \subset \ker_{L^p}(d_1^*)$, we have

$$Y_2 \cap \ker_{L^p}(d_1) = Y_2 \cap \ker_{L^p}(d_1) \cap \ker_{L^p}(d_1^*) = Y_2 \cap \ker_{L^p}(\Delta_{H_1}) = \{0\},$$

because $Y_2$ complements $Z \oplus Y_1 = \text{im}_{L^p}(d_0) \cap \ker_{L^p}(\Delta_{H_1})$ in $\text{im}_{L^p}(d_0)$. Therefore, we get (63).

The proof of (64) is completely analogous. It remains to show (65). At first, we have

$$X_1 \oplus Z \oplus Y_1 \oplus \ker_{-n}(\Delta_{H_1}) \subset \ker_{1-n}(\Delta_{H_1})$$

from construction. To show equality, it suffices to show

$$(X_2 \oplus Y_2) \cap \ker_{1-n}(\Delta_{H_1}) = \{0\}.$$ 

From Proposition 4.3, we obtain

$$(X_2 \oplus Y_2) \cap \ker_{1-n}(\Delta_{H_1}) = (X_2 \oplus Y_2) \cap \ker_{L^p}(\Delta_{H_1}) = (X_2 \oplus Y_2) \cap \ker_{L^p}(d_1) \cap \ker_{L^p}(d_1^*)$$

and (60), (63) and (64) show that the intersection on the right hand side is the zero space. This finishes the proof. 

This lemma allows us to treat the cases of small and large Hölder exponents simultaneously:

**Corollary 4.9.** Let $p \in [n, \infty)$ and $q \in (1, \frac{n}{n-1}]$ such that $1 = \frac{1}{p} + \frac{1}{q}$. With the notation of Lemma 4.8, we have identifications

$$H^1_p(M) \cong Y_1 \oplus \ker_{-n}(\Delta_{H_1}) = \ker_{1-n}(\Delta_{H_1}) \cap \text{im}_{L^p}(d_0),$$

$$H^1_q(M) \cong X_1 \oplus \ker_{-n}(\Delta_{H_1}) = \ker_{1-n}(\Delta_{H_1}) \cap \text{im}_{L^p}(d_0^*).$$

**Proof.** The description of $H^1_p(M)$ follows from (61), (63) and (65). For $H^1_q(M)$, recall that

$$\ker_{L^p}(d_1) = \text{Ann}(\text{im}_{L^p}(d_0^*)), \quad \text{im}_{L^p}(d_0) = \text{Ann}(\ker_{L^p}(d_1^*)�,$$

which allows us to identify

$$H^1_q(M) = \frac{\ker_{L^p}(d_1)}{\text{im}_{L^p}(d_0)} = \frac{\text{Ann}(\text{im}_{L^p}(d_0^*))}{\text{Ann}(\ker_{L^p}(d_1^*))} \cong \left(\frac{\ker_{L^p}(d_1^*)}{\text{im}_{L^p}(d_0^*)}\right)^\ast,$$

where $\ast$ denotes the dual space. From (62), (64) and (65), we get

$$\frac{\ker_{L^p}(d_1^*)}{\text{im}_{L^p}(d_0^*)} \cong X_1 \oplus \ker_{-n}(\Delta_{H_1}) = \ker_{1-n}(\Delta_{H_1}) \cap \text{im}_{L^p}(d_0^*).$$

In particular, the quotient on the left hand side is finite-dimensional and thus isomorphic to its dual space. This finishes the proof. \qed
5. Heat Flows on General ALE Manifolds

5.1. The Hodge-Laplacian on the exterior algebra. The following theorem is Corollary 1.23.

Theorem 5.1 (Heat kernel and derivative estimates for the Hodge-Laplacian). Let \((M, g)\) be an ALE manifold and suppose that \(H_1(0) = 0\). Then, \(e^{-t\Delta_H}\) satisfies almost Euclidean heat kernel estimates and derivative estimates of degree 0.

Proof. Let \(\omega \in \ker L^2(\Delta_H)\). By assumption, \(\omega\) is not a one-form and by Hodge duality, it is not an \(n-1\)-form either. By Proposition 4.3, this implies \(\omega \in O_\infty(r^{-n})\). The assertion then follows from Theorem 1.8 and Theorem 1.14.

Remark 5.2. In the next two subsections, we demonstrate that these results can be substantially improved if we restrict to functions and one-forms. By Hodge duality, the same estimates also hold for \(n\)-forms and \((n-1)\)-forms, respectively.

5.2. The Hodge-Laplacian on one-forms. Here, we are going to prove Corollary 1.26 (i) and (ii). For convenience, we will in the following use the notation \(\Delta_{H_1} := \Delta_H|_{T^*M}\).

Theorem 5.3. Let \((M, g)\) be an ALE manifold and suppose that \(H_1(0) = 0\). Then, \(e^{-t\Delta_{H_1}}\) satisfies Euclidean heat kernel estimates and strong derivative estimates of degree 0.

Proof. Because \(\Delta_{H_1} = \nabla^*\nabla + \text{Ric}\) (66) and \(\ker L^2(\Delta_{H_1}) = H_1(0) = 0\), the first assertion follows from Remark 1.9. Furthermore, we already know that we have derivative estimates of degree zero. Thus it remains to show that

\[
\|\nabla^k \circ e^{-t\Delta_{H_1}} \omega\|_{L^p} \leq Ct^{-\frac{p}{2}} \|\omega\|_{L^p}
\]

for all \(p \in \left(\frac{2}{k}, \infty\right)\) and \(k \geq 1\). Let \(\omega_t = e^{-t\Delta_{H_1}} \omega\). Corollary 2.9 then implies

\[
\|\nabla^k \omega_t\|_{L^p} \leq C(\|d + d^*\|^k \omega_t\|_{L^p} + \|\omega_t\|_{L^\infty}) \leq C \left(t^{-\frac{k}{2}} + t^{-\frac{k}{2}}\right) \|\omega\|_{L^p} \leq Ct^{-\frac{k}{2}}
\]

for \(t \geq t_0\) which finishes the proof.

Theorem 5.4. Let \((M, g)\) be an ALE manifold with nonnegative Ricci curvature. Then \(e^{-t\Delta_{H_1}}\) satisfies weak derivative estimates of degree 1.

Proof. First let us see that \((M, g)\) has only one end: Suppose that there were at least two, then \((M, g)\) contains a line, i.e. a geodesic that is minimizing between each of its points. Because \(\text{Ric} \geq 0\), the Cheeger-Gromoll splitting theorem implies that \((M, g)\) splits isometrically as a product \((\mathbb{R} \times N, dr^2 + h)\). However, if \((N, h) \neq (\mathbb{R}^{n-1}, g_{eucl})\), curvature would not fall off at infinity which contradicts the assumption that \((M, g)\) is ALE.

Due to (66), the assertion follows from Corollary 3.30, because we assumed \(\text{Ric} \geq 0\).

Remark 5.5. If \((M, g)\) is Ricci-flat and obtains a parallel spinor, the latter result can be slightly improved. This will be discussed in Subsection 6.5 below.

5.3. The Laplace-Beltrami operator on functions. In this subsection, we are going to prove the two assertions of Corollary 1.25.

Theorem 5.6 (Derivative estimates). Let \((M^n, g)\) be an ALE manifold with \(H_1(M) = 0\) and \(\Delta\) be its Laplace-Beltrami operator. Then, \(e^{-t\Delta}\) satisfies Euclidean heat kernel estimates and strong derivative estimates of degree 1.
Proof. The first assertion follows from Remark 1.9. Because the differential $\nabla = d : C^\infty(M) \to C^\infty(M,T^*M)$ satisfies
\[ d \circ \Delta = \Delta_{H_1} \circ d, \quad d^* \circ d = \Delta, \quad d \circ d^* \leq \Delta_{H_1}, \]
the second assertion now follows from Proposition 1.15 and Theorem 5.3.

**Theorem 5.7** (Derivative estimates). Let $(M^n, g)$ be an ALE manifold with $H_1(M) = 0$, which is not AE. Let $\Delta$ be its Laplace-Beltrami operator. Then, $e^{-\Delta}$ satisfies derivative estimates of degree 2.

Proof. Let us recall a few more facts for the Laplace operator, see e.g. [Pa 13, Sec. 9] for details. On an asymptotically conical manifold $(M^n, g)$ with link $(L, g_L)$,
\[ \Delta : W^{k+2,\beta}_\delta(M) \to W^{k,\beta}_\delta(M) \]
is Fredholm for $\delta \in \mathbb{R} \setminus \mathcal{E}_L$, where the exceptional set is given by
\[ \mathcal{E}_L = \left\{ \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \lambda} \mid \lambda \in \text{Spec}(L) \right\}. \]

If $(M^n, g)$ is ALE, i.e. asymptotically conical with link $(L, g_L) = (S^{n-1}/\Gamma, g_{st})$ and we have
\[ \text{Spec}(S^n/\Gamma, g_{st}) \subset \{ k(n+k-2) \mid k \in \mathbb{N} \}. \]
However, if $\Gamma$ is nontrivial, the well-known Lichnerowicz-Obata eigenvalue estimate (see [Oba 62]) implies that $(n-1) \notin \text{Spec}(S^n/\Gamma, g_{st})$. For this reason, we have
\[ \mathcal{E}_L \subset \mathcal{E}_2 \setminus \{ 1-n, 1 \} \]
in this case. In particular any harmonic $u \in C^\infty(M)$ with $u = o(r^2)$ at infinity is bounded since there are no exceptional values in $(0, 2)$. Furthermore, $u$ is constant: We have $\Delta_{H_1} du = d\Delta u$ and due to elliptic regularity, $du = O_{\infty}(r^{-1})$. Proposition 4.3 then implies that $du = O(r^{1-n})$, so that
\[ du \in \ker_{L^2}(\Delta_{H_1}) = \mathcal{H}^1(M) = \{ 0 \}. \]
Summing up, we have shown
\[ (d + d^*)^k u = 0, \quad u = o(r^k) \quad \Rightarrow \quad \nabla^k u = 0, \]
for $k = 1, 2$, which is condition (16). Theorem 1.19 implies that $e^{-t\Delta}$ has weak derivative estimates of degree 2. In particular we have
\[ \|\nabla^k u\|_{L^p} \leq Ct^{-\frac{\pi}{k}}\|u_0\|_{L^p} \quad (67) \]
for $u = e^{-t\Delta}u_0$ and pairs $(p, k)$ satisfying
\[ p \in (1, n) \cup (n, \infty), \quad k = 1, \]
\[ p \in \left( 1, \frac{n}{2} \right) \cup \left( \frac{n}{2}, n \right) \cup (n, \infty), \quad k = 2, \]
\[ p \in \left( \frac{n}{k}, \frac{n}{k-1} \right) \cup \left( \frac{n}{k-1}, \frac{n}{k-2} \right), \quad k > 2. \]
In order to finish the proof of the theorem, we have to establish (67) also for the gap points in these intervals. In fact, since 1 is no longer exceptional, the proof of Theorem 1.19 also works for the nonexceptional value $\delta = k - \frac{n}{2} = 1$. Hence we get (67) also for $k \geq 2$ and $p = \frac{n}{k-1}$. For the remaining cases $k \geq 1$ and $p = \frac{n}{k-1}$, interpolation shows that
\[ \|\nabla^k u\|_{L^p} \leq \|\nabla^{k-1} u\|_{L^p}^{1/2} \|\nabla^{k+1} u\|_{L^p}^{1/2} \leq Ct^{-\frac{\pi}{k}}\|u_0\|_{L^p}, \]
where the second inequality follows from cases which are already covered. \(\square\)
5.4. The classical Dirac operator. Let \((M^n, g)\) be an ALE spin manifold and \(\mathcal{D}_S\) be the classical Dirac operator acting on sections of the spinor bundle \(S\). The following theorem is the first part of Corollary 1.27.

**Theorem 5.8** (Derivative estimates). Let \((M, g)\) be an ALE spin manifold with nonnegative scalar curvature and one end. Then, \(e^{-t(\mathcal{D}_S)^2}\) satisfies almost euclidean heat kernel estimates and weak derivative estimates of degree 1.

**Proof.** Due to the well known Weitzenböck formula

\[
(\mathcal{D}_S)^2 = \nabla^* \nabla + \frac{1}{4} \text{scal},
\]

the result follows from the assumption \(\text{scal} \geq 0\) and Corollary 3.30.

6. Heat flows on ALE manifolds with a parallel spinor

Throughout this section, we assume that \((M, g)\) is a simply-connected ALE manifold with a parallel spinor. For convenience, we assume that \((M, g) \neq (\mathbb{R}^n, g_{\text{eucl}})\). These assumptions have various consequences:

- \((M, g)\) is Ricci-flat.
- \((M, g)\) has irreducible holonomy: Otherwise, \((M, g) = (\mathbb{R} \times N, dr^2 + h)\) but this contradicts the assumption of the manifold to be ALE unless it is flat. Consequently,

\[
\text{Hol}(M, g) \in \{ \text{SU}(n/2), \text{Sp}(n/4), \text{Spin}(7) \}.
\]

- \(M\) is even-dimensional (therefore, we excluded the case of holonomy \(G_2\)): If the dimension was odd, then the group \(\Gamma \in \text{SO}(n)\) at infinity is trivial. However, this implies that \((M, g)\) is AE and contains a line, i.e. a geodesic that is minimizing between all of its points. In this situation, \(\text{Ric} = 0\) and the Cheeger-Gromoll splitting theorem imply that it splits as \((M, g) = (\mathbb{R} \times N, dr^2 + h)\) which we already excluded.

- \((M, g)\) has only one end. Otherwise, it contains a line and Cheeger-Gromoll would again lead to a contradiction.

- \(\mathcal{H}_L(M) = \ker_{L^2}(\Delta_{H_2}) = 0\). This is due to the maximum principle, because \(\Delta_{H_2} = \nabla^* \nabla\).

- \((M, g)\) admits at most finite quotients which are again ALE: If \((N, h)\) is an ALE manifold with \((M, g)\) as its universal cover, then we also have a covering map \(\pi : M_\infty \rightarrow N_\infty\). Because \(M\) has only one end, \(M_\infty\) is connected. Since \(\pi_1(N_\infty)\) is finite, \(\pi\) is a finite cover which extends to a finite cover \(\pi : M \rightarrow N\). Therefore \(N = M/G\) with \(G\) being a finite group.

**Remark 6.1.** By restricting to \(G\)-invariant sections, all the heat kernel estimates we are going to establish in this section do also descend to \(M/G\). Therefore, we may drop the assumption \(\pi_1(M) = \{0\}\) and state the results as in Subsection 1.2.5.

**Remark 6.2.** All known Ricci-flat ALE manifolds satisfy (69) and thus carry a parallel spinor. Moreover, all these groups actually appear as holonomy groups of Ricci-flat ALE manifolds, see [Kro89, Joy99, Joy00, Joy01]. It is an open question whether there are other examples, c.f. [BKN89, p. 315].

6.1. The twisted Dirac operator on vector-spinors. Here, we are going to prove Corollary 1.24 (i). We start this subsection with a short exposition on the twisted Dirac operator which is based on [HS19]. We refer to this paper for further details. We consider the bundle \(S \otimes T^*_c M\) of
Definition 6.3. Let $S \otimes T^*_C M$ be equipped with the twisted Dirac operator $D_{T^*_M}$.

(i) The operator

$$P := \text{pr}_{S_{3/2}} \circ \nabla : C^\infty(M, S) \to C^\infty(M, S_{3/2})$$

is called Penrose operator or twistor operator.

(ii) The operator

$$Q := \text{pr}_{S_{3/2}} \circ D_{T^*_M}|_{S_{3/2}} : C^\infty(M, S_{3/2}) \to C^\infty(M, S_{3/2})$$

is called Rarita-Schwinger operator.

With respect to the decomposition $S \otimes T^*_C M = S_{1/2} \oplus S_{3/2}$, the operator $D_{T^*_M}$ is written as

$$D_{T^*_M} = \begin{pmatrix} 2 - n & 2i \circ P^* \\ 2P \circ i^{-1} & Q \end{pmatrix},$$

where $D_S$ is the classical Dirac operator on spinors. For a general Riemannian spin manifold, $D_{T^*_M}$ decomposes as

$$(D_{T^*_M})^2 = \begin{pmatrix} (2 - n)^2 & i \circ D_S \circ i^{-1} + \frac{1}{n} i \circ P^* \circ P \circ i^{-1} \\ \frac{4}{n^2} i \circ D_S \circ P^* \circ P \circ i^{-1} + 4Q \circ P \circ i^{-1} & 4 - 2n \frac{2}{n^2} i \circ D_S \circ P^* \circ P \circ i^{-1} + 4Q \circ i^{-1} \\ Q^2 + \frac{4}{n} P \circ P^* \circ P \circ i^{-1} & Q^2 + \frac{4}{n} P \circ P^* \circ P \circ i^{-1} \end{pmatrix}.$$

We have the Weitzenböck formula

$$P^* \circ P = \frac{n - 1}{n} D^2_S - \frac{\text{scal}}{4},$$

and the formula

$$2 - \frac{n}{n} P \circ D_S + Q \circ P = \frac{1}{2} \left( \text{Ric} - \frac{\text{scal}}{n} \right),$$

where we consider a trace-free endomorphism on $TM$ as a map $h : C^\infty(S) \to C^\infty(S_{3/2})$ defined by $h(s) = \sum_i h_{ij} e_j \cdot s \otimes e^*_i$. By introducing the standard Laplacians $\Delta_{S_{1/2}} \in \text{Diff}(S_{1/2})$ and $\Delta_{S_{3/2}} \in \text{Diff}(S_{3/2})$, we get the Weitzenböck formulas

$$D^2_S = \Delta_{S_{1/2}} + \frac{\text{scal}}{8},$$

$$Q^2 + \frac{4}{n} P \circ P^* = \Delta_{S_{3/2}} + \frac{\text{scal}}{8} + \text{Ric}^{3/2},$$

where $\text{Ric}^{3/2} = \text{pr}_{S_{3/2}} \circ \text{id} \otimes \text{Ric}|_{S_{3/2}}$. In particular, if $(M, g)$ is Ricci-flat, we obtain

$$(D_{T^*_M})^2 = \begin{pmatrix} i \circ \Delta_{S_{1/2}} \circ i^{-1} & 0 \\ 0 & \Delta_{S_{3/2}} \end{pmatrix}.$$
Proposition 6.4. Let \( \psi \in \ker(\Delta_{S^{1/2}}) \) and suppose that \( \psi \in L^p \) for some \( p < \infty \). Then, \( \psi \in O_\infty(r^{-n}) \). Furthermore,
\[
\ker_{L^2}(\Delta_{S^{1/2}}) = \ker_{L^2}(D^2_{T^*M}) = \ker_{L^2}(D_{T^*M}) \cap \ker_{L^2}(Q).
\]

Proof. Recall that
\[
\text{Hol}(M, g) \in \{ \text{SU}(n/2), \text{Sp}(n/4), \text{Spin}(7) \}.
\]
In all these cases, bundle isometries were constructed in [HS19]. If \( n = 2m \) and \( \text{Hol}(M^n, g) = \text{SU}(m) \) (i.e. \( (M, g) \) is Calabi-Yau), we have (c.f. [HS19, Sec. 4.6])
\[
S_{3/2} \cong \bigoplus_{k=0}^m (\Lambda^{1-k} T^* c M \oplus \Lambda^{n-k,1} T^* c M) \oplus \bigoplus_{k=0}^m \Lambda^{0,k} T^* c M.
\]
If \( n = 4m \) and \( \text{Hol}(M^n, g) = \text{Sp}(m) \) (i.e. \( (M, g) \) is hyperkähler), we have (c.f. [HS19, Sec. 4.7])
\[
S_{3/2} \cong \bigoplus_{k=0}^m 2(n-k+1)(\Lambda^{0,k+1} T^* c M \oplus \Lambda^{k-1} T^* c M \oplus \Lambda^{k,1} T^* c M) \oplus \bigoplus_{k=0}^m (n-k+1) \Lambda^{0} T^* c M.
\]
Finally, if \( n = 8 \) and \( \text{Hol}(M^n, g) = \text{Spin}(7) \), we have (c.f. [HS19, Sec. 4.9])
\[
S_{3/2} \cong T^*_c M \oplus \Lambda^1 \oplus T^*_c M \oplus \Lambda_3 \oplus T^*_c M \oplus \Lambda_2 \oplus T^*_c M.
\]
In all of these cases, \( \Delta_{S^{1/2}} \) coincides with \( \Delta_H \) via the bundle isometry. Note that [HS19] considers compact manifolds but these identifications also work in the noncompact setting as they are purely built on representation-theoretic arguments. Because \( H_1(M) = \{ 0 \} \), Proposition 4.3 \( \omega \in \mathcal{O}_\infty(r^{-n}) \) for any \( \omega \in C^\infty(M, \Lambda^2 \omega) \) with \( \omega \in L^p \) for some \( p < \infty \). By splitting into real and imaginary part, the same assertion also holds for \( \omega \in C^\infty(M, \Lambda^* c M) \). Thus the first assertion of the proposition follows from the existence of these bundle isometries which identify \( \Delta_{S^{1/2}} \) and \( \Delta_H \).

We furthermore have
\[
\ker_{L^2}(D^2_{T^*M}) = \ker_{L^2}(\Delta_{S^{1/2}}) \oplus \ker_{L^2}(\Delta_{S^{3/2}}) = \ker_{L^2}(\Delta_{S^{3/2}}).
\]

The first equality follows from the diagonal form of \( D^2_{T^*M} \). Moreover, (68) and (73) yield \( \Delta_{S^{1/2}} = (\Delta_{S})^2 = \nabla^* \nabla \) in this situation. Because \( (M, g) \) has only one end, the maximum principle implies that \( \ker_{L^2}(\Delta_{S^{1/2}}) \) is trivial which proves the second equality. Furthermore, because any \( \psi \in \ker_{L^2}(D^2_{T^*M}) \) satisfies \( \psi \in \mathcal{O}_\infty(r^{-n}) \), we may use integration by parts to conclude that
\[
\ker_{L^2}(D^2_{T^*M}) \subseteq \ker_{L^2}(D_{T^*M})
\]
and the converse inclusion is trivial. To finish the proof, it remains to show
\[
\ker_{L^2}(\Delta_{S^{3/2}}) = \ker_{L^2}(P^*) \cap \ker_{L^2}(Q).
\]
However, this identity is clear from (74) and integration by parts, which is possible due to the first assertion of this proposition. \( \square \)

Theorem 6.5. The heat flows of \( D^2_{T^*M} \) and \( \Delta_{S^{3/2}} \) satisfy heat kernel and derivative estimates of degree 0.

Proof. By the proof of the previous proposition, we have \( \Delta_{S^{3/2}} \cong \Delta_H \) via a parallel isomorphism of vector bundles. Furthermore, we have \( D^2_{T^*M} \cong (\Delta_{S^{1/2}}, \Delta_{S^{3/2}}) \cong ((\Delta_S)^2, \Delta_H) \). Thus, the assertion follows from Theorem 5.8 and Theorem 5.1. \( \square \)
6.2. The classical Dirac operator revisited. In this subsection, we are going to prove the second assertion in Corollary 1.27.

**Lemma 6.6.** We have $D_{T^c\cdot M} \circ \nabla = \nabla \circ D_S$.

**Proof.** Using a local orthonormal basis, we may write
\[
\nabla \varphi = \sum_i \nabla_{e_i} \varphi \otimes e_i^*
\]
and calculate, using the curvature identity in [Gin09, Lem. 1.2.4],
\[
D_{T^c\cdot M} \nabla \varphi = \sum_{i,j} e_j \cdot \nabla_{e_j, e_i} \varphi \otimes e_i^* = \sum_{i,j} [\nabla_{e_i} (e_j \cdot \nabla_{e_i} \varphi) \otimes e_i^* + e_j \cdot R_{e_j, e_i} \varphi \otimes e_i^*] = \nabla (D_S \varphi) + \sum_i \text{Ric}(e_i) \cdot \varphi \otimes e_i^*.
\]
The result follows from Ric = 0. \qed

**Lemma 6.7.** We have
\[
\nabla^* \nabla \leq (D_S)^2, \quad \nabla \circ \nabla^* \leq \frac{n}{2} (D_{T^c\cdot M})^2.
\]

**Proof.** The first identity is immediate from (68). For proving the second inequality, we use the decomposition $S \oplus T^c\cdot M = S_{1/2} \oplus S_{3/2}$ to write
\[
\nabla = (\text{pr}_{S_{1/2}}, \text{pr}_{S_{3/2}}) \circ \nabla = (i \circ D_S, P).
\]
For $\psi \in C^\infty_c(M, S)$, written as $\psi = \psi_{1/2} + \psi_{3/2}$ with respect to this decomposition, we conclude
\[
\nabla^* \psi = D_S \circ i^{-1} \psi_{1/2} + P^* \psi_{3/2}.
\]
An application of the triangle inequality and integration by parts yields
\[
(\nabla \circ \nabla^* \psi, \psi)_{L^2} = \|\nabla^* \psi\|_{L^2}^2 \leq 2 \|
abla \circ \nabla^* \psi\|_{L^2} \|\nabla \psi\|_{L^2} + 2 \|P^* \psi_{3/2}\|_{L^2}^2
\]
\[
\leq 2 \|D_S \circ i^{-1} \psi_{1/2}\|_{L^2}^2 + 2 \|P^* \psi_{3/2}\|_{L^2}^2 + \frac{n}{2} \|Q \psi_{3/2}\|_{L^2}^2
\]
\[
\leq \frac{n}{2} (i \circ (D_S)^2 \circ i^{-1} (\psi_{1/2}), \psi_{1/2})_{L^2} + \frac{n}{2} (\|Q (Q^{1/2} \psi_{3/2}) + \frac{4}{n} P \circ P^* (\psi_{3/2}), \psi_{3/2}\|_{L^2}^2
\]
\[
= \frac{n}{2} ((D_{T^c\cdot M})^2 \psi, \psi)_{L^2},
\]
where we used (73), (74) and (75). \qed

**Theorem 6.8.** On a Ricci-flat spin manifold with a parallel spinor, the operator $e^{-t(D_S)^2}$ satisfies derivative estimates of degree 1.

**Proof.** This is a consequence of Lemma 6.6, Lemma 6.7 and Theorem 6.5. \qed

6.3. The Rarita-Schwinger operator. Here, we are going to prove Corollary 1.24 (ii).

**Proposition 6.9.** We define two closed subspaces
\[
X_p = \overline{P(C^\infty_c(M))^P}, \quad Y_p = \ker_{L^p}(P^*).
\]
Then for every $p \in (1, \infty)$, we have $X_p \oplus Y_p = L^p(M, S_{3/2})$ and
\[
Q^2|_{X_p} = \left(\frac{n-2}{n}\right)^2 \Delta_{S_{3/2}}, \quad Q^2|_{Y_p} = \Delta_{S_{3/2}}.
\]
Proof. To show (76), we use (71)-(74), which simplify in the Ricci-flat setting. Applying (72) twice and using (73) yields
\[ Q^2 \circ P = \left(\frac{n-2}{n}\right)^2 P \circ (\mathcal{D}_S)^2 = \left(\frac{n-2}{n}\right)^2 P \circ \Delta_{S_{1/2}}. \]
Composing (74) with \( P \) and using (71),(73) and the above identity yields
\[ \Delta_{S_{1/2}} \circ P = Q^2 \circ P + \frac{4}{n} P \circ P^* \circ P = \left(\frac{n-2}{n}\right)^2 P \circ \Delta_{S_{1/2}} + \frac{n-1}{n} \cdot \frac{4}{n} P \circ \Delta_{S_{1/2}} = P \circ \Delta_{S_{1/2}}. \]
These two formulas together imply
\[ Q^2 \circ P = \left(\frac{n-2}{n}\right)^2 \Delta_{S_{1/2}} \circ P \]
and by taking the closure in \( L^p \) we get \( Q^2|_{X_p} = \left(\frac{n-2}{n}\right)^2 \Delta_{S_{1/2}} \). The identity \( Q^2|_{Y_p} = \Delta_{S_{1/2}} \) is immediate from (74).
It remains to prove \( X_p \oplus Y_p = L^p(M, S_{1/2}) \). We are now going to show \( X_p \cap Y_p = \{0\} \) for every \( p \in (1, \infty) \). At first, (76) immediately implies \( X_p \cap Y_p \subset \ker_L(\Delta_{S_{1/2}}) \). Thus, Proposition 6.4 implies that any \( \psi \in X_p \cap Y_p \) satisfies \( |\psi| = O_{\infty}(r^{-n}) \) and \( P^* \psi = 0 \). Because \( \psi \in X_p \), we have a sequence \( \sigma_i \in C_c^\infty(M, S) \) such that \( P \sigma_i \to \psi \) in \( L^p \). Let \( q \in (1, \infty) \) be the conjugate Hölder exponent. Then \( \psi \in L^q \cap L^2 \) because \( |\psi| = O_{\infty}(r^{-n}) \). Consequently,
\[ \|\psi\|_{L^2}^2 = \langle \psi, \psi \rangle_{L^2} - (P^* \psi, \sigma_i)_{L^2} = \langle \psi, \psi - P \sigma_i \rangle_{L^2} \leq \|\psi\|_{L^q} \|\psi - P \sigma_i\|_{L^p} \to 0 \]
which shows that \( \psi = 0 \), hence \( X_p \cap Y_p = \{0\} \). To show that \( X_p \oplus Y_p = L^p \) it suffices to show that the annihilator \( \text{Ann}(X_p \oplus Y_p) \subset L^q(S_{1/2}) \) is vanishing. In fact, we have
\[ Y_q = \ker_L(P^*) = \text{Ann}(P(C_c^\infty(M, S))) = \text{Ann}(X_p) \]
and applying the annihilator to this equation and using the fact that \( L^p \) is a reflexive Banach space yields
\[ \text{Ann}(Y_q) = X_p. \]
Therefore,
\[ \text{Ann}(X_p \oplus Y_p) = \text{Ann}(X_p) \cap \text{Ann}(Y_p) = Y_q \cap X_q = \{0\} , \]
which implies that \( X_p \oplus Y_p = \text{Ann}(\{0\}) = L^p \). \( \square \)

Theorem 6.10. The heat flow \( e^{-tQ^2} \) preserves the splitting in Proposition 6.9 splitting and satisfies almost Euclidean heat kernel estimates and derivative estimates of degree 0.

Proof. In the proof of Proposition 6.9, we have shown \( P \circ \Delta_{S_{1/2}} = \Delta_{S_{1/2}} \circ P \), which obviously implies \( P^* \circ \Delta_{S_{1/2}} = \Delta_{S_{1/2}} \circ P^* \). Therefore \( X_p \) and \( Y_p \) are both invariant under the heat flow of \( \Delta_{S_{1/2}} \). Due to Proposition 6.9 again,
\[ e^{-tQ^2}|_{X_p} = e^{-t\left(\frac{n-2}{n}\right)^2\Delta_{S_{1/2}}}|_{X_p}, \quad e^{-tQ^2}|_{Y_p} = e^{-t\Delta_{S_{1/2}}}|_{Y_p} \]
and therefore, \( e^{-tQ^2} \) also preserves this splitting. The second assertion is immediate from Theorem 6.5. \( \square \)
6.4. **The Lichnerowicz Laplacian.** Here, we are going to prove Corollary 1.24 (iii).

**Theorem 6.11.** The heat flow of the Lichnerowicz Laplacian satisfies almost Euclidean heat kernel estimates and derivative estimates of degree 0.

**Proof.** Let $\sigma \in C^\infty(S)$ be a parallel spinor normalized such that $|\sigma| = 1$. Consider the following endomorphism

$$\Phi : C^\infty(S^2M) \to C^\infty(S \otimes T^*_CM), \quad h \mapsto \sum_{i,j} h_{ij} e_i \cdot \sigma \otimes e^*_j.$$ 

It is easy to see that

$$\langle \Phi(h), \Phi(k) \rangle = \langle h, k \rangle, \quad |\nabla^k(\Phi(h))| = |\nabla^k h|, \quad k \in \mathbb{N}.$$ 

In other words, $\Phi$ is an isometric embedding and its image forms a parallel subbundle of $S \otimes T^*_CM$. The key formula was established by Wang [Wan91] (and independently in [DWW05]) and states that

$$\Phi \circ \Delta_L = (\mathcal{D}_{T^*M})^2 \circ \Phi.$$  \hfill (77)

Therefore

$$\Phi \circ e^{-t\Delta_L} = e^{-t(\mathcal{D}_{T^*M})^2} \circ \Phi.$$  \hfill (78)

In other words, $\Delta_L$ coincides with $(\mathcal{D}_{T^*M})^2$ via the parallel embedding $\Phi$. The assertion follows now directly from Theorem 6.5. $\square$

The second theorem in this subsection concerns an improved decay for the linearized de-Turck vector field and the linearized Ricci curvature along the heat flow of the Lichnerowicz Laplacian. Given two arbitrary metrics $\tilde{g}, \hat{g}$, the de Turck vector field is

$$V(\tilde{g}, \hat{g}) = g^{ij}(\tilde{\Gamma}^k_{ij} - \hat{\Gamma}^k_{ij}),$$

and it is used to define the Ricci de Turck flow which is a strictly parabolic variant of the Ricci flow. Let

$$\delta : C^\infty(M, S^2M) \to C^\infty(M, T^*M), \quad (\delta h)_k = -g^{ij} \nabla_i h_{jk},$$

$$\delta^* : C^\infty(M, T^*M) \to C^\infty(M, S^2M), \quad (\delta^* \omega)_{ij} = \frac{1}{2} (\nabla_i \omega_j + \nabla_j \omega_i)$$

be the divergence and its formal adjoint, respectively. Let furthermore

$$G : C^\infty(M, S^2M) \to C^\infty(M, S^2M), \quad G(h) = h - \frac{1}{2} \text{tr} g \cdot h \cdot g$$

be the gravitational operator and $\sharp : C^\infty(M, T^*M) \to C^\infty(M, TM)$ be the musical isomorphism. The linearization of $V$ in the first component can now be expressed in terms of these operators as

$$DV(h) := \frac{d}{dt}|_{t=0} V(g + th, g) = -\sharp \circ \delta \circ G(h).$$

Moreover, one computes (using e.g. the formulas in [Bes08, Thm. 1.174])

$$DL_V(h) := \frac{d}{dt}|_{t=0} L_{V(g + th, g)}(g + th) = -2 \cdot \delta^* \circ \delta \circ G(h),$$

$$DRic(h) := \frac{d}{dt}|_{t=0} Ric_{g + th} = \frac{1}{2} \Delta_L h - \delta^* \circ \delta \circ G(h) = \frac{1}{2}(\Delta_L + DL_V)(h)$$

The key tool to get good estimates for these operators under the heat flow is the following lemma:
Lemma 6.12. Let $\Delta_H$ the Hodge Laplacian on one-forms, $\Delta_L$ the Lichnerowicz Laplacian, $\delta : C^\infty(S^2 M) \to C^\infty(T^* M)$ and $\delta^*$ its adjoint. Then we have

$$\Delta_L \circ \delta^* = \delta^* \circ \Delta_H, \quad \delta \circ \Delta_L = \Delta_H \circ \delta,$$

$$\delta \circ \delta^* \leq \Delta_H, \quad \delta^* \circ \delta \leq \Delta_L.$$

Proof. Recall that $0 = \Phi^* \text{Ric}_g = \text{Ric}_{\Phi^* g}$ for any diffeomorphism $\Phi$. Linearizing this equation at $g$, and noticing that $\delta^* \omega = \frac{1}{2} L_{\omega^*} g$, we get

$$0 = D\text{Ric}(\delta^* \omega) = \frac{1}{2} \Delta_L(\delta^* \omega) - \delta^*(\delta + \frac{1}{2} \nabla \text{tr}) \delta^* \omega = \frac{1}{2} \Delta_H(\delta^* \omega) - \frac{1}{2} \delta^* \Delta_H \omega,$$

for all one-forms $\omega$, where we have used that

$$\Delta_H = (2\delta + \nabla \text{tr}) \delta^*,$$

which uses $\text{Ric} = 0$. This proves the first assertion, the second assertion follows by the equality of the formal adjoints. Using the last equation for $\Delta_H$, note that

$$\delta \circ \delta^* \nabla f = \frac{1}{2} \Delta_H \nabla f - \frac{1}{2} \nabla \text{tr} (\delta^* \nabla f) = \Delta_H \nabla f, \quad f \in C^\infty(M),$$

$$\delta \circ \delta^* \omega = \frac{1}{2} \Delta_H \omega \in \ker(\delta), \quad \omega \in C^\infty(T^* M) \cap \ker(\delta).$$

Thus, $\delta \circ \delta^*$ preserves the $L^2$-orthogonal decomposition

$$H^k(M, T^* M) = \nabla(H^{k+1}(M))^{H^k} \oplus \ker_{H^k}(\delta),$$

with respect to which we have $\delta \circ \delta^* = \text{diag}(\Delta_H, \frac{1}{2} \Delta_H)$. In particular, $\delta \circ \delta^* \leq \Delta_H$. For the second inequality, we use the formulas from above to compute

$$\delta^* \circ \delta(\delta^* \nabla f) = \frac{1}{2} \delta^* (\Delta_H \nabla f) = \frac{1}{2} \Delta_L (\delta^* \nabla f), \quad f \in C^\infty(M),$$

$$\delta^* \circ \delta(\delta^* \omega) = \frac{1}{2} \delta^* \circ \Delta_H \omega = \frac{1}{2} \Delta_L (\delta^* \omega), \quad \omega \in C^\infty(T^* M) \cap \ker(\delta).$$

These equations show that $\Delta_L$ and $\delta^* \circ \delta$ both preserve the splitting

$$H^k(M, S^2 M) = \delta^* \nabla(H^{k+2}(M))^{H^k} \oplus \delta^*(\ker_{H^{k+1}}(\delta))^{H^k} \oplus \ker_{H^k}(\delta).$$

The splitting is $L^2$-orthogonal for the following reason: The first two spaces are both orthogonal to the third one. Using integration by parts and the fact that $\delta \circ \delta^*$ preserves the above splitting of $H^k(T^* M)$, one easily sees that the first two factors are also orthogonal to each other. Because we see that $\delta^* \delta = \text{diag}(\frac{1}{2} \Delta_L, \frac{1}{2} \Delta_L, 0)$ with respect to this decomposition, we get the desired inequality. \hfill \square

Theorem 6.13. We have, for all $p \in (1, \infty)$,

$$\|D V \circ e^{-t \Delta_L}\|_{p \to p} \leq C t^{-\frac{1}{2}},$$

$$\|D L_V \circ e^{-t \Delta_L}\|_{p \to p} \leq C t^{-1},$$

$$\|D \text{Ric} \circ e^{-t \Delta_L}\|_{p \to p} \leq C t^{-1}.$$

Proof. Because the endomorphism, $G \in C^\infty(M, \text{End}(S^2 M))$ commutes with $\Delta_L$, we have

$$D V \circ e^{-t \Delta_L} = G \circ \delta \circ G \circ e^{-t \Delta_L} = G \circ \delta \circ e^{-t \Delta_L} \circ G.$$

Because of Lemma 6.12, we can apply Theorem 3.13 to get

$$\|D V \circ e^{-t \Delta_L}\|_{p \to p} \leq C \|\delta \circ e^{-t \Delta_L}\|_{p \to p} \leq C t^{-\frac{1}{2}}.$$
where we used that $\delta$ and $G$ are both pointwise bounded, hence bounded on $L^p$. For the proof of the second inequality, we write
\[ DL_V \circ e^{-t\Delta_L} = -2\delta^* \circ \delta \circ G \circ e^{-t\Delta_L} = -2\delta^* \circ e^{-\frac{t}{2}\Delta_L} \circ \delta \circ e^{-\frac{t}{2}\Delta_L} \circ G. \]

Again due to Lemma 6.12, Theorem 3.13 and boundedness of $G$, we get
\[ \|DL_V \circ e^{-t\Delta_L}\|_{p \to p} \leq C \|\delta^* \circ e^{-\frac{t}{2}\Delta_L}\|_{p \to p} \|\delta \circ e^{-\frac{t}{2}\Delta_L}\|_{p \to p} \leq Ct^{-1}. \]

For the proof of the last estimate, we first use (77) and (78) to get
\[ \Phi \circ \Delta_L \circ e^{-t\Delta_L} = (D_{T^*M})^2 \circ e^{-t(D_{T^*M})^2} \circ \Phi = (D_{T^*M} \circ e^{-\frac{t}{2}(D_{T^*M})^2})^2 \circ \Phi. \]
Because the map $\Phi : C^\infty(M, S^2 M) \to C^\infty(M, S \otimes T^* M)$ is an isometric embedding, Theorem 3.13 yields
\[ \|\Delta_L \circ e^{-t\Delta_L}\|_{p \to p} \leq \|D_{T^*M} \circ e^{-\frac{t}{2}(D_{T^*M})^2}\|_{p \to p}^2 \leq Ct^{-1}. \]

Because $2\text{Ric} = -\Delta_L + DL_V$,
\[ \|\text{Ric} \circ e^{-t\Delta_L}\|_{p \to p} \leq \frac{1}{2} \left( \|DL_V \circ e^{-t\Delta_L}\|_{p \to p} + \|\Delta_L \circ e^{-t\Delta_L}\|_{p \to p} \right) \leq Ct^{-1}, \]
which finishes the proof. \hfill \Box

6.5. The Hodge Laplacian on one-forms revisited. In this final subsection, we are proving Corollary 1.26 (iii).

**Theorem 6.14.** The heat flow of the Hodge Laplacian on one-forms satisfies derivative estimates of degree 1.

**Proof.** Via the canonical splitting $T^2 M = S^2 M \oplus \Lambda^2 M$ of 2 tensors into the symmetric and antisymmetric part, the covariant derivative $\nabla : C^\infty(M, T^* M) \to C^\infty(M, T^2 M)$ splits as
\[ \nabla \omega \cong (\delta^* \omega, d\omega) \]
and we have
\[ \delta^* \circ \Delta_H = \Delta_L \circ \delta^*, \quad d \circ \Delta_H = \Delta_H \circ d \]
and
\[ \delta \circ \delta^* \leq \Delta_H, \quad \delta^* \circ \delta \leq \Delta_L, \quad d^* \circ d \leq \Delta_H, \quad d \circ d^* \leq \Delta_H. \]
The result now follows from Proposition 1.15, Theorem 5.1 and Theorem 6.11. \hfill \Box

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