ADAPTIVE TESTS FOR PARAMETER CHANGES IN ERGODIC DIFFUSION PROCESSES FROM DISCRETE OBSERVATIONS

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ABSTRACT. We consider the adaptive test for the parameter change in discretely observed ergodic diffusion processes based on the cusum test. Using two test statistics based on the two quasi-log likelihood functions of the diffusion parameter and the drift parameter, we perform the change point tests for both diffusion and drift parameters of the diffusion process. It is shown that the test statistics have the limiting distribution of the sup of the norm of a Brownian bridge. Simulation results are illustrated for the 1-dimensional Ornstein-Uhlenbeck process.

1. INTRODUCTION

We consider a $d$-dimensional diffusion process $\{X_t\}_{t \geq 0}$ satisfying the stochastic differential equation:

$$
\begin{align*}
\text{d}X_t &= b(X_t, \beta)\text{d}t + a(X_t, \alpha)\text{d}W_t, \\
X_0 &= x_0,
\end{align*}
$$

(1.1)

where parameter space $\Theta = \Theta_A \times \Theta_B$, which is a compact subset of $\mathbb{R}^p \times \mathbb{R}^q$, $\theta = (\alpha, \beta) \in \Theta$ is an unknown parameter and $\{W_t\}_{t \geq 0}$ is a $d$-dimensional standard Wiener process. The diffusion coefficient $a : \mathbb{R}^d \times \Theta_A \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ and the drift coefficient $b : \mathbb{R}^d \times \Theta_B \rightarrow \mathbb{R}^d$ are known except for the parameter $\theta$, and the true parameter $\theta_0 = (\alpha_0, \beta_0)$ belongs to $\text{Int} \Theta$. We assume that the solution of (1.1) exists, and $P_0$ and $E_\theta$ denote the law of the solution and the expectation with respect to $P_0$, respectively.

We deal with the testing problem for parameter change in diffusion processes from discrete observations $\{X_{t_i}\}_{i=0}^n$, where $t_i^n = ih_n$ and $\{h_n\}$ is a positive sequence such that $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$. We further assume $nh_n^2 \rightarrow 0$. In order to analyze stochastic phenomena, we construct a statistical model with statistical inference for unknown parameters. In some cases, the values of the parameters of the model may change due to some influence. For example, if we consider a model of the stock price fluctuation, then the parameters of the statistical model may change due to factors such as interest rates, economic conditions, or political trends. It is important to see if the values of the parameters of this model have changed. That is, we want to determine if there are any changes in such paths. As we see from the paths shown in Figure[1] the parameters of this model may have changed. In particular, it would be nice to be able to determine whether there is a change in the paths as shown in Figure[2]. However, it seems difficult to determine whether there are changes in these paths. In this paper, we show that our proposed test statistics detect whether these paths have changed and the tests also tell us whether either the diffusion parameter or the drift parameter changes.

Change point problems for diffusion processes has been developed by many researchers. Kutoyants (2004), Lee et al. (2006), Negri and Nishiyama (2012) and Tsukuda (2017) treated the problem of testing for a change of the drift parameter in continuously observed ergodic diffusion processes. On the other hand, De Gregorio and Iacus (2008), Song and Lee (2009), Lee (2011) and Song (2020) studied the problem of testing for a change of the diffusion parameter for discretely observed diffusion processes, see also Iacus and Yoshida (2012) for the estimation problem for the change point of the volatility in a stochastic differential equation. Song and Lee (2009) considered the testing problem for a change of diffusion parameter $\alpha$ for the following stochastic differential equation with known drift function $b(x, \beta) = b(x)$ under the assumption $nh_n^2 \rightarrow \infty$, $nh_n^2 \rightarrow 0$ ($4 < p < q$). Furthermore, Song (2020) considered the test statistic for a change in the diffusion parameter $\alpha$ of the stochastic differential equation with $a(x, \alpha) = \alpha$ under $nh_n^2 \rightarrow 0$. Negri and

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Key words and phrases. Adaptive test, Brownian bridge, cusum test, diffusion processes, discrete observations, test for parameter change.
Figure 1. Sample paths of the 1-dimensional Ornstein-Uhlenbeck process $dX_t = -\beta(X_t - \gamma)dt + \alpha dW_t$ whose parameter changes from $(\alpha, \beta, \gamma) = (1, 1, 1)$ to $(3, 1, 1)$ (left) and changes to $(1, 1, -1)$ (right) at $t = 50$.

Figure 2. Sample paths of the 1-dimensional Ornstein-Uhlenbeck process $dX_t = -\beta(X_t - \gamma)dt + \alpha dW_t$ whose parameter from $(\alpha, \beta, \gamma) = (1, 1, 1)$ to $(1.05, 1, 1)$ (upper right), $(1, 2, 1)$ (lower left), and $(1, 1, 0.3)$ (lower right) at $t = 50$. In the upper left figure, the parameter does not change with $(\alpha, \beta, \gamma) = (1, 1, 1)$.

Nishiyama (2017) proposed the test statistics for changes of the diffusion parameter or the drift parameter in the stochastic differential equation. In this paper, we construct the two kinds of adaptive test statistics for the diffusion parameter $\alpha$ and the drift parameter $\beta$. Using the proposed test statistics, we can determine whether either the diffusion parameter or the drift parameter changes.
This paper is organized as follows. In Section 2, we state the main results. We propose the adaptive test statistics and show that the test statistics have limiting distribution of the sup of the norm of a Brownian bridge. In Section 3, examples and simulation studies are given. In Section 4, we provide the proofs.

2. Main Results

Let \( A = aa^T, \Delta X_i = X_i^n - X_{i-1}^n, (\Delta X_i)^{\otimes 2} = (\Delta X_i)(\Delta X_i)^T \), where \( T \) is the transpose. Let \( C_{k}^l(\mathbb{R}^d \times \Theta) \) be the space of all functions \( f \) satisfying the following conditions:

(i) \( f \) is continuously differentiable with respect to \( x \in \mathbb{R}^d \) up to order \( k \) for all \( \theta \in \Theta \);
(ii) \( f \) and all its \( x \)-derivatives up to order \( k \) are \( \ell \) times continuously differentiable with respect to \( \theta \in \Theta \);
(iii) \( f \) and all derivatives are of polynomial growth in \( x \in \mathbb{R}^d \) uniformly in \( \theta \in \Theta \), i.e., \( g \) is of polynomial growth in \( x \in \mathbb{R}^d \) uniformly in \( \theta \in \Theta \) if, for some \( C > 0 \), we have

\[
\sup_{\theta \in \Theta} \|g(x, \theta)\| \leq C(1 + \|x\|)^C.
\]

We assume the following conditions:

[A1] There exists a constant \( C \) such that for any \( x, y \in \mathbb{R}^d \),

\[
\sup_{\alpha \in \Theta_A} \|a(x, \alpha) - a(y, \alpha)\| + \sup_{\beta \in \Theta_B} \|b(x, \beta) - b(y, \beta)\| \leq C\|x - y\|.
\]

[A2] \( \sup_{\ell} \mathbb{E}_\theta \left[ \|X_i^k\right] < \infty \) for all \( k \geq 0 \) and \( \theta \in \Theta \).

[A3] \( \inf_{\Theta} \det (A(x, \alpha)) > 0 \).

[A4] \( a \in C_{4,2}^k(\mathbb{R}^d \times \Theta_A) \) and \( b \in C_{4,2}^k(\mathbb{R}^d \times \Theta_B) \).

[A5] The solution of (1.1) is ergodic with its invariant measure \( \mu_\theta \) such that

\[
\int_{\mathbb{R}^d} \|x\|^kd\mu_\theta(x) < \infty \quad \text{for all} \quad k \geq 0 \quad \text{and} \quad \theta \in \Theta.
\]

2.1. Test for the diffusion parameter.

We consider the following testing problem:

\[
\begin{align*}
H_0^\alpha & : \alpha_0 \text{ does not change over } 0 \leq t \leq nh_n, \\
H_1^\alpha & : \text{not } H_0^\alpha.
\end{align*}
\]

Now, we additionally assume the following condition:

[A6] Under \( H_0^\alpha \), there exists an estimator \( \hat{\alpha}_n \) such that

\[
\sqrt{n}(\hat{\alpha}_n - \alpha_0) = O_p(1).
\]

Let

\[
U_n^{(1)}(\alpha) = -\frac{1}{2} \sum_{i=1}^{n} \left( \text{tr} \left( A^{-1}(X_{i-1}^n, \alpha) \frac{(\Delta X_i)^{\otimes 2}}{h_n} \right) + \log \det A(X_{i-1}^n, \alpha) \right).
\]

Note that

\[
\partial_\alpha U_n^{(1)}(\alpha) = \frac{1}{2} \sum_{i=1}^{n} \text{tr} \left( A^{-1}(X_{i-1}^n, \alpha) \partial_\alpha A(X_{i-1}^n, \alpha) \left( A^{-1}(X_{i-1}^n, \alpha) \frac{(\Delta X_i)^{\otimes 2}}{h_n} - I_d \right) \right)
\]

and \( \frac{1}{\sqrt{n}} \partial_\alpha U_n^{(1)}(\alpha_0) \) has asymptotic normality, let

\[
\hat{\eta}_i = \text{tr} \left( A^{-1}(X_{i-1}^n, \hat{\alpha}_n) \frac{(\Delta X_i)^{\otimes 2}}{h_n} \right) = \sum_{\ell_1, \ell_2=1}^{d} \left( A^{-1}(X_{i-1}^n, \hat{\alpha}_n) \right)_{\ell_1, \ell_2} (\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}.
\]

The test statistic for \( \alpha \) is as follows:

\[
T_n^{\alpha} = \frac{1}{\sqrt{2dn}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \hat{\eta}_i - \frac{k}{n} \sum_{i=1}^{n} \hat{\eta}_i \right|.
\]

For \( k = 1, 2, \ldots, \) let \( B_k^n \) be a \( k \)-dimensional standard Brownian bridge.
Theorem 1  Suppose that [A1]-[A6] hold. Then under $H_0^\beta$, as $nh_n^2 \to 0$,
\[
T_n^\alpha \overset{d}{\to} \sup_{0 \leq s \leq 1} |B_1^0(s)|.
\]

Remark 1  The test statistic $T_n^\alpha$ is a multidimensional version of the test statistic proposed in Remark on page 842 of Lee (2011). The asymptotic distribution of the test statistic $T_n^\alpha$ is shown in Theorem I. Moreover, it follows from Theorem I below that the test is consistent.

2.2. Test for the drift parameter.

We make the following assumption:

[A7] $\alpha_0$ does not change over $0 \leq t \leq nh_n$.

Then we consider the following testing problem:
\[
\begin{aligned}
H_0^\beta &: \beta_0 \text{ does not change over } 0 \leq t \leq nh_n, \\
H_1^\beta &: \text{not } H_0^\beta.
\end{aligned}
\]

Now, we additionally assume the following condition:

[A8] Under $H_0^\beta$, there exists an estimator $(\hat{\alpha}_n, \hat{\beta}_n)$ such that
\[
\sqrt{n}(\hat{\alpha}_n - \alpha_0) = O_p(1), \quad \sqrt{nh_n}(\hat{\beta}_n - \beta_0) = O_p(1).
\]

Let
\[
U_n^{(2)}(\beta|\alpha) = -\frac{1}{2} \sum_{i=1}^n \frac{1}{h_n} \left( A^{-1}(X_{t^n_{i-1}}, \alpha) \left( X_{t^n_i} - X_{t^n_{i-1}} - h_n b(X_{t^n_{i-1}}, \beta) \right) \right)^{\otimes 2}.
\]

Note that
\[
\partial_\beta U_n^{(2)}(\beta|\alpha) = \sum_{i=1}^n \partial_\beta b(X_{t^n_{i-1}}, \beta)^T A^{-1}(X_{t^n_{i-1}}, \alpha) \left( X_{t^n_i} - X_{t^n_{i-1}} - h_n b(X_{t^n_{i-1}}, \beta) \right),
\]
and
\[
\frac{1}{\sqrt{nh_n}} \partial_\beta U_n^{(2)}(\beta|\alpha_0) \text{ has asymptotic normality, let}
\]
\[
\hat{\xi}_i = \left( 1_d a^{-1}(X_{t^n_{i-1}}, \hat{\alpha}_n) \left( X_{t^n_i} - X_{t^n_{i-1}} - h_n b(X_{t^n_{i-1}}, \hat{\beta}_n) \right) \right).
\]

The test statistic for $\beta$ is as follows:
\[
T_{1,n}^\beta = \frac{1}{\sqrt{nh_n}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \hat{\xi}_i - \frac{k}{n} \sum_{i=1}^{n} \hat{\xi}_i \right|.
\]

Theorem 2  Suppose that [A1]-[A5], [A7] and [A8] hold. Then under $H_0^\beta$, as $nh_n^2 \to 0$,
\[
T_{1,n}^\beta \overset{d}{\to} \sup_{0 \leq s \leq 1} |B_1^0(s)|.
\]

$T_{1,n}^\beta$ is the simple test statistic, but the test may not be consistent, see Section 3 below. In this case, we consider another test statistic. Let
\[
\hat{\xi}_i = \partial_\beta b(X_{t^n_{i-1}}, \hat{\beta}_n)^T A^{-1}(X_{t^n_{i-1}}, \hat{\alpha}_n) \left( X_{t^n_i} - X_{t^n_{i-1}} - h_n b(X_{t^n_{i-1}}, \hat{\beta}_n) \right),
\]
\[
\mathcal{I}_n = \frac{1}{n} \sum_{i=1}^{n} \partial_\beta b(X_{t^n_{i-1}}, \hat{\beta}_n)^T A^{-1}(X_{t^n_{i-1}}, \hat{\alpha}_n) \partial_\beta b(X_{t^n_{i-1}}, \hat{\beta}_n).
\]

Note that $\mathcal{I}_n$ is a consistent estimator of
\[
\mathcal{I} := \int_{\mathbb{R}^d} \partial_\beta b(x, \beta_0)^T A^{-1}(x, \alpha_0) \partial_\beta b(x, \beta_0) d\mu_{\theta_0}(x).
\]

The test statistic for $\beta$ is as follows:
\[
T_{2,n}^\beta = \frac{1}{\sqrt{nh_n}} \max_{1 \leq k \leq n} \left\| \mathcal{I}_n^{-1/2} \left( \sum_{i=1}^{k} \hat{\xi}_i - \frac{k}{n} \sum_{i=1}^{n} \hat{\xi}_i \right) \right\|.
\]

We additionally assume the following condition:

[A9] There exists an integer $m \geq 3$ such that $nh_n^{m/(m-1)} \to \infty$ and $b \in C_{\uparrow}^{4,m+1}(\mathbb{R}^d \times \Theta_B)$. 

Theorem 3 Suppose that [A1]-[A5], [A7], [A8] and [A9] hold. Then under $H_0^\beta$, as $nh_n^2 \to 0$,

$$T_{2,n}^\beta \to_d \sup_{0 \leq s \leq 1} \| B_q^\beta(s) \|.$$ 

We assume the following condition instead of [A9]:

[A9] There exists an integer $M \geq 1$ such that $b \in C_{t,M+1}^4(\mathbb{R}^d \times \Theta_B)$ and $\partial_{\beta,\ell_M+1} \cdots \partial_{\beta,1} b(x, \beta) = 0$ for $1 \leq \ell_1, \ldots, \ell_{M+1} \leq q$.

Corollary 4 Suppose that [A1]-[A5], [A7], [A8] and [A9'] hold. Then under $H_0^\beta$, as $nh_n^2 \to 0$,

$$T_{2,n}^\beta \to_d \sup_{0 \leq s \leq 1} \| B_q^\beta(s) \|.$$ 

Remark 2 Since $h_n \to 0$, $nh_n \to \infty$ and $nh_n^2 \to 0$, if $h_n = O(n^{-\alpha})$ for some $\alpha \in (\frac{1}{2}, 1)$, then there exists an integer $m > \frac{1}{1-\alpha} > 2$ such that $nh_n^{m/(m-1)} \to \infty$. Therefore, in [A9] of Theorem 3, we make the assumption of the smoothness of the drift coefficient $b$ with respect to $\beta$ up to order $m+1 \geq 4$ when the drift coefficient $b$ is general. In particular, if $b \in C_{t,M}^4(\mathbb{R}^d, \Theta_B)$, then [A9] is satisfied. In contrast, Corollary 4 indicates that when the drift coefficient $b(x, \beta) = -\beta x$, we set $M = 1$ in [A9'] and there is no need to assume $m \geq 3$ in [A9].

2.3. The powers of tests.

First, we consider the power of the test for the diffusion parameter $\alpha$, that is, the following testing problem: $H_0^\alpha : \alpha_0$ does not change over $0 \leq t \leq nh_n$, $H_1^\alpha :$ There exists $0 < t^* < 1$ such that

$$\alpha_0 = \begin{cases} \alpha_1^*, & 0 \leq t \leq [nt^*]h_n, \\ \alpha_2^*, & [nt^*]h_n < t \leq nh_n, \end{cases}$$

where $\alpha_1^* \neq \alpha_2^*$. Now, we assume the following conditions:

[B1] Under $H_1^\alpha$, there exist $\alpha_* \in \Theta_A$ and an estimator $\hat{\alpha}_n$ such that

$$\hat{\alpha}_n - \alpha_* = o_p(1).$$

Set

$$F(\alpha) = \int_{\mathbb{R}^d} \text{tr} \left( A^{-1}(x, \alpha)A(x, \alpha) \right) d\mu_n(x).$$

[B2] $F(\alpha_1^*) \neq F(\alpha_2^*)$ under $H_1^\alpha$.

Let $0 < \epsilon < 1$ and $w_1(\epsilon)$ denote the upper-$\epsilon$ point of $\sup_{0 \leq s \leq 1} \| B_k^\alpha(s) \|$, that is, $P(\sup_{0 \leq s \leq 1} \| B_k^\alpha(s) \| > w_k(\epsilon)) = \epsilon$.

Theorem 5 Assume [A1]-[A4], [B1] and [B2]. Then, under $H_1^\alpha$,

$$P \left( T_n^\alpha > w_1(\epsilon) \right) \to 1.$$ 

Hence, if [B1] and [B2] are satisfied, then the test $T_n^\alpha$ is consistent.

In the following, we assume [A7]. We consider the power of the test for the drift parameter $\beta$, that is, the following testing problem: $H_0^\beta : \beta_0$ does not change over $0 \leq t \leq nh_n$, $H_1^\beta :$ There exists $0 < t^* < 1$ such that

$$\beta_0 = \begin{cases} \beta_1^*, & 0 \leq t \leq [nt^*]h_n, \\ \beta_2^*, & [nt^*]h_n < t \leq nh_n, \end{cases}$$

where $\beta_1^* \neq \beta_2^*$. Now, we assume the following conditions:

[B3] Under $H_1^\beta$, there exist $\beta_* \in \Theta_B$ and estimators $\hat{\alpha}_n$ and $\hat{\beta}_n$ such that

$$\sqrt{n}(\hat{\alpha}_n - \alpha_0) = O_p(1), \quad \sqrt{n}(\hat{\beta}_n - \beta_*) = O_p(1).$$

[B3'] Under $H_1^\beta$, there exist $\beta_* \in \Theta_B$ and estimators $\hat{\alpha}_n$ and $\hat{\beta}_n$ such that

$$\sqrt{n}(\hat{\alpha}_n - \alpha_0) = O_p(1), \quad \hat{\beta}_n - \beta_* = o_p(1).$$

Let

$$G(\beta) = \int_{\mathbb{R}^d} 1_T a^{-1}(x, \alpha_0)(b(x, \beta) - b(x, \beta_*)) d\mu_{(\alpha_0, \beta)}(x),$$

$$H(\beta) = \int_{\mathbb{R}^d} \partial_\beta(x, \beta_*)^T A^{-1}(x, \alpha_0)(b(x, \beta) - b(x, \beta_*)) d\mu_{(\alpha_0, \beta)}(x).$$
Theorem 6  Assume [A1]-[A4], [A7], [B3'] and [B4]. Then, under $H_1^\beta$,  
$$P\left(T_{1,n}^\beta > w_1(\epsilon)\right) \rightarrow 1.$$ 

Theorem 7  Assume [A1]-[A4], [A7], [A9], [B3] and [B5]. Then, under $H_1^\beta$,  
$$P\left(T_{2,n}^\beta > w_q(\epsilon)\right) \rightarrow 1.$$ 

Corollary 8  Assume [A1]-[A4], [A7], [A9'], [B3'] and [B5]. Then, under $H_1^\beta$,  
$$P\left(T_{2,n}^\beta > w_q(\epsilon)\right) \rightarrow 1.$$ 

Remark 3  Theorems 6, 7 and Corollary 8 indicate that the tests $T_{1,n}^\beta$ and $T_{2,n}^\beta$ are consistent under some regularity conditions. Theorem 7 and Corollary 8 indicate that the test $T_{2,n}^\beta$ is consistent when the drift coefficient $b$ satisfies [A9] and [A9'], respectively. In particular, for the drift coefficient $b$ satisfying [A9], it is necessary to assume $\sqrt{nh_n}(\beta_n - \beta_*) = O_p(1)$ in [B3]. When the drift coefficient $b$ satisfies [A9'], it is enough to assume $\hat{\beta}_n - \beta_* = o_p(1)$ in [B3']. For the maximum likelihood type estimators of the misspecified diffusion processes, see Uchida and Yoshida (2011).

Remark 4  In Theorems 6, 7 and Corollary 8 it is not necessary to assume $nh_n^2 \rightarrow 0$, but it is required to construct estimators $\hat{\alpha}_n, \hat{\beta}_n$ that satisfies $\sqrt{n}(\hat{\alpha}_n - \alpha_0) = O_p(1)$ and $\sqrt{nh_n}(\hat{\beta}_n - \beta_*) = O_p(1)$ in [B3], or estimator $\hat{\alpha}_n$ that satisfies $\sqrt{n}(\hat{\alpha}_n - \alpha_0) = O_p(1)$ in [B3'].

3. Examples and simulation results

In this section, we evaluate the asymptotic performance of the test statistics $T_n^\alpha$, $T_{1,n}^\beta$ and $T_{2,n}^\beta$ through a simulation study. In this study, we consider the 1-dimensional Ornstein-Uhlenbeck process (OU process):

$$dX_t = -\beta(X_t - \gamma)dt + \sigma dW_t, \quad X_0 = 1. \quad (\alpha, \beta > 0, \gamma \in \mathbb{R})$$

(3.1)

The data $\{X_{at}\}_{a=0}^n$ are obtained discretely with sampling interval $h_n = n^{-1/3}$, so that $nh_n = n^{1/3} \rightarrow \infty$.

Song and Lee (2009), Lee (2011) and Song (2020) conduct the simulations for the test statistic of the diffusion coefficient. On the other hand, we also perform the simulations for the test statistic of the drift coefficient as well as the diffusion coefficient. In addition, Negri and Nishiyama (2017) perform the simulations for the test statistic of the diffusion and drift parameters at the same time. In our case, however, we first test the diffusion parameter, and if we determine that there is no change in the diffusion parameter, then we test the drift parameter. They are the adaptive tests for the changes of both the drift and the diffusion parameters. As far as we know, this is the first numerical simulations of the adaptive tests for the change point problem of an ergodic diffusion process. For details of the adaptive tests based on the Wald type test statistics and the Rao type ones and the likelihood ratio type ones for ergodic diffusion processes, see Kitagawa and Uchida (2014).

Under the above setting with $a(x, \alpha) = \alpha$ and $b(x, \beta) = -\beta(x - \gamma)$ for $\beta = (\beta, \gamma)$, we have

$$\hat{\eta}_i = \frac{(X_{t_i}^n - X_{t_{i-1}}^n)^2}{h_n \hat{\alpha}_n^2},$$

$$T_n^\alpha = \frac{1}{\sqrt{2n}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \hat{\eta}_i - \frac{k}{n} \sum_{i=1}^{n} \hat{\eta}_i \right|,$$

$$\hat{\xi}_i = \frac{X_{t_i}^n - X_{t_{i-1}}^n}{\hat{\alpha}_n},$$

$$T_{1,n}^\beta = \frac{1}{\sqrt{nh_n}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \hat{\xi}_i - \frac{k}{n} \sum_{i=1}^{n} \hat{\xi}_i \right|,$$

$$\hat{\xi}_i = \frac{-(X_{t_{i-1}}^n - \hat{\gamma}_n)}{\hat{\beta}_n} \frac{X_{t_i}^n - X_{t_{i-1}}^n + h_n \hat{\beta}_n(X_{t_{i-1}}^n - \hat{\gamma}_n)}{\hat{\alpha}_n^2},$$

$$\hat{\eta}_i = \frac{(X_{t_i}^n - X_{t_{i-1}}^n)^2}{h_n \hat{\alpha}_n^2}.$$
We consider the empirical size of situation: and the number of data is large. (3.1) is recorded. This is repeated 10 times. As a result, we have

\[ T_{2,n}^\beta = \frac{1}{\sqrt{nh_n}} \max_{1 \leq k \leq n} \left\| T_n^{-1/2} \left( \sum_{i=1}^k \hat{\zeta}_i - k \frac{n \hat{\zeta}}{n} \right) \right\|, \]

where \((\hat{\alpha}_n, \hat{\beta}_n)\) is an estimator satisfying the assumption \([A6]\) or \([A8]\), and we construct the estimator from

\[ U_n^{(1)}(\alpha) = -\frac{1}{2} \sum_{i=1}^n \frac{(X_{t_i}^\alpha - X_{t_{i-1}}^\alpha)^2}{h_n \alpha^2} - n \log \alpha, \quad U_n^{(2)}(\beta|\alpha) = -\frac{1}{2} \sum_{i=1}^n \frac{(X_{t_i}^\alpha - X_{t_{i-1}}^\alpha + h_n \beta (X_{t_i}^\alpha - \gamma))^2}{h_n \alpha^2}, \]

\[ \hat{\alpha}_n = \arg \sup_\alpha U_n^{(1)}(\alpha), \quad \hat{\beta}_n = \arg \sup_\beta U_n^{(2)}(\beta|\hat{\alpha}_n), \]

see, for example, Kessler (1995, 1997), Uchida and Yoshida (2012) and Kaino and Uchida (2018). We have, from Theorems 1, 2 and Corollary 4

\[ T_{n}^\alpha \xrightarrow{d} \sup_{0 \leq s \leq 1} |B_0^\alpha(s)|, \quad T_{1,n}^\beta \xrightarrow{d} \sup_{0 \leq s \leq 1} |B_0^\beta(s)|, \quad T_{2,n}^\beta \xrightarrow{d} \sup_{0 \leq s \leq 1} \|B_2^\beta(s)\|. \tag{3.2} \]

In order to evaluate the empirical sizes and powers of \(T_{n}^\alpha, T_{1,n}^\beta, T_{2,n}^\beta\), we consider three cases as follows:

**Case 1**: Neither parameter changes.

**Case 2**: The diffusion parameter \(\alpha\) changes.

**Case 3**: The diffusion parameter \(\alpha\) does not change, but the drift parameter \((\beta, \gamma)\) changes.

For each of the above cases, we change parameters at \(\alpha, \beta, \gamma\) goes to 1 under the following situation:

(i) the only parameter \(\alpha\) changes from \(\alpha^*_1 = 1\) to \(\alpha^*_2 \in \{1.01, 1.05, 1.1, 1.5\}\).

Lee (2011) also performed this simulation, but we consider the cases where the parameter change is small and the number of data is large.

In Case 3, after confirming that \(T_{n}^\alpha \xrightarrow{d} \sup_{0 \leq s \leq 1} |B_0^\alpha(s)|\), it is verified whether \(T_{1,n}^\beta\) and \(T_{2,n}^\beta\) are consistent.

We consider the empirical size of \(T_{n}^\alpha\) and the powers of \(T_{1,n}^\beta\) and \(T_{2,n}^\beta\) under the following situations:

(i) the parameter \(\beta\) only changes from \(\beta^*_1 = 1\) to \(\beta^*_2 \in \{1.1, 1.5, 3.5\}\);

(ii) the parameter \(\gamma\) only changes from \(\gamma^*_1 = 1\) to \(\gamma^*_2 \in \{0.9, 0.5, 0, -1\}\);

(iii) the parameter \((\beta, \gamma)\) changes from \((\beta^*_1, \gamma^*_1) = (1, 1)\) to \((\beta^*_2, \gamma^*_2) \in \{(3.0, 5), (3, 0), (5, 0.5), (5, 0)\}\).

\(\beta\) and \(\gamma\) in OU process (3.1) represent the speed of reversion and the mean value as \(nh_n \to \infty\), respectively. In (i) and (ii), we verify the consistency of the test when each parameter changes, and in (iii), we also verify the consistency of the test when both \(\beta\) and \(\gamma\) change.

First, let us consider the consistency of the test based on \(T_{n}^\alpha\). The invariant measure of the solution in (3.1) is \(\mu_\theta \sim N(\gamma, \frac{\alpha^2}{2\beta})\), \(\theta = (\alpha, \beta, \gamma)\), and we have

\[ F(\alpha) = \int_\mathbb{R} \frac{\alpha^2}{2\beta} \frac{d\mu_\theta(x)}{\alpha^2} = \frac{\alpha^2}{2\beta}. \]

Therefore, noting that \(\alpha > 0\), \(F(\alpha^*_1) \neq F(\alpha^*_2)\) for \(\alpha^*_1 \neq \alpha^*_2\), and the test \(T_{n}^\alpha\) has consistency according to Theorem 5.
Next, we investigate the consistency of the test \( T_{1,n}^\beta \). For the drift parameter \( \beta_1^\ast = (\beta_1^\ast, \gamma_1^\ast) \), we have

\[
G(\beta_1^\ast) = \int_{\mathbb{R}} \frac{1}{\alpha_0} (b(x, \beta_1^\ast) - b(x, \bar{\beta}_n))d\mu_{\theta_1}(x) = \int_{\mathbb{R}} \frac{1}{\alpha_0} ((-\beta_1^\ast - \bar{\beta}_n)x + (\beta_1^\ast \gamma_1^\ast - \bar{\beta}_n \gamma_1))(x)
\]

\[
= \frac{1}{\alpha_0}(-(\gamma_1^\ast - \bar{\gamma}_n) + (\beta_1^\ast \gamma_1^\ast - \bar{\beta}_n \gamma_1))
\]

\[
= \frac{\bar{\beta}_n}{\alpha_0}(\gamma_1^\ast - \bar{\gamma}_n),
\]

where \( \theta_1^\ast = (\alpha_0, \beta_1^\ast, \gamma_1^\ast) \).

We construct the estimators \( \hat{\alpha}_n \) and \( \hat{\beta}_n \) in the same way as above. Since

\[
\frac{1}{n h_n} \left( U_{n}^{(2)}(\beta | \hat{\alpha}_n) - U_{n}^{(2)}(\beta | \hat{\alpha}_n) \right) \rightarrow^p -\frac{1}{2}(t^*Q(\beta, \beta_1^\ast) + (1 - t^*)Q(\beta, \beta_1^\ast)) =: Q(\beta),
\]

where

\[
Q(\beta, \beta_1^\ast) = \int_{\mathbb{R}} \frac{(b(x, \beta) - b(x, \beta_1^\ast))^2}{\alpha_0}d\mu_{\theta_1}(x) = \int_{\mathbb{R}} \frac{((-\beta + \beta_1^\ast)x + (\beta \gamma - \beta_1^\ast \gamma_1))^2}{\alpha_0}d\mu_{\theta_1}(x),
\]

if we set \( \beta_\ast = \text{arg sup}_\beta Q(\beta) \), then \( \beta_n \rightarrow^p \beta_\ast \) under \( R_1^\beta \).

Since \( \partial_\beta Q(\beta_\ast) = 0 \),

\[
\partial_\gamma Q(\beta, \beta_1^\ast) = t^* \partial_\gamma Q(\beta, \beta_1^\ast) + (1 - t^*)\partial_\gamma Q(\beta, \beta_1^\ast) = t^* \frac{2\beta_1^\ast}{\alpha_0}(\gamma - \gamma_1^\ast) + (1 - t^*) \frac{2\beta_1^\ast}{\alpha_0}(\gamma - \gamma_2^\ast)
\]

\[
= \frac{2\beta_1^\ast}{\alpha_0}(\gamma - (t^* \gamma_1^\ast + (1 - t^*) \gamma_2^\ast)),
\]

\[
\partial_\beta Q(\beta, \beta_1^\ast) = \frac{2}{\alpha_0} \int_{\mathbb{R}} (-x + \gamma)((-\beta + \beta_1^\ast)x + (\beta \gamma - \beta_1^\ast \gamma_1^\ast))d\mu_{\theta_1}(x) = \frac{2}{\alpha_0} \left( \beta \left( \frac{\alpha_0^2}{2\beta_1^\ast} + (\gamma - \gamma_1^\ast)^2 \right) - \frac{\alpha_0^2}{2} \right),
\]

and

\[
\partial_\beta Q(\beta) = t^* \partial_\beta Q(\beta, \beta_1^\ast) + (1 - t^*)\partial_\beta Q(\beta, \beta_1^\ast)
\]

\[
= \frac{2}{\alpha_0} \left( \beta \left( t^* \left( \frac{\alpha_0^2}{2\beta_1^\ast} + (\gamma - \gamma_1^\ast)^2 \right) + (1 - t^*) \left( \frac{\alpha_0^2}{2\beta_2^\ast} + (\gamma - \gamma_2^\ast)^2 \right) \right) - \frac{\alpha_0^2}{2} \right),
\]

we have

\[
\gamma_\ast = t^* \gamma_1^\ast + (1 - t^*) \gamma_2^\ast,
\]

\[
\bar{\beta}_\ast = \frac{\gamma_\ast}{t^* \beta_1^\ast + (1 - t^*) \beta_2^\ast}
\]

\[
= \frac{1}{t^* \beta_1^\ast + (1 - t^*) \beta_2^\ast + \frac{\alpha_0^2}{\alpha_0^2} t^*(1 - t^*) \gamma_1^\ast - \gamma_2^\ast)^2} > 0.
\]

If \( \beta_1^\ast \neq \beta_2^\ast \) and \( \gamma_1^\ast = \gamma_2^\ast \), then

\[
G(\beta_1^\ast) - G(\beta_2^\ast) = \frac{\bar{\beta}_\ast}{\alpha_0}((\gamma_1^\ast - \bar{\gamma}_n) - (\gamma_2^\ast - \bar{\gamma}_n)) = \frac{\bar{\beta}_\ast}{\alpha_0}(\gamma_1^\ast - \gamma_2^\ast) = 0,
\]

and [B4] does not hold. If \( \gamma_1^\ast \neq \gamma_2^\ast \), then

\[
G(\beta_1^\ast) - G(\beta_2^\ast) = \frac{\bar{\beta}_\ast}{\alpha_0}(\gamma_1^\ast - \gamma_2^\ast) \neq 0,
\]

and the test \( T_{1,n}^\beta \) has consistency according to Theorem 6.

Furthermore, we study the consistency of the test \( T_{2,n}^\beta \). One has

\[
H(\beta_1^\ast) = \frac{1}{\alpha_0} \int_{\mathbb{R}} \frac{1}{\beta_\ast} \left( -\frac{(x - \gamma_\ast)}{\beta_\ast} \right) ((-\beta_1^\ast x + \beta_1^\ast \gamma_1^\ast) - (-\bar{\beta}_\ast x + \bar{\beta}_\ast \gamma_\ast))d\mu_{\theta_1}(x)
\]
\[
\frac{1}{\alpha_0} \int_{\mathbb{R}} \left( -(x - \bar{\gamma}_*) \right) \left( -\left( \beta_1^* - \bar{\beta}_* \right) x + (\beta_1^* \gamma_*^* - \bar{\beta}_* \gamma_*) \right) d\mu_{\theta_1}(x)
\]
\[
= \frac{1}{\alpha_0} \int_{\mathbb{R}} \left[ \left( \beta_1^* - \bar{\beta}_* \right) x^2 - (\beta_1^* \gamma_*^* - \bar{\beta}_* \gamma_*) x + \bar{\gamma}_*(\beta_1^* \gamma_*^* - \bar{\beta}_* \gamma_*) \right] d\mu_{\theta_1}(x)
\]
\[
= \frac{1}{\alpha_0^2} \left( \beta_1^* - \bar{\beta}_* \right) \left( \left( \gamma_*^* \right)^2 + \frac{\alpha_2^2}{\alpha_0^2} \right) - (\beta_1^* \gamma_*^* - \bar{\beta}_* \gamma_*) \left( \beta_1^* \gamma_*^* - \bar{\beta}_* \gamma_* \right) + \bar{\gamma}_*(\beta_1^* \gamma_*^* - \bar{\beta}_* \gamma_*)
\]
\[
= \frac{1}{\alpha_0^2} \left( \frac{\alpha_2^2}{2} \left( 1 - \frac{1}{\beta_1^*} \right) - \bar{\beta}_*(\gamma_*^* - \bar{\gamma}_*)^2 \right) (\bar{\beta}_* x^2 + \bar{\beta}_* \gamma_* x + \bar{\gamma}_*(\beta_1^* \gamma_*^* - \bar{\beta}_* \gamma_*))
\]
and
\[
H(\beta_1^*) - H(\beta_2^*) = \frac{\bar{\beta}_*}{\alpha_0^2} \left( -\frac{\alpha_2^2}{2} \left( 1 - \frac{1}{\beta_1^*} - \frac{1}{\beta_2^*} \right) (\gamma_*^* - \bar{\gamma}_*) \right).
\]

If \( \gamma_*^* \neq \bar{\gamma}_* \), noting that \( \bar{\beta}_* > 0 \), then
\[
H(\beta_1^*, \beta_1^* \gamma_*^*) - H(\beta_2^*, \beta_2^* \gamma_*^*) \neq 0.
\]

If \( \beta_1^* \neq \beta_2^* \) and \( \gamma_*^* = \bar{\gamma}_* \), then
\[
H(\beta_1^*, \beta_1^* \gamma_*^*) - H(\beta_2^*, \beta_2^* \gamma_*^*) = -\frac{\bar{\beta}_*}{2} \left( \frac{1}{\beta_1^*} - \frac{1}{\beta_2^*} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \neq 0.
\]

Therefore, the test \( T_{2,n}^\beta \) is consistent by Corollary 8.

Figures 37 show the sample paths in Case 1, (i) of Case 2 and (i)-(iii) of Case 3, and Figures 8-20 show the histograms of \( T_{n}^\alpha, T_{1,n}^\beta, \) and \( T_{2,n}^\beta \) in Case 1 and the probability density function of \( \sup_{0 \leq s \leq 1} |B_0^0(s)| \) and \( \sup_{0 \leq s \leq 1} \|B(s)\| \) obtained by the simulation in the above cases. Figures 21-33 show the empirical distribution function of \( T_{n}^\alpha, T_{1,n}^\beta, \) and \( T_{2,n}^\beta \) and the cumulative distribution function of \( \sup_{0 \leq s \leq 1} |B_0^0(s)| \) and \( \sup_{0 \leq s \leq 1} \|B(s)\| \).

Table 1 shows the empirical sizes of \( T_{n}^\alpha, T_{1,n}^\beta, \) and \( T_{2,n}^\beta \) in Case 1 when \( n = 8 \times 10^3, 1.25 \times 10^5 \) and \( 10^6 \). According to Table 1, the empirical sizes of \( T_{n}^\alpha, T_{1,n}^\beta, \) and \( T_{2,n}^\beta \) are around 0.1 for each \( n \) and \( (\alpha, \beta, \gamma) \). Furthermore, from Figures 8-10 and Figures 21-23 we can see (3.2).

Table 2 shows the empirical powers of \( T_{n}^\alpha \) in Case 2. According to Table 2, if \( n = 10^6 \), the changes as shown in Figure 2 is detected. We can confirm this from Figures 11 and 24.

Tables 3-5 show the empirical sizes of \( T_{n}^\alpha \) and empirical sizes of \( T_{1,n}^\beta, \) and \( T_{2,n}^\beta \) in Case 3. First, in (i)-(iii) of Case 3, \( T_{n}^\alpha \) are about 0.1, and we can confirm \( T_{n}^\alpha \rightarrow \sup_{0 \leq s \leq 1} |B_0^0(s)| \) from Figures 12, 15, 18, 25, 28, and 31. It can be seen from Table 3 that \( T_{1,n}^\beta \) does not give good results in (i) of Case 3. As seen above, since \( T_{n}^\alpha \) does not satisfy the sufficient condition \([4.1]\) for having consistency in (i) of Case 3, it can be guessed that \( T_{1,n}^\beta \) may not have consistency in (i) of Case 3. In contrast, \( T_{2,n}^\beta \) gives good results in (i) of Case 3. We can confirm them from Figures 13, 14, 26, and 27. Similarly, when \( \gamma \) changes in (ii) of Case 3, we can see \( T_{n}^\alpha \rightarrow \sup_{0 \leq s \leq 1} |B_0^0(s)| \) from Table 4, Figures 15 and 28 and the consistency of \( T_{1,n}^\beta, \) \( T_{2,n}^\beta \), and \( T_{2,n}^\beta \) from Figures 16, 17, 29, and 30. Moreover, in (iii) of Case 3, we can confirm \( T_{n}^\alpha \rightarrow \sup_{0 \leq s \leq 1} |B_0^0(s)| \) from Table 5, Figures 18 and 31 and the consistency of \( T_{1,n}^\beta, \) \( T_{2,n}^\beta \), and \( T_{2,n}^\beta \) from Figures 19, 20, 32, and 33.
Table 1. Empirical sizes of $T^n_\alpha$, $T^{\beta}_1,n$ and $T^{\beta}_2,n$ in Case 1

| $n$     | $nh_n$ | $h_n$ | $(\alpha, \beta, \gamma)$ | (1, 1, 1) | (0.5, 1, 0) | (1.5, 1.5, 1) | (2, 3, 0.5) |
|---------|--------|-------|---------------------------|---------|-------------|--------------|------------|
| $8.0 \times 10^3$ | 20     | $2.5 \times 10^{-3}$ | $T^n_\alpha$ | 0.106  | 0.107       | 0.107        | 0.107      |
|         |        |       | $T^{\beta}_1,n$ | 0.095  | 0.078       | 0.091        | 0.097      |
|         |        |       | $T^{\beta}_2,n$ | 0.091  | 0.083       | 0.087        | 0.090      |
| $1.25 \times 10^5$ | 50     | $4.0 \times 10^{-4}$ | $T^n_\alpha$ | 0.100  | 0.099       | 0.099        | 0.100      |
|         |        |       | $T^{\beta}_1,n$ | 0.093  | 0.102       | 0.109        | 0.115      |
|         |        |       | $T^{\beta}_2,n$ | 0.098  | 0.095       | 0.091        | 0.106      |
| $10^6$  | 100    | $10^{-4}$ | $T^n_\alpha$ | 0.119  | 0.118       | 0.119        | 0.118      |
|         |        |       | $T^{\beta}_1,n$ | 0.106  | 0.104       | 0.107        | 0.102      |
|         |        |       | $T^{\beta}_2,n$ | 0.097  | 0.096       | 0.095        | 0.095      |

Table 2. Empirical powers of $T^n_\alpha$ in (i) of Case 2

| $n$     | $nh_n$ | $h_n$ | $\alpha^*_1 = 1 \rightarrow \alpha^*_2$ | 1.01 | 1.05 | 1.1 | 1.5 |
|---------|--------|-------|----------------------------------------|-----|-----|----|----|
| $8.0 \times 10^3$ | 20     | $2.5 \times 10^{-3}$ | $T^n_\alpha$ | 0.144  | 0.864       | 1.000        | 1.000     |
| $1.25 \times 10^5$ | 50     | $4.0 \times 10^{-4}$ | $T^n_\alpha$ | 0.741  | 1.000       | 1.000        | 1.000     |
| $10^6$  | 100    | $10^{-4}$ | $T^n_\alpha$ | 1.000  | 1.000       | 1.000        | 1.000     |

Table 3. Empirical sizes of $T^n_\alpha$ and empirical powers of $T^{\beta}_1,n$ and $T^{\beta}_2,n$ in (i) of Case 3

| $n$     | $nh_n$ | $h_n$ | $\beta^*_1 = 1 \rightarrow \beta^*_2$ | 1.1 | 1.5 | 3  | 5  |
|---------|--------|-------|----------------------------------------|----|----|---|----|
| $8.0 \times 10^3$ | 20     | $2.5 \times 10^{-3}$ | $T^n_\alpha$ | 0.106  | 0.107       | 0.102        | 0.102      |
|         |        |       | $T^{\beta}_1,n$ | 0.095  | 0.112       | 0.208        | 0.296      |
|         |        |       | $T^{\beta}_2,n$ | 0.100  | 0.152       | 0.532        | 0.918      |
| $1.25 \times 10^5$ | 50     | $4.0 \times 10^{-4}$ | $T^n_\alpha$ | 0.099  | 0.101       | 0.101        | 0.097      |
|         |        |       | $T^{\beta}_1,n$ | 0.101  | 0.119       | 0.204        | 0.296      |
|         |        |       | $T^{\beta}_2,n$ | 0.119  | 0.247       | 0.906        | 0.999      |
| $10^6$  | 100    | $10^{-4}$ | $T^n_\alpha$ | 0.119  | 0.118       | 0.118        | 0.116      |
|         |        |       | $T^{\beta}_1,n$ | 0.108  | 0.129       | 0.220        | 0.302      |
|         |        |       | $T^{\beta}_2,n$ | 0.126  | 0.351       | 0.998        | 1.000      |
Table 4. Empirical sizes of $T_n^\alpha$ and empirical powers of $T_{1,n}^\beta$ and $T_{2,n}^\beta$ in (ii) of Case 3

| $n$       | $nh_n$ | $h_n$ | $\gamma_1^* = 1 \rightarrow \gamma_2^*$ |
|-----------|--------|-------|----------------------------------------|
| $8.0 \times 10^3$ | 20     | $2.5 \times 10^{-3}$ | 0.106 | 0.106 | 0.104 | 0.102 |
|           |        |       | $T_n^\alpha$             | 0.103 | 0.170 | 0.287 | 0.210 |
|           |        |       | $T_{1,n}^\beta$          | 0.093 | 0.118 | 0.158 | 0.075 |
| $1.25 \times 10^5$ | 50     | $4.0 \times 10^{-4}$ | 0.100 | 0.100 | 0.100 | 0.100 |
|           |        |       | $T_n^\alpha$             | 0.108 | 0.378 | 0.852 | 0.990 |
|           |        |       | $T_{1,n}^\beta$          | 0.102 | 0.277 | 0.621 | 0.641 |
| $10^6$    | 100    | $10^{-4}$ | 0.119 | 0.119 | 0.119 | 0.119 |
|           |        |       | $T_n^\alpha$             | 0.121 | 0.680 | 0.996 | 1.000 |
|           |        |       | $T_{1,n}^\beta$          | 0.115 | 0.574 | 0.980 | 1.000 |

Table 5. Empirical sizes of $T_n^\alpha$ and empirical powers of $T_{1,n}^\beta$ and $T_{2,n}^\beta$ in (iii) of Case 3

| $n$       | $nh_n$ | $h_n$ | $(\beta_1^*, \gamma_1^*) = (1, 1) \rightarrow (\beta_2^*, \gamma_2^*)$ |
|-----------|--------|-------|------------------------------------------|
| $8.0 \times 10^3$ | 20     | $2.5 \times 10^{-3}$ | (3, 0.5) | (3, 0) | (5, 0.5) | (5, 0) |
|           |        |       | $T_n^\alpha$               | 0.102 | 0.102 | 0.100 | 0.101 |
|           |        |       | $T_{1,n}^\beta$            | 0.458 | 0.758 | 0.585 | 0.860 |
|           |        |       | $T_{2,n}^\beta$            | 0.676 | 0.735 | 0.961 | 0.962 |
| $1.25 \times 10^5$ | 50     | $4.0 \times 10^{-4}$ | 0.100 | 0.100 | 0.096 | 0.096 |
|           |        |       | $T_n^\alpha$               | 0.720 | 0.992 | 0.808 | 0.996 |
|           |        |       | $T_{1,n}^\beta$            | 0.989 | 1.000 | 0.999 | 1.000 |
| $10^6$    | 100    | $10^{-4}$ | 0.118 | 0.119 | 0.116 | 0.116 |
|           |        |       | $T_n^\alpha$               | 0.931 | 1.000 | 0.956 | 1.000 |
|           |        |       | $T_{1,n}^\beta$            | 1.000 | 1.000 | 1.000 | 1.000 |
Figure 3. Sample paths of the 1-dimensional OU process (3.1) with $(\alpha, \beta, \gamma) = (1, 1, 1), (0.5, 1, 0), (1.5, 1.5, -1), \text{ and } (2, 3, 0.5), \text{ in the order of upper left, upper right, lower left and lower right (Case 1).}$

Figure 4. Sample paths of the 1-dimensional OU process (3.1) whose parameter changes from $\alpha = 1 \text{ to } 1.01, 1.05, 1.1 \text{ and } 1.5, \text{ in the order of upper left, upper right, lower left and lower right ((i) of Case 2).}$
Figure 5. Sample paths of the 1-dimensional OU process (3.1) where the parameter changes from $\beta = 1$ to 1.1, 1.5, 3 and 5, whose parameter in the order of upper left, upper right, lower left and lower right ((i) of Case 3).

Figure 6. Sample paths of the 1-dimensional OU process (3.1) whose parameter changes from $\gamma = 1$ to 0.9, 0.5, 0 and $-1$, whose parameter in the order of upper left, upper right, lower left and lower right ((ii) of Case 3).
Figure 7. Sample paths of the 1-dimensional OU process (3.1) whose parameter changes from \((\beta, \gamma) = (1, 1)\) to \((3, 0.5), (3, 0), (5, 0.5)\) and \((5, 0)\), whose parameter in the order of upper left, upper right, lower left and lower right ((iii) of Case 3).

Figure 8. Histograms of \(T^\alpha_n\) when \(n = 8 \times 10^3, 1.25 \times 10^5\) and \(10^6\) in order from the first row and \((\alpha, \beta, \gamma) = (1, 1, 1), (0.5, 1, 0), (1.5, 1.5, -1)\) and \((2, 3, 0.5)\) in order from the first column (Case 1). The red line is the probability density function of \(\sup_{0 \leq s \leq 1} |B^\alpha_1(s)|\).
Figure 9. Histograms of $T_{1,n}^\alpha$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\alpha, \beta, \gamma) = (1, 1, 1), (0.5, 1, 0), (1.5, 1.5, -1)$ and $(2, 3, 0.5)$ in order from the first column (Case 1). The red line is the probability density function of $\sup_{0 \leq s \leq 1} |B_1^s|$.

Figure 10. Histograms of $T_{2,n}^\beta$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\alpha, \beta, \gamma) = (1, 1, 1), (0.5, 1, 0), (1.5, 1.5, -1)$ and $(2, 3, 0.5)$ in order from the first column (Case 1). The red line is the probability density function of $\sup_{0 \leq s \leq 1} \|B_2^s\|$.
Figure 11. Histograms of $T_n^α$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $α$ changes from 1 to 1.01, 1.05, 1.1 and 1.5 in order from the first column (i) of Case 2). The red line is the probability density function of $\sup_{0 \leq s \leq 1} |B_1^i(s)|$.

Figure 12. Histograms of $T_n^α$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $β$ changes from 1 to 1.1, 1.5, 3 and 5 in order from the first column ((i) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} |B_1^i(s)|$. 
Figure 13. Histograms of $T_{1,n}^{\beta}$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\beta$ changes from 1 to 1, 1.5, 3 and 5 in order from the first column ((ii) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} |B_1(t)|$.

Figure 14. Histograms of $T_{2,n}^{\beta}$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\beta$ changes from 1 to 1, 1.5, 3 and 5 in order from the first column ((ii) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} |B_2(t)|$. 
Figure 15. Histograms of $T_n^s$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\gamma$ changes from 1 to 0.9, 0.5, 0 and $−1$ in order from the first column ((i) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} |B^s_1(s)|$.

Figure 16. Histograms of $T_{1,n}^s$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\gamma$ changes from 1 to 0.9, 0.5, 0 and $−1$ in order from the first column ((ii) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} |B^s_1(s)|$. 
Figure 17. Histograms of $T_{2,n}^3$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\gamma$ changes from 1 to 0.9, 0.5, 0 and −1 in order from the first column ((i) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} \|B_2^3(s)\|$.

Figure 18. Histograms of $T_{n}^\beta$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\beta, \gamma)$ changes from (1, 1) to (3, 0.5), (3, 0), (5, 0.5) and (5, 0) in order from the first column ((i) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} |B_1^0(s)|$. 
Figure 19. Histograms of $T_{1,n}^3$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\beta, \gamma)$ changes from (1, 1) to (3, 0.5), (3, 0), (5, 0.5) and (5, 0) in order from the first column ((ii) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} B_{1}^3(s)$.

Figure 20. Histograms of $T_{2,n}^3$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\beta, \gamma)$ changes from (1, 1) to (3, 0.5), (3, 0), (5, 0.5) and (5, 0) in order from the first column ((iii) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} \|B_{2}^3(s)\|$.
Figure 21. Empirical distribution functions of $T_\alpha^n$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\alpha, \beta, \gamma) = (1, 1, 1), (0.5, 1, 0), (1.5, 1.5, -1)$ and $(2, 3, 0.5)$ in order from the first column (Case 1). The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} |B_1^n(s)|$.

Figure 22. Empirical distribution functions of $T_1^n$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\alpha, \beta, \gamma) = (1, 1, 1), (0.5, 1, 0), (1.5, 1.5, -1)$ and $(2, 3, 0.5)$ in order from the first column (Case 1). The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} |B_1^n(s)|$. 
Figure 23. Empirical distribution functions of $T_{2n}^\beta$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\alpha, \beta, \gamma) = (1, 1, 1), (0.5, 1, 0), (1.5, 1.5, -1)$ and $(2, 3, 0.5)$ in order from the first column (Case 1). The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} \| B_2^\beta(s) \|$.

Figure 24. Empirical distribution functions of $T_n^\alpha$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\alpha$ changes from 1 to 1.01, 1.05, 1.1 and 1.5 in order from the first column ((i) of Case 2). The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} | B_1^\alpha(s) |$. 
Figure 25. Empirical distribution functions of $T_n^\alpha$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\beta$ changes from 1 to 1, 1.5, 3 and 5 in order from the first column (i) of Case 3. The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} |B_1^\beta(s)|$.

Figure 26. Empirical distribution functions of $T_{1,n}^\beta$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\beta$ changes from 1 to 1, 1.5, 3 and 5 in order from the first column (ii) of Case 3. The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} |B_1^\beta(s)|$.
Figure 27. Empirical distribution functions of $T_{2,n}^\beta$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\beta$ changes from 1 to 1.1, 1.5, 3 and 5 in order from the first column ((iii) of Case 3). The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} \| B_2^\beta(s) \| \cdot$

Figure 28. Empirical distribution functions of $T_n^\alpha$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\gamma$ changes from 1 to 0.9, 0.5, 0 and $-1$ in order from the first column ((i) of Case 3). The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} | B_1^\gamma(s) |$. 
Figure 29. Empirical distribution functions of $T_{1,n}^\beta$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\gamma$ changes from 1 to 0.9, 0.5, 0.0 and $-1$ in order from the first column ((ii) of Case 3). The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} |B_1^\beta(s)|$.

Figure 30. Empirical distribution functions of $T_{2,n}^\beta$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $\gamma$ changes from 1 to 0.9, 0.5, 0.0 and $-1$ in order from the first column ((iii) of Case 3). The red line is the probability density function of $\sup_{0 \leq s \leq 1} \|B_2^\beta(s)\|$ obtained by simulation.
Figure 31. Empirical distribution functions of $T_n^\alpha$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\beta, \gamma)$ changes from $(1, 1)$ to $(3, 0.5), (3, 0), (5, 0.5)$ and $(5, 0)$ in order from the first column ((i) of Case 3). The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} |B_1^0(s)|$.

Figure 32. Empirical distribution functions of $T_{1,n}^\beta$ when $n = 8 \times 10^3, 1.25 \times 10^5$ and $10^6$ in order from the first row and $(\beta, \gamma)$ changes from $(1, 1)$ to $(3, 0.5), (3, 0), (5, 0.5)$ and $(5, 0)$ in order from the first column ((ii) of Case 3). The red line is the cumulative distribution function of $\sup_{0 \leq s \leq 1} |B_1^0(s)|$. 
4. Proofs

Let $\mathcal{G}_{i-1}^n = \sigma\left(\left\{W_s\right\}_{s \leq t_i^n}\right)$, and $C, C' > 0$ denote universal constants. If $f$ is a function on $\mathbb{R}^d \times \Theta$, $f_{i-1}(\theta)$ denotes the value $f(X_{t_{i-1}^n}, \theta)$. If $\{u_n\}$ is a positive sequence, $R$ denotes a function on $\mathbb{R}^d \times \mathbb{R}_+ \times \Theta$ for which there exists a constant $C > 0$ such that

$$
\sup_{\theta \in \Theta} \|R(x, u_n, \theta)\| \leq u_n C(1 + \|x\|)^C.
$$

Let $R_{i-1}(u_n, \theta) = R(X_{t_{i-1}^n}, u_n, \theta)$.

**Lemma 1** (Kessler, 1997) Suppose that $[A1] - [A4]$ hold. Then for $\ell, \ell_1, \ell_2, \ell_3, \ell_4 = 1, \ldots, d$,

$$
\mathbb{E}_0 [(\Delta X_{i})_{\ell}]^{(\mathcal{G}_{i-1}^n)} = h_n b_{i-1}(\theta) + R_{i-1}(h_n^2, \theta),
$$

(4.1)

$$
\mathbb{E}_0 [(\Delta X_{i})_{\ell}^4(\Delta X_{i})_{\ell_2}^2 | \mathcal{G}_{i-1}^n] = h_n A_{i-1}^{\ell_1, \ell_2}(\alpha) + R_{i-1}(h_n^2, \theta),
$$

(4.2)

$$
\mathbb{E}_0 [(\Delta X_{i})_{\ell}^4(\Delta X_{i})_{\ell_2}^2(\Delta X_{i})_{\ell_4}^2 | \mathcal{G}_{i-1}^n] = h_n^2 \left( A_{i-1}^{\ell_1, \ell_2} A_{i-1}^{\ell_3, \ell_4}(\alpha) + A_{i-1}^{\ell_1, \ell_2} A_{i-1}^{\ell_3, \ell_4}(\alpha) + A_{i-1}^{\ell_3, \ell_4} A_{i-1}^{\ell_2, \ell_3}(\alpha) + R_{i-1}(h_n^3, \theta) \right).
$$

(4.3)

Let

$$
\eta_i = \text{tr} \left( A_{i-1}^{-1}(\alpha_0) \frac{(\Delta X_{i})^{\otimes 2}}{h_n} \right), \quad \kappa(x, \alpha) = 1_{d}^T a^{-1}(x, \alpha), \quad \xi_i = \kappa_{i-1}(\alpha_0)(\Delta X_{i} - h_n b_{i-1}(\beta_0)),
$$

$$
\zeta_i = \partial_{\beta} b_{i-1}(\beta_0)^T A_{i-1}^{-1}(\alpha_0) (\Delta X_{i} - h_n b_{i-1}(\beta_0)).
$$

**Lemma 2** Suppose that $[A1] - [A4]$ hold. Then,

$$
\mathbb{E}_{\theta_0} \left[ \eta_i | \mathcal{G}_{i-1}^n \right] = d + R_{i-1}(h_n, \theta),
$$

(4.4)

$$
\mathbb{E}_{\theta_0} \left[ \eta_i^2 | \mathcal{G}_{i-1}^n \right] = d^2 + 2d + R_{i-1}(h_n, \theta),
$$

(4.5)
\[ \mathbb{E}_{\theta_0} [\xi_i | \mathcal{F}_i^{n-1}] = R_{i-1}(h_n^2, \theta), \quad (4.6) \]
\[ \mathbb{E}_{\theta_0} [\xi_i^2 | \mathcal{F}_i^{n-1}] = dh_n + R_{i-1}(h_n^2, \theta), \quad (4.7) \]
\[ \mathbb{E}_{\theta_0} [\xi_i | \mathcal{F}_i^{n-1}] = R_{i-1}(h_n^2, \theta), \quad (4.8) \]
\[ \mathbb{E}_{\theta_0} [\xi_i^2 | \mathcal{F}_i^{n-1}] = h_n \partial_\beta b_{i-1}(\beta_0)^T A_{i-1}^{-1}(\alpha_0) \partial_\beta b_{i-1}(\beta_0) + R_{i-1}(h_n^2, \theta). \quad (4.9) \]

Proof. Let us prove (4.4) to (4.9). We have, from (4.1) to (4.3),
\[
\mathbb{E}_{\theta_0} [\eta_i | \mathcal{F}_i^{n-1}] = \mathbb{E}_{\theta_0} \left[ \text{tr} \left( A_{i-1}^{-1}(\alpha_0) \frac{(\Delta X_i)^{\otimes 2}}{h_n^2} \right) \right] = \mathbb{E}_{\theta_0} \left[ A_{i-1}^{-1}(\alpha_0) \frac{(\Delta X_i)^{\otimes 2}}{h_n^2} \right] = \text{tr} \left( A_{i-1}^{-1}(\alpha_0)(A_{i-1}(\alpha_0) + R_{i-1}(h_n, \theta)) \right).
\]

\[
\mathbb{E}_{\theta_0} [\eta_i^2 | \mathcal{F}_i^{n-1}] = \mathbb{E}_{\theta_0} \left[ \sum_{\ell_1, \ell_2 = 1}^d (A_{i-1}^{-1})^{\ell_1, \ell_2} (A_{i-1})^{\ell_1, \ell_2} (\alpha_0) \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}(\Delta X_i)^{\ell_3}(\Delta X_i)^{\ell_4}}{h_n^2} \right] = \mathbb{E}_{\theta_0} \left[ (\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}(\Delta X_i)^{\ell_3}(\Delta X_i)^{\ell_4} \right].
\]

Lemma 3 (Song and Lee, 2009) Suppose that [A1], [A2], [A5] hold and a function \( f \) on \( \mathbb{R}^d \times \Theta \) satisfies

(i) \( f \) is continuous in \( \theta \in \Theta \) for all \( x \in \mathbb{R}^d \);
(ii) \( \partial_x f \) exists and \( f, \partial_x f \) are of polynomial growth in \( x \in \mathbb{R}^d \) uniformly \( \theta \in \Theta \).

Moreover, if \( nh_n^2 \to \infty \) for some \( 1 < r < 2 \), then, under \( H_0^r \) (or [A7] and \( H_0^2 \)), as \( nh_n^2 \to 0 \),
\[
\max_{[1/k^r] \leq t \leq \infty} \sup_{\theta \in \Theta} \left\| \frac{1}{k} \sum_{i=1}^k f(X_{t,i}^n, \theta) - \int_{\mathbb{R}^d} f(x, \theta) d\mu_{\theta_0}(x) \right\| \xrightarrow{\Delta} 0.
\]
Lemma 4 Suppose that [A1], [A2], [A5] hold and $f$ satisfies the conditions (i), (ii) in Lemma 3. Then, under $H_0^n$ (or [A7] and $H_0^b$), as $nh_n^2 \to 0$,

$$
\frac{1}{n} \max_{1 \leq k \leq n} \sup_{\theta \in \Theta} \left\| \sum_{i=1}^{k} f(X_{i-1}^n, \theta) - \frac{k}{n} \sum_{i=1}^{n} f(X_{i-1}^n, \theta) \right\| = o_p(1).
$$

Proof. If $nh_n \to \infty$, $nh_n^2 \to 0$, there exists $1 < r < 2$ such that $nh_n \to \infty$. Then, it follows from Lemma 3 that

$$
\frac{1}{n} \max_{1 \leq k \leq \lfloor n^{1/r} \rfloor} \sup_{\theta \in \Theta} \left\| \sum_{i=1}^{k} f_i(\theta) - \frac{k}{n} \sum_{i=1}^{n} f_i(\theta) \right\|
$$

$$
\leq \frac{1}{n} \left( \sum_{i=1}^{\lfloor n^{1/r} \rfloor} \sup_{\theta \in \Theta} \| f_i(\theta) \| + \frac{n^{1/r}}{n} \sum_{i=1}^{\lfloor n^{1/r} \rfloor} \sup_{\theta \in \Theta} \| f_i(\theta) \| \right)
$$

$$
= \frac{n^{1/r}}{n} \left( \frac{1}{\lfloor n^{1/r} \rfloor} \sum_{i=1}^{n^{1/r}} \sup_{\theta \in \Theta} \| f_i(\theta) \| + \frac{1}{n} \sum_{i=1}^{n} \sup_{\theta \in \Theta} \| f_i(\theta) \| \right) = o_p(1)
$$

and

$$
\frac{1}{n} \max_{\lfloor n^{1/r} \rfloor \leq k \leq n} \sup_{\theta \in \Theta} \left\| \sum_{i=1}^{k} f_i(\theta) - \frac{k}{n} \sum_{i=1}^{n} f_i(\theta) \right\|
$$

$$
= \frac{1}{n} \max_{\lfloor n^{1/r} \rfloor \leq k \leq n} \sup_{\theta \in \Theta} \left\| \frac{k}{n} \sum_{i=1}^{k} f_i(\theta) - \frac{k}{n} \sum_{i=1}^{n} f_i(\theta) \right\|
$$

$$
\leq \frac{1}{n} \max_{\lfloor n^{1/r} \rfloor \leq k \leq n} \left( \frac{k}{n} \sum_{i=1}^{k} f_i(\theta) - \int_{\mathbb{R}^d} f(x, \theta) d\mu_0(x) \right) + \frac{1}{n} \max_{\lfloor n^{1/r} \rfloor \leq k \leq n} \left( \frac{k}{n} \sum_{i=1}^{k} f_i(\theta) \right) + \frac{1}{n} \max_{\lfloor n^{1/r} \rfloor \leq k \leq n} \left( \frac{k}{n} \sum_{i=1}^{k} f_i(\theta) \right)\to 0.
$$

Therefore, we obtain

$$
\frac{1}{n} \max_{1 \leq k \leq n} \sup_{\theta \in \Theta} \left\| \sum_{i=1}^{k} f_i(\theta) - \frac{k}{n} \sum_{i=1}^{n} f_i(\theta) \right\|
$$

$$
\leq \frac{1}{n} \max_{1 \leq k \leq \lfloor n^{1/r} \rfloor} \sup_{\theta \in \Theta} \left\| \sum_{i=1}^{k} f_i(\theta) - \frac{k}{n} \sum_{i=1}^{n} f_i(\theta) \right\| + \frac{1}{n} \max_{\lfloor n^{1/r} \rfloor \leq k \leq n} \left( \frac{k}{n} \sum_{i=1}^{k} f_i(\theta) \right)\to 0.
$$

$$
= o_p(1).
$$

Lemma 5 Suppose that [A1]-[A5] hold and $f$ satisfies the conditions (i), (ii) in Lemma 3. Then, under $H_0^n$ (or [A7] and $H_0^b$), as $nh_n^2 \to 0$,

$$
\frac{1}{nh_n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f(X_{i-1}^n, \theta_0)(\Delta X_i)^{\ell} - \frac{k}{n} \sum_{i=1}^{n} f(X_{i-1}^n, \theta_0)(\Delta X_i)^{\ell} \right\| = o_p(1),
$$

(4.10)

$$
\frac{1}{nh_n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f(X_{i-1}^n, \theta_0)(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2} - \frac{k}{n} \sum_{i=1}^{n} f(X_{i-1}^n, \theta_0)(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2} \right\| = o_p(1).
$$

(4.11)

Proof. If we prove

$$
\frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_i(\theta_0)\frac{(\Delta X_i)^{\ell}}{h_n} - \sum_{i=1}^{k} E_{\theta_0} \left[ f_i-1(\theta_0) \frac{(\Delta X_i)^{\ell}}{h_n} \right] \right\| = o_p(1),
$$

(4.12)

$$
\frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_i(\theta_0) E_{\theta_0} \left[ \frac{(\Delta X_i)^{\ell}}{h_n} \right] g_i^{n-1} - k \int_{\mathbb{R}^d} f(x, \theta_0) b(x, \beta_0) d\mu_0(x) \right\| = o_p(1),
$$

(4.13)
then we have

$$\frac{1}{nh_n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0)(\Delta X_i)^e - \frac{k}{n} \sum_{i=1}^{n} f_{i-1}(\theta_0)(\Delta X_i)^e \right\| \leq \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0) (\Delta X_i)^e - k \int_{\mathbb{R}^d} f(x, \theta_0) b^e(x, \beta_0) d\mu_{\theta_0}(x) \right\|$$

$$+ \frac{1}{n} \max_{1 \leq k \leq n} \left\| k \int_{\mathbb{R}^d} f(x, \theta_0) b^e(x, \beta_0) d\mu_{\theta_0}(x) - \frac{k}{n} \sum_{i=1}^{n} f_{i-1}(\theta_0) (\Delta X_i)^e \right\|$$

$$\leq \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0) (\Delta X_i)^e - k \int_{\mathbb{R}^d} f(x, \theta_0) b^e(x, \beta_0) d\mu_{\theta_0}(x) \right\|$$

$$+ \frac{1}{n} \max_{1 \leq k \leq n} \left\| n \int_{\mathbb{R}^d} f(x, \theta_0) b^e(x, \beta_0) d\mu_{\theta_0}(x) - \sum_{i=1}^{n} f_{i-1}(\theta_0) (\Delta X_i)^e \right\|$$

$$\leq \frac{2}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0) (\Delta X_i)^e - k \int_{\mathbb{R}^d} f(x, \theta_0) b^e(x, \beta_0) d\mu_{\theta_0}(x) \right\| = o_p(1).$$

Hence, we obtain (4.10). Let us prove (4.12) and (4.13).

**Proof of (4.12).** Set

$$T_i = f_{i-1}(\theta_0) \left( \frac{(\Delta X_i)^e}{h_n} - E_{\theta_0} \left[ \frac{(\Delta X_i)^e}{h_n} | \mathcal{F}_{i-1}^n \right] \right), \quad \mathcal{M}_k = \sum_{i=1}^{k} T_i.$$ 

Note that, from Lemma \[\square\] $E_{\theta_0}[||T_i||^2 | \mathcal{F}_{i-1}^n] = R_{i-1}(h_n^{-1}, \theta)$. It follows from Theorem 2.11 of Hall and Heyde (1980) that

$$\frac{1}{n^2} E_{\theta_0} \left[ \max_{1 \leq k \leq n} ||\mathcal{M}_k||^2 \right] \leq C \left( E_{\theta_0} \left[ \sum_{i=1}^{n} E_{\theta_0}[||T_i||^2 | \mathcal{F}_{i-1}^n] \right] + E_{\theta_0} \left[ \max_{1 \leq k \leq n} ||T_k||^2 \right] \right) \leq \frac{C'}{nh_n} \to 0.$$ 

Hence, we obtain

$$\frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0) (\Delta X_i)^e \right\| = \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0) (\Delta X_i)^e - E_{\theta_0} \left[ f_{i-1}(\theta_0) (\Delta X_i)^e \right] \right\| = o_p(1).$$

**Proof of (4.13).** We have, from Lemmas \[\square\],

$$\frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0) \mathbb{E}_{\theta_0} \left[ \frac{(\Delta X_i)^e}{h_n} | \mathcal{F}_{i-1}^n \right] - k \int_{\mathbb{R}^d} f(x, \theta_0) b^e(x, \beta_0) d\mu_{\theta_0}(x) \right\|$$

$$= \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0)(b_{i-1}^e(\beta_0) + R_{i-1}(h_n, \theta)) - k \int_{\mathbb{R}^d} f(x, \theta_0) b^e(x, \beta_0) d\mu_{\theta_0}(x) \right\|$$

$$\leq \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0)b_{i-1}^e(\beta_0) - k \int_{\mathbb{R}^d} f(x, \theta_0) b^e(x, \beta_0) d\mu_{\theta_0}(x) \right\|$$

$$+ \frac{1}{n} \max_{1 \leq k \leq n} \left\| R_{i-1}(h_n, \theta) \right\|$$

$$\leq \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} f_{i-1}(\theta_0)b_{i-1}^e(\beta_0) - \frac{k}{n} \sum_{i=1}^{n} f_{i-1}(\theta_0)b_{i-1}^e(\beta_0) \right\|$$

$$+ \frac{1}{n} \max_{1 \leq k \leq n} \left\| \frac{k}{n} \sum_{i=1}^{n} f_{i-1}(\theta_0)b_{i-1}^e(\beta_0) - k \int_{\mathbb{R}^d} f(x, \theta_0) b^e(x, \beta_0) d\mu_{\theta_0}(x) \right\| + o_p(1)$$
Therefore, it is enough to show

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} f_i(\theta_0) b_{i-1}(\beta_0) - \frac{k}{n} \sum_{i=1}^{n} f_i(\theta_0) b_{i-1}(\beta_0) \right\| \\
+ \left\| \frac{1}{n} \sum_{i=1}^{n} f_i(\theta_0) b_{i-1}(\beta_0) - \int_{\mathbb{R}^d} f(x, \theta_0) b_0(x, \beta_0) d\mu_{\theta_0}(x) \right\| + o_p(1)
\]

= o_p(1).

This completes the proof of (4.10). In the same way, we have (4.11).

\[
\textbf{Proof of Theorem 11} \text{ Let }
\eta_i = \text{tr} \left( A_{i-1}^{-1}(\alpha_0) \frac{(\Delta X_i)^{\otimes 2}}{h_n} \right) = \sum_{\ell_1, \ell_2 = 1}^{d} \left( A_{i-1}^{-1}(\alpha_0) \right)^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n}.
\]

By Taylor expansion, we have

\[
(A_{i-1}^{-1}(\tilde{\alpha}_n))^{\ell_1, \ell_2} = (A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2} + \partial_\alpha (A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2} (\tilde{\alpha}_n - \alpha_0) + (\tilde{\alpha}_n - \alpha_0)^T A_{i-1}^{\ell_1, \ell_2} (\tilde{\alpha}_n - \alpha_0),
\]

where

\[
A_{i-1}^{\ell_1, \ell_2} = \int_0^1 (1 - u) \partial^2_\alpha (A_{i-1}^{-1}(\alpha_0 + u(\tilde{\alpha}_n - \alpha_0)))^{\ell_1, \ell_2} du.
\]

Then, we can express

\[
\hat{\eta}_i = \sum_{\ell_1, \ell_2 = 1}^{d} \left( A_{i-1}^{-1}(\tilde{\alpha}_n) \right)^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n}
\]

\[
= \sum_{\ell_1, \ell_2 = 1}^{d} \left( (A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2} + \partial_\alpha (A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2} (\tilde{\alpha}_n - \alpha_0) + (\tilde{\alpha}_n - \alpha_0)^T A_{i-1}^{\ell_1, \ell_2} (\tilde{\alpha}_n - \alpha_0) \right) \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n}
\]

\[
= \eta_i + \left( \frac{1}{\sqrt{n}} \sum_{\ell_1, \ell_2 = 1}^{d} \partial_\alpha (A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} \right) \sqrt{n}(\tilde{\alpha}_n - \alpha_0)
\]

\[
+ \left( (\sqrt{n}(\tilde{\alpha}_n - \alpha_0))^T \left( \frac{1}{n} \sum_{\ell_1, \ell_2 = 1}^{d} A_{i-1}^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} \right) \right) \sqrt{n}(\tilde{\alpha}_n - \alpha_0)
\]

\[
= \eta_i + \frac{1}{\sqrt{n}} H_{1,i} \left( \sqrt{n}(\tilde{\alpha}_n - \alpha_0) \right) + \frac{1}{n} \left( \sqrt{n}(\tilde{\alpha}_n - \alpha_0) \right)^T H_{2,i} \left( \sqrt{n}(\tilde{\alpha}_n - \alpha_0) \right).
\]

Therefore, it is enough to show

\[
\frac{1}{\sqrt{2dn}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \eta_i - \frac{k}{n} \sum_{i=1}^{n} \eta_i \right\| \xrightarrow{d} \sup_{0 \leq s \leq 1} |B_1^0(s)|,
\]

(4.14)

\[
\frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} H_{1,i} - \frac{k}{n} \sum_{i=1}^{n} H_{1,i} \right\| = o_p(1),
\]

(4.15)

\[
\frac{1}{n^{3/2}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} H_{2,i} - \frac{k}{n} \sum_{i=1}^{n} H_{2,i} \right\| = o_p(1).
\]

(4.16)

\[
\textbf{Proof of (4.14)} \text{ Let } B_d(t) \text{ be a } d\text{-dimensional standard Brownian motion. } X_n(\cdot) \xrightarrow{w} X(\cdot) \text{ in } \mathbb{D}[0,1] \text{ denotes that } X_n(\cdot) \text{ weakly converges to } X(\cdot) \text{ in the Skorohod space on } [0,1]. \text{ If we prove }
\]

\[
U_n(s) := \frac{1}{\sqrt{2dn}} \sum_{i=1}^{\lceil ns \rceil} (\eta_i - d) \xrightarrow{w} B_1^1(s) \text{ in } \mathbb{D}[0,1],
\]

(4.17)

then, it follows from the continuous mapping theorem that

\[
\frac{1}{\sqrt{2dn}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \eta_i - \frac{k}{n} \sum_{i=1}^{n} \eta_i \right\| = \frac{1}{\sqrt{2dn}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} (\eta_i - d) - \frac{k}{n} \sum_{i=1}^{n} (\eta_i - d) \right\|.
\]
Hence (4.21) holds. Since
\[ \sum_{i=1}^{k} (q_i - d) - \frac{k}{n} \frac{1}{\sqrt{2dn}} \sum_{i=1}^{n} (\eta_i - d) \]
\[ = \max_{1 \leq k \leq n} \left| \frac{1}{\sqrt{2dn}} \sum_{i=1}^{k} (q_i - d) - \frac{k}{n} \frac{1}{\sqrt{2dn}} \sum_{i=1}^{n} (\eta_i - d) \right| \]
\[ = \sup_{0 \leq s \leq 1} \left| \frac{1}{\sqrt{2dn}} \sum_{i=1}^{[ns]} (q_i - d) - \frac{[ns]}{n} \frac{1}{\sqrt{2dn}} \sum_{i=1}^{n} (\eta_i - d) \right| \]
\[ = \sup_{0 \leq s \leq 1} \left| \mathcal{U}_n(s) - \frac{[ns]}{n} \mathcal{U}_n(1) \right| \]
\[ \Rightarrow \sup_{0 \leq s \leq 1} \left| B_1(s) - s B_1(1) \right| = \sup_{0 \leq s \leq 1} |B_1^0(s)|. \]

Therefore we obtain (4.22). This completes the proof of (4.14).

Let us prove (4.17). It is enough to show
\[ \frac{1}{\sqrt{2dn}} \sum_{i=1}^{[ns]} (\eta_i - d - \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n]) \xrightarrow{w} B_1(s) \text{ in } \mathbb{D}[0,1], \tag{4.18} \]
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n] = o_p(1). \tag{4.19} \]

From Lemma [1]
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_{i-1}(h_n, \theta) = \sqrt{n h_n^2} \frac{1}{n} \sum_{i=1}^{n} R_{i-1}(1, \theta) = o_p(1). \]

Hence we have (4.19).

According to Corollary 3.8 of McLeish (1974), we obtain (4.18) if we prove
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \mathbb{E}_{\theta_0} \left[ (\eta_i - d - \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n]) \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n] \right] = o_p(1), \tag{4.20} \]
\[ \frac{1}{2dn} \sum_{i=1}^{[ns]} \mathbb{E}_{\theta_0} \left[ (\eta_i - d - \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n])^2 | \mathcal{F}_{i-1}^n \right] \xrightarrow{p} s, \tag{4.21} \]
\[ \frac{1}{n^2} \sum_{i=1}^{[ns]} \mathbb{E}_{\theta_0} \left[ (\eta_i - d - \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n])^4 | \mathcal{F}_{i-1}^n \right] = o_p(1) \tag{4.22} \]

for all $s \in [0,1]$.

(4.20) is obvious. From Lemma [1] we have
\[ \mathbb{E}_{\theta_0} \left[ (\eta_i - d - \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n])^2 | \mathcal{F}_{i-1}^n \right] = \mathbb{E}_{\theta_0} \left[ (\eta_i - d + R_{i-1}(h_n, \theta))^2 | \mathcal{F}_{i-1}^n \right] \]
\[ = \mathbb{E}_{\theta_0} \left[ \eta_i^2 - 2d \eta_i + d^2 + R_{i-1}(h_n, \theta) | \mathcal{F}_{i-1}^n \right] \]
\[ = \mathbb{E}_{\theta_0} \left[ \eta_i^2 | \mathcal{F}_{i-1}^n \right] - 2d \mathbb{E}_{\theta_0} \left[ \eta_i | \mathcal{F}_{i-1}^n \right] + d^2 + R_{i-1}(h_n, \theta) \]
\[ = 2d + R_{i-1}(h_n, \theta) \]
and
\[ \frac{1}{2dn} \sum_{i=1}^{[ns]} \mathbb{E}_{\theta_0} \left[ (\eta_i - d - \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n])^2 | \mathcal{F}_{i-1}^n \right] = \frac{[ns]}{2d} \frac{1}{n} \sum_{i=1}^{[ns]} (2d + R_{i-1}(h_n, \theta)) \xrightarrow{p} s. \]

Hence (4.21) holds. Since $\mathbb{E}_{\theta_0} [\eta_i^4 | \mathcal{F}_{i-1}^n] = R_{i-1}(1, \theta)$,
\[ \frac{1}{n^2} \sum_{i=1}^{[ns]} \mathbb{E}_{\theta_0} \left[ (\eta_i - d - \mathbb{E}_{\theta_0} [\eta_i - d | \mathcal{F}_{i-1}^n])^4 | \mathcal{F}_{i-1}^n \right] \leq \frac{C}{n^2} \sum_{i=1}^{[ns]} \mathbb{E}_{\theta_0} \left[ \eta_i^4 + d^4 + R_{i-1}(h_n^4, \theta) | \mathcal{F}_{i-1}^n \right] \]
\[ = \frac{1}{n} \cdot \frac{1}{n} \sum_{i=1}^{[ns]} R_{i-1}(1, \theta) = o_p(1). \]

Therefore we obtain (4.22). This completes the proof of (4.14).
Proof of (4.15). Noting that
\[ H_{1,i} = \sum_{\ell_1, \ell_2 = 1}^{d} \partial_\alpha \left( A_{i-1}^{-1}(\alpha_0) \right)^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n}, \]
we have, by using Lemma [5]
\[
\frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} H_{1,i} - \sum_{i=1}^{n} H_{1,i} \right\| \\
\leq \sum_{\ell_1, \ell_2 = 1}^{d} \frac{1}{nh_n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \partial_\alpha \left( A_{i-1}^{-1}(\alpha_0) \right)^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} - \frac{k}{n} \sum_{i=1}^{n} \partial_\alpha \left( A_{i-1}^{-1}(\alpha_0) \right)^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} \right\| \\
= o_p(1).
\]

Proof of (4.16). Since \( \alpha_0 \in \text{Int} \Theta_A \), there exists an open neighborhood \( O_{\alpha_0} \) of \( \alpha_0 \) such that \( O_{\alpha_0} \subset \Theta_A \). If \( \hat{\alpha}_n \in O_{\alpha_0} \), then
\[
\| H_{2,i} \| = \left\| \sum_{\ell_1, \ell_2 = 1}^{d} A_{i-1}^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} \right\| \leq \sum_{\ell_1, \ell_2 = 1}^{d} \sup_{\alpha \in \Theta_A} \left\| \partial_\alpha^2 \left( A_{i-1}^{-1}(\alpha) \right)^{\ell_1, \ell_2} \right\| \frac{|(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} \right\|
\]
and
\[
E_{\theta_0} \left[ \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2 = 1}^{d} \sup_{\alpha \in \Theta_A} \left\| \partial_\alpha^2 \left( A_{i-1}^{-1}(\alpha) \right)^{\ell_1, \ell_2} \right\| \frac{|(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} \right] \\
\leq \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2 = 1}^{d} E_{\theta_0} \left[ \sup_{\alpha \in \Theta_A} \left\| \partial_\alpha^2 \left( A_{i-1}^{-1}(\alpha) \right)^{\ell_1, \ell_2} \right\|^2 \right]^{1/2} \left[ \left( \frac{|(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} \right) \right]^{1/2} \\
\leq \frac{C}{n^{1/2}} \rightarrow 0.
\]
Hence, from [A6], we have \( P_{\theta_0} (\hat{\alpha}_n \in O_{\alpha_0}) \rightarrow 1 \) and, for all \( \epsilon > 0 \),
\[
P_{\theta_0} \left( \frac{1}{n^{3/2}} \sum_{i=1}^{n} \| H_{2,i} \| \geq \epsilon \right) \\
\leq P_{\theta_0} \left( \frac{1}{n^{3/2}} \sum_{i=1}^{n} \| H_{2,i} \| \geq \epsilon \cap \{ \hat{\alpha}_n \in O_{\alpha_0} \} \right) + P_{\theta_0} \left( \frac{1}{n^{3/2}} \sum_{i=1}^{n} \| H_{2,i} \| \geq \epsilon \cap \{ \hat{\alpha}_n \notin O_{\alpha_0} \} \right) \\
\leq P_{\theta_0} \left( \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2 = 1}^{d} \sup_{\alpha \in \Theta_A} \left\| \partial_\alpha^2 \left( A_{i-1}^{-1}(\alpha) \right)^{\ell_1, \ell_2} \right\| \frac{|(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} \right] \geq \epsilon \cap \{ \hat{\alpha}_n \in O_{\alpha_0} \} \\
+ P_{\theta_0} (\hat{\alpha}_n \notin O_{\alpha_0}) \\
\leq P_{\theta_0} \left( \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2 = 1}^{d} \sup_{\alpha \in \Theta_A} \left\| \partial_\alpha^2 \left( A_{i-1}^{-1}(\alpha) \right)^{\ell_1, \ell_2} \right\| \frac{|(\Delta X_i)^{\ell_1}(\Delta X_i)^{\ell_2}}{h_n} \right] \geq \epsilon \right) \\
\leq 1 \frac{C}{\epsilon n^{1/2}} + P_{\theta_0} (\hat{\alpha}_n \notin O_{\alpha_0}) \rightarrow 0.
\]
Therefore we obtain
\[
\frac{1}{n^{3/2}} \sum_{i=1}^{n} \| H_{2,i} \| = o_p(1)
\]
and
\[
\frac{1}{n^{3/2}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} H_{2,i} - \frac{k}{n} \sum_{i=1}^{n} H_{2,i} \right\| \leq \frac{1}{n^{3/2}} \max_{1 \leq k \leq n} \left( \sum_{i=1}^{k} \| H_{2,i} \| + \frac{k}{n} \sum_{i=1}^{n} \| H_{2,i} \| \right) \\
\leq 2 \frac{n^{3/2}}{n^{3/2}} \sum_{i=1}^{n} \| H_{2,i} \| = o_p(1).
\]
Proof of Theorem 2. Let
\[ \kappa(x, \alpha) = 1^T a^{-1}(x, \alpha), \quad \xi_i = \kappa_{i-1}(\alpha_0)(\Delta X_i - h_n b_{i-1}(\beta_0)). \]

By the Taylor expansion, we have
\[ \kappa_{i-1}^\ell(\alpha_n) = \kappa_{i-1}(\alpha_0) + \partial_\alpha \kappa_{i-1}^\ell(\alpha_0)(\hat{\alpha}_n - \alpha_0) + (\hat{\alpha}_n - \alpha_0)^T \kappa_i^\ell(\hat{\alpha}_n - \alpha_0), \]

where
\[ \kappa_i^\ell = \int_0^1 (1-u) \partial^2_\alpha \kappa_{i-1}^\ell(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)) du. \]

Then, we can express
\[
\hat{\xi}_i = \kappa_{i-1}(\hat{\alpha}_n) \left( \Delta X_i - h_n b_{i-1}(\hat{\beta}_n) \right) \\
= \sum_{\ell=1}^d \kappa_{i-1}^\ell(\hat{\alpha}_n) \left( \Delta X_i - h_n b_{i-1}(\hat{\beta}_n) \right)^\ell \\
= \sum_{\ell=1}^d \left( \kappa_{i-1}^\ell(\alpha_0) + \partial_\alpha \kappa_{i-1}^\ell(\alpha_0)(\hat{\alpha}_n - \alpha_0) + (\hat{\alpha}_n - \alpha_0)^T \kappa_i^\ell(\hat{\alpha}_n - \alpha_0) \right) \left( \Delta X_i - h_n b_{i-1}(\hat{\beta}_n) \right)^\ell \\
+ \left( \frac{1}{\sqrt{n}} \sum_{\ell=1}^d \partial_\alpha \kappa_{i-1}^\ell(\alpha_0) \left( \Delta X_i - h_n b_{i-1}(\hat{\beta}_n) \right)^\ell \right) \sqrt{n}(\hat{\alpha}_n - \alpha_0) \\
+ \left( \sqrt{n}(\hat{\alpha}_n - \alpha_0)^T \left( \frac{1}{n} \sum_{\ell=1}^d \kappa_i^\ell \left( \Delta X_i - h_n b_{i-1}(\hat{\beta}_n) \right)^\ell \right) \sqrt{n}(\hat{\alpha}_n - \alpha_0) \right) \\
= \xi_i - \left( \frac{1}{\sqrt{n}} \sum_{\ell=1}^d \kappa_{i-1}^\ell(\alpha_0) \partial_\beta b_{i-1}^\ell(\beta_0) \right) \sqrt{n h_n}(\hat{\beta}_n - \beta_0) \\
- \left( \sqrt{n h_n}(\hat{\beta}_n - \beta_0) \right)^T \left( \frac{1}{n} \sum_{\ell=1}^d \kappa_i^\ell \left( \Delta X_i - h_n b_{i-1}(\hat{\beta}_n) \right)^\ell \right) \sqrt{n h_n}(\hat{\beta}_n - \beta_0) \\
+ \left( \frac{1}{\sqrt{n}} \sum_{\ell=1}^d \partial_\alpha \kappa_{i-1}^\ell(\alpha_0) \left( \Delta X_i \right)^\ell \right) \sqrt{n}(\hat{\alpha}_n - \alpha_0) - \left( \frac{h_n}{\sqrt{n}} \sum_{\ell=1}^d \partial_\alpha \kappa_{i-1}^\ell(\alpha_0) b_{i-1}^\ell(\beta_n) \right) \sqrt{n}(\hat{\alpha}_n - \alpha_0) \\
+ \left( \sqrt{n}(\hat{\alpha}_n - \alpha_0)^T \left( \frac{1}{n} \sum_{\ell=1}^d \kappa_i^\ell \left( \Delta X_i - h_n b_{i-1}(\hat{\beta}_n) \right)^\ell \right) \sqrt{n}(\hat{\alpha}_n - \alpha_0) \right). \\
\]

Therefore, it is enough to show
\[
\frac{1}{\sqrt{d n h_n}} \max_{1 \leq \ell \leq k} \sum_{i=1}^k \xi_i - \frac{k}{n} \sum_{i=1}^n \xi_i \xrightarrow{d} \sup_{a \leq s \leq 1} |B_1^0(s)|, \tag{4.23}
\]
\[
\frac{1}{n} \max_{1 \leq \ell \leq k} \left\| Q_{1,i} - \frac{k}{n} \sum_{i=1}^n Q_{1,i} \right\| = o_p(1), \tag{4.24}
\]
\[
\frac{1}{n \sqrt{n h_n}} \max_{1 \leq \ell \leq k} \left\| Q_{2,i} - \frac{k}{n} \sum_{i=1}^n Q_{2,i} \right\| = o_p(1), \tag{4.25}
\]
\[
\frac{1}{n^{1/k}} \max_{1 \leq i \leq n} \left| \sum_{i=1}^{n} Q_{3,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{3,i} \right| = o_p(1), \quad \text{(4.26)}
\]
\[
\frac{1}{n^{1/k}} \max_{1 \leq i \leq n} \left| \sum_{i=1}^{k} Q_{4,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{4,i} \right| = o_p(1), \quad \text{(4.27)}
\]
\[
\frac{1}{n^{1/k}} \max_{1 \leq i \leq n} \left| \sum_{i=1}^{k} Q_{5,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{5,i} \right| = o_p(1). \quad \text{(4.28)}
\]

Proof of (4.23). If we prove
\[
\mathcal{V}_n(s) := \frac{1}{\sqrt{dh_n}} \sum_{i=1}^{[ns]} \xi_i \xrightarrow{w} B_1(s) \quad \text{in } D[0,1], \quad \text{(4.29)}
\]
we see from the continuous mapping theorem that
\[
\frac{1}{\sqrt{dh_n}} \max_{1 \leq i \leq n} \left| \sum_{i=1}^{k} \xi_i - \frac{k}{n} \sum_{i=1}^{n} \xi_i \right| = \max_{1 \leq k \leq n} \left| \frac{1}{\sqrt{dh_n}} \sum_{i=1}^{[ns]} \xi_i - \frac{1}{n} \sqrt{dh_n} \sum_{i=1}^{n} \xi_i \right|
\]
\[
= \sup_{0 \leq s \leq 1} \left| \mathcal{V}_n(s) - \frac{[ns]}{n} \mathcal{V}_n(1) \right|
\]
\[
\xrightarrow{d} \sup_{0 \leq s \leq 1} \left| B_1(s) - sB_1(1) \right| = \sup_{0 \leq s \leq 1} \left| B_1^0(s) \right|.
\]

Let us prove (4.29). It is enough to prove
\[
\frac{1}{\sqrt{dh_n}} \sum_{i=1}^{[ns]} \left( \xi_i - \mathbb{E}_{\theta_0} \left[ \xi_i | \mathcal{G}_{i-1}^n \right] \right) \xrightarrow{w} B_1(s) \quad \text{in } D[0,1], \quad \text{(4.30)}
\]
\[
\frac{1}{\sqrt{nh_n}} \sum_{i=1}^{n} \mathbb{E}_{\theta_0} \left[ \xi_i | \mathcal{G}_{i-1}^n \right] = o_p(1). \quad \text{(4.31)}
\]

From Lemma 1
\[
\frac{1}{\sqrt{nh_n}} \sum_{i=1}^{n} \mathbb{E}_{\theta_0} \left[ \xi_i | \mathcal{G}_{i-1}^n \right] = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^{n} R_{-1}(h_n^2, \theta) = \sqrt{nh_n^3} \cdot \frac{1}{n} \sum_{i=1}^{n} R_{-1}(1, \theta) = o_p(1).
\]

Hence we have (4.31).

It follows from Corollary 3.8 of McLeish (1974) that one has (4.30) if we prove
\[
\frac{1}{\sqrt{nh_n}} \sum_{i=1}^{[ns]} \left[ \mathbb{E}_{\theta_0} \left[ \xi_i - \mathbb{E}_{\theta_0} \left[ \xi_i | \mathcal{G}_{i-1}^n \right] \right] \right] = o_p(1), \quad \text{(4.32)}
\]
\[
\frac{1}{dh_n} \sum_{i=1}^{[ns]} \left[ \mathbb{E}_{\theta_0} \left[ \xi_i - \mathbb{E}_{\theta_0} \left[ \xi_i | \mathcal{G}_{i-1}^n \right] \right]^2 \right] \xrightarrow{p} s, \quad \text{(4.33)}
\]
\[
\frac{1}{nh_n^2} \sum_{i=1}^{[ns]} \left[ \mathbb{E}_{\theta_0} \left[ \xi_i - \mathbb{E}_{\theta_0} \left[ \xi_i | \mathcal{G}_{i-1}^n \right] \right] \right] = o_p(1), \quad \text{(4.34)}
\]

for all \( s \in [0,1] \).

(4.32) is trivial. From Lemma 1
\[
\frac{1}{dh_n} \sum_{i=1}^{[ns]} \left[ \mathbb{E}_{\theta_0} \left[ \xi_i - \mathbb{E}_{\theta_0} \left[ \xi_i | \mathcal{G}_{i-1}^n \right] \right]^2 \right] = \frac{1}{nh_n} \sum_{i=1}^{[ns]} \left( \mathbb{E}_{\theta_0} \left[ \xi_i^2 | \mathcal{G}_{i-1}^n \right] + R_{-1}(h_n^2, \theta) \right)
\]
\[
= \frac{[ns]}{n} \frac{1}{d[ns]h_n} \sum_{i=1}^{[ns]} \left( dh_n + R_{-1}(h_n^2, \theta) \right) \xrightarrow{p} s,
\]
i.e., (4.33) holds. Since $E_{0} [\xi | \varphi_{-1}^{n}] = R_{i-1}(h_{n}^{2}, \theta)$,

$$
\frac{1}{(nh_{n})^{2}} \sum_{i=1}^{[ns]} E_{0} \left[ \left( \xi_{i} - E_{0} [\xi | \varphi_{-1}^{n}] \right)^{4} \right] \leq \frac{C}{(nh_{n})^{2}} \sum_{i=1}^{[ns]} E_{0} \left[ \xi_{i}^{4} + R_{i-1}(h_{n}^{2}, \theta) | \varphi_{i-1}^{n} \right]
$$

$$
= \frac{1}{(nh_{n})^{2}} \sum_{i=1}^{[ns]} R_{i-1}(h_{n}^{2}, \theta)
$$

$$
= \frac{1}{n} \sum_{i=1}^{[ns]} R_{i-1}(1, \theta) = o_{p}(1).
$$

Therefore we obtain (4.34). This completes the proof of (4.23).

Proofs of (4.24) and (4.26). Noting that

$$
Q_{1,i} = - \sum_{\ell=1}^{d} \kappa_{i-1}(\alpha_{0}) \partial_{\beta} b_{i-1}^{\ell}(\beta_{0}),
$$

$$
Q_{3,i} = \sum_{\ell=1}^{d} \partial_{\alpha} \kappa_{i-1}(\alpha_{0})(\Delta X_{i})^{\ell},
$$

we have, by using Lemma 4 and 5

$$
\frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} Q_{1,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{1,i} \right\|
$$

$$
\leq \sum_{\ell=1}^{d} \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \kappa_{i-1}(\alpha_{0}) \partial_{\beta} b_{i-1}^{\ell}(\beta_{0}) - \frac{k}{n} \sum_{i=1}^{n} \kappa_{i-1}(\alpha_{0}) \partial_{\beta} b_{i-1}^{\ell}(\beta_{0}) \right\|
$$

$$
= o_{p}(1)
$$

and

$$
\frac{1}{n \sqrt{h_{n}}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{n} Q_{3,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{3,i} \right\|
$$

$$
\leq \sum_{\ell=1}^{d} \frac{1}{n \sqrt{h_{n}}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{n} \partial_{\alpha} \kappa_{i-1}(\alpha_{0})(\Delta X_{i})^{\ell} - \frac{k}{n} \sum_{i=1}^{n} \partial_{\alpha} \kappa_{i-1}(\alpha_{0})(\Delta X_{i})^{\ell} \right\|
$$

$$
= o_{p}(1).
$$

Proofs of (4.25), (4.27) and (4.28). Since $\theta_{0} \in \text{Int} \Theta$, there exists an open neighborhood $O_{\theta_{0}}$ of $\theta_{0}$ such that $O_{\theta_{0}} \subset \Theta$. Note that

$$
Q_{2,i} = \sum_{\ell=1}^{d} \kappa_{i-1}(\alpha_{0}) \int_{0}^{1} (1 - u) \partial_{\beta}^{2} b_{i-1}^{\ell}(\beta_{0} + u(\hat{\beta}_{n} - \beta_{0})) du,
$$

$$
Q_{4,i} = - \sum_{\ell=1}^{d} \partial_{\alpha} \kappa_{i-1}(\alpha_{0}) b_{i-1}(\hat{\beta}_{n}),
$$

$$
Q_{5,i} = \sum_{\ell=1}^{d} \kappa_{i}^{\ell} \left( \Delta X_{i} - h_{n} b_{i-1}(\hat{\beta}_{n}) \right)^{\ell}
$$

$$
= \sum_{\ell=1}^{d} \int_{0}^{1} (1 - u) \partial_{\alpha}^{2} \kappa_{i-1}(\alpha_{0} + u(\hat{\alpha}_{n} - \alpha_{0})) du \left( \Delta X_{i} - h_{n} b_{i-1}(\hat{\beta}_{n}) \right)^{\ell}.
$$

If $\hat{\beta}_{n} \in O_{\theta_{0}}$, then

$$
\| Q_{2,i} \| \leq \sum_{\ell=1}^{d} \sup_{\alpha \in \Theta_{x}} | \kappa_{i-1}(\alpha) | \sup_{\beta \in \Theta_{x}} \left\| \partial_{\beta}^{2} b_{i-1}^{\ell}(\beta) \right\|,
$$

$$
\| Q_{4,i} \| \leq \sum_{\ell=1}^{d} \sup_{\alpha \in \Theta_{x}} \| \partial_{\alpha} \kappa_{i-1}(\alpha) \| \sup_{\beta \in \Theta_{x}} \| b_{i-1}^{\ell}(\beta) \|,
$$

$$
\| Q_{5,i} \| \leq \sum_{\ell=1}^{d} \left( \Delta X_{i} - h_{n} b_{i-1}(\hat{\beta}_{n}) \right)^{\ell}.
$$
Hence, in the same way as the proof of (4.16), we see from Proof of Theorem 3.

Then, we can express

\[
C_0 = A_i - \sqrt{n} \to 0.
\]

and

\[
E_{\theta_0} \left[ \frac{1}{n\sqrt{nh_n}} \sum_{i=1}^{d} \sup_{\alpha \in \Theta_A} \left\| \partial_t^2 \kappa_i^\alpha \right\| \left( \left| (\Delta X_t)^\alpha \right| + h_n \sup_{\beta \in \Theta_B} \left\| b_{i-1}^\beta \right\| \right) \right]
\]

\[
\leq \frac{C}{n\sqrt{nh_n}} \sum_{i=1}^{d} \sup_{\alpha \in \Theta_A} \left\| \partial_t^2 \kappa_i^\alpha \right\| \left( \left| (\Delta X_t)^\alpha \right| + h_n \sup_{\beta \in \Theta_B} \left\| b_{i-1}^\beta \right\| \right)^{1/2}
\]

\[
\leq C' \to 0.
\]

Hence, in the same way as the proof of (4.16), we see from [A8] that

\[
\frac{1}{n\sqrt{nh_n}} \sum_{i=1}^{n} \| Q_{2,i} \| = o_p(1), \quad \frac{1}{n\sqrt{nh_n}} \sum_{i=1}^{n} \| Q_{4,i} \| = o_p(1), \quad \frac{1}{n\sqrt{nh_n}} \sum_{i=1}^{n} \| Q_{5,i} \| = o_p(1).
\]

**Proof of Theorem 3**

Let

\[
\zeta_i = \partial \beta b_i^{-1}(\beta_0)^T A_i^{-1}(\alpha_0) (\Delta X_i - h_n b_i^{-1}(\beta_0)).
\]

By the Taylor expansion, we have

\[
(A_i^{-1}(\hat{\alpha}_n))^{\ell_1,\ell_2} = (A_i^{-1}(\alpha_0))^{\ell_1,\ell_2} + \partial \alpha (A_i^{-1}(\alpha_0))^{\ell_1,\ell_2} (\hat{\alpha}_n - \alpha_0) + (\hat{\alpha}_n - \alpha_0)^T A_i^{-1}(\alpha_0) (\hat{\alpha}_n - \alpha_0),
\]

where

\[
A_i^{-1} = \int_0^1 (1-u) \partial^2 \left( A_i^{-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)) \right)^{\ell_1,\ell_2} du.
\]

Then, we can express

\[
\hat{\zeta}_t = \sum_{\ell_1,\ell_2=1}^{d} \partial t_t b_{i-1}^{\ell_1}(\hat{\beta}_n) \left( A_i^{-1}(\hat{\alpha}_n) \right)^{\ell_1,\ell_2} \left( \Delta X_i - h_n b_i^{-1}(\hat{\beta}_n) \right)_{\ell_1,\ell_2}
\]

\[
= \sum_{\ell_1,\ell_2=1}^{d} \partial t_t b_{i-1}^{\ell_1}(\hat{\beta}_n) \left( A_i^{-1}(\alpha_0) \right)^{\ell_1,\ell_2} \left( \Delta X_i - h_n b_i^{-1}(\hat{\beta}_n) \right)_{\ell_1,\ell_2}
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{\ell_1,\ell_2=1}^{d} \partial t_t b_{i-1}^{\ell_1}(\hat{\beta}_n) \partial \alpha \left( A_i^{-1}(\alpha_0) \right)^{\ell_1,\ell_2} \left( \Delta X_i - h_n b_i^{-1}(\hat{\beta}_n) \right)_{\ell_1,\ell_2} \sqrt{n}(\hat{\alpha}_n - \alpha_0)
\]

\[
+ \left( \sqrt{n}(\hat{\alpha}_n - \alpha_0) \right)^T \left( \frac{1}{n} \sum_{\ell_1,\ell_2=1}^{d} \partial t_t b_{i-1}^{\ell_1}(\hat{\beta}_n) A_i^{-1}(\alpha_0) \left( \Delta X_i - h_n b_i^{-1}(\hat{\beta}_n) \right)_{\ell_1,\ell_2} \right) \sqrt{n}(\hat{\alpha}_n - \alpha_0)
\]

\[
= \sum_{\ell_1,\ell_2=1}^{d} \partial t_t b_{i-1}^{\ell_1}(\hat{\beta}_n) \left( A_i^{-1}(\alpha_0) \right)^{\ell_1,\ell_2} \left( \Delta X_i - h_n b_i^{-1}(\beta_0) \right)_{\ell_1,\ell_2}
\]

\[
+ h_n \sum_{\ell_1,\ell_2=1}^{d} \partial t_t b_{i-1}^{\ell_1}(\hat{\beta}_n) \left( A_i^{-1}(\alpha_0) \right)^{\ell_1,\ell_2} \left( b_{i-1}^{\ell_2}(\beta_0) - b_{i-1}^{\ell_2}(\hat{\beta}_n) \right)
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{\ell_1,\ell_2=1}^{d} \partial t_t b_{i-1}^{\ell_1}(\hat{\beta}_n) \partial \alpha \left( A_i^{-1}(\alpha_0) \right)^{\ell_1,\ell_2} \left( \Delta X_i - h_n b_i^{-1}(\hat{\beta}_n) \right)_{\ell_1,\ell_2} \sqrt{n}(\hat{\alpha}_n - \alpha_0)
\]
\[
+ (\sqrt{n}(\hat{\alpha}_n - \alpha_0))^T \left( \frac{1}{n} \sum_{\ell_1, \ell_2=1}^d \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\hat{\beta}_n) A_{\ell_1-1}^{\ell_2, \ell_2} \left( \Delta X_i - h_n b_{\ell_1-1}(\hat{\beta}_n) \right) \right)^{\ell_2} \sqrt{n}(\hat{\alpha}_n - \alpha_0)
\]
\[=: J_1 + J_2 + J_3 + J_4.\]

We set
\[
A \otimes x^{\otimes k} = \sum_{\ell_1, \ldots, \ell_k=1}^q A^{\ell_1, \ldots, \ell_k} x^{\ell_1} \cdots x^{\ell_k}, \quad \text{for } A \in \mathbb{R}^q \otimes \cdots \otimes \mathbb{R}^q, \ x \in \mathbb{R}^q.
\]

Note that, by the Taylor expansion,
\[
\partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\hat{\beta}_n) = \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\beta_0) + \sum_{j=1}^{m-1} \frac{1}{(nh_n)^{j/2}} \partial_{\beta_\ell_1} \partial_{\beta_\ell_2} b_{\ell_1-1}^{\ell_2}(\beta_0) \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes j}
\]
\[\quad + \frac{1}{(nh_n)^{m/2}} B^{\ell_1, \ell_2}_{m, i-1} \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes m},\]

where
\[
B^{\ell_1, \ell_2}_{m, i-1} = \frac{1}{(m-1)!} \int_0^1 (1-u)^{m-1} \partial_{\beta_\ell_1} \partial_{\beta_\ell_2} b_{\ell_1-1}^{\ell_2}(\beta_0 + u(\hat{\beta}_n - \beta_0)) \, du,
\]
\[
\partial_{\beta_\ell_1} \partial_{\beta_\ell_2} b_{\ell_1-1}^{\ell_2}(x, \beta) = \left( \partial_{\beta_\ell_1} \partial_{\beta_\ell_2} b_{\ell_1-1}^{\ell_2}(x, \beta) \right)_{\ell_1, \ldots, \ell_k} \in \mathbb{R}^q \otimes \cdots \otimes \mathbb{R}^q.
\]

Then,
\[
J_1 = \sum_{\ell_1, \ell_2=1}^d \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\hat{\beta}_n) \left( A_{\ell_1-1}^{\ell_2, \ell_2}(\alpha_0) \right)^{\ell_1, \ell_2} \left( \Delta X_i - h_n b_{\ell_1-1}(\beta_0) \right)^{\ell_2}
\]
\[= \sum_{\ell_1, \ell_2=1}^d \left( \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\beta_0) + \sum_{j=1}^{m-1} \frac{1}{(nh_n)^{j/2}} \partial_{\beta_\ell_1} \partial_{\beta_\ell_2} b_{\ell_1-1}^{\ell_2}(\beta_0) \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes j}
\]
\[\quad + \frac{1}{(nh_n)^{m/2}} B^{\ell_1, \ell_2}_{m, i-1} \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes m} \right) \left( A_{\ell_1-1}^{\ell_2, \ell_2}(\alpha_0) \right)^{\ell_1, \ell_2} \left( \Delta X_i - h_n b_{\ell_1-1}(\beta_0) \right)^{\ell_2}
\]
\[=: \zeta^{\ell_1} + \sum_{j=1}^{m-1} \frac{1}{(nh_n)^{j/2}} Y^{\ell_1, \ell_2}_{j, i-1} \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes j} + \frac{1}{(nh_n)^{m/2}} Y^{\ell_1, \ell_2}_{m, i-1} \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes m}
\]

and
\[
J_2 = h_n \sum_{\ell_1, \ell_2=1}^d \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\hat{\beta}_n) \left( A_{\ell_1-1}^{\ell_2, \ell_2}(\alpha_0) \right)^{\ell_1, \ell_2} \left( b_{\ell_1-1}^{\ell_2}(\beta_0) - b_{\ell_1-1}^{\ell_2}(\hat{\beta}_n) \right)
\]
\[= h_n \sum_{\ell_1, \ell_2=1}^d \left( \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\beta_0) + \frac{1}{\sqrt{nh_n}} \partial_{\beta_\ell_1} \partial_{\beta_\ell_2} b_{\ell_1-1}^{\ell_2}(\beta_0) \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right) \right.
\]
\[\quad + \frac{1}{\sqrt{nh_n}} B^{\ell_1, \ell_2}_{2, i-1} \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes 2} \left( A_{\ell_1-1}^{\ell_2, \ell_2}(\alpha_0) \right)^{\ell_1, \ell_2} \left( b_{\ell_1-1}^{\ell_2}(\beta_0) - b_{\ell_1-1}^{\ell_2}(\hat{\beta}_n) \right)
\]
\[= h_n \sum_{\ell_1, \ell_2=1}^d \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\beta_0) \left( A_{\ell_1-1}^{\ell_2, \ell_2}(\alpha_0) \right)^{\ell_1, \ell_2} \left( b_{\ell_1-1}^{\ell_2}(\beta_0) - b_{\ell_1-1}^{\ell_2}(\hat{\beta}_n) \right)
\]
\[+ \frac{\sqrt{h_n}}{n} \left( \sum_{\ell_1, \ell_2=1}^d \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\beta_0) \left( A_{\ell_1-1}^{\ell_2, \ell_2}(\alpha_0) \right)^{\ell_1, \ell_2} \left( b_{\ell_1-1}^{\ell_2}(\beta_0) - b_{\ell_1-1}^{\ell_2}(\hat{\beta}_n) \right) \right) \sqrt{nh_n}(\hat{\beta}_n - \beta_0)
\]
\[+ \frac{1}{n} \left( \sum_{\ell_1, \ell_2=1}^d B^{\ell_1, \ell_2}_{2, i-1} \left( A_{\ell_1-1}^{\ell_2, \ell_2}(\alpha_0) \right)^{\ell_1, \ell_2} \left( b_{\ell_1-1}^{\ell_2}(\beta_0) - b_{\ell_1-1}^{\ell_2}(\hat{\beta}_n) \right) \right) \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes 2}
\]
\[= h_n \sum_{\ell_1, \ell_2=1}^d \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\beta_0) \left( A_{\ell_1-1}^{\ell_2, \ell_2}(\alpha_0) \right)^{\ell_1, \ell_2} \left( -\frac{1}{\sqrt{nh_n}} \partial_{\beta_\ell_1} b_{\ell_1-1}^{\ell_2}(\beta_0) \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)
\]
\[- \frac{1}{nh_n} \int_0^1 (1 - u) \partial_{\beta} b_{i_1}^{t_2} (\beta_0 + u(\hat{\beta}_n - \beta_0)) du \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes 2} \]

\[- \frac{1}{n} \left( \sum_{t_1, t_2 = 1}^d \partial_\beta \partial_\beta b_{i_1}^{t_2} (\beta_0 + u(\hat{\beta}_n - \beta_0)) \right) \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes 2} \]

\[+ \frac{1}{n} \left( \sum_{t_1, t_2 = 1}^d B_{i_1}^{T t_2} (A_{i_1}^{-1}(\alpha_0))^{t_1, t_2} b_{i_1}^{t_2} (\hat{\beta}_n - \beta_0) \right) \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes 2} \]

\[= \frac{\sqrt{h_n}}{n} Z_{i_1}^{\ell_1} \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right) + \frac{1}{n} Z_{i_1}^{\ell_1} \otimes \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^{\otimes 2}. \]

Moreover,

\[J_3 \equiv \frac{1}{\sqrt{n}} \sum_{t_1, t_2 = 1}^d \left( \partial_\beta b_{i_1}^{t_2} (\beta_0) + \frac{1}{\sqrt{nh_n}} \int_0^1 \partial_\beta \partial_\beta b_{i_1}^{t_2} (\beta_0 + u(\hat{\beta}_n - \beta_0)) du \right) \sqrt{\beta_0} \alpha_0 \]

\[\cdot \partial_\alpha (A_{i_1}^{-1}(\alpha_0))^{t_1, t_2} \left( \Delta X_i - h_n b_{i_1} (\hat{\beta}_n) \right)^{t_2} \sqrt{n}(\hat{\alpha}_n - \alpha_0) \]

\[= \frac{1}{\sqrt{n}} \sum_{t_1, t_2 = 1}^d \partial_\beta b_{i_1}^{t_2} (\beta_0) \partial_\alpha (A_{i_1}^{-1}(\alpha_0))^{t_1, t_2} (\Delta X_i)^{t_2} \sqrt{n}(\hat{\alpha}_n - \alpha_0) \]

\[- \frac{h_n}{\sqrt{n}} \sum_{t_1, t_2 = 1}^d \partial_\beta b_{i_1}^{t_2} (\beta_0) \partial_\alpha (A_{i_1}^{-1}(\alpha_0))^{t_1, t_2} b_{i_1} (\hat{\beta}_n) \sqrt{n}(\hat{\alpha}_n - \alpha_0) \]

\[+ \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right) \left( \frac{1}{n} \sum_{t_1, t_2 = 1}^d \int_0^1 \partial_\beta \partial_\beta b_{i_1}^{t_2} (\beta_0 + u(\hat{\beta}_n - \beta_0)) du \right) \]

\[\cdot \partial_\alpha (A_{i_1}^{-1}(\alpha_0))^{t_1, t_2} (\Delta X_i - h_n b_{i_1} (\hat{\beta}_n))^\alpha \sqrt{n}(\hat{\alpha}_n - \alpha_0) \]

\[= \frac{1}{\sqrt{n}} Z_{i_1}^{\ell_1} \left( \sqrt{n}(\hat{\alpha}_n - \alpha_0) \right) + \frac{h_n}{\sqrt{n}} Z_{i_1}^{\ell_1} \left( \sqrt{n}(\hat{\alpha}_n - \alpha_0) \right) \]

\[+ \frac{1}{n} \left( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \right)^T Z_{i_1}^{\ell_1} \left( \sqrt{n}(\hat{\alpha}_n - \alpha_0) \right), \]

\[J_4 \equiv \frac{1}{n} \left( \sqrt{\beta_0} \alpha_0 \right) Z_{i_1}^{\ell_1} \left( \sqrt{n}(\hat{\alpha}_n - \alpha_0) \right). \]

Note that \( I_n \xrightarrow{p} I \) and

\[T_{\beta_n} = \frac{1}{\sqrt{nh_n}} \max_{1 \leq k \leq n} \left\| \left( \sum_{i=1}^k \hat{z}_i - \frac{k}{n} \sum_{i=1}^n \hat{z}_i \right) \right\| \leq \frac{1}{\sqrt{nh_n}} \max_{1 \leq k \leq n} \left\| \left( I_n^{-1/2} T_{\beta_n}^{1/2} \right) I_n^{-1/2} \left( \sum_{i=1}^k \hat{z}_i - \frac{k}{n} \sum_{i=1}^n \hat{z}_i \right) \right\|. \]

Therefore, it is enough to show

\[\frac{1}{\sqrt{nh_n}} \max_{1 \leq k \leq n} \left\| \left( I_n^{-1/2} T_{\beta_n}^{1/2} \right) I_n^{-1/2} \left( \sum_{i=1}^k \hat{z}_i - \frac{k}{n} \sum_{i=1}^n \hat{z}_i \right) \right\| \xrightarrow{\text{d}} \sup_{0 \leq s \leq 1} \| B_{\beta_n}^0 (s) \|, \]  

\[\frac{1}{(nh_n)^{(j+1)/2}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k \gamma_{i, j} - \frac{k}{n} \sum_{i=1}^n \gamma_{i, j} \right\| = o_p(1), \quad (1 \leq j \leq m - 1) \]  

\[\frac{1}{(nh_n)^{(m+1)/2}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k \gamma_{i, m} - \frac{k}{n} \sum_{i=1}^n \gamma_{i, m} \right\| = o_p(1), \]  

\[\frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k \zeta_{i, 1} - \frac{k}{n} \sum_{i=1}^n \zeta_{i, 1} \right\| = o_p(1), \]  

\[\frac{1}{n \sqrt{nh_n}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k \zeta_{i, 2} - \frac{k}{n} \sum_{i=1}^n \zeta_{i, 2} \right\| = o_p(1). \]
\[
\frac{1}{n^{3/2}h_n} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} Z_{i,i}^{l} - \frac{k}{n} \sum_{i=1}^{n} Z_{i,i}^{l} \right| = o_p(1),
\]
(4.43)

Proof of (4.35). If we prove
\[
W_n(s) := \frac{1}{\sqrt{nh_n}} \sum_{i=1}^{[ns]} I^{-1/2} \zeta_i \xrightarrow{w} B_q(s) \quad \text{in } D[0,1],
\]
(4.44)
then, it follows from the continuous mapping theorem that
\[
\frac{1}{\sqrt{nh_n}} \max_{1 \leq k \leq n} \left| I^{-1/2} \left( \sum_{i=1}^{k} \zeta_i - \frac{k}{n} \sum_{i=1}^{n} \zeta_i \right) \right| = \max_{1 \leq k \leq n} \left| \frac{1}{\sqrt{nh_n}} \sum_{i=1}^{k} I^{-1/2} \zeta_i - \frac{k}{n} \frac{1}{\sqrt{nh_n}} \sum_{i=1}^{n} I^{-1/2} \zeta_i \right|
\]
\[
= \sup_{0 \leq s \leq 1} \left| W_n(s) - \frac{[ns]}{n} W_n(1) \right|
\]
\[
\frac{1}{\sqrt{nh_n}} \sum_{i=1}^{[ns]} I^{-1/2} \zeta_i \xrightarrow{w} B_q(s) - sB_q(1) = \frac{\sup_{0 \leq s \leq 1} \| B_q(s) - sB_q(1) \|}{\| B_q(1) \|}.
\]
(4.45)

Let us prove (4.44). It is sufficient to show
\[
\frac{1}{\sqrt{nh_n}} \sum_{i=1}^{[ns]} I^{-1/2} \left( \zeta_i - \mathbb{E}_{\theta_0} [\zeta_i | \mathfrak{g}_i^{n-1}] \right) \xrightarrow{w} B_q(s) \quad \text{in } D[0,1],
\]
(4.46)

From Lemma 1
\[
\frac{1}{\sqrt{nh_n}} \sum_{i=1}^{n} \mathbb{E}_{\theta_0} [\zeta_i | \mathfrak{g}_i^{n-1}] = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^{n} R_{i-1}(h_n^2, \theta) = \sqrt{nh_n} \cdot \frac{1}{n} \sum_{i=1}^{n} R_{i-1}(1, \theta) = o_p(1).
\]
Hence we have (4.46).

According to the Cramér-Wold theorem and Corollary 3.8 of McLeish (1974), we obtain (4.45) if we show that for all \( c \in \mathbb{R}^q \),
\[
\frac{1}{\sqrt{nh_n}} \sum_{i=1}^{[ns]} c^T I^{-1/2} \left( \zeta_i - \mathbb{E}_{\theta_0} [\zeta_i | \mathfrak{g}_i^{n-1}] \right) \xrightarrow{w} c^T B_q(s) \quad \text{in } D[0,1],
\]
i.e.,
\[
\frac{1}{\sqrt{nh_n}} \sum_{i=1}^{[ns]} \left\| \mathbb{E}_{\theta_0} \left[ c^T I^{-1/2} \left( \zeta_i - \mathbb{E}_{\theta_0} [\zeta_i | \mathfrak{g}_i^{n-1}] \right) \right] \mathfrak{g}_i^{n-1} \right\| = o_p(1),
\]
(4.47)
\[
\frac{1}{nh_n} \sum_{i=1}^{[ns]} \mathbb{E}_{\theta_0} \left[ c^T I^{-1/2} \left( \zeta_i - \mathbb{E}_{\theta_0} [\zeta_i | \mathfrak{g}_i^{n-1}] \right) \right] \mathfrak{g}_i^{n-1} \xrightarrow{p} \| c \|^2 s,
\]
(4.48)
\[
\frac{1}{(nh_n)^2} \sum_{i=1}^{[ns]} \mathbb{E}_{\theta_0} \left[ c^T I^{-1/2} \left( \zeta_i - \mathbb{E}_{\theta_0} [\zeta_i | \mathfrak{g}_i^{n-1}] \right) \right]^4 \mathfrak{g}_i^{n-1} = o_p(1),
\]
(4.49)
Therefore we obtain (4.49). This completes the proof of (4.35).

Noting that

\[
\mathcal{Y}_{j,i}^f = \sum_{\ell_1, \ell_2 = 1}^d \partial_{\beta_1} \partial_{\beta_2} b_{i-1}^j(\beta_0)(A_{i-1}^-1(\alpha_0))^{\ell_1, \ell_2}(\Delta X_i - h_n b_{i-1}(\beta_0))^{\ell_2}
\]

\[
= \sum_{\ell_1, \ell_2 = 1}^d \partial_{\beta_1} \partial_{\beta_2} b_{i-1}^j(\beta_0)(A_{i-1}^-1(\alpha_0))^{\ell_1, \ell_2}(\Delta X_i)^{\ell_2} - h_n \sum_{\ell_1, \ell_2 = 1}^d \partial_{\beta_1} \partial_{\beta_2} b_{i-1}^j(\beta_0)(A_{i-1}^-1(\alpha_0))^{\ell_1, \ell_2} b_{i-1}(\beta_0),
\]

\[
\mathcal{Z}_{j,i}^f = - \sum_{\ell_1, \ell_2 = 1}^d \partial_{\beta_2} b_{i-1}^j(\beta_0)(A_{i-1}^-1(\alpha_0))^{\ell_1, \ell_2} \partial_{\beta_1} b_{i-1}^j(\beta_0),
\]

\[
\mathcal{Z}_{j,i}^z = \sum_{\ell_1, \ell_2 = 1}^d \partial_{\beta_1} b_{i-1}^j(\beta_0) \partial_{\alpha_1}(A_{i-1}^-1(\alpha_0))^{\ell_1, \ell_2}(\Delta X_i)^{\ell_2},
\]

we have, by using [A9], Lemmas 4 and 5

\[
\frac{1}{(nh_n)^{(j+1)/2}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k \mathcal{Y}_{j,i}^f \right\| \leq \frac{1}{nh_n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k \partial_{\beta_1} \partial_{\beta_2} b_{i-1}^j(\beta_0)(A_{i-1}^-1(\alpha_0))^{\ell_1, \ell_2}(\Delta X_i)^{\ell_2} \right\|
\]
\[ -\frac{k}{n} \sum_{i=1}^{n} \partial^{2}_{\beta} \partial^{2}_{\beta} b^{\ell_{1}}_{i-1}(\beta_{0})(A_{i-1}^{-1}(\alpha_{0}))^{\epsilon_{1},\epsilon_{2}}(\Delta X_{i})^{\ell_{2}} \]

\[ + \sum_{\ell_{1},\ell_{2}=1}^{d} \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \partial^{2}_{\beta} \partial^{2}_{\beta} b^{\ell_{1}}_{i-1}(\beta_{0})(A_{i-1}^{-1}(\alpha_{0}))^{\epsilon_{1},\epsilon_{2}} b^{\ell_{2}}_{i-1}(\beta_{0}) \right\| \]

\[ = o_{p}(1), \]

\[ \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \mathcal{Z}_{2,i}^{\ell} - \frac{k}{n} \sum_{i=1}^{n} \mathcal{Z}_{1,i}^{\ell} \right\| \leq \sum_{\ell_{1},\ell_{2}=1}^{d} \frac{1}{n} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \partial^{2}_{\beta} b^{\ell_{1}}_{i-1}(\beta_{0})(A_{i-1}^{-1}(\alpha_{0}))^{\epsilon_{1},\epsilon_{2}} \partial^{2}_{\beta} b^{\ell_{2}}_{i-1}(\beta_{0}) \right\| \]

\[ = o_{p}(1), \]

and

\[ \frac{1}{n \sqrt{h_{n}}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{n} \mathcal{Z}_{3,1}^{\ell} - \frac{k}{n} \sum_{i=1}^{n} \mathcal{Z}_{3,i}^{\ell} \right\| \]

\[ = \sum_{\ell_{1},\ell_{2}=1}^{d} \frac{1}{n \sqrt{h_{n}}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \partial^{2}_{\beta} b^{\ell_{1}}_{i-1}(\beta_{0}) \partial_{\alpha}(A_{i-1}^{-1}(\alpha_{0}))^{\epsilon_{1},\epsilon_{2}} (\Delta X_{i})^{\ell_{2}} \right\| \]

\[ = o_{p}(1). \]

**Proofs of (4.37), (4.39), (4.41), (4.42) and (4.43):** Since \( \theta_{0} \in \text{Int} \Theta \), there exists an open neighborhood \( \Theta_{0} \) of \( \theta_{0} \) such that \( \Theta_{0} \subset \Theta \). Note that

\[ \mathcal{Y}_{m,i}^{\ell} = \sum_{\ell_{1},\ell_{2}=1}^{d} \mathcal{B}_{m,i-1}^{\ell_{1}}(A_{i-1}^{-1}(\alpha_{0}))^{\epsilon_{1},\epsilon_{2}}(\Delta X_{i} - h_{n} b_{i-1}(\beta_{0}))^{\ell_{2}} \]

\[ = \sum_{\ell_{1},\ell_{2}=1}^{d} \frac{1}{(m-1)!} \int_{0}^{1} (1-u)^{m} \partial^{\ell_{1}}_{\beta} \partial^{\ell_{2}}_{\beta} b^{\ell_{1}}_{i-1}(\beta_{0}) \partial_{\alpha}(A_{i-1}^{-1}(\alpha_{0}))^{\epsilon_{1},\epsilon_{2}} (\Delta X_{i} - h_{n} b_{i-1}(\beta_{0}))^{\ell_{2}} du, \]

\[ \mathcal{Z}_{2,i}^{\ell} = -\sum_{\ell_{1},\ell_{2}=1}^{d} \partial^{2}_{\beta} b^{\ell_{1}}_{i-1}(\beta_{0})(A_{i-1}^{-1}(\alpha_{0}))^{\epsilon_{1},\epsilon_{2}} \int_{0}^{1} (1-u) \partial^{2}_{\beta} b^{\ell_{2}}_{i-1}(\beta_{0} + u(\hat{\beta}_{n} - \beta_{0})) du \]

\[ + \sum_{\ell_{1},\ell_{2}=1}^{d} \partial^{2}_{\beta} \partial^{2}_{\beta} b^{\ell_{1}}_{i-1}(\beta_{0}) \partial_{\alpha}(A_{i-1}^{-1}(\alpha_{0}))^{\epsilon_{1},\epsilon_{2}} (\Delta X_{i} - h_{n} b_{i-1}(\beta_{0}))^{\ell_{2}} \]

\[ \mathcal{Z}_{3,i}^{\ell} = \sum_{\ell_{1},\ell_{2}=1}^{d} \partial^{2}_{\beta} b^{\ell_{1}}_{i-1}(\beta_{0}) \partial_{\alpha}(A_{i-1}^{-1}(\alpha_{0}))^{\epsilon_{1},\epsilon_{2}} (\Delta X_{i} - h_{n} b_{i-1}(\hat{\beta}_{n}))^{\ell_{2}}. \]
\[ Z_{0,i}^t = \sum_{\ell_1, \ell_2=1}^{d} \partial_{\beta^0} b_{\ell_1-1}^t (\hat{\beta}_n) A_{\ell_1-1}^t (\Delta X_i - h_n b_{\ell_1-1}(\hat{\beta}_n))^{\ell_2} \]
\[ = \sum_{\ell_1, \ell_2=1}^{d} \partial_{\beta^0} b_{\ell_1-1}^t (\hat{\beta}_n) \int_0^t (1-u) \partial_{\beta^0}^2 (A_{\ell_1-1}^t (\alpha_0 + u(\hat{\alpha}_n - \alpha_0)))^{\ell_1, \ell_2} du \left( \Delta X_i - h_n b_{\ell_1-1}(\hat{\beta}_n) \right)^{\ell_2}. \]

If \( \hat{\beta}_n = (\hat{\alpha}_n, \hat{\beta}_n) \in \mathcal{G}_0 \), then

\[ \| Y_{m,s}^t \| \leq \sum_{\ell_1, \ell_2=1}^{d} \sup_{\beta^0 \in \Theta_B} \| \partial_{\beta^0} \partial_{\beta^0} b_{\ell_1-1}^t (\beta) \| \sup_{\alpha \in \Theta_A} \left| (A_{\ell_1-1}^t (\alpha))^{\ell_1, \ell_2} \right| \left| (\Delta X_i - h_n b_{\ell_1-1}(\beta_0))^{\ell_2} \right|, \]
\[ \| Z_{2,i}^t \| \leq \sum_{\ell_1, \ell_2=1}^{d} \left( \sup_{\beta^0 \in \Theta_B} \| \partial_{\beta^0} b_{\ell_1-1}^t (\beta) \| \sup_{\alpha \in \Theta_A} \left| (A_{\ell_1-1}^t (\alpha))^{\ell_1, \ell_2} \right| \sup_{\beta \in \Theta_B} \| \partial_{\beta} b_{\ell_1-1}^t (\beta) \| \right. \]
\[ + \sup_{\beta \in \Theta_B} \| \partial_{\beta} \partial_{\beta} b_{\ell_1-1}^t (\beta) \| \sup_{\alpha \in \Theta_A} \left| (A_{\ell_1-1}^t (\alpha))^{\ell_1, \ell_2} \right| \sup_{\beta \in \Theta_B} \| \partial_{\beta} b_{\ell_1-1}^t (\beta) \| \]
\[ \left. + 2 \sup_{\beta \in \Theta_B} \| \partial_{\beta}^2 \partial_{\beta} b_{\ell_1-1}^t (\beta) \| \sup_{\beta \in \Theta_B} |b_{\ell_1-1}^t(\beta)| \right), \]
\[ \| Z_{4,i}^t \| \leq \sum_{\ell_1, \ell_2=1}^{d} \sup_{\beta \in \Theta_B} |\partial_{\beta} b_{\ell_1-1}^t(\beta)| \sup_{\alpha \in \Theta_A} \left| (A_{\ell_1-1}^t(\alpha))^{\ell_1, \ell_2} \right| \sup_{\beta \in \Theta_B} |b_{\ell_1-1}^t(\beta)|, \]
\[ \| Z_{5,i}^t \| \leq \sum_{\ell_1, \ell_2=1}^{d} \sup_{\beta \in \Theta_B} \| \partial_{\beta} b_{\ell_1-1}^t(\beta) \| \sup_{\alpha \in \Theta_A} \left| (A_{\ell_1-1}^t(\alpha))^{\ell_1, \ell_2} \right| \left| (\Delta X_i)^{\ell_2} + h_n \sup_{\beta \in \Theta_B} |b_{\ell_1-1}^t(\beta)| \right|, \]
\[ \| Z_{6,i}^t \| \leq \sum_{\ell_1, \ell_2=1}^{d} \sup_{\beta \in \Theta_B} |\partial_{\beta} b_{\ell_1-1}^t(\beta)| \sup_{\alpha \in \Theta_A} \left| (A_{\ell_1-1}^t(\alpha))^{\ell_1, \ell_2} \right| \left| (\Delta X_i)^{\ell_2} + h_n \sup_{\beta \in \Theta_B} |b_{\ell_1-1}^t(\beta)| \right|, \]
\[ E_{\Theta_B} \left[ \frac{1}{(nh_n)^{(m+1)/2}} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2=1}^{d} \sup_{\beta \in \Theta_B} \| \partial_{\beta} \partial_{\beta} b_{\ell_1-1}^t(\beta) \| \sup_{\alpha \in \Theta_A} \left| (A_{\ell_1-1}^t(\alpha))^{\ell_1, \ell_2} \right| \| (\Delta X_i - h_n b_{\ell_1-1}(\beta_0))^{\ell_2} \| \right] \]
\[ \leq \frac{1}{(nh_n)^{(m-1)/2}} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2=1}^{d} E_{\Theta_B} \left[ \sup_{\beta \in \Theta_B} \| \partial_{\beta} \partial_{\beta} b_{\ell_1-1}^t(\beta) \|^2 \sup_{\alpha \in \Theta_A} \left| (A_{\ell_1-1}^t(\alpha))^{\ell_1, \ell_2} \right|^2 \right]^{1/2} \]
\[ \leq C \frac{\sqrt{h_n}}{(nh_n)^{(m-1)/2}} \rightarrow 0, \]
\[ E_{\Theta_B} \left[ \frac{1}{n^{3/2}h_n} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2=1}^{d} \sup_{\beta \in \Theta_B} \| \partial_{\beta} \partial_{\beta} b_{\ell_1-1}^t(\beta) \| \sup_{\alpha \in \Theta_A} \left| (A_{\ell_1-1}^t(\alpha))^{\ell_1, \ell_2} \right| \left| (\Delta X_i)^{\ell_2} + h_n \sup_{\beta \in \Theta_B} |b_{\ell_1-1}^t(\beta)| \right| \right] \]
\[ \leq C \sqrt{h_n} \rightarrow 0, \]
\[ E_{\Theta_B} \left[ \frac{1}{n^{3/2}h_n} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2=1}^{d} \sup_{\beta \in \Theta_B} \| \partial_{\beta} \partial_{\beta} b_{\ell_1-1}^t(\beta) \| \sup_{\alpha \in \Theta_A} \left| \partial_{\alpha} (A_{\ell_1-1}^t(\alpha))^{\ell_1, \ell_2} \right| \left| (\Delta X_i)^{\ell_2} + h_n \sup_{\beta \in \Theta_B} |b_{\ell_1-1}^t(\beta)| \right| \right] \]
\[ \leq C \sqrt{h_n} \rightarrow 0, \]
\[ E_{\Theta_B} \left[ \frac{1}{n^{3/2}h_n} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2=1}^{d} \sup_{\beta \in \Theta_B} \| \partial_{\beta} \partial_{\beta} b_{\ell_1-1}^t(\beta) \| \sup_{\alpha \in \Theta_A} \left| \partial_{\alpha} (A_{\ell_1-1}^t(\alpha))^{\ell_1, \ell_2} \right| \left| (\Delta X_i)^{\ell_2} + h_n \sup_{\beta \in \Theta_B} |b_{\ell_1-1}^t(\beta)| \right| \right] \]
\[ \leq C \sqrt{h_n} \rightarrow 0. \]
Proof of Theorem 5.

\[ \leq \frac{C}{n^{3/2}h_n} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2=1}^{d} \mathbb{E}_{\theta_0} \left[ \sup_{\beta \in \Theta_B} |\partial_\beta \partial_\beta b_{i-1}^\ell (\beta)| \sup_{\alpha \in \Theta_A} |\partial_\alpha (A_{i-1}^{-1}(\alpha))^{\ell_1, \ell_2}| \right]^{1/2} \]

\[ \leq \frac{C'}{\sqrt{nh_n}} \to 0, \]

\[ \mathbb{E}_{\theta_0} \left[ \frac{1}{n^{3/2}h_n} \sum_{i=1}^{n} \sum_{\ell_1, \ell_2=1}^{d} \sup_{\beta \in \Theta_B} |\partial_\beta b_{i-1}^\ell (\beta)| \sup_{\alpha \in \Theta_A} |\partial_\alpha (A_{i-1}^{-1}(\alpha))^{\ell_1, \ell_2}| \right]^{1/2} \]

\[ \leq \frac{C'}{\sqrt{nh_n}} \to 0. \]

Hence, in the same way as the proof of (4.16), it follows from [A8] that

\[ \frac{1}{(nh_n)^{(m+1)/2}} \sum_{i=1}^{n} \|Y_{m,i}^\ell\| = o_p(1), \quad \frac{1}{n^{3/2}h_n} \sum_{i=1}^{n} \|Z_{2,i}^\ell\| = o_p(1), \quad \frac{\sqrt{h_n}}{n} \sum_{i=1}^{n} \|Z_{1,i}^\ell\| = o_p(1), \]

\[ \frac{1}{n^{3/2}h_n} \sum_{i=1}^{n} \|Z_{0,i}^\ell\| = o_p(1), \quad \frac{1}{n^{3/2}h_n} \sum_{i=1}^{n} \|Z_{6,i}^\ell\| = o_p(1). \]

\[ \square \]

Proof of Corollary 4. Note that by [A9], we can show (4.37) in the proof of Theorem 3. If we assume [A9'], then one has

\[ \frac{1}{(nh_n)^{(M+1)/2}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} Y_{M,i}^\ell - \frac{k}{n} \sum_{i=1}^{n} Y_{M,i}^\ell \right\| = 0 \]

corresponding to (4.37). Therefore, this corollary is shown as in the proof of Theorem 3. \[ \square \]

Proof of Theorem 5. If we prove

\[ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \hat{\eta}_i \overset{p}{\to} F(\alpha_1^*), \]

\[ \frac{1}{n-[nt^*]} \sum_{i=[nt^*]+1}^{n} \hat{\eta}_i \overset{p}{\to} F(\alpha_2^*), \]

then, we see from [B2] that

\[ \frac{1}{n} \sum_{i=1}^{n} \hat{\eta}_i = \frac{[nt^*]}{n} \sum_{i=1}^{[nt^*]} \hat{\eta}_i + \frac{n-[nt^*]}{n} \sum_{i=[nt^*]+1}^{n} \hat{\eta}_i \]

\[ \overset{p}{\to} t^*F(\alpha_1^*) + (1-t^*)F(\alpha_2^*) \]

and

\[ \frac{1}{n} \sum_{i=1}^{[nt^*]} \hat{\eta}_i - \frac{[nt^*]}{n} \sum_{i=1}^{[nt^*]} \hat{\eta}_i = \frac{[nt^*]}{n} \left( \sum_{i=1}^{[nt^*]} \hat{\eta}_i - \frac{1}{n} \sum_{i=1}^{n} \hat{\eta}_i \right) \]

\[ \overset{p}{\to} t^*(F(\alpha_1^*) - (t^*F(\alpha_1^*) + (1-t^*)F(\alpha_2^*))) = t^*(1-t^*)(F(\alpha_1^*) - F(\alpha_2^*)) \neq 0, \]
From Lemma 1, we have
\[ T_n^\alpha = \frac{1}{\sqrt{2dn}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \hat{\eta}_i - \frac{k}{n} \sum_{i=1}^n \hat{\eta}_i \right| = \frac{1}{\sqrt{2dn}} \sup_{0 \leq s \leq 1} \left| \sum_{i=1}^{[ns]} \hat{\eta}_i - \frac{[ns]}{n} \sum_{i=1}^n \hat{\eta}_i \right| \geq \frac{1}{\sqrt{2dn}} \left| \sum_{i=1}^{[nt^*]} \hat{\eta}_i - \frac{[nt^*]}{n} \sum_{i=1}^n \hat{\eta}_i \right| \]
\[ = \sqrt{\frac{n}{2d}} \left| \frac{1}{n} \sum_{i=1}^{[nt^*]} \hat{\eta}_i - \frac{1}{n} \sum_{i=1}^n \hat{\eta}_i \right| \rightarrow \infty. \]

Therefore, we have \( P(T_n^\alpha > w_1(\epsilon)) \rightarrow 1. \)

Let us show (4.50). We can express
\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \hat{\eta}_i \]
\[ = \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \text{tr} \left( A_{i-1}^{-1}(\hat{\alpha}_n) \frac{(\Delta X_i)^{\otimes 2}}{h_n} \right) \]
\[ = \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1, \ell_2=1}^d \left( A_{i-1}^{-1}(\hat{\alpha}_n) \right)^{\ell_1, \ell_2} (\Delta X_i)^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_1, \ell_2}}{h_n} \]
\[ = \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \text{tr} \left( A_{i-1}^{-1}(\hat{\alpha}_n) \frac{(\Delta X_i)^{\otimes 2}}{h_n} \right) + \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1, \ell_2=1}^d \int_0^1 \partial_\alpha (A_{i-1}^{-1}(\hat{\alpha}_n + u(\hat{\alpha}_n - \alpha_*)))^{\ell_1, \ell_2} du (\hat{\alpha}_n - \alpha_*) \frac{(\Delta X_i)^{\ell_1, \ell_2}}{h_n} \]
\[ = \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \text{tr} \left( A_{i-1}^{-1}(\hat{\alpha}_n) \frac{(\Delta X_i)^{\otimes 2}}{h_n} \right) \]
\[ + \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \left( \sum_{\ell_1, \ell_2=1}^d \int_0^1 \partial_\alpha (A_{i-1}^{-1}(\hat{\alpha}_n + u(\hat{\alpha}_n - \alpha_*)))^{\ell_1, \ell_2} du \frac{(\Delta X_i)^{\ell_1, \ell_2}}{h_n} \right) (\hat{\alpha}_n - \alpha_*. \]

From Lemma 1 we have
\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} E_{\alpha_1} \left[ \text{tr} \left( A_{i-1}^{-1}(\hat{\alpha}_n) \frac{(\Delta X_i)^{\otimes 2}}{h_n} \right) \varphi_{i-1}^n \right] = \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \text{tr} \left( A_{i-1}^{-1}(\hat{\alpha}_n) E_{\alpha_1} \left[ \frac{(\Delta X_i)^{\otimes 2}}{h_n} \varphi_{i-1}^n \right] \right) \]
\[ = \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \text{tr} \left( A_{i-1}^{-1}(\hat{\alpha}_n) A_{i-1}(\alpha_*) + R_{i-1}(h_n, \theta) \right) \]
\[ \rightarrow P \int_{\mathbb{R}^d} \text{tr}(A^{-1}(x, \alpha_*)) A(x, \alpha_*) \, d\mu_{\alpha_*}(x) = F(\alpha_*^1). \]

and
\[
\frac{1}{[nt^*]^2} \sum_{i=1}^{[nt^*]} E_{\alpha_1} \left[ \left( \text{tr} \left( A_{i-1}^{-1}(\hat{\alpha}_n) \frac{(\Delta X_i)^{\otimes 2}}{h_n} \right) \right)^2 \varphi_{i-1}^n \right] \]
\[ = \frac{1}{[nt^*]^2} \sum_{i=1}^{[nt^*]} E_{\alpha_1} \left[ \sum_{\ell_1, \ell_2=1}^d \sum_{\ell_3, \ell_4=1}^d (A_{i-1}^{-1}(\hat{\alpha}_n))^{\ell_1, \ell_2} (A_{i-1}^{-1}(\hat{\alpha}_n))^{\ell_3, \ell_4} \frac{(\Delta X_i)^{\ell_1, \ell_2}(\Delta X_i)^{\ell_3, \ell_4}(\Delta X_i)^{\ell_1, \ell_2}}{h_n^2} \varphi_{i-1}^n \right] \]
\[ = \frac{1}{[nt^*]^2} \sum_{i=1}^{[nt^*]} \sum_{\ell_1, \ell_2=1}^d \sum_{\ell_3, \ell_4=1}^d (A_{i-1}^{-1}(\hat{\alpha}_n))^{\ell_1, \ell_2} (A_{i-1}^{-1}(\hat{\alpha}_n))^{\ell_3, \ell_4} E_{\alpha_1} \left[ \frac{(\Delta X_i)^{\ell_1, \ell_2}(\Delta X_i)^{\ell_3, \ell_4}(\Delta X_i)^{\ell_1, \ell_2}}{h_n^2} \varphi_{i-1}^n \right] \]
\[ = \frac{1}{[nt^*]^2} \sum_{i=1}^{[nt^*]} \sum_{\ell_1, \ell_2=1}^d \sum_{\ell_3, \ell_4=1}^d (A_{i-1}^{-1}(\hat{\alpha}_n))^{\ell_1, \ell_2} (A_{i-1}^{-1}(\hat{\alpha}_n))^{\ell_3, \ell_4} R_{i-1}(1, \theta) \]}
\[ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} R_{i-1}(1, \theta) \xrightarrow{p} 0. \]

Therefore, from Lemma 9 of Genon-Catalot and Jacod (1993),
\[ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \text{tr} \left( A_{i-1}^{-1}(\tilde{\alpha}_*) \frac{(\Delta X_i)^{\otimes 2}}{h_n} \right) \xrightarrow{p} F(\alpha_*^2). \] (4.52)

On the other hand, from \([\text{B1}]\) and
\[ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{t_1, t_2=1}^{d} \int_0^1 \partial_\alpha (A_{i-1}^{-1}(\tilde{\alpha}_* + u(\hat{\alpha}_n - \tilde{\alpha}_*)))^{t_1, t_2} du \frac{(\Delta X_i)^{t_1}(\Delta X_i)^{t_2}}{h_n} = O_p(1), \]

we have
\[ \left( \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{t_1, t_2=1}^{d} \int_0^1 \partial_\alpha (A_{i-1}^{-1}(\tilde{\alpha}_* + u(\hat{\alpha}_n - \tilde{\alpha}_*)))^{t_1, t_2} du \frac{(\Delta X_i)^{t_1}(\Delta X_i)^{t_2}}{h_n} \right) (\hat{\alpha}_n - \tilde{\alpha}_*) = o_p(1). \] (4.53)

Hence, it follows from (4.52) and (4.53) that we obtain (4.50). In the same way, noting that \(\alpha_0 = \alpha_*^2\) if \([nt^*] + 1 \leq i \leq n\), we have (4.51), which completes the proof of Theorem 6.

**Proof of Theorem 6** If we prove
\[ \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \xi_i \xrightarrow{p} G(\beta_1^*), \] (4.54)

\[ \frac{1}{(n - [nt^*])h_n} \sum_{i=[nt^*]+1}^{n} \xi_i \xrightarrow{p} G(\beta_2^*), \] (4.55)

then, it follows from \([\text{B4}]\) that
\[ \frac{1}{nh_n} \sum_{i=1}^{n} \xi_i = \frac{[nt^*]}{n} \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \xi_i + \frac{n - [nt^*]}{n} \frac{1}{(n - [nt^*])h_n} \sum_{i=[nt^*]+1}^{n} \xi_i \xrightarrow{p} t^*G(\beta_1^*) + (1 - t^*)G(\beta_2^*) \]

and
\[ \frac{1}{nh_n} \sum_{i=1}^{[nt^*]} \xi_i - \frac{[nt^*]}{n} \frac{1}{nh_n} \sum_{i=1}^{n} \xi_i = \frac{[nt^*]}{n} \left( \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \xi_i - \frac{1}{nh_n} \sum_{i=1}^{n} \xi_i \right) \xrightarrow{p} t^*(G(\beta_1^*) - (t^*G(\beta_1^*) + (1 - t^*)G(\beta_2^*))) \]
\[ = t^*(1 - t^*)(G(\beta_1^*) - G(\beta_2^*)) \neq 0, \]

i.e.,
\[ T_{1,n}^\beta = \frac{1}{\sqrt{dh_n}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \hat{\xi}_i - \frac{k}{n} \sum_{i=1}^{n} \hat{\xi}_i \right| = \frac{1}{\sqrt{dh_n}} \sup_{0 \leq s \leq 1} \left| \sum_{i=1}^{[ns]} \xi_i - \frac{[ns]}{n} \sum_{i=1}^{n} \xi_i \right| \geq \frac{1}{\sqrt{dh_n}} \left| \sum_{i=1}^{[nt^*]} \xi_i - \frac{[nt^*]}{n} \sum_{i=1}^{n} \xi_i \right| \xrightarrow{p} \sqrt{\frac{nh_n}{d}} \left| \frac{1}{nh_n} \sum_{i=1}^{[nt^*]} \xi_i - \frac{1}{nh_n} \sum_{i=1}^{n} \xi_i \right| \rightarrow \infty. \]

Hence, one has \( P(T_{1,n}^\beta > w_1(\epsilon)) \rightarrow 1. \)

Let us show (4.54). We can express
\[ \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \xi_i \]
From Lemma 1, we have

\[ \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\hat{\alpha}_n) (X_{t_i^n} - X_{t_{i-1}^n} - h_nb_{i-1}(\hat{\beta}_n)) \]

\[ = \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \left( \kappa_{i-1}(\alpha_0) + (\hat{\alpha}_n - \alpha_0)^T \int_0^1 \partial_\alpha \kappa_{i-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)) du \right) (X_{t_i^n} - X_{t_{i-1}^n} - h_nb_{i-1}(\hat{\beta}_n)) \]

\[ = \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0) \frac{\Delta X_i}{h_n} - \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0) b_{i-1}(\hat{\beta}_n) \]

\[ + \sqrt{n}(\hat{\alpha}_n - \alpha_0)^T \left( \frac{1}{\sqrt{n}h_n[n^*_t]} \sum_{i=1}^{[nt^*]} \int_0^1 \partial_\alpha \kappa_{i-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)) du \left( \frac{\Delta X_i}{\sqrt{h_n}} - \sqrt{h_nb_{i-1}(\hat{\beta}_n)} \right) \right). \]

From Lemma 1 we have

\[ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \mathbb{E}_{\theta_1} \left[ \kappa_{i-1}(\alpha_0) \frac{\Delta X_i}{h_n} \right] = \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0)(b_{i-1}(\beta_1^n) + R_{i-1}(h_n, \theta)) \]

\[ \overset{p}{\rightarrow} \int_{\mathbb{R}^d} \kappa(x, \alpha_0)b(x, \beta_1^n) d\mu_{\theta_1}(x) \]

and

\[ \frac{1}{[nt^*]^2} \sum_{i=1}^{[nt^*]} \mathbb{E}_{\theta_1} \left[ \left( \kappa_{i-1}(\alpha_0) \frac{\Delta X_i}{h_n} \right)^2 \right] = \frac{1}{[nt^*]^2 h_n} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0) \mathbb{E}_{\theta_1} \left[ (\Delta X_i)^2 \frac{\Delta X_i}{h_n} \right] \kappa_{i-1}(\alpha_0)^T \]

\[ = \frac{1}{[nt^*]h_n} \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} R_{i-1}(1, \theta) \overset{p}{\rightarrow} 0. \]

Therefore, from Lemma 9 of Genon-Catalot and Jacod (1993),

\[ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0) \frac{\Delta X_i}{h_n} \overset{p}{\rightarrow} \int_{\mathbb{R}^d} \kappa(x, \alpha_0)b(x, \beta_1^n) d\mu_{\theta_1}(x). \] (4.56)

On the other hand, from [B3]

\[ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0) \int_0^1 \partial_\beta b_{i-1}(\hat{\beta}_n + u(\hat{\beta}_n - \tilde{\beta}_n)) du = O_p(1), \]

\[ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \int_0^1 \partial_\alpha \kappa_{i-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0))^T du \left( \frac{\Delta X_i}{\sqrt{h_n}} - \sqrt{h_nb_{i-1}(\hat{\beta}_n)} \right) = O_p(1), \]

we have

\[ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0) b_{i-1}(\hat{\beta}_n) \]

\[ = \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0) \left( b_{i-1}(\hat{\beta}_n) + \int_0^1 \partial_\beta b_{i-1}(\hat{\beta}_n + u(\hat{\beta}_n - \tilde{\beta}_n)) du (\hat{\beta}_n - \tilde{\beta}_n) \right) \]

\[ = \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0) b_{i-1}(\hat{\beta}_n) + \left( \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \kappa_{i-1}(\alpha_0) \int_0^1 \partial_\beta b_{i-1}(\hat{\beta}_n + u(\hat{\beta}_n - \tilde{\beta}_n)) du \right) (\hat{\beta}_n - \tilde{\beta}_n) \]

\[ \overset{p}{\rightarrow} \int_{\mathbb{R}^d} \kappa(x, \alpha_0)b(x, \tilde{\beta}_n) d\mu_{\theta_1}(x), \] (4.57)

and

\[ \sqrt{n}(\hat{\alpha}_n - \alpha_0)^T \left( \frac{1}{\sqrt{n}h_n[n^*_t]} \sum_{i=1}^{[nt^*]} \int_0^1 \partial_\alpha \kappa_{i-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0))^T du \left( \frac{\Delta X_i}{\sqrt{h_n}} - \sqrt{h_nb_{i-1}(\hat{\beta}_n)} \right) \right) = O_p(1). \] (4.58)
Hence, from (4.56), (4.57) and (4.58), we obtain (4.54). In the same manner, noting that $\beta_0 = \beta_2^*$ if $[nt^*] + 1 \leq i \leq n$, we have (4.55), which completes the proof of Theorem 6. \qed

**Proof of Theorem 7** If we prove

\[
\frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \hat{\zeta}_i \xrightarrow{p} H(\beta_1^*), \tag{4.59}
\]

\[
\frac{1}{(n-[nt^*])h_n} \sum_{i=[nt^*]+1}^{n} \hat{\zeta}_i \xrightarrow{p} H(\beta_2^*), \tag{4.60}
\]

we obtain, from [B5],

\[
\frac{1}{nh_n} \sum_{i=1}^{n} \hat{\zeta}_i = \frac{[nt^*]}{n} \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \hat{\zeta}_i + \frac{n-[nt^*]}{n} \frac{1}{(n-[nt^*])h_n} \sum_{i=[nt^*]+1}^{n} \hat{\zeta}_i \xrightarrow{p} t^*H(\beta_1^*) + (1-t^*)H(\beta_2^*)
\]

and

\[
\frac{1}{nh_n} \sum_{i=1}^{[nt^*]} \hat{\zeta}_i - \frac{[nt^*]}{n} \frac{1}{nh_n} \sum_{i=1}^{n} \hat{\zeta}_i = \frac{[nt^*]}{n} \left( \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \hat{\zeta}_i - \frac{1}{nh_n} \sum_{i=1}^{n} \hat{\zeta}_i \right) \xrightarrow{p} t^*(H(\beta_1^*) - t^*H(\beta_1^*) + (1-t^*)H(\beta_2^*)) = t^*(1-t^*)(H(\beta_1^*) - H(\beta_2^*)) \neq 0,
\]

i.e.,

\[
T_{2,n}^{\beta} = \frac{1}{\sqrt{nh_n}} \max_{1 \leq k \leq n} \left\| \mathcal{I}^{-1/2} \left( \sum_{i=1}^{k} \hat{\zeta}_i - \frac{k}{n} \sum_{i=1}^{n} \hat{\zeta}_i \right) \right\| \rightarrow \infty.
\]

Let us show (4.59). One has that from [A9],

\[
\frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \hat{\zeta}_i^2
\]

\[
= \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \frac{1}{d} \sum_{\ell_1, \ell_2 = 1}^{d} \partial_{\beta t} b_{\ell_1}^t(\hat{\beta}_n) (A_{t-1}^{-1}(\hat{\alpha}_n))^{\ell_1,\ell_2} (X_{\ell_1}^{t_i} - X_{\ell_2}^{t_{i-1}} - h_n b_{\ell_2}^{t_{i-1}}(\hat{\beta}_n))^{\ell_2}
\]

\[
= \frac{1}{[nt^*]h_n} \sum_{i=1}^{[nt^*]} \frac{1}{d} \sum_{\ell_1, \ell_2 = 1}^{d} \partial_{\beta t} b_{\ell_1}^t(\hat{\beta}_n) \left( (A_{t-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} + \int_0^1 \partial_{\alpha} (A_{t-1}^{-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)))^{\ell_1,\ell_2} du(\hat{\alpha}_n - \alpha_0) \right)
\]

\[
\cdot ((\Delta X_i)^{\ell_2} - h_n b_{\ell_2}^{t_{i-1}}(\hat{\beta}_n))
\]

\[
= \frac{1}{[nt^*]} \sum_{\ell_1, \ell_2 = 1}^{d} \partial_{\beta t} b_{\ell_1}^t(\hat{\beta}_n) (A_{t-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} \frac{\Delta X_i^{\ell_2} h_n}{[nt^*]}
\]

\[
- \frac{1}{[nt^*]} \sum_{\ell_1, \ell_2 = 1}^{d} \partial_{\beta t} b_{\ell_1}^t(\hat{\beta}_n) (A_{t-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} b_{\ell_2}^{t_{i-1}}(\hat{\beta}_n)
\]
It follows from Lemma 1 that for $f \in C^{r+1}_t(\mathbb{R}^d \times \Theta_B)$,

$$
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1, \ell_2 = 1}^{d} \partial_{\beta \ell_1} b_{\ell_1-1}(\hat{\beta}_n) \left( \frac{(\Delta X_i)^{\ell_2}}{\sqrt{h_n}} - \sqrt{h_n} b_{\ell_1-1}(\hat{\beta}_n) \right)
\cdot \int_0^1 \partial_u (A_{i-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)))^{\ell_1, \ell_2} du \sqrt{n} \hat{\alpha}_n - \alpha_0)
\cdot \int_0^1 \partial_u (A_{i-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)))^{\ell_1, \ell_2} du \sqrt{n} \hat{\alpha}_n - \alpha_0)
$$

$$
= \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1, \ell_2 = 1}^{d} \partial_{\beta \ell_1} b_{\ell_1-1}(\hat{\beta}_n) \left( \frac{(\Delta X_i)^{\ell_2}}{\sqrt{h_n}} - \sqrt{h_n} b_{\ell_1-1}(\hat{\beta}_n) \right)
\cdot \int_0^1 \partial_u (A_{i-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)))^{\ell_1, \ell_2} du \sqrt{n} \hat{\alpha}_n - \alpha_0)
$$

$$
\cdot \int_0^1 \partial_u (A_{i-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)))^{\ell_1, \ell_2} du \sqrt{n} \hat{\alpha}_n - \alpha_0)
$$

It follows from Lemma 1 that for $f \in C^{r+1}_t(\mathbb{R}^d \times \Theta_B)$,

$$
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \mathbb{E}_{\theta_i} \left[ f_{i-1}(\hat{\beta}_n)(A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_2}}{h_n} \right] g_{i-1}^n
$$

$$
= \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} f_{i-1}(\hat{\beta}_n)(A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2} b_{\ell_1-1}(\hat{\beta}_n) + R_{i-1}(h_n, \theta)
$$

$$
\mathbb{E}_{\theta_i} \left[ f(x, \hat{\beta}_n)(A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2} b_{\ell_1-1}(\hat{\beta}_n) d\mu_{\theta_i}(x) \right]
$$

and

$$
\frac{1}{[nt^*]^2} \sum_{i=1}^{[nt^*]} \mathbb{E}_{\theta_i} \left[ f_{i-1}(\hat{\beta}_n)(A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2} \frac{(\Delta X_i)^{\ell_2}}{h_n} \right]^2 g_{i-1}^n
$$

$$
= \frac{1}{[nt^*]^2} \sum_{i=1}^{[nt^*]} (f_{i-1}(\hat{\beta}_n)(A_{i-1}^{-1}(\alpha_0))^{\ell_1, \ell_2})^2 \mathbb{E}_{\theta_i} \left[ \frac{(\Delta X_i)^{\ell_2}}{h_n} \right] g_{i-1}^n
$$
Therefore, by Lemma 9 of Genon-Catalot and Jacob (1993),

\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} f_{i-1}(\hat{\beta}_n) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} \frac{(\Delta X_i)^{\ell_1,\ell_2}}{h_n} \xrightarrow{p} \int_{\mathbb{R}^d} f(x, \hat{\beta}_n) (A^{-1}(x, \alpha_0))^{\ell_1,\ell_2} b^2(x, \beta_n^*) d\mu_\theta(x),
\]

From this and [B3], we obtain

\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1,\ell_2=1}^d \left( \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n) \sum_{j=1}^{m-1} \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n) \otimes (\beta_n - \beta_n^*) \right) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} \frac{(\Delta X_i)^{\ell_1,\ell_2}}{h_n} \xrightarrow{p} \int_{\mathbb{R}^d} \partial_{\beta \ell} b^1(x, \hat{\beta}_n) (A^{-1}(x, \alpha_0))^{\ell_1,\ell_2} b^2(x, \beta_n^*) d\mu_\theta(x),
\]

i.e.,

\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} \frac{(\Delta X_i)^{\ell_1,\ell_2}}{h_n} \xrightarrow{p} \int_{\mathbb{R}^d} \partial_{\beta \ell} b^1(x, \hat{\beta}_n)^T A^{-1}(x, \alpha_0) b(x, \beta_n^*) d\mu_\theta(x). \tag{4.61}
\]

On the other hand, from [B3] and

\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1,\ell_2=1}^d \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} \int_0^1 \partial_{\beta \ell} b_{\ell_1,\ell_2}^2 (\beta_n + u(\beta_n - \beta_n^*)) d\mu = O_p(1),
\]

\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1,\ell_2=1}^d b_{\ell_1,\ell_2}^1 (\beta_n) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} \int_0^1 \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n + u(\beta_n - \beta_n^*)) d\mu = O_p(1),
\]

\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1,\ell_2=1}^d \int_0^1 \partial_{\beta \ell} b_{\ell_1,\ell_2}^2 (\beta_n + u(\beta_n - \beta_n^*)) d\mu_\theta(A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} \frac{(\Delta X_i)^{\ell_1,\ell_2}}{h_n} = O_p(1),
\]

\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1,\ell_2=1}^d \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n) (\frac{(\Delta X_i)^{\ell_1,\ell_2}}{h_n} - \sqrt{h_n} b_{\ell_1,\ell_2}^2 (\beta_n)) \int_0^1 \partial_\alpha(A_{i-1}^{-1}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)))^{\ell_1,\ell_2} d\mu = O_p(1),
\]

we have

\[
\frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1,\ell_2=1}^d \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} b_{\ell_1,\ell_2}^2 (\beta_n)
\]

\[
= \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1,\ell_2=1}^d \left( \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n) + \int_0^1 \partial_{\beta \ell} b_{\ell_1,\ell_2}^2 (\beta_n + u(\hat{\beta}_n - \beta_n^*)) d\mu_\theta(\hat{\beta}_n - \beta_n^*) \right) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2}
\]

\[
\left( b_{\ell_1,\ell_2}^1 (\beta_n) + \int_0^1 \partial_{\beta \ell} b_{\ell_1,\ell_2}^2 (\beta_n^* + u(\hat{\beta}_n - \beta_n^*)) d\mu_\theta(\hat{\beta}_n - \beta_n^*) \right)
\]

\[
= \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1,\ell_2=1}^d \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} b_{\ell_1,\ell_2}^2 (\beta_n)
\]

\[
+ \frac{1}{[nt^*]} \sum_{i=1}^{[nt^*]} \sum_{\ell_1,\ell_2=1}^d \left( \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} \int_0^1 \partial_{\beta \ell} b_{\ell_1,\ell_2}^2 (\beta_n + u(\hat{\beta}_n - \beta_n^*)) d\mu_\theta(\hat{\beta}_n - \beta_n^*) \right)
\]

\[
+ b_{\ell_1,\ell_2}^2 (\beta_n) (A_{i-1}^{-1}(\alpha_0))^{\ell_1,\ell_2} \int_0^1 \partial_{\beta \ell} b_{\ell_1,\ell_2}^1 (\beta_n + u(\hat{\beta}_n - \beta_n^*)) d\mu_\theta(\hat{\beta}_n - \beta_n^*)
\]
and

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