Distilling Non-Locality

Manuel Forster, Severin Winkler, and Stefan Wolf
Computer Science Department, ETH Zürich, CH-8092 Zürich, Switzerland
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Two parts of an entangled quantum state can have a correlation in their joint behavior under measurements that is unexplainable by shared classical information. Such correlations are called non-local and have proven to be an interesting resource for information processing. Since non-local correlations are more useful if they are stronger, it is natural to ask whether weak non-locality can be amplified. We give an affirmative answer by presenting the first protocol for distilling non-locality in the framework of generalized non-signaling theories. Our protocol works for both quantum and non-quantum correlations. This shows that in many contexts, the extent to which a single instance of a correlation can violate a CHSH inequality is not a good measure for the usefulness of non-locality. A more meaningful measure follows from our results.

When two separated parts of a quantum state are measured in fixed bases, then the outcomes can show a correlation. Whereas this may be surprising from a physical point of view, it is not from the standpoint of information: such correlations could be explained by randomness shared when the two particles were generated.

If one considers, however, different possible measurement settings on the two sides, then correlations of a stronger kind can arise, which are unexplainable by shared randomness only [1]: This is non-locality.

Quantum mechanics is non-local but not maximally so. There are stronger correlations still in accordance with the non-signaling postulate of relativity [2]. This fact motivated the study of so-called generalized non-signaling theories [3, 4] in which quantum correlations are a special case. Following this general approach to non-locality, we study correlations between the joint behavior of the two ends of a bipartite input-output system, characterized by a conditional probability distribution $P(ab|xy)$. Let $x$ and $a$ be the input and output on the left-hand side of the system, and $y$ and $b$ the corresponding values on the right-hand side.

We call such a system local if it is explainable by shared classical information. On the other hand, it is signaling if it allows for message transmission in either direction.

John Bell has given properties that local systems have, namely certain inequalities they must obey. Hence, violation of such an inequality is a witness of non-locality. In the case where both inputs and both outputs are binary, the only such inequality (up to symmetries) is the so-called CHSH (after Clauser, Horne, Shimony, Holt) inequality [3]. Furthermore, the set of eight CHSH inequalities is complete for binary systems in the sense that if none of them is violated, then the system is local.

In this letter we restrict ourselves to the state space of binary input-binary output non-signaling systems. We refer to [4] for a detailed description of this set.

Non-local correlations are not only a fascinating phenomenon, but have as well been shown to be an interesting resource for information processing. Examples are device-independent secrecy of quantum cryptography [6] and non-local computation [7]. Furthermore, the existence of non-locality that is super-quantum to some extent would have dramatic consequences on communication complexity [8], and device-independent secrecy of quantum cryptography [9], but this result is independent of ours.

The extent by which a Bell inequality, e.g., CHSH, is violated can be taken as a measure for non-locality. Not surprisingly, non-locality is a more useful resource, the stronger it is. For instance, the violation of CHSH gives a lower bound to the uncertainty of a third party about the output bits of a non-signaling system, which is better the stronger the violation is.

Motivated by these facts, we study the problem of whether non-locality can be amplified: Can stronger non-locality be obtained from a number of weakly non-local systems? We consider protocols for non-locality distillation executed by two parties having access to weakly non-local systems. The parties on the two sides can carry out arbitrary operations on their pieces of information, but they cannot communicate.

Note that such protocols should not be confused with protocols for entanglement distillation: There, the input and output are (weakly and strongly, respectively) entangled quantum states, and the allowed operations are classical communication and local quantum operations. The existence of certain entanglement distillation protocols without communication is known [10], but this result is independent of ours.

There are several known impossibility results on non-locality distillation. First, it is not possible to create non-locality from locality, i.e., to pass the Bell bound [1]. Second, there exists no non-locality distillation which can pass the Tsirelson bound [11] if the non-local systems
can be simulated by quantum mechanics. Third, a simple inductive argument shows that a system that exhibits the algebraically maximal possible CHSH violation cannot be obtained from weaker ones. Fourth, it has been shown recently that the CHSH violation of two copies of isotropic systems cannot be distilled. And finally, it has been proven that there exists an infinite number of isotropic systems for which non-locality distillation cannot be achieved.

An open question which remains is whether non-locality can be distilled at all. We answer this question affirmatively.

**Main Result.** There exists a protocol which allows the distillation of certain, both quantum-mechanically achievable and unachievable, binary non-local systems.

**DEFINITIONS**

A binary input–output system characterized by a conditional probability distribution $P(ab|xy)$ is non-signaling if one cannot signal from one side to the other by the choice of the input. This means that the marginal probabilities $P(a|x)$ and $P(b|y)$ are independent of $y$ and $x$, respectively, i.e.,

$$\sum_b P(ab|xy) = \sum_b P(ab|xy') \equiv P(a|x) \forall a, x, y, y',$$

$$\sum_a P(ab|xy) = \sum_a P(ab|x'y) \equiv P(b|y) \forall b, x, x', y.$$

When using a non-signaling system, a party receives its output immediately after giving its input, independently of whether the other has given its input already. This prevents the parties from signaling by delaying their inputs.

If appropriate we represent a system by its probability distribution $P(ab|xy)$ in matrix notation as

$$
\begin{bmatrix}
P(00|00) & P(01|00) & P(10|00) & P(11|00) \\
P(00|01) & P(01|01) & P(10|01) & P(11|01) \\
P(00|10) & P(01|10) & P(10|10) & P(11|10) \\
P(00|11) & P(01|11) & P(10|11) & P(11|11)
\end{bmatrix}.
$$

Given $P(ab|xy)$ ($P$) we define the set of four correlation functions:

$$X_{xy}(P) = P(00|xy) + P(11|xy) - P(01|xy) - P(10|xy),$$

for $xy = 00, 01, 10, 11$. The corresponding system is local if and only if its correlation functions satisfy the following CHSH inequalities:

$$|X_{xy}(P) + X_{xy}(P) + X_{xy}(P) - X_{xy}(P)| \leq 2,$$

for $xy = 00, 01, 10, 11$. (We use $\bar{x}$ and $\bar{y}$ to indicate bit flips, that is, $\bar{0} = 1$ and $\bar{1} = 0$.)

In order to measure the non-locality of a system we will use the maximal violation of a CHSH inequality:

**Definition 1.** We define the CHSH non-locality of a binary input, binary output system $P$ as

$$NL[P] := \max_{xy} |X_{xy}(P) + X_{xy}(P) + X_{xy}(P) - X_{xy}(P)|,$$

Note that $NL[P] > 2$ indicates that the correlation $P$ violates CHSH and is therefore called non-local.

Quantum mechanics predicts violations of the CHSH inequalities up to $2\sqrt{2}$. However, this bound is only necessary. The necessary and sufficient condition for a set of four numbers to be reached by quantum mechanics was found by Landau and Tsirelson (see also Masanes).

**Lemma 1.** A set of correlation functions $X_{xy}$, $xy = 00, 01, 10, 11$, can be reached by a quantum state and some local observables if and only if they satisfy the following four inequalities:

$$|\arcsin X_{xy} + \arcsin X_{xy} + \arcsin X_{xy} - \arcsin X_{xy}| \leq \pi.$$

Using the terms introduced above we formally define a non-locality distillation protocol as follows:

**Definition 2.** A non-locality-distillation protocol is executed by two parties (Alice and Bob) without communication. It simulates a binary input/binary output system $P^n$ by classical (local) operations on $n$ non-local resource systems $P$, such that $NL[P^n] > NL[P] > 2$.

**RESULTS**

In the following we present a non-locality-distillation protocol and distillable non-local resource systems. We will also present resource systems that are measurable on a quantum state and can be used by our protocol to distill (quantum) non-locality.

We define the protocol $NDP_n(P)$ on $n$ non-signaling systems $P$ between Alice and Bob as follows: On inputs $x$ to Alice and $y$ to Bob the parties input $x$ and $y$ to all $n$ systems in parallel and receive outputs $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, respectively. The parties then locally compute their output bits as $a = \sum_{i=1}^n a_i \pmod{2}$ for Alice and $b = \sum_{i=1}^n b_i \pmod{2}$ for Bob. The whole protocol is illustrated in more detail in Figure

For $0 < \varepsilon \leq 1$ we define the following non-signaling system

$$P_\varepsilon = \begin{bmatrix}
1/2 & 0 & 0 & 1/2 \\
1/2 & 0 & 0 & 1/2 \\
1/2 & 0 & 0 & 1/2 \\
1/2 - \varepsilon/2 & \varepsilon/2 & \varepsilon/2 & 1/2 - \varepsilon/2
\end{bmatrix}$$
For we obtain only guarantees that the correlation functions
Protocol NDP
1
put/binary output system system behaves like a PR-box [2] and with probability
locality
Proof of Theorem 1. Therefore, we have established
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= 3 − X_{11}(P_\varepsilon) = 3 − (1 − 2\varepsilon)^n − 1, which implies NL[P^{n}_\varepsilon] > NL[P_\varepsilon].

FIG. 1: The final outputs are a simple exclusive-or of all the
outputs obtained from a parallel usage of the available non-
local resource systems.

as our non-local distillation resource with CHSH non-
lability NL[P_\varepsilon] = 3 − (1 − 2\varepsilon) > 2. With probability \varepsilon this
system behaves like a PR-box [2] and with probability
locality
Proof of Theorem 1. Obviously, NDP_n(P_\varepsilon) describes
only classical, local operations on Alice’s and Bob’s side. Furthermore, NDP_n(P_\varepsilon) simulates another binary input/binary output system \tilde{P}_\varepsilon with CHSH non-locality

\begin{align*}
NL[P^{n}_\varepsilon] &= X_{00}(P^{n}_\varepsilon) + X_{01}(P^{n}_\varepsilon) + X_{10}(P^{n}_\varepsilon) - X_{11}(P^{n}_\varepsilon) \\
&= 3 - X_{11}(P^{n}_\varepsilon) \\
&= 3 - (P^{n}_\varepsilon(00|11) + P^{n}_\varepsilon(11|11) - P^{n}_\varepsilon(01|11) - P^{n}_\varepsilon(10|11)).
\end{align*}

Here, we used that X_{00}(P^{n}_\varepsilon), X_{01}(P^{n}_\varepsilon), X_{10}(P^{n}_\varepsilon) are constant functions reaching the algebraic maximum of 1. Analogously to \tilde{P}_\varepsilon, let P^{n-1}_\varepsilon denote the system simulated by NDP_{n-1}(P_\varepsilon). Using

\begin{align*}
P^{n}_\varepsilon(00|11) &= P^{n}_\varepsilon(11|11) \\
&= (1/2 - \varepsilon/2)(P^{n-1}_\varepsilon(00|11) + P^{n-1}_\varepsilon(11|11)) + \varepsilon/2(P^{n-1}_\varepsilon(01|11) + P^{n-1}_\varepsilon(10|11)),
\end{align*}

\begin{align*}
P^{n}_\varepsilon(01|11) &= P^{n}_\varepsilon(10|11) \\
&= \varepsilon/2(P^{n-1}_\varepsilon(00|11) + P^{n-1}_\varepsilon(11|11)) + (1/2 - \varepsilon/2)(P^{n-1}_\varepsilon(01|11) + P^{n-1}_\varepsilon(10|11))
\end{align*}

we derive

\begin{align*}
NL[P^{n}_\varepsilon] &= 3 - (1 - 2\varepsilon)(P^{n-1}_\varepsilon(00|11) + P^{n-1}_\varepsilon(11|11) - P^{n-1}_\varepsilon(01|11) - P^{n-1}_\varepsilon(10|11)) \\
&= 3 - (1 - 2\varepsilon)X_{11}(P^{n}_\varepsilon).
\end{align*}

Therefore, we have established

\begin{align*}
NL[P^{n}_\varepsilon] &= 3 - X_{11}(P^{n}_\varepsilon) = 3 - (1 - 2\varepsilon)(P^{n}_\varepsilon - 1) \\
&= 3 - (1 - 2\varepsilon)^n - 1 X_{11}(P_\varepsilon) = 3 - (1 - 2\varepsilon)^n.
\end{align*}

For 0 < \varepsilon < 1/2 we can guarantee 3 − (1 − 2\varepsilon)^n > 3 − (1 − 2\varepsilon)^n − 1, which implies NL[P^{n}_\varepsilon] > NL[P_\varepsilon].

In the limit we have lim_{n \to \infty} NL[P^{n}_\varepsilon] = lim_{n \to \infty} 3 − (1 − 2\varepsilon)^n = 3.

Note that the presented systems are not quantum-
physically realizable. This allows our protocol to pass the
Tsirelson bound using P_\varepsilon with 0 < \varepsilon \leq \sqrt{2} − 1 as system resources. In the following we show that non-
lability distillation is also possible for systems available in quantum mechanics. We therefore introduce a more

\begin{align*}
P_{\varepsilon,\delta} &= \begin{bmatrix}
1/2 - \delta/2 & \delta/2 & \delta/2 & 1/2 - \delta/2 \\
1/2 - \delta/2 & 2/2 & \delta/2 & 1/2 - \delta/2 \\
1/2 - \delta/2 & \delta/2 & \delta/2 & 1/2 - \delta/2 \\
1/2 - \varepsilon/2 & \varepsilon/2 & \varepsilon/2 & 1/2 - \varepsilon/2
\end{bmatrix}
\end{align*}

This system has CHSH non-locality 3(1 − 2\varepsilon) − (1 − 2\varepsilon). For \delta = 0 we have P_{\varepsilon,\delta} = P_\varepsilon.

Note that we have chosen the two example resource
systems because of their simplicity. This should not sug-
ject that these exact systems are the only systems dis-
tillable by our protocol. Obviously the distillability of a
system with the presented protocol does only depend on
its correlation functions and not on the marginals.

\begin{align*}
NL[P^{n}_\varepsilon,\delta] &= X_{00}(P^{n}_\varepsilon,\delta) + X_{01}(P^{n}_\varepsilon,\delta) + X_{10}(P^{n}_\varepsilon,\delta) - X_{11}(P^{n}_\varepsilon,\delta) \\
&= 3(1 - 2\delta)^n - (1 - 2\varepsilon)^n.
\end{align*}

We can find values \alpha and 0 < \beta < \varepsilon < 1/2 (for example, \alpha = 2, \beta = 0.01, \delta = 0.002) such that P_{\varepsilon,\delta} is at the same
time distillable, i.e.,

3(1 − 2\delta)^n − (1 − 2\varepsilon)^n > 3(1 − 2\varepsilon) − (1 − 2\varepsilon)

and a quantum system, i.e.,

\begin{align*}
|3\arcsin(1 - 2\delta) - \arcsin(1 - 2\varepsilon)| &\leq \pi, \\
|\arcsin(1 - 2\delta) + \arcsin(1 - 2\varepsilon)| &\leq \pi.
\end{align*}

Lemma 1 only guarantees that the correlation functions of P_{\varepsilon,\delta} are obtainable by quantum mechanics. But Al-
ice and Bob can make their outputs locally uniform such that the correlation functions are preserved using shared randomness. Thus P_{\varepsilon,\delta} is a quantum system if its correlation functions are obtainable by quantum mechanics.

Therefore, we can achieve NL[P^{n}_\varepsilon,\delta] > NL[P_\varepsilon,\delta], which means that non-locality has been distilled with quantum systems as resources.
A natural follow up question concerns the maximum non-locality our protocol can distill using the quantum systems presented above.

Optimal parameters $n, \varepsilon, \delta$ maximize the term $NL[P_{\varepsilon,\delta}^n] = 3(1 - 2\varepsilon)^n - (1 - 2\varepsilon)^n$ with respect to the conditions that $NL[P_{\varepsilon,\delta}^n] > NL[P_{\varepsilon,\delta}]$ and that $P_{\varepsilon,\delta}$ is a quantum system (Lemma 1). The maximal non-locality that can be distilled by NDP$_n(P_{\varepsilon,\delta})$ is

$$NL[P_{\varepsilon,\delta}^{n_{\max}}] = 1 + \sqrt{2},$$

where $n_{\max} = 2, \varepsilon_{\max} \simeq 0.30866$ and $\delta_{\max} \simeq 0.03806$.

**A NEW MEASURE OF NON-LOCALITY**

The possibility of distillation motivates the definition of a new measure for non-locality, namely the maximal CHSH violation achievable from many realizations of a given system by any distillation protocol.

As an example application consider the computation of the non-locally distributed version of the AND function: Two separated parties are given inputs $x_1, x_2$ and $y_1, y_2$, respectively and have to find outputs $a$ and $b$, such that the probability of obtaining

$$a \oplus b = (x_1 \oplus y_1) \land (x_2 \oplus y_2)$$

is maximal. Quantum mechanics allows no advantage over the optimal, classical strategy [2]. Rearranging [2] yields a strategy with success probability directly related to the CHSH violation of a given resource system. By non-locality distillation copies of our arbitrarily weak non-local system $P_\varepsilon$ a higher success probability above the quantum bound can be reached. This illustrates that distillable systems like $P_\varepsilon$ – although located arbitrarily “close” to the quantum bound – are a stronger computational resource than any quantum system. Therefore, we obtain a separation of quantum and post-quantum correlations below the Tsirelson bound in terms of information processing power.

**CONCLUSION**

We have shown that non-locality of binary-input binary-output systems, measured by how strongly the CHSH inequality is violated, can be amplified. More precisely, we have shown that certain systems which violate CHSH arbitrarily weakly (achieving the value $2 + 2\varepsilon$), but that are nevertheless not realizable by quantum physics, can be distilled.

Furthermore, we show that even certain quantum-mechanically achievable systems can be distilled: Interestingly, the achievable limit by our protocol is then the exact mean $(1 + \sqrt{2})$ between the classical (2) and the quantum $(2\sqrt{2})$ bounds.

Our result complements previous ones, stating that the distillability of non-locality of two isotropic systems is impossible [12] and at most very limited in general [13]. Isotropic systems are an important special case because they are the worst case with respect to distillability, i.e., every non-signaling system can be turned into an isotropic system such that non-locality is preserved using shared randomness only (this transformation is known as depolarization [17]). Therefore, these non-distillable isotropic systems cannot be used to simulate the distillable resources defined here. In other words, bipartite isotropic and non-isotropic non-signaling (and quantum) systems are in general inequivalent correlations, although they exhibit the same violation of CHSH.

The possibility of distillation motivates the definition of a new measure for non-locality. Clearly, this measure is significant in any context where non-locality is used as a resource for information processing, and where the number of realizations available is not limited to one.

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