UNIFORMLY WELL-POSED HYBRIDIZED DISCONTINUOUS GALERKIN/HYBRID MIXED DISCRETIZATIONS FOR BIOT’S CONSOLIDATION MODEL

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Abstract. We consider the quasi-static Biot’s consolidation model in a three-field formulation with the three unknown physical quantities of interest being the displacement $u$ of the solid matrix, the seepage velocity $v$ of the fluid and the pore pressure $p$. As conservation of fluid mass is a leading physical principle in poromechanics, we preserve this property using an $H$(div)-conforming ansatz for $u$ and $v$ together with an appropriate pressure space. This results in Stokes and Darcy stability and exact, that is, pointwise mass conservation of the discrete model.

The proposed discretization technique combines a hybridized discontinuous Galerkin method for the elasticity subproblem with a mixed method for the flow subproblem, also handled by hybridization. The latter allows for a static condensation step to eliminate the seepage velocity from the system while preserving mass conservation. The system to be solved finally only contains degrees of freedom related to $u$ and $p$ resulting from the hybridization process and thus provides, especially for higher-order approximations, a very cost-efficient family of physics-oriented space discretizations for poroelasticity problems.

We present the construction of the discrete model, theoretical results related to its uniform well-posedness along with optimal error estimates and parameter-robust preconditioners as a key tool for developing uniformly convergent iterative solvers. Finally, the cost-efficiency of the proposed approach is illustrated in a series of numerical tests for three-dimensional test cases.

1. Introduction

Poroelastic models describing the mechanical behaviour of fluid saturated porous media find a wide range of applications in many different fields of science, medicine and engineering. The theory of poroelasticity was initially conceived by Maurice Anthony Biot who, in the period between 1935 and 1962, see e.g. [5, 6], proposed a soil consolidation model to calculate the settlement of structures placed on fluid-saturated porous soils.

Recently, interest in Biot’s consolidation equations has been revived due to their newly discovered applications in medicine, see e.g. [49] and [18], where they have been studied in the context of human cancellous bone samples and risk factors associated with the early stages of Alzheimer’s disease, respectively. Their numerical solution has consequently been a subject of active research. One major challenge is that the parameters involved in Biot’s model can vary over many orders of magnitude and, therefore, it is vital that not only the variational formulation of the problem is stable but also that the iterative solution method is uniformly convergent over the whole range of admissible model parameter values.

A rigorous stability and convergence analysis for finite element (FE) approximations of the two-field formulation of Biot’s equations where the velocity field has been eliminated from the unknowns has first been presented in [37, 38]. The derived a priori error estimates are valid for...
both semidiscrete and fully discrete formulations, where the backward Euler method is used for time-discretization and inf-sup stable finite elements are used for space discretization.

Other recent developments in discretizing Biot-type models are related to the stabilization of conforming methods [46], stable finite volume methods [40], discretizations for total-pressure-based formulations [32, 41], including conservative discontinuous finite volume and mixed schemes [29], enriched Galerkin methods [15, 34], space-time finite element approximations [4], and methods for two-phase flow and non-linear extensions of the Biot problem [34, 45], to mention only a few. Finally, and, nevertheless, important in the context of the present research, are the extensions of abovementioned discretization techniques to multicompartamental (multiple network) poroelasticity problems presented in [33, 22].

The subject of the study in this paper is the standard three-field formulation of Biot’s model in which the unknown fields are the displacement, seepage velocity and fluid pressure. Discretizations based on three-field-formulation have originally been proposed in [42, 43] where continuous-in-time and discrete-in-time error estimates have been proved. This approach has also been extended to discontinuous Galerkin approximations of the displacement field in [44] and other nonconforming approximations, e.g., using modified rotated bilinear elements [52], or Crouzeix-Raviart elements for the displacements in [25]. More recently, in [24], a family of strongly mass conserving discretizations based on the $H(\text{div})$-conforming discontinuous Galerkin (DG) discretization of the displacement field has been suggested and its parameter-robust stability and near best approximation properties proven. Time-dependent error estimates for the same family of discretizations have been proved in [26]. Note that these approaches are based on the inf-sup stability of the corresponding Stokes discretization scheme which were originally stated in [10, 12, 11] and the Brinkman problem [28, 27].

Hybridization techniques have been applied to discretizations of Biot’s model in the recent works [14] and [39]. Whereas in [14] the authors introduced a hybridized $H(\text{div})$-conforming DG method for the two-field formulation, the work [39] starts from a lowest-order conforming stabilized discretization of the three-field formulation and uses hybridization for the flow subsystem as it was first presented in [1].

The aim of the present work is the construction, analysis and numerical testing of a new family of higher-order mass-conserving hybridized/hybrid mixed FE discretizations for the three-field formulation of Biot’s model. The main focus lies on a well-posedness analysis in properly scaled norms resulting in estimates with constants that are independent of any problem parameters. As a consequence, we obtain norm-equivalent preconditioners and optimal near best approximation estimates.

The paper is structured as follows. In Section 2 the governing equations are stated and the three-field formulation of Biot’s model is discussed. Its semi-discretization in time by the implicit Euler method along with a proper rescaling of the parameters results in a static boundary value problem and is presented in Section 3. The latter then is discretized in space by a new family of hybridized discontinuous Galerkin/hybrid mixed methods while addressing the advantages of this approach. The main theoretical results follow in Section 4 where the uniform boundedness and the parameter-robust inf-sup stability of the underlying bilinear form are proven to be independent of all model and discretization parameters. Furthermore, the corresponding parameter-robust preconditioners and error estimates are provided. In Section 5 the theoretical results of this paper are complemented by a series of numerical tests assessing the approximation quality and cost efficiency of these preconditioners for the proposed family of higher-order hybridized discontinuous Galerkin/hybrid mixed discretizations.

2. Problem formulation

2.1. Governing equations. We consider a porous medium, which is linearly elastic, homogeneous, isotropic and saturated by an incompressible Newtonian fluid. Then Biot’s consolidation model,
see \cite{51, 5}, for a bounded Lipschitz domain $\Omega \in \mathbb{R}^d$, $d \in \{2, 3\}$,
\begin{align}
(1a) \quad -\text{div}(2\tilde{\mu}\epsilon(u)) - \lambda \text{div} u + \alpha \nabla p & = \tilde{f}, \quad \text{in } \Omega \times (0, T), \\
(1b) \quad \frac{\partial}{\partial t}(S_0 p + \text{div} u) - \text{div}(\nabla p) & = \tilde{g}, \quad \text{in } \Omega \times (0, T),
\end{align}
relates the deformation $u$ and the fluid pressure $p$ for a given body force density $\tilde{f}$ and mass source or sink $\tilde{g}$. For convenience, we assume a scalar conductivity coefficient $K$. In this work, we use bold symbols to denote vector- or tensor-valued quantities, e.g., $\epsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^T)$ denoting the symmetric gradient. Further, $\tilde{\lambda}$ and $\tilde{\mu}$ are the Lamé parameters, $\alpha$ is the Biot-Willis parameter and $S_0$ the constrained specific storage coefficient.

The three-field \cite{42, 44} formulation is based on the primary variables $(u, w, p)$, i.e.,
\begin{align}
-\text{div} \sigma & = \tilde{f}, \quad \text{in } \Omega \times (0, T), \\
K^{-1} w + \nabla p & = 0, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial}{\partial t}(S_0 p + \text{div} u) + \text{div} w & = \tilde{g}, \quad \text{in } \Omega \times (0, T),
\end{align}
where $w$ denotes the seepage velocity, $\tilde{\sigma} := 2\tilde{\mu}\epsilon(u) + \tilde{\lambda}\text{div}(u)I$ is the total stress and $\sigma = \tilde{\sigma} - \alpha p I$ the effective stress. If not mentioned otherwise, we assume homogeneous Dirichlet boundary conditions for the displacement $u$ and homogeneous Neumann conditions for the pressure $p$. In this context, let $H^1_0(\Omega), H_0(\text{div}, \Omega)$ denote the standard vector-valued Sobolev spaces where the subscript 0 refers to homogeneous essential boundary conditions. Further, let $L^2_0(\Omega)$ denote the space of square integrable functions with zero mean value. Following the standard procedure, one derives the weak formulation: Find $(u, w, p) \in H^1_0(\Omega) \times H_0(\text{div}, \Omega) \times L^2_0(\Omega)$ such that
\begin{align}
(2a) \quad \check{a}(u, v) - (\alpha p, \text{div} v) & = (\tilde{f}, v), \quad \forall v \in H^1_0(\Omega), \\
(2b) \quad (K^{-1} w, z) - (p, \text{div} z) & = 0, \quad \forall z \in H_0(\text{div}, \Omega), \\
(2c) \quad -(\alpha \text{div} \partial_t u, q) - (\text{div} w, q) - (S_0 \partial_t p, q) & = -(\tilde{g}, q), \quad \forall q \in L^2_0(\Omega),
\end{align}
where
\begin{align}
(3) \quad \check{a}(u, v) := 2\tilde{\mu} \int_{\Omega} \varepsilon(u) : \varepsilon(v) \, dx + \tilde{\lambda} \int_{\Omega} \text{div} u \cdot \text{div} v \, dx.
\end{align}
Finally, system (2) is completed with suitable initial conditions $u(\cdot, 0) = u_0(\cdot)$ and $p(\cdot, 0) = p_0(\cdot)$.

3. Hybridized discontinuous Galerkin/hybrid mixed discretizations of the Biot problem

3.1. Strongly mass-conserving discretization of the Biot problem. The starting point for this subsection is a family of strongly mass-conserving discretizations of the three-field formulation of the quasi-static Biot model based on a discontinuous Galerkin (DG) formulation for the mechanics subproblem, as proposed in \cite{24}. After time discretization by the implicit Euler scheme, the method for the arising static problem in each time step can be expressed as follows:

Find the time-step functions $(u^k, w^k, p^k) := (u(x, t_k), w(x, t_k), p(x, t_k)) \in H^1_0(\Omega) \times H_0(\text{div}, \Omega) \times L^2_0(\Omega)$ which solve the following system of equations
\begin{align}
(4a) \quad \check{a}(u^k, v) - (\alpha p^k, \text{div} v) & = (\tilde{f}^k, v), \quad \forall v \in H^1_0(\Omega), \\
(4b) \quad (K^{-1} w^k, z) - (p^k, \text{div} z) & = 0, \quad \forall z \in H_0(\text{div}, \Omega), \\
(4c) \quad -(\alpha \text{div}(u^k - u^{k-1}), q) - \tau(\text{div} w^k, q) - (S_0(p^k - p^{k-1}), q) & = -(\tilde{g}^k, q), \quad \forall q \in L^2_0(\Omega),
\end{align}
where $\tau$ is the time-step parameter and $\tilde{f}^k = \tilde{f}(x, t_k), \tilde{g}^k = \tilde{g}(x, t_k)$. 

For the space discretization, consider a shape-regular triangulation \( T_h \) whose set of facets are denoted by \( F_h \). We introduce the following finite element spaces

\[
U_h := \{ v \in H_0(\text{div}, \Omega) : v|_T \in U(T), \ T \in T_h \},
\]

\[
W_h := \{ z \in H_0(\text{div}, \Omega) : z|_T \in W(T), \ T \in T_h \},
\]

\[
P_h := \{ q \in L^2_0(\Omega) : q|_T \in P(T), \ T \in T_h \}.
\]

The local spaces \( U(T), W(T), P(T) \) are either \( \text{BDM}_{\ell}(T) \), \( \text{RT}_{\ell-1}(T) \), \( P_{\ell-1}(T) \) or by \( \text{BDM}_\ell(T), \text{RT}_{\ell-1}(T), \) \( P_{\ell-1}(T) \) where \( \text{BD}(F)M_\ell(T) \), \( \text{RT}_{\ell-1}(T), \) and \( P_{\ell-1}(T) \) denote the local Brezzi-Douglas-(Fortin-)Marini space of order \( \ell \), the Raviart-Thomas space of order \( \ell - 1 \), and full polynomials of degree \( \ell - 1 \), respectively. A definition of these local spaces can be found, for example, in [7].

We present the definitions of some trace operators next. Let \( F = \partial T_1 \cap \partial T_2 \) be a common facet of two adjacent elements \( T_1, T_2 \subset T_h \), and let \( n_1, n_2 \) be the corresponding outward pointing unit normal vectors. For any interior facet \( F \not\subset \partial \Omega \) and element-wise smooth and scalar-valued function \( q \), vector-valued function \( v \) and tensor-valued function \( \tau \), their averages and jumps on the facet \( F \) are defined by

\[
\{ v \} = \frac{1}{2} (v_1 \cdot n_1 - v_2 \cdot n_2), \quad \{ \tau \} = \frac{1}{2} (\tau_1 n_1 - \tau_2 n_2), \quad [q] = q_1 - q_2, \quad [v] = v_1 - v_2,
\]

where the subscript \( i \), \( i = 1, 2 \), with the functions \( q, v \) and \( \tau \) refers to their evaluation on \( T_i \cap F \).

For any boundary facet \( F \subset \partial \Omega \), these quantities are given as

\[
\{ v \} = v|_F \cdot n, \quad \{ \tau \} = \tau|_F n, \quad [q] = q|_F, \quad [v] = v|_F.
\]

With these definitions at hand, the formulation of the method is as follows: Find \( (u_h, w_h, p_h) \in U_h \times W_h \times P_h \), such that

\[
(\text{5a}) \quad a_h(u_h, v_h) - (p_h, \text{div} v_h) = (f, v_h), \quad \forall v_h \in U_h,
\]

\[
(\text{5b}) \quad (R^{-1} w_h, z_h) - (p_h, \text{div} z_h) = 0, \quad \forall z_h \in W_h,
\]

\[
(\text{5c}) \quad - (\text{div} u_h, q_h) - (\text{div} w_h, q_h) - (S p_h, q_h) = (g, q_h), \quad \forall q_h \in P_h.
\]

This system has been derived by dividing system (4) by \( 2\mu \) and, additionally, equation (4b) by the time step size \( \tau \) and furthermore by applying the substitutions \( u_h = \alpha u^k_h, w_h = \tau w^k_h, p_h = \alpha^2 p^k_h \). The right-hand sides in (5) are

\[
f = \alpha \tilde{f}(x, t_k)/2\mu \text{ and } g = (\tau \tilde{g}(x, t_k) - \alpha \text{div}(u_h(x, t_{k-1}))) - c_0 p(x, t_{k-1}))/2\mu,
\]

\[
a_h(u_h, v_h) := a^\text{DG}_h(u_h, v_h) + \lambda \int \text{div} u_h \text{div} v_h \, dx
\]

and

\[
(\text{6}) \quad \lambda := \frac{\lambda}{2\mu}, \quad R := \frac{2\mu \tau}{\alpha^2} K > 0, \quad S := \frac{2\mu S_0}{\alpha^2}.
\]

Note that the discrete bilinear form \( a_h(\cdot, \cdot) \) is obtained from scaling the bilinear form in (3) by \( 1/2\mu \). We denote the tangential component of any vector field on a facet by its symbol with a subscript \( t \). Then the symmetric interior penalty Galerkin (SIPG) bilinear form \( a^\text{DG}_h(\cdot, \cdot) \) is defined as

\[
a^\text{DG}_h(u, v) := \sum_{T \in T_h} \int_K e(u) : e(v) \, dx - \sum_{F \in F_h} \int_F \{ e(u) \} \cdot [w_t] \, ds
\]

\[
- \sum_{F \in F_h} \int_F \{ e(v) \} \cdot [u_t] \, ds + \sum_{F \in F_h} \eta \epsilon^2 h_F^{-1} \int_F [u_t] \cdot [v_t] \, ds
\]

with a sufficiently large stabilization parameter \( \eta \) independent of all model parameters, i.e., \( \lambda, R, S \), and discretization parameters \( h \) and \( \tau \). Note that in this paper the constants in all parameter robust estimates are independent of model and discretization parameters.
3.2. Hybridized DG method. When dealing with Stokes-type problems, $H(\text{div})$-conforming discretizations possess several advantages over $H^1$-conforming discretizations. This is mainly due to the fact that they allow for a suitable approximation of the incompressibility constraint which results in favorable properties such as pointwise divergence-free solutions and pressure robustness, see, e.g., [12, 10]. However, the incorporation of (tangential) continuity in standard DG schemes leads to a significantly increased number of (globally) coupled degrees of freedom (dof). To overcome this, in hybridized DG methods, one decouples element unknowns by introducing additional unknowns on the facets through which (tangential) continuity is imposed weakly, see, e.g., [9, 35].

In the context of an $H(\text{div})$-conforming hybridized DG discretization, one introduces an additional space

$$
\tilde{U}_h := \{ \tilde{u} \in L^2(\mathcal{F}_h) : \tilde{u}|_{F} \in P_{\ell}(F) \text{ and } \tilde{u}|_{F} \cdot n = 0, F \in \mathcal{F}_h; \quad \tilde{u} = 0 \text{ on } \partial \Omega \}
$$

for the approximation of the tangential trace of the displacement field $u$. Here, $L^2(\mathcal{F}_h)$ denotes the space of vector-valued square integrable functions on the skeleton $\mathcal{F}_h$ and $P_{\ell}(F)$ the vector-valued polynomial space of order $\ell$ on each facet $F \in \mathcal{F}_h$. We replace the bilinear form $a^\text{DG}((\cdot, \cdot))$ defined in (7) by $a^\text{HDG}((\cdot, \cdot))$ given by

$$
a^\text{HDG}((u, \tilde{u}), (v, \tilde{v})) := \sum_{T \in \mathcal{T}_h} \left[ \int_T \epsilon(u) : \epsilon(v) \, dx + \int_{\partial T} \epsilon(u)n \cdot (\tilde{v} - v)_t \, ds 
+ \int_{\partial T} \epsilon(v)n \cdot (\tilde{u} - u)_t \, ds + \eta \ell^2 h^{-1} \int_{\partial T} (\tilde{u} - u)_t \cdot (\tilde{v} - v)_t \, ds \right],
$$

where $(u, \tilde{u}), (v, \tilde{v}) \in \mathcal{U}_h := U_h \times \tilde{U}_h$. Our approach for exactly divergence-free hybridized discontinuous Galerkin methods will be based on [35] as well as its improvements presented in [31, 30]. The resulting method for the Biot problem now reads as: Find $((u_h, \tilde{u}_h), (w_h, p_h)) \in \mathcal{U}_h \times \mathcal{W}_h \times P_h$, such that

$$(9a) \quad a_h((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) - (p_h, \text{div}w_h) = (f, v_h), \quad \forall (v_h, \tilde{v}_h) \in \mathcal{U}_h,$$

$$(9b) \quad (R^{-1}w_h, z_h) - (p_h, \text{div}z_h) = 0, \quad \forall z_h \in \mathcal{W}_h,$$

$$(9c) \quad - (\text{div}u_h, q_h) - (\text{div}w_h, q_h) - (Sp_h, q_h) = (g, q_h), \quad \forall q_h \in P_h,$$

where

$$
a_h((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) := a^\text{HDG}((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) + \lambda \int_\Omega \text{div}u_h \text{div}v_h \, dx
$$

and $a^\text{HDG}((\cdot, \cdot), (\cdot, \cdot))$ is defined in (8).

3.3. A family of hybridized DG/hybrid mixed methods. In this subsection, we enrich the hybridization idea by additionally introducing a hybrid mixed formulation for the flow subproblem. While the stability analysis presented in [24, 21] uses properly scaled $H(\text{div})$ and an $L^2$ norms for the flow subproblem, we hybridize the latter one in the present work. This approach has the advantage that when solving the full saddle-point problem with some preconditioned iterative method, one needs to invert a div-grad type operator instead of a grad-div operator in order to apply the preconditioner which is easier and more cost-efficient in general. Note that the solution of the hybridized system is the same as that of the non-hybridized one.

The additional hybridization step can be expressed as follows. First, one enforces the normal continuity of the velocity by a Lagrange multiplier. To this end, we introduce the following finite element spaces

$$
\mathcal{W}_h^- := \{ z \in L^2(\Omega) : z|_T \in \mathcal{W}(T), T \in \mathcal{T}_h \}, \quad \mathcal{P}_h := \prod_{F \in \mathcal{F}_h} P_{l-1}(F), \quad \mathcal{T}_h := P_h \times \mathcal{P}_h,
$$
where \( W(T) \) can be chosen in the same way as before. Here, the space \( W_h^- \) is simply a discontinuous version of the space \( W_h^+ \). Further, note that \( \hat{P}_h \) is chosen as the normal trace space of \( W_h \), e.g., in the case \( W(T) = RT_0 \) (thus \( l - 1 = 0 \)) the normal traces on each facet are constant and so correspondingly we also choose \( \hat{P}_h \) to be defined as facet wise constants. Based on these spaces, we next define for all \( w_h \in W_h^- \) and \( (p_h, \hat{p}_h) \in \hat{P}_h \) the bilinear form

\[
b((p_h, \hat{p}_h), w_h) = \sum_{T \in T_h} \left( \int_T \text{div} w_h p_h \, dx - \int_{\partial T} w_h \cdot n \hat{p}_h \, ds \right).
\]

This bilinear form can be interpreted as a distributional version of the inner product \( \langle (\text{div} w_h, p_h) \rangle \) since functions in \( W_h^- \) are not normal continuous. Therefore, variational problem (9), when using a hybrid mixed formulation of the flow subproblem, is expressed as: Find \( ((u_h, \hat{u}_h), w_h, (p_h, \hat{p}_h)) \in U_h \times W_h^- \times \hat{P}_h \), such that

\[
\begin{align}
(12a) \quad & a_h((u_h, \hat{u}_h), (v_h, \hat{v}_h)) - (p_h, \text{div} v_h) = (f, v_h), \quad \forall (v_h, \hat{v}_h) \in U_h, \\
(12b) \quad & (R^{-1} w_h, z_h) - b((p_h, \hat{p}_h), z_h) = 0, \quad \forall z_h \in W_h^-,
\end{align}
\]

\[
(12c) \quad - (\text{div} u_h, q_h) - b((q_h, \hat{q}_h), w_h) - (Sp_h, q_h) = (g, q_h), \quad \forall (q_h, \hat{q}_h) \in \hat{P}_h.
\]

Note that if we test this system with the test function \( ((0, 0), 0, (0, \hat{q}_h)) \), we obtain

\[
b((0, \hat{p}_h), w_h) = \sum_{T \in T_h} \int_{\partial T} w_h \cdot n \hat{p}_h \, ds = \sum_{F \in F_h} \int_F [w_h \cdot n] \hat{p}_h \, ds = 0.
\]

Hence, by choosing \( \hat{q}_h = [w_h \cdot n] \) on each facet \( F \in F_h \), the above equation demonstrates that the velocity solution of (12) is normal continuous, i.e. \( w_h \in W_h^- \).

In the next section, we extend the parameter-robust stability results from [24, 21] to the hybridized three-field formulation given by systems (9) and (12).

4. Parameter-robust stability, preconditioners and optimal error estimates

4.1. Parameter-robust well-posedness.

4.1.1. Parameter-dependent norms. First, let us recall the norms previously used in the parameter robust stability analysis presented in [24]. These are, for the infinite dimensional spaces \( U, W, P \),

\[
\begin{align}
(13a) \quad & \|v\|_U^2 := \|v\|_0^2 + \lambda \|\text{div} v\|_0^2, \\
(13b) \quad & \|z\|_{W^+}^2 := R^{-1} \|z\|_0^2 + \gamma^{-1} \|\text{div} z\|_0^2, \\
(13c) \quad & \|z\|_{W^-}^2 := R^{-1} \|z\|_0^2, \\
(13d) \quad & \|q\|_P^2 := \gamma \|q\|_0^2,
\end{align}
\]

where the parameter \( \gamma \) can be defined as \( \gamma := \lambda_0^{-1} + R + S \approx \max \{\lambda_0^{-1}, R, S\} \), with \( \lambda_0 = \max \{1, \lambda\} \approx 1 + \lambda \), or exactly as in [24] where \( \gamma \) has been defined as \( \gamma := \max \{\min \{\lambda, (R^{-1})^{-1}\}^{-1}, S\} \). Due to the non-conformity of the DG discretization, the norm for the discrete displacement space \( U_h \) is based on the standard DG norm

\[
(14) \quad \|v_h\|_{DG}^2 := \sum_{T \in T_h} \|\nabla v_h\|_{0,T}^2 + \sum_{F \in F_h} h^{-1}_F \|(v_h)_{\mid T}\|_{0,F}^2 + \sum_{T \in T_h} h^2_F \|v_h\|_{2,T}^2
\]

and defined by

\[
(15) \quad \|v_h\|_{U_h}^2 := \|v_h\|_{DG}^2 + \lambda \|\text{div} v_h\|_0^2.
\]

Next, we introduce the hybridized discontinuous Galerkin (HDG) norm

\[
(16) \quad \|(v_h, \hat{v}_h)\|_{HDG}^2 := \sum_{T \in T_h} \left( \|\nabla v_h\|_{0,T}^2 + h^{-1}_T \|(\hat{v}_h - v_h)_{\mid T}\|_{0,\partial T}^2 + h^2_T \|v_h\|_{2,T}^2 \right),
\]
based on which we can define a discrete norm on the extended displacement space $\mathcal{U}_h$, i.e.,
\begin{equation}
\|(v_h, \hat{v}_h)\|_{\mathcal{T}_h}^2 := \|(v_h, \hat{v}_h)\|_{\text{HDG}}^2 + \lambda \|\text{div}v_h\|_0^2.
\end{equation}
Moreover, we define the following discrete norm on the extended pressure space $\mathcal{P}_h$
\begin{equation}
\|(q_h, \hat{q}_h)\|_{\mathcal{P}_h}^2 := \sum_{T \in \mathcal{T}} (\|\nabla q_h\|_{0,T}^2 + h_T^{-1}\|\hat{q}_h - q_h\|_{0,\partial T}^2 + h_T^2|q_h|_{2,T}^2),
\end{equation}
\begin{equation}
\|(q_h, \hat{q}_h)\|_{\mathcal{P}_h}^2 := R\|(q_h, \hat{q}_h)\|_{\text{HDG}}^2 + \gamma \|q_h\|_0^2,
\end{equation}
where
\begin{equation}
\gamma = S + \frac{1}{\lambda} \simeq \max \left\{S, \frac{1}{\lambda_0}\right\}.
\end{equation}
Finally, we consider the following two product spaces
\begin{align}
\mathbf{X}_h &:= \mathcal{U}_h \times \mathbf{W}_h \times \mathbf{P}_h, \\
\overline{\mathbf{X}}_h &:= \mathcal{U}_h \times \mathbf{W}_h^- \times \mathcal{P}_h
\end{align}
equipped with the norms
\begin{align}
\|(v_h, \hat{v}_h, z_h, q_h)\|_{\mathbf{X}_h}^2 &:= \|(v_h, \hat{v}_h)\|_{\mathbf{U}_h}^2 + \|z_h\|_{\mathbf{W}}^2 + \|q_h\|_{\mathcal{P}}^2, \\
\|(v_h, \hat{v}_h, z_h, q_h)\|_{\overline{\mathbf{X}}_h}^2 &:= \|(v_h, \hat{v}_h)\|_{\mathbf{U}_h}^2 + \|z_h\|_{\mathbf{W}_h^-}^2 + \|(q_h, \hat{q}_h)\|_{\mathcal{P}_h}^2
\end{align}
in the context of problems (9) and (12), respectively.

4.1.2. Uniform well-posedness of the time-discrete problem. The well-posedness of the three-field formulation (2) on the continuous and discrete levels has been addressed and answered in [53, 54, 50, 19] using semi-group theory and Galerkin discretization methods. After time discretization by an implicit or semi-implicit time integration scheme, the continuous three-field formulation results in a variational problem of the form: Find $x \in X$ such that
\begin{equation}
\mathcal{A}(x, y) = \mathcal{F}(y), \quad \forall y \in X := U \times W \times P,
\end{equation}
where $\mathcal{A}(x, y) := a(u, v) - (p, \text{div}v) + (R^{-1}w, z) - (p, \text{div}z) - (\text{div}u, q) - (\text{div}w, q) - (Sp, q)$ and $\mathcal{F}(\cdot) \in X'$ denotes a corresponding linear form, which depends on the time integrator.

As it is well known, the abstract variational problem (22) is well-posed under the following necessary and sufficient conditions, see [3].

**Theorem 1.** Assume that $\mathcal{F} \in X'$ and the bilinear form $\mathcal{A}(\cdot, \cdot)$ in (22) satisfies the following conditions:
- $\mathcal{A}(\cdot, \cdot)$ is bounded, i.e., there exists a constant $C > 0$ such that
\begin{equation}
\mathcal{A}(x, y) \leq C \|x\|_X \|y\|_X, \quad \forall x, y \in X;
\end{equation}
- There exists a constant $\beta > 0$ such that
\begin{equation}
\inf_{x \in X} \sup_{y \in X} \frac{\mathcal{A}(x, y)}{\|y\|_X} \geq \beta > 0.
\end{equation}

Then there exists a unique solution $x^* \in X$ of the variational problem (22). Further, the solution satisfies the stability estimate
\begin{equation}
\|x^*\|_X \leq \sup_{y \in X} \frac{\mathcal{F}(y)}{\|y\|_X} =: \|\mathcal{F}\|_{X'}.
\end{equation}
Besides for the establishment of well-posedness on the continuous and discrete levels, boundedness, i.e., property (23), and inf-sup stability, i.e., property (24), is crucial in the error analysis and for the construction of preconditioners and iterative solution methods for the algebraic problems arising from the discretization of (22). Furthermore, aiming at parameter-independent error, or near-best approximation estimates and parameter-robust preconditioners, it is essential that the constants $C$ and $\beta$ in (23) and (24) are independent of any physical (model) and discretization parameters.

**Definition 1.** We call problem (22) uniformly well-posed on its parameter space (or, in short, uniformly well-posed) under the norm $\| \cdot \|_X$ if the conditions of Theorem 1 are satisfied and the constants $C$ and $\beta$ in (23) and (24) do not depend on any of the problem parameters.

**Remark 1.** The parameter space is the space of all problem parameters, i.e., physical parameters of the continuous mathematical model but also discretization parameters when $A(\cdot, \cdot)$ represents a semi- or fully discrete problem.

Uniform well-posedness of the time-discrete problem resulting from the three-field formulation of Biot’s consolidation model has first been proven in [24] using the norm $\| \cdot \|_{\bar{U}_h}$.

\[
\| (v, z, q) \|_{\bar{U}_h}^2 := \| v \|_{\bar{U}}^2 + \| z \|_{W}^2 + \| q \|_{p}^2
\]

where $\| \cdot \|_{U}$, $\| \cdot \|_{W}$, $\| \cdot \|_{p}$ are defined in (13). In the remainder of Subsection 4.1, we extend the uniform well-posedness analysis from [24, 21] to the three-field formulations (9) and (12).

### 4.1.3. Hybridized DG method

Following the approach presented in [24], we will show that problem (9) is uniformly well-posed. Initially, we rewrite (9) in the form: Find $\bar{x}_h := ((u_h, \hat{u}_h), (w_h, p_h)) \in \bar{U}_h \times W_h \times P_h := X_h$, such that

\[
\bar{A}_h(\bar{x}_h, \bar{y}_h) = \bar{F}_h(\bar{y}_h), \quad \forall \bar{y}_h \in X_h,
\]

where with $\bar{y}_h := ((v_h, \hat{v}_h), (z_h, q_h))$ we have

\[
\bar{A}_h(\bar{x}_h, \bar{y}_h) := a_h((u_h, \hat{u}_h), (v_h, \hat{v}_h)) - (p_h, \text{div} v_h) + (R^{-1} w_h, z_h)
\]

\[
- (p_h, \text{div} z_h) - (\text{div} u_h, q_h) - (\text{div} w_h, q_h) - (S p_h, q_h),
\]

\[
(27b) \quad \bar{F}_h(\bar{y}_h) := (f, v_h) + (g, q_h),
\]

and $a_h(\cdot, \cdot), \beta \geq 0$ is defined in (10). Next, we recall two auxiliary results crucial for establishing the main result of this subsection.

**Lemma 2.** The following discrete inf-sup condition

\[
(28) \quad \inf_{q_h \in P_h} \sup_{(v_h, \hat{v}_h) \in \bar{U}_h} \frac{(\text{div} v_h, q_h)}{\| (v_h, \hat{v}_h) \|_{HDG}} \geq \beta_{S,d} > 0,
\]

holds where $\| \cdot \|_{HDG}$ is the HDG norm defined in (16).

**Proof.** As shown, for example, in [23, 20], the following inf-sup condition holds true:

\[
(29) \quad \inf_{q_h \in P_h} \sup_{v_h \in U_h} \| (v_h) \|_{HDG} \geq \beta_{S,d} > 0.
\]

Moreover, for all $v_h \in U_h$ there exists $\hat{v}_h \in \bar{U}_h$ such that $\| v_h \|_{HDG} \geq C \| (v_h, \hat{v}_h) \|_{HDG}$ with a constant $C$ depending only on mesh regularity. Combining the latter estimate with (29) yields (28). \hfill \Box

The proof of the following theorem also makes use of the boundedness and coercivity of the bilinear form $a_h^{HDG}(\cdot, \cdot), \beta \geq 0$ on $U_h$ defined in (8), i.e.,

\[
(30) \quad |a_h^{HDG}((u_h, \hat{u}_h), (v_h, \hat{v}_h))| \leq C_a \| (u_h, \hat{u}_h) \|_{HDG} \| (v_h, \hat{v}_h) \|_{HDG}
\]
for all \((u_h, \hat{u}_h), (v_h, \hat{v}_h) \in \overline{U}_h\) and
\[ a_h^{\text{HDG}}((u_h, \hat{u}_h), (u_h, \hat{u}_h)) \geq C_c \|(u_h, \hat{u}_h)\|_{\text{HDG}}^2 \] for all \((u_h, \hat{u}_h) \in \overline{U}_h\), see e.g. [35, 30].

**Theorem 3.** Problem (26)–(27) is uniformly well-posed under the norm \(\|\cdot\|_{\overline{X}_h}\) defined in (21a), that is,
\[ \overline{A}(\overline{x}_h, \overline{y}_h) \leq \overline{C}\|\overline{x}_h\|_{\overline{X}_h}\|\overline{y}_h\|_{\overline{X}_h} \quad \forall \overline{x}_h, \overline{y}_h \in \overline{X}_h, \]
\[ \inf_{\overline{x}_h} \sup_{\overline{y}_h} \frac{\overline{A}(\overline{x}_h, \overline{y}_h)}{\|\overline{x}_h\|_{\overline{X}_h}\|\overline{y}_h\|_{\overline{X}_h}} \geq \overline{\beta} > 0. \]

**Proof.** To show (32) one uses Cauchy-Schwarz inequality, the continuity of the bilinear form \(a_h((\cdot, \cdot), (\cdot, \cdot))\) in the norm \(\|\cdot\|_{\overline{U}_h}\) on \(\overline{U}_h\), i.e.,
\[ |a_h((u_h, \hat{u}_h), (v_h, \hat{v}_h))| \leq \overline{C}_a \|(u_h, \hat{u}_h)\|_{\overline{U}_h}\|(v_h, \hat{v}_h)\|_{\overline{U}_h} \quad \forall (u_h, \hat{u}_h), (v_h, \hat{v}_h) \in \overline{U}_h, \]
which follows from (30) as well as the definitions of \(a_h((\cdot, \cdot), (\cdot, \cdot))\) and \(\|\cdot\|_{\overline{U}_h}\) and \(\|\cdot\|_{\overline{X}_h}\), see (10), (17) and (21a), respectively.

The proof of (33) follows exactly the lines of the proof of Theorem 4.4 in [24] replacing the DG bilinear form (7) by the HDG bilinear form (8) and the DG norm (14) by the HDG norm (16).

**4.1.4. Hybridized DG/hybrid mixed method.** Consider the HDG/hybrid mixed method for the three-field formulation as stated in (12). To prove the uniform well-posedness of this fully discrete problem, as we did with (9), we rewrite (12) in the form: Find \(\overline{\theta}_h := ((u_h, \hat{u}_h), w_h, (p_h, \hat{p}_h)) \in \overline{U}_h \times W_h \times \overline{P}_h =: \overline{X}_h\) such that
\[ \overline{A}_h(\overline{\theta}_h, \overline{\theta}_h) = \overline{F}_h(\overline{\theta}_h), \quad \forall \overline{\theta}_h \in \overline{X}_h, \]
where with \(\overline{\theta}_h := ((\nu_h, \hat{\nu}_h), z_h, (q_h, \hat{q}_h))\) we have
\[ \overline{A}_h(\overline{\theta}_h, \overline{\theta}_h) := a_h((u_h, \hat{u}_h), (\nu_h, \hat{\nu}_h)) - (p_h, \text{div}v_h) + (R^{-1}w_h, z_h) \]
\[ - b((p_h, \hat{p}_h), z_h) - (\text{div}u_h, q_h) - (q_h, \hat{q}_h), w_h) - (Sp_h, q_h), \]
(36a)
\[ \overline{F}_h(\overline{\theta}_h) := (f, v_h) + (g, q_h), \]
(36b)
and \(a_h((\cdot, \cdot), (\cdot, \cdot))\) and \(b((\cdot, \cdot), \cdot)\) are defined in (10) and (11), respectively. Before proving the main theorem, we need another auxiliary result given by the following lemma.

**Lemma 4.** There holds the following discrete inf-sup condition
\[ \inf_{(q_h, \hat{q}_h) \in \overline{P}_h} \sup_{z_h \in V_h} \frac{b((q_h, \hat{q}_h), z_h)}{\|z_h\|_{HDG} \|(q_h, \hat{q}_h)\|_{HDG}} \geq \overline{\beta}_{D,d} > 0 \]
where \(\|(\cdot, \cdot)\|_{HDG}\) is defined in (18a).

**Proof.** A direct proof of (37) can be readily constructed, similarly as for the inf-sup condition in [16, 17], using the definition of the degrees of freedom for the Raviart-Thomas space, see [7], and standard scaling arguments.

Such inf-sup conditions with mesh-dependent norms are widely used in structural mechanics, see, e.g., [13].

**Theorem 5.** Problem (35)–(36) is uniformly well-posed under the norm \(\|\cdot\|_{\overline{X}_h}\) defined in (21b).
We start with proving the boundedness of the bilinear form \( \overline{A}_h(\cdot, \cdot) \), i.e.,
\[
\overline{A}_h(\underline{v}_h, \underline{g}_h) \leq \overline{C} ||\underline{v}_h||_{\overline{X}_h} ||\underline{g}_h||_{\overline{X}_h} \quad \forall \underline{v}_h, \underline{g}_h \in \overline{X}_h.
\]

First we note that
\[
b((p_h, \hat{p}_h), w_h) = \sum_{T \in \mathcal{T}_h} \int_T \text{div} w_h p_h \, dx - \int_{\partial T} w_h \cdot n \hat{p}_h \, ds \\
= \sum_{T \in \mathcal{T}_h} \int_T -w_h \cdot \nabla p_h \, dx - \int_{\partial T} w_h \cdot n (\hat{p}_h - p_h) \, ds \\
= \sum_{T \in \mathcal{T}_h} \int_T -w_h \cdot \nabla p_h \, dx - \int_{\partial T} h w_h \cdot n \frac{1}{h} (\hat{p}_h - p_h) \, ds \\
\leq \sqrt{||w_h||_0^2 + \sum_{T \in \mathcal{T}_h} h ||w_h \cdot n||_{\partial T} ||(p_h, \hat{p}_h)||_{\text{HDG}}} \\
\leq C_b ||w_h||_0 ||(p_h, \hat{p}_h)||_{\text{HDG}},
\]
where we have used standard scaling arguments in the last step of (39), i.e. the constant \( C_b \) depends only on the mesh regularity.

Further, using the continuity of the bilinear form \( a_h((\cdot, \cdot), (\cdot, \cdot)) \) on \( \overline{U}_h \), i.e. (34), the definitions of the norms \( ||\cdot||_{\overline{U}_h}, ||\cdot||_{\overline{W}_h}, ||\cdot||_{\overline{P}_h}, \) and \( ||\cdot||_{\overline{X}_h} \), see (17), (13b), (18b) and (21b), respectively, and applying the Cauchy-Schwarz inequality and also estimate (39), one gets
\[
\overline{A}_h(\underline{v}_h, \underline{g}_h) := a_h((u_h, \hat{u}_h), (v_h, \hat{v}_h)) - (p_h, \text{div} v_h) + (R^{-1} w_h, z_h) \\
- b((p_h, \hat{p}_h), z_h) - (\text{div} u_h, q_h) - b((q_h, \hat{q}_h), w_h) - (Sp_h, q_h) \\
\leq C_a ||(u_h, \hat{u}_h)||_{\overline{U}_h} ||(v_h, \hat{v}_h)||_{\overline{U}_h} + \lambda^{-1/2} ||p_h||_0 \lambda^{1/2} ||\text{div} v_h||_0 \\
+ R^{-1/2} ||w_h||_0 R^{-1/2} ||z_h||_0 + C_h R^{1/2} ||(p_h, \hat{p}_h)||_{\text{HDG}} R^{-1/2} ||z_h||_0 \\
+ \lambda^{1/2} ||\text{div} u_h||_0 \lambda^{-1/2} ||q_h||_0 + C_h R^{1/2} ||(q_h, \hat{q}_h)||_{\text{HDG}} R^{-1/2} ||w_h||_0 \\
+ S^{1/2} ||p_h||_0 S^{1/2} ||q_h||_0 \\
\leq C_a ||(u_h, \hat{u}_h)||_{\overline{U}_h} ||(v_h, \hat{v}_h)||_{\overline{U}_h} + ||(p_h, \hat{p}_h)||_{\overline{P}_h} ||(v_h, \hat{v}_h)||_{\overline{U}_h} \\
+ ||w_h||_{\overline{W}_h} ||z_h||_{\overline{W}_-} + C_b ||(p_h, \hat{p}_h)||_{\overline{P}_h} ||z_h||_{\overline{W}_-} \\
+ ||(u_h, \hat{u}_h)||_{\overline{U}_h} ||(q_h, \hat{q}_h)||_{\overline{P}_h} + C_b ||(q_h, \hat{q}_h)||_{\overline{P}_h} ||w_h||_{\overline{W}_-} \\
+ ||(p_h, \hat{p}_h)||_{\overline{P}_h} ||(q_h, \hat{q}_h)||_{\overline{P}_h} \\
\leq \overline{C} \left( ||(u_h, \hat{u}_h)||_{\overline{U}_h} + ||w_h||_{\overline{W}_-} + ||(p_h, \hat{p}_h)||_{\overline{P}_h} \right) \\
\times \left( ||(v_h, \hat{v}_h)||_{\overline{U}_h} + ||z_h||_{\overline{W}_-} + ||(q_h, \hat{q}_h)||_{\overline{P}_h} \right).
\]

Next we prove the inf-sup condition
\[
\inf_{\overline{v}_h \in \overline{X}_h} \sup_{\overline{g}_h \in \overline{X}_h} \frac{\overline{A}_h(\overline{v}_h, \overline{g}_h)}{||\overline{v}_h||_{\overline{X}_h} ||\overline{g}_h||_{\overline{X}_h}} \geq \frac{\overline{C}}{\beta} > 0
\]
which immediately follows if for all \( \overline{v}_h \in \overline{X}_h \) we can find \( \overline{g}_h = \overline{g}_h(\overline{v}_h) \) such that
\[
||\overline{g}_h||_{\overline{X}_h} \leq \overline{C} ||\overline{v}_h||_{\overline{X}_h}
\]
Note that the existence of the discrete inf-sup conditions (28) and (37). With this particular choice, we first verify (42). To begin with

\[ \| \frac{1}{\sqrt{\lambda_0}} (\mathbf{u}_{h,0}, \hat{u}_h) \|_{\Omega_h}^2 = \| \frac{1}{\sqrt{\lambda_0}} (\mathbf{u}_{h,0}, \hat{u}_h) \|_{\text{HDG}}^2 + \lambda_0 (\text{div} \left( \frac{1}{\sqrt{\lambda_0}} \mathbf{u}_{h,0} \right), \text{div} \left( \frac{1}{\sqrt{\lambda_0}} \mathbf{u}_{h,0} \right) ) \]

\[ \leq \frac{1}{\lambda_0} \beta^{-2}_{S,d} \frac{1}{\lambda_0} \| p_{h,0} \|_0^2 + \frac{1}{\lambda_0} \| p_h \|_0^2 \]

\[ \leq \left( \frac{1}{\lambda_0} \beta^{-2}_{S,d} + 1 \right) \| p_h \|_0^2 \]

\[ \leq \left( \frac{1}{\lambda_0} \beta^{-2}_{S,d} + 1 \right) \| (p_h, \hat{p}_h) \|_{\Omega_h}^2, \]

from which we conclude

\[ \| (\mathbf{v}_h, \hat{v}_h) \|_{\Omega_h} \leq \delta \| (\mathbf{u}_h, \hat{u}_h) \|_{\Omega_h} + \beta^{-2}_{S,d} + 1)^{\frac{1}{2}} \| (p_h, \hat{p}_h) \|_{\Omega_h}. \]

Next,

\[ \| z_h \|_{W^{-}} \leq \delta \| w_h \|_{W^{-}} + R \| w_{h,0} \|_{W^{-}} \]

\[ \leq \delta \| w_h \|_{W^{-}} + \delta \| w_{h,0} \|_{0} \]

\[ \leq \delta \| w_h \|_{W^{-}} + \delta \| w_{h,0} \|_{\text{HDG}} \]

\[ \leq \delta \| w_h \|_{W^{-}} + \beta^{-1}_{D,d} \| (p_h, \hat{p}_h) \|_{\Omega_h}. \]

Finally,

\[ \| (q_h, \hat{q}_h) \|_{\Omega_h} \leq \delta \| (p_h, \hat{p}_h) \|_{\Omega_h}. \]

The bounds (48), (49) and (50) together imply (42) with \( \bar{C}_h = [2(\delta^2 + \beta^{-2}_{S,d} + \beta^{-2}_{D,d} + 1)]^{\frac{1}{2}} \).
What remains is to verify (43):

$$\mathcal{A}_h((u_h, \dot{u}_h), w_h, (p_h, \dot{p}_h)) = a_h^{\text{HDG}}((u_h, \dot{u}_h), (v_h, \dot{v}_h)) + \lambda (\text{div} u_h, \text{div} v_h)
- b((p_h, \dot{p}_h), z_h) - (\text{div} u_h, q_h)
- b((q_h, \dot{q}_h), w_h) - (S p_h, q_h)
= \delta a_h^{\text{HDG}}((u_h, \dot{u}_h), (u_h, \dot{u}_h)) - \frac{1}{\sqrt{\lambda_0}} (\text{div} u_h, \text{div} u_h, (u_{h,0}, \dot{u}_{h,0})) + \delta \lambda (\text{div} u_h, \text{div} u_h) - \frac{\lambda}{\sqrt{\lambda_0}} (\text{div} u_h, \text{div} u_{h,0}) - (p_h, \text{div} u_h)
+ \frac{1}{\sqrt{\lambda_0}} (p_h, \text{div} u_{h,0}) + \delta R^{-1}(w_h, w_h) + (w_h, w_{h,0})
+ R \Vert (p_h, \dot{p}_h) \Vert_{\text{HDG}}^2 + \delta (\text{div} u_h, p_h) + \delta (S p_h, p_h)
\geq \delta a_h^{\text{HDG}}((u_h, \dot{u}_h), (u_h, \dot{u}_h)) - \frac{1}{2} \frac{1}{\lambda_0} \varepsilon_1 a_h^{\text{HDG}}((u_h, \dot{u}_h), (u_{h,0}, \dot{u}_{h,0})) + \delta \lambda (\text{div} u_h, \text{div} u_h)
- \frac{1}{2} \varepsilon_1 a_h^{\text{HDG}}((u_h, \dot{u}_h), (u_{h,0}, \dot{u}_{h,0})) + \delta \lambda (\text{div} u_h, \text{div} u_h)
- \frac{1}{2} \varepsilon_1 \lambda (\text{div} u_h, \text{div} u_h) - \frac{1}{2} \varepsilon_2 \lambda_0 (\text{div} u_{h,0}, \text{div} u_{h,0})
+ \frac{1}{\lambda_0} (p_h, p_h) + \delta R^{-1}(w_h, w_h) - \frac{1}{2} \varepsilon_3 R^{-1}(w_h, w_h)
- \frac{1}{2} \varepsilon_3 R (w_{h,0}, w_{h,0}) + R \Vert (p_h, \dot{p}_h) \Vert_{\text{HDG}}^2 + \delta (S p_h, p_h)
\geq \left( \delta - \frac{1}{2} \frac{1}{\lambda_0} \varepsilon_1 \right) a_h^{\text{HDG}}((u_h, \dot{u}_h), (u_h, \dot{u}_h)) + \left( \delta - \frac{1}{2} \varepsilon_2 \right) \lambda (\text{div} u_h, \text{div} u_h)
+ \left( \delta S + \frac{1}{\lambda_0} - \frac{1}{2} \varepsilon_2 \lambda_0 - \frac{1}{2} \varepsilon_1 C a \beta_{S,d}^{-2} \right) (p_h, p_h)
+ \left( \delta - \frac{1}{2} \varepsilon_3 \right) R^{-1}(w_h, w_h) + \left( 1 - \frac{1}{2} \varepsilon_3 \beta_{D,d}^{-2} \right) R \Vert (p_h, \dot{p}_h) \Vert_{\text{HDG}}^2.
$$

By choosing $\varepsilon_1 = \frac{1}{2} C a^{-1} \beta_{S,d}^2$, $\varepsilon_2 = \frac{1}{2}$, $\varepsilon_3 = \beta_{D,d}^2$ the last inequality becomes

$$\mathcal{A}_h((u_h, \dot{u}_h), w_h, (p_h, \dot{p}_h)) \geq (\delta - C a \beta_{S,d}^{-2}) a_h^{\text{HDG}}((u_h, \dot{u}_h), (u_h, \dot{u}_h)) (p_h, p_h)
+ \left( \delta S + \frac{1}{\lambda_0} - \frac{1}{2} \frac{1}{\lambda_0} - \frac{1}{2} \frac{1}{\lambda_0} \varepsilon_1 C a \beta_{S,d}^{-2} \right) (p_h, p_h)
+ \left( \delta - \frac{1}{2} \varepsilon_3 \right) R^{-1}(w_h, w_h) + \left( 1 - \frac{1}{2} \varepsilon_3 \beta_{D,d}^{-2} \right) R \Vert (p_h, \dot{p}_h) \Vert_{\text{HDG}}^2.
$$

For $\delta \geq \max \left\{ \frac{3}{2}, \frac{1}{2 \lambda_0} + C a \beta_{S,d}^{-2} \right\}$, we finally obtain

$$\mathcal{A}_h((u_h, \dot{u}_h), w_h, (p_h, \dot{p}_h)) \geq \frac{1}{2} (\Vert (u_h, \dot{u}_h) \Vert_{\text{HDG}}^2 + \lambda \Vert \text{div} u_h \Vert^2
+ \Vert S \frac{1}{\lambda_0} \Vert_{\text{HDG}}^2 + R \Vert (p_h, \dot{p}_h) \Vert_{\text{HDG}}^2 + R^{-1} \Vert w_h \Vert_{\text{HDG}}^2)
\geq \frac{1}{2} \left( \Vert (u_h, \dot{u}_h) \Vert_{\text{HDG}}^2 + \Vert (p_h, \dot{p}_h) \Vert_{\text{HDG}}^2 + \Vert w_h \Vert_\text{W}^2 \right),
$$

utilizing $a_h^{\text{HDG}}((u_h, \dot{u}_h)) \geq C e \Vert (u_h, \dot{u}_h) \Vert_{\text{HDG}}^2.$

\[\square\]
4.2. Uniform preconditioners. The results from the previous subsection imply a “mapping property” that is the basis for defining uniform preconditioners. Here, we discuss norm-equivalent (block-diagonal) preconditioners which fall into this category.

Consider a uniformly well-posed problem of the form (22) where \( A : X \rightarrow X' \) is a linear operator, i.e., \( A \in \mathcal{L}(X, X') \), \( F \in X' \) for a given Hilbert space \( X \), e.g., \( X := U \times W \times P \) or \( X := \overline{X}_h = \overline{U}_h \times \overline{W}_h \times \overline{P}_h \). Here we assume that \( A \) and \( F \) are defined via the bilinear and linear forms \( A(\cdot, \cdot) \), \( F(\cdot) \), or \( \mathcal{A}_h(\cdot, \cdot) \), \( \mathcal{F}_h(\cdot) \), or \( \overline{\mathcal{A}}_h(\cdot, \cdot) \), \( \overline{\mathcal{F}}_h(\cdot) \), cf. (22), (27), (36). Let us write equation (22) in operator form, i.e.,

\[
 Ax = F \in X'
\]

and define the linear operator \( B : X' \rightarrow X \), i.e., \( B \in \mathcal{L}(X', X) \) by

\[
 (BG, y)_X = (G, y), \quad \forall G \in X', y \in X,
\]

where \( (\cdot, \cdot)_X \) is the inner product inducing the norm \( \| \cdot \|_X \), that is, \( \| y \|_X = (y, y)^{1/2} \), or, equivalently, \( B^{-1} : X \rightarrow X' \), \( B^{-1} \in \mathcal{L}(X, X') \) by

\[
 (B^{-1} x, y)_X = (x, y)_X, \quad \forall x, y \in X,
\]

which implies

\[
 (B^{-1} x, x)_X = (x, x)_X = \| x \|^2_X, \quad \forall x \in X.
\]

In practice, the latter relation is often replaced by the weaker condition

\[
 (B^{-1} x, x) \approx \| x \|^2_X,
\]

for which reason the preconditioner \( B \) is also referred to as a norm-equivalent preconditioner, cf. [36]. The symbol “\( \approx \)” stands for a norm equivalence, uniform with respect to all problem parameters.

Since (23) and (24) are in the norm \( \| \cdot \|_X \), we conclude for the operators \( BA \in \mathcal{L}(X, X) \) and \( (BA)^{-1} \in \mathcal{L}(X, X) \) the following bounds:

\[
 \| BA \|_{\mathcal{L}(X, X)} = \sup_{x, y} \frac{(BAx, y)_X}{\| x \|_X \| y \|_X} = \sup_{x, y} \frac{(Ax, y)_X}{\| x \|_X \| y \|_X} = \sup_{x, y} \frac{\mathcal{A}(x, y)_X}{\| x \|_X \| y \|_X} \leq C,
\]

\[
 (\| (BA)^{-1} \|_{\mathcal{L}(X, X)})^{-1} = \inf_x \frac{1}{\sup_y \frac{(BA^{-1} x, y)_X}{\| x \|_X \| y \|_X}} = \inf_x \sup_y \frac{(BAx, y)_X}{\| x \|_X \| y \|_X}
\]

\[
 = \inf_x \sup_y \frac{(Ax, y)_X}{\| x \|_X \| y \|_X} = \inf_x \sup_y \frac{\mathcal{A}(x, y)_X}{\| x \|_X \| y \|_X} \geq \beta.
\]

Finally, (59) and (60) together imply that the condition number \( \kappa \) of the preconditioned operator \( BA \in \mathcal{L}(X, X) \) is uniformly bounded by a constant that does not depend on any problem parameters, i.e.,

\[
 \kappa(BA) := \| BA \|_{\mathcal{L}(X, X)} \| (BA)^{-1} \|_{\mathcal{L}(X, X)} \leq \frac{C}{\beta}.
\]

4.3. Optimal error estimates. The uniform well-posedness that we have established in Theorem 5 for the hybridized/hybrid mixed discretization implies near best approximation estimates, which we state next. For the following statements let \( (u, w, p) \) be the exact solution of the continuous problem (2) assuming that

\[
 u \in H^1_0(\Omega) \cap H^2(T_h), \quad w \in H^1_0(\text{div}, \Omega), \quad \text{and} \quad p \in H^1(\Omega) \cap H^2(T_h),
\]

where \( H^m(T_h) := \{ v \in L^2(\Omega) : v|_T \in H^m(T) \ \forall T \in T_h \} \) is the broken Sobolev space of order \( m \). Further let \( \overline{u} := (u, \hat{u}) \) and \( \overline{p} := (p, \hat{p}) \) with \( \hat{u} := u|_{T_h} \) and \( \hat{p} := p|_{T_h} \).
Theorem 6. Consider problem (35)–(36) as a discretization of the continuous problem (2) in three-field formulation and assume that the exact solution fulfills (62). Then the following near-best approximation result holds with a constants $\overline{C}_{e,uv}, \overline{C}_{e,p}$ independent of all problem parameters:

\[
\|\mathbf{u} - \mathbf{uh}\|_U + \|\mathbf{w} - \mathbf{w}_h\| - \overline{C}_{e,uv} \left( \inf_{\mathbf{v}_h \in U_h} \|\mathbf{u} - \mathbf{v}_h\|_U + \inf_{z_h \in W_h} \|\mathbf{w} - z_h\| \right)
\]

(63)

\[
\|\mathbf{p} - \mathbf{ph}\|_P \leq \overline{C}_{e,p} \left( \inf_{\mathbf{v}_h \in U_h} \|\mathbf{u} - \mathbf{v}_h\|_U + \inf_{z_h \in W_h} \|\mathbf{w} - z_h\| + \inf_{\mathbf{p}_h \in P_h} \|\mathbf{p} - \mathbf{ph}\|_P \right)
\]

(64)

Proof. The proof follows the lines of the proof of Theorem 5.2 in [24]. □

Remark 2. An analogous result to Theorem 6 is also valid for the discrete problem (26)–(27) if one replaces the spaces $W^-, W_h^-, P_h$ by $W, W_h$ and $P_h$ and the corresponding norms $\|\cdot\|_W$ and $\|\cdot\|_P$ by $\|\cdot\|_W^-$ and $\|\cdot\|_P^-$. The result is then a consequence of Theorem 3.

In the following let $\Pi_{P_h}(\cdot) = (\Pi_{P_h}(\cdot), \Pi_{P_h}(\cdot)) \in P_h$ be the standard element and facet-wise $L^2$-projection. Using the proper, well known (see [7, 2, 35]) interpolation operators and standard arguments, one can derive the following optimal error estimates from the above best approximation results.

Theorem 7. Consider problem (35)–(36) as a discretization of the continuous problem (2) in three-field formulation. Beside (62) we assume that the exact solution fulfills the regularity estimate $(u, w, p) \in H^m(T_h) \times H^{m-1}(T_h) \times H^{m-1}(T_h)$. Then there hold the following error estimates with a constants $\overline{C}_{e,uv}, \overline{C}_{e,p}$ independent of all problem parameters:

\[
\|\mathbf{u} - \mathbf{uh}\|_U + \|\mathbf{w} - \mathbf{w}_h\| - \overline{C}_{e,uv} h^{-s}(|u|_{H^{s+1}(T_h)} + \lambda^\frac{1}{2} |\operatorname{div}(u)|_{H^s(T_h)} + R^\frac{1}{2} |w|_{H^s(T_h)})
\]

\[
\|\mathbf{p} - \mathbf{ph}\|_P \leq \overline{C}_{e,p} h^{s-1}(|u|_{H^{s+1}(T_h)} + \lambda^\frac{3}{2} |\operatorname{div}(u)|_{H^{s+1}(T_h)} + R^{\frac{3}{2}} |w|_{H^{s+1}(T_h)} + R^\frac{3}{2} |p|_{H^{s+1}(T_h)} + \gamma^\frac{1}{2} |p|_{H^{s+1}(T_h)})
\]

where $s := \min\{l, m - 1\}$.

Remark 3. Assuming enough regularity of the exact solution, Theorem 7 shows that the projected error $\|\Pi_{P_h}\mathbf{p} - \mathbf{ph}\|_P$ converges with one order higher than $\|\mathbf{p} - \mathbf{ph}\|_P$. This super convergence property of (hybrid) mixed methods is well known in the literature, see for example [28, 27].

4.4. Implementation aspects and static condensation. In order to solve the discrete system, we employ static condensation of the local degrees of freedom. These are given by the dof introduced through the discontinuous approximation spaces $W_h^-$ and $P_h$. One can also eliminate the local $H(\operatorname{div})$-conforming element bubbles of the space $U_h$. However, for ease of representation, we only consider the lowest order case $l = 1$, hence, no bubbles for the displacement are present. In the following, we use the same symbols $\mathbf{uh} := (\mathbf{uh}, \hat{\mathbf{uh}}), \mathbf{w}_h, p_h$ and $\hat{\mathbf{ph}}$ for the representation of the coefficients of the corresponding discrete finite element solutions. Then (12) can be written as

\[
\begin{pmatrix}
A_{\Pi} & 0 & B_u^T & 0 & 0 \\
0 & M_w & B_w^T & 0 & 0 \\
B_u & B_w & -M_p & 0 & 0 \\
0 & B_w & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{uh} \\
\mathbf{w}_h \\
p_h \\
\hat{\mathbf{ph}}
\end{pmatrix}
= \begin{pmatrix} f_h \\
0 \\
g_h \\
0 \end{pmatrix},
\]

where $f_h$ represent the corresponding vector of the right hand side $(f, v_h)$ and $g_h$ the vector of $(g, q_h)$. Further, $A_{\Pi}, B_u, M_w, M_p, B_w$ and $\hat{B}_w$ denote the operators, or their corresponding matrix representations, defined via the bilinear forms $a_h((\mathbf{uh}, \mathbf{uh}), (\mathbf{v}_h, \mathbf{v}_h)), (\mathbf{uh}, \mathbf{uh}), (\mathbf{v}_h, \mathbf{v}_h))$, $(- \mathbf{div} \mathbf{uh}, g_h), (R^{-1}\mathbf{w} - \mathbf{z}_h), (S\mathbf{ph}, q_h)$,
b((q_h, 0), w_h) and b((0, q_h), w_h), respectively. From the second line we see that we can eliminate w_h using w_h = M_w^{-1}(-\hat{B}_w 1_p h - \hat{B}_w 1_p h). Then the third line gives p_h = -(M_p + B_w M_w^{-1} B_w^{-1})^{-1}(-B_u \bar{u}_h + B_w M_w^{-1} \hat{B}_w 1_p h). Thus, we have the following system to solve

\begin{equation}
(A \quad B^T) \begin{pmatrix} \bar{u}_h \\ \hat{p}_h \end{pmatrix} = \begin{pmatrix} f_h \\ g_h \end{pmatrix},
\end{equation}

with

\begin{align*}
A &:= A_p + \hat{B}_w^T (M_p + B_w M_w^{-1} B_w^{-1})^{-1} B_u, \\
B &:= -\hat{B}_w M_w^{-1} B_w^{-1} (M_p + B_w M_w^{-1} B_w^{-1})^{-1} B_u, \\
C &:= \hat{B}_w M_w^{-1} B_w^{-1} (M_p + B_w M_w^{-1} B_w^{-1})^{-1} B_w M_w^{-1} \hat{B}_w + \hat{B}_w M_w^{-1} \hat{B}_w.
\end{align*}

Note that M_w, M_p and (M_p + B_w M_w^{-1} B_w^{-1}) are all block diagonal, thus locally invertible. Further, the latter operator is equivalent to a (scaled) H^1-like norm on \hat{P}_h. By means of norm equivalent preconditioning, cf. equation (58), we now follow two different approaches. The first preconditioner we investigate is based on a block system that decouples mechanics from the flow problem, and, additionally, the velocity from the fluid pressure. The latter is achieved by introducing an HDG bilinear form on \hat{P}_h for the discretization of div(R\nabla p) as given in the original equation (1) (where K was replaced due to scaling by R). Henceforth, let \hat{M}_p denote the matrix representation of the scaled bilinear form (\gamma p_h, q_h). Then we define the operator

\[ B := \begin{pmatrix} A_p & 0 & 0 & 0 \\ 0 & M_w & 0 & 0 \\ 0 & 0 & -\hat{M}_p - A_p & -B_p^T \\ 0 & 0 & -B_p & -A_p \end{pmatrix}^{-1}. \]

where A_p, B_p and \hat{A}_p correspond to the bilinear forms given by

\begin{align*}
\hat{a}_p(p_h, q_h) &:= R \sum_{T \in \mathcal{T}_h} \int_T \nabla p_h \cdot \nabla q_h \, dx + \int_{\partial T} (-\nabla p_h \cdot n q_h - \nabla q_h \cdot n p_h) + \eta_p l^2 h^{-1} \eta_p M_p q_h \, ds, \\
\hat{b}_p(p_h, \hat{q}_h) &:= R \sum_{T \in \mathcal{T}_h} \int_T \nabla p_h \cdot n \hat{q}_h - \eta_p l^2 h^{-1} p_h \hat{q}_h \, ds, \\
\hat{a}_p(\hat{p}_h, \hat{q}_h) &:= R \sum_{T \in \mathcal{T}_h} \int_{\partial T} \eta_p l^2 h^{-1} \hat{p}_h \hat{q}_h \, ds,
\end{align*}

respectively, where \eta_p is again a sufficiently large stabilization parameter. Note that the combined bilinear form \hat{a}_p(p_h, q_h) + \hat{b}_p(p_h, \hat{q}_h) + \hat{b}_p(q_h, \hat{p}_h) + \hat{a}_p(\hat{p}_h, \hat{q}_h) is the HDG bilinear form mentioned above which is continuous and elliptic with respect to ||\cdot||_{H^1}. Similarly, as before, we can eliminate the local variables to obtain the following preconditioner

\begin{equation}
\begin{pmatrix} A_p & 0 \\ 0 & -(A_p + B_p (\hat{M}_p^{-1} + A_p^{-1}) B_p^T) \end{pmatrix}
\end{equation}

for the condensed system (65), where we have again made use of A_p being block diagonal and invertible. Further, note that both blocks on the diagonal are H^1-type systems. Thus, standard solvers, such as, for example, an algebraic multigrid method for the lowest order system and a “balancing domain decomposition with constraints” (BDDC) preconditioner, the latter featuring robustness in the polynomial degree, can be used.

The second block diagonal preconditioner we test still satisfies the norm equivalence (58), but decouples only the mechanics and flow problems, hence, keeps the hybrid mixed formulation of the
velocity pressure system. The block diagonal operator preconditioner is then given by

\[
\mathcal{B} := \begin{pmatrix}
A_{11} & 0 & 0 & 0 \\
0 & M_w & 0 & -\hat{M}_p \\
0 & 0 & B_{w} & 0 \\
0 & \hat{B}_w & 0 & 0
\end{pmatrix}^{-1}
\]

Following similar steps as above, the preconditioner for the condensed system is

\[
(A_{00} - \tilde{C})^{-1},
\]

(67)

where \(\tilde{C}\) is the same as \(C\) with \(M_p\) replaced by \(\tilde{M}_p\). The advantage of the preconditioner defined by (67), as demonstrated below in Section 5.2, is that the subsystem for the pressure variable does not require a stabilization parameter \(\eta_p\) which in general affects the condition number.

5. Numerical results

In this section, we present several numerical examples to validate our theoretical findings. First, we test for the expected orders of convergence for a problem with a constructed solution increasing the degree of the FE approximation. Second, we study the parameter-robustness of the proposed preconditioners. Finally, we discuss the cost efficiency of our modified methods. All numerical examples are implemented within the finite element library Netgen/NGSolve, see [47, 48] and www.ngsolve.org.

5.1. Convergence of the hybridized/hybrid mixed method. Here we discuss the convergence orders of the errors of the methods introduced in this work. Note, however, that we only consider the discretization given by (12) since the solution is the same as of (9).

5.1.1. 2D example. We solve problem (12) on the spatial domain \(\Omega = (0,1)^2\) and choose the right hand side \(f\) and \(g\) such that the exact solutions are given by

\[
\mathbf{u} := (-\partial_y \phi, \partial_x \phi), \quad p := \sin(\pi x) \sin(\pi y) - p_0,
\]

with the potential \(\phi = x^2(1-x)^2y^2(1-y)^2\) and \(p_0 \in \mathbb{R}\) is chosen such that \(p \in L^2_0(\Omega)\). For simplicity, we choose the constants \(K = 1\), \(\mu = 1\), \(S_0 = 1\), and \(\alpha = 1\). Further, we set \(\lambda = c\) with an arbitrary constant \(c \in \mathbb{R}^+\) since the exact and discrete solutions are exactly divergence-free.

In Table 1 we have displayed several discrete errors and their estimated order of convergence (oeC) for the discretization of problem (12) for varying polynomial orders \(l = 1, 2, 3, 4\). Whereas the \(H^1\)-seminorm error of the displacement \(\mathbf{u}_h\) and the pressure \(p_h\) converge with the expected (see Theorem 7) order \(O(\varepsilon^l)\) and \(O(\varepsilon^{l-1})\), respectively, the corresponding \(L^2\)-norm errors converge with order \(O(\varepsilon^{l+1})\) and \(O(\varepsilon^l)\). This can be shown by a standard Aubin-Nitsche duality argument whenever the considered problem is sufficiently regular, see for example [7]. Note also that the \(L^2\)-norm error \(||\nabla p + R^{-1}w_h||_0\) of the discrete velocity \(\mathbf{w}_h\) converges with optimal order \(O(\varepsilon^l)\). In the lowest order case where we have a piece-wise constant approximation of the pressure \(p_h\), we do not present the \(H^1\)-semi norm error of the pressure since the gradient \(\nabla p_h\) vanishes locally on each element.

5.1.2. 3D example. We solve problem (12) on the spatial domain \(\Omega = (0,1)^3\) and choose the right hand side \(f\) and \(g\) such that the exact solutions are given by

\[
\mathbf{u} := \text{curl}(\phi, \phi, \phi), \quad p := \sin(\pi x) \sin(\pi y) \sin(\pi z) - p_0,
\]

with the potential \(\phi = x^2(1-x)^2y^2(1-y)^2z^2(1-z)^2\) and \(p_0 \in \mathbb{R}\) is chosen such that \(p \in L^2_0(\Omega)\). The parameters \(\lambda, \mu, S_0, \alpha, K\) are chosen as in the two-dimensional example.
Parameter-robustness of the preconditioners. We observe that the preconditioner defined in (66) with a fixed stabilization parameter $\eta_p = 10$ for variations of the parameters $R^{-1}, \lambda, S$. In Figure 2 we plot the number of iterations for the same example using the preconditioner defined in (67). Although both preconditioners show the expected robustness as predicted by the analysis presented in Section 4.1, we see that the results with (67) demonstrate improvement upon those with (66). Besides resulting in a smaller number of iterations, the second preconditioner (67) is substantially more robust with respect to the polynomial degree $l$. We emphasize that the definition of (66) includes a proper scaling of the interior penalty stabilization parameter with respect to the polynomial order given by $O(l^2)$. Although a different (smaller) stabilization parameter might lead to better results--we have fixed $\eta_p = 10$ here--the analysis unfortunately only shows that $\eta_p$ has to be chosen sufficiently large (see [2]), its optimal choice is difficult. Therefore, it is obvious that the mixed formulation, which is known to result in a minimal stabilization, as used in (67), is preferable.

| $l$ | $||\nabla u - \nabla u_h||_0$ (eoc) | $||u - u_h||_0$ (eoc) | $||\nabla p - \nabla p_h||_0$ (eoc) | $||p - p_h||_0$ (eoc) | $||\nabla p + R^{-1}w_h||_0$ (eoc) |
|-----|----------------------------------|-----------------|-------------------------------|-----------------|----------------------------------|
| 1   | 5.3$\times$10^{-2} (2) | 5.5$\times$10^{-3} (2) | 1.8$\times$10^{-1} (2) | 2.9$\times$10^{-1} (2) | 9.7$\times$10^{-1} (2) |
| 2   | 4.9$\times$10^{-2} (3) | 4.9$\times$10^{-3} (3) | 1.8$\times$10^{-1} (3) | 5.7$\times$10^{-1} (3) | 5.7$\times$10^{-1} (3) |
| 3   | 2.2$\times$10^{-2} (4) | 1.1$\times$10^{-3} (4) | 8.1$\times$10^{-2} (4) | 2.8$\times$10^{-2} (4) | 2.8$\times$10^{-2} (4) |
| 4   | 1.1$\times$10^{-2} (5) | 2.6$\times$10^{-4} (5) | 4.0$\times$10^{-2} (5) | 1.4$\times$10^{-2} (5) | 1.4$\times$10^{-2} (5) |
| 5   | 5.4$\times$10^{-3} (6) | 6.3$\times$10^{-5} (6) | 2.0$\times$10^{-2} (6) | 7.1$\times$10^{-2} (6) | 7.1$\times$10^{-2} (6) |
| 6   | 2.7$\times$10^{-3} (7) | 1.6$\times$10^{-5} (7) | 1.0$\times$10^{-2} (7) | 3.6$\times$10^{-2} (7) | 3.6$\times$10^{-2} (7) |
| 7   | 4.6$\times$10^{-3} (8) | 5.1$\times$10^{-3} (8) | 1.5$\times$10^{-1} (8) | 2.9$\times$10^{-1} (8) | 2.9$\times$10^{-1} (8) |
| 8   | 1.1$\times$10^{-2} (9) | 5.9$\times$10^{-4} (9) | 3.1$\times$10^{-2} (9) | 8.2$\times$10^{-2} (9) | 8.2$\times$10^{-2} (9) |
| 9   | 3.0$\times$10^{-3} (10) | 7.4$\times$10^{-5} (10) | 3.8$\times$10^{-2} (10) | 2.7$\times$10^{-2} (10) | 2.7$\times$10^{-2} (10) |
| 10  | 7.7$\times$10^{-4} (11) | 9.3$\times$10^{-6} (11) | 1.9$\times$10^{-2} (11) | 6.9$\times$10^{-3} (11) | 6.9$\times$10^{-3} (11) |
| 11  | 1.9$\times$10^{-4} (12) | 1.2$\times$10^{-6} (12) | 9.6$\times$10^{-3} (12) | 1.8$\times$10^{-3} (12) | 1.8$\times$10^{-3} (12) |
| 12  | 4.9$\times$10^{-5} (13) | 1.5$\times$10^{-7} (13) | 4.8$\times$10^{-3} (13) | 4.4$\times$10^{-4} (13) | 4.4$\times$10^{-4} (13) |

Table 1. The $H^1$-seminorm and the $L^2$-norm errors of the discrete displacement $u_h$ and the discrete pressure $p_h$ and the $L^2$-norm errors of the discrete velocity $w_h$ for different polynomial degrees $l = 1, 2, 3, 4$ for the two-dimensional example.
and find problems: Find $u \in H^1(\Omega)$ such that

$$-\text{div}(a(u)) = f,$$

with a given right hand side $f$ and $\Omega = (0, 1)^3$. We solve this problem on a given triangulation with 166 elements either with an $H(\text{div})$-conforming DG or HDG method, i.e. setting $\lambda = 0$ we have the problems: Find $u_h \in U_h$ such that

$$a_{h}^{\text{DG}}(u_h, v_h) = (f, v_h) \quad \forall v_h \in u_h,$$

and find $(u_h, \hat{u}_h) \in U_h$ such that

$$a_{h}^{\text{HDG}}((u_h, \hat{u}_h), (v_h, \hat{v}_h)) = (f, v_h), \quad \forall (v_h, \hat{v}_h) \in \overline{U}_h.$$

In Table 3 we compare the values

| dof | number of unknowns, | |
|-----|----------------------|---|
| 48  | 5.2 \times 10^{-3}   | 4.9 \times 10^{-4} |
| 384 | 2.6 \times 10^{-3}   | 1.7 \times 10^{-4} |
| 3072| 1.3 \times 10^{-3}   | 5.2 \times 10^{-5} |
| 24576| 6.6 \times 10^{-4}  | 1.4 \times 10^{-5} |

Table 2. The $H^1$-seminorm and the $L^2$-norm errors of the discrete displacement $u_h$ and the discrete pressure $p_h$ and the $L^2$-norm errors of the discrete velocity $\v_h$ for different polynomial degrees $l = 1, 2, 3$ for the three-dimensional example.
Figure 2. Robustness of the preconditioner defined in (67)

cdof: number of coupling unknowns,
nze: number of non-zero entries in thousands of the resulting system matrix,

for varying polynomial degrees \( l = 1, \ldots, 6 \) which correspond to the local order of approximation of \( u_h, \hat{u}_h \) in \( \text{BDM}_l(T)/P_{l-1}^\perp(F) \), for all \( T \in T_h \) and all \( F \in F_h \). Here \( P_{l}^\perp(F) \) is the space of polynomials of order \( l \) that are orthogonal to the normal vector, see the definition of the space \( \hat{U}_h \) in Section 3.2.

First, note that, due to the coupling between element unknowns in the DG method, no static condensation can be applied, i.e. \( \text{dof} = \text{cdof} \). When solving the linear system one is particularly interested in the number of non-zero entries. As we can see, the HDG method clearly outperforms the DG method in case of higher order approximation (\( l \geq 4 \)). In the low order cases, the additional unknowns introduced by the new facet unknowns dominate, and thus no improvement can be expected.

Remark 4. The HDG method can further be improved by means of another technique, called “projected jumps”, which was introduced in [35]. This modification allows to further decrease the coupling of the HDG method without affecting its approximation properties. This essentially compensates the overhead of the HDG method in the low order cases by reducing the polynomial degree of the space of \( \hat{u}_h \) to \( P_{l-1}^\perp(F) \) and adding consistent projections in the bilinear form. Although we do not discuss these modifications here, we include the corresponding numbers in Table 3 in the rows denoted by PHDG. Note that the well-posedness theory and the robustness of the preconditioners obtained in this work also hold for the PHDG method.

|        | \( l = 1 \) | \( l = 2 \) | \( l = 3 \) | \( l = 4 \) | \( l = 5 \) | \( l = 6 \) |
|--------|-----------|-----------|-----------|-----------|-----------|-----------|
| DG     | 834 834 65 | 2664 2664 454 | 6100 6100 1945 | 11640 11640 6238 | 19782 19782 16512 | 31024 31024 38106 |
| HDG    | 2502 2502 193 | 6000 5004 770 | 11660 8340 2140 | 19980 12510 4815 | 31458 17514 9438 | 46592 23352 16779 |
| PHDG   | 1390 1390 59 | 4332 3336 342 | 9436 6116 1151 | 17200 9730 2913 | 28122 14178 6185 | 42700 19460 11652 |

Table 3. \( \text{dof}, \text{cdof} \) and \( \text{nze} \) of the system matrix of the DG, HDG and PHDG methods for different polynomial degrees \( l \).

5.3.2. Mixed vs hybrid mixed methods. We continue discussing the modifications introduced in Section 3.3 with regards to the following Darcy model problem: Find \((w, p) \in H(\text{div})(\Omega) \times L^2(\Omega) \) such
that
\[ \mathbf{w} + \nabla p = 0, \]
\[ \text{div}(\mathbf{w}) = g, \]
for a given right hand side \( g \) on the domain \( \Omega = (0,1)^3 \). We use the same mesh as in the previous section, and consider the problems: Find \((\mathbf{w}_h, p_h) \in W_h \times P_h\) such that
\[ (\mathbf{w}_h, z_h) - (p_h, \text{div} z_h) = 0, \quad \forall z_h \in W_h, \]
\[ -\text{div}(\mathbf{w}_h, q_h) = -(g, q_h), \quad \forall q_h \in P_h, \tag{70a} \]
and find \((\mathbf{w}_h, (p_h, \tilde{p}_h)) \in W^-_h \times P^-_h\) such that
\[ (\mathbf{w}_h, z_h) - b((p_h, \tilde{p}_h), z_h) = 0, \quad \forall z_h \in W^-_h, \]
\[ -b((q_h, \tilde{q}_h), \mathbf{w}_h) = -(g, q_h), \quad \forall (q_h, \tilde{q}_h) \in P^-_h. \tag{71a} \]

Note that equation (70) only allows a static condensation of the following degrees of freedom: all local (element-associated) degrees of freedom of the space \( W_h \), i.e. element-wise basis functions with a vanishing normal trace, and all high-order (considering a standard \( L^2 \)-Dubiner basis) basis functions of \( P_h \) such that element-wise constant basis functions remain in the system. In contrast to this, system (71) allows us to eliminate all degrees of freedom associated with the basis functions of the spaces \( W^-_h \) and \( P_h \). In Table 4, we again present the corresponding numbers as discussed above, where \( M \) represents the discretization of (70), and \( HM \) of (71). Here, the order \( l \) corresponds to the local approximation polynomial degree of \( \mathbf{w}_h, p_h, \tilde{p}_h \) in \( RT_\ell(T)/P_\ell(T)/P_\ell(F) \), for all \( T \in T_h \) and all \( F \in F_h \). Further, we observe that the hybrid mixed method produces always a smaller number of non-zero entries than the standard mixed method although the difference is negligible. However, the main purposes of hybridization are a reduction of the number of coupling dof and obtaining a condensed system, cf. (67), which is symmetric positive definite, see also [9, 8]. The latter allows us to use preconditioners for \( H^1 \)-elliptic problems like standard algebraic multigrid methods.

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