INTRODUCTION

We shall introduce and study quadratic entry locus varieties, a class of projective algebraic varieties whose extrinsic and intrinsic geometry is very rich.

Let us recall that, for an irreducible non-degenerate variety \( X \subset \mathbb{P}^n \) of dimension \( n \geq 1 \), the secant defect of \( X \), denoted by \( \delta(X) \), is the difference between the expected dimension and the effective dimension of the secant variety \( S_X \subset \mathbb{P}^n \) of \( X \), that is \( \delta(X) = 2n + 1 - \dim(S_X) \). This is an important projective invariant measuring the dimension of the entry locus of \( X \) described by the points on \( X \) spanning secant lines passing through a general point \( p \in \mathbb{P}X \); see Section 1.

Many examples appearing in different settings suggested the definition of quadratic entry locus manifold of type \( \delta \), briefly \( \mathcal{Q}_E \mathcal{L} \)-manifold of type \( \delta \). These are smooth varieties \( X \subset \mathbb{P}^n \) for which \( \delta_p \) is a smooth quadric hypersurface of dimension \( = \delta(X) \), whose linear span in \( \mathbb{P}^n \) is the locus of secant lines to \( X \) passing through \( p \in \mathbb{P}X \) general. We also consider \( \mathcal{Q}_E \mathcal{L} \)-manifolds of type \( \delta \), that is smooth varieties \( X \subset \mathbb{P}^n \) for which \( \delta_p \) is the union of smooth quadric hypersurfaces of dimension \( = \delta(X) \), \( p \in \mathbb{P}X \) general; see Section 1 for more details. When \( \delta = 0 \) and \( n \geq 2n + 1 \) the above conditions do not impose particular geometric restrictions on \( X \). On the contrary \( \mathcal{Q}_E \mathcal{L} \)-manifolds \( X \subset \mathbb{P}^{2n+1} \) of type \( \delta = 0 \) are linearly normal and rational, see [CMR], while in [IR2] it is proved that every \( \mathcal{Q}_E \mathcal{L} \)-manifold of type \( \delta > 0 \) is rational. The notion of \( \mathcal{Q}_E \mathcal{L} \)-manifold was also motivated by the remark that a lot of secant defective smooth varieties with special geometric properties and/or with extremal tangential behaviour are \( \mathcal{Q}_E \mathcal{L} \)-manifolds: varieties defined by quadratic equations having enough linear syzygies, for example satisfying condition \( N_2 \) of Green (Proposition 1.4), homogeneous varieties, secant defective or not; Scorza Varieties (and in particular Severi Varieties; this property being an essential ingredient for their classification, cf. [Z2] and Section 3); centers of special Cremona transformations of type \( \varepsilon_2(z) \) ([ESB] and Section 4), varieties whose dual variety is small ([E1] and [IR2]) and which are not hypersurfaces. Furthermore, if \( n = 2;3 \) any smooth secant defective variety \( X \subset \mathbb{P}^n \) with \( SX \subset \mathbb{P}^n \) ( \( \mathbb{P}^n \) is a \( \mathcal{Q}_E \mathcal{L} \)-manifold; see [Se,S1,F1].

By definition \( \mathcal{Q}_E \mathcal{L} \)-manifolds of type \( \delta > 0 \) are very special examples of rationally connected varieties [KMM, Ko, De]. Gaetano Scorza was the first who realized the link between secant defective varieties and rational connectedness in the pioneering papers [S1, S3], where, ante litteram, the condition of rational connectedness by conics appears for \( \mathcal{V}_2 \), and the condition of rational connectedness by conics for \( \mathcal{V}_3 \) is an important family of conics and, for \( \delta \geq 2 \), of lines produced by the quadratic entry loci via the modern tools of deformation theory of rational curves on a manifold and via their parameter spaces. For \( \delta 

The most important results for \( \mathcal{Q}_E \mathcal{L} \)-manifolds of type \( \delta \geq 2 \) are consequences of the study of the projective geometry of the subvariety \( Y_x \subset \mathbb{P}^{n-1} \), especially for \( \delta > 3 \) when \( Y_x \) is also irreducible. This is not
surprising since the family of lines on such an $X \in \mathbb{P}^N$ is the minimal covering family of rational curves in the sense of Mori; see [Mo], [HM1], [HM2], [HM3], the surveys [H1], [H2] and the recent [HK] for many interesting connections between the projective geometry of analogous of $Y_x$, the so called variety of minimal rational tangents, and geometrical properties of Fano manifolds.

With regard to the projective geometry of $Y_x \in \mathbb{P}^{n-1}$, we prove a key result asserting that, for $\mathbb{P}E \mathbb{L}$-manifolds of type $3$, the variety $Y_x \in \mathbb{P}^{n-1}$ is a $\mathbb{Q}E \mathbb{L}$-manifold of type $2$ such that $S(Y_x) = \mathbb{P}^{n-1}$, see Theorem 2.3 part (d). This allows the inductive definition of some varieties naturally attached to a $\mathbb{L}E \mathbb{L}$-manifold of type $3$, see Section 2 which can be considered as algebro-geometric analogues of the successive projective differential forms of an arbitrary irreducible projective variety; [GH], [H]. If $r_x = \sup f(2N : 2x + 1g, for every $k = 1, \ldots, r_2$, we set $X^k = Y_x (x^k)$; where $2X^k$ is a general point and where $X^0 = X$ (see Section 2 for more precise definitions). Thus for every $k = 1, \ldots, r_2$, the variety $X^k$ is a $\mathbb{Q}E \mathbb{L}$-manifold of type $2k$. Performing the calculations on the dimension of the $X^k$’s we arrive at the surprising Divisibility Property that $2^r_x$ divides $n$ and at other strong restrictions, Theorem 2.8. This result can be considered as a generalized secant analogue of the famous Landman Parity Theorem for the dual defect $\text{def}(X)$ of a variety $X = \mathbb{P}^N : n \equiv \text{def}(X) \pmod{2}$, if $\text{def}(X) \equiv 0 \pmod{2}$; see [E1] Theorem 2.4]. The repeated applications of the Divisibility Theorem in the paper will allow unitary and simple proofs of many important theorems regarding projective varieties with very special geometric properties and it was also applied, together with its consequences proved here, in [Fu], Section 4] to obtain a new Linear Normality Bound for $\mathbb{P}E \mathbb{L}$-manifolds and a new characterization of smooth quadric hypersurfaces.

For example we can easily classify $\mathbb{P}E \mathbb{L}$-manifolds of type $3$ in Corollary 3.1 obtaining only five known examples and answering a problem posed in [KS] 0.12.6. We also provide a two line proof that $\mathbb{P}E \mathbb{L}$-manifolds of type $3 = \frac{n}{2}$ appear only in dimension 2, 4, 8 or 16, see the proof of Corollary 3.2. In particular this last result implies that Severi varieties appear only in these dimensions. This was the key and most difficult point for getting their classification, which after this step is quite easy at least today, see the almost self-contained and short proof of the classification in Corollary 3.2. The classification of Severi varieties was the main result of [Z1] (see also [LV], [L1], [Ch]) and produced a great impact for its deep connections with Hartshorne Conjecture and also with other areas of mathematic such as the classification of composition algebras over a field. Here, via our approach, we generalize this to the classification of $\mathbb{P}E \mathbb{L}$-manifolds of type $3 = n/2$, see loc. cit., and we present an interesting connection between the classification of Severi Varieties and Hartshorne Conjecture which, as far as we know, was not noticed before, see Remark 3.3.

In Section 4 using also [H1], we obtain some strong restrictions for the existence of special Cremona transformations $\mathbb{P}^N \rightarrow \mathbb{P}^N$ of type $(2; d)$, whose base loci are linearly normal $\mathbb{Q}E \mathbb{L}$-manifolds of type $3 = \frac{n+1}{2}$ with notable properties, as shown in Proposition 4.2. Our applications to special Cremona transformations concern the complete classification of those of type $(2; 3)$, Corollary 4.4 of those of type $(2; 5)$, Corollary 4.5 and of all special Cremona transformation in $\mathbb{P}^{2n+1}$, Corollary 4.6 where $n$ is the dimension of the base locus.

1. Definitions, Preliminary Results and Examples

We work over the field of complex numbers, unless otherwise stated. Let $X \in \mathbb{P}^N$ be a closed irreducible subvariety, which we will always suppose to be non-degenerate. From now on $n = \dim X + 1$ and $N = n + 1$.

Let $X \in \mathbb{P}^N$ be an irreducible, non-degenerate projective variety of dimension $n$. Let

$$S_X = \frac{S}{[x^k y^l] \mathbb{P}^N} \times \mathbb{P}^N$$

be the secant variety to $X$. 
Clearly \( \dim (SX) = m \in \mathbb{N} \geq 2n + 1g \). If the equality \( \dim (SX) = 2n + 1 \) holds, then \( X = \mathbb{P}^N \) is said to be non-defective. If \( \dim (SX) < 2n + 1 \), then \( X = \mathbb{P}^N \) is said to be secant defective, or simply defective and \( \mathcal{X} = 2n + 1 \). \( \dim (SX) \) is called the secant defect of \( X \). 

For \( p \in SX \) and \( x \), the closure of the locus of couples of distinct points on \( X \) spanning secant lines passing through \( p \) is called the entry locus of \( X \) with respect to \( p \in SX \) and it will be indicated by \( \mathcal{E}_p \). The closure of the secant lines to \( X \) passing through \( p \) is a cone over \( \mathcal{E}_p \), let us call it \( C_p \). If \( X = \mathbb{P}^N \) is smooth, then \( \mathcal{E}_p = C_p \setminus X \) as schemes for \( p \in SX \) general, see for example [FR, Lemma 4.5]. Moreover it is easy to see that for \( p \in SX \) general, \( C_p \) is equidimensional of dimension equal to \( \mathcal{X} \). Thus \( \dim (\mathcal{E}_p \cap X) = \mathcal{X} + 1 \). In general \( \mathcal{E}_p \) (and hence a fortiori \( C_p \)) is not irreducible.

For an irreducible variety \( X = \mathbb{P}^N \) and for \( x \) and \( y \in X \), the (embedded) projective tangent space at \( x \) and \( y \) is denoted by \( T_x \) and the affine (or Zariski) tangent space at \( x \) by \( T_x \).

**Definition 1.1.** (cf. also [ZZ, KS, IR1]) An irreducible, non-degenerate projective variety \( X = \mathbb{P}^N \) is said to be a local quadratic entry locus variety of type \( 0 \), briefly an \( LQEL \)-variety of type \( 0 \), if, for general \( x ; y \in X \) distinct points and for general \( p \in \mathbb{P}^N \), the union of the irreducible components of the entry locus of \( p \) passing through \( x \) and through \( y \) is a quadric hypersurface of dimension \( = \mathcal{X} \). in the given embedding \( X \subset \mathbb{P}^N \). Equivalently an \( LQEL \)-variety of type \( 0 \) is an irreducible projective variety \( X = \mathbb{P}^N \) if through two general points there passes a quadric hypersurface of dimension \( = \mathcal{X} \) contained in \( X \).

An irreducible projective variety \( X = \mathbb{P}^N \) is said to be a quadratic entry locus variety of type \( 0 \), briefly a \( LQEL \)-variety of type \( 0 \), if for general \( p \in SX \) the entry locus \( \mathcal{E}_p \) is a quadric hypersurface of dimension \( = \mathcal{X} \).

An irreducible, non-degenerate projective variety \( X = \mathbb{P}^N \) is said to be a conic-connected variety, briefly a \( CC \)-variety, if through two general points of \( X \) there passes an irreducible conic contained in \( X \).

If \( X = \mathbb{P}^N \) is also smooth, we shall use the terms \( LQEL \)-manifold, \( QEL \)-manifold, respectively \( CC \)-manifold.

**Lemma 1.2.** Let \( X \) be an \( LQEL \)-manifold with \( \mathcal{X} > 0 \) and let \( x ; y \in X \) be general points. There is a unique quadric hypersurface of dimension \( \mathcal{X} \), say \( Q_{xy} \), passing through \( x ; y \) and contained in \( X \). Moreover, \( Q_{xy} \) is irreducible.

**Proof.** Uniqueness follows from the fact that the general entry locus passing through two general points is smooth at these points by Terracini Lemma (see e.g. [FR, Proposition 3.3]). To see that \( Q_{xy} \) is irreducible, we may assume \( \mathcal{X} = 1 \) by passing to general hyperplane sections (see Proposition 1.3 below). Assume first that \( X \) is covered by lines passing through \( x \). Being also smooth, \( X \) is a linear space, so it is not an \( LQEL \)-manifold. Otherwise, after suitable normalization, the family of conics through \( x \) is generically smooth and the result follows.

A monodromy argument shows that for an \( LQEL \)-variety the general entry locus \( \mathcal{E}_p \) is a union of quadric hypersurfaces of dimension \( \mathcal{X} \). Moreover, the general entry locus of a \( QEL \)-manifold, or each irreducible component of the general entry locus of an \( LQEL \)-manifold, is a smooth quadric hypersurface, see the arguments in [FR] pp. 964–966.

A \( QEL \)-variety of type \( 0 \) is clearly an \( LQEL \)-variety of type \( 0 \) and an \( LQEL \)-variety of type \( 1 \) is a \( CC \)-variety but the converses are not true. Every variety \( X = \mathbb{P}^N \) with \( \mathcal{X} = 0 \) is a \( LQEL \)-variety of type \( = 0 \), while \( QEL \)-varieties of type \( = 0 \) are those for which through a general point of \( SX \) there passes a unique secant line. This last condition is especially relevant for \( N = 2n + 1 \); see [CMR] for example. One constructs examples of \( LQEL \)-manifolds which are not \( QEL \)-manifolds by projecting isomorphically a \( QEL \)-manifold \( X = \mathbb{P}^N \) of type \( 1 \) and with \( N > 2n + 1 \) into \( \mathbb{P}^{2n+1} \). Indeed, consider a general \( L = \mathbb{P}^{2n+1} \) in \( \mathbb{P}^N \) and let \( L : \mathbb{P}^{99k} \mathbb{P}^{2n+1} \) be the projection from \( L \). Then \( SX \setminus L = \emptyset \) and the the restriction of \( L \) to \( SX \) is a finite surjective morphism onto \( \mathbb{P}^{2n+1} \) of degree \( d = 2 \) equal to \( \deg (SX) \), while the restriction of \( L \) to \( X \) is an isomorphism onto \( \mathbb{P}^{2n+1} \). Thus a general \( q \in SX = \mathbb{P}^{2n+1} \) has exactly \( d \) preimages via \( L \) on \( SX \), let us say \( p_1 ; \ldots ; p_d \), which are also general points on \( SX \).
The entry locus \( \bar{X} \) contains as irreducible components of dimension \( = \bar{X} \) the projection of the quadric hypersurfaces \( \mathcal{Q} X \). Moreover, it is not difficult to see that the irreducible components of \( \bar{X} \) of dimension \( \bar{X} \) are exactly the quadric hypersurfaces \( \mathcal{Q} X \), \( i = 1, \ldots, d \).

Let us collect some consequences of the definitions in the following Proposition, whose proof is left to the reader, see also [R2] Proposition 1.3.

**Proposition 1.3.** Let \( X = \mathbb{P}^N \) be an irreducible, non-degenerate projective variety and let, if it exists, \( X_0 = \mathbb{P}^N, M \) be an isomorphic projection of \( X = \mathbb{P}^N \). Then

1. If \( X = \mathbb{P}^N \) and if \( X = \mathcal{Q} E L \)-manifold, then \( X = \mathbb{P}^N \) is linearly normal;
2. \( X^0 = \mathbb{P}^N \) is an \( \mathcal{Q} E L \)-manifold if and only if \( X = \mathcal{Q} E L \)-manifold;
3. If \( X = \mathbb{P}^N \) is an \( \mathcal{Q} E L \)-manifold of type \( 1 \), then a general hyperplane section \( \mathcal{E} = \mathbb{P}^N \) is an \( \mathcal{Q} E L \)-manifold of type \( \mathcal{E} = \mathcal{L} \).
4. \( X = \mathbb{P}^N \) is an \( \mathcal{Q} E L \)-variety of type \( n \) if and only if \( N = n + 1 \) and \( \mathbb{P}^{n+1} \) is a quadric hypersurface.

We describe a lot of examples of \( \mathcal{Q} E L \)-manifolds to see their ubiquity among projective varieties with very special geometric properties, as cited in the introduction. Another interesting class, not contained in the one described below, is considered in Section 4.

**Proposition 1.4.** ([Vc], [HKS]) A smooth non-degenerate variety \( X = \mathbb{P}^N \), scheme theoretically defined by quadratic equations whose Koszul syzygies are generated by linear ones, is a \( \mathcal{Q} E L \)-manifold.

There are several possible equivalent definitions of the projective second fundamental form \( J \mathcal{I}_X \mathcal{X} \mathcal{P} (S^2 (\Gamma_x X)) \) of an irreducible projective variety \( X = \mathbb{P}^N \) at a general point \( x \neq X \), see for example [IL], Section 3.2 and end of Section 3.5]. We shall use the one related to tangential projections, as in [IL], Remark 3.2.11.

Suppose \( X = \mathbb{P}^N \) is non-degenerate, as always, let \( x = \mathbb{P}^N \) be a general point and consider the projection from \( T_x X \) onto a disjoint \( \mathbb{P}^n \).

\[
(1.1) \quad x : X = \mathbb{P}^N \quad \mathbb{P}^n.
\]

The map \( x \) is associated to the linear system of hyperplane sections cut out by hyperplanes containing \( T_x X \), or equivalently by the hyperplane sections of \( X = \mathbb{P}^N \) singular at \( x \).

Let \( B_x X = \mathcal{E} \) be the blow-up of \( X \) at \( x \), let

\[
E = \mathcal{P} (\Gamma_x X) = \mathbb{P}^{n-1} = B_x X
\]

be the exceptional divisor and let \( H \) be a hyperplane section of \( X = \mathbb{P}^N \). The induced rational map \( e_x : B_x X \to \mathbb{P}^{N-1} \) is defined as a rational map along \( E \) since \( X = \mathbb{P}^N \) is not a linear space; see for example the argument in [E1], 2.1 (a)]. The restriction of \( e_x \) to \( E \) is given by a linear system in \( j \), \( (H) = 2E, j \).

**Definition 1.5.** The second fundamental form \( J \mathcal{I}_x \mathcal{X} \mathcal{P} (S^2 (\Gamma_x X)) \) of an irreducible non-degenerate variety \( X = \mathbb{P}^N \) of dimension \( n \) at a general point \( x \neq X \) is the non-empty linear system of quadric hypersurfaces in \( \mathcal{P} (\Gamma_x X) \) defining the restriction of \( e_x \) to \( E \).

Clearly \( J \mathcal{I}_x \mathcal{X} \mathcal{P} (S^2 (\Gamma_x X)) \) consists of asymptotic directions, i.e. of lines directed to lines having a contact of order at least two with \( X \) at \( x \). For example when \( X = \mathbb{P}^N \) is defined by equations of degree at most two, the base locus of the second fundamental form consists of points giving tangent lines contained in \( X \) and passing through \( x \) so that it is exactly the locus of lines through \( x \) and contained in \( X \), \( Y_x = \mathcal{P} (\Gamma_x X) = E \), which will be defined in Proposition 2.2.

**Lemma 1.6.** Let \( X = \mathbb{P}^N \) be a smooth irreducible non-degenerate variety, and assume \( x \) is not linear. The irreducible components of the closure of a general fibre of \( x \) are not linear.
Proof. We may assume by passing to linear sections $\mathcal{X} = 1$. Let $l$ be a line, passing through a general point $y \in X$, which is an irreducible component of the closure of a general fibre of $x$. By Terracini Lemma $T_y (\mathcal{X}) \setminus T_y (\mathcal{X})$ is a point, say $p_{x,y}$. Since $1 \in \mathcal{H}_x (\mathcal{X}) \setminus T_y (\mathcal{X})$, $\mathbb{P}_{x,y} 2 1$. By symmetry there is also a line $\ell$ in $\mathcal{H}_x (\mathcal{X}) \setminus x 1 X$, $x 2 \ell$, $p_{x,y} 2 \ell$. So, $1$ [$\ell$ is a conic contained in the plane $\mathcal{H}_x y p_{x,y} \ell$ passing through $x_1 y$. Reasoning as in the proof of Lemma [7.2] we find a contradiction.

2. Qualitative properties of $\mathbb{L}_\mathbb{Q} \mathbb{E} \mathbb{L}$-manifolds

We describe the conics naturally appearing on $\mathbb{L}_\mathbb{Q} \mathbb{E} \mathbb{L}$-manifolds of type $\mathbb{P} > 0$ and relate them to intrinsic invariants; see also [IR2].

Theorem 2.1. Let $X$ be an $\mathbb{L}_\mathbb{Q} \mathbb{E} \mathbb{L}$-manifold of type $\mathbb{I}$. Then:

1. The variety $X$ is rationally connected, so that it is a simply connected manifold such that $H^0 (\mathcal{X}^m) = 0$ for every $m \geq 1$ and $H^1 (\mathcal{O}_X) = 0$ for every $1 > 0$.
2. There exists on $X$ an irreducible family of conics $\mathcal{C}$ of dimension $2n + 3$, whose general member is smooth. This family describes an open subset of an irreducible component of the Hilbert scheme of conics on $X$.
3. Given a general point $x \in X$, let $\mathcal{C}_x$ be the family of conics in $\mathcal{C}$ passing through $x$. Then $\mathcal{C}_x$ has dimension $n + 2$, equal to the dimension of the irreducible components of $\mathcal{C}_x$ describing dense subsets of $X$.
4. Given two general points $x_1 y \in X$, the locus $Q_{x_1 y}$ of the family $\mathcal{C}_{x_1 y}$ of smooth conics in $\mathcal{C}$ passing through $x$ and $y$ is a smooth quadric hypersurface of dimension $1$. The family $\mathcal{C}_{x_1 y}$ is irreducible and of dimension $1$.
5. For a general conic $C$ in $\mathcal{C}_x$, $X \setminus K_x = C = n + 1$.
6. A general conic $C \subset X$, $\mathcal{C}_x$ intersects $T_x X$ only at $x$. Moreover the tangent lines to smooth conics contained in $X$ and passing through $x$ describe an open subset of $\mathbb{P} (\mathcal{T}_x X)$.

Proof. The variety $X$ is clearly rationally connected. The conclusions of part (1) are contained in [KMM Proposition 2.5] and also in [De] Corollary 4.18.

Part (4) is the definition of an $\mathbb{L}_\mathbb{Q} \mathbb{E} \mathbb{L}$-variety. Indeed, the plane spanned by every conic through $x$ and $y$ contains the line $\mathcal{H}_x yi$, which is a general secant line to $X$. Thus a conic through $x$ and $y$ is contained in the entry locus of every $p 2 \mathcal{H}_x yi$ not on $X$, so that for $p 2 \mathcal{H}_x yi$ general it is contained in the smooth quadric hypersurface $Q_{x_1 y}$ through $x$ and $y$. Thus there exists a smooth conic passing through $x$ and a general point $y \in X$. In particular, there exists an irreducible family of smooth conics passing through $x$, let us say $\mathcal{C}_x$, whose members describe a dense subset of $X$. Since the number of irreducible components of the Hilbert scheme of conics in $X$ is finite, we get that

$$T_x \subset \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1} (a_i)$$

is ample for $C 2 C_2^1$ general; see for example [Ko II.10.1]. Hence $a_i > 0$ for every $i = 1; \ldots; n$, and $\mathcal{N}_{C = X}$ is ample being a quotient of $T_x \subset \mathcal{O}$. Thus $C_1$ is smooth at $C 2 C_2^1$ and of dimension

$$H^0 (\mathcal{N}_{C = X} (1)) = H^0 (\mathcal{N}_{C = X} (1)) = 2 = 2 + \sum_{i=1}^{X} a_i = K_x C 2^1.$$

Let $\mathcal{G}_x$ be the universal family over $\mathcal{C}_x$ and let $\mathcal{G}_x \subset X$ be the tautological morphism. By part (4) and the theorem on the dimension of the fibers we get that $\dim (\mathcal{G}_x) = n + 1$. Thus

$$n + 2 = \dim (\mathcal{C}_x) = K_x C 2.$$
Proposition 2.2. Theorem 2.3. The proof is for reader's convenience.

Let \( C \times X \) be a general smooth conic passing through \( x \times X \). In particular \( C \) is not completely contained in \( T_x X \setminus X \) since the conics through \( x \) cover \( X \). Suppose there exists a point \( z \in T_x X \setminus C \), \( z \not\in x \). The line \( l_x z_i \) is contained in \( T_x X \) and also in the plane generated by \( C \), let us say \( =: \mathcal{C} \). Since the conic \( C \) is smooth at \( x \) and since \( C \) is not contained in \( T_x X \), we deduce \( l_x z_i \cap T_x X = T_x C \), that is \( l_x z_i \cap T_x C \) is smooth.

Then the line \( l_x z_i \) would cut \( C \) in at least 3 points, counted with multiplicities, which is clearly impossible.

We saw that \( T X \overset{\oplus}{_{i=1}} O_p : (a_i) \), with \( a_i > 0 \) for \( C \setminus C_x \) general. Consider the map \( x : C \mapsto P(\Gamma_x X) \), which associates to a conic in \( C_x \) its tangent line at \( x \). The closure of the image of \( x \) in \( P(\Gamma_x X) \) has dimension \( n - 1 \). Indeed by \([\mathbf{Ko}, \text{II.3.4}]\) the dimension of \( X \cap C \) at \( x \) is equal to \# \( a_i > 0 \geq 1 \), finishing the proof.

On an \( LQ E L \)-manifold of type \( 2 \) there are also lines coming from the entry loci and we proceed to investigate them. The following result is essentially well known, see also \([\mathbf{H}], \text{Proposition 1.5}]\); we recall its proof for reader's convenience.

**Proposition 2.2.** Let \( X \mapsto P^n \) be a smooth irreducible variety. Then:

1. The Hilbert scheme of lines passing through a general point \( x \times X \), if not empty, is smooth and can be identified with a smooth not necessarily irreducible variety \( Y_x \), \( P^{n-1} = P(\Gamma_x X) \).
2. If \( Y_x \) is 1; : : : ; \( m \), \( m \) are the irreducible components of \( Y_x \), then we have
   \[ \dim (Y_x^1) + \dim (Y_x^p) \leq 2 \quad \text{for every } 1 \leq p \leq m \]

**Proof.** We argue essentially as in the proof of Theorem 2.1. Let \( L \) be a line through the very general point \( x \times X \). Every such line \( L \) is free, see for example \([\mathbf{Ko}, \text{II.3.11}]\), so that \( T X \overset{\oplus}{_{i=1}} O_p : (a_i(L)) \), with \( a_n(L) = 0 \) and \( a_{n-1}(L) = 1 \) \( \text{if } a_1(L) = 2 \). Moreover \( a_1(L) = 0 \) because \( TP^1 \) is a subbundle of \( T X \). On the other hand, \( TP^n \overset{\oplus}{_{i=1}} O_p : (a_i(L)) \), with \( a_n(L) = 0 \) \( \text{and } a_{n-1}(L) = 1 \) \( \text{if } a_1(L) = 2 \). Moreover, the arbitrary line \( L \) is a standard (or minimal) curve in the sense of Mori Theory. It follows that the map which associates to each line through \( x \) its tangent direction is a closed embedding so that we can identify the Hilbert scheme of lines through \( x \) with a variety \( Y_x \mapsto P^{n-1} = P(\Gamma_x X) \), which is smooth. Indeed, \( N_x = \sum_{j=1}^{n} \overline{L_j}^1 \) contains \( T X \) as a subbundle so that \( a_n(L) = 0 \) and \( a_{n-1}(L) = 1 \) \( \text{for every } j = 1; \ldots; n \), being the quotient of a locally free sheaf generated by global sections. Therefore \( h^1 N_x = 0 \) and \( Y_x \) is smooth at the point corresponding to \( L \). Since \( L \) was an arbitrary line through \( x \), \( Y_x \) is smooth. The conditions on the dimension of two irreducible components simply say that these components cannot intersect in \( P^n \).

Now we prove a fundamental result on the geometry of lines on an \( LQ E L \)-manifold, which, via the study of the projective geometry of \( Y_x \mapsto P^{n-1} \) and of its dimension, will yield significant obstructions for the existence of \( LQ E L \)-manifolds of type \( 3 \). The most relevant part for future applications is part (4), (d). Part (1) holds more generally for every smooth secant defective variety.

**Theorem 2.3.** Suppose that \( X \mapsto P^n \) is an \( LQ E L \)-manifold of type \( 3 \).

1. (\([\mathbf{SS}], \text{p. 282, Opere Complete, vol. I}]\)) If \( 1 \), then
   \[ \mathcal{E}_X : P(\Gamma_x X) \mapsto P^{n-1} \]
   is dominant, so that \( \dim (Y_x^1) = N - 1 \) and \( N \geq \frac{n(n+3)}{2} \).
2. If \( 2 \), the smooth, not necessarily irreducible, variety \( Y_x \mapsto P^{n-1} \) is non-degenerate and it consists of irreducible components of the base locus scheme of \( Y_x \). Moreover, the closure of the irreducible component of a general fiber of \( \mathcal{E}_X \) passing through a general point \( p \in P(\Gamma_x X) \) is a linear space \( P^{n-1} \), cutting scheme-theoretically \( Y_x \) in a quadric hypersurface of dimension \( 2 \).
(3) If $Y_x \subseteq \mathbb{P}(\mathcal{T}_x X)$ is irreducible and if $2 \leq X \leq \mathbb{P}(\mathcal{I}_x X)$ and $Y_x \subseteq \mathbb{P}(\mathcal{T}_x X)$ is a Q-EL-manifold of type $X$, then $S Y_x = \mathbb{P}(\mathcal{I}_x X)$ and $Y_x \subseteq \mathbb{P}(\mathcal{T}_x X)$ is a Q-EL-manifold of type $X$.

(4) If $3 \leq \mathcal{I}_x X$, then

a) $\mathcal{P} \subseteq \mathcal{I}_x X \implies \mathcal{L} \cap \mathcal{I}_x X (1) = 1$.

b) For any line $\mathcal{L} \subseteq X$, $k_x \cap (\mathcal{L} = \mathcal{L} \cap \mathcal{I}_x X)$, where $k_x$ is the index of the Fano manifold $X$. In particular $n + 1 \equiv 0 \mod 2$; that is $n \equiv 0 \mod 2$.

c) There exists on $\mathcal{L}$ an irreducible family of lines of dimension $\frac{n+1}{2}$ such that for a general $\mathcal{L}$ in this family

$$\mathbb{T}_X \mathfrak{L} = \mathcal{O}_{\mathcal{L}^1} (\mathcal{L}) \mathcal{O}_{\mathcal{L}^1} (\mathcal{L})^{\frac{n+1}{2}} \mathcal{O}_{\mathcal{L}^1} (\mathcal{L})^{\frac{n+1}{2}}$$

\[ \text{d) If } x \leq 2 \text{ is general, then } Y_x \subseteq \mathbb{P}(\mathcal{I}_x X) \text{ is a Q-EL-manifold of dimension } \frac{n+1}{2} \text{, of type } X. \]

**Proof.** Part (1) is classical and as we said above holds for every smooth secant defective variety. Since its proof is self-contained and elementary for Q-EL-varieties, we include it for the reader’s convenience. It suffices to show that, via the restriction of $e_x$, the exceptional divisor $E = \mathcal{P}(\mathcal{I}_x X)$ dominates $W_x \subseteq \mathbb{P}^n \times \mathbb{P}^n$. Take a general point $y \leq 2 \times X$. By part (6) of Theorem 2.1, there exists a conic $C_{x,y}$, through $x$ and $y$, cutting $T_x X$ only at $x$. Thus $(C_{x,y}) = x (y)$, $W_x$ is a general point and clearly $e_x (\mathcal{P}(\mathcal{I}_x C_{x,y})) = x (y)$. Therefore the restriction of $e_x$ to $E$ is dominant as a map to $W_x \subseteq \mathbb{P}^n \times \mathbb{P}^n = N \times N$, yielding $\dim (\mathcal{I}_x E) = N \equiv 1$. In particular $N \equiv 1 = \dim (\mathcal{I}_x X).$ (2.14: 2.14) (2.14: 2.14) (2.14: 2.14)

Suppose now on $2$. If $y \leq 2 \times X$ is a general point and if $C_{x,y}$ is a smooth conic through $x$ and $y$, then $\mathcal{P}(\mathcal{I}_x C_{x,y})$ is a general point of $\mathcal{P}(\mathcal{I}_x X)$, by Theorem 2.1 part (6). Consider the unique quadric hypersurface $Q_{x,y}$ of the unique quadric hypersurface $Q_{x,y}$ of dimension $2$ through $x$ and $y$, the irreducible conic through $x$ and $y$ of the entry locus of a general $p \leq 3 \times X$. Then $C_{x,y} \subseteq X$ and $T_x \subseteq C_{x,y} \subseteq X$, $T_x \subseteq C_{x,y} \subseteq X$, $T_x \subseteq C_{x,y} \subseteq X$. Take a line $L_x$ through $x$ and contained in $Q_{x,y}$, which can be thought of as a point of $Y_x \subseteq \mathbb{P}(\mathcal{I}_x X)$. The plane $\mathcal{P}(\mathcal{I}_x X) \subseteq X$ is contained in $T_x \subseteq C_{x,y}$ so that it cuts $Q_{x,y}$ at least in another line $L_x^0$, clearly different from $T_x \subseteq C_{x,y}$. Thus $T_x \subseteq C_{x,y}$ belongs to the pencil generated by $L_x$ and $L_x^0$, which projectivized in $\mathfrak{P}(\mathcal{I}_x X)$ means that through the general point $p \in \mathcal{P}(\mathcal{I}_x C_{x,y}) \subseteq \mathfrak{P}(\mathcal{I}_x X)$ there passes the secant line $\mathcal{P}(\mathcal{I}_x L_x) \subseteq \mathcal{P}(\mathcal{I}_x L_x)$ to $Y_x$. Therefore $Y_x \subseteq \mathcal{P}(\mathcal{I}_x X)$ is non-degenerate and the join of $Y_x$ with itself equals $\mathbb{P}(\mathcal{I}_x X)$. For an irreducible $Y_x \subseteq \mathcal{P}(\mathcal{I}_x X)$, this means exactly $Y_x = \mathbb{P}(\mathcal{I}_x X)$. The scheme $T_x \subseteq Q_{x,y} \subseteq Q_{x,y}$ is a quadric cone with vertex $x$ and base a smooth quadric hypersurface of dimension $2$. The lines in $T_x \subseteq Q_{x,y} \subseteq Q_{x,y}$ describe a smooth quadric hypersurface of dimension $2$ of $Y_x \subseteq \mathcal{P}(\mathcal{I}_x X)$, whose linear span $\mathfrak{P}(\mathcal{I}_x X) = \mathbb{P}(\mathcal{I}_x X)$ passes through $e_x (y) \in \mathcal{P}(\mathcal{I}_x C_{x,y})$. Since $e_x : \mathcal{P}(\mathcal{I}_x X) \times \mathbb{P}^n \to \mathcal{P}(\mathcal{I}_x X) \times \mathbb{P}^n$ is given by a linear system of quadrics vanishing on $Y_x$, the whole $\mathcal{P}(\mathcal{I}_x X)$ is contracted by $e_x$ to $e_x (y)$. The closure of the irreducible component of $e_x (e_x (y))$ passing through $x$ has dimension $n = \dim (\mathcal{I}_x X) = 1$ so that it coincides with $\mathfrak{P}(\mathcal{I}_x X) = \mathbb{P}(\mathcal{I}_x X)$. This also shows that $Y_x$ is an irreducible component of the support of the base loci scheme of $\mathcal{P}(\mathcal{I}_x X)$ and also that, when irreducible, $Y_x \subseteq \mathbb{P}(\mathcal{I}_x X)$ is a Q-EL-manifold of type $2$. Indeed in this case $x \leq 2 \subseteq Y_x$ is a general point and every secant or tangent line to the smooth irreducible variety $Y_x$ passing through $x$ is contracted by $e_x$, since the quadrics in $\mathcal{P}(\mathcal{I}_x X)$ vanish on $Y_x$. Thus every secant line through $x$ is contained in $\mathfrak{P}(\mathcal{I}_x X)$ and the entry locus with respect to $x$ is exactly $\mathfrak{P}(\mathcal{I}_x X)$. This concludes the proof of parts (2) and (3).

Suppose $3 \leq \mathcal{I}_x X$ and let us concentrate on part (4). Item (a) follows directly from Barth–Larsen Theorems, [BL], but we provide a direct proof using the geometry of Q-EL-varieties. There are lines through a general point $x \leq 2 \times X$, for example the ones constructed from the family of entry loci. Reasoning as in Proposition 2.22 we get

$$\mathbb{T}_X \mathfrak{L} = \mathcal{O}_{\mathcal{L}^1} (\mathcal{L}) \mathcal{O}_{\mathcal{L}^1} (\mathcal{L})^{\frac{n+1}{2}} \mathcal{O}_{\mathcal{L}^1} (\mathcal{L})^{\frac{n+1}{2}}$$
for every line $L$ through $x$. Thus if such a line comes from a general entry locus, we get

$$2 + m(L) = K_x, \quad L = \frac{K_x}{2} = \frac{n}{2};$$

yielding $m(L) = \frac{n}{2} - 2$.

We define $R_x$ to be the locus of points on $X$ which can be joined to $x$ by a connected chain of lines whose numerical class is $\frac{1}{2} + 1 \in C$ a general conic. By construction we get $R_x = X$, so that the Picard number of $X$ is one by [Ko IV.3.13.3]. Since the variety $X$ is simply connected being rationally connected, see Theorem 2.2, we deduce $\dim(X) = 1$. Thus $X$ is a Fano variety, $X$ is equidimensional of dimension $\frac{n}{2}$, and for every line $L$ through $x$. We claim that $X$ is irreducible.

Indeed, if there were two irreducible components $Y_1, Y_2$ of $X$ and obtain $i_1 + i_2 = 2$ almost clear (and well known) that $\dim(X) = 2$.

By Proposition 1.4, we know that $\dim(X) = 2$.

Example 2.4. (Segre varieties) $X = \mathbb{P}^{1} \times \mathbb{P}^{m} \times \mathbb{P}^{3}, l, m, 1, m, 1, are \ Fano \ manifolds \ of \ type \ (2, 2)$

By Proposition 1.4 we know that $X = \mathbb{P}^{1} \times \mathbb{P}^{m} \times \mathbb{P}^{3}, l, m, 1, m, 1, are \ Fano \ manifolds \ of \ type \ (2, 2)$

The fact that $X_1 \cap X_2 = \mathbb{P}(\mathbb{P}^{1} \times \mathbb{P}^{m} \times \mathbb{P}^{3})$ is a Fano manifold, clearly with $2$, and we calculate its type, that is we determine $X_1 \cap X_2 = \mathbb{P}(\mathbb{P}^{1} \times \mathbb{P}^{m} \times \mathbb{P}^{3})$.

The locus of lines through a point $x$ is easily described, being the union of the two linear spaces of the rulings through $x$, that is $X_1 \cap X_2 = \mathbb{P}(\mathbb{P}^{1} \times \mathbb{P}^{m} \times \mathbb{P}^{3})$. Indeed, if there were two irreducible components $Y_1, Y_2$ of $X$ and obtain $i_1 + i_2 = 2$.

Example 2.5. (Grassmann varieties of lines $G(1, 2)$) $\mathbb{P}^{3}$ is projectively equivalent to the Segre variety $\mathbb{P}^{1} \times \mathbb{P}^{2}$.

Moreover, $G(1, 2)$ is an arbitrary line, yielding $K_x = (r + 1)H, H$ an hyperplane section. Finally $r + 1 = 2\frac{2}{r + 1}$ by Theorem 2.3, that is $r = 4$.

Example 2.6. (Spinor variety $S^{10}$ and $\mathbb{C}$ variety $X$) $\mathbb{P}^{26}$ as Fano manifolds). Let us analyze the 10-dimensional spinor variety $S^{10}$.

It is scheme theoretically defined by 10 quadratic forms defining a 10-dimensional quadric hypersurface and the closure of every fiber is a $\mathbb{P}^{3}$ cutting $X$ along a smooth quadric hypersurface, see for example [LSB]. In particular $X_1 = 6$ and $X$ is a Fano manifold of index $i(X) = 8 = n$ such that $\mathbb{P}(\mathbb{C} X) = ZH X (1)$. It is a so called Mukai variety with $\mathbb{C}(1) = 1$ and by the above description it is a Fano manifold of type $= 6$. For every $L$ the variety $X = X_1 \cap ZH X (1)$ is a variety of dimension $\frac{n}{r + 1}$.

We can begin the process with the 16-dimensional variety $X = E_6 \times \mathbb{P}^{26}$, a Fano manifold of index $i(X) = 12$ with $l_2(X) = 1$ and with $X = 8$. This is a Fano manifold of index $= 8$, being the center of a (2, 2) special Cremona transformation, see [LSB] and Proposition 4.2. By applying the above constructions one obtains $X^1 = X_1 \cap ZH X (1) = \mathbb{P}^{10}$, see [Z2 IV.1]. One could also apply [Mii], since $X_1 \times \mathbb{P}^{15}$ has dimension 10 and type $= 6$ so that it is a Fano manifold of index $i(X) = (n + 2) = 8 = n$.

Hence $X^2 = X_1 \cap ZH X (1) = \mathbb{P}^{10}$ and finally $X^3 = X_1 \cap ZH X (1) = \mathbb{P}^{10}$.
The examples discussed above and the results of Theorem 2.3 suggest to iterate the process, whenever possible, of attaching to an LQ E L-manifold of type 3 a non-degenerate Q E L-manifold $Y_4 \subset \mathbb{P}^n$ of type 2 such that $SY_4 = \mathbb{P}^{n - i}$. If $r = 1$ is the largest integer such that $2r \equiv 1 \pmod{2}$, and if $X \subset \mathbb{P}^n$ is a LQ E L-manifold of type 2, then the process can be iterated $r$ times, obtaining Q E L-manifolds of type $2k \equiv 3$ for every $k = 1, \ldots, r$.

**Definition 2.7.** Let $X \subset \mathbb{P}^n$ be a LQ E L-manifold of type 3. Let

$$x_k = \sup\{2r : 2r + 1 \equiv \frac{1}{2} \pmod{2}\}.$$ 

For every $k = 1, \ldots, x_k$, 1, we define inductively

$$X^k = X^k (z_0 ; \ldots ; z_k, 1) = Y_{z_k}, \quad \alpha^k (z_0 ; \ldots ; z_k, 2);$$

where $z_i \in X^i, i = 0, \ldots, x_k$, 1, is a general point and where $X^0 = X$.

The process is well defined by Theorem 2.3 since for every $k = 1, \ldots, x_k$, 1, the variety $X^k$ is a Q E L-manifold of type $\alpha^k = 2k \equiv 3$. The Q E L-manifold $X^k$ depends on the choices of the general points $z_0 ; \ldots ; z_k, 1$ used to define it. The type and dimensions of the $X^k$'s are well defined and we are interested in the determination of these invariants.

The following result is crucial for the rest of the paper. Its proof is a direct consequence of part (4), d) of Theorem 2.3.

**Theorem 2.8.** Let $X \subset \mathbb{P}^n$ be a LQ E L-manifold of type 3. Then:

1. For every $k = 1, \ldots, x_k$; the variety $X^k \subset \mathbb{P}^n$ is a Q E L-manifold of type $\alpha^k = 2k \equiv 3$; \quad (mod $Z^k$)

2. $2^{x_k}$ divides $n$, that is $n \equiv 0 \pmod{2^{x_k}}$.

**Remark 2.9.** Much weaker forms of the Divisibility Theorem were proposed in [OH, Theorem 0.2] after long computations with Chern classes.

The hypothesis 3 is clearly sharp for the congruence established in part (2) of Theorem 2.3, or for its weaker form proved in part (4) of Theorem 2.3. Indeed for the Segre varieties $X_{2n} \subset \mathbb{P}^1 \times \mathbb{P}^{n - 1}$ of odd dimension $n = 1 + m$, of odd dimension $n = 1 + m$, we have $\alpha_{2n} = 2; (2)$. See Example 2.4.

It is worthwhile remarking that the above result is not true for arbitrary smooth secant defective varieties having $\alpha = 3$ neither in the weaker form of a parity result. One can consider smooth non-degenerate complete intersections $X \subset \mathbb{P}^n$ with $2n - 2$ and such that $n \equiv 1 \pmod{2}$. It is easy to see that for an arbitrary non-degenerate smooth complete intersection $X \subset \mathbb{P}^n$ with $2n + 1$, we have $\alpha = 2; (2)$. If $n \equiv 2 \pmod{2}$, then $\alpha = 2 + 1 \equiv 3 \pmod{2}$ and $\alpha = 1 \equiv 1 \pmod{2}$.

Infinite series of secant defective smooth varieties $X \subset \mathbb{P}^n$ of dimension $n$ with $SY_4 \subset \mathbb{P}^n$, $\alpha = 3$ and such that $n \equiv 0 \pmod{2}; \alpha \equiv 3 \pmod{2}$ can be constructed in the following way. Take $Z \subset \mathbb{P}^n$ a smooth Q E L-manifold of type 4 and dimension $n$ such that $Z \subset \mathbb{P}^{n + 1}$. Consider a $\mathbb{P}^{n + 1}$ containing the previous $\mathbb{P}^n$ as a hyperplane, take $p \subset \mathbb{P}^{n + 1}$ and let $Y = \mathbb{P}^n$ be the cone over $Z$ of vertex $p$. If $W \subset \mathbb{P}^{n + 1}$ is a general hypersurface of degree $d > 1$, not passing through $p$, then $X = W \setminus Y \subset \mathbb{P}^{n + 1}$ is a smooth non-degenerate variety of dimension $n$ such that $X_{2n} \subset \mathbb{P}^{n + 1}$, $(\mathbb{P}^{n + 1})$. Thus $\alpha = 1 \equiv 1 \pmod{3}$ and $\alpha = 3 \equiv 1 \pmod{2}$.

One can take, for example, $Z_n = G (L_2 + 1) \subset \mathbb{P}^{n + 1}$, $n = 8$, which are Q E L-manifolds of dimension $n = 8$ and type $\alpha = 4$ such that $Z_n \subset \mathbb{P}^{n + 1}$.
3. Some Classification Results

In this section we classify various important classes of $LQEL$-manifolds. [KI] contains the complete classification of $CC$-manifolds with $(X) = 2$ and hence that of $LQEL$-manifolds of type $= 1; 2$.

Let us recall that a non-degenerate smooth projective variety $X \subset P^+_n$ is said to be a "Hartshorne variety" if it is not a complete intersection, where as always $n = \dim (X)$. It is worth remarking that there exist Hartshorne varieties different from the ones described in item ii) and iii) of Corollary 3.1 below, see for example [E2 Proposition 1.9]. This last fact was kindly pointed out to me by Giorgio Ottaviani.

The first relevant application of the Divisibility Property is the following classification of $LQEL$-manifolds of type $> \frac{1}{2}$, which answers a problem posed in [KS 0.12.6].

**Corollary 3.1.** Let $X \subset P^N$ be an $LQEL$-manifold of type $with \frac{1}{2} < n$. Then $X \subset P^N$ is projectively equivalent to one of the following:

i) the Segre 3-fold $P^1 \times P^2 \subset P^5$;

ii) the Plücker embedding $G(1; 4) \subset P_9$;

iii) the 10-dimensional spinor variety $S^{10} \subset P_{15}$;

iv) a general hyperplane section of $G(1; 4) \subset P^9$;

v) a general hyperplane section of $S^{10} \subset P_{15}$.

In particular, $G(1; 4) \subset P^9$ and $S^{10} \subset P_{15}$ are the only $LQEL$-manifolds, modulo projective equivalence, which are also Hartshorne varieties.

**Proof.** By assumption $\frac{1}{2} > 0$. If $\frac{1}{2} > 2$, then $n = 3$ and $m = 2 = n - 1$. Therefore $N = 5$ and $X$ is projectively equivalent to the Segre 3-fold $P^1 \times P^2 \subset P^5$ by Proposition 3.4 below.

From now on we can assume $\frac{1}{2} > 2$. Then $X \subset P^N$ is a Fano manifold with $P \cong (X) = ZH (1)$. By Theorem 2.3 there exists an integer $m$ such that $n = \frac{1}{2} + m 2^\ell$ and since $2 > n$ by hypothesis, we have

\begin{equation}
\frac{1}{2} > m 2^\ell.
\end{equation}

Suppose $2x_2 + 2$. From $2x_2 + 2 > m 2^\ell$ it follows $m = 1$ and $x_2 = 2$. Hence either $\ell = 4$ and $n = 6$ and $X \subset P^N$ is a Fano manifold of index $i(X) = (n + 1) = 2 = n - 1$ or $\ell = 6$ and $n = 10$ and $X \subset P^N$ is a Fano manifold as above and of index $i(X) = (n + 1) = 2 = n - 1$. In the first case by [F2 Theorem 8.11] we get case ii).

In the second case we apply [Mu], obtaining case iii).

Suppose $2x_2 + 1$. From $2x_2 + 1 > m 2^\ell$ it follows $m = 1$ and $x_2 = 2$. Therefore either $\ell = 3$ and $m = 1 + m 2^\ell$ or $\ell = 5$ and $n = 1 + m 2^\ell$. Reasoning as above, we get cases iv) and v).

To prove the last part let us recall that for a non-degenerate smooth variety $X \subset P^+_n$ necessarily $S^X = P^+_n$; see [Z2 V.1.13]. Thus $(X) = \frac{1}{2} + 1 > \frac{1}{2}$ and applying the first part we deduce that we are either in case ii) or iii) or that $\frac{1}{2} + 1 = n$, i.e. $n = 2$. In the last case $X \subset P^3$ would be a quadric surface which is a complete intersection and hence not a Hartshorne variety. This concludes the proof.

Let us recall that a smooth non-degenerate irreducible variety $X \subset P^+_{n+2}$ of dimension $n$ such that $S_X (P^+_{n+2})$ is called a "Severi variety", cf. [Z1], [Z2].

Another interesting application of Theorem 2.3 is the classification of $LQEL$-manifolds of type $= \frac{1}{2}$. For such varieties we get immediately that $n = 2; 3; 8 \text{ or } 16$ and among them we find Severi varieties. Indeed, by [Z2] IV.2.1, IV.3.1, IV.2.2], see also [Ru], Severi varieties are $QEL$-manifolds of type $= \frac{1}{2}$. Once we know that $n = 2; 3; 8 \text{ or } 16$, it is rather simple to classify Severi varieties, see also [Z2 IV.4] and [L1]. For $n = 2; 4$ the result is classical and well known while in our approach the $n = 8$ case follows from the classification of Mukai manifolds, [Mu]. The less obvious case is $n = 16$ where we apply a result from [L2], see also [L3], after an easy reduction via Corollary 3.1. What is notable, in our opinion, is not the fact that this proof is short, easy, natural, immediate and almost self-contained but the perfect parallel between our argument based on the Divisibility Theorem and some proofs of Hurwitz Theorem on the dimension of composition algebras over a field such as the one contained in [Lal V.5.10], see also [Cu Chap. 10. Sec. 36]. Surely this connection is well
known today, see [Z2] pg. 89–91, but the other proofs of the classification of Severi varieties did not make this parallel so transparent. Moreover in Remark 3.3 below we shall explain an interesting relation between our approach to the classification of Severi varieties and Hartshorne Conjecture on Complete Intersection.

About this result and the word "generalization" we would like to quote Herman Weyl: "Before you can generalize, formalize and axiomatize, there must be a mathematical substance", [We]. There is no doubt that the mathematical substance in this problem is entirely due to Fyodor Zak, who firstly brilliantly solved it in [Z1].

**Corollary 3.2.** Let $X \subset P^N$ be an LQEL-manifold of type $\frac{12}{7}$. Then $n = 2; 4; 8$ or 16 and $X \subset P^N$ is projectively equivalent to one of the following:

i) the cubic scroll $S(1; 2) \subset P^4$;

ii) the Veronese surface $\pi_2(P^2) \subset P^5$ or one of its isomorphic projection in $P^7$;

iii) the Segre 4-fold $P^1 \times P^3 \subset P^7$;

iv) a general 4-dimensional linear section $X \subset P^7$ of $G(1; 4) \subset P^9$;

v) the Segre 4-fold $P^2 \subset P^8$ or one of its isomorphic projections in $P^7$;

vi) a general 8-dimensional linear section $X \subset P^{13}$ of $S^{10} \subset P^{15}$;

vii) the Plücker embedding $G(1; 5) \subset P^{14}$ or one of its isomorphic projection in $P^{13}$;

viii) the $E_8$-variety $X \subset P^{26}$ or one of its isomorphic projection in $P^{25}$;

ix) a 16-dimensional linearly normal rational variety $X \subset P^{25}$, which is a Fano variety of index 12 with $S X = P^{25}$, def$(X) = 0$ and such that the base locus of $\mathcal{I}_{X, x} \mathfrak{j} Z_x \subset P^{15}$, is the union of a 10-dimensional spinor variety $S^{10} \subset P^{15}$ with $C_2 S^{10} \subset P^7$, $P^{15} \cap S^{10}$.

In particular, a Severi variety $X \subset P^{2n+2}$ is projectively equivalent to a linearly normal variety as in ii), v), vii) or viii).

**Proof.** By assumption $n$ is even. If $n < 6$, then $n = 2$ or $n = 4$. If $n = 2$, the conclusion is well known, see [Sc] or Proposition 3.4. If $n = 4$, then $n = 2 = 2 n = 2$. If $H$ is a hyperplane section and if $C \subset 2 H$ is a general conic, then $K_X + 3 H$ (hence $C = n + 2 n = 0$ by part (5) of Theorem 2.1). Suppose $X \subset P^N$ is a scroll over a curve, which is rational by Theorem 2.1. Since for a rational normal scroll either $S X = P^{25}$ or $\dim(S X) = 2 n + 1$, we get $N = \dim(S X) = 2 n + 1 = 7$ so that $X \subset P^7$ is a rational normal scroll of degree 4, which is the case described in iii). If $X \subset P^N$ is not a scroll over a curve, $K_X + 3 X$ is generated by global sections, see [Z1] Theorem 1.4, and since through two general points of $X$ there passes such a conic, we deduce $K_X = 3 H$. Thus $X \subset P^N$ is a del Pezzo manifold, getting cases iii), iv) or v) by [Z2] Theorem 8.11.

Suppose from now on $n = 6$, $n = 2$, and hence that $X \subset P^N$ is a Fano manifold with $\mathcal{I}_{X, x} \mathfrak{j} Z + 3 H$ (hence $C = n + 2 n = 0$ by part (5) of Theorem 2.1). By Theorem 2.8 $Z^i(x)$ divides $n = \frac{n}{2}$ so that $Z^{i+1}$ divides $n$ and $= \frac{n}{2}$ is even. By definition of $r_X$, $\frac{n}{2} = 2 r_X + 2$, so that, for some integer $m \geq 1$, $m 2^{i+1} = n = 4 (6 m + 1)$:

Therefore either $r_X = 1$ and $n = 8$, or $r_X = 3$ and $n = 16$. In the first case we get that $X \subset P^N$ is a Fano manifold as above and of index $i(x) = n + 2 = 6 = n = 2$ and we are in cases vi) and vii) by [Mu]. In the remaining cases $Y_x \subset P^{15}$ is a 10-dimensional LQEL-manifold of type $= 6$ so that $Y_x \subset P^{15}$ is projectively equivalent to $S^{10} \subset P^{15}$ by Corollary 3.1. Furthermore

$N = 16 = \dim(\mathcal{I}_{X, x} \mathfrak{j} Z + 1) + h^0(I_{S^{10}}(2)) = 10$.

Thus either $N = 26$ and $Y_x \subset S^{10} \subset P^{15}$ is the base locus of the second fundamental form or one easily sees that we are in case ix). If $\mathcal{I}_{X, x} \mathfrak{j} Z = h^0(I_{S^{10}}(2)) \mathfrak{j} j$ we are in case viii) by [L2], see also [L3].

The next remark is somehow hidden in the proof above and, at least to our knowledge, it seems to have not been noticed before.
Remark 3.3. The famous Hartshorne Conjecture on Complete Intersections asserts that a smooth $n$-dimensional variety $X \subset P^n$ is a complete intersection if $n > \frac{2d}{3}$, see [1a]. We recalled above that the classification of $LQEL$-manifolds of type $\frac{n}{2}$ was known classically for $n = 2; 4$. Thus suppose $n < 6$ and that $X \subset P^n$ is a $LQEL$-manifold of type $\frac{n}{2} \leq 3$. In particular $N \dim (S_X) = \frac{3n}{2} + 1$. Then, by Theorem 2.3, $Y_X \subset P^{n-1}$ is a non-degenerate $QEL$-manifold of dimension $\frac{3n}{2} - 2$ and type $\frac{n}{2} = 2, 1$, which is not a complete intersection since

$$h^0 (I_{Y_X}(O)) \leq \dim (S_X) + 1 = \frac{n}{2} + 1 > \frac{n}{4} + 1 = \coxim (Y_X; P^{n-1});$$

Moreover we have that

$$\frac{3n}{4} - 2 = \dim (V_X) \geq \frac{2(n - 1)}{3}$$

so that the existence of a $LQEL$-manifold of type $\frac{n}{2}$ and of dimension $n > 16$ would have produced a counterexample to Hartshorne Conjecture on Complete Intersections. Obviously the interesting fact in the above proof is that we can prove directly $n = 3; 4; 6$, if $n = 6$, without invoking Hartshorne Conjecture.

We collect below some classification results used previously in this section, whose proof is straightforward.

Proposition 3.4. Let $X \subset P^n$ be an $LQEL$-manifold of type $= \frac{n}{2}$, $n = 2$ or $n = 3, N \geq 5$ and $X \subset P^n$ is projectively equivalent to one of the following:

1. $P^1 \times P^2 \subset P^5$ Segre embedded, or one of its hyperplane sections;
2. the Veronese surface $\gamma (P^3) \subset P^5$ or one of its isomorphic projections in $P^6$.

Proof. Theorem 2.3 part(4) yields $n = 2$. Thus either $n = 2$ and $n = 1$, or $n = 3$ and $n = 2$. Suppose first $n = 2$ and let $C \subset X$ be an irreducible conic coming from a general entry locus. Since $\kappa_X = C = 3$ by Theorem 2.1, we get $C^2 = 1$ via adjunction formula. Moreover, $h^1 (O_X) = 0$ by Theorem 2.1, so that $h^0 (O_X (C)) = h^0 (O_{P^1} (1)) + 1 = 3$ and $C$ is base point free. The morphism $\gamma : \gamma : X \subset P^2$ is dominant and birational since $\deg (O) = C^2 = 1$. Moreover, sends a conic $C$ into a line so that $\gamma : P^2 \cong K$ is given by a sublinear system of $\gamma (P^2 (O))$ of dimension at least four and the conclusion for $n = 2$ is now immediate. If $n = 3$, we get the conclusion by passing to a hyperplane section, taking into account Proposition 1.3.

4. Special Cremona Transformations of Type $(2; d)$

In this section we apply the Divisibility Theorem and the classification of conic-connected manifolds contained in [11] to the classification of special Cremona transformations of type $(2; d)$, $d \geq 3$. Special Cremona transformations of type $(2; d)$ were classified in [ESB] as those for which the base locus scheme is one of the four Severi varieties.

Definition 4.1. (cf. [SR, STII, ESB]) A special Cremona transformation $\gamma : P^N \cong K P^N$ of type $(d; d)$ is a birational map given by forms of degree $d \geq 2$, whose base locus scheme is a smooth irreducible variety and whose inverse is given by forms of degree $d_2 \geq 2$.

Proposition 4.2. Let $X \subset P^n$ be the center of a special Cremona transformation $\gamma : P^N \cong K P^N$ of type $(2; d)$. Then:

1. $X \subset P^n$ is a $QEL$-manifold of type $\frac{n+2}{d} = 2$ such that $S_X \subset P^n$ is a hypersurface of degree $2d - 1$;
2. $X \subset P^n$ is projectively normal;
3. if $X \subset P^n$ is a scroll over a curve $C$, then $N = 2n + 2$ and $q(C) > 0$;
4. if $N = 2n + 1$ and if $H$ is a hyperplane section, then $\gamma X + (n - 1)H$ is generated by global sections, unless $n = 2$ and $X \subset P^5$ is projectively equivalent to the Veronese surface.
Proof. By [ESB] Proposition 2.5, b), the variety $X \subseteq \mathbb{P}^N$ is non-degenerate. By [ESB] Proposition 2.3, $S X \subseteq \mathbb{P}^N$ is a hypersurface of degree $2d$, and $X \subseteq \mathbb{P}^N$ is a $\mathbb{Q}L$-manifold of type

$$
= 2n + 1 \quad \dim (S X) = 2n + 1 \quad N + 1 = 2n \quad N + 2 = (2n \quad N + 3) \quad 1 = \frac{n + 2}{d} \quad 1;
$$

where the last equality follows from [ESB] Lemma 2.4, b), see also [MR]. This yields

\[ X \quad \text{or} \quad \text{odd.} \]

Part i) is proved.

Part ii) is an improvement of the main theorem of [BEL], using the same proof and taking into account that on $\mathbb{B} \subseteq (\mathbb{P}^N)$, the linear system $\mathcal{I}_n \subseteq \mathcal{O}_p \mathbb{P}_n$. The exceptional divisor, is generated by global sections and big, see also [MR]. This yields $H^2(I(X, k)) = 0$ for every $i > 0$ and for every $k \geq 2$. Part ii) follows.

Let us consider part iii). Since $S X \subseteq (\mathbb{P}^N)$ is a hypersurface by part i), the secant variety does not fill the ambient space. For linearly normal scrolls over curves $\dim (S X) = m \in \mathbb{N}$, so that $N = 2n + 2$. If $g(C) = 0$, then $X \subseteq \mathbb{P}^{2n + 2}$ would be a rational normal scroll for which $h^0(I(X, 2)) = \frac{2n + 2(n + 3)}{2} > 2n + 3$, in contrast with $h^0(I(X, 2)) = 2n + 3$. By [ESB] Proposition 2.5.

By [ESB] Theorem 1.4, the linear system $\mathcal{K}_X + (n - 1)H \subseteq \mathcal{O}_X$ is spanned by global sections, unless $X \subseteq \mathbb{P}^N$ is a scroll over a curve. If $X \subseteq \mathbb{P}^N$ is the center of a special Cremona transformation, then it is linearly normal by part i) and it has codimension at least 2, so that $\mathcal{K}_X + (n - 1)H \subseteq \mathcal{O}_X$ is generated by global sections unless $X \subseteq \mathbb{P}^N$ is a scroll over a curve or it is projectively equivalent to the Veronese surface in $\mathbb{P}^5$. If $N \geq 2n + 1$, the first case is excluded by part iii), while the last case exists since quadrics through a Veronese surface define a special Cremona transformation of $\mathbb{P}^5$, see Proposition 2.4.

We are in position to prove a general result on special Cremona transformations of type $(\mathbb{P}^d; d, 3$. It says that centers $X \subseteq \mathbb{P}^N$ of such Cremona transformations are very rare, that most of them are $\mathbb{Q}L$-manifolds of type $= 0$ and that there are no examples with $= 1$. For $d + 3$ no example of special Cremona transformation of type $(\mathbb{P}^d; d$ whose center has $= 2$ is known, see also Remark 4.7.

Theorem 4.3. Let $X \subseteq \mathbb{P}^N$ be a $\mathbb{Q}L$-manifold of type and dimension $n$, which is the center of a special Cremona transformation $: \mathbb{P}^n \rightarrow \mathbb{P}^d$ of type $(\mathbb{P}^d; d, 3$.

(i) If $\frac{n + 2}{d} = 1, 3$, then $n$ and $d$ are even numbers.

(ii) For odd $d$, $N = 2d$, $n = d$ and the center of is a $\mathbb{Q}L$-manifold $X \subseteq \mathbb{P}^d$ of type $= 0$.

(iii) For even $d$, $4$, either $N = 2d$, $n = d$ and the center of is a $\mathbb{Q}L$-manifold $X \subseteq \mathbb{P}^d$ of type $= 0$ or $X \subseteq \mathbb{P}^N$ is a Fano manifold of even dimension $n$ with $\mathbb{P} \subseteq \mathbb{P}^N$ and index $\frac{n + 2}{d}$, which is a $\mathbb{Q}L$-manifold of type $= 2$.

Proof. Suppose $3$ and $n = 2k + 1$, $r_X = 1$. Then from $2r_X + 1 = \frac{n + 2}{d} = 1$, the last equality coming from Proposition 4.2 we get $n = 2k + 1$ so that $n$ is even, contradicting the Divisibility Theorem.

Therefore $= 2k + 2$ is even as soon as $3$ and $n$ is also even by the Divisibility Theorem. From $2r_X + 2 = \frac{n + 2}{d} = 1$, we get $d(2r_X + 3) = n + 2$ so that $2d$ divides $d$ since it divides $n$ and since $2r_X + 3$ is odd. Part i) is proved.

Suppose $= 2$, that is $n = 3d$. 2. By part i) of Proposition 4.2, $X \subseteq \mathbb{P}^N$ is linearly normal, so that [IR] Theorem 2.2) yields that either $X \subseteq \mathbb{P}^N$ is projectively equivalent to $\mathbb{P}^1 \subseteq \mathbb{P}^n$, $\mathbb{P}^n$ Segre embedded or $X \subseteq \mathbb{P}^N$ is a Fano manifold of even dimension $n$ with $\mathbb{P} \subseteq \mathbb{P}^N$, and index $\frac{n + 2}{d}$. The first case is excluded because $S X \subseteq \mathbb{P}^n$ is a hypersurface only for $= 2$ and $n = 4$, contradicting $n = 3d$.

In the second case $n = 3d$ is even, forcing $d$ even.

Let us suppose $n + 2 = 2d$, that is $= 1$. Then $X \subseteq \mathbb{P}^{4d}$ would be a linearly normal $\mathbb{Q}L$-manifold of even dimension $2d$, of type $= 1$ and such that $S X \subseteq \mathbb{P}^d$ is a hypersurface, see Proposition...
Therefore a special Cremona transformation whose center has type \( 1 \) cannot exist by [RI] Theorem 2.2.

The following classification results appear to be new.

**Corollary 4.4.** Let \( X \subset \mathbb{P}^N \) be the center of a special Cremona transformation of type \((2; 3)\). Then \( N = 4 \), \( X \subset \mathbb{P}^4 \) is projectively equivalent to a linearly normal elliptic curve of degree 5, and \( X \) is of type \((2; 3)\), since \( d = 2 \) and \( n = 2 + 1 = 3 \). By Theorem 4.3 we have to consider only the case \( N = 4 \), \( d = 0 \) and \( X \subset \mathbb{P}^4 \) a smooth linearly normal curve. Let \( \sigma : \mathbb{P}^3 \to \mathbb{P}^4 \) be such a transformation. Take 3 general quadrics hypersurfaces \( Q_1; Q_2; Q_3 \in \mathbb{H}^0(I_X(\mathcal{O}))) \) such that \( Q_1 \setminus Q_2 \setminus Q_3 = X \). As schemes, where \( C = \mathcal{O}(L), L \subset \mathbb{P}^4 \) a general line. Since \( X \) is of type \((2; 3)\), \( \deg(C) = 3 \) so that \( X \subset \mathbb{P}^5 \) is an elliptic normal curve of degree 5. The linear system \( \mathbb{H}^0(I_X(\mathcal{O})) \) defines a special Cremona transformation for example by Proposition 1.4. A nice geometrical description of \( X \) and of \( X \) is contained in [SR, Chapter 8, Section 5].

**Corollary 4.5.** Let \( X \subset \mathbb{P}^N \) be the center of a special Cremona transformation of type \((2; 5)\). Then \( N = 8 \) and \( X \subset \mathbb{P}^8 \) is a Fano 3-fold of degree 13 and sectional genus 8, projection of a point of degree 14 and sectional genus 8 Fano 3-fold \( Y \subset \mathbb{P}^9 \).

**Proof.** By Theorem 4.3 we have to consider only the case \( N = 8 \), \( d = 0 \) and \( X \subset \mathbb{P}^8 \) a smooth linearly normal 3-fold defined by nine quadratic equations. In [MR] it is proved that such a 3-fold is as in the statement, concluding the proof.

Let \( : \mathbb{P}^N \to \mathbb{P}^N \) be a special Cremona transformation of type \((d_1; d_2)\), having a base locus \( X \subset \mathbb{P}^N \) of dimension \( n \). By [ESB, Proposition 2.3] the locus of \( d_1 \)-secant lines to \( X \), \( \sigma : X \to \mathbb{P}^N \), is an irreducible hypersurface of degree \( d_1 \), \( \sigma : X \to \mathbb{P}^N \), clearly contained in \( S \subset \mathbb{P}^N \). Thus \( N = 1 = \dim(S, X) \) \( \dim(S, X) = 2n + 1 \). Moreover, by the Trisecant Lemma, \( N = 2n + 2 \) if and only if \( d_1 = 2 \) and \( X \subset \mathbb{P}^{n+2} \) is a \( Q \subset \mathbb{E} \) manifold of type \( 0 \). Now we consider the case \( N = 2n + 1 \).

**Corollary 4.6.** Let \( X \subset \mathbb{P}^{2n+1} \) be the center of a special Cremona transformation \( \sigma : \mathbb{P}^{2n+1} \to \mathbb{P}^{2n+1} \). Then either \( d_1 = 2 = d_2 \) or \( X \subset \mathbb{P}^{2n+1} \) is projectively equivalent to the Veronese surface \( \mathcal{V}(2; 2) \subset \mathbb{P}^5 \). The inverse transformation is special and of the same kind; or \( d_1 = 3, X \subset \mathbb{P}^3 \) is a non-hyperelliptic curve of genus 3, degree 6 and the inverse transformation is special and of the same kind.

**Proof.** By [ESB, Lemma 2.4] (see also [CK]) we get the following relation \( 2 + n = d_2(2 + d_1)n + 2 \). If \( d_1 = 3 \), then \( n = 1 \) and \( d_1 = 3 \) and the conclusion is well known; see for example [SR, VIII.4.3] or [MR, Proposition 3.1]. If \( d_1 = 2 \), then \( n \) is even and \( X \subset \mathbb{P}^{2n+1} \) is projectively normal \( Q \subset \mathbb{E} \) manifold of type \( 1 \) by Proposition 4.2, so that we can apply [RI] Theorem 2.2. Quadrics through a Veronese surface in \( \mathbb{P}^5 \) define a special Cremona transformation whose inverse is of the same kind. Indeed in this case : \( \mathbb{P}^5 \to \mathbb{P}^5 \) can be interpreted, modulo projective transformations, as the map associating to a plane conic its dual conic (see also [SR, p. 188]. [ESB, Theorem 2.8] or apply directly Proposition 1.4).

**Remark 4.7.** The argument used in the proof of Theorem 4.3 also yields a bound \( n = n_0(d) \) for the dimension of special Cremona transformations of type \((2; d)\), \( d \) even, which is less sharp than in the odd case. Indeed, if \( d = 4 \) is even and if \( n = 2n_0 + 2 \), then the Divisibility Theorem implies that \( 2n_0 \) divides \( d(2n_0 + 3) \), bounding \( n_0 \).

In [HKS, Section 5] a series of Cremona transformations is built \( d : \mathbb{P}^{2(d+1)} \to \mathbb{P}^{2(d+1)} \) of type \((2; d)\) for any \( d \) even. There are not defined along a scheme \( X_d \) of dimension \( d \), which is irreducible for \( d = 3 \). For \( d = 3; 4; 5 \) this scheme is known to be smooth so that for \( d = 3; 5 \) there exists examples of special Cremona
transformations as in case (ii) of the above theorem, as we saw in the previous corollaries. The cases $d = 3 \text{ and } 4$ are classical and were described and studied in [SR, ST1, ST2]. Thus the above theorem cannot be improved and there could exist examples of special Cremona transformations of type $2(2|d)$ with $d = 0$ also for $d = 6$.

The Severi varieties of dimension 8 and 16 yield examples of special Cremona transformations of type $2(2|2)$ with centers of type $3$. One could ask if for $d = 4$ there exists a special Cremona transformation of type $2(2|d)$ with even $d > 0$ and whose center is a Fano variety of the kind described in Theorem 4.3. The known examples point towards a negative answer to this question, so that one can conjecture that also for even $d = 4$ necessarily $n = d/2$ and $N = 2(d - 1)$. It could be also true that the above series are the only possible examples for every (odd) $d = 3$, the first interesting cases to be considered being $d = 4$ and $d = 6$, due to Corollaries 4.4 and 4.5.

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