Nondissipative force acting on a vortex in Fermi superfluids and superconductors is calculated from the spectral flow of fermion zero modes in the vortex core. The spectral flow effect casts a new light on the problem of mutual friction in superfluids. We demonstrate that, for vortices in an isotropic system, the reactive mutual friction parameter $D'(T)$ is negative in the collisionless regime and it is positive in the hydrodynamic regime. This agrees with $^3$He-B measurements [1] (T.D.C. Bevan, et al, Phys. Rev. Lett., 74, 750 (1995)), which thus give the experimental verification of the spectral flow effect in the vortex dynamics. General expression for all 3 nondissipative forces (Magnus, Iordanskii and spectral flow) is presented in the whole temperature range.

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Spectral flow of fermions is the topological phenomenon related to anomalies in the relativistic quantum field theory. The inhomogeneous vacuum induces a spectral flow, which carries particles from negative energy states of the vacuum to positive energy states. As a result, the corresponding conserved quantity (viz, charge) is transferred from a coherent vacuum motion into incoherent fermionic degrees of freedom, which is visualized as creation of charge from the vacuum. It was shown in [2,3] that the same phenomenon takes place in superfluid $^3$He-A, where the gap in the quasiparticle spectrum has topologically stable point nodes, resulting in a spectral flow of Bogoliubov quasiparticles through the nodes when the condensate evolves in time. This leads to transfer of a linear momentum from the coherent superfluid motion into the normal component of the liquid, i.e., to a force between the superfluid and normal components.

A mutual friction force of the same type arises for a Fermi superfluid or superconductor even without gap nodes, if quantized vortices are present [4]. The singularity at the vortex axis plays the same role as gap nodes in A-phase of $^3$He, while the evolution of the condensate, which leads to the spectral flow, is realized by the motion of the vortex with respect to the heat bath. A microscopic analysis of the spectral flow in vortices was also presented in [5], but the hydrodynamic and collisionless regimes were not distinguished there which led to the contradictory conclusion. Here we consider the influence of the spectral flow on a low-dissipative vortex dynamics in both regimes for isotropic Fermi systems such as $s$-wave superconductors and $^3$He-B. For superconductors, we consider a simple case of Galilean invariant system, i.e., we ignore energy-band effects in metals and assume that the mean free path is longer than the coherence length (clean case). We compare the theoretical mutual friction force in $^3$He with the experimental results of Ref. [1]. The observed change of sign of the parameter $D'$ following the change of regimes illustrates the essential contribution of the spectral flow effect into the vortex dynamics.

**Nondissipative forces in Bose superfluids.** There is a big difference between dynamics of quantized vorticity in Bose and Fermi superfluids. In Bose liquids the nondissipative force acting on the vortex moving with respect to superfluid and normal components of the liquid is simply a sum of the Magnus forces from the superfluid and normal components:

$$
\mathbf{F}_{nd} = \kappa \hat{\mathbf{z}} \times \tilde{\rho}_s(T)(\mathbf{v}_L - \mathbf{v}_s) + \kappa \hat{\mathbf{z}} \times \tilde{\rho}_n(T)(\mathbf{v}_L - \mathbf{v}_n).
$$

(1)

Here $\mathbf{v}_L$ is the vortex velocity, $\mathbf{v}_s$ and $\mathbf{v}_n$ are the superfluid and normal velocities; $\kappa \hat{\mathbf{z}}$ is the circulation vector. For Bose superfluids $\kappa = 2\pi N\hbar/m$, where $N$ is the vortex winding number, and $m$ is the bare mass of a boson. The densities $\tilde{\rho}_s(T)$ and $\tilde{\rho}_n(T)$ of the normal and superfluid components, respectively, can be anisotropic tensors.
Eq. (1) can be rearranged to comply with the Landau picture of a superfluid, where the motion of a superfluid vacuum of the total density $\rho$ with the superfluid velocity $v_s$ is supplemented with the dynamics of elementary excitations moving with the velocity $v_n - v_s$. The force becomes

$$F_{nd} = \kappa \hat{z} \times \rho (v_L - v_s) + \kappa \hat{z} \times \bar{\rho}_n(T) (v_s - v_n)$$

$$= F_{\text{Magnus}} + F_{\text{Iordanskii}}$$

(2)

Now the Magnus force is defined as if the vortex moves with respect to the superfluid vacuum. The Iordanskii force results from elementary excitations outside the vortex core: the vortex line provides them with an Aharonov-Bohm potential.

The Iordanskii force in Eq. (2) is a sum of elementary forces which act on individual particles, $\partial_t \mathbf{p} = (\nabla \times v_s) \times \mathbf{p}$. Here $\mathbf{p}$ is the quasiparticle momentum and the vorticity $\nabla \times v_s = \kappa \hat{z} \delta_2(\mathbf{r})$ is concentrated in the vortex core "tube". One has

$$- \sum_{\mathbf{p}} f \partial_t \mathbf{p} = - \int \frac{d^3p}{(2\pi)^3} \kappa \hat{z} \times \mathbf{p} f[\mathbf{E}_p + \mathbf{p} \cdot (v_s - v_n)]$$

$$= \kappa \hat{z} \times \bar{\rho}_n(T) (v_n - v_s).$$

(3)

Here $f(E)$ is the distribution function of elementary excitations, Doppler shifted due to the counterflow $v_n - v_s$. The Iordanskii force has the same origin as the Aharonov-Bohm effect for spinning cosmic strings.

Adding a dissipative friction force to Eq. (1), one obtains a conventional expression for the balance of forces acting on the vortex

$$\kappa \rho_s (v_s - v_L) \times \hat{z} + D (v_n - v_L) - D' (v_n - v_L) \times \hat{z} = 0.$$  

(4)

Here the term with $D$ is associated with the friction while the term with $D'$ is due to the reactive force $F_{nd}$. Comparison with Eq. (1) gives the reactive parameter

$$D' = - \kappa \bar{\rho}_n(T).$$

(5)

Our consideration is valid when dissipation can be neglected compared to the reactive terms, $D \ll D'$. It is only under this condition the nondissipative forces do not depend on details of the core structure and of the quasiparticle kinetics, and are defined by general characteristics of the vortex and the superfluid. Eq. (5) coincides with microscopic calculations (see review).

**Nondissipative forces in Fermi superfluids.** Now we consider Fermi systems such as superfluid phases of $^3\text{He}$ and superconductors. For the latter we assume a large magnetic-field penetration depth compared to the intervortex distance: in this case the magnetic field can be treated as homogeneous. We also assume that the magnetic induction $B$ is much smaller than the upper critical field $H_{c2}$, so that vortex cores do not overlap.

In Fermi superfluids, there is a new contribution to $F_{nd}$ in addition to Eq. (2) (now $\kappa = \pi N \hbar/m$ and $m$ is the bare mass of a fermion):

$$F_{nd} = F_{\text{Magnus}} + F_{\text{Iordanskii}} + F_{\text{sp. flow}},$$

(6)

This contribution is related to the chiral fermion zero modes, which are absent in Bose superfluids.

The spectrum of single-fermionic excitations in a vortex, $E_n(p_z, Q)$, depends on the momentum projection $p_z$ on the vortex axis; the orbital quantum number $Q$, integer or half of odd integer, which corresponds to the generalized angular momentum conserved in an axisymmetric vortex; and $n$ denotes radial and spin quantum numbers. The
interlevel distance of bound states $\partial E_n/\partial Q = \omega_n$ is small compared to the gap amplitude $\Delta(T)$ of fermions in bulk: $\omega_n \sim \Delta^2(T)/E_F$. This spectrum has anomalous (chiral) branches $E_0(p_z, Q)$ whose number $N_{zm}$ is related to the vortex winding number $N_{zm} = 2N$ according to the index theorem \[1\]. As a function of $Q$, each anomalous branch crosses zero of energy an odd number of times and runs through both discrete and continuous spectrum from $E_0 = -\infty$ to $E_0 = +\infty$. Any other branch either does not cross zero of energy at all or crosses it an even number of times. For low-energy bound states, the spectrum of the chiral branch is linear in $Q$. For the most symmetric vortices, for example, \[8\]

$$E_0(p_z, Q) = Q\omega_0(p_z).$$

Due to an odd number of crossings of zero, the chiral branch can result in a momentum exchange between fermions localized in the vortex core and fermions in the heat bath. The corresponding contribution to the force thus depends on $v_n - v_L$. In contrast to the other two forces, this contribution does depend on the core structure and on the quasiparticle kinetics: It is determined by the parameter $\omega_0\tau$, where $\tau$ is the lifetime of fermions. Two limiting cases are important, the collisionless and hydrodynamic limits: they correspond to $\omega_0\tau \gg 1$ and $\omega_0\tau \ll 1$, respectively. In both cases dissipation is small \[13\]\[14\] and the results should thus not be sensitive to the core structure of the vortex.

**Hydrodynamic limit.** In the hydrodynamic limit $\omega_0\tau \ll 1$ the interlevel spacing is smaller than the level width $1/\tau$ and the spectrum $E_0(p_z, Q)$ can be considered as a function of the continuous parameter $Q$: it crosses zero at some value of $Q$ [at $Q = 0$ for the spectrum of Eq. \[7\]]. When the vortex moves with respect to the heat bath (normal component), the velocity difference $v_n - v_L$ induces a flow of quasiparticles from negative levels to positive levels of the spectrum $E_0(p_z, Q)$. Since the angular momentum evolves as $Q \rightarrow Q + (r(t) \times p) \cdot \dot{z} = Q + t((v_L - v_n) \times p) \cdot \dot{z}$, the number of levels crossing zero energy per unit time is $\partial_t Q = (v_L - v_n) \cdot (p \times \dot{z})$. Each level bears the linear momentum $p$, therefore, the total flux of the linear momentum from the vortex to the heat bath is

$$\dot{p} = \sum p (-\frac{\partial f}{\partial Q}) \partial_t Q = -\frac{1}{2} \sum_{n,Q} \frac{\partial f(E_n)}{\partial Q}$$

$$\times \int_{-p_F}^{p_F} \frac{dp_z}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} p \left[((v_L - v_n) \times p) \cdot \dot{z}\right]$$

$$= \pi N \frac{p_F^3}{3\pi^2} \dot{z} \times (v_L - v_n). \quad (8)$$

Here we used that only zero modes contribute the sum $-\sum_{n,Q}(\partial f(E_n)/\partial Q) = \sum_n(f(E_n = -\infty) - f(E_n = \infty)) = 2N$ as $Q$ together with $E_0(Q)$ run from $-\infty$ to $+\infty$. Thus the spectral flow contribution to $F_{nd}$ in the hydrodynamic limit is

$$F_{sp, flow} = \kappa \dot{z} \times C_0(v_n - v_L), \quad \omega_0\tau \ll 1, \quad (9)$$

where the anomaly parameter $C_0$ is expressed in terms of the Fermi momentum $p_F$ and coincides with the mass density of the normal state:

$$C_0 = mp_F^3/3\pi^2. \quad (10)$$

The spectral flow of the fermion zero modes within the vortex core is a realization of the Callan-Harvey mechanism of the anomaly cancellation in the relativistic quantum field theories \[13\]\[14\].

**Collisionless limit in uncharged Fermi superfluids.** In the collisionless limit $\omega_0\tau \gg 1$ the level width is much smaller than the interlevel distance and the spectral flow along the discrete levels is suppressed. The spectral flow becomes possible only for $E_0(p_z, Q)$ above the gap in the region of the continuous spectrum. Since only the levels in the range $E_n \in \left(-\Delta - \frac{1}{\omega_0\tau}, -\Delta + \frac{1}{\omega_0\tau}\right)$ contribute, only the zero modes $n = 0$ are relevant. The number $N_{zm}$ is expressed in terms of the Fermi momentum $p_F$ and coincides with the mass density of the normal state:

$$C_0 = mp_F^3/3\pi^2. \quad (10)$$
\( \Delta(T) < E_0(p_z, Q) < \infty \) are "fluid" the spectral flow takes place only at nonzero \( T \) due to the thermal tail of Fermi function \( f(E_n) \) in Eq.(8). It is reduced by the factor \([f(-\Delta(T)) - f(-\infty)] - [f(\Delta(T)) - f(\infty)] = 1 - \tanh[\Delta(T)/2T]\) as compared to the hydrodynamic limit (here we assume that the gap is isotropic).

Thus for an isotropic pair-correlated system, such as \(^3\)He-B, which has an isotropic gap \( \Delta(T) \),

\[
F_{\text{sp.flow}} = \kappa C_0 \left[ 1 - \tanh \left( \frac{\Delta(T)}{2T} \right) \right] \hat{z} \times (\mathbf{v}_n - \mathbf{v}_L), \quad \omega_0 \tau \gg 1. \tag{11}
\]

**Spectral flow in superconductors.** For a charged Fermi-system the levels above the gap are also discrete due to quantization in the magnetic field, the interlevel distance being the cyclotron frequency \( \omega_c \). Note that \( \omega_c \ll \omega_0 \) due to the condition \( B \ll H_{c2} \). The levels can be considered as a continuum only under the condition \( \omega_c \tau \ll 1 \). The results Eqs.(9) and (11) are thus reproduced for superconductors in the limit \( \omega_c \tau \ll 1 \). In the extreme collisionless limit, when \( \omega_c \tau \gg 1 \), the spectrum is discrete everywhere and the spectral flow is completely suppressed. Hence,

\[
F_{\text{sp.flow}} = \kappa C_0 \left[ 1 - \tanh \left( \frac{\Delta(T)}{2T} \right) \right] \hat{z} \times (\mathbf{v}_n - \mathbf{v}_L), \quad \omega_0 \tau \ll 1, \quad \omega_c \tau \ll 1; \tag{12}
\]

\[
F_{\text{sp.flow}} = \kappa C_0 \left( 1 - \tanh \left( \frac{\Delta(T)}{2T} \right) \right) \hat{z} \times (\mathbf{v}_n - \mathbf{v}_L), \quad \omega_0 \tau \gg 1, \quad \omega_c \tau \ll 1; \tag{13}
\]

\[
F_{\text{sp.flow}} = 0, \quad \omega_0 \tau \gg 1, \quad \omega_c \tau \gg 1. \tag{14}
\]

**Parameter \( D' \) for \(^3\)He-B.** Let us consider first the effect of the spectral flow on the mutual friction parameter \( D' \) in \(^3\)He-B. The dissipative force, proportional to \( D' \), is small in both limits, \( \omega_0 \tau \ll 1 \) and \( \omega_0 \tau \gg 1 \) \([14]\). Comparing Eq.(1) with Eqs.(12) and (13), we find that the reactive parameter in the hydrodynamic limit is

\[
D'_{\text{hydro}} = \kappa |C_0 - \rho_n(T)|, \quad \omega_0 \tau \ll 1. \tag{15}
\]

Here it is used that the normal density \( \rho_n \) in \(^3\)He-B is isotropic. In the collisionless limit we find from Eq. (11)

\[
D'_{\text{coll}} = \kappa \left[ C_0 \left( 1 - \tanh \left( \frac{\Delta(T)}{2T} \right) \right) - \rho_n(T) \right], \quad \omega_0 \tau \gg 1. \tag{16}
\]

Eq.(15) was obtained in \([1] \) in the limit \( T \to 0 \) and was extended to nonzero \( T \) in \([8,17]\). In the weak coupling approximation the anomaly parameter \( C_0 \approx \rho_0 \), since \((\rho - C_0) \approx \rho \Delta^2(T)/E_F^2 \ll \rho \).

If one neglects the difference between \( C_0 \) and \( \rho_0 \),

\[
D'_{\text{hydro}}(\kappa \rho_0(T)), \quad \omega_0 \tau \ll 1, \tag{17}
\]

\[
D'_{\text{coll}}(\kappa \rho_0(T) - \rho \tanh \left( \frac{\Delta(T)}{2T} \right)), \quad \omega_0 \tau \gg 1. \tag{18}
\]

Eq. (17) in the limit \( T \ll T_c \) was derived for superconductors in \([13]\) and for \(^3\)He in \([14]\); it was extended for higher temperatures in \([13]\). Eq. (18) was first obtained for superconductors in \([18,19]\) by calculating the force produced by excitations scattered by moving vortex. The same result has been reproduced in \([13]\) using the kinetic equation. The classical results of Bardeen and Stephen \([20]\) and of Nozieres and Vinen \([21]\) can also be compared with Eqs. (17) and (18). The former corresponds to the hydrodynamic limit \( D' = \kappa \rho_0 \). The result of \([21]\), namely that the only force left at \( T = 0 \) is the Magnus force, remains valid in a collisionless limit, when both the Iordanskii force and the spectral flow disappear. Its generalization which extrapolates between the two limits \([22]\) does not contain the spectral flow term \( 1 - \tanh[\Delta(T)/2T] \) and thus is valid only at \( T = 0 \).

Applying the Eqs. (17,18) to \(^3\)He-B one finds that these two regimes take place in the regions \( T \ll T_c \) and \( T_c - T \ll T_c \). For \( T \) close to \( T_c \) the hydrodynamic limit is always realized since \( \omega_0 \sim \Delta^2(T)/E_F \to 0 \), and the reactive parameter is
positive $D' \approx \kappa \rho_s(T)$. At low $T \ll \Delta(0)$ the collisionless limit is always reached, hence the parameter $D'$ is negative according to Eq.(18). Indeed, the ratio of $D'$ values for these two regimes is

$$D'_{\text{coll}}/D'_{\text{hydro}} = 1 - \frac{\rho}{\rho_s(T)} \tanh \frac{\Delta(T)}{2T} < 0 .$$

This dependence is shown in Fig.1. The sign reversal of $D'$ is clearly seen in the oscillating membrane experiment in $^3$He-B [1].

Using the results of Ref. [15], we can write a phenomenological expression which interpolates between the hydrodynamic and collisionless limits:

$$D \approx \kappa \rho_s \frac{\omega_0 \tau}{1 + \omega_0^2 \tau^2},$$

$$D' \approx \kappa \left[ C_0 - \rho_n(T) - \frac{\omega_0^2 \tau^2}{1 + \omega_0^2 \tau^2} C_0 \tanh \frac{\Delta(T)}{2T} \right].$$

The temperature dependences of $D(T)$ and $D'(T)$ can be visualized using a simple model expression for the lifetime \cite{14} $\tau \sim (E_F/T^2) \exp[\Delta(T)/T]$, so that the parameter $\omega_0 \tau$ is approximated by

$$\omega_0 \tau = \alpha [\Delta(T)/T]^2 \exp[\Delta(T)/T],$$

where $\alpha$ is a fitting constant. The plot in Fig. 2 qualitatively agrees with the experimental temperature dependence found in [1]. We see that $D'/\kappa \rho_s$ approaches its negative collisionless-limit asymptote of Eq.(19) at low $T$ and tends to its positive hydrodynamic-limit asymptote $D'/\kappa \rho_s = 1$ at high $T$. The observed sign reversal of $D'$ is the experimental illustration of the spectral flow force in the vortex dynamics. Eq. (21) suggests that for $T \to 0$, one always has $D' \approx -\kappa \rho_n$ since $\rho_n/\rho \gg 1 - \tanh[\Delta(0)/2T]$, and $(\omega_0 \tau)^{-2}$ decreases faster than $\rho_n$.

**Superconductors.** Eq. (21) can also be used for investigation of the sign of the Hall effect in superconductors. The Hall conductivity in the limit $\omega_c \tau \ll 1$ is

$$\sigma_H = \frac{c^2 (\kappa \rho_s - D')/\Phi_0 B}{mB \left[ \rho - C_0 + C_0 \frac{\omega_0^2 \tau^2}{1 + \omega_0^2 \tau^2} \tanh \frac{\Delta(T)}{2T} \right]}.$$

where $\Phi_0 = \pi c/e$ is the magnetic flux quantum. If $C_0 > \rho$ the sign reversal of $\sigma_H$ can occur in the hydrodynamic regime as proposed in Ref. [24].

**Conclusion.** We found a spectral flow contribution to the nondissipative force acting on a moving vortex in hydrodynamic and collisionless limits. Due to the spectral flow the reactive mutual friction parameter $D'$ has different signs in these two limits. This sign reversal was observed in the $^3$He-B experiments on the transition between the two regimes, realized at different temperatures. This confirms the essential role of the spectral flow effect in the vortex dynamics of Fermi superfluids and superconductors.

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FIG. 1. Theoretical dependence of the ratio $D'_{\text{coll}}/D'_{\text{hydro}}$ from Eq. (19) for an isotropic superfluid. This ratio is always negative. The gap function $\Delta(T)$ and the superfluid density $\rho_s(T)$ are taken in the weak-coupling BCS approximation. $F_1$ is the Fermi-liquid parameter which enters $\rho_s$ in a Fermi superfluid. For $^3$He, the values of $F_1$ correspond to pressures $p = 0$, $p = 12$, and $p = 24$ bar, respectively.

FIG. 2. Qualitative temperature dependences of $D/\kappa \rho_s$ and $D'/\kappa \rho_s$ from Eq. (21) in a simple model of Eq. (22). The fitting parameter $\alpha = 0.02$. 