Nonorientable, incompressible surfaces in punctured-torus bundles over $S^1$

Józef H. Przytycki$^{1,2}$

Dedicated to Maite Lozano on the occasion of her 70th birthday

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Abstract
We classify incompressible, $\partial$-incompressible, nonorientable surfaces in punctured-torus bundles over $S^1$. We use the ideas of Floyd, Hatcher, and Thurston. The main tool is to put our surface in the “Morse position” with respect to the projection of the bundle into the basis $S^1$.

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1 Introduction

This paper is devoted to the classification, up to isotopy, of incompressible, $\partial$-incompressible, nonorientable surfaces in punctured-torus bundles over $S^1$. We also give a partial classification of nonorientable, incompressible (not necessarily $\partial$-incompressible) surfaces. In the proof we use the ideas of Hatcher and Thurston [4] and of Floyd and Hatcher [3]. The main tool is to put our surface in the “Morse position” with respect to the projection of the bundle into the basis $S^1$. Then we make careful and very laborious analysis of critical points.

We work in the smooth category, however all the results can be proven in the PL category as well.

Definition 1.1 (a) Let $M$ be a 3-manifold and $F$ a surface which is either properly embedded in $M$ or contained in $\partial M$. We say that $F$ is compressible in $M$ if one of the following conditions is satisfied:

(i) $F$ is a 2-sphere which bounds a 3-cell in $M$, or
(ii) $F$ is a 2-cell and either $F \subset \partial M$ or there is a 3-cell $X \subset M$ such that $F \subset \partial X$ and $\partial X \subset (F \cup \partial M)$, or
(iii) there is a 2-cell $D \subset M$ such that $D \cap F = \partial D$ and $\partial D$ is not contractible in $F$.

\[1\] The paper was the second part of the author’s doctoral dissertation prepared at Columbia University under the supervision of Professor Joan Birman [10].

\[\checkmark\] Józef H. Przytycki
przytyck@gwu.edu

1 Department of Mathematics, The George Washington University, Washington, DC 20052, USA
2 University of Gdańsk, Gdańsk, Poland
We say that $F$ is incompressible in $M$ if it is not compressible.

(b) Let $F$ be a submanifold of a manifold $M$. We say that $F$ is $\pi_1$-injective in $M$ if the inclusion induced homomorphism from $\pi_1(F)$ to $\pi_1(M)$ is an injection.

(c) Let $F$ be a surface properly embedded in a compact 3-manifold $M$, and $\partial_0 M$ a component of $\partial M$. We say that $F$ is $\partial$-incompressible along $\partial_0 M$ if there is no 2-disk $D \subset M$ such that: $\partial D \subset (\partial_0 M \cup F)$, $D \cap F = \alpha$ is an arc in $\partial D$, $D \cap \partial_0 M = \beta$ is an arc in $\partial D$. Furthermore $\alpha \cap \beta = \partial \alpha = \partial \beta$ and $\alpha \cup \beta = \partial D$, and $\alpha$ is not parallel to $\partial F$ in $F$. We say that $F$ is $\partial$-incompressible in $M$ if $F$ is $\partial$-incompressible along each component of $\partial M$.

2 Classification theorems.

In this chapter we prove our main theorem on the structure of incompressible surfaces in a punctured-torus bundle over a circle with a hyperbolic monodromy map. We start from the basic construction of Farey diagram which can be used to describe the action of the projective special linear group $PSL(2, \mathbb{Z})$ on the set of fractions $\mathbb{Q} \cup \{\infty\}$.

**Definition 2.1** (a) [3, 4] The following graph, $W'$, placed in a disc, is called the diagram of $PSL(2, \mathbb{Z})$ or the Farey diagram. The set of vertices of $W'$ is the set $W = \mathbb{Q} \cup \{\infty\}$, where $\mathbb{Q}$ is the set of rational numbers. Two vertices $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in W$ are joined by an edge if and only if $\text{det} \left( \begin{array}{cc} p_1 & p_2 \\ q_1 & q_2 \end{array} \right) = \pm 1$ (see Fig. 1a).

(b) Let $W_0 = \{ \frac{p}{q} : p$ and $q$ are odd $\}$,

$W_1 = \{ \frac{p}{q} : q$ is even $\}$,

$W_2 = \{ \frac{p}{q} : p$ is even $\}$.

We define $\overline{W}$ (respectively $\overline{W}_0, \overline{W}_1,$ and $\overline{W}_2$) to be the graph with vertices $W$ (resp. $W_0, W_1$ and $W_2$) and such that two vertices $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ in $W$ (resp. $W_0, W_1$ and $W_2$) are joined by an edge if and only if $\text{det} \left( \begin{array}{cc} p_1 & p_2 \\ q_1 & q_2 \end{array} \right) = \pm 2$ (see Fig. 1b).

We need the following, well-known fact, related, via Cayley graph, to the fact that matrices $\left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right)$ and $\left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right)$ generate a free subgroup of $PSL(2, \mathbb{Z})$ (for this classical result see e.g. [9], Sec. 2.3).

**Proposition 2.2** The graph $\overline{W}$ is a forest of three connected components, that is: $\overline{W} = \overline{W}_0 \cup \overline{W}_1 \cup \overline{W}_2$ and $\overline{W}_i$ $(i = 0, 1, \text{or } 2)$ is a tree.

**Proof** The first part follows from the observation:

If $\text{det} \left( \begin{array}{cc} p_1 & p_2 \\ q_1 & q_2 \end{array} \right) = \pm 2$ and $(p_i, q_i) = 1 (i = 1, 2)$, then either $p_1, p_2, q_1, q_2$ are odd numbers or $p_1$ and $p_2$ are even or $q_1$ and $q_2$ are even. To prove the second part of Proposition 2.2 consider the edge-path $\frac{1}{2}, \frac{1}{2}, \frac{p_3}{q_3}, \frac{p_4}{q_4}, \ldots, \frac{p_n}{q_n}$...in $\overline{W}_1$ which is minimal (i.e. $\frac{p_{i+1}}{q_{i+1}} \neq \frac{p_{i-1}}{q_{i-1}}$).

Then $p_i \leq p_{i+1}$ and $q_i \leq q_{i+1}$. This follows by induction on $i$. Namely $\frac{1}{2}$ and $\frac{1}{2}$ satisfy these inequalities and we have $\frac{p_{i+1}}{q_{i+1}} = \frac{p_{i-1} + 2kp_i}{q_{i-1} + 2kqi} = \frac{|p_{i-1} + 2kp_i|}{|q_{i-1} + 2kqi|}$ for some integer $k$ depending on $i, k \neq 0$. For $k > 0$, $p_{i+1} = p_{i-1} + 2kp_i \geq p_i$, $q_{i+1} = q_{i-1} + 2kqi > q_i$ and for negative $k$ (say $-k = k' > 0$) we have $p_{i+1} = 2k'p_i - p_{i-1} \geq p_i$ and $q_{i+1} = 2k'q_i - q_{i-1} > q_i$. The above inequalities imply that no minimal edge path starting at $\frac{1}{2}, \frac{1}{2}$ contains a cycle. Generally...
\[ \bar{W} \] does not contain any cycle by homogeneity of \( \bar{W} \). Connectivity of \( \bar{W}_i \) \((i = 0, 1, 2)\) may be proved by induction.

The following known facts\(^2\) can be formulated using a notion of a minimal edge-path in \( \bar{W} \).

**Theorem 2.3** Let \( F \) be an incompressible surface in \( T^2 \times I \), Then either

(a) \( F \) is isotopic to a saturated annulus (i.e. annulus of type \((\gamma) \times I\) for some nontrivial simple closed curve \( \gamma \) in \( T^2 \)), or

(b) \( F \) is an annulus or torus parallel to the boundary, or

(c) \( F \) is isotopic to a nonorientable manifold uniquely determined by two different slopes \( p_0 q_0 \) and \( p_1 q_1 \) where the determinant of \( \begin{bmatrix} p_0 & p_1 \\ q_0 & q_1 \end{bmatrix} \) is even and \( F \cap (T^2 \times \{i\}) \) is a curve of slope \( \frac{p_i}{q_i} \) \((i = 0, 1)\). The genus of such a surface\(^3\) is equal to the period of the minimal edge-path from \( \frac{p_0}{q_0} \) to \( \frac{p_1}{q_1} \) in \( \bar{W} \).

**Remark 2.4** In the case of a surface which is nonorientable and an isotopy whose restriction to the boundary is the identity we have to know additionally the intersection number modulo 2 of \( F \) with the arc \( \{\ast\} \times I \), where \( \{\ast\} \) is a fixed point on \( T^2 \), to determine \( F \).

**Corollary 2.5** Each incompressible, non-parallel to the boundary surface in a solid torus \( S^1 \times D^2 \) is determined, up to isotopy by a slope \( \frac{p}{q} \in W_1 \). The genus of such a surface is equal to the period of the minimal edge-path from \( \frac{1}{0} \) to \( \frac{p}{q} \). If the period is > 0 then the surface is nonorientable and \( \partial \)-compressible.

**Corollary 2.6** \([1,16]\) Let \( L(q, p) \) be a lens space. Then

(i) if \( q \) is odd then \( L(q, p) \) does not contain any incompressible surface,

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\(^2\) In some form one can find it in \([1,16]\); it is also written in my PhD thesis \([10]\) and in \([6]\).

\(^3\) If \( F \) is a connected sum of \( k \) copies of a projective space and \( b \) discs \((F = \#_k RP^2 \# \#_b D^2)\) then we say that \( F \) has genus \( k \) and \( b \) boundary components. In particular, the Klein bottle has genus 2.
(ii) if \( q \) is even then \( L(q, p) \) contains exactly one incompressible surface, which is nonorientable, and of genus equal to the period of the minimal edge-path joining \( 0 \) to \( \frac{p}{q} \) in \( W_1 \).

**Proposition 2.7** Let \( M \) be an irreducible 3-manifold, and \( F \) a closed, 2-sided, incompressible surface in \( \text{int} M \). Let \( M' \) be a manifold obtained from \( M \) cut open along \( F \) (\( M' \) may be connected or not), and \( F' = F_1 \sqcup F_2 \) “a trace” of \( F \) in \( M' \). Let \( S \) be a properly embedded surface in \( M \), which is transverse to \( F \).

Further, let \( S' \) be a surface obtained from \( S \) by cutting open \( M \) along \( F \) (that is we delete \( F \) and compactify \( M - F \) by two copies of \( F \)). Then:

(a) If \( S' \) is incompressible in \( M' \) and \( \partial \)-incompressible along \( F' \) then \( S \) is incompressible in \( M \).

(a') If \( S' \) is incompressible, \( \partial \)-incompressible in \( M' \) then \( S \) is incompressible, \( \partial \)-incompressible in \( M \).

(b) If \( S \) is incompressible in \( M \) and \( M' \) has two components (\( M_1 \) and \( M_2 \)) then \( S \) can be deformed by isotopy in such a way (the new embedding is still denoted by \( S \)) that \( S' \) is incompressible and \( S' \cap M_1 \) is \( \partial \)-incompressible along \( F' \cap M_1 \). If we assume additionally that \( S \) is \( \partial \)-incompressible, we can conclude also that \( S' \cap M_1 \) is \( \partial \)-incompressible.

If \( \partial M \) consists of tori, Proposition 2.7 and Theorem 2.3 give us:

**Proposition 2.8** Let \( M \) be a compact, irreducible 3-manifold with \( \partial M \) equal to a collection of tori \( T_1, \ldots, T_k \). Let \( V_1, \ldots, V_k \) be small regular neighborhoods of \( T_1, \ldots, T_k \) and \( M' = M - \text{int} \bigcup_{i=1}^{k} V_i \) (\( M' \) is homeomorphic to \( M \) and \( V_i \) to \( T^2 \times [0, 1] \) for each \( i \)). Then each incompressible, non-parallel to the boundary surface \( S \) (\( \partial S \neq \emptyset \)) properly embedded in \( M \) can be obtained by gluing together an incompressible, \( \partial \)-incompressible, non-parallel to the boundary surface in \( M' \), and incompressible, non-parallel to the boundary surfaces in \( V_1, \ldots, V_k \) (they are described in Theorem 2.3). In particular, if in addition \( S \) is orientable or has more than one boundary component on each \( T_i \), then \( S \) is \( \partial \)-incompressible (see [18,19]).

The converse to Proposition 2.8 is false, i.e. even if \( S \) allows a decomposition as above it could be compressible (see Example 3.1).

Now, we will use the above ideas to classify nonorientable, incompressible, \( \partial \)-incompressible surfaces in a punctured-torus bundle over \( S^1 \) (compare remarks in the preliminary version of [3] circulating around 1980 when I was a PhD student at Columbia).

First we introduce some models and constructions.

Let \( F \) be a torus with a hole. We will use two standard models for \( F \):

(a) either as a square with the opposite edges identified and a hole cut in the middle, or

(b) as a square \( [-1, 1] \times [-1, 1] \) with corners cut and the level \( x = -1 \) identified with the level \( x = 1 \), and the level \( y = -1 \) identified with the level \( y = 1 \) (Fig. 2)
Nonorientable, incompressible surfaces in punctured-torus

Similarly we will use two “cube” models for $F \times I$ as a product with interval of models of $F$; see Fig. 3.

If a surface $S$ is embedded in $F \times I$ such that the restriction of the natural projection $p : F \times I \to I$ to $S$ (that is $p/S$) is a Morse function, then instead of drawing a saddle we will schematically draw the projection of $S$ in the neighborhood of the saddle onto the level of the saddle (see Figs. 4, 5 or 6).

From now on, we assume that each monodromy map $\phi$ is of the form $\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \geq 0$ (each hyperbolic matrix is conjugate to a matrix of such a form).

Recall that elements of $\text{SL}(2, \mathbb{Z})$ can be divided into three classes depending on its trace:

(e) elliptic if $|\text{tr}(\phi)| < 2$,

(p) parabolic if $|\text{tr}(\phi)| = 2$,

(h) hyperbolic if $|\text{tr}(\phi)| > 2$, or equivalently $\phi$ has two different real eigenvalues.

**Construction 2.9** Let $\phi : F \to F$ be a self-homeomorphism of $F$, defined by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{Z})$, i.e. $A$ takes a vector $(y \ x)$ to $(ay + bx \ cy + dx)$ (in particular the slope $1 \ 0$ to $a \ b$ and the slope $0 \ 1$ to $c \ d$).

Let $\gamma$ be an edge-path edge-path in $W$ (minimal or not) with the successive vertices $\ldots, a^{i-1}_b, a^i_0, a^i_1, \ldots$. Assume that $\gamma$ is $\phi$-invariant, with period $k$, that is $\phi(a^i_b) = a^{i+k}_b$ for all $i$. Now to each such $\phi$-invariant edge-path $\gamma$, we associate the family of surfaces in $M_\phi = F \times \mathbb{R}/\sim$ where $(x, t) \sim (\phi(x), t + 1)$.

(a) First construct surface $S_\gamma \in F \times \mathbb{R}$. Let $F_t = F \times \{t\} \subseteq F_{i/k} = \text{the standard circle of slope } a^i_b$. The saddles are on the levels $\frac{1}{k}(i + \frac{1}{2})$, for all $i$.

Consider a saddle with the slope $\frac{1}{2}$ below the critical point and the slope $\frac{1}{2}$ above the critical point (Fig. 4).

The saddle between $\frac{a_i}{b_y}$ and $\frac{a_{i+1}}{b_y+1}$ in $S_\gamma$ is obtained from that in Fig. 4 by applying the homeomorphism given by a linear isomorphism which takes the slopes $\frac{1}{2}$ to $\frac{a_i}{b_y}$ and $\frac{1}{2}$ to $\frac{a_{i+1}}{b_y+1}$. Now we define a surface $S_\gamma^\partial = S_\gamma/\phi^\partial$, in a punctured-torus bundle over $S^1$ with monodromy $\phi$.

(b) Let $S_\gamma^\partial$ be a surface obtained from $S_\gamma^\partial$ by modifying $S_\gamma$ between levels $F_0$ and $F_1$ by adding one saddle and one horizontal boundary component; see Fig. 5 (compare Observation 2.18).

As will be shown later, in Observation 2.19, $S_\gamma^\partial$ is incompressible but not $\partial$-incompressible.

**Construction 2.10** We will define a surface $S_\gamma(\varepsilon_1, \ldots, \varepsilon_k)$ associated to a $\phi$-invariant edge-path $\gamma$ (minimal or not) in $\overline{W}$ of period $k$, and an element $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in (\mathbb{Z}_2)^k$. 

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Fig. 4 The saddle is changing a circle of slope $\frac{1}{0}$ to a circle of slope $\frac{1}{2}$

Fig. 5 Adding a saddle and a horizontal boundary component

Fig. 6 Saddles between slopes $\frac{1}{0}$ and $\frac{1}{2}$

Let \(\ldots \frac{a_i-1}{b_i-1}, \frac{a_i}{b_i}, \frac{a_i+1}{b_i} \ldots\) be the successive vertices of \(\gamma\) with \(\phi\left(\frac{a_i}{b_i}\right) = \frac{a_i+1}{b_i} \), and \(\varepsilon_{k+i} = \varepsilon_i\), for all \(i\). First we construct surface \(\tilde{S}_\gamma(\varepsilon_1, \ldots, \varepsilon_k)\) in \(F \times \mathbb{R}\). \(\tilde{S} \cap F_i\) is the standard arc of slope \(\frac{a_i}{b_i}\). The saddles are on the levels \(\frac{1}{2}(i + \frac{1}{2})\), for all \(i\).

Consider two schemes of saddles:

Both saddles lead from arcs of slope $\frac{1}{0}$ to arcs of slope $\frac{1}{2}$.

Assume \(a_i, b_i, a_{i+1}, b_{i+1} \geq 0\). The saddle of type \(\varepsilon_i = 0\) (resp. \(\varepsilon_i = 1\)) which leads from an arc of slope \(\frac{a_i}{b_i}\) to an arc of slope \(\frac{a_{i+1}}{b_{i+1}}\) is obtained from that of Fig. 6(i) (resp. Fig. 6(ii)) by applying the homeomorphism given by the linear isomorphism which takes a vector \((x, y)\) to \(\left(\frac{a_i y + \frac{1}{2}(a_{i+1} - a_i)x}{b_i y + \frac{1}{2}(b_{i+1} - b_i)x}, \frac{a_i + 1}{b_i} \right)\)

in particular the slopes $\frac{1}{0}$ to $\frac{a_i}{b_i}$ and $\frac{1}{2}$ to $\frac{a_{i+1}}{b_{i+1}}$.

Now we can finish the construction of \(\tilde{S}_\gamma(\varepsilon_1, \ldots, \varepsilon_k)\). Namely, we construct the saddle of type \(\varepsilon_i\) between levels \(F_i\) and \(F_{i+1}\) (first, for \(\frac{a_i}{b_i}, \frac{a_{i+1}}{b_{i+1}} \geq 0\), later using the \(\phi\)-invariability of \(\tilde{S}_\gamma(\varepsilon_1, \ldots, \varepsilon_k)\)).

Finally we define \(S_\gamma(\varepsilon_1, \ldots, \varepsilon_k) = \tilde{S}_\gamma(\varepsilon_1, \ldots, \varepsilon_k)/\phi\).

Notice that the surfaces \(\tilde{S}_\gamma(\varepsilon_1, \ldots, \varepsilon_k)\) for given \(\gamma\) are carried by branched surface \(\tilde{\Sigma}(\gamma)\) where

\[
\tilde{\Sigma}(\gamma) = \bigcup_{i=-\infty}^{\infty} F_{i(i+1/2)/k} \cup \bigcup_{i=-\infty}^{\infty} P_i,
\]
Fig. 7 There are two arcs of slopes $\frac{1}{0}$ and $\frac{0}{1}$ under the saddle, and two arcs of slopes $\frac{1}{2}$ and $\frac{1}{2}$ above the saddle.

$$\begin{array}{c}
1 \\
0 \\
0 \\
1 \\
\end{array}$$

where $P_i = \alpha_i \times \left[\frac{i-\frac{1}{k}}{k}, \frac{i+\frac{1}{k}}{k}\right]$ and $\alpha_i$ is a standard arc of slope $\frac{a_i}{b_i}$. That is, our branched surface contains a punctured-torus on each saddle (critical) level and appropriate slope curves on other levels (compare [4]).

Recall the result of [3] that if $\gamma$ is a $\phi$-invariant edge-path in the diagram of $PSL(2, \mathbb{Z})$ then we can uniquely assign to $\gamma$ a $\phi$-invariant surface $\tilde{S}_\gamma$ in $F \times \mathbb{R}$. Let $S_\gamma = \tilde{S}_\gamma/\phi$ (this definition is slightly different than that of [3]). Let further $\tilde{S}_\gamma = \partial N(S_\gamma)$ where $N(S_\gamma)$ is a tubular neighborhood of $S_\gamma$, for $\gamma$ of odd period (so $S_\gamma$ nonorientable).

**Definition 2.11** We define, here, a new graph, which we call the special graph. The set of vertices of the special graph consists of ordered pairs of slopes ($\frac{a}{b}, \frac{c}{d}$) which satisfy: det

$$\begin{vmatrix}
a & c \\
b & d \\
\end{vmatrix} = \pm 1.$$  

Two vertices ($\frac{a_1}{b_1}, \frac{c_1}{d_1}$) and ($\frac{a_2}{b_2}, \frac{c_2}{d_2}$) are joined by an edge if and only if either

(i) $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and det $\begin{vmatrix} c_1 & c_2 \\
d_1 & d_2 \\
\end{vmatrix} = \pm 2$, or

(ii) $\frac{c_1}{d_1} = \frac{c_2}{d_2}$ and det $\begin{vmatrix} a_1 & a_2 \\
b_1 & b_2 \\
\end{vmatrix} = \pm 2$.

An edge-path $\gamma$ in the special graph defines two edge-paths in the graph $\bar{W}$. Namely if $\gamma = \ldots, (\frac{a_1}{b_1}, \frac{c_1}{d_1}), (\frac{a_2}{b_2}, \frac{c_2}{d_2}), \ldots$ then $\gamma_1 = \ldots, (\frac{a_1}{b_1}, \frac{c_1}{d_1}), (\frac{a_2}{b_2}, \frac{c_2}{d_2}), \ldots$ and $\gamma_2 = \ldots, (\frac{a_1}{b_1}, \frac{c_1}{d_1}), (\frac{a_2}{b_2}, \frac{c_2}{d_2}), \ldots$ (we allow here, for simplicity, repetitions of consecutive slopes).

We say that an edge-path $\gamma$ in the special graph is minimal if the associated edge-paths $\gamma_1$ and $\gamma_2$ are minimal in $\bar{W}$. We say that $\gamma$ is $\phi$-invariant if $\phi(\gamma) = \gamma$ or $-\gamma$ ($-\gamma$ is obtained from $\gamma$ by changing the order of slopes in each vertex of $\gamma$).

**Construction 2.12** We define a surface, $S_\gamma^{sp}$, associated to a $\phi$-invariant (minimal or not) edge path $\gamma$ in the special graph. Let $k$ be a period of $\gamma$ (i.e. $\phi((\frac{a_i}{b_i}, \frac{c_i}{d_i})) = (\frac{a_{i+k}}{b_{i+k}}, \frac{c_{i+k}}{d_{i+k}})$ for all $i$, or $\phi((\frac{a_i}{b_i}, \frac{c_i}{d_i})) = (\frac{a_{i+k}}{b_{i+k}}, \frac{c_{i+k}}{d_{i+k}})$ for all $i$). First construct surface $\tilde{S}_\gamma^{sp}$ in $F \times \mathbb{R}$. Let $F_i = F \times \{t\}$. $\tilde{S}_\gamma^{sp}$ consists of two arcs: one of them of slope $\frac{a_i}{b_i}$ and the other of slope $\frac{c_i}{d_i}$. The saddles are on the levels $\frac{1}{k}(i + \frac{1}{2})$, for all $i$.

Consider the following saddle (invariant under the matrix $-Id = \begin{bmatrix} -1 & 0 \\
0 & -1 \end{bmatrix}$):

The saddle (in $\tilde{S}_\gamma^{sp}$) associated to an edge ($\frac{a_i}{b_i}, \frac{c_i}{d_i}$) of the form $\frac{a_{i+1}}{b_{i+1}}, \frac{c_{i+1}}{d_{i+1}}$ is obtained from that on Fig. 7 by applying the homeomorphism defined by the linear isomorphism given by:

(i) if the edge is of the type (i) of Definition 2.11:

the slope of $\frac{a}{b}$ is taken to $\frac{a}{b}$, $\frac{1}{b}$ to $\frac{a}{d}$ and $\frac{1}{2}$ to $\frac{a_{i+1}}{d_{i+1}}$ = $\frac{a_{i+2}}{d_{i+2}}$;

(ii) if the edge is of the type (ii) of Definition 2.11:

the slope of $\frac{0}{1}$ is taken to $\frac{a_i}{b_i}$, $\frac{1}{0}$ to $\frac{a_i}{b_i}$ and $\frac{1}{2}$ to $\frac{a_i+1}{b_i+1}$ = $\frac{a_i+2}{b_i+2}$.
Now we define \( S_{\gamma}^{\phi} = \tilde{S}_{\gamma/\phi} \). If \( \phi(\gamma) = \gamma \) then \( S_{\gamma}^{\phi} \) consists of two components and if \( \phi(\gamma) = -\gamma \) then \( S_{\gamma}^{\phi} \) is connected.

Let \( \gamma \) be a \( \phi \)-invariant edge-path in the special diagram with \( \phi(\gamma) = -\gamma \). \( \gamma \) uniquely determines \( \phi^2 \)-invariant edge-paths \( \gamma_1 \) and \( \gamma_2 \) in the diagram of \( \text{PSL}(2, \mathbb{Z}) \):

\[
\gamma = \ldots, (a_2^{b_2}, c_0^{d_0}), (a_1^{b_1}, c_1^{d_1}), \ldots, (a_k^{b_k}, c_k^{d_k}), \ldots \]

where

\[
\phi(a_i^{b_i}, c_i^{d_i}) = (c_i^{a_i+k}, a_i^{d_i+k}), \text{ then } \gamma_1' = \ldots, a_0^{b_0}, a_1^{b_1}, \ldots, a_{2k}^{c_{2k}}, \ldots
\]

where

\[
a_i' = \begin{cases} \frac{a_i}{b_i}, & \text{when } 0 \leq i \leq k, \\ \frac{c_i-k}{d_i-k}, & \text{when } k \leq i \leq 2k. \end{cases}
\]

\( \gamma_2' \) is defined similarly, using \( -\gamma \) in the place of \( \gamma \) (so \( \gamma_2' = \phi(\gamma_1') \)). In fact \( S_{\gamma_1'} \cup S_{\gamma_2'} \) is the boundary of a regular neighborhood of the (non-connected) lifting of \( S_{\gamma}^{\phi} \) to \( M_{\phi^2} \).

Now we are ready to formulate our main theorem.

**Theorem 2.13** Let \( M_{\phi} \) be a punctured-torus bundle over \( S^1 \) with a hyperbolic monodromy map \( \phi \) (as in [3], for convenience, we shall usually not distinguish the open manifold \( M_{\phi} \) from its natural compactification obtained by adding a boundary torus). Then:

(a) Each closed, connected, incompressible surface in \( M_{\phi} \) is either

(i) a torus parallel to the boundary, or

(ii) isotopic to one of nonorientable surfaces \( S_{\gamma}^c \), where \( \gamma \) is a minimal, \( \phi \)-invariant edge-path in \( \overline{W} \).

(b) Each connected, incompressible surface, \( S \), in \( M_{\phi} \) with \( \partial S \) parallel to the boundary of a fiber is either an annulus parallel to \( \partial M_{\phi} \), or

(i) isotopic to a fiber (then \( \partial \)-incompressible), or

(ii) isotopic to one of nonorientable surfaces \( S_{\gamma}^\partial \), where \( \gamma \) is a minimal, \( \phi \)-invariant edge-path in \( \overline{W} \).

(c) Each connected, incompressible, \( \partial \)-incompressible surface, \( S \), in \( M_{\phi} \) with \( \partial S \neq \emptyset \) transverse to each fiber is either

(i) isotopic to one of the surfaces \( S_{\gamma} \) indexed by a minimal, \( \phi \)-invariant edge-path \( \gamma \) in the diagram of \( \text{PSL}(2, \mathbb{Z}) \), or to \( \tilde{S}_{\gamma} \ldots \text{ where } N(\gamma) \) is a tubular neighborhood of \( S_{\gamma} \) (where \( N(\gamma) \) is a tubular neighborhood of \( S_{\gamma} \) in the diagram of \( \text{PSL}(2, \mathbb{Z}) \), or

(ii) isotopic to one of the surfaces \( S_{\gamma}(\varepsilon_1, \ldots, \varepsilon_k) \), where \( \gamma \) is a minimal \( \phi \)-invariant edge-path in \( \overline{W} \), or

(iii) isotopic to a surface \( S_{\gamma}^{\phi} \) associated to a minimal, \( \phi \)-invariant edge-path \( \gamma \) in the special graph with \( \phi(\gamma) = -\gamma \).

**Definition 2.14** Consider a symbol \( \gamma[\varepsilon_1, \ldots, \varepsilon_k] \) where \( \gamma \) is a minimal, \( \phi \)-invariant edge-path in \( \overline{W} \) of period \( k \) and \( [\varepsilon_1, \ldots, \varepsilon_k] \in (\mathbb{Z}_2)^k \). This symbol uniquely determines a \( \phi \)-invariant (minimal or not) edge-path \( \gamma' \) in the diagram of \( \text{PSL}(2, \mathbb{Z}) \):

If \( \gamma \) is defined by a sequence \( \ldots, a_{-2}^{b_{-2}}, a_0^{b_0}, a_2^{b_2}, \ldots, a_{2k}^{b_{2k}}, \ldots \) where \( \phi(a_i^{b_i}) = a_i^{b_i+2k} \), then \( \gamma' \) is defined by the sequence of vertices \( \ldots, a_{-1}^{b_{-1}}, a_0^{b_0}, a_{2k-1}^{b_{2k-1}}, a_{2k}^{b_{2k}}, \ldots \) where
Fig. 8  Possibility of 2-saddles on the same level

\[ \frac{a_{2i+1}}{b_{2i+1}} = \begin{cases} \frac{1}{2} (a_{2i+2} - a_{2i}) & \text{if } \varepsilon_{i+1} = 0 \\ \frac{1}{2} (a_{2i+2} + a_{2i}) & \text{if } \varepsilon_{i+1} = 1 \end{cases} \]

This formula is valid for \( \frac{a_{2i}}{b_{2i}} \geq 0 \). We use the assumption that \( \gamma' \) is \( \phi \)-invariant, to get all vertices of \( \gamma \) (compare the remark before Construction 2.9). In fact \( \gamma' \) is associated with the boundary of a regular neighborhood of \( S_{\gamma} (\varepsilon_1, \ldots, \varepsilon_k) \) in \( M_\phi \).

In considerations below, we consider a part of \( \gamma \) in one period with vertices \( \frac{a_i}{b_i} \geq 0 \).

Let \( \sigma_i = \)

\[ \begin{cases} 1 & \text{if } \frac{a_{2i+2}}{b_{2i+2}} > \frac{a_{2i}}{b_{2i}} \text{ i.e. the edge } \frac{a_{2i}}{b_{2i}}, \frac{a_{2i+2}}{b_{2i+2}} \text{ goes to the left in the diagram of } PSL(2, \mathbb{Z}) \\ -1 & \text{if } \frac{a_{2i+2}}{b_{2i+2}} < \frac{a_{2i}}{b_{2i}} \text{ i.e. the edge } \frac{a_{2i}}{b_{2i}}, \frac{a_{2i+2}}{b_{2i+2}} \text{ goes to the right in the diagram of } PSL(2, \mathbb{Z}) \end{cases} \]

To complete the definition we introduce an equivalence relation among symbols \( \gamma [\varepsilon_1, \ldots, \varepsilon_k] \) by elementary equivalences:

\[ \gamma [\varepsilon_1, \ldots, \varepsilon_i, 1-\varepsilon_{i+1}, \ldots, \varepsilon_k] \sim \gamma [\varepsilon_1, \ldots, 1-\varepsilon_i, 1-\varepsilon_{i+1}, \ldots, \varepsilon_k] \]

if \( \frac{a_{2i+2}}{b_{2i+2}} = \frac{a_{2i-2}+a_{2i}}{b_{2i-2}+b_{2i}} \) and either

(i) \( \varepsilon_i = \varepsilon_{i+1} = 0 \) and \( \sigma_{i-1} = -\sigma_i \) or

(ii) \( \varepsilon_i = 1 - \varepsilon_{i+1} \) and \( \sigma_{i-1} = \sigma_i \).

**Example 2.15** (Compare [4]) Consider a pair of consecutive essential saddles of a surface \( S_{\gamma} (\varepsilon_1, \ldots, \varepsilon_k) \) with \( \gamma \) a minimal, invariant edge-path in \( \overline{W} \). Suppose the relative heights of these two saddles can be reversed by an isotopy of the surface supported between the levels slightly below the first saddle and slightly above the second, which does not introduce any other critical points. If the two saddles are put on the same level, then there are only three possible configurations, up to level preserving isotopy and change of coordinates (Fig. 8):

(a) Symbol \( \gamma [\varepsilon_1, \ldots, \varepsilon_k] \) remains unchanged.

(b) The pictures change from (Fig. 9)

\( \gamma \) is not changed and the change of \( [\varepsilon_1, \ldots, \varepsilon_k] \) reflects the equivalence (i) from Definition 2.14.

(c) The pictures change from (Fig. 10)

\( \gamma \) is not changed and the change of \( [\varepsilon_1, \ldots, \varepsilon_k] \) reflects the equivalence (ii) from Definition 2.14.

To complete Theorem 2.13, we need:
Proposition 2.16 Let $\gamma$ be a minimal, $\phi$-invariant edge-path in either the diagram of $\text{PSL}(2, \mathbb{Z})$, or $\mathbb{W}$, or in the special graph. Then surfaces $S_\gamma, \bar{S}_\gamma, S^c_\gamma, S^\partial_\gamma, S_\gamma(\varepsilon_1, \ldots, \varepsilon_k)$ and $S^{sp}_\gamma$ are incompressible (if defined) and furthermore surfaces $S_\gamma, \bar{S}_\gamma, S_\gamma(\varepsilon_1, \ldots, \varepsilon_k)$ and $S^{sp}_\gamma$ are $\partial$-incompressible. Two surfaces from the above are isotopic if and only if the following conditions are satisfied:

(i) the surfaces are associated with the same $\gamma$ (up to sign, in the case of $S^{sp}_\gamma$),
(ii) they are in the same class ($S_\gamma, \bar{S}_\gamma, S^c_\gamma, S^\partial_\gamma, S_\gamma(\varepsilon_1, \ldots, \varepsilon_k)$ or $S^{sp}_\gamma$) and
(iii) $[\varepsilon'_1, \ldots, \varepsilon'_k] \sim [\varepsilon''_1, \ldots, \varepsilon''_k]$ if we deal with surfaces of type $S_\gamma(\varepsilon_1, \ldots, \varepsilon_k)$ (see Definition 2.14).
Let $g(S)$ denote the genus of a surface $S$, $b(S)$ the number of boundary curves and $sl(S)$ the slope of $\partial S$. In this last case we have to establish a coordinate system of $H_1(\partial M_\phi)$ to have slope well defined. The second generator, longitude, of $H_1(\partial M_\phi)$ is determined by the boundary of a fiber (with the clockwise orientation; see Fig. 11). To define the first generator, meridian, of $H_1(\partial M_\phi)$ we have to consider two cases (taking into account the fact that for hyperbolic $\phi$, $|tr(\phi)| > 2$):

(a) $tr\phi > 0$; so $\phi$ has two positive eigenvalues. Then the restriction of $\phi$ to the boundary of a fiber ($\partial F$ is understood to be the set of angles) has four fixed points, say $\pm \alpha_1$ and $\pm \alpha_2$.

Now the image, under projection $F \times \mathbb{R} \to M_\phi$, of the straight line in $\partial F \times \mathbb{R}$ which joins $(\alpha_1, 0)$ and $(\alpha_1, 1)$ is a circle which determines the first generator of $H_1(\partial M_\phi)$.

(b) $tr\phi < 0$; so $\phi$ has two negative eigenvalues. Then the restriction of $-\phi$ to $\partial F$ has four fixed points, say $\pm \alpha_1$ and $\pm \alpha_2$, so, in particular, $\phi(\alpha_1) = -\alpha_1$. Let $\lambda$ be the curve in $\partial F \times \mathbb{R}$ given by the equation $z = e^{\pi it}$ where $z \in \partial F$ and $t \in \mathbb{R}$ (so $\lambda$ joins $(\alpha_1, 0)$ and $(-\alpha_1, 1)$ with a negative half twist with respect to the chosen orientation of $\partial F$). The image of $\lambda$ under projection $F \times \mathbb{R} \to M_\phi$ determines the first generator of $H_1(\partial M_\phi)$.

The slope of a curve on $\partial M_\phi$ is defined to be the second coordinate of the curve.
| $S$          | $k$      | $tr\phi$ | $g(S)$                             | $b(S)$       | $sl(S)$                      | orientation     |
|------------|----------|----------|------------------------------------|--------------|------------------------------|-----------------|
| $S_\gamma$ | Odd      | Positive | $k + 1$                           | 1            | $\frac{L-R}{4}$            | Nonorientable   |
| $S_\gamma$ | Odd      | Negative | $k + 1$                           | 1            | $\frac{L-R+2}{4}$          | Nonorientable   |
| $S_\gamma$ | Even     | Positive | $\frac{k}{2} - \frac{b(S)}{2} + 1$ | $gcd(L - R, 4)$ | $\frac{L-R}{4}$            | Orientable      |
| $S_\gamma$ | Even     | Negative | $\frac{k}{2} - \frac{b(S)}{2} + 1$ | $gcd(L - R + 2, 4)$ | $\frac{L-R+2}{4}$          | Orientable      |
| $\tilde{S}_\gamma$ | Odd      | Positive | $k$                               | 2            | $\frac{L-R}{4}$            | Orientable      |
| $\tilde{S}_\gamma$ | Odd      | Negative | $k$                               | 2            | $\frac{L-R+2}{4}$          | Orientable      |
| $S_\gamma^c$ | Any      | Any      | $2 + k$                           | 0            | none                         | Nonorientable   |
| $S_\gamma^p$ | Any      | Any      | $2 + k$                           | 1            | $\frac{1}{2}$              | Nonorientable   |
| $S_\gamma (e_1, \ldots, e_k)$ | Any      | Positive | $k + 2 - b(S)$                    | $gcd((\Sigma \sigma_i e_i), 2)$ | $\frac{\Sigma_{i=1}^{k} \sigma_i e_i}{2}$ | Nonorientable   |
| $S_\gamma (e_1, \ldots, e_k)$ | Any      | Negative | $k + 2 - b(S)$                    | $gcd((\Sigma \sigma_i e_i) + 1, 2)$ | $\frac{(\Sigma_{i=1}^{k} \sigma_i e_i) + 1}{2}$ | Nonorientable   |
| $S_\gamma^{sp}$ | Any      | Positive | $k + 2 - b(S)$                    | $gcd(L_{\gamma'} - R_{\gamma'}, 4)$ | $\frac{L_{\gamma'} - R_{\gamma'}}{4}$ | Nonorientable   |
| with        |          |          |                                    |              |                              |                 |
| $\phi(\lambda) = -\lambda$ | Any      | Negative | $k + 2 - b(S)$                    | $gcd(L_{\gamma'} - R_{\gamma'}, 4)$ | $\frac{L_{\gamma'} - R_{\gamma'} + 4}{8}$ | Nonorientable   |
So we have two possibilities:

1. The slope is undefined in some level $F_{t_0}$; that is $F_{t_0} \cap S$ has no essential circle.
   Now, the proof as in [3] works even in the case of nonorientable surfaces. The case 1. describes tori which are parallel to the boundary.

2. The slope is defined on each level.
   We call a saddle essential if the slope does in fact change (the only possible type of essential saddle is sketched in Fig. 12). In fibers $F_t$ near an essential saddle trivial circles can be eliminated by isotopy of $S$. Let $F_{t_1}, \ldots, F_{t_k}$ be levels just below essential saddles, and $F'_{t_1}, \ldots, F'_{t'_k}$, just above essential saddles. We assume that there are no more saddles between $F_{t_i}$ and $F'_{t_i}$. Let $n = \sum n_i$ where $n_i$ is the number of peripheral circles in $F_{t_i} \cap S$.
   We prove Theorem 2.13 (a) by induction on $n$. 
I. Let \( n = 0 \). We present the region between \( F_i' \) and \( F_{i+1} \) as a cube with opposite lateral faces identified and the open neighborhood of the central vertical axis deleted (Fig. 13). We can assume that circles of \( F_i' \cap S \) and \( F_{i+1} \cap S \) have slope \( \frac{1}{0} \) and, because \( n = 0 \), both are disjoint from the rectangle \( R \) of Fig. 13a.

Trivial circles of the intersection of \( S \) and a lateral face can be eliminated by isotopy of \( S \). An arc with endpoints on the same vertical edge can be eliminated too. So, we have the situation as in Fig. 13b.

Now we consider \( R \cap S \). Again, the circles of \( R \cap S \) can be eliminated; and the arcs with the endpoints on the right edge of \( R \) can be eliminated too (Fig. 14).

Thus we can assume that \( R \cap S = \emptyset \).

Therefore we can conclude that no saddle occurs between \( F_i' \) and \( F_{i+1} \). This ends the proof of the case when \( n = 0 \).

II. Now let us assume that for each number less than \( n > 0 \) Theorem 2.13 (a) is proven. Consider an embedded surface \( S \) with the number of peripheral circles equal to \( n \). We will isotope \( S \) to decrease \( n \). Analyzing the situation, as before, we have (Fig. 15): If \( S \) is connected and \( n > 0 \), then for some \( i \) the corresponding \( R \) must contain a "corner". The corner arc gives the following saddle, Fig. 16b, which we will push in the direction of the corner.

Now we apply the fact that such a saddle commutes with an essential saddle (Fig. 17) to decrease \( n \). We decrease \( n \) till all unessential saddles are in one box but then \( n = 0 \), otherwise \( S \) is compressible or not connected. If \( \gamma \) is not minimal then \( S_{\gamma}^C \) is easily seen to be compressible.

It ends the Proof of Theorem 2.13 (a). \( \square \)

Proof of Theorem 2.13(b) Let \( S \) be a connected, incompressible surface in \( \mathcal{M}_\phi \) with \( \partial S \) parallel to the boundary of a fiber. Considerations similar to those in case (a) lead us to conclusion that we can rearrange the saddles in such a way that all unessential saddles and boundary curves lie between some essential saddles, say \( s_1 \) and \( s_2 \) (in the case of only one essential saddle \( s_1 = s_2 \)). Now if a pair of consecutive saddles looks as in Fig. 18 (each of them "adds" one horizontal boundary component):
Then the surface is $\partial$-compressible and therefore compressible (Proposition 2.8). Thus the region between essential saddles $s_1$ and $s_2$ looks as follows (Fig. 19):

The following observations end the proof of Theorem 2.13 (b) (It will remain only to show that $S^\partial_\gamma$ is compressible when $\gamma$ is not minimal; but it is easy).

**Observation 2.18** (Changing of a direction of an unessential saddle).

Consider a part of consecutive saddles (in $\tilde{S} \subset F \times \mathbb{R}$):

First of them (on the level $i + \frac{1}{2}$) is unessential, and “adds” a horizontal boundary component (Fig. 20a) and second (on the level $i + \frac{3}{2}$) is essential and changes the slope from $\frac{1}{0}$ to $\frac{1}{2}$ (Fig. 20b).

Then $\tilde{S}$ can be isotoped (the isotopy is the identity map outside the segment $F \times [i, i + 2]$) in such a way that the new position of $\tilde{S}$ in $F \times [i, i + 2]$ is defined by the saddles:
The proof follows from the fact that one can change the order of the unessential saddle (Figs. 20a or 21a) and the essential saddle (Fig. 20b); as shown in Fig. 22.

Observation 2.19 The surface $S^0_{\gamma}$ is $\partial$-compressible when $\gamma$ is a minimal edge-path of positive period in $\overline{W}$. $S^0_{\gamma}$ can be obtained from each surface of type $S_{\gamma}(\varepsilon_1, \ldots, \varepsilon_k)$ with $b(S_{\gamma}(\varepsilon_1, \ldots, \varepsilon_k)) = 1$ by construction described in Proposition 2.8. Namely: consider $S_{\gamma}(\varepsilon_1, \ldots, \varepsilon_k) \subset M_{\Phi}$ with $b(S_{\gamma}(\varepsilon_1, \ldots, \varepsilon_k)) = 1$. $sl(S_{\gamma}(\varepsilon_1, \ldots, \varepsilon_k))$ is of type $\frac{2m+1}{2}$. 
Consider $S_0$, a Mobius band with a hole, embedded properly in $T^2 \times [0, 1]$ such that $S_0 \cap T^2 \times \{0\}$ is a curve of slope $\frac{2m+1}{2}$ and $S_0 \cap T^2 \times \{1\}$ is a curve of slope $\frac{1}{2}$. $S_0$ is incompressible (compare Theorem 2.3). Now we glue $(M_\phi, S_\gamma(\epsilon_1, \ldots, \epsilon_k))$ with $(T^2 \times [0, 1], S_0)$ along $(\partial M_\phi, \partial S_\gamma(\epsilon_1, \ldots, \epsilon_k))$ and $(T^2 \times \{0\}, T^2 \times \{1\} \cap S_0)$. The new manifold with $S_\gamma(\epsilon_1, \ldots, \epsilon_k) \cup S_0$ embedded is homeomorphic to $(M_\phi, S_\partial)$ (see Fig. 23).

Proof of Theorem 2.13 (c) Let $S \subset M_\phi$ be a compact, incompressible, $\partial$-incompressible surface with $\partial S \neq \emptyset$, the circles of $\partial S$ not being isotopic to fibers in $\partial M_\phi$. Then $S$ can be isotoped so that $\partial S$ is transverse to the fibers in $\partial M_\phi$ and the bundle projection is a Morse function on $S$. In a non-critical fiber $F_t$, the arcs of $S \cap F_t$ must all be non-trivial in $H_1(F_t, \partial F_t)$, since a homologically trivial arc would bound a disk on $F_t$ (so $F$ would be $\partial$-compressible) (Fig. 24).

The same number of arcs with a defined slope is on each non-critical level.

As before [3] it can not happen that $S \cap F_t$ has three-slope configuration. So the only possible changes of slope are of type (for better visualization we draw two models of a punctured-torus) (Fig. 25):

If the number of curves (on each non-critical level and of each slope) is even then an o-essential saddle can not occur and we deal with the case (c)(i) of Theorem 2.13 studied in [3] (we allow $S$ to be nonorientable). Let the number of the curves of some slope be odd. Because $\phi$ is hyperbolic, the o-essential saddle has to occur. We can eliminate circles (the only possible ones are trivial) near o-essential saddles by isotopy of $S$. Let $F_{t_1}, \ldots, F_{t_k}$
Fig. 25 Saddle (a) called an $e$-essential and (b) called $o$-essential saddle

Fig. 26 a $F_{i}'$—the bottom face of the cube; b $F_{i+1}'$—the top face of the cube

be levels just below $o$-essential saddles and $F_{i}', \ldots, F_{i+k}'$, just above the $o$-essential saddles. There are no more saddles between $F_{i}$ and $F_{i}'$ ($i = 1, 2, \ldots, k$). $F_{i} \cap S$ (resp. $F_{i}' \cap S$) consists of one arc, say $\gamma_{i}^{-}$ (resp. $\gamma_{i}^{+}$) of slope $\frac{a_{i}}{b_{i}}$ (resp. $\frac{a_{i+1}}{b_{i+1}}$), which plays a role in the $i$th saddle and $k_{i}$ ($k_{i} \geq 0$) arcs of slope $\frac{a_{i}'}{b_{i}'}$ such that $|\det \begin{bmatrix} a_{i} & a_{i}' \\ b_{i} & b_{i}' \end{bmatrix}| = |\det \begin{bmatrix} a_{i+1} & a_{i}' \\ b_{i+1} & b_{i}' \end{bmatrix}| = 1$. ($\frac{a_{i}'}{b_{i}'}$ is not uniquely determined by $\frac{a_{i}}{b_{i}}$ and $\frac{a_{i+1}}{b_{i+1}}$; in fact $\frac{a_{i}'}{b_{i}'} = \frac{1}{4}(a_{i+1} \pm a_{i})$). Consider the region between $F_{i}'$ and $F_{i+1}'$ (decomposed as a cube as in the proof of Theorem 2.13 (a) or (b)). After an appropriate choice of coordinates we can assume that $\frac{a_{i+1}}{b_{i+1}} = \frac{p}{T}$ (for some $p \geq 0$). See Fig. 26.

Now we use the following observation.

If $\gamma, \gamma_{1}, \ldots, \gamma_{2k}$ ($k \geq 0$) are (all) curves of a given slope in $S \cap F_{i}$ (for some noncritical level $i$) and $\gamma$ lies in the middle of the curves (Fig. 27), then if $S_{a}$ is the nearest essential saddle to level $F_{i}$ (from above or below) then either $\gamma$ is not changed in $S_{a}$ (it is always the case if $k > 0$) or $S_{a}$ is $o$-essential. Furthermore the condition that $\gamma$ is in the middle is preserved. From this observation and Proposition 2.1 of [3] it follows that, after isotopy, one can assume that in the region between $F_{i}'$ and $F_{i+1}'$ the curve $\gamma_{i}^{+}$ (and $\gamma_{i+1}^{-}$) is not “involved” in any saddles. Therefore each connected component of the lifting of $S$ to $\tilde{S}$ in $F \times \mathbb{R}$ (which contains an $o$-saddle) is of the forms $\tilde{S}_{\gamma}(\varepsilon_{1}, \ldots, \varepsilon_{k})$. Now we have two possibilities for connected $S$:
(i) on some (so each) non-critical level $t$, $F_t \cap S$ consists of one curve. Then we deal with the case (ii) of Theorem 2.13(c),
(ii) on some (so each) non-critical level $t$, $F_t \cap S$ consists of two curves. Then two components of $S$ are interchanged by $\phi$ and we deal with the case (iii) of Theorem 2.13(c).

The proof of Theorem 2.13(c) will be completed if we show that for a non-minimal $\gamma$, $S$ is not incompressible, $\partial$-incompressible.

Consider the case of $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$ with $\gamma$ determined by vertices ... $a_{i-1}/b_{i-1}, a_0/b_0, a_1/b_1, \ldots$. There are just two possibilities for successive o-saddles yielding $a_i/b_i = a_{i+2}/b_{i+2}$, up to a change of coordinates (Fig. 28):

In the first of these two sequences, $S$ is clearly compressible. In the second, $S$ is $\partial$-compressible. This can be seen after one puts both saddles on the same level (Fig. 29).

Consider the case of $S_\gamma^{sp}$. If $\gamma$ is not minimal then $\gamma_1$ or $\gamma_2$ is not minimal. Then we can find consecutive vertices in $\gamma$ such that $(a_i/b_i, a_{i+2}/b_{i+2}) = (a_i/b_i, c_{i+2}/d_{i+2})$ (assume for simplicity that the first slope is going for and back so the second is unchanged $c_i/d_i = c_{i+1}/d_{i+1} = c_{i+2}/d_{i+2}$).
Because $\phi$ is hyperbolic so $\gamma$ has the period at least 2. Therefore the surface corresponding to $(a_i/b_i, c_i/d_i), (a_{i+1}/b_{i+1}, c_{i+1}/d_{i+1}), (a_{i+2}/b_{i+2}, c_{i+2}/d_{i+2})$ allows the obvious compressing disk which is a compressing disk of $S_{\gamma}^{\partial}$ (compare the upper part of Fig. 28).

This ends the proof of Theorem 2.13; it still remains to prove Proposition 2.16.

**Remark 2.20** If we allow $S$ to be disconnected then we can get more incompressible, $\partial$-incompressible surfaces, for example surfaces $S_{\gamma}^{\partial}$ in $M_{\phi}$ with $\phi(\gamma) = \gamma$ (compare Example 3.1).

Proof of Proposition 2.16:

(a) A case of a closed surface.

Consider the following properties of a closed, connected surface $S_{\gamma}^{\partial}$ in $M_{\phi}$:

(i) Each circle of the intersection of $S_{\gamma}^{\partial}$ with a non-critical fiber $F_t$ which is trivial in $F_t$ bounds a disk in $S_{\gamma}^{\partial}$.

(ii) $S_{\gamma}^{\partial} \cap F_t$, where $F_t$ is a non-critical fiber does not contain circles parallel to $\partial F_t$.

(iii) On each non-critical level $F_t$ there is exactly one slope (and it is represented by an odd number of circles) and the sequence of these slopes in $S_{\gamma}^{\partial}$ traces out the vertex sequence of the given minimal, invariant edge-path $\gamma \subset W$.

We claim that properties (i)-(iii) are preserved by any isotopy of $S_{\gamma}^{\partial}$. If this is so then, the proposition follows (compare [3]). For suppose $S_{\gamma}^{\partial}$ was compressible. Let $D$ be a compressing disk $D \cap S_{\gamma}^{\partial} = \partial D$. A small sub-disk $D'$ can be isotoped to lie in a fiber $F_t$. The shrinking of $D$ to $D'$ extends to an isotopy of $S_{\gamma}^{\partial}$ to $S_{\gamma}^{\partial'}$. Condition (i) implies that $\partial D'$ bounds a disk in $S_{\gamma}^{\partial'}$, so $\partial D$ bounds a disk in $S_{\gamma}^{\partial}$.

To prove the claim, consider a generic isotopy of $S_{\gamma}^{\partial}$. At any time during this isotopy, the projection to $S^1$ will be a Morse function, except for the following isolated phenomena:

(A) a saddle and a local maximum (or minimum) are introduced or canceled in a region containing no other critical points,

(B) a pair of critical points interchange levels.

(A) cannot affect conditions (i)-(iii),

(B) cannot affect conditions (i)-(iii) when one or both of the nondegenerate critical points are of index 0 or 2.

Thus the only case left to check is when two saddles interchange heights. Each of the two saddles has one of the forms (Fig. 30):

If both saddles are of type (a), up to the level preserving isotopy and change of coordinates, then we have the following two possibilities (Fig. 31):

Interchanging heights of the saddles in these cases preserves (i)–(iii). In all other cases one can check that the conditions (i)–(ii) are preserved (to prove that condition (ii) is
preserved it is useful to consider the fundamental group of a part of $S^c_\gamma$ consisting of a segment containing both saddles).

For (iii), if neither saddle before the interchange was essential then the same is true after the interchange, so the edge-path is preserved (however a pair of circles with defined slope can be created or canceled; see Fig. 32).

If just one saddle before the interchange is essential it holds after the interchange also (it follows from the fact that $\gamma$ lies on the tree; see Fact 2.2). Thus the edge-path is preserved and hence the condition (iii). It ends the part (a) of the proof of Proposition 2.16.

(b) The case of one horizontal boundary component. Consider the following properties of the connected surface $S$ with exactly one, horizontal boundary component:

(i) Each circle of the intersection of $S$ with a non-critical fiber $F_t$ which is trivial in $F_t$ bounds a disk in $S$,

(ii) Each circle of the intersection of $S$ with a non-critical fiber $F_t$ which is parallel to $\partial F_t$ in $F_t$ is parallel to $\partial S$ in $S$,

(iii) On each non-critical level $F_t$ there is exactly one slope and it is represented by an odd number of circles. The sequence of these slopes (in $S$) traces out the vertex sequence of the given minimal edge-path $\gamma \subset \overline{W}$.

We can prove, similarly as in the previous case, that properties (i)-(iii) are preserved by any isotopy of $S$ (rel $\partial S$). Thus the part (b) of the proof of Proposition 2.16 is completed.

\[\square\]

**Remark 2.21** (a) and (b) can be derived from the fact that the incompressible surfaces in $T^2$ bundle over $S^1$ with a hyperbolic monodromy map are classified up to isotopy by invariant, minimal edge-paths in $\overline{W}$ (and element of $\mathbb{Z}_2$); see Theorem 2.3 and Remark 2.4.
The case of vertical boundary components.
Consider the following properties of a connected surface, $S$, with vertical boundary components:

(i) $S$ has no trivial arcs on any non-critical level,
(ii) Each non-critical level circle of $S$ bounds a disk in $S$,
(iii) On each non-critical level there is exactly one arc of a defined slope, and the slope sequence of $S$ traces out the vertex sequence of a given minimal edge-path $\gamma \subset \overline{W}$.

We claim that properties (i)-(iii) are preserved by any isotopy of $S$ (rel $\partial S$) and that the only operation (under generic isotopy) which could reverse the type of an essential saddle is interchanging the relative heights of this saddle and another saddle and we are in the same situation as Example 2.15. The considerations are similar to those of part (a) and (b) and [3,4], and we omit them.

From the considerations above it follows that $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$ is incompressible and now we will prove $\partial$-incompressibility. If $b(S_\gamma(\epsilon_1, \ldots, \epsilon_k)) = 2$ then $\partial$-incompressibility follows from condition (iii) and Proposition 2.8.

If $b(S_\gamma(\epsilon_1, \ldots, \epsilon_k)) = 1$, we consider the lifting of $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$ to $M_{\phi^2}$ and the lifted manifold has two boundary components and it is incompressible, so $\partial$-incompressible, so $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$ is $\partial$-incompressible.

To prove that $S^{sp}_\gamma$ is incompressible, $\partial$-incompressible we consider the lifting of this surface to $M_{\phi^2}$. Some connected component of the lifted surface is of type $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$ where $\gamma_1$ is the $\phi^2$-invariant, minimal edge-path in $\overline{W}$ associated to the special edge-path $\gamma$ (see Definition 2.11). Because $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$ is incompressible, $\partial$-incompressible so $S^{sp}_\gamma$ is incompressible, $\partial$-incompressible. This ends the proof of Proposition 2.16. □

Proof (Proof of Proposition 2.17) The calculations are standard. The only one which is more troublesome is the computation of $sl(S_\gamma(\epsilon_1, \ldots, \epsilon_k))$ and $sl(S^{sp}_\gamma)$. The computation of $sl(S_\gamma(\epsilon_1, \ldots, \epsilon_k))$ reduces to the computation of $sl(S_\gamma'(\epsilon_1, \ldots, \epsilon_k))$ where $S_\gamma'$ is the boundary of a tubular neighborhood of $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$; see Definition 2.14. The computation of $sl(S^{sp}_\gamma)$ reduces to the computation of $sl(S_\gamma'(\epsilon_1, \ldots, \epsilon_k))$ where $S_\gamma'$ is the boundary of a tubular neighborhood of some connected component of the lifting of $S^{sp}_\gamma$ to $M_{\phi^2}$. □

One could hope to extend our classification to the case of incompressible, but not $\partial$-incompressible surfaces, however some additional difficulties are involved (compare Example 3.1) and we stop on Theorem 2.13 and Proposition 2.8.
If we drop the assumption about hyperbolicity of the monodromy map, we will deal either with a periodic monodromy so with a Seifert fibered space (for this case see [12]) or with a reducible monodromy map [17]. The latter case may be studied by using Proposition 2.8 and the knowledge about incompressible surfaces in a 2-punctured disk bundle over $S^1$ (compare [2] and [12]).

In [11] we study with details the case of nonorientable, incompressible surfaces of genus 3 embedded in manifolds obtained from punctured-torus bundles over $S^1$ by capping off the torus in the boundary.

### 3 Example of surfaces in $M_{\phi^k}$

We illustrate our classification theorems on the rather general example.

**Example 3.1** Consider manifold $M_{\phi^k}$ where $\phi = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \alpha^2 \beta^2$; see Fig. 33 (here $\tilde{\alpha} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are standard generators of $SL(2, \mathbb{Z})$; compare [3]).

Because $\phi^k$, modulo $\mathbb{Z}_2$, has three eigenvalues we have three minimal, invariant edge-paths in $W_i$ for $i = 1, 2$ or 0 (compare [3]; the first version). Therefore we have three (up to isotopy) closed, non-parallel to the boundary, incompressible surfaces in $M_{\phi^k}$ (see Theorem 2.13a and Proposition 2.16): $S_{\gamma_1}^c$, $S_{\gamma_2}^c$, and $S_{\gamma_0}^c$. Now consider each $\gamma_i$ independently.

1. Consider $\gamma_1 \subset \overline{W}_1$. It is determined by the vertices:

   $\ldots, a_0 = 0, b_0, a_2 = 2, b_2 = \frac{12}{5}, \ldots, a_{2k} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}^k (0, \ldots, 0) \ldots$

   We have $a_{2l+2} = \frac{a_{2l+2} + 2\epsilon a_{2l}}{b_{2l+2} + 2\epsilon b_{2l}}$ where $\epsilon = -3$ so $(2\epsilon \neq \pm 2)$, so each symbol $[\epsilon_1, \ldots, \epsilon_k] \in (\mathbb{Z}_2)^k$ gives us a different surface $S_{\gamma_1}(\epsilon_1, \ldots, \epsilon_k)$ (see Definition 2.14 and Proposition 2.16).

   **Slopes are:** $sl(S_{\gamma_1}(\epsilon_1, \ldots, \epsilon_k)) = \frac{1}{2} \Sigma \epsilon_i$. Boundary of a tubular neighborhood of each $S_{\gamma_1}(\epsilon_1, \ldots, \epsilon_k)$ is incompressible (compare Definition 2.14).

   Now let us assume that $\Sigma \epsilon_i$ is odd, so $b(S_{\gamma_1}(\epsilon_1, \ldots, \epsilon_k)) = 1$

   Consider the construction from Proposition 2.8 (as in Observation 2.19) with $S_0$ a Mobius band with a hole; $S_0 \subset [0, 1] \times T^2 (0, 1] \times T^2$ will be glued to $M_{\phi^k}$ along $[0] \times T^2$ and $\partial M_{\phi^k}$). $S_0$ is determined uniquely by the slopes $\frac{1}{2} \Sigma \epsilon_1$ on $[0] \times T^2$ and $\frac{1}{2} \Sigma \epsilon_2$ on $[0] \times T^2$. The construction from Observation 2.19 can be used to realize the two slopes, and $S_0$ is glued to $M_{\phi^k}$ along $[0] \times T^2$ and $\partial M_{\phi^k}$.

---

Fig. 33 Part of $PSL(2\mathbb{Z})$ diagram
\( \{1\} \times T^2 \). It leads us to the surface \( \tilde{S}_{\gamma_1} \) (see Observation 2.19) independently of a choice of \( (\varepsilon_1, \ldots, \varepsilon_k) \) with \( \Sigma \varepsilon_i \) odd. Hence we obtain the following examples:

(i) if \( (\varepsilon_1, \ldots, \varepsilon_k) \neq (\varepsilon_1', \ldots, \varepsilon_k') \) and \( \Sigma \varepsilon_i = \Sigma \varepsilon_i' = \) odd number then the construction gives us examples of incompressible, \( \delta \)-incompressible surfaces in \( M_{\varphi_t} \) which are not isotopic but which after adding the “collar” \( (I \times T^2, S_0) \) become isotopic (however still incompressible).

(ii) if \( \Sigma \varepsilon_i \neq \Sigma \varepsilon_i' \) (both numbers odd) and we add to the surface \( S_{\gamma_1}(\varepsilon_1, \ldots, \varepsilon_k) \) the “collar” \( ([0, 1] \times T^2, S_1) \) where \( S_1 \) is the unique incompressible Klein bottle in \( [0, 2] \times T^2 \) with two holes given by slopes \( \frac{1}{2} (\Sigma \varepsilon_i) \) in \([0] \times T^2 \) and \( \frac{1}{2} \Sigma \varepsilon_i' \) in \([2] \times T^2 \) (see Theorem 2.3). In such a way we construct the compressible surface promised in Proposition 2.8. To see that the constructed surface (say \( S \)) is compressible we can use equality:

\[
S = S_{\gamma_1}(\varepsilon_1, \ldots, \varepsilon_k) \cup S_1 = S_{\gamma_1}(\varepsilon_1, \ldots, \varepsilon_k) \cup S_1' \cup S_1'',
\]

where \( S_1' \) in \([0, 1] \times T^2 \) is given by the slopes \( \frac{1}{2} \Sigma \varepsilon_i \) in \([0] \times T^2 \) and \( \frac{1}{2} \Sigma \varepsilon_i' \) in \([1] \times T^2 \) and \( S_1'' \) in \([1, 2] \times T^2 \) is determined by the slopes \( \frac{1}{2} \Sigma \varepsilon_i' \) in \([1] \times T^2 \) and \( \frac{1}{2} \Sigma \varepsilon_i' \) in \([2] \times T^2 \). \( S \) is isotopic to \( S_{\gamma_1}(\varepsilon_1, \ldots, \varepsilon_k') \cup S_2 \cup S_1'' \) where \( S_2 \) in \([0, 1] \times T^2 \) is determined by the slopes \( \frac{1}{2} \Sigma \varepsilon_i' \) in \([0] \times T^2 \) and \( \frac{1}{2} \Sigma \varepsilon_i' \) in \([1] \times T^2 \). \( S \) is compressible because \( S_2 \cup S_1'' \) is compressible (nonorientable surface in \([0, 2] \times T^2 \) with the same slopes in \([0] \times T^2 \) and \([2] \times T^2 \); see Theorem 2.3).

(b) \( \gamma_2 \subset \overline{W}_2 \), \( \gamma_2 \) is determined by the vertices

\[
\ldots, a_0 = \frac{1}{17}, a_2 = \frac{1}{17}, \ldots, a_{26} = \phi_2(\frac{1}{17}), \ldots \quad (\text{see Fig. 34}).
\]

There are only trivial relations among symbols \( (\varepsilon_1, \ldots, \varepsilon_k) \), similar to the case of \( \gamma_1 \), and

\[
sl(S_{\gamma_2}(\varepsilon_1, \ldots, \varepsilon_k)) = -\frac{1}{2} \Sigma \varepsilon_i.
\]

Incompressible surfaces \( S_{\gamma_1}(0, \ldots, 0) \) and \( S_{\gamma_2}(0, \ldots, 0) \) are disjoint and non-isotopic but the boundaries of their tubular neighborhoods are parallel.

(c) \( \gamma_0 \subset \overline{W}_0 \); see Fig. 35.

\( \gamma_0 \) is determined by the vertices

\[
\ldots, a_0 = \frac{1}{17}, a_2 = \frac{1}{3}, a_4 = \frac{7}{3}, a_6 = \frac{17}{17}, \ldots, a_{22} = \frac{4}{2} a_{20}, a_{22} = \frac{4}{2} a_{20} + \frac{2}{2} a_{20}, \ldots \quad (\text{see Fig. 34}).
\]

Definition 2.14 gives us relations among symbols \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2k}) \) which identify symbol \( (\ldots 0^{i-th}, 0 \ldots) \) with \( (\ldots 1^{i-th}, 1 \ldots) \). It gives us \( 2k + 1 \) non-isotopic surfaces \( S_{\gamma_0}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2k}) \), \( sl(S_{\gamma_0}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2k})) = \frac{1}{2} \Sigma (-1)^{i+1} \varepsilon_i \). \( S_{\gamma_0}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2k}) \) is

\[ \text{Fig. 34} \quad \text{The paths } \gamma_1 \text{ and } \gamma_2 \]
Fig. 35 The path $\gamma_0 \in \tilde{W}_0$ has vertices ..., $\frac{1}{7}$, $\frac{3}{7}$, $\frac{7}{3}$, $\frac{13}{7}$, ...

not $\pi_1$-injective (the boundary of a tubular neighborhood of $S_{\gamma_0}(\epsilon_1, \epsilon_2, \ldots, \epsilon_{2k})$ is compressible; see Fig. 35).

(d) Each minimal, $\phi^k$-invariant edge-path in the diagram of $PSL(2, \mathbb{Z})$ is of even period, so it leads to an orientable manifold (see Proposition 2.17).

4 Incompressible surfaces and skein modules

Last time, before Maite-fest, I visited Zaragoza in February 1986; I was already then thinking about the Jones polynomial and its generalizations (e.g. Conway algebras). Soon after, in April 1987, I discovered skein modules of 3-manifolds [13]. Immediately I thought that incompressible surfaces have an important role in creating torsion in skein modules [5,14,15]. In particular I asked:

Conjecture 4.1 If $M$ is a submanifold of a rational homology sphere and it does not contain a closed, oriented incompressible surface then its Homflypt skein module $S_3(M)$ is free and isomorphic to the symmetric tensor algebra over module spanned by conjugacy classes of nontrivial elements of the fundamental group, $S_3(M) = SR\hat{\pi}^0$.

I will leave it to readers to think of this and other possible relations of incompressible surfaces and torsion of skein modules.

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