On the stability of scalar-vacuum space-times

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We study the stability of static, spherically symmetric solutions to the Einstein equations with a scalar field as the source. We describe a general methodology of studying small radial perturbations of scalar-vacuum configurations with arbitrary potentials $V(\phi)$, and in particular space-times with throats (including wormholes), which are possible if the scalar is phantom. At such a throat, the effective potential for perturbations $V_{\text{eff}}$ has a positive pole (a potential wall) that prevents a complete perturbation analysis. We show that, generically, (i) $V_{\text{eff}}$ has precisely the form required for regularization by the known S-deformation method, and (ii) a solution with the regularized potential leads to regular scalar field and metric perturbations of the initial configuration. The well-known conformal mappings make these results also applicable to scalar-tensor and $f(R)$ theories of gravity. As a particular example, we prove the instability of all static solutions with both normal and phantom scalars and $V(\phi) \equiv 0$ under spherical perturbations. We thus confirm the previous results on the unstable nature of anti-Fisher wormholes and Fisher’s singular solution and prove the instability of other branches of these solutions including the anti-Fisher “cold black holes”.

1. Introduction

Phantom field configurations have become a subject of particular interest since the discovery of the accelerated expansion of our Universe and its explanation in the framework of general relativity by the existence of dark energy (DE), a source of gravity of unknown nature with the pressure to density ratio $w < -1/3$. Numerous observations lead to estimates of $w$ around $-1$, which corresponds to a cosmological constant, but values smaller than $-1$ are still admissible and, moreover, preferable for describing an increasing acceleration. The most recent estimates read $w = -1.10 \pm 0.14 \ (1\sigma)$ \textsuperscript{[1]} (according to the 7-year WMAP data) and $w = -1.069^{+0.091}_{-0.092}$ \textsuperscript{[2]} (mainly from data on type Ia supernovae from the SNLS3 sample).

The possible existence of phantom DE is certainly only one of potentially viable explanations of the observations (numerous DE models are described, e.g., in the reviews \textsuperscript{[3–7]}), but if we take it as a working hypothesis, it is natural to expect that there are manifestations of DE in local objects and phenomena. The simplest of them can be described by static, spherically symmetric solutions to Einstein-scalar equations where the scalar field has an unusual sign of kinetic energy (a phantom scalar, by definition). In the case of a massless scalar it is the so-called anti-Fisher solution, a phantom analogue of Fisher’ solution \textsuperscript{[8]} for an ordinary minimally coupled massless scalar field. The anti-Fisher solution consists of three branches, one of them, termed Branch C in this paper,
represents wormholes [9,10], the others, A and B, also have properties of interest, and some of them
describe so-called cold black holes [11,12], which possess horizons of infinite area and zero Hawking
temperature; in all such solutions there are throats (minima of the spherical radius), but, unlike
wormholes, here beyond the throat one does not find an asymptotically flat region.

A number of solutions for phantom scalar fields with nonzero self-interaction potentials are also
known, among them are wormholes with flat and AdS asymptotic behaviours and a specific class
of regular black holes called black universes [13,14]. These are configurations where a possible
explorer, after crossing the event horizon, gets into an expanding universe instead of a singularity,
and moreover, this expanding universe, being initially highly anisotropic, eventually isotropizes and
approaches a de Sitter mode of expansion at late times. The stability of all such models under small
perturbations is an important test of their possible viability [15].

The purpose of this paper is to present a methodology of studying small radial perturbations of
scalar-vacuum configurations with any potential \( V(\phi) \) (see (1)), and in particular space-times with
throats. The difficulty with the latter consists in the fact that the effective potentials \( V_{\text{eff}}(x) \) for
perturbations (not to be confused with the self-interaction potential \( V(\phi) \)) always, in any gauge,
possess a singularity at the throat, which prevents a complete perturbation analysis. It is for this
reason that some previous stability studies of anti-Fisher solutions [11,16] did not find any unstable
mode whereas a numerical perturbation analysis of Shinkai and Hayward [17] revealed an instability
of the simplest representative of this family of solutions, the Ellis massless wormhole [9,10].

Gonzalez et al. [18], analyzing the stability of anti-Fisher wormholes, made a proper substitution
in the perturbation equation (a special case of the so-called S-deformation [19,20]) and regularized
the effective potential \( V_{\text{eff}} \). As a result, they found an exponentially growing mode with nonzero
perturbation of the throat radius, thus showing that the anti-Fisher wormholes are unstable. We
show here that a similar methodology can be applied to more general self-gravitating scalar field
configurations including those with arbitrary self-interaction potentials \( V(\phi) \). To this end, we prove
that, generically, (i) the effective potential \( V_{\text{eff}} \) has precisely the form required for regularization by
S-deformation and (ii) any solution of the transformed wave equation with a regularized potential
leads to a regular perturbation of the background static configuration.

As a particular example, we study the stability of all anti-Fisher solutions under spherically
symmetric perturbations. We prove the instability of Branch A and B solutions and confirm the
conclusions of [18] for Branch C solutions (wormholes).

2. Perturbation equations

2.1. Preliminaries

Consider a self-gravitating, minimally coupled scalar field with an arbitrary self-interaction potential
in general relativity. The Lagrangian is (up to a constant factor)

\[
L = \sqrt{-g} \left( R + \epsilon g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - 2V(\phi) \right),
\]

where \( \epsilon = 1 \) for a normal scalar field with positive kinetic energy and \( \epsilon = -1 \) for a phantom scalar field. Other notations are usual, the gravitational constant is absorbed in the definitions of \( \phi \) and
\( V(\phi) \). The field equations are

\[
\epsilon \Box \phi + V_\phi = 0,
\]

\[
R^\nu_\mu = -\epsilon \phi_\mu \phi^\nu + \delta^\nu_\mu V(\phi),
\]

where \( V_\phi \equiv dV/d\phi \).
The general spherically symmetric metric may be written in the form
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{2\gamma} dt^2 - e^{2\alpha} du^2 - e^{2\beta} d\Omega^2, \]
where \( \alpha, \beta, \gamma \) are functions of the radial coordinate \( u \) and the time coordinate \( t \) and \( d\Omega^2 = d\theta^2 + \sin^2 \theta \ d\phi^2 \). We will also use the notation \( r = e^\beta \) for the areal radius, such that \( 4\pi r^2 \) is the area of coordinate spheres \( u = \text{const}, \ t = \text{const} \). There remains a coordinate freedom in the \((u,t)\) subspace, which in general corresponds to choosing a reference frame preserving the spherical symmetry. In the static case, there is a reference frame such that there is no \( t \)-dependence, and then the coordinate freedom concerns the choice of the \( u \) coordinate.

Curvature singularities for static, spherically symmetric space-times are entirely determined using the Kretschamnn scalar, which, for the metric (4), may be written as a sum of squares,
\[ K = 4K_1^2 + 8K_2^2 + 8K_3^2 + 4K_4^2, \]
with
\[
K_1 = R_{01}^0 = -e^{-\alpha - \gamma} (\gamma' e^\gamma - \alpha'), \\
K_2 = R_{02}^0 = R_{03}^0 = -e^{-2\alpha} \beta' \gamma', \\
K_3 = R_{12}^{13} = R_{13}^{13} = -e^{-\alpha - \beta} (\beta' e^\beta - \alpha'), \\
K_4 = R_{23}^{23} = e^{-2\beta} - e^{-2\alpha} \beta'^2,
\]
where the prime denotes \( d/du \). The structure of Eq. (5) indicates that an infinite value of any \( K_i \) at some value of \( u \) implies the presence of a singularity at this \( u \). All these expressions for \( K_i \) are invariant with respect to the choice of the \( u \) coordinate.

We will assume that a certain static, spherically symmetric solution to Eqs. (2) and (3) is known and study its stability under small spherically symmetric perturbations. We thus consider, instead of \( \phi(u) \), a perturbed unknown function
\[ \phi(u, t) = \phi(u) + \delta \phi(u, t) \]
and similarly for the metric functions \( \alpha, \beta, \gamma \), where \( \phi(u) \), etc., are taken from the static solutions.

Preserving only linear terms with respect to time derivatives, we can write all the nonzero component of the Ricci tensor and the time-time component of the Einstein tensor as
\[
R_0^0 = e^{-2\gamma}(\ddot{\alpha} + 2\dot{\beta}) - e^{-2\alpha}[\gamma'' + \gamma'(\gamma' - \alpha' + 2\beta')], \\
R_1^1 = e^{-2\gamma} \dot{\alpha} - e^{-2\alpha}[\gamma'' + 2\beta'' + \gamma' + 2\beta' - \alpha'(\gamma' + 2\beta')], \\
R_2^2 = e^{-2\beta} + e^{-2\gamma} \dot{\beta} - e^{-2\alpha}[\beta'' + \beta'(\gamma' - \alpha' + 2\beta')], \\
R_{01} = 2[\dot{\beta}' + \dot{\beta} - \dot{\alpha}' - \dot{\beta'} \gamma'], \\
G_0^0 = e^{-2\alpha}[2\beta'' + \beta'(3\beta' - 2\alpha') - e^{-2\beta},
\]
where dots and primes denote \( \partial/\partial t \) and \( \partial/\partial u \), respectively.

### 2.2. General form of the field equations

The zero-order (i.e., static) scalar, \((0_0), (1_1), (2_2)\) components of Eqs. (3) and the Einstein equation \( G_0^0 = \ldots \) are
\[ \phi'' + \phi'(\gamma' + 2\beta' - \alpha') = \epsilon e^{2\alpha} V_\phi, \]
The first-order perturbed equations (scalar, $R_{01} = \ldots$, $R_{2} = \ldots$, and $G_{0}^{0} = \ldots$) read

\begin{align*}
    e^{2\alpha-2\gamma}\delta \phi - \delta \phi'' & = -\delta \phi'(\gamma' + 2\beta' - \alpha') - \phi'(\delta \gamma' + 2\delta \beta' - \delta \alpha') + \epsilon \delta(e^{2\alpha}V_{\phi}) = 0, \\
    \delta \beta' + \beta'\delta \beta - \beta'\delta \alpha - \gamma'\delta \beta & = -\frac{1}{2}\epsilon \phi'\delta \phi, \\
    \delta(e^{2\alpha-2\beta}) + e^{2\alpha-2\gamma}\delta \beta - \delta \beta'' - \delta \beta'(\gamma' + 2\beta' - \alpha') & = \delta(e^{2\alpha}V), \\
    - \delta(e^{2\alpha-2\beta}) + \delta \beta'' + 6\beta'\delta \beta - 2\beta'\delta \alpha - 2\alpha'\delta \beta & = -\epsilon \phi'\delta \phi - \delta(V e^{2\alpha}).
\end{align*}

Eq. (18) may be integrated in $t$; since we are interested in time-dependent perturbations, we omit the appearing arbitrary function of $u$ describing static perturbations and obtain

\begin{equation}
    \delta \beta' + \beta'\delta \beta - \gamma'\delta \beta = -\frac{1}{2}\epsilon \phi'\delta \phi.
\end{equation}

Let us note that we have two independent forms of arbitrariness: one is the freedom of choosing a radial coordinate $u$, the other is a perturbation gauge, or, in other words, a reference frame in the perturbed space-time, which can be expressed in imposing a certain relation for $\delta \alpha$, $\delta \beta$, etc. In what follows we will employ both kinds of freedom. All the above equations have been written in the most universal form, without coordinate or gauge fixing.

### 2.3. Gauge $\delta \beta \equiv 0$

This is technically the simplest gauge, in particular, it is convenient for considering usual black hole perturbations, but causes certain difficulties when applied to wormholes and other configurations with throats. The reason is that the assumption $\delta \beta = 0$ leaves invariable the throat radius, while perturbation must in general admit its time dependence [11]. This problem will be discussed below.

With $\delta \beta = 0$, Eq. (21) expresses $\delta \alpha$ in terms of $\delta \phi$:

\begin{equation}
    2\beta'\delta \alpha = \epsilon \phi'\delta \phi.
\end{equation}

Eq. (19) expresses $\delta \gamma' - \delta \alpha'$ in terms of $\delta \alpha$ and $\delta \phi$:

\begin{equation}
    \beta'(\delta \gamma' - \delta \alpha') = 2e^{2\alpha-2\beta}\delta \alpha - \delta(e^{2\alpha}V).
\end{equation}

Substituting all this into (17), we obtain the wave equation

\begin{equation}
    e^{2\alpha-2\gamma}\delta \phi - \delta \phi'' - \delta \phi'(\gamma' + 2\beta' - \alpha') + U\delta \phi = 0,
\end{equation}

where

\begin{equation}
    U \equiv e^{2\alpha}\left\{\epsilon(V - e^{-2\beta})\frac{\phi'^{2}}{\beta^{2}} + \frac{2\phi'}{\beta}V_{\phi} + \epsilon V_{\phi\phi}\right\}.
\end{equation}

Before proceeding with a study of the wave equation, let us make sure that all the remaining Einstein equations hold as a consequence of (22), (17) and (19) and do not lead to any new restrictions. Consider the component (20) (the constraint equation). It now takes the form

\begin{equation}
    2\delta \alpha e^{2\alpha-2\beta} + 2\beta'\delta \alpha' = \epsilon \phi'\delta \phi + \delta(V e^{2\alpha}).
\end{equation}
This equation holds automatically owing to the zero-order equations and (22). Indeed, a substitution of $\delta\alpha$ from (22) brings (25) to the form

$$\frac{\delta\phi}{\beta} \left[ \beta'\phi'' - \phi'\beta'' + e^{2\alpha-2\beta}\phi' - e^{2\alpha}V\phi' - \epsilon e^{2\alpha}V\phi' \right] = 0,$$

(26)

Now, substituting $\phi''$ and $\beta''$ from (12) and (15), respectively, we see that all terms cancel, i.e., the equation does hold. Furthermore, the Einstein equation $G^1_1 = \ldots$ holds as a consequence of (19) and (20); lastly, the equation $G^2_2 = \ldots$ holds due to the Bianchi identity $\nabla_\rho G^\rho_1 = 0$ and the corresponding property of the stress-energy tensor of the scalar field.

So we can return to Eq. (24). Passing on to the “tortoise” coordinate $x$ introduced according to

$$du/dx = e^{\gamma-\alpha}$$

(27)

and changing the unknown function $\delta\phi \mapsto \psi$ according to

$$\delta\phi = \psi(x,t) e^{-\beta}, \quad \Leftrightarrow \quad \psi(x,t) = r\delta\phi,$$

(28)

we reduce the wave equation to its canonical form, also called the master equation for radial perturbations:

$$\ddot{\psi} - \psi_{xx} + V_{\text{eff}}(x)\psi = 0,$$

(29)

(the index $x$ denotes $d/dx$), with the effective potential

$$V_{\text{eff}}(x) = e^{2\gamma-2\alpha}[U + \beta'' + \beta'^2 + \beta'(\gamma' - \alpha')].$$

(30)

This effective potential was previously obtained in other notations for $\epsilon = -1$ in [21]. A further substitution

$$\psi(x,t) = y(x) e^{i\omega t}, \quad \omega = \text{const},$$

(31)

which is possible because the background is static, leads to the Schrödinger-like equation

$$y_{xx} + [\omega^2 - V_{\text{eff}}(x)]y = 0.$$

(32)

If there is a nontrivial solution to (32) with $\text{Im}\,\omega < 0$ satisfying some physically reasonable conditions at the ends of the range of $u$ (in particular, the absence of ingoing waves), then the static system is unstable since $\delta\phi$ can exponentially grow with $t$. Otherwise our static system is stable in the linear approximation. Thus, as usual in such studies, the stability problem is reduced to a boundary-value problem for Eq. (32) — see, e.g., [11,18,22–26].

Note that all the above relations are written without fixing the background radial coordinate $u$.

2.4. Gauge-invariant perturbations

To be sure that we are dealing with real perturbations of the static background rather than purely coordinate effects, it is necessary to construct gauge-invariant quantities.

Small coordinate transformations $x^a \mapsto x^a + \xi^a$ in the $(t,u)$ subspace can be written as

$$t = \bar{t} + \Delta t(t,u), \quad u = \bar{u} + \Delta u(t,u),$$

(33)

where $\Delta t$ and $\Delta u$ are supposed to be small. Any scalar quantity with respect to such transformations, such as, e.g., $\phi(t,u)$ acquires an increment:

$$\Delta\phi = \dot{\phi}\Delta t + \phi'\Delta u \approx \phi'\Delta u$$

(34)
in the linear approximation since both $\dot{\phi}$ and $\Delta t$ are small. The quantity $r$, also being a scalar in the $(t,u)$ subspace (a 2-scalar, for short), behaves in the same way. If there are perturbations $\delta \phi$ and $\delta r$, the transformation (33) changes them as follows:

$$
\delta \phi \mapsto \delta \phi = \delta \phi + \phi' \Delta u,
$$

$$
\delta r \mapsto \delta r = \delta r + r' \Delta u.
$$

(35)

It then follows that the combination

$$
\psi_1 \equiv r' \delta \phi - \phi' \delta r,
$$

(36)

is invariant under the transformation (33), or gauge-invariant. Recall that the prime here denotes $d/du$ in the background static configuration.

One can notice that combinations constructed like (36) from any 2-scalars (for example, such as $e^\phi$ and $\beta = \ln r$, or two different linear combinations of $\phi$ and $r$) are also gauge-invariant. Moreover, gauge-invariant is $\psi_1$ multiplied by any 2-scalar or any combination of background quantities which are known and fixed functions of $u$.

The physical properties of perturbations must not depend on which gauge-invariant quantity $\psi$ is chosen to describe them. Meanwhile, with different $\psi$, the effective potentials will, in general, also be different. However, given a specific background configuration, in order that the theory be consistent, these different potentials should lead to the same perturbation spectrum.

Due to gauge invariance of $\psi_1$, equations that govern it may be written in any admissible gauge, in particular, the gauge $\delta \beta = 0$, and Eq. (29) for $\psi$ then may be considered as a result of substituting $\psi = (r/r') \psi_1 = r \delta \phi$ in a manifestly correct equation for $\psi_1 = r' \delta \phi$.

On the similar problem regarding cosmological perturbations and the definition of the corresponding gauge invariants, see, e.g., the review [27].

### 2.5. Regularized potential near a throat

The gauge $\delta \beta = 0$ (the same as $\delta r = 0$) is suitable for describing the perturbations at any points except those where $r' = 0$: these are throats and other critical points of $r(u)$. Indeed [11], putting $\delta r = 0$, we forbid perturbations of the throat radius, while there is no physical reason for that. Technically, this restriction manifests itself in a generically infinite value of the potential $U(u)$ in Eq. (24) and consequently in $V_{\text{eff}}$ involved in the wave equation (29). Throats are only possible in the case $\epsilon = -1$, and, provided $U/r^2 < 1$ at such a throat ($u = u_{th}$), the potential has there a wall of infinite height, with the generic behavior $V_{\text{eff}} \sim 1/(u - u_{th})^2$ near the throat since we have there generically $r'(u) \sim u - u_{th}$. As a result, perturbations are actually independent at different sides of the throat, necessarily turn to zero at the throat itself, and we thus partly lose information on their possible properties. Such an incomplete treatment has led to a conclusion that anti-Fisher wormholes [9, 10] (that is, wormhole solutions to Eqs. (2), (3) with $V \equiv 0$ and $\epsilon = -1$) were stable under spherically symmetric perturbations. A similar conclusion was made in [11] concerning such wormholes and cold black holes by using another (harmonic) gauge, $\delta \alpha = 2 \delta \beta + \delta \gamma$, which does not lead to a pole in the effective potential, but, as follows from our further consideration in this paper, this analysis was also incomplete.

It could seem that the above difficulty only concerns the gauge $\delta \beta = 0$. However, due to the gauge-invariant nature of Eq. (29) (to be verified below), it is clear that the problem is inherent to the background geometry itself, and the pole in the effective potential always emerges at a throat, if any.
A way of avoiding the restriction $\delta \beta(u_{\text{th}}) = 0$ is connected with the so-called $S$-deformations of the potential $V_{\text{eff}}$. This method was used in [19, 20] for transforming a partly negative potential to a positive-definite one in master equations for perturbations of higher-dimensional black holes. Using this method, Gonzalez et al. [18] transformed a singular potential to a nonsingular one for perturbations of the anti-Fisher wormholes and discovered the existence of an exponentially growing mode, showing that such wormholes are unstable. We will try to formulate a similar scheme suitable for the more general field system (1).

Consider a wave equation of the type (29)

$$\ddot{\psi} - \psi_{xx} + W(x)\psi = 0,$$

with an arbitrary potential $W(x)$ (whose specific example is the above potential $V_{\text{eff}}$). If there is a function $S(x)$ such that $W(x)$ is presented in the form

$$W(x) = S^2(x) + S_x,$$

then Eq. (37) is rewritten as follows:

$$\ddot{\psi} + (\partial_x + S)(-\partial_x + S)\psi = 0.$$  (39)

Now, if we introduce the new function

$$\chi = (-\partial_x + S)\psi,$$  (40)

then, applying the operator $-\partial_x + S$ to the left-hand side of Eq. (39), we obtain the following wave equation for $\chi$:

$$\ddot{\chi} - \chi_{xx} + W_{\text{reg}}(x)\chi = 0,$$  (41)

with the new effective potential

$$W_{\text{reg}}(x) = -S_x + S^2 = -W(x) + 2S^2.$$  (42)

If a static solution $\psi_s(x)$ of Eq. (37) is known, so that $\psi_{s,xx} = W(x)\psi_s$, then we can choose

$$S(x) = \psi_{s,x}/\psi_s$$  (43)

to carry out the above transformation.

Generically, the function $U$ in (24) and hence the potential (30) behave near a throat as $r'^{-2} \sim (u - u_{\text{th}})^{-2} \sim x^{-2}$, where, without loss of generality, we put $x = 0$ at the throat. Assuming that the potential $W(x)$ behaves in such a way, let us look if a transition to $W_{\text{reg}}$ can really remove this singularity. Above all, we see that according to (42), such removal is only possible if $W \to +\infty$ as $x \to 0$ since we are dealing with real quantities. Thus a potential wall in $W(x)$ can be removed but a potential well cannot.

A positive pole $W \sim x^{-2}$ can be removed in $W_{\text{reg}}$ if

$$S \approx -1/x \quad \Rightarrow \quad \psi_s \propto 1/x.$$  (44)

It is a necessary condition for regularizing the potential. Besides, to avoid a singularity of $W_{\text{reg}}$, $\psi_s$ must be nonzero in the whole range of $u$. Moreover, according to (42), it is clear that near the throat $x = 0$ in this case

$$W(x) \approx 2/x^2.$$  (45)
Thus, to be regularized by the procedure described, the pole in $W(x)$ must behave as (45). Let us show that this is generically the case for the potential $W = V_{\text{eff}}(x)$ given by Eq. (30).

Suppose such a generic situation, so that

(i) the function $\beta(x)$ in the background metric is expanded near its minimum (the throat) in powers of $x$ as follows:

$$\beta(x) = \beta_0 + \frac{1}{2}\beta_2 x^2 + \frac{1}{6}\beta_3 x^3 + \ldots \quad (46)$$

where $\beta_{0,2,3}$ are constants;

(ii) the background quantity $\phi'(x) \neq 0$ at $x = 0$.

Here and till the end of the section, we use the coordinate freedom to choose the “tortoise” radial coordinate $u = x$ specified by the condition $\alpha = \gamma$. All functions are considered as power series in $x$ at small $x$.

Let us estimate $V_{\text{eff}}$ (30). The term that determines the pole at $x = 0$ is

$$W_{\text{pole}}(x) = e^{2\gamma}(V - e^{-2\beta \phi'^2/\beta^2}), \quad (47)$$

where we have put $\epsilon = -1$ since throats are possible only in this case. By (46),

$$\beta_x = \beta_2 x + \frac{1}{2}\beta_3 x^2 + \ldots \quad (48)$$

Now we use the equations governing the static configuration: from (15) it follows that $e^{2\gamma}(e^{-2\beta} - V) = \beta_{xx} x$ at $x = 0$, and then from (16) we find that $\phi^2_x = 2\beta_{xx} x$ at $x = 0$. Thus from assumption (ii) it follows $\beta_2 \neq 0$. Substituting all this to (47), we find that it behaves precisely as required in (45).

Thus we have shown that, for a (generic) throat in a solution to Eqs. (12)–(16), the effective potential $V_{\text{eff}}$ for spherically symmetric perturbations satisfies the necessary condition for regularization by the above method.

Whether or not this regularization really works and leads to a regular boundary-value problem for the perturbations, should be investigated for specific background configurations. (In particular, one should also take into account the other singular term in the potential $V_{\text{eff}}$, proportional to $V_{\phi}/\beta_x$: the terms $\propto 1/x$ evidently depend on the finite part of $S(x)$.)

A positive example of such a study, concerning anti-Fisher wormholes, is known from [18]; we here show that the other two branches of the anti-Fisher solution are also such examples.

2.6. Regular perturbations near a throat

Suppose we have found a solution $\chi(x, t)$ to Eq. (41), satisfying the appropriate boundary conditions. The function $\chi$ is regular at $x = 0$ since the potential $W_{\text{reg}}$ is regular there. If $\chi$ is a growing function of $t$, it probably indicates an instability of the initial static configuration; but it is indeed the case only if this $\chi(x, t)$ creates regular perturbations of the metric functions $\alpha$, $\beta$, $\gamma$ and the scalar field $\phi$.

Let us look how it happens. Given $\chi(x, t)$, a solution to (37), or (29), is found as $\psi = (\partial_x + S)\chi$. Generically, $\chi$ is finite at $x = 0$ while $S \approx 1/x$ at small $x$, hence $\psi \sim 1/x$, and according to (28) we obtain $\delta \phi \to \infty$ at the throat. This result is in fact quite natural since the relation (28) corresponds to the gauge $\delta r = 0$, in which the throat radius is fixed, whereas we were seeking perturbations
with nonzero $\delta r$ on the throat. So it is necessary to pass on to another gauge, which is easily done due to gauge invariance of the quantity $\psi$ given by
\[ \psi = r\delta\phi - \frac{r\phi_x}{r_x}\delta r. \] (49)

Namely, a finite expression for $\delta r$ is obtained in the gauge $\delta\phi = 0$ provided $\phi_x(0) \neq 0$ since then
\[ \delta r = -\frac{r_x}{r\phi_x}\psi, \] (50)
while the product $r_x\psi$ is finite. It remains to find $\delta\alpha$ and $\delta\gamma$ from the perturbation equations in the gauge $\delta\phi = 0$. From Eqs. (18) and (17) we find
\[ \beta_x\delta\alpha = \delta\beta_x + \delta\beta(\beta_x - \gamma_x), \] (51)
\[ \delta\gamma_x = \delta\alpha_x - 2\delta\beta_x - \frac{2\epsilon}{\phi_x}V_\phi e^{2\alpha}\delta\alpha, \] (52)
but here we are again facing a problem: according to (51), in general $\delta\alpha$ diverges at the throat where $\beta_x = 0$. This divergence is only avoided if the right-hand side of (51) behaves like $\beta_x \sim u - u_{th} \sim x$.

Surprisingly, it is the case in a generic situation, as can be verified in a general form using near-throat expansions. Indeed, let us preserve the above assumptions (i) and (ii) and assume, in addition, that

(iii) the function $\chi(x,t)$ that solves Eq. (41) is finite and nonzero at $x = 0$,
(iv) the function $S(x)$ behaves at small $x$ according to (44).

Our task is to estimate the right-hand side of Eq. (51) in the order $O(x^0)$: if it is zero, it means that $\beta_x\delta\alpha \sim x \sim \beta_x$ and thus $\delta\alpha(0)$ is finite; the remaining metric perturbation $\delta\gamma$ is then found from (52) and is also finite.

Taking $\delta\beta = \delta r/r$ from (50) and substituting $\psi$ as $\psi = \chi_x + S\chi \sim -\chi_0/x$, where $\chi_0 = \chi(0)$, we find
\[ \delta\beta = \frac{\chi_0 e^{-\beta}}{\phi_x}(\beta_2 + \frac{1}{2}\beta_3x + \ldots). \]
Then we substitute this expression to (51) to obtain
\[ \beta_x\delta\alpha \approx \frac{\chi_0 e^{-\beta}}{\phi_x^2}\left[-\phi_{xx}\beta_2 + \frac{1}{2}\phi_x(\beta_3 - 2\gamma_x\beta_2)\right], \]
where all quantities are taken at $x = 0$. Now, $\phi_{xx}$ can be expressed from the background equation (12), $\beta_{xx}$ from (15); we can use the fact that $\beta_2 = \beta_{xx}(0)$ etc., and we can also ignore all terms proportional to $\beta_x$. After these substitutions we finally obtain
\[ \beta_x\delta\alpha \propto e^{2\gamma}V_\phi \left[\epsilon_{xx} + \frac{1}{2}\phi_x^2\right]_{x=0}. \]
But the expression in the square brackets vanishes due to the difference of Eqs. (13) and (14), which proves that $\beta_x\delta\alpha = O(x)$ and thus $\delta\alpha(0)$ is finite.

We conclude that under the generic assumptions (i)–(iv), regularization of the potential $V_{\text{eff}}$ always leads to finite perturbations of the background static solution.
3. Instabilities of the Fisher and anti-Fisher solutions

3.1. The static solutions

Let us recall the well-known static, spherically symmetric solutions to the field equations (2) and (3) for zero potential, \( V \equiv 0 \). In the case \( \epsilon = +1 \) this solutions was found by I.Z. Fisher in 1948 [8] and afterwards repeatedly re-discovered. For \( \epsilon = -1 \) the corresponding solution was first obtained, to our knowledge, by Bergmann and Leipnik [28] and also repeatedly re-discovered. But these authors used the curvature coordinates [i.e., the condition \( u \equiv r \) in terms of the metric (4)], which are not well suited for the problem, and maybe therefore they did not give a clear interpretation of the solutions.

The solution can be written jointly for \( \epsilon = \pm 1 \) if one uses the harmonic coordinate \( u = v \) in the metric (4), corresponding to the coordinate condition \( \alpha(v) = \gamma(v) + 2\beta(v) \) [10]. The scheme of obtaining it is as follows. In these coordinates, the scalar field equation and the \((0)\) component of (3) read simply \( \phi'' = 0 \) and \( \beta'' = 0 \), respectively (the prime here stands for \( d/dv \)), so that, choosing the zero point of \( \phi \) and the time scale, without loss of generality we can write \( \phi = Cv \) and \( \gamma = -mv \) with \( C, m = \text{const} \). Furthermore, a sum of the components \((0)\) and \((2)\) of (3) leads to the Liouville equation \( (\beta + \gamma)'' = e^{2\beta+2\gamma} \) whose solution is

\[
e^{-\beta-\gamma} = s(k, v) := \begin{cases} 
  k^{-1} \sinh kv, & k > 0 \\
  v, & k = 0 \\
  k^{-1} \sin kv, & k < 0.
\end{cases}
\]  

As a result, the solution as a whole reads

\[
ds^2 = e^{-2mv}dt^2 - e^{2mv} \left[ \frac{dv^2}{s^2(k, v)} + d\Omega^2 \right], \quad \phi = Cv,
\]  

where the integration constants \( m \) (the Schwarzschild mass), \( C \) (the scalar charge) and \( k \) are related by the equality

\[
2k^2 \text{sign} k = 2m^2 + \epsilon C^2,
\]  

obtained after substituting all found functions to the Einstein equation \( G^1_1 = \ldots \) which does not contain second-order derivatives in \( v \) and is an integral of other equations.

The coordinate \( v \) is defined in the whole range \( v > 0 \) for \( k \geq 0 \) and in the range \( 0 < v < \pi/|k| \) for \( k < 0 \). The value \( v = 0 \) in all cases corresponds to flat spatial infinity, so that at small \( v \) the spherical radius is \( r(v) \approx 1/v \), and the metric becomes approximately Schwarzschild.

In the case \( k > 0 \), it is helpful to pass over to the quasiglobal coordinate coordinate \( u \) (defined by the condition \( \alpha + \gamma = 0 \) in (4)) by the transformation

\[
e^{-2kv} = 1 - 2k/u =: P(u),
\]  

and the solution takes the form

\[
ds^2 = P^a dt^2 - P^{-a} du^2 - P^{1-a} u^2 d\Omega^2, \quad \phi = -\frac{C}{2k} \ln P(u),
\]  

with the constants related by

\[
a = m/k, \quad a^2 = 1 - \epsilon C^2/(2k^2).
\]  

The Fisher solution [8] corresponds to \( \epsilon = +1 \), hence according to (55), it consists of a single branch \( k > 0 \) and, in (57), \(|a| < 1 \). It is defined in the range \( u > 2k \), and \( u = 2k \) is a naked central
(r = 0) singularity which is attractive for m > 0 and repulsive for m < 0. The Schwarzschild solution is restored at C = 0, a = 1 for m > 0 and at C = 0, a = −1 for m < 0.

The solution for ε = −1 (that is, for a phantom scalar field) is conveniently termed the anti-Fisher solution, by analogy with de Sitter and anti-de Sitter. According to three variants of the function (53), this solution splits into three branches with the following properties.

Branch A, k > 0: the solution again has the form (57), but now |a| > 1. For m < 0, that is, a < −1, we have, just as in the Fisher solution, a repulsive central singularity at u = 2k. The situation is, however, drastically different for a > 1.

Indeed, the spherical radius r in all such cases has a finite minimum at u = u_{th} = (a + 1)k, corresponding to a throat of the size
\[ r(u_{th}) = r_{th} = k(a + 1)^{(a+1)/2}(a - 1)^{(1-a)/2}, \]
and tends to infinity as u → 2k. Moreover, for a = 2, 3, ... the metric exhibits a horizon of order a at u = 2k and admits a continuation to smaller u [12]. A peculiarity of such horizons is their infinite area. Such asymptotically flat configurations with horizons of infinite area have been termed cold black holes (CBHs) [11] since all of them have zero Hawking temperature. The throat radius (59) does not coincide with the Schwarzschild mass m = ak (in usual units, half the Schwarzschild radius 2Gm/c²) but is of the same order of magnitude.

Furthermore, it can be verified using (6) [12] that the metric (57) has a curvature singularity at u = 2k if a < 2 (except for the Schwarzschild case a = 1), a finite curvature if a = 2 and zero curvature if a > 2.

For non-integer a > 2, the qualitative behavior of the metric as r → 2k is the same as near a horizon of infinite area, but a continuation beyond it is impossible due to non-analyticity of the function P^a(u) at u = 2k. Since geodesics terminate there at a finite value of the affine parameter, this is a space-time singularity (a singular horizon as it is named in [11]) even though the curvature invariants tend there to zero.

Branch B, k = 0: the solution is defined in the range u ∈ ℝ⁺ and is rewritten in terms of the quasiglobal coordinate u = 1/v as follows:
\[ ds^2 = e^{-2m/u}dt^2 - e^{2m/u}[du^2 + u^2dΩ^2], \quad φ = C/u. \]
As before, u = ∞ is a flat infinity, while at the other extreme, u → 0, the behavior is different for positive and negative mass. Thus, for m < 0, u = 0 is a singular center (r = 0), while for m > 0, r → ∞ and all Kᵢ → 0 as u → 0. This is again a singular horizon: despite the vanishing curvature, the non-analyticity of the metric in terms of u makes its continuation impossible. The throat occurs at u = m and has the size e · m, e being the base of natural logarithms.

Branch C, k < 0: the solution describes a wormhole with two flat asymptotics at v = 0 and v = π/|k|. The metric has the form [9, 10]
\[ ds^2 = e^{-2mv}dt^2 - \frac{k^2 e^{2mv}}{\sin^2(kv)} \left[ \frac{k^2}{\sin^2(kv)} du^2 + dΩ^2 \right] = e^{-2mv}dt^2 - e^{2mv}[du^2 + (k² + u²)dΩ²], \]
where v is expressed in terms of the quasiglobal coordinate u, defined on the whole real axis, by \( \tilde{u} = \cot^{-1}(u/k) \), where we have denoted −k = \( \tilde{k} > 0 \). If m > 0, the wormhole is attractive for ambient test matter at the first asymptotic (u → ∞) and repulsive at the second one (u → −∞), and vice versa in case m < 0. For m = 0 one obtains the simplest possible wormhole solution,
sometimes called the Ellis wormhole, although Ellis [9] actually discussed these solutions with any \( m \).

The wormhole throat occurs at \( u = m \) and has the size
\[
  r_{th} = (m^2 + \mathbf{k}^2)^{1/2} \exp \left( \frac{m}{k} \cot^{-1} \frac{m}{k} \right).
\]  

### 3.2. Perturbations: the Fisher solution

For a massless scalar field \([V \equiv 0 \text{ in (1)}]\), the effective potential in (29) takes the following form in terms of the quasiglobal coordinate \( u \):
\[
  V_{\text{eff}} = -\epsilon \frac{A\phi'^2}{r'^2} + A \frac{r''}{r} + A' \frac{r'}{r} = -\epsilon \frac{A\phi'^2}{r'^2} + \frac{A}{r^2} - \frac{A^2 r'^2}{r^2},
\]
where \( A(u) := e^{2\gamma(u)} = e^{-2\alpha(u)} \), and the second equality in (63) follows from Eq. (15).

Calculating \( V_{\text{eff}} \) for the solution (57), we find a common expression for both \( \epsilon = +1 \) and \( \epsilon = -1 \):
\[
  V_{\text{eff}}(u) = kp^{2a} 2au^3 - 3(1 + a)^2 k u^4 + 2(3 + 4a + 3a^2 + 2a^3)k^2 u - (1 + a)^4 k^3 u^2 (u - (1 + a))^2 (u - 2k)^2.
\]

Since in the Fisher solution \( a < 1 \), the binomial \( u - k(1 + a) \) is positive at all \( u \geq 2k \), and the only singularity in \( V_{\text{eff}} \) is \( u \to 2k \), coinciding with the singularity of the background solution. Near the singularity, at which according to (27) we can put \( x = 0 \), \( V_{\text{eff}}(u) \sim -1/(4x^2) \), a negative pole in agreement with [26] and many subsequent papers.

The boundary condition at spatial infinity \((u \to \infty, x \to \infty)\) is natural: \( \delta \phi \to 0 \), or \( \psi \to 0 \). For \( u \to 2k \), where the background field \( \phi \) tends to infinity, the boundary condition is not so evident. In [26] and other papers, dealing with minimally coupled or dilatonic scalar fields, the minimal requirement was used providing the validity of the perturbation scheme:
\[
  |\delta \phi / \phi| < \infty.
\]
(The requirement of absence of ingoing waves then does not lead to further restrictions.) Under this boundary condition it is easy to conclude that there are solutions to the Schrödinger-like equation (32) with any \( \omega^2 < 0 \), which means that the static field configuration is unstable, in agreement with the previous work [26] (see also [12] for details).

### 3.3. Perturbations: the anti-Fisher solution

**Branch A**

The effective potential has the same form (64), but now, since \( a > 1 \) (we restrict ourselves to this case providing \( m > 0 \)), the potential has a positive pole at \( u = u_{th} = k(a + 1) > 2k \), the throat in the background configuration.

To remove this singularity, we were able to find the following simple static solution to Eq. (29):
\[
  \psi_{s+}(u) \propto r(u) \frac{au - u_{th}}{u - u_{th}},
\]
(66)

Applying the technique described in Sec. 2.5 with these \( \psi_s(u) \), we the new effective potential \( W_{\text{reg}} = W_A(u) \):
\[
  W_A(u) = \left( 1 - \frac{2k}{u} \right)^{2a} \frac{N(u)}{u^2(u - 2k)^2 (au - u_{th})^2},
\]
\[
  N(u) = 3(1 + a)^4 k^4 - 2a(6 + 19a + 16a^2 + 3a^3)k^2 u
  + 3a^2(9 + 10a + a^2)k^2 u^2 - 2a^2(4 + 5a)k^3 u^3 + 2a^2 u^4.
\]  

(67)
The potential (67) has no singularities at \( r > 2k \) and is partly negative, which should in general lead to an instability. To prove it, one can use the method of time-domain integration [29] allowing for following the time evolution of the perturbations under prescribed initial and boundary conditions. The latter, as usual, must provide the absence of ingoing waves from the boundary, and in our case it is sufficient to simply require \( \psi \to 0 \) and \( \chi \to 0 \) as \( x \to \pm \infty \) for all three branches of the anti-Fisher solution. The reason is that the effective potential \( V_{\text{eff}} \) as well as the regularized potentials vanish at large \( |x| \), and therefore modes of interest, those with \( \omega^2 < 0 \), should exponentially decay at large \( |x| \).

Examples of plots for the potential (67) and the results of time-domain integration are shown in Fig. 1. By fitting of the profile we find that the perturbations grow approximately as \( \psi \propto e^{0.25t/m} \).

In the limit \( a \to 1 \) the regularized potential still has a negative gap, however, at \( a = 1 \) no growing mode is observed, and a stationary solution dominates at late times (Fig. 2), while at any finite \( a - 1 \) the perturbation does grow. This shows how the instability is “dying out” when approaching the Schwarzschild solution. One can recall that in the genuine Schwarzschild case there is no scalar field, and the modes we are considering here simply do not exist.

Figure 1: The regularized effective potential \( W_A \) (left panel) and the time-domain profiles (right panel) for the Branch A solutions with \( a = 3/2 \) (blue), \( a = 2 \) (green), \( a = 3 \) (red). Smaller values of \( a \) correspond to deeper potential wells and a more rapid growth of perturbations.

Figure 2: The potential \( W_A \) (left panel) and the time-domain profile (right panel) for the Branch A solution with \( a = 1 \).
Figure 3: The potential $W_B$ (left panel) and the time-domain profile (right panel) for the Branch B solution.

**Branch B**

For the solution (60), the effective potential (63)

$$V_{\text{eff}}(u) = m \exp \left(-\frac{4m}{u} \right) \frac{2u^3 - 3mu^2 + 4m^2u - m^3}{(u - m)^2u^4}$$

(68)

is singular at $u_{\text{th}} = m$. We again find a static solution to (29)

$$\psi_{s0}(u) \propto \frac{u^2 e^{m/u}}{u - m}$$

(69)

and perform the transformation described in Sec. 2.5.

The regularized effective potential has the form of an inverse potential barrier:

$$W_B(u) = \exp \left(-\frac{4m}{u} \right) \frac{3m^2 - 10mu + 2u^2}{u^4}.$$  

(70)

The perturbations grow approximately as $\psi \propto e^{0.23t/m}$ (see an example in Fig. 3).

**Branch C**

For the wormhole solution (61), the potential (63) has the form

$$V_{\text{eff}}(u) = e^{-4mu} \frac{2k^4 + k^2(3m^2 - 2mu + 3u^2) - m(m^3 - 4m^2u + 3mu^2 - 2u^3)}{(m - u)^2(k^2 + u^2)^2}.$$ 

(71)

It again exhibits a positive pole on the throat. To regularize it, we can take any of the two static solutions

$$\psi_{s-1}(u) \propto \frac{r(u)}{u - m}[k^2 + m^2 - mv(k^2 + mu)],$$

(72)

$$\psi_{s-2}(u) \propto \frac{r(u)}{u - m}(k^2 + mu).$$

(73)

The first of them coincides with the solution found by González et al. in [18]. As a result, we reproduce their regular potential

$$W_{C1}(u) = e^{-4mu} \frac{N_1(u)}{(k^2 + u^2)^2(k^2(mv - 1) + m^2(uv - 1))^2}.$$ 

(74)
where
\[
N_1(u) = -3k^6(mv - 1)^2 + m^4(3m^2 - 10mu + 2u^2)(uv - 1)^2 \\
+ k^2m^3(uv - 1)(8u + 2m^2v + m(5 - 15uv)) + k^4m(mv - 1)(2u + m^2v + m(9 - 12uv)).
\]

A further investigation revealing an unstable mode of perturbations is described in detail in [18] (Fig. 4).

The second static solution (73) can be used to demonstrate that, despite the different regularized potentials, the physical result, namely, the perturbation growth rate remains the same. Indeed, the regularized potential produces by the solution (73) reads
\[
W_{C2}(u) = e^{-4mu} \frac{-3k^6 + k^4m(m - 12u) + k^2m^2(2m - 15u)u + m^2u^2(3m^2 - 10mu + 2u^2))}{(k^2 + mu)^2(k^2 + u^2)^2}, \tag{75}
\]

and leads to perturbations growing as shown in Fig. 5, at the same rate as in Fig. (4).

4. Concluding remarks

We have demonstrated the instability of all anti-Fisher solutions to the Einstein-scalar equations under spherically symmetric perturbations, thus confirming and extending the conclusions of [18]
made for Branch C solutions (wormholes). It turns out that in almost all cases the characteristic time of perturbation growth is of the order of the time needed for a light signal to cover a distance equal to the throat radius.

We have found out that the S-deformation method of regularizing the effective potential for the perturbations, having a positive pole at a throat (a minimum of the spherical radius \( r(u) \)), used in [18], is applicable to any static, spherically symmetric configurations of self-gravitating scalar fields with self-interaction potentials. We have shown in a general form that, under some generic assumptions [items (i)–(iv) in Sections 2.5 and 2.6], the potential \( V_{\text{eff}} \) near the throat has the form admitting regularization, and that a regular mode found as a solution to the regularized equation leads to a regular perturbation of the initial configuration. The latter circumstance is quite non-trivial because regularity is required not only for solutions to the master equation but also for all metric coefficients.

However, application of this methodology to specific solutions with self-interaction potentials \( V(\phi) \neq 0 \) faces the problem of explicitly finding the function \( S(x) \) satisfying Eq. (38), or, equivalently, a proper static solution to the master equation (a zero mode). We hope to extend this study to some configurations with nonzero \( V(\phi) \) in the near future.

One more important extension of the presently obtained results should be mentioned: Eqs. (2), (3) are well known to be not only the equations of general relativity but also the Einstein-frame equations of the general scalar-tensor and \( f(R) \) theories of gravity, see, e.g., [30–33]. Note that in \( f(R) \) theories, the Einstein frame always contains a scalar field with nonzero \( V(\phi) \). The observable physics in such theories is usually described in the corresponding Jordan conformal frame, whose metric differs from that of the Einstein frame only by a conformal factor that varies from theory to theory. A transition from one frame to the other is, from the viewpoint of differential equations, simply a substitution, therefore the stability study can safely be performed in the Einstein frame. Then, if the conformal factor is everywhere regular and nonzero (including the end points of the coordinate range), the conformal mapping preserves the boundary conditions for regular perturbations, and the stability conclusions obtained in the framework of general relativity are readily extended to the corresponding solutions of these generalized theories. In particular, we can assert that all vacuum static, spherically symmetric solutions in scalar-tensor theories, connected with the (anti-)Fisher solutions by everywhere regular conformal factors, are unstable.

However, in many cases the conformal factors bear nontrivial features, i.e., somewhere blow up or vanish, and this can affect the boundary conditions for the perturbation equations; in any such case a separate study is necessary. Consider, for instance, the counterpart of Fisher’s solution (57) in the Brans-Dicke (BD) scalar-tensor theory, where the Jordan-frame metric is \( g^J_{\mu \nu} = (1/\Phi)g_{\mu \nu} \); \( \Phi = \exp(\phi/\sqrt{\omega + 3/2}) \) is the BD scalar field, \( \omega > -3/2 \) is the BD coupling constant, and \(|a| < 1\). It is the so-called Brans class 1 solution. In both frames, the value \( u = 2k \) is a naked singularity (see more details on these solutions in [11, 34]). In the stability study, to formulate a boundary condition at this singularity, for Fisher’s solution we have used the minimal requirement (65), \(|\delta \phi/\phi| < \infty\), providing the validity of the perturbation scheme, and we then concluded that the background solution is unstable. In the BD picture, it is more reasonable to require that the perturbed conformal factor \( 1/\Phi \) behave not worse than the unperturbed one, i.e., \(|\delta \Phi/\Phi| < \infty\). However, since \( \delta \Phi/\Phi \sim \delta \phi \), we arrive at the condition \(|\delta \phi| < \infty\) which is more restrictive then (65) if \( \phi \to \infty \), and this made us conclude in [11] that this BD solution is stable.

Another example of using the conformal mapping between Jordan and Einstein frames for stability studies can be found in [22, 23], where the instability was proved for electrically neutral and charged wormholes supported by nonminimally (in particular, conformally) coupled scalar fields [10, 23, 35]. In this case, there is a drastic difference between the manifold structures in the
two conformal frames: in the Einstein frame, without an electric charge, it is the Fisher solution with the metric is \( g_{\mu\nu} \) that has a singularity at \( u = 2k \). In Jordan’s, this singularity is removed due to the conformal factor, the solution is continued beyond this (now regular) sphere and has a flat asymptotic at the other end. Such wormholes proved to be unstable [22,23], but their instability is of quite different nature than that described in this paper and in [18]: it is related to a negative pole of the effective potential (64) at \( u = 2k \) for Fisher’s solution (57), \( \epsilon = +1 \) and a similar singularity in its counterpart with an electric charge.

These examples show that there remains quite a lot of work in studying the stability of more complicated solutions of scalar-tensor and \( f(R) \) theories.

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