Viscosity solutions of contact Hamilton-Jacobi equations without monotonicity assumptions

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Abstract

This paper deals with: (1) the generalized additive eigenvalue problem

\[ H(x, u(x), Du(x)) = c, \quad x \in M, \]

where the unknown is a pair \((c, u)\) of a constant \(c \in \mathbb{R}\) and a function \(u\) on \(M\) for which \(u\) is a backward weak KAM solution (or equivalently, viscosity solution) of the above equation; (2) the long-time behavior of viscosity solutions of the Cauchy problem for

\[ w_t(x, t) + H(x, w(x, t), Dw(x, t)) = c, \quad (x, t) \in M \times (0, +\infty). \]

We assume \(H = H(x, u, p)\) satisfies Tonelli conditions in the argument \(p \in T^*_x M\) and the uniform Lipschitz condition in the argument \(u \in \mathbb{R}\).

First, we provide three necessary and sufficient conditions for the existence of backward weak KAM solutions of the stationary equation for a given \(c \in \mathbb{R}\). Second, we analyse the structure of the set \(\mathcal{C}\) of all real numbers \(c\)'s for which the stationary equation admits backward weak KAM solutions. Third, the long-time behavior of viscosity solutions of the Cauchy problem for the evolutionary equation is studied.

Implicit Lax-Oleinik semigroups \(\{T^\pm_t\}_{t \geq 0}\) play an essential role in this paper. The most important novelty in this work is to reveal several new aspects of existence and long-time behavior of viscosity solutions of the above equations without monotonicity assumptions \(\frac{\partial H}{\partial u} \geq 0\) (or, \(\frac{\partial H}{\partial u} \leq 0\)): the properties of the operator \(T^+_t \circ T^-_t\) is used to provide a necessary and sufficient condition for the existence of viscosity solutions of the stationary equation; three constants are introduced to analyse the structure of \(\mathcal{C}\); the information of \(w\) in finite time partly determines the long-time behavior of \(w\).

Keywords. Hamilton-Jacobi equations, weak KAM theory, viscosity solutions, additive eigenvalue problem, long-time behavior problem

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1 Introduction

1.1 The purpose of this paper

Let $M$ be a closed (compact, without boundary), connected and smooth manifold of dimension $n$. We shall write everything in a local coordinate system: $x = (x_1, \ldots, x_n)$ are the coordinates.

In this article we consider: (1) The generalized additive eigenvalue problem

$$H(x, u(x), Du(x)) = c, \quad x \in M. \quad (E_c)$$
Here the unknown is a pair \((c, u)\) of a constant \(c \in \mathbb{R}\) and a function \(u\) on \(M\) for which \(u\) satisfies (\(E_c\)) in the sense of viscosity. (2) The long-time behavior of the Cauchy problem

\[
\begin{align*}
    w_t(x, t) + H(x, w(x, t), Dw(x, t)) &= c, & (x, t) \in M \times (0, +\infty), \\
    w(x, 0) &= \varphi(x), & x \in M.
\end{align*}
\]

(1.1)

Here \(w\) represents the real-valued unknown function on on \(M \times [0, +\infty)\). Note that for any \(c \in \mathbb{R}\), Cauchy problem (1.1) admits a unique viscosity solutions under (H1)-(H3) below. The symbol \(D\) in equations (\(E_c\)) and (1.1) denotes the spatial gradient.

Throughout this paper, we will work under the following assumptions imposed on the Hamiltonian \(H\). Let \(H = H(x, u, p)\) be a \(C^3\) function on \(T^*M \times \mathbb{R}\) with \((x, p) \in T^*M\) and \(u \in \mathbb{R}\), satisfying

- **(H1)** Strict convexity: the second partial derivative \(\frac{\partial^2 H}{\partial p^2}(x, u, p)\) is positive definite as a quadratic form for all \((x, u, p) \in T^*M \times \mathbb{R}\);

- **(H2)** Superlinearity: \(H(x, u, p)\) is superlinear in \(p\) for all \((x, u) \in M \times \mathbb{R}\);

- **(H3)** Lipschitz continuity: there exists \(\lambda > 0\) such that \(|\frac{\partial H}{\partial u}(x, u, p)| \leq \lambda\) for all \((x, u, p) \in T^*M \times \mathbb{R}\).

Conditions (H1) and (H2) are called Tonelli conditions, under which Tonelli formulated his clean and modern direct approach to minimum problems more than one hundred years ago.

- **Generalized additive eigenvalue problem.** Let \(G\) be a Hamiltonian defined on \(T^*M\). Finding solutions \((c, u)\) of equation \(G(x, Du(x)) = c\) is a well-known problem, called the cell (or, corrector) problem. Under a set of standard assumptions, the real number \(c\) is unique, for which the equation has viscosity solutions. Let \(\mathcal{G}_t\) be the solution operator of the corresponding evolutionary equation \(w_t(x, t) + G(x, Dw(x, t)) = 0\). Then \(u\) is a viscosity solution of \(G(x, Du(x)) = c\) if and only if \(\mathcal{G}_t u = u - ct\) for all \(t \geq 0\). Since the relation \(\mathcal{G}_t u = u - ct\) looks like a nonlinear eigenvalue problem, finding solutions \((c, u)\) of \(G(x, Du(x)) = c\) is also called an additive eigenvalue problem. The additive eigenvalue \(c\) determines the effective Hamiltonian in the homogenization of Hamilton-Jacobi equations [15][23]. An interesting dynamical feature of \(c\) was discovered by weak KAM theory [21], where \(c\) is called Mane’s critical value, and a link between viscosity solutions of \(G(x, Du(x)) = c\) and Aubry sets, Mather sets of Hamiltonian systems generated by \(G\) was established. Under Tonelli conditions, Conterras et al. [10] provided a representation formula for \(c\):

\[
c = \inf_{f \in C^\infty(M)} \sup_{x \in M} H(x, Df(x)).
\]

The above infimum is not a minimum. This formula still holds true when \(C^\infty(M)\) is replaced by \(C^{1,1}(M)\), \(C^1(M)\), or \(\text{Lip}(M)\) and inf is replaced by min, see [4][16].

Now come back to our problem (\(E_c\)). It is a well-known fact that equation (\(E_c\)) has viscosity solutions for each real number \(c\), provided \(H\) satisfies stronger conditions: (H1), (H2) and \(0 < \delta \leq \frac{H(\cdot, u, p) - H(\cdot, \overline{u}, \overline{p})}{\lVert p - \overline{p} \rVert^2} < \infty\).
\( \frac{\partial H}{\partial u} \leq \lambda \). In this case, for each \( c \in \mathbb{R} \) there is a unique viscosity solution of \((E_c)\). However, this is not the case even for the decreasing case \(-\lambda \leq \frac{\partial H}{\partial u} \leq -\delta < 0\). The nonuniqueness phenomenon appears in the following simple example: 

\[-u(x) + \frac{1}{2} |Du(x)|^2 = 0, \quad x \in S,\]

where \( S := \left(-\frac{1}{2}, \frac{1}{2}\right) \) denotes the unit circle. Let \( u_1 \) be the even 1-periodic extension of \( u(x) = \frac{1}{2} x^2 \) in \([0, \frac{1}{2}]\). Then both \( u_1 \) and \( u_2 \equiv 0 \) are viscosity solutions. Seen in this light, studying the generalized additive eigenvalue problem \((E_c)\) under (H1)-(H3) is not a straightforward task at all.

The first aim of this paper is: for a given \( c \in \mathbb{R} \), to provide necessary and sufficient conditions for the existence of viscosity solutions of equation \((E_c)\). The second aim of this paper is to analyse the structure of the set of all real numbers \( c \)'s for which equation \((E_c)\) admits viscosity solutions.

- **Long-time behavior of the Cauchy problem.** The study of the long-time behavior problem of viscosity solutions of equation

\[ w_t(x, t) + G(x, Dw(x, t)) = 0 \tag{1.2} \]

has a long history. It is clear that \((c, u)\) is a solution of the additive eigenvalue problem \((E_c)\) if and only if \( u(x) - ct \) is a solution of \((1.2)\). Such a function \( u(x) - ct \) is called an asymptotic solution of \((1.2)\). An interesting question is: does any solution \( w(x, t) \) of \((1.2)\) converge to an asymptotic solution? We refer to, for example, \([3, 13, 20, 25, 27, 31, 32]\) for previous results and developments in this direction. Especially, the weak KAM approach contributes to the study on this problem, where the Lax-Oleinik semigroup plays an important role.

However, to the best of our knowledge, there are few results on the long-time behavior problem for equation \((1.1)\) without monotonicity condition \( \frac{\partial H}{\partial u} \geq 0 \) (or, \( \frac{\partial H}{\partial u} \leq 0 \)). The third aim of this paper is to discuss the long-time behavior of viscosity solutions of \((1.1)\) under (H1)-(H3).

Our method is dynamical in nature and inspired by the deep connection between contact Hamilton-Jacobi equations \((E_c)\), \((1.1)\) and contact Hamiltonian system

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p}(x, u, p), \\
\dot{p} &= -\frac{\partial H}{\partial x}(x, u, p) - \frac{\partial H}{\partial u}(x, u, p)p, \\
\dot{u} &= \frac{\partial H}{\partial p}(x, u, p) \cdot p - H(x, u, p).
\end{align*} \tag{1.3}
\]

The authors of \([35, 36, 37, 38]\) discussed the weak KAM and Aubry-Mather aspects of contact Hamiltonian systems from variational principles, dynamical properties of action minimizing orbits, to weak KAM solutions, viscosity solutions of stationary and evolutionary contact Hamilton-Jacobi equations. The tools used in the present paper mainly come from those papers.

### 1.2 Solution semigroups and weak KAM solutions

The key tools used in this paper are solution semigroups. Before stating our results, we take a look at the semigroups first.

- **Contact Lagrangians.** The contact Lagrangian \( L(x, u, \dot{x}) \) associated with \( H(x, u, p) \) is defined by

\[ L(x, u, \dot{x}) := \sup_{p \in T^*_x M} \left\langle (\dot{x}, p)_x - H(x, u, p) \right\rangle, \quad (x, \dot{x}) \in TM, \ u \in \mathbb{R}. \]
Since $H$ satisfies (H1)-(H3), it is direct to check that $L$ satisfies:

(L1) Strict convexity: the second partial derivative $\frac{\partial^2 L}{\partial x^2}(x, u, \dot{x})$ is positive definite as a quadratic form for all $(x, u, \dot{x}) \in TM \times \mathbb{R}$;

(L2) Superlinearity: $L(x, u, \dot{x})$ is superlinear in $\dot{x}$ for all $(x, u) \in M \times \mathbb{R}$;

(L3) Lipschitz continuity: there exists $\lambda > 0$ such that $|\frac{\partial L}{\partial u}(x, u, \dot{x})| \leq \lambda$ for all $(x, u, \dot{x}) \in TM \times \mathbb{R}$.

**Solution semigroups.** Following Fathi’s weak KAM theory [18, 19, 20, 21] for classical Hamiltonian systems, under assumptions (H1)-(H3) the authors of [36] introduced two semigroups of operators associated with the contact Lagrangian $L$, denoted by $\{T_t^-\}_{t \geq 0}$ and $\{T_t^+\}_{t \geq 0}$. Let us recall definitions of $\{T_t^\pm\}_{t \geq 0}$ here. For each $\varphi \in C(M)$, denote by $(x, t) \mapsto T_t^- \varphi(x)$ the unique continuous function on $(x, t) \in M \times [0, +\infty)$ such that

$$T_t^- \varphi(x) = \inf_\gamma \left\{ \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), T^-_\tau \varphi(\gamma(\tau)), \dot{\gamma}(\tau)) \, d\tau \right\},$$

where the infimum is taken among curves $\gamma \in C^a([0, t], M)$ with $\gamma(t) = x$. We call $\{T_t^-\}_{t \geq 0}$ the backward solution semigroup for equation

$$w_t(x, t) = H(x, w(x, t), Dw(x, t)) = 0. \quad (1.4)$$

The function $(x, t) \mapsto T_t^- \varphi(x)$ is the unique viscosity solution of equation (1.4) with the initial condition $w(x, 0) = \varphi(x)$. Similarly, one can define another semigroup of operators $\{T_t^+\}_{t \geq 0}$, called the forward solution semigroup by

$$T_t^+ \varphi(x) = \sup_\gamma \left\{ \varphi(\gamma(t)) - \int_0^t L(\gamma(\tau), T^+_\tau \varphi(\gamma(\tau)), \dot{\gamma}(\tau)) \, d\tau \right\},$$

where the supremum is taken among curves $\gamma \in C^a([0, t], M)$ with $\gamma(0) = x$. It is direct to see that the function $(x, t) \mapsto T_t^+ \varphi(x)$ is the unique viscosity solution to

$$\begin{cases} w_t(x, t) - H(x, w(x, t), Dw(x, t)) = 0, \\ w(x, 0) = \varphi(x), \end{cases}$$

and that the function $(x, t) \mapsto -T_t^+ \varphi(x)$ is the unique viscosity solution to

$$\begin{cases} w_t(x, t) + H(x, -w(x, t), -Dw(x, t)) = 0, \\ w(x, 0) = -\varphi(x). \end{cases}$$

**Weak KAM solutions and viscosity solutions.** A fixed point of $\{T_t^-\}_{t \geq 0}$ (resp. $\{T_t^+\}_{t \geq 0}$) is called a backward (resp. forward) weak KAM solution of equation

$$H(x, u(x), Du(x)) = 0, \quad x \in M, \quad (E_0)$$
or BWKAM (resp. FWKAM) solution for short. Denote by $S_{-}$ (resp. $S_{+}$) the set of all BWKAM (resp. FWKAM) solutions of equation $(E_0)$. It is straightforward to show that a BWKAM solution of equation $(E_0)$ is a viscosity solution of equation $(E_0)$, see for example [21], [36]. Conversely, if $u$ is a viscosity solution of equation $(E_0)$, then both $T_{-t}u$ and $u$ are viscosity solutions of (1.4). By the uniqueness of viscosity solutions of the Cauchy problem

$$
\begin{cases}
  w_t(x, t) + H(x, w(x, t), Dw(x, t)) = 0, \\
  w(x, 0) = u(x),
\end{cases}
$$

we deduce that $T_{-t}u = u$ for all $t \geq 0$, i.e., $u$ is a BWKAM solution of equation $(E_0)$. Thus, BWKAM solutions coincide with viscosity solutions under (H1)-(H3).

1.3 Main results

• Necessary and sufficient conditions for the existence of BWKAM solutions of $(E_c)$. Using the solution semigroups, we get three necessary and sufficient conditions for the existence of BWKAM solutions of equation $(E_c)$.

Define

$$
C_{\pm} := \{ \varphi \in C(M) : T_{t}^{\pm} \varphi \geq \varphi, \ \forall t \geq 0 \}.
$$

**Proposition 1.** Equation $(E_0)$ has BWKAM solutions if and only if both $C_{-}$ and $C_{+}$ are non-empty.

A direct consequence of Proposition 1 is stated as follows.

**Corollary 1.** Let $c \in \mathbb{R}$. Equation $(E_c)$ has BWKAM solutions if and only if both $C_{-}^c$ and $C_{+}^c$ are non-empty, where

$$
C_{\pm}^c := \{ \varphi \in C(M) : T_{t}^{\pm,c} \varphi \geq \varphi, \ \forall t \geq 0 \},
$$

and $\{T_{t}^{-,c}\}_{t \geq 0}$ (resp. $\{T_{t}^{+,c}\}_{t \geq 0}$) denotes the backward (resp. forward) solution semigroup associated with $L + c$.

This is the first necessary and sufficient condition for the existence of BWKAM solutions of equation $(E_c)$. We will weaken the condition $C_{\pm}^c \neq \emptyset$ by deepening the properties of the solution semigroups later. In fact, one will see that Proposition 1 is a direct consequence of Theorem 1 below, but we will still give an independent proof of Proposition 1.

If either $S_{-}$ or $S_{+}$ is non-empty, then we obtain the following result, which is very useful in the proofs of Proposition 1 and Theorem 1.

**Proposition 2.**

1. For each $u \in S_{-}$, the uniform limit $\lim_{t \to +\infty} T_{t}^{+} u =: v$ exists and $v \in S_{+}$.

2. For each $v \in S_{+}$, the uniform limit $\lim_{t \to +\infty} T_{t}^{-} v =: u$ exists and $u \in S_{-}$.
It follows from Proposition 2 that $S_− \neq \emptyset \iff S_+ \neq \emptyset$.

For any $t > 0$, we define

$$D^t_\pm := \{ \varphi \in C(M) : T^\pm_t \varphi \geq \varphi \}.$$

**Theorem 1.** Equation (E$_0$) has BWKAM solutions if and only if both $\bigcup_{t > 0} D^-_t$ and $\bigcup_{t > 0} D^+_t$ are non-empty.

Consequently, we have that

**Corollary 2.** Let $c \in \mathbb{R}$. Equation (E$_c$) has BWKAM solutions if and only if both $\bigcup_{t > 0} D^t_{-, c}$ and $\bigcup_{t > 0} D^t_{+, c}$ are non-empty, where

$$D^t_{\pm, c} := \{ \varphi \in C(M) : T^\pm_{t, c} \varphi \geq \varphi \}, \quad t > 0.$$

We provide here another necessary and sufficient condition for the existence of BWKAM solutions of equation (E$_c$). It is a direct consequence of Theorem 3 (3) and Proposition 12 below.

**Proposition 3.** Let $c \in \mathbb{R}$. Equation (E$_c$) has BWKAM solutions if and only if there exists $\varphi \in C(M)$ such that for each $t > 0$,

$$\{ x \in M : T^{-c}_{t} \varphi(x) = \varphi(x) \} \neq \emptyset.$$

Let

$$\mathcal{C} := \{ c \in \mathbb{R} : \text{equation (E$_c$) has BWKAM solutions} \}.$$ 

We call it the admissible set for equation (E$_c$).

**Structure of the set** $\mathcal{C}$. We aim to describe the set $\mathcal{C}$ here. Before that, we need to introduce three constants (may be $\pm \infty$) determined by $H$. Lip($M$) stands for the space of Lipschitz continuous functions on $M$. SCL($M$) stands for the set of all functions which are semiconcave on $M$ with a linear modulus. See for example, [6, Definition 1.1.1] for the definition of semiconcave functions with linear modulus. Let Dom($u$) denote the domain of definition of $Du$, i.e., the set of the points $x$ where the derivative $Du(x)$ exists. We define

$$c_1 := \inf_{u \in \text{Lip}(M)} \sup_{x \in \text{Dom}(Du)} H(x, u(x), Du(x)),$$

$$c_2 := \sup_{u \in \text{C}^\infty(M)} \inf_{x \in M} H(x, u(x), Du(x)),$$

$$c_3 := \sup_{u \in \text{SCL}(M)} \inf_{x \in \text{Dom}(Du)} H(x, u(x), Du(x)).$$

(1.6)

Note that $c_1, c_2, c_3 \in [-\infty, +\infty]$. In view of a result by Czarnecki and Rifford [12], it is direct to show that $c_1 = \inf_{u \in \text{C}^\infty(M)} \sup_{x \in M} H(x, u(x), Du(x))$. See Lemma 2 below.
Figure 1: By definition, it is direct to see that $c_2 \leq c_3$. By [36, Theorem 1.2], one can deduce that $c_1 \leq c_3$. So far we do not know the ordering relation between $c_1$ and $c_2$. We will analyse several examples in Section 4.

Theorem 2. Let $c_1$, $c_2$ and $c_3$ be as in (1.6), and let $\mathcal{C}$ be as in (1.5). Then

1. $\mathcal{C}$ is connected, i.e., a non-empty interval;
2. $(c_1, c_2) \subset \mathcal{C}$;
3. $\mathcal{C} \subset [c_1, c_3]$.

Remark 1. Figure 1 helps to understand Theorem 2. Under the same assumptions imposed in this paper, the existence of solutions $(c, u)$ of $(E_c)$ was proven in [36], $\mathcal{C} \neq \emptyset$. But, the structure of the set $\mathcal{C}$ was not discussed there. Let us give more explanation of Theorem 2:

- $\mathcal{C}$ is connected. Denote by $c_l$ and $c_r$ the two endpoints of $\mathcal{C}$ with $c_l \leq c_r$. It means that $\mathcal{C}$ is one of the following: $(c_l, c_r)$, $[c_l, c_r)$, $[c_l, c_r]$, $(c_l, c_r]$. In fact, each case can happen: let $H(x, u, p) := \|p\|_x^2 + f(u)$, where $f(u)$ is a smooth function on $\mathbb{R}$ with $|f'(u)| \leq \lambda$. If $\text{Ran}(f) = (a, b)$, then $\mathcal{C} = (a, b)$. If $\text{Ran}(f) = [a, b]$, then $\mathcal{C} = [a, b]$. If $\text{Ran}(f) = (a, b]$, then $\mathcal{C} = (a, b)$. If $\text{Ran}(f) = [a, b)$, then $\mathcal{C} = [a, b)$. Here, $a, b \in \mathbb{R}$ with $a \leq b$, and $\text{Ran}(f)$ denotes the range of $f$.

- If $c_1 < c_2$, then the open interval $(c_1, c_2)$ is contained in $\mathcal{C}$. If $c_1 > c_2$, then $(c_1, c_2) = \emptyset$ and thus item (2) still holds true. We are not sure whether $c_1 > c_2$ can happen. See Section 4 for more details.

- Since $\text{SCL}(M) \subset \text{Lip}(M)$ (see, for instance, [6, Theorem 2.1.7]), by definition, it is clear that $c_2 \leq c_3$. We will prove in Section 3 that there is no viscosity subsolutions of equation $(E_c)$ when $c < c_1$, and that there is no viscosity solutions of equation $(E_c)$ when $c > c_3$. So, in view of the non-emptiness of $\mathcal{C}$, one can deduce that $c_1 \leq c_3$.

- We will show what $c_1$, $c_2$ and $c_3$ are in several examples in Section 4.

- Long-time behavior of the Cauchy problem (1.1).

Theorem 3. Let $\varphi \in C(M)$ and let $c \in \mathbb{R}$.
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(1) if there is $t_0 > 0$ such that $T_{t_0}^{-c} \varphi \geq \varphi$ (resp. $T_{t_0}^{-c} \varphi \leq \varphi$), then for any $s \in [0, t_0]$, 
\[ \lim_{n \to +\infty} T_{nt_0+s}^{-c} \varphi(x) = +\infty \text{ (resp. } \lim_{n \to +\infty} T_{nt_0+s}^{-c} \varphi(x) = -\infty \text{) uniformly on } x \in M, \]
or \[ \lim_{n \to +\infty} T_{nt_0+s}^{-c} \varphi(x) = u(x, s) \text{ for all } x \in M \text{ and all } s \in \mathbb{R}, \]
where $u(x, s)$ is a classical solution of equation (1.1a) which is $t_0$-periodic in time.

(2) if there is $t_0 > 0$ such that $T_{t_0}^{-c} \varphi > \varphi$ (resp. $T_{t_0}^{-c} \varphi < \varphi$), then \[ \lim_{t \to +\infty} T_t^{-c} \varphi(x) = +\infty \text{ (resp. } \lim_{t \to +\infty} T_t^{-c} \varphi(x) = -\infty \text{) uniformly on } x \in M, \]
or \[ \lim_{t \to +\infty} T_t^{-c} \varphi(x) = \varphi_\infty(x) \text{ uniformly on } x \in M, \]
where $\varphi_\infty$ is a BWKAM solution of equation (E).

(3) if for any $t > 0$, 
\[ \{ x \in M : T_t^{-c} \varphi(x) = \varphi(x) \} \neq \emptyset, \]
then $|T_t^{-c} \varphi(x)| \leq K_\varphi$ for all $(x, t) \in M \times [0, +\infty)$ and for some constant $K_\varphi > 0$ depending on $\varphi$.

Remark 2. In view of items (1), (2) of Theorem [3] it is natural to ask: when there is $t_0 > 0$ such that $T_{t_0}^{-c} \varphi \geq \varphi$, can the case “$\lim_{n \to +\infty} T_{nt_0+s}^{-c} \varphi(x) = u(x, s)$ for all $x \in M$ and all $s \in \mathbb{R}$, where $u(x, s)$ is a viscosity solution of equation (1.1a) which is $t_0$-periodic in time”, really happen?

We provide an example here to give a positive answer to the above question. This example helps to understand the relation between items (1) and (2) of Theorem [3].

Let $H(x, u, p) = p^2 - p + f(u) - 1$ for all $x \in S = (0, 2\pi]$, $u \in \mathbb{R}$, $p \in \mathbb{R}$, where $f$ is a smooth function on $\mathbb{R}$ with $|f'(u)| \leq 2$ and $f(u) = u^2$ for $|u| \leq 1$. Then it is clear that $w(x, t) = \sin(x + t)$ is a classical solution of
\[ w_t + (w_x)^2 - w_x + f(w) - 1 = 0, \quad x \in S, \quad t \in [0, +\infty). \]

By the uniqueness of viscosity solutions of the Cauchy problem, $T_t^{-} \sin(x) = \sin(x + t)$ for all $x \in S, t \in [0, +\infty)$. Thus, $T_{2\pi n}^{-} \sin(x) = \sin x$ for all $x \in S$, and \[ \lim_{n \to +\infty} T_{2\pi n + s}^{-} \sin(x) = \sin(x + s), \quad x \in S, \quad s \in \mathbb{R}. \]

Example 1. Let $H(x, u, p)$ be a $C^3$ function defined on $T^* M \times \mathbb{R}$ satisfying (H1), (H2) and $0 < \delta \leq \frac{\partial H}{\partial u} \leq \lambda$. It is well-known that under these assumptions equation (E) has a unique viscosity solution, denoted by $\bar{u}$. Furthermore, for any $\varphi \in C(M)$, $\lim_{t \to +\infty} T_t^{-} \varphi = \bar{u}$, see for example [37].

We will explain here how to get the above convergence result from Theorem [3](2). For any given $\varphi \in C(M)$, there is a constant $B_\varphi > 0$ such that \[ u'(x) := \bar{u}(x) - B_\varphi < \varphi(x) < \bar{u}(x) + B_\varphi := u''(x), \quad \forall x \in M. \quad (1.7) \]

By definition, for any $t > 0$, any $x \in M$,
\[ T_t^{-} u'(x) = \inf_{\gamma} \left\{ \bar{u}(\gamma(0)) + \int_{0}^{t} L(\gamma(\tau), T_{\tau}^{-}(\bar{u} - B_\varphi)(\gamma(\tau)), \dot{\gamma}(\tau)) \, d\tau \right\} - B_\varphi, \]
where the infimum is taken among curves $\gamma \in C^{ac}([0, t], M)$ with $\gamma(t) = x$. By Proposition 9 below, for any $\tau > 0$ and any $x \in M$, we have

$$T^-_\tau (\bar{u} - B_\varphi)(x) < T^-_\tau \bar{u}(x).$$

Note that $\frac{\partial L}{\partial u} < 0$ and

$$\bar{u}(x) = T^-_\tau \bar{u}(x) = \inf_{\gamma} \left\{ \bar{u}(\gamma(0)) + \int_0^t L(\gamma(\tau), \bar{u}(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau \right\}, \quad \forall x \in M, \forall \tau > 0,$n

where the infimum is taken among curves $\gamma \in C^{ac}([0, t], M)$ with $\gamma(t) = x$. Then, we get that

$$T^-_\tau u'(x) = \inf_{\gamma} \left\{ \bar{u}(\gamma(0)) + \int_0^t L(\gamma(\tau), T^-_\tau (\bar{u} - B_\varphi)(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau \right\} - B_\varphi$$

$$> \inf_{\gamma} \left\{ \bar{u}(\gamma(0)) + \int_0^t L(\gamma(\tau), T^-_\tau \bar{u}(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau \right\} - B_\varphi$$

$$= \inf_{\gamma} \left\{ \bar{u}(\gamma(0)) + \int_0^t L(\gamma(\tau), \bar{u}(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau \right\} - B_\varphi$$

$$= \bar{u}(x) - B_\varphi = u'(x), \quad \forall x \in M, \forall \tau > 0.$$

Similarly, one can deduce that

$$T^-_\tau u''(x) < u''(x), \quad \forall x \in M, \forall \tau > 0.$$

Note that

$$T^-_\tau u''(x) > \bar{u}(x), \quad T^-_\tau u'(x) < \bar{u}(x), \quad \forall x \in M, \forall \tau > 0.$$

Then by Theorem 3(2), one get that

$$\lim_{t \to +\infty} T^-_\tau u''(x) = \lim_{t \to +\infty} T^-_\tau u'(x) = \bar{u}(x), \quad \forall x \in M.$$

Recall (1.7), we obtain that

$$\lim_{t \to +\infty} T^-_\tau \varphi(x) = \bar{u}(x), \quad \forall x \in M.$$

Remark 3. In view of Theorem 3, one can deduce that all the complicated phenomenons appear only in case (3) of Theorem 3. We will go on to study the long-time behavior problem in this direction in future work.

When $c \notin \mathcal{C}$, the long-time behavior of $T^-_\tau \varphi$ is quite clear:

Theorem 4. Let $\varphi \in C(M)$ and let $c \in \mathbb{R}$ with $c \notin \mathcal{C}$. Then

1. if $c < c_r$, then $\lim_{t \to +\infty} T^-_\tau \varphi(x) = -\infty$ uniformly on $x \in M$. 

Viscosity solutions of contact Hamilton-Jacobi equations without monotonicity assumptions

(2) if \( c > c_l \), then \( \lim_{t \to +\infty} T_t^{-c} \varphi(x) = +\infty \) uniformly on \( x \in M \).

- **Some historical remarks.** Hamilton-Jacobi equations have been first introduced in classical mechanics, but find applications in many other fields of mathematics. The theory of viscosity solutions of Hamilton-Jacobi equations was introduced in the early 80’s by Crandall and Lions [8], Crandall, Evans and Lions [9]. It provides a suitable PDE framework for studying Hamilton-Jacobi equations which does not have classical solutions. For a good introductory book on viscosity solutions, we refer readers to [2]. The theory of viscosity solutions has been extensively studied and refined by many authors, and, among the numerous contributions in the literature, we would like to point out that the weak KAM theory for Tonelli Hamiltonian systems opened a way to study viscosity solutions of the corresponding Hamilton-Jacobi equations using the dynamical information of action minimizing orbits of Hamiltonian systems. We refer readers to [1, 5, 11, 13, 16, 17, 22, 25, 28, 33, 34] and the references therein for more details on this topic. Along this line, it is natural to consider whether one can use weak KAM type results for contact Hamiltonian systems to study viscosity solutions of contact Hamilton-Jacobi equations (\( E_c \)) and (1.1). For weak KAM aspects for contact Hamiltonian systems, we refer readers to [14, 30, 37, 38]. Under assumptions imposed in this paper, it was shown in [36] the existence of solutions \((c, u)\) of equation (\( E_c \)). See [26] for a similar result using traditional PDE methods. We aim to refine and deepen the results in [36] in the present paper. We still use dynamical tools from the weak KAM theory for contact Hamiltonian systems satisfying (H1)-(H3). These assumptions will be weakened in a forthcoming paper.

The rest of this paper is organized as follows. We recall some results on the weak KAM theory for contact Hamiltonian systems in Section 2. Section 3 is devoted to the proofs of Propositions 1, 2 and Theorem 1. In Section 4, we prove Theorem 2 and provide several examples. In Section 5, we show Theorems 3, 4 and Proposition 3. The proofs of some preliminary results are given in the appendix.

# 2 Preliminaries

## 2.1 Notations

We write as follows a list of symbols used throughout this paper.

- We choose, once and for all, a \( C^\infty \) Riemannian metric on \( M \). It is classical that there is a canonical way to associate to it a Riemannian metric on \( TM \). We use the same symbol \( d \) to denote the distance function defined by the Riemannian metric on \( M \) and the distance function defined by the Riemannian metric on \( T^*M \). We use the same symbol \( || \cdot ||_x \) to denote the norms induced by the Riemannian metrics on \( T_xM \) and \( T^*_xM \) for \( x \in M \), and by \( \langle \cdot , \cdot \rangle_x \) the canonical pairing between the tangent space \( T_xM \) and the cotangent space \( T^*_xM \).

- \( C^k(M) (k \in \mathbb{N}) \) stands for the function space of \( k \)-times continuously differentiable functions on \( M \), and \( C^\infty(M) := \bigcap_{k=0}^{\infty} C^k(M) \). And \( || \cdot ||_\infty \) denotes the supremum norm on these spaces.
• \( C^{ac}([a, b], M) \) stands for the space of absolutely continuous curves \([a, b] \to M\).

• \( \text{Lip}(M) \) stands for the space of Lipschitz continuous functions on \( M \).

• Denote by \( \text{SCL}(M) \) the set of all functions which are semiconcave on \( M \) with a linear modulus.

• \( Du(x) = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) \) and \( Dw(x, t) = (\frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n}) \).

• Let \( u \in \text{Lip}(M) \). Denote by \( \text{Dom}(u) \) the set of all points \( x \in M \) where \( Du(x) \) exists.

• \( S_- \) (resp. \( S_+ \)) denotes the set of all BWKAM (resp. FWKAM) solutions of equation \((E_0)\).

• Let \( \Phi_t \) denote the local flow of contact Hamiltonian system \((1.3)\).

• \( \text{cl}(A) \) denotes the closure of a set \( A \subset T^*M \times \mathbb{R} \).

• \( \text{co}(A) \) denotes the convex hull of a set \( A \subset T^*M \times \mathbb{R} \).

• \( h_{x_0,u_0}(x, t) \) (resp. \( h_{x_0,u_0}^c(x, t) \)) denotes the forward (resp. backward) implicit action function associated with \( L \).

• \( h_{x_0,u_0}^c(x, t) \) (resp. \( h_{x_0,u_0}^c(x, t) \)) denotes the forward (resp. backward) implicit action function associated with \( L + c \), where \( c \in \mathbb{R} \).

• \( \{T^{-}_t\}_{t \geq 0} \) (resp. \( \{T^{+}_t\}_{t \geq 0} \)) denotes the backward (resp. forward) solution semigroup associated with \( L \).

• \( \{T^{-}_t, c\}_{t \geq 0} \) (resp. \( \{T^{+}_t, c\}_{t \geq 0} \)) denotes the backward (resp. forward) solution semigroup associated with \( L + c \), where \( c \in \mathbb{R} \).

• \( C_\pm := \{ \varphi \in C(M) : T^c_t \varphi \geq \varphi, \quad \forall t \geq 0 \} \).

• \( C^c_\pm := \{ \varphi \in C(M) : T^{+,-,c}_t \varphi \geq \varphi, \quad \forall t \geq 0 \} \).

• \( D^c_\pm := \{ \varphi \in C(M) : T^c_t \varphi \geq \varphi, \quad t > 0 \} \).

• \( D^c_{\pm, c} := \{ \varphi \in C(M) : T^{+,-,c}_t \varphi \geq \varphi, \quad t > 0 \} \).

• \( \mathcal{C} := \{ c \in \mathbb{R} : \text{equation } (E_c) \text{ has BWKAM solutions} \} \).

• We use \( \varphi, \psi, \varphi', \psi' \) to denote generic functions in \( C(M) \) not necessarily the same in any two proofs, and use \( \varphi_\infty, \psi_\infty, \varphi'_\infty, \psi'_\infty \) to denote limit functions of \( T^{\pm,c}_t f \) as \( t \to +\infty \), with \( f = \varphi, \psi, \varphi', \psi' \).
2.2 Weak KAM theory for contact Hamiltonian systems

In the rest of this section we recall some definitions and basic results in the weak KAM theory for contact Hamiltonian system \((1.3)\), where an implicit variational principle plays an essential role. Most of the results in this section can be found in \([35,36,37,38]\).

- **Variational principles.** First recall implicit variational principles for contact Hamiltonian system \((1.3)\), which connects contact Hamilton-Jacobi equations and contact Hamiltonian systems.

**Proposition 4.** For any given \(x_0 \in M, u_0 \in \mathbb{R}\), there exist two continuous functions \(h_{x_0,u_0}(x,t)\) and \(h^{x_0,u_0}(x,t)\) defined on \(M \times (0, +\infty)\) satisfying

\[
h_{x_0,u_0}(x,t) = u_0 + \inf_{\gamma(0) = x_0} \int_0^t L(\gamma(\tau), h_{x_0,u_0}(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau, \quad (2.1)
\]

\[
h^{x_0,u_0}(x,t) = u_0 - \inf_{\gamma(t) = x, \gamma(0) = x_0} \int_0^t L(\gamma(\tau), h^{x_0,u_0}(\gamma(\tau), t-\tau), \dot{\gamma}(\tau)) d\tau, \quad (2.2)
\]

where the infimums are taken among the Lipschitz continuous curves \(\gamma : [0, t) \to M\). Moreover, the infimums in \((2.1)\) and \((2.2)\) can be achieved. If \(\gamma_1\) and \(\gamma_2\) are curves achieving the infimums \((2.1)\) and \((2.2)\) respectively, then \(\gamma_1\) and \(\gamma_2\) are of class \(C^1\). Let

\[
x_1(s) := \gamma_1(s), \quad u_1(s) := h_{x_0,u_0}(\gamma_1(s), s), \quad p_1(s) := \frac{\partial L}{\partial x}(\gamma_1(s), u_1(s), \dot{\gamma_1}(s)),
\]

\[
x_2(s) := \gamma_2(s), \quad u_2(s) := h^{x_0,u_0}(\gamma_2(s), t-s), \quad p_2(s) := \frac{\partial L}{\partial x}(\gamma_2(s), u_2(s), \dot{\gamma_2}(s)).
\]

Then \((x_1(s), u_1(s), p_1(s))\) and \((x_2(s), u_2(s), p_2(s))\) satisfy equations \((1.3)\) with

\[
x_1(0) = x_0, \quad x_1(t) = x, \quad \lim_{s \to 0^+} u_1(s) = u_0,
\]

\[
x_2(0) = x, \quad x_2(t) = x_0, \quad \lim_{s \to t^-} u_2(s) = u_0.
\]

We call \(h_{x_0,u_0}(x,t)\) (resp. \(h^{x_0,u_0}(x,t)\)) a forward (resp. backward) implicit action function associated with \(L\) and the curves achieving the infimums in \((2.1)\) (resp. \((2.2)\)) minimizers of \(h_{x_0,u_0}(x,t)\) (resp. \(h^{x_0,u_0}(x,t)\)). The relation between forward and backward implicit action functions is as follows: for any given \(x_0, x \in M, u_0, u \in \mathbb{R}\) and \(t > 0\),

\[
h_{x_0,u_0}(x,t) = u \quad \text{if and only if} \quad h^{x,u}(x_0,t) = u_0. \quad (2.3)
\]

See \([7]\) for another formulation of variational principles from the optimal control point of view. This viewpoint is strongly reminiscent of Herglotz’ variational principle \([24]\). The following result is a direct consequence of \((2.3)\).

**Proposition 5.** For any \(x, y \in M, any u \in \mathbb{R}\) and any \(t > 0\),
(1) \( h_{y,h^x,u}(y,t)(x,t) = u \),

(2) \( h_{y,h^x,u}(y,t)(x,t) = u \).

- Implicit action functions. We now collect some basic properties of implicit action functions.

**Proposition 6.**

1. **(Monotonicity).** Given \( x_0 \in M, u_0, u_1, u_2 \in \mathbb{R}, \) Lagrangians \( L_1 \) and \( L_2 \) satisfying (L1)-(L3),
   
   (i) if \( u_1 < u_2 \), then \( h_{x_0,u_1}(x,t) < h_{x_0,u_2}(x,t), \) for all \( (x,t) \in M \times (0, +\infty) \);

   (ii) if \( L_1 < L_2 \), then \( h_{x_0,u_0}^{L_1}(x,t) < h_{x_0,u_0}^{L_2}(x,t), \) for all \( (x,t) \in M \times (0, +\infty) \) where \( h_{x_0,u_0}^{L_i}(x,t) \) denotes the forward implicit action function associated with \( L_i, i = 1, 2 \).

2. **(Lipschitz continuity).** The function \( (x_0, u_0, x, t) \mapsto h_{x_0,u_0}(x,t) \) is Lipschitz continuous on \( M \times [a, b] \times M \times [c, d] \) for all real numbers \( a, b, c, d \) with \( a < b \) and \( 0 < c < d \).

3. **(Minimality).** Given \( x_0, x \in M, u_0 \in \mathbb{R} \) and \( t > 0 \), let \( S_{x_0,u_0}^{x,t} \) be the set of the solutions \( (x(s), u(s), p(s)) \) of (1.3) on \([0, t]\) with \( x(0) = x_0, x(t) = x, u(0) = u_0 \). Then
   
   \[
   h_{x_0,u_0}(x,t) = \inf \{ u(t) : (x(s), u(s), p(s)) \in S_{x_0,u_0}^{x,t}, \forall (x,t) \in M \times (0, +\infty) \}.
   \]

4. **(Markov property).** Given \( x_0 \in M, u_0 \in \mathbb{R}, \)

   \[
   h_{x_0,u_0}(x,t+s) = \inf_{y \in M} h_{y,h_{x_0,u_0}(y,t)}(x,s)
   \]

   for all \( s, t > 0 \) and all \( x \in M \). Moreover, the infimum is attained at \( y \) if and only if there exists a minimizer \( \gamma \) of \( h_{x_0,u_0}(x,t+s) \) with \( \gamma(t) = y \).

5. **(Reversibility).** Given \( x_0, x \in M \) and \( t > 0 \), for each \( u \in \mathbb{R} \), there exists a unique \( u_0 \in \mathbb{R} \)

   such that

   \[
   h_{x_0,u_0}(x,t) = u.
   \]

**Proposition 7.**

1. **(Monotonicity).** Given \( x_0 \in M \) and \( u_1, u_2 \in \mathbb{R}, \) Lagrangians \( L_1, L_2 \) satisfying (L1)-(L3),

   (i) if \( u_1 < u_2 \), then \( h_{x_0,u_1}(x,t) < h_{x_0,u_2}(x,t), \) for all \( (x,t) \in M \times (0, +\infty) \);

   (ii) if \( L_1 > L_2 \), then \( h_{x_0,u_0}^{L_1}(x,t) < h_{x_0,u_0}^{L_2}(x,t), \) for all \( (x,t) \in M \times (0, +\infty) \), where \( h_{x_0,u_0}^{L_i}(x,t) \) denotes the backward implicit action function associated with \( L_i, i = 1, 2 \).

2. **(Lipschitz continuity).** The function \( (x_0, u_0, x, t) \mapsto h_{x_0,u_0}(x,t) \) is Lipschitz continuous on \( M \times [a, b] \times M \times [c, d] \) for all real numbers \( a, b, c, d \) with \( a < b \) and \( 0 < c < d \).
(3) (Maximality). Given \( x_0, x \in M, u_0 \in \mathbb{R} \) and \( t > 0 \), let \( S^{x_0,u_0}_{x,t} \) be the set of the solutions \( (x(s), u(s), p(s)) \) of \((1.3)\) on \([0,t]\) with \( x(0) = x, x(t) = x_0, u(t) = u_0 \). Then
\[
h^{x_0,u_0}(x,t) = \sup\{u(0) : (x(s), u(s), p(s)) \in S^{x_0,u_0}_{x,t}\}, \quad \forall (x,t) \in M \times (0, +\infty).
\]

(4) (Markov property). Given \( x_0 \in M, u_0 \in \mathbb{R} \),
\[
h^{x_0,u_0}(x,t+s) = \sup_{y \in M} h^{y,h^{x_0,u_0}(y,t)}(x,s)
\]
for all \( s, t > 0 \) and all \( x \in M \). Moreover, the supremum is attained at \( y \) if and only if there exists a minimizer \( \gamma \) of \( h^{x_0,u_0}(x, t+s) \), such that \( \gamma(t) = y \).

(5) (Reversibility). Given \( x_0, x \in M \), and \( t > 0 \), for each \( u \in \mathbb{R} \), there exists a unique \( u_0 \in \mathbb{R} \) such that
\[
h^{x_0,u_0}(x,t) = u.
\]

We will use the following result to prove Proposition 8 below. See Appendix 6.1 for the proof.

**Proposition 8.** Let \((x(t), u(t)) : \mathbb{R} \to M \times \mathbb{R}\) be a locally Lipschitz curve satisfying
\[
u(t_2) = h_{x(t_1),u(t_1)}(x(t_2), t_2 - t_1)
\]
for all \( t_1, t_2 \in \mathbb{R} \) with \( t_1 < t_2 \). Then \( x(t) \) is of class \( C^1 \). Let \( p(t) := \frac{\partial L}{\partial x}(x(t), u(t), \dot{x}(t)) \). Then \((x(t), u(t), p(t))\) is a solution of \((1.3)\). Moreover, for each \( t_1, t_2 \in \mathbb{R} \) with \( t_1 < t_2 \), \( x(t)|_{[t_1,t_2]} \) is a minimizer of \( h_{x(t_1),u(t_1)}(x(t_2), t_2 - t_1) \).

• **Solution semigroups.** We collect some basic properties of the solution semigroups.

**Proposition 9.** Let \( \varphi, \psi \in C(M) \).

1. (Monotonicity). If \( \psi < \varphi \), then \( T^\pm_t \psi < T^\pm_t \varphi, \forall t \geq 0 \).

2. (Local Lipschitz continuity). The function \((x,t) \mapsto T^\pm_t \varphi(x)\) is locally Lipschitz on \( M \times (0, +\infty) \).

3. \((e^M\)-expansiveness\). \( \|T^\pm_t \varphi - T^\pm_t \psi\|_\infty \leq e^M \cdot \| \varphi - \psi \|_\infty, \forall t \geq 0 \).

4. (Continuity at the origin). \( \lim_{t \to 0^+} T^\pm_t \varphi = \varphi \).

5. (Representation formula). For each \( \varphi \in C(M) \),
   
   (i) \( T^-_t \varphi(x) = \inf_{y \in M} h_{y,\varphi(y)}(x,t), \quad \forall (x,t) \in M \times (0, +\infty) \); 

   (ii) \( T^+_t \varphi(x) = \sup_{y \in M} h_{y,\varphi(y)}(x,t), \quad \forall (x,t) \in M \times (0, +\infty) \).
We say that $u \in C(M)$ is called a backward weak KAM solution of $(E_0)$ if

1. for each continuous piecewise $C^1$ curve $\gamma : [t_1, t_2] \to M$, we have
   
   $$u(\gamma(t_2)) - u(\gamma(t_1)) \leq \int_{t_1}^{t_2} L(\gamma(s), u(\gamma(s)), \dot{\gamma}(s))ds; \quad (2.4)$$

2. for each $x \in M$, there exists a $C^1$ curve $\gamma : (-\infty, 0] \to M$ with $\gamma(0) = x$ such that
   
   $$u(x) - u(\gamma(t)) = \int_t^0 L(\gamma(s), u(\gamma(s)), \dot{\gamma}(s))ds, \quad \forall t < 0. \quad (2.5)$$

Similarly, a function $v \in C(M)$ is called a forward weak KAM solution of $(E_0)$ if it satisfies (1) and for each $x \in M$, there exists a $C^1$ curve $\gamma : [0, +\infty) \to M$ with $\gamma(0) = x$ such that

$$v(\gamma(t)) - v(x) = \int_t^0 L(\gamma(s), v(\gamma(s)), \dot{\gamma}(s))ds, \quad \forall t > 0. \quad (2.6)$$

We say that $u$ in (2.4) is a dominated function by $L$, denoted by $u \prec L$. We call curves satisfying (2.5) (resp. (2.6)), $(u, L, 0)$-calibrated curves (resp. $(v, L, 0)$-calibrated curves).

**Proposition 10.**

1. $u \in S_-$ if and only if $T^-_t u = u$ for all $t \geq 0$.

2. $v \in S_+$ if and only if $T^+_t v = v$ for all $t \geq 0$.

The following result will be useful for the proof of Proposition 2. We give the proof in the Appendix 6.2, since it is quite lengthy.

**Proposition 11.** Let $u \in S_-$. Given any $x \in M$, if $\gamma : (-\infty, 0] \to M$ is a $(u, L, 0)$-calibrated curve with $\gamma(0) = x$, then $(\gamma(t), u(\gamma(t)), p(t))$ satisfies equations (1.3) on $(-\infty, 0)$, where $p(t) = \frac{\partial L}{\partial \dot{x}}(\gamma(t), u(\gamma(t)), \dot{\gamma}(t))$. Moreover, we have

$$(\gamma(t+s), u(\gamma(t+s)), Du(\gamma(t+s))) = \Phi_s(\gamma(t), u(\gamma(t)), Du(\gamma(t)), \quad \forall t, s < 0,$$

and

$$H(\gamma(t), u(\gamma(t)), \frac{\partial L}{\partial x}(\gamma(t), u(\gamma(t)), \dot{\gamma}(t))) = 0, \quad \forall t < 0.$$
Remark 4. A similar result holds true for \( v \in S_+ \): let \( v \in S_+ \). Given any \( x \in M \), if \( \gamma : [0, +\infty) \to M \) is a \( (v, L, 0) \)-calibrated curve with \( \gamma(0) = x \), then \( (\gamma(t), v(\gamma(t)), p(t)) \) satisfies equations (1.3) on \( (0, +\infty) \), where \( p(t) = \frac{\partial}{\partial z} (\gamma(t), v(\gamma(t)), \gamma(t)) \). Moreover, we have

\[
(\gamma(t + s), v(\gamma(t + s)), Dv(\gamma(t + s))) = \Phi_s(\gamma(t), v(\gamma(t)), Dv(\gamma(t))), \quad \forall t, s > 0.
\]

Since the proof of the above result is quite similar to the one of Proposition 11, we omit it.

Let \( u \in S_- \) and \( v \in S_+ \). In view of Lemma 5 in the Appendix 6.2, both \( u \) and \( v \) are Lipschitz continuous.

3 Existence of BWKAM solutions

3.1 More on solution semigroups

Proposition 12. Let \( \varphi \in C(M) \). If the function \( (x, t) \mapsto T_t^- \varphi(x) \) is bounded on \( M \times [0, +\infty) \), then \( \varphi_\infty(x) := \lim_{t \to +\infty} T_t^- \varphi(x) \) is a BWKAM solution of \((E_0)\).

Proof. Let \( K_1 \) be a positive constant such that

\[
|T_t^- \varphi(x)| \leq K_1, \quad \forall (x, t) \in M \times [0, +\infty). \tag{3.1}
\]

Recall Proposition 6(2), i.e., the function \( (x_0, u_0, x, t) \mapsto h_{x_0,u_0}(x, t) \) is Lipschitz on \( M \times [a, b] \times M \times [c, d] \) for all real numbers \( a, b, c, d \) with \( a < b \) and \( 0 < c < d \).

First we show that \( \{T_t^- \varphi(x)\}_{t \geq 0} \) is equi-Lipschitz on \( M \). Denote by \( l_1 > 0 \) a Lipschitz constant of the function \( (x_0, u_0, x) \mapsto h_{x_0,u_0}(x, 1) \) on \( M \times [-K_1, K_1] \times M \). From Proposition 9(6)(i), we have

\[
|T_t^- \varphi(x) - T_t^- \varphi(y)| \leq \sup_{z \in M} |h_{z,T_t^- \varphi(z)}(x, 1) - h_{z,T_t^- \varphi(z)}(y, 1)|
\]

for all \( x, y \in M \), and all \( t > 1 \). In view of (3.1), the above inequality implies that

\[
|T_t^- \varphi(x) - T_t^- \varphi(y)| \leq l_1 \cdot d(x, y).
\]

Then let \( \varphi_\infty(x) := \lim_{t \to +\infty} T_t^- \varphi(x) \). We show that \( \varphi_\infty \) is a fixed point of \( \{T_t^-\}_{t \geq 0} \). Since \( \{T_t^- \varphi(x)\}_{t \geq 1} \) is equi-Lipschitz on \( M \), it is easy to see that

\[
\lim_{t \to +\infty} \inf_{s \geq t} T_s^- \varphi(x) = \varphi_\infty(x) \quad \text{uniformly on } x \in M. \tag{3.2}
\]

For each \( t > 0 \) and each \( x \in M \), we get

\[
\varphi_\infty(x) = \lim_{\sigma \to +\infty} \inf_{s \geq \sigma} T_{s+t}^- \varphi(x)
= \lim_{\sigma \to +\infty} \inf_{s \geq \sigma} h_{y,T_s^- \varphi(y)}(x, t)
= \lim_{\sigma \to +\infty} \inf_{y \in M} h_{y,\inf_{s \geq \sigma} T_s^- \varphi(y)}(x, t)
= \lim_{\sigma \to +\infty} g_\sigma(x, t) \tag{3.3}
\]
where the first equality comes from the definition of $\varphi_{\infty}$, the second one comes from Proposition 9(6)(i) and the third one comes from Proposition 6(1).

Denote by $l_t > 0$ a Lipschitz constant of the function $(x_0, u_0, x) \mapsto h_{x_0,u_0}(x, t)$ on $M \times [-K_1 - \|\varphi_{\infty}\|_\infty; K_1 + \|\varphi_{\infty}\|_\infty] \times M$. Note that for each $\sigma > 0$, we have

$$
|g_\sigma(x, t) - T^{-}\varphi_{\infty}(x)| = |\inf_{y \in M} h_{y, \inf s \geq \sigma T^{-}\varphi(y)}(x, t) - \inf_{y \in M} h_{y, \varphi_{\infty}(y)}(x, t)| = l_t \cdot \sup_{y \in M} |\inf_{s \geq \sigma} T^{-}\varphi(y) - \varphi_{\infty}(y)|.
$$

(3.4)

Combining (3.2) and (3.4), we deduce that $\lim_{\sigma \rightarrow +\infty} g_\sigma(x, t) = T^{-}\varphi_{\infty}(x)$, which together with (3.3), implies that $T^{-}\varphi_{\infty}(x) = \varphi_{\infty}(x)$, $\forall (x, t) \in M \times [0, \infty)$.

Since the function $(x, t) \mapsto T^{-}\varphi_{\infty}(x)$ is a viscosity solution of evolutionary equation (1.4), and thus the function $x \mapsto \varphi_{\infty}(x)$ is a BWKAM solution of equation (E).

**Proposition 13.** Let $\varphi \in C(M)$.

(1) $T^{-}_t \circ T^+_t \varphi \geq \varphi$, $\forall t > 0$,

(2) $T^+_t \circ T^{-}_t \varphi \leq \varphi$, $\forall t > 0$.

**Proof.** For any $x \in M$ and $t > 0$, we have that

$$
T^{-}_t \circ T^+_t \varphi(x) = \inf_{y \in M} h_{y, \varphi_{\infty}(y)}(x, t) \geq \inf_{y \in M} h_{y, h_{x, \varphi_{\infty}}(y, t)}(x, t) = \varphi(x),
$$

and

$$
T^+_t \circ T^{-}_t \varphi(x) = \sup_{y \in M} h_{y, \varphi_{\infty}(y)}(x, t) \leq \sup_{y \in M} h_{y, h_{x, \varphi_{\infty}}(y, t)}(x, t) = \varphi(x).
$$

The following result is a direct consequence of the above proposition.

**Corollary 3.** Let

$$
\Psi^-_t := T^{-}_t \circ T^+_t, \quad \Psi^+_t := T^+_t \circ T^{-}_t.
$$

Then for any $\varphi \in C(M)$, and any $t > 0$,

$$
\Psi^-_t \varphi \geq \varphi, \quad \Psi^+_t \varphi \leq \varphi.
$$
3.2 Proof of Proposition 2

Before showing Proposition 2, we need to prove some preliminary results.

**Proposition 14.** Let \( u, v \in C(M) \) and let \( t \geq 0 \). Then \( v \leq T_t^{-} u \) if and only if \( T_t^{+} v \leq u \).

**Proof.** If \( v \leq T_t^{-} u \) for some \( t \geq 0 \), we will show \( T_t^{+} v \leq u \). It is clear that \( T_0^{+} v(x) = v(x) \). Fix \((x,t) \in M \times (0, +\infty)\). By Proposition 9(5)(ii), we have

\[
T_t^{+} v(x) = \sup_{y \in M} h_{y,v}(y)(x,t).
\]

It suffices to prove that \( h_{y,v}(y)(x,t) \leq u(x) \) for all \( y \in M \). Let \( \varphi(y) := h_{y,v}(y)(x,t) \) for all \( y \in M \). Then by (2.3), we have that \( v(y) = h_{x,\varphi(y)}(y,t) \) for all \( y \in M \). In view of \( v \leq T_t^{-} u \) and Proposition 9(5)(i), for each \( y \in M \), we get that

\[
v(y) \leq T_t^{-} u(y) = \inf_{z \in M} h_{z,u}(z)(y,t),
\]

which implies \( v(y) \leq h_{x,u}(x)(y,t) \), that is, \( h_{x,\varphi(y)}(y,t) \leq h_{x,u}(x)(y,t) \). By Proposition 6(1)(i), we have \( \varphi(y) \leq u(x) \) for each \( y \in M \).

The converse implication can be proved in a similar manner.

The following result is a direct consequence of the above proposition.

**Corollary 4.** Solution semigroups \( \{ T_t^{-} \}_{t \geq 0} \) and \( \{ T_t^{+} \}_{t \geq 0} \) preserve the set of viscosity subsolutions of equation (E0).

**Proposition 15.** Let \( u \in S_- \). For each \( x \in M \), let \( \gamma : (-\infty, 0] \to M \) be a \( (u,L,0) \)-calibrated curve with \( \gamma(0) = x \). Then, \( T_t^{+} u(\gamma(-t)) = u(\gamma(-t)) \) for all \( t \geq 0 \).

**Proof.** For any given \( t > 0 \), let \( z = \gamma(-t) \) and \( u_t = u(z) \). By Proposition 14, we only need to prove \( T_t^{+} u(z) \geq u_t \). By Proposition 9(5)(ii), we have

\[
T_t^{+} u(z) = \sup_{y \in M} h_{x,u}(y)(z,t) \geq h_{x,u}(x)(z,t).
\]

So, it suffices to show \( h_{x,u}(x)(z,t) \geq u_t \). By Proposition 11 \((\gamma(s), u(\gamma(s)), p(s)) \) satisfies equations (1.3) on \((-\infty, 0)\), where \( p(s) = \frac{\partial}{\partial s} (\gamma(s), u(\gamma(s)), \gamma(s)) \). Let \( u(s) := u(\gamma(s-t)) \) for \( s \in [0,t] \). Then \( u(0) = u(\gamma(-t)) = u_t \) and \( u(t) = u(\gamma(0)) = u(x) \). By Proposition 7(3), it is clear that

\[
h_{x,u}(t)(z,t) \geq u_t.
\]

This completes the proof.

**Corollary 5.** Let \( u \in S_- \). The family of functions \( \{ T_t^{+} u \}_{t \geq 2} \) is uniformly bounded and equi-Lipschitz on \( M \).
Proof. In order to prove the corollary, we proceed in two steps.

Step 1: we first prove the uniform boundedness of \( \{T^+_tu\}_{t>2} \). By Proposition \ref{prop:uniformboundedness} and the compactness of \( M \), the function \((x,t) \mapsto T^+_tu(x)\) is bounded from above on \( M \times [0, +\infty) \).

On the other hand, since \( u \in S_- \), then for any given \( y \in M \), there is a \((u, L, 0)\)-calibrated curve \( \gamma : (-\infty, 0] \to M \) with \( \gamma(0) = y \). By Proposition \ref{prop:gamma1}, \( T^+_tu(\gamma(t)) = u(\gamma(t)) \) for all \( t > 0 \). For any \( t > 1 \) and any \( x \in M \), from Proposition \ref{prop:gamma2}(i), we deduce that

\[
T^+_tu(x) = T^+_1 \circ T^+_t u(x) = \sup_{z \in M} h^z : T^+_t u(z)(x, 1) \\
\geq h^{\gamma(0)}, T^+_t u(\gamma(t))(x, 1) \\
= h^{\gamma(t), u(\gamma(-t))}(x, 1).
\]

By Proposition \ref{prop:gamma1}(2), the function \( h^\cdot(\cdot, 1) \) is bounded on \( M \times [-\|u\|_{\infty}, \|u\|_{\infty}] \times M \). Thus, the function \((x,t) \mapsto T^+_tu(x)\) is bounded from below on \( M \times (1, +\infty) \).

Step 2: we show the equi-Lipschitz property of \( \{T^+_tu\}_{t>2} \). Denote by \( K_2 > 0 \) a constant such that \( \|T^+_tu\|_{\infty} \leq K_2 \) for all \( t > 1 \). In view of Proposition \ref{prop:gamma2}(ii), for any \( x, y \in M \), we get that

\[
|T^+_tu(x) - T^+_tu(y)| = |\sup_{z \in M} h^z : T^+_t u(z)(x, 1) - \sup_{z \in M} h^z : T^+_t u(z)(y, 1)| \\
\leq \sup_{z \in M} |h^z : T^+_t u(z)(x, 1) - h^z : T^+_t u(z)(y, 1)|.
\]

By Proposition \ref{prop:gamma1}(2), the function \( h^\cdot(\cdot, 1) \) is Lipschitz on \( M \times [-K_2, K_2] \times M \) with a Lipschitz constant \( \kappa_1 > 0 \), and thus we get that

\[
|T^+_tu(x) - T^+_tu(y)| \leq \kappa_1 d(x, y), \quad \forall t > 2.
\]

Proof of Proposition \ref{prop:uniformboundedness} (1) By Proposition \ref{prop:uniformboundedness} and Corollary \ref{cor:uniformboundedness}, the uniform limit \( \lim_{t \to +\infty} T^+_tu \) exists. Define

\[
v := \lim_{t \to +\infty} T^+_tu.
\]

It follows from Proposition \ref{prop:uniformboundedness} that, for any given \( t \geq 0 \), we get that

\[
\|T^+_tu - T^+_tv\|_{\infty} \leq e^{t} \|T^+_su - v\|_{\infty}, \quad \forall s > 0.
\]

Letting \( s \to +\infty \), we have

\[
T^+_tv(x) = v(x), \quad \forall x \in M.
\]

By Proposition \ref{prop:uniformboundedness}(2), we deduce that \( v \in S_+ \). The proof of Proposition \ref{prop:uniformboundedness}(1) is complete.

(2) Now we turn to the proof of Proposition \ref{prop:uniformboundedness}(2). Since the proof of (2) is quite similar to the one of (1), we only sketch the strategy of the proof.

Let \( v \in S_+ \). By similar arguments used in the proofs of Propositions \ref{prop:uniformboundedness} \ref{prop:gamma1} and Corollary \ref{cor:uniformboundedness} one can show that
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(a) $T_t^- v \geq v$ for all $t \geq 0$.

(b) For each $x \in M$, let $\gamma : [0, +\infty) \to M$ be a $(v, L, 0)$-calibrated curve with $\gamma(0) = x$. Then, $T_t^- v(\gamma(t)) = v(\gamma(t))$ for all $t \geq 0$.

(c) The family of functions $\{T_t^- v\}_{t>2}$ is uniformly bounded and equi-Lipschitz on $M$.

Using the above three facts, we deduce that the uniform limit $\lim_{t \to +\infty} T_t^- v =: u$ exists and that $u \in S_-$.

The proof of Proposition 2 is now complete.

3.3 Proof of Proposition 1

Proof of Proposition 1 Let $u$ be a BWKAM solution of equation $(E_0)$. Then by Proposition 2, the function

$$v := \lim_{t \to +\infty} T_t^+ u$$

is well-defined and is a FWKAM solution of equation $(E_0)$. So, $T_t^- u = u$ and $T_t^+ v = v$ for all $t \geq 0$, and thus $C_\pm \neq \emptyset$.

On the other hand, since $C_\pm \neq \emptyset$, one can take $\varphi \in C_-$ and $\psi \in C_+$. In view of $\varphi \leq T_t^- \varphi$ for all $t \geq 0$, if $\lim_{t \to +\infty} T_t^- \varphi(x_0) < +\infty$ for some $x_0 \in M$, then for each $x \in M$, we have

$$T_t^- \varphi(x) = (T_1^0 \circ T_{t-1}^-) \varphi(x) = \inf_{y \in M} h_{y,T_t^{-1} \varphi(y)}(x,1) \leq h_{x_0,T_t^{-1} \varphi(x_0)}(x,1) \leq h_{x_0,A}(x,1) < +\infty,$$

where $A = \lim_{t \to +\infty} T_t^- \varphi(x_0)$. So, there are two possibilities: (i) $\lim_{t \to +\infty} T_t^- \varphi(x) = \varphi_\infty(x)$ for all $x \in M$ and for some $\varphi_\infty \in C(M)$; (ii) $\lim_{t \to +\infty} T_t^- \varphi(x) = +\infty$ for all $x \in M$.

For case (i), since $\varphi \leq T_t^- \varphi \leq \varphi_\infty$, then by Proposition 12 we deduce that $\varphi_\infty$ is a BWKAM solution of $(E_0)$.

For case (ii), we assert that $\lim_{t \to +\infty} T_t^- \varphi(x) = +\infty$ uniformly on $x \in M$. Suppose not. Then there are a constant $A' > 0$, a sequence $\{t_n\} \nearrow +\infty$ and a sequence $\{x_n\}$ such that

$$T_{t_n}^- \varphi(x_n) \leq A'.$$

Note that for any $x \in M$ and any $t > 1$,

$$T_t^- \varphi(x) = T_1^- \circ T_{t-1}^- \varphi(x) \leq h_{y,T_t^{-1} \varphi(y)}(x,1), \quad \forall y \in M.$$

For any $t > 1$, there is $n_0$ such that $t_{n_0} > t - 1$ and thus

$$T_t^- \varphi(x) \leq h_{x_{n_0},T_{t-n_0}^- \varphi(x_{n_0})}(x,1) \leq h_{x_{n_0},A'}(x,1) \leq \max_{x,y \in M} h_{y,A'}(x,1),$$

implying that $T_t^- \varphi(x)$ is bounded from above, a contradiction.
Thus, we deduce that for any $t > T$ of \( (\text{Proposition 12}) \), we have \( \lim_{t \to \infty} \phi_t \leq \phi_0 \). Since \( \phi \in C_\alpha \), then \( \phi_0 \in C_\alpha \). In view of Proposition \( 14 \), we deduce that \( T_t^+ \phi_0 \leq \phi_0 \) for all \( t > 0 \). Note that \( T_t^+ \phi_0 \geq T_t^+ \psi \geq \psi \). So, the function

\[
v(x) := \lim_{t \to +\infty} T_t^+ \phi_0(x)
\]

is well-defined. By Proposition \( 9 \) (6)(ii), we have

\[
|T_t^+ \phi_0(x) - T_t^+ \phi_0(y)| \leq \sup_{z \in M} |h^{z,T_{t-1}^+ \phi_0}(x,1) - h^{z,T_{t-1}^+ \phi_0}(y,1)|
\]

for all \( x, y \in M \), and all \( t > 1 \). In view of the boundedness of \( T_t^+ \phi_0 \), we deduce that \( \{T_t^+ \phi_0(x)\}_{t \geq 0} \) is equi-Lipschitz. So, \( v(x) = \lim_{t \to +\infty} T_t^+ \phi_0(x) \) uniformly on \( M \). And then \( T_t^+ v = v \) for all \( t \geq 0 \), i.e., \( v \) is a FWKAM solution of equation \( (E_0) \). From Proposition \( 2 \) we know that \( \lim_{t \to +\infty} T_t^+ v \) exists and is a BWKAM solution of equation \( (E_0) \).

\( \square \)

3.4 Proof of Theorem \( 1 \)

**Lemma 1.** Let \( \phi \in C(M) \). If \( \phi \leq T_t^{-c} \phi \) (resp. \( \phi \geq T_t^{-c} \phi \)) for all \( t \geq 0 \), then either

\[
\lim_{t \to +\infty} T_t^{-c} \phi(x) = +\infty \quad \text{(resp.} \quad \lim_{t \to +\infty} T_t^{-c} \phi(x) = -\infty) \]

uniformly on \( x \in M \), or

\[
\lim_{t \to +\infty} T_t^{-c} \phi(x) = \phi_\infty(x)
\]

uniformly on \( x \in M \), where \( \phi_\infty(x) \) is a BWKAM solution of \( (E_0) \).

**Proof.** We divide the proof in two steps.

\textbf{Step 1:} First, we consider the case \( \phi \leq T_t^{-c} \phi \) for all \( t \geq 0 \). In view of \( \phi \leq T_t^{-c} \phi \), if \( \lim_{t \to +\infty} T_t^{-c} \phi(x_0) = B < +\infty \) for some \( x_0 \in M \), then for each \( x \in M \), we have

\[
T_t^{-c} \phi(x) = (T_t^{-c} \circ T_t^{-c})(y) = \inf_{y' \in M} h_{y',T_t^{-c} \phi(y)}^c(x,1) \leq h_{y_0,T_t^{-c} \phi(x_0)}^c(x,1) \leq h_{y_0,B}^c(x,1) < +\infty.
\]

Thus, we deduce that for any \( t > 1 \),

\[
-\|\phi\|_\infty \leq T_t^{-c} \phi(x) \leq \max_{y \in M} h_{y,B}^c(y',1), \quad \forall x \in M.
\]

\( (3.6) \)

So, there are two possibilities: (i) \( \lim_{t \to +\infty} T_t^{-c} \phi(x) = \phi_\infty(x) \) for all \( x \in M \). In view of \( (3.6) \), by Proposition \( 12 \) \( \lim_{t \to +\infty} T_t^{-c} \phi(x) = \phi_\infty(x) \) uniformly on \( x \in M \), and \( \phi_\infty \) is a BWKAM solution of \( (E_0) \). (ii) \( \lim_{t \to +\infty} T_t^{-c} \phi(x) = +\infty \) for all \( x \in M \). Next, we prove \( \lim_{t \to +\infty} T_t^{-c} \phi(x) = +\infty \) uniformly on \( x \in M \). Suppose not. Then there are \( K_0 > 0 \), \( \{t_n\} \nearrow +\infty \) and \( x_n \in M \), such that

\[
T_{t_n}^{-c} \phi(x_n) \leq K_0.
\]
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Then for any \( n \in \mathbb{N} \), any \( x \in M \)

\[
T_{t_{n+1}}^{-c} \varphi(x) \leq h_{x_n,T_{t_{n+1}}^{-c} \varphi(x_n)}^c(x, 1) \leq h_{x_n,K_0}^c(x, 1) \leq \max_{y',y'' \in M} h_{y',K_0}^c(y'', 1) < +\infty,
\]

which contradicts \( \lim_{t \to +\infty} T_t^{-c} \varphi(x) = +\infty \) for all \( x \in M \).

**Step 2:** Second, we deal with the case \( \varphi \geq T_t^{-c} \varphi \) for all \( t \geq 0 \). If \( \lim_{t \to +\infty} T_t^{-} \varphi(x_0) = B' > -\infty \) for some \( x_0 \in M \), then for any \( y \in M \), we have that

\[
v_t := h_{y,T_t^{-c} \varphi(y)}^c(x_0, 1) \geq T_{t+1}^{-c} \varphi(x_0) \geq B'.
\]

So, we get that

\[
h_{x_0-v_t}^c(y, 1) = T_t^{-c} \varphi(y).
\]

Thus, one can deduce that for any \( t > 0 \)

\[
\|\varphi\|_\infty \geq \varphi(y) \geq T_t^{-c} \varphi(y) \geq h_{x_0,B'}^c(y, 1) \geq -\|h_{x_0,B'}^c(\cdot, 1)\|_\infty, \quad \forall y \in M.
\]

(3.7)

So, there are two possibilities: (i) \( \lim_{t \to +\infty} T_t^{-} \varphi(x) = \varphi'_\infty(x) \) for all \( x \in M \). In view of (3.7), by Proposition \ref{prop:liminf}, \( \lim_{t \to +\infty} T_t^{-c} \varphi(x) = \varphi'_\infty(x) \) uniformly on \( x \in M \), and \( \varphi'_\infty \) is a BWKAM solution of \((E_c)\). (ii) \( \lim_{t \to +\infty} T_t^{-c} \varphi(x) = +\infty \) for all \( x \in M \). Next, we prove \( \lim_{t \to +\infty} T_t^{-c} \varphi(x) = -\infty \) uniformly on \( x \in M \). Suppose not. Then there are \( K'_0 < 0 \), \( \{t'_n\} \nearrow +\infty \) and \( x'_n \in M \), such that

\[
T_{t'_n}^{-c} \varphi(x'_n) \geq K'_0.
\]

Given any \( y \in M \), let

\[
v'_n := h_{y,T_{t'_n-1}^{-c} \varphi(y)}^c(x'_n, 1) \geq T_{t'_n}^{-c} \varphi(x'_n) \geq K'_0.
\]

By (2.3),

\[
h_{x'_n,v'_n}^c(y, 1) = T_{t'_n-1}^{-c} \varphi(y).
\]

Thus,

\[
T_{t'_n-1}^{-c} \varphi(y) \geq h_{x'_n,K'_0}^c(y, 1) \geq \min_{z,z' \in M} h_{x'_n,K'_0}^c(z', 1) > -\infty,
\]

which contradicts \( \lim_{t \to +\infty} T_t^{-c} \varphi(y) = -\infty \).

The proof is complete.

The following result is a direct consequence of Lemma \ref{lem:convex}. We omit the proof.

**Corollary 6.** Let \( \varphi \in C(M) \). If \( \varphi \leq T_{t_0}^{-c} \varphi \) (resp. \( \varphi \geq T_{t_0}^{-c} \varphi \)) for some \( t_0 > 0 \), then either

\[
\lim_{n \to +\infty} T_{nt_0}^{-c} \varphi(x) = +\infty \quad (\text{resp.} \quad \lim_{n \to +\infty} T_{nt_0}^{-c} \varphi(x) = -\infty)
\]

uniformly on \( x \in M \), or \( \lim_{n \to +\infty} T_{nt_0}^{-c} \varphi(x) = \varphi_\infty(x) \) uniformly on \( x \in M \), where \( \varphi_\infty(x) \in \text{Lip}(M) \).
Proof of Theorem 1. We only need to prove that if $D_{±} \neq \emptyset$, then equation (E₀) has viscosity solutions. Let $\varphi \in C(M)$ and $t_1 > 0$ be such that $T^{-}_{t_1} \varphi \geq \varphi$. Let $\psi \in C(M)$ and $t_2 > 0$ be such that $T^{-}_{t_2} \psi \geq \psi$.

Since $T^{-}_{t_1} \varphi \geq \varphi$, then by Corollary 6, either $\lim_{n \to +\infty} T^{-}_{nt_1} \varphi(x) = +\infty$ uniformly on $x \in M$, or $\lim_{n \to +\infty} T^{-}_{nt_1} \varphi(x) = \varphi_\infty(x)$ uniformly on $x \in M$, where $\varphi_\infty(x) \in \text{Lip}(M)$.

Case 1: If $\lim_{n \to +\infty} T^{-}_{nt_1} \varphi(x) = \varphi_\infty(x)$ uniformly on $x \in M$, then for any $s \in [0, t_1]$, $\lim_{n \to +\infty} T^{-}_{nt_1 + s} \varphi(x) = T^{-}_{s} \varphi_\infty(x)$. Hence, $T^{-}_{t} \varphi(x)$ is bounded on $M \times [0, +\infty)$. Then by Proposition 12,

$$\varphi'(x) := \liminf_{t \to +\infty} T^{-}_{t} \varphi(x)$$

is a BWKAM solution of equation (E₀).

Case 2: If $\lim_{n \to +\infty} T^{-}_{nt_1} \varphi(x) = +\infty$ uniformly on $x \in M$, then there is $n_1 \in \mathbb{N}$ such that $T^{-}_{n_1 t_1} \varphi > \psi$, and $T^{-}_{n_1 t_1} \varphi > \varphi$. Choose $k_1, k_2 \in \mathbb{N}$ such that

$$s_0 := \frac{k_1}{k_2} t_2 - n_1 t_1 > 0$$

small enough with

$$T^{-}_{n_1 t_1 + s_0} \varphi > \psi, \quad T^{-}_{n_1 t_1 + s_0} \varphi > \varphi.$$ 

Let $t_0 := k_2(n_1 t_1 + s_0) = k_1 t_2$. Then

$$T^{-}_{t_0} \varphi > \varphi, \quad T^{-}_{t_0} \varphi > \psi, \quad T^{+}_{t_0} \psi \geq \psi.$$ 

Then by Proposition 14,

$$T^{+}_{t_0} \varphi \leq \varphi.$$ 

Let $\varphi' = T^{-}_{t_0} \varphi$. Then by Proposition 13, we have that

$$T^{+}_{t_0} \varphi' = T^{+}_{t_0} \circ T^{-}_{t_0} \varphi \leq \varphi,$$

and

$$T^{+}_{t_0} \varphi' \geq T^{+}_{t_0} \psi \geq \psi.$$ 

Therefore, we get that

$$\psi \leq T^{+}_{nt_0} \psi \leq T^{+}_{nt_0} \varphi' \leq \varphi.$$ 

So, $\{T^{+}_{nt_0} \psi\}_{n \in \mathbb{N}}$ is bounded, and thus the uniform limit

$$\lim_{n \to \infty} T^{+}_{nt_0} \psi =: \psi_\infty$$ 

exists. And for any $s \in [0, t_0]$,

$$\lim_{n \to \infty} T^{+}_{nt_0 + s} \psi(x) =: T^{+}_{s} \psi_\infty(x), \quad x \in M.$$
It follows that the function \((x, t) \mapsto T_t^+ \psi(x)\) is bounded on \(M \times [0, +\infty)\). Let

\[
\psi'_\infty(x) := \lim_{t \to +\infty} \sup_t T_t^+ \psi(x).
\]

We assert that \(\psi'_\infty\) is a FWKAM solution of equation (3.10). If the assertion is true, then by Proposition 2, one can deduce that \(S_\infty \neq \emptyset\).

Next we show that \(\psi'_\infty\) is a fixed point of \(\{T_t^+\}_{t \geq 0}\). Since \(\{T_t^+ \psi(x)\}_{t \geq 0}\) is equi-Lipschitz on \(M\), it is easy to see that

\[
\lim_{t \to +\infty} \sup_{s \geq t} T_s^+ \psi(x) = \psi'_\infty(x)
\]

uniformly on \(x \in M\). (3.8)

For each \(t > 0\) and each \(x \in M\), we get

\[
\psi'_\infty(x) = \lim_{\sigma \to +\infty} \sup_{s \geq \sigma} T_s^+ \psi(x)
\]

\[
= \lim_{\sigma \to +\infty} \sup_{s \geq \sigma} \sup_{y \in M} h^{y, T_s^+ \psi(y)}(x, t)
\]

\[
= \lim_{\sigma \to +\infty} \sup_{y \in M} h^{y, \sup_{s \geq \sigma} T_s^+ \psi(y)}(x, t)
\]

\[
=: \lim_{\sigma \to +\infty} g'_\sigma(x, t).
\]

Denote by \(\kappa_t > 0\) a Lipschitz constant of the function \((x, u_0, x) \mapsto h^{x_0, u_0}(x, t)\) on \(M \times [-K_3 - \|\psi'_\infty\|_{\infty}, K_3 + \|\psi'_\infty\|_{\infty}] \times M\). Note that for each \(\sigma > 0\), we have

\[
|g'_\sigma(x, t) - T_t^+ \psi'_\infty(x)| = \sup_{y \in M} h^{y, \sup_{s \geq \sigma} T_s^+ \psi(y)}(x, t) - \sup_{y \in M} \psi'_\infty(y)(x, t)|
\]

\[
\leq \kappa_t \cdot \sup_{y \in M} \sup_{s \geq \sigma} |T_s^+ \psi(y) - \psi'_\infty(y)|.
\]

(3.10)

Combining (3.8) and (3.10), we deduce that \(\lim_{\sigma \to +\infty} g'_\sigma(x, t) = T_t^+ \psi'_\infty(x)\), which together with (3.9), implies that

\[
T_t^+ \psi'_\infty(x) = \psi'_\infty(x), \quad \forall (x, t) \in M \times [0, +\infty).
\]

The proof is complete.
4 Analysis of the admissible set

4.1 Proof of Theorem 2

Before proving Theorem 2, we recall a approximation and regularity result of Lipschitz functions by Czarnecki and Rifford [12].

Proposition 16. Let \( f \in \text{Lip}(M) \). Then there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( C^\infty(M) \) such that
\[
\lim_{n \to +\infty} \|f_n - f\|_\infty = 0
\]
and
\[
\lim_{n \to +\infty} d_{\text{Haus}}(\text{graph}(Df_n), \text{graph}(\partial f)) = 0.
\]

Here, \( \partial f(x) \) denotes Clarke’s generalized gradient of \( f \) at \( x \):
\[
\partial f(x) = \text{co} \{ \zeta \mid \exists (x_n)_{n \in \mathbb{N}} \subset \text{Dom}(Df), x_n \to x, Df(x_n) \to \zeta, n \to \infty \}
\]
which is non-empty by Rademacher’s Theorem. Let \( S_1 \) and \( S_2 \) be two non-empty closed subsets of \( T^*M \),
\[
d_{\text{Haus}}(S_1, S_2) := \sup \left\{ \sup_{(x,p) \in S_1} d_S(x,p), \sup_{(x,p) \in S_2} d_S(x,p) \right\}
\]
denotes the Hausdorff distance, where \( d_S(x,p) = \inf_{(x',p') \in S} d((x,p), (x',p')) \). And \( \text{graph}(\partial f) := \{(x,p) \in T^*M : p \in \partial f(x)\} \).

In view of Rademacher’s theorem, \( M \setminus \text{Dom}(Du) \) is negligible. Since \( \|Du(x)\|_x \) is bounded by the Lipschitz constant of \( u \) and \( H \) is of class \( C^3 \), then the following two lemmas are direct consequences of Proposition 16. We omit the proofs.

Lemma 2. Let \( c \in \mathbb{R} \). If \( u \in \text{Lip}(M) \) satisfies \( H(x,u(x),Du(x)) \leq c \) for a.e. \( x \in M \), then for each \( \varepsilon > 0 \) there is a \( C^\infty \) function \( u_\varepsilon \) such that \( H(x,u_\varepsilon(x),Du_\varepsilon(x)) \leq c + C_{u,\varepsilon} \cdot \varepsilon \) for all \( x \in M \), where \( C_{u,\varepsilon} > 0 \) depends only on \( u, \varepsilon \) and is increasing in \( \varepsilon \).

Lemma 3.
\[
c_1 := \inf_{u \in \text{Lip}(M)} \sup_{x \in \text{Dom}(Du)} H(x,u(x),Du(x)) = \inf_{u \in C^\infty(M)} \sup_{x \in M} H(x,u(x),Du(x)).
\]

Proof of Theorem 2. By definition \( c_2 \leq c_3 \). By [36], there exists a constant \( c \) such that equation \( (E_c) \) admits viscosity solutions, i.e., \( \mathcal{C} \neq \emptyset \).

Step 1: We aim to show that if \( c < c_1 \), then equation \( (E_c) \) has no viscosity subsolutions. Assume by contradiction that there is a viscosity subsolution of equation \( (E_c) \). By classical results on viscosity
solutions, we have that \( u \) is Lipschitz on \( M \) and satisfies \( H(x, u(x), Du(x)) \leq c \) for a.e. \( x \in M \). Thus,
\[
c_1 = \inf_{w \in \text{Lip}(M)} \sup_{x \in \text{Dom}(Dw)} H(x, w(x), Dw(x)) \leq c,
\]
a contradiction.

**Step 2:** If equation \( (E_c) \) admits a viscosity solution \( u \), then \( H(x, u(x), Du(x)) = c \) for a.e. \( x \in M \).
Thus, one can get that
\[
c_3 = \sup_{w \in \text{SCL}(M)} \inf_{x \in \text{Dom}(Dw)} H(x, w(x), Dw(x)) \geq c.
\]
By Step 1, Step 2 and \( \mathcal{C} \neq \emptyset \), it is clear that \( c_1 \leq c_3 \).

**Step 3:** For \( c \in (c_1, c_2) \), we now show that equation \( (E_c) \) has viscosity solutions. Our strategy is to show sets \( \mathcal{C}_\pm^c \) are non-empty.

Since \( c_1 < c \), by Lemma 3 there is \( u \in C^\infty(M) \) such that
\[
\sup_{x \in M} H(x, u(x), Du(x)) < c,
\]
i.e., \( u \) is a smooth subsolution of equation \( (E_c) \). Let \( w(x, t) := T^{-c}_t u(x) \). Then \( w \) is the unique viscosity solution of \( w_t + H(x, w, Dw) = c \) with \( w(x, 0) = u(x) \).

Notice that \( u \) is of class \( C^\infty \). In view of the classical method of characteristics, there is \( t^* > 0 \) such that \( w \in C^\infty(M \times (-t^*, t^*)) \) and \( w(x, 0) = u(x), Dw(x, 0) = Du(x) \) for all \( x \in M \). So, we get that
\[
w_t(x, 0) = c - H(x, u(x), Du(x)) > 0,
\]
and thus for \( t > 0 \) small enough,
\[
T^{-c}_t u(x) \geq u(x), \quad \forall x \in M.
\]
By Proposition 9(1), we have that
\[
T^{-c}_t u \geq u, \quad \forall t \geq 0,
\]
implying that \( \mathcal{C}_-^c \) is non-empty.

Since \( c < c_2 \), by definition there is \( \varphi \in C^\infty(M) \) such that
\[
H(x, \varphi(x), D\varphi(x)) > c, \quad \forall x \in M.
\]
By method of characteristics again, there is \( s^* > 0 \) such that \( w'(x, t) := T^{-c}_t \varphi(x) \in C^\infty(M \times (-s^*, s^*)) \) and \( w'(x, 0) = \varphi(x), Dw'(x, 0) = D\varphi(x) \) for all \( x \in M \).

Since
\[
H(x, w'(x, 0), Dw'(x, 0)) = H(x, \varphi(x), D\varphi(x)) > c, \quad \forall x \in M,
\]
then there is $t_0 \in (0, s^*)$ such that for each $t \in (0, t_0)$, we get that

$$H(x, w'(x, t), Dw'(x, t)) > c, \quad \forall x \in M.$$ 

So, $w'(x, t) = c - H(x, w'(x, t), Dw'(x, t)) < 0$ and thus,

$$w'(x, -t) > w'(x, 0) = \varphi(x), \quad \forall x \in M. \quad (4.1)$$

We assert that for $\delta \in (0, t_0)$,

$$w'(x, -\delta) \leq T_{\delta}^{+c}\varphi(x), \quad \forall x \in M. \quad (4.2)$$

In fact, for $\delta > 0$ small enough and for each $x \in M$, let $(x(t), u(t), p(t))$ be the solution of (1.3) with $x(0) = x$, $u(0) = w'(x, -\delta)$ and $p(0) = Dw'(x, -\delta)$. Then $u(\delta) = \varphi(y)$ for some $y \in M$. In view of Proposition (6.3), we have that

$$\varphi(y) = u(\delta) \geq h_{x,w'(x,-\delta)}^c(y, \delta).$$

From (2.3) and Proposition (7.1), we get that

$$h_{y,\varphi(y)}^c(x, \delta) \geq w'(x, -\delta).$$

In view of Proposition (7.4),

$$T_{\delta}^{+c}\varphi(x) \geq w'(x, -\delta), \quad \forall x \in M.$$ 

Combining (4.1) and (4.2), we get that

$$T_{\delta}^{+c}\varphi \geq \varphi, \quad \forall t \geq 0,$$

which shows that $C_{\delta}^c$ is non-empty.

**Step 4:** For any $c \in (c', c'')$, we need to show equation (E) has viscosity solutions. Since $c' \in \mathfrak{C}$, there is $\varphi \in C(M)$ such that $T_{t}^{-c'}\varphi = \varphi$ for all $t \geq 0$. In view of $c > c'$, Proposition (6.1)(ii) and Proposition (9.5)(i), we get that

$$T_{t}^{-c'}\varphi \geq T_{t}^{-c'}\varphi \geq \varphi, \quad \forall t \geq 0,$$

which means that $C_{\delta}^c$ is non-empty.

Since $c'' \in \mathfrak{C}$, then equation $H(x, u(x), Du(x)) = c''$ admits BWKAM solutions and thus admits FWKAM solutions. Let $\psi \in C(M)$ be a FWKAM solution. Then $T_{t}^{+c''}\psi = \psi$ for all $t \geq 0$. By $c'' > c$, Proposition (7.1)(ii) and Proposition (9.5)(ii), one can deduce that

$$T_{t}^{+c}\psi \geq T_{t}^{+c'}\psi = \psi,$$

i.e., $C_{\delta}^c$ is non-empty. Hence, by Corollary (1) one can deduce that equation (E) has viscosity solutions.

The proof of Theorem (2) is complete.
4.2 Examples

Let us discuss several illustrative examples of contact Hamiltonians satisfying (H1)-(H3) and describe the corresponding $c_1$, $c_2$, $c_3$. We discuss genuine contact Hamiltonians in the first two examples, while a classical Hamiltonian is studied in the last example. The classical case can be regarded as the critical case. Let $h(x, p)$ denote a generic Tonelli Hamiltonian on $T^*M$.

Example 2 ($c_1 = -\infty$, $c_2 = c_3 = +\infty$). Let $H(x, u, p) = f(x)u + h(x, p)$ for all $(x, u, p) \in T^*M \times \mathbb{R}$, where $f$ is a smooth function on $M$.

(i) If $f(x) > 0$ for all $x \in M$, then for any $a < 0$,

$$\sup_{x \in M} (f(x)a + h(x, 0)) \leq \sup_{x \in M} (f(x)a) + \sup_{x \in M} h(x, 0) = a \inf_{x \in M} f(x) + \sup_{x \in M} h(x, 0).$$

Letting $a \to -\infty$, we get that $c_1 = -\infty$. Similarly, for any $a > 0$, we have that

$$\inf_{x \in M} (f(x)a + h(x, 0)) \geq \inf_{x \in M} (f(x)a) + \inf_{x \in M} h(x, 0) = a \inf_{x \in M} f(x) + \inf_{x \in M} h(x, 0).$$

Letting $a \to +\infty$, we get that $c_2 = +\infty$ and thus $c_3 = +\infty$.

(ii) For case $f(x) < 0$ for all $x \in M$, one can get the same results in a similar manner.

Example 3. Let $H(x, u, p) = V(u) + h(x, p)$ for all $(x, u, p) \in T^*M \times \mathbb{R}$, where $V(u)$ is a smooth function on $\mathbb{R}$ and $\|V'\|_{\infty} \leq \lambda$.

(i) ($c_1 \in \mathbb{R}$, $c_2 = c_3 = +\infty$). Assume, in addition, $V$ is bounded from below and $\sup_{u \in \mathbb{R}} V(u) = +\infty$. Since $V(u) + h(x, p)$ is bounded from below, then

$$c_1 = \inf_{u \in \text{Lip}(M)} \sup_{x \in \text{Dom}(Du)} V(u) + h(x, Du(x)) > -\infty,$$

and

$$c_1 \leq V(a) + \sup_{x \in M} h(x, 0), \quad \forall a \in \mathbb{R}.$$

Thus, $c_1 \in \mathbb{R}$. Note that,

$$\sup_{a \in \mathbb{R}} V(a) + \inf_{x \in M} h(x, 0) = +\infty.$$

Thus, we get that

$$c_2 = \sup_{u \in C^{\infty}(M)} \inf_{x \in M} (V(u) + h(x, Du(x))) = +\infty = c_3.$$
(ii) \((c_1 = -\infty, c_2, c_3 \in \mathbb{R})\). Assume, in addition, \(V\) is bounded from above and \(\inf_{u \in \mathbb{R}} V(u) = -\infty\).

\[
C_1 \leq \inf_{a \in \mathbb{R}} \{V(a) + \sup_{x \in M} h(x, 0)\} = \inf_{a \in \mathbb{R}} V(a) + \sup_{x \in M} h(x, 0) = -\infty.
\]

Notice that for any \(u \in C^\infty(M)\),

\[
\inf_{x \in M} (V(u(x)) + h(x, Du(x))) \leq \sup_{x \in M} V(u(x)) + \inf_{x \in M} h(x, Du(x)) \leq \sup_{x \in M} V(u(x)) + h(y, 0),
\]

where \(y\) is an arbitrary point in \(M\) with \(Du(y) = 0\). Hence, we deduce that \(c_2 \in \mathbb{R}\). In view of properties of semiconcave functions \([6]\), by completely the same arguments one can get \(c_3 \in \mathbb{R}\).

Example 4 \((c_1 = c_2 = c_3 = 0)\). Let \(H(x, u, p) = \|p\|_x^2\) for all \((x, u, p) \in T^*M \times \mathbb{R}\). For each \(u \in C^\infty(M)\),

\[
\inf_{x \in M} \|Du(x)\|_x^2 = 0,
\]

which implies that

\[
c_2 = \sup_{u \in C^\infty(M)} \inf_{x \in M} \|Du(x)\|_x^2 = 0,
\]

and

\[
c_3 = \sup_{u \in \text{SCL}(M)} \inf_{x \in \text{Dom}(Du)} \|Du(x)\|_x^2 = 0.
\]

By definition, it is direct to see that

\[
c_1 = \inf_{u \in C^\infty(M)} \sup_{x \in M} \|Du(x)\|_x^2 = 0.
\]

In this example, \(C = \{0\}\).

5 Long-time behavior of solutions of the Cauchy problem

5.1 Proof of Theorem 3

Proof of Theorem 3 We will split the proof into three steps.

Step 1: if there is \(t_0 > 0\) such that \(\varphi \leq T_{t_0}^{-c} \varphi\), then by Corollary \(\varphi\) either \(\lim_{n \to +\infty} T_{t_0}^{-c} \varphi(x) = +\infty\) uniformly on \(x \in M\), or, \(\lim_{n \to +\infty} T_{-t_0}^{-c} \varphi(x) = \varphi(x)\) uniformly on \(x \in M\), where \(\varphi(x)\) is a Lipschitz continuous function on \(M\).

Case (i): If \(\lim_{n \to +\infty} T_{-t_0}^{-c} \varphi(x) = +\infty\) uniformly on \(x \in M\), then we assert that for any \(s \in [0, t_0]\),

\[
\lim_{n \to +\infty} T_{-t_0 + s}^{-c} \varphi(x) = +\infty
\]
for all \( x \in M \). Note that if there are \( s_0 \in [0, t_0] \) and \( x_0 \in M \) such that
\[
\lim_{n \to \infty} T^{-c}_{n t_0 + s_0} \varphi(x_0) = A'' < +\infty,
\]
then for all \( x \in M \),
\[
T^{-c}_{n t_0 + s_0} \varphi(x) \leq h^{-c}_{x_0, A''}(x, t_0) \leq \max_{y, y'' \in M} h^{-c}_{y, A''}(y'', t_0) < +\infty.
\]

So, in order to show the above assertion, we can assume by contradiction that there is \( s_0 \in [0, t_0] \) such that
\[
\lim_{n \to \infty} T^{-c}_{n t_0 + s_0} \varphi(x) = \varphi^s_\infty(x), \quad \forall x \in M,
\]
where \( \varphi^s_\infty(x) \) is a function defined on \( M \). It is clear that \( \{T^{-c}_{n t_0 + s_0} \varphi(x)\}_n \) is bounded by a constant \( K > 0 \). And thus, by similar arguments used in the proof of Proposition \[2\] \( \{T^{-c}_{n t_0 + s_0} \varphi(x)\}_n \) is equi-Lipschitz. Therefore,
\[
\lim_{n \to \infty} T^{-c}_{n t_0 + s_0} \varphi(x) = \varphi^s_\infty(x),
\]
uniformly on \( x \in M \), and \( \varphi^s_\infty \in \text{Lip}(M) \). Note that
\[
\|T^{-c}_{t_0 - s_0} \circ T^{-c}_{n t_0 + s_0} \varphi - T^{-c}_{t_0 - s_0} \varphi^s_\infty\| \leq e^{\lambda t_0} \|T^{-c}_{n t_0 + s_0} \varphi - \varphi^s_\infty\|.
\]

Thus, we get that
\[
+\infty = \lim_{n \to \infty} T^{-c}_{t_0 - s_0} \circ T^{-c}_{n t_0 + s_0} \varphi(x) = T^{-c}_{t_0 - s_0} \varphi^s_\infty(x),
\]
a contradiction. We have proved the assertion.

Next, we prove for any \( s \in [0, t_0] \), \( \lim_{t \to +\infty} T^{-c}_{t_0 + s} \varphi(x) = +\infty \) uniformly on \( x \in M \). Suppose not. Then there are \( s' \in [0, s_0], K_0 > 0, \{n_k\} \nearrow +\infty \) and \( x_k \in M \), such that
\[
T^{-c}_{n_k t_0 + s'} \varphi(x_k) \leq K_0.
\]
Then for any \( k \in \mathbb{N} \), any \( x \in M \)
\[
T^{-c}_{(n_k + 1)t_0 + s'} \varphi(x) \leq h^{-c}_{x_k, T^{-c}_{n_k t_0 + s'} \varphi(x_n)} (x, t_0) \leq h^{-c}_{x_k, K_0} (x, t_0) \leq \max_{y', y'' \in M} h^{-c}_{y', K_0} (y'', t_0) < +\infty,
\]
which contradicts \( \lim_{n \to \infty} T^{-c}_{n t_0 + s'} \varphi(x) = +\infty \) for all \( x \in M \).

Case (ii) : If \( \lim_{n \to +\infty} T^{-c}_{n t_0} \varphi(x) = \varphi_\infty(x) \) for all \( x \in M \), where \( \varphi_\infty(x) \) is a Lipschitz continuous function on \( M \), then for any \( s \in \mathbb{R} \),
\[
\lim_{n \to +\infty} T^{-c}_{n t_0 + s} \varphi(x) = T^{-c}_{s} \varphi_\infty(x) =: u(x, s).
\]

It is clear that \( u(x, s + t_0) = u(x, s) \) for all \( s \in \mathbb{R} \), and that \( u(x, s) \) is a viscosity solution of (1.5a).

The proof for the case \( \varphi \geq T^{-c}_{t_0} \varphi \) for some \( t_0 > 0 \) is quite similar and thus we omit it.
Step 2: if there is $t_0 > 0$ such that $\varphi < T_{t_0}^{-c}\varphi$, then there is $t_1 > 0$ close enough to $t_0$ such that $t_1/t_0$ is an irrational number and $\varphi < T_{t_1}^{-c}\varphi$. Since $t_1/t_0$ is an irrational number, then for any $s \in [0, t_0]$, any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there are $m_0, m_1 \in \mathbb{N}$ with $m_0, m_1 > N$, such that

$$|m_1 t_1 - (m_0 t_0 + s)| < \varepsilon.$$  \hspace{1cm} (5.1)

By the result obtained in Theorem 3(1), then $\lim_{n \to +\infty} T_{nt_0+s}^{-c}\varphi(x) = +\infty$ for all $x \in M$ and all $s \in \mathbb{R}$, or $\lim_{n \to +\infty} T_{nt_0+s}^{-c}\varphi(x) = u(x, s)$ for all $x \in M$ and all $s \in \mathbb{R}$. If $\lim_{n \to +\infty} T_{nt_0+s}^{-c}\varphi(x) = u(x, s)$ for all $x \in M$ and all $s \in [0, t_0]$, then in view of (5.1), we get that $\lim_{n \to +\infty} T_{nt_1}^{-c}\varphi(x) =: \varphi'_\infty(x)$ for some $\varphi'_\infty \in \text{Lip}(M)$. By (5.1) again, $\varphi'_\infty(x) = u(x, s)$ for all $x \in M$ and $s \in [0, t_0]$. Thus, $\varphi'_\infty$ is a BWKAM solution of $(E_c)$.

The proof for the case $\varphi > T_{t_0}^{-c}\varphi$ for some $t_0 > 0$ is quite similar and thus we omit it.

Step 3: if for any $t > 0$, there are $x_1, x_2 \in M$ such that $T_{t}^{-c}\varphi(x_1) > \varphi(x_1)$ and $T_{t}^{-c}\varphi(x_2) < \varphi(x_2)$, then for any $t > 0$, there is $x_t \in M$ such that $T_{t}^{-c}\varphi(x_t) = \varphi(x_t)$. Note that

$$T_{t}^{-c}\varphi(x) = T_{1}^{-c} \circ T_{t-1}^{-c}\varphi(x) \leq h_{x_{t-1},x_{t-1}^{-c}\varphi(x_{t-1})}(x,1) = h_{x_{t-1},x_{t-1}^{-c}\varphi(x_{t-1})}(x,1).$$

Thus, $T_{t}^{-c}\varphi(x)$ is bounded from above. Note that for any $y \in M$,

$$h_{y,T_{t}^{-c}\varphi(y)}(x_{t+1},1) \geq T_{t}^{-c}\varphi(x_{t+1}) = \varphi(x_{t+1}),$$

which implies that $T_{t}^{-c}\varphi(y) \geq h_{c,x_{t+1}^{-c}\varphi(x_{t+1})}(y,1)$ for all $y \in M$. Therefore, we get that for any $t > 1$, any $y \in M$,

$$\min_{(z,z')} h_{c,z^{-c}\varphi(z)}(z',1) \leq h_{c,x_{t+1}^{-c}\varphi(x_{t+1})}(y,1) \leq T_{t}^{-c}\varphi(y) \leq \max_{(z,z')} h_{c,z^{-c}\varphi(z)}(z',1).$$

Hence, $|T_{t}^{-c}\varphi(y)|$ is bounded on $M \times [0, +\infty)$. \hfill \qed

5.2 Proof of Proposition 3

Proof of Proposition 3: If $u$ is a BWKAM solution of equation $(E_c)$, then $T_t^{-c}u = u$ for all $t \geq 0$. Thus, for each $t \geq 0$,

$$\{x \in M : T_t^{-c}u(x) = u(x)\} \neq \emptyset.$$

On the other hand, if there is $\varphi \in C(M)$ such that

$$\{x \in M : T_t^{-c}\varphi(x) = \varphi(x)\} \neq \emptyset, \quad \forall t \geq 0,$$

then by Theorem 2(3), we deduce that $|T_t^{-c}\varphi(x)|$ is bounded on $M \times [0, +\infty)$. Thus, from Proposition 12 we get that $\varphi_{\infty}(x) := \lim_{t \to +\infty} T_t^{-c}\varphi(x)$ is a BWKAM solution of $(E_c)$. \hfill \qed
5.3 Proof of Theorem 4

Proof of Theorem 4

We show items (1), (2) in order.

Step 1: Since \( c < c_r \), one can choose \( c_0 \in \mathcal{C} \) with \( c < c_0 \). By the definition of \( \mathcal{C} \), equation \( H(x, u(x), Du(x)) = c_0 \) has BWKAM solutions. Let \( \varphi \) be an arbitrary BWKAM solution of the above equation. Then, \( T_t^{c_0} \varphi = \varphi \) for all \( t > 0 \). In view of \( c < c_0 \), we have that

\[
T_t^{-c_0} \varphi \leq \varphi, \quad \forall t > 0.
\]

By Lemma 1, either \( \lim_{t \to +\infty} T_t^{-c_0} \varphi(x) = -\infty \) uniformly on \( x \in M \), or \( \lim_{t \to +\infty} T_t^{-c_0} \varphi(x) = \varphi_\infty(x) \) uniformly on \( x \in M \), where \( \varphi_\infty(x) \) is a BWKAM solution of \( (E) \). Since \( c \notin \mathcal{C} \), then

\[
\lim_{t \to +\infty} T_t^{-c_0} \varphi(x) = -\infty
\]

uniformly on \( x \in M \).

For any \( t > 0 \), any \( y \in M \), there is \( z \in M \) such that

\[
T_t^{-c_0} \varphi(y) = h_{z,\varphi(y)}^{c}(y, t).
\]

Then,

\[
T_t^{+c}(T_t^{-c_0} \varphi)(z) \geq h_{c}^{y,T_t^{-c_0} \varphi(y)}(z, t) = \varphi(z).
\]

It means that for any \( t > 0 \), there is \( x_t \in M \) such that

\[
T_t^{+c}(T_t^{-c_0} \varphi)(x_t) \geq \varphi(x_t).
\]

For any \( x \in M \), any \( t > 1 \),

\[
T_t^{+c}(T_{t-1}^{-c_0} \varphi)(x) = T_1^{+c} \circ (T_{t-1}^{+c} \circ T_{t-1}^{-c_0} \varphi)(x) \geq h_{c}^{x_1^{-1},\varphi(x_1)}(x, 1) \geq \inf_{y,y' \in M} h_{c}^{y,y'}(y', 1) =: B_1.
\]

By (5.2), there is \( t_0 > 1 \) such that \( T_{t_0}^{-c_0} \varphi(x) < B_1 \) for any \( x \in M \). Let \( \psi_0 := T_{t_0}^{-c_0} \varphi \). Then

\[
T_{t_0}^{+c} \psi_0(x) > \psi_0.
\]

(5.3)

For any given \( \psi \in C(M) \), if there is \( t_1 > 0 \) such that \( T_{t_1}^{-c_0} \psi \geq \psi \), then by Theorem 1 and (5.3), one can deduce that \( c \in \mathcal{C} \), a contradiction. Thus, for any \( t > 0 \), there is \( y_t \in M \) such that

\[
T_t^{-c_0} \psi(y_t) < \psi(y_t).
\]

(5.4)

Thus, for any \( x \in M \), any \( t > 1 \),

\[
T_t^{-c_0} \psi(x) = T_1^{-c_0} \circ T_{t-1}^{-c_0} \psi(x) \leq h_{y_{t-1},\psi(y_{t-1})}^{c}(x, 1) \leq \max_{y,y'' \in M} h_{y,y''}^{c}(y'', 1) =: B_2.
\]

(5.5)

We assert that there exists \( s_0 > 0 \) such that

\[
T_{s_0}^{-c_0} \psi < \psi.
\]
Suppose not. By (5.4), we know that for any \( t > 0 \), there is \( z_t \in M \) such that

\[ T_t^{-c}(z_t) = \psi(z_t). \]

Then, for any \( t > 1 \), any \( x \in M \),

\[ T_{t-1}^{-c}(x) \geq h_c^{z_t, \psi(z_t)}(x, 1) \geq \min_{y', y'' \in M} h_c^{y', \psi(y')}(y'', 1) =: B_3. \tag{5.6} \]

Proposition 12, (5.5) and (5.6) imply that

\[ \psi_{\infty}(x) := \liminf_{t \to +\infty} T_t^{-c}\psi(x) \]

is a BWKAM solution of \((E_c)\), a contradiction. So, the above assertion is true. Then by Theorem 3 (2), we deduce that either \( \lim_{t \to +\infty} T_t^{-c}\varphi(x) = -\infty \) uniformly on \( x \in M \), or \( \lim_{t \to +\infty} T_t^{-c}\varphi(x) = \varphi_{\infty}(x) \) uniformly on \( x \in M \), where \( \varphi_{\infty}(x) \) is a BWKAM solution of equation \((E_c)\). Since \( c \notin \mathcal{C} \), we get that \( \lim_{t \to +\infty} T_t^{-c}\varphi(x) = -\infty \) uniformly on \( x \in M \).

**Step 2:** For \( c > c_t \) with \( c \notin \mathcal{C} \), we aim to show that for any \( \psi \in C(M) \), \( \lim_{t \to +\infty} T_t^{-c}\psi(x) = +\infty \) uniformly on \( x \in M \). By Theorem 3 (2), we only need to show that there is \( s_0 > 0 \) such that \( T_{s_0}^{-c}\psi > \psi \). Assume by contradiction that for any \( t > 0 \), there is \( x_t \in M \) such that

\[ T_t^{-c}\psi(x_t) \leq \psi(x_t). \]

Then

\[ T_t^{-c}\psi(x) = T_1^{-c} \circ T_{t-1}^{-c}\psi(x) \leq h_c^{x_{t-1}, \psi(x_{t-1})}(x, 1) \leq \max_{y', y'' \in M} h_c^{y', \psi(y')}(y'', 1) =: B_4. \tag{5.7} \]

We assert that for any \( t > 0 \), there is \( y_t \in M \) such that

\[ T_t^{-c}\psi(y_t) = \psi(y_t). \tag{5.8} \]

If the assertion is true, then for any \( t > 1 \), any \( x \in M \),

\[ T_{t-1}^{-c}\psi(x) \geq h_c^{y_t, \psi(y)}(x, 1) \geq \min_{y', y'' \in M} h_c^{y', \psi(y')}(y'', 1) =: B_6. \tag{5.9} \]

Proposition 12, (5.7) and (5.9) imply that

\[ \psi_{\infty}(x) := \liminf_{t \to +\infty} T_t^{-c}\psi(x) \]

is a BWKAM solution of \((E_c)\), a contradiction.

So, it suffices to show the assertion. Assume by contradiction that there is \( t_0 > 0 \), such that for any \( x \in M \),

\[ T_{t_0}^{-c}\psi(x) < \psi(x). \]
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By Theorem 3(2) again, we get that

$$\lim_{t \to +\infty} T_t^{-c} \psi(x) = -\infty$$  \hspace{1cm} (5.10)

uniformly on $x \in M$.

For any $t > 0$, any $y \in M$, there is $z \in M$ such that

$$T_t^{-c} \psi(y) = h_z \psi(z, t).$$

Then,

$$T_t^{+c}(T_t^{-c} \psi)(z) \geq h_z \psi(y, t) = \psi(z).$$

It means that for any $t > 0$, there is $z_t \in M$ such that

$$T_t^{+c}(T_t^{-c} \psi)(z_t) \geq \psi(z_t).$$

For any $x \in M$, any $t > 1$,

$$T_t^{+c}(T_{t-1}^{-c} \psi)(x) = T_{t-1}^{+c} \circ (T_{t-1}^{+c} \circ T_{t-1}^{-c} \psi)(x) \geq h^{z_{t-1}, \psi(z_{t-1})} \geq \inf_{y', y'' \in M} h^y_{z_t}(y', 1) = B_6.$$

By (5.10), there is $t_1 > 1$ such that $T_{t_1-1}^{-c} \psi(x) < B_5$ for any $x \in M$. Let $\psi_1 := T_{t_1-1}^{-c} \psi$. Then

$$T_{t_1}^{+c} \psi_1(x) > \psi_1.$$  \hspace{1cm} (5.11)

Since $c > c_l$, one can choose $c_0 \in C$ with $c > c_0$. By the definition of $C$, equation $H(x, u, Du) = c_0$ has BWKAM solutions. Let $\varphi$ be an arbitrary viscosity solution of the above equation. Then, $T_t^{-c_0} \varphi = \varphi$ for all $t \geq 0$. In view of $c > c_0$, we have that

$$T_t^{-c_0} \varphi \geq \varphi, \hspace{0.5cm} \forall t \geq 0.$$  

By Lemma 1, either $\lim_{t \to +\infty} T_t^{-c_0} \varphi(x) = +\infty$ uniformly on $x \in M$, or $\lim_{t \to +\infty} T_t^{-c_0} \varphi(x) = \varphi_\infty(x)$ uniformly on $x \in M$, where $\varphi_\infty(x)$ is a BWKAM solution of (E). Since $c \notin C$, then

$$\lim_{t \to +\infty} T_t^{-c_0} \varphi(x) = +\infty$$  \hspace{1cm} (5.12)

uniformly on $x \in M$. In view of (5.11), (5.12) and Theorem 1, we deduce that equation (E) has BWKAM solutions, a contradiction. So, the assertion (5.8) holds ture.

The proof is now complete.

6 Appendix

6.1 Proof of Proposition 8

In order to prove the proposition 8, we provide a preliminary lemma.
Lemma 4. Given any \( x_0, x \in M, u_0 \in \mathbb{R} \) and \( t > 0 \), let \( \gamma : [0, t] \to M \) be a minimizer of \( h_{x_0, u_0}(x, t) \). Then for each \( t_0 \in (0, t) \), there is a unique minimizer of \( h_{x_0, u_0}(\gamma(t_0), t_0) \).

Proof. Since \( \gamma \) is a minimizer of \( h_{x_0, u_0}(x, t) \), then \( \gamma|_{[0, t_0]} \) is a minimizer \( h_{x_0, u_0}(\gamma(t_0), t_0) \). If there is another minimizer \( h_{x_0, u_0}(\gamma(t_0), t_0) \), denoted by \( \alpha \), then we will show that \( \alpha = \gamma|_{[0, t_0]} \). Let

\[
\beta(s) := \begin{cases} 
\alpha(s), & s \in [0, t_0], \\
\gamma(s), & s \in [t_0, t]. 
\end{cases}
\]

Then we get

\[
h_{x_0, u_0}(x, t) = h_{x_0, u_0}(\gamma(t_0), t_0) + \int_{t_0}^{t} L(\gamma(s), h_{x_0, u_0}(\gamma(s), s), \dot{\gamma}(s)) ds
\]

\[
= h_{x_0, u_0}(\alpha(t_0), t_0) + \int_{t_0}^{t} L(\gamma(s), h_{x_0, u_0}(\gamma(s), s), \dot{\gamma}(s)) ds
\]

\[
= u_0 + \int_{0}^{t_0} L(\alpha(s), h_{x_0, u_0}(\alpha(s), s), \dot{\alpha}(s)) ds + \int_{t_0}^{t} L(\gamma(s), h_{x_0, u_0}(\gamma(s), s), \dot{\gamma}(s)) ds
\]

\[
= u_0 + \int_{0}^{t} L(\beta(s), h_{x_0, u_0}(\beta(s), s), \dot{\beta}(s)) ds,
\]

which implies that \( \beta \) is a minimizer of \( h_{x_0, u_0}(x, t) \). From Proposition 4, \( \gamma \) and \( \beta \) are both of class \( C^1 \). Therefore, we have \( \dot{\gamma}(t_0) = \dot{\beta}(t_0) \). By Proposition 4 and the uniqueness of solutions of initial value problem of ordinary differential equations, we have \( \alpha(s) = \gamma(s) \) for all \( s \in [0, t_0] \), which completes the proof.

Proof of Proposition 8. We divide the proof in two steps.

Step 1: Given any \( t_1, t_2 \in \mathbb{R} \) with \( t_1 < t_2 \) and \( t_0 \in (t_1, t_2) \), since \( (x(t), u(t)) \) is globally minimizing, then we have

\[
u(t_2) = h_{x(t_1), u(t_1)}(x(t_2), t_2 - t_1),
\]

\[
u(t_2) = h_{x(t_0), u(t_0)}(x(t_2), t_2 - t_0),
\]

\[
u(t_0) = h_{x(t_1), u(t_1)}(x(t_0), t_0 - t_1).
\]

It follows that

\[
h_{x(t_1), u(t_1)}(x(t_2), t_2 - t_1) = h_{x(t_0), u(t_0)}(x(t_2), t_2 - t_0) = h_{x(t_0), h_{x(t_1), u(t_1)}(x(t_0), t_0 - t_1)}(x(t_2), t_2 - t_0).
\]

In view of Proposition 6(4), there is a minimizer of \( h_{x(t_1), u(t_1)}(x(t_2), t_2 - t_1) \), denoted by \( \gamma \), such that \( \gamma(t_0) = x(t_0) \).
Step 2: From the above arguments, there exists a minimizer $\alpha$ of $h_{x(t_1),u(t_1)}(x(t_2 + 1), t_2 - t_1 + 1)$ such that $x(t_2) = \alpha(t_2)$. By Lemma 5, $\alpha\big|_{[t_1,t_2]}$ is the unique minimizer of $h_{x(t_1),u(t_1)}(x(t_2), t_2 - t_1)$. By the arguments used in Step 1 again, $x(s) = \alpha(s)$ for all $s \in [t_1, t_2]$. Thus, by Proposition 4 and the arbitrariness of $t_1$ and $t_2$ with $t_1 < t_2$, $x(t)$ is of class $C^1$ for $t \in \mathbb{R}$, and $(x(t), u(t), p(t))$ is a solution of (1.3), where $p(t) := \frac{\partial L}{\partial x}(x(t), u(t), \dot{x}(t))$. Since $\dot{u}(t) = L(x(t), u(t), \dot{x}(t))$, it is easy to see that $x(t)|_{[t_1,t_2]}$ is a minimizer of $h_{x(t_1),u(t_1)}(x(t_2), t_2 - t_1)$.

\[ \square \]

6.2 Proof of Proposition 11

Lemma 5. If $\varphi < L$, then $\varphi$ is Lipschitz continuous on $M$.

Proof. For each $x, y \in M$, let $\gamma : [0, d(x, y)] \to M$ be a geodesic of length $d(x, y)$, parameterized by arclength and connecting $x$ to $y$. Since $M$ is compact and $\varphi$ is continuous, then

$$A_1 := \max_{x \in M} |\varphi(x)| \quad A_2 := \sup\{ L(x, u, \dot{x}) \mid x \in M, |u| \leq A_1, \|\dot{x}\|_x = 1 \}$$

are well-defined. Since $\|\dot{\gamma}(s)\|_{\gamma(s)} = 1$ for each $s \in [0, d(x, y)]$, we have $L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \leq A_2$. Then by $\varphi < L$,

$$\varphi(\gamma(d(x, y))) - \varphi(\gamma(0)) \leq \int_0^{d(x, y)} L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s))ds \leq \int_0^{d(x, y)} A_2ds = A_2d(x, y).$$

We finish the proof by exchanging the roles of $x$ and $y$.

\[ \square \]

Lemma 6. Let $\varphi < L$ and let $\gamma : [a, b] \to M$ be a $(\varphi, L, 0)$-calibrated curve. If $\varphi$ is differentiable at $\gamma(t)$ for some $t \in (a, b)$, then we have

$$H(\gamma(t), \varphi(\gamma(t)), D\varphi(\gamma(t))) = 0, \quad D\varphi(\gamma(t)) = \frac{\partial L}{\partial \dot{x}}(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t)).$$

Proof. By Lemma 5, $\varphi$ is Lipschitz continuous on $M$. We first show that at each point $x \in M$ where $D\varphi(x)$ exists, we have

$$H(x, \varphi(x), D\varphi(x)) \leq 0. \quad (6.1)$$

For any given $\varphi \in T_xM$, let $\alpha : [0, 1] \to M$ be a $C^1$ curve such that $\alpha(0) = x, \dot{\alpha}(0) = v$. By $\varphi < L$, for each $t \in [0, 1]$, we have

$$\varphi(\alpha(t)) - \varphi(\alpha(0)) \leq \int_0^t L(\alpha(s), \varphi(\alpha(s)), \dot{\alpha}(s)))ds.$$
Dividing by $t > 0$ and let $t \to 0^+$, we have $\langle D\varphi(x), v \rangle \leq L(x, \varphi(x), v)$, which implies $H(x, \varphi(x), D\varphi(x)) = \sup_{v \in T_x M} \langle D\varphi(x), v \rangle - L(x, \varphi(x), v) \leq 0$. Thus, (6.1) holds.

If $\varphi$ is differentiable at $\gamma(t)$ for some $t \in (a, b)$, then for each $t' \in [a, b]$ with $t \leq t'$, we have $\varphi(\gamma(t')) - \varphi(\gamma(t)) = \int_t^{t'} L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s))ds$, since $\gamma : [a, b] \to M$ is a $(\varphi, L, 0)$-calibrated curve. Dividing by $t' - t$ and let $t' \to t^+$, we have $\langle D\varphi(\gamma(t)), \dot{\gamma}(t) \rangle = L(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t))$. Thus, we have

$$H(\gamma(t), \varphi(\gamma(t)), D\varphi(\gamma(t))) \geq \langle D\varphi(\gamma(t)), \dot{\gamma}(t) \rangle \gamma(t) - L(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t)) = 0,$$

which together with (6.1) implies $H(\gamma(t), \varphi(\gamma(t)), D\varphi(\gamma(t))) = 0$ and

$$\langle D\varphi(\gamma(t)), \dot{\gamma}(t) \rangle \gamma(t) = H(\gamma(t), \varphi(\gamma(t)), D\varphi(\gamma(t))) + L(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t)).$$

In view of Legendre transform, we get

$$D\varphi(\gamma(t)) = \frac{\partial L}{\partial x}(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t)).$$

This completes the proof. \qed

**Lemma 7.** Given any $a > 0$, let $\varphi \prec L$ and let $\gamma : [-a, a] \to M$ be a $(\varphi, L, 0)$-calibrated curve. Then $\varphi$ is differentiable at $\gamma(0)$.

**Proof.** It suffices to prove the lemma for the case when $M = U$ is an open subset of $\mathbb{R}^n$. Set $x = \gamma(0)$. In order to prove the differentiability of $u$ at $x$, we only need to show for each $y \in U$, there holds

$$\limsup_{\lambda \to 0^+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \leq \frac{\partial L}{\partial x}(x, \varphi(x), \dot{\gamma}(0)) \cdot y \leq \liminf_{\lambda \to 0^+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda}. \quad (6.2)$$

For $\lambda > 0$ and $0 < \varepsilon \leq a$, define $\gamma_\lambda : [-\varepsilon, \varepsilon] \to U$ by $\gamma_\lambda(s) = \gamma(s) + \frac{s + \varepsilon}{\varepsilon} y$. Then $\gamma_\lambda(0) = x + \lambda y$ and $\gamma_\lambda(-\varepsilon) = (\varepsilon)$. Since $u \prec L$ and $\gamma : [-a, a] \to M$ is a $(\varphi, L, 0)$-calibrated curve, we have

$$\varphi(x + \lambda y) - \varphi(\gamma(-\varepsilon)) \leq \int_{-\varepsilon}^0 L(\gamma_\lambda(s), \varphi(\gamma_\lambda(s)), \dot{\gamma}_\lambda(s))ds,$$

and

$$\varphi(x) - \varphi(\gamma(-\varepsilon)) = \int_{-\varepsilon}^0 L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s))ds.$$

It follows that

$$\frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \leq \frac{1}{\lambda} \int_{-\varepsilon}^0 \left(L(\gamma_\lambda(s), \varphi(\gamma_\lambda(s)), \dot{\gamma}_\lambda(s)) - L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s))\right)ds.$$

By Lemma 5, there exists $K > 0$ such that
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\[ |\varphi(\gamma_\lambda(s)) - \varphi(\gamma(s))| \leq K\|\gamma_\lambda(s) - \gamma(s)\| = K \cdot \frac{s + \varepsilon}{\varepsilon} \cdot \lambda\|y\|, \]

which implies

\[
\limsup_{\lambda \to 0^+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \leq \int_{-\varepsilon}^{0} \left( \frac{s + \varepsilon}{\varepsilon} \cdot \frac{\partial L}{\partial x}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \cdot y \\
+ K \frac{s + \varepsilon}{\varepsilon} \left| \frac{\partial L}{\partial \varphi}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \right| \|y\| \\
+ \frac{1}{\varepsilon} \frac{\partial L}{\partial x}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \cdot y \right) ds.
\]

If we let \( \varepsilon \to 0^+ \), we get the first inequality in (6.2).

Define \( \gamma_\lambda : [0, \varepsilon] \to M \) by \( \gamma_\lambda(s) = \gamma(s) + \frac{\varepsilon - s}{\varepsilon} \lambda y \). We have

\[
\varphi(\gamma(\varepsilon)) - \varphi(x + \lambda y) \leq \int_{0}^{\varepsilon} L(\gamma_\lambda(s), \varphi(\gamma_\lambda(s)), \dot{\gamma}_\lambda(s)) ds,
\]

\[
\varphi(\gamma(\varepsilon)) - \varphi(x) = \int_{0}^{\varepsilon} L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) ds.
\]

It follows that

\[
\frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \geq \frac{1}{\lambda} \int_{0}^{\varepsilon} \left( L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) - L(\gamma_\lambda(s), \varphi(\gamma_\lambda(s)), \dot{\gamma}_\lambda(s)) \right) ds,
\]

which implies

\[
\liminf_{\lambda \to 0^+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \geq \int_{0}^{\varepsilon} \left( \frac{s - \varepsilon}{\varepsilon} \frac{\partial L}{\partial x}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \cdot y \\
+ K \frac{s - \varepsilon}{\varepsilon} \left| \frac{\partial L}{\partial \varphi}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \right| \|y\| \\
+ \frac{1}{\varepsilon} \frac{\partial L}{\partial x}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \cdot y \right) ds.
\]

Letting \( \varepsilon \to 0^+ \), we obtain the second inequality in (6.2). This completes the proof. \( \square \)

**Proof of Proposition 11.** Let \( u(t) := u(\gamma(t)) \) for \( t \leq 0 \). We assert that for each \( s, t < 0 \) with \( s < t \), there holds

\[
u(t) = h_{\gamma(s), u(s)}(\gamma(t), t - s).
\]

If the assertion is true, then by Proposition 8 \( (\gamma(t), u(t), p(t)) \) satisfies equations (1.3) on \((-\infty, 0)\), where \( p(t) = \frac{\partial L}{\partial x}(\gamma(t), u(t), \dot{\gamma}(t)) \). Now we prove the assertion. Since \( u \) is a backward
weak KAM solution, then we have $T_\sigma^- u(x) = u(x), \forall x \in M, \forall \sigma \geq 0$. Recall that $T_\sigma^- u(x) = \inf_{y \in M} h_{y,u(y)}(x, \sigma)$ for all $\sigma > 0$. Given any $s < t \leq 0$, we get

$$u(\tau) \leq h_{\gamma(s),u(s)}(\gamma(\tau), \tau - s), \quad \forall \tau \in (s, t].$$

(6.4)

Since $\gamma : (-\infty, 0] \to M$ is a $(u, L, 0)$-calibrated curve, then we have

$$u(t) - u(s) = \int_s^t L(\gamma(\tau), u(\tau), \dot{\gamma}(\tau)) d\tau,$$

which together with (6.4) implies

$$u(t) \geq u(s) + \int_s^t L(\gamma(\tau), h_{\gamma(s),u(s)}(\gamma(\tau), \tau - s), \dot{\gamma}(\tau)) d\tau \geq h_{\gamma(s),u(s)}(\gamma(t), t - s).$$

By (6.4) again, we have $u(t) = h_{\gamma(s),u(s)}(\gamma(t), t - s)$. Hence, (6.3) holds.

By Lemma 6 and Lemma 7, $u$ is differentiable at $\gamma(t)$ for any $t < 0$ and

$$Du(\gamma(t)) = \frac{\partial L}{\partial x}(\gamma(t), u(\gamma(t)), \dot{\gamma}(t)).$$

Hence, $(\gamma(t + s), u(\gamma(t + s)), Du(\gamma(t + s))) = \Phi_s(\gamma(t), u(\gamma(t)), Du(\gamma(t))), \forall t, s < 0$. In view of Lemma 6, we have

$$H(\gamma(t), u(\gamma(t)), \frac{\partial L}{\partial x}(\gamma(t), u(\gamma(t)), \dot{\gamma}(t))) = 0, \quad \forall t < 0,$$

which completes the proof. \qed

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