NECESSARY OPTIMALITY CONDITIONS FOR AN INTEGRO-DIFFERENTIAL BOLZA PROBLEM VIA DUBOVITSKII-MILYUTIN METHOD

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A tribute to Helmut Maurer, Urszula Ledzewicz and Heinz Schăttler.

Abstract. In the paper, we derive a maximum principle for a Bolza problem described by an integro-differential equation of Volterra type. We use the Dubovitskii-Milyutin approach.

1. Introduction. The aim of the paper is to derive optimality conditions for the optimal control problem described by the nonlinear integro-differential equation of Volterra type

\[
\begin{align*}
    x'(t) + \int_a^t \Phi(t, \tau, x(\tau), u(\tau))d\tau &= f(t, x(t), v(t)), \quad t \in J := [a, b] \text{ a.e.,} \\
    x(a) &= 0
\end{align*}
\]  

(1)

with constraints

\[
u \in M, \quad v \in N
\]  

(2)

and the nonlinear performance index of Bolza type

\[
F_0(x, u) = \int_a^b f_0(t, x(t), u(t), v(t))dt + g_0(x(b))
\]  

(3)

where \( \Phi : P_{\Delta} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, f : J \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n, f_0 : J \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}, g_0 : \mathbb{R}^n \to \mathbb{R}, M \subset \mathbb{R}^m, N \subset \mathbb{R}^r \) and

\[
P_{\Delta} = \{(t, \tau) \in J \times J; \quad \tau \leq t\}.
\]

Integro-differential equations have been introduced by Volterra in work [15]. In paper [16], he used such equations to study some electromagnetic phenomena and

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The second author has a great satisfaction to be a supervisor of the doctoral dissertation of Professor Urszula Ledzewicz.

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in [17], [18] - elasticity ones. Actually, integro-differential equations occur in many branches of physics to describe the objects the state of which depends on the history. Most of existing results concern the existence, uniqueness of solutions as well as the continuous dependence of solutions on parameters and numerical methods. In our paper we deal with an optimal control problem of Bolza type for the system described by such an equation. Our aim is to derive a Pontriagin maximum principle with the aid of the Dubovitskii-Milyutin approach. A systematic exposition of the Dubovitskii-Milyutin method introduced in papers [2] and [3], is contained in the book [4]. Some generalization of this method to the case of many equality constraints described by linear operators has been proposed by Professor Urszula Ledzewicz in her doctoral dissertation. These results and some further ones in this field have been published, among others, in [10] - [13].

Our paper consists of three main parts. In the first part, we recall basics of the Dubovitskii-Milyutin method. In the second part, we derive differentiability properties of the cost functional $F_0$ and an operator $(F)$ describing equality constraints. In the third part, we derive a maximum principle for the problem under consideration. To the best of our knowledge, such a maximum principle has not been obtained up to now and the Dubovitskii-Milyutin approach was not applied to the problem (1)-(2)-(3) by other authors.

2. Preliminaries. In this section, we give some basics concerning Dubovitskii-Milyutin method. They can be found in [4].

Let $X$ be a linear topological space with the dual space (the space of linear continuous functionals on $X$) denoted by $X'$. If $K$ is a cone in $X$ with vertex at the point $0$ (by a cone with vertex at $0$ we mean a set $K$ such that $tK = K$ for any $t > 0$), then the conjugate cone $K^*$ we define by

$$K^* = \{ f \in X'; f(x) \geq 0 \text{ for any } x \in K \}.$$

Of course, the conjugate cone is the convex cone with vertex at $0$.

Basic notions in the Dubovitskii-Milyutin method are the following.

**Definition 2.1.** Let $F : X \to \mathbb{R}$ be a functional. We say that a vector $h \in X$ is a direction of decrease of functional $F$ at a point $x_0$ if there exist a neighborhood $U$ of $h$, $\varepsilon_0 > 0$ and $\alpha < 0$ such that

$$F(x_0 + \varepsilon h) \leq F(x_0) + \varepsilon \alpha$$

for $\varepsilon \in (0, \varepsilon_0)$, $h \in U$.

**Definition 2.2.** Let $Q$ be a subset of $X$. We say that a vector $h \in X$ is a feasible direction for $Q$ at a point $x_0 \in Q$ if there exist a neighborhood $U$ of $h$ and $\varepsilon_0 > 0$ such that

$$x_0 + \varepsilon h \in Q$$

for $\varepsilon \in (0, \varepsilon_0)$, $h \in U$.

**Definition 2.3.** Let $Q$ be a subset of $X$. We say that a vector $h \in X$ is a tangent direction to $Q$ at a point $x_0 \in Q$ if there exist $\varepsilon_0 > 0$ and mapping $r : (0, \varepsilon_0) \to X$ such that

$$x_0 + \varepsilon h + r(\varepsilon) \in Q$$

for $\varepsilon \in (0, \varepsilon_0)$ and $\frac{r(\varepsilon)}{\varepsilon} \to 0$ (i.e. for any neighborhood $U$ of $0$ there exists $\varepsilon_1 > 0$ such that $\frac{r(\varepsilon)}{\varepsilon} \in U$ for $\varepsilon \in (0, \varepsilon_1)$; when $X$ is a Banach space, we write in such a case $r(\varepsilon) = o(\varepsilon)$).
One can show that the set of directions of decrease of functional \( F \) at a point \( x_0 \) and the set of feasible directions for the set \( Q \) at a point \( x_0 \) are open cones as well as the set of tangent directions to a set \( Q \) at a point \( x_0 \) is a cone.

Now, assume \( X \) is locally convex and consider the problem

\[
\begin{align*}
F(x) &\rightarrow \min, \\
x &\in Q := \bigcap_{i=1}^{n+1} Q_i,
\end{align*}
\]

where \( F : X \to \mathbb{R}, Q_i \subset X, i = 1, ..., n+1. \) We say that \( x_0 \in Q \) is a local minimum point of \( F \) if there exists a neighborhood \( V \) of \( x_0 \) such that

\[
F(x_0) = \min_{x \in Q \cap V} F(x).
\]

The central result of the Dubovitskiĭ-Milyuţin approach is the following theorem.

**Theorem 2.4 (Dubovitskiĭ-Milyuţin).** Let \( x_0 \) be a local minimum point for problem (4). Assume that the cone \( K_0 \) of directions of decrease of functional \( F \) at \( x_0 \) is nonempty and convex, cones \( K_i, i = 1, ..., n \), of feasible directions for the sets \( Q_i \) at \( x_0 \) are nonempty and convex and the cone \( K_{n+1} \) of tangent directions for the set \( Q_{n+1} \) at \( x_0 \) is nonempty and convex. Then there exist functionals \( f_i \in K_i^* \), \( i = 0, 1, ..., n+1, \) not all identically zero, such that

\[
f_0 + f_1 + \ldots + f_n + f_{n+1} = 0
\]

and if \( \bigcap_{i=1}^{n+1} K_i \neq \emptyset \), then \( f_0 \neq 0 \).

The introduced cones have nice characterizations in some situations.

**Proposition 1.** If \( X \) is a Banach space, \( F : X \to \mathbb{R} \) a functional differentiable at \( x_0 \) in Frechet sense, then the cone \( K_d \) of directions of decrease of functional \( F \) at \( x_0 \) has the form

\[
K_d = \{ h \in X; F'(x_0)h < 0 \}.
\]

**Theorem 2.5 (Lusternik).** If \( X, Y \) are Banach spaces, \( P : X \to Y \) operator of class \( C^1 \), \( P(x_0) = 0 \) and \( \text{Im} P'(x_0) = Y \), then the cone \( K_i \) of tangent directions for the set \( \{ x \in X; P(x) = 0 \} \) at \( x_0 \) is a subspace of the form

\[
K_i = \{ h \in X; P'(x_0)h = 0 \}.
\]

**Proposition 2.** If \( Q \) is a convex set in linear topological space \( E \), then the cone \( K_f \) of feasible directions for the set \( Q \) at a point \( x_0 \in \overline{Q} \) is convex and has the form

\[
K_f = \{ \rho(\text{Int}Q - x_0); \ \rho > 0 \}.
\]

We shall also use the following characterizations of the conjugate cones.

**Proposition 3.** Let \( E \) be a linear topological space, \( f \in E' \) and \( K = \{ x \in E; f(x) > 0 \} \). Then

\[
K^* = \begin{cases} 
E' & \text{if } f = 0 \\
\{ \lambda f; \ \lambda \geq 0 \} & \text{if } f \neq 0
\end{cases}.
\]

**Proposition 4.** Let \( Q \) be a convex closed set in linear topological space \( E \) and \( x_0 \in Q \). If \( \text{Int}Q \neq \emptyset \), then

\[
K_f^* = \{ g \in E'; g(x) \geq g(x_0) \text{ for } x \in Q \},
\]

where \( K_f \) is the cone of feasible directions for \( Q \) at \( x_0 \).

We shall also use the following lemma (see [9]).
Lemma 2.6. If $X, Y$ are Banach spaces and $\Lambda : X \to Y$ is linear bounded operator such that $\text{Im} \Lambda = Y$, then $(\text{ker} \Lambda)^\perp = \text{Im} \Lambda^*$ where $\Lambda^* : Y' \to X'$ is adjoint operator to $\Lambda$ and $(\text{ker} \Lambda)^\perp$ is the set of linear continuous functionals on $X$ vanishing on ker $\Lambda$.

The set $(\text{ker} \Lambda)^\perp$ is named the anulator of the subspace ker $\Lambda$.

3. Bolza problem. Let us consider the following problem

\begin{align}
F_0(x, u, v) & \to \min, \\
F(x, u, v) & = 0 \\
(u, v) & \in U \times V 
\end{align}

where

\[ F_0 : AC_0^2 \times L_m^\infty \times L_r^\infty \ni (x, u, v) \mapsto \int_a^b f_0(t, x(t), u(t), v(t))dt + g_0(x(b)) \in \mathbb{R}, \]

\[ F : AC_0^2 \times L_m^\infty \times L_r^\infty \ni (x, u, v) \mapsto x'(t) + \int_a^t \Phi(t, x(t), u(t))d\tau - f(t, x(t), v(t)) \in L^2, \]

with the set of solutions $AC_0^2 = AC_0^2(J, \mathbb{R}^n)$ (set of absolutely continuous functions possessing squared integrable derivatives, vanishing at $t = 0$) and the sets of functional parameters (controls)

\[ U = L_m^\infty(J, M) \subset L_m^\infty(J, \mathbb{R}^m) \]

\[ V = L_r^\infty(J, N) \subset L_r^\infty(J, \mathbb{R}^r) \]

consisting of essentially bounded functions taking their values in the sets $M \subset \mathbb{R}^m$, $N \subset \mathbb{R}^r$, respectively. On the sets $M$, $N$ we assume that they are convex closed with non-empty interiors.

3.1. Differential properties of $F$. In paper [5], an integro-differential equation of Volterra type of fractional order $\alpha \in (0, 1)$ (with derivatives in Riemann-Liouville sense) is investigated. Results on the existence and uniqueness of a solution have been obtained there as well as continuous differentiability property of the mapping assigning the corresponding solution to any functional parameters. Our initial problem (1) is a limit case of the system considered in [5], for $\alpha = 1$. The same proof as that of Lemma 4.1 from [5] (with $\alpha = 1$) gives the following theorem.

Theorem 3.1. If functions $\Phi$, $f$ satisfy the conditions

(A1) $\Phi(\cdot, x, u)$ is measurable on $P_\Delta$ for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; $\Phi(t, r, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m$ for $(t, r) \in P_\Delta$ a.e.

(A2) there exist functions $a_\Phi \in L^2(P_\Delta, \mathbb{R}_0^+)$, $\omega_\Phi \in C(\mathbb{R}_0^+ \times \mathbb{R}_0^+, \mathbb{R}_0^+)$ such that

\[ |\Phi(t, r, x, u)|, |\Phi_x(t, r, x, u)|, |\Phi_u(t, r, x, u)| \leq a_\Phi(t, r)\omega_\Phi(|x|, |u|) \]

for $(t, r) \in P_\Delta$ a.e., $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.

(B1) $f(\cdot, x, u)$ is measurable on $J$ for any $x \in \mathbb{R}^n$, $v \in \mathbb{R}^r$; $f(t, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^r$ for $t \in J$ a.e.

(B2) there exist functions $a_f \in L^2(J, \mathbb{R}_0^+)$, $\omega_f \in C(\mathbb{R}_0^+ \times \mathbb{R}_0^+, \mathbb{R}_0^+)$ such that

\[ |f(t, x, v)|, |f_x(t, x, v)|, |f_v(t, x, v)| \leq a_f(t)\omega_f(|x|, |v|) \]

for $t \in J$ a.e., $x \in \mathbb{R}^n$, $v \in \mathbb{R}^r$. 

then the mapping
\[ F : AC^2_0 \times L^\infty_{\sigma} \times L^\infty_r \rightarrow L^2_n, \]
\[ F(x, u, v) = x'(t) + \int_a^t \Phi(t, s, x(s), u(s))ds - f(t, x(t), v(t)) \]
is continuously differentiable in Gâteaux (equivalently, in Fréchet) sense and the
mapping
\[ \partial F(x, u, v) : AC^2_0 \times L^\infty_{\sigma} \times L^\infty_r \rightarrow L^2_n, \]
\[ \partial F(x, u, v) = \xi(t) + \int_a^t \Phi_x(t, \tau, x(\tau), u(\tau))\xi(\tau)d\tau \]
\[ + \int_a^t \Phi_u(t, \tau, x(\tau), u(\tau))g(\tau)d\tau - f_x(t, x(t), v(t))\xi(t) - f_v(t, x(t), v(t))h(t) \]
is the differential of \( F \) at \( (x, u, v) \).

In an analogous way as in [5, proofs of Lemma 4.2 and Lemma 6.1] (see also [7],
[8, Proof of Lemma 7]) one can obtain

**Theorem 3.2.** If assumptions of Theorem 3.1 are satisfied and the growth conditions
concerning \( \Phi_x \), \( f_x \) are replaced by the following ones

(A3) there exist a function \( c_\Phi \in L^2(P_\Delta, \mathbb{R}^p_0) \) and a constant \( C_\Phi \geq 0 \) such that
\[ |\Phi_x(t, \tau, x, u)| \leq c_\Phi(t, \tau)\omega_\Phi(|x|, |u|) \]
\[ \int_a^t c_\Phi^2(t, s)ds \leq C_\Phi \]
for \( (t, \tau) \in P_\Delta \) a.e., \( x \in \mathbb{R}^n \), \( v \in \mathbb{R}^r \),

(B3) there exists a constant \( c_f \geq 0 \) such that
\[ |f_x(t, x, v)| \leq c_f\omega_f(|x|, |v|) \]
for \( t \in J \) a.e., \( x \in \mathbb{R}^n \), \( v \in \mathbb{R}^r \),

then the partial differential \( F'_x(x, u, v) : AC^2_0 \rightarrow L^2_n \) given by
\[ F'_x(x, u, v)\xi = \xi'(t) + \int_a^t \Phi_x(t, \tau, x(\tau), u(\tau))\xi(\tau)d\tau - f_x(t, x(t), v(t))\xi(t) \]
is bijective.

**Corollary 1.** Under assumptions of Theorem 3.2, the differential \( \partial F(x, u, v) : AC^2_0 \times L^\infty_{\sigma} \times L^\infty_r \rightarrow L^2_n \) is surjective for any \( (x, u, v) \in AC^2_0 \times L^\infty_{\sigma} \times L^\infty_r \).

**Proof.** It is sufficient to consider the triples \( (\xi, 0, 0) \in AC^2_0 \times L^\infty_{\sigma} \times L^\infty_r \).

3.2. Differential property of \( F_0 \). Let us consider the functional \( F_0 : AC^2_0 \times L^\infty_{\sigma} \times L^\infty_r \rightarrow \mathbb{R} \) given by (3). In a similar way, as in the case of Theorem 3.1, one obtains

**Proposition 5.** If the function \( f_0 : J \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \rightarrow \mathbb{R} \) satisfies the conditions

(C1) \( f_0(\cdot, x, u, v) \) is measurable on \( J \) for all \( (x, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \)

(C2) \( f_0(t, \cdot, \cdot, \cdot) \) is of class \( C^1 \) on \( \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \) for \( t \in J \) a.e.

(C3) there exist functions \( a_{f_0} \in L^1(J, \mathbb{R}^p_0) \), \( \omega_{f_0} \in C(\mathbb{R}^p_0 \times \mathbb{R}^p_0 \times \mathbb{R}^p_0, \mathbb{R}^p_0) \) such that
\[ |f_0(t, x, u, v)|, |(f_0)_x(t, x, u, v)|, |(f_0)_u(t, x, u, v)|, |(f_0)_v(t, x, u, v)| \]
\[ \leq a_{f_0}(t)\omega_{f_0}(|x|, |u|, |v|) \]
for \( t \in J \) a.e., \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( v \in \mathbb{R}^r \).
then the functional
\[ AC_0^2 \times L^\infty_m \times L^\infty_r \ni (x, u, v) \mapsto \int_a^b f_0(t, x(t), u(t), v(t))dt \in \mathbb{R} \]

is differentiable in Frechet sense and its differential at a point \((x, u, v)\) is given by
\[ ((\xi, g, h)) \mapsto \int_a^b (f_0)_x(t, x(t), u(t), v(t))\xi(t)dt + \int_a^b (f_0)_u(t, x(t), u(t), v(t))g(t)dt + \int_a^b (f_0)_v(t, x(t), u(t), v(t))h(t)dt \]

for \((\xi, g, h) \in AC_0^2 \times L^\infty_m \times L^\infty_r\).

Clearly, we have

**Proposition 6.** If the function \(g : \mathbb{R}^n \to \mathbb{R}\) is continuously differentiable, then the mapping
\[ AC_0^2 \times L^\infty_m \times L^\infty_r \ni (x, u, v) \mapsto g(x(b)) \in \mathbb{R} \]

is differentiable in Frechet sense and the differential at a point \((x, u, v)\) is given by
\[ ((\xi, g, h)) \mapsto g_x(x(b))\xi(b) \]

for \((\xi, g, h) \in AC_0^2 \times L^\infty_m \times L^\infty_r\).

Thus,

**Corollary 2.** If assumptions of Propositions 5 and 6 are satisfied, then the functional \(F_0\) given by (3) is differentiable in Frechet sense and its differential \(\partial F_0(x, u, v)\) at a point \((x, u, v)\) is given by
\[ \partial F_0(x, u, v)(\xi, g, h) = \int_a^b (f_0)_x(t, x(t), u(t), v(t))\xi(t)dt + \int_a^b (f_0)_u(t, x(t), u(t), v(t))g(t)dt + \int_a^b (f_0)_v(t, x(t), u(t), v(t))h(t)dt + g_x(x(b))\xi(b) \]

for \((\xi, g, h) \in AC_0^2 \times L^\infty_m \times L^\infty_r\).

4. Maximum principle.

4.1. Conjugate cones. Let us fix a point \((x_*, u_*, v_*) \in AC_0^2 \times L^\infty_m \times L^\infty_r\). Using the results of Sections 2, 3 we assert that the cone \(K_d\) of directions of decrease of functional \(F_0\) at \((x_*, u_*, v_*)\) has the form
\[ K_d = \{(\xi, g, h) \in AC_0^2 \times L^\infty_m \times L^\infty_r; \int_a^b (f_0)_x(t, x_*(t), u_*(t), v_*(t))\xi(t)dt \]
\[ + \int_a^b (f_0)_u(t, x_*(t), u_*(t), v_*(t))g(t)dt \]
\[ + \int_a^b (f_0)_v(t, x_*(t), u_*(t), v_*(t))h(t)dt + g_x(x(b))\xi(b) < 0 \} \]

Of course, this set is convex. Moreover, it is nonempty if and only if \(\partial F_0(x_*, u_*, v_*) \neq 0\). When \(K_d \neq \emptyset\), then the conjugate cone \(K_d^*\) is given by
\[ K_d^* = \{-\lambda_0 \partial F_0(x_*, u_*, v_*); \lambda_0 \geq 0\}. \]
Now, let us consider the cone \( K_t \) of tangent directions for the set 
\[
\{ (x, u, v) \in AC_0^2 \times L_m^\infty \times L_r^\infty ; \ F(x, u, v) = x'(t) \\
+ \int_a^t \Phi(t, s, x(s), u(s)) ds - f(t, x(t), v(t)) = 0 \}
\]
at the point \((x_*, u_*, v_*)\). From the Lusternik theorem it follows that 
\[
K_t = \ker \partial F(x_*, u_*, v_*).
\]
Of course, \( K_t \) is a nonempty subspace. Consequently, 
\[
K^*_t = (\ker \partial F(x_*, u_*, v_*))^\perp
\]
where \((\ker F'(x_*, u_*, v_*))^\perp\) is the set of linear continuous functionals on \(AC_0^2 \times L_m^\infty \times L_r^\infty \) vanishing on \( \ker F'(x_*, u_*, v_*) \). From the Lemma 2.6 it follows that 
\[
(\ker \partial F(x_*, u_*, v_*))^\perp = \im((\partial F(x_*, u_*, v_*))^*)
\]
So, 
\[
K^*_t = \im((\partial F(x_*, u_*, v_*))^*) = \{ g \in (AC_0^2 \times L_m^\infty \times L_r^\infty)^\prime ; \text{there exists } \lambda \in L^2 \text{ such that} \}
\[
g(\xi, g, h) = \int_a^b \lambda(t)(\xi'(t) + \int_a^t \Phi_x(t, \tau, x_*(\tau), u_*(\tau))\xi(\tau) d\tau - f_x(t, x_*(t), v_*(t))\xi(t) \\
+ \int_a^t \Phi_u(t, \tau, x_*(\tau), u_*(\tau))g(\tau) d\tau - f_v(t, x_*(t), v_*(t))h(t)) dt
\]
for any \((\xi, g, h) \in AC_0^2 \times L_m^\infty \times L_r^\infty \).

To finish this section, let us write the constraint \((u, v) \in U \times V \) in the form 
\[
(x, u, v) \in Q := \{ (x, u, v) \in AC_0^2 \times L_m^\infty \times L_r^\infty ; \ u \in U, \ v \in V \} = AC_0^2 \times U \times V
\]
Since \( \text{Int} Q \neq \emptyset \), therefore (cf. Proposition 2) the cone \( K_f \) of feasible directions for the set \( Q \) at the point \((x_*, u_*, v_*) \in Q \) is the following 
\[
K_f = \{ \rho(\text{Int}(AC_0^2 \times U \times V) - (x_*, u_*, v_*)) ; \ \rho > 0 \}
\]
\[
= \{ \rho(AC_0^2 \times \text{Int} U \times \text{Int} V) - (x_*, u_*, v_*) ; \ \rho > 0 \}.
\]
Consequently, it is nonempty and convex. From Proposition 4 it follows that conjugate cone \( K^*_f \) has the form 
\[
K^*_f = \{ g_1 = (\varsigma, \mu, \varkappa) \in (AC_0^2 \times L_m^\infty \times L_r^\infty)^\prime ; \text{there exists } \lambda \in L^2 \text{ such that} \}
\[
\varsigma(x) + \mu(u) + \varkappa(v) = g_1(x, u, v) \geq g_1(x_*, u_*, v_*)
\]
\[
= (\varsigma(x_*) + \mu(u_*) + \varkappa(v_*)) \text{ for } (x, u, v) \in Q
\]
\[
= \{ (0, \mu, \varkappa) \in (AC_0^2 \times L_m^\infty \times L_r^\infty)^\prime ; \mu(u) + \varkappa(v) \geq \mu(u_*) + \varkappa(v_*) \text{ for } (u, v) \in U \times V \}.
\]

4.2. Thesis. From Dubovitskii-Milyutin Theorem it follows that if \((x_*, u_*, v_*) \in AC_0^2 \times L_m^\infty \times L_r^\infty \) is a local minimum point for the problem under consideration, then there exist functionals 
\[
g_0 \in K^*_d, \ g_1 \in K^*_f, \ g_2 \in K^*_t,
\]
not all identically zero, such that 
\[
g_0 + g_1 + g_2 = 0.
\]
In other words, there exist \( \lambda_0 \leq 0, \mu \in (L^\infty_m)' = \mathcal{L}^\infty = (L^\infty_r)' \), \( \lambda \in L^2 \), not all zero, such that

\[
\lambda_0 \partial F_0(x_*, u_*, v_*) (\xi, g, h) + \mu(g) + \nu(h) = 0
\]

for \( (\xi, g, h) \in AC_0^2 \times L^\infty_m \times L^\infty_r \) and

\[
\mu(u) + \nu(v) \geq \mu(u_*) + \nu(v_*)
\]

for \( (u, v) \in U \times V \), i.e.,

\[
\lambda_0 \left( \int_a^b (f_0)_x(t, x_*(t), u_*(t), v_*(t)) \xi(t) dt + \int_a^b (f_0)_u(t, x_*(t), u_*(t), v_*(t)) g(t) dt \right) + \int_a^b \left( \int_a^t (f_0)_v(t, x_*(t), u_*(t), v_*(t)) h(t) dt + g_x(x(b)) \xi(b) \right) + \mu(g) + \nu(h)
\]

\[
+ \int_a^b \lambda(t) (\xi'_t(t) + \int_a^t \Phi_x(t, \tau, x_*(\tau), u_*(\tau)) \xi(\tau) d\tau - f_x(t, x_*(t), v_*(t)) \xi(t))
\]

\[
+ \int_a^t \Phi_u(t, \tau, x_*(\tau), u_*(\tau)) g(\tau) d\tau - f_v(t, x_*(t), v_*(t)) h(t) dt = 0
\]

for \( (\xi, g, h) \in AC_0^2 \times L^\infty_m \times L^\infty_r \) and

\[
\mu(u) + \nu(v) \geq \mu(u_*) + \nu(v_*)
\]

for \( (u, v) \in U \times V \).

Taking points \( (\xi, g, h) = (\xi, 0, 0) \in AC_0^2 \times L^\infty_m \times L^\infty_r \) in (6), we obtain

\[
\lambda_0 \left( \int_a^b (f_0)_x(t, x_*(t), u_*(t), v_*(t)) \xi(t) dt + \lambda_0 g_x(x(b)) \xi(b) \right) + \int_a^b \lambda(t) (\xi'(t) + \int_a^t \Phi_x(t, \tau, x_*(\tau), u_*(\tau)) \xi(\tau) d\tau - f_x(t, x_*(t), v_*(t)) \xi(t)) dt = 0
\]

for \( \xi \in AC_0^2 \), i.e.,

\[
\int_a^b \left( \lambda_0 (f_0)_x(t, x_*(t), u_*(t), v_*(t)) - \lambda(t) f_x(t, x_*(t), v_*(t)) \right) \xi(t) dt
\]

\[
+ \int_a^b \lambda(t) \int_a^t \Phi_x(t, \tau, x_*(\tau), u_*(\tau)) \xi(\tau) d\tau dt + \lambda_0 g_x(x(b)) \xi(b)
\]

\[
= - \int_a^b \lambda(t) \xi'(t) dt.
\]

Let us observe that using Fubini theorem we obtain

\[
\int_a^b \lambda(t) \int_a^t \Phi_x(t, \tau, x_*(\tau), u_*(\tau)) \xi(\tau) d\tau dt
\]

\[
= \int_a^b \int_a^t \lambda(t) \Phi_x(t, \tau, x_*(\tau), u_*(\tau)) \xi(\tau) d\tau dt
\]

\[
= \int_a^b \xi(t) \int_a^b \lambda(t) \Phi_x(t, \tau, x_*(\tau), u_*(\tau)) dt d\tau.
\]
So, (9) gives
\[
\int_a^b (\lambda_0(f_0)_x(\tau, x_*(\tau), u_*(\tau), v_*(\tau)) - \lambda(\tau)f_x(\tau, x_*(\tau), v_*(\tau))) \xi(\tau) d\tau
\]
(10)
\[
+ \int_a^b \xi(\tau) \int_\tau^b \lambda(t) \Phi_x(t, \tau, x_*(\tau), u_*(\tau)) dt d\tau + \lambda_0 g_x(x(b)) \xi(b)
\]
\[
= - \int_a^b \lambda(\tau) \xi'(\tau) d\tau
\]
for any \( \xi \in AC^2_0 \). In particular,
\[
\int_a^b (\lambda_0(f_0)_x(\tau, x_*(\tau), u_*(\tau), v_*(\tau)) - \lambda(\tau)f_x(\tau, x_*(\tau), v_*(\tau)))
\]
(11)
\[
+ \int_\tau^b \lambda(t) \Phi_x(t, \tau, x_*(\tau), u_*(\tau)) dt \xi(\tau) d\tau = - \int_a^b \lambda(\tau) \xi'(\tau) d\tau
\]
for any \( \xi \in AC^2_0 = \{ \xi \in AC^2_0; \xi(b) = 0 \} \). Using the Du Bois-Reymond lemma we assert that the function \( \lambda \) is absolutely continuous (more precisely, it has an absolutely continuous representant) and
\[
\lambda'(\tau) - \int_\tau^b [\Phi_x(t, \tau, x_*(\tau), u_*(\tau))]^T \lambda(t) dt
\]
(12)
\[
= \lambda_0(f_0)_x(\tau, x_*(\tau), u_*(\tau), v_*(\tau)) - \lambda(\tau)f_x(\tau, x_*(\tau), v_*(\tau)) \]
for \( \tau \in J \ a.e. \) Replacing in (10) the integral \( \int_a^b \lambda(\tau) \xi'(\tau) d\tau \) by \( \lambda(b)\xi(b) - \int_a^b \lambda'(\tau) \xi(\tau) d\tau \) we obtain
\[
\int_a^b (\lambda_0(f_0)_x(\tau, x_*(\tau), u_*(\tau), v_*(\tau)) - \lambda(\tau)f_x(\tau, x_*(\tau), v_*(\tau))
\]
\[
+ \int_\tau^b \lambda(t) \Phi_x(t, \tau, x_*(\tau), u_*(\tau)) dt - \lambda'(\tau) \xi(\tau) d\tau + \lambda_0 g_x(x(b)) \xi(b) = -\lambda(b)\xi(b)
\]
i.e. (cf. (12))
\[
\lambda_0 g_x(x(b)) \xi(b) = -\lambda(b)\xi(b)
\]
for any \( \xi \in AC^2_0 \). Thus, if we choose a function \( \xi(t) = t \), then we have
\[
\lambda(b) = -\lambda_0 g_x(x(b)).
\]
On the other hand, taking in (6) points \( (\xi, g, h) = (0, g, h) \in AC^2_0 \times L_\infty \times L_\infty \), we obtain
\[
\lambda_0 \int_a^b (f_0)u(t, x_*(t), u_*(t), v_*(t))g(t) dt
\]
\[
+ \lambda_0 \int_a^b (f_0)v(t, x_*(t), u_*(t), v_*(t))h(t) dt + \mu(g) + \varphi(h)
\]
\[
+ \int_a^b \lambda(t) \left( \int_a^t \Phi_u(t, \tau, x_*(\tau), u_*(\tau)) g(\tau) d\tau - f_v(t, x_*(t), v_*(t)) h(t) \right) dt = 0
\]
for \( (g, h) \in L_\infty \times L_\infty \). The above condition and (7) imply the following inequality
\[
-\lambda_0 \int_a^b (f_0)u(t, x_*(t), u_*(t), v_*(t)) u(t) dt - \lambda_0 \int_a^b (f_0)v(t, x_*(t), u_*(t), v_*(t)) v(t) dt
\]
for any \((u, v) \in U \times V\), i.e.

\[
\int_a^b (\lambda_0(f_0)_u(t, x_s(t), u_s(t), v_s(t)) - \lambda(t) \int_a^t \Phi_u(t, \tau, x_s(\tau), u_s(\tau))d\tau, \\
- \lambda_0(f_0)_v(t, x_s(t), u_s(t), v_s(t)) + \lambda(t)f_v(t, x_s(t), v_s(t)))(u(t), v(t))d\tau
\]

\[
\geq \int_a^b (\lambda_0(f_0)_u(t, x_s(t), u_s(t), v_s(t)) - \lambda(t) \int_a^t \Phi_u(t, \tau, x_s(\tau), u_s(\tau))d\tau, \\
- \lambda_0(f_0)_v(t, x_s(t), u_s(t), v_s(t)) + \lambda(t)f_v(t, x_s(t), v_s(t)))(u(t), v(t))d\tau
\]

Applying “\(L^\infty(J, M \times N)\)”-version of [6, Lemma 6] we assert: for \(t \in J\) a.e.,

\[
(\lambda_0(f_0)_u(t, x_s(t), u_s(t), v_s(t)) - \lambda(t) \int_a^t \Phi_u(t, \tau, x_s(\tau), u_s(\tau))d\tau, \\
- \lambda_0(f_0)_v(t, x_s(t), u_s(t), v_s(t)) + \lambda(t)f_v(t, x_s(t), v_s(t)))(u, v)
\]

\[
\geq (\lambda_0(f_0)_u(t, x_s(t), u_s(t), v_s(t)) - \lambda(t) \int_a^t \Phi_u(t, \tau, x_s(\tau), u_s(\tau))d\tau, \\
- \lambda_0(f_0)_v(t, x_s(t), u_s(t), v_s(t)) + \lambda(t)f_v(t, x_s(t), v_s(t)))(u_s(t), v_s(t))
\]

for any \((u, v) \in M \times N\).

Let us also point out that \(\lambda_0 \neq 0\). Indeed, let us assume that \(\lambda_0 = 0\). Then from (8) it follows that \(\lambda = 0\) because the operator

\[
AC^2_0 \ni \xi \mapsto \xi'(t) + \int_a^t \Phi_x(t, \tau, x_s(\tau), u_s(\tau))d\tau - f_x(t, x_s(t), v_s(t))\xi(t) \in L^2
\]

is surjective (see Corollary 1)). So, from (6) it follows that

\[
\mu(g) + \nu(h) = 0
\]

for any \((g, h) \in L^{\infty}_m \times L^{\infty}_r\). It means that \(\mu = 0\) and \(\nu = 0\) and contradicts to the fact that \(\lambda_0 \leq 0\), \(\mu \in (L^{\infty}_m)'\), \(\nu \in (L^{\infty}_r)'\), \(\lambda \in L^2\) are not all zero. Consequently, we may assume that \(\lambda_0 = -1\).

4.3. Final result. So, we have proved the following maximum principle

**Theorem 4.1** (maximum principle). Let the sets \(M \subset \mathbb{R}^m, N \subset \mathbb{R}^r\) be convex closed with non-empty interiors. If \((x_s, u_s, v_s) \in AC^2_0 \times L^{\infty}_m \times L^{\infty}_r\) is a local minimum point for problem (1)-(2)-(3), assumptions of Theorems 3.1, 3.2 and Propositions...
5, 6 are satisfied, then

\[
((f_0)_u(t,x_*(t),u_*(t),v_*(t)) - \lambda(t) \int_a^t \Phi_u(t,\tau,x_*(\tau),u_*(\tau))d\tau)u \\
+ ((f_0)_v(t,x_*(t),u_*(t),v_*(t)) + \lambda(t)f_u(t,x_*(t),v_*(t)))v \\
geq ((f_0)_u(t,x_*(t),u_*(t),v_*(t)) - \lambda(t) \int_a^t \Phi_u(t,\tau,x_*(\tau),u_*(\tau))d\tau)u_*(t) \\
+ ((f_0)_v(t,x_*(t),u_*(t),v_*(t)) + \lambda(t)f_u(t,x_*(t),v_*(t)))v_*(t)
\]
for any \( t \in J \) a.e., \((u,v) \in M \times N\), where \( \lambda \) is an absolutely continuous solution to the Cauchy problem of Volterra type

\[
\lambda'(t) - \int_\tau^b [\Phi_x(t,\tau,x_*(\tau),u_*(\tau))]^T \lambda(t)dt \\
= -(f_0)_x(\tau,x_*(\tau),u_*(\tau),v_*(\tau)) - \lambda(\tau)f_x(\tau,x_*(\tau),v_*(\tau))
\]
for \( \tau \in J \) a.e.,

\[
\lambda(b) = g_x(x(b)).
\]

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