A Dozen Problems, Questions and Conjectures about Positive Scalar Curvature

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Abstract

Unlike manifolds with positive sectional and with positive Ricci curvatures which aggregate to modest (roughly) convex islands in the vastness of all Riemannian spaces, the domain \( \{ \mathcal{S} > 0 \} \) of manifolds with positive scalar curvatures protrudes in all direction as a gigantic octopus or an enormous multi-branched tree. Yet, there are certain rules to the shape of \( \{ \mathcal{S} > 0 \} \) which limit the spread of this domain but most of these rules remain a guesswork.

In the present paper we collect a few "guesses" extracted from a longer article, which is still in preparation: 100 Questions, Problems and Conjectures around Scalar Curvature.

Some of these "guesses" are presented as questions and some as conjectures. Our formulation of these conjectures is not supposed to be either most general or most plausible, but rather maximally thought provoking.

1 Definition of Scalar Curvature.

The scalar curvature of a \( C^2 \)-smooth Riemannian manifold \( X = (X, g) \), denoted \( Sc = Sc(X) = Sc(X, g) = Sc(g) \) is a continuous function on \( X \), written as
$Sc(X)(x)$ and $Sc(g)(x)$, $x \in X$, which is uniquely characterised by the following four properties.

1. **Additivity under Cartesian-Riemannian Products**.

$$Sc(X_1 \times X_2, g_1 \oplus g_2) = Sc(X_1, g_1) + Sc(X_2, g_2),$$

where this equality is understood point-wise,

$$Sc(X_1 \times X_2)(x_1, x_2) = Sc(X_1)(x_1) + Sc(X_2)(x_2).$$

2. **Scale covariance**.

$$Sc(X, \lambda^2 \cdot g) = \lambda^2 \cdot Sc(X).$$

Thus, for instance, since $(\mathbb{R}^n, g_0)$ is isometric to $(\mathbb{R}^n, \lambda^2 \cdot g_0)$ for the Euclidean metric $g_0$,

$$Sc(\mathbb{R}^n) = 0$$

and

$$vol(\lambda B_x(X, \varepsilon)) > vol(B_{\lambda x'}(X', \varepsilon))$$

for all sufficiently small $\varepsilon > 0$.

This volume inequality, in agreement with 1, is additive under Riemannian products: if

$$vol(B_{x_i}(X, \varepsilon)) > vol(B_{x'_i}(X'_i, \varepsilon)), \text{ for } \varepsilon \leq \varepsilon_0,$$

and for all points $x_i \in X_i$ and $x'_i \in X'_i$, $i = 1, 2$, then

$$vol_n(B_{(x_1, x_2)}(X_1 \times X_2, \varepsilon_0)) > vol_n(B_{(x'_1, x'_2)}(X'_1 \times X'_2, \varepsilon_0))$$

for all $(x_1, x_2) \in X_1 \times X_2$ and $(x'_1, x'_2) \in X'_1 \times X'_2$.

This follows from the Pythagorean formula

$$\text{dist}_{X_1 \times X_2} = \sqrt{\text{dist}_{X_1}^2 + \text{dist}_{X_2}^2},$$

and the Fubini theorem applied to the “fibrations” of balls over balls:

$$B_{(x_1, x_2)}(X_1 \times X_2, \varepsilon_0)) \to B_{x_1}(X_1, \varepsilon_0) \text{ and } B_{(x'_1, x'_2)}(X'_1 \times X'_2, \varepsilon_0)) \to B_{x'_1}(X'_1, \varepsilon_0),$$

where the fibers are balls of radii $\varepsilon \in [0, \varepsilon_0]$ in $X_2$ and $X'_2$.

4. **Normalisation/Convention for 2-spheres.** The unit sphere $S^2 = S^2(1)$ has constant scalar curvature 2 (twice the sectional curvature).

It is an elementary exercise to prove the following.

*1 The function $Sc(X, g)(x)$ which satisfies 1-4 exists and unique;

*2 The unit spheres and the hyperbolic spaces with sect.curv = -1 satisfy

$$Sc(S^n(1)) = n(n - 1) \text{ and } Sc(H^n(-1)) = -n(n - 1).$$
Thus,  
\[ Sc(S^n(1) \times H^n(-1)) = 0 = Sc(\mathbb{R}^n), \]
which implies that the volumes of the small balls in \( S^n(1) \times H^n(-1) \) are "very close" to the volumes of the Euclidean 2n-balls.

\[ * \quad \text{The scalar curvature of a Riemannian manifold } X \text{ is equal to the sum of the sectional curvatures at the bivectors of an orthonormal frame in } X, \]
\[ Sc(X)(x) = \sum_{i,j} c_{ij}, \quad i,j = 1, ..., n. \]

For example, all compact Riemannian symmetric spaces \( X \), except for the \( n \)-torus \( T^n \), have \( Sc(X) > 0 \), while \( T^n \), being covered by \( \mathbb{R}^n \), has \( Sc(T^n) = 0 \).

It may be tempting to take the above \( \bullet_1 - \bullet_4 \) for a definition of scalar curvature for singular metric spaces \( X \). In fact, it may work for \( X \) with moderate singularities, e.g. for Alexandrov’s spaces with sectional curvatures bounded from below (see [1]), where the properties of the so defined scalar curvature must be comparable to what is observed in the smooth case (see section 7).

Yet, volumes of balls to not touch the heart of the scalar curvature; we suggest an alternative in section 7.

2 Soft and Hard Facets of Scalar Curvature.

We are not so much concerned with the scalar curvature \( Sc(X) \) per se, but rather with the effect of lower scalar curvature bounds on the geometry and the topology of \( X \), where, for instance, the inequality "\( Sc(X) > 0 \)" can be defined by saying that

all sufficiently small balls \( B_x(\varepsilon) \subset X, \varepsilon \leq \varepsilon_0(x) > 0 \), have the volumes smaller than the volumes of the equidimensional Euclidean \( \varepsilon \)-balls.

Then "\( Sc(X) \geq 0 \)" is defined as
\[ Sc(X) > -\varepsilon \] for all \( \varepsilon > 0 \).

Similarly
"\( Sc(X) \geq \sigma \), \( \sigma > 0 \), is equivalent the volumes of \( B_x(\varepsilon) \) in \( X \) being smaller than the volumes of the \( \varepsilon \)-balls in the Euclidean spheres \( S^n(R) \) of radii \( R > \sqrt{(n(n-1))/\sigma} \),
and \( Sc(X) \geq -\sigma \) is expressed by
the bound on the volumes of \( B_x(\varepsilon) \) by those of the \( \varepsilon \)-balls in the hyperbolic spaces with constant the sectional curvatures \( < -\sigma/n(n-1) \).

Alternatively, "\( Sc(X) \geq -\sigma \)" can be defined with no reference to hyperbolic spaces by the reduction to the case \( \sigma = 0 \) and appealing to the relation
\[ Sc(X \times S^m(R)) \geq 0 \] for \( R = \sqrt{(m(m-1))/\sigma} \),
where one may use any \( m \geq 2 \) one likes.

\[ ^1 \text{Remarkably, this sum is independent of the frame by the Pythagorean theorem.} \]
\[ ^2 \text{[1] S.Alexander, V. Kapovitch, A.Petrunin, Alexandrov geometry,} \]
\[ \text{https://www.math.psu.edu/petrunin/} \]
Although the key role of the scalar curvature in general relativity was established by Hilbert’s variational derivation of the Einstein equation more than a century ago (see [2]) the significance of $\text{Sc}(X)$ in the global geometry and in topology remained obscure until 1963, when André Lichnerowicz (see [3]) showed that the inequality $\text{Sc}(X) > 0$ imposes non-trivial constraints on the topology of $X$.

For instance, Lichnerowicz’ theorem implies that

*If $m$ is even, then smooth complex projective hypersurfaces $X \subset \mathbb{C}P^{m+1}$ (these have real dimension $\text{dim}(X) = 2m$) of degrees $\geq m + 2$, e.g. $X \subset \mathbb{C}P^3$ given by the equation $x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$, admit no metrics with $\text{Sc} > 0$.\n
*This follows from the Atiyah-Singer formula for the (Atiyah-Singer)-Dirac operator $D$ confronted with (what is now called) the Schroedinger-Lichnerowicz-(Weitzenboeck-Bochner) identity.

In fact, the index formula implies that the index of $D$ on these manifolds does not vanish\footnote{This formula says in the present case that $\text{Ind}(D) = \hat{A}(X)$ where $\hat{A}(X)$ is a particular Pontryagin number of $X$.} and, consequently, there are non-zero harmonic spinors on these $X$ (i.e. solutions $s$ of $D(s) = 0$), while the Schroedinger-Lichnerowicz-(Weitzenboeck-Bochner) identity

$$D^2 = \nabla\nabla^* + \frac{1}{4}\text{Sc},$$

shows that closed manifolds with $\text{Sc} > 0$ admit no harmonic spinors.

Eleven years later, Nigel Hitchin\footnote{N. Hitchin, Harmonic Spinors , Adv. in Math. 14 (1974), 1-55.} used a more sophisticated 1971 version of the Atiyah-Singer index theorem which yields harmonic spinors on some exotic spheres $\Sigma^n$ (which are homeomorphic but not diffeomorphic to the ordinary spheres $S^n$) of dimensions $n = 8k + 1$ and $n = 8k + 2$ and which, together with the Schroedinger-Lichnerowicz’ identity, implies that

there is no metrics with $\text{Sc} > 0$ on these $\Sigma^n$.

Then Stefan Stolz, elaborating on the earlier work by several authors, showed that there are

no further obstructions to the existence of metrics with $\text{Sc} > 0$ on simply connected manifolds of dimension $\geq 5$ besides those delivered by the index theorem\footnote{S. Stolz. Simply connected manifolds of positive scalar curvature, Ann. of Math. (2) 136 (1992), 511-540.}

For instance

all simply connected manifolds of dimensions $n = 3, 5, 6, 7\mod 8$ admit metrics with positive scalar curvatures.

The proof of this theorem, which relies on surgery of manifolds with $\text{Sc} > 0$ and on the cobordism theory, suggests that manifolds with positive scalar curvature are almost as soft as smooth manifolds with no geometric constraints.
imposed on them. But the grand picture of scalar curvature in all its beauty unravels when one looks beyond this “almost”.

(The opposite inequality \( Sc(X) < 0 \) is truly and fully soft and, unlike \( Sc > 0 \), has no influence on the topology and global geometry of \( X \) what-so-ever (see [6]).

A manifestly rigid property of \( Sc > 0 \) can be already seen in the following corollary to Schoen-Yau solution of the Riemannian positive mass conjecture in relativity (see [7]).

**Solution of the Geroch Conjecture**

The Euclidean metric \( g_0 \) on \( \mathbb{R}^3 \) (which has \( Sc(g_0) = 0 \)) admits no non-trivial compactly supported perturbations \( g \) with \( Sc(g) \geq 0 \).

Namely, if a smooth Riemannian metric \( g \) on the Euclidean space \( \mathbb{R}^3 \) has \( Sc(g) > 0 \) and if \( g \) is equal to \( g_0 \) outside a compact subset in \( \mathbb{R}^3 \), then \( Sc(g) = 0 \); moreover, \( g \) is Riemannian flat, that is \( (\mathbb{R}^3, g) \) is isometric to \( (\mathbb{R}^3, g_0) \).

This result has been refined and generalised in a variety of directions (see below and also [13] and [21] at the end of the next section and references therein) but the rigidity of \( Sc > 0 \) we are after, albeit related to the above, is of different nature. In fact what we look for is

a structurally organised set of (desirably sharp) geometric inequalities satisfied by manifolds with \( Sc > 0 \), more generally, with \( Sc \geq \sigma \).

Also, we search for a general category (or categories) of spaces, or other kind of objects, which would satisfy (certain classes of) such inequalities.

**Additional Remarks and References.**

Geroch conjecture has been validated in all dimensions:

The Euclidean metrics on \( \mathbb{R}^n \) for all \( n \) admit no non-trivial compactly supported perturbations with \( Sc \geq 0 \).

This (trivially) follows, for instance, from non-existence of metrics with \( Sc > 0 \) on the \( n \)-tori where the latter can be most easily proved by applying the index theorem to suitably “twisted” Dirac operators.

Witten suggested a different way of using the Dirac operator in the context of the positive mass problem, where the index theorem is replaced by a direct proof of harmonic stability of parallel spinors on \( \mathbb{R}^n \) under certain perturbations of the Euclidean metric.

By a similar method, Min-Oo (see [8]) proved that

the hyperbolic metric \( g_0 \) on the real hyperbolic space \( H^3_\mathbb{R} \) admits non-trivial compactly supported perturbations \( g \) with \( Sc(g) \geq -n(n-1) = Sc(g_0) \).

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8{[6]} J. Lohkamp, Metrics of negative Ricci curvature, Annals of Mathematics, 140 (1994), 655-683.

9{[7]} R. Schoen and S.-T. Yau, On the proof of the positive mass conjecture in general relativity, Commun. Math. Phys. 65, (1979). 45-76.

10Attribution of this simplified podescribessitive mass conjecture to Robert Geroch is made in the above cited paper by Schoen and Yau.

In fact, the full Riemannian positive mass conjecture which describes possible asymptotic behaviours of metrics with \( Sc > 0 \) on \( \mathbb{R}^3 \) (and on \( \mathbb{R}^n \) for this matter) which are close (rather than equal) to the Euclidean metric at infinity follows from this Geroch conjecture according to J. Lohkamp, Scalar curvature and hammocks, Math. Ann. 313 (1999), 385-407.

11{[8]} M. Min-Oo, Scalar curvature rigidity of asymptotically hyperbolic spin manifolds., Math. Ann. 285, 527-539 (1989)
Apparently, it is unknown if other symmetric spaces of non-compact types admit compactly supported perturbations of their Riemannian metrics which would increase scalar curvature.

3 Bounds on the Uryson Width, Slicing Area and Filling Radius.

A. Conjecture. Let $X$ be an $n$-dimensional Riemannian manifold with scalar curvature bounded from below by

$$Sc(X) \geq n(n - 1) = Sc(S^n).$$

Then the $(n - 1)$-dimensional Uryson width of $X$ is bounded by a universal constant.

This means that there exists a continuous map from $X$ to an $(n - 1)$-dimensional polyhedral space $P$,

$$f : X \to P = P^{n-1},$$

such that the pullbacks of all points have controllably bounded diameters, namely,

$$\text{diam}_X(f^{-1}(p)) \leq \text{const} \text{ for all } p \in P.$$

for some universal constant $\text{const} > 0$ possibly (and undesirably) depending on $n$.

This conjecture says, in effect, that $n$-dimensional manifolds $X$ with $Sc(X) \geq \sigma > 0$ "topologically spread" in at most $n - 1$ directions.

In fact, one expects that these $X$ spread only in $n - 2$ direction which can be formulated as follows.

A++. Conjecture. The above $X$ admits a continuous map $f$ to an $(n - 2)$-dimensional polyhedral space $P$, such that $\text{diam}_X(f^{-1}(p)) \leq \text{const}^* \text{ for all } p \in P$.

But the most attractive (and least tenable) is the conjecture $A_{++}$ below which claims that closed manifolds with $Sc \geq \sigma > 0$ can be sliced by surfaces with small areas according the following definition.

Slicings and Waists. An $m$-sliced $n$-cycle, $m \leq n$, is an $n$-dimensional pseudomanifold $P = P^n$ partitioned into $m$-slices $P_q \subset P$, which are the pullbacks of the points of a simplicial map $\varphi : P \to Q$ where $Q$ is an $(n - m)$-dimensional pseudomanifold and where all pullbacks $P_q = \varphi^{-1}(q) \subset P$ have $\dim(P_q) \leq m$, $q \in Q$.

(Sometimes one insists that $\varphi$ must be proper, hence, with compact pullbacks $\varphi^{-1}(q)$, even if $P$ is non-compact.)

The $m$-waist (mod 2), denoted $\text{waist}_m(h)$, of a homology class $h \in H_n(X; \mathbb{Z}_2)$ is

$$\text{the infimum of the numbers } w,$$

such that $X$ receives a Lipschitz map from a compact $m$-sliced cycle, $\phi : P^n \to X$, which represent $h$, i.e.

$$\phi_*[P] = h.$$
and the

*the images of all slices in X have m-volumes ≤ w,*

where these ”volumes of the images” are counted with multiplicities (which is unneeded for generically 1-1 maps.)

**A_{++} Conjecture.** Let X be a closed n-dimensional Riemannian manifold the scalar curvature of which is bounded from below as earlier:

\[ Sc(X) ≥ n(n−1)(= Sc(S^n)). \]

Then the slicing area of the fundamental homology class \([X] ∈ H_n(X;\mathbb{Z}_2)\) is bounded by

\[ \text{waist}_2[X] ≤ \text{const}_{++}. \]

(Ideally, one expects \(\text{const}_{++} = \text{waist}_2(S^n) = \frac{4\pi}{\sqrt{2}}\) by an Almgren’s theorem.)

The above conjectures can be interpreted as saying that X contains ”many” small subsets of dimensions 1 and/or 2.

For instance, A implies that that X contains a topologically significant/representative family of 1-dimensional subsets (graphs) with diameters \(\lesssim \frac{1}{\sqrt{\sigma}}\).

This suggests the following.

**(a) Conjecture.** If \(Sc(X) ≥ \sigma > 0\) and if X is a closed (compact without boundary) manifold, then X contains a closed minimal geodesic of length \(≤ \frac{\text{const}_{++}}{\sqrt{\sigma}}\), or, at least, a stationary one-dimensional \(\mathbb{Z}_2\)-current of diameter (better length) \(≤ \frac{\text{const}_{++}}{\sqrt{\sigma}}\).

And A_{++} actually implies the following.

**(a_{++}) Conjecture.** Closed manifolds X with \(Sc(X) ≥ \sigma > 0\) contain closed minimal surfaces (i.e. stationary two-dimensional \(\mathbb{Z}_2\)-currents) of areas \(≤ \frac{\text{const}_{++}}{\sigma}\).

Below is a weaker version of A which already imposes non-trivial topological constraints on X.

**A Conjecture.** If \(Sc(X) ≥ n(n−1)\) then the filling radius of X is bounded by

\[ \text{fil.rad}(X) ≤ \text{const}_-. \]

*Definition of fil.rad.* If \(X = (X,g)\) is closed Riemannian manifold then the filling radius is equal to the infimum of \(R > 0\), such that the cylinder \(X^* = X × [0,1)\) admits a Riemannian metric \(g^*\) with the following three properties.

1. the restriction of \(\hat{g}\) to \(X = X_0 × \{0\} ⊂ X × [0,1) = X^*\) is equal to g; moreover,

\[ \text{dist}_{g^*}(X) = \text{dist}_g. \]

This means that the \(g\)-shortest curves in X between all pairs of points in X minimise the \(g^*\)-lengths of such curves in \(X^* \supset X\).

2. All points in \(X^*\) lie within distance at most \(R\) from X,

\[ \text{dist}_{g^*}(x^*,X) ≤ R \text{ for all } x^* ∈ X^*. \]
The \( n \)-dimensional volumes of the submanifolds \( X \times \{ t \} \subset X \times [0,1) = X^\times \), \( t < 1 \), with respect to \( g^* \) vanish in the limit for \( t \to 0 \),

\[
\text{vol}(X \times \{ t \}) \to 0 \text{ for } t \to 1.
\]

(The equivalence of this definition to the usual one follows from the the filling volume inequality see \cite{9} and references therein).

Then the filling radius of a compact manifold \( X \) with boundary – our manifolds may, a priori, have boundaries and/or to be incomplete – is defined as \( \text{fil.rad} \) of the double of \( X \) along the boundary and \( \text{fil.rad} \) of an open \( X \) is defined via exhaustions of \( X \) by compact submanifolds.

It is obvious that \( A_+ \Rightarrow A \Rightarrow A_- \) and that \( A_+ \) is optimal in a way.

Indeed, the product \( X_r = X_0 \times S^2(r) \), where \( X_0 \) is, a compact manifold and \( S^2(r) \) is the 2-sphere of small radius \( r \to 0 \), (these spheres have \( Sc(S^2(r)) = \frac{2}{r^2} \)), has \( Sc(X_r) \geq \left( \frac{2}{r^2} - \text{const}_{X_0} \right) \to +\infty \), while the \((n-2)\)-dimensional size/spread of \( X_r \) is as large as that of \( X_0 \).

Also one knows (see \cite{17} at the end of this section and references therein) that

\[
A_{++} \Rightarrow A_-.
\]

(It is plausible in view of \cite{18} that \( A_{++} \Rightarrow A_+ \).

On the other hand, it is not hard to show that if the if the isometry group of a Riemannin manifold \( \hat{X} \) acts cocompactly on \( \hat{X} \), i.e \( \hat{X}/\text{isom}(\hat{X}) \) is compact, and if \( \hat{X} \) is contractible, then

\[
\text{fil.rad}(\hat{X}) = \infty.
\]

Therefore, \( A_- \) yields the following topological \( Sc \times 0 \)-non-existence corollary.

**B. Conjecture.** Closed manifolds \( X \) with contractible universal coverings \( \hat{X} \) admit no metrics with \( Sc > 0 \).

(Granted \( B \), the non-strict inequality \( Sc(X) \geq 0 \) implies that \( X \) Ricci flat by Kazdan-Warner’s perturbation theorem (see \cite{10} \cite{13}). And since \( \hat{X} \) is contractible, the universal covering \( \hat{X} \) is isometric to the Euclidean space \( \mathbb{R}^n \), \( n = \text{dim}(X) \), by the Cheeger-Gromoll splitting theorem.)

**Remarks and References.**

However plausible, none of the \( A \)-conjectures (above dimension 2) has been confirmed except for \( A_+ \) for 3-manifolds \( X \) with (apparently non-sharp) constant \( \text{const}_+ = 2\pi\sqrt{6} \) (see \cite{14} below).

On the other hand \( B \) is known to hold for many manifolds \( X \), starting from the case of \( n \)-tori due to Schoen and Yau. Later \( B \) was proven by a use of twisted Dirac operators\cite{14} for several classes of manifolds with "large" universal coverings including those \( X \) which admit metrics with non-positive sectional curvatures.

\[12\] L. Guth, Notes on Gromov’s systolic estimate, Geom Dedicata (2006) 123:113-129.

\[13\] If a metric \( g_0 \) with \( Sc \geq 0 \) can’t be perturbed to \( g \) with \( Sc(g) > 0 \), then \( \text{Ricci}(g) = 0 \).

\[10\] J Kazdan, F. Warner, Existence and Conformal Deformation of Metrics With Prescribed Gaussian and Scalar Curvatures, Annals of Mathematics, 101, # 2. (1975), pp. 317-331.

\[14\] This means: Dirac operators with coefficients in some (possibly infinite dimensional) vector bundles.
Below are a few relevant papers where one can find further references.

[11] Yau, S.T., and Schoen, R. On the Structure of Manifolds with positive Scalar Curvature. Manuscripta mathematica 28 (1979): 159-184.

[12] J. Lohkamp, The Higher Dimensional Positive Mass Theorem II, (2016) arXiv:1612.07505.

[13] R. Schoen, S.T. Yau, Positive Scalar Curvature and Minimal Hypersurface Singularities, (2017) arXiv:1704.05490.

In [11], the authors introduced their method of induction descent by minimal hypersurfaces and proved non-existence of metrics with $Sc > 0$ on the $n$-torus and, more generally, on $n$-dimensional manifolds $X$ which admit smooth maps $X \to \mathbb{T}^{n-2}$, such that the homology classes in $H_2(X)$ represented by the pull backs of generic points are non-spherical.

Originally, this method was limited to $n \leq 7$, but the techniques developed in [12] and [13] apparently remove this limitation.

[14] M. Gromov, H. B. Lawson, Jr., Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Publ. Math. IHS 58 (1983), 295-408.

In this paper besides above mentioned $A_3$, for 3-manifolds, we rule out complete metrics with $Sc > 0$ on certain classes of manifolds, including

closed orientable $n$-dimensional spin manifolds $X$ which admit continuous maps to complete manifolds $Y$ with non-positive sectional curvatures, such that the fundamental classes $[X] \in H_n(X)$ go to non-zero classes in $H_n(Y)$ under these maps.

[15] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures, in Proc of 1993 Conf. in Honor of the Eightieth Birthday of I. M. Gelfand, Functional Analysis on the Eve of the 21st Century: Volume I Progress in Mathematics, (1996) pp 1-213, Vol. 132.

This paper presents a geometric perspective on the Dirac operator and soap bubble methods in the study of scalar curvature and related problems.

[16] S. Markvorsen, M. Min-Oo, Global Riemannian Geometry: Curvature and Topology, 2012 Birkhuser.

A chapter in this book offers a friendly introduction to the Dirac operator methods in the $Sc > 0$ problems.

[17] L. Guth, Metaphors in systolic geometry. In: Proceedings of the International Congress of Mathematicians. 2010, Volume II, pp. 745-768.

[18] L. Guth, Volumes of balls in Riemannian manifolds and Uryson width. Journal of Topology and Analysis, Vol. 09, No. 02, pp. 195-219 (2017).

These two papers and references therein give a fair idea of results and ideas around the filling radius.

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15 This trivially implies non-existence of compactly supported perturbations with $Sc > 0$ of the Euclidean metric on $\mathbb{R}^n$.

16 A manifold of dimension $n \geq 3$ is spin if the restrictions of the tangent bundle $T(X)$ to all immersed surfaces in $X$ are trivial bundles.

Most (all?) known non-existence results for $Sc > 0$ obtained for spin manifolds more or less automatically generalise to manifolds whose universal coverings are spin, i.e where $T(X)$ trivialises on all immersed 2-spheres in $X$.

17 Also see MinOo, K-Area, mass and asymptotic geometry, http://ms.mcmaster.ca/minoo/mypapers/crm

9
The authors of these two papers are concerned with topological versions of $A$, for certain classes of manifolds $X$.

This is survey of topological obstructions to metrics with $Sc > 0$ on spin manifolds $X$ expressed in terms of indices of Dirac operators twisted with $C^*$-algebras of $\pi_1(X)$.

Also obstructions for 4-dimensional manifolds $X$ with non-vanishing Seiberg-Witten invariants due to Taubes and LeBrun are described in this paper.

This is an overview of waists and related invariants which may bear some relevance to $Sc \geq \sigma$.

4 Extremality and Rigidity with Positive Scalar Curvature.

The proof(s) of the above $A$-conjectures (let them be only approximately true) would require constructions of certain maps or spaces which makes these conjectures difficult.

What is easier is getting upper bounds on the “size” of an $X$ with $Sc(X) \geq \sigma > 0$ by proving lower bounds on dilations of topologically significant maps from $X$ to (more or less) standard manifolds $Y$.

The first sharp bound of this kind was proved in

- [23] M. Llarull, Sharp estimates and the Dirac Operator, Math. Ann. 310 (1998), 55-71,

followed by

- [24] M. Min-Oo, Scalar Curvature Rigidity of Certain Symmetric Spaces, Geometry, Topology and Dynamics (Montreal, PQ, 1995), CRM Proc. Lecture Notes, 15, Amer. Math. Soc., Providence, RI, 1998, pp. 127-136.

and by

- [25] S. Goette and U. Semmelmann, Scalar curvature estimates for compact symmetric spaces, Differential Geom. Appl. 16(1), (2002) 65-78, where further references can be found.

What is proven in these papers can be expressed in the the following terms.

Extremality/Rigidity. A Riemannian metric $g$ on a manifold $Y$ is called length extremal if it
can’t be enlarged without making the scalar curvature smaller somewhere. Namely, the inequalities

\[ \text{Sc}(g) \geq \text{Sc}(g_0) \text{ and } g \geq g_0 \]

for a Riemannian metric \( g \) on \( Y \) imply

\[ \text{Sc}(g) = \text{Sc}(g). \]

Then the stronger implication

\[ [\text{Sc}(g) \geq \text{Sc}(g_0)] \& [g \geq g_0] \Rightarrow [g = g_0] \]

is qualified as length rigidity of \( g_0 \).

**CY-Example.** If a closed manifold \( Y \) admits no metric with \( \text{Sc} > 0 \), then all \( g_0 \) with \( \text{Sc}(g_0) = 0 \) are extremal according to this definition. Instances of such scalar flat manifolds are flat Riemannian manifolds (with universal coverings \( \mathbb{R}^n \)) and also (simply connected) hypersurfaces \( Z \subset \mathbb{C} \mathbb{P}^{n+1} \) of degree \( n+2 \) and even \( n \), with Ricci flat Calabi-Yau metrics, where non-existence of metrics with \( \text{Sc} > 0 \) on these \( Z \) follows from the Lichnerowicz theorem.

Next, define area extremality and area rigidity by relaxing the inequality \( g \geq g_0 \), which says in effect that \( \text{length}_g(C) \geq \text{length}_{g_0}(f(C)) \) for all smooth curves \( C \subset Y \), to

\[ \text{area}_g(\Sigma) \geq \text{area}_{g_0}(f(\Sigma)) \]

for all smooth surfaces \( \Sigma \subset Y \), where the extremality and rigidity requirements remains the same: \( \text{Sc}(g) = \text{Sc}(g) \) and \( g = g_0 \).

Stronger versions of these extremalities and rigidities allow modifications of the topology as well as geometry of \( Y \), where the role of “topologically modified” \( Y \) are played by a Riemannian manifold \( X = (X, g) \) and a map \( f : X \rightarrow Y \), where the above inequalities are understood as

\[ \text{Sc}(g)(x) \geq \text{Sc}(g)(f(x)), \text{length}_g(C) \geq \text{length}_{g_0}(f(C)) \]

and

\[ \text{area}_g(\Sigma) \geq \text{area}_{g_0}(f(\Sigma)) \]

correspondingly.

Accordingly, the required conclusion for extremality is

\[ \text{Sc}(g)(x) = \text{Sc}(g)(f(x)), \]

while both, the length and the area rigidities, signify that

\[ \text{length}_g(C) = \text{length}_{g_0}(f(C)) \]

for all smooth curves \( C \subset X \).

Of course, these definitions makes sense only for particular topological classes of manifolds \( X \) and maps \( f \), such for instance as the class \( \{ \text{deg} \neq 0 \} \) of orientable manifolds of dimension \( n = \text{dim}(Y) \) and \( C^2 \)-smooth maps with non-zero degrees.

\[ \text{Extremal manifolds define, in a way, the boundary of the domain } \{ \text{Sc} \geq 0 \} \text{ of manifolds with } \text{Sc} \geq 0. \]

\[ \text{The condition } \text{Sc}(g_0) = 0 \text{ implies } g_0 \text{ Ricci}(g_0) = 0 \text{ on these } Y \text{ by the Kazdan-Warner perturbation theorem, see [10] in section 3.} \]
C. Problem. Find verifiable criteria for extremality and rigidity, decide which manifolds admit extremal/rigid metrics and describe particular classes of extremal/rigid manifolds.

For instance,

do all closed manifolds which admits metrics with $Sc \geq 0$ also admit (length) extremal metrics?

More specifically, prove (disprove?) the following.

C.1. Conjecture. All compact Riemannin symmetric spaces are area extremal in the class $\{DEG \neq 0\}$ and those which have $\text{Ricci} > 0$ (this is equivalent to absence of local $\mathbb{R}$ factors, and to finiteness of fundamental group) are area rigid in this class.

This conjecture was proved by Larull (see [23] above) in the case $Y = S^n$, under the additional assumption of $X$ being spin [21].

Then Min-Oo [24] proved area extremality for Hermitian symmetric spaces in the class $\{SPIN, DEG \neq 0\}$, where the maps $f : X \to Y$, besides having degrees $\neq 0$, must be spin [21].

This was generalised by Goette and Semmelmann [25] who proved

area extremality in $\{SPIN, DEG \neq 0\}$ of compact (here it means closed) Kähler manifolds with $\text{Ricci} \geq 0$, rigidity for $\text{Ricci} \geq 0$.

Moreover, they establish

area rigidity in $\{SPIN, DEG \neq 0\}$ of certain (non-Hermitian) compact symmetric spaces including those with non-vanishing Euler characteristics and also of Riemannian metrics on $S^{2m}$ with positive curvature operators.

These extremality and rigidity theorems are proven in the non-Kählerian cases by sharply evaluating the contribution from $\mathcal{f}(\mathcal{S}^1(Y))$ in the Schroedinger-Lichnerowicz formula for the Dirac operator on $X$ twisted with the $\mathcal{f}$-pullback of the spinor$^*$ bundle $\mathcal{S}^1(Y)$ which is, in the case where $\chi(Y) \neq 0$ is confronted with the index theorem.

(The case of odd dimensional spheres $S^{2m-1}$, which depends on an additional argument(s) applied to maps $X \times S^1 \to S^{2m}$ [22] seems to apply only to metrics on $S^{2m-1}$ with constant sectional curvatures.)

Since $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$, there are at most two isomorphism classes of vector bundles with rank $\geq 3$ over connected surfaces $\Sigma$ (exactly two for closed $\Sigma$), where the trivial bundle is called spin and where bundles of rank $< 3$ are spin if their Whitney sums with trivial bundles are spin. An orientable vector bundle $V$ of over a topological space $B$ is spin if the pullbacks of $V$ under continuous maps $\phi : \Sigma \to B$ for all surfaces $\Sigma$ are spin. A manifold $X$ is spin if its tangent bundle is spin.

The spin condition is necessary for the definition of the Dirac operator on $X$ but some twisted Dirac operators make sense on non-spin manifolds.

A map $f : X \to Y$ is spin if the pullbacks $\mathcal{f}(T(X))$ for maps of surfaces, $\phi : \Sigma \to X$, satisfy

$[\phi(\mathcal{f}(T(X))) \text{ is spin}] \Leftrightarrow [(\phi \circ f)(T(Y)) \text{ is spin}]$ for all $\Sigma$ and $f$. Equivalently, a map $f$ between orientable manifolds is spin if the Whitney sum $T(X) \oplus f^*(T(Y))$ is spin.

Obviously, the identity map $id : Y \to Y$ is spin and if $Y$ is spin, e.g. $Y = S^n$, then $[f : X \to Y \text{ is spin}] \Leftrightarrow [X \text{ is spin}]$.

22Larull uses the product metric on $X \times S^1$, where his calculation applies even though the scalar curvature $Sc(X \times S^1)$, which is $\geq Sc(S^{2m-1})$, may be smaller than $Sc(S^{2m})$.

Alternatively, one can use the spherical suspension metric $g^S$ (of $g$ on $X$) on (the bulk of) $X \times S^1$, which has $Sc(g^S) \geq Sc(S^{2m})$ and thus allows a formal reduction of the $2m - 1$ case
And in the Kähler case, this is done with the "virtual square root" of the canonical (complex) line bundle on $Y$ instead of $S^*(Y)$.

**Spin or non-Spin?** In all of the above cases one can replace the spin condition for $f : X \to Y$ by this condition for the corresponding map between the universal coverings, $\tilde{f} : \tilde{X} \to \tilde{Y}$, where a version of Atiyah’s $L^2$-index theorem applies.

Probably, "spin" can be removed all together in these theorems but this seems beyond reach of the present day methods.

On the other hand, the spin condition is essential for the extremality in the class $\{\text{SPIN}, \deg \hat{A} \neq 0\}$ where the dimension of $X$ can be greater than $n = \dim(Y)$ and where the condition $\deg(f) \neq 0$ is replaced by $\deg_A(f) \neq 0$, where the $\hat{A}$-degree $\deg_A(f)$ stands for the $\hat{A}$-genus of the $f$-pull back of a generic point $y \in Y$,

$$\deg_A(f) = \hat{A}(f^{-1}(y)).$$

(Here, strictly speaking, $f$ must be smooth; if $f$ is just continuous, this applies to a smooth approximation of $f$, where the so defined $\hat{A}$-degree does not depend on a choice of approximation.).

This implies for instance, that the products of the above $Y$, e.g. of $Y = S^n$ by the Calabi-Yau manifolds with $\hat{A} \neq 0$, e.g with $Z$ from the above CY-example are area extremal in the class $\{\text{SPIN}, \deg_X \hat{A} \neq 0\}$ as well as in the class $\{\text{SPIN}, \deg_X \hat{A} \neq 0\}$ where spin condition is delegated to $\tilde{f} : \tilde{X} \to \tilde{Y}$.

Notice, however, that neither simply connected Calabi-Yau manifolds $Z$ themselves nor their products by $Y$ are extremal in the class $\{\text{SPIN}, \deg_X \hat{A} \neq 0\}$, at least if $\dim(Z) \geq 5$.

Indeed the connected sums $X = Z \# (-Z)$, where "-" stands for the reversal of orientation and where the obvious map $Z \# (-Z) \to Z$ has degree 1, admit metrics with $Sc > 0$ by Stolz’ theorem mentioned in section 2. sleeker

It seems that the there are two divergent, yet interconnected by bridges, branches in the tree of $Sc(X) \geq 0$, where a smoother and sleeker one involves differential structure and depends on spin, while the other one is made of rougher staff such as the homotopy classes of $X$.

Probably, the second branch can be transplanted to a harsh world inhabited by singular spaces but fully cleaning off spin from this branch is by no means easy even for smooth $X$.

**Extremality and Rigidity of Products.** It seems not hard to show that the Riemannian products of the area extremal/rigid manifolds in the above examples are area extremal/rigid which suggests to the following.

23 Apparently, no single case of extremality of a closed simply connected manifold $X$ of dimension $n \geq 3$ is amenable to the the minimal hypersurface techniques, except, may be(?) for $X = S^3$.

24 The smooth branch is manifested by $\hat{A}$ and the mod 2 $\alpha$-invariant in the index formula while the rough branch is represented by the Chern character and supported by minimal hypersurfaces.

25 I have not verified the proof in detail at this point.
C2. Question. Are the Riemannian products of all area extremal/rigid manifolds area extremal/rigid?

Smoothing Lipschitz Maps. The length extremal/rigid manifolds in some homotopy class of smooth maps remain extremal/rigid in the corresponding class of Lipschitz maps $f$.

This can be proven by a smooth approximation of these $f$ with a minor change of their length dilations.

But this is unclear for the area extremality and/or area rigidity, since, conceivably all smooth approximation $f'$ of a Lipschitz map $f : X \to Y$ may have $\text{area}(f'(\Sigma)) \gg \text{area}(f(\Sigma))$ for some $\Sigma$.

Normalisation by Scalar Curvature: Extremality/Sc and Rigidity/Sc. A map $f : X \to Y$ between Riemannian manifolds $X = (X, g)$ and $Y = (X, g_0)$ with positive scalar curvatures, $\text{Sc}(g), \text{Sc}(g_0) > 0$, is called length decreasing/Sc if it decreases the length of the curves measured in the metrics $\text{Sc}(X)^{-1}g$ and $\text{Sc}(g_0)^{-1}g_0$, i.e. if it decreases the integrals of $\sqrt{\text{Sc}}$ over all curves in $X$. Similarly one understand decrease/Sc of areas of surfaces $\Sigma \subset X$ under maps $X \to Y$, etc.

Accordingly, one defines length/area extremality/Sc of a $Y$ as non existence of strictly length/area decreasing/Sc maps $X \to Y$ in a given class of manifolds and maps, while the rigidity/Sc signifies that all length/area non-increasing/Sc maps $f : X \to Y$ are homotheties (similarities) with respect to the original metrics, i.e. $f^*(g_0) = \text{const} \cdot g$.

Since the “contribution of the twist” to the Schroedinger-Lichnerowicz formula for the twisted Dirac operator on $X$ scales as $\text{Sc}(X)^{-1}$, the arguments from the above cited papers based on this formula deliver the corresponding extremality/Sc and rigidity/Sc results. (This was pointed out in [26].)

Category $\mathcal{R}_{+}/\text{Sc}$. Let this be the category of Riemannian manifolds with $\text{Sc} > 0$ and length (alternatively, area) non-increasing/Sc maps.

C3. Question. How much of the geometry of spaces with $\text{Sc} > 0$ can be reconstructed in the category theoretic language of $\mathcal{R}_{+}/\text{Sc}$?

Extremality beyond $\text{Sc} \geq 0$. The condition $\text{Sc}(g) \geq 0$ may be not indispensable for extremality of $g$.

For instance, the double of the unit hyperbolic disk is (kind of) extremal for the natural $C^0$-continuous metric on it and there are similar high dimensional examples. But it is unclear if such metrics are ever smooth.

Relativisation of Non-existence Theorems for $\text{Sc} > 0$. Let $Y$ be a closed length or area extremal or rigid manifold in some class of smooth manifolds $X$ and smooth maps $f : X \to Y$, where this class is invariant under homotopies of maps.

Then, most (all?) known Dirac operator obstructions to the existence of metrics with $\text{Sc} > 0$ on closed manifolds $X_0$ naturally extend to similar obstructions to the existence of (strict) area decreasing/Sc maps in certain homotopy invariant classes of maps $X \to Y$, including $X = X_0 \times Y \to Y$ for $(x_0, y) \mapsto y$.

\[26\] It may (or may not) be worthwhile to normalise by $g \sim n(n - 1)\text{Sc}(X)^{-1}g$, $n = \text{dim}(X)$, and see what happens for $n \to \infty$.

\[27\] [26] M.Listing, The Scalar curvature on compact symmetric spaces, arXiv:1007.1832, 2010 - arxiv.org.
For instance, one knows that \((\co)\text{homologically symplectic manifolds} \ X_0\) with \(\pi_2(X_0) = 0\) admit no metrics with \(Sc > 0\) and the proof of this (see [15] cited in the previous section) also implies that words

*If \(Y\) is the above area-extremal manifold, e.g. \(Y = S^n\), then no homologically symplectic\(^{28}\) map \(f : X \to Y\), which, moreover, induces an isomorphism \(\pi_2(X) \to \pi_2(Y)\), can be strictly area decreasing/Sc.*

This suggests the following.

**C. Conjecture.** Let \(g\) be a metric on \(X\) and \(f_0 : X \to Y\) be a (smooth?) strictly length (area?) decreasing/Sc map in this class. Then there exists a smooth map \(f : X \to Y\), which, moreover, induces an isomorphism \(\pi_2(X) \to \pi_2(Y)\), such that the \(f\)-pullback submanifold \(f^{-1}(y_0) \in X\) admits a metric with \(Sc > 0\).

Also other properties, e.g. extremality, of manifolds \(X\) with \(Sc(X) > 0\) may have counterparts for length and area decreasing/Sc maps \(X \to Y\) and, furthermore, for foliations on \(X\).

**C. Question.** Are infinite dimensional counterparts of compact symmetric spaces, e.g. the Hilbert sphere \(S^\infty\), extremal/rigid in some class(es) of perturbations of their metrics?

## 5 Extremality and Gap Extremality of Open manifolds.

Let \(U \subset Y\) be an open subset in a extremal or rigid Riemannian manifold \(Y\) where the extremality/rigidity for this \(Y\) follows by the twisted Dirac operator argument from the previous section. Then the same argument yields the following.

* If the complement \(Z = Y \setminus U\) is non-empty, yet LC-negligible (explained below) then no complete orientable Riemannian manifold admits a smooth area non-increasing/Sc map \(f : X \to U\), which has non-zero degree\(^{29}\) and the lift of which to the universal coverings, \(\tilde{f} : \tilde{X} \to \tilde{U}\), is spin.

**LC-negligible Sets.** A piecewise smooth polyhedral subset \(Z\) in a Riemannian manifold \(Y\) is called LC-negligible if the Levi-Civita connection on the tangent bundle of \(X\) restricted to \(Z\) is split trivial.

- finite subsets in \(Y\) are LC-negligible;
- piecewise smooth graphs \(Z \subset Y\) with trivial monodromies around the cycles, e.g. disjoint unions of trees, are LC-negligible;
- simply connected isotropic (e.g. Lagrangian) submanifolds in Kähler manifolds are LC-negligible.

\(^{28}\)A smooth proper map between orientable manifold, \(f : X \to Y\), is homologically symplectic if the difference of the dimensions \(n_0 = n - m\) for \(n = \dim(X)\) and \(m = \dim(Y)\) is even and if there exists a closed 2-form \(\omega\) on \(X\) such that the integrals of \(\omega^n\) over the \(f\)-pullbacks of generic points \(y \in Y\) do not vanish.

In other words, the real fundamental cohomology class \([X]^n \in H^n_{\text{comp}}(X; \mathbb{R})\) with compact support is equal to the \(n\)-product of the \(f\)-pullback of \([Y]^n \in H^n_{\text{comp}}(Y; \mathbb{R})\) and the \(\frac{n}{2}\)-th-power of the class \([\omega] \in H^2(X; \mathbb{R})\),

\[ [X]^n = f^*([Y]^n) \cdot [\omega]^{\frac{n}{2}}. \]

\(^{29}\)Maps \(f : X \to Y\) of non-zero degree, by definition, must be equidimensional and proper.
This definition extends to general closed subsets $Z$, such as Cantor sets, for instance, by requiring that the monodromies along smooth curves $C$ in the $\varepsilon$-neighbourhoods of $Y$ are $o(\varepsilon \cdot \text{length}(C))$ as $\varepsilon \to 0$ but the geometry behind this definition needs to be clarified.

**D1. Problem.** Study essential properties, such as the Hausdorff dimensions, of these subsets $Z \subset Y$ and find cases (if there are any) where * remains valid for small, yet non-LC-negligible $Z \subset Y$, e.g., for (generic) smooth curves $Z$ in $Y$.

Notice in this regard that a simple surgery type argument (see Stolz’ paper [5] cited in section 2 and references therein) shows that

- if $Z$ is equal to the $k$-skeleton $T^k$ of a smooth triangulation $\mathcal{T}$ of a compact Riemannian manifold $(Y, g_0)$, for $k \geq 2$, then $U = Y \setminus Z$ admits a complete metric $g \geq g_0$ with $\text{Sc}(g) \geq \sigma_0 = \sigma_0(Y, Z) > 0$.

Moreover, it is easy to show that

the complements $U_\varepsilon = Y \setminus T^k_\varepsilon$ of the $k$-skeleta of the “standard fat” $\varepsilon$-refinement[30] of $\mathcal{T}$ admit complete Riemannian metrics $g_\varepsilon \geq g$ the scalar curvatures of which for $k \geq 2$ satisfy

$$\text{Sc}(g_\varepsilon) \geq \text{const} \frac{1}{\varepsilon^2}$$

for some constant $\text{const} = \text{const}(Y, \mathcal{T}) > 0$.

Thus * fails to be true, for $Z = T^k_\varepsilon$, $k \geq 2$, and small (how small?) $\varepsilon$.

On the other hand, the torical band width inequality from the next section shows that if, for instance, $Z$ is a codimension two torus in $Y$, e.g., $Z = T^2 \subset S^4$, then the complement $U = Y \setminus Z$ admits no complete metrics with $\text{Sc} \geq \sigma > 0$ whatsoever and the same applies to a large (how large) class of codimension two polyhedra $Z \subset Y$ with contractible universal coverings.

Non-existence of complete metrics $g \geq g_0$ with $\text{Sc} > \sigma_0$ on the above $U = (U, g_0)$ with $\text{Sc}(g_0) = \sigma_0$ may be interesting in its own right but this can’t be regarded as extremality of $g_0$, since a comparison of the manifolds $(U, g_0)$, which have bounded diameters with their competitors $(U, g)$ of infinite size is patently unfair. The true extremity issue for these $U$, thus, remains unresolved.

**D2. Question.** Do there ever exist length extremal domains $U \subset Y$, $U \neq Y$, in closed connected Riemannian manifolds $Y$ of dimensions $\geq 3$?

For instance, is the the sphere $S^3$ minus a point (or the 3-torus minus a point) extremal?

We still do not know the answer but, on the other hand, the following warped product construction sometimes delivers examples of both complete and non-complete extremal and rigid manifolds (compare §12 in [14] cited in section 3 and [27] cited below).

Let $Y_0 = (Y_0, g_0)$ be a Riemannian manifold with constant scalar curvature $\sigma_0$ and let $g_1 = \varphi^2 g_0 + dt^2$ be a Riemannian metric on $Y_1 = Y \times (l_-, l_+)$ for $-\infty \leq l_- < l_+ \leq \infty$, for some smooth function $\varphi = \varphi(t) > 0$ for $l_- < t < l_+$.

[30] It is more practical to start with a cubilation $\mathcal{T}$ of $Y$ which can be canonically $\varepsilon$-refined for $\varepsilon = \frac{1}{i}$, $i = 2, 3, \ldots$, by subdividing each $m$-cube into $i^m$-sub-cubes in an obvious way.
Then, by elementary calculation,

\[ Sc(g) = \frac{\sigma_0}{\varphi^2} - 2n\frac{\varphi''}{\varphi} - n(n-1)\frac{\varphi'^2}{\varphi^2}, \quad \text{where } n = \dim(Y_0).\]

Now, let \( g \) have constant scalar curvature, say \( Sc(g_1) = \sigma_1 \) for a given \( \sigma_1 \geq 0 \) and prescribe: \( \varphi(0) = 1 \) and \( \varphi'(0) = 0 \).

Then, regarded as an ODE and rewritten as

\[ f'' = \frac{1}{2}(n+1)f'^2 + \frac{n-1}{2n}\left(\frac{\varphi_0}{\varphi}\right) + \frac{\sigma_1}{2n} \quad \text{for } f = \log \varphi,\]

admits a unique solution \( f \) on some maximal (extremal) open interval \((l_1^{ext}, l_2^{ext})\) beyond which the solution does not extend.

**Examples.**
(a) If \( Y_0 = S^n \) and \( \sigma_1 = n(n+1) \), then \( Y_1 \) is equal to \( S^{n+1} \) minus two opposite points.
(b) If \( Y_0 = \mathbb{R}^n \) and \( \sigma_1 = 0 \), then \( Y_1 = \mathbb{R}^{n+1} \).
(c) If \( Y_0 = \mathbb{R}^n, \sigma_1 = n(n+1) = Sc(S^{n+1}) \) and \( n = 1 \), then \( Y_1 \) is equal to the universal covering of \( S^2 \) minus two opposite points.

In general, the manifold \((Y_1, g_1)\) is uniquely characterised by the following three properties.

\[ \dim_1 \text{ The scalar curvature of } Y_1 \text{ is everywhere equal to } n(n+1) \text{ for } n = \dim(Y_1) - 1. \]

\[ \dim_2 \text{ The isometry group of } Y_1 \text{ is } Iso(\mathbb{R}^n) = O(n) \rtimes \mathbb{R}^n \text{ times } \mathbb{Z}_2. \text{ (This } \mathbb{Z}_2 \text{ corresponds to the involution } t \leftrightarrow -t.) \]

\[ \dim_3 \text{ The band width of } Y_1 \text{ is } \frac{2\pi}{n+1}, \text{ where this width is understood in the present case as the distance between the two (one point) boundary components of } Y_1 \text{ in the metric completion } \bar{Y}_1 \supset Y_1. \]

(The band-like shape of \( Y_1 \) is best seen for \( \dim(Y_1) = 2 \), where this \( Y_1 \) is equal to the universal covering of the doubly punctured sphere \( S^2 \).)

Alternatively, one might say that the in-radius of \( Y_1 \) is equal to \( \frac{\pi}{n+1} \):

\[ \text{there are closed compact balls in } Y_1 \text{ of all radii } R < \frac{\pi}{n+1} \text{ but no ball of radius } \geq \frac{\pi}{n+1} \text{ is compact.} \]

**Gap Extremality.** We do not know if the above spheres minus pairs of points are extremal for \( n \geq 2 \) but the Euclidean spaces \( \mathbb{R}^n \) are definitely not length extremal starting from \( m = 2 \).

In fact, there are (obvious, \( O(m) \)-invariant) metrics \( g \geq g_{\text{Eucl}} \) on \( \mathbb{R}^m \) with \( Sc(g_1) > 0 \) for all \( m \geq 2 \).

On the other hand,

\[ \text{(*) no metric } g \geq g_{\text{Eucl}} \text{ on } \mathbb{R}^m \text{ may have } Sc(g) \geq \varepsilon > 0. \text{(See [15] cited in section 3.)} \]

This suggests the following weaker version of extremality for non-compact manifolds which we call gap extremality.

A metric \( g_0 \) on \( Y \) is \( \varepsilon \)-gap length extremal if no \( g \geq g_0 \) on \( Y \) satisfies

\[ Sc(g) - Sc(g_0) > \varepsilon. \]

Then \( g_0 \) is called gap length extremal if it is \( \varepsilon \)-gap length extremal for all \( \varepsilon > 0 \) (0-gap extremal=extremal).
Similarly one defines area gap extremality and gap extremality for classes of maps \( f : X \to Y \). (But I am not certain what a workable definition of normalised gap extremality/Sc should be.)

Whenever the twisted Dirac operator argument from the previous section yields area extremality of a closed manifold \( Y \), e.g. if \( Y = S^n \) or \( Y = \mathbb{C}P^n \), this argument, combined with that from [15] (cited in section 3) for \( \mathbb{R}^m \), also delivers

\[ \text{(**) gap area extremality of } Y_m = Y \times \mathbb{R}^m \text{ for all } m = 1, 2, ..., \text{ as well as this extremality for smooth proper spin maps } f : X \to Y_m \text{ of non-zero degrees.} \]

if a smooth proper spin map \( f : X \to Y_m \) of non-zero degree decreases the areas of all surfaces \( \Sigma \subset X \), then, given \( \varepsilon > 0 \), there exists a point \( x \in X \), such that

\[ \text{Sc}(X)(x) - \text{Sc}(Y')(f(x)) < \varepsilon. \]

**D3. Question.** Does gap extremality is always stable under \( Y \sim Y \times \mathbb{R}^m \)? (Beware of \( \dim(Y) = 4 \).)

One can’t discard of \( \varepsilon \) for \( m \geq 2 \) but the true area (or, at least length) extremality of \( Y' = Y \times \mathbb{R} \) (that allows \( \varepsilon = 0 \)) may be provable by some twisted Dirac operator argument. For instance, if \( Y = T^n \) this follows from theorem 6.12 in [14] (cited in section 3). Alternatively, one might use minimal hypersurfaces and soap bubble in \( X \) the \( f \)-images of which separate the two ends in \( Y' = Y \times \mathbb{R} \) but then one would face a possibility of non-compact minimal hyper surfaces in \( X \) and would be obliged to resort to imposing extra assumptions on \( X \), e.g. uniform two sided bounds on the sectional curvatures of \( X \).

Finally, let us look at the manifold \( Y_1 \), which has the band width \( 2\pi n \), in the above Example (c).

It is plausible that this \( Y_1 \) is length gap extremal but not length extremal starting from \( D = \dim(Y_1) = 3 \).

And what we definitely know is that the quotient space \( Y_1/\mathbb{Z} = T^n \times (\frac{-\pi n}{n+1}, \frac{\pi n}{n+1}) \), \( n + 1 = \dim(Y_1) \), is length extremal.

We shall see the reason for this in the next section, where we shall also explain the current status of the rigidity problem for these manifolds.

## 6 Bounds on Widths of Bands with Positive Curvatures.

Let us start with the following question which, on the surface of things, has nothing to do with scalar curvature.

Given a smooth \( n \)-dimensional manifold \( X \) immersed into a complete Riemannian manifold \( Y \) denote by \( rad^+(X \to Y) \) the maximal \( R \), such that the normal exponential map

\[ \exp^+: T^+(X) = T(Y)|_X \oplus T(X) \to Y, \]

A smooth map \( X \to Y \) is an immersion if it is a diffeomorphism of small neighbourhoods in \( X \) to smooth submanifolds in \( Y \).
is locally injective on the subbundle $B_i(R)(X) \subset T^i(X)$ of open normal $R$-balls $B^R_{x^n}(R) \subset T^i(X)$, $x \in X$.

(If the ambient space $Y = \mathbb{R}^n$, then $\text{rad}^i(X; \mathbb{R}^n)$ is equal to the reciprocal of the supremum of the principal curvatures of $X$.)

Take the supremum of these radii over all immersions $f : X \rightarrow Y$, set

$$
suprad^i(X; Y) = \sup_{f} \text{rad}^i(X \rightarrow Y)
$$

and let

$$
suprad^i_N(X) = \sup_{f_o} \text{rad}^i(X \rightarrow \mathbb{R}^N),
$$

where the latter "sup" is taken over all immersions $f_o$ from $X$ to the unit ball $B^N(1) \subset \mathbb{R}^N$.

(The notation $suprad^i(X; B^N(1))$ would be unjustified, since the image of the exponential map may be not contained in $B^N(1)$.)

**E1. Problem.** Evaluate $suprad^i_N(X)$ in terms of the topology of $X$.

**Examples.** (a) It is obvious that $suprad^i_N(X) \leq 1$ for all closed manifolds $X$, where the equality holds if and only if $X$ is diffeomorphic to $S^n$ and $N > n$.

(b) Let $X_k$ be diffeomorphic to the product of $k$ spheres,

$$
X_k = S^{n_1} \times \ldots \times S^{n_k}, \ n_k \geq 1.
$$

Then

$$
suprad^i_N(X) \geq \frac{1}{\sqrt{k}} \text{ for all } N \geq (n_1 + 1) + \ldots + (n_k + 1).
$$

But we do not know, for instance, whether

$$
suprad^i_N(X_k) \rightarrow 0 \text{ for } N = \text{dim}(X_k) + 1 \text{ and } k \rightarrow \infty.
$$

or, on the contrary, if

$$
suprad^i_N(X) \geq \rho_0
$$

for all manifolds $X$, (e.g. for all $X_k$) all sufficiently large $N \geq N(X)$ and some universal constant $\rho_0 > 0$, say $\rho_0 = 0.001$.

All known upper bounds on $suprad^i_N(X)$ — am I missing something obvious? exclusively apply to manifolds $X$ which admit no metrics with $Sc > 0$.

A simple way to obtain such a bound is as follows.

1. Scale $B^N(1) \rightarrow B^N\left(\frac{1}{2}\right)$, project $B^N\left(\frac{1}{2}\right)$ to $S^n$ from the south pole of $S^n$ and observe that this distorts the curvatures of submanifolds $X$ in the ball $B^N(1)$ by a finite amount independent of $X$ and $N$.

2. Apply the Gauss formula to $X \rightarrow S^n$ and thus show that the supremum of the principal curvatures of $X$ in $S^n$ satisfies

$$
\sup \text{curv}(X \rightarrow S^n) \geq \frac{\sqrt{N - 1}}{N - n}
$$

and therefore,

$$
suprad^i_N(X) \leq \text{const} \cdot \frac{N - n}{\sqrt{N - 1}}
$$

for all $n$-dimensional manifolds $X$ which admit no metrics with $Sc > 0$ and for some constant $\text{const} \leq 100$. (See [27] cited below for details.)
It follows, for instance, that there are exotic spheres $\Sigma^n$ of dimensions $n = 9, 17, 25, 33, \ldots$, such that

$$\text{suprad}_{n+1}(\Sigma^n) \leq \frac{100}{\sqrt{n-1}}.$$  

but one has no idea how sharp this inequality is and if there are similar inequalities for exotic spheres which admit metrics with $Sc > 0$.

The above also applies to tori $T^n$, since these admit no metrics with $Sc > 0$ either, but here the following better (but, probably, still very far from being sharp) inequality is available.

$$\text{suprad}_{n+1}(T^n) \leq \frac{2\pi}{n+1}.$$  

This is proven again by passing to $S^{n+1}$, where all we use of the geometry of $S^{n+1}$ is the inequality $Sc(S^{n+1}) \geq n(n+1)$. (Isn’t it amazing that there is no apparent direct proof of a much stronger bound on $\text{rad}(T^n \subset B^{n+1}(1))$.)

Namely, the above bound on $\text{suprad}_{n+1}(T^n)$ trivially follows from the following.

### Torical Band Width Inequality

Let $g$ be a metric with $Sc(g) \geq n(n+1)$ on the torical band (cylinder) $T^n \times [-1, 1]$. Then the distance between the two boundary components of this band satisfies

$$(O_\ast < \frac{2\pi}{n+1})$$

$$\text{dist}_g(T^n \times \{-1\}, T^n \times \{1\}) < \frac{2\pi}{n+1}.$$  

This is proven in [27] with a relative version of the Schoen-Yau minimal hypersurface method.

Besides a bound on $\text{suprad}_{n+1}(T^n)$, the inequality $\left[ O_\ast < \frac{2\pi}{n+1} \right]$ (trivially) implies that

the warped product metric $\varphi^2(t)g_{T^n} + dt^2$ on $T^n \times (-\frac{\pi}{n+1}, \frac{\pi}{n+1})$ with $Sc = n(n+1)$, which was introduced in the previous section, is length extremal.

Also, the argument in [27] yields length rigidity of this metric for $n \leq 6$, while the general case needs an elaboration on recent results on ”irrelevance of singularities” of minimal hypersurfaces proved in the papers [12] and/or [13] cited in section 3.

### 7 Extremality and Rigidity of Convex Polyhedra.

Let $P \subset \mathbb{R}^n$ be a compact convex polyhedron with non-empty interior, let $Q_i \subset P$, $i \in I$, denote its $(n-1)$-faces and let

$$\omega_{ij}(P) = \omega(Q_i, Q_j)$$

denote its dihedral angles.

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[27] M. Gromov, Metric Inequalities with Scalar Curvature.

[http://www.ihes.fr/~gromov/PDF/Inequalities-July2017.pdf](http://www.ihes.fr/~gromov/PDF/Inequalities-July%202017.pdf)
Say that $P$ is extremal if all convex polyhedra $P'$ which are combinatorially equivalent to $P$ and which have

$$
\angle_{ij}(P') \leq \angle_{ij}(P) \text{ for all } i, j \in I,
$$

necessarily satisfy

$$
\angle_{ij}(P') = \angle_{ij}(P).
$$

It is known – the proof is elementary – that the simplices and the rectangular solids are extremal and also all $P$ with $\angle_{ij}(P) \leq \pi/2$ are extremal.

But it is unclear (at least to the present author) what are (if any) non-extremal $P$.

What we are truly interested in, however, is extremality (and rigidity) of $P$ under transformations which keep the faces $Q_i$ convex (rather than flat) or, even better, mean convex, i.e. keeping their mean curvatures non-negative.

Thus, we say that $P$ is mean convexly extremal if there is no $P' \subset \mathbb{R}^n$ diffeomorphic to $P$ and such that

- the faces $Q'_i \subset P'$ corresponding to all $Q_i \subset P$ have mean.curv$(Q'_i) \geq 0$,
- the dihedral angles of $P'$, that are the angles between the tangent spaces $T_{p'}(Q'_i)$ and $T_{p'}(Q'_j)$ at the points $p'$ on the $(n-2)$-faces $Q'_{ij} = Q'_i \cap Q'_j$, satisfy

$$
\angle_{ij}(P') \leq \angle_{ij}(P),
$$

- this angle inequality is strict at some point, i.e. there exits $p'_0 \in Q'_{ij}$ in some $Q'_{ij}$, such that

$$
\angle(T_{p'_0}(Q'_i),T_{p'_0}(Q'_j)) < \angle_{ij}(P).
$$

$F_1$. Question. Are all extremal convex polyhedra $P$ mean convexly extremal?

It is not even known if the regular 3-simplex is mean convexly extremal, but

the mean convex extremality of the $n$-cube

follows by developing the cube $P$ into a complete (orbi-covering) manifold $\hat{P}$ homeomorphic to $\mathbb{R}^n$ by reflecting $P$ in the faces, approximating the natural continuous Riemannin metric on $\hat{P}$ by a smooth one with $Sc \geq \varepsilon > 0$ (see [28] and appealing to gap extremality of $\mathbb{R}^n$ stated in section 5.

And the same argument yields (see [28]) the following

[⋆] Let a Riemannin metric $g$ on the $n$-cube $P$ satisfy:

* $_0 Sc(g) \geq 0.$
* $_1 mean.curv_g(Q_i) \geq 0,$
* $_2 \angle_{ij}(P,g) \leq \pi/2.$

Then, necessarily, $Sc(g) = 0$, mean.curv$g(Q_i) = 0$ and $\angle_{ij}(P,g) = \pi/2$.

Probably, these equalities imply that $P$ is isometric to a Euclidean rectangular solid but the approximation/smoothing is no good for proving this kind of rigidity.

\[33\] M. Gromov Dirac and Plateau Billiards in Domains with Corners, Cent. Eur. J. Math. 12(8) pp 1109-1156, (2014).
The main merit of [*] is that it provides a test for $Sc \geq 0$ in all Riemannian manifolds $X$:

$Sc(X) \geq 0$ if and only if no cubical domain $P \subset X$ satisfies

$$[\text{mean.curv}_g(Q_i) > 0] \& [\angle_{ij}(P,g) \leq \frac{\pi}{2}]$$

This suggests a possibility of defining $Sc(X) \geq 0$ for some singular spaces, $X$, e.g. for Alexandrov spaces with sectional curvatures bounded from below.

**F2. Conjecture** All known (and expected) properties of Riemannian manifolds with $Sc \geq 0$, which have no "spin" attached to their formulations, generalise to Alexandrov’s spaces.

For instance, most probably,

if an $n$-dimensional Alexandrov space $X$ with curvatures bounded from below has $Sc > 0$ at all regular points $x \in X$, (or if the volumes of all infinitesimally small balls in $X$ are bounded by the volumes of such Euclidean balls) then

every continuous map from $X$ to a space $Y$ with $CAT(0)$ universal covering (i.e. an Alexandrov’s space with non-positive sectional curvatures) contracts to an $(n-1)$-dimensional subset in $Y$.

If true, this would imply that (suitably defined) harmonic maps $X \to Y$ must necessarily have $(n-1)$-dimensional images, which suggests a (non-local?) Weitzenboeck-Bochner type formula in this context and a definition of $Sc > 0$ via spectral properties of small (large?) balls (cubes?) in $X$.

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