Open loop amplitudes and causality to all orders and powers from the loop-tree duality

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Multiloop scattering amplitudes describing the quantum fluctuations at high-energy scattering processes are the main bottleneck in perturbative quantum field theory. The loop-tree duality opens multiloop scattering amplitudes to non-disjoint tree dual amplitudes by introducing as many on-shell conditions on the internal propagators as independent loop momenta, and is realized by modifying the usual infinitesimal imaginary prescription of Feynman propagators. Remarkably, non-causal singularities of the unintegrated amplitudes are explicitly cancelled in the dual representation, while the causal and anomalous threshold, soft and collinear singular structures emerge clearly in a compact region of the loop three-momenta, enabling a simultaneous computation with the extra emission real matrix elements through suitable momentum mappings. Based on the original formulation of the loop-tree duality, we present in this letter very compact and definite dual representations of a series of multiloop topologies with arbitrary powers of the Feynman propagators. These expressions are sufficient to describe any scattering amplitude up to three-loops, and their clear recurrence structure allows to conjecture other topologies with more complex combinatorics. Causal and infrared singularities are also manifestly characterized in these expressions.

INTRODUCTION

Precision modeling of fundamental interactions relies mostly on perturbative Quantum Field Theory (pQFT). Quantum fluctuations in pQFT are encoded by Feynman diagrams with closed loop circuits. These loop diagrams are the main bottleneck to achieve higher perturbative orders and therefore more precise theoretical predictions for high-energy colliders [1,2]. Whereas loop integrals are defined in the Minkowski space of the loop four-momenta, the loop-tree duality (LTD) [3,22] exploits the Cauchy residue theorem to reduce the dimensions of the integration domain by one unit in each loop. In the most general version of LTD the loop momentum component that is integrated out is arbitrary [3,4]. In numerical implementations [7,8,10,11,13,14,16,21,22] and asymptotic expansions [12,17], it is convenient to select the energy component because the remaining integration domain, the loop three-momenta, is Euclidean.

LTD opens any loop diagram to a forest (a sum) of non-disjoint trees by introducing as many on-shell conditions on the internal loop propagators as the number of loops, and is realized by modifying the usual infinitesimal imaginary prescription of the Feynman propagators. The new propagators with modified prescription are called dual propagators. LTD at higher orders proceeds iteratively, or in words of Feynman [23,24], by opening the loops in succession. While the position of the poles of Feynman propagators in the complex plane is well defined, i.e. the positive (negative) energy modes feature a negative (positive) imaginary component due to the momentum independent $+i0$ imaginary prescription, the dual prescription of dual propagators is momentum dependent. Therefore, after applying LTD to the first loop, the position of the poles in the subsequent loop momenta moves up and down on the real axis. The solution found in Ref. [4,5] was to reshuffle the imaginary components of the dual propagators by using a general identity that relates dual with Feynman propagators in such a way that propagators entering the second and successive applications of LTD are Feynman propagators only. This procedure requires to reverse the momentum flow of a few subsets of propagators in order to keep a coherent momentum flow in each LTD round.

Recent papers have proposed alternative dual representations [19,22]. In Ref. [19,20], an average of all the possible momentum flows is proposed, which requires a cumbersome calculation of symmetry factors. In Ref. [21,22], the Cauchy residue theorem is applied iteratively by keeping track of the actual position of the poles in the complex plane. The procedure requires to close the Cauchy contours at infinity from either below or above the real axis in order to cancel the de-
In the next sections, we will derive the LTD representation of the multiloop scattering amplitude in Eq. (4), and will present explicit expressions for several general topologies to all orders and arbitrary powers of the Feynman propagators.

Beyond one-loop, any loop subtopology involves at least two loop lines that depend on the same loop momentum. We define the dual function as

$$G_D(s; t) = -2\pi i \sum_{i_s \in s} \text{Res}(G_F(s, t), \text{Im}(q_{i_s, 0}) < 0) ,$$ 

(5)

where $G_F(s, t)$ represents the product of the Feynman propagators that belong to the two sets $s$ and $t$. Each of the Feynman propagators can be raised to an arbitrary power. Notice that in Eq. (5) only the propagators that belong to the set $s$ are set consecutively on-shell, and the Cauchy contour is closed always from below the real axis. For single power propagators and $s = t$, Eq. (5) provides the usual dual function at one loop [3]

$$G_D(s) = -\sum_{i_s \in s} \delta(q_{i_s}) \prod_{j_s \in s, j_s < i_s} \frac{1}{(q_{i_s, 0}^2 + k_{j_s i_s, 0}^2 - (q_{j_s, 0}^2)^2)},$$

(6)

with $k_{j_s i_s} = q_{j_s} - q_{i_s}$, and $\delta(q_{i_s}) = 2\pi i \theta(q_{i_s, 0})\delta(q_{i_s}^2 - m_{i_s}^2)$ selecting the on-shell positive energy mode, $q_{i_s, 0} > 0$. If some of the Feynman propagators are raised to multiple powers, then Eq. (5) leads to heavier expressions [5] but the location of the poles in the complex plane is the same as in the single power case.

Then, we construct the nested residue involving several sets of momenta

$$G_D(1, \ldots, r; n) = -2\pi i \sum_{i_r \in r} \text{Res}(G_D(1, \ldots, r - 1; r, n), \text{Im}(q_{i_r, 0}) < 0).$$

(7)

In Eqs. (5) and (7), we can also introduce numerators and define the corresponding unintegrated open dual amplitudes $A^{(L)}_D(1, \ldots, r; n)$ by replacing the Feynman propagators by the integrand of Eq. (4) (see e.g. Ref. [13] at two-loops). Also, the energy component of the loop momenta can be replaced by the scalar product $\eta \cdot q_{i_r}$, with $\eta$ a future-like vector, to generalize the nested residue to an arbitrary coordinate system as in the original formulation of LTD [3]. With this compact notation, we express very easily the dual representation of benchmark multiloop scattering amplitude topologies to all orders.

**MAXIMAL LOOP TOPOLOGY**

The maximal loop topology (MLT), see Fig. 1, is defined by $L$-loop topologies with $n = L + 1$ sets of propagators, where the momenta of the propagators belonging to the first $L$ sets depend on one single loop momentum, $q_{i_s} = \ell_s + k_{s}$, with $s \in \{1, \ldots, L\}$, and the momenta of the extra set, denoted by $n$, are a linear combination of all the loop momenta, $q_{i_n} = -\sum_{s=1}^{L} \ell_s + k_{i_n}$. The minus sign in front of the sum
is imposed by momentum conservation. The momenta $k_{1a}$ and $k_{1b}$ are linear combinations of external momenta. At two loops ($n = 3$), this is the only possible topology.

The LTD representation of the multiloop MLT amplitude, starting from two loops, is extremely simple and symmetric

$$A_{\text{MLT}}^{(L)}(1, \ldots, n) = \int_{\ell_1, \ldots, \ell_L} \frac{1}{N} \sum_{i=1}^{N} A_{D}^{(L)}(1, \ldots, i-1, i+1, \ldots, n; i),$$

with $A_{D}^{(L)}(\mathbf{2}, \ldots, \mathbf{n}; 1)$ and $A_{D}^{(L)}(1, \ldots, n-1; n)$ as the first and the last elements of the sum, respectively. In each term there are $n - 1$ on-shell propagators, and the multiloop amplitude is topologically open into non-disjoint trees. Moreover, there is no dependence on the position of the poles in the complex plane. The bar in $\pi$ indicates that the momentum flow of the set $s$ is reversed ($q_\pi \to -q_\pi$), which is equivalent to selecting the on-shell modes with negative energy of the original momentum flow. The LTD representation in Eq. (8) is displayed graphically in Fig. 1. This very compact expression is proven by induction, and represents the basic building block entering other topologies.

The causal behavior of Eq. (8) is also clear and manifest. The dual representation in Eq. (8) becomes singular when one or more off-shell propagators eventually become on-shell and generate a disjoint tree dual subamplitude. If these propagators belong to a set where there is already one on-shell propagator then the discussion reduces to the one-loop case [6]. We do not comment further on this case. The interesting case occurs when the propagator becoming singular involves the set with all the propagators off-shell [15]. For example, the first element of the sum in Eq. (8) features all the propagators in the set 1 off-shell. One of those propagators might become on-shell, and there are two potential solutions, one with positive energy and another with negative energy, depending on the magnitude and direction of the external momenta [6, 13]. The solution with negative energy represents a singular configuration where there is at least one on-shell propagator in each set. Therefore, the amplitude splits into two disjoint trees, with the momenta over the causal on-shell cut pointing to the same direction. Abusing notation:

$$A_{D}^{(L)}(\mathbf{2}, \ldots, \mathbf{n}; 1) \rightarrow A_{D}^{(L)}(\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}).$$

The causal singularities collapse to finite segments for massless particles leading to infrared singularities and are bounded by the magnitude of the external momenta, thus enabling the simultaneous generation of the tree contributions describing real emissions of extra radiation through suitable mappings of momenta, as defined in four-dimensional unsubtraction (FDU) [9-11]. A similar situation happens with the last term of the sum in Eq. (8) that features a potential causal singularity when all the on-shell momenta are aligned in the opposite direction over the causal on-shell cut, $A_{D}^{(L)}(1, \ldots, n-1; n) \rightarrow A_{D}^{(L)}(1, \ldots, n)$.

Also remarkable is the special case where each loop set consists of one single propagator. At two loops, this is for example the sunrise diagram. Then, Eq. (8) for single power propagators reduces to the extremely compact expression

$$A_{\text{MLT}}^{(L)}(1, \ldots, n) = \int_{\ell_1 \ldots \ell_L} \frac{1}{2} \left( \frac{1}{\lambda_{1,n}^+} - \frac{1}{\lambda_{1,n}^-} \right),$$

with $\lambda_{1,n}^+ = \pm \sum_{i=1}^{n} q_{1,i}^{(+)} + k_{0,n}$, with $k_{n} = \sum_{i=1}^{n} q_{i}$, and $\int_{\ell_j} = -\mu^{4-d} \int d^{d-1} f_{j}/(2\pi)^{d-1}/(2q_{j,0}^{ (+)})$. The most notable property of this expression is that it is explicitly free of unphysical singularities, and the causal singularities occur, as expected, when either $\lambda_{1,n}^+$ or $\lambda_{1,n}^-$ vanishes, depending on the sign of the energy component of $k_{n}$, in the loop three-momenta region where the on-shell energies are bounded, $q_{1,0}^{(+)} < |k_{0,n}|$. This property also holds for powered propagators, with then $\lambda_{1,n}^+$ and $\lambda_{1,n}^-$ raised to specific powers.

**NEXT-TO-MAXIMAL LOOP TOPOLOGY**

The next multiloop topology in complexity, see Fig. 2, contains one extra set of momenta, denoted by 12, that depends on the sum of two loop momenta, $q_{12} = -\ell_1 - \ell_2 + k_{12}$. We
call it next-to-maximal loop topology (NMLT). This topology appears for the first time at three loops, i.e. $L = n - 1$ with $n \geq 4$, and its LTD representation is given by the compact and factorized expression

$$A_{\text{NMLT}}^{(L)}(1, \ldots, n, 12) = A_{\text{MLT}}^{(2)}(1, 2, 12) \otimes A_{\text{MLT}}^{(L-2)}(3, \ldots, n)$$

$$+ A_{\text{MLT}}^{(1)}(1, 2) \otimes A_{\text{MLT}}^{(0)}(12) \otimes A_{\text{MLT}}^{(L-1)}(3, \ldots, \pi).$$

(12)

The first term on the r.h.s. of Eq. (12) represents a convolution of the two-loop subtopology involving the sets $(1, 2, 12)$ with the rest of the amplitude, such that two propagators in this subtopology are set simultaneously on-shell. In the second term on the r.h.s. of Eq. (12), the set $12$ remains off-shell while there are on-shell propagators in either 1 or 2, and all the inverted sets from 3 to $n$ contain on-shell propagators. For example, at three loops ($n = 4$), these convolutions are interpreted as

$$A_{\text{MLT}}^{(2)}(1, 2, 12) \otimes A_{\text{MLT}}^{(1)}(3, 4)$$

$$= \int_{\ell_1, \ell_2, \ell_3} \left( A_{D}^{(3)}(2, 12; 1, 3) + A_{D}^{(3)}(1, 12; 2, 3) + A_{D}^{(3)}(1, 2; 12, 3) + (1 \leftrightarrow 3) \right),$$

(13)

and

$$A_{\text{MLT}}^{(1)}(1, 2) \otimes A_{\text{MLT}}^{(0)}(12) \otimes A_{\text{MLT}}^{(2)}(3, 4)$$

$$= \int_{\ell_1, \ell_2, \ell_3} \left( A_{D}^{(3)}(2, 12; 1, 3) + A_{D}^{(3)}(1, 2; 12, 3) \right).$$

(14)

In total, the number of terms generated by Eq. (12) is $3n - 4$.

Causal thresholds and infrared singularities are then determined by the singular structure of the $A_{\text{MLT}}^{(2)}(1, 2, 12)$ subtopology, and by the singular configurations that split the NMLT topology into two disjoint tree amplitudes with all the on-shell momenta aligned over the causal cut. Again, the singular surfaces in the loop three-momenta space are limited by the external momenta, and all the non-causal singular configurations that arise in individual contributions undergo dual cancellations.

**NEXT-TO-NEXT-TO-MAXIMAL LOOP TOPOLOGY**

The last multiloop topology that we consider explicitly is the next-to-next-to-maximal loop topology (N$^2$MLT), shown in Fig. 3. At three loops, it corresponds to the so-called Mercedes-Benz topology. Besides the 12-set, there is another set denoted by 23 with $q_{23} = -\ell_2 - \ell_3 + k_{23}$. Its LTD representation is given by the following convolution of factorized subtopologies

$$A_{\text{NMLT}}^{(L)}(1, \ldots, n, 12, 23)$$

$$= A_{\text{NMLT}}^{(3)}(1, 12, 23) \otimes A_{\text{MLT}}^{(L-3)}(4, \ldots, n)$$

$$+ A_{\text{MLT}}^{(2)}(1 \cup 23, 2, 3 \cup 12) \otimes A_{\text{MLT}}^{(L-2)}(4, \ldots, \pi).$$

(15)

The sets $(1, 2, 3, 12, 23)$ form a NMLT subtopology. Therefore, the first term on the r.h.s. of Eq. (15) is obtained iteratively from Eq. (12). Explicitly,

$$A_{\text{NMLT}}^{(3)}(1, 12, 23) = A_{\text{MLT}}^{(2)}(1, 12, 2) \otimes A_{\text{MLT}}^{(1)}(3, 23)$$

$$+ A_{\text{MLT}}^{(1)}(1, 12) \otimes A_{\text{MLT}}^{(0)}(2) \otimes A_{\text{MLT}}^{(2)}(3, 23).$$

(16)

The last term on the r.h.s. of Eq. (15) is fixed by the condition that the sets $(2, 3, 23)$ can not generate a disjoint subtree. The second term in the r.h.s. of Eq. (15) contains a two-loop subtopology made of five sets of momenta, $A_{\text{MLT}}^{(2)}(1 \cup 23, 2, 3 \cup 12)$, which is dualized through Eq. (8). The total number of terms generated by Eq. (15) is $8(n - 2)$. As for the NMLT, the causal singularities of the N$^3$MLT topology are determined by its subtopologies and by the singular configurations that split the open dual amplitude into disjoint trees with all the on-shell momenta aligned over the causal cut. Any other singular configuration is entangled among dual amplitudes and cancels.

![FIG. 3. Next-to-next-to-maximal loop topology (left) and its convoluted dual representation (right).](image-url)

Finally, let us comment on more complex topologies at higher orders. Consider for example the multiloop topology made of one MLT and two two-loop NMLT subtopologies. This case appears for the first time at four loops. This topology is open into non-disjoint trees by leaving three loop sets off-shell and by introducing on-shell conditions in the others under certain conditions; either one off-shell set in each subtopology, or two in one NMLT subtopology and one in the other with on-shell propagators in all the sets of the MLT subtopology. Once the loop amplitude is open into trees, the singular causal structure is determined by the causal singularities of its subtopologies, and all the entangled non-causal singularities of the forest cancel.

**CONCLUSIONS**

We have reformulated the loop-tree duality at higher orders and have obtained very compact open-into-tree analytical representations of selected loop topologies to all orders.
These loop-tree dual representations exhibit a factorized cascade form in terms of simpler subtopologies. Since this factorized structure is imposed by the opening into non-disjoint trees and by causality, we conjecture that it holds to all loop orders and topologies. Remarkably, specific multiloop configurations are described by extremely compact dual representations which are, moreover, free of unphysical singularities. We have tested this property with several topologies. Therefore, we also conjecture that analytic dual representations in terms of only causal denominators are always plausible.

The explicit expressions presented in this letter are sufficient to describe any scattering amplitude up to three loops. Other topologies that appear for the first time at four loops and beyond have been anticipated, and will be presented in a forthcoming publication. This reformulation allows for a direct and efficient application to physical scattering processes, and is also advantageous to unveil formal aspects of multiloop scattering amplitudes.

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