A VOEVODSKY MOTIVE ASSOCIATED TO A LOG SCHEME

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Abstract. For each fs log scheme \((X, M_X)\) over a field \(k\) we construct a geometrical Voevodsky motive \([X]^{\log} \in DM_{gm}(k, \mathbb{Q})\). We prove that, for \(k = \mathbb{C}\), the Betti realization of \([X]^{\log}\) is the log Betti cohomology of \((X, M_X)\). We give applications to motivic tubular neighborhoods, limit motives and the monodromy filtrations.

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1. Introduction

Let \(X\) be a scheme over a field \(k\). Recall that a log structure on \(X\) is a sheaf of commutative monoids \(\mathcal{M}\) together with a homomorphism \(\alpha : \mathcal{M} \to \mathcal{O}_X\) such that \(\alpha^{-1}(\mathcal{O}_X^*) \simeq \mathcal{O}^*\). A pair \((X, \mathcal{M})\) consisting of a scheme and a log structure is called a log scheme. Usually, requires some finiteness conditions on a monoid \(\mathcal{M}\) so we will always assume that all log structures are
Fine and saturated (fs) log schemes form the category $\text{Sch}_{fs}^{\log}/k$ which contains $\text{Sch}/k$ as a full subcategory.

It is well known that several Weil cohomology theories can be extended to the category of log schemes. The key result in this area is the paper of Kato and Nakayama [KN99]. The authors define for each fs log scheme $(X, \mathcal{M})$ over $\mathbb{C}$ a certain topological space $X^{\log}$ which is now called the Kato-Nakayama space. Then the log Betti cohomology of $X$ can be defined as $H^\text{Betti}_{\log}(X^{\log}, \mathbb{Z})$. Kato and Nakayama also proved several comparison theorems between log étale, log De Rham and log Betti cohomology that generalize the classical comparison results.

One can expect the existence of the motive whom’s realizations recover various log cohomology theories. Unfortunately, at the moment, there is no such construction. There are several variants of the construction of the logarithmic motivic category (for example [P16], [How] or [IKNU19]) but in all cases the connection with the Voevodsky category remains at the level of conjectures.

In this article we define a geometrical Voevodsky motive for each fs log scheme. Namely, let $[-] : \text{Sch}/k \rightarrow D\text{M}_{gm}(k, \mathbb{Q})$ be the functor which maps $X \in \text{Sch}/k$ to the homological motive $[X]$. We extend $[-]$ to the $(\infty, 1)$-functor

$$[-]^{\log} : \text{Sch}_{fs}^{\log}/k \rightarrow D\text{M}_{gm}(k, \mathbb{Q}), \quad (X, \mathcal{M}_X) \mapsto [X]^{\log},$$

which has the following basic properties:

1. Let us endow the category $\text{Sch}_{fs}^{\log}/k$ with Cartesian monoidal structure. Then $[-]^{\log}$ is monoidal. That is for each log scheme $(X, \mathcal{M})$ the motive $[X]^{\log}$ has the structure of $E^\infty$-coalgebra.

2. For $k \subset \mathbb{C}$ the Betti realization of $[X]^{\log}$ is quasi-isomorphic to the cohomology of Kato-Nakayama space:

$$R^\text{Betti}([X]^{\log}) \cong \text{Sing}^*(X^{\log})_{\mathbb{Q}}.$$

In particular, the log Betti cohomology $H^*_\text{Betti}(X^{\log}, \mathbb{Q})$ can be endowed with the canonical mixed Hodge structure.

**Example 1.1.** Let $i : D \hookrightarrow X$ be a normal crossing divisor in a smooth variety $X$. Then $X$ can be endowed with the canonical log structure $\mathcal{M}_X := \mathcal{O}_X \cap j_* \mathcal{O}_{X-D}$. Let $\mathcal{M}_D$ be the restriction of the log structure to $D$. Let us define the motivic punctured tubular neighborhood $[PTN_X D]$ as the homotopy fiber

$$\text{Cone}([D] \oplus [X - D] \rightarrow [X])[-1].$$

1In particular, $\mathcal{M}$ is finitely generated and $\mathcal{M}/\mathcal{O}^*$ has no torsion. See [Og18] for more details.

2here $D\text{M}_{gm}(k, \mathbb{Q})$ is the category of geometrical Voevodsky motives.
We prove that
\[(1.1) \quad [(X, \mathcal{M}_X)]^{\log} \simeq [X - D] \quad \text{and} \quad [(D, \mathcal{M}_D)]^{\log} \simeq [PTN_X D].\]

We expect that our construction will be useful for computing the limit motive functor. That is, let \(f : X \to C\) be a proper semi-stable degeneration over a smooth curve \(C/k\). Let \(X_s\) be the special fiber. Let us fix a local parameter \(t\) around \(s \in C\). By the properties of \([-]^{\log}\) the map of the motivic punctured tubular neighborhood
\[\text{PTN}_X(X_s) \to \text{PTN}_C(s) \simeq \mathbb{G}_m\]
is a homomorphism of \(\mathbb{E}_\infty\)-coalgebras. We define
\[\text{LM}_f := [\text{PTN}_X(X_s)]^* \otimes_{[\mathbb{G}_m]^*} \mathbb{Q}.\]

Recall that the limit motive is \(\Upsilon_{id,C}([X_\eta]^*)\) where
\[\Upsilon_{id,C} : \text{DM}_{et}(k(\eta), \mathbb{Q}) \to \text{DM}_{et}(k(s), \mathbb{Q})\]
is the functor of motivic unipotent nearby cycles (see [Ayo07]). We prove the following

**Proposition 1.2.** There is a canonical quasi-isomorphism
\[\text{ML}_f \simeq \Upsilon_{id,C}([X_\eta]^*).\]

Let us explain the generic picture behind Proposition 1.2 in the case \(k = \mathbb{C}\). Let \(\psi_f : D(\text{Sh}(X_\eta^{an}, \mathbb{Q})) \to D(\text{Sh}(X_s^{an}, \mathbb{Q}))\) be the functor of nearby cycles and \(\Delta\) be a small disk around \(s \in C\). Let us choose the punctured tubular neighborhood \(\text{PTN}_{X^{an}}(X_s^{an})\) such that the map \(f : \text{PTN}_{X^{an}}(X_s^{an}) \to \Delta\) is a fiber bundle. Then there is the quasi-isomorphism
\[(1.2) \quad \text{Sing}^*(\text{PTN}_{X^{an}}(X_s^{an}), \mathbb{Q}) \otimes_{\text{Sing}^*(\Delta, \mathbb{Q})} \mathbb{Q} \simeq R\Gamma(X_s, \psi_f \mathbb{Q}).\]

One can think about the proposition as the motivic version of (1.2).

In [St76], Steenbrink endowed \(\psi_f \mathbb{Q}\) with the structure of a mixed Hodge complex. Let us denote this mixed complex by \(\psi_f \mathbb{Q}^{Hdg}\). It follows from the property (2) that the Betti realization of \(\text{LM}_f\) is quasi-isomorphic to \(R\Gamma(X_0, \psi_f \mathbb{Q})\). We propose that this quasi-isomorphism can be lifted on the level of Hodge complexes.

**Conjecture 1.3.** There is a quasi-isomorphism of mixed Hodge complexes
\[R_{Hodge}(\text{LM}_f) \simeq R\Gamma(X_s, \psi_f \mathbb{Q}^{Hdg}).\]

Recall that the cohomology of \(R\Gamma(X_s, \psi_f \mathbb{Q}^{Hdg})\) is the limit Hodge structure of the family \(f\). Let us denote this Hodge structure by \((H^n(X_t, \mathbb{Q}), W_*, F^\bullet_{lim})\).

---

3here \(\eta\) is a generic point of \(C\).

4here \(X_t\) is a fiber of \(f\) at the complex point \(t \neq s\).
Now, suppose that the monodromy \( M : H^n_\text{Betti}(X_t, \mathbb{Q}) \to H^n_\text{Betti}(X_t, \mathbb{Q}) \) has only one Jordan block. Then the limit Hodge structure has a Hodge-Tate type.

**Conjecture 1.4.** (Kerr, Griffiths, Green) Suppose that the degeneration \( f \) is defined over \( k \subset \mathbb{C} \). Then the limit Hodge structure \( (H^n(X_t, \mathbb{Q}), W_\bullet, F_\text{lim}^\bullet) \) can be lifted to a mixed Tate motive over \( k \).

Conjecturally above mentioned motive is a cohomology of \( \text{LM}_f \) with respect to motivic t-structure.

The conjecture is based on the following arithmetic considerations. Let \( (V, W_\bullet, F^\bullet) \) be a Hodge-Tate structure. Following Goncharov [G99], one can define the period operator \( P : V_\mathbb{C} \to V_\mathbb{C} \) and, after choosing a basis with respect to \( W_\bullet \), the period matrix

\[
\begin{pmatrix}
1 & p_{01} & p_{02} & \cdots \\
0 & 1 & p_{12} & \cdots \\
& \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1
\end{pmatrix}.
\]

Let \( \mathbb{Q}(p_{ij}) \) be the subfield of \( \mathbb{C} \) generated by the entries of the period matrix. Note that the matrix is defined only up to a multiplication by a non-degenerate rational matrix. So the extension \( \mathbb{Q}(p_{ij}) \) is well defined.

Now, let \( \{Y_t\} \) be the family of Calabi-Yau varieties which is mirror dual to the universal family of quintic 3-folds. In [GGK], Kerr, Griffiths and Green considered as an example of \( f \) the family \( \{Y_t\} \) together with the canonical parameter \( t = t_{\text{can}} \). Firstly, they deduced from Conjecture 1.4 that \( \mathbb{Q}(p_{ij}) = \mathbb{Q}(\zeta(3)/(2\pi i)^3) \). Then the authors computed the periods using mirror symmetry. More concretely, they proved that, in a suitable basis, the period matrix has the form

\[
\begin{pmatrix}
1 & 0 & \frac{25}{12} & -\frac{200\zeta(3)}{(2\pi i)^3} \\
0 & 1 & -\frac{5}{2} & \frac{25}{12} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

We hope that our construction is related to the Conjecture 1.4. In particular, Conjecture 1.4 follows from Conjecture 1.3 in the case when the motive of all irreducible components of \( X_s \) together with all intersections are mixed Tate motives.

Now, let us sketch the construction of the functor \([-]^{\log} \). Suppose \( (X, M \to \mathcal{O}_X) \) is a fs log scheme over \( k = \mathbb{C} \). Firstly, let us explain how to compute rational Betti cohomology of

\footnote{Note that the degeneration \( \{Y_t\} \) is defined over \( \mathbb{Q} \).}
$X^{\log}$ in terms of the sheaf $\mathcal{M}$. Let us denote by $\exp(\alpha)$ the complex

$$
\cdots \longrightarrow 0 \overset{(-1)}{\longrightarrow} \mathcal{O}_{X, an} \overset{\exp}{\longrightarrow} \mathcal{M}_{an}^{gr} \overset{(0)}{\longrightarrow} \mathcal{O}_{an} \overset{(1)}{\longrightarrow} \mathcal{O}_{an} \overset{(2)}{\longrightarrow} \cdots
$$

Here $gr$ means group completion, $an$ means analytification and the differential is given by the composition $\mathcal{O}_{X, an} \overset{\exp}{\longrightarrow} \mathcal{O}_{X, an}^{*} \longrightarrow \mathcal{M}_{an}^{gr}$. The following fact was discovered by Steenbrink (in the case of a log smooth scheme).

**Proposition 1.5.** Let $\pi : X^{\log} \longrightarrow X^{an}$ be the canonical map. Then there is the quasi-isomorphism

$$R\pi_\ast \mathbb{Q} \simeq \text{colim}_n S^n(\exp(\alpha)\mathbb{Q}).$$

Here the colimit is taken along the maps $S^{n-1}(\exp(\alpha)\mathbb{Q}) \rightarrow S^n(\exp(\alpha)\mathbb{Q})$ induced by the inclusion $\mathbb{Z} \rightarrow \exp(\alpha)$.

Now, we define the motivic analog of the complex $\exp(\alpha)$. Let us extend the sheaf $\mathcal{M}^{gr}$ on the big étale site of $X$ by the rule

$$\mathcal{M}^{gr}(Y \overset{g}{\longrightarrow} X) := \Gamma(Y, (f_{log}^\ast \mathcal{M})^{gr})$$

where $f_{log}^\ast \mathcal{M}$ is the pullback of log structure. Let us take $\mathbb{A}^1$-invariant replacement and stabilize by $\Omega^1_{(\mathbb{G}_m, 1)}$. As the result we get the object of $DA_{et}(X, \mathbb{Z})$ which we will denote by $M^{gr}_{\mathbb{Z}}$. This construction turns out to be functorial. So the inclusion $\mathcal{O}^* \hookrightarrow \mathcal{M}^{gr}$ induces the map $\mathbb{Z}(1)[1] \rightarrow M^{gr}_{\mathbb{Z}}$ and we can define the motivic sheaf

$$\text{colim}_n S^n(M^{gr}(-1)[-1]) \in DA_{et}(X, \mathbb{Q}).$$

Here $M^{gr} := M^{gr}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $S^n$ are motivic symmetric powers [Maz10]. Let $f : X \rightarrow \text{Spec}(k)$ be the canonical morphism. We define the motive $[X]^{log}$ by the formula

$$[X]^{log} := \mathbb{D} Rf_\ast(\text{colim}_n S^n(M^{gr}(-1)[-1])).$$

Note that the construction makes sense for any based field $k$.

Observe that the motive of a log scheme depends only on the group completion $\mathcal{M}^{gr}$. This observation allows to extend the functor $[-]^{log}$ on the category of virtual logarithmic schemes (which was introduced in [How]). A **virtual log structure** on a scheme $X$ is a sheaf of abelian group $\mathcal{L}$ together with an inclusion $\mathcal{O}^* \hookrightarrow \mathcal{L}$. Of course, the most important example is $\mathcal{L} = \mathcal{M}^{gr}$. Another series of examples arises from virtual log schemes over log point $pt_{log}$. Namely, let $Y \rightarrow pt_{log}$ be a morphism of log schemes. In contrast to $Sch_{ft}^{Log}/k$, in the category of virtual log schemes there is the morphism $\text{Spec}(k) \rightarrow pt_{log}$. So we can define the fiber product $\tilde{Y} := Y \times_{pt_{log}} \text{Spec}(k)$. In particular, in the case of a degeneration $f$, the choosing

\[\text{DA}_{et}(X, \mathbb{Z})\] is the category of étale motivic sheaves [Ayo14].
of parameter gives the morphism \((X_s, M_{X_s}) \to pt_{log}\). Then the motive \([\tilde{X}_s]^{log}\) is dual to \(LM_f\) (see Proposition [15.8]).

**Acknowledgments.** I would like to thank my supervisor Prof. Vadim Vologodsky for consistent support and guidance during the running of this project. I am also grateful to Artem Prikhodko and Vova Shaidurov for the discussions that helped me a lot. Finally, I should separately thank Ravil Gabdurakhmanov and Vova Shaidurov for helping me edit this text. The work was supported by the Russian Science Foundation, grant 21-11-00153.

2. **Presheaf of virtual log structures**

**Definition of virtual log structures.** Let \(X \in \text{Sch}/k\) be a scheme of finite type. Let us define the category \(v\text{Log}_X\) in the following way: the objects of \(v\text{Log}_X\) are pairs \((F, i: O^* \hookrightarrow F)\) where \(F\) is an étale sheaf of abelian groups on \(X\) and \(i\) is an inclusion, the morphisms of \(v\text{Log}_X\) are maps \(\mu: F_1 \to F_2\) which commutes with the inclusions of \(O^*_X\).

For the pair \((F, i)\) let us define the ghost sheaf as the factor \(\tilde{F} = F/O^*\). We will call the category \(v\text{Log}_X\) the category of all virtual logarithmic structures on \(X\).

**Functoriality.** Let \(f: X \to Y\) be a morphism of schemes. For a virtual log structure \(F_Y \in v\text{Log}_Y\) let us define the pullback of \(F_Y\) as a colimit of the diagram

\[
\begin{array}{ccc}
\tilde{F}_Y & \xrightarrow{f^{-1}(i)} & F_Y \\
\downarrow & & \downarrow \\
O^*_X & \xrightarrow{i} & f^*F_Y
\end{array}
\]

Notice that \(f^{-1}\) is exact. So \(f^{-1}(i)\) and \(\tilde{i}\) are inclusions. Observe that this construction is functorial by \(F_Y\). So we constructed the functor \(f^*: v\text{Log}_Y \to v\text{Log}_X\). Moreover, by construction for any \(f: X \to Y\) and \(g: Z \to X\) we have the natural isomorphism \((fg)^* \cong g^*f^*\). So we can define the presheaf of categories (contravariant pseudofunctor)

\[
v\text{Log}: \text{Sch}/k \to \text{Cat}
\]

**Coproduct.** The category of virtual log structures \(v\text{Log}_X\) admits a coproduct of any objects. For two virtual log structures \((F_1, i_1)\) and \((F_1, i_1)\) let us denote their coproduct by \((F_1 \ast F_2, i_1 \ast i_2)\). It is easy to see that the sheaf \(F_1 \ast F_2 \in \text{Sh}(X_{et}, \mathcal{Z})\) can be computed as a pushout:

\[
F_1 \ast F_2 := F_1 \oplus_{O^*} F_2.
\]
Remark 2.1. Notice that for any morphism \( f : X \rightarrow Y \) the pullback functor \( f^* \) commute with coproduct. Indeed, let us consider the diagram

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
\mathcal{F}_2 \ar[r] & f^{-1}(\mathcal{F}_1 \ast \mathcal{F}_2) \\
\mathcal{O}^* \ar[u] & f^{-1}\mathcal{F}_1 \ar[l] & f^*(\mathcal{F}_1 \ast \mathcal{F}_2) \\
\mathcal{O}^* \ar[u] & f^*\mathcal{F}_2 \ar[l] & f^*\mathcal{F}_1 \\
\end{array}
\end{array}
\]

Notice that the union of front and right squares is cocartesian square. Moreover the front square is a pushout by definition. Using the pushout lemma we conclude that the right square is cocartesian. Observe that the upper square is cocartesian because \( f^{-1} \) is exact. Hence the union of left and bottom squares is cocartesian. But the left square is a pushout by definition. So using the pushout lemma again we get that the bottom square is cocartesian.

Remark 2.2. (Group completion of a sheaf). Let \( \mathcal{X} \) be a site and \( \text{Sh}(\mathcal{X}, \text{cMon}) \) be a category of sheaves of commutative monoids on \( \mathcal{X} \). The natural inclusion \( \text{inc} : \text{Sh}(\mathcal{X}, \text{Ab}) \rightarrow \text{Sh}(\mathcal{X}, \text{cMon}) \) admits the left adjoint called group completion \( \text{gr} : \text{Sh}(\mathcal{X}, \text{cMon}) \rightarrow \text{Sh}(\mathcal{X}, \text{Ab}) \). This functor can also be described in the following way. Notice that the diagram

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
\text{Sh}(\mathcal{X}, \text{cMon}) \ar[r] & \text{PSh}(\mathcal{X}, \text{cMon}) \\
\text{Sh}(\mathcal{X}, \text{Ab}) \ar[u] \ar[r] & \text{PSh}(\mathcal{X}, \text{Ab}) \\
\end{array}
\end{array}
\]

is commutative. So for any presheaf of monoids \( M \) we have \((M^{\text{gr}})^\# \simeq (M^\#)^{\text{gr}} \) where \( \# \) is a sheafification. On the other hand, it is easy to see that the group completion \( \text{gr} : \text{PSh}(\mathcal{X}, \text{cMon}) \rightarrow \text{PSh}(\mathcal{X}, \text{Ab}) \) can be computed pointwise: \( M^{\text{gr}}(U) = (M(U))^{\text{gr}} \). So for any sheaf of monoids \( \mathcal{M} \) we have

\[
\mathcal{M}^{\text{gr}} \simeq (i(\mathcal{M})^\#)^{\text{gr}} \simeq (i(\mathcal{M})^{\text{gr}})^\#.
\]

The virtualization of a log structure. Let \( \mathcal{M} \xrightarrow{\alpha} \mathcal{O} \) be an integral log structure on \( X \) (see [Og18] for the definition). By the definition of log structures we have the inclusion \( i: \mathcal{O}^* \hookrightarrow \mathcal{M} \). Moreover, any map of log structures \( f: \mathcal{M}_1 \rightarrow \mathcal{M}_2 \) preserves such inclusions. For the log structure \( \mathcal{M} \xrightarrow{\alpha} \mathcal{O} \) let us define the virtualization as a pair \((\mathcal{M}^{\text{gr}}, i^{\text{gr}})\). By Proposition 1.1.3. of [Og18] the sheaf of monoids \( \mathcal{M} \) is integral iff \( \mathcal{M}(U) \) is integral for any \( U \). So the composition \( \mathcal{O}^*(U) \hookrightarrow \mathcal{M}(U) \hookrightarrow (\mathcal{M}(U))^{\text{gr}} \) is a monomorphism. Finally, the sheafification is exact. So \( i^{\text{gr}} \) is an inclusion.
Observe that we constructed the functor \( v_X : \mathbf{Log}^\text{inf}_X \rightarrow \mathbf{vLog}_X \), \((\mathcal{M} \rightarrow \mathcal{O}) \mapsto (\mathcal{M}^g, i^g)\). We will call this functor the virtualization functor.

**Warning.** Most of the virtual log structures in we are interested are virtualization of ordinary log structures. So further we will abuse the notation and will denote by \( \mathcal{M}^g_X \) an arbitrary virtual log structure on \( X \).

**Fine and saturated virtual log structures.** The following lemma shows what natural analogs of fs log schemes exist in virtual log geometry.

**Lemma 2.3.** Let \( X \) be a fine logarithmic scheme. Then the sheaf \( \mathcal{M}^g_X \) is constructible. Moreover if \( X \) is also saturated then \( \mathcal{M}^g_X \) is torsion-free.

**Proof:** First of all, let us show that \( \mathcal{M}^g_X \) is constructible. The proof is almost the same as the proof of constructability of the monoidal sheaf \( \mathcal{M}_X \) (see [SS06]). The problem is local so we may assume that \( X = \text{Spec}(k[P]) \) together with the canonical log structure. Let \( p_i \) be generators of \( P \) for \( 1 \leq i \leq r \). For \( J \subset \{1, ..., r\} \) let \( R_J := k[P][p^{-1}_i]/I \) where the set \( \{i_k\} := \{1, ..., r\} \setminus J \) and the ideal is generated by all \( p_j \) with \( j \in J \). Then each \( X_J := \text{Spec}(R_J) \) is a locally closed subset and \( X \simeq \bigcup J X_J \). Moreover, for each geometric point \( x_j \in X_J \) the stack \( \mathcal{M}_{X,x} \) equals the sharp monoid \( (P + \sum_{i \in J} \mathbb{Z}p_i)/G \), where \( G \subset (P + \sum_{i \in J} \mathbb{Z}p_i) \) is the subgroup of invertible elements. Finally, it follows from the Remark 2.2 and left adjointness of group completion that \( \mathcal{M}^g_{X,x} \simeq (\mathcal{M}_{X,x})^g \).

Now, suppose that \( X \) is saturated. Let us use two facts:

(i) for the canonical log structure on \( \text{Spec}(k[P]) \) the map \( \mathcal{M} \rightarrow \mathcal{O}_X \) is an inclusion.

(ii) for f.s. log structure \( \mathcal{M} \rightarrow \mathcal{O}_X \) on any \( X \) and geometric point \( x \in X \) the monoid \( \mathcal{M}_x \) is fine and saturated.

Again, we may assume that \( X = \text{Spec}(k[P]) \). Observe that a sheaf of abelian group \( F \) is torsion-free if \( f_\pi \) is torsion-free for any geometric point \( \pi \in X \). Let \( a \in (\mathcal{M}_x)^g \) and \( a^n = 1 \). Notice that \( 1 \in \mathcal{M}_x \). So \( a \in \mathcal{M}_x \) by (ii). Let \( b \in \mathcal{M}_x \) be a lift of \( a \) and \( g \in \mathcal{M}(U) \) be a corresponding section. Then \( g^n = f \) for some invertible function \( f \) on \( U \). By (i) we know that \( g \in \mathcal{O}(U) \). But \( g \cdot (g^{-1}f^{-1}) = 1 \). So \( g \in \mathcal{O}^*(U) \) and \( a = 1 \). □

**Definition 2.4.** Let us define the category of fs virtual log structures \( \mathbf{vLog}^\text{fs}_X \) as the full subcategory of \( \mathbf{vLog}_X \) contained all virtual log structures \((\mathcal{F}_x, i)\) with constructible and torsion-free ghost sheaf \( \mathcal{F}_X \).

**Remark 2.5.** Observe that \( \mathcal{F}^i_{\mathcal{F}_X} \simeq f^{-1}\mathcal{F}_X \). So for any morphism \( f : Y \rightarrow X \) the pullback functor \( f^* \) maps \( \mathbf{vLog}^\text{fs}_X \) to \( \mathbf{vLog}^\text{fs}_X \). On the other hand, by construction of coproduct we have \( \mathcal{F}_1 \mathcal{F}_2 \simeq \mathcal{F}_1 \oplus \mathcal{F}_2 \). So \( \mathbf{vLog}^\text{fs}_X \) is closed under coproduct into \( \mathbf{vLog}_X \). Both of these facts allow us to consider the collections of categories \( \mathbf{vLog}^\text{fs}_X \) as the presheaf of monoidal categories (with cocartesian monoidal structure) on \( \text{Sch}/k \). We will denote this presheaf by \( \mathbf{vLog}^\text{fs}_X \).
3. Category of virtual log schemes

**Grothendieck construction.** Let \( C : \text{Sch}^{\text{op}} / k \to \text{Cat} \) be a presheaf of categories. Let us denote by \( \int_{\text{Sch} / k} C \) the corresponding Grothendieck construction. Objects of \( \int_{\text{Sch} / k} C \) are pairs \((X, c_X)\) with \(c_X \in C(X)\), the morphisms of \( \int_{\text{Sch} / k} C \) are pairs \((f : X \to Y, f^# : c_X \to f^* c_Y)\).

We will also assume that the following two conditions are satisfied:

\((\ast)\) For each \( X \in \text{Sch} / k \) the terminal object \( *_X \) of \( C(X) \) exists and \( f^* *_Y = *_X \) for any \( f : X \to Y \).

\((\times)\) For each \( X \in \text{Sch} / k \) the product of any two objects of \( C(X) \) exists and commutes with \( f^* \).

**Remark 3.1.** Notice that \((v\text{Log})^{\text{op}}\) and \((v\text{Log}^{\text{fs}})^{\text{op}}\) are presheaves of categories satisfying properties \((\ast)\) and \((\times)\). Indeed, the property \((\times)\) satisfied by Remark 2.1. On the other hand, in both cases we have \( *_X = (\mathcal{O}_X^*, \text{id}) \).

Note that due to the property \((\ast)\) the canonical projection

\[ U : \int_{\text{Sch} / k} C \to \text{Sch} / k ; \quad (X, c_X) \mapsto X, \]

admits the right adjoint

\[ T : \text{Sch} / k \to \int_{\text{Sch} / k} C ; \quad X \mapsto (X, *_X). \]

Indeed, if \((f, f^#) : (X, c_X) \to T(Y)\) is a morphism then \( f^# : c_X \to *_X \). So \( f^# \) is unique and \( \text{Hom}_{\int_{\text{Sch} / k} C}(X, T(Y)) = \text{Hom}_{\text{Sch} / k}(U(X), Y) \). Also notice that \( T \) is fully faithful and \( U \cdot T = \text{Id}_{\text{Sch} / k} \).

**Example 3.2.** *(The category of log schemes)* Let \( \text{Log} \) be the presheaf of all logarithmic structures on \( \text{Sch} / k \). Then \( \text{Sch}^{\text{Log}} / k := \int_{\text{Sch} / k} \text{Log}^{\text{op}} \) is the categories of all log schemes. In this case we have the pair of adjoint functors \( T : \text{Sch} / k \rightleftarrows \text{Sch}^{\text{Log}} / k : U \) where \( T \) maps \( X \) to trivial log schemes \( \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X \). Notice that \( f^* \) commutes with coproducts in \( \text{Log}_X \) because pullbacks and coproducts are the particular cases of fiber products in the category \( \text{Sch}^{\text{Log}} / k \) (see \[Og18\]).

**Category of fs virtual logarithmic schemes.** Let us define the category of fs virtual logarithmic schemes \( \text{Sch}_{v\text{Log}}^{\text{fs}} / k \) as the Grothendieck construction \( \int_{\text{Sch} / k}(v\text{Log}^{\text{fs}})^{\text{op}} \). More explicitly this category can be described as the category of triples \((X, \mathcal{M}^{gr}_X, \mathcal{O}_X^* \hookrightarrow \mathcal{M}^{gr}_X)\) where \( \mathcal{M}^{gr}_X \) is torsion-free and constructible. The morphisms can be defined as pairs \((f, f^#)\) where \( f^# : f^* \mathcal{M}^{gr}_Y \to \mathcal{M}^{gr}_X \) commutes with the inclusions of \( \mathcal{O}_X^* \). Again, we have the pair of adjoint functors \( T : \text{Sch} / k \rightleftarrows \text{Sch}_{v\text{Log}}^{\text{fs}} / k : U \) and \( T(X) = (X, \mathcal{O}_X^*, \text{id}_{\mathcal{O}_X^*}) \).
Lemma 3.4. The Cartesian product of any objects exists in the category $\text{Sch}_{\text{fs}}^{v\text{Log}}/k$. Here $\alpha$ and we can use the property $(\times)$ and the morphism $(\alpha): (X,\mathcal{M}_X) \to (Y,\mathcal{M}_Y)$ by the following way: an integral log scheme $\text{Sch}_{\text{fs}}^{v\text{Log}}/k$ exists and can be computed as a pair $(X \times_k Y, \pi_X^*c_X \times \pi_Y^*c_Y)$. Indeed, the pair of maps $(f, f^\#): (Z,c_Z) \to (X,c_X)$ and $(g,g^\#): (Z,c_Z) \to (Y,c_Y)$ gives rise to the unique maps $h: Z \to X \times_k Y$ and $\mu: c_Z \to f^*c_X \times g^*c_Y$. But $f^*c_X \times g^*c_Y \simeq (h^*\pi_X^*c_X) \times (h^*\pi_Y^*c_Y)$ and we can use the property $(\times)$. So we get

**Remark 3.3. (alternative description of morphisms)** One can also describe morphisms of $\text{Sch}_{\text{fs}}^{v\text{Log}}/k$ as pairs of maps $f: X \to Y$ and $f^\#: \mathcal{M}_Y^{gr} \to f_*\mathcal{M}_X^{gr}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{O}_Y & \longrightarrow & \mathcal{M}_Y^{gr} \\
\downarrow & & \downarrow f^\# \\
f_*\mathcal{O}_X & \longrightarrow & f_*\mathcal{M}_X^{gr}
\end{array}
$$

is commutative. Indeed, if we have the pair $(f, f^\#)$ then we can compose $f^\#$ with the canonical map $f^{-1}\mathcal{M}_Y^{gr} \to f^*\mathcal{M}_Y^{gr}$ and use the $f^{-1} - f_*$ adjunction. On the other hand, suppose we have a pair $(f, f^\#)$. Then again by $f^{-1} - f_*$ adjunction there is the commutative diagram

$$
\begin{array}{ccc}
f^{-1}\mathcal{O}_Y & \longrightarrow & f^{-1}\mathcal{M}_Y^{gr} \\
\downarrow & & \downarrow \\
\mathcal{O}_X & \longrightarrow & \mathcal{M}_X^{gr}
\end{array}
$$

So we get the map $f^*\mathcal{M}_Y^{gr} \to \mathcal{M}_X^{gr}$ satisfying the required properties.

**Cartesian product.** Let $C: \text{Sch}/k^{\text{op}} \to \text{Cat}$ be a presheaf of categories satisfying the property $(\times)$. Then the product of any two objects $(X,c_X)$ and $(Y,c_Y)$ of $\int_{\text{Sch}/k} C$ exists and can be computed as a pair $(X \times_k Y, \pi_X^*c_X \times \pi_Y^*c_Y)$. Indeed, the pair of maps $(f, f^\#): (Z,c_Z) \to (X,c_X)$ and $(g,g^\#): (Z,c_Z) \to (Y,c_Y)$ gives rise to the unique maps $h: Z \to X \times_k Y$ and $\mu: c_Z \to f^*c_X \times g^*c_Y$. But $f^*c_X \times g^*c_Y \simeq (h^*\pi_X^*c_X) \times (h^*\pi_Y^*c_Y)$ and we can use the property $(\times)$. So we get

**Lemma 3.4.** The Cartesian product of any objects exists in the category $\text{Sch}_{\text{fs}}^{v\text{Log}}/k$.

**The virtualization of a schemes.** Let $\text{Sch}_{\text{int}}^{\text{Log}}/k$ be the category of integral log schemes and $\text{Sch}_{\text{fs}}^{v\text{Log}}/k := \int_{\text{Sch}/k} \text{vLog}^{\text{op}}$ be the category of all virtual log schemes. Then we can define the (schematic) virtualization functor

$$
(3.1) \quad v: \text{Sch}_{\text{int}}^{\text{Log}}/k \to \text{Sch}_{\text{fs}}^{v\text{Log}}/k
$$

by the following way: an integral log scheme $(X,\mathcal{M}_X \to \mathcal{O}_X)$ maps to $(X,\mathcal{O}_X \leftrightarrow \mathcal{M}_X^{gr})$ and the morphism $(X \xrightarrow{f} Y,\mathcal{M}_Y \xrightarrow{\alpha} f_*\mathcal{M}_X)$ maps to the pair $f$ and $\alpha': \mathcal{M}_Y^{gr} \to f_*\mathcal{M}_X^{gr}$. Here $\alpha'$ can be defined by the following way. Let $i_X$ (resp. $i_Y$) be a canonical inclusion of category presheaves of monoids to the category of sheaves of monoids on $X$ (resp. $Y$). Then the map $\alpha$ gives rise to the map of presheaves of group $(i_Y\mathcal{M}_Y)^{gr} \to f_(i_X\mathcal{M}_X)^{gr}$. Notice that $\mathcal{M}_X^{gr}$ is the sheafification of $(i_X\mathcal{M}_X)^{gr}$. So we can define $\alpha'$ as the sheafification of the map$(i_Y\mathcal{M}_Y)^{gr} \to f_*\mathcal{M}_X^{gr}$. 
Relation with fs log schemes. The category $\text{Sch}^\text{vLog}_{fs}/k$ is closely related with the category of fs log schemes $\text{Sch}^\text{Log}_{fine}/k$ and $\text{Sch}^\text{vLog}_{fs}/k$. Namely, thanks to Lemma 2.3, the restriction of (3.1) on the category $\text{Sch}^\text{vLog}_{fs}/k$ gives the virtualization functor

$$v : \text{Sch}^\text{Log}_{fs}/k \rightarrow \text{Sch}^\text{vLog}_{fs}/k$$

that maps an fs log schemes $(X, \mathcal{M} \xrightarrow{\Delta} \mathcal{O}_X)$ to the fs virtual log scheme $(X, \mathcal{O}_X^* \hookrightarrow \mathcal{M}^{gr})$.

Remark 3.5. We will not use this further, but it should be noted that all the constructed schematic virtualization functors commute with products.

4. Free $\mathbb{E}_\infty$-algebras

Suppose we have a monoidal $\infty$-category $C^\otimes$ with monoidal unit $I$. Let $C_I/\,_{\infty}$ be the under category. One can define on $C_I/_\infty$ the monoidal structure (see formula 4.1). Let $I \rightarrow A$ be an object of $C_I/_\infty$. In this section we explain how to construct a free $\mathbb{E}_\infty$-algebra $\text{Sym}^*(A)$.

Throughout this section we will use the notion of $\infty$-operads which was introduced by Lurie in [HA]. We will denote by $\text{Op}(\infty)$ the quasi-category of $\infty$-operads.

Let $\text{Fin}_*$ be the category of pointed finite sets. Recall that a symmetric monoidal $\infty$-category is a quasi-category $C$ together with a cocartesian fibration $q : C \rightarrow \text{N}(\text{Fin}_*)$ such that for any $n$ we have the natural equivalence $C\otimes_{[n]} \simeq (C\otimes_{[1]})^n$ induced by the set of maps $\{\rho^i : [n] \rightarrow [1]\}_{1 \leq i \leq n}$. For any $\infty$-operad $\mathcal{O}$ let us denote by $\text{Alg}_{\mathcal{O}}(C) = \text{Map}_{\text{Op}(\infty)}(\mathcal{O}, C^\otimes)$ the $\infty$-category of $\mathcal{O}$-algebras. Note that by definition $\text{Alg}_{\mathcal{O}}(C)$ is a full subcategory of $\text{Fun}_{\text{N}(\text{Fin}_*)}(\mathcal{O}, C^\otimes)$.

Remark 4.1. Let $\mathcal{O} = \text{N}(\text{Fin}_*) = \mathbb{E}_\infty$. Let $C := C\otimes_{[1]} \simeq \text{Fun}_{\text{N}(\text{Fin}_*)}([1], C^\otimes)$ be the underlying $\infty$-category of $C^\otimes$. Then the map of simplicial sets $[1] \rightarrow \text{N}(\text{Fin}_*)$ induces the forgetful functor

$$U : \text{CAlg}(C) \rightarrow C.$$

Suppose that infinity direct sums are exist in $C$ and for any object $c$ the functor $\otimes c$ commutes with colimits. Then by Proposition 3.1.3.13 of [HA] the forgetful functor admits left-adjoint $S^*$ which can be constructed using coinvariants of symmetric groups $\Sigma_n$. Namely, let $A \in \text{Ob}(C)$. Then

$$S^*(A) = \bigsqcup_n (A^\otimes_n)_{h\Sigma_n}$$

together with the multiplication induced by the isomorphism

$$(A^\otimes_n) \otimes (A^\otimes_m) \simeq A^\otimes_{n+m}.$$
The category of pointed objects. Note that the inclusion of marked point to $[1]$ induce the functor

$$C^\otimes_{[0]} \simeq \Delta^0 \to C^\otimes_{[1]}.$$ 

Will will called the associated object the monoidal unit and denote by $I$.

Let $C_I$ be the under category. In the case then $C$ is the nerve of a 1-category one can define the monoidal structure on $C_I$ by the rule

$$\tag{4.1} (I \to A) \otimes (I \to B) := (I \simeq I \otimes I \to A \otimes B).$$

In general, we can induce the monoidal structure from the category of $\mathcal{E}_0$-algebras of $C$.

Following [HA] let us denote by $\text{Fin}_{\text{inj}}^*$ the subcategory of $\text{Fin}_*$ spanned by all objects together with those morphisms $f : [m] \to [n]$ such that $f^{-1}([i])$ has at most one element for $1 \leq i \leq n$. The nerve $N(\text{Fin}_{\text{inj}}^*)$ is an $\infty$-operad, which we will denote by $E_0$. Note that we have the unique map $[0] \to [1]$. The corresponding restriction

$$\text{Alg}_{E_0}(C) \longrightarrow \text{Fun}_{N(\text{Fin}_*)}(\Delta^1, C^\otimes)$$

is an equivalence of categories by Proposition 2.1.3.9 of [HA]. So $\text{Alg}_{E_0}(C) \simeq C_I$. Note that the category of $\infty$-operads can be equipped with natural tensor product $\otimes$ (see Section 2.2.5 of [HA]). The corresponding monoidal structure is closed so we can equipped $\text{Alg}_{E_0}(C)$ with the structure of $\infty$-operads. By Proposition 1.8.19 and Example 1.8.20 of [DAGIII] $\text{Alg}_{E_0}(C)^\otimes$ is the symmetric monoidal category. It follows from Lemma 4.3 that the corresponding tensor product on $\text{Alg}_{E_0}(C)^\otimes$ coincide with (4.1).

**Remark 4.2.** (Forgetful functors). Let Triv be the subcategory of $\text{Fin}_*$ whose objects are the objects of $\text{Fin}_*$, and whose morphisms are the inert morphisms in $\text{Fin}_*$. Let $\text{Triv}^\otimes = N(\text{Triv})$. Then the inclusion $\text{Triv} \subseteq \text{Fin}_*$ unduces the functor $\text{Triv}^\otimes \to N(\text{Fin}_*)$ which exhibits $\text{Triv}^\otimes$ as an $\infty$-operad.

Let $\Delta^0 \to \text{Triv}$ be the functor corresponding to $[1] \in \text{Ob}(\text{Triv})$. Then by Remark 2.1.3.6 of [HA] the restriction functor

$$\text{Alg}_{\text{Triv}}(C^\otimes) \longrightarrow \text{Fun}_{N(\text{Fin}_*)}([1], C^\otimes)$$

is a trivial Kan fibration. So we can consider any map of $\infty$-operads as a forgetful functor.

**Lemma 4.3.** The forgetful functor $F : C_I \to C$ is monoidal.

Proof: The natural restriction

$$\text{Map}_{\text{Op}(\infty)}(\mathcal{E}_0, C^\otimes) \to \text{Fun}_{N(\text{Fin}_*)}([1], C^\otimes)$$
factors through $\text{Fun}_{N(\text{Fin})}(\Delta^1, \mathcal{C}^\otimes)$. So it suffice to check that the forgetful functor $\text{Alg}_{E_0}(\mathcal{C})^\otimes \to \mathcal{C}^\otimes$ is monoidal. Now, we can use Proposition 1.8.19 of [DAGIII]. □

Let us denote by $\mathcal{U}$ the forgetful functor $\text{CAlg}(\mathcal{C}_I/ \to \mathcal{C}_I/ \rightarrow \text{CAlg}(\mathcal{C})$.

**Lemma 4.4.** The diagram

\[
\begin{array}{ccc}
\text{CAlg}(\mathcal{C}_I/) & \xrightarrow{\text{CAlg}(F)} & \text{CAlg}(\mathcal{C}) \\
\downarrow U & & \downarrow U \\
\mathcal{C}_I/ & \xrightarrow{F} & \mathcal{C}
\end{array}
\]

is commutative. Moreover, the functor $\text{CAlg}(F)$ is an equivalence of categories.

**Proof:** By Lemma 4.3 $F$ induced by the monoidal functor $F^\otimes : \mathcal{C}^\otimes_{I/} \to \mathcal{C}^\otimes_{I/}$. So the diagram can be rewritten as

\[
\begin{array}{ccc}
\text{Map}_{\text{Op}(\infty)}(\mathcal{E}_0, \mathcal{C}^\otimes_{I/}) & \xrightarrow{(F^\otimes)^*} & \text{Map}_{\text{Op}(\infty)}(\mathcal{E}_0, \mathcal{C}^\otimes) \\
\downarrow i^* & & \downarrow i^* \\
\text{Map}_{\text{Op}(\infty)}(\text{Triv}^\otimes, \mathcal{C}^\otimes_{I/}) & \xrightarrow{(F^\otimes)^*} & \text{Map}_{\text{Op}(\infty)}(\text{Triv}^\otimes, \mathcal{C}^\otimes)
\end{array}
\]

where $i$ is the inclusion $\text{Triv} \hookrightarrow N(\text{Fin}_*)$.

Let us prove that $\text{CAlg}(F)$ is an equivalence. Note that the monoidal structure on $\text{Op}(\infty)$ is closed and $\text{Triv}^\otimes$ is the monoidal unit of $\text{Op}(\infty)$. So

\[\text{CAlg}(\mathcal{C}_I/) \simeq \text{CAlg}(\mathcal{C})_{I/}\]

and $\text{CAlg}(F)$ coincides with the forgetful functor

\[\text{CAlg}(\mathcal{C})_{I/} \rightarrow \text{CAlg}(\mathcal{C}).\]

But $I$ is an initial object of $\text{CAlg}(\mathcal{C})$. □

**Monoidal adjunction.** Let $\mathcal{C}^\otimes$ and $\mathcal{D}^\otimes$ be symmetric monoidal $\infty$-categories and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Suppose that $F$ is monoidal i.e. $F$ can be extended to the co-cartesian fibration of $\infty$-operads

\[\begin{array}{ccc}
\mathcal{C}^\otimes & \xrightarrow{F^\otimes} & \mathcal{D}^\otimes \\
\downarrow \Delta & & \downarrow \\
\Delta & & \Delta
\end{array}\]

Suppose that $F$ admits right-adjoint $R$. Then by Corollary 7.3.2.7. of [HA] $F^\otimes$ is also admits right-adjoint $G^\otimes$. Moreover, $G^\otimes$ is a map of $\infty$-operads (so $G^\otimes$ is lax-monoidal).

Now, let $\mathcal{O}^\otimes$ be an $\infty$-operad. Then applying $\text{Alg}_{\mathcal{O}}(-)$ we get the pair of functors

\[F : \text{Alg}_{\mathcal{O}}(C) \rightleftarrows \text{Alg}_{\mathcal{O}}(D) : G.\]
We claim that $F$ is left-adjoint to $G$. Indeed, by Joyal (Definition 4.0.1 of [RV15]), we can define the adjunction for an $(\infty,2)$-category $C$ using a unit and counit satisfying the triangle identities. By Remark 4.4.5. of [RV15], in the case $C = \text{Cat}_{\infty}$ we get the Lurie’s definition of adjunction [HTT]. So applying $\text{Alg}(\cdot)$ to the unit and counit of adjunction $(F \otimes, G \otimes)$ we get the unit and counit of adjunction $(F,G)$ which satisfy the triangle identities by functoriality.

Suppose that $f : O_1 \otimes \rightarrow O_2 \otimes$ is a map of operads. Then we get the natural transformation $f^* : \text{Alg}_{O_1} \rightarrow \text{Alg}_{O_2}$. So $F$ and $G$ commute with $f^*$ and $f^*$ maps the (co)unit of adjunction to the (co)unit of adjunction.

Remark 4.5. In particular, for $O^\otimes := E_\infty$ we get the adjunction

$$F : \text{CAlg}(F) \leftrightarrows \text{CAlg}(D) : G.$$ 

Applying $\text{Alg}(\cdot)$ to the map of operads $\text{Triv} \otimes \rightarrow E_\infty$ we conclude that the forgetful functors $U_C : \text{CAlg}(C) \rightarrow C$ and $U_D : \text{CAlg}(D) \rightarrow D$ commutes with $F$ and $G$. So if $A$ be a $E_\infty$-algebra then the unit of adjunction $U_C A \rightarrow GF(U_C A)$ can be extended to the homomorphism of algebras. The similar statement is true for the counit.

**Free algebra of pointed object.**

**Theorem 4.6.** Suppose that infinity direct sums is exist in $C$ and for any object $c$ the functor $\otimes c$ commutes with colimits. Then

1) The forgetful functor $U$ admits left-adjoint $\text{Sym}^*$. Moreover, for any object $I \rightarrow A$ we have

$$\text{Sym}^*(A) = I \otimes S^*(I) S^*(A).$$

2) let us consider the diagram

$$\cdots \rightarrow (A^{\otimes n})_{h\Sigma_n} \rightarrow (A^{\otimes n+1})_{h\Sigma_{n+1}} \rightarrow \cdots$$

(4.3)

where the maps induced by $v$. Then there is the weak equivalence in $C$:

$$\text{Sym}^*(A) \simeq \text{colim}_n (A^{\otimes n})_{h\Sigma_n}.$$ 

**Proof:** Let us prove the first statement. Let us denote by $x$ the map induced by the unit of adjunction $I \rightarrow S^*(I)$. Note $I$ has the natural structure of $E_{\infty}$-algebra. Let us denote by $\text{ev}_1$ the map $S^*(I) \rightarrow I$ adjoint to $\text{id}_I$. Observe that the composition

$$I \xrightarrow{x} S^*(I) \xrightarrow{\text{ev}_1} I$$

is equal to $\text{id}_I$.

Now, let $\text{CAlg}(S^*(I) - \text{mod})$ be the category of $S^*(I)$-algebras. Note that the functor $U$ factors through $\text{CAlg}(S^*(I) - \text{mod})$. Indeed, we have $\text{CAlg}(C_{1/}) \simeq \text{CAlg}(C)$ by Lemma 4.4 and $\text{CAlg}(S^*(I) - \text{mod}) \simeq \text{CAlg}(C)_{S^*(1)}$ by Corollary 3.4.1.7 of [HA]. The map $\text{ev}_1$ induce
the functor
\[ CAlg(C) \xrightarrow{ev_1^*} CAlg(CS^*_I)/ \]
\[ (I \xrightarrow{1} A) \mapsto (S^*(I) \xrightarrow{ev_1} I \xrightarrow{1} A) \]

The map \( x \) induce the functor \( u' \):
\[ CAlg(CS^*_I)/ \xrightarrow{U_{S^*_I}/} C^*/ \xrightarrow{x^*} C_I/ \]
\[ (S^*(I) \xrightarrow{\to} B) \mapsto (I \xrightarrow{\to} S^*(I) \xrightarrow{\to} B). \]

But \( ev_1 \cdot x = id_I \) so \( u' \cdot ev_1^* \simeq U \).

On the other hand, by results of [HA] (see Theorem 4.5.3.1 and Corollary 4.2.3.7) and Remark 4.5 the functor \( ev_1 \) is right-adjoint to relative tensor product \( I \otimes_{S^*(I)} - \). The functor \( u' \) is also admits left-adjoint which given by \( S^*/ \). This is the particular case of slicing of adjoint functors (see Proposition 5.2.5.1 of [HA]). So the functor \( I \otimes_{S^*(I)} S^*(-) \) is the left-adjoint to \( U \).

Now, let us prove the second statement. Let \( I \xrightarrow{v} A \) be an object of \( C_I/ \). Note that the colimit of any diagram can be presented as the cofiber. In the case of the diagram 4.3 we get
\[ \text{cofib}(S^*(A) \xrightarrow{(1-v)} S^*(A)). \]

Let us rewrite \( S^*(I) \) as \( S^*(I) \otimes_{S^*(I)} S^*(A) \). Then we have
\[ \text{id} \otimes (1-v) = (1-x) \otimes \text{id}. \]

So it suffice to check that the triangle
\[ S^*(I) \otimes_{S^*(I)} S^*(A) \xrightarrow{(1-x)\otimes \text{id}} S^*(I) \otimes_{S^*(I)} S^*(A) \xrightarrow{1} I \otimes_{S^*(I)} S^*(A) \]
is exact.

Note that the functor
\[ - \otimes_{S^*(I)} S^*(A) : C \rightarrow C \]
preserves colimits. Indeed, for any \( S^*(I) \)-algebra \( B \) the relative tensor product is given by the geometric realization of the two-sides bar construction \( \text{Bar}_{S^*(I)}(B, S^*(A)) \) (see Section 4.4 of [HA]). Geometric realizations commute with colimits. On the other hand, colimits in the category of simplicial objects can be computed pointwise. Note that
\[ \text{Bar}_{S^*(I)}(B, S^*(A))_n = B \otimes S^*(I)^{\otimes n} \otimes S^*(A). \]

So the commutation of \( - \otimes_{S^*(I)} S^*(A) \) with colimits follows from the commutation of \( \otimes \) with colimits.

Finally, observe that \( \text{cofib}(S^*(I) \xrightarrow{1-x} S^*(I)) \simeq I \). Indeed, \( (I^{\otimes n})_{\Sigma_n} \simeq I \) so the diagram 4.3 for \( A = I \) coincide with ...

\[ \xrightarrow{\text{id}} I \xrightarrow{\text{id}} I \xrightarrow{\text{id}} ... \ \square. \]
Relation with symmetric powers. Let $C^\otimes$ be as above. Suppose that $C^\otimes$ is stable, $\mathbb{Q}$-linear and idempotent-complete (in the sense of Section 4.4.5 of [HTT]). Let $A$ be an object of $C^\otimes$. Note that the $n$-th symmetric group $\Sigma_n$ acts on $A^\otimes n$. So we have the endomorphism $\varphi := \frac{1}{n!}\Sigma_{\sigma_i \in \Sigma_n} \sigma_i$ of $A^\otimes n$. Now, define the n-th symmetric power $S^n(A)$ as the image $\text{im}(\varphi)$.

Lemma 4.7. The composition

$$S^n(A) \hookrightarrow A^\otimes n \to (A^\otimes n)_{h\Sigma_n}$$

is a weak equivalence.

Firstly, let us explain what do we mean by $(A^\otimes n)_{h\Sigma_n}$. Note that the finite set $[n]$ induces the inclusion $B\Sigma_n \hookrightarrow \text{Fin}_*$. Applying $\text{Fun}(\text{N}(\text{Fin}_*), (-, C^\otimes))$ we get the functor

$$\text{CAlg}(C) \to \text{Rep}_{\Sigma_n}(C) = \text{Fun}(B\Sigma_n, C)$$

which maps $A$ to $A^\otimes n$. Note that $C$ can be considered as the full subcategory of constant functors. The corresponding inclusion $C \to \text{Rep}_{\Sigma_n}(C)$ preserves limits so admits left-adjoint

$$(-)_{h\Sigma_n} : \text{Rep}_{\Sigma_n}(C) \to C$$

which is called homotopy coinvariants.

Note that the element $\frac{1}{n!}\Sigma_{\sigma_i \in \Sigma_n} \sigma_i$ is an idempotent in the sense of Section 4.4.5 of [HTT]. So we have the decomposition

$$A^\otimes n \simeq S^n(A) \oplus B$$

in $C$ where $B = \ker(\frac{1}{n!}\Sigma_{\sigma_i})$.

Note that in this case we have the fixed homotopies

$$\frac{1}{n!}\Sigma_{\sigma_i|_{S^n(A)}} \simeq id_{S^n(A)} \quad \frac{1}{n!}\Sigma_{\sigma_i|_{B}} \simeq 0.$$

Using this one can check that $S^n(A)$ and $B$ are $\Sigma_n$-equivariant objects.

The proof of Lemma: Note that the action on $S^n(A)$ is trivial. So $(S^n(A))_{h\Sigma_n} \simeq S^n(A)$. It suffice to show that $B_{h\Sigma_n} = 0$. By stability of $C$ is equivalent to

$$H^0(\text{Map}_C(B_{h\Sigma_n}, c) \simeq H^0(\text{Map}_{\text{Rep}_{\Sigma_n}(C)}(B, c) = 0$$

for any $c \in C$. By definition the element of $H^0(\text{Map}_{\text{Rep}_{\Sigma_n}(C)}(B, c)$ is a map $\phi : B \to c$ and homotopies $\phi \cdot \sigma \simeq \phi$ for any $\sigma \in \Sigma_n$. So

$$0 \not\sim \phi \cdot \Sigma_{\sigma_i} \simeq \phi \cdot \Sigma_{\sigma_i} \not\sim n! \phi. \quad \square$$

5. Rational cohomology of a Kato-Nakayama space

Let $(X, \alpha : \mathcal{O}_X \to \mathcal{M}_X)$ be a fs log analytic space. For any point $x \in X$ let us define the map $\text{arg} : \mathcal{O}_{X,x}^* \to S^1; f \mapsto \text{arg}(f(x))$. Following [Og18], let us construct the topological
space $X^{\log}$. The point of $X^{\log}$ is the pairs $(x, \varphi)$ where $x$ is a complex point of $X$ and $\varphi : M_{X,x}^{gr} \rightarrow S^1$ is a map such that the following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{O}_{X,x}^* & \longrightarrow & M_{X,x}^{gr} \\
\downarrow & & \downarrow \varphi \\
S^1 & \longrightarrow & M_{X,x}^{gr}
\end{array}
$$

Notice that we have the projection $\pi : X^{\log} \rightarrow X$. For any open $U \subseteq X$ and $m \in M_{X,x}^{gr}(U)$ let us define the map $arg(m) : \pi^{-1}(U) \rightarrow S^1$

$$(x, \varphi) \mapsto \varphi(m_x).$$

The topology on $X^{\log}$ is given the weak topology defined by the map $\pi : X^{\log} \rightarrow X$ and the family of maps $arg(m)$. The topological space $X^{\log}$ is called the Kato-Nakayama space of $(X, \alpha)$. Note that $X^{\log}$ does not depend on the map $\mathcal{O}_X \rightarrow M_X$, but only on the group completion $M_{X}^{gr}$.

By definition, the Betti cohomology of a fs log analytic space $(X, \alpha)$ is a singular cohomology of $X^{\log}$. So for any abelian group $A$ we have

$$H^*_B((X, \alpha), A) \overset{def}{=} H^*(X, Rf_*A) \simeq H^*(X, R\pi_*A)$$

where $f$ is the map $X^{\log} \rightarrow pt$.

Suppose we have a description of $R\pi_*A$ in terms of the log structure on $X$. Then one could define the Betti cohomology independently of the construction of $X^{\log}$. The first approximation to such the description can be given by the following lemma proved by Kato and Nakayama.

**Lemma 5.1.** (Lemma (1.5) of [KN99]) For any abelian group $A$ we have a canonical isomorphism

$$R^q\pi_*A \simeq A \otimes_\mathbb{Z} \mathcal{M}_X^{gr}.$$

Now, let us give the explicit description of $R\pi_*A$ in the case $A = \mathbb{Q}$. Let $(X, \alpha)$ be as above. Let us denote by $\exp(\alpha)$ the complex

$$
\begin{array}{ccccccc}
... & \rightarrow & 0 & \rightarrow & \mathcal{O}_X & \rightarrow & M_{X}^{gr} & \rightarrow & 0 & \rightarrow & ...
\end{array}
$$

where the differential given by the composition $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \hookrightarrow M_{X}^{gr}$. Note that the inclusion of zero cohomology $1_{\exp(\alpha)} : \mathbb{Q} \rightarrow \exp(\alpha)\mathbb{Q}$ allows to consider $\exp(\alpha)\mathbb{Q} = \exp(\alpha) \otimes \mathbb{Q}$ as an object of the under category $\mathcal{C}h^*(\text{Sh}(X^{an}, \mathbb{Q}))\mathbb{Q}$. Then the maps

$$
\exp(\alpha)^{\otimes n} \otimes \mathbb{Q} \xrightarrow{id \otimes 1_{\exp(\alpha)}} \exp(\alpha)^{\otimes n}
$$
induce the inclusion
\[ S^{n-1}(\exp(\alpha)_\mathbb{Q}) \to S^n(\exp(\alpha)_\mathbb{Q}). \]
We put
\[ (5.1) \quad \text{Sym}^*(\exp(\alpha)_\mathbb{Q}) := \colim_n S^n(\exp(\alpha)_\mathbb{Q}). \]
By results of the previous section (5.1) is a free $E_\infty$-algebra in $Ch^\bullet(Sh(X^\text{an}, \mathbb{Q}))$.\]

**Theorem 5.2.** There exist a natural quasi-isomorphism of $E_\infty$-algebras $R\pi_*\mathbb{Q} \simeq \text{Sym}^*(\exp(\alpha)_\mathbb{Q})$.

Firstly, let us construct the quasi-isomorphism $\exp(\alpha) \simeq \tau_{\leq 1}R\pi_*\mathbb{Z}$. For a topological space $Y$ let us denote by $C(Y)$ the sheaf of continuous $Y$-valued functions on $X^\log$. Note that the sheaf $C(\mathbb{R})$ is soft. For $x \in X^\text{an}$ let us denote by $i$ and $\phi$ be the inclusions $x \hookrightarrow X^\text{an}$ and $\pi^{-1}(x) \hookrightarrow X^\log$. By Proposition 1.2.5. of [Og18] $X^\log$ is a Hausdorff space. So $\pi$ is universally closed and we have the proper base change theorem. Then
\[ i^{-1}R^1\pi_*C(\mathbb{R}) \simeq R^1\pi_*((\phi^{-1}C(\mathbb{R}))) \simeq H^1(\pi^{-1}(x), C(\mathbb{R})) = 0 \]
for any $x \in X^\text{an}$.

**Lemma 5.3.** Let
\[ \mathbb{Z} \longrightarrow A \longrightarrow B \]
be an exact sequence of sheaves and $R^1\pi_*A = 0$. Then $\tau_{\leq 1}R\pi_*\mathbb{Z} \simeq \pi_*(\text{Cone}(A \longrightarrow B))$.

**Proof:** Note that Cone commutes with $\pi_*$ so we have the diagram with exact rows
\[
\begin{array}{c}
\pi_*B[-1] \longrightarrow \pi_*Cone(A \to B) \longrightarrow \pi_*A \\
\downarrow \quad \downarrow \quad \downarrow \\
R\pi_*B[-1] \longrightarrow R\pi_*Z \longrightarrow R\pi_*A
\end{array}
\]
It remains to use the long exact sequences of cohomology \( \square \).

Let $\exp : C(\mathbb{R}) \longrightarrow C(S^1)$ be the map induces by $\exp : \mathbb{R} \longrightarrow S^1$, $a \mapsto e^{2\pi i a}$.

Notice that the sequence of sheaves
\[ \mathbb{Z} \longrightarrow C(\mathbb{R}) \longrightarrow C(S^1) \]
is exact.

**Remark 5.4.** For $U \in X^\log$ let us compute the cokernel of homomorphism $\text{Map}(U, \mathbb{R}) \longrightarrow \text{Map}(U, S^1)$. Note that a map $\phi \in \text{Map}(U, S^1)$ can be lifted to $\text{Map}(U, \mathbb{R})$.
iff \(|\phi| = 0 \in [U, S^1] \simeq H^1(U, \mathbb{Z})\). Indeed, let \(\mathbf{1} : X \rightarrow S^1\) be a constant map with the value 1. Suppose that \(\phi\) is homotopy equivalent to \(\mathbf{1}\). But \(\exp(0) = 1\) so we can use the homotopy lifting properties:

\[
\begin{array}{ccc}
X & \xrightarrow{0} & \mathbb{R} \\
\downarrow & & \downarrow \exp \\
X \times I & \xrightarrow{\phi \times 1} & S^1.
\end{array}
\]

Hence the sheaf \(R^1\pi_*\mathbb{Z}\) is a sheafification of the presheaf \(W \mapsto H^1(\pi^{-1}W, \mathbb{Z})\).

By the Lemma we have the quasi-isomorphism \(\tau_{\leq 1} R\pi_* \mathbb{Z} \simeq \pi_* \text{Cone}(\exp)\). Let us define the map of complexes

\[
(5.2) \quad \mathcal{O} \xrightarrow{\exp} \mathcal{M}^{gr} \xrightarrow{\exp} \pi_* C(\mathbb{R}) \xrightarrow{\exp} \pi_* C(S^1). \]

by the rule \(f \in \mathcal{O}(U) \mapsto \pi^* \text{Ref} f, \; h \in \mathcal{M}^{gr}(U) \mapsto \{ h^* : (x, \phi_x) \mapsto \phi_x(h_x) \}\).

**Lemma 5.5.** The map \((5.2)\) induces quasi-isomorphism

\(\exp(\alpha) \simeq \tau_{\leq 1} R\pi_* \mathbb{Z}\).

**Proof:** It suffice to check that the induced map \(\overline{\mathcal{M}}^{gr}_X \rightarrow R^1\pi_* \mathbb{Z}\) is an isomorphism. Let \(x \in X^{\text{an}}\) be a closed point. Let us consider the commutative diagram

\[
\begin{array}{ccc}
\colim_{x \in V} \mathcal{O}(V) & \xrightarrow{\exp} & \colim_{x \in V} \mathcal{M}^{gr}(V) \\
\downarrow \text{Res} & & \downarrow \text{Res} \\
\mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^{*} \oplus \mathcal{M}^{gr}_x \\
\end{array}
\]

where \(\text{Res}\) are the restriction maps. Applying \(H^1\) we get the commutative diagram

\[
\begin{array}{ccc}
\colim_{x \in V} \mathcal{M}^{gr}(V) & \xrightarrow{\exp} & \colim_{x \in V} H^1(\pi^{-1}V, \mathbb{Z}) \\
\downarrow \text{Res} & & \downarrow \text{Res} \\
\mathcal{M}^{gr}_x & \rightarrow & H^1(\pi^{-1}(x), \mathbb{Z})
\end{array}
\]

Let \(\bar{\pi}\) be the map \(\pi^{-1}(x) \rightarrow x\). Then by Proper base change theorem the restriction map

\[
(R^1\pi_* \mathbb{Z})_x = \colim_{x \in V} H^1(\pi^{-1}V, \mathbb{Z}) \xrightarrow{\text{Res}} H^1(\pi^{-1}(x), \mathbb{Z}) = R^1\bar{\pi}_* \mathbb{Z}
\]
is an isomorphism. So we may assume that $X^\text{an}$ is a point.
In this case the map $\overline{M}_X \to R^1\pi_*Z$ can be expressed as the composition
\begin{equation}
\overline{M}_X \simeq \mathbb{Z}^r \to \text{Map}((S^1)^r, S^1) \to [(S^1)^r, S^1]
\end{equation}
$(a_1, \ldots, a_r) \mapsto ((\phi_1, \ldots, \phi_r) \mapsto (\phi_1^{a_1}, \ldots, \phi_r^{a_r}))$.
Note that the map (5.3) is defined by the images of $(0, \ldots, 1, \ldots, 0)$ which correspond to the projections
\begin{equation}
\pi_i : S^1 \times \ldots \times S^1 \to S^1.
\end{equation}
It remains to note that $[\pi_i]$ generate $H^1((S^1)^r, \mathbb{Z})$. □

The proof of Theorem 5.2: Let $A^\otimes, B^\otimes$ be a symmetric monoidal categories and $F^\otimes : A^\otimes \rightleftarrows B^\otimes : G^\otimes$ be a monoidal adjunction. Then by Remark 4.5 we have the pairs of adjoint functors
\begin{equation}
F : A_{\downarrow I_A} \rightleftarrows B_{\downarrow I_B} : G,
\end{equation}
\begin{equation}
F : CAlg(A_{\downarrow I_A}) \rightleftarrows CAlg(B_{\downarrow I_B}) : G.
\end{equation}
Suppose that $A \in CAlg(A_{\downarrow I_A})$ is an algebra and $a \xrightarrow{\varphi} A$, $Sym(a) \xrightarrow{\psi} A$ is a pair of adjoint morphisms. Then $F(\varphi)$, $F(\psi)$ are also adjoint. Indeed, $F$ commutes with the forgetful functors $U_A : CAlg(A_{\downarrow I_A}) \to A_{\downarrow I_A}$ and $U_B : CAlg(B_{\downarrow I_B}) \to B_{\downarrow I_B}$ by Remark 4.5. So it suffice to show that $F$ preserves the unit of adjunction $\text{Id} \to U : Sym^*$. Let $a \in A_{\downarrow I_A}$. Then the map $a \to Sym^*(a)$ can be expressed as the canonical morphism
\begin{equation}
a \to \text{colim}_n (a^\otimes)_{h\Sigma_n}.
\end{equation}
Note that $F$ is monoidal and preserves colimits. So $F$ commutes with $(-^\otimes)_{h\Sigma_n}$ and maps the canonical morphism (5.4) to the canonical morphism.

Now, let $\tau_{\leq 1} R\pi_* Q \to R\pi_* Q$ be the canonical inclusion. Note that $R\pi_*$ is lax-monoidal so $R\pi_* Q$ is an algebra with the unit $R^0\pi_* Q = Q$. So we get the map
\begin{equation}
Sym^*(\tau_{\leq 1} R\pi_* Q) \to R\pi_* Q.
\end{equation}
To complete the proof, it suffices to show that the map is a quasi-isomorphism on any fiber. Let $i : x \to X^\text{an}$ be a point. By the previous $Sym^*(\tau_{\leq 1} R\pi_* Q)_x \to (R\pi_* Q)_x$ is adjoint to $\tau_{\leq 1} R\pi_* Q_x \to R\pi_* Q_x$. So using the proper base change we may assume that $X^\text{an}$ is a point. In this case, $X^{\text{log}}$ is a torus, $exp(\alpha) \simeq \mathbb{Z} \oplus \mathbb{Z}[n][-1]$ for some $n$ and $Sym^*(exp(\alpha))_Q \simeq \Lambda^*(\mathbb{Q}^n)$. So (5.5) is the map $\Lambda^*(\mathbb{Q}^n) \to R\pi_* Q$ induced by the inclusion of the one-dimension cycles $Q^n[-1] \to R\pi_* Q$. It is well known that such a map is a quasi-isomorphism. □.
6. From log structures to motivic sheaves

**Extension on a smooth étale site.** Let $X$ be a scheme of finite type over $k$. The natural inclusion of site $X_{et} \hookrightarrow \text{Sm}/X_{et}$ give rise to the pair of functors

$$\text{Ex}_X : \text{Sh}(X_{et}, \mathbb{Z}) \xrightarrow{\sim} \text{Sh}_{et}(\text{Sm}/X, \mathbb{Z}) : \text{Res}_X.$$ 

They are given as follows: If $\mathcal{F} \in \text{Sh}_{et}(\text{Sm}/X, \mathbb{Z})$, we have $\text{Res}_X \mathcal{F}(X') = \mathcal{F}(X')$ for étale $X$-schemes $X'$. If $\mathcal{G}$ is a sheaf on $X_{et}$, then the sheaf $\text{Ex}_X(\mathcal{G})$ is associated to the presheaf

$$Z \mapsto \lim_{\rightarrow} \mathcal{G}(X')$$

on $\text{Sm}/X$, where the limit extends over diagrams of the form

$$\begin{array}{ccc}
Z & \xrightarrow{f} & X' \\
\downarrow^\text{et} & & \downarrow^\text{et} \\
X & & 
\end{array}$$

By results of ([SGA4], Section VII, 4.0, 4.1) we have

**Lemma 6.1.** The functor $\text{Ex}_X$ is left adjoint to $\text{Res}_X$ and exact. Moreover, $\text{Ex}_X$ commutes with the inverse image functors $f^{-1}$ for any $f$.

Now, we want to extend any virtual log structure on $X_{et}$ to the étale sheaf on $\text{Sm}/X_{et}$. Namely, for the log structure $\mathcal{M}^{gr}$ on $X$ and smooth $f:Y \rightarrow X$ let us define the sheaf

$$Y \xrightarrow{\text{Ex}_X^{log} \mathcal{M}^{gr}(\downarrow_f)} \Gamma(Y, f^* \mathcal{M}^{gr})$$

where $f^*$ is a pullback of log structures.

Observe that $\text{Ex}_X^{log}$ can be defined by the more functorial way. Let

$$\mathcal{M}^{gr}_X, \mathcal{O}^{*}_X : v\text{Log}_X \rightarrow \text{Sh}(X_{et}, \mathbb{Z})$$

be the functors which maps the log structure $\mathcal{O}^{*} \rightarrow \mathcal{M}^{gr}$ to the sheaf $\mathcal{M}^{gr}$ and $\mathcal{O}^{*}$. Let us denote by $\text{Ex}_X(\mathcal{M}^{gr}_X)$ and $\text{Ex}_X(\mathcal{O}^{*}_X)$ the compositions with $\text{Ex}_X$. Observe that we have the natural transformation

$$\text{Ex}_X(\mathcal{O}^{*}_X) \rightarrow \text{Ex}_X(\mathcal{M}^{gr}_X).$$

Also, let us denote by $\mathcal{O}^{*}_{\text{sm}/X}$ the constant functor $v\text{Log}_X \rightarrow \text{Sh}_{et}(\text{Sm}/X, \mathbb{Z})$ with value $\mathcal{O}^{*}_{\text{sm}/X}$ (here $\mathcal{O}^{*}_{\text{sm}/X}$ is an étale sheaf of inverse function on $\text{Sm}/X$). For any morphism of scheme $f:X \rightarrow Y$, the natural map $f^{-1} \mathcal{O}^{*}_Y \rightarrow \mathcal{O}^{*}_X$ gives rise to the natural transformation

$$\text{Ex}_X(\mathcal{O}^{*}_X) \rightarrow \mathcal{O}^{*}_{\text{sm}/X}.$$
Now we can define \( \text{Ex}^\log \) as the pushout

\[
\begin{array}{c}
\text{Ex}_X(O_X^*) \longrightarrow \text{Ex}_X(M_X^{gr}) \\
\downarrow \quad \downarrow \\
O_{\text{Sm}/X}^* \longrightarrow \text{Ex}_X^\log
\end{array}
\]

Notice that the map \( \text{Ex}_X(O_X^*)(\phi) \longrightarrow \text{Ex}_X(M_X^{gr})(\phi) \) is an inclusion for any virtual log structure \( \phi \). So the map \( \text{Ex}_X(O_X^*)(\phi) \longrightarrow \text{Ex}_X(M_X^{gr})(\phi) \) is an inclusion in the abelian category \( \text{Fun}(v\text{Log}_X, \text{Sh}_{\text{et}}(\text{Sm}/X, \mathbb{Z})) \) and we have the short exact sequence

\[
\text{Ex}_X(O_X^*) \to O_{\text{Sm}/X}^* \oplus \text{Ex}_X(M_X^{gr}) \to \text{Ex}_X^\log.
\]

**The natural transformation** \( \text{Ex}^\log \). Let \( f : Y \to X \) be a morphism of schemes. Let us construct the functorial map \( \psi_f : f^{-1}\text{Ex}_X^\log \longrightarrow \text{Ex}_Y^\log f^* \). Recall that we defined \( f^* M_X^{gr} \) as the cocartesian square. So applying \( \text{Ex} \) and using commutation \( \text{Ex} \) with \( f^{-1} \) we get the square

\[
\begin{array}{c}
f^{-1}\text{Ex}_X(O_X^*) \longrightarrow f^{-1}\text{Ex}_X(M_X^{gr}) \\
\downarrow \quad \downarrow \\
\text{Ex}_Y(O_Y^*) \longrightarrow \text{Ex}_Y(f^* M_X^{gr})
\end{array}
\]

On the other hand, let us consider the diagram

\[
\begin{array}{c}
O_X^* \longrightarrow f_* O_Y^* \\
\downarrow id \quad \downarrow id \\
\text{Res}_X(O_{\text{Sm}/X}^*) \longrightarrow f_* \text{Res}_Y(O_{\text{Sm}/Y}^*)
\end{array}
\]

where the horizontal maps induced by \( f \). Using commutation \( \text{Res} \) with \( f_* \) and adjunction \( \text{Res} - \text{Ex}, f_* - f^{-1} \) we get the commutative diagram

\[
\begin{array}{c}
f^{-1}\text{Ex}_X(O_X^*) \longrightarrow \text{Ex}_Y(O_Y^*) \\
\downarrow \quad \downarrow \\
f^{-1}O_{\text{Sm}/X}^* \longrightarrow O_{\text{Sm}/Y}^*
\end{array}
\]

So we have the commutative diagram

\[
\begin{array}{c}
f^{-1}\text{Ex}_X(O_X^*) \longrightarrow f^{-1}O_{\text{Sm}/X}^* \oplus f^{-1}\text{Ex}_X(M_X^{gr}) \\
\downarrow \quad \downarrow \\
\text{Ex}_Y(O_Y^*) \longrightarrow O_{\text{Sm}/Y}^* \oplus \text{Ex}_Y(f^* M_X^{gr})
\end{array}
\]

(6.2)
and we define $\psi_f : f^{-1}\text{Ex}^\log_X(\mathcal{M}^{gr}_X) \to \text{Ex}^\log_Y(f^*\mathcal{M}^{gr}_X)$ as the map of cokernels.

**Proposition 6.2.** The map $\psi_f$ is an isomorphism.

**Proof:** Let us consider the diagram 6.1. By the exactness of $\text{Ex}$ the square 6.1 is cocartesian and the horizontal maps are inclusions. So the map between the cokernels of horizontal arrows is an isomorphism. By Snake lemma we immediately get that the canonical maps

$$\ker(f^{-1}\text{Ex}_X(\mathcal{O}^*_X) \to \text{Ex}_Y(\mathcal{O}^*_Y)) \to \ker(f^{-1}\text{Ex}_X(\mathcal{M}^{gr}_X) \to \text{Ex}_Y(f^*\mathcal{M}^{gr}_X))$$

$$\text{coker}(f^{-1}\text{Ex}_X(\mathcal{O}^*_X) \to \text{Ex}_Y(\mathcal{O}^*_Y)) \to \text{coker}(f^{-1}\text{Ex}_X(\mathcal{O}^*_X) \to \text{Ex}_Y(\mathcal{O}^*_Y))$$

are isomorphisms. So

$$\ker(\psi_f) \simeq \ker(f^{-1}\mathcal{O}^*_{\text{Sm}/X} \to \mathcal{O}^*_{\text{Sm}/Y})$$

$$\text{coker}(\psi_f) \simeq \text{coker}(f^{-1}\mathcal{O}^*_{\text{Sm}/X} \to \mathcal{O}^*_{\text{Sm}/Y})$$

and we can use Corollary A.2.

**Corollary 6.3.** Then the collection of functors $\text{Ex}^\log_X$ gives rise to the natural transformation of contravariant pseudofunctors

$$\text{Ex}^\log : \text{vLog} \to \text{Sh}_{\text{et}}(\text{Sm/}/, \mathbb{Q}).$$

**From log structures to motivic sheaves.** Let $R$ be a commutative ring and $S$ be a scheme. Let us denote by $\text{DA}^{\text{eff}}_{\text{et}}(S, R)$ and $\text{DA}^{\text{eff}}_{\text{et}}(S, R)$ be the categories of effective and spectral étale motivic sheaves on $S$ (see [Ayo14] for definition, see also [Ayo07] and [CD19] for more details). Notice that for any morphism $f : S' \to S$ we have the inverse image functor $f^* : \text{DA}^{\text{eff}}_{\text{et}}(S, R) \to \text{DA}^{\text{eff}}_{\text{et}}(S', R)$ which can be define as the Kan extension of the composition

$$\text{Sm}/S \xrightarrow{\times_S S'} \text{Sm}/S' \longrightarrow \text{DA}^{\text{eff}}_{\text{et}}(S', R).$$

We can also define the inverse image functor $f^* : \text{DA}^{\text{eff}}_{\text{et}}(S, R) \to \text{DA}^{\text{eff}}_{\text{et}}(S', R)$ by the same way. Moreover, by construction (see [CD19]) $\text{DA}^{\text{eff}}_{\text{et}, R}$ and $\text{DA}^{\text{eff}}_{\text{et}, R}$ can be considered as the presheaves of $\infty$-categories

$$S \mapsto \text{DA}^{\text{eff}}_{\text{et}}(S, R); \quad S \mapsto \text{DA}^{\text{eff}}_{\text{et}}(S, R)$$

$$(f : S' \to S) \mapsto f^*.$$

**Remark 6.4.** For any $S \in \text{Sch}/k$ we have the $\mathbb{A}^1$-localization functor

$$L_{\mathbb{A}^1} : \text{D}(\text{Sh}_{\text{et}}(\text{Sm}/S, R)) \to \text{DA}^{\text{eff}}_{\text{et}}(S, R)$$

and the infinity $\mathbb{G}_m$-suspension functor

$$\Sigma^\infty_{\mathbb{G}_m} : \text{DA}^{\text{eff}}_{\text{et}}(S, R) \to \text{DA}^{\text{eff}}_{\text{et}}(S, R).$$
By construction $L_{\mathbb{A}^1}$ and $\Sigma_{\mathbb{G}_m}^\infty$ commute with inverse image functors. So we have the composition of natural transformations

$$\Sigma_{\mathbb{G}_m}^\infty \cdot L_{\mathbb{A}^1} : D(\text{Sh}_{et}(\text{Sm}/-, \mathbb{R})) \rightarrow DA_{et, \mathbb{R}}$$

**Definition 6.5.** Let $X = (\underline{X}, \mathcal{M}^{gr}_X)$ be a fs log scheme. Let us denote by $\mathcal{M}^{gr}_X$ the image of $\mathcal{M}^{gr}_X$ under the composition

$$\mathcal{M}^{gr}_X \rightarrow \text{Sh}_{et}(\text{Sm}/X, \mathbb{R}) \hookrightarrow D(\text{Sh}_{et}(\text{Sm}/X, \mathbb{R})) \rightarrow DA_{et}(X, \mathbb{R}).$$

The motivic sheaves $\mathcal{M}^{gr}_X$ can be considered as a motivic analog of log structures. They play a critical role further in the text.

7. The homological motive of a log scheme

Now, let $(X, \mathcal{M}^{gr})$ be a virtual log scheme. Let $R = \mathbb{Q}$. By the previous, the inclusion $\mathcal{O}^* \hookrightarrow \mathcal{M}^{gr}$ induces the map $\mathbb{Q}(1)[1] \rightarrow \mathcal{M}^{gr}$. It makes $\mathcal{M}^{gr}(-1)[-1]$ an object of the under category $DA_{et}(X, \mathbb{Q})/\mathbb{Q}$. So we have a free algebra

$$\mathcal{Q}^{log}_X := \text{Sym}^*(\mathcal{M}^{gr}(-1)[-1]).$$

Let $f$ be the canonical morphism $X \rightarrow \text{Spec}(k)$. By analogy with Theorem 5.2 we define the homological motive of $(X, \mathcal{M}^{gr})$ by the formula

$$[X]^{log} := \text{Hom}(Rf_! \mathcal{Q}^{log}_X, \mathbb{Q}).$$

**Remark 7.1.** In [Del02], Deligne defined for any $\mathbb{Q}$-linear, monoidal and idempotent complete category the notions of Schur functors. In particular, we have well-defined motivic symmetric powers $S^n : DA_{et}(X, \mathbb{Q}) \rightarrow DA_{et}(X, \mathbb{Q})$. By Lemma 4.7.1, we get the weak equivalence

$$\text{Sym}^*(\mathcal{M}^{gr}(-1)[-1]) \simeq \text{colim}_n S^n(\mathcal{M}^{gr}(-1)[-1]).$$

So the motivic sheaf $\mathcal{Q}^{log}_X$ can be defined without using of any higher algebra.

Let $(g, g^\#) : (X, \mathcal{M}^{gr}_X) \rightarrow (Y, \mathcal{M}^{gr}_Y)$ be the morphism of virtual log schemes. Let us construct the map $[X]^{log} \rightarrow [Y]^{log}$. Firstly, note that (6.3) maps $g^*\mathcal{M}^{gr}_Y$ to $g^*\mathcal{M}^{gr}_Y$ by Corollary 6.3 and Remark 6.4. Hence $g^\#$ maps to

$$S^g(g^*\mathcal{M}^{gr}_Y(-1)[-1]) \rightarrow S^g(\mathcal{M}^{gr}_X(-1)[-1]).$$

On the other hand, $g^*$ is monoidal so $S^g(\mathcal{M}^{gr}_Y(-1)[-1]) \simeq g^*S^g(\mathcal{M}^{gr}_Y(-1)[-1])$ and we get the map

$$g^*\text{Sym}^*(\mathcal{M}^{gr}_Y(-1)[-1]) \rightarrow \text{Sym}^*(\mathcal{M}^{gr}_X(-1)[-1]).$$
Then, by adjunction, we have
\[ \text{Sym}^* (M^\heartsuit_Y (-1)[-1]) \rightarrow Rg_* \text{Sym}^* (M^\heartsuit_X (-1)[-1]). \]
and it remains to apply \( Rf_* \) and \( \text{Hom}(\cdot, \mathbb{Q}) \).

**The motive of trivial log structure.** Let \((X, \mathcal{O}^*)\) be a log scheme with trivial log structure. Then
\[ [X]^\log \simeq [X] \]
where \([X]\) is the homological motive of the underlying scheme \(X\). Indeed, it suffices to check that \( Rf_* \mathbb{Q}^\log_X \simeq \text{Hom}([X], \mathbb{Q}) \). Let us note that \( Rf_* \mathbb{Q}^\log_X \simeq Rf_* \mathbb{Q}_X \) and use Yoneda Lemma. For any smooth \(Y\) over \(k\) we have
\[ \text{Hom}_{DM}(\mathbb{Q}(-q)[-p]\otimes [X], \mathbb{Q}) \simeq \text{Hom}_{DA}(X \times_k Y, \mathbb{Q}(-q)[-m]). \]
On the other hand,
\[ \text{Hom}_{DM}(\mathbb{Q}(-q)[-m], Rf_* \mathbb{Q}_X) \simeq \text{Hom}_{DA}(X \times_k Y, \mathbb{Q}(q)[m]). \]

**The motive of a fs log scheme.** Let \(X\) be a fs log scheme. Recall that in Section 3 we constructed virtualization functor
\[ v : \text{Sch}^\log_{/k} \rightarrow \text{Sch}^v\log_{/k} \]
which maps a log scheme \((X, \mathcal{M} \rightarrow \mathcal{O})\) to the pair \((X, \mathcal{O}^* \hookrightarrow \mathcal{M}^\heartsuit)\). We define the Kato-Nakayama motive of \(X\) as the motive of the virtualization
\[ [X]^\log := [v(X)]^\log. \]

**Functoriality.** Note that we construct the functor
\[ (7.1) \quad \text{Sch}^v\log_{/k} \rightarrow \text{Ho}(DM(k, \mathbb{Q})) \]
to the triangulated category of Voevodsky motives. On the other hand, \(DM(k, \mathbb{Q})\) has the natural DG-enrichment \[BV06\] and, consequently, can be considered as a stable \((\infty, 1)\)–category.

**Proposition 7.2.** The functor \((10.4)\) gives rise to the \(\infty\)-functor
\[ (7.2) \quad [-]^\log : \text{Sch}^v\log_{/k} \rightarrow DM(k, \mathbb{Q}). \]
It suffice to show the dual functor \((X, \mathcal{M}^\heartsuit) \mapsto Rf_* \mathbb{Q}^\log_X\) can be lifted on the level of \(\infty\)-category. To prove the Proposition let us reformulate the construction of \[(10.4).\] Let us
consider the 1-functors\(^7\)

\[
\begin{align*}
\text{Ho}(\text{DA}_{\text{et},\mathbb{Q}}) & : X \mapsto \text{Ho}(\text{DA}_{\text{et}}(X, \mathbb{Q}), \ f \mapsto f^*; \\
\text{Ho}(\text{DA}_{\text{et},\mathbb{Q}})^{\text{op}} & : X \mapsto \text{Ho}(\text{DA}_{\text{et}}(X, \mathbb{Q})^{\text{op}}, \ f \mapsto Rf_*).
\end{align*}
\]

Here \(\text{Ho}\) means the homotopy category. Combining Corollary 6.3 and Remark 6.4 we get the natural transformation of \(\infty\)-functors \(\Psi : \text{vLog}^{\text{fs}} \rightarrow \text{DA}_{\text{et},\mathbb{Q}}\) which is defined as the composition

\[
\begin{array}{c}
\text{vLog}^{\text{fs}} \xrightarrow{\text{Ex}^{\log} \otimes \mathbb{Q}} \text{Sh}_{\text{et}}(\text{Sm}/-, \mathbb{Q}) \xrightarrow{\bigcirc} \text{D}(\text{Sh}_{\text{et}}(\text{Sm}/-, \mathbb{Q})) \xrightarrow{\Sigma^\infty L_\ast} \text{DA}_{\text{et},\mathbb{Q}}.
\end{array}
\]

So we have the natural transformation \(\text{Ho}(\Psi) : \text{vLog}^{\text{fs}} \rightarrow \text{Ho}(\text{DA}_{\text{et},\mathbb{Q}})\)

Let us define the natural transformation \(R\Gamma^{\text{mot}} : \text{Ho}(\text{DA}_{\text{et},\mathbb{Q}})^{\text{op}} \rightarrow \text{DA}_{\text{et}}(k, \mathbb{Q}); \)

\((F \in \text{DA}_{\text{et}}(X, \mathbb{Q})) \mapsto Rf_* F.\)

Here \(\text{DA}_{\text{et}}(k, \mathbb{Q})\) is a constant sheaf and \(f\) is a canonical morphism \(X \rightarrow \text{Spec}(k)\).

Now, we have the commutative diagram

\[
\begin{array}{c}
\int_{\text{Sch}/k^{\text{op}}} \text{Hom}(\text{DA}_{\text{et},\mathbb{Q}})^{\text{op}} \xrightarrow{\text{Ex}^{\log} \otimes \mathbb{Q}} \text{Sh}_{\text{et}}(\text{Sm}/-, \mathbb{Q}) \rightarrow \text{D}(\text{Sh}_{\text{et}}(\text{Sm}/-, \mathbb{Q})) \rightarrow \text{DA}_{\text{et},\mathbb{Q}}.
\end{array}
\]

To construct \([-\]^{log}\) it remains to use the \(\infty\)-category version of Grothendieck construction

**Proof of Proposition 7.2** Let us consider the \(\infty\)-functors\(^8\)

\[
\begin{align*}
\text{DA}_{\text{et},\mathbb{Q}} : \text{Sch}/k^{\text{op}} & \rightarrow \text{Pr}^R; \quad X \mapsto \text{DA}_{\text{et}}(X, \mathbb{Q}), \ f \mapsto f^*; \\
\text{DA}_{\text{et},\mathbb{Q}}^{\text{op}} : \text{Sch}/k & \rightarrow \text{Pr}^L; \quad X \mapsto \text{DA}_{\text{et}}(X, \mathbb{Q})^{\text{op}}, \ f \mapsto Rf_*.
\end{align*}
\]

By construction, \(R\Gamma^{\text{mot}}\) can be lifted to the natural transformation of \(\infty\)-functors \(R\Gamma^{\text{mot}} : \text{DA}_{\text{et},\mathbb{Q}}^{\text{op}} \rightarrow \text{DA}_{\text{et}}(k, \mathbb{Q}).\) Then we define \([-\]^{log}\) as the composition

\[
\begin{array}{c}
\int_{\text{Sch}/k^{\text{op}}} \text{DA}_{\text{et},\mathbb{Q}}^{\text{op}} \xrightarrow{\varphi} \int_{\text{Sch}/k} \text{DA}_{\text{et},\mathbb{Q}}^{\text{op}} \xrightarrow{\text{Sym}^\ast \Psi} \text{Sch}/k \times \text{DA}_{\text{et}}(k, \mathbb{Q}) \xrightarrow{\text{pr}_2 \Psi} \text{DA}_{\text{et}}(k, \mathbb{Q}).
\end{array}
\]

\(^7\)from the category of schemes to the category of all 1-categories

\(^8\)here \(\text{Pr}^*\) mean the category of presentable categories together with the morphisms given by continuous and cocontinuous functors
Here $\int$ mean the $(\infty, 1)$-Grothendieck construction (see Appendix [3] and $\varphi$ is the equivalence of categories (see Proposition [3, 6]). □.

**Constructability of $[X]^\text{log}$**. Let $X$ be a scheme. Recall that a motivic sheaf on $X$ is called constructible if it can be presented as a finite colimits of $Y_i(n_i)[m_i]$ for some $Y_i \in \text{Sm}/X$ and $n_i, m_i \in \mathbb{Z}$. For any $X$ constructible motivic sheaves form full subcategory of $\text{DA}_{et}(X, \mathbb{Q})$ which is stable under six operations.

**Theorem 7.3.** For any fs virtual log scheme $(X, \mathcal{M}^{gr})$ the motivic sheaf $\mathbb{Q}^\text{log}_X$ is constructible.

**Proof:** Firstly, suppose that $X$ is normal. The category of constructible motivic sheaves closed under tensor product and direct summands. So it suffice to check that $\mathcal{M}^{gr}$ is constructible. The motivic sheaf $\mathcal{M}^{gr}$ is the extension of $\mathbb{Q}(1)[1]$ by $\mathcal{M}^{gr} = \Sigma_{\mathbb{G}_m}^\infty L_{A^1} \text{Ex}_X(\mathcal{M}^{gr})$ (see Section 6). So it suffice to check that $\mathcal{M}^{gr}$ is constructible. The six operations preserve constructible objects. Hence we may assume that $\mathcal{M}^{gr}$ is a local system. For normal schemes the fiber functor $\text{Fib}_x : \text{LocSys}(X)_\mathbb{Q} \xrightarrow{\sim} \text{Rep}_{\text{cont}}(\pi^\text{et}_1(X, x))_\mathbb{Q}$ is an equivalence of categories between finite type local systems and finite continuous representations. So the action of $\pi^\text{et}_1(X, x)$ on $\mathcal{M}^{gr}$ is continuous and the stabilizer of each element is an open subgroup. Hence the kernel $\pi^\text{et}_1(X, x) \rightarrow \text{Aut}(\mathcal{M}^{gr})$ is open and the image is a finite group. Consequently, there is the Galois cover $f : Y \rightarrow X$ such that the action of $\pi^\text{et}_1(X, x)$ on the stack $\mathcal{M}^{gr}_y$ factors through $\text{Aut}(Y/X)$.

Now, using the equivalence of categories we conclude that $\mathcal{M}^{gr} = \bigoplus_i \text{Fib}_x^{-1}(\rho_i)$ where $\rho_i$ are some irreducible representations of $\text{Aut}(Y/X)$. So $\mathcal{M}^{gr} = \bigoplus_i [\rho_i]$ where $[\rho_i] := \Sigma_{\mathbb{G}_m}^\infty L_{A^1} \text{Ex}_X(\text{Fib}_x^{-1}(\rho_i))$. But for any irreducible representation $\rho_i$ the corresponding motivic sheaf $[\rho_i]$ is constructible. Indeed, for the representable local system $\mathbb{Q}[Y]$ we have

$$\text{Fib}_x \mathbb{Q}[Y] = \mathbb{Q}[\text{Aut}(Y/X)] \simeq \bigoplus_j \rho_j^{\text{dim}(\rho)}$$

there $j$ runs of all irreducible representations of $\text{Aut}(Y/X)$. On the other hand, the motivic sheaf $[Y]$ is constructible.

For arbitrary $X$ let us use induction by $\text{dim}(X)$. Let $\text{dim}(X) = 0$. By results of [CD19], any motivic sheaf admits h-descent. So $\text{DA}_{et}(X, \mathbb{Q}) \simeq \text{DA}_{et}(X_{\text{red}}, \mathbb{Q})$ and we may assume that $X$ is reduced. Then $X = \sqcup_m \text{Spec}(L_m)$ where $L_m/k$ are algebraic extensions. So $\mathbb{Q}^\text{log}_X = \oplus_m \mathbb{Q}_{\text{Spec}(L_m)}^\text{log}$.

Suppose we check the statement for all $Y$ with $\text{dim}(Y) < n$. Let $X^\nu$ be a normalization of $X$. The canonical morphism $\varphi : X^\nu \rightarrow X$ is birational. So we have an abstract blow-up...
Note that $Q_{log}^{gr}$ is a motivic sheaf so it admits $h$-descent. Moreover, the functor $\mathcal{M}_{X}^{gr} \to Q_{X}^{log}$ commutes with inverse images. Hence we get the exact triangle

$$R(i\phi)^{*}Q_{Z'}^{log} \to i_{*}Q_{Z}^{log} \oplus R\phi_{*}Q_{X}^{log} \to Q_{X}^{log}$$

where dim($Z'$) and dim($Z$) less than $n$. □

Note that for $X = \text{Spec}(k)$ the subcategory of constructible motivic sheaves is equivalent to the category of geometrical Voevodsky motives $DM_{gm}(k, \mathbb{Q})$. As a consequence we get

**Corollary 7.4.** For any log scheme $(X, \mathcal{M}^{gr})$ the motive $[X]^{log}$ belongs to $DM_{gm}(k, \mathbb{Q})$.

Proof: The six operations preserve constructability . □

**The structure of $E_{\infty}$-coalgebra on $[X]^{log}$**. Note that the construction of $Q_{X}^{log}$ is functorial so we have the functor

$$(7.4) \quad \text{vLog}/X^{\mathcal{M}^{gr} \to \Omega \text{gm} \mathcal{M}^{gr}} \to \text{DA}_{et}(X, \mathbb{Q})_{\mathbb{Q}/} \to \text{CAlg}^{*}(\text{DA}_{et}(X, \mathbb{Q})).$$

Now, let us consider $\text{vLog}/X$ as the monoidal category together with cocartesian monoidal structure and $\text{CAlg}(\text{DA}_{et}(X, \mathbb{Q}))$ as the monoidal category relatively to the tensor product of algebras.

**Proposition 7.5.** The functor $(7.4)$ is monoidal.

Proof: Note that the coproduct of two objects $(\mathbb{Q} \to E), (\mathbb{Q} \to F)$ of $\text{DA}_{et}(X, \mathbb{Q})_{\mathbb{Q}/}$ coincide with the colimit of the diagram

$$\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
F
\end{array} \quad \begin{array}{c}
\text{DA}_{et}(X, \mathbb{Q})_{\mathbb{Q}/} \\
\downarrow \\
\text{CAlg}(\text{DA}_{et}(X, \mathbb{Q})).
\end{array}$$

in $\text{DA}_{et}(X, \mathbb{Q})$. So let us equip $\text{DA}_{et}(X, \mathbb{Q})_{\mathbb{Q}/}$ with cocartesian monoidal structure. Then the first functor in the composition $(7.4)$ is monoidal. On the other hand, by Proposition 3.2.4.7. of [HA] the monoidal structure on $\text{CAlg}(\text{DA}_{et}(X, \mathbb{Q}))$ induced by $\otimes$ is cocartesian and $\text{Sym}^{*}$ commutes with colimits. So the second functor in the composition $(7.4)$ is also monoidal. □

Let us endow the category of virtual log schemes $\text{Sch}_{vlog}^{\text{vLog}}/k$ with Cartesian monoidal structure.

**Theorem 7.6.** The $\infty$-functor $[-]^{log} : \text{Sch}_{vlog}^{\text{vLog}}/k \to DM_{gm}(k, \mathbb{Q})$ is monoidal.
Proof: By Remark 4.5 for any morphism of schemes $f : X \to Y$ we have the adjoint pair $Rf_* : \text{CAlg}(\text{DA}_c(X, \mathbb{Q})) \rightleftarrows \text{CAlg}(\text{DA}_c(Y, \mathbb{Q})) : f^*$. Note that the anti-equivalence
$$\text{Hom}(-, \mathbb{Q}) : \text{DM}_{gm}(k, \mathbb{Q}) \to \text{DM}_{gm}(k, \mathbb{Q})$$
maps algebras to coalgebras. So we can define the new contravariant functor
\begin{equation}
(7.5)
\text{Sch}^{v\text{Log} / k} \to \text{coCAlg}(\text{DM}_{gm}(k, \mathbb{Q}))
\end{equation}
replaced in the diagram (7.3) $\text{DA}_{et}$ with $\text{CAlg}(\text{DA}_{et})$.

By Example 3.2.4.5 of [HA] the motivic tensor product $\otimes$ induces on $\text{coCAlg}(\text{DM}_{gm}(k, \mathbb{Q}))$ the structure of symmetric monoidal category. The functor $[-]^{\log}$ can be recovered as the composition of (7.5) with the forgetful functor $\text{coCAlg}(\text{DM}_{gm}(k, \mathbb{Q})) \to \text{DM}_{gm}(k, \mathbb{Q})$. So it suffice to check that (7.5) is monoidal.

Note that the monoidal structure on $\text{coCAlg}(\text{DM}_{gm}(k, \mathbb{Q})) = \text{Alg}(\text{DM}_{gm}(k, \mathbb{Q})^{op})^{op}$ is Cartesian by Proposition 3.2.4.7. of [HA]. So it remains to show that (7.5) commutes with products or, equivalently, check that the functor
$$X \mapsto ([X]^{\log})^*$$
maps a product of log schemes to the tensor product of algebras. Let $X, Y$ be virtual log schemes. Observe that the functors
$$\text{vLog} / X \overset{M_{gr}}{\longrightarrow} \text{Gr}_m \overset{M_{gr}}{\longrightarrow} \text{DA}_{et}(X, \mathbb{Q})_{Q/} \overset{\text{Sym}^*}{\longrightarrow} \text{CAlg}(\text{DA}_{et}(X, \mathbb{Q}))$$
is monoidal for each $X$ (Proposition 7.5) and commute with inverse image functors. So we have the isomorphism of algebras
$$Q_{X \times Y}^{\log} \cong Q_X^{\log} \otimes Q_Y^{\log}.$$

On the other hand, Cisinski proved the Künneth formula for motivic sheaves (see [C21]). So we have
\begin{equation}
(7.6)
Rf_*(Q_X^{\log} \otimes Q_Y^{\log}) \cong R\alpha_X^* Q_X^{\log} \otimes R\alpha_Y^* Q_Y^{\log}
\end{equation}
where $f, \alpha_X, \alpha_Y$ are canonical morphisms to Spec$(k)$. It suffice to show that (7.6) is an isomorphism of $\mathbb{E}_{\infty}$-algebras.

Lemma 7.7. Let $A$ and $B$ be $\mathbb{E}_{\infty}$-algebras in $\text{DA}_{et}(X \times Y, \mathbb{Q})$. Then the canonical map
$$\psi : Rf_* A \otimes Rf_* B \to Rf_* (A \otimes B)$$
can be extended to the homomorphism of $\mathbb{E}_{\infty}$-algebras.

Proof: Note that $\psi$ is adjoint to $\varepsilon_A \otimes \varepsilon_B \in \text{Hom}(f^* Rf_* A \otimes f^* Rf_* B, A \otimes B)$ and apply the Remark 4.5.
Using Remark 4.5 and the Lemma we can define the homomorphism of algebras \( \mu \) as the composition:

\[
\begin{align*}
\rho_{\alpha X}^* (R\pi_{\alpha X}^* \pi_X^* \mathbb{Q}_{\log}^X) \otimes \rho_{\alpha Y}^* (R\pi_{\alpha Y}^* \pi_Y^* \mathbb{Q}_{\log}^Y) & \xrightarrow{\sim} Rf_* \pi_X^* \mathbb{Q}_{\log}^X \otimes Rf_* \pi_Y^* \mathbb{Q}_{\log}^Y \\
\rho_{\alpha X}^* (\eta) \otimes \rho_{\alpha Y}^* (\eta) & \xrightarrow{\psi} Rf_* (\mathbb{Q}_{\log}^X \boxtimes \mathbb{Q}_{\log}^Y).
\end{align*}
\]

Finally, comparing the diagram 7.7 with the construction of the isomorphism 7.6 (see [C21]) we conclude that \( \mu \) is an isomorphism of algebras. □

8. Betti realization of \([X]^{\log}\)

Let \( k \) be a field of characteristic zero together with the embedding \( \sigma : k \hookrightarrow \mathbb{C} \). Let us fix the ring of coefficients \( R \). In the paper [Ayo10], Ayoub constructed for any \( X \) the Betti realization of motivic sheaves

\[
\text{Betti}_X : \text{DA}_{et}(X, R) \to \text{D}(\text{Sh}(X^{an}, R))
\]

Let us fix \( X \). The construction of \( \text{Betti}_X \) is the following. Let us consider the diagram

\[
\begin{align*}
\text{D}(\text{Sh}_{et}(\text{Sm}/X, R)) & \xrightarrow{\text{An}^*} \text{D}(\text{Sh}_{top}(\text{Sm}/X^{an}, R)) \\
\text{DA}_{et}^{\text{eff}}(X, R) & \xrightarrow{\text{An}^*} \text{DA}_{et}^{\text{eff}}(X, R) \quad \xrightarrow{\text{Res}_{X^an}^X \cdot i} \quad \text{D}(\text{Sh}(X^{an}, R)) \\
\text{DA}_{et}(X, R) & \xrightarrow{\text{An}^*} \text{DA}_{an}(X, R)
\end{align*}
\]

Here we denote by \( \text{An}^* \) the Kan extensions of the analytification functor

\[
\text{Sm}/X \to \text{Sm}/X^{an}, \ X \mapsto (X \times_k \mathbb{C})^{an}.
\]

By results of [Ayo10] the functors \( \Sigma^\infty_{D} \) and \( \text{Res}_{X^an}^X \cdot i \) are equivalences of categories. So we can define \( \text{Betti}_X \) as the composition \( \text{Res}_{X^an}^X \cdot i \cdot \Omega^\infty_{D} \cdot \text{An}^* \).

Let us denote by \( \text{DA}_{et,c}(X, R) \) the subcategory of compact objects. The main result of [Ayo10] is the following.

**Theorem 8.1. (Theorem 3.19 of [Ayo10]).** The restriction of \{\text{Betti}_X\} on the subcategory of constructible objects

\[
\text{Betti} : \text{DA}_{et,c} \to \text{D}(\text{Sh}(-^{an}, R))
\]

preserves six operations.

**Remark 8.2.** The original theorem consists the categories \( \text{SH}_{2R,c}(X) \) rather than \( \text{DA}_{et,c}(X, R) \).

By construction \( \text{SH}_{2R,c}(X) = \lim_{\text{NC}} \text{H}_0_{\text{A}, 1 \text{-Nis}}(\text{PSh}(\text{Sm}/X, \mathfrak{M}) \). One can put \( \mathfrak{M} = R - \text{Mod} \).
On the other hand, the proof of the theorem in the étale case tautologically coincides with the case of Nisnevich topology.

**The Betti realization of** $Q^\log_X$ **and** $[X]^\log$. Let $R = \mathbb{Z}$. For a virtual log structure $\alpha : \mathcal{O}^* \to \mathcal{M}^\text{gr}$ on $X$ we will denote by $\mathcal{M}^\text{gr}_Z$ the motivic sheaf $\Sigma_\infty L_\eta^! E^{\log}_X(\mathcal{M}^\text{gr})$. Note that $\mathcal{M}^\text{gr} = \mathcal{M}^\text{gr}_Z \otimes \mathbb{Q}$.

Let $\alpha : \mathcal{O}^*_\text{an} \to \mathcal{M}^\text{gr}_\text{an}$ be the analytification. Recall that we denoted by $\exp(\alpha)$ the complex

$$
\ldots \to 0 \to \mathcal{O}_{\text{an}} \to \mathcal{M}^\text{gr}_{\text{an}} \to 0 \to \ldots
$$

where the differential given by the composition $\mathcal{O}_{\text{an}} \xrightarrow{\exp} \mathcal{O}^*_\text{an} \hookrightarrow \mathcal{M}^\text{gr}_{\text{an}}$.

**Theorem 8.3.** There is the natural quasi-isomorphism $\text{Betti}_X(\mathcal{M}^\text{gr}_Z)[-1] \simeq \exp(\alpha)$.

Let $\text{An}^* : \text{Sh}_{\text{et}}(\text{Sm}/X) \to \text{Sh}_{\text{top}}(\text{Sm}/X^\text{an})$ be the Kan extension of the analytification functor. For any $W \in \text{Sm}/X^\text{an}$ we have the map

$$
\text{colim}_{W \xrightarrow{g} Y^\text{an}} \mathcal{O}^*(Y) \to \mathcal{O}^*_\text{an} ;
$$

$$(f \in \mathcal{O}^*(Y)) \mapsto g^* f.
$$

We define the map

$$
\psi : \text{An}^*(\mathcal{O}^*_\text{Sm}/X) \to \mathcal{O}^*_\text{Sm}/X^\text{an}
$$

as the sheafification of (8.2).

**Lemma 8.4.** The map $\psi$ is an isomorphism.

*Proof:* The analytification functor commutes with fiber product. Consequently, $\text{An}^*$ is exact. In Appendix A for any abelian algebraic group $G$, we defined the complex of sheaves $Q_{\geq -1}(G)$ with $0(Q_{\geq -1}(G)) = G$. Note that $\psi$ can be extended to the map of complexes

$$
Q_{\geq -1}(G_m) \to Q_{\geq -1}(G^\text{an}_m)
$$

which induced by the map

$$
\text{colim}_{W \xrightarrow{f} Y^\text{an}} Z[G_m](Y) \xrightarrow{\psi'} Z[G^\text{an}_m](W) ;
$$

$$
\Sigma_i a_i [f_i] \mapsto \Sigma_i a_i [g^* f_i].
$$

Note that $\text{An}^*$ is exact and consequently, commutes with cohomology. So it suffice to check that $\psi'$ is an isomorphism. But $\psi'$ is the canonical isomorphism between $\text{An}^*(h_{G_m})$ and $h_{G^\text{an}_m}$ which can be define by the same way for any Kan extension and any presentable object. □
Let us consider the small analytic and the smooth analytic sites of $X^{\text{an}}$. As in the case of the small and the smooth étale sites there exists the pair of adjoint functors

$$\text{Ex}_{X}^{\text{an}} : \text{Sh}(X^{\text{an}}, R) \rightleftarrows \text{Sh}_{\text{top}}(\text{Sm}/X^{\text{an}}, R) : \text{Res}_{X}^{\text{an}}.$$ 

So we can define the functor

$$\text{Ex}_{\text{an}}^{\log} : \text{vLog}/X \rightarrow \text{Sh}_{\text{top}}(\text{Sm}/X^{\text{an}}, \mathbb{Z})$$

in the same way as in Section 6. Namely, we define $\text{Ex}_{\text{an}}^{\log} \mathcal{M}_{\text{an}}^{\text{gr}}$ as the colimit of the diagram

$$\begin{array}{ccc}
\text{Ex}_{X}^{\text{an}} \mathcal{O}_{X}^{*} & \longrightarrow & \text{Ex}_{X}^{\text{an}} \mathcal{M}_{X}^{\text{gr}} \\
\downarrow & & \downarrow \\
\mathcal{O}_{X}^{*} & \rightarrow & \mathcal{O}_{\text{Sm}/X^{an}}^{*}.
\end{array}$$

Now, let us prove that the map \textbf{8.3} induces the canonical isomorphism

$$\phi : \text{An}^{*}(\text{Ex}_{\text{an}}^{\log} \mathcal{M}_{\text{an}}^{\text{gr}}) \rightarrow \text{Ex}_{\text{an}}^{\log} \mathcal{M}_{\text{an}}^{\text{gr}}.$$ 

Indeed, by construction of $\text{Ex}_{\text{an}}^{\log}$ there is the pushout

$$\begin{array}{ccc}
\text{An}^{*} \text{Ex}_{X}(\mathcal{O}^{*}) & \longrightarrow & \text{An}^{*} \text{Ex}_{X}(\mathcal{M}^{\text{gr}}) \\
\downarrow & & \downarrow \\
\text{An}^{*} \mathcal{O}_{\text{Sm}/X}^{*} & \longrightarrow & \text{An}^{*} \text{Ex}_{\text{an}}^{\log} \mathcal{M}_{\text{an}}^{\text{gr}}
\end{array}$$

On the other hand, let us consider the diagram

$$\begin{array}{ccc}
\text{Ex}_{X}^{\text{an}} \text{An}^{*}(\mathcal{O}^{*}) & \longrightarrow & \text{Ex}_{X}^{\text{an}} \text{An}^{*}(\mathcal{M}^{\text{gr}}) \\
\downarrow & & \downarrow \\
\text{Ex}_{X}^{\text{an}} \mathcal{O}_{X}^{*} & \longrightarrow & \text{Ex}_{X}^{\text{an}} \mathcal{M}_{X}^{\text{gr}} \\
\downarrow & & \downarrow \\
\mathcal{O}_{\text{Sm}/X^{an}}^{*} & \rightarrow & \text{Ex}_{\text{an}}^{\log} \mathcal{M}_{\text{an}}^{\text{gr}}
\end{array}$$

Here the top square is cocartesian by construction of $\mathcal{M}_{\text{an}}^{\text{gr}}$ and the bottom square is cocartesian by construction of $\text{Ex}_{\text{an}}^{\log}$. So the large square is also cocartesian and the existence of $\phi$ follows from the Lemma.

\textbf{Lemma 8.5.} The morphisms

\textbf{(8.6)} \quad \text{An}^{*} \text{Ex}_{X}(\mathcal{O}_{\text{small}}^{*}) \rightarrow \text{An}^{*}(\mathcal{O}_{\text{Sm}/X}^{*})
and

\[ \text{Ex}_X^\text{an} \text{An}^*(\mathcal{O}_{\text{small}}^*) \to \text{Ex}_X^\text{an} (\mathcal{O}_{\text{an,small}}^*) \to \mathcal{O}_{\text{Sm}/X}^* \]

are the same up to the canonical isomorphism \( \text{8.3} \).

Proof: Note that \( \text{Ex}_X \), \( \text{Ex}_X^\text{an} \) and \( \text{An}^* \) commute with sheafification. So it suffice to check the statement on the level of preshaves.

Let \( W \) belongs to \( \text{Sm}/X^\text{an} \). Then the map \( \text{8.6} \) induces by the composition

\[
\text{colim}_{W \to U \times X} (\text{colim}_{U \to V^\text{an}} \mathcal{O}^*(V \xrightarrow{\text{et}} X)) \to \text{colim}_{W \to U \times X} \mathcal{O}_{\text{an}}^*(U) \to \mathcal{O}_{\text{an}}^*(W)
\]

where the both maps induced by pullbacks of functions. Note that \( W \) is smooth over \( X \). So we can drop the external colimit and assume that \( U = \text{im}(W) \):

\[
\text{colim}_{\text{im}(W) \to V^\text{an}} \mathcal{O}^*(V \xrightarrow{\text{et}} X) \to \mathcal{O}_{\text{an}}^*(\text{im}(W)) \to \mathcal{O}_{\text{an}}^*(W).
\]

On the other hand, the arrow \( \text{8.7} \) is the sheafification of the map

\[
\text{colim}_{W \to Y^\text{an}} \mathcal{O}^*(V \xrightarrow{\text{et}} X) \to \text{colim}_{W \to Y^\text{an}} \mathcal{O}^*(Y \xrightarrow{\text{sm}} X))
\]

where the internal colimit taken by the collection of the diagrams

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{id}} & Y \\
\downarrow & & \downarrow \\
Y & & \\
\end{array}
\]

Note that this collection coincides with

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{et}} & V \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]

So we can rewrite the arrow \( \text{8.7} \) as the map

\[
\text{colim}_{W \to Y^\text{an}} \mathcal{O}^*(V \xrightarrow{\text{et}} X) \to \text{colim}_{W \to Y^\text{an}} \mathcal{O}^*(Y \xrightarrow{\text{sm}} X).
\]

Finally, let us consider the diagram

\[
\begin{array}{ccc}
\text{colim}_{W \to Y^\text{an}} \mathcal{O}^*(V \xrightarrow{\text{et}} X) & \xrightarrow{\text{id}} & \text{colim}_{W \to Y^\text{an}} \mathcal{O}^*(Y \xrightarrow{\text{sm}} X) \\
\downarrow & & \downarrow \\
\text{colim}_{\text{im}(W) \to V^\text{an}} \mathcal{O}^*(V \xrightarrow{\text{et}} X) & \to & \mathcal{O}_{\text{an}}^*(\text{im}(W)) \to \mathcal{O}_{\text{an}}^*(W)
\end{array}
\]

where the upper horizontal arrow is \( \text{8.6} \), the lower horizontal arrows is \( \text{8.7} \) and the right vertical arrow is \( \text{8.2} \). Now, one can check that the diagram is commutative. □
Let $X$ and $\alpha$ be as above. Let us define the complex $\mathcal{E}xp(\alpha) \in \text{Ch}^\bullet(\text{Sh}_{\text{top}}(\text{Sm}/X^\text{an}))$:

$$
\cdots \longrightarrow 0 \longrightarrow \mathcal{O}_{\text{Sm}/X^\text{an}} \xrightarrow{\exp} \mathcal{E}xp^\log \mathcal{M}_{\text{an}} \longrightarrow 0 \longrightarrow \cdots
$$

Let us define the analytic motivic sheaf $[\mathcal{E}xp(\alpha)]$ using the functors $L_D$ and $\Sigma_\infty^\circ$.

Note that the functor $\text{Res} \cdot i \cdot \Omega^\infty_D : \text{DA}_{\text{an}}(X, \mathbb{Z}) \longrightarrow \text{D}(\text{Sh}(X^\text{an}, \mathbb{Z}))$ maps $[\mathcal{E}xp(\alpha)]$ to $\exp(\alpha)$. Indeed, by previous there exists the canonical map

$$
\text{Ex}^\text{an}(\exp(\alpha)) \longrightarrow \mathcal{E}xp(\alpha)
$$

which is a quasi-isomorphism. But the functor $\Sigma_\infty^\circ \cdot L_D \cdot \text{Ex}^\text{an}$ is adjoint to $\text{Res} \cdot i \cdot \Omega^\infty_D$ and hence inverse.

**Proof of Theorem 8.3** Observe that the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_{\text{Sm}/X^\text{an}} \\
\downarrow & & \downarrow \exp \\
\text{An}^*(\text{Ex}^\log \mathcal{M}^\text{gr}) & \xrightarrow{\phi} & \text{Ex}^\log \mathcal{M}^\text{gr}_{\text{an}}
\end{array}
$$

define the map of complexes

$$
\text{An}^*(\text{Ex}^\log \mathcal{M}^\text{gr})[-1] \xrightarrow{\phi'} \mathcal{E}xp(\alpha).
$$

Note that $\text{An}^*$ commutes with $L_-$ and $\Sigma_\infty^\circ$. So it suffice to show that $\phi'$ is an $\mathbb{A}^1$-equivalence or equivalently check that

$$
L_D(\text{Cone}(\phi')) \simeq L_D(\mathcal{O}_{\text{Sm}/X^\text{an}}) = 0.
$$

Indeed, let us consider the commutative diagram
where $m$ is the multiplication. This diagram defines the $\mathbb{A}^1$-homotopy between $\text{id}_O$ and the zero endomorphism. □

Now, let $X$ and $\alpha$ be as above, $X^{\log}$ be the corresponding Kato-Nakayama space and $\pi : X^{\log} \rightarrow X^{an}$ be the canonical map.

**Theorem 8.6.** There is the canonical quasi-isomorphisms

$$\text{Betti}_X(Q_X^{\log}) \simeq R\pi_*Q; \quad [X]^{\log} \simeq \text{Sing}_*(X^{\log}, Q).$$

*Proof:* Each $Q_X^{\log}$ is constructible. So the second quasi-isomorphism can be derived from the first using the six operations. Note that the functor $\text{Betti}_X$ is monoidal. So we have

$$\text{Betti}_X Q_X^{\log} \simeq \text{Betti}_X S^n(\Omega_{G_m} M^\text{gr}) \simeq S^n(\text{Betti}_X \Omega_{G_m} M^\text{gr}) \simeq S^n(\text{Betti}_X M^\text{gr}[-1])$$

and

$$S^n(\text{Betti}_X M^\text{gr}[-1]) \simeq S^n(\text{Betti}_X (M^\text{gr}[-1] \otimes Q) \simeq S^n(\exp(\alpha) \otimes Q).$$

Hence $\text{Sym}^*(M^\text{gr}(-1)[-1]) \simeq \text{Sym}^*(\exp(\alpha) \otimes Q)$ by Theorem 4.6. It remains to use Theorem 5.2 □

9. **Logarithmic $h$-descent**

In analogy to Kato’s logarithmic geometry, we will say that a morphism of virtual log schemes $(f, f^\#)$ is strict if $f^\#$ is an isomorphism.

**Definition 9.1.** We will say that the morphism $f : Y \rightarrow X$ of virtual log schemes is a *strict $h$-cover* if the underlying morphism $Y \rightarrow X$ is a $h$-cover and $f$ is strict. Let $\mathcal{F}$ be a presheaf of complexes on $\text{Sch}_{\text{vLog}}/k$. We will say that $\mathcal{F}$ *admits strict $h$-descent* if for any strict $h$-cover $f : Y \rightarrow X$ the canonical map of complexes

$$\mathcal{F}(X) \rightarrow \text{Tot}^\bullet(\mathcal{F}(Y) \rightarrow \mathcal{F}(Y \times_X Y) \rightarrow \mathcal{F}(Y \times_X Y \times_X Y) \rightarrow ...)$$

is a quasi-isomorphism.

**Definition 9.2.** A *logarithmic abstract blow-up* is a Cartesian square

\[
\begin{array}{ccc}
Z' & \xrightarrow{i'} & X' \\
\downarrow{p_Z} & & \downarrow{p} \\
Z & \xrightarrow{i} & X
\end{array}
\]

in $\text{Sch}_{\text{vLog}}/k$ such that the maps $i$ and $i'$ are strict, the map $p|_{X' \setminus Z'} : X' \setminus Z' \rightarrow X \setminus Z$ is an isomorphism in $\text{Sch}_{\text{vLog}}/k$ and the underlying square of ordinary schemes is an abstract blow-up square.
Let us suppose that all morphism in square (4.1) are strict. In this case we will call such a square the strict log abstract blow-up. In the case then underlying maps \( p : X' \to X \) and \( p_z : Z' \to Z \) are identity maps we will say that the square (4.1) is geometrically trivial.

**Remark 9.3.** Notice that any morphism of virtual log schemes \((f, f^#) : (X, M^r_X) \to (Y, M^r_Y)\) can be decomposed as \((X, M^r_X) \xrightarrow{(\text{id}, f^#)} (X, f^* M^r_Y) \xrightarrow{(f, \text{id})} (Y, M^r_Y)\). So the log abstract blow-up (4.1) can be decomposed into strict and geometrically trivial parts:

\[
\begin{array}{ccc}
(Z', M^r_{Z'}) & \xrightarrow{(\text{id}, p^#_Z)} & (Z', p^*_{Z'} M^r_{Z'}) \\
\downarrow & & \downarrow \\
(Z, M^r_Z) & \xrightarrow{(\text{id}, \text{id})} & (Z, M^r_Z)
\end{array}
\]

\[
\begin{array}{ccc}
(X', M^r_{X'}) & \xrightarrow{(\text{id}, p^#)} & (X', p^* M^r_X) \\
\downarrow & & \downarrow \\
(X, M^r_X) & \xrightarrow{(\text{id}, \text{id})} & (X, M^r_X)
\end{array}
\]

**Example 9.4.** Let us denote by \( A^n_{\log} \) the virtualization of canonical log structure on \( \text{Spec}(k[\mathbb{N}^n]) \). Notice that \( A^n_{\log} = A^1_{\log} \times \cdots \times A^1_{\log} \) and \( M^r_{A^1_{\log}} = j_* \mathcal{O}_{\mathbb{G}_m}, \ \overline{M}^r_{A^1_{\log}} = i_* \mathbb{Z} \) where \( pt \xrightarrow{i} A^1 \xrightarrow{j} \mathbb{G}_m \). Let \( pt_{\log} \) be a virtualization of a standard log point. Notice that \( M^r_{pt_{\log}} = k^* \oplus \mathbb{Z} \) and the virtualization of an inclusion \( pt_{\log} \xrightarrow{i} A^1_{\log} \) is the strict morphism.

Contrasting with ordinary logarithmic geometry the morphism \( pt_{\log} \xrightarrow{i} A^1_{\log} \) is a section of the projection \( A^1_{\log} \xrightarrow{p} pt_{\log} \) which arises from the generator of the group \( H^0_{et}(\overline{M}^r_{A^1_{\log}}) = \mathbb{Z} \).

It is easy to see that for any \( X \in \text{Sch}^{v\log}_{/k} \) the square

\[
\begin{array}{ccc}
X \times pt_{\log} & \xrightarrow{i} & X \times A^1_{\log} \\
\downarrow & & \downarrow \\
X & \xrightarrow{i} & X \times A^1
\end{array}
\]

is a geometrically trivial log abstract blow-up.

**Definition 9.5.** Let \( F \) be a presheaf of complexes on \( \text{Sch}^{v\log}_{/k} \). We will say that \( F \) admits logarithmic h-descent if \( F \) admits strict h-descent and for any log abstract blow-up of the form (9.1) there is the exact triangle of complexes

\[
F(X) \to F(X') \oplus F(Z) \to F(Z').
\]

The main result of this section is the following:

**Theorem 9.6.** Let \( F \) and \( F' \) are the presheaves of complexes on \( \text{Sch}^{v\log}_{/k} \) and \( \varphi : F \to F' \) be a map. Suppose that both \( F \) and \( F' \) admit logarithmic h-descent and for any \( X \times pt_{\log} \) with \( X \in \text{Sch}_{/k} \) the map

\[
\varphi_{X \times pt_{\log}} : F(X \times pt_{\log}) \to F'(X \times pt_{\log})
\]

is a quasi-isomorphism. Then \( \varphi_Y \) is a quasi-isomorphism for any virtual \( fs \) logarithmic scheme \( Y \).

**Proof:** Notice that by additivity we may assume that \( F' = 0 \). Let us denote by \( \mathcal{C} \) the full subcategory of \( \text{Sch}^{v \Log}_{fs}/k \) generated by all virtual log schemes of the form \( X \times pt^n_{\log} \). The proof of the theorem contain several steps. At the \( n \)-th step we define the category \( \mathcal{C}(n) \) such that \( \mathcal{C} \subset \mathcal{C}(n) \subset \mathcal{C}(n-1) \) and check that acyclicity of \( F(X) \) for any \( X \in \mathcal{C}(n) \) entails acyclicity of \( F \).

**Step 1:** For an ordinary scheme \( X \) let us fix open immersion \( j : U \hookrightarrow X \), local system \( \Lambda \) on \( U \) and an element \( \xi \) of \( \text{Ext}^1(j_#!\Lambda, \mathcal{O}_X^*) \). We will denote by \( j_#X_\xi \) the virtual log scheme corresponding to \( \xi \). Let \( \mathcal{C}(1) \) be the category containing all \( j_#X_\xi \). Now, let us suppose that \( F|_{\mathcal{C}(1)} \) is acyclic. Then \( F(Y) \) is acyclic for any \( Y \in \text{Sch}^{v \Log}_{fs}/k \). Indeed, we can check that statement using induction by dimension of \( Y \). Let \( \text{dim}(Y) = 0 \). Then any constructible sheaf on \( Y \) is a local system and \( F(Y) \cong 0 \). Now, suppose that for all \( Y \in \text{Sch}^{v \Log}_{fs}/k \) with \( \text{dim}(Y) < n \) we have \( F(Y) \cong 0 \). Let \( X \) be an \( fs \) virtual log scheme with \( \text{dim}(X) = n \). Recall that the ghost sheaf \( M_{gr}^p_X \) is constructible. Let \( ... \subset Z_2 \subset Z_1 \subset X \) be the corresponding stratification and let \( U = X \setminus Z_1 \). So \( \Lambda = \overline{M_{gr}^p_X}|_U \) is a local system and we can define the log scheme \( j_#X_\xi \) by the Cartesian diagram

\[
\begin{array}{ccc}
M^p_{j_#X_\xi} & \rightarrow & j_#\Lambda \\
\downarrow & & \downarrow \\
M^p_X & \rightarrow & M^p_X
\end{array}
\]

By definition we have the log abstract blow-up

\[
\begin{array}{ccc}
Z_1 & \rightarrow & X \\
\downarrow & & \downarrow \\
Z_1 & \rightarrow & j_#X_\xi
\end{array}
\]

But \( F(Z_1), F(Z_1) \) and \( F(j_#X) \) are acyclic so \( F(X) \cong 0 \).

**Step 2:** Now, let \( \mathcal{C}(2) \) be the category generated by all \( j_#Y_\xi \) with constant \( \Lambda \) and \( \xi = 0 \). We should check that if \( F|_{\mathcal{C}(2)} \) is acyclic then \( F \) is acyclic. By previous it suffice to show that \( F|_{\mathcal{C}(1)} \) is acyclic.

Let \( j_#X_\xi \) be a log scheme corresponding to some \( \Lambda \) and \( \xi \). Using strict \( h \)-descent we may assume that \( X \) is normal. So, by definition, for any geometric point \( \pi \) the fiber \( \Lambda_{\pi} \) is a finitely generated abelian group. The action of \( \pi^e_1(U, \pi) \) is continuous so the stabilizer of each element is an open subgroup. Hence the kernel \( \pi^e_1(U, \pi) \rightarrow \text{Aut}(\Lambda_{\pi}) \) is open and the image is a finite group. So there exist the Galois cover \( \overline{U} \rightarrow U \) such that \( \pi^*\Lambda \) is constant.
Now, let $X' \xrightarrow{\nu} X$ be the normalization of $\tilde{U}$ in $X$. Observe that $\nu$ is a $h$-cover. Indeed, by Lemma 29.53.15 of \textit{Stacks} $\nu$ is finite and $\text{Im}(\nu) \supset \text{Im}(\tilde{U}) = U$. So $\nu$ is a finite surjective morphism. Notice that a normalization commute with smooth base change (see Lemma 37.17.2 of \textit{Stacks}). Moreover, the normalization of $\tilde{U}$ in $U$ coincide with $U$ because $\pi$ is finite. So $X' \times_X U \simeq \tilde{U}$ and $v^{-1}(j_\# \Lambda) \simeq \tilde{j}_\#(\pi^{-1} \Lambda) = \tilde{j}_\#(\mathbb{Z}^n)$. Here $\tilde{j}$ is an open inclusion of $\tilde{U}$ into $X'$.

The log structure $v^* \mathcal{M}_{j_\# X}$ corresponding to some element $\xi$ of

$$\text{Ext}^1(\tilde{j}_\# \mathbb{Z}^n, \mathcal{O}_{X'}) \simeq \text{Ext}^1(\mathbb{Z}^n, \mathcal{O}_{\tilde{U}}) = H^1_{\text{Zar}}(\tilde{U}, \mathcal{O}^*)$$

So there exists a Zariski cover $\{W_i \rightarrow X'\}$ such that $\xi|_{W_i} = 0$. Now, let us compose the normalization with $h$-cover $X' \xrightarrow{\nu} X$ and the Zariski cover $\{W_i \rightarrow X'\}$. We get the $h$-cover $Y \rightarrow X$ such that for any product $Y \times_X \ldots \times_X Y$ the pullback of the log structure has the form $j_\# \xi$ with constant $\Lambda$ and $\xi = 0$. So using strict $h$-descent we get $F(j_\# X, \xi) = 0$.

**Step 3:** Let us prove the main statement. For all $j_\# X, \xi$ with constant $\Lambda = \mathbb{Z}^n$ and $\xi = 0$ there exist the log abstract blow-up

$$\mathbb{Z}_1 \times pt^n \log \longrightarrow X \times pt^n \log$$

So we can use induction by dimension of $X$. □

### 10. Logarithmic motivic cohomology

Let us use the notation of section 6. Let $R$ be a commutative ring.

**Definition 10.1.** For $q = 0, 1$ we define the complexes $R\Gamma(X, R^{log}(q))$ by the formulas

$$R\Gamma(X, R^{log}(q)) := \text{Hom}^\bullet_{\text{DAet}(X, R)}([X], [X]); \quad R\Gamma(X, R^{log}(1)) := \text{Hom}^\bullet_{\text{DAet}(X, R)}([X], \mathcal{M}_{X}^{gr}).$$

and the logarithmic motivic cohomology as

$$H^{p,q}_{log}(X, R) := H^p(R\Gamma(X, R^{log}(q))).$$

Now, let us assume that $R = \mathbb{Q}$. In this case we can define $H^{p,q}_{log}$ for $q > 1$ using motivic symmetric powers.

**Remark 10.2.** (motivic symmetric powers) Let $S$ be a scheme. For $n > 1$ and $F \in \text{DAet}(S, \mathbb{Q})$ let us consider the endomorphism $\varphi := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$ of $F^{\otimes n}$. Notice that $\varphi^2 = \varphi$ and, consequently, the motivic sheaf $F^{\otimes n}$ can be canonically factorized as the direct sum

$$F^{\otimes n} \simeq \text{im}(\varphi) \oplus \ker(\varphi).$$
The direct term $\text{im}(\varphi)$ is usually denoted by $S^n F$ and called the n-th symmetric power of $F$ (see [Maz10] for details). The factorization [10.1] gives rise to the endofunctor

$$S^n : \text{DA}_{et}(S, \mathbb{Q}) \to \text{DA}_{et}(S, \mathbb{Q})$$

of triangulated categories. Notice that this functor is not triangulated.

On the other hand, the n-th symmetric power can be considered as the DG-functor. Indeed, we can define the map of complexes $\text{Hom}^\bullet(E, F) \to \text{Hom}^\bullet(S^n E, S^n F)$ as the composition of the map $\text{Hom}^\bullet(E, F) \to \text{Hom}^\bullet(E \otimes^n, F \otimes^n)$ with the canonical projection $\text{Hom}^\bullet(E \otimes^n, F \otimes^n) \to \text{Hom}^\bullet(S^n E, S^n F)$. It is easy to see that such a collection of maps preserve compositions and associativity.

We can also define the motivic exterior powers $\Lambda^n$ replace $\varphi$ with $\frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{|\sigma|} \sigma$. Note that $S^n(F[-1]) \simeq \Lambda^n F[-n]$ for any $F$ and $n$.

**Definition 10.3.** Using motivic symmetric powers we put

$$R\Gamma(X, Q^{\log}(q)) \overset{\text{def}}{=} \text{Hom}^\bullet_{DA_{et}(X, \mathbb{Q})}(\{X\}, S^q(M^\text{gr}_{X)[-1])).$$

and

$$H^{p,q}_{\text{log}}(X, \mathbb{Q}) \overset{\text{def}}{=} H^p(R\Gamma(X, Q^{\log}(q))).$$

Further, for simplicity, we will use the sheaves $\Lambda^q M^\text{gr}_X$ instead $S^q(M^\text{gr}_{X}[-1])$.

**Functoriality.** Let us fix $q$. Let $(f, f^\#) : (X, \mathcal{M}^\text{gr}_X) \to (Y, \mathcal{M}^\text{gr}_Y)$ be the morphism of virtual log schemes. Now, we explain how to construct the map

$$R\Gamma(f, f^\#) : R\Gamma(Y, Q^{\log}(q)) \to R\Gamma(X, Q^{\log}(q)).$$

Firstly, note that (6.3) maps $f^* M^\text{gr}_Y$ to $f^* M^\text{gr}_Y$ by Corollary 6.3 and Remark 6.4. Hence $f^\#$ maps to $\Lambda^q f^* M^\text{gr}_Y \to \Lambda^q M^\text{gr}_X$.

On the other hand, $f^*$ is monoidal so $\Lambda^q f^* M^\text{gr}_Y \simeq f^* \Lambda^q M^\text{gr}_Y$ and there exists the map $\Lambda^q M^\text{gr}_Y \to Rf_* \Lambda^q M^\text{gr}_X$.

It remains to apply the functor $\text{Hom}^\bullet_{DA_{et}(Y, \mathbb{Q})}(\{Y\}, -)$ and use the natural quasi-isomorphism

$$\text{Hom}^\bullet_{DA_{et}(Y, \mathbb{Q})}(M(Y), Rf_* \Lambda^q M^\text{gr}_X) \simeq \text{Hom}^\bullet_{DA_{et}(X, \mathbb{Q})}(f^* M(Y), \Lambda^q M^\text{gr}_X).$$

Note that we construct the set of functors

$$[R\Gamma(-, Q^{\log}(q))] : \text{Sch}^{\text{vLog}}/k \to \text{D}(\text{Vect}_Q);$$

$$H^{p,q}_{\text{log}} : \text{Sch}^{\text{vLog}}/k \to \text{Vect}_Q.$$
Moreover, after choosing a suitable model structure, (10.4) can be lifted to the presheaf of complexes

\[ R\Gamma(-, Q^{\log}(q)) : \text{Sch}^{v\log}_k \to \text{Ch}^\bullet(\text{Vect}_Q). \]

Indeed, it is equivalent to show that (10.4) can be lifted to a \((\infty, 1)\)-functor \(\text{Sch}^{v\log}_k \to \text{Ch}^\bullet(\text{Vect}_Q)\). By analogy with the construction of \([-]^{\log}\), it can be done using the following diagram

\[
\begin{array}{ccc}
\int_{\text{Sch}^{v\log}_k} \text{DA}_{et, Q}^{op} & \xrightarrow{\varphi} & \int_{\text{Sch}^{v\log}_k} \text{DA}_{et, Q}^{op} \\
\downarrow f^{Aq}\Psi & & \downarrow f^{RT} \\
\text{Sch}^{v\log}_k & \xrightarrow{R\Gamma(-, Q^{\log}(q))} & \text{D}(\text{Vect}_Q)
\end{array}
\]

Again, here \(\int\) mean the \((\infty, 1)\)-Grothendieck construction (see Appendix B) and \(\varphi\) is the equivalence of categories from Proposition B.6.

\[ A^1\text{-invariance and log h-descent.} \]

Note that the presheaves \(R\Gamma(-, Q^{\log}(q))\) is \(A^1\)-invariant by construction. It should also be noted that \(R\Gamma(-, Q^{\log}(q))\) admit log h-descent. Indeed, the presheaves \(R\Gamma(-, Q^{\log}(q))\) admit strict h−descent iff for any \(X \in \text{Sch}^{v\log}_k\) the \(G_m\)-spectrum \(A^q\text{M}_{gr}^X\) admits h−descent. But any \(G_m\)-spectrum \(E \in \text{DA}_{et}(X, Q)\) admits h−descent by results of [CD19]. Let

\[
\begin{array}{ccc}
Z' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X
\end{array}
\]

be a log abstract blow-up square. We should check that the sequence

\[
R\Gamma(X, Q^{\log}(q)) \to R\Gamma(X', Q^{\log}(q)) \oplus R\Gamma(Z, Q^{\log}(q)) \to R\Gamma(Z', Q^{\log}(q))
\]

is an exact triangle. The presheaf \(R\Gamma(-, Q^{\log}(q))\) admits strict h−descent so we may assume that the square (10.5) is geometrically trivial (i.e \(X' = X\) and \(Z' = Z\)). Let us denote by \(i\) the inclusion \(Z \hookrightarrow X\) and let \(j : U \hookrightarrow X\) be the inclusion of the complement. It suffice to show that the sequence

\[
A^q\text{M}_{gr}^X \to A^q\text{M}_{gr}^{X'} \oplus i_* A^q\text{M}_{gr}^Z \to i_* A^q\text{M}_{gr}^{Z'}
\]
is an exact triangle. Applying the localization sequence we get the map of exact triangle
\[(10.7)\]
\[
\begin{array}{ccc}
j_\# \Lambda^q M^\text{gr}_U & \longrightarrow & \Lambda^q M^\text{gr}_X \\
\downarrow & & \downarrow \\
j_\# \Lambda^q M^\text{gr}_U' & \longrightarrow & \Lambda^q M^\text{gr}_X'.
\end{array}
\]

By definition of log abstract blow-up the left horizontal map is a quasi-isomorphism. Hence the exactness of (10.6) follows formally from (10.7).

**Weight one motivic cohomology.** In ordinary theory of motivic cohomology we have \(H^{p-1}(X, \mathbb{Q}) = H^{p-1}(X, \mathcal{O}^* \otimes_\mathbb{Z} \mathbb{Q})\) for a smooth scheme \(X\). One would expect a similar result for the motivic cohomology of a log schemes. At least for smooth and log smooth \(X\) we have the following result.

**Proposition 10.4.** Let \(X = (\underline{X}, \mathcal{O}^* \xrightarrow{\alpha} \mathcal{M}^\text{gr}_X)\) be a smooth and log smooth scheme. Let us apply the natural transformation \(\text{Ex}_{\mathbb{A}^1}^\text{log} \mathcal{M}^\text{gr}_X \otimes \mathbb{Q} \longrightarrow \Omega_{G_{\text{rm}}}^\infty \Sigma_{G_{\text{rm}}}^\infty \mathcal{L}_{\mathbb{A}^1} \text{Ex}_{\mathbb{A}^1}^\text{log} \mathcal{M}^\text{gr}_X\) to \(\alpha\). Then the map
\[
\text{Ex}_{\mathbb{A}^1}^\text{log} \mathcal{M}^\text{gr}_X \otimes \mathbb{Q} \longrightarrow \Omega_{G_{\text{rm}}}^\infty \Sigma_{G_{\text{rm}}}^\infty \mathcal{L}_{\mathbb{A}^1} \text{Ex}_{\mathbb{A}^1}^\text{log} \mathcal{M}^\text{gr}_X
\]
is a quasi-isomorphism in \(D(\text{Sh}_{\text{et}}(\text{Sm}/X, \mathbb{Q}))\). In particular,
\[
H^{p-1}(X, \mathbb{Q}) \simeq H^{p-1}_{\text{et}}(X, \mathcal{M}^\text{gr}_X \otimes \mathbb{Q}).
\]

**Proof:** Notice that \(\text{Ex}_{Y}(i_* \mathbb{Q}) = i_* \mathbb{Q}\) by construction and the functor
\[
i_* : \text{Sh}_{\text{et}}(\text{Sm}/D, \mathbb{Q}) \longrightarrow \text{Sh}_{\text{et}}(\text{Sm}/Y, \mathbb{Q})
\]
is exact (because for any \(W\) over \(Y\) the functor \(\text{Res}_W(i_*)\) is exact). So
\[
H^p(Y, \text{Ex}_Y(i_* \mathbb{Q})) = H^p_{\text{et}}(D, \mathbb{Q}) = \begin{cases} \mathbb{Q} & p = 0 \\ 0 & p > 0 \end{cases}
\]
and \(\text{Ex}_Y(i_* \mathbb{Q})\) is \(\mathbb{A}^1\)-invariant. Using étale descent we may assume that the log structure \(\alpha\) admits a global chart. So it suffice to check that the units of adjunction
\[
\eta_{\mathcal{O}^*} : \mathcal{O}^* \rightarrow \Omega_{G_{\text{rm}}}^\infty \Sigma_{G_{\text{rm}}}^\infty \mathcal{O}^*
\]
\[
\eta_{i_* \mathbb{Q}} : i_* \mathbb{Q} \rightarrow \Omega_{G_{\text{rm}}}^\infty \Sigma_{G_{\text{rm}}}^\infty i_* \mathbb{Q}
\]
are isomorphisms.

Let \(F\) be \(\mathcal{O}^*\) or \(i_* \mathbb{Q}\). Using the localization and Voevodsky Cancellation theorem we see that, in both cases, \(F \simeq \Omega_{G_{\text{rm}}}^\infty \Sigma_{G_{\text{rm}}}^\infty F\). Moreover, in both cases \(\text{End}_{\text{DA}_{\text{eff}}}(X, \mathbb{Q})(F) \simeq \mathbb{Q}\). So \(\eta_F\) is an isomorphism or 0. The second case is impossible because \(\eta_F\) corresponds to \(\text{id}_{\Sigma_{G_{\text{rm}}}^\infty F}\) which generates \(\text{End}_{\text{DA}_{\text{et}}}(X, \mathbb{Q})(\Sigma_{G_{\text{rm}}}^\infty F)\). \(\square\)
11. Motivic cohomology of a log projective line

In this section we prove $\mathbb{P}^1_{\log}$-invariance of log motivic cohomology. By definition $\mathbb{P}^1_{\log}$ is the gluing of $\mathbb{A}^1_{\log}$ and $\mathbb{A}^1$ along $\mathbb{G}_m$. We have the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}^1_{\log} & \xrightarrow{pl_{\log}} & \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \xrightarrow{j} & \mathbb{A}^1
\end{array}
$$

Let $\mathcal{O}_{\mathbb{P}^1}^* \hookrightarrow \mathcal{M}^{gr}$ be the virtual log structure on $\mathbb{P}^1$ associated with $\mathbb{P}^1_{\log}$. Notice that $\mathcal{M}^{gr} = j_* \mathcal{O}_{\mathbb{A}^1}^*$. Let us give the explicit description of the motivic sheaves $\Lambda_q^{\mathcal{M}^{gr}}$.

**Lemma 11.1.** Let $Rj_* : \text{DA}_{et}^\text{eff}(\mathbb{A}^1, \mathbb{Q}) \rightarrow \text{DA}_{et}^\text{eff}(\mathbb{P}^1, \mathbb{Q})$ be the direct image functor. Then

$$
\Omega_{\mathcal{M}^{gr}}^\infty \simeq Rj_* \mathcal{O}_{\mathbb{A}^1}^*.
$$

**Proof:** By Yoneda Lemma we should check that

$$
\text{Hom}_{\text{DA}_{et}^\text{eff}(\mathbb{P}^1)}(Y, \mathcal{M}^{gr}) \simeq \text{Hom}_{\text{DA}_{et}^\text{eff}(\mathbb{A}^1)}(Y \times_{\mathbb{P}^1} \mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^*)
$$

for any $Y \in \text{Sm}/\mathbb{P}^1$. Then by Proposition 10.3 it suffice to show that $R^k j_* \mathcal{O}_{\mathbb{A}^1}^* \in \text{Sh}(\mathbb{P}^1_{\text{et}}, \mathbb{Q})$ vanish for $k > 0$. Moreover, by smoothness of $\mathbb{P}^1$ it enough to show $R^1 j_* \mathcal{O}_{\mathbb{A}^1}^* = 0$. The sheaf $R^1 j_* \mathcal{O}_{\mathbb{A}^1}^*$ is an étale sheafification of the presheaf

$$
\mathcal{Pic} : X \mapsto \text{Pic}(X \times_{\mathbb{P}^1} \mathbb{A}^1).
$$

Let us denote by $\mathcal{Pic}$ the presheaf which maps $X$ to $\text{Pic}(X)$. Let us consider the short exact sequence $\mathcal{O}_{\mathbb{P}^1}^* \rightarrow j_* \mathcal{O}_{\mathbb{A}^1}^* \rightarrow i_* \mathcal{Q}$. Applying long exact sequence of cohomology we get the surjection of presheaves $\mathcal{Pic} \rightarrow \mathcal{Pic}$. But the Zariski sheafification of $\mathcal{Pic}$ is zero. Indeed, by definition of plus construction we have

$$
\text{Pic}^+(X) = \text{holim}(\ker(\Pi_i \text{Pic}(U_i) \xrightarrow{\partial U_i} \Pi_{i,j} \text{Pic}(U_i \cap U_j)))
$$

where colimit taken by all Zariski covers of $X$. Let $\varphi \in \ker(\partial U_i)$ for some cover. Suppose $\varphi$ is represented by collection of vector bundles $E_i$ on $U_i$. For each $U_i$ there exist a Zariski cover $\{W_{ki} \rightarrow U_i\}$ such that $E_i|W_{ki}$ are trivial. So $\varphi$ maps to zero by the canonical map $\ker(\partial U_i) \rightarrow \ker(\partial W_{ki}[,\varphi] = 0$ and $\text{Pic}^+(X) = 0$. □

**Corollary 11.2.** The motivic spectrum $\mathcal{M}^{gr}$ coincide with $Rj_* \mathbb{Q}(1)[1]$.

**Proof:** This is a formal consequence of Verdier duality and the localization property. Let us denote by $\pi : \mathbb{P}^1 \rightarrow \text{Spec}(k)$ the map to the point. By Verdier duality

$$
\mathbb{Q}(1) \simeq \pi^* \mathbb{Q}(1) \simeq \pi^! \mathbb{Q}[-2].
$$
So $i^*\mathbb{Q}(1)[1] \simeq i^!\pi^!\mathbb{Q}[-1] = \mathbb{Q}[-1]$ and

$$\Hom_{\DAet(\mathbb{P}^1)}(i_\ast\mathbb{Q}[-1], \mathbb{Q}(1)[1]) \simeq \Hom_{\DAet(k)}(\mathbb{Q}[-1], \mathbb{Q}[-1]) = \mathbb{Q}.$$  

Hence there exists only one non-split extension $\mathbb{Q}(1)[1] \to \mathcal{F} \to i_\ast\mathbb{Q}$.

On the other hand, applying the localization to the motivic sheaf $\mathbb{Q}(1)[1] \in \DAet(\mathbb{P}^1)$ we get the triangle $i_\ast\mathbb{Q}[-1] \to \mathbb{Q}(1)[1] \to Rj_\ast\mathbb{Q}(1)[1]$ and the non-split extension

$$\mathbb{Q}(1)[1] \to Rj_\ast\mathbb{Q}(1)[1] \to i_\ast\mathbb{Q}.$$  

So if $M^{gr} \neq Rj_\ast\mathbb{Q}(1)[1]$ then $M^{gr}$ and, consequently, $\Omega_\infty M^{gr}$ are split. We get the contradiction with Lemma 11.1. □

Now, let $A$ be an abelian category and

$$K^0 \to K^1 \to ... \to K^{n-1} \to K^n$$  

be a complex concentrated in the finite numbers of components. Let

$$F^nK^\bullet \subset F^{n-1}K^\bullet \subset ... \subset F^0K^\bullet = K^\bullet$$

be the silly filtration with

$$F^jK^\bullet = \begin{cases} K^i & \text{if } j \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{E} : D(A) \to \mathcal{T}$ be a triangulated functor to some triangulated category. Suppose that for $m < n$ we have $\mathcal{E}(K^0) = \mathcal{E}(K^1) = ... = \mathcal{E}(K^m) = 0$. Then $\mathcal{E}(K^\bullet) \simeq \mathcal{E}(F^{m+1}K^\bullet)$. This assertion allow us to construct the Koszul complexes and compute the external powers $\Lambda^qM^{gr}$.

Let us consider the complex $\mathcal{O}_Q^* \to \mathcal{M}_Q^{gr} \in \Ch^\bullet(\Shet(\Sm/\mathbb{P}^1, \mathbb{Q}))$. The $q$-th exterior power of $\mathcal{O}_Q^* \to \mathcal{M}_Q^{gr}$ is the complex

$$S^q\mathcal{O}_Q^* \to S^{q-1}\mathcal{O}_Q^* \otimes \mathcal{M}_Q^{gr} \to ... \to \mathcal{O}_Q^* \otimes \Lambda^{q-1}\mathcal{M}_Q^{gr} \to \Lambda^q\mathcal{M}_Q^{gr}.$$  

Notice that any sheaf of vector spaces are flat. It follows that the functor $\Ch^\bullet(\Shet(\Sm/\mathbb{P}^1, \mathbb{Q})) \to D(\Shet(\Sm/\mathbb{P}^1, \mathbb{Q}))$ is monoidal and commute with $\Lambda^q$ and $S^q$. Using this and monoidality of $\Sigma_\infty^L L_{A^1}$ we get

$$\Sigma_\infty^L L_{A^1}(S^{q-k}\mathcal{O}_Q^* \otimes \Lambda^k\mathcal{M}_Q^{gr}) \simeq \Lambda^{q-k}\mathbb{Q}(1)[n] \otimes \Lambda^kM^{gr} = 0$$

for $k < q - 1$. So by the previous

$$\Lambda^q(\Cone(\mathbb{Q}(1)[1] \to M^{gr})) \simeq \Cone(\Lambda^{q-1}M^{gr}(1) \to \Lambda^qM^{gr}).$$  

But $\Lambda^q(\Cone(\mathbb{Q}(1)[1] \to M^{gr})) \simeq \Lambda^q(i_\ast\mathbb{Q})$ and $\Lambda^q(i_\ast\mathbb{Q}) = 0$. Indeed, $j^*\Lambda^q(i_\ast\mathbb{Q}) = \Lambda^q(j^*i_\ast\mathbb{Q}) = 0$ and $i^*\Lambda^q(i_\ast\mathbb{Q}) = \Lambda^q(\mathbb{Q}) = 0$. So we proved the following
Corollary 11.3. For $M^\text{gr}$ the same as before we have $\Lambda^q M^\text{gr} \simeq Rj_* \mathbb{Q}(q)[q]$. In particular,

$$H^p_q(\mathbb{P}^1_\text{log}, \mathbb{Q}) \simeq H^p_q(\mathbb{A}^1, \mathbb{Q}) \simeq H^p_q(\text{Spec}(k), \mathbb{Q}).$$

The last part of the Corollary can be generalized as $\mathbb{P}^1_\text{log}$-invariance of log motivic cohomology.

Theorem 11.4. Let $\pi : \mathbb{P}^1 \to \text{Spec}(k)$ be the canonical map. Then for any virtual fs log scheme $X$ the maps

$$H^p_q(X, \mathbb{Q}) \xrightarrow{\pi^*} H^p_q(X \times \mathbb{P}^1_\text{log}, \mathbb{Q})$$

are isomorphisms.

Proof: It enough to show that the morphism $(j_X, j_X^#) : X \times \mathbb{A}^1 \to X \times \mathbb{P}^1_\text{log}$ induced an isomorphism on the motivic cohomology. Using resolution of singularities and Theorem 9.6 we may assume that $X = X \times pt^n_\text{log}$ for some $n$ and $X$ smooth over $k$. Let us denote by $M^\text{gr}$ the motivic spectrum associated with $M^\text{gr}_X \times \mathbb{P}^1_\text{log}$. By construction, the map $(j_X, j_X^#)^*$ is adjoint to the map

$$j_X^* \Lambda^q M^\text{gr} \rightarrow \mathbb{Q}(q)[q]$$

So we have the commutative diagram

$$\begin{array}{ccc}
\Lambda^q M^\text{gr} & \xrightarrow{(j_X, j_X^#)^*} & Rj_* \mathbb{Q}(q)[q] \\
\xrightarrow{\eta} & \Downarrow & \\
Rj_* (j_X^* \Lambda^q M^\text{gr}) & \xrightarrow{\eta} & Rj_* (\mathbb{Q}(q)[q])
\end{array}$$

The vertical map is an isomorphism. So it enough to show that for any $p$ the map

$$\eta_p : \text{Hom}_{DA_{\text{et}}}(\mathbb{Q}[-p], \Lambda^q M^\text{gr}) \rightarrow \text{Hom}_{DA_{\text{et}}}(\mathbb{Q}[-p], Rj_* (j_X^* \Lambda^q M^\text{gr}))$$

is an isomorphism.

Suppose that $n = 0$ and $X = X$. Then applying smooth base change (combine A.5.1.4 and A.5.1.5 of [CD19]) to the diagram

$$\begin{array}{ccc}
X \times \mathbb{A}^1 & \xrightarrow{\tau_{X^1}} & \mathbb{A}^1 \\
\downarrow j_X & & \downarrow j \\
X \times \mathbb{P}^1 & \xrightarrow{\tau_{\mathbb{P}^1}} & \mathbb{P}^1
\end{array}$$

we get $\Lambda^q M^\text{gr} \simeq Rj_* \mathbb{Q}(q)[q]$. So $\eta$ is a quasi-isomorphism.

Now, let $n > 0$ and $X = X \times pt^n_\text{log}$. In this case $M^\text{gr} \simeq Rj_{X*} \mathbb{Q}(1)[1] \oplus \mathbb{Q}^n$. For any direct sum $A \oplus B$ in any tensor $\mathbb{Q}$-linear category $C$ we have the isomorphism $\Lambda^q(A \oplus B) \simeq \bigoplus_j \Lambda^j A \otimes \Lambda^{q-j} B$. For any direct sum $A \oplus B$ in any tensor $\mathbb{Q}$-linear category $C$ we have the isomorphism $\Lambda^q(A \oplus B) \simeq \bigoplus_j \Lambda^j A \otimes \Lambda^{q-j} B$
(see [Del02]). So it suffice to show that $\eta_*$ is an isomorphism for each summand. If $j \neq 0$ then

$$\Lambda^j(Rj_!\mathbb{Q}(1)[1] \otimes \Lambda^{q-j}(\mathbb{Q}^n) = Rj_!\mathbb{Q}(j)[j] \otimes \mathbb{Q}^{(q^n)}_{q^n}).$$

If $j = 0$ then $\Lambda^q(\mathbb{Q}^n) \simeq \mathbb{Q}^{(q^n)}$. We should check that

$$\eta_* : \text{Hom}_{DA\text{et}}(\mathbb{Q}^{[−p]}, \mathbb{Q}) \to \text{Hom}_{DA\text{et}}(\mathbb{Q}^{[−p]}, Rj_!j_*\mathbb{Q})$$

is an isomorphism for any $p$. Note that the groups $\text{Hom}_{DA\text{et}}(\mathbb{Q}^{[−p]}, i_X^! \mathbb{Q})$ are zero for any $p$. Indeed, by Verdier duality $i_X^! \mathbb{Q} = \mathbb{Q}^{(-1)}$. So $\text{Hom}_{DA\text{et}}(\mathbb{Q}^{[−p]}, i_X^! \mathbb{Q}) = H^p(X, \mathbb{Q}^{(-1)})$ and $H^p(X, \mathbb{Q}^{(-1)}) = 0$

for any smooth $X$ and any $p$. Now, use the localization sequence $Rj_!i_X^! \mathbb{Q} \to \mathbb{Q} \to Rj_!j_*X_! \mathbb{Q}$.

\[ \square \]

12. Log motivic cohomology as a motivic sheaf

Logarithmic motivic sheaves. We will say that a presheaf $E : \text{Sch}_{\text{isLog}}^{\text{vLog}}/k \to \text{Ch}^\bullet(\mathbb{Q})$ is an effective logarithmic motivic sheaf if it is $\mathbb{A}^1$-invariant, $\mathbb{P}^1_{\text{log}}$-invariant and admits logarithmic $h$-descent.

By result of Section 10 and Theorem 11.4 we get

**Proposition 12.1.** The presheaves $R\Gamma(−, \mathbb{Q}^{\log}(q))$ are effective logarithmic motivic sheaves.

Let us denote by $\text{PSh}(\text{Sch}_{\text{isLog}}^{\text{vLog}}/k, \text{Ch}^\bullet(\mathbb{Q}))$ the category of presheaves of complexes. As an ordinary theory of motivic sheaves we can define the functor

$$\Omega_{G_m} : \text{PSh}(\text{Sch}_{\text{isLog}}^{\text{vLog}}/k, \text{Ch}^\bullet(\mathbb{Q})) \to \text{PSh}(\text{Sch}_{\text{isLog}}^{\text{vLog}}/k, \text{Ch}^\bullet(\mathbb{Q}))$$

by the rule

$$\Omega_{G_m} E(X) = \text{Cone}(E(X \times \mathbb{G}_m) \xrightarrow{E(1)} E(X))[−1].$$

**Definition 12.2.** A logarithmic motivic sheaf is the following data:

1) For each $n \in \mathbb{Z}$ an effective log motivic sheaf $E_n$.

2) For each $n \in \mathbb{Z}$ a map $\varphi_n : E_n \to \Omega_{G_m} E_{n+1}$ which is a quasi-isomorphism.

Let us explain how to construct the log motivic sheaf with

$$E_n = \begin{cases} R\Gamma(−, \mathbb{Q}^{\log}(n))[n], & n \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let us denote by $pt_{\text{log}}$ the virtualization of standard log point. Namely, this is the pair $(\text{Spec}(k), k^* \to k^* \oplus \mathbb{Z} \to \mathbb{Z})$. One can defined another endofunctor of $\text{PSh}(\text{Sch}_{\text{isLog}}^{\text{vLog}}/k, \text{Ch}^\bullet(\mathbb{Q}))$:

$$\Omega_{pt_{\text{log}}} : \text{PSh}(\text{Sch}_{\text{isLog}}^{\text{vLog}}/k, \text{Ch}^\bullet(\mathbb{Q})) \to \text{PSh}(\text{Sch}_{\text{isLog}}^{\text{vLog}}/k, \text{Ch}^\bullet(\mathbb{Q}))$$
\[ \Omega_{pt_{log}} E(X) = \text{Cone}(E(X \times pt_{log}) \xrightarrow{E(1)} E(X))[{-1}] \]

Here the morphism \( \text{Spec}(k) \xrightarrow{1} pt_{log} \) correspond to the projection \( pr : k^* \oplus \mathbb{Z} \rightarrow k^* \).

**Remark 12.3.** Let \( A \) be an abelian category. Let

\[ A \rightarrow A \oplus B \rightarrow B \]

be a split exact sequence of complexes. Then \( \text{Cone}(\pi_B)[-1] \equiv A \oplus \text{Cone}(id_B)[-1] \). So we have the inclusion \( i_A : A \hookrightarrow \text{Cone}(\pi_B)[-1] \) which is a quasi-isomorphism.

For fix \( n \geq 0 \) let us construct the map

\[ \phi_n : R\Gamma(-, \mathbb{Q}_{log}^n(n))[n] \rightarrow \Omega_{pt_{log}} R\Gamma(-, \mathbb{Q}_{log}(n+1))[n+1]. \]

It suffice to construct the map

\[ \Lambda^n M^\mathbb{G}_m^X \rightarrow \text{Cone}(\Lambda^{n+1}(M^\mathbb{G}_m^X \oplus \mathbb{Q})) \rightarrow \Lambda^{n+1} M^\mathbb{G}_m^X[-1] \]

natural by \( X \). Note that

\[ \Lambda^{n+1}(M^\mathbb{G}_m^X \oplus \mathbb{Q}) \simeq \bigoplus_{i+j=n+1} \Lambda^i(M^\mathbb{G}_m^X) \otimes \Lambda^j(\mathbb{Q}) = \Lambda^{n+1}(M^\mathbb{G}_m^X) \oplus \Lambda^n(M^\mathbb{G}_m^X) \]

and the map \( \Lambda^{n+1}(pr) \) is the projection on the first term. So \( \ker(\Lambda^{n+1}(pr)) = \Lambda^n M^\mathbb{G}_m^X \). Hence by Remark 12.3 we have the canonical inclusion \( \phi_n(X) : \Lambda^n M^\mathbb{G}_m^X \hookrightarrow \text{Cone}(\Lambda^{n+1}(pr))[{-1}] \).

**Remark 12.4.** Let \( DA_{et}(X, \mathbb{Q}) \) be the category of rational étale motivic sheaves on \( X \) (see [Ayo14] and [CD19] for the definition). Note that \( DA_{et}(X, \mathbb{Q}) \) can be construct as a Bousfield localization of the category of complexes \( \text{Ch}^\bullet(\text{Sp}_{\mathbb{G}_m}(\text{Sh}_{et}(\text{Sm}/X, \mathbb{Q}))) \). Here \( \text{Sp}_{\mathbb{G}_m}(\text{Sh}_{et}(\text{Sm}/X, \mathbb{Q})) \) is the abelian category of sheaves of \( \mathbb{G}_m \)-spectra (see [CD19] or [Ayo14] for details). By construction of Bousfield localization we have the inclusion \( DA_{et}(X, \mathbb{Q}) \hookrightarrow \text{Ch}^\bullet(\text{Sp}_{\mathbb{G}_m}(\text{Sh}_{et}(\text{Sm}/X, \mathbb{Q}))) \) which commute with homotopy limits. This allow us to define the functor

\[ \text{Mor}(DA_{et}(X, \mathbb{Q})) \rightarrow DA_{et}(X, \mathbb{Q}); \ (E \xrightarrow{f} E') \mapsto \text{Cone}(f)[-1] \]

using the standard construction of mapping cone.

Let \( A^1_{log} \) be a virtual log line - the virtualization of a unique smooth log scheme associated with the divisor \( 0 \in A^1 \). Now, let us consider the composition

\[ \mathbb{G}_m \hookrightarrow A^1_{log} \rightarrow pt_{log} \]
of the origin complement inclusion with the projection from Example 9.4. For any $X \in \text{Sch}_{fs}^{\log}/k$ we have the commutative diagram

$$\begin{array}{c}
X \times \mathbb{G}_m \longrightarrow X \times \mathbb{A}^1_{\log} \\
\downarrow \quad \downarrow \\
X \longrightarrow X \times pt_{\log}
\end{array}$$

So the morphism gives rise to the natural transformation

(12.1) \hspace{1cm} \Omega_{pt_{\log}} \longrightarrow \Omega_{\mathbb{G}_m}.

Composing this transformation with $\{\phi_n\}$ we get the set of maps

$$\varphi_n : R\Gamma(-, \mathbb{Q}^{log}(n))[n] \longrightarrow \Omega_{\mathbb{G}_m} R\Gamma(-, \mathbb{Q}^{log}(n+1))[n+1].$$

**Theorem 12.5.** The data $\{R\Gamma(-, \mathbb{Q}^{log}(n))[n], \varphi_n\}$ is a logarithmic motivic sheaf.

Each $\varphi_n$ is a quasi-isomorphism by construction. So it suffice to prove the following

**Proposition 12.6.** Let $E$ be a log motivic sheaf. Then applying natural transformation (12.1) to $E$ we get the quasi-isomorphism

$$\Omega_{pt_{\log}} E \sim \Omega_{\mathbb{G}_m} E.$$

Proof: Note that the projection $\mathbb{A}^1_{\log} \longrightarrow pt_{\log}$ admits a section $i : pt_{\log} \longrightarrow \mathbb{A}^1_{\log}$. Applying $E$ to the log abstract blow-up square

$$\begin{array}{c}
X \times pt_{\log} \xrightarrow{i_X} X \times \mathbb{A}^1_{\log} \\
\downarrow \quad \downarrow \\
X \longrightarrow X \times \mathbb{A}^1
\end{array}$$

we get that $E(i_X)$ is a quasi-isomorphism.

On the other hand, let us consider the strict Zariski cover of $\mathbb{P}^1_{\log}$:

$$\mathbb{A}^1_{\log} \amalg \mathbb{A}^1 \longrightarrow \mathbb{P}^1_{\log}$$

with $\mathbb{A}^1_{\log} \cap \mathbb{A}^1 = \mathbb{G}_m$. By strict $h$–descent for any $X$ we have the exact sequence

$$E(X \times \mathbb{P}^1_{\log}) \longrightarrow E(X \times \mathbb{A}^1_{\log}) \oplus E(X \times \mathbb{A}^1) \longrightarrow E(X \times \mathbb{G}_m).$$

The map $E(X \times \mathbb{P}^1_{\log}) \longrightarrow E(X \times \mathbb{A}^1)$ is a quasi-isomorphism because $E$ is a log motivic sheaf. This implies that $E(X \times \mathbb{A}^1_{\log}) \longrightarrow E(X \times \mathbb{G}_m)$ is a quasi-isomorphism. □
**Restriction on a classical scheme.** Let us fix a virtual log scheme \(X \in \text{Sch}_{\text{Is}}^{v \log}/k\). Note that the morphism \(X \xrightarrow{f} \text{Spec}[k]\) induces the inclusion of categories

\[(12.2) \quad \text{Sm}/X \hookrightarrow \text{Sch}_{\text{Is}}^{v \log}/k\]

which maps \((Y \xrightarrow{g} X)\) to \((Y, g^\ast \mathcal{M}^g_X)\) and \(\phi : Y_1 \to Y_2\) to the strict morphism. Moreover, étale covers map to strict étale covers. Hence (12.2) is a map of sites and we have the pair of adjoint functors

\[f_*^{\text{log}} : \text{Sh}((\infty, 1), \text{st}, h)(\text{Sch}_{\text{Is}}^{v \log}/k, \mathbb{Q}) \rightleftarrows \text{Sh}((\infty, 1), \text{et})(\text{Sm}/X, \mathbb{Q}) : f^\ast\]

Here \(f_*^{\text{log}} F(Y, g, X) = F(Y, g^\ast \mathcal{M}^g_X)\) so we can naturally consider \(f_*^{\text{log}}\) as the restriction.

Now, let \(E\) be an effective log motivic sheaf. By construction \(f_*^{\text{log}} E\) is \(\Lambda^1\)-invariant. Note that effective motivic sheaf on \(X\) admits étale descent (see [CD19]). So \(f_*^{\text{log}} E\) belongs to \(DA_{et}^{\text{eff}}(X, \mathbb{Q})\).

Finally, let \(\{E_n, \varphi_n\}\) be a log motivic sheaf. Note that \(f_*^{\text{log}}\) commutes with \(\Omega_{\text{Gr}}\). So \(\{f_*^{\text{log}} E_n, f_*^{\text{log}}(\varphi_n)\}\) is an object of \(DA_{et}(X, \mathbb{Q})\).

**The motivic sheaf representing log motivic cohomology.** Let \(\{R\Gamma(-, Q^{\log}(n))[n], \varphi_n\}\) be a log motivic sheaf from Theorem 12.5. For \((X \xrightarrow{f} k) \in \text{Sch}_{\text{Is}}^{v \log}/k\) let us denote by \(Q_X(\log)\) the restriction \(\{f_*^{\text{log}} R\Gamma(-, Q^{\log}(n))[n], f_*^{\text{log}}(\varphi_n)\}\). Note that by construction we have

\[\text{Hom}_{DA_{et}(X, \mathbb{Q})}(X, Q_X(\log)(q)[p]) \simeq H^{p,q}_\log(X, \mathbb{Q}).\]

Now, let us give the explicit description of the motivic sheaf \(Q_X(\log)\).

**Theorem 12.7.** Let \(n\) be the maximal rank of \(\overline{M}_{X,x}^g\) where \(x\) runs over all closed points of \(X\). Then we have the natural weak equivalence \(Q_X(\log) \simeq S^n(\Omega_{\text{Gr}}^n M^g)\).

**Proof:** Firstly, let us check that \(S^n(\Omega_{\text{Gr}}^n M^g) \simeq \Omega_{\text{Gr}}^n A^n(M^g)\). Note that \(S^n([-1]) = (\Lambda^n)[-n]\). Then it suffice to prove the following

**Lemma 12.8.** \(S^n(A(-1)) \simeq (S^n A)(-n)\) for any \(A \in DA_{et}(X, \mathbb{Q})\).

**Proof:** By the results of Deligne (see also [Maz10], Prop. 1.4) for any \(A, B \in DA_{et}(X, \mathbb{Q})\) we have

\[S^n(A \otimes B) \simeq \bigoplus_{|\lambda|=n} S^\lambda(A) \otimes S^\lambda(B).\]

Using this and the isomorphisms

\[Q^{\otimes n} \simeq S^n Q \quad Q^{(1)^{\otimes n}} \simeq S^n(Q^{(1)}) \quad Q(1) \otimes Q(-1) \simeq Q\]

one can check that \(Q(-1)^{\otimes n} \simeq S^n(Q(-1))\). So we get

\[S^n(A \otimes Q(-1)) \simeq \bigoplus_{|\lambda|=n} S^\lambda(A) \otimes S^\lambda(Q(-1)) \simeq S^n A \otimes Q(-n).\]

\(\square\)
Note that $Q_X(\log)$ is an omega $\Omega_{G_m}$-spectrum with $(Q_X(\log))_k = \Omega^\infty_{G_m} A^k M^{gr}$. For any $\Omega_{G_m}$-spectrum $E$ we have $E_k = \Omega^\infty_{G_m} \Sigma^k_{G_m} E$. So it suffice to check that

$$\Omega^\infty_{G_m} A^k M^{gr} \simeq \Omega^\infty_{G_m} \Sigma^k_{G_m} \Omega^n_{G_m} A^n M^{gr}.$$ 

For $k \leq n$ the isomorphism induce by the set of maps $\{\varphi_l\}_{l \in \mathbb{N}}$ (see Theorem 12.5). For $k > n$ it suffice to construct the set of isomorphism

$$\Sigma_{G_m} A^m M^{gr} \simeq \Lambda^{m+1} M^{gr}$$

for any $m \geq n$. Let

$$\Lambda^m M^{gr}(1)[1] \xrightarrow{\tau_m} \Lambda^{m+1} M^{gr}$$

be the map induce by the $\Sigma_m$-equivariant map

$$(M^{gr})^\otimes m \otimes \mathbb{Q}(1)[1] \xrightarrow{id \otimes \varphi} (M^{gr})^\otimes (m+1).$$

Observe that $\tau_m$ is exactly the rightmost differential in the Koszul complex $\Lambda^{m+1}(\text{Cone}(\mathbb{Q}(1)[1] \to M^{gr}))$. So we have

$$\text{Cone}(\tau_m) \simeq \Sigma^\infty_{G_m} \text{Ex}_{Q,X} A^{m+1}(\mathcal{M}^{gr}_X)$$

(see the proof of Corollary 11.3 for more details). But $\Lambda^{m+1}(\mathcal{M}^{gr}_X) = 0$ for $m + 1 > n$. □

**Corollary 12.9.** We have the natural weak equivalence $Q_X(\log) \simeq \text{Sym}^*(M^{gr}(-1)[-1])$. In particular,

$$H^{p,q}_{\log}(X, \mathbb{Q}) \simeq \text{Hom}_{DM(k, \mathbb{Q})}([X]^{log}, \mathbb{Q}(q)[p]).$$

**Proof:** By Theorem 4.6 and Lemma 4.7 we have $\text{Sym}^*(\Omega_{G_m} M^{gr}) \simeq \text{colim} S^k(\Omega_{G_m} M^{gr})$. Then, using Lemma 12.8 and the isomorphisms 12.3 we conclude that $\text{colim} S^k(\Omega_{G_m} M^{gr}) \simeq S^n(\Omega_{G_m} M^{gr})$ where $n$ is the maximal rank of $\mathcal{M}^{gr}_{X,x}$. □

### 13. Log smooth schemes and motivic tubular neighborhoods

**Motivic cohomology of log smooth schemes.** Let $X$ be a log smooth scheme over $k \subset \mathbb{C}$. Denote by $X^*$ the maximal open subset with trivial log structure. Suppose that the underlying scheme $\underline{X}$ is smooth. Then the Kato-Nakayama space $X^{log}$ can be constructed as so-called oriented real blow-up (see [LGM] for more details). In particular, $X^{log}$ is a smooth manifold with boundary and the interior of $X^{log}$ coincides with $X^*$. Consequently the natural inclusion

$$X^* \hookrightarrow X^{log}$$

is a homotopy equivalence.

Hence it would be natural to expect that the logarithmic motivic cohomology of $X^*$ and $X$ are isomorphic. In fact, the stronger statement is true.
**Theorem 13.1.** Let $k$ be an arbitrary field. Let $\mathcal{F}$ be an effective log motivic sheaf. Then the induced map

$$\mathcal{F}(X) \to \mathcal{F}(X^*)$$

is a quasi-isomorphism for any smooth and log smooth $X$ over $k$.

Further, we will use the following description of log smooth schemes.

**Lemma 13.2.** (Corollary 4.8 of [K96]) Let $k$ and $X$ be as above. Then, locally in étale topology, there exist charts $f_i : X \to A^m_{\log} \times A^m_{\log}$ with étale $f_i$. So the log structure $\mathcal{M}_X$ on $X$ is the canonical log structure associated with some normal crossing divisor $D \subset X$.

**Remark 13.3.** Let $\mathcal{F}$ be as above. Let $G^n_m \hookrightarrow A^m_{\log}$ be the strict open immersion. Then for any $Y \in \text{Sch}_{\text{Log}/k}$ the map

$$\mathcal{F}(Y \times A^m_{\log}) \to \mathcal{F}(Y \times G^n_m)$$

is a quasi-isomorphism. Indeed, let us consider the strict Zariski cover of $\mathbb{P}^1_{\log}$:

$$\mathbb{A}^1_{\log} \amalg \mathbb{A}^1_{\log} \to \mathbb{P}^1_{\log}$$

with $\mathbb{A}^1_{\log} \cap \mathbb{A}^1_{\log} = G_m$. By strict $h$-descent for any $Y$ we have the exact sequence

$$\mathcal{F}(Y \times \mathbb{P}^1_{\log}) \to \mathcal{F}(Y \times \mathbb{A}^1_{\log}) \oplus \mathcal{F}(Y \times \mathbb{A}^1_{\log}) \to \mathcal{F}(Y \times G^n_m).$$

The map $\mathcal{F}(Y \times \mathbb{P}^1_{\log}) \to \mathcal{F}(Y \times \mathbb{A}^1_{\log})$ is a quasi-isomorphism. This implies that $\mathcal{F}(Y \times \mathbb{A}^1_{\log}) \to \mathcal{F}(Y \times G^n_m)$ is a quasi-isomorphism.

So we checked the statement for $n=1$. But (13.1) is the product of $G^n_m \hookrightarrow A^m_{\log}$ and we have the chain of quasi-isomorphisms

$$\mathcal{F}(Y \times G^n_m) \to \mathcal{F}(Y \times G^{m-1}_m \times A^1_{\log}) \to \cdots \to \mathcal{F}(Y \times G^k_m \times \mathbb{A}^1_{\log}) \to \cdots \to \mathcal{F}(Y \times A^n_{\log}).$$

Now, let $A^n_{\log} \setminus 0 \hookrightarrow A^n_{\log}$ be the strict open immersion.

**Lemma 13.4.** The map $\mathcal{F}(Y \times A^n_{\log} \setminus 0) \to \mathcal{F}(Y \times A^n_{\log} \setminus 0)$ is a quasi-isomorphism.

**Proof:** By Remark 13.3 the composition $G^n_m \hookrightarrow A^n_{\log} \setminus 0 \hookrightarrow A^n_{\log}$ becomes a quasi-isomorphism after applying $\mathcal{F}(Y \times -)$. So it enough to check that

$$\mathcal{F}(Y \times A^n_{\log} \setminus 0) \to \mathcal{F}(Y \times G^n_m)$$

is a quasi-isomorphism. For $n = 1$, (13.2) is identity. Let $n > 1$. It suffice to show that there is Zariski covers $\bigsqcup_{i=1}^n U_i \to G^n_m$ and $\bigsqcup_{i=1}^n V_i \to A^n_{\log} \setminus 0$ such that for any index $i_1, \ldots, i_k$ with $k \leq n$ we have

$$\mathcal{F}(Y \times (V_{i_1} \cap \cdots \cap V_{i_k})) \simeq \mathcal{F}(Y \times (U_{i_1} \cap \cdots \cap U_{i_k})).$$
Now, let $V_i = \mathbb{A}^n \setminus \{x_i = 0\}$, $U_i = \mathbb{G}_m^n$ and use induction by $n$. □

**Lemma 13.5.** Let $X$ be a smooth and log smooth scheme with canonical divisor $D = D_1 \cup \ldots \cup D_r$. Let us denote by $Z$ the intersection $D_1 \cap \ldots \cap D_r$. Then the induced map

(13.3) \[ F(X) \to F(X \setminus Z) \]

is a quasi-isomorphism.

**Proof:** The question is local so we may assume that $X$ admits an étale chart $f : X \to \mathbb{A}^{n-r} \times \mathbb{A}^r_{\log}$. So $Z = f^{-1}(0)$. In this case, Morel and Voevodsky [MV99] construct the common Nisnevich neighborhood of $Z \subset X$ and the zero section $0_Z$ of $N_X Z$. Let us denote such a neighborhood by $U$ and define the log structure on $U$ as the pullback along the map $U \to X \xrightarrow{f} \mathbb{A}^{n-r} \times \mathbb{A}^r_{\log}$. Then we get the commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{pr_1} & X \\
\downarrow f & & \downarrow f \\
\mathbb{A}^{n-r} \times \mathbb{A}^r_{\log} & \xrightarrow{id} & X \times \mathbb{A}^r_{\log}
\end{array}
\]

with strict arrows, $pr_1^{-1}(Z) \simeq Z$ and $pr_2^{-1}(0_Z) \simeq Z$. Consequently, we have two log Nisnevich squares

\[
\begin{array}{ccc}
Z \times \mathbb{A}^r_{\log} & \xrightarrow{0} & U \setminus Z \\
\downarrow & & \downarrow \\
Z \times \mathbb{A}^r_{\log} & \xrightarrow{U} & X \setminus Z
\end{array}
\]

Applying $F$ and using the Lemma [13.3] we see that the map $F(U) \to F(U \setminus Z)$ is a quasi-isomorphism. So the map (13.3) is also a quasi-isomorphism. □

**Proof of Theorem 13.1:**
Let $D = D_1 \cup \ldots \cup D_r$ be a canonical divisor of $X$ and $r$ be a number of smooth components. For $r = 1$ the statement follows from Lemma [13.5]. Let $r > 1$. By Lemma [13.5] it enough to prove that the map

$F(X \setminus Z) \to F(X^*)$

is a quasi-isomorphism. So it suffice to find Zariski covers $\bigsqcup_{i=1}^n U_i \to X^*$ and $\bigsqcup_{i=1}^n V_i \to X \setminus Z$ such that for any index $i_1, \ldots, i_k$ with $k \leq n$ we have

$F(V_{i_1} \cap \ldots \cap V_{i_k}) \simeq F(U_{i_1} \cap \ldots \cap U_{i_k})$. 

Now, let $V_i = X \setminus D_i, U_i = X^*$ and use induction by $r$. □

**Corollary 13.6.** Let $X$ be a log smooth scheme over $k$. Suppose that the underlying scheme $\underline{X}$ is smooth. Then the natural map

$$[X^*] \to [X]^{\text{log}}$$

is a weak equivalence.

**Proof:** Let us use Yoneda Lemma. Note that by construction for any smooth $Y \in \text{Sm}/k$ and any log scheme $(Z, M^a_Z)$ we have

$$(13.4) \quad \text{Hom}(Y(-q)[-p], \mathbb{D}([Z]^{\text{log}})) \simeq H^{p,q}_{\text{log}}(Y \times Z, \mathbb{Q}).$$

But $H^{p,q}_{\text{log}}(Y \times X, \mathbb{Q}) \simeq H^{p,q}(Y \times X^*, \mathbb{Q})$ by Theorem 13.1 (let $F := H^{p,q}_{\text{log}}(Y \times -, \mathbb{Q})$). □

**Remark 13.7.** Using Yoneda lemma and Corollary 12.9 one can check that the functor $[-]^{\text{log}}$ admits descent under log abstract blow-up. That is, for any log abstract blow-up square

$$
\begin{array}{ccc}
Z' & \to & X' \\
\downarrow & & \downarrow \\
Z & \to & X
\end{array}
$$

we have the exact triangle

$$[Z']^{\text{log}} \to [Z]^{\text{log}} \oplus [X']^{\text{log}} \to [X]^{\text{log}}.$$  

**Motivic tubular neighborhood.** Let $\underline{X}$ be a smooth scheme, $\underline{D}$ be a normal crossing divisor and $\underline{U} = \underline{X} \setminus \underline{D}$. Let us denote by $X$ the associated log scheme. Let $D := (\underline{D}, M_X|_D)$. Note that the square

$$(13.5) 
\begin{array}{ccc}
D & \to & X \\
\downarrow & & \downarrow \\
\underline{D} & \to & \underline{X}
\end{array}
$$

is a log abstract blow-up square. So by Remark 13.7 and Corollary 13.6 we have the exact triangle

$$[D]^{\text{log}} \to [\underline{D}] \oplus [\underline{U}] \to [X].$$

This allows us to interpret the motive $M^{\text{log}}(D)$ as the *motivic punctured tubular neighborhoods of $\underline{D}$ in $\underline{X}$. Let us denote the punctured tubular neighborhoods by $\text{PTN}_{\underline{X}D}$. Note that we also have the canonical *exponential map*

$$(13.6) \quad \text{exp} : \text{PTN}_{\underline{X}D} \to \underline{U}$$

which induce by the strict closed immersion of log schemes $D \hookrightarrow X$. 
Remark 13.8. Suppose \( k \) admits resolution of singularities. Let \( Y \) be a smooth scheme and \( Z \) be an arbitrary closed subscheme. Then by Hironaka principalization theorem ([H64], see also Theorem 3.35 of [K09]) there exists the abstract blow-up

\[
D \longrightarrow X \\
\downarrow \\
Z \longrightarrow Y
\]

(13.7)

where \( X \) and \( D \) as above and \( p^{-1}(Y \setminus Z) \simeq X \setminus D \). Then combining [13.5] and [13.7] we get the log abstract blow-up

\[
D \longrightarrow X \\
\downarrow \\
Z \longrightarrow Y
\]

and, consequently, the exact triangle

\[
[D]^{log} \longrightarrow [Z] \oplus [Y \setminus Z] \longrightarrow [Y].
\]

This allows us to interpret \([D]^{log}\) as the motivic punctured tubular neighborhoods of \( Z \) in \( Y \).

14. Limit motives via log geometry

Unipotent local systems. Let \( \tilde{\Delta} := \{z \in \mathbb{C} | 0 < |z| \leq 1 \} \) be a punctured disk. Let us denote by \( DLocSys(\tilde{\Delta})_\mathbb{Q} \) the full subcategory of \( D(\mathcal{Sh}(\tilde{\Delta}, \mathbb{Q})) \) containing the complexes \( K^\bullet \) such that \( \mathcal{H}^i(K^\bullet) \) are local systems. Let \( DLocSys^{uni}(\tilde{\Delta})_\mathbb{Q} \) be the full subcategory of \( DLocSys(\tilde{\Delta})_\mathbb{Q} \) generated by the trivial local system \( \mathbb{Q} \). By abstract nonsense we have

\[
DLocSys^{uni}(\tilde{\Delta})_\mathbb{Q} \simeq A - Mod
\]

(14.1)

where \( A = REnd(\mathbb{Q}) \simeq C^*(S^1, \mathbb{Q}) \) is the \( E_\infty \)-algebra of cochains. Indeed, for any derived local system \( M^\bullet \) we can constructed the \( C^*(S^1, \mathbb{Q}) \)-module \( RHom(\mathbb{Q}, M^\bullet) \). On the other hand, let \( BC^*(S^1, \mathbb{Q}) \) be the category with one object and the ring of endomorphisms \( C^*(S^1, \mathbb{Q}) \). Then the inclusion \( BC^*(S^1, \mathbb{Q}) \) gives rise to the Yoneda extension

\[
A - Mod \longrightarrow DLocSys^{uni}(\tilde{\Delta})_\mathbb{Q}.
\]

One can check that this functors form the equivalence of categories.

Let \( i: pt \hookrightarrow \tilde{\Delta} \) be the inclusion of a point. Then we have the inverse image functor

\[
i^*: DLocSys(\tilde{\Delta})_\mathbb{Q} \longrightarrow D(\mathcal{Sh}(pt, \mathbb{Q})) \simeq D(\mathbb{Q} - Vect)
\]
and we can restrict \( i^* \) on \( DLocSys^{uni}(\Delta)_{\mathbb{Q}} \). On the other hand, the map \( i \) induces the map of \( \mathbb{E}_\infty \)-algebras

\[
(14.2) \quad C^*(S^1, \mathbb{Q}) \longrightarrow C^*(pt, \mathbb{Q}) \simeq \mathbb{Q}.
\]

The map (14.2) defines the functor

\[
- \otimes_{C^*(S^1, \mathbb{Q})} C^*(pt, \mathbb{Q}) : A-\text{Mod} \longrightarrow D(\mathbb{Q}-\text{Vect})
\]

Note that \(- \otimes_{C^*(S^1, \mathbb{Q})} C^*(pt, \mathbb{Q})\) and \(i^*\) commute with colimits. The both maps

\[
REnd_{DLocSys(\Delta)_{\mathbb{Q}}}(Q) \longrightarrow End_{\mathbb{Q}}(Q)
\]

\[
REnd_A(A) \longrightarrow End_{\mathbb{Q}}(Q)
\]

coincide with (14.2). Moreover, this maps differ only by the isomorphism

\[
REnd_{DLocSys(\Delta)_{\mathbb{Q}}}(Q) \simeq REnd_A(A)
\]

induced by the equivalence (14.1). So \( i^* \) and \(- \otimes_{C^*(S^1, \mathbb{Q})} C^*(pt, \mathbb{Q})\) are isomorphic.

**Nearby cycles as a bar-construction.** Let \( C \) be a smooth complex curve with marked point 0. Let \( f : X \longrightarrow C \) be a semi-stable degeneration with a special fiber \( X_s \). Suppose that \( f \) is proper and let \( \tilde{f} : \tilde{X} \longrightarrow \tilde{C} \) be the restriction of \( f \) on \( \tilde{X} := X \setminus X_s \). Then by Ehresmann’s theorem \( \tilde{f} \) is a fiber bundle. Let \( t \) be a point of \( \tilde{C} \). Using proper base change we conclude that \( R^n\tilde{f}_*\mathbb{Q} \) is a local system with the fiber \( H^n(\tilde{X}_t, \mathbb{Q}) \). Now, let us chose a disk \( \Delta \subset C \) centered at \( s \). For \( X' := f^{-1}(\Delta) \) the map \( f : X' \longrightarrow \Delta \) is also proper semi-stable degeneration. Note the \( X' \) can be considered as a tubular neighborhood of \( X_s \) in \( X \). By the previous the complex \( R\tilde{f}_*\mathbb{Q} \) belongs to \( DLocSys(\Delta)_{\mathbb{Q}} \).

**Lemma 14.1.** \( R\tilde{f}_*\mathbb{Q} \) is a unipotent derived local system.

**Proof:** Note that \( R^m\tilde{f}_*\mathbb{Q} = H^n(X_t, \mathbb{Q}) \) is zero for sufficiently large \( m \). So it enough to check that \( R^h\tilde{f}_*\mathbb{Q} \) belongs to \( DLocSys^{uni}(\Delta)_{\mathbb{Q}} \). Let us denote by \( \gamma \) the monodromy matrix acts on \( H^n(X_t, \mathbb{Q}) \). Note that \( \gamma \) is a unipotent operator. Now, let \( M \) be an arbitrary local system with unipotent monodromy \( \gamma \). Then \( M \) belongs to \( DLocSys^{uni}(\Delta)_{\mathbb{Q}} \). Indeed, \((\gamma - 1)^k = 0\) so all eigenvalues are 1. Representing \( M \) as a direct sum we can assume that \( \gamma \) is a Jordan block. Then we can use the induction by dimension of \( \gamma \). \( \square \)

Note that the equivalence \( DLocSys^{uni}(\Delta)_{\mathbb{Q}} \longrightarrow A-\text{Mod} \) maps \( R\tilde{f}_*\mathbb{Q} \) to the complex

\[
RHom_{D(Sh(\Delta, \mathbb{Q})))}(Q, R\tilde{f}_*\mathbb{Q}) \simeq RHom_{D(Sh(X', \mathbb{Q})))}(Q, \mathbb{Q}) \simeq C^*(X', \mathbb{Q})
\]

such that the structure of \( C^*(S^1, \mathbb{Q}) \)-module on \( R\tilde{f}_*\mathbb{Q} \) induces by the homomorphism of \( \mathbb{E}_\infty \)-algebras

\[
C^*(\Delta, \mathbb{Q}) \longrightarrow \tilde{f}_* C^*(X', \mathbb{Q}).
\]
So for \( f : X \rightarrow C \) and \( t : pt \hookrightarrow \hat{\Delta} \hookrightarrow \hat{C} \) we have
\[
(14.3) \quad t^*(R\hat{f}_*\mathbb{Q}) \simeq C^*(\hat{X}', \mathbb{Q}) \otimes_{C^*(S^1, \mathbb{Q})} C^*(t, \mathbb{Q})
\]
where \( \hat{X}' \) is a punctured tubular neighborhood of \( X_0 \subset X \).

Now, let \( \psi_f : D(\text{Sh}(\hat{X}, \mathbb{Q})) \rightarrow D(\text{Sh}(X_s, \mathbb{Q})) \) be the nearby cycles functor. For semi-stable degeneration \( f \) we have
\[
H^*(X_s, \psi_f \mathbb{Q}) \simeq H^*(X_t, \mathbb{Q}).
\]

So, using (14.3) we get
\[
(14.4) \quad R\Gamma(X_s, \psi_f \mathbb{Q}) \simeq C^*(\hat{X}', \mathbb{Q}) \otimes_{C^*(S^1, \mathbb{Q})} C^*(t, \mathbb{Q}).
\]

**The limit motive of a semi-stable degeneration.** Let \( k \) be an arbitrary field. Again, let \( C \) be a smooth curve over \( k \) and \( f : X \rightarrow C \) be a semi-stable degeneration. Let us associated with \((X, X_s)\) the canonical log structure. Let \( \mathcal{M}^\text{gr}_{X_s} \) be the restriction of this structure on \( X_s \).

Note that the map \( f \) induces the morphism of virtual logarithmic schemes
\[
(14.5) \quad f : (X_s, \mathcal{M}^\text{gr}_{X_s}) \rightarrow pt_{\text{log}}.
\]

By 7.6 the map \([X_s]^{\text{log}} \rightarrow [pt_{\text{log}}]^{\text{log}} = [G_m] \) is a homomorphism of \( E_{\infty} \)-coalgebras. Let \( t : \text{Spec}(k) \hookrightarrow G_m \) be a closed point of \( G_m \). Then, replicating (14.3) we define the geometrical motive
\[
\text{LM}_f := ([X_0]^{\text{log}})^* \otimes_{[G_m]^*} [\text{Spec}(k)].
\]

We will call the motive \( \text{LM}_f \) the **Voevodsky limit motive of \( f \).**

Now, let us fix an inclusion \( \tau : k \rightarrow \mathbb{C} \). Let Betti : \( \text{DM}^\text{gm}(k, \mathbb{Q}) \rightarrow \text{D}(\mathbb{Q} - \text{Vect}) \) be the associated Betti realization.

**Proposition 14.2.** There exists the canonical quasi-isomorphism
\[
\text{Betti}((\text{LM}_f)) \simeq R\Gamma(X_s, \psi_f \mathbb{Q})
\]

induced by (14.3).

**Proof:** Let \( X' \) and \( \hat{X}' \) be as above. Note that the inclusion \( X'_s \hookrightarrow X' \) induces the canonical log analytic structure on \( X' \). So we have the commutative diagram
\[
\begin{array}{cccc}
C^*(X'_s^{\text{log}}, \mathbb{Q}) & \rightarrow & C^*(X'^{\text{log}}, \mathbb{Q}) & \rightarrow & C^*(\hat{X}', \mathbb{Q}) \\
\uparrow & & \uparrow & & \uparrow \\
C^*(([pt_{\text{log}}]^{\text{log}}, \mathbb{Q}) & \rightarrow & C^*(\Delta^{\text{log}}, \mathbb{Q}) & \rightarrow & C^*(\hat{\Delta}, \mathbb{Q})
\end{array}
\]
where each map is a homomorphism of \( E_{\infty} \)-algebras. Combining Corollary 4.17. of [PS08] with Theorem 5.2 we conclude that the right-upper and the right-lower horizontal arrows are quasi-isomorphisms.
Moreover, Schmid proved \cite{Sch73} that the data \( X' \rightarrow C^*(X'^{\log}, \mathbb{Q}) \) admits descent under virtual logarithmic blow-ups. Indeed, using Theorem 5.2 we can check this by the same way as in the Section 10. So we get the exact triangle

\[
C^*(X', \mathbb{Q}) \rightarrow C^*(X'^{\log}, \mathbb{Q}) \oplus C^*(X'^{\log}_0, \mathbb{Q}) \rightarrow C^*(X^{'0}_{log}, \mathbb{Q}).
\]

But the inclusion \( X'^{\log}_0 \rightarrow X' \) is a homotopy equivalence. So the left-upper and the left-lower horizontal arrows are quasi-isomorphisms.

It remains to note that, by Proposition 8.6 \( C^*((pt_{log})^{\log}, \mathbb{Q}) \rightarrow C^*(X^{'0}_{log}, \mathbb{Q}) \) is the (covariant) Betti realization of the map \([G_m]^* \rightarrow [X^{'0}]^{log*}\). \(\Box\)

15. Motivic monodromy filtration

**Limit Hodge structure.** Let \( \Delta \) be a complex unit disk and \( f : X \rightarrow \Delta \) be a proper semi-stable degeneration. By Ehresmann’s theorem the restriction of \( f \) on \( X - X_0 \) is a fiber bundle over \( \hat{\Delta} \). So \( f \) defines the variation of Hodge structures over \( \hat{\Delta} \). Namely, let us fix \( t \in \hat{\Delta} \) and \( n \in \mathbb{N} \). We will denote by \( \hat{\mathcal{D}} \) the completed local period domain. Recall that \( \hat{\mathcal{D}} \) is the space of flags \( \ldots \subset F^p \subset \ldots \subset F^1 \subset F^0 = H^n(X_t, \mathbb{C}) \) with the property \( \dim(F^p/F^{p-1}) = h^{p,q}(X_t) \).

Note that \( f \) is proper so \( Rf_*\mathbb{Q} \) defines the local system over \( \hat{\Delta} \) with the fibers \( H^n(X_t, \mathbb{Q}) \). Then we can define the local period map \( \Phi : \Delta \rightarrow \hat{\mathcal{D}} \)

\[
t' \in \hat{\Delta} \mapsto \{ \text{filtration on } H^n(X_t, \mathbb{C}) \text{ induced by the Hodge filtration on } H^n(X_{\nu'}), \mathbb{C} \}.\}
\]

Note that an isomorphism \( H^n(X_t, \mathbb{C}) \simeq H^n(X_{\nu'}, \mathbb{C}) \) depends on the choosing of a path so the map \( \Phi \) is multi-valued. However, one can ask is it possible to continue \( \Phi \) holomorphically on all \( \Delta \)?

Let \( \gamma \) be a monodromy matrix which act on \( H^n(X_t, \mathbb{C}) \) and \( N = -\log(\gamma) \). Then we can define \( \Psi(z) := \exp(zN)\Phi(z) \) where \( z \) is the coordinate on the universal cover of \( \hat{\Delta} \). Note that \( \Psi(z) = \Psi(z+1) \) so we have well-defined map \( \Psi : \hat{\Delta} \rightarrow \hat{\mathcal{D}} \). Then \( \Psi \) can be holomorphically continue on all \( \Delta \).

Observe that the extension of \( \Psi \) defines the new filtration \( F^\bullet_{lim} \) on \( H^n(X_t, \mathbb{C}) \) given by \( \Psi(0) \).

On the other hand, by Borel results, the monodromy matrix \( \gamma \) is unipotent. So there is some \( k \) such that \( N^k = 0 \) and \( N \) defines one more filtration

\[
\ldots \subset 0 \subset W_0 \subset \ldots \subset W_{2k} = H^n(X_t, \mathbb{Q})
\]

with the properties:

- for each \( r \) we have \( N(W_r) \subset W_{r-2} \);
- the operator \( N \) defines the isomorphisms \( \text{Gr}^W_r(H^n(X_t, \mathbb{Q})) \simeq \text{Gr}^W_{r-2}(H^n(X_t, \mathbb{Q})) \).

Moreover, Schmid proved \cite{Sch73} that the data \( (H^n(X_t, \mathbb{Z}), W_\bullet, F^\bullet_{lim}) \) defines the mixed Hodge structure. This Hodge structure called the **limit Hodge structure**.
The Steenbrink construction. There is another, pure algebraic construction of the limit Hodge structure proposed by Steenbrink (see \cite{St76, St95} and \cite{PS08}). Namely, he constructed two filtered bicomplexes \((C^{••}, W(M)^{••})_r\) and \((A^{••}, W(M)^{••}_{C,r})\) with \(C^{••} \in \text{Bicom}(\text{Sh}(X_0, \mathbb{Q}))\) and \(A^{••} \in \text{Bicom}(\text{Sh}(X_0, \mathbb{C}))\), the filtered quasi-isomorphism
\[
\text{Tot}(C^{••} \otimes \mathbb{C}) \simeq \text{Tot}(A^{••})
\]
and the Hodge filtration on \(\text{Tot}(A^{••})\) given by the partial totalizations
\[
F^n A = \text{Tot}(s_{>p} A^{••}); \quad s_{>r} A^{i,j} = \begin{cases} A^{i,j}, & j \geq p \\ 0, & \text{otherwise.} \end{cases}
\]

Steenbrink proved that the data
\[
(15.1) \quad \psi_f^\mathbb{Z}^{\text{Hdg}} := (\psi_f^\mathbb{Z}, (\text{Tot}(C^{••}, \text{Tot}(W(M)^{••})), (\text{Tot}(A^{••}), \text{Tot}(W(M)^{••}_{C,r})), FA))
\]
forms the mixed Hodge complex of sheaves over \(X_0\). Then the mixed Hodge structure on \(\mathbb{H}^n(X_0, \psi_f^\mathbb{Z}^{\text{Hdg}})\) is isomorphic to the Schmid limit Hodge structure on \(H^n(X_t, \mathbb{Q})\).

Let us describe the bicomplexes \(C^{••}\) and \(W(M)^{••}\) explicitly. Note that we have the canonical log structure on \(X\). Let \(M \to \mathcal{O}_{X_0}\) be the restriction of this structure on \(X_0\). Then we have the associated exponential complex \(\exp\) given as the cone of the map
\[
\mathcal{O}_{X_0\exp} \xrightarrow{\exp} \mathcal{M}^{\text{gr}}_{X_0\exp}.
\]
Moreover, \(f\) induced the morphism of smooth log analytic spaces \((X^{an}, X_0^{an}) \to (\Delta, 0)\). The restriction of this morphism on \(X_0\) gives the morphism of virtual log analytic spaces
\[(X_0^{an}, \mathcal{O}_{an} \hookrightarrow \mathcal{M}^{\text{gr}}_{an}) \to pt_{\log}.\] So we get the element \(t \in \Gamma(X_0^{an}, \mathcal{M}^{\text{gr}}_{an})\).

Let us define the following bicomplexes:

the bicomplex \(W^{••}\)
\[
... \to 0 \xrightarrow{-1} S^r(\exp)[1] \xrightarrow{\Lambda t} S^{r+1}(\exp)[2] \xrightarrow{\Lambda t} S^{r+2}(\exp)[3] \xrightarrow{\Lambda t} ... 
\]

where the horizontal differential given by
\[
(15.2) \quad S^i(\mathcal{O}_{an}) \otimes \Lambda^j(\mathcal{M}^{gr}_{an}) \xrightarrow{id \otimes \Lambda t} S^i(\mathcal{O}_{an}) \otimes \Lambda^{j+1}(\mathcal{M}^{gr}_{an})
\]
and \(W^{p,q}_r = 0\) for \(p < 0\);

the bicomplex \(W^{••}(M)_r\)
\[
... \to 0 \xrightarrow{-1} S^{r+1}(\exp)[1] \xrightarrow{d} S^{r+3}(\exp)[2] \xrightarrow{d} S^{r+5}(\exp)[3] \xrightarrow{d} ... 
\]

where the differential \(d\) given by the composition of \(15.2\) with the inclusion
\[
(15.3) \quad \mathbb{Q} \otimes S^i(\exp) \hookrightarrow S^{i+1}(\exp)
\]
$a \otimes w \mapsto a \cdot w$.

Here $\tilde{W}(M)^{r,p} = S^{r+2p+1}(\exp)[p]$ and $\tilde{W}(M)^{r,q} = 0$ for $p < 0$.

Note that we can also define $W(M)^{r,p}$ for $r < -1$ if we put

$$S^k(-) = 0 \text{ for } k < 0.$$  

Let $\bar{C}^{r,p}$ be the bicomplex

$$... \longrightarrow 0 \longrightarrow \text{Sym}^*(\exp)[1] \xrightarrow{\Lambda^1} \text{Sym}^*(\exp)[2] \xrightarrow{\Lambda^1} \text{Sym}^*(\exp)[3] \xrightarrow{\Lambda^1} ...$$

Note that we have the canonical inclusions

$$W_r^{r,p} \hookrightarrow \tilde{W}(M)^{r,p}$$

$$\tilde{W}(M)^{r,p} \hookrightarrow \tilde{W}(M)^{r,p}$$

$$W_r^{r,p} \hookrightarrow \bar{C}^{r,p}$$

given by compositions of (15.3) Let us define

$$C^{r,p} := \bar{C}/W_0$$

$$W(M)^{r,p} := \text{im}(\tilde{W}(M)^{r,p} \hookrightarrow C^{r,p}).$$

Now, we should make a few remarks. Let $N$ be the number of irreducible components of $X_0$. Note that $\tilde{W}(M)^{r,p} \subseteq W_r^{r,p}$ if $r + 2n + 1 \leq n$. So $\tilde{W}(M)^{r,p} = 0$ for $n \leq -r - 1$. Let $n > -r$ and $r < -(N - 1)$. Then $r + 2n + 1 > N$ and

$$(\text{Gr}_r^{W(M)}(C^{r,p}))^n = \Lambda^{r+2n+1}M^{gr}[-r-n] = 0.$$  

On the other hand, let $n = -r$ and $r < -(N - 1)$. Then $-r + 1 > N$ and $(\text{Gr}_r^{W(M)}(C^{r,p}))^n = \Lambda^{-r+1}M^{gr}[-r-n] = 0$. Finally, assume that $r > N - 1$. Then again $(\text{Gr}_r^{W(M)}(C^{r,p}))^n = 0$. So we get

**Lemma 15.1.** The inclusions $W(M)^{r-1} \subset W(M)^{r}$ are quasi-isomorphisms for $r \leq -(N - 1)$ and $r \geq N - 1$. Moreover, for $r \geq N - 1$ the canonical inclusions $W(M)^{r,p} \hookrightarrow C^{r,p}$ are quasi-isomorphisms.

**Proof:** It remains to check that the canonical inclusion $\tilde{W}(M)^{N-1} \hookrightarrow \bar{C}$ is a quasi-isomorphism. Using the spectral sequence of a bicomplex, it can be reduce to the fact that the maps $S^{N-1+2n+1}(\exp) \longrightarrow \text{Sym}^*(\exp)$ are quasi-isomorphisms. \( \square \)

Let $s_{\geq N-1}C^{r,p}$ be the complex with $s_{\geq N-1}C^{n,p} = C^{n,p}$ for $n \geq N - 1$ and $s_{\geq N-1}C^{n,p} = 0$ otherwise. Using the same arguments as in the proof of the lemma one can check that the canonical inclusion $s_{\geq N-1}C^{r,p} \hookrightarrow C^{r,p}$ is a quasi-isomorphism. Indeed, it suffice th check
that the complexes $W(M)^n_{N-1}$ are quasi-isomorphic to zero for $n > N - 1$. But
\[ W(M)^n_{N-1} = \text{Cone}(S^n(exp) \to S^{r+2n+1}(exp)) \sim 0 \]
for $n \leq N$.

Remark 15.2. Using the same arguments we can also replace the complex $A^{\bullet\bullet}$ with $s_{\geq N-1}A^{\bullet\bullet}$. Then, analyzing the graded pieces, we conclude that the Steenbrink limit Hodge complex can be replaced with the data
\[(\text{Tot}(s_{\geq N-1}C), \text{Tot}(s_{\geq N-1}W(M)), (\text{Tot}(s_{\geq N-1}A), \text{Tot}(s_{\geq N-1}W(M)_C), FA)).\]

![Diagram](image-url)

**Picture 9.1.** The non-zero elements of the complexes $s_{\geq N-1}W(M)_r$.

**The complexes $W_r$.** Now, again let $k$ be an arbitrary field, $C/k$ be a smooth curve and $f : X \to C$ be a proper semi-stable degeneration. Let $O^*_X \hookrightarrow M^{gr}_{X_s}$ be the pullback of the log structure on $X_s$. Again, the map $f : (X_s, M^{gr}_{X_s}) \to pt_{log}$ defines the global section $t \in \Gamma(X_s, M^{gr}_{X_s})$.

Let us abuse the notation and denote by $M^{gr}$ the extension $\text{Ext}^1(M^{gr})$ on the smooth étale site $Sm/X_0$. Further, we will denote by $f_n$ the rightmost differential in the Koszul complex
(15.4) \[ S^{n+1}(O^* \to M^{gr}) = \cdots \to O^* \otimes \Lambda^n(M^{gr}) \to \Lambda^{n+1}(M^{gr}). \]
Let $W_r$ be the complex
\[ \cdots \to 0 \to \Lambda^rM^{gr}_0 \overset{\Lambda}{\to} \Lambda^{r+1}M^{gr}_1 \overset{\Lambda}{\to} \Lambda^{r+2}M^{gr}_2 \overset{\Lambda}{\to} \cdots \]
with differentials given by the multiplication on $t$ and $W^n_r = 0$ for $n \leq 0$ and $n > N$.

Note that the compositions
\[ f_j \circ f_j \circ \cdots \circ f_j : O^* \otimes (M^{gr})^n \to (M^{gr})^{n+1}. \]

which is induced by the inclusion $O^* \otimes (M^{gr})^n \hookrightarrow (M^{gr})^{n+1}$. 

---

\[ \text{Cone}(S^n(exp) \to S^{r+2n+1}(exp)) \sim 0 \]
induce the maps
\[ \alpha_{r,r+i} : O^{r \otimes i} \otimes W_r \to W_{r+i}. \]

**The complexes \( \tilde{W}(M)_r \).** Let \( \tilde{W}(M)_r \) be the complex
\[ \cdots \to O^{r \otimes N-1} \otimes \Lambda^{r+1} M^{gr} \to O^{r \otimes N-2} \otimes \Lambda^{r+3} M^{gr} \to \cdots \to \Lambda^{r+2(N-1)+1} M^{gr} \to 0 \cdots \]
where the differential given by the composition of the multiplication by \( t \)
\[ O^{r \otimes i} \otimes \Lambda^j M^{gr} \longrightarrow O^{r \otimes i} \otimes \Lambda^j M^{gr} \]
with the differential of the Koszul complex
\[ (O^{r \otimes i-1}) \otimes O^* \otimes \Lambda^j M^{gr} \longrightarrow O^{r \otimes i-1} \otimes \Lambda^{j+2} M^{gr} \]
We define \( \tilde{W}(M)_r^n = 0 \) for \( n < 0 \), \( n > N-1 \) and \( \tilde{W}(M)_r^n = O^{r \otimes N-1-n} \otimes \Lambda^{r+2n+1} M^{gr} \)
otherwise.

Note that the diagram
\[ O^{r \otimes i+j} \otimes \Lambda^k M^{gr} \longrightarrow O^{r \otimes i+j} \otimes \Lambda^{k+1} M^{gr} \]
\[ (id \otimes f_{k+j-1}) \cdots (id \otimes f_k) \]
\[ O^{r \otimes i} \otimes \Lambda^k M^{gr} \longrightarrow O^{r \otimes i-1} \otimes \Lambda^{k+2} M^{gr} \]

is commutative. So we have the maps
\[ \psi_{r,r+i} : O^{r \otimes N+i} \otimes W_r \to \tilde{W}(M)_{r+i}. \]

On the other hand, one can define the morphisms of complexes
\[ \varphi_r : O^* \otimes \tilde{W}(M)_r \to \tilde{W}_{r+1}(M), \]
using the commutative diagram
\[ O^{r \otimes N} \otimes \Lambda^{r+1} M^{gr} \longrightarrow O^{r \otimes N-1} \otimes \Lambda^{r+3} M^{gr} \longrightarrow O^{r \otimes N-2} \otimes \Lambda^{r+5} M^{gr} \longrightarrow \cdots \]
\[ \begin{array}{c}
\downarrow f_{r+1} \\
\downarrow f_{r+3} \\
\downarrow f_{r+5} \\
\end{array} \]
\[ O^{r \otimes N-1} \otimes \Lambda^{r+2} M^{gr} \longrightarrow O^{r \otimes N-2} \otimes \Lambda^{r+4} M^{gr} \longrightarrow O^{r \otimes N-3} \otimes \Lambda^{r+6} M^{gr} \longrightarrow \cdots \]

**Motivic monodromy filtration.** Let \( W(M)_{N-1} \) be the cokernel of the map
\[ id \otimes \psi_0, N : O^{r \otimes 2N-1} \otimes W_0 \to \tilde{W}(M)_{N-1}. \]

We define the increasing filtration
\[ \cdots \subset W(M)_{-N+1} \subset W(M)_{-N+2} \subset \cdots \subset W(M)_{N-2} \subset W(M)_{N-1} \]
with
\[ W(M)_r := \text{im}(O^{r \otimes N-1-r} \otimes \tilde{W}(M)_r \longrightarrow W(M)_{N-1}) \]
Remark 15.3. Note that $\text{Gr}_r^{W(M)}(W(M)_{N-1}) = \bigoplus_{n \geq 0, -r} W(M)_n^{i}/W(M)_{n-1}^{i}[-n]$. Indeed, same as in the Steenbrink construction, the $n$-th differential of $W(M)$ factors through the map $\mathcal{O}^{*} \otimes W(M)_{r+1}^{n+1} \xrightarrow{\partial_{r+1}} W(M)_{r+1}^{n+1}$. So all differentials of the complex $\text{Gr}_r^{W(M)}(W(M)_{N-1})$ are zero. On the other hand, by the same argument as in the proof of Lemma 15.1, we get $(\text{Gr}_r^{W(M)}(W(M)_{N-1}))^n = 0$ for $n < -r$.

Now, let us define the stable motivic sheaves $[W(M)_{r}] \in \text{DA}_{\text{et}}(X_0, \mathbb{Q})$ by the formula

$$[W(M)_{r}] := \Sigma_{G_m}^{\infty} L_{A^1}(W(M)_{r})(-2(N-1))[-2(N-1)]$$

and define

$$[\text{Gr}_r^{W(M)}(W(M)_{N-1})] := \Sigma_{G_m}^{\infty} L_{A^1}(\text{Gr}_r^{W(M)}(W(M)_{N-1}))(2(N-1))[-2(N-1)].$$

Note that we have an exact sequences

$$[W(M)_{r-1}] \rightarrow [W(M)_{r}] \rightarrow [\text{Gr}_r^{W(M)}(W(M)_{N-1})].$$

One can think about sheaves $[W(M)_{r}]$ as the motivic analog of the filtration $W(M)_{N-1} \supset W(M)_{N-2} \ldots$ Now, we will find out some fundamental properties of the motives $[W(M)_{r}]$ and $[\text{Gr}_r^{W(M)}(W(M)_{N-1})]$.

Lemma 15.4. The map $\mathcal{O}^{*} \otimes \Lambda^j \mathcal{M}^{\text{gr}} \rightarrow \text{im}(\mathcal{O}^{*} \otimes \Lambda^j \mathcal{M}^{\text{gr}} \rightarrow \Lambda^{i+j} \mathcal{M}^{\text{gr}})$ is a $\Lambda^1$-homotopy equivalence for any $i$ and $j$.

Proof: It suffice to check that $L_{A^1}(\ker(f_{i+j-1} \cdots f_j)) = 0$. If $i = 1$ we can consider the Koszul complex

$$S^{j+1}(\mathcal{O}^* \rightarrow \mathcal{M}^{\text{gr}}) = \ldots \rightarrow S^{k+1} \mathcal{O}^* \otimes \Lambda^{j-k} \mathcal{M}^{\text{gr}} \xrightarrow{\partial_{j-k}} S^{k} \mathcal{O}^* \otimes \Lambda^{j-k+1} \mathcal{M}^{\text{gr}} \rightarrow \ldots$$

Then we have the exact sequences

$$\text{im}(\partial_{j-k}) \rightarrow S^{k} \mathcal{O}^* \otimes \Lambda^{j-k+1} \mathcal{M}^{\text{gr}} \rightarrow \ker(\partial_{j-k-1})$$

and

$$\text{im}(\partial_{j-1}) \rightarrow S^{j} \mathcal{O}^* \otimes \Lambda^{j-1} \mathcal{M}^{\text{gr}} \rightarrow \ker(f_j)$$

But $L_{A^1} S^{k} \mathcal{O}^* = \Lambda^{k}(\mathbb{Q})(k)[k] = 0$.

If $i > 1$ note that $f_{i+j+1} \cdots f_j$ factors through the map

$$\lambda_{i,j} : \Lambda^i \mathcal{O}^* \otimes \Lambda^j \mathcal{M}^{\text{gr}}$$

$$\eta \otimes \omega \mapsto \eta \wedge \omega.$$
Applying the snake lemma to the diagram

\[
\begin{array}{c}
\ker(f_{i+j-1} \ldots f_j) \longrightarrow O^* \otimes \Lambda^j M^{gr} \longrightarrow \im(f_{i+j-1} \ldots f_j) \\
\downarrow \quad \downarrow \\
\ker(\lambda_{i,j}) \longrightarrow \Lambda^i O^* \otimes \Lambda^j M^{gr} \longrightarrow \im(\lambda_{i,j})
\end{array}
\]

we conclude that the left vertical arrow is a $\mathbb{A}^1$-homotopy equivalence. Now, let us consider two Koszul complexes

\[
S^{i+j}(O^* \xrightarrow{id+i} O^* \oplus M^{gr}) = \ldots \rightarrow S^{k+1} O^* \otimes \Lambda^{j-k}(O^* \oplus M^{gr}) \xrightarrow{\partial^k} S^k O^* \otimes \Lambda^{j-k+1}(O^* \oplus M^{gr}) \rightarrow \ldots;
\]

\[
S^{i+j}(O^* \xrightarrow{id} O^*) = \ldots \rightarrow S^{k+1} O^* \otimes \Lambda^{j-k}(O^*) \xrightarrow{\partial^k} S^k O^* \otimes \Lambda^{j-k+1}(O^* M) \rightarrow \ldots;
\]

Then the map $\lambda_{i,j}$ can be given as the composition

\[
\Lambda^i O^* \otimes \Lambda^j M^{gr} \longrightarrow \Lambda^{i+j}(O^* \oplus M^{gr})
\]

\[
\downarrow
\]

\[
\Lambda^{i+j} M^{gr}
\]

so the kernel of $\lambda_{i,j}$ is the intersection of $\im(O^* \otimes \Lambda^{i+j-1}(O^* \oplus M^{gr}) \xrightarrow{\partial^k} \Lambda^{i+j}(O^* \oplus M^{gr}))$ with $\Lambda^i O^* \otimes \Lambda^j M^{gr}$. Note that the preimage of $\Lambda^i O^* \otimes \Lambda^j M^{gr}$ is the direct sum

\[
(O^* \otimes \Lambda^{i-1} O^* \otimes \Lambda^j M^{gr}) \oplus (O^* \otimes \Lambda^i O^* \otimes \Lambda^{j-1} M^{gr}).
\]

So $\ker(\lambda_{i,j})$ is the sum of kernels $\ker(\partial^k) = \ker(O^* \otimes \Lambda^i O^* \otimes \Lambda^{j-1} M^{gr} \longrightarrow \Lambda^{i+1} O^* \otimes \Lambda^{j-1} M^{gr})$ and $\ker(f_j) = \ker(O^* \otimes \Lambda^{i-1} O^* \otimes \Lambda^j M^{gr} \longrightarrow \Lambda^{i-1} O^* \otimes \Lambda^{j+1} M^{gr})$. But both of them are $\mathbb{A}^1$-homotopy equivalent to zero by the previous. □

Now, let us describe the graded pieces $[\Gr^W_r(W(M)_{N-1})]$ explicitly. We will denote by $D_i$ the irreducible components of $X_0 = D_1 \cup \ldots D_N$. Let us introduce the notation

\[
D_J := \bigcap_{j \in J} D_j \\
D \{m\} := \bigcup_{|J|=m} D_J
\]

and denote by $a_m$ the canonical inclusions $D \{m\} \hookrightarrow X_0$.

**Proposition 15.5.** For any $r$ we have

\[
[\Gr^W_r(W(M)_{N-1})] \simeq \bigoplus_{n \geq 0, -r} a_{r+2n+1} \mathbb{Q}_{D \{r+2n+1\}}(-n - r)[-2n - r].
\]

\footnote{Here $i$ is the canonical inclusion $O^* \hookrightarrow M^{gr}$.}
Proof: Firstly, observe that $\mathcal{M}^{gr} = a_{1*}\mathbb{Q}_{D(1)}$ and $\Lambda^j(a_{1*}\mathbb{Q}_{D(1)}) = a_{j*}\mathbb{Q}_{D(j)}$. So by Remark 15.3 it suffice to construct an $\mathbb{A}^1$-homotopy equivalence

\[
\mathcal{O}^{*\otimes 2(N-1) - n - r} \otimes \Lambda^{r + 2n + 1}\mathcal{M}^{gr} \longrightarrow (\mathcal{W}(M)^n_r/\mathcal{W}(M)_{r - 1}^n).
\]

Let $\beta$ be the canonical map $\hat{W}(M)_r^n \rightarrow \mathcal{W}(M)_r^n$. Let us consider the diagram

\[
\begin{array}{ccc}
\ker(\beta \cdot \varphi_{N - 2}...\varphi_{r - 1}) & \longrightarrow & \mathcal{O}^{*\otimes N - 1 - r + 1} \otimes \hat{W}(M)^n_{r - 1} \longrightarrow \mathcal{W}(M)^n_{r - 1} \\
& \downarrow \hat{\varphi} & \downarrow \text{id} \otimes \varphi_{r - 1} \\
\ker(\beta \cdot \varphi_{N - 2}...\varphi_r) & \longrightarrow & \mathcal{O}^{*\otimes N - 1 - r} \otimes \hat{W}(M)_r^n \longrightarrow \mathcal{W}(M)_r^n
\end{array}
\]

Then 15.5 can be defined as the map between cokernel of $\beta \cdot \varphi_{N - 2}...\varphi_{r - 1}$ and $\mu$. So it suffice to check that $\hat{\varphi}$ is an $\mathbb{A}^1$-homotopy equivalence. Let us consider the diagram

\[
\begin{array}{ccc}
\ker(\beta \varphi_{N - 2}...\varphi_j) & \longrightarrow & \mathcal{O}^{*\otimes N - 1 - j} \otimes \hat{W}(M)^n_j \longrightarrow \text{im}(\beta \varphi_{N - 2}...\varphi_j) \\
& \downarrow \varphi_{N - 2}...\varphi_j & \downarrow \\
\ker(\beta) & \longrightarrow & \hat{W}(M)^n_{N - 1} \longrightarrow \mathcal{W}(M)^n_{N - 1}
\end{array}
\]

Suppose we have $\alpha \in \hat{W}(M)^n_{N - 1}$ and $\beta(\alpha) = 0$. Then $\alpha \in \text{im}(\mathcal{O}^{*\otimes 2(N - 1) - n} \otimes \Lambda^n \mathcal{M}^r \rightarrow \ldots)$. Hence $\alpha \in \text{im}(\mathcal{O}^{*\otimes 2(N - 1) - j - n} \otimes \Lambda^j + 2n + 1 \mathcal{M}^r \rightarrow \ldots)$ and there is some $\tilde{\alpha}$ such that $\varphi_{N - 2}...\varphi_j(\tilde{\alpha}) = \alpha$. Then using commutativity of the diagram we conclude that $\tilde{\alpha} \in \ker(\beta \varphi_{N - 2}...\varphi_j)$. So by the snake lemma the left vertical arrow is a surjection. Then for any $j$ we get the diagram with exact lines

\[
\begin{array}{ccc}
\ker(\varphi_{N - 2}...\varphi_j) & \longrightarrow & \ker(\beta \varphi_{N - 2}...\varphi_j) \\
& \downarrow & \downarrow \\
\ker(\varphi_{N - 2}...\varphi_{j + 1}) & \longrightarrow & \ker(\beta \varphi_{N - 2}...\varphi_{j + 1}) \\
& \downarrow & \downarrow \\
\ker(\varphi_{N - 2}...\varphi_j) & \longrightarrow & \ker(\beta)
\end{array}
\]

Note that $\ker(\varphi_{N - 2}...\varphi_j) = \ker(f_{N - 1 + 2n,...j + 2n + 1})$. So the Proposition follows from Lemma 15.4. \[ \square \]

Relative projection formula. Suppose $f : X \rightarrow Y$ is a proper morphism of schemes. Let $B \rightarrow Rf_*C$ and $B \rightarrow A$ be homomorphisms of motivic $\mathbb{E}_\infty$-algebras. Then by Remark 15.4 and monoidality of $f^*$ the maps $f^*B \rightarrow C$ and $f^*B \rightarrow f^*A$ are also homomorphisms of motivic $\mathbb{E}_\infty$-algebras. So we can form two relative tensor products $A \otimes_B Rf_*C$ and $f^*A \otimes_{f^*B} C$.

**Proposition 15.6.** We have the canonical weak equivalence

\[
A \otimes_B Rf_*C \rightarrow Rf_*(f^*A \otimes_{f^*B} C).
\]

Proof: Note that $f^*$ is monoidal so $f^*Rf_*C$ has the natural structure of $f^*B$-algebra. Moreover, the counit of adjunction $\varepsilon_C : f^*Rf_*C \rightarrow C$ is a morphism of $f^*B$-algebras. Indeed,
by Remark 4.5, $\varepsilon_C$ is a homomorphism of algebras. The homomorphism $f^*B \to C$ is adjoint to the homomorphism $B \to Rf_*C$. So the diagram

$$
\begin{array}{ccc}
  C & \xrightarrow{\varepsilon_C} & f^*Rf_*C \\
\downarrow & & \downarrow \\
  f^*B & & 
\end{array}
$$

is commutative. Now, let us define the map [15.6] as the composition

$$A \otimes_B Rf_*C \xrightarrow{\eta} Rf_*f^*(A \otimes_B Rf_*C) \approx Rf_*(f^*A \otimes f^*B \otimes Rf_*C) \xrightarrow{Rf_*(\text{id} \otimes \varepsilon)} Rf_*(f^*A \otimes f^*C).
$$

Note that $f$ is proper so $Rf_*$ commutes with geometric realizations. Each maps in the composition can be lifted on the level of the corresponding simplicial sets. So we get the map

$$(15.7) \quad (A \otimes_B Rf_*)^* \xrightarrow{\eta} Rf_*(f^*A \otimes f^*B)^*.
$$

For fixed $n$ we have

$$A \otimes B^\otimes n \otimes Rf_*C \xrightarrow{\eta} Rf_*f^*(A \otimes B^\otimes n \otimes Rf_*C) \xrightarrow{f^*} Rf_*(f^*A \otimes f^*B^\otimes n \otimes B^\otimes C) \xrightarrow{Rf_*(\text{id} \otimes \varepsilon)} Rf_*(f^*A \otimes f^*C).
$$

Note that this maps induces by the exchange transformation $Ex(f^*, \otimes)$ (see [CD19], Section 1.1.31). So, using projection formula, we conclude that (15.7) is a weak equivalence of simplicial objects. □

The limit log scheme. By the relative projection formula we get

$$LM_f \simeq Rf_*(\mathbb{Q} \otimes \text{Sym}^*(\mathbb{Q} \oplus \mathbb{Q}(-1) \otimes \mathbb{Q}(-1))); \text{Sym}^*(Mgr(-1) \otimes \mathbb{Q}(-1) \otimes \mathbb{Q}(-1))).
$$

But $\text{Sym}^*$ is monoidal and commutes with colimits. So

$$
\mathbb{Q} \otimes \text{Sym}^*(\mathbb{Q} \oplus \mathbb{Q}(-1) \otimes \mathbb{Q}(-1)) \otimes \text{Sym}^*(\tilde{M}^\text{gr}(-1) \otimes \mathbb{Q}(-1)) \simeq \text{Sym}^*(\tilde{M}^\text{gr}(-1) \otimes \mathbb{Q}(-1))
$$

where $\tilde{M}^\text{gr} := \mathbb{Q} \oplus \mathbb{Q}(-1) \otimes \mathbb{Q}(-1)$. Note that $\tilde{M}^\text{gr} = \sum_{\mathbb{G}_m} L_{\mathbb{A}^1} \text{Ex}^\text{log}(\tilde{M}^\text{gr} \otimes \mathbb{Q})$ where $\tilde{M}^\text{gr}$ is the colimit of the diagram

$$
\begin{array}{ccc}
  O^* & \xrightarrow{t} & M^\text{gr} \\
\uparrow & & \\
O^* \oplus \mathbb{Z}
\end{array}
$$
Moreover, one can note that the sheaf \( \tilde{\mathcal{M}}^{gr} \) corresponds to the fiber product

\[
(X_s, O^* \leftarrow \mathcal{M}^{gr})
\]

\[
\downarrow f
\]

\[
X_s \rightarrow X_s \times pt_{log}
\]

in the category of virtual log schemes. Let us denote the fiber product by \( X_{lim} \).

**Remark 15.7.** Note that the fiber product make sense only in the context of virtual log geometry. Indeed, there is no morphisms \( X_s \rightarrow X_s \times pt_{log} \) in the classical category of logarithmic schemes.

By the previous we get

**Proposition 15.8.** The relative projection formula induces the canonical weak equivalence

\[
LM_f^* \simeq [X_{lim}]^{log}.
\]

In particular, any inclusion \( k \subset C \) induces the isomorphism

\[
H^*(X_{lim}^{log}, \mathbb{Q}) \simeq H^*(X_t, \mathbb{Q}).
\]

Observe that the sheaf \( \tilde{\mathcal{M}}^{gr} \) is closely related to the complexes \( \mathcal{W}_r \).

**Lemma 15.9.** For any \( n \) we have

\[
\text{im}(\Lambda^n \mathcal{M}^{gr} \xrightarrow{\Lambda t} \Lambda^{n+1} \mathcal{M}^{gr}) \simeq \ker(\Lambda^{n+1} \mathcal{M}^{gr} \xrightarrow{\Lambda t} \Lambda^{n+2} \mathcal{M}^{gr}) \simeq \Lambda^n \tilde{\mathcal{M}}^{gr}.
\]

As a consequence,

\[
H^0(\mathcal{W}_r) = \Lambda^r \tilde{\mathcal{M}}^{gr}, \ H^{N-1}(\mathcal{W}_r) = \Lambda^{r+N-1} \tilde{\mathcal{M}}^{gr} \text{ and } H^n(\mathcal{W}_r) = 0 \text{ for } n \neq 0, N - 1.
\]

**Remark 15.10.** Suppose we have two complexes \( A, B \in \text{Comp}^\bullet(Sh_{et}(Sm/X_0, \mathbb{Q})) \) together with filtrations \( F^\bullet A, F^\bullet B \). Let \( g : A \rightarrow B \) be a filtered morphism. If the induced maps \( \text{Gr}_r^{FA}(A) \rightarrow \text{Gr}_r^{FB}(B) \) are \( \mathbb{A}^1 \)-homotopy equivalences then so is \( g \). In particular, \( L_{k^1}A \simeq 0 \) if \( L_{k^1}\text{Gr}^{FA}(A) \simeq 0 \).

Now, let us construct a quasi-isomorphism \( \text{Sym}^*(\tilde{\mathcal{M}}^{gr}(-1)[-1]) \simeq [W(M)_{N-1}]. \) Firstly, let us consider the diagram

\[
\begin{array}{ccc}
\Lambda^{N-1} \tilde{\mathcal{M}}^{gr} & \rightarrow & \mathcal{W}_N \\
\uparrow & & \uparrow \\
0 & \rightarrow & \mathcal{O}^* \otimes \mathcal{W}_0 \\
\downarrow & & \downarrow \\
\Lambda^{2N-1} \tilde{\mathcal{M}}^{gr} & \rightarrow & \Lambda^{N-1} \tilde{\mathcal{M}}^{gr}
\end{array}
\]

The lines of the diagram is exact by Lemma 15.9. The right vertical arrow is a composition of \( \mathbb{A}^1 \)-homotopy equivalences. So the map \( \Lambda^{N-1} \tilde{\mathcal{M}}^{gr} \rightarrow \text{Cone}(\mathcal{O}^* \otimes \mathcal{W}_0 \rightarrow \mathcal{W}_N) \) is a \( \mathbb{A}^1 \)-homotopy equivalence.
On the other hand, note that the map
\[ \mathcal{O}^\otimes_{N-1} \otimes W_N \to \tilde{W}(M)_{N-1} \]
is a $A^1$-homotopy equivalence. Indeed, let we consider the silly filtration on $\mathcal{O}^\otimes_{N-1} \otimes W_N$ and $\tilde{W}(M)_{N-1}$. Then on the n-th graded pieces we get
\[ \text{id} \otimes (f_{N+n} \cdots f_{N+2n-1}) : \mathcal{O}^\otimes_{N-1} \otimes \Lambda^{N+n} \mathcal{M}^{\text{gr}} \to \mathcal{O}^\otimes_{N-1-n} \otimes \Lambda^{N+2n} \mathcal{M}^{\text{gr}}. \]
Again, this map is a composition of $A^1$-homotopy equivalences and we can use Remark 15.10. So we proved

**Proposition 15.11.** We have a chain of homotopy equivalences

\[ \mathcal{O}^\otimes_{N-1} \otimes \Lambda^{N-1} \mathcal{M}^{\text{gr}} \to \text{Cone}(\mathcal{O}^\otimes_{2N-1} \otimes W_0 \to W_N) \]
\[ \text{Cone}(\mathcal{O}^\otimes_{2N-1} \otimes W_0 \to \tilde{W}(M)_{N-1}) \to W(M)_{N-1} \]

which induces the quasi-isomorphism
\[ \text{Sym}^*(\tilde{M}^{\text{gr}}(-1)[-1]) \simeq [W(M)_{N-1}]. \]

**Motivic logarithm of monodromy.** Firstly, let us recall the classical picture. In [PS08], Steenbrink defined the morphism $\nu : \psi_f^{Hdg} \mathbb{Q} \to \psi_f^{Hdg} \mathbb{Q}(-1)$ which maps $W(M)_r$ to $W(M)_{r-2}$. On the level of the complex $\text{Tot}(C^{\bullet\bullet})$ the construction is the following.

Let $K^{\bullet\bullet}$ be a bicomplex. Note that the operation $K^{p,q} \mapsto K^{p+1,q-1}$ does not change the totalization. Let us define the map of bicomplexes by the rule

\[ 0 \to S^r[1] \to S^{r+2}[2] \to \ldots \]
\[ S^{r-1}[0] \to S^{r-2}[1] \to S^{r-4}[2] \to \ldots \]

So we get the map
\[ \text{Tot}(\tilde{W}(M)_r) \to \text{Tot}(\tilde{W}(M)_{r-2}). \]

We also have the map of bicomplexes

\[ 0 \to S^0[1] \to S^1[2] \to \ldots \]
\[ S^0[0] \to S^1[1] \to S^2[2] \to \ldots \]

which induces the map
\[ \text{Tot}(W_0) \to \text{Tot}(W_0). \]
Then, taken limit by \( r \), we get the endomorphism
\[
\text{Tot}(C^{\bullet, \bullet}) \xrightarrow{\nu} \text{Tot}(C^{\bullet, \bullet})
\]
such that \( \nu(W(M)_r) \subset W(M)_{r-2} \). Steenbrink proved that \( 2\pi i \nu \) acts on \( H^k(X_0, \text{Tot}(C^{\bullet, \bullet})) \simeq H^k(X_t, \mathbb{Q}) \) as a logarithm of monodromy \( \log(\gamma) \).

It should be noted that \( \nu \) is nilpotent. Indeed, \( \nu^{r+p+1} \) maps \( S^{r+2p+1}(\exp) / S^p(\exp) \) to \( S^{r+2p+1}(\exp) / S^r(\exp) = 0 \). But \( S^{r+2N+1}(\exp) / S^N(\exp) \) is quasi-isomorphic to 0. So \( \nu^{r+N+1} = 0 \) in the homotopy category.

Now, let us define a motivic analog of \( \nu \). Note that we have the map of complexes
\[
\tilde{W}(M)_r \to \mathcal{O}^* \otimes \tilde{W}(M)_{r-2}[1]
\]
which can be defined using the diagram
\[
\begin{array}{ccccccc}
0 & \to & \mathcal{O}^{*N-1} \otimes \Lambda^{r+1}M^g & \to & \mathcal{O}^{*N-2} \otimes \Lambda^{r+3}M^g & \to & \ldots \\
\downarrow & & \downarrow{id} & & \downarrow{id} & & \\
\mathcal{O}^{*N} \otimes \Lambda^{r-2+1}M^g & \to & \mathcal{O}^{*N-1} \otimes \Lambda^{r+2+3}M^g & \to & \mathcal{O}^{*N-2} \otimes \Lambda^{r-2+5}M^g & \to & \ldots
\end{array}
\]
Let us rewrite this as the map
\[
\mathcal{O}^* \otimes (\mathcal{O}^{*N-1-r} \otimes \tilde{W}(M)_r) \to (\mathcal{O}^{*N-1-r+2} \otimes \tilde{W}(M)_{r-2})[1].
\]
We also have the map
\[
\mathcal{O}^* \otimes (\mathcal{O}^{*2(N-1)} \otimes \mathcal{W}_0) \to (\mathcal{O}^{*2(N-1)} \otimes \mathcal{W}_0)[1]
\]
defined by the rule
\[
\begin{array}{ccccccc}
0 & \to & \mathcal{O}^{*2(N-1)} \otimes \mathcal{O}^* & \to & \mathcal{O}^{*2(N-1)} \otimes (\mathcal{O}^* \otimes M^g) & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}^{*2(N-1)} & \to & \mathcal{O}^{*2(N-1)} \otimes M^g & \to & \mathcal{O}^{*2(N-1)} \otimes \Lambda^2M^g & \to & \ldots
\end{array}
\]
Here the vertical arrows induced by the rightmost differential in the Koszul complex (15.4). So we get the morphism
\[
\mathcal{O}^* \otimes \mathcal{W}(M)_{N-3} \xrightarrow{\nu} \mathcal{W}(M)_{N-1}[1]
\]
which maps \( \mathcal{O}^* \otimes \mathcal{W}(M)_r \) to \( \mathcal{W}(M)_{r-2}[1] \). It suffice to apply \( \Sigma^\infty_{\mathfrak{fr}} \), \( L_{\mathbb{A}^1} \) and \( - \otimes \mathbb{Q}(-1)[-1] \) and get the maps
\[
[W(M)_r] \xrightarrow{\nu} [W(M)_{r-2}][-1].
\]
Moreover, using Proposition 15.11 we can define the \textit{motivic logarithm of monodromy}
\[
\text{Sym}^*(\tilde{M}^g(-1)[-1]) \xrightarrow{\nu} \text{Sym}^*(\tilde{M}^g(-1)[-1])(-1)
\]
such that all the diagrams

\[
\begin{array}{ccc}
\text{Sym}^*(\tilde{M}^r(-1)[-1])) & \xrightarrow{\nu} & \text{Sym}^*(\tilde{M}^r(-1)[-1]))(-1) \\
[\mathcal{W}(M)_r] & \xrightarrow{\nu} & [\mathcal{W}(M)_{r-2}][-1] \\
\end{array}
\]

is commutative. Notice that \(\nu\) is nilpotent by the same arguments as in the classical case.

16. Relations with Ayoub’s nearby cycles

Specialization systems. Let \(B\) be a scheme together with closed and open subsets \(Z, U \subset B\). Following Ayoub \([\text{Ays}07]\), a specialization system is a collection of functors

\[Sp_f : DA_{et}(X_U, \mathbb{Q}) \to DA_{et}(X_Z, \mathbb{Q}),\]

one for each \(B\)-scheme \(f : X \to B\), satisfied a certain list of axioms. In particular, for any composition \(Y \xrightarrow{g} X \xrightarrow{f} B\) we have the natural transformations

\[\alpha_g : g^* Sp_f \to Sp_fg^*, \quad \beta_g : Sp_f Rg_* \to Rg_* Sp_fg\]

such that \(\alpha_g\) is an equivalence when \(g\) is smooth and \(\beta_g\) is an equivalence when \(g\) is proper.

Let \(\varphi : B' \to B\) be a morphism of scheme. Let \(Z' := \varphi^{-1}(Z)\) and \(U' := \varphi^{-1}(U)\). Using \(f\), one can restrict a specialization system from \((Z, B, U)\) to \((Z', B', U')\) by an obvious way.

Now, let us consider the following category:

- the objects given by pairs \((C, D)\) where \(C\) is curve over \(k\) and \(D\) is a divisor;
- the morphisms between \((C, D)\) and \((C', D')\) given by morphisms \(f : C \to C'\) such that \(f^{-1}(D') = D\).

Lemma 16.1. Suppose we choose for each \((D, C, C \setminus D)\) a specialization system \(Sp^{(C,D)}\) such that the collection \(\{Sp^{(C,D)}\}\) is coherent under restrictions. Then the collection can be recovered using \(Sp^{(\mathbb{A}^1,0)}\).

Proof: Firstly, note that the collection uniquely defined by \(Sp^{(C,D)}\) with affine \(C\). Indeed, let \(W_j \hookrightarrow C\) be an open affine cover of an arbitrary \(C\) and \(f : X \to C\) be a \(C\)-scheme. Let us denote by \(\nu_j\) the inclusions \(X_{W_j} \hookrightarrow X\). Then

\[\nu_j^* Sp_f^{(C,D)} \simeq Sp_{f\nu_j}^{(C,D)} \nu_j^* = Sp_{f\nu_j}^{(W_j,D_j)} \nu_j^*.
\]

So we can recover \(Sp^{(C,D)}\) by gluing together \(Sp_{f\nu_j}^{(W_j,D_j)} \nu_j^*\).

On the other hand, for affine \(C\) we can choose \(g : C \to \mathbb{A}^1\) such that \(D = g^{-1}(0)\). Then \(Sp^{(C,D)}\) is the restriction of \(Sp^{(\mathbb{A}^1,0)}\). \(\square\)
Motivic nearby cycles. Let $(C, s)$ be a curve with a marked point. In [Ayo07], Ayoub constructed \textit{unipotent motivic nearby cycles} $\Upsilon_*$ as a specialization system over $(s, C, \hat{C})$. By the previous it suffice to define $\Upsilon_*$ for $C = \mathbb{A}^1$, $s = 0$.

Let $f : X \to \mathbb{A}^1$ be an $\mathbb{A}^1$-scheme. Let us consider the diagram

\[
\begin{array}{ccc}
\mathbb{G}_m & \xrightarrow{id} & \mathbb{G}_m \\
\downarrow \Delta & & \\
\mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{1} & \mathbb{G}_m
\end{array}
\]

This is a diagram of coalgebras in $Sm/\mathbb{G}_m$. Note that the canonical functor $Sm/\mathbb{G}_m \to DA_{et}(\mathbb{G}_m, \mathbb{Q})$ is monoidal. So we get the diagram of $E_\infty$-coalgebras in $DA_{et}(\hat{X}, \mathbb{Q})$

\[
\mathbb{Q}_{\hat{X}} \xrightarrow{\Delta} \mathbb{Q}_{\hat{X}} \oplus \mathbb{Q}_{\hat{X}}(1)[1] \xleftarrow{} \mathbb{Q}_{\hat{X}}
\]

and the cosimplicial object $\text{coBar}_{\mathbb{Q}_{\hat{X}} \oplus \mathbb{Q}_{\hat{X}}(1)[1]}((\mathbb{Q}_{\hat{X}}, \mathbb{Q}_{\hat{X}})^*)$. Let $\mathcal{F}$ be a motivic sheaf over $\hat{X}$. Then, following Ayoub,

$$\Upsilon_f\mathcal{F} := |i^* Rj_* (\text{Hom}(\text{coBar}_{\mathbb{Q}_{\hat{X}} \oplus \mathbb{Q}_{\hat{X}}(1)[1]}((\mathbb{Q}_{\hat{X}}, \mathbb{Q}_{\hat{X}})^* \mathcal{F}))|.$$

\begin{remark}
Let $U$ be a Zariski neighborhood of $s$. Let us consider the composition $X_U \xrightarrow{g} X \xrightarrow{f} C$. Then $\Upsilon_f(\mathcal{F}) = g^* \Upsilon_f(\mathcal{F}) \simeq \Upsilon_f(g^* \mathcal{F})$. So we get well defined functor

$$\Upsilon_f : DA_{et}(X_\eta, \mathbb{Q}) \to DA_{et}(X_\eta, \mathbb{Q}).$$

\end{remark}

The equivalence of construction. Let $f : X \to C$ be a proper semi-stable degeneration. Now, we want to construct a natural weak equivalence between the limit motive $LM_f$ and $\Upsilon_{id}([X_\eta]^*)$. Let us start from the following

\begin{lemma}
Let $(X, D)$ be a log smooth scheme with smooth $X$. Let $j : X - D \hookrightarrow X$ be the open immersion. Then there is a natural weak equivalence

$$\text{Sym}^*(\mathcal{M}_{(X, D)}^{gr}(-1)[-1]) \simeq Rj_* \mathbb{Q}.$$

\end{lemma}

\begin{proof}
We have the natural isomorphism of algebras $j^* \text{Sym}^*(\mathcal{M}_{(X, D)}^{gr}(-1)[-1]) \simeq \mathbb{Q}$. So we get the homomorphism $\text{Sym}^*(\mathcal{M}_{(X, D)}^{gr}(-1)[-1]) \to Rj_* \mathbb{Q}$. Let $Y$ be a scheme smooth over $X$. Then, by Corollary \ref{13.6} and the construction of functoriality of $H_{log}^{p,q}$, the map

$$\text{Hom}(Y(q)[p], \text{Sym}^*(\mathcal{M}_{(X, D)}^{gr}(-1)[-1])) \to \text{Hom}(Y(q)[p], Rj_* \mathbb{Q})$$

\end{proof}
is a quasi-isomorphism for any \( p \) and \( q \). □

By the previous, the algebra \( \text{Sym}^\ast(\tilde{M}^{gr}(-1)[-1]) \) is equivalent to the bar construction associated with the diagram

\[
\begin{array}{ccc}
\text{Sym}^\ast(\tilde{M}^{gr}_{X_0}(-1)[-1]) & \mu^\ast \\
\mu & \\
\mathbb{Q}X_0 \oplus \mathbb{Q}X_0(-1)[-1] & \mathbb{Q}X_0
\end{array}
\]

Note that \( f \) comes from the global section of \( M^{gr}(X,D) \). So the diagram can be extended to the diagram

\[
\begin{array}{ccc}
Rj_*\mathbb{Q}_{\tilde{X}} & \\
\mu & \\
\mathbb{Q}X \oplus \mathbb{Q}X(-1)[-1] & \mathbb{Q}X
\end{array}
\]

So we get

\[
\text{Sym}^\ast(\tilde{M}^{gr}(-1)[-1]) \simeq i^\ast(Rj_*\mathbb{Q}_{\tilde{X}} \otimes [G_m]_X^\ast \mathbb{Q}X) \simeq |i^\ast(Rj_*\mathbb{Q}_{\tilde{X}} \otimes [G_m]_X \mathbb{Q}X)|.
\]

Moreover,

\[
i^1((Rj_*\mathbb{Q}_{\tilde{X}}) \otimes [G_m]_X^\ast \mathbb{Q}X) = i^\ast(Rj_*\mathbb{Q}_{\tilde{X}} \otimes (\mathbb{Q} \oplus \mathbb{Q}(-1)[-1])^\otimes n \otimes \mathbb{Q}) \simeq \oplus i^1Rj_*\mathbb{Q}(-k)[-k] = 0.
\]

So

\[
(Rj_*\mathbb{Q}_{\tilde{X}}) \otimes [G_m]_X^\ast \mathbb{Q}X \simeq Rj_*j^\ast((Rj_*\mathbb{Q}_{\tilde{X}}) \otimes [G_m]_X^\ast \mathbb{Q}X) \simeq Rj_*(\mathbb{Q} \otimes [G_m]_X^\ast \mathbb{Q})^\ast
\]

and

\[
(16.1) \quad \text{Sym}^\ast(\tilde{M}^{gr}(-1)[-1]) \simeq |i^\ast Rj_*(\mathbb{Q} \otimes [G_m]_X^\ast \mathbb{Q})^\ast|
\]

Here the relative tensor product is the product associated with the diagram

\[
\begin{array}{ccc}
\mathbb{Q}_{\tilde{X}} & \\
\mu & \\
\mathbb{Q}X \oplus \mathbb{Q}X(-1)[-1] & \mathbb{Q}X
\end{array}
\]

**Theorem 16.4.** Suppose that \( k \) satisfies the Beilinson-Soulé vanishing conjecture. Then

\[
\text{Sym}^\ast(\tilde{M}^{gr}(-1)[-1]) \simeq \Upsilon_f(\mathbb{Q})
\]

and, consequently,

\[
\text{LM}_f \simeq \Upsilon_{id}([X_\eta]^\ast).
\]

**Proof:** It suffice construct the first equivalence. By Remark 16.2 we may assume that \( C \) is affine. Let us choose a function \( g \) on \( C \) with the property \( g^{-1}(0) = s \) and denote
by $\varphi$ the composition $X \xrightarrow{f} C \xrightarrow{g} \mathbb{A}^1$. Note that $\Upsilon_f \mathbb{Q}$ can be given by the formula $|R^\varphi_j(\text{Bar}_{\mathbb{Q}\bar{X}} \mathbb{Q}\bar{X}(-1)[-1])(Q_{\bar{X}}, Q_{\bar{X}}^*)|$ where the bar construction associated with the diagram

\[
\begin{array}{ccc}
Q_{\bar{X}} & \xrightarrow{\Delta^*} & Q_{\bar{X}} \\
\downarrow & & \downarrow \\
Q_{\bar{X}} \oplus Q_{\bar{X}}(-1)[-1] & \longrightarrow & Q_{\bar{X}}
\end{array}
\]

(16.3)

Then, according [16.1] it enough to check that the diagrams [16.2] and [16.3] define equivalent bar constructions.

Note that the map $\Delta^*$ is dual to the global section $\varphi = g \cdot f$ of sheaf $\mathcal{O}^*$ on $\bar{X}$. On the other hand, the map $\mu$ corresponding to the map $((id, \varphi) : \mathcal{O}^* \oplus \mathbb{Z} \to \mathcal{O}^*$. The both map can be represented as the pullbacks from $\mathbb{G}_m$ under $\varphi$.

Remark 16.5. Let $C^\otimes$ be a monoidal $\infty$-category. Let $A \xrightarrow{\alpha} A'$ be a weak equivalence of $\mathbb{E}_\infty$-algebras. Suppose we have the diagrams

\[
\begin{array}{ccc}
B & \xrightarrow{g} & C \\
f \downarrow & & \downarrow f \\
A & \xrightarrow{g'} & A'
\end{array}
\]

of $\mathbb{E}_\infty$-algebras such that $g' \alpha = g$, $B' = B \otimes_A A'$ and $f' = f \otimes_A A'$. Then $B' \otimes_A C \simeq B \otimes_A C$ as objects of $C^\otimes$. Indeed, by results of ([HA], Section 4.4.3), $B' \otimes_A C = (B \otimes_A A') \otimes_A C \simeq B \otimes_A (A' \otimes_A C) \simeq B \otimes_A C$.

Remark 16.6. Let $B$ and $B'$ be as above. Then $B \simeq B'$ in $C^\otimes$. In fact, $\alpha : A \to A'$ induce the weak equivalence $B \simeq A \otimes_B A' \rightarrow A' \otimes_B B$.

Remark 16.7. Note that the unit of adjunction $\text{Sym}^*([G_m]^*_{G_{\mathbb{m}}}) \xrightarrow{\epsilon} [G_m]^*_{G_{\mathbb{m}}}$ is a weak equivalence of algebras. Indeed, by conservativity of the forgetful functor it suffice to check that $\epsilon$ is a weak equivalence of the motivic sheaves. But the natural inclusion $[G_m]^*_{G_{\mathbb{m}}} \rightarrow \text{Sym}^*([G_m]^*_{G_{\mathbb{m}}})$ is an equivalence and we can use the triangle identities.

Let us consider the automorphism

\[
(1, -1) : \mathbb{Q} \oplus \mathbb{Q}(-1)[-1] \longrightarrow \mathbb{Q} \oplus \mathbb{Q}(-1)[-1].
\]

By the Remark [16.7] it is an automorphism of algebras. For any module $M$ let us denote by $M \otimes_{(1, -1)} [G_m]^*$ the tensor product $M \otimes_{\mathbb{Q} \oplus \mathbb{Q}(-1)[-1]} \mathbb{Q} \oplus \mathbb{Q}(-1)[-1]$ with respect to $(1, -1)$. 

Then $\mathbb{Q} \otimes_{(1,-1)} [\mathbb{G}_m]^* \simeq \mathbb{Q}$ and $(1,t) \otimes_{(1,-1)} [\mathbb{G}_m]^* = (1,-t)$ in the homotopy category of $E_\infty$-algebras. Indeed, the first equivalence follows from Remark 16.6. To prove the second equality, let us consider the diagram

$$
\begin{array}{ccc}
\mathbb{Q} & \xrightarrow{a} & \mathbb{Q} \simeq \mathbb{Q} \otimes_{(1,-1)} [\mathbb{G}_m]^* \\
(1,t) & \uparrow & \uparrow (b,c) \\
\mathbb{Q} \oplus \mathbb{Q}(-1)[-1] & \xrightarrow{(1,-1)} & \mathbb{Q} \oplus \mathbb{Q}(-1)[-1].
\end{array}
$$

Note that $\mathbb{Q} \otimes_{(1,-1)} [\mathbb{G}_m]^*$ is an algebra with unit. So $a = b = 1$. Then, by commutativity, $c = -t$.

Now, one can apply Remark 16.5 to prove the Theorem 16.4. Namely, we should check that the homomorphisms $\mathbb{Q}_{\mathbb{G}_m} \oplus \mathbb{Q}_{\mathbb{G}_m}(-1)[-1] \xrightarrow{(1,t)^*} \mathbb{Q}_{\mathbb{G}_m}$ and $\mathbb{Q}_{\mathbb{G}_m} \oplus \mathbb{Q}_{\mathbb{G}_m}(-1)[-1] \xrightarrow{(1,-t)} \mathbb{Q}_{\mathbb{G}_m}$ are coincide in the homotopy category of $E_\infty$-algebras. Since the algebra $[\mathbb{G}_m]^*_m$ is free it suffice to check that $(1,t)^* = (1,-t)$ in the homotopy category of motivic sheaves $DA_{et}(\mathbb{G}_m, \mathbb{Q}) \setminus \mathbb{Q}$. Then the Theorem follows from the Lemma.

**Lemma 16.8.** Suppose that $k$ satisfies the Beilinson-Soulé vanishing conjecture. Let $f : \mathbb{Q} \rightarrow \mathbb{Q}(1)[1]$ be a morphism in $DA_{et}(\mathbb{G}_m, \mathbb{Q})$. Let $\hat{f}$ and $f(-1)[-1]$ be the images of $f$ in $\text{Hom}(\mathbb{Q}(-1)[-1], \mathbb{Q})$ under the functors $\text{Hom}(-, \mathbb{Q})$ and $- \otimes \mathbb{Q}(-1)[-1]$. Then $\hat{f} = - f(-1)[-1]$.

**Proof:** Let $DTM(\mathbb{G}_m)_\mathbb{Q} \subset DA_{et}(\mathbb{G}_m, \mathbb{Q})$ be the triangulated subcategory of Tate motives. By result of [Lev93], there is weight t-structure on $DTM(\mathbb{G}_m)_\mathbb{Q}$ together with tensor exact functors

$$
gr^W_q : DTM(\mathbb{G}_m)_\mathbb{Q} \rightarrow T_q$$

where $T_q \simeq D(Vect_\mathbb{Q})$ is a full triangulated subcategory of $DTM(\mathbb{G}_m)_\mathbb{Q} \subset DA_{et}(\mathbb{G}_m, \mathbb{Q})$ generated by $\mathbb{Q}(q)[n]$, $n \in \mathbb{Z}$.

By assumption, $\mathbb{G}_m$ is also satisfies Beilinson-Soulé conjecture. Indeed,

$$H^p(\mathbb{G}_m, \mathbb{Q}(q)) = H^{p-1}(k, \mathbb{Q}(q-1)) \oplus H^p(k, \mathbb{Q}(q)).$$

So we can apply Theorem 1.4 of [Lev93] and get the following:

- There is the non-degenerate t-structure on $DTM(\mathbb{G}_m)_\mathbb{Q}$ with the heart $TM(\mathbb{G}_m)_\mathbb{Q}$ which consists $M \in DTM(\mathbb{G}_m)_\mathbb{Q}$ such that all $gr^W_i(A) \simeq \mathbb{Q}(-i)^{n_i}$ for some $n_i \in \mathbb{N}$;
- The category $TM(\mathbb{G}_m)_\mathbb{Q}$ is closed under extensions and

$$\text{Ext}^1_{TM(\mathbb{G}_m)_\mathbb{Q}}(B, A) \simeq \text{Hom}_{DA_{et}(\mathbb{G}_m, \mathbb{Q})}(B[-1], A);$$

- The fiber functor $\oplus \text{id} : TM(\mathbb{G}_m)_\mathbb{Q} \rightarrow Vect_\mathbb{Q}$ is a faithful exact tensor functor. By the same arguments as [DG05], the group $\text{Aut}(\oplus \text{id})$ is isomorphic to $\mathbb{G}_m \ltimes U$ where $U$ is a pro-unipotent algebraic group.
Now, let $\mathbb{Q}(1) \to A \to \mathbb{Q}$ be the extension associated with $f$ and let
\[
\mathbb{Q} \to \check{A} \to \mathbb{Q}(-1); \quad \mathbb{Q} \to A(-1) \to \mathbb{Q}(-1)
\]
be the extensions associated with $\check{f}$ and $f(-1)[-1]$. It suffice to construct an appropriate map of the extensions in the category of representations of $G = \mathbb{G}_m \ltimes U$. Let $(\lambda, u)$ be an element of $\mathbb{G}_m \ltimes U$. Let $(v_1, v_2)$ be the basis of $A$ such that $v_1$ corresponds to the inclusion $\mathbb{Q}(1) \to A$. By construction, the action of $G$ preserves the weight filtration so
\[
\rho(\lambda, u) = \begin{pmatrix} \lambda & a(\lambda, u) \\ 0 & 1 \end{pmatrix}.
\]
Then the action of $(\lambda, u)$ on $A(-1)$ is given by $\begin{pmatrix} 1 & a(\lambda, u) \lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix}$. Finally, the action on $\check{A}$ given by
\[
(\rho(\lambda, u)^{-1})^T = \begin{pmatrix} \lambda^{-1} & 0 \\ -a(\lambda, u) \lambda^{-1} & 1 \end{pmatrix}
\]
where $\delta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponds to the inclusion $\mathbb{Q} \to \check{A}$. So we have the map of extensions
\[
\begin{array}{ccc}
\mathbb{Q} & \longrightarrow & A(-1) \longrightarrow \mathbb{Q}(-1) \\
\downarrow & & \downarrow B \\
\mathbb{Q} & \longrightarrow & \check{A} \longrightarrow \mathbb{Q}(-1)
\end{array}
\]
where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with respect to the basis $v_1, v_2$ and $\delta_1, \delta_2$.

**Appendix A. The inverse image of an algebraic group**

Suppose that $G$ is an abelian algebraic group over $k$. Then the for each $X \in \text{Sch}/k$ the group scheme $G \times_k X$ defines the sheaf $G_X \in \text{Sh}_{et}(\text{Sm}/X, \mathbb{Z})$. Suppose that there is a morphism $f: Y \longrightarrow X$. For any scheme $U$ smooth over $X$ and $\varphi \in G(U)$ the composition
\[
(U \xrightarrow{\varphi} G) \mapsto (U \times_X Y \longrightarrow U \xrightarrow{\varphi} G)
\]
define the map
\[
f_\varphi : G_X \longrightarrow f_* G_Y
\]
where $f_* : \text{Sh}_{et}(\text{Sm}/Y, \mathbb{Z}) \longrightarrow \text{Sh}_{et}(\text{Sm}/X, \mathbb{Z})$. Let $f_\varphi^* \check{G}$ be the adjoint map.

**Theorem A.1.** Suppose that $G$ is smooth over $k$. Then for any morphism $f: Y \longrightarrow X$ the map $f_\varphi^* : f^{-1} G_X \longrightarrow G_Y$ is an isomorphism.
Proof: Let \( G \) be an abelian group. Let \( I \subset \mathbb{Z}[G] \) be the subgroup generated by \([0]\) and \( J \subset \mathbb{Z}[G \times G] \) be the subgroup generated by all \([[0, g]] \) and \([[g, 0]] \) for any \( g \in G \). Let us define the complex \( Q_{\leq -1}^*(G) \):

\[
\mathbb{Z}[G \times G]/J \xrightarrow{d} \mathbb{Z}[G]/I
\]

where \( d([((g_1, g_2))]) = [g_1] + [g_2] - [g_1 + g_2] \).

Now, let \( S \) be a scheme. Then we can define the complex of presheaves \( Q_{\leq -1}^*(G_X) \) on \( \text{Sm}/S \) by the rule

\[
U \mapsto Q_{\leq -1}^*(G(U)).
\]

Note that \( H^0(Q_{\leq -1}^*(G)) = G \) for any \( G \). So \( H^0(Q_{\leq -1}^*(G_X)) = G_X \). For any \( f : Y \to X \) we have the canonical map

(A.1) \( Q_{\leq -1}^*(G_X) \to f_*Q_{\leq -1}^*(G_Y) \)

induced by \( \mathbb{Z}[f_G] \) and \( \mathbb{Z}[f_{G \times G}] \). The map \( f_G \) is obtained from (A.1) by applying of the functor \( H^0 \). So it suffice to check that the map adjoint to (A.1) is an isomorphism.

Observe that \( f^{-1}\mathbb{Z}[G_X] = \mathbb{Z}[G_Y] \) by definition of \( f^{-1} \). Moreover, \( \mathbb{Z}[f_G] \) and \( \mathbb{Z}[f_{G \times G}] \) are the units of adjunction. The map

(A.2) \( \mathbb{Z}[G_X]/I_X \to f_*(\mathbb{Z}[G_Y]/I_Y) \)

is obtained by applying of the functor \( H^0 \) to the morphism of complexes

\[
\begin{array}{ccc}
\mathbb{Z}[p_X] & \xrightarrow{f_*[p_Y]} & f_*\mathbb{Z}[p_Y] \\
\downarrow{[0]} & & \downarrow{[0]} \\
\mathbb{Z}[G_X] & \xrightarrow{f_*[f_G]} & f_*\mathbb{Z}[G_Y]
\end{array}
\]

The horizontal map are the units of adjunction. So the adjoint morphism is an identity and (A.2) is an isomorphism. Analogically, the map

(A.3) \( \mathbb{Z}[G_X \times G_X]/J_X \to f_*(\mathbb{Z}[G_Y \times G_Y]/J_Y) \)

is obtained from the morphism of complexes

\[
\begin{array}{ccc}
\mathbb{Z}[G_X] \oplus \mathbb{Z}[G_X] & \xrightarrow{\mathbb{Z}[f_G] \oplus \mathbb{Z}[f_G]} & f_*(\mathbb{Z}[G_Y] \oplus \mathbb{Z}[G_Y]) \\
\downarrow{i_{(0, 1)} + i_{(1, 0)}} & & \downarrow{} \\
\mathbb{Z}[G_X \times G_X] & \xrightarrow{\mathbb{Z}[f_{G \times G}]} & f_*\mathbb{Z}[G_Y \times G_Y]
\end{array}
\]

So the map adjoint to (A.3) is an isomorphism by the same arguments. \( \square \)
Corollary A.2. For a scheme $S$ let us denote by $\mathcal{O}_{Sm/S}^*$ the étale sheaf of invertible functions on $Sm/S$. Let $f: Y \to X$ be a morphism and $f^*: f^{-1}\mathcal{O}_{Sm/X}^* \to \mathcal{O}_{Sm/Y}^*$ be the canonical map. Then the map $f^*$ is an isomorphism.

Appendix B. $(\infty, 1)$-Grothendieck construction

In classical category theory the Grothendieck construction provide the equivalence of categories between the category of Grothendieck fibration over the category $C$ and the category of pseudofunctors $\text{Fun}(C^{\text{op}}, \text{Cat})$. The similar pictures exists in the word of infinity categories. The notation of Cartesian fibrations of simplicial sets is a direct analog of Grothendieck fibrations of 1-categories. Moreover, for any simplicial set $S$ we have the canonical equivalence of $\infty$-categories $\text{Fun}_\infty(S^{\text{op}}, \text{Cat}_{(\infty, 1)}) \simeq \text{Cart}_S$ where $\text{Cart}_S$ is the category of Cartesian fibrations over $S$. This equivalence can be describe using the straightening and unstraightening functors:

$$\text{Un}^+: \text{Fun}_\infty(S^{\text{op}}, \text{Cat}_{(\infty, 1)}) \rightleftarrows \text{Cart}_S: \text{St}^+$$

The interested reader can find all details in Chapter 3.2 of [HTT]. We will abuse the notation and denote by $\int_C F$ the value of functor $\text{Un}^+$ on the presheaf $F$. In fact, we will only consider the case when $S$ is the nerve of a 1-category $C$.

Relative nerve construction. Let $F: C \to S\text{Set}$ be a 1-functor. We define the simplicial set $N_F(C)$ as follows. Let $J$ be a linear order set with $J = |n|$. One can consider $J$ as a category. We will denote by $\Delta^J$ the associated simplicial set. The n-cell of $N_F(C)$ contained the following data:

(RN1) A functor $\sigma$ from $J$ to $C$.

(RN2) For every nonempty subset $J' \subseteq J$ having a maximal element $j_{\max}'$, a map $\tau(J'): \Delta^{J'} \to F(\sigma(j_{\max}'))$

together with the condition

(RN3) For nonempty subsets $J'' \subseteq J' \subseteq J$, with maximal elements $j_{\max}'' \in J'', j_{\max}' \in J'$, the diagram

$$\begin{array}{ccc}
\Delta^{J''} & \longrightarrow & F(\sigma(j_{\max}'')) \\
\downarrow & & \downarrow \\
\Delta^{J'} & \longrightarrow & F(\sigma(j_{\max}'))
\end{array}$$

is required to commute.

The simplicial set $N_F(C)$ is called the relative nerve of $F$. Suppose $F$ takes value in weak Kan complexes. Then $N_F(C)$ can be identified with $\int_C F$ ([HTT], Section 3.2.5).
Remark B.1. Note that a 0-simplex of \( N_F(C) \) is a pair \((c, Y)\) where \( c \in \text{Ob}(C) \) and \( Y \) is a 0-simplex of \( F(c) \). Moreover, a 1-simplex of \( N_F(C) \) is also a pair which contains a morphism \( \alpha : c_1 \to c_2 \) of \( C \) and a 1-simplex \( F(\alpha)(Y_1) \to Y_2 \). So the relative nerve compatible with the classical Grothendieck construction.

Example B.2. Let \( S \) be a simplicial set and \( S : C \to \text{SSet} \) be the constant functor. Then \( N_S(C) \) is the Cartesian product \( S \times N(C) \). Indeed, the maps from the condition (RN2) is just a point of \( S|_J \). By (RN3) all such maps unique defined by the one map \( \tau(J) \).

Presentable fibrations. Following [HTT] we will call a map of simplicial sets \( p : X \to S \) a presentable fibration if it is a Cartesian and cocartesian fibration such that each fiber \( X_s = X \times_S \{s\} \) is a presentable \( \infty \)-category.

Proposition B.3. (Proposition 5.5.3.3. of [HTT])

- Let \( p : X \to S \) be a Cartesian fibration of simplicial sets, classified by a map \( \chi : S^{\text{op}} \to \text{Cat}_{(\infty,1)} \). Then \( p \) is a presentable fibration if and only if \( \chi \) factors through \( \text{Pr}^R \subseteq \text{Cat}_{(\infty,1)} \).
- Let \( p : X \to S \) be a cocartesian fibration of simplicial sets, classified by a map \( \chi : S \to \text{Cat}_{(\infty,1)} \). Then \( p \) is a presentable fibration if and only if \( \chi \) factors through \( \text{Pr}^L \subseteq \text{Cat}_{(\infty,1)} \).

Corollary B.4. (Corollary 5.5.3.4. of [HTT].) For every simplicial set \( S \), there is a canonical bijection \([S, \text{Pr}^L] \simeq [S^{\text{op}}, \text{Pr}^R]\) where \([S, C]\) denotes the collection of equivalence classes of objects of \( \text{Fun}_{\infty}(S, C) \).

For \( F : C \to \text{Pr}^R \) let us denote by \( F^{\text{adj}} \) the corresponding functor \( C^{\text{op}} \to \text{Pr}^L \). Note that \( F^{\text{adj}} \) can be described as follows:

- \( F^{\text{adj}}(c) = F(c) \) for any \( c \in C \);
- \( F^{\text{adj}}(\alpha) \) is right-adjoint to \( F(\alpha) \) for any \( \alpha : c_1 \to c_2 \). Further, we will denote \( F^{\text{adj}}(\alpha) \) and \( F(\alpha) \) by \( \alpha^* \) and \( \alpha_* \).

According to the Corollary, \( \int_C(F^{\text{op}}) \) and \( (\int_{C^{\text{op}}} F^{\text{adj}})^{\text{op}} \) are equivalent as \( \infty \)-categories.

Remark B.5. Let \( S = \Delta^1 \). Then the data \( F : \Delta^1 \to \text{Pr}^R \) is equivalent to the pair of categories \( F(0) \) and \( F(1) \) together with a pair of adjoint functors \( \alpha^* \) and \( \alpha_* : \alpha^* : F(1) \to F(0) : \alpha_* \).

Then the corresponding relative nerves can be described as follows:
• the objects of $\int_{\Delta^1}(F^{op})$ are the disjoint union of objects $F(0)$ and $F(1)$. For any $x, y \in Ob(\int_C(F^{op}))$ we have

$$\text{Hom}(x, y)^\bullet = \begin{cases} \emptyset & \text{if } x \in F(1), y \in F(0) \\ \text{Hom}_{F(1)}(y, x)^\bullet & \text{if } x, y \in F(i) \\ \text{Hom}_{F(0)}(\alpha_*(y), x)^\bullet & \text{if } x \in F(0), y \in F(1) \end{cases}$$

• the objects of $(\int_{\Delta^1}F^{adj})^{op}$ are the disjoint union of objects $F(0)$, $F(1)$ and $\text{Hom}(x, y)^\bullet = \begin{cases} \emptyset & \text{if } x \in F(1), y \in F(0) \\ \text{Hom}_{F(1)}(y, x)^\bullet & \text{if } x, y \in F(i) \\ \text{Hom}_{F(0)}(\alpha_*(y), x)^\bullet & \text{if } x \in F(0), y \in F(1) \end{cases}$

Note that for any $x \in F(0)$ and $y \in F(1)$ we have the canonical map

$$(B.1) \quad \text{Hom}_{F(1)}(\alpha_*(x)) \rightarrow \text{Hom}_{F(0)}(\alpha_*(y), x)$$

which is a weak equivalence of simplicial sets. Moreover, the maps $[B.1]$ define the functor

$$\int_{\Delta^1}(F^{op}) \rightarrow (\int_{\Delta^1}F^{adj})^{op}$$

which is an equivalence of $\infty$-categories.

**Construction of the equivalence.** Let $C$ be a 1-category and $F : N(C) \rightarrow \text{Pr}^R$ be a functor. Let us construct the canonical equivalence of categories $\int_C(F^{op})$ and $(\int_{C^{op}}F^{adj})^{op}$. First of all, observe that by (RN3) the functors $\tau(J')$ factor through the simplicial subsets

$$\text{Hom}_{F^{op}(\sigma(J_{max}^{\prime}))(\alpha_*(\tau(j_{min}^{\prime})), \tau(j_{max}^{\prime}))} \subset F(\sigma(j_{max}^{\prime})).$$

Here $\alpha_*$ is defined as $F^{op}(\sigma(j_{min}^{\prime} \rightarrow j_{max}^{\prime})$ and

$$\tau(j_{min}^{\prime}) : \Delta^0 \rightarrow F^{op}(\sigma(j_{min}^{\prime}))$$

$$\tau(j_{max}^{\prime}) : \Delta^0 \rightarrow F^{op}(\sigma(j_{max}^{\prime}))$$

are the morphisms corresponded to the inclusions $j_{min}^{\prime} \subset J' \subset J$, $j_{max}^{\prime} \subset J' \subset J$.

With this in mind we can define the morphisms

$$\varphi_n : N_{F^{op}}(C)_n \rightarrow N_{F^{adj}}(C^{op})_n.$$  

For $n = 0$ this is identity map. For $n > 0$ we map the data (RN1) and (RN2) to the following data:

(RN1$^{adj}$) the same functor $\sigma$ from $J^{op}$ to $C^{op}$. 


(RN2adia) For every nonempty subset \( J^{\text{op}} \subseteq J^{\text{op}} \) having the maximal element \( j'_{\text{min}} \), the map 
\[
\tau(J^{\text{op}})^{\vee} : \Delta^{J'} \to F(\sigma(j'_{\text{min}}))
\]
which define as the composition
\[
\begin{array}{ccc}
\Delta^{J'} & \xrightarrow{\tau(J^{\text{op}})^{\vee}} & F(\sigma(j'_{\text{min}})) \\
\downarrow & & \downarrow \\
\Hom_{F(\sigma(j'_{\text{max}}))}(\tau(j'_{\text{max}}), \alpha^* \tau(j'_{\text{min}})) & \xrightarrow{\varphi} & \Hom_{F(\sigma(j'_{\text{min}}))}(\alpha^* \tau(j'_{\text{max}}), \tau(j'_{\text{min}}))
\end{array}
\]
Here \( \alpha^* \) def = \( F(\text{adj}(\sigma(j'_{\text{max}}} \to j'_{\text{min}}))) \) (recall that we replace \( J \) with \( J^{\text{op}} \)). Notice that the condition (RN3) for data (RN1adia) and (RN2adia) automatically satisfied.

**Proposition B.6.** The map \( \varphi \) is an equivalence of \( \infty \)-categories.

**Proof:** It suffice to check that for any pair \( (c_1, Y_1) \) and \( (c_2, Y_2) \) the map
\[
\varphi : \Hom_{F(C)(F^{\text{op}})}((c_1, Y_1), (c_2, Y_2)) \to \Hom_{F(C^{\text{op}})^{\text{op}}}(c_1, Y_1), (c_2, Y_2))
\]
is a weak equivalence of simplicial sets. Note that we have the the commutative diagram of spaces
\[
\begin{array}{ccc}
\Hom_{F(C)(F^{\text{op}})}((c_1, Y_1), (c_2, Y_2)) & \xrightarrow{\varphi} & \Hom_{F(C^{\text{op}})^{\text{op}}}(c_1, Y_1), (c_2, Y_2)) \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
\Hom_C(c_1, c_1)
\end{array}
\]
and \( \varphi \) maps \( \pi_1^{-1}(\alpha) \) to \( \pi_2^{-1}(\alpha) \) for any \( \alpha \). Moreover, \( \pi_1^{-1}(\alpha) \) and \( \pi_1^{-1}(\alpha') \) lie in different connection components for different \( \alpha \) and \( \alpha' \). Hence it suffice to prove that \( \varphi \) induces the weak equivalence
\[
\Delta^1 \times_C \Hom_{F^{\text{op}}(F^{\text{op}})}((c_1, Y_1), (c_2, Y_2)) \to \Delta^1 \times_C \Hom_{F^{\text{op}}(F^{\text{op}})^{\text{op}}}(c_1, Y_1), (c_2, Y_2))
\]
for any \( \alpha : \Delta^1 \to C \).

On the other hand, for any 1-functor \( C' \to C \) the fiber product \( C' \times_C \int_C F \) compatible with the inverse image \( C' \to C \to F \Pr^R \). So we may assume that \( C = \Delta^1 \). In this case the statement follows from Remark B.5.

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