ZERO-FREE STRIPS FOR THE RIEMANN ZETA-FUNCTION DERIVED FROM THE PRIME NUMBER THEOREM

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Abstract. We use the Prime Number Theorem to prove the existence of zero-free strips for the Riemann-zeta function. Precisely, we prove that there exists \( \delta > 0 \) for which if \( 0 \leq r < \delta \) then \( \zeta(s) \neq 0 \) for \( \text{Re}(s) > 1 - r \).

1. Introduction

Let \( s = \sigma + it \), \( \zeta(s) \), as usual, the Riemann zeta-function and \( \mu \) the M"obius function, that is, the multiplicative function with \( \mu(n) = (-1)^{\omega(n)} \), if \( n \) is square free and \( \mu(n) = 0 \) otherwise. Here \( \omega(n) \) denotes the number of distinct prime dividing \( n \).

It is well known that \( \mu(n) \) is deeply related to the Riemann hypothesis (RH) and also to the Prime Number Theorem (PNT). For instance, RH is equivalent to

\[
\sum_{n \leq x} \mu(n) = O(x^{1/2 + \epsilon}),
\]

for all \( \epsilon > 0 \), whereas the PNT is equivalent to [2 p. 248]

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0.
\]

Moreover, any improvements in the zero-free region for \( \zeta(s) \) will immediately imply improvements in the error term of the prime number theorem. For example, if the Riemann Hypothesis is true then we obtain

\[
\pi(x) = \int_{2}^{x} \frac{du}{\log(u)} + O(\sqrt{x} \log(x)),
\]

and this last is not only implied by the RH but actually implies the RH itself.

The well-known identity

\[
1 \overline{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \text{Re}(s) > 1,
\]

that connects the Riemann-zeta function and the M"obius function can be seen as the main object of this paper. Indeed, as we show in our main result (see Theorem 3.1), it is possible to use it combined with the PNT to show that there exist zero-free strips for \( \zeta(s) \) inside the critical strip. Precisely, we prove that there exists

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\( \delta > 0 \) for which \( \zeta(s) \) has no zeros in the half-plane \( \text{Re}(s) > 1 - r \), for \( 0 \leq r < \delta \). Consequently, due to the functional equation [2, p. 329]
\[
\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin(\pi s/2) \zeta(1-s), \quad s \neq 1,
\]
it implies that \( \zeta(1-s) \neq 0 \), where \( s = 1 - r + it \) and \( 0 \leq r < \delta \). For more information on zero-free regions for the Riemann-zeta function we refer to [11], where an explicit version of the Vinogradov-Korobov zero free region is obtained.

The identity
\[
\zeta(2s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \quad \text{Re}(s) > 1,
\]
where \( \lambda(n) \) denotes the Liouville function [2, p.22], can also be used to obtain the same results that are presented in this paper.

2. Auxiliary results

Let us begin this section with the main tool that will be used throughout this paper, which is an analogue of Landau’s theorem concerning Dirichlet series with non-negative coefficients [2, Lemma 15.1].

**Lemma 2.1.** Suppose that \( G(u) \) is bounded Riemann-integrable function on every compact interval \([1, a]\) and that \( G(u) \) does not change sign for all \( u \) large. Let \( \sigma_c \) denote the infimum of those \( \sigma \) for which \( \int_{1}^{\infty} G(u) u^{-\sigma} \, du \) converges. Then the function
\[
\varphi(s) = \int_{1}^{\infty} G(u) u^{-s} \, du
\]
is analytic in the half-plane \( \text{Re}(s) > \sigma_c \) but not at \( s = \sigma_c \).

Our first result is just an observation derived from the identity [11] and an application of the PNT.

**Lemma 2.2.** Let \( \mu(n) \) be the Möbius function and \( a(n) \) be the arithmetic function defined as
\[
a(n) = \begin{cases} 
0, & n = 1 \\
\mu(n), & n \geq 2.
\end{cases}
\]
We have that,
\[
\frac{1}{\zeta(s)} - 1 = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \text{Re}(s) > 1
\]
and
\[
\sum_{n=1}^{\infty} \frac{a(n)}{n} = -1.
\]

**Proof.** Since \( \lambda(1) = 1 \), it is immediate that
\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 + \sum_{n=2}^{\infty} \frac{\mu(n)}{n^s} = 1 + \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \text{Re}(s) > 1.
\]
The conclusion of the proof follows from the definition of \( a(n) \) and from the fact that the PNT is equivalent to
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0.
\]
that is,
\[ \sum_{n=1}^{\infty} \frac{a(n)}{n} = -1. \]

Let \( a(n) \) be as in Lemma 2.2 and

\[ A_r(x) = \sum_{n \leq x} \frac{a(n)}{n^r}, \]

with \( r > 0 \) and \( x \geq 1 \). It is clear that the PNT is equivalent to

\[ \lim_{x \to \infty} A_1(x) = -1. \]

A first consequence that can be extracted from (2.2) and Lemma 2.2 is an alternative proof for the well-known fact that \( \zeta(s) \neq 0 \) for \( \text{Re}(s) > 1 \). Indeed, by partial summation

\[ \left( -1 + \frac{1}{\zeta(s)} \right) (s - 1)^{-1} = \int_1^\infty A_1(u)u^{-s-1}du, \text{ Re}(s) > 1. \]

Since \( \lim_{x \to \infty} A_1(x) = -1 \), it is clear that \( A_1(x) < 0 \) for all \( x \) large. Also note that, since the inequalities

\[ \frac{1}{\sigma - 1} < \zeta(\sigma) < \frac{\sigma}{\sigma - 1} \]

hold for all \( \sigma > 0 \) [2, p.25], the left hand side of (2.3) has a pole at \( \sigma = 1 \) but is analytic on the real line for \( \sigma > 1 \). Hence, by an application of Lemma 2.1, (2.3) holds for \( \text{Re}(s) > 1 \) and both sides of this identity are analytic in this half-plane. Therefore, \( \zeta(s) \neq 0 \) for \( \text{Re}(s) > 1 \).

3. Main result

The main result of this paper is basically an extension of the arguments presented above, in our alternative proof for the fact that \( \zeta(s) \neq 0 \), for \( \text{Re}(s) > 1 \). Different from works on zero-free regions (see for instance [1] and references therein), here it is proved the existence of a fixed zero-free strip for the Riemann-zeta function, in the critical strip \( 0 < \text{Re}(s) < 1 \), which does not shrink as \( |t| \to \infty \).

The main result of this paper is as follows.

**Theorem 3.1.** There exists \( \delta > 0 \) for which if \( 0 \leq r < \delta \) then

\[ (s - 1 + r)^{-1} \left( \frac{1}{\zeta(s)} - 1 \right) = \int_1^\infty A_{1-r}(u)u^{-s-r}du, \]

for \( \text{Re}(s) > 1 - r \), and both sides of the equality are analytic in this half-plane. In particular, \( \zeta(s) \) has no zeros in the half-plane \( \text{Re}(s) > 1 - r \).

**Proof.** Let \( x > 1 \) be arbitrarily fixed and \( A_r(x), r > 0 \) as in (2.1). Since

\[ f(r) = A_r(x), r > 0, \]

is continuous, for any given \( \epsilon > 0 \), there exists \( \delta > 0 \) for which if \( |r - r_1| < \delta \) then \( |f(r) - f(r_1)| < \epsilon \). Let \( r > 0 \) and \( r_1 = 1 - r \). If \( 0 \leq r < \delta \) then \( |f(1) - f(1-r)| < \epsilon \). That is,

\[ A_1(x) - \epsilon < A_{1-r}(x) < A_1(x) + \epsilon. \]
Also, from (2.2), there exists $M > 1$ such that, if $x > M$ then $-1 - \epsilon < A_1(x) < -1 + \epsilon$. Hence, if $0 \leq r < \delta$ then

$$-1 - 2\epsilon < A_{1-r}(x) < -1 + 2\epsilon,$$

for all $x > M$. Thus, by choosing $\epsilon > 0$ sufficiently small we have that there exists $\delta > 0$ and $M > 1$ such that $A_{1-r}(x) < 0$ for all $x > M$, whenever $0 \leq r < \delta$.

For this $\epsilon, \delta$ and any $0 \leq r < \delta$, by partial summation we obtain

$$\sum_{n \leq x} \frac{a(n)}{n^s} = A_{1-r}(x)x^{-s+1-r} + (s - 1 + r) \int_1^x A_{1-r}(u)u^{-s-r} du.$$

Since

$$\lim_{x \to \infty} A_{1-r}(x)x^{-s+1-r} = 0,$$

whenever $\text{Re}(s) > 1$, it follows that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = (s - 1 + r) \int_1^{\infty} A_{1-r}(u)u^{-s-r} du,$$

holds in the half-plane $\text{Re}(s) > 1$. By (1.1) we obtain that

$$(s - 1 + r)^{-1} \left( \frac{1}{\zeta(s)} - 1 \right) = \int_1^{\infty} A_{1-r}(u)u^{-s-r} du,$$

for $\text{Re}(s) > 1$. Here, the left hand side of the above identity is analytic on the real line for $\sigma > 1 - r$ but has a pole at $\sigma = 1 - r$, in view of (2.4). Hence, by Lemma 2.1 the above identity holds in the half-plane $\text{Re}(s) > 1 - r$ and both sides are analytic in this half-plane. In particular, we can conclude that $\zeta(s) \neq 0$ for $\text{Re}(s) > 1 - r$.

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