Energy and entropy of relativistic diffusing particles

Z. Haba
Institute of Theoretical Physics, University of Wroclaw, 50-204 Wroclaw, Plac Maxa Borna 9, Poland
PACS:05.10.Gg,05.20.Dd,98.80.Jk

Abstract
We discuss energy-momentum tensor and the second law of thermodynamics for a system of relativistic diffusing particles. We calculate the energy and entropy flow in this system. We obtain an exact time-dependence of energy, entropy and free energy of a beam of photons in a reservoir of a fixed temperature.

1 Introduction
A probabilistic description of a particle in an environment of some other particles is well-known in the non-relativistic classical and quantum mechanics [1]. The criteria for a diffusive approximation of such systems have been discussed for a long time. The diffusive approximation determines a single particle probability distribution. Then, we may consider a stream of particles with the same probability distribution and apply methods of statistical mechanics for a thermodynamic or hydrodynamic description of such a system. This is the well-known model of non-equilibrium thermodynamics [2][3].

A relativistic diffusion approximation is less developed (for a review see [4][5]). There are prospects for applications mainly in heavy ion collisions, plasma physics (ordinary electron-ion plasma and quark-gluon plasma) and in astrophysics [6]. There are obstacles in such an approach because relativistic dynamics of multiparticle systems encounters some conceptual problems. Quantum field theory gives a relativistic description of scattering processes. However, it encounters difficulties with a notion of interacting particle systems at a finite time. Nevertheless, if we interpret the Wigner function as the probability distribution then relativistic transport equations can be derived from quantum field theory [7][8]. A diffusion approximation to such equations has been applied in
high energy physics [9][10][11][12]. In astrophysics a diffusion approximation to a description of light moving through a space filled with an electron gas has found wide spread applications [6].

Putting aside the problem of a derivation of a diffusion equation from multi-particle dynamics we can ask the question about its mathematical form if it is to satisfy the postulates of the relativistic mechanics. The mathematical problem of a relativistic diffusion has been posed by Schay [13] and Dudley [14]. It comes out that if the particle mass is defined as the square of the four-momentum and the diffusion is to be a Markov process then the answer is unique. The generator of the diffusion must be defined on the mass-shell as the second order relativistic invariant differential operator. We usually are interested in systems which are close to equilibrium (otherwise a mathematical description would not be feasible). The notion of an equilibrium is relativistically covariant but not invariant. If we assume the usual postulates of equilibrium statistical mechanics then the reversible relativistic Markov process is uniquely determined by the detailed balance principle. We have discussed relativistic diffusions with an equilibrium distribution in [15][16].

Many particle systems can also be described approximately by relativistic hydrodynamics. The hydrodynamics can in principle be derived from mechanics or through an intermediate approximation step from diffusive dynamics. The form of the energy-momentum tensor (as expressed by fluid velocity) constitutes the basic assumption of the relativistic hydrodynamics [17][18][19]. We discuss the energy-momentum tensor in diffusion theory (sec.3) but do not attempt to derive its hydrodynamic form as expressed by fluid velocity. The energymomentum tensor is defined by the particle’s probability distribution on the phase space. It is not conserved because the energy of a stream of particles is dissipated in a medium. The diffusion equations determine the time evolution of the energy-momentum tensor. The definition of the probability distribution and an equilibrium distribution allows to define the relative entropy (sec.4)(the relative entropy for a class of relativistic diffusions is discussed also in [20][21]) which in the theory of non-relativistic diffusions has been related to free energy [22] and subsequently to the entropy and internal energy. The relation of the relative entropy to the thermodynamics of non-relativistic diffusing systems has been discussed in [22] [23][24]. In this letter we show such a relation for relativistic diffusions. We obtain equations for a time evolution of the energy and entropy. The case of a diffusion of massless particles is exactly soluble. We have shown [25] that the diffusion of massless particles can be considered as a linear approximation to the non-linear Kompaneets diffusion [26]. In this letter (sec.5) we obtain exact results for a time evolution of energy and entropy of radiation within a linear relativistic diffusion theory.
2 The relativistic diffusion

Following [13][14][15] we begin with a proper time evolution of a function $\phi$ on the phase space

$$\partial_\tau \phi = G \phi = (p^\mu \partial_\mu + A)\phi,$$

where $A$ is the second order differential operator defined on the mass-shell

$$p_\mu p^\mu = p_0^2 - p^2 = m^2 c^2.$$

Differentiation over space-time coordinates has an index $x$ whereas differentiation without an index concerns momenta, space-time indices are denoted by Greek letters whereas spatial indices by Latin letters. The probability density $\Phi$ evolves according to an adjoint equation

$$\partial_\tau \Phi = G^* \Phi = (-p^\mu \partial_\mu + A^*)\Phi,$$

resulting from

$$\int dxdp \phi(x,p)\Phi(x,p) = \int dxdp \phi(x,p)\Phi(x,p).$$

The probability density $\Phi$ of a particle in a laboratory frame is independent of $\tau$ and (from eq.(2)) is the solution of the equation

$$G^* \Phi = 0.$$  

If we choose the spatial momenta $p$ as coordinates on the mass-shell then the $O(3,1)$ invariant diffusion generator reads

$$2\gamma^{-2} A_0 = \triangle_H = (\delta_{jk} + m^{-2}c^{-2}p_j p_k) \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} + 3m^{-2}c^{-2}p_k \frac{\partial}{\partial p_k}. $$

We shall also use $\kappa^2 = m^{-2}c^{-2}\gamma^2$ as the diffusion constant. A diffusion generated by $A_0$ does not have a finite equilibrium measure. In order to achieve an equilibrium we add a friction term $R = R_j \partial^j$ to $A_0$. Now,

$$A = A_0 + R_j \partial^j \equiv A_{jk} \partial^j \partial^k + B_j \partial^j,$$

where the drift $B$ is

$$B_k \equiv R_k + \frac{3\kappa^2}{2} p_k.$$  

In an electromagnetic field we still add the force

$$G = \frac{e}{mc} F_{\mu\nu} p^\nu \partial^\mu,$$
(where $e$ is an electric charge and $F_{\mu\nu}$ is the electromagnetic field strength tensor).

We assume that there exists a time independent equilibrium solution $\Phi_E$ of eq.(4). If the Markov process is reversible [27] then the probability distribution $\Phi$ and the invariant measure $\Phi_E$ satisfy the detailed balance condition [22] which determines the drift $R$ as a function of the invariant measure. Applying basic assumptions of the classical relativistic statistical mechanics we can conclude that the equilibrium distribution of free relativistic particles is (the J"uttner distribution [28])

$$\Phi_E = p_0^{-1} \exp(-\beta cp_0).$$  \hfill (9)

where $\beta^{-1} = k_B T$, $k_B$ is the Boltzmann constant and $T$ is the temperature. $\Phi_E$ determines the drift

$$R_j = -\frac{\kappa^2}{2} p_j \beta c p_0.$$ \hfill (10)

We can also find an equilibrium measure of a particle in an electric field. Under the assumption that only the spatial components $F_{0j} = -e \partial_x^j V$ of $F_{\mu\nu}$ are different from zero we have

$$\Phi_V^E = p_0^{-1} \exp(-\beta (cp_0 + eV)).$$ \hfill (11)

Then, the drift is

$$R^V_j = -\frac{\kappa^2}{2} p_j \beta c p_0 + \frac{e}{mc} p_0 \partial_x^j V$$ \hfill (12)

(it could be obtained by an addition of the force (8) to the friction (10)).

The limit $m \to 0$ is particularly simple. It can be obtained as the limit $m \to 0$ of the time evolution $\exp(\tau m^2 \mathcal{G})$ [25]. Then, eq.(4) reads

$$|p| \partial_t \Phi = \frac{c^2 \kappa^2}{2} \partial_i \partial^k p_j p_k \Phi - \frac{3c^2 \kappa^2}{2} \partial_i p_j \Phi + \frac{\kappa^2}{2} \beta c^2 \partial_i p_j |p| \Phi + |p| \nabla_x \Phi$$ \hfill (13)

We write

$$\Phi = \Phi_E \Psi$$ \hfill (14)

and $r = |p|$. Then, the equation for $\Psi(|p|, n, x)$ is related to the Bessel diffusion [29]

$$\partial_t \Psi = \frac{c \kappa^2}{2} r \partial_r^2 \Psi + \left( \frac{3c \kappa^2}{2} - \frac{\beta c^2 \kappa^2}{2} r \right) \partial_r \Psi - cn \nabla_x \Psi$$ \hfill (15)

where $n = p|p|^{-1}$ is a fixed vector.

### 3 Energy-momentum tensor

We define a configuration space density

$$\rho(x) = \int d|p| \Phi(x, |p|).$$ \hfill (16)
The particle density current is defined by

\[ N^\mu = \int d\mathbf{p} \Phi(x, \mathbf{p}). \]  
(17)

From the transport equation (4) we derive the current conservation

\[ \partial_\mu N^\mu = 0. \]  
(18)

As a consequence

\[ Z = \int d\mathbf{x} N^0 = \int d\mathbf{x} \int d\mathbf{p} \rho_0 \Phi = \text{const.} \]  
(19)

We can apply eq.(19) to normalize the probability distribution.

We define the energy-momentum tensor \([8]\)

\[ T_{\mu \nu}(x) = \int d\mathbf{p} p_\mu p_\nu \Phi(x, \mathbf{p}). \]  
(20)

The energy-momentum tensor gives a covariant description of a continuous distribution of the energy and momentum. \(T_0^\mu\) has the meaning of the four-momentum density current. Then

\[ P_\mu = \int d\mathbf{x} T_0^\mu, \]

determines the four-momentum of a stream of particles. We can write

\[ T_{0 \mu} = \int d\mathbf{p} \Omega(\mathbf{p}, \mathbf{x}) p_\mu \]  
(21)

where the physical meaning of \(T_{0 \mu}\) and \(P_\mu\) imply that \(\Omega(\mathbf{p}, \mathbf{x}) = \Phi(\mathbf{p}, \mathbf{x}) \rho_0\) is the phase space probability distribution (after a normalization).

We obtain simple equations for the divergence of the kinetic energy-momentum in an electromagnetic field at \(\beta = 0\) (no friction). Then,

\[ \partial^\mu T_{\mu \nu} = \frac{3 \kappa^2}{2} N_\nu + \frac{e}{mc} F_{\nu \sigma} N^\sigma. \]  
(22)

If friction is present then we consider a diffusion in the electric field \(E_k = -\partial_k V\) (in such a case eq.(4) has an equilibrium solution (11)). Then, from eq.(4)

\[ \partial^\mu T_{\mu 0} = \frac{3 \kappa^2}{2} N_0 - \frac{\kappa^2}{2} \beta c T_{00} + \frac{\kappa^2}{2} \beta m^2 c^3 \rho + \frac{e}{m} \partial_k V N_k \]  
(23)

and

\[ \partial^\mu T_{\mu k} = \frac{3 \kappa^2}{2} N_k - \frac{\kappa^2}{2} \beta c T_{0k} + \frac{e}{mc} \partial_k V N^0. \]  
(24)
Eqs. (23)-(24) say that the flow of energy and momentum is determined by the currents.

The kinetic energy can be defined as

\[ W = cP_0 = c \int dx T_{00} = c \int dx p \Phi p_0^2. \]  

(25)

Then, from eq. (23)

\[ \partial_0 W = 3ZV - \frac{\kappa^2}{2} \left( 2 - V \right) \beta c W + \frac{\kappa^2}{2} \beta m^2 c^4 \int dx \rho + \frac{ec}{m} \int dx \nabla V N \]  

(26)

where the normalization constant (19) in the case of a potential \( V \) is denoted \( ZV \).

It follows from eq. (26) that if there is no friction (\( \beta = 0 \)) and no potential (\( V = 0 \)) then the energy grows linearly in time. If we include the external potential \( V \) into the energy

\[ W_V = \int dx (cT_{00} + cp_0 V \Phi) \equiv \int dp \Omega(p)(cp_0 + eV) \equiv W + e(V). \]  

(27)

then eq. (26) reads

\[ \partial_0 W_V = 3ZV - \frac{\kappa^2}{2} \beta c W + \frac{\kappa^2}{2} \beta m^2 c^4 \int dx \rho. \]  

(28)

We show that the energy \( W_V \) is bounded if \( V \) is bounded and \( \beta > 0 \). Let

\[ \nu = \beta \kappa^2 c^2. \]  

(29)

We write eq. (28) in an integral form

\[ W_V(t) = \exp(-\frac{\nu}{2} t) W_V(t = 0) + \frac{\kappa^2}{2} \beta c W + \frac{\kappa^2}{2} \beta m^2 c^4 \int dx \rho + e\beta(V) ds. \]  

(30)

Then, from eq. (28) and the inequality

\[ \int dx \rho \leq \int dp dx (mc)^{-1} p_0 \Phi = (mc)^{-1} ZV \]

we have

\[ W_V \leq \exp(-\frac{\nu}{2} t) W_V(t = 0) + \beta^{-1} \left( 1 - \exp(-\frac{\nu}{2} t) \right) ZV (3 + \beta mc^2 + e\beta v), \]  

(31)

where \( v = \sup|V| \).
The relative entropy (also called Kullback-Leibler entropy) determines a distance between two probability measures. Define the relative entropy of two unnormalized probability distributions $\Phi$ and $\Phi_E$ ($Z$ and $Z_E$ are the normalization constants) as

$$S_K(\Phi; \Phi_E) = Z^{-1} \int dx dp p_0 \Phi \ln \left( Z^{-1} \Phi(Z_E)^{-1} Z_E \right),$$  \hspace{1cm} (32)$$

where

$$Z_E = \int dx dp \exp(-\beta p_0)$$ \hspace{1cm} (33)$$

(the system must be in a finite volume if $Z_E$ is to be finite). It is known that

$$S_K(\Phi; \Phi_E) \geq 0.$$ \hspace{1cm} (34)$$

An easy calculation using the transport equation (4) gives

$$\partial_0 S_K(\Phi; \Phi_E) = -Z^{-1} \int dx dp \Phi A_{jk} \partial_j \ln Q \partial_k \ln Q \leq 0,$$ \hspace{1cm} (35)$$

where

$$Q = \Phi(Z_E)^{-1}$$ \hspace{1cm} (36)$$

and $A_{jk}$ is defined in eq.(6). Eq.(35) holds true for any two probability distributions solving the same Fokker-Planck equation [30]. It follows from eqs.(34)-(35) that $S_K(\Phi; \Phi_E)$ is a non-negative function monotonically decreasing to zero at the equilibrium $\Phi_E$ (however, the gradients in eq.(35) do not depend on the potential $V$). For non-relativistic diffusions the properties of the relative entropy are known for a long time [22][30][31][32] (the decrease of the relative entropy for a class of relativistic diffusions has been shown in [20][21]).

The relative entropy in an electric field is

$$S_K(\Phi; \Phi_E^V) = Z^{-1} \int dx dp p_0 \Phi \ln \left( Z^{-1} \Phi(Z_E^V)^{-1} Z_E \exp(\beta(c p_0 + e V)) \right).$$ \hspace{1cm} (37)$$

Here

$$Z_E^V = \int dx dp \exp(-\beta(c p_0 + e V)).$$ \hspace{1cm} (38)$$

If we have a single particle distribution $\Phi$ then we can define the entropy current

$$S^\mu(\Phi) = -k_B \int \frac{dp}{p_0} p_\mu(p_0 \Phi) \ln (p_0 \Phi)$$ \hspace{1cm} (39)$$

and the Boltzmann entropy

$$S(\Phi) = \int dx S^0(\Phi).$$ \hspace{1cm} (40)$$
From the definitions (25),(32) and (40) we obtain the relation

\[ ZS_K(\Phi; \Phi_E) = -k_B^{-1}S + \beta W + Z \ln(Z_E Z^{-1}) \]  

(41)

We define the free energy

\[ \mathcal{F} = \beta^{-1} ZS_K(\Phi; \Phi_E) - Z \beta^{-1} \ln(Z E Z^{-1}) \]  

(42)

and in a potential \( V \)

\[ \mathcal{F}_V = \beta^{-1} ZV_S K(\Phi; \Phi_V) - (Z V^{-1}) \beta^{-1} \ln(Z V E (Z E^{-1})) \]  

(43)

Then, from eqs.(25),(40) and (42) we obtain the basic thermodynamic relation

\[ TS = W - \mathcal{F}. \]  

(44)

\( Z \) has the meaning of the mean number of modes. We can define the energy per mode

\[ w = Z^{-1}W \]  

(45)

\[ f = Z^{-1}\mathcal{F} \] and \( s = Z^{-1}S \). Then, the thermodynamic equality (44) reads

\[ Ts = w - f \]  

(46)

We can calculate the time derivative of the entropy

\[ T \partial_0 S = \partial_0 W - \partial_0 \mathcal{F} \]

\[ = \frac{3c^2}{2} Z - \frac{5c^2}{2} \beta \epsilon W + \frac{5c^2}{2} \beta m^2 c^4 \int d^2 x \rho + \int d^2 p d^2 x A_{jk} \partial_j \ln Q \partial_k \ln Q \]  

(47)

\( Q = \Phi \Phi_E^{-1} \). The negative contribution to the change of entropy comes from the energy loss into the surroundings caused by the friction in the medium (see the discussion of the entropy balance in [2][3]). The question whether the entropy is increasing or not depends on the relative strength of the terms on the rhs of eq.(47). The last term on the rhs of eq.(47) tends to zero when the equilibrium is approached. The sum of first three terms also tends to zero. In order to prove this we take the equilibrium limit on the rhs of eq.(47) (here \( z = \int d^2 p \exp(-\beta p_0) \))

\[ \lim_{t \to \infty} k^{-2} \partial_0 S = \frac{3c}{2} z - \frac{1}{2} \beta c \int d^2 p_0 \exp(-\beta p_0) + \frac{1}{2} \beta m^2 c^4 \int d^2 p_0^-1 \exp(-\beta p_0) = 0 \]  

(48)

The result (48) comes from direct calculations of the integrals (with a use of some identities for modified Bessel functions \( K_\nu \)). We can see that at large time there is a subtle cancellation of the first three terms with the last one on the rhs of eq.(48). Such an entropy balance is characteristic for a system exchanging energy (and entropy) with the surroundings of the system until it achieves an equilibrium [2][3]. The problem whether the entropy \( S \) grows or
not may depend on the initial conditions, on the temperature and on time. If we choose the initial condition \( \Phi \) so that the initial \( \mathcal{W} \) is small and far from its equilibrium value then the rhs of eq.(47) will be positive for a small time and will continue to be positive until it approaches zero when \( t \to \infty \) and \( \Phi \to \Phi_E \). In the opposite case of large initial \( \mathcal{W} \) the entropy begins to decrease at \( t = 0 \) and it may continue doing so as time goes on. In the Appendix we consider an example of a beam of particles prepared at \( t = 0 \) in the equilibrium state of temperature \( T' \). We calculate the rhs of eq.(47) at \( t = 0 \). If \( T' > T \) the energy and entropy start to decrease (although \(-\mathcal{F}\) grows) because of the dissipation of the energy in the reservoir. In the opposite case \( T' < T \) the beam acquires the energy and the entropy from the reservoir (hence, \( \partial_0 S > 0 \)). The beam can gain the entropy till it achieves the equilibrium at the temperature \( T \). In the next section we prove that this is exactly so in a soluble model of diffusing photons.

We could calculate the time derivative of the entropy directly from the definition (40)

\[
\partial_0 S(\Phi) = -k_B \int \left( (p \nabla + A^*) \Phi \right) \ln(p \Phi) - k_B \int (p \nabla + A^*) \Phi. \quad (49)
\]

Eq.(49) shows (as \( A^* \Phi_E = 0 \)) that

\[
\partial_0 S(\Phi_E) = 0 \quad (50)
\]

(no entropy production in an equilibrium). The conclusion (50) is in agreement with (48) because \( \Phi \to \Phi_E \) as \( t \to \infty \).

5 Thermodynamics of diffusing photons

We have shown in \([25] \,[33]\) that a diffusion of photons can be described by the limit \( m \to 0 \) of the diffusion equation. This equation can be solved explicitly. We write \( \Phi = \Phi_E \Psi \). We choose the initial condition \( \Psi \) for eq.(15) in the form of the Laplace transform \((n = p|p|^{-1})\)

\[
\Psi(x, p) = \Psi(x, |p|, n) = \int_0^\infty d\sigma \tilde{\Psi}(x, \sigma, n) \exp(-\sigma c|p|). \quad (51)
\]

First, we choose \( \tilde{\Psi} = \delta(\sigma - \lambda) \). Let

\[
A(\lambda, t) = \left( 1 + \lambda \beta^{-1} (1 - \exp(-\nu t/2)) \right), \quad (52)
\]

then the solution of eq.(15) with the initial condition \( \exp(-c\lambda|p|) \) is \([29] \,[16]\)

\[
\Psi_t^\lambda(p) = A(\lambda, t)^{-3} \exp\left(-A(\lambda, t)^{-1} c\lambda \exp(-\nu t/2)|p|\right). \quad (53)
\]
In general,
\[ \Psi_t(x, |p|, n) = \int_0^\infty d\sigma \tilde{\Psi}(x - nct, \sigma, n) \Psi^*_t(p). \] (54)

From eqs.(52)-(54) we obtain the limit
\[
\lim_{t \to \infty} \int dxdp \Phi_t(x, p) M(p) = \int d\Psi E \Phi E \left( \frac{\int dxdp \Psi(x, p) \exp(-c\beta |p|)}{\int d\Psi \exp(-c\beta |p|)} \right)^{-1}.
\] (55)

Let
\[ M(p) = p_0 H(p). \] (56)

Then, eq.(55) can be expressed in a form which looks like a standard limit of mean values
\[
\lim_{t \to \infty} \int dxdp \Phi_t(x, p) p_0 H(p) \left( \frac{\int dxdp \Psi(x, p)}{\int d\Psi \Phi E \Phi E} \right)^{-1} = \int d\Psi E \Phi E \left( \frac{\int d\Psi \Phi E \Phi E}{\int d\Psi \phi E} \right)^{-1}.
\] (57)

We can study the time dependence of thermodynamic functions explicitly using the explicit solution of the diffusion equation. When \( m = 0 \) and \( V = 0 \) then we obtain a closed equation (28) for \( W \) with the solution
\[ W(t) = \exp\left(-\frac{\nu^2}{2}t\right) W(t = 0) + 3 \frac{\beta}{Z} - 1 \left(1 - \exp\left(-\frac{\nu^2}{2}t\right)\right). \] (58)

We study in more detail the case
\[ \Phi^\lambda = \Phi = p_0^{-1} \exp(-\beta' |p|) = \Phi E \exp(-c\lambda |p|), \] (59)

(in a finite volume which we set as 1) where
\[ \lambda = \beta' - \beta. \] (60)

The state \( \Phi^\lambda \) describes a beam of photons coming from a source of a fixed temperature \( T' \). We are interested in a time evolution of this beam when passing through a medium of a fixed temperature \( T \). We assume that the interaction of photons with the medium can be described approximately as the diffusion. Such an approximation has been established by Kompaneets in a model of an interaction of photons with an electron gas [26]. In eq.(58) for the model (59) we have
\[ W(t = 0) = c \int d\Psi^2 p_0^2 \Phi E = 24\pi c(c\beta + c\lambda)^{-4} = 24\pi c^{-3} \beta^{-4} \] (61)

(the Stefan-Boltzmann law) and
\[ Z = \int d\Psi \Phi E = 8\pi (c\beta + c\lambda)^{-3} = 8\pi c^{-3} \beta^{-3}. \] (62)
Hence

\[ W(t) = 24\pi c^{-3} \beta^{-1} \beta' - 3 - 24\pi \lambda \exp\left(-\frac{\nu}{2} t\right) c^{-3} \beta^{-1} \beta' - 3 - 24\pi \lambda \exp\left(-\frac{\nu}{2} t\right). \]  (63)

and

\[ \beta^4 (24\pi)^{-1} Z c^3 S_K(\Phi_E', \Phi_E) = -\beta' \ln \left(1 + \lambda \beta^{-1} (1 - \exp(-\frac{\nu}{2} t))\right) - \lambda \exp(-\frac{\nu}{2} t). \]  (64)

Hence,

\[ \partial_t F = -12\pi c^{-3} \lambda^2 \nu \beta^{-1} \beta' - 4 \exp(-\nu t) \left(1 + \lambda \beta^{-1} (1 - \exp(-\frac{\nu}{2} t))\right)^{-1}, \]  (65)

\[ \partial_t W = 12\pi \lambda c^{-1} \kappa^2 \beta' - 4 \exp(-\frac{\nu}{2} t) \]  (66)

and

\[ \partial_t S = 12\pi \lambda \kappa^2 c^{-1} \beta' - 3 \exp(-\frac{\nu}{2} t) \left(1 + \lambda \beta^{-1} \exp(-\frac{\nu}{2} t)\right)^{-1}, \]  (67)

Eq. (67) means that the change of the entropy has the same sign as the change of energy (66), i.e., if the temperature of the system is higher than the temperature of the reservoir then the entropy of the system is decreasing, because the energy is flowing into the reservoir. If the temperature of the reservoir is higher than the temperature of the system then the entropy of the system is increasing.

The total change of the entropy of the system plus the reservoir (the entropy production) is described by \[ -\partial_t F = T \partial_t S - \partial_t W \geq 0. \]

Note that the energy per mode \( w(t) \) (45) tends from its initial value \( w(t = 0) = 3\beta' - 1 \) to the equilibrium limit \( w(t = \infty) = 3\beta' - 1 \) in accordance with the equipartition theorem for massless particles.

### 6 Discussion

We have extended the diffusion model of non-equilibrium thermodynamics [2] to relativistic diffusions. The model describes a balance of entropy and energy: the exchange of the entropy with the surroundings and the entropy production inside the system. The time derivative of the relative entropy (with an opposite sign) is interpreted as the entropy production in the total system consisting of the relativistic diffusing particles and the reservoir [2][3][23]. Such models can be useful in a description of a beam (fluid) of relativistic particles moving in a medium. As an example we could consider a motion of light through the space filled with electrons [6][26] (this could include the initial stage of the Big Bang [34]); for a discussion of some other applications of non-equilibrium radiation theory see [35]. We could consider also a diffusive motion of a heavy quark in the plasma of gluons and light quarks. Such a model of diffusion (after an extension to open systems) can be tested in the LHC experiments [12]. The entropy exchange can play an important role in a description of heavy ion collisions [36].
In this letter we have discussed a diffusion equation with a probability distribution tending to the classical relativistic (Jüttner) equilibrium distribution. We could consider in the diffusion equations (2) the drifts leading to the quantum equilibrium distributions. However, in such a case the thermodynamics will change substantially, e.g., the entropy for Bose-Einstein particles should have the form

\[ S = -\int dx dp p_0 \Phi \ln p_0 \Phi + \int dx dp (1 + p_0 \Phi) \ln(1 + p_0 \Phi) \]

instead of eq.(39). The classical statistical thermodynamics is valid in the limit of large \( \beta c p_0 \). If this condition is not satisfied then quantum phenomena appear. In the photon case there are bunching (enhancement) effects resulting from Bose-Einstein statistics described by non-linear terms in the Boltzmann equation [6][26] (there is the Pauli blocking factor in the case of the Fermi-Dirac statistics). Only in the limit of low photon density these non-linear terms can be neglected. A discussion of thermodynamics with quantum equilibrium distributions requires non-linear relativistic diffusion equations which will be studied in forthcoming publications.

7 Appendix

Let us assume that the initial state \( \Phi'_E \) is an equilibrium state but with a temperature \( T' \) different from the temperature \( T \) of the reservoir

\[ \Phi'_E = \exp(-\beta' c p_0) \]

We are unable to obtain an exact time evolution of energy and entropy of a stream of massive particles. However, we can obtain their evolution for a small time calculating the time derivatives at \( t = 0 \) from eqs.(26) and (35). First, for the initial distribution \( \Phi'_E \)

\[ \partial_t W(t = 0) = \frac{4 \pi \kappa^2}{c^2} Z - \frac{4 \pi \kappa^2}{c^2} \beta^2 W_0 + \frac{2 \pi \kappa^2}{c^2} \beta m^2 c^5 \int dx \rho \]

\[ = 6 \pi \kappa^2 \beta^{-3} (1 - \frac{\beta}{\beta'}) K_2(m c^2 \beta')(m c^2 \beta')^2 \]

Then, after a calculation of \( \partial_0 S_K(\Phi'_E, \Phi_E) \) from eq.(35) we obtain

\[ \partial_0 F(t = 0) = -6 \pi \kappa^2 \beta^{-3} m^2 (1 - \frac{\beta}{\beta'})^2 K_2(m c^2 \beta') \]

(where \( K_\nu \) denotes the modified Bessel function of order \( \nu \)). The limit \( m \to 0 \) of these formulas agrees with eqs.(65)-(67) at \( t = 0 \) as \( K_2(x) \simeq 2x^{-2} \) for a small \( x \).
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