On logics extended with embedding-closed quantifiers

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Abstract

We study first-order as well as infinitary logics extended with quantifiers upwards closed under embeddings. In particular, we show that if a chain of quasi-homogeneous structures is sufficiently long then a given formula of such a logic is eventually equivalent to a quantifier-free formula. We use this fact to produce a number of undefinability results for logics with embedding-closed quantifiers. In the final section we introduce an Ehrenfeucht-Fraïssé game that characterizes the $L_{\infty\omega}(Q_{\text{emb}})$-equivalence between structures, where $Q_{\text{emb}}$ is the class of all embedding-closed quantifiers. In conclusion, we provide an application of this game illustrating its use.

1 Introduction

In this paper we focus our attention on a certain class of logics whose expressive power is greater than that of first-order logic (denoted by $L_{\omega\omega}$), the central logic in mathematics. Whilst $L_{\omega\omega}$ has well-developed model theory due to its many convenient properties, it has a downside in that its expressive power is rather limited. Many natural mathematical statements, for example "there are infinitely many", cannot be expressed in $L_{\omega\omega}$. This motivates study of alternative logics.

Mostowski was one of the first to suggest in [6] the idea of expanding $L_{\omega\omega}$ with formulas of the form $Q_\alpha x \varphi(x)$ which are interpreted so that $\mathfrak{A} \models Q_\alpha x \varphi(x)$ if, and only if, there are at least $\aleph_\alpha$ elements $a$ with $\mathfrak{A} \models \varphi(a)$. This idea broadened the notion of quantifier giving rise to many interesting logics defined in a similar way. The current definition of generalized quantifier is due to Lindström [5]. We describe it in more detail in Section 2.

In short, every generalized quantifier $Q$ corresponds to some property of structures. Suppose $\mathcal{L}$ is a logic closed under substitution and $P$ is a property not expressible in $\mathcal{L}$. By adding quantifier $Q_P$ we get the smallest extension
of \( \mathcal{L} \) satisfying certain closure conditions that can express \( P \). The properties of the new logic \( \mathcal{L}(Q_P) \) can differ substantially from those of \( \mathcal{L} \) and may thus become an interesting object of study.

In the present work we shall concentrate on extensions of logics \( \mathcal{L}_{\omega \omega} \), \( \mathcal{L}_{\infty \omega} \) and \( \mathcal{L}_{\omega \infty} \) (the finite variable logic) with generalized quantifiers \( Q \) that satisfy the following restriction: for all structures \( \mathfrak{A} \in \text{Str}[\tau_Q] \), if \( \mathfrak{A} \in Q \) and \( \mathfrak{A} \) is embeddable into \( \mathfrak{B} \) then \( \mathfrak{B} \in Q \). We call such quantifiers embedding-closed and denote the class of all embedding-closed quantifiers by \( Q_{\text{emb}} \).

Call a structure \( \mathfrak{A} \) quasi-homogeneous if every isomorphism between finitely generated substructures of \( \mathfrak{A} \) can be extended to an embedding of \( \mathfrak{A} \) into itself. This weakens the usual notion of homogeneity which deals with automorphisms instead of embeddings. The notion of embedding-closed quantifier arises naturally when we observe that in order to guarantee that logic \( \mathcal{L}_{\infty \omega} \) extended with a set of quantifiers \( Q \) has quantifier elimination, the quantifiers in \( Q \) should be closed under embeddings.

In [2], Dawar and Grädel showed that \( \mathcal{L}_{\omega \infty} \) extended with finitely many embedding-closed quantifiers of finite width has a 0-1 law meaning that on finite structures such logic can only express properties that hold in almost all finite structures. Our aim is to study further limits of the expressive power of logics with embedding-closed quantifiers that are not implied by a 0-1 law. These include for example indefinability of properties of infinite structures and structures with function symbols. In this article we provide two methods that make it possible. The first method involves construction of a certain chain of quasi-homogeneous structures. The second method is based on the Ehrenfeucht-Fraïssé game that we develop in order to characterize \( \mathcal{L}_{\infty \omega}(Q_{\text{emb}}) \)-equivalence between structures. In the article we apply these methods to produce a number of undefinability results.

The article is structured as follows. In Section 2 we introduce preliminary notions. In Section 3 we describe basic properties of embedding-closed quantifiers that will be needed later, and give some examples. Before moving to our own major results, we show that \( \mathcal{L}_{\infty \omega}(Q) \), where \( Q \) is a finite set of embedding-closed quantifiers of finite width, has a 0-1 law. We do this in Section 4. The proof concerning 0-1 law was originally given in [2]. In Section 5 we introduce the notion of quasi-homogeneity and show that if a chain of quasi-homogeneous structures is sufficiently long then the truth value of a given sentence of a logic with embedding-closed quantifiers is eventually preserved. This in turn allows us to obtain some undefinability results. The section has two subsections, one of which is devoted to the undefinability of properties of finite structures and another deals with infinite structures. In Section 6 we describe the embeddability game that characterizes \( \mathcal{L}_{\infty \omega}(Q_{\text{emb}}) \)-equivalence of a given pair of structures. We close the section with an ap-
plication of the game that allows us to show that for each \( n < \omega \) there is a first-order sentence of quantifier rank \( n \) that is not expressible by any sentence of \( \mathcal{L}_{\infty \omega}(Q_{\text{emb}}) \) of quantifier rank \(< n \).

## 2 Preliminaries

A *signature* \( \tau \) consists of relation, function and constant symbols,

\[
\tau = \{ R, \ldots, f, \ldots, c, \ldots \}.
\]

We denote by \( \text{ar}(R) \) and \( \text{ar}(f) \) the arities of relation and function symbols. A \( \tau \)-structure \( \mathfrak{A} \) is a sequence

\[
\mathfrak{A} = (A, R^A, \ldots, f^A, \ldots, c^A),
\]

where \( A \) is a set that we call the *universe* of \( \mathfrak{A} \), and \( R^A \subseteq A^{\text{ar}(R)} \) are interpretations of symbols of \( \tau \). We denote the class of all \( \tau \)-structures by \( \text{Str}[\tau] \).

A *logic* is a pair \((\mathcal{L}, \models_{\mathcal{L}})\), where \( \mathcal{L} \) is a function mapping signatures \( \tau \) to a class \( \mathcal{L}[\tau] \) of \( \mathcal{L} \)-formulas of signature \( \tau \), and \( \models_{\mathcal{L}} \) is a binary relation between \( \tau \)-structures and formulas of \( \mathcal{L}[\tau] \). A logic must satisfy certain obvious natural conditions that we will not describe here. Free variables \( x_1, \ldots, x_n \) of formulas in \( \mathcal{L}[\tau] \) can be seen as extra constant symbols not in \( \tau \). All logics that we will consider in this article have formulas with at most finite number of free variables. Notation \( \varphi(x_1, \ldots, x_n) \) means that all free variables of \( \varphi \) are in the set \( \{x_1, \ldots, x_n\} \). Given a structure \( \mathfrak{A} \in \text{Str}[\tau] \) and an \( n \)-tuple \( \pi \in A^n \), we write \( \mathfrak{A}, \pi \models \varphi \) to mean that \( \mathfrak{A} \models \varphi \) when every \( x_i \) is interpreted as \( a_i \). We denote the number of free variables of \( \varphi \) by \( \text{frvar}(\varphi) \). If \( \psi \) is a formula with free variables then \( \mathfrak{A} \models \psi \) means that \( \mathfrak{A}, \pi \models \psi \) for all \( \pi \in A^{\text{frvar}(\psi)} \). A formula without free variables is called *sentence*.

A \( \tau \)-term is either a variable, a constant symbol of \( \tau \), or a string of the form \( F(t_1, \ldots, t_n) \), where \( F \in \tau \) is a function symbol and all \( t_i \) are terms. An \( \mathcal{L}[\tau] \)-formula is *atomic* if it has form \( R(t_1, \ldots, t_n) \) or \( t = s \), where \( R \in \tau \) is a relation symbol and \( t_1, \ldots, t_n, t, s \) are \( \tau \)-terms. A *literal* is an atomic formula or a negation of an atomic formula. An *atomic \( n \)-type* of \( \tau \) is a set \( \Phi \) of literals of \( \tau \) in variables \( x_1, \ldots, x_n \) such that there is a \( \tau \)-structure \( \mathfrak{A} \) and \( n \)-tuple \( \pi \) of elements in \( A \) with

\[
\Phi = \{ \varphi : \mathfrak{A}, \pi \models \varphi \text{ and } \varphi \text{ is a literal of } \tau \}.
\]

If we work with a logic \( \mathcal{L} \) which for each atomic type \( \Phi \) has a formula \( t \) equivalent to \( \bigwedge \Phi \) then \( t \) is also called an atomic type.
Lemma 2.1. Let $\vartheta$ be a quantifier-free $\tau$-formula in $n$ free variables. Then there exists a set $T$ of atomic $n$-types of $\tau$ such that

$$\models \vartheta \iff \bigvee_{\Phi \in T} \bigwedge_{\varphi \in \Phi} \varphi.$$  

In this article we will consider the following logics. We assume that the reader is familiar with the first-order logic $L_{\omega \omega}$ and the related notions. Let $\kappa$ be a cardinal. The logic $L_{\kappa \omega}$ is allowed to have conjunctions over sets of formulas of cardinality $< \kappa$. The logic $L_{\omega \infty}$ can have conjunctions over arbitrary sets of formulas. Formulas of the logic $L_{\omega \infty}$ (finite variable logic) are exactly those of $L_{\omega \infty}$ that use at most finite number of variables.

Suppose $L$ is a logic and $\tau, \sigma$ are signatures where $\sigma$ has only relation symbols. An $L$-interpretation of $\sigma$ in $\tau$ is a sequence $(\Psi, (\psi_R)_{R \in \sigma})$, where each $\psi_R$ is an $L[\tau]$-formula that has exactly $\text{ar}(R)$ free variables, and $\Psi$ is a function $\text{Str}[\tau] \to \text{Str}[\sigma]$ such that for each $A \in \text{Str}[\tau]$, the universe of $\Psi(A)$ is $A$ and

$$R^{\Psi(A)} = \{ \overline{a} \in A^{\text{ar}(R)} : A \models \psi_R(\overline{a}) \}$$

for all $R \in \sigma$.

Let $C \subseteq \text{Str}[\sigma]$ be a class of structures closed under isomorphism. The logic $L_{\kappa \omega}(Q_C)$ is the smallest extension of $L_{\kappa \omega}$ closed under negation, conjunctions of cardinality $< \kappa$ and application of the existential quantifier $\exists$ such that for all $L[\sigma]$-interpretations $(\Psi, (\psi_i)_{i < \delta})$ there is a $L(Q_C)[\tau]$-formula $\chi$ such that

$$A, \overline{a} \models \chi \iff \Psi(A), \overline{a} \models \psi_i$$

for all $A \in \text{Str}[\tau]$ and tuples $\overline{a}$ of elements in $A$.

Lemma 2.2. Let $Q$ be a set of quantifiers, $\kappa$ a cardinal and $L = L_{\kappa \omega}(Q) \lor L = L_{\omega \infty}(Q)$. Suppose $(\Psi, (\psi_R)_{R \in \sigma})$ is an $L$-interpretation of $\sigma$ in $\tau$. Then for each $L[\sigma]$-formula $\varphi$ there is a $L[\tau]$-formula $\varphi^*$ such that

$$A, \overline{a} \models \varphi^* \iff \Psi(A), \overline{a} \models \varphi$$

for all $A \in \text{Str}[\tau]$ and tuples $\overline{a}$ of elements in $A$.

Proof. Replace all atomic subformulas $R(\overline{a})$ of $\varphi$ with formulas $\psi_R(\overline{a})$ to get $\varphi^*$. \qed
3 Embedding-closed quantifiers

Definition 3.1. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be structures of the same signature \( \tau \). An injection \( f: A \to B \) is an embedding of \( \mathfrak{A} \) into \( \mathfrak{B} \) if

1. \( f(c^A) = c^B \) for all constant symbols \( c \in \tau \),
2. \( \bar{a} \in R^A \iff f(\bar{a}) \in R^B \) for all relation symbols \( R \in \tau \) and tuples \( \bar{a} \) in \( A \),
3. \( fF^A(\bar{a}) = F^B(f(\bar{a})) \) for all function symbols \( F \in \tau \) and tuples \( \bar{a} \) in \( A \).

The notation \( \mathfrak{A} \leq \mathfrak{B} \) means that \( \mathfrak{A} \) is embeddable into \( \mathfrak{B} \). A class \( K \) of \( \tau \)-structures is embedding-closed if \( \mathfrak{A} \in K \) and \( \mathfrak{A} \leq \mathfrak{B} \) imply \( \mathfrak{B} \in K \). We say that a quantifier \( Q \) is embedding-closed if its defining class is embedding-closed. We denote by \( Q_{\text{emb}} \) the class of all embedding-closed quantifiers.

Lemma 3.2. Let \( \tau \) be a signature, \( (\varphi_\alpha)_{\alpha<\kappa} \) quantifier-free \( \tau \)-formulas and \( Q \) an embedding-closed quantifier of width \( \kappa \). The formula \( Q(x_\alpha \varphi_\alpha)_{\alpha<\kappa} \) is preserved by embeddings.

Proof. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be \( \tau \)-structures. Suppose that \( (\mathfrak{A}, \bar{a}) \models Q(x_\alpha \varphi_\alpha)_{\alpha<\kappa} \) and \( f: A \to B \) is an embedding. Then \( f \) is also an embedding of \( (A, (\varphi_\alpha^A, \bar{a})_{\alpha<\kappa}) \) into \( (B, (\varphi_\alpha^B, f(\bar{a}))_{\alpha<\kappa}) \) since quantifier-free formulas are preserved by embeddings, so \( (\mathfrak{B}, f(\bar{a})) \models Q(x_\alpha \varphi_\alpha)_{\alpha<\kappa} \) since \( Q \) is embedding-closed.

Note that instead of requiring the quantifiers to be closed upwards under embeddings, we could use the downwards closure to get the equivalent class of quantifiers. Call a quantifier \( Q \) substructure-closed if from \( \mathfrak{A} \in K_Q \) and \( \mathfrak{B} \leq \mathfrak{A} \) follows \( \mathfrak{B} \in K_Q \), and denote the class of all substructure-closed quantifiers by \( Q_{\text{sub}} \). The expressive power of \( Q_{\text{sub}} \) is clearly the same as that of \( Q_{\text{emb}} \) since the complement \( Q^* \) of an embedding-closed quantifier \( Q \) is substructure-closed, so

\[ \mathfrak{A} \models Q(x_\alpha \varphi_\alpha)_{\alpha<\kappa} \iff \mathfrak{A} \models -Q^*(x_\alpha \varphi_\alpha)_{\alpha<\kappa}. \]

Next we present some examples of well-known properties and quantifiers that are either embedding-closed or are definable in the logic \( L_{\infty\omega}(Q_{\text{emb}}) \). We use notation \( Q^{\dagger} \) to denote the closure of the quantifier \( Q \) under embeddings. In other words \( Q^{\dagger} \) is the smallest embedding-closed quantifier containing \( Q \).

Examples 3.3. 1. Let \( \tau = \{ U \} \) be a signature consisting of a single unary relation symbol. The existential quantifier \( \exists \) corresponds to the class of structures \( \{ (A, U) \in \text{Str}[\tau]: U \neq \emptyset \} \), so it is embedding-closed.
2. Let $\alpha$ be an ordinal. The defining class of the cardinality quantifier $Q_{\alpha}$ is $\{(A, U) \in \text{Str}[\tau] : |U| \geq \aleph_\alpha \}$ which is clearly embedding-closed.

3. For each $n < \omega$, let $\sigma_n = \{M_n\}$ be a signature consisting of a single $n$-ary relation symbol. The Magidor-Malitz quantifier $Q_{\alpha}^n$, whose defining class is

$$\{(A, M_n) \in \text{Str}[\sigma_n] : \text{there is } C \subseteq A \text{ with } |C| \geq \aleph_\alpha \text{ and } C^n \subseteq M_n \}$$

is embedding-closed.

4. The well-ordering quantifier $Q^W$, whose defining class is the class of all well-orders, is substructure-closed, so by our earlier remark it can be defined with embedding-closed quantifiers.

5. The equivalence quantifier $Q_{\alpha}^E$, whose defining class consists of all structures $(A, E)$ with $E$ an equivalence relation on $A$ which has at least $\aleph_\alpha$ equivalence classes, can be defined by the sentence

$$(Q_{\alpha}^E)^{cl}xyE(x, y) \land 'E \text{ is an equivalence relation'}.$$ 

6. Many graph properties are embedding- or substructure-closed. Examples include $k$-colorability, being a forest, completeness, planarity, having a cycle, and many others.

7. This is an example of a graph property that is not embedding- or substructure-closed but is however definable in $L_{\infty\omega}(Q)$ for a certain $Q \in Q_{\text{emb}}$. The property in question is connectedness of a graph. Let $\sigma = \{R, B, E\}$ be a coloured graph -signature with symbols $R$ and $B$ standing for red and blue. Let $C \subseteq \text{Str}[\sigma]$ consist of all the graphs in which for every blue-red pair $(x, y)$ of vertices there is a path between $x$ and $y$. Put $D = C^{cl}$. Then for all graphs $G$,

$$G \models \forall xyQ_{D}stu(s = x, t = y, E(u, v))$$

if, and only if, $G$ is connected.

As we will show below, there exist properties not definable in $L_{\infty\omega}(Q_{\text{emb}})$. These include among others equicardinality of sets (Example 5.24), and completeness and cofinality of an ordering (Examples 5.18 and 5.19).
4 0-1 law

In [2], Dawar and Grädel showed that logic $L_{\omega_1\omega}$ extended with finitely many embedding-closed graph quantifiers of finite width has a 0-1 law. We start our investigation of embedding-closed quantifiers with demonstration of this proof.

The notation $\mu(P) = 1$ means that the asymptotic probability of a property $P$ is 1. A structure $\mathfrak{A}$ is homogeneous if every isomorphism between finitely generated substructures of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$. Let $\tau$ be a relational signature. The random $\tau$-structure is the unique homogeneous countable $\tau$-structure into which any finite $\tau$-structure can be embeded. The following is a well-known fact:

**Theorem 4.1.** If $P$ is $L_{\omega_1\omega}$-definable property that is true in the random structure then $\mu(P) = 1$.

**Lemma 4.2.** Let $\tau$ be a finite relational signature, $\mathfrak{A}$ a finite $\tau$-structure and $\overline{a}$ a tuple of elements of $A$ with $|\overline{a}| = n$. Suppose that $t$ is the atomic type of $\overline{a}$ and set

$$P = \{ \mathfrak{B} \in \text{Str}[\tau]: \text{for all } \overline{b} \in B^n, \text{ if } \mathfrak{B} \models t(\overline{b}) \text{ then } (\mathfrak{A}, \overline{a}) \leq (\mathfrak{B}, \overline{b}) \}.$$ 

Then $\mu(P) = 1$.

**Proof.** Denote the random $\tau$-structure by $\mathfrak{R}$. The structure $\mathfrak{A}$ is embeddable into $\mathfrak{R}$ by, say, an embedding $f$. Let $\overline{b}$ be a tuple in $R$ whose atomic type is $t$. Since $\mathfrak{R}$ is homogeneous, there is an automorphism $h$ that takes $\overline{a}$ to $\overline{b}$. Thus, $h \circ f$ is an embedding of $(\mathfrak{A}, \overline{a})$ into $(\mathfrak{R}, \overline{b})$, so $P$ holds in the random structure, and since $P$ is $L_{\omega_1\omega}$-definable, we have $\mu(P) = 1$. \qed

**Lemma 4.3** ([2]). Let $\tau$ be a finite relational signature, $Q$ embedding-closed quantifier of finite width $k$ and $\psi_0, \ldots, \psi_{k-1}$ quantifier-free $\tau$-formulas. There is a quantifier-free $\tau$-formula $\vartheta$ such that $\forall \overline{x} (\vartheta \leftrightarrow Q(\overline{x}_i)_{i<k})$ has asymptotic probability 1.

**Proof.** Write $\varphi := Q(\overline{x}_i)_{i<k}$, and set

$$\vartheta := \bigvee \{ t: t \text{ is an atomic type and } (\mathfrak{A}, \overline{a}) \models t \land \varphi \text{ for some finite } \tau \text{-structure } \mathfrak{A} \text{ and tuple } \overline{a} \}.$$ 

We clearly have $\mathfrak{A} \models \forall \overline{x}(\varphi \rightarrow \vartheta)$ for all finite $\tau$-structures $\mathfrak{A}$. For the other direction, if $\vartheta$ is an empty disjunction then $\varphi$ defines the empty relation on all finite structures thus being equivalent to a quantifier-free formula. Therefore, assume that $\vartheta$ is not empty.
For each type \( t \) in \( \vartheta \), choose a pair \((A, a)_t\) such that \((A, a)_t \models t \land \varphi\) and \( A \) is finite. Such pairs exist by the definition of \( \vartheta \). Let \( B \) be a finite \( \tau \)-structure such that \((B, \vec{b}) \models \forall \varphi\). Then there is a tuple \( \vec{b} \) in \( B \) such that \((B, \vec{b}) \models \vartheta \land \neg \varphi\). Let \( t \) be the atomic type of \( \vartheta \). Then \((A, a)_t \models \vartheta \land \varphi\) and \((A, a)_t \models t \land \varphi\) and \( A \) is finite. Let \( B \) be a finite \( \tau \)-structure such that \( B \models \exists x(\vartheta \land \neg \varphi)\). Then there is a tuple \( b \) in \( B \) such that \((B, b) \models \vartheta \land \neg \varphi\). Let \( t \) be the atomic type of \( b \). Then \((A, a)_t \models \vartheta \land \varphi\), so if \((A, a)_t \leq (B, b)\) then by Lemma 3.2, \((B, b) \models \varphi\) which is contradiction. Thus, \((A, a)_t \not\leq (B, b)\), so by Lemma 4.2, \( \mu(\exists x(\vartheta \land \neg \varphi)) = 0\), so \( \mu(\forall \varphi(\vartheta \rightarrow \varphi)) = 1\). □

**Theorem 4.4** ([2]). Let \( \tau \) be a finite relational signature and \( Q \) a finite set of embedding-closed quantifiers of finite width. For any \( \tau \)-formula \( \varphi \in L_{\omega_1}^\omega(Q) \) there is a quantifier-free \( \tau \)-formula \( \vartheta \) such that \( \forall \vartheta(\vartheta \leftrightarrow \varphi) \) has asymptotic probability 1.

**Proof.** Let \( k \) be a natural number. There are, up to logical equivalence, finitely many quantifier-free formulas that use only variables \( x_0, \ldots, x_{k-1} \). Let \( \psi_0, \ldots, \psi_{l-1} \) be an enumeration of all \( L_{\omega_1}^k \)-formulas of the form \( Q(\vec{y}_i)_{1< n} \) with all \( \vartheta_i \) quantifier-free. Note that \( l \) is finite. By Lemma 4.3 for each \( i < l \) there is a quantifier free formula \( \chi_i \) such that \( \forall \vartheta(\psi_i \leftrightarrow \chi_i) \) has asymptotic probability 1. For every \( i < l \), let \( C_i \) be the set of all isomorphism types of finite structures on which \( \forall \vartheta(\psi_i \leftrightarrow \chi_i) \) is true. Put \( C = \bigcap_{i<l} C_i \). Then \( \mu(C) = 1 \), since \( l \) is finite and \( \mu(C_i) = 1 \) for all \( i \).

Now we can show that for all \( \varphi \in L_{\omega_1}^k(Q) \) there is a quantifier-free formula \( \vartheta \) such that \( \forall \vartheta(\vartheta \leftrightarrow \varphi) \) for all \( \varphi \in C \) from which the claim follows. We use induction on the structure of \( \varphi \). If \( \varphi \) is quantifier-free, there is nothing to prove. It is also clear that the claim holds for \( \varphi = \neg \alpha \) and for \( \varphi = \bigwedge_{i \in I} \alpha_i \) if it holds for \( \alpha \) and all \( \alpha_i \), respectively. Assume that \( \varphi = Q(\vec{y}_i)_{1< n} \) and the claim holds for all \( \alpha_i \). By the induction hypothesis, there are quantifier-free formulas \( \vartheta_i \) such that

\[
\forall \vartheta(\varphi \leftrightarrow Q(\vec{y}_i)_{1< n})
\]

for all \( \varphi \in C \). Since \( Q(\vec{y}_i)_{1< n} = \psi_m \) for some \( m < l \), we have \( \forall \vartheta(\varphi \leftrightarrow \chi_m) \) on all structures of \( C_m \), and therefore on \( C \), because \( C \subseteq C_m \). □

**Corollary 4.5** ([2]). For any finite set \( Q \) of embedding-closed quantifiers of finite width, the logic \( L_{\omega_1}^\omega(Q) \) has a 0-1 law.
5 Quantifier elimination for $\mathcal{L}_{\infty\omega}(Q_{\text{emb}})$ and some undefinability results

In this section we introduce a method that allows us to produce some undefinability results for logics with embedding-closed quantifiers that cannot be established by using a 0-1 law. For instance we will show that equicardinality cannot be defined in $\mathcal{L}_{\infty\omega}(Q_{\text{emb}})$.

**Definition 5.1.** A structure $\mathfrak{A}$ is **homogeneous** if every isomorphism between finitely generated substructures of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$. We say that $\mathfrak{A}$ is **quasi-homogeneous** if every isomorphism between finitely generated substructures of $\mathfrak{A}$ can be extended to an embedding of $\mathfrak{A}$ into itself.

Note that a structure $\mathfrak{A}$ is homogeneous (quasi-homogeneous) if and only if for all tuples $\vec{a}$ and $\vec{b}$ of the same atomic type there is an automorphism (embedding) of $\mathfrak{A}$ taking $\vec{a}$ to $\vec{b}$. It is clear that every countable quasi-homogeneous structure is homogeneous. Let $\mathfrak{R} = (R \setminus \{r\}, \leq)$ be the usual ordering of real numbers with some number $r$ removed. This "punctured" real line is an example of a structure that is quasi-homogeneous but not homogeneous.

**Definition 5.2.** Suppose $\mathcal{L}$ is a logic. We say that a structure $\mathfrak{A}$ has **quantifier elimination for** $\mathcal{L}$ if for all formulas $\varphi \in \mathcal{L}$ there is a quantifier-free formula $\vartheta$ such that $\mathfrak{A} \models \forall \vec{x}(\vartheta \leftrightarrow \varphi)$.

**Theorem 5.3.** A $\tau$-structure $\mathfrak{A}$ has quantifier elimination for $\mathcal{L}_{\infty\omega}(Q_{\text{emb}})$ if and only if it is quasi-homogeneous.

**Proof.** Assume for simplicity that $\tau$ is relational signature. The proof can be generalized in a straightforward way to signatures with constant and function symbols. Suppose first that $\mathfrak{A}$ has quantifier elimination. Let $\vec{a} = (a_1, \ldots, a_k)$ and $\vec{b} = (b_1, \ldots, b_k)$ be tuples of elements of $A$ having the same atomic type. We want to find embedding of $\mathfrak{A}$ into itself that maps $\vec{a}$ to $\vec{b}$. Let $\tau' = \tau \cup \{P\}$ where $P$ is a new relation symbol of arity $k$. Define a $\tau'$-structure $\mathfrak{A}'$ by setting $\mathfrak{A}' \upharpoonright \tau = \mathfrak{A}$ and $P^{\mathfrak{A}'} = \{\vec{a}\}$. Let $Q$ be a quantifier whose defining class is

$$K_Q = \{ \mathfrak{B} \in \text{Str}[\tau'] : \mathfrak{A}' \text{ is embeddable into } \mathfrak{B} \},$$

and suppose $\varphi \in \mathcal{L}_{\infty\omega}(Q_{\text{emb}})[\tau]$ is the next formula:

$$\varphi(\vec{z}) := Q((\vec{x} R)_{R \in \tau}, \exists P = \vec{z}).$$
Then $\mathfrak{A} \models \varphi(\overline{a})$ and, since $\mathfrak{A}$ has quantifier elimination and $\overline{a}$ and $\overline{b}$ have the same atomic type, we have $\mathfrak{A} \models \varphi(\overline{b})$, so there is an embedding $f$ of $\mathfrak{A}$ into $(A, (S^A)_{s \in T}, \{\overline{b}\})$ which clearly is a wanted embedding.

For the other direction, assume that $\mathfrak{A}$ is quasi-homogeneous. Let $Q \in \mathcal{Q}_{\text{emb}}$ and suppose $(\psi_R)_{R \in \tau_Q}$ are quantifier-free formulas. Now set $\varphi := Q(\varphi_R)_{R \in \tau_Q}$ and denote by $k$ the number of free variables of $\varphi$. Let

$$\vartheta = \bigvee \{t : t \text{ is an atomic type and for some } a \in A^k, (A, a) \models \varphi \land t\}.$$ 

Let $\overline{b} \in A^k$ and $(\mathfrak{A}, \overline{b}) \models \vartheta$. Then $(\mathfrak{A}, \overline{b}) \models t$ and $(\mathfrak{A}, \overline{a}) \models \varphi \land t$ for some $\overline{a} \in A^k$ with atomic type $t$. Since $\mathfrak{A}$ is quasi-homogeneous, there is an embedding $f$ of $(\mathfrak{A}, \overline{a})$ into $(\mathfrak{A}, \overline{b})$. Since $\mathfrak{A}, \overline{a} \models \varphi$, we have $(A, (\psi^A_R)_{R \in \tau_Q}) \in K_Q$, so $(A, (\psi^A_R)_{R \in \tau_Q}) \in K_Q$ because $Q$ is embedding-closed and $\psi_R$ are quantifier-free. Thus, $\mathfrak{A} \models \forall \overline{t}(\vartheta \rightarrow \varphi)$, and since clearly $\mathfrak{A} \models \forall \overline{t}(\varphi \rightarrow \vartheta)$, we have $\mathfrak{A} \models \forall \overline{t}(\vartheta \leftrightarrow \varphi)$.

Thus, by using induction, we can eliminate quantifiers in all formulas $\varphi \in \mathcal{L}_{\infty\omega}(\mathcal{Q}_{\text{emb}})$.

5.1 The finite case

In this subsection, we will consider logic $\mathcal{L}_{\infty\omega}^\omega$ (finite variable logic) extended with finite number of embedding-closed quantifiers of finite width. We will show that in a countably infinite chain of quasi-homogeneous relational structures, a formula of such a logic is eventually equivalent to a quantifier-free formula. This will allow us to show, among others, that certain properties of finite structures are not definable in such a logic.

**Lemma 5.4.** Let $\tau$ be a finite relational signature, $Q$ an embedding-closed quantifier of width $n < \omega$ and $\varphi = Q(x_i \psi_i)_{i < n}$ where all $\psi_i$ are quantifier-free $\tau$-formulas. Let $(\mathfrak{A}_i)_{i < \omega}$ be a chain of quasi-homogeneous structures. Then there is a natural number $k$ and a quantifier-free $\tau$-formula $\vartheta$ such that

$$\mathfrak{A}_i \models \forall \overline{t}(\varphi \leftrightarrow \vartheta)$$

for all $k \leq i$.

**Proof.** For each $i < \omega$, let

$$T_i = \{t : t \text{ is an atomic type and } (\mathfrak{A}_i, \overline{a}) \models \varphi \land t \text{ for some } \overline{a}\}.$$ 

Since all $\mathfrak{A}_i$ are quasi-homogeneous, it follows from Theorem 5.3 and Lemma 2.1 that

$$\mathfrak{A}_i \models \forall \overline{t}(\varphi \leftrightarrow \bigvee T_i).$$
Now let \( i \leq j < \omega \) and \( t \in T_i \). We have \((A_i, \bar{a}) \models \varphi \land t\) for some \( \bar{a} \), so \((A_j, \bar{a}) \models \varphi \land t\) since both \( \varphi \) and \( t \) are preserved by embeddings. Thus, \( t \in T_j \), so \( T_i \subseteq T_j \) always when \( i \leq j \). Since there are finitely many distinct atomic \( n \)-types, the chain \((T_i)_{i<\omega}\) reaches its maximum at some \( k < \omega \). Then \( \vartheta = \bigvee T_k \) is a quantifier-free \( \tau \)-formula we want.

**Theorem 5.5.** Let \( \tau \) be a finite relational signature, \( Q \) a finite set of embedding-closed quantifiers of finite width and \( \varphi \in L^{\omega}_\infty(\mathcal{Q})[\tau] \). Suppose that \((A_i)_{i<\omega}\) is a chain of quasi-homogeneous structures. Then there is a natural number \( k_\varphi \) and a quantifier-free \( \tau \)-formula \( \vartheta_\varphi \) such that

\[ A_i \models \forall \varphi (\varphi \leftrightarrow \vartheta_\varphi) \]

for all \( k_\varphi \leq i \).

**Proof.** Let \( m \) be the number of variables used in the formula \( \varphi \). Let \( \psi_0, \ldots, \psi_l \) be an enumeration of all (up to equivalence) the \( \tau \)-formulas in at most \( m \) variables having form \( Q(\bar{x}, \vartheta_i)_{i<n} \) with \( Q \in Q \) and all \( \vartheta_i \) quantifier-free. Note that \( l \) is finite. By Lemma 5.4 for each \( \psi_i \) there is \( k_i < \omega \) and a quantifier-free \( \tau \)-formula \( \vartheta_i \) such that

\[ A_j \models \forall \psi_i (\psi_i \leftrightarrow \vartheta_i) \]

when \( j \geq k_i \). Let \( k = \max \{ k_i \}_{i \leq l} \). We claim that we can set \( k_\varphi := k \). We prove the claim by induction on the structure of the formula \( \varphi \). The cases \( \varphi \) is atomic, \( \varphi = \neg \alpha \) and \( \varphi = \bigwedge_{i \in I} \alpha_i \) are clear. Suppose \( \varphi = Q(\bar{x}, \vartheta_i)_{i<n} \) and the claim holds for all \( \alpha_i \). Then there are quantifier-free \( \tau \)-formulas \( \vartheta_i \) such that

\[ A_j \models \forall \varphi (\varphi \leftrightarrow Q(\bar{x}, \vartheta_i)_{i<n}) \]

when \( j \geq k \). Thus, if \( j \geq k \) then \( \varphi \) is equivalent to some \( \psi_r \) and therefore to some quantifier-free formula \( \vartheta \), so we can set \( \vartheta_\varphi := \vartheta \).

In Section 4 above, we saw that the logic \( L^{\omega}_\infty \) extended with finitely many embedding-closed quantifiers of finite width has a 0-1 law which implies undefinability of certain properties, like having even cardinality, in such a logic. By using Theorem 5.5 we can determine further properties of finite structures that are not definable in a logic of this kind. In order to apply the theorem, however, we first need to know which structures are homogeneous. This question has been studied to some extent (a survey can be found in [4]). Finite homogeneous structures have been classified completely at least in the cases of finite graphs [3], groups [1] and rings [7]. In addition, it is easy to see that all unary structures are homogeneous.
Example 5.6. According to [4], the only finite homogeneous (undirected) graphs are up to complement

1. $P_e = (\{0,1,2,3,4\}, \{(i,j) : |i-j| \in \{1,4\}\})$ (pentagon),
2. $K_3 \times K_3$,
3. $I_m[K_n]$ with $m, n < \omega$,

where $K_n$ is the complete graph of $n$ vertices and $I_m[G]$ consists of $m$ disjoint copies of $G$.

It is easy to see that $I_m[K_n] \leq I_{m'}[K_{n'}]$ if and only if $m \leq m'$ and $n \leq n'$.

Thus, for example, the graph properties "there is a clique of even cardinality" or "there are more cliques than there are vertices in any clique" are not definable in $L^\omega_{\infty\omega}(Q)$ for any finite set $Q$ of embedding-closed quantifiers of finite width.

Example 5.7. Let $\tau = \{U,V\}$ be a signature with $U$ and $V$ unary relational symbols. The quantifier corresponding to the class

$I = \{ \mathfrak{A} \in \text{Str}[\tau] : |U^\mathfrak{A}| = |V^\mathfrak{A}| \}$

of $\tau$-structures is known as Härtig quantifier. Let $Q$ be a finite set of embedding-closed quantifiers of finite width. The following is an easy observation showing that the Härtig quantifier is not definable in $L^\omega_{\infty\omega}(Q)$ even if we consider only finite structures.

For all $i < \omega$ define a $\tau$-structure $\mathfrak{A}_i$ by setting $A_i = \{0,\ldots,i\}$ and letting $U^{\mathfrak{A}_i}$ to be the set of all even and $V^{\mathfrak{A}_i}$ of all odd numbers of $A_i$. Then $\mathfrak{A}_i \leq \mathfrak{A}_{i+1}$ for all $i < \omega$, and $|U^{\mathfrak{A}_i}| = |V^{\mathfrak{A}_i}|$ if and only if $i$ is odd. Thus, it follows from Theorem 5.5 that the class

$\{ \mathfrak{A} \in \text{Str}[\tau] : \mathfrak{A} \text{ is finite and } |U^\mathfrak{A}| = |V^\mathfrak{A}| \}$

is not definable in the logic $L^\omega_{\infty\omega}(Q)$. In the next subsection, we will show that in fact the Härtig quantifier is not definable in the logic $L_{\infty\omega}(Q_{\text{emb}})$ as well.

As we saw in Example 5.6, there are few homogeneous finite graphs, so we usually cannot directly apply Theorem 5.5 in studying definability of graph properties. The following theorem shows that this situation can be remedied to some extent by using interpretations.
Theorem 5.8. Let $\tau$ and $\sigma$ be finite relational signatures and $Q$ a finite set of embedding-closed quantifiers of finite width. Write $\mathcal{L} = \mathcal{L}_{\infty \omega}^{\infty}(Q)$. Suppose $(A_i)_{i<\omega}$ is a chain of quasi-homogeneous $\tau$-structures and $(\Psi, (\psi_R)_{R \in \sigma})$ is an $\mathcal{L}$-interpretation of $\sigma$ in $\tau$. Let $\varphi$ be a $\mathcal{L}[\sigma]$-sentence. Then there is $k < \omega$ such that

$$\Psi(A_i) \models \varphi \iff \Psi(A_j) \models \varphi$$

for all $i, j \geq k$.

Proof. By Lemma 2.2, there is an $\mathcal{L}[\tau]$-sentence $\varphi^*$ such that

$$A_i \models \varphi^* \iff \Psi(A_i) \models \varphi.$$ 

By Theorem 5.5, the truth value of $\varphi^*$ stabilizes in the chain $(A_i)_{i<\omega}$ after some $k < \omega$, so the same must happen in the sequence $(\Psi(A_i))_{i<\omega}$ as well. $\square$

Example 5.9. A graph $G$ is regular if every vertex of $G$ has the same number of neighbours. We will show that regularity of a graph is not definable in the logic $\mathcal{L} = \mathcal{L}_{\infty \omega}^{\infty}(Q_1, \ldots, Q_n)$ where all $Q_i \in Q_{\text{emb}}$ have finite width. Let $\tau$ be the same signature and $(A_i)_{i<\omega}$ a chain of $\tau$-structures as in Example 5.7. Let $\sigma = \{E\}$ where $E$ is a relation symbol and suppose $(\Psi, \psi)$ is an interpretation of $\sigma$ in $\tau$ where

$$\psi(x, y) := (U(x) \land U(y)) \lor (V(x) \land V(y)).$$

Then for all $k < \omega$,

$$\Psi(A_k) = \begin{cases} K_{\frac{k+1}{2}} + K_{\frac{k+1}{2}+1} & \text{if } k \text{ is odd} \\ K_{\frac{k}{2}} + K_{\frac{k}{2}+1} & \text{if } k \text{ is even,} \end{cases}$$

where every $K_m$ is the complete graph on $m$ vertices and $+$ means disjoint union. Thus, a graph $\Psi(A_k)$ is regular if and only if $k$ is odd, so by Theorem 5.8 regularity is not definable in $\mathcal{L}$.

Example 5.10. If we allow $\tau$ to have function symbols then Theorem 5.5 does not hold. Let $\tau$ be the signature of groups. A formula $\varphi \in \mathcal{L}_{\infty \omega}^{\infty}[\tau]$,

$$\varphi(x, y) := \bigvee_{k<\omega} (x^k = 1 \land y^k = 1),$$

says that $x$ and $y$ have the same order, and a formula $\chi \in \mathcal{L}_{\infty \omega}^{\infty}[\tau]$,

$$\chi(x, y) := \bigvee_{k<\omega} y = x^k,$$
says that $y$ is in the subgroup generated by $x$. For each $n < \omega$, set

$$G_{2n} = \prod_{i \leq n} C^2_{p_i},$$

$$G_{2n+1} = \prod_{i \leq n} C^2_{p_i} \times C_{p_{n+1}},$$

where $p_i$ is the $i$:th prime number. Then $(G_n)_{n<\omega}$ is a chain of homogeneous groups, but $G_{2n} \models \psi$ and $G_{2n+1} \not\models \psi$ for all $n$, where

$$\psi := \forall x \exists y (\neg \chi(x, y) \land \varphi(x, y)).$$

If $(\mathfrak{A}_i)_{i<\omega}$ is a chain of finite homogeneous $\tau$-structures for a finite relational signature $\tau$ then clearly $L^{\omega}_\omega(Q) \equiv L^{\omega}_\omega$ over the structures in this chain for any finite set $Q$ of embedding-closed quantifiers of finite width. Theorems 5.12 and 5.13 say more:

**Lemma 5.11.** Let $L = L^{\omega}_\omega$ or $L = L^{\omega}_\omega$. Suppose $\sigma = \{R_0, \ldots, R_{n-1}\}$ is a relational signature, $\mathfrak{A}_0, \ldots, \mathfrak{A}_{k-1}$ are finite $\sigma$-structures and $\psi_0, \ldots, \psi_{n-1} \in L[\tau]$ are formulas with $\text{frvar}(\psi_i) = \text{ar}(R_i)$ for all $i$. Then there is an $L[\tau]$-sentence $\varphi$ such that

$$\mathfrak{B} \models \varphi \iff \mathfrak{A}_j \leq (B, (\psi^\mathfrak{B}_i)_{i<n}) \text{ for some } j < k$$

for all $\mathfrak{B} \in \text{Str}[\tau]$.

**Theorem 5.12.** Let $L = L^{\omega}_\omega$ or $L = L^{\omega}_\omega$. Suppose $\tau$ is a finite relational signature and $C$ is a class of finite homogeneous $\tau$-structures without infinite antichains. Let $Q$ be a finite set of embedding-closed quantifiers of finite width. Then over the class $C$, we have $L \equiv L(Q)$.

**Proof.** For each formula $\varphi \in L(Q)[\tau]$ we construct inductively an $L[\tau]$-formula $\varphi^*$ that is equivalent to $\varphi$ over structures of $C$ as follows. Set $\varphi^* = \varphi$ if $\varphi$ is atomic, $(\neg \varphi)^* = \neg \varphi^*$ and $(\Lambda_{i \in I} \varphi_i)^* = \Lambda_{i \in I} \varphi_i^*$. For the case involving a quantifier, suppose $Q \in Q$ and $\tau_Q = \{R_0, \ldots, R_{n-1}\}$. Let

$$\Theta = \{(\vartheta_0, \ldots, \vartheta_{n-1}) \in L[\tau]^n : \text{every } \vartheta_i \text{ is quantifier-free}$$

and $\text{frvar}(\vartheta_i) - \text{ar}(R_i)$ is the same for all $i\}.$

For each $\overline{\vartheta} \in \Theta$, we use notation $n(\overline{\vartheta}) = \text{frvar}(\vartheta_i) - \text{ar}(R_i)$ and define

$$K_{\overline{\vartheta}} = \{(A, (\vartheta^A_i)_{i<n}) : \mathfrak{A} \in C \text{ and } \overline{\vartheta} \in A^n(\overline{\vartheta}) \} \cap K_Q.$$
We claim that all antichains of $K_\vartheta$ are finite. Assume towards contradiction that $D \subseteq K_\vartheta$ is an infinite antichain. For each atomic $n(\vartheta)$-type $t$, let

$$D_t = \{(A, (\vartheta_i^A, a_i))_{i<n} \in D : a_i \text{ has atomic type } t\}.$$ 

Then $D$ is the union of all $D_t$, and since there are finitely many atomic $n(\vartheta)$-types, one of $D_t$ must be an infinite antichain. Now, since all $\vartheta_i$ are quantifier-free, we have

$$\mathfrak{A}, \overline{\alpha} \leq \mathfrak{B}, \overline{\beta} \Rightarrow (A, (\vartheta_i^A, a_i))_{i<n} \leq (B, (\vartheta_i^B, b_i))_{i<n},$$

so there is an infinite antichain $D'$ whose elements are pairs $(\mathfrak{A}, \overline{\alpha})$ with $\mathfrak{A} \in C$ and $\overline{\alpha}$ having atomic type $t$. However, if $\mathfrak{A}, \overline{\alpha} \leq \mathfrak{B}, \overline{\beta}$ since $\mathfrak{B}$ is homogeneous, so $\mathcal{F} = \{\mathfrak{A} : (\mathfrak{A}, \overline{\alpha}) \in D'\}$ is an infinite antichain of structures of $C$, contradiction. Thus, the claim holds.

For all $\vartheta \in \Theta$, let $M_\vartheta$ be the set of all minimal structures of $K_\vartheta$, $M = \bigcup_{\vartheta \in \Theta} M_\vartheta$ and $N = \{\mathfrak{B} \in K_Q : \mathfrak{B} \leq \mathfrak{A} \text{ for some } \mathfrak{A} \in M\}$. Then $N$ is (up to isomorphism) finite since minimal structures form an antichain and $\Theta$ is finite. We claim that for all formulas $\psi_i \in L[\tau]$, structures $\mathfrak{A} \in C$ and tuples $\overline{\alpha}$ of elements in $A$, $\mathfrak{A}, \overline{\alpha} \models Q(\overline{\tau}, \psi)_{i<n}$

$$\Leftrightarrow \mathfrak{B} \leq (A, (\psi_i^A, a_i))_{i<n} \text{ for some } \mathfrak{B} \in N.$$ 

To prove the claim, let $\mathfrak{A}, \overline{\alpha} \models Q(\overline{\tau}, \psi)_{i<n}$. Then there is $\overline{\vartheta} = (\vartheta_0, \ldots, \vartheta_{n-1}) \in \Theta$ such that $\psi_i^A = \psi_i^A$ for all $i < n$, so $(A, (\psi_i^A, a_i))_{i<n} \in K_\vartheta$ and further $\mathfrak{B} \leq (A, (\psi_i^A, a_i))_{i<n}$ for some $\mathfrak{B} \in M_\vartheta \subseteq N$. The other direction follows from the fact that $Q$ is embedding-closed. Thus, by Lemma 5.11 over the class $C$, $Q(\overline{\tau}, \psi)_{i<n}$ is equivalent to an $L[\tau]$-formula. 

**Theorem 5.13.** Suppose $L = L_\omega^\omega$ or $L = L_\omega^{<\omega}$. Let $\tau$ be a finite relational signature, $C$ a class of finite homogeneous $\tau$-structures, $Q$ a finite set of embedding-closed quantifiers of finite width, and $(\Psi, \psi_1, \ldots, \psi_n)$ an $L$-interpretation of $\sigma$ in $\tau$. Then $L \equiv L(Q)$ over the class $\{\Psi(\mathfrak{A}) : \mathfrak{A} \in C\}$.

**Proof.** Let $Q \in Q$ and suppose $\varphi_1, \ldots, \varphi_k$ are $L[\sigma]$-formulas. Let

$$K = \{(A, (\varphi_i^{\Psi(\mathfrak{A})}))_{R \in \tau_Q} \in K_Q : \mathfrak{A} \in C\}.$$
According to Lemma 2.2 for each $\varphi_R$ there is an $\mathcal{L}[\tau]$-formula $\varphi^*_R$ such that

\[
\Psi(\mathfrak{A}), \overline{\tau} \models \varphi_R \iff \mathfrak{A}, \overline{\tau} \models \varphi^*_R
\]

for all $\tau$-structures $\mathfrak{A}$ and tuples $\overline{\tau}$ of elements in $A$. Thus,

\[
K = \{(A, (\varphi^*_R(\mathfrak{A}), \overline{\tau}))_{R \in \mathcal{Q}} \in K_Q : \mathfrak{A} \in C\},
\]

so we can show in the same way as in the proof of Theorem 5.12 that the set $\mathcal{M}$ of all minimal structures of $K$ is finite up to isomorphism.

**Example 5.14.** For any given finite set $\mathcal{Q}$ of embedding-closed quantifiers of finite width, we have $\mathcal{L}_{\omega\omega}(\mathcal{Q}) \equiv \mathcal{L}_{\omega\omega}$ and $\mathcal{L}_{\omega\omega}(\mathcal{Q}) \equiv \mathcal{L}_{\omega\omega}$ over, among others, the following classes:

1. The class of all finite $\tau$-structures for a unary $\tau$.
2. The class of all finite homogeneous graphs.
3. For every $n < \omega$ the class of all finite equivalence relations that have less than $n$ equivalence classes.
4. For any given finite set $p_1, \ldots, p_n$ of prime numbers, the class of all groups of the form

\[
C_{p_1}^{k_1} \times \cdots \times C_{p_n}^{k_n},
\]

where $C_{p_i}^{k_i}$ is the direct product of $k_i$ cyclic groups of order $p_i^{k_i}$.

**Example 5.15.** By Theorem 5.12 and Example 5.14.4 the group property of being the direct product of an even number of groups is not definable in $\mathcal{L}_{\omega\omega}(\mathcal{Q})$ for any finite set $\mathcal{Q}$ of embedding-closed quantifiers of finite width.

### 5.2 The infinite case

We can generalize Theorem 5.5 to signatures and sets of embedding-closed quantifiers of arbitrary cardinality.

**Theorem 5.16.** Let $\tau$ be a signature, $\kappa$ a cardinal, $\mathcal{Q}$ a set of embedding-closed quantifiers of width $< \kappa$ and $\varphi \in \mathcal{L}_\infty(\mathcal{Q})$ a $\tau$-formula. Let $\lambda$ be a regular cardinal such that $2^{\tau|\mathcal{Q}|} \cdot |\mathcal{Q}| \cdot \kappa < \lambda$. Then for all chains $(\mathfrak{A}_\alpha)_{\alpha < \lambda}$ of quasi-homogeneous structures there is a cardinal $\mu_\varphi$ and a quantifier-free $\tau$-formula $\vartheta_\varphi$ such that

\[
\mathfrak{A}_\alpha \models \forall \varphi (\varphi \iff \vartheta_\varphi)
\]

for all $\mu_\varphi \leq \alpha$. 

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Proof. Let $Q \in \mathbb{Q}_{\text{emb}}$ and $\varphi = Q(\varpi_i \vartheta_i)_{i<\delta}$ where all $\delta_i$ are quantifier-free $\tau$-formulas. We can see in the same way as in the proof of Lemma 5.4 that in a $\tau$-structure $\mathfrak{A}$ the formula $\varphi$ is equivalent to $\bigvee T_\mathfrak{A}$, where $T_\mathfrak{A}$ is a set of atomic types of $\tau$. In addition, if $\mathfrak{A} \leq \mathfrak{B}$ then $T_\mathfrak{A} \subseteq T_\mathfrak{B}$. Thus, since there are at most $2^{\|\tau\|+\aleph_0}$ atomic types of $\tau$, if $\lambda$ is a regular cardinal greater than $2^{\|\tau\|+\aleph_0}$ and $(\mathfrak{A}_i)_{i<\lambda}$ is a chain of quasi-homogeneous $\tau$-structures then there is a cardinal $\kappa < \lambda$ and a quantifier-free $\tau$-formula $\vartheta$ that is equivalent to $\varphi$ in structures $\mathfrak{A}_i$ with $i \geq \kappa$.

The number of formulas of the form $Q(\varpi_i \vartheta_i)_{i<\delta}$, with $Q \in \mathbb{Q}$ and all $\vartheta_i$ quantifier-free, is at most $2^{\|\tau\|+\aleph_0} \cdot \|\mathbb{Q}\| \cdot \kappa$. The rest follows in the same way as in the proof of Theorem 5.5.

Lemma 5.17. If $\mathfrak{A}$ and $\mathfrak{B}$ are bi-embeddable quasi-homogeneous $\tau$-structures then $\mathfrak{A} \equiv_{\text{emb}} \mathfrak{B}$.

Proof. We can build a chain of arbitrary length in which structures $\mathfrak{A}$ and $\mathfrak{B}$ alternate. By Theorem 5.16 the truth value of any sentence $\varphi \in L_{\infty\omega}(\mathbb{Q}_{\text{emb}})$ is eventually preserved in this chain, so $\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi$.

Example 5.18. Let $\eta = (0, 1)$, that is $\eta$ is the open real line interval between 0 and 1, and $\xi = \eta \setminus \{\frac{1}{2}\}$. Then $\eta$ and $\xi$ are both quasi-homogeneous and bi-embeddable, so $\eta \equiv_{\text{emb}} \xi$. Thus, the completeness of an ordering is not definable in $L_{\infty\omega}(\mathbb{Q}_{\text{emb}})$.

Example 5.19. Suppose $\aleph_\alpha$ is a regular cardinal. Let $\eta = \omega_{\omega_\alpha}$, the set of all functions $\omega_\alpha \to \omega_\alpha$, and $\xi = \omega \times \eta$, both ordered lexicographically. The orderings $\eta$ and $\xi$ are quasi-homogeneous and bi-embeddable, hence $\eta \equiv_{\text{emb}} \xi$. Therefore, for any ordinal $\beta$, the property of having cofinality $\aleph_\beta$ is not definable in $L_{\infty\omega}(\mathbb{Q}_{\text{emb}})$.

Definition 5.20. Let $\tau$ be a signature and $K$ a class of $\tau$-structures. We say that $K$ is neat if the following holds:

1. Every structure in $K$ is quasi-homogeneous.
2. The embeddability relation defines a partial well-ordering on $K$.
3. Every antichain in $K$ is a set.
4. $K$ is closed under unions of chains.

Remark 5.21. Vopěnka’s principle (VP) is a statement independent of ZFC saying that every proper class of structures has two structures such that one is embeddable into another. If we assume VP then clearly the condition 3. in the definition of neat class always holds.
Definition 5.22. We write \( A \leq_L B \) if

\[(A, \bar{a}) \models \varphi \iff (B, \bar{a}) \models \varphi\]

for all tuples \( \bar{a} \) in \( A \) and \( \tau \)-formulas \( \varphi \) in \( L \).

Theorem 5.23. Let \( \tau \) be a signature and \( K \) a neat class of \( \tau \)-structures. For each cardinal \( \kappa \) there is a cardinal \( \lambda \) such that \( \forall A, B \in K \) and \( |A|, |B| \geq \lambda \) imply \( A \equiv_{L_{\omega\omega}(Q)} B \) for any set \( Q \) of embedding-closed quantifiers of width \( < \kappa \).

Proof. Define sets \( P_\alpha \subseteq K \), where \( \alpha \) is an ordinal, as follows.

\[
P_0 = \emptyset, \quad P_{\alpha+1} = \{ A \in K \setminus P_\alpha : B < A \Rightarrow B \in P_\alpha \},
\]

for all ordinals \( \alpha \) and limit ordinals \( \gamma \). Then each \( P_\alpha \) contains chains of length \( \alpha \). By Theorem 5.16 for each sentence \( \varphi \in L_{\kappa\omega}(Q)[\tau] \) there is a cardinal \( \mu_\varphi \) such that

\[
A \models \varphi \iff B \models \varphi
\]

if \( A, B \notin P_\mu \) and \( A \leq B \). Let

\[
\mu = \sup \{ \mu_\varphi : \varphi \in L_{\kappa\omega}(Q)[\tau] \}.
\]

Then

\[
\lambda = \sup \{ |A|^+ : A \in P_\mu \}
\]

is such that \( A \leq B \) and \( |A|, |B| \geq \lambda \) imply \( A \leq_{L_{\omega\omega}(Q)} B \).

Example 5.24. It can be shown that the class of all \( \tau \)-structures for a unary signature \( \tau \) is neat, so the Härtig and Rescher quantifiers are not definable in \( L_{\infty\omega}(Q_{\text{emb}}) \).

6 Embeddability game

In this section we will introduce a game characterizing relation \( \equiv_{\text{emb}} \). The embeddability game is played on two structures \( A \) and \( B \) of the same signature by two players, Spoiler and Duplicator. A position in the game is a tuple \( (A, \bar{a}, B, \bar{b}) \), where \( \bar{a} \) and \( \bar{b} \) are tuples of elements of \( A \) and \( B \), respectively. The game proceeds in rounds and starts from the position \( (A, \emptyset, B, \emptyset) \). Suppose that \( n \) rounds of the game have been played and the position is

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$(\mathfrak{A}, \tau, \mathfrak{B}, \overline{b})$. First Duplicator chooses embeddings $f: A \rightarrow B$ and $g: B \rightarrow A$ such that $f\overline{a} = \overline{b}$ and $g\overline{b} = \overline{a}$. If there are no such embeddings then Spoiler wins the game. Otherwise Spoiler selects a natural number $k$ and a tuple $\overline{r} \in A^n$ or $\overline{d} \in B^n$. This completes the round, and the game continues from the position $(\mathfrak{A}, \overline{ar}, \mathfrak{B}, \overline{bfr})$ or $(\mathfrak{A}, \overline{agd}, \mathfrak{B}, \overline{bd})$ depending on whether Spoiler chose $\overline{r} \in A^n$ or $\overline{d} \in B^n$. Duplicator wins the game if and only if the game goes on infinitely.

We write $\mathfrak{A} \overset{\text{emb}}{\simeq} \mathfrak{B}$ if Duplicator wins the embeddability game on $\mathfrak{A}$ and $\mathfrak{B}$, and $\mathfrak{A} \overset{k}{\simeq}_{\text{emb}} \mathfrak{B}$ if Duplicator does not lose in the first $k$ rounds. We write $\mathfrak{A} \equiv_{\text{emb}} \mathfrak{B}$ if $\mathfrak{A}$ and $\mathfrak{B}$ agree on all sentences of $\mathcal{L}_{\infty\omega}(\mathbb{Q}_{\text{emb}})$. Notation $\mathfrak{A} \equiv_{\text{emb}}^k \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ agree on all the sentences of $\mathcal{L}_{\infty\omega}(\mathbb{Q}_{\text{emb}})$ whose quantifier rank is $\leq k$.

Remark 6.1. Let $\tau$ be a signature, $\mathfrak{A}$ and $\mathfrak{B}$ $\tau$-structures and $\overline{a} \in A^n$ and $\overline{b} \in B^n$. A position $(\mathfrak{A}, \overline{a}, \mathfrak{B}, \overline{b})$ is equivalent to the position $(\mathfrak{A}', \emptyset, \mathfrak{B}, \emptyset)$, where $\mathfrak{A}'$ and $\mathfrak{B}'$ are structures of signature $\tau$ expanded with new constant symbols $c_1, \ldots, c_n$ with interpretations $c_i' = a_i$ and $c_i'' = b_i$ for all $i$. Thus for brevity we will use signature expansions instead of writing positions explicitly.

Theorem 6.2. Let $\tau$ be signature and $\mathfrak{A}$ and $\mathfrak{B}$ $\tau$-structures. For all natural numbers $k \geq 1$, $\mathfrak{A} \overset{k}{\simeq}_{\text{emb}} \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_{\text{emb}}^k \mathfrak{B}$.

Proof. We use induction on $k$. Supppose first that $\mathfrak{A} \overset{1}{\simeq}_{\text{emb}}^1 \mathfrak{B}$. Then $\mathfrak{A} \leq \mathfrak{B}$ and $\mathfrak{B} \leq \mathfrak{A}$, so $\mathfrak{A} \equiv_{\text{emb}}^1 \mathfrak{B}$ by Lemma 5.2. Assume next that $\mathfrak{A} \equiv_{\text{emb}}^1 \mathfrak{B}$. Let $\mathfrak{A}'$ be $\mathfrak{A}$ with functions and constants replaced by corresponding relations. Let $Q$ be the smallest embedding-closed quantifier containing $\mathfrak{A}'$. For each symbol of $\tau$ define $\varphi_R := R$ for all relation symbols $R \in \tau$, $\varphi_f := f(\overline{r}) = y$ for all function symbols $f \in \tau$ and $\varphi_c := x = c$ for all constant symbols $c \in \tau$. Then $\mathfrak{A} \models Q(\tau_{\mathfrak{A}'})_{\delta \in \kappa}$, and since $\mathfrak{A} \equiv_{\text{emb}}^1 \mathfrak{B}$, we have $\mathfrak{B} \models Q(\tau_{\mathfrak{B}'})_{\delta \in \kappa}$, so $\mathfrak{A} \leq \mathfrak{B}$. In the same way we prove that $\mathfrak{B} \leq \mathfrak{A}$, so $\mathfrak{A} \overset{1}{\simeq}_{\text{emb}} \mathfrak{B}$. The base step of induction is thus proved. Assume now that the claim holds for $k$.

Suppose first that $\mathfrak{A} \overset{k+1}{\simeq}_{\text{emb}} \mathfrak{B}$, and let $\mathfrak{A} \models Q(\tau_{\delta} \psi_{\delta})_{\delta \in \kappa}$, where $\text{qr}(\psi_{\delta}) \leq k$ for all $\delta \in \kappa$ and $Q$ is an embedding-closed quantifier. Let $f$ be an embedding $A \rightarrow B$ that Duplicator can choose in the first round. Then by the induction hypothesis,

$$\mathfrak{A} \models \psi_{\delta}(\overline{a}) \iff \mathfrak{B} \models \psi_{\delta}(f\overline{a})$$

for all tuples $\overline{a}$ of elements of $A$ and $\delta \in \kappa$. Thus $f$ is an embedding

$$(A, (\psi_{\delta})_{\delta \in \kappa}) \rightarrow (B, (\psi_{\delta})_{\delta \in \kappa}),$$

so $\mathfrak{B} \models Q(\tau_{\delta} \psi_{\delta})_{\delta \in \kappa}$. The other direction is proved in the same way. Since every $\mathcal{L}(Q_{\text{emb}})$-sentence of quantifier rank $k + 1$ is a Boolean combination
(with possibly infinite conjunctions or disjunctions) of sentences of the form \( Q(\pi_b \psi_b) \) with \( qr(\psi_b) \leq k \), it follows that \( \mathfrak{A} \equiv_{k+1} \mathfrak{B} \).

For the other direction, assume that \( \mathfrak{A} \not\equiv_{k+1} \mathfrak{B} \). We denote by \( \mathcal{F}_A \) the set of all embeddings \( A \to B \), and by \( \mathcal{F}_B \) the set of all embeddings \( B \to A \). Then for each pair \( (f, g) \in \mathcal{F}_A \times \mathcal{F}_B \), there is \( \overline{a} \in A^{<\omega} \) or \( \overline{b} \in B^{<\omega} \) such that Spoiler wins on \( (\mathfrak{A}, \overline{a}, \mathfrak{B}, f(\overline{a})) \) or on \( (\mathfrak{A}, \overline{b}, \mathfrak{B}, \overline{b}) \). Thus, by the induction hypothesis, for each pair of embeddings \( (f, g) \) there is a formula \( \psi_f \) or a formula \( \psi_g \), both of quantifier rank \( \leq k \), such that

\[
(*): \mathfrak{A} \models \psi_f(\overline{a}) \iff \mathfrak{B} \models \psi_f(f(\overline{a})) \quad \text{or} \quad \mathfrak{A} \models \psi_g(\overline{b}) \iff \mathfrak{B} \models \psi_g(g(\overline{b}))
\]

for some \( \overline{a} \in A^{<\omega} \) or \( \overline{b} \in B^{<\omega} \). Let \( Q_A \) be the smallest embedding-closed quantifier containing the structure \( (A, (\psi^A_h)_{h \in \mathcal{F}_A}) \), and \( Q_B \) be the smallest embedding-closed quantifier containing the structure \( (B, (\psi^B_h)_{h \in \mathcal{F}_B}) \). Then \( \mathfrak{A} \models Q_A(\overline{a}) \psi_h \) \( h \in \mathcal{F}_A \) and \( \mathfrak{B} \models Q_B(\overline{b}) \psi_h \) \( h \in \mathcal{F}_B \). Assume to the contrary that \( \mathfrak{B} \models Q_A(\overline{a}) \psi_h \) \( h \in \mathcal{F}_A \) and \( \mathfrak{A} \models Q_B(\overline{b}) \psi_h \) \( h \in \mathcal{F}_B \). Then there are embeddings

\[
f: (A, (\psi^A_h)_{h \in \mathcal{F}_A}) \to (B, (\psi^B_h)_{h \in \mathcal{F}_B}) \quad \text{and} \quad g: (B, (\psi^B_h)_{h \in \mathcal{F}_B}) \to (A, (\psi^A_h)_{h \in \mathcal{F}_A}),
\]

so there are embeddings \( f \) and \( b \) such that

\[
\mathfrak{A} \models \psi_f(\overline{a}) \iff \mathfrak{B} \models \psi_f(f(\overline{a})) \quad \text{and} \quad \mathfrak{A} \models \psi_g(\overline{b}) \iff \mathfrak{B} \models \psi_g(g(\overline{b}))
\]

for all \( \overline{a} \) and \( \overline{b} \), which contradicts \( (*) \). Thus, \( \mathfrak{B} \not\models Q_A(\overline{a}) \psi_h \) \( h \in \mathcal{F}_A \) or \( \mathfrak{A} \not\models Q_B(\overline{b}) \psi_h \) \( h \in \mathcal{F}_B \), which completes the induction step. \( \square \)

**Example 6.3.** Let \( E_0 \) be an equivalence relation with countably infinite number of \( E_0 \)-classes, and suppose that each \( E_0 \)-class has cardinality \( \aleph_1 \). Let \( E_1 \) satisfy the same conditions with exception of having one \( E_1 \)-class of cardinality \( \aleph_0 \). Then \( E_0 \) and \( E_1 \) are bi-embeddable, so \( E_0 \equiv_{1}^{\text{emb}} E_1 \).

Let \( f: E_0 \to E_1 \) and \( g: E_1 \to E_0 \) be embeddings. Let \( [a]_{E_1} \) be the \( E_1 \)-class of cardinality \( \aleph_0 \), and suppose Spoiler chooses the embedding \( g \) and the element \( a \). It is easy to see that there is no embedding of \( E_0 \) into \( E_1 \) that maps \( g(a) \) to \( a \), since the restriction of such an embedding to an \( E_0 \)-class must be included in some \( E_1 \)-class, and \( [g(a)]_{E_0} = \aleph_1 \) and \( [a]_{E_1} = \aleph_0 \). Thus, Duplicator loses the second round, so \( E_0 \not\equiv_{2}^{\text{emb}} E_1 \).
**Example 6.4.** For each $2 \leq n < \omega$, let $\tau_n = \{ E, \leq, \eta_1, \ldots, \eta_n \}$ be a relational signature with symbols $E$ and $\leq$ binary and all $\eta_i$ unary, and $\varphi_n \in L_{\omega, \omega}[\tau_n]$ the following sentence:

$$\varphi_2 := \forall x_1 \in \eta_1 \exists x_2 \in \eta_2 E(x_1, x_2),$$

$$\varphi_n := \begin{cases} \exists x_1 \in \eta_1 \forall x_2 \in \eta_2 \cdots \exists x_n \in \eta_n \psi_n(x_1, \ldots, x_n) & \text{if } n \text{ is odd,} \\ \forall x_1 \in \eta_1 \exists x_2 \in \eta_2 \cdots \exists x_n \in \eta_n \psi_n(x_1, \ldots, x_n) & \text{if } n \text{ is even,} \end{cases}$$

where

$$\psi_n := E(x_1, x_2) \land E(x_2, x_3) \land \cdots \land E(x_{n-2}, x_{n-1}) \rightarrow E(x_{n-1}, x_n).$$

We will define $\tau_n$-structures $\mathfrak{A}_n$ and $\mathfrak{B}_n$ so that $\mathfrak{A}_n \models \varphi_n$, $\mathfrak{B}_n \not\models \varphi_n$ and $\mathfrak{A}_n \equiv_{nmb} \mathfrak{B}_n$. Every structure $\mathfrak{A} \in \{ \mathfrak{A}_n, \mathfrak{B}_n \}$ consists of $n$ disjoint orderings $\eta_1^{\mathfrak{A}}, \ldots, \eta_n^{\mathfrak{A}}$. For each $1 \leq i < n$, $\eta_i^{\mathfrak{A}}$ is isomorphic to the ordering $((\omega^n)^{i-1}, \leq)$, where $\omega^n$ is the set of all functions $\omega \rightarrow \omega$, $(\omega^n)^i$ is the cartesian product of $i$ copies of $\omega^2$ and $\leq$ is lexicographic ordering. The ordering $\eta_i^{\mathfrak{A}}$ is isomorphic to $((\omega^n)^{i-1}, \leq)$. We will denote by $x_i^{\mathfrak{A}}$ the element of $\eta_i^{\mathfrak{A}}$ that corresponds to $x \in (\omega^n)^i$ ($x \in (\omega^n)^{i-1}$ if $i = n$). We will sometimes write $x_m$ instead of $x_m^{\mathfrak{A}}$ if $\mathfrak{A}$ is clear from the context.

The relation $E$ connects elements of $\eta_i$ with elements of $\eta_{i+1}$. The symbol’s $E$ interpretation is what differentiates the structures $\mathfrak{A}_n$ and $\mathfrak{B}_n$. We define it recursively as follows. For the base case $n = 2$, let $E^{\mathfrak{A}_2}$ connect elements $x_1^{\mathfrak{A}_2}$ and $y_1^{\mathfrak{A}_2}$ if, and only if, $x = y$. We define $E^{\mathfrak{B}_2}$ in the same way except we remove the pair $(1^{\mathfrak{A}_2}_1, 1^{\mathfrak{B}_2}_1)$ from it. Then clearly $\mathfrak{A}_2 \models \varphi_2$ and $\mathfrak{B}_2 \not\models \varphi_2$.

Now suppose $n > 2$ and we defined $\mathfrak{A}_m$ and $\mathfrak{B}_m$ for all $m < n$. In both $\mathfrak{A}_n$ and $\mathfrak{B}_n$, let $E$ connect every $x_m$ with all the elements of the form $(x, y)_m$. Assume that $n$ is odd. Choose some element in $\eta_1^{\mathfrak{A}_n}$, say $1^{\mathfrak{A}_n}_1$, and for all $i > 1$, let $E^{\mathfrak{A}_n}$ connect elements $(1^{\mathfrak{A}_n}, x_1, \ldots, x_r)_{n-1}$ and $(1^{\mathfrak{B}_n}, y_1, \ldots, y_s)_{n+1}$ if, and only if, $E^{\mathfrak{A}_{n-1}}$ connects elements $(x_1, \ldots, x_r)_{n-1}^{\mathfrak{A}_{n-1}}$ and $(y_1, \ldots, y_s)_{n+1}^{\mathfrak{B}_{n+1}}$. For all other elements of $A_n$, let $E^{\mathfrak{A}_n}$ connect $(u, x_1, \ldots, x_r)_{n-1}^{\mathfrak{A}_n}$ and $(v, y_1, \ldots, y_s)_{n+1}^{\mathfrak{B}_n}$, $u \neq 1^{\mathfrak{A}_n}$ if, and only if, $u = v$ and $E^{\mathfrak{A}_{n-1}}$ connects elements $(x_1, \ldots, x_r)_{n-1}^{\mathfrak{A}_{n-1}}$ and $(y_1, \ldots, y_s)_{n+1}^{\mathfrak{B}_{n+1}}$. In the same way, let $E^{\mathfrak{B}_n}$ connect $(u, x_1, \ldots, x_r)_{n}^{\mathfrak{A}_n}$ and $(v, y_1, \ldots, y_s)_{n+1}^{\mathfrak{B}_n}$, $u \neq 1^{\mathfrak{B}_n}$, if, and only if, $u = v$ and $E^{\mathfrak{B}_{n-1}}$ connects elements $(x_1, \ldots, x_r)_{n-1}^{\mathfrak{B}_{n-1}}$ and $(y_1, \ldots, y_s)_{n+1}^{\mathfrak{B}_{n+1}}$.

Thus, informally, if $n$ is odd then in the structure $\mathfrak{A}_n$, the element $1^{\mathfrak{A}_n}_1$ is connected by $E$ to a copy of the structure $\mathfrak{A}_{n-1}$, and the rest of elements of $\eta_1^{\mathfrak{A}_n}$ are each connected to a copy of the structure $\mathfrak{B}_{n-1}$. Similarly, every element of $\eta_1^{\mathfrak{B}_n}$ is connected to a copy of the structure $\mathfrak{B}_{n-1}$. 

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If \( n > 2 \) is even, then we can define structures \( \mathfrak{A}_n \) and \( \mathfrak{B}_n \) in a similar way so that every element in \( \eta^{\mathfrak{A}_n}_1 \) is connected by \( E \) to a copy of the structure \( \mathfrak{A}_{n-1} \), and every element in \( \mathfrak{B}_n \) is connected to a copy of the structure \( \mathfrak{B}_{n-1} \) and the rest of elements of \( \eta^{\mathfrak{B}_n}_1 \) are each connected to a copy of the structure \( \mathfrak{A}_{n-1} \). The definition of structures \( \mathfrak{A}_n \) and \( \mathfrak{B}_n \) is now complete.

It can be easily verified that \( \mathfrak{A}_n \equiv \mathfrak{A}_n^{\mathfrak{B}_n} \) and if \( n \) is odd then there is an embedding \( g_n : B_n \to A_n \) such that \( \mathfrak{A}_n, g \mathfrak{B} \equiv \mathfrak{A}_n^{\mathfrak{B}_n} \mathfrak{A}_n \), and if \( n \) is even then there is an embedding \( f_n : A_n \to B_n \) such that \( \mathfrak{A}, \mathfrak{B} \equiv \mathfrak{A}_n^{\mathfrak{B}_n} \mathfrak{B}, f \mathfrak{A} \). The claim implies that in order to progress in the game Spoiler must choose the embedding \( f : A_n \to B_n \) if \( n \) is odd and \( g : B_n \to A_n \) if \( n \) is even.

Consider first the base case \( n = 2 \). Let \( g_2 : B_2 \to A_2 \) be a function

\[
\begin{align*}
    f_2(x_{\eta_1}) &= (11x)_{\eta_1} \text{ if } 1\mathfrak{U} < x, \\
    f_2(x_{\eta_1}) &= (0x)_{\eta_1} \text{ if } x < 1\mathfrak{U}, \\
    f_2(1\mathfrak{U}_{\eta_1}) &= 1\mathfrak{T}_{\eta_1}, \\
    f_2(10\mathfrak{U}_{\eta_2}) &= 10\mathfrak{U}_{\eta_2},
\end{align*}
\]

and \( f_2 : A_2 \to B_2 \) a function that maps all \( x_{\eta_1} \in A_2 \) to \( 0x_{\eta_1} \). Then \( f_2 \) and \( g_2 \) are embeddings, so \( \mathfrak{A}_2 \equiv_{\mathfrak{B}_2} \mathfrak{B}_2 \). To prove the second part of the claim, suppose that Spoiler selected the embedding \( f \) and elements \( a_{\eta_1}^1, \ldots, a_{\eta_k}^k \in A_2, a_1 \leq \cdots \leq a_k \). Put \( a_0 = \mathfrak{U} \) and let

\[
\begin{align*}
    X_i &= \{ x \in A_2 : a_i \leq x < a_{i+1} \text{ (} a_i \leq x \text{ if } i = k \} \\
    Y_i &= \{ x \in B_2 : f_2(a_i) \leq x \leq f_2(a_{i+1}) \text{ (} f_2(a_i) \leq x \text{ if } i = k \}
\end{align*}
\]

for all \( i \leq k \). Then \( \mathfrak{A}_2[X_i \cong \mathfrak{B}_2]Y_i \) for all \( i < k \), \( \mathfrak{A}_2[X_k \cong \mathfrak{A}_2] \) and \( \mathfrak{B}_2[Y_k \cong \mathfrak{B}_2] \) from which the claim follows.

Now let \( n > 2 \) and assume that the claim holds for all \( m < n \). Assume first that \( n \) is odd. Let \( f_n : A_n \to B_n \) be a function that maps each \( x_{\eta_i}^m \) to \( x_{\eta_i}^m \), and for all \( i > 1 \), each \( (x_1, \ldots, x_r)_{\eta_i}^m \) to \( (x_1, y_2, \ldots, y_r)_{\eta_i}^m \), where \( (y_2, \ldots, y_r) \) is such that \( f_{n-1}(x_2, \ldots, x_r)_{\eta_i}^{n-1} = (y_2, \ldots, y_r)_{\eta_i}^{n-1} \). Then \( f_n \) is an embedding. We define an embedding \( g_n : B_n \to A_n \) by setting \( g_n(x_{\eta_i}^m) = 0x_{\eta_i}^m \) for all \( x \in \omega^\omega \) and \( g_n(x_1, \ldots, x_r)_{\eta_i}^m = (0x_1, y_2, \ldots, y_r)_{\eta_i}^m \), where \( (y_2, \ldots, y_r) \) is such that \( g_{n-1}(x_2, \ldots, x_r)_{\eta_i}^{n-1} = (y_2, \ldots, y_r)_{\eta_i}^{n-1} \).

Suppose that Duplicator selects embeddings \( f_n \) and \( g_n \) defined above in the game on \( (\mathfrak{A}_n, \mathfrak{B}_n) \). If Spoiler chooses embedding \( g_n \) and elements \( b_1, \ldots, b_s \in B_n \) then let

\[
C = \{ x \in \omega^\omega : x_{\eta_i} = b_i \text{ for some } i, \text{ or } (x, y_1, \ldots, y_l)_{\eta_j} = b_i \text{ for some } y_1, \ldots, y_l, i \text{ and } j > 1 \}.
\]
Write $C = \{c_1, \ldots, c_s\}$ and assume that $c_1 \leq \cdots \leq c_s$. Put $c_0 = 0$. For all $i \leq s$, set

$$Y_i = \{ x \in B_n : x = (y_1, \ldots, y_r)_n \text{ and } c_i \leq y_i < c_{i+1} \text{ (} c_i \leq y_i \text{ if } i = s) \},$$

$$X_i = \{ x \in A_n : x = (y_1, \ldots, y_r)_n \text{ and } g_n(c_i) \leq y_i < g_n(c_{i+1}) \}.$$

Then $A_n|X_i \equiv B_n|Y_i$ for all $i < s$, $A_n|X_s \equiv A_n$ and $B_n|Y_s \equiv B_n$, so the situation stays the same as in the beginning of the round. Thus, $A_n, g_n \equiv_{\text{emb}} B_n, \overline{f}$.

Suppose now that Spoiler selects embedding $f_n$ and elements $a_1, \ldots, a_r \in A_n$. Define sets $C \subseteq \omega^\omega$, $X_i \subseteq A_n$ and $Y_i \subseteq B_n$ in the same way as above. Let $p$ be such that $c_p \leq 1 \overline{f} \leq c_{p+1}$. If $c_p \not= 1 \overline{f}$ then $A_n|X_i \equiv B_n|Y_i$ for all $i \not= p$, $A_n|X_p \equiv A_n$ and $B_n|Y_p \equiv B_n$. If $c_p = 1 \overline{f}$ then the next round of the game is reduced to the first round in the game on $(A_n|X_p, B_n)$ which in turn is reduced to the game on $(A_{n-1}, B_{n-1})$. Thus, by the induction hypothesis, $A_n \equiv_{\text{emb}} B_n$. We show that the claim holds for $n$ in case it is even in a similar way.

Example 6.3 implies the next theorem:

**Theorem 6.5.** For each $n < \omega$ there is a first-order sentence $\varphi$ of quantifier rank $n$ that is not expressible by any $L_{\omega,\omega}(Q_{\text{emb}})$-sentence of quantifier rank $< n$.

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