Loop Fayet-Iliopoulos terms in $T^2/\mathbb{Z}_2$ models: instability and moduli stabilization

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Abstract

We study Fayet-Iliopoulos (FI) terms of six-dimensional supersymmetric Abelian gauge theory compactified on a $T^2/\mathbb{Z}_2$ orbifold. Such orbifold compactifications can lead to localized FI-terms and instability of bulk zero modes. We study 1-loop correction to FI-terms in more general geometry than the previous works. We find induced FI-terms depend on the complex structure of the compact space. We also find the complex structure of the torus can be stabilized at a specific value corresponding to a self-consistent supersymmetric minimum of the potential by such 1-loop corrections, which is applicable to the modulus stabilization.
1 Introduction

Effective theory of the superstring includes various dimensional objects, i.e., branes. Branes are important components for particle phenomenology. Branes can break the supersymmetry (SUSY) and realize the chiral spectrum \([1,4]\). They can be a source of generations of matter fields, and flavor structure \([5,6]\). Anti-branes can induce the positive cosmological constant \([7]\). Such a brane mode behaves as a localized mode in the effective theory. Therefore it is important to investigate interactions between bulk fields and localized operators \([8,9]\).

The FI-term in supersymmetric Abelian gauge theory was introduced as a source of spontaneous SUSY breaking at first \([10]\). Later it was shown that FI-term is not only a source of a SUSY breaking, but have vast implication for theoretical particle physics. The FI-term is prohibited by local SUSY unless the gauge group is related to \(U(1)_R \) \([11,13]\) or associated with non-linear terms \([14]\). Especially in higher dimensional supersymmetric theory, it is related to anomaly \([15]\), and introduces an instability of bulk superfields \([16,17]\). (See also \([18]\).)

Even in the higher dimensional theory, the bulk FI-term is prohibited by local SUSY, but the FI-term localized at special points, i.e. orbifold fixed points can appear \([19]\). Such a FI-term is called localized FI-term. The localized FI-term is induced by quantum corrections in orbifold compactification even if the FI-term is set to zero at the tree level \([20]\). This is formally calculated by infinite sum of all KK-modes of fields which have charges of the corresponding \(U(1)\). In the trivial background without the localized FI-term, mode expansion of the bulk fields is given by plane waves. Their infinite sum converges to the Dirac delta function. Hence the localized FI-term is induced. Since it is localized, the FI-term induces a local potential for bulk fields. To cancel the FI-term, the vacuum expectation value of the auxiliary field must also be localized. It affects the wave function profile of the bulk field. For the model of five-dimensional Abelian gauge theory compactified by \(S^1/Z_2\), the localized FI-term induces localization of bulk zero modes at the fixed points, and rejects all the massive mode profiles from fixed points \([16,17]\). Similar results are obtained also for six-dimensional SUSY compactified by \(T^2/Z_2\) orbifold \([21]\). Thus it is a quite general consequence for higher dimensional SUSY compactified by orbifolds.

If the value of the localized FI-term is not zero, vacuum expectation values (VEVs) of the auxiliary fields are shifted. The massive modes can not penetrate to the fixed points in this 1-loop corrected vacuum. Hence 1-loop corrections to the FI-term are only due to the zero mode. The zero mode is localized at the fixed points, and reproduces the localized FI-term, but it is not the same as that of the infinite sum of the plane wave. Bulk contribution is not canceled by brane mode contributions in general, and the FI-term receives further corrections. Thus this background is unstable. In our previous work we investigated this instability for the \(S^1/Z_2\) compactification model \([22]\). In the present paper we investigate instability for \(T^2/Z_2\) compactification. Toroidal compactification is a more realistic compactification for phenomenology; it has concrete stringy origin \([1]\). It also can realize chiral spectrum of the Standard Model (SM). (See e.g. Refs. \([23,24]\).) The
localized FI-term on toroidal orbifold may affect the flavor structure of the SM [25]. As well as $S^1/Z_2$ compactification, loop correction of the FI-term can lead to the instability of 1-loop corrected vacuum. We find that the instability is related to the complex structure of the torus. There are some applications for moduli stabilization and extra dimensional models.

This paper is organized as follows. In section 2, we examine the localized FI-term and zero mode of bulk scalar field in six-dimensional SUSY gauge theory compactified by $T^2/Z_2$ orbifold, whose geometry is described by arbitrary value of the complex structure modulus $\tau (\in \mathbb{C})$. The localized FI-term is induced by quantum corrections, and it leads to nonzero VEV of the auxiliary field. It affects equations of motion for bulk fields and their wave function profiles. In Section 3, we focus on the untilted torus, i.e., $\tau$ is pure imaginary, and recalculate the 1-loop corrections to FI-term in the SUSY vacuum which has nonzero VEV of the auxiliary field. We see that 1-loop corrections can cause the instability of the SUSY vacuum. In Section 4, we extend the consequences in section 3 to the torus that has arbitrary value of $\tau$. We find these quantum corrections depend on $\tau$. We show the complex structure modulus must take a specific value for the cancellation between loop corrections from bulk and brane modes. In other words, modulus stabilization of the complex structure is realized. Section 5 is devoted to our conclusion. In Appendix A we study the validity of our evaluation of the FI terms. In Appendix B we show our regularization. In Appendix C we show the modular transformation of elliptic theta functions.

2 Localized FI-terms on $T^2/Z_2$ model

In this section, we evaluate the localized FI-term induced by quantum corrections in the $T^2/Z_2$ orbifold, whose geometry is described by arbitrary value of the complex structure modulus $\tau (\in \mathbb{C})$. Then, we calculate the zero mode of KK-expansion in the supersymmetric vacuum that is changed by the localized FI-term.

Before describing the multiplets that are contained in $T^2/Z_2$ model, we describe the torus $T^2$ and orbifold action of $Z_2$. We define the orthogonal coordinates of $T^2$ as $x_5, x_6$, and we denote the two-dimensional metric by $g_{ij}$:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (i, j = 5, 6).$$

(2.1)

The coordinates $(x_5, x_6)$ have two periodic boundary conditions:

$$\begin{cases} x_5 \sim x_5 + 2\pi R \\ x_5 \sim x_5 + 2\pi R \text{Re}\,\tau, \quad x_6 \sim x_6 + 2\pi R \text{Im}\,\tau \end{cases},$$

(2.2)

where we introduced complex structure modulus $\tau (\in \mathbb{C})$. We define the $Z_2$ orbifold action as

$$Z_2 : (x_5, x_6) \to (-x_5, -x_6),$$

(2.3)
Non-orthogonal coordinates

\[ x'_5 = x_5 - \frac{\text{Re} \tau}{\text{Im} \tau} x_6 \]
\[ x'_6 = \frac{1}{\text{Im} \tau} x_6 \]

Complex coordinates

\[ Rz = x'_5 + \tau x'_6 \]
\[ R\bar{z} = x'_5 + \bar{\tau} x'_6 \]

boundary conditions

\[ x'_5 \sim x'_5 + 2\pi R \]
\[ x'_6 \sim x'_6 + 2\pi R \]

Then we can define the complex coordinates \((z, \bar{z})\) as

\[ Rz \equiv x'_5 + \tau x'_6 \]
\[ R\bar{z} \equiv x'_5 + \bar{\tau} x'_6. \]

From now on, we use the notation of indices as \(M, N \in \{0, 1, 2, 3, 4, 5\}\), \(\mu, \nu \) run from 0 to 3, and \(i, j, m, n \) run from 5 to 6. Also, we use the indices with prime \(M', N', i', j'\), whose vector, tensor and derivative mean the ones in the non-orthogonal coordinates \((x'_5, x'_6)\).

We summarize the relation of the coordinates and the metrics in Table 1.

|                     | Non-orthogonal | Complex       |
|---------------------|----------------|---------------|
| coordinates         |                |               |
| \( x'_5 = x_5 - \frac{\text{Re} \tau}{\text{Im} \tau} x_6 \) | \( Rz = x'_5 + \tau x'_6 \) |               |
| \( x'_6 = \frac{1}{\text{Im} \tau} x_6 \) | \( R\bar{z} = x'_5 + \bar{\tau} x'_6 \) |               |
| boundary conditions |                |               |
| \( x'_5 \sim x'_5 + 2\pi R \) | \( z \sim z + 2\pi \) |               |
| \( x'_6 \sim x'_6 + 2\pi R \) | \( z \sim z + 2\pi \tau \) |               |
| metric              | \( g_{i'j'} = \begin{pmatrix} 1 & \text{Re} \tau \\ \text{Re} \tau & |\tau|^2 \end{pmatrix} \) | \( g_{mn} = \frac{1}{R^2} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \) |

Table 1: Coordinates and metrics on the torus

and there are four fixed points: \((0, 0), (\pi R, 0), (\pi R \text{ Re} \tau, \pi R \text{ Im} \tau) \) and \((\pi R(1+\text{Re} \tau), \pi R \text{ Im} \tau)\). Hereafter, these fixed points are denoted by \(z_1, z_2, z_3 \) and \(z_4\), respectively.

Here, we introduce non-orthogonal coordinates \((x'_5, x'_6)\) which are along the lattice vectors of the torus. In these non-orthogonal coordinates, the two periodic boundary conditions can be represented as

\[
\begin{cases}
    x'_5 \sim x'_5 + 2\pi R \\
    x'_6 \sim x'_6 + 2\pi R
\end{cases}
\]  \hspace{1cm} (2.4)

Then we can define the complex coordinates \((z, \bar{z})\) as \(Rz \equiv x'_5 + \tau x'_6\) and \(R\bar{z} \equiv x'_5 + \bar{\tau} x'_6\). From now on, we use the notation of indices as \(M, N \in \{0, 1, 2, 3, 4, 5\}\), \(\mu, \nu \) run from 0 to 3, and \(i, j, m, n \) run from 5 to 6. Also, we use the indices with prime \(M', N', i', j'\), whose vector, tensor and derivative mean the ones in the non-orthogonal coordinates \((x'_5, x'_6)\).

We summarize the relation of the coordinates and the metrics in Table 1.

We consider six-dimensional SUSY gauge theory defined below the cutoff scale \(\Lambda\). Six-dimensional SUSY \(U(1)\) gauge theory is described by four-dimensional \(N = 2\) supermultiplets: Abelian vector multiplet and hypermultiplets. In addition to the \(N = 2\) multiplets, we can introduce brane modes at the fixed points. The brane modes preserve \(N = 1\) SUSY and we assume that they consist of only chiral multiplets, i.e., there are no extra gauge fields. We introduce brane modes \(\Phi_I = (\phi_I, \psi_I)\) at each fixed point \(z_I\). The multiplets are summarized as follows:

- **bulk mode:**
  \[
  \begin{cases}
    \mathcal{N} = 2 \text{ Abelian vector multiplet} \\
    \text{hypermultiplet} = \{\text{real scalars } A_i, \text{ hyperino } \zeta\}
  \end{cases}
  \]
- **brane mode:**
  chiral multiplet = \{complex scalar \(\phi_I\), Weyl fermion \(\psi_I\) \}.

We should pay attention to the auxiliary field in \(\mathcal{N} = 2\) Abelian vector multiplet. It is decomposed into an \(\mathcal{N} = 1\) vector multiplet and a single chiral multiplet. The auxiliary field \(D\) of the \(\mathcal{N} = 1\) vector multiplet is given by a linear combination of the part of the auxiliary field \(\vec{D}\) and the field strength \(F_{56}\). We choose \(D = -D_3 + F_{56}\) basis in this paper.

The \(Z_2\) orbifold action is defined to preserve this four-dimensional \(\mathcal{N} = 1\) structure, e.g., the parity assignment to \(D_3\) is even and those to other two auxiliary fields \(D_1\) and \(D_2\) are
odd. We also introduce two complex scalar fields $\phi_+$ and $\phi_-$, which are linear combination of the real scalars of the hypermultiplet. $\phi_+$ is parity even and $\phi_-$ is parity odd.$^1$

In this paper, we discuss the localized FI-term and the SUSY vacuum in the non-orthogonal coordinates $(x'_5, x'_6)$ and the complex coordinates $(z, \bar{z})$. However, the Lagrangian and four-dimensional effective potential in their coordinates are rather complicated. Therefore we write them in the orthogonal coordinates. The bosonic Lagrangian in the orthogonal coordinates $(x_5, x_6)$ is written as follows:

$$\mathcal{L} = -\frac{1}{4} F_{MN} F^{MN} + i \Omega \Gamma^M \partial_M \Omega + \frac{1}{2} \bar{b}^2 + \sum_{\pm} (D_M \phi_\pm \partial^M \phi_\mp + g \phi_\pm q \phi_\pm D_3) + \cdots$$

$$+ \sum_{I=1}^{4} \delta(x_5 - x'_5) \delta(x_6 - x'_6) \left[ D_\mu \phi_I^\dagger D^\mu \phi_I + g \phi_I^\dagger q_I \phi_I (-D_3 + F_{56}) + \cdots \right], \quad (2.5)$$

where

$$D_M \phi_\pm = \partial_M \phi_\pm \pm ig \phi_\pm A_M.$$ 

The quantities $q$ and $q_I$ are charges of the hypermultiplet and the brane modes respectively. $g$ is the gauge coupling constant. Four-dimensional effective potential is represented as follows:

$$V_{4d} = \int dx_5 dx_6 \left[ 2g^2 |\phi^\dagger q_\phi_\phi_\phi |^2 + \sum_{\pm} (D_5 \phi_\pm + i D_6 \phi_\pm)^\dagger (D_5 \phi_\pm + i D_6 \phi_\pm) \right]$$

$$+ \frac{1}{2} (F_{56} - \xi - g(\phi^\dagger_+ q_\phi_+ - \phi^\dagger_- q_\phi_-) - g \sum_I \phi^\dagger_I q_I \phi_I \delta^{(2)}(x_5 - x'_5, x_6 - x'_6))^2$$

$$- \frac{1}{2} (D_3 - \xi - g(\phi^\dagger_+ q_\phi_+ - \phi^\dagger_- q_\phi_-) - g \sum_I \phi^\dagger_I q_I \phi_I \delta^{(2)}(x_5 - x'_5, x_6 - x'_6))^2$$

$$- \frac{1}{2} (D_1 + g \phi^\dagger_+ q_\phi_- + g \phi^\dagger_- q_\phi_+)^2 - \frac{1}{2} (D_2 + ig \phi^\dagger_+ q_\phi_- - ig \phi^\dagger_- q_\phi_+)^2 \right]. \quad (2.6)$$

Here, for the discussion of localization, we include the contributions of FI-term $\mathcal{L}_{FI} = \xi (-D_3 + F_{56})$.

From (2.6), the SUSY conditions are written by

$$D_3 = F_{56} = \xi + g(\phi^\dagger_+ q_\phi_+ - \phi^\dagger_- q_\phi_-) + g \sum_I \phi^\dagger_I q_I \phi_I \delta^{(2)}(x_5 - x'_5, x_6 - x'_6), \quad (2.7)$$

$$\phi^\dagger_+ q_\phi_- = 0, \quad D_5 \phi_\pm + i D_6 \phi_\pm = 0. \quad (2.8)$$

We study the situation where the $U(1)$ is unbroken, i.e., $\langle \phi_\pm \rangle = \langle \phi_I \rangle = 0$. Then the SUSY background solution is as follows:

$$\langle F_{56} \rangle = \xi(x_5, x_6). \quad (2.9)$$

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$^1$For precise calculation, see [21].
We also obtain the equation of motion (EOM) for the scalar fields $\phi_{\pm}$ in terms of the compact directions:

(zero mode) : $(D_5 + iD_6)\phi_{\pm} = 0$, \hspace{1cm} (2.10)

(massive mode) : $(-D_5 + iD_6)(D_5 + iD_6)\phi_{\pm} = \lambda\phi_{\pm}$. \hspace{1cm} (2.11)

The SUSY background solution and EOM of zero modes in the non-orthogonal coordinates are represented simply as follows:

(SUSY condition) : $\langle F_{5'6'} \rangle = |\text{Im}\tau|\xi(x'_5, x'_6)$, \hspace{1cm} (2.12)

(zero mode EOM) : $(\tau D_5' - D_6')\phi_{\pm}(x'_5, x'_6) = 0$. \hspace{1cm} (2.13)

By evaluating (2.12) and (2.13), we will see that, when the FI-term exists, the zero mode of bulk mode is localized on fixed point $z = z_I$, that is similar to [21].

2.1 KK-modes and 1-loop FI-term when $\xi = 0$

We calculate the FI-term induced by 1-loop corrections of the scalar fields $\phi_{\pm}$. As the first step, we use the mode expansions in the SUSY vacuum with $\xi = 0$. In the SUSY vacuum with $\xi = 0$, the EOMs (2.10) and (2.11) become

$$\partial\bar{\partial}\phi_{\pm}(z, \bar{z}) = \frac{R^2}{4}\lambda\phi_{\pm}(z, \bar{z}),$$ \hspace{1cm} (2.14)

where we represent them in the complex coordinates $(z, \bar{z})$ as $Rz \equiv x'_5 + \tau x'_6$ and $R\bar{z} \equiv x'_5 + \bar{\tau}x'_6$. The periodic boundary conditions are rewritten as

$$z \sim z + 2\pi, \quad \bar{z} \sim \bar{z} + 2\pi\tau.$$ \hspace{1cm} (2.15)

The general solutions of EOMs are given by

$$\phi_{\pm}(z, \bar{z}) = Ae^{cz - c\bar{z}},$$ \hspace{1cm} (2.16)

where $A$ is complex constant, and $c$, $c'$ are also complex constants satisfying

$$cc' = -\frac{R^2}{4}\lambda.$$ \hspace{1cm} (2.17)

By imposing the boundary conditions for these solutions, the complex constants $c$, $c'$ are quantized:

$$2\pi(c - c') = 2\pi in \quad (n \in \mathbb{Z}),$$ \hspace{1cm} (2.18)

$$2\pi(c\tau + c'\bar{\tau}) = 2\pi i\ell \quad (\ell \in \mathbb{Z}).$$ \hspace{1cm} (2.19)

Thus the solutions that satisfy the boundary conditions are represented as follows:

$$\phi_{\pm,n\ell}(z, \bar{z}) = A_{n\ell}e^{\frac{1}{2}\frac{R^2}{(\text{Im}\tau)^2}\left\{nRe\tau - \ell\right\}^2 + \left\{n\text{Im}\tau\right\}^2},$$ \hspace{1cm} (2.20)

$$\lambda = -\frac{1}{R^2(\text{Im}\tau)^2}\left\{nRe\tau - \ell\right\}^2 + \left\{n\text{Im}\tau\right\}^2.$$ \hspace{1cm} (2.21)
In the coordinates \((x_5', x_6')\), these can be more simple form as
\[
\phi_{\pm,n\ell}(x_5', x_6') = A_{n\ell} e^{i\frac{n}{R}x_5' + \frac{\ell}{R}x_6'}. \tag{2.22}
\]
Thus, the independent solutions are written by
\[
\phi_{\pm,n\ell}(x_5', x_6') = \left\{ \sin \left( \frac{n}{R}x_5' \right) \sin \left( \frac{\ell}{R}x_6' \right), \cos \left( \frac{n}{R}x_5' \right) \cos \left( \frac{\ell}{R}x_6' \right), \cos \left( \frac{n}{R}x_5' \right) \sin \left( \frac{\ell}{R}x_6' \right), \sin \left( \frac{n}{R}x_5' \right) \cos \left( \frac{\ell}{R}x_6' \right) \right\}. \tag{2.23}
\]

Now, under the action \(Z_2\) they behave as
\[
\phi_+(-x_5', -x_6') = \phi_+(x_5', x_6'), \tag{2.25}
\]
\[
\phi_-(-x_5', -x_6') = -\phi_-(x_5', x_6'). \tag{2.26}
\]
Thus, we obtain mode expansions of the bulk scalars:
\[
\phi_{+,n\ell}^1(x_5', x_6') = A_\lambda \sin \left( \frac{n}{R}x_5' \right) \sin \left( \frac{\ell}{R}x_6' \right), \tag{2.27}
\]
\[
\phi_{+,n\ell}^2(x_5', x_6') = A_\lambda \cos \left( \frac{n}{R}x_5' \right) \cos \left( \frac{\ell}{R}x_6' \right), \tag{2.28}
\]
\[
\phi_{-,n\ell}^1(x_5', x_6') = A_\lambda \cos \left( \frac{n}{R}x_5' \right) \sin \left( \frac{\ell}{R}x_6' \right), \tag{2.29}
\]
\[
\phi_{-,n\ell}^2(x_5', x_6') = A_\lambda \sin \left( \frac{n}{R}x_5' \right) \cos \left( \frac{\ell}{R}x_6' \right), \tag{2.30}
\]
where the normalization factor \(A_\lambda\) is \(1/\pi R \sqrt{\text{Im} \tau}\) for \(\lambda \neq 0\). Zero modes are constant solutions. They are
\[
\phi_{+,00} = A_0 \quad (A_0 = 1/2\pi R \sqrt{\text{Im} \tau}), \tag{2.31}
\]
\[
\phi_{-,00} = 0. \tag{2.32}
\]

1-loop diagrams contributing to the FI-term are written as the Figure \[\text{in the case of } S^1/Z_2\] \(^2\) We can evaluate the divergent part of the FI-term that is induced by 1-loop diagrams of bulk scalars:
\[
\xi_{\text{bulk}}(x_5', x_6') = \text{tr}(q) \left( \frac{\Lambda^2}{16\pi^2} + \frac{1}{4} \frac{\Lambda^2}{16\pi^2} g^{i'j'} \partial_{i'} \partial_{j'} \right) \sum_{n,l=0}^{\infty} \left\{ |\phi_{+,n\ell}^1|^2 + |\phi_{+,n\ell}^2|^2 - |\phi_{-,n\ell}^1|^2 - |\phi_{-,n\ell}^2|^2 \right\}
\]
\[
= \text{tr}(q) \left( \frac{\Lambda^2}{16\pi^2} + \frac{1}{4} \frac{\Lambda^2}{16\pi^2} g^{i'j'} \partial_{i'} \partial_{j'} \right) \frac{1}{4|\text{Im} \tau|} \sum_{f,p} \delta(x_5' - x_5^f) \delta(x_6' - x_6^p), \tag{2.33}
\]

\(^2\)The loop diagram around which the scalars \(\phi \pm\) run induces only the linear term of \(D_3\). The same contribution to the linear term of \(F_5\) arise from fermion’s loop as same as the \(\partial_5 \Sigma\) in the \(S^1/Z_2\) model unless the SUSY is broken.
where the second derivative \( g^{ij'} \partial_i \partial_j \) is originated from rewriting of the log divergent term \( \frac{1}{4} \lambda \ln \Lambda^2 \) using the EOM. In the second row, we used the Fourier expansion of the Dirac delta function:

\[
\delta(y) = \frac{1}{\pi R} + \frac{2}{\pi R} \sum_{n>0} \cos \left( \frac{2ny}{R} \right) (-\pi R < y < \pi R). \tag{2.34}
\]

Note that the factor \( 1/|\text{Im}\tau| \) is multiplied, which comes from \( \sqrt{\det g_{ij'}} = |\text{Im}\tau| \) when we normalize the wave function. The FI-term is localized at the fixed points of the orbifold. Thus, considering the contributions from the brane modes, we obtain the 1-loop induced FI-term:

\[
\begin{align*}
\xi(x'_{5}, x'_{6}) &= \xi_{\text{bulk}} + \xi_{\text{brane}} \\
&= \frac{1}{|\text{Im}\tau|} \sum_{I, f, p} \left( \xi_I + \xi'' g^{ij'} \partial_i \partial_j' \delta(x'_5 - x'_{5l}) \delta(x'_6 - x'_{6l}) \right) \delta(y), \\
&= g \Lambda^2 \left( \frac{1}{16\pi^2} \text{tr}(q) + \text{tr}(q_I) \right), \quad \xi'' = \frac{g \ln \Lambda^2}{4 \frac{1}{16\pi^2}} \frac{1}{4} \text{tr}(q). \tag{2.35}
\end{align*}
\]

2.2 Zero Mode when \( \xi \neq 0 \)

On the untilted orbifold, i.e. \( \text{Re}\tau = 0 \), it has been shown that the zero mode of scalar field is localized at the fixed points by the localized FI-term [21]. Here, we show that the FI-term localizes the zero mode of scalar field similarly at the fixed points in the general \( T^2/Z_2 \) orbifold with arbitrary \( \tau \).

From (2.12) and (2.13), the SUSY conditions and the EOMs of the zero modes for the bulk scalar fields are represented by

\[
\begin{align*}
\langle F_{5' 6'} \rangle &= |\text{Im}\tau| \xi(x'_5, x'_6), \\
(\tau \mathcal{D}_{5'} - \mathcal{D}_{6'}) \phi_{\pm,0}(x'_{5}, x'_{6}) &= 0. \tag{2.37, 2.38}
\end{align*}
\]
We concentrate on the parity even mode. We write explicitly them by the derivatives \( \partial_{x'_5}, \partial_{x'_6} \) and gauge fields \( A_{x'_5}, A_{x'_6} \):

\[
\partial_{x'_5} \langle A_{x'_5} \rangle - \partial_{x'_6} \langle A_{x'_6} \rangle = |\text{Im} \tau| \xi(x'_5, x'_6), \tag{2.39}
\]

\[
\left\{ (\tau \partial_{x'_5} - \partial_{x'_6}) + igq(\tau \langle A_{x'_5} \rangle - \langle A_{x'_6} \rangle) \right\} \phi_{+0}(x'_5, x'_6) = 0. \tag{2.40}
\]

Here, we consider the following gauge fixing:

\[
\begin{cases}
A_{x'_5} = (\text{Im} \tau)^{-1}(\text{Re} \tau \partial_{x'_5} - \partial_{x'_6}) W \\
A_{x'_6} = (\text{Im} \tau)^{-1}|\tau|^2 \partial_{x'_5} - \text{Re} \tau \partial_{x'_6}) W.
\end{cases} \tag{2.41}
\]

By taking the gauge (2.41), the SUSY condition and EOM become

\[
\frac{1}{\text{Im} \tau} (|\tau|^2 \partial_{x'_5}^2 - 2\text{Re} \tau \partial_{x'_5} \partial_{x'_6} + \partial_{x'_6}^2) \langle W \rangle = |\text{Im} \tau| \xi(x'_5, x'_6), \tag{2.42}
\]

\[
\left\{ (\tau \partial_{x'_5} - \partial_{x'_6}) - gq(\tau \partial_{x'_5} - \partial_{x'_6}) \langle W \rangle \right\} \phi_{+0}(x'_5, x'_6) = 0. \tag{2.43}
\]

In the complex coordinates \( R z = x'_5 + \tau x'_6 \) and \( R \bar{z} = x'_5 + \bar{\tau} x'_6 \), the derivatives \( \partial_z, \partial_{\bar{z}} \) are given by

\[
\left( \begin{array}{c} \partial_z \\ \partial_{\bar{z}} \end{array} \right) = -\frac{R}{\tau - \bar{\tau}} \left( \begin{array}{cc} \bar{\tau} & -1 \\ -\tau & 1 \end{array} \right) \left( \begin{array}{c} \partial_{x'_5} \\ \partial_{x'_6} \end{array} \right).	ag{2.44}
\]

Eqs. (2.42) and (2.43) are written as follows:

\[
\partial \bar{\partial} \langle W \rangle = \frac{R^2}{4} \xi, \tag{2.45}
\]

\[
\left\{ \partial \bar{\partial} - gq(\partial \bar{\partial} \langle W \rangle) \right\} \phi_{+0}(z, \bar{z}) = 0, \tag{2.46}
\]

where the 1-loop FI-term (2.35) and (2.36) is represented in the complex coordinate as

\[
\xi(z, \bar{z}) = \frac{2}{R^2} \sum_{I,f.p.} (\xi_I + \xi''_I \frac{4}{R^2} \partial \bar{\partial}) \delta^{(2)}(z - z_I), \tag{2.47}
\]

\[
\xi_I = g \frac{\Lambda}{16\pi^2} \left( \frac{1}{4} \text{tr}(q) + \text{tr}(q_I) \right), \quad \xi''_I = g \frac{\ln \Lambda^2}{4} \frac{1}{16\pi^2} \frac{1}{4} \text{tr}(q), \tag{2.48}
\]

which is transformed by the relation \( \frac{4}{R^2} \partial \bar{\partial} = g^{ij} \partial_i \partial_j \) and a factor that arises from the delta function in transformation to complex coordinate. From (2.45) and (2.47), we can split the SUSY solution into two parts:

\[
\langle W \rangle = \langle W' \rangle / 2 + \langle W'' \rangle, \tag{2.49}
\]

\[
\partial \bar{\partial} \langle W' \rangle = \sum_{I,f.p.} \xi_I \delta^{(2)}(z - z_I), \quad \langle W'' \rangle = \frac{2}{R^2} \sum_{I,f.p.} \xi''_I \delta^{(2)}(z - z_I). \tag{2.50}
\]

\(^3\)Obviously the parity odd modes have no zero mode.

\(^4\)Considering \( \text{Re} \tau = 0 \) and the differences of scale between \( x_6 \) and \( x'_6 \), we see that this gauge (2.41) intrinsically corresponds to the gauge in [21].
The equation for $\langle W' \rangle$ is the Poisson equation for the source $\xi_I$ at the fixed points. Because the periodic boundary conditions become $z \sim z + 2\pi$, $z \sim z + 2\pi \tau$ in the complex coordinate, the solution of the $\langle W' \rangle$ equation becomes as follows:

$$\langle W' \rangle = \frac{1}{2\pi} \sum_I \xi_I \left[ \ln \left| \vartheta_1 \left( \frac{z - z_I}{2\pi} \tau \right) \right|^2 - \frac{1}{2\pi \text{Im} \tau} \left( \text{Im}(z - z_I) \right)^2 \right]. \quad (2.51)$$

Here, $\vartheta_1(z|\tau)$ is the elliptic theta function, and our convention is given by

$$\vartheta_{ab}(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+a/2)^2 \tau + 2\pi i (n+a/2)(z+b/2)}, \quad (2.52)$$

$$\vartheta_1(z|\tau) \equiv -\vartheta_{11}(z,\tau), \ \vartheta_2(z|\tau) \equiv \vartheta_{10}(z,\tau), \ \vartheta_3(z|\tau) \equiv \vartheta_{00}(z,\tau), \ \vartheta_4(z|\tau) \equiv \vartheta_{01}(z,\tau). \quad (2.53)$$

Using this background solution, the solution of the EOM (2.46) can be formally represented by

$$\phi_{+0}(z,\bar{z}) = f(z)e^{gq\langle W \rangle}. \quad (2.54)$$

The holomorphic function $f(z)$ must be constant because it is a periodic holomorphic function. Then the zero mode of $\phi_+$ is represented as follows:

$$\phi_{+0}(z,\bar{z}) = f \prod_{I.f.p.} \left| \vartheta_1 \left( \frac{z - z_I}{2\pi} \tau \right) \right|^{gq\xi_I/2\pi} \exp \left\{ - \frac{gq\xi_I}{8\pi^2 \text{Im} \tau} \left( \text{Im}(z - z_I) \right)^2 + \frac{2gq\xi''}{R^2} \delta^{(2)}(z - z_I) \right\}. \quad (2.55)$$

Since this wave function includes the Dirac delta function in the argument of exponential, it is not well defined. The Dirac delta function implies that this wave function has a serious divergence at the fixed points. Integral of the wave function on any small region including a fixed point seems to be divergent. On the other hands, wave functions must be canonically normalized. Thus this serious divergence must be canceled by the normalization factor $f$. As a result, normalized wave function would be a localized mode at the fixed points such as the Dirac delta function. We can check that such a localized mode appears in an explicit regularization scheme for the case of $S^1/Z_2$ compactification [16,22]. We assume that it is true for $T^2/Z_2$ models, too.

### 2.3 1-loop FI-term when $\xi \neq 0$

Calculation of the 1-loop FI-term is affected by the zero mode localization. It implies the instability of the supersymmetric vacuum for the $S^1/Z_2$ model [22]. Such a vacuum instability may happen in the present $T^2/Z_2$ model. Thus we should reevaluate the 1-loop FI-term again with the background given by (2.49), (2.50) and (2.51), and we should examine how stable configurations for the brane mode are.
In our evaluation, we make the following two assumptions:

Assumption 1: The massive mode profiles of the bulk scalar are excluded at the fixed points.

Assumption 2: Corrections to the FI-term can be evaluated by the square values of wave functions at the fixed points only.

The first assumption means that the induced FI-term can be evaluated by the zero mode of the bulk scalar field only. It is true for the $S^1/Z_2$ model.\(^5\) The second assumption means that the ratio of the 1-loop FI-term at each fixed point $z = z_I$ is equal to the ratio of $|\phi_{+,0}(z_I, \bar{z}_I)|^2$. We evaluate the validity of the second assumption in Appendix A.

As mentioned in the previous subsection, the zero mode of bulk scalar field is localized at fixed points because of the factor\(^6\):

$$\prod_{I:f.p.} \exp \left\{ \frac{2gg\xi''}{R^2} \delta^{(2)}(z - z_I) \right\}. \tag{2.56}$$

The factor is divergent at fixed points. The normalization of the wave function leads to exclusion of the wave function in the bulk region except the fixed points. In the rest of this section, we evaluate the ratio of the zero mode of bulk scalar field at the fixed points. From (2.55), the zero mode at the fixed point $z = z_I$ is written as below:

$$\phi_{+,0}(z_I, \bar{z}_I) = f \exp \left\{ \frac{2gg\xi''}{R^2} \delta^{(2)}(0) \right\} |\vartheta_1(0|\tau)|^{gq\xi_I/2\pi} \times \prod_{J:f.p.(\neq I)} |\vartheta_1\left(\frac{z_I - z_J}{2\pi}\right)|^{gq\xi_J/2\pi} \exp \left\{ - \frac{gg\xi_I}{8\pi^2\text{Im}\tau} \{|\text{Im}(z_I - z_J)|^2\} \right\}. \tag{2.57}$$

Here, we defined $\xi_{\text{min}}$ as $\xi_{\text{min}} = \xi_I$ and $I_* = \text{arg min}_{I:f.p.}\{\text{sgn}(g)\xi_I\}$. Because $|\vartheta_1(0|\tau)|$ vanishes, for the sake of finite representation, we redefine the normalization factor:

$$f' \equiv f \exp \left\{ \frac{2gg\xi''}{R^2} \delta^{(2)}(0) \right\} |\vartheta_1(0|\tau)|^{gq\xi_{\text{min}}/2\pi}. \tag{2.58}$$

Then, the zero mode at the fixed point is represented as

$$\phi_{+,0}(z_I, \bar{z}_I) = f' |\vartheta_1(0|\tau)|^{gq(\xi_I - \xi_{\text{min}})/2\pi} \times \prod_{J:f.p.(\neq I)} |\vartheta_1\left(\frac{z_I - z_J}{2\pi}\right)|^{gq\xi_J/2\pi} \exp \left\{ - \frac{gg\xi_I}{8\pi^2\text{Im}\tau} \{|\text{Im}(z_I - z_J)|^2\} \right\}. \tag{2.59}$$

If $\xi_I$ is not equal to $\xi_{\text{min}}$, because of the suppression of $|\vartheta_1(0|\tau)|$, the wave function must be zero at $z = z_I$. Thus the parts $|\vartheta_1(0|\tau)|^{gq(\xi_I - \xi_{\text{min}})/2\pi}$ affect only whether the wave function has nonzero value at the fixed points or not. The ratio of the zero mode of bulk scalar field is evaluated in [21]. However, the evaluation was performed except the small regions that contain the fixed points, and the analysis near the fixed points are very difficult.
fields can be practically evaluated by
\[ r_I \equiv \prod_{J:f.p.(\neq I)} \left| \vartheta_1 \left( \frac{z_I - z_J}{2\pi} \right) \right|^{gq\xi_j/2\pi} \exp \left\{ - \frac{gq\xi_j}{8\pi^2\text{Im}(z_I - z_J)^2} \right\}, \]  
(2.59)

where \( I = 1, ..., 4 \). In the complex coordinates \((z, \bar{z})\), the fixed points are
\[ z_I = \{ 0, \pi, \pi \tau, \pi(1 + \tau) \}. \]  
(2.60)

Then, \( \{\text{Im}(z_I - z_J)^2\} \) is summarized in Table 2. We define \( T_I \) as
\[ T_I \equiv \prod_{J:f.p.(\neq I)} \left| \vartheta_1 \left( \frac{z_I - z_J}{2\pi} \right) \right|^{gq\xi_j/2\pi}, \]  
(2.61)

which is the elliptic theta function part of \( r_I \). From (2.60), we find
\[ T_I = \begin{pmatrix} \frac{1}{8} \times |\vartheta_1(-\frac{1}{2}|\tau)|^{\xi_2} \times |\vartheta_1(-\frac{1+\tau}{2}|\tau)|^{\xi_3} \times |\vartheta_1(-\frac{1+3\tau}{2}|\tau)|^{\xi_4} \end{pmatrix}^{gq/2\pi} \]
\[ \begin{pmatrix} |\vartheta_1(\frac{1}{2}|\tau)|^{\xi_1} \times |\vartheta_1(\frac{1+2\tau}{2}|\tau)|^{\xi_2} \times 1 \times |\vartheta_1(-\frac{1+3\tau}{2}|\tau)|^{\xi_4} \end{pmatrix}^{gq/2\pi} \]
\[ \begin{pmatrix} |\vartheta_1(\frac{1+\tau}{2}|\tau)|^{\xi_1} \times |\vartheta_1(\frac{1+3\tau}{2}|\tau)|^{\xi_2} \times |\vartheta_1(\frac{1+3\tau}{2}|\tau)|^{\xi_3} \times 1 \end{pmatrix}^{gq/2\pi} \],

where the first, second, third and fourth rows correspond to \( T_1, T_2, T_3 \) and \( T_4 \) respectively.

The elliptic theta function \( \vartheta_1 \) satisfies the following relations:
\[ \vartheta_1(v + 1|\tau) = -\vartheta_1(v|\tau), \]  
(2.62)
\[ \vartheta_1(v + \tau|\tau) = -e^{-i\pi(2v+\tau)}\vartheta_1(v|\tau), \]  
(2.63)

and the elliptic theta functions \( \vartheta_i(i = 2, 3, 4) \) are related to \( \vartheta_1 \) as
\[ \vartheta_1\left( \frac{1}{2} \right) = \vartheta_2(0|\tau), \]  
(2.64)
\[ \vartheta_1\left( \frac{\tau}{2} \right) = i e^{-i\pi \tau/4} \vartheta_4(0|\tau), \]  
(2.65)
\[ \vartheta_1\left( \frac{1+\tau}{2} \right) = \vartheta_2\left( \frac{\tau}{2} \right) = e^{-i\pi \tau/4} \vartheta_4(0|\tau). \]  
(2.66)

| \{\text{Im}(z_I - z_J)^2\} | | J = 1 | J = 2 | J = 3 | J = 4 |
|---|---|---|---|---|
| \( I = 1 \) | 0 | 0 | \( \pi^2(\text{Im}\tau)^2 \) | \( \pi^2(\text{Im}\tau)^2 \) |
| \( I = 2 \) | 0 | 0 | \( \pi^2(\text{Im}\tau)^2 \) | \( \pi^2(\text{Im}\tau)^2 \) |
| \( I = 3 \) | \( \pi^2(\text{Im}\tau)^2 \) | \( \pi^2(\text{Im}\tau)^2 \) | 0 | 0 |
| \( I = 4 \) | \( \pi^2(\text{Im}\tau)^2 \) | \( \pi^2(\text{Im}\tau)^2 \) | 0 | 0 |

Table 2: \( \{\text{Im}(z_I - z_J)^2\} \).
Therefore, by using \( \vartheta_i(0|\tau) \) \( (i = 2, 3, 4) \), \( T_I \) is simply rewritten as

\[
T_I = \begin{pmatrix}
|\vartheta_2(0|\tau)|^{\xi_2} \times |\vartheta_3(0|\tau)|^{\xi_3} \times |\vartheta_4(0|\tau)|^{\xi_4} \times e^{\frac{i\pi}{4}(\xi_3 + \xi_4)} & \frac{gq}{2\pi} \\
|\vartheta_2(0|\tau)|^{\xi_1} \times |\vartheta_3(0|\tau)|^{\xi_3} \times |\vartheta_4(0|\tau)|^{\xi_4} \times e^{\frac{i\pi}{4}(\xi_3 + \xi_4)} & \frac{gq}{2\pi} \\
|\vartheta_2(0|\tau)|^{\xi_4} \times |\vartheta_3(0|\tau)|^{\xi_3} \times |\vartheta_4(0|\tau)|^{\xi_1} \times e^{\frac{i\pi}{4}(\xi_3 + \xi_4)} & \frac{gq}{2\pi} \\
|\vartheta_2(0|\tau)|^{\xi_4} \times |\vartheta_3(0|\tau)|^{\xi_3} \times |\vartheta_4(0|\tau)|^{\xi_1} \times e^{\frac{i\pi}{4}(\xi_3 + \xi_4)} & \frac{gq}{2\pi}
\end{pmatrix}.
\tag{2.67}
\]

The value of the zero mode at the fixed points is evaluated as

\[
\phi_{+,0}(z_I, \bar{z}_I) = f'|\vartheta_1(0|\tau)|^{gq(\xi_1 - \xi_{\min})/2}\pi \begin{pmatrix}
|\vartheta_2(0|\tau)|^{\xi_2} \times |\vartheta_3(0|\tau)|^{\xi_3} \times |\vartheta_4(0|\tau)|^{\xi_4} \times e^{\frac{i\pi}{4}(\xi_3 + \xi_4)} & \frac{gq}{2\pi} \\
|\vartheta_2(0|\tau)|^{\xi_1} \times |\vartheta_3(0|\tau)|^{\xi_3} \times |\vartheta_4(0|\tau)|^{\xi_4} \times e^{\frac{i\pi}{4}(\xi_3 + \xi_4)} & \frac{gq}{2\pi} \\
|\vartheta_2(0|\tau)|^{\xi_4} \times |\vartheta_3(0|\tau)|^{\xi_3} \times |\vartheta_4(0|\tau)|^{\xi_1} \times e^{\frac{i\pi}{4}(\xi_3 + \xi_4)} & \frac{gq}{2\pi} \\
|\vartheta_2(0|\tau)|^{\xi_4} \times |\vartheta_3(0|\tau)|^{\xi_3} \times |\vartheta_4(0|\tau)|^{\xi_1} \times e^{\frac{i\pi}{4}(\xi_3 + \xi_4)} & \frac{gq}{2\pi}
\end{pmatrix}.
\tag{2.68}
\]

These values transform each other by the modular symmetry. The modular symmetry is generated by two elements, \( S \) and \( T \), and these generators transform the modulus \( \tau \) as

\[
S: \tau \rightarrow -\frac{1}{\tau}, \quad T: \tau \rightarrow \tau + 1.
\tag{2.69}
\]

The elliptic theta functions transform each other by \( S \) and \( T \), and transformation behavior is shown in Appendix C. The \( S \) transforms zero mode values at \( z_1 \) and \( z_4 \), and \( z_2 \) and \( z_3 \), i.e. \( \phi_{+,0}(z_1, \bar{z}_1) \leftrightarrow \phi_{+,0}(z_4, \bar{z}_4) \) and \( \phi_{+,0}(z_2, \bar{z}_2) \leftrightarrow \phi_{+,0}(z_3, \bar{z}_3) \). On the other hand, the \( T \) transforms zero mode values at \( z_1 \) and \( z_2 \), and \( z_3 \) and \( z_4 \), i.e. \( \phi_{+,0}(z_1, \bar{z}_1) \leftrightarrow \phi_{+,0}(z_2, \bar{z}_2) \) and \( \phi_{+,0}(z_3, \bar{z}_3) \leftrightarrow \phi_{+,0}(z_4, \bar{z}_4) \). When \( \xi_1 = \xi_2 = \xi_3 = \xi_4 \), the above zero mode profile is invariant under the modular symmetry.

### 3 Stability of SUSY vacua on untilted torus

In the previous section, we have finished the preparations to calculate the localized FI-term in the new SUSY background, where the VEV of \( F^2 \) has nonzero value. In the configurations where the 1-loop FI-term is not induced, the bulk mode contribution cancels the brane mode contributions. Thus we examine configurations where the cancellation occurs. Under the second assumption, the 1-loop FI-term that is induced by the bulk mode can be evaluated by \( r_\tau^2 \). In the configurations where the cancellation cannot occur, the 1-loop FI-term changes the supersymmetric vacuum further, which leads to the instability of SUSY vacuum.

In this section, we investigate the stability of the SUSY vacuum in the untilted torus, i.e. \( \text{Re} \tau = 0 \). In the untilted torus, except for the differences from the scale of \( x_6 \) and \( x_\prime^6 \), the zero mode profile \( \phi_{+,0} \) and gauge field \( W \) coincide with the ones in 21.
3.1 Completely symmetric configuration

Firstly, we consider the completely symmetric configuration of brane charges, i.e. \( q_1 = q_2 = q_3 = q_4 \). We assume the sum of \( U(1) \) charges is set to zero, which means that the bulk charge is four times as big as that of the localized charge: \( q = -4q_1 \). Furthermore, we assume the tree level Lagrangian has no FI-term and \( \langle F_{5'6'} \rangle = 0 \). Then, from (2.35) and (2.36), we obtain the 1-loop induced FI-term:

\[
\xi = \xi_{\text{bulk}} + \xi_{\text{brane}} = 2 \frac{R^2}{f.p.} \left( \xi_I + \xi'' \frac{4}{R^2} \delta^{(2)}(z - z_I) \right),
\]

(3.1)

\[
\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0, \quad \xi'' = \frac{gg \ln \Lambda^2}{16 \pi^2}.
\]

(3.2)

Solving the D-flat condition (2.37) in the gauge (2.41), the SUSY background solution is corrected by 1-loop effects as

\[
\langle W \rangle = 2 \frac{R^2}{f.p.} \xi'' \delta^{(2)}(z - z_I). \]

(3.3)

In the new SUSY background, we recompute the zero mode of \( \phi_+ \). The zero mode can be evaluated from (2.68):

\[
\phi_{+,0}(z, \bar{z}) = \frac{\sqrt{2}}{R} \sqrt{\frac{1}{4} \sum_{f.p.} \delta^{(2)}(z - z_I)},
\]

(3.4)

where we denoted the square root of the delta function as a conventional function that means the wave function is localized at the fixed points and canonically normalized.

Substituting the KK-modes of the bulk fields in (2.33), we obtain the 1-loop FI-term again. From the assumption 1 in section 2.3, the massive modes do not contribute the 1-loop FI-term. Then we can evaluate the contribution of the bulk fields from (3.11):

\[
\xi_{\text{bulk}} = g \frac{\Lambda^2}{16 \pi^2} \frac{1}{4} \frac{2}{R^2} \sum_{f.p.} \delta^{(2)}(z - z_I). \]

(3.5)

The contribution of the brane fields is unchanged, and is written as

\[
\xi_{\text{brane}} = g \frac{\Lambda^2}{16 \pi^2} \frac{2}{R^2} \sum_{f.p.} q_I \delta^{(2)}(z - z_I).
\]

(3.6)

As a result, we obtain the quantum correction to the FI-term in the new SUSY background,

\[
\xi(z, \bar{z}) = \xi_{\text{bulk}} + \xi_{\text{brane}} = 0.
\]

(3.7)

The quantum correction vanishes. In the SUSY background, the bulk zero mode shields the brane charges completely, so the SUSY vacuum does not shift further, i.e. it is stable vacuum.
3.2 Partially symmetric configuration

Next, we consider a partially symmetric configuration where the brane fields have the \( U(1) \) charge as \( q_1 = 0 \) and \( q_2 = q_3 = q_4 \). We assume the sum of \( U(1) \) charges is set to zero, which means that the bulk charge is three times as big as that of the localized charge: \( q = -3q_2 \). Furthermore, we assume the tree level Lagrangian has vanishing FI-term and \( \langle F_{\alpha'\alpha'} \rangle = 0 \). The 1-loop induced FI-term is calculated as

\[
\xi = \xi_{\text{bulk}} + \xi_{\text{brane}}
\]

\[
= \frac{2}{R^2} \sum_{l.f.p.} (\xi_l + \xi'' \frac{4}{R^2} \partial \bar{\partial}) \delta^2(z - z_l),
\]

\[
\xi_1 = \kappa, \quad \xi_2 = \xi_3 = \xi_4 = -\kappa/3 \quad \left( \kappa \equiv \frac{1}{4} g q \frac{\Lambda^2}{16\pi^2} \right),
\]

\[
\xi'' = \frac{g q \ln \Lambda^2}{16 \times 16\pi^2}.
\]

Solving the D-flat condition (2.37) in the gauge (2.41), the SUSY background solution is corrected by 1-loop effects as

\[
\langle W \rangle = \frac{1}{4\pi} \sum_{l.f.p.} \xi_l \left[ \ln \left| \vartheta_1 \left( \frac{z - z_l}{2\pi} \right) \right|^2 - \frac{1}{2\pi \Im \tau} \{ \Im(z - z_l) \}^2 \right] + \frac{2}{R^2} \sum_{l.f.p.} \xi'' \delta^2(z - z_l).
\]

In the new SUSY background, we recompute the zero mode of \( \phi_+ \). The zero mode can be evaluated from (2.68):

\[
\phi_{+,0}(z, \bar{z}) = \sqrt{2} R \left[ \frac{\vartheta_2(0|\tau)}{\vartheta_2(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_2) + \left[ \frac{\vartheta_4(0|\tau)}{\vartheta_4(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_3) + \left[ \frac{\vartheta_3(0|\tau)}{\vartheta_3(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_4) + \left[ \frac{\vartheta_4(0|\tau)}{\vartheta_4(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_4).
\]

Substituting the KK-modes of the bulk fields in (2.33), we obtain the 1-loop FI-term again. The contribution of the bulk fields is evaluated by (3.11):

\[
\xi_{\text{bulk}} = g q \frac{\Lambda^2}{16\pi^2} \frac{2}{R^2} \left[ \frac{\vartheta_2(0|\tau)}{\vartheta_2(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_2) + \left[ \frac{\vartheta_4(0|\tau)}{\vartheta_4(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_3) + \left[ \frac{\vartheta_3(0|\tau)}{\vartheta_3(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_4)
\]

\[
\times \left[ \frac{\vartheta_2(0|\tau)}{\vartheta_2(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_2) + \left[ \frac{\vartheta_4(0|\tau)}{\vartheta_4(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_3) + \left[ \frac{\vartheta_3(0|\tau)}{\vartheta_3(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_4)
\]

\[
\quad \times \left[ \frac{\vartheta_4(0|\tau)}{\vartheta_4(0|\tau)} \right]^{4qqc/3\pi} \delta^2(z - z_4).
\]

The contribution of the brane fields is unchanged, and is written as

\[
\xi_{\text{brane}} = g q \frac{\Lambda^2}{16\pi^2} \frac{2}{R^2} \sum_{l.f.p.} q_l \delta^2(z - z_l).
\]
the charges of brane modes | stability of the vacuum
---|---
$q_1 = q_2 = q_3 = q_4$ | stable
$q_1 = q_2 = q_3 \neq q_4$ | unstable
$q_1 = q_2 \neq q_3 = q_4$ | stable
$\{q_1 = q_2 \neq q_3, q_4\}$ and $\{q_3 \neq q_4\}$ | unstable
$q_I \neq q_J (I \neq J)$ | unstable

Table 3: Stable and unstable configurations of brane modes.

As a result, we obtain the quantum correction to the FI-term in the new SUSY background,

$$\xi(z, \bar{z}) = \xi_{\text{bulk}} + \xi_{\text{brane}} \neq 0.$$  \hfill (3.15)

The quantum correction does not vanish. Therefore, the SUSY vacuum shifts further by the 1-loop FI-term, i.e. it is unstable vacuum. Furthermore, we find that, unless we induce the fine-tuned FI-term at tree level, any stable vacuum can not exist in the partially symmetric configuration.

### 3.3 Stable and instable configurations

We have examined the stability of the SUSY vacuum in the two configurations: completely symmetric one and partially symmetric one. The former has the supersymmetric vacuum, but the latter does not.

As the end of this section, we show stability of various configurations in Table 3. In all of these examples, we consider that the bulk mode has a charge $q$ which cancels the charges of brane modes, i.e., $q + \sum_I q_I = 0$. The first and second rows correspond to the results in the section 3.1 and 3.2, respectively. In the table, the “stable” means that the FI-term is not induced in a new SUSY vacuum. On the other hand, the “unstable” means that the FI-term is always induced in new SUSY vacuum.

It is always possible to introduce a tree FI-term which makes zero mode profile of the bulk field shield the brane charges completely. If such a fine-tuned FI-term is available, unstable configurations can be stabilized. Otherwise, unstable configurations do not have any stable supersymmetric vacuum.

### 4 Stability of SUSY vacua on tilted torus

Next, we examine the stability of SUSY vacuum in the tilted torus $T^2/Z_2$, i.e. $\text{Re} \tau \neq 0$. Basically, the results of the stability are the same as those of untilted torus. This is predicted from the formula of the zero mode (2.55) and (2.68). However, profiles of zero modes generally depend on the background geometry. In more general background, such an instability might be cancelled. Especially, the partially symmetric configuration leads different results and very interesting consequence.
4.1 Stable configuration and Moduli stabilization

From (2.68), we can calculate the FI-term that is induced by 1-loop corrections in general backgrounds. We are interested in the partially symmetric configuration, i.e., the charges of three brane modes are same and the charge of the other one is zero. Similar to section 3.2, we concentrate on the configuration that the charges of brane modes in the fixed points $z = z_2, z_3, z_4$ are same for concreteness. We set the charge of the bulk mode as it is three times as big as that of the localized charge, that is needed for $\sum_I \xi_I = 0$. (See Figure 2.)

In the SUSY vacuum with $\langle F_{5'6'} \rangle = 0$, the 1-loop induced FI-term is written by

$$\xi_1 = \kappa, \quad \xi_2 = \xi_3 = \xi_4 = -\kappa/3, \quad \xi'' \neq 0,$$

(4.1)

where $\kappa \equiv \frac{1}{16} g q \frac{\Lambda^2}{16\pi^2}$.

The FI-term corrects the SUSY vacuum as $\langle F_{5'6'} \rangle = |\text{Im}\tau| \xi(x'_5, x'_6)$. Again we evaluate the 1-loop FI-term in the new SUSY vacuum. Since (4.1) satisfies $\xi_{\text{min}} = \xi_2 = \xi_3 = \xi_4$, the $\phi_{+,0}(z_I, \bar{z}_I)$ becomes

$$\phi_{+,0}(z_I, \bar{z}_I) = f' \begin{pmatrix} 0 \\ \vartheta_2(0|\tau)|^{\kappa} \times |\vartheta_3(0|\tau)|^{-\kappa/3} \times |\vartheta_4(0|\tau)|^{-\kappa/3} \\ |\vartheta_2(0|\tau)|^{-\kappa/3} \times |\vartheta_3(0|\tau)|^{-\kappa/3} \times |\vartheta_4(0|\tau)|^{\kappa} \\ |\vartheta_2(0|\tau)|^{-\kappa/3} \times |\vartheta_3(0|\tau)|^{\kappa} \times |\vartheta_4(0|\tau)|^{-\kappa/3} \\ \lambda q/2\pi \end{pmatrix}.$$

Therefore, the ratio of zero mode profiles at fixed points is given by

$$|\phi_{+,0}(z_1)|^2 : |\phi_{+,0}(z_2)|^2 : |\phi_{+,0}(z_3)|^2 : |\phi_{+,0}(z_4)|^2 = 0 : |\vartheta_2(0|\tau)|^{\frac{4\log 5}{8\pi}} : |\vartheta_4(0|\tau)|^{\frac{4\log 5}{8\pi}} : |\vartheta_3(0|\tau)|^{\frac{4\log 5}{8\pi}}.$$

The 1-loop FI-term by bulk mode in the new vacuum would be induced as this ratio at fixed points. In order not to generate the 1-loop FI-term in the new vacuum, the 1-loop
FI-term by bulk mode must cancel the contribution from the brane modes. The 1-loop FI-term by the brane modes is the same in the fixed points $z = z_2, z_3, z_4$. We obtain the following stability condition\(^6\)

$$|\vartheta_2(0|\tau)| = |\vartheta_3(0|\tau)| = |\vartheta_4(0|\tau)|.$$ \hspace{1cm} (4.2)

These conditions cannot be satisfied in the case of $\text{Re}\tau = 0$. This is because we insisted that this configuration is unstable in the untilted torus at section 3.3. However, in the tilted torus, the configuration can be stable, i.e., the complex structure $\tau$ that satisfies the conditions (4.2) exists.

By using of modular transformation behavior of elliptic theta functions as shown in Appendix C, we find that the complex structure, e.g. $\tau = e^{i\pi/3}$ satisfies the above condition (4.2). The point $\tau = e^{i\pi/3}$ is on the boundary of the fundamental domain of the modular group. Thus, in the torus which has the complex structure $\tau = e^{i\pi/3}$, the 1-loop induced FI-term in the new SUSY vacuum vanishes. Accordingly the configuration of three brane modes has a stable vacuum.

Especially, the 1-loop FI-term, which cannot vanish in the new vacuum, generates a D-term potential:

$$V_D \propto \int dx_5^\prime dx_6^\prime \sqrt{\det g_{ij}^\prime} (\xi + \cdots)^2.$$ \hspace{1cm} (4.3)

Moreover, the $\xi$ contains the divergent term of cutoff $\Lambda$, so the D-term potential would be dominant. Thus, we consider that the $\tau$ would be stabilized in the value that cancels the 1-loop FI-term in the new SUSY vacuum.

### 4.2 Stabilized complex structure

For example, in the configuration of three brane modes, we insist that the complex structure is stabilized dynamically in $\tau = e^{i\pi/3}$ by the potential $V_D$.

We show the stable configuration in Figure 3. In this configuration, there are the brane modes in the fixed points except the origin, and the localization of bulk mode occurred at the fixed points except the origin, too. Figure 3 shows when the complex structure $\tau$ is a stable one, the positional relations of fixed points where the brane mode exists are equidistant each other. We expect that the complex structure is stabilized in such a way that the fixed points where the brane modes exist, have symmetric positional relations. Otherwise there are no stable SUSY vacuum, and SUSY or gauge symmetry would be broken.

Four-dimensional CP can be embedded into proper Lorentz transformation in higher dimensional theory, where extra dimensions are also reflected \([26][31]\). For example, in six dimensional theory, four-dimensional CP is combined with the reflection,

$$z \rightarrow -\bar{z},$$ \hspace{1cm} (4.4)

\[^6\]For other combinations of three fixed points where the three brane modes exist, the same conditions appear.
Figure 3: Torus of $\tau = e^{i\pi/3}$

so as to be embedded into six-dimensional proper Lorentz transformation. Under the above reflection, the modulus transforms

$$\tau \rightarrow -\bar{\tau}.$$  \hfill (4.5)

Thus, when $\text{Re}\tau = 0$, CP is conserved. For other values of $\text{Re}\tau$, CP can be broken. Hence, the value $\tau = e^{i\pi/3}$ has implication in CP violation physics.  

5 Conclusion

We have investigated the quantum corrections to FI-terms in six-dimensional SUSY Abelian gauge theory compactified on the $T^2/Z_2$ orbifold.

In the $S^1/Z_2$ orbifold, the localization of bulk zero mode causes the instability of the vacuum. Similarly, the localization of bulk zero mode happens in the untilted $T^2/Z_2$ model, too [21]. We found that the new supersymmetric vacuum which is changed by 1-loop FI-term can be unstable in untilted compactification. The instability is related to the configuration of brane modes and their $U(1)$ charges. We have shown that the 1-loop correction vanishes for the completely symmetric configurations, but it is not true for the asymmetric configurations. It is because the zero mode profile and brane charges cancel each other for the former case, but it does not happen for the latter case. Therefore, in the asymmetric configurations the vacuum receives further corrections and is unstable. If we put a fine-tuned FI-term in the tree level Lagrangian, we can realize a stable vacuum even for asymmetric configuration. In such a stable vacuum, zero mode profile shields the brane charges completely and their corrections are canceled each other. This result is the same as the one derived on the $S^1/Z_2$ orbifold [22].

If theory has modular symmetry, the transformation (4.5) is meaningful up to the modular symmetry. (See e.g. [32–34].) That implies that CP is conserved at the values of $\tau$ at the boundary of the fundamental domain including $\tau = e^{i\pi/3}$.

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7 If theory has modular symmetry, the transformation (4.5) is meaningful up to the modular symmetry. (See e.g. [32–34].) That implies that CP is conserved at the values of $\tau$ at the boundary of the fundamental domain including $\tau = e^{i\pi/3}$. 

18
As opposed to the $S^1/Z_2$ orbifold, the complex structure exists in the $T^2/Z_2$ orbifolds. The 1-loop FI-term depends on the complex structure, i.e., the complex structure associates with the instability of the vacuum. Especially, we can stabilize the complex structure $\tau$ by using the cancellation of 1-loop FI-term that is induced in a new supersymmetric vacuum. We have considered the configuration with three brane modes, that exist in each of three fixed points and have same charge and $U(1)$ coupling as an example. We have found that the complex structure $\tau$ is stabilized at the value $e^{i\pi/3}$, which makes the three fixed points equidistant each other. We expect that the stabilization mechanism which is caused by the cancellation of 1-loop FI-term occurs in more general orbifolds, and the stabilized complex structures make the positions of fixed points more symmetric. It contrasts with the traditional moduli stabilization mechanism by three form flux [35–37].

The examination of the vacuum instability of the $T^2/Z_2$ orbifolds with magnetic fluxes is much valuable. It is known that, by tuning magnetic fluxes, wave function profiles of bulk fields are changed and the number of the chiral zero modes increases [23, 38–40]. It is interesting to extend our analysis to the $T^2/Z_2$ orbifolds with magnetic fluxes. That would lead to the flavor structure different from magnetized orbifold models without FI terms [24, 41, 42]. In magnetized orbifold models, zero modes transform each other under the modular symmetry [43–45]. That is a flavor symmetry. In addition, our FI term has already non-trivial behavior under the modular symmetry. Thus, it is interesting to study localized FI terms from the viewpoint of modular flavor models [46] and their modulus stabilization [34, 47].

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A Validity about the Evaluation of FI-term

To investigate the instability of supersymmetric vacuum, we evaluated the 1-loop FI-term by the wave function of bulk zero mode only at fixed points $z_I$. Here we describe its validity.

The solution of the bulk zero mode is represented by (2.55):

$$
\phi_{+,0}(z, \bar{z}) = f e^{gq(W)}
$$

$$
= f \prod \left| \vartheta_1 \left( \frac{z - z_I}{2\pi} \right) \right|^{gq\xi_I/2\pi} \exp \left\{ - \frac{gq\xi_I}{8\pi^2 \mathrm{Im}\tau} \left( \mathrm{Im}(z - z_I) \right)^2 + \frac{2gq\xi''}{R^2} \delta^{(2)}(z - z_I) \right\}.
$$

(A.1)
Therefore, the probability density is given as follows:

\[ |\phi_{+0}(z, \bar{z})|^2 = |f|^2 \prod_{I, f.p.} |\varphi_1\left(\frac{z - z_I}{2\pi}\right)|^{2gq\xi_I/2\pi} \exp \left\{ -\frac{gq\xi_I}{4\pi^2\text{Im}\tau} \{\text{Im}(z - z_I)\}^2 + \frac{4gq\xi''}{R^2}\delta^{(2)}(z - z_I) \right\}. \]  

(A.2)

The factor

\[ \exp \left\{ \frac{4gq\xi''}{R^2}\delta^{(2)}(z - z_I) \right\} \]  

(A.3)

is exponential function of delta function. It has stronger singularity on the point where delta function has a singularity. Now, we define \( \Delta^{(2)} \) as

\[ \Delta^{(2)}(z - z_I) \equiv \exp \left\{ \frac{4gq\xi''}{R^2}\delta^{(2)}(z - z_I) \right\}, \]  

(A.4)

and we decompose \( |\phi_{+0}(z, \bar{z})|^2 \) as follows:

\[ |\phi_{+0}(z, \bar{z})|^2 = |f|^2 \prod_{I, f.p.} |\psi_I(z, \bar{z})|^2 \Delta^{(2)}(z - z_I), \]  

(A.5)

\[ \Delta^{(2)}(z - z_I) \equiv \exp \left\{ \frac{4gq\xi''}{R^2}\delta^{(2)}(z - z_I) \right\}, \]  

(A.6)

\[ |\psi_I(z, \bar{z})|^2 \equiv |\psi_1\left(\frac{z - z_I}{2\pi}\right)|^{2gq\xi'/2\pi} \exp \left\{ -\frac{gq\xi_I}{4\pi^2\text{Im}\tau} \{\text{Im}(z - z_I)\}^2 \right\}. \]  

(A.7)

From the localization of \( |\phi_{+0}(z, \bar{z})|^2 \), we expect that the probability density can be represented by delta function:

\[ |\phi_{+0}(z, \bar{z})|^2 = \sum_{I, f.p.} C_I \delta^{(2)}(z - z_I). \]  

(A.8)

Then, because the integral of \( \delta^{(2)}(z) \) is equal to 1, the coefficients \( C_I \) are calculated by the surface integral of \( |\phi_{+0}(z, \bar{z})|^2 \) on smallness areas \( R_I \) which are around fixed point \( z_I \):

\[ C_I = \int_{R_I} d^2z |\phi_{+0}(z, \bar{z})|^2. \]  

(A.9)

From (A.5), the coefficients are rewritten by

\[ C_I = \int_{R_I} d^2z |f|^2 \prod_{J, f.p.} |\psi_J(z, \bar{z})|^2 \Delta^{(2)}(z - z_J) \]  

\[ = \int_{R_I} d^2z |f|^2 |\psi_I(z, \bar{z})|^2 \Delta^{(2)}(z - z_I) \prod_{J \neq I} |\psi_J(z, \bar{z})|^2 \Delta^{(2)}(z - z_J). \]  

(A.10)
Because $\Delta^{(2)}(z - z_I)$ behaves like

$$\Delta^{(2)}(z - z_I) = \begin{cases} \infty & (z = z_I) \\ 1 & (z \neq z_I) \end{cases}, \quad (A.11)$$

and we defined $R_I$ in such a way that the region has only one fixed point, we can evaluate $C_I$ as follows:

$$C_I = \int_{R_I} d^2z |f|^2 |\psi_I(z, \bar{z})|^2 \Delta^{(2)}(z - z_I) \prod_{J \neq I} |\psi_J(z, \bar{z})|^2$$

$$= \int_{R_I} d^2z |f|^2 \Delta^{(2)}(z - z_I) \prod_{J: f.p.} |\psi_J(z, \bar{z})|^2. \quad (A.12)$$

Now, we define $\Gamma$ as

$$\Gamma \equiv \int_{T^2} d^2z \Delta^{(2)}(z - z_I). \quad (A.13)$$

This quantity is divergent and by using $\Gamma$, we can express $\Delta^{(2)}$ by delta function.

$$\Delta^{(2)}(z - z_I) = \Gamma \times \begin{cases} \infty/\Gamma & (z = z_I) \\ 1/\Gamma & (z \neq z_I) \end{cases}$$

$$= \Gamma \times \begin{cases} \infty & (z = z_I) \\ 0 & (z \neq z_I) \end{cases} \rightarrow \Gamma \delta^{(2)}(z - z_I). \quad (A.14)$$

Because $\prod_{J: f.p.} |\psi_J(z, \bar{z})|^2$ is finite, the coefficients $C_I$ become

$$C_I = \int_{R_I} d^2z |f|^2 \Gamma \delta^{(2)}(z - z_I) \prod_{J: f.p.} |\psi_J(z, \bar{z})|^2$$

$$= |\tilde{f}|^2 \prod_{J: f.p.} |\psi_J(z_I, \bar{z}_I)|^2 \quad (|\tilde{f}|^2 \equiv |f|^2 \Gamma). \quad (A.15)$$

Therefore, the evaluation of the coefficients $C_I$ by the wave function of zero mode only at the fixed point $z_I$, is reasonable.

---

8 $\Gamma$ is defined by integral over all region of torus, and $\Delta^{(2)}$ has dependence of the fixed point on only argument $(z - z_I)$. Sift of integral variable considered, the $\Gamma$ is independent of the fixed point i.e. constant.

9 We can check this concretely with a regularization for $\Delta^{(2)}$ (see Appendix B).
B Behaver of a regularized $\Delta^{(2)}(z - z_I)$

Now, we will check that, when we regularize the delta function $\delta^{(2)}(z - z_I)$, the function $\Delta^{(2)}(z - z_I)/\Gamma$ shows property of delta function. The $\Delta^{(2)}(z - z_I)$ and $\Gamma$ are defined as

\[
\Delta^{(2)}(z - z_I) \equiv \exp \left\{ \frac{4gq'\xi}{R^2} \delta^{(2)}(z - z_I) \right\} \quad (B.1)
\]
\[
\Gamma \equiv \int_{T^2} d^2 z \Delta^{(2)}(z - z_I). \quad (B.2)
\]

Here, for the simplicity we set $z_I = 0$ and transform the coordinates to $(x,y)$ plane. Then, we write the delta function in the $(x,y)$ coordinates that has peak on $x, y = 0$ as $\delta^{(2)}(x,y)$.

We regularize the delta function as follows (see Figure 4):

\[
\delta^{(2)}_\rho(x, y) = \begin{cases} 
\frac{3}{\pi \rho^2} \left( 1 - \sqrt{x^2 + y^2}/\rho \right) & (\sqrt{x^2 + y^2} \leq \rho), \\
0 & (\sqrt{x^2 + y^2} > \rho). 
\end{cases} \quad (B.3)
\]

Figure 4: A regularization of $\delta^{(2)}(x,y)$

For this regularization, we can check $\int dxdy \delta^{(2)}_\rho(x,y) = 1$ immediately:

\[
\int dxdy \delta^{(2)}_\rho(x, y) = \int dr \int d\theta r \delta^{(2)}_\rho(r, \theta) \\
= \int_0^\rho dr \int d\theta r \frac{3}{\pi \rho^2} \left( 1 - r/\rho \right) \\
= 2\pi \int_0^\rho dr \frac{3}{\pi \rho^2} \left( r - r^2/\rho \right) \\
= 2\pi \frac{3}{\pi \rho^2} \times \frac{\rho^3}{6} = 1. \quad (B.4)
\]
From this regularization, $\Delta^{(2)}_{\rho}(x, y)$ is represented as follows:

$$
\Delta^{(2)}_{\rho}(x, y) = \begin{cases} 
\exp\left[\frac{3k}{\pi \rho^3} \left(1 - \sqrt{x^2 + y^2}/\rho\right)\right] \left(\sqrt{x^2 + y^2} \leq \rho\right), \\
1 \left(\sqrt{x^2 + y^2} > \rho\right),
\end{cases}
$$

where $k$ is constant which includes the cutoff $\Lambda$ and so on. Therefore, $\Gamma$ can be calculated as

$$
\Gamma \equiv \int dxdy \Delta^{(2)}_{\rho}(x, y)
= \int_0^{\rho} dr \int d\theta \frac{e^{\frac{3k}{\pi \rho^3}(1-r/\rho)}}{r} + \int_\rho^\infty dr \int d\theta r
= 2\pi \left(\frac{\pi \rho^3}{3k}\right)^2 \left\{e^{\frac{3k}{\pi \rho^3}} - \left(1 + \frac{3k}{\pi \rho^2}\right)\right\} + (2\pi R)^2 \text{Im} \tau - \pi \rho^2.
$$

Here, because the $\Delta^{(2)}_{\rho}$ has nonzero value for the region of $r > \rho$, we treated the two-dimensional space as a torus, and evaluated the integral in the region. Thus, in the limit where $\rho$ is much smaller than 1, the leading term is

$$
\Gamma \simeq 2\pi \left(\frac{\pi \rho^3}{3k}\right)^2 e^{\frac{3k}{\pi \rho^3}}.
$$

Hence, in the limit $\rho \to 0$, the $\Delta^{(2)}_{\rho}(x, y)/\Gamma$ behaves as

$$
\Delta^{(2)}_{\rho}(x, y)/\Gamma = \begin{cases} 
\exp\left[\frac{3k}{\pi \rho^3} \left(1 - \sqrt{x^2 + y^2}/\rho\right)\right]/\Gamma \left(\sqrt{x^2 + y^2} \leq \rho\right), \\
1/\Gamma \left(\sqrt{x^2 + y^2} > \rho\right),
\end{cases}
$$

$$
\simeq \begin{cases} 
\frac{1}{2\pi} \left(\frac{3k}{\pi \rho^3}\right)^2 \exp\left[\frac{3k}{\pi \rho^3} \left(1 - \sqrt{x^2 + y^2}/\rho\right) - \frac{3k}{\pi \rho^2}\right] \left(\sqrt{x^2 + y^2} \leq \rho\right), \\
\frac{1}{2\pi} \left(\frac{3k}{\pi \rho^3}\right)^2 e^{-\frac{3k}{\pi \rho^2}} \left(\sqrt{x^2 + y^2} > \rho\right),
\end{cases}
$$

$$
= \begin{cases} 
\frac{1}{2\pi} \left(\frac{3k}{\pi \rho^3}\right)^2 \exp\left[-\frac{3k}{\pi \rho^4} \sqrt{x^2 + y^2}\right] \left(\sqrt{x^2 + y^2} \leq \rho\right), \\
\frac{1}{2\pi} \left(\frac{3k}{\pi \rho^3}\right)^2 e^{-\frac{3k}{\pi \rho^2}} \left(\sqrt{x^2 + y^2} > \rho\right),
\end{cases}
$$

$$
\rho \to 0 \quad \begin{cases} 
\infty \left(\sqrt{x^2 + y^2} = 0\right), \\
0 \left(\sqrt{x^2 + y^2} \neq 0\right).
\end{cases}
$$

By the definition, the integral of $\Delta^{(2)}_{\rho}(x, y)/\Gamma$ is equal to 1. Therefore, in the limit where $\rho$ is much smaller than 1, $\Delta^{(2)}_{\rho}(x, y)/\Gamma$ behaves like delta function.
C  Modular symmetry of elliptic theta functions

Here, we summarize modular symmetry of elliptic theta functions. Under the $S$ transformation, they satisfy the relations,

$$\begin{align*}
\vartheta_1(0|-1/\tau) &= -i\sqrt{-i\tau}\vartheta_1(0|\tau), \\
\vartheta_3(0|-1/\tau) &= \sqrt{-i\tau}\vartheta_3(0|\tau), \\
\vartheta_4(0|-1/\tau) &= \sqrt{-i\tau}\vartheta_2(0|\tau).
\end{align*}$$

(C.1)

Also, under the $T$ transformation, they satisfy the relations,

$$\begin{align*}
\vartheta_1(0|\tau_1) &= e^{\pi i/4}\vartheta_1(0|\tau), \\
\vartheta_3(0|\tau_1) &= \vartheta_4(0|\tau), \\
\vartheta_4(0|\tau_1) &= \vartheta_3(0|\tau).
\end{align*}$$

(C.2)

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