BOUNDS OF HAUSDORFF MEASURES
OF TAME SETS

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To the memory of Professor Nguyễn Hữu Đức

Abstract

In this paper we present some bounds of Hausdorff measures of objects definable in o-minimal structures: sets, fibers of maps, inverse images of curves of maps, etc. Moreover, we also give some explicit bounds for semi-algebraic or semi-Pfaffian cases, which depend only on the combinatoric data representing the objects involved.

Keywords o-minimal structures, Hausdorff measures

2010 Mathematics Subject Classification 14J17, 14P10, 53C65

1 Introduction

Considering the upper bounds for the lengths of curves contained in a disk, the areas of surfaces in a ball, or generally, the Hausdorff measures of subsets of a ball, one can see that if the numbers of points of the intersections of the curves or the surfaces with the generic lines are bounded, then their lengths or areas could

This research is supported by Vietnam’s National Foundation for Science and Technology Development (NAFOSTED).
be estimated (see Figure 1 for an example). Note that, spirals or oscillations do not have finite numbers of points of intersections with generic lines, so they can have infinite lengths in certain disks (see Figure 2 for an example). The objects of o-minimal structures have the finiteness of number of connected components (see [4], [7], [3] and [13]), and integral-geometric methods allow us to estimate Hausdorff measures of sets via the numbers of connected components of the intersections of the sets with generic affine subspaces of appropriate dimensions (see [8] and Figure 3 for examples).

Figure 1: $l \leq 2r$ \hspace{1cm} $l \leq 4dr$

Figure 2: $l = \infty$

Figure 3: $l \leq \sum_i \sum_j(l_{1,i} + l_{2,j})$
For these reasons, in this paper, we shall use integral-geometric methods to give some estimates of Hausdorff measures of objects definable in o-minimal structures: sets, fibers of maps, inverse images of curves of maps, etc. They can be considered as a generalization and refinement of some results of [11]. Moreover, we also give some explicit bounds for semi-algebraic and semi-Pfaffian cases (relying on the results in [1], [9], [12], [10], and [25]), which depend only on the combinatoric data representing the objects involved. These relate to some results in [23], [24], [5] and [6].

In section 2 we shall give some definitions. The results and examples will be stated and proved in sections 3 - 6.

2 Definitions

We give here some definitions and notations that will be used later.

**Definition 2.1.** An *o-minimal structure* on the real field \((\mathbb{R}, +, \cdot)\) is a sequence \(\mathcal{D} = (\mathcal{D}_n)_{n \in \mathbb{N}}\) such that the following conditions are satisfied for all \(n \in \mathbb{N}\):

- \(\mathcal{D}_n\) is a Boolean algebra of subsets of \(\mathbb{R}^n\).
- If \(A \in \mathcal{D}_n\), then \(A \times \mathbb{R}\) and \(\mathbb{R} \times A \in \mathcal{D}_{n+1}\).
- If \(A \in \mathcal{D}_{n+1}\), then \(\pi(A) \in \mathcal{D}_n\), where \(\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n\) is the projection on the first \(n\) coordinates.
- \(\mathcal{D}_n\) contains \(\{x \in \mathbb{R}^n : P(x) = 0\}\), for every polynomial \(P \in \mathbb{R}[X_1, \ldots, X_n]\).
- Each set in \(\mathcal{D}_1\) is a finite union of intervals and points.

A set belonging to \(\mathcal{D}\) is said to be *definable* (in that structure). *Definable maps* in structure \(\mathcal{D}\) are maps whose graphs are definable sets in \(\mathcal{D}\).

The class of semi-algebraic sets and the class generated by semi-Pfaffian sets ([12] and [21]) are examples of such structures, and there are many interesting classes of sets which have been proved to be o-minimal. For important properties of o-minimal structures we refer the readers to [4], [7], [3], [13] and [21]. Note that by Cell Decomposition [1, Chapter 3 Theorem 2.11], the dimension of a definable set \(A\) is defined by

\[\dim A = \max \{\dim C : C \text{ is a } C^1 \text{ submanifold contained in } A\}\]

In this note we fix an o-minimal structure on \((\mathbb{R}, +, \cdot)\). “Definable” means definable in this structure.
Definition 2.2. Let $A \subset \mathbb{R}^m$ be a semi-algebraic set represented by $A = \bigcup_{i=1}^{p} \bigcap_{j=1}^{s_i} A_{ij}$, where each $A_{ij}$ has the form:

$$\{x \in \mathbb{R}^m : p_{ij}(x) \star 0\},$$

where $p_{ij}$ is a polynomial of degree $d_{ij}$ and $\star \in \{>,=,<\}$.

The set of data $D = D(A) = (m, p, s_1, \ldots, s_p, (d_{ij}, i = 1, \cdots, p, j = 1, \cdots, s_i))$ is called the diagram of the set $A$.

Definition 2.3. A Pfaffian chain of length $l \geq 0$ and degree $\alpha \geq 1$ in an open domain $U \subseteq \mathbb{R}^m$ is a sequence of analytic functions $f = (f_1, \ldots, f_l)$ in $U$ satisfying a system of Pfaffian equations

$$\frac{\partial f_i}{\partial x_j}(x) = P_{ij}(x, f_1(x), \ldots, f_i(x)), \forall x \in U \ (1 \leq i \leq l, \ 1 \leq j \leq m).$$

where $P_{ij}$ are polynomials of degree not exceeding $\alpha$.

We say that $q$ is a Pfaffian function of degree $\beta$ with the Pfaffian chain $f$ if there exists a polynomial $Q$ of degree not exceeding $\beta$ such that

$$q(x) = Q(x, f_1(x), \ldots, f_l(x)), \forall x \in U.$$

Let $\mathcal{P} = \{p_1, \ldots, p_s\}$ be a set of Pfaffian functions. A quantifier-free formula (QF formula) with atoms in $\mathcal{P}$ is constructed as follows:

- An atom is of the form $p_i \star 0$, where $1 \leq i \leq s$ and $\star \in \{>,=,<\}$. It is a QF formula;

- If $\Phi$ and $\Psi$ are QF formulae, then their conjunction $\Phi \land \Psi$, their disjunction $\Phi \lor \Psi$, and the negation $\neg \Phi$ are QF formulae.

A set $A \subseteq U$ is called semi-Pfaffian if there exists a finite set $\mathcal{P}$ of Pfaffian functions and a QF formula $\Phi$ with atoms in $\mathcal{P}$ such that

$$A = \{x \in U : \Phi(x)\}.$$ 

Let $A$ be a semi-Pfaffian set as above. Then the format of $A$ is the set of data $F = F(A) = (m, l, \alpha, \beta, s)$, where $m$ is the number of variables, $l$ is the length of $f$, $\alpha$ is the maximum of the degrees of the polynomials $P_{ij}$, $\beta$ is the maximum of the degrees of the functions in $\mathcal{P}$, and $s$ is the number of the functions in $\mathcal{P}$. 
**Definition 2.4.** Let $m$ be a positive integer. For each $k \in \{0, \ldots, m\}$, let $\mathcal{H}^k(A)$ denote the $k$-dimensional Hausdorff measure of $A \subset \mathbb{R}^m$. Let $O^*(m, k)$ denote the space of all orthogonal projections of $\mathbb{R}^m$ onto $\mathbb{R}^k$, i.e.

$$O^*(m, k) = \{ p \mid p : \mathbb{R}^m \to \mathbb{R}^k \text{ linear and } p \circ p^* = \text{id}_{\mathbb{R}^k} \}.$$ 

The orthogonal group $O(m)$ acts transitively on $O^*(m, k)$ through right multiplication. This action induces a unique invariant measure $\theta^*_{m,k}$ over $O^*(m, k)$ with $\theta^*_{m,k}[O^*(m, k)] = 1$.

**The Cauchy-Crofton formula.** Since definable sets can be partitioned into finitely many $C^1$ submanifolds, by [8, Theorems 2.10.15 and 3.2.26], for every $k$-dimensional definable bounded subset $A$ of $\mathbb{R}^m$, we have

$$\mathcal{H}^k(A) = c(m, k) \int_{O^*(m, k)} \int_{\mathbb{R}^k} \#(A \cap p^{-1}(y)) dy d\theta^*_{m,k} p,$$

where $c(m, k) = \frac{\Gamma(m+1)\Gamma(k+1)}{\Gamma(k+1)\Gamma(m-k+1)}$, and $\Gamma(s) = \int_0^{+\infty} t^{-s-1} dt$ $(s > 0)$.

### 3 Uniform bounds of the Betti numbers of the fibers

**Proposition 3.1.** Let $f : A \to \mathbb{R}^n$ be a continuous definable map. Let $i \in \mathbb{N}$. Then there exists a positive number $M_i$, such that the $i$-th Betti numbers of the fibers of $f$ are bounded by $M_i$

$$B_i(f^{-1}(y)) \leq M_i, \text{ for all } y \in \mathbb{R}^n.$$

In particular, the numbers of connected components of the fibers of $f$ are uniformly bounded.

Moreover, if $f$ is semi-algebraic (resp. semi-Pfaffian), then $M_i$ only depends on the diagram (resp. the format) of $f$.

**Proof.** The first part follows from Hardt’s Trivialization Theorem [4, Chapter 9 Theorem 1.2]. When $f$ is semi-algebraic or semi-Pfaffian, the last assertion follows from [1], [9] or [12], [25], [10].
4 Hausdorff measures of definable sets

Let $A$ be a subset of $\mathbb{R}^m$. For each $k \in \{0, \ldots, m\}$, define

$$B_{0,m-k}(A) = \sup \{B_0(A \cap p^{-1}(y)) : p \in O^*(m, k), y \in \mathbb{R}^k\}$$

Note that if $A$ is definable, then applying Proposition 3.1 to the canonical projection

$$\{(x, p, y) \in A \times O^*(m, k) \times \mathbb{R}^k : p(x) = y\} \to \{(p, y) \in O^*(m, k) \times \mathbb{R}^k\},$$

we get the boundedness of $B_{0,m-k}(A)$. Moreover, if $A$ is semi-algebraic or semi-Pfaffian, then $B_{0,m-k}(A)$ is bounded by an explicit constant depending only on the diagram or the format of $A$ (see the examples below).

**Theorem 4.1.** Let $A, B$ be definable subsets of $\mathbb{R}^m$. Suppose $B$ is compact, $\dim A = k$, and $A \subset B$. Then

$$\mathcal{H}^k(A) \leq c(m, k)B_{0,m-k}(A) \sup_{p \in O^*(m, k)} \mathcal{H}^k(p(B)).$$

If moreover $A, B$ are semi-algebraic or semi-Pfaffian sets, then

$$\mathcal{H}^k(A) \leq C \sup_{p \in O^*(m, k)} \mathcal{H}^k(p(B)),$$

where $C$ is a constant depending only on the diagram or the format of $A$.

**Proof.** Let $p \in O^*(m, k)$. Set

$$S_p(d) = \{w \in \mathbb{R}^k : \dim(A \cap p^{-1}(w)) = d\}.$$

Applying [4, Chapter 4 Corollary 1.6], we have

$$\dim(A \cap p^{-1}(S_p(d))) = \dim(S_p(d)) + d.$$

Furthermore,

$$\dim(A \cap p^{-1}(S_p(d))) \leq \dim A = k.$$

So if $\dim(S_p(d)) = k$ then

$$d \leq 0.$$
Therefore, for each \( p \in O^*(m, k) \), \( \dim(A \cap p^{-1}(w)) \leq 0 \), for all \( w \in \mathbb{R}^k \) outside a definable set of dimension less than \( k \). By the Cauchy-Crofton formula, we get
\[
\mathcal{H}^k(A) = c(m, k) \int_{O^*(m, k)} \int_{\mathbb{R}^k} \#(A \cap p^{-1}(w)) dW_{m,k}^w \geq \int_{O^*(m, k)} \int_{\mathbb{R}^k} \#(A \cap p^{-1}(w)) dW_{m,k}^w \geq c(m, k) B_{0,m-k}(A) \int_{O^*(m, k)} \int_{\mathbb{R}^k} 1_{p(B)} dW_{m,k}^w \geq c(m, k) B_{0,m-k}(A) \sup_{p \in O^*(m, k)} \mathcal{H}^k(p(B)).
\]

The last assertion is followed by Proposition 1.

**Corollary 4.2** (c.f. [24] and [6]). Let \( A \) be a definable subset of \( \mathbb{R}^m \) of dimension \( k \). Then for any ball \( B^m_r \) of radius \( r \) in \( \mathbb{R}^m \),
\[
\mathcal{H}^k(A \cap B^m_r) \leq c(m, k) B_{0,m-k}(A) \text{Vol}_k(B^k_1) r^k.
\]

**Proof.** From Theorem 4.1, we get
\[
\mathcal{H}^k(A \cap B^m_r) \leq c(m, k) B_{0,m-k}(A) \mathcal{H}^k(B^k_r) = c(m, k) B_{0,m-k}(A) \text{Vol}_k(B^k_1) r^k.
\]

**Example 4.3.**

**Algebraic case.** When \( A \subset \mathbb{R}^m \) is a \( k \)-dimensional algebraic set of degree \( d \), then
\[
\mathcal{H}^k(A \cap B^m_r) \leq c(m, k) d \text{Vol}_k(B^k_1) r^k.
\]

In particular, when \( A \) is an algebraic curve of degree \( d \) in the plane, then the length \( l(A \cap B^2_r) \leq c(2, 1) d 2r = \pi dr \).

**Semi-algebraic case.** Generally, when \( A \subset \mathbb{R}^m \) is a \( k \)-dimensional semi-algebraic set of diagram \( D = (m, p, d, s) \), then
\[
\mathcal{H}^k(A \cap B^m_r) \leq c(m, k) B_0(D) \text{Vol}_k(B^k_1) r^k,
\]
where \( B_0(D) = \frac{2^m}{m!} \sum_{i=1}^p (d_i s_i)^m + O(s_i^{m-1}) \) , with \( d_i = \max_{1 \leq j \leq s_i} d_{ij} \) and \( m \) considered fixed.
**Semi-Pfaffian case.** We say that $U$ is a domain of bounded complexity $\gamma$ for the Pfaffian chain $f = (f_1, \ldots, f_l)$ if there exists a function $g$ of degree $\gamma$ in the chain $f$ such that the sets $\{g \geq \varepsilon\}$ form an exhausting family of compact subsets of $U$ for $\varepsilon \ll 1$. We call $g$ an exhausting function for $U$.

Let $A$ be a $k$-dimensional semi-Pfaffian set defined by a fixed Pfaffian chain $f = (f_1, \ldots, f_l)$ of degree $\alpha$ in a domain $U \subseteq \mathbb{R}^m$ with format $(m, l, \alpha, \beta, s)$, where $U$ is a domain of bounded complexity $\gamma$ for $f$. Using \cite[Remark 1.30, Theorem 2.25, and Remark 2.26]{25}, and applying Corollary 4.2, we get

$$H^k(A \cap B^m_r) \leq c(m, k)(4s + 1)^d \mathcal{V}(m, l, \alpha, \beta^*, \gamma) \operatorname{Vol}_k(B^k_1)r^k,$$

where

$$\mathcal{V}(m, l, \alpha, \beta^*, \gamma) = 2^{\frac{l(l-1)}{2}} \beta^*(\alpha + \beta^* - 1)^{m-1} \frac{\gamma}{2}[m(\alpha + \beta^* - 1) + \gamma + \min(m, l)\alpha]^l,$$

with $\beta^* = \max(\beta, \gamma)$.

### 5 Uniform bounds of Hausdorff measures of definable fibers

Let $f : A \to \mathbb{R}^n$ be a definable map, where $A \subseteq \mathbb{R}^m$. For each $k \in \{0, \ldots, \dim A\}$, let

$$I_k(f) = \{y \in \mathbb{R}^n : \dim f^{-1}(y) \leq k\}.$$

Then, by \cite[Chapter 4 Corollary 1.6]{25}, $I_k(f)$ is definable. Let

$$B_{0, m-k}(f) = \sup\{B_0(f^{-1}(y) \cap p^{-1}(w) \cap B^m_r(a)) : y \in I_k(f), p \in O^*(m, k), w \in \mathbb{R}^k, a \in \mathbb{R}^m, r > 0\}.$$

Note that applying Proposition 3.1 to the canonical projection

$$(x, y, p, w, a, r) \in \mathbb{R}^m \times \mathbb{R}^n \times O^*(m, k) \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} :$$

$$x \in A, y \in I_k(f), f(x) = y, p(x) = w, \|x - a\| \leq r \}$$

$$\rightarrow \{(y, p, w, a, r) \in \mathbb{R}^n \times O^*(m, k) \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}\},$$

we have the boundedness of $B_{0, m-k}(f)$. When $f$ is semi-algebraic (resp. semi-Pfaffian), then $B_{0, m-k}(f)$ is bounded by a constant depending only on the diagram (resp. format) of $f$. 
Theorem 5.1. Let \( f : A \to \mathbb{R}^n \) be a continuous definable map, where \( A \) is a compact subset of \( \mathbb{R}^m \). Then for each \( k \in \{0, \ldots, \dim A\} \), we have

\[
\mathcal{H}^k(f^{-1}(y)) \leq c(m, k)B_{0,m-k}(f) \sup_{p \in O^*(m,k)} \mathcal{H}^k(p(A)), \quad \text{for all } y \in I_k(f).
\]

In particular, if \( f \) is semi-algebraic or semi-Pfaffian map, then

\[
\mathcal{H}^k(f^{-1}(y)) \leq C_k \sup_{p \in O^*(m,k)} \mathcal{H}^k(p(A)), \quad \text{for all } y \in I_k(f),
\]

where \( C_k \) is a constant depending only on the diagram or the format of \( f \).

Proof. By [4, Chapter 4 Proposition 1.5 and Corollary 1.6], for each \( p \in O^*(m,k) \) and \( y \in I_k(f) \), \( \dim(f^{-1}(y) \cap p^{-1}(w)) \leq 0 \), for all \( w \in \mathbb{R}^k \) outside a definable set of dimension less than \( k \). By the Cauchy-Crofton formula, when \( y \in I_k(f) \), we get

\[
\mathcal{H}^k(f^{-1}(y)) = c(m, k) \int_{O^*(m,k)} \int_{\mathbb{R}^k} \#(f^{-1}(y) \cap p^{-1}(w))dwd\theta_{m,k}^p
\]

\[
\leq c(m, k)B_{0,m-k}(f) \int_{O^*(m,k)} \int_{\mathbb{R}^k} 1_{p(A)}dwd\theta_{m,k}^p
\]

\[
\leq c(m, k)B_{0,m-k}(f) \sup_{p \in O^*(m,k)} \mathcal{H}^k(p(A)).
\]

If \( f \) is semi-algebraic or semi-Pfaffian, then using the note above we have the last assertion. \( \square \)

Corollary 5.2. Let \( f : A \to \mathbb{R}^n \) be a continuous definable map, where \( A \subset \mathbb{R}^m \). Then for each \( k \in \{0, \ldots, \dim A\} \) and for any ball \( B_r^m \) of radius \( r \) in \( \mathbb{R}^m \),

\[
\mathcal{H}^k(f^{-1}(y) \cap B_r^m) \leq c(m, k)B_{0,m-k}(f)\text{Vol}_k(B_1^k)r^k, \quad \text{for all } y \in I_k(f).
\]

In particular, if \( f \) is semi-algebraic or semi-Pfaffian map, then

\[
\mathcal{H}^k(f^{-1}(y) \cap B_r^m) \leq C_k r^k, \quad \text{for all } y \in I_k(f),
\]

where \( C_k \) is a constant depending only on the diagram or the format of \( f \).

Example 5.3. Let \( \alpha_1, \ldots, \alpha_q \in \mathbb{N}^m \). Consider the family of algebraic surfaces in the positive orthant determined by the ‘fewnomials’ having only at most the monomials \( x_1^{\alpha_i}, i = 1, \ldots, q \):

\[
A = \{(x, a) : x = (x_1, \ldots, x_m) \in \mathbb{R}^m, a = (a_1, \ldots, a_q) \in \mathbb{R}^q, \}
\]
Let $f$ be the projection $(x, a) \mapsto a$ and $A_a = A \cap f^{-1}(a)$. When $k = m - 1$, and $\dim A_a \leq m - 1$ from the theorem we have the following estimates:

**Estimate 1.** Since $A_a$ is a semi-algebraic set of diagram $(m, 1, m+1, (1, \ldots, 1, d))$, with $d = \max_i |\alpha_i|$, using the Oleinik-Petrovskii-Thom-Milnor bound (see [17], [20], [16]), we get

$$\mathcal{H}^{m-1}(A_a \cap B_r^m) \leq c(m, m - 1)B_0(D(A_a))\text{Vol}_{m-1}(B_1^{m-1})r^{m-1},$$

where $B_0(D(A_a)) \leq \frac{1}{2}(m + d)(m + d - 1)^{m-1}$.

**Estimate 2.** Using the Khovanskii bound [12], Chapter III Corollary 5, we get

$$\mathcal{H}^{m-1}(A_a \cap B_r^m) \leq c(m, m - 1)B_0(f)\text{Vol}_{m-1}(B_1^{m-1})r^{m-1},$$

where $B_0(f) \leq 2^{\frac{q(q-1)}{2}}(2m)^{m-1}(2m^2 - m + 1)^q$.

A family $(C_q)_{q \in Q}$ is called a definable family of definable curves in $B \subset \mathbb{R}^n$ if there exists a definable map $\gamma : Q \times [0, 1] \to B$, $\gamma(q, t) = \gamma_q(t)$, such that for each $q \in Q$, $\gamma_q : [0, 1] \to B$ is continuous, injective and $C_q = \gamma_q([0, 1])$.

Let $\Phi^1$ denote the set of all odd, strictly increasing $C^1$ definable bijection from $\mathbb{R}$ to $\mathbb{R}$ and flat at 0.

**Theorem 5.4.** Let $f : A \to \mathbb{R}^n$ be a continuous definable map, and $A \subset \mathbb{R}^m$ be a compact set. Then for each $k \in \{0, \ldots, \dim A\}$, compact definable subset $B$ of $I_k(f)$ and definable family of definable curves $(C_q)_{q \in Q}$ in $\mathbb{R}^n$, there exists $\varphi \in \Phi^1$, such that

$$\mathcal{H}^{k+1}(f^{-1}(C_q \cap B)) \leq \varphi^{-1}(\mathcal{H}^1(C_q)), \text{ for all } q \in Q.$$

In particular, if $f$ is semi-algebraic, and $(C_q)_{q \in Q}$ is a semi-algebraic family of semi-algebraic curves, then there exist $C, \alpha > 0$ such that

$$\mathcal{H}^{k+1}(f^{-1}(C_q \cap B)) \leq C(\mathcal{H}^1(C_q))^\alpha, \text{ for all } q \in Q.$$

First we have:

**Lemma 5.5.** Let $h : B \to \mathbb{R}^m$ be a continuous definable map, and $B$ be a compact subset of $\mathbb{R}^n$. Then there exists $\psi \in \Phi^1$, such that

$$\mathcal{H}^1(h(C_q \cap B)) \leq \psi^{-1}(\mathcal{H}^1(C_q)), \text{ for all } q \in Q.$$
Proof of Lemma 5.5. To prove the lemma for a family \((C_q)_{q \in Q}\) of curves in \(B\), applying [7, C.17] and the uniform bounds [7, 4.4] to the family 
\[\{t \in [0,1] : \exists i, h_i \circ \gamma_q \text{ is not strictly monotone on any neighbourhood of } t \}\}_{q \in Q},
\]
where \(h = (h_1, \cdots, h_m)\), we have \(\psi_1 \in \Phi^1\) such that
\[\mathcal{H}^1(h(C_q)) \leq \psi_1^{-1}(\mathcal{H}^1(C_q)), \text{ for all } q \in Q.\]

For a family of definable curves in \(\mathbb{R}^n\), the number of connected components of \(C_q \cap B\) is uniform bounded by \(M\), for all \(q \in Q\). Therefore, denoting the connected components of \(C_q \cap B\) by \(C_q,i\) and applying the above case, we get
\[\mathcal{H}^1((f|_{A_j})^{-1}(C_q \cap B)) \leq \varphi_j^{-1}(\mathcal{H}^1(C_q)), \text{ for all } q \in Q,\]
where \(\varphi \in \Phi^1\) with \(\varphi^{-1} \geq \sum_{j=1}^J \varphi_j^{-1}\).

Proof of Theorem 5.4. The proof of the theorem is an adaptation of that of [11, Theorem 5].

For \(k = 0\): Since \(B\) is compact and the fibers of \(f\) over \(B\) are finite, by Trivialization [4, Chapter 9 Theorem 1.2], \(f^{-1}(B) = \bigcup_{j=1}^J A_j\), where \(A_j\) is a definable compact set, and \(f|_{A_j}\) is injective. For each \(j \in \{1, \ldots, J\}\), applying the lemma to \((f|_{A_j})^{-1}\), we get \(\psi_j \in \Phi^1\), such that
\[\mathcal{H}^1((f|_{A_j})^{-1}(C_q \cap B)) \leq \varphi_j^{-1}(\mathcal{H}^1(C_q)), \text{ for all } q \in Q.\]

So
\[\mathcal{H}^1(f^{-1}(C_q \cap B)) \leq \sum_j \varphi_j^{-1}(\mathcal{H}^1(C_q)) \leq \varphi^{-1}(\mathcal{H}^1(C_q)), \text{ for all } q \in Q,\]
where \(\varphi \in \Phi^1\) with \(\varphi^{-1} \geq \sum_{j=1}^J \varphi_j^{-1}\).

For \(k \geq 1\): let \(G_k(\mathbb{R}^m)\) denote the Grassmannian of \(k\)-dimensional linear subspaces of \(\mathbb{R}^m\). Define
\[\text{dist}(L, L') = \sup\{d(x, L') : x \in L, \|x\| = 1\}, \text{ for } L \in G_k(\mathbb{R}^m), L' \in G_l(\mathbb{R}^m).\]

Let \(\pi : \mathbb{R}^m \to \mathbb{R}^k\) denote the canonical projection. Choose a finite subset \(I\) of \(O(m)\) and \(\delta > 0\), so that for each \(L \in G_k(\mathbb{R}^m)\), there exists \(g \in I\) so that
\[\text{dist}(L, (\pi \circ g)^{-1}(0)) > \delta.\]
By [14] we can choose a stratification $S$ of $A$ satisfying Whitney’s condition (a), so that for each $S \in S$, rank$f|_S$ is constant and either $f(S) \subset I_k(f)$ or $f(S) \cap I_k(f) = \emptyset$. Let $J = \{S \in S: \dim S - \rank f|_S = k\}$. We can refine the stratification so that for each $g \in I$ and $T \in J$, the definable function

$$d(T, g)(x) = \dist(T_x(T \cap f^{-1}(f(x))), (\pi \circ g)^{-1}(0)) - \delta$$

has a constant sign on $T$.

For each $S \in S \setminus J$ we have $\dim(S \cap f^{-1}(y)) \leq k - 1$ for all $y \in I_k(f)$, therefore, $\mathcal{H}^{k+1}(f^{-1}(C_q \cap I_k(f)) \cup_{T \in J} T) = 0$ whenever $q \in Q$.

For each $T \in J$, there is a $g_T \in I$ so that $d(T, g_T)$ is positive on $T$. Hence, by Whitney’s condition (a), $\dim(f^{-1}(y) \cap (\pi \circ g)^{-1}(w) \cap \text{cl}(T)) \leq 0$, for all $y \in I_k(f), w \in \mathbb{R}^k$.

For each $g \in I$, let $A_g = \cup\{\text{cl}(T): T \in J, g_T = g\}$. Using the coarea formula [8, Theorem 3.2.22 (3)] and applying case $k = 0$ with $A := A_g$, $f := (f, \pi \circ g)|_{A_g}$, and $((C_q \times w))_{(q, w) \in Q \times \mathbb{R}^k}$ for the family of curves, we get

$$\mathcal{H}^{k+1}(f^{-1}(C_q \cap B)) \leq \sum_{g \in I} \mathcal{H}^{k+1}(g(A_g \cap f^{-1}(C_q \cap B)))$$

$$= \sum_{g \in I} \int_{g(A_g \cap f^{-1}(C_q \cap B))} d\mathcal{H}^{k+1}$$

$$= \sum_{g \in I} \int \mathcal{H}^{1}(g(A_g \cap f^{-1}(C_q \cap B)) \cap \pi^{-1}(w))dw$$

$$= \sum_{g \in I} \int \mathcal{H}^{1}(A_g \cap f^{-1}(C_q \cap B)) \cap g^{-1}(\pi^{-1}(w)))dw$$

$$= \sum_{g \in I} \int \mathcal{H}^{1}(A_g \cap (f, \pi \circ g)^{-1}[(C_q \times w) \cap (B \times \pi(g(A)))]))dw$$

$$\leq \sum_{g \in I} \int 1_{\text{cl}(A)} \varphi^{-1}(\mathcal{H}^{1}(C_q))dw$$

$$\leq \sum_{g \in I} \mathcal{H}^{k}(\pi \circ g(A)) \varphi^{-1}(\mathcal{H}^{1}(C_q))$$

$$\leq \varphi^{-1}(\mathcal{H}^{1}(C_q)), \text{ for all } q \in Q,$$

where $\varphi \in \Phi^1$ of the form $\varphi(t) = \varphi(t/K)$.

If $f$ is a semi-algebraic map, then by the Łojasiewicz inequality $\varphi$ has the form $\varphi^{-1}(y) = C\|y\|^\alpha$. \hfill $\Box$

Note that the above estimate is ‘effective’, not explicit.
Example 5.6.

a) Applying the theorem to the family of segments, we get \( \varphi \in \Phi^1 \), such that
\[
\mathcal{H}^{k+1}(f^{-1}([y, z] \cap B)) \leq \varphi^{-1}(\|y - z\|),
\]
whenever \( y, z \in \mathbb{R}^n \).

In particular, if \( f \) is a semi-algebraic or semi-Pfaffian map, then there exist \( C, \alpha > 0 \) such that
\[
\mathcal{H}^{k+1}(f^{-1}([y, z] \cap B)) \leq C\|y - z\|^\alpha.
\]

b) In general, for the semi-algebraic case one cannot choose \( C \) depending only on the diagram of \( f \), or \( \alpha = 1 \) in the estimate of the preceding theorem, e.g. for \( f_k(x) = kx^n \) with \( n \geq 2, k > 0 \),
\[
\mathcal{H}^1(f_k^{-1}([0, y])) = \frac{1}{\sqrt{k}} \sqrt[n]{y}, \text{ for every } y > 0.
\]

c) Let \( f(x) = e^{-|x|} \). Then \( f \) is definable in the o-minimal structure \( \mathbb{R}_{exp} \), and
\[
f^{-1}([0, y]) = [0, -\frac{1}{\ln |y|}].
\]
Since \( \frac{1}{y^\alpha \ln |y|} \to \infty \), when \( y \to 0 \), there does not exist \( C, \alpha > 0 \) so that \( \mathcal{H}^1(f^{-1}([0, y])) \leq C|y|^\alpha \) for all \( y \in [0, 1] \).

6 Morse-Sard’s Theorem

**Theorem 6.1.** Let \( f : A \to \mathbb{R}^n \) be a definable map. Suppose \( A = \cup_{i \in I} C_i \) is a finite union of \( C^1 \) definable manifolds \( C_i \), such that \( f|_{C_i} \) is of class \( C^1 \). For each \( s \in \mathbb{N} \) and \( i \in I \), let
\[
\Sigma_s(f, C_i) = \{ x \in C_i : \text{rank } df|_{C_i}(x) < s \} \text{ and } \Sigma_s(f, A) = \bigcup_{i \in I} \Sigma_s(f, C_i)
\]
Then \( C_s(f, A) = f(\Sigma_s(f, A)) \) is a definable set of dimension \( < s \). In particular, \( \mathcal{H}^s(C_s(f, A)) = 0 \).

**Proof.** The proof is similar to [13]. □

7 Remarks

The results in this paper still hold true for tame sets (see [7], [18], [19] for the definitions) with global changing to local. Applying theorems 1 and 2, one can get the explicit estimates for sub-Pfaffian case (see [10]).
References

[1] S.Basu, R.Pollack, and M.-F.Roy. An asymptotically bound on the number of semi-algebraically connected components of realizable sign conditions. 2009, available at arXiv: math/0603256v3, 24 pages.

[2] R.Benedetti and J.J.Risler. Real algebraic and semi-algebraic sets. Actualités Mathématiques, Hermann, Paris, 1990.

[3] M.Coste. An Introduction to O-minimal Geometry. Doctorato di Ricerca in Matematica, Dip. Mat. Pisa. Instituti Editoriali e Poligrafici Internazionali, 2000.

[4] L.van den Dries. Tame Topology and O-minimal Structures. LMS Lecture Notes, Cambridge University Press, 1997.

[5] L.van den Dries. Limit sets o-minimal structures. Proceedings of the RAAG summer school, Lisbon 2003: O-minimal structures, 2005: 172-215.

[6] D. D'Acunto and K. Kurdyka. Bounds for gradient trajectories and geodesic diameter of real algebraic sets. Bull. London Math. Soc, 2006, 38, no.6: 951-965.

[7] L.van den Dries and C.Miller. Geometric Categories and O-minimal Structures. Duke Math. J. 1996, 84, No 2: 497-540.

[8] H.Federer. Geometric measures theory. Springer-Verlag, 1969.

[9] A.Gabrielov and N.Vorobjov. Betti numbers of semi-algebraic sets defined by quantifier-free formulae. Discrete Comput Geom, 1997, 33: 395-401.

[10] A.Gabrielov, N.Vorobjov and T.Zell. Betti numbers of semi-algebraic sub-Pfaffian sets. J. London Math. Soc, 2004, (2) 69: 27-43.

[11] R.M.Hardt. Some analytic bounds for subanalytic sets. Differential geometric control theory (Houghton, Mich, 1982): 259-267, Progr. Math. 27, Birkhauser Boston, Boston, Mass., 1983.

[12] A.G.Khovanski. Fewnomials. Translations of mathematical monographs 88, AMS, Providence RI, 1991.

[13] T.L.Loi. Tame topology and Tarski-type systems. Vietnam J. Math, 2003, 31:2: 127-136.

[14] T.L.Loi. Stratification of families of functions definable in o-minimal structures. Acta Math. Vietnam, 2002, Vol. 27, 2: 239-244.

[15] T.L.Loi. Transversality theorem in o-minimal structures. Compositio Math, 2008, 144: 1227-1234.

[16] J.Milnor. On the Betti numbers of real varieties. Proc.Amer.Math.Soc, 1964, 15: 275-280.

[17] O.A.Oleinik and I.G.Petrovskii. On the topology of real algebraic hypersurfaces. (Russian) Izv.Acad.Nauk SSSR, 1949, 13: 389-402.

[18] M.Shiota. Geometry of Subanalytic and Semialgebraic Sets. Progress in Math., Vol. 150, Birkhäuser, Boston, 1997.
[19] B. Tessier. Tame and stratified objects. Geometric Galois Actions, 1. Around Grothendieck’s esquisse d’un programme. London. Math. Soc. Lecture Note Series 242, 1997: 231-242.

[20] R. Thom. Sur l’homologie des variétés algébriques réelles. Differential and combinatorial topology, Princeton Univ. Press, Princeton, 1965: 255-265.

[21] A.J. Wilkie. A Theorem of The Complement and Some New O-minimal Structures. Sel. Math., 1999, New ser. 51: 397-421.

[22] H. Whitney. A Function not Constant on Connected Set of Critical Points. Duke Math. J, 1935, 1: 514-517.

[23] Y. Yomdin. Metric properties of semialgebraic sets and mappings and their applications in smooth analysis. (Proceedings of the Second International Conference on Algebraic Geometry, La Rabida, Spain, 1984, J.M. Aroca, T. Sahcez-Geralda, J.L. Vicente, eds.), Travaux en Course, Hermann, Paris, 1987: 165-183.

[24] Y. Yomdin and G. Comte. Tame Geometry with Application in Smooth Analysis. LNM vol. 1834, 2004.

[25] T. Zell. Quantitative study of semi-Pfaffian sets [PhD thesis]. School of Mathematics, Georgia Institute of Technology, 2003.