Research Article

Strong Convergence for Hybrid S-Iteration Scheme

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Abstract: We establish a strong convergence for the hybrid S-iterative scheme associated with nonexpansive and Lipschitz strongly pseudo-contractive mappings in real Banach spaces.

1. Introduction and Preliminaries

Let $E$ be a real Banach space and let $K$ be a nonempty convex subset of $E$. Let $J$ denote the normalized duality mapping from $E$ to $2^E^*$ defined by

$$J(x) = \{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \|f^*\| = \|x\| \}, \quad \forall x, y \in E,$$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We will denote the single-valued duality map by $j$.

Let $T : K \to K$ be a mapping.

**Definition 1.** The mapping $T$ is said to be Lipschitzian if there exists a constant $L > 1$ such that

$$\|Tx - Ty\| \leq L \|x - y\|, \quad \forall x, y \in K.$$  \hspace{1cm} (2)

**Definition 2.** The mapping $T$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K.$$  \hspace{1cm} (3)

**Definition 3.** The mapping $T$ is said to be pseudocontractive if for all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2.$$  \hspace{1cm} (4)

**Definition 4.** The mapping $T$ is said to be strongly pseudocontractive if for all $x, y \in K$, there exists $k \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2.$$  \hspace{1cm} (5)

Let $K$ be a nonempty convex subset $C$ of a normed space $E$.

(a) The sequence $\{x_n\}$ defined by, for arbitrary $x_1 \in K$,

\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 1,
\end{align*}

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, is known as the Ishikawa iteration process [1].

If $\beta_n = 0$ for $n \geq 1$, then the Ishikawa iteration process becomes the Mann iteration process [2].

(b) The sequence $\{x_n\}$ defined by, for arbitrary $x_1 \in K$,

\begin{align*}
x_{n+1} &= Ty_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 1,
\end{align*}

where $\{\beta_n\}$ is a sequence in $[0, 1]$, is known as the S-iteration process [3, 4].

In the last few years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz strongly pseudocontractive mappings using the Ishikawa iteration scheme (see, e.g., [1]). Results which had
been known only in Hilbert spaces and only for Lipschitz mappings have been extended to more general Banach spaces (see, e.g., [5–10] and the references cited therein).

In 1974, Ishikawa [1] proved the following result.

**Theorem 5.** Let $K$ be a compact convex subset of a Hilbert space $H$ and let $T : K \to K$ be a Lipschitzian pseudocontractive mapping. For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by

$$
\begin{align*}
& x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Ty_n, \\
& y_n = (1 - \beta_n) x_n + \beta_n Tx_n, \\
& n \geq 1,
\end{align*}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying

(i) $0 \leq \alpha_n \leq 1$

(ii) $\lim_{n \to \infty} \beta_n = 0$

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly at a fixed point of $T$.

In [6], Chidume extended the results of Schu [9] from Hilbert spaces to the much more general class of real Banach spaces and approximated the fixed points of (strongly) pseudocontractive mappings.

In [11], Zhou and Jia gave the more general answer of the question raised by Chidume [5] and proved the following.

If $X$ is a real Banach space with a uniformly convex dual $X^*$, $K$ is a nonempty bounded closed convex subset of $X$, and $T : K \to K$ is a continuous strongly pseudocontractive mapping, then the Ishikawa iteration scheme converges strongly at the unique fixed point of $T$.

In this paper, we establish the strong convergence for the hybrid $S$-iterative scheme associated with nonexpansive and Lipschitz strongly pseudocontractive mappings in real Banach spaces. We also improve the result of Zhou and Jia [11].

### 2. Main Results

We will need the following lemmas.

**Lemma 6** (see [12]). Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, one has

$$
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, J(x + y) \rangle,
$$

\forall j \in J(x + y).

**Lemma 7** (see [10]). Let $\{\rho_n\}$ be nonnegative sequence satisfying

$$
\rho_{n+1} \leq (1 - \theta_n) \rho_n + \omega_n,
$$

where $\theta_n \in [0, 1], \sum_{n=1}^{\infty} \theta_n = \infty$, and $\omega_n = o(\theta_n)$. Then

$$
\lim_{n \to \infty} \rho_n = 0.
$$

The following is our main result.

**Theorem 8.** Let $K$ be a nonempty closed convex subset of a real Banach space $E$, let $S : K \to K$ be nonexpansive, and let $T : K \to K$ be Lipschitz strongly pseudocontractive mappings such that $p \in F(S) \cap F(T) = \{x \in K : Sx = Tx = x\}$ and

$$
\|x - Sy\| \leq \|Sx - Sy\|, \quad \forall x, y \in K,
$$

$$
\|x - Ty\| \leq \|Tx - Ty\|, \quad \forall x, y \in K.
$$

Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

(iv) $\sum_{n=1}^{\infty} \beta_n = \infty$,

(v) $\lim_{n \to \infty} \beta_n = 0$.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence iteratively defined by

$$
\begin{align*}
& x_{n+1} = Sy_n, \\
& y_n = (1 - \beta_n) x_n + \beta_n Tx_n, \\
& n \geq 1.
\end{align*}
$$

Then the sequence $\{x_n\}$ converges strongly at the common fixed point $p$ of $S$ and $T$.

**Proof.** For strongly pseudocontractive mappings, the existence of a fixed point follows from Delmling [13]. It is shown in [11] that the set of fixed points for strongly pseudocontractions is a singleton.

By (v), since $\lim_{n \to \infty} \beta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$
\beta_n \leq \min \left\{ \frac{1}{4k}, \frac{1 - k}{(1 + L)(1 + 3L)} \right\},
$$

where $k < 1/2$. Consider

$$
\|x_{n+1} - p\|^2 = \langle x_{n+1} - p, j(x_{n+1} - p) \rangle
$$

$$
= \langle Sy_n - p, j(x_{n+1} - p) \rangle
$$

$$
= \langle Tx_{n+1} - p, j(x_{n+1} - p) \rangle
$$

$$
+ \langle Sy_n - Tx_{n+1}, j(x_{n+1} - p) \rangle
$$

$$
\leq k \|x_{n+1} - p\|^2 + \|Sy_n - Tx_{n+1}\| \|x_{n+1} - p\|,
$$

which implies that

$$
\|x_{n+1} - p\| \leq \frac{1}{1 - k} \|Sy_n - Tx_{n+1}\|,
$$

where

$$
\|Sy_n - Tx_{n+1}\| \leq \|Sy_n - Ty_n\| + \|Ty_n - Tx_{n+1}\|
$$

$$
\leq \|x_n - Sy_n\| + \|x_n - Ty_n\| + \|Ty_n - Tx_{n+1}\|
$$

$$
\leq \|x_n - Sy_n\| + \|Tx_n - Ty_n\| + \|Ty_n - Tx_{n+1}\|
$$

$$
\leq \|x_n - Sy_n\| + L \left( \|x_n - y_n\| + \|y_n - x_{n+1}\| \right),
$$

$$
\|y_{n+1} - x_{n+1}\| \leq \|y_{n+1} - x_n\| + \|x_n - x_{n+1}\| = \|y_{n+1} - x_n\| + \|x_n - Sy_n\| \leq \|y_{n+1} - x_n\| + \|Sx_n - Sy_n\|,
$$

$$
\lim_{n \to \infty} \beta_n = 0.
$$

The following is our main result.
and consequently from (16), we obtain
\[
\|S_{y_n} - T_{x_{n+1}}\| \leq (1 + L) \|S_{x_n} - S_{y_n}\| + 2L \|x_n - y_n\|
\]
\[
\leq (1 + 3L) \|x_n - y_n\|
\]
\[
= (1 + 3L) \beta_n \|x_n - T_{x_n}\|
\]
\[
\leq (1 + L) (1 + 3L) \beta_n \|x_n - p\|.
\]
Substituting (18) in (15) and using (13), we get
\[
\|x_{n+1} - p\| \leq \frac{(1 + L)(1 + 3L)}{1 - k} \beta_n \|x_n - p\|
\]
\[
\leq \|x_n - p\|. \tag{19}
\]
So, from the above discussion, we can conclude that the sequence \(\{x_n - p\}\) is bounded. Since \(T\) is Lipschitzian, so \(\{Tx_n - p\}\) is also bounded. Let \(M_1 = \sup_{n \geq 1} \|x_n - p\| + \sup_{n \geq 1} \|Tx_n - p\|\). Also by (ii), we have
\[
\|x_n - y_n\| = \beta_n \|x_n - T_{x_n}\|
\]
\[
\leq M_1 \beta_n \rightarrow 0 \tag{20}
\]
as \(n \rightarrow \infty\), implying that \(\{x_n - y_n\}\) is bounded, so let \(M_2 = \sup_{n \geq 1} \|x_n - y_n\| + M_1\). Further,
\[
\|y_n - p\| \leq \|y_n - x_n\| + \|x_n - p\|
\]
\[
\leq M_2, \tag{21}
\]
which implies that \(\{y_n - p\}\) is bounded. Therefore, \(\{Ty_n - p\}\) is also bounded.

Set
\[
M_3 = \sup_{n \geq 1} \|y_n - p\| + \sup_{n \geq 1} \|Ty_n - p\|.
\]
Denote \(M = M_1 + M_2 + M_3\). Obviously, \(M < \infty\).

Now from (12) for all \(n \geq 1\), we obtain
\[
\|x_{n+1} - p\|^2 = \|S_{y_n} - T_{x_n} - p\|^2 \leq \|y_n - p\|^2, \tag{23}
\]
and by Lemma 6, we get
\[
\|y_n - p\|^2 = \|(1 - \beta_n) x_n + \beta_n T_{x_n} - p\|^2
\]
\[
= \|(1 - \beta_n) (x_n - p) + \beta_n (T_{x_n} - p)\|^2
\]
\[
\leq (1 - \beta_n)^2 \|x_n - p\|^2 + 2\beta_n \langle T_{x_n} - p, j(y_n - p) \rangle
\]
\[
= (1 - \beta_n)^2 \|x_n - p\|^2 + 2\beta_n \langle T_{x_n} - p, j(y_n - p) \rangle
\]
\[
+ 2\beta_n \langle T_{x_n} - T_{y_n}, j(y_n - p) \rangle
\]
\[
\leq (1 - \beta_n)^2 \|x_n - p\|^2 + 2k\beta_n \|y_n - p\|^2
\]
\[
+ 2\beta_n \|T_{x_n} - T_{y_n}\| \|y_n - p\|
\]
\[
\leq (1 - \beta_n)^2 \|x_n - p\|^2 + 2k\beta_n \|y_n - p\|^2
\]
\[
+ 2ML\beta_n \|x_n - y_n\|, \tag{24}
\]
which implies that
\[
\|y_n - p\|^2 \leq \frac{(1 - \beta_n)^2}{1 - 2k\beta_n} \|x_n - p\|^2 + \frac{2ML\beta_n}{1 - 2k\beta_n} \|x_n - y_n\|
\]
\[
\leq (1 - \beta_n) \|x_n - p\|^2 + 4ML\beta_n \|x_n - y_n\|. \tag{25}
\]
because by (13), we have \((1 - \beta_n)/(1 - 2k\beta_n) \leq 1\) and \((1/(1 - 2k\beta_n)) \leq 2\). Hence, (23) gives us
\[
\|x_{n+1} - p\|^2 \leq (1 - \beta_n) \|x_n - p\|^2 + 4ML\beta_n \|x_n - y_n\|. \tag{26}
\]
For all \(n \geq 1\), put
\[
\rho_n = \|x_n - p\|, \quad \theta_n = \beta_n, \quad \omega_n = 4ML\beta_n \|x_n - y_n\|,
\]
then according to Lemma 7, we obtain from (26) that
\[
\lim_{n \to \infty} \|x_n - p\| = 0. \tag{28}
\]
This completes the proof.

\(\square\)

**Corollary 9.** Let \(K\) be a nonempty closed convex subset of a real Hilbert space \(H\), let \(S : K \rightarrow K\) be nonexpansive, and let \(T : K \rightarrow K\) be Lipschitz strongly pseudocontractive mappings such that \(p \in F(S) \cap F(T)\) and the condition (C). Let \(\{\beta_n\}\) be a sequence in \([0, 1]\) satisfying the conditions (iv) and (v).

For arbitrary \(x_1 \in K\), let \(\{x_n\}\) be a sequence iteratively defined by (12). Then the sequence \(\{x_n\}\) converges strongly at the common fixed point \(p\) of \(S\) and \(T\).

**Example 10.** As a particular case, we may choose, for instance, \(\beta_n = 1/n\).

**Remark 11.** (1) The condition (C) is not new and it is due to Liu et al. [14].

(2) We prove our results for a hybrid iteration scheme, which is simple in comparison to the previously known iteration schemes.

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