Two-Loop Superstrings in Hyperelliptic Language I: the Main Results

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Abstract

Following the new gauging fixing method of D’Hoker and Phong, we study two-loop superstrings in hyperelliptic language. By using hyperelliptic representation of genus 2 Riemann surface we derive a set of identities involving the Szegö kernel. These identities are used to prove the vanishing of the cosmological constant and the non-renormalization theorem point-wise in moduli space by doing the summation over all the 10 even spin structures. Modular invariance is maintained at every stage of the computation explicitly. The 4-particle amplitude is also computed and an explicit expression for the chiral integrand is obtained. We use this result to show that the perturbative correction to the $R^4$ term in type II superstring theories is vanishing at two loops.

In this paper, a summary of the main results is presented with detailed derivations to be provided in two subsequent publications.

1 Introduction

Although we believe that superstring theory is finite in perturbation at any order \cite{1,2,3,4}, a rigorous proof is still lacking despite great advances in the covariant formulation of superstring perturbation theory à la Polyakov. The main problem is the presence of supermoduli and modular invariance in higher genus. At two loops these problems were solved explicitly by using the hyperelliptic formalism in a series of papers \cite{5,6,7,8,9}. The explicit result was also used by Iengo \cite{10} to prove the vanishing of perturbative correction to the $R^4$ term \cite{11} at two loop, in agreement with the indirect argument of Green and Gutperle \cite{12}, Green, Gutperle and Vanhove \cite{13}, and Green and Sethi \cite{14} that the $R^4$ term does not receive perturbative contributions beyond one loop. Recently, Stieberger and Taylor \cite{15} also used the result of \cite{8} to prove the vanishing of the heterotic two-loop $F^4$ term. For some closely related works we refer the reader to the reviews \cite{16,17}. In the general case, there is no satisfactory solution. For a review of these problem we refer the reader to \cite{18,19}.

Recently two-loop superstring was studied by D’Hoker and Phong. In a series of papers \cite{20,21,22,23} (for a recent review see \cite{19}), D’Hoker and Phong found an unambiguous and slice-independent two-loop superstring measure on moduli space for even spin structure from first principles.
Although their result is quite explicit, it is still a difficult problem to use it in actual computation. In [22], D’Hoker and Phong used their result to compute explicitly the chiral measure by choosing the split gauge and proved the vanishing of the cosmological constant and the non-renormalization theorem [24, 4]. They also computed the four-particle amplitude in another forthcoming paper [25]. Although the final results are exactly the expected, their computation is quite difficult to follow because of the use of theta functions.\footnote{In [26], the two-loop 4-particle amplitude was also computed by using theta functions. Its relation with the previous explicit result [8] has not been clarified.} Also modular invariance is absurd in their computations because of the complicated dependence between the 2 insertion points (the insertion points are also spin structure dependent).

In the old works [5, 6, 7, 8] on two-loop superstrings, one of the author (with Iengo) used the hyperelliptic representation to do the explicit computation at two loops which is quite explicit and modular invariance is manifest at every stage of the computations. So it is natural to do computations in this language by using the newly established result. As we will report in this paper and in more detail in [27, 28], everything is quite explicit in hyperelliptic language although the algebra is a little bit involved.

By using the hyperelliptic language we derive a set of identities involving the Szegö kernel (some identities were already derived in [6, 7, 8]). These identities are used to prove the vanishing of the cosmological constant and the non-renormalization theorem point-wise in moduli space by doing the summation over all the 10 even spin structures. Modular invariance is maintained at every stage of the computation explicitly. The 4-particle amplitude is also computed and an explicit expression for the integrand is obtained. We use this result to show that the perturbative correction to the $R^4$ term in type II superstring theories is vanishing at two loops, confirming the computation of Iengo [10] and the the conjecture of Green and Gutperle [12]. We leave the proof of the equivalence between the new result and the old result as a problem of the future.

Here we also note that D’Hoker and Phong have also proved that the cosmological constant and the 1-, 2- and 3-point functions are zero point-wise in moduli space [24]. They have also computed the 4-particle amplitude [25]. The agreement of the results from these two different gauge choices and different methods of computations would be another proof of the validity of the new supersymmetric gauge fixing method at two loops.
2 Genus 2 hyperelliptic Riemann surface

First we remind that a genus-\( g \) Riemann surface, which is the appropriate world sheet for one and two loops, can be described in full generality by means of the hyperelliptic formalism.\(^2\) This is based on a representation of the surface as two sheet covering of the complex plane described by the equation:

\[
y^2(z) = \prod_{i=1}^{2g+2} (z - a_i),
\]

(1)

The complex numbers \( a_i, (i = 1, \ldots, 2g + 2) \) are the \( 2g + 2 \) branch points, by going around them one passes from one sheet to the other. For two-loop \((g = 2)\) three of them represent the moduli of the genus 2 Riemann surface over which the integration is performed, while the other three can be arbitrarily fixed. Another parametrization of the moduli space is given by the period matrix.

At genus 2, by choosing a canonical homology basis of cycles we have the following list of 10 even spin structures:

\[
\delta_1 \sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim (a_1a_2a_3|a_4a_5a_6), \quad \delta_2 \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \sim (a_1a_2a_4|a_3a_5a_6),
\]

\[
\delta_3 \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim (a_1a_2a_5|a_3a_4a_6), \quad \delta_4 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim (a_1a_2a_6|a_3a_4a_5),
\]

\[
\delta_5 \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \sim (a_1a_3a_4|a_2a_5a_6), \quad \delta_6 \sim \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \sim (a_1a_3a_5|a_2a_4a_6),
\]

\[
\delta_7 \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sim (a_1a_3a_6|a_2a_4a_5), \quad \delta_8 \sim \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \sim (a_1a_4a_5|a_2a_3a_6),
\]

\[
\delta_9 \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \sim (a_1a_4a_6|a_2a_3a_5), \quad \delta_{10} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim (a_1a_5a_6|a_2a_3a_4).
\]

We will denote an even spin structure as \((A_1A_2A_3|B_1B_2B_3)\). By convention \(A_1 = a_1\). For each even spin structure we have a spin structure dependent factor from determinants which is given as follows \[5\):

\[
Q_\delta = \prod_{i<j} (A_i - A_j)(B_i - B_j).
\]

\[\text{(2)}\]

\[\text{Some early works on two loops computation by using hyperelliptic representation are [29, 30, 31, 32, 33, 34, 35] which is by no means the complete list.}\]
This is a degree 6 homogeneous polynomials in $a_i$.

At two loops there are two odd supermoduli and this gives two insertions of supercurrent at two different points $x_1$ and $x_2$. Previously the chiral measure was derived in [36, 18] by a simple projection from the supermoduli space to the even moduli space. This projection does’t preserve supersymmetry and there is a residual dependence on the two insertion points. This formalism was used in [5, 6, 7, 8]. In these papers we found that it is quite convenient to choose these two insertion points as the two zeroes of a holomorphic abelian differential which are moduli independent points on the Riemann surface. In hyperelliptic language these two points are the same points on the upper and lower sheet of the surface. We denote these two points as $x_1 = x_+ (on$ the upper sheet) and $x_2 = x_− (on$ the lower sheet). We will make these convenient choices again in this paper and [27, 28].

3 Some conventions and useful formulas

In what follows we will give some formulas which will be used later. First all the relevant correlators are given by

$$\langle \psi^\mu(z) \psi^\nu(w) \rangle = -\delta^{\mu\nu}G_{1/2}([\delta](z, w) = -\delta^{\mu\nu}S_\delta(z, w),$$

$$\langle \partial_z X^\mu(z) \partial_w X^\nu(w) \rangle = -\delta^{\mu\nu} \partial_z \partial_w \ln E(z, w),$$

$$\langle b(z)c(w) \rangle = +G_2(z, w),$$

$$\langle \beta(z)\gamma(w) \rangle = -G_3/2 [\delta](z, w),$$

(3)

where

$$S_\delta(z, w) = \frac{1}{z - w} \frac{u(z) + u(w)}{2\sqrt{u(z)u(w)}},$$

(4)

$$u(z) = \prod_{i=1}^3 \left( \frac{z - A_i}{z - B_i} \right)^{1/2},$$

(5)

$$G_2(z, w) = -H(w, z) + \sum_{a=1}^3 H(w, p_a) x_a(z, z),$$

(6)

$$H(w, z) = \frac{1}{2(w - z)} \left( 1 + \frac{y(w)}{y(z)} \right) \frac{y(w)}{y(z)},$$

(7)

$^3$We follow closely the notation of [21].
\[ G_{3/2}[\delta](z, w) = -P(w, z) + P(w, q_1)\psi^*_1(z) + P(w, q_2)\psi^*_2(z), \quad (8) \]

\[ P(w, z) = \frac{1}{\Omega(w)} S_\delta(w, z) \Omega(z), \quad (9) \]

where \( \Omega(z) \) is an abelian differential satisfying \( \Omega(q_{1,2}) \neq 0 \). These correlators were adapted from [37]. \( \omega_\alpha(z, w) \) are defined in [20] and \( \psi^*_1, \psi^*_2(z) \) are the two holomorphic \( \frac{3}{2} \)-differentials. When no confusion is possible, the dependence on the spin structure \( [\delta] \) will not be exhibited.

In order to take the limit of \( x_{1,2} \to q_{1,2} \) we need the following expansions:

\[ G_{3/2}(x_2, x_1) = \frac{1}{x_1 - q_1} \psi^*_1(x_2) - \psi^*_1(x_2)f_{3/2}^{(1)}(x_2) + O(x_1 - q_1), \quad (10) \]

\[ G_{3/2}(x_1, x_2) = \frac{1}{x_2 - q_2} \psi^*_2(x_1) - \psi^*_2(x_1)f_{3/2}^{(2)}(x_1) + O(x_2 - q_2), \quad (11) \]

for \( x_{1,2} \to q_{1,2} \). By using the explicit expression of \( G_{3/2} \) in (8) we have

\[ f_{3/2}^{(1)}(q_2) = -\frac{\partial_q S(q_1, q_2)}{S(q_1, q_2)} + \partial \psi^*_2(q_2), \quad (12) \]

\[ f_{3/2}^{(2)}(q_1) = \frac{\partial_q S(q_2, q_1)}{S(q_1, q_2)} + \partial \psi^*_1(q_1) = f_{3/2}^{(1)}(q_2)|_{q_1 \leftrightarrow q_2}. \quad (13) \]

The quantity \( \psi^*_\alpha(z) \)'s are holomorphic \( \frac{3}{2} \)-differentials and are constructed as follows:

\[ \psi^*_\alpha(z) = (z - q_\alpha)S(z, q_\alpha) \frac{y(q_\alpha)}{y(z)}, \quad \alpha = 1, 2. \quad (14) \]

For \( z = q_{1,2} \) we have

\[ \psi^*_\alpha(q_\beta) = \delta_{\alpha, \beta}, \quad (15) \]

\[ \partial \psi^*_1(q_2) = -\partial \psi^*_2(q_1) = S(q_1, q_2) = \frac{i}{4} S_1(q), \quad (16) \]

\[ \partial \psi^*_1(q_1) = \partial \psi^*_2(q_2) = -\frac{1}{2} \Delta_1(q), \quad (17) \]

\[ \partial^2 \psi^*_1(q_1) = \partial^2 \psi^*_2(q_2) = \frac{1}{16} S^2_1(q) + \frac{1}{4} \Delta^2_1(q) + \frac{1}{2} \Delta_2(q), \quad (18) \]

where

\[ \Delta_n(x) \equiv \sum_{i=1}^{6} \frac{1}{(x - a_i)^n}, \quad (19) \]

\[ S_n(x) \equiv \sum_{i=1}^{3} \left[ \frac{1}{(x - A_i)^n} - \frac{1}{(x - B_i)^n} \right], \quad (20) \]
for \( n = 1, 2 \). This shows that \( \partial \psi^*_\alpha(q_{\alpha+1}) \) and \( \partial^2 \psi^*_\alpha(q_{\alpha}) \) are spin structure dependent.

Other explicit formulas for \( \partial_z \partial_w \ln E(z, w) \) will be given in [28].

4 The chiral measure: the result of D’Hoker and Phong

The chiral measure obtained in [20, 21, 22, 23] after making the choice \( x_\alpha = q_\alpha (\alpha = 1, 2) \) is

\[
\mathcal{A}[\delta] = i \mathcal{Z} \left\{ 1 + \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4 + \mathcal{X}_5 + \mathcal{X}_6 \right\},
\]

\[
\mathcal{Z} = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \rangle}{\det \omega_i \omega_j(p_a)},
\]

and the \( \mathcal{X}_i \) are given by:

\[
\mathcal{X}_1 + \mathcal{X}_6 = \frac{\zeta_1 \zeta_2}{16 \pi^2} \left[ -\left\langle \psi(q_1) \cdot \partial X(q_1) \psi(q_2) \cdot \partial X(q_2) \right\rangle 
\right.

\left. - \partial_{q_1} G_2(q_1, q_2) \partial \psi^*_1(q_2) + \partial_{q_2} G_2(q_2, q_1) \partial \psi^*_2(q_1)
\right.

\left. + 2 G_2(q_1, q_2) \partial \psi^*_1(q_2) f^{(1)}_3(q_2) - 2 G_2(q_2, q_1) \partial \psi^*_2(q_1) f^{(2)}_3(q_1) \right] ,
\]

\[
\mathcal{X}_2 + \mathcal{X}_3 = \frac{\zeta_1 \zeta_2}{8 \pi^2} S_\delta(q_1, q_2)
\]

\[
\times \sum_{a=1}^3 \tilde{\omega}_a(q_1, q_2) \left[ \langle T(\tilde{p}_a) \rangle + \tilde{B}_2(\tilde{p}_a) + \tilde{B}_{3/2}(\tilde{p}_a) \right] ,
\]

\[
\mathcal{X}_4 = \frac{\zeta_1 \zeta_2}{8 \pi^2} S_\delta(q_1, q_2) \sum_{a=1}^3 \left[ \partial_{p_a} \partial_{q_1} \ln E(p_a, q_1) \tilde{\omega}^*_a(q_2)
\right.

\left. + \partial_{p_a} \partial_{q_2} \ln E(p_a, q_2) \tilde{\omega}^*_a(q_1) \right] ,
\]

\[
\mathcal{X}_5 = \frac{\zeta_1 \zeta_2}{16 \pi^2} \sum_{a=1}^3 \left[ S_\delta(p_a, q_1) \partial_{p_a} S_\delta(p_a, q_2)
\right.

\left. - S_\delta(p_a, q_2) \partial_{p_a} S_\delta(p_a, q_1) \right] \omega_a(q_1, q_2) .
\]
Furthermore, $\tilde{B}_2$ and $\tilde{B}_{3/2}$ are given by
\begin{align}
\tilde{B}_2(w) &= -2 \sum_{a=1}^{3} \partial_p \partial_a \ln E(p_a, w) \varpi^*_a(w), \tag{26}
\tilde{B}_{3/2}(w) &= \sum_{a=1}^{2} \left( G_2(w, q_a) \partial_{q_a} \psi_{a}^*(q_a) + \frac{3}{2} \partial_{q_a} G_2(w, q_a) \psi_{a}^*(q_a) \right). \tag{27}
\end{align}

In comparing with the results given in [22], we have written $X_2, X_3$ together and we didn’t split $T(w)$ into different contributions. We also note that in eq. (23) the three arbitrary points $\tilde{p}_a$ ($a = 1, 2, 3$) can be different from the three insertion points $p_a$’s of the $b$ ghost field. The symbol $\tilde{\varpi}_a$ is obtained from $\varpi_a$ by changing $p_a$’s to $\tilde{p}_a$’s. In the following computation we will take the limit of $\tilde{p}_1 \to q_1$ or $q_2$. In this limit we have $\tilde{\varpi}_{2, 3}(q_1, q_2) = 0$ and $\tilde{\varpi}_1(q_1, q_2) = -1$. This choice greatly simplifies the formulas and also makes the summation over spin structure doable (see [27, 28] for more details).

5 The chiral measure in hyperelliptic language

The strategy we will follow is to isolate all the spin structure dependent parts first. As we will show in the following, the spin structure dependent factors are just $S_1(q), S_2(q), S_3^1(q), S_1(p_0)$ and the Szegő kernel if we also include the vertex operators. Before we do this we will first write the chiral measure in hyperelliptic language and take the limit of $\tilde{p}_1 \to q_1$. The full computations and the complete results will be presented in [27, 28]. Here we only present the singular terms and other terms which depend on the spin structure.

First we have
\begin{align}
T_{\beta\gamma}(w) = -\frac{3/2}{(w - q_1)^2} - \frac{\partial \psi_{1}^{*}(q_1)}{w - q_1} - \frac{1}{8} \Delta_1^2(q) - \frac{1}{32} S_1^{2}(q) + O(w - q_1). \tag{28}
\end{align}

In this limit, the dependence on the abelian differential $\Omega(z)$ drops out. These singular terms are cancelled by similar singular terms in $\tilde{B}_{3/2}(w)$. By explicit computation we have:
\begin{align}
\tilde{B}_{3/2}(w) &= \frac{3/2}{(w - q_1)^2} + \frac{\partial \psi_{1}^{*}(q_1)}{w - q_1} - \frac{1}{4} \Delta_1^2(q) + \frac{3}{4} \Delta_2(q) \\
&- \left( \frac{1}{p_1 - q} \frac{(q - p_2)(q - p_3)}{(p_1 - p_2)(p_1 - p_3)} \Delta_1(q) + ... \right).
\end{align}
where \( \beta \gamma \) indicates two other terms obtained by cyclicly permutating \((p_1, p_2, p_3)\). By using the above explicit result we see that the combined contributions of \(T_{\beta\gamma}(w)\) and \(\tilde{B}_{3/2}(w)\) are non-singular in the limit of \(w \to q_1\). We can then take \(\tilde{p}_1 \to q_1\) in \(X_2 + X_3\). In this limit only \(a = 1\) contributes to \(X_2 + X_3\). This is because \(\tilde{w}_{2,3}(q_1, q_2) = 0\) and \(\tilde{w}_1(q_1, q_2) = -1\).

Apart from the factor \(\frac{c_1 c_2}{16 \pi^2}\), we have the following form of the left part of integrand for the \(n\)-particle amplitude (by combining the chiral measure and the left part of the vertex operators):

\[
\begin{align*}
\mathcal{A}_1 + \mathcal{A}_6 &= -\langle \psi(q_1) \cdot \partial X(q_1) \psi(q_2) \cdot \partial X(q_2) \prod_i V_i \rangle \\
&\quad - (\partial_{q_1} G_2(q_1, q_2) + \partial_{q_2} G_2(q_2, q_1)) S(q_1, q_2) \langle \prod_i V_i \rangle \\
&\quad + 2(G_2(q_1, q_2) + G_2(q_2, q_1)) \times (\partial \psi^*(q_1) S(q_1, q_2) - \partial_q S(q_1, q_2)) \langle \prod_i V_i \rangle, \\
\mathcal{A}_2 + \mathcal{A}_3 &= -2 S(q_1, q_2) \left\{ \langle (T_X(q_1) + T_\psi(q_1)) \prod_i V_i \rangle \\
&\quad + (T_{\beta\gamma}(q_1) + T_{bc}(q_1) + \tilde{B}_{3/2}(q_1) + \tilde{B}_2(q_1)) \langle \prod_i V_i \rangle \right\}, \\
\mathcal{A}_4 &= -2 \sum_{a=1}^3 \left[ \partial_{p_a} \partial_{q_1} \ln E(p_a, q_1) - \partial_{p_a} \partial_{q_2} \ln E(p_a, q_2) \right] \\
&\quad \times w^*_a(q_1) S(q_1, q_2) \langle \prod_i V_i \rangle, \\
\mathcal{A}_5 &= \sum_{a=1}^3 \frac{1}{(q_1 - p_a)^2} w_a(q_1, q_2) S(p_a+, p_a-) \langle \prod_i V_i \rangle.
\end{align*}
\]

In this paper and in \([27, 28]\), we consider only the massless particle from the Neveu-Schwarz sector and the left part of the vertex operator is

\[
V_i(k_i, \epsilon_i; z_i, \bar{z}_i) = (\epsilon_i \cdot \partial X(z_i) + i k_i \cdot \psi(z_i) \epsilon_i \cdot \psi(z_i)) e^{i k_i \cdot X(z_i, \bar{z}_i)}.
\]

Because the vertex operator doesn’t contain any ghost fields, all terms involving ghost fields can be explicit computed which we have done in the above. For the computation of amplitudes of other kinds of particles (like fermions),
one either resorts to supersymmetry or can use similar method which was used in [38, 39] to compute the fermionic amplitude.

From the above results we see that all the spin structure dependent parts (for the cosmological constant) are as follows:

$$c_1 S_1(q) + c_2 S_2(q) + c_3 S_1^3(q) + \sum_{a=1}^{3} d_a S_1(p_a),$$

(35)

where $c_{1,2,3}$ and $d_a$’s are independent of spin structure. In computing the $n$-particle amplitude there are more spin structure factors coming from the correlators of $\psi$. These are explicitly included in eqs. (30)–(33).

6 The vanishing of the cosmological constant and non-renormalization theorem

The vanishing of the cosmological constant is proved by using the following identities:

$$\sum_\delta \eta_\delta Q_\delta S_n(x) = 0,$$

(36)

$$\sum_\delta \eta_\delta Q_\delta S_1^3(x) = 0,$$

(37)

for $n = 1, 2$ and arbitrary $x$.

For the non-renormalization theorem we need more identities. By modular invariance we can easily prove the following “vanishing identities”:

$$\sum_\delta \eta_\delta Q_\delta \left\{ \frac{u(z_1)}{u(z_2)} - \frac{u(z_2)}{u(z_1)} \right\} = 0,$$

(38)

$$\sum_\delta \eta_\delta Q_\delta \left\{ \frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} - \frac{u(z_3)u(z_4)}{u(z_1)u(z_2)} \right\} = 0,$$

(39)

$$\sum_\delta \eta_\delta Q_\delta \left\{ \frac{u(z_1)}{u(z_2)} + \frac{u(z_2)}{u(z_1)} \right\} S_n(x) = 0, \quad n = 1, 2,$$

(40)

$$\sum_\delta \eta_\delta Q_\delta \left\{ \frac{u(z_1)}{u(z_2)} - (-1)^n \frac{u(z_2)}{u(z_1)} \right\} (S_1(x))^n = 0, \quad n = 1, 2, 3.$$  

(41)

These identities can be proved by modular invariance and simple “power counting”. To prove the vanishing of the 3-particle amplitude we also need a
“non-vanishing identity”. This and other identities needed in the 4-particle amplitude computations are summarized as follows:

\[
\sum_{\delta} \eta_{\delta} Q_{\delta} \left\{ \frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} - (-1)^n \frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} \right\} \left( S_m(x) \right)^n
\]

\[
= \frac{2P(a) \prod_{i=1}^{2} \prod_{j=3}^{4} (z_i - z_j) \prod_{i=1}^{4} (x - z_i)}{y^2(x) \prod_{i=1}^{4} y(z_i)} \times C_{n,m},
\]

where

\[
C_{1,1} = 1,
\]

\[
C_{2,1} = -2(\tilde{z}_1 + \tilde{z}_2 - \tilde{z}_3 - \tilde{z}_4),
\]

\[
C_{1,2} = \Delta_1(x) - \sum_{k=1}^{4} \tilde{z}_k,
\]

\[
C_{3,1} = 2\Delta_2(x) - \Delta_1^2(x) + 2\Delta_1(x) \sum_{k=1}^{4} \tilde{z}_k
\]

\[+ 4 \sum_{k<l} \tilde{z}_k \tilde{z}_l - 12(\tilde{z}_1 + \tilde{z}_2)(\tilde{z}_3 + \tilde{z}_4),
\]

\[
\tilde{z}_k = \frac{1}{x - z_k},
\]

\[
P(a) = \prod_{i<j} (a_i - a_j).
\]

\[C_{1,1} \text{ and } C_{1,2} \text{ were derived in [8]. We will not derive these formulas here and refer the reader to [28]. You will find some other interesting identities also in [27]. Although other values of } n, m \text{ also gives modular invariant expressions, the results are quite complex.}^4
\]

Fortunately we only need to use the above listed results.

By using these formulas we have:

\[
\sum_{\delta} \eta_{\delta} Q_{\delta} S_\delta(x, z_1) S_\delta(z_1, z_2) S_\delta(z_2, z_3) \partial_x S_\delta(z_3, x) S_1(x)
\]

\[
= - \frac{P(a)}{16y^2(x)} \prod_{i=1}^{3} \frac{x - z_i}{y(z_i)},
\]

We note that the above formula is invariant under the interchange $z_i \leftrightarrow z_j$.

\[^4\text{This is partially due to the non-vanishing of the summation over spin structures when we set } z_1 = z_3 \text{ or } z_1 = z_4, \text{ etc.}\]
By using this result and other “vanishing identities” given in eqs. (38)–(41), we proved the vanishing of the cosmological constant and the non-renormalization theorem at two loops (see [27] for details).

7 The 4-particle amplitude

The 4-particle amplitude can also be computed explicitly. The final result for the chiral integrand is:

\[
\mathcal{A} = K(k_i, \epsilon_i) \langle (\partial X(q_1) + X(q_2)) \cdot (\partial X(q_1) + \partial X(q_2)) : \prod_{i=1}^{4} e^{ik_i \cdot X(z_i, \bar{z}_i)} \prod_{i=1}^{4} \frac{q - z_i}{y(z_i)} \rangle
\]

\[
= \frac{K(k_i, \epsilon_i)}{\prod_{i=1}^{4} y(z_i)} \prod_{i<j} \exp[-k_i \cdot k_j \ G(z_i, z_j)]
\]

\[
\times (s(z_1z_2 + z_3z_4) + t(z_1z_4 + z_2z_3) + u(z_1z_3 + z_2z_4)),
\]

(50)

where \( K(k_i, \epsilon_i) \) is the standard kinematic factor appearing at tree level and one loop computations [1, 7, 8]. \( G(z_i, z_j) \) is the scalar Green function which is given in terms of the prime form \( E(z_i, z_j) \) as follows:

\[
G(z, w) = -\ln |E(z, w)|^2 + 2\pi \text{Im} \int_w^z \omega \ (\text{Im}\Omega)_{i,j}^{-1} \text{Im} \int_z^w \omega. \]

(51)

Here in eq. (50) we also included the factor \( Z \) and used the explicit correlators for \( \langle \partial X(z)\partial X(w) \rangle \) and \( \langle \partial X(z)X(w, \bar{w}) \rangle \) given in [29] [1] (see [28] for details). As it is expected, the find result is independent on the insertion points \( q_1,2 \) and \( p_a \)'s.

For type II superstring theory the complete integrand is

\[
\mathcal{A} = c_{II} K(k_i, \epsilon_i) \langle (\partial X(q_1) + \partial X(q_2)) \cdot (\partial X(q_1) + \partial X(q_2)) : \prod_{i=1}^{4} e^{ik_i \cdot X(z_i, \bar{z}_i)} \prod_{i=1}^{4} \frac{q - z_i}{y(z_i)} \rangle
\]

\[
= c_{II} \frac{K(k_i, \epsilon_i)}{2\prod_{i=1}^{4} |y(z_i)|^2} \prod_{i<j} \exp[-k_i \cdot k_j \ G(z_i, z_j)]
\]

\[
\times |s(z_1z_2 + z_3z_4) + t(z_1z_4 + z_2z_3) + u(z_1z_3 + z_2z_4)|^2,
\]

(52)
which is independent the left-mover insertion points $q_{1,2}$ and also the right part insertion points $\tilde{q}_{1,2}$.

The amplitude is obtained by integrating over the moduli space. At two loops, the moduli space can be parametrized either by the period matrix or three of the six branch points. We have

$$A_{II} = c_{II} K(k_i, \epsilon_i) \int \frac{\prod_{i=1}^{6} d^2 a_i}{T^5} \frac{dV_{pr}}{\prod_{i<j} |a_i - a_j|^2} \times \prod_{i=1}^{4} \frac{d^2 z_i}{y(z_i)^2} \prod_{i<j} \exp \left[ -k_i \cdot k_j G(z_i, z_j) \right] \times |s(z_1 z_2 + z_3 z_4) + t(z_1 z_4 + z_2 z_3) + u(z_1 z_3 + z_2 z_4)|^2,$$

where $dV_{pr} = \frac{d^2 a_i d^2 a_j d^2 a_k}{|a_{ij} a_{jk} a_{ki}|^2}$ is a projective invariant measure and $c_{II}$ is a constant which should be determined by factorization or unitarity (of the $S$-matrix).

An immediate application of the above result is to study the perturbative correction to the $R^4$ term at two loops. In the low energy limit $k_i \to 0$, the chiral integrand is 0 apart from the kinematic factor because of the extra factors of $s$, $t$ and $u$ in eq. (53). This confirms the explicit computation of Iengo [10] by using the old result [8, 7], and it is in agreement with the indirect argument of Green and Gutperle [12], Green, Gutperle and Vanhove [13], and Green and Sethi [14].

The finiteness of the amplitude can also be checked. We refer the reader to [28] for details.

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