EXPONENTIALS OF NON-SINGULAR SIMPLICIAL SETS

VEGARD FJELLBO AND JOHN Rognes

Abstract. A simplicial set is non-singular if the representing map of each non-degenerate simplex is degreewise injective. The simplicial mapping set $X^K$ has $n$-simplices given by the simplicial maps $\Delta[n] \times K \to X$. We prove that $X^K$ is non-singular whenever $X$ is non-singular. It follows that non-singular simplicial sets form a cartesian closed category with all limits and colimits, but it is not a topos.

1. Introduction

Let $sSet$ be the category of simplicial sets, and let $nsSet$ denote its full subcategory of non-singular simplicial sets, i.e., those $X$ such that for each non-degenerate simplex $x \in X_n$ the representing map $\bar{x}: \Delta[n] \to X$ is degreewise injective. The geometric realization $|X|$ of each non-singular simplicial set admits a well-defined PL (piecewise-linear) structure, and the category $nsSet$ plays a key role in the passage between simplicial sets and PL manifolds in the proof of the stable parametrized $h$-cobordism theorem [WJR13, Thm. 0.1, §3.4].

The inclusion $U: nsSet \to sSet$ admits a left adjoint $D: sSet \to nsSet$, called desingularization, and the adjunction unit $\eta_X: X \to UD X$ is degreewise surjective. The category $nsSet$ has all (small) limits and colimits, which are preserved by $U$ and $D$, respectively. Let $(Sd, Ex)$ denote Kan’s adjoint pair [Kan57] of endofunctors of $sSet$. The first author has exhibited a model structure on the category $nsSet$, and furthermore shown that the adjunction $DSDd_2: sSet \rightleftarrows nsSet: Ex^2U$ defines a Quillen equivalence from the standard model structure on simplicial sets [Fje]. The proofs of these two results depend on knowing that the endofunctor $X \mapsto X \times \Delta[1]$ of $nsSet$ preserves all colimits, and one purpose of the present paper is to establish this fact.

For any simplicial sets $X$ and $K$ let $X^K$ be the simplicial mapping set, with $n$-simplices the set of maps $\Delta[n] \times K \to X$. Our main result follows.

Theorem 1.1. Let $X$ and $K$ be any two simplicial sets. If $X$ is non-singular, then so is $X^K$.

It follows that $X \mapsto X^K$ restricts to an endofunctor of $nsSet$. This implies the following generalization of the aforementioned fact.

Proposition 1.2. Let $K$ be any non-singular simplicial set. The endofunctor $X \mapsto X \times K$ of $nsSet$ preserves all colimits.

The proof of Theorem 1.1 follows easily from the following special case, which also directly implies the case $K = \Delta[1]$ of Proposition 1.2.

Proposition 1.3. If $X$ is non-singular, then so is $X^{\Delta[1]}$. 

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We may restate Theorem 1.1 by saying that the non-singular simplicial sets form an exponential ideal in the cartesian closed category \([ML98 \, \text{§IV.6}]\) of simplicial sets. The adjunction \((D, U)\) exhibits \(\text{sSet}\) as a reflective full subcategory \([ML98 \, \text{§IV.3}]\) of \(\text{nsSet}\), which is closed under exponentiation in the sense of \([Day72]\). In this situation, Day's reflection theorem \([Day72 \, \text{Thm. 1.2, Cor. 2.1}]\) shows that the reflector \(\text{D}: \text{sSet} \to \text{nsSet}\) preserves finite products, making \(\text{nsSet}\) a cartesian closed category.

**Proposition 1.4.** Desingularization \(\text{D}: \text{sSet} \to \text{nsSet}\) preserves finite products.

**Remark 1.5.** The category \(\text{nsSet}\) is not a topos in the sense of \([ML98 \, \text{§IV.10}]\), because it does not admit a subobject classifier \(t: \Delta[0] \to \Omega\). Here \(\Omega\) would have to consist of precisely two elements, so \(\Omega\) would be at most 1-dimensional, and could not classify all the subobjects of \(\Delta[2]\). This is related to the fact that desingularization does not in general preserve equalizers, as the example of the two maps \(\Delta[0] \rightrightarrows \Delta[2]/\delta_1 \Delta[1]\) illustrates.

We give the proof of Proposition 1.3 in Section 2 and deduce the remaining results in Section 3.

### 2. Rigidity of prisms

Informally, Proposition 1.3 asserts that maps \(\Phi: \Delta[n] \times \Delta[1] \to X\) from prisms to non-singular simplicial sets are very rigid.

We recall some terminology and notation before turning to the proofs. For each \(n \geq 0\) let \([n]\) denote the totally ordered set \(\{0 < 1 < \cdots < n\}\). Following \([FP90 \, \text{§4.1}]\), we shall refer to the functions \(\alpha: [m] \to [n]\) such that \(\alpha(i) \leq \alpha(j)\) for all \(i \leq j\) as operators. These are the objects and morphisms of the category \(\Delta\). For a simplicial set \(X: [n] \mapsto X_n\), we write \(x \in X_n\) for the value of the operator \(\alpha: [m] \to [n]\) on a simplex \(x \in X_n\). The standard \(n\)-simplex \(\Delta[n]\) is the simplicial set \([m] \mapsto \Delta([m],[n])\) represented by \([n]\).

An injective operator is said to be a face operator, and a surjective operator is said to be a degeneracy operator. Special face operators are the elementary face operators \(\delta^i: [n - 1] \to [n]\) that omit the element \(i\), and the vertex operators \(\varepsilon^i: [0] \to [n]\) that hit the element \(i\). Special degeneracy operators are the elementary degeneracy operators \(\sigma^i: [n + 1] \to [n]\) that send \(i\) and its successor \(i + 1\) to \(i\). Usually, we omit the superscript in the notation.

A face operator or degeneracy operator is proper if it is not the identity. A simplex \(x\) is a (proper) face of a simplex \(y\) if \(x = y\mu\) for a (proper) face operator \(\mu\). Analogously, \(x\) is a (proper) degeneracy of \(y\) if \(x = y\rho\) for a (proper) degeneracy operator \(\rho\). A simplex is degenerate if it is a proper degeneracy of some simplex. Otherwise, it is said to be non-degenerate.

By the Eilenberg–Zilber lemma \([FP90 \, \text{Thm. 4.2.3}]\) any simplex \(x\) in a simplicial set \(X\) can be uniquely expressed as a degeneration \(x = x^d x^d\) of a non-degenerate simplex. We call the non-degenerate simplex \(x^d\) the non-degenerate part of \(x\), and will refer to the degeneracy operator \(x^d\) as the degenerate part of \(x\). By the Yoneda lemma, the \(n\)-simplices \(x\) of a simplicial set \(X\) are in natural bijective correspondence with the simplicial maps \(\tilde{x}: \Delta[n] \to X\). The map \(\tilde{x}\) is the representing map of \(x\).

**Lemma 2.1.** Let \(x \in X_n\) be any simplex. The representing map \(\tilde{x}: \Delta[n] \to X\) is degreewise injective if and only if the \(n + 1\) vertices \(x \varepsilon_0, \ldots, x \varepsilon_n \in X_0\) are pairwise distinct.

**Lemma 2.2.** Let \(x\) be a simplex in a non-singular simplicial set \(X\), and suppose that \(x \varepsilon_k = x \varepsilon_l\) for some \(k < l\). Then the degenerate part \(x^d\) of \(x\) factors uniquely through the proper degeneracy operator \(\sigma_k \cdots \sigma_{l-1}\).
Proof. The representing map of the non-degenerate part $x^2$ is degreewise injective, since $X$ is non-singular, so its vertices are pairwise distinct. It follows that $x^2(k) = x^2(l)$. Since $x^2$ is order-preserving, it also follows that $x^2(k) = x^2(j)$ for all $k \leq j \leq l$. Let $\rho = \sigma_0 \cdots \sigma_{l-1}$. Then $x^2(i) = x^2(j)$ whenever $\rho(i) = \rho(j)$, and this implies that $x^2 = (x^2 \rho)\mu$, where $\mu$ is any choice of section to $\rho$. Thus the asserted factorization exists. Its uniqueness is automatic, since $\rho$ is surjective. 

Proof of Proposition 1.3 Suppose that $X$ is non-singular. We must show that each non-degenerate $n$-simplex $\Phi$ in the simplicial mapping set $X^{\Delta[1]}$ has $n + 1$ distinct vertices $\Phi_{\varepsilon_0}, \ldots, \Phi_{\varepsilon_n}$. Equivalently, we must show that if the $k$-th and $l$-th vertices of an $n$-simplex $\Phi$ are equal, for some $0 \leq k < l \leq n$, then $\Phi$ is degenerate. This follows from the two lemmas below.

Lemma 2.3. Suppose that $X$ is non-singular and $\Phi$ is an $n$-simplex in $X^{\Delta[1]}$ such that $\Phi_{\varepsilon_k} = \Phi_{\varepsilon_l}$, for some $0 \leq k < l \leq n$. Then

$$\Phi_{\varepsilon_k} = \Phi_{\varepsilon_j} = \Phi_{\varepsilon_l}$$

for all $k \leq j \leq l$.

Lemma 2.4. Suppose that $X$ is non-singular and $\Phi$ is an $n$-simplex in $X^{\Delta[1]}$ such that $\Phi_{\varepsilon_k} = \Phi_{\varepsilon_{k+1}}$, for some $0 \leq k < n$. Then there is an $(n - 1)$-simplex $\Psi$ in $X^{\Delta[1]}$ for which $\Phi = \Psi_{\sigma_k}$, exhibiting $\Phi$ as a degenerate simplex.

We introduce some more notation before proving these lemmas. By definition, an $n$-simplex in $X^{\Delta[1]}$ is a simplicial map

$$\Phi: \Delta[n] \times \Delta[1] \rightarrow X.$$

Here, the prism $\Delta[n] \times \Delta[1]$ is generated by the non-degenerate $(n + 1)$-simplices

$$\gamma_j^{n+1}: \Delta[n + 1] \rightarrow \Delta[n] \times \Delta[1],$$

for $0 \leq j \leq n$, given by

$$\gamma_j^{n+1}(i) = \begin{cases} (i, 0) & \text{for } 0 \leq i \leq j, \\ (i - 1, 1) & \text{for } j + 1 \leq i \leq n + 1. \end{cases}$$

Viewing $\Delta[n] \times \Delta[1]$ as the nerve of the partially ordered set $[n] \times [1]$, these generators can be seen as maximal length paths in the diagram below.

$$\begin{array}{ccc}
(0, 1) & \cdots & (j, 1) & (j + 1, 1) & \cdots & (n, 1) \\
(0, 0) & \cdots & (j, 0) & (j + 1, 0) & \cdots & (n, 0)
\end{array}$$

In particular, they satisfy the relations

$$(2.1) \quad \gamma_j^{n+1} \delta_{j+1} = \gamma_j^{n+1} \delta_{j+1}$$

for $0 \leq j < n$. Conversely, to specify $\Phi$ it suffices to give its values $\Phi_{\gamma_j^{n+1}}$ on these $n + 1$ generators, subject to the $n$ relations $(\Phi_{\gamma_j^{n+1}}) \delta_{j+1} = (\Phi_{\gamma_{j+1}^{n+1}}) \delta_{j+1}$.

Proof of Lemma 2.3. Let $X$ be non-singular and let $\Phi$ be an $n$-simplex in $X^{\Delta[1]}$ with $\Phi_{\varepsilon_k} = \Phi_{\varepsilon_l}$, where $0 \leq k < l \leq n$. The vertex operators $\varepsilon_i: \Delta[0] \rightarrow \Delta[1]$ for $i \in \{0, 1\}$ induce maps $\varepsilon_i^*: X^{\Delta[1]} \rightarrow X^{\Delta[0]} \cong X$. Let $x_i = \varepsilon_i^* \Phi$ in $X_n$ be represented by the composite

$$x_i: \Delta[n] \cong \Delta[n] \times \Delta[0] \xrightarrow{1 \times \varepsilon_i} \Delta[n] \times \Delta[1] \xrightarrow{\Phi} X,$$

restricting $\Phi$ to the bottom (for $i = 0$) or the top (for $i = 1$) of the prism. The hypothesis on $\Phi$ implies that $x_i \varepsilon_k = x_i \varepsilon_l$ in $X_0$, so by Lemma 2.2 we can factor
the degenerate part \( x_i^1 \) of \( x_i \) through \( \sigma_k \cdots \sigma_{l-1} \), so that \( x_i = y_i \sigma_k \cdots \sigma_{l-1} \) for some \((n + k - l)\)-simplices \( y_i \) of \( X \).

Consider any \( j \) with \( k \leq j < l \), let \( \mu : [1] \to [n] \) be the face operator given by \( \mu(0) = j \) and \( \mu(1) = j + 1 \), and view the 1-simplex \( \Phi \mu \) in \( X \Delta[1]\) as the map \( \Delta[1] \times \Delta[1] \to X \) indicated by the following square.

The factorization of \( \sigma_k \) shows that \( x_i \) through \( \sigma_k \) shows that \( x_i \delta_j = x_i \delta_{j+1} \), for each \( i \). Hence each 2-simplex \( z_i \) does not have pairwise distinct vertices, and must therefore be degenerate, since \( X \) is non-singular. By Lemma 2.2 we must have \( z_0 = w_0 \sigma_1 \) and \( z_1 = w_1 \sigma_0 \) for some 1-simplices \( w_i \). More precisely, we must have \( w_0 = z_0 \delta_2 = \Phi \varepsilon_j \) and \( w_1 = z_1 \delta_0 = \Phi \varepsilon_{j+1} \).

It follows that the diagonal 1-simplex in the figure is simultaneously equal to \( \delta_0 \delta_1 = (\Phi \varepsilon_j) \sigma_1 \delta_1 = \Phi \varepsilon_j \) and to \( z_1 \delta_1 = (\Phi \varepsilon_{j+1}) \sigma_0 \delta_1 = \Phi \varepsilon_{j+1} \). This proves that \( \Phi \varepsilon_j = \Phi \varepsilon_{j+1} \) are equal as vertices in \( X \Delta[1] \).

**Proof of Lemma 2.4.** Let \( X \) be non-singular and let \( \Phi \) be an \( n \)-simplex in \( X \Delta[1] \) with \( \Phi \varepsilon_k = \Phi \varepsilon_{k+1} \), where \( 0 \leq k < n \). We will construct an \((n-1)\)-simplex \( \Psi \) in \( X \Delta[1] \) with \( \Phi = \Psi \sigma_k \). Equivalently, we must define \( \Psi : \Delta[n-1] \times \Delta[1] \to X \) so as to make the right hand triangle commute in the diagram below.

\[
\begin{array}{ccc}
\Delta[n+1] & \xrightarrow{\gamma_{n+1}} & \Delta[n] \times \Delta[1] \\
\downarrow{\sigma_k \times 1} & & \downarrow{\Phi} \\
\Delta[n-1] \times \Delta[1] & \xrightarrow{\Psi} & X
\end{array}
\]

The triangle will commute if \( \Phi \gamma_{n+1}^j = \Psi(\sigma_k \times 1) \gamma_{n+1}^j \) for each \( 0 \leq j \leq n \), since the simplices \( \gamma_{n+1}^0, \ldots, \gamma_{n+1}^n \) generate the prism \( \Delta[n] \times \Delta[1] \). Here

\[
(\sigma_k \times 1) \gamma_{n+1}^j = \begin{cases} 
\gamma_{n+1}^j \sigma_{k+1} & \text{for } 0 \leq j \leq k, \\
\gamma_{n+1}^{j-1} \sigma_k & \text{for } k < j \leq n.
\end{cases}
\]

Should \( \Psi \) exist, it must therefore satisfy

\[
\Phi(\gamma_{n+1}^j) = \begin{cases} 
\Psi(\gamma_{n+1}^j) \sigma_{k+1} & \text{for } 0 \leq j \leq k, \\
\Psi(\gamma_{n+1}^j) \sigma_k & \text{for } k < j \leq n.
\end{cases}
\]

Observing that \( \delta_{k+1} \) is a section to both \( \sigma_k \) and \( \sigma_{k+1} \), we are led to define a function \( \psi : \{\gamma_{n+1}^0, \ldots, \gamma_{n+1}^n\} \to X_n \) by

\[
\psi(\gamma_{n+1}^j) = \begin{cases} 
\Phi(\gamma_{n+1}^j) \delta_{k+1} & \text{for } 0 \leq j \leq k, \\
\Phi(\gamma_{n+1}^j) \delta_{k+1} & \text{for } k \leq j \leq n-1,
\end{cases}
\]

which specifies where \( \Psi \) must send the generators \( \gamma_{n+1}^0, \ldots, \gamma_{n+1}^{n-1} \) of \( \Delta[n-1] \times \Delta[1] \), should it exist. Note that for \( j = k \) the relation

\[
\Phi(\gamma_{n+1}^{n+1}) \delta_{k+1} = \Phi(\gamma_{n+1}^{n+1} \delta_{k+1}) = \Phi(\gamma_{n+1}^{n+1} \delta_{k+1}) = \Phi(\gamma_{n+1}^{n+1}) \delta_{k+1}
\]
holds, by (2.1), so $\psi(\gamma^k_j)$ is unambiguously defined. To verify that $\Psi(\gamma^k_j) = \psi(\gamma^k_j)$ for $0 \leq j \leq n - 1$ defines a map $\Psi : \Delta[n - 1] \times \Delta[1] \to X$, it is (necessary and) sufficient to confirm the relations

$$\Psi(\gamma^k_j) \delta_{j+1} = \psi(\gamma^k_{j+1}) \delta_{j+1}$$

for $0 \leq j < n - 1$. We separate the proof of (2.3) into two cases.

First, for $0 \leq j < k$ we use the general rule $\delta_{k+1} \delta_{j+1} = \delta_{j+1} \delta_k$ for $j < k$, together with (2.1), to see that

$$\Psi(\gamma^k_j) \delta_{j+1} = \Phi \gamma^k_{j+1} \delta_{k+1} \delta_{j+1} = \Phi \gamma^k_{j+1} \delta_{j+1} \delta_k$$

is equal to

$$\Psi(\gamma^k_{j+1}) \delta_{j+1} = \Phi \gamma^k_{j+1} \delta_{k+1} \delta_{j+1} = \Phi \gamma^k_{j+1} \delta_{j+1} \delta_k$$

Second, for $k \leq j < n - 1$ we use the general rule $\delta_{k+1} \delta_{j+1} = \delta_{j+2} \delta_{k+1}$ for $k \leq j$, together with (2.1), to see that

$$\Psi(\gamma^k_j) \delta_{j+1} = \Phi \gamma^k_{j+1} \delta_{k+1} \delta_{j+1} = \Phi \gamma^k_{j+1} \delta_{j+2} \delta_{k+1}$$

is equal to

$$\Psi(\gamma^k_{j+1}) \delta_{j+1} = \Phi \gamma^k_{j+1} \delta_{k+1} \delta_{j+1} = \Phi \gamma^k_{j+1} \delta_{j+2} \delta_{k+1}$$

This concludes the verification of (2.3), giving us a well-defined map $\Psi$.

It still remains to argue that $\Phi = \Psi(\sigma_k \times 1)$, and this is where we use the hypotheses on $X$ and $\Phi$. It suffices to check that the equation

$$\Phi \gamma^k_{j+1} = \Psi(\sigma_k \times 1) \gamma^k_{j+1}$$

holds for $0 \leq j \leq n$. Again, we separate the proof into two cases.

First, for $0 \leq j \leq k$ we must show that the $(n + 1)$-simplex $z_j = \Phi(\gamma^k_{j+1})$ in $X$ is equal to

$$\Psi(\sigma_k \times 1) \gamma^k_{j+1} = \Psi \gamma^k_{j+1} \sigma_{k+1} = \Phi(\gamma^k_{j+1}) \delta_{k+1} \sigma_{k+1} = z_j \delta_{k+1} \sigma_{k+1}$$

where we have used the calculation (2.2). The vertices $z_j \in \Delta n + 1$ and $z_j \in \Delta n + 2$ in $X$ are equal to $\varepsilon^j_0(\Phi \varepsilon_k)$ and $\varepsilon^j_1(\Phi \varepsilon_{k+1})$, respectively, hence are equal by the assumption that $\Phi \varepsilon_k = \Phi \varepsilon_{k+1}$. It follows by Lemma 2.2 that $z_j = w_j \sigma_{k+1}$ for some $n$-simplex $w_j$ in $X$, since $X$ is non-singular. This immediately implies that $z_j \delta_{k+1} \sigma_{k+1} = w_j \sigma_{k+1} \delta_{k+1} \sigma_{k+1} = w_j \sigma_{k+1} \sigma_{k+1} = z_j$, hence $z_j \delta_{k+1} \sigma_{k+1}$ is a section to $\sigma_{k+1}$.

Second, $k < j \leq n$ we must show that the $(n + 1)$-simplex $z_j = \Phi(\gamma^k_{j+1})$ in $X$ is equal to

$$\Psi(\sigma_k \times 1) \gamma^k_{j+1} = \Psi \gamma^k_{j+1} \sigma_k = \Phi(\gamma^k_{j+1}) \delta_{k+1} \sigma_k = z_j \delta_{k+1} \sigma_k$$

The vertices $z_j \notin \Delta n$ and $z_j \notin \Delta n + 1$ in $X$ are equal to $\varepsilon^j_0(\Phi \varepsilon_k)$ and $\varepsilon^j_1(\Phi \varepsilon_{k+1})$, respectively, hence are themselves equal. It follows by Lemma 2.2 that $z_j = w_j \sigma_k$ for some $n$-simplex $w_j$ in $X$. This implies that $z_j \delta_{k+1} \sigma_k = w_j \sigma_k \delta_{k+1} \sigma_k = w_j \sigma_k = z_j$, hence $z_j \delta_{k+1} \sigma_k$ is a section to $\sigma_k$. This concludes our verification of (2.4), proving that $\Phi$ is a degenerate simplex of $X \Delta[1]$.

3. Outstanding proofs

Proof of Theorem 1.1. Let $X$ be any non-singular simplicial set. By Proposition 1.3 and induction, $X \Delta[1]^n$ is non-singular, for each $n \geq 0$. The inclusion $i : \Delta[n] \to \Delta[1]^n$ sending $j \in [n]$ to $(1, \ldots, 1, 0, \ldots, 0) \in [1]^n$ (with $j$ copies of 1) admits a retraction $r : \Delta[1]^n \to \Delta[n]$ sending $(k_1, \ldots, k_n)$ to the largest index $j$ such that $k_j = 1$. Hence $r^* : X \Delta[n] \to X \Delta[1]^n$ is split injective, and shows that $X \Delta[n]$ is non-singular.
For any simplicial set $K$, we can find a simplicial set $L = \coprod_{\alpha} \Delta[n_{\alpha}]$ and a degreewise surjection $s : L \to K$. The induced map $s^* : X^K \to X^L \cong \prod_{\alpha} X^{\Delta[n_{\alpha}]}$ is then degreewise injective, and exhibits $X^K$ as a simplicial subset of a product of non-singular simplicial sets. It follows that $X^K$ is non-singular. □

Proof of Proposition 1.2. When $X$, $K$, and $Y$ are non-singular, so that $X \times K$ and $Y^K$ are non-singular by Theorem 1.1, the natural bijection $\text{sSet}(X \times K, Y) \cong \text{sSet}(X, Y^K)$ restricts to a natural bijection $\text{nsSet}(X \times K, Y) \cong \text{nsSet}(X, Y^K)$. Hence the endofunctor $X \mapsto X \times K$ of $\text{nsSet}$ is a left adjoint, and preserves all colimits. □

Proof of Proposition 1.4. Let $X$ and $Y$ be any simplicial sets. Recall that each adjunction unit $\eta_Z : Z \to DZ$ is degreewise surjective. Let $a : D(X \times Y) \to DX \times DY$ be induced by the two projections from $X \times Y$. The composite $X \times Y \xrightarrow{\eta_X \times \eta_Y} D(X \times Y) \xrightarrow{a} DX \times DY$ is then equal to $\eta_X \times \eta_Y$, so $a$ is degreewise surjective. The right adjoint $X \to D(X \times Y)^Y$ of $\eta_X \times \eta_Y$ factors through $\eta_X : X \to DX$, since $D(X \times Y)^Y$ is non-singular by Theorem 1.1. Hence there is a unique factorization $X \times Y \xrightarrow{\eta_X \times 1} DX \times Y \xrightarrow{b} D(X \times Y)$ of $\eta_X \times \eta_Y$. Similarly, there is a unique factorization $DX \times Y \xrightarrow{1 \times \eta_Y} DX \times DY \xrightarrow{c} D(X \times Y)$ of $b$, again by Theorem 1.1. It follows that $\text{can}_{X \times Y} = c(\eta_X \times \eta_Y) = \eta_X \times \eta_Y$, so that $ca = 1$, which proves that $a$ is an isomorphism. □

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Department of Mathematics, University of Oslo, Norway
Email address: vegard.fjellbo@gmail.com

Department of Mathematics, University of Oslo, Norway
Email address: rogenes@math.uio.no