Type III Spacetime with Closed Timelike Curves

Faizuddin Ahmed
Hindustani Kendriya Vidyalaya, Guwahati-05, Assam, India
Current Affiliation : National Academy Gauripur, Assam, India, 783331

Abstract
We present a cyclic symmetric space-time, admitting closed time-like curves (CTCs) which appear after a certain instant of time, i.e., a time-machine space-time. These closed time-like curves evolve from an initial spacelike hypersurface on the plane $z = \text{constant}$ in a causally well-behaved manner. The space-time discussed here is free-from curvature singularities and a 4D generalization of the Misner space in curved space-time. The matter field is of pure radiation with a negative Cosmological constant.

1 Introduction
One of the most intriguing aspects of Einstein’s theory of gravitation is that solutions of Field Equations admit closed time-like curves (CTC). Presence of CTC in a space-time lead to time-travel which violate the causality condition. The first one being the Gödel’s space-time \cite{1} which admit closed time-like curves (CTC) everywhere and an eternal time-machine space-time. There are a considerable number of space-times in literature that admitting closed time-like curves have been constructed. A small sample would be \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21}. One way of classifying such causality violating space-times would be to categorize the metrics as either eternal time-machine in which CTC always exist (in this class would be \cite{1, 2}), or as time-machine space-times in which CTC appear after a

\footnote{faizuddinahmed15@gmail.com ; faiz4U.enter@rediffmil.com}
certain instant of time. In the latter category would be the one discussed in [18, 19, 20]. Many of the models, however, suffer from one or more severe drawbacks. For instance, in some of these solutions, for example [13, 14, 20] the weak energy condition (WEC) is violated indicating unrealistic matter-energy content and some other solutions have singularities.

Among the time-machine space-times, we mention two: the first being Ori’s compact core [17] which is represented by a vacuum metric locally isometric to pp waves and second, which is more relevant to the present work, the Misner space [22] in 2D. This is essentially a two dimensional metric (hence flat) with peculiar identifications. The Misner space is interesting in the context of CTC as it is a prime example of a space-time where CTC evolve from causally well-behaved initial conditions.

The metric for the Misner space [22]

$$ds^2_{Misn} = -2 dt dx - t dx^2.$$  \hspace{1cm} (1)

where $-\infty < t < \infty$ but the co-ordinate $x$ periodic. The metric (1) is regular everywhere as $\text{det} g = -1$ including at $t = 0$. The curves $t = t_0$, where $t_0$ is a constant, are closed since $x$ is periodic. The curves $t < 0$ are spacelike, but $t > 0$ are time-like and the null curves $t = t_0 = 0$ form the chronology horizon. The second type of curves, namely, $t = t_0 > 0$ are closed time-like curves (CTC). This metric has been the subject of intense study and quite recently, D. Levanony et al [23] have studied the motion of extended bodies in the 2D Misner space and its flat 4D generalizations. A non-flat 4D space-time, satisfying all the energy conditions, but with causality violating properties of the Misner space, primarily that CTC evolve smoothly from a initially causally well-behaved stage, would be physically more acceptable as a time-machine space-time.

In this paper, we shall attempt to show that causality violating curves appear in non-vacuum space-time with comparatively simple structure. In section 2, we analyze the space-time, in section 3, the matter distribution and
the energy condition, in section 4, the space-time is classified and discussed its kinematical properties and concluding one in section 5.

2 Analysis of the space-time

Consider the following metric

\[ ds^2 = 4 r^2 dr^2 + e^{2 \alpha r^2} (dz^2 - t d\phi^2 - 2 dt d\phi) + 4 \beta z r e^{-\alpha r^2} dr d\phi, \]  

(2)

where \( \phi \) coordinate is assumed periodic \( 0 \leq \phi \leq \phi_0 \), with \( \alpha \) is an integer and \( \beta > 0 \) is real number. We have used co-ordinates \( x^1 = r, \ x^2 = \phi, \ x^3 = z \) and \( x^4 = t \). The ranges of the other co-ordinates are \( t, z \in (-\infty, \infty) \) and \( 0 \leq r < \infty \). The metric has signature (+, +, +, -) and the determinant of the corresponding metric tensor \( g_{\mu\nu} \), \( \det g = -4 r^2 e^{6 \alpha r^2} \). The non-zero components of the Einstein tensor are

\[ G_{\mu}^{\mu} = 3 \alpha^2 \]

\[ G_{\phi}^{\phi} = -\frac{1}{2} e^{-6 \alpha r^2} \beta^2 \]  

(3)

Consider an azimuthal curves \( \gamma \) defined by \( r = r_0, \ z = z_0 \) and \( t = t_0 \), where \( r_0, z_0, t_0 \) are constants, then we have from the metric (2)

\[ ds^2 = -t e^{2 \alpha r^2} d\phi^2 \]  

(4)

These curves are null for \( t = 0 \), spacelike throughout for \( t = t_0 < 0 \), but become time-like for \( t = t_0 > 0 \), which indicates the presence of closed timelike curves (CTC). Hence CTC form at a definite instant of time satisfy \( t = t_0 > 0 \).

It is crucial to have analysis that the above CTC evolve from an spacelike \( t = \text{constant} \) hypersurface (and thus \( t \) is a time coordinate)[17]. This can be ascertained by calculating the norm of the vector \( \nabla_\mu t \) (or by determining the sign of the component \( g^{tt} \) in the inverse metric tensor \( g^{\mu\nu} \))[17]. We find from (2) that

\[ g^{tt} = t e^{-2 \alpha r^2} + \beta^2 z^2 e^{-6 \alpha r^2} \]  

(5)
A hypersurface $t = \text{constant}$ is spacelike provided $g^{tt} < 0$ for $t = t_0 < 0$, but become time-like provided $g^{tt} > 0$ for $t = t_0 > 0$. Here we choose the $z -$ planes defined by $z = z_0$, ($z_0$, a constant equal to zero) such that the above condition is satisfied. Thus the spacelike $t = \text{constant} < 0$ hypersurface can be chosen as initial conditions over which the initial may be specified.

There is a Cauchy horizon for $t = t_0 = 0$ called Chronology horizon which separates the causal and non-causal of the space-time. Hence the space-time evolves from a partial Cauchy hypersurface (initial spacelike hypersurface) in a causally well-behaved manner, upto a moment, i.e., a null hypersurface $t = 0$ and CTC form at a definite instant of time on $z = \text{constant}$ plane.

Consider the Killing vector $\eta = \partial_\phi$ for metric (2) which has the normal form

$$\eta^\mu = (0, 1, 0, 0) \quad (6)$$

Its co-vector is

$$\eta_\mu = \left(2 \beta z r e^{-\alpha r^2}, -t e^{2\alpha r^2}, 0, -e^{2\alpha r^2}\right) \quad (7)$$

(6) satisfies the Killing equation $\eta_{\mu;\nu} + \eta_{\nu;\mu} = 0$. For cyclicly symmetric metric, the norm $\eta_\mu \eta^\mu$ of the Killing vector is spacelike, closed orbits [24, 25, 26, 27, 28]. We note that

$$\eta_\mu \eta^\mu = -t e^{2\alpha r^2} \quad (8)$$

which is spacelike for $t < 0$, closed orbits ($\phi$ co-ordinate being periodic).

An important note is that the Riemann tensor $R_{\mu\nu\rho\sigma}$ can be expressed in terms of metric tensor $g_{\mu\nu}$ as

$$R_{\mu\nu\rho\sigma} = k \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right) \quad (9)$$

where $k = -\alpha^2$ for the space-time (2).

Another important note is that if we take $\beta = 0$, then the space-time represented by (2) is maximally symmetric vacuum space-time and locally isometric anti-de Sitter space in four-dimension. One can easily show by
a number of transformations the standard form of locally isometric $AdS_4$
metric \[ ds^2 = \frac{3}{(-\Lambda)x^2} (-dt^2 + dx^2 + d\phi^2 + dz^2) \] (10)
where one of the co-ordinate $\phi$ being periodic.

3 Matter distribution of the space-time and the energy condition

The Einstein’s Field Equations taking into account the cosmological constant
\[ G^{\mu\nu} + \Lambda g^{\mu\nu} = T^{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4 \] (11)
Consider the energy-momentum tensor that of pure radiation field \[ T^{\mu\nu} = \rho n^\mu n^\nu \] (12)
where $n^\mu$ is the null vector defined by
\[ n^\mu = (0, 0, 0, 1) \] (13)
The non-zero component of the energy-momentum tensor
\[ T^t_\phi = -\rho e^{2\alpha r^2} \] (14)
Equating Field Equations (11) using (3) and (14) we get
\[ \Lambda = -3\alpha^2 \] (15)
\[ \rho = \frac{1}{2} \beta^2 e^{-8\alpha r^2}, \quad 0 \leq r < \infty \] (16)
The energy-density of pure radiation or null dust decreases exponentially with $r$ and vanish at $r \to \pm \infty$. The matter field pure radiation satisfy the energy condition and the energy density $\rho$ is always positive.
4 Classification and kinematical properties of the space-time

For classification of the spacetime (2), we can construct the following set of null tetrads \((k, l, m, \bar{m})\) as

\[ k_\mu = (0, 1, 0, 0) \]  
\[ l_\mu = \left(-2 \beta z r e^{-\alpha r^2}, \frac{t}{2} e^{2\alpha r^2}, 0, e^{2\alpha r^2}\right) \]  
\[ m_\mu = \frac{1}{\sqrt{2}} (2 r, 0, i e^{\alpha r}, 0) \]  
\[ \bar{m}_\mu = \frac{1}{\sqrt{2}} (2 r, 0, -i e^{\alpha r^2}, 0) \]

where \(i = \sqrt{-1}\). The set of null tetrads above are such that the metric tensor for the line element (2) can be expressed as

\[ g_{\mu\nu} = -k_\mu l_\nu - l_\mu k_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu. \]

The vectors (17)—(20) are null vectors and are orthogonal except for \(k_\mu l_\mu = -1\) and \(m_\mu \bar{m}_\mu = 1\). Using this null tetrad above we have calculated the five Weyl scalars

\[ \Psi_3 = -\frac{i \alpha \beta e^{-2\alpha r^2}}{2 \sqrt{2}} \]
\[ \Psi_4 = -\frac{1}{4} \beta e^{-2\alpha r^2} \left(i + 2 \alpha z e^{\alpha r^2}\right) \]

are non-vanishing, while \(\Psi_0 = \Psi_1 = \Psi_3 = 0\). The spacetime represented by (2) is of type III in the Petrov classification scheme. Note that the non-zero Weyl scalars \(\Psi_3\) and \(\Psi_4\) are finite at \(r \to 0\) and vanish as \(r \to \pm\infty\) indicating asymptotic flatness of the spacetime (2). The metric (2) is free-from curvature singularities. The curvature invariant known as Kretschmann scalar is given by

\[ R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 24 \alpha^4 \]  

(23)
and the curvature scalar

\[ R = -12 \alpha^2 \]  

are constant being non-zero.

Using the null tetrad \( \text{(17)} \) we have calculated the Optical scalars \( \text{[30]} \) the expansion, the twist and the shear and they are

\[
\Theta = \frac{1}{2} k_{;\mu} = 0, \\
\omega^2 = \frac{1}{2} k_{[\mu;\nu]} k^{\mu;\nu} = 0, \\
\sigma \bar{\sigma} = \frac{1}{2} k_{(\mu;\nu)} k^{\mu;\nu} - \Theta^2 = 0
\]

and the null vector \( \text{(17)} \) satisfy the geodesics equation

\[ k_{\mu;\nu} k^{\nu} = 0. \]  

Thus the spacetime represented by \( \text{(2)} \) is non-diverging, have shear-free null geodesics congruence. One can easily show that for constant \( r \) and \( z \), the metric \( \text{(2)} \) reduces to conformal Misner space in 2D

\[ ds^2_{\text{confo}} = \Omega ds^2_{\text{Misn}} \]

where \( \Omega = e^{-2\alpha r^2} \) is a constant.

## 5 Conclusion

Our primary motivation in this paper is to write down a metric for a spacetime that incorporates the Misner space and its causality violating properties and to classify it. The solution presented here is non-vacuum, cyclicly symmetric metric \( \text{(2)} \) and serves as a model of time-machine spacetime in the sense that CTC appear at a definite instant of time on the \( z - \text{plane} \). Most of the CTC spacetimes violate one or more energy conditions or unrealistic matter source and are unphysical. The model discussed here is free-from all these problems and matter distribution is of pure radiation field with negative cosmological constant satisfying the energy condition.
Appendix

The above metric (2) can be transform to a simple form as follows. Let us perform the following transformation

\[ r \to \sqrt{r}', \]  

(28)

into the metric (2), one will get

\[ ds^2 = dr'^2 + e^{2\alpha r'} (dz^2 - t d\phi^2 - 2 dt d\phi) + 2 \beta z e^{-\alpha r'} dr' d\phi. \]  

(29)

Finally doing another transformation

\[ r' \to \frac{1}{\alpha} \ln \rho \]  

(30)

into the metric (29), we get the following line element

\[ ds^2 = \frac{d\rho^2}{\alpha^2 \rho^2} + \rho^2 dz^2 + \rho^2 (-t d\phi^2 - 2 dt d\phi) + \frac{2 \beta \rho}{\alpha^2 \rho^2} d\rho d\phi. \]  

(31)

Or if one does the following transformation

\[ r' \to \frac{1}{\alpha} \ln (\alpha \rho) \]  

(32)

into the metric (29), we get the following line element

\[ ds^2 = \frac{d\rho^2}{\alpha^2 \rho^2} + \alpha^2 \rho^2 dz^2 + \alpha^2 \rho^2 (-t d\phi^2 - 2 dt d\phi) + \frac{2 \beta \rho}{\alpha^2 \rho^2} d\rho d\phi. \]  

(33)

Or if one does the following transformation

\[ r' \to -\frac{1}{\alpha} \ln (\alpha \rho) \]  

(34)

into the metric (29), we get the following line element

\[ ds^2 = \frac{1}{\alpha^2 \rho^2} (d\rho^2 - t d\phi^2 - 2 dt d\phi + dz^2) - 2 \beta \rho d\rho d\phi. \]  

(35)

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