ON THE LENGTH OF A PARTIAL INDEPENDENT TRANSVERSAL IN A MATROIDAL LATIN SQUARE

DANIEL KOTLAR AND RAN ZIV

Computer Science Department, Tel-Hai College, Upper Galilee 12210, Israel

Abstract. We suggest and explore a matroidal version of the Brualdi - Ryser conjecture about Latin squares. We prove that any \( n \times n \) matrix, whose rows and Columns are bases of a matroid, has an independent partial transversal of length \( \lceil \frac{2n}{3} \rceil \). We show that for any \( n \), there exists such a matrix with a maximal independent partial transversal of length at most \( n - 1 \).

1. INTRODUCTION

A Latin Square of order \( n \) is an \( n \times n \) array \( L \) with entries taken from the set \{1, \ldots, n\}, where each entry appears exactly once in each row or column of \( L \). A partial transversal of size \( k \) of a Latin square \( L \) is a subset of \( k \) different entries of \( L \), where no two of them lie in the same row or column.

The maximal size of a partial transversal in \( L \) will be denoted here by \( t(L) \) and the minimal size of \( t(L) \), over all Latin squares \( L \) of order \( n \), will be denoted by \( T(n) \).

It has been conjectured by Ryser [10] that \( T(n) = n \) for every odd \( n \) and by Brualdi [4] (see also [2] p. 255) that \( T(n) = n - 1 \) for every even \( n \). Although these conjectures are still unsettled, a consistent progress has been made towards its resolution: Koksm [8] proved that for \( n \geq 3 \), \( T(n) \geq \lceil \frac{2n + 1}{3} \rceil \). This bound was improved by Drake [5] to \( T(n) \geq \lceil \frac{3n}{4} \rceil \) for \( n > 7 \), and again by de Veris and Wieringa [3] who obtained a lower bound of \( \lceil \frac{(4n - 3)}{5} \rceil \) for \( n \geq 12 \). Woolibright [14] showed that \( T(n) \geq \lceil n - \sqrt{n} \rceil \). A similar result was obtained independently by Brouwer, de Vries and Wieringa [1]. Recently, Hatami and Shor [11] proved that \( T(n) \geq n - O(\log^2 n) \). See also a recent comprehensive survey by Wanless [11].

The aim of this note is to suggest and explore a matroidal version of the Brualdi-Ryser conjectures. For basic texts on matroids the reader is referred to Welsh [12], Oxley [9] and White [13].

Definition 1.1. Let \((M, S)\) be a matroid \( M \) on a ground set \( S \). A matroidal Latin square (abbreviated MLS) of degree \( n \) over \((M, S)\) is an \( n \times n \) matrix \( A \) whose entries are elements of \( S \), where each row or column of \( A \) is a base of \( M \).

Notice that a matroidal Latin square reduces to a Latin square if \( M \) is a partition matroid. We mention that according to a well-known conjecture of Rota [7] every
set of \( n \) bases of a matroid of rank \( n \) can be arranged to form an MLS of degree \( n \) so that its rows consist of the original bases.

**Definition 1.2.** An independent partial transversal of an MLS \( A \) is an independent subset of entries of \( A \) where no two of them lie in the same row or column of \( A \).

We propose the following analogue of Brualdi’s conjecture:

**Conjecture 1.3.** Every MLS of degree \( n \) has an independent partial transversal of size \( n - 1 \).

In view of Ryser’s conjecture, it is natural to ask whether in Conjecture 1.3 an independent partial transversal of size \( n \) exists whenever \( n \) is odd. Theorem 3.1 asserts that this is not the case.

2. A LOWER BOUND FOR A MAXIMAL INDEPENDENT PARTIAL TRANSVERSAL

Let \( A = (a_{ij})_{i,j=1}^n \) be an MLS of degree \( n \) over a matroid \( M \). Let \( T \) be an independent partial transversal of size \( t \). Without loss of generality we may assume that the elements of \( T \) are the first \( t \) elements of the main diagonal of \( A \). That is

\[
A = \begin{pmatrix}
B & C \\
D & E
\end{pmatrix}
\]

where \( B, C, D \) and \( E \) are sub-matrices of \( A \) of dimensions \( t \times t, t \times (n-t), (n-t) \times t \) and \( (n-t) \times (n-t) \) respectively, and \( T \) constitutes the main diagonal of \( B \). If \( T \) is of maximal length, then \( t \geq \lceil n/2 \rceil \). Otherwise \( \dim(E) \geq n-t > t = \dim(T) \) and thus \( E \) would contain an element that is not spanned by \( T \) and hence can be added to \( T \), contradicting the maximality of \( T \). In order to show that \( t \geq \lceil 2n/3 \rceil \) we shall need the following lemma:

**Lemma 2.1.** Let \( X \) be a finite set and let \( s > |X|/2 \). Let \( X_1, \ldots, X_s \) be a family of subsets of \( X \), each of size at least \( s \). Then there exists some \( X_i \), all of whose elements appear in other subsets in the family.

**Proof.** Let \( Y_1 \) be the set of elements in \( X \) that appear in exactly one of the subsets \( X_1, \ldots, X_s \) and let \( Y_2 \) be the set of elements in \( X \) that appear in at least two of the subsets \( X_1, \ldots, X_s \). Let \( k_1 = |Y_1| \) and \( k_2 = |Y_2| \). Assume, by contradiction, that each \( X_i \) contains at least one element of \( Y_1 \). Then \( k_1 \geq s \) and thus

\[
k_2 \leq |X| - k_1 \leq |X| - s < |X|/2
\]

(since \( s > |X|/2 \)). If, for some \( i \), \( |X_i \cap Y_1| = 1 \) then \( |X_i \cap Y_2| \geq s - 1 \) and thus \( k_2 \geq s - 1 \geq |X|/2 - 1 \). It follows that \( k_2 \geq |X|/2 \), contradicting (2.2). It follows that for all \( i \), \( |X_i \cap Y_1| \geq 2 \). Then \( k_1 \geq 2s \) and thus \( k_2 \leq |X| - k_1 \leq |X| - 2s < |X| - |X| = 0 \), which is absurd. This proves the lemma.

**Theorem 2.2.** Let \( A \) be an MLS of degree \( n \) over a matroid \( M \). Then \( A \) contains an independent partial transversal of size \( \lceil 2n/3 \rceil \).

**Proof.** We use the notations from the beginning of Section 2. Since \( T \) is maximal, all the elements in the sub-matrix \( E \) are spanned by \( T \). Let \( T_E \) be the minimal subset of \( T \) that spans \( E \) (this set is unique since \( T \) is independent.) Since \( \dim(E) \geq n-t \) then \( |T_E| \geq n-t \) and thus \( |T \setminus T_E| \leq t - (n-t) = 2t - n \). Since each row of \( A \) is a base and all the elements of \( E \) are spanned by \( T \), each row of the sub-matrix \( D \) contains a subset of size \( n-t \) that complement \( T \) to a base. In particular,
each row of $D$ contains at least $n - t$ elements that are not spanned by $T$. Let $X = \{1, \ldots, t\}$ be the set of indices of the columns of $D$. For each of the $n - t$ rows in $D$ we define a subset $X_i \subseteq X, i = t + 1, \ldots, n$, in the following way: $j \in X_i$ if and only if the $j$th element of the $i$th row of $A$ is not spanned by $T$. It follows that $|X_i| \geq n - t$ for all $i = t + 1, \ldots, n$. Now assume, by contradiction, that $t < 2n/3$. Then $n - t > n/3 > t/2$. So we have a set $X$ of size $t$ and $n - t$ subsets $X_{t+1}, \ldots, X_n$, each of size at least $n - t$, such that $n - t > t/2$. Let $s = n - t$. By Lemma 2.3 we conclude that there exists a subset $X_i$ of whose elements are contained in other subsets in the family $X_{t+1}, \ldots, X_n$. This means that there is a row in $D$ containing at least $n - t$ elements that are not spanned by $T$ and for each such element there exists another element in the same column in $D$ that is not spanned by $T$. It follows that $D$ contains at least $n - t$ columns, each containing at least two elements that are not spanned by $T$. Since $t < 2n/3$ we have that $|T \setminus T_E| \leq 2t - n < n/3 < n - t$. So there exists $j \leq t$ such that (1) $a_{jj} \in T_E$ and (2) the $j$th column of $D$ contains at least two elements that are not spanned in $T$. Let $x \in E$ be such that its support (i.e., its minimal spanning set) in $T$ contains $a_{jj}$ and let $y$ and $z$ be two elements in the $j$th column of $D$ that are not spanned by $T$. We may assume that $x$ and $y$ are not in the same row (otherwise we take $z$ instead of $y$). Since $T \cup \{y\}$ is independent, and the support of $x$ in $T$ contains $a_{jj}$, it follows that $T \setminus \{a_{jj}\} \cup \{y\}$ does not span $x$ and thus $S \setminus \{a_{jj}\} \cup \{x, y\}$ is an independent partial transversal in $A$ of length $t + 1$, contrary to the maximality of $T$. Thus $t$ must be at least $[2n/3]$.

3. An upper bound of size $n - 1$ for an MLS of degree $n$

It is well known that for any even $n$ there exist Latin squares of order $n$ with no transversal of size $n$. The following theorem shows that for any $n$ there exists an MLS of degree $n$ with no independent transversal of size $n$.

**Theorem 3.1.** Let $v_1, v_2, \ldots, v_n$ be a basis of a vectorial matroid of rank $n$. Then the matrix $A = (a_{ij})_{i,j=1}^n$, whose elements are $a_{ii} = v_1$, for $i = 1, \ldots, n$, and $a_{ij} = v_i - v_j$, for $1 \leq i \neq j \leq n$, is an MLS of order $n$ with no independent transversal of size $n$.

**Proof.** We leave it to the reader to check that the rows and columns of $A$ are independent. Let $T$ be a transversal of size $n$ in $A$. We show that $T$ is not independent. If $T$ does not contain elements of the main diagonal of $A$, then, since each row and column is represented exactly once among the elements of $T$, the sum of the elements of $T$ is 0, and $T$ is not independent. Thus we may assume that $T$ meets the main diagonal exactly once. Let $a_{ii} = v_i \in T$. If $i = 1$ then the sum of the elements of $T - a_{11}$ is 0. If $i > 1$, then $v_i$ is not spanned by $T$, so $T$ is not a basis, and thus, is not independent.

**References**

1. A.E. Brouwer, A.J. de Vries, and R.M.A. Wieringa, *A lower bound for the length of partial transversals in a Latin square*, Nieuw Arch. Wiskd. 24 (1978), no. 3, 330–332.
2. R.A. Brualdi and H.J. Ryser, *Combinatorial matrix theory*, Cambridge University Press, 1991.
3. A.J. de Vries and R.M.A. Wieringa, *Een ondergrens voor de lengte van een partiele transversaal in een Latijns vierkant*, preprint.
4. J. Dénes and A.D. Keedwell, *Latin squares and their applications*, Academic Press, New York, 1974.
5. D.A. Drake, *Maximal sets of Latin squares and partial transversals*, J. Statist. Plann. Inference 1 (1977), 143–149.
6. P. Hatami and P. W. Shor, *A lower bound for the length of a partial transversal in a Latin square*, J. Combin. Theory A 115 (2008), 1103–1113.
7. R. Huang and G-C. Rota, *On the relations of various conjectures on Latin squares and straightening coefficients*, Discrete Mathematics 128 (1994), 225–236.
8. K.K. Koksma, *A lower bound for the order of a partial transversal in a Latin square*, J. Combin. Theory 7 (1969), 94–95.
9. J. Oxley, *Matroid theory*, 2 ed., Oxford University Press, 2011.
10. H.J. Ryser, *Neuere probleme der kombinatorik*, Vorträge über Kombinatorik, Oberwolfach, Mathematisches Forschungsinstitute (Oberwolfach, Germany), July 1967, pp. 69–91.
11. I. M. Wanless, *Transversals in Latin squares: A survey*, Surveys in Combinatorics, London Mathematical Society Lecture Note Series, vol. 392, pp. 403–437, Cambridge University Press, 2011.
12. D. Welsh, *Matroid theory*, Academic Press, London, 1976.
13. N. White (ed.), *Encyclopedia of mathematics and its applications, theory of matroids*, vol. 26, Cambridge University Press, 1986.
14. D.E. Woolbright, *An n × n Latin square has a transversal with at least n – √n distinct elements*, J. Combin. Theory A 24 (1978), 235–237.