Massive states as the relevant deformations of gravitating branes

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Five-dimensional theories manifesting spontaneous brane generation are discussed in a gravitational context. Without gravity, the IR dynamics of the brane fluctuation below the brane tension scale is described by an effective theory for the Nambu-Goldstone bosons. When gravity is properly taken into account the long distance dynamics changes. The spontaneous breaking of local translational invariance triggers the formation of massive representations via the Higgs mechanism and induces the appearance of new mass scales in the IR. These scales can in principle depend on other fundamental parameters besides the brane tension and the Planck scale. In noncompact extra dimensions the massive states are found to be scalar bound states. We obtain explicit expressions for their propagator and show that their masses depend on the brane width and are thus much heavier than expected. We present an exactly solvable model which captures the main features of the gravitational system.

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I. INTRODUCTION AND RESULTS

Spontaneous breaking of continuous symmetries predicts the existence of Nambu-Goldstone modes as relevant field coordinates for the low energy dynamics. If the symmetry is gauged a massive representation emerges in place of them due to the Higgs phenomenon. The mass scale generated by this mechanism may represent an important threshold for the IR theory, as the Fermi coupling in the weak interactions.

An analogous consideration holds for the case of spontaneous breaking of translational invariance. If the spontaneous breaking refers to a compact coordinate a Higgs phenomenon is expected to involve the graviphoton \[1\]. A rigorous analysis of the nonlinear realization of the local space-time symmetries shows that the Nambu-Goldstone boson kinetic term (the Nambu-Goto action) does not provide any mass term for the graviphoton \[2\], as naively expected by analogy with internal gauge theories. The mass term arises from additional operators that one can build out of the relevant IR variables. These operators are essential for the self-consistency of the description since they encode the presence of the symmetry breaking, and thus ensure continuity of the observable as gravity is switched off. In this phenomenological approach the mass of the graviphotons cannot be rigorously connected to other fundamental scales of the theory (such as the brane tension and the 4d Planck mass) and remains a free parameter. Its determination is crucial for a realistic study of the phenomenology of these models.

The main purpose of the paper is to analyze the emerging of this gravity-induced scale in an exactly tractable framework. We will study a class of gravitational models in which a scalar field develops a nontrivial background along a single space-like direction. Without loss of generality we consider the 5d lagrangian

\[
\mathcal{L} = \sqrt{-g} \left( -M^2 R + g^{AB} \partial_A \Phi \partial_B \Phi - V(\Phi) + \ldots \right) \tag{1}
\]

where the dots refer to some unspecified theory coupled to the scalar \(\Phi\) (and gravity). The 5-dimensional coordinates are denoted as \(x^\mu, y\), with \(A, B = 0, 1, 2, 3, 5\), Greek indeces referring to the unbroken coordinates, while \(y\) denotes the broken spacial direction. The metric signature is ”mostly minus”.

The \(y\)-dependent vacuum expectation value \(\langle \Phi \rangle \equiv \Phi_0(\dot{y})\) is assumed to trigger the localization of light modes of the above unspecified theory around some point of the \(y\)-direction, which we conventionally choose as the origin. This effectively realizes a brane world with a nontrivial dynamics trapped on it. In order to keep our discussion as general as possible we will not write down any explicit function \(\Phi_0\), nor any action for the would-be localized fields. We simply stress that natural candidates for \(\Phi_0\) are the kink shape proposed in \[8\] (in this case the dots in \[1\] would stand for fermionic fields), or one of the scalars described in \[6\] (in this case the dots in \[1\] would stand for gauge fields). For simplicity the background geometry preserves 4d-Poincare invariance

\[
\text{ds}^2 = a^2 \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \tag{2}
\]

where \(a = e^{A}\) and \(A = A(y)\).

The equations of motion derived from \[1\] reduce to the following independent conditions

\[
\ddot{\Phi}_0^2 = -3M^2 \ddot{A},
\]

\[
V(\Phi_0) = -3M^3 (4\dot{A}^2 + \ddot{A}), \tag{3}
\]

where \(\ddot{f} \equiv d^2f/dy^2\). We see that the first equation \[3\] requires \(\ddot{A} \leq 0\), this being a general consequence of the energy conditions \[5\]. Such a constraint implies that no regular solution of the equations of motion is admitted on a circle (we are restricting our analysis to two-derivatives theories). Being interested in the study of translationally invariant theories, we are forced to consider a noncompact extra dimension \(y\). This, by itself, may not represent a serious problem for our study because the authors of \[6\] succeeded in making sense of these theories, at least for...
what concerns the spin-2 sector\(^1\). Similarly, we will be able to obtain an effective description of the brane fluctuations. Another feature of noncompact extra dimensions is more subtle: the graviphotons are not dynamical fields. This raises up another question: what about the gravity-free model. Under some simplifying assumptions (in this case the scalar-gravity system can be captured by an exactly solvable model. Under some simplifying assumptions (in this set up the requirement \( k < w \), where \( k \) is the curvature scale and \( w \) is the inverse brane width, is not an option), we will be able to extract an explicit expression for the resonant propagator. The resonance couplings to brane localized currents are those of the zero mode of the global symmetry, except for corrections of order \( O(k/w) \). In the realistic limit \( k \ll w \), and at energies below the scale \( w \), the resonance represents the only relevant deformation of the brane, very much like the zero mode of the gravity-free model.

In this class of models the resonance mass \( m_R \) plays an analogous role as the graviphoton mass in compact extra dimensions: for energies much larger than \( m_R \) one recovers the Nambu-Goldstone boson dynamics via a generalization of the equivalence theorem while, at scales much smaller, the information about the spontaneous brane generation is encoded in nonrenormalizable operators suppressed by the scale \( m_R \).

In the following we will analyze in detail the physics of the brane fluctuations (the \( \Phi \) excitations) and their relevance on the brane-localized dynamics, with and without gravity. Even though the class of backgrounds considered here are to some extent special, as we have seen, we believe it can shed light on the effect of the gauging of space-time symmetries in more general scenarios.

\section{The Linearized Theory without Gravity}

In order to render the paper self-consistent we review some of the basic properties of the nongravitational theory. Without loss of generality we consider the potential \( V = (\delta_\Phi W)^2 \), with \( W \) an arbitrary function of \( \Phi \) and \( \delta_\Phi \) indicating derivative with respect to the field. The system admits degenerate constant vacua which, by convention, have zero energy. These solutions have \( \delta_\Phi W = 0 \), and consequently \( \delta_\Phi^2 V = 2(\delta_\Phi W)^2 \geq 0 \), while nontrivial solutions satisfy \( \Phi_0 = \delta_\Phi W \).

We will quantize our theory on the background \( \Phi_0(y) \) which, as anticipated in the introduction, is supposed to trigger the formation of a brane at \( y = 0 \). A prototypical example which will be used as a reference is the kink background

\[ \Phi_0 = v \tanh(wy) \tag{4} \]

which follows from

\[ W = wv\Phi \left( 1 - \frac{\Phi^2}{3w^2} \right). \tag{5} \]

The spectrum of the field \( \Phi \) is obtained by studying the linearized equation for the fluctuation \( \phi \equiv \Phi - \Phi_0 \), which is easily found to be

\[ \left( \partial_\mu \partial^\mu - \partial_\mu^2 + \frac{\Phi_0}{\Phi} \right) \phi = 0, \tag{6} \]

where, because of the \( y \) dependence of the background, the relation \( \delta_\Phi^2 V(\Phi_0) = 2\Phi_0/\Phi_0 \) holds. The eigenmodes \( \phi_m(y) \) are solutions of eq. (4) provided we replace \( \partial_\mu \partial^\mu \rightarrow -m^2 \). They form a complete set in the space of square integrable functions and can be used to expand in Kaluza-Klein modes the 5d field \( \phi(x,y) = \sum_m \phi_m(y)Q_m(x) \) and write an effective 4d lagrangian:

\[ \mathcal{L}_{4d} = \sum \delta N_m (\partial_\mu Q_m \partial^\mu Q_m - m^2 Q_m^2), \tag{7} \]

where \( N_m = \int dy \phi_m^2 \).

The spectrum of the brane fluctuation generally contains a zero mode \( Q_0 \) with wavefunction \( \phi_0 \propto \Phi_0 \) (we take to be normalizable), possible discrete eigenvalues, and a continuum starting at a threshold defined by the potential \( \delta_\Phi^2 V \). The interacting terms are obtained as usual from the convolutions of the 5d profiles and, after the fields \( Q_m \) have been properly normalized, they turn out to depend on inverse powers of the normalizations \( N_m \).

An important comment is in order. In the absence of gravity the system is truly translational invariant. A global shift \( y \rightarrow y + \xi \) changes the nontrivial background configuration \( \Phi_0(y) \) into a new vacuum solution. Promoting the parameter to a 4d field \( \xi(x) \) we can identify it with the Nambu-Goldstone mode. Although the Lorentz rotations orthogonal to \( y \) are spontaneously broken, as well, they do not act independently on the vacuum \( \Phi_0 \) so that a single massless mode is predicted. The Nambu-Goldstone boson cannot be found as a dynamical mode in the linearized approach. Indeed, even though at infinitesimal level in the symmetry transformation it coincides with the zero mode \( Q_0(x) \), ensuring its mass is exactly zero, at a nonlinear level all of the \( Q_m \) have a nontrivial overlap with it. Hence the fields \( Q_m \) are seen

\(^1\) These theories are expected to provide a sensible effective description of the brane physics even though an apparent inconsistency arises in the evaluation of the self-couplings.\[3\] [4].
to acquire a potential, while the Nambu-Goldstone boson interacts only via derivative couplings. The strength of these couplings follows from the Nambu-Goldstone boson normalization, which also coincides with $N_0$,

$$\int dy \dot{\Phi}_0^2 = \int dy \partial_\mu W = \Delta W, \quad (8)$$

where we defined the topological charge $\Delta W = W(+\infty) - W(-\infty)$. This quantity measures the energy density of the dimensionally reduced system:

$$\rho[\Phi_0] = \int dy \left( \dot{\Phi}_0^2 + V \right) = 2\Delta W, \quad (9)$$

i.e. the brane tension.

A description of the dimensionally reduced theory can be obtained using an effective approach that makes no reference to the physics responsible for the generation of the defect \[10\]. This is a very powerful approach if we ignore the dynamics responsible for the brane generation, but it cannot tell much about the relevance of the excitations ignored in the description. Since we have at our disposal an explicit model, an effective description in terms of a Nambu-Goldstone boson is not convenient and we will adopt the language of the linearized theory of the fluctuations. Making contact between these two approaches is not immediate because the Nambu-Goldstone boson is a composite state of the fields $\Phi_0$ and we will adopt the language of the linearized theory of the fluctuations. The pole appears in the real axis and thus corresponds to a physical massless particle. The residue $4w$ sets the strength of a typical interaction between the zero mode and arbitrary brane-localized currents.

$$\phi_m = \dot{\Phi}_0 f_m$$

so that the eigenvalue equation reads

$$-\ddot{f}_m - 2\frac{\dot{\Phi}_0}{\Phi_0} \dot{f}_m = m^2 f_m. \quad (10)$$

The convenience of this parametrization is that we extracted the zero mode profile and we can now simplify the eigenvalue problem without losing the main dynamical ingredient. In order to achieve this it is necessary to make some assumptions on the background configuration. We will assume that the defect is exponentially localized as $\Phi_0 \sim e^{-2w|y|}$, i.e. the zero mode is normalizable, and that $\Phi_0(0)$ is odd, i.e. the zero mode has a sharp peak on the brane. Under our assumptions we can write

$$-\ddot{f}_m + 4w \text{sign}(y) \dot{f}_m = m^2 f_m \quad (11)$$

from which it follows, except for a normalization, the general solution

$$f_m = e^{2w|y|} \left( \cos(\mu|y|) + \beta_m \sin(\mu|y|) \right) \quad (12)$$

with $\mu^2 = m^2 - 4w^2$. The boundary condition on the brane $\dot{f}(0) = 0$ and normalizability fully determine the spectrum. The simplified problem clearly predicts a zero mode $f_0 = \text{const}$, and a continuum $m > 2w$ of delta function normalizable modes.

For later convenience we also define the brane to brane propagator as the $\phi$ two-point function $G(p^2, y, y')$ in 4d momentum space and for $y = y' = 0$. From its formal definition it follows $G(p^2, y, 0) \propto \phi_p(y)$ for any $y \neq 0$. Since $\phi_m$ is smooth everywhere and satisfies trivial boundary conditions at the origin, the normalization reads $G(p^2, 0, 0) = 1$. This specifies the solution

$$G(p^2, y, 0) = \frac{\phi_p(y)}{\phi_p(0)} \quad (13)$$

up to boundary conditions in the asymptotic region $|y| \to \infty$. We can integrate the poles of the discrete spectrum by requiring an asymptotic exponential behavior for the Green’s function. This is done by imposing $\beta_p = i$ in the approximate expressions obtained above. Using our parity assumption ($\Phi_0(0) = 0$) and expanding in $p/w$ keeping the dominant contribution only, we see that the brane to brane correlator acquires a pole at $p^2 = 0$:

$$G(p^2, 0, 0) \sim \frac{4w}{p^2}. \quad (14)$$

The pole appears in the real axis and thus corresponds to a physical massless particle. The residue $4w$ is of the same order as the bare couplings.

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2 Because of the composite nature of the Nambu-Goldstone boson the states $Q_m$, for $m \neq 0$, cannot decouple in the limit $w \to \infty$. Their integration leads to corrections to the $Q_0$ potential which are of the same order as the bare couplings.

3 Under our assumptions, a possible next to higher localized mode would have odd parity and would not be crucial for the effective brane dynamics.
III. THE LINEARIZED SCALAR SECTOR WITH GRAVITY

Smooth scalar backgrounds triggering brane generation can be found in a gravitational context in terms of a function $W$ \cite{11}. The solutions for (23) read $\Phi_0 = \delta_4 W$ and $-3M^3\ddot{A} = W$ provided

$$ V = \left( \frac{\delta W}{\delta \Phi} \right)^2 - \frac{W^2}{3M^3}. \tag{15} $$

For example, the kink background is defined by the same $W$ introduced in the previous section. Its backreaction gives rise to a warp factor (assuming $A(y)$ even and choosing the normalization $A(0) = 0$)

$$ A = - \frac{2}{9M^3} \left( \log \cosh(wy) + \frac{1}{4} \tanh(wy)^2 \right). \tag{16} $$

We can interpret this solution as a smooth realization of the Randall-Sundrum geometry. When we explore energies of order $p^2 \ll w^2$, the brane looks like an infinitely thin defect and the warp factor can be approximated as $A = -k|y|$, with

$$ k = \frac{2}{9M^3} w = \frac{\Delta W}{M^3}. \tag{17} $$

The second equality holds up to numerical factors and it is completely general. The relation $k \ll w$ is thus forced by the requirement of a sensible semiclassical approach to gravity.

When translational symmetry is made local a shift of the vacuum along the broken direction no longer identifies a dynamical perturbation. As anticipated in the introduction, no Higgs mechanism is expected since the graviphotons are found to be unphysical perturbations in these backgrounds. The spontaneous breaking will modify the pure scalar sector of the gravitational theory. The analysis of the latter has been carried out in detail in \cite{12,13} (for a generalization to nonminimal couplings see \cite{14}). Let us review the main conclusions and, for convenience, express the results in the conformal coordinate $z$, $dy = adz$.

The nontrivial background induces a mixing between the field excitation $\phi = \Phi - \Phi_0$ and the scalar components of the metric. In the longitudinal gauge and at linearized level the perturbed line element reads

$$ ds^2 = a^2[(1 + F)\eta_{\mu\nu}dx^\mu dx^\nu - (1 - S)dz^2]. \tag{18} $$

The additional $\delta_{AB}$ components include tensor and vector fields which play no role in the diagonalization of the scalar sector. They give rise to a continuous spectrum of massive spin-2 fields, and a normalizable zero mode identified with the graviton. We refer the reader to the literature for details on the spin-2 dynamics, in the following we will discuss the pure spin-0 sector.

The scalar sector is subject to two constraints. In the longitudinal gauge the first requires $S = 2F$, while the second is the formal statement that the fields $F$ and $\phi$ are not independent fluctuations. For arbitrary gauge choices this reads:

$$ a^2(F'' + A'S) = \frac{2}{3M^3} a^2 \Phi_0 \phi', \tag{19} $$

where a prime indicates derivative with respect to $z$. We conclude that the system admits a single independent 5d scalar fluctuation.

The fields $F$ and $\phi$ satisfy two dynamical equations in addition to the constraint \cite{19}. These are more elegantly expressed in terms of gauge invariant variables \cite{12} as:

$$ (A^+ F - \partial_\mu \partial^\mu) G = 0 \tag{20} $$

and the diffeomorphism invariant variables are defined in the longitudinal gauge as

$$ G = a^{3/2}(\phi - \frac{\Phi_0}{2A'} F), \tag{22} $$

$$ U = a^{3/2} F. \tag{21} $$

In order to analyze the spectrum we decompose the 5d fields $F, \phi$ in Kaluza-Klein modes $F(x, z) = \sum_m F_m(z)Q_m(x)$ and $\phi(x, z) = \sum_m \phi_m(z)Q_m(x)$, where the eigenfunctions $F_m(z), \phi_m(z)$ satisfy eqs. \cite{19} and \cite{20} with $\partial_\mu \partial^\mu \rightarrow -m^2$. Because of \cite{19} the equations \cite{20} are equivalent for $m \neq 0$. More precisely, for any eigenvalue $m$ (including $m = 0$) the eigenfunctions are related by

$$ A^- G_m = -\frac{3}{2} m^2 U_m. \tag{23} $$

In order to complete the eigenvalue problem for the modes $F_m, \phi_m$ and determine the spectrum we need to specify the normalization condition. The 4d kinetic terms for the scalars $F, \phi$, and $S$ in the longitudinal gauge is $\cite{12,13}$

$$ a^3 \left( \partial_\mu \phi \partial^\mu \phi - \frac{3}{2} M^3 \partial_\mu F (\partial^\mu F - \partial^\mu S) \right). \tag{24} $$

Substituting the condition $S = 2F$ we can immediately derive the normalization of the dynamical fields $Q_m \cite{13}$:

$$ N_m = \int dz a^3 \left[ \frac{\phi_m^2}{2} + \frac{3}{2} M^3 F_m^2 \right] \tag{25} $$

$$ = \frac{3M^3}{2} \int dy \left[ \frac{2}{a^3} M^3 Y_m^2 + Y_m^2 \right], $$

where in the second equality we changed back to the coordinate $y$. In the last expression the definition $Y_m = a^2 F_m$
and the constraint \( \Phi_m = -2\dot{\Phi}_0 \phi_m / 3 \), have been used.

The authors \([13]\) have proven the hermiticity of the quadratic action in the fluctuations under very general boundary conditions. This ensures that the Kaluza-Klein expansion is meaningful. The 4d dependent coefficients \( Q_m(x) \) play the role of the physical 4d fields and satisfy the dispersion relation \( (\partial_\mu \Phi^\mu + m^2)Q_m = 0 \). One can thus write a free lagrangian formally equivalent to \([4]\). The crucial difference between these two theories is the normalization (25).

Both the boundary and the divergent nature of \( N_0 \) hold for any zero mode solution of the scalar system. To see this explicitly we use equation (26) to recast eq. (25) in the form

\[
N_m = \frac{3M^3}{4} \int dy \left[ -\frac{m^2}{A} \left( \frac{Y_m}{a^2} \right)^2 - \partial_y \left( \frac{Y_m \dot{Y}_m}{a^2A} \right) \right]. \tag{29}
\]

By plugging the solutions (27) into (29) we see that \( N_0 \propto e^{-2A(\infty)} \) for any \( \beta_1,2 \), and therefore diverges. This is tantamount to say that the zero mode decouples from the effective theory. We conclude that the spectrum reduces to a continuum starting at zero mass.

### IV. SCALAR BOUND STATES

The gravitational theory describes a drastically different spectrum compared to the one of the globally symmetric model. The zero mode predicted by the nongravitational theory disappears and no brane perturbation is expected to mediate a long range force. Nevertheless, the brane can still be excited at arbitrary small scales since the continuum starts at zero momentum. In this section we will see that this continuum forms a bound state that appears as a resonant mode to a 4d observer residing on the brane.

Resonant states are found as the eigenvalue problem with complex momentum. They were first identified in the context of alpha decay by Gamow by imposing asymptotic outgoing wave behavior on the wavefunctions. An explicit expression of the resonant condition for our system will be presented later on. For the moment let us stress that the only field manifesting a resonant behavior is \( \phi \).

To derive an explicit expression for the dynamical equation for \( \phi_m \) we can differentiate (26) and use (19). As anticipated in section II this equation can be conveniently written as a function of \( f \), with \( \phi = \Phi_0 f \). The expression is rather involved and reads:

\[
\partial_y \left( \frac{f e^{-2A}}{1 + \frac{m^2}{2A} e^{-2A}} \right) = -2\dot{A} f e^{-2A}. \tag{30}
\]

The equation thus obtained is gauge invariant. Indeed, eq. (22) tells us that in the longitudinal gauge the field \( \phi \) coincides, up to a factor, to the frame-independent fluctuation \( 2A' \Phi + \Phi_0 \dot{U} \). The implications of eq. (30) are therefore physical and do not depend on the gauge choice.

An exact solution of eq. (30) cannot be found even for the simplest backgrounds, but we can find an approximate solution by matching two expressions for \( f_m \), namely \( f_< \), defined for \( 0 < |y| < y_* \), and \( f_> \), defined for \( |y| > y_* \). The distance \( y_* \) is of order the brane width \( 1/w \). For localized defects and \( m \neq 0 \) the terms \( m^2/A \gg 1 \) as soon as we move away from the brane. Near the brane, on the other hand, the smoothness of the background requires \( \dot{A} \approx A \propto \hat{\Phi} \approx 0 \) and leads to a condition for \( f_< \).

Our approximate expressions are:

\[
-\dot{f}_m - \dot{f}_m (4A + 2\frac{\dot{A}}{A}) = m^2 e^{-2A} f_> \tag{31}
\]
In the limit \( y_s \to 0 \), in which the brane appears infinitely thin \( A \approx -k|y_s| \), and assuming exponentially localized scalar defects \( \Phi \sim e^{-2k|y|} \), eqs. (31) can be compactly written as

\[-\ddot{f} - 2A\dot{f} = m^2 e^{-2A} f.\]  

Because of the localization of the defect the mixing between \( \phi \) and the scalars \( F, S \) reduces to a brane effect and translates into the boundary condition at \( y = 0 \). The latter corresponds to a localized positive mass term and will play a role in the determination of the resonant condition. Notice that the boundary condition could have been deduced from the would-be massless profile \( \Phi_0 \propto e^{-2A} \) found in the previous section. As the above derivation makes it clear, however, the simplified eigenvalue problem is a reasonable approximation for massive modes only.

The model described by the equation (32) is a particular example of a more general class of theories that will be introduced in the following section. The system (32) can be used as a toy model for the description of our nontrivial set up, having the great advantage of being exactly solvable. The solution, up to a normalization, is

\[ f_m = e^{\nu k|y|} \left( J_\nu \left( \frac{m}{k} e^{k|y|} \right) + \beta_m Y_\nu \left( \frac{m}{k} e^{k|y|} \right) \right) \]  

where \( J_\nu \) and \( Y_\nu \) are the Bessel function of order \( \nu \) of first and second kind respectively. Imposing the boundary condition at the brane and estimating \( N_m \) using the normalization (25) (for \( m \neq 0 \) the normalization is dominated by the \( \Phi \) integral) we find

\[ \beta_m = -\frac{m J_{\nu-1} (\frac{m}{k}) - 2J_\nu (\frac{m}{k})}{2Y_{\nu-1} (\frac{m}{k}) - 2Y_\nu (\frac{m}{k})} \]  

\[ f_m \propto \sqrt{\frac{m}{k}} e^{\nu k|y|} \left( J_\nu \left( \frac{m}{k} e^{k|y|} \right) + \beta_m Y_\nu \left( \frac{m}{k} e^{k|y|} \right) \right) \]  

where the proportionality factors are numbers depending on the scale entering \( \Phi_0 \) (and the brane tension \( \Delta W \)), which we can simply ignore.

A plot of the spectral density at \( y = 0 \), i.e. \( \phi_m^2 (0) \), reveals the presence of a continuum starting at zero, this is very suppressed up to an energy of the order \( m^2 \sim kw \), where it develops a very sharp peak near the pole of \( \beta_m \), and finally stabilizes at a scale \( m > w \). The peak is the remnant of the delta function localized zero mode density of the global theory which, remarkably, has been shifted to a nonzero mass by gravity. A similar result has been previously found in [8]. In order to understand more deeply the nature of this peak we isolate it by demanding for the appearance of resonant modes.

Imposing outgoing wave boundary conditions on (33) (setting \( \beta_m = i \)) we find the explicit form of the resonant state of our model (32)

\[ \phi_R = \Phi_0 e^{i\nu k|y|} H^{(1)}_\nu \left( \frac{m}{k} e^{k|y|} \right), \]  

where \( H^{(1)}_\nu = J_\nu + iY_\nu \) is the Hankel function of the first kind. The boundary condition on the brane fixes completely the eigenvalue at a complex \( m = m_R - i\Gamma_R/2 \). If the pole is located in the lower half plane, i.e. \( \Gamma_R > 0 \), an asymptotically outgoing wave manifests the characteristic exponential decay in time (and, typically, an exponential divergence in space) and the quantities \( m_R \) and \( \Gamma_R \) can be identified as the mass and the width of the resonance, respectively. The resonant condition for our system finally reads:

\[ \frac{m}{k} H^{(1)}_{\nu-1} \left( \frac{m}{k} \right) = 2. \]  

This condition admits a single solution at a scale \( m_R^2 \sim kw \), corresponding to a complex pole in the scalar propagator. A similar procedure applied to \( F \) (or, equivalently, to \( G \) and \( U \)) provides no solution.

An explicit analytical identification of the root \( m = m_R - i\Gamma_R/2 \) is not possible since, strictly speaking, the Bessel functions cannot be approximated by an expansion in either the small or large argument limit for \( m^2 \sim kw \). Nevertheless a numerical investigation shows that for sufficiently large \( \nu \) the approximation of small argument works quite well. This approximation allows us to obtain expressions similar to those of [15] for the resonant propagator, which will be used to estimate the pole in the following. The reliability of the small argument graph we draw should not come as a surprise, since it is not possible to estimate the pole in the following. The reliability of the small argument approximation should not come as a surprise, since it is natural in the framework studied by [15] since they considered a case analog to \( w \leq k \), which in our framework is not a reliable limit.
with \( g_m^{a,b} = \phi_m, F_m, S_m \) (see [8] for an explicit expression of this matrix in the light-cone gauge). Under the simplifying assumption of an exponentially localized defect, the mixing between the scalar states becomes a pure brane effect and the Green’s function \( G_{\phi\phi} \) is completely determined by the eigenmode \( \phi_p \) and its boundary conditions. Following an analog procedure as that used in section II we obtain an expression for the brane to brane propagator as:

\[
G_{\phi\phi}(p^2, 0, 0) = \frac{1}{\phi_p(0) \phi_p(\nu) - 2k}.
\]

(38)

Setting \( \phi_p = \phi_R \) in the above expression and using (37) we are able to isolate a pole

\[
G_{\phi\phi}(p^2, 0, 0) \sim \frac{1}{p^2 - m_R^2 + ip \Gamma_R(p)}
\]

(39)

where we have defined:

\[
\left(\frac{m_R}{k}\right)^2 \approx 4(\nu - 1)
\]

\[
\frac{\Gamma_R(m_R)}{m_R} \approx \frac{\pi/4}{\Gamma(\nu - 1)\Gamma(\nu)} \frac{m_R}{2k}(2^{(\nu - 1)}).
\]

(40)

As already emphasized, eqs. (40) are only approximate, though reliable, expressions of the pole. Eqs. (39) and (40) represent the main results of the paper. Eqs. (39) and (41) can be derived. On the other hand, the same line of reasoning applied to the solution found in [8] using the light-cone gauge reveals the presence of a resonant mode described by an equation similar to (36), with \( \nu \rightarrow 1 \) in this case the solution has mass and width of comparable magnitude, and both proportional to the curvature scale. In addition, in the light-cone gauge the authors [8] found that the potential mediated by the continuum of light modes in the regime \( kr \gg 1 \) is suppressed by \( 1/(kr)^2 \) with respect to the long range \( 1/r \), while in both the longitudinal and the axial gauge the suppression is much more significant (at least \( 1/(kr)^{2\nu} \)).

One should not be worried in finding different predictions for the resonant pole: the resonance appears in the gauge-dependent propagator of the field \( \phi \). Only in the longitudinal gauge the dynamical equation for \( \phi \) provides gauge-invariant information.

V. AN EXACTLY SOLVABLE MODEL

The appearance of resonant modes in place of the discrete states of the global theory resembles the situation
considered in [13]. In that paper the authors studied the effect of the noncompact Randall-Sundrum warping on chiral fermions localized with the mechanism described in [3]. They found that if we supply the fermionic field with a bulk mass, then the massless localized state of the nongravitational theory disappears and the continuum generates a resonant mode with mass proportional to the small bulk mass and an exponentially suppressed width. We will now show that a similar situation is realized in the spin-0 sector of the gravitational theory considered in the present paper.

Consider a bulk scalar field $\psi$ fluctuating in a gravitational background

$$\mathcal{L} = \sqrt{-g} \left( g^{AB} \partial_A \psi \partial_B \psi - m_0^2 \psi^2 \right).$$  \hfill (43)

In order to localize light modes on the brane we introduce an attractive delta function mass term. Let us first discuss the theory in flat space and define $m_0^2 = 4(w^2 - w_0 \delta(y))$. The mass eigenvalue problem for the scalar wavefunctions $\psi_m$ can be written as:

$$\int dy \psi_m(y) \psi_{m'}(y) = \delta_{m,m'}$$ \hfill (44)

$$-\psi_m + 4(w^2 - w_0 \delta(y)) \psi_m = m_0^2 \psi_m,$$

where $w_0$ is assumed to be a free parameter of order $w$. The system (44) admits a continuum starting at the threshold $2w$ and a single localized mode. The normalizable wavefunction for the latter is $\psi = e^{-\mu y}$ and has a mass squared $4w^2 - \mu^2$. The parameter $\mu$ is determined by the boundary conditions as $\psi(0) = -2w_0 \psi(0)$, which gives $\mu = 2w_0$. We see that normalizability enforces $w_0 \geq 0$ while absence of tachyons $w_0 \leq w$ (the delta function potential cannot be too attractive).

As gravity is switched on eq. (44) receive corrections both in the potential and the kinetic term. For a geometry of the Randall-Sundrum form $ds^2 = e^{-2k|y|} dx^\mu dx^\nu - dy^2$ we have

$$\int dy e^{-2k|y|} \psi_m(y) \psi_{m'}(y) = \delta_{m,m'}$$ \hfill (45)

$$-\psi_m + 4w \delta \psi_m + 4(\bar{w} - w_0 \delta(y)) \psi_m = m_0^2 e^{2k|y|} \psi_m$$

where now both $\bar{w}$ and $w_0$ may contain curvature corrections of order $k/w$ with respect to $w$ and $w_0$ respectively. Under the assumption $w^2 > w_0 (w_0 + 2w)$ the spectrum has no tachionic modes, while the requirement $\bar{w} = w_0 (w_0 + 2w)$ is the necessary condition for the existence of a localized massless mode.

The model [15] typically admits resonant modes. Imposing outgoing boundary conditions we find:

$$\frac{m H_{\nu-1}(\frac{k}{\bar{w}})}{k H_{\nu}^{(1)}(\frac{k}{\bar{w}})} = \left( \nu - 2 - \frac{2 \bar{w}_0}{k} \right)$$ \hfill (46)

$$\nu = 2 \sqrt{1 + \frac{\bar{w}_0^2}{k^2}}$$

Thin resonances exist only if the right hand side of the first equation (46) is positive and, not surprisingly, if $\bar{w}_0 \neq 0$. The former requirement is the same condition ensuring absence of tachionic excitations. Proceeding as in the previous section we approximately find a solution for $m = m_R - i \Gamma_R/2$, where

$$\left( \frac{m_R}{k} \right)^2 \approx 2(\nu - 1) \left( \nu - 2 - \frac{\bar{w}_0}{k} \right)$$ \hfill (47)

$$\frac{\Gamma_R(m_R)}{m_R} \approx \frac{\pi/4}{\Gamma(\nu-1)\Gamma(\nu)} \left( \frac{m_R}{2k} \right)^{2(\nu-1)}.$$

The theory analyzed in the text is found to be a particular example of [15] with $\bar{w}_0 = w(w+2k)$ and $w_0 = w - k$ (see eq. (32) in the text). Despite the involved expression for the $\delta_{m}$ norm obtained in section III, it is easy to verify that the $F_2$ integral in eq. (25) is convergent for any $m \neq 0$ and, thus, the normalization condition for the fluctuations $\delta_{m}$ is effectively of the same form as [13]. As gravity decouples $(k = 0) \bar{w}, \bar{w}_0 \rightarrow w$ and the model predicts a localized massless mode$. Because of the mixing with gravity, an $O(k/w)$ correction to the mass term is induced and the mode disappears from the spectrum leaving a resonance in its place.

Using this simplified picture one can also deduce the fate of a possibly massive localized mode. An inspection of (17) shows that, in the limit $k \ll w$ and for $w_0 = O(\bar{w})$, a resonance still appears at $m_R \sim w$ but quickly becomes wider as its mass increases. For example the choice $\bar{w}_0 = w(w+2k)$ and $w_0 = w$ leads to the prediction of a broad resonance with mass $m_0^2 = 8w^2(1-w_0/w)$, that should be compared with the flat space-time result $4w_0^2(1 - w_0^2/w^2)$. Although this choice seems quite arbitrary, one can verify with techniques similar to those used in the bulk of the paper that it provides a good approximation of the next to higher state of the kink background.

VI. SUMMARY AND CONCLUSION

We analyzed in detail the scalar fluctuations around nontrivial gravitational backgrounds triggered by the space-dependent vacuum expectation value of a scalar field $\Phi$. Assuming the scalar defect localizes light states around a particular region of the noncompact extra dimension, we focused on the role of gravity on the brane-localized physics.

In the globally symmetric version of the model, the dimensionally reduced theory of the $\Phi$ fluctuations de-

$5$ The kink background [3] reproduces the above potential in the thin brane limit $w \rightarrow \infty$. For an observer at a distance $y > 1/w$, where $1/w$ characterizes the thickness of the defect, the background can be simplified by the approximation $\tanh(wy) \approx \text{sign}(y)$. From the identities $\text{sign}(y)^2 = 1$ and $\partial_y \text{sign}(y) = 2 \delta(y)$ we derive $\Phi_0/\Phi_0 \approx -2w_0(2\delta(y)/w - 2)$.\]
scribes a localized zero mode, possibly discrete eigenvalues, and a continuum interacting with the physics residing on the brane. Including the gravitational field no scalar bound state persists, but the continuum of physical spin-0 fields is found to generate massive resonances in the 2-point function of the scalar $\Phi$. Because of the frame-dependent nature of the latter, the resonances turn out to be gauge dependent entities. We extracted frame-independent predictions by working in the longitudinal gauge.

Under some reasonable assumptions we realized that the main features of the coupled scalar-gravity system can be described by an exactly calculable model. We found an explicit expression for the resonant propagator and showed that the heavier the resonance the wider its peak. At distances much larger than the brane width the only possibly relevant brane deformation is the approximately stable lightest resonance. Its couplings to brane localized currents are those of the zero mode of the global theory except for small corrections, but its nonvanishing mass introduces new scales in the IR theory. We have shown that this new scale depends on the brane width.

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