A lifting method for hyperbolic equations with $\delta$-singularities

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Abstract. In this paper, we propose a method to lift the solution space of hyperbolic equations with $\delta$-singularities such that dealing with the annoying $\delta$-singularities directly is avoided. Thus, the easily implemented finite difference schemes can be employed conveniently for solving such problems with $\delta$-singularities. In particular, we consider a fifth-order nonlinear finite difference scheme that can capture discontinuities robustly. To demonstrate the effectiveness of the proposed method, numerical examples are presented. Additionally, a novel splitting method is also proposed to extend the lifting method to high dimensions.

1. Introduction

Hyperbolic equations involving $\delta$-singularities are often used as models in science and engineering. Since in most cases to obtain analytical solutions is impractical, numerical methods are necessitated to investigate the underlying dynamics. However, these equations are difficult to approximate numerically due to the $\delta$-singularities, especially for finite difference methods. Many numerical methods rely on techniques that regularize the singularities with non-singular terms [1-3]. Nevertheless, these regularizations may smear the singularities severely and thus lead to large errors in the approximation. Therefore, we are in favour of approximating the original equations without modifications. In this paper, we first consider the study of the one-dimensional hyperbolic problem

$$\begin{align*}
\frac{\partial u}{\partial t} + f(u) \frac{\partial u}{\partial x} &= s(x), \\
u(x,0) &= u_0(x),
\end{align*}$$

(1)

where the source term $s(x)$, the initial value $u_0(x)$ or the solution $u(x,t)$ may contain $\delta$-singularities. Then we extend the proposed method to high dimensions by proposing a new splitting method.

So far, some useful numerical methods for solving Eq. (1) are available. In [4], the convergence of a class of numerical schemes was studied. In [5], the authors considered the case with source terms containing $\delta$-singularities and studied the convergence property of a finite volume scheme. Later on, they developed further a class of high resolution methods in [6]. In [7], the resonance behavior of the case with point source was investigated. The authors in [8] developed discontinuous Galerkin (DG) methods for solving Eq. (1). They also obtained error estimates in the smooth region away from singularities for the scalar one-dimensional linear case. This is the earliest result that we can find in the literature about numerical error estimates for problems described by Eq. (1) with $\delta$-singularities. The DG method was then applied to solving Krause’s consensus models and pressureless Euler equations.
[9] Recently, an arbitrary Lagrangian-Eulerian DG method was developed in [10] for solving Eq. (1), showing that sharper numerical solutions can be obtained.

Since finite difference methods are easy to derive and implement, we are concerned with the problem that how to solve Eq. (1) by using high-order finite difference methods without modifications of \( \delta \)-singularities. Here, motivated by the close relation between conservation laws and Hamilton-Jacobi equations [11], we introduce a new quantity \( \varphi(x,t) \) such that \( \varphi_t + f(\varphi) = S(x) \). Thus, we can lift the solution space of Eq. (1) and then consider the corresponding problem

\[
\begin{aligned}
\varphi_t + f(\varphi) &= S(x), \\
\varphi(x,0) &= \varphi_0(x),
\end{aligned}
\]

where \( S'(x) = s(x) \) and \( \varphi'(x) = u_0(x) \). It is easy to note that differentiating Eq. (2) once results in Eq. (1). The \( \delta \)-singularities appearing in Eq. (1) are translated into sign functions in Eq. (2). Since there are no \( \delta \)-singularities in Eq. (2) anymore, dealing with the annoying \( \delta \)-singularities is avoided and traditional shock-capturing finite difference schemes can be applied directly. In particular, we consider in this paper a fifth-order scheme in the framework of the weighted compact nonlinear schemes (WCNS), initially developed in [12], which have been successfully applied in the field of computational fluid dynamics [13].

The rest of this paper is organized as follows. In Sect. 2, the numerical scheme for solving the problem (1) is described. Then some numerical examples are employed in Sect. 3 to demonstrate the effectiveness of the proposed method. Additionally, we propose a splitting method in Sect. 4 to extend the lifting method to high dimensions, taking a two-dimensional case as an example. Finally, conclusions are drawn in Sect. 5.

2. Numerical scheme

The computational domain is denoted by \([x_L, x_R]\) and divided into \(N\) cells equally. Flux points are placed at the edges of the cells, denoted by \(x_{L1} = x_{L2} = \ldots = x_{L(N+1)/2} = x_L\). Within each cell, a solution point is placed at the center, denoted by \(x_j\), with \(j = 1, 2, \ldots, N\). For the described staggered grid, the algorithm of finite difference schemes can be summarized as the following four steps: (i) Get the left and right values of \( \varphi \) (denoted by \( \varphi'_{j+1/2} \) and \( \varphi'_{j-1/2} \)) at flux points \(x_{j+1/2}\) by interpolating the values \( \varphi_j \) at solution points; (ii) Compute the left and right derivatives (denoted by \( \varphi'_{j,L} \) and \( \varphi'_{j,R} \)) by deriving difference schemes using the values \( \varphi'_{j+1/2} \) and \( \varphi'_{j-1/2} \), respectively; (iii) Evaluate the flux \( f_j \) at solution points \(x_j\) by using an appropriate numerical flux, resulting in a system of ordinary differential equations, denote by

\[
\frac{d\Phi}{dt} = R(\Phi), \quad \Phi = [\varphi_1, \varphi_2, \ldots, \varphi_N]^T,
\]

which can be integrated by some time-marching schemes; (iv) At the end, we get the approximate values \( u_j \) by evaluating the derivative \( (\varphi_j) \) according to the relation \( u = \varphi \). In this paper, we just set \( u_j = \varphi'_{j,L} \).

2.1. Interpolation scheme

At a time level, suppose that the values \( \varphi_j \) at solution points \( x_j \) are known. In addition, it is assumed that there are no discontinuities near the boundaries of the computational domain. Therefore, nonlinear shock-capturing interpolation schemes are only used in the interior domain. For simplicity, we only
present the scheme for obtaining the left values \( \phi_{j+1/2}^L \), while the right values \( \phi_{j+1/2}^R \) can be obtained in the same way by using the symmetry property of the grid with respect to its center.

Suppose further that the value of \( \phi_{1/2}^L \) is also known, which may be determined by a given boundary condition (This will be discussed later in Sect. 3). Then the fifth-order left boundary interpolation schemes are determined to be

\[
\phi_{1/2}^L = -\frac{1}{7} \phi_{1/2}^L + \frac{5}{8} \phi_1 + \frac{5}{8} \phi_2 - \frac{1}{8} \phi_3 + \frac{1}{56} \phi_4, \\
\phi_{3/2}^L = \frac{3}{35} \phi_{1/2}^L - \frac{1}{4} \phi_1 + \frac{3}{4} \phi_2 + \frac{9}{20} \phi_3 - \frac{1}{28} \phi_4,
\]

where the coefficients can be obtained by simply using the method of Lagrangian interpolation (e.g. in [14]).

For the interior interpolation scheme, we can first derive three third-order schemes:

\[
\phi_{j+1/2}^{L,(0)} = \frac{3}{8} \phi_{j-2} - \frac{5}{4} \phi_{j-1} + \frac{15}{8} \phi_j, \\
\phi_{j+1/2}^{L,(1)} = -\frac{1}{8} \phi_{j-1} + \frac{3}{4} \phi_j + \frac{3}{8} \phi_{j+1}, \\
\phi_{j+1/2}^{L,(2)} = \frac{3}{4} \phi_{j+1} - \frac{1}{8} \phi_{j+2}.
\]

Then the fifth-order nonlinear interpolation scheme can be expressed as

\[
\phi_{j+1/2}^L = \sum_{k=0}^{2} \omega_k \phi_{j+1/2}^{L,(k)}, \quad 3 \leq j \leq N - 2,
\]

where

\[
\omega_k = \frac{\tilde{\omega}_k}{\sum_{l=0}^{2} \tilde{\omega}_l}, \quad \tilde{\omega}_k = \frac{\gamma_k}{(\beta_k + \epsilon)^2}, \quad k = 0,1,2,
\]

following the recipe of the classical WENO scheme derived in [15]. Here \( \gamma_k \) are linear optimal weights with values

\[
\gamma_0 = \frac{1}{16}, \quad \gamma_1 = \frac{5}{8}, \quad \gamma_2 = \frac{5}{16},
\]

\( \beta_k \) are smoothness indicators defined by

\[
\beta_0 = (\phi_{j-2} - 2 \phi_{j-1} + \phi_j)^2 + \frac{1}{4} (\phi_{j-2} - 4 \phi_{j-1} + 3 \phi_j)^2, \\
\beta_1 = (\phi_{j-1} - 2 \phi_j + \phi_{j+1})^2 + \frac{1}{4} (\phi_{j-1} - \phi_{j+1})^2, \\
\beta_2 = (\phi_j - 2 \phi_{j+1} + \phi_{j+2})^2 + \frac{1}{4} (3 \phi_j - 4 \phi_{j+1} + \phi_{j+2})^2,
\]

and \( \epsilon \) is a small parameter chosen to be \( \epsilon = 10^{-6} \) here, avoiding the denominator in Eq. (10) becoming zero. For more details of the introduced nonlinear interpolation scheme, the reader is referred to [12].

For the right boundary, we just use the following fifth-order linear interpolation schemes:

\[ \phi_{N-1/2}^j = -\frac{5}{128}\phi_{N-4}^j + \frac{7}{32}\phi_{N-3}^j - \frac{35}{64}\phi_{N-2}^j + \frac{35}{32}\phi_{N-1}^j + \frac{35}{128}\phi_N^j, \quad (15) \]
\[ \phi_{N+1/2}^j = \frac{35}{128}\phi_{N-4}^j - \frac{45}{32}\phi_{N-3}^j + \frac{189}{64}\phi_{N-2}^j - \frac{105}{32}\phi_{N-1}^j + \frac{315}{128}\phi_N^j. \quad (16) \]

2.2. Difference scheme

According to the theory developed in [16], the order of boundary schemes can be one order lower than the interior for first-order hyperbolic equations to attain global order of accuracy as the interior scheme. Thus, we set the boundary difference schemes to be fourth-order. Here, the left ones read as
\[
\phi_{L}^{j} = \frac{1}{h} \left( -\frac{11}{12}\phi_{L}^{j+1} + \frac{17}{24}\phi_{L}^{j+3/2} + \frac{3}{8}\phi_{L}^{j+5/2} + \frac{5}{24}\phi_{L}^{j+7/2} + \frac{1}{2}\phi_{L}^{j+1/2} \right),
\]
\[
\phi_{L}^{j-1} = \frac{1}{h} \left( \frac{1}{24}\phi_{L}^{j+1} - \frac{9}{8}\phi_{L}^{j+3/2} + \frac{9}{8}\phi_{L}^{j+5/2} - \frac{1}{24}\phi_{L}^{j+1/2} \right),
\]

where \( h \) stands for the length of the grid cell. The interior difference scheme is set to be central sixth-order, which can be written as
\[
\phi_{j,L}^{j} = \frac{3}{128}\phi_{j-5/2}^{j} - \frac{25}{128}\phi_{j-3/2}^{j} - \frac{75}{64}\phi_{j+1/2}^{j} - \frac{1}{h}, \quad 3 \leq j \leq N - 2
\]
\[
(19)
\]

The right boundary schemes are fourth-order as the left ones, expressed as
\[
\phi_{N-1,L}^{j} = \frac{1}{h} \left( \frac{1}{24}\phi_{N-5/2}^{j} - \frac{9}{8}\phi_{N-3/2}^{j} + \frac{9}{8}\phi_{N-1/2}^{j} - \frac{1}{24}\phi_{N+1/2}^{j} \right),
\]
\[
\phi_{N,L}^{j} = \frac{1}{h} \left( \frac{1}{24}\phi_{N-5/2}^{j} + \frac{5}{24}\phi_{N-3/2}^{j} - \frac{3}{8}\phi_{N-1/2}^{j} - \frac{17}{24}\phi_{N-1/2}^{j} + \frac{11}{12}\phi_{N+1/2}^{j} \right)
\]
\[
(20)
\]

2.3. Time-marching scheme

Knowing the values \( \phi_{j,L}^{j} \) and \( \phi_{j,R}^{j} \), we applied the local Lax-Friedrichs splitting to compute the flux as
\[
f_{j}^{'} = f^{+}(\phi_{j+1/2}^{j}) + f^{-}(\phi_{j-1/2}^{j}),
\]
\[
(22)
\]

where \( f^{+}(u) = \frac{1}{2} |f(u) + f'(u) u| \). Then we apply the third-order strong stability preserving (SSP) Runge-Kutta scheme [17,18] to solve the system (3), expressed as
\[
\begin{align*}
\Phi^{(1)} &= \Phi^{\circ} + \Delta t R(\Phi^{\circ}), \\
\Phi^{(2)} &= \frac{3}{4}\Phi^{\circ} + \frac{1}{4}\Phi^{(1)} + \Delta t R(\Phi^{(1)}), \\
\Phi^{n+1} &= \frac{1}{3}\Phi^{\circ} + \frac{2}{3}\Phi^{(2)} + \Delta t R(\Phi^{(2)}). 
\end{align*}
\]
\[
(23)
\]

3. Numerical experiments

Some numerical examples presented in [8] are reconsidered in this section to show the effectiveness of the proposed method. In order to keep the order of the spatial scheme in smooth region, the time step \( \Delta t = h^2 \) is chosen to implement the aforementioned SSP Runge-Kutta scheme for all examples.
3.1. Singular initial condition

We begin with considering the linear scalar problem

\[
\begin{aligned}
    u_t + u_x &= 0, \quad (x,t) \in [0,\pi] \times [0,1], \\
    u(x,0) &= \sin 2x + \delta(x - 0.5), \\
    u(0,t) &= u(\pi,t),
\end{aligned}
\]

whose exact solution is

\[ u(x,t) = \sin 2(x - t) + \delta(x - t - 0.5). \tag{25} \]

In this case, the corresponding lifting problem (2) reads as

\[
\begin{aligned}
    \varphi_t + \varphi_x &= 0, \quad (x,t) \in [0,\pi] \times [0,1], \\
    \varphi(x,0) &= -\frac{1}{2} \cos 2x + \frac{1}{2} \text{sign}(x - 0.5), \\
    \varphi_x(0,t) &= \varphi_x(\pi,t),
\end{aligned}
\]

with the analytical solution

\[ \varphi(x,t) = -\frac{1}{2} \cos 2(x - t) + \frac{1}{2} \text{sign}(x - t - 0.5). \tag{27} \]

Here \( \text{sign}(x) \) represents the sign function of \( x \).

To implement the algorithm presented in Sect. 2, we handle the periodic boundary condition of the derivative by using the governed equation and obtain \( \varphi_x(0,t) = \varphi_x(\pi,t) \). This equation can be integrated to yield

\[ \varphi(0,t) = \varphi(\pi,t) + \varphi(0,0) - \varphi(\pi,0) = \varphi(\pi,t) - 1, \tag{28} \]

where we have used the initial condition in the last equality. Thus, we can first approximate the value of \( \varphi(\pi,t) \) by Eq. (16) and then set the approximate left boundary value as \( \varphi_{N/2}^L = \varphi_{N+1/2}^L - 1 \), according to Eq. (28). Since the advection is from the left to the right, the right values of \( \varphi \) are not needed for this example.

As we can see from Figure 1, the numerical solution of \( \varphi \) matches well with the exact solution (27). Compared the numerical solution of \( u \) with that in [8] (see Figure 1 therein), large oscillations have been eliminated around the delta peak. To show the accuracy of the scheme away from the singularity, we evaluate the numerical errors in the region \([0,0.8] \cup [1.2,\pi]\) and obtain the test of accuracy as showed in Table 1, indicating that the design fifth-order convergence rate is achieved approximately in the smooth region.

| \( N \) | \( L^2\)-error | order | \( L^\infty\)-error | order | \( L^2\)-error | order | \( L^\infty\)-error | order |
|---|---|---|---|---|---|---|---|---|
| 50 | 5.74E-04 | – | 2.27E-03 | – | 1.25E-02 | – | 4.91E-02 | – |
| 100 | 1.30E-04 | 2.14 | 9.95E-04 | 1.19 | 5.50E-03 | 1.18 | 3.93E-02 | 0.32 |
| 200 | 6.35E-06 | 4.36 | 6.56E-05 | 3.92 | 7.61E-04 | 2.85 | 8.02E-03 | 2.29 |
| 300 | 7.15E-07 | 5.39 | 7.88E-06 | 5.23 | 1.06E-04 | 4.86 | 1.30E-03 | 4.48 |
| 400 | 9.47E-08 | 7.02 | 1.45E-06 | 5.88 | 2.34E-05 | 5.26 | 3.66E-04 | 4.42 |
3.2. Singular source term
The second example is a problem with singular source term

\[
\begin{cases}
    u_t + [(x+1)u]_x = \delta(x-c), & x \in [0,1.5], \\
    u(x,0) = 0, \\
    u(0,t) = 0,
\end{cases}
\]  

which admits the exact solution

\[
u(x,t) = \frac{1}{1+x}[H(x-c) - H(x+1-(c+1)e^t)].
\]  

Here \( c \) is a constant, and \( H(x) \) denotes the Heaviside function defined by \( H(x) = 1 \) for \( x \geq 0 \) and \( H(x) = 0 \) for \( x < 0 \).

The lifting problem (2) corresponds to Eq. (29) is

\[
\begin{cases}
    \varphi_t + (x+1)\varphi_x = 0.5\text{sign}(x-c), & x \in [0,1.5], \\
    \varphi(x,0) = 0, \\
    \varphi_(0,t) = 0,
\end{cases}
\]  

whose exact solution is

\[
\varphi(x,t) = \frac{1}{2} \int_0^t \text{sign}((x+1)e^{\tau-t} - 1-c) d\tau \begin{cases}
    -t/2, & x \leq c, \\
    -t/2 - \ln \frac{1+c}{1+x}, & c < x < (1+c)e^t - 1, \\
    t/2, & x \geq (1+c)e^t - 1.
\end{cases}
\]  

As Example 3, the boundary condition \( \varphi_x(0,t) = 0 \) is translated into

\[
\varphi_{v2} = \varphi(0,t) = \varphi(0,0) - \frac{1}{2} t = -\frac{1}{2} t.
\]  

Here we set \( c = \pi/20 \) to produce numerical solutions as depicted in Figure 2. As we can see, numerical solutions match well with the exact solutions (32) and (30). Oscillations are reduced compared with those in [8].
3.3. Rendez-vous algorithm
The so-called rendez-vous algorithm is described by
\[
\begin{cases}
\rho_t + F_x = 0, & x \in [0,1], \\
\rho(x,0) = 1,
\end{cases}
\]  
where \( \rho \) represents the density, and \( F \) the flux that is defined by
\[
F(x,t) = v(x,t) \rho(x,t)
\]  
with the velocity given as
\[
v(x,t) = \int_{\infty}^{x} (y-x) \xi(y-x) \rho(y,t) dy.
\]  
Here \( \xi(x) = \chi_{[-R,R]} \) denotes the indicator function of the interval \([-R,R]\), where \( R \) is a constant. In [19] Canuto et al. proved that \( \delta \)-singularities arise in the solution to Eq. (34) when \( t \to \infty \). Additionally, the number of delta peaks depends on the value of \( R \).

The lifting problem (2) corresponding to Eq. (34) is
\[
\begin{cases}
\phi_t + v(x,t) \phi_x = 0, & x \in [0,1], \\
\phi(x,0) = x,
\end{cases}
\]  
where \( v(x,t) \) can be computed as
\[
v(x,t) = \int_{-R}^{R} z \rho(z+x,t) dz
\]  
\[
= \int_{-R}^{R} z \phi_z(z+x,t) dz
\]  
\[
= R \phi(x+R,t) + \phi(x-R,t) - \int_{-R}^{R} \phi(z+x,t) dz.
\]  
In practice, we need to know the approximate value of \( v(x_j,t) \). Thus it is convenient to choose the spatial step such that \( x_j \pm R \) are also solution points. For case with \( R = 0.02 \), this is satisfied by setting the spatial step to be \( h = 1/800 \) (i.e. \( N = 800 \)). Here the integral appearing in Eq. (38) is approximated by using a rectangular quadrature formula. As depicted in Figure 3 for \( t = 1000 \).
stairs are observed in the solution of $\varphi$, yielding 22 $\delta$-singularities appearing in the solution of $u$. This result is consistent with those in [8,10,19]. However, some small oscillations can be observed around the $\delta$-singularities. These defects may be cured by using some positivity-preserving limiters, which is left for a future work.

Figure 3. Numerical solutions to the Rendez-vous algorithm and corresponding lifting problem at $t=1000$; $N=800$. (a) Numerical solution to problem (37); (b) numerical solution to problem (34).

4. Extension to high dimensions
In the previous presentation, the effectiveness of the lifting method for dealing with one-dimensional hyperbolic problems with $\delta$-singularities has been demonstrated. However, it is not clear how the method could be extended to high dimensions. To the best of our knowledge, a similar relation between conservation laws and Hamilton-Jacobi equations is not available in high dimensions. In this section, we aim to propose a novel splitting method to handle this. For simplicity, we confine the description to the following two-dimensional hyperbolic problem

$$\begin{align*}
\varphi_t + f(\varphi_x + \varphi_y) &= s(x, y), \\
\varphi(x, y, 0) &= u_0(x, y).
\end{align*}$$

As the one-dimensional case (1), the source term $s(x, y)$, the initial value or the solution may contain $\delta$-singularities. However, we should note that the splitting method described below is applicable for higher dimensions as well.

4.1. Splitting method
Without surprise, the solution space of the two-dimensional case (39) cannot be lifted directly like the one-dimensional case. Instead, we introduce a new vector $\Psi(x, y, t) = [\varphi, \varphi']^T$, such that the divergence $\text{div}(\Psi) = \varphi_x + \varphi_y = u$. Then the hyperbolic equation in Eq. (40) can be obtained by first differentiating the following splitting equations with respect to $x$ and $y$, respectively, and then adding them up:

$$\begin{align*}
\varphi_t + f(\varphi_x + \varphi_y) &= S_1(x, y), \\
\psi_t + g(\varphi_x + \varphi_y) &= S_2(x, y),
\end{align*}$$

where it is required that $(S_1)_x + (S_2)_y = s(x, y)$. In practice, we may set $(S_1)_x = (S_2)_y = s(x, y)/2$ and $\varphi_x = \psi_x = u_0(x, y)/2$. Then the initial conditions of Eq. (40) can be obtained by integrals. In the following example, we adopt this choice to produce numerical results.
4.2. Numerical example

To verify the proposed splitting method, we consider the two-dimensional linear scalar problem [8]

\[
\begin{align*}
    u_t + u_x + u_y &= 0, \\
    u(x, y, 0) &= \sin(x + y) + \delta(x + y - 2\pi) + \delta(x + y),
\end{align*}
\]

whose exact solution is

\[
    u(x, y, t) = \sin(x + y - 2t) + \delta(x + y - 2\pi - 2t) + \delta(x + y - 2t).
\]

For this problem, the corresponding splitting system (40) can be written as

\[
\begin{pmatrix}
    1 & 0 & 0 & 1 \\
    0 & 1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    \varphi \\
    \psi
\end{pmatrix}
+ \begin{pmatrix}
    0 & 1 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    \varphi \\
    \psi
\end{pmatrix}
= 0,
\]

(43)

where \( \Psi = [\varphi, \psi]^T \). As stated above, the initial condition is set to be

\[
    \varphi(x, y, 0) = \psi(x, y, 0) = \frac{1}{2} \cos(x + y) + \frac{1}{4} \text{sign}(x + y - 2\pi) + \frac{1}{4} \text{sign}(x + y),
\]

(44)

such that \( \varphi_x + \varphi_y = u \).

Since the coefficient matrices of Eq. (43) admit nonnegative eigenvalues, the corresponding semi-discretized equations can be expressed as

\[
\frac{d\Psi_{j,k}}{dt} + A(\Psi_{j,k})^T + B(\Psi_{j,k})_L = 0,
\]

(45)

where the superscript ‘L’ denotes the left biased values with respect to the corresponding partial derivatives. Here the computational domain is set to be \((x, y) \in [0, 2\pi] \times [0, 2\pi]\) with periodic boundary conditions, which are treated as the case in Sect. 3.1. As shown in Figure 4, the delta peaks are captured well by the proposed method, demonstrating its effectiveness.

![Figure 4. Numerical solutions to the problem (41) at \( t = 0.5 \); \( N_x \times N_y = 400 \times 400 \). (a) Solution in two dimensions; (b) one-dimensional cut-plot of the two-dimensional result along the line \( x = y \).](image-url)
5. Conclusions
To solve hyperbolic equations involving $\delta$-singularities by using finite difference schemes, we have proposed in this paper a lifting method to avoid dealing with $\delta$-singularities directly. This method results in problems without $\delta$-singularities appearing in the equations or the solutions, allowing us to apply conveniently the shock-capturing schemes for hyperbolic conservation laws. Particularly, we considered a fifth-order nonlinear finite difference scheme and showed its advantage in reducing oscillations around singularities or discontinuities in the solutions. Additionally, a novel splitting method has also been proposed to extend the lifting method to high dimensions.

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