Nonlinearity of the non-Abelian lattice gauge field theory according to the spectra of Kolmogorov-Sinai entropy and complexity

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Abstract

Yang-Mills fields are an important part of the non-Abelian space theory describing the properties of quark-gluon plasma. The dynamics of the classical fields are given by the Hamiltonian equations of motion, which contain the member of the field strength tensor SU(2) \[29\]. This system exhibits chaotic behavior \[30, 2, 23\].

The homogeneous Yang-Mills equation includes the quadratic part of the field strength tensor \(F^a_{\mu\nu}\) expressed in Minkowski space, which was determined by the fields \(A^a_{\mu}\). The dynamics of the classical Yang-Mills field equations is arisen from the Hamiltonian SU(2) field tensor so that the total energy remains constant and fulfills the Gaussian law \[6\].

The microcanonical equations of motion are solved on a lattice \(N^d\), which shows chaotic dynamics and we research the time-dependent entropy-energy relation on the lattice, which can be shown by the spectrum of Kolmogorov-Sinai entropy and the statistical complexity \[28\].

1 Introduction

In order to know the microscopic mechanisms of high-energy physics, non-Abelian gauge theory provides an appropriate theoretical model. It plays an important role in understanding non-equilibrium processes where energy and momentum are in local equilibrium. Within the framework of perturbative quantum field theory the equilibrium and transport processes are studied.

One of the results of the study of nonlinearity, showed the system to be chaotic \[30\]. This dynamical quantity is well characterized by the Kolmogorov-Sinai entropy and the statistical complexity, we study the dependence of the systems on energy and time range.

The relation between average energy and Kolmogorov entropy was first introduced in \[17\] for pure SU(2) Yang-Mills systems. The finite size extrapolated initial evolution and as a function of the scaled energy was researched
in [15]. The Kolmogorov-Sinai entropy was investigated by Pesin form. The Lyapunov exponent was extrapolated \((N \to \infty)\) by monodromy matrix and the scaling properties were studied at a given time range.

The concept of entropy was introduced by Clausius [7] in the thermodynamics given an atomic interpretation in the foundational works of the statistical mechanics and gas by Boltzmann [9], Gibbs [10] and others. This plays important role in the many-particle physics, nonequilibrium process. Shannon made of entropy on the theory of information[35]. We apply the information-theoretic formalism i.e. entropy forms are a function of probability distributions. We can be translated to a physical context through the concept of microstate.

**BGS** entropy was later by other quantities in dynamical system (Kolmogorov-Sinai entropy [22]) information theory (Renyi entropy \(R_q[33]\)) and statistical physics (Tsallis entropy \(T_q[37]\)).

Remark that \(S_{BGS} T_q R_q\) considered as special cases of the \((h, \phi)\) entropies [34] for the study of asymptotic probability distributions. These quantities were generalized to quantum information theory [27]. This includes the Neumann entropy [31] and a quantum version of Tsallis' and Renyi entropies, which have been applied for example to the detection of quantum entanglement [27].

The structure of the article: In the second section we introduced the field theory by Feynman path integral and considered the gauge fields. These quantities play an important role in particle and statistical physics. In the third section, we discretized these quantities on the lattice by parallel transporter and Wilson action. In the fourth section we researched the spectrum of Kolmogorov-Sinai entropy and Statistical complexity on the homogenous Yang-Mills fields considering the dynamics by monodromy matrix.

## 2 Path integral in the particle physics

Numerous representations of the field theory are known, Schrödinger wave mechanics resp. Heisenberg operator algebra. One of the best-known methods of quantum production with the Feynman path integral. The advantage of this method is that the analogy between statistical physics and particle physics can be easily drawn. It is well applicable to the formulation of the gauge theory and also accurately reproduces its symmetries. This method is briefly described through quantum mechanics.
2.1 Field theory is described by path integral

2.1.1 Feynman path integral

Using canonical transformations in classical and quantum mechanics, the action is a general function of the canonical transformation. Dirac [14] applied this procedure in the quantum mechanics to the Hamiltonian function \( H \) at time \( t \) in the \( q' \) state, respectively. At the moment \( T \) for a system where the transient amplitude is:

\[
\langle q'|q_T \rangle \sim \exp \left( \frac{i}{\hbar} \int_T^t L dt \right). \tag{1}
\]

On a finite time interval \( T - t \), the range \( T - t \) is divided into \( N \) infinitesimal time intervals, \( t_a = t + a \varepsilon \), \( N \varepsilon = T - t \). Let \( q_a = q_{t_a} \) for all \( t_a \). We apply the \( \int dq|q >< q| = 1 = 1 \) correlation than it is following:

\[
\langle q'|q_T \rangle = \int dq_1 dq_2 \ldots dq_{N-1} \langle q'|q_1 \rangle \langle q_1|q_2 \rangle \ldots \langle q_{N-1}|q_T \rangle.
\]

The transient amplitude prescribed by the path integral introduced by Feynman:

\[
\langle q'|q_{t+\delta t} \rangle = \lim_{N \to \infty} \left( \text{const.} \right)^N \int \left( \prod_{i=1}^{N-1} dq_i \right) \left( \prod_{i=1}^{N-1} dp_i \right) \times \\
\exp \left( -\frac{i}{\hbar} \int_T^t dt L(q, \dot{q}) \right) = \equiv \int Dq Dp \exp \left( -\frac{i}{\hbar} S(t, T, q) \right). \tag{2}
\]

The boundary conditions are the value of orbit at the initial and at the final moment. The above expression gives the probability amplitude of the particle, assuming that it was \( t \) moment in \( q' \) state and at time \( T \) was in \( q \) state. The transient amplitude is expressed as the sum of each of the possible orbits, which begins in \( q \) at time \( T \) and ends in \( q' \) at time \( t \), weighted by the exponential expression \( (-\frac{i}{\hbar} S) \) for each trajectory.

The expression of the transient amplitude for Hamiltonian systems can be described as

\[
\langle q'|q_T \rangle = \int \cdots \int Dq Dp \exp \left( i \int_T^t d\tau \left[ p \frac{dq}{d\tau} - H(p, \langle q \rangle) \right] \right), \tag{3}
\]
where \( <q> \) is the average of \( q \) over a given interval.

### 2.1.2 Relation between the statistical physics and particle physics

Statistical mechanics is closely related to the Feynman path integral of quantum mechanics. Creutz showed in 1977 [12] that the transfer matrix method simplifies the problem of quadratic functions operator diagonalization in Hilbert space.

The Lagrange function of the free nonrelativistic particle, which measure is \( m \) moving in potential \( V(x) \) (imaginaries time lattice):

\[
L(x, \dot{x}) = K(\dot{x}) + V(x), \quad K(\dot{x}) = \frac{1}{2}m\dot{x}^2.
\]

The action function of any trajectory is following

\[
S = \int dt L(\dot{x}(t), x(t)), \quad (4)
\]

with which we can specify with the path integral:

\[
Z = \int [dx(t)] \exp(-S).
\]

The integral is for all trajectories \( x(t) \). The time component of the lattice is discretized. Investigate the trajectories over the entire \( \tau \) time interval, which is decomposed into discrete time slices of length \( a = \tau/N \). The coordinate for the \( i \)-th time slice is \( x_i \). The time derivative \( x \) is approximated with the difference of the neighbors:

\[
S = a \sum_i \left[ \frac{1}{2}m \left( \frac{x_{i+1} - x_i}{a} \right)^2 + V(x_i) \right]. \quad (5)
\]

Expression (5.45) is written with \( x_i \) coordinates using the \( Z \) integral approximation:

\[
Z = \int \left( \prod_i dx_i \right) \exp(-S).
\]

Equation (5.47) is no different than the shape of partition functions in a statistical physical system.

The procedure that leads from the path integral to the expression of the quantum mechanical Hilbert space in three steps is: First, we define the path
integral on a time-like lattice. We construct the transfer matrix in Hilbert space. We finally take the logarithm of the transfer matrix, where the linear term expresses the temporal evolution of the system. If the \( i \)-th eigenvalue of the transfer matrix is \( \lambda_i \), then \( Z = \sum \lambda_i^N \). Since the number of time slices goes to infinity, therefore, the expression can be characterized by the maximum self-values \( \lambda_0 \):

\[
Z = \lambda_0^N \left[ 1 + O \left( \exp \left( -N \ln \left( \frac{\lambda_0}{\lambda_1} \right) \right) \right) \right].
\]

### 2.2 Gauge fields

Several introductions of the gauge fields are known. The simplest way is an extension of the Abelian gauge theory describing the electromagnetic field. The components of the antisymmetric tensor are electromagnetic fields, which are four-dimensional vectors:

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \mu, \nu = 0, 1, 2, 3.
\]

The space-time indices are denoted by \( \mu, \nu \), and the group generators by \( \alpha, \beta, \gamma \). Yang and Mills [40] proposed (1954) to assign the isospin index \( A_\mu \) and \( F_{\mu\nu} \):

\[
A_\mu \rightarrow A_\mu^\alpha \quad F_{\mu\nu} \rightarrow F_{\mu\nu}^\alpha \quad \alpha = 1, \ldots, N,
\]

a further antisymmetric term is added to the expression and the shape of \( F_{\mu\nu} \) is:

\[
F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + g_0 C^{\alpha\beta\gamma} A_\mu^\beta A_\nu^\gamma,
\]

where \( g_0 \) is the bare coupling constant, \( C^{\alpha\beta\gamma} \) is the structural constant of the Lie algebra of a \( G \) Lie group. Here we use only uniter groups, the fundamental representation of the \( G \) group. We parameterize the elements of \( G \) with the set of generators \( g = \exp(\mathbf{i} \omega^\alpha \lambda^\alpha) \), where \( \omega^\alpha \) is the set of parameters and \( \lambda^\alpha \) is the set of Hermitian matrices that generalize the group. The structure constants are defined by the following context:

\[
[\lambda^\alpha, \lambda^\beta] = \mathbf{i} C^{\alpha\beta\gamma} \lambda^\gamma.
\]

The generators are orthonormal: \( \text{tr}(\lambda^\alpha \lambda^\beta) = \frac{1}{2} \delta^{\alpha\beta} \). The simplest non-Abelian theory uses the \( SU(2) \) group, which is generalized by Pauli matrices \( \lambda^\alpha = \ldots \)
\[ C^{\alpha \beta \gamma} = \varepsilon^{\alpha \beta \gamma}. \]

The Maxwell equations can be derived from the Lagrange density:
\[ \mathcal{L} = \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + j_\mu A_\mu, \]

where \( j_\mu \) is the external source as the electrodynamic fields. The non-Abelian Lagrange density starts in the same way, except for the amount for isospin and \( F_{\mu \nu} \) contains an additional member. The classical equation of motion of electrodynamics is the equation \( \partial_\mu F^{\mu \nu} = j_\nu \). In the non-Abelian theory \( (D_\mu F^{\mu \nu})^\alpha = j_\nu^\alpha \). Here are the covariance derivatives:
\[ (D_\mu F^{\mu \nu})^\alpha = \partial_\mu F^{\alpha \mu \nu} + g_0 C^{\alpha \beta \gamma} A^{\beta \mu} F^{\gamma \mu \nu}. \]

The non-Abelian analog of current conservation follows
\[ D_\mu j_\mu = 0. \]

Second definition of gauge fields uses the local symmetry of the action function. Gauge symmetry of electrodynamics: \( A_\mu + \partial_\mu \Lambda \), where the gauge function \( \Lambda(x) \) is an arbitrary function of the space-time coordinates. In the case of non-Abelian, \( A_\mu \) is transformed as follows: \( A_\mu \rightarrow g^{-1} A_\mu g + ig_0^{-1} \partial_\mu g \), where \( g \) is an element of a suitably chosen group of gauges. In electrodynamics, this transformation is written by a simple phase: \( g(x) = \exp(-ig_0 \Lambda(x)) \). This is the so-called \( U(1) \) gauge theory of electrodynamics. Then, using the transformation formula given above, the transformation of \( F_{\mu \nu} \) can be simply written: \( F_{\mu \nu} \rightarrow g^{-1} F_{\mu \nu} g \). The gauge transformation of the covariance derivative can be given in a similar form.

A third possible introduction to gauge theory is phase theory (Mandelstam (1962)[20], Yang (1975)).

In this article we mention the introduction of gauge theories to canonical Hamiltonian formalism following Steven Weinberg (1965) [38].

**Group** In this article, we apply some basic properties of the invariant integral introduced by Haar [18] in Wilson on compact Lie groups. Haar-measure satisfies the following condition:
\[ \int_G f(U) dU = \int_G f(U^{-1}) dU. \]

In the case of \( G = SU(2) \) the group elements can be parameterized in the following way:
\[ U = x^0 1 + ix \tau = \left( \begin{array}{cc} x^0 + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & x^0 - ix^3 \end{array} \right). \]
The parameters $x^i$ must be sufficient to satisfy the condition:
\[
\det U = x^2 = (x^0)^2 + \vec{x}^2 = 1.
\]
that specifies the $S^3$ key. In the case of numerical calculation, we used the quaternion representation $x_0, x_1, x_2, x_3$, because the runtime is faster and the memory requirement is smaller than the matrix representation.

3 Lattice Field Theory

3.1 Discrete Parallel transporter

Consider hypercubic lattice of size $a$ and the regularization of the continuous Euclidean lattice. The scalar field $\phi(x)$ is interpreted on the lattice point. Local gauge transformation is following:
\[
\phi(x) \rightarrow \phi'(x) = \Lambda^{-1}(x)\phi(x).
\]
In this case, the nearest non-zero lattice spacing $a$ must be introduced on the hypercube grid.

The elementary parallel transporters are closely connected by the links $b$, which connect the neighboring points. Let $x$ be an arbitrary point on the lattice. Nearest neighbour points can be written in the form $x + a\hat{\mu}$, where $\mu = 1, 2, 3, 4$ and $\hat{\mu}$ denotes the $\mu$-th unit vector. The links from $x$ to $x + a\hat{\mu}$ can be denoted by the following ordered pair: $b = (x + a\hat{\mu}, x) \equiv (x, \mu)$. The parallel transporter can be described by the link $b$:
\[
U(b) \equiv U(x + a\hat{\mu}, x) \equiv U_{x\mu} \in G, \quad (8)
\]
where $G$ is the gauge group. The link thus introduced satisfies the corresponding properties of the parallel transporter. Arbitrary path $C = b_n \circ b_{n-1} \circ \cdots \circ b_1$ corresponds to the parallel transporter $U(b) = U(b_n) \cdots U(b_1) \equiv \prod_{b \in C} U(b)$ on lattice, which describes the link variables. These are denoted by $\{U(b)\}$\textsuperscript{25}. Transformation of link variables is following:
\[
U'(y, x) = \Lambda^{-1}(y)U(y, x)\Lambda(x),
\]
where $\Lambda \in SU(N)$ and the size of matrix is $N \times N$. We define the covariance derivative:
\[
D_\mu \phi(x) = \frac{1}{a} (U^{-1}(x, \mu)\phi(x + a\hat{\mu}) - \phi(x)).
\]
The term of derivatives are substituted by covariate derivatives in the kinetic expression:

\[
\frac{1}{2} \sum_x a^4 D_\mu \Phi D_\mu \Phi = -a^2 \sum_{\langle xy \rangle} \phi(x) U(x, y) \phi(y) + 4a^2 \sum_x \phi(x)^2.
\]

The smallest closed loop on the lattice is called a plaque. A plaque is enclosed by 4 links and it contains the following points: \(x, x + a\hat{\mu}, x + a\hat{\mu} + a\hat{\nu}, x + a\hat{\nu}\), denoted by \(p = (x; \mu\nu)\). The corresponding parallel transporter can be written in the following form:

\[
U_p \equiv U_{x_{\mu\nu}} \equiv U(x, x + a\hat{\nu}) U(x + a\hat{\nu}, x + a\hat{\mu} + a\hat{\nu}) \times U^{\dagger}(x + a\hat{\mu} + a\hat{\nu}, x + a\hat{\mu}) U^{\dagger}(x + a\hat{\mu}, x),
\]

which we call the plaque variables. Wilson’s suggestion \[21, 39\] is to write the theoretical definition of a simple lattice gauge with the plaque variables:

\[
S[U] = \sum_p S_p(U_p),
\]

that is, the action is summed for all \(p\), i.e. \(\sum_p = \sum_x \sum_{1 \leq \mu, \nu \leq 4}\) means. The action is written on the elementary plaque (showing only one direction):

\[
S_p(U_p) = \beta \left\{1 - \frac{1}{N} \text{Retr} U_p\right\}.
\]

### 3.2 Wilson action, lattice Hamilton

Wilson action is gauge invariant quantity because \(\text{tr} U'_p = \text{tr} U_p\) is appropriately chosen for group \(SU(N)\), further real and positive. We consider the Yang-Mills action by the Wilson action. We introduced the vector potential: \(A_\mu(x) = -igA^b_\mu(x)T_b\). Lie-algebra value vector field was defined on the lattice:

\[
U(x, \mu) \equiv \exp(-aA_\mu(x)) = 1 - aA_\mu(x) + \frac{a^2}{2} A_\mu(x)^2 + \ldots
\]

we apply \(A_\nu(x + a\hat{\mu}) = A_\nu(x) a\Delta^f_{\mu} A_\nu(x)\) where \(\Delta^f_{\mu} f(x) = \frac{1}{a} (f(x + a\hat{\mu}) - f(x))\).

The Campbell-Baker-Hausdorff expression:

\[
\exp(x) \exp(y) = \exp(x + y + \frac{1}{2}[x, y] + \ldots)
\]

therefore we get:

\[
U_{x_{\mu\nu}} = \exp\left(-a^2 G_{\mu\nu}(x)\right), \quad \text{where} \quad G_{\mu\nu}(x) = F_{\mu\nu}(x) + O(a)
\]

\[
F_{\mu\nu}(x) = \Delta^f_{\mu} A_\nu(x) - \Delta^f_{\nu} A_\mu(x) + [A_\mu(x), A_\nu(x)].
\]
Therefore
\[ 1 - \frac{1}{N} \text{Retr} U_p = 2 \text{tr} 1 + a^4 \text{tr} (F_{\mu\nu}(x))^2 + O(a^5), \]
since \( \text{tr} G_{\mu\nu}(x) = 0 \) and \( \sum_p \text{tr} (F_{\mu\nu}(x))^2 = \frac{1}{2} \sum_{x\mu\nu} \text{tr} (F_{\mu\nu}(x))^2 \). We get the following expression of the Wilson action:
\[ S = -\frac{\beta}{4N} \sum_x a^4 \text{tr} F_{\mu\nu}(x) F^{\mu\nu}(x) + O(a^5). \] (11)

Because, the leading member coincides with the Yang-Mills action for small \( a \) if \( \beta = \frac{2N}{g^2} \) and \( g \) correspond to the bare coupling constant of the lattice theory. We split the action into time-space components
\[ S = \frac{2}{g^2} \sum_{p_t} (N - \text{tr} U_{p_t}) - \frac{2}{g^2} \sum_{p_s} (N - \text{tr} U_{p_s}), \] (12)
where \( g \) is the continuous limitation value of the coupling constant, the (-) sign is derived from the Minkovski space-time structure. The Taylor series of \( U_{p_t} \) is explained in time-dependent term
\[ U_{p_t} = U(t)U^\dagger(t + a_t) = UU^\dagger + a_t U\dot{U}^\dagger + \frac{a_t^2}{2} U\ddot{U}^\dagger + \ldots, \]
The expressions appear in the Wilson Action:
\[ N - \text{tr} U_{p_t} = -\frac{a_t^2}{2} \text{tr} (U\ddot{U}^\dagger) + O(a_t^3) \text{ correction.} \]
Since \( UU^\dagger = 1 \), trace disappears \( N \). It follows from the first derivative of this term that \( \text{tr} (U\ddot{U}^\dagger) = 0 \) and the second derivative is \( \ddot{U}U^\dagger + 2\dot{U}\dot{U}^\dagger + U\dddot{U}^\dagger = 0 \). Therefore the Hamiltonian lattice action is following:
\[ \Delta S_H = \frac{2}{g^2} \left( \frac{a_t^2}{2} \sum_i \text{tr} (\dot{U}_i\ddot{U}_i^\dagger) - \sum_{ij} (N - \text{tr} (U_{ij})) \right). \] (13)
The generalized discretized ansatz can be written:
\[ S = a_t \sum_t a_s^3 \sum_s L. \]
The scaled Hamilton density is able to write in the following form.

\[ a_t H = \frac{2}{g^2} \left( \frac{a_t^2}{2} \sum_{x,i} \text{tr} \left( \dot{U}_{x,i}, \dot{U}^\dagger_{x,i} \right) + \sum_{x,ij} \left( N - \text{tr} \left( U_{x,ij} \right) \right) \right), \quad (14) \]

namely

\[ H = a_s^3 \sum_s \left( \text{tr} \left( \dot{U}, \frac{\partial L}{\partial \dot{U}} \right) - L \right). \]

On the lattice, the gauge field can be specified by configuring the link variables. The expected value of the quantity denoted by the \{U(b)\} \equiv U and \Theta(\{U(b)\}) link variables:

\[ \langle \Theta \rangle = \frac{1}{Z} \int \prod_b dU(b) \Theta \exp(-S(U)), \quad (15) \]

where \( Z = \int \prod_b dU(b) \exp(-S(U)) \) and \( S(U) \) are Wilson actions. If we introduce \( \phi(x) \) the field "material" is given by the corresponding integral:

\[ \langle \Theta \rangle = \frac{1}{Z} \int \prod_b dU(b) \prod_x d\phi(x) \Theta \exp(-S(U, \phi)). \]

In these expressions, the integration measures \( dU(b) \), must be chosen to be gauge invariant. During the gauge transformation it is written:

\[ U'(x, y) = \Lambda^{-1}(x) U(x, y) \Lambda(y) \]

because the action is invariant: \( dU = dU', S(U) = S(U') \).

### 3.3 Lattice Yang-Mills Theory

In the following, we use Hamiltonian formulation of the classical lattice SU(2) gauge theory \[31\]. The Hamilton function is considering:

\[ H' = \frac{g^2 aH}{4} = \sum_{x,i} \frac{a^2}{4} \text{tr} \left( \dot{U}_{x,i}, \dot{U}^\dagger_{x,i} \right) + \sum_{x,ij} \left[ 1 - \frac{1}{2} \text{tr} U_{x,ij} \right], \]

where \( U_{x,i} \) is the group element SU(2), this term means the \( x + a \mathbf{e}_i \) link pointing in the \( i \) direction starting at \( x = (x_1, x_2, x_3) \) on the lattice. \( U_{x,ij} \) denotes the elementary plaque which is expressed by link \( U_{x,ij} = U_{x,i} U_{x+i,j} U^\dagger_{x,i+j} U^\dagger_{x,j} \).
lying in the plane stretched by the elementary vectors \( \mathbf{i} \) and \( \mathbf{j} \) starting at \( \mathbf{x} \). We apply the link variables only in the expressions \( H \):

\[
H = \sum_{x,\mathbf{i}} \left[ \frac{1}{2} \langle \dot{U}_{x,i}, \dot{U}_{x,i} \rangle + \left( 1 - \frac{1}{4} \langle U_{x,i}, V_{x,i} \rangle \right) \right],
\]

where the complement link variable \( V_{x,i}(U) \) is following:

\[
V_{x,i} = \frac{1}{4} \sum_{(\ell,s):((-\ell_{i},s),(-\ell_{j},s))} U_{x+\ell_{i}} \U_{x+\ell_{i}+s_{-1}} \U_{x+\ell_{i}+s_{-1}}, \text{ where}
\]

\( i, j, k \) are the unit vectors of the three-dimensional lattice.

In the gauge field section (2.2) we introduced the quaternion representation, which is defined in the following way on a lattice:

\[
\mathbf{U} = u_{0} \mathbf{1} + i\mathbf{\tau} \mathbf{u} \quad \mathbf{U} = \left( u_{0} + iu_{3}, iu_{1} + u_{2}, iu_{1} - u_{2}, u_{0} - iu_{3} \right).
\]

The equations of motion are derived from the Hamiltonian function:

\[
\dot{\mathbf{U}} = \mathbf{P}, \quad \dot{\mathbf{P}} = \mathbf{V} - \langle \mathbf{U}, \mathbf{V} \rangle \mathbf{U} - \langle \mathbf{P}, \mathbf{P} \rangle \mathbf{U},
\]

where \( \langle \mathbf{P}, \mathbf{P} \rangle = \frac{1}{2} \sum_{j} P_{j}P_{j} \).

The lattice equation of motion [4] follows:

\[
U_{t+1} - U_{t-1} = 2h(P_{t}^{*} - \epsilon U_{t}^{*}), \quad P_{t+1} - P_{t-1} = 2h(V(U_{t}^{*}) - \mu U_{t}^{*} + \epsilon P_{t}^{*}), \text{ where}
\]

\[
\epsilon = \frac{\langle U_{t}^{*}, P_{t}^{*} \rangle}{\langle U_{t}^{*}, U_{t}^{*} \rangle}, \quad \mu = \frac{\langle V(U_{t}^{*}), U_{t}^{*} \rangle + \langle P_{t}^{*}, P_{t}^{*} \rangle}{\langle U_{t}^{*}, U_{t}^{*} \rangle}, \quad \text{and}
\]

\( U_{t}^{*} = aU_{t+1} + bU_{t} + cU_{t-1} \).

The quantities \( \epsilon, \mu \) denote the Lagrange multipliers. The energy of the Hamiltonian system was constant during the movement. A periodic boundary condition was used to solve the system of equations. The color charge was defined following:

\[
\Gamma_{i} = \sum_{l_{+}} P_{l} \U_{i}^{\dagger} - \sum_{l_{-}} U_{l}^{\dagger} P_{i}, \quad i = 1, \ldots N
\]
The measure of change is written by this term:

\[
\dot{\Gamma}_i = \sum_{\Gamma_{l+}} (\mathcal{V}U^\dagger - \langle \mathcal{V}, U \rangle) 1,
\]

where \( P_1 = QU_1 \) and \( P_1 = QU_1 \) and \( P_n = U_{n-1}^\dagger P_{n-1}U_n, 1 < n < N \). The condition of neutrality formulated as

\[
Q - F^\dagger QF = 0, \quad \text{tr} Q = 0,
\]

from which it follows

\[
Q = \frac{q}{2} (F^\dagger - F), \quad \text{where} \quad F = \prod_{i=1}^{N-1} U_i \quad \text{oriented product}
\]

the initial color charge is \( Q \) and the final state is \(-F^\dagger QF\).

## 4 Nonlinearity

In this section, we numerically determined the Lyapunov spectrum on the three-dimensional lattice of the SU(2) Yang-Mills field. The spectra of Kolmogorov-Sinai entropy

are studied by the eigenvalues of the monodromy matrix from the classical chaotic dynamics to extrapolate on a lattice with a large size limit.

**Monodromy matrix** We consider a periodic orbit of the energy \( E \), with initial phase space coordinates \((p = p_0, x = x_0)\) and final coordinates \((p = p_0, x = x_0)\). We study the behavior of the neighborhood path of the periodic orbits, how these trajectories develop in the case of small transverse perturbation.

This means the same situation when considering the deviation of flow on the Poincare surface of the section transverse to the path. Then the relation between the initial \(\{\delta y_0i, \delta p_0i\}\) and final state \(\{\delta y_1, \delta p_1\}\) deviation is following:

\[
\delta y_i = \sum_{j=1}^{d-1} \left( \frac{\partial y_i}{\partial y_{0i}} \right) \delta y_{0j} + \left( \frac{\partial y_i}{\partial p_{0i}} \right) \delta p_{0j} = \sum_{j=1}^{d-1} A_{ij} \delta y_{0j} + C_{ij} \delta p_{0j}
\]

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and
\[ \delta p_i = \sum_{j=1}^{d-1} \left( \frac{\partial p_i}{\partial y_{0j}} \right) \delta y_{0j} + \left( \frac{\partial p_i}{\partial p_{0j}} \right) \delta p_{0j} = \sum_{j=1}^{d-1} C_{ij} \delta y_{0j} + D_{ij} \delta p_{0j} \]

It is written by matrix form:
\[
\begin{pmatrix} \delta y \\ \delta p \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \delta y_0 \\ \delta p_0 \end{pmatrix} = \mathbf{M} \begin{pmatrix} \delta y_0 \\ \delta p_0 \end{pmatrix},
\]

(19)

where \( \delta \mathbf{y} \) and \( \delta \mathbf{p} \) are \((d-1) \times 1\) dimensional column matrices, and \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \) are \((d-1) \times (d-1)\) dimensional square matrices where \( A_{ij}, B_{ij}, C_{ij}, D_{ij} \) matrix elements. This \((2d - 2) \times (2d - 2)\) dimensional square matrix \( \mathbf{M} \) means the monodromy matrix according to the equation motion [32].

The shape of the monodromy matrix by the lattice equations of motion [15] is following
\[
\mathbf{M} = \begin{pmatrix} \frac{\partial \dot{U}}{\partial U} & \frac{\partial \dot{U}}{\partial P} \\ \frac{\partial \dot{P}}{\partial U} & \frac{\partial \dot{P}}{\partial P} \end{pmatrix}.
\]

(20)

We write down each partial derivative by the equation of motion:
\[
\frac{\partial \dot{U}^a}{\partial U^b} = 0, \quad \frac{\partial \dot{U}^a}{\partial P^b} = \delta_{ab},
\]
\[
\frac{\partial \dot{P}^a}{\partial U^b} = \frac{\partial V^a}{\partial U^b} - \left( \sum_{c=1}^{N} U_c \frac{\partial V^c}{\partial U^b} \right) U^a - V^b U^a - \sum_{c=1}^{N} (U_c V^c + P_c P^c) \delta_{ab},
\]
\[
\frac{\partial \dot{P}^a}{\partial P^b} = -2P^b U^a, \quad \text{where}
\]
\[
\frac{\partial V^a}{\partial U^b} = \sum_{l=1}^{N} \frac{\partial V^a}{\partial U^b} \begin{pmatrix} l \\ 1 \end{pmatrix}, \quad \text{where} \ N = 12, \ \alpha_q, \beta_q = 0, 1, 2, 3.
\]

The shape of the characteristic equation is then:
\[
\det \left[ \begin{pmatrix} 0 & 1 \\ \frac{\partial P}{\partial U} & \frac{\partial P}{\partial P} \end{pmatrix} - \Lambda_1 \right] = 0.
\]

(21)

We showed the stability of the trajectories along the trajectory in the vicinity of any point on the \((U, P)\) phase space. The time evolution of a small \((\delta U, \delta P)\) perturbation is determined by the monodromy matrix. Among the eigenvalues of the stability matrix, real and positive quantities indicate an exponential departure of adjacent trajectories, i.e., motion is unstable. At the long-term limit, the Lyapunov exponents are obtained from the eigenvalues.
4.1 Spectra of the maximal Lyapunov exponent

We investigated the ergodization of the SU(2) lattice gauge theory due to classical chaotic dynamics [15]. The Lyapunov exponents are calculated on larger lattice ($N = 2, 3, 4, 5, 6, 7$) and their extrapolation to $N \to \infty$ in the thermodynamic limit. We obtained a good approximation of the real maximal Lyapunov spectrum for time-evolving field configurations by the monodromy matrix. The initial configurations are randomized we choose according to the Haar measure and the total energy.

The $L_i$ Lyapunov exponent is expressed with eigenvalues $\Lambda_i$ of monodromy matrix:

$$L_i = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Lambda_i(t) \, dt \quad i = 1, \ldots, f,$$

(22)

where $\Lambda_i(t)$ is the solution of the characteristic equation:

$$\det[\Lambda_i(t)1 - M(t)] = 0,$$

(23)

in which $M$ is the linear stability matrix, $f$ is the number of degrees of freedom. Conservative dynamical systems satisfy the Liouville theorem: $\sum_{i=0}^f L_i = 0$. In numerical calculations, we use the definition of the discrete Lyapunov spectrum

$$L'_i = \langle \Lambda_i \rangle^{(n)} = \frac{1}{n} \sum_{j=1}^n \Lambda_i(t_{j-1}), \quad i = 1, \ldots, f,$$

(24)

where $t_j$ is the time series during the orbital evolution of the gauge field configuration.

The quantities $L'_i$ are extrapolated to a long-term ($N \to \infty$) limit with fixed time steps. We assumed it converges to the $L_i$ Lyapunov exponent for a noncompact configuration space. As we shall discuss the numerical results.

The monodromy matrix method uses only the development of time in a gauge field and allows us to know the behavior of the Lyapunov spectrum as a function of time.

In the numerical simulation the total number of degrees of freedom $f = 4 \times 3 \times N^3 = 12 \times N^3$, where the group element $SU(2)$ is represented by 4 real quaternions (thus the phase space has a dimension of $2f = 24N^3$). Due to the conditions of survival (unity, orthogonality), the number of physically relevant degrees of independent freedom decreases [5].
The spectrum of the $2f \times 2f$ stability matrix although rare, is large enough to determine the eigenvalue with sufficient accuracy. Since it requires $O(f^2)$ memory to determine eigenvalues, $N = 7$ ($2f = 24N^3 = 8232$ dimensional phase space) was the maximum size of the system, which could be examined by the capacity of the computer, which is due to the fact that the Hamiltonian system is conservative (energy is time-independent).

Our goal is to determine the spectrum of maximal Lyapunov exponent depending on the energy and time and we consider the scaling behavior of this system.

Therefore the first step we extrapolated the real maximal Lyapunov exponent ($N \to \infty$) to the thermodynamical limit from the dataset, which is determined for $N = 2, 3, 4, 5, 6, 7$ at the different energies $g^2 a E \in [0.0, 0.8]$ range considering the finite-size scaling.

The scaling of the maximal Lyapunov exponent by energy has long been the subject of research [30], linear behavior has been observed in the long-term boundary case. At low energy experienced $L_0 \sim E^{\frac{1}{4}}$. The trend of further study leads to relation to $L_0 \sim E$.

Figure (4.1) show the real maximal Lyapunov exponent’s dependence on time and energy.

The scaling of short-time maximal Lyapunov exponents with energy is plotted, before reaching saturation. If the time is increasing the value of the Lyapunov exponent becomes larger and remains linear, but the saturation occurs at earlier time step. In the case of long-time evolution, the scaling of maximal Lyapunov exponent becomes more logarithmic. In the long-time, the behavior of the maximal real part of the monodromy matrix eigenvalues shows a definitely sublinear scaling with the energy per degree of freedom. It can be assumed that too long a trajectory and the compactness of the configuration space cause the investigated eigenvalues, which is the Hamiltonian lattice field theory artifact. In the following, we will refer to linear scaling.

The behavior of the real part of the maximum Lyapunov exponent obtained from the eigenvalues of the monodromy matrix is is not yet complete. The best fit $E^{\frac{1}{4}}$, but the logarithm also fits well. It can be assumed that too long a trajectory and the compactness of the configuration space cause the investigated eigenvalues, which is the Hamiltonian lattice field theory artifact. In the following, we will refer to linear scaling.

Figure (4.1) shows the extrapolation of the maximum Lyapunov exponent
to the thermodynamic limit at different energies. The fits proved to be almost linear, provided they scale to a finite size:

\[ L_0 \sim \frac{1}{\sqrt{f}} \sim N^{-\frac{3}{2}}. \]

This accords to sampling ergodic states [8]. At high energy, the extrapolated \( L_0 \) takes on a larger value than the value obtained for the finite \( N \) from the simulations.

### 4.2 Spectrum of Kolmogorov-Sinai Entropy

The relation between average energy and Kolmogorov-Sinai entropy was first discussed in \[30\] for the simple SU(2) Yang-Mills system.

We define the Kolmogorov-Sinai entropy by the term Pesin:

\[ h^{KS} = \sum_i L_i \Theta(L_i), \tag{25} \]

where the value of the function \( \Theta(x) \) is 1 if the argument is positive and 0 otherwise. The dimension of the quantity \( h^{KS} \) is a rate (1/time). Therefore, the change in entropy density can be given on an \( N \times N \times N \) lattice by normalizing the Kolmogorov-Sinai entropy:

\[ \langle S \rangle = \frac{h^{KS}}{\text{Re}(L_0)N^3}. \tag{26} \]

The state equation can be derived from the simulations of the dynamics. The finite-size scaling is extrapolated to infinity (\( \frac{1}{N} \to 0 \)) on the lattice. We consider the dependence of the Kolmogorov-Sinai entropy on energy. This
leads to the state equation, which is the $S(E)$ entropy-energy relation in the thermodynamic limit.

Finally, we obtained the normalized Kolmogorov-Sinai entropy from the extrapolated $L_i$ data, which depend only slightly on the initial values and scale linearly according to the energy.

$$< S > - b \log(g^2Ea) + c,$$

where $b, c \in \mathbb{R}$.

The closest is the relation of the ideal gas $S \sim \log E$. This is the best fit for inverse temperatures:

$$\frac{1}{T} = \frac{\partial < S >}{\partial E} \sim 0.5 \frac{E}{E}$$

Thus the equipartition, i.e. the energy per degree of freedom:

$$E = \frac{1}{2}kT$$

The dependence of the normalized entropy on energy and time is shown in Figure (2). Within the lower and middle energy interval, entropy showed no anomalous behavior for a finite lattice size.

Apart from the initial oscillation, the normalized entropy becomes constant over time within the studied energy interval.

Since the Kolmogorov-Sinai entropy was determined from the Lyapunov exponents with the form Pesin, the lattice calculation artifact experienced in the numerical determination of the Lyapunov exponents manifest in the Kolmogorov-Sinai entropy spectrum. Figure (2) shows the relation of entropy-energy is changing between the short-time and long-time range.

In our case lattice SU(2) system $S(E)$ curve would show a first-order two-phase structure containing a break somewhere [36]. To decide this, we need to filter out lattice artifacts and reduce entropy fluctuations to give a clear answer.

5 Spectrum of Statistical Complexity

5.1 Statistical Complexity

The family of statistical complexity measures $C$ is defined on the basis of the functional product form $C = H \cdot Q$ for difference disorder $H$ and disequilibrium $Q$ measures on the probabilistic space.
Figure 2: Entropy spectrum $S$ depends on the $t/a$ and $g^2aE$.

The information measure $\mathcal{L}$ is able to characterize by a given probability distribution $P = \{p_j, j = 1, \ldots, N\}$, and this quantity corresponds to the measure of uncertainty of a physical process. The amount of disorder $H$ is defined:

$$H[P] = \mathcal{L}[P]/\mathcal{L}_{\text{max}},$$

where $\mathcal{L}_{\text{max}} = \mathcal{L}[P_e]$ and $P_e = \{1/N, \ldots, 1/N\}$ is the uniform distribution which is maximizing the information measure ($0 \geq H \geq 1$). In order to consider the idea of statistical complexity, it is necessary to define the disequilibrium $Q$. The measure of this quantity is investigated by some kind of distance $D$ fulfills the equal probability distribution $P_e$. The disequilibrium $Q$ is as follows:

$$Q[P] = Q_0D[P, P_e],$$

where $Q_0$ is a normalization factor ($0 \leq Q \leq 1$). This concept characterizes the structure of the systems is larger than zero if there are more likely states among the available situations. Therefore, we take the following functional form for the statistical complexity measure:

$$C[P] = H[P] \cdot Q[P].$$

This quantity $C[P]$ characterizes the amount of information stored in the systems and its disequilibrium together [27]. The definition of this notion is divided into three classes: (c1) this is the monotonous increase in the function of entropy; (c2) it fulfills a convex function including a maximal $C_{\text{max}}$ at the probability distribution $P_e$ and the minimal $C_{\text{min}}$ occurs at the extreme values of the entropy, i.e., the $H = 0$ or $H = 1$; and (c3) the third part is the monotonous decrease with increasing entropy [27].
The two extreme situations can be understood as follows: (i) Each set of sequences has the same probability distribution. All of them accept the information stored in an equal measure similar to the ideal gas\[24\].

(ii) If we study a system that corresponds to some kind of symmetries properties and distance, then it is able to write by minimal information as spacial solid-state material or symmetry in quantum mechanics or the pattern is completely disordered.

The statistical complexity comments with the scale because it was introduced in a finite system. At each scale of investigation, a new set of available simulated series occurs with its corresponding probability distribution \(P\); therefore, the complexity is changing.

The statistical mechanics is often interested in isolated systems characterized by an initial optional and discrete uniform probability distribution\[11\]. During the dynamic of isolated systems and their statistical complexity can not gain arbitrary values in a \(C_{\text{LMC}}\) vs \(H\) map. These are limited by certain values of minimum and maximum bounds of complexity.

We use the Shannon entropy measure and Euclidean distance on the probability space as the statistical complexity was investigated by Lopez-Ruiz, Manchini, and Calbet (LMC)\[27\].

**Information measure** We consider the Shannon logarithmic information measure on the \(P \equiv \{p_1, \ldots, p_N\}\) discrete probability distribution in this article as follows:

\[
\mathcal{L}[P] = - \sum_{j=1}^{N} p_j \log(p_j)
\] (30)

The maximal value \(\mathcal{L}_{\text{max}}\) is calculated by the uniform probability \(P_e = \{\frac{1}{N}, \ldots, \frac{1}{N}\}\) fulfilling this criterion \(\sum_{j=1}^{N} p_j = 1\) therefore, \(\mathcal{L}_{\text{max}} = \ln N\). If \(\mathcal{L}[P] = 0\), it means that the possible outcomes \(j\) whose probabilities are given by \(p_j\) will actually take place. The knowledge of the process is corresponded by the probability distribution is maximal in this case. Anyway this quantity becomes largest for a uniform distribution, when \(\mathcal{L}[P] = S_{\text{max}}\). These two extreme criteria correspond to the (i) perfect order and (ii)maximum randomness as trivial ones.

**Disequilibrium** Evidently, the Euclidean statistical distance is taken to express the quantity \(D\), i.e., the quadratic distance between the probability
distributions of each state to the equiprobability. If \( D \) is the Euclidean norm in \( \mathbb{R}^N \), we get

\[
D_E(P, P_e) = \| P - P_e \| = \sum_{i=1}^{N} (p_i - p_e)^2
\]

(31)

where \( p_e = 1/N \). The maximum disequilibrium is gained for overwhelming simulation sequences with \( p_e \sim 1 \) and \( D \to 1 \) for increasing \( N \), while this quantity disappears \( D \sim 0 \) for \( p_i \sim 1/N \) for all \( i \). In other probability distribution, the value of the disequilibrium \( D \) will vary between these two extreme rates. Then, the form of the normalization factor of the Euclidean statistical distance fulfills \( Q_0 = \frac{N}{N-1} \).

**Entropy** There are several definitions of entropy. In this article, entropy is defined by classical discrete and finite probability distributions. The next definitions are considering, which are important in the statistical complexity and physics:

One of the most important is the Boltzmann-Gibbs-Shannon entropy [9] (BGS) following

\[
S_{\text{BGS}}(p_1 \ldots p_w) = k \sum_{i=1}^{w} p_i \ln \frac{1}{p_i},
\]

(32)

where \( k = 1.3807 \times 10^{-23} \text{J/K} \) is the Boltzmann constant and \( w \) means the number of microstates and \( p_i \) is the probability associated with the \( i \)-th microstate. If \( k = 1 \) then it means the entropy of information theory and \( S_{\text{BGS}} \) measures the average information by the outcomes of a random variable with probability distribution \( \{p_1 \ldots p_w\} \).

We remark:

- Tsallis entropy [37]:
  \[
  T_q(p_1 \ldots p_w) = \frac{1}{1-q} \left( \sum_{i=1}^{w} p_i^q - 1 \right) \quad q \in \mathbb{R}, q \neq 1
  \]

- Renyi entropy [33]:
  \[
  R_q(p_1 \ldots p_w) = \frac{1}{1-q} \ln \left( \sum_{i=1}^{w} p_i^q \right) \quad q \geq 0, q \neq 1
  \]

The Tsallis and Renyi entropies are related to the BGS entropy through the limit \( \lim_{q \to 1} T_q(p_1 \ldots p_w) = \lim_{q \to 1} R_q(p_1 \ldots p_w) = S_{\text{BGS}}(p_1 \ldots p_w) \), where \( \{p_1 \ldots p_w\} w \geq 2 \).

The uniqueness of \( S_{\text{BGS}} \) is derived from the four Khinchin-Shannon axioms.

We introduce the generalized entropy [1]. Let \( P \) be the set of probability mass distribution \( \{p_1 \ldots p_w\} \) for all \( w > 2 \). For any function \( H : P \to \mathbb{R}^+ \), the Shannon-Khinchin axioms for an entropy term \( H \) are the following:
(1): Continuity: \( H(p_1 \ldots p_w) \) depends continuously on all variables for each \( w \).

(2): Maximality: For all \( w \)

\[
H(p_1 \ldots, p_w) \leq H \left( \frac{1}{w} \ldots \frac{1}{w} \right)
\]

(3): Expensability: For all \( w \) and \( 1 \leq i \leq w \),

\[
H(0, p_1 \ldots p_w) = H(p_1 \ldots p_0 p_{i+1} \ldots p_w) = H(p_1 \ldots p_i, p_{i+1} \ldots p_w)
\]

(4): Separability (or strong additivity): For all \( w, u \)

\[
H(p_{11} \ldots p_{1u}, p_{21} \ldots p_{2u} \ldots p_{w1} \ldots p_{wu})
\]

\[
= H(p_1, p_2 \ldots p_w) + \sum_{i=1}^{w} p_i H \left( \frac{p_{i1}}{p_i}, \frac{p_{i2}}{p_i} \ldots \frac{p_{iu}}{p_i} \right),
\]

where \( p_i = \sum_{j=1}^{u} p_{ij} \).

As a function \( F : P \to \mathbb{R}^+ \) satisfying axioms (1)-(4) is necessarily of the form \( F(p_1 \ldots p_w) = kS_{BGS}(p_1 \ldots p_w) \) for every \( w \), where \( k \) is a positive constant [20].

We define nonnegative functions \( P \) fulfilling axioms (1)-(3) these are called generalized entropies [19]. In the simplest situation, a generalized entropy has the sum property [13]:

\[
F_g(p_1 \ldots p_w) = \sum_{i=1}^{w} g(p_i)
\]

with \( g : [0, 1] \to \mathbb{R}^+ \).

We consider the next propositions:

(i.) Symmetry \( F_g(p_1 \ldots p_w) \) is invariant under perturbation of \( p_1 \ldots p_w \).

(ii.) \( F_g \) satisfies axiom (1) if and only if \( g \) continuous.

(iii.) If \( F_g \) satisfies axiom (2), then

\[
\sum_{i=1}^{w} g(p_i) \leq wg \left( \frac{1}{w} \right)
\]

for all \( w \geq 2 \) and \( p_1 \ldots p_w \) with \( p_1 + \cdots + p_w = 1 \).
(iv.) \( g \) is concave, then \( F_g \) satisfies axiom (2).
(v.) \( F_g \) satisfies axiom (3) if and only if \( g(0) = 0 \).

Note that Proposition (iv) follows from the symmetry and concavity of \( F_g \) (since the unique maximum of \( F_g \) must occur at equal probabilities).

We conclude from Propositions (ii), (iv), and (v), that for \( F_g \) to be a generalized entropy, the following three conditions suffice:

(g1) \( g \) is continuous  
(g2) \( g \) is concave  
(g3) \( g(0) = 0 \)

As in [19] we say that the macroscopic statistical system is admissible if it is described by a generalized entropy \( F_g \) of the form in equation (33) such that \( g \) verifies condition (g1)-(g3).

5.2 Complexity of the lattice Yang-Mills equation

In the section (5.1) we introduced the statistical complexity which is based on the probability distribution providing a statistical estimation of the series of dynamical systems. There are \( N \) finite different elements on the sequence \( \{x_1, x_2, \ldots, x_N\} \) corresponding to the set of discrete probability distribution \( p \equiv \{p_1, p_2, \ldots, p_N\} \), where \( p_i := P(x_i) \), \( \sum_{i=1}^{N} p_i = 1 \), and \( p_i > 0 \) for all \( i \).

We study the time evolution of the gauge field by the Yang-Mills equation on the lattice. Random initial values are chosen which fulfill the constraint (unitarity, orthogonality, and energy). The length of trajectory is taken as \( m = 10000 \), the subsequent along the orbit is \( n = 2 \). The lattice size was chosen \( N = 2, 3, 4, 5, 6, 7 \).

The state of the gauge field at time \( t \) contains all \( U_{x,i} \) links on a lattice of size \( a \). The number of link is \( \text{dim} \times N^3 \). The lattice gauge field configuration characterizes the state at a given time by the links altogether.

The value of entropy (27), disequilibrium (28), and the statistical complexity (29), can be determined by the simulation unambiguously. Since the probability distribution of element is discontinuous in three-dimensional gauge space, some complexity and disequilibrium values do not appear for certain entropy quantities.

In the Figure (3) the complexity \( C \) as a function \( g^2 \alpha E \) and \( H \) (left) is represented, the lattice size \( N = 7 \). The spectrum of \( C \) is finite and limited but not necessarily a unique function of \( H \) and there exists a convex boundary and larger inner structure between the minimal value \( C_{\text{min}} \) and the maximal value \( C_{\text{max}} \) for different energy range \([0.075,0.07]\). The inner structure can be seen better in Figure (3(right), where the complexity \( C \) dependence on the entropy \( H \) is shown. The fit of maximal values \( C \) becomes to zero at \( H \sim 0 \).
and \( H \sim 1 \) and the curve is convex on the interval \( H \in [0, 1] \). The minimal and maximal boundary is increasing as the energy is growing.

In the Figure (4) the complexity \( C \) as a function \( H \) and \( D \) (left) and \( C \) depends on \( H \) (right) are plotted. The spectrum of the complexity \( C \) is able to measure by the term LMC complexity, it contains all points at the range of energy \([0.075, 0.7]\). This behavior of complexity belongs to a class \( (c2) \). The curve of the complexity shows a larger inner structure near to the \( C_{\text{max}} \). Because the number of points on a long trajectory is finite, \( C \) is a function \( H \) shows scaling behavior, i.e., the bigger complexity appears at less entropy with a larger discrete probability distribution. The disequilibrium \( D \) (right) features a local maximum near to a small value of the entropy, and, therefore, the complexity \( C \) has a peak at this range of entropy and approaches zero as \( H \) increases.

6 Summary

The SU(2) lattice gauge field is characterized for a long time trajectory on the lattice by the statistical complexity. It contains a large inner structure, therefore it is able to consider in the thermodynamical limit \( N \to \infty \) to gain quantitative behavior.

In this article, we considered Hamiltonian function on lattice gauge theory in especially the maximal real Lyapunov spectrum of the non-Abelian gauge theory. The spectra of Kolmogorov-Sinai entropy were studied as a function of energy and lattice size approaching the thermodynamical limit for SU(2) lattice gauge theory. Long time evolution of the equation of motion for gauge fields was characterized by statistical complexity on a probability space.
Figure 3: left: Complexity spectra $C$ as a function of the $H$ and $0 < g^2\alpha E < 1$. right: $C$ depends on $H$ and the lattice size $N = 7, n = 2$.

Figure 4: left: Complexity $C$ as a function of the $H$ and $D$, $0 < g^2\alpha E < 1$. right: $C$ is function as $D$ on the lattice size $N = 7, n = 2$. 
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