ON THE SOLUTION OF A RIESZ EQUILIBRIUM PROBLEM
AND INTEGRAL IDENTITIES FOR SPECIAL FUNCTIONS

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Abstract. The aim of this note is to provide a full space quadratic external field extension of a classical result of Marcel Riesz for the equilibrium measure on a ball with respect to Riesz $s$-kernels. We address the case $s = d - 3$ for arbitrary dimension $d$, in particular the logarithmic kernel in dimension 3. The equilibrium measure for this full space external field problem turns out to be a radial arcsine distribution supported on a ball with a special radius. As a corollary, we obtain new integral identities involving special functions such as elliptic integrals and more generally hypergeometric functions. It seems that these identities are not found in the existing tables for series and integrals, and are not recognized by advanced mathematical software. Among other ingredients, our proofs involve the Euler–Lagrange variational characterization, the Funk–Hecke formula, the Weyl regularity lemma, the maximum principle, and special properties of hypergeometric functions.

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1. Introduction and main results

The goal of this note is to provide a full space quadratic external field extension (Theorem 1.4 below) of a classical result of Marcel Riesz (Theorem 1.1 below) for the equilibrium measure on a ball in arbitrary dimensions with respect to Riesz s-kernels, including the logarithmic kernel. The equilibrium measure turns out to be a radial arcsine distribution. As corollaries, we obtain new integral identities involving special functions such as elliptic integrals and more generally hypergeometric functions; see, for example, Corollaries 1.3, 1.5, and 1.6 below. It seems that these identities are not found in the existing tables for series and integrals, and are not recognized by advanced mathematical software.

Before we present our results and identities, we recall some basic notions from potential theory. Throughout this note, we denote by $\sigma$ the uniform probability measure on $S$, the Euclidean dimension, which is always a positive integer, and by $s \in (-2, +\infty)$ the Riesz parameter. For $x \in \mathbb{R}^d$, $x \neq 0$, the Riesz $s$-kernel is defined by

$$K_s(x) := \begin{cases} \text{sign}(s) |x|^{-s} & \text{if } -2 < s < 0 \text{ or } s > 0, \\ \log |x| & \text{if } s = 0, \end{cases}$$

(1.1)

where $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$ is the Euclidean norm. It is the Coulomb or Newton kernel if $s = -2$. Let $\mathcal{M}_1$ be the set of probability measures on $\mathbb{R}^d$ and let $V : \mathbb{R}^d \mapsto (-\infty, +\infty]$ be a lower semicontinuous function, which will play the role of an external field. In this note we only deal with either an external field constant on a centered ball and infinite outside the ball, or with a quadratic external field of the form $V(\cdot) = \gamma |\cdot|^2$, $\gamma > 0$. The energy of $\mu \in \mathcal{M}_1$ with external field $V$ is defined by

$$I(\mu) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (K_s(x-y) + V(x) + V(y)) \mu(dx)\mu(dy) \in (-\infty, +\infty].$$

(1.2)

For $s \in (-2, d)$, with our choices of $V$, the integrand in the double integral in (1.2) is bounded below, $I$ is strictly convex on $\mathcal{M}_1$ and lower semicontinuous with compact level sets. It has a unique global minimizer called the “equilibrium measure” $\mu_{eq} \in \mathcal{M}_1$; in other words,

$$I(\mu_{eq}) = \min_{\mu \in \mathcal{M}_1} I(\mu) \quad \text{and} \quad I(\mu) > I(\mu_{eq}) \quad \text{for all } \mu \neq \mu_{eq}, \mu \in \mathcal{M}_1. \quad (1.3)$$

Moreover, $\mu_{eq}$ is compactly supported with finite energy $I(\mu_{eq}) < +\infty$. We refer to [23] and [5] for more details. If $s < 0$, then $K_s$ is not singular and we could have $I(\mu) < \infty$ for a $\mu \in \mathcal{M}_1$ having Dirac masses; in particular $\mu_{eq}$ could conceivably have Dirac masses. In contrast, if $s \geq 0$ then $K_s$ is singular and $I(\mu) = +\infty$ whenever $\mu$ has Dirac masses; consequently $\mu_{eq}$ cannot have such masses.

We first recall a classical result of M. Riesz for the equilibrium measure with constant external field in a closed ball and infinite outside the ball. For $R > 0$, let

$$B_R := \{x \in \mathbb{R}^d : |x| \leq R\} \quad \text{and} \quad S_R := \{x \in \mathbb{R}^d : |x| = R\}$$

denote the ball and sphere of radius $R$ centered at the origin. In particular $S_1 = S^{d-1}$ is the unit sphere, with surface area $|S^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$. For a subset $S$ of $\mathbb{R}^d$, we denote, when it makes sense, by $\sigma_S$ the uniform probability measure on $S$ (normalized trace of Lebesgue measure).

**Theorem 1.1** (Riesz [31]). Suppose that $d \in \{2, 3, 4, \ldots\}$ and $V = \begin{cases} 0 & \text{on } B_R, \\ +\infty & \text{outside } B_R, \end{cases}$ $R > 0$.

- If $-2 < s \leq d - 2$, then $\mu_{eq} = \sigma_{S_R}$.
- If $d - 2 < s < d$, then $\mu_{eq}$ is the probability measure

$$\mu_{eq}(dx) = \frac{\Gamma(1 + \frac{s}{2})}{R^d \pi^{\frac{d}{2}} \Gamma(1 + \frac{s}{2})} \begin{cases} 1_{|x| \leq R} & \text{if } s \neq -2, \\ \log |x| & \text{if } s = -2, \end{cases} \quad (1.4)$$

\[\int_{|x| \leq R} \frac{1}{(R^2 - |x|^2)^{\frac{d}{2}}} \, dx = \frac{2\Gamma(1 + \frac{s}{2})}{R^d \Gamma(1 + \frac{s}{2}) \Gamma(\frac{d}{2})} \frac{r^{d-1} 1_{r \leq R}}{(R^2 - r^2)^{\frac{d}{2}}} \, dr \, d\sigma_{S_1},\]

1In other words, $K_s$ is conditionally strictly positive in the sense of Bochner, see for instance [5, Section 4.4].

2We follow the probability theory standard and equip the convex set $\mathcal{M}_1$ with the topology of weak convergence with respect to continuous and bounded test functions, in other words the weak-* convergence.
where $dx$ and $dr$ denote the Lebesgue measures on $\mathbb{R}^d$ and on $[0, +\infty)$ respectively. Moreover, the equilibrium potential $U^{\mu_{eq}}$ satisfies, for $x \in B_R$,
\begin{equation}
U^{\mu_{eq}}(x) := (K_* \mu_{eq})(x) = \int_{\mathbb{R}^d} K_s(x-y) \mu_{eq}(dy) = I(\mu_{eq}) = \frac{\Gamma(1 + \frac{s}{d}) \Gamma(d-s)}{R^d \Gamma(\frac{d}{2})}.
\end{equation}

The case $d - 2 < s < d$ in Theorem 1.1 is a direct consequence of the following lemma.

**Lemma 1.2** (Riesz formula [31]). If $d \in \{2, 3, 4, \ldots\}$, $0 \leq d - 2 < s < d$, and $R > 0$, then for $x \in B_R$,
\begin{equation}
\int_{\mathbb{R}^d} \frac{|x-y|^{-s}}{(R^2 - |y|^2)^{\frac{s}{2}}} I_{|y| \leq R} dy = \frac{\pi^{\frac{d}{2} + 1}}{\Gamma(\frac{d}{2}) \sin(\frac{\pi s}{d})}.
\end{equation}

The proof of Theorem 1.1 and Lemma 1.2 can be found, together with some geometric aspects, in the works of M. Riesz [30, p. 438–439] and [31, § 16, Eq. (1)], where it is mentioned that the cases $d = 1, 2, 3$ were already considered by Pólya and Szegő in [29]. It can also be found in the book [23, § II.3.13, p. 163–164, and Appendix p. 399–400], and is stated in [5, Eq. (4.6.13)]. The proof sketched by Riesz, with a bit more detail by Landkoj, involves first a geometric inversion transforming the integral on the ball into an integral on its complement, and second a trigonometric substitution which has a geometric interpretation, both steps being inspired by the analytic-geometric techniques used classically for elliptic integrals since the eighteenth century. For the reader’s convenience, a detailed proof of Lemma 1.2 is given in Appendix C. After publication of the present article in Journal of Mathematical Analysis and Applications, we have found that Dyda, Kuznetsov, and M. Kwaśnicki gave in [13] an alternative analytic (and short!) proof of Lemma 1.2. Since this proof is somewhat hidden by the Meijer G-function material in [13], we give, in Appendix D of the present arXiv post-publication update, a straight version of this analytic proof, without using Meijer G-functions.

Our first result, Corollary 1.3, is a simple consequence of Theorem 1.1. It relates an equilibrium measure of potential theory with an integral identity for special functions (here a $2F_1$ hypergeometric function). Before stating it, let us recall the Newton binomial series
\begin{equation}
\frac{1}{(1 - z)^n} = \sum_{n=0}^{\infty} (\alpha)_n \frac{z^n}{n!}, \quad \alpha, z \in \mathbb{C}, \quad |z| < 1,
\end{equation}

where $(\alpha)_n := \alpha(\alpha + 1) \cdots (\alpha + n - 1)$ is the Pochhammer symbol for the rising factorial, with the convention $(\alpha)_0 := 1$ if $\alpha \neq 0$. If $\Re(\alpha) > 0$, then $(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)$. More generally, the hypergeometric function with parameters $(a_1, \ldots, a_p) \in \mathbb{C}^p$ and $(b_1, \ldots, b_q) \in \mathbb{C}^q$, at $z \in \mathbb{C}$, $|z| < 1$, is given (when it makes sense) by the series
\begin{equation}
pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.
\end{equation}

Special choices of the parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ allow us to recover many special functions, for instance $2F_1(1, 1; 2; z) = (1 - z)^{-1}$, $2F_1(1, 1; 2; -z) = \frac{\log(1+z)}{z}$, and $2F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z^2\right) = \frac{\arcsin(z)}{z}$. Actually one of the main historical motivations for the introduction and study of hypergeometric functions is the unification of as many as possible special functions via series expansions. For instance the complete elliptic integral of first and second kind, $K$ and $E$ respectively, satisfy for $z \in [0, 1]$,
\begin{equation}
K(z) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - z \sin^2(\theta)}} = \int_0^1 \frac{dt}{\sqrt{1 - t^2 \sqrt{1 - t^2}}} = \frac{\pi}{2} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)
\end{equation}
and
\begin{equation}
E(z) := \int_0^{\frac{\pi}{2}} \sqrt{1 - z \sin^2(\theta)} d\theta = \int_0^1 \frac{1 - t^2}{\sqrt{1 - t^2}} dt = \frac{\pi}{2} 2F_1\left(- \frac{1}{2}, \frac{1}{2}; 1; z\right).
\end{equation}

They can be extended to the complex plane, with a branch cut discontinuity running from 1 to $\infty$. For more basic facts about these functions, we refer for instance to the classical books [17, 20]. Here we only remark that $K, E \geq 0$ on the interval $[0, 1]$, $K(0) = E(0) = \frac{\pi}{2}$, $K(1) = \infty$, and $E(1) = 1$.

The following identity is an easy consequence of Theorem 1.1 (see Section 2.1 and Remark 2.4).
Corollary 1.3 (Special function identity). For \( d \in \{2, 3, 4, \ldots\}, \ d - 2 < s < d, \ \lambda \in [0, 1], \)
\[
\int_{0}^{1} 2F_{1} \left( \frac{s}{4}, \frac{s + 2}{4}; \frac{d}{2}; \frac{4r^{2}\lambda^{2}}{(\lambda^{2} + r^{2})^{2}} \right) \frac{d^{d-1}}{(\lambda^{2} + r^{2})^{s+1}} \, dx = \frac{\pi}{2sin(\frac{\pi}{2}(d-s))}. \quad (1.11)
\]

Here are some special cases worth noting for the hypergeometric function in (1.11):

- for \((d, s) = (2, 1)\) we get \(2F_{1} \left( \frac{1}{4}, \frac{3}{4}; \frac{d}{4}; \frac{1}{2} \right) = 2F_{1} \left( \frac{1}{4}, \frac{1}{4}; 1; \frac{1}{2} \right) = \frac{2K(x)}{\pi\sqrt{2 + x}}.
- for \((d, s) = (3, 2)\) we get \(2F_{1} \left( \frac{1}{4}, \frac{3}{4}; \frac{d}{4}; \frac{1}{2} \right) = 2F_{1} \left( \frac{1}{4}, 1; \frac{3}{2}; \frac{1}{2} \right) = \frac{\tanh^{-1}(\sqrt{2})}{\sqrt{2}}.
- for \((d, s) = (5, 4)\) we get \(2F_{1} \left( \frac{1}{4}, \frac{3}{4}; \frac{d}{4}; \frac{1}{2} \right) = 2F_{1} \left( 1; \frac{3}{2}; \frac{1}{2} \right) = \frac{3(\sqrt{2}\tanh^{-1}(\sqrt{2}))}{2^\frac{5}{2}}.

Our main potential theoretic result is the following external field version of Theorem 1.4:

Theorem 1.4 (Main result). Suppose that \( d \in \{2, 3, 4, \ldots\) and \( s = d - 3, \namely\)
\[(d, s) \in \{(2, -1), (3, 0), (4, 1), \ldots\}.\]

Let \( R := \left( \frac{c_{d,s}d^{-3\sqrt{\pi}}}{4\gamma} \frac{\Gamma(d/2)}{\Gamma(\frac{d+1}{2})} \right)^{\frac{1}{d-1}}, \) where \( c_{d,s} := \begin{cases} \frac{|s|(d-2-s)}{d-2} & \text{if } s \neq 0, \\ d-2 & \text{if } s = 0. \end{cases} \quad (1.12)\]

If \( V = \gamma |x|^{2}, \gamma > 0, \) then the equilibrium measure \( \mu_{eq} \) for the minimum energy problem on \( \mathbb{R}^{d} \) with kernel \( K_{e} \) and external field \( V \) is the “radial arcsine distribution”
\[
\mu_{eq}(dx) = \frac{\Gamma(d/2)}{\pi^{d/2}R^{d-1}} \frac{1_{|x| \leq R}}{\sqrt{R^{2} - |x|^{2}}} \, dx = \frac{2\Gamma(d/2)}{\pi^{d/2}R^{d-1}} \frac{r^{d-1}1_{r \leq R}}{\sqrt{R^{2} - r^{2}}} \, dr \, d\sigma_{1}, \quad (1.13)
\]
where \( dx \) and \( dr \) are the Lebesgue measures on \( \mathbb{R}^{d} \) and on \([0, \infty)\) respectively. Moreover, this \( \mu_{eq} \) is also the equilibrium measure in Theorem 1.4, with \( s = d - 1 \) and \( R \) as in (1.12).

Theorem 1.4 is proved in Section 2.2.

Several extensions of Theorem 1.3 for more general \((d, s)\) or \( V \) are considered in 10.

\[
\begin{array}{cccccc}
\frac{\pi}{8} & \approx 0.392699 & 1 & \frac{1}{\sqrt{3}} & \approx 0.57735 & \frac{1}{2} & \frac{1}{2}\sqrt{3}\pi & \approx 0.665335 & 1 \\

\end{array}
\]

Table 1. Some special values of the critical radius \( R \) in Theorem 1.4 with \( \gamma = 1.\)

Table 1 gives values of the radius \( R \) in (1.12) for \( \gamma = 1 \) and various values of \( d. \) For \( \gamma = 1, \) integer \( d \geq 2, \) the function \( d \mapsto R \) achieves its minimum \( \approx 0.392699 \) at \( d = 2 \) \( (s = -1) \) and its maximum \( \approx 1.04747 \) at \( d = 16 \) \( (s = 13) \), and these values are the unique extreme points. Regarding high dimensional behavior or asymptotic analysis, we have \( \lim_{d=s+3 \to \infty} R = 1. \)

As we shall verify, Theorem 1.4 yields the following integral formulas.

Corollary 1.5 (Integral formula). Let \( d \in \{2, 3, 4, \ldots\) and \( \lambda \in [0, 1]. \) Then
\[
\int_{0}^{1} S_{d-3} \left( \frac{4\lambda r}{(\lambda + r)^{2}} \left( \lambda + r \right)^{3-d} \right)^{d-1} \, dr = \frac{\pi^{\frac{d}{2}}\Gamma(d/2)}{2^{d+1}\Gamma(3/2)} \left( \frac{3}{d-1} \right)^{\lambda^{2} + 1}, \quad (1.14)
\]
where
\[
S_{s}(z) := \int_{0}^{\frac{\pi}{2}} \frac{\sin^{s+1}(2\alpha) d\alpha}{\left( 1 - z \sin^{2}(\alpha) \right)^{\frac{1}{2}}} = \int_{0}^{\frac{\pi}{2}} \frac{t^{s+1}(1-t^{2})^{\frac{1}{2}}}{(1-t^{2})^{s+1}} \, dt = \frac{\Gamma(s+2)^{2}}{2\Gamma(s+2)} 2F_{1} \left( \frac{s+2}{2}; \frac{s}{2}; s+2; z \right). \quad (1.15)
\]

\footnote{It is worth noting that \( 4r^{2}\lambda^{2} \leq (\lambda^{2} + r^{2})^{2} \) with equality if and only if \( \lambda = r, \) so that the radius of convergence of the \( 2F_{1} \) in (1.11), which is equal to 1, is reached in the interior of the interval of integration over \( r \) when \( \lambda \in [0, 1). \)}
Equivalently, for \( d \in \{2, 3, 4, \ldots \} \) and \( \lambda \in [0, 1] \),
\[
\int_0^1 {}_2F_1\left( \frac{d-1}{2}, \frac{d-3}{2}; d-1; 1 - \frac{4\lambda r}{(\lambda + r)^2} \right) \frac{(\lambda + r)^{3-d} r^{d-1}}{\sqrt{1-r^2}} \, dr = \frac{\pi}{4} \left( \frac{3}{d} - 1 \right) \lambda^2 + 1.
\] (1.16)

Corollary 1.5 is proved in Section 2.3.

The formula (1.16) comes from the Euler–LaGrange characterization related to Theorem 1.3. Numerical experiments suggest that (1.14) and (1.15) remain valid whenever the parameter \( d > 1 \) is real.

It is tempting to regard \( S_2 \) as a special function in its own right. When \((d, s) = (2, -1)\), it becomes the complete elliptic integral of the second kind, namely \( S_{-1} = E \), and (1.14) becomes (1.19) below.

**Corollary 1.6** (More integral formulas\(^4\)). For \( \lambda \in [0, 1] \),
\[
\int_0^1 ((\lambda + r)^2 \log(\lambda + r) - (\lambda - r)^2 \log(\lambda - r)) \frac{r \, dr}{\sqrt{1-r^2}} = \frac{\pi}{3} \left( \frac{\lambda^3}{3} + (1 - \log 2) \lambda \right),
\] (1.17)
\[
\int_0^1 ((\lambda + r) \log(\lambda + r) - (\lambda - r) \log(\lambda - r)) \frac{r \, dr}{\sqrt{1-r^2}} = \frac{\pi}{2} \left( \frac{\lambda^2}{2} + \frac{1}{2} \log 2 \right),
\] (1.18)
\[
\int_0^1 E\left( \frac{4\lambda r}{(\lambda + r)^2} \right) \frac{(\lambda + r) \, dr}{\sqrt{1-r^2}} = \frac{\pi^2}{8} \left( \frac{\lambda^2}{2} + 1 \right),
\] (1.19)
\[
\int_0^1 K\left( \frac{4\lambda r}{(r + \lambda)^2} \right) \frac{(\lambda - r) \, dr}{\sqrt{1-r^2}} = \frac{\pi^2}{8} \left( \frac{3\lambda^2}{2} - 1 \right),
\] (1.20)
where \( E \) and \( K \) are the special functions defined in (1.14) and (1.15).

Corollary 1.6 is proved in Section 2.3 by applying further transformations to the Euler–LaGrange conditions of Theorem 1.4 in the special cases \( (d, s) \in \{(2, -1), (3, 0)\} \).

To the best of our knowledge, the formulas provided by Corollaries 1.3, 1.5, and 1.6 are not found in the existing catalogs of identities and tables for series and integrals such as [14], [28], and [6], and are not recognized by advanced software such as Maplesoft Maple and Wolfram Mathematica. However it is worth noting that these softwares do recognize the first parts of (1.17) and (1.18) in terms of \( \text{$_3F_2$} \):
\[
\int_0^1 \frac{(\lambda + r)^2 \log(\lambda + r)}{\sqrt{1-r^2}} \, dr
= \frac{\pi}{3} \text{$_3F_2$} \left( \frac{1}{2}, 1, 1; 2, 3; \frac{1}{\lambda} \right) - \frac{2\pi}{45\lambda^2} \text{$_3F_2$} \left( 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{\lambda} \right) + \frac{\pi \lambda}{4} + \frac{3\lambda(2\lambda + \pi) + 4}{6} \log(\lambda) + 1
\] (1.21)
and
\[
\int_0^1 \frac{(\lambda + r) \log(\lambda + r)}{\sqrt{1-r^2}} \, dr
= \frac{32\pi}{96\lambda^2} \text{$_3F_2$} \left( 1, 1, 1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{\lambda} \right) - 3\pi \text{$_3F_2$} \left( 1, 1, \frac{3}{2}, 2; 3, \frac{1}{\lambda} \right) + 24\lambda^2((4\lambda + \pi) \log(\lambda) + \pi)
\] (1.22)

**2. Proofs**

2.1. Proofs of Corollary 1.3. Let us consider the settings of Theorem 1.1 in the case \( d - 2 < s < d \). By scaling we can assume without loss of generality that \( R = 1 \). Then, using the Funk–Hecke formula (1.11), we get, for \( x \in \mathbb{R}^d \), \( x \neq 0 \), denoting \( \lambda := |x| \) and \( C := \frac{\Gamma(1 + \frac{s}{d})}{\pi^{\frac{s}{2}} \Gamma(1 + \frac{d}{2})} \),
\[
U^{\mu=\alpha}(x) = C \int_{|y| \leq 1} \frac{|x - y|^{-s}}{(1 - |y|^2)^{\frac{d-s}{2}}} \, dy
= C \int_0^1 \left( \int_{S_1} \frac{(\lambda^2 + r^2 - 2\lambda r \frac{r}{|r|} \cdot u)^{-\frac{s}{2}}}{(1 - r^2)^{\frac{s}{2}}} \, r^{d-1} \, du \right) \, dr.
\]
\(^4\)Note that \( \frac{4\lambda r}{(\lambda + r)^2} = \frac{4x}{(1+x)^2} \) with \( x := \frac{r}{\lambda} \). The map \( x \mapsto \frac{4x}{(1+x)^2} \) is the Landen transform of elliptic integrals.
\[
= C\tau_{d-1}|S_1| \int_0^1 \left( \int_{-1}^1 \frac{(1-t^2)^{d-3}}{(\lambda^2 + r^2 - 2\lambda r t)^{\frac{d}{2}}} dt \right) \frac{r^{d-1}}{(1-r^2)^{\frac{d}{2}}} dr.
\]

Now, since \(2\lambda \in [0,1/2]\), using the Newton binomial series (1.7),
\[
\int_{-1}^1 \frac{(1-t^2)^{\frac{d}{2}}}{(\lambda^2 + r^2 - 2\lambda r t)^{\frac{d}{2}}} dt = \frac{1}{(\lambda^2 + r^2)^{\frac{d}{2}}} \sum_{n=0}^{\infty} \left(\frac{s}{2}\right)_{2n} \frac{4r^2\lambda^2}{(2n)!} \int_{-1}^1 (1-t^2)^{\frac{d}{2}+2n} dt
\]
\[= \frac{1}{(\lambda^2 + r^2)^{\frac{d}{2}}} \sum_{n=0}^{\infty} \left(\frac{s}{2}\right)_{2n} \frac{4r^2\lambda^2}{(2n)!} \left(\frac{\Gamma(d+1)\Gamma(n+\frac{1}{2})}{\Gamma(\frac{d}{2}+n)}\right)
\]
\[= \frac{\sqrt{\pi}\Gamma(d+1)}{\Gamma(\frac{d}{2})} \frac{\left(\frac{d+s-1}{2}\right)_{d+2}}{(\lambda^2 + r^2)^{\frac{d}{2}}},
\]
where we have used the identities (related to the Legendre duplication formula (A.2))
\[
\left(\frac{s}{2}\right)_{2n} = 2^{2n} \left(\frac{s}{4}\right)_n \left(\frac{s}{4} + \frac{1}{2}\right)_n \quad \text{and} \quad 2^{2n} \left(\frac{1}{2}\right)_n = \frac{(2n)!}{n!}.
\]

Note that we could alternatively proceed as in the proof of Lemma 1.2 and use the Euler integral formula (A.10). Next, using the fact that \(\tau_{d-1} = \frac{\Gamma(d+1)}{\sqrt{\pi}(\frac{d}{2})^{d+1}}\), we get, for \(x \in \mathbb{R}^d, x \neq 0\),
\[
U^{\mu_{eq}}(x) = C|S_1| \int_0^1 2\Gamma(\frac{d+s-1}{2}) \frac{\left(\frac{d+s-1}{2}\right)_{d+2}}{(\lambda^2 + r^2)^{\frac{d}{2}}} \frac{r^{d-1}}{(1-r^2)^{\frac{d}{2}}} dr.
\]
(This last integral may be compared with extensions of the Beta integral in [15] Th. 5.1 and [33]).

Now, the Euler–Lagrange conditions (A.14) and the continuity of \(U^{\mu_{eq}}\) give that this quantity is constant on \(B_1\). Finally the value of the constant can be obtained from (1.6). \(\Box\)

2.2. Proof of Theorem 1.4

We split the proof into several subsections.

2.2.1. Computation of critical radius and candidate equilibrium measure. From the uniqueness property we know that the equilibrium measure \(\mu_{eq}\) is radially symmetric. Let us make a succession of assumptions to extract a candidate for \(\mu_{eq}\), and we will then check that it indeed satisfies the Euler–Lagrange conditions (A.14). We start by observing from (A.13) that, for \(x \in S_* := \text{supp}(\mu_{eq})\),
\[
\int K_s(x-y)\mu_{eq}(dy) + \gamma|x|^2 = c.
\]
Applying the Laplacian operator to (2.1) and assuming it can be taken inside the integral, we get from (2.1) and (A.15) that for all \(x\) in the interior of \(S_*\),
\[
-\cd_s \int K_{s+2}(x-y)\mu_{eq}(dy) + 2\gamma d = 0.
\]
In our case \(s = d - 3\), so \(c_{d,s} = c_{d,d-3}\) is equal to \(|d-3|/d\) if \(d \neq 3\) while it is equal to 1 if \(d = 3\).

Next suppose that \(S_* = B_R\) for some \(R > 0\). Let \(\nu_R\) be the equilibrium measure for the minimum energy problem on \(B_R\) with kernel \(K_{s+2} = |r|^{-(s+2)}\) and \(V = 0\). Observing that \(\nu_R\) is the dilation by a factor of \(R\) of \(\nu_1\), we see from Theorem 1.1 that \(\nu_R\) is the “radial arcsine distribution”; in other words, the measure
\[
\nu_R(dx) = \frac{C_{d,R}}{\sqrt{R^2 - |x|^2}} 1_{|x| \leq R} dx, \quad \text{where} \quad C_{d,R} = \frac{2\Gamma(d+1)}{|S_1|\sqrt{\pi}\Gamma(d/2)^{d-1}} \frac{\Gamma(d+1)}{R^{d-1}\pi^{d/2}}.
\]
In particular, the support of \(\nu_R\) is all of \(B_R\). Next, by definition of \(\nu_R\), the associated Euler–Lagrange conditions state that, for some constant \(W_R\),
\[
\int K_{s+2}(x-y)\nu_R(dx) = W_R, \quad \text{for} \ y \in B_R.
\]
As \( d = s + 3 \) we obtain (using \( y = 0 \in B_R \))

\[
W_R = \frac{W_1}{R^{s+2}} = \frac{W_1}{R^{d-1}} \quad \text{and} \quad W_1 = C_{d,1}|S_1| \int_0^1 \frac{1 - (s+2)r^{d-1}}{\sqrt{1-r^2}} \, dr = \sqrt{\pi} \frac{\Gamma \left( \frac{d+1}{2} \right)}{\Gamma \left( \frac{d}{2} \right)}.
\]  

(2.5)

To derive the value of \( R \) we first integrate (2.2) with respect to \( \nu_R(dx) \) and swap the integrals, assuming that this is legal, giving

\[
-c_{d,s} \int \left( \int K_{s+2}(x-y) \nu_R(dx) \right) \mu_{eq}(dy) + 2\gamma d = 0.
\]  

(2.6)

Then, using (2.4) and (2.5) in (2.6), we get

\[
\frac{c_{d,s}\sqrt{\pi} \Gamma \left( \frac{d+1}{2} \right)}{\Gamma \left( \frac{d}{2} \right)} R^{-s-2} = 2\gamma d.
\]

Finally, from the formula \( z \Gamma(z) = \Gamma(z+1) \) with \( z = d/2 \), we derive the desired formula for \( R \), namely

\[
R = \left( \frac{c_{d,d-3}\sqrt{\pi} \Gamma \left( \frac{d+1}{2} \right)}{4\Gamma \left( \frac{d+1}{2} \right)} \right)^{\frac{1}{d-4}} = \left( \frac{c_{d+s,3}\sqrt{\pi} \Gamma \left( \frac{s+1}{2} \right)}{4\Gamma \left( \frac{s+1}{2} \right)} \right)^{\frac{1}{d-4}}.
\]  

(2.7)

See also Remark 2.4 for an alternative way to compute this critical value of \( R \).

2.2.2. Euler – Lagrange characterization. The probability measure \( \nu_R \) in (2.3) with \( R \) as in (2.7) satisfies the Frostman conditions (A.14) with kernel \( K_s \) and \( V = \gamma |\cdot|^2 \) thanks to Lemma 2.1 below.

**Lemma 2.1 (Potentials).** Let \( R \) be as in (2.7) and define \( \Phi := K_s \ast \nu_R + \gamma |\cdot|^2 \). Then,

- \( \Phi \) is continuous on \( \mathbb{R}^d \);
- \( \Phi = \Phi(0) = 0 \) on \( B_R \);
- \( \Phi \geq 0 \) outside \( B_R \).

**Proof.** As \( K_s \ast \nu_R \) is radially symmetric, so is \( \Phi \). Thus we define for any \( \lambda \geq 0 \),

\[
\varphi(\lambda) := \Phi(\lambda \overline{x}) \quad \text{for any} \quad \overline{x} \in \mathbb{R}^d \quad \text{with} \quad |\overline{x}| = 1.
\]  

(2.8)

Using from Lemma [A.1] that \( K_s \ast \nu_R \in L^1_{\text{loc}}(\mathbb{R}^d, dx) \) and we can swap the Laplacian and the Riesz potential, we get

\[
\Delta \Phi = -c_{s+3,s}K_{s+2} \ast \nu_R + 2d\gamma.
\]  

(2.9)

Moreover, \( \nu_R \) and the radius \( R \) have been chosen in the preceding subsection precisely in such a way that on \( \text{int}(B_R) := \{ x \in \mathbb{R}^d : |x| < R \} \),

\[
\Delta \Phi = -c_{s+3,s}K_{s+2} \ast \nu_R + 2d\gamma = 0.
\]  

(2.10)

**Continuity of \( \Phi \).** At this step, let us remark that when \( d < 6 \), Lemma [A.1] (iv) gives that \( K_s \ast \nu_R \) is continuous on \( \mathbb{R}^d \) since \( \frac{d-6}{4} \geq 2 > p > d/(d-s) = d/3 \) (recall that \( s = d-3 \)).

Actually \( K_s \ast \nu_R \) is continuous on \( \mathbb{R}^d \) for arbitrary dimension \( d \). Indeed, this can be checked directly, in the case \( (d,s) = (2,-1) \), since \( K_{-1} \) is not singular. This can also be checked directly in the case \( (d,s) = (3,0) \) from the formula provided by Lemma [B.1]. Finally, in the case \( s = d-3 > 0 \), using Lemma [B.1] and Lemma [B.2] and the change of variable \( r = \sin(\theta) \) to remove the singularity at the edge \( r = 1 \), we get, for \( x \in \mathbb{R}^d \), with \( \lambda = |x|/R \), for some constant \( C_s > 0 \),

\[
(K_s \ast \nu_R)(x) = C_s \int_0^{\pi} 2F_1 \left( \frac{s}{2} + 1, \frac{s}{2}; s + 2; -\frac{4\lambda \sin(\theta)}{(\lambda + \sin(\theta))^2} \right) \sin(\theta)^{s+2} \frac{\sin(\theta)^{s+2}}{(\lambda + \sin(\theta))^{s+2}} \sin(\theta) d\theta.
\]  

(2.11)

The continuity of \( K_s \ast \nu_R \) follows then from the uniform continuity of the hypergeometric function. Indeed, by [A.1] the series that defines \( 2F_1(a,b;c;z) \) converges absolutely for \( z \in [0,1] \) (and remarkably for \( z = 1 \)) when \( c - a - b > 0 \) as in our case \( c - a - b = 1 \). The hypergeometric function \( 2F_1 \) \( \left( \frac{3}{2} + 1, \frac{3}{2}; s + 2; z \right) \) is uniformly continuous on \([0,1]\) since it is clearly analytic on \([0,1]\) and it is also continuous at \( z = 1 \); the latter assertion follows from Abel’s Limit Theorem [1 Sec. 2.5] and the fact that \( 2F_1 \left( \frac{1}{2} + 1, \frac{3}{2}; s + 2; 1 \right) \) is finite. Furthermore \( \lambda = 0 \) is not a problem as soon as we establish the fact that \( \Phi \) is harmonic (in fact constant) in the unit disk (see below!).
Constantness on $B_R$. It follows from Lemma 2.2.

Behavior outside $B_R$. Since $\varphi$ defined in (2.8) is continuous on $[0, +\infty)$ and differentiable on $(1, +\infty)$, the Frostman condition outside $B_R$ is realized if we show that $\varphi'(\lambda) \geq 0$ for $\lambda > 1$.

Let us first consider the case $(d, s) = (2, -1)$. By Lemma B.3 $\varphi$ is convex on $[1, +\infty)$ since

$$
\varphi'(\lambda) = R \sqrt{1 - \frac{s}{2\lambda}} - R \frac{\arcsin \left( \frac{\lambda}{2s} \right)}{2\lambda} + 2\gamma R^2 \geq 0 - \frac{\pi}{8\gamma} \frac{\lambda^2}{2} + 2\frac{\pi^2}{64\gamma} = 0, \quad \lambda > 1.
$$

(2.12)

Moreover since (B.8) gives $\lim_{\lambda \to +\infty} \varphi'(\lambda) = -\frac{\pi}{16\gamma} \frac{\lambda^2}{2} + 2\frac{\pi^2}{64\gamma} = 0$, it follows that $\varphi'(\lambda) \geq 0$ when $\lambda > 1$.

Finally, consider the case $(d, s) = (3, 0)$. By Lemma B.4 if $\lambda > 1$,

$$
\varphi'(\lambda) = \frac{2\sqrt{\lambda^2 - 1}}{3\gamma \lambda^2} \geq 0.
$$

(2.13)

Let us now consider the case $s = d - 3 > 0$. We can rewrite (2.11) as

$$
\varphi(\lambda) = C_s \int_0^1 h(\lambda, r) \frac{r^{s-1}}{\sqrt{1 - r^2}} \, dr,
$$

(2.14)

where

$$
h(\lambda, r) := 2F_1 \left( \frac{s}{2} + 1, \frac{s}{2}; s + 2; \frac{4\lambda r}{(\lambda + r)^2} \right) (\lambda + r)^{-s}
$$

(2.15)

$$
= 2F_1 \left( \frac{s}{4} + \frac{s}{4}; \frac{s}{2} + 2; \frac{4\lambda^2 r^2}{(\lambda^2 + r^2)^2} \right) (\lambda^2 + r^2)^{-\frac{s}{2}}
$$

(2.16)

where the last equality comes from the quadratic transformation (A.6). Differentiating (2.15) with

$$
z = \frac{4\lambda r}{(\lambda + r)^2}, \quad \frac{\partial z}{\partial \lambda} = \frac{4r(\lambda - r)}{(\lambda + r)^3}
$$

and using the derivative formula (A.9) for $2F_1$, we get

$$
\frac{\partial h}{\partial \lambda}(\lambda, r) = sr(r - \lambda)(\lambda + r)^{-s-3} 2F_1 \left( \frac{s}{2} + 2, \frac{s}{2} + 1; s + 3; \frac{4\lambda r}{(\lambda + r)^2} \right)
$$

$$
- s(\lambda + r)^{-s-1} 2F_1 \left( \frac{s}{2} + 1, \frac{s}{2}; s + 2; \frac{4\lambda r}{(\lambda + r)^2} \right).
$$

(2.17)

The only potential difficulties are when the argument of the hypergeometric functions is $z = 1$. As before $z \in [0, 1]$ and $z = 1 \iff \lambda = r$. The parameters in the first hypergeometric function in (2.17) satisfy $c - a - b = 0$, so by the property (A.5) of $2F_1$, we get

$$
\lim_{\lambda \to r^+} (r - \lambda) 2F_1 \left( \frac{s}{2} + 2, \frac{s}{2}; s + 3; \frac{4\lambda r}{(\lambda + r)^2} \right) = 0.
$$

As before, the second hypergeometric function in (2.17) has parameters which satisfy $c - a - b = 1 > 0$. Thus $\frac{\partial h}{\partial \lambda}(\lambda, r)$ is uniformly continuous for $r \in [0, 1]$ and $\lambda \geq 0$, so by the Leibniz integral rule

$$
\varphi'(\lambda) = C_s \int_0^1 \tilde{\varphi}'(\lambda, \sin(\theta)) \sin(\theta)^{d-1} \, d\theta
$$

and $\varphi'(\lambda)$ is continuous for $\lambda \geq 0$. In particular

$$
\lim_{\lambda \to +1^+} \varphi'(\lambda) = \varphi'(1).
$$

Let us show now that $\varphi'(\lambda) \geq 0$ for $\lambda \geq 0$.

Since $s + 2 = d - 1 > d - 2$, the function $K_{s+2} \ast \nu_R$ is subharmonic outside the support $B_R$ of $\nu_R$, see for instance [23, Th. I.1.4 p. 66]. Since it is continuous everywhere in $\mathbb{R}^d$ even at $\infty$, it follows by the maximum principle applied on the complement of $B_R$, that for $|x| \geq R$,

$$
I(\nu_R) \geq U^{\nu_R}(x) = \int_{\mathbb{R}^d} \frac{1}{|x - y|^{d+2}} \nu_R(\, dy)
$$

(2.18)
(equality holds for $|x| = R$ by Theorem 1.1. It follows by using (1.5) and (2.10) that $\Delta \Phi(x) \geq 0$ for $|x| > R$. Next, using the radial form of the Laplacian,

$$\frac{1}{\lambda^{d-1}} (\lambda^{d-1} \varphi'(\lambda))' \geq 0 \quad \text{for} \quad \lambda > 1.$$  

Thus $\int_0^\lambda [\tau^{d-1} \varphi'(\tau)]' \, d\tau \geq 0$ for $\lambda \geq \rho > 1$, and so

$$\lambda^{d-1} \varphi'(\lambda) \geq \rho^{d-1} \varphi'(\rho).$$

Finally, letting $\rho \to 1^+$ we get, as $\varphi'(1) = 0$,

$$\varphi'(\lambda) \geq 0 \quad \text{for} \quad \lambda \geq 1.$$  

We also know that $\varphi$ is constant for $0 \leq \lambda \leq 1$, so $\varphi'(\lambda) \geq 0$ for $\lambda \geq 0$. □

**Lemma 2.2** (Laplacian inversion or Liouville lemma). Let $\Phi : \text{int}(B_R) \to \mathbb{R}$, $d \geq 2$, $R > 0$. If

- (local integrability) $\Phi \in L^1_\text{loc}(d\nu)$;
- (weak harmonicity) $\Delta \Phi = c$ for a constant $c$, in the sense of Schwartz distributions;
- (radial symmetry) $\Phi$ is equal to a constant on the sphere $S_r$ for all $r < R$;

then $\Phi$ is $C^\infty$ and is given by $\Phi = \frac{c}{2\pi} |\cdot|^2 + \Phi(0)$. In particular $\Phi$ is constant when $c = 0$.

Related statements can be found in [32 Sec. 0.3] for $d = 2$, and in [33 Th. 3.3, Ch. III, p. 183].

**Remark 2.3** (Extension). Lemma 2.2 extends to the case where $\Delta \Phi$ is a $C^\infty$ radial function on $B_R$, say $\Delta \Phi = A(|\cdot|)$. Indeed the same proof gives $\Phi = \Phi(0) + B(|\cdot|)$, where $B$ solves $rB''(r) + (d-1)B'(r) = A(r)$, $0 < r < R$, with $B(0) = B'(0) = 0$, which gives $B(r) = \int_0^r u^{d-1} A(v)dv$.

Thus

$$B(r) = \int_0^r (k(r) - k(v))u^{d-1}A(v)dv \quad \text{with} \quad k(v) := \begin{cases} \frac{c^{\frac{2-d}{2}}}{d-2} & \text{if} \ d \neq 2 \\ \log(v) & \text{if} \ d = 2. \end{cases}$$

If $A$ is a polynomial of degree $m$, then $B$ is a polynomial of degree $m + 2$, while if $A$ is a hypergeometric series, then $B$ is also a hypergeometric series. For an arbitrary integer $m \geq 1$, repeating this procedure gives a symmetric polynomial in $d$ variables $\Phi$ such that $\Delta^m \Phi = c$. See for instance [16] and references therein for a link with Jacobi and Zernike orthogonal polynomials and hypergeometric functions.

**Proof of Lemma 2.2**. By a version of the Weyl lemma expressing the Hörmander hypoellipticity of the Laplacian operator, see for instance Stroock’s expository note [30], we get that $\Phi$ is $C^\infty(\text{int}(B_R))$. Next, by radial symmetry $\Phi(x) = \psi(r)$ where $r = |x|$. Using $\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \partial_r + \frac{1}{r^2} \Delta_{S_1}$ we get that

$$c = \Delta \Phi(x) = \psi''(r) + \frac{d-1}{r} \psi'(r) = \frac{(r^{d-1} \psi'(r))'}{r^{d-1}},$$

and thus $r^{d-1} \psi'(r) = \frac{c}{d-2} r^{d}$ (note that we use here the fact that $d > 1$ to get that $r^{d-1} \psi'(r) = 0$ when $r \to 0$). Hence $\psi(r) = \frac{c}{2d} r^2 + \psi(0)$. Note that $\Delta \Phi = c = \Delta(\frac{c}{2d} |\cdot|^2)$ gives $\Delta(\Phi - \frac{c}{2d} |\cdot|^2) = 0$. □

### 2.2.3. Completion of proof. To complete the proof of Theorem 1.1 note that we can reinterpret (2.2) as Frostman conditions (A.14): $\mu_{eq} = \nu_R$ is seen as an equilibrium measure for kernel $\tilde{K} = K_{s+2}$ with external field $\tilde{V}$ equal to 0 on $B_R$ and to $+\infty$ outside, connecting with Theorem 1.1. □

**Remark 2.4** (Alternative motivation of the Frostman condition on $B_R$). Following [15 Lemma 2.3] or [8 Sec. 4], Riesz’s formula (1.6) with $d \geq 2$, $d - 2 < s < d$, $R > 0$ gives, using (A.15) and (A.17), for $x \in \text{int}(B_R)$,

$$\Delta \int_{|y| \leq R} \frac{dy}{|x-y|^{s-2}(R^2 - |y|^2)^{\frac{d-2}{2}}} = \int_{|y| \leq R} \frac{c_{d,s-2} dy}{|x-y|^{s}(R^2 - |y|^2)^{\frac{d-2}{2}}} = \frac{c_{d,s-2} \pi^{d-2}}{\Gamma(\frac{d}{2}) \sin((d-s)\frac{\pi}{2})}.$$  

(2.19)
Now, inverting the Laplacian as in Lemma 2.2 and using Lemma 2.1 for continuity at the boundary, we get
\[
\int_{|y| \leq R} \frac{dy}{|x-y|^{s-2}(R^2 - |y|^2)^{d-2}} = \frac{\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2}) \sin((d-s)\pi)} \left( \frac{c_{d,s-2}|x|^2}{2d} + \frac{d-s}{2}R^2 \right).
\] (2.20)

Replacing \( s - 2 \) by \( s \) gives, for \( d \geq 2 \) and \( d - 4 < s < d - 2 \), \( R > 0 \), \( x \in B_R \),
\[
\int_{|y| \leq R} \frac{dy}{|x-y|^{s}(R^2 - |y|^2)^{d-2}} = \frac{\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2}) \sin((d-s)\pi)} \left( \frac{c_{d,s}|x|^2}{2d} - \frac{d-s-2}{2}R^2 \right).
\] (2.21)

The left-hand side can be normalized using the fact that for \( 0 < \beta < 1 \),
\[
Z_\beta := \int_{|y| \leq R} \frac{dy}{|x-y|^{s}\sqrt{R^2 - |y|^2}} = |S| R^{d-2}\beta \int_0^1 \frac{r^{d-1} dr}{(1-r^2)^\beta} = \frac{R^{d-2}\beta \pi^{\frac{d}{2}} \Gamma(1-\beta)}{\Gamma(1-\beta + \frac{d}{2})}.
\] (2.22)

Indeed, with \( d - s = 3 \) and \( \beta = \frac{d-s}{2} - 1 = \frac{1}{2} \), we get \( Z_\frac{1}{2} = \frac{R^{d+s+4}}{2R^d\Gamma(\frac{d+2}{2})} \), and (2.21) gives
\[
\frac{\Gamma(\frac{d+2}{2})}{R^{d+s} \pi^\frac{d+1}{2}} \int_{|y| \leq R} \frac{dy}{|x-y|^{s}\sqrt{R^2 - |y|^2}} + \frac{\Gamma(\frac{d+2}{2}) \sqrt{\pi} e^{\frac{s}{4}}}{4R^{d+2} \Gamma(\frac{d+4}{2})} |x|^2 = \frac{\Gamma(\frac{d+2}{2}) \sqrt{\pi}}{2R^d \Gamma(\frac{d+2}{2})}.
\] (2.23)

When \( R \) is equal to the critical value \( \frac{d-1}{2} \), the prefactor of \( |x|^2 \) in (2.22) is equal to \( \gamma \) and \( (d,s) \) becomes the Frostman condition on \( B_R \) for Theorem 1.4. Note also that taking \((d,s) = (3,0)\) in (2.23) is allowed but produces a trivial kernel inside the integral in the left-hand side. It is also possible to take \( d = 3 \) and \( s \to 0 \) while keeping \( s \neq 0 \), and use, for \( x \neq 0 \),
\[
\lim_{s \to 0} \frac{1}{s} c_{3,s} = 1 \quad \text{and} \quad \lim_{s \to 0} \left( \frac{1}{s} - \frac{1}{s} \right) |x|^s - \frac{1}{s} = \lim_{s \to 0} \frac{1}{s} |x|^s - \frac{1}{s} = -\log|x|.
\] (2.24)

to recover the logarithmic kernel in this case. In another direction, note also that repeating the process that we used to get (2.21) to reach higher powers provides a family of generalizations of (2.21) involving Jacobi polynomials in the right-hand side, and even more generally hypergeometric \( 2F_1 \) functions, see for instance [16, 8, 19], producing potential extensions of Theorem 1.3.

Remark 2.5 (Hypergeometric formulas outside \( B_R \). As pointed out by the referee we can give hypergeometric formulas for the integrals \((1.11)\) and \((1.12)\) when \( \lambda = |x|/R \geq 1 \). More precisely, for \( d \in \{2,3,4,\ldots\}, d - 2 < s < d, \lambda \geq 1 \),
\[
\int_{|y| \leq R} \frac{|x-y|^{-s}}{(R^2 - |y|^2)^{d-2}} dy = \frac{1}{\lambda^s} \left( \frac{\pi^{\frac{d}{2}} \Gamma(s-d+2)}{\Gamma(s+2)} \right) 2F_1 \left( \frac{s}{2}, \frac{s-d+2}{2}; \frac{s+2}{2}; \frac{1}{\lambda^2} \right)
\] (2.25)

and, using the function \( S_{d-3} \) defined in (1.15),
\[
\int_{r^d} \frac{4\lambda r}{(\lambda + r)^2} (\lambda + r)^{d-3} dr = \frac{1}{2\lambda^{d-3}} \left( \frac{\Gamma(d+1)}{\Gamma(d-1)} \right) 2F_1 \left( \frac{d-3}{2}, \frac{d-1}{2}; \frac{d+1}{2}; \frac{1}{\lambda^2} \right).
\] (2.26)

Indeed, to get (2.25), we start from the integral in the left hand side of (1.11) by using, for \( 0 \leq r < \lambda \),
\[
2F_1 \left( \frac{s}{2}, \frac{s+2}{2}; \frac{d}{2}; \frac{4\lambda^2 r^2}{(\lambda^2 + r^2)^2} \right) = \frac{(\lambda^2 + r^2)^\frac{d}{2}}{\lambda^s} 2F_1 \left( \frac{s}{2}, \frac{s-d+2}{2}; \frac{d}{2}; \frac{r^2}{\lambda^2 \lambda^s} \right).
\] (2.27)

which comes from the quadratic transformation \( \text{A.47} \) with \( z = r^2/\lambda^2 < 1 \). With this replacement the left-hand side of \( 1.11 \) takes the form of the following Euler beta integral formula (see [20] and \[3\] eq. (2.2.2))
\[
\int_0^\infty u^{a-1}(1-u)^{b-1} 2F_1(a_1,a_2;b_1;tu) du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} 2F_1 \left( a_1, a_2; b_1; \gamma; t \right),
\] (2.28)
which comes from the quadratic transformation \( (A.8) \) after cancellation of upper and lower \( _2F_2 \) parameters. Similarly, to get \( (2.20) \), we start from the integral on the left-hand side of \( (1.14) \) by using, for \( 0 \leq r < \lambda \), the relation\(^5\)

\[
2F_1\left( \frac{s}{2} + \frac{3}{2}; s + 2; \frac{4\lambda r}{(\lambda + r)^2} \right) = \frac{(\lambda + r)^s}{\lambda^s} 2F_1\left( \frac{s}{2}, -\frac{1}{2}; \frac{s + 3}{2}; \frac{r^2}{\lambda^2} \right),
\]

which comes from the quadratic transformation \( (A.8) \) with \( z = r/\lambda < 1 \). This leads to \( (2.26) \) via \( (2.28) \).

2.3. Proof of Corollary 1.5 The function \( \varphi \) defined in \( (2.8) \) is continuous on \([0, 1]\) and differentiable on \((0, 1)\). From the proof of Theorem 1.4, the Frostman condition states that \( \varphi \) is constant and equal to \( \varphi(0) \) on \( [0, 1] \), namely \( \varphi'(\lambda) = 0 \) for \( \lambda \in (0, 1) \). Now the formula \( (1.14) \) in Corollary 1.5 comes from the combination of equation \( \varphi(\lambda) = \varphi(0) \) together with the formulas for \( \varphi \) provided by Lemmas \( B.1 \) and \( B.2 \) of Appendix \( B \). Note that \( (1.14) \) is trivial when \( d = 3 \). The formula \( (1.16) \) is obtained from \( (1.14) \) by using \( (1.15) \) and the Legendre duplication formula \( (A.2) \).

2.4. Proof of Corollary 1.6 Let us keep the notation used in Appendix \( B \). First of all, the formulas \( (1.17) \) \( (1.18) \) in Corollary 1.6 come from the Frostman condition \( \varphi(\lambda) = \varphi(0) \) and its reformulation \( \varphi'(\lambda) = 0 \), and the formula for \( \varphi \) provided by Lemma \( B.1 \). The formula \( (1.19) \) in Corollary 1.6 is obtained by further reformulating \( \varphi \) when \( (d, s) = (2, -1) \) in terms of special functions using Lemma \( B.3 \) below. The formula for \( \varphi' \) provided by this lemma gives

\[
\int_0^1 \left( \lambda + r \right) E \left( \frac{4\lambda r}{(r + \lambda)^2} \right) + (\lambda - r) K \left( \frac{4\lambda r}{(r + \lambda)^2} \right) \frac{r \, dr}{\sqrt{1 - r^2}} = \frac{\pi^2}{4} \lambda^2.
\]

Next, following \( [22, 21, 2] \), the Landen transform for \( E \) and \( K \) gives, for \( z \in [-1, 1] \),

\[
K\left( \frac{4z}{(1 + z)^2} \right) = (1 + z) K(z^2) \quad \text{and} \quad E\left( \frac{4z}{(1 + z)^2} \right) = \frac{2}{1 + z} E(z^2) - (1 - z) K(z^2).
\]

Now, the formula \( (1.20) \) of Corollary 1.6 comes by combining \( (1.19) \) and \( (2.30) \) with \( z = \xi \).

APPENDIX A. USEFUL TOOLS

Let us recall the Euler reflection formula for the Gamma function, valid for \( z \notin \{-1, -2, \ldots\} \),

\[
\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.
\]

and the Legendre duplication formula, valid for \( 2z \notin \{-0, -1, -2, -3, \ldots\} \),

\[
\sqrt{\pi} \Gamma(2z) = 2^{2z - 1} \Gamma(z) \Gamma\left( z + \frac{1}{2} \right).
\]

A.1. Hypergeometric Identities.

- The hypergeometric function \( _2F_1 \) can be written as (see [12] (15.2(i)))

\[
_2F_1\left( a, b; c; z \right) := \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a + k) \Gamma(b + k)}{\Gamma(c + k)} \frac{z^k}{k!}.
\]

- If \( \Re(c - a - b) > 0 \) then \( (A.3) \) converges absolutely for \( |z| \leq 1 \) and (see [12] (15.4.20))

\[
_2F_1\left( a, b; c; 1 \right) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}.
\]

- If \( c = a + b \) then ([12] (15.4.21))

\[
\lim_{z \to 1^-} \frac{2F_1\left( a, b; a + b; z \right)}{- \log(1 - z)} = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)}
\]

\(^5\)The relation \( (2.29) \) can also be derived using Erdélyi [13, 3.3(13)] which is Whipple’s relation between the Legendre functions of the first and second kinds, rewritten in terms of \( _2F_1 \). Both the first-kind \( P_\nu^m \) and the second-kind \( Q_\nu^m \) have \( _2F_1 \) representations; see [12] 14.3.6 and 14.3.7.
• Quadratic transformation (see [13], (15.8.13) or [14], 2.11.4]): if $|\text{phase}(1 - z)| < \pi$ then
  \[ 2F_1\left(\frac{a}{2}, \frac{1}{2} + \frac{a}{2} \mid \frac{1}{2} + b; \frac{z^2}{(1 - z)^2}\right) = (1 - \frac{z}{2})^a 2F_1(a, b; 2b; z). \] (A.6)

• Quadratic transformation ([14], 2.11.34)): if $0 \leq z \leq 1$ then
  \[ 2F_1\left(\frac{a}{2}, \frac{a + 1}{2}; a - b + 1; \frac{4z}{(1 + z)^2}\right) = (1 + z)^a 2F_1(a, b; a - b + 1; z) \] (A.7)

(there is a typo in [14], 2.11.34]): $a - b - 1$ has been corrected here to $a - b + 1$.

• Quadratic transformation (see [14], 2.11.5)): if $0 \leq z \leq 1$ then
  \[ 2F_1\left(a, b; 2b; \frac{4z}{(1 + z)^2}\right) = (1 + z)^2a 2F_1\left(a, a + \frac{1}{2} - b; b + \frac{1}{2}; z^2\right). \] (A.8)

• Derivative formula (see [12], (15.5.1)):
  \[ \frac{d}{dz} 2F_1(a, b; c; z) = \left(\frac{ab}{c}\right) 2F_1(a + 1, b + 1; c + 1; z). \] (A.9)

• Euler integral formula (see [3], p. 4-5] and [12], (15.6.1)):
  \[ 2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 \frac{u^{b-1}(1 - u)^{c-b-1}}{(1 - zu)^a} du, \] (A.10)

provided that $\Re(b) > 0$, $\Re(c) > 0$, and $|\text{phase}(1 - z)| < \pi$.

A.2. Funk–Hecke formula. Let $d \geq 2$ and $\sigma_{S_1}$ denote the uniform probability measure on the unit centered sphere $S_1 = \{x \in \mathbb{R}^d : |x| = 1\}$. Then, for $z \in \mathbb{R}^d$ with $|z| = 1$,

\[ \int_{S_1} f(z \cdot x) \sigma_{S_1}(dx) = \tau_{d-1} \int_0^\pi f(\cos(\theta)) \sin^{d-2}(\theta) d\theta = \tau_{d-1} \int_{-1}^1 f(t)(1 - t^2)^{\frac{d-1}{2}} \frac{dt}{\sqrt{1 - t^2}} \] (A.11)

where

\[ \tau_{d-1} := \left(\int_0^\pi \sin^{d-2}(\theta) d\theta\right)^{-1} = \left(\int_{-1}^1 (1 - t^2)^{\frac{d-1}{2}} \frac{dt}{\sqrt{1 - t^2}}\right)^{-1} = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \Gamma\left(\frac{d-1}{2}\right). \] (A.12)

The Funk–Hecke formula (A.11) is a useful tool to reduce multivariate integrals into univariate integrals. It gives the projection on any diameter of the uniform law on the sphere. If $X$ is a random vector in $\mathbb{R}^d$ uniformly distributed on $S_1$ then for $z \in S_1$, the law of $z \cdot X$ has density $\tau_{d-1}(1 - t^2)^{\frac{d-1}{2}} \mathbf{1}_{t \in [-1, 1]}$. This is an arcsine law when $d = 2$, a uniform law when $d = 3$, a semicircle law when $d = 4$, and more generally, for an arbitrary $d \geq 2$, the image by the map $u \mapsto \sqrt{u}$ of the beta law $\text{Beta}\left(\frac{1}{2}, \frac{d-1}{2}\right)$. We refer to [27], p. 18 or [6], Eq. (5.1.9) p. 197] for a proof.

A.3. Euler–La grille characterization of equilibrium measure (Frostman conditions). For $\mu \in \mathcal{M}_1$ such that $K_s(x)1_{|x| > 1}1_{s \leq 0} \in L^1(\mu)$, we define the $s$-Riesz potential at point $x \in \mathbb{R}^d$ by

\[ U^s(x) := (K_s * \mu)(x) = \int K_s(x - y)\mu(dy) \in (-\infty, +\infty]. \] (A.13)

The Euler–Lagrange characterization of the equilibrium measure $\mu_{eq}$, also known as Frostman conditions in potential theory, states that a necessary and sufficient condition for such an element $\mu$ of $\mathcal{M}_1$ to be an equilibrium measure is that for some finite constant $c$ we have (see, for example, [28])

\[ U^s + V \begin{cases} \leq c & \text{on the support of } \mu \\ \geq c & \text{quasi-everywhere on } \mathbb{R}^d \end{cases} \] by “quasi-everywhere” we mean except on a set for which every probability measure supported on it has infinite energy. This condition holds everywhere when $V$ is continuous. It is customary to say that $c$ is the modified Robin constant and we have $c = \int U^s_{eq} d\mu_{eq} + \int V d\mu_{eq} = I(\mu_{eq}) - \int V d\mu_{eq}$.\]
A.4. Integrability and regularity of Riesz potentials. The following Lemma summarizes key regularity properties of the Riesz kernel, some of which are classical. We give a proof for the reader’s convenience. On this topic, we also refer to the works of Mizuta such as [21, 23, 26].

Lemma A.1 (Integrability and regularity of Riesz potentials). (i) $K_s \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$ if and only if $s = 0$ or $s \neq 0$ and $s < d$.

(ii) If $s < d - 2$ then, in the sense of distributions, and in the sense of functions on $\{x \in \mathbb{R}^d : x \neq 0\}$,

$$
\Delta K_s = -c_{d,s} K_{s+2} \quad \text{where} \quad c_{d,s} := \begin{cases} |s|(d - 2 - s) & \text{if } s \neq 0 \\ d - 2 & \text{if } s = 0 \end{cases}. \quad (A.15)
$$

(iii) Suppose that $s = 0$ or $s \neq 0$ and $s < d$. Let $\mu$ be a compactly supported probability measure on $\mathbb{R}^d$. Then the following function is well defined and belongs to $L^1_{\text{loc}}(\mathbb{R}^d, dx)$:

$$
x \in \mathbb{R}^d \mapsto (K_s * \mu)(x) := \int K_s(x - y) \mu(dy). \quad (A.16)
$$

Moreover, in the sense of distributions,

$$
\Delta(K_s * \mu) = (\Delta K_s) * \mu = -c_{d,s} K_{s+2} * \mu. \quad (A.17)
$$

(iv) Suppose that $s = 0$ or $s \neq 0$ and $s < d$. If $\mu$ is a compactly supported probability measure on $\mathbb{R}^d$ such that $\mu(dx) = f(x)dx$, $f \in L^p(\mathbb{R}^d, dx)$, and $p > d/(d - s)$, then $K_s * \mu$ is continuous on $\mathbb{R}^d$.

Note that $K_{s+2} \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$ implies $s + 2 < d$; in other words $s < d - 2$. Furthermore, the condition $s < d - 2$ is sharp for (A.15), indeed; in the sense of distributions, we have $\Delta K_{d-2} = -c_d \delta_0$ (Coulomb kernel). This suggests defining $K_0 := \delta_0$ to make the formula (A.15) valid for the critical case $s = d - 2$, provided that we also set $c_{d,d-2} := c_d$.

We remark that (A.15) is a special case of (A.17) which corresponds to taking $\mu = \delta_0$ and that (A.17) goes beyond [35, Eq. (7) p. 118] and [18, Eq. (85) p. 136]. Note also that the distribution $\Delta \mu$ equals the convolution $(\Delta \delta_0) * \mu$, see [31] end of Ch. VI, Sec. 3; notably eq. (VI. 3; 34–35)]. From this point of view, it follows that (A.17) is a consequence of the associative law for convolution of three distributions, two of which have compact support, see [34, Ch. VI, Sec. 3, Th. VII] and [23, Lemma 0.6]. We give however a direct short proof of (A.17) below.

Proof of Lemma A.1. Proof of (i). It suffices to check local integrability in the neighborhood of the origin. We have

$$
\int_{|x| \leq 1} |K_s(x)| dx = \begin{cases} 2\pi \int_0^1 r^{-s+d-1} dr < \infty & \text{if } s \neq 0 \text{ and } d > s \\ 2\pi \int_0^1 \log(r)r^{d-1} dr < \infty & \text{if } s = 0 \end{cases}.
$$

Proof of (ii). On $\mathbb{R}^d \setminus \{0\}$, the function $K_s$ is $C^\infty$ and a computation reveals that

$$
\Delta K_s = -c_{d,s} K_{s+2}.
$$

It follows that this equality also holds in the sense of distributions for test functions supported away from the origin. For general test functions, we proceed by integration by parts by parts outside a centered ball of small radius. Namely, let $\varphi$ be a compactly supported $C^\infty$ test function, and let $\varepsilon > 0$. By the Green integration by parts formula for the open set $\{x \in \mathbb{R}^d : |x| > \varepsilon\}$, denoting $n(x) = -|x|^{-1}$ the inner unit normal vector to the sphere $\{x \in \mathbb{R}^d : |x| = \varepsilon\}$ at the point $x$,

$$
\int_{|x| \geq \varepsilon} \Delta \varphi(x) K_s(x) dx - \int_{|x| \geq \varepsilon} \varphi(x) \Delta K_s(x) dx
$$

\[= \int_{|x| = \varepsilon} K_s(x) \nabla \varphi(x) \cdot n(x) d\sigma_\varepsilon(x) - \int_{|x| = \varepsilon} \varphi(x) \nabla K_s(x) \cdot n(x) d\sigma_\varepsilon(x).\]
If \( s \neq 0 \) and \( d > s + 1 \) then

\[
\int_{|x| = \varepsilon} K_s(x) \nabla \varphi(x) \cdot n(x) d\sigma_\varepsilon(x) = e^{-s} \int_{|x| = \varepsilon} \nabla \varphi(x) \cdot n(x) d\sigma_\varepsilon(x) = e^{-s} O(\varepsilon^{d-1}) = O(\varepsilon^{-s+d-1}) = o_{\varepsilon \to 0+}(1),
\]

while, using \( \nabla K_s(x) = -s x|x|^{-(s+2)} = -s x K_{s+2}(x) \) and \( x \cdot n_x = -|x| \), if \( d > s + 2 \),

\[
\int_{|x| = \varepsilon} \varphi(x) \nabla K_s(x) \cdot n(x) d\sigma_\varepsilon(x) = s \varepsilon^{1-(s+2)} \int_{|x| = \varepsilon} \varphi(x) d\sigma_\varepsilon(x) = s \varepsilon^{1-(s+2)} O(\varepsilon^{d-1}) = O(\varepsilon^{d-(s+2)}) = o_{\varepsilon \to 0+}(1).
\]

Finally a careful analysis reveals that the conditions on \( d \) are the same in the case \( s = 0 \).

**Proof of (iii).** If \( s < 0 \), then \( K_s \leq 0 \), and hence \( K_s * \mu \) is well defined and takes its values in \([-\infty, 0]\). Similarly, if \( s > 0 \), then \( K_s \geq 0 \), and hence \( K_s * \mu \) is well defined and takes its values in \([0, +\infty]\). If \( s = 0 \) then \( K_0 \mathbf{1}_{|\cdot| \leq 1} \geq 0 \) while \( \sup_{\mathbb{R}^d} K_0 \mathbf{1}_{|\cdot| \geq 1}/\log(1 + |\cdot|) < \infty \), hence \( K_0 * \mu \) is well defined and takes its values in \((\infty, +\infty]\). Next, by the Fubini–Tonelli theorem, for \( R > 0 \), using (i) and the compactness of support of \( \mu \) (note that this can be weakened into integrability of \( \log(1 + |\cdot|) \mathbf{1}_{s=0} \)),

\[
\int \int |K_s(x-y)|1_{|x| \leq R} \mu(dy) dx = \left( \int |K_s(x)|1_{|x| \leq R} dx \right) \mu(dy) < \infty.
\]

It follows that \( K_s * \mu \) belongs to \( L^1_{\text{loc}}(\mathbb{R}^d, dx) \).

For the differentiability, let \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be a \( C^\infty \) and compactly supported test function. By the Fubini–Tonelli theorem, the Green integration by parts formula, and (ii), we have

\[
\int (K_s * \mu)(x) \Delta \varphi(x) dx = \int \left( \int K_s(x-y) \mu(dy) \right) \Delta \varphi(x) dx = \int \left( \int K_s(x-y) \Delta \varphi(x) dx \right) \mu(dy)
\]

\[
= -c_{d,s} \int \left( \int K_{s+2}(x-y) \varphi(x) dx \right) \mu(dy)
\]

\[
= -c_{d,s} \int \varphi(x) \left( \int K_{s+2}(x-y) \mu(dy) \right) dx
\]

\[
= -c_{d,s} \int \varphi(x) (K_{s+2} * \mu)(x) dx.
\]

**Proof of (iv).** For the continuity, we follow closely the cutoff argument used in [9, Lem. 4.3], see also [24, Th. 1], [23], and [26, Sec. 5.3]. Namely, let us consider first the case \( s > 0 \). For \( n \geq 1 \) and \( x \in \mathbb{R}^d \), let us define

\[
R_n(x) := \int f(y) K_s(x-y) 1_{|K_s(x-y)| \leq n} dy
\]

and

\[
T_n(x) := (K_s * \mu)(x) - R_n(x) = \int f(y) K_s(x-y) 1_{|K_s(x-y)| \geq n} dy.
\]

By the dominated convergence theorem, \( R_n \) is continuous on \( \mathbb{R}^d \). Let us show now that \( \lim_{n \to \infty} T_n = 0 \) uniformly on compact subsets, which will prove the continuity of \( K_s \). Let \( q := p/(p-1) \) be the Hölder conjugate exponent of \( p \). Now, by the Hölder inequality, using the fact that \( K_s = |\cdot|^{-s}, s > 0 \),

\[
0 \leq T_n(x) = \int f(y) \frac{1_{|x-y| \leq n^{-1/s}}}{|x-y|^q} dy \leq \|f\|_{p,B(x,1)} \varepsilon_n^{1/q}
\]

where \( B(x, r) := \{ x \in \mathbb{R}^d : |x| \leq r \} \) is the closed centered ball of radius \( r \), where \( \| \cdot \|_{p,C} \) denotes the \( L^p \) norm with respect to the trace of the Lebesgue measure on \( C \), where

\[
\varepsilon_n := |\mathbb{S}^{d-1}| \int_0 n^{-1/s} \frac{dr}{r^{q-2d+1}}.
\]
and where $|S|^{d-1}$ is the surface area of the unit sphere $\{x \in \mathbb{R}^d : |x| = 1\}$. The condition $p > d/(d-s)$, which is equivalent to $qs - d + 1 < 1$, ensures that $\varepsilon_n$ is finite for all $n$ and that $\lim_{n \to \infty} \varepsilon_n = 0$. Hence, if $C \subset \mathbb{R}^d$ is a compact set, then, denoting $C_1 := \{x \in \mathbb{R}^d : \text{dist}(x, C) \leq 1\}$, we have

$$\sup_{x \in C} |T_n(x)| \leq \|f\|_{p,K_1} \varepsilon_n^{1/q} \to 0,$$

which completes the proof of the continuity of $K_s \ast \mu$. The case $s < 0$ is entirely similar up to a sign. It remains to examine the case $s = 0$. Let us write $K_0 = K_0^+ - K_0^-$ with $K_0^+ \geq 0$, namely $K_0^+ = -\log |\cdot|_{1 \leq |\cdot| \leq 1}$ and $K_0^- = \log |\cdot|_{|\cdot| > 1}$. To establish the continuity of $K_0^+ \ast \mu$ we write

$$0 \leq T_n^+(x) := \int f(y) \log \frac{1}{|x-y|} 1_{|x-y| \leq 1} \rho_1 \rho_{-\varepsilon_n} dy \leq \|f\|_{p,B(1,1)} (\varepsilon_n^{1/q})$$

where

$$\varepsilon_n^+ = -|S|^{d-1} \int_0^{a-n} r^{d-1} \log(r) dr \to 0.$$

On the other hand, the continuity of $K_0^- \ast \mu$ follows from that of $K_0^+$. Hence $K_0 \ast \mu$ is continuous.

**Appendix B. Key formulas for potential plus external field**

Let $d, s, \mu_{eq} = \nu R$, and $\sigma_{S_1}$ be as in Theorem 1.4. For $x \in \mathbb{R}^d$, the quantity $\Phi(x) := (K_s \ast \mu_{eq})(x) + \gamma |x|^s$ depends only on $\lambda := |x|/R$ and we can define

$$\varphi(\lambda) := \Phi(x) = U^{\mu_{eq}}(x) + \gamma |x|^2 = U^{\mu_{eq}}(x) + \gamma R^2 \lambda^2. \quad (B.1)$$

The following lemmas provide key formulas for the potential plus external field $\varphi$.

**Lemma B.1** (Integral formula for potential). For $\lambda \geq 0$, denoting $c_d := \frac{2 \text{sign}(d-3)\Gamma(d+1)}{\pi \Gamma(d/2)},$

$$\varphi(\lambda) = \begin{cases} \frac{1}{R^{d-3}} \left( c_d \int_0^{\pi} \left( \int_0^{\pi} \frac{\sin^{d-2}(\theta)}{(\lambda^2 - 2r\lambda \cos(\theta) + r^2)^{\frac{d-1}{2}}} \frac{d\theta}{\sqrt{1 - r^2}} \right) \frac{d^d - 1}{\sqrt{1 - r^2}} \right) + \gamma R^{d-1} \lambda^2 & \text{if } d \neq 3 \\ \int_0^\lambda (\lambda + r)^{d-2} \log(\lambda + r) - (\lambda - r)^{d-2} \log(\lambda - r) \frac{dr}{\pi \lambda} - \log R + \frac{1}{2} + \gamma R^2 \lambda^2 & \text{if } d = 3 \end{cases}.$$

Note that $\gamma R^{d-1}$ does not depend on $\gamma$.

**Proof.** By the Funk–Hecke formula (A.11), for $x \in \mathbb{R}^d$, $s \neq 0$, with $C_d := \text{sign}(s) \frac{\Gamma(d/2)}{\pi \Gamma(d/2)}$

$$U^{\mu_{eq}}(x) = C_d \int_{|y| \leq 1} \frac{dy}{|x - R y|^{d-1}} \sqrt{1 - |y|^2}$$

$$= C_d |S_1| \int_0^1 \int_{S_1} \frac{\sigma_{S_1} (dy) r^{d-1} dr}{(2r^2 - 2r R(x, y) + r^2 R^2)^{\frac{d-1}{2}}} \sqrt{1 - \frac{r^2}{R^2}}$$

$$= C_d |S_1| \int_0^1 \left( \int_0^\lambda \frac{(1 - t^2)^{\frac{d-1}{2}}}{(\lambda^2 - 2t\lambda + t^2)^{\frac{d-1}{2}}} \frac{dt}{\sqrt{1 - t^2}} \right) \frac{d^d - 1}{\sqrt{1 - t^2}} dr$$

$$= \frac{2 \text{sign}(s) \Gamma(d+1)}{\pi \Gamma(d/2) R^{d-3}} \int_0^\lambda \left( \int_0^\pi \frac{\sin^{d+1}(\theta)}{(\lambda^2 - 2\lambda \cos(\theta) + r^2)^{\frac{d-1}{2}}} \frac{d\theta}{\sqrt{1 - r^2}} \right) \sqrt{1 - r^2} dr,$$  \hspace{1cm} (B.2)

while if $s = 0$ ($d = 3$),

$$U^{\mu_{eq}}(x) = -\frac{1}{\pi^2} \int_{|y| \leq 1} \frac{\log |x - R y|}{\sqrt{1 - |y|^2}} dy$$

$$= -\frac{2}{\pi} \int_0^1 \left( \int_{S_2} \frac{\log(\lambda^2 R^2 - 2r R(x, y) + r^2 R^2)}{\sigma_{S_1} (dy) r^{d-1} dr} \right) \frac{d^2 - 1}{\sqrt{1 - r^2}} dr$$

$$= -\frac{1}{\pi} \int_0^1 \left( 4 \log R + \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \log(\lambda^2 - 2\lambda t + r^2) dt \right) \frac{d^2 - 1}{\sqrt{1 - r^2}} dr.$$
Finally we observe that
\[ \int_{-1}^{1} \log(\lambda^2 - 2r\lambda + r^2) dt = \frac{(\lambda + r)^2 \log(\lambda + r) - (\lambda - r)^2 \log |\lambda - r|}{r\lambda} - 2. \]

**Lemma B.2** (Landen transform and a special function). For \( d \in \{2, 3, \ldots\}, \lambda \geq 0, \) and \( r \in [0,1], \)
\[ \int_0^\pi \frac{\sin^{d-2}(\theta)}{(\lambda^2 - 2r\lambda \cos(\theta) + r^2)^\frac{d-2}{2}} d\theta = \frac{2^{d-1}}{(\lambda + r)^{d-3}} S_{d-3}\left(\frac{4\lambda r}{(\lambda + r)^2}\right), \]
where for \( z \in [0,1], \)
\[ S_d(z) := \int_0^\pi \frac{\sin^{d+1}(\alpha) \cos^{d+1}(\alpha)}{(1 - z \sin^2(\alpha))^\frac{d+1}{2}} d\alpha = \int_0^\pi \frac{\sin^{d+1}(\alpha) \cos^{d+1}(\alpha)}{(1 - z \sin^2(\alpha))^\frac{d+1}{2}} d\alpha \]
\[ = \frac{2^{d+2}}{(\lambda^2 + r^2)^\frac{d+1}{2}} \int_0^\pi \frac{\sin^{d+1}(\alpha) \cos^{d+1}(\alpha)}{(1 - \rho_1(1 - 2 \sin^2(\alpha)))^\frac{d+1}{2}} d\alpha \]
\[ = \frac{2^{d+2}}{(\lambda^2 + r^2)^\frac{d+1}{2}} \int_0^\pi \frac{\sin^{d+1}(\alpha) \cos^{d+1}(\alpha)}{(1 - \frac{2\rho_1}{1 - \rho_1} \sin^2(\alpha))^\frac{d+1}{2}} d\alpha \]
\[ = \frac{2^{d+2}}{(\lambda + r)^{d+1}} S_d\left(\frac{\rho_2}{1 - \rho_1}\right). \]

Proof. We set \( \rho_1 := \frac{2\lambda^2}{\lambda^2 + r^2}, \rho_2 := \frac{2\rho_1}{1 - \rho_1} = \frac{4\lambda^2}{(\lambda + r)^2}, \) which gives \((\lambda^2 + r^2)(1 + \rho_1) = (\lambda + r)^2.\) Using the change of variable \( \theta = 2\alpha, \) and \( \cos(\theta) = 1 - 2\sin^2(\alpha), \) \( \sin(\theta) = 2\sin(\alpha)\cos(\alpha), \) we get
\[ \int_0^\pi \frac{\sin^{d+1}(\theta)}{(\lambda^2 - 2r\lambda \cos(\theta) + r^2)^\frac{d+1}{2}} d\theta = \frac{1}{(\lambda^2 + r^2)^\frac{d+1}{2}} \int_0^\pi \frac{\sin^{d+1}(\theta) \cos^{d+1}(\theta)}{(1 - \rho_1 \cos(\theta))^\frac{d+1}{2}} d\theta \]
\[ = \frac{2^{d+2}}{(\lambda^2 + r^2)^\frac{d+1}{2}} \int_0^\pi \frac{\sin^{d+1}(\theta) \cos^{d+1}(\theta)}{(1 - \rho_1(1 - 2 \sin^2(\alpha)))^\frac{d+1}{2}} d\theta \]
\[ = \frac{2^{d+2}}{(\lambda^2 + r^2)^\frac{d+1}{2}} \int_0^\pi \frac{\sin^{d+1}(\theta) \cos^{d+1}(\theta)}{(1 - \frac{2\rho_1}{1 - \rho_1} \sin^2(\alpha))^\frac{d+1}{2}} d\theta \]
\[ = \frac{2^{d+2}}{(\lambda + r)^{d+1}} S_d\left(\frac{\rho_2}{1 - \rho_1}\right). \]

But for \( z \in [0,1], \)
\[ S_d(-z) = \int_0^\pi \frac{\sin^{d+1}(\alpha) \cos^{d+1}(\alpha)}{(1 + z \sin^2(\alpha))^\frac{d+1}{2}} d\alpha = \frac{1}{(1 + z)^\frac{d+1}{2}} \int_0^\pi \frac{\sin^{d+1}(\alpha) \cos^{d+1}(\alpha)}{(1 - \frac{2\rho_1}{1 - \rho_1} \sin^2(\alpha))^\frac{d+1}{2}} d\alpha = \frac{1}{(1 + z)^\frac{d+1}{2}} S_d\left(\frac{\rho_2}{1 - \rho_1}\right). \]

In particular, with \( z = \frac{2\rho_1}{1 - \rho_1}, \) we get \( 1 + z = \frac{1 + \rho_1}{1 - \rho_1} \) and \( z = \frac{2\rho_1}{1 - \rho_1} = \rho_2; \) therefore
\[ S_d\left(\frac{z}{1 + z}\right) = \frac{2\rho_1}{1 - \rho_1} = \rho_2 \]

\[ S_d(z) = \frac{1}{2\pi} \int_0^1 \frac{u}{z} \frac{1 - u^2}{(1 - zu)^2} du \] and the Euler integral formula \((A.10).\) Note that we could alternatively proceed as in the proof of Corollary \( \Box \) via the Newton binomial series \((A.17).\)

**Lemma B.3** \((d = 2, s = -1).\) If \((d, s) = (2, -1),\) then
\[ \varphi(\lambda) = -\frac{1}{4\gamma} \int_0^1 (\lambda + r) E\left(\frac{4\lambda r}{(\lambda + r)^2}\right) \frac{r}{\sqrt{1 - r^2}} dr + \frac{\pi^2}{64\gamma} \lambda^2, \quad \lambda \geq 0, \]
\[ \varphi'(\lambda) = \frac{1}{8\gamma} \int_0^1 \left[ \left(1 + \frac{r}{\lambda}\right) E\left(\frac{4\lambda r}{(\lambda + r)^2}\right) + (1 - \frac{r}{\lambda}) K\left(\frac{4\lambda r}{(\lambda + r)^2}\right) \right] \frac{r}{\sqrt{1 - r^2}} dr + \frac{\pi^2}{32\gamma} \lambda, \quad \lambda > 0, \]
\[ \varphi(\lambda) = -\frac{\pi}{8\gamma} \lambda_2 F_1\left[ -\frac{1}{2}, \frac{3}{2}, \frac{1}{\lambda^2}\right] + \frac{\pi^2}{64\gamma} \lambda^2, \quad \lambda \geq 1 \]
\[ \varphi'(\lambda) = -\frac{\pi}{16\gamma} \left(1 - \frac{1}{\lambda^2} - \arcsin\left(\frac{1}{\lambda}\right)\right) + \frac{\pi^2}{32\gamma} \lambda, \quad \lambda > 1. \]

Proof. By combining Lemmas \( B.1 \) and \( B.2 \) with \( d = 2, \) we get, for \( \lambda \geq 0, \)
\[ \varphi(\lambda) = -\frac{R}{2\pi} \int_0^1 (\lambda + r) E\left(\frac{4\lambda r}{(\lambda + r)^2}\right) \frac{r}{\sqrt{1 - r^2}} dr + \gamma R^2 \lambda^2. \]
Next, by using the well-known ordinary differential equations (for $0 < z < 1$)

$$K'(z) = \frac{E(z) - (1 - z)K(z)}{2(1 - z)z} \quad \text{and} \quad E'(z) = \frac{E(z) - K(z)}{2z} \quad (B.4)$$

we get, after some algebra,

$$\varphi'(\lambda) = -\frac{R}{\pi} \int_0^1 \left[ \left( 1 + \frac{r}{\lambda} \right) E \left( \frac{4\lambda r}{(r + \lambda)^2} \right) + \left( 1 - \frac{r}{\lambda} \right) K \left( \frac{4\lambda r}{(r + \lambda)^2} \right) \right] \frac{r}{\sqrt{1 - r^2}} dr + 2\gamma R^2 \lambda. \quad (B.5)$$

By combining (1.10) with the quadratic transformation (A.8) we get, for $z \in [0, 1]$,

$$(1 + z)E\left( \frac{4z}{(1 + z)^2} \right) = (1 + z)^2 F_2F_1 \left( -\frac{1}{2}, \frac{1}{2}, 1; \frac{1}{2}, 1; \frac{4z}{(1 + z)^2} \right) = \pi^2 F_2F_1 \left( -\frac{1}{2}, -\frac{1}{2}, 1; z^2 \right). \quad (B.6)$$

Thus (B.3) gives, with $z = \frac{r}{\lambda} \in [0, 1]$, $r \in [0, 1]$, and $\lambda > 1$,

$$2 \pi \int_0^1 \left( 1 + \frac{r}{\lambda} \right) E \left( \frac{4\lambda r}{(\lambda + r)^2} \right) \frac{r}{\sqrt{1 - r^2}} dr = \frac{2}{\pi} \int_0^1 (1 + z)E \left( \frac{4z}{(1 + z)^2} \right) \frac{r}{\sqrt{1 - r^2}} dr$$

$$= \int_0^1 2F_1 \left( -\frac{1}{2}, -\frac{1}{2}, 1; z^2 \right) \frac{r}{\sqrt{1 - r^2}} dr$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2^2})^n}{n!^2} \left( \frac{1}{\lambda} \right)^{2n} \int_0^1 \frac{1}{\sqrt{1 - r^2}} dr$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2^2})^n}{n!^2} \left( \frac{1}{\lambda} \right)^{2n} 2F_1 \left( -\frac{1}{2}, -\frac{1}{2}, 3; \left( \frac{1}{\lambda} \right)^2 \right),$$

and therefore, using (B.3), we get, when $\lambda > 1$,

$$\varphi(\lambda) = -R\lambda 2F_1 \left( -\frac{1}{2}, -\frac{1}{2}, 3; \frac{1}{\lambda^2} \right) + \gamma R^2 \lambda^2, \quad (B.7)$$

hence

$$\varphi'(\lambda) = -R 2F_1 \left( -\frac{1}{2}, -\frac{1}{2}, 3; \frac{1}{\lambda^2} \right) + \frac{1}{R \lambda^2} 2F_1 \left( \frac{1}{2}, \frac{1}{2}, 5; \frac{5}{\lambda^2} \right) + 2\gamma R^2 \lambda$$

$$= -R \left( \frac{\lambda + \sqrt{\lambda^2 - 1}}{\lambda^2} \right) + 2\gamma R^2 \lambda. \quad (B.8)$$

Finally, it remains to recall that $R = \frac{\sqrt{3}}{\gamma^3}$.

**Lemma B.4 (d = 3, s = 0).** If $(d, s) = (3, 0)$ then, for $\lambda \geq 0$,

$$\varphi(\lambda) = \frac{1 + \log(3\gamma)}{2} + \frac{1}{2\pi \lambda} \int_0^1 \left( (\lambda + r)^2 \log((\lambda + r)^2) - (\lambda - r)^2 \log((\lambda - r)^2) \right) r \sqrt{1 - r^2} dr + \frac{\lambda^2}{3}$$

Moreover, if $\lambda \geq 1$,

$$\varphi(\lambda) = -\frac{1}{2} + \frac{\log(3\gamma)}{2} + \frac{\lambda^2 + 2}{3\lambda} \sqrt{\lambda^2 - 1} - \log \left( \frac{\lambda + \sqrt{\lambda^2 - 1}}{2} \right).$$

**Proof.** From Lemma B.1 and with the formulas $(b \in (-a, a))$

$$\int_{-1}^1 \log(a + bt) dt = \frac{(a + b) \log(a + b) - (a - b) \log(a - b)}{b} - 2$$

and

$$\int_0^1 \frac{r^2}{\sqrt{1 - r^2}} dr = \frac{\pi}{4},$$

with $a = \lambda^2 + r^2$ and $b = 2r\lambda$, we obtain, for $\lambda \geq 0$,

$$\varphi(\lambda) = \frac{1}{2} - \log(R) - \int_0^1 \frac{(\lambda + r)^2 \log((\lambda + r)^2) - (\lambda - r)^2 \log((\lambda - r)^2)}{2\pi \lambda} \frac{r}{\sqrt{1 - r^2}} dr + \gamma R^2 \lambda^2.$$ 

Recall that $R = \frac{\sqrt{3}}{\gamma^3}$. It follows that if $\lambda > 1$,

$$\varphi(\lambda) = \frac{1 + \log(3\gamma)}{2} - \frac{1}{\pi} \int_0^1 \log((\lambda + r)^2) - \log((\lambda - r)^2) \frac{r}{\sqrt{1 - r^2}} dr$$

$$= -\frac{2}{\pi} \int_0^1 \log((\lambda + r)^2) + \log((\lambda - r)^2) \frac{r^2}{\sqrt{1 - r^2}} dr.$$
\[-\frac{1}{\pi \lambda} \int_0^1 \log(\lambda + r) - \log(\lambda - r) \sqrt{1 - r^2} dr.\]

These three last integrals can be explicitly computed and we obtain the desired formula. \qed

**Appendix C. Proof of Riesz formula due to Riesz**

C.1. **Cross-ratio.** Recall that in projective geometry, the cross-ratio (birapport in French) of four distinct points \(z_1, z_2, z_3, z_4\) on the Riemann sphere \(\mathbb{C} \cup \{\infty\}\) is defined by

\[
[z_1, z_2; z_3, z_4] = \frac{z_3 - z_1}{z_3 - z_2} \frac{z_4 - z_1}{z_4 - z_2} = \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)},
\]

where each length is removed from the formula if it involves the point at infinity. The following lemma is a classical and important result of projective geometry.

**Lemma C.1** (Cross-ratio invariance). The cross-ratio is invariant under the Möbius transform

\[
z \mapsto \frac{az + b}{cz + d}, \quad ad - bc \neq 0,
\]

and thus its modulus is invariant under the “conjugated Möbius transform” \(z \mapsto \frac{az + b}{cz + d}, ad - bc \neq 0\).

C.2. **Inversions.** In \(\mathbb{R}^d, d \geq 1\), the inversion with center \(x_0\) and radius \(R > 0\) is the transform that maps \(x \neq x_0\) to \(T(x)\) on the half line started from \(x_0\) and passing through \(x\), in such a way that

\[
|x - x_0| |T(x) - x_0| = R^2.
\]

The circle centered at \(x_0\) and of radius \(R\) is pointwise invariant under the transformation in the sense that all its elements are fixed points of the transformation. The transformation maps the interior of this circle to its exterior, and vice versa. In projective geometry, this transformation is extended to the \(d\)-dimensional sphere by mapping \(x_0\) to the point at infinity \(\infty\), and vice versa. We have

\[
T(x) - x_0 = \frac{R^2}{|x - x_0|^2} (x - x_0),
\]

which exchanges \(x_0\) and \(\infty\). In dimension \(d = 2\), using complex numbers, \(T(z) - z_0 = R^2/(z - z_0)\), which is a special case of the conjugated Möbius transform \(z \mapsto \frac{az + b}{cz + d}\). It is worth mentioning that inversions are geometric transformations at the basis of the Kelvin transform of functions \(\mathbb{R}^d \to \mathbb{R}\).

**Lemma C.2** (Classical properties of inversions). Let \(T\) be the inversion of \(\mathbb{R}^d, d \geq 1\), with center \(x_0 \in \mathbb{R}^d\) and radius \(R > 0\). Then we have the following properties.

1. For all \(x\), \(|x - T(x)| = \frac{|R^2 - |x - x_0|^2|}{|x - x_0|}\).
2. For all \(x, y\), \(|T(x) - T(y)| = R^2 \frac{|x - y|}{|x - x_0| |y - x_0|}\).
3. As differential forms \(\frac{dT(x)}{|T(x) - x_0|^d} = \frac{dx}{|x - x_0|^d}\).
4. The modulus of the cross-ratio of distinct coplanar points \(x_1, x_2, x_3, x_4\) is invariant under \(T\).

**Proof.** We can assume without loss of generality that \(x_0 = 0\).

1. Since \(0, x, T(x)\) are aligned with 0 at the edge we have

\[
|x - T(x)| = ||x| - |T(x)|| = \left| |x| - \frac{R^2}{|x|}\right| = \frac{|x|^2 - R^2}{|x|}.
\]

2. We have

\[
|T(x) - T(y)|^2 = |T(x)|^2 + |T(y)|^2 - 2|T(x), T(y)|
= \frac{R^4}{|x|^2} + \frac{R^4}{|y|^2} - 2\frac{R^4}{|x|^2 |y|^2} \langle x, y \rangle = \frac{R^4}{|x|^2 |y|^2} |x - y|^2.
\]
(3) We have \( \text{Jac}(T) (x) = \frac{R^2}{|x|^2} (I_d + u \otimes v) \), \( u = \frac{x}{|x|} \), \( v = -2 \frac{x}{|x|} \), which gives then

\[
| \det \text{Jac}(T)(x) | = \left( \frac{R^2}{|x|^2} \right)^d \left( \frac{|T(x)|}{|x|} \right)^d,
\]

via the “matrix determinant lemma” \( \det(A + u \otimes v) = (1 + u \cdot A^{-1} v) \det(A) \), the determinant analogue of the Sherman–Morrison formula \( (A + u \otimes v)^{-1} = A^{-1} - \frac{A^{-1} u \otimes v A^{-1}}{1 + v \cdot A^{-1} u} \).

(4) Follows from the fact that \( T \) restricted to the plane is a conjugated Möbius transform.

\[
\square
\]

C.3. \textbf{Intersecting chords.} The \textit{intersecting chords theorem} in Euclidean (planar) geometry states that if \( AA^* \) and \( BB^* \) are two chords of a circle, intersecting at the point \( M \), see Figure 1, then

\[
AM \times MA^* = BM \times MB^*.
\]

Indeed, the triangles \( A^*MB \) and \( AMB^* \) are similar, identical up to rotation and scaling, more precisely they have two equal angles: \( A^*MB = AMB^* \) (opposite angles) and \( MA^*B = MB^*A \) (subtend the same arc).

\[
\text{Figure 1. Intersecting chords of a circle, } AA^* \text{ and } BB^* \text{ in the first two pictures, } xx^* \text{ and } zz^* \text{ for the third. On the two last pictures, the chords } BB^* \text{ and } zz^* \text{ are diameters of the circle. On the right, } x, y \in \mathbb{R}^d, d \geq 2, |x| = r, |y| < r, x^* \text{ is aligned with } x \text{ and } y, y \text{ separates } x \text{ and } x^*.
\]

Suppose now that the circle has center \( O \), radius \( r \), that \( BB^* \) is a diameter, and that \( M \) belongs to the segment \( OB \) (instead \( OB^* \)). Then \( BM = r - OM \) while \( MB^* = OM + r \) and thus

\[
BM \times MB^* = (r - OM)(OM + r) = r^2 - OM^2.
\]

In Euclidean geometry, this quantity is known as the Laguerre power of the point \( M \) with respect to the circle. We deduce immediately the following lemma.

\textbf{Lemma C.3} (Intersecting chords). \textit{For every chord } \( AA^* \text{ of a circle with center } O \text{ and radius } r \text{, intersecting an arbitrary diameter at point } M \), see Figure 4, \textit{we have}

\[
AM \times MA^* = r^2 - OM^2.
\]

C.4. \textbf{Riesz geometric argument.} The argument is essentially two-dimensional and involves projective geometry. Fix \( r > 0 \) and \( x, y \in \mathbb{R}^d, d \geq 2, |y| < r \). Let us define the map \( S : x \mapsto S(x) = x^* \) where \( x^* \in \mathbb{R}^d \) is the point aligned with \( x, y \) such that \( y \) separates \( x \) and \( x^* \) and

\[
|x - y| |y - x^*| = r^2 - |y|^2.
\]

The map \( S \) is the composition of an inversion centered at \( y \) of radius \( \sqrt{r^2 - |y|^2} \) and the central symmetry centered at \( y \) (recall that \( y \) separates \( x \) and \( x^* \)). Moreover, by Lemma C.3 see also Figure 4 we have \( |x| = r \) if and only if \( |x^*| = r \), namely then centered sphere of radius \( r \) is globally invariant under \( S \). The points \( y \) and \( \infty \) are mapped to each other by \( S \).

Let \( T \) be the inversion centered at the origin and with radius \( r \). By Lemma C.2 the modulus of the cross-ratio of the coplanar points \( x, T(y), y, T(x) \) satisfies

\[
|[x, T(y); y, T(x)]| = \frac{|x - y| |T(x) - T(y)|}{|x - T(x)| |y - T(y)|} = \frac{|x - y|^2 r^2}{|r^2 - |x|^2| |r^2 - |y|^2|}.
\]
Note that since \( x, y, x^\ast \) are aligned, the points \( x, y, x^\ast, T(x), T(y) \) are coplanar.

**Lemma C.4** (Commutation). \( S \) and \( T \) commute.

This is related to the fact that \( S \) leaves globally invariant the fixed points (circle) of \( T \).

**Proof.** Using complex coordinates \( T(z) = r^2/z \) while \( T(z) - z_0 = -(r^2 - |z_0|^2)/(z - z_0) \), where \( z_0 \) stands for \( y \). Now we have

\[
T(S(z)) = \frac{r^2}{z_0 - z} = \frac{r^2(z - z_0)}{z_0 z - r^2} \quad \text{and} \quad S(T(z)) = z - \frac{r^2 - |z_0|^2}{r^2 - z_0 z} = r^2(z_0 - z).
\]

\( \square \)

Since \( S \) is the composition of an inversion and a central symmetry, it is a special case of a conjugate Möbius transform, and then, by Lemma C.1 \( ||x, T(y); y, T(x)|| = ||S(x), S(T(y)); S(y), S(T(x))|| \). Since \( S \) and \( T \) commute (Lemma C.3), we have, using Lemma C.2 for the final step,

\[
||x, T(y); y, T(x)|| = ||S(x), T(S(y)); S(y), T(S(x))|| = ||x^\ast, T(\infty); \infty, T(x^\ast)|| = ||x^\ast, 0; \infty, T(x^\ast)||
\]

\[
\frac{|T(x^\ast)|}{|T(x^\ast) - x^\ast|} = \frac{|T(\ast)|}{|\ast - |x^\ast|^2|} = \frac{r^2}{|r^2 - |x^\ast|^2|}.
\]

It follows that in the case \( |x| < r \) (in other words \( |x^\ast| > r \)) we get (recall that \( |y| < r \))

\[
\frac{|x - y|^2}{(r^2 - |x|^2)(r^2 - |y|^2)} = \frac{1}{|x^\ast|^2 - r^2} \quad \text{hence} \quad \frac{1}{(r^2 - |x^\ast|^2)|x - y|^{-\alpha}} = \frac{(r^2 - |y|^2)^{\alpha}}{(|x^\ast|^2 - r^2)^{\alpha}}.
\]

Finally, using this formula, we get, for all \( y \in \mathbb{R}^d, |y| \leq r \), and all \( \alpha \geq 0, d \geq 2 \),

\[
I(y) := \int_{|x| < r} \frac{dx}{|x - y|^{\alpha}} = (r^2 - |y|^2)^{\frac{\alpha}{2}} \int_{|x^\ast| > r} \frac{dx^\ast}{(|x^\ast|^2 - r^2)^{\frac{\alpha}{2}}|x^\ast - y|^\alpha},
\]

where the differential identity \( \frac{dx}{|x - y|^\alpha} = \frac{dx^\ast}{|x^\ast - y|^\alpha} \) comes from Lemma C.2 applied to \( S \) which is not an inversion but which is the composition of an inversion with an isometry (central symmetry).

Using spherical coordinates with \( \rho = |x| \) and the Funk--Hecke formula (A.11) we get

\[
I(y) = (r^2 - |y|^2)^{\frac{\alpha}{2}} \int_{r^1 < \rho < r} \frac{d\rho \rho^{d-1} \sin^{d-1}(\theta) \Gamma\left(d-\frac{1}{2}\right)}{\sqrt{\rho^2 - 2\rho_1 \cos(\theta) + 1} \Gamma\left( \frac{d}{2} \right)} d\theta
\]

\[
= |S_1| \frac{\Gamma\left( \frac{d}{2} \right)}{\sqrt{\pi} \Gamma\left(d-\frac{1}{2}\right)} \int_{r^1}^\infty \rho^{d-1} \frac{1}{(\rho^2 - 2\rho_1 \cos(\theta) + 1)^{\frac{\alpha}{2}}} \sin^{d-1}(\theta) \Gamma\left(d-\frac{1}{2}\right) d\theta \int_{r_2}^\infty \rho_1^{d-1} \sin^{d-1}(\theta) \Gamma\left(d-\frac{1}{2}\right) d\rho_1 = |S_1| \frac{\Gamma\left( \frac{d}{2} \right)}{\sqrt{\pi} \Gamma\left(d-\frac{1}{2}\right)} \int_{r^1}^\infty \frac{\rho^{d-1} \Gamma\left(d-\frac{1}{2}\right)}{(\rho^2 - 2\rho_1 \cos(\theta) + 1)^{\frac{\alpha}{2}}} d\theta.
\]

where \( r := r_1 |y| \) and \( \rho := r_1 |y| \). Note that \( r_1 \geq 1 \) and \( \rho_1 > 1 \).

**C.5. Trigonometric change of variable.** Let us show that for \( d > 1 \) and \( \rho_1 > 1 \),

\[
i_d := \int_0^\pi \frac{\sin^{d-2}(\theta)}{\rho^2 - 2\rho_1 \cos(\theta) + 1} d\theta = \frac{\rho_1^{2-d}}{\rho_1^2 - 1} \int_0^\pi \sin^{d-2}(\alpha) d\alpha = \frac{\rho_1^{2-d}}{\rho_1^2 - 1} \Gamma\left( \frac{d}{2} \right).
\]

We first give a historical geometrical argument. We then give in Remark C.5 an analytic argument using properties of the Gegenbauer polynomials. The second equality in (C.2) follows from the fact that the middle integral becomes an Euler beta integral after the change of variable \( u = \sin(\alpha) \). To prove the first equality in (C.2), we follow [23], p. 400, and we use the change of variable

\[
\frac{\sin(\theta)}{\sqrt{\rho^2 - 2\rho_1 \cos(\theta) + 1}} = \frac{\sin(\alpha)}{\rho_1}.
\]
Therefore, we obtain, noting that 

\[ \rho_i^2 - 2 \rho_i \cos(\theta) + 1 = A(\alpha)^2 \]  

where \( A(\alpha) = \sqrt{\rho_i^2 - \sin^2(\alpha) + \cos(\alpha)} \), hence 

\[ 2 \rho_i \sin(\theta) d\theta = 2A(\alpha)A'(\alpha) d\alpha \]  

and by using the formula for the change of variable this gives 

\[ d\theta = \frac{A'(\alpha)}{\sin(\alpha)} d\alpha = -\frac{\sin(\alpha)}{\sqrt{\rho_i^2 - \sin^2(\alpha)}} d\alpha = -\left( \frac{\sqrt{\rho_i^2 - \sin^2(\alpha) + \cos(\alpha)}}{\sqrt{\rho_i^2 - \sin^2(\alpha)}} \right) d\alpha. \]

Therefore, we obtain, noting that \( \theta = 0 \iff \alpha = \pi \) and \( \theta = \pi \iff \alpha = 0 \) (see Figure 2),

\[ i_d = \int_0^\pi \left( \frac{\sin(\alpha)}{\rho_i} \right)^{d-2} \frac{1}{(\cos(\alpha) + \sqrt{\rho_i^2 - \sin^2(\alpha)})^2} \sqrt{\rho_i^2 - \sin^2(\alpha)} \; d\alpha \]

\[ = \int_0^\pi \left( \frac{\sin(\alpha)}{\rho_i} \right)^{d-2} \frac{1}{\cos(\alpha) + \sqrt{\rho_i^2 - \sin^2(\alpha)}} \frac{1}{\sqrt{\rho_i^2 - \sin^2(\alpha)}} d\alpha \]

\[ = \int_0^\pi \frac{1}{\rho_i^{d-2}(1-\rho_i^2)} \int_0^\pi (\sin(\alpha))^{d-2} \left( \frac{\cos(\alpha)}{\sqrt{\rho_i^2 - \sin^2(\alpha)}} - 1 \right) d\alpha \]

where the last equality follows from the antisymmetry of \( \cos \) around \( \pi/2 \). This proves (C.2).

**Remark C.5 (Proof of (C.2) using Gegenbauer polynomials).** Let \( \rho = \frac{1}{\rho_i} < 1 \). Using the generating function for Gegenbauer polynomials 

\[ (1-2\rho \cos(\theta) + \rho^2)^\frac{d}{2} = \sum_{n=0}^{\infty} C_n^{(\frac{d}{2})} (\cos(\theta)) \rho^n \]

gives

\[ \int_0^\pi \frac{\sin^{d-2} \theta}{(1-2\rho \cos(\theta) + \rho^2)^\frac{d}{2}} \; d\theta = \sum_{n=0}^{\infty} \rho^n \int_0^\pi \sin^{d-2} \theta \; C_n^{(\frac{d}{2})} (\cos(\theta)) \; d\theta. \]  \hspace{1cm} (C.3)

The integral on the right-hand side vanishes for odd degree \( n \) since the Gegenbauer polynomials are odd functions of \( \cos \). For even degree \( n = 2k \), the integral can be computed as

\[ \int_0^\pi \sin^{d-2} \theta \; C_{2k}^{(\frac{d}{2})} (\cos(\theta)) \; d\theta = \int_0^\pi \sin^{d-2} \theta \; d\theta = \sqrt{\pi} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})}. \]  \hspace{1cm} (C.4)

To establish (C.4), use the recurrence relation [12] (18.9.7)

\[ C_{2k}^{(\frac{d}{2})} (x) = C_{2k-2}^{(\frac{d}{2})} (x) + \frac{2k + \frac{d}{2} - 1}{\frac{d}{2} - 1} C_{2k-1}^{(\frac{d}{2})} (x) \]

and integrate against \( \sin^{d-2} \theta \) to produce

\[ \int_0^\pi \sin^{d-2} \theta \; C_{2k}^{(\frac{d}{2})} (\cos(\theta)) \; d\theta = \int_0^\pi \sin^{d-2} \theta \; C_{2k-2}^{(\frac{d}{2})} (\cos(\theta)) \; d\theta + \frac{2k + \frac{d}{2} - 1}{\frac{d}{2} - 1} \int_0^\pi \sin^{d-2} \theta \; C_{2k-1}^{(\frac{d}{2})} (\cos(\theta)) \; d\theta. \]

The second integral on the right-hand side vanishes by orthogonality, so

\[ \int_0^\pi \sin^{d-2} \theta \; C_{2k}^{(\frac{d}{2})} (\cos(\theta)) \; d\theta = \int_0^\pi \sin^{d-2} \theta \; C_{2k-2}^{(\frac{d}{2})} (\cos(\theta)) \; d\theta, \]

with repeated application giving the first equality in (C.4). Using (C.4) in (C.3) then gives

\[ \int_0^\pi \frac{\sin^{d-2} \theta}{(1-2\rho \cos(\theta) + \rho^2)^\frac{d}{2}} \; d\theta = \sqrt{\pi} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} \sum_{k=0}^{\infty} \rho^{2k} = \sqrt{\pi} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} \frac{1}{1 - \rho^2}. \]

The substitution \( \rho = \frac{1}{\rho_i} \) then gives (C.2).

---

\[ \text{It is mentioned in [23] p. 400] that this change of variable was suggested S.I. Greenberg. Nevertheless such geometric reasoning goes back at least to the works on elliptic integrals of the 19-th century, see [17].} \]
Figure 2. Geometric interpretation of the $\theta$ to $\alpha$ change of variables for $i_d$. The angles and distances are $ACB = \theta$, $CBA = \alpha$, $CB = 1$ and $CA = \rho_1$. The right-angled triangle $ABQ$ has hypotenuse $AB$, thus

$$AB^2 = BQ^2 + AQ^2 = \sin^2(\theta) + (AC - QC)^2 = \sin^2(\theta) + (\rho_1 - \cos(\theta))^2 = \rho_1^2 - 2\rho_1 \cos(\theta) + 1.$$ 

The sine rule then gives

$$\frac{\sin(\alpha)}{\rho_1} = \frac{\sin(\theta)}{\sqrt{\rho_1^2 - 2\rho_1 \cos(\theta) + 1}}.$$ 

On the other hand, we also have

$$\sqrt{\rho_1^2 - 2\rho_1 \cos(\theta) + 1} = AB = AP + PB = \sqrt{\rho_1^2 - \sin^2(\alpha) + \cos(\alpha)}.$$ 

C.6. Conclusion. By combining (C.1) and (C.2), using the successive changes of variables $t = \rho_2^2 - r_1^2$, $t_1 = t/(r_1^2 - 1)$, and $u = 1/(1 + t_1)$, and the Euler reflection formula (A.1), we get

$$I(y) = |S_1|(r_1^2 - 1)^{\frac{d}{2}} \int_1^\infty \frac{\rho_1 d\rho_1}{(\rho_1^2 - r_1^2) \sqrt{(\rho_1^2 - 1)}}$$

$$= |S_1| r_1^2 \frac{d}{2} \int_0^\infty \frac{dt}{t^2 (t + r_1^2 - 1)}$$

$$= |S_1| r_1^2 \frac{d}{2} \int_0^1 \frac{u^{\frac{d}{2} - 1} du}{(1 - u)^{\frac{d}{2}}} = |S_1| \frac{\Gamma(\frac{d}{2}) \Gamma(1 - \frac{d}{2})}{\Gamma(\frac{d}{2}) \sin(\frac{\pi}{2} \frac{d}{2})} = \frac{\pi \frac{d}{2} + 1}{\Gamma(\frac{d}{2}) \sin(\frac{\pi}{2} \frac{d}{2})}.$$ 

This completes the proof of (1.6) and thus of Lemma 1.2.

Appendix D. Alternative analytic proof of Riesz formula

D.1. The Mellin transform and Riesz potentials. Quoting [11] Ch. 12, we recall that the Fourier transform pair may be written in the form

$$A(\theta) := \int_{\mathbb{R}} a(t) e^{i\theta t} dt, \quad \alpha < \Im \theta < \beta, \quad \text{and} \quad a(t) = \frac{1}{2\pi} \int_{\mathbb{R}+i\mathbb{R}} A(\theta) e^{-i\theta t} d\theta, \quad \alpha < c < \beta.$$ 

The Mellin transform and its inverse follow if we introduce the variable changes

$$z = i\theta, \quad x = e^t, \quad f(x) = a(\log(x)),$$

so that we obtain the reciprocal pair of integral transforms, for $f : (0, +\infty) \to \mathbb{R}$,

$$F(z) := \int_0^\infty f(x) x^{z-1} dx, \quad \alpha < \Re z < \beta, \quad \text{and} \quad f(x) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} F(z) x^{-z} dz, \quad \alpha < c < \beta. \quad (D.1)$$

Display (D.1) exhibits the Mellin transform followed by its inversion formula. The integral defining the transform normally exists only in the strip $\alpha < \Re z < \beta$; therefore the inversion contour must be placed in this strip. For convenience we also denote by $Mf = F$ the Mellin transform of $f$. 


The Mellin transform of $x \mapsto e^{-x}$ is the Euler Gamma function $\Gamma$. Its poles are $0, -1, -2, -3, \ldots$. The Mellin transform of $x \mapsto (1 - x)^{b-1}$ at a point $z$ is\[ \int_0^\infty x^{z-1} (1 - x)^{b-1} dx = \text{Beta}(z, b). \]

Lemma D.1 (Riesz potential of radial functions). Suppose that\[ x \in \mathbb{R}^d \mapsto f(x) = \varphi(|x|^2), \]
where $\varphi : \mathbb{C} \to \mathbb{R}$ is given as the absolutely convergent inverse Mellin transform\[ \varphi(r) := \frac{1}{2\pi i} \int_{\lambda+i\mathbb{R}} \mathcal{M} \varphi(z) r^{-z} dz, \quad \text{for some } \lambda \in \mathbb{R}. \]
If $0 < \alpha < 2\lambda < d$, the Riesz potential $(|\cdot|^{-\alpha} * f)(x)$ is well defined for $x \neq 0$, and\[ (|\cdot|^{-\alpha} * f)(x) = \psi(|x|^2), \]
where\[ \psi(r) := \frac{1}{2\pi i} \int_{\lambda-\delta+i\mathbb{R}} \Gamma(z) \Gamma\left(\frac{d-2\alpha}{2} - z\right) \mathcal{M} \varphi(z + \frac{\alpha}{2}) r^{-z} dz. \]
In other words, the Mellin transform of $\psi$ satisfies\[ \mathcal{M} \psi(z) = \frac{\pi^\delta \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{d-2\alpha}{2} - z\right)}{\Gamma\left(\frac{d-2\alpha}{2}\right) \Gamma\left(\frac{d}{2} + z\right) \Gamma\left(\frac{d}{2} - z\right)} \mathcal{M} \varphi(z + \frac{\alpha}{2}). \]

Proof. This is \cite{[13]} Prop. 2 with $V \equiv 1$ (and $l = 0$), see also \cite{[13]} eq. (7). The idea is to use the inverse Mellin transform of $\varphi$ to reduce the problem, via the Fubini theorem, to the computation of the Riesz potential of inverse powers of the norm, which is immediate from the semigroup property of the Riesz integral formula (1.6) for $\varphi$, the Fubini theorem, and the semigroup property for Riesz kernels reads, for all $\alpha, \beta \in \mathbb{C}$ such that $\Re \alpha, \Re \beta > 0$ and $\Re \alpha + \Re \beta < d$,
\begin{equation}
|\cdot|^{-\alpha} * |\cdot|^{-\beta} = \frac{c_d(\alpha) c_d(\beta)}{c_d(\alpha + \beta)} |\cdot|^{-(\alpha + \beta)} \quad \text{where } c_d(z) := \frac{2\pi^\delta \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d-\alpha}{2}\right)}.
\end{equation}

Now, by the inverse Mellin transform of $\varphi$, the Fubini theorem, and the semigroup property,
\begin{align*}
(|\cdot|^{-\alpha} * f)(x) &= \frac{1}{2\pi i} \int_{\lambda+i\mathbb{R}} (|\cdot|^{-\alpha} * |\cdot|^{-2z}) \mathcal{M} \varphi(z) dz \\
&= \frac{1}{2\pi i} \int_{\lambda+i\mathbb{R}} \frac{c_d(\alpha) c_d(d - 2z)}{c_d(d + \alpha - 2z)} \mathcal{M} \varphi(z) |\cdot|^{-2z} dz \\
&= \frac{1}{2\pi i} \int_{\lambda-\delta+i\mathbb{R}} \frac{c_d(\alpha) c_d(d - \alpha - 2w)}{c_d(d - 2w)} \mathcal{M} \varphi(w + \frac{\alpha}{2}) |\cdot|^{-2w} dw.
\end{align*}

D.2. Analytic proof of Riesz integral formula. The Riesz integral formula (1.6) for $R = 1$, $x \in B_1$, $x \neq 0$, is a special case of \cite{[13]} Cor. 4 with $V \equiv 1$, $l = 0$, $\alpha = s - d$, $\delta = d$, $\rho = \sigma = -\frac{d-s}{2}$, $0 < s < d - 2$ (implies $\sigma > -1$), see also eq. (7). Let us give the proof extracted from there. With $\varphi(r) := (1 - r)^{-\alpha}$, we have $\mathcal{M} \varphi(z) = \int_0^1 r^{z-1} (1 - r)^{-\alpha} dr = \text{Beta}(z, 1 - \frac{d-s}{2})$, and by Lemma D.1 with $\alpha = d - s$,
\[ \mathcal{M} \psi(z) = \frac{\pi^\delta \Gamma\left(\frac{d-\alpha}{2}\right) \Gamma\left(\frac{d}{2} - \frac{d-s}{2}\right)}{\Gamma\left(\frac{d}{2} + z\right) \Gamma\left(\frac{d}{2} - z\right)} \frac{\Gamma(z) \Gamma\left(\frac{d-s}{2} - z\right)}{\Gamma\left(\frac{d-s}{2} - z\right)}. \]

\[\text{Recall the definition of the Euler Beta function } \text{Beta}(a, b) := \int_0^1 t^{a-1} (1 - t)^{b-1} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}.\]
Now, if the vertical line \( \lambda + i \mathbb{R} \) separates the poles of \( z \mapsto \Gamma(z) \) and the poles of \( z \mapsto \Gamma(\frac{x}{2} - z) \), then

\[
\frac{1}{2\pi i} \int_{\lambda+i\mathbb{R}} \frac{\Gamma(z)\Gamma(\frac{x}{2} - z)}{\Gamma(\frac{x}{2} - z)\Gamma(z+1)} x^{-z} dz = \sum_{k=0}^{\infty} \text{Residue}_{z=-k} \left( \frac{\Gamma(z)\Gamma(\frac{x}{2} - z)}{\Gamma(\frac{x}{2} - z)\Gamma(z+1)} x^{-z} \right) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{x}{2} + k)\Gamma(\frac{x}{2} - k)}{\Gamma(\frac{x}{2} + k)\Gamma(-k+1)} x^k \text{Residue}_{z=-k}(\Gamma(z)) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{x}{2} + k)}{\Gamma(\frac{x}{2} + k)\Gamma(-k+1)} (-x)^k = \frac{\Gamma(\frac{x}{2})}{\Gamma(\frac{x}{2})}. \]

**Remark D.2** (Meijer G-functions). A key point in the proof above is the computation of the inverse Mellin transform of a certain ratio of products of Gamma functions (Mellin transform of \( \psi \)). This is actually the definition of Meijer G-functions. Following [13], if \( f(x) := (1 - |x|^2)^n \times F_1(a, b; c; 1 - |x|^2) = \varphi(|x|^2) \) where \( \varphi(r) = (1 - r)^n \times F_1(a, b; c; r) \), then it is possible, by using the same method as above, to express \( \varphi \) as a Meijer G-function, and to deduce that the Riesz potential of \( f \) on the unit ball is equal to another Meijer G-function, which reduces to a hypergeometric function in certain cases.

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8The \( \Gamma \) function has no zeros, indeed a zero leads via \( \Gamma(z) = (z-1)\Gamma(z-1) \) to infinitely many zeros to the left, and then via \( \Gamma(z)\Gamma(1-z) = \sin(\pi z) \) to infinitely many poles to the right, which contradicts the analyticity of \( \Gamma \) on \( \mathbb{R} \setminus \mathbb{Z} \).

The \( \Gamma \) function is meromorphic on the complex plane; its poles are the non-positive integers, and are simple. Moreover, \( \text{Residue}_{z=-n}(\Gamma(z)) := \lim_{z \to -n}(z-(-n))\Gamma(z) = \frac{(-1)^n}{n!} \), which follows from \( (z+n)\Gamma(z) = \frac{\Gamma(z+n+1)}{\Gamma(1)} \).
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