Regularity for fully nonlinear elliptic equations with oblique boundary conditions

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Abstract In this paper, we obtain a series of regularity results for viscosity solutions of fully nonlinear elliptic equations with oblique derivative boundary conditions. In particular, we derive the pointwise $C^{\alpha}$, $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity. As byproducts, we also prove the A-B-P maximum principle, Harnack inequality, uniqueness and solvability of the equations.

Keywords Oblique derivative problem · Fully nonlinear elliptic equations · Viscosity solutions · Boundary regularity

Mathematics Subject Classification (2000) 35J25 · 35B65 · 35J60 · 35D40

1 Introduction

We study the regularity of viscosity solutions to the following fully nonlinear elliptic equation with oblique boundary condition:

\[
\begin{aligned}
F(D^2 u) &= f & \text{in} & \Omega; \\
\beta \cdot Du + \gamma u &= g & \text{on} & \Gamma \subset \partial \Omega,
\end{aligned}
\]

(1.1)

where $\Omega \subset \mathbb{R}^n$ is a bounded domain; $\Gamma \in C^1$ and is relatively open to $\partial \Omega$; $f$ is a function defined in $\Omega$; and $\beta$ (vector valued), $\gamma$ and $g$ are functions defined...
on $\Gamma$. Here “Oblique” means that $|\beta \cdot n| \geq \delta_0 > 0$ on $\Gamma$, where $n$ denotes the inner normal of $\Omega$. We call $\Gamma$ the oblique boundary. Since the sign of $\gamma$ is not required, without loss of generality, throughout this paper, we always assume that

$$\beta \cdot n \geq \delta_0$$

for some positive constant $\delta_0$. In addition, we always assume that $F$ is uniformly elliptic, i.e., there exist positive constants $\lambda$ and $\Lambda$ such that

$$\lambda \|N\| \leq F(M + N) - F(M) \leq \Lambda \|N\|, \quad \forall M, N \in S^n, \quad N \geq 0,$$

where $S^n$ denotes the set of $n \times n$ symmetric matrices; $N \geq 0$ means the nonnegativeness and $\|N\|$ is the spectral radius of $N$. Since we deal with the viscosity solutions of (1.1), it is convenient for us to assume that $f, \beta, \gamma$ and $g$ are continuous throughout this paper.

Few regularity but some a priori estimates and existence results are known for (1.1), where $\Gamma = \partial \Omega$ and $\gamma \leq 0$ are needed. For linear equations, the existence of $C^{2,\alpha}$ solutions are obtained by Green’s representation, the method of freezing the coefficients and method of continuity (cf. [3, Chapter 6]). The work of Lieberman [8] covered above results and only required the $C^{1,\alpha}$ smoothness on $\partial \Omega$. Safonov [13, 14] extended this results to the nonlinear Bellman equations, that is, he proved the existence of $C^{2,\alpha}$ solutions as $\partial \Omega \in C^{1,\alpha}$ for (1.1). In 1982, Lieberman [7] obtained the existence of solutions for quasi-linear equations based on a fixed point theorem and the solvability for linear equations. By a priori estimates and the method of continuity, Lieberman and Trudinger [10] proved the existence and uniqueness of $C^{2,\alpha}$ solutions for fully nonlinear elliptic equations with nonlinear oblique boundary conditions as $\partial \Omega \in C^4$.

Since 1980s, the notion of viscosity solutions has been applied widely in the study of non-divergent equations especially of fully nonlinear elliptic equations; and some important interior regularity results and global results with Dirichlet boundary conditions have been obtained (see [1] and [2] and references therein). As for applications of viscosity solutions to oblique derivative problems, Ishii [4] obtained the existence and uniqueness of viscosity solutions for fully nonlinear elliptic equations. Milakis and Silvestre [12] proved the $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity results for fully nonlinear equations with Neumann boundary data on flat boundaries. For more a priori estimates of the oblique derivative problems, we refer the reader to the book [9].

In this paper, we derive a series of regularity results for viscosity solutions of (1.1). In particular, the $C^{\alpha}, C^{1,\alpha}$ and $C^{2,\alpha}$ boundary regularity are deduced:

**Theorem 1.1** ($C^{\alpha}$ regularity) **Let $u$ satisfy**

$$\begin{cases}
  u \in S(\lambda, \Lambda, f) & \text{in } \Omega; \\
  \beta \cdot Du + \gamma u = g & \text{on } \Gamma.
\end{cases}$$

**Then for any $\Omega' \subset \subset \Omega \cup \Gamma$, $u \in C^{\alpha}(\Omega')$ and**

$$\|u\|_{C^{\alpha}(\Omega')} \leq C \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Gamma)} \right),$$
where $0 < \alpha < 1$ depends only on $n$, $\lambda$, $\Lambda$ and $\delta_0$, and $C$ depends also on $\|\gamma\|_{L^\infty(\Gamma)}$, $\Omega'$ and $\Omega$.

**Theorem 1.2 ($C^{1,\alpha}$ regularity)** Let $u$ be a viscosity solution of (1.1) and $0 < \alpha < \bar{\alpha}$ where $0 < \bar{\alpha} < 1$ is a constant depending only on $n$, $\lambda$, $\Lambda$ and $\delta_0$. Suppose that $\beta, \gamma, g \in C^\alpha(\bar{\Gamma})$ and $f$ satisfies

$$
\left( \frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0) \cap \Omega} |f|^n \right)^{\frac{1}{n}} \leq C_f r^{\alpha - 1} \forall x_0 \in \Omega, \forall r > 0.
$$

Then for any $\Omega' \subset \subset \Omega \cup \Gamma$, $u \in C^{1,\alpha}(\Omega')$ and

$$
\|u\|_{C^{1,\alpha}(\Omega')} \leq C \left( \|u\|_{L^\infty(\Omega)} + C_f + \|g\|_{C^\alpha(\Gamma)} + |F(0)| \right), \tag{1.2}
$$

where $C$ depends only on $n$, $\lambda$, $\Lambda$, $\delta_0$, $\alpha$, $\|\beta\|_{C^\alpha(\Gamma)}$, $\|\gamma\|_{C^\alpha(\Gamma)}$, $\Omega'$ and $\Omega$.

**Theorem 1.3 ($C^{2,\alpha}$ regularity)** Let $F$ be convex, $u$ be a viscosity solution of (1.1) and $0 < \alpha < \bar{\alpha}$ where $0 < \bar{\alpha} < 1$ is a constant depending only on $n$, $\lambda$, $\Lambda$ and $\delta_0$. Suppose that $\Gamma \in C^{1,\alpha}$, $\beta, \gamma, g \in C^{1,\alpha}(\bar{\Gamma})$ and $f \in C^\alpha(\bar{\Omega})$.

Then for any $\Omega' \subset \subset \Omega \cup \Gamma$, $u \in C^{2,\alpha}(\Omega')$ and

$$
\|u\|_{C^{2,\alpha}(\Omega')} \leq C \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{1,\alpha}(\Gamma)} + |F(0)| \right),
$$

where $C$ depends only on $n$, $\lambda$, $\Lambda$, $\delta_0$, $\alpha$, $\|\beta\|_{C^{1,\alpha}(\Gamma)}$, $\|\gamma\|_{C^{1,\alpha}(\Gamma)}$, $\Omega'$ and $\Omega$.

The following is the outline of this paper. First, the Alexandrov-Bakel’man-Pucci type maximum principle and boundary Harnack type inequality are presented as basic tools. Then $C^{\alpha}$ regularity follows clearly. The $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity for viscosity solutions of (1.1) are obtained by approximating the original problem by model problems at each scales whose regularity can be dealt with and which should be designed properly. We point out here that we do not need flatten the boundary, which is different from the references mentioned above. This paper also contains uniqueness and existence results of (1.1), which will be used to solve our model problems.

The paper is organized in the following way. We introduce the Alexandrov-Bakel’man-Pucci (A-B-P for short) maximum principle and the boundary Harnack inequality in Section 2, which are the basic tools to attack the regularity for (1.1). Based the boundary Harnack inequality, the pointwise $C^{\alpha}$ regularity follows. The proof for the A-B-P maximum principle has been motivated by [12], where the authors deal with the Neumann problems. Combining the A-B-P maximum principle, the interior Harnack inequality and the barrier technique, we derive the boundary Harnack inequality. The barrier is adopted from [10].

A Jensen’s type uniqueness of viscosity solutions is presented in Section 3. That is, the subsolution minus the supersolution is also a “subsolution”. This leads to the uniqueness of viscosity solutions by combining with the A-B-P maximum principle, where a mixed boundary value problem is considered. In addition, we also prove the existence result by Perron’s method, which will
respectively, where higher regularities will be frequently used in this paper.

Based on the previous results, we show the pointwise $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity for the restriction solution. Finally, from the boundary $C^{2,\alpha}$ regularity of the restriction solution, we deduce the conclusion.

Before the end of this section, we introduce the following notations which will be frequently used in this paper.

**Notation 1.1**

1. \(\{e_i\}_{i=1}^n\): the standard basis of \(\mathbb{R}^n\), i.e., \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0)\).
2. Given \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\), we may rewrite \(x = (x', x_n)\) where \(x' = (x_1, x_2, \ldots, x_{n-1})\).
3. \(S^n\): the set of \(n \times n\) symmetric matrices and \(\|A\| := \text{the spectral radius of } A\) for any \(A \in S^n\).
4. \(R_n^+ := \{x \in \mathbb{R}^n | x_n > 0\}\).
5. Given \(\beta \in \mathbb{R}^n\) with \(\beta_n \neq 0\), denote by \(x_\beta = (x_{\beta,1}, x_{\beta,2}, \ldots, x_{\beta,n-1})\) the projection of \(x\) to the hyperplane \(\{x_n = 0\}\) along the direction \(\beta\), i.e., \(x_{\beta,i} = x_i - (\beta_i/\beta_n)x_n\) for \(i = 1, 2, \ldots, n-1\). Clearly, \(|x - (x_\beta',0)| = (|\beta|/|\beta_n|)|x_n|\).
6. \(B_r(x_0) := \{x \in \mathbb{R}^n | |x - x_0| < r\}\), \(B_r := B_r(0)\) and \(B^+_r(x_0) := B_r(x_0) \cap R_n^+\).
7. \(T_r(x_0) := \{(x',0) | x' \in \mathbb{R}^{n-1}, |x' - x_0'| < r\}\) and \(T_r := T_r(0)\).
8. \(B_{r,h} := B_R(- (R - h)e_n)\), where \(R\) satisfies \((R - h)^2 + r^2 = R^2\).
9. \(B^+_{r,h}\): the spherical cap with the base radius \(r\) and height \(h (h \leq r)\), i.e., \(B^+_{r,h} := B_{r,h} \cap R_n^+\).
10. \(A^c\) : the complement of \(A\), \(\bar{A}\) : the interior of \(A\) and \(\bar{A}^c\) : the closure of \(A\), \(\forall A \subset \mathbb{R}^n\).
11. \(\text{dist}(A,B) := \inf\{|x - y| | x \in A, y \in B\}, \forall A,B \subset \mathbb{R}^n\).
12. \(a^+ := \max\{0,a\}\) and \(a^- := -\min\{0,a\}\).
13. \(n(x_0)\): the inner normal of \(\Omega\) at \(x_0 \in \partial \Omega\).
14. \(I\) : the unit matrix.
15. Given a function \(\varphi\). We may use \(\varphi_i\) or \(D_i \varphi\) to denote \(\partial \varphi / \partial x_i\). Similarly, \(\varphi_{ij}\) and \(D_{ij} \varphi\) denote \(\partial^2 \varphi / \partial x_i \partial x_j\).
16. \(D_\varphi := (\varphi_1, \ldots, \varphi_n), D_\varphi := (\varphi_1, \ldots, \varphi_{n-1}), D^2 \varphi := (\varphi_{ij})_{n \times n}\) and \(D^2 \varphi := (\varphi_{ij})_{(n-1) \times (n-1)}\).
17. For \( k \geq 0 \), \( \varphi \) is called \( C^{k,\alpha} \) at \( x_0 \) if there exists a polynomial \( P \) of degree \( k \) such that \( |\varphi(x) - P(x)| \leq K|x - x_0|^{k+\alpha} \) for any \( x \). Then, \( D^\zeta \varphi(x_0) := D^\zeta P(x_0) \) where \( \zeta \) is a multi index, \( [\varphi]_{C^{k,\alpha}(x_0)} := K \) and \( \|\varphi\|_{C^{k,\alpha}(x_0)} := K + \sum_{|\zeta| \leq k} |D^\zeta \varphi(x_0)| \).

18. For viscosity solutions, we use the notations \( \bar{S}(\lambda, \Lambda, f) \), \( S(\lambda, \Lambda, f) \), \( M^+(M, \lambda, \Lambda) \), \( M^-(M, \lambda, \Lambda) \) etc. as in [1].

### 2 A-B-P maximum principle, Harnack inequality and \( C^\alpha \) regularity

In this section, we introduce some notations and present some elementary results concerning the viscosity solutions of oblique derivative problems. We say that \( u \) touches \( v \) by above at \( x_0 \) in \( \Omega \) if \( u \geq v \) in \( \Omega \) and \( u(x_0) = v(x_0) \).

Similarly, we have the definition for touching by below. Now, we give the definition of viscosity solutions.

**Definition 2.1** Let \( u \) be upper (lower) semicontinuous in \( \Omega \cup \Gamma \). We say that \( u \) is a viscosity subsolution (supersolution) of (1.1) if for any \( \varphi \in C^2(\Omega \cup \Gamma) \) touching \( u \) by above (below) at \( x_0 \) in \( \Omega \cup \Gamma \), we have that

\[
F(D^2\varphi(x_0)) \geq (\leq) f(x_0) \text{ if } x_0 \in \Omega
\]

and

\[
\beta(x_0) \cdot D\varphi(x_0) + \gamma(x_0)\varphi(x_0) \geq (\leq) g(x_0) \text{ if } x_0 \in \Gamma. \tag{2.1}
\]

If \( u \in C(\Omega \cup \Gamma) \) is both subsolution and supersolution, we call it a viscosity solution.

**Remark 2.1**

(i) We may write \( u \in USC(\Omega \cup \Gamma) \) (\( u \in LSC(\Omega \cup \Gamma) \)) for short if \( u \) is upper (lower) semicontinuous in \( \Omega \cup \Gamma \).

(ii) Touching in \( \Omega \cup \Gamma \) can be replaced by touching in a neighborhood of \( x_0 \) (see [1, Proposition 2.4]). In this case, \( \varphi \in C^2(\Omega \cup \Gamma) \) in the definition can be replaced by that \( \varphi \) is a paraboloid (a polynomial of degree 2).

(iii) The notion of viscosity solutions for oblique derivative problems was introduced first by P.-L. Lions [11].

Now we present a closedness result concerning the viscosity solutions.

**Proposition 2.1** Suppose that \( \Gamma \in C^2 \). Let \( \{u_m\} \) satisfy

\[
\begin{cases}
u_m \in S(\lambda, \Lambda, f) & \text{in } \Omega; \\
\beta \cdot Du_m + \gamma u_m = g & \text{on } \Gamma.
\end{cases}
\]

Suppose that \( u_m \to u \) uniformly in any compact subset of \( \Omega \cup \Gamma \). Then

\[
\begin{cases}
u \in S(\lambda, \Lambda, f) & \text{in } \Omega; \\
\beta \cdot Du + \gamma u = g & \text{on } \Gamma.
\end{cases}
\]
Proof $u \in S(\lambda, \Lambda, f)$ in $\Omega$ is classical (see Proposition 2.9 [1]). We only prove that $\beta \cdot Du + \gamma u \geq g$ on $\Gamma$ and the proof for the other direction is similar. Let $P$ be a paraboloid touching $u$ by above in a neighborhood of $x_0 \in \Gamma$. Without loss of generality, we assume that $x_0 = 0$ and $n(0) = e_n$. Let $\varphi$ denote the representation function of $\Gamma$ near 0 with $\varphi(0) = 0$ and $D\varphi(0) = 0$. We need to prove that $\beta(0) \cdot DP(0) + \gamma(0)P(0) \geq g(0)$. Suppose not. Then

$$P_n(0) < \frac{1}{\beta_n(0)} (g(0) - \gamma(0)P(0) - \beta'(0) \cdot D^2P(0)) = A. \quad (2.2)$$

By the continuity of $P_n$, there exist $\tau, \eta > 0$ such that

$$P_n(x) < A - \eta \quad \forall x \in B_\tau.$$ 

By Taylor’s formula, for any $x \in \bar{\Omega} \cap B_\tau$,

$$P(x) = P(x', \varphi(x')) + P_n(x', \varphi(x'))(x_n - \varphi(x')) + \frac{1}{2}P_n(x_n - \varphi(x'))^2 \leq P(x', \varphi(x')) + (A - \eta)(x_n - \varphi(x'))^2 + \frac{1}{2}P_n(x_n - \varphi(x'))^2.$$

By the boundedness of $P_n$, for any $N > 0$ there exists $\tau' > 0$ such that

$$P(x) \leq P(x', \varphi(x')) + (A - \eta/2)(x_n - \varphi(x')) - N(x_n - \varphi(x'))^2 \quad \forall x \in \bar{\Omega} \cap B_{\tau'}.$$ 

The constant $N$ is to be chosen later. Here, we choose $\tau'$ small enough such that $\bar{\Omega} \cap B_{\tau'} \subset \subset \bar{\Omega} \cup \Gamma$ and

$$\beta_n \geq \delta_0/2 \quad \text{on} \quad \Gamma \cap B_{\tau'}.$$ 

For $\varepsilon > 0$, let $\psi_\varepsilon(x) = P(x', \varphi(x')) + (A - \eta/2)(x_n - \varphi(x')) - N(x_n - \varphi(x'))^2 + \varepsilon |x|^2$. Then

$$\psi_\varepsilon \geq P \geq u \quad \text{in} \quad \bar{\Omega} \cap B_{\tau'}, \quad \text{and} \quad \psi_\varepsilon(0) = P(0) = u(0).$$

Since $u_m \to u$ uniformly in $\bar{\Omega} \cup \Gamma$, there exists $m_0$ large enough such that

$$\|u_{m_0} - u\|_{L^\infty(\bar{\Omega} \cap B_{\tau'})} \leq \frac{1}{2}(\varepsilon \tau')^2.$$ 

Then

$$|\psi_\varepsilon(0) - u_{m_0}(0)| \leq \frac{1}{2}(\varepsilon \tau')^2 \quad \text{and} \quad |\psi_\varepsilon - u_{m_0}| > \frac{1}{2}(\varepsilon \tau')^2 \quad \text{on} \quad \partial(\bar{\Omega} \cap B_{\tau'}) \setminus \Gamma.$$ 

Hence, $\psi_\varepsilon + c_0$ touches $u_{m_0}$ by above at some $\bar{x} \in \bar{\Omega} \cap B_{\tau'}$ for a proper $c_0$ with $|c_0| \leq (\varepsilon \tau')^2/2$.

By a calculation, the second derivatives of $\psi_\varepsilon$ are

$$\psi_{\varepsilon,ij} = \begin{cases} (-A + \eta/2 - 2N\varphi + 2N x_n)\varphi_{ij} - 2N\varphi_i \varphi_j + P_{ij} + P_n \varphi_{ij} + P_{\gamma n} \varphi_i + P_{\nu} \varphi_{ij} + 2\varepsilon, & i, j < n, \\ 2N\varphi_i, & i < n, j = n, \\ -2N + 2\varepsilon, & i = j = n. \end{cases}$$
Note that $\varphi(0) = 0$ and $D\varphi(0) = 0$. Then by choosing $N$ large enough and $\tau'$ small enough, we have
\[ M^+(D^2\psi) < -\|f\|_{L^\infty(\Omega)} \text{ in } \Omega \cap B_{\tau'}.
\]
Combining with $M^+(D^2u_m) \geq f$, we know that $\tilde{x} \notin \Omega$, i.e., $\tilde{x} \in \Gamma$. By the definition of viscosity solutions, we have
\[ \beta(\tilde{x}) \cdot D\psi_\varepsilon(\tilde{x}) + \gamma(\tilde{x})\psi_\varepsilon(\tilde{x}) \geq g(\tilde{x}).
\]
By a calculation (note that $\tilde{x}_n = \varphi(\tilde{x}')$),
\[ D\psi_\varepsilon = (A - \eta/2)(-D\varphi, 1) - 2\varepsilon\tilde{x} + (D_P + P_n D\varphi, 0) \text{ at } \tilde{x}.
\]
Hence, at $\tilde{x}$
\[ g \leq (\eta/2 - A)\beta' D\varphi + \beta_n(A - \eta/2) - 2\varepsilon\beta\tilde{x} + \beta'(D_P + P_n D\varphi) + \gamma u.
\]
Note that $D\varphi(0) = 0$ and $|\tilde{x}| \leq \sqrt{\varepsilon}\tau'$. Taking $\varepsilon$ small enough, we have
\[ g \leq \beta_n(A - \eta/4) + \beta' D_P + \gamma u \text{ at } \tilde{x}.
\]
By the definition of $A$,
\[ g(\tilde{x}) = \frac{\beta_n(\tilde{x})}{\beta_n(0)}(g(0) - \gamma(0)P(0) - \beta'(0) \cdot D_P(0)) - \beta_n(\tilde{x})\eta/4 + \beta'(\tilde{x}) \cdot D_P(\tilde{x}) + + \gamma(\tilde{x}) u(\tilde{x}).
\]
By the continuity of each ingredient in above equation and taking $\varepsilon$ small enough, we have
\[ -\beta_n(\tilde{x})\eta/8 \geq 0
\]
which is impossible by recalling (2.3). $\square$

Next, we intend to present an Alexandrov-Bakel’man-Pucci type maximum principle for oblique derivative problems following the idea of [12].

**Lemma 2.1** Let $\Omega \subset B_1$ and $u$ satisfy
\[
\begin{cases}
  u \in \overline{S}(\lambda, A, f) & \text{in } \Omega; \\
  \beta \cdot Du \leq g & \text{on } \Gamma.
\end{cases}
\]
Suppose that there exists a direction $\xi \in \partial B_1$ such that $\beta : \xi \geq \delta_1$ on $\Gamma$. Then
\[ \sup_{\Omega} u^- \leq \sup_{\partial\Omega \setminus \Gamma} u^- + C \max g^+ + C\|f^+\|_{L^\infty(\Omega \cap \{u=\Gamma_u\})},
\]
where $\Gamma_u$ is the convex envelop of $u$ in $\Omega$ and $C$ depends only on $n$, $\lambda$, $A$ and $\delta_1$. 

5. **Proof:** The proof follows the idea of [12] by considering the signed distance function and using the comparison principle. 

6. **Remark:** The assumption $\beta : \xi \geq \delta_1$ on $\Gamma$ ensures that the maximum principle holds in the direction $\xi$. 

7. **Application:** The maximum principle can be used to establish regularity results for fully nonlinear elliptic equations with oblique boundary conditions. 

8. **References:** [12] A. Bakel’man, The maximum principle in the theory of quasilinear equations, Izv. Akad. Nauk SSSR Ser. Mat., 1965. 


Proof Without loss of generality, let $\xi$ be $e_n$. We assume that $u \geq 0$ on $\partial \Omega \setminus \Gamma$. Otherwise, we may consider $\tilde{u} := u - \inf_{\partial \Omega \setminus \Gamma} u$. Let $M := \sup_{\Omega} u$ and

$$A := \left\{ A \in \mathbb{R}^n | A_n \delta_1 > 2 \max_{\Gamma} g, |A'| \leq \frac{\delta_1}{2} A_n, |A| \leq \frac{M}{4} \right\}. \quad (2.5)$$

For any $A \in A$, $A \cdot x + c_0$ touches $u$ by below at some $x_0 \in \bar{\Omega}$ for a proper $c_0 \in \mathbb{R}$. Since $|A| \leq M/4$ and $\Omega \subset B_1$, $x_0 \notin \partial \Omega \setminus \Gamma$. If $x_0 \in \Gamma$, then

$$g(x_0) \geq A_n \beta_n(x_0) + A' \cdot \beta'(x_0) \geq A_n \delta_1 - \frac{\delta_1}{2} A_n = \frac{\delta_1}{2} A_n > \max_{\Gamma} g,$$

which is a contradiction. Hence $x_0 \in \Omega$, i.e., $A \in \nabla \Gamma_u(\Omega)$.

If $\max_{\Gamma} g > \frac{\delta_1}{16} M$, then

$$\sup_{\Omega} u - M \leq \frac{16}{\delta_1} \max_{\Gamma} g. \quad (2.6)$$

Otherwise,

$$\left\{ A \in \mathbb{R}^n | A_n > \frac{M}{8}, |A'| \leq \frac{\delta_1}{2} A_n, |A| \leq \frac{M}{4} \right\} \subset A \subset \nabla \Gamma_u(\Omega).$$

From the proof of [1, Theorem 3.2], we have that $\Gamma_u \in C^{1,1}(\Omega)$ and

$$\|f^+\|_{L^n(\{u = \Gamma_u\})} \geq C|\nabla \Gamma_u(\Omega)|,$$

where $C$ depends only on $n$, $\lambda$ and $\Lambda$. Hence,

$$\|f^+\|_{L^n(\{u = \Gamma_u\})} \geq CM^n. \quad (2.7)$$

Combining (2.6) and (2.7), we conclude that

$$\sup_{\Omega} u - M \leq C \max_{\Gamma} g^+ + C\|f^+\|_{L^n(\{u = \Gamma_u\})}, \quad (2.8)$$

where $C$ depends only on $n$, $\lambda$, $\Lambda$ and $\delta_1$.

Now, the full version of A-B-P maximum principle follows easily:

**Theorem 2.1 (A-B-P)** Let $\Omega \subset B_1$ and $u$ satisfy

$$\begin{cases}
  u \in \bar{S}(\lambda, \Lambda, f) \quad \text{in} \; \Omega; \\
  \beta \cdot Du + \gamma u \leq g \quad \text{on} \; \Gamma.
\end{cases}$$

Suppose that $\gamma \leq 0$ on $\Gamma$ and there exists $\xi \in \partial B_1$ such that $\beta \cdot \xi \geq \delta_1$ on $\Gamma$. Then

$$\sup_{\Omega} u - \sup_{\partial \Omega \setminus \Gamma} u - C \max_{\Gamma} g^+ + C\|f^+\|_{L^n(\{u = \Gamma_u\})},$$

where $C$ depends only on $n$, $\lambda$, $\Lambda$ and $\delta_1$. 

\[ \Box \]
Proof Let \( v = \min\{u, 0\} \). Then (note that \( \gamma \leq 0 \) and \( v \leq 0 \))

\[
\begin{cases}
v \in S(\lambda, \Lambda, f) & \text{in } \Omega; \\
\beta \cdot Dv \leq g^+ & \text{on } \Gamma.
\end{cases}
\]

Hence, by Lemma 2.1, we have

\[
\sup_{\Omega} u^- = \sup_{\Omega} v^- \leq \sup_{\partial \Omega \setminus \Gamma} v^- + C \max_{\Gamma} g^+ + C \| f^+ \|_{L^\infty(\{v = v_\gamma\})} \]

\[
= \sup_{\partial \Omega \setminus \Gamma} u^- + C \max_{\Gamma} g^+ + C \| f^+ \|_{L^\infty(\{u = u_\gamma\})}.
\]

\[\Box\]

Remark 2.2 The hypotheses \( \beta \cdot n \geq \delta_0 \) and \( \beta \cdot \xi \geq \delta_1 \) on \( \Gamma \) imply that the A-B-P maximum principle holds when \( \Gamma \) is a “small” portion of \( \partial \Omega \) (see also [15]).

Through above A-B-P maximum principle, we obtain the boundary Harnack type inequality (see also [10]).

Theorem 2.2 (Boundary Harnack inequality) Let \( u \geq 0 \) satisfy

\[
\begin{cases}
u \in S(\lambda, \Lambda, f) & \text{in } \Omega; \\
\beta \cdot Du + \gamma u = g & \text{on } \Gamma.
\end{cases}
\]

Suppose that \( \gamma \leq 0 \) and \( \beta_n \geq \delta_1 \) on \( \Gamma \). In addition, assume that

\[
\Gamma = \{ (x', x_n) \in B_1 \mid x_n = \varphi(x') \}, \{ (x', x_n) \in B_1 \mid x_n > \varphi(x') \} \subset \Omega \quad (2.9)
\]

and \( \varphi \) satisfies

\[
\varphi(0) = 0 \quad \text{and} \quad D\varphi(0) = 0. \quad (2.10)
\]

Then there exist constants \( 0 < \rho < 1 \) and \( C \) depending only on \( n, \lambda, \Lambda, \delta_1 \) and \( \|\gamma\|_{L^\infty(\Gamma)} \), and a constant \( 0 < R_0 < 1 \) depending also on the \( C^1 \) continuity modulus of \( \Gamma \) such that for any \( R \leq R_0 \),

\[
\sup_{\tilde{G}(R)} u \leq C \left( \inf_{\tilde{G}(R/4)} u + R \| f \|_{L^\infty(\tilde{G})} + R \| g \|_{L^\infty(\Gamma)} \right), \quad (2.11)
\]

where \( \tilde{G}(R) := \{ x \in \Omega \mid |x'| < R, x_n < \rho R \} \) and \( \tilde{G}(R) := \{ x \in \Omega \mid |x'| < R, x_n < 3\rho R \} \).

Proof Take \( \rho \) small enough such that

\[
\rho < \frac{\delta_1}{16(1 + \|\gamma\|_{L^\infty(\Gamma)})}. \quad (2.12)
\]
and
\[ M^+ \left( \begin{pmatrix} 2I & 0 \\ 0 & -\frac{1}{2\rho^2} \end{pmatrix} \right) < -1. \]  
(2.13)

Since \( \varphi(0) = 0 \) and \( D\varphi(0) = 0 \), we take \( R_0 \) small enough such that
\[ |\varphi(x')| \leq \frac{\rho}{16} |x'| \quad \forall |x'| \leq R_0. \]  
(2.14)

Then for any \( R \leq R_0/2 \), \( \text{dist}(\tilde{G}(R), \partial \Omega) \geq CR \). By the interior Harnack inequality (see [1, Theorem 4.3] with a proper scaling)
\[ \sup_{\tilde{G}(R)} u \leq C \left( \inf_{\tilde{G}(R)} u + R\|f\|_{L^n(\Omega)} \right), \]
we only need to prove
\[ \inf_{\tilde{G}(R)} u \leq C \left( \inf_{\tilde{G}(R/4)} u + R\|f\|_{L^n(\Omega)} + R\|g\|_{L^\infty(\Gamma)} \right). \]  
(2.15)

Let \( A := \inf_{\tilde{G}(R)} u \). Set
\[ w_1(x) = 2\rho R - x_n, w_2(x) = 2 - \left( \frac{x_n}{2\rho R} \right)^2 - \frac{x_n^2}{2\rho R} + \frac{|x'|^2}{R^2} \]
and
\[ w(x) = u(x) + \frac{1}{\delta_1} \|g\|_{L^\infty(\Gamma)} w_1(x) + \frac{1}{4} Aw_2(x) - A. \]

Combining (2.12), (2.13), (2.14) with the definition of \( A \), it is easy to verify that
\[ \begin{cases} 
  w \in \tilde{S}(\lambda/n, A, f) & \text{in } G(2R); \\
  w \geq 0 & \text{on } \partial G(2R) \setminus \Gamma; \\
  \beta \cdot Dw + \gamma w \leq 0 & \text{on } \Gamma.
\end{cases} \]

By the A-B-P maximum principle,
\[ w \geq -CR\|f\|_{L^n(\Omega)}, \]
i.e.,
\[ u + CR\|f\|_{L^n(\Omega)} + CR\|g\|_{L^\infty(\Gamma)} \geq A - \frac{1}{4} Aw_2 \quad \text{in } G(2R). \]

Then
\[ u + CR\|f\|_{L^n(\Omega)} + CR\|g\|_{L^\infty(\Gamma)} \geq \frac{A}{4} \quad \text{in } G(R/4), \]
which is (2.15). \( \square \)

Based on above boundary Harnack inequality, the boundary pointwise Hölder estimate follows standardly (see [3, Theorem 8.22 and Theorem 9.31]):
Lemma 2.2 Let $u$ satisfy

$$\begin{aligned}
    u &\in S(\lambda, A, f) \quad \text{in } \Omega; \\
    \beta \cdot Du &= g \quad \text{on } \Gamma
\end{aligned}$$

and $x_0 \in \Gamma$ such that $\text{dist}(x_0, \partial\Omega \setminus \Gamma) > 1$.

Then $u$ is $C^{\alpha_0}$ at $x_0$. Precisely, for any $r \leq \bar{C}^{-1}$,

$$\|u - u(x_0)\|_{L^\infty(\partial\Omega \cap B_r(x_0))} \leq C \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)} + \|g\|_{L^\infty(\Gamma)} \right) r^{\alpha_0}, \quad (2.16)$$

where $0 < \alpha_0 < 1$ and $C$ depend only on $n, \lambda, A$ and $\delta_0$, and $\bar{C}$ depends also on the $C^1$ modulus of $\Gamma$ at $x_0$.

Remark 2.3 The condition $\text{dist}(x_0, \partial\Omega \setminus \Gamma) > 1$ is not an essential assumption and “1” can be replaced by any positive constant. Then, we obtain the scaling version of (2.16).

Then we have the following pointwise $C^\alpha$ estimate:

Theorem 2.3 Let $u$ satisfy

$$\begin{aligned}
    u &\in S(\lambda, A, f) \quad \text{in } \Omega; \\
    \beta \cdot Du + \gamma u &= g \quad \text{on } \Gamma
\end{aligned}$$

and $x_0 \in \Gamma$ such that $\text{dist}(x_0, \partial\Omega \setminus \Gamma) > 1$.

Then $u$ is $C^{\alpha_0}$ at $x_0$. Precisely, for any $r \leq \bar{C}^{-1}$,

$$\|u - u(x_0)\|_{L^\infty(\partial\Omega \cap B_r(x_0))} \leq C \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)} + \|g\|_{L^\infty(\Gamma)} \right) r^{\alpha_0}, \quad (2.17)$$

where $0 < \alpha_0 < 1$ depends only on $n, \lambda, A$ and $\delta_0$; $C$ depends also on $\|\gamma\|_{L^\infty(\Gamma)}$ and $\bar{C}^{-1}$ depends also on the $C^1$ modulus of $\Gamma$ at $x_0$.

Proof Rewrite the equations as

$$\begin{aligned}
    u &\in S(\lambda, A, f) \quad \text{in } \Omega; \\
    \beta \cdot Du &= g - \gamma u \quad \text{on } \Gamma.
\end{aligned}$$

Then from Lemma 2.2, $u$ is $C^{\alpha_0}$ at $x_0$ and (2.17) holds. $\square$

Combining Theorem 2.3 with the interior Hölder estimate, the boundary local Hölder estimate (Theorem 1.1) follows easily (see [1, Proposition 4.10]).
3 Uniqueness and existence of viscosity solutions

In this section, we derive the Jensen’s type uniqueness of viscosity solutions which will be also used to prove the $C^{1,\alpha}$ regularity in a spherical cap in next section. That is, the subsolution minus the supersolution is also a “subsolution”. This leads to the uniqueness of viscosity solutions by combining with the A-B-P maximum principle. In addition, we prove an existence result which will be also used to construct auxiliary functions in later sections. The results of this section have been motivated by [2] and [4].

We start with the following notations (see [2] or [4]). For $u$ defined on $\overline{\Omega}$ and $x_0 \in \overline{\Omega}$, let

\[ J^{2,+}u(x_0) := \left\{ (p, A) \in \mathbb{R}^n \times \mathbb{S}^n \mid u(x+h) \leq u(x) + p \cdot h + \frac{1}{2} h^T A h + o(|h|^2) \right\} \]

for $x+h \in \overline{\Omega}$ as $h \to 0$

and

\[ J^{2,-}u(x_0) := \left\{ (p, A) \in \mathbb{R}^n \times \mathbb{S}^n \mid u(x+h) \geq u(x) + p \cdot h + \frac{1}{2} h^T A h + o(|h|^2) \right\} \]

for $x+h \in \overline{\Omega}$ as $h \to 0$.

We also define the following:

\[ \overline{J}^{2,+}u(x_0) := \left\{ (p, A) \mid \text{there exist a sequence } (x_m, p_m, A_m) \text{ such that} \right\} \]

\[ (p_m, A_m) \in J^{2,+}u(x_m), x_m \to x_0, p_m \to p \text{ and } A_m \to A \} . \]

$\overline{J}^{2,-}u(x_0)$ is defined similarly.

Upon above notations, we have the following observation:

**Proposition 3.1** Suppose that $\Gamma \in C^2$. Then $u$ is a viscosity subsolution of (1.1) if and only if

\[ F(A) \geq f(x_0) \quad \forall x_0 \in \Omega, \forall (p, A) \in \overline{J}^{2,+}u(x_0) \quad (3.1) \]

and

\[ \begin{cases} F(A) \geq f(x_0) \\ \text{or} \\ \beta(x_0) \cdot p + \gamma(x_0)u(x_0) \geq g(x_0) \end{cases} \quad \forall x_0 \in \Gamma, \forall (p, A) \in J^{2,+}u(x_0). \quad (3.2) \]

**Proof** The “only if” part is trivial and we only prove the “if” part. It is obvious that $F(D^2u(x_0)) \geq f(x_0)$ in the viscosity sense if $x_0 \in \Omega$. Hence, we only need
to consider the case \( x_0 \in \Gamma \), let \( P \) be a paraboloid touching \( u \) by above at \( x_0 \). We need to prove
\[
\beta(x_0) \cdot DP(x_0) + \gamma(x_0)P(x_0) \geq g(x_0).
\]

Suppose not. Without loss of generality, we assume that \( x_0 = 0 \) and \( n(0) = e_n \). Let \( \varphi \) denote the representation function of \( \Gamma \) near 0 with \( \varphi(0) = 0 \) and \( D\varphi(0) = 0 \). Then by an argument similar to the proof of Proposition 2.1, for any \( N > 0 \), there exist \( \eta, \tau' > 0 \) such that
\[
P(x) \leq P(x', \varphi(x')) + (A - \eta/2)(x_n - \varphi(x')) - N(x_n - \varphi(x'))^2 \quad \forall x \in \bar{\Omega} \cap B_{\tau'}.
\]
where \( A = (g(0) - \gamma(0)P(0) - \beta(0) \cdot D\varphi(0))/\beta_n(0) \). Here, we choose \( \tau' \) small enough such that \( \Omega \cap B_{\tau'} \subset \subset \Omega \cup \Gamma \) and \( \beta_n \geq \delta_0/2 \) on \( \Gamma \cap B_{\tau'} \). For \( \varepsilon > 0 \), let \( \psi_{\varepsilon}(x) = P(x', \varphi(x')) + (A - \eta/2)(x_n - \varphi(x')) - N(x_n - \varphi(x'))^2 + \varepsilon|x|^2 \). Then
\[
(D\psi_{\varepsilon}(0), D^2\psi_{\varepsilon}(0)) \in J^{2,+}u(0).
\]
Hence, we have
\[
F(D^2\psi_{\varepsilon}(0)) \geq f(0)
\]
or
\[
\beta(0) \cdot D\psi_{\varepsilon}(0) + \gamma(0)\psi_{\varepsilon}(0) \geq g(0).
\]
Similar to the argument in the proof of Proposition 2.1, by choosing \( N \) large enough and \( \varepsilon \) small enough, we obtain a contradiction. \( \square \)

**Remark 3.1** In some context, that (3.1) and (3.2) hold is the definition of viscosity subsolution (cf. [2, Definition 7.4]). This proposition states that these two definitions are equivalent if \( \Gamma \in C^2 \).

We introduce the following two results (see [4, Theorem 4.1] and [4, Lemma 3.1]).

**Proposition 3.2** Let \( \Omega \) be a bounded domain, \( \partial \Omega \supset \Gamma \in \mathcal{C}^2 \) be relatively open and \( \beta \in \mathcal{C}^2(\overline{\Gamma}) \) with \( \beta \cdot n \geq \delta_0 \) on \( \Gamma \). Given \( x_0 \in \Gamma \), there are positive constants \( r_0, C \) and a family of \( \{w_{\varepsilon}\}_{\varepsilon > 0} \) of \( C^{1,1} \) functions on \( B_{r_0}(x_0) \times B_{r_0}(x_0) \) such that for \( \varepsilon > 0 \) and \( x, y \in B_{r_0}(x_0) \),
\[
\begin{align*}
  w_\varepsilon(x, x) &\leq \varepsilon, \\
  w_\varepsilon(x, y) &\geq \frac{|x - y|^2}{8\varepsilon}, \\
  \beta(x) \cdot D_x w_\varepsilon(x, y) &\leq C \left( \frac{|x - y|^2}{\varepsilon} + \varepsilon \right) \quad \text{if} \quad x \in \Gamma \quad \text{and} \quad y \in \bar{\Omega}, \\
  \beta(y) \cdot D_y w_\varepsilon(x, y) &\leq C \left( \frac{|x - y|^2}{\varepsilon} + \varepsilon \right) \quad \text{if} \quad y \in \Gamma \quad \text{and} \quad x \in \bar{\Omega},
\end{align*}
\]

(3.3)
and
\[
\left( Dw_\varepsilon(x,y), \frac{C}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C \frac{|x-y|^2}{\varepsilon} + \varepsilon \right) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right) \in J^{2,+} w_\varepsilon(x,y).
\]

(3.4)

**Proposition 3.3** Let \( u \in USC(\Omega), v \in LSC(\Omega) \) and \( w(x,y) = u(x) - v(y) \) for \( x, y \in \Omega \). Assume that
\[
\left( p, q, \frac{\Gamma}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \nu \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right) \in J^{2,+} w(\tilde{x}, \tilde{y})
\]
for some \( p, q \in \mathbb{R}^n, \tilde{x}, \tilde{y} \in \Omega, \tau > 0 \) and \( \nu > 0 \). Then there are \( X, Y \in \mathbb{R}^n \) for which
\[
-3\tau \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} X - \nu I & 0 \\ 0 & Y - \nu I \end{pmatrix} \leq 3\tau \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}
\]
and
\[
(p, X) \in J^{2,+} u(\tilde{x}) \quad \text{and} \quad (-q, Y) \in J^{2,-} v(\tilde{y}).
\]

Combining above two results and the doubling variable arguments (see [2, Section 3]), we prove a Jensen’s type uniqueness result.

**Theorem 3.1** Suppose that \( \Gamma \in C^2 \) and \( \beta \in C^2(\hat{\Gamma}) \). Let \( u \) and \( v \) satisfy
\[
\begin{cases}
F(D^2u) \geq f_1 & \text{in } \Omega; \\
\beta \cdot Du + \gamma u \geq g_1 & \text{on } \Gamma
\end{cases}
\]
and
\[
\begin{cases}
F(D^2v) \leq f_2 & \text{in } \Omega; \\
\beta \cdot Dv + \gamma v \leq g_2 & \text{on } \Gamma.
\end{cases}
\]

Then
\[
\begin{cases}
u \in S(\lambda/n, A, f_1 - f_2) & \text{in } \Omega; \\
\beta \cdot D(u - v) + \gamma (u - v) \geq g_1 - g_2 & \text{on } \Gamma.
\end{cases}
\]

**Proof** Let \( P \) be a paraboloid touching \( u - v \) by above at \( x_0 \in \Omega \). We need to prove
\[
M^+(D^2P, \lambda/n, A) \geq f_1(x_0) - f_2(x_0).
\]

Let \( P_{x_0} = P + \varepsilon_0 |x - x_0|^2, w(x,y) = u(x) - v(y) - P_{x_0}(y), \varphi_{\alpha}(x,y) = \alpha |x-y|^2/2 \) (\( \alpha > 0 \)) and \( (x_\alpha, y_\alpha) \) be a maximum point of \( w - \varphi_{\alpha} \). Then
\[
\left( \alpha(x_\alpha - y_\alpha), \alpha(y_\alpha - x_\alpha), \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \right) \in J^{2,+} w(x_\alpha, y_\alpha).
\]

Applying Proposition 3.3 with \( \tau = \alpha \) and \( \nu = 0 \), we conclude that there exist \( X_\alpha, Y_\alpha \in \mathbb{R}^n \) such that
\[
(\alpha(x_\alpha - y_\alpha), X_\alpha) \in J^{2,+} u(x_\alpha), \quad (\alpha(x_\alpha - y_\alpha), Y_\alpha) \in J^{2,-}(v + P_{x_0})(y_\alpha)
\]
Recall that $x_0$ is the unique maximum point of $u - v - P_{x_0}$. Then it is easy to verify that $x_0 \to x_0$ and $y_0 \to x_0$ as $\alpha \to +\infty$ (see [2, Lemma 3.1]). This implies that $x_0, y_0 \in \Omega$ for $\alpha$ large enough. Since $F(D^2u) \geq f_1$ and $F(D^2v) \leq f_2$, we have that $F(A) \geq f_1(x_0)$ and $F(Y_0 - D^2P - 2\varepsilon_0 I) \leq f_2(y_0)$. Note that (3.6) implies $X_0 \leq Y_0$. Hence,

$$f_2(y_0) \geq F(Y_0 - D^2P - 2\varepsilon_0 I) \geq F(Y_0 - M^+(D^2P, \lambda/n, A) - 2\alpha \varepsilon_0) \geq F(X_0) - M^+(D^2P, \lambda/n, A) - 2\alpha \varepsilon_0 \geq f_1(x_0) - M^+(D^2P, \lambda/n, A) - 2\alpha \varepsilon_0. $$

Let $\alpha \to \infty$ and it follows that

$$M^+(D^2P, \lambda/n, A) \geq f_1(x_0) - f_2(x_0) - 2\alpha \varepsilon_0. $$

Next, let $\varepsilon_0 \to 0$ and (3.5) follows.

In the following, we consider the boundary case. Let $P$ be a paraboloid touching $u - v$ by above at $x_0 \in \Gamma$. Without loss of generality, we assume that $x_0 = 0$ and $n(0) = e_n$. Let $\varphi$ denote the representation function of $\Gamma$ near 0 with $\varphi(0) = 0$ and $D\varphi(0) = 0$. We need to prove

$$\beta(0) \cdot DP(0) + \gamma(0)P(0) \geq g_1(0) - g_2(0). \tag{3.7}$$

Suppose not. Then

$$P_n(0) < (g_1(0) - g_2(0) - \gamma(0)P(0) - \beta(0) \cdot DP(0)) / \beta_n(0) := A_1 - A_2, $$

where $A_1 = (g_1(0) - \gamma(0)u(0)) / \beta_n(0)$ and $A_2 = (g_2(0) - \gamma(0)v(0) + \beta(0) \cdot DP(0)) / \beta_n(0)$. By the continuity of $P_n$, there exist $\tau > 0$ and $\eta > 0$ such that

$$P_n(x) < A_1 - A_2 - 3\eta \quad \forall x \in \Gamma \cap B_\tau. $$

By Taylor’s formula, for any $x \in \bar{\Omega} \cap B_\tau$,

$$P(x) = P(x', \varphi(x')) + P_n(x', \varphi(x'))(x_n - \varphi(x')) + \frac{1}{2}P_{nn}(x_n - \varphi(x'))^2 \leq P(x', \varphi(x')) + (A_1 - A_2 - 3\eta)(x_n - \varphi(x')) + \frac{1}{2}P_{nn}(x_n - \varphi(x'))^2. $$

By the boundedness of $P_{nn}$, for any $N > 0$ there exists $\tau' > 0$ such that

$$P(x) \leq P(x', \varphi(x')) + (A_1 - A_2 - 2\eta)(x_n - \varphi(x')) - 2N(x_n - \varphi(x'))^2 \quad \forall x \in \bar{\Omega} \cap B_{\tau'}. $$

The constant $N$ is large enough to be chosen later. We choose $\tau' < r_0$ small enough such that $\bar{\Omega} \cap B_{\tau'} \subset \Omega \cup \Gamma$ and $\beta_n \geq \delta_0/2$ on $\Gamma \cap B_{\tau'}$ where $r_0$ is as in Proposition 3.2.
Let \( \psi(x) = (A_1 - \eta)(x_n - \varphi(x')) - N(x_n - \varphi(x'))^2 \). For \( \epsilon_1 > 0 \), let \( \psi_{\epsilon_1}(x) = P'(x', \varphi(x')) - (A_2 + \eta)(x_n - \varphi(x')) - N(x_n - \varphi(x'))^2 + \epsilon_1|x|^2 \). Then

\[
\psi + \psi_{\epsilon_1} > P \geq u - v \text{ in } \Omega \cap B_{\tau} \setminus \{0\} \text{ and } \psi + \psi_{\epsilon_1} = P = u - v \text{ at } 0.
\]

Next, for \( \theta > 0 \), let

\[
\tilde{u}(x) = u(x) + \theta^2 x_n - \theta |x|^2 - \psi(x)
\]

and

\[
\tilde{v}(x) = v(x) - \theta^2 x_n + \psi_{\epsilon_1}(x).
\]

Finally, for \( \epsilon > 0 \), let

\[
\Phi_{\epsilon}(x, y) = \tilde{u}(x) - \tilde{v}(y) - w_{\epsilon}(x, y),
\]

where \( w_{\epsilon}(x, y) \) is as in Proposition 3.2. Then by (3.3),

\[
-\epsilon \leq \Phi_{\epsilon}(0, 0) \leq \Phi_{\epsilon}(x_{\epsilon}, y_{\epsilon}) = u(x_{\epsilon}) + \theta^2 x_{\epsilon,n} - \theta |x_{\epsilon}|^2 - \psi(x_{\epsilon}) - (v(y_{\epsilon}) - \theta^2 y_{\epsilon,n} + \psi_{\epsilon_1}(y_{\epsilon})) - w_{\epsilon}(x_{\epsilon}, y_{\epsilon}) \leq u(x_{\epsilon}) - v(y_{\epsilon}) - \theta |x_{\epsilon} - \theta \epsilon_n|^2 + \theta^3 - \theta^2(x_{\epsilon} - y_{\epsilon}) \cdot \epsilon_n - \frac{|x_{\epsilon} - y_{\epsilon}|^2}{8\epsilon} - \psi(x_{\epsilon}) - \psi_{\epsilon_1}(y_{\epsilon}),
\]

where \( (x_{\epsilon}, y_{\epsilon}) \) is a maximum point of \( \Phi_{\epsilon} \). It follows that for \( \epsilon_1 \) and \( \theta \) fixed,

\[
|x_{\epsilon} - y_{\epsilon}| \rightarrow 0 \text{ as } \epsilon \rightarrow 0
\]

and

\[
\lim_{\epsilon \rightarrow 0} \frac{|x_{\epsilon} - y_{\epsilon}|^2}{8\epsilon} \leq \theta^3, \quad x_{\epsilon}, y_{\epsilon} \rightarrow B_{\theta \epsilon_n} \text{ as } \epsilon \rightarrow 0.
\]

We choose \( \epsilon \) and \( \theta \) small enough such that \( x_{\epsilon}, y_{\epsilon} \in \hat{\Omega} \cap B_{\tau'} \).

Since \( (x_{\epsilon}, y_{\epsilon}) \) is a maximum point of \( \Phi_{\epsilon} \), we have

\[
\tilde{u}(x_{\epsilon} - \tilde{v}(y_{\epsilon}) \leq w_{\epsilon}(x_{\epsilon}, y_{\epsilon}) + \tilde{u}(x_{\epsilon}) - \tilde{v}(y_{\epsilon}) - w_{\epsilon}(x_{\epsilon}, y_{\epsilon}) \forall x, y \in \hat{\Omega} \cap B_{\tau'}.
\]

Using (3.4) in Proposition 3.2, we have

\[
\left( (p, q), \frac{C}{\epsilon} \begin{pmatrix} 1 & -I \\ -I & I \end{pmatrix} + Cs \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right) \in J^2_{\epsilon} \tilde{w}(x_{\epsilon}, y_{\epsilon}),
\]

where \( \tilde{w}(x, y) = \tilde{u}(x) - \tilde{v}(y), p = D_{x}w_{\epsilon}(x_{\epsilon}, y_{\epsilon}), q = D_{y}w_{\epsilon}(x_{\epsilon}, y_{\epsilon}) \) and \( s = (|x_{\epsilon} - y_{\epsilon}|^2/\epsilon + \epsilon) \). By Proposition 3.3, there are \( X, Y \in S^n \) such that

\[
\left( X - CsI, 0 \right) \leq \frac{3C}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}
\]

and

\[
(p, X) \in J^2_{\epsilon} \tilde{w}(x_{\epsilon}) \text{ and } (-q, -Y) \in J^2_{\epsilon} \tilde{v}(y_{\epsilon}).
\]
By the definition of \( \bar{u}, \bar{v}, \psi \) and \( \psi_{\varepsilon_1} \), we have

\[
(\bar{p}, X + 2\theta I + \bar{M}) \in \bar{J}^{2,+}u(x_{\varepsilon}) \tag{3.10}
\]

and

\[
(\bar{q}, -Y - \bar{M} - 2\varepsilon_1 I) \in \bar{J}^{2,-}v(y_{\varepsilon}) \tag{3.11}
\]

where

\[
\bar{p} = p - \theta^2 e_n + 2\theta x_{\varepsilon} + \left((A_1 - \eta) - 2N(x_{\varepsilon,n} - \varphi(x'_{\varepsilon}))\right)(-D\varphi(x'_{\varepsilon}), 1),
\]

\[
\bar{q} = - q + \theta^2 e_n + \left((A_2 + \eta) + 2N(y_{\varepsilon,n} - \varphi(y'_{\varepsilon}))\right)(-D\varphi(y'_{\varepsilon}), 1) - 2\varepsilon_1 y_{\varepsilon}
\]

\[\begin{align*}
\bar{M}_{ij} &= \begin{cases} 
- A_1 + \eta - 2N\varphi + 2Nx_{\varepsilon,n}\varphi_{ij} - 2N\varphi_i\varphi_j, & i, j < n, \\
2N\varphi_i, & i < n, j = n, \\
- 2N, & i = j = n.
\end{cases}
\end{align*}\]

and

\[
\tilde{M}_{ij} = \begin{cases} 
(A_2 + \eta - 2N\varphi + 2Ny_{\varepsilon,n})\varphi_{ij} - 2N\varphi_i\varphi_j + P_{ij}, & i, j < n, \\
+ P_{in}\varphi_j + P_{jn}\varphi_i + P_{nn}\varphi_{ij}, & i < n, j = n, \\
2N\varphi_i, & i = j = n.
\end{cases}
\]

If \( x_{\varepsilon} \in \Gamma \), by (3.10) and the definition of viscosity solutions,

\[
\beta(x_{\varepsilon}) \cdot \bar{p} + \gamma(x_{\varepsilon})u(x_{\varepsilon}) \geq g_1(x_{\varepsilon}), \tag{3.12}
\]

i.e.,

\[
\begin{align*}
\beta(x_{\varepsilon}) \cdot p - \theta^2 \beta_n(x_{\varepsilon}) + 2\theta \beta(x_{\varepsilon}) \cdot x_{\varepsilon} \\
+ \left((A_1 - \eta) - 2N(x_{\varepsilon,n} - \varphi(x'_{\varepsilon}))\right)(-\beta'(x_{\varepsilon}) \cdot D\varphi(x'_{\varepsilon}) + \beta_n(x_{\varepsilon})) \\
+ \gamma(x_{\varepsilon})u(x_{\varepsilon}) \geq g_1(x_{\varepsilon}).
\end{align*}
\]

By (3.3) and (3.8),

\[
\beta(x_{\varepsilon}) \cdot p \leq C \left( \frac{|x_{\varepsilon} - y_{\varepsilon}|^2}{\varepsilon} + \varepsilon \right) \leq C(\theta^3 + \varepsilon)
\]

if \( \varepsilon \) and \( \theta \) are small. On the other hand, since \( \varphi(0) = 0, D\varphi(0) = 0, \beta_n(0) \geq \delta_0 \) and \( x_{\varepsilon} \to 0 \) as \( \varepsilon, \theta \to 0 \) (recall (3.8)), by choosing \( \varepsilon \) and \( \theta \) small enough, we have (for \( N \) fixed)

\[
(A_1 - \eta/2)\beta_n(x_{\varepsilon}) + \gamma(x_{\varepsilon})u(x_{\varepsilon}) \geq g_1(x_{\varepsilon}).
\]
Since $A_1 = (g_1(0) - \gamma(0)u(0))/\beta_4(0)$, by choosing $\varepsilon$ and $\theta$ small enough and the continuity of $u$, $\beta$, $\gamma$ and $g_1$, we have

$$-\eta \delta_0/4 \geq 0,$$

which is a contradiction. Thus, $x_\varepsilon \in \Omega$ and

$$F(X + 2\theta I + \bar{M}) \geq f_1(x_\varepsilon). \quad (3.13)$$

By a similar argument for $y_\varepsilon$, we have that $y_\varepsilon \in \Omega$ and

$$F(-Y - \bar{M} - 2\varepsilon I) \leq f_2(y_\varepsilon). \quad (3.14)$$

Note that (3.9) implies,

$$X - CsI \leq -Y + CsI, \quad \text{i.e., } X - 2CsI \leq -Y.$$

Then from (3.13) and (3.14),

$$f_2(y_\varepsilon) \geq F(-Y - \bar{M} - 2\varepsilon I)$$

$$\geq F(-Y) - M^+(\bar{M}, \lambda/n, A) - 2nA\varepsilon_1$$

$$\geq F(X - 2CsI) - M^+(\bar{M}, \lambda/n, A) - 2nA\varepsilon_1$$

$$\geq F(X) - M^+(\bar{M}, \lambda/n, A) - 2nA\varepsilon_1 - 2nACs$$

$$\geq f_1(x_\varepsilon) - M^+(\bar{M}, \lambda/n, A) - M^+(\bar{M}, \lambda/n, A) - 2nA(\varepsilon_1 + Cs + \theta).$$

Recall the definitions of $\bar{M}$ and $\bar{M}$. By choosing $N$ large enough (independent of $\theta$ and $\varepsilon$), and $\theta$ and $\varepsilon$ small enough, we obtain a contradiction (note that $\varphi(0) = 0$, $D\varphi(0) = 0$ and $x_\varepsilon, y_\varepsilon \to 0$ as $\varepsilon, \theta \to 0$). Therefore, (3.7) holds.  \[ \square \]

**Remark 3.2** If $u$ or $v$ belongs to $C^1$, the conditions $\Gamma \in C^2$ and $\beta \in C^2(\bar{\Gamma})$ can be removed; and $\beta \cdot D(u - v) + \gamma(u - v) \geq g_1 - g_2$ on $\Gamma$ can be verified directly by the definition of viscosity solutions.

Combining above theorem with the A-B-P maximum principle, we derive a uniqueness result:

**Theorem 3.2** Let $\Gamma \in C^2$, $\beta \in C^2(\bar{\Gamma})$, $\gamma \leq 0$ and $\varphi \in C(\partial \Omega \setminus \Gamma)$. Suppose that there exists $\xi \in \partial B_1$ such that

$$\beta \cdot \xi \geq \delta_1 \text{ on } \Gamma.$$

Then there exists at most one viscosity solution of

$$\begin{cases}
F(D^2u) = f & \text{in } \Omega; \\
\beta \cdot Du + \gamma u = g & \text{on } \Gamma; \\
u = \varphi & \text{on } \partial \Omega \setminus \Gamma.
\end{cases}$$
Next, we use Perron’s method to prove an existence result for fully nonlinear elliptic equations with a “small” portion of oblique boundary. 

**Theorem 3.3** Let $Γ ∈ C^2$, $γ ≤ 0$ and $φ ∈ C(∂Ω \setminus Γ)$. Suppose that there exists $ξ ∈ ∂B_1$ such that 

$$\beta \cdot ξ ≥ δ_1 \text{ on } Γ.$$  

(3.15) 

In addition, suppose that $Ω$ satisfies an exterior cone condition at any $x ∈ ∂Ω \setminus Γ$ and satisfies an exterior sphere condition at any $x ∈ Γ \cap (∂Ω \setminus Γ )$.

Then there exists a unique viscosity solution $u ∈ C(Ω)$ of 

$$\begin{align*}
F(D^2u) &= f \quad \text{in } Ω; \\
\beta \cdot Du + γu &= g \quad \text{on } Γ; \\
u &= φ \quad \text{on } ∂Ω \setminus Γ.
\end{align*}$$  

(3.16) 

**Proof** We assume that $F(0) = 0$. Otherwise, by the uniform ellipticity, there exists $t ∈ R$ such that $F((t\delta_{nn}) = 0$ and $|t| ≤ |F(0)|/λ$. Let $G(M) := F(M + t\delta_{nn})$ and then $G(0) = 0$. Hence, $\tilde{u} + tx_n^2/2$ is the unique solution of (3.16) where $\tilde{u}$ is the unique solution of 

$$\begin{align*}
G(D^2\tilde{u}) &= f \quad \text{in } Ω; \\
β \cdot D\tilde{u} + γ\tilde{u} &= -tx_nβ_n - tx_n^2γ \quad \text{on } Γ; \\
\tilde{u} &= φ - tx_n^2/2 \quad \text{on } ∂Ω \setminus Γ.
\end{align*}$$ 

From now on, we assume that $F(0) = 0$. Let 

$$V := \{ v ∈ USC(Ω) \mid v \text{ is a subsolution of (3.16) with } v ≤ φ \text{ on } ∂Ω \setminus Γ \}. $$

By choosing proper positive constants $K_1$, $K_2$ and $K_3$, $K_1x_n^2 + K_2x_n - K_3 ∈ V$ and hence $V$ is nonempty. Set 

$$w(x) = \sup_{v ∈ V} v(x), w^*(x) = \lim_{y → x} w(y) \text{ and } w_*(x) = \lim_{y → x} w(y).$$

Then, $w^* ∈ USC(Ω)$, $w_* ∈ LSC(Ω)$ and $w_* ≤ w ≤ w^*$. Next, we divide the proof into three steps. 

**Step 1.** We prove that $w^*$ is a subsolution. 

For any $\tilde{x} ∈ Ω$ and paraboloid $P$ touching $w^*$ by above at $\tilde{x}$, we need to prove that $F(D^2P) ≥ f(\tilde{x})$. Suppose not. Then there exists $ε > 0$ such that 

$$F(D^2P + 2εI) < f(\tilde{x}) - 2ε.$$  

(3.17) 

By the definition of $w^*$, there exist $\{x_k\} ⊂ Ω$ and $\{v_k\} ⊂ V$ such that 

$$x_k → \tilde{x} \text{ and } w^*(\tilde{x}) = \lim_{k → ∞} v_k(x_k).$$

Then there exists $r > 0$, for $k$ large enough, such that $x_k ∈ B_{r/2}(\tilde{x}) ⊂ B_r(\tilde{x}) \subset Ω$, $|f(x) - f(\tilde{x})| ≤ ε$ for any $x ∈ B_r(\tilde{x})$ and 

$$|P(x_k) + ε|x_k - \tilde{x}|^2 - v_k(x_k)| \leq |P(x_k) - P(\tilde{x})| + ε|x_k - \tilde{x}|^2 + |P(\tilde{x}) - v_k(x_k)| < \frac{1}{2} r^2.$$
On the other hand, \( P + \varepsilon |x - \bar{x}|^2 - v_k \geq \varepsilon r^2 \) on \( \partial B_r(\bar{x}) \) (note that \( v_k \leq w \leq w^* \)). Hence, \( P + \varepsilon |x - \bar{x}|^2 + c_0 \) touches \( v_k \) by above at some \( x^* \in B_r(\bar{x}) \) for a proper constant \( c_0 \). Since \( v_k \) is a subsolution,

\[
F(D^2 P + 2\varepsilon I) \geq f(x^*) \geq f(\bar{x}) - \varepsilon,
\]

which contradicts with (3.17).

Next, we consider the case \( \bar{x} \in \Gamma \). Without loss of generality, we assume that \( \bar{x} = 0 \) and \( n(0) = e_r \). Let \( \varphi \) denote the representation function of \( \Gamma \) near 0 with \( \varphi(0) = 0 \) and \( D\varphi(0) = 0 \). We need to prove that \( \beta(0) \cdot DP(0) + \gamma(0)P(0) \geq g(0) \). Suppose not. Then

\[
P_n(0) < \frac{1}{\beta_n(0)}(g(0) - \gamma(0)P(0) - \beta'(0) \cdot D_nP(0)) := A. \tag{3.18}
\]

By the continuity of \( P_n \), there exist \( \tau, \eta > 0 \) such that

\[
P_n(x) < A - \eta \quad \forall x \in B_\tau.
\]

By Taylor’s formula, for any \( x \in \bar{\Omega} \cap B_\tau \),

\[
P(x) = P(x', \varphi(x')) + P_n(x', \varphi(x'))(x_n - \varphi(x')) + \frac{1}{2}P_{nn}(x_n - \varphi(x'))^2
\leq P(x', \varphi(x')) + (A - \eta)(x_n - \varphi(x')) + \frac{1}{2}P_{nn}(x_n - \varphi(x'))^2.
\]

By the boundedness of \( P_{nn} \), for any \( N > 0 \) there exists \( \tau' > 0 \) such that

\[
P(x) \leq P(x', \varphi(x')) + (A - \eta/2)(x_n - \varphi(x')) - N(x_n - \varphi(x'))^2 \quad \forall x \in \bar{\Omega} \cap B_{\tau'}.
\]

The constant \( N \) is large enough to be chosen later. Here, we choose \( \tau' \) small enough such that \( \Omega \cap B_{\tau'} \subset \subset \bar{\Omega} \cup \Gamma \) and \( \beta_n \geq \delta_0/2 \) on \( \Gamma \cap B_{\tau'} \). For \( \varepsilon > 0 \), let \( \psi_\varepsilon(x) = P(x', \varphi(x')) + (A - \eta/2)(x_n - \varphi(x')) - N(x_n - \varphi(x'))^2 + \varepsilon|x|^2 \). Then

\[
\psi_\varepsilon > P \geq w^* \quad \text{in} \quad \Omega \cap B_{\tau'} \quad \text{and} \quad \psi_\varepsilon(0) = P(0) = w^*(0).
\]

Similar to the interior case, there exist \( \{x_k\} \subset \bar{\Omega} \cup \Gamma \) and \( \{v_k\} \subset V \) such that

\[
x_k \to 0 \quad \text{and} \quad w^*(0) = \lim_{k \to \infty} v_k(x_k).
\]

Then for \( k \) large enough,

\[
x_k \in \bar{\Omega} \cap B_{\varepsilon \tau'/2} \quad \text{and} \quad \psi_\varepsilon - v_k \geq \varepsilon^3(\tau')^2 \quad \text{on} \quad \partial B_{\varepsilon \tau'/2} \cap \Omega
\]

and

\[
|v_k(x_k) - \psi_\varepsilon(x_k)| \leq |v_k(x_k) - \psi_\varepsilon(0)| + |\psi_\varepsilon(0) - \psi_\varepsilon(x_k)| \leq \frac{1}{2} \varepsilon^3(\tau')^2.
\]

Then \( \psi_\varepsilon + c_0 \) touches \( v_k \) by above at some \( x^* \in \bar{\Omega} \cap B_{\varepsilon \tau'} \) for a proper \( c_0 \). If \( x^* \in \Omega \), then

\[
F(D^2 \psi_\varepsilon(x^*)) \geq f(x^*)
\]
which is impossible by taking $N$ large enough as in Theorem 3.1. If $x^* \in \Gamma$, then
\begin{equation}
\beta(x^*) \cdot D\psi(x^*) + \gamma(x^*) \psi(x^*) \geq g(x^*). \tag{3.19}
\end{equation}
As in Theorem 3.1, by recalling the definition of $A$ (see (3.18)) and the continuity of the functions in (3.19), we obtain a contradiction.

From above arguments, we conclude that $w^*$ is a subsolution. Hence, $w^* \in V$ and it follows that $w^* \leq w$. Recall that $w^* \geq w$. Hence, $w = w^*$ is a subsolution.

**Step 2.** We prove that $w_*$ is a supersolution.

Suppose not. Then there exist $\tilde{x} \in \Omega \cup \Gamma$ and a paraboloid $P$ touching $w_*$ by below at $\tilde{x}$ such that

\[ F(D^2 P) > f(\tilde{x}) \text{ if } \tilde{x} \in \Omega \]

and
\[ \beta(\tilde{x}) \cdot DP(\tilde{x}) + \gamma(\tilde{x}) P(\tilde{x}) > g(\tilde{x}) \text{ if } \tilde{x} \in \Gamma. \]

If $\tilde{x} \in \Omega$, take $\varepsilon_0 > 0$ small such that
\[ F(D^2 P - 2\varepsilon_0 I) > f(\tilde{x}). \tag{3.20} \]
Take $r > 0$ small such that $B_r(\tilde{x}) \subset \subset \Omega$. Let
\[ \psi_{\varepsilon_0} = P - \varepsilon_0 |x - \tilde{x}|^2 + \frac{1}{2} \varepsilon_0 r^2. \]
Then
\[ w \geq w_* \geq \psi_{\varepsilon_0} \text{ on } \partial B_r(\tilde{x}) \text{ and } w_*(\tilde{x}) < \psi_{\varepsilon_0} (\tilde{x}), \]
which implies that there exists $x_1 \in B_r(\tilde{x})$ such that
\[ w(x_1) < \psi_{\varepsilon_0}(x_1). \tag{3.21} \]
Define
\[ \bar{w} = \begin{cases} 
\max(w, \psi_{\varepsilon_0}) & \text{if } x \in B_r(\tilde{x}); \\
w & \text{if } x \notin B_r(\tilde{x}). 
\end{cases} \]
It is easy to verify that $\bar{w}$ is a subsolution (recall that $w$ is a subsolution). Hence, $\bar{w} \leq w$. In particular, $\psi_{\varepsilon_0}(x_1) \leq \bar{w}(x_1) \leq w(x_1)$ which contradicts with (3.21).

If $\tilde{x} \in \Gamma$, similar to previous arguments, we assume that $\tilde{x} = 0$ and $n(0) = e_n$. Then, there exist $\eta, \tau' > 0$ such that
\[ P(x) > P(x', \varphi(x')) + (A + \eta/2)(x_n - \varphi(x')) + N(x_n - \varphi(x'))^2 \forall x \in \Omega \cap B_{\tau'}. \]
where $A := (g(0) - \gamma(0) P(0) - \beta'(0) \cdot D\psi(0)) / \beta_n(0)$. We choose $\tau'$ small enough such that $\Omega \cap B_{\tau'} \subset \subset \Omega \cup \Gamma$ and $\beta_n \geq \delta_0 / 2$ on $\Gamma \cap B_{\tau'}$. For $\varepsilon > 0$, let $\psi_{\varepsilon}(x) := P(x', \varphi(x')) + (A + \eta/2)(x_n - \varphi(x')) + N(x_n - \varphi(x'))^2 - \varepsilon |x|^2 - \varepsilon \tau'^2 / 2.$
By the definition of \( w_\ast \), and the continuity of \( \psi_\varepsilon \), there exists \( x^\ast \in \bar{\Omega} \cap B_{\tau'} \), such that
\[
 w(x^\ast) < w_\ast(0) + \frac{1}{4}\varepsilon \tau'^2 = \psi_\varepsilon(0) - \frac{1}{4}\varepsilon \tau'^2 \leq \psi_\varepsilon(x^\ast).
\] (3.22)

By taking \( \tau' \) small and \( N \) large (as in Theorem 3.1), we have
\[
 F(D^2\psi_\varepsilon) \geq f \text{ in } \Omega \cap B_{\tau'}.
\] (3.23)

Similarly, by the continuity of \( DP, \beta, \gamma \) and \( g \),
\[
 \beta \cdot D\psi_\varepsilon + \gamma \psi_\varepsilon \geq g \text{ on } \Gamma \cap B_{\tau'},
\] (3.24)
for \( \tau' \) and \( \varepsilon \) small enough.

Define
\[
 \bar{w} = \begin{cases} 
 \max(w, \psi_\varepsilon) & \text{if } x \in \bar{\Omega} \cap B_{\tau'}; \\
 w & \text{if } x \notin \bar{\Omega} \cap B_{\tau'}.
\end{cases}
\]

From (3.23) and (3.24), we have that \( \psi_\varepsilon \) is a subsolution in \( \bar{\Omega} \cap B_{\tau'} \). Note that
\[
 w \geq w_\ast \geq \psi_\varepsilon \text{ on } \partial B_{\tau'} \cap \Omega.
\]

Thus, \( \bar{w} \) is a subsolution of (3.16) and hence \( \bar{w} \in V \). This implies \( \bar{w} \leq w \) which contradicts with (3.22).

From above arguments, we conclude that \( w_\ast \) is a supersolution of (3.16).

**Step 3.** We construct the barriers on the Dirichlet boundary.

Recall (3.15). Without loss of generality, we assume that \( \xi = e_n \). Then \( \beta_n \geq \delta_1 \) on \( \Gamma \). Given \( x_1 \in \partial\Omega \setminus \bar{\Gamma} \), if \( x_1 \notin \bar{\Gamma} \), then there exists \( r > 0 \) such that \( B_r(x_1) \cap \Gamma = \emptyset \). Since \( \Omega \) satisfies the exterior cone condition at \( x_1 \), there exists a function \( v_1 \) such that \( v_1(x_1) = 0, v_1 > 0 \) in \( \Omega \setminus \{x_1\} \) and \( F(D^2v_1) \leq -1 \) in \( \Omega \cap B_r(x_1) \).

On the other hand, let \( v_2 = -K_1x_n^2 - K_2x_n + K_3 \). By choosing proper constants \( K_1, K_2 \) and \( K_3 \), we have
\[
\begin{cases} 
 F(D^2v_2) \leq -\|f\|_{L^\infty(\Omega)} & \text{in } \Omega; \\
 \beta \cdot Dv_2 + \gamma v_2 \leq -\|g\|_{L^\infty} - \|\gamma\|_{L^\infty}\|\varphi\|_{L^\infty} & \text{on } \Gamma; \\
 v_2 \geq 1 & \text{in } \Omega.
\end{cases}
\] (3.25)

Take \( K \) large enough such that \( Kv_1 > v_2 \) on \( \Gamma \) and let \( v = \inf\{Kv_1, v_2\} \). Then \( v \) is a supersolution of (3.16) with \( v(x_1) = 0 \) and \( v > 0 \) on \( \Omega \setminus x_1 \).

Since \( \varphi \) is continuous at \( x_1 \), for any \( \varepsilon > 0 \), there exists a constant \( k \) large enough such that
\[
 w_1 := \varphi(x_1) - \varepsilon - kv \leq \varphi(x) \leq \varphi(x_1) + \varepsilon + kv := w_2 \text{ on } \partial\Omega \setminus \Gamma.
\] (3.26)
Recall (3.25) and it is easy to verify that $w_1$ is a subsolution and $w_2$ is a supersolution. Hence, $w_1 \in V$ and then we have

$$w_1 \leq w_* \leq w \quad \text{on} \quad \Omega.$$ 

By the A-B-P maximum principle and the uniqueness result Theorem 3.1,

$$w = w^* \leq w_2 \quad \text{on} \quad \Omega.$$ 

Thus

$$w_1 \leq w_* \leq w = w^* \leq w_2 \quad \text{on} \quad \Omega.$$ 

Since $\varepsilon$ is arbitrary, it follows that $w_*(x_1) = w(x_1) = w^*(x_1) = \varphi(x_1)$.

Now, we consider the case $x_1 \in \bar{\Gamma}$. Since $\Omega$ satisfies the exterior sphere condition at $x_1$, let $B_{r_1}(y) \ (r_1 < 1)$ be the ball such that $B_{r_1}(y) \cap \Omega = \{x_1\}$. Let $v_3(x) = r_1^{-p} - |x - y|^{-p}$ and $v = \inf\{Kv_3, v_2\}$ ($v_2$ is as above). Similar to above arguments, we conclude that $w_*(x_1) = w(x_1) = w^*(x_1) = \varphi(x_1)$.

In consequence, we have

$$w_* = w = w^* \quad \text{on} \quad \partial\Omega \setminus \Gamma.$$ 

From the A-B-P maximum principle, we conclude that

$$w_* = w = w^* \quad \text{on} \quad \bar{\Omega}$$ 

and hence, $w \in C(\bar{\Omega})$ is a viscosity solution. \qed

As a special case of Theorem 3.3, the existence of viscosity solutions in a spherical cap is presented in the following. This existence will be used to construct auxiliary functions in later sections.

**Corollary 3.1** There exists a unique viscosity solution $u \in C(\bar{B}_{1,h_0}^+)$ of

$$\begin{cases}
F(D^2u) = 0 & \text{in} \quad B_{1,h_0}^+; \\
\beta \cdot Du = 0 & \text{on} \quad T_1; \\
u = \varphi & \text{on} \quad \partial B_{1,h_0}^+ \setminus T_1; 
\end{cases} \quad (3.27)$$

where $\beta \in C^2(\bar{T}_1)$, $\varphi \in C(\partial B_{1,h_0}^+ \setminus T_1)$ and $h_0 > 0$ is small enough such that

$$\beta(x) \cdot n(y) < 0 \quad \forall x \in T_1 \quad \text{and} \quad y \in \partial B_{1,h_0}^+ \setminus T_1. \quad (3.28)$$

Furthermore, if $\varphi \in C^\alpha(\partial B_{1,h_0}^+ \setminus T_1)$, then $u \in C^\alpha(\bar{B}_{1,h_0}^+)$ and

$$\|u\|_{C^\alpha(\partial B_{1,h_0}^+ \setminus T_1)} \leq C\|\varphi\|_{C^\alpha(\partial B_{1,h_0}^+ \setminus T_1)}, \quad (3.29)$$

where $\hat{\alpha} = \alpha/2$ and $C$ depends only on $n, \lambda, \Lambda, \delta_0$ and $h_0$.

**Remark 3.3** From now on, unless stated otherwise, $h_0$ is always chosen (depending only on $\delta_0$) such that (3.28) holds.
Proof We only need to prove the Hölder estimate. Given \( x_1 \in \partial B^+_1(h_0) \), \( \varphi \in C^\alpha(\partial B^+_1(h_0) \setminus T_1) \), then (recall (3.28)) 
\[
K\|\varphi\|_{C^K(\partial B^+_1(h_0) \setminus T_1)}(n(1) \cdot (x - x_1))^{\frac{\alpha}{2}} + \varphi(x_1) \text{ and } -K\|\varphi\|_{C^K(\partial B^+_1(h_0) \setminus T_1)}(n(x_1) \cdot (x - x_1))^{\frac{\alpha}{2}} + \varphi(x_1)
\]
are supersolution and subsolution respectively for large \( K \) depending only on \( n, \lambda, A \) and \( \delta_1 \).

From the A-B-P maximum principle, we obtain
\[
|u(x) - \varphi(x_1)| \leq K\|\varphi\|_{C^K(\partial B^+_1(h_0) \setminus T_1)}|n(x_1) \cdot (x - x_1)|^{\frac{\alpha}{2}} \leq K\|\varphi\|_{C^K(\partial B^+_1(h_0) \setminus T_1)}|x - x_1|^{\frac{\alpha}{2}},
\]
which implies the Hölder continuous up to the Dirichlet boundary. Combining with the Hölder continuous up to the oblique boundary (see Lemma 2.2), we obtain the full result (3.29). \( \square \)

4 Regularity for the model problem

In this section, we derive \( C^{1,\alpha} \) and \( C^{2,\alpha} \) regularity for the model problem, i.e.,
the homogenous equations in a spherical cap. In addition to the importance of themselves, these regularity will be used to approximate the solution of (1.1) in different scales and attack the regularity of the solution. Precisely, the model problem is
\[
\begin{align*}
\{ & F(D^2u) = 0 \quad \text{in } B^+_1(h_0); \\
& \beta \cdot Du = 0 \quad \text{on } T_1,
\end{align*}
\]
where \( \beta \) is a constant vector with \( \|\beta\| = 1 \).

From the uniqueness result obtained in last section, the following \( C^{1,\alpha} \) regularity for the model problem is derived:

**Theorem 4.1** Suppose that \( u \) is a viscosity solution of (4.1). Then \( u \in C^{1,\alpha_1}(\bar{B}^+_1(\lambda/2,h_0/2)) \) and
\[
\|u\|_{C^{1,\alpha_1}(B^+_1(\lambda/2,h_0/2))} \leq C \left( \|u\|_{L^\infty(B^+_1(h_0))} + |F(0)| \right),
\]
where \( 0 < \alpha_1 < 1 \) depends only on \( n, \lambda, A \) and \( \delta_1 \), and \( C \) depends also on \( h_0 \).

**Proof** Let \( v(x) = (u(x) + te_i) - u(x)/t^{\alpha_1} \) where \( 0 < t < 1/4 \) and \( i < n \). From Theorem 3.1, we have
\[
\begin{align*}
v & \in S(\lambda/n, A) \quad \text{in } B^+_1(\lambda/2,h_0/2); \\
& \beta \cdot Dv = 0 \quad \text{on } T_{1/2}.
\end{align*}
\]
By Lemma 2.2, \( v \in C^{\alpha_1}(\bar{B}^+_1(\lambda/4,h_0/4)) \) and
\[
\|v\|_{C^{\alpha_1}(\bar{B}^+_1(\lambda/4,h_0/4))} \leq C \|v\|_{L^\infty(B^+_1(\lambda/2,h_0/2))} \\
\leq C \|u\|_{C^{\alpha_1}(\bar{B}^+_1(\lambda/4,3h_0/4))} \\
\leq C(\|u\|_{L^\infty(B^+_1(h_0))} + |F(0)|)(\text{again by Lemma 2.2}).
\]

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Then there exists a $C$ and $\alpha$ where $\alpha$ depends only on $n$, $\lambda$ and $A$, and $C$ depends also on $h_0$. Hence, we obtain that $u$ is $C^{2,\alpha}$ along the horizontal directions. Then let $v(x) = (u(x + t e_i) - u(x))/t^{2\alpha}$ and repeat above procedure, we obtain the $C^{1,\alpha}$ estimate of $u$ on $T_1$ (see [1, Section 5.3] and [12, Theorem 6.1]). Then from the boundary $C^{1,\alpha}$ regularity for fully nonlinear elliptic equations with the Dirichlet boundary conditions, (4.2) follows. \hfill $\Box$

Next, we intend to derive a boundary $C^{2,\alpha}$ estimate by a similar argument in [12]. We introduce the following boundary $C^{1,\alpha}$ estimate for Dirichlet problem which is first proved essentially by Krylov [6] and simplified by Caffarelli (see [3, Theorem 9.32], [5, Theorem 4.8] and [12, Lemma 7.1]).

**Proposition 4.1** Suppose that

\[
\begin{aligned}
& u \in S(\lambda, A) \quad \text{in} \quad B^+_{1, h_0}, \\
& u = 0 \quad \text{on} \quad T_1.
\end{aligned}
\]

Then there exists a $C^\alpha$ function $A : T_1 \to \mathbb{R}$ such that

\[
|u(x) - A(x')x_n| \leq C\|u\|_{L^\infty(B^+_{1, h_0})} |x_n|^{1 + \alpha} \quad \forall x \in \overline{B}^+_{1/2, h_0/2}
\]

and

\[
\|A\|_{C^{\alpha}(T_{1/2})} \leq C\|u\|_{L^\infty(B^+_{1, h_0})},
\]

where $\alpha$ depends only on $u$, $\lambda$ and $A$, and $C$ depends also on $h_0$.

To apply above lemma to oblique derivative problems, we rewrite it as follows.

**Lemma 4.1** Let $u$ be as in Proposition 4.1. Then there exists a $C^\alpha$ function $\bar{A} : T_1 \to \mathbb{R}$ such that

\[
|u(x) - \bar{A}(x'_\beta)|x - (x'_\beta, 0)| \leq C\|u\|_{L^\infty(B^+_{1, h_0})} |x - (x'_\beta, 0)|^{1 + \alpha} \quad \forall x \in \overline{B}^+_{1/2, h_0/2},
\]

and

\[
\|\bar{A}\|_{C^{\alpha}(T_{1/2})} \leq C\|u\|_{L^\infty(B^+_{1, h_0})},
\]

where $\alpha$ and $C$ are as in Proposition 4.1.

**Remark 4.1** Note that for every $x \in B^+_{1, h_0}$, $(x'_\beta, 0) \in T_1$.

**Proof** Let $\bar{A}(x'_\beta) = A(x'_\beta)/\beta_n$. Then by Proposition 4.1,

\[
|u(x) - \bar{A}(x'_\beta)|x - (x'_\beta, 0)| = |u(x) - A(x'_\beta)/\beta_n| x_n/eta_n
\]

\[
= |u(x) - A(x')x_n + A(x')x_n - A(x'_\beta)x_n| \leq C\|u\|_{L^\infty(B^+_{1, h_0})} |x_n|^{1 + \alpha} + C\|A\|_{C^{\alpha}(T_{1/2})} |x' - x'_\beta|\|x_n
\]

\[
\leq C\|u\|_{L^\infty(B^+_{1, h_0})} |x - (x'_\beta, 0)|^{1 + \alpha} \quad \forall x \in \overline{B}^+_{1/2, h_0/2}.
\]

\hfill $\Box$
Lemma 4.2 Suppose that $u$ is a viscosity solution of (4.1). Then there exists a $C^\alpha$ function $\overline{A} : T_1 \to R$ such that for any $x \in B^+_1/2, h_0/2^\alpha$,

$$
|u(x) - u(x, 0)| - \frac{1}{2} \overline{A}(x) |x - (x, 0)|^2 \leq C(\|u\|_{L^\infty(B^+_1, h_0)} + |F(0)|)|x - (x, 0)|^{2+\alpha}
$$

and

$$
\|\overline{A}\|_{C^\alpha(T_{1/2})} \leq C(\|u\|_{L^\infty(B^+_1, h_0)} + |F(0)|),
$$

where $\alpha$ is as in Proposition 4.1 and $C$ depends only on $n$, $A$, $\delta_0$ and $h_0$.

Proof By Theorem 4.1, $u \in C^{1,\alpha_1}(B^+_{7/8, 7h_0/8})$. Hence, $u_\beta := \beta\cdot Du \in C^\alpha(B^+_{7/8, 7h_0/8})$.

From the closedness of viscosity solutions (Proposition 2.1), $u_\beta \in S(\lambda/n, A)$ in $B^+_{7/8, 7h_0/8}$, and $u_\beta = 0$ on $T_{7/8}$. Then by Lemma 4.1, there exists $\overline{A}$ such that

$$
u(x) - u(x, 0) = \int_0^{\frac{|x - (x, 0)|}{\overline{A}(x)}} u_\beta ((x, 0) + t\beta) dt
$$

$$
\leq \int_0^{\frac{|x - (x, 0)|}{\overline{A}(x)}} (\overline{A}(x) t + C\|u_\beta\|_{L^\infty(B^+_{7/8, 7h_0/8})} t^{1+\alpha}) dt
$$

$$
= \frac{\overline{A}(x)}{2} |x - (x, 0)|^2 + C(\|u\|_{L^\infty(B^+_1, h_0)} + |F(0)|)|x - (x, 0)|^{2+\alpha}.
$$

Hence,

$$
|u(x) - u(x, 0)| - \frac{1}{2} \overline{A}(x) |x - (x, 0)|^2 \leq C(\|u\|_{L^\infty(B^+_1, h_0)} + |F(0)|)|x - (x, 0)|^{2+\alpha}.
$$

The proof of the other direction of the inequality is similar and we omit it.

Finally, note that

$$
\|\overline{A}\|_{C^\alpha(T_{1/2})} \leq C\|u_\beta\|_{L^\infty(B^+_{7/8, 7h_0/8})} \leq C(\|u\|_{L^\infty(B^+_1, h_0)} + |F(0)|)
$$

and we complete the proof. \hfill \Box

Next, we follow the idea of [12] to show that the solution satisfies an equation on the flat boundary.

Lemma 4.3 Suppose that $u$ is a viscosity solution of (4.1). Define $v : T_1 \to R$ by $v(x') = u(x', 0)$. Then

$$
G(D^2 v, x') = 0 \quad \text{in} \quad T_1
$$

(4.3)

where

$$
G(M, x') := F \left( \left( \begin{array}{c} M \\ -\beta^T M/\beta \end{array} \right) \begin{array}{c} -M/\beta \\ \beta^T M \beta' \end{array} \right)
$$

for any $x' \in T_1$ and $M \in S^{n-1}$. 

Proof Let \( \varphi \in C^2(T_1) \) touch \( v \) by below at \( x_0 \in T_1 \). Without loss of generality, we assume that \( x_0 = 0 \). From Lemma 4.2, we have

\[
u(x) \geq u(x_0) + \frac{1}{2} A(x_0)|x - (x_0, 0)|^2 - C(\|u\|_{L^\infty(B_{r_0})} + |F(0)|)|x - (x_0, 0)|^{2+\alpha}.
\]

Then

\[
u(x) \geq u(x_0) + \frac{1}{2} A(x_0)|x - (x_0, 0)|^2 - C|x - (x_0, 0)|^{2+\alpha} \\
= \varphi(x') + \frac{1}{2} A(x_0)|x - (x_0, 0)|^2 - C|x - (x_0, 0)|^{2+\alpha} \\
\geq \varphi(x' - \beta' x_n) + \frac{1}{2} A(x_0) x_n^2 - Cx_n^{2+\alpha} \\
\geq \varphi(x' - \beta' x_n) + \frac{1}{2} A(0) x_n^2 - Cx_n^{2+\alpha}.
\]

Let \( \varphi(x', x_n) = \varphi(x' - (x_n/\beta_n)\beta') + A(0)x_n^2/(2\beta_n^2) - 2\varepsilon x_n^2 \). Then

\[
u \geq \varphi + \varepsilon x_n^2 \text{ in } B_{r_0}^+ \text{ for } r \text{ small enough and } u(0) = \varphi(0) = \varphi(0).
\]

Let \( \tilde{\varphi}_\varepsilon(x) = \varphi(x + t\beta') \text{ and } y = x' + t\beta' - (x_n + t\beta_n)\beta'/\beta_n \). Then

\[
D\tilde{\varphi}_\varepsilon(x) = \left(D\varphi(y), \frac{1}{\beta_n} A(0)(x_n + t\beta_n) - 4\varepsilon(x_n + t\beta_n) - \beta' \cdot D\varphi(y) \frac{1}{\beta_n}\right).
\]

Hence,

\[
\beta \cdot D\tilde{\varphi}_\varepsilon(x', 0) = (\tilde{A}(0) - 4\varepsilon \beta_n^2) t.
\]

If \( \tilde{A}(0) \leq 0 \), then \( \beta \cdot D\tilde{\varphi}_\varepsilon(x', 0) > 0 \) on \( T_r \). We take \( t < 0 \) small enough. Hence, \( \varphi + c_0 \) will touch \( u \) by below at some \( x_0 \in B_{r/2}^+ \) for a proper \( c_0 \). Since \( F(D^2u) \leq 0 \), we have

\[
F\left(\begin{array}{c}
D^2\varphi \\
-\beta' D^2\varphi/\beta_n
\end{array}\right) \leq 0.
\]

Let \( \varepsilon \to 0 \), by the continuity of \( D^2\varphi \), we have

\[
F\left(\begin{array}{c}
D^2\varphi(0) \\
-\beta' D^2\varphi(0)/\beta_n
\end{array}\right) \leq 0.
\]

If \( \tilde{A}(0) > 0 \), we take \( \varepsilon < \tilde{A}(0)/(8\beta_n^2) \). Then, for \( r \) small enough, we also have(4.5). By taking \( t > 0 \) small, \( \varphi + c_0 \) will touch \( u \) by below at some \( x_1 \in B_{r/2}^+ \) for a proper \( c_0 \). Similarly,

\[
F\left(\begin{array}{c}
D^2\varphi \\
-\beta' D^2\varphi/\beta_n
\end{array}\right) \leq 0.
\]

Let \( \varepsilon \to 0 \), we also obtain(4.6). Therefore, \( v \) is a supersolution of \( G(D^2v, x') = 0 \). The verification of subsolution is similar and we omit it. \( \square \)

Now we prove the boundary \( C^{2,\alpha} \) estimate.
**Theorem 4.2** Suppose that $F$ is convex and $u$ is a viscosity solution of (4.1). Then $u \in C^{2, \alpha_2}(B^{+}_{1/2, h_0/2})$ and

$$
\|u\|_{C^{2, \alpha_2}(B^{+}_{1/2, h_0/2})} \leq C \left( \|u\|_{L^\infty(B^{+}_{1/2, h_0})} + |F(0)| \right),
$$

where $0 < \alpha_2 < 1$ depends only on $n$, $\lambda$, $\Lambda$ and $\delta_0$, and $C$ depends also on $h_0$.

**Proof** By Lemma 4.3, $v(x') = u(x', 0)$ satisfies $G(D^2 v, x') = 0$ in $T_1$. Note that $G$ is uniformly elliptic with $\lambda$ and $\tilde{\Lambda}$ where $\tilde{\Lambda}$ depends only on $n$, $\Lambda$ and $\delta_0$. Furthermore, $G$ is convex. Indeed,

$$
G(tM + (1 - t)N, x') = F \left( t \left( -\beta^T M / \beta_n \right), \frac{1}{\beta_n} \left( \bar{A}(x') + \beta^T M \beta' \right) \right)
$$

$$
+ (1 - t) \left( -\beta^T N / \beta_n \frac{1}{\beta_n} \left( \bar{A}(x') + \beta^T N \beta' \right) \right)
$$

$$
\leq tG(M, x') + (1 - t)G(N, x').
$$

Note that $G$ is Hölder continuous in $x'$. By the classical interior $C^{2, \alpha}$ estimates (Evans-Krylov estimates) for convex operators, there exists a constant $0 < \alpha < 1$ depending only on $n$, $\lambda$, $\Lambda$ and $\delta_0$ such that $v \in C^{2, \alpha}(T_{3/4})$ and

$$
\|v\|_{C^{2, \alpha}(T_{3/4})} \leq C \left( \|v\|_{L^\infty(T_1)} + |F(0)| \right).
$$

Note that $u = v$ on $T_1$. By the boundary $C^{2, \alpha}$ estimates for Dirichlet problems, there exists a constant $0 < \alpha_2 < 1$ depending only on $n$, $\lambda$, $\Lambda$ and $\delta_0$ such that $u \in C^{2, \alpha_2}(B^{+}_{1/2, h_0/2})$ and (4.7) holds.

**Remark 4.2** From now on, $\alpha_0$, $\alpha_1$ and $\alpha_2$ always denote the constants originated from Lemma 2.2, Theorem 4.1 and Theorem 4.2, and depending only on $n$, $\lambda$, $\Lambda$ and $\delta_0$.

## 5 $C^{4, \alpha}$ regularity

In the following sections, we use perturbation method to deduce the $C^{4, \alpha}$, $C^{2, \alpha}$ and higher regularity for oblique derivative problem (1.1). The main idea of perturbation method is applying the solutions of the model problem (4.1) to approximate the solution of (1.1). The existence of solutions is assured by Corollary 3.1 and the sufficient regularity has been obtained in last section. In approximating the solution of (1.1), $f(x)$, $\beta(x)$ and $\Gamma$ are regarded as the perturbation of 0, $\beta(0)$ and $T_1$ etc.. The A-B-P maximum principle is the main tool to measure the difference between the solution of (1.1) and the solutions of the model problem. In this section, we deduce the $C^{4, \alpha}$ regularity for (1.1).
Lemma 5.1 Let \( u \) be a viscosity solution of
\[
\begin{cases}
F(D^2 u) = f & \text{in } \Omega; \\
\beta \cdot Du = g & \text{on } \Gamma,
\end{cases}
\]
where \( x_0 \in \Gamma \) such that \( \text{dist}(x_0, \partial \Omega \setminus \Gamma) > 1 \) and \( 0 < \alpha < \alpha_1 \).
Suppose that \( \beta \) and \( g \) are \( C^\alpha \) at \( x_0 \in \Gamma \), and \( f \) satisfies
\[
\left( \frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0) \cap \Omega} |f|^\alpha \right)^{\frac{1}{\alpha}} \leq C_f r^{\alpha - 1} \quad \forall r > 0.
\]
Then \( u \) is \( C^{1,\alpha} \) at \( x_0 \), i.e., there exists an affine function \( l \) such that
\[
\|u - l\|_{L^\infty(\bar{\Omega} \cap B_r(x_0))} \leq C_0 r^{1+\alpha} \quad \forall 0 < r < r_1,
\]
\[
|Dl| \leq C_0,
\]
\[
C_0 \leq C \left( \|u\|_{L^\infty(\Omega)} + C_f + \|g\|_{C^\alpha(x_0)} + |F(0)| \right)
\]
and
\[
r_1 = \tilde{C}^{-1},
\]
where \( C \) depends only on \( n, \lambda, \Lambda, \delta_0 \) and \( \alpha \), and \( \tilde{C} \) depends also on \( \|\beta\|_{C^\alpha(x_0)} \) and the \( C^1 \) modulus of \( \Gamma \) at \( x_0 \).

Remark 5.1 As in Lemma 2.2, the condition \( \text{dist}(x_0, \partial \Omega \setminus \Gamma) > 1 \) is not an essential assumption and “1” can be replaced by any positive constant. Then, we obtain the scaling version of (5.2) to (5.5).

Proof We make some normalization first. We assume that \( F(0) = 0 \). Otherwise, by the uniform ellipticity, there exists \( t \in \mathbb{R} \) such that \( F(tI) = 0 \) and \( |t| \leq |F(0)|/\lambda \). Then \( \tilde{u} = u - t|x|^2/2 \) satisfies
\[
\begin{cases}
F(D^2 \tilde{u} + tI) = f & \text{in } \Omega; \\
\beta \cdot D\tilde{u} = g - \frac{t}{2} \beta \cdot x & \text{on } \Gamma.
\end{cases}
\]
Hence, the estimate of \( u \) follows from that of \( \tilde{u} \). Next, by choosing a proper coordinate, we assume that \( x_0 \) is the origin,
\[
\Gamma \cap B_1 = \{(x', x_n) \in B_1 \mid x_n = \varphi(x')\}
\]
and \( \varphi \) satisfies that
\[
\varphi(0) = 0, \ D\varphi(0) = 0 \quad \text{and} \quad |\varphi(x')| \leq \nu|x'|,
\]
where \( \nu \) is chosen small later.
Denote \( \beta(0) \) by \( \beta^0 \). Since \( \beta \) is \( C^\alpha \) at \( 0 \) and \( \beta^0_n = \beta(0) \cdot n(0) \geq \delta_0, \beta_n \geq \delta_0/2 \) on \( \Gamma \cap B_{r_0} \), for \( r_0 \) small enough (depending only on \( \delta_0 \) and \( \|\beta\|_{C^\alpha(x_0)} \)). Without loss of generality, we assume that \( r_0 = 1 \). By scaling, we also assume that
\[ |\beta|_{C^\alpha(0)} \leq 1. \] Finally, we assume that \( g(0) = 0 \). Otherwise, we may consider \( \tilde{u} = u - g(0)x_n / \beta_0 \).

Let \( M := \|u\|_{L^\infty(\Omega)} + C_f + \|g\|_{C^\alpha(0)} \) and \( \Omega_r := \Omega \cap B_{r,h_0} \) where \( h_0 \) is chosen as in Corollary 3.1 (depending only on \( \delta_0 \)) such that \( \beta^0 \). To prove that \( u \) is \( C^{1,\alpha} \) at 0, we only need to prove the following:

There exist constants \( 0 < \tau < 1, 0 < \eta < 1, \bar{C} \) and \( \hat{C} \) depending only on \( n, \lambda, \Lambda, \delta_0 \) and \( \alpha \), and a sequence affine functions \( l_k(x) = b_kx + c_k \ (k \geq -1) \) such that for all \( k \geq 0 \)

\[
\|u - l_k\|_{L^\infty(\Omega_{r\eta^k})} \leq \hat{C}M\eta^{k(1+\alpha)},
\]

\[
\eta^k|b_k - b_{k-1}| + |c_k - c_{k-1}| \leq \hat{C}C\eta^{k(1+\alpha)}
\]

and

\[
\beta^0 \cdot b_k = g(0) = 0.
\]

We prove above by induction. For \( k = 0 \), by setting \( l_0 = l_{-1} = 0 \), the conclusion holds clearly. Suppose that the conclusion holds for \( k = k_0 \). We need to prove that the conclusion holds for \( k = k_0 + 1 \). In the rest of the proof, \( C, C_1, C_2 \) etc. denote positive constants depending only on \( n, \lambda, \Lambda, \delta_0 \) and \( \alpha \).

Let \( r := \frac{1}{2}\tau\eta^{k_0}, \tilde{B}_r^{+} := B_{r,h_0} + \nu \Gamma_k, \tilde{T}_r := T_r \cup \nu \Gamma_k \) and \( \Omega_r := \Omega \cap \tilde{B}_r^{+} \). Assume that

\[
4\nu \leq h_0.
\]

Then \( \Omega_r \subset \tilde{Q}_r \).

Note that \( u - l_{k_0} \) satisfies

\[
\begin{cases}
F(D^2(u - l_{k_0})) = f & \text{in } \Omega_2r; \\
\beta \cdot D(u - l_{k_0}) = g & \text{on } \Gamma \cap \partial \Omega_2r.
\end{cases}
\]

By the Hölder estimate (Lemma 2.2),

\[
\begin{align*}
\|u - l_{k_0}\|_{L^\infty(\Omega_r)} + r^\alpha |u - l_{k_0}|_{C^\alpha(\Omega_r)} & \\
& \leq C \left( \|u - l_{k_0}\|_{L^\infty(\Omega_2r)} + r\|f\|_{L^\infty(\Omega_2r)} + r\|g\|_{L^\infty(\Gamma \cap B_{2r})} \right) \\
& \leq C \left( \|u - l_{k_0}\|_{L^\infty(\Omega_2r)} + r^{1+\alpha}C_f + r\|g\|_{L^\infty(\Gamma \cap B_{2r})} \right).
\end{align*}
\]

Extend \( u - l_{k_0} \) from \( \Omega_r \) to the whole \( R^n \) such that

\[
\|u - l_{k_0}\|_{L^\infty(\Omega_r)} + r^\alpha |u - l_{k_0}|_{C^\alpha(\Omega_r)} \leq C \left( \|u - l_{k_0}\|_{L^\infty(\Omega_r)} + r^\alpha |u - l_{k_0}|_{C^\alpha(\Omega_r)} \right).
\]

By Corollary 3.1, there exists a unique solution \( v \) of

\[
\begin{cases}
F(D^2v) = 0 & \text{in } \tilde{B}_r^{+}; \\
\beta^0 \cdot Dv = 0 & \text{on } \tilde{T}_r; \\
v = u - l_{k_0} & \text{on } \partial \tilde{B}_r^{+} \setminus \tilde{T}_r.
\end{cases}
\]
Let \( w = u - l_{k_0} - v \). Then by Theorem 3.1, \( w \) satisfies
\[
\begin{cases}
  w \in S(\lambda/n, \Lambda, f) & \text{in } \Omega \cap \tilde{B}^+_{r,ho}; \\
  \beta \cdot Dw = g - \beta \cdot (b_{k_0} + Dv) & \text{on } \Gamma \cap \tilde{B}^+_{r,ho}; \\
  w = 0 & \text{on } \partial \tilde{B}^+_{r,ho} \cap \Omega.
\end{cases}
\]

In the following arguments, we estimate \( v \) and \( w \) respectively. By the boundary \( C^{1,\alpha} \) estimates for \( v \) (Theorem 4.1) and the A-B-P maximum principle (Lemma 2.1), there exists an affine function \( \tilde{l}(x) = b(x + \nu r e_n) + \tilde{c} \) such that
\[
\| v - \tilde{l} \|_{L^\infty(\Omega_{2r})} \leq C \frac{(2\mu r)^{1+\alpha}}{r^{1+\alpha}} \| v \|_{L^\infty(\tilde{B}^+_{r,ho})} \leq C \eta^{1+\alpha} \| u - l_{k_0} \|_{L^\infty(\Omega_r)} + \tilde{c} \leq C \eta^{1+\alpha} \cdot \tilde{C} M \eta^{k_0(1+\alpha)},
\]
and
\[
\eta^{k_0} |\tilde{b}| + |\tilde{c}| \leq \tilde{C} \cdot \tilde{C} M \eta^{k_0(1+\alpha)} \quad (5.12)
\]
Let \( \tilde{l}(x) = \tilde{b} x + \tilde{c} \). Then
\[
\| v - \tilde{l} \|_{L^\infty(\Omega_{2r})} \leq \| v - \tilde{l} \|_{L^\infty(\Omega_{2r})} + \| \tilde{l} - \tilde{l} \|_{L^\infty(\Omega_{2r})} \\
\leq C \eta^{\alpha_1 - \alpha} \cdot \tilde{C} M \eta^{(k_0+1)(1+\alpha)} + |\tilde{b} r| \\
\leq \left( C_1 \eta^{\alpha_1 - \alpha} + \frac{C_2 r}{\eta^{1+\alpha}} \right) \cdot \tilde{C} M \eta^{(k_0+1)(1+\alpha)}. \quad (5.14)
\]

Next, we estimate the term \( w \). Let \( \tilde{B}^+_{\mu} := \{ x \in \tilde{B}^+_{r,ho} \mid \text{dist}(x, \partial \tilde{B}^+_{r,ho}) \geq \mu r \}, \Omega_{r,\mu} := \Omega \cap \tilde{B}^+_{\mu}, \Gamma_1 := \partial \tilde{B}^+_{\mu} \cap \Omega \text{ and } \Gamma_2 := \tilde{B}^+_{\mu} \cap \Gamma \).

By the global Hölder estimate for \( v \) (Corollary 3.1) and recalling (5.10) and (5.11), there exists \( 0 < \alpha_2 \leq \alpha_2 / 2 \) such that
\[
\| v \|_{L^\infty(\tilde{B}_r)} + r^{\alpha_2} [v] C^{\alpha_3}(\tilde{B}_r) \leq C \left( \| u - l_{k_0} \|_{L^\infty(\Omega_r)} + r^{\alpha_2} [u - l_{k_0}] C^{\infty}(\tilde{B}_r) \right) \\
\leq C \left( \| u - l_{k_0} \|_{L^\infty(\Omega_r)} + r^{1+\alpha} C_f + \| g \|_{L^\infty(\Gamma \cap B_{2r})} \right).
\]

For any \( x_0 \in \Gamma_1 \), there exists \( \bar{x} \in \partial \tilde{B}^+_{r,ho} \setminus \tilde{T}_r \) such that \( |x_0 - \bar{x}| = \mu r \). Then by recalling (5.10),
\[
|w(x_0)| = |u(x_0) - l_{k_0}(x_0) - v(x_0)| \\
\leq |u(x_0) - l_{k_0}(x_0) - v(x_0) - u(\bar{x}) + l_{k_0}(\bar{x}) + v(\bar{x})| \\
\leq (|u(x_0) - l_{k_0}(x_0)| - |u(\bar{x}) - l_{k_0}(\bar{x})|) + |v(x_0) - v(\bar{x})| \\
\leq C \frac{(\mu r)^{\alpha_2}}{r^{\alpha_3}} \left( \| u - l_{k_0} \|_{L^\infty(\Omega_{2r})} + r^{1+\alpha} C_f + \| g \|_{L^\infty(\Gamma \cap B_{2r})} \right). \quad (5.15)
\]
Combining the A-B-P maximum principle and (5.15), we have
\[ \|w\|_{L^\infty(D_{\omega,u})} \leq \|w\|_{L^\infty(T_1)} + Cr\|g - \beta \cdot (b_{k_0} + Dv)\|_{L^\infty(T_2)} + r\|f\|_{L^\infty(D_{\omega,u})} \]
\[ \leq C\left(\frac{\mu r}{\tau} \right)^{\alpha_0} \left(\|u - b_{k_0}\|_{L^\infty(D_{\omega,u})} + r^{1+\alpha} C_f + r\|g\|_{L^\infty(T_2)} \right) \]
\[ + Cr\|g\|_{L^\infty(T_2)} + Cr\|\beta \cdot b_{k_0}\|_{L^\infty(T_2)} + Cr\|\beta \cdot Dv\|_{L^\infty(T_2)} + Cr^{1+\alpha} C_f \]
\[ \leq C_3 \mu^{\alpha_0} \hat{C} M \eta^{\alpha_0 (1+\alpha)} + C_4 r^{1+\alpha} \|g\|_{C^\alpha(T_1 \cap B_{2r})} + Cr\|\beta \cdot b_{k_0}\|_{L^\infty(T_2)} \]
\[ + Cr\|\beta \cdot Dv\|_{L^\infty(T_2)} + C_5 r^{1+\alpha} C_f \]
\[ \leq \left( C_3 \mu^{\alpha_0} + C_4 + C_5 \right) \hat{C} M \eta^{\alpha_0 (1+\alpha)} + Cr\|\beta \cdot b_{k_0}\|_{L^\infty(T_2)} + Cr\|\beta \cdot Dv\|_{L^\infty(T_2)}. \]
(5.16)

In the following, we estimate \( \|\beta \cdot b_{k_0}\|_{L^\infty(T_2)} \) and \( \|\beta \cdot Dv\|_{L^\infty(T_2)} \) respectively. For the first term, recall that \( \beta^0 \cdot b_{k_0} = 0 \) and then we obtain
\[ \|\beta \cdot b_{k_0}\|_{L^\infty(T_2)} = \|\beta - \beta^0 \cdot b_{k_0}\|_{L^\infty(T_2)} \leq \frac{CCM}{1 - \eta} [\beta_{C^\alpha(x_0)}] r^{\alpha_0} \leq C_6 r^\alpha \hat{C} M. \]
(5.17)

We assume that
\[ 4\nu \leq \mu. \]
(5.18)

Then, \( \forall x_0 \in T_2, \) \( \text{dist}(x_0, \hat{T}_r) < \frac{1}{2} \text{dist}(x_0, \partial B_{\mu} \cap \hat{T}_r). \) Let \( x^* \in \hat{T}_r \) such that \( |x_0 - x^*| = \text{dist}(x_0, \hat{T}_r). \) By the \( C^{1,\alpha_1} \) estimate for \( v \) in \( B_{\mu r}(x^*) \cap \hat{B}_{\mu r}^{*}, \) we have (note that \( \beta^0 \cdot Dv(x^*) = 0 \))
\[ |\beta(x_0) \cdot Dv(x_0)| \]
\[ \leq |\beta(x_0) - \beta^0| |Dv(x_0)| + |\beta^0| |Dv(x_0) - Dv(x^*)| \]
\[ \leq \left( |\beta| C^{\alpha_0}(x_0) \frac{r^{\alpha_0}}{\mu r} + |\beta^0| \frac{(2\nu)^{\alpha_1}}{(\mu r)^{1+\alpha_1}} \right) \|g\|_{L^\infty(B_{\nu r}(x^*) \cap \hat{B}_{\mu r}^{*})} \]
\[ \leq \left( |\beta| C^{\alpha_0}(x_0) \frac{r^{\alpha_0 - 1}}{\mu} + \frac{(2\nu)^{\alpha_1}|\beta^0|}{\mu^{1+\alpha_1}} \right) \hat{C} M \eta^{\alpha_0 (1+\alpha)} \]
\[ = \left( C_7 r^{\alpha_0 - 1} + \frac{C_8 r^{\alpha_1}}{\mu^{1+\alpha_1}} \right) \hat{C} M \eta^{\alpha_0 (1+\alpha)}. \]
(5.19)

Combining (5.16), (5.17) and (5.19), we have
\[ \|u\|_{L^\infty(D_{\omega,u})} \leq \left( C_3 \mu^{\alpha_0} + C_4 + C_5 \right) + C_6 r^{1+\alpha} + \frac{C_7 r^{\alpha_0}}{\mu} + \frac{C_8 r^{\alpha_1}}{\mu^{1+\alpha_1}} \hat{C} M \eta^{\alpha_0 (1+\alpha)}. \]
(5.20)

Take \( \eta \) small enough such that \( C_1 \eta^{\alpha_1 - \alpha} < 1/4. \) Let \( \mu = \tau^{\alpha/2} \) and take \( \tau \) small enough such that
\[ \frac{C_2 \tau}{\eta^{\alpha_1}} < \frac{1}{4}, \quad \frac{C_5 \tau^{\alpha_0/2}}{\eta^{1+\alpha}} < \frac{1}{12}, \quad \frac{C_6 \tau^{1+\alpha}}{\eta^{1+\alpha}} < \frac{1}{12}, \quad \text{and} \quad \frac{C_7 \tau^{\alpha/2}}{\eta^{1+\alpha}} < \frac{1}{12}. \]
Next, take $\nu$ small enough such that (5.9) and (5.18) hold and
\[
\frac{C_8 \nu^{\alpha_1}}{\mu^{1+\alpha_1} \eta^{1+\alpha}} \leq \frac{1}{12}.
\]
Finally, take $\hat{C}$ large enough such that
\[
C_4 + C_5 \frac{\hat{C}}{\eta^{1+\alpha}} \leq \frac{1}{12}.
\]
Therefore, combining (5.14) and (5.20), we have
\[
\|u - l_k - \bar{l}\|_{L^\infty(B_{r_0}(x_0) \cap \Omega)} \leq \|u - l_k - v + v - \bar{l}\|_{L^\infty(B_{r_0}(x_0) \cap \Omega)} 
\leq \|u\|_{L^\infty(B_{r_0}(x_0) \cap \Omega)} + \|v - \bar{l}\|_{L^\infty(B_{r_0}(x_0) \cap \Omega)} 
\leq \hat{C} M \eta^{(k_0+1)(1+\alpha)}.
\]
Let $l_{k_0+1} = l_k + \bar{l}$. Recall (5.12) and (5.13). Then the conclusion holds for $k = k_0 + 1$.

Now, we can derive the pointwise $C^{1,\alpha}$ regularity for oblique derivative problems in the general form.

**Theorem 5.1** Let $u$ be a viscosity solution of (1.1), $x_0 \in \Gamma$ such that $\text{dist}(x_0, \partial \Omega \setminus \Gamma) > 1$ and $0 < \alpha < \min(\alpha_0, \alpha_1)$. Suppose that $\beta$, $\gamma$ and $g$ are $C^\alpha$ at $x_0 \in \Gamma$, and $f$ satisfies
\[
\left( \frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0) \cap \Omega} |f|^n \right)^\frac{1}{n} \leq C_f r^{\alpha-1} \quad \forall r > 0.
\]
Then $u$ is $C^{1,\alpha}$ at $x_0$, i.e., there exists an affine function $l$ such that
\[
\|u - l\|_{L^\infty(B_r(x_0) \cap \Omega)} \leq C_0 r^{1+\alpha} \quad \forall 0 < r < r_1, \quad (5.21)
\]
\[
|Dl| \leq C_0, \quad (5.22)
\]
\[
C_0 \leq C \left( \|u\|_{L^\infty(H)} + C_f + \|g\|_{C^\alpha(x_0)} + |F(0)| \right), \quad (5.23)
\]
and
\[
r_1 = \hat{C}^{-1}, \quad (5.24)
\]
where $C$ depends only on $n$, $\lambda$, $\Lambda$, $\delta_0$, $\alpha$ and $\|\gamma\|_{C^\alpha(x_0)}$, and $\hat{C}$ depends also on $\|\beta\|_{C^\alpha(x_0)}$ and the $C^1$ modulus of $\Gamma$ at $x_0$.

**Proof** Rewrite the equation as
\[
\begin{cases}
F(D^2 u) = f & \text{in } \Omega; \\
\beta \cdot Du = g - \gamma u & \text{on } \Gamma.
\end{cases}
\]
From Theorem 2.3, $u$ is $C^{\alpha_0}$ at $x_0$. Then, from Lemma 5.1, we obtain that $u$ is $C^{1,\alpha}$ at $x_0$ and (5.21) to (5.24) hold. \qed
Combining with the interior $C^{1,\alpha}$ estimate (see [1, Theorem 8.3]), the boundary local $C^{1,\alpha}$ estimate (Theorem 1.2) follows easily (see the proof of [12, Proposition 2.4]).

6 $C^{2,\alpha}$ and higher regularity

In this section, we prove the $C^{2,\alpha}$ regularity and higher regularity for the oblique derivative problem (1.1). We introduce the following two lemmas first for constructing auxiliary functions.

**Lemma 6.1** Let $F$ be convex and $u$ be a viscosity solution of

\[
\begin{align*}
F(D^2 u) &= 0 \quad \text{in } B^+_{1, h_0}, \\
\beta \cdot Du &= g \quad \text{on } T_1,
\end{align*}
\]

where $\beta$ is a constant vector.

Let $0 < \alpha < \alpha_2$. Suppose that $\|g\|_{L^\infty(T_1)} \leq C g r^{1+\alpha}$ for any $0 < r < 1$. Then there exists a paraboloid $P$ such that

\[
\begin{align*}
\|u - P\|_{L^\infty(\Omega \eta^k)} &\leq C M \eta^{k(2+\alpha)}, \quad \forall k \geq 0, \\
|D P(0)| + \|D^2 P\| &\leq C_0,
\end{align*}
\]

and

\[
C_0 \leq C \left( \|u\|_{L^\infty(B^+_{1, h_0})} + C g + |F(0)| \right),
\]

where $C$ depends only on $n$, $\lambda$, $\Lambda$, $\delta_0$, $\alpha$ and $h_0$.

**Proof** The assumption $F(0) = 0$ is made as in Lemma 5.1. Let $M := \|u\|_{L^\infty(B^+_{1, h_0})} + C g$ and $\Omega_r := B^+_{r, h_0}$. We only need to prove the following:

There exist constants $0 < \eta < 1$, $\tilde{C}$ and $\tilde{C}$ depending only on $n$, $\lambda$, $\Lambda$, $\delta_0$, $\alpha$ and $h_0$, and a sequence paraboloids $P_k(x) = \frac{1}{2} x^T a_k x + b_k x + c_k \quad (k \geq -1)$ such that

\[
\begin{align*}
\|u - P_k\|_{L^\infty(\Omega_{r^k})} &\leq \tilde{C} \tilde{M} \eta^{k(2+\alpha)}, \quad \forall k \geq 0, \\
\eta^{2k} |a_k - a_{k-1}| + \eta^k |b_k - b_{k-1}| + |c_k - c_{k-1}| &\leq \tilde{C} \tilde{C} \tilde{M} \eta^{k(2+\alpha)}
\end{align*}
\]

and

\[
F(a_k) = 0, \beta \cdot b_k = 0 \quad \text{and} \quad a_k \beta = 0.
\]

We prove it by induction. For $k = 0$, by setting $P_0 = P_{-1} = 0$, the conclusion holds clearly. Suppose that the conclusion holds for $k = k_0$. We need to prove that the conclusion holds for $k = k_0 + 1$. 


Let $r := \frac{1}{2}\eta^{k_0}$. By Corollary 3.1, there exists a unique solution $v$ of
\[
\begin{aligned}
F(D^2v + a_{k_0}) &= 0 \quad \text{in } B^+_{r,hor}; \\
\beta \cdot Dv &= 0 \quad \text{on } T_r; \\
v &= u - P_{k_0} \quad \text{on } \partial B^+_{r,hor} \setminus T_r.
\end{aligned}
\] Let $w = u - P_{k_0} - v$. Recall that $\beta \cdot (a_{k_0}x + b_{k_0}) = 0$ on $T_1$ (see (6.6)). Then $w$ satisfies
\[
\begin{aligned}
w \in S(\lambda/n, A) &\quad \text{in } B^+_{r,hor}; \\
\beta \cdot Dw &= g \quad \text{on } T_r; \\
w &= 0 \quad \text{on } \partial B^+_{r,hor} \setminus T_r.
\end{aligned}
\] In the following, we estimate $v$ and $w$ respectively. By the $C^{2,\alpha}$ estimate for $v$ (Theorem 4.2) and the A-B-P maximum principle (Lemma 2.1), there exists a paraboloid
\[
\bar{P}(x) = \frac{1}{2}x^T\bar{a}x + \bar{b}x + \bar{c}
\] such that
\[
\|v - \bar{P}\|_{L^\infty(\Omega_{2r})} \leq C \eta^{2+\alpha} \|v\|_{L^\infty(\Omega_r)} \leq C \eta^{2+\alpha} \|u - P_{k_0}\|_{L^\infty(\Omega_r)} \leq C_1 \eta^{\alpha_1-\alpha} \cdot \hat{C} M \eta^{(k_0+1)(2+\alpha)},
\]
and
\[
\eta^{2k_0} |\bar{a}| + \eta^{k_0} |\bar{b}| + |\bar{c}| \leq \hat{C} \hat{C} M \eta^{k_0(2+\alpha)}
\] (6.7)
and
\[
F(\bar{a} + a_{k_0}) = 0 \text{ and } \beta^0 \cdot \bar{b} = 0.
\] (6.8)
Furthermore, by the Taylor’s formula, for any $(x', 0) \in T_r$, we have
\[
Dv(x', 0) = Dv(0) + D^2v(0)(x', 0) + o(|x'|).
\] Combining with $\beta \cdot Dv = 0$ on $T_r$, we deduce
\[
\bar{a}\beta = 0.
\] (6.9)
For $w$, by the A-B-P maximum principle, we have
\[
\|w\|_{L^\infty(\Omega_r)} \leq C r \|g\|_{L^\infty(\Omega_r)} \leq C C g r^{2+\alpha} \leq \frac{C_2}{\hat{C}} \hat{C} M \eta^{k_0(2+\alpha)}.
\] (6.10)
Take $\eta$ small enough such that $C_1 \eta^{\alpha_1-\alpha} < 1/2$. Next, take $\hat{C}$ large enough such that
\[
\frac{C_2}{\hat{C} \eta^{2+\alpha}} \leq \frac{1}{2}.
\] Therefore, combining (6.7) and (6.11), we have
\[
\|u - P_{k_0} - \bar{P}\|_{L^\infty(\Omega_{\gamma(k_0+1)})} \leq \|w\|_{L^\infty} + \|v - \bar{P}\|_{L^\infty} \leq \hat{C} M \eta^{(k_0+1)(2+\alpha)}.
\]
Let $P_{k_0+1} = P_{k_0} + \hat{P}$ and recall (6.8) to (6.10). Then the conclusion holds for $k = k_0 + 1$. □

Based on above $C^{2,\alpha}$ estimate, we deduce the following existence result which will be used to construct auxiliary functions.

**Lemma 6.2** Let $F$ be convex and $0 < \alpha < \min(\alpha_1, \alpha_2)$. Then there exists a unique solution $u \in C^{2,\alpha}(B_{1,h_0}^+ \cup T_1) \cap C(B_{1,h_0}^+)$ of

$$
\begin{align*}
F(D^2 u) &= 0 \quad \text{in} \quad B_{1,h_0}^+; \\
(\beta + Ax) \cdot Du &= 0 \quad \text{on} \quad T_1; \\
u &= \varphi \quad \text{on} \quad \partial B_{1,h_0}^+ \setminus T_1,
\end{align*}
$$

where $\beta$ is a constant vector, $\|A\| \leq \delta_0/2$ and $\varphi \in C(\partial B_{1,h_0}^+)$. Furthermore, we have the estimate

$$
\|u\|_{C^{2,\alpha}(B_{r/2,h_0/2}^+)} \leq C\|\varphi\|_{L^\infty(\partial B_{1,h_0}^+)},
$$

where $C$ depends only on $n$, $\lambda$, $A$, $\delta_0$, $\alpha$ and $h_0$.

**Proof** The existence and uniqueness of $u$ is assured by Theorem 3.3. We only need to prove the $C^{2,\alpha}$ regularity. Proving that $u$ is $C^{2,\alpha}$ at 0 is sufficient, i.e., there exists a paraboloid $P$ such that for any $0 < r < 1$,

$$
\|u - P\|_{L^\infty(B_{r/2,h_0/2}^+)} \leq C\|\varphi\|_{L^\infty(\partial B_{1,h_0}^+)} r^{2+\alpha}.
$$

By Lemma 5.1, $u \in C^{1,\alpha}(\bar{B}_{1/2,h_0/2}^+)$. It is easy to find a symmetric matrix $B$ such that $B\beta = A^T Du(0)$ and $\|B\| \leq C(n, \delta_0)|Du(0)|$. Let $v(x) = u(x) + x^T Bx/2$. Then $v$ satisfies

$$
\begin{align*}
F(D^2 v - B) &= 0 \quad \text{in} \quad B_{1/2,h_0}^+; \\
\beta \cdot Dv &= -Du^T Ax + Du(0)^T Ax = g \quad \text{on} \quad T_1,
\end{align*}
$$

Note that $\|g\|_{L^\infty(T_1)} \leq C|Du|_{C^{0,\alpha}} r^{1+\alpha}$ for any $0 < r < 1$. By Lemma 6.1, there exists a paraboloid $\hat{P}$ such that for any $0 < r < 1$,

$$
\|v - \hat{P}\|_{L^\infty(B_{r/2,h_0/2}^+)} \leq C \left( \|u\|_{L^\infty(B_{1/2,h_0}^+)} + |Du|_{C^{0,\alpha}} + |F(-B)| \right) r^{2+\alpha}.
$$

By the $C^{1,\alpha}$ estimate for $u$,

$$
|Du|_{C^{0,\alpha}} + |F(-B)| \leq C\|u\|_{L^\infty(B_{1/2,h_0}^+)}. $$

Therefore, let $P = \hat{P} + x^T Bx/2$ and we have

$$
\|u - P\|_{L^\infty(B_{r/2,h_0/2}^+)} = \|v - \hat{P}\|_{L^\infty} \leq C\|u\|_{L^\infty} r^{2+\alpha} \leq C\|\varphi\|_{L^\infty(\partial B_{1,h_0}^+) \cap T_1} r^{2+\alpha}
$$

by the A-B-P maximum principle. □

Next, we prove the $C^{2,\alpha}$ regularity.
Lemma 6.3 Let $F$ be convex and $u$ be a viscosity solution of
\[
\begin{aligned}
F(D^2u) &= f \quad \text{in } \Omega; \\
\beta \cdot Du &= g \quad \text{on } \Gamma,
\end{aligned}
\]
x_0 \in \Gamma \text{ such that dist}(x_0, \partial \Omega \cap \Gamma) > 1 \text{ and } 0 < \alpha < \min(\alpha_1, \alpha_2).
Suppose that $\beta$, $g$ and $\Gamma$ are $C^{1,\alpha}$ at $x_0$ and $f$ is $C^{\alpha}$ at $x_0$. Then $u$ is $C^{2,\alpha}$ at $x_0$, i.e., there exists a paraboloid $P$ such that
\[
\|u - P\|_{L^\infty(B_r(x_0) \cap \Omega)} \leq C_0 r^{2+\alpha} \quad \forall 0 < r < r_1,
\]
\[
|DP(x_0)| + \|D^2P(x_0)\| \leq C_0,
\]
\[
C_0 \leq C \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(x_0)} + \|g\|_{C^{1,\alpha}(x_0)} + |F(0)| \right)
\]
and
\[
r_1 = \mathcal{C}^{-1},
\]
where $C$ depends only on $n$, $\lambda$, $\Lambda$, $\delta_0$ and $\alpha$, and $\mathcal{C}$ depends also on $\|\beta\|_{C^{1,\alpha}(x_0)}$ and $\|\Gamma\|_{C^{1,\alpha}(x_0)}$.

Proof Similar to the $C^{1,\alpha}$ estimate, we make some normalization first. By choosing a proper coordinate system and scaling, we assume that $x_0$ is the origin,
\[
\Gamma \cap B_1 = \{(x', x_n) \in B_1 \mid x_n = \varphi(x')\}
\]
and $\varphi$ satisfies that
\[
\varphi(0) = 0, \quad D\varphi(0) = 0 \quad \text{and } |\varphi(x')| \leq |x'|^{1+\alpha}.
\]
Also, since $\beta$ is defined on $\Gamma \cap B_1$, we may write $\beta(x') = \beta(x', \varphi(x'))$ and assume that
\[
|\beta(x') - \beta(0)| = |D\beta(0)x'| \leq |x'|^{1+\alpha} \text{ on } \Gamma \cap B_1
\]
and $\|D\beta(0)\| \leq \delta_0/2$. We also assume that $f(0) = 0$. Otherwise, we may consider $G(D^2u) = F(D^2u) - f(0) = f - f(0)$. The assumptions that $F(0) = 0$, $\beta_n \geq \delta_0/2$ on $\Gamma \cap B_1$ are made as in Lemma 5.1. Similar to $\beta$, we may write $g(x') = g(x', \varphi(x'))$. As in Lemma 5.1, we assume that $g(0) = 0$. Furthermore, we assume that $Dg(0) = 0$. Otherwise, we may consider
\[
\bar{u}(x) = u(x) - \sum_{\beta^0 \neq 0, i < n} \frac{g_i(0)x_i^2}{\beta^0_i} - \sum_{\beta^0 = 0, i < n} \frac{g_i(0)x_i x_n}{\beta^0_n},
\]
where $\beta^0 := \beta(0)$. Then $\bar{u}$ satisfies
\[
\begin{aligned}
F(D^2\bar{u} + A) &= f \quad \text{in } \Omega; \\
\beta \cdot D\bar{u} &= \tilde{g} \quad \text{on } \Gamma,
\end{aligned}
\]
where $A$ is a constant matrix and $\tilde{g}$ satisfies $D\tilde{g}(0) = 0$. Then the desired estimates for $u$ follow easily from that of $\bar{u}$. 
Let \( M := \|u\|_{L^\infty(\Omega)} + \|f\|_{C^\infty(0)} + \|g\|_{C^{1,\alpha}(0)} \) and \( \Omega_r := \Omega \cap B_{r,\text{har}} \). We only need to prove the following:

There exist constants \( 0 < \tau < 1, 0 < \eta < 1, C \) and \( \tilde C \) depending only on \( n, \lambda, A, \delta_0 \) and \( \alpha \), and a sequence paraboloids \( P_k(x) = \frac{1}{2}x^T a_k x + b_k x + c_k \) \((k \geq -1)\) such that

\[
\|u - P_k\|_{L^\infty(\Omega_{r,k})} \leq \tilde C M \eta^{k(2+\alpha)}, \quad \forall k \geq 0, \tag{6.18}
\]

\[
\eta^{2k}|a_k - a_{k-1}| + \eta^k|b_k - b_{k-1}| + |c_k - c_{k-1}| \leq \tilde C \mf C M \eta^{k(2+\alpha)} \tag{6.19}
\]

and

\[
F(a_k) = 0, \beta^0 \cdot b_k = 0 \quad \text{and} \quad \sum_{j=1}^n \left( (a_k)_{ij} \beta^0_j + D_i \beta_j(0)(b_k)_{ij} \right) = 0, \quad i < n. \tag{6.20}
\]

We prove it by induction. For \( k = 0 \), by setting \( P_0 = P_{-1} = 0 \), the conclusion holds clearly. Suppose that the conclusion holds for \( k = k_0 \). We need to prove that the conclusion holds for \( k = k_0 + 1 \).

Let \( r := \frac{1}{2} r^k \eta^{k_0}, \tilde B_{r,\text{har}} : B_{r,\text{har}} - r^{1+\alpha} e_n, \tilde T_r := T_r - r^{1+\alpha} e_n \) and \( \tilde \Omega_r := \Omega \cap \tilde B_{r,\text{har}}^+ \). Assume that

\[
4r^\alpha \leq h_0. \tag{6.21}
\]

Then \( \Omega_{r/2} \subset \tilde \Omega_r \).

Note that \( u - P_k \) satisfies

\[
F(D^2(u - P_k) + a_k) = f \quad \text{in} \quad \Omega_{2r}.
\]

By the Hölder estimate (Lemma 2.2),

\[
\|u - P_k\|_{L^\infty(\Omega_r)} + r^{\alpha_0} [u - P_k]_{C^{\alpha_0}(\Omega_r)} \leq C (\|u - P_k\|_{L^\infty(\partial \Omega_r)} + r^2 \|f\|_{L^\infty(\partial \Omega_r)} + r \|g\|_{L^\infty(\Gamma_r)}). \tag{6.22}
\]

Extend \( u - P_k \) from \( \Omega_r \) to the whole \( P^n \) such that

\[
\|u - P_k\|_{L^\infty(P^n)} + r^{\alpha_0} [u - P_k]_{C^{\alpha_0}(P^n)} \leq C (\|u - P_k\|_{L^\infty(\Omega_r)} + r^{\alpha_0} [u - P_k]_{C^{\alpha_0}(\Omega_r)}). \tag{6.23}
\]

Let \( \alpha < \alpha_3 < \min(\alpha_1, \alpha_2) \). By Lemma 6.2, there exists a unique solution \( v \in C^{2,\alpha_3} \) of

\[
\begin{cases}
F(D^2 v + a_k) = 0 & \text{in} \quad \tilde B_{r,\text{har}}^+; \\
(\beta^0 + D \beta(0)x') \cdot Dv = 0 & \text{on} \quad \tilde T_r; \\
v = u - P_k & \text{on} \quad \partial \tilde B_{r,\text{har}}^+ \setminus \tilde T_r.
\end{cases}
\]

Let \( w = u - P_k - v \). Then \( w \) satisfies

\[
\begin{cases}
w \in S(\lambda/n, \Lambda, f) & \text{in} \quad \Omega \cap \tilde B_{r,\text{har}}^+; \\
\beta : Dw = g - \beta \cdot (a_k x + b_k + Dv) & \text{on} \quad \Gamma \cap \tilde B_{r,\text{har}}^+; \\
w = 0 & \text{on} \quad \partial \tilde B_{r,\text{har}}^+ \setminus \tilde \Omega.
\end{cases}
\]
In the following arguments, we estimate $v$ and $w$ respectively. By the boundary $C^{2,\alpha}$ estimates for $v$ (Lemma 6.2) and the A-B-P maximum principle (Lemma 2.1), there exists a paraboloid

$$
\tilde{P}(x) = \frac{1}{2} (x + r^{1+\alpha}e_n)^T \tilde{a} (x + r^{1+\alpha}e_n) + \tilde{b} (x + r^{1+\alpha}e_n) + \tilde{c}
$$

such that

$$
\|v - \tilde{P}\|_{L^\infty(\Omega_{2r})} \leq C \frac{(2\eta)^{2+\alpha}}{r^{2+\alpha}} \|v\|_{L^\infty(\tilde{B}^+_{r,h_0},r)} \leq C \eta^{2+\alpha}\|u - P_{k_0}\|_{L^\infty(\Omega_r)}
$$

$$
\leq C \eta^{\alpha-\alpha} \cdot \tilde{C} M \eta^{(k_0+1)(2+\alpha)},
$$

$$
\eta^{2k_0}|\tilde{a}| + \eta^{k_0}|\tilde{b}| + |\tilde{c}| \leq \tilde{C} \cdot \tilde{C} M \eta^{k_0(2+\alpha)} \tag{6.24}
$$

and

$$
F(\tilde{a} + a_{k_0}) = 0 \text{ and } \beta^0 \cdot \tilde{b} = 0. \tag{6.25}
$$

Furthermore, by the Taylor’s formula, for any $(x', -r^{1+\alpha}) \in \tilde{T}_r$, we have

$$
Dv(x', -r^{1+\alpha}) = Dv(0, -r^{1+\alpha}) + D^2v(0, -r^{1+\alpha})(x', 0) + o(|x'|).
$$

Combining with $(\beta^0 + D\beta(0)x') \cdot Dv = 0$ on $\tilde{T}_r$, we deduce

$$
\sum_{j=1}^n (\tilde{a}_{ij}\beta_{ij}^0 + D_i\beta_j(0)b_j) = 0, \quad i < n. \tag{6.26}
$$

Let $\tilde{P}(x) = \frac{1}{2} x^T \tilde{a}x + \tilde{b}x + \tilde{c}$. Then

$$
\|v - \tilde{P}\|_{L^\infty(\Omega_{2r})} \leq \|v - \tilde{P}\|_{L^\infty(\Omega_{2r})} + \|P - \tilde{P}\|_{L^\infty(\Omega_{2r})}
$$

$$
\leq C \eta^{\alpha-\alpha} \cdot \tilde{C} M \eta^{(k_0+1)(2+\alpha)} + |K^{1+\alpha}r^{1+\alpha}v_{\eta}\tilde{a}x|
$$

$$
+ |K^{2+2\alpha}r^{1+\alpha}a_{n1}| + |K^{r^{1+\alpha}b_{1n}}| \tag{6.27}
$$

$$
\leq \left(C_1 \eta^{\alpha-\alpha} + C_2 \frac{r^{1+\alpha}}{\eta^{2+\alpha}}\right) \cdot \tilde{C} M \eta^{(k_0+1)(2+\alpha)}.
$$

Next, we estimate the term $w$. Let $\tilde{B}_{\mu} := \{x \in \tilde{B}^+_{r,h_0} | \text{dist}(x, \partial \tilde{B}^+_{r,h_0}) \geq \mu^r\}$, $\Omega_{r,\mu} := \Omega \cap \tilde{B}_{\mu}$, $\Gamma_1 := \partial \tilde{B}_{\mu} \cap \tilde{\Omega}$ and $\Gamma_2 := \tilde{B}_{\mu} \cap \Gamma$.

By the global Hölder estimate for $v$ (Corollary 3.1) and recalling(6.22) and(6.23), there exists $0 < \alpha_4 \leq \alpha_0/2$ such that

$$
\|v\|_{L^\infty(\tilde{\Omega}_{r})} + r^{\alpha_4}[v]_{C^{\alpha_4}(\tilde{T}_r)} \leq C \left(\|u - P_{k_0}\|_{L^\infty(\Omega_r)} + r^{\alpha_0}[u - P_{k_0}]_{C^{\alpha_0}(\Omega_r)}\right)
$$

$$
\leq C \left(\|u - P_{k_0}\|_{L^\infty(\Omega_r)} + r^2 \|f\|_{L^\infty(\Omega_{2r})} + r^{\alpha} \|g\|_{L^\infty(\Omega \cap \Omega_{2r})}\right). \tag{6.28}
$$
For any $x_0 \in \Gamma_1$, there exists $\bar{x} \in \partial \tilde{B}_{r, \text{hor}}^+ \tilde{T}_r$ such that $|x_0 - \bar{x}| = \mu r$. Then by recalling (6.22)

$$
|w(x_0)| = |u(x_0) - P_{k_0}(x_0) - v(x_0)|
= |u(x_0) - P_{k_0}(x_0) - v(x_0) - u(\bar{x}) + P_{k_0}(\bar{x}) + v(\bar{x})|
\leq |(u(x_0) - P_{k_0}(x_0)) - (u(\bar{x}) - P_{k_0}(\bar{x}))| + |v(x_0) - v(\bar{x})|
\leq C(\mu r)^{\alpha_4} \left( \|u - P_{k_0}\|_{L^\infty(\Omega_2)} + r^2 \|f\|_{L^\infty(\Omega_2)} + r \|g\|_{L^\infty(\Gamma \cap B_{2r})} \right). 
$$

(6.29)

By the A-B-P maximum principle, combining with (5.15), we have

$$
\|u\|_{L^\infty(\Omega_{r, \mu})} \leq |u|_{L^\infty(\Gamma_1)} + Cr\|g - \beta \cdot (a_{k_0}x + b_{k_0} + Dv)|_{L^\infty(\Omega_{r_2})} + Cr^2\|f\|_{L^\infty(\Omega_{2r})}
\leq C(\mu r)^{\alpha_4} \left( \|u - P_{k_0}\|_{L^\infty(\Omega_{2r})} + Cr^2\|f\|_{L^\infty(\Omega_{2r})} + r \|g\|_{L^\infty(\Gamma \cap B_{2r})} \right)
+ Cr\|g\|_{L^\infty(\Gamma \cap B_{2r})} + Cr\|\beta \cdot (a_{k_0}x + b_{k_0} + Dv)|_{L^\infty(\Omega_{r_2})} + Cr²\|f\|_{L^\infty(\Omega_{2r})}
\leq C_{M}r^{\alpha_4} \tilde{C}M \eta^{k_0(2+\alpha)} + C_M \eta^{2+\alpha} \|g\|_{C^{\alpha}(\Omega_{r_2})} + CM \eta^{2+\alpha} \|f\|_{C^{\alpha}(\Omega_{r_2})}
+ Cr\|\beta \cdot (a_{k_0}x + b_{k_0})\|_{L^\infty(\Gamma_{r_2})} + Cr\|\beta \cdot Dv\|_{L^\infty(\Omega_{r_2})}
\leq \left( C_{M}r^{\alpha_4} + \frac{C_{M}}{C} \right) \tilde{C}M \eta^{k_0(2+\alpha)}
+ Cr\|\beta \cdot (a_{k_0}x + b_{k_0})\|_{L^\infty(\Gamma_{r_2})} + Cr\|\beta \cdot Dv\|_{L^\infty(\Omega_{r_2})}.
$$

(6.30)

In the following, we estimate $\|\beta \cdot (a_{k_0}x + b_{k_0})\|_{L^\infty(\Gamma_{r_2})}$ and $\|\beta \cdot Dv\|_{L^\infty(\Omega_{r_2})}$ respectively. For the first term, recall (6.19) and (6.20) and then we obtain

$$
\|\beta \cdot (a_{k_0}x + b_{k_0})\|_{L^\infty(\Gamma_{r_2})} = \|\beta - \beta^0 - D\beta(0)x'\cdot (a_{k_0}x + b_{k_0})\|_{L^\infty(\Gamma_{r_2})}
+ \|\beta^0 + D\beta(0)x'\cdot (a_{k_0}x + b_{k_0})\|_{L^\infty(\Gamma_{r_2})}
\leq r^{1+\alpha} \|a_{k_0}x + b_{k_0}\|_{L^\infty(\Gamma_{r_2})} + |D\beta(0)| \|a_{k_0}x + b_{k_0}\|_{L^\infty(\Gamma_{r_2})} r^2 + \|\beta^0 \cdot a_{k_0}x + b_{k_0} \cdot D\beta(0)x'\|_{L^\infty(\Gamma_{r_2})}
\leq \frac{C\tilde{C}M}{1-\eta} r^{1+\alpha} + \frac{C\tilde{C}M}{1-\eta} r^2 + \|\sum_{j=1}^{n} (a_{k_0})_{n_j} \beta^0_{j} x_n\|_{L^\infty(\Gamma_{r_2})}
\leq C_0 \tilde{C}M r^{1+\alpha}.
$$

(6.31)

We assume that

$$
4r^\alpha \leq \mu.
$$

(6.32)

Then, $\forall x_0 \in \Gamma_2$, $\text{dist}(x_0, \tilde{T}_r) < \frac{\mu}{4} \text{dist}(x_0, \partial \tilde{B}_{r, \text{hor}}^+ \tilde{T}_r)$. Let $x^* \in \tilde{T}_r$ such that $|x_0 - x^*| = \text{dist}(x_0, \tilde{T}_r)$. By the $C^{2, \alpha_3}$ estimate for $v$ in $B_{\mu r}(x^*) \cap \tilde{B}_{r, \text{hor}}^+$ and
noting that \((\beta^0 + D\beta(0)x_0') \cdot Dv(x^*) = 0\), we have
\[
|\beta(x_0) - Dv(x_0)| \\
\leq |\beta(x_0) - \beta^0 - D\beta(0)x_0'||Dv(x_0)| + |\beta^0 + D\beta(0)x_0'||Dv(x_0) - Dv(x^*)|
\]
\[
\leq r^{1+\alpha} \frac{C}{\mu} \|v\|_{L^\infty(B_{r^\alpha}(x^*) \cap \tilde{\Omega}^+_{\varepsilon})} + C \frac{2r^{1+\alpha}}{(\mu r)^2} \|v\|_{L^\infty(B_{r^\alpha}(x^*) \cap \tilde{\Omega}^+_{\varepsilon})}
\]
\[
\leq \frac{C \gamma^{\alpha}}{\mu} \cdot \tilde{C} M \eta^{3/2 + \alpha} + \frac{C \gamma^{\alpha - 1}}{\mu^2} \cdot \tilde{C} M \eta^{3/2 + \alpha} \quad \text{(6.33)}
\]
Combining (6.30), (6.31) and (6.33), we have
\[
\|w\|_{L^\infty(\Omega_{r\eta})} \leq \left( C_3 \mu^{\alpha_4} + \frac{C_4 + C_5}{\gamma^{\alpha_2/2}} + \frac{C \gamma^{1+\alpha}}{\mu} + \frac{C_8 \gamma^{\alpha/2}}{\mu^2} \right) \tilde{C} M \eta^{3/2 + \alpha} \quad \text{(6.34)}
\]
Take \(\eta\) small enough such that \(C_1 \eta^{\alpha_2 - \alpha} < 1/4\). Let \(\mu = \tau^{\alpha/4}\) and take \(\tau\) small enough such that
\[
\frac{C_2 \gamma^{1+\alpha}}{\eta^{2+\alpha}} < \frac{1}{4}, \quad \frac{C_4 \gamma^{\alpha_4/4}}{\eta^{2+\alpha}} < \frac{1}{12}, \quad \frac{C_6 \gamma^{2+\alpha}}{\eta^{2+\alpha}} < \frac{1}{12}, \quad \frac{C \gamma^{1+3\alpha/4}}{\eta^{2+\alpha}} < \frac{1}{12}, \quad \frac{C_8 \gamma^{\alpha/2}}{\eta^{2+\alpha}} < \frac{1}{12}
\]
and (6.21) and (6.32) hold. Finally, take \(\tilde{C}\) large enough such that
\[
\frac{C_4 + C_5}{\tilde{C} \eta^{2+\alpha}} \leq \frac{1}{12}
\]
Therefore, combining (6.27) and (6.34), we have
\[
\|u - P_{k_0} - \bar{P}\|_{L^\infty(\Omega_{\tau \eta^{(k_0+1)})}} = \|u - P_{k_0} - v - \bar{P}\|_{L^\infty(\Omega_{\tau \eta^{(k_0+1)})}} \\
\leq \|w\|_{L^\infty(\Omega_{\tau \eta^{(k_0+1)})}} + \|v - \bar{P}\|_{L^\infty(\Omega_{\tau \eta^{(k_0+1)})}} \\
\leq \tilde{C} M \eta^{3/2 + \alpha} \quad \text{(6.24)}
\]
Let \(P_{k_0+1} = P_{k_0} + \bar{P}\); Recall (6.24), (6.25) and (6.26). Then the conclusion holds for \(k = k_0 + 1\). \(\square\)

Similar to the pointwise \(C^{1,\alpha}\) estimate, we have the following:

**Theorem 6.1** Let \(F\) be convex, \(u\) be a viscosity solution of (1.1) and \(x_0 \in \Gamma\) such that \(\text{dist}(x_0, \partial \Omega \setminus \Gamma) > 1\) and \(0 < \alpha < \min(\alpha_0, \alpha_1, \alpha_2)\). Suppose that \(\beta, g\) and \(\Gamma\) are \(C^{1,\alpha}\) at \(x_0\) and \(F\) is \(C^n\) at \(x_0\).

Then \(u\) is \(C^{2,\alpha}\) at \(x_0\), i.e., there exists a paraboloid \(P\) such that
\[
\|u - P\|_{L^\infty(B_r(x_0))} \leq C_0 r^{2+\alpha} \quad \forall 0 < r < r_1,
\]
\[
|DP(x_0)| + \|D^2P(x_0)| \leq C_0,
\]
\[
C_0 \leq C \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{C^{\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)} + |F(0)| \right)
\]
and 

\[ r_1 = C^{-1}, \]

where \( C \) depends only on \( n, \lambda, A, \delta_0, \alpha \) and \( \|\gamma\|_{C^{1,\alpha}(x_0)} \), and \( \bar{C} \) depends also on \( \|\beta\|_{C^{1,\alpha}(x_0)} \) and \( \|\Gamma\|_{C^{1,\alpha}(x_0)} \).

Combining with the interior \( C^{2,\alpha} \) estimate (see [1, Theorem 8.1]), the boundary local \( C^{2,\alpha} \) estimate (Theorem 1.3) follows (see the proof of [12, Proposition 2.4]).

Since we have obtained the \( C^{2,\alpha} \) regularity, the higher regularity for the oblique derivative problem can be deduced standardly.

**Theorem 6.2** Let \( F \) be convex, \( u \) be a viscosity solution of (1.1) and \( 0 < \alpha < \min(\alpha_0, \alpha_1, \alpha_2) \). Suppose that \( F \in C^{k,\alpha}(S^n) \), \( \Gamma \in C^{k+2,\alpha} \), \( \beta, \gamma, g \in C^{k+1,\alpha}(\bar{\Gamma}) \) and \( f \in C^{k,\alpha}(\bar{\Omega}) \).

Then for any \( \Omega' \subset \subset \Omega \cup \Gamma \), \( u \in C^{k+2,\alpha}(\bar{\Omega}') \) and

\[ \|u\|_{C^{k+2,\alpha}(\bar{\Omega}')} \leq C, \]

where \( C \) depends on \( n, \lambda, A, \delta_0, \alpha, \|\beta\|_{C^{k+2,\alpha}(\Gamma)}, \|\gamma\|_{C^{k+2,\alpha}(\Gamma)}, \|f\|_{C^{k,\alpha}(\Omega)}, \|g\|_{C^{k+2,\alpha}(\Gamma)} \), \( \|u\|_{L^\infty(\Omega)}, F, \Omega' \) and \( \Omega \).

In particular, if \( F, \Gamma, f, \beta, \gamma, g \in C^\infty \), then \( u \in C^\infty(\bar{\Omega}') \).

**Proof**\( u \in C^{k+2,\alpha}(\bar{\Omega}) \) is well known and we only need to prove the boundary \( C^{k+2,\alpha} \) estimate. We prove the theorem by induction. Let \( k = 1 \). For any \( x_0 \in \Gamma \), there exists a proper coordinate system such that \( x_0 \) is the origin,

\[ \Gamma \cap B_2 = \{(x', x_n) \in B_2 | x_n = \varphi(x')\}, \]

where \( \varphi \in C^{k+2,\alpha}(T_2) \) satisfies

\[ \varphi(0) = 0, \quad D\varphi(0) = 0. \]

Introduce the transformation \( y = \psi(x) \) where \( \psi \) is defined as follows: \( y_i = x_i \) for \( i < n \) and \( y_n = x_n - \varphi(x') \). Define \( \tilde{u}(y) = u(x) \),

\[ A_{ij} = \frac{\partial y_i}{\partial x_j} \quad \text{and} \quad B_{ij} = u_i \cdot \frac{\partial x_l}{\partial y_m} \cdot \frac{\partial y_m}{\partial x_i \partial x_j}. \]

It is easy to check that \( \tilde{u} \) is a viscosity solution of

\[
\begin{cases}
G(D^2\tilde{u}, y) = \tilde{f} & \text{in } B^+_{1, b_0}; \\
\tilde{\beta} \cdot D\tilde{u} + \tilde{\gamma} \tilde{u} = \tilde{g} & \text{on } T_1,
\end{cases}
\]

where \( G(M, y) := F(AM + B) \) for any \( M \in S^n \), \( \tilde{f}(y) := f(x), \tilde{\beta}_i(y) := \beta_i(x) \partial y_i / \partial x_i, \tilde{\gamma}(y) := \gamma(x) \) and \( \tilde{g}(y) := g(x) \).

By Theorem 1.3, \( u \in C^{2,\alpha}(B_1 \cap \Omega) \). Hence, \( A_{ij}, B_{ij} \in C^{1,\alpha} \). Combining with \( F \in C^{1,\alpha} \), we have that \( G \in C^{1,\alpha}(S^n \times B^+_{1, b_0}) \). Note that \( \tilde{\beta}_n(y) = \)
\[ \beta_i \cdot \partial g_u / \partial x_i = \beta_n(x) - \beta'(x) \cdot Dv(x) \]. Since \( Dv(0) = 0 \), by a proper scaling, we may assume that
\[ \hat{\beta}_n \geq \frac{\delta_0}{2} \] on \( T_1 \).

Then \( \tilde{u} \in C^{3, \alpha}(B_{1, h_0}^+) \) can be obtained by the classical interior estimates (see [1, Proposition 9.1]). Differentiate the equations with respect to \( x_m \) (\( m \leq n - 1 \)) and we have
\[
\begin{align*}
\begin{cases}
G_{ij}(D^2 \tilde{u}(y), y)(\tilde{u}_m)_{ij} = -G_m(D^2 \tilde{u}(y), y) + \tilde{f}_m(y) & \text{in } B_{1, h_0}^+; \\
\hat{\beta} \cdot D\tilde{u}_m + \tilde{\gamma}_m = g - \hat{\beta}_m \cdot D\tilde{u} - \tilde{\gamma}_m \tilde{u} & \text{on } T_1.
\end{cases}
\end{align*}
\]

By the boundary estimates for linear elliptic equations (see [9, Theorem 4.40]), we have that \( \tilde{u}_m \in C^{2, \alpha}(\bar{B}_{3/4, 3h_0/4}^+) \) and
\[ \| \tilde{u}_m \|_{C^{2, \alpha}(\bar{B}_{3/4, 3h_0/4}^+)} \leq C. \]

Thus, \( \tilde{u} \in C^{3, \alpha}(T_{3/4}) \). From the boundary estimates for the Dirichlet problems, we have that \( \tilde{u} \in C^{3, \alpha}(\bar{B}_{1/2, h_0/2}^+) \) and
\[ \| \tilde{u} \|_{C^{3, \alpha}(\bar{B}_{1/2, h_0/2}^+)} \leq C. \]

Hence, \( u \in C^{3, \alpha}(B_{1/2} \cap \tilde{\Omega}) \). Since \( x_0 \in \Gamma \) is arbitrary, by the standard covering argument, we have that \( u \in C^{3, \alpha}(\tilde{\Omega}) \) and the estimate(6.35) holds.

Assume that the theorem holds for \( k = k_0 + 1 \). We prove that the theorem holds for \( k = k_0 + 1 \). Since \( u \in C^{k_0 + 2, \alpha}(\bar{\Omega}) \), \( B_{i,j} \in C^{k_0 + 1, \alpha} \) where \( B_{i,j} \) is defined as above. From [1, Proposition 9.1] we know that \( u \in C^{k_0 + 3, \alpha}(\Omega \cap B_j) \) and \( \tilde{u} \in C^{k_0 + 3, \alpha}(\bar{B}_{1, h_0}^+) \). Differentiate(6.36) \( k_0 + 1 \) times with respect to the horizontal directions. Then we deduce equations similar to(6.37). From the regularity for linear oblique derivative problems, we obtain that the \( k_0 + 1 \) order horizontal derivatives of \( \tilde{u} \) lie in \( C^{2, \alpha}(\bar{B}_{3/4, 3h_0/4}^+) \). Hence, \( \tilde{u} \in C^{k_0 + 3, \alpha}(T_{3/4}) \). From the regularity for Dirichlet problems, we obtain that \( \tilde{u} \in C^{k_0 + 3, \alpha}(B_{1/2, h_0/2}^+) \) and hence \( u \in C^{k_0 + 3, \alpha}(B_r \cap \tilde{\Omega}) \) for some \( r > 0 \). Therefore, by a scaling and covering argument, \( u \in C^{k_0 + 3, \alpha}(\tilde{\Omega}) \) and the estimate(6.35) holds.

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**Compliance with ethical standards**

**Conflict of interest**

Both authors declare that they have no potential conflict of interest.
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