Representing objects for $G$-crossed products

Ofir David

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel
E-mail address: ofirdav@tx.technion.ac.il

Abstract

Let $G$ be a finite group of order $n$ and $\mathbb{F}$ a field of characteristic zero. Let $M_n(\mathbb{F})$ be graded with the $G$-crossed product grading and let $\Omega = \mathbb{F} \langle X \rangle / \text{Id}_G(M_n(\mathbb{F}))$ be the relatively free algebra. We use $\Omega$ to construct representing objects for $G$-crossed products over field extensions of $\mathbb{F}$ and show that it is essentially isomorphic to other constructions of representing objects for crossed products given by Snider, Rosset and Saltman. We then investigate the fraction field of the center of these representing objects and determine how close it is to being a rational extension of $\mathbb{F}$ for certain families of groups.

1 Introduction

Given a field $\mathbb{F}$, one of its most important invariants is its Brauer group, denoted by $Br(\mathbb{F})$, which up to an equivalence relation consists of simple algebras with center $\mathbb{F}$. It is well known that any $\mathbb{F}$-central simple algebra $A$ is equivalent to a crossed product over over $\mathbb{F}$, namely there is a finite group $G$, a $G$-Galois extension of fields $L/\mathbb{F}$ and a 2-cocycle $\alpha \in Z^2(G, L^\times)$ such that $A \sim L^\alpha G$. Furthermore, the equivalence classes of central simple algebras which are split by the extension $L$ are in one to one correspondence with elements of $H^2(G, L^\times)$. Thus, in order to study the Brauer group, we can restrict ourselves to the study of crossed products over the field $\mathbb{F}$.

As each division algebra is Brauer equivalent to a crossed product, one may ask whether each division algebra is actually isomorphic to a crossed product. This claim turns to be true for certain families of fields, for example local and global fields (this is the Brauer-Hasse-Noether theorem), although it is false in general. The first counter example is due to Amitsur in [4]. For this purpose, Amitsur constructed a division algebra $D_n$ of dimension $n^2$ over its center (where $8 \mid n$ or $p^2 \mid n$ for an odd prime $p$) which had the following property – if $D_n$ is a $G$-crossed product, then any central simple algebra of dimension $n^2$ over a field of characteristic zero is also a $G$-crossed product with the same group $G$. He then constructed two division algebras $A_1, A_2$ of dimension $n^2$ over their centers, such that $A_i$ is a crossed product only with a group $G_i$ for $i = 1, 2$, and $G_1$ isn’t isomorphic to $G_2$. Hence $D_n$ cannot be a $G$-crossed product for any group $G$.

The algebra $D_n$, called the generic division algebra of dimension $n^2$, continued to play an important role in the study of the Brauer group. The idea is that there are certain nice properties which hold for $D_n$ if and only if they hold for any central simple algebra of dimension $n^2$.

Let us recall another such property. A central simple algebra is called a cyclic algebra if it is isomorphic to a crossed product with a cyclic group. The Merkurjev-Suslin theorem [26] states that
over a field $\mathbb{F}$ which contains enough roots of unity, every central simple algebra is equivalent to a tensor product of cyclic algebras, or in other words $Br(\mathbb{F})$ is generated by the classes of cyclic algebras. While Merkurjev-Suslin’s theorem uses $K$-cohomology, another approach to this problem prior to their proof was through generic algebras. More precisely, if $D_n$ is equivalent to a product of cyclic algebras, then so is every central simple algebra of dimension $n^2$ over a field extension of $\mathbb{F}$.

One approach to obtain such a result is using Bloch’s theorem. Fix an integer $m \in \mathbb{N}$ and suppose that every $[A] \in Br(\mathbb{F})$ of order dividing $m$ is equivalent to a product of cyclic algebras, $m$ is invertible in $\mathbb{F}$ and $\mathbb{F}$ contains a primitive $m$-th root of unity. Then Bloch’s theorem states that for $x$ transcendental over $\mathbb{F}$, any $[B] \in Br(\mathbb{F}(x))$ of order dividing $m$ is also Brauer equivalent to a product of cyclic algebras. We conclude that if $Br(\mathbb{F})$ is generated by cyclic algebras and the center of $D_n$ is a rational extension of $\mathbb{F}$ (i.e. purely transcendental), then $D_n$ is Brauer equivalent to a product of cyclic algebras. For more details and a proof of Bloch’s theorem using the Auslander-Brumer Faddeev theorem see [12], and for Bloch’s original proof see [10].

The last argument shows that the center $Z_n$ of $D_n$ is significant in the study of Brauer groups, or more precisely we would like to know if $Z_n/\mathbb{F}$ is a rational extension or not.

Over the rational field $\mathbb{Q}$, the first to calculate a transcendental basis for $Z_2$ was Sylvester in [40]. It was later reproved by Procesi who further studied the fields $Z_n$ (see [29]). Later Formanek used Procesi’s method to give a positive solution for $n = 3, 4$ (see [14, 15] and also [16]). Le Bruyn and Bessenrodt in [9] proved that for $n = 5, 7$ the field $Z_n$ is a stably rational extension of $\mathbb{F}$ and Beneish [7] gave another more elementary proof for these primes. Schofield [37], Katsylo [21] and Saltman [35] showed that $Z_{nm}$ is stably equivalent to the fraction field of $Z_n \otimes Z_m$ whenever $n, m$ are coprime. Finally, Saltman showed in [34] that $Z_p$ is retract rational over $\mathbb{F}$ for $p$ prime, and this result can be extended to product of distinct primes. For more information on this see Le Bruyn [23] (the definitions for the different types of rationality are given in beginning of 3).

The goal of this paper is to construct “generic” crossed products, and determine how far their centers are from being a rational extensions of the base field. There are several constructions for such objects in the literature. Our construction adapts a construction similar to the standard generic division algebra, which uses polynomial identities. More precisely, let $G$ be a finite group and consider the elementary grading on $M_n(\mathbb{F})$ induced by the tuple $\tilde{g} = (g_1, ..., g_n) \in G^n$, such that each element of $G$ appear exactly once in $\tilde{g}$, i.e. the $g$ homogeneous part of $M_n(\mathbb{F})$ is spanned by $\{e_{i,j} \mid g_1^{-1}g_j = g\}$. This grading is called the crossed product grading. Let $\mathbb{F}\langle X_G \rangle$ be the free graded algebra over the noncommuting indeterminates $X_G = \{x_{g,i} \mid g \in G, i \in \mathbb{N}\}$, and set $I = Id_G(M_n(\mathbb{F})) \leq \mathbb{F}\langle X_G \rangle$ to be the ideal of graded identities of $M_n(\mathbb{F})$ with the crossed product grading. Our representing objects are suitable central localization of the relatively free graded algebra $\mathbb{F}\langle X_G \rangle/I$. Another equivalent construction will be given using generic graded matrices.

As we shall see, every $G$-crossed product $A$ will be twisted forms of $M_n(\mathbb{F}^{ad})$, namely $A \otimes_\mathbb{F} \mathbb{F}^{ad} \cong M_n(\mathbb{F}^{ad})$ as graded algebras. We note here that constructions of generic algebras using graded identities can be generalized to classes of twisted forms of other graded simple algebras (see [1], [2]). We also refer the reader to Aljadeff and Karasik’s [3] where a set of generators for $Z(\mathbb{F}\langle X_G \rangle/I)$ is given (in this work we are interested in the fraction field of this center).

Similar to the non-graded case, many nice properties are inherited from the generic crossed product to any other crossed product over a field extension of $\mathbb{F}$. In particular, Bloch’s theorem is still applicable, given that the center is a rational extension of the base field (for details, see [38]). In addition, if the center is retract rational over the base field, then the class of $G$-crossed products has a lifting property (i.e. crossed product over a residue field of a local ring can be lifted to the local
ring). For more details on retract rational extensions and their connection to the center of generic algebras, we refer the reader to Saltman’s [34].

As in Formanek’s work [16], the center of the generic crossed product can be described using $G$-lattices. More precisely, these lattices are achieved by taking the kernel of $\partial_2$ in the bar resolution of $G$, which we denote by $\text{Eu}(G)$. This lattice plays a central role in the theory of lattice invariants by determining when the flasque lattices are exactly the invertible lattices. This result is due to Endo and Miyata [11] and its formulation appears in 4.

In this paper we follow Saltman’s notation used in [34], and in particular the word “generic” has a different meaning than the one used in “generic division algebra”. Since the usual meaning of “generic” will be the central localization of densely representing objects, we shall call them the localized representing algebras.

Denote by $F(G)$ the center of the localized representing crossed product. Using lattice invariants methods we were able to prove the stable rationality and retract rationality for several families of groups.

**Theorem 1.** Let $G$ be a finite group of exponent $m$, and let $\mathbb{F}$ be a field of characteristic zero containing a primitive root of unity of order $m$. Then

1. If every $p$-Sylow subgroup of $G$ is cyclic, then $F(G)/\mathbb{F}$ is a retract rational extension.
2. If $G = \langle \sigma, \tau \mid \sigma^n = \tau^{2m} = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ where $(n, 2m) = 1$, then $F(G)/\mathbb{F}$ is stably rational.
3. If $G$ is either cyclic, or $G$ is a product of cyclic groups $C_2 \times C_{2d}$ where $d$ is odd, then $F(G)/\mathbb{F}$ is stably rational.

The next result is the counterpart of its non-graded version, where the rationality problem can be reduced to the prime power case.

**Theorem 2.** Let $G \cong H \times K$ where $H, K$ are groups of coprime orders. Then $F(G)$ is stably equivalent to the fraction field of $F(H) \otimes F(K)$.

The paper is organized as follows.

In section §2 we recall the definition of a generalized crossed product and some of its properties. After giving the definitions of graded algebras and their graded polynomial identities, we construct densely representing objects for crossed products using the graded relatively free algebra.

In the first part of section §3 we define the types of rationality extensions, and give an initial description for the center of the localized representing crossed products, using $G$-lattice methods. The second part is dedicated to the current constructions of crossed product representing objects given in the literature. The three constructions which are dealt with, are the Snider-Rosset construction [38, 30] which uses free presentations of groups, and two constructions due to Saltman given in [34] and [36] which uses generic 2-cocycles. In particular, we show that the localized representing objects in this paper and the ones mentioned above are essentially isomorphic, except for one of Saltman’s construction, which is slightly different.

Finally, in the last section we determine the rationality extension type of $F(G)/\mathbb{F}$ as given in the theorems above.
2 Constructing representing objects using graded identities

2.1 $G$-crossed products

Fix a field $F$ of characteristic zero. All the rings in this paper are assumed to be $F$-algebras, and the homomorphisms are $F$-algebra homomorphisms.

Let $R$ be an algebra and $G$ a finite group. A $G$-grading on an $R$-algebra $A$ is a decomposition $A = \bigoplus_{g \in G} A_g$ as an $R$-module such that $A_g \cdot A_h \subseteq A_{gh}$ for every $g, h \in G$. The elements in $A_g$ are called homogeneous of degree $g$. An algebra homomorphism $\varphi : A \to B$ between two $G$-graded algebras is called graded if $\varphi(A_g) \subseteq B_g$ for any $g \in G$.

A prime example and the main interest of this paper are the $G$-crossed products. Usually, these crossed products are defined over Galois extensions of field, though in this paper we require the generalized definition of Galois extensions of rings, which we now briefly describe. For a full treatment of Galois extension of commutative rings and the crossed products defined over them, we refer the reader to [27] and [28].

**Definition 3** (Galois Extension). Let $R \subseteq S$ be an extension of unital commutative rings, and let $G$ be a finite subgroup of $\text{Aut}(S)$. We say that $S$ is a $G$-Galois extension of $R$ if

1. $S$ is a faithful $R$-algebra.
2. $S^G = R$.
3. There are $x_1, ..., x_n, y_1, ..., y_n \in S$ for some $n \in \mathbb{N}$ such that $\sum_i x_i g(y_i) = \delta_{c,g}$ where $\delta_{c,g} = 1$ if $g = e$ and zero otherwise.

In case $R$ is a field, which is our main interest, its $G$-Galois extensions are always induced from $H$-Galois field extensions for subgroups $H$ of $G$.

Let us recall the precise definition of the $\text{Ind}_{H}^{G}(-)$ functor. Let $H \leq G$ be a subgroup and let $T = \{t_1, ..., t_m\}$ be left coset representatives of $H$ in $G$. For $1 \leq i \leq m$ and $\sigma \in G$ write $\sigma \cdot t = t_{\sigma(i)} \cdot h_{\sigma,i}$ where $1 \leq \sigma(i) \leq m$ and $h_{\sigma,i} \in H$. For an $H$-Galois extension $S'/R$ define $\text{Ind}_{H}^{G}(S')$ to be the product of $m$ copies of $S$ written as $\prod_{i=1}^{m} t_i \cdot S$ ($t_i$ is just a place holder) where the $G$ action is defined by

$$\sigma \left( \sum_{i=1}^{m} t_i s_i \right) = \sum_{i=1}^{m} t_{\sigma(i)} \cdot h(s_i),$$

and identify $R$ as the subalgebra $\{(\sum_{i=1}^{m} t_i) \cdot r \mid r \in R\}$. The extension $\text{Ind}_{H}^{G}(S')/R$ is a $G$-Galois extension, and for a field $R$, any $G$-Galois extension has the form $\text{Ind}_{H}^{G}(L)/R$ where $L$ is an $H$-Galois field extension of $R$. In particular, $\text{Ind}_{c}^{G}(R)$ is just $|G|$ copies of $R$ on which $G$ acts by permuting the idempotents.

**Definition 4** ($G$-Crossed Product). Let $S$ be a $G$-Galois extension of $R$ and let $c \in Z^2(G, S^\times)$ be a 2-cocycle, namely $c$ is a function from $G \times G$ to $S^\times$ that satisfies

for all $g_1, g_2, g_3 \in G$ \[ c(g_1, g_2)c(g_1g_2, g_3) = g_1 \cdot (c(g_2, g_3)) \cdot c(g_1, g_2g_3). \]

Consider the $R$-module $\bigoplus_{g \in G} S \cdot \epsilon_g$ with a multiplication defined by

for all $\alpha, \beta \in S$, $g, h \in G$ \[ \alpha \epsilon_g \beta \epsilon_h = \alpha g(\beta) \epsilon_{gh} = \alpha g(\beta) c(g, h) \epsilon_{gh}. \]
This algebra is called a \textit{G-crossed product} and is denoted by $\Delta(S/R, G, c)$. It has a natural $G$-grading defined by $(\Delta(S/R, G, c))_g = S \epsilon_g$.

Crossed products over Galois extensions of fields are essential in the study of central simple algebras. Their generalized versions play a similar role in the study of Azumaya algebras and have many similar properties as the crossed products over field extensions.

In particular, cohomologous 2-cocycles produce isomorphic crossed products, so we can assume that the two cocycles are always normalized, namely $\epsilon_c$ is the unity of $\Delta(S/R, G, c)$. We thus identify $S$ with $S \cdot \epsilon_c$ and $R$ with $R \cdot \epsilon_c$. Under this notation, the crossed product $\Delta(S/R, G, c)$ is an Azumaya algebra central over $S^G = R$.

It is well known that any crossed product over a Galois extension of fields is central simple, and therefore after suitable scalar extension it becomes a matrix algebra. Since crossed products have a natural group grading, it induces a grading on the matrix algebra which we now describe.

\textbf{Definition 5} (Elementary Grading). Let $A = M_n(R)$ and $\bar{g} = (g_1, ..., g_n) \in G^n$ be a tuple of length $n$. The \textit{elementary grading} on $A$ induced by $\bar{g}$ is defined by $A_h = \text{span}_R \{ E_{i,j} \mid g_i^{-1}g_j = h \}$, where $E_{i,j}$ is the matrix with 1 in the $(i, j)$ coordinate and zero elsewhere. In case $|G| = n$ and each element of $G$ appears exactly once in $\bar{g}$, the induced elementary grading is called the \textit{crossed product grading}.

\textbf{Remark 6}. Note that reordering the tuple $\bar{g}$ produces graded isomorphic algebras.

Let $M_n(R)$ have the crossed product grading with a tuple $\bar{g} = (g_1, ..., g_n) \in G^n$. Identifying $G$ with $\{1, ..., n\}$, we write $E_{g_i, g_j}$ instead of $E_{i,j}$. Let $S$ be the subalgebra of diagonal matrices in $M_n(R)$ and set $P_\bar{g}$ to be the permutation matrix $P_\bar{g} := \sum_{h \in G} E_{h, h g}$. Letting $G$ act on $S$ by $g(s) = P_\bar{g} s P_\bar{g}^{-1}$ for $g \in G$ and $s \in S$ we get that $S \cong \text{Ind}^G_c(R)$ as a $G$-Galois extension and $M_n(R) \cong \Delta(S/R, G, c, 1)$ with $(M_n(R))_g = S \cdot P_\bar{g}$. From now on we will always assume that the grading on $M_n(R)$ is the crossed product grading.

Similar to the standard case, if $\Delta := \Delta(S/R, G, c)$ is any crossed product, then $S$ splits $\Delta$, meaning that $\Delta \otimes_R S \cong M_G(S)$ as graded algebras. Suppose $R$ is an integral domain and $L$ its field of fractions. Then $S \otimes_R L/L$ is $G$-Galois and therefore $S \otimes_R L/L \cong \text{Ind}^G_c(E)$ for some subgroup $H \leq G$ and $H$-Galois extension $E/L$. It follows that $S \otimes_R E \cong \text{Ind}^G_c(E)$ and it can be shown that $\Delta \otimes_R E \cong M_n(E)$ as graded algebras. This fact connects the study of crossed products (and central simple algebras in general) to the study of polynomial identities.

### 2.2 Graded polynomial identities

Let $G$ be a finite group, $R$ a unital commutative ring of characteristic zero, and $A = \bigoplus_{g \in G} A_g$ a $G$-graded $R$-algebra such that $R \subseteq A_e$. For each $g \in G$ let $X_g := \{ x_{g,i} \mid i \in \mathbb{N} \}$ be a set of countably infinite many noncommuting indeterminates and set $X = \bigcup_{g \in G} X_g$. Denote by $R \langle X_G \rangle$ the free unital $R$-algebra on the noncommuting indeterminates $X_G$. This algebra admits a natural $G$ grading by defining the degree of a monomial $\prod x_{g_i, i}$ to be $\prod g_i$ and constants from $R$ to be homogenous of degree 0. A polynomial $f(x_{g_1, i_1}, ..., x_{g_n, i_n}) \in R \langle X_G \rangle$ is called a \textit{graded polynomial identity} of $A$ if $f(a_1, ..., a_n) = 0$ for all choices of $a_i \in A_{g_i}$. Denote by $Id^G_\mathbb{Z}(A)$ the set of all graded identities of $A$. 

5
in \( R \langle X_G \rangle \). The set \( \text{Id}_G(A) \) is a graded \( T \)-ideal in \( R \langle X_G \rangle \), namely it is a graded ideal that is closed under graded endomorphisms of \( R \langle X_G \rangle \).

Suppose that \( R \) is an integral domain and \( \mathbb{L} \) its field of fractions. Let \( A, B \) be \( G \)-graded \( R \)-algebras such that the natural homomorphism \( A \to A \otimes_R \mathbb{L} \) is injective (which is the case for crossed products over \( R \)) and similarly for \( B \). In this case it is well known that \( \text{Id}_G(A_L) = \text{Id}_G(A) \otimes_R \mathbb{L} \) and \( \text{Id}_G(A) = R \langle X_G \rangle \cap \text{Id}_G(A_L) \) and the same holds for \( B \). It follows that \( \text{Id}_G(A) = \text{Id}_G(B) \) if and only if \( \text{Id}_G(A_L) = \text{Id}_G(B_L) \). These properties also hold if \( R \) is a field and \( \mathbb{L} \) is any field extension of \( R \). Combining these two results we see that if \( R \) is an integral domain and \( R \subseteq \mathbb{L} \) is a field, then \( \text{Id}_G(A) = \text{Id}_G(B) \) if and only if \( \text{Id}_G(A)_{R'} = \text{Id}_G(B)_{R'} \), which proves the following.

**Corollary 7.** Let \( A = \Delta(S/R, G, c) \in \mathcal{C}(G, F) \) such that \( R \) is a unital integral domain, and let \( M_{\langle G \rangle}(R) \) be the matrix algebra with the crossed product grading. Then \( \text{Id}_G(A) = \text{Id}_G(M_{\langle G \rangle}(R)) \).

The last corollary shows that all \( G \)-crossed product over an integral domain \( R \) have the same \( T \)-ideal of identities \( \text{Id}_G(M_{\langle G \rangle}(R)) \). The next goal is to build the most general object in this class.

For a general \( T \)-ideal \( I \) in \( R \langle X_G \rangle \), define the \( I \)-relatively free algebra to be \( R \langle X_G \rangle / I \). This algebra is free in the sense that for any algebra \( A \) with \( I \subseteq \text{Id}_G(A) \), any graded map \( \phi : X_G \to A \) can be uniquely extended to a graded homomorphism \( \varphi : R \langle X_G \rangle / I \to A \). In particular, taking \( I = 1 \), it follows that all the \( G \)-crossed products over \( R \) are homomorphic images of \( \Omega_G := R \langle X_G \rangle / 1 \). In particular we are interested in \( \Omega := \Omega_{G} = R \langle X_G \rangle / \text{Id}_G(M_{\langle G \rangle}(R)) \) (note that \( \Omega \otimes_F R \cong \Omega_R \)). Later on, our main interest will be to describe the field of fractions of the center of \( \Omega \), i.e. quotient of nonzero central polynomials of \( M_{\langle G \rangle}(F) \).

Another useful presentation of \( \Omega \) is defined as follows. Let \( \Gamma = \{ u_{g, h, i} \mid g, h \in G, \ i \in \mathbb{N} \} \) be a set of commuting indeterminates over \( F \). Define \( U_{g, i} \in M_n(F[\Gamma]) \) to be a generic homogeneous \( g \)-matrix by setting

\[
(U_{g, i})_{h_1, h_2} = \begin{cases} 
  u_{h_1, h_2, i} & h_2 = h_1 g \\
  0 & \text{else}
\end{cases}
\]

Let \( \Omega' = \mathbb{F}[U_{g, i}] \) be the \( \mathbb{F} \)-subalgebra of \( M_n(F[\Gamma]) \) generated by the graded generic matrices \( \{ U_{g, i} \mid g \in G, \ i \in \mathbb{N} \} \). In a similar manner to the nongraded case, the homomorphism \( \psi : \Omega \to \Omega' \) defined by \( \psi(x_{g, i}) = U_{g, i} \) is an isomorphism of graded \( \mathbb{F} \) algebras, hence we denote both algebras by \( \Omega \).

### 2.3 Constructing representing objects

In the following we use Saltman’s notations given in [34], adapted to the case of crossed products. Let \( G \) be a finite group, \( \mathbb{F} \) a field of characteristic zero and let \( \mathcal{C}(G, F) \) be the class of all \( G \)-crossed products \( \Delta(S/R, G, c) \) such that \( R \) is an \( \mathbb{F} \)-algebra.

Let \( A_1 = \Delta(S_i/R_i, G, c_i) \in \mathcal{C} \) for \( i = 1, 2 \). A homomorphism \( \varphi : A_1 \to A_2 \) is always assumed to satisfy \( \varphi(R_1) \subseteq R_2 \). While we do not require for it so be graded, most homomorphism dealt with here are graded. We say that \( A_2 \) is a specialization of \( A_1 \) or that \( A_1 \) specializes to \( A_2 \), if there is a homomorphism \( \varphi : R_1 \to R_2 \) such that \( A_1 \otimes_{R_1} R_2 \cong A_2 \). To emphasize the role of \( \varphi \), we write \( A_1 \otimes_{\varphi} R_2 \) instead of \( A_1 \otimes_{R_1} R_2 \).
The class $C(G, \mathbb{F})$ is a special class of $\mathbb{F}$-classes as defined in [34]. Its defining properties are given by the next lemma.

**Lemma 8.** The class $C(G, \mathbb{F})$ has the following properties.

1. (Closure under specialization) Let $A = \Delta(S/R, G, c) \in C(G, \mathbb{F})$ and $\varphi : R \to R'$ an $\mathbb{F}$-algebra homomorphism. Then $A \otimes_R R' \in C(G, \mathbb{F})$.

2. (Finitely defined) Let $A = \Delta(S/R, G, c) \in C(G, \mathbb{F})$ and let $R'$ be a ring $\mathbb{F} \subseteq R' \subseteq R$. Then there is a $B = \Delta(S''/R'', G, c')$ such that $R' \subseteq R'' \subseteq R$, $R''$ is finitely generated over $R'$ and $B \otimes_{R''} R \cong A$.

**Proof.** 1. This is clear and is left to the reader.

2. By Theorem 1.2 in [32] there is a $G$-Galois extensions $S_1/R_1$ such that $S_1 \otimes_{R_1} R \cong S$ and $R' \subseteq R_1 \subseteq R$ is finitely generated over $R'$. Taking $R'' = R_1 \langle c(g, h) \mid g, h \in G \rangle$ we get that $R''$ is finitely generated over $R$ and $B = \Delta(S_1 \otimes_{R_1} R''/R'', G, c)$ satisfies $B \otimes_{R''} R \cong A$. \hfill $\Box$

**Remark 9.** Other important classes that satisfy this type of properties which also appear in Saltman’s work in [34] are the class of all ring Galois extensions with a fixed group $G$, and the class of all Azumaya algebras of degree $n^2$ for some fixed $n \in \mathbb{N}$.

**Definition 10.** Let $A = \Delta(S/R, G, c) \in C$.

1. The algebra $A$ is called a representing object in $C$ if $R$ is an integral domain affine over $\mathbb{F}$ (i.e. finitely generated) and it specializes to every $B \in C$ which is central over a field.

2. A representing object $A$ is said to be densely representing if for any $0 \neq r \in R$, the crossed product $A[r^{-1}] := A \otimes_R R[r^{-1}]$ is also a representing object.

3. A representing object $A$ is said to be a generic object if $R$ has the form $\mathbb{F}[x_1, ..., x_n][r^{-1}]$, namely a polynomial ring localized in a single polynomial.

As we saw before, the relatively free algebras $\Omega := \mathbb{F}(X_G)/I_{\Delta}(M_n(\mathbb{F}))$ is a natural candidates to be representing objects, except that it is not finitely generated. To overcome this, define $\Omega_r$ to be the $\mathbb{F}$-subalgebra of $\Omega$ generated by $\Gamma_r = \{U_{g,i} \mid g \in G, 1 \leq i \leq r\}$ (or alternatively by $\{x_{g,i} \mid g \in G, 1 \leq i \leq r\}$). Unfortunately, these algebras are not crossed products, although they become such after inverting their nonzero central elements, as is showed next. It follows from part (2) of the preceding lemma, that it is enough to invert a single element, thus producing crossed products affine over $\mathbb{F}$.

**Remark 11.** From now on we write $U_g$ instead of $U_{g,1}$ and $u_{h,g}$ instead of $u_{h,g,1}$. Note also that $U_g U_h U_{gh}^{-1}$ are diagonal elements in $M_n(\mathbb{F}(\Gamma))$.

Since $U_c$ is diagonal with distinct entries, the only matrices which commute with $U_c$ are diagonal. If in addition they commute with $U_g$ for all $g \in G$, then clearly they must be scalar. In particular the center of algebras containing $\Omega_1$ (e.g. the algebras $\Omega_r$) are just the scalar matrices.

The next step is to show that after central localization, the $c$ component becomes a field.
Lemma 12. Define a $G$ action on $F(\Gamma)$ by setting $\sigma(u_{g,h,i}) = u_{\sigma g, \sigma h, i}$. Then for each $a = \sum_{g \in G} a_g E_{g,g} \in \Omega_e$ we have $a_g = g(a_e)$. In particular, the nonzero homogeneous elements of $\Omega_e$ are invertible in $M_n(F(\Gamma))$.

Proof. Let $h_1, \ldots, h_m \in G$ such that $\prod_{j=1}^m h_j = e$. For every $g \in G$ we have

$$\left(\prod_{j=1}^m U_{h_j,i_j}\right)_{g,g} = \prod_{j=1}^m u_{g\Pi_{k<j} h_k, g\Pi_{k<j} h_k, i_j} = \prod_{j=1}^m g \left(\prod_{k<j} u_{\Pi_{k<j} h_k, \Pi_{k<j} h_k, i_j}\right) = g \left(\prod_{j=1}^m U_{h_j,i_j}\right)_{e,e}.$$

As $\Omega_e$ is spanned over $F$ by 1 and elements of the form $\prod U_{h_j,i_j}$ where $\prod h_j = e$, it follows that $a_g = g(a_e)$ for each $a = \sum_{g \in G} a_g E_{g,g} \in \Omega_g$ and $g \in G$. In particular, if $a \neq 0$, then all its entries on the diagonal are nonzero, hence $a$ is invertible in $M_n(F(\Gamma))$. Note that the elements $\{U_g\}_{g \in G}$ are invertible in $M_n(F(\Gamma))$, so an element $b \in \Omega_e$ is nonzero if and only if $U_{g^{-1}b} \in \Omega_e$ is nonzero and the result follows.

Let $E_r$ be the fraction field of $Z(\Omega_r)$ and set $\Omega'_r = \Omega_r \otimes_{Z(\Omega)} E_r$, which is viewed as an $E_r$-subalgebra of $M_n(F(\Gamma))$, and note that $E_r$ is the set of diagonal matrices in $\Omega'_r$.

Lemma 13. The algebra $\Omega'_r = \Omega_r \otimes_{Z(\Omega)} E_r$ is a $G$-crossed product over a $G$-Galois extension of fields. In addition, there is some $0 \neq s \in Z(\Omega_r)$ such that $\Omega_r[s^{-1}]$ is a $G$-crossed product.

Proof. Recall that a graded algebra $A$ is called graded prime if for any homogeneous elements $b, c \in A$ we have $bAc = 0$ if and only if $b = 0$ or $c = 0$. In particular, $\Omega_e$ is graded prime by the preceding lemma. By a graded analog of Posner’s theorem given in [6], since $\Omega_r$ is graded prime, the algebra $\Omega'_r$ is graded simple over its center $E_r$. In particular $\Omega'_r \otimes_{E_r} F(\Gamma)$ is graded simple and therefore the homomorphism $\Omega'_r \otimes_{E_r} F(\Gamma) \to M_n(F(\Gamma))$ is injective and can be easily seen to be an isomorphism. It follows that $\dim_{E_r}(\langle \Omega'_r \rangle_c) = \dim_{F(\Gamma)}(\langle \Omega'_r \otimes F(\Gamma) \rangle_c) = n$, and since $L_r := (\Omega'_r)_e$ is a finite dimensional integral domain over a field $E_r$ it is also a field.

For each $g \in G$ we have $0 \neq U_g^{-1} \in L_r$, so that $U_g$ is invertible in $\Omega'_r$. We thus see that $g(\lambda) := U_g \lambda U_g^{-1}$ is a faithful $G$-action defined on $L_r$, with $L_r^G = E_r$, so that $L_r/E_r$ is a $G$-Galois extension of fields. We conclude that $\Omega'_r \cong \Delta(L_r/E_r, G, c)$ where the 2-cocycle is $c(g, h) = U_g U_h U_{gh}^{-1} \in L_r^\times$.

As mentioned in Lemma 8, crossed product are finitely defined, so there is some finitely generated $Z(\Omega_r)$-algebra $R \subseteq E_r$ such that $\Omega_r \otimes Z(\Omega_r) R$ is already a $G$-crossed product. Thus choosing $s \in Z(\Omega_r)$ such that all the generators of $R$ are in $Z(\Omega_r) \langle s^{-1} \rangle$ we get that $\Omega_r[s^{-1}]$ is also a $G$-crossed product (by closure to specializations).

Remark 14. The construction above, which takes a relatively free algebra and turns it into a representing object can be performed in a much more general setting. In particular, a similar construction can be given for other $G$-simple algebras (and not just crossed products) using Artin Procesi lemma (For example see [2]).

We finish with the proof that these algebra are densely representing objects.
Lemma 15. Let $0 \neq s \in Z(\Omega_r)$ such that $\tilde{\Omega}_r = \Omega_r [s^{-1}]$ is in $\mathcal{C}(G, F)$. Then $\tilde{\Omega}_r$ is a densely representing object.

Proof. Let $A = \Delta(S/\mathbb{K}, G, c) \in \mathcal{C}(G, F)$ where $\mathbb{K}$ is a field and recall that $Id^{\mathbb{K}}_G(A) = Id^F_G(M_n(F)) \otimes_{\mathbb{K}} \mathbb{K}$. Since $0 \neq s \in \Omega$ is not a graded identity of $A$, there is a surjective graded homomorphism from $\Omega$ to $A$ such that the image of $s$ is nonzero, and in particular it is in the center of $A$, which is $\mathbb{K}$. As the elements in $\mathbb{K} \setminus \{0\}$ are invertible, this homomorphism can be extended to $\Omega [s^{-1}]$. Denote its restriction to $\tilde{\Omega}_r$ by $\psi$ and the restriction to $Z(\tilde{\Omega}_r) = Z(\Omega_r) [s^{-1}]$ by $\varphi$.

By the properties of the class $\mathcal{C}(G, F)$ we get that $\tilde{\Omega}_r \otimes_{\varphi} \mathbb{K}$ is a $G$-crossed product central over a field and therefore a central simple algebra. Thus the induced map $\psi \otimes id : \left( \tilde{\Omega}_r \otimes_{\varphi} \mathbb{K} \right) \rightarrow A$ sending $a \otimes \lambda$ to $\psi(a)\lambda$ is injective. Both algebras $\tilde{\Omega}_r \otimes_{\varphi} \mathbb{K}$ and $A$ are have dimension $|G|^2$ over $\mathbb{K}$, and therefore $\psi \otimes id$ must also be surjective. Thus, there is a graded isomorphism $\tilde{\Omega}_r \otimes_{\varphi} \mathbb{K} \cong A$ which shows that $\tilde{\Omega}_r$ is a representing object.

Let $0 \neq t \in Z(\Omega_r) = Z(\Omega_r) [s^{-1}]$. Writing $t = \frac{a}{b}$ with $a \in Z(\Omega_r)$, we see that $\tilde{\Omega}_r [t^{-1}] = \tilde{\Omega}_r [u^{-1}] = \tilde{\Omega}_r [(su)^{-1}]$. Since $\Omega_r [(su)^{-1}] \in \mathcal{C}(G, F)$, the previous argument shows that $\tilde{\Omega}_r [t^{-1}]$ is a representing object, and hence $\tilde{\Omega}_r$ is a densely representing object.

\[ \square \]

3 Localized representing objects

Let $\tilde{\Omega}_0$ be any densely representing object of the class $\mathcal{C}(G, F)$ and let $\tilde{\Omega}$ be the algebra obtains from $\tilde{\Omega}_0$ by inverting all of its nonzero central elements. We call such an algebra a localized representing object. As mentioned in the introduction, our main interest is to study the center of $\tilde{\Omega}$, and determine how close it is to being a rational extension of $F$.

There are many nonisomorphic densely representing objects for $\mathcal{C}(G, F)$. Indeed, by definition, for every nonzero central element $s$ of $\tilde{\Omega}_0$, the algebra $\tilde{\Omega}_0 [s^{-1}]$ is again a densely representing object. While this process produce many nonisomorphic algebras, all of them become isomorphic after inverting their center. A well known generalization of Swan’s lemma ([39], Lemma 8) states that if $A$ and $B$ are Azumaya central over integral domains which become isomorphic after inverting the center, then $A \cong B$ for some $0 \neq a \in Z(A)$ and $0 \neq b \in Z(B)$. Since we deal with densely representing objects, after inverting a single central element, they remain densely representing. Hence, the “right” equivalence between densely representing objects is that their central localized versions are isomorphic. We thus restrict ourselves to the study of the localized representing object.

Before we continue, we give some definitions and notation needed to describe the centers of the localized representing objects.

Definition 16. Let $L/\mathbb{K}$ be a finitely generated field extension. Then:

- $L$ is called a rational extension of $\mathbb{K}$ if $L = \mathbb{K}(x_1, ..., x_n)$ for some algebraically independent indeterminates $\{x_1, ..., x_n\}$.
- $L$ is called a stably rational extension of $\mathbb{K}$ if $L(y_1, ..., y_m)$ is rational over $\mathbb{K}$ for some $\{y_1, ..., y_m\}$ algebraically independent over $\mathbb{L}$.
- $L$ is called retract rational extension of $\mathbb{K}$ if $L$ is the fraction field of some $\mathbb{K}$-algebra $A$, and there are homomorphisms $i : A \rightarrow \mathbb{K}[x_1, ..., x_n] [s^{-1}]$ and $\pi : \mathbb{K}[x_1, ..., x_n] [s^{-1}] \rightarrow A$ such that $\pi \circ i = id_A$.
L is called a unirational extension of \( K \) if \( L \) can be embedded in a rational extension of \( K \).

We always have

\[
\text{rational} \subseteq \text{stably rational} \subseteq \text{retract rational} \subseteq \text{unirational}.
\]

We remark here that the inclusions above are all proper, although the examples are not trivial.

**Definition 17.** Two finitely generated field extensions \( L_1, L_2/K \) are called stably isomorphic (over \( K \)) if there are \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_k \) algebraically independent over \( L_1, L_2 \) respectively such that \( L_1(x_1, \ldots, x_m) \cong L_2(y_1, \ldots, y_k) \) as \( K \)-algebras. Two crossed products are called stably isomorphic if their centers are stably isomorphic.

The type of fields considered in this work are described using \( G \)-lattices. For that, we assume that the reader is familiar with the basic definitions and results in \( G \)-lattices and their field invariants, as appearing for example in [22] chapters 2 and 5, or [24] chapter 2. Let us recall the main definitions and results needed in this work.

**Definition 18.** Let \( G \) be a finite group.

- A \( G \)-module is called a \( G \)-lattice if it isomorphic as an abelian group to \( \mathbb{Z}^m \) for some \( m \in \mathbb{N} \).
- A \( G \)-lattice \( M \) is called a permutation lattice if \( M \) has a \( \mathbb{Z} \)-basis which is \( G \)-invariant.

Let \( M \) be a \( G \)-lattice, \( K \) a field with a \( G \)-action (possibly trivial) and set \( K[M] \) to be their group algebra. Since \( M \cong \mathbb{Z}^m \) as an abelian group, \( K[M] \) is just the algebra of Laurent polynomial in \( m \) variables, and in particular an integral domain. The \( G \)-action on \( K \) and \( M \) induce a \( G \)-action on \( K[M] \) and on its field of fractions \( K(M) \). We are interested in fields of the form \( K(M)^G \).

The importance of permutation lattices is given in the next two results, both of which can be found in [22] (Prop. 2.8 and Theorem 2.12).

**Proposition 19 (Masuda).** Let \( K/K^G \) be a \( G \)-Galois extension of field and \( P \) a permutation lattice. Then \( K(P)^G \) is a rational extension of \( K^G \).

**Theorem 20.** Let \( 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow P \longrightarrow 0 \) be an exact sequence of \( G \)-lattices where \( M_1 \) is a faithful \( G \)-lattice, \( P \) a permutation lattice and \( K \) a field with a trivial \( G \)-action. Then \( K(M_2)^G \) is rational extension of \( K(M_1)^G \).

### 3.1 The fraction field of \( Z(\Omega_r) \)

Let us return to the representing objects constructed in the previous section. Fix \( r \in \mathbb{N} \) and let \( E_r \) be the fraction field of \( Z(\Omega_r) \). As we have seen in the previous section, we have \( \Omega_r' := \Omega_r \otimes_{Z(\Omega_r)} E_r \cong \Delta(\mathbb{L}_r/E_r, G, c) \) where \( \mathbb{L}_r/E_r \) is a \( G \)-Galois extension of fields and \( c(g, h) = U_gU_h^{-1} \).

The aim of this section is to describe \( E_r = \mathbb{L}_r^G \), and we start by examining \( \mathbb{L}_r \). The field \( \mathbb{L}_r \) is the fraction field of \( (\Omega_r)_r' \), hence can be viewed as a set of diagonal matrices. Define \( \varphi : \mathbb{L}_r \to \mathbb{F}(\Gamma_r) \) by sending a diagonal matrix \( a = \sum a_g E_{g, g} \) to \( a_e \). Since \( \mathbb{L}_r \) is a field we get that \( \mathbb{L}_r \) is isomorphic to \( \varphi(\mathbb{L}_r) \), which we describe instead of \( \mathbb{L}_r \). Recall from Lemma 12 that if \( a = \sum a_g E_{g, g} \in (\Omega_r)_r' \), then \( a_g = g(a_e) \), which also holds for the fraction field \( \mathbb{L}_r \). If follows that

\[
\text{for all } \sigma \in G \quad \varphi(\sigma(a)) = \varphi(\sigma(\sum a_g E_{g, g})\sigma^{-1}) = \varphi(\sum a_{g\sigma}E_{g, g}) = a_e = \sigma(a_e) = \sigma(\varphi(a)),
\]

10
so the induced $G$-action on $\varphi(\mathbb{L}_r)$ is the one defined by $\sigma(u_{g,h,i}) = u_{sg,sh,i}$.

Let $\Gamma_r = \{ u_{g,h,i} \mid g,h \in G, 1 \leq i \leq r \} \subseteq \Gamma$ be algebraically independent indeterminates. Letting $B(G, r)$ be the free abelian group over the set $\hat{\Gamma}_r = \{ u_{g,h,i} \mid g,h \in G, 1 \leq i \leq r \}$ written additively with a $G$-action induced by $\sigma(u_{g,h,i}) = u_{sg,sh,i}$, we get that $\mathbb{F}(B(G, r)) \cong \mathbb{F}(\Gamma_r)$ as $G$-fields.

If $h_i \in G$ such that $\prod_{j=1}^{n} h_j = e$ and $1 \leq i_j \leq r$, then $\prod_{j=1}^{n} \hat{u}_{h_j,i_j}$ is homogeneous of degree $e$ and

$$
\left( \prod_{j=1}^{n} u_{h_j,i_j} \right)_e = u_{e,i_1,i_1} u_{h_1,h_2,i_2} \cdots \hat{u}_{(h_1 h_2 \cdots h_{n-1}, h_1 h_2 \cdots h_n),i_n,i_n}.
$$

By taking quotients of such elements, it follows that $\varphi(\mathbb{L}_r)$ is generated by elements of the form $u_{g_1,g_2,g_3} u_{g_2,g_3,g_4} \cdots u_{g_{n-1},g_n}$ where $g_i \in G$ and $1 \leq i_j \leq r$. Let us make this more precise.

Define $\alpha : B(G, r) \to ZG$ by setting $\alpha(u_{g,h,i}) = g - h$, and set $Eu(G, r) = \ker(\alpha)$. Denoting by $\epsilon : ZG \to Z$ the augmentation map $\epsilon(\sum a_g g) = \sum a_g$, we get an exact sequence of $G$-lattices

$$
0 \to Eu(G, r) \to B(G, r) \xrightarrow{\alpha} ZG \xrightarrow{\epsilon} Z \to 0.
$$

Clearly $Eu(G, r)$ is generated as a group by the elements in

$$
X_r = \left\{ \sum_{j=1}^{n} \hat{u}_{g_j,i_j} \mid g_1, \ldots, g_n \in G, \ g_{n+1} = g_1, \ 1 \leq i_j \leq r \text{ for all } j \right\},
$$

and under the isomorphism $\mathbb{F}(B(G, r)) \cong \mathbb{F}(\Gamma_r)$ we get that $\mathbb{F}(Eu(G, r)) \cong \varphi(\mathbb{L}_r)$ as $G$-fields. It follows that $E_r \cong \mathbb{F}(Eu(G, r))^G$.

If $M$ is a $G$-lattice, then $\mathbb{F}(M)/\mathbb{F}$ is a rational extension with $\text{tr.deg}_\mathbb{F}(\mathbb{F}(M)) = \text{rank}(M)$. This leads to the next corollary.

**Corollary 21.** The field $\mathbb{L}_r = \mathbb{F}(Eu(G, r))$ is rational over $\mathbb{F}$ of transcendence degree $|G|^2 r - |G| + 1$ and $E_r = \mathbb{L}_r^G$ is unirational with the same transcendence degree.

**Proof.** Using the exact sequence above, we get that

$$
0 = \text{rank}(Eu(G, r)) - \text{rank}(B(G, r)) + \text{rank}(ZG) = \text{rank}(A(G, r)) - |G|^2 r + |G| - 1,
$$

so that $\text{rank}(A(G, r)) = |G|^2 r - |G| + 1$.

The field extension $\mathbb{L}_r$ over $E_r = \mathbb{L}_r^G$ is finite, so the extension $E_r/\mathbb{F}$ is unirational and $\text{tr.deg}_\mathbb{F}(E_r) = \text{tr.deg}_\mathbb{F}(\mathbb{L}_r)$. \hfill $\Box$

**Remark 22.** A similar exact sequence appears in the theorems of Procesi and Formanek for the ungraded case (see [16] for details). More precisely, let $B_{n,r}$, $P_n$, $Q$ be the free abelian groups over $\{ u_{i,j,k} \mid 1 \leq i, j \leq n, 1 \leq k \leq r \}$, $\{ v_i \mid 1 \leq i \leq n \}$ and $\{ q \}$ respectively, and define an $S_n$ action on them by $\sigma(u_{i,j,k}) = \hat{u}_{\sigma(i),\sigma(j),k}$, $\sigma(v_i) = v_{\sigma(i)}$ and $\sigma(q) = q$ for all $\sigma \in S_n$. We have a similar exact sequence of $S_n$-lattices

$$
0 \to A_{n,r} \to B_{n,r} \xrightarrow{\alpha} P_n \xrightarrow{\epsilon} Q \to 0,
$$

where $\epsilon(v_i) = 1$ and $\alpha(u_{i,j,k}) = v_j - v_i$. The center of the generic division algebra generated by $r$ generic $n \times n$ matrices is given by $\mathbb{F}(A_{n,r})^{S_n}$. If $|G| = n$, then the embedding $G \hookrightarrow S_n$ via the left regular action, makes the $S_n$-lattices above into $G$-lattices. With this notation, $A_{n,r}$ is isomorphic to $A(G, r)$ as $G$-modules, and therefore $E_r \cong \mathbb{F}(A_{n,r})^G$.\hfill $\Box$
Finally, Procesi’s result stating that it is enough to take only two generic matrices still holds in this graded case, and actually we have a slightly better result.

**Lemma 23.** The field $\mathbb{E}_r$ is rational over $\mathbb{E}_1$ of transcendence degree $(r - 1)|G|^2$. In particular, the central localizations of $\Omega_r$ are all stably isomorphic.

**Proof.** Let $\sum_{j=1}^n \tilde{u}_{g_j,g_{j+1},i_j} \in X_r$, so that $g_j \in G$, $g_1 = g_{n+1}$ and $1 \leq i_j \leq r$. Then

$$\sum_{j=1}^n \tilde{u}_{g_j,g_{j+1},i_j} = \sum_{j=1}^n [\tilde{u}_{g_j,g_{j+1},i_j} - \tilde{u}_{g_{j+1},g_j,i_j}] + \sum_{j=1}^n \tilde{u}_{g_{j+1},g_j,i_j}.$$

Letting $Y_r = \{\tilde{u}_{g,h,i} - \tilde{u}_{h,g,i} \mid g, h \in G, 2 \leq i \leq r\}$, it follows that $Eu(G, r) \cong \text{span}_\mathbb{Z}(X_1) \oplus \text{span}_\mathbb{Z}(Y_r) \cong Eu(G, 1) \oplus \text{span}_\mathbb{Z}(Y_1)$ as $G$-lattices. In addition, $Y_r$ is a stable $G$-basis for $\text{span}_\mathbb{Z}(Y_r)$, so it is a permutation lattice. The first claim now follows from Theorem 20.

Recall that the central localization of $\Omega_r$ is the crossed product $\Delta(L_r/\mathbb{E}_r, G, c)$ where $c(g, h) = U_g U_h U_{gh}^{-1}$, so in particular the 2-cocycle is already defined over the field $\mathbb{E}_1$. Thus, the first part of the lemma implies that $\Delta(L_r/\mathbb{E}_r, G, c) \cong \Delta(L_1/\mathbb{E}_1, G, c) \otimes_{\mathbb{E}_1} \mathbb{E}_1(\zeta_1, ..., \zeta_{(r-1)|G|^2})$ where the $\zeta_i$ are algebraically independent, and the second claim follows.

The $G$ action on the field $\mathbb{F}(Eu(G, r))$ is described by the $G$-action on a chosen $\mathbb{Z}$-basis for $Eu(G, r)$. In order to give a more intuitive way to choose a $\mathbb{Z}$-basis, we describe another presentation of the lattices $Eu(G, r)$ which uses cycles in directed graphs. For convenience, we restrict the discussion to $r = 1$, and write $Eu(G) := Eu(G, 1)$, $B(G) := B(G, 1)$. The generalization for an arbitrary $r$ is left to the reader.

Denote by $\Lambda(G)$ the full directed Cayley graph of $G$, namely the graph with the vertices $V(G) = \{v_g \mid g \in G\}$ and edges $\text{Edge}(G) = \{u_{g,gs} \mid g, s \in G\}$, where $u_{g,gs}$ is a directed edge from $v_g$ to $v_{gs}$. Under these notations, the elements of $X_1$, which generate $Eu(G)$, are “cycles” in the graph $\Lambda(G)$. We wish to make this notion more precise.

Define a $G$-action of graph isomorphisms on $\Lambda(G)$ by $\sigma(v_g) = v_{\sigma g}$ and $\sigma(u_{g,gs}) = u_{\sigma g,\sigma gs}$. This action induces a $G$-action on the free group $\text{Hom}(\text{Edge}(G), \mathbb{Z})$ of all $\mathbb{Z}$-weighted functions on the set of edges of $\Lambda(G)$. Let $\{\tilde{u}_{g,gs} \mid g, s \in G\}$ be the basis of this lattice defined by

$$\tilde{u}_{g,gs}(u_{g',gs'}) = \begin{cases} 1 & \text{if } g = g', s = s' \\ 0 & \text{otherwise} \end{cases}.$$

We clearly have $\text{Hom}(\text{Edge}(G), \mathbb{Z}) \cong B(G)$ as $G$-lattices via the map defined by $\tilde{u}_{g,gs} \mapsto \tilde{u}_{g,gs}$. The “cycle” property of the elements in $X_1$ is defined as follows.

**Definition 24.** Let $Gr(V, E)$ be a directed graph. A weight function $w : E \to \mathbb{Z}$ is called an Eulerian cycle if for each vertex $v_0 \in V$ we have

$$\sum_{v \in V} w(v_0, v) = \sum_{v \in V} w(v, v_0),$$

where $w(u, v) = 0$ if $(u, v) \notin E$. Denote by $Eu(Gr)$ the free abelian group of all Eulerian cycles in $Gr$. 

12
If $Gr$ has a $G$ action on it, then $Eu(Gr)$ is a $G$-lattice. In particular, $Eu(\Lambda(G))$ is a $G$-sublattice of $B(G)$, which corresponds to $Eu(G)$.

Call an Eulerian cycle $w : E \rightarrow \mathbb{Z}$ simple if the edges $\{(u, v) \mid |w(u, v)| = 1\}$ constitute a simple cycle in the undirected underlying graph of $Gr$. Clearly, any simple cycle $C$ in the undirected underlying graph of $Gr$ corresponds (up to a sign) to a simple Eulerian cycle, hence we identify between these two objects.

Let $Gr'$ be a subgraph of $Gr$ defined on the same set of vertices, such that for each edge $(w, v)$ not appearing in $Gr'$, there is a simple Eulerian cycle $\hat{C}_{(w, v)}$, which contains $(w, v)$ and all of its other edges are in $Gr'$. It follows that $Eu(Gr) = Eu(Gr') \oplus \left( \bigoplus_{(w, v) \notin Gr'} \mathbb{Z}\hat{C}_{(w, v)} \right)$. This method is used to decompose our $G$-lattices and find nice $\mathbb{Z}$-bases for them, as we shall see next.

**Lemma 25.** Let $G$ be a finite group acting on a directed graph $Gr(V, E)$. Let $Gr'(V, E')$ be a subgraph of $Gr(V, E)$ such that

1. The graphs $Gr$ and $Gr'$ are connected as undirected graphs.
2. $G$ acts on $E \setminus E'$ freely.
3. $G$ acts faithfully on $Eu(Gr')$.

Then $Eu(Gr) \cong Eu(Gr') \oplus (\mathbb{Z}G)^{|E \setminus E'|}$ and $\mathbb{F}(Eu(Gr))^G$ is a rational extension of $\mathbb{F}(Eu(Gr'))^G$.

**Proof.** Let $\{(w_i, v_i)\}_{i=1}^m \subseteq E \setminus E'$ be a set of representatives of the $G$ orbits in $E \setminus E'$ (where $m = \frac{|E| - |E'|}{|G|}$). For each $i$, choose a simple Eulerian cycle $\hat{C}_{(w_i, v_i)}$ which contains $(w_i, v_i)$ and otherwise is supported on $Gr'$ (such a cycle exists by the hypothesis). If $(w, v) \in E \setminus E'$, then there is some $g \in G$ and $1 \leq i \leq m$ such that $(w, v) = g(w_i, v_i)$, and we define $\hat{C}_{(w, v)} = g \left( \hat{C}_{(w_i, v_i)} \right)$. Since $E \setminus E'$ is a free $G$-set, the elements $\hat{C}_{(w, v)}$ are well defined, and are supported on $Gr'$ except for the corresponding edge $(w, v)$. More importantly, the set $\left\{ \hat{C}_{(w, v)} \mid (w, v) \in E \setminus E' \right\}$ is a free $G$-set, which produces a $G$-lattice decomposition

$$Eu(Gr) = Eu(Gr') \oplus \left( \bigoplus_{(w, v) \notin E'} \mathbb{Z}\hat{C}_{(w, v)} \right) \cong Eu(Gr') \oplus (\mathbb{Z}G)^m.$$ 

By the assumption, $G$ acts faithfully on $Gr'$ so by Theorem 20 we get that $\mathbb{F}(Eu(Gr))^G$ is rational over $\mathbb{F}(Eu(Gr'))^G$. \hfill $\square$

**Remark 26.** The last lemma can be used to give another proof for Lemma 23 when considering the lattices $Eu(G, r)$ as Eulerian cycles on Cayley graphs where each edge appears $r$ times.

A special case of subgraphs of $\Lambda(G)$ arise from Cayley graph with respect to some generating set $S$ of $G$. For $g, s \in G$, an edge $u_{g, gs} \in \text{Edge}(G)$ is said to be of degree $s$. Let $\Lambda(G, S)$, $B(G, S)$ and $Eu(G, S)$ be the respective Cayley graph, weighted functions and Eulerian cycles in $\Lambda(G)$ restricted to edges with degrees in $S$ (note that since $S$ generates $G$, the graph $\Lambda(G, S)$ is connected). Viewing
that \( \{ \\}\). Let us mention two results from the constructions given so
\[ \left( \begin{array}{c} \{ \\} \\ \{ \\}\setminus \{ e \}\end{array} \right) \]
so our field is a “generic” \( G \)-Galois extension together with a “generic” 2-cocycle. We now have that \( \mathbb{F}(\mathbb{E}(G)) = \mathbb{F}(\mathbb{Z}G)(c,gh) \mid g, h \in G \), so our field is a “generic” \( G \)-Galois extension together with a “generic” 2-cocycle.

**Remark 28.** One of the uses for representing objects is to give an upper bound to the essential
dimension (for the definition, see [8]). Let us mention two results from the constructions given so far. Both of these results appear in [25], though they were achieved there by different methods.

1. For a generating set \( S \) of \( G \), consider the subalgebra \( \Omega_S \) of \( \Omega_1 \) generated by \( \{ U_s \mid s \in S \} \). This is a \( G \)-graded algebra with \( \langle \Omega_S \rangle_g \neq 0 \) for any \( g \in G \) (since \( S \) generates \( G \)). The fraction field of \( \langle \Omega_S \rangle \) is just \( \mathbb{F}(\mathbb{E}(G, S)) \) which is \( G \)-Galois over \( \mathbb{F}(\mathbb{E}(G, S))^G \), as long as \( G \) acts faithfully on \( \mathbb{E}(G, S) \) (which is the case for \( |S| \geq 2 \)). Similarly to the computations in the last section, after a suitable central localization we get that \( \Omega_S \left[ t^{-1} \right] \) is a \( G \) crossed product. In addition, the central localization of \( \Omega_1 \) is just a scalar extension of \( \Omega_S \left[ t^{-1} \right] \), so it follows that \( \Omega_S \left[ t^{-1} \right] \) is also a densely representing object.

\[ B(G, S) \] as a sublattice of \( B(G, G) \equiv B(G) \) we get that \( \mathbb{E}(G, S) = \mathbb{E}(G, G) \cap B(G, S) \) and we have the exact sequence

\[
\begin{array}{c}
0 \rightarrow \mathbb{E}(G, S) \rightarrow \hat{B}(G, S) \stackrel{\alpha}{\rightarrow} \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0
\end{array}
\]

The last lemma now gives the following.

**Corollary 27.** Let \( S \subseteq G \) be a generating set. Then \( \mathbb{E}(G, S) \cong \mathbb{E}(G, S) \otimes (\mathbb{Z}G)^{|G| - |S|} \). If in addition \( |S| \geq 2 \), then \( G \) acts faithfully on \( \mathbb{E}(G, S) \) so \( \mathbb{F}(\mathbb{E}(G, S))^G \) is rational over \( \mathbb{F}(\mathbb{E}(G, S))^G \).

A special case of the decomposition above is when \( S = G \setminus \{ e \} \), namely \( \Lambda(G, S) \) is the full directed graph without self loops. Setting \( \mathbb{E}(G) \equiv \mathbb{E}(G, G) \setminus \{ e \} \), the corollary shows that \( \mathbb{E}(G) \cong \mathbb{E}(G) \otimes \mathbb{Z}G \) as \( G \)-lattices. Consider the spanning tree \( T \) of \( \Lambda(G, G \setminus \{ e \}) \) constituting the edges \( \{ (e, g) \mid e \neq g \in G \} \) and set \( \hat{c}(g, h) = \hat{u}_{e,g} \pm \hat{u}_{g,gh} \in \mathbb{E}(G) \) for \( e \neq g, h \in G \) and \( \hat{c}(g, h) = 0 \) otherwise. A proof similar to the one in the previous lemma shows that \( \mathbb{E}(G) = \mathbb{E}(T) \otimes \text{span}_{\mathbb{Z}}(\hat{c}(g, h) \mid e \neq g, h \in G) \). Since \( T \) is a tree, we have that \( \mathbb{E}(T) = 0 \), hence \( \{ \hat{c}(g, h) \mid e \neq g, h \in G \} \) is a \( \mathbb{Z} \)-basis for \( \mathbb{E}(G) \). The G-action is defined by

\[
\sigma(\hat{c}(g, h)) = \hat{c}(\sigma, g) + \hat{c}(\sigma, h) + \hat{c}(\sigma, gh),
\]

so \( \hat{c}(g, h) \) is a generic (additive) 2-cocycle.

The decomposition \( \mathbb{E}(G) \cong \mathbb{Z}G \otimes \mathbb{E}(G) \) is quite intuitive when considering the field \( \mathbb{F}(\mathbb{E}(G)) \). Let \( \{ z_g \mid g \in G \} \) be algebraically independent indeterminates over \( \mathbb{F} \) and set \( \mathbb{L} = \mathbb{F}(z_g \mid g \in G) \). The field \( \mathbb{L} \) has a natural G-action defined by \( h(z_g) = z_{gh} \) for all \( h, g \in G \) which make it isomorphic to \( \mathbb{F}(\mathbb{Z}G) \) as \( G \)-fields. Recall that the G-Galois extension \( \mathbb{L}/\mathbb{L}^G \) is used to construct representing objects in the class of G-Galois extensions. The main ingredient of the construction is the normal basis theorem which state that if \( \mathbb{L}/\mathbb{L}^G \) is any G-Galois extension, then there is some \( \lambda \in \mathbb{L}^G \) such that \( \{ g(\lambda) \mid g \in G \} \) is an \( \mathbb{F}^G \) basis for \( \mathbb{L}^G \) (for more details see [33]).

Let \( \{ c, gh \mid e \neq g, h \in G \} \) be algebraically independent indeterminates over \( \mathbb{L} \) and define a G-action on the field \( \mathbb{L}(c,gh) \) by

\[
\sigma(c,gh) = \frac{c(\sigma, g)c(\sigma g, h)}{c(\sigma, gh)},
\]

where \( c(e,g) = c(ge, e) = 1 \) for all \( g \in G \). We now have that \( \mathbb{F}(\mathbb{E}(G)) = \mathbb{F}(\mathbb{Z}G)(c,gh) \mid g, h \in G \), so our field is a “generic” G-Galois extension together with a “generic” 2-cocycle.
After inverting all the nonzero central elements of $\Omega_S$, its center is just $F(Eu(G,S))^G$, and therefore it has transcendence dimensions

$$\text{rank}(Eu(G,S)) = |E(G,S)| - |V(G,S)| + 1 = |G| |S| - |G| + 1 = |G| (|S| - 1) + 1.$$ 

This gives an upper bound to the essential dimension of $\mathcal{C}(F, G)$.

2. In the nongraded case, it is well known that the central simple algebras of dimension $n^2$ can be classified using the cohomology group $H^1(G(F/F), GL_n(F))$, where $GL_n(F)$ is the automorphism group of $M_n(F)$ and $F$ is the algebraic closure of $F$.

A similar process is also true in the graded case. Consider the $G$-crossed product grading on $A = M_n(F)$. Clearly, if $a \in A^G_n \cong F^{e_{-1}}$ is an invertible diagonal matrix, then the conjugation by $a$ is a graded isomorphism of $A$, and scalar matrices act as the identity. Another isomorphism is given by conjugation by a permutation matrix $P_{gh} = \sum_{h \in G} E_{gh,h}$. These permutation matrices form a group isomorphic to $G$ and acts on $A^G_n$ by permuting the elements in the diagonal. Thus, the semidirect product $A^G_n \rtimes G$ acts on $A$, and actually, by [20], this group is isomorphic to the group of graded automorphisms of $A$. Once again, as in the nongraded case, it follows that the $F$-forms of $M_n(F)$ are classified by $H^1(G(F/F), A^G_n \rtimes G)$.

### 3.2 Representing objects in the literature

In this section we describe three constructions of densely representing objects for crossed products already appearing in the literature. Two of them are essentially the same as the one constructed in the previous section (namely, their localized versions are stably isomorphic), and the third is slightly different. We denote by $\Delta_G$ the localized representing object (up to a stable isomorphism) constructed in the previous sections.

We start with two constructions due to Saltman. In his paper [34] (pages 190-194), Saltman constructed densely representing objects for similar classes of $G$-crossed product. Let us recall their definition.

**Definition 29.** Let $L$ be a fixed $G$-Galois extension of $F$. Denote by $\mathcal{C}(G, L/F)$ all the $G$-crossed products of the form $\Delta((L \otimes_R R)/R, G, d)$ where $R$ is a commutative unital $F$-algebra and $d \in Z^2(G, (L \otimes_R R))^G$.

Saltman’s representing object for the class $\mathcal{C}(G, L/F)$ is $\Delta(S/S^G, G, c(g, h))$ where $S = L [Eu^-(G)]$ (or in other words, $S$ is $L$ adjoined by a generic 2-cocycle).

Note that if instead of a fixed $G$-Galois extension $L/F$ we take a “generic” $G$-Galois extension $F(ZG)/F(ZG)^G$, the resulting object (after inverting the center) is isomorphic to $\Delta_G$.

As the definition implies, Saltman’s representing objects depend on the choice of the $G$-Galois extension $L/F$, although Saltman was able to show that certain properties of the classes $\mathcal{C}(G, L/F)$ depend only on the group $G$. In particular, he proved that $L (Eu^-(G))^G$ is a retract rational extension of $F$ if and only if $Eu^-(G)$ is an invertible $G$-lattice (i.e. direct summand of a permutation lattice) if and only if every $p$-Sylow subgroup of $G$ is cyclic.
Saltman’s second construction is given in [36] and is defined as follows. Let $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ be the augmentation map with $I_G = \ker(\epsilon)$. Choose an exact sequence of $G$-lattices

$$0 \longrightarrow M \longrightarrow P \longrightarrow I_G \longrightarrow 0$$

such that $P$ is a free $G$ module and $G$ acts on $M$ faithfully (if this is not the case, it can always be fixed by taking $M \oplus \mathbb{Z}G$ and $P \oplus \mathbb{Z}G$ instead of $M$ and $P$ respectively). The corresponding long exact sequences give the isomorphism

$$H^2(G, M) \cong H^1(G, I_G) \cong H^0(\mathbb{Z}/\mathbb{Z})/\epsilon(\mathbb{Z}/\mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}.$$

Thus, $H^2(G, M)$ is cyclic and we choose a 2-cocycle representative $\hat{d}$ for one of its generators. Saltman’s representing object for this sequence is $\Delta_M = \Delta(\mathbb{F}(M)/\mathbb{F}(M)^G, G, d)$, where $d$ is the 2-cocycle in $\mathbb{F}(M)$ corresponding to the 2-cocycle $\hat{d}$ in $M$.

Saltman showed that $\Delta_M$ is representing in the following sense. Given any crossed product $\Delta(\mathbb{K}/\mathbb{F}, G, d)$, there is a field $\mathbb{K}' \supseteq \mathbb{K} \cdot \mathbb{F}(M)^G$ such that $\mathbb{K}'/\mathbb{K}$ is a rational extension and $\Delta_M \otimes \mathbb{F}(M)^G \mathbb{K}' \cong A \otimes_{\mathbb{K}} \mathbb{K}'$.

Suppose that $M, P$ and $M', P'$ are two pairs of $G$-lattices satisfying Saltman’s requirements. By Schanuel’s lemma we have that $M \oplus P' \cong M' \oplus P$. Using Theorem 20 we get that $\mathbb{F}(M \oplus P')^G = \mathbb{F}(M' \oplus P)^G$ is rational over $\mathbb{F}(M)^G$ and over $\mathbb{F}(M')^G$ concluding that $\Delta_M$ is stably isomorphic to $\Delta_{M'}$. In particular, since equation (1) satisfies Saltman’s conditions, these objects are stably isomorphic to $\Delta_G$.

We remark that Saltman didn’t use graded identities in his work, and instead proved that these objects are densely representatives using a generic 2-cocycle inside $Eu^{-1}(G)$ which generates $H^2(G, Eu^{-1}(G))$.

The last construction considered here was given independently by Snider [38] and Rosset [30]. Let us recall its definition.

Let $G$ be a finite group, and $F$ a finitely generated free group with an epimorphism $\pi : F \to G$. By the Nielsen-Schreier theorem the group $R := \ker(\pi)$ is also finitely generated and free. The commutator subgroup $[R, R]$ of $R$ is normal in $F$, so defining $\bar{R} = F/\ker([R, R])$ and $\bar{F} = F/\ker([R, R])$, we have the exact sequence

$$1 \longrightarrow \bar{R} \longrightarrow \bar{F} \overset{\pi}{\longrightarrow} G \longrightarrow 1.$$  

Any section $\phi : G \to \bar{F}$ induces a $G$-action on $\bar{R}$ by $g(r) = \phi(g)r\phi(g)^{-1}$ and since $\bar{R}$ is commutative, this action is independent of the choice of the section $\phi$. Putting all of these facts together we get that $\bar{R}$ is a $G$-lattice, so we can define $\mathbb{F}(\bar{R})$ and $\mathbb{F}(\bar{R})^G$.

Let $A$ be a $G$-module and

$$1 \longrightarrow A \longrightarrow E \overset{\rho}{\longrightarrow} G \longrightarrow 1$$

an extension of $G$ by $A$. Since $F$ is a free group, there is $\phi : F \to E$ such that $\rho \circ \phi = \pi$. This induces a homomorphism $\bar{\phi} : \bar{F} \to E$ such that the following diagram commutes

$$\begin{array}{ccc}
1 & \longrightarrow & \bar{R} \\
\downarrow & & \downarrow \bar{\phi} \\
1 & \longrightarrow & \bar{F} \\
\bar{\phi} & \circ & \bar{\phi} \\
1 & \longrightarrow & E \\
\rho & \circ & \phi \\
1 & \longrightarrow & G \\
\end{array}$$
If $L^o G = \Delta(L/\mathbb{K}, G, \alpha)$ is a $G$-crossed product where $L, \mathbb{K}$ are fields, then the 2-cocycle $\alpha$ determines an exact sequence

$$1 \longrightarrow L^\times \longrightarrow E \longrightarrow G \longrightarrow 1$$

where $E$ can be identified with the group $\{\lambda U_g \mid \lambda \in L^\times, \ g \in G\} \leq (L^o G)^\times$. Clearly $E$ is a spanning set for $L^o G$, so the homomorphism $\phi : \hat{F} \to E$ can be extended to an algebra homomorphism $\tilde{\phi} : \mathbb{F}[\hat{F}] \to L^o G$. The group $\hat{F}$ is a torsion free abelian by finite group, and therefore $\mathbb{F}[\hat{F}]$ has a division ring of fraction which we denote by $\mathbb{F}(\hat{F})$. This algebra is considered by Snider and Rosset as the generic $G$-crossed product, where its generic property comes from the process we described above.

Consider the free group $F = \langle z_g \mid g \in G \rangle$ and the homomorphism $\pi : F \to G$ defined by $\pi(z_g) = g$. In this case, the group $R$ consists of all the words $\prod z_{g_i}^i$ such that $\prod g_{i}^i = e$. This words should be viewed as paths in the Cayley graph $\Lambda(G)$ where a letter $z_g$ corresponds to moving along an edge of degree $g$. Thus, the condition that $\prod g_{i}^i = e$ corresponds to the path being a cycle. After taking the quotient modulo $[R,R]$, we get “commutative” cycles as those that appear in the abelian group $Eu(G)$. This induces an isomorphism $\Delta_G \cong \mathbb{F}^{(\hat{F}^t)}$, where the proof is left to the reader.

4 The rationality of $\mathbb{F}(Eu(G))^G$ over $\mathbb{F}$.

In this section we try to determine how close is $\mathbb{F}(Eu(G))^G$ to be a rational extension of $\mathbb{F}$. As in the previous section, we assume familiarity with with the basic definitions and results in $G$-lattices as appearing in [22] or [24]. In particular we use $H\hat{1}$ to denote the Tate cohomology groups, $[M]$ to denote the stable permutation equivalence class of $M$, and $[M]^{fl}$ to denote the permutation equivalence class of the flasque lattice appearing in a flasque resolution of $M$.

Before we start with the computations, let us recall the connection of the lattices $Eu(G)$ to the theory of field invariants.

Recall that for a generating set $S \subseteq G$ we have the exact sequences

$$0 \longrightarrow I_G \longrightarrow ZG \xrightarrow{\epsilon} Z \longrightarrow 0$$

$$0 \longrightarrow Eu(G,S) \longrightarrow B(G,S) \xrightarrow{\alpha} I_G \longrightarrow 0.$$  

We also write $Eu(G) = Eu(G,G)$ and $Eu^-(G) = Eu(G,G\setminus \{e\})$. Since $ZG$ and $B(G,S)$ are $\hat{G}$-free modules for every subgroup $\hat{G} \leq G$, the corresponding long exact sequences of Tate cohomology gives

$$\hat{H}^1(\hat{G}, Eu(G,S)) \cong \hat{H}^0(\hat{G}, I_G) \cong \hat{H}^{-1}(\hat{G}, Z) = 0.$$  

Thus $Eu(G,S)$ is a coflasque module, and the second sequence above is a coflasque resolution of $I_G$, or by taking the dual we have that $[I_G]_{fl} = [Eu(G,S)^*]$. In [11], Endo and Miyata proved that there is an equality of sets coflasque $= \text{flasque} = \text{invertible}$ for $G$-lattices if and only if $[I_G]_{fl}$ is invertible, which is of course equivalent to $Eu(G,S)$ being invertible. Another equivalent condition given by Endo and Miyata is that $G$ is a $Z$-group, namely that every Sylow subgroup of $G$ is cyclic.

Groups with cyclic Sylow subgroup also appear in Saltman’s representing objects for crossed products in [34]. More precisely, let $L/L^G$ be a $G$-Galois extension and $Eu^-(G) = Eu(G,G\setminus \{e\})$.  

17
Saltman used Brauer theory in order to prove that the field $L(Eu^-(G))^G$ (which is the center of Saltman’s localized representing object) is a retract rational extension of $\mathbb{L}^G$ if and only if $G$ is a $Z$-group if and only if $fl(Eu^-(G))$ is in the same equivalence class as some invertible lattice. Since $Eu(G)$ is coflasque, this last condition is also equivalent to $Eu(G)$ being invertible.

On the other hand, suppose that $G$ is “far” from being a $Z$-group in the sense that it has some Sylow subgroup $P$ which is not a rank 1 or 2 abelian group. Since $H^2(G, Eu(G, S)) \cong H^1(G, I_G) \cong H^0(G, \mathbb{Z}) \cong Z/|G|Z$, we see that it contains an element of order $|P|$. Saltman used this to show that $F(Eu(G, S))^G$ is not retract rational over $F$ and in particular it is not stably rational (Theorem 12.17 in [36]).

We note that here we consider the extension $F(Eu(G, S))^G/F$ where $G$ acts on $F$ trivially and not $L(Eu(G, S))^G/L^G$ where $L/L^G$ is a $G$-Galois extension, so we cannot use directly the invertibility of $Eu(G, S)$ in case $G$ is a $Z$-group. Nevertheless, we begin by investigating $Z$-groups.

### 4.1 Groups with cyclic $p$-Sylow subgroups

Each $Z$-group $G$ is isomorphic to a semidirect product $C_n \rtimes C_m$ where $C_n, C_m$ are cyclic of coprime orders $n$ and $m$ respectively it is well known (see [18], section 9.4). We start with the simplest case, namely cyclic groups.

**Lemma 30.** For $G$ cyclic, the lattice $Eu(G)$ is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}G)^{|G|^{-1}}$.

**Proof.** Letting $\sigma$ be a generator of $G$ we get that $Eu(G) \cong Eu(G, \{\sigma\}) \oplus (\mathbb{Z}G)^{|G|^{-1}}$ by Corollary 27. The Cayley graph $\Lambda(G, \{\sigma\})$ is just a cycle of length $|G|$ so $Eu(G, \{\sigma\}) \cong \mathbb{Z}$ and the lemma follows. □

**Remark 31.** By Theorem 20, the field $F(\mathbb{Z} \oplus (\mathbb{Z}G)^{|G|^{-1}})^G$ is rational over $F(\mathbb{Z}G)^G$. A result by Fischer [13] states that if $G$ is abelian with exponent $m$ and $F$ contains a primitive $m$-th root of unity, then $F(\mathbb{Z}G)^G$ is rational over $F$ (we generalize this result in Lemma 42). Thus, if $F$ contains a primitive $|G|$-th root of unity for $G$ cyclic, then $F(Eu(G))^G$ is rational over $F$. For $F = \mathbb{Q}$, even the extensions $\mathbb{Q}(\mathbb{Z}C_n)^{C_n}/\mathbb{Q}$ are not always rational, with the first counter example given by Swan in [39].

Let $G$ be an arbitrary $Z$-group, so in particular $Eu(G)$ is invertible from Endo-Miyata’s theorem (another proof for this is given in Corollary 36). Thus, if $L/L^G$ is $G$-Galois, then $L(Eu(G))^G/L^G$ is a retract rational extension. A similar result holds in our case.

**Lemma 32.** Let $G$ be a $Z$-group. Let $\zeta$ be a primitive $t$-th root of unity, where $t$ is the highest power of 2 dividing $|G|$. If $Gal(F(\zeta)/F)$ is cyclic, then the field $F(M)^G$ is retract rational over $F$ for any invertible faithful $G$-lattice $M$. In particular $F(Eu(G))^G/F$ is retract rational.

**Proof.** Given any invertible lattice $M$ and a $G$-Galois extension $L/L^G$, Saltman showed that $L(M)^G/L^G$ is a retract rational extension ([34], Theorem 3.14). In addition, if $Gal(F(\zeta)/F)$ is cyclic and $G$ is a $Z$-group, he proved that for $L = F(ZG)$ the extension $L^G/F$ is also retract rational (see [33], Theorems 2.1, 3.5, 5.3 and [34] Theorem 3.12). In particular we get that $F \subseteq F(\mathbb{Z}G)^G \subseteq F(\mathbb{Z}G \oplus M)^G$ is a tower of retract rational extensions, which by [19] means that $F(\mathbb{Z}G \oplus M)^G/F$ is also a retract rational extension. Since $M$ is faithful and $ZG$ is permutation, Proposition 19 shows that $F(\mathbb{Z}G \oplus M)^G/F(M)^G$ is rational and it follows that $F(M)^G/F$ is retract rational. □

Now that we know that $F(Eu(G))^G/F$ is a retract rational extension given that $F$ contains enough roots of unity, we turn to ask if it is a stably rational extension. One such result was given by Snider in [38]. Snider proved that if $G = \langle \sigma, \tau \mid \sigma^n = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ is a dihedral group with $n$
odd and $F$ contains a primitive $n$-th root of unity, then $\mathbb{F}(Eu(G))^G/F$ is stably rational (actually, by changing Snider’s final step in his proof, it can be shown that the extension is purely rational). As mentioned before, any $Z$-group is a semidirect product $\langle \sigma \rangle \rtimes \langle \tau \rangle$. Note that $\langle \tau \rangle$ acts on $\langle \sigma \rangle$ by conjugation, and in the dihedral case the action is faithful. In what follows, we shall prove that we can always reduce the question to the case where the $\langle \tau \rangle$ action is faithful.

For the rest of this section we fix the following notation.

Let $G = \langle \sigma, \tau \mid \sigma^n = r^m = e, \tau^{-1} \sigma \tau = \sigma^r \rangle$ where $(n, m) = 1$ and $r \in \mathbb{Z}$ satisfying $r^m \equiv_n 1$. Let $\langle \tau \rangle$ act on $\langle \sigma \rangle$ by conjugation and let $K$ be the kernel of this action, so in particular $K \subseteq Z(G)$. Note that if $m_0$ is the order of $r$ in $\mathbb{Z}/nz$, then $K = \langle \tau^{m_0} \rangle$. It follows that $G \cong G/K = \langle \sigma, \hat{\tau} \mid \hat{\sigma}^n = \hat{r} = e, \hat{\sigma}\hat{\tau}\hat{\sigma}^{-1} = \hat{\sigma}^r \rangle$ and $\langle \hat{\tau} \rangle$ acts faithfully on $\langle \hat{\sigma} \rangle$. We also assume that $\sigma^r \neq \sigma$, and in particular $n > 2$, since otherwise $G$ is cyclic.

Our next goal is to show that $F(Eu(G))^G$ is stably isomorphic to $F(Eu(G))^G$. We start by investigating the restriction of $Eu(G, \{\sigma, \tau\})$ to the subgroups $\langle \sigma \rangle$ and $\langle \tau \rangle$.

**Lemma 33.** The restriction of $Eu(G, \{\sigma, \tau\})$ to $\langle \sigma \rangle$ (resp. $\langle \tau \rangle$) is isomorphic to $\mathbb{Z} \oplus P$ where $P$ is a free $\langle \sigma \rangle$-module (resp. $\langle \tau \rangle$-module).

*Proof.* Let $Gr'$ be the subgraph of $\hat{B}(G, \{\sigma, \tau\})$ containing the edges $E' = E_1 \cup E_2$ where $E_1 = \{(\sigma^i, \sigma^{i+1}) \mid 0 \leq i \leq n - 1\}$ and $E_2 = \{(\sigma^i r^j, \sigma^{i+1} r^{j+1}) \mid 0 \leq i \leq n - 1, 0 \leq j \leq m - 2\}$. Clearly, $Gr'$ is a connected subgraph of $\hat{B}(G, \{\sigma, \tau\})$ which is $\sigma$ stable. This graph is basically a cycle of length $n + 1$ with $n$ paths going out of it ($E_2$). In particular, each Eulerian cycle on it is supported on $E_1$, so $Eu(Gr') \cong \mathbb{Z}$. Thus Lemma 25 shows that $Eu(G)_{C_n} \cong \mathbb{Z} \oplus P$ where $P$ is a free $\langle \sigma \rangle$-module. The proof for the subgroup $\langle \tau \rangle$ is similar. □

**Lemma 34.** Let $G$ be a group, $H \leq G$ a subgroup and $M$ a $G$-lattice. Let $M_1 = ZG/H \otimes_Z A$ and $M_2 = ZG \otimes_Z A_H$ be the $G$-lattices with the $G$-action induced by $g'(gH \otimes a) = g'gH \otimes g'a$ and $g'(g \otimes a) = (g'g) \otimes a$, respectively. Then the map $\varphi_H : M_1 \to M_2$ defined by $\varphi_H(gH \otimes a) = g \otimes g^{-1}a$ is an isomorphism of $G$-lattices.

*Proof.* The proof is straight forward and is left to the reader. □

If $H$ is either $\langle \sigma \rangle$ or $\langle \tau \rangle$, then from the previous two lemmas we have

$$ZG/H \otimes_Z Eu(G, \{\sigma, \tau\}) \cong ZG \otimes_Z Eu(G, \{\sigma, \tau\})_H \cong ZG/H \oplus Q$$

where $Q$ is a free $G$-module.

For a subgroup $H \leq G$ denote by $\varepsilon_{G/H}$ the augmentation map from $ZG/H$ to $Z$ defined by $\varepsilon_{G/H}(\sum a_g gH) = \sum a_g$ and set $I_{G/H} = \ker(\varepsilon_{G/H})$.

**Lemma 35.** Define $\varepsilon = \varepsilon_{G/\langle \sigma \rangle} \oplus \varepsilon_{G/\langle \tau \rangle} : ZG/\langle \sigma \rangle \oplus ZG/\langle \tau \rangle \to Z$ and set $K_{\varepsilon} = \ker(\varepsilon)$. Then we have exact sequences

$$0 \to K_{\varepsilon} \to ZG/\langle \sigma \rangle \oplus ZG/\langle \tau \rangle \overset{\varepsilon}{\to} Z \to 0$$

$$0 \to I_{G/\langle \sigma \rangle} \oplus I_{G/\langle \tau \rangle} \to K_{\varepsilon} \to Z \to 0$$

where the first sequence splits.
Proof. Since \((a, m) = 1\), there exist \(a, b \in \mathbb{Z}\) such that \(am + bn = 1\). Define \(\iota : \mathbb{Z} \to \mathbb{Z}/(a) \oplus \mathbb{Z}/(b)\) by \(\iota(1) = \left(\sum_{0}^{m-1} \sigma^{i} \langle \sigma \rangle, \sum_{0}^{n-1} \sigma^{j} \langle \tau \rangle\right)\). Since \(\iota(1)\) is \(G\)-invariant, \(\iota\) is \(G\)-equivariant and \(\iota(1) = am + nb = 1\), so the first sequence splits.

Clearly, as an abelian group \(K_{\varepsilon} \cong I_{G/(\sigma)} \oplus I_{G/(\tau)} \oplus \mathbb{Z}(e \langle \sigma \rangle - e \langle \tau \rangle)\). Since

\[g(e \langle \sigma \rangle - e \langle \tau \rangle) = (g - e) \langle \sigma \rangle - (g - e) \langle \tau \rangle + (e \langle \sigma \rangle - e \langle \tau \rangle)\]

we get that \(K/(I_{G/(\sigma)} \oplus I_{G/(\tau)}) \cong \mathbb{Z}\) with the trivial \(G\)-action. 

For the rest of this section, denote \(M := Eu(G, \{\sigma, \tau\})\) and recall that it is stably permutation equivalent to \(Eu(G, S)\) for any generating set \(S\) of \(G\). Tensoring the split sequence from the last lemma by \(M\), we conclude the following.

**Corollary 36.** The exact sequence

\[
\begin{array}{ccccccc}
0 & \longrightarrow & K_{\varepsilon} \otimes_{\mathbb{Z}} M & \longrightarrow & (\mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau)) \otimes_{\mathbb{Z}} M & \stackrel{\varepsilon \otimes id}{\longrightarrow} & M & \longrightarrow & 0
\end{array}
\]

splits and the module in the middle is a permutation module which is isomorphic to \(\mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau) \oplus P\), where \(P\) is a free \(G\) module. In particular, \(M\), \(K_{\varepsilon} \otimes_{\mathbb{Z}} M\) and \(Eu(G, S)\) are invertible modules for every generating set \(S\) of \(G\).

Since the exact sequence in the corollary above splits, we can reverse the arrows and get that \([M]^f = [K_{\varepsilon} \otimes_{\mathbb{Z}} M]\). Our next step is to find another such exact sequence which is easier to work with.

Taking the Tate cohomology for a subgroup \(H\) of \(G\) we get an exact sequence

\[
\begin{array}{ccccccc}
\hat{H}^0(H, (\mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau)) \otimes_{\mathbb{Z}} M) & \longrightarrow & \hat{H}^0(H, M) & \longrightarrow & \hat{H}^1(H, K_{\varepsilon} \otimes_{\mathbb{Z}} M) = 0
\end{array}
\]

On the other hand \((\mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau)) \otimes_{\mathbb{Z}} M \cong \mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau) \oplus P\) where \(P\) is a free \(G\)-module, hence

\[
\hat{H}^0(H, (\mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau)) \otimes_{\mathbb{Z}} M) \cong \hat{H}^0(H, \mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau) \oplus P) \cong \hat{H}^0(H, \mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau)).
\]

Thus, if \(P'\) is any permutation lattice and \(\pi : \mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau) \oplus P' \to M\) is surjective such that the restriction to \(\mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau)\) is \(\varepsilon \otimes id\), then the left most homomorphism of

\[
\begin{array}{ccccccc}
\hat{H}^0(H, \mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau) \oplus P') & \longrightarrow & \hat{H}^0(H, M) & \longrightarrow & \hat{H}^1(H, \ker(\pi)) & \longrightarrow & \hat{H}^1(H, \mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau) \oplus P') = 0
\end{array}
\]

will be surjective, and therefore \(\hat{H}^1(H, \ker(\pi)) = 0\) for every subgroup \(H \leq G\), or in other words \(\ker(\pi)\) is coflasque. Since \(M\) is invertible, we get that \(\pi\) splits, and therefore \(\ker(\pi) \oplus M \cong \mathbb{Z}/(\sigma) \oplus \mathbb{Z}/(\tau) \oplus P'\), making \(\ker(\pi)\) into an invertible \(G\)-lattice and \([M]^f = [\ker(\pi)]\).

Recall that \(1 \leq r \leq n-1\) is the integer such that \(\sigma \tau = \tau \sigma^r\). Define the Eulerian cycles \(S, T, A\) to be

\[
S := e \to \sigma \to \cdots \to \sigma^{n-1} \to e
\]

\[
T := e \to \tau \to \cdots \to \tau^{m-1} \to e
\]

\[
A := (e \to \sigma \to \sigma \tau) - (e \to \tau \to \tau \sigma \to \cdots \to \tau \sigma^r).
\]

20
Clearly we have that $\sigma(S) = S$ and $\tau(T) = T$. Let $\hat{S}, \hat{T}$ and $\hat{A}$ be generators for $\mathbb{Z}G/\langle \sigma \rangle, \mathbb{Z}G/\langle \tau \rangle$ and $\mathbb{Z}G$ respectively (as $\mathbb{Z}G$-modules), and define $\pi : \mathbb{Z}G/\langle \sigma \rangle \oplus \mathbb{Z}G/\langle \tau \rangle \oplus \mathbb{Z}G \to M$ by setting $\pi(\hat{S}) = S$, $\pi(\hat{T}) = T$ and $\pi(\hat{A}) = A$.

Denote by $N_{\sigma}, N_{\tau}$ the norm elements $N_{\sigma} = \sum_{i=0}^{n-1} \sigma^i$ and $N_{\tau} = \sum_{j=0}^{m-1} \tau^j$.

**Lemma 37.** The homomorphism $\pi : \mathbb{Z}G/\langle \sigma \rangle \oplus \mathbb{Z}G/\langle \tau \rangle \oplus \mathbb{Z}G \to M$ is surjective and $\ker(\pi)$ is spanned by the elements

\[
V_g = g \left( \sum_{j=0}^{m-1} \tau^j \sum_{i=0}^{r^j-1} \sigma^i(\hat{A}) + \frac{r^m - 1}{n} \cdot (\hat{S}) + \hat{T} - \sigma(\hat{T}) \right)
\]

\[
U_{\tau^j} = \tau^j \left( N_{\sigma}(\hat{A}) - \hat{S} + r\tau(\hat{S}) \right)
\]

where $g \in G$ and $0 \leq j \leq m - 1$.

**Proof.** For convenience, we sketch the proof for $S_3 \cong C_3 \times C_2$. The general proof is a similar and is left to the reader.

The Cayley graph of $S_3$ with respect to $\{\sigma, \tau\}$ is

\[
\begin{array}{c}
\sigma & \sigma^2 & \tau(A) & \tau(\sigma) & \tau(\sigma^2) & \tau(\sigma)(A) & \tau(\sigma^2)(A) & \tau(\sigma^3) & \tau(A)\\
\sigma & \sigma^2 & \tau(A) & \tau(\sigma) & \tau(\sigma^2) & \tau(\sigma)(A) & \tau(\sigma^2)(A) & \tau(\sigma^3) & \tau(A)\\
\sigma & \sigma^2 & \tau(A) & \tau(\sigma) & \tau(\sigma^2) & \tau(\sigma)(A) & \tau(\sigma^2)(A) & \tau(\sigma^3) & \tau(A)\\
\end{array}
\]

Note that some of the vertices and edges appear more than one time to make the illustration clearer.

By definition, $S, T$ and $A$ are in $\text{Im}(\pi)$, so to show surjectivity it is enough to show that they generate $M$.

Suppose that $a \in M$ is an Eulerian cycle which contains the edge $\sigma^2 \to \tau \sigma$ with coefficient $\lambda$. We can remove this edge by moving to $a - \lambda\sigma(\hat{A})$. Similarly we can remove the edge $\sigma \to \tau \sigma^2$ with the use of $A$, and without adding back the edge $\sigma^2 \to \tau \sigma$. We can do the same trick in the top row and therefore we are left only with the edges of the form $\sigma^i \to \sigma^{i+1}$, $\tau \sigma^i \to \tau \sigma^{i+1}$ and $\tau^j \to \tau^{j+1}$. Clearly, this Eulerian cycle is a linear combination of $S, \tau(S)$ (horizontal lines) and $T$ (left vertical line), concluding that $a \in \text{Im}(\hat{\pi})$.

We now turn to study the kernel $\ker(\pi)$. Clearly the norms $N_{\sigma}, N_{\tau}$ satisfy $N_{\sigma} \sigma = \sigma N_{\sigma} = N_{\sigma}$ and $N_{\tau} \tau = \tau N_{\tau} = N_{\tau}$. Since $\langle \sigma \rangle$ is normal in $G$, the norm $N_{\sigma}$ is central in $\mathbb{Z}G$.

By definition, the Eulerian cycle $A$ has $+1$ on the right and bottom edges and $-1$ on the left and top edges. We thus see that $A + \sigma(\hat{A}) + \sigma^2(\hat{A})$ is zero on the edges of degree $\tau$ (which point up) and is making one cycle $e \to \sigma \to \sigma^2 \to e$ on the bottom lines and twice the cycle $\tau \to \tau \sigma \to \tau \sigma^2 \to \tau$ in the middle with a minus sign. In other words we have that $A + \sigma(\hat{A}) + \sigma^2(\hat{A}) = S - 2\tau(\hat{S})$, so that $U_{\tau^0} \in \ker(\pi)$. Similarly, the sum $A + \tau(\hat{A}) + \tau(\sigma)(\hat{A})$ equals the square with the edges $e \to \sigma$, $\sigma(T) = \sigma \to \tau \sigma^2 \to \sigma$ with plus sign and the edges $T = e \to \tau \to e$ and $e \to \sigma \to \sigma^2 \to e \to \sigma$ with a minus sign. The top and the bottom edges gives us one cycle $e \to \sigma \to \sigma^2 \to e$ (where $1 = \frac{r^m - 1}{n}$.

21
for \( r = m = 2 \) and \( n = 3 \). We conclude that \( A + (\tau(A) + \tau\sigma(A)) = \sigma(T) - T - S \) and therefore \( V_\epsilon \in \ker(\pi) \).

Let \( N = \text{span} \{ U_g, V_{ij} \mid g \in G, 0 \leq j \leq m - 1 \} \). If \( a \in \ker(\pi)/N \), then using the element \( U_{\tau,i} \) and \( V_g \) we can find a representative for him of the form

\[
x = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} \alpha_{j,i} \tau^j \sigma^i(\hat{A}) \right) + \beta \cdot \hat{T} + \gamma \cdot \hat{S}
\]

where \( \alpha_j, \beta, \gamma \in \mathbb{Z} \). For any \( 0 \leq j \leq m - 1 \) and \( 1 \leq i \leq n - 1 \), the only elements in \( \{ \tau^{j+i} \sigma^i A \} \cup \{ S, T \} \) that touch the edge \( \langle \tau^j \sigma^i, \tau^j \sigma^i \tau \rangle \) are \( \tau^j \sigma^i(A) \) (with a minus sign) and \( \tau^j \sigma^i - 1 \) \( A \) (with a plus sign). We conclude that \( \alpha_{j,i} = \alpha_{j,i-1} \), so setting \( \alpha_j = \alpha_{j,0} \) we see that

\[
x = \sum_{j=0}^{m-1} \alpha_j \tau^j N_\sigma(\hat{A}) + \beta \cdot \hat{T} + \gamma \cdot \hat{S}.
\]

Using the elements \( U_{\tau} \), we can find another representative of the form

\[
x' = \sum_{j=0}^{m-1} \alpha_j \tau^j \left( \hat{S} - r \tau \hat{S} \right) + \beta \cdot \hat{T} + \gamma \cdot \hat{S} = \beta \hat{T} + \sum_{j=1}^{m-1} (\alpha_j - ra_{j-1}) \tau^j \hat{S} + (\alpha_0 - ra_{m-1} + \gamma) \hat{S}.
\]

Since \( \pi(x') = 0 \) and the elements \( T, \{ \tau^j S \}_{1}^{n} \) are linearly independent, \( x' \) must be zero and therefore \( \ker(\pi) = N \).

We are now ready to show that \( H^0(\hat{G}, \mathbb{Z}G \oplus \mathbb{Z}\langle \sigma \rangle \oplus \mathbb{Z}\langle \tau \rangle) \xrightarrow{\epsilon^*} H^0(\hat{G}, M) \) is surjective. In order to do this we need to find suitable \( \langle \sigma \rangle \) and \( \langle \tau \rangle \) permutation bases for \( M \).

Let

\[
\Gamma_S = \{ S \} \cup \{ \tau^j \sigma^i(A) \mid 1 \leq j \leq m - 1, 0 \leq i \leq n - 1 \} \cup \{ \sigma^i(T) \mid 0 \leq i \leq n - 1 \}
\]

\[
\Gamma_T = \{ T \} \cup \{ \tau^j \sigma^i(A) \mid 0 \leq j \leq m - 1, 0 \leq i \leq n - 2 \} \cup \{ \tau^j(S) \mid 0 \leq j \leq n - 1 \}
\]

and let \( M_S, M_T \) be the groups generated by \( \Gamma_S \) and \( \Gamma_T \) respectively. The sets \( \Gamma_S \) and \( \Gamma_T \) are \( \sigma \) and \( \tau \)-invariant respectively, so \( M_S \) and \( M_T \) are \( \sigma \) and \( \tau \)-lattices respectively. Notice that both these sets are of cardinality \( nm + 1 \) which is equal to \( \text{rank}(M) \).

We have \( 0 = \pi(V_\epsilon) \in A + M_S \) so \( A \in M_S \), and since this is a \( \sigma \) lattice we get that \( \sigma^j(A) \in M_S \) for all \( i \in \mathbb{Z} \). Similarly we have \( 0 = \pi(U_{\tau,j}) \in \tau^{j+1}S + M_S \) and since \( S \in \Gamma_S \) we have that \( \tau^j(S) \in M_S \) for all \( j \). We conclude that \( M_S = M \), and a similar process shows that \( M_T = M \). In addition, since \( |\Gamma_S| = |\Gamma_T| = nm + 1 = \text{rank}(M) \) we get that \( \Gamma_S \) and \( \Gamma_T \) are \( \mathbb{Z} \)-bases for \( M \).

**Lemma 38.** The restriction of \( \epsilon \otimes \text{id} \) to \( \mathbb{Z}G/\langle \sigma \rangle \oplus \mathbb{Z}G/\langle \tau \rangle \) sends \( \langle \sigma \rangle \) to \( S \) and \( \langle \tau \rangle \) to \( T \) for the choice of bases \( \Gamma_S \) and \( \Gamma_T \).

**Proof.** Let

\[
\hat{\Gamma}_S = \{ \tau^j \langle \sigma \rangle \otimes \tau^j a \mid 0 \leq j \leq m - 1, a \in \Gamma_S \} \subseteq \mathbb{Z}G/\langle \sigma \rangle \otimes \mathbb{Z} M.
\]

Since \( \tau^j(\hat{\Gamma}_S) \) is a \( \mathbb{Z} \)-basis for \( M \) for every \( j \in \mathbb{Z} \), we get that \( \hat{\Gamma}_S \) is a \( \mathbb{Z} \)-basis for \( \mathbb{Z}G/\langle \sigma \rangle \otimes \mathbb{Z} M \). It is also easy to see that \( \hat{\Gamma}_S \) is \( G \)-stable. Taking \( \hat{\Gamma}_S' = \{ \tau^j \langle \sigma \rangle \otimes \tau^j S \mid 0 \leq j \leq m - 1 \} \), we get that \( \hat{\Gamma}_S \setminus \hat{\Gamma}_S' \) is \( G \)-free because \( \hat{\Gamma}_S \setminus \{ S \} \) is \( \langle \sigma \rangle \)-free. Thus, the copy of \( \mathbb{Z}G/\langle \sigma \rangle \) in \( \mathbb{Z}G/\langle \sigma \rangle \otimes \mathbb{Z} M \) corresponds to \( \mathbb{Z}\hat{\Gamma}_S' \), and the image of \( \langle \sigma \rangle \otimes S \) in \( \mathbb{Z} \otimes M \) is 1 \( \otimes S = S \). A similar process shows that the image of \( \langle \tau \rangle \) is \( T \).
Corollary 39. We have $M \oplus \ker(\pi) \cong \mathbb{Z}G \oplus \mathbb{Z}G/\langle \sigma \rangle \oplus \mathbb{Z}G/\langle \tau \rangle$, and in particular $\ker(\pi)$ is invertible with $|M|^f = [\ker(\pi)]$.

Proof. Since $\pi$ and $\varepsilon \otimes id$ coincide on their respective restriction to $\mathbb{Z}G/\langle \sigma \rangle \oplus \mathbb{Z}G/\langle \tau \rangle$, the lattice $\ker(\pi)$ is coflasque. Since $M$ is invertible and $\ker(\pi)$ coflasque, the exact sequence

$$0 \rightarrow \ker(\pi) \rightarrow \mathbb{Z}G \oplus \mathbb{Z}G/\langle \sigma \rangle \oplus \mathbb{Z}G/\langle \tau \rangle \rightarrow M \rightarrow 0$$

splits.

We now turn to study the structure of $\ker(\pi)$. Recall that

$$V_g = g \left( \sum_{j=0}^{m-1} \tau^j \sum_{i=0}^{r^j-1} \sigma^i(\hat{A}) + \frac{m-1}{n} \cdot (\hat{S}) + \hat{T} - \sigma(\hat{T}) \right)$$

$$U_{\tau^j} = \tau^j \left( N_{\sigma}(\hat{A}) - \hat{S} + r\tau(\hat{S}) \right).$$

The elements $\{\tau^j N_{\sigma}(\hat{A})\}_{j=0}^{m-1}$ are $\mathbb{Z}$ linearly independent and $U_\varepsilon$ is $\sigma$ invariant, hence $M_0 = \text{span}_\mathbb{Z} \{U_{\tau^j}\}$ is a submodule of $\ker(\pi)$ isomorphic to $\mathbb{Z}G/\langle \sigma \rangle$. Obviously $M_0 \subseteq \ker(\pi) \cap \langle \hat{A}, \hat{S} \rangle_{\mathbb{Z}G}$ and we wish to show that this is actually an equality.

Lemma 40. There is an equality $M_0 = \ker(\pi) \cap \langle \hat{A}, \hat{S} \rangle_{\mathbb{Z}G}$ and $\ker(\pi)/M_0 \cong I_{G/\langle \tau \rangle}$. 

Proof. Let $x \in \ker(\pi) \cap \langle \hat{A}, \hat{S} \rangle_{\mathbb{Z}G}$. As in the previous lemma, the coefficients of $\tau^j \sigma^i(A)$ for $j$ fixed are the same, so $x$ has the form

$$\sum_{j=0}^{m-1} \alpha_j \tau^j N_{\sigma} \hat{A} + \sum_{j=0}^{m-1} \beta_j \tau^j \hat{S} \equiv M_0 \sum_{j=0}^{m-1} \alpha_j \tau^j \left( \hat{S} - r\tau(\hat{S}) \right) + \sum_{j=0}^{m-1} \beta_j \tau^j \hat{S} = \sum_{j=0}^{m-1} (\alpha_j - r\alpha_{j-1} + \beta_j) \tau^j \hat{S}$$

where the subtraction in the indices is modulo $m$. Since $\pi(x) = 0$ and $\{\tau^j \hat{S}\}_{j=1}^m$ are linearly independent, we get that $x \equiv M_0 0$, so $M_0 = \ker(\pi) \cap \langle \hat{A}, \hat{S} \rangle_{\mathbb{Z}G}$. Since

$$\ker(\pi)/M_0 \leq \mathbb{Z}G/\langle \sigma \rangle \oplus \mathbb{Z}G/\langle \tau \rangle/\langle \hat{A}, \hat{S} \rangle \cong \mathbb{Z}G/\langle \tau \rangle$$

and $\ker(\pi)/M_0$ is generated by the images $\{g(\hat{T} - \sigma(\hat{T}))\}_{g \in G}$ of $\{V_g\}_{g \in G}$, it follows that $\ker(\pi)/M_0 \cong I_{G/\langle \tau \rangle}$. 

Let $P$ be any permutation $G$-lattice with a surjection $\psi : P \to I_{G/\langle \tau \rangle}$. We have the following
exact diagram

\[
\begin{array}{cccccc}
0 & 0 & & & & \\
\ker(\psi) & \rightarrow & \ker(\psi) & & & \\
0 & \rightarrow & ZG/\langle \sigma \rangle & \rightarrow & E & \rightarrow \rightarrow & P & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & ZG/\langle \sigma \rangle & \rightarrow & \ker(\pi) & \rightarrow I_{G/\langle \tau \rangle} & \rightarrow & 0 & \rightarrow & 0 \\
0 & & & & & & & & & \\
0 & & & & & & & & & \\
\end{array}
\]

where \( E = \ker(\pi) \times I_{G/\langle \tau \rangle} \), \( P \). Since \( P \) and \( ZG/\langle \sigma \rangle \) are permutation lattices, the middle row splits and hence \( E \) is a permutation lattice. The lattice \( \ker(\pi) \) is invertible and in particular flasque, so the middle column is a flasque resolution and therefore \( |M|^H = |\ker(\psi)|^H \).

**Corollary 41.** The field \( \mathbb{F}(ZG \oplus Eu(G, \{\sigma, \tau\})^G \) is stably isomorphic to \( \mathbb{F}(ZG \oplus \ker(\psi))^G \) (over \( \mathbb{F}(ZG)^G \)).

We remind the reader that \( M \) was defined in a similar manner, as the kernel of a surjection from a permutation lattice onto \( I_G \).

Let \( \psi : ZG \rightarrow I_{G/\langle \tau \rangle} \) be defined as \( g \rightarrow (\sigma - e)\hat{T} \), where we continue to use the notation \( \hat{T} \) instead of \( \langle \tau \rangle \). The kernel of \( \psi \) can also be viewed as a lattice of Eulerian cycles. Let \( \Lambda(G/\langle \tau \rangle) \) be the graph on the vertices \( \{ g\hat{T} \mid g \in G \} \) and the edges \( \{ g\hat{T} \rightarrow g\sigma\hat{T} \mid g \in G \} \). There is a natural \( G \)-action on the graph, which induces a \( G \) action on the Eulerian cycles \( Eu^{-}(G/\langle \tau \rangle) := Eu(Gr) \). Similar to the standard case, we get that \( Eu^{-}(G/\langle \tau \rangle) \cong \ker(\psi) \) as \( G \)-modules.

Contrary to the standard case, if \( \tau^j \) commutes with \( \sigma \), then \( \tau^j \) acts trivially on \( \ker(\psi) \). Recall that since \( \langle \sigma \rangle \) is normal in \( G \), the subgroup \( \langle \tau \rangle \) acts on \( \langle \sigma \rangle \) by conjugation, and we denote by \( K \) the kernel of this action. This subgroup is exactly the kernel of the action of \( G \) on the Cayley graph described above. It follows that \( G/K \) acts faithfully on \( Eu^{-}(G/\langle \tau \rangle) \). Since \( \langle \tau \rangle \) acts faithfully on \( ZG/\langle \sigma \rangle \), we get from Theorem 20 the stable rational isomorphisms

\[
\mathbb{F}(ZG \oplus Eu(G, \{\sigma, \tau\}))^G \cong \mathbb{F}(ZG \oplus Eu^{-}(G/\langle \tau \rangle))^G \sim \mathbb{F}(ZG \oplus ZG/\langle \sigma \rangle \oplus Eu^{-}(G/\langle \tau \rangle))^G \sim \mathbb{F}(ZG/\langle \sigma \rangle \oplus Eu^{-}(G/\langle \tau \rangle))^G.
\]

The next step is to mod out \( K \) from \( G \), and we do this by generalizing Fischer’s theorem.

**Lemma 42.** Let \( H \) be an abelian group of exponent \( m \) acting on a field \( \mathbb{K} \), such that \( \mathbb{K}^H \) contains an \( m \)-th primitive root of unity. Suppose that the kernel of the action is \( H_0 \leq H \), then \( \mathbb{K}(ZH)^H \) is rational over \( \mathbb{K}^{H/H_0} \).

**Proof.** Write \( \mathbb{K}(ZH) = \mathbb{K}(x_h \mid h \in H) \). Since \( \mathbb{K}^H \) contains a root of unity of the order of the exponent of \( H \), we get that \( H^* := Hom(H, \mathbb{K}^\times) = Hom(H, (\mathbb{K}^H)^\times) \). For any \( \varphi \in H^* \) define \( x_{\varphi} = \sum_h \varphi(h^{-1})x_h \). We have that \( \mathbb{K}(x_h \mid h \in H) = \mathbb{K}(x_\varphi \mid \varphi \in H^*) \) and \( g(x_\varphi) = \varphi(g)x_\varphi \). Let \( \psi : \prod_{|H|} \mathbb{Z} \rightarrow H_0^* \) be defined by

\[
\psi(k_1, ..., k_m)(h) = \prod \varphi_i^{k_i}(h).
\]

24
Remark of unity, then \( F \) is a stable rational extension. The kernel of \( \psi \) is a subgroup of \( \prod_{|H|} \mathbb{Z} \) of finite index \( |H_0| \), so it is also free of rank \( m \). If \( \bar{k} = (k_1, ..., k_m) \) and \( x^{(\bar{k})} = \prod x_{\bar{\varphi}_i}^{k_i} \), then \( g(x^{(\bar{k})}) = \psi(\bar{k})(g) \cdot x^{(\bar{k})} \).

Let \( \bar{k}^{(j)} = (k_1^{(j)}, ..., k_m^{(j)}) \); \( j = 1, ..., m \) be a basis for \( \ker(\psi) \) and set \( x^{(j)} := \prod x_{\bar{\varphi}_i}^{k_i^{(j)}} \) for \( 1 \leq j \leq m \).

By the previous argument we have that \( K(x^{(1)}, ..., x^{(m)}) \subseteq K(x_{\varphi_1}, ..., x_{\varphi_m})^{H_0} \).

If \( \varphi_i \mid H_0 \) has order \( t \) in \( H_0 \), then \( x_{\varphi_i} \) is an invariant monomial, and therefore

\[
\left[ K(x^{(1)}, ..., x^{(m)}) : K(x^{(1)}, ..., x^{(m)}) \right] \leq t.
\]

Using the decomposition of \( H_0 \) into cyclic groups and adding each of their generators, we get that

\[
\left| H_0 \right| = \left[ K(x_{\varphi_1}, ..., x_{\varphi_m}) : K(x_{\varphi_1}, ..., x_{\varphi_m})^{H_0} \right] \leq \left[ K(x^{(1)}, ..., x^{(m)}) \right] \leq |H_0|,
\]

and the equality \( K(x_{\varphi_1}, ..., x_{\varphi_m})^{H_0} = K(x^{(1)}, ..., x^{(m)}) \) follows (up to this point we followed Fischer’s proof).

For each \( 1 \leq j \leq m \) and \( h \in H/H_0 \) we have \( h(x^{(j)}) = \zeta_{h,j} x^{(j)} \) for elements (roots of unity) \( \zeta_{h,j} \in K \). For any fixed \( j \), the map \( h \mapsto \zeta_{h,j} \) is a 1-cocycle in \( Z^1(H/H_0, K^x) \), which by Hilbert 90 has the form \( \zeta_{h,j} = \frac{a_j}{h(a_j)} \) for some \( a_j \in K^x \). It follows that \( y^{(j)} = x^{(j)} a_j \) are \( H/H_0 \) invariant and algebraically independent. The proof is now finished by noting that

\[
K(x^{(1)}, ..., x^{(m)})^{(H/H_0)} = K(y^{(1)}, ..., y^{(m)})^{(H/H_0)} = K^{(H/H_0)}(y^{(1)}, ..., y^{(m)}).
\]

\[\square\]

**Theorem 43.** Suppose that \( F \) contains a primitive \( |G| \)-th root of unity. Then the field \( F(ZG/\sigma) \oplus Eu^{-}(G/(\tau))^{G} \) is stably isomorphic to \( F(Eu^{-}(G/(\tau)))^{G} = F(Eu^{-}(G/(\tau)))^{G/K} \). In particular, \( F(Eu(G))^{G} \) is stably isomorphic to \( F(Eu(G/K))^{G/K} \).

**Proof.** Write \( K = F(Eu^{-}(G/(\tau))) \). Since \( \langle \sigma \rangle \) acts trivially on \( ZG/\sigma \), we have that \( K(ZG/\sigma)^{\langle \sigma \rangle} = K^{\langle \sigma \rangle}(ZG/\sigma)^{\sigma} \). Thus, for the first claim it is enough to show that \( (K^{\langle \sigma \rangle}(ZG/\sigma))^{\sigma} \) is stably isomorphic to \( K^{\sigma}(ZG/\sigma) \), which is solved by the previous lemma.

We already proved that \( F(Eu(G))^{G} \) is stably isomorphic to \( F(ZG/\sigma) \oplus Eu^{-}(G/(\tau))^{G} \), and therefore stably isomorphic to \( F(Eu^{-}(G/(\tau))^{G/K} \). Since \( K \leq \langle \tau \rangle \) we have that \( (G/K)/(\tau/K) \cong G/\tau \) so \( F(Eu(G/K))^{G/K} \) is also stably isomorphic to \( F(Eu(G/\tau))^{G/\tau} \) which finishes the proof. \[\square\]

As we mentioned before, if \( G = \langle \sigma, \tau \mid \sigma^n = \tau^{2m} = e, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \) with \((n, 2m) = 1\), then \( G/K \cong D_{2n} \) where \( n \) is odd. Thus, the last lemma together with Snider’s proof of stable rationality for such dihedral group imply the stable rationality for the group \( G \).

**Corollary 44.** If \( G = \langle \sigma, \tau \mid \sigma^n = \tau^{2m} = e, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \) and \( F \) contains a primitive \( 2nm \) root of unity, then \( F(Eu(G))^{G}/F \) is stably rational.

**Remark 45.** In [11], Endo and Miyata also proved that for the groups appearing in the last corollary, the module \( I_G^\lambda \) is a quasi-permutation lattice, which is equivalent to \( Eu(G, S) \) being stably permutation. In particular, this means that \( F(Eu(G))^{G} \) is stably isomorphic to \( F(ZG)^G \), so that \( F(ZG)^G/F \) is a stable rational extension.
4.2 Abelian groups of rank 2

In this section we fix \( G = C_m \times C_n \) where \( C_m = \langle \sigma \rangle , \ C_n = \langle \tau \rangle \) are cyclic groups of order \( m,n \) respectively. Since \( \{ \sigma, \tau \} \) is a generating set, we have \( Eu(G) \cong Eu(G, \{ \sigma, \tau \}) \oplus P \) for some free \( G \)-module \( P \). Unless \( G \) is cyclic, it has noncyclic Sylow subgroups, so Endo and Miyata’s theorem [11] that was mentioned in the previous section states that \( Eu(G, \{ \sigma, \tau \}) \) is not invertible. In particular, we cannot use the approach from the previous section since \( \mathbb{F}(Ev(G)) \) is not rational over \( \mathbb{F}(P) \).

On the other hand, \( G \) acts faithfully on \( Eu(G, \{ \sigma, \tau \}) \) and \( P \) is a permutation lattice, so \( \mathbb{F}(Ev(G)) \) is rational over \( \mathbb{F}(Eu(G, \{ \sigma, \tau \})) \). We thus investigate the last field instead of \( \mathbb{F}(Ev(G)) \).

Define \( A,S,T \in Eu(G, \{ \sigma, \tau \}) \) to be the Eulerian cycles

\[
S := e \to \sigma \to \cdots \to \sigma^{n-1} \to e \\
T := e \to \tau \to \cdots \to \tau^{m-1} \to e \\
A := (e \to \sigma \to \sigma \tau) - (e \to \tau \to \tau \sigma).
\]

Similar to the proof in the case of \( Z \) groups in the previous section, it can be shown that \( Eu(G, \{ \sigma, \tau \}) \) is generated as a \( ZG \)-module by \( S,T,A \). Let \( M \) be the sublattice generated by \( A \).

**Lemma 46.** The lattice \( M \) is isomorphic to \( I_G^Z \) and \( Eu(G, \{ \sigma, \tau \})/M \cong Z \oplus Z \) is a permutation lattice. In particular, the field \( \mathbb{F}(Ev(G, \{ \sigma, \tau \})) \) is rational over \( \mathbb{F}(I_G^Z) \).

**Proof.** Suppose that \( \sum_{g \in G} k_g g(A) = 0 \) in \( M \). Note that \( g(A) \) and \( g \sigma(A) \) are the only elements containing the edge \( g \sigma \to g \sigma \tau \), and with opposite signs, and therefore \( k_g = k_{g \sigma} \). Similarly we have that \( k_g = k_{g \tau} \) for every \( g \in G \). It follows that \( \sum_{g \in G} k_g g(A) = k \sum_{g \in G} g(A) \), for some \( k \in Z \), hence \( M \cong ZG/\langle \sum_{g \in G} g \rangle \cong I_G^Z \).

Let \( L := Ev(G, \{ \sigma, \tau \})/M \). The lattice \( L \) is generated as a \( ZG \) module by the images \( \tilde{S}, \tilde{T} \) of \( S \) and \( T \) respectively. These images are \( G \)-invariant since

\[
\tau(S) - S = \sum_{i=0}^{n-1} \sigma^i(A), \quad \sigma(S) = S \\
\sigma(T) - T = \sum_{j=0}^{m-1} \tau^j(A), \quad \tau(T) = T.
\]

It follows that \( L \) has a trivial \( G \)-action. Since \( \text{rank} \ (Ev(G, \{ \sigma, \tau \})/M) = \text{rank} \ (Ev(G, \{ \sigma, \tau \})) - \text{rank} (M) = 2 \), we conclude that \( L \cong Z \oplus Z \).

The last claim now follows from Theorem 20. \( \square \)

**Theorem 47.** The field \( \mathbb{F}(Ev(G)) \) is rational over \( \mathbb{F} \) for the Klein four group \( G \).

**Proof.** We proved that \( \mathbb{F}(Ev(G)) \) is rational over \( \mathbb{F}(I_G^Z) \). The rationality of \( \mathbb{F}(I_G^Z) \) over \( \mathbb{F} \) was proved by Hajja and Kang in [17] (case 5). \( \square \)

**Remark 48.** The rationality of \( \mathbb{F}(Ev(G)) \) over \( \mathbb{F} \) for the Klein four group was proved directly by Snider in [38].

In the next section we will show that if \( \mathbb{F}(Ev(G)) \) is stably rational over \( \mathbb{F} \) for groups \( G_1,G_2 \) of coprime orders, then \( \mathbb{F}(Ev(G)) \) is also stably rational for \( G = G_1 \times G_2 \). We can thus conclude the following.
Theorem 49. Let $F$ be a field such that $F(Eu(C_m))^{C_{2m}}/F$ is stably rational where $m$ is odd (for example, if $F$ contains a primitive $m$-th root of unity). Then $F(Eu(G))^G/F$ is stably rational for $G = C_2 \times C_{2m}$.

Remark 50. In [31], it was shown that when $F$ contains a primitive $m$-th root of unity, for $m$ odd, then any $(C_2 \times C_{2m})$-crossed product over is actually a cyclic algebra.

4.3 $G = H \times K$ for groups $H, K$ of coprime orders

Let $L/L^G$ be a $G$-Galois extension of fields and $\alpha \in Z^2(G, L^\times)$ a 2-cocycle. Define $L_0(I_G)$ to be the field $L(y_g \mid e \neq g \in G)$ where the $y_g$ are algebraically independent with the $G$-Galois action

$$h(y_g) = \frac{y_{hg}}{y_h} \alpha(h, g),$$

where we denote $y_e = 1$. Note that the $G$-action can be rewritten as $\frac{y_h \alpha(g)}{y_h} = \alpha(h, g)$.

For fixed elements $\lambda_g \in L^\times$, $e \neq g \in G$, let $z_g = \lambda_g y_g$. Then clearly $L_0(I_G) = L(z_g \mid e \neq g \in G)$ with the $G$-action defined by $\frac{\lambda_h(z_g)}{z_g} = \lambda_h \frac{\alpha(h, \lambda_g)}{z_g}$, so in particular the field $L_0(I_G)$ is a function of the cohomology class of $\alpha$.

It can also be shown that $L_0(I_G)$ is a generic splitting field of $\Delta = \Delta(L/L^G, G, \alpha)$ in the sense of Amitsur [5]. We need only one property of such fields which we now prove.

Lemma 51. Let $\Delta = \Delta(L/L^G, G, \alpha)$ and where $\Delta$ is split. Then $L_0(I_G)^G$ is stably rational over $L^G$.

Proof. Since $\Delta$ splits, the cocycle $\alpha$ is cohomologous to 1. As mentioned above, we may assume that the action is defined by $h(y_g) = \frac{y_{\lambda_g}}{y_h}$, or in other words, $L_0(I_G) \cong L_1(I_G) \cong L(I_G)$ where the correspondence is $g - e \leftrightarrow y_g$. The module $I_G$ is part of the exact sequence

$$0 \rightarrow I_G \rightarrow ZG \xrightarrow{\epsilon} Z \rightarrow 0,$$

so by Theorem 20 we get that $L(ZG)^G$ is rational over $L(I_G)^G$. On the other hand $L(ZG)^G$ is rational over $L^G$ since $ZG$ is a permutation lattice and $L/L^G$ is Galois, and the lemma follows.

We now turn to prove that if $F(Eu(H))^H$ and $F(Eu(K))^K$ are stably rational over $F$ and $H, K$ are of coprime orders, then so is $F(Eu(G))^G$ for $G = H \times K$.

This was proved in the nongraded case by Katsylo [21], Schofield [37] and Saltman [35]. We will adapt Saltman’s proof for the graded case.

Since $H^2(K, L^\times) = 0$ by Hilbert 90, we have the inflation restriction exact sequence

$$0 \rightarrow H^2(G/K, (L^\times)^K) \xrightarrow{inf} H^2(G, L^\times) \xrightarrow{res} H^2(K, L^\times)^{G/K}.$$  

Since $|H| = |G/K|$ and $|K|$ are coprime, we can find $a, b \in Z$ such that $a|G/K| + b|K| = 1$. If $\alpha \in H^2(G, L^\times)$ is any 2-cocycle, then we can write $\alpha = \alpha_H \cdot \alpha_K$ where $\alpha_H = \alpha^{[G/K]}$ and $\alpha_K = \alpha^{[K]}$. The group $H^2(K, L^\times)$ is $|K|$-torsion, so $res(\alpha_K) = 0$, and therefore $\alpha_H = inf(\beta_H)$ for some $\beta_H \in H^2(G/K, (L^\times)^K)$. Switching the roles of $H$ and $K$ we get that $\alpha_K$ is an inflation of some $\beta_K \in H^2(G/H, (L^\times)^H)$. This induces a decomposition

$$\Delta(L/L^G, G, \alpha) = \Delta(L^K/L^G, H, \beta_H) \otimes_{L^G} \Delta(L^H/L^G, K, \beta_K).$$
Let \( E \) be a field with a \( G \)-action. Let \( E(c(g, h) \mid e \neq g, h \in G) \) be a field extension of \( E \) such that \( \{c(g, h) \mid e \neq g, h \in G\} \) are algebraically independent over \( E \). Define a \( G \)-action on it by 
\[
\sigma(c(g, h)) = \frac{c(\sigma g, \sigma h)}{c(g, h)}
\]
for every \( \sigma, g, h \in G \), where we denote \( c(g, e) = c(e, g) = 1 \) for all \( g \in G \). We call this field a generic \( G \)-2-cocycle extension of \( E \), and denote it by \( E(c) \). Under this notation, the field \( E(c) \) is just \( E(U(G, G \setminus \{e\})) \).

**Lemma 52.** Let \( L/L^G \) be a \( G \)-Galois extension \( \gamma \in Z^2(G, L^\times) \) and let \( c \) be a generic \( G \)-2-cocycle. Then we have the following:

1. \( L\alpha,\gamma(I_G) \cong L\alpha(c)(I_G) \).
2. \( L\alpha(c)(I_G)^G \) is a rational extension of \( L^G \).

**Proof.**

1. The action on \( L\alpha(c)(I_G) \) is defined by 
\[
\frac{y_hh(y_g)}{y_o} = \alpha(h, g)c(h, g).
\]
Setting \( c'(h, g) = c(h, g)\alpha(h, g) \), we clearly get that \( \{c'(h, g) \mid e \neq g, h \in G\} \) are algebraically independent over \( L \), so \( c' \) is a generic \( G \)-2-cocycle and \( L\alpha(c)(I_G) \cong L\alpha(c')(I_G) \).

2. The \( G \)-action on \( L\alpha(c)(I_G) \) is defined by 
\[
h(y_g) = \frac{y_hh(y_o)}{y_h}c(h, g).
\]
It follows that
\[
L\alpha(c)(I_G) = L\alpha(c(g, h) \mid e \neq g, h \in G)(y_g) = L\alpha(c(g, h) \mid e \neq g, h \in G)(y_g) \cong L((ZG)^{G[G-1]})^G.
\]
Since \( L/L^G \) is \( G \)-Galois and \( ZG \) is a permutation lattice, we get that \( L\alpha(c)(I_G)^G = L((ZG)^{G[G-1]})^G \) is rational over \( L^G \).

\( \square \)

Part (2) in the last lemma can be also viewed in the notations of Eulerian cycles. While the generic 2-cocycle corresponds to the Eulerian cycles in the Cayley graph, the elements \( y_g \) corresponds to the edges \( (e, g) \) for \( g \in G \) which constitute a spanning tree. Clearly, any edge in the graph can be written as a cycle minus edges in a spanning tree, so the Eulerian cycles together with a spanning tree gives us all the edges in the graph, which is a free \( G \)-set.

We now turn to decompose 2-cocycles on \( G \) into two 2-cocycles on \( H \) and \( K \).

Let \( c \) be a generic \( G \)-2-cocycle and let \( \alpha, \beta \) be inflations of generic \( H \cong G/K \) and \( K \cong G/H \) 2-cocycles. More precisely, the elements \( \{\alpha(h_1, h_2) \mid e \neq h_1, h_2 \in H\} \) are algebraically independent, \( \alpha(h_1g_1, h_2g_2) = \alpha(h_1, h_2) \) for any \( h_1, h_2 \in H \) and \( g_1, g_2 \in K \), and the \( G \)-action is defined by 
\[
g(\alpha(h_1, h_2)) = \alpha(h_1, h_2) \quad \text{for all } g \in K, \ h_1, h_2 \in H
\]
\[
h_1(\alpha(h_2, h_3)) = \frac{\alpha(h_1, h_2)\alpha(h_1h_2, h_3)}{\alpha(h_1, h_2h_3)},
\]
and similarly define \( \beta \).

We will show that \( F(\alpha, \beta, c)_{\alpha\beta^e}(I_G)^G \) is rational over \( F(c)^G \) and over \( F(\alpha, \beta)^G \). The last field is the fraction field of \( F(\alpha)^H \otimes_F F(\beta)^K \), and this will finish the proof.
The first part follows from the previous two lemmas. More precisely we have $F(\alpha, \beta)(c)_{\alpha, \beta}(I_G)^G = F(\alpha, \beta)(c)_{\alpha, \beta}(I_G) \cong F(\alpha, \beta)(c)_{\alpha, \beta}(I_G)$ which is rational over $F(\alpha, \beta)^G$.

Since $\alpha, \beta$ are two generic 2-cocycles for $H, K$ respectively, we expect that $\alpha \cdot \beta$ will have similar properties as a generic $G$ 2-cocycle. Indeed, the next lemma shows that the previous lemma still holds in this case.

**Lemma 53.** Let $L/L^G$ be a $G$-Galois extension where $G = H \times K$ with $|H|, |K| = 1$ and $\gamma \in Z^2(G, L^x)$. Let $\alpha, \beta$ be the inflations of the $H$ and $K$ generic 2-cocycles. Then the following holds:

1. $L(\alpha, \beta)_{a, \beta}(I_G) \cong L(\alpha, \beta)_{a, \beta}(I_G)$.

2. $L(\alpha, \beta)_{a, \beta}(I_G)^G$ is stably isomorphic to $L^G$.

**Proof.** 1. As we mentioned in the beginning of this section, the inflation restriction exact sequence produces a decomposition of $\gamma$ into $\gamma_H \cdot \gamma_K$ where $\gamma_H \in \text{inf}(Z^2(G/K, (L^x)^K))$ and $\gamma_K \in \text{inf}(Z^2(G/H, (L^x)^H))$. Letting $y_g$ be the indeterminates corresponding to $I_G$ in $L(\alpha, \beta)_{a, \beta}(I_G)$, we have the action

$$h_1 \tau_1(y_{h_2 \tau_2}) = \frac{y_{h_1 h_2 \tau_1 \tau_2}}{y_{h_1 \tau_1}}(h_1 \tau_1, h_2 \tau_2)\beta(h_1 \tau_1, h_2 \tau_2)\gamma(h_1 \tau_1, h_2 \tau_2)$$

for all $h_1, h_2 \in H$ and $\tau_1, \tau_2 \in K$. As in the previous lemma, defining $\alpha(g_1, g_2) = \alpha(g_1, g_2)\gamma_H(g_1, g_2)$ and $\beta(g_1, g_2) = \beta(g_1, g_2)\gamma_K(g_1, g_2)$ for $g_1, g_2 \in G$, we get that $\alpha$ and $\beta$ are inflation of generic 2-cocycles in $H, K$ respectively, which show that $L(\alpha, \beta)_{a, \beta}(I_G) \cong L(\alpha, \beta)_{a, \beta}(I_G)$.

2. Consider the field $K = L(\alpha, \beta)_{a, \beta}(I_G/K)\beta(I_G/H)_{a, \beta}(I_G)$, where the indeterminates of $I_G/K$, $I_G/H$ and $I_G$ are denoted by $z_h$, $w_\tau$ and $y_g$ respectively. The 2-cocycle $\alpha \cdot \beta$ splits in $L(\alpha, \beta)_{a, \beta}(I_G/K)\beta(I_G/H)$. Indeed, setting $\lambda_{h, \tau} = z_h w_\tau$ for $h \in H$ and $\tau \in K$ we get that $\alpha \beta(g_1, g_2) = \frac{\lambda_{g_1} \gamma(g_1, g_2)}{\lambda_{g_2}}$, where we use the fact that the $z_h$ are $K$-trivial and the $w_\tau$ are $H$-trivial. We conclude that $K \cong L(\alpha, \beta)_{a, \beta}(I_G/K)\beta(I_G/H)$, so $G^K$ is rational over $L(\alpha, \beta)_{a, \beta}(I_G/K)\beta(I_G/H)^G$.

Clearly, we have $K = L(\alpha, \beta)_{a, \beta}(I_G/K)\beta(I_G/H)$, and we wish to show that $\alpha$ splits in $L(\alpha, \beta)_{a, \beta}(I_G)^K$. Let $a, b \in Z$ such that $|H| a + |K| b = 1$, and set $\tilde{z}_h = z_h \cdot \left( \prod_{\tau \in K} \tau(y_{h_\tau}) \right)^{-b}$ for each $h \in H$ and note that $\tilde{z}_h$ is $K$ invariant. For $h, h' \in H$ we get that

$$h'(\tilde{z}_h) = \left[ \tilde{z}'_h \frac{\tilde{z}'h}{\tilde{z}'h'}(h', h) \right] \left[ \prod_{\tau} \frac{y_{\tau \lambda}(h', h')}{y_{\tau \lambda}(h', h)}(h, h') \right] = \frac{\tilde{z}_h}{\tilde{z}_{h'}}(h, h')^b$$

Since $\alpha$ is an inflation of an $H$ 2-cocycle, we get that $\alpha^{a|H|} \sim 1$ where both cocycles are in $L(\alpha, \beta)^K$. Thus, we can find $\lambda_h \in L(\alpha, \beta)^K$ such that $\alpha(h_1, h_2) = \frac{\lambda_h h_1(\lambda_h)}{\lambda_h h_2}$ for all $h_1, h_2 \in H$. 29
Taking $\bar{z}_h = \frac{z_h}{\lambda_h}$ we get that (1) $\bar{z}_h$ are $K$ invariant and (2) $H$ acts on them by $h'(\bar{z}_h) = \frac{\bar{z}_h}{\lambda_{h'}}$.

or in other words $K \cong L(\alpha, \beta)_{\alpha, \beta} (I_G/H)(I_G/K)$. By Lemma 51, we know that $K^H$ is stably isomorphic to $K = L(\alpha, \beta)_{\alpha, \beta} (I_G/H)^H$, and actually it can be shown that this stably isomorphism uses $K$ invariant indeterminates, so we actually have that $K^G$ is stably isomorphic to $L(\alpha, \beta)_{\alpha, \beta} (I_G/H)^G$.

Repeating this process for $K$, we get that $K^G$ is stably isomorphic to $L(\alpha, \beta)_{\alpha, \beta} (I_G)^G$ and we are done. 

We now have all the ingredients for the proof of Theorem 2.

Proof. (of Theorem 2) Let $c$ be a generic $G$ 2-cocycle where $G = H \times K$ with $|H|, |K| = 1$. Let $\alpha, \beta$ be the inflations of the $H$ and $K$ generic 2-cocycles. Consider the field $\mathbb{K} = F(\alpha, \beta, c)_{\alpha, \beta, c}(I_G)^G$.

From part (1) in Lemma 52 we get that

$$F(\alpha, \beta, c)_{\alpha, \beta, c}(I_G) = (F(\alpha, \beta))(c)_{\alpha, \beta, c}(I_G)$$

and from part (2) we get that $(F(\alpha, \beta))(c)_{\alpha, \beta}(I_G)^G$ is stably isomorphic to $F(\alpha, \beta)^G$. Lemma 53 shows similarly that $F(\alpha, \beta, c)_{\alpha, \beta, c}(I_G)^G$ is stably isomorphic to $F(c)^G$.

Finally, we have

$$|G| = [F(\alpha, \beta) : F(\alpha)_{\alpha, \beta}^G] \leq [F(\alpha, \beta) : F(\alpha)^H F(\beta)^K] [F(\alpha) F(\beta)^K : F(\alpha)^H F(\beta)^K]$$

$$ \leq [F(\beta) : F(\beta)^K] [F(\alpha) : F(\alpha)^H] = |K||H| = |G|,$$

so there are equalities everywhere. In particular $F(\alpha, \beta)^G = F(\alpha)^H \cdot F(\beta)^K$, which is isomorphic to the fraction field of $F(\alpha)^H \otimes F(\beta)^K$ (since $F(\alpha) \cap F(\beta) = F$). It follows that $F(c)^G$ is stably isomorphic to the fraction field of $F(\alpha)^H \otimes F(\beta)^K$. 

\begin{thebibliography}{9}

[1] E. Aljadeff, D. Haile and M. Natapov, \textit{Graded identities of matrix algebras and the universal graded algebra}, Trans. Amer. Math. Soc. \textbf{362}(06):(2010), 3125–3147.

[2] E. Aljadeff and Y. Karasik, \textit{On generic g-graded algebras}, In preparation .

[3] E. Aljadeff and Y. Karasik, \textit{Crossed products and their central polynomials}, J. Pure Appl. Algebra \textbf{217}(9):(2013), 1634–1641.

[4] S. A. Amitsur, \textit{On central division algebras}, Israel J. Math. \textbf{12}(4):(1972), 408–420.

[5] S. A. Amitsur, \textit{Generic splitting fields}, in F. M. J. v. Oystaeyen and A. H. M. J. Verschoren, editors, Brauer Groups in Ring Theory and Algebraic Geometry, number 917 in Lecture Notes in Mathematics, pages 1–24, Springer Berlin Heidelberg (1982).

[6] I. N. Balaba, \textit{Graded prime PI-Algebras}, J. Math. Sci. \textbf{128}(6):(2005), 3345–3349.

[7] E. Beneish, \textit{Induction theorems on the stable rationality of the center of the ring of generic matrices}, Trans. Amer. Math. Soc. \textbf{350}(9):(1998), 3571–3585.

\end{thebibliography}
[8] G. Berhuy and G. Favi, *Essential dimension: a functorial point of view (after a. merkurjev)*, Documenta Math 8(106):(2003), 279–330.

[9] C. Bessenrodt and L. Le Bruyn, *Stable rationality of certain $PGL_n$-quotients*, Invent. Math. 104(1):(1991), 179–199.

[10] S. Bloch, *Torsion algebraic cycles, $k_2$, and brauer groups of function fields*, Bull. Amer. Math. Soc 80(5):(1974), 941–945.

[11] S. Endo and T. Miyata, *On a classification of the function fields of algebraic tori*, Nagoya Math J. 56:(1975), 85–104.

[12] B. Fein and M. Schacher, *Brauer groups of rational function fields over global fields*, in M. Ker- vraire and M. Ojanguren, editors, Groupe de Brauer, number 84 in Lecture Notes in Mathematics, pages 46–74, Springer Berlin Heidelberg (1981).

[13] E. Fischer, *Die isomorphie der invariantenkorper der endlichen abelschen gruppen lineareer transformationen*, Nachrichten von der Gesellschaft der Wissenschaften zu Gottingen, Mathematisch-Physikalische Klasse pages 77–80.

[14] E. Formanek, *The center of the ring of 3x3 generic matrices*, Linear Multilinear Algebra 7(3):(1979), 203–212.

[15] E. Formanek, *The center of the ring of 4x4 generic matrices*, J. Algebra 62(2):(1980), 304–319.

[16] E. Formanek, *The Polynomial Identities and Invariants of n x n Matrices*, American Mathematical Society (1992).

[17] M. Hajja and M.-C. Kang, *Finite group actions on rational function fields*, J. Algebra 149(1):(1992), 139–154.

[18] M. Hall, *The theory of groups*, AMS Chelsea Pub., Providence, R.I. (1999).

[19] M.-C. Kang, *Retract rational fields*, J. Algebra 349(1):(2012), 22–37.

[20] Y. Karasik and Y. Shpigelman, *On the codimension sequence of g-simple algebras*, arXiv:1312.4167 [math] .

[21] P. I. Katsylo, *Stable rationality of the field of invariants of linear representations of the groups PSL6 and PSL12*, Mathematical notes of the Academy of Sciences of the USSR 48(2):(1990), 751–753.

[22] L. Le Bruyn, *Permutation modules and rationality problems 1*.

[23] L. Le Bruyn, *Centers of generic division algebras, the rationality problem 1965-1990*, Israel J. Math. 76(1):(1991), 97–111.

[24] M. Lorenz, *Multiplicative Invariant Theory*, Springer, 2005 edition (2005).

[25] M. Lorenz and Z. Reichstein, *Lattices and parameter reduction in division algebras*, arXiv:math/0001026v1 [math.RA] .
[26] A. S. Merkur’ev and A. A. Suslin, *Cohomology of severi-brauer varieties and the norm residue homomorphism*, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya 46(5):(1982), 1011–1046.

[27] F. D. Meyer and E. Ingraham, *Separable Algebras over Commutative Rings*, Springer, 1971 edition (1971).

[28] M. Orzech and C. Small, *The Brauer Group of Commutative Rings*, M. Dekker, New York (1975).

[29] C. Procesi, *Non-commutative affine rings*, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8) 8:(1967), 237–255.

[30] S. Rosset, *Group extensions and division algebras*, J. Algebra 53(2):(1978), 297–303.

[31] L. H. Rowen and D. J. Saltman, *Diheral algebras are cyclic*, Proc. Amer. Math. Soc. 84(2):(1982), 162–164.

[32] D. J. Saltman, *Azumaya algebras with involution*, J. Algebra 52(2):(1978), 526–539.

[33] D. J. Saltman, *Generic galois extensions and problems in field theory*, Adv. Math. 43(3):(1982), 250–283.

[34] D. J. Saltman, *Retract rational fields and cyclic galois extensions*, Israel J. Math. 47(2):(1984), 165–215.

[35] D. J. Saltman, *A note on generic division algebras*, Abelian groups and noncommutative rings, Contemporary Math 130:(1992), 385–394.

[36] D. J. Saltman, *Lectures on Division Algebras*, American Mathematical Soc. (1999).

[37] A. Schofield, *Matrix invariants of composite size*, J. Algebra 147(2):(1992), 345–349.

[38] R. Snider, *Is the brauer group generated by cyclic algebras?*, in D. Handelman and J. Lawrence, editors, *Ring theory*, volume 734 of Lecture Notes in Mathematics, pages 279–301, Springer Berlin / Heidelberg (1979).

[39] R. G. Swan, *Invariant rational functions and a problem of steenrod*, Invent. Math. 7(2):(1969), 148–158.

[40] J. J. Sylvester, *On the involution of two matrices of the second order*, The Collected Mathematical Paper 4.