Identification of generalized Cobb-Douglas production functions with multiplicative errors in variables

I L Sandler\textsuperscript{1}, D V Ivanov\textsuperscript{2}, M Yu Lifshits\textsuperscript{1} and A N Diligenskaya\textsuperscript{1}

\textsuperscript{1}Samara State Technical University, 244, Molodogvardejskaya St., Samara, 443100, Russia
\textsuperscript{2}Samara State University of Economics, 141, Sovetskoi Armii St., Samara, 443090, Russia

E-mail: dvi85@list.ru

Abstract. The article discusses the identification of a class of generalized Cobb-Douglas production functions with multiplicative errors in all variables. The article proposes a generalization of Cobb-Douglas production functions in the presence of memory for input and output variables. Parametrization of noise in the form of multiplicative noise in all observable variables is proposed. The logarithmic transformation of such production functions leads to the need to solve the problem of estimating the parameters of a linear difference equation in the presence of additive errors in all variables. To identify the parameters of production functions, a modification of the method of total least squares was used. Computational experiments have shown high accuracy of parameter estimation based on the proposed algorithm.

1. Introduction

Currently, production is being actively developed methods of modeling socio-economic systems. Production functions are the main tools used to model socio-economic systems. The parameters of production functions are not known in advance, so they have to be estimated from statistical data. Socio-economic models usually contain a stochastic component that makes it difficult to estimate the parameters of such models.

The Cobb-Douglas function and their modifications have found the widest application among production functions [1]. There are known methods for estimating the parameters of Cobb-Douglas production functions in the presence of multiplicative and additive noise [2-4].

In [5] nonparametric methods of identification of dynamic production functions are considered. In the article [5], the production function is described by a non-linear homoscedastic ARX (autoregressive with exogenous input) process.

This article proposes a Cobb-Douglas function with memory. It should be noted that the authors are not aware of any work on the estimation of production functions in which the stochastic component is interpreted as an error in variables. This article discusses the estimation of the parameters of a production function in the presence of multiplicative noise in variables.

2. Problem statement

Economic and mathematical modeling widely uses the method of constructing mathematical models in the form of a non-uniform production function (PF) of Cobb-Douglas:
The production function of Cobb-Douglas can take into account the influence of scientific and technological progress:

\[ Y(t) = A \cdot K(t)^\alpha L(t)^\beta e^{\tau}, \]

where \( Y \) is the release of the final product; \( K(t) \) - capital resources, \( L(t) \) - labor resources, \( \alpha, \beta \) - characteristic of resource use efficiency - elasticity indicator, \( A \) - scale factor of transformation technology, \( \tau \) - factor of influence of scientific and technological progress.

The Cobb-Douglas production function assumes that the output of the final product does not depend on the previous values of capital and labor resources. This article proposes to use the generalized Cobb-Douglas function using past values of capital and labor resources. Let us write the equation for the dynamic Cobb-Douglas function:

\[ Y(t) = A \cdot K(t) \cdot K(t - m \cdot \Delta t)^{\alpha_m} \cdot L(t - m \cdot \Delta t)^{\beta_m} e^{\tau_i}. \]

For discrete values, function (3) will take the form:

\[ Y_i = A \cdot K_i \cdot K_i^{\alpha_m} \cdot L_i^{\beta_m} e^{\tau_i}. \]

The values of capital and labor resources are usually known with errors; we will assume that measurements with multiplicative noise are available:

\[ \tilde{Y}_i = Y_i \cdot \exp\left(\xi^{(Y)}_i\right), \quad \tilde{K}_i = K_i \cdot \exp\left(\xi^{(K)}_i\right), \quad \tilde{L}_i = L_i \cdot \exp\left(\xi^{(L)}_i\right), \]

where sequences of independent random variables \( \{\tilde{Y}_i\}, \{\tilde{K}_i\}, \{\tilde{L}_i\} \) with

\[ E\left(\xi^{(Y)}_i\right) = 0, \quad E\left(\xi^{(K)}_i\right) = 0, \quad E\left(\xi^{(L)}_i\right) = 0, \]

\[ E\left(\xi^{(Y)}_i\right)^2 = \sigma^2_\gamma, \quad E\left(\xi^{(K)}_i\right)^2 = \sigma^2_K, \quad E\left(\xi^{(L)}_i\right)^2 = \sigma^2_L. \]

Let us logarithm the equation (4):

\[ \ln\left(\tilde{Y}_i\right) + \sum_{m=1}^{r_i} \gamma_m \ln(\tilde{Y}_{i-m}) = \ln(A) + \sum_{m=0}^{r_i} \alpha_m \ln(\tilde{K}_{i-m}) + \sum_{m=0}^{n_i} \beta_m \ln(\tilde{L}_{i-m}) + \tau_i. \]

Let us write the equation for the output of the final product at a given time:

\[ \ln\left(\tilde{Y}_i\right) = -\sum_{m=1}^{r_i} \gamma_m \ln(\tilde{Y}_{i-m}) + \ln(A) + \sum_{m=0}^{r_i} \alpha_m \ln(\tilde{K}_{i-m}) + \sum_{m=0}^{n_i} \beta_m \ln(\tilde{L}_{i-m}) + \tau_i. \]

Applying the property of the logarithm to observations with noise, we get:

\[ \ln\left(\tilde{Y}_i\right) = \ln(\tilde{Y}_i) + \xi^{(Y)}_i, \quad \ln(\tilde{K}_i) = \ln(K_i) + \xi^{(K)}_i, \quad \ln(\tilde{L}_i) = \ln(L_i) + \xi^{(L)}_i. \]

It is required to estimate the parameters of equation (6) using noisy sequences of observations \( \{\tilde{Y}_i\}, \{\tilde{K}_i\}, \{\tilde{L}_i\} \).

### 3. Identification algorithm based on total least square

We will define estimates in the form of minimizing the criterion:
\[
\sum_{i=1}^{N} \left( \ln \left( Y_i \right) - \varphi_i^T \theta \right)^2 \min_{\theta} \frac{1}{1 + \theta^T D \theta},
\]  
(8)

where \( \theta = (\gamma_1 \ldots \gamma_{\tau} ; \alpha_1 \ldots \alpha_{\tau} ; \beta_1 \ldots \beta_{\tau} ; \tau \ln (A))^T \),
\[
\varphi_i^T = \left( \ln \left( \tilde{Y}_i \right) \ldots \ln \left( \tilde{Y}_{i-\tau} \right) \right) \begin{bmatrix} i \end{bmatrix} \left( \ln \left( \tilde{K}_i \right) \ldots \ln \left( \tilde{K}_{i-\tau} \right) \right) \begin{bmatrix} i \end{bmatrix} \ldots \begin{bmatrix} i \end{bmatrix} \left( \ln \left( \tilde{L}_i \right) \ldots \ln \left( \tilde{L}_{i-\tau} \right) \right) \begin{bmatrix} i \end{bmatrix},
\]
\[
D = \begin{bmatrix} I_{\tau} & 0 & 0 & 0 \\
0 & I_{\tau} & 0 & 0 \\
0 & 0 & I_{\tau} & 0 \\
0 & 0 & 0 & 0_{2} \\
\end{bmatrix}.
\]

The minimum criterion (8) can be found as a solution to the biased normal system:
\[
\hat{\theta} = \left( \Phi^T \Phi - \sigma^2 I \right)^{-1} \Phi^T \tilde{Y},
\]
(9)

where
\[
\Phi = \begin{bmatrix} \ln \left( \tilde{Y}_1 \right) \ldots 0 & \ln \left( \tilde{K}_1 \right) \ldots 0 & \ln \left( \tilde{Y}_1 \right) \ldots 0 \\
\ln \left( \tilde{Y}_2 \right) \ldots 0 & \ln \left( \tilde{K}_2 \right) \ldots 0 & \ln \left( \tilde{Y}_2 \right) \ldots 0 \\
\vdots & \vdots & \vdots \\
\ln \left( \tilde{Y}_N \right) & \ln \left( \tilde{K}_N \right) & \ln \left( \tilde{Y}_N \right) \\
\end{bmatrix},
\]
\[
\begin{bmatrix} \ln \left( \tilde{Y}_{i-1} \right) \ldots \ln \left( \tilde{Y}_{i-N} \right) \end{bmatrix}^T,
\]
\[
\sigma^2 = \frac{1}{\lambda_{\max} \left( \left( \Phi^T \Phi \right)^{-1} D \right)},
\]
\[
\lambda_{\max} \left( \left( \Phi^T \Phi \right)^{-1} D \right) \text{ is maximum eigenvalue of the matrix } \left( \Phi^T \Phi \right)^{-1} D.
\]
\[
\tilde{D} = \begin{bmatrix} 1 & 0 \\
0 & \tilde{D} \end{bmatrix},
\]

4. Test Example

The generalized Cobb-Douglas production function is described by the equation:
\[
Y_i = 0.9 \ln \left( \tilde{Y}_{i-1} \right) + 0.3 \ln \left( K_i \right) + 0.7 \ln \left( K_{i-1} \right) + 0.4 \ln \left( L_i \right) + 0.5 \ln \left( L_{i-1} \right) + 0.1i. \quad (10)
\]

After the logarithmic transformation, the generalized Cobb-Douglas production function (10) is described by the equation:
\[
\ln \left( \tilde{Y}_i \right) = 1 + 0.9 \ln \left( \tilde{Y}_{i-m} \right) + 0.3 \ln \left( \tilde{K}_i \right) + 0.7 \ln \left( \tilde{K}_{i-1} \right) + 0.4 \ln \left( \tilde{L}_i \right) + 0.5 \ln \left( \tilde{L}_{i-1} \right) + 0.1i. \quad (11)
\]

The vector of true parameters is \( \theta = (0.9 \quad 0.3 \quad 0.7 \quad 0.4 \quad 0.5 \quad 0.1 \quad 1)^T \).

Sequences \( \{K_i\}, \{L_i\} \) are white noise. The number of observations is \( N = 48 \).

Signal-to-Noise Ratio for function (11) is defined by:
\[ SNR_y = \frac{\hat{\sigma}_y^{(Y)}}{\sigma_y} = 0.02, \quad SNR_x = \frac{\hat{\sigma}_x^{(K)}}{\sigma_x} = 0.02, \quad SNR_L = \frac{\hat{\sigma}_L^{(L)}}{\sigma_L} = 0.02. \]

The algorithms were compared by root-mean-square error (RMSE) of the coefficient estimation vector:

\[ \delta \theta = \sqrt{\frac{\| \theta - \hat{\theta} \|^2}{\| \theta \|^2}} \times 100\%. \]

Table 1 shows errors in estimating the parameters according to least-square (LS) estimate and the estimate based on total least square (TLS).

| True Parameters | LS    | TLS   |
|-----------------|-------|-------|
| \( \gamma_1 = 0.9 \) | 0.8816 | 0.8883 |
| \( \alpha_1 = 0.4 \) | 0.3624 | 0.3532 |
| \( \alpha_2 = 0.5 \) | 0.7677 | 0.7573 |
| \( \beta_1 = 0.3 \) | 0.3442 | 0.3457 |
| \( \beta_2 = 0.7 \) | 0.5547 | 0.5529 |
| \( \tau = 0.1 \) | 0.1390 | 0.1238 |
| \( \delta \theta \) | 0.1009 | 0.0675 |

5. Conclusion
The results of the test example show that the proposed modification of total least square based for estimation production function parameters can significantly improve the accuracy of estimating system parameters.

The direction of further research is the development of a recursive version of the algorithm for estimating the parameters of generalized Cobb-Douglas production functions [7].

When estimating parameters based on real data, the condition of interference independence is often violated. Methods based on instrumental variables for models with errors in variables can be developed to estimate interference with autocorrelated interference [8].

References
[1] Cobb C W and Douglas P H 1928 A theory of production American Econ. Rev. 8 139–1965
[2] Felipe J and Adams F G 2005 The estimation of the Cobb-Douglas function: A retrospective review Eastern Econ. J. 31 427–445
[3] Stephen M G and Richard E Q 1970 The Estimation of Cobb-Douglas Type Functions with Multiplicative and Additive Int. Economic Review 11(2) 251-257
[4] Kelejian H H 1972 The Estimation of Cobb-Douglas Type Functions with Multiplicative and Additive Errors: A Further Analysis Int. Economic Review 13 179-182
[5] Hossain M M, Majumder A I and Basak T 2012 An Application of Non-Linear Cobb-Douglas Production Function to Selected Manufacturing Industries in Bangladesh Open J. of Statistics 2 460-468. DOI: http://dx.doi.org/10.4236/ojs.2012.4058
[6] Koshkin G and Kitayeva A 2011 Nonparametric Identification of Static and Dynamic Production Functions Int. J. of Applied Mathematics 276-280
[7] Ivanov D V, Sandler I L and Kozlov E V 2018 Identification of fractional linear dynamical systems with autocorrelated errors in variables by generalized instrumental variables *IFAC PapersOnLine* 51(32) 580-584. DOI: https://doi.org/10.1016/j.ifacol.2018.11.485

[8] Ivanov D V and Katsyuba O A 2010 Recursive identification dynamic systems with errors-in-variables using nonlinear least square *Proc. of the IASTED Int. Conf. on Automation, Control, and Information Technology-Control, Diagnostics, and Automation (ACITCDA 2010)* pp 267-271