Higher-Order Derivative SUSY in Quantum Mechanics with Large Energy Shifts

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Abstract

Within the framework of second order derivative (one dimensional) SUSYQM we discuss particular realizations which incorporate large energy shifts between the lowest states of the spectrum of the superhamiltonian (of Schrödinger type). The technique used in this construction is based on the "gluing" procedure. We study the limit of infinite energy shift for the charges of the Higher Derivative SUSY Algebra, and compare the results with those of the standard SUSY Algebra. We conjecture that our results can suggest a construction of a toy model where large energy splittings between fermionic and bosonic partners do not affect the SUSY at low energies.

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1. Introduction

First we write down for completeness the essential formalism of one dimensional Higher Derivative SUSY Quantum Mechanics (HSUSYQM) developed in [1] and compare it with standard SUSYQM. The Standard SUSYQM is defined in terms of the charges $Q^\pm$ and the superhamiltonian $H$ with $Q^- = (Q^+)^\dagger$

\[
(Q^\pm)^2 = 0, \quad [H, Q^\pm] = 0, \quad \{Q^+, Q^\pm\} = H \tag{1}
\]

The one-dimensional representation ($-\infty < x < +\infty$) is realized by the $2 \times 2$ supercharges

\[
Q^+ = \begin{pmatrix} 0 & 0 \\ q^- & 0 \end{pmatrix} \quad \text{and} \quad Q^- = \begin{pmatrix} 0 & q^+ \\ 0 & 0 \end{pmatrix} \tag{3}
\]

where

\[
q^\pm = \mp \partial + W(x), \quad \partial \equiv \frac{\partial}{\partial x} \tag{4}
\]

and

\[
H = (-\partial^2 + W^2)1 - \sigma_3 W' \tag{5}
\]

where $\sigma_3$ is the Pauli matrix.

The spectrum of $H$ is clearly non-negative. The Second Order extension of this algebra was recently elaborated in [1]. Further investigations concerning scattering and applications to radial problems are represented in [2]. The main difference coming from the second order derivatives in the charges $q^\pm$

\[
q^+ = (q^-)^\dagger = \partial^2 + \{f(x), \partial\} + \phi(x), \tag{5}
\]

consists in the fact that the anticommutator of supercharges generates a non Schrödinger type of non negative operator $K$ :

\[
\{Q^+, Q^-\} = K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} = \begin{pmatrix} q^+q^- & 0 \\ 0 & q^-q^+ \end{pmatrix}
\]

such that

\[
(Q^\pm)^2 = 0, \quad [K, Q^\pm] = 0, \quad \{Q^+, Q^-\} = K \tag{6}
\]

It is instructive to try to parametrize the charges $q^\pm$ in terms of a product of (first derivative) standard charges\footnote{Here we do not discuss the conditions in general which allow this reparametrization [1], [2] but we will only consider the case in which it is possible to solve this problem introducing the gluing condition as discussed later.} written in terms of ordinary superpotentials $W_1$ and $W_2$ (see Eq.(5)) ,

\[
q^+ = q_1^+ q_2^+ = (-\partial + W_1(x))(-\partial + W_2(x)) = \partial^2 + 2f\partial + f^2 + f' - W^2 + W', \tag{8}
\]
\( q^- = q_2^- q_1^- = \partial^2 - 2f \partial + f^2 - f' - W^2 + W', \)  \hspace{1cm} (9)

with the relations
\[ W_1 + W_2 = -2f, \quad W_1 - W_2 = 2W, \quad \phi = f^2 - W^2 + W'. \]  \hspace{1cm} (10)

In the context of a discussion of physical applications, a relevant case is the one in which \( K \) is a (second) order polynomial in a Schrödinger like Hamiltonian operator \( H \) (both operators being 2x2 superoperators). Such a realization is provided by the following construction [1],[2].

Take two standard supersymmetric hamiltonians:
\[
\begin{pmatrix}
h^{(1)} & 0 \\
0 & h
\end{pmatrix}, \quad \begin{pmatrix}
q_1^+ q_1^- & 0 \\
0 & q_1^- q_1^+
\end{pmatrix}, \quad \begin{pmatrix}
h & 0 \\
0 & h^{(2)} - \lambda
\end{pmatrix} = \begin{pmatrix}
q_2^+ q_2^- - \lambda & 0 \\
0 & q_2^- q_2^+ - \lambda
\end{pmatrix},
\]  \hspace{1cm} (11)

impose the ”gluing” condition
\[ q_2^+ q_2^- - \lambda - q_1^- q_1^+ = -W_1'' - W_2^2 - W_2' + W_2^2 - \lambda = 0, \quad \lambda > 0 \]  \hspace{1cm} (12)

and consider the resulting Superhamiltonian \( H \) (obtained by deleting the intermediate hamiltonian \( h \)):
\[
H = \begin{pmatrix}
h^{(1)} & 0 \\
0 & h^{(2)} - \lambda
\end{pmatrix}.
\]  \hspace{1cm} (13)

Its hamiltonian partners \( h_1, h_2 - \lambda \) are intertwined by the original higher order supercharges or equivalently :
\[
[H, Q^\pm] = 0
\]

The spectra of the two partners in \( H \) are the same except for the lowest states and the wave functions are formally connected as in standard SUSYQM the supercharges being now of higher order. In addition it turns out [1] that the superalgebra takes the form
\[
K = \{Q^+, Q^-\} = H(H + \lambda)
\]

2. Examples

We want to discuss different possibilities for the spectrum of \( H \) which should exhibit a large energy shift between the lowest energy states of the partner hamiltonians in \( H \). We discuss, in particular, cases in which the lowest eigenvalue of \( h^{(2)} - \lambda \) is \(-\lambda\) so that we have to study the zero modes of \( q_2^+ \). The general expression reads:
\[
\Psi_{\pm\lambda}^{(2)}(x) = \exp(\int^x dy \ W_2(y)). \]  \hspace{1cm} (14)

The condition of normalizability can be related to the asymptotic properties of \( W_2 \) for \( x \rightarrow \pm \infty \):
\[
W_{2, +\infty} < 0, \quad W_{2, -\infty} > 0. \]  \hspace{1cm} (15)

More detailed information on the spectrum of \( H \) also depends on \( W_1 \).

EXAMPLE 1.
Let us assume

$$W_1^2 + W_1' = \alpha^2$$

The gluing condition, Eq.(12), reads

$$W_2^2 - W_2' = \lambda + \alpha^2$$

Its general nonsingular solution is

$$W_2(x) \equiv W_2^{(0)}(x) = -\kappa \tanh(\kappa x + \gamma), \quad (16)$$

where $\gamma$ is arbitrary real constant and $\kappa \equiv \sqrt{\lambda + \alpha^2}$. The asymptotic properties of $W_2^{(0)}(x)$

$$W_{2,\pm\infty} = \mp \kappa$$

are in agreement with the normalizability condition, Eq.(15).

Let us choose

$$W_1 = \pm \alpha$$

The spectrum of $H$ consists of a degenerate continuum spectrum starting at $\alpha^2$. There is a non degenerate ground state with eigenvalue $E = -\lambda$ whose wave function, Eq.(14), is:

$$\Psi_{-\lambda}^{(2)}(x) = \cosh^{-1}(\kappa x + \gamma), \quad (17)$$

potential

$$V^{(2)}(x) = \alpha^2 - \frac{2\kappa^2}{\cosh^2(\kappa x + \gamma)}$$

can be recognized as a familiar reflectionless potential.

A second possibility for $W_1$ is allowed i.e.

$$W_1(x) = \alpha \cdot \tanh(\alpha x)$$

The continuum spectrum is not modified but now $h^{(1)}$ acquires a zero energy (non degenerate) bound state while the deeply bound state of its partner as well as $V^{(2)}$ remain unmodified.

EXAMPLE 2

An alternative choice for potential $V^{(1)}$ is given by

$$V^{(1)} = W_1^2 - W_1' = \alpha^2.$$ 

We only study the non trivial solution (the trivial would not be new in respect to the one given in Example 1)

$$W_1(x) = -\alpha \cdot \tanh(\alpha x + \beta), \quad \beta = \text{Const} \quad (18)$$

4As a remark we would like to notice that the second possibility of Example 1 can provide a case where $q$-deformed SUSY techniques could be used to generate the same partnership as the higher order derivative charges used in the present note.

Finally we would like to make also another observation i.e. that the same problem also can be formulated as a partnership between two systems associated with two hamiltonians $h^{(1)} - \alpha^2$ and $h^{(2)} - (\lambda + \alpha^2)$, respectively. The role of the intertwining operators is taken now simply by the dilatation operator (without any further differential operator), the energies as well as the wave function’s normalization constants turn out to be rescaled.

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for which the gluing condition Eq.(12) leads to

\[ W_2^2 - W_2' - (\lambda + \alpha^2) + \frac{2\alpha^2}{\cosh^2(\alpha x + \beta)} = 0. \]  (19)

In this case the continuum part of the spectrum \((E \geq \alpha^2)\) of \(H\) is also degenerate. The intermediate potential

\[ V = W_1^2 + W_1' = \alpha^2 - \frac{2\alpha^2}{\cosh^2(\alpha x + \beta)} = W_2^2 - W_2' - \lambda \]  (20)

has precisely one bound state (with zero energy):

\[ \Psi_{E=0}(x) = q_1^{-1} \Psi_{E=0}^{(1)}(x). \]

Therefore operator \(q_2^+ q_2^-\) has the only bound state \(E = +\lambda\) (see Eq.(12)) but its zero-energy solution, which may be obtained from non-normalizable function \(\Psi_{E=0}^{(1)}(x)\),

\[ \Psi_0(x) = \exp(-\int W_2(y)dy) = q_1^{-1} \Psi_{E=0}^{(1)}(x) = (\partial + W_1)(c_1 e^{\kappa x} + c_2 e^{-\kappa x}) \]  (21)

is also non-normalizable for all constants \(c_1, c_2\).

Now, by means of Darboux transformation \((21)\), we can write the general solution of the gluing equation \((19)\):

\[ W_2(x) = -\partial \ln \Psi_0(x) = -\partial \ln[(\partial - \alpha \tanh \alpha x + \beta)(c_1 e^{\kappa x} + c_2 e^{-\kappa x})]. \]  (22)

If we are interested in nonsingular solutions \(W_2(x)\) it is necessary to take \(c_1 c_2 < 0\), i.e.

\[ W_2(x) = \alpha \tanh(\alpha x + \beta) + \frac{\lambda \tanh(\kappa x + c)}{\alpha \tanh(\alpha x + \beta) \tanh(\kappa x + c) - \kappa}, \quad c = const, \]  (23)

which leads to the normalizable eigenfunction of \((h^{(2)} - \lambda)\)

\[ \Psi_{E=0}^{(2)}(x) = \exp(\int W_2(y)dy) \]  (24)

with the eigenvalue \(E = -\lambda\). Let us put the constant \(c = 0\) for simplicity. Thus the fermionic component of Hamiltonian \(H\), which depends on the real parameter \(\rho\), has exactly two bound states with wave functions

\[ \Psi_{E=0}^{(2)}(x) = (\partial + W_2) \exp(\int W_1(y)dy) \]  (25)

and \(\Psi_{E=0}^{(2)}(x)\) (see Eq.(24)) where \(W_{1,2}\) were defined in Eqs.(18), (23).

For the purposes of analysis given in the next Section let us compare the behavior of superpotentials \(W_2(x)\) and \(W_2^{(0)}(x)\) of the previous Examples for large values of \(\kappa\) (see Eqs.(16), (23)). First of all it is possible to choose the parameter \(\gamma\) in Eq.(16) such that

\[ W_2(x = 0) = W_2^{(0)}(x = 0). \]  (26)
At arbitrary finite \( x \) and \( (| x | \kappa) \gg 1 \) we see that

\[
W_2(x) \sim W_2^{(0)}(x) \sim \mp \kappa
\]

depending on the sign of \( x \). More of that in the same limit we have

\[
W_2(x) - W_2^{(0)}(x) \sim O(\kappa^{-1}). \tag{27}
\]

Before ending this section we would like to make additional comments about the behaviour of the higher order charges in the large \( \lambda \) limit where naively one can expect the higher order theory to become effectively the first order (standard) SUSY because \( K \) would become equal to \( H \). What emerges from the previous examples is that for very large values of \( \lambda \), i.e. disregarding in this limit the very deep level, the spectrum has indeed the standard feature of SUSYQM where one can eliminate or add or retain the lowest bound state.

For a better understanding let us consider the one dimensional scattering problem on the line, i.e. \( x \in (-\infty, +\infty) \), for the Hamiltonians of Eq.(13). The scattering wave function for \( h^{(1)} \) fulfils the asymptotic conditions

\[
\Psi_{k,-\infty}^{(1)} = e^{ikx} + R^{(1)}(k)e^{-ikx} \tag{28}
\]

and

\[
\Psi_{k,\infty}^{(1)} = T^{(1)}(k)e^{ikx} \tag{29}
\]

where \( R^{(1)}(k), T^{(1)}(k) \) are the reflection and transmission coefficients respectively.

The ladder operators \( q^\pm \), Eq.(4), are asymptotically expressed as

\[
q^- = +\partial + W^-; \quad q^+ = -\partial + W^+
\]

with

\[
W^\pm = \lim_{x \to \pm\infty} W(x).
\]

The asymptotic scattering wave function of the partner Hamiltonian \( (h^2 - \lambda) \) is written in the same way. The relations between the two scattering problems for HSSQM can be deduced by iteration from first order standard SUSY (see for instance [4]):

\[
T^{(2)}(k) = T^{(1)}(k) \frac{(k - iW_2^+)(k - iW_1^+)}{(k - iW_2^-)(k - iW_1^-)}
\]

\[
R^{(2)}(k) = R^{(1)}(k) \frac{(k + iW_2^-)(k + iW_1^-)}{(k - iW_2^-)(k - iW_1^-)} \tag{30}
\]

One can insert the asymptotic values of the superpotentials discussed in the previous examples and obtain that in the large \( \lambda \) limit the relations reduce effectively to the ones of a first order theory but for the extra phase of \( \pi \). Also the addition or suppression of a bound state in the partner relationship becomes transparent from the pole structure of the multiplicative factor relating the scattering observables.

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5 Note however that in the scattering (e.g. transmission coefficient) one can recognize the effects of this state in agreement with Levinson’s theorem [4].
Formally we can rescale the charges \( q^{\pm} \), Eqs.(9), (10), by \( \sqrt{\lambda} \) and obtain the finite limiting operators

\[
\tilde{q}^- = \lim_{\lambda \to \infty} q^- / \sqrt{\lambda} = \left( \lim_{\lambda \to \infty} W_2(x) / \sqrt{\lambda} \right) (+\partial + W_1(x)) = -\epsilon(x)(+\partial + W_1(x)), \\
\tilde{q}^+ = (\tilde{q}^-)^\dagger = -(-\partial + W_1(x))\epsilon(x)
\]

with \( \epsilon(x) \) the standard step function: \( \epsilon(x) = +1 \) for \( x > 0 \) and \( \epsilon(x) = -1 \) for \( x < 0 \). One can thus obtain new supercharges with the following algebra

\[
\{ \tilde{Q}^+, \tilde{Q}^- \} = H + H^2 / \lambda
\]

and the limiting Hamiltonian

\[
\lim_{\lambda \to \infty} (H + H^2 / \lambda) = H_\infty
\]

is unitary equivalent to the Hamiltonian of the standard SUSYQM:

\[
h^{(1)}_{\infty} = h^{(1)} = -\partial^2 + W_1^2 - W_1' ; \\
h^{(2)}_{\infty} = \epsilon(x)(-\partial^2 + W_1^2 + W_1')\epsilon(x).
\]

Evidently the appearance of the unitary rotation \( \epsilon(x) \) gives rise to the phase observed in Eqs.(30) in scattering coefficients. We notice that superficially it does not coincide with the limit

\[
\lim_{\lambda \to \infty} (h^{(2)} - \lambda) = -\partial^2 + W_1^2 + W_1' - 2\sqrt{\lambda} \cdot \delta(x).
\]

Nevertheless they are equal for all \( x \neq 0 \) and generate the same phase of \( \pi \) in scattering characteristics.

As compared to the paper [8] we keep track of the existence of deep level in wave functions of \( h^{(2)}_{\infty} \) and expect them to have bounce-like behavior at the origin which however can be easily smeared out by the rotation \( \Psi^{(2)}(x) \to \epsilon(x)\Psi^{(2)}(x) \) for \( h^{(2)}_{\infty} \). Thus we impose the boundary conditions opposite to [8].

3. General Case

The previous examples which we have illustrated suggest how to formulate a general approximation scheme for arbitrary \( W_1(x) \) which by assumption is independent on \( \lambda \) and sufficiently regular at the origin.

We shall employ the Quasilinearization Method [8] for the solution of gluing condition, Eq.(12),

\[
W_2^2 - W_2' = W_1^2 + W_1' + \lambda,
\]

which can provide us with \( 1/\lambda \)-type expansion:

\[
W_2(x) = W_2^{(0)}(x) + W_2^{(1)}(x) + W_2^{(2)}(x) + ....
\]
The procedure of solution can be started from the solution $W_2(0)(x)$, Eq.(16) of Example 1, where $\alpha^2$ is now replaced by the value of

$$W_2^2(0) + W_1'(0) = \rho$$

at the origin (with $\rho$ not necessarily non negative). Thus $W_2(0)(x)$ has the correct behaviour at small $x$ and the other terms in the expansion (35) equal zero at the origin by construction.

The solution for $W_2(1)(x)$ is obtained by a Quasilinearization of Eq.(12):

$$2W_2(0)^2W_2(1) - W_2(1)' - (W_2^2 + W_1' - \rho) = 0,$$

(36)

$$W_2(1)(x) = A - \exp(2\int_0^x dy W_2(0)(y)) \left[ \int_0^x (W_2^2 + W_1' - \rho) \exp(-2\int_0^y dz W_2(0)^2(z))dy \right].$$

(37)

When approximating $(W_2^2 + W_1' - \rho)$ in the integral by a constant value $\nu$ one obtains for $x\sqrt{\lambda} \rightarrow \pm \infty$ that

$$W_2(1)(x) \rightarrow \mp \frac{\nu}{2\sqrt{\lambda} + \rho}$$

(compare this result with Eq.(27) of Example 2). Iterating this procedure to obtain $W_2(2)$ one has to solve again the same type of equation as before, Eq.(36):

$$2(W_2(0)^2 + W_2(1)^2)W_2(2) + (W_2(1))^2 - W_2(2)' = 0.$$

(38)

The resulting sum, Eq.(35), shows the expected dependence on $\lambda$ of its various terms, i.e.

$$W_2 = 1/\sqrt{\lambda}[A + B/\lambda + C/\lambda^2 + ...].$$

In particular, the potential $V(2)(x)$ can be derived in terms of $V_1$ and $W_{1,2}$:

$$V_2(x) = V_1(x) + 2(W_1(x) + W_2(x))'.$$

(39)

For large values of $\lambda$ $V(2)(x)$ acquires an attractive ”intense” (proportional to $\sqrt{\lambda}$) $\delta$-type singularity, as in Eq.(34). As before the finite energy properties of such Hamiltonian are the same for unitarily rotated regular Hamiltonian, Eq.(33).

The spectrum does not in general exhibit any new features for the two lowest bound states in respect to the ones provided by Examples 1 and 2.

4. Conclusions

The theoretical scheme we have illustrated involves an effective non negative operator of the form $H + H^2/\lambda$ which belongs to a HSUSY algebra:

$$\{\hat{Q}^+, \hat{Q}^- \} = H + H^2/\lambda$$

The results we have obtained may be of some relevance within the context of effective approximation schemes for the ”resolvent operator” $1/(\lambda - H)$ (involving similar operators) which are generally to be interpreted ($\lambda$ large) as high energy approximations [7].

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Extensions to higher orders (larger than two) do not seem to present specific difficulties, leading to a Higher-order algebra involving polynomials of $H^{[1]}$, $H^{[2]}$. They can be relevant in the approximation of relativistic Hamiltonians in the nonrelativistic limit ($c \to \infty$):

$$c\sqrt{m^2c^2 + H} \sim mc^2 + H/2m - H^2/8m^3c^2 + H^3/16m^5c^4,$$

possessing the Higher-Order SUSY.

We are thus able to provide toy models in which a finite amount of levels is shifted down in energy in such a way that the remaining part of the spectrum can be disregarded because it is associated to very large energies. However we stress that this high energy part of the spectrum preserves the appropriate boson-fermion degeneracy important for the ultraviolet behavior of the theory. Note that this mechanism for lifting the original degeneracy is somewhat complementary to the more standard one in which some levels are pushed up in energy becoming thereby effectively unobservable. In particular we would like to stress that our framework cannot simply be readjusted to accommodate negative values of $\lambda$ corresponding to a positive energy shift, this is so because the corresponding superpotentials would become periodic and have no definite asymptotic behavior.

We have seen that the presence of deep level simulated by $\delta$-type potential is revealed in the discontinuity of wave functions at the origin which can be equivalently rotated away in the alternative description, Eq. (33) (our approach is different from [5]).

From an algebraic point of view what we have done is to associate the algebra of the Supercharges to some type of enveloping algebra. The inclusion of the quasihamiltonian $K/\lambda$ which incorporates $H^2/\lambda$ and higher order terms (if the corresponding generalization is implemented) while preserving SUSY requires to introduce second (higher) order charges.

The possibility to obtain a large lifting of degeneracy for the lowest state of $H$ may be of some interest as a very simple example where energy scales associated to fermions and bosons (as SUSY partners) can be vastly different without necessarily breaking a generalized HSUSY algebra.

Finally we would like to stress that the physical effects we discuss would remain essentially unchanged in the case of $\lambda$ large but finite thus eliminating any possible problems arising from singularities in the limit $\lambda \to \infty$.

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