On movable singularities of Garnier systems

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Abstract

We study movable singularities of Garnier systems using the connection of the latter with Schlesinger isomonodromic deformations of Fuchsian systems.

§1. What is Painlevé VI equations and Garnier systems?

We start with the Painlevé VI (PVI) equation

$$\frac{d^2 u}{dt^2} = \frac{1}{2} \left( \frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t} \right) \left( \frac{du}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t} \right) \frac{du}{dt} + \frac{u(u-1)(u-t)}{t^2(t-1)^2} \left( \alpha + \beta \frac{t}{u^2} + \gamma \frac{t-1}{(u-1)^2} + \delta \frac{t(t-1)}{(u-t)^2} \right),$$

the second order ODE for a complex function $u(t)$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are constants.

However, simply giving the explicit equation seems to be the least helpful introduction to it. For the purposes of this paper it is more convenient to look at PVI as at

- the equation for an apparent (fifth) singularity of isomonodromic family of second order scalar Fuchsian equations with the four singularities $t, 0, 1, \infty$;
- the most general second order ODE with the Painlevé property;
- the equation controlling isomonodromic deformations of certain rank 2 Fuchsian systems with the four singularities $t, 0, 1, \infty$.

Let us recall the first two viewpoints in more details (the last one will appear in §3). The monodromy of a linear differential equation

$$\frac{d^p u}{dz^p} + b_1(z) \frac{d^{p-1} u}{dz^{p-1}} + \ldots + b_p(z) u = 0$$

with singularities $a_1, \ldots, a_n \in \mathbb{C}$ (which are the poles of the coefficients) can be defined as follows. In a neighbourhood of a non-singular point $z_0$ we consider a basis $(u_1, \ldots, u_p)$ in the solution space of the equation (2). Analytic continuations of the functions $u_1(z), \ldots, u_p(z)$ along an arbitrary loop $\gamma$ outgoing from $z_0$ and lying in $\mathbb{C} \setminus \{a_1, \ldots, a_n\}$ transform the basis $(u_1, \ldots, u_p)$ into a (in general case different) basis $(\tilde{u}_1, \ldots, \tilde{u}_p)$. The two bases are related by means of a non-singular transition matrix $G_\gamma$ corresponding to the loop $\gamma$:

$$(u_1, \ldots, u_p) = (\tilde{u}_1, \ldots, \tilde{u}_p) G_\gamma.$$

The map $[\gamma] \mapsto G_\gamma$ (which depends only on the homotopy class $[\gamma]$ of the loop $\gamma$) defines the representation

$$\chi : \pi_1(\mathbb{C} \setminus \{a_1, \ldots, a_n\}, z_0) \longrightarrow \text{GL}(p, \mathbb{C})$$

of the fundamental group of the space $\mathbb{C} \setminus \{a_1, \ldots, a_n\}$ in the space of non-singular complex matrices of size $p$. This representation is called the monodromy of the equation (2).

A singular point $a_i$ of the equation (2) is said to be regular if any solution of the equation has a polynomial (with respect to $1/|z - a_i|$) growth near $a_i$. Linear differential equations with regular singular points only are called Fuchsian.
A. Poincaré \[13\] has established that the number of parameters determining a Fuchsian equation of order \(p\) with \(n\) singular points is less than the dimension of the space of representations \(\chi\), if \(p > 2, n > 2\) or \(p = 2, n > 3\) (see also \[1\], pp. 158–159). Hence in the construction of a Fuchsian equation with the given singularities and monodromy there arise so-called *apparent* singularities, at which the coefficients of the equation have poles but the solutions are single-valued meromorphic functions. In the case \(p = 2, n = 4\) \(\{a_1, a_2, a_3, a_4\} = \{t, 0, 1, \infty\}\) the number of such singularities equals one. If we move a little the singularity \(z = t\) so that the monodromy of the equation preserves (this is an *isomonodromy* property which is defined precisely in the next paragraph), the apparent (fifth) singularity \(w(t)\) will move satisfying \(P_{VI}\) (this was first obtained by R. Fuchs \[5\]).

The equation (11) has three fixed singular points – 0, 1, \(\infty\). Its movable singularities (which depend on the initial conditions) can be poles only. In other words, any local solution of the Garnier system, extend to meromorphic functions. In the case \(\alpha = 0\) their orders do not exceed two (see, for instance, \[8\], Ch. VI, \(\S\) 6).

Extending the first of the above three viewpoints to general case of \(n + 3\) singularities \(a_1, \ldots, a_n, 0, 1, \infty\), R. Garnier \[6\] has obtained the system \(G_n(\theta)\) depending on \(n + 3\) complex parameters \(\theta_1, \ldots, \theta_{n+2}, \theta_\infty\). This is a completely integrable system of non-linear partial differential equations of second order. Later it was written down by K. Okamoto \[12\] in an equivalent Hamiltonian form

\[
\frac{\partial u_i}{\partial a_j} = \frac{\partial H_j}{\partial v_i}, \quad \frac{\partial v_i}{\partial a_j} = -\frac{\partial H_j}{\partial u_i}, \quad i, j = 1, \ldots, n, \tag{3}
\]

with certain Hamiltonians \(H_i = H_i(a, u, v, \theta)\) rationally depending on \(a = (a_1, \ldots, a_n), u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n), \theta = (\theta_1, \ldots, \theta_{n+2}, \theta_\infty)\). (Here \(u_1(a), \ldots, u_n(a)\) are apparent singular points of a certain isomonodromic family of second order Fuchsian equations with singularities \(a_1, \ldots, a_n, 0, 1, \infty\).) In the case \(n = 1\) the Garnier system \(G_1(\theta_1, \theta_2, \theta_3, \theta_\infty)\) is an equivalent (Hamiltonian) form of \(P_{VI}\) \[1\], where

\[
\alpha = \frac{1}{2} \theta_2^2, \quad \beta = -\frac{1}{2} \theta_2^2, \quad \gamma = \frac{1}{2} \theta_3^2, \quad \delta = \frac{1}{2} (1 - \theta_1^2).
\]

For \(n > 1\) the Garnier system generically does not satisfy the Painlevé property. However, due to Garnier’s theorem, the elementary symmetric polynomials \(\sigma_i(u_1(a), \ldots, u_n(a))\), depending on local solutions of the Garnier system, extend to meromorphic functions \(F_i(a)\) on the universal cover \(Z'\) of the space \((\mathbb{C} \setminus \{0, 1\})^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}\). Our addition to this theorem consists in some estimates for orders of irreducible components of the polar loci of the functions \(F_i\) (Theorem 2, Proposition 1).

\[\S2. \text{Isomonodromic deformations of Fuchsian systems}\]

Let us include a Fuchsian system

\[
\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i^0}{z - a_i^0}\right) y, \quad B_i^0 \in \text{Mat}(p, \mathbb{C}), \quad \sum_{i=1}^{n} B_i^0 = 0, \tag{4}
\]

of \(p\) equations with singularities \(a_1^0, \ldots, a_n^0\) into a family

\[
\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i(a)}{z - a_i}\right) y, \quad B_i(a^0) = B_i^0, \quad \sum_{i=1}^{n} B_i(a) = 0, \tag{5}
\]
of Fuchsian systems holomorphically depending on the parameter \( a = (a_1, \ldots, a_n) \in D(a^0) \), where \( D(a^0) \) is a disk of small radius centered at the point \( a^0 = (a^0_1, \ldots, a^0_n) \) of the space \( \mathbb{C}^n \setminus \bigcup_{i \neq j} \{ a_i = a_j \} \).

One says that the family (5) is isomonodromic (or it is an isomonodromic deformation of the system (4), if for all \( a \in D(a^0) \) the monodromies

\[
\chi: \pi_1(\overline{\mathbb{C}} \setminus \{ a_1, \ldots, a_n \}) \to \text{GL}(p, \mathbb{C})
\]

of the corresponding systems are the same. This means that for every value \( a \) there exists a fundamental matrix \( Y(z, a) \) of the corresponding system (5) that has the same monodromy for all \( a \in D(a^0) \). This matrix \( Y(z, a) \) is called an isomonodromic fundamental matrix.

Is it always possible to include the system (4) into an isomonodromic family of Fuchsian systems? The answer is positive. For instance, if the matrices \( B_i(a) \) satisfy the Schlesinger equation \[14\]

\[
dB_i(a) = -\sum_{j=1, j \neq i}^{n} \frac{[B_i(a), B_j(a)]}{a_i - a_j} \, d(a_i - a_j),
\]

then the family (5) is isomonodromic (in this case it is called the Schlesinger isomonodromic family).

Due to Malgrange’s theorem \[10\], for arbitrary initial conditions \( B_i(a^0) = B^0_i \) the Schlesinger equation has the unique solution \( \{ B_1(a), \ldots, B_n(a) \} \) in some disk \( D(a^0) \), and the matrices \( B_i(a) \) can be extended to the universal cover \( Z \) of the space \( \mathbb{C}^n \setminus \bigcup_{i \neq j} \{ a_i = a_j \} \) as meromorphic functions. Thus, the Schlesinger equation satisfies the Painlevé property.

Recall, that a function \( f \) is meromorphic on \( Z \), if it is holomorphic on \( Z \setminus P \), can not be extended to \( P \) holomorphically and is presented as a quotient \( f(a) = \varphi(a)/\psi(a) \) of holomorphic functions in a neighbourhood of every point \( a^0 \in P \) (hence, \( \psi(a^0) = 0 \)). Thus, \( P \subset Z \) is an analytic set of codimension one (it is defined locally by the equation \( \psi(a) = 0 \)), which is called the polar locus of the meromorphic function \( f \). The points of this set are divided into poles (at which the function \( \varphi \) does not vanish) and ambiguous points (at which \( \varphi = 0 \)).

One can also define a divisor of a meromorphic function. Denote by \( A = N \cup P \) the union of the set \( N \) of zeros and polar locus \( P \) of the function \( f \). Any regular point \( a^0 \) of the set \( A \) can belong to only one irreducible component of \( N \) or \( P \). Thus, one can define the order of this component as the degree (taken with “+”, if \( a^0 \in N \), and with “−”, if \( a^0 \in P \)) of the corresponding factor in the decomposition of the function \( \varphi \) or \( \psi \) into irreducible factors. Then the divisor of the meromorphic function \( f \) is the pair \( (A, \kappa) \), where \( \kappa = \kappa(a) \) is an integer-valued function on the set of regular points of \( A \) (which takes a constant value on each its irreducible component, this value is equal to the order of a component). The pair \( (P, \kappa) \) is called the polar divisor of the meromorphic function \( f \). By \( (f)_\infty \) we will mean the restriction of \( \kappa \) on regular points of \( P \).

Let us return to the Schlesinger equation. The polar locus \( \Theta \subset Z \) of the extended matrix functions \( B_1(a), \ldots, B_n(a) \) is called the Malgrange \( \Theta \)-divisor \[4\]. If we consider the system (4) as an equation for horizontal sections of the logarithmic connection \( \nabla_0 \) (with singularities \( a^0_1, \ldots, a^0_n \) in

\[1\] Under small variations of the parameter \( a \) there exist canonical isomorphisms of the fundamental groups \( \pi_1(\overline{\mathbb{C}} \setminus \{ a_1, \ldots, a_n \}) \) and \( \pi_1(\overline{\mathbb{C}} \setminus \{ a^0_1, \ldots, a^0_n \}) \) generating canonical isomorphisms

\[
\text{Hom}(\pi_1(\overline{\mathbb{C}} \setminus \{ a_1, \ldots, a_n \}), \text{GL}(p, \mathbb{C}))/\text{GL}(p, \mathbb{C}) \cong \text{Hom}(\pi_1(\overline{\mathbb{C}} \setminus \{ a^0_1, \ldots, a^0_n \}), \text{GL}(p, \mathbb{C}))/\text{GL}(p, \mathbb{C})
\]

of the spaces of conjugacy classes of representations for the above fundamental groups; this allows one to compare \( \chi \) for various \( a \in D(a^0) \).

\[2\] In view of the above definition of a divisor, here the term ”divisor” is not precise enough.
the holomorphically trivial vector bundle $E_0$ of rank $p$, then the set $\Theta$ corresponds to those points $\alpha^*$, where the bundle $E_{\alpha^*}$ associated to the parameter $\alpha^*$ in the isomonodromic deformation $(E_a, \nabla_a)_{a \in \mathbb{Z}}$ of $(E_0, \nabla_a)$ is not holomorphically trivial (see details in [10]).

In what follows we will use the theorem of Bolibrukh [2], [4] (the proof also can be found in [7]) describing a general solution of the Schlesinger equation near the $\Theta$-divisor in the case $p = 2$. For the polar locus $P \subset \mathbb{Z}$ of a function $f$ meromorphic on $\mathbb{Z}$, and $\alpha^* \in P$, let us denote by $\Sigma_{\alpha^*}(f)$ the sum of orders of all irreducible components of $P \cap D(\alpha^*)$.

**Theorem 1.** Let the monodromy of the two-dimensional ($p = 2$) Schlesinger isomonodromic family $\{E_a\}$ be irreducible, $\alpha^*$ an arbitrary point of $\Theta$ and $E_{\alpha^*} \cong \mathcal{O}(k) \oplus \mathcal{O}(-k)$. Then $\Sigma_{\alpha^*}(B_i) \geq -2k$, $i = 1, \ldots, n$.

**Remark 1.** As known [4], $2k \leq n - 2$. Thus, the estimate of Theorem 1 can be written in the form $\Sigma_{\alpha^*}(B_i) \geq 2 - n$, furthermore $\Sigma_{\alpha^*}(B_i) \geq 3 - n$ in the case of odd $n$.

The following auxiliary lemma will be used later.

**Lemma 1.** Consider a two-dimensional Schlesinger isomonodromic family of the form

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i(a)}{z-a_i}\right) y, \quad \sum_{i=1}^{n} B_i(a) = K = \text{diag}(\theta, -\theta), \quad \theta \in \mathbb{C},$$

and the function $b(a) = \sum_{i=1}^{n} b_{i}^{12}(a) a_i$, where $b_{i}^{12}(a)$ are the upper-right elements of the matrices $B_i(a)$ respectively. Then the differential of the function $b(a)$ is given by the formula

$$db(a) = (2\theta + 1) \sum_{i=1}^{n} b_{i}^{12}(a) da_i.$$

**Proof.** The differential $db(a)$ has the form

$$db(a) = \sum_{i=1}^{n} a_i db_{i}^{12}(a) + \sum_{i=1}^{n} b_{i}^{12}(a) da_i.$$

To find the first of the two latter summands, let us use the Schlesinger equation for the matrices $B_i(a)$. Then we have

$$\sum_{i=1}^{n} a_i dB_i(a) = - \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} a_i \left[ B_i(a), B_j(a) \right] \frac{a_i - a_j}{a_i - a_j} d(a_i - a_j) = - \sum_{i=1}^{n} \sum_{j=1, j > i}^{n} [B_i(a), B_j(a)] d(a_i - a_j) =$$

$$= - \sum_{i=1}^{n} \left[ B_i(a), \sum_{j=1, j \neq i}^{n} B_j(a) \right] da_i = - \sum_{i=1}^{n} [B_i(a), K] da_i.$$

The upper-right element of the latter matrix 1-form is equal to $\sum_{i=1}^{n} 2\theta b_{i}^{12}(a) da_i$, hence $\sum_{i=1}^{n} a_i db_{i}^{12}(a) = 2\theta \sum_{i=1}^{n} b_{i}^{12}(a) da_i$, and $db(a) = (2\theta + 1) \sum_{i=1}^{n} b_{i}^{12}(a) da_i$. $\square$

§3. Schlesinger isomonodromic deformations and Garnier systems

Let us recall the relationship between Schlesinger isomonodromic deformations and Garnier systems.
Consider a two-dimensional Schlesinger isomonodromic family

\[
\frac{dy}{dz} = \left( \sum_{i=1}^{n+2} \frac{B_i(a)}{z-a_i} \right) y, \quad B_i(a^0) = B_i^0 \in \text{sl}(2, \mathbb{C}),
\]  

(6)
of Fuchsian systems with singular points \(a_1, \ldots, a_n, a_{n+1} = 0, a_{n+2} = 1, a_{n+3} = \infty\) which depends holomorphically on the parameter \(a = (a_1, \ldots, a_n) \in D(a^0)\), where \(D(a^0)\) is a disk of small radius centered at the point \(a^0\) of the space \((\mathbb{C} \setminus \{0,1\})^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}\). Denote by \(\pm \beta_i\) the eigenvalues of the matrices \(B_i(a)\) respectively. Recall that the isomonodromic deformation preserves the eigenvalues of the residue matrices \(B_i(a)\). As follows from the Schlesinger equation, the matrix residue at the infinity is constant. We assume that it is a diagonalisable matrix, i.e., \(\sum_{i=1}^{n+2} B_i(a) = -B_\infty = \text{diag}(-\beta_\infty, \beta_\infty)\).

By Malgrange's theorem the matrix functions

\[
B_i(a) = \begin{pmatrix} b^{11}_i(a) & b_i(a) \\ b^{21}_i(a) & b^{22}_i(a) \end{pmatrix}
\]
can be extended to the universal cover \(Z'\) of the space \((\mathbb{C} \setminus \{0,1\})^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}\) as meromorphic functions (holomorphic off the analytic subset \(\Theta\) of codimension one).

Denote by \(B(z, a)\) the coefficient matrix of the family \([6]\). Since the upper-right element of the matrix \(B_\infty\) equals zero, for every fixed \(a\) the same element of the matrix \(z(z-1)(z-a_1) \cdots (z-a_n)B(z, a)\) is a polynomial \(P_n(z, a)\) of degree \(n\) in \(z\). We denote by \(u_1(a), \ldots, u_n(a)\) the roots of this polynomial and define the functions \(v_1(a), \ldots, v_n(a)\):

\[
v_j(a) = \sum_{i=1}^{n+2} \frac{b^{11}_i(a) + \beta_i}{u_j(a) - a_i}, \quad j = 1, \ldots, n.
\]

Then the following statement takes place: the pair \((u(a), v(a)) = (u_1, \ldots, u_n, v_1, \ldots, v_n)\) satisfies the Garnier system \([3]\) with the parameters \(2\beta_1, \ldots, 2\beta_{n+2}, 2\beta_\infty - 1\) (see proof of Proposition 3.1 from \([12]\), or \([9]\), Cor. 6.2.2 (p. 207)).

One can express the coefficients of the polynomial \(P_n(z, a)\) in terms of the upper-right elements \(b_i(a)\) of the matrices \(B_i(a)\). Let

\[
\sigma_1(a) = \sum_{i=1}^{n+2} a_i, \quad \sigma_2(a) = \sum_{1 \leq i < j \leq n+2} a_i a_j, \ldots, \sigma_{n+1}(a) = a_1 \cdots a_n
\]

be the elementary symmetric polynomials in \(a_1, \ldots, a_n, a_{n+1} = 0, a_{n+2} = 1, a_{n+3} = \infty\) and \(Q(z) = \prod_{i=1}^{n+2} (z-a_i)\). Then

\[
P_n(z, a) = \sum_{i=1}^{n+2} b_i(a) \frac{Q(z)}{z-a_i} =: b(a) z^n + f_1(a) z^{n-1} + \ldots + f_n(a)
\]

(recall that \(\sum_{i=1}^{n+2} b_i(a) = 0\)). By the Viète theorem one has

\[
b(a) = \sum_{i=1}^{n+2} b_i(a)(-\sigma_1(a) + a_i) = \sum_{i=1}^{n+2} b_i(a) a_i = \sum_{i=1}^{n} b_i(a) a_i + b_{n+2}(a),
\]

\[
f_1(a) = \sum_{i=1}^{n+2} b_i(a)(\sigma_2(a) - \sum_{j=1, j \neq i}^{n+2} a_i a_j) = \sum_{1 \leq i < j \leq n+2} (b_i(a) + b_j(a)) a_i a_j.
\]
In the similar way,
\[ f_k(a) = (-1)^k \sum_{1 \leq i_1 < \ldots < i_{k+1} \leq n+2} (b_{i_1}(a) + \ldots + b_{i_{k+1}}(a))a_{i_1} \ldots a_{i_{k+1}} \]
for each \( k = 1, \ldots, n \).

Alongside formulae for the transition from a two-dimensional Schlesinger isomonodromic family with \( sl(2, \mathbb{C}) \)-residues to a Garnier system, there also exist formulae for the inverse transition (see [12], Prop. 3.2). This allows to suggest some addition to Garnier’s theorem (which claims that the elementary symmetric polynomials \( F_i(a) = \sigma_i(u_1(a), \ldots, u_n(a)) \) of solutions of a Garnier system are meromorphic on \( Z' \)).

By the linear monodromy of a solution of a Garnier system we will mean the monodromy of the corresponding two-dimensional Schlesinger isomonodromic family.

**Theorem 2.** Let \((u(a), v(a))\) be a solution of the Garnier system \([3]\) that has an irreducible linear monodromy, and \( \Delta_i \) denotes the polar locus of the function \( F_i \), \( i = 1, \ldots, n \). Then

a) in the case \( \theta_\infty = 0 \) and \( u_i(a) \neq u_j(a) \) for \( i \neq j \), one has \( \Sigma_\alpha(F_i) \geq -n - 1 \) for any point \( a^* \in \Delta_i \);

b) in the case \( \theta_\infty \neq 0 \) one has \( \Sigma_\alpha(F_i) \geq -n \) for any point \( a^* \in \Delta_i \), may be, with the exception of some subset \( \Delta^0 \subset \Delta_i \) of positive codimension (in any case \( (F_i)_\infty \geq -n \)).

**Proof.** Consider the family \([4]\) with the irreducible monodromy corresponding to the given solution, and the functions \( b(a), f_1(a), \ldots, f_n(a) \) constructed by the residue matrices \( B_i(a) \). By the Viète theorem, \( F_i(a) = (-1)^i f_i(a)/b(a) \). Due to Theorem 1 and Remark 1, for each function \( f_i \) and any point \( a^* \) of the \( \Theta \)-divisor of the family \([4]\) one has \( \Sigma_\alpha(f_i) \geq -n - 1 \).

By Lemma 1 we have \( db(a) = -\theta_\infty \sum_{i=1}^n b_i(a)da_i \), where \( \theta_\infty = 2\beta_\infty - 1 \).

a) In the case \( \theta_\infty = 0 \) one has \( db(a) \equiv 0 \) for all \( a \in Z' \), hence \( b(a) \equiv \text{const} \neq 0 \). Indeed, if \( b(a) \equiv 0 \), then \( P_n(z, a) \) is a polynomial of degree \( n - 1 \) in \( z \), and \( u_i(a) \equiv u_j(a) \) for some \( i \neq j \), which contradicts the conditions of the theorem. Thus, \( \Sigma_\alpha(F_i) = \Sigma_\alpha(f_i) \geq -n - 1 \) in this case.

b) In the case \( \theta_\infty \neq 0 \)
\[
\begin{align*}
b_i(a) &= -\frac{1}{\theta_\infty} \frac{\partial b(a)}{\partial a_i}, \quad i = 1, \ldots, n; \\
b_{n+2}(a) &= b(a) - \sum_{i=1}^n b_i(a)a_i, \quad b_{n+1}(a) = -b_{n+2}(a) - \sum_{i=1}^n b_i(a).
\end{align*}
\]

Thus, if the function \( b \) is holomorphic at a point \( a' \in Z' \), so are the functions \( b_i \), \( i = 1, \ldots, n + 2 \), and hence, the functions \( f_i \). Therefore, the points \( a^* \in \Delta_i \) can be of two types: such that \( b(a^*) = 0 \) (then \( \Sigma_\alpha(F_i) \geq -1 \), since the function \( b \) is irreducible\([3]\)) or that belong to the polar locus \( \Delta \subset \Theta \) of the function \( b \).

Denote by \( \Delta^0 \subset \Delta \) the set of ambiguous points of the function \( b \). Then in a neighbourhood of any point \( a^* \in \Delta \setminus \Delta^0 \) it can be presented in the form
\[
b(a) = \frac{h(a)}{\tau_1^{j_1}(a) \ldots \tau_r^{j_r}(a)}, \quad j_1 \geq 1, \ldots, j_r \geq 1,
\]

\(^3\)Indeed, if for some \( a' \in \{b(a) = 0\} \) one has \( \sum_{i=1}^n b_i(a')da_i = 0 \) and \( b_1(a') = \ldots = b_n(a') = 0 \). Taking into consideration the relations \([3]\), one gets also \( b_{n+2}(a') = 0 \) and \( b_{n+1}(a') = 0 \). This contradicts the irreducibility of the monodromy of the family \([4]\).
where the functions \( \tau_i, h \) are holomorphic near \( a^* \), \( h(a^*) \neq 0 \), furthermore \( \tau_i \) are irreducible at \( a^* \), just as

\[
f_i(a) = \frac{g(a)}{\tau_1^{k_1}(a) \ldots \tau_r^{k_r}(a)}, \quad k_1 + \ldots + k_r \leq n + 1,
\]

where the function \( g \) is holomorphic near \( a^* \). Thus,

\[
\frac{f_i(a)}{b(a)} = \frac{g(a)}{h_1(a) \ldots h_s(a) \tau_1^{j_1}(a) \ldots \tau_r^{j_r}(a)} = \frac{g(a)/h(a)}{\tau_1^{k_1-j_1}(a) \ldots \tau_r^{k_r-j_r}(a)},
\]

therefore,

\[
\Sigma_{\alpha}(F_i) = -\sum_{\alpha}(k_\alpha - j_\alpha) \geq -n
\]

(the sum is taken with respect to such indices \( \alpha \) that \( k_\alpha - j_\alpha > 0 \)), which proves the first part of the statement b).

In a neighbourhood of a point \( a^* \in \Delta^0 \) the decompositions \( [5], [9] \) take place for the functions \( b, f_i \) respectively, but \( h(a^*) = 0 \). However, due to the irreducibility of \( b \), all irreducible factors of \( h \) in its decomposition \( h(a) = h_1(a) \ldots h_s(a) \) near \( a^* \) are distinct (we can assume also that none of \( h_i \) coincides with some of \( \tau_i \)). Since \( k_l - j_l \leq n \) for all \( l = 1, \ldots, r \), the second part of the statement b) follows from the decomposition

\[
\frac{f_i(a)}{b(a)} = \frac{g(a)}{h_1(a) \ldots h_s(a) \tau_1^{j_1}(a) \ldots \tau_r^{j_r}(a)}.
\]

\[\Box\]

**Remark 2.** As follows from Remark 1, in all estimates of Theorem 2 one can substitute \( n \) by \( n - 1 \) in the case of even \( n \).

In particular, the polar loci of the functions \( F_1(a) = u_1(a) + u_2(a) \) and \( F_2(a) = u_1(a)u_2(a) \), where \( (u_1, u_2, v_1, v_2) \) is a solution of the Garnier system \( G_2(\theta_1, \ldots, \theta_4, \theta_\infty) \) corresponding to a two-dimensional Schlesinger isomonodromic family with five singular points and irreducible monodromy, are analytical submanifolds with \( (F_i)_\infty \geq -2 \). (Note that a bundle \( E_{a^*} \) corresponding to a point \( a^* \) of the \( \Theta \)-divisor of this family has the form \( E_{a^*} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1) \), which implies the regularity of the \( \Theta \)-divisor; see \([3], [7]\))

M. Mazzocco \([11]\) has shown that the solutions of the Garnier system \( [3] \), that have reducible linear monodromy, are classical functions (in each variable, in sense of Umemura \([15]\) and can be expressed via solutions of Lauricella hypergeometric equations. We discuss this case in more details in the following section.

**§4. Garnier systems and Lauricella hypergeometric equations**

Consider the *Lauricella hypergeometric equation* \( E_D(\alpha, \beta_1, \ldots, \beta_n, \gamma) \)

\[
a_i(1 - a_i) \frac{\partial^2 u}{\partial a_i^2} + (1 - a_i) \sum_{j=1}^{n} a_j \frac{\partial^2 u}{\partial a_i \partial a_j} + (\gamma - (\alpha + \beta_i + 1)a_i) \frac{\partial u}{\partial a_i} - \beta_i \sum_{j=1}^{n} a_j \frac{\partial u}{\partial a_j} - \alpha \beta_i u = 0, \quad i = 1, \ldots, n;
\]

\[
(a_i - a_j) \frac{\partial^2 u}{\partial a_i \partial a_j} + \beta_i \frac{\partial u}{\partial a_i} - \beta_j \frac{\partial u}{\partial a_i} = 0, \quad i, j = 1, \ldots, n.
\]
This is a system of linear partial differential equations of second order for a complex function
\( u(a_1, \ldots, a_n) \), where \( \alpha, \beta_1, \ldots, \beta_n, \gamma \in \mathbb{C} \) are constants. The system is defined on the space
\( B = (\mathbb{C} \setminus \{0, 1\})^n \setminus \bigcup_{i \neq j} \{a_i = a_j\} \). As shown in [9] (proof of Proposition 9.1.4, p.249), the vector-function
\( y(a) = (u, a_1 \frac{\partial u}{\partial a_1}, \ldots, a_n \frac{\partial u}{\partial a_n})^\top \), \( a = (a_1, \ldots, a_n) \in B \), satisfies a completely integrable linear Pfaffian system
\[
\frac{dy}{\omega y}, \quad y(a) \in \mathbb{C}^{n+1}.
\] (10)

Therefore, the set of solutions of a Lauricella hypergeometric equation near every point \( a \in B \) forms an \((n + 1)\)-dimensional vector space, and solutions can be extended holomorphically to the universal cover \( \mathbb{C}' \) of the space \( B \).

Further we recall how particular solutions of certain Garnier systems are connected with Lauricella hypergeometric equations and study movable singularities of such solutions.

As we noted, for \( n > 1 \) the Garnier system \([3]\) generically does not satisfy the Painlevé property (coordinates \( u_1, \ldots, u_n \) of its solution are defined as roots of a polynomial of degree \( n \)), but it can be transformed by a certain (symplectic) transformation \((a, u, v, H) \mapsto (s, q, p, \tilde{H})\),
\[
\sum_{i=1}^n (p_i dq_i - \tilde{H}_i ds_i) = \sum_{i=1}^n (v_i du_i - \tilde{H}_i da_i),
\]
into a Hamiltonian system \( \mathcal{H}_n(\theta) \)
\[
\frac{\partial q_j}{\partial s_i} = \frac{\partial \tilde{H}_i}{\partial p_j}, \quad \frac{\partial p_j}{\partial s_i} = -\frac{\partial \tilde{H}_i}{\partial q_j}, \quad i, j = 1, \ldots, n,
\]
satisfying the Painlevé property (see [9], Ch. III, §7). Recall this transformation introducing the functions
\[
M_i(a, u) = -\frac{(a_i - u_1) \cdots (a_i - u_n)}{\prod_{j=1, j \neq i}^{n+2} (a_i - a_j)}, \quad M^{k,i}(a, u) = \frac{u_k (u_k - 1)(u_k - a_1) \cdots (u_k - a_n)}{(u_k - a_i) \prod_{j=1, j \neq k}^{n+2} (u_k - u_j)},
\]
i, k = 1, \ldots, n. (11)

Then the transformation \((a, u, v) \mapsto (s, q, p)\) is given by the formulae
\[
s_i = \frac{a_i}{a_i - 1}, \quad q_i = -a_i M_i, \quad p_i = (1 - a_i) \sum_{k=1}^n \frac{M^{k,i} v_k}{u_k (u_k - 1)}, \quad i = 1, \ldots, n,
\] (12)
furthermore
\[
v_i = \sum_{k=1}^n \frac{q_k p_k}{u_i - a_k}, \quad i = 1, \ldots, n,
\] (13)
while the new Hamiltonians
\[
\tilde{H}_i = -(1 - a_i)^2 \left( H_i + \sum_{j=1}^n p_j \frac{\partial q_j}{\partial a_i} \right) = \frac{1}{s_i (s_i - 1)} \left( \sum_{j,k=1}^n E_{jk}^i(s, q)p_j p_k - \sum_{j=1}^n F_j^i(s, q)p_j + \varkappa q_i \right)
\]
are polynomial in \((q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)\). Here \( E_{jk}^i(s, q) \) and \( F_j^i(s, q) \) are polynomials in \( q \) of third and second degrees respectively, and \( \varkappa = \frac{1}{4} \left( \sum_{i=1}^{n+2} \theta_i - 1 \right)^2 - \theta_{\infty}^2 \).

In the case \( \varkappa = 0 \) (i. e., \( \sum_{i=1}^{n+2} \theta_i - 1 = \pm \theta_{\infty} \)) the system \( \mathcal{H}_n(\theta) \) has solutions of the form \((q, 0)\), where \( q \) is a solution of the system
\[
s_i (s_i - 1) \frac{\partial q_i}{\partial s_i} = -F_j^i(s, q), \quad i, j = 1, \ldots, n.
\] (14)
Since the right side of this system consists of polynomials of second degree in \( q \), the system may be considered as a several-variables generalization of the classical Riccati equation. Similarly to the classical case, the system (14) can be linearized by a suitable change of unknown. Exactly, according to [9] (Th. 9.2.1, p.252), a solution \((q_1, \ldots, q_n)\) of (14) can be presented in the form

\[
q_i(s) = \frac{s_i(s_i - 1)}{\sum_{i=1}^{n+2} \theta_i - 1} \left( \frac{\theta_i}{s_i - 1} + \frac{1}{f \partial_s \theta_i} \right), \quad i = 1, \ldots, n,
\]

where \( f(s) \) is an arbitrary solution of the Lauricella hypergeometric equation \( E_D(1-\theta_{n+2}, \theta_1, \ldots, \theta_n, \sum_{i=1}^{n+1} \theta_i) \).

The function \( f \) is irreducible at its zeros (if \( f(s^*) = 0 \) and \( \frac{\partial f}{\partial s_i} (s^*) = \ldots = \frac{\partial f}{\partial s_n} (s^*) = 0 \), then \( f \equiv 0 \) due to the uniqueness of solution of the system (10)), therefore a solution \((q_1, \ldots, q_n)\) of the system (14), as well as any linear combination \( Q \) of \( q_i \) (with holomorphic coefficients), are meromorphic on \( \mathbb{Z}' \) and the polar locus of \( Q \) is an analytical submanifold with \((Q)_\infty = -1\).

Now consider a solution \((u, 0)\) of the Garnier system (3) corresponding to a solution \((q, 0)\) of the system \( \mathcal{H}_n(\theta) \), with \( \sum_{i=1}^{n+2} \theta_i - 1 = \pm \theta_\infty \) (note that \( v = 0 \iff p = 0 \), by (12) and (13)). The elementary symmetric polynomials \( F_i(a) = \sigma_i(u_1(a), \ldots, u_n(a)) \) are expressed via linear combinations of \( q_i \) with holomorphic coefficients. Indeed, let

\[
Q_i(a) := \prod_{j=1, j \neq i}^{n+2} (a_i - a_j), \quad i = 1, \ldots, n + 2.
\]

Then, as follows from the formulae (11) and (12),

\[
a_i^n - F_1(a)a_i^{n-1} + \ldots + (-1)^n F_n(a) = \frac{Q_i(a)}{a_i} - q_i, \quad i = 1, \ldots, n.
\]

Thus, the vector \((-F_1, \ldots, (-1)^n F_n)\) is a solution of the system of linear equations with the coefficients matrix whose determinant is the Vandermonde determinant.

The above reasonings lead to the following statement.

**Proposition 1.** Let \((u(a), 0)\) be a solution of the Garnier system (3) with \( \sum_{i=1}^{n+2} \theta_i - 1 = \pm \theta_\infty \) (this solution has the reducible linear monodromy). Then the polar loci of the functions \( F_i \) are analytical submanifolds and \((F_i)_\infty = -1\).

M. Mazzocco [11] has shown that any solution of the Garnier system (3) having reducible linear monodromy can be expressed classically via particular solutions from the above proposition. We hope that a careful reading of her article will lead to the obtaining of concluding estimates for the elementary symmetric polynomials depending on an arbitrary solution of (3).

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