Non-degenerate metrics, hypersurface deformation algebra, non-anomalous representations and density weights in quantum gravity

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Abstract

Classical General Relativity is a dynamical theory of spacetime metrics of Lorentzian signature. In particular the classical metric field is nowhere degenerate in spacetime. In its initial value formulation with respect to a Cauchy surface the induced metric is of Euclidian signature and nowhere degenerate on it. It is only under this assumption of non-degeneracy of the induced metric that one can derive the hypersurface deformation algebra between the initial value constraints which is absolutely transparent from the fact that the inverse of the induced metric is needed to close the algebra. This statement is independent of the density weight that one may want to equip the spatial metric with.

Accordingly, the very definition of a non-anomalous representation of the hypersurface deformation algebra in quantum gravity has to address the issue of non-degeneracy of the induced metric that is needed in the classical theory. In the Hilbert space representation employed in Loop Quantum Gravity (LQG) most emphasis has been layed to define an inverse metric operator on the dense domain of spin network states although they represent induced quantum geometries which are degenerate almost everywhere. It is no surprise that demonstration of closure of the constraint algebra on this domain meets difficulties because it is a sector of the quantum theory which is classically forbidden and which lies outside the domain of definition of the classical hypersurface deformation algebra. Various suggestions for addressing the issue such as non-standard operator topologies, dual spaces (habitats) and density weights have been proposed to address this issue with respect to the quantum dynamics of LQG.

In this article we summarise these developments and argue that insisting on a dense domain of non-degenerate states within the LQG representation may provide a natural resolution of the issue thereby possibly avoiding the above mentioned non-standard constructions.

1 Introduction

One possible approach to quantum gravity is via the Hamiltonian or canonical formulation [1]. This so called initial value formulation is widely used in mathematical general relativity [2] and numerical relativity [3] with recent spectacular success in predicting e.g. black hole merger templates [4]. The canonical approach is also the fundament of Loop Quantum Gravity (LQG) [5]. LQG derives its name from the fact that GR can be formulated in terms of Yang-Mills like non-Abelian connection variables [6] and thus methods from lattice QCD [7] (specifically Wilson loops) are employed in the quantisation.

A central ingredient of the Hamiltonian approach is the abstract hypersurface deformation algebra \( \mathfrak{h} \). As shown in the seminal paper [8], every generally covariant Lagrangian (e.g. GR with any type of matter) has a singular Legendre transform, leading to initial value constraints whose Poisson algebra is a representation of \( \mathfrak{h} \). The algebra \( \mathfrak{h} \) is isomorphic to the Lie algebra of spacetime diffeomorphisms when the equations of motion (e.g. Einstein equations and Bianchi identities) hold. As an abstract algebra it can be defined as follows: Under the usual assumption of global hyperbolicity without which the initial value formulation is ill-posed, the spacetime manifold \( M \) is diffeomorphic to \( \mathbb{R} \times \sigma \). The freedom in choosing this diffeomorphism and thus setting up an
initial value formulation is encoded by a scalar (“lapse”) function $n \in N$ and a (“shift”) vector field $u \in U$ on $\sigma$ respectively which depend parametrically on the time coordinate $t$ that defines a foliation of $M$ into leaves $\Sigma_t$ which are all diffeomorphic to $\sigma$. Then the fundamental $\mathfrak{h}$ algebra reads (we absorb any constant such as Newton's constant into $C,D$)

$$\{D(u), D(v)\} = -D([u,v]), \quad \{D(u), C(n)\} = -C(u[N]), \quad \{C(m), C(n)\} = -D(q^{-1} [M \, dN - N \, dM]) \quad (1.1)$$

Here $[u,v]$ is the lie bracket of vector fields and $u[N]$ the action of $u$ considered as a derivation on the scalar functions. The first two relations in (1.1) therefore just depend on the differentiable structure on $\sigma$. The last relation of (1.1) however also depends on a metric tensor field $q$ on $\sigma$ whose inverse features into (1.1). It is therefore a prerequisite for the very definition of $\mathfrak{h}$ that $q$ be invertible everywhere on $\sigma$ and at all times. In GR the physical meaning of $q$ is of course to be the pullback of the spacetime metric $g$ to $\sigma$ (at the respective time). Because of this, (1.1) is strictly speaking not a Lie algebra because its right hand side cannot be expressed as linear combinations of $C,D$ with (so called structure) constant coefficients, rather those coefficients are themselves functions of the dynamical fields. It is therefore customary to call them structure functions rather than structure constants.

A classical representation of (1.1) as a Poisson bracket algebra is generated by a generally covariant Lagrangian such as GR with any matter coupling including a possible cosmological constant. In this case $C,D$ acquire the meaning of Hamiltonian and spatial diffeomorphism constraint respectively. Their Poisson brackets with the fields underlying the canonical formulation are equivalent to the Lagrangian equations of motion (e.g. Einstein equations and Bianchi identities). In quantum gravity, one is interested in a quantum representation of (1.1) by commutators of operators defined on a common (i.e. independent of $N,u$), dense and invariant domain $D$ of a Hilbert space $\mathcal{H}$ which is supposed to implement the canonical (anti-) commutation relations among the the fields (such as $q$ and its conjugate momentum $p$) such that (we absorb $\hbar$ into $C,D$)

$$[D(u), D(v)] = -i \, D([u,v]), \quad [D(u), C(n)] = -i \, C(u[N]), \quad [C(m), C(n)] = -i \, D(q^{-1} [M \, dN - N \, dM])$$

(1.2)

where $D(x)\, , \, C(x)\, ;\, x \in \sigma$ together with $q(x)$ and $q(x)^{-1}$ have become operator valued distributions, $C(n) = \int_{\sigma} dx n(x) \, C(x)$ and similar for $D(u)$. Therefore (1.2) is not completely defined in terms of $C,D$ alone but also requires in addition information about what should be done with the operator valued distribution $q^{-1}$, e.g. ordering issues have to be addressed. In particular, the appearance of the inverse of $q$ requires that both $q, q^{-1}$ are well defined on $D$. Moreover, the quantum Einstein or Wheeler-DeWitt equations [9] are the conditions on distributions (linear functionals) $l$ on $D$ such that

$$l[C(n) \psi] = l[D(u) \psi] = 0$$

(1.3)

for all $n \in N$, $u \in U \psi \in D$. Then the validity of (1.2), without anomalous terms not in the linear span of $C,D$, are “integrability conditions” in order that (1.3) holds. The solutions of (1.3) are in general not elements of $\mathcal{H}$ unless zero is in the joint pure point spectrum of all $C(n), D(n)$. To find an inner product on these “generalised zero eigenvectors” is then an additional task that one has to carry out. If (1.2) was a true Lie algebra one could use the theory of rigged Hilbert spaces [11]. Since it is not, one has to resort to different methods, for example [12] which replaces the set of all $C(n), D(u)$ by a single (“master”) constraint $M$ to which the theory of rigged Hilbert spaces may then be applied.

This rough sketch of the general canonical quantisation programme now must be implemented concretely. This is what has been done in LQG. Inspired by the fact that fermionic matter requires to work with Vielbeins and connections rather than metrics, a connection formulation [6] has been introduced. A rigorous Hilbert space representation of the CCR for geometry [12] and the matter CCR and CAR [13] has been defined which is designed to formulate the quantum dynamics non-perturbatively given the perturbative non-renormalisability of quantum gravity and thus is not one of the standard Fock representations. The difference between the LQG representation and the standard Fock representation can be seen for instance in the fact that the former is non not separable, that only finite but no infinitesimal unitary quantum diffeomorphisms $U(\varphi)$ can be defined [14] (technically, 1 parameter groups $s \mapsto \varphi^u_s$ of diffeomorphisms are not strongly continuous), that area and volume operators [15] can be defined at all and that their spectra are pure point and that connections themselves are ill defined while
their holonomies are bounded operators. That these geometrical operators and their generalisations such as triads [10] can be defined in the LQG representation turns out to be very important in order to define the quantum dynamics, i.e. the operators $C(n)$ [17], because literally all geometry and matter matter contributions including the cosmological constant depend on them.

The definition [17] of the operators $c(n)$ on the dense domain $D$ given by the span of an ONB of $H$ known as spin network functions (SNWF) [18] involves two steps: First a regularisation introducing a UV regulator $\epsilon$ necessary in order that the connections involved in $c(n)$ are replaced by holonomies which yields operators $c_\epsilon(n)$. Secondly, the limit $\epsilon \to 0$ is taken in an operator topology which makes use of the existence of a space diffeomorphism invariant distributions $L$ on $D$ proved in [14] resulting in regulator free operators $c(n)$. More in detail

$$c_\epsilon(n) \to c(n) \iff l[c_\epsilon(n)\psi] \to l[c(n)\psi] \ \forall \ l \in L, \ \psi \in D \ \ (1.4)$$

It should be stressed that these operators are still defined on $D$, that their commutator does not vanish but that their commutator annihilates $L$ and that its commutator with finite diffeomorphisms $U(\varphi)$ yields another operator $c(n,\varphi)$ up to a diffeomorphism. Stricly speaking therefore (1.2) is not implemented, it cannot, as the operator $D(u)$ does not exist. However, (1.2) is replaced by

$$U(\varphi) U(\varphi') = U(\varphi \circ \varphi'), \quad U(\varphi) C(n) U(\varphi)^{-1} = U(\phi_{\varphi,n}) C(\varphi^* n)$$

$$[C(m), C(n)] = -i \sum_\varphi [U(\varphi) - 1_H], \quad F'(m,n,\varphi) \ \ (1.5)$$

Here $\phi_{\varphi,n}$ is a diffeomorphism that depends on both $\varphi, n$ and $F'(m,n,\varphi) = -F'(n,m,\varphi)$ are “structure operators”. The sum is formally over all diffeomorphisms but the structure operators vanish except for finitely many when acting on $D$. The algebra (1.5) is therefore consistent in the sense that the joint kernel of the set of commutators of its generators is contained in the joint kernel of the set of its generators which is a minimal requirement, we call it mathematical anomaly freeness in what follows.

At first this appears to be as close as one may hope to get that $D(u)$ is not at one’s disposal and that (1.5) is a suitable substitute for (1.2) under these circumstances. In fact, the optimum that one could hope for would be

$$U(\varphi) U(\varphi') = U(\varphi \circ \varphi'), \quad U(\varphi) C(n) U(\varphi)^{-1} = C(\varphi^* n), \quad [C(m), C(n)] = -i \sum_\varphi F(m,n,\varphi) [U(\varphi) - 1_H], \quad (1.6)$$

where the “structure operators” $F(m,n,\varphi)$ qualify as quantisations of the structure functions $q^{-1}(m \ dn - n \ dm)$. The actual situation (1.5) agrees with the wish list (1.6) as far as the first relation is concerned, they agree with respect to the second up to a diffeomorphism but the last relation in (1.5) is manifestly violated: The $F'$ do not qualify as a possible realisation of the $F$. To see this one quantises the operator corresponding to $D(q^{-1}(mdn - ndm)$ independently [19] and notices that the structure operators $F,F'$ differ. To bring them to match, the second action by $c(n)$ would need to be non-trivial at the new excitations of a SNWF that a first action by $c(n)$ has created. However, due to the properties of the volume operator used in the quantisation of $c(n)$ in [17] that second action is trivial. Changing that property in the current set-up would create even a mathematical anomaly.

One can summarise the situation therefore by saying that [17] is free of mathematical anomalies but that it displays a physical anomaly in the sense just described, namely that the algebra of generators closes under taking commutators, however with the wrong structure functions. This is of course inacceptable: Using a crude ananalogy from the theory of finite dimensional Lie algebras, suppose that one were to quantise the Lie algebra $A$ but ended up with a representation of the Lie algebra $B$ of the same dimension. Then one would wrongly define physical states to be those that are invariant under the symmetries (gauge transformations) generated by $B$ rather than those of $A$ (consider the examples of $A = so(1,3)$ and $B = so(4)$).

To improve the situation one must obviously modify $c(n)$ within the LQG representation or, more drastically, change the LQG representation and start from the beginning. In this paper we focus on efforts that do not change
the LQG representation but mention that currently Hamiltonian renormalisation methods are being developed \[20\] which potentially will change the LQG representation. These non representation changing techniques are based on so-called “habitats” \[21\]. A habitat is a space of distributions \(L'\) on \(\mathcal{D}\) which is different from the space \(L\) of diffeomorphism invariant distributions. One tries to define a dual representation \(C'(n)\) of \(C(n)\) on \(L'\) by taking

\[
[C'(n)L'](\psi) := \lim_{\epsilon \to 0} L'[C_n(\psi)](\epsilon)
\]

for all \(\psi \in \mathcal{D}\). Provided the limit exists (in the sense of complex numbers) it defines a new distribution on \(\mathcal{D}\) and if the space \(L'\) is carefully chosen, then \(L'\) is invariant under the \(C'(n)\). In that case one can take commutators and compute the algebra of dual operators.

Note the difference between \(1.4\) and \(1.7\): In \(1.4\) the space \(L\) is used to define a topology on the unbounded operators on \(\mathcal{H}\) with common, dense, invariant domain \(\mathcal{D}\) of the Hilbert space \(\mathcal{H}\) while in \(1.7\) one defines a new space of operators on a different space \(L'\) which however is not Hilbert space, it does not come with an inner product. Therefore, the algebra of the \(c(n)\) is purely algebraic and is not equipped with any obvious topology while the \(c(n)\) come equipped with any of the topologies that descend from \(\mathcal{H}\) and \(\mathcal{D}\). Note also that by definition \(c(n)\) does not commute with diffeomorphisms and thus \(L\) itself does not qualify as an invariant space \(L'\) while it maybe a subspace thereof.

In \[21\] such an invariant space \(L'\) was found and it was shown that the algebra of the \(C'(n)\) is Abelian on this \(L'\). Unfortunately, the difference between the \(C(n)\) and the \(C'(n)\) has been confused repeatedly in the literature \[22\] and has wrongly lead to the statement that the \(C(n)\) only close “on shell” (i.e. on the space \(L\) annihilated by the operators \(U(\varphi) - 1_{\mathcal{H}}\)) and commute. Not only is this mathematically impossible because the \(C(n)\) or rather their duals \(C'(n)\) do not preserve \(L\) but it is also technically wrong as the \(C(n)\) are defined densely on \(\mathcal{D}\), hence they are defined manifestly “off shell” and there their commutator does not vanish. See the extensive discussion in the second book of \[5\] and the recent review \[20\].

To work with non-trivial habitats appears to be attractive because in the topology defined by \(1.4\) the resulting limit operators \(C(n)\) carry a large amount of quantisation ambiguities. While the joint kernel of the \(C(n)\), \(U(\varphi) - 1_{\mathcal{H}}\) that one is eventually interested in is insensitive to most of these ambiguities it maybe hoped for that the \(C'(n)\) have less ambiguities. Therefore an ambiguous programme has been recently launched \[23\] \[24\] which also aims to find a non-trivial representation of the algebra generated by the \(C'(n)\) such that \([C'(m), C'(n)]\) has no physical anomaly. A common feature of these developments is that \(C'(n)\) is replaced by \(\tilde{C}'(n)\) corresponding to quantisations (on certain dual spaces \(L'\)) of classical \(\tilde{C}(n)\) where \(\tilde{C}\) has non-standard density weight. This non-standard density weight is absolutely crucial in order to avoid the Abelian character of the algebra of the \(C'(n)\).

It was shown in \[23\] that non-trivial choices of \(L'\) exist which make the algebra of the \(\tilde{C}'(n)\) for Euclidian signature vacuum GR without cosmological constant free of physical anomalies. This is a most astonishing and non-trivial result. It triggers the following questions:

1. The detailed action of \(\tilde{C}(n)\) at finite regulator and the choice of \(L'\) have to be matched carefully to each other in order that the regulator can be removed for the dual operator \(\tilde{C}'(N)\) on \(L'\) and its action be non-anomalous. This raises the question how many suitable such choices do exist and what the residual amount of ambiguity is.

2. While for density weight unity the norm of the states in the image of the operator \(C(N)\) converge as the regulator is removed, for the non standard density operator \(\tilde{C}(N)\) that limit is divergent. This is the price to pay in order that the commutator of the \(\tilde{C}'(N)\) be non-Abelian on the chosen \(L'\). While one may argue that the \(\tilde{C}'(N)\) are “more fundamental” than the \(\tilde{C}(N)\) one may wonder whether one cannot have natural density weight unity without Abelian \(C'(N)\) on suitable different \(L'\).

3. The non-standard density weight does not allow to switch to physical Lorentzian signature, or admit a non-vanishing cosmological constant or non-trivial matter coupling as also stated in \[17\] \[23\]. We will review the reasons for this in section \[2\]. Thus one presumably needs to combine \[17\] \[23\] in some non-trivial way and resort to density weight one.
Note that the whole strategy of solving the constraints in quantum theory rather than classically (gauge fixing) maybe criticised as being beyond practicability for complicated theories such as GR: After all, the quantum constraints have to be regularised and densely defined, the regulator removed, solutions to be found, the solutions to be equipped with a new inner product, observables to be defined on that new physical Hilbert space. All of that can be avoided using gauge fixing the classical theory and one works directly just with observables and the physical Hilbert space. That gauge fixings are usually plagued by global issues (Gribov copies) appears to be higher order problem given the immense difficulties in solving the constraints in quantum theory. Yet, the concerns about anomaly freeness of the constraints can not entirely be ignored in such a reduced phase space approach. This is because pieces of the constraints are building blocks of the physical Hamiltonian that drives the physical time evolution of observables and in that sense their quantisation ambiguities reappear, it is just that the anomaly issue is absent. On the other hand, typically gauge fixing conditions are phrased as coordinate conditions on scalar configuration degrees of freedom $q$ and one then solves the constraints for the respective conjugate momenta $p = -\hbar$ which are scalar densities of weight one. Thus we see that even in the gauge fixed theory density weight unity is the natural choice. See [26] for natural implementations of gauge fixing and the naturality of the density weight one choice.

In this article we wish to complement the debate about an anomaly free implementation of $\mathfrak{h}$ by communicating the following observations:

1. That, with minimal physical assumptions, the standard density weight of [17] is the only viable one.
2. That the algebra of the $C'(n)$ of [21] is Abelian on the chosen $L'$ is physically correct.
3. That the reason for this Abelian character is the neglectance of the implicit assumption about $\mathfrak{h}$, namely that $q$ be invertible.
4. That a non-Abelian algebra of the $C'(n)$ on suitable $L'$ and maybe even of the $C(n)$ on suitable $D$ maybe possible with standard density weight on states on which $q$ is invertible.

The presumptions expressed in item [4.] rest on section 5 of the present paper and on our companion paper [25] where it is shown that when quantum non-degeneracy is taken into account the apparent tension between density unity and non-trivial quantum $\mathfrak{h}$ disappears at least in those theories.

The architecture of this article is as follows:

In section 2 we show, independent of the arguments of [12], why an LQG like representation in quantum gravity is dynamically preferred. We also repeat from [17] why density weight one of $C$ is the only viable choice under physically well motivated assumptions and using again dynamical input.

In section 3 we show why current calculations of the algebra of the $C(n)$ or $C'(n)$ are inconclusive and the result of the computations of [21] are not at all surprising: Current calculations are performed in sectors of $\mathcal{H}$ or spaces of distributions $L'$ which do not qualify as quantisations of classical geometries in which the classical algebra $\mathfrak{h}$ is defined. This is because current calculations are performed in regimes with quantum geometries that are degenerate almost everywhere. In other words, we promote the point of view that quantum geometrical non-degeneracy be a central ingredient in the very definition of $\mathfrak{h}$. In doing so, the Abelian character of the dual action on certain spaces of distributions [21] may disappear by itself, without changing the density weight, thereby avoiding the issues mentioned above. Also, many of the results of [17] can probably be transferred to the non-degenerate sector.

In section 4 we show how the non-degeneracy condition can potentially be taken care of in the Hamiltonian renormalisation of LQG [20]. This programme is still its infancy. In a preliminary calculation we consider a certain set of coherent non-degenerate states based on [27] and compute expectation values of the commutators $[C(m), C(n)]$ with $C(m)$ chosen as in [28]. Exploiting the freedom in the choice of these states one can get the expectation value of the difference between the commutator and the quantisation of the Poisson bracket as small as one wishes.

In section 5 we consider the toy model of parametrised field theory [29]. It has been quantised by LQG methods before and it was demonstrated to have a non-anomalous algebra with non-standard density weight on
some \( L' \) as above and standard density weight on \( \mathcal{H} \) if an addition a renormalisation step is invoked. Here we employ a new LQG like representation which is better geared towards quantum non-degeneracy and we show that \( C'(n) \), \( D'(u) \) with standard density weight can be represented on a certain space \( L' \) without anomalies (but including the central charge of the Virasoro algebra).

In section [5] we summarise and conclude.

In closing, we stress that this is mostly conceptual work. We deliberatively neglect many technical details in order not to draw attention away from the main line of argument. However, all the technical details can be found in the original articles quoted along with.

## 2 CCR and CAR representations of LQG type

In any QFT is quite important that one finds a ground state of the corresponding Hamiltonian in order that the dynamics defined by it can be constructed. For instance, choosing a Fock representation not precisely geared to the Hamiltonian \( H \) of a free Klein Gordon field of mass \( M \) makes the quantum dynamics ill defined. If one considers the Hamiltonian constraints \( C(n) \) of GR (with or without matter) and looks for a representation of the CCR and CAR on a Hilbert space such that \( C(n) \) be densely defined on a suitable common invariant domain \( \mathcal{D} \) thereof it is well motivated to try to find a representation based on a cyclic vector \( \Omega \) whose excitations create \( \mathcal{D} \). A peculiarity of \( C(n) \) is that every single piece of it depends non-trivially on the induced metric \( q \) and/or its inverse (bosons) or the co-triad \( e \) (square root of \( q \)) or its inverse. Accordingly, one has good chances to have a ground state of the \( C(n) \) at one’s disposal if one manages to build a representation based on a cyclic state \( \Omega \) annihilated by \( e \) or suitable aggregates formed from it. In LQG one considers the variables

\[
E^a_j := \sqrt{\det(q)} \ e^a_j, \quad q_{ab} = \delta_{jk} \ e^j_a \ e^k_b, \quad e^a_j \ e^k_a = \delta^k_j
\]  

(2.1)

where \( a, b, c, \ldots = 1, 2, 3 \) are tensor indices w.r.t. \( \sigma \) while \( a, b, c, \ldots = j, k, l \) are tensor indices w.r.t. \( \text{su}(2) \). Canonically conjugate to \( E^a_j \) is a \( \text{su}(2) \) connection \( A^a_j \) which captures information about the extrinsic curvature of the Cauchy surfaces

\[
\{ E^a_j (x), A^b_k (y) \} = \delta^a_b \ \delta^k_j \ \delta(x, y)
\]  

(2.2)

where we have set the Newton constant equal to unity.

To find a representation of the CCR corresponding to (2.2) we consider the Weyl algebra defined by the Weyl elements

\[
W(f, F) := \exp(i \ [ < E, f > + < F, A > ]), \quad < E, f > := \int d^3 x \ E^a_j (x) f^j_a (x), \quad < F, A > := \int d^3 x \ F^a_j (x) A^j_a (x), \quad (2.3)
\]

where we leave the nature of the test functions \( f, F \) unspecified for the moment. Then we define \( \mathcal{D} \) as the linear span of the \( W(f, F)\Omega \) where \( \Omega \) is annihilated by \( E \) as motivated above, that is

\[
W(f, 0)\Omega = \Omega
\]  

(2.4)

It then follows immediately from the Weyl relations

\[
< \Omega, W(0, F) \Omega >_{\mathcal{H}} = < \Omega, W(f, 0) W(0, F) W(f, 0)^{-1} \Omega >_{\mathcal{H}} = \exp(-i < F, f >) < \Omega, W(0, F) \Omega >_{\mathcal{H}}
\]  

(2.5)

for any \( F, f \) therefore automatically

\[
< \Omega, W(0, F) \Omega >_{\mathcal{H}} = \delta_{F, 0}
\]  

(2.6)

where \( \delta_{F, 0} \) is the Kronecker \( \delta \). Hence the algebraic structure of \( c(n) \) leads in a few lines to a representation of the Narnhofer-Thirring type [32]. The Hilbert space is the completion of the span \( \mathcal{D} \) of the \( W(0, F)\Omega \). The Weyl elements \( W(f, 0) \) act continuously, in fact diagonally, on those by multiplication by \( \exp(-i < F, f >) \) while the \( W(0, G) \) act discontinuously by shifting \( F \) to \( F + G \). In fact, by Stone’s theorem, also \( < E, f > \) is well defined and acts by multiplication by \( - < F, f > \). The Hilbert space is not separable whenever the set of admitted \( F \) is not countable.
The representation theory of (2.2) is really as simple as that as soon as we agree that \( \Omega \) be annihilated by \( E \), with no conditions on \( f, F \) except that \( < F, f > \) should be a well defined number. This still allows \( f, F \) to be distributions on \( \sigma \) subject to the condition that their singularity structure is weighted in such a way that \( < F, f > \) be well defined. We now show that the algebraic form of the Hamiltonian constraints uniquely dictates 1. the smearing dimensions of \( A, E \), 2. the density weight of the Hamiltonian constraint and 3. in any dimension. For the formulation of quantum gravity in connection variables for all dimensions see [33].

**Theorem 2.1.**
Consider quantum gravity in \( D + 1 \) spacetime dimensions in a representation of Narnhofer-Thirring type as above. Suppose that there is at least a cosmological constant in addition to the vacuum contribution. Then the smearing dimension of \( A, E \) are \( 1, D-1 \) respectively and the density weight of the Hamiltonian constraint must be unity.

**Proof.**
In \( D \) spatial dimensions we have with \( E_j^a = \sqrt{\det(q)} e_j^a \) that \( |\det(E)| = \det(q)^{(D-1)/2} \). Therefore the cosmological term of the density weight \( w \) Hamiltonian constraint is given by

\[
\Lambda \int_\sigma d^Dx \, n(x) \, |\det(E)|^{w/(D-1)}(x)
\]

while the kinetic term of the vacuum contribution contains the term

\[
\int_\sigma d^Dx \, n(x) \, \text{Tr}([A \cdot E] \, [A \cdot E]) \, |\det(E)|^{(w-2)/(D-1)}
\]

where \( A \cdot E \) denotes some contraction of tensorial and Lie algebra indices which depend on \( D \) and which is not important for the proof.

Now \( E \) is diagonal on the \( W[(0, F)] \) with eigenvalue \( F \). Thus in (2.8) the action of the operator produces in particular the term \( \text{Tr}([A \cdot F] \, [A \cdot F]) \) which we have to quantise e.g. in terms of \( \sin(< F \Box, A >) \) in a Riemann sum regularisation of the integral by summing over cells \( \Box \) of coordinate volume \( \epsilon \) and where \( F \Box \) is the restriction of \( F \) to such a cell. This means that for such a cell the contribution of (2.8) reads schematically

\[
n(p) \, \text{Tr}([\sin(< F \Box, A >)] \, [\sin(< F \Box, A >)]) \, \epsilon^{-D} \, |\det(F)|^{(w-2)/(D-1)}
\]

where \( p \) is the center of the cell.

Thus if \( A \) is smeared in \( k = 0, 1, .., D \) dimensional submanifolds, then \( F \) contains \( D - k \) \( \delta \) distributions and correspondingly \( \det(F) \) will contain \( D(D-k) \) \( \delta \) distributions, isotropically wrt direction dependence, that is, it will contain \( D - k \) \( \delta \) distributions in each coordinate direction. From (2.9) we see that \( k = D \) is not possible because then the limit \( \epsilon \to 0 \) would be singular, rather the \( D \) factors of \( \epsilon \) in the denominator must turn into something finite upon replacing the denominator by an integral

\[
\int_\Box d^Dx \, |\det(F)|^{(2-w)/(D-1)}(x)
\]

and the integrand must have the singularity structure of the \( \delta \) distribution in \( D \) dimensions in order to remain finite as \( \epsilon \to 0 \).

Likewise the action of (2.7) also just replaces \( E \) by \( F \). In order that this term be also finite we obviously must have equal powers of the determinant

\[
\frac{w}{D-1} = \frac{2-w}{D-1} \implies w = 1
\]

and in order to produce the \( \delta \) distribution in \( D \) dimensions we must have

\[
(D-k)\frac{w}{D-1} = 1 \implies k = 1
\]
It follows that for $D = 3$ the functions $F, f$ smear $A, E$ effectively in 1 and 2 dimensions respectively, i.e. they are concentrated on 1 and 2 dimensional submanifolds respectively (that is curves $c$ and surfaces $S$ respectively). If one wants in addition that $W(0, F), W(0, f)$ transform covariantly under SU(2) gauge transformations then one considers instead of $W(F, 0)$ holonomies $H(c)$ of $A$ along curves $c$ and instead of $W(0, f)$ fluxes $\Phi_f(S)$ of $\text{Tr}(E_f)$ through surfaces $S$ where $f$ is an su(2) valued function on $S$. In this way one arrives naturally at the holonomy flux algebra and its LQG representation [12] using mostly dynamical input.

The discussion reveals that choosing density weight $w \neq 1$ while keeping smearing dimensions 1, 2 for $F, f$ as is done in [23] makes both terms (2.7) and (2.8) formally diverge for $w > 1$ and trivial for $w < 1$ in the limit $\epsilon \to 0$. For the cosmological term that can be rigorously shown e.g. in the weak operator topology of the LQG representation because the operators $\Phi_f(S)$ and not only their exponentials exist. For (2.8) this argument cannot be made because while one can replace (2.8) by a Riemann sum approximant of the structural form

$$e^{-3(w-1)} \sum_v n(v) \text{Tr}([H_\epsilon(v) - H_\epsilon(v)^{-1}]^2 \Phi_\epsilon(v))^2 | \det(\Phi_\epsilon(v))|^{w/2-1}$$

(2.13)

where $H_\epsilon(v), \Phi_\epsilon(v)$ denote holonomies and fluxes respectively localised in cubes of coordinate volume $\epsilon^3$ and centre $v \in \sigma$, the limit $\epsilon \to 0$ does not exist say in the weak operator topology because the holonomies are not weakly continuous in the LQG representation. In [23] one picks the non-standard density weight $w = 4/3$ which yields a prefactor $\epsilon^{-1}$ in front of the sum in (2.13). We will see the motivation for doing this in the subsequent sections.

For the time being, we note that for $w = 1$ a term like (2.13) does converge in the following non-standard operator topology: Let $l \in L$ a diffeomorphism invariant distribution on $D$ the linear span of the $W(0, F)\Omega$ or the Pol($\{H\})\Omega$ where Pol denotes polynomials of holonomies. Let $\psi \in D$. Then the operator $O_\epsilon(n)$ corresponding to (2.13) can be defined and evaluated on $\psi \in D$ [17]. Then $O_\epsilon(n)$ converges to the operator $O(n)$ densely defined on $D$ if

$$\lim_{\epsilon \to 0} l[(O_\epsilon(n) - O(n)) \psi] = 0 \forall \psi \in D, \ l \in L$$

(2.14)

One finds that the limit is trivial and one may pick any fixed $\epsilon = \epsilon_0$ and set $O(n) = O_{\epsilon_0}(n)$. The reason why this works is because $w = 1$: This makes the whole construction diffeomorphism covariant and changing $\epsilon$ can be absorbed into a diffeomorphism to which $l$ is insensitive. This does not work for any other density weight.

We close this section by mentioning that a similar argument as above applies for all the matter content of the standard model and uniquely fixes the smearing dimensions whenever one of the members of the canonical pair annihilates the vacuum $\Omega$, see [13, 17] for details.

3 Non-degenerate states and density weight

To understand the apparent tension between the natural density one weight of $C(x)$ and proper representation of of the hypersurface deformation algebra $\mathfrak{h}$ in the LQG representation and its relation to the non-degeneracy condition, it is necessary to go into more details. The rest of the paper considers the case $D = 3$ only.

As outlined in the previous section, it is well motivated to work in a representation in which the vacuum is annihilated by the 2d smeared operator corresponding to $E_f^\dagger$. The corresponding LQG Hilbert space is equipped with an ONB known as SNWF. These are certain polynomial functions of an arbitrary set of holonomies along 1d oriented curves called edges $e$ that intersect nowhere except in their endpoints called vertices $v$. The edges are labelled by half integral spin quantum numbers $j_e$ while the vertices are labelled by intertwiners $t_v$ between the corresponding irreducible representations meeting at $v$. Hence a SNWF $T_s$ is labelled by a spin network $s = \{\gamma, j, t\}$ where $\gamma$ is a finite, oriented graph and $j, t$ are the collection of $j_e, t_v$.

The classical function $C$ on the phase space coordinatised by $A, E$ is polynomial in $A$ but not a polynomial in $E$. Rather it also depends on integer inverse powers of $|\det(E)|^{1/2}$. It is a non-trivial result in LQG that the integral of $|\det(E)|^{1/2}$ over 3d submanifolds $R$ is a well defined, in fact essentially s.a. operator known as the volume operator $V(R)$ whose dense domain is given by the span $D$ of the SNWF. We use here as in [17] the version of $V(R)$ due to Ashtekar and Lewandowski because only with this version the operator $C(n)$ defined
below annihilates vertices with co-planar tangents of adjacent edges and only this operator passes the triad test.\[10\], i.e. it implements the classical non-polynomial identity

$$E^a_j(x) = \frac{1}{2} \epsilon^{abc} \epsilon_{jkl} \text{sgn}(\det(V(R), A(x))) V(R), A_j^a(x) \{V(R), A_j^k(x)\}$$ \hspace{1cm} (3.1)

for any \(x \in R\). If co-planar vertices are not annihilated then \(C(n)\) is not densely defined on \(\mathcal{D}\).

To define inverse powers of \(|\det(E)|^{1/2}\) one now uses classical approximate identities of the form \[17\]

$$q^3 V(R)^{1-3(1-q)} \approx \int_R d^3x \det(V(R), A(x))$$ \hspace{1cm} (3.2)

for \(0 < q < 2/3\) to obtain \(V(R)^{-p}\), \(0 < p < 2\) for small regions \(R\) of coordinate volume \(\epsilon^3\) and approximates the integral in \((3.2)\) in terms of Poisson brackets with three holonomies. This rewrites negative powers of \(V(R)\) in terms of positive powers and commutators with holonomies and the latter operators are well defined in the LQG representation.

To tame the highly non-polynomial expressions that appear in \(C\) in its quantisation one proceeds as follows \[17\] (for illustrational purposes we consider here only the Euclidian part of the Lorentzian constraint and only the geometry contribution, Lorentzian part, cosmological constant and matter part can be treated by similar methods \[17\]): We partition the manifold into cells \(\square_p\) of coordinate volume \(\epsilon^3\) and centre \(p\) and approximates the integral involved in \(C(n)\) as the Riemann sum \(\sum_p n(p) C(\square_p)\) where \(C(\square_p)\) is the integral of \(C(x)\) over \(\square_p\). Then one approximates the "electrical part" of integrand of \(C(\square_p)\) in terms of quantities of the form \((3.2)\) with \(R = \square_p\) and \(A\) replaced by holonomies along edges of coordinate length \(\epsilon\) and the "magnetic part" in terms of holonomies along loops enclosing a surface of coordinate area \(\epsilon^2\). When one now acts on a SNWF \(T_\gamma\) over a graph \(\gamma\), due to the properties of the volume operator mentioned, for sufficiently small \(\epsilon\) one finds \(C(\square_p) T_\gamma = 0\) unless \(\square_p\) contains a vertex \(v \in V\) where \(V\) is the vertex set of \(\gamma\). This is because \(V(R)\) acts non-trivially only at at least 4-valent non-co-planar gauge invariant vertices \(v\) or at least 3-valent non-coplanar non gauge invariant vertices \(v\) and only if \(v \in R\).

One further approximates \(p = v\) for such \(p\) with \(v \in \square_p\), and chooses the edges involved in the electric part to be beginning segments \(s(e)\) of edges \(e \in E\) adjacent to \(v\) and the loops involved in the magnetic part along pairs \(s'(e), s'(e')\) with \(e \neq e'\) adjacent to \(v\) connected by an "arc" \(a_{v,e,e'}\) so that a loop \(a_{v,e,e'}^{\epsilon} = s'(e) a_{v,e,e'}^{\epsilon} s'(e')^{-1}\) results with \(\alpha_{v,e,e'}^{\epsilon} = [\alpha_{v,e,e'}^{\epsilon}]^{-1}\) which does not intersect \(\gamma\) except at interior points of \(e, e'\). Then the final regulated operator acts on SNWF as

$$C^\epsilon(n) T_\gamma = \sum_v n(v) \sum_{e \cap e' = v} C_{v,e,e'}^{\epsilon} T_\gamma$$ \hspace{1cm} (3.3)

In the non-standard topology mentioned above, we may choose for each \(\gamma\) a sufficiently small \(\epsilon(\gamma)\) meeting the conditions used in the derivation of \((3.3)\) and then the limit \(\epsilon \rightarrow 0\) becomes trivial and amounts to define the limit operator by

$$C(n) T_\gamma = \sum_v n(v) \sum_{e \cap e' = v} C_{v,e,e'} T_\gamma, \quad C_{v,e,e'} := C_{v,e,e'}^{\epsilon(\gamma)}$$ \hspace{1cm} (3.4)

Since \(\epsilon(\gamma)\) is quite arbitrary, \((3.5)\) suffers from that arbitrariness, however, when looking for solutions of the constraints \(C(n), U(\varphi) - 1\), that is, distributions which vanish on \(C(n) T_\gamma, [U(\varphi) - 1] T_\gamma\) for all \(n, \varphi, T_\gamma\) most of these ambiguities are washed away because \((3.5)\) is diffeomorphism covariant: Because the density weight is unity, by construction operators resultiung from different choices of \(\epsilon(\gamma)\) are related by a diffeomorphism so that the joint kernel is the same. Differences can only occur from different diffeomorphism equivalence classes of the loops \(a_{v,e,e'}\).

We can now proceed to compute the commutator of the \([C(m), C(n)]\). One finds

$$\frac{1}{2} \sum_{v \neq v'} [m(v) n(v') - n(v) m(v')] \sum_{e \cap e' = v} \sum_{f \cap f' = v'} [1 - U(\varphi_{v,e,e'; v',f,f'})] C_{\gamma_{v,e,e'; v',f,f'}, \gamma_{v',f,f'}; T_\gamma}$$ \hspace{1cm} (3.5)

where, again due to the properties of the volume operator, the sums only involve vertices and edges of \(\gamma\) and not the new edges and vertices generated by the arcs. The second graph is \(\gamma_{v',f,f'} = \gamma \cup a_{v',f,f'}\) and the diffeomorphism
displayed preserves $\gamma$ but takes care of the fact that the second action depends on whether one first acted at $v$ or $v'$. It was also used that the two $C$ expressions displayed commute for $v \neq v'$.

On the other hand, the direct quantisation of the classical $K(m, n) = \{C(m), C(n)\}$ leads to an operator with similar action and properties. It is given by [19]

$$K(m, n) = \sum_v \sum_{\epsilon \mathcal{W} = v} [m(v)n(v + s(\epsilon)) - n(v)m(v + s(\epsilon))][U(\varphi_{\gamma, v, e} - 1)|Q^{ee'}T_{\gamma} \quad (3.6)$$

where $s(\epsilon) = s'(\gamma')(\epsilon)$ is again the segment of $\epsilon$ beginning at $v$ and $Q^{ee'}$ is a geometrical operator which quantises $q^{-1}$ by the methods above independent of the choice $\epsilon(\gamma)$ and $\varphi_{\gamma, v, e} = \varphi_{\gamma', v, e}'$ is a diffeomorphism that acts non-trivially only in the $\epsilon$ cube. Also this operator contributes only at the vertices of $\gamma$. One could be content with this if the diffeomorphisms involved in (3.5) and (3.6) would depend on the same data and if $Q^{ee'}$ could be related to the two $C$ expressions in (3.5). This, however, not possible because the vertices $v, v'$ belong to $\gamma$ while the vertex $v + s(\epsilon)$ does not and because the two $C$ expressions mutually commute and thus are not able to produce an operator resembling $Q^{ee'}$ via a commutator. In order to improve this the second action of $C$ would need to involve also the new vertices that the first action creates via the arcs but this leads to a mathematical anomaly. This is in more detail what we meant by closure with the wrong structure functions, while both (3.5) and (3.6) annihilate the space $L$ of diffeomorphism invariant distributions.

We now switch to the framework of [21] where one tries to take the limit $\epsilon \to 0$ using dual action of $C^c(n)$ on a subspace $L'$ of distributions on $D$. One picks the $l \in L'$ of the general form

$$l = \sum_{s'} l(s') < T_{s'}, \cdot \rangle \quad (3.7)$$

where the sum is over all SNW labels $s'$ with certain coefficients $l(s')$. Then

$$l[C^c(n)T_{\gamma}] = \sum_v n(v) \sum_{e \mathcal{W} = v} \sum_{s' S_{s'}^{(\gamma, v, e, e')} C_{\gamma, v, e, e'} T_{s'} > (3.8)$$

where $S_{s'}^{(\gamma, v, e, e)}$ is the set of all SNW into which $C_{\gamma, v, e, e'} T_{s'}$ decomposes which are finitely many.

The observation is now the same that makes the non standard operator topology limit work: The matrix elements $< T_{s'}, C_{\gamma, v, e, e'} T_{s} >$ do not depend on $\epsilon$ because the inner product is diffeomorphism invariant (the diffeomorphisms act unitarily). Therefore, the only $\epsilon$ dependence in (3.8) rests in the coefficient $l(s')$. If we choose $L'$ to consist of those $l$ such that $l(s')$ is a continuous functional of $s' = (\gamma', j', v')$ with respect to its graph entry $\gamma'$ (e.g. a function of edge length with respect to a background metric) then the limit $\epsilon \to 0$ can be carried out. Setting $l_{\gamma}(T_{s}) := \lim_{\epsilon} l[C^c(n)T_{\gamma}]$ then a dual operator $C'(n) l := l_{\gamma}$ is defined. It is now easy to see that $[C'(m), C'(n)] = 0$ because the commutator depends on different vertices and $L'$ consists of jointly continuous functions. One can even generalise this to more general actions of $C(n)$ which also involve the new vertices created [21]. Note that $L'$ contains $L$ (for which $l(s)$ is the constant function for $s$ in the same diffeomorphism equivalence class).

For the same choice of $L'$ we can construct the dual action of (3.6) in the limit $\epsilon \to 0$. The result trivially vanishes if the lapse functions $m, n$ are continuous by the same argument. Thus we get consistently $[C'(m), C'(n)] = K'(m, n) = 0$.

This latter observation motivates the non-standard density weight: Similar to [23] consider $w = \frac{7}{6}$. It leads to an additional $\epsilon^{-1}$ factor in front of an expression the limit of whose action on $L'$ is finite, see (2.13). Classically we have for $w = 7/6$ that $C_w = C \sqrt{(\det(q))^{1/4}}$ and by the relations that $\{C_w(m), C_w(n)\} = -D(q^{-1}(mdn - ndm)(\det(q))^{1/6}) =: K_w(m, n)$ so that $K_w(m, n)$ can be approximated by an expression of the symbolic form

$$\sum_v \text{Tr}([H_\epsilon(v) - H_\epsilon(v)^{-1}]^2 \Phi_\epsilon(v)^3) \frac{m(v)n(v + \epsilon) - n(v)m(v + \epsilon)}{\epsilon} |\det(\Phi_\epsilon(v))|^{-5/6} \quad (3.9)$$
which can be quantised using (3.2) and above Riemann sum techniques on $D$. Now to define $K'_{w}(m, n)$ on some $L'$ will involve the limit

$$
\lim_{\epsilon \to 0} \frac{m(v)n(v + s^{t}(e)) - n(v)m(v + s^{t}(e))}{\epsilon} = \epsilon^{a}(0) \left[ m \partial_{a}n - n \partial_{a}m \right](v)
$$

(3.10)

where $t \mapsto e(t)$, $e(0) = v$ denotes the parametrisation of $e$ and thus produces exactly the combination of lapse functions that one has in the classical theory. The challenge then consists in quantising $C'_{w}(m)$ for $w = 7/6$ in such a way that $[C'_{w}(m), C'_{w}(n)] = i K'_{w}(m, n)$ on suitable $L'$ which involves letting $C'_{w}(m)$ act on the new vertices it creates and to interpret combinations such as $\hat{u}_{n,j} := nE^{a}_{j} \left[ \det(E) \right]^{-5/12}$ as three phase space dependent shift vector fields called electric shifts which motivates to let $\hat{C}'_{w}$ act on SNWF in a similar way as a linear combination of finite diffeomorphism operators would do, but with operator valued coefficients. The associated deformations caused by these diffeomorphisms have to be chosen carefully in order that the $\epsilon^{-1}$ factor in (2.13) causes no singularity. The choice of $L'$ needs some form of analytic structure which is fed in by using dependence of $l(s')$ on a background metric.

There is a price to pay whatever choice of $w$ one makes: For $w = 1$ the norm of (3.3) in the LQG Hilbert space is independent of $\epsilon$ and finite (due to diffeomorphism covariance of the construction and this in fact motivates the non-standard topology) while the norm of (3.5) in the LQG Hilbert space converges to zero for continuous $M, N$ as $\epsilon \to 0$. For $w = 7/6$ it is opposite: The norm in the LQG Hilbert space of the analog of (3.3) diverges as $\epsilon \to 0$ while the norm of the analog of (3.5) in the LQG Hilbert space converges to a finite limit which correctly depends on the wanted combinations of derivatives $M \partial N - N \partial M$.

It is now time to unveil the reason for why with standard density weight $w = 1$ the dual algebra $[C'(m), C'(n)] = 0$ must be Abelian for the choice of $L'$ made in [21] and its generalisations. In other words its Abelian nature is physically correct. To see this we note that $L'$ is a space of distributions over $D = D_{SNWF}$, the finite linear span of SNWF. This is a dense and invariant domain for $C(n)$ because $C(n)$ acts only at the vertices of a graph and the graphs involved in SNWF are finite, the number of SNWF involved in $\psi \in D$ is finite. Therefore there is a substantial difference between the quantum state $C'(n)\psi$, $\psi \in D$ and the classical expression $C'(n)$ (the Riemann sum regularisation of $C(n)$ sketched above): The former is a sum over a finite (say $N$) number of cells where $N$ is the number of vertices involved in $\psi$ while $C'(n)$ is an infinite sum (for non compact $\sigma$; for compact $\sigma$ the number of cells still grows indefinitely as $\epsilon \to 0$). We pick $\psi = T_{\gamma}$ and denote the situation symbolically as

$$
C'(n) T_{\gamma} = \sum_{v \in V} n(v) C(\square_{v}^{\epsilon}) T_{\gamma}, \quad C'(n) = \sum_{p \in P} n(p) C(\square_{p}^{\epsilon})
$$

(3.11)

where $P$ denotes the set of centre points used in the partition into cubes $\square_{p}^{\epsilon}$ of $\sigma$. Computing commutators and Poisson brackets respectively yields again symbolically

$$
[C'(m), C'(n)] T_{\gamma} = \frac{1}{2} \sum_{v, v' \in V} \left[ m(v)n(v') - m(v')n(v) \right] [C(\square_{v}^{\epsilon}), C(\square_{v'}^{\epsilon})] T_{\gamma}
$$

$$
\{C'(m), C'(n)\} = \frac{1}{2} \sum_{p, p' \in P} \left[ m(v)n(v') - m(v')n(v) \right] \{C(\square_{p}^{\epsilon}), C(\square_{p'}^{\epsilon})\}
$$

(3.12)

Let us compute the classical Poisson brackets explicitly for the explicit form used in [17]

$$
C(\square_{p}^{\epsilon}) := \sum_{a,b,c=1}^{3} e^{abc} \text{Tr}([H_{ab}(p) - H_{ba}(p)]H_{c}(p) \{V(\square_{p}^{\epsilon}), H_{c}(p)^{-1}\})
$$

(3.13)

where $H_{a}(p)$ is the holonomy from $p$ in direction $a$ by one unit of $\epsilon$ and $H_{ab}(p) = H_{a}(p) H_{b}(p + \delta_{a}) H_{a}(p + \delta_{b})^{-1} H_{b}(p)^{-1}$ is a plaquette holonomy while $V(\square_{p}^{\epsilon}) = \left[ \det(\delta \Phi(p)) \right]^{1/2}$ where $[\delta \Phi]^{a}_{j}(p) = \Phi^{a}_{j}(p) - \Phi^{a}_{j}(p - \delta_{a})$ and $\Phi^{a}_{j}(p)$ is the gauge covariant flux based at $p$ [34] through the boundary face of the two cubes with centres
\[ p, p + \delta_{\alpha} \text{ with co-normal in } a \text{ direction and in direction } j \text{ wrt an ONB basis } \tau_j \text{ of } \mathfrak{su}(2) \text{ wrt the trace metric.} \]

Since by construction

\[ \{ \Phi_c^{\alpha}(p), H_b(p') \} = \delta_{p,p'} \delta_b^{\alpha} [\tau_j H_a(p)] \]

one finds to order \( \epsilon^3 \)

\[ \{ C(\Box_{p}^{\epsilon}), C(\Box_{p'}^{\epsilon}) \} = - \sum_a \sum_{\sigma=\pm} \sigma \delta_{p',p+\sigma \delta_{\alpha}} D_{\sigma}^{\alpha}(p), \quad D_{\sigma}^{\alpha}(p) = \sum_{b,c} \Tr(H_{bc}(p) \Phi^{c}(p)) \frac{\Tr(\Phi_{\alpha}^{a}(p) \Phi^{b}(p))}{V(\Box_{p}^{\epsilon})^2} \]

so that to leading order in \( \epsilon \)

\[ \{ C^{\epsilon}(m), C^{\epsilon}(n) \} = - \frac{1}{2} \sum_{p \in P} \sum_a [m (\partial_{\alpha}^{\epsilon} n) - n (\partial_{\alpha}^{\epsilon} m)](p) \frac{m(p+\delta_{\alpha}) - m((p-\delta_{\alpha})}{V(\Box_{p}^{\epsilon})} \]

with the lattice derivative \( (\partial_{\alpha}^{\epsilon} m)(p) = m(p+\delta_{\alpha}) - m((p-\delta_{\alpha}) \). If we take the limit \( \epsilon \to 0 \) of (3.16) then we recover precisely \( -D[q^{-1}(m \, dn - ndm)] \) because 1. \( D_{\sigma}^{\alpha}(p) \) is of order \( \epsilon^2 \), 2. \( m \partial_{\alpha}^{\epsilon} n - n \partial_{\alpha}^{\epsilon} m \) is of order \( \epsilon \) and 3. mostly importantly for the main argument of this work the number of terms in the sum grows as \( \epsilon^{-3} \). In order that this holds, we need all the contributions \( p \in P \) in order that the Riemann sum \( \sum_{p \in P} \epsilon^3 \) turns into the integral \( \int_{\sigma} d^3x \). We also need that \( V(\Box_{p}^{\epsilon}) > 0 \) for all \( p, \epsilon \), i.e. that the classical metric is regular.

Let us now mirror this with the quantum computation in (3.12). Under the assumption that the commutator between the \( v, v' \) contributions does not vanish identically we expect it to be non-vanishing at most when \( v, v' \) are next neighbour vertices in the graph \( \gamma \) since the quantum operator is constructed from local expressions. Let \( \mathcal{N}_v \) be the set of next neighbour vertices \( v' \) of \( v \in V \). Then we obtain

\[ [C^{\epsilon}(m), C^{\epsilon}(n)] \, T_{\gamma} = \frac{1}{2} \sum_{v \in V} \sum_{v' \in \mathcal{N}_v} [m(v)n(v') - m(v')n(v)] \, [C(\Box_{v}^{\epsilon}), C(\Box_{v'}^{\epsilon})] \, T_{\gamma} \]

Now even if in the best case the commutator left in (3.17) is turned into a linear combination of diffeomorphism operators, there is no chance to match with (3.16) because the number of terms involved is finite. Therefore, as for generic graphs \( \gamma \) the next neighbours of \( v \) are all away from \( v \) by far more than \( \epsilon \), for such \( \gamma \) (3.17) trivially vanishes (perhaps modulo a diffeomorphism but that diffeomorphism has nothing to do with the diffeomorphism involved in \( K(m, n) \)). As this is automatically the case for sufficiently small \( \epsilon \), there is no chance to match (3.16) and (3.17).

In other words, even if in a semiclassical limit we have that the commutator in (3.17) turns into the Poisson bracket (3.15), that contribution is of order \( \epsilon^2 \) while \( m(v)n(v') - m(v')n(v) \) is of order \( \epsilon \) and we need an order of \( 1/\epsilon^3 \) terms to make the semiclassical limit non vanishing, but there are only finitely many, namely \( N \) of them.

Yet in other words, even if one would get an equality of the form \( [C^{\epsilon}(m), C^{\epsilon}(n)] \, T_{\gamma} = K^{\epsilon}(m, n)T_{\gamma} \), we would find \( K^{\epsilon}(m, n) \equiv 0 \) on any suitable \( L' \) as soon as \( K^{\epsilon}(m, n)T_{\gamma} \) is of the form \( \sum_{v,v'} [m(v)n(v') - m(v')n(v)] K_{\gamma,v,v'}T_{\gamma} \) with \( K_{\gamma,v,v'} \) having diffeomorphism invariant matrix elements between a finite number of SNWF, with \( m, n \) and with \( l \in L' \) having continuous coefficients \( l(T_{\gamma}) \).

The discussion reveals that if one wants to avoid this triviality and if one does not want to change the density weight, which as we showed in section 2 is problematic, then one must avoid that the sum over \( P \) collapses to a sum over \( V \) in (3.11). More precisely, several conditions must be met at the same time in order that the quantum computation (3.17) comes as close as possible to the classical computation (3.16). In the classical Riemann sum computation (5.16) three things are happening simultaneously and are matched to each other: First a discretisation of space by cells, second a discretisation of the phase space labelled by those cells and third a discretisation of cell constraints by functions of the cell variables. In the quantum computation (5.17) these three steps are also applied to the constraint operator but not to the quantum state. The quantum state still lives in the continuum and it is not subject to discretisation. Thus it is defined by a continuum of configuration quantum degrees of freedom (in the connection representation and spin degrees of freedom in the Fourier transformed representation). Then two effects bring the classical and quantum computation drastically out of balance: first, due to the fact that SNWF are highly degenerate, the quantum constraint is “blind” for almost all of the cells,
namely those that do not contain a vertex. Second, even for the cells that contain a vertex, the set of degrees of freedom that are changed on the quantum state by the action of the discretised constraint contain new ones with respect to which it was not already excited. In the literature this is referred to as “graph changing action”. The fluctuations of these new excitations are therefore not controlled by the state one acts upon and thus avoid e.g. application of coherent state techniques.

To avoid the first effect, the state one acts upon should be non-degenerate. To avoid the second effect, the discretised constraint should act on an invariant subspace of states.

A first proposal that meets these two conditions is the the algebraic quantum gravity (AQG) programme. There one works with a single, fundamental, infinite abstract graph that can be embedded arbitrarily densely (i.e. with arbitrarily small but finite spatial resolution) everywhere on $\sigma$ and the constraints, which are considered as regulator free, preserve that fundamental abstract graph, but not all its subgraphs. Thus AQG is like a lattice gauge theory with the difference that the quantum state one acts upon decides about how densely the abstract graph is embedded. Since the abstract constraints of AQG do not close under commutators, the AQG framework was embedded into the master constraint programme which replaces all spatial diffeomorphism and Hamiltonian constraints by a single one so that anomalies are of no immediate concern.

A second proposal that meets these conditions is the Hamiltonian renormalisation programme. Here one still works with a Hilbert space $\mathcal{H}$ of concrete (embedded) states, however, the states in $\mathcal{H}$ are projected into subspaces $\mathcal{H}_\epsilon = \mathcal{P}_\epsilon \mathcal{H}$ in a controlled way. The control consists in a partial order on the set $\mathcal{E}$ of labels $\epsilon$ with respect to which it is directed. The $\mathcal{P}_\epsilon$ arise as $\mathcal{P}_\epsilon = J_\epsilon^1 J_\epsilon^J$ where $J_\epsilon : \mathcal{H}_\epsilon \to \mathcal{H}$ is an isometric injection $J_\epsilon^1 J_\epsilon = 1_{H_\epsilon}$ which are fixed points of a renormalisation flow which grants that $\mathcal{H}$ is the inductive limit of the $\mathcal{H}_\epsilon$ which may or may not coincide with the LQG Hilbert space. The discretised constraints are also subject to renormalisation and yield a consistent family $C_\epsilon(n)$ at the fixed point (if it exists) in the sense that

$$C_\epsilon(n) = J_\epsilon^1 C(n) J_\epsilon$$

(3.19)

The $C_\epsilon(n)$ must not close under commutators, even if the $C(n)$ do. An example where the validity of this procedure has been recently demonstrated is parametrised field theory which among other things also displays a non-trivial realisation of the hypersurface deformation algebra, in fact for density weight two rather than one. On the other hand, in the corresponding Hilbert space representation (a Fock representation) the metric operator does not annihilate the vacuum so that all Fock states are non-degenerate. We therefore revisit PFT with density weight one and with degenerate vacuum in section 4 and show that nevertheless one can get the algebra to close.

A third proposal is to consider the representation of the holonomy flux algebra different from the LQG representation. It modifies it by a condensate $\langle \Omega^0, \Phi_f(S) \Omega^0 \rangle = \Phi_f^0(S)$ where $\Phi^0$ is a classical electric field. Choosing $\Phi^0$ to correspond to a non-degenerate metric, then $\Omega^0$ is a non-degenerate vacuum state in the sense of the next section. To see whether in this representation we can hope to make progress wrt the representation of $\mathfrak{h}$ we consider formally $C(n) w[F] \Omega_0$ which formally can be written

$$\int d^3 x \ n \ B^a_j \ [e(\Phi_0 + F)]^j_a \ w[F] \Omega_0, \ e(G)^j_a = \frac{1}{2} \epsilon^{ijkl} \epsilon_{abc} \frac{G^j_k G^c_l}{|\det(G)|^{1/2}}$$

(3.20)

Note that $F$ is a distribution with the singularity structure of a $\delta$ distribution in 2 dimensions while $\Phi_0$ is smooth. Also the magnetic field itself is ill-defined. To regularise we use a Riemann sum approximation of the integral by a sum over $\epsilon$ sized cells $\square$ with centre $p_\epsilon$ and $\epsilon^2 B^a_j$ replaced by $B^a_j(\square) := \text{Tr}(H(\alpha_0^a))\tau_j$ where $\alpha_0^a$ is an appropriate loop located in $\square$ in the coordinate plane transversal to the $a$ direction. In order that the denominator $|\det(\Phi_0 + F)|^{1/2}$ turns into something finite, we integrate it over $\square$ which in the limit $\epsilon \to 0$ makes the $\Phi_0$ dependence disappear from the denominator. The numerator then depends schematically on the term.
\[ \epsilon^4 (\Phi_0 + F)^2 \] where \( \epsilon^4 \) comes from the left over \( \epsilon \) of the measure factor \( \epsilon^2 \) not absorbed by \( B \) and the fact that we have to multiply both numerator and denominator by \( \epsilon^2 \) when we integrate the denominator over \( \Box \). Thus the \( \Box \) contribution to the numerator contains the three terms, schematically

\[ B(\Box) \left[ \Phi_0(\Box)^2 + 2\Phi_0(\Box)(F(\Box) + F(\Box)^2) \right] \] (3.21)

where \( \Phi_0(\Box), F(\Box) \) are integrals over faces of \( \Box \). Suppose we use some of the methods of [17] to define \( |\det(F(\Box))|^{-1/2} \) which thus lets only those \( \Box \) contribute that contain a vertex of the graph. Then the contribution of the first two terms in (3.21) to the norm of (3.20) vanishes in the limit \( \epsilon \to 0 \) because the number of contributing \( \Box \) is constant so that altogether nothing has changed as compared to the situation without condensate. To change something, as argued above, all \( \Box \) must contribute, thus \( F \) itself must be excited already everywhere so that the denominator is finite for every \( \Box \). Thus it appears that \( \Phi_0 \) by itself is not sufficient to achieve a non-anomalous \( h \).

These qualitative arguments suggest that we need to consider \( F \) that correspond to an everywhere excited quantum geometry. This should define a new dense domain different from the finite linear span of spin network functions. In the next section we investigate qualitatively how this might be achieved by combination of coherent state and renormalisation methods.

4 Qualitative investigation of \( h \) in Hamiltonian renormalisation of LQG

As mentioned before and reviewed in [20] in Hamiltonian renormalisation we construct a sequence labelled by \( s \) of families labelled by \( \epsilon \) of triples \( (\mathcal{H}_\epsilon^{(s)}, J^{(s)}_{\epsilon,\kappa(\epsilon)}, C^{(s)}(n)) \) where \( \epsilon' := \kappa(\epsilon) \leq \epsilon \) is a fixed element wrt the partial order \( \geq \) (dictating how many degrees of freedom of the finer theory labelled by \( \epsilon' \) are integrated out to reach the coarser theory labelled by \( \epsilon \) ) and the entries of the triple are Hilbert spaces, isometric embeddings \( J^{(s+1)}_{\epsilon'} : \mathcal{H}^{(s+1)}_{\epsilon} \to \mathcal{H}^{(s)}_{\epsilon} \) and constraints respectively. This isometry condition together with the prescription

\[ C^{(s+1)}(n) := [J^{(s+1)}_{\epsilon'}]^* C^{(s)}(J^{(s+1)}_{\epsilon'}), \epsilon' = \kappa(\epsilon) \] (4.1)

defines a renormalisation flow starting from an initial triple that uses a classical discretisation such as the Riemann sum approximations that were used in section 3. While the initial triple suffers from many ambiguities, the intuition collected from statistical physics examples gives rise to the hope that theories labelled by different ambiguity parameters flow into the same fixed point family \( \mathcal{H}_\epsilon, J_{\epsilon,\epsilon'}, C(n) \) defining a continuum Hilbert space \( \mathcal{H} \) as the inductive limit of the \( \mathcal{H}_\epsilon \) and continuum operators \( C(n) \) such that there exist isometric embeddings \( J_\epsilon : \mathcal{H}_\epsilon \to \mathcal{H} \) with

\[ J_\epsilon = J_{\epsilon'} J_{\epsilon'}, \quad C(n) = J_{\epsilon}^\dagger C(n) J_\epsilon \] (4.2)

The \( C(n) \) must not close even if the \( C(n) \) do close since

\[ [C(m), C(m)] = J_{\epsilon}^\dagger [C(m), C(n)] J_\epsilon - J_{\epsilon}^\dagger \{C(m) [1_H - P_\epsilon] C(n) - C(n) [1_H - P_\epsilon] C(m)\} J_\epsilon \] (4.3)

where \( P_\epsilon = J_{\epsilon} J_{\epsilon}^\dagger \) is the projection of the continuum theory into a subspace isomorphic to the discretised theory at resolution \( \epsilon \). The correction terms proportional to \( 1_H - P_\epsilon \) are expected to converge to zero as \( \epsilon \to 0 \) e.g. in the weak operator topology on \( \mathcal{H} \), that is, given \( \psi, \psi' \in \mathcal{H} \) the correction terms in (4.3) are expected to vanish at fixed \( \psi, \psi' \) when sandwiched between \( P_\epsilon \psi, P_\epsilon \psi' \) provided that \( 1_H - P_\epsilon \) itself converges to zero when sandwiched between \( \psi, \psi' \). These expectations are met in PFT [35].

The strategy to check for anomaly freeness of \( h \) in Hamiltonian renormalised LQG would therefore be as follows:

Step 1:

Start with initial families of Hilbert spaces \( (\mathcal{H}_{\epsilon}^{(0)} := L_2(d\nu_{\epsilon}^{(0)}), \mathcal{A}_\epsilon) \) defined as square integrable functions with respect to some measure \( \nu_{\epsilon}^{(0)} \) on a space of connections \( \mathcal{A}_\epsilon \) as well as with discretised constraints \( C_{\epsilon}^{(0)}(n) \) and discretised “would be” commutators \( K_{\epsilon}^{(0)}(m, n) \) thereof.
Step 2:
Construct the flow of these where isometry translates into a flow of measures $s \mapsto \nu^s_\epsilon$.

Step 3
Compute the corresponding fixed points and check $\hbar$.

Since the completion of step 3 will be very difficult in practice because the computation of the fixed point will require a large number of iterations of the renormalisation step, we consider the following algebra check after a finite number $s$ of iterations:

Step 3$_s$,$N$:
Consider the matrix elements of the combination

$$[C^{(s)}(m), C^{(s)}(n)] - K^{(s)}(m,n)$$

between $P^{(s)}_{\epsilon_N} \psi_{\epsilon_N}^I$, $P^{(s)}_{\epsilon'_N} \psi_{\epsilon'_N}^I$ for fixed $\psi_{\epsilon_N}^I$, $\psi_{\epsilon'_N}^I \in H^{(s)}_{\epsilon_N}$ where $\epsilon'_N = \kappa^N(\epsilon)$ is the $N$-fold refined theory and $P^{(s)}_{\epsilon'_N} = J^{(s)}(J^{(s)}_{\epsilon'_N})^\dagger$. Here the fixed $N$ should be as large a number as practically possible. Then, if these matrix elements become smaller as $s$ increases and as $\epsilon$ decreases, one would have strong evidence for convergence to the fixed point and vanishing of the anomaly. For $s, N \to \infty$ this step becomes step 3.

These steps have not been carried out yet for LQG not even in the weakened version (4.4). However, we want to sketch at least how one starts the flow, say for the case that $\sigma$ is compact with periodic boundary conditions. By the Bieberbach theorem [38] we may assume w.l.o.g. that $\sigma$ is a 3-torus. First we do not consider all possible graphs but only those which can be sensibly labelled by a controllable set $E$ and such that the discretised classical degrees of freedom labelled by $\epsilon \in E$ still separate the points of the classical phase space. For instance, these could be cubical lattices $\gamma_\epsilon$ in $\sigma$ where $\epsilon' \leq \epsilon$ iff $\gamma_\epsilon$ is a sublattice of $\gamma'_\epsilon$ and $\epsilon$ could be the lattice spacing with respect to some coordinates on $\sigma$. We discretise holonomies on the edges of $\gamma_\epsilon$ and fluxes on a similarly chosen dual cubical cell complex $\gamma^*_\epsilon$. Then we take some discretisations $C^I_l(n_I)$ of the $C^I_l(n_I)$ where

$$C^I = f^{ij}_k B^a_j e^{ij}_k, B^a_j = \epsilon^{abc} P^b_{\epsilon a} \delta_{jk}, e^{ij}_a = q_{ab} \delta_j k f^{ij}_k = \left( \begin{array}{cc} 2 \delta^i_k & I = 0 \\ \epsilon^{ij} m & I = l \end{array} \right)$$

are the density weight one building blocks of the extended master constraint [11, 28]. For $I = l$ these have been reused more recently with non-standard density weight and were called "electric diffeomorphisms" [23, 24] there. As pointed out in [11, 28] and as follows from the general results established in [17] in contrast to $D(u)$ the constraints $\tilde{C}(N) = D(u)_{u=\bar{N}, E/\sqrt{\det(q)}}$ which are classically equivalent to $D(u)$ for non-degenerate $q$ can be quantised in the LQG representation and by (4.5) display a more balanced structure as far as the algebraic structure of all 4 constraints is concerned and which has the advantage that not only the exponentiated version of $D(u)$ exists in the quantum theory. The price to pay is that now all constraints close with structure functions only and these are classically well defined only when the metric is non-degenerate. We denote them as

$$\{C^I_l(m_I), C^J_l(n_J)\} =: C_K(f^K_{IJ}(m_I, n_J); q) =: K(m, n)$$

We now proceed as in [28]: We introduce holonomy flux variables on $\gamma_\epsilon$ and discretise the constraints using them, see [28] for details, resulting in classical functions $C^I_l(n_I)$. We do the same with the right hand side of (4.7) resulting in $K_\epsilon(m, n)$. By construction we have to leading order in $\epsilon$

$$\{C^I_l(m_I), C^I_l(n_I)\} = K_\epsilon(m, n)$$

and (4.7) converges to (4.5) pointwise $Z$ on the phase space.

Then we quantise $C^I_l(n_I), K_\epsilon(m, n)$ with all flux depending variables ordered to the right [28] and denote the resulting operators on $H^{(0)}_\epsilon$ by $C^{(0)}_l(m_I), K^{(0)}_\epsilon(m, n)$ that start the renormalisation flow. Here $H^{(0)}_\epsilon$ is $L_2(SU(2), d\mu_H)N_\epsilon$, where $\mu_H$ is the Haar measure, $N_\epsilon = |E(\gamma_\epsilon)|$ the number of edges of $\gamma$, the fluxes being quantised as right invariant vector fields on corresponding copies if $SU(2)$ and the holonomies as multiplication operators corresponding to that copy. Then we wish to study

$$\Delta^{(0)}_\epsilon(m, n) := [C^{(0)}_l(m_I), C^{(0)}_l(n_J)] - i K^{(0)}_\epsilon(m, n)$$

(4.8)
As we showed above, (4.8) must not vanish, not even for the fixed point family. However, one may hope that (4.8) is small in a suitable operator topology for small $\epsilon$ even for the initial “naive” discretisation. We consider the following topology which can be argued to be as close as possible to the topology of pointwise convergence on the classical on phase space: We consider the coherent states $\psi^e_{\epsilon} \in \mathcal{H}^{(0)}_\epsilon$ which take as an input a point $Z$ in the classical continuum phase space, map it to elements of $g_\epsilon \in \text{SL}(2, \mathbb{C})$, one for each edge of $\gamma_\epsilon$, take a coherent superposition of SNWF over a single edge weighted by corresponding irreducible representations of SU(2) analytically continued to SL(2, $\mathbb{C}$) and by a Gaussian in the corresponding spin quantum number and finally one takes the tensor product over all edges and normalises the result. The construction of the $\text{SL}(2, \mathbb{C})$ element and the Gaussian factor are not randomly chosen but in fact follow from the complexifier machinery [27]. In more detail one constructs

$$\psi_{\epsilon, Z}(A) = \sum_{2j=0}^{\infty} (2j + 1) e^{-t_j(j+1)} \text{Tr}(\pi_j(g_\epsilon(Z) H_\epsilon(A)^{-1}))$$ \hfill (4.9)

where $Z = (A^0, E^0)$ is a point in the classical phase space, $g_\epsilon(Z) = \exp(iE_j^0(S_\epsilon)))$ $H_\epsilon(A_0) \in \text{SL}(2, \mathbb{C})$ where $S_\epsilon$ is the face in $\gamma_\epsilon^*$ dual to $e$. Then

$$\psi_{\epsilon}(Z) := \prod_e \frac{\psi_{\epsilon}}{||\psi_{\epsilon}||} \hfill (4.10)$$

These coherent states are known to be sharply peaked on points $Z_\epsilon$ where $Z_\epsilon$ encodes the discretised variables constructed from $Z$ and restricted to the edges and faces of $\gamma_\epsilon$ and $\gamma_\epsilon^*$ respectively (basically the collection of the $g_\epsilon(Z)$). Also there is a measure $\rho_\epsilon$ on the cotangent bundle phase space $\Gamma_\epsilon = \text{SL}(2, \mathbb{C})^{N_\epsilon} \cong T^*(SU(2)^{N_\epsilon})$ coordinatised by $Z_\epsilon$ which leads to a resolution of unity

$$\int_{\Gamma_\epsilon} d\rho_\epsilon(Z) \psi_{\epsilon}(Z) < \psi_{\epsilon}(Z), >_{\mathcal{H}^{(0)}} = 1_{\mathcal{H}^{(0)}} \hfill (4.11)$$

Then

$$\Delta^{(0)}_{\epsilon}(m, n)\psi_{\epsilon}(Z) = \int d\rho_\epsilon(Z') < \psi_{\epsilon}(Z'), \Delta_{\epsilon}(m, n)\psi_{\epsilon}(Z) > \psi_{\epsilon}(Z')$$ \hfill (4.12)$$

Due to sharp peakedness

$$< \psi_{\epsilon}(Z'), \Delta^{(0)}_{\epsilon}(m, n)\psi_{\epsilon}(Z) > = < \psi_{\epsilon}(Z), \Delta_{\epsilon}(m, n)\psi_{\epsilon}(Z) > < \psi_{\epsilon}(Z'), \Delta^{(0)}_{\epsilon}(m, n)\psi_{\epsilon}(Z) >$$ \hfill (4.13)

plus corrections in $\epsilon$ which are subleading provided that the estimates performed for similar operators in [28] carry over to the present case.

Then

$$\Delta_{\epsilon}(m, n)\psi_{\epsilon}(Z) = < \psi_{\epsilon}(Z), \Delta_{\epsilon}(m, n)\psi_{\epsilon}(Z) > \psi_{\epsilon}(Z)$$ \hfill (4.14)$$

plus corrections in $\epsilon$ which are subleading provided the above assumptions hold. It remains to compute the expectation values

$$< \psi_{\epsilon}(Z), \Delta_{\epsilon}(m, n)\psi_{\epsilon}(Z) >$$ \hfill (4.15)$$

Again, this kind of calculation has been carried out in [28] already and one finds

$$< \psi_{\epsilon}(Z), \Delta^{(0)}_{\epsilon}(m, n)\psi_{\epsilon}(Z) > = [\Delta_{\epsilon}(m, n)](Z), \quad \Delta_{\epsilon} = \{C^{(0)}_\epsilon(m_1), C^{(0)}_\epsilon(n_1)\} - K_{\epsilon}(m, n) \hfill (4.16)$$

plus corrections in $\epsilon$ which are subleading provided the above assumptions hold. The latter quantity is known to converge to zero at fixed $Z$ as $\epsilon$ tends to zero.

This shows that, modulo the above reservations, for sufficiently small $\epsilon$ the matrix elements of $\Delta^{(0)}_{\epsilon}(m, n)$ are almost diagonal and approach the classically discretised value, the latter approaching the classical continuum integral. The validity of the calculation and the estimates alluded to rest on the assumption that $Z$ encodes a non-degenerate metric. Note that coherent states on $\gamma_\epsilon$ are non-degenerate in the sense of non-vanishing volume expectation values for regions containing a vertex of $\gamma_\epsilon$. This is the case even if $Z$ is degenerate but in
this case the $\epsilon$ corrections mentioned above are not subleading. For details the reader is referred to \cite{27,28}. This underlines once more the importance of the non-degeneracy condition even at the quantum level.

More details of the concrete calculation sketched above will appear elsewhere. Note however, that exact closure can only be expected for the continuum operator which is the critical theory one tries to produce from the renormalisation flow.

5 Anomaly free, density one parametrised field theory

In the first subsection we motivate and define the density weight one model, in the second we define the LQG inspired Hilbert space representation in which quantum non-degeneracy is manifest, in the third we define the regularised constraints, in the fourth we remove the regulator on dual constraints wrt a suitable habitat (space of distributions), in the fifth we verify the anomaly representation of these dual constraints, in the sixth we illustrate how in the degenerate representations considered in \cite{30,31} this model would yield an anomalous algebra and pin point that indeed the anomaly is caused by degeneracy and in the seventh we comment on the construction of a physical Hilbert space for this model.

5.1 Motivation and definition of the model

The constraints of 1+1 dimensional parametrised field theory \cite{29,30,31} can be written as

$$\hat{D} = \Pi \Phi' + P T' + Y X', \quad \hat{C} = \frac{1}{2}[\Pi^2 + (\Phi')^2] + P X' + Y T'$$

(5.1)

where $T, X$ are the embedding fields with conjugate momenta $P, T$, the massless Klein-Gordon field pulled back by $T, X$ and its conjugate momentum are denoted as $\Phi, \Pi$. A prime denotes derivation with respect to the angular variable $x \in [0,1)$ and a dot derivation with respect to the time foliation parameter $t \in \mathbb{R}$. We have e.g.

$$\{\Pi(x), \Phi(y)\} = \delta(x - y)$$

(5.2)

where $\delta(x)$ is the 1-periodic delta-distribution. The constraints satisfy the classical hypersurface deformation algebra $\mathfrak{h}$

$$\{\hat{D}(u), \hat{D}(v)\} = -\hat{D}([u,v]), \quad \{\hat{D}(u), \hat{C}(\tilde{n})\} = -\hat{C}([u,\tilde{n}]), \quad \{\hat{C}(m), \hat{C}(n)\} = -\hat{D}([\tilde{m},\tilde{n}]), \quad [u,v] = u v' - v u'$$

(5.3)

This does not resemble the form that $\mathfrak{h}$ has in GR. There are two reasons for this which are due to the peculiarity of two spacetime dimensions. First, the constraints (5.1) have density weight two rather than one, because tensors of rank $(a,b)$ are scalar densities of weight $b - a$. Second, (5.3) does not display any structure functions. This is because a spatial metric $q$ is just a scalar density of weight two, hence the density weight one scalar constraints would be $C = \sqrt{q}^{-1} \hat{C}$ and their Poisson brackets would yield $q^{-1} D = q^{-2} \hat{D}$ instead, which explains why there is no $q$ dependence in the Poisson brackets of the $\hat{C}$. Note also that the smearing functions are naturally vector fields i.e. scalar densities of weight $-1$.

Accordingly, to shed light on the complex of questions that concerns us in the present work - density weights, structure functions, anomaly freeness, non-degeneracy, habitats - the form of the constraints (5.1) is not useful. To make the analogy with (Euclidian) GR manifest we relabel the canonical pairs

$$A_1 := T, \quad E_1 := P; \quad A_2 := X, \quad E_2 := Y$$

(5.4)

and rewrite the constraints in these variables however with density weight unity for the Hamiltonian constraint

$$D := \Pi \Phi' + A'_1 E_1 + A'_2 E_2, \quad C = \frac{1}{2}[\Pi^2 + (\Phi')^2] + E_1 A'_2 + E_2 A'_1 |E_1 E_2|^{-1/2}$$

(5.5)

There is no curvature in one dimension but $A'_1, A'_2$ can be considered as a substitute depending like a curvature on the derivative of the “connections” $A_1, A_2$ (which are actually scalar fields). We also have introduced the density two valued metric $q := |E_1 E_2|$. The constraints (6.1) and (5.2) are classically equivalent iff $q > 0$ i.e. if that metric is non-degenerate.
We compute (note that $u, v$ are scalar densities of weight $-1$ while $m, n$ are scalar densities of weight zero)
\[
\{D(u), D(v)\} = -D([u, v]), \quad \{D(u), C(n)\} = -C(u[n]),
\]
\[
\{C(m), C(n)\} = -D([m \, dn - n \, dm] \, q^{-1}) + C([m \, dn - n \, dm]) \frac{1}{2} (\frac{E_1^1}{E_2^2} + \frac{E_2^1}{E_1^2}) q^{-1/2}
\]
which resembles $\hbar$ of GR more closely than $(5.3)$ because $u, v, D$ and $m, n, C$ assume their standard density weight and because the $C(n)$ close with non-trivial structure functions. In some sense these structure functions are even more complicated than the ones of GR, hence the quantisation of this model in a LQG inspired representation will be a rather stringent test of the validity of the viewpoint that density weight one and non degeneracy are intimately connected while yielding anomaly freeness on suitable habitats.

5.2 LQG inspired Hilbert space representation

In its density weight two versions, the constraints $(5.1)$ are naturally quantised in a Fock representation. This option is not available for the density weight one version $(5.5)$ because an operator valued distribution such as
\[
Q := |E_1^1 E_2^2|^{1/2}
\]
is ill-defined in that Fock representation, at least as far as the geometry sector is concerned. We thus adopt the “hybrid quantisation approach” employed in Loop Quantum Cosmology (LQC) \[39\] and consider the usual Fock representation $\mathcal{H}_F$ for the matter sector \[29, 35\] and an LQG inspired representation $\mathcal{H}_G$ for the geometry sector. The total representation space is then the tensor product $\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_G$.

As motivated in section 1 the corresponding geometry vacuum $\Omega_G$ is annihilated by the “electric flux operators” $E[f]$ and the representation is discontinuous with respect to the “holonomy operators” $H[g]$ where $f, g$ are a pair of scalar smearing function and ($I = 1, 2$)
\[
E[f] = \int_{[0,1]} dx \, f_1(x) \, E^I(x), \quad H[g] = \exp(i \int_{[0,1]} dx \, g^I(x) \, A^I_1(x)),
\]
That the derivative of $A^I_1$ instead of $A_I$ appears in $H[g]$ is justified by the fact that the spatial manifold $\sigma = [0, 1]$ is a loop, i.e. a circle, hence by “Stokes theorem” the holonomy $H(g)$ is the exponential of the “magnetic flux”. Explicitly we have in terms of the original variables that
\[
\delta <g^I, A^I_1>_{L^2([0,1])} = - <g^I, A_I> + T(1) \, g^I(1) - T(0) \, g^I(0) + X(1) \, g^2(1) - X(0) \, g^2(0)
\]
Physically we have $T(1) = T(0), \, X(1) = X(0) + 1$ as $X$ is an angular coordinate. If we impose, as we will do in what follows, that $T(0), X(0)$ assume fixed values then the boundary term of the variation of $(5.9)$ vanishes even if $g^I$ is not periodic as we will consider below. Thus we may use that $\delta <g^I, A^I_1> = - <g^I, \delta A_I>$.

Then the representation is completely defined by the relations
\[
E[f]^* = E[f], \quad H[g]^* = H[-g], \quad H[g] \, H[\tilde{g}] = H[g + \tilde{g}], \quad [E[f], E[\tilde{f}]] = 0, \quad [H[g], H[\tilde{g}]] = 0,
\]
\[
[H[f], H[g]] = <f, g^I > \quad H[g], \quad [E[f], H[g]] = 0, \quad <\Omega_G, H[g] \Omega_G > = \delta_{g,0}
\]
which clearly resembles the LQG representation. The vector $\Omega_G$ is cyclic and the span of the “Weyl states” $H(g) \Omega_G$ is dense (thus replacing the spin network functions).

In this representation it is much simpler to find non-degenerate states according to the definition in section ??, than in the LQG representation. This is because the Weyl states $H[g] \Omega_G$ already diagonalise the “volume operator”
\[
V(O) := \int_O dx \, Q(x)
\]
for any open $O \subset [0, 1]$ namely
\[
V(O) \, H[g] \Omega_G = V_g(O), \quad H[g] \Omega_G, \quad V_g(O) := \int_O dx \, Q_g(x), \quad Q_g(x) := |g^1 \, g^2|^{1/2}(x)
\]
thus $H(g) \Omega_G$ is non-degenerate iff $g^1, g^2$ are nowhere vanishing in $[0, 1)$. We call such $g$ also non-degenerate. Thus non-degenerate $g$ are strictly monotonous and thus are not periodic but “angular functions” which is why we required $\delta A_I(0) = 0$ above.
5.3 Constraint regularisation

We formally apply the approximation of the integral by intervals, with all dependence on $E^1, E^2$ ordered to the right, to tensor products of Fock states with non-degenerate $g$

$$D[u] \psi_F \otimes H[g] \Omega_G = \left[ \int dx \ u(x) \ [d(x) \otimes 1_G + 1_F \otimes \{A'_1 \ g^{(1)} + A'_2 \ g^{(2)}\}(x)] \right] \psi_F \otimes H[g] \Omega_G$$

$$C[n] \psi_F \otimes H[g] \Omega_G = \left[ \int dx \ n(x) \ [h(x) \otimes 1_G + 1_F \otimes \{A'_1 \ g^{(1)} + A'_2 \ g^{(2)}\}(x)] \right] Q_g(x)^{-1} \psi_F \otimes H[g] \Omega_G$$

$$d(x) =: \Pi(x) \Phi'(x) ; , \ h(x) =: \frac{1}{2} [\Pi(x)^2 + (\Phi'(x))^2] \ :$$  \hspace{1cm} (5.13)

where $\ : \ :$ denotes the Fock space normal ordering.

As the representation is irregular with respect to the connection, the objects $A'_1(x)$ in (5.13) do not even exist as operator valued distributions. We therefore use the classical identity

$$A'_1(x) = \lim_{N \to \infty} \lim_{s \to 0} \frac{H[j_{I_2}^{s,N}]}{is} - 1, \ [j_{I_2}^{s,N}]^J(y) = s \delta_J^I \delta_N(x,y), \ \delta_N(x,y) = \sum_{k \in \mathbb{Z}, |k| \leq N} e^{2\pi i (x-y)}$$  \hspace{1cm} (5.14)

to regularise (5.13) as

$$D_{s,N}[u] \psi_F \otimes H[g] \Omega_G = \left[ \int dx \ u(x) \ [d(x) \otimes 1_G + 1_F \otimes \{g^{(1)} \frac{H[j_{I_1}^{s,N}]}{is} - 1 + g^{(2)} \frac{H[j_{I_2}^{s,N}]}{is} - 1\}(x)] \right] \psi_F \otimes H[g] \Omega_G$$

$$\hat{C}_{s}[n] \psi_F \otimes H[g] \Omega_G = \left[ \int dx \ n(x) \ [h(x) \otimes 1_G + 1_F \otimes \{g^{(2)} \frac{H[j_{I_1}^{s,N}]}{is} - 1 + g^{(1)} \frac{H[j_{I_2}^{s,N}]}{is} - 1\}(x)] \right] Q_g(x)^{-1} \times$$

$$\psi_F \otimes H[g] \Omega_G$$  \hspace{1cm} (5.15)

This expression is still formal because the Lebesgue integral over vectors of the form $F(x) \ H[g_x] \Omega_G$ with $F$ a continuous function and $g_x = g_y$ iff $x = y$ has zero norm in $\mathcal{H}_G$. We thus in addition introduce a Riemann sum approximation of the integral by intervals $\square$ of coordinate size $\epsilon$ and centre $p\square$ and obtain

$$D_{s,N,\epsilon}[u] \psi_F \otimes H[g] \Omega_G = \epsilon \sum_{\square} [u \ [d \otimes 1_G + 1_F \otimes \{g^{(1)} \frac{H[j_{I_1}^{s,N}]}{is} - 1 + g^{(2)} \frac{H[j_{I_2}^{s,N}]}{is} - 1\}]_{x=p\square} \psi_F \otimes H[g] \Omega_G$$

$$\hat{C}_{s,\epsilon}[n] \psi_F \otimes H[g] \Omega_G = \epsilon \sum_{\square} [n \ [h \otimes 1_G + 1_F \otimes \{g^{(2)} \frac{H[j_{I_1}^{s,N}]}{is} - 1 + g^{(1)} \frac{H[j_{I_2}^{s,N}]}{is} - 1\}] \ Q_g^{-1} \ |x=p\square] \psi_F \otimes H[g] \Omega_G$$  \hspace{1cm} (5.16)

and the sum is finite as $[0,1]$ is compact.

5.4 Regulator removal

We consider $\mathcal{D}$, the finite linear span of the $b \otimes w[g]$ where the countable system of states $b$ provides an orthonormal Fock basis of $\mathcal{H}_F$ and $w[g] := H[g] \Omega_G$. Then an algebraic distribution $l \in \mathcal{D}^*$ is of the form

$$l = \sum_{b,g} l(b,g) < b \otimes w[g], \ : >_{\mathcal{H}}$$  \hspace{1cm} (5.17)
Taking the notation
\[ \int_s \] and which either have support on non-degenerate\( g \) only. This domain is certainly non-trivial, any smooth function \( F \) in \( N \) variables of the form \( \langle j_k, g \rangle \) for smooth functions \( j_k, k = 1, \ldots, N \), is in this domain, even under any order of constraint actions.

**Diffeomorphism-Diffeomorphism**

### 5.5 Algebra of dual constraints

The domain of definition of \( D'[u] \), \( C'(n) \) is given by functionals \( l(b, g) \) which are functionally differentiable with respect to \( g \) and which either have support on non-degenerate \( g \) or, as we motivated frequently in this paper, are only tested with respect to non-degenerate \( g \). In order that we can compute commutators we need this domain to be invariant, hence \( l \) should at least be twice functionally differentiable and its support should not be changed by taking functional derivatives or alternatively higher derivatives should be tested with non-degenerate \( g \) only. This domain is certainly non-trivial, any smooth function \( F \) in \( N \) variables of the form \( \langle j_k, g \rangle \) for smooth functions \( j_k, k = 1, \ldots, N \), is in this domain, even under any order of constraint actions.
We have

\[ [D'[u] D'[v]](b, g) = \sum_b <d[u] \hat{b}, b > [D'[v]](\hat{b}, g) - i < u, (g^{1'} \delta_{g^1} + g^{2'} \delta_{g^2}) > [D'[v]](b, g) \]

\[ = \sum_b <d[u] \hat{b}, b > \left\{ \sum_b <d[v] \hat{b}, b > l(\hat{b}, g) - i < u, (g^{1'} \delta_{g^1} + g^{2'} \delta_{g^2}) > l(\hat{b}, g) \right\} \]

\[ = \sum_b <d[v] \hat{b}, d[u] > l(\hat{b}, g) \]

\[ - i \sum_b \left\{ <d[u] \hat{b}, b > < v, (g^{1'} \delta_{g^1} + g^{2'} \delta_{g^2}) > + <d[v] \hat{b}, b > < u, (g^{1'} \delta_{g^1} + g^{2'} \delta_{g^2}) > l(\hat{b}, g) \right\} \]

\[ - < u, (g^{1'} \delta_{g^1} + g^{2'} \delta_{g^2}) > < v, (g^{1'} \delta_{g^1} + g^{2'} \delta_{g^2}) > l(\hat{b}, g) \]

where we used the completeness relation on the Fock basis \( b \) and the symmetry of the normal ordered Fock operators \( d[u], d[v] \). Upon taking the commutator we see that the second and third term in (5.23) cancel as they are symmetric in \( u, v \). The first terms in (5.23) combine into

\[ \sum_b <d[v] \hat{b}, d[u] > l(\hat{b}, g) \]

where we made again use symmetry of the Fock operators. The fourth terms can be worked out assuming that second functional derivatives commute, using the fundamental functional derivatives

\[ \frac{\delta g''(x)}{\delta g'(y)} = \delta^{i j}_x \partial_x \delta(x, y) \]

and that integrations by parts does not create boundary terms which can be granted e.g. by assuming that \( u, v \) vanish there. Then the commutator of the fourth terms is found to be

\[ < u v' - v' u, (g^{1'} \delta_{g^1} + g^{2'} \delta_{g^2}) > l(\hat{b}, g) \]

Since in the Fock representation we have by construction

\[ [d[v], d[u]] = -i (d[v u' - u v'] + c_{DD}(v, u)) \]

with the central term \( c_{DD}(v, u) \) of the Virasoro algebra one finds altogether

\[ [D'[u], D'[v]] = i (D'[u v' - v u'] + c_{DD}(u, v)) \]

which is an anti-representation of the diffeomorphism algebra. This is because taking commutators in the dual space reverses order. Note also that we only obtain one central term, not three. This is because the geometrical sector is not normal ordered with respect to the Fock annihilators but rather wrt the geometric “annihilators” \( E^I \).

**Diffeomorphism-Hamiltonian**

The other commutators are more complicated to compute but follow the same pattern. We have using again completeness and symmetry

\[ [D'[u], C'[n]](b, g) = \sum_b <\hat{b}, [h[n \frac{n}{Q_g}], d[u]] > b > l(\hat{b}, g) \]

\[ - i \sum_b <\hat{b}, h[u, g^{1'} \delta_{g^1} + g^{2'} \delta_{g^2}], n \frac{n}{Q_g}] > b > l(\hat{b}, g) \]

\[ - [u, g^{1'} \delta_{g^1} + g^{2'} \delta_{g^2}], n \frac{n}{Q_g}] > l(\hat{b}, g) \]
As compared to the previous calculation one needs
\[
\frac{\delta}{\delta g(x)} |g''(y)|^{-1/2} = -\frac{\delta_f}{2} \frac{1}{g''(y)} \left[ \frac{1}{|g''(y)|^{1/2}} [\partial_y \delta(x,y)] \right]
\tag{5.30}
\]

Using the Virasoro algebra
\[
[d[u], h(n)] = -i(h(u \hat{n}' - u' \hat{n}) + c_{DC}(u, \hat{n})
\tag{5.31}
\]
and (5.30) one finds that the first and second term after several cancellations of terms combine to
\[
i \sum_{\hat{b}} \langle \hat{b}, h[\frac{n'}{Q_g}] \rangle > b \rangle l(\hat{b}, g) + c_{DC}(u, \frac{n}{Q_g}) l(b, g)]
\tag{5.32}
\]

Note that (the prime denotes dual action and not derivation w.r.t. x0)
\[
l[b \otimes V(O)^{-1} w(g)] = V_g(O)^{-1} l(b, g) = [1_F \otimes V(O)^{-1}] l[b \otimes w(g)]
\tag{5.33}
\]

The third term in (5.29) yields using (5.30) after a longer but elementary calculation
\[
< u n', g^{1r} \delta_g^2 + g^{2r} \delta_g^1 > l(b, g)
\tag{5.34}
\]

Thus altogether
\[
[D'[u], C'[n]] = i [C'(u n') + c_{DC}(u, n [Q']^{-1})]
\tag{5.35}
\]

Of independent interest is the fact that the central term is no longer central but depends on the inverse dual volume density $Q'$ as follows from (5.30) and (5.32).

**Hamiltonian-Hamiltonian**

We have by already familiar methods
\[
[[C'(m), C'(n)] l(b, g) = \sum_{\hat{b}} \langle \hat{b}, h[\frac{n}{Q_g}], h[\frac{m}{Q_g}] \rangle > b \rangle l(\hat{b}, g)
\]
\[
+i \sum_{\hat{b}} \{ \langle h[\frac{n}{Q_g}] g^{1r} \delta_g^2 + g^{2r} \delta_g^1 >, \frac{m}{Q_g} \rangle \} > b \rangle l(\hat{b}, g)
\]
\[
- \langle \frac{m}{Q_g} g^{1r} \delta_g^2 + g^{2r} \delta_g^1 >, \frac{n}{Q_g} \rangle g^{1r} \delta_g^2 + g^{2r} \delta_g^1 > \} l(b, g)
\tag{5.36}
\]

Using the Virasoro algebra
\[
[h(\tilde{m}), h(\tilde{n})] = i [d(\tilde{m} \tilde{n} - \tilde{n} \tilde{m}) + c_{CC}(\tilde{m}, \tilde{n})]
\tag{5.37}
\]

the first term in (5.36) becomes
\[
i \sum_{\hat{b}} \langle \hat{b}, d[\frac{m n' - m' n}{Q_g^2}] - c_{CC}(\frac{m}{Q_g}), h[\frac{n}{Q_g}] \rangle > b \rangle l(\hat{b}, g)
\tag{5.38}
\]

The middle terms in (5.36) are computed using (5.30) to be
\[
- \frac{i}{2} \sum_{\hat{b}} \langle \hat{b}, h[(m n' - m' n) \frac{g^{1r}}{g^{2r}} + \frac{g^{2r}}{g^{1r}} \frac{Q_g^{-1}}{Q_g}] > b \rangle l(\hat{b}, g)
\tag{5.39}
\]

Finally, a tedious but straightforward calculation which uses (5.30) yields the last term in (5.36) to be
\[
< \frac{m n' - m' n}{Q_g^2} [1 - \frac{1}{2} \frac{g^{1r}}{g^{2r}} + \frac{g^{2r}}{g^{1r}}] \{ g^{1r} \delta_g^2 + g^{2r} \delta_g^1 > \} l(b, g)
\tag{5.40}
\]

Combining all three contributions we get
\[
[C'(m), C'(n)] = i [D'(\frac{m n' - m' n}{[Q']^2}) + c_{CC}(\frac{m}{Q'}, \frac{n}{Q'}) - \frac{1}{2} C'(\frac{m n' - m' n}{Q'}) \frac{E^{1r}}{E^{2r}}] + \frac{E^{2r}}{E^{1r}})
\tag{5.41}
\]

where $Q', E'''$ denotes dual action of $Q, E'$ and it is understood that these objects are ordered to the outmost left when acting on a distribution $l$ in order to correctly reproduce (5.39).
5.6 Anomaly freeness and non-degeneracy

Comparing the classical relations (5.6) with (5.28), (5.35) and (5.41) we see that we get an anomaly free anti-representation on non-degenerate distributions including the correct central extensions caused by the Fock normal ordering of the matter sector. If one goes through the quantum calculation in detail, it is reassuring to see that all of the classical Poisson bracket relations that were used in the derivation of (5.6) are being reused in the quantum computation.

As the computation so far is only meaningful on non-degenerate distributions one may wonder whether it can be extended to degenerate distributions. A drastic case is the LQG like representation of PFT considered in [30] [31] which is a purely degenerate representation. In the language used here, the functions $g^I$ used in [30] [31] were not smooth but rather piecewise constant, namely characteristic functions of intervals of the circle multiplied by constants, that is, step functions with support on left closed and right open intervals partitioning $[0, 1)$. The derivative of a step function is zero almost everywhere and has a delta distribution singularity at the interval ends. Accordingly the corresponding holonomies actually are excited only at finitely many points $v \in V(g)$ where $V(g)$ denotes the “vertices” of the step function $g$. If $O$ is an open interval of the circle then on such a Weyl function $w[g]$ we get the eigenvalue

$$V_g(O) = \sum_{v \in O \cap V(g)} V_g(v), \quad V_g(v) := |s_v^1, s_v^2|^{1/2}, \quad s_v^I = g^I(v') - g^I(v) \tag{5.42}$$

where $v' \in V(g)$ is the left neighbour of $v \in V(g)$. In order to define the operator $V(O)^{-1}$ we may e.g. use Poisson bracket identities as in [31] or, for the purposes of this paper sufficient, simply Tychonov regularisation $V(O)^{-1} := \lim_{\delta \to 0} \frac{V(O)}{V(O)+\delta}$. As a result the eigenvalues of $V(O)$ on $w[g]$ are simply given by $V_g(O)^{-1}$ if $V_g(O) > 0$ and zero otherwise. Let us use this definition of $V(O)^{-1}$ in (5.13) and focus on the geometry part of both $D(u), C(g)$. Then the operators $E^I$ applied to $w[g]$ restrict the integrals involved in $D(u), C(n)$ to a sum over $v \in V(g)$. This requires to regularise the objects

$$\sum_v u(v) \sum_I A^I(v) s_v^I w[g], \quad \sum_v n(v) \frac{A^I(v) s_v^2 + A^I_2(v) s_v^1}{Q(v)} w[g] \tag{5.43}$$

If $O_\epsilon(v)$ is an open interval of coordinate length $\epsilon$ and centre $v$, a possible regularisation of (5.43) in terms of step function smearing functions is

$$\frac{1}{i \epsilon} \sum_v u(v) [w[g_{\epsilon,v}]) - 1] w[g], \quad -i \sum_v n(v) \frac{Q(v)}{Q_g(v)} [w[g_{\epsilon,v}]) - 1] w[g], \tag{5.44}$$

where

$$g^I_{\epsilon,v}(x) = \chi_{O_\epsilon(v)}(v) s_v^I, \quad g^I_{\epsilon,v}(x) = \chi_{O_\epsilon(v)}(v) s_v^2, \quad g^I_{\epsilon,v}(x) = \chi_{O_\epsilon(v)}(v) s_v^1 \tag{5.45}$$

and $\chi_O$ is the characteristic function of $O$. Thus (5.44), which replaces (5.16) maps the span $\mathcal{D}$ of $w[g]$ with step functions $g$ to itself.

The algebraic dual of this $\mathcal{D}$ consists of distributions

$$l = \sum_g l(g) < w[g], > \tag{5.46}$$

where the sum is now restricted to step functions. Computing the duals of (5.44) on (5.46) and taking the limit $\epsilon \to 0$ yields a finite result for $D'(u)$ if $l$ is differentiable in the sense that $\lim_{\epsilon \to 0} \frac{l(g+g_{\epsilon,v}) - l(g)}{\epsilon}$ exists. But then $C'(n) \equiv 0$. This happens, precisely because $w[g]$ is degenerate almost everywhere. Indeed, we see that in the Riemann sum approximation (5.16) for sufficiently small $\epsilon$ only those cells $\Box$ contribute which contain precisely one vertex $v \in V(g)$ thus resulting in (5.44). The missing sum over all cells when taking $\epsilon \to 0$ on the dual, which returned the Riemann sum into an integral in passing from (5.18) to (5.19), is missing here and prevents a non-trivial action of $C'(n)$. This is a concrete demonstration of the importance of non-degeneracy.
One may argue that in this model one should therefore use the density two constraints $\tilde{D}, \tilde{C}$. Indeed, as is well known, by taking linear combinations, those constraints are equivalent to two commuting diffeomorphism constraint algebras. While this is true, the purpose of this model with standard density weight for the constraints was to mimic the situation in GR as close as possible where for no choice of density weight it is possible to avoid the structure functions and where density weight one is uniquely selected as the universal choice as demonstrated in section 3.

We could now proceed as in section ?? and use linear combinations of parallel translates of step function states $\psi_{b]g}$ to a dense set of points on the circle with coefficients such that the linear combination is normalisable. This would then be non-degenerate states which do not leave the realm of step functions as smearing functions in the Weyl operators. We leave this for future investigations but remark that working with smooth and non degenerate $g$ rather than step functions is at least much more convenient.

5.7 Solutions and physical Hilbert space

Given the explicit form of the dual constraints (5.23) on the domain of $l$ with $l(b,g)$ functionally differentiable wrt $g$ we ask for solutions $l$ and a Hilbert space structure thereon. In contrast to a Fock quantisation of $\tilde{D}, \tilde{C}$ the above domain does not carry an obvious Hilbert space structure. While not necessary, see below, we may supply such a structure as follows: The set of labels $b$ is discrete as Fock spaces are separable while the set of labels $g$ is continuous. We may therefore consider a Hilbert space of coefficients $l(b,g)$ which are square summable wrt $b$ and square integrable wrt $g$ and then measure $\mu$ on the space $\mathcal{G}$ of $g$ which is supported on non-degenerate $g$. That is, the space of $l$ may be given the Hilbert space structure $\mathcal{H}_F \otimes L_2(d\mu, \mathcal{G})$ where $\mathcal{H}_F$ is the matter Fock space.

With respect to such a measure the $D'(u), C'(n)$ are not symmetric operators and they should not be because of the non-trivial structure functions by the argument given in [40]. Due to the central extension, there can be no strong solutions with respect to such a Hilbert space structure which is also true for the usual density weight two Fock space quantisation. However, one can construct weak solutions in the usual way because the matter parts $d, h$ of the constraints are still symmetric and it is only those that cause the central extension. As the purpose of the present model was just to illustrate the drastic effect of non-degeneracy we do not go into further details here and leave the issue of solutions for possible future research.

In closing, we remark that the procedure of first having defined a representation of the CCR among $A_I, E^I$ which is irregular for $A_I$ and a vacuum representation for $E^I$ just to find out that a dual representation in which $E^I$ acts by multiplication by $g^I$ and $A'_I$ by functional derivation with respect to $g^I$ is better suited to formulate the dynamics apparently could have been avoided altogether as one could have started with that latter representation right away. The point is, however, if we had started with the \textquoteleft g-representation\textquoteright then we should have had to worry about the measure $\mu$ on $\mathcal{G}$ right away and we should have had to make $E^I, A'_I$ self-adjoint with respect to that measure. The indirect method frees us from doing that because a Hilbert space structure is only needed on the space of solutions of the constraints.

6 Conclusion and Outlook

In the present work we have illustrated that in canonical quantum gravity density weight one of the constraints is not only natural as far as spatial diffeomorphism covariance is concerned but also dynamically selected.

We have shown that it comes at no surprise that testing the density weight one algebra on degenerate states leads to anomalous (dual, i.e. on spaces of distributions) representations thereof.

Furthermore, we have shown in an example that an anomaly free implementation of the hypersurface deformation algebra $\mathfrak{h}$ in natural representations of the CCR in which the spatial metric annihilates the vacuum (which is therefore the most degenerate state imaginable) is still possible if one does not neglect the fact that in the classical theory the algebra $\mathfrak{h}$ only exists when the metric is non-degenerate. Thus, the new point of view advertised here is that the representation of the algebra $\mathfrak{h}$ is required to hold only in the sector of the Hilbert space in which the quantum metric is non-degenerate. If the span of such states is dense and invariant then checking the algebra on this sector is sufficient.

We thus propose that to identify a non degenerate sector in LQG becomes an integral part of looking for an anomaly free representation of $\mathfrak{h}$. Such a sector can hopefully be extracted, e.g., by using renormalisation
methods as follows: One first defines a family of theories at finite resolutions and can identify non-degenerate states at finite resolutions since the number of degrees of freedom is (locally) finite. Then, if the fixed point family of the renormalisation flow exists, the family of finite resolution non-degenerate states is consistent and defines finite resolution projections of continuum non-degenerate states. These qualify as the continuum states on which to test the continuum algebra $\mathfrak{h}$.

For the much simpler $U(1)^3$ quantum gravity model we show in our companion paper [25] that one can actually complete all the steps of the quantisation and establish the anomaly freeness of $\mathfrak{h}$ even without renormalisation and without using dual spaces. Quantum non-degeneracy is crucial for this to be possible. It is therefore conceivable that this mechanism also works in realistic quantum gravity.

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