Effective long wavelength scalar
dynamics in de Sitter

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Abstract. We discuss the effective infrared theory governing a light scalar’s long wavelength dynamics in de Sitter spacetime. We show how the separation of scales around the physical curvature radius $k/a \sim H$ can be performed consistently with a window function and how short wavelengths can be integrated out in the Schwinger-Keldysh path integral formalism. At leading order, and for time scales $\Delta t \gg H^{-1}$, this results in the well-known Starobinsky stochastic evolution. However, our approach allows for the computation of quantum UV corrections, generating an effective potential on which the stochastic dynamics takes place. The long wavelength stochastic dynamical equations are now second order in time, incorporating temporal scales $\Delta t \sim H^{-1}$ and resulting in a Kramers equation for the probability distribution - more precisely the Wigner function - in contrast to the more usual Fokker-Planck equation. This feature allows us to non-perturbatively evaluate, within the stochastic formalism, not only expectation values of field correlators, but also the stress-energy tensor of $\phi$. 
1 Introduction

The quantum dynamics of a scalar field $\phi$ in de Sitter spacetime geometry is among the simplest examples of a quantum field theory in a curved spacetime. It is also highly relevant physically as it forms the basis of understanding quantum fluctuations during inflation. Nevertheless, it is still far from being completely understood. The main issues revolve around the behaviour of the field’s correlations on long wavelengths and over long times. These correlations are usually referred to as exhibiting infrared (IR) and secular divergences, first recognized in [1]. Similar divergences are found in the theory of inflationary perturbations when curvature correlations are computed beyond the leading order. A comprehensive review of these issues and extensive references to the relevant works can be found in [2] - see also [3] for a concise view on related issues. Given the central position of this physical model for our current conception the early universe, a complete elucidation of its IR physics seems pertinent and could shed further light into the theory of inflationary perturbations where also gravitational fluctuations are relevant.

A major conceptual advance in understanding the IR behaviour of $\phi$ in (quasi-)de Sitter was the realization by Starobinsky that, owing to the non-oscillatory behavior of long wavelength modes, the quantum dynamics of $\phi$ on super-Hubble scales reduces to that of a classical field in the presence of a stochastic force [4, 5]. Although physically appealing, the foundations of this idea remained somewhat obscure. A significant step to clarify the relationship of this picture to the underlying Quantum Field Theory (QFT) was taken by Woodard [6] who demonstrated that stochastic inflation reproduces the leading order scale factor logarithms appearing in perturbative QFT computations of coincident expectation values. Unlike the QFT framework however, the stochastic picture allows one to re-sum the uncontrolled perturbative expansion and obtain finite answers.
The comparison of the stochastic formalism to the full QFT computation was further studied in [7–11] and its results for the final equilibrium distribution for $\phi$ agree, where comparisons can be made, with other computational methods such as Euclidean computations or use of the non-perturbative renormalization group [12–16]. In [9, 10] it was shown that the full QFT and the stochastic descriptions of the field’s dynamics coincide exactly in their diagrammatic expansions in the IR and hence all expectation values of powers of $\phi$ computed in the two frameworks should also coincide in the IR. The appropriateness of the stochastic framework for the describing the IR effective theory in inflation was recently also advocated in [17].

Many of the above works include (favorable) comparisons of computations in the stochastic formalism to corresponding computations using the full QFT machinery. Given the evidence in favour of its validity, the stochastic picture has been extensively used by cosmologists studying inflation [18–42]. Nevertheless, its basis on the full quantum theory has remained somewhat obscure since, apart from Starobinsky’s original argument which was limited to the leading order, no first principles derivation has been put forward. It has therefore been considered by many field theorists as a heuristic tool at best whose results would need to be checked against more refined computations. Furthermore some misunderstandings exist in the literature regarding the nature of the stochastic force which, unlike more common system-environment computations, exists even for a free theory, being the result of a time dependent system-environment split.

In this work we further elucidate the framework of the stochastic theory of $\phi$ during inflation by explicitly computing the IR Effective Field Theory which results after integrating out short wavelength fluctuations. We use the appropriate Schwinger-Keldysh amphichronous path integral, extending the results of [45] beyond the leading order. By “short wavelength” we mean fluctuations with physical wavelengths below the curvature scale of de Sitter, $k/a > H$. An initial formulation along similar lines was given by [46] and a related approach was discussed in [47]. The IR effective theory we derive here is fully renormalized in the UV and consists of a generalization of Starobinsky’s stochastic inflation, extended to include quantum corrections from sub-Hubble quantum fluctuations. As expected, the latter shift the potential on which the stochastic evolution takes place. Once these UV quantum fluctuations are properly accounted for, the theory is IR safe with IR resummations afforded by the stochastic dynamics. Our formulation also accounts correctly for processes that operate on timescales of order $\Delta t \gtrsim H^{-1}$, unlike Starobinsky’s original formulation which corresponds to the long time/strong friction limit $\Delta t \gg H^{-1}$.

On the technical side, our methodology is somewhat similar to a Wilsonian integrating out of fast/short wavelength dynamics to obtain the relevant slow/long wavelength theory, adopted to the peculiarities of a time dependent split between the two sectors. We expect that a generalization of this approach will be possible and useful in obtaining the IR effective theory for more complicated field theories in (quasi-)de Sitter spacetime, for example including gravitational fluctuations - see [48] for a first attempt in this direction.

2 Long wavelength stochastic dynamics

In this section we systematically derive the long wavelength effective field theory, obtaining Starobinsky’s stochastic inflation along with its leading order quantum corrections which

\footnote{For a description of stochastic dynamics due to a standard system-environment interaction in an inflationary setting see [43, 44]}

-2-
shift the IR potential from its bare form. After briefly reviewing the CTP path integral we proceed to separate scales through the use of a window function. We then explicitly integrate out the short wavelength sector using the Schwinger-Keldysh formalism, obtaining the long wavelength effective action in a gradient expansion which includes the leading order quantum corrections. Our discussion is fairly general and relies on the infrared enhancement of the propagator on long separations relative to the Minkowski result. We then specialize in de Sitter, deriving the IR effective potential for a light scalar with a quartic bare interaction potential.

2.1 The CTP formalism

The real time evolution of the scalar field from given initial conditions, without reference to a future asymptotic state, is described by the so-called Schwinger-Keldysh formalism employing the amphichronous or closed-time-path (CTP) path integral \([49, 50]\). This replaces the scalar potential.

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Sitter, deriving the IR effective potential for a light scalar with a quartic bare interaction

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which describes evolution forward in time from \(t = 0\) to \(t = T\) with an initial density matrix \(\rho(0)\) and evolution operator \(U_{J_2}\) in the presence of an external current \(J_1\), and then evolution backwards in time in the presence of a different current \(J_2\). The boundary conditions implied in (2.1) involve the initial density matrix \(\rho\) (or some given state in the past, e.g. the vacuum) and that \(\phi_1 = \phi_2\) at some point in the future after any possible time of interest. The latter condition results from inserting a complete set of field eigenstates at \(T\) in (2.3). This type of generat functional, and the corresponding doubling of fields in (2.1), is necessary for obtaining the correct causal dynamical evolution for expectation values of field operators \([49]\). The boundary conditions implicit in (2.1) determine the choice of the Green function matrix which satisfies,

\[
\begin{align*}
(\nabla^2 - m^2) D_{ab}(x, x') = \delta_a^b \delta(x, x').
\end{align*}
\]

In the Schwinger \(\phi_{1,2}\) basis with both field indices lowered,

\[
D_{ab}(x, x') = \begin{pmatrix} D_{11}(x, x') & D_{12}(x, x') \\ D_{21}(x, x') & D_{22}(x, x') \end{pmatrix} = -i \begin{pmatrix} \langle T \phi(x) \phi(x') \rangle & \langle \phi(x') \phi(x) \rangle \\ \langle \phi(x) \phi(x') \rangle & \langle T \phi(x) \phi(x') \rangle \end{pmatrix}.
\]

Instead of using \(\phi_1\) and \(\phi_2\), it’s possible to change the field description via

\[
\phi' = U \phi.
\]

\(2\) There is also an additional, subdominant, component of the stochastic noise, see (A.11).
For example, it is often helpful to employ the “classical” and “quantum” fields of the Keldysh basis
\[ \phi = \frac{\phi_1 + \phi_2}{2}, \quad \phi_q = \phi_1 - \phi_2, \]  
(2.7)
with transformation matrix
\[ U_a^{\ b} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \]  
(2.8)
In this basis,
\[ C^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  
(2.9)
and accordingly
\[ D_{-1ab}(x,x') = \begin{pmatrix} 0 & (\nabla^2 - m^2) \delta(x,x') \\ (\nabla^2 - m^2) & 0 \end{pmatrix} \]  
(2.10)
The Green function (2.5) in the Keldysh basis is
\[ D_{ab} = \begin{pmatrix} -iD_K(x,x') & D_R(x,x') \\ D_A(x,x') & 0 \end{pmatrix} = -i \begin{pmatrix} \frac{1}{2} \langle \{ \phi(x), \phi(x') \} \rangle & \langle [\phi(x), \phi(x')] \rangle \theta(t - t') \\ \langle [\phi(x'), \phi(x)] \rangle \theta(t' - t) & 0 \end{pmatrix}, \]  
(2.11)
where \( D_{R(A)} \) is the retarded (advanced) Green function and \( D_K \) is the Keldysh (or Hadamard) component of the propagator. The coupling to currents reads in the Keldysh basis
\[ J^{\dagger} \phi = J \phi_q + J \phi_2. \]  
(2.12)
while
\[ \overline{V}_I = \frac{\partial V_I}{\partial \phi_q} + \sum_{m=1}^{\infty} \frac{V_I^{(2m+1)}}{2^m (2m+1)!} \phi_q^{2m+1}. \]  
(2.13)
It is worth noting that classical physics is obtained in the limit \( \phi_q \to 0 \). Keeping \( \mathcal{O}(\phi_q^2) \) terms corresponds to making the dynamics stochastic (and possibly incorporating stochastic initial conditions).

Perturbation theory is built similarly to the more usual in-out path integral, using the matrix \( D \) as the propagator, with one notable difference. Setting \( V_I \) and \( J \) to zero we immediately have
\[ Z_0[0] = \int D\phi \exp \left( \frac{i}{2} \int \phi^\dagger D^{-1} \phi \ d\mu \right) = 1 \]  
(2.14)
since it is simply \( U_0 U_0^\dagger = 1 \) for unitary evolution. This implies that we must have
\[ \text{Tr} \ln \begin{pmatrix} -iD_K(x,x') & D_R(x,x') \\ D_A(x,x') & 0 \end{pmatrix} = 0 \]  
(2.15)
in any representation of the propagator which may be used to define the above trace. Unlike the in-out vacuum-to-vacuum amplitude used in scattering computations, \( Z_0[0] = 1 \) and there is no need to factor out vacuum bubbles as they are automatically zero. Indeed, unitarity of the time evolution operator \( U_J(T,0) \) implies a stronger result for the full, interacting theory,
\[ Z[J]|_{J_q=0} = 1, \]  
(2.16)
and the effective action constructed from 1PI Feynman diagrams will also vanish when \( \phi_q = 0 \).
2.2 Long - short split

We now wish to split the quantum field into two parts: field configurations $\phi_<$ whose Green function is $D_<$, and field configurations $\phi_>$ with corresponding Green function $D_>$ such that

$$\phi(x) = \phi_>(x) + \phi_<(x),$$

(2.17)

and

$$D(x, x') = D_>(x, x') + D_<(x, x').$$

(2.18)

We do not need to specify at this point exactly how (2.18) is achieved and the manipulations below are valid for any such split of the propagator. The fluctuations of the fields $\phi_<$ and $\phi_>$ are by definition governed by the propagators $D_<$ and $D_>$ respectively. The fields $\phi_<$ and $\phi_>$ are simply dummy variables associated with path integrals weighted by $D_<$ and $D_>$ and do not need to be separately determined - see [51] for further explanation of this point.

Using the following identity for Gaussian integrals

$$\int_{-\infty}^{+\infty} dx \exp\left(-\frac{x^2}{2\alpha} - V(x)\right) = \sqrt{\frac{2\pi\beta\gamma}{\alpha}} \int_{-\infty}^{+\infty} dy dz \exp\left(-\frac{y^2}{2\beta} - \frac{z^2}{2\gamma} - V(y + z)\right),$$

(2.19)

where $x = y + z$ and $\alpha = \beta + \gamma$, we can then split the path integral as

$$Z[J] = \int D\phi_<>D\phi_> \exp i \int \left[ \frac{1}{2} \phi^>_D D^{-1}_> \phi_> + \frac{1}{2} \phi^<_D D^{-1}_< \phi_<_ - \nabla_1 (\phi_< + \phi_<) + J^1 (\phi_< + \phi_>) \right] d\mu.$$

(2.20)

The field integrals (2.1) and (2.20) equivalent and define the same perturbative expansions [52].

We can utilize the above formal discussion for our problem by explicitly specifying the split (2.18) to reflect the separation into long wavelength and short wavelength modes. We can achieve this by using a window function $W(x, x')$ smoothing out short wavelength perturbations. Its complementary window function $\overline{W}(x, x')$, will filter out long wavelength fluctuations, satisfying

$$W(x, x') + \overline{W}(x, x') = 1 \delta(x, x').$$

(2.21)

Here, long and short are defined with respect to some smoothing scale which could be both spatial and temporal. We normalize $\int W(x, x') d\mu(x') = 1$ so that $\int \overline{W}(x, x') d\mu(x') = 0$. We take the $D_<$ propagator to be

$$D_<(x, x') = \int d\mu(x)d\mu(y) W(x, y) D(y, z) W(z, x') \equiv WDW,$$

(2.22)

where the integrations over spacetime are implicit in the final expression. Therefore, from (2.21) $D_>$ is given by

$$D_> = DW + \overline{W}D - WDW.$$

(2.23)

For any reasonably sharp window function, $D_>$ and $D_<$ will have a negligible overlap for scales sufficiently different from the smoothing scale.

Let us emphasize here that using a smooth window function $W$ does not partition the function space in which the field $\phi$ lives into fields with strictly short and strictly long modes. An exact step function in $k$-space, and only such a projector function, would be required for this. Therefore, the ‘integration variables’ $\phi_<$ and $\phi_>$, integrated over in the path integral
contain all wavelengths in their integration measure. However, by construction \( D_\sigma \) is suppressed on short scales and hence configurations of \( \phi_\sigma \) with short wavelength components have suppressed contributions to the path integral due to a correspondingly large exponent. Similarly, the propagator (2.23) ensures that long wavelength configurations do not contribute in the \( D\phi_\sigma \) path integral. In this sense \( \phi_\sigma \) and \( \phi_\sigma \) with short wavelength components have suppressed contributions to the path integral due to a correspondingly large exponent. Similarly, the propagator (2.23) ensures that long wavelength configurations do not contribute in the \( D\phi_\sigma \) path integral. In this sense \( \phi_\sigma \) and \( \phi_\sigma \) can meaningfully be considered as long and short wavelength fields respectively. Note that this formulation differs somewhat from using a direct convolution of \( \phi \) with a window function as was originally done in [46, 47], bringing this approach more in line with common notions of renormalization, see e.g. [51]. More importantly, as we will see the split of the fields into long and short wavelength components is time dependent in inflation. Given this time dependence, the common practice of convolving the field with a window function does not offer itself for a clear understanding of the integration measure’s split into long and short wavelength sectors: \( D\phi \to D\phi_\sigma D\phi_\sigma \).

2.3 The effective IR action

We now wish to integrate out the “short wavelength” field \( \phi_\sigma \), obtaining an effective action for the IR sector described by \( \phi_\sigma \). The relatively small field gradients in the IR sector suggest that a derivative expansion in the IR fields is appropriate, and the most important contribution to the effective action should be an effective potential term. After integrating out the high energy modes we will be left with a generating function, \[
Z[J] = \int D\phi_\sigma e^{iS_\sigma[\phi_\sigma] + \frac{1}{2} \int d\mu J^\dagger \phi_\sigma},
\]
(2.24)
with the low energy effective action defined by
\[
e^{iS_\sigma[\phi_\sigma]} = \left\{ \exp \left[ \frac{1}{2} \phi_\sigma^\dagger D_\sigma^{-1} \phi_\sigma d\mu \right] \right\} \int D\phi_\sigma \left\{ \exp \left[ \frac{1}{2} \phi_\sigma^\dagger D_\sigma^{-1} \phi_\sigma - \nabla I[\phi_\sigma + \phi_\sigma] \right] \right\} d\mu.
\]
(2.25)
The one-loop approximation can be obtained by expanding the interaction terms up to quadratic order,
\[
S_I[\phi_\sigma + \phi_\sigma] \approx S_I[\phi_\sigma] + \phi_\sigma^a \frac{\delta S_I}{\delta \phi_\sigma^a} + \frac{1}{2} \phi_\sigma^a \frac{\delta^2 S_I}{\delta \phi_\sigma^a \delta \phi_\sigma^b} \phi_\sigma^b,
\]
(2.26)
where \( S_I \) is the integral of the potential \( -\nabla I \) and repeated indices on the product of two functions imply an integration over spacetime. Collecting the terms involving the high frequency fields together forms a shifted propagator \( G_\sigma \),
\[
iG_\sigma^{ab} = \left( D_\sigma^{-1} + \frac{\delta^2 S_I}{\delta \phi_\sigma^a \delta \phi_\sigma^b} \right)^{-1},
\]
(2.27)
which depends on the slowly varying background \( \phi_\sigma^a \). The Gaussian integration over \( \phi_\sigma \) gives
\[
S_\sigma[\phi_\sigma] \approx \int \frac{1}{2} \phi_\sigma^\dagger D_\sigma^{-1} \phi_\sigma d\mu + S_I[\phi_\sigma] - \frac{i}{2} \frac{\delta S_I}{\delta \phi_\sigma^a} G_\sigma^{ab} \frac{\delta S_I}{\delta \phi_\sigma^b}.
\]
(2.28)
The final two terms in the effective action are the leading order contributions from the short wavelength modes. We consider the logarithmic term below, dropping the last term as it is
higher order in the non-linear coupling. Further discussion of the final term can be found in the appendix.

The logarithm term is fairly intractable as it stands, but simplifies if we use the Keldysh representation and expand to first order in the variable $\phi_q<$, where (using the inverse metric $C_{ab}$ for convenience)

$$\phi_{<a} = \phi_< + \frac{1}{2} C_{1a} \phi_q<.$$  \hfill (2.29)

The first two terms in the expansion of the logarithm of (2.27) are,

$$i \frac{1}{2} \log \det(G>) \approx i \frac{1}{2} \log \det(G>[\phi]) + \frac{1}{4} G_{>ab} \frac{\delta^3 S_I}{\delta \phi_<^a \delta \phi_<^b \delta \phi_<^c} \epsilon_{1c} \phi_q<.$$  \hfill (2.30)

In the CTP formalism, all terms vanish when $\phi_q$ vanishes (see the discussion in 2.1), and the first term is therefore zero. After substituting in the interaction potential and uncondensing the notation we have

$$i \frac{1}{2} \log \det(G>) \approx \frac{1}{2} \int d\mu(x) D_K>(x, x) \frac{\partial^3 V_I}{\partial \phi_<^a} \phi_q<(x).$$  \hfill (2.31)

At equal time coordinate, the Keldysh propagator $D_K$ is equal to the shifted Feynman propagator $G_{>F}$,

$$i \frac{1}{2} \log \det(G>) \approx \frac{1}{2} \int d\mu(x) G_{>F}(x, x) \frac{\partial^3 V_I}{\partial \phi_<^a} \phi_q<(x).$$  \hfill (2.32)

Regularisation of the coincident limit of the Feynman propagator can be done as described in the next section.

The low-energy effective action can be expressed in a more suggestive way by introducing the one loop correction to the de Sitter space effective potential $\Delta V$,

$$\int d\mu(x) \Delta V = i \frac{1}{2} \log \det G_{>F}.$$  \hfill (2.33)

The determinant can be expanded in powers of $\phi_q<$ as above, and when compared to (2.32) we find

$$i \frac{1}{2} \log \det(G>) \approx \frac{1}{2} \int d\mu(x) \left[ \Delta V \left( \phi + \frac{\phi_q<}{2} \right) - \Delta V \left( \phi - \frac{\phi_q<}{2} \right) \right].$$  \hfill (2.34)

The quantity inside the integral is simply $\Delta V(\phi_1) - \Delta V(\phi_2)$. Recalling that $S_I$ is the integral of $V(\phi_1) - V(\phi_2)$, we see that the logarithmic term can be included in the effective action,

$$S_<(\phi_<) \approx \frac{1}{2} \phi_<^a D_<^{-1ab} \phi_<^b + S_I[\phi_<],$$  \hfill (2.35)

where we make the replacement $V_I \to V_I + \Delta V$ in $S_I$.

2.4 de Sitter effective potential

Calculation of the effective potential from (2.35) requires the computation of $G_F^>(x, x)$, where $G_F(x, x')$ is the familiar Feynman propagator. In de Sitter space computing $G_F(x, x')$ is routine, apart from the fact that the operator has been combined with a window function which suppresses the IR modes. In what follows we will express the effective potential in
terms of the result for the full theory (without the window function) plus a correction term, and evaluate the latter numerically. The result for the full theory is presented first.

Consider a scalar field potential
\[ V = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4. \] (2.36)

Equation (A.9) relates the effective potential to coincident limits of the shifted Feynman propagator \( G_F \). This is the free theory propagator with mass \( \tilde{m} \equiv m(\phi) \) given by
\[ \tilde{m}^2(\phi) = m^2 + 3\lambda \phi^2. \] (2.37)

The de Sitter propagator [53, 54] has a closed form expression in terms of a hypergeometric function,
\[ G_F(x, x') = \frac{\Gamma(3/2 + \nu)\Gamma(3/2 - \nu)H^2}{16\pi^2} F[3/2 + \nu, 3/2 - \nu, 2; \cos^2(H\sigma/2)], \] (2.38)
where \( \nu^2 = 9/4 - \tilde{m}^2/H^2 \). We have taken the geodesic separation \( \sigma \) to be spacelike, for which \( G_F \) does not have an imaginary part. In the limit \( \sigma \to 0 \), the divergent terms are removed to give the regularised coincident limit \( \text{reg} \, G_F \),
\[ \text{reg} \, G_F = \frac{H^2}{16\pi^2} \left( \tilde{m}^2 - 2 \right) \left[ \psi_{\text{poly}}(3/2 + \nu) + \psi_{\text{poly}}(3/2 - \nu) - \ln \left( \frac{\mu_R^2}{R H^2} \right) \right], \] (2.39)
where \( \psi_{\text{poly}} \) is the polygamma function\(^3\). Ambiguity in subtracting logarithmic divergences results in the term containing the renormalisation scale \( \mu_R \). The one loop correction to the effective potential in de Sitter space is then
\[ \Delta V_{\text{des}} = \frac{H^2}{32\pi^2} \int \left( \frac{\tilde{m}^2}{H^2} - 2 \right) \left[ \psi_{\text{poly}}(3/2 + \nu) + \psi_{\text{poly}}(3/2 - \nu) - \ln \left( \frac{\mu_R^2}{H^2} \right) \right] d\tilde{m}^2, \] (2.40)
since \( d\tilde{m}^2 = V''d\phi \). This indefinite integral contains an undetermined constant, but gives all the important terms which depend on \( \phi \).

In order to apply the window functions we use a spatially flat coordinate system for de Sitter, with metric
\[ ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \] (2.41)
In these coordinates, the Feynman propagator has a mode expansion,
\[ G_F(x, x') = \begin{cases} \sum_k u_k(t) \pi_k(t') e^{ik(x-x')} & t > t' \\ \sum_k \pi_k(t) u_k(t') e^{ik(x-x')} & t < t' \end{cases} \] (2.42)
where \( \sum_k \) is the momentum-space measure in flat space and the modes are Hankel functions,
\[ u_k(t) = \frac{1}{2} \sqrt{\pi} H^{-1/2} a^{-3/2} H^{(1)}(k/a H). \] (2.43)

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\(^3\)Adiabatic regularisation, described for example in [66], removes an additional term related to the conformal anomaly. We prefer to recover the conformal anomaly from the variation of \( \mu_R \), and perform what might be regarded as a ‘minimal’ subtraction in \( G_F(x, x) \).
We will take a window function which selects the small momentum modes, 

\[ W(t, x, t', x') = \delta(t - t') \sum_k W_k(k, t) e^{i k \cdot (x - x')} \]  

where \( W(k, t) \to 0 \) for large \( k \).

The propagator with the IR modes removed is then 

\[ G_F(x, x) = \text{reg} G_F(x, x) - \sum_k W(k, t)^2 u_k(t) \bar{u}_k(t), \]

where the term involving the window function \( W \) is finite in the coincident limit provided that \( W(t, k) \) falls off sufficiently rapidly at large \( k \). The correction to the potential for the long range modes is then 

\[ \Delta V = \Delta V_{\text{deS}} - \frac{1}{2} \int d\tilde{m}^2 \sum_k W(k, t)^2 u_k(t) \bar{u}_k(t). \]  

The effective IR potential term \( \Delta V \) has been calculated numerically for a gaussian window function 

\[ W(k, t) = \exp \left( -\frac{k^2}{a^2 H^2} \right) \]

and plotted in figure 1. The left hand plot is against \( m(\phi) \), as defined in (2.37), for selected values of the renormalisation scale. The right-hand plot is specialised to \( V = \lambda \phi^4/4 \). In principle, we could remove the renormalisation scale by using the renormalisation group to relate the running parameters in the theory to measurable parameters. However, since the model is not related to any particular particle model at this stage, it is convenient to treat the renormalisation scale as an extra parameter.

The left-hand plot in figure 1 shows the effect of suppressing the IR modes in the short wavelength propagator contributing to the effective potential. The full de Sitter result has a logarithmic divergence in the limit \( \tilde{m} \to 0 \),

\[ \Delta V_{\text{deS}} \sim \frac{3H^4}{16\pi^2} \ln \frac{\tilde{m}^2}{H^2} = \frac{\tilde{m}^2 H^2}{16\pi^2} \left\{ \ln \frac{H^2}{\mu_R^2} - 2\gamma + \frac{10}{3} \right\} + \ldots, \]

where \( \gamma = 0.577216 \ldots \) is Euler’s constant. Suppressing the IR modes has done the job of removing the leading logarithmic \( H^4 \ln m^2 \) term. At the opposite extreme of mass, the plot rapidly approaches the flat-space result as \( m \) is increased beyond \( m = H \),

\[ \Delta V_{\text{flat}} = \frac{\tilde{m}^4}{64\pi^2} \left\{ \ln \frac{\tilde{m}^2}{\mu_R^2} - \frac{1}{2} \right\}, \]

The logarithm in the flat space result becomes large for small mass, leading to a breakdown in perturbation theory. This can be corrected by using the renormalisation group corrected potential.

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4The chosen window function smooths in space, or momentum, but not in time, hence the temporal delta function. This is the choice made in all works on stochastic inflation so far and explicitly breaks de Sitter invariance. Our formalism does not strictly require this and it would be interesting to formulate the theory using a de Sitter invariant window function which smooths time as well as space.
The result of truncating the full de Sitter one loop potential by simply dropping the logarithmic term is plotted in $\Delta V_{\text{deS}}$ truncated. There is a significant qualitative difference between this and $\Delta V$, which shows that suppressing the IR modes, as we have done, does more than just remove the large logarithmic terms in the potential. The leading order behaviour of the truncated theory is given by the second term in (2.48), which is negative for $\mu_R \sim H$. Suppressing the IR modes gives a positive $\tilde{m}^2$ term for $\mu_R \sim H$ which dominates the one-loop potential in the right-hand plot as $\phi \to 0$. The value of the effective mass at $\phi = 0$ can be varied by changing the renormalisation scale.

2.5 The IR dynamics

In order to complete our construction of the IR dynamics we also need a convenient representation for the operator inverse of $D_<$ which appears in the effective action (2.28). We can exploit the fact that de Sitter correlations decay at most with a mild power law at large distances [9], while $D_>$ is by construction small above the smoothing scale. Hence, when acting on long wavelength fields, we can formally write

$$D_\leq^{-1} = \frac{1}{D - D_>} = D^{-1} + D_D^{-1} D_0^{-1} + \ldots$$

where in the second line we used (2.23) and the operator series on the r.h.s is understood to act on functions with only long wavelength support. Thus we find that the dominant contribution to the long wavelength action, expressed in the Keldysh basis, is

$$\phi^D D_\leq^{-1} \phi \simeq (\phi_c, \phi_q) \left( \begin{array}{c} 0 & D \\ D & 0 \end{array} \right) \left( \begin{array}{c} \phi_c \\ \phi_q \end{array} \right) - (\phi_c, \phi_q) \left( \begin{array}{c} 0 \\ \tilde{W} D_A \tilde{W} \end{array} \right) \left( \begin{array}{c} \phi_c \\ \phi_q \end{array} \right) ,$$

where $W_{DK} = W_{KD}$.
where $\mathcal{D} = \nabla^2 - m^2$. We have removed the subscript $\prec$ to simplify notation, and spacetime integrations are implicit in the elements of the second matrix. Any terms where $W$ is directly convolved with $\phi$ only provide higher order spatial gradient terms which are subdominant and were dropped from (2.51). Noting that $D_A$ and $D_R$ are suppressed on long wavelengths compared to the Keldysh propagator $D_K$, the dominant contribution to the second term of (2.51) comes from the $\phi^2_q$ term. A feature of the Keldysh formalism is that such terms are equivalent to adding a stochastic force $\xi(x)$ [50] to the equation of motion with correlator

$$\langle \xi(x)\xi(x') \rangle \equiv N(x, x') = \int \mathcal{D}x \mathcal{D}x' W(x, y) D_K(y, z) W(z, x') d\mu(y) d\mu(z). \quad (2.52)$$

This corresponds to the well known Starobinsky noise term [4].

Rescaling $\psi = a^3 \phi_q$, the exponent in the path integral of the IR theory is written as

$$iS[\phi, \phi_q] = \int dt d^3x \left[ \frac{1}{2} \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \left( \begin{array}{cc} 0 & \left( \nabla^2 - m^2 \right) \\ \left( \nabla^2 - m^2 \right)^\dagger & \left( \nabla^2 - m^2 \right) \end{array} \right) \left( \begin{array}{c} \phi \\ \psi \end{array} \right) - \frac{\partial V_{\text{eff}}}{\partial \phi} \psi \right] + 2 \sum_{m=1}^{\infty} \frac{V_{\text{eff}}^{(2m+1)}}{(2m+1)!} \left( \begin{array}{c} \psi \\ \psi \end{array} \right)^{2m+1} \left( \frac{H}{a} \right)^{6m} \quad (2.53)$$

where

$$-\nabla^2 = \partial_t^2 + 3H \partial_t - H^2 a^{-2} \partial_x^2$$

$$-\nabla^2 = \partial_t^2 - 3\partial_t H - H^2 a^{-2} \partial_x^2. \quad (2.54)$$

Note that the anti-damping in (2.55) results from absorbing the $a^3$ factor in $\psi$ which makes the operator non-self-adjoint. It is clear that powers of $\psi$ higher than 2 are highly suppressed by inverse powers of $a$, inducing semi-classical dynamics in the IR. The only remnant of quantum fluctuations is the stochastic $\psi^2$ term. Note that this argument holds for the fully interacting theory, relying on the growth of the volume of the system and not on the behaviour of the linear mode functions on long wavelengths [55], although the latter is related to the former. This justifies the approximation we made earlier in (2.30) as the neglected terms would end up being suppressed. We have thus arrived at the following action on long wavelengths

$$iS[\phi, \psi] \simeq -i \int dt d^3x \left[ \psi \left( -\nabla^2 \phi + m^2 \phi + \frac{\partial V_{\text{eff}}}{\partial \phi} \right) \right] - \frac{1}{2} \int d^3x d^3x' dt dt' \psi(x) N(x, x') \psi(x'), \quad (2.56)$$

which describes stochastic Langevin dynamics for $\phi$

$$-\nabla^2 \phi + m^2 \phi + \frac{\partial V_{\text{eff}}}{\partial \phi} = \xi(t, x) \quad (2.57)$$

with the $\xi$ correlator given by (2.52).

The noise kernel $N(x, x')$ depends of course on the window function but any physical results should be independent of this choice. The original formulation of stochastic inflation by Starobinsky, which neglected the field acceleration $\ddot{\phi}$, used a sharp step function in $k$ space to define the long wavelength system. Here we use the gaussian window function,

$$W'(k, t) = \exp \left( -\frac{k^2}{a^2 H^2} \right). \quad (2.58)$$
In momentum space, the noise kernel (2.52) becomes
\[ \mathcal{N}(x, x') = \sum_k e^{i \mathbf{k} \cdot (x - x')} \mathcal{D}_k \mathcal{D}_l e^{\mathbf{W}(k, t)u_k(t)\bar{u}_l(t')\mathbf{W}(k, t')} \tag{2.59} \]
where
\[ \mathcal{D} = -\partial_t^2 - 3H \partial_t - a^{-2}k^2 - m^2. \tag{2.60} \]
The window functions pick out the small-k limit, where we can approximate the modes (2.43) by,
\[ u_k(t)\bar{u}_k(t') \approx \frac{H^2}{2k^3} \left( \frac{k}{a(t) \bar{H}} \right) \frac{1}{\pi t} \left( \frac{k}{a(t') \bar{H}} \right). \tag{2.61} \]
For small mass \( m \ll H \), the integration results in
\[ \mathcal{N}(x, x') = \frac{3H^6}{2\pi^2} \text{sech} H(t - t') \int F_1(-5/2, 3/2; \kappa^2 |x - x'|^2) e^{-\kappa^2 |x - x'|^2}, \tag{2.62} \]
where \( F_1 \) is a confluent hypergeometric function and
\[ \kappa = \frac{1}{2} H \text{sech} H(t - t') e^{H(t+t')/2}. \tag{2.63} \]
We can get an understanding about the IR behaviour in various limits. For sub-horizon scales \( |x - x'| < 1/(aH) \), the noise resembles white noise with a smoothed delta-function kernel,
\[ \mathcal{N}(x, x') \sim \frac{9H^5}{4\pi^2} \delta(t - t'). \tag{2.64} \]
For super-horizon separations the noise correlations are exponentially suppressed. This behaviour is repeated for other types of window functions. Hence, on large scales, \( k < aH \), evolution becomes quasi-local in space and different regions of physical size \( \Delta r \sim 1/H \) are acted upon by spatially uncorrelated noise \( \xi \). We can thus approximate
\[ \mathcal{N}(x, x') \sim \mathcal{N}(x, x', t) \delta(t - t'), \tag{2.65} \]
where the spatial kernel is
\[ \mathcal{N}(x, x', t) \sim \frac{9H^5}{4\pi^2} \theta \left( 1 - aH |x - x'| \right). \tag{2.66} \]
Having found the noise statistics, we shall now examine the long-range dynamics for the free field theory. Since only \( k < aH \) modes are included in \( \mathcal{D}_k \), the propagator functions become approximately local in space [6]. If we let \( \mathcal{G}^R(t, t') \) be the Green function for the stochastic system (2.57) dropping the gradient terms, then
\[ D_k^R(x, x') = \frac{9H^5}{4\pi^2} \int_0^{+\infty} d\tau \mathcal{G}^R(t, \tau)\mathcal{G}^A(\tau, t') \theta \left( 1 - a(\tau)H |x - x'| \right) + \mathcal{F}_0(x, x') \tag{2.67} \]
where \( \mathcal{F}_0 \) is a function satisfying the equation of motion, resulting from the arbitrariness of inverting \( D_k^{-1} \). Its choice reflects a choice of initial state. For \( \Delta x = 0 \), (2.67) gives the coincident correlation function as
\[ \langle \phi(t, x)\phi(t', x) \rangle = D_k^R(t, x, t', x) = \frac{3H^4}{8\pi^2m^2} \left( e^{-\frac{m^2}{2H^2} |t - t'|} - \frac{m^2}{9H^2} e^{-3H|t - t'|} \right), \tag{2.68} \]
where we chose $F_0$ such that the correlation function assumes a unique form depending only on $|t - t'|$. For non-zero comoving separation $\Delta x$ we have

$$D^<_{K}(x, x') = \frac{9H^5}{4\pi^2} \int_0^{t_*} d\tau G^R(t, \tau)G^A(\tau, t') + F_0(t, t', \Delta x)$$  \hspace{1cm} (2.69)$$

where $t_* = -H^{-1} \ln(H |\Delta x|)$, and we assumed that $H |\Delta x| < 1$, i.e. the points are initially within the horizon. Regardless of the specific choice of $F_0$, relation (2.69) reproduces via stochastic dynamics the asymptotic result for free quantum fields in the Bunch-Davies de Sitter vacuum,

$$\langle \phi(t, x)\phi(t', x') \rangle \sim \frac{3H^4}{8\pi^2m^2} \left( a(t)a(t')H^2 |\Delta x|^2 \right)^{-\frac{3m^2}{2H^2}}.$$  \hspace{1cm} (2.70)$$

3 The Kramers Equation for $\phi$

The benefits of using the stochastic dynamics become apparent when we consider the interacting theory. It is possible to find probability distributions for $\phi$ at a spatial point $x$ and use these to calculate non-perturbative expectation values of functions of $\phi$ in the stochastic theory. By the previous construction, these will be excellent approximations to the expectation values of the full quantum theory. As a further step, which is less well known, one can construct joint probability distributions of $\phi$ and $\dot{\phi}$ in the stochastic theory in the form of a Wigner function which satisfies a Kramers equation. Kramers equations were first applied for the study of cosmological IR fluctuations in [32].

As a starting point, recall the definition of the Wigner function for a coordinate $x$ and momentum $p$ with density matrix $\hat{\rho}(t)$,

$$W(x, p, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dq \langle x + \frac{q}{2} | \hat{\rho}(t) | x - \frac{q}{2} \rangle e^{-ipq/\hbar}.$$  \hspace{1cm} (3.1)$$

For the IR dynamical field we introduce the Wigner functional using a path integral over spatial fields $\psi$,

$$W[\phi, v, t] = \frac{1}{2\pi} \int_{-\infty}^{\infty} D\psi \rho[\phi, \psi, t] e^{-i \int d^4x \psi v}.$$  \hspace{1cm} (3.2)$$

We have found that the the leading-order IR dynamics is given by a path integral over the long-range spacetime fields $\phi'$ and $\psi'$, where the prime will be used now to clearly distinguish spacetime fields from purely spatial fields. We will keep careful track of the field values held fixed at time $t$, and add these as superscript indices on the integral sign. The density matrix is given by a path integral

$$\rho[\phi, \psi, t] = \int e^{-i \int d^4x \left[ \psi' (\dot{\phi}' + 3H\phi' - a^{-2} \nabla^2 \phi' + dV/\phi') + i \int d^3x \dot{\phi}' \right]} K[\psi'],$$  \hspace{1cm} (3.3)$$

where spatial coordinates have been suppressed and the fields $\phi'(t) = \phi$ and $\psi'(t) = \psi$ are held fixed at time $t$. The noise kernel

$$K[\psi] = \exp \left( -\frac{1}{2} \int d^4x d^4x' \psi(x)N(x, x')\psi(x') \right),$$  \hspace{1cm} (3.4)$$

with $N(x, x')$ given by (2.64). Normalisation constants are dropped since they play no further role. The boundary conditions on the fields are also set by some predetermined
initial conditions at time $t_i$ as well as the fixed field values at time $t_i$. Because the Lagrangian is second order in derivatives, the boundary conditions require the addition of the boundary term $\psi$ in the exponent to cancel boundary terms from the variation of the Lagrangian.

The Wigner function satisfies a Fokker-Planck equation, which we will now obtain. First, introduce an auxiliary spacetime field $v$ to replace $\dot{\phi}$ and a Lagrange multiplier field $z$,

$$\rho(\phi, \psi, t) = \int d^3x d^3x' D\phi D\psi Dv Dz' K[\psi'] e^{-i \int_{t_i}^t d^4x [z(\phi' - v') - \psi(\phi' + 3Hv' + \delta U/\delta \phi')]} + i \int d^3x \psi(t)v,$$

where we introduced the total potential energy

$$U = \int d^3x \left( V_{\text{eff}} + \frac{1}{2} \left( \nabla \phi \right)^2 a^2 \right).$$

Note that eq. (3.2) for the Wigner functional integrates out the field $\psi$, so this is no longer fixed. However, the auxiliary field $v$ does not get integrated, hence

$$W[\phi, v, t] = \int d^3x d^3x' D\phi D\psi Dv Dz' K[\psi'] e^{i \int_{t_i}^t d^4x [z(\phi' - v') - \psi(\phi' + 3Hv' + \delta U/\delta \phi')]}.$$ 

The path integral can then be put into canonical form,

$$W[\phi, v, t] = \int d^3x d^3x' D\phi D\psi Dv Dz' e^{i \int_{t_i}^t d^4x [z(\phi' - v') - \psi(\phi' + 3Hv' + \delta U/\delta \phi')]},$$

which has momenta $(z, -\psi)$ and a “pseudo-Hamiltonian”

$$H_{\text{stoc}} = \int d^3x d^3x' \mathcal{H}(\psi, z, v, \phi),$$

where

$$\mathcal{H} = \left[ \psi \left( 3Hv + \frac{\delta U}{\delta \phi} \right) + vz \right] \delta(x - x') - i \frac{1}{2} \psi(t, x) N(x, x') \psi(t, x').$$

By setting $\psi = i\delta/\delta v$ and $z = -i\delta/\delta \phi$ and choosing an operator ordering in the pseudo-Hamiltonian, the Wigner function $W$ will satisfy a corresponding “Schrödinger” equation $i\partial_t W = \mathcal{H} W$, or equivalently

$$\partial_t W = \frac{1}{2} \int d^3x d^3x' N(x, x') \frac{\delta^2 W}{\delta v(x) \delta v(x')}$$

$$+ \int d^3x \left[ 3H \left\{ \frac{\delta}{\delta v(x)} v(x) + \frac{\delta U}{\delta \phi(x)} \frac{\delta}{\delta v(x)} - v(x) \frac{\delta}{\delta \phi(x)} \right\} \right] W.$$ 

This is a functional equation, structurally similar to a Kramers equation [56], for the field $\phi$ and its velocity $v$. It is clear that for any fixed comoving separation $\Delta x$ the diffusion term only acts for a finite time with the classical drift terms dominating the later evolution. Hence when fixed comoving points are dragged outside the horizon they no longer experience stochastic diffusion relative to each other, allowing for long wavelength correlations such as the ones observed in the CMB to be preserved. Continuous action of the noise takes place only for $\Delta x = 0$, or for fixed physical sub-horizon separations.
Expectation values of functions of $\phi(x)$ and $v(x)$ can be computed using $\mathcal{W}$. The coincident field variance

$$\langle \phi^2(x) \rangle_{stoc} \equiv \int DvD\phi \phi^2(x)\mathcal{W}$$

(3.12)

is of particular interest. Using (3.11) it can be shown to satisfy

$$\left( \partial_t^2 + 3H \partial_t \right) \langle \phi^2 \rangle_{stoc} = 2\left( \frac{v^2}{a^2} - \frac{1}{a^2} \langle \nabla \phi \rangle^2 \right)_{stoc} - 2\left\langle \frac{dV_{eff}}{d\phi} \right\rangle_{stoc},$$

(3.13)

where we used $\nabla \langle \ldots (x) \rangle_{stoch} = 0$ for any coincident stochastic expectation value computed with a probability distribution functional of $\phi(x)$. We thus see that the stochastic IR theory formally reproduces the operator equation

$$-\nabla^2 \langle \hat{\phi}^2 \rangle = 2\left( -\nabla \mu \hat{\phi} \nabla \mu \hat{\phi} \right) - 2\left\langle \hat{\phi} dV_{eff} d\hat{\phi} \right\rangle,$$

(3.14)

resulting from the operator equations of motion for $\hat{\phi}$, for translationally invariant states with $\nabla \langle \hat{\phi}^2 \rangle = 0$. Note that in the stochastic version $V$ is replaced by the long wavelength effective potential $V_{eff}$ and operator averages are replaced by probabilistic expectation values weighted by $\mathcal{W}[\phi(x)]$. To further demonstrate the correspondence between the stochastic and the quantum theory, we need to compare the first terms on the rhs of (3.13) and (3.14). We do this below, after we derive a simplification to (3.11).

The full solution to equation (3.11) contains information for expectation values of field products at separate spacetime points. Leaving a detailed study for future work, we obtain here a simpler expression that allows for the computation of coincident expectation values. We can use a derivative expansion of the non-local Wigner functional $\mathcal{W}$, keeping just the leading term

$$\ln \mathcal{W} \approx \omega \int d^3x \ln W(\phi, v, t),$$

(3.15)

where $\omega$ is a constant factor, roughly the inverse of the volume over which gradients have been smoothed out $\omega \sim H^{-3}$. This approximation should be adequate for computing coincident expectation values since in the IR theory, and for points closer than $\Delta r \sim H^{-1}$, gradient terms are negligible by construction since there are no short wavelength modes. We then have

$$\frac{1}{W} \frac{\delta W}{\delta \phi} = \omega \frac{\partial W}{\partial \phi},$$

(3.16)

and

$$\frac{1}{W} \frac{\delta^2 W}{\delta v(x) \delta v(x')} = \omega \frac{W}{W^2} \left( \omega \frac{\partial W}{\partial v(x)} \frac{\partial W}{\partial v(x')} - \frac{\partial W}{\partial v(x)} \frac{\partial W}{\partial v(x')} \delta(x - x') \right) + \omega \frac{\partial^2 W}{\partial v(x)^2} \delta(x - x').$$

(3.17)

The fact that the spatial gradients of the IR field are negligible on subhorizon scales results in a spatial localisation of the noise kernel in (2.66), allowing for a simplification of the “diffusion” term. We can pick the constant $\omega$ such that the first two terms in (3.17) cancel to leading order in spatial gradients and then

$$\frac{1}{W} \int d^3x d^3x' N(x, x') \frac{\delta^2 W}{\delta v(x) \delta v(x')} \approx \frac{9H^5}{4\pi^2} \int d^3x \frac{\partial^2 W}{\partial v^2}$$

(3.18)
The equation for the Wigner functional thus reduces to an ordinary Kramers equation for $W$,
$$
\partial_t W = \left[ \frac{9H^5}{8\pi^2} \partial^2 + 3H \frac{\partial}{\partial v} v + \frac{dV_{\text{eff}}}{d\phi} \frac{\partial}{\partial v} - v \frac{\partial}{\partial \phi} \right] W. \tag{3.19}
$$
Interestingly, (3.19) has been found previously by starting from general properties of a one-dimensional stochastic field in de Sitter space [57, 58]. Kramers equations were also used to study IR fluctuations in inflation in [32]. We have arrived at this result by a series of steps starting from the full quantum field theory in de Sitter space.

Let us now remark on the evolution of $\langle \phi^2 \rangle$ as described by QFT and the stochastic IR theory. In QFT, in the de Sitter invariant Bunch-Davies vacuum, we have the renormalized value $\langle \nabla_\mu \hat{\phi} \nabla_\nu \hat{\phi} \rangle = -g_{\mu\nu}3H^4/32\pi^2$ [59] which implies
$$
(\partial_t^2 + 3H\partial_t) \langle \hat{\phi}^2 \rangle = \frac{3H^4}{4\pi^2} - 2 \langle \hat{\phi} \frac{dV}{d\phi} \rangle . \tag{3.20}
$$
The corresponding result is automatically obtained in our stochastic formalism. One notes that on a timescale $\Delta t \sim H^{-1}$ the stochastic velocity $v$ reaches an equilibrium distribution
$$
W(\phi, v) \propto e^{-\Omega v^2}, \tag{3.21}
$$
where $\Omega = 8\pi^2/3H^4$, implying that
$$
\langle v^2 \rangle_{\text{eq}} = \frac{3H^4}{8\pi^2}. \tag{3.22}
$$
Since gradient terms are absent in the stochastic formalism for subhorizon separations, we have
$$
(\partial_t^2 + 3H\partial_t) \langle \phi^2 \rangle_{\text{stoc}} = \frac{3H^4}{4\pi^2} - 2 \langle \hat{\phi} \frac{dV_{\text{eff}}}{d\phi} \rangle_{\text{stoc}}, \tag{3.23}
$$
in correspondence to the QFT result. We see that including the stochastic velocity $v$ naturally recovers the correct growth of the field variance $\langle \hat{\phi}^2 \rangle$ [59–62] within the stochastic formalism, with $\langle v^2 \rangle_{\text{eq}}$ replacing the QFT $\langle (\nabla \phi)^2 \rangle$ term in the Bunch-Davies vacuum. It is worth noting that the relation $\langle \nabla_\mu \hat{\phi} \nabla_\nu \hat{\phi} \rangle = \frac{1}{4}g_{\mu\nu}\langle (\nabla \phi)^2 \rangle$ cannot be imposed for classical stochastic fields as it would imply $\langle \hat{\phi}^2 \rangle = -\frac{1}{3}\langle (\partial_t \hat{\phi})^2 \rangle$ which is impossible for expectation values of positive quantities with respect to a probability distribution. One is then led to conclude that either a) the stochastic formalism could be amended to impose this symmetry, or b) an effective decoherence occurs which necessarily breaks de Sitter symmetry [37, 63, 64].

Equation (3.19) admits a unique normalizable time-independent equilibrium solution
$$
W(\phi, v) = \mathcal{A}e^{-\Omega(\frac{1}{2}v^2 + V_{\text{eff}})}, \tag{3.24}
$$
where $\mathcal{A}$ is a normalisation constant. At equilibrium
$$
\langle v^2 \rangle_{\text{eq}} = \langle \hat{\phi} \frac{dV_{\text{eff}}}{d\phi} \rangle_{\text{eq}} = \frac{3H^4}{8\pi^2}, \tag{3.25}
$$
as it should be from (3.23).\footnote{Note that this implies equipartition $\langle V \rangle = \frac{1}{2}\langle v^2 \rangle_{\text{eq}}$ for a free field, which is consistent with a thermal nature for the state.} Full equilibrium is established over much longer timescales that depend on the parameters of the potential. For example, the time to reach the equilibrium distribution $\Delta t_{\text{equil}} \sim H/m^2$ for $V = \frac{1}{2}m^2\phi^2$ or $\Delta t_{\text{equil}} \sim 1/\sqrt{\lambda}H$ for $V = \frac{1}{4}\lambda\phi^4$. It would be interesting to check these timescales against a one-loop QFT calculation.
4 The stress-energy tensor

As an example of the stochastic formalism and the use of the Wigner function, we shall consider the expectation value of the stress-energy tensor for a light scalar field. We therefore introduce

\[ \langle T_{\mu\nu} \rangle = \left\langle \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} (\nabla \phi)^2 + V \right] \right\rangle. \] (4.1)

It immediately becomes apparent that the full solution to (3.11), also containing information about the spatial variation of \( \phi \) is needed to reliably calculate the expectation values of spatial derivative terms using the stochastic formalism. However it is possible to reach some conclusions using the reduced Wigner function (3.24) assuming that the system finds itself in a de Sitter invariant state. We will also comment on the ways in which the field can be thought of developing a ‘dynamical mass’ [5, 13, 65] from the interactions.

When discussing expectation values at coincident spacetime points, de Sitter invariance corresponds to using the equilibrium distribution (3.24), since de Sitter invariant expectation values are time independent. We can then write for the expectation value of \( T_{\mu\nu} \)

\[ \langle T_{\mu\nu} \rangle = \frac{1}{4} g_{\mu\nu} \langle T_{\rho\rho} \rangle. \] (4.2)

We can also express the expectation value as a functional derivative

\[ \langle T_{\rho\rho} \rangle = g_{\rho\sigma} \frac{2i}{Z} \frac{\delta Z}{\delta g_{\rho\sigma}}, \] (4.3)

where we previously reduced \( Z \) to the generating functional for the IR theory in (2.24). We have to take a little care because the regularization has introduced a scale dependence, so that there are long-range and short-range contributions to the expectation value,

\[ \langle T_{\rho\rho} \rangle = \langle T_{\rho\rho} \rangle_{\text{eq}} + \langle T_{\rho\rho} \rangle_{\text{anom}}. \] (4.4)

The first term above is to be computed using the equilibrium probability distribution or Wigner functional while the second term is an anomalous term related to the variation of the renormalization scale. Variation of the metric in the effective action gives\(^6\)

\[ \langle T_{\rho\rho} \rangle_{\text{eq}} = \langle - (\nabla \phi_\bot)^2 - 4V(\phi_\bot) \rangle, \] (4.5)

and these IR terms can be evaluated using the stochastic theory. As we discussed above, the spatial gradients are absent in the stochastic formalism when computing coincident expectation values and, numerically, \( \langle \nabla_\mu \hat{\phi} \nabla_\mu \hat{\phi} \rangle = -\langle v^2 \rangle_{\text{eq}} \). We are thus left with just

\[ \langle T_{\rho\rho} \rangle_{\text{eq}} = \langle v^2 - 4V \rangle_{\text{eq}} = \langle \phi V' - 4V \rangle_{\text{eq}}, \] (4.6)

where we noted that the equilibrium distribution (3.24) implies that

\[ \langle v^2 \rangle_{\text{eq}} = \langle \phi V' \rangle_{\text{eq}}. \] (4.7)

The anomalous trace is given by variation of the renormalisation scale

\[ \langle T_{\rho\rho} \rangle_{\text{anom}} = \mu_{R} \frac{d}{d\mu_{R}} \Delta V_{\text{des}}. \] (4.8)

\(^6\)In the CTP formalism, this involves introducing two metrics \( g_{1\mu\nu} \) and \( g_{2\mu\nu} \) associated with the fields \( \phi_1 \) and \( \phi_2 \).
Figure 2. The plot shows the value of $\langle T_{00} \rangle$ for a light scalar field of mass $m$ and $\lambda = 0.6$ in de Sitter space with expansion rate $H$. The total result is the sum of the stochastic theory $\langle T_{00} \rangle_{\text{eq}}$ and the anomalous trace contribution.

for $\Delta V_{\text{deS}}$ given in (2.40). The full result for the anomalous trace of a free scalar field of mass $\tilde{m}$ is [66],

$$
\langle T^{\rho \sigma}_{\text{anom}} \rangle = \frac{1}{16\pi^2} \left( \frac{1}{15} H^4 - \frac{1}{2} (\tilde{m}^2 - 2H^2)^2 \right).
$$

If we use a shifted mass $\tilde{m}$ depending on $\phi$, then we take the stochastic average of the anomalous trace.

For a specific computation, consider an effective long range potential

$$
V_{\text{eff}} = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4
$$

with a small effective mass $m \ll H$. In this case,

$$
\langle T_{\mu \nu} \rangle_{\text{eq}} = -\frac{1}{4} g_{\mu \nu} m^2 \langle \phi^2 \rangle_{\text{eq}}
$$

where from (3.24)

$$
\langle \phi^2 \rangle_{\text{eq}} = \frac{1}{Z} \int_{-\infty}^{\infty} d\phi \phi^2 e^{-\Omega V}
$$

and $Z$ is a normalisation constant. The values of $\langle T_{00} \rangle_{\text{eq}}$ and $\langle T_{00} \rangle$ have been plotted in figure 2, for the case $\lambda = 0.6$ and $\tilde{m}^2 = 3\lambda \phi^2$. The effective mass can be varied by changing the renormalisation scale.

Approximate results can be obtained by introducing

$$
I(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ e^{-x^2/2 - gx^4},
$$

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then

\[ \langle \phi^2 \rangle_{\text{eq}} = \frac{3H^4}{8\pi^2 m^2} + \frac{3H^4}{2\pi^2 m^2} \frac{d \ln I}{d \ln g}, \]  

(4.14)

where

\[ g = \frac{3\lambda H^4}{32 m^4}, \]  

(4.15)

The leading terms in \( \langle \phi^2 \rangle \) are

\[ \langle \phi^2 \rangle_{\text{eq}} \sim \begin{cases} \frac{3H^4}{8\pi^2 m^2} & 3\lambda \ll \frac{32\pi^2 m^4}{H^4} \\ \frac{\sqrt{3\Gamma(3/4)^2} H^2}{2\pi^2} \frac{2m^2}{\sqrt{\lambda}} & 3\lambda \gg \frac{32\pi^2 m^4}{H^4} \end{cases} \]  

(4.16)

The comparison of these two expressions leads to the idea of an ‘effective mass’ of order \( \lambda^{1/4} H \) in the \( m \to 0 \) limit [13, 65]. However, this does not play the role of a true mass in the following sense: if we look at the ‘00’ component stress-energy tensor we obtain

\[ \langle T_{00} \rangle_{\text{eq}} \sim \begin{cases} \frac{3H^4}{32\pi^2} & 3\lambda \ll \frac{32\pi^2 m^4}{H^4} \\ \frac{\sqrt{3\Gamma(3/4)^2} H^2 m^2}{8\pi^2} \frac{1}{\sqrt{\lambda}} & 3\lambda \gg \frac{32\pi^2 m^4}{H^4} \end{cases} \]  

(4.17)

In the \( m \to 0 \) limit \( \langle T_{00} \rangle_{\text{eq}} \to 0 \), whereas a free scalar field with a mass \( m \sim \lambda^{1/4} H \) would give a quite different result \( \langle T_{00} \rangle_{\text{eq}} \sim H^4 \). The above expression also shows that IR fluctuations do not contribute to the energy density of a massless self interacting scalar field; all such contributions come from UV fluctuations, see figure 2.

5 Discussion

We presented a detailed derivation of the effective IR theory of a light scalar field \( \phi \) in de Sitter spacetime. By explicitly integrating out subhorizon modes, we obtained the stochastic dynamics first discussed by Starobinsky, as well as the next-to-leading-order corrections to the original stochastic picture. A new feature is that the dynamics is second order in time, meaning that both \( \phi \) and its velocity are included, with the latter equilibrating to a Maxwell distribution on a timescale \( \Delta t \sim 1/H \). The origin of the dominant stochastic term is the time dependence of the split between long and short modes. Including self interactions that couple long and short modes results in the long wavelength field evolving on a shifted effective potential whose renormalized form we compute. Self-interactions also result in subdominant stochastic forces of the kind encountered when there is a system-environment coupling. We hope our derivation further clarifies some existent misunderstandings on the origin of Starobinsky’s noise term which is present even if there is no self-coupling.

We also derived a functional Kramers equation for the long wavelength Wigner functional which can be used to compute spacetime expectation values of fields and field velocities. When restricted to coincident expectation values, the equation can be simplified to an ordinary Kramers equation, admitting an equilibrium solution that takes the Maxwell - Boltzmann form for the field averaged over a spatial volume \( \sim \frac{4}{9} H^{-3} \) and at a temperature \( T = H/2\pi \) and an effective potential resulting from integrating out the UV.
If the IR action (2.56) is used to define Feynman rules, individual diagrams will exhibit the same IR divergences as the full QFT diagrams. However, the stochastic theory can also be formulated in terms of a Wigner functional providing a probability distribution wrt which IR finite expectation values can be computed. It will be interesting to see if the methods developed here will be applicable to more complicated theories, including gravitational fluctuations, in the hope of resolving the IR issues they exhibit. It may also be of interest to use the dynamical stochastic formalism to investigate symmetry restoration in de Sitter studied in [14, 39, 67, 68].

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A Deriving the effective action in the Keldysh Basis

We present here the derivation of the effective action, unpacking the index notation of section 2.3 in the Keldysh basis. For our computations it suffices to only consider terms linear in $\phi_q$ in $V_I$ (2.13)

$$V_I \simeq V'_I \phi_q,$$

and

$$D^{-1ab} + \frac{\delta S_I}{\delta \phi_{<a} \delta \phi_{<b}} \simeq \begin{pmatrix} 0 & \nabla^2 - \tilde{m}^2 \\ \nabla^2 - \tilde{m}^2 & 0 \end{pmatrix} \delta(x - x') + \begin{pmatrix} \frac{\partial^3 V_I}{\partial \phi_{<1} \phi_{<2} \phi_{<3}} \\ 0 \end{pmatrix} \delta(x - x')$$

$$\equiv D^{-1ab} + V^{ab}_3,$$

where

$$\tilde{m}^2(\phi_{<}) = m^2 - V''(\phi_{<}).$$

We therefore have

$$-\frac{i}{2} \ln \det(G_{>}) \simeq \frac{i}{2} \Tr \ln \left(1 + \tilde{D}_{>} V_3\right)$$

since $\Tr \ln \tilde{D}_{>} = 0$, see (2.14, 2.15), and we obtain

$$-\frac{i}{2} \ln \det(G_{>}) \simeq \frac{1}{2} \int d\mu(x) D_K^< (x, x) \frac{\partial^3 V_I}{\partial \phi_{<3} \phi_{<1} \phi_{<2}} \phi_{<}(x).$$

The coincident limit of the $D^<_K$ propagator corresponds to the same limit of the more familiar Feynman propagator and hence harbours the same UV divergence.

The contribution (A.6) to the effective action adds a correction $\Delta V$ to the interacting potential experienced by the $\phi_{<}$ field

$$V_{I_{\text{eff}}} (\phi_{<}) = V_I (\phi_{<}) + \Delta V (\phi_{<}),$$

and this enters the effective Keldysh action through

$$\nabla_{I_{\text{eff}}} (\phi_{<}) \simeq \frac{\partial V_{I_{\text{eff}}}}{\partial \phi_{<3} \phi_{<1} \phi_{<2}}.$$

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\( \Delta V \) must therefore satisfy

\[
\frac{\partial \Delta V}{\partial \phi_\ast} = \frac{1}{2} D_K^\ast(x,x) \frac{\partial^3 V_I}{\partial \phi_\ast^3} .
\] (A.9)

Note that \( D_K^\ast \) also depends on \( \phi_\ast \) through (A.4).

The last term in (2.28) adds to the effective action

\[
\Delta S_2 = - \int \psi_\ast V_I'' D_R^\ast V' + \frac{i}{2} \int \psi_\ast V_I'' D_K^\ast V_I'' \psi_\ast ,
\] (A.10)

where integration over the two arguments of \( D \) is implied. The second term in \( \Delta S_2 \) corresponds to a stochastic force \( f \) on the long wavelength modes with a coloured correlation function

\[
\langle f(x) f(x') \rangle = V_I''(x) D_k(x, x') V_I''(x').
\] (A.11)

This force acts in addition to the Starobinsky noise term discussed in the main text and results from the short wavelength fluctuations due to the self coupling of the field. The first term in (A.10) adds an a priori non-local force term to the equations of motion. However, by restricting \( D_R^\ast \) to timescales not significantly longer than \( \Delta t \sim H^{-1} \) we have \( D_R^\ast \sim \frac{1}{m^2} e^{-3H(t-t') \Theta(t-t') \delta(x-x')} \) and hence we obtain an addition to the effective potential of \( \phi_\ast \) of the form

\[
\Delta V \approx - \int V_I' \frac{V_I''}{9H^2} d\phi ,
\] (A.12)

which is second order in the non-linear coupling.

B Long time/strong friction limit and the Fokker-Planck equation

The more familiar Fokker-Planck equation can be obtained by standard analysis [56] when considering timescales \( \Delta t \gg H^{-1} \). Reference [32] provides a detailed exposition of this procedure in cosmologically relevant examples. A simple approach amounts to setting \( H^{-1} \rightarrow 0 \). Using 1/3\( H \) as an expansion parameter it is possible to obtain

\[
W(\phi, v, t) = W_0(\phi) - \frac{v}{3H} \left( \frac{\partial W_0}{\partial \phi} + dU \Omega W_0 \right) + W_1(\phi, t) \frac{1}{3H} + O\left( \frac{1}{9H^2} \right) e^{-\Omega v^2/2} ,
\] (B.1)

where

\[
\frac{\partial}{\partial t} W_1 = \frac{1}{\Omega} \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} + \Omega \frac{dV}{d\phi} \right) W_0 .
\] (B.2)

Integrating over \( v \) we see that, to \( O(1/3H) \),

\[
\frac{\partial P}{\partial t} = \frac{1}{3H} \Omega \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} + \Omega \frac{dV}{d\phi} \right) P ,
\] (B.3)

where \( P(\phi, t) = \int dv W(\phi, v, t) \). Equation (B.3) is the more familiar Starobinsky-Fokker-Planck equation for \( \phi \). Hence Starobinsky’s original stochastic equation can be used when processes with time scales \( \Delta t \sim H^{-1} \) are neglected.
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