A WAY TO DISCOVER MAXWELL’S EQUATIONS
THEORETICALLY

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The Coulomb force, established in the rest frame of a source-charge $Q$, when transformed to a new frame moving with a velocity $\vec{V}$ has a form $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$, where $\vec{E} = \vec{E}'_{\parallel} + \gamma \vec{E}'_{\perp}$ and $\vec{B} = \frac{\vec{V}}{c^2} \times \vec{E}$ and $\vec{E}'$ is the electric field in the rest frame of the source. The quantities $\vec{E}$ and $\vec{B}$ are then manifestly interdependent. We prove that they are determined by Maxwell’s equations, so they represent the electric and magnetic fields in the new frame and the force $\vec{F}'$ is the well known from experiments Lorentz force. In this way Maxwell’s equations may be discovered theoretically for this particular situation of uniformly moving sources. The general solutions of the discovered Maxwell’s equations lead us to fields produced by accelerating sources.

Key words: Maxwell’s equations, Lorentz transformation, Coulomb’s law.

1. INTRODUCTION

Maxwell’s equations,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$  \hspace{1cm} (1)

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \frac{\vec{J}}{c^2\varepsilon_0}$$  \hspace{1cm} (2)
\[ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \]  
\[ \vec{\nabla} \cdot \vec{B} = 0 \]

and the Lorentz force law
\[ \vec{F} = q\vec{E} + q\vec{v} \times \vec{B}, \]

are generalizations based on many different experiments on the forces between electric charges and currents. The Lorentz force law is usually introduced as an assumption separate from the field equations. Nevertheless the entire structure appears to be Lorentz covariant. Since the theory of relativity establishes some relationships among Maxwell’s equations and between electric and magnetic fields, this suggest that the equations of classical electrodynamics could be derived from fewer experimental data by using relativistic arguments.

In this paper we want to reveal such a possibility. It is our desire to introduce Maxwell’s equations in a manner which more clearly exhibits the role of special relativity. Namely, we show that to find out Maxwell’s equations it is enough to know from experiments only Coulomb’s law (established in the rest frame of the source-charge). Using the Lorentz transformation of coordinates we are able to determine the force and its components (the fields \( \vec{E} \) and \( \vec{B} \)) produced by the source-charge when it is in motion with constant velocity. We prove that defined within this formalism fields \( \vec{E} \) and \( \vec{B} \) undergo Maxwell’s equations. In this way the experimental way of discovering Maxwell’s equations may be replaced to some extent by a theoretical study.

Certainly, our method cannot be applied to accelerating sources because we do not know the Lorentz transformation between accelerating frames. Surprisingly Maxwell’s equations we derive for the special case, i.e. charges in uniform motion, have also much more general solutions depending on the acceleration of sources. The discovered theoretically laws lead us then to the completely general electromagnetic fields. Nevertheless one must be aware that while the physical correctness of the solutions of Maxwell’s equations for uniformly moving charges is logically guarantied by our reasoning (provided our space-time structure is Minkowskian), validity of the general solutions requires experimental verification.

Let us explain it in more detail in the context of the known views on the issue. It may seem that the results presented in this paper are in contradiction to the opinion of Feynman.
In his lectures he wrote that the statement that "all of electrodynamics can be deduced solely from Lorentz transformation and Coulomb’s law (...) is completely false."[1] Next he explains that to obtain Maxwell’s fields one must assume the retardation of interactions and that the electromagnetic potentials do not depend on acceleration of source. The textbook by W. G. V. Rosser [2] is the most famous work in which the development of Maxwell’s fields is performed on the basis of these assumptions. The equivalent set of assumptions is used by Frisch and Wilets [3] in their paper. If concerns our work, the standpoint we represent is essentially different. First of all, we do not derive directly Maxwell’s fields but Maxwell’s equations. Next we argue that the general Maxwell’s fields can be obtained simply as the solutions of the discovered equations. Certainly, in such an attitude no assumptions of those ones listed by Feynman must be separately postulated; simply they are embodied in the general solutions of Maxwell’s laws. It follows then that, contrary to the Feynman point of view, the proposed in this paper way to Maxwell’s equations, and next to Maxwell’s fields, requires only knowledge of Coulomb’s law and the Lorentz transformation.

But this is not the end of the story. In a sense Feynman is right. Namely, although one can derive Maxwell’s equations from merely Coulomb’s law and the Lorentz transformation, the general solutions of the derived Maxwell equations are not equivalent to the assumed laws. The final result, i.e. fields \( \vec{E} \) and \( \vec{B} \) depending on the acceleration of source, exceeds the situation of uniformly moving sources we consider to derive Maxwell’s equations and "suggests" also that the most general force law is the Lorentz one instead of Coulomb’s law (even if the source is at rest, a nonzero acceleration-dependent magnetic field is produced). But let us emphasize: it is only a "suggestion". While the fields \( \vec{E} \) and \( \vec{B} \) for sources moving with constant velocity are well defined through the Lorentz transformation of Coulomb’s law, the acceleration dependent electric and magnetic fields are completely different and cannot be expressed by means of Coulomb’s force. In the context of our reasoning the latter ones occur as a bonus but not necessary physically correct. This is why they need to be confirmed by additional experiments.

Concluding, actually "all of electrodynamics" does not amount to Coulomb’s law and the Lorentz transformation but at the same time we show in this paper that the Maxwell theory can be theoretically discovered (to be later verified experimentally) merely on the basis of
these laws.

The presented here approach at some points may resemble the Rosser [2] method of developing Maxwell’s equation for uniformly moving sources. However Rosser derives each Maxwell’s equation in a different manner. Gauss’ law is verified by means of integration of the explicit expression for $\vec{E}$. Faraday’s law is checked by integration and differentiation of the explicit formulas on $\vec{E}$ and $\vec{B}$. In turn, Gauss’ law for magnetism is proved on the basis of a field-line picture. The Ampere-Maxwell law is developed using purely vector analysis but its inhomogeneous form is derived only for the macroscopic (“averaged”) charge distributions and fields. In contrast to it our method is purely algebraic and the same for all Maxwell’s equations. In this way one may clearly see the role of Coulomb’s law and the Lorentz transformation occurring on the background of the vector identities, which is not so apparent in the method of Rosser. It seems also that mathematically our method is more concise and strict, especially if one compares the way how the transformation of forces is performed and how the important relations between the time and spatial derivatives are introduced (Rosser uses a pictorial method).

One can find an axiomatic development of the laws of electrodynamics in the work of Lehmberg [4]. At crucial points, however, the author performs generalization of Gauss’ law to a Lorentz-covariant law, which is a weak point of theoretical argumentations since generalizing means simply guessing. Similar methods are presented in the papers of Krefetz [5] and Kobe [6]. Maxwell’s fields are also derived by Tessman [7] who uses Coulomb’s law and a set of several additional assumptions (among them is Newton’s third law for steady state charge distributions). In the textbook by Purcell [8] electric and magnetic fields are developed for a special configuration of steady currents using the force transformation. A heuristic way to introduce Maxwell’s equations is shown in the textbook of M. Schwartz [9]. We believe that our deductive method which does not require neither extraordinary postulates nor any generalizations to get Maxwell’s equations is a nice exemplification of a power of theoretical reasoning based solely on the properties of the Lorentz transformations. The generalizations we need to justify entirely Maxwell’s theory are not required for obtaining Maxwell’s equations, as it is done in the cited papers, but refer only to the validity of the general solutions of the independently discovered laws.
2. DEFINITIONS

Assume in a system of reference \( S' \) a force \( \vec{F}' \) is exerted on a body. From the point of view of some observer placed in an another frame \( S \) in the same situation the measured force is \( \vec{F} \). If the frame \( S' \) has a velocity \( \vec{V} \) with respect to the system \( S \), then the relation between the forces is (see Appendix):

\[
\vec{F} = \vec{F}' || + \gamma \vec{F}' \perp + \gamma \vec{v} \times \left( \frac{\vec{V}}{c^2} \times \vec{F}' \right),
\]

where \( \gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2} \), \( \vec{v} \) is a velocity of the body measured in the system \( S \) and the indices \( || \) and \( \perp \) refer to the directions parallel and perpendicular to the velocity \( \vec{V} \). The force \( \vec{F} \) may be rewritten in the following form:

\[
\vec{F} = \vec{E} + \vec{v} \times \vec{B},
\]

where:

\[
\vec{E} = \vec{F}' || + \gamma \vec{F}' \perp
\]

and

\[
\vec{B} = \gamma \left( \frac{\vec{V}}{c^2} \times \vec{F}' \right).
\]

Note however that we can write the definition of \( \vec{B} \) also as follows:

\[
\vec{B} = \gamma \left( \frac{\vec{V}}{c^2} \times \vec{F}' \right) = \gamma \left( \frac{\vec{V}}{c^2} \times \frac{\vec{E} \perp \gamma}{c} \right),
\]

and finally get:

\[
\vec{B} = \frac{\vec{V}}{c^2} \times \vec{E}.
\]

The equation (7) with definitions (8) and (11) are quite general because we have not introduced any restrictions on the force \( \vec{F}' \).

Consider now interaction between two charged point particles. Let a charge \( Q \) be a source of force \( \vec{F}' \) that acts on a test charge \( q \). Let the charge \( Q \) be at rest in some inertial system \( S' \). The force \( \vec{F}' \) acting on the particle \( q \) at position \( \vec{r}' \) is given by the Coulomb law:

\[
\vec{F}' = \frac{qQ}{4\pi\varepsilon_0} \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3}.
\]
where \( \vec{r}'_Q \) it is the position of the charge \( Q \) in the frame \( S' \). If we express this force by means of electric field intensity \( \vec{E}' \)

\[
\vec{F}' = q\vec{E}',
\]  

(13)

the Coulomb law may be written in the equivalent form of Gauss’ law as:

\[
\vec{\nabla}' \cdot \vec{E}' = \frac{\rho' (\vec{r}')}{\epsilon_0},
\]  

(14)

where \( \rho' (\vec{r}') = Q\delta (\vec{r}' - \vec{r}'_Q) \). Eq. (12) shows clearly that the force \( \vec{F}' \) (so also \( \vec{E}' \)) is independent of the velocity of the test charge \( q \). Also the Coulomb force (and the field \( \vec{E}' \)) in the frame \( S' \) is static in the sense that it does not depend explicitly on time \( t' \).

Using Eq. (13) and definitions (8) and (11) we can now rewrite Eq. (7) as follows:

\[
\vec{F} = q\vec{E} + q\vec{v} \times \vec{B},
\]  

(15)

where

\[
\vec{E} = \vec{E}' || + \gamma \vec{E}' \perp
\]  

(16)

and

\[
\vec{B} = \frac{\vec{V}}{c^2} \times \vec{E}.
\]  

(17)

In our reasoning the charges \( q \) and \( Q \) are treated as scaling factors invariant under the Lorentz transformation. This assumption follows from the experimental data which show that charge is a velocity-independent quantity.

Because \( \vec{E}' \) does not depend on the test charge velocity, from the definitions (16) and (17) follows that the fields \( \vec{E} \) and \( \vec{B} \) as functions of \( \vec{E}' \) also do not depend on the test particle velocity. The force given by Eq. (15) separates then on the term that does not depend on the charge \( q \) velocity \( \vec{v} \), i.e. \( q\vec{E} \) and the term velocity-dependent: \( q\vec{v} \times \vec{B} \) determining the force perpendicular to the velocity of moving particle. Eq. (15) seems then to represent the Lorentz force law. To prove it is true we must show that the vectors \( \vec{E} \) (which we will call the electric field intensity in the frame \( S \)) and \( \vec{B} \) (which defines the magnetic field induction in the frame \( S \)) satisfy Maxwell’s equations. Validity of the Coulomb law in the rest frame of the source charge \( Q \) (which we will use in the form of the Gauss’ law (14)) is the only assumption which is needed to perform this task.
3. GAUSS' LAW FOR ELECTRICITY: $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

Let us calculate the divergence $\nabla \cdot \vec{E}$ for the field $\vec{E}$ in an arbitrary system of reference $S$. Since the operator $\nabla$ transforms as a vector under the spatial rotation of coordinates, the divergence $\nabla \cdot \vec{E}$ is a scalar function which value does not depend on the orientation of the axes of the system of reference. So, we choose the $x-$axis along the velocity $\vec{V}$ and apply the Lorentz transformation in its simple form: $x' = \gamma(x - Vt)$, $y' = y$, $z = z'$. From the relation (16) we have the components of electric field $\vec{E}$:

$$E_x = E'_x, \quad E_y = \gamma E'_y, \quad E_z = \gamma E'_z$$

Thus:

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial x'}{\partial x} \frac{\partial E'_x}{\partial x'} + \gamma \frac{\partial E'_y}{\partial y'} + \gamma \frac{\partial E'_z}{\partial z'} = \gamma \nabla' \cdot \vec{E}' = \frac{1}{\epsilon_0} Q \delta(r' - r'_Q),$$

(19)

where we have dropped the term $\frac{\partial E'_x}{\partial t'}$ because $E'_x$ does not depend on $t'$. But:

$$\delta(r' - r'_Q) = \delta(x' - x'_Q)\delta(y' - y'_Q)\delta(z' - z'_Q) = \delta(\gamma(x - x_Q))\delta(y - y_Q)\delta(z - z_Q) =$$

$$= \frac{1}{\gamma} \delta(x - x_Q)\delta(y - y_Q)\delta(z - z_Q) = \frac{1}{\gamma} \delta(r - r_Q),$$

(20)

where we have used the identity:

$$\delta(f(x)) = \sum_n \frac{\delta(x - x_n)}{|f'(x_n)|}$$

(21)

Inserting the result Eq. (20) into Eq. (19) we conclude that the field $\vec{E}$ measured in the arbitrary frame $S$ satisfies the Gauss’ law:

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} Q \delta(r - r_Q)$$

(22)

because $Q \delta(r - r_Q)$ is simply the charge density $\rho(r)$ of a point particle $Q$ in the frame $S$.

Note that instead of using the Dirac-delta function one may get the same result by considering the density $\rho'$ to be defined as $\rho' = Q/(dx'dy'dz')$. Next, due to the Lorentz contraction $dx' = \gamma dx$ we get $\rho' = Q/(\gamma dx dy dz) = \rho/\gamma$, which corresponds to the relation given in Eq. (20).
4. AMPERE-MAXWELL LAW: \( \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \frac{\gamma}{c^4} \vec{E} \)

Using the following vector identity:

\[ \vec{\nabla} \times (\vec{C} \times \vec{D}) = \vec{C} (\vec{\nabla} \cdot \vec{D}) - \vec{D} (\vec{\nabla} \cdot \vec{C}) + (\vec{D} \cdot \vec{\nabla}) \vec{C} - (\vec{C} \cdot \vec{\nabla}) \vec{D} \]  (23)

we have for the field \( \vec{B} \):

\[ \vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\frac{\vec{V}}{c^2} \times \vec{E}) = \frac{1}{c^2} \vec{V} (\vec{\nabla} \cdot \vec{E}) - \frac{1}{c^2} \vec{E} (\vec{\nabla} \cdot \vec{V}) + \frac{1}{c^2} (\vec{E} \cdot \vec{\nabla}) \vec{V} - \frac{1}{c^2} (\vec{V} \cdot \vec{\nabla}) \vec{E} \]  (24)

Because \( \vec{V} \) is a constant parameter of the Lorentz transformation (formally it may be treated as a constant "field") hence the second and the third term on the right hand side of the last equation vanish. So we get:

\[ \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \vec{V} (\vec{\nabla} \cdot \vec{E}) - \frac{1}{c^2} (\vec{V} \cdot \vec{\nabla}) \vec{E}. \]  (25)

First we find the equivalent form for the operator \( \vec{V} \cdot \vec{\nabla} \). The Lorentz transformation of spatial coordinates for an arbitrary direction of the velocity \( \vec{V} \) is [10]:

\[ \vec{r}' = \vec{r} + \frac{\gamma - 1}{V^2} (\vec{V} \cdot \vec{r}) \vec{V} - \gamma \vec{V} t, \]  (26)

and for the time coordinate:

\[ t' = \gamma \left( t - \frac{\vec{V} \cdot \vec{r}}{c^2} \right). \]  (27)

It follows that:

\[ \frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x} \frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x} \frac{\partial}{\partial z'} + \frac{\partial t'}{\partial x} \frac{\partial}{\partial t'} = \frac{\partial}{\partial x'} + \frac{\gamma - 1}{V^2} V_x (\vec{V} \cdot \vec{\nabla}') - \frac{\gamma V_x}{c^2} \frac{\partial}{\partial t'} \]  (28)

and similarly for the remaining components:

\[ \frac{\partial}{\partial y} = \frac{\partial y'}{\partial y} + \frac{\gamma - 1}{V^2} V_y (\vec{V} \cdot \vec{\nabla}') - \frac{\gamma V_y}{c^2} \frac{\partial}{\partial t'} \]  (29)

\[ \frac{\partial}{\partial z} = \frac{\partial z'}{\partial z} + \frac{\gamma - 1}{V^2} V_z (\vec{V} \cdot \vec{\nabla}') - \frac{\gamma V_z}{c^2} \frac{\partial}{\partial t'} \]  (30)

Using the above relation one gets:

\[ \vec{V} \cdot \vec{\nabla} = \gamma \vec{V} \cdot \vec{\nabla}' - \gamma \beta^2 \frac{\partial}{\partial t'} \]  (31)
where $\beta = V/c$. However on the basis of Eq. (27) we have also that:

$$
\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} + \frac{\partial y'}{\partial t} \frac{\partial}{\partial y'} + \frac{\partial z'}{\partial t} \frac{\partial}{\partial z'} + \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} = -\gamma \vec{V} \cdot \vec{\nabla}' + \gamma \frac{\partial}{\partial t'}.
$$

(32)

Hence:

$$
\gamma \vec{V} \cdot \vec{\nabla}' = \gamma \frac{\partial}{\partial t'} - \frac{\partial}{\partial t}.
$$

(33)

Inserting this result into Eq. (31) we find that:

$$
\vec{V} \cdot \vec{\nabla} = \frac{1}{\gamma} \frac{\partial}{\partial t} - \frac{\partial}{\partial t}.
$$

(34)

Coming back to Eq. (25) we obtain:

$$
\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \vec{V}(\vec{\nabla} \cdot \vec{E}) + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} - \frac{1}{\gamma c^2} \frac{\partial \vec{E}}{\partial t'}.
$$

(35)

Using the definition (16) and remembering that $\vec{E}'$ does not depend on time $t'$ we conclude that the last term is equal to zero. Recalling also the earlier derived Maxwell equation (22) and noting that $\vec{V} \rho$ is the current density $\vec{j}$ we finally arrive at the subsequent Maxwell equation:

$$
\vec{\nabla} \times \vec{E} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \frac{\vec{j}}{c^2 \epsilon_0}.
$$

(36)

5. FRADAY’S LAW: \( \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \)

Using Eqs (28)- (30) it is easy to show that:

$$
\vec{\nabla} \times \vec{E} = \gamma - \frac{1}{V^2} \vec{V} \cdot \vec{\nabla}'(\vec{V} \times \vec{E}) + \vec{\nabla}' \times \vec{E} - \frac{\gamma_c}{c^2} \frac{\partial}{\partial t'}(\vec{V} \times \vec{E}).
$$

(37)

But recalling Eq. (33) we have from (37):

$$
\vec{\nabla} \times \vec{E} = -\frac{\gamma - \frac{1}{V^2}}{\gamma} \frac{\partial}{\partial t}(\vec{V} \times \vec{E}) + \vec{\nabla}' \times \vec{E} + \left(1 - \frac{\gamma_c}{c^2}\right) \frac{1}{\gamma} \frac{\partial}{\partial t'}(\vec{V} \times \vec{E}).
$$

(38)

The last term is zero because $\vec{E}$ does not depend explicitly on $t'$. So:

$$
\vec{\nabla} \times \vec{E} = -\frac{\gamma - \frac{1}{V^2}}{\gamma} \frac{\partial}{\partial t}(\vec{V} \times \vec{E}) + \vec{\nabla}' \times \vec{E}.
$$

(39)
Let us now calculate $\vec{\nabla}' \times \vec{E}$. Firstly, from the definition (16) follows that:

$$\vec{E} = \vec{E}' + (\gamma - 1)\vec{E}_\perp'.\quad (40)$$

Also we can write that:

$$\vec{E}'_\perp = \frac{1}{V^2}(\vec{V} \times \vec{E}') \times \vec{V} \quad (41)$$

So we have:

$$\vec{\nabla}' \times \vec{E} = \vec{\nabla}' \times \vec{E}' + \frac{\gamma - 1}{V^2} \vec{\nabla}' \times [(\vec{V} \times \vec{E}') \times \vec{V}] \quad (42)$$

The field $\vec{E}'$ is an electrostatic field determined by the Coulomb law for which we have: $\vec{\nabla}' \times \vec{E}' = 0$. In turn, we can rewrite the remaining term using identity (23). Thus the last equation is:

$$\vec{\nabla}' \times \vec{E} = \frac{\gamma - 1}{V^2}[(\vec{\nabla}' \times \vec{E}')(\vec{\nabla}' \cdot \vec{V}) - \vec{V}(\vec{\nabla}' \cdot (\vec{V} \times \vec{E}')) +
+(\vec{V} \cdot \vec{\nabla}')(\vec{V} \times \vec{E}') - (\vec{V} \times \vec{E}') \cdot \vec{\nabla}'\vec{V}].\quad (43)$$

The first term in brackets is zero and, because of constancy of the "field" $\vec{V}$, also the last term vanish. It turns out also that the second term is zero since from the well know vector identity we have:

$$\vec{\nabla}' \cdot (\vec{V} \times \vec{E}') = \vec{E}' \cdot (\vec{\nabla}' \times \vec{V}) - \vec{V} \cdot (\vec{\nabla}' \times \vec{E}') = 0.\quad (44)$$

Thus there remains only the third term which we can evaluate using Eq. (33). As the result we get:

$$\vec{\nabla}' \times \vec{E} = \frac{\gamma - 1}{V^2} \left(\frac{\partial}{\partial t'} - \frac{1}{\gamma} \frac{\partial}{\partial t}\right) (\vec{V} \times \vec{E}') = -\frac{\gamma - 1}{\gamma V^2} \frac{\partial}{\partial t}(\vec{V} \times \vec{E}')\quad (45)$$

because the differentiation over $t'$ gives zero. But:

$$\vec{V} \times \vec{E}' = \vec{V} \times \vec{E}'_\perp = \vec{V} \times \frac{\vec{E}_\perp}{\gamma} = \frac{1}{\gamma} \vec{V} \times \vec{E}.\quad (46)$$

So Eq. (45) may be written as:

$$\vec{\nabla}' \times \vec{E} = -\frac{\gamma - 1}{\gamma^2 V^2} \frac{\partial}{\partial t}(\vec{V} \times \vec{E}).\quad (47)$$

Inserting the last result into Eq. (39) we find that:

$$\vec{\nabla} \times \vec{E} = -\left(\frac{\gamma - 1}{\gamma V^2} + \frac{\gamma - 1}{\gamma^2 V^2}\right) \frac{\partial}{\partial t}(\vec{V} \times \vec{E}) = -\frac{1}{c^2} \frac{\partial}{\partial t}(\vec{V} \times \vec{E})\quad (48)$$
Recalling the definition \((17)\) we see that Eq. \((48)\) represents the Maxwell equation:

\[
\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{49}
\]

\[\vec{\nabla} \cdot \vec{B} = 0 \]

6. GAUSS’ LAW FOR MAGNETISM: \(\vec{\nabla} \cdot \vec{B} = 0\)

Now it is easy to prove the remaining Maxwell equation. Namely,

\[
\vec{\nabla} \cdot \vec{B} = \frac{1}{c^2} \vec{V} \cdot (\vec{V} \times \vec{E}) = \frac{1}{c^2} \vec{E} \cdot (\vec{\nabla} \times \vec{V}) - \frac{1}{c^2} \vec{V} \cdot (\vec{\nabla} \times \vec{E}). \tag{50}
\]

The first term is zero and with help of Eq. \((49)\) we get:

\[
\vec{\nabla} \cdot \vec{B} = \frac{1}{c^4} \vec{V} \cdot \left( \frac{\partial}{\partial t} (\vec{V} \times \vec{E}) \right) = \frac{1}{c^4} \frac{\partial}{\partial t} \vec{V} \cdot (\vec{V} \times \vec{E}). \tag{51}
\]

It is evident that \(\vec{V} \cdot (\vec{V} \times \vec{E})\) is zero, so:

\[
\vec{\nabla} \cdot \vec{B} = 0, \tag{52}
\]

which ends our job.

7. DISCUSSION

1. Our definition of magnetic field \(\vec{B}\) (Eq. \((17)\)) seems to differ from that one gets as a solution of Maxwell’s equations:

\[
\vec{B} = \frac{1}{c} \vec{n}_{ret} \times \vec{E} \tag{53}
\]

However, it is well known that the solution for the field \(\vec{E}\) in case of constant velocity of source \(\vec{V}\) is a vector proportional to the instantaneous radius vector \(\vec{R} \equiv \vec{r} - \vec{r}_Q\). This vector is connected with the retarded vector radius \(\vec{R}_{ret} \equiv \vec{r} - \vec{r}_{ret}\) by the equality \(\vec{R} = \vec{R}_{ret} + \vec{V}_{ret}/c\).

It follows that \(\vec{V} = (\vec{R}_{ret} - \vec{R})c/R_{ret}\). Inserting this formula into the definition given by Eq. \((17)\) we arrive at the equality \((53)\), where \(\vec{n}_{ret} \equiv \vec{R}_{ret}/R_{ret}\).
2. Our reasoning is performed for the special case of source-charges moving with constant velocity. We know that besides the fields determined by such sources Maxwell’s equations offer us as their general solutions fields that additionally depend on the acceleration of source-charges. This may be regarded as an unexpected auxiliary discovery suggested by our reasoning. We emphasize that it does not follow from our reasoning that the general solutions are physically correct because we have neither defined the fields \( \vec{E} \) and \( \vec{B} \) for accelerating sources nor have proved that in this case the force have the form of the Lorentz force. Existence of such fields and forces is merely suggested by the derived results. It means that one must verify this fact experimentally. Forces produced by accelerating charges should agree with the Lorentz force law (15) in which inserted are the general solutions of Maxwell’s equations. So far all the experimental data confirm that the Lorentz force law and the discovered here theoretically general solutions of Maxwell’s equations are valid.

To avoid confusion let us mention that the relations between electromagnetic fields in different systems of reference given in Eqs. (16) and (17) do not apply to the general case of accelerating sources. The reason is that in case of non-uniformly moving charges at a space-time point \((\vec{x}', t')\) in the rest frame of the source, if only at the retarded moment the source accelerated, there exist a magnetic field \( \vec{B}' \). The force in such a frame is not then the Coulomb one but Lorentzian force \( \vec{F}' = q\vec{E}' + q\vec{v}' \times \vec{B}' \). The transformation (6) allows us to find the correct relations between the fields \((\vec{E}, \vec{B})\) and \((\vec{E}', \vec{B}')\):

\[
\vec{E} = \vec{E}' + \gamma \vec{E}'_\perp - \gamma \vec{V} \times \vec{B}',
\]

\[
\vec{B} = \vec{B}' + \gamma \vec{B}'_\perp + \frac{\gamma}{c^2} \vec{V} \times \vec{E}'.
\]

Eqs. (16) and (17) are then only a special case of the above ones (for the field \( \vec{B}' \) equal to zero).

Similarly, the reader should be aware that the correct general relation between the fields \( \vec{E} \) and \( \vec{B} \) is not (17) but (53).

3. We have derived Maxwell’s equations for a single point charge. Because these equations are linear and thanks to the Principle of Superposition the same equations are correct for electromagnetic fields produced by any distribution of many charges.

4. One may be tempted to apply our method to the gravitational Newton force which
formally may seem to be identical to Coulomb’s force and obtain Maxwell’s equation for the gravitational interactions. This procedure would however be wrong because if one wants to consider the gravitational force in the realm of relativity (without using the General Theory of Relativity) to get correct results it must be assumed that the gravitational mass depends on the velocity of moving body. It is not the case if concerns the charge $Q$ which is velocity-independent. The method presented in this work applies then only to the electromagnetic forces.

8. CONCLUSIONS

Coulomb’s law together with the Lorentz transformation of force have led us to electromagnetic fields produced by uniformly moving sources. These fields have appeared to be governed by Maxwell’s equations. In turn, if Maxwell’s equations are solved, they deliver us a wider class of solutions that could apply to arbitrarily moving sources. Experiments confirm this unexpected prediction. In this way the reasoning presented in this paper is an example of a scientific discovery based solely on the theoretical argumentation. However, as for any kind of theoretical results, their experimental verification is indispensable.

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APPENDIX

Let a reference system $S'$ moves from the point of view of some other system $S$ with constant velocity $\vec{V}$. In the reference system $S'$ the equation of motion is:

$$\vec{F}' = \frac{d\vec{p}'}{dt'}$$  \hspace{1cm} (56)

and in the system $S$:

$$\vec{F} = \frac{d\vec{p}}{dt}.$$  \hspace{1cm} (57)

Each of these equations may be decomposed on the components along and the directions parallel and perpendicular to the velocity $\vec{V}$:

$$\vec{F}'_\parallel = \frac{d\vec{p}'_\parallel}{dt'}, \quad \vec{F}'_\perp = \frac{d\vec{p}'_\perp}{dt'}$$  \hspace{1cm} (58)

and

$$\vec{F}_\parallel = \frac{d\vec{p}_\parallel}{dt}, \quad \vec{F}_\perp = \frac{d\vec{p}_\perp}{dt}. $$  \hspace{1cm} (59)

To find the relation between forces $\vec{F}$ and $\vec{F}'$ let us recall the Lorentz transformation for momentum and time:

$$\vec{p}'_\parallel = \gamma \left( \vec{p}_\parallel - E\frac{\vec{V}}{c^2} \right),$$

$$\vec{p}'_\perp = \vec{p}_\perp$$  \hspace{1cm} (60)

and

$$t' = \gamma \left( t - \frac{\vec{r} \cdot \vec{V}}{c^2} \right), $$  \hspace{1cm} (61)

where $\gamma = (1 - V^2/c^2)^{-1/2}$. We have then that:

$$\frac{dt}{dt'} = \frac{1}{\gamma \left( 1 - \vec{v} \cdot \vec{V}/c^2 \right)},$$  \hspace{1cm} (62)

where $\vec{v} = d\vec{r}/dt$ is a velocity of body measured in the frame $S$. First, using the second equation of (60) and Eq. (62), we find that:

$$\vec{F}'_\perp = \frac{dt}{dt'} \frac{d\vec{p}_\perp}{dt} = \frac{1}{\gamma \left( 1 - \vec{v} \cdot \vec{V}/c^2 \right)} F_\perp.$$  \hspace{1cm} (63)
In turn, for the parallel component of force we obtain from the first equation of (60) and Eq. (62) that:

\[ \vec{F}_\parallel' = \gamma \frac{dt}{d\tau} \left( \vec{p}_\parallel - E \frac{\vec{V}}{c^2} \right) = \frac{1}{1 - \vec{v} \cdot \vec{V}/c^2} \left( \vec{F}_\parallel - \vec{F} \cdot \vec{v} \frac{\vec{V}}{c^2} \right), \]  

(64)

where we have substituted \( dE/dt = \vec{F} \cdot \vec{v} \). Now using the well known vector identity:

\[ \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \]  

(65)

we get:

\[ \vec{F} \cdot \vec{v} \frac{\vec{V}}{c^2} = \vec{v} \times \left( \frac{\vec{V}}{c^2} \times \vec{F} \right) + \left( \frac{\vec{v} \cdot \vec{V}}{c^2} \right) \vec{F}. \]  

(66)

Thus Eq. (64) may be written as:

\[ \vec{F}_\parallel' = \frac{1}{1 - \vec{v} \cdot \vec{V}/c^2} \left( \vec{F}_\parallel - \frac{\vec{v} \cdot \vec{V}}{c^2} \vec{F}_\parallel - \frac{\vec{v} \cdot \vec{V}}{c^2} \vec{F}_\perp - \vec{v} \times \left( \frac{\vec{V}}{c^2} \times \vec{F} \right) \right), \]  

(67)

or in a simpler form:

\[ \vec{F}_\parallel' = \vec{F}_\parallel - \gamma \frac{\vec{v} \cdot \vec{V}}{c^2} \vec{F}_\perp - \frac{\vec{v} \times \left( \frac{\vec{V}}{c^2} \times \vec{F} \right)}{1 - \vec{v} \cdot \vec{V}/c^2}. \]  

(68)

However, using Eq. (63) we get:

\[ \vec{V} \times \vec{F} = \vec{V} \times \vec{F}_\perp = \gamma \left( 1 - \frac{\vec{v} \cdot \vec{V}}{c^2} \right) \vec{V} \times \vec{F}_\perp = \gamma \left( 1 - \frac{\vec{v} \cdot \vec{V}}{c^2} \right) \vec{V} \times \vec{F}'. \]  

(69)

Because of this the third term on the right hand side of Eq. (68) can be much simplified and we have:

\[ \vec{F}_\parallel' = \vec{F}_\parallel - \gamma \frac{\vec{v} \cdot \vec{V}}{c^2} \vec{F}_\perp - \gamma \vec{v} \times \left( \frac{\vec{V}}{c^2} \times \vec{F}' \right). \]  

(70)

Finally then from the last equation we obtain:

\[ \vec{F}_\parallel = \vec{F}_\parallel' + \gamma \frac{\vec{v} \cdot \vec{V}}{c^2} \vec{F}_\perp + \gamma \vec{v} \times \left( \frac{\vec{V}}{c^2} \times \vec{F}' \right) \]  

(71)

and from Eq. (63):

\[ \vec{F}_\perp = \gamma \left( 1 - \frac{\vec{v} \cdot \vec{V}}{c^2} \right) \vec{F}_\perp'. \]  

(72)
Adding up the last two equations we find that:

\[ \vec{F} = \vec{F}'_{\parallel} + \gamma \vec{F}'_{\perp} + \gamma \vec{v} \times \left( \frac{\vec{V}}{c^2} \times \vec{F}' \right), \]  

(73)

which is the desired relation between the forces \( \vec{F} \) and \( \vec{F}' \).

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