Discrete Wigner Functions from Informationally Complete Quantum Measurements

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Abstract

Wigner functions provide a way to do quantum physics using quasiprobabilities, that is, “probability” distributions that can go negative. Informationally complete POVMs, a subject that has been developed more recently than Wigner’s time, are less familiar but provide wholly probabilistic representations of quantum theory. We study how these two classes of structure relate and the art of interconverting between them. Pushing Wigner functions to their limits, in a suitably quantified sense, reveals a new way in which the Symmetric Informationally Complete quantum measurements (SICs) are significant.

1 Introduction

In the practical course of doing physics, managing coordinate systems is an important skill. It is helpful to know how to choose a basis that makes a problem as simple as possible, and it is beneficial to understand what can and cannot be eliminated by a clever choice of reference frame. “One good coordinate system may be worth more than a hundred blue-in-the-face arguments,” a colleague advises us [1]. To that end, this article will explore two particular classes of normalized operator bases and the relations between them.

We will work in the quantum theory of finite-dimensional systems familiar from the study of quantum information and computation [2]. In this theory, each physical system is associated with a Hilbert space \( \mathcal{H}_d \simeq \mathbb{C}^d \), where the dimension \( d \) can be taken as a physical characteristic of the system. For example, in a quantum computer containing \( N \) qubits, the dimension is \( d = 2^N \). A quantum state, the means of expressing the preparation of a system, is an operator on this Hilbert space that is positive semidefinite and has a trace of unity. Measurements that one can perform upon a system are represented mathematically as positive-operator-valued measures (POVMs), which are resolutions of the identity into positive semidefinite operators:

\[
\sum_{i=1}^{n} E_i = I.
\]  

(1)

Each effect \( E_i \) corresponds to a possible outcome of the measurement \( \{E_i\} \), and the probability of that outcome is calculated by the Born Rule:

\[
p(E_i) = \text{tr} (\rho E_i).
\]  

(2)

That is, probabilities are Hilbert-Schmidt inner products between the operators that stand for outcomes and for preparations (or priors, in more probabilistic language). If the set of effects \( \{E_i\} \) spans the space of Hermitian operators, than any such operator can be expressed as a list of inner products. Such a POVM is informationally complete, since any quantum-mechanical calculation about the system can be done in terms of it. In order to be informationally complete, a POVM must provide a basis, and so it must contain at least \( d^2 \) effects. An informationally-complete POVM with exactly \( d^2 \) effects is a minimal informationally complete measurement, or MIC.
We can equivalently express a POVM \{E_i\} as a set of quantum states and associated weights, 
\[ E_i := e_i \rho_i, \tag{3} \]
where the weights \{e_i\} are just the traces of the \{E_i\}. The Born Rule then implies that these weights are, up to a normalization constant, just the probabilities of the POVM outcomes given the quantum state of maximal indifference, the garbage state \( \rho = \frac{1}{d} I \). If a MIC has all its weights equal, then normalization requires them to all be \( \frac{1}{d} \), and this MIC maps the garbage state to the flat probability distribution \( (\frac{1}{d^2}, \cdots, \frac{1}{d^2}) \). Such a MIC is said to be unbiased.

Much of the community’s interest in MICs has focused upon the fascinating special case of the symmetric informationally complete measurements, the SICs [3–6]. A SIC is an unbiased MIC where each effect is proportional to a rank-1 projector — so, each outcome of the measurement is specified by a ray in the Hilbert space — and the inner products between any two effects are constant:
\[ \text{tr} E_i E_j = \frac{1}{d^2} \delta_{ij} + \frac{1}{d+1}. \tag{4} \]
SICs have proved in many ways optimal among MICs [7–12].

The present paper is a close sequel to an earlier publication from our group [13], and we will refer to that work on occasion. Our plan for the remainder of this paper is as follows. In §2, we will use the MIC concept to introduce probabilistic and quasiprobabilistic representations of quantum theory. Then, in §3, we will review the fundamentals of frame theory, a subject that provides useful tools for exploring MICs, Wigner functions and the relations between them. Next, §4 will explain how to construct an unbiased Wigner function given an unbiased MIC. Here, we will prove Theorem 1, which bounds the distance between an unbiased MIC and an unbiased Wigner function. Theorem 2 will then show the special role that SICs play in these considerations. §5 will follow up on the previous section with a proposal for generalizing beyond the unbiased case to arbitrary MICs. We will then study a conceptual inverse of sorts in §6, where we will meet a procedure for constructing an equiangular MIC given a Wigner function.

## 2 Probability and Quasiprobability

Because a MIC is by definition informationally complete, it can serve as a reference measurement. In other words, any MIC has the property that if an agent has written a probability distribution over its \( d^2 \) possible outcomes, she can then compute the probabilities that she should assign to the outcomes of any other measurement.

We can illustrate the meaning of this by comparing and contrasting it with classical particle mechanics. There, a “reference measurement” would just be an experiment that reads off the system’s phase-space coordinates, i.e., the positions and momenta of all the particles making up the system. Any other experiment, such as observing the total kinetic energy, is in principle a coarse-graining of the information that the reference measurement itself provides.

To develop the analogy, consider the following scenario [14]. An agent Alice has a physical system of interest, and she plans to carry out either one of two different, mutually exclusive laboratory procedures upon it. In the first protocol, she will drop the system directly into a measuring apparatus and thereby obtain an outcome. In the second protocol, she will cascade her measurements, sending the system through a reference measurement and then, in the next stage, feeding it into the device from the first protocol. Probability theory in the abstract provides no consistency conditions between Alice’s expectations for these two protocols. Different circumstances, different probabilities! Let \( P \) denote her probability assignments for the consequences of following the two-step procedure and \( Q \) those for the single-step protocol. Then, writing \( \{H_i\} \) for the possible outcomes of the reference measurement and \( \{D_j\} \) for those of the other,
\[ P(D_j) = \sum_i P(H_i) P(D_j|H_i). \tag{5} \]
This much is just logic, or more specifically speaking, a consequence of Dutch-book coherence [9, 15]. It is known as the Law of Total Probability (LTP). However, the claim that
\[ Q(D_j) = P(D_j) \tag{6} \]
is an assertion of physics, above and beyond probabilistic self-consistency. It codifies in probabilistic language the idea that the classical ideal of a reference measurement simply reads off the system’s pre-existing phase-space coordinates, or data equivalent thereto.

In quantum physics, life is very different. Instead of taking a weighted average of the \( \{P(D_j|H_i)\} \) as in the LTP, Alice instead uses a mapping

\[
Q(D_j) = \mu (\{P(H_i)\}, \{P(D_j|H_i)\}),
\]

where the exact form of the function \( \mu \) depends upon her choice of MIC. Conveniently, quantum theory is only so nonclassical that \( \mu \) is a bilinear form, rather than some much more convoluted function.

We can write our equations more compactly by introducing a vector notation, in which omitted subscripts imply that an entire vector or matrix is being treated as an entity. Then the LTP has the expression

\[
P(D) = P(D|H)P(H),
\]

while the quantum relation, the Born Rule, is

\[
Q(D) = P(D|H)\Phi P(H), \text{ with } [\Phi^{-1}]_{ij} := \frac{1}{\text{tr } H_j} \text{tr } H_i H_j.
\]

The matrix \( \Phi \) depends upon the MIC, but it is always a column quasistochastic matrix, meaning its columns sum to one but may contain negative elements [12]. In fact, \( \Phi \) must contain negative entries; this follows from basic structural properties of quantum theory [16]. As a consequence, \( \Phi P(H) \) is a quasiprobability. Considering the operation of \( \Phi \) on \( P(H) \) as a single term results in an equation algebraically equivalent to the LTP aside from the appearance of negativity in the last term. The same thing happens if we regard \( \Phi \) as acting to the left on \( P(H|D) \), but now the negativity has been relegated to the first term. For an unbiased MIC, \( \Phi \) is proportional to the inverse Gramian of the MIC and we can do the same thing in yet another way by “splitting the map down the middle” and acting with one factor in either direction:

\[
Q(D) = \left( P(D|H)\Phi^{1/2} \right) \left( \Phi^{1/2} P(H) \right).
\]

This has the intriguing consequence that Euclidean orthogonality coincides with Hilbert–Schmidt orthogonality: The final probability \( Q(D_j) \) is zero exactly when the two transformed vectors are orthogonal with respect to the ordinary dot product. In summary, for a given MIC, there is a certain “gauge freedom” about where the negativity can occur if we wish to massage (9) into a form which fits together in exactly the same way as the LTP [17].

This brings us into the territory of quasiprobability representations of quantum theory constructed on orthogonal operator bases [18, 19], which were motivated in the first place by the demand that we cast the Born Rule structurally analogous to the LTP. For our purposes, a Wigner function is defined by an orthogonal Hermitian operator basis \( \{F_i\} \) constrained to sum to the identity: \( \sum_i F_i = I \). We could more fully refer to these as “minimal discrete Wigner functions”, but in what follows, we will generally omit the additional qualifiers since we will not be discussing quasiprobability representations of quantum theory that use overcomplete operator bases. The sum condition is chosen so that \( \text{tr } \rho F_i \) may be interpreted as a quasiprobability. Because it is orthogonal, the Gram matrix of a Wigner function is diagonal, so \( \text{tr } F_i F_j = c_i \delta_{ij} \) for some constants \( \{c_i\} \). Given any (not necessarily orthogonal) basis \( \{F_i\} \) for a finite-dimensional space, its dual basis is the unique basis \( \{\tilde{F}_i\} \) such that \( \text{tr } \tilde{F}_i F_j = \delta_{ij} \). So, we must have \( F_i = c_i \tilde{F}_i \) in the present case. That is, the dual basis elements are proportional to the corresponding basis element. Contrast this with MICs, where such a proportionality never holds [13]. The sum constraint enforces \( \text{tr } \tilde{F}_i = 1 \) for all \( i \) and, if we define \( f_i := \text{tr } F_i \) to be the weights of the Wigner function in analogy with MICs, it is easy to see that the basis and dual basis proportionality is equal to the weights of the Wigner function, \( c_i = f_i \).

To summarize, the Gram matrix of a Wigner function is given by \( \text{tr } F_i F_j = f_i \delta_{ij} \) and its dual basis satisfies \( F_i = f_i \tilde{F}_i \).

For the remainder of this paper, we will discuss the relation between MICs and Wigner functions and explore how to interconvert between them. We should reveal that we have an agenda in this enterprise — the authors would like to see a shift in attention away from Wigner functions and toward MICs in the quantum
information and computation communities. The reason for preferring MICs to Wigner functions is that MICs furnish true probabilistic representations of quantum theory, and, notwithstanding the occasionally esoteric disputes over the philosophy of probabilities, without an explicit description of an experimental context, quasiprobabilities lack the direct operational interpretation that probabilities have. Because of this and because MICs and Wigner functions turn out to be naturally related, we believe MICs will ultimately prove to be the more illuminating class of structures to study.

Before we proceed, we note two operations which send any Wigner representation to other elements of the equivalence class of Wigner representations with the same weights. Firstly, it is easy to see that conjugation by any unitary \( U \), that is, \( F_i \rightarrow UF_iU^\dagger \) for all \( i \), is another representation with the same weights. In addition to this unitary invariance of Wigner representations with a certain set of weights, the elementwise affine transformations

\[
F^s_i := -F_i + \frac{2f_i}{d}I
\]

also accomplish this goal. It’s clear that the Wigner-function sum condition is satisfied, and a simple calculation demonstrates

\[
\text{tr} \left( -F_i + \frac{2f_i}{d}I \right) \left( -F_j + \frac{2f_j}{d}I \right) = \text{tr} F_i F_j - \frac{2f_i}{d} \text{tr} F_i - \frac{2f_j}{d} \text{tr} F_j + \frac{4f_if_j}{d^2} \text{tr} I = f_i \delta_{ij} .
\]

For any Wigner function \( \{F_i\} \), we refer to \( \{F^s_i\} \) as the shifted Wigner function.

### 3 The Briefest Course in Frame Theory

For a Hilbert space \( \mathcal{H} \), let \( \mathcal{L}(\mathcal{H}) \) denote the set of self-adjoint linear operators on \( \mathcal{H} \). This set is a Hilbert space itself. Throughout the paper we will make use of a convenient way to represent an operator in \( \mathcal{L}(\mathcal{H}) \) as a vector in \( \mathcal{H} \otimes \mathcal{H} \), sometimes called vectorizing the operator:

\[
|A\rangle\rangle := \sum_i (A \otimes I)|i\rangle|i\rangle ,
\]

where \( \{|i\rangle\} \) is an orthonormal basis in \( \mathcal{H} \). One may easily check that the vectorized operator inner product is equal to the standard Hilbert–Schmidt inner product, so

\[
|G|_{ij} = \langle E_i|E_j \rangle .
\]

Working with vectorized operators gives a concise expression to some helpful entities from frame theory, which studies generalizations of bases. For example, the synthesis operator for a MIC is the matrix

\[
V := \begin{pmatrix}|E_1\rangle\rangle \cdots |E_d\rangle\rangle \end{pmatrix}
\]

and the frame operator takes the form

\[
S :=VV^\dagger = \sum_i |E_i\rangle\langle E_i| .
\]

When the frame operator is proportional to the identity on the space where it lives, the frame is called tight; and if the proportionality constant is unity, the tight frame is normalized. In general, the Gram matrix can be written as \( G = V^\dagger V \), from which it follows that the Gram matrix and the frame operator are isospectral.

Depending on context, we sometimes think of \( S \) as an operator in \( \mathcal{L}(\mathcal{L}(\mathcal{H})) \) and sometimes as an operator in \( \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \). Substituting into (16) the relation between the basis and its dual reveals that the frame operator maps the one to the other:

\[
S|E_j\rangle = |E_j\rangle .
\]

Moreover, we have a couple of handy resolutions of the identity:

\[
\sum_i |\tilde{E}_i\rangle\langle E_i| = \sum_i |E_i\rangle\langle \tilde{E}_i| = I_{d^2} .
\]
The inverses of the frame operator and Gram matrix are given by the frame operator and Gram matrix of the dual basis. Several ways to express these are as follows:

\[ [G^{-1}]_{ij} = \langle \bar{V}^\dagger \bar{V} \rangle_{ij} = \text{tr} \bar{E}_i \bar{E}_j = \langle \langle \bar{E}_i | \bar{E}_j \rangle \rangle, \]  

(19)

and

\[ S^{-1} = \bar{V} \bar{V}^\dagger = \sum_i |\bar{E}_i\rangle\langle \bar{E}_i| \, . \]  

(20)

This is a good place to clear up a confusion that we have encountered a few times during conferences, when people from different subfields try to communicate. A MIC is a basis for the \( d^2 \)-dimensional space \( \mathcal{L}(\mathcal{H}_d) \). It is not overcomplete, but exactly complete, having just the right number of elements to span the operator space while keeping itself a linearly-independent set. A rank-1 MIC, for which \( E_i = e_i |\psi_i\rangle\langle \psi_i| \) is specified by a set of weights \( \{e_i\} \) and a set of vectors \( \{ |\psi_i\rangle \} \). These vectors are \( d^2 \) in number and live within \( \mathcal{H}_d \), so if we tried to use them as a basis for that space, they would be overcomplete. The vectors onto which the MIC elements \( \{E_i\} \) project can be considered a frame for the \( d \)-dimensional space \( \mathcal{H}_d \). We will not, however, rely upon this perspective in the current paper.

## 4 Unbiased Wigner Functions from Unbiased MICs

Unbiased Wigner functions, where \( f_i = 1/d \) for all \( i \), are of special interest for the same reason unbiased MICs are — they represent the state of maximal indifference by a flat distribution. Furthermore, most of the well-known quasiprobability representations of quantum theory are unbiased when expressed in this formalism [16,20–22]. An unbiased Wigner function forms a tight frame because its frame operator \( S = \frac{1}{d} I \) is proportional to the identity. A MIC, on the other hand, cannot be a tight frame; if its frame operator were proportional to the identity, its Gram matrix would have to be as well and a MIC cannot be an orthogonal basis [13]. However, as we will see, there is a very natural way to relate MICs and Wigner functions which makes use of the canonical tight frame concept introduced above.

Because \( dG \) is doubly stochastic, \( \frac{1}{\sqrt{d}} G^{-1/2} \) is doubly quasistochastic. Also note that

\[ [G^{-1}]_{jk} = \langle \bar{E}_j | \bar{E}_k \rangle = \langle \bar{E}_j | S^{-1} E_k \rangle = \sum_i \langle \bar{E}_j | S^{-1/2} | E_i \rangle \langle \bar{E}_i | S^{-1/2} | E_k \rangle \]  

\[ \implies [G^{-1/2}]_{ij} = \langle \bar{E}_i | S^{-1/2} | E_j \rangle \, . \]  

(21)

With these facts in hand, we may see that a rescaling of the canonical tight frame of an unbiased MIC is a Wigner function which we call the \textit{principal} Wigner function:

\[ |F_i\rangle = \frac{1}{\sqrt{d}} S^{-1/2} |E_i\rangle = \frac{1}{\sqrt{d}} \sum_j [G^{-1/2}]_{ij} |E_j\rangle \, . \]  

(22)

Substituting the result of (21) into the second equivalence of (22) and resolving the identity that appears demonstrates the equivalence of the two expressions for \( F_i \). The Wigner-function sum identity follows from the quasistochasticity of \( \frac{1}{\sqrt{d}} G^{-1/2} \) and the POVM sum condition. Orthogonality is also straightforward:

\[ \langle F_i | F_j \rangle = \frac{1}{d} \langle E_i | S^{-1/2} S^{-1/2} | E_j \rangle = \frac{1}{d} \langle E_i | \bar{E}_j \rangle = \frac{1}{d} \delta_{ij} \, . \]  

(23)

If we define the symmetric orthogonal matrix

\[ V := -I + \frac{2}{d^2} J \, , \]  

(24)

where \( J \) is the Hadamard identity, that is, the matrix of all 1s, one may also verify that the shifted Wigner function of an unbiased MIC is related to another square root of \( G^{-1} \) as follows:

\[ |F_i S\rangle = \frac{1}{\sqrt{d}} \sum_j [G^{-1/2} V]_{ij} |E_j\rangle \, . \]  

(25)
How does all of this fit in with quasiprobability representations discussed in the literature? We note three points of contact. First, the Wigner functions most familiar from the literature are in fact the quasiprobabilities obtained from the principal Wigner function of a MIC first constructed by Marcus Appleby [23]. The Appleby MIC can be constructed in any odd dimension $d$. The elements $\{E_{k,l}\}$ of this MIC are labeled by ordered pairs of integers $k, l \in \{0, \ldots, d-1\}$, and each element has rank $(d+1)/2$. Together, the elements of the Appleby MIC comprise an orbit under the action of the Weyl–Heisenberg group. The principal Wigner function of the Appleby MIC is given by

$$F_{k,l} = (d+1)E_{k,l} - \frac{1}{d} I.$$  \hfill(26)

When $d$ is prime, this Wigner function coincides with the one introduced by Wootters [20].

Secondly, in the SIC case, we can write

$$E_i := \frac{1}{d} \Pi_i, \quad \text{with} \quad \text{tr} \Pi_i \Pi_j = \frac{d\delta_{ij} + 1}{d+1},$$  \hfill(27)

and so we have

$$\frac{1}{\sqrt{d}} [C^{-1/2}_{\text{SIC}}]_{ij} = \sqrt{d+1} \delta_{ij} + \frac{1}{d^2}(1 - \sqrt{d+1}),$$  \hfill(28)

which gives the following principal and shifted Wigner functions:

$$F_j = \frac{1}{d} \left( \sqrt{d+1} \right) \Pi_j + \frac{1}{d^2} \left( 1 - \sqrt{d+1} \right) I \quad \text{and} \quad F^S_j = -\frac{1}{d} \left( \sqrt{d+1} \right) \Pi_j + \frac{1}{d^2} \left( 1 + \sqrt{d+1} \right) I.$$  \hfill(29)

These are the two Wigner functions identified by Zhu [18] for a different reason.\footnote{Zhu calls unbiased Wigner functions “NQPRs” and prefers to report the dual basis elements. In his notation, \(Q_j^+ = dF_j\) and \(Q_j^- = dF^S_j\).} Given a Wigner function, the\hfill
ceiling negativity\hfill
of a quantum state $\rho$ is the magnitude of the most negative entry in the quasiprobability vector that represents $\rho$. Maximizing the ceiling negativity over all quantum states yields the ceiling negativity of the Wigner function. Zhu proved that the principal and shifted principal Wigner functions associated with a SIC provide, respectively, the\hfill
lower and upper bounds\hfill
on the ceiling negativity over all unbiased Wigner functions in dimension $d$. This orthogonalization procedure sets Zhu’s result in a broader conceptual context: Zhu’s Wigner functions are the output of applying to a SIC a procedure that works for any MIC.

Our third point of contact pertains to the case of a single qubit. In dimension $d = 2$, a SIC is defined by a set of four projectors $\{\Pi_j\}$, and taking the state orthogonal to each of these,

$$\Pi^\perp_j := I - \Pi_j,$$  \hfill(30)

yields another SIC. Together, the two satisfy

$$\text{tr} \Pi_j \Pi^\perp_k = \frac{1}{3} (1 - \delta_{jk}),$$  \hfill(31)

and the principal Wigner function of one is the shifted principal Wigner function of the other. As Zhu notes, the principal Wigner function of a qubit SIC is equivalent to Wootters’ Wigner function for a qubit, up to an overall unitary transformation and permutation. Choosing the proper SIC — that is, picking a tetrahedron with the correct orientation in the Bloch sphere — reproduces Wootters’ qubit Wigner function exactly [18].

We will return to the question of connecting our work with that of Wootters. First, we prove two theorems that will help situate SICs among MICs more generally when it comes to constructing Wigner functions.

Much of the significance of an unbiased MIC’s principal Wigner function can be understood through frame theory. The principal Wigner function of an unbiased MIC is the canonical tight frame of that MIC. The canonical tight frame is the closest normalized tight frame to the original frame in the sense of minimizing the least squares error [24]. Likewise, the principal Wigner function of an unbiased MIC is the closest unbiased Wigner function to the MIC. Frame theory has no well-defined notion of a tight frame farthest from a given frame, but Wigner functions are more constrained than general frames, thanks to their origins as MICs. Consequently, there is also a distinguished unbiased Wigner function farthest from a given MIC, as the next theorem shows. Let $\| \cdot \|$ denote the Hilbert–Schmidt norm, $\| A \| := \sqrt{\text{tr} A^\dagger A}$.\hfill

1
Theorem 1. Let \( \{E_i\} \) be an unbiased MIC and \( \{F_j\} \) be an unbiased Wigner function. Let \( \lambda_k \) be the \( k \)th eigenvalue of the MIC’s frame operator \( S \). Then
\[
\sum_k \left( \sqrt{\lambda_k} - \sqrt{1/d} \right)^2 \leq \sum_i |E_i - F_i|^2 \leq \sum_k \left( \sqrt{\lambda_k} + \sqrt{1/d} \right)^2 - \frac{4}{d},
\]
where the lower bound is saturated iff \( \{F_j\} \) is the principal Wigner function of \( \{E_i\} \) and the upper bound is saturated iff \( \{F_j\} \) is the shifted principal Wigner function of \( \{E_i\} \).

Proof. For the lower bound we closely follow the proof of Theorem 3.2 in [24]. Then
\[
\sum_i |E_i - F_i|^2 = \sum_i \mathrm{tr} (E_i - F_i)(E_i - F_i)
\]
\[
= \sum_i (\mathrm{tr} E_i^2 + \mathrm{tr} F_i^2 - 2\mathrm{tr} E_i F_i)
\]
\[
= \sum_k \lambda_k + d - 2 \sum_i \mathrm{tr} E_i F_i,
\]
so we need to prove the inequality
\[
\sum_j \mathrm{tr} E_j F_j = \sum_j \langle E_j | F_j \rangle \leq \frac{1}{\sqrt{d}} \sum_k \sqrt{\lambda_k}.
\]
Let \( \{|u_i\} \) be an orthonormal basis of eigenvectors of \( S \). Inserting the resolution of the identity, we equivalently need to show that
\[
\sum_j \langle E_j | u_k \rangle \langle u_k | F_j \rangle \leq \frac{1}{\sqrt{d}} \sum_k \sqrt{\lambda_k}.
\]
It would suffice if we could show
\[
\Re \sum_j \langle E_j | u_k \rangle \langle F_j | u_k \rangle \leq \frac{1}{\sqrt{d}} \sqrt{\lambda_k}, \quad \forall k.
\]
Thus, using the Cauchy–Schwarz inequality and the fact an unbiased Wigner function is a tight frame, we have
\[
\Re \sum_j \langle E_j | u_k \rangle \langle F_j | u_k \rangle \leq \left| \sum_j \langle E_j | u_k \rangle \langle F_j | u_k \rangle \right| \leq \left( \sum_j |\langle E_j | u_k \rangle|^2 \right)^{1/2} \left( \sum_j |\langle F_j | u_k \rangle|^2 \right)^{1/2}
\]
\[
= \sqrt{\langle u_k | S | u_k \rangle} \sqrt{\langle u_k | (1/d)I | u_k \rangle} = \frac{1}{\sqrt{d}} \sqrt{\lambda_k},
\]
with equality iff there is a real positive constant \( c_k \) such that
\[
\langle E_j | u_k \rangle = c_k \langle F_j | u_k \rangle, \quad \forall j.
\]
Substituting into (37) reveals \( c_k = \sqrt{d}\lambda_k \), hence the lower bound is saturated iff
\[
|F_j| = \sum_k |u_k \rangle \langle u_k | F_j \rangle = \sum_k \frac{1}{\sqrt{d}\lambda_k} |u_k \rangle \langle u_k | E_j \rangle = \frac{1}{\sqrt{d}} \sum_k \frac{1}{\sqrt{\lambda_k}} |u_k \rangle \langle u_k | S^{1/2}S^{-1/2}E_j \rangle
\]
\[
= \frac{1}{\sqrt{d}} \sum_k |u_k \rangle \langle u_k | S^{-1/2}E_j \rangle = \frac{1}{\sqrt{d}} S^{-1/2}E_j.
\]
For the upper bound, we can proceed in an analogous way:

\[
\sum_i |E_i - F_i|^2 = \sum_i \left| E_i + F_i^S - \frac{2}{d^2} I \right|^2 \\
= \text{tr} \left( E_i + F_i^S - \frac{2}{d^2} I \right) \left( E_i + F_i^S - \frac{2}{d^2} I \right) \\
= \sum_i |E_i + F_i^S|^2 - \frac{4}{d} \\
\leq \sum_k \left( \sqrt{\lambda_k} + \sqrt{1/d} \right)^2 - \frac{4}{d}.
\]

(40)

The inequality follows from the same reasoning as in equations (33) through (39). And so, it is saturated iff \( \{F_j\} \) is the principal Wigner function of \( \{E_i\} \), that is, iff \( \{F_j\} \) is the shifted principal Wigner function of the MIC \( \{E_i\} \).

As the next theorem demonstrates, the smallest and largest that these bounds can be occurs when the unbiased MIC is a SIC. The minimal lower bound result complements the prior discovery that SICs are the closest MICs can come to being orthogonal bases [10]. Consequently, the next theorem supports the intuition that SICs are the natural relaxation of the notion of an orthogonal operator basis which fits into the cone of positive semidefinite operators.

**Theorem 2.** Let \( \{E_i\} \) be an unbiased MIC and \( \{F_j\} \) be an unbiased Wigner function. Then

\[
\frac{d-1}{d} \left( d + 2 - 2\sqrt{d+1} \right) \leq \sum_i |E_i - F_i|^2 \leq \frac{d-1}{d} \left( d + 2 + 2\sqrt{d+1} \right)
\]

(41)

where the lower bound is saturated iff \( \{E_i\} \) is a SIC and \( \{F_j\} \) is its principal Wigner function and the upper bound is saturated iff \( \{E_i\} \) is a SIC and \( \{F_j\} \) is its shifted principal Wigner function.

**Proof.** The matrix \( dG \) is doubly stochastic for an unbiased MIC, so the maximal eigenvalue of \( S \) is always \( 1/d \). Furthermore, because the diagonal entries of an unbiased MIC’s Gram matrix are bounded above by \( 1/d^2 \), we also know that \( \text{tr} \ S \leq 1 \). It is then straightforward to perform a constrained optimization to see that the bounds in (32) achieve their extreme values when

\[
\lambda = \left( \frac{1}{d}, \frac{1}{d(d+1)}, \ldots, \frac{1}{d(d+1)} \right).
\]

(42)

Plugging this spectrum in to (32) gives the upper and lower bounds in (41). Such a spectrum occurs iff the MIC is a SIC, a fact that is easy to derive from Lemma 1 in [12].

The construction of principal Wigner functions from unbiased MICs provides a Wigner function with appealing symmetry properties for all \( N \)-qubit systems. Let \( \{E_i : i = 1, \ldots, 4\} \) be a qubit SIC. Up to the weighting factor, each effect is a rank-1 projector, and the set of four such projectors can be portrayed as a regular tetrahedron inscribed in the Bloch sphere [4]. A tensorhedron MIC [13] is a POVM whose elements are tensor products of operators chosen from the qubit SIC \( \{E_i\} \):

\[
E_{i_1, \ldots, i_N} = E_{i_1} \otimes \cdots \otimes E_{i_N}.
\]

(43)

Up to an overall unitary conjugation, every qubit SIC is covariant under the Pauli group, and so every tensorhedron MIC has an \( N \)-qubit Pauli symmetry. Furthermore, any \( N \)-qubit tensorhedron MIC has a principal Wigner function.

We can identify the principal Wigner function of the \( N \)-qubit tensorhedron MIC with a known quantity, thanks to a pair of convenient facts. The first applies when a basis is group covariant, i.e., when it can be produced by taking the orbit of an initial element under the action of a group, thereby making \( d^2 \) elements out of one. If a basis is group covariant then its dual basis inherits that covariance, and it is straightforward
to show that the principal Wigner function — the structure “halfway in between the two” produced by acting with the square root of the frame operator — does so as well. This follows readily from the observation that if \( U_j \) and \( U_k \) are unitaries in the group that generates the basis, then the tensor product \( U_j \otimes U_k \) commutes with the frame operator. Second, the transformations of dualizing and orthogonalizing both respect the tensor product. If \( \{ E_i \} \) is a MIC and \( \{ \tilde{E}_j \} \) its dual, then
\[
\text{tr} \left( E_i \otimes E_k \right) \left( \tilde{E}_j \otimes \tilde{E}_l \right) = \delta_{ij} \delta_{kl},
\]
and so the dual of the tensor-product basis \( \{ E_{ik} \} \) is the tensor products of the original dual elements. Therefore, the frame operator also respects the tensor product, as does the orthogonalization of the MIC to produce a Wigner function.

From these observations, we can conclude that the principal Wigner function of the \( N \)-qubit tensorhedron MIC is exactly the \( N \)-qubit Wigner function proposed by Wootters [20].

One of the mysteries of the SICs is that, in all known cases, the SICs are group covariant. The definition of a SIC does not mention group covariance anywhere — the only symmetry in it is the equality of the inner products — and so the fact that the known SICs are all group covariant might be a subtle consequence we do not yet understand, or it might be an accident of convenience. We do know, thanks to Zhu, that in prime dimensions, if a SIC is group covariant then it must be covariant under the Weyl–Heisenberg group specifically [25, 26]. This leaves open the cases of dimensions that are higher prime powers or products of distinct primes. And in dimension \( d = 8 \), there exists in addition to the Weyl–Heisenberg SICs the class of Hoggar-type SICs, which are related to the octonions and are covariant under the three-qubit Pauli group [27–31]. All of these SICs can be converted to unbiased Wigner functions in the manner described above, and the resulting Wigner functions will inherit the group-covariance properties of the original SICs. Therefore, for a three-qubit system, the construction of principal Wigner functions from unbiased MICs furnishes three inequivalent Wigner functions of interest: Wootters, Weyl–Heisenberg, and Hoggar. The Wootters version is distinguished by particularly nice permutation symmetry properties [32].

Wootters [20] spends a good deal of time exploring the triple products of his Wigner function, which in our notation are
\[
\Gamma_{jkl} = d^2 \text{tr} \left( F_j F_k F_l \right).
\]
These can of course be defined for any Wigner function. Of particular note is the case where the Wigner function is the principal Wigner function of a SIC, because the SIC triple products \( \text{tr} \Pi_j \Pi_k \Pi_l \) are remarkable numbers [11,33–36]. We have that
\[
d^3 \text{tr} \left( F_j F_k F_l \right) = \pm (d + 1)^{3/2} \text{tr} \left( \Pi_j \Pi_k \Pi_l \right)
+ \left( 1 - \sqrt{d + 1} \right) \left( \delta_{jk} + \delta_{kl} + \delta_{lj} \right)
- \frac{1}{d^2 \sqrt{d + 1}} \left( 2 \sqrt{d + 1} + (d + 1)(d - 2) \right).
\]
(46)
For Wootters’ definition of the discrete Wigner function, the triple products can be found using the geometry of the finite affine plane on \( d^2 \) points. Essentially, one takes the triangle formed by three points in that phase space, and the triple product depends upon the “area” of it. Specifically, the triple product (for the case of odd prime dimension) is given by the complex exponential
\[
\Gamma_{jkl} = \frac{1}{d} \exp \left( \frac{4\pi i}{d} A_{jkl} \right).
\]
(47)
This leads naturally to an interpretation of the triple products in terms of geometric phases. The larger the enclosed area, the greater the geometric phase. The SIC triple products, and thus by extension those of their associated Wigner functions, have rich number- and group-theoretic properties [30,31,33–36]. What these properties imply for the Wigner functions derived from SICs is largely an open question. For early results in this vein, see Theorems 6 and 13 of [11].

5 Proposed Generalization for Arbitrary MICs

We present one way to extend the association between Wigner functions and MICs discussed above beyond the unbiased case. Let \( S := \sum_i \frac{1}{\sqrt{\epsilon_i}} |E_i\rangle\langle E_i| \) be the frame operator for the rescaled basis \( \{ \frac{1}{\sqrt{\epsilon_i}} E_i \} \), define
Φ := AG⁻¹ where [A]ij = eiδij, and denote by \( \sqrt{Φ} \) the square root of Φ with all positive eigenvalues.² Then the principal Wigner function is given by

\[
|F_i\rangle = S^{-1/2}|E_i\rangle = \sum_j [\sqrt{Φ}]_{ij}|E_j\rangle .
\]

The equivalence of the two expressions for the principal Wigner function follows in the same way as in the unbiased case with the versions of (21) and (22) suited to the rescaled basis rather than the original MIC. One can see from its inverse that Φ is column quasistochastic and thus that \( \sqrt{Φ} \) is as well. With this fact, it is easy to see that the sum condition of a Wigner function is satisfied. For orthogonality, we directly compute

\[
\langle F_i|F_j\rangle = \langle E_i|S^{-1}|E_j\rangle = \sqrt{e_i e_j} \left( \sum_i E_i \right) \langle E_i|S^{-1}|E_j\rangle = \sqrt{e_i e_j} \delta_{ij} ,
\]

from which \( f_i = e_i \) for all \( i \) also follows.

Our convention has the advantages that it preserves the weights of the MIC and reduces to equation (22) in the unbiased case, but it remains to be seen if this is the best generalization available. Part of the issue is that it is harder to judge the merits of this definition than the unbiased version due to the lack of biased Wigner functions in the literature. Perhaps more tellingly, however, a numerical search we conducted in dimension 2 revealed that Theorem 1 does not generalize to the biased case for this convention. This is legitimately frustrating, because Theorem 1 is the primary reason for singling out the principal Wigner function in the unbiased case. However, the frame operator for the rescaled basis has independently arisen in a different context pertaining to quantum state tomography, suggesting a deeper significance [37]. In the absence of a better convention which generalizes Theorem 1 exactly, there might be a satisfying modification of the notion of “closest Wigner function” which suits this convention. We offer one such modification with a conjecture:

**Conjecture 1.** The closest and farthest Wigner functions to a MIC are always unitarily related to the principal Wigner function and shifted Wigner function respectively.

### 6 From Wigner functions to Equiangular MICs

We have discussed how to produce Wigner functions given MICs, but not the reverse. If we wish to advocate more attention be paid to MICs, it would be nice to discover an association so clean that we could claim that someone’s favorite Wigner function was “really” a particular MIC in disguise. At a glance, however, this appears to be quite difficult. Should the whole unitary freedom from the previous sections map to a single MIC? Should the principal and shifted Wigner functions both map to the same MIC? Naively, given an unbiased Wigner function, we could try to invert the matrix multiplication in Eq. (22) and obtain \( |E_i\rangle \) in terms of \( |F_i\rangle \). However, without an independent expression for the frame operator, we lack the information to carry this out.

From the way we’ve presented it in the previous sections, it may feel like the set of Wigner functions is larger and more unwieldy than the set of MICs, but this isn’t so clear. They both consist of Hermitian matrices constrained to sum to the identity, so the question boils down to whether it is more restrictive to focus only on orthogonal sets or to require every element be a member of the cone of positive semidefinite operators. To temper these intuitions, take \( \{ \Pi_i : i = 1, \ldots, d^2 \} \) to comprise a SIC, and consider the one-parameter family of MICs made by mixing that SIC with the identity operator:

\[
E_i = \frac{\beta}{d} \Pi_i + \frac{1 - \beta}{d^2} I , \quad -\frac{1}{d - 1} \leq \beta \leq 1 .
\]

It is easy to verify that the principal Wigner functions are the same for any two \( \beta \) values of the same sign. In other words, there are instances where different MICs map to the same Wigner function. Incidentally,

²To see that \( \sqrt{Φ} \) is well defined, note that Φ has the same spectrum as the manifestly positive semidefinite matrix \( A^{1/2}G^{-1}A^{1/2} \). One may check that \( \sqrt{Φ} = A^{1/2} (A^{1/2}G^{-1}A^{1/2})^{1/2} A^{-1/2} \).
the MIC with the lowest allowed $\beta$ value in this family,

$$E_i = \frac{1}{d(d-1)}(-\Pi_i + I),$$

has, as its principal Wigner function, the shifted Wigner function of the SIC.

It is nevertheless possible to construct a MIC from a Wigner function in a simple way, potentially deserving special attention, following a procedure suggested by Marcus Appleby. The idea is to shift up the eigenvalues of every element of a Wigner function to be nonnegative and then uniformly rescale to preserve the identity sum condition. Explicitly, if $\{F_i\}$ is a Wigner function and $\lambda_i$ is the minimal (i.e., most negative) eigenvalue of $F_i$, then

$$E_i := \frac{F_i - \lambda_i I}{1 - \sum_j \lambda_j}$$

is a MIC. One nice thing about this convention is that it turns any group covariant Wigner function into an equiangular MIC, that is, a MIC whose Gramian has one constant value along the diagonal and another constant value everywhere else. To see this, first note that a group covariant Wigner function is necessarily unbiased and that every element has the same spectrum. So, we can write $\lambda_i = \lambda$ in this case and compute:

$$\text{tr } E_i E_j = \left(\frac{1}{1 - d^2 \lambda}\right)^2 \text{tr } (F_i - \lambda I)(F_j - \lambda I) = \left(\frac{1}{1 - d^2 \lambda}\right)^2 (\text{tr } F_i F_j - \lambda \text{tr } F_i - \lambda \text{tr } F_j + d\lambda^2)
= \left(\frac{1}{1 - d^2 \lambda}\right)^2 \left(\frac{1}{d} \delta_{ij} - \frac{2\lambda}{d} + d\lambda^2\right) = \frac{\delta_{ij} - 2\lambda + d^2\lambda^2}{d - 2d^3\lambda + d^5\lambda^2},$$

and so we can see that it takes one value when $i = j$ and another when $i \neq j$.

The composition of the principal Wigner function mapping with the one just described is a map which takes a MIC to a MIC. If we start with a group covariant MIC, the output will be an equiangular MIC as noted above. One may verify that subsequent applications of the composed mapping sends the equiangular MIC to itself, that is, the output for a group covariant input is a fixed point of the mapping. This operation also sends a SIC to itself, boosting the significance of the convention due to the importance of SICs to the study of MICs and Wigner functions. We end this section with a numerical observation: It seems that after sufficiently many applications of the combined operation, given any input MIC whatsoever, one arrives at a MIC which is a fixed point of the operation.

7 Conclusions

A MIC is a basis for the operator space $\mathcal{L}(\mathcal{H}_d)$, or in other words, a coordinate system for doing quantum mechanics. Because MIC elements are required to be positive semidefinite, no MIC can ever be an orthogonal basis; the closest that a MIC can come is by being a SIC [10]. Prior work has shown that this expresses how much of the oddity of quantum theory is an artifact of coordinates, versus what is the unavoidable residuum of nonclassicality. If one abandons direct operational meaning in terms of probabilities, then one can push basis elements outside of the positive semidefinite cone and achieve orthogonality. In this paper, we have shown well-defined procedures for doing so, and we have quantified how far an orthogonalized basis — a Wigner function — can deviate from the original MIC.

While negative quasiprobabilities do not have direct operational meaning as probabilities do, they can be made meaningful in combination with additional data. Of particular relevance is the discovery that negativity can be a resource for quantum computation [22, 38]. This is a theme that is compelling in broad strokes, while the details are still being hashed out. Grasping the variety of Wigner functions, and how they relate to the most economical of probabilistic representations of quantum theory, may prove helpful in advancing our understanding of this intriguing subject.

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