Tridiagonal pairs of $q$-Racah type, the Bockting operator $\psi$, and $L$-operators for $U_q(L(sl_2))$

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Abstract

We describe the Bockting operator $\psi$ for a tridiagonal pair of $q$-Racah type, in terms of a certain $L$-operator for the quantum loop algebra $U_q(L(sl_2))$.

Keywords. Bockting operator, tridiagonal pair, Leonard pair.

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1 Introduction

In the theory of quantum groups there exists the concept of an $L$-operator; this was introduced in [20] to obtain solutions for the Yang-Baxter equation. In linear algebra there exists the concept of a tridiagonal pair; this was introduced in [13] to describe the irreducible modules for the subconstituent algebra of a $Q$-polynomial distance-regular graph. Recently some authors have connected the two concepts. In [1, 3] Pascal Baseilhac and Kozo Koizumi use $L$-operators for the quantum loop algebra $U_q(L(sl_2))$ to construct a family of finite-dimensional modules for the $q$-Onsager algebra $O_q$; see [2, 4–6] for related work. A finite-dimensional irreducible $O_q$-module is essentially the same thing as a tridiagonal pair of $q$-Racah type [9, Section 12], [23, Section 3]. In [22, Section 9], Kei Miki uses similar $L$-operators to describe how $U_q(L(sl_2))$ is related to the $q$-tetrahedron algebra $\boxtimes_q$. A finite-dimensional irreducible $\boxtimes_q$-module is essentially the same thing as a tridiagonal pair of $q$-geometric type [15, Theorem 2.7], [16, Theorems 10.3, 10.4]. Following Baseilhac, Koizumi, and Miki, in the present paper we use $L$-operators for $U_q(L(sl_2))$ to describe the Bockting operator $\psi$ associated with a tridiagonal pair of $q$-Racah type. Before going into detail, we recall some notation and basic concepts. Throughout this paper $\mathbb{F}$ denotes a field. Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. For an $\mathbb{F}$-linear map $A : V \to V$ and a subspace $W \subseteq V$, we say that $W$ is an eigenspace of $A$ whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V | Av = \theta v\}$; in this case $\theta$ is called the eigenvalue of $A$ associated with $W$. We say that $A$ is diagonalizable whenever $V$ is spanned by the eigenspaces of $A$.

Definition 1.1. (See [13] Definition 1.1.) Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. By a tridiagonal pair (or TD pair) on $V$ we mean an ordered pair of $\mathbb{F}$-linear maps $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions:
(i) Each of $A, A^*$ is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \tag{1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V^*_i\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that

$$AV^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1} \quad (0 \leq i \leq \delta), \tag{2}$$

where $V^*_{-1} = 0$ and $V^*_{\delta+1} = 0$.

(iv) There does not exist a subspace $W \subseteq V$ such that $AW \subseteq W, A^*W \subseteq W, W \neq 0, W \neq V$.

We refer the reader to [12,13,17] for background on TD pairs, and here mention only a few essential points. Let $A, A^*$ denote a TD pair on $V$, as in Definition [11]. By [13, Lemma 4.5] the integers $d$ and $\delta$ from (1) and (2) are equal; we call this common value the diameter of $A, A^*$. An ordering of the eigenspaces for $A$ (resp. $A^*$) is called standard whenever it satisfies (1) (resp. (2)). Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of $A$. By [13, Lemma 2.4] the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no further ordering is standard. A similar result holds for the eigenspaces of $A^*$. Until the end of this section fix a standard ordering $\{V_i\}_{i=0}^d$ (resp. $\{V^*_i\}_{i=0}^d$) of the eigenspaces for $A$ (resp. $A^*$). For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta^*_i$) denote the eigenvalue of $A$ (resp. $A^*$) for the eigenspace $V_i$ (resp. $V^*_i$). By construction $\{\theta_i\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{F}$. Moreover $\{\theta^*_i\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{F}$. By [13, Theorem 11.1] the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of $i$ for $2 \leq i \leq d - 1$. For this constraint the solutions can be given in closed form [13, Theorem 11.2]. The “most general” solution is called $q$-Racah, and will be described shortly.

We now recall the split decomposition [13, Section 4]. For $0 \leq i \leq d$ define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}).$$

For notational convenience define $U_{-1} = 0$ and $U_{d+1} = 0$. By [13, Theorem 4.6] the sum $V = \sum_{i=0}^d U_i$ is direct. By [13, Theorem 4.6] both

$$U_0 + U_1 + \cdots + U_i = V_0^* + V_1^* + \cdots + V_i^*,$$

$$U_i + U_{i+1} + \cdots + U_d = V_0 + V_1 + \cdots + V_{d-i}$$

for $0 \leq i \leq d$. By [13, Theorem 4.6] both

$$\begin{align*}
(A - \theta_{d-i}I)U_i & \subseteq U_{i+1}, \\
(A^* - \theta^*_iI)U_i & \subseteq U_{i-1}
\end{align*} \quad (3)$$

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for $0 \leq i \leq d$.

We now describe the $q$-Racah case. Pick a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. We say that $A, A^*$ has $q$-Racah type whenever there exist nonzero $a, b \in \mathbb{F}$ such that both

$$
\theta_i = aq^{2i-d} + a^{-1}q^{d-2i}, \quad \theta_i^* = bq^{2i-d} + b^{-1}q^{d-2i}
$$

for $0 \leq i \leq d$. For the rest of this section assume that $A, A^*$ has $q$-Racah type. For $1 \leq i \leq d$ we have $q^{2i} \neq 1$; otherwise $\theta_i = \theta_0$. Define an $\mathbb{F}$-linear map $K : V \to V$ such that for $0 \leq i \leq d, U_i$ is an eigenspace of $K$ with eigenvalue $q^{d-2i}$. Thus

$$(K - q^{d-2i}I)U_i = 0 \quad (0 \leq i \leq d),$$

where $I : V \to V$ is the identity map. Note that $K$ is invertible. For $0 \leq i \leq d$ the following holds on $U_i$:

$$aK + a^{-1}K^{-1} = \theta_{d-i}I.$$  \hspace{1cm} (6)

Define an $\mathbb{F}$-linear map $R : V \to V$ such that for $0 \leq i \leq d$, $R$ acts on $U_i$ as $A - \theta_{d-i}I$. By (6),

$$A = aK + a^{-1}K^{-1} + R.$$ \hspace{1cm} (7)

By the equation on the left in (3),

$$RU_i \subseteq U_{i+1} \quad (0 \leq i \leq d).$$ \hspace{1cm} (8)

We now recall the Bockting operator $\psi$. By [8, Lemma 5.7] there exists a unique $\mathbb{F}$-linear map $\psi : V \to V$ such that both

$$\psi U_i \subseteq U_{i-1} \quad (0 \leq i \leq d),$$

$$\psi R - R\psi = (q - q^{-1})(K - K^{-1}).$$ \hspace{1cm} (9, 10)

The known properties of $\psi$ are described in [7-9]. Suppose we are given $A, A^*, R, K$ in matrix form, and wish to obtain $\psi$ in matrix form. This can be done using [5, 9, 10] and induction on $i$. The calculation can be tedious, so one desires a more explicit description of $\psi$. In the present paper we give an explicit description of $\psi$, in terms of a certain $L$-operator for $U_q(L(sl_2))$. According to this description, $\psi$ is equal to $-a$ times the ratio of two components for the $L$-operator. Theorem 5.4 is our main result.

The paper is organized as follows. In Section 2 we review the algebra $U_q(L(sl_2))$ in its Chevalley presentation. In Section 3 we recall the equitable presentation for $U_q(L(sl_2))$. In Section 4 we discuss some $L$-operators for $U_q(L(sl_2))$. In Section 5 we use these $L$-operators to describe $\psi$. 

3
2 The quantum loop algebra $U_q(L(\mathfrak{sl}_2))$

Recall the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \}$ and natural numbers $\mathbb{N} = \{0, 1, 2, \ldots \}$. We will be discussing algebras. An algebra is meant to be associative and have a 1. Recall the field $\mathbb{F}$. Until the end of Section 4, fix a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$. Define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{Z}.$$ 

All tensor products are meant to be over $\mathbb{F}$.

Definition 2.1. (See [10, Section 3.3].) Let $U_q(L(\mathfrak{sl}_2))$ denote the $\mathbb{F}$-algebra with generators $E_i, F_i, K_i^{\pm 1} (i \in \{0, 1\})$ and relations

$$
\begin{align*}
K_i K_i^{-1} &= 1, & K_i^{-1} K_i &= 1, \\
K_0 K_1 &= 1, & K_1 K_0 &= 1, \\
K_i E_i &= q^2 E_i K_i, & K_i F_i &= q^{-2} F_i K_i, \\
K_i E_j &= q^{-2} E_j K_i, & K_i F_j &= q^2 F_j K_i, & i \neq j, \\
E_i F_j - F_j E_i &= \delta_{ij} K_i - K_i^{-1}, \\
E_i^3 E_j - [3]_q E_i^2 E_j E_i + [3]_q E_i E_j E_i^2 - E_j E_i^3 &= 0, & i \neq j, \\
F_i^3 F_j - [3]_q F_i^2 F_j F_i + [3]_q F_i F_j F_i^2 - F_j F_i^3 &= 0, & i \neq j.
\end{align*}
$$

We call $E_i, F_i, K_i^{\pm 1}$ the Chevalley generators for $U_q(L(\mathfrak{sl}_2))$.

Lemma 2.2. (See [18, p. 35].) We turn $U_q(L(\mathfrak{sl}_2))$ into a Hopf algebra as follows. The coproduct $\Delta$ satisfies

$$
\begin{align*}
\Delta(K_i) &= K_i \otimes K_i, & \Delta(K_i^{-1}) &= K_i^{-1} \otimes K_i^{-1}, \\
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \Delta(F_i) &= 1 \otimes F_i + F_i \otimes K_i^{-1}.
\end{align*}
$$

The counit $\varepsilon$ satisfies

$$
\varepsilon(K_i) = 1, \quad \varepsilon(K_i^{-1}) = 1, \quad \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0.
$$

The antipode $S$ satisfies

$$
S(K_i) = K_i^{-1}, \quad S(K_i^{-1}) = K_i, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i.
$$

We now discuss the $U_q(L(\mathfrak{sl}_2))$-modules.

Lemma 2.3. (See [10, Section 4].) There exists a family of $U_q(L(\mathfrak{sl}_2))$-modules

$$
V(d, t) \quad 0 \neq d \in \mathbb{N}, \quad 0 \neq t \in \mathbb{F} \quad (11)
$$

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with this property: \( V(d, t) \) has a basis \( \{ v_i \}_{i=0}^d \) such that
\[
\begin{align*}
K_1 v_i &= q^{d-2i} v_i & (0 \leq i \leq d), \\
E_1 v_i &= [d - i + 1] q v_{i-1} & (1 \leq i \leq d), \\
F_1 v_i &= [i + 1] q v_{i+1} & (0 \leq i \leq d - 1), \\
K_0 v_i &= q^{2i-d} v_i & (0 \leq i \leq d), \\
E_0 v_i &= t[i + 1] q v_{i+1} & (0 \leq i \leq d - 1), \\
F_0 v_i &= t^{-1}[d - i + 1] q v_{i-1} & (1 \leq i \leq d).
\end{align*}
\]

The module \( V(d, t) \) is irreducible provided that \( q^{2i} \neq 1 \) for \( 1 \leq i \leq d \).

**Definition 2.4.** Referring to Lemma 2.3, we call \( V(d, t) \) an evaluation module for \( U_q(L(\mathfrak{sl}_2)) \). We call \( d \) the diameter. We call \( t \) the evaluation parameter.

**Example 2.5.** For \( 0 \neq t \in \mathbb{F} \) the \( U_q(L(\mathfrak{sl}_2)) \)-module \( V(1, t) \) is described as follows. With respect to the basis \( v_0, v_1 \) from Lemma 2.3, the matrices representing the Chevalley generators are
\[
\begin{align*}
E_1 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \quad F_1 : \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \quad K_1 : \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \\
E_0 : \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, & \quad F_0 : \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}, & \quad K_0 : \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.
\end{align*}
\]

**Lemma 2.6.** (See [19] p. 58.) Let \( U \) and \( V \) denote \( U_q(L(\mathfrak{sl}_2)) \)-modules. Then \( U \otimes V \) becomes a \( U_q(L(\mathfrak{sl}_2)) \)-module as follows. For \( u \in U \) and \( v \in V \),
\[
\begin{align*}
K_i(u \otimes v) &= K_i(u) \otimes K_i(v), \\
K_i^{-1}(u \otimes v) &= K_i^{-1}(u) \otimes K_i^{-1}(v), \\
E_i(u \otimes v) &= E_i(u) \otimes v + K_i(u) \otimes E_i(v), \\
F_i(u \otimes v) &= u \otimes F_i(v) + F_i(u) \otimes K_i^{-1}(v).
\end{align*}
\]

**Definition 2.7.** (See [11] p. 110.) Up to isomorphism, there exists a unique \( U_q(L(\mathfrak{sl}_2)) \)-module of dimension 1 on which each \( u \in U_q(L(\mathfrak{sl}_2)) \) acts as \( \varepsilon(u) I \), where \( \varepsilon \) is from Lemma 2.2. This \( U_q(L(\mathfrak{sl}_2)) \)-module is said to be trivial.

**Proposition 2.8.** (See [22] Theorem 3.2.) Assume that \( \mathbb{F} \) is algebraically closed with characteristic zero, and \( q \) is not a root of unity. Let \( V \) denote a nontrivial finite-dimensional irreducible \( U_q(L(\mathfrak{sl}_2)) \)-module on which each eigenvalue of \( K_1 \) is an integral power of \( q \). Then \( V \) is isomorphic to a tensor product of evaluation \( U_q(L(\mathfrak{sl}_2)) \)-modules.

### 3 The equitable presentation for \( U_q(L(\mathfrak{sl}_2)) \)

In this section we recall the equitable presentation for \( U_q(L(\mathfrak{sl}_2)) \). Let \( \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} \) denote the cyclic group of order 4. In a moment we will discuss some objects \( X_{ij} \). The subscripts \( i, j \) are meant to be in \( \mathbb{Z}_4 \).
Lemma 3.1. (See [14, Theorem 2.1], [22, Proposition 4.2].) The algebra $U_q(L(\mathfrak{sl}_2))$ has a presentation by generators
\begin{equation}
X_{01}, \ X_{12}, \ X_{23}, \ X_{30}, \ X_{13}, \ X_{31}
\end{equation}
and the following relations:
\begin{align*}
X_{13}X_{31} &= 1, \quad X_{31}X_{13} = 1, \\
\frac{qX_{01}X_{12} - q^{-1}X_{12}X_{01}}{q - q^{-1}} &= 1, \\
\frac{qX_{12}X_{32} - q^{-1}X_{32}X_{12}}{q - q^{-1}} &= 1,
\end{align*}
\begin{align*}
qX_{23}X_{30} - q^{-1}X_{30}X_{23} &= 1, \\
\frac{qX_{30}X_{01} - q^{-1}X_{01}X_{30}}{q - q^{-1}} &= 1, \\
\frac{qX_{01}X_{13} - q^{-1}X_{13}X_{01}}{q - q^{-1}} &= 1,
\end{align*}
\begin{align*}
qX_{31}X_{12} - q^{-1}X_{12}X_{31} &= 1, \\
\frac{qX_{23}X_{31} - q^{-1}X_{31}X_{23}}{q - q^{-1}} &= 1, \\
\frac{qX_{13}X_{30} - q^{-1}X_{30}X_{13}}{q - q^{-1}} &= 1,
\end{align*}
\begin{align*}
X_{i,i+1}X_{i+2,i+3} - [3]q^2X_{i,i+1}X_{i+2,i+3}X_{i,i+1} + [3]q^2X_{i,i+1}X_{i+2,i+3}X_{i,i+1}^2 - X_{i+2,i+3}X_{i,i+1} = 0.
\end{align*}
An isomorphism with the presentation in Definition 2.1 sends
\begin{align*}
X_{01} &\mapsto K_0 + q(q - q^{-1})K_0F_0, & X_{12} &\mapsto K_1 - (q - q^{-1})E_1, \\
X_{23} &\mapsto K_1 + q(q - q^{-1})K_1F_1, & X_{30} &\mapsto K_0 - (q - q^{-1})E_0, \\
X_{13} &\mapsto K_1, & X_{31} &\mapsto K_0.
\end{align*}
The inverse isomorphism sends
\begin{align*}
E_1 &\mapsto (X_{13} - X_{12})(q - q^{-1})^{-1}, & E_0 &\mapsto (X_{31} - X_{30})(q - q^{-1})^{-1}, \\
F_1 &\mapsto (X_{31}X_{23} - 1)q^{-1}(q - q^{-1})^{-1}, & F_0 &\mapsto (X_{13}X_{01} - 1)q^{-1}(q - q^{-1})^{-1}, \\
K_1 &\mapsto X_{13}, & K_0 &\mapsto X_{31}.
\end{align*}
Note 3.2. For notational convenience, we identify the copy of $U_q(L(\mathfrak{sl}_2))$ given in Definition 2.1 with the copy given in Lemma 3.1 via the isomorphism given in Lemma 3.1.

Definition 3.3. Referring to Lemma 3.1, we call the generators (12) the equitable generators for $U_q(L(\mathfrak{sl}_2))$.

Lemma 3.4. (See [24, Theorem 3.4].) From the equitable point of view the Hopf algebra $U_q(L(\mathfrak{sl}_2))$ looks as follows. The coproduct $\Delta$ satisfies
\begin{align*}
\Delta(X_{13}) &= X_{13} \otimes X_{13}, \\
\Delta(X_{31}) &= X_{31} \otimes X_{31},
\end{align*}
\begin{align*}
\Delta(X_{01}) &= (X_{01} - X_{31}) \otimes 1 + X_{31} \otimes X_{01}, \\
\Delta(X_{12}) &= (X_{12} - X_{13}) \otimes 1 + X_{13} \otimes X_{12}, \\
\Delta(X_{23}) &= (X_{23} - X_{13}) \otimes 1 + X_{13} \otimes X_{23}, \\
\Delta(X_{30}) &= (X_{30} - X_{31}) \otimes 1 + X_{31} \otimes X_{30}.
\end{align*}
The counit $\varepsilon$ satisfies
\begin{align*}
\varepsilon(X_{13}) &= 1, & \varepsilon(X_{31}) &= 1, & \varepsilon(X_{01}) &= 1, \\
\varepsilon(X_{12}) &= 1, & \varepsilon(X_{23}) &= 1, & \varepsilon(X_{30}) &= 1.
\end{align*}
The antipode $S$ satisfies
\begin{align*}
S(X_{31}) &= X_{13}, & S(X_{13}) &= X_{31}, \\
S(X_{01}) &= 1 + X_{13} - X_{13}X_{01}, & S(X_{12}) &= 1 + X_{31} - X_{31}X_{12}, \\
S(X_{23}) &= 1 + X_{31} - X_{31}X_{23}, & S(X_{30}) &= 1 + X_{13} - X_{13}X_{30}.
\end{align*}
4 Some $L$-operators for $U_q(L(sl_2))$

In this section we recall some $L$-operators for $U_q(L(sl_2))$, and describe their basic properties.

We recall some notation. Let $\Delta$ denote the coproduct for a Hopf algebra $H$. Then the opposite coproduct $\Delta^{\text{op}}$ is the composition

$$\Delta^{\text{op}} : H \xrightarrow{\Delta} H \otimes H \xrightarrow{r \otimes s \mapsto s \otimes r} H \otimes H.$$

**Definition 4.1.** (See [22, Section 9.1].) Let $V$ denote a $U_q(L(sl_2))$-module and $0 \neq t \in \mathbb{F}$. Consider an $\mathbb{F}$-linear map

$$L : V \otimes V(1,t) \to V \otimes V(1,t).$$

We call this map an $L$-operator for $V$ with parameter $t$ whenever the following diagram commutes for all $u \in U_q(L(sl_2))$:

$$\begin{array}{ccc}
V \otimes V(1,t) & \xrightarrow{\Delta(u)} & V \otimes V(1,t) \\
\downarrow L & & \downarrow L \\
V \otimes V(1,t) & \xrightarrow{\Delta^{\text{op}}(u)} & V \otimes V(1,t)
\end{array}$$

**Definition 4.2.** (See [22, Section 9.1].) Let $V$ denote a $U_q(L(sl_2))$-module and $0 \neq t \in \mathbb{F}$. Consider any $\mathbb{F}$-linear map

$$L : V \otimes V(1,t) \to V \otimes V(1,t).$$

For $r, s \in \{0, 1\}$ define an $\mathbb{F}$-linear map $L_{rs} : V \to V$ such that for $v \in V$,

$$L(v \otimes v_0) = L_{00}(v) \otimes v_0 + L_{10}(v) \otimes v_1,$$

$$L(v \otimes v_1) = L_{01}(v) \otimes v_0 + L_{11}(v) \otimes v_1.$$  \hspace{2cm} (14, 15)

Here $v_0, v_1$ is the basis for $V(1, t)$ from Lemma 2.3

**Lemma 4.3.** Referring to Definition 4.2, the map (13) is an $L$-operator for $V$ with parameter
if and only if the following equations hold on $V$:

\[
\begin{align*}
K_1L_{00} &= L_{00}K_1, & K_1L_{01} &= q^{-2}L_{01}K_1, \\
K_1L_{10} &= q^2L_{10}K_1, & K_1L_{11} &= L_{11}K_1; \\
L_{00}E_1 - qE_1L_{00} &= L_{10}, & L_{01}E_1 - qE_1L_{01} &= L_{11} - L_{00}K_1, \\
L_{10}E_1 - q^{-1}E_1L_{10} &= 0, & L_{11}E_1 - q^{-1}E_1L_{11} &= -L_{10}K_1; \\
F_1L_{00} - q^{-1}L_{00}F_1 &= L_{01}, & F_1L_{01} - qL_{01}F_1 &= 0, \\
F_1L_{10} - q^{-1}L_{10}F_1 &= L_{11} - K_0L_{00}, & F_1L_{11} - qL_{11}F_1 &= -K_0L_{01}; \\
K_0L_{00} &= L_{00}K_0, & K_0L_{01} &= q^2L_{01}K_0, \\
K_0L_{10} &= q^{-2}L_{10}K_0, & K_0L_{11} &= L_{11}K_0; \\
L_{00}E_0 - q^{-1}E_0L_{00} &= -tL_{01}K_0, & L_{01}E_0 - q^{-1}E_0L_{01} &= 0, \\
L_{10}E_0 - qE_0L_{10} &= tL_{00} - tL_{11}K_0, & L_{11}E_0 - qE_0L_{11} &= tL_{01}; \\
F_0L_{00} - qL_{00}F_0 &= -t^{-1}K_1L_{10}, & F_0L_{01} - q^{-1}L_{01}F_0 &= t^{-1}L_{00} - t^{-1}K_1L_{11}, \\
F_0L_{10} - qL_{10}F_0 &= 0, & F_0L_{11} - q^{-1}L_{11}F_0 &= t^{-1}L_{10}.
\end{align*}
\]

Proof. This is routinely checked. □

**Example 4.4.** (See [21 Appendix], [22 Proposition 9.2].) Referring to Definition 4.2, assume that $V$ is an evaluation module $V(d, \mu)$ such that $q^{2i} \neq 1$ for $1 \leq i \leq d$. Consider the matrices that represent the $L_{rs}$ with respect to the basis $\{v_i\}_{i=0}^d$ for $V(d, \mu)$ from Lemma 2.3. Then the following are equivalent:

(i) the map (13) is an $L$-operator for $V$ with parameter $t$;

(ii) the matrix entries are given in the table below (all matrix entries not shown are zero):

| operator | $(i, i - 1)$-entry | $(i, i)$-entry | $(i - 1, i)$-entry |
|----------|-------------------|--------------|-------------------|
| $L_{00}$ | 0 | $\frac{q^{i-1}-\mu^{-1}q^{d-i}}{q-q^{-1}}$ | $\xi$ | 0 |
| $L_{01}$ | $[i]_q q^{1-i} \xi$ | 0 | 0 |
| $L_{10}$ | 0 | 0 | $[d-i+1]_q q^{j-d} \mu^{-1} t \xi$ |
| $L_{11}$ | 0 | $\frac{q^{i-d-1}-\mu^{-1}q^{d-i}}{q-q^{-1}}$ | $\xi$ | 0 |

Here $\xi \in \mathbb{F}$.

**Lemma 4.5.** (See [22 Proposition 9.3].) Let $U$ and $V$ denote $U_q(L(gl_2))$-modules, and consider the $U_q(L(sl_2))$-module $U \otimes V$ from Lemma 2.6. Let $0 \neq t \in \mathbb{F}$. Suppose we are given $L$-operators for $U$ and $V$ with parameter $t$. Then there exists an $L$-operator for $U \otimes V$ with parameter $t$ such that for $r, s \in \{0, 1\}$,

\[
L_{rs}(u \otimes v) = L_{r0}(u) \otimes L_{0s}(v) + L_{r1}(u) \otimes L_{1s}(v)
\]

$u \in U, \ v \in V.$ \hspace{1cm} (16)
Proof. For \( r, s \in \{0, 1\} \) define an \( \mathbb{F} \)-linear map \( L_{rs} : U \otimes V \to U \otimes V \) that satisfies (16). Using (16) and Lemma 2.6 one checks that the \( L_{rs} \) satisfy the equations in Lemma 4.3. The result follows by Lemma 4.3.

**Corollary 4.6.** Adopt the notation and assumptions of Proposition 2.8. Then for \( 0 \neq t \in \mathbb{F} \) there exists a nonzero \( L \)-operator for \( V \) with parameter \( t \).

*Proof.* By Proposition 2.8 along with Example 4.4 and Lemma 4.5.

## 5 TD pairs and \( L \)-operators

In Section 1 we discussed a TD pair \( A, A^* \) on \( V \). We now return to this discussion, adopting the notation and assumptions that were in force at the end of Section 1. Recall the scalars \( q, a, b \) from (4). Recall the map \( K \) from above (5).

**Proposition 5.1.** (See [17, p. 103].) Assume that \( \mathbb{F} \) is algebraically closed with characteristic zero, and \( q \) is not a root of unity. Then the vector space \( V \) becomes a \( U_q(L(\mathfrak{sl}_2)) \)-module on which

\[
A = ax_{01} + a^{-1}x_{12}, \quad A^* = bX_{23} + b^{-1}X_{30}.
\]

*Proof.* This is how [17, p. 103] looks from the equitable point of view.

**Note 5.2.** The \( U_q(L(\mathfrak{sl}_2)) \)-module structure from Proposition 5.1 is not unique in general.

We now investigate the \( U_q(L(\mathfrak{sl}_2)) \)-module structure from Proposition 5.1. Recall the map \( R \) from above (7).

**Lemma 5.3.** Assume that the vector space \( V \) becomes a \( U_q(L(\mathfrak{sl}_2)) \)-module on which \( K = X_{31}, K^{-1} = X_{13} \) and

\[
A = ax_{01} + a^{-1}X_{12}, \quad A^* = bX_{23} + b^{-1}X_{30}.
\]

*Proof.* (i) In line (7) eliminate \( A, K, K^{-1} \) using the assumptions of the present lemma.

(ii) Evaluate the right-hand side of (17) using the identifications from Lemma 3.1 and Note 3.2.

We now present our main result. Recall the Bockting operator \( \psi \) from (9), (10).
Theorem 5.4. Assume that the vector space $V$ becomes a $U_q(L(sl_2))$-module on which $K = X_{31}$, $K^{-1} = X_{13}$ and

$$A = aX_{01} + a^{-1}X_{12}, \quad A^* = bX_{23} + b^{-1}X_{30}.$$

Consider an $L$-operator for $V$ with parameter $a^2$. Then on $V$,

$$\psi = -a(L_{00})^{-1}L_{01}$$  \hspace{1cm} (19)

provided that $L_{00}$ is invertible.

Proof. Let $\hat{\psi}$ denote the expression on the right in (19). We show $\psi = \hat{\psi}$. To do this, we show that $\hat{\psi}$ satisfies (9), (10). Concerning (9), by Lemma 4.3 the equation $K_0\hat{\psi} = q^2\hat{\psi}K_0$ holds on $V$. By Lemma 3.1, Note 3.2, and the construction, we obtain $K_0 = X_{31} = K$ on $V$. By these comments $K\hat{\psi} = q^2\hat{\psi}K$ on $V$. By this and (5) we obtain $\hat{\psi}U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. So $\hat{\psi}$ satisfies (9). Next we show that $\hat{\psi}$ satisfies (10). Since $L_{00}$ is invertible and $K_0K_1 = 1$ it suffices to show that on $V$,

$$L_{00}(\hat{\psi}R - R\hat{\psi}) = (q - q^{-1})L_{00}(K_0 - K_1).$$  \hspace{1cm} (20)

By this and (18) it suffices to show that on $V$,

$$aqL_{00}(\hat{\psi}K_0F_0 - K_0F_0\hat{\psi}) - a^{-1}L_{00}(\hat{\psi}E_1 - E_1\hat{\psi}) + L_{00}(K_1 - K_0) = 0.$$  \hspace{1cm} (21)

We examine the terms in (21). By Lemma 4.3 and the construction, the following hold on $V$:

$$L_{00}\hat{\psi}K_0F_0 = -aL_{01}K_0F_0$$

$$= -aq^{-2}K_0L_{01}F_0$$

$$= -aq^{-1}K_0(F_0L_{01} - a^{-2}L_{00} + a^{-2}K_1L_{11})$$

and

$$L_{00}K_0F_0\hat{\psi} = K_0L_{00}F_0\hat{\psi}$$

$$= q^{-1}K_0(a^{-2}K_1L_{10} + F_0L_{00})\hat{\psi}$$

$$= q^{-1}K_0(a^{-2}K_1L_{10})\hat{\psi} - aF_0L_{01}$$

and

$$L_{00}\hat{\psi}E_1 = -aL_{01}E_1$$

$$= -a(qE_1L_{01} + L_{11} - L_{00}K_1)$$

$$= -a(qE_1L_{01} + L_{11} - K_1L_{00})$$

and

$$L_{00}E_1\hat{\psi} = (L_{10} + qE_1L_{00})\hat{\psi}$$

$$= L_{10}\hat{\psi} - qaE_1L_{01}$$

and

$$L_{00}K_1 = K_1L_{00}, \quad L_{00}K_0 = K_0L_{00}.$$

To verify (21), evaluate its left-hand side using the above comments and simplify the result using $K_0K_1 = 1$. The computation is routine, and omitted. We have shown that $\hat{\psi}$ satisfies (10). The result follows.
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