A COMPUTABLE ANALYSIS OF VARIABLE WORDS THEOREMS

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Abstract. The Carlson-Simpson lemma is a combinatorial statement occurring in the proof of the Dual Ramsey theorem. Formulated in terms of variable words, it informally asserts that given any finite coloring of the strings, there is an infinite sequence with infinitely many variables such that for every valuation, some specific set of initial segments is homogeneous. Friedman, Simpson, and Montalban asked about its reverse mathematical strength. We study the computability-theoretic properties and the reverse mathematics of this statement, and relate it to the finite union theorem. In particular, we prove the Ordered Variable word for binary strings in ACA₀.

1. Introduction

Let İ(N)k and İ(N)∞ denote the set of partitions of İ into exactly k and infinitely many non-empty pieces, respectively. For X ∈ İ(N)∞, (X)k is the set of all Y ∈ İ(N)k which are coarser than X.

Statement 1.1 (Dual Ramsey theorem). DRTk is the statement “If İ(N)k is colored with finitely many Borel colors, then there is some X ∈ İ(N)∞ such that (X)k is monochromatic”.

The Dual Ramsey theorem was proven by Carlson and Simpson [1], and studied from a reverse mathematical viewpoint by Slaman [10], Miller and Solomon [6] and Dzhafarov et al. [3]. In this paper, we shall focus on a combinatorial lemma used by Carlson and Simpson to prove the Dual Ramsey theorem. This lemma can be formulated in terms of variable words.

Definition 1.2 (Variable word). An infinite variable word on a finite alphabet A is an ω-sequence W of elements of A ∪ {x₁ : i ∈ İ} in which all variables occur at least once, and finitely often. A finite variable word is an initial segment of an infinite variable word. A finite or infinite variable word is ordered if moreover all occurrences of xi come before any occurrence of xi+1. Given a = a₀a₁...ak−1 ∈ A<ω, we let W(a) denote the finite A-string obtained by replacing xi with ai in W and then truncating the result just before the first occurrence of xi.

Statement 1.3 (Variable word theorem). VW(n, r) is the statement “If A<ω is colored with r colors for some alphabet A of cardinality n, there exists an infinite variable word W such that {W(ā) : ā ∈ A<ω} is monochromatic. OVW(n, r) is the same statement as VW(n, r) but for ordered variable words.

In this paper, we study the computability-theoretic properties of the variable word theorems using the framework of reverse mathematics.¹

1.1. Reverse mathematics. Reverse mathematics is a vast foundational program aiming to determine the optimal axioms to prove ordinary theorems. It uses the framework of second-order arithmetics, with a base theory RCA₀ consisting of the axioms of Robinson arithmetic, the Σ¹₀ induction scheme and the Δ¹₀ comprehension scheme. The system RCA₀ arguably captures computable mathematics. Starting from a proof-theoretic perspective, modern reverse mathematics tend to be seen as a framework to analyse the computability-theoretic features of theorems. Among the distinguished statements, let us mention weak König’s lemma (WKL), asserting that every infinite binary tree has an infinite path, the arithmetic comprehension axiom (ACA), and the Π¹₁ comprehension axiom (Π¹₁CA), consisting of the comprehension scheme restricted to arithmetic and Π¹₁ formulas, respectively. See Simpson [9] for reference book on classical reverse mathematics.

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The statements studied within this framework are mainly of the form $(\forall X) [\Phi(X) \rightarrow (\exists Y) \Psi(X, Y)]$, where $\Phi$ and $\Psi$ are arithmetic formulas with set parameters, and can be considered as problems. Given a statement $P$ of this form, a set $X$ such that $\Phi(X)$ holds is an instance of $P$, and a set $Y$ such that $\Psi(X, Y)$ holds is a solution to the $P$-instance $X$. In this paper, we shall consider exclusively statements of this kind.

Friedman and Simpson [4], and later Montalban [7], asked about the reverse mathematical strength of the ordered variable word. The statement $OVW(k, \ell)$ is known to be provable in $RCA_0 + \Pi^1_1 CA$. Our main result is a direct combinatorial proof of $OVW(2, \ell)$ in $RCA_0 + ACA$.

**Theorem 1.4.** For every $\ell \geq 2$, $RCA_0 + ACA \vdash OVW(2, \ell)$.

On the lower bound hand, Miller and Solomon [6] constructed a computable instance $c$ of $OVW(2, 2)$ with no $\Delta^0_2$ solution, and deduced that $RCA_0 + WKL$ does not prove $OVW(2, 2)$. Indeed, seeing the instance $c$ of $OVW(2, 2)$ as an instance of $VW(2, 2)$, and noticing that the jump of a solution to $VW(2, 2)$ gives a solution to $OVW(2, 2)$, on deduce that $c$ has no low $VW(2, 2)$-solution. In this paper, we improve their lower bound by constructing a computable instance of $OVW(2, 2)$ whose solutions are of DNC degree relative to $\emptyset'$.

1.2. **Organization of the paper.** In section 2, we shall give a simple proof of the ordered variable word for binary strings ($OVW(2, \ell)$) using the finite union theorem. Then, in section 3 we provide a direct combinatorial proof of the same statement over $RCA_0 + ACA$. In section 4, we give a new lower bound on the strength of $OVW(2, \ell)$ using a computable version of Lovasz Local Lemma. Finally, in section 5, we improve a lower bound of Townser about a combinatorial lemma in his proof of the finite union theorem.

1.3. **Notation.** Given two sets $A$ and $B$, we write $A < B$ for the formula $(\forall x \in A)(\forall y \in B)x < y$. Given a set $A$, we write $A^{<\omega}$ for the set of finite $A$-valued strings. In particular, $2^{<\omega}$ is the set of binary strings. We denote by $P_{fin}(N)$ the collection of finite non-empty subsets of $N$. Given two strings $\sigma, \tau \in A^{<\omega}$, $\sigma * \tau$ denotes their concatenation. We may also write $\sigma \tau$ when there is no ambiguity. Given a string or a sequence $X$ and some $n \in \omega$, we write $X|n$ for the initial segment of $X$ of length $n$. In particular, $X|0$ is the empty string, written $\varepsilon$.

2. A SIMPLE PROOF OF THE ORDERED VARIABLE WORD THEOREM FROM THE FINITE UNION THEOREM

Simpson first noted a relation between Hindman’s theorem and the Carlson-Simpson lemma [1]. In this section, we give a formal counterpart to his observation by giving a simple proof of $OVW(2, \ell)$ using the Finite Union Theorem, a statement known to be equivalent to Hindman’s theorem. A variation of the proof below was used by Dzhafarov et al. [3] to give an upper bound to the Open Dual Ramsey’s theorem. A direct combinatorial proof of $OVW(2, \ell)$ in $RCA_0 + ACA$ will be given in the next section.

**Definition 2.1.** An IP collection is an infinite collection of finite sets $I \subseteq P_{fin}(N)$ which is closed under non-empty finite unions and contains an infinite subcollection of pairwise disjoint sets.

Note that any IP collection $I$ necessarily contains an infinite $I$-computable sequence $S_0 < S_1 < \ldots$.

**Statement 2.2** (Finite union theorem). For every $r \in N$, $\text{FUT}_r$ is the statement “For every coloring $c : P_{fin}(N) \rightarrow \ell$, there is a monochromatic IP collection”. $\text{wFUT}^2_\ell$ is the statement “For every coloring $c : P_{fin}(N) \times N \rightarrow \ell$, there is an IP collection $I$ and a color $i < \ell$ such that $c(S, \min T) = i$ for every $S < T \in I$.”

**Theorem 2.3.** $RCA_0 + \text{FUT}_\ell \rightarrow \text{wFUT}^2_\ell$.

**Proof.** Let $f : P_{fin}(N) \times N \rightarrow \ell$ be an instance of $\text{wFUT}^2_\ell$. Note that over $RCA_0$, $\text{FUT}_\ell \rightarrow ACA$ and $ACA \rightarrow COH$. Let $R$ be a sequence of set defined for every $S \in P_{fin}(N)$ and $i < \ell$ by $R_{S,i} = \{n \in N : f(S, n) = i\}$. Apply COH to $R$ to obtain an infinite $R$-cohesive set $C$. In particular, for every $S \in P_{fin}(N)$, $\lim_{n \in C} f(S, n)$ exists.

Let $I = \{S \in P_{fin}(N) : \min S \in C\}$. Note that $I$ is an IP collection. Let $\tilde{f} : I \rightarrow \ell$ be defined by $\tilde{f}(S) = \lim_{n \in C} f(S, n)$. $\tilde{f}$ is a $\Delta^0_2$ instance of $\text{FUT}_\ell$, so by the finite union theorem, there is an IP collection $J \subseteq I$ and a color $i \in \ell$ such that for every $S \in J$, $\tilde{f}(S) = \lim_{n \in C} f(S, n) = i$. Note that for every $S \in J$, $\min S \in C$. Therefore, by $f$-computably thinning-out the set $J$, we obtain an IP collection $K \subseteq J$ such that for every $S < T \in K$, $f(S, \min T) = i$. $\square$

**Theorem 2.4.** $RCA_0 + \text{wFUT}^2_\ell \rightarrow OVW(2, \ell)$.
Proof. Let \( f : 2^{<\omega} \to \ell \) be an instance of \( \text{OVW}(2, \ell) \). Define an instance \( g : P_{\text{fin}}(\mathbb{N}) \times \mathbb{N} \to \ell \) of \( \text{wFUT}_2^\ell \) as follows: Given some \( S \in P_{\text{fin}}(\mathbb{N}) \) and \( n \in \mathbb{N} \), if \( \max S < n \), then set \( g(S, n) = f(\sigma) \), where \( \sigma \) is the binary string of length \( n \) defined by \( \sigma(i) = 1 \) iff \( i \in S \). If \( n \leq \max S \), set \( g(S, n) = 0 \). By \( \text{wFUT}_2^\ell \), there is an IP collection \( I \) and a color \( i < \ell \) such that \( g(S, \min I) = i \) for every \( S < T \in I \). Compute from \( I \) an infinite increasing sequence of pairwise disjoint finite sets \( F_0 < F_1 < \ldots \). Let \( W \) be the infinite variable word defined by \( W(n) = x_i \) if \( n \in F_i \) for some \( i < \omega \), and \( W(n) = 0 \) otherwise. The variable word \( W \) and the sequence of the \( F_i \)'s is a solution to the instance \( f \) of \( \text{OVW}(2, \ell) \). \( \square \)

Note that if we impose infinite variable words to contain only the bit 0 or variables that take values in \( \{0, 1\} \), as it is the case for the instance constructed in the previous theorem, then \( \text{OVW}(2, \ell) \) is equivalent to Hindman’s theorem over \( \text{RCA}_0 \).

Corollary 2.5. \( \text{RCA}_0 + \text{ACA}^+ \rightarrow \text{OVW}(2, \ell) \).

Proof. Immediate since \( \text{ACA}^+ \rightarrow \text{FUT}_\ell \rightarrow \text{wFUT}_2^\ell \rightarrow \text{OVW}(2, \ell) \) over \( \text{RCA}_0 \). \( \square \)

3. A proof of the Ordered Variable Word theorem in ACA

The proof of the previous section gave a very coarse computability-theoretic upper bound of the Ordered Variable Word theorem in terms of \( \omega \)-jumps. In this section, we give a direct combinatorial proof of \( \text{OVW}(2, \ell) \) in \( \text{RCA}_0 + \text{ACA} \). Actually, every PA degree relative to \( \emptyset' \) is sufficient to compute a solution of a computable instance of \( \text{OVW}(2, \ell) \). We thereby answer a question of Miller and Solomon [0].

Theorem 3.1. For every \( \ell \in \omega \), every computable instance \( c \) of \( \text{OVW}(2, \ell) \), every PA degree over \( \emptyset' \) computes a solution to \( c \).

A formalization of Theorem 3.1 yields a proof of Theorem 1.4.

Proof of Theorem 1.4. The proof of Theorem 3.1 can be formalized within \( \text{RCA}_0 + \text{ACA} \). Indeed, the arguments require only arithmetical induction to be carried out, and every model of \( \text{RCA}_0 + \text{ACA} \) is a model of the statement “For every set \( X \), there is a set of PA degree over the jump of \( X \).” \( \square \)

Let us first introduce some notation. For a finite set \( F \) and a string \( \sigma \in 2^{<\omega} \) let \( \sigma_F \) be the binary string of length \( |\sigma| \) defined by \( \sigma_F(i) = \sigma(i) \) if \( i \notin F \), and \( \sigma_F(i) = 1 - \sigma(i) \) otherwise. Let \( \leq_{\text{lex}} \) denote the shortlex order on \( \omega^{<\omega} \), that is, the order with the shortest length first, and with the strings of same length sorted lexicographically.

In what follows, fix a coloring \( \tilde{c} : 2^{<\omega} \rightarrow \ell \), and a string \( \rho \in 2^{<\omega} \).

The main combinatorial lemma we use is Lemma 3.3. As a warm up, we first prove the following lemma 3.2. which is a consequence of Lemma 3.4 and the proof is somehow similar but much simpler. In the following lemma, one may think of \( \rho_{\tilde{P}} \) as a finite variable word, where the positions at \( \tilde{P} \) are replaced by a same variable kind.

Lemma 3.2. For any \( P \subseteq \{0, \ldots, |\rho| - 1\} \) with \((\forall n \in P)[|\rho(n)| = 0] \land |P| \geq \ell \), there exist two subsets \( P' < \tilde{P} \) of \( P \) with \( \tilde{P} \neq \emptyset \) such that \( \tilde{c}(\alpha) = \tilde{c}(\alpha_{\tilde{P}}) \) where \( \alpha = \rho_{\tilde{P}} \).

Proof. Suppose \( P = \{p_0 < \cdots < p_m - 1\} \). Let \( \ell_0, \ldots, \ell_m \) be defined by \( \ell_i = \tilde{c}(\rho_{\{p_0, \ldots, p_i-1\}}) \). In particular, \( \ell_0 = \tilde{c}(\rho) \). Since \( |P| = m \geq \ell \), so among \( \ell_0, \ldots, \ell_m \), there must exists \( i < j \) such that \( \ell_i = \ell_j \). Let \( P' = \{p_0, \ldots, p_{i-1}\} \) (if \( i = 0 \) then \( P' = \emptyset \)), and \( \tilde{P} = \{p_i, \ldots, p_{j-1}\} \), let \( \alpha = \rho_{\tilde{P}} \). Clearly \( P' < \tilde{P} \) and \( \tilde{P} \neq \emptyset \). It is also easy to see that \( \tilde{c}(\alpha) = \ell_i = \ell_j = \tilde{c}(\alpha_{\tilde{P}}) \). \( \square \)

We now prove a technical lemma used in the proof of our main combinatorial lemma (Lemma 3.4). The sequence in the following lemma is obtained by a simple greedy algorithm, with finitely many resets.

Lemma 3.3. There exists a nonempty set of colors \( L \subseteq \{0, 1, \ldots, \ell - 1\} \), \( |L| + 1 \) many sets of binary strings \( \Gamma_0 = \{\tau^n\}_{n \in L}, \Gamma_1 = \{\tau^n\}_{n \in L^2}, \ldots, \Gamma_{|L|} = \{\tau^n\}_{n \in L^{|L|+1}} \), such that, letting

\[
\tilde{\eta} = \max L * \max L * \cdots * \max L
\]

and letting \( \tilde{\rho} = \tau^{\tilde{\eta}} * 0 \), the following holds:

1. \( \rho < \Gamma_0 \) and \( \tau^n < \tau^\beta \Leftrightarrow \eta <_{\text{lex}} \beta \);
\[(2) \hat{\rho}(|\tau|) = 0 \text{ for all } \tau \in \Gamma_0, i \leq |L|;\]

\[(3) \text{for all } i \leq |L|, \eta \in L^{i+1}, \text{ let } \eta_0 < \eta_1 < \cdots < \eta_i - 1 \text{ denote all nonempty predecessors of } \eta, \text{ let } \forall Q = \{[\tau_0], |\tau_1|, \ldots, |\tau_{i-1}|\} \text{ if } i = 0 \text{ then } Q = \emptyset, \text{ then } \hat{c}(\tau^Q_i) = \eta(i);\]

\[(4) \text{let } P = \{[\tau_{i-1}]\}_{\eta \subseteq L^{i+1}}, \text{ for all subset } Q \text{ of } P, \text{ all } \tau \succeq \hat{\rho}, \hat{c}(\tau_Q) \in L.\]

Moreover, \(\Gamma_i, i \leq |L|\) is computable in the jump of \(\hat{c}\), uniformly in \(\rho\).

Proof. We firstly show how to find \(\Gamma_0\). Start with \(L = \{0, 1, \ldots, \ell - 1\}\). At step 1, try to find a string \(\tau \in 2^{<\omega}\) such that \(\hat{c}(\tau^0) = 0\) and let \(\tau^0 = \rho\). Then try to find a \(\tau\) such that \(\hat{c}(\tau^0) = 1\) and let \(\tau^1 = \tau^0\). Generally, after \(\tau^i\) is found, try to find \(\tau\) such that \(\hat{c}(\tau^i) = j + 1\) and let \(\tau^{i+1} = \tau^i\) if \(\tau\) is found. If during the above process, after \(\tau^i\) is found, there is no \(\tau\) such that \(\hat{c}(\tau^i) = j + 1\), then we start all over again with \(\rho\) replaced by \(\rho_1 = \tau^0\) and with \(L\) replaced by \(L \setminus \{j + 1\}\).

Generally, given a set of colors \(L\) and after \(\tau^i\) is found, let \(\eta\) be the immediate successor (with respect to \(\leq_{\text{lex}}\) order restricted to \(L\)-strings) of \(\beta\), let \(\eta < \eta_1 < \cdots < \eta_i - 1\) denote all nonempty predecessors of \(\eta\), let \(Q = \{[\tau_{i-1}], |\tau_i|, \ldots, |\tau_{i-1}|\}\) if \(i = 0\) then \(Q = \emptyset\), we try to find \(\tau\) such that \(\hat{c}(\tau_0) = \eta(|\eta| - 1)\). If such a string \(\tau\) does not exists then we start all over again with \(\rho\) replaced by \(\tau^0\) and \(L\) replaced by \(L \setminus \{\eta(|\eta| - 1)\}\). If such \(\tau\) exists then let \(\tau^i = \tau^0\).

Note that we have to start over for at most \(\ell - 1\) times before we ultimately succeed since there are \(\ell\) colors in total. It is plain to check all the four items. Also note that the sequence \(\Gamma_0, \ldots, \Gamma_{|L|}\) is \(\hat{c}\)-computable since we only need to use the jump of \(\hat{c}\) to know whether the next \(\tau^i\) can be found.

Lemma 3.4. There exists a string \(\hat{\rho} \succeq \rho\) and a finite set \(P \subseteq \{[\rho], \ldots, [\hat{\rho}] - 1\}\) with \(\forall i \in P[\hat{\rho}(i) = 0]\) such that for all \(\sigma \succeq \hat{\rho}\) there exists two subsets \(P' \prec \hat{P}\) of \(P\) with \(\hat{P} \neq \emptyset\) such that, letting \(\alpha = \sigma_{P'}, \hat{c}(\alpha) = \hat{c}(\alpha_{\hat{P}}) = \hat{c}(\alpha_{\min \hat{P}}).\) Moreover, \(|P'| < \ell^{i+2}\) and \(\hat{\rho}, \hat{P}\) are computable in the jump of \(\hat{c}\), uniformly in \(\rho\).

Proof. Let \(L\) and \(\hat{\rho}\) satisfy Lemma 3.3. We claim that \(\hat{\rho} \succeq \rho\) and by item 2 of Lemma 3.3 that \(\forall i \in P[\hat{\rho}(i) = 0]\).

Fix an arbitrary \(\sigma \succeq \hat{\rho}\). We now how to construct \(P'\) and \(\hat{P}\). Define \(\ell_0, \ldots, \ell_{|L|}\) and \(p_0, \ldots, p_{|L|}\) inductively by \(\ell_0 = \hat{c}(\sigma), \ell_i + 1 = \hat{c}(\sigma_{[p_0, p_1, \ldots, p_i]}), \) and \(p_i = [\tau_{\ell_0, \ldots, \ell_i}]\) (where \(\tau_{\ell_0, \ldots, \ell_i} \in \Gamma_i\)). Since \(\ell_0, \ldots, \ell_{|L|} \in L\) by item 4 of Lemma 3.3, there is some \(i < j \leq |L|\) such that \(\ell_i = \ell_j\). Let \(P' = \{p_0, \ldots, p_{i-1}\}\) if \(i = 0\) then \(P' = \emptyset\), \(\hat{P} = \{p_i, \ldots, p_{|L|}\}\), and \(\alpha = \sigma_{P'}\). We claim that \(\hat{c}(\alpha) = \hat{c}(\alpha_{\hat{P}}) = \hat{c}(\alpha_{\min \hat{P}})\). Note that \(\min \hat{P} = p_i = [\tau_{\ell_0, \ldots, \ell_i}]\). Therefore \(\alpha_{\min \hat{P}} = \tau_{\ell_0, \ldots, \ell_i}\). By item 3 of Lemma 3.3 we have \(\hat{c}(\tau_{\ell_0, \ldots, \ell_i}) = \ell_i\). Meanwhile, by definition of \(\ell_i, \hat{c}(\sigma_{P'}) = \hat{c}(\alpha) = \ell_i\). By definition of \(\ell_j, \hat{c}(\sigma_{P' \cup \hat{P}}) = \hat{c}(\alpha_{\hat{P}}) = \ell_j\). Thus, \(\hat{c}(\alpha) = \hat{c}(\alpha_{\hat{P}}) = \hat{c}(\alpha_{\min \hat{P}})\).

We say that \((\hat{\rho}, \hat{P})\) is \(\hat{c}\)-valid if \(P\) and \(\hat{P}\) satisfy Lemma 3.3. We say that \((P', \hat{P})\) witnesses \(\hat{c}\)-validity of \((\hat{\rho}, \hat{P})\) for \(\sigma \succeq \hat{\rho}\) if \(P' \prec \hat{P} \subseteq P\), and letting \(\alpha = \sigma_{P'}, \hat{c}(\alpha) = \hat{c}(\alpha_{P}) = \hat{c}(\alpha_{\min \hat{P}})\). Before proving Theorem 3.1 we start with the following simpler version.

Theorem 3.5. For every \(\ell \in \omega\), every computable degree \(c : 2^{<\omega} \rightarrow \ell\) of \(OV(2, \ell)\), every PA degree over \(\emptyset''\) computes a solution to \(c\).

Proof. It suffices to compute, given a PA degree relative to \(\emptyset''\), an infinite binary sequence \(Y \in 2^\omega\) together with a sequence of finite sets \(P_0 \prec P_1 \prec \cdots\) with \(\forall \in \omega[\forall n \in P_\ell]Y(n) = 0\) such that the following holds: Let \(\text{Position} = \{\min P_i : i \geq 1\}\). There is some \(\ell < \ell\) such that for all subset \(J\) of \(\omega\), letting \(\hat{P}_J = \bigcup_{i \in J} P_i\), then we have, \((\forall i \in \text{Position}[c(Y_{P_i} | p) = \ell].\)

Using Lemma 3.4 we first construct a \(\emptyset''\)-computable sequence of strings \(\hat{\rho}_0 < \hat{\rho}_1 < \cdots\), a sequence of finite sets \(P_i \subseteq \{[\hat{\rho}_0], \ldots, [\hat{\rho}_i]\} - 1\) and a sequence of colorings \(c_i : [\hat{\rho}_i]^{-\omega} \rightarrow \ell_i\) inductively as follows. \(\hat{\rho}_0 = \epsilon\) and \(c_0 = c\). Given \(\hat{\rho}_i\) and \(c_i : [\hat{\rho}_i]^{-\omega} \rightarrow \ell_i\), let \(\hat{\rho}_{i+1} \succeq \hat{\rho}_i\) and \(P_i \subseteq \{[\hat{\rho}_0], \ldots, [\hat{\rho}_{i+1}]\} - 1\) be such that \((\hat{\rho}_{i+1}, P_i)\) is \(c_i\)-valid, and let \(c_i+1\) be the coloring of \([\hat{\rho}_{i+1}]^{-\omega}\) which on \(\sigma \succeq \hat{\rho}_{i+1}\) associates \((P_{i+1}, \hat{P}_{j})\) such that \((P_{i+1}, \hat{P}_J)\) witnesses \(c_{i+1}\)-validity of \((\hat{\rho}_{i+1}, P_i)\) for \(\sigma\), and \(c_i(\sigma_{P_{i+1}}) = \ell_{i+1}\). If there are multiple such tuples, take the least one, in some arbitrary order. Note that the range of \(c_i\) is some finite set \(\ell_i\).
We now analyze for \( \sigma \geq \tilde{\rho}_i \) what \( c_1(\sigma) = \langle P'_i, \tilde{\rho}_j \rangle \) means. Note that elements of \( L_i, i \in \omega \) admit a natural partial order \( \prec \) as follows: for \( \langle P'_0, \tilde{\rho}_0, j_0 \rangle \in L_i, \langle P'_1, \tilde{\rho}_1, j_1 \rangle \in L_{i+1}, \langle P'_1, \tilde{\rho}_1, j_1 \rangle \) is an immediate successor of \( \langle P'_0, \tilde{\rho}_0, j_0 \rangle \) if and only if \( j_1 = \langle P'_0, \tilde{\rho}_0, j_0 \rangle \). Clearly every \( j \in L_i \) admit a unique immediate predecessor.

**Claim 3.6.** Fix some \( n \geq 1 \), and let \( \tilde{\ell} \prec \langle P'_0, \tilde{\rho}_0, j_0 \rangle \prec \cdots \prec \langle P'_{n-1}, \tilde{\rho}_{n-1}, j_{n-1} \rangle = c_n(\sigma) \). Let \( P' = \bigcup_{i \leq n-1} P'_i \) and \( \alpha = \sigma_{P'} \). Then for any subset \( J \) of \( \{0, \ldots, n-1\} \),

\[
(\forall p \in \{ \min \tilde{\rho} : 1 \leq j \leq n-1 \} \cup \{ |\alpha| \}) (c(\alpha_{\tilde{\rho}_j} | p) = \tilde{\ell}) .
\]

**Proof.** First we prove the claim for \( n = 1 \). By definition of \( c_1(\sigma) = \langle P'_0, \tilde{\rho}_0, j_0 \rangle \), letting \( \beta = \sigma_{P'_0}, c_0(\beta) = \langle P'_0 \rangle \) is computable in the given PA degree relative to \( P' \). Clearly \( |\beta| \min \tilde{\rho}_i = \min \tilde{\rho}_i \) means. Note that elements of \( \tilde{\rho} \) are computable and \( |\beta| \min \tilde{\rho}_i \) means. Since \( |\beta| \min \tilde{\rho}_i \) means.

So the claim holds for \( n = 1 \). Suppose now the claim holds for \( n - 1 \).

Suppose \( c_n(\sigma) = \langle P'_{n-1}, \tilde{\rho}_{n-1}, j_{n-1} \rangle \). Let \( \beta = \sigma_{P'_{n-1}} \). We have \( c_{n-1}(\beta) = \langle \beta_{P'_{n-1}} \rangle = c_{n-1}(\beta) \min \tilde{\rho}_{n-1} = j_{n-1} = \langle P'_{n-2}, \tilde{\rho}_{n-2}, j_{n-2} \rangle \). As \( c_{n-1}(\beta) = \langle P'_{n-2}, \tilde{\rho}_{n-2}, j_{n-2} \rangle \) and as \( \tilde{\ell} \prec \langle P'_{n-2}, \tilde{\rho}_{n-2}, j_{n-2} \rangle \), by induction hypothesis, for any subset \( J \) of \( \{0, \ldots, n-2\} \) we have:

\[
(\forall p \in \{ \min \tilde{\rho} : 1 \leq j \leq n-2 \} \cup \{ |\beta| \min \tilde{\rho}_{n-1} \}) (c(\beta_{\tilde{\rho}_j} | p) = \tilde{\ell}) .
\]

Let \( \beta' = \beta_{P_{n-1}} \). As \( c_{n-1}(\beta') = \langle P'_{n-2}, \tilde{\rho}_{n-2}, j_{n-2} \rangle \) and as \( \tilde{\ell} \prec \langle P'_{n-2}, \tilde{\rho}_{n-2}, j_{n-2} \rangle \), by induction hypothesis, for any subset \( J \) of \( \{0, \ldots, n-2\} \) we have:

\[
(\forall p \in \{ \min \tilde{\rho} : 1 \leq j \leq n-2 \} \cup \{ |\beta| \min \tilde{\rho}_{n-1} \}) (c(\beta_{\tilde{\rho}_j} | p) = \tilde{\ell}) .
\]

Or equivalently, for any subset \( J \) of \( \{0, \ldots, n-1\} \) we have:

\[
(\forall p \in \{ \min \tilde{\rho} : 1 \leq j \leq n-1 \} \cup \{ |\beta| \min \tilde{\rho}_{n-1} \}) (c(\beta_{\tilde{\rho}_j} | p) = \tilde{\ell}) .
\]

Now from 3.1 3.2 and 3.4 we deduce that for any subset \( J \) of \( \{0, \ldots, n-1\} \) we have:

\[
(\forall p \in \{ \min \tilde{\rho} : 1 \leq j \leq n-1 \} \cup \{ |\beta| \min \tilde{\rho}_{n-1} \}) (c(\beta_{\tilde{\rho}_j} | p) = \tilde{\ell}) .
\]

which completes the proof of the claim.

Let \( T_0 \) be the \( \psi' \)-computable set of all \( \gamma \) such that \( (\forall i \leq |\gamma|)(\gamma(i) \in L_i) \). Then, let \( T \) be the downward closure of the set \( T_0 \) by the prefix relation. The tree \( T \) is infinite by construction of the strings \( \tilde{\rho}_i \), the colors \( c_i \) and the sets \( P_i : a \) witness for the \( c_i \)-validity of \( \langle \tilde{\rho}_i, P_i \rangle \) for \( \rho_i \). Let \( \rho \) be a witness for the \( c_{\gamma} \)-validity of \( \langle \tilde{\rho}_{|\gamma|-1} \rangle \). Then, let \( T \) be the downward closure of the set \( T_0 \) by the prefix relation. The tree \( T \) is infinite by construction of the strings \( \tilde{\rho}_i \), the colors \( c_i \) and the sets \( P_i : a \) witness for the \( c_i \)-validity of \( \rho_i \) for \( \rho_i \) yields a node of \( T_0 \) of length \( i \). The tree \( T \) is also \( \psi' \)-computably bounded, and \( \psi' \)-computable. Let \( j_0 * (P'_0, \tilde{\rho}_0, j_0) * (P'_1, \tilde{\rho}_1, j_1) * \cdots \) be an infinite path through \( T \) computed by any PA degree over \( \psi' \). By construction, \( (P'_i, \tilde{\rho}_i, j_i) \prec (P'_{i+1}, \tilde{\rho}_{i+1}, j_{i+1}) \). Let \( X = \bigcup_{i \in \omega} \tilde{\rho}_i, P' = \bigcup_{i \in \omega} P'_i \) and let \( Y = X_{\psi'}. \) Clearly \( (\forall i \leq n \in \tilde{\rho}_i) |Y(n)| = 0 \) and \( Y \) is computable in the given PA degree relative to \( \psi' \). Therefore, letting \( Position = \{ \min \tilde{\rho} : i \geq 1 \} \), it suffices to show that for all subsets \( J \) of \( \omega \),

\[
(\forall p \in Position) (c(Y_{\tilde{\rho}_j} | p) = j_0) .
\]

Without loss of generality, suppose \( p = \min \tilde{\rho}_n \) and \( J \subseteq \{0, \ldots, n-1\} \). Since \( j_0 * (P'_0, \tilde{\rho}_0, j_0) * (P'_1, \tilde{\rho}_1, j_1) * \cdots \) is an infinite path through \( T \), there must exist some \( N > n \) such that \( c_N(\tilde{\rho}_{N+1}) = (P'_{N-1}, \tilde{\rho}_{N-1}, j_{N-1}) \). Let \( \sigma = \tilde{\rho}_{N+1}, \alpha = \sigma_{P'}. \) Clearly \( \alpha \prec Y \cup \{ \alpha \} \supset p. \) Moreover, by Claim 3.5, \( c(\alpha_{\tilde{\rho}_j} | p) = j_0. \) Thus \( c(Y_{\tilde{\rho}_j} | p) = j_0. \)

Finally, we slightly modify the proof of Theorem 3.5 to derive Theorem 3.1.

\[ \square \]
Proof of Theorem 3.1. The main point is to make the tree $T$ $\Psi'$-computable. To ensure this, after we obtain $\tilde{\rho}_i, c_i$, we do not directly go to $\tilde{\rho}_{i+1}$. Instead, we $\Psi'$-compute $\tilde{\rho}_i^0 < \tilde{\rho}_i^1 < \cdots < \tilde{\rho}_i^i$ such that $\tilde{\rho}_i > \tilde{\rho}_{i+1}$ and $c_i(\{\tau : \tau \geq \tilde{\rho}_i^i\}) \subseteq c_i(\{\tilde{\rho}_i^0, \cdots, \tilde{\rho}_i^i\})$. Then we $\Psi'$-compute $\tilde{\rho}_{i+1} > \tilde{\rho}_i^i$ as in the proof of Theorem 3.5. Note that this indeed can be achieved using $\Psi'$ since $c_i$ is computable. Define $T$ to be the set of all $\gamma$ such that $(\forall i \leq |\gamma|) (\gamma(i) \in L_i \cap \gamma(i) \not\prec \gamma(i+1))$, and either $|\gamma| = 1$ or $\gamma \in L_0$ or there exists $\tilde{\rho}_{|\gamma|-1}$ with $c_{|\gamma|-1}(\tilde{\rho}_{|\gamma|-1}) = (\gamma(|\gamma| - 1))$. It is easy to see that $T$ is $\Psi'$-computable since $c_i$ is computable for all $i$ and the sequences $(c_i : i \in \omega)$ and $(\tilde{\rho}_i^i : i \in \omega, v \leq r_i)$ are $\Psi'$-computable.

Now we show that $T$ is the tree. Suppose $\gamma \in T$, $|\gamma| = n + 1$ with $n \geq 1$, and $c_n(\tilde{\rho}_n^n) = \gamma(n) = (P', \tilde{\rho}, j) \in L_n$. We claim that $\gamma|n \in T$. If $n = 1$, then $\gamma|1 \in L_0 \subseteq T$. Otherwise, let $(Q', \tilde{\rho}, k) \in L_{n-1}$ be the predecessor of $(P', \tilde{\rho}, j)$. We need to show that there exists $\tilde{\rho}_{n-1}$ such that $c_{n-1}(\tilde{\rho}_{n-1}) = (Q', \tilde{\rho}, k)$. $c_n(\sigma) = (P', \tilde{\rho}, j)$ implies that, letting $\alpha = \sigma \rho'$, $c_{n-1}(\alpha) = c_{n-1}(\alpha \min \tilde{\rho}) = j = (Q', \tilde{\rho}, k)$. Note that $\alpha \geq \tilde{\rho}_{n-1}$ since $P' > \tilde{\rho}_{n-1}$. Therefore there exists $\tilde{\rho}_{n-1}$ such that $c_{n-1}(\tilde{\rho}_{n-1}) = (Q_{n-1}', \tilde{\rho}_{n-1}, k)$. It follows that $\gamma|n \in T$ and that $T$ is a tree. Any PA degree relative to $\Psi'$ computes an infinite path through $T$. The rest of the proof goes exactly the same as Theorem 3.5. □

We now give an alternative proof of Theorem 3.1 based on the definitional complexity of the solutions of $c$.

Second proof of Theorem 3.1. Let $P_0, P_1, \ldots$ be the $\Psi'$-computable sequence defined in the proof of Theorem 3.5. We have seen that there exists an infinite ordered variable word such that the $n$th variable kind appears before the position $\max P_n$. Let $T$ be the tree of all finite ordered variable words which are finite solutions to $c$ and such that the $n$th variable appears before the position $\max P_n$. By the previous observation, the tree is infinite. $\Psi'$-computable, and $\Psi'$-computably bounded. Any PA degree relative to $\Psi'$ computes an infinite variable word which, by construction of $T$, is a solution to $c$. This completes the proof of Theorem 3.1. □

Note that the above proof can be slightly modified to obtain a proof of a sequential version of the ordered variable word.

Statement 3.7. SeqOVW$(n, \ell)$ is the statement “If $c_0, c_1, \ldots$ is a sequence of $\ell$-colorings of a fixed alphabet $A$ of cardinality $n$, there exists a variable word $W$ such that for every $i \in \omega$ and every $\tilde{b} \in A^i$, $\{W(\tilde{b}a) : \tilde{a} \in A^{<\infty}\}$ is monochromatic for $c_i$.”

Theorem 3.8. For every computable instance $c_0, c_1, \ldots$ of SeqOVW$(2, \ell)$, every PA degree relative to $\Psi'$ computes a solution to $c$.

Proof. The proof is similar to Theorem 3.1. Using Lemma 3.4, we first construct a $\Psi'$-computable sequence of strings $\hat{p}_0 \prec \hat{p}_1 \prec \cdots$, a sequence of finite sets $P_i \subseteq \{\hat{p}_i, \cdots, \hat{p}_i - 1\}$ and a sequence of colorings $d_i : [\hat{p}_i]^i \rightarrow L_i$ inductively as follows. $\hat{p}_0 = \varepsilon$ and $d_0 = c_0$. Given $\hat{p}_i$ and $d_i : [\hat{p}_i]^i \rightarrow L_i$, let $\hat{p}_{i+1} \supseteq \hat{p}_i$ and $P_i \subseteq \{\hat{p}_i, \cdots, \hat{p}_{i+1} - 1\}$ be such that $(\hat{p}_{i+1}, P_i)$ is $d_i$-valid, and let $d_{i+1}$ be the coloring of $[\hat{p}_{i+1}]^i$ which on $\sigma \geq \hat{p}_{i+1}$ associates $(P', \tilde{P}, j, k)$ such that $(P', \tilde{P})$ witnesses $d_i$-validity of $(\hat{p}_{i+1}, P_i)$ for $\sigma$, $d_{i+1}(\sigma \rho') = j$ and $c_{i+1}(\sigma \rho') = k$. Note that the main difference with the previous construction is that we handle more and more colorings among $c_0, c_1, \ldots$ at each level. The remainder of the proof is the same as in Theorem 3.1. □

The theorem above is optimal in that, in order to obtain the following reversal.

Theorem 3.9. There is a computable instance $c_0, c_1, \ldots$ of SeqOVW$(2, \ell)$, such that every solution is of PA degree relative to $\Psi'$.

Proof. Let $R_0, R_1, \ldots$ be a uniformly computable sequence of sets such that for every $e$, if $\Phi_e^\Psi'(e) \downarrow = 0$ then $R_e$ is finite, and if $\Phi_e^\Psi'(e) \downarrow = 1$ then $R_e$ is cofinite. In particular, any function $f : \omega \rightarrow 2$ such that $f(e)$ gives a side of $R_e$ which is infinite, is DNC relative to $\Psi'$, hence of PA degree relative to $\Psi'$. Let $c_i : 2^{<\infty} \rightarrow 2$ be defined by $c_i(\sigma) = 1 \text{ if } |\sigma| \in R_i$, and let $W$ be a solution to $c_i$, that is, a variable word $W$ such that for every $i \in \omega$ and every $\tilde{b} \in A^i$, $\{W(\tilde{b}a) : \tilde{a} \in A^{<\infty}\}$ is monochromatic for $c_i$. We claim that $W$ computes such a function $f$. Given $e$, let $f(e) = c_e(W(\tilde{b}))$, where $\tilde{b} \in 2^e$ is arbitrary (this is well-defined, since $c_e(\tilde{b})$ depends only on the length of $\tilde{b}$). By definition of $W$, $\{W(\tilde{b}a) : \tilde{a} \in A^{<\infty}\}$ is monochromatic for $c_e$, the color of $c_e(W(\tilde{b}))$ appears infinitely often in $R_e$. Therefore, $W$ is of PA degree relative to $\Psi'$. This completes the proof. □
4. A Difficult Instance of the Ordered Variable Word Theorem

Miller and Solomon [6] constructed a computable instance of OVW(2, 2) with no $\Delta^0_0$ solution. In this section, we strengthen their proof by constructing a computable instance of OVW(2, 2) such that every solution is of DNC degree relative to $\emptyset'$, using a significantly simpler argument.

The proof makes an essential use of a computable version of Lovasz Local Lemma proven by Rumnyantsev and Shen [8]. The idea of using Lovasz Local Lemma to analyze the computability-theoretic strength of problems in reverse mathematics comes from Csima and Hirschfeldt [3] who proved that a version of Hindman’s theorem for subtractions is not computably true.

**Definition 4.1.** Fix a countable set of variables $x_0, x_1, \ldots$. A (disjunctive) clause $C$ is a tuple of the form $(x_{i_k} = i_k)$, with $i_1, \ldots, i_k < 2$. The length of $C$ is the integer $k$. An infinite CNF formula is an infinite conjunction of disjunctive clauses. An infinite CNF formula $\bigwedge_n C_n$ is computable if the function which given $n$ outputs a code for $C_n$ is computable, and the set of $n$ such that $C_n$ contains the variable $x_j$ is uniformly computable in $j$.

**Theorem 4.2** (Rumnyantsev and Shen [8]). For every $\alpha \in (0, 1)$, there exists some $N \in \omega$ such that every computable infinite CNF where each variable appears in at most $2^{\alpha n}$ clauses of size $n$ (for every $n$) and all clauses have size at least $N$, has a computable satisfying assignment.

**Theorem 4.3.** There is a computable instance $c$ of OVW(2, 2) and a computable function $h : \omega \to \omega$ such that if $\Phi^\emptyset_e$ outputs a finite variable word in which the first $h(e)$ variables kinds occur, then $\Phi^\emptyset_e$ is not extendible into an infinite solution to $e$.

**Proof.** Fix $\alpha = 0.5$, and let $N$ be the threshold of Theorem 4.2. For every index $e$ and stage $s$, we interpret $\Phi_e^\emptyset[s]$ as a finite variable word $W_{e,s}$ with exactly $N + e$ variable kinds, and where a new variable occurs right after $W_{e,s}$. Such a variable word induces a binary tree $T_{e,s}$ with $2^{N+e}$ leaves. Let $L_{e,s}$ be the set of leaves of $T_{e,s}$, that is, the set of all instantiations of the variable word $W_{e,s}$. Moreover, all the leaves of $T_{e,s}$ have the same length $n_{e,s}$.

The idea is the following: since the variable word is ordered and a new variable kind occurs right after $W_{e,s}$, no variable among the first $N + e$ variables can occur after $W_{e,s}$. If $W$ is a solution to $c$ with initial segment $W_e = \lim_n W_{e,s}$ for some color $i$, then $W$ must be homogeneous for $c$ for every instance of the variables, so in particular when setting all the variables after the $N + e$ first ones to 0. Hence, there must be infinitely many strings $\tau$ such that for every $\sigma \in \lim_n L_{e,s}$, $c(\sigma \tau) = i$. By ensuring that for infinitely many $\tau$, there is some $\sigma \in L_{e,|\tau|}$ such that $c(\sigma \tau) \neq i$, we force $W_e$ not to be a solution to $e$ for color $i$.

Fix a countable collection of variables $(x_\rho : \rho \in 2^{<\omega})$. Each variable $x_\rho$ corresponds to the color of the string $\rho$. Given some $s \in \omega, \tau \in 2^{<\omega}$ and some $i < 2$, if $n_{e,s} + |\tau| = s$, then let $C_{e,s,\tau,i}$ be the disjunctive $2^{N+e}$-clause

$$\bigvee \{x_{\sigma \tau} = i : \sigma \in L_{e,s}\}.$$

And let $C$ be the conjunction

$$\bigwedge_{n_{e,s} + |\tau| = s} \{C_{e,s,\tau,i} : e \in \omega, \tau \in 2^{<\omega}, i < 2\}.$$

This infinite CNF formula is clearly computable. Clearly $C_{e,s,\tau,i}$ has length $2^{N+e}$. Note that for every $\rho, e$, there exists at most one $\tau$ such that $(3 \sigma \in L_{e,|\rho|})(\sigma \tau = \rho)$. Therefore, each variable $x_\rho$ appears in at most 2 clauses of length $2^{N+e}$, namely, $C_{e,|\rho|,0}$ and $C_{e,|\rho|,1}$, where $\tau$ is such that $(3 \sigma \in L_{e,|\rho|})(\sigma \tau = \rho)$. Therefore, this formula satisfies the conditions of Theorem 4.2 and has a computable assignment $c : 2^{<\omega} \to 2$. By construction, letting $h(e) = N + e + 1$, the formula ensures that if $\Phi_e^\emptyset$ outputs a finite variable word in which the first $h(e)$ variables kinds occur, then $\Phi_e^\emptyset$ is not extendible into an infinite solution to $c$.

**Definition 4.4.** A function $f : \omega \to \omega$ is diagonally non-computable relative to $X$ (or $X$-dnc) if for every $e$, $f(e) \neq \Phi_e^X(e)$.

**Corollary 4.5.** There is a computable instance $c$ of OVW(2, 2) such that every solution is of $\emptyset'$-dnc degree.

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2 Personal communication
Proof. Let \( c \) and \( h \) be as in Theorem 4.3. For every \( e \), let \( \alpha_e \) be a computable bijection from the finite variable words in which the first \( h(e) \) variable kinds occur, to the set of the integers. By Kleene’s fixpoint theorem, there is a computable function \( g : \omega \to \omega \) such that for every \( e \), \( \Phi_{g(e)}^{\omega} = \Phi_{\alpha_e}^{\omega}(\Phi_{\alpha_e}^{\omega}(e)) \).

Let \( W \) be a solution to \( c \), that is, an infinite variable word. Let \( f \) be the \( W \)-computable function defined by \( f(e) = \alpha_{g(e)}(w_e) \), where \( w_e \) is the first initial segment of \( W \) in which the first \( h(g(e)) \) variable kinds occur. We claim that \( f \) is \( \vartheta' \)-dnc. Indeed, given \( e \in \omega, w_e \neq \Phi_{g(e)}^{\omega} \), so

\[
f(e) = \alpha_{g(e)}(w_e) \neq \alpha_{g(e)}(\Phi_{g(e)}^{\omega}(e)) = \Phi_{\vartheta}^{\omega}(e)
\]

This completes our proof.

We conclude this section with a small computational observation about \( \text{OVW}(2,2) \) based on the syntactical form of the statement.

Definition 4.6. A function \( g : \omega \to \omega \) dominates \( f : \omega \to \omega \) if \( (\forall x)f(x) < g(x) \). A function \( f : \omega \to \omega \) is hyperimmune if it is not dominated by any computable function. A Turing degree is hyperimmune-free if it does not contain any hyperimmune function.

Lemma 4.7 (Folklore). Let \( P \) be a statement of the form \( (\forall X)\Phi(X) \to (\exists Y)\Psi(X, Y) \) where \( \Phi \) is an arbitrary predicate, and \( \Psi \) is a \( \Pi^1_1 \) predicate. For every computable instance \( I \) of \( P \), if \( I \) has a solution of hyperimmune-free degree, then every PA degree computes a solution to \( I \).

Proof. Let \( I \) be a computable \( P \)-instance with a solution \( S \) of hyperimmune-free degree. In particular, there is a computable function \( g : \omega \to \omega \) such that \( (\forall x)S(x) < g(x) \). Let \( T \subseteq \omega^{<\omega} \) be the computably bounded tree defined by

\[
T = \{ \sigma \in \omega^{<\omega} : (\forall i < |\sigma|)\sigma(i) < g(i) \land \Psi(I, \sigma) \}
\]

In particular, \( S \in |T| \), so the tree is infinite. Moreover, any \( R \in |T| \) is a solution to \( I \), and any PA degree computes a member of \( |T| \). This completes the proof.

Corollary 4.8. There is a computable instance of \( \text{OVW}(2,2) \) such that every solution is of hyperimmune degree.

Proof. First, note that the statement \( \text{OVW}(2,2) \) is of the form of Lemma 4.7. Let \( c : 2^{<\omega} \to 2 \) be the computable instance of \( \text{OVW}(2,2) \) with no low solution constructed by Miller and Solomon [6] or by Theorem 4.3. Letting \( d \) be a low PA degree, \( d \) computes no solution to \( c \), hence by Lemma 4.7, every solution to \( c \) is of hyperimmune degree.

It is still unknown whether there is a computable instance of \( \text{OVW}(2,2) \) such that every solution is PA over \( \vartheta' \), or even just computes \( \vartheta' \). In particular the following questions remain open:

Question 4.9. Does \( \text{OVW}(2,2) \) imply \( \text{ACA}_0 \)?

Question 4.10. Is there a computable instance of \( \text{OVW}(2,2) \) such that the measure of oracles computing a solution to it is null?

5. Lower bounds on the Finite Union Theorem

Towsner [11] gave a combinatorially simple proof of the Finite Union Theorem in \( \text{ACA}^+ \). He introduced the notion of full-match, and proved its existence over \( \text{ACA} \) in [11] Lemma 2.7. However, the existence of a full-match is not known to be equivalent to \( \text{ACA} \). This notion is the cornerstone of Towsner’s proof, as having low\(_n\) full-matches for some fixed \( n \) would be sufficient to obtain arithmetical solutions to the Finite Union Theorem. In this section, we improve the lower bound of the existence of a full-match by showing that it cannot be proven in \( \text{WKL} \).

Definition 5.1. Fix a coloring \( c : \mathcal{P}_{\text{fin}}(\mathbb{N}) \to r \). Let \( B \subseteq \mathcal{P}_{\text{fin}}(\mathbb{N}) \) be a finite collection, and let \( I \subseteq \mathcal{P}_{\text{fin}}(\mathbb{N}) - B \) be an IP collection.

(i) \( B \) left-matches \( I \) if for every \( S \in I \), there is some \( B \in B \) such that \( c(B) = c(B \cup S) \).
(ii) \( B \) right-matches \( I \) if for every \( S \in I \), there is some \( B \in B \) such that \( c(S) = c(B \cup S) \).
(iii) \( B \) full-matches \( I \) if for every \( S \in I \), there is some \( B \in B \) such that \( c(B) = c(B \cup S) = c(S) \).
Note that in Towsner’s paper [11], a right-match is called a half-match. A full-match is both a left-match and a right-match, but the converse is not true in general.

**Statement 5.2.** LM-FUT$_r$ denotes the statement “For every coloring $c : \mathcal{P}_{fin}(\mathbb{N}) \to r$, there is a finite collection $\mathcal{B} \subseteq \mathcal{P}_{fin}(\mathbb{N})$ and an IP collection $\mathcal{I} \subseteq \mathcal{P}_{fin}(\mathbb{N})$ – $\mathcal{B}$ such that $\mathcal{B}$ left-matches $\mathcal{I}$. The statements RM-FUT$_r$ and RM-FUT$_{\mathbb{N}}$ are defined accordingly for the notions of right-match and full-match.

Towsner [11] Lemma 2.5] proved RM-FUT$_r$ over RCA$_0$. He also constructed in [11] Theorem 3.8] a computable instance of LM-FUT$_2$ with no computable solution, therefore showing that RCA$_0 \not\vDash$ LM-FUT$_2$.

We now adapt his proof to show that RCA$_0 + \text{WKL} \not\vDash$ LM-FUT$_2$, by proving that LM-FUT$_2$ implies the existence of hyperimmune functions.

The following theorem combines the techniques of Towsner [11] Theorem 3.8] and Csima and Mileti [2] Theorem 4.1]. A familiarity with the mentioned proofs will simplify drastically the understanding of the proof of Theorem 5.3.

**Theorem 5.3.** There is a computable instance of LM-FUT$_2$ such that every solution is of hyperimmune degree.

**Proof.** The strategy is the following: We will build a computable coloring $c : \mathcal{P}_{fin}(\mathbb{N}) \to 2$ such that for every finite collection $\mathcal{B}$ which left-matches an IP collection $\mathcal{I}$ we satisfy the following requirement for each $e \in \mathbb{N}$:

$\mathcal{R}_e$: If $\Phi_e$ is total, then there is an input $k_e$ such that $\mathcal{I} \cap [k_e, \Phi_e(k_e)] = \emptyset$.

Here, $[a, b] = \{a, a + 1, \ldots, b\}$. Suppose we have constructed $c$, and let $(\mathcal{B}, \mathcal{I})$ be such that $\mathcal{B}$ left-matches $\mathcal{I}$. Let $p_2$ be the $\mathcal{I}$-computable function which on input $n$ searches for a set $S \in \mathcal{I}$ such that $\min S > n$ and outputs $\max S$. For every $e \in \mathbb{N}$, $p_2(k_e) > \Phi_e(k_e)$. Therefore $p_2$ is not dominated by any computable function, hence is hyperimmune. We now explain how to build the coloring $c$.

Using a movable marker procedure, define for every $e \in \mathbb{N}$ and at every stage $s$ two inputs $x_{e,s}, y_{e,s} > e$ ordered as follows

$$x_{0,s} < y_{0,s} < x_{1,s} < y_{1,s} < \ldots$$

Moreover, if $\Phi_{e,s}(x_{e,s}) \downarrow$, then $\Phi_{e,s}(x_{e,s}) < y_{e,s}$, and if $\Phi_{e,s}(y_{e,s}) \downarrow$, then $\Phi_{e,s}(y_{e,s}) < x_{e+1,s}$. We can show by induction that each marker is moved at most finitely many times, and therefore $x_e = \lim x_{e,s}$ and $y_e = \lim y_{e,s}$ exist for every $e$. For every $e, s \in \mathbb{N}$, let $X_{e,s} = [x_{e,s}, \Phi_{e,s}(x_{e,s})]$ if $\Phi_{e,s}(x_{e,s}) \downarrow$, and $X_{e,s} = \emptyset$ otherwise. The set $Y_{e,s}$ is defined accordingly. According to the previous observation, $X_e = \lim X_{e,s}$ and $Y_e = \lim Y_{e,s}$ exist for every $e \in \mathbb{N}$, and satisfy

$$x_{0,s} < y_{0,1} < X_{1,s} < Y_{1,s} < \ldots$$

The construction will work as follows. Using the combinatorics of Towsner [11] Theorem 3.8] (to be explained later), for every $e$, we will pick two sets $S \subseteq X_{e,s}$ and $T \subseteq Y_{e,s}$, and will ensure that $\{S, T, U\} \not\subseteq \mathcal{I}$ for every solution $(\mathcal{B}, \mathcal{I})$ to $c$, and cofinitely many finite sets $U$. If $\Phi_e$ is partial, our requirement is vacuously satisfied. If $\Phi_e$ is total, at some finite stage $t$, there will be some input $k_{S,T} \in \mathbb{N}$ such that $\Phi_{e,t}(k_{S,T}) \downarrow$ and $\{S, T, U\} \not\subseteq \mathcal{I}$ for every $U \subseteq [k_{S,T}, \Phi_{e,t}(k_{S,T})]$. This way, we will have ensured that if $S$ and $T$ both belong to $\mathcal{I}$, then $\mathcal{I} \cap [k_{S,T}, \Phi_{e,t}(k_{S,T})] = \emptyset$. Since this is checked at a finite stage $t$, we then pick the next pair $S' \subseteq X_{e,t}$ and $T' \subseteq Y_{e,t}$ and do the same procedure, and so on until we have consumed all the pairs over $X_{e,t}$ and $Y_{e,t}$. Therefore, we will have ensured that either $\mathcal{I} \cap X_e = \emptyset$, or $\mathcal{I} \cap Y_e = \emptyset$, or there is some $S \subseteq \mathcal{I} \cap X_e$ and $T \subseteq \mathcal{I} \cap Y_e$ such that $\mathcal{I} \cap [k_{S,T}, \Phi_{e,t}(k_{S,T})] = \emptyset$.

We say that a requirement $\mathcal{R}_e$ is ready at stage $s$ if $\Phi_{e,s}(x_{e,s}) \downarrow$ and $\Phi_{e,s}(y_{e,s}) \downarrow$. Given a pair of finite sets $S^0 < S^1$, $\mathcal{R}_e$ is $(S^0, S^1)$-satisfied for $\mathcal{R}_e$ at stage $s$ if there is some $k > S^1$ such that $\Phi_{e,s}(k) \downarrow$ and for every set $A < S^0$, there is some $u < 2$ such that for every $B \subseteq [k, \Phi_{e,s}(k)]$, $c(A) \neq c(A \cup S^u \cup B)$. In other words, $\mathcal{R}_e$ is $(S^0, S^1)$-satisfied at stage $s$ if for every solution $(\mathcal{B}, \mathcal{I})$ to $c$ such that $S^0, S^1 \in \mathcal{I}$, $\mathcal{I} \cap [k, \Phi_{e,s}(k)] = \emptyset$. Note that a requirement may be ready at some stage, but not a later stage, whereas if $S^0 < S^1$ is satisfied for $\mathcal{R}_e$ at some stage, then will always remain satisfied. A requirement $\mathcal{R}_e$ is satisfied at stage $s$ if either $\Phi_e$ is partial, or it is $(S^0, S^1)$-satisfied for every pair $S^0 \subseteq X_{e,s}$ and $S^1 \subseteq Y_{e,s}$.

At any stage $s$ of the construction and for every $e < s$ such that $\mathcal{R}_e$ is ready, we will have distinguished two sets $S^0_s \subseteq X_{e,s}$ and $S^1_s \subseteq Y_{e,s}$ which currently receive attention. They are chosen two be the least pair (in an arbitrary fixed order) which is not yet satisfied for $\mathcal{R}_e$. The following two definitions are defined by Towsner [11] Theorem 3.8].
A primary $s$-decomposition of a set $B$, where $s = \max B$ is a tuple $i, u, Z, D$ such that $B = Z \cup S^u_{e,s} \cup D$, $Z < S^u_{e,s} < D$, neither $Z$ nor $D$ contains $S^{1-i}_{t,u}$ as a subsequence, and there is no primary $s$-decomposition of $D$. Clearly, there is at most one primary $s$-decomposition of $B$.

We say that $B$ contains $e$ with polarity $v$ if there is a primary max $B$-decomposition $i, u, Z, D$ of $B$ with either $e = i$ and $v = u$, or $e$ contained in $Z$ with polarity $|v - u|$. Observe that whenever $B$ contains $e$, $B = Z \cup S^u_{e,s} \cup D$ for some $t \leq \max B$.

We now define our coloring stages. At stage $s$, supposed we have already decided $c(B'_s)$ whenever $\max B'_s < s$. Let $B$ be such that $\max B = s$. If $B$ has a primary $s$-decomposition $B = Z \cup S^u_{e,s} \cup D$, we set $c(B) = c(Z)$ if $u = 0$ and $c(B) \neq c(Z)$ if $u = 1$. If there is no primary $s$-decomposition of $B$, we set $c(B) = 0$. This completes the construction. We now turn to the verification.

**Claim 5.4.** For any total Turing functional $\Phi_e$, all $S \subseteq X_e$ and all $T \subseteq Y_e$, $R_e$ is $(S, T)$-satisfied at some stage.

**Proof.** Suppose not. Let $S \subseteq X_e$ and $T \subseteq Y_e$ be the least pair (in the previously fixed order) such that $R_e$ is not $(S, T)$-satisfied at any stage. Let $s$ be a stage such that for all $t \geq s$, $X_{s,t} = X_e$, $Y_{s,t} = Y_e$, and $R_e$ is $(S', T')$-satisfied for all the previous pairs $(S', T')$. By construction, $S^0_{s,t} = S$ and $S^1_{s,t} = T$.

It is easy to see that for any $B \geq s$, there is a $v_B$ such that $A \cup S^u_{e,s} \cup B$ contains $e$ with polarity $|v_B - u|$ for all $A < B$. We now prove by induction on the length of $B$ that for all $B > s_1$ and $A < s^0_{e,t}$, $c(A \cup S^s_{e,t} \cup B) = c(A)$ and $c(A \cup S^{1-s}_{e,t} \cup B) = c(A)$. Given $u < 2$, let $D = A \cup S^u_{e,s} \cup B$. In particular, $D$ admits a primary max $B$-decomposition $Z \cup S^u_{e,s} \cup B'$. If we just have $e = i$, then $v_B = 0$ and the claim follows immediately from the decomposition of the coloring. Otherwise, we have two cases. In the first case, $u' = 0$. Then $c(D) = c(Z)$ and $Z$ contains $e$ with polarity $|v_B - u'|$. By induction hypothesis applied to $Z \setminus (A \cup S^u_{e,s})$, $c(D) = c(Z) = c(A)$ if $u = v_B$ and $c(D) = c(Z) \neq c(A)$ if $u \neq v_B$. In the second case, $u' = 1$. Then $c(D) = c(D)$ and $Z$ contains $e$ with polarity $1 - |v_B - u'|$. By induction hypothesis applied to $Z \setminus (A \cup S^u_{e,s})$, $c(D) \neq c(Z) \neq c(A)$ if $u = v_B$, so $c(D) = c(A)$, and $c(D) \neq c(Z) = c(A)$ if $u \neq v_B$. This completes the induction and the proof of the claim. 

Let $(B, I)$ be a solution to $c$.

**Claim 5.5.** For every total Turing functional $\Phi_e$, there is an input $k$ such that $I \cap [k, \Phi_e(k)] = \emptyset$.

**Proof.** By the padding lemma, we can assume that for all $A \in B$, $A < e$. If $I \cap [x_e, \Phi_e(x_e)] = \emptyset$ or $I \cap [y_e, \Phi_e(y_e)] = \emptyset$, the we are done. Otherwise, let $S \subseteq X_e$ and $T \subseteq Y_e$ be such that $S, T \in I$. By assumption, $x_{e,s}, y_{e,s} > e$ for every $s \in \mathbb{N}$. It follows that $A < e < S$. By the previous claim, $R_e$ is $(S, T)$-satisfied at some stage. Unfolding the definition, there is some $k > T$ such that $\Phi_e(k) \downarrow$ and for every $A < S$ (and in particular for every $A \in B$), either for every $B \subseteq [k, \Phi_e(k)]$, $c(A) \neq c(A \cup S \cup B)$, or for every $B \subseteq [k, \Phi_e(k)]$, $c(A) \neq c(A \cup T \cup B)$. In any case, $I \cap [k, \Phi_e(k)] = \emptyset$. This completes the proof of the claim.

As explained above, it follows that $I$ is of hyperimmune-degree. This completes the proof of Theorem 5.5. 

**Corollary 5.6.** RCA$_0 + WKL \not\vdash LM$-FUT$_2$

**Proof.** By the relativized Hyperimmune-free Basis Theorem 5.3, this is a model of RCA$_0 + WKL$ with only hyperimmune-free degrees. By Theorem 5.3, this is not a model of LM-FUT$_2$. 

Blass, Hirst and Simpson built a simple computable instance of the finite union theorem such that every solution computes the halting set: Given a finite set $S = \{x_0 < x_1 < \cdots < x_n\}$, a gap $(x_i, x_{i+1})$ is large if $\emptyset'[x_i] \neq \emptyset'[x_{i+1}], [x_i]$, and is small otherwise. A gap $(x_i, x_{i+1})$ is very small in $S$ if $\emptyset'[x_i] \neq \emptyset'[x_{i+1}]$. Letting $SG(S)$ and $VSG(G)$ denote the number of small gaps and of very small gaps in $S$, respectively, their coloring is simply defined by $c(S) = VSG(S) \mod 2$. Given an IP collection $I$ homogeneous for $c$, they proved that $SG(S)$ is even for every $S \in I$, and that the gap $(\max S, \min T)$ is large for every $S < T \in I$. However, for this same instance, one can easily construct a computable full-match $(B, I)$ with $|B| = 2$ as follows. Let $B = \{B_0, B_1\}$, where $SG(B_0) \mod 2 = SG(B_1) \mod 2 = 0$, $VSG(B_0)$ is odd, and $VSG(B_1)$ is even. Then $B$ full-matches $P_{fin}(\mathbb{N}) - B$. This motivates the following question.

**Question 5.7.** Does LM-FUT$_2$ or FM-FUT$_2$ imply ACA over RCA$_0$?
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