Abstract

In this article we consider Sturm-Liouville operator with \(q \in W^2_1[0, 1]\) and Dirichlet boundary conditions. We prove that if the set \(\{(n\pi)^2 : n \in \mathbb{N}\}\) is a subset of the spectrum of the Sturm-Liouville operator with Dirichlet boundary conditions, then \(q = 0\) a.e.

Key words: Sturm-Liouville operators; Dirichlet boundary conditions; inverse spectral theory

1 Introduction

Denote by \(L(q)\) the operator generated in \(L^2(0, 1)\) by the expression

\[-y'' + q(x)y,\]  \hspace{1cm} (1.1)

and by the boundary conditions

\[y(1) = y(0) = 0\] \hspace{1cm} (1.2)

where \(q(x) \in W^2_1[0, 1]\) is a real-valued function and \(q^{(k)}(0) = q^{(k)}(1), k = 0, 1\). In [1], they obtained asymptotic formulas of arbitrary order for eigenfunctions and eigenvalues of the Dirichlet boundary value problem (1.1) with (1.2) when \(q(x)\) is a complex-valued summable function.

In this point, we examine Ambarzumyan theorem in [2]. In 1929, Ambarzumyan [2] proved the following theorem as the first theorem in inverse spectral theory:
Theorem 1 If \( \{n^2 : n = 0, 1, \ldots\} \) is the spectrum of the Neumann boundary condition with Sturm-Liouville operator, then \( q = 0 \) a.e.

In the Pöschel-Trubowitz [3] works, the \( \sigma = \{n^2 : n \in \mathbb{N}\} \) spectrum for the Dirichlet problem corresponds to zero and they showed that there are infinitely many \( L^2 \) potentials near zero. For the Dirichlet problem, when the spectrum is zero, the potential does not have to be zero. Therefore, the Ambarzumyan’s theorem is not valid (see [4]).

In [4], putting an additional condition on the potential they expanded the classical Ambarzumyan’s theorem for the Sturm-Liouville equation to the general separated boundary conditions. They obtained the following this theorem (see [4, Theorem 1.1]).

Theorem 2 The Sturm-Liouville problem

\[-y'' + qy = \lambda y\]

such that

\[y(0) \cos \alpha + y'(0) \sin \alpha = 0\]
\[y(\pi) \cos \beta + y'(\pi) \sin \beta = 0\]

where \( q \in L^1(0, \pi), \alpha, \beta \in [0, \pi) \). For the Sturm-Liouville problem, assume that \( \alpha = \beta \neq \frac{\pi}{2} \). Then \( \sigma = \{n^2 : n \in \mathbb{N}\} \) and the potential function \( q \) satisfies

\[\int_0^\pi q(x) \cos 2(x - \alpha) \, dx = 0\]

if and only if \( q = 0 \) a.e.

In [5], Yurko also proved the following generalization theorem of the Ambarzumyan theorem on wide classes of the arbitrary self-adjoint boundary conditions and self-adjoint differential operators (see [5, Theorem 3]).

Theorem 3 Let

\[\lambda_0 = \tilde{\lambda}_0 + \frac{(\tilde{q}\tilde{y}_0, \tilde{y}_0)}{(\tilde{y}_0, \tilde{y}_0)};\]

where \( \tilde{y}_0(x) \) is an eigenfunctions \( \tilde{L} \) related to \( \tilde{\lambda}_0 \). Then

\[q(x) = \tilde{q}(x) + \lambda_0 - \tilde{\lambda}_0 \quad a.e. \ on \ (0, 1).\]

Kiraç [6] proved for the Sturm-Liouville operators with quasi-periodic boundary conditions without putting any condition on \( q \) the potential and \( q \) can be determined from a single spectrum. Thus he get the classical Ambarzumyan’s theorem. This result is the following (see [6, Theorem 1]).
**Theorem 4** If first eigenvalue of the operator $L_t(q)$ for any fixed number $t$ in $[0, 2\pi)$ is not less than the value of $\min\{t^2, (2\pi - t)^2\}$ and the spectrum $S(L_t(q))$ contains the set $\{(2n\pi - t)^2 : n \in \mathbb{N}\}$, then $q = 0$ a.e.

Kiraç [7, Theorem 1.1] proved the following inverse spectral result for periodic and anti-periodic boundary conditions:

**Theorem 5** Denote the $n$th instability interval by $\ell_n$, and suppose that $\ell_n = o(n^{-2})$ as $n \to \infty$. Then the following two assertions hold:

(i) If $\{(n\pi)^2 : n \text{ even and } n > n_0\}$ is a subset of the periodic spectrum of the Hill operator then $q = 0$ a.e. on $(0, 1)$,

(ii) If $\{(n\pi)^2 : n \text{ odd and } n > n_0\}$ is a subset of the anti-periodic spectrum of the Hill operator then $q = 0$ a.e. on $(0, 1)$.

Given $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$, a sufficiently large positive integer such that

$$\ell_n < c\epsilon n^{-2} \quad \text{for all} \quad n > n_0(\epsilon).$$

In this article, we prove the following inverse spectral result:

**Theorem 6** Let $q(x) \in W_1^2[0, 1]$ and $q^{(k)}(0) = q^{(k)}(1)$, $k = 0, 1$ hold. Then the following assertion satisfies:

If $\{(n\pi)^2 : n \in \mathbb{N}\}$ is a subset of the spectrum of the operator (1.1)-(1.2) then $q = 0$ a.e. on $(0,1)$.

### 2 Preliminaries

We shall consider the problem (1.1)-(1.2). Considering Theorem 1 in [1], the eigenvalues of the Sturm-Liouville operator for $m \geq N$ without loss of generality assumption $c_0 = 0$ and

$$c_m = \int_0^1 q(x) \cos m\pi x \, dx \quad (2.1)$$

such that we obtain asymptotic formula for $\{\lambda_m\}$ eigenvalues

$$\lambda_m = (m\pi)^2 + c_0 - c_{2m} + O\left(\frac{\ln |m|}{m}\right) \quad (2.2)$$

form taking into account that satisfying

$$\lambda_m = (m\pi)^2 + o(1). \quad (2.3)$$
Here we denote by $N$ a large positive integer. Using this formula, for all $k \neq m$, $k = 0, 1, \ldots$ the following inequality holds

$$|\lambda_m - (\pi k)^2| > |(m - k)\pi||m + k| - c_1 m^{1/2} > c_2 m$$  \hspace{1cm} (2.4)

for $m \geq N$, where we denote by $c_n$, $n = 1, 2, \ldots$, positive constants whose exact value is not essential. To obtain the asymptotic formula for eigenvalues $\lambda_m$ corresponding to the normalized eigenfunctions $\Psi_m(x)$ of $L(q)$ we use (2.4) and the following relation

$$(\lambda_N - (\pi m)^2)(\Psi_N(x), \sin m\pi x) = (q(x)\Psi_N(x), \sin m\pi x).$$  \hspace{1cm} (2.5)

Here

$$\Psi_m(x) = \sqrt{2} \sin m\pi x + O\left(\frac{1}{m}\right).$$  \hspace{1cm} (2.6)

$$|(q(x)\Psi_N(x), \sin m\pi x)| < 4M$$  \hspace{1cm} (2.7)

satisfy for $\forall m, \forall N \gg 1$, where $M = \int_0^1 |q(x)| \, dx$.

Now, the expansion

$$\Psi_N(x) = \sum_{m_1 > -m}^{\infty} 2(\Psi_N(x), \sin (m + m_1)\pi x)\sin(m + m_1)\pi x$$

of $\Psi_N(x)$ by the orthonormal basis $\{\sqrt{2} \sin (m + m_1)\pi x : m_1 > -m\}$ has the formula

$$\Psi_N(x) = \sum_{m_1 > -m}^{n} 2(\Psi_N(x), \sin (m + m_1)\pi x)\sin(m + m_1)\pi x + g(x)$$

where $\sup_{x \in [0, 1]} |g(x)| < \frac{c_3}{n}$.

This formula is substituted in the $(q(x)\Psi_N(x), \sin m\pi x)$ expression and as $n \to \infty$ we obtain

$$(q(x)\Psi_N(x), \sin m\pi x) = \sum_{m_1 > -m}^{\infty} 2(q(x), (\sin (m + m_1)\pi x)(\sin m\pi x))(\Psi_N(x), \sin(m + m_1)\pi x).$$  \hspace{1cm} (2.8)

By doing the necessary operations in (2.8), we obtain the following equation (see [1, p.155])

$$(q(x)\Psi_N(x), \sin m\pi x) = \sum_{m_1 = 1}^{\infty} c_{m_1}(\Psi_N(x), \sin (m + m_1)\pi x)$$

$$+ \sum_{m_1 = 1}^{\infty} c_{m_1}(\Psi_N(x), \sin (m - m_1)\pi x).$$

Substituting this equality into (2.5) we get

$$(\lambda_m - (\pi m)^2)(\Psi_N(x), \sin m\pi x) = \sum_{m_1 = -\infty}^{\infty} c_{m_1}(\Psi_N(x), \sin (m + m_1)\pi x)$$  \hspace{1cm} (2.9)
where \( c_m = \int_0^1 q(x) \cos m\pi x \, dx \). Also, \( c_m = c_{-m}, \, c_m \to 0 \) as \( |m| \to \infty \).

Now we isolate the terms that contain the expression \((\Psi_N(x), \sin(m + m_1)\pi x)\) to the right of (2.9). For this we have \( m + m_1 \) instead of \( N \) and \( m \) in (2.9). Hence we obtain the following equation

\[
(\lambda_m - (\pi m)^2)(\Psi_m(x), \sin m\pi x) = -c_{2m}(\Psi_m(x), \sin m\pi x)
+ \sum_{m_1, m_2 = -\infty}^{\infty} \frac{c_{m_1}c_{m_2}(\Psi_m(x), \sin(m + m_1 + m_2)\pi x)}{\lambda_m - (\pi(m + m_1))^2}.
\]

(2.10)

Again, we isolate the terms for which \( m_1 + m_2 = 0, -2m \). First, \( -m_1 \) is written instead of \( m_2 \) and use the equality \( c_{-m_1} = c_{m_1} \) and the last sum of this equation is isolated for which \( m_1, m_2, m_3 = 0, -2m \) (see (12), (13) of [1]) we get the following lemma for our notation.

**Lemma 7** (see [1]) The eigenvalue \( \lambda_m \) of the operator \( L(q) \) satisfies the asymptotic formula

\[
\lambda_m = (m\pi)^2 + c_0 - c_{2m} + a_1(\lambda_m) - b_1(\lambda_m) + a_2(\lambda_m) - b_2(\lambda_m) + R_3
\]

(2.11)

where \( q(x) \) is a real-valued summable function and \( c_m = \int_0^1 q(x) \cos m\pi x \, dx \),

\[
a_1(\lambda_m) = \sum_{m_1 = -\infty}^{\infty} \frac{c_{m_1}c_{m_1}}{\lambda_m - (\pi(m + m_1))^2}
\]

(2.12)

\[
b_1(\lambda_m) = \sum_{m_1 = -\infty}^{\infty} \frac{c_{m_2}c_{m_1}c_{m_1}}{\lambda_m - (\pi(m + m_1))^2}
\]

(2.13)

\[
a_2(\lambda_m) = \sum_{m_1, m_2 = -\infty}^{\infty} \frac{c_{m_1}c_{m_2}c_{m_1} + c_{m_2}}{\prod_{t=1,2}[\lambda_m - (\pi(m + m_t))^2]}
\]

(2.14)

\[
b_2(\lambda_m) = \sum_{m_1, m_2 = -\infty}^{\infty} \frac{c_{m_1}c_{m_2}c_{m_1} + c_{m_2}}{\prod_{t=1,2}[\lambda_m - (\pi(m + m_t))^2]}
\]

(2.15)

\[
R_3 = \sum_{m_1, m_2, m_3 = -\infty}^{\infty} \frac{c_{m_1}c_{m_2}c_{m_3}(q(x)\Psi_m(x), \sin(m + m_1 + m_2 + m_3)\pi x)}{\prod_{t=1,2,3}[\lambda_m - (\pi(m + m_t))^2]}.
\]

(2.16)

Here, using the following equality

\[
\frac{1}{-m_1(2m + m_1)} = \frac{1}{2m} \left( \frac{1}{2m + m_1} - \frac{1}{m_1} \right)
\]

we get the relation

\[
\sum_{m_1 \neq 0, -2m} \frac{1}{-m_1(2m + m_1)} = O \left( \frac{\ln|m|}{m} \right).
\]

(2.17)
In [1], obtained the following estimate (see (19) of [1], [8])

\[ R_3 = O \left( \left( \frac{\ln |m|}{m} \right)^3 \right). \]  

(2.18)

3 Main Results

In our next notations we will need the following relation. Let us see that

\[ c_{2m_1} = \int_0^1 \tilde{q}(x) e^{-i(2m_1)\pi x} \, dx \]  

(3.1)

and

\[ c_{2m_1+1} = \int_0^1 \hat{q}(x) e^{-i(2m_1+1)\pi x} \, dx, \]  

(3.2)

equalities are satisfying by using the even and odd functions

\[ \tilde{q}(x) = \frac{q(x) + q(1 - x)}{2}, \]  

(3.3)

\[ \hat{q}(x) = \frac{q(x) - q(1 - x)}{2}. \]  

(3.4)

in (2.1), respectively of the \( q \) function in \([0, 1]\). Really,

\[ \int_0^1 \tilde{q}(x) e^{-i(2m_1)\pi x} \, dx = \int_0^1 q(x) + q(1 - x) \cos 2m_1 \pi x \, dx \]

\[ - i \int_0^1 q(x) + q(1 - x) \sin 2m_1 \pi x \, dx. \]  

(3.5)

By using \( 1 - x = u \) change of variable in both integrals on the right-hand side of the equation (3.5), we get (3.1). Similarly, we obtain (3.2).

Lemma 8 Let \( q(x) \in W_1^1[0, 1], \, q(0) = q(1) \) and \( c_0 = 0 \).

The following asymptotic estimates are valid for the relation in (2.15) and (2.16):  

\[ b_1(\lambda_m) = o(m^{-2}), \]

\[ b_2(\lambda_m) = o(m^{-2}). \]

PROOF. Now, for the proof of lemma by using (2.5), (2.7) and (2.4), we obtain the following asymptotic estimate (see [8])

\[ \sum_{m \in \mathbb{Z}} |(\psi_m(x), \sin m\pi x)|^2 < \sum_{m \in \mathbb{Z}} \frac{(4M)^2}{m_1(2m + m_1)^2}. \]
Arguing as in [7, Theorem 1.2] arbitrary a constant $C$ with by using (2.3), (2.4) and (3.6), we obtain that

$$
\sum_{m_1 \neq 0, -2m} \left| \frac{1}{\lambda_m - (m + m_1)^2 \pi^2} - \frac{1}{m^2 \pi^2 - (m + m_1)^2 \pi^2} \right| \leq C \cdot o(1) \sum_{m_1 \neq 0, -2m} |m_1|^{-2} |2m + m_1|^{-2} = o(m^{-2}).
$$

First, we prove $b_1(\lambda_m)$. For this, we write taking into account that (3.7) and the sum on the right-hand-side of equation (2.13) by using equalities (3.1) and (3.2), as following (see [9, Lemma 3], [10,7]).

$$
b_1(\lambda_m) = \sum_{m_1 = -\infty}^{\infty} \frac{c_{m_1} c_{2m + m_1}}{\lambda_m - (\pi(m + m_1))^2},
$$

by using (2.3) and with together equality

$$
\lambda_m - (\pi(m + m_1))^2 = -m_1(2m + m_1)\pi^2 + o(1),
$$

$$
b_1(\lambda_m) = \frac{1}{\pi^2} \sum_{m_1 = -\infty}^{\infty} \frac{c_{m_1} c_{2m + m_1}}{-m_1(2m + m_1)} + o(m^{-2})
$$

$$
= \frac{1}{\pi^2} \sum_{m_1 \neq 0, -2m} \frac{c_{2m_1} c_{2m_1 + 2m_1}}{-2m_1(2m + 2m_1)} + \frac{1}{\pi^2} \sum_{m_1 \neq 0, -2m} \frac{c_{2m_1 + 1} c_{2m_1 + 2m_1 + 1}}{-(2m_1 + 1)(2m + 2m_1 + 1)} + o(m^{-2})
$$

$$
= - \int_0^1 (\tilde{Q}(x) - \tilde{Q}_0)^2 e^{2m \pi x} dx - \int_0^1 (\hat{Q}(x) - \hat{Q}_0)^2 e^{2m \pi x} dx + o(m^{-2})
$$

$$
= \frac{1}{i \pi 2m} \int_0^1 2(\tilde{Q}(x) - \tilde{Q}_0) \tilde{q}(x) e^{i2m \pi x} dx
$$

$$
+ \frac{1}{i \pi 2m} \int_0^1 2(\hat{Q}(x) - \hat{Q}_0) \hat{q}(x) e^{i2m \pi x} dx + o(m^{-2})
$$

where

$$
\tilde{Q}(x) - \tilde{Q}_0 = \sum_{2m_1 \neq 0} \tilde{Q}_{m_1} e^{i(2m_1)\pi x}, \quad \hat{Q}(x) - \hat{Q}_0 = \sum \hat{Q}_{m_1} e^{i(2m_1 + 1)\pi x}
$$

$$
\tilde{Q}_{m_1} =: (\tilde{Q}(x), e^{i(2m_1)\pi x}) = \frac{c_{2m_1}}{i \pi (2m_1)}, \quad 2m_1 \neq 0,
$$

$$
\hat{Q}_{m_1} =: (\hat{Q}(x), e^{i(2m_1 + 1)\pi x}) = \frac{c_{2m_1 + 1}}{i \pi (2m_1 + 1)},
$$

are denote the Fourier coefficients with respect to the systems \{e^{i(2m_1)\pi x} : m_1 \in \mathbb{Z}\} and \{e^{i(2m_1 + 1)\pi x} : m_1 \in \mathbb{Z}\} of the functions $Q(x) = \int_0^x \tilde{q}(t) dt$ and
\( \hat{Q}(x) = \int_0^x \hat{q}(t) \, dt \). Here, since \( \hat{Q}(1) = \int_0^1 \frac{q(t) + q(1-t)}{2} \, dt \), we get \( \hat{Q}(1) = c_0 = 0 \) if \( q(1-t) = u \) use variable change. Similarly, \( \hat{Q}(1) = 0 \). Now, using integration by parts, together by \( \hat{Q}(1) = 0 \) and \( \hat{Q}(1) = 0 \), yields

\[
\begin{align*}
\frac{b_1(\lambda_m)}{2\pi^2m^2} \int_0^1 (q^2(x) + (\hat{Q}(x) - \hat{Q}_0) \hat{q}'(x)) e^{i2m\pi x} \, dx \\
+ \frac{1}{2\pi^2m^2} \int_0^1 (q^2(x) + (\hat{Q}(x) - \hat{Q}_0) \hat{q}'(x)) e^{i2m\pi x} \, dx + o(m^{-2}).
\end{align*}
\]

(3.11)

Since \( \tilde{q}(x) \) and \( \hat{q}(x) \) are absolutely continuous a.e., \( (\tilde{q}^2(x) + (\tilde{Q}(x) - \tilde{Q}_0) \tilde{q}'(x)) \in L^1[0,1] \) and \( (q^2(x) + (\hat{Q}(x) - \hat{Q}_0) \hat{q}'(x)) \in L^1[0,1] \). By using the Riemann-Lebesgue lemma, we obtain that

\[
b_1(\lambda_m) = o(m^{-2}).
\]

(3.12)

Now, by using (2.15) we prove that (see [7])

\[
b_2(\lambda_m) = o(m^{-2}).
\]

(3.13)

By considering that \( \tilde{q}(x) \) and \( \hat{q}(x) \) are absolutely continuous a.e., we obtain \( c_{m_1}c_{m_2}c_{2m+m_1+m_2} = o(m^{-1}) \) (see [10]). This, together with (3.11) and (3.12), we write the sum on the right-hand-side of equation (2.15), as follows.

\[
b_2(\lambda_m) = \sum_{m_1,m_2=\ldots=\infty \atop m_1,m_1+m_2 \neq 0, -2m} \frac{c_{m_1}c_{m_2}c_{2m+m_1+m_2}}{\prod_{t=1,2} \left[ \lambda_m - (\pi(m+m_t))^2 \right]} \\
= \frac{1}{\pi^4} \sum_{m_1,m_2=\ldots=\infty \atop m_1,m_1+m_2 \neq 0, -2m} \frac{c_{m_1}c_{m_2}c_{2m+m_1+m_2}}{-m_1(2m+m_1)(-m_1-m_2)(2m+m_1+m_2)} + o(m^{-2})
\]

\[
|b_2(\lambda_m)| = o(m^{-1}) \sum_{m_1,m_2} \frac{1}{| -m_1(2m+m_1)(-m_1-m_2)(2m+m_1+m_2)|} \\
= o(m^{-1})O \left( \left( \frac{\ln|m|}{m} \right)^2 \right) = o(m^{-2}).
\]

Thus, the estimate of (3.13) is proved.

**Lemma 9** Let \( q(x) \in W^2_2[0,1] \), \( q^{(k)}(0) = q^{(k)}(1), \ k = 0,1 \) and \( c_0 = 0 \). For all sufficiently large \( m \), we obtain following estimates

\[
a_1(\lambda_m) = \frac{1}{(2m)^2\pi^2} \int_0^1 q^2(x) \, dx + o(m^{-2}),
\]

(3.14)

\[
a_2(\lambda_m) = o(m^{-2}).
\]

(3.15)
**PROOF.** In a similar way, arguing as in $b_1(\lambda_m)$, let us prove that $a_1(\lambda_m)$. For this, we write considering that (3.7) and (3.8) and the sum on the right-hand-side of equation (2.12) by using equalities (3.1) and (3.2), as following.

\[
a_1(\lambda_m) = \sum_{m_1 = -\infty}^{\infty} \frac{c_{m_1}c_{m_1}}{\lambda_m - (\pi(m + m_1))^2}
\]

\[
= \frac{1}{\pi^2} \sum_{m_1 = -\infty}^{\infty} \frac{c_{m_1}c_{m_1}}{-m_1(2m + m_1)} + o(m^{-2})
\]

\[
= \frac{1}{\pi^2} \sum_{2m_1 \neq 0, -2m} \frac{c_{2m_1}c_{2m_1}}{-2m_1(2m + 2m_1)}
+ \frac{1}{\pi^2} \sum_{2m_1 + 1 \neq -2m} \frac{c_{2m_1+1}c_{2m_1+1}}{2} - (2m_1 + 1)(2m + 2m_1 + 1) + o(m^{-2}).
\]

(3.16)

It also follows from [7 Lemma 2.2] (see [11 Lemma 2.3]) that we get in our expression,

\[
a_1(\lambda_m)
\]

\[
= \frac{2}{\pi^2} \sum_{2m_1 \neq 0, 2m_1 \neq 0} \frac{c_{2m_1}c_{2m_1}}{(2m + 2m_1)(2m - 2m_1)}
+ \frac{2}{\pi^2} \sum_{2m_1 + 1 \neq 0} \frac{c_{2m_1+1}c_{2m_1+1}}{(2m + 2m_1 + 1)(2m - 2m_1 - 1)} + o(m^{-2})
\]

\[
= - \int_0^1 (\tilde{G}^+(x, m) - \tilde{G}^+_0(m))^2 e^{i(-4m)x} dx
- \int_0^1 (\tilde{G}^+(x, m) - \tilde{G}^+_0(m) + o(m^{-2}))^2 e^{i(-4m)x} dx + o(m^{-2})
\]

\[
= - \int_0^1 (\tilde{G}^+(x, m) - \tilde{G}^+_0(m))^2 e^{i(-4m)x} dx
- \int_0^1 (\tilde{G}^+(x, m) - \tilde{G}^+_0(m))^2 e^{i(-4m)x} dx + o(m^{-2})
\]

(3.17)

where

\[
\tilde{G}^\pm_{m_1}(m) = (\tilde{G}^\pm(x, m), e^{i(2m_1)x}) = \frac{c_{2m_1 \pm (2m_1)}}{i\pi(2m_1)}
\]

\[
\hat{G}^\pm_0(m) = (\hat{G}^\pm(x, m), e^{i(2m_1+1)x}) = \frac{c_{2m_1+1 \pm (2m_1)}}{i\pi(2m_1 + 1)} + \frac{2}{(2m_1 + 1)^2\pi^2} \int_0^1 \hat{q}(t)e^{i(2m)x} dt
\]

(3.18)
for $2m_1 \neq 0$ are denote the Fourier coefficients with respect to system $\{e^{i(2m_1)\pi x} : m_1 \in \mathbb{Z}\}$ and $\{e^{i(2m_1+1)\pi x} : m_1 \in \mathbb{Z}\}$ of the functions

$$
\hat{G}^\pm(x, m) = \int_0^x \hat{q}(t) e^{\mp i(-2m)\pi t} dt - c_{\pm(2m)} x,
$$

and

$$
\hat{G}^\pm(x, m) = \int_0^x \hat{q}(t) e^{\mp i(-2m)\pi t} dt - x \int_0^1 \hat{q}(t) e^{\mp i(-2m)\pi t} dt \quad (3.19)
$$

Taking into account the equalities (see (3.19))

$$
\hat{G}^\pm(x, m) - \hat{G}_0^\pm(m) = \sum_{2m_1 \neq -2m} \frac{c_{2m_1}}{i\pi(2m_1 \mp (-2m))} e^{i(2m_1 \mp (-2m))\pi x} + o(m^{-2}).
$$

(3.20)

Here, in the second expression of (3.20), the second term on the right-hand side of the equality is uniform in $x$. Considering the [9, Lemma 1] and (3.19) we have following the estimates

$$
\hat{G}^\pm(x, m) - \hat{G}_0^\pm(m) = \hat{G}^\pm(x, m) - \int_0^1 \hat{G}^\pm(x, m) dx = o(1),
$$

$$
\hat{G}^\pm(x, m) - \hat{G}_0^\pm(m) = \hat{G}^\pm(x, m) - \int_0^1 \hat{G}^\pm(x, m) dx = o(1), \quad \text{as} \quad m \to \infty
$$

(3.21)

uniformly in $x$.

Taking into account the equalities (see (3.19))

$$
\hat{G}^\pm(1, m) = \hat{G}^\pm(0, m) = 0, \quad \hat{G}^\pm(1, m) = \hat{G}^\pm(0, m) = 0,
$$

(3.22)

and since $\hat{q}(x)$ and $\hat{q}(x)$ are absolutely continuous a.e., using the integration by parts obtain for the right-hand side of (3.17), the value

$$
a_1(\lambda_m) = \frac{1}{(2m)^2 \pi^2} \left[ \int_0^1 \hat{q}^2 + \int_0^1 (\hat{G}^+(x, m) - \hat{G}^+_0(m)) \hat{q}'(x) e^{i(-2m)\pi x} dx \right] - \frac{3{|c_{-2m}|}^2}{2\pi^2 (2m)^2}
$$

$$
+ \frac{1}{(2m)^2 \pi^2} \left[ \int_0^1 \hat{q}^2 + \int_0^1 (\hat{G}^+(x, m) - \hat{G}^+_0(m)) \hat{q}'(x) e^{i(-2m)\pi x} dx \right] + o(m^{-2})
$$

for sufficiently large $m$. Thus, by using the Riemann - Lebesgue lemma, together with $(\hat{G}^+(x, m) - \hat{G}^+_0(m)) \hat{q}'(x) \in L^1[0, 1]$ and $(\hat{G}^+(x, m) - \hat{G}^+_0(m)) \hat{q}'(x) \in L^1[0, 1]$, we get (3.14).

Now, let’s prove that $a_2(\lambda_m) = o(m^{-2})$. Similarly, together with (2.14), (3.7) and (3.8), we get

$$
a_2(\lambda_m) = \sum_{m_1, m_2 = -\infty}^{\infty} \prod_{l=1,2} \frac{c_{m_1} c_{m_2} c_{m_1 + m_2}}{\lambda_m - (\pi(m + m_l))^2}
$$
where

\[ \pi^{-4}c_{m_1}c_{m_2}c_{m_1+m_2} \frac{m_1(2m+m_1)(-m_1-m_2)(2m+m_1+m_2)}{m_1} + o(m^{-2}). \]  

(3.23)

Arguing as in [9, Lemma 4] and [7], using the summation variant \( m_2 \) to impress the previous \( m_1 + m_2 \) in (3.23), we write (3.23) in the formula

\[ a_2(\lambda_m) = \frac{1}{\pi^4(2m)^2} \sum_{j=1}^{4} S_j, \]  

(3.24)

Here, the forbidden indices in the sums take the form of \( m_1, m_2 \neq 0, -2m \). By the equality

\[ \frac{1}{-k(2m + k)} = \frac{1}{2m} \left( \frac{1}{2m + k} - \frac{1}{k} \right) \]

we have

\[ a_2(\lambda_m) = \frac{1}{\pi^4(2m)^2} \sum_{j=1}^{4} S_j, \]

where

\[ S_1 = \sum_{m_1, m_2} \frac{c_{m_1}c_{m_2-m_1}c_{m_2}}{m_1m_2}, \quad S_2 = \sum_{m_1, m_2} \frac{c_{m_1}c_{m_2-m_1}c_{m_2}}{-m_2(2m+m_1)} \]

\[ S_3 = \sum_{m_1, m_2} \frac{c_{m_1}c_{m_2-m_1}c_{m_2}}{-m_1(2m+m_2)}, \quad S_4 = \sum_{m_1, m_2} \frac{c_{m_1}c_{m_2-m_1}c_{m_2}}{(2m+m_1)(2m+m_2)}. \]

Now, using the first equality in (3.10), integration by parts and the assumption \( c_0 = 0 \) which means \( \hat{Q}(1) = 0 \), we get for even that

\[ \hat{S}_1 = \pi^2 \int_0^1 (\hat{Q}(x) - \hat{Q}_0)^2 \hat{q}(x) dx = 0. \]

Similarly, taking into account (3.10) and (3.18)-(3.22), we obtain for even by the Riemann-Lebesgue lemma, the following relations

\[ \hat{S}_2 = -\pi^2 \int_0^1 (\hat{Q}(x) - \hat{Q}_0) (\hat{G}^+(x,m) - \hat{G}^+_0(m)) \hat{q}(x) e^{i\pi(-2m)x} dx = o(1), \]

\[ \hat{S}_3 = -\pi^2 \int_0^1 (\hat{Q}(x) - \hat{Q}_0) (\hat{G}^-(x,m) - \hat{G}^-_0(m)) \hat{q}(x) e^{i\pi(2m)x} dx = o(1) \]

and with the first equality in (3.21),

\[ \hat{S}_4 = \pi^2 \int_0^1 (\hat{G}^-(x,m) - \hat{G}^-_0(m)) (\hat{G}^+(x,m) - \hat{G}^+_0(m)) \hat{q}(x) dx = o(1). \]

In the same way, using the second equality in (3.10), integration by parts and the assumption \( c_0 = 0 \) which means \( \tilde{Q}(1) = 0 \), we obtain for odd that

\[ \hat{S}_1 = \pi^2 \int_0^1 (\hat{Q}(x) - \hat{Q}_0)^2 \hat{q}(x) dx = 0. \]
Similarly, considering (3.10) and (3.18)-(3.22), we get for odd by the Riemann-Lebesgue lemma, the following relations

\[ \hat{S}_2 = -\pi^2 \int_0^1 (\hat{Q}(x) - \hat{Q}_0) (\hat{G}^+(x, m) - \hat{G}_0^+(m)) \hat{q}(x) e^{i\pi(-2m)x} \, dx = o(m^{-2}), \]

\[ \hat{S}_3 = -\pi^2 \int_0^1 (\hat{Q}(x) - \hat{Q}_0) (\hat{G}^-(x, m) - \hat{G}_0^-(m)) \hat{q}(x) e^{i\pi(2m)x} \, dx = o(m^{-2}) \]

and with the second equality in (3.21),

\[ \hat{S}_4 = \pi^2 \int_0^1 (\hat{G}^-(x, m) - \hat{G}_0^-(m)) (\hat{G}^+(x, m) - \hat{G}_0^+(m)) \hat{q}(x) \, dx = o(m^{-2}). \]

Thus, (3.24) proved.

**Proof of Theorem 6.** First, considering (2.2) we find \( c_0 = 0 \) since

\[ \lambda_m = (m\pi)^2 + c_0 + o(1). \]

By substituting the estimates obtained in \( c_0 = 0,\ c_{2m} = o(m^{-2}), \) Lemma 8 and Lemma 9 in Lemma 7, we obtain the form

\[ \lambda_m = (m\pi)^2 + \frac{1}{(2m)^2\pi^2} \int_0^1 q^2(x) \, dx + o(m^{-2}). \]

From the hypothesis,

\[ \int_0^1 q^2(x) \, dx = 0, \]

so \( q = 0 \) a.e.

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