Universally Optimal Privacy Mechanisms for Minimax Agents

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ABSTRACT
A scheme that publishes aggregate information about sensitive data must resolve the trade-off between utility to information consumers and privacy of the database participants. Differential privacy [5] is a well-established definition of privacy—this is a universal guarantee against all attackers, whatever their side-information or intent. In this paper, we present a universal treatment of utility based on the standard minimax rule from decision theory [13] (in contrast to the utility model in [8], which is Bayesian).

In our model, information consumers are minimax (risk-averse) agents, each possessing some side-information about the query, and each endowed with a loss-function which models their tolerance to inaccuracies. Further, information consumers are rational in the sense that they actively combine information from the mechanism with their side-information in a way that minimizes their loss. Under this assumption of rational behavior, we show that for every fixed count query, a certain geometric mechanism is universally optimal for all minimax information consumers. Additionally, our solution makes it possible to release query results at multiple levels of privacy in a collusion-resistant manner.

1. INTRODUCTION
Privacy Mechanisms: Agencies such as medical establishments, survey agencies, governments use and publish aggregate statistics about individuals; this can have privacy implications. Consider the query: *Q: How many adults from San Diego contracted the flu this October?* The government can use the query result to track the spread of flu, and drug companies can use it to plan production of vaccines. However, knowledge that a specific person contracted the flu could be used to deny her health insurance based on the rationale that she is susceptible to disease. As discussed in [20], and as is exemplified by [21, 16], seemingly benign data publications can have privacy implications. Thus, it is important to think rigorously about privacy. The framework of differential privacy [3] does this, and is applicable widely (see Section 2.8).

Mechanisms guarantee differential privacy by perturbing results – they add random noise to the query result, and guarantee protection against all attackers, whatever their side-information or intent (see Section 2.1 for a formal definition).

Our Utility Model: The addition of noise increases privacy but intuitively reduces utility of the query result. To understand this privacy-utility trade-off, we propose a formal decision-theoretic model of utility. Decision-theory is a widely applied field that provides mathematical foundations for dealing with preferences under uncertainty. The use of decision theory in this context is appropriate because, as we discussed above, mechanisms guarantee differential privacy by introducing uncertainty.

In our model of utility (see Section 2.3 for details), the user of information, i.e. the information consumer has side-information—for instance, knowledge of the population of San Diego is an upper bound on the result of the query Q. It has a loss-function that expresses its tolerance to inaccuracy. It is rational in the sense that it combines information from the mechanism with its side-information optimally with respect to its personal loss-function. It is risk-averse in the sense that it would like to minimize worst-case loss over all scenarios.1

Given the privacy parameter, the loss-function and the side-information of an information consumer it is possible to identify an optimal mechanism—a mechanism that is differentially private and that maximizes its utility. See Section 2.4.3 for an algorithm to find such a mechanism.

Non-Interactive Settings: Very often aggregate statistics, like answers to Q, are published in mass media as opposed to following a query-response form [18]. In such cases neither the information consumer nor its loss-function and side-information are known in advance. Thus it seems hard to identify the optimal mechanism for a information consumer.

Nevertheless, we show that it is possible to deploy an optimal mechanism without knowledge of the information consumer’s parameters. Furthermore, this mechanism is uni-

1Ghosh et al. [8] propose a model with most of these features, but assumes that information consumers are Bayesian and have a prior over the query-result.
versally optimal for all information consumers, no matter what their side-information or loss-function.

How can we identify the optimal mechanism without knowledge of the information consumer's parameters? The apparent paradox is resolved by relying on the information consumers' rationality, i.e., each information consumer uses its personal loss-function and side-information to actively transform the output of the deployed mechanism. For a certain class of queries called count queries, when the deployed mechanism is a certain geometric mechanism, this transformation is effective enough to result in the optimal mechanism for the information consumer—a fact that we will establish via linear-algebraic proof techniques.

Multiple Levels of Privacy: We also show how to simultaneously release the query result at different levels of privacy to different information consumers. This is useful, for instance, when we want to construct two versions of the report on flu statistics, one which prioritizes utility for the eyes of government executives, and a publicly available Internet version that prioritizes privacy.

A naive solution is to perturb the query results differently, independently adding differing amounts of noise each time. The drawback is that consumers at different levels of privacy can collude and combine their results to cancel the noise (as in Chernoff bounds). An alternate way is to correlate the noise added to different outputs. We give an algorithm to achieve this that makes the data release collusion-resistant.

In this paper we focus on a single query; the complexity comes from a rich model of consumer preferences, where we consider different utility functions for each consumer and optimize for each of them. [1, 10, 9] exploit similarities between the queries to obtain extension to multiple queries with good utility guarantees. However, they do not consider a rich consumer preference model. Our results could be used as a building block while answering multiple queries.

2. MODEL AND RESULTS
We gave an informal description of our model and results in the Introduction. In this section, we formally define our model and discuss the main results. The proofs of the results are presented in Sections 3, 4.

2.1 Privacy Mechanisms and Differential Privacy
A database is a collection of rows, one per individual. Each row is drawn from an arbitrary domain $D$; for instance, in our running example, a row of the database has the name, age, address and medical records of a single individual. A database with $n$ rows is thus drawn from the domain $D^n$.

We will focus on a class of queries, called count queries, that frequently occur in surveys: Given a predicate $p : D \rightarrow \{\text{True, False}\}$, the result of a count query is the number of rows that satisfy this predicate, a number between 0 and the database size, $n$. $Q$ is an example of a count query with the predicate: individual is an adult residing in in San Diego, who contracted flu this October. Though simple in form, count queries are expressive because varying the predicate naturally yields a rich space of queries.

We guarantee differential privacy to protect information of individual database participants. Differential privacy is a standard, well-accepted definition of privacy [3] that has been applied to query privacy [7, 5, 17], privacy preserving machine learning [1, 11] and economic mechanism design [15]. A fixed count query maps the database $d$ to a number which belongs to the set $N$. A privacy mechanism $M$ for a fixed count query is a probabilistic function that maps a database $d \in D^n$ to the elements of the set $N = \{0 \cdots n\}$. These can be represented, for each $d \in D^n$, by $\{m_{d,r}\}_{r \in N}$, which gives for each database $d \in D^n$ the probability that $M$ outputs $r$. For the database $d$, the mechanism releases a perturbed result by sampling from the distribution $\{m_{d,r}\}_{r \in N}$.

The Geometric Mechanism [8] is a simple example of a privacy mechanism. It is a discrete version of the Laplace mechanism from [5].

Definition 1 ($\alpha$-Geometric Mechanism). When the true query result is $f(d)$, the mechanism outputs $f(d) + Z$. $Z$ is a random variable distributed as a two-sided geometric distribution: $\Pr[Z = z] = \frac{1 - \alpha^{z-1}}{1 - \alpha}$ for every integer $z$.

![Figure 1: The probability distribution on outputs given by the Geometric Mechanism for $\alpha = 0.2$ and query result 5.](image)

Informally, a mechanism satisfies differential privacy if it induces similar output distributions for every two databases that differ only in a single individual’s data, thereby ensuring that the output is not sensitive to any one individual’s data. Formally, differential privacy is defined as follows [5]:

Given a privacy parameter $\alpha \in [0,1]$ and two database $d_1, d_2 \in D^n$ that differ in at most one individual’s data, a mechanism $M$ is $\alpha$-differentially private, if for all elements $r$ in the range of the mechanism: $
\frac{1}{\alpha} \cdot x_{d_1,r} \geq x_{d_2,r} \geq \alpha \cdot x_{d_1,r}$.

Thus any attack on an individual’s privacy that can be constructed using the perturbed query result with this individual present in the database can also be constructed, with a similar success rate, without this individual present in the database. See Sec [5, 12] for details of such semantics of differential privacy.
The parameter $\alpha$ can be varied in the interval $[0, 1]$ to vary the strength of the privacy guarantee—when $\alpha = 0$, the above definition is vacuous and there is no privacy, whereas when $\alpha = 1$, we effectively insist on absolute privacy—the query result cannot depend on the database because we require distributions over perturbed results to be identical for neighboring databases.

### 2.2 Oblivious Mechanisms

We will focus in this paper on a class of privacy mechanisms that are oblivious. A mechanism is oblivious if it sets up an identical distribution over outputs for every two databases that have the same unperturbed query result. Naturally, an implementation of an oblivious mechanism need only have access to the true query result—the input—and can be oblivious to the database itself. An oblivious mechanism for count queries can be expressed by the set of probability masses for every $i \in N: \{x_{i,r}\}_{r \in N}$, where $x_{i,r}$ is the probability that the mechanism outputs $r$ when the true result is $i$. Appendix A shows that this restriction to oblivious mechanisms is without loss of generality. The geometric mechanism (Definition 1) only depends on the query result $f(d)$ and not on the database $d$ itself; so it is a oblivious mechanism.

The query result for a count query can change by at most one when we change any one row of the database, so we can rewrite the definition of differential privacy as follows:

**Definition 2 (Differential Privacy for Count Queries).** An oblivious mechanism for count queries for $\alpha \in [0, 1]$ is $\alpha$-differentially private if for all $i \in \{0 \ldots n - 1\}$, $r \in N$:

$$\frac{1}{\alpha} x_{i,r} \geq x_{i+1,r} \geq \alpha x_{i,r}.$$ 

Observe that the geometric mechanism is $\alpha$-differentially private because for two adjacent inputs $i, i+1 \in N$, and any output $r \in N$, $\frac{x_{i,r}}{x_{i+1,r}} \in [\alpha, 1/\alpha]$.

### 2.3 Minimax Information Consumers

We now discuss our model of an information consumer’s utility. The loss-function $l(i, r) : N \times N \to \mathbb{R}$ specifies the loss of the information consumer, given the mechanism outputs $r$ when the true result is $i$. We only assume that the loss-function is monotone non-decreasing in $|i - r|$, for every $i$. That is, the consumer becomes unhappier as the returned answer is further away from the true result.

Consider some examples of valid loss-functions: The loss-function $l(i, r) = |i - r|$ quantifies the mean error—for our query $Q$, this loss-function may be a reasonable one for the government who want to keep track of the rise of flu. The loss-function $l(i, r) = (i - r)^2$ quantifies the variance in the error—this may be reasonable for a drug company who wants to ensure that they don’t over-produce or under-produce the flu drug. The loss-function $l(i, r) = \begin{cases} 0 & \text{if } i = r \\ 1 & \text{if } i \neq r \end{cases}$ measures the frequency of error.

Additionally, we will assume that the information consumer has side information $S \subseteq N$, i.e., the information consumer knows that the query result cannot fall outside the set $S$. For instance, knowledge of the population of San Diego yields an upper-bound on the query result. The drug company may also know how many people bought its flu drug this month, yielding a lower bound on the query result.

For any specific input $i$, the loss-function $l$ allows us to evaluate the information consumer’s dis-utility as the expected loss over the coin tosses of the mechanism: $\sum_{r \in N} l(i, r) \cdot x_{i,r}$. To quantify the overall loss, we follow the minimax decision rule, i.e., we take the worst-case loss over all inputs in the set $S$ [13]. This amounts to the information consumers being risk-averse. Hence, the dis-utility of the mechanism $x$ to the consumer $c$ is:

$$L(x) = \max_{i \in S} \sum_{r \in N} l(i, r) \cdot x_{i,r}$$ (1)

### 2.4 Interactions of Information Consumers with Mechanisms

As mentioned in the Introduction, information consumers actively interact with the mechanism to induce a new mechanism; we now discuss the mechanics of this interaction.

#### 2.4.1 Motivation

The following example argues why a rational information consumer will not accept the mechanism’s output at face value.

**Example 1.** Recall the query $Q$ defined in the Introduction. Suppose that the information consumer is a drug company, who knows that $l$ individuals in San Diego bought its flu drug in the month of October. Thus the query result $Q$ must be at least $l$; the information consumer cannot conclude that the query result is exactly $l$ because some individuals with flu may have bought a competitor’s drug, or bought no drug at all. Thus it has side-information $S = \{l \ldots n\}$.

Suppose we deploy the geometric mechanism for the query $Q$. This mechanism returns with non-zero probability outputs outside the set $\{l \ldots n\}$. Such outputs are evidently incorrect to the information consumer, and naturally it makes sense for the information consumer to map these results within the set $\{l \ldots n\}$. Though it is not clear what the best way of doing so is, a reasonable rule may be to re-interpret results less than $l$ as $l$, and results larger than $n$ as $n$.

#### 2.4.2 Feasible Interactions

Before we discuss the optimal way for an information consumer to interact with the mechanism, we describe the space of feasible interactions. On receiving a query result $r$ from the mechanism, the consumer can reinterpret it as a different output. This reinterpretation can be probabilistic and can be represented by a set of probability masses $\{T_{r,r'} : r' \in N\}$ which gives for each result $r$, the probability that the consumer will reinterpret it as the output $r'$. Such an interaction induces a new mechanism for the user. Suppose the deployed mechanism is represented by the set of probability masses $\{y_{i,r} : i, r \in N\}$, and the induced mechanism as the probability masses $\{x_{i,r} : i, r \in N\}$, then $x_{i,r'} = \sum_{r \in N} y_{i,r} \cdot T_{r,r'}$. We formalize this in a definition.

**Definition 3 (Derivability).** Given two mechanism $x$ and $y$, we say that mechanism $x$ can be derived from $y$ if and
only if, for every \( r \in N \), there exists a set of probability masses \( \{T_{i,r'} : r' \in N \} \) such that for every \( i, r' \in N \) :

\[
x_{i,r'} = \sum_{r \in N} y_{i,r} \cdot T_{r,r'}.
\]

### 2.4.3 Optimal Interactions

Given a deployed mechanism \( y \), the optimal interaction \( T^\ast \) is one that minimizes the information consumer’s maximum loss on the induced mechanism. The optimal interaction can be computed using a simple linear program. There are \( n^2 \) variables: one for each \( T_{r,r'} \in N \). The objective function is obtained by minimizing the loss to the consumer if it uses interaction \( T^\ast \). The constraints are obtained from the fact for each \( r \), the entries \( T_{r,r'} \) form a probability distribution and hence sum up to 1 and that all entries of \( T^\ast \) are positive. The actual linear program is given as:

\[
\begin{align*}
\text{minimize} & \quad \max_{i \in S} \sum_{r \in N} x_{i,r} \cdot l(i,r) \\
x_{i,r} & = \sum_{r' \in N} y_{i,r'} \cdot T_{r,r'}^\ast \quad \forall i \in N, \forall r \in N \\
T_{r,r'}^\ast & = 1 \quad \forall r \in N \\
T_{r,r'}^\ast & \geq 0 \quad \forall r \in N, \forall r' \in N
\end{align*}
\]

### 2.5 Optimal Mechanism for a Single Known Information Consumer

Identifying the optimal mechanism for a specific consumer reduces to the following: Identify a level of privacy \( \alpha \) with which to release the result. Find the consumer’s loss-function and side-information. Identify an \( \alpha \)-differentially private mechanism such that the mechanism induced by the consumer’s optimal interaction (as described in the previous section), has the best possible utility.

In the case of a single information consumer, we can obvi-ate the need for the information consumer to reinterpret the deployed mechanism’s output: Suppose there is a mechanism \( y \) with post-processing \( T \) that induces a mechanism \( x \). Clearly, presenting \( x \) directly to the information consumer yields at least as much utility for it. All we have to ensure is that \( x \) is \( \alpha \)-differentially private, and a simple proof (omitted) shows that this is indeed so.

Thus, to identify the optimal mechanism for a specific information user, it suffices to search over \( \alpha \)-differential mechanisms. For a given consumer \( c \) with loss-function

\[
L(l, S) = \max_{i \in S} \sum_{r \in N} x_{i,r} \cdot l(i,r)
\]

and privacy parameter \( \alpha \), the optimal differentially private mechanism \( M_\alpha \) is the solution to a simple linear program. Like in the previous section, there are \( n^2 \) variables one for each matrix entry of the mechanism \( x \). The objective is to minimize the user’s loss function. The constraints are obtained by the facts that

1. \( x \) is differentially private. So the variables \( x_{i,r} \) must satisfy Definition 2
2. For each input \( i \), elements \( x_{i,r} \) form a probability distribution and hence sum up to 1.

3. All elements \( x_{i,r} \) are positive

Writing this as an optimization problem we get:

\[
\begin{align*}
\text{minimize} & \quad \max_{i \in S} \sum_{r \in N} x_{i,r} \cdot l(i,r) \\
x_{i,r} - \alpha \cdot x_{i+1,r} & \geq 0 \quad \forall i \in N \setminus \{n\}, \forall r \in N \\
\alpha \cdot x_{i,r} - x_{i+1,r} & \leq 0 \quad \forall i \in N \setminus \{n\}, \forall r \in N \\
\sum_{r \in N} x_{i,r} & = 1 \quad \forall i \in N \\
x_{i,r} & \geq 0 \quad \forall i \in N, \forall r \in N
\end{align*}
\]

We can convert it into a Linear Program, the solution of which gives us \( x^\ast \).

\[
\begin{align*}
\text{minimize} & \quad d \\
d & = \sum_{r \in N} x_{i,r} \cdot l(i,r) \geq 0 \quad \forall i \in S \\
x_{i,r} - \alpha \cdot x_{i+1,r} & \geq 0 \quad \forall i \in N \setminus \{n\}, \forall r \in N \\
\alpha \cdot x_{i,r} - x_{i+1,r} & \leq 0 \quad \forall i \in N \setminus \{n\}, \forall r \in N \\
\sum_{r \in N} x_{i,r} & = 1 \quad \forall i \in N \\
x_{i,r} & \geq 0 \quad \forall i \in N, \forall r \in N
\end{align*}
\]

To deploy this mechanism \( x^\ast \), we first compute the true query result, say \( i \), then sample the perturbed result \( r \) from the distribution \( \{x^\ast_{i,r} : \forall r \in N\} \), and release the sampled result. Table 1(a) gives an example of a optimal mechanism for a particular information consumer.

### 2.6 Optimal Mechanism for Multiple Unknown Information Consumers

How can we extend the results of the previous section to multiple consumers? The naive solution is to identify and separately deploy the optimal mechanism for each information consumer as described in the previous section.

There are two reasons why this is undesirable. First, the naive solution results in the release of several re-randomizations of the query result—this allows colluding consumers to combine their results and cancel out the noise leading to a degradation in privacy; see [15] for a discussion,

Second, solving the linear program that identifies the optimal mechanism for a user requires the knowledge of the consumer’s parameters; knowledge that is often unavailable when the decision of which mechanism to deploy is made. Consider a report published on the Internet. It is not clear who the information consumers are going to be.

Our main result works around these issues successfully.

**Theorem 1.** Consider a database \( d \), count query \( q \), \( k \) users and privacy levels \( \alpha_1 < \ldots < \alpha_k \). There exists a mechanism \( M \) that constructs \( k \) results \( r_1, \ldots, r_k \), and releases result \( r_i \) to the \( i \)th information consumer, such that:
We now describe the release mechanism $M$. The $i$th stage of the mechanism $M_i$ is just the $\alpha_i$-geometric mechanism. We shall prove in Lemma 3, that for any $\alpha > \beta$, the $\alpha$-geometric mechanism can be derived from the $\beta$-geometric mechanism: that is there is an implementable mechanism $T_{\alpha, \beta}$ such that if we use $T_{\alpha, \beta}$ to reinterpret results given by the $\beta$-geometric mechanism, we get the $\alpha$-geometric mechanism. The query results $r_i$ are not generated independently of each other, they are obtained by successive perturbations: the result $r_i$ of mechanism $M_i$ is given as input to the mechanism $T_i = T_{\alpha_i, \alpha_{i+1}}$. Hence, the $(i+1)$th stage mechanism $M_{i+1}$ is just the $\alpha_{i+1}$-geometric mechanism. This specifies how the noise added to the query results is correlated. We describe the mechanism formally in Algorithm 1. In Section 4.1 we show that it is collusion-resistant.

Consumer $i$ interacts optimally with the published query result $r_i$ to get a result tailored specifically for it. In Section 4.2, we prove that the interaction yields optimal utility for the consumer. The main idea is that the optimal mechanism can be factored into two parts – The first is a database specific mechanism which has access to the database but not to the user parameters. In our case this is the $\alpha_i$-geometric mechanism. The second is the user specific mechanism, which has access to the user loss-function and side-information and the perturbed query result (given by the first mechanism), but not to the database itself. Table 1 shows these two factors of the optimal mechanism discussed in Section 2.5.

We briefly discuss proof techniques: Section 3 completely characterizes mechanisms derivable from the geometric mechanism using linear algebraic techniques. Section 4 applies this characterization twice: the first application shows that a $\alpha$-geometric mechanism can be derived by re-randomizing the output of a $\beta$-geometric mechanism so long as $\alpha > \beta$. The second application shows that the mechanism induced by the interaction of a rational information consumer with the geometric mechanism is an optimal solution to the linear program mentioned in Section 2.5.

| 2/3 | 5/17 | 1/25 | 1/98 |
| 1/6 | 7/11 | 7/44 | 2/49 |
| 2/49 | 7/44 | 7/11 | 1/6 |
| 1/98 | 1/25 | 5/17 | 2/3 |

(a) The Optimal Mechanism

| 4/3 | 1/4 | 1/16 | 1/48 |
| 1/3 | 1 | 1/4 | 1/12 |
| 1/12 | 1/4 | 1 | 1/3 |
| 1/48 | 1/16 | 1/4 | 4/3 |

(b) $G_{3,4}$

Mechanism with access to the database. Mechanism with access to the user parameters.

Table 1: This shows the optimal mechanism for a consumer $c$ with loss-function $l(i, r) = |i - r|$ and side-information $S = \{0, 1, 2, 3\}$. $n = 3, \alpha = 1/4$.

2.7 Comparison with Bayesian Information Consumers

An alternative to the Minimax decision rule is the Bayesian decision rule. Ghosh et al. [8] prove an analogous result to Theorem 1 for all Bayesian information consumers. We briefly compare the models and the proof techniques.

The main distinction between the two models is their treatment of side-information. The Bayesian model requires agents to have a prior over all possible scenarios (true query results). Often, in practice, agents do not behave consistent with the preferences of the Bayesian model, perhaps because they find it hard to come up with meaningful priors [14, Example 6.B.2, B.3], or are genuinely risk-averse [14, Section 6.3].

As discussed in [8], Bayesian information consumers employ deterministic post-processing, unlike minimax information consumers which employ randomized post-processing (For example, see Table 1). Handling this extra complexity requires us to construct a broader characterization of mechanisms derivable from the geometric mechanism—Section 3 presents a complete characterization in terms of a simple condition on the probability masses $x_{i-1,j}$, $x_{i,j}$, $x_{i+1,j}$. Our proof avoids the LP based techniques and counting arguments of [8], and consequentially strictly generalizes and gives a simpler proof of the main result of that paper. In addition, our characterization enables us to release data at multiple levels of privacy in a collusion-resistant manner.

2.8 Related Work

A recent thorough survey of the state of the field of differential privacy is given in [4]. Dinur and Nissim [2], Dwork et al. [6] establish upper-bounds on the number of queries that can be answered with reasonable accuracy. Most of the differential privacy literature circumvents these impossibility results by focusing on interactive models where a mechanism supplies answers to only a sub-linear (in $n$) number of queries. Count queries (e.g. [2, 7]) and more general queries (e.g. [5, 17]) have been studied from this perspective.

Hardt and Talwar [9] give tight upper and lower bounds on the amount of noise needed to ensure differential privacy for $d$ non-adaptive linear queries, where the database is a vector in $\mathbb{R}^n$. Hay et al. [10] give a way to increase accuracy of answering multiple related queries while ensuring that the query results follow consistency constraints.

Blum et al. [1] focus attention to count queries that lie in a restricted class; they obtain non-interactive mechanisms that provide simultaneous good accuracy (in terms of worst-
In this section we give a characterization of all mechanisms guarantees about differential privacy. However, they use random output per-

Our formulation of the multiple privacy levels is similar to Xiao et al. [22]. However, they use random output perturbations for preserving privacy, and do not give formal guarantees about differential privacy.

3. CHARACTERIZING MECHANISMS DERIVABLE FROM THE GEOMETRIC MECHANISM

In this section we give a characterization of all mechanisms that can be derived from the geometric mechanism. Recall that differential privacy imposes conditions on every two consecutive entries \((x_1, x_2)\) of every column: \(x_1 \geq \alpha x_2\) (and \(x_2 \geq \alpha x_1\)). Our characterization imposes syntactically similar conditions on every three consecutive entries \((x_1, x_2, x_3)\) in a column: \((x_2 - \alpha \cdot x_3) \geq \alpha (x_1 - \alpha \cdot x_2)\). Neither condition implies the other. This characterization is both necessary and sufficient for any differentially private mechanism to be derivable from the geometric mechanism.

We defined feasible consumer interactions in Section 2.4.2. A slightly different way of representing these is to arrange the probability masses in a \(n \times n\) matrix \((T_{r,r'})_{r,r' \in N}\). We say that a matrix is (row) stochastic if the sum of elements in each row is 1 and all elements are non-negative. We say that a matrix is a generalized (row) stochastic matrix if the sum of the elements in each row is 1, but with no condition on individual entries. If the deployed mechanism is given by the matrix \(y\), and the reinterpretation by the matrix \(T\), then the new mechanism is given by the matrix \(x = y \cdot T\).

We define a version of the Geometric Mechanism whose range is restricted to \(\{0, \ldots, n\}\), which will be easier to work with since it can be easily represented as a matrix.

**Definition 4** (Range-Restricted Geometric Mechanism). For a given privacy parameter \(\alpha\), when the true query result is \(k \in [0, n]\), the mechanism outputs \(Z(k)\) where \(Z(k)\) is a random variable with the following distribution for each integer \(z\):

\[
Pr[Z(k) = z] = \begin{cases} 
\frac{1}{1 + \alpha} & \text{if } z \in \{0, n\} \\
\frac{1 - \alpha}{1 + \alpha} & \frac{1 - \alpha}{z} & \text{if } 0 < z < n \\
0 & \text{otherwise.} 
\end{cases}
\]

This mechanism is equivalent to the geometric mechanism in the sense that we can derive this from the geometric mechanism and derive the geometric mechanism from its range-restricted version. We shall refer to both as the Geometric Mechanism and denote the matrix by \(G_{\alpha}\). (Table 2).

For ease of notation, we shall denote by \(G'_{\alpha}\) the matrix obtained by multiplying the columns 1 and \(n\) of \(G_{\alpha}\) by \((1 + \alpha)\) and all other entries by \(\frac{1 + \alpha}{1 - \alpha}\). Table 2 shows the matrices of \(G_{\alpha}\) and \(G'_{\alpha}\). We are now ready to state the characterization.

**Theorem 2.** Suppose \(M\) is any oblivious differentially private mechanism. Then \(M\) can be derived from the geometric mechanism if and only if every three consecutive entries \(x_1, x_2, x_3\) in any column of \(M\) satisfy \((x_2 - \alpha x_3) \geq \alpha (x_1 - \alpha x_2)\).

The key insight is to think of each column in \(M\) and in \(G_{\alpha}\) as a vector. Looking at the problem through this linear algebraic lens, we see that deriving \(M\) from \(G_{\alpha}\) amounts to proving that each column of \(M\) lies in the convex hull of the (vectors which form the) columns of \(G_{\alpha}\). In Lemma 1, we show that \(G_{\alpha}\) is non-singular, hence each column of \(M\) can be represented as a linear combination of columns of \(G_{\alpha}\).

**Lemma 1.** \(\det(G_{\alpha}) > 0\).

**Proof.** Since \(G'_{\alpha}\) can be obtained by multiplying each entry in the first and last column of \(G_{\alpha}\) by \((1 + \alpha)\) and entries in all other columns by \(\frac{1 + \alpha}{1 - \alpha}\), \(\det G'_{\alpha} = (1 + \alpha)^2 \frac{1 + \alpha}{1 - \alpha} n^{-2} \det G_{\alpha}\). Hence, we only need to prove that \(\det G_{\alpha} > 0\). We prove this by induction on \(n\). For \(n = 2\), we explicit calculation yields \(G'_{\alpha} = (1 - \alpha^2)\). For the general case, perform the column transformation \(C_1 \longleftarrow C_1 - \alpha C_2\) on \(G'_{\alpha}\). Expanding on the first column gives us \(\det G'_{\alpha} = (1 - \alpha^2) \det G'_{\alpha-1}\). Hence, by induction, \(\det G'_{\alpha} = (1 - \alpha^2)^{n-1}\.

We need to show that each column of \(M\) is actually a convex combination of columns of \(G\). We can write \(M = G_{\alpha} \cdot T\) for some matrix \(T\). Hence, \(T = G_{\alpha}^{-1} \cdot M\). Note that \(G_{\alpha}\) and \(M\) are both generalized stochastic matrices. Since the set of all non-singular generalized stochastic matrices forms a group [19], \(G_{\alpha}\) is a generalized stochastic matrix. And since generalized stochastic matrices are closed under multiplication, \(T\) is also a generalized stochastic matrix and is uniquely defined. All we need to prove is that all entries in \(T\) are non-negative. We shall use Cramer’s Rule to calculate the entries of \(T\) and complete the proof.

Given a \(n \times n\) matrix \(G\) and a vector \(x = (x_1, \ldots, x_n)^t\), define \(G(i, x)\) as the matrix where the \(i^{th}\) column of \(G\) has been replaced by \(x\).

Let \(t_j\) be the \(j^{th}\) column of \(T\). \(t_{i,j}\) denotes the \(i, j\) entry in \(T\). Observe that, \(\det \frac{G_{\alpha}}{G_{\alpha-1}} (i, m_j)\). By Cramer’s Rule, we get that \(t_{i,j} = \frac{\det G_{\alpha}(i, m_j)}{\det G_{\alpha}(i, m_j)}\). To calculate this, we shall explicitly calculate the value of \(\det G_{\alpha}(i, m_j)\).

**Lemma 2.** Given \(G_{\alpha}\) and a vector \(x = (x_1, \ldots, x_n)^t\):

1. \(\det G_{\alpha}(1, x) > 0\ if\ x_1 > \alpha x_2\)
2. \(\det G_{\alpha}(n, x) > 0\ if\ x_n > \alpha x_{n-1}\)
3. \(\det G_{\alpha}(i, x) > 0\ if\ and\ only\ if\ (x_3 - \alpha x_1) \geq \alpha (x_3 - \alpha x_2) : For\ 2 \leq i \leq n - 1\)
Hence, when $M$ satisfies the condition that for every three consecutive entries $x_1, x_2, x_3$ in any column $(x_2 - \alpha x_1) \geq \alpha(x_3 - \alpha x_2)$, then $s_{i,j} \geq 0$ for all $i, j$. This proves that $M$ can be derived from the geometric mechanism.

To prove the converse, suppose that there is a column $c$ and row $i$ of $M$ such that $((1+\alpha^2)m_{i,j} - \alpha(m_{i-1,j} + m_{i+1,j})) < 0$, then $s_{i,c} = \det G(i,m_c)/\det G < 0$. This says that $M$ cannot be derived from $G$. This completes the proof of Theorem 2.

We now prove Lemma 2, using similar column transformations as we used in Lemma 1 to calculate $\det G_{n,\alpha}(i, x)$ for an arbitrary vector $x$.

**Lemma 2.** Given $G_{n,\alpha}$ and a vector $x = (x_1, \ldots, x_n)^t$:

1. $\det G_{n,\alpha}(1, x) > 0$ iff $x_1 > \alpha x_2$
2. $\det G_{n,\alpha}(n, x) > 0$ iff $x_n > \alpha x_{n-1}$
3. $\det G_{n,\alpha}(i, x) > 0$ if and only if $(x_2 - \alpha x_1) \geq \alpha(x_3 - \alpha x_2)$: For $2 \leq i \leq n - 1$

**Proof.** We will prove the above properties for $G'_{n,\alpha}$. Since, $G'_{n,\alpha}$ is obtained from $G_{n,\alpha}$ by multiplying columns with positive reals, the properties above will continue to hold for $G_{n,\alpha}$. We divide the proof into cases depending on the value of $i$:

1. $i = 1$ : We repeatedly do the column transformation $C_\alpha \leftarrow C_\alpha - \alpha C_{n-1}$ to get that $\det G'_{n,\alpha}(1, x) = (1 - \alpha^2)^{n-2} x_1^\alpha x_2^1 = (1 - \alpha^2)^{n-2} (x_1 - \alpha x_2)$. Hence, $\det G'_{n,\alpha}(1, x) > 0 \iff (x_1 > \alpha x_2)$.
2. $i = n$ : We can do the same column transformations to get that $\det G'_{n,\alpha}(n, x) = (1 - \alpha^2)^{n-2} x_{n-1}^\alpha x_n = (1 - \alpha^2)^{n-2} (x_n - \alpha x_{n-1})$. Hence, $\det G'_{n,\alpha}(n, x) > 0 \iff (x_n > \alpha x_{n-1})$.
3. $2 \leq i \leq n - 1$ : Similarly, for the general case we get that $\det G'_{n,\alpha}(i, x) = (1 - \alpha^2)^{n-3} x_{i-1}^\alpha x_i^\alpha = (1 - \alpha^2)^{n-3} (1 + \alpha^2)x_i - \alpha(x_{i-1} + x_{i+1}))$. Hence, $\det G'_{n,\alpha}(i, x) > 0 \iff (x_2 - \alpha x_1) \geq \alpha(x_3 - \alpha x_2)$.

4. **APPLICATIONS OF THE CHARACTERIZATION**

We show two applications of the characterization result of Theorem 2. The first one gives us a way to simultaneously release data to consumers at different levels of privacy. As a second application we show how to obtain a optimal mechanism for an information consumer without knowing its parameters.

4.1 **Information-Consumers at Different Privacy Levels**

Suppose we want to release the answer of the query to different information consumers. We represent the level of privacy of a consumer $c$ by the privacy parameter $\alpha_c$. Given true result $r$, we will release $r_c$ to consumer $c$ such that the mechanism is $\alpha_c$-differentially private. We expect that consumers at different levels of privacy do not share query results with each other which is enforced via, say, non-disclosure agreements. Even when they do share data, we want our mechanism to be collusion-resistant and not leak privacy—the colluding group should not get any more information about the database than the consumer with access to the least private result i.e., the one with the smallest $\alpha$.

We now describe a mechanism that achieves this. The next lemma gives us a way to “add” more privacy to an existing geometric mechanism.

**Lemma 3.** For two privacy parameters $\alpha \leq \beta$, the geometric mechanism $G_{n,\beta}$ can be derived from the mechanism $G_{n,\alpha}$ i.e., there exists a stochastic matrix $T_{\alpha,\beta}$ such that $G_{n,\beta} = G_{n,\alpha} \cdot T_{\alpha,\beta}$.

**Proof.** Theorem 2 states that $G_{n,\beta}$ can be derived from $G_{n,\alpha}$ and only if for every three consecutive entries $x_1, x_2, x_3$ in any column of $G_{n,\beta}$, $(x_2 - \alpha x_1) \geq \alpha(x_3 - \alpha x_2)$. We check this condition for each of the three forms that consecutive entries in each row of $G_{\beta,n}$ can have:

1. $(\beta^i, \beta^{i+1}, \beta^{i+2}) : (1 + \alpha^2)\beta^{i+1} - \alpha(\beta^i + \beta^{i+2}) = \beta^i(\beta + \alpha^2\beta - \alpha - \alpha\beta^2) = \beta^i\beta(\beta - \alpha)(1 - \alpha\beta) > 0$.
2. $(\beta, 1, \beta) : (1 + \alpha^2)1 - \alpha(\beta + \beta) = 1 + \alpha^2 - 2\alpha > (1 - \alpha)^2 > 0$.
3. $(\beta^{i+2}, \beta^{i+1}, \beta^i) : (1 + \alpha^2)\beta^{i+1} - \alpha(\beta^i + \beta^{i+2}) = \beta^i(\beta + \alpha^2\beta - \alpha - \alpha\beta^2) = \beta^i(\beta - \alpha)(1 - \alpha\beta) > 0$.

This shows that $T_{\alpha,\beta} = G_{n,\alpha}^{-1} \cdot G_{n,\beta}$ is a stochastic matrix.
Algorithm 1: Releasing Query Result to Consumers at Multiple Levels of Trust.

Input: True Query Result $r$, $k$ privacy levels given by parameters $a_1 < a_2 < \ldots < a_k$.
Output: Query Results $r_1, r_2, \ldots, r_k$ to be released.

Define $T_i = G_{a_i,n}$.

for $1 \leq i \leq k$ do

Compute post-processing matrix $T_{i+1}$ such that $G_{a_{i+1},n} = G_{a_i,n} \cdot T_{i+1}$.

end

By Lemma 3, each $T_i$ is a stochastic matrix. Hence, we can think of $T_i$ as a mechanism – Given any input $k$ we sample from the probability distribution given by the $k^{th}$ row of $T_i$ which we represent by $T_i(k)$.

Let $r_0 = r$.

for $1 \leq i \leq k$ do

$r_i = T_i(r_{i-1})$ is obtained by treating $r_{i-1}$ as the true query output and applying mechanism $T_i$ to it.

end

Release the query results $r_1, r_2, \ldots, r_k$ to consumers at privacy levels $a_1, \ldots, a_k$.

The release mechanism is given in Algorithm 1. We conclude the section by proving that Algorithm 1 is collusion-resistant.

Lemma 4. Any subset $C = \{c_1 < \cdots < c_l\} \subseteq \{1, \ldots, k\}$ of colluding information consumers who have access to query results $R(C) = \{r_{c_1}, \ldots, r_{c_l}\}$, released at privacy levels $a_{c_1}, \ldots, a_{c_l}$ respectively, can only reconstruct as much information about the database $d$ by combining their results as $c_1$ can working alone.

Proof. The matrix $G_{n,a_1}$ and post-processing matrices $T_{a_{c_1}, a_{c_1+1}}$ can be calculated by anyone. Hence, given the random coin tosses made by the algorithm, Lemma 3 shows that $r_{c_j}$ can be obtained from $r_{c_j}$ for $c_j > c_l$. Given $r_{c_1}$, having access to $R(C)$ can at most reveal information about these coin tosses that Algorithm 1 made. Since, these coin tosses do not depend on the database, any information about the database that is reconstructed from $R(C)$ can also be reconstructed by consumer $c_1$ (who has access to result $r_{c_1}$) alone.

4.2 Universal Utility Maximizing Mechanisms

We now prove that if we deploy the geometric mechanism (Definition 4), then the interaction of every information consumer will yield a mechanism that is optimal for that consumer. Since, the geometric mechanism is not dependent on any information consumer’s loss-function or side information, it is simultaneously optimal for all of them.

Our result proves that all optimal mechanisms can be derived from the geometric mechanism. However, there do exist differentially private mechanisms (which are not optimal for any information consumer) that cannot be derived from the geometric mechanism. We give an example of such a mechanism in Appendix B.

The first part of the proof shows that every two adjacent rows of every optimal mechanism must satisfy certain condition; if it does not, we can perturb the mechanism in a way to yield a differentially private mechanism with strictly better utility. The second part of the proof leverages this lemma and the characterization from Theorem 2 to complete the proof of Theorem 1.

Lemma 5. For every monotone loss-function $L(l,S) = \max_{r \in S} \sum_{i \in N} l((i,r) \cdot x_{i,r})$, there exists an optimal mechanism $x$ such that for every two adjacent rows $i, i+1$ of this mechanism, there exist column indices $c_1$ and $c_2$ such that:

1. $\forall j \in 1\ldots c_1 : \alpha x_{i,j} = x_{i+1,j}$
2. $\forall j \in c_2\ldots n : \alpha x_{i_j} = x_{i+1,j}$
3. Either $c_2 = c_1 + 1$ or $c_2 = c_1 + 2$.

Proof. We define the function $L' : M \rightarrow R$ given by $L'(x) = \sum_{r \in N} \sum_{l \in L} x_{i,r} \cdot |i - r|$. Consider the total order $\succ$ on $R^2$ given by $(a,b) \succ (c,d) \iff \{(a > c) \text{ or } (a = c \text{ and } b > d)\}$. Let $x$ be an optimal mechanism for the loss-function $(L, L')$ according to the order defined above. The idea here is that there are a few mechanisms that optimize $L$ and using $L'$ we isolate the ones with the property that we want. We prove by contradiction that $x$ satisfies the constraints given above.

Assume otherwise. Then there exist rows $i, i+1$ and columns $j, k; k > j$ such that $\alpha x_{ij} < x_{i+1,j}$ and $\alpha x_{i+1,k} < x_{i,k}$. We shall construct a differentially private mechanism $y$ for which $(L_y, L'_y)$ is strictly smaller than $(L_x, L'_x)$ which is a contradiction since we assumed that $x$ minimized $(L, L')$.

We divide the proof into two cases: $i \leq (j+k)/2$ and $i > (j+k)/2$. Consider the case $i \leq (j+k)/2$ first. For $i' \in \{1\ldots i\}$ set $y_{i',j} = x_{i',j} + \delta x_{i',k}$ and $y_{i',k} = (1 - \delta)x_{i',k}$. For all other values set $y_{i,m} = x_{i,m}$. We first show that $y$ is a differentially private mechanism. Let the set of changed elements $C = \{y_{i,m} : m \in \{j,k\} \text{ and } i \leq i'\}$. The set of unchanged elements $U$ is all the remaining $y_{i,m}$. All privacy constraints involving elements only from $U$ are satisfied since they were satisfied in $M$. The privacy constraints involving only elements in $C$ continue to hold since they are the same linear combinations of corresponding elements from $M$. We only need to check that the privacy constraints are satisfied when one element is from $C$ and another from $U$. But this only happens for $y_{i,j}, y_{i+1,j}$ and $y_{i,k}, y_{i+1,k}$. By assumption, $\alpha x_{ij} < x_{i+1,j}$ and $\alpha x_{i+1,k} < x_{i,k}$ can be chosen a small enough $\delta$ such that $\alpha y_{ij} = \alpha x_{ij} + \delta x_{i,k} < x_{i+1,j} = y_{i+1,j}$ and $\alpha y_{ik} = \alpha (1 - \delta)x_{i,k} < x_{i+1,k} = y_{i+1,k}$. Also, for $m \in \{j,k\}$, $y_{i,m} = x_{i,m} > x_{i+1,m} = y_{i+1,m}$. This proves that $y$ satisfies differential privacy.
The total loss \( L = \max_{r \in S} \sum_{i \in N} l(i, r) \cdot x_{i,r} \) and since so \( L_x \geq L_y \). Also \( \sum_{i} \sum_{r} x_{i,r} \cdot |i - r| > \sum_{i} \sum_{r} y_{i,r} \cdot |i - r| \). This means that \((L_x, L'_x) > (L_y, L'_y)\). But \( x \) was an optimal mechanism with respect to \( \geq \). This gives us a contradiction.

The proof for the case \( i > (j + k)/2 \) is similar. For \( i \geq i' \) set \( y_{i',k} \leftarrow x_{i',k} + \delta x_{i',j} \) and \( y_{i,j} \leftarrow (1 - \delta) x_{i,j} \). The same arguments as above now hold for this definition of \( \delta \) as well.

We are now ready to prove Theorem 1. We state it again for convenience.

**Theorem 1.** Consider a database \( d \), count query \( q \), \( k \) consumers and privacy levels \( \alpha_1 < \ldots < \alpha_k \). There exists a mechanism \( M \) that constructs \( k \) results \( r_1 \ldots r_k \), and releases result \( r_i \) to the \( i \)th information consumer, such that:

1. (Collusion-Resistance) Mechanism \( M \) is \( \alpha_i \)-differentially private for any set \( I \) of colluding information consumers who combine their results. Here, \( C \subseteq \{1 \ldots k\} \) and \( i' = \min \{ j : j \in C \} \).

2. (Simultaneous Utility Maximization) Suppose that the \( i \)th consumer is rational and interacts optimally with the mechanism (as described in Section 2.4.1), then its utility is equal to that of the differentially private mechanism tailored specifically for it (the mechanism from Section 2.5).

Proof. Algorithm 1 is used to deploy geometric mechanism at different levels of privacy. Lemma 3 shows that it is always possible to deploy geometric mechanism this way. This proves that the deployed mechanisms are differentially private. Lemma 4 proves that the release is \( \alpha_{i,j} \)-differentially private even for any set \( C \) of colluding consumers, where \( i' = \min \{ j : j \in C \} \). This completes the proof of part 1.

To prove part 2, we concentrate on a single trust level with privacy parameter \( \alpha \). We prove the result by contradiction. Assume there is an information consumer \( c \) with loss-function \( l \) and side-information \( S \), whose interaction with \( G_{n,\alpha} \) does not optimize its loss. Let \( M \) be an optimal differentially private mechanism for \( c \) that satisfies Lemma 5. Since, \( c \) cannot optimize its loss by interacting with \( G_{n,\alpha} \), \( M \) cannot be derived from the geometric mechanism. We prove that this implies that \( M \) is infeasible which is a contradiction.

We know from Theorem 2 that there exists a column \( j \) of \( M \) and rows \( i, i+1, i+2 \), such that the three entries \( x_{i,j}, x_{i+1,j}, x_{i+2,j} \) satisfy

\[
(1 + \alpha^2) x_{i+1,j} - \alpha (x_{i,j} + x_{i+2,j}) < 0.
\]

Recall the pattern of every pair of adjacent rows of \( M \) from Lemma 5. Let \( k \) be the unique column that satisfies \( \alpha x_{i,k} < x_{i+1,k} \) and \( \alpha x_{i+1,k} < x_{i+2,k} \), or if there is no such column, let it be the last column such that \( \alpha x_{i,j} = x_{i+1,j} \). Let \( a = \sum_{l<k} x_{i,l}, b = x_{i,k}, b' = x_{i+1,k}, b'' = x_{i+2,k} \) and \( c = \sum_{l>k} x_{i,l} \). Rewrite Equation (2) to get:

\[
0 \leq x_{i+1,j} - \alpha x_{i+2,j} < \alpha (x_{i,j} - \alpha x_{i+1,j}) \implies x_{i,j} > \alpha x_{i+1,j}. \quad \text{Thus, by Lemma 5, } k \geq j. \quad \text{We now claim that:}
\]

\[
(1 + \alpha^2) b' - \alpha (b + b'') < 0
\]

This is true from Equation (2) if \( k = j \). Otherwise rewrite Equation (2) to get

\[
0 \leq x_{i+1,j} - \alpha x_{i+2,j} < \alpha (x_{i+2,j} - \alpha x_{i+1,j}) \implies x_{i+2,j} > \alpha x_{i+1,j}. \quad \text{Thus, by Lemma 5, it must be that } \alpha \cdot b' = b. \quad \text{Further, by privacy } b \geq \alpha b' \text{ and so, } b > \alpha^2 b'. \quad \text{This proves the claim.}
\]

Because \( M \) is a generalized stochastic matrix, \( \sum_{i} x_{i,l} = \sum_{i} x_{i+1,l} = 1. \) Thus, \( a + b + c = 1 \) and \( \alpha \cdot a + b' + c/a = 1. \) Using these equations, we have:

\[
a = \frac{1 - b - \alpha + b' \alpha}{1 - \alpha^2} \quad \text{and} \quad c = \frac{\alpha - \alpha^2 + b \alpha^2 - b' \alpha}{1 - \alpha^2}
\]

We now prove that \( M \) is not feasible.

\[
\sum_{l} x_{i+2,l} \geq a^2 \cdot a + b''^2 + c/\alpha^2 = \frac{\alpha^3 - b \alpha^3 - b' \alpha^4 + b'' \alpha^4 - b'' \alpha^3}{\alpha (1 - \alpha^2)} + \frac{1 - \alpha + \alpha^2}{\alpha} + \frac{(b + b'')(\alpha - b' (1 + \alpha^2))}{\alpha} > 1
\]

The first step is from Equation (2) and Lemma 5, the second is by Equation (4), the third is by rearranging and the fourth holds because the first summand is always at least 1 and the second is strictly positive by Equation (3).

**5. CONCLUSION**

We give a minimax model of utility for information consumers that is prescribed by decision theory. We show that for any particular count query, the geometric mechanism is simultaneously optimal for all consumers, assuming that consumers interact rationally with the output of the mechanism. This is particularly useful in publishing aggregate statistics, like the number of flu infections in a given region, to a wide unknown audience, say on the Internet.

An open question is to investigate whether similar guarantees are possible for multiple queries and other types of queries.

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[3] C. Dwork. Differential privacy. In *Proceedings of the 33rd Annual International Colloquium on Automata,*
For every minimax information consumer with loss-function \( l \), Fix a database size \( n \).

While natural mechanisms (such as the Laplace mechanism and side information \( S \subset D \) from \([5]\)) are usually oblivious, we now justify this restriction from first principles. Specifically, we show that for every information consumer with a loss-function over databases and side information over query results, there exists a oblivious loss-function and side-information, such that the optimal utility with the oblivious loss-function is no more than the optimal utility with the non-oblivious loss-function.

Consider a non-oblivious mechanism \( x \). For the minimax information consumer with loss-function \( l \) over databases and side information \( S \subset \{0, 1, \ldots, n\} \), the utility of this mechanism is given by

\[
\max_{d \in S \subseteq D^n} \sum_{r \in N} x_{d,r} \cdot l(f(d), r)
\]

The following lemma proves that obliviousness is without loss of generality i.e. there always exists an oblivious mechanism whose loss is lower than or equal to the loss of the best non-oblivious mechanism.

\textbf{Lemma 6.} Fix a database size \( n \geq 1 \) and privacy level \( \alpha \). For every minimax information consumer with loss-function \( l \) and side information \( S \subset \{0, 1, \ldots, n\} \), there is an \( \alpha \)-differentially private mechanism that minimizes the objective function (5) and is also oblivious.
Proof. We shall now construct a differentially privacy mechanism \( x' \) that is oblivious and whose loss is not greater than the loss of \( x \). This will prove our assertion.

We construct a partition \( E \) of all the databases, according to the query output. All databases that have the same query output belong to the same subset of the partition. For a database \( d \), let \( E(d) = \{ d' : q(d) = q(d') \} \). For \( r \in \mathbb{N} \) and \( d \in D^n \), define \( x_{E(d),r} = \text{avg}_{d' \in E(d)} x_{d',r} \). It is clear that \( x' \) is an oblivious mechanism.

First we show that \( x' \) is \( \alpha \)-differentially private. Fix two databases \( d_1, d_2 \in D^n \) such that \( d_1 \) and \( d_2 \) differ in exactly one row; We need to show that \( \alpha x_{d_1,r} \leq x_{d_2,r} \). Assume \( f(d_1) \neq f(d_2) \), otherwise the proof is trivial.

For any database of \( E(d_1) \), we can generate all its neighbors (databases that differ in exactly one row) in \( E(d_2) \) by enumerating all the ways in which we can change the query result by exactly 1. For instance when \( f(d_1) = f(d_2) + 1 \), pick one of the \( n - f(d_1) \) rows that satisfy the predicate in \( d_1 \) and change its value to one of those that violates the predicate. This is identical for all databases of \( E(d_1) \), and so for all \( d \in E(d_1), \) the number of neighbors of \( d \) that belong to the set \( E(d_2) \) is the same (does not vary with \( d \)). Similarly, for all \( d \in E(d_2) \), the number of neighbors of \( d \) that belong to the set \( E(d_1) \) is the same.

Consider the following set of inequalities that hold because \( x \) is \( \alpha \)-differentially private: \( d \in E(d_1), d' \in E(d_2), \) where \( d_1 \) and \( d_2 \) are neighbors, \( \alpha x_{d,r} \leq x_{d',r} \). By the argument in the above paragraph, all the databases in \( E(d_1) \) appear equally frequently in the left-hand-side of the above inequality and all the databases in \( E(d_2) \) equally frequently in the right-hand-side. Summing the inequalities and recalling the definition of \( x' \) completes the proof of privacy.

Now we show that \( x' \) does not incur more loss than \( x \). The loss for \( x' \) is given by \( \max_{d \in S \subset D^n} \sum_{r \in \mathbb{N}} x_{E(d),r} \cdot l(f(d),r) \). Suppose the worst loss for \( x' \) occurs for the partition \( E(d_1) \).

\[
L(x') = \sum_{r \in \mathbb{N}} x'_{E(d_1),r} \cdot l(f(d_1),r) = \sum_{r \in \mathbb{N}} \text{avg}_{d \in E(d_1)} x_{d,r} \cdot l(f(d),r) \leq \max_{d \in E(d_1)} \sum_{r \in \mathbb{N}} x_{d,r} \cdot l(f(d),r) \leq \max_{d \in S \subset D^n} \sum_{r \in \mathbb{N}} x_{d,r} \cdot l(f(d),r) = L(x).
\]

This completes the proof.

\[\square\]

**B. A DIFFERENTIALLY PRIVATE MECHANISM THAT IS NOT DERIVABLE FROM THE GEOMETRIC MECHANISM**

Consider the mechanism \( M \) given by the following matrix. \( M(i,j) \) gives the probability of returning \( j \) when the true query result is \( i \). We can verify that \( M \) is \( \frac{1}{2} \)-differentially private.

\[
M = \begin{bmatrix}
1/9 & 2/9 & 4/9 & 2/9 \\
2/9 & 1/9 & 2/9 & 4/9 \\
4/9 & 2/9 & 1/9 & 2/9 \\
13/18 & 1/9 & 1/18 & 1/9
\end{bmatrix}
\]

We claim that \( M \) cannot be derived from the geometric mechanism. We can explicitly calculate \( G_{3,\frac{1}{2}}^{-1} \cdot M \) to see that \( M \) is not derivable from the geometric. Instead we shall use the characterization from Theorem 2. If we look at elements \( M(0,1), M(1,1), M(2,1), \) then \( (1 + \alpha^2)M(1,1) - \alpha(M(0,1) + M(2,1)) = 1.25 \times \frac{1}{9} - \frac{1}{2} \times (\frac{2}{9} + \frac{2}{9}) = -\frac{0.75}{9} \). This proves that \( M \) cannot be derived from \( G_{3,\frac{1}{2}} \).