Single-Electron Parametron: Reversible Computation in a Discrete State System

Konstantin K. Likharev\textsuperscript{1} and Alexander N. Korotkov\textsuperscript{1,2}

\textsuperscript{1}Department of Physics, State University of New York, Stony Brook, NY 11794-3800

\textsuperscript{2}Institute of Nuclear Physics, Moscow State University, Moscow 119899 GSP, Russia

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Abstract

We have analyzed energy dissipation in a digital device ("Single-Electron Parametron") in which discrete degrees of freedom are used for presenting digital information. If the switching speed is not too high, the device may operate reversibly (adiabatically), and the energy dissipation $E$ per bit may be much less than the thermal energy $k_B T$. The energy-time product $E \tau$ is, however, much larger than Planck’s constant $\hbar$, at least in the standard “orthodox” model of single-electron tunneling, which was used in our calculations.
Computation using any real physical system leads to the dissipation of energy, because of unavoidable coupling between the degrees of freedom which present information, and the environment. In most practical electronic devices (such as semiconductor transistor circuits), energy is dissipated at some rate even in static state, i.e. in the absence of information processing. Some prospective digital devices, such as various Single-Flux-Quantum and Single-Electron logics, however, do not involve static power consumption, because they present conservative systems where digital information is coded by the choice of a local minimum of potential energy. In this case the energy dissipation is proportional to the number of logic operations. If such a conservative system is switched irreversibly (e.g., as in the RSFQ logic), the energy loss $E$ per one logic operation is of the order of energy barrier $W$ separating the states. The barrier should be sufficiently high to make the probability $p \sim \exp(-W/k_B T)$ of thermally-induced errors low enough, so that at the physically irreversible computation $E_{\text{min}} \sim k_B T \ln(1/p) \gg k_B T$.

Some conservative systems, e.g. the Parametric Quantron, are flexible enough to allow independent adjustment of the potential tilt and barrier height. As has been shown earlier, this flexibility allows physically reversible (adiabatic) switching of the system with $E \ll k_B T$, if the information content of the system remains intact. If, however, information is lost during the operation (informationally irreversible computation), the minimum energy loss is at least $E_{\text{min}} = k_B T \ln 2$ per each lost bit. Thus only at completely (physically and informationally) reversible computation, $E$ may be made much less $k_B T$. In this case, $E$ scales as $\tau^{-1}$, where $\tau$ is the switching time, so that the product $E\tau$ is fixed.

$E\tau$ has the dimensionality of Planck’s constant, so the natural question is whether quantum mechanics imposes any fundamental lower bound on this product. A quantitative analysis of a particular reversible system (the Parametric Quantron) has shown that $E\tau$ may be much less than $\hbar$. The analysis has, however, used the assumption that the potential energy is a function of a continuous degree of freedom (in that particular case the Josephson phase $\phi$). To our knowledge, a similar analysis has never before been carried out for any system presenting information with discrete states.
Recently we suggested a system (“Single-Electron Parametron”) based on correlated single-electron tunneling (see, e.g. Ref. 3), which may be used for reversible computation using a discrete degree of freedom – the electric charge $Q$. The goal of the present work was to find the minimum energy dissipation $\mathcal{E}$ for this system, and relate it to the switching time $\tau$ and to the error probability $p$. We have shown, in particular, that within the “orthodox” theory of single-electron tunneling, $\mathcal{E}\tau$ is always considerably larger than $\hbar$.

Figure 1a shows the possible structure of a unit cell of the system. It consists of at least three small conducting islands (with capacitances $C \ll e^2/k_B T$), with the middle island slightly shifted in $y$-direction. Tunnel barriers with small conductances ($G \ll e^2/\hbar$) allow direct transfer of electrons only between the neighboring islands. The system is biased by a periodic “clock” electric field $E(t)$ perpendicular to axis $x$. Let us consider the conceptually simplest case when the cell is charged as a whole by a single extra electron. (For practice, the operation using electron-hole pairs may be beneficial, but for our present discussion both versions are identical.)

When the vertical component $E_y(t)$ of the field is lower than a certain threshold value $E_t$, the extra electron is kept inside the middle island. The energy diagram for this “OFF” state is shown at the top of Fig. 1b. As $E_y$ is increased, tunneling of the electron from the middle island into one of the edge islands becomes energy advantageous at $E_y > E_t$. If the system is completely $x-$symmetric, this results in spontaneous symmetry breaking, so that the direction of the resulting electric dipole moment of the system is random: $P_x = Qd_{\text{eff}}$, $d_{\text{eff}} \approx d$, $Q = \pm e$. However, if the symmetry is broken by a weak additional external field $E_x$ (say, induced by the dipole moment of a similar neighboring cell), the direction of the electron tunneling and hence the sign of $P_x$ is predetermined by this field. The middle frame in Fig. 1b shows this “OFF→ON switching” stage. Finally, when $E_y$ is well above $E_t$, the electron is trapped inside one of the edge islands, even if the “signal” field $E_x$ now favors its transfer in the opposite edge direction (“ON” state, the bottom frame in Fig. 1b). In this state, electron transfer may only be achieved via a higher-order “co-tunneling” process; the probability of this process may be made negligibly small by either decreasing the tunnel
conductance or by inserting a few additional islands into the cell. If this parasitic process is negligible, the cell has a fixed dipole moment and may serve as a robust source of signal field $E_x$ for similar neighboring cells. After this source has been used, the system is reset into the “OFF” state during the corresponding part of the clock cycle, when $E(t)$ drops below $E_t$ again.

It is evident that the operation of the cell is quite similar to that of the Parametric Quantron except now the information is presented by a discrete variable, $Q = \pm e$. Similarly to the Parametric Quantron, the Single-Electron Parametron may be used for reversible transfer and processing of information. For example, Figure 2 shows a possible structure of a shift register. In each neighboring cell, the extra charge sign alternates, while the direction of the middle island shift within plane $yz$ is changed by $\Theta = \pi - 2\pi/M$, $M > 2$ (in Fig. 2, $M = 3$). The clock field $E(t)$ has a fixed magnitude $E > E_t$, but rotates within plane $yz$, providing periodic switching ON and OFF of the cells, with the phase shift $2\pi/M$ between the neighboring cells. At an appropriate choice of $E$ and the distance between the cells, the orientation of the dipole moment of the cells in ON state determines the direction of the field $E_x$ and hence the direction of electron tunneling in the neighboring cell which is being switched OFF→ON. As a result, the information is being re-written from cell to cell, and thus transferred over $M$ cells each clock period. Reversible logic operations may be implemented in a similar way, e.g. by using majority gates with additional output cells.

Within the “orthodox” theory all properties of the system may be found from solution of the system of master equations for the probabilities $p_i(t)$ to find the extra electron in the middle ($i = m$), left ($i = l$), and right ($i = r$) islands:

$$\frac{d}{dt} p_i = \sum_{j=m,l,r} (p_j \Gamma_{ji} - p_i \Gamma_{ij}), \quad \sum_i p_i(t) = 1, \quad p_m(0) = 1,$$

where in our case the tunneling rate matrix $\Gamma$ has only four nonvanishing components:

$$\Gamma_{mr}^\pm = \frac{\pm GW}{e^2\{1 - \exp[\mp W/k_B T]\}}, \quad \Gamma_{ml}^\pm = \frac{\pm G(W - \Delta)}{e^2\{1 - \exp[\pm (W - \Delta)/k_B T]\}}. \tag{2}$$

Here $\Gamma_{ij}^+ \equiv \Gamma_{ji}, \Gamma_{ij}^- \equiv \Gamma_{ji}$, while $W(t) \approx E_y(t) d' + const$ is the energy difference between the charge configurations with the extra electron in the middle and right islands. We will
accept that near the decision-making point of the ON→OFF switching ($t \approx 0$ in Fig. 1c) the difference is a linear function of time: $W = \alpha t$. $\Delta \approx 2dE_x$ is the energy difference between the left and right ON states. In order to operate with a low error probability $p \ll 1$, at the decision-making moment this difference should be large enough. Without the loss of generality, we may assume $\Delta > 0$; then $p$ can be found from the solution of the master equation as $p_l(\infty)$, while the average energy $\mathcal{E}(t)$ dissipated by moment $t$ can be calculated as

$$
\mathcal{E}(t) = \int_{-\infty}^{t} \{ W(t) \frac{dp_r}{dt} + [W(t) - \Delta] \frac{dp_l}{dt} \} dt;
$$

we will be mostly interested in the net dissipation $\mathcal{E} \equiv \mathcal{E}(\infty)$.

The solution of equations (1)–(3) yields the following results. With an accuracy sufficient for our final result, the total error probability $p$ may be calculated as a maximum of probabilities of the thermal and dynamic errors. The thermal error may occur due to thermally-activated tunneling to the wrong state (in our case, $l$), and its probability $p_{\text{therm}}$ may always be expressed as $\exp(-\Delta/k_BT)$. (Because of that, we will restrict our discussion to the limit $\Delta \gg k_BT$.) The dynamic error occurs when the switching speed $\alpha$ is too high, and the system remains in the initial (symmetric) state up to the moment when tunneling to the upper energy level becomes possible. If $\delta \equiv \alpha e^2/G\Delta k_BT \gg 1$, the dynamic error dominates and its probability is given by expression

$$
p_{\text{dyn}} = K\gamma \exp\left(-\frac{1}{2\gamma}\right), \quad K = \frac{1}{2\gamma} - \frac{\sqrt{\pi}}{4\gamma^{3/2}} \exp\left(\frac{1}{4\gamma}\right)[1 - \text{Erf}\left(\frac{1}{2\sqrt{\gamma}}\right)] = 1 + \sum_{n=1}^{\infty} \frac{(2n + 1)!}{n!} (-\gamma)^n,
$$

where $\gamma \equiv \alpha e^2/G\Delta^2$. In order to keep $p_{\text{dyn}} \ll 1$, $\gamma$ should be much smaller than 1, so that one can use Eq. (3) with $K = 1$.

Energy dissipation depends on another dimensionless parameter, $\beta \equiv \alpha e^2/(Gk_BT)^2 = (\Delta/k_BT)^2\gamma = (\Delta/k_BT)\delta$. Like $\gamma$ and $\delta$, parameter $\beta$ is also proportional to the switching speed $\alpha$, but is much larger than both of them (because $\Delta \gg k_BT$) and may be comparable to unity. In the low-speed limit $\beta \ll 1$, the switching process is adiabatic. It consists of
numerous tunneling events (back and forth between \(m\) and \(r\)) taking place within the energy interval \(\sim k_B T\) around the point \(W(t) = 0\). In this case \(E = \kappa \beta k_B T\), where

\[
\kappa = \int_{-\infty}^{\infty} e^x (e^x - 1) dx \approx 0.426, \tag{5}
\]

so that for this (reversible) process \(E \ll k_B T\). Notice that \(E = \kappa \beta k_B T = \kappa \alpha e^2/G k_B T\) decreases when temperature increases.

Our model allows not only to calculate the net dissipation \(E\), but also follow the time dynamics of energy transfer between the system and the environment (“heat bath”) during the switching process. During the first half of the process (when \(W(t) \leq 0\)) the energy \(E_1 \equiv -E(0) = T \ln 2 \gg E\) is borrowed from the heat bath (which, hence, is cooled), while virtually the same amount \(E_2 \equiv E - E(0)\) is returned back to the heat bath during the second half of the process \((W(t) \geq 0)\). This exchange is directly related as \(E(t) = T \Delta S(t)\) to the temporal increase and consequent decrease of the entropy corresponding to the degree of freedom used to code information (in this particular case, the polarization \(P_x\)). At the moment when \(W = 0\), the system may be in either of two states \((p_m = p_r = 1/2)\), i.e. \(\Delta S = k_B \ln 2\) has been acquired in the comparison with the definite initial state \((p_m = 1, p_r = 0)\). By the end of the switching \((W \gg k_B T)\) the entropy is restored to the initial value since the state is definite again \((p_m = 0, p_r = 1)\). Finite switching speed decreases \(E_1\) and increases \(E_2\) (see the dotted lines in Fig. 3).

In the limit \(\beta \gg 1\), the speed of energy change is so high that switching may take place only at \(W > 0\), but within a much larger interval of energies: \(\Delta W \sim \beta^{1/2} k_B T\). The average energy dissipation for this (irreversible) process is of the same order, i.e. much larger than \(k_B T\) and independent of temperature: \(E = (\pi \beta/2)^{1/2} k_B T = (\pi e^2 \alpha/2G)^{1/2}\). The results of numerical calculation of \(E\) for intermediate values of \(\beta\) are presented by the solid line in Fig. 3.

Taking into account that at \(p \ll 1\) the parameter \(\tau = \Delta/\alpha\) may be considered as the duration of the switching process (Fig. 1c), all our asymptotic results may be summarized as follows:
\[ \mathcal{E}_\tau = \frac{\hbar}{G R_Q} \times \begin{cases} 
0.67 \ln \frac{1}{p}, & \text{for } \delta, \beta \ll 1, \\
1.97 \beta^{-1/2} \ln \frac{1}{p}, & \text{for } \delta \ll 1 \ll \beta, \\
2.78 \left( \ln \frac{1}{2p \ln(1/p)} \right)^{1/2}, & \text{for } 1 \ll \delta, \beta, 
\end{cases} \]

where \( R_Q = \pi \hbar/2e^2 \approx 6.45 \text{k}\Omega \) is the quantum unit of resistance. Since the orthodox theory is valid only at \( G R_Q \ll 1 \), within this theory \( \mathcal{E}_\tau \gg \hbar \) for any switching speed.

To summarize, we have shown that reversible computation with the energy dissipation \( \mathcal{E} \) per bit well below \( k_B T \) may be implemented in a physical system with discrete states. The quantum bound for the product \( \mathcal{E}_\tau \), obtained within our concrete model is, however, much higher than that obtained earlier for a system with continuous degrees of freedom.\[6\]

Apparently the \( \hbar \)-limit for \( \mathcal{E}_\tau \) may be overcome in the case of islands with discrete spectra of electron energies,\[3,4\] though this may require an exponentially high energy barrier \( W \) during the ON state of the cell. A quantitative analysis of this opportunity is in progress.

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FIGURES

FIG. 1. The Single-Electron Parametron: (a) 3-island version of the system, (b) its energy diagram for three values of the clock field $E_y$, and (c) energy of the extra electron in various islands as a function of time, close to the decision-making moment $t \approx 0$.

FIG. 2. Top (left) and side (right) views of a shift register using an array Single-Electron Parametron cells. Clock field $E(t)$ rotates in $yz$ plane. Digital bits are coded by positions of the extra charges in ON state of the cells, and are propagated from the top to the bottom, over $M = 3$ cells during one clock period.

FIG. 3. Components of the energy exchange between the Parametron and the heat bath as functions of the process speed $\alpha = dW/dt$. Dotted lines: average energy flow $\mathcal{E}_1$ from the heat bath to the Parametron during the first half of the process ($W \leq 0$) and the average flow $\mathcal{E}_2$ from the device back into the heat bath during its second half ($W \geq 0$), respectively. Solid line: net energy dissipation $\mathcal{E} = \mathcal{E}_2 - \mathcal{E}_1$. Dashed lines show the low-speed (adiabatic) and high-speed (diabatic) asymptotes of the function $\mathcal{E}(\alpha)$ – see formulas in the text.
Fig. 1
OFF -> ON

State ON

State OFF

State ON

State OFF

State ON

State ON

Fig. 2
\[ \beta = \alpha e^2 / G(k_B T)^2 \]

Fig. 3