ESSENTIAL SELF-ADJOINTNESS OF MAGNETIC SCHRÖDINGER OPERATORS ON LOCALLY FINITE GRAPHS

OGNJEN MILATOVIC

Abstract. We give sufficient conditions for essential self-adjointness of magnetic Schrödinger operators on locally finite graphs. Two of the main theorems of the present paper generalize recent results of Torki-Hamza.

1. Introduction and the Main Results

1.1. The setting. Let $G = (V, E)$ be an infinite graph without loops and multiple edges between vertices. By $V = V(G)$ and $E = E(G)$ we denote the set of vertices and the set of unoriented edges of $G$ respectively. In what follows, the notation $m(x)$ indicates the degree of a vertex $x$, that is, the number of edges that meet at $x$. We assume that $G$ is locally finite, that is, $m(x)$ is finite for all $x \in V$.

In what follows, $x \sim y$ indicates that there is an edge that connects $x$ and $y$. We will also need a set of oriented edges $E_0 := \{[x, y], [y, x] : x, y \in V \text{ and } x \sim y\}$. (1.1)

The notation $e = [x, y]$ indicates an oriented edge $e$ with starting vertex $o(e) = x$ and terminal vertex $t(e) = y$. The definition (1.1) means that every unoriented edge in $E$ is represented by two oriented edges in $E_0$. Thus, there is a two-to-one map $p : E_0 \to E$. For $e = [x, y] \in E_0$, we denote the corresponding reverse edge by $\hat{e} = [y, x]$. This gives rise to an involution $e \mapsto \hat{e}$ on $E_0$.

To help us write formulas in unambiguous way, we pick an orientation on each edge by specifying a subset $E_s$ of $E_0$ such that $E_0 = E_s \cup \hat{E}_s$ (disjoint union), where $\hat{E}_s$ denotes the image of $E_s$ under the involution $e \mapsto \hat{e}$. Thus, we may identify $E_s$ with $E$ by the map $p$.

In the sequel, we assume that $G$ is connected, that is, for any $x, y \in V$ there exists a path $\gamma$ joining $x$ and $y$. Here, $\gamma$ is a sequence $x_1, x_2, \ldots, x_n \in V$ such that $x = x_1$, $y = x_n$, and $x_j \sim x_{j+1}$ for all $1 \leq j \leq n - 1$. The length of a path $\gamma$ is defined as the number of edges in $\gamma$.

The distance $d(x, y)$ between vertices $x$ and $y$ of $G$ is defined as the number of edges in the shortest path connecting the vertices $x$ and $y$. Fix a vertex $x_0 \in V$ and define $r(x) := d(x_0, x)$. The $n$-neighborhood $B_n(x_0)$ of $x_0 \in V$ is defined as

$$\{x \in V : r(x) \leq n\} \cup \{e = [x, y] \in E_s : r(x) \leq n \text{ and } r(y) \leq n\}. \quad (1.2)$$

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In what follows, $C(V)$ is the set of complex-valued functions on $V$, and $C(E_n)$ is the set of functions $Y: E_0 \to \mathbb{C}$ such that $Y(e) = -Y(\bar{e})$. The notations $C_c(V)$ and $C_c(E_n)$ denote the sets of finitely supported elements of $C(V)$ and $C(E_n)$ respectively.

In the sequel, we assume that $V$ is equipped with a weight $w: V \to \mathbb{R}^+$. By $\ell_w^2(V)$ we denote the space of functions $f \in C(V)$ such that $\|f\| < \infty$, where $\|f\|$ is the norm corresponding to the inner product

$$(f, g) := \sum_{x \in V} w(x)f(x)\overline{g(x)}. \quad (1.3)$$

Additionally, we assume that $E$ is equipped with a weight $a: E_0 \to \mathbb{R}^+$ such that $a(e) = a(\bar{e})$ for all $e \in E_0$. This makes $G = (G, w, a)$ a weighted graph weights $w$ and $a$.

1.2. **Magnetic Schrödinger Operator.** Let $U(1) := \{z \in \mathbb{C}: |z| = 1\}$ and $\sigma: E_0 \to U(1)$ with $\sigma(\bar{e}) = \overline{\sigma(e)}$ for all $e \in E_0$, where $\overline{\cdot}$ denotes the complex conjugate of $z \in \mathbb{C}$.

We define the magnetic Laplacian $\Delta_\sigma: C(V) \to C(V)$ on the graph $(G, w, a)$ by the formula

$$(\Delta_\sigma u)(x) = \frac{1}{w(x)} \sum_{e \in O_x} a(e)(u(x) - \sigma(\bar{e})u(t(e))), \quad (1.4)$$

where $x \in V$ and

$$O_x := \{e \in E_0: o(e) = x\}. \quad (1.5)$$

For the case $a \equiv 1$ and $w \equiv 1$, the definition (1.4) is the same as in Dodziuk–Mathai [10]. For the case $\sigma \equiv 1$, see Sy–Sunada [29] and Torki–Hamza [30].

Let $q: V \to \mathbb{R}$, and consider a Schrödinger-type expression

$$Hu := \Delta_\sigma u + qu \quad (1.6)$$

We give sufficient conditions for $H|_{C_c(V)}$ to be essentially self-adjoint in the space $\ell_w^2(V)$. We first state the main results, and in Section 2 we make a few remarks concerning the existing work on the essential self-adjointness problem on locally finite graphs.

**Theorem 1.3.** Assume that $(G, w, a)$ is an infinite, locally finite, connected, oriented, weighted graph with $w(x) \equiv w_0$, where $w_0 > 0$ is a constant. Additionally, assume that there exists a constant $C \in \mathbb{R}$ such that $q(x) \geq -C$ for all $x \in V$. Then, the operator $H|_{C_c(V)}$ is essentially self-adjoint in $\ell_w^2(V)$.

In the next theorem, we will need the following additional assumption on the graph $G$.

**Assumption (A)** Assume that

$$\lim_{n \to \infty} \frac{m_n a_n}{n^2} = 0, \quad (1.7)$$

where

$$m_n := \max_{x \in B_n(x_0)} (m(x)) \quad \text{and} \quad a_n := \max_{x \in B_n(x_0)} \left( \max_{e \sim x} \left( \frac{a(e)}{w(x)} \right) \right), \quad (1.8)$$

where $B_n(x_0)$ as in (1.2), and $e \sim x$, with $e \in E_n$ and $x \in V$, indicates that $t(e) = x$ or $o(e) = x$. 


Theorem 1.4. Assume that $(G, w, a)$ is an infinite, locally finite, connected, oriented, weighted graph. Assume that the Assumption (A) is satisfied. Additionally, assume that there exists a constant $C \in \mathbb{R}$ such that
\[
(Hu, u) \geq -C\|u\|^2 \quad \text{for all } u \in C_c(V),
\] (1.9)
where $(\cdot, \cdot)$ and $\| \cdot \|$ are as in (L.3). Then, the operator $H|_{C_c(V)}$ is essentially self-adjoint in $\ell^2_w(V)$.

In the next theorem, we will need the notion of weighted distance on $G$. Let $a : E_0 \to \mathbb{R}^+$ be as in (1.4). Following Colin de Verdière, Torki-Hamza, and Truc [5], we define the weighted distance $d_{w,a}$ on $G$ as follows:
\[
d_{w,a}(x, y) := \inf_{\gamma \in P_{x,y}} L_{w,a}(\gamma),
\] (1.10)
where $P_{x,y}$ is the set of all paths $\gamma : x = x_1, x_2, \ldots, x_n = y$ such that $x_j \sim x_{j+1}$ for all $1 \leq j \leq n - 1$, and the length $L_{w,a}(\gamma)$ is computed as follows:
\[
L_{w,a}(\gamma) = \sum_{j=1}^{n-1} \sqrt{\min\{w(x_j), w(x_{j+1})\} / a([x_j, x_{j+1}]).}
\] (1.11)
We say that the metric space $(G, d_{w,a})$ is complete if every Cauchy sequence of vertices has a limit in $V$.

In what follows, we say that $G$ is a graph of bounded degree if there exists a constant $N > 0$ such that $m(x) \leq N$ for all $x \in V$.

Theorem 1.5. Assume that $(G, w, a)$ is an infinite, locally finite, connected, oriented, weighted graph. Assume that $G$ is a graph of bounded degree. Assume that $(G, d_{w,a})$ is a complete metric space. Additionally, assume that $H$ satisfies (1.9). Then, the operator $H|_{C_c(V)}$ is essentially self-adjoint in $\ell^2_w(V)$.

Remark 1.6. Let $d_{w,a}$ be as in (1.10). It is easily seen that if $G$ is a graph of bounded degree and if (L.7) is satisfied, then $(G, d_{w,a})$ is complete.

The following example describes a graph $G$ of bounded degree such that $(G, d_{w,a})$ is complete and (L.7) is not satisfied.

Examples. (i) Denote $\mathbb{Z}_+ := \{1, 2, 3, \ldots\}$, and consider the graph $G_1 = (V, E)$ with $V = \mathbb{Z}_+ \cup \{0\}$ and $E = \{[n - 1, n] : n \in \mathbb{Z}_+\}$. Define $a([n - 1, n]) = n$ and $w(n - 1) = \frac{1}{n}$, for all $n \in \mathbb{Z}_+$.

Since $w(x)$ is not constant, we cannot use Theorem [L.3] in this example.

Let $K \in \mathbb{Z}_+$ and let $m_K$ and $a_K$ be as in (1.8) with $n = K$ and $x_0 = 0$. We have $m_K = 2$ and $a_K = (K + 1)^2$. Thus,
\[
\lim_{K \to \infty} \frac{m_K a_K}{K^2} = 2,
\]
and (L.7) is not satisfied. Thus, in this example, we cannot use Theorem [L.4].
Fix $K_0 \in \mathbb{Z}_+ \cup \{0\}$, and let $K > K_0$. For $x_0 = K_0$ and $x = K$, by (1.10) we have
\[
d_{w,a}(x_0, x) = \sum_{n=K_0}^{K-1} \frac{1}{\sqrt{(n+1)(n+2)}} \to \infty, \quad \text{as } K \to \infty.
\]
Thus, the metric $d_{w,a}$ is complete. Additionally, the graph $G_1$ has bounded degree. By Theorem 1.5 the operator $\Delta_{\sigma}|_{C_c(V)}$ is essentially self-adjoint in $\ell^2_w(V)$.

The following example describes a graph of unbounded degree such that (1.7) is satisfied.

(ii) Consider $G_2 = (V, E)$, where $V = \{x_0, x_1, x_2, \ldots \}$. The vertices are arranged in a “triangular” pattern so that $x_0$ is in the first row, $x_1$ and $x_2$ are in the second row, $x_3$, $x_4$, and $x_5$ are in the third row, and so on. The vertex $x_0$ is connected to $x_1$ and $x_2$. The vertex $x_i$, where $i = 1, 2$, is connected to every vertex $x_j$, where $j = 3, 4, 5$. The pattern continues so that each of $k$ vertices in the $k$-th row is connected to each of $k+1$ vertices in the $(k+1)$-th row. Define $a(e) \equiv 1$ for all $e \in E$. For every vertex $x$ in the $n$-th row, define $w(x) = \frac{n-1}{2}$.

Since $w(x)$ is not constant, we cannot use Theorem 1.3. Since $G_2$ does not have a bounded degree, we cannot use Theorem 1.5.

Let $K \in \{1, 2, \ldots \}$. Let $m_K$ and $a_K$ be as in (1.8) with $n = K$ and $x_0$ as in this example. We have $m_K = 2K + 2$ and $a_K = \sqrt{K + 1}$. Thus,
\[
\lim_{K \to \infty} \frac{m_K a_K}{K^2} = 0,
\]
and (1.7) is satisfied. By Theorem 1.4 the operator $\Delta_{\sigma}|_{C_c(V)}$ is essentially self-adjoint in $\ell^2_w(V)$.

Remark 1.7. In the context of a not necessarily complete graph of bounded degree, a sufficient condition for essential self-adjointness of $\Delta_{\sigma}|_{C_c(V)}$ in $\ell^2_w(V)$ is given by Colin de Verdière, Torki-Hamza, and Truc [6, Theorem 3.1]. In the case $q \equiv 0$, Theorem 1.5 is contained in [6, Theorem 3.1].

2. Background of the Problem

In the context of a locally finite graph $G = (V, E)$, recently there has been a lot of interest in the operator
\[
(\Delta u)(x) = \frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e)(u(x) - u(t(e))),
\]
where $x \in V$ and $\mathcal{O}_x$ is as in (1.5).

In many spectral-theoretic investigations of $\Delta$ and $\Delta + q$, where $q: V \to \mathbb{R}$ is a real-valued function, it is helpful to have a self-adjoint operator. Thus, finding sufficient conditions for essential self-adjointness of $\Delta$ and $\Delta + q$ is an important problem in analysis on locally finite graphs. Note that $\Delta$ in (2.1), also known as physical Laplacian, is generally an unbounded operator in $\ell^2_w(V)$. Putting $w \equiv 1$ and $a \equiv 1$ in (2.1) and dividing by the degree function $m(x)$, we get the normalized Laplacian, which is a bounded operator on $\ell^2_w(V)$, with inner product as in (1.3) with $w(x) = m(x)$. The normalized Laplacian has been studied extensively; see, for instance, Chung [4] and Mohar–Woess [21].
In the discussion that follows, the local finiteness assumption is understood, unless specified otherwise. The essential self-adjointness of $\Delta|_{C_c(V)}$, where $\Delta$ is as in (2.1) with $w \equiv 1$ and $a \equiv 1$, was proven by Wojciechowski [33] and Weber [31]. For $\Delta$ as in (2.1) with $w \equiv 1$, the essential self-adjointness of $\Delta|_{C_c(V)}$ was proven by Jorgensen [14] (see also Jorgensen–Pearse [15]). With regard to Theorem 1.3 of the present paper, Torki-Hamza [30] proved the essential self-adjointness of $(\Delta + q)|_{C_c(V)}$, where $\Delta$ is as in (2.1) with $w \equiv c_0$ and $q \geq -c_1$, where $c_0 > 0$ and $c_1 \in \mathbb{R}$ are constants. The results of Wojciechowski [33], Weber [31], and Jorgensen [14] on the essential self-adjointness of $\Delta$ and the result of Torki-Hamza [30] on the essential self-adjointness of $(\Delta + q)|_{C_c(V)}$ with $q \geq -c_1$, where $c_1$ is a constant, are all contained in Keller–Lenz [17] and Keller–Lenz [18].

Under the assumption (1.7) above, the essential self-adjointness of $(d\delta + \delta d)|_{\Omega_0(G)}$, where $\Omega_0(G)$ denotes finitely supported forms $\alpha \in C(V) \oplus C(E)$, was proven by Masamune [19]. Additionally, Masamune [19] studied $L^p$-Liouville property for non-negative subharmonic forms on $G$.

In the context of a graph of bounded degree, Torki-Hamza [30] made an important link between the essential self-adjointness of $(\Delta + q)|_{C_c(V)}$, where $\Delta$ is as in (2.1) with $w \equiv 1$, and completeness of the weighted metric $d_{1,a}$ in (1.10) above; namely, if $d_{1,a}$ is complete and if $(\Delta + q)|_{C_c(V)}$ is semi-bounded below, then $(\Delta + q)|_{C_c(V)}$ is essentially self-adjoint on the space $\ell^2_w(V)$ with $w \equiv 1$. Theorem 1.5 of the present paper extends this result to the operator (1.6).

For a study of essential self-adjointness of $(\Delta + q)|_{C_c(V)}$ on a metrically non-complete graph, see Colin de Verdière, Torki-Hamza, and Truc [5]. Adjacency matrix operator on a locally finite graph was studied in Golénia [12]]. For a study of the problem of deficiency indices for Schrödinger operators on a locally finite graph, see Golénia–Schumacher [13].

Kato’s inequality for $\Delta_\sigma$ as in (1.3), with $w \equiv 1$ and $a \equiv 1$, was proven in Dodziuk–Mathai [10] and used to study asymptotic properties of the spectrum of a certain discrete magnetic Schrödinger operator. For a study of essential self-adjointness of the magnetic Laplacian on a metrically non-complete graph, see Colin de Verdière, Torki-Hamza, and Truc [6]. A different model for discrete magnetic Laplacian was given by Susch [28]. In the model of [28], the essential self-adjointness of a semi-bounded below discrete magnetic Schrödinger operator was proven.

Dodziuk [8], Wojciechowski [33], Wojciechowski [34], and Weber [31] explored connections between stochastic completeness and the essential self-adjointness of $\Delta$. For extensions to the more general context of Dirichlet forms on discrete sets, see Keller–Lenz [17] and Keller–Lenz [18]. For a related study of random walks on infinite graphs, see Dodziuk [7], Dodziuk-Karp [9], Woess [32], and references therein.

Finally, we remark that the problem of essential self-adjointness of Schrödinger operators on infinite graphs has a strong connection to the corresponding problem on non-compact Riemannian manifolds; see Gaffney [11], Oleinik [22], Oleinik [23], Braverman [1], Shubin [25], Shubin [26], and [2].
3. Preliminaries

In what follows, \( d: C(V) \to C(E_s) \) is the standard differential
\[
du(e) := u(t(e)) - u(o(e)).
\]
The deformed differential \( d_\sigma: C(V) \to C(E_s) \) is defined as
\[
(d_\sigma u)(e) := \overline{\sigma(e)u(t(e)) - u(o(e))}, \quad \text{for all } u \in C(V),
\]
where \( \sigma \) is as in (1.4).

The deformed co-differential \( \delta: C(E_s) \to C(V) \) is defined as follows:
\[
(\delta Y)(x) := \frac{1}{w(x)} \sum_{e \in E_s, \, t(e) = x} \sigma(e)a(e)Y(e) - \frac{1}{w(x)} \sum_{e \in E_s, \, o(e) = x} a(e)Y(e),
\]
for all \( Y \in C(E_s) \), where \( \sigma, w, \) and \( a \) are as in (1.4).

Let \( \ell^2_a(E_s) \) denote the space of functions \( F \in C(E_s) \) such that \( \|F\| < \infty \), where \( \|F\| \) is the norm corresponding to the inner product
\[
(F, G) := \sum_{e \in E_s} a(e)F(e)\overline{G(e)}.
\]

For a general background on the theory of magnetic Laplacian on graphs, see Mathai–Yates [20] and Sunada [27].

**Lemma 3.1.** The following equality holds:
\[
(d_\sigma u, Y) = (u, \delta Y), \quad \text{for all } u \in \ell^2_w(V), \ Y \in C_c(E_s),
\]
where \((\cdot, \cdot)\) on the left-hand side (right-hand side) denotes the inner product in \( \ell^2_a(E_s) \) (in \( \ell^2_w(V) \)).

**Proof.** Using (3.1) and (3.2) we have
\[
(u, \delta Y) = \sum_{x \in V} u(x) \left( \sum_{e \in E_s, \, t(e) = x} a(e)\overline{\sigma(e)Y(e)} - \sum_{e \in E_s, \, o(e) = x} a(e)Y(e) \right)
\]
\[
= \sum_{e \in E_s} a(e)u(t(e))\overline{\sigma(e)Y(e)} - \sum_{e \in E_s} a(e)u(o(e))Y(e)
\]
\[
= \sum_{e \in E_s} a(e)(\overline{\sigma(e)u(t(e))} - u(o(e)))Y(e) = (d_\sigma u, Y).
\]
The convergence of the sums is justified by observing that only finitely many \( x \in V \) contribute to the sum as \( Y \) has finite support. \( \square \)

Using the definitions (3.1) and (3.2) together with the properties \( a(\bar{e}) = a(e), \ \sigma(\bar{e}) = \sigma(e), \) and \( |\sigma(e)| = 1 \), which hold for all \( e \in E_0 \), one can easily prove the following lemma.

**Lemma 3.2.** The equality \( \delta d_\sigma u = \Delta_\sigma u \) holds for all \( u \in C(V) \).
The following lemma follows easily from Lemma 3.2 and (3.3).

Lemma 3.3. The operator $\Delta_\sigma |_{C_c(V)}$ is symmetric in $\ell^2_w(V)$:

$$(\Delta_\sigma u, v) = (u, \Delta_\sigma v), \quad \text{for all } u, v \in C_c(V).$$

Lemma 3.4. For all $u, v \in C(V)$ the following property holds:

$$(\Delta_\sigma (uv))(x) = (\Delta_\sigma u)(x)v(x)$$

$$+ \frac{1}{w(x)} \sum_{e \in O_x} a(e)\sigma(\hat{e})u(t(e))(v(x) - v(t(e))),$$

(3.4)

where $x \in V$ and $O_x$ is as in (1.5).

Proof. Using the definition (1.4) we have

$$(\Delta_\sigma (uv))(x) = \frac{1}{w(x)} \sum_{e \in O_x} a(e)(u(x)v(x))$$

$$- \frac{1}{w(x)} \sum_{e \in O_x} a(e)\sigma(\hat{e})u(t(e)v(t(e))).$$

(3.5)

Adding and subtracting

$$\frac{1}{w(x)} \sum_{e \in O_x} a(e)\sigma(\hat{e})u(t(e)v(x)}$$

on the right-hand side of (3.5) and grouping the terms appropriately, we get (3.4). □

In the proof of the following proposition, we will use a technique similar to Shubin [26, Section 5.1], Masamune [19], and Torki-Hamza [30].

Proposition 3.5. Assume that $u \in \ell^2_w(V)$ and $Hu = 0$. Then the following holds for all $\phi \in C_c(V)$:

$$(H(u\phi), u\phi)$$

$$= \sum_{e \in E_s} a(e)\sigma_1(\hat{e})[u_1(t(e))u_1(o(e)) + u_2(t(e))u_2(o(e))](\phi(o(e)) - \phi(t(e)))^2$$

$$+ \sum_{e \in E_s} a(e)\sigma_2(\hat{e})[-u_1(o(e))u_2(t(e)) +$$

$$+ u_1(t(e))u_2(o(e))](\phi(o(e)) - \phi(t(e)))^2,$$

(3.6)

where $u_1 := Re u$, $u_2 := Im u$, $\sigma_1 := Re \sigma$, and $\sigma_2 := Im \sigma$.

Proof. Using (3.4) with $v = \phi$, we obtain

$$(H(u\phi))(x) = (Hu)(x)\phi(x)$$

$$+ \frac{1}{w(x)} \sum_{e \in O_x} a(e)\sigma(\hat{e})u(t(e))(\phi(x) - \phi(t(e))).$$

(3.7)
Taking the inner product $(\cdot, \cdot)$ with $u\phi$ on both sides of (3.7), we obtain:

\[
(H(u\phi), u\phi) = (\phi(Hu), u\phi) \\
+ \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma(\tilde{e})u(t(e))(\phi(x) - \phi(t(e)))u(x)\phi(x).
\]  

(3.8)

Taking the real parts on both sides of (3.8), we get

\[
(H(u\phi), u\phi) = \text{Re } (\phi(Hu), u\phi) \\
+ \text{Re} \left( \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma(\tilde{e})u(t(e))(\phi(x) - \phi(t(e)))u(x)\phi(x) \right).
\]  

(3.9)

Since $\sigma(\tilde{e}) = \overline{\sigma(e)}$, it follows that $\sigma_1(\tilde{e}) = \sigma_1(e)$ and $\sigma_2(\tilde{e}) = -\sigma_2(e)$. Substituting $u = u_1 + iu_2$, $\sigma = \sigma_1 + i\sigma_2$ and $Hu = 0$ in (3.9) leads to

\[
(H(u\phi), u\phi) = J_1 + J_2,
\]  

(3.10)

where

\[
J_1 := \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma_1(\tilde{e})[u_1(t(e))u_1(x) + \\
+ u_2(t(e))u_2(x)](\phi^2(x) - \phi(x)\phi(t(e))),
\]

and

\[
J_2 := \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma_2(\tilde{e})[-u_1(x)u_2(t(e)) + \\
+ u_1(t(e))u_2(x)](\phi^2(x) - \phi(x)\phi(t(e))).
\]

In each of the sums $J_1$ and $J_2$ an edge $e = [x, y] \in \mathcal{E}_0$ occurs twice: once as $[x, y]$ and once as $[y, x]$. Since $a([x, y]) = a([y, x])$, $\sigma_1([x, y]) = \sigma_1([y, x])$, and $\sigma_2([x, y]) = -\sigma_2([y, x])$, it follows that the expressions

\[
a(e)\sigma_1(\tilde{e})(u_1(t(e))u_1(x) + u_2(t(e))u_2(x))
\]

\[
= a([x, y])\sigma_1([y, x])(u_1(y)u_1(x) + u_2(y)u_2(x))
\]

and

\[
a(e)\sigma_2(\tilde{e})(-u_1(x)u_2(t(e)) + u_1(t(e))u_2(x))
\]

\[
= a([x, y])\sigma_2([y, x])(-u_1(x)u_1(y) + u_1(y)u_2(x))
\]

are invariant under the involution $e \mapsto \tilde{e}$. Hence, in the sum $J_1$, the contribution of $e = [x, y]$ and $\tilde{e} = [y, x]$ together is

\[
a(e)\sigma_1(\tilde{e})(u_1(t(e))u_1(x) + u_2(t(e))u_2(x))(\phi(x) - \phi(t(e)))^2.
\]  

(3.11)

In the sum $J_2$, the contribution of $e = [x, y]$ and $\tilde{e} = [y, x]$ together is

\[
a(e)\sigma_2(\tilde{e})(-u_1(x)u_2(t(e)) + u_1(t(e))u_2(x))(\phi(x) - \phi(t(e)))^2.
\]  

(3.12)

Using (3.11) and (3.12), we can rewrite (3.10) to get (3.5). □
We now give the definitions of minimal and maximal operators associated with the expression (1.6).

3.6. Operators $H_{\min}$ and $H_{\max}$. We define the operator $H_{\min}$ by the formula

$$H_{\min} u := H u, \quad \text{Dom}(H_{\min}) := C_c(V).$$

(3.13)

Since $q$ is real-valued, the following lemma follows easily from Lemma 3.3.

Lemma 3.7. The operator $H_{\min}$ is symmetric in $\ell^2_w(V)$.

We define $H_{\max} := (H_{\min})^*$, where $T^*$ denotes the adjoint of operator $T$. We also define $D := \{ u \in \ell^2_w(V) : Hu \in \ell^2_w(V) \}$.

Lemma 3.8. The following hold: $\text{Dom}(H_{\max}) = D$ and $H_{\max} u = Hu$ for all $u \in D$.

Proof. Suppose that $v \in D$. Then, for all $u \in C_c(V)$ we have

$$(H_{\min} u, v) = (\Delta_{\sigma} u + qu, v) = (u, \Delta_{\sigma} v + qv).$$

Since $(\Delta_{\sigma} v + qv) \in \ell^2_w(V)$, by the definition of the adjoint we obtain $v \in \text{Dom}((H_{\min})^*)$ and $(H_{\min})^* v = \Delta_{\sigma} v + qv$. This shows that $D \subset \text{Dom}((H_{\min})^*)$ and $(H_{\min})^* v = Hv$ for all $v \in D$.

Suppose that $v \in \text{Dom}((H_{\min})^*)$. Then, there exists $z \in \ell^2_w(V)$ such that

$$(\Delta_{\sigma} u + qu, v) = (u, z), \quad \text{for all } u \in C_c(V).$$

(3.14)

Since $(\Delta_{\sigma} u + qu, v) = (u, \Delta_{\sigma} v + qv)$ and since $C_c(V)$ is dense in $\ell^2_w(V)$, from (3.14) it follows that $\Delta_{\sigma} v + qv = z = (H_{\min})^* v$. This shows that $\text{Dom}((H_{\min})^*) \subset D$. Thus, we have shown that $D = \text{Dom}((H_{\min})^*)$ and $(H_{\min})^* v = Hv$ for all $v \in D$. \hfill \Box

4. Proof of Theorem 1.3

We begin with a version of Kato’s inequality for discrete magnetic Laplacian. For the original version in the setting of differential operators, see Kato [16]. In the case $w \equiv 1$ and $a \equiv 1$, the following lemma was proven in Dodziuk–Mathai [10].

Lemma 4.1. Let $\Delta$ and $\Delta_{\sigma}$ be as in (2.1) and (1.4) respectively. Then, the following pointwise inequality holds for all $u \in C(V)$:

$$|u| \cdot \Delta|u| \leq \text{Re} \, (\Delta_{\sigma} u \cdot \overline{u}),$$

(4.1)

where $\text{Re} \, z$ denotes the real part of a complex number $z$.

Proof. Using (2.1), (1.4), and the property $|\sigma(\tilde{e})| \leq 1$, we obtain

$$(|u| \cdot \Delta|u|)(x) - \text{Re} \, (\Delta_{\sigma} u \cdot \overline{u})(x)$$

$$= \frac{1}{w(x)} \sum_{e \in O_x} a(e) \text{Re} (\sigma(\tilde{e}) u(t(e)) \overline{u(x)} - |u(x)||u(t(e))|) \leq 0,$$

and the lemma is proven. \hfill \Box
Continuation of the Proof of Theorem 1.3
Without loss of generality, we may assume $w(x) \equiv w_0 = 1$. By adding a constant to $q$, we may assume that $q(x) \geq 1$, for all $x \in V$. Let $H_{\text{min}}$ and $H_{\text{max}}$ be as in Section 3.6.

Since $H_{\text{min}} = H|_{C_c(V)}$ is symmetric and since $(H_{\text{min}} u, u) \geq \|u\|^2$, for all $u \in C_c(V)$, the essential self-adjointness of $H_{\text{min}}$ is equivalent to the following statement: $\ker(H_{\text{max}}) = \{0\}$; see Reed–Simon [24, Theorem X.26]. Let $u \in \text{Dom}(H_{\text{max}})$ satisfy $H_{\text{max}} u = 0$:

$$\Delta \sigma + q)u = 0. \quad (4.2)$$

By (4.1) and (4.2) we get the pointwise inequality

$$|u| \cdot \Delta |u| \leq \text{Re} \left( \Delta \sigma u \cdot \overline{u} \right) = \text{Re} \left( -qu \cdot \overline{u} \right) = -q |u|^2 \leq -|u|^2. \quad (4.3)$$

Rewriting (4.3) we obtain the pointwise inequality

$$|u| (\Delta |u| + |u|) \leq 0,$$

which leads to

$$0 \geq (\Delta |u|)(x) + |u(x)| = \sum_{e \in \partial x} a(e) (|u(x)| - |u(t(e))|) + |u(x)|, \quad (4.4)$$

for all $x \in V$.

From here on, the argument is the same as in Torki-Hamza [30, Theorem 3.1]. Assume that there exists $x_0 \in V$ such that $|u(x_0)| > 0$. Then, by (4.4) with $x = x_0$, there exists $x_1 \in V$ such that $|u(x_0)| < |u(x_1)|$. Using (4.4) with $x = x_1$, we see that there exists $x_2 \in V$ such that $|u(x_2)| > |u(x_1)|$. Continuing like this, we get a strictly increasing sequence of positive real numbers $|u(x_n)|$. But this contradicts the fact that $|u| \in \ell^2_w(V)$. Hence, $|u| \leq 0$ for all $x \in V$. In other words, $u = 0$. \hfill \Box

5. Proof of Theorem 1.4

In what follows, we will use a sequence of cut-off functions.

5.1. Cut-off functions. Fix a vertex $x_0 \in V$, and define

$$\phi_n(x) := \left( \left( \frac{2n - r(x)}{n} \right) \lor 0 \right) \land 1, \quad x \in V, \quad n \in \mathbb{Z}_+, \quad (5.1)$$

where and $r(x) = d(x_0, x)$ is as in Section 1.1.

As shown in Masamune [19, Proposition 3.2], the sequence $\{\phi_n\}_{n \in \mathbb{Z}_+}$ satisfies the following properties:

(i) $0 \leq \phi_n(x) \leq 1$, for all $x \in V$;

(ii) $\phi_n(x) = 1$ for $x \in B_n(x_0)$, and $\phi_n(x) = 0$ for $x \notin B_{2n}(x_0)$;

(iii) $\sup_{e \in E_x} |(d\phi_n)(e)| \leq \frac{1}{n}$.

Continuation of the Proof of Theorem 1.4. We will use a technique similar to Shubin [26, Section 5.1], Masamune [19], and Torki-Hamza [30].
Since \( H \) satisfies (1.9), without loss of generality, we may add \((C + 1)I\) to \( H \) and assume that
\[
(Hv, v) \geq \|v\|^2, \quad \text{for all } v \in C_c(V). \tag{5.2}
\]
Since \( H_{\text{min}} = H|_{C_c(V)} \) is symmetric and satisfies (5.2), the essential self-adjointness of \( H_{\text{min}} \) is equivalent to the following statement: \( \ker(H_{\text{max}}) = \{0\} \); see Reed–Simon [24, Theorem X.26].

Let \( u \in \text{Dom}(H_{\text{max}}) \) satisfy \( H_{\text{max}}u = 0 \). Let \( \phi_n \) be as in Section 5.1. Starting from (3.6) with \( \phi = \phi_n \) and using the properties (ii) and (iii) of \( \phi_n \), together with \( |\sigma_1| \leq 1 \) and \( |\sigma_2| \leq 1 \), we get the following estimate:
\[
(H(u\phi_n), u\phi_n) \leq \frac{1}{n^2} \sum_{e \in B_{2n}(x_0)} a(e)(u_1^2(t(e)) + u_1^2(o(e)) + u_2^2(t(e)) + u_2^2(o(e))), \tag{5.3}
\]
where \( B_{2n}(x_0) \) is as in property (ii) of \( \phi_n \).

By (1.8) and (5.3) we obtain
\[
(H(u\phi_n), u\phi_n) \leq \frac{m_{2n}a_{2n}}{n^2} \sum_{x \in B_{2n}(x_0)} w(x)((u_1(x))^2 + (u_2(x))^2)
\leq \frac{m_{2n}a_{2n}}{n^2} \|u\|^2. \tag{5.4}
\]
Since \( \phi_n u \in C_c(V) \), the inequality (5.2) is satisfied with \( v = \phi_n u \). Combining (5.4) and (5.2) we get
\[
\|u\phi_n\|^2 \leq \frac{m_{2n}a_{2n}}{n^2} \|u\|^2. \tag{5.5}
\]
We now take the limit as \( n \to \infty \) in (5.5). Using the assumption (1.7) and the definition of \( \phi_n \), we obtain \( \|u\|^2 \leq 0 \). This shows that \( u = 0 \). \( \square \)

6. Proof of Theorem 1.5

In the case \( w \equiv 1 \), the following family of cut-off functions was constructed in Torki-Hamza [30].

6.1. Family of cut-off functions. Fix \( x_0 \in V \). For \( R > 0 \) define
\[
U_R := \{x \in V: d_{w,a}(x_0, x) \leq R\}, \tag{6.1}
\]
where \( d_{w,a} \) is as in (1.10). Define
\[
\psi_R := \min\{1, d_{w,a}(x, V \setminus U_{R+1})\}. \tag{6.2}
\]
The family \( \psi_R \) satisfies the following properties:
(i) \( \psi_R(x) \equiv 1 \), for all \( x \in U_R \); (ii) \( \psi_R(x) \equiv 0 \), for all \( x \in V \setminus U_{R+1} \); (iii) \( 0 \leq \psi_R \leq 1 \), for all \( x \in V \);
(iv) \( \psi_R \) has finite support;
(v) \( \psi_R \) is a Lipschitz function with Lipschitz constant 1.

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It is easy to see that the properties (i), (ii), (iii) and (v) hold. To prove property (iv), we will show that \( U_{R+1} \) is finite. Clearly, \( U_{R+1} \) is a closed and bounded set. With \( d_{w,a} \) defined as in [1.10], it follows that \( (V,d_{w,a}) \) is a length space in the sense of Burago–Burago–Ivanov [3, Section 2.1]. Additionally, we know by hypothesis that \( (V,d_{w,a}) \) is complete. Thus, by [3, Theorem 2.5.28] the set \( U_{R+1} \) is compact. Suppose that there exists a sequence of vertices \( \{x_n\}_{n \in \mathbb{Z}^+} \subset U_{R+1} \). Since \( U_{R+1} \) is compact, there exists a subsequence, which we again denote by \( \{x_n\}_{n \in \mathbb{Z}^+} \), such that \( x_n \to x \) and \( x \in U_{R+1} \). Let \( F = \{y_1, y_2, \ldots, y_s\} \) be the set of all vertices \( y \in V \) such that there is an edge connecting \( y \) and \( x \). The set \( F \) is finite since \( G \) is locally finite. Let \( k_0 = \max\{n: x_n \in F\} \) (if there is no \( x_n \) such that \( x_n \in F \), we take \( k_0 = 0 \)). Take \( \varepsilon > 0 \) such that \( \varepsilon < \min_{1 \leq j \leq s} (d_{w,a}(y_j, x)) \). Then there exists \( n_0 \in \mathbb{Z}^+ \) such that \( d_{w,a}(x_n, x) < \varepsilon \) for all \( n \geq n_0 \). Take \( K \in \mathbb{Z}^+ \) such that \( K > \max\{k_0, n_0\} \). Clearly, \( d_{w,a}(x_K, x) < \varepsilon \). Since \( (V,d_{w,a}) \) is a complete locally compact length space, by [3, Theorem 2.5.23] there is a shortest path \( \gamma \) connecting \( x_K \) and \( x \). This means that the length \( L_{w,a}(\gamma) \) of the path \( \gamma \) satisfies

\[
L_{w,a}(\gamma) = d_{w,a}(x_K, x) < \varepsilon.
\] (6.3)

Since \( x_K \not\in F \), there is no edge connecting \( x_K \) and \( x \). Hence, the path \( \gamma \) will contain a vertex \( y_j \in F \). Thus, \( L_{w,a}(\gamma) > d_{w,a}(y_j, x) > \varepsilon \), and this contradicts \( \varepsilon \). Hence, the set \( U_{R+1} \) is finite.

**Continuation of the Proof of Theorem 1.5.** We adapt the technique of Torki-Hamza [30] to our setting.

As in the proof of Theorem 1.4, without the loss of generality, we will assume \( (5.2) \) and show that \( \ker(H_{\text{max}}) = \{0\} \). Let \( u \in \text{Dom}(H_{\text{max}}) \) satisfy \( H_{\text{max}}u = 0 \). Using (3.6) with \( \phi = \psi_R \), we get

\[
(H(u\psi_R), u\psi_R) = \frac{1}{2} \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma_1(\tilde{e})[u_1(t(e))u_1(o(e)) + u_2(t(e))u_2(o(e))] (\psi_R(o(e)) - \psi_R(t(e)))^2 + \frac{1}{2} \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma_2(\tilde{e})[u_1(o(e))u_2(t(e)) + u_1(t(e))u_2(o(e))] (\psi_R(o(e)) - \psi_R(t(e)))^2,
\] (6.4)

where \( \mathcal{O}_x \) is as in (1.5). Using the inequality \( 2\alpha\beta \leq \alpha^2 + \beta^2 \), properties \( |\sigma_1| \leq 1 \) and \( |\sigma_2| \leq 1 \), and the invariance of \( a(e) \) and

\[
(u_j^2(t(e)) + u_j^2(o(e)))(\psi_R(o(e)) - \psi_R(t(e)))^2, \quad j = 1, 2
\]

under involution \( e \mapsto \tilde{e} \), we get

\[
(H(u\psi_R), u\psi_R) \leq \frac{1}{2} \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)(u_1^2(o(e)) + u_2^2(o(e)))(\psi_R(o(e)) - \psi_R(t(e)))^2.
\] (6.5)
Using properties (i), (ii) and (v) of $\psi_R$, (6.3) leads to

\[
(H(u\psi_R), u\psi_R) \leq \frac{1}{2} \sum_{x \in U_{R+1}\setminus U_R} \sum_{e \in \Omega_x} a(e) |u(o(e))|^2 (d_{w,a}(o(e), t(e)))^2.
\]

(6.6)

By (1.10) and (1.11) it follows that

\[
d_{w,a}(o(e), t(e)) \leq \sqrt{\frac{w(a(e))}{a(e)}}.
\]

(6.7)

Using (6.6), (6.7), and bounded degree assumption on $G$, we get

\[
(H(u\psi_R), u\psi_R) \leq \frac{N}{2} \sum_{x \in U_{R+1}\setminus U_R} w(x)|u(x)|^2.
\]

(6.8)

By property (iv) of $\psi_R$, it follows that $\psi_R u \in C_c(V)$; hence, the inequality (5.2) is satisfied with $v = \psi_R u$. Combining (6.8) and (5.2) we get

\[
\|u\psi_R\|^2 \leq \frac{N}{2} \sum_{x \in U_{R+1}\setminus U_R} w(x)|u(x)|^2.
\]

(6.9)

We now take the limit as $R \to \infty$ in (6.9). Using the definition of $\psi_R$ and the assumption $u \in l^2_w(V)$, we obtain $\|u\|^2 \leq 0$. This shows that $u = 0$. \hfill \Box

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTH FLORIDA, JACKSONVILLE, FL 32224, USA
E-mail address: omilatov@unf.edu