\( q \)-Ultraspherical Polynomials for \( q \) a Root of Unity

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Abstract

Properties of the \( q \)-ultraspherical polynomials for \( q \) being a primitive root of unity are derived using a formalism of the \( so_q(3) \) algebra. The orthogonality condition for these polynomials provides a new class of trigonometric identities representing discrete finite-dimensional analogs of \( q \)-beta integrals of Ramanujan.

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1. Introduction. There are many applications of quantum algebras, or \( q \)-analogues of Lie algebras in mathematical physics. In particular, it is well known that the corresponding representation theory is related to \( q \)-special functions and \( q \)-orthogonal polynomials. In this Letter we investigate from this point of view a class of \( q \)-ultraspherical polynomials for \( q \) a root of unity.

The monic \( q \)-ultraspherical polynomials satisfy three-term recurrence relation

\[
P_{n+1}(x) + u_n P_{n-1}(x) = x P_n(x), \quad n = 1, 2, \ldots
\]

and initial conditions \( P_0(x) = 1, \ P_1(x) = x \), where coefficients \( u_n \) have the form

\[
u_n = \frac{(1 - q^n)(1 - q^{n+2\beta-1})}{(1 - q^{n+\beta})(1 - q^{n+\beta-1})}.
\]

These polynomials are well investigated for \( 0 \leq q \leq 1 \). The case of \( q \) a root of unity was considered only in the nontrivial limits \( \beta \to 0 \) or \( 1 \) leading to the so-called sieved ultraspherical polynomials. In these limits the recurrence coefficients \( u_n > 0 \) for all \( n > 0 \), so that the sieved polynomials are orthogonal on an interval of real line with some continuous measure. Consider what happens if \( \beta \neq 0,1 \) and \( q \) is \( N \)th primitive root of unity, \( q = \exp 2\pi ip/N, \ (p, N) = 1 \). Then \( u_n \) can be rewritten in the form

\[
u_n = \frac{\sin \omega n \sin \omega (n + 2\beta - 1)}{\sin \omega (n + \beta) \sin \omega (n + \beta - 1)},
\]

where \( \omega = \pi p/N \). In order for \( P_n(x) \) to possess a positive weight function (i.e. to be “classical” orthogonal polynomials) \( u_n \) must be positive. Reality of \( u_n \) implies reality of the parameter \( \beta \). Then (3) cannot

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satisfy the positivity condition for all $n > 0$. One has to analyze thus the finite-dimensional cases when $u_0 = u_M = 0$ for some natural $M \geq 2$ and $u_n > 0$ for $0 < n < M$ (then $M$ is the dimension of Jacobi matrix defining the eigenvalue problem (1)).

The possibility to construct finite-dimensional classical orthogonal polynomials on trigonometric grids (including those generated by $q$ a root of unity) was mentioned in [3], however no explicit examples were presented. For examples of non-standard $q$-special functions at $q^N = 1$ related to the Lamé-type differential equations, see [7].

Recently we have found that a special class of $q$-ultraspherical polynomials for $q^N = 1$ can be obtained by “undressing” of the finite-dimensional Chebyshev polynomials [3, 4], existence of the more complicated cases has been pointed out as well. Here we describe $q$-ultraspherical polynomials with positive measure when $q$ is the $p = 1$ primitive root of unity, i.e. when

$$q = \exp 2\pi i/N, \quad N = 2, 3, \ldots .$$

(4)

The main tool in our analysis will be the theory of representations of quantum algebra $so_q(3)$ introduced in [10, 11]. The connection of this algebra with some (finite-dimensional) class of $q$-ultraspherical polynomials for real $q$ was mentioned in [13].

2. Restrictions upon the Parameters. Consider restrictions upon the parameters $M$ and $\beta$ necessary for positivity of $u_n$ in the interval $0 < n < M$. There are two principally different situations. The first possibility is $M = N$, when dimension of representation coincides with $N$. Then the positivity condition leads to the following restrictions upon $\beta$ (all the inequalities are modulo $N$):

$$-1/2 < \beta < 0, \quad 0 < \beta < 1, \quad 1 < \beta < 3/2.$$  

(5)

Note that the ends of intervals do not belong to this class. Indeed, there are ambiguities in choosing the coefficients $u_n$ for $\beta = 0, 1$. The cases $\beta = -1/2, 3/2$ lead to $u_M = 0$ for $M = 2, N - 2$, which is not allowed since we demand the dimension of representation to be equal to $N$.

The second possibility arises if $2\beta$ is a natural number:

$$2\beta = j = 1, 2, 3, \ldots , N - 1.$$  

(6)

We define coefficients $u_n$ in the case $\beta = 1$ as $u_0 = u_{N-1} = 0, u_1 = \ldots = u_{N-2} = 1$. Then the dimension of representations defined by (3) is determined by the general formula $M = N + 1 - j$. We note by passing that the even $j$ cases describe undressing of the “discrete finite well” in discrete quantum mechanics [13]. The case $j = 1$ corresponds to the $q$-Legendre polynomials because $u_n$ reduce to the ordinary Legendre polynomials’ recurrence coefficients for $N \to \infty$. This system has an interesting physical application in the Azbel-Hofstadter problem of electron on a two-dimensional lattice in a magnetic field [14].

It is not difficult to see that (5) and (6) are the only choices of $\beta$ providing positive $u_n$ (we skip the trivial case $\beta = -1/2$ when $M = 2$ and assume $\beta \neq 0$).

3. $so_q(3)$ Algebra and Its Representations. Consider the associative algebra of three generators defined by commutation relations

$$[K_0, K_1]_\omega = K_2, \quad [K_1, K_2]_\omega = -K_0, \quad [K_2, K_0]_\omega = -K_1,$$  

(7)

where $[A, B]_\omega = e^{i\omega/2}AB - e^{-i\omega/2}BA$ denotes so-called $q$-mutator and $\omega = \pi/N$. Note that if $K_0$ and $K_1$ are hermitian operators then $K_2$ should be anti-hermitian. Note also that for $N \to \infty$ we get the ordinary rotation algebra $so(3)$, which justifies the name $so_q(3)$ for the algebra (7). This algebra was introduced (in a slightly different form) in 1986 by Odesskii [10] who considered its representations without discussion of unitarity. Its relations to quantum algebra $su_q(2)$ and $q$-special functions have been discussed in [11, 12].

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A non-compact version of this algebra $so_q(2,1)$ for real $q$ appeared to be quite useful as an algebra describing dynamical symmetry of special classes of discrete reflectionless potentials [8].

Construct the unitary finite-dimensional representations of $so_q(3)$ by the method, proposed in [12]. It is easy to see that there exists an orthonormal basis $|n\rangle$ for which the operator $K_0$ is diagonal whereas the operator $K_1$ is two-diagonal:

$$K_0|n\rangle = \lambda_n|n\rangle, \quad K_1|n\rangle = a_{n+1}|n+1\rangle + a_n|n-1\rangle,$$

where $\lambda_n$ are eigenvalues of operator $K_0$ and $a_n$ are matrix elements of a representation. Substituting (8) into (7) we find

$$\lambda_n = \frac{\cos \omega(n + \beta)}{\sin \omega}, \quad n = 0, 1, \ldots, M - 1,$$

$$a_n^2 = \frac{\nu - \cos \omega(n + \beta)\cos \omega(n + \beta - 1)}{4 \sin^2 \omega \sin \omega(n + \beta) \sin \omega(n + \beta - 1)},$$

where $\nu$ is the eigenvalue of Casimir operator of $so_q(3)$ algebra which can be represented in the form

$$Q = \frac{1}{2}(K_2\tilde{K}_2 + \tilde{K}_2K_2) - \cos \omega(K_0^2 + K_1^2),$$

where $\tilde{K}_2 = [K_0, K_1]_\omega$. The condition $a_0 = 0$ gives $\nu = \cos \omega \beta \cos \omega(\beta - 1)$, and then

$$a_n^2 = \frac{\sin \omega n \sin \omega(n + 2\beta - 1)}{4 \sin^2 \omega \sin \omega(n + \beta) \sin \omega(n + \beta - 1)}$$

coincide up to a constant factor with (9). Hence, all unitary irreducible finite-dimensional representations of the $so_q(3)$ algebra for $q = \exp 2\pi i/N$ are exhausted by two possibilities: (8) of the dimension $N$ and (6) of a smaller dimension.

We introduce for brevity the following terminology: the representations (5) will be called “complementary series”; the cases (6) will be called “integer series” for $\beta = 1, 2, 3, \ldots$ and “half-integer series” for $\beta = 1/2, 3/2, \ldots$ (the case $\beta = 1/2$ is exceptional, for it belongs to both complementary and half-integer series). Such definitions are supported by the fact that eigenvalues of the Casimir operator (11) are continuous and quantized for the complementary and integer (half-integer) series respectively (cf. with the Lorentz algebra).

4. $q$-Ultraspherical Polynomials as Overlap Coefficients. $q$-Ultraspherical polynomials arise as overlap coefficients between two distinct bases in the space of $so_q(3)$ representations. Indeed, because $K_1$ is hermitian, we can choose another orthonormal basis $|s\rangle$ for which the operator $K_1$ is diagonal:

$$K_1|s\rangle = \mu_s|s\rangle,$$

$so_q(3)$ is a special case of the $AW(3)$ algebra describing symmetries of the Askey-Wilson polynomials [15]. From general properties of the latter algebra we know that the operator $K_0$ cannot be more than tridiagonal in this dual basis:

$$K_0|s\rangle = d_{s+1}|s+1\rangle + d_s|s-1\rangle + b_s|s\rangle,$$

where $s = 0, \ldots, M - 1$. The explicit form of matrix elements $d_s$ and $b_s$ depends on the representation series of $so_q(3)$. For integer and half-integer series we have the spectrum

$$\mu_s = \frac{\cos \omega(s + j/2)}{\sin \omega},$$

where $j$ is an integer. Thus, for integer and half-integer series $\mu_s$ are quantized functions of $s$. For complementary series $\mu_s$ are continuous functions of $s$.
where $j = 2\beta = 1, 2, 3, \ldots$, and the matrix elements

$$d_s^2 = \frac{\sin \omega s \sin \omega(s + j)}{4\sin^2 \omega \sin \omega(s + j/2) \sin \omega(s - 1 + j/2)}, \quad b_s \equiv 0.$$  \hfill (16)

This case is symmetric – matrix elements in the bases $|n\rangle$ and $|s\rangle$ are identical.

For the complementary series we have

$$\mu_s = \frac{\cos(\omega(s + 1/2))}{\sin \omega}, \quad s = 0, 1, \ldots, N - 1,$$

$$d_s^2 = \frac{\sin \omega(s - \beta + 1/2) \sin \omega(s + \beta - 1/2)}{4\sin^2 \omega \sin \omega(s + 1/2) \sin \omega(s - 1/2)}, \quad s = 1, 2, \ldots, N - 2,$$  \hfill (18)

$$b_0 = b_{N-1} = \frac{\sin \omega(1/2 - \beta)}{2\sin \omega \sin \omega/2}, \quad d_0 = d_{N-1} = b_1 = b_2 = \ldots = b_{N-2} = 0. \quad \hfill (19)$$

Note that the coefficients $d_n$ do not truncate automatically at the ends of the index intervals, one has to do it by hands. In this case the operator $K_0$ is tridiagonal instead of being two-diagonal as it is expected from the symmetry of $so_q(3)$ (after the replacement $K_2 \rightarrow iK_2$ this algebra looks totally symmetric under the cyclic permutations). This anomaly shows that permutations of generators are not necessarily unitary automorphisms of the “Cartesian” quantum algebras.

For any series: integer, half-integer or the complementary one, we can find the overlap coefficients between two bases $(s|n)$. These coefficients can be factorized, $(s|n) = (s|0)S_n(\mu_s)$, where $S_n(\mu_s)$ are symmetric polynomials satisfying the three-term recurrence relation

$$a_{n+1}S_{n+1} + a_nS_{n-1} = \mu_S S_n, \quad S_0 = 1, \quad S_1 = \mu_S/a_1.$$  \hfill (20)

Polynomials $S_n(\mu_s), \ n > 1$ are connected with $P_n(x)$ \cite{1} by the simple relation

$$P_n(x) = \sqrt{u_1u_2 \cdots u_n}S_n(\mu_s), \quad x = 2\mu_s \sin \omega.$$  \hfill (21)

Existence of the algebraic interpretation of $q$-ultraspherical polynomials in terms of the $so_q(3)$ algebra allows to calculate the weight function for these polynomials. Omitting the details (we use the method described, e.g. in \cite{3}), we present the final result.

For the integer and half-integer series the weight function has the form

$$w_s(j) = \sin \omega(s + j/2) \prod_{l=1}^{j-1} \sin \omega(s + l), \quad s = 0, 1, \ldots, N - j.$$  \hfill (22)

For the complementary series we have

$$w_s(2\beta) = w_0 \frac{\sin \omega(s + 1/2)}{\sin \omega/2} \prod_{l=0}^{s-1} \frac{\sin \omega(\beta + 1/2 + l)}{\sin \omega(-\beta + 3/2 + l)}, \quad s = 1, 2, \ldots, N - 1,$$  \hfill (23)

where $w_0$ is (undefined) value of the weight function at the points $s = 0$ and $s = N - 1$.

Clearly both weight functions \cite{22} and \cite{23} are known up to a normalization constant. This constant can not be found directly from the representation theory of $so_q(3)$ algebra and should be calculated separately. For the integer and half-integer series it is possible to derive weight functions together with normalization constants with the help of Darboux transformations.
5. Darboux Transformation and Normalization Constants. Let $P_n^{(j)}(x)$ be the monic $q$-ultraspherical polynomials belonging to integer or half-integer series. Write the orthogonality relation in the form

$$\sum_{s=0}^{N-j} P_n^{(j)}(x_s)P_n^{(j)}(x_s)w_s(j) = h_n(j)\delta_{nm},$$

(24)

where $h_n(j)$ are normalization constants which have to be found, and $w_s(j)$ is given by (22). Recall that for the taken series $x_s(j) = 2 \cos \omega(s + j/2)$.

The crucial observation is that the polynomials $P_n^{(j+2)}(x)$ and $P_n^{(j)}(x)$ are related to each other by the Darboux transformation [9] which is equivalent in our case to the transition to symmetric kernel polynomials [10]:

$$P_n^{(j+2)}(x_s(j + 2)) = \frac{P_n^{(j)}(x_s(j)) - A_n(j)P_n^{(j)}(x_s(j))}{x_s^2(j) - x_0^2(j)},$$

(25)

where

$$A_n(j) = \frac{P_n^{(j)}(x_0(j))}{P_n^{(j)}(x_0(j))}. \quad (26)$$

From the theory of kernel polynomials [10] it follows that the weight function is transformed as

$$w_s(j + 2) = w_{s+1}(j)(x_0^2(j) - x_{s+1}(j))/4,$$

(27)

and the normalization constants are transformed as

$$h_n(j + 2) = h_n(j)A_n(j)/4. \quad (28)$$

In our case the factor $A_n(j)$ can be easily found

$$A_n(j) = \frac{\sin \omega(n + j + 1)\sin \omega(n + j)}{\sin \omega(n + j/2)\sin \omega(n + j/2)}. \quad (29)$$

Then using (29), (28) and starting from $j = 2$ and $j = 1$ we obtain explicit expressions for the normalization constant in the case of integer series, $j = 2k$:

$$h_n(2k) = \frac{h_0(2)s_{n+k+1}s_{n+k+2} \ldots s_{n+2k-1}}{4^k s_{n+1}s_{n+2} \ldots s_{n+k-1}}, \quad n = 0, 1, \ldots, N - 2k, \quad (30)$$

where $s_n \equiv \sin \omega n$, and in the case of half-integer series, $j = 2k + 1$:

$$h_n(2k + 1) = \frac{h_0(1)s_1^2s_2^2 \ldots s_{n+1}s_{n+2} \ldots s_{n+2k}}{4^k s_1s_2s_3 \ldots s_{n+k-1}/2s_{n+k+1}/2}, \quad n = 0, 1, \ldots, N - 2k - 1, \quad (31)$$

where we assume that for $n = 0$ the product $s_1^2s_2^2 \ldots s_n^2$ is replaced by 1. The only coefficients left to be determined are $h_0(2)$ and $h_0(1)$. But these constants can be calculated directly. Indeed, for $j = 2$ we have the finite-dimensional Chebyshev polynomials

$$P_n(x_s) = \frac{\sin \omega(n + 1)(s + 1)}{\sin \omega(s + 1)}.$$

(32)

The weight function in this case is $w_s(2) = \sin^2 \omega(s + 1)$. Hence

$$h_0(2) = \sum_{s=0}^{N-2} \sin^2 \omega(s + 1) = N/2. \quad (33)$$

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The \( q \)-Legendre polynomials case \( j = 1 \) is characterized by the surprisingly simple weight function, \( w_s(1) = \sin \omega(s + 1/2) \), from which we derive

\[
h_0(1) = \sum_{s=0}^{N-1} \sin \omega(s + 1/2) = \frac{1}{\sin \omega/2}.
\] (34)

Thus we have calculated \( h_n(j) \) for both integer (30) and half-integer (31) series.

The \( n = 0 \) values of the normalization constants \( h_n(j) \) provide non-trivial trigonometric identities of the form \( \sum_{r=0}^{M-1} w_r(j) = h_0(j) \). For \( j = 2k \) we get

\[
\sum_{r=0}^{N-2k} s_{r+k}s_{r+1}s_{r+2}\cdots s_{r+2k-1} = 2N s_{k+1}s_{k+2}\cdots s_{2k-1} \\
4s_1s_2\cdots s_{k-1},
\] (35)

where \( k = 2, 3, \ldots, [N/2] \). For \( j = 2k + 1 \) we get

\[
\sum_{r=0}^{N-2k-1} s_{r+k+1/2}s_{r+1}s_{r+2}\cdots s_{r+2k} = \frac{s_1s_2\cdots s_{2k}}{4s_{1/2}s_{3/2}\cdots s_{k-1/2}s_{k+1/2}}.
\] (36)

where \( k = 1, 2, \ldots, [N/2] \). These identities can be considered as discrete finite-dimensional analogs of the \( q \)-beta integrals of Ramanujan (the latter are defined for real \( q \), see e.g. [3]). Of course, in the case of \( q \)-ultraspherical polynomials we get only special cases of such identities. Their generalizations are obtained if we deal with general trigonometric (four-parameter) analogs of the Askey-Wilson polynomials. Existence of similar simple relations for the representations of complementary series is an interesting open problem which we hope to address in the future.

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