ON THE STRONG LAW OF LARGE NUMBERS FOR $L$-STATISTICS WITH DEPENDENT DATA

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Abstract. The strong law of large numbers for linear combinations of functions of order statistics ($L$-statistics) based on weakly dependent random variables is proven. We also establish the Glivenko–Cantelli theorem for $\varphi$-mixing sequences of identically distributed random variables.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of random variables with the common distribution function $F$. Let us consider the $L$-statistic

$$L_n = \frac{1}{n} \sum_{i=1}^{n} c_{ni} h(X_{n:i}),$$

where $X_{n:1} \leq \ldots \leq X_{n:n}$ are the order statistics based on the sample $\{X_i, i \leq n\}$, $h$ is a measurable function called a kernel, $c_{ni}$, $i = 1, \ldots, n$, are some constants called weights.

The aim of this paper is to establish the strong law of large numbers (SLLN) for $L$-statistics (1) based on sequences of weakly dependent random variables. The similar problems were considered in the papers [1] and [2], where the SLLN was proved for aforementioned $L$-statistics based on stationary ergodic sequences. For example, in [2] the case of linear kernels ($h(x) = x$) and asymptotic regular weights was considered, i.e.

$$c_{ni} = n \int_{(i-1)/n}^{i/n} J_n(t) \, dt,$$

with $J_n$ denoting an integrable function. In addition, the existence of a function $J$ such that for all $t \in (0, 1)$

$$\int_{0}^{t} J_n(s) \, ds \to \int_{0}^{t} J(s) \, ds$$
was imposed there. The statistics (1) with linear kernels and regular weights, i.e. $J_n \equiv J$ in (2), were considered in [1]. In the present paper we relax the regularity assumption on $c_{ni}$ and, furthermore, consider the $L$-statistics (1) based on both stationary ergodic sequences and $\varphi$-mixing sequences. We also do not impose monotonicity of the kernel in (1). Note, that if $h$ is a monotonic function, then the $L$-statistic (1) can be represented as a statistic
\[ \frac{1}{n} \sum_{i=1}^{n} c_{ni} Y_{n,i}, \]
based on a sample \{\(Y_i = h(X_i), i \leq n\)\} (see [3] for more detail).

As an auxiliary result we obtain the Glivenko–Cantelli theorem for $\varphi$-mixing sequences.

2. Notations and Results

2.1. Assumptions and notations. We first introduce our main notations. Let $F^{-1}(t) = \inf\{x : F(x) \geq t\}$ be the quantile function corresponding to the distribution function $F$ and let $U_1, U_2, \ldots$ be a sequence of uniformly distributed on $[0,1]$ random variables. Due to the fact that joint distributions of random vectors $(X_{n:1}, \ldots, X_{n:n})$ and $(F^{-1}(U_{n:1}), \ldots, F^{-1}(U_{n:n}))$ coincide, we have that
\[ L_n = \frac{1}{n} \sum_{i=1}^{n} c_{ni} H(U_{n:i}), \]
where $H(t) = h(F^{-1}(t))$, and $\frac{d}{d}$ denotes the equality in distribution. Let us consider a sequence of functions $c_n(t) = c_{ni}$, $t \in ((i-1)/n, i/n]$, $i = 1, \ldots, n$, $c_n(0) = c_{n1}$.

It is not difficult to see that in this case we have:
\[ L_n = \int_{0}^{1} c_n(t) H(G_n^{-1}(t)) \, dt, \]
where $G_n^{-1}$ is the quantile function corresponding to the empirical distribution function $G_n$ based on the sample \{\(U_i, i \leq n\)\}. We also introduce the following notation:
\[ \mu_n = \int_{0}^{1} c_n(t) H(t) \, dt, \]
\[ C_n(q) = \begin{cases} \frac{1}{n} \sum_{i=1}^{n} |c_{ni}|^q & \text{if } 1 \leq q < \infty, \\ \max_{1 \leq i \leq n} |c_{ni}| & \text{if } q = \infty. \end{cases} \]

Further we will use the following conditions on the weights $c_{ni}$ and the function $H$:
(i) the function $H$ is continuous on $[0,1]$ and $\sup_{n \geq 1} C_n(1) < \infty$.
(ii) $E|h(X_1)|^p < \infty$ and $\sup_{n \geq 1} C_n(q) < \infty$ ($1 \leq p < \infty$, $1/p + 1/q = 1$).

Assumptions (i) and (ii) guarantee the existence of $\mu_n$. We also note that $C_n(\infty) = \|c_n\|_\infty = \sup_{0 \leq t \leq 1} |c_n(t)|$ and $C_n(q) = \|c_n\|_q^q = \int_0^1 |c_n(t)|^q \, dt$ for $1 \leq q < \infty$. 


2.2. SLLN for ergodic and stationary sequences. Let us formulate our main statement for stationary ergodic sequences.

**Theorem 1.** Let \( \{X_n, n \geq 1\} \) be a strictly stationary and ergodic sequence and let either (i) or (ii) hold. Then, as \( n \to \infty \),

\[
L_n - \mu_n \to 0 \quad \text{a. s.} \tag{3}
\]

**Remark.** Let us consider the case of regular weights:

\[
c_{ni} = n \int_{(i-1)/n}^{i/n} J(t) \, dt.
\]

Then

\[
L_n = \sum_{i=1}^{n} H(U_{ni}) \int_{(i-1)/n}^{i/n} J(t) \, dt = \int_{0}^{1} J(t) H(G_n^{-1}(t)) \, dt.
\]

Hence, assuming \( c_n(t) = J(t) \) in Theorem 1, we have

\[
L_n \to \int_{0}^{1} J(t) H(t) \, dt \quad \text{a. s.}
\]

Also note that the convergence \( \mu_n \to \mu, |\mu| < \infty \), yields that \( L_n \to \mu \) a. s. In particular, if \( c_n(t) \to c(t) \) uniformly in \( t \in [0,1] \), then \( \mu_n \to \int_{0}^{1} c(t) H(t) \, dt \).

Without the requirement that the coefficients \( c_{ni} \) are regular one can easily construct an example where the assumptions of Theorem 1 are satisfied, but the sequence \( c_n(t) \) does not converge in any reasonable sense to a limit function. Let, for simplicity, \( h(x) = x \) and let \( X_1 \) be uniformly distributed on \([0,1]\). Set \( c_{ni} = (i-1)\delta_n, 1 \leq i \leq k \), and \( c_{ni} = (2k-i)\delta_n, k+1 \leq i \leq 2k \), \( k = k(n) = \lfloor n^{1/2} \rfloor \), \( \delta_n = n^{-1/2} \). Thus, the function \( c_n(t) \) is defined on the interval \([0,2k/n]\). On the remaining part of \([0,1]\) we extend \( c_n(t) \) periodically with period \( 2k/n \): \( c_n(t) = c_n(t-2k/n), 2k/n \leq t \leq 1 \) (see also [3] p. 138). Note that \( 0 \leq c_n(t) \leq 1 \). One can show that in this case \( \mu_n \to 1/4 \). In view of this fact we have that the assumptions of Theorem 1 are satisfied and, consequently,

\[
L_n \to 1/4 \quad \text{a. s.}
\]

2.3. SLLN for \( \varphi \)-mixing sequences. We will now formulate our main statement for mixing sequences. Let us define the mixing coefficients:

\[
\varphi(n) = \sup_{k \geq 1} \sup_{A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+n}} \left| P(B|A) - P(B) \right| > 0
\]

where \( \mathcal{F}_1^k \) and \( \mathcal{F}_{k+n} \) denote the \( \sigma \)-fields generated by \( \{X_i, 1 \leq i \leq k\} \) and \( \{X_i, i \geq k+n\} \) respectively. The sequence \( \{X_i, i \geq 1\} \) is called \( \varphi \)-mixing (uniform mixing) if \( \varphi(n) \to 0 \) as \( n \to \infty \).

**Theorem 2.** Let \( \{X_n, n \geq 1\} \) be a \( \varphi \)-mixing sequence of identically distributed random variables such that

\[
\sum_{n \geq 1} \varphi^{1/2}(n) < \infty, \tag{4}
\]

and let any of the conditions (i) or (ii) hold. Then the statement (3) remains true.
The proof of Theorem 2 essentially uses the result of the Lemma 1 below. The statement (a) of Lemma 1 is the SLLN for \( \varphi \)-mixing sequences. The statement (b) is a Glivenko–Cantelli-type result for \( \varphi \)-mixing sequences and is of independent interest. We note that neither in Theorem 2 nor in Lemma 1 we do not assume the stationarity of the sequence \( \{X_n\} \).

**Lemma 1.** Let \( \{X_n, n \geq 1\} \) be a \( \varphi \)-mixing sequence of identically distributed random variables such that the statement (4) holds. Then

(a) for any function \( f \) such that \( \mathbb{E}|f(X_1)| < \infty \),

\[
\frac{1}{n} \sum_{i=1}^{n} f(X_i) \rightarrow \mathbb{E}f(X_1) \quad a. s.
\] (5)

(b)

\[
\sup_{-\infty < x < \infty} |F_n(x) - F(x)| \rightarrow 0 \quad a. s.,
\] (6)

where \( F_n \) is the empirical distribution function based on the sample \( \{X_i, i \leq n\} \).

3. Proofs

3.1. Proof of Theorem 1.

**Lemma 2.** Let the function \( H \) be continuous on \([0, 1]\). Then

\[
\sup_{0 \leq t \leq 1} |H(G_n^{-1}(t)) - H(t)| \rightarrow 0 \quad a. s.
\] (7)

**Proof of Lemma 2.** Using the equality

\[
\sup_{0 \leq t \leq 1} |G_n^{-1}(t) - t| = \sup_{0 \leq t \leq 1} |G_n(t) - t|
\]

(see, for example, [4, p. 95]) and the Glivenko–Cantelli theorem for stationary ergodic sequences, we get

\[
\sup_{0 \leq t \leq 1} |G_n^{-1}(t) - t| \rightarrow 0 \quad a. s.,
\]

i. e. \( G_n^{-1}(t) \rightarrow t \) a. s. uniformly in \( t \in [0, 1] \) as \( n \rightarrow \infty \). Since the function \( H \) is uniformly continuous on the compact \([0, 1]\), it follows that \( H(G_n^{-1}(t)) \rightarrow H(t) \) a. s. uniformly in \( t \in [0, 1] \). This concludes the proof.

Let the condition (i) hold. Now, by Lemma 2,

\[
|L_n - \mu_n| \leq \int_{0}^{1} |c_n(t)||H(G_n^{-1}(t)) - H(t)| \, dt
\]

\[
\leq C_n(1) \sup_{0 \leq t \leq 1} |H(G_n^{-1}(t)) - H(t)| \rightarrow 0 \quad a. s.
\]

Consequently, the proof of Theorem 1 for the first case is complete.

**Lemma 3.** Let \( \mathbb{E}|h(X_1)|^p < \infty \). Then

\[
\int_{0}^{1} |H(G_n^{-1}(t)) - H(t)|^p \, dt \rightarrow 0 \quad a. s.
\] (8)
Proof of Lemma 3. First note that the set of all continuous on the interval $[0, 1]$ functions is everywhere dense in $L_p[0, 1]$, $1 \leq p < \infty$. Therefore, for any $\varepsilon > 0$ and any function $f \in L_p[0, 1]$ there exists a continuous on $[0, 1]$ function $f_\varepsilon$ such that $\int_0^1 |f(t) - f_\varepsilon(t)|^p dt < \varepsilon$. Since $\mathbb{E}|h(X_1)|^p = \int_0^1 |H(t)|^p dt < \infty$, this implies that there exists a continuous on $[0, 1]$ function $H_\varepsilon$ such that

$$\int_0^1 |H(t) - H_\varepsilon(t)|^p dt < \varepsilon/2.$$ 

Further,

$$\int_0^1 |H(G_n^{-1}(t)) - H(t)|^p dt \leq 3^{p-1} \int_0^1 |H(t) - H_\varepsilon(t)|^p dt + 3^{p-1} \int_0^1 |H(G_n^{-1}(t)) - H_\varepsilon(G_n^{-1}(t))|^p dt + 3^{p-1} \int_0^1 |H_\varepsilon(G_n^{-1}(t)) - H_\varepsilon(t)|^p dt. \quad (9)$$

From Lemma 2 it follows that $H_\varepsilon(G_n^{-1}(t)) \to H_\varepsilon(t)$ a. s. uniformly in $t$ as $n \to \infty$. Hence, the last integral on the right hand side of (9) converges to zero a. s. as $n \to \infty$. Now let us consider the second integral. By ergodic theorem for stationary sequences,

$$\int_0^1 |H(G_n^{-1}(t)) - H_\varepsilon(G_n^{-1}(t))|^p dt = \frac{1}{n} \sum_{i=1}^n |H(U_i) - H_\varepsilon(U_i)|^p \to_{\text{a. s.}} \mathbb{E}|H(U_1) - H_\varepsilon(U_1)|^p$$

$$= \int_0^1 |H(t) - H_\varepsilon(t)|^p dt < \varepsilon/2.$$

Consequently,

$$\limsup_{n \to \infty} \int_0^1 |H(G_n^{-1}(t)) - H(t)| dt < 3^{p-1} \varepsilon \quad \text{a. s.}$$

Since $\varepsilon$ is arbitrary, we obtain (8).

Now let the assumption (ii) hold. Using Hölder’s inequality, we get

$$|L_n - \mu_n| \leq C_n^{1/q} \left( \int_0^1 |H(G_n^{-1}(t)) - H(t)|^p dt \right)^{1/p}$$

for $p > 1$,

and

$$|L_n - \mu_n| \leq C_n(\infty) \int_0^1 |H(G_n^{-1}(t)) - H(t)| dt$$

for $p = 1$.

The statement (3) follows from Lemma 3. This completes the proof of Theorem 1.
3.2. **Proof of Theorem 2.** We now prove Lemma 1. Note that for any measurable function $f$ the sequence $\{f(X_n), n \geq 1\}$ has its $\varphi$-mixing coefficient bounded by the corresponding coefficient of the initial sequence, since for any measurable $f$ the $\sigma$-field generated by $\{f(X_n), n \geq 1\}$ is contained in the $\sigma$-field generated by $\{X_n, n \geq 1\}$. Therefore, if the sequence $\{X_n, n \geq 1\}$ is $\varphi$-mixing, then so is the sequence $\{f(X_n), n \geq 1\}$. Hence, the condition (4) holds for mixing coefficients of the sequence $\{f(X_n), n \geq 1\}$. The statement (5) follows from the SLLN for $\varphi$-mixing sequences (see [5, p. 200]).

The statement (6) is an immediate corollary of (5) and classical Glivenko–Cantelli theorem.

The proof of Theorem 2 is similar to the proof of Theorem 1. Indeed, the statement (7) follows from the Glivenko–Cantelli theorem (6); using the SLLN (5), we get the statement (8). Thus the proof of Theorem 2 is complete.

**References**

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