AREA PRESERVING DIFFEOMORPHISMS AND $W_\infty$ SYMMETRY IN A 2 + 1 CHERN-SIMONS THEORY

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Abstract
We discuss the $W_\infty$ symmetry in the 2 + 1 gauge theory with the Chern-Simons term. It is shown that the generators of this symmetry act on the ground state as the canonical transformations in the phase space. We shall also discuss the analogy between discrete states in $c = 1$ string theory and Landau level states in 2 + 1 gauge theory with Chern-Simons term.
1 Introduction

The $W_\infty$ algebras $[1]$ can be obtained after taking $N \to \infty$ limit of the Zamolodchikov’s $W_N$ algebras ($W_2 = V$ is the Virasoro algebra) $[2]$, which are not Lie algebras at finite $N > 2$ due to the non-linear terms in the commutation relations. In the $N \to \infty$ limit the structure of the commutators is much more simpler $[3]$ and, depending on the limiting procedure which is not uniquely defined, one gets different algebras. One particular limit $[1]$ is known as $w_\infty$ algebra

\[ [w_m^{(i)}, w_n^{(j)}] = ((j - 1)m - (i - 1)n)w_{m+n}^{(i+j-2)} \]  

(1)

where $w_m^{(i)}$ is a generator of conformal spin $i$.

It is amusing that the algebra (1) is an algebra of area-preserving (or symplectic) diffeomorphisms of the two-dimensional manifolds, for example it is an infinite-dimensional algebra of canonical transformation in a two-dimensional phase space $(p, q)$. It is interesting that $w_\infty$ algebra generates the symmetry of the relativistic membranes after gauge fixing - this symmetry and its connection with the $SU(\infty)$ were considered in $[4]$.

Recently it was found $[5]$ that $w_\infty$ symmetry is the dynamical symmetry for discrete states $[6]$ of the two dimensional strings $[7]$ and area preserving diffeomorphisms are the canonical transformations acting on the phase space of the inverted harmonic oscillator which is arised in the $c = 1$ matrix models $[8]$.

In this letter we would like to discuss another interesting physical system where area preserving diffeomorphisms and $w_\infty$ symmetry are arising as canonical transformations - this is topologically massive gauge theory $[9]$, i.e. $2 + 1$-dimensional gauge theory with the Chern-Simons term. It was shown $[10]$ that the Hilbert space of the theory is a direct product of the massive gauge particles Hilbert space (one free massive particle in the most simple $U(1)$ case) and some quantum-mechanical Hilbert space. In the $U(1)$ case this QM Hilbert space is the product of the $g$ copies (for a genus $g$ Riemann surface) of the Hilbert space for the Landau problem on the torus. In the infinite mass limit all levels except the first one are decoupled as well as massive particles Hilbert space and we have only the first Landau level. It is easy to

\footnote{for $SL(2;R)/U(1)$ cosets the infinite-dimensional symmetries were considered in $[11]$}
see that the first Landau level becomes the phase space for the pure Chern-Simons theory and canonical transformations acting on the phase space are nothing but area preserving diffeomorphisms.

The organization of the paper is as follows. In section 2, which bears essentially a review character, we consider the canonical phase space of the topologically massive gauge theory and describe canonical quantization and Landau levels picture. We shall also discuss analogy between discrete states in $c = 1$ string theory and quantum mechanical states in gauge theory. In section 3 the area-preserving diffeomorphisms and $\omega_{\infty}$ algebra will be considered.

## 2 Canonical quantization of the $2 + 1$ TMGT

Let us consider the most simple case of the abelian gauge theory with the action [9]:

$$S_{U(1)} = -\frac{1}{4\gamma} \int \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \frac{k}{8\pi} \int \epsilon_{\mu\nu\lambda} A_{\mu} \partial_{\nu} A_{\lambda} \quad (2)$$

To perform canonical quantization one chooses the $A_0 = 0$ gauge. One can represent vector-potential on the plane as:

$$A_i = \partial_i \xi + \epsilon_{ij} \partial_j \chi \quad (3)$$

Substituting this decomposition into constraint

$$\frac{1}{\gamma} \partial_i \dot{A}_i + \frac{k}{4\pi} \epsilon_{ij} F_{ij} = 0, \quad (4)$$

one gets $\partial^2 \dot{\xi} = (k\gamma/2\pi) \partial^2 \chi$. Neglecting all possible zero modes we put $\dot{\xi} = (k\gamma/2\pi) \chi = (M/2) \chi$. Substituting this constraint into action (2) one gets the action

$$S = \frac{1}{2\gamma} \int (\partial_i \chi)^2 - (\partial^2 \chi)^2 - M^2 \chi \partial^2 \chi \quad (5)$$

which becomes the free massive particle action for the field $\Phi = \sqrt{\partial^2 / \gamma} \chi$:

$$S = \frac{1}{2} \int \dot{\Phi}^2 - (\partial_t \Phi)^2 - M^2 \Phi^2 \quad (6)$$
To get this free action it was extremely important to use the constraint (4). However there are some field configurations which are not affected by this constraint. It is easy to see that on the plane the constant fields $A_i(x, t) = A_i(t)$ are not affected by the constraint (4) - because both terms $F_{ij}$ and $\partial_i E_i$ are zero for space-independent vector potential (but not electric field $E_i = A_i$). For this field one gets the quantum mechanical (no coordinate dependence) Lagrangian:

$$L = \frac{1}{2\gamma} \dot{A}_i^2 - \frac{k}{8\pi} \epsilon_{ij} A_i \dot{A}_j$$

which describes the particle on the plane $A_1, A_2$ in magnetic field - the famous Landau problem.

Let us consider the configuration and phase spaces of this problem. From Lagrangian (7) one gets the canonical momenta:

$$P_i = \frac{\partial L}{\partial \dot{A}_i} = \frac{1}{\gamma} \dot{A}_i + \frac{k}{8\pi} \epsilon_{ij} A_j$$

with the usual commutation relations (or Poisson brackets in the classical limit $[,] \rightarrow -i\{,\}$)

$$[P_i, P_j] = [A_i, A_j] = 0; \quad [P_i, A_j] = -i\delta_{ij}$$

The canonical Landau Hamiltonian and the eigenvalues $E_n$ are

$$H = -\frac{\gamma}{2} \left( \frac{\partial}{\partial A_i} - i \frac{k}{8\pi} \epsilon_{ij} A_j \right)^2 \quad E_n = (n + 1/2)M$$

where the gap equals to the topological mass $M = \gamma k/4\pi$.

Let us note that variables ($A_1$ and $A_2$) belong to the configuration space, however if reduced to the first Landau level the configuration space is transformed into the phase space. It is easy to see that in the limit $\gamma \rightarrow \infty$ (which corresponds to the reduction on the first Landau level) $P_i = \frac{k}{8\pi} \epsilon_{ij} A_j$ and one of the coordinates becomes a conjugate momentum for another one. Thus ($A_1, A_2$) plane is the configuration space for the topologically massive gauge theory and becomes the phase space for pure Chern-Simons theory which is an exactly solvable 2+1 dimensional topological field theory [10]. This picture of the Landau levels were considered in [11] and it was proved, that due to
the degeneracy on each Landau level one gets long-range part of the gauge propagator in TMGT (the propagator in pure CS theory), in spite of the fact that there are no massless particles in theory. In other words, the Hilbert space of the pure Chern-Simons theory (i.e. in the limit $M \to \infty$) is the first Landau level.

Is it possible to consider a constant gauge field as a physical, i.e. gauge invariant variable in the theory? The answer is positive for 2-dimensional Riemann surfaces different from a sphere. It is well-known that any one-form $A$ can be uniquely decomposed according to Hodge theorem as

$$A = d\xi + \delta\chi + A$$
$$dA = \delta A = 0$$

which generalizes the decomposition (3); the harmonic form $A$ equals

$$A = \sum_{p=1}^{g} (A^p \alpha_p + B^p \beta_p)$$

where $\alpha_p$ and $\beta_p$ are canonical harmonic 1-forms on the Riemann surface of genus $g$. It is well known that there are precisely $2g$ harmonic 1-forms for any genus $g$ Riemann surface and one gets $g$ copies of the Landau problem for each conjugate pair $A^p, B^p$. The Hamiltonian for each pair is the usual Landau Hamiltonian where $A, B$ corresponds to $A_1, B_2$. There is also some dependence on the moduli of the Riemann surface due to the $F^2$ term in (2) dependence on metric $g_{\mu\nu}$. It is easy to see that for $g_{00} = 1$ and $g_{ij} = \rho(x) h_{ij}(\tau)$ the $F^2_{0i}$ term does not depend on conformal factor $\rho$. Let us consider dependence on the moduli $\tau$ in the most simple case of a torus where $\tau$ is a complex number and metric $h_{ij}$ can be parametrized as

$$h^{ij} = \frac{1}{(Im \tau)^2} \begin{pmatrix} 1 & Re \tau \\ Re \tau & |\tau|^2 \end{pmatrix}$$
$$h_{ij} = \begin{pmatrix} |\tau|^2 & -Re \tau \\ -Re \tau & 1 \end{pmatrix}$$

and $h = det h_{ij} = (Im \tau)^2$. Lagrangian takes the form

$$L = \frac{1}{2\gamma} \sqrt{h} h^{ij} \dot{A}_i \dot{A}_j - \frac{k}{8\pi} \epsilon^{ij} A_i \dot{A}_j$$

(14)
which can be transformed to diagonal form \([0]\) for new fields

\[
\mathbf{A}_{(a)} = e^i_{(a)} \mathbf{A}_i
\]  

(15)

where zweibein \(e^i_{(a)}\) defines the metric \(h^{ij} = e^i_{(a)} e^j_{(b)} \delta^{(a)(b)}\) and \(e^{(a)(b)} e^i_{(a)} e^j_{(b)} \sim \epsilon^{ij}\). It is easy to find that

\[
\begin{pmatrix}
\mathbf{A}_{(1)} \\
\mathbf{A}_{(2)}
\end{pmatrix} =
\begin{pmatrix}
1 & \text{Re} \tau \\
0 & \text{Im} \tau
\end{pmatrix}
\begin{pmatrix}
\mathbf{A}_1 \\
\mathbf{A}_2
\end{pmatrix}
\]  

(16)

In terms of the new variables the Lagrangian \((14)\) takes the form

\[
L = \frac{1}{2\gamma \text{Im} \tau} \mathbf{A}_i^2 - \frac{k}{8\pi \text{Im} \tau} \epsilon^{(i)(j)} A_{(i)} \dot{A}_{(j)}
\]  

(17)

and we see that the Chern-Simons coefficient depends on moduli: \(k \rightarrow k/\text{Im} \tau\). However the mass gap is unchanged because \(\gamma\) is also changed \(\gamma \rightarrow \gamma \text{Im} \tau\) and \(M = \gamma k/4\pi\) does not depend on \(\tau\).

Thus we get the Landay problem on the plane \((\mathbf{A}_{(1)}, \mathbf{A}_{(2)})\). However we forgot about large gauge transformations acting on the quantum-mechanical coordinates \(\mathbf{A}_i \rightarrow \mathbf{A}_i + 2\pi N_i\), where \(N_i\) are integers. This is due to the fact that gauge-invariant objects - Wilson lines \(W(C) = \exp(i \oint_C A_\mu dx^\mu)\), are not changed under these transformations (we normalize here the lengths of all the fundamental cycles to one) and one can consider torus \(0 \leq \mathbf{A}_i < 2\pi\) with the area \((2\pi)^2\). However after we consider the new variables \(A_{(i)}\) one gets the torus (see \((16)\)) generated by the shifts \(2\pi\) and \(2\pi \tau\) with an area \(S = (2\pi)^2 \text{Im} \tau\).

Let us note that being reduced to the first Landau level this torus becomes the phase space - thus for the consistent quantization this area must be proportional to the integer (the total number of the states must be integer). It is known that the density of states \(\rho\) on Landau level equals to \(H/2\pi\), where \(H\) is a magnetic field. In our case the ”magnetic field” in \((\mathbf{A}_{(1)}, \mathbf{A}_{(2)})\) plane can be easily obtained from \((17)\) and equals to \(\mathcal{H} = (k/4\pi \text{Im} \tau)\), thus the total number of states will be \(\bar{N} = (1/2\pi)(k/4\pi \text{Im} \tau) \times (2\pi)^2 \text{Im} \tau = k/2\).

and does not depend on \(\tau\) but only on \(k\). One can factorize over whole large gauge transformations only for even \(k\), for general \(k = 2m/n\) only the shifts \(\mathbf{A}_i \rightarrow \mathbf{A}_i + n N_i\), are compatible with the dynamics and one gets \(0 \leq \mathbf{A}_i < 2\pi n\) so the total area is now \((2\pi n)^2\) and the number of states will be \(2mn\).
In the case of the genus $g$ Riemann surface one gets $g$ conjugate pairs $A^p, B^p$, $p = 1, \cdots, g$. After diagonalization one finds that there are $g$ copies of the Landau problem and the total Hilbert space of the abelian topologically massive gauge theory

$$H = H_\Phi \otimes \prod_{i=1}^{g} H_A$$

is the product of the free massive particles Hilbert space and $g$ copies of the Landau problem’s Hilbert space. One can consider the non-abelian theories in a similar way.

Let us also note some analogy between gauge fields (12) and discrete states in string theory [7]. The physical states in the string theory must satisfy the Virasoro constraints modulo gauge transformations - in a complete analogy with the constraint (4) in a gauge theory. As was stressed by Polyakov in [7] the discrete states in $c = 1$ string theory exists in the string theory only because Virasoro constraints $L_n|\psi> = 0$, $n > 0$ do not act effectively on some excited states, contrary to the generic case when the only non-trivial solutions of the Virasoro constraints are gauge artifacts. The simplest example is provided by the level one operator in open string theory (we are using here Klebanov and Polyakov [5] notation)

$$e_\mu(p, \epsilon) \partial X^\mu \exp(ipX + \epsilon \phi)$$

where $\phi$ and $X$ are the Liouville and matter fields and $X^\mu = (\phi, X)$. The conditions that this operator is physical are the follows:

$$(f_\mu + b_\mu)e_\mu(f) = 0; \quad f_\mu(f_\mu + b_\mu) = 0$$

where $f_\mu = (\epsilon, p)$, $b_\mu = (2, 0)$ and the signature is $(+, -)$. For generic $p$ the only solution of this constraints is the pure gauge $e_\mu(f) \sim f_\mu$. However for $f_\mu = 0$ or $-b_\mu$ one constraint disappears and it is possible to show that one can gauged away only $e_0$ component, but not $e_1$. In the same way all other states at higher levels can be obtained.

It is evident that this picture is extremely similar to the one which was considered here - when the quantum-mechanical degrees of freedom (12) were not affected by the gauge constraint (4) and becomes the new physical degrees of freedom. Thus we have very intriguing analogy between $c = 1$ strings with a $1 + 1$ dimensional physical field (massless ”tachyon”) and discrete states and topologically massive gauge theory with a $2 + 1$ dimensional physical field (topologically massive ”photon”) and quantum-mechanical degrees of
freedom (states on the Landau levels). What is extremely interesting in this analogy is the $w_\infty$ symmetry which exists for both discrete states [5, 8] and states on Landau level as we shall see in the next section.

3 Canonical transformation on the Landau level and $w_\infty$ algebra

Let us consider the Landau Hamiltonian

$$H = \frac{\gamma}{2}(P_i - \frac{k}{8\pi} \epsilon_{ij} A_j)^2$$  \hspace{1cm} (20)

where $A_i$ and $P_j$ are the canonical coordinates in the four-dimensional phase space. In order to study the symmetries of the problem in more details let us remember some well-known facts about canonical transformations (see, for example [12]). By definition canonical transformations are diffeomorphisms of the phase space which preserve the symplectic structure $\omega = \sum dq^l \wedge dp^l$. It is possible to show that conservation of the symplectic structure leads to the Liouville theorem about the conservation of the phase space volume during the time.

The canonical transformations are usually defined by the generation function depending on both old ($p$ or $q$) and new ($P$ and $Q$) phase space coordinates, for example one can consider arbitrary $F(q, Q)$ and put

$$p_i = \partial F/\partial q_i; \quad P_i = -\partial F/\partial Q_i$$  \hspace{1cm} (21)

It is easy to see that $P, Q$ are new canonical coordinates. There is however another representation, namely one can consider evolution with respect to some "Hamiltonian" $\Phi(p, q)$ (which is arbitrary function on the phase space and has nothing common with the physical Hamiltonian). The change in quantities $p$ and $q$ during this evolution may itself be regarded as a series of canonical transformations. Let $p$ and $q$ be the values of the canonical variables at time $t$ and $P$ and $Q$ are their values at another time $t + \tau$. The latter are some function of the former, depending on $\tau$ as on parameter

$$Q = Q(q, p; \tau), \quad P = P(q, p; \tau)$$  \hspace{1cm} (22)

These formulae can be considered as the canonical transformation from the old coordinates $p, q$ to the new ones $P, Q$. This representation is convenient.
for the infinitesimal transformation, when $\tau \to 0$. In this case using Hamiltonian equations of motion with "Hamiltonian" $\Phi(p, q)/\tau$ one gets

$$Q_i = q_i + \dot{q}_i \tau = q_i + \{q_i, \Phi\}; \quad P_i = p_i + \dot{p}_i \tau = p_i + \{p_i, \Phi\}$$

(23)

where

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i}$$

(24)

is the usual Poisson brackets.

Let us now consider canonical transformations acting on the Landau problem phase space. The general canonical transformations are acting on the whole four-dimensional phase space and after quantization they will mix different Landau levels. However there is a special subgroup of the canonical transformations acting on the two-dimensional subspace of the phase space commuting with the Hamiltonian. This means that this transformations do not mix different Landau levels and thus acting on each Landau level as on two-dimensional phase space. To define this group let us notice that Hamiltonian (20) depends on two variables $a$ and $a^+$

$$a^+ = 2P_\bar{z} + i \frac{k}{8\pi} A_\bar{z}; \quad a = 2P_z - i \frac{k}{8\pi} A_z; \quad [a, a^+] = -i \{a, a^+\} = \frac{k}{2\pi}$$

(25)

and

$$H = \frac{\gamma}{4}(aa^+ + a^+a)$$

(26)

Here $A_{z,\bar{z}} = A_1 \pm i A_2$ and $P_{z,\bar{z}} = -i \partial/\partial A_{z,\bar{z}}$ are the corresponding conjugate momenta. There is another pair $b$ and $b^+$ commuting with $a$ and $a^+$

$$b^+ = 2P_\bar{z} + i \frac{k}{8\pi} A_\bar{z}; \quad b = 2P_z - i \frac{k}{8\pi} A_z; \quad [b, b^+] = -i \{b, b^+\} = \frac{k}{2\pi}$$

(27)

Now what means the restriction on the first level - this is to the pure Chern-Simons theory, i.e. taking the limit $\gamma \to \infty$. In this limit one gets from (8)

$$2P_\bar{z} = P_1 + iP_2 = -i \frac{k}{8\pi} A_\bar{z}; \quad 2P_z = P_1 - iP_2 = i \frac{k}{8\pi} A_z$$

(28)
and then

\[ a = a^+ = 0; \quad b^+ = i \frac{k}{4\pi} A_z; \quad b = -i \frac{k}{4\pi} A_z \]  \hspace{1cm} (29)

The physical meaning of this reduction is the follows - operators \( a \) and \( a^+ \) acting on the state at the given level \( n \) shift it to \( n \pm 1 \). To be at a given Landau level we must put these operators to zero after which \( b \) and \( b^+ \) play the role of the coordinate on the reduced phase space (which is the total phase space in the pure Chern-Simons theory) and one can consider the canonical transformations on this space.

As we know from (23) and (24) the canonical transformations acting on the two-dimensional phase space \((q, p)\) are defined by

\[ \delta q = \{q, W(p, q)\} = \frac{\partial W(p, q)}{\partial p}; \quad \delta p = \{p, W(p, q)\} = -\frac{\partial W(p, q)}{\partial q} \]  \hspace{1cm} (30)

where \( W(p, q) \) is an arbitrary function. The fact that these transformations preserves the area can be easily checked using the fact that the general infinitesimal area-preserving diffeomorphism takes the form

\[ \xi^a \rightarrow \xi^a + \epsilon^a(\xi); \quad \partial_a \epsilon^a = 0 \]  \hspace{1cm} (31)

where \( \epsilon^a = (q, p) \). General solution of \( \partial_a \epsilon^a = 0 \) is the sum of the two terms

\[ \epsilon^a(\xi) = \epsilon^{ab}_c \partial_b W(\xi) + \sum_{i=1}^{b_1} c_i \epsilon^a \]  \hspace{1cm} (32)

where first term describes infinite number (all possible functions \( W(\xi) \)) of the local co-exact solutions and the second term describes the finite number (given by the first Betti number \( b_1 \)) of the harmonic forms on two-dimensional phase space. It is easy to see that diffeomorphisms generated by the first term are nothing but canonical transformations (30). Let us note that because our phase space is torus there are also two (for torus the first Betti number \( b_1 = 2 \)) global translations \( P_a = -i\partial_a \).

Any function \( f \) on the phase space is transformed under the canonical transformation generated by \( W \) according to the rule \( w f = \delta_W f = \{f, W\} \), where \( w \) is the operator corresponding to function \( W(\xi) \). Using the Jacobi identity \( \{\{f, W_1\}W_2\} - \{\{f, W_2\}W_1\} + \{\{W_1, W_2\}f\} = 0 \) one can check
that algebra of the area-preserving diffeomorphisms is given by the Poisson brackets

\[ [w_1, w_2]f = [\delta W_1, \delta W_2]f = \delta_{\{W_1, W_2\}} f \]  \hspace{1cm} (33)

Let us define our torus phase space \( T^2 \) to be a square with both sides equal to \( 2\pi \) (generalization for the general case is straightforward and will be done in the end). Any function \( W \) can be written in terms of the complete set of harmonics

\[ W_{\vec{n}} = \exp(i\vec{n}\vec{\xi}) \]  \hspace{1cm} (34)

where \( \vec{n} = (n_1, n_2) \) with integers \( n_1, n_2 \) and \( \vec{\xi} = (\xi_1, \xi_2) \) are coordinates on the torus. One gets the commutation relations for operators \( w_{\vec{n}} \) computing the Poisson bracket for \( W_{\vec{n}} \), \( [\ ] \),

\[ [w_{\vec{m}}, w_{\vec{n}}] = (\vec{m} \times \vec{n}) \; w_{\vec{m}+\vec{n}} \]  \hspace{1cm} (35)

where \( \vec{a} \times \vec{b} = a_1 b_2 - a_2 b_1 \). One can see that (33) is equivalent to the the commutation relations for the \( w_\infty \) algebra (1) identifying \( w_{\vec{m}} = w_{m_1,m_2} \) with \( w_{m_1,m_2}^{(m_2+1)} \).

Let us note that \( w_\infty \) algebra for planar electrons in the magnetic field was discussed recently in [13] as dynamical symmetry for the Quantum Hall (QH) system. Formally this is the same Landau problem ( the only difference is that in [13] phase space was a plane, however one can consider the QH system on a torus). However physically it is a complete different problem - in our case we are dealing with gauge field configurations - in the QH case - with the many-body fermion system.

After quantization we get instead of (34) the quantum version

\[ W_{n,\bar{n}} = \exp\left(\frac{2\pi}{k}(nb^+ - \bar{n}b)\right) = \exp\left(\frac{2\pi}{k}nb^+\right)\exp\left(-\frac{2\pi}{k}\bar{n}b\right); \]

\[ [W_{n,\bar{n}}, W_{m,\bar{m}}] = (\exp(-\frac{2\pi}{k}n\bar{m}) - \exp(-\frac{2\pi}{k}m\bar{n}))(W_{n+m,n+m}) = \]

\[ -2i\sin\frac{2\pi}{k}(n_1 m_2 - n_2 m_1) \exp(-\frac{2\pi}{k}(n_1 m_1 + n_2 m_2))W_{n+m,n+m} \]  \hspace{1cm} (36)

\[ 4\text{Let us note that this algebra has a central extension (which does not exist in the case of the area preserving diffeomorphisms on the sphere) } [w_{\vec{m}}, w_{\vec{n}}] = (\vec{m} \times \vec{n}) \; w_{\vec{m}+\vec{n}} + \vec{a} \vec{m} \delta_{\vec{n}+\vec{n},0} \]
Here $n(\bar{n}) = n_1 \pm in_2$ where $n_1, n_2$ are integers and the classical limit corresponds to $k \to \infty$. Let us also note that after rescaling

$$\mathcal{W}_{n, \bar{n}} = \exp\left(\frac{\pi}{k}(n_1^2 + n_2^2)\right)\tilde{\mathcal{W}}_{n, \bar{n}}$$

one can recover from (37) the Fairlie-Fletcher-Zachos (FFZ) algebra (see D.B. Fairlie et al in [4])

$$[\tilde{\mathcal{W}}_{n, \bar{n}}, \tilde{\mathcal{W}}_{m, \bar{m}}] = -2i \sin \frac{2\pi}{k}(n_1 m_2 - n_2 m_1)\tilde{\mathcal{W}}_{n+m, \bar{n}+\bar{m}}$$

It is also interesting to know how this algebra will act on the first Landau level, which is the quantum analog of the classical phase space. To define this action one need to know the wave functions on the first Landau level for torus which was studied in [14]. One starts from the general wave function on the first Landau level for Hamiltonian (20)

$$\Psi(A_1, A_2) = \exp\left(-ik\frac{8\pi}{4\pi}A_1A_2 + ik\frac{8\pi}{4\pi}pA_1 - k\frac{8\pi}{4\pi}(A_2 - p)^2\right) =$$

$$\exp\left(-\frac{k}{16\pi}A_z A_{\bar{z}}\right)\exp\left(-\frac{k}{8\pi}p^2 + \frac{k}{16\pi}A_z^2 + i\frac{kp}{4\pi}A_z\right)$$

where $p$ is some constant - "momentum" in $A_1$ direction. Let us note that except the first factor the wave function depends only on holomorphic combination $A_{\bar{z}} = A_1 - iA_2$. This can be obtained from the fact that any wave function on first Landau level is annihilated by the operator $a$ (see (25))

$$\frac{\partial}{\partial A_z}\Psi(A_1, A_2) + \frac{k}{16\pi}A_z \Psi(A_1, A_2) = 0;$$

$$\Psi(A_1, A_2) = \exp\left(-\frac{k}{16\pi}A_z A_{\bar{z}}\right)\Phi(A_{\bar{z}})$$

To get the correct wave functions for simplest case of $\tau = i$ (see (16)) from (39) one have to sum over all $p = 4\pi n/k$. It is easy to see that for even $k$ there are $k/2$ different classes $p = 4\pi r/k + 2\pi n$, $r = 1, \cdots k/2$; $n \in Z$ which gives $k/2$ basic wave functions.\footnote{In general rational case $k = 2m/n$ it will be $mn$ basis vectors, but here we shall not consider the general case} Thus the basic wave functions can be chosen as...
\[
\Psi(A_1, A_2)_r = \exp\left(-\frac{k}{16\pi} A_z A_{\bar{z}}\right) \exp\left(\frac{k}{16\pi} A_{\bar{z}}^2\right) \times \\
\sum_n \exp\left[-\frac{\pi k}{2} (n + 2r/k)^2 + \frac{i k A_{\bar{z}}}{2} (n + 2r/k)\right] = \\
\exp\left(-\frac{k}{16\pi} A_z A_{\bar{z}}\right) \exp\left(\frac{k}{16\pi} A_{\bar{z}}^2\right) \theta\left[\begin{array}{c} 2r/k \\ 0 \end{array}\right] \left(\frac{k A_{\bar{z}}}{4\pi}\right) \left(\frac{ik}{2}\right) 
\]

where the theta functions are defined as

\[
\theta\left[\begin{array}{c} \alpha \\ \beta \end{array}\right] (z|\tau) = \sum_n \exp[i \pi \tau(n + \alpha)^2 + 2\pi i (n + \alpha)(z + \beta)] 
\]

Now let us consider the action of the generators \((36)\) on the wave functions \((41)\). From \((27)\) and \((40)\) one can see that

\[
b \Psi(A_1, A_2) = \exp\left(-\frac{k}{16\pi} A_z A_{\bar{z}}\right)(-2i \frac{\partial}{\partial A_{\bar{z}}})\Phi(A_{\bar{z}})
\]

\[
b^+ \Psi(A_1, A_2) = \exp\left(-\frac{k}{16\pi} A_z A_{\bar{z}}\right)(\frac{ik}{4\pi} A_{\bar{z}})\Phi(A_{\bar{z}}) 
\]

and then \(W_{n,\bar{n}}\) effectively acts only on holomorphic factors \(\Phi_r(A_{\bar{z}})\):

\[
W_{n,\bar{n}}\Phi_r(A_{\bar{z}}) = \exp\left(\frac{i}{2} n A_{\bar{z}}\right) \exp\left(\frac{4\pi i}{k} \bar{n} \frac{\partial}{\partial A_{\bar{z}}}\right) \Phi_r(A_{\bar{z}}) = \\
\exp\left(\frac{i}{2} n A_{\bar{z}}\right) \Phi_r(A_{\bar{z}} + \frac{4\pi i}{k} \bar{n}) 
\]

Using \((41)\) we get after some calculation \((n(\bar{n}) = n_1 \pm n_2)\)

\[
W_{n,\bar{n}}\Phi_r(A_{\bar{z}}) = \exp\left(\frac{\pi}{k} (n_1^2 + n_2^2)\right) \exp\left(\frac{2\pi i}{k} n_2(n_1 + 2r)\right) \Phi_{r+n_1}(A_{\bar{z}}) 
\]

It is amusing that to get unitary representation, i.e. to preserve the normalization of the wave functions \(\Phi_r(A_{\bar{z}})\) we must rescale \(W_{n,\bar{n}}\) according to \((37)\) - then the nonunitary exponential factor \(\exp(\frac{\pi}{k} (n_1^2 + n_2^2))\) disappears.

Thus we demonstrated that the ground state (first Landau level) wave functions in the topologically massive gauge theory (not only in the pure Chern-Simons case with infinite mass gap) form a unitary representation of
the FFZ algebra \([38]\). Let us note that states at higher Landau levels which can be obtained from our ground state wave functions \(\Psi_r\) by the action of the \(a^+\) operator \([25]\)

\[
\Psi_r'(A_1, A_2) = \frac{(a^+)^l}{\sqrt{l!}} \Psi_r(A_1, A_2)
\]

form unitary equivalent representations because generators \(\tilde{W}_n, \bar{n}\) are built from \(b\) and \(b^+\) operators only and thus commute with \(a\) and \(a^+\).

It is interesting to note that one can construct the second \(W\) algebra from \(a, a^+\) operators in the complete analogy with \([36]\) - then one gets the direct products of two \(W_\infty\) algebras: \(W_\infty^a \otimes W_\infty^b\). The second ("b") algebra acts on each Landau level, the first ("a") algebra mixes the level - it is the coherent states \(|\alpha >, \sim \exp(\alpha a^+)|0 >,\) which are transformed in a simple way under the action of \(W_\infty^a\) generators.

Finally let us write some formulas in the case of a general modular parameter \(\tau\). Using \([16]\) and \([17]\) one get instead of \([36]\)

\[
W_{n,\bar{n}} = \exp\left(\frac{2\pi Im\tau}{k} nb^+\right)\exp\left(-\frac{2\pi Im\tau}{k}\bar{n}b\right)
\]

where \(n = (-i\tau n_1 + in_2)/Im\tau\), \(\bar{n} = (i\bar{\tau}n_1 - in_2)/Im\tau\) and \([b, b^+] = k/2\pi Im\tau\).

The commutator equals to

\[
[W_{n,\bar{n}}, W_{m,\bar{m}}] = \exp\left\{-\frac{2\pi}{kIm\tau}(|\tau|^2 n_1 m_1 + n_2 m_2 - Re(\tau(n_2 m_1 + n_1 m_2)))\right\} \times
-2i \sin \frac{2\pi}{k}(n_1 m_2 - n_2 m_1)W_{n+m,n+m}
\]

and after renormalization \(W_{n,\bar{n}} = \exp(\pi|n_2 - \tau n_1|^2/kIm\tau)\tilde{W}_{n,\bar{n}}\) one gets the same FFZ algebra \([38]\).

The basic wave functions are now \((r = 1, \cdots k/2)\)

\[
\Psi_r(A_1, A_2) = \exp\left(-\frac{k}{16\pi Im\tau}A_z A_\bar{z}\right)\exp\left(\frac{k}{16\pi Im\tau}A_\bar{z}^2\right) \times
\sum_n \exp\left[i\frac{\pi k\tau}{2}(n + 2r/k)^2 + i\frac{kA_\bar{z}}{2}(n + 2r/k)\right] = \exp\left(-\frac{k}{16\pi}A_z A_\bar{z}\right)\exp\left(\frac{k}{16\pi Im\tau}A_\bar{z}^2\right) \theta \left[\begin{array}{c} 2r/k \\ 0 \end{array}\right] \left[\begin{array}{c} kA_\bar{z} \\ \frac{4\pi}{2} k\tau \end{array}\right]
\]
where $A_z = A_1 + \tau A_2$, $A_{\bar{z}} = A_1 + \bar{\tau} A_2$ and the theta-functions were defined in (12).

One can check that generators (48) act on wave functions (50) as

$$\mathcal{W}_{n,\bar{n}} \Phi_r(A_{\bar{z}}) = \exp(\pi |n_2 - \tau n_1|^2/k \Im \tau) \exp(i\phi) \Phi_{r+n_1}(A_{\bar{z}})$$

- in a complete agreement with (45). After rescaling to the $\tilde{\mathcal{W}}_{n,\bar{n}}$ the exponential factor $\exp(\pi |n_2 - \tau n_1|^2/k \Im \tau)$ disappears and we again get unitary representation. Let us also note that these wave functions (to be more precise the holomorphic parts) can be also obtained in the geometric quantization of the pure Chern-Simons theory [10], [15] and give the conformal blocks for the $c = 1$ conformal field theory (for more details see [15]). Let us note that even in a case of a finite mass gap $M = k\gamma/4\pi$ one gets the same ground state (first Landau level) wave functions.

4 Conclusion

We discussed in this letter the classical and quantum canonical symmetry in the $2+1$ gauge theory with a Chern-Simons term and found that one can consider canonical transformation on the reduced Chern-Simons phase space (which becomes after the quantization the first Landau level) as the $w_\infty$ algebra, which after quantization becomes FFZ $W_\infty$ algebra (38). This algebra (we shall call it $W^b_\infty$ algebra) commute with the Hamiltonian $\mathcal{H} \sim (a a^+ + a^+ a)$ and thus acts independently on each Landau level. The ground state wave functions form the unitary representation of this algebra - as well as wave functions at any Landay level. One can construct another $W$ algebra from $a, a^+$ operators and gets: $W^a_\infty \otimes W^b_\infty$. The second (“b”) algebra acts on each Landau level, the first (“a”) algebra mixes the level and acts in a simple form onto the coherent states $|\alpha >_r \sim \exp(\alpha a^+)|0 >_r$.

It is interesting to find if there any connection between this algebra and $W_\infty$ arising in the $c = 1$ strings and corresponding matrix models [3]-[8]. Let also note that one can get $W_N$-algebra from the $SU(N)$ Chern-Simons theory [17] - thus one can get $W_\infty$ for the $SU(\infty)$ Chern-Simons theory. It is interesting and open question if there is any connection between these two different $W_\infty$ structures: the first one which is connected with the canonical transformations - area preserving diffeomorphisms acting in the phase space (or Hilbert space after quantization) and the second one which was considered...
in [10] and is connected with the general coordinate transformations in the usual space (2 + 1 space-time).

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