ON THE ADMISSIONIBILITY OF LINEAR STOCHASTIC VOLterra OPERATORS

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Abstract. Conditions guaranteeing convergence of linear stochastic Volterra operators are studied. Necessary and sufficient conditions for mean square convergence are established, while almost sure convergence of the linear operator is shown to imply mean square convergence. Sufficient conditions for almost sure convergence of the stochastic linear operator are established. The sharpness or necessity of these conditions is explored by means of examples.

1. INTRODUCTION

This article studies convergence properties of Itô-Volterra integrals of the form
\[(\mathcal{H}f)(t) := \int_0^t H(t, s)f(s)\,dB(s)\] (1.1)
where \(H\) is a deterministic Volterra kernel and \(f\) is a deterministic function on \([0, \infty)\). We require certain continuity and regularity properties on \(H\) and \(f\) which simplify our analysis and ensure the existence of \(\mathcal{H}f\) for every appropriate \(f\). The result we have found of most use is to determine, for fixed sample path, under which conditions \(\mathcal{H}\) takes the space of bounded continuous functions on \([0, \infty)\) into the space of bounded continuous functions on \([0, \infty)\) with a limit at infinity.

This may be thought of as an analogue of the theory of admissibility of (deterministic) linear continuous Volterra operators, especially in the important case where the operator takes \(BC(0, \infty)\) into itself, or when \(\mathcal{H}\) takes \(BC\) into \(BC^1\).

Corduneanu has done significant work on the general theory of admissibility for Volterra integral operators (see [6] and [7]). One motivation for the development of such an admissibility theory in the deterministic case is to give precise asymptotic information regarding the solutions of integral, differential and integro-differential equations. Corduneanu [8] contains a comprehensive survey of progress up to 1991, while further developments in this theory are due to Cushing, Miller and others.

More recently, admissibility of continuous linear Volterra operators has been used to determine asymptotic behaviour of a nonlinear integro-differential equation with infinite memory in Appleby, Győri and Reynolds [9] in this journal. Parallel results are also available in discrete time: indeed, recent results on the theory of admissibility of Volterra operators is discrete time, together with applications to Volterra summation equations, include Győri and Reynolds [9] and Song and Baker [15].

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Reynolds [13] has established results which characterise certain admissible pairs of spaces, as well as connecting the recent dynamical systems literature with parallel, earlier work in the theory of linear operators.

Analogous to the deterministic setting, integrals of the form (1.1) arise in the analysis of asymptotic growth rates of solutions of affine stochastic Volterra equations. To illustrate we define some notation. For a function \( x : [-\tau, \infty) \to \mathbb{R} \), we define the segment of \( x \) at time \( t \geq 0 \) as the function
\[
x_t : [-\tau, \infty) \to \mathbb{R} : x_t(u) = x(t + u).
\]
Consider the affine stochastic functional equation
\[
dX(t) = L(t, X_t)dt + \sigma(t)dB(t), \quad t \geq 0; \quad X(0) = x_0 \in \mathbb{R},
\]
(1.2)
where \( L \) is a linear functional and \( \sigma \) is a continuous function. The associated resolvent equation arises from setting \( \sigma \equiv 0 \), giving
\[
r'(t) = L(t, r_t), \quad t \geq 0; \quad r(0) = 1; \quad r(t) = 0, \quad t < 0.
\]
(1.3)
Providing both (1.2) and (1.3) have well-defined solutions then \( X \) may be expressed in terms of \( r \), i.e.
\[
X(t) = r(t)x_0 + \int_0^t r(t-s)\sigma(s)dB(s), \quad t \geq 0.
\]
(1.4)
If one were to scale the solution of (1.2) by a growing or decaying term then stochastic Volterra integrals of the form (1.1) arise. Establishing results about the convergence of these integrals (i.e. the integrals in (1.1)) therefore amounts to determining very precisely their almost sure asymptotic rates of growth or decay. The authors propose to follow this programme of research in later works. Another work where the scaled solution of a stochastic integral equation tends to non-trivial limit includes Appleby [1].

The main results of this article concerning mean square convergence and almost sure convergence of (1.2) are given in Section 2. The proofs of all results are given in subsequent sections.

1.1. Preliminaries. Let \( \mathbb{R} \) be the set of real numbers. We denote by \( \mathbb{R}_+ \) the half-line \([0, \infty)\). If \( d \) is a positive integer, \( \mathbb{R}^d \) is the space of \( d \)-dimensional column vectors with real components and \( \mathbb{R}^{d_1 \times d_2} \) is the space of all \( d_1 \times d_2 \) real matrices.

Let \( BC(\mathbb{R}_+; \mathbb{R}^{d_1 \times d_2}) \) denote the space matrices whose elements are bounded continuous functions. The abbreviation \( a.e. \) stands for almost everywhere. The space of continuous and continuously differentiable functions on \( \mathbb{R}_+ \) with values in \( \mathbb{R}^{d_1 \times d_2} \) is denoted by \( C(\mathbb{R}_+; \mathbb{R}^{d_1 \times d_2}) \) and \( C^1(\mathbb{R}_+; \mathbb{R}^{d_1 \times d_2}) \) respectively. While \( C^{1,0}(\Delta; \mathbb{R}^{d_1 \times d_2}) \) represents the space of functions which are continuously differentiable in their first argument and continuous in their second argument, over some two-dimensional space \( \Delta \). The space of \( p \)-integrable functions is denoted by
\[
L^p(\mathbb{R}_+; \mathbb{R}) := \{ f : \mathbb{R}_+ \to \mathbb{R} : \int_0^\infty |f(s)|^p ds < +\infty \}.
\]
For any vector \( x \in \mathbb{R}^d \) the norm \( \| \cdot \| \) denotes the Euclidean norm, \( \| x \|^2 = \sum_{j=1}^d |x_j|^2 \). While for matrices, \( A = (a_{i,k}) \in \mathbb{R}^{n \times d} \), we use the Frobenius norm, i.e.
\[
\|A\|_F^2 = \sum_{i=1}^n \sum_{k=1}^d |a_{i,k}|^2.
\]
As both \( \mathbb{R}^d \) and \( \mathbb{R}^{n \times d} \) are finite dimensional Banach spaces all norms are equivalent in the sense that for any other norm, \( \| \cdot \| \), one can find universal constants \( d_1(n, d) \leq...
\(d_2(n, d)\) such that
\[
d_1 \|A\|_F \leq \|A\| \leq d_2 \|A\|_F.
\]
Thus there is no loss of generality in using the Euclidean and Frobenius norms, which for ease of calculation, are used throughout the proofs of this paper. Moreover we remark that the Frobenius norm is a \textit{consistent matrix norm}, i.e. for any \(A \in \mathbb{R}^{n_1 \times n_2}, B \in \mathbb{R}^{n_2 \times n_3}\)
\[
\|AB\|_F \leq \|A\|_F \|B\|_F.
\]

We define the following modes of convergence:

\textbf{Definition 1.} The \(\mathbb{R}^n\)-valued stochastic process \(\{X(t)\}_{t \geq 0}\) converges in mean-square to \(X_\infty\) if
\[
\lim_{t \to \infty} \mathbb{E}[\|X(t) - X_\infty\|^2] = 0.
\]

\textbf{Definition 2.} If there exists a \(\mathbb{P}\)-null set \(\Omega_0\) such that for every \(\omega \notin \Omega_0\) the following holds
\[
\lim_{t \to \infty} X(t, \omega) = X_\infty(\omega),
\]
then we say \(X\) converges almost surely (a.s.) to \(X_\infty\).

\section{Stochastic Limit Relation}

\subsection{Mean Square Convergence}
Let \(B(t) = \{B_1(t), B_2(t), ..., B_d(t)\}\) be a vector of mutually independent standard Brownian motions. We consider the following hypotheses: let \(\Delta \subset \mathbb{R}^2\) be defined by \(\Delta = \{(t, s): 0 \leq s \leq t < +\infty\}\) and
\[
H : \Delta \to \mathbb{R}^{n \times n} \text{ is continuous. (2.1)}
\]
We first characterise, for \(f \in C([0, \infty); \mathbb{R}^{n \times d})\) with bounded norm, the convergence of the stochastic process \(X_f = \{X_f(t) : t \geq 0\}\) defined by
\[
X_f(t) = \int_0^t H(t, s)f(s) dB(s), \quad t \geq 0
\]
to a limit as \(t \to \infty\) in \textit{mean-square}.

Before discussing this convergence, we note that (2.1) is sufficient to guarantee that \(X_f(t)\) is a well-defined \(\mathcal{F}^B(t)\)-adapted random variable for each fixed \(t\). Therefore the family of random variables \(\{X_f(t) : t \geq 0\}\) is well-defined, and \(X_f\) is indeed a process. Condition (2.1) also guarantees that \(\mathbb{E}[X_f(t)^2] < +\infty\) for each \(t \geq 0\). Since \(f \mapsto X_f\) is linear, and the family \((X_f(t))_{t \geq 0}\) is Gaussian for each fixed \(f\), the limit should also be Gaussian and linear in \(f\), as well as being an \(\mathcal{F}^B(\infty)\)-measurable random variable. Therefore, a reasonably general form of the limit should be
\[
X_f^* := \int_0^{\infty} H_\infty(s)f(s) dB(s),
\]
where we would expect \(H_\infty\) to be a function independent of \(f\). In our first main result, we show that \(X_f(t) \to X_f^*\) in mean square as \(t \to \infty\) for each \(f\).

\textbf{Theorem 3.} Suppose that \(H\) obeys (2.1). Then the statements
\begin{enumerate}[(A)]
\item There exists \(H_\infty \in C([0, \infty); \mathbb{R}^{n \times n})\) such that \(\int_0^{\infty} \|H_\infty(s)\|^2 ds < +\infty\) and
\[
\lim_{t \to \infty} \int_0^t \|H(t, s) - H_\infty(s)\|^2 ds = 0. \tag{2.4}
\]
\item There exists \(H_\infty \in C([0, \infty); \mathbb{R}^{n \times n})\) such that for each \(f \in BC(\mathbb{R}_+; \mathbb{R}^{n \times d})\),
\[
\lim_{t \to \infty} \mathbb{E} \left[ \left\| \int_0^t H(t, s)f(s) dB(s) - \int_0^{\infty} H_\infty(s)f(s) dB(s) \right\|^2 \right] = 0 \tag{2.5}
\]
\end{enumerate}
are equivalent.
In the deterministic admissibility theory, the assumptions for convergence are given in a different form from (2.4), c.f. e.g. Theorem A.1 from [3]. Our next result shows that the natural analogues of those assumptions are equivalent to (2.4).

Proposition 1. Suppose that $H$ obeys (2.1). Then the following are equivalent:

(A) $H$ obeys (2.4);

(B) There exists $H_{\infty} \in C([0, \infty); \mathbb{R}^{n \times n})$ such that

$$\lim_{T \to \infty} \limsup_{t \to \infty} \int_t^T \|H(t, s)\|^2 \, ds = 0,$$

$$\lim_{t \to \infty} \int_0^T \|H(t, s) - H_{\infty}(s)\|^2 \, ds = 0, \quad \text{for every } T > 0. \quad (2.7)$$

2.2. Necessary Condition for Almost Sure Convergence. We now consider the almost sure convergence of $X(f)(t)$ as $t \to \infty$ to a limit. Our next main result shows that if we have convergence in an a.s. sense, we must also have convergence in a mean square sense.

Theorem 4. Suppose that $H$ obeys (2.1) and there exists $H_{\infty} \in C([0, \infty); \mathbb{R}^{n \times n})$ such that for each $f \in BC([0, \infty); \mathbb{R}^{n \times d})$,

$$\lim_{t \to \infty} \int_0^t H(t, s)f(s) \, dB(s) = \int_0^\infty H_{\infty}(s)f(s) \, dB(s), \quad \text{a.s.} \quad (2.8)$$

Then (2.4) and (2.5) hold.

Theorem 3 is concerned with moment behaviour of $X(f)(t) = \int_0^t H(t, s)f(s) \, dB(s)$, indeed the continuity of these moments is guaranteed by the assumption (2.1). In Theorem 4 the condition (2.8) may implicitly impose continuity of the sample paths of $X(f)$. The issue of continuous sample paths of $X(f)$ is addressed in Lemma 2.D. of Berger and Mizel [4]. Specifically, let $H$ obey (2.1). Suppose that $H$ obeys a Hölder continuity condition of the following form: there exists a function $K(s)$ and a constant $\alpha > 0$ such that

$$\int_0^T |K(s)|^2 \, ds < +\infty$$

and

$$|H(t_2, s) - H(t_1, s)| \leq K(s)(t_2 - t_1)^\alpha, \quad \text{for } 0 \leq s \leq t_1 \leq t_2 \leq T. \quad (2.9)$$

Since $H$ is continuous, it follows that there exist constants $\epsilon > 0$, $D > 0$ such that

$$\sup_{t \in [0, T]} \int_0^T |H(t, s)|^{2+\epsilon} \, ds \leq D.$$

Lemma 2.D. of [4] now guarantees that a continuous version of

$$\int_0^t H(t, s)f(s) \, dB(s)$$

exists on $[0, T]$.

Remark 1. Therefore we have shown that (2.4) is a necessary condition for a.s. convergence. It is of course natural to then ask whether (2.4) is sufficient. We show by a simple example that in general additional conditions are needed in order for (2.8) to hold. It is further noted that the assumed continuity and structure of $H$ in Examples 5 and 6 immediately gives the continuity of the sample paths of $\int_0^t H(t, s)f(s) \, dB(s)$, and that the sufficient condition (2.9) is not needed.
Example 5. Suppose that $H^3 : [0, \infty) \to \mathbb{R}$ and $H_\infty : [0, \infty) \to \mathbb{R}$ are continuous functions, and define

$$H(t, s) = H_\infty(s) + H^3(t), \quad (t, s) \in \Delta.$$ 

Then $H$ is continuous. Suppose also that $H_\infty \in L^2([0, \infty); \mathbb{R})$. By Theorem 3 it follows that

$$\lim_{t \to \infty} \sqrt{t}H^3(t) = 0$$

is necessary and sufficient for (2.5). It is also a necessary condition for (2.8). If one further supposes that $H^3$ obeys

$$\lim_{t \to \infty} \sqrt{t \log \log t} |H^3(t)| = 0,$$

then (2.5) holds.

Obviously the conditions (2.11) and (2.10) do not coincide; in fact, (2.11) implies (2.10). This provides an example of the veracity of Theorem 4 which can be verified independently of the general proof of that result.

We note also that it is very difficult to relax (2.11) and still have the integral $\int_0^t H(t, s)f(s)dB(s)$ tending to a limit a.s. as $t \to \infty$. Indeed, there exist functions $H^3$ which do not obey (2.11), and so must satisfy

$$\limsup_{t \to \infty} \sqrt{t \log \log t} |H^3(t)| > 0,$$

for which

$$P\left[ \lim_{t \to \infty} \int_0^t H(t, s)dB(s) \text{ exists} \right] = 0,$$

while at the same time we still have (2.5).

A choice of $H^3$ which satisfies these conditions can readily be made. Consider a continuous function $H^3$ which obeys $H^3(n) = 1/\sqrt{n \log \log (n + 2)}$ for all integers $n \geq 1$ but for which $\sqrt{t}H^3(t) \to 0$ as $t \to \infty$ and $\limsup_{t \to \infty} \sqrt{t \log \log t} |H^3(t)| < +\infty$.

The next example shows that sometimes the conditions which give mean square convergence and a.s. convergence are the same.

Example 6. Suppose that $H^3 : [0, \infty) \to \mathbb{R}$ and $H_\infty : [0, \infty) \to \mathbb{R}$ are continuous functions, and define

$$H(t, s) = H_\infty(s)H^3(t), \quad (t, s) \in \Delta.$$ 

Then $H$ is continuous. Suppose also that $H_\infty \in L^2([0, \infty); \mathbb{R})$. Therefore, by Theorem 5 we have (2.5) if and only if

$$\lim_{t \to \infty} H^3(t) = 1.$$ 

We know by Theorem 5 that this condition is also necessary for a.s. convergence. Indeed it is also sufficient for (2.8).

2.3. Sufficient Conditions for Almost Sure Convergence. We now investigate sufficient conditions for a.s. convergence for functions $H$ which need not necessarily be of the form

$$H(t, s) = \sum_{j=1}^{n} H_j(s)H_j^3(t), \quad (t, s) \in \Delta,$$

and which are covered by explicit and direct calculations in Examples 5 and 6.
Theorem 7. Suppose that $H$ obeys $(2.11)$ and also that $H \in C^{1,0}(\Delta; \mathbb{R}^{n \times n})$. Suppose also that there exists $H_\infty \in C([0, \infty); \mathbb{R}^{n \times n})$ such that \( \int_0^\infty \|H_\infty(s)\|^2 \, ds < +\infty \) and
\[
\lim_{t \to \infty} \int_0^t \|H(t, s) - H_\infty(s)\|^2 \, ds \cdot \log t = 0, \tag{2.13}
\]
and
\[
\text{There exists } q \geq 0 \text{ and } c_q > 0 \text{ such that }
\int_0^t \|H_1(t, s)\|^2 \, ds \leq c_q (1 + t)^{2q}, \quad \|H(t, t)\|^2 \leq c_q (1 + t)^{2q}. \tag{2.14}
\]
Then $H$ obeys $(2.8)$.

Remark 2. Condition $(2.13)$ implies a rate of decay of \( \int_0^t \|H(t, s) - H_\infty(s)\|^2 \, ds \) to zero as $t \to \infty$. This strengthens the hypothesis $(2.3)$ which is known, by Theorem 4, to be necessary.

Remark 3. We comment now on the continuity of the sample paths of the process \( \int_0^t H(t, s) f(s) \, dB(s) \) in Theorem 7. For simplicity we discuss only the scalar case. This continuity of the sample paths is assured by the derivative condition $(2.14)$.

Fix $T > 0$ and let $0 \leq s \leq t_1 \leq t_2 \leq T$. Then, as $H \in C^{1,0}(\Delta; \mathbb{R})$, by the Mean Value Theorem we get
\[
|H(t_2, s) - H(t_1, s)| = |H_1(t^*, s)| |t_2 - t_1|,
\]
for some $t^* = t^*(s) \in [t_1, t_2]$. Next, define $K(s) := \sup_{t_1 \leq t \leq t_2} |H_1(t, s)|$. This is well-defined and finite by the continuity of $H_1$. Therefore
\[
|H(t_2, s) - H(t_1, s)| \leq K(s) |t_2 - t_1|, \quad \text{for all } 0 \leq s \leq t_1 \leq t_2 \leq T,
\]
which is $(2.8)$ with $\alpha = 1$. Note moreover that the continuity of $s \mapsto K(s)$ ensures that
\[
\int_0^T |K(s)|^2 \, ds < +\infty,
\]
and therefore all the conditions of Lemma 2.D. of Berger and Mizel [4] are satisfied.

Remark 4. While the pointwise bound on $\|H(t, t)\|$ in $(2.14)$ may appear quite mild, one may prefer an integral condition to this pointwise bound as this would allow $\|H(t, t)\|$ to potentially have “thin spikes” of larger than polynomial order. Scrutiny of the proof of Theorem 7 reveals that the condition $\|H(t, t)\|^2 \leq c_q (1 + t)^{2q}$ can be replaced by
\[
\lim_{k \to \infty} \int_{k^\theta}^{(k+1)^\theta} \|H(s, s)\|^2 \, ds \cdot \log k = 0, \quad \text{for } 0 < \theta < 1/(1 + 2q), \tag{2.15}
\]
where the limit is taken through the integers. Nevertheless for simplicity we retain the condition on $\|H(t, t)\|$ in the statement of Theorem 7.

Example 8. In light of Examples 6 and 7 (both of whose hypotheses lie outside the scope of Theorem 7 in spite of the conclusion of that theorem still holding) one might query the sharpness of $(2.13)$. In fact, a condition of the form $(2.13)$ is, to some extent, necessary.

Suppose, for example that $H(t, s) = e^{-(t-s)} \sigma(s)$, $0 \leq s \leq t$, where $t \mapsto \sigma(t)$ is a continuous function. Define the process
\[
Y(t) = \int_0^t H(t, s) \, dB(s), \quad t \geq 0.
\]
Then, in the context of Theorem 3 or 4, \( H_\infty(s) = 0 \). Moreover, if \( Y \) converges to a limit almost surely we note from (2.5) that the limit should be zero, and by Theorem condition (2.1) must hold. Hence, we must have that \( \lim_{t \to \infty} \int_0^t H(t, s)^2 \, ds = 0 \). It can be shown that this happens if and only if \( \int_0^{t+1} \sigma(s)^2 \, ds \to 0 \) as \( t \to \infty \).

Suppose, moreover, that \( \lim_{t \to \infty} \int_0^{t+1} \sigma(s)^2 \, ds \cdot \log t = L \in (0, \infty) \). Then (2.13) is violated. As \( Y \) is the unique adapted process satisfying the stochastic differential equation \( dY(t) = -Y(t) + \sigma(t) dB(t) \), then it is a consequence of Theorem 4.1 of Appleby, Cheng and Rodkina \([2]\) that if \( L \in (0, \infty) \) then \( Y \) is bounded and does not tend to a limit or otherwise \( L = +\infty \) and \( Y \) is unbounded, i.e. the conclusion of Theorem 7 does not hold.

Moreover, (2.13) may in some sense be regarded as sharp for \( Y \) tending to a limit. From Theorem 4.1 of \([2]\) it is known that if \( \int H(t, s)^2 \, ds \cdot \log n \to 0 \) as \( n \to \infty \), \( \lim_{n \to \infty} \int_0^{n+1} \sigma(s)^2 \, ds \cdot \log n \to 0 \). This implies (2.13). On the other hand, if \( \int_0^{n+1} \sigma(s)^2 \, ds \cdot \log n \to 0 \) as \( n \to \infty \), we have by part (A) of Theorem 4.1 of \([2]\) it follows that \( Y(t) \to 0 \) as \( t \to \infty \) a.s. Therefore, in this case, we see that (2.13) is necessary and sufficient for convergence.

If \( Y \) is bounded we have that \( \int_0^{t+1} \sigma(s)^2 \, ds \to 0 \) as \( t \to \infty \). Therefore imposing a polynomial growth bound on \( \sigma \) is not restrictive. Under this restriction and the condition \( \lim_{t \to \infty} \sigma(t)^2 \log t = L \in (0, \infty) \) we have that (2.13) is violated but (2.14) holds. Hence (2.13) is chiefly responsible for convergence in this case.

3. Proofs of Admissibility Results

The following proofs are given for scalar valued functions. The multi-dimensional results may be obtained by considering matrix functions component-wise (in such calculations it is often convenient to use the Frobenius norm due to Itô’s isometry).

Remark 5. If (2.5) holds, it is implicit that the stochastic integral

\[
\int_0^\infty H_\infty(s) f(s) \, dB(s)
\]

exists for every \( f \in BC([0, \infty); \mathbb{R}) \), and in particular this holds in the case when \( f(s) = 1 \) for all \( s \geq 0 \). Therefore we have that \( \int_0^\infty H_\infty(s) dB(s) \) exists. Therefore, by the martingale convergence theorem, we have that \( H_\infty \in L^2([0, \infty); \mathbb{R}) \).

Proof of Theorem 3. It is not difficult to see using Itô’s isometry that

\[
\mathbb{E} \left[ \left( \int_0^t \int_0^\infty H(t, s) f(s) dB(s) - \int_0^\infty H_\infty(s) f(s) dB(s) \right)^2 \right]
\]

\[
= \int_0^t \left( H(t, s) - H_\infty(s) \right)^2 f(s)^2 \, ds + \int_t^\infty H_\infty(s)^2 f(s)^2 \, ds,
\] (3.1)

where the independence of the elements of the Brownian vector and of stochastic integrals over non-overlapping intervals has been used.

Firstly we show that (A) implies (B). Now as \( f \in BC([0, \infty); \mathbb{R}) \),

\[
\mathbb{E} \left[ \left( \int_0^t H(t, s) f(s) dB(s) - \int_0^\infty H_\infty(s) f(s) dB(s) \right)^2 \right]
\]

\[
\leq \left( \int_0^t (H(t, s) - H_\infty(s))^2 \, ds + \int_t^\infty H_\infty(s)^2 \, ds \right) \sup_{s \geq 0} |f(s)|^2.
\]
Taking the limit as $t \to \infty$, then by hypothesis both terms on the right--hand side of the above inequality tend to zero, and so we obtain (2.5). Suppose to the contrary that (B) holds. By Remark 5 we have that $H_\infty \in L^2([0, \infty); \mathbb{R})$. Rearranging (3.1) with $f(s) = 1$ for all $s \geq 0$,\[
abla T \int_0^t \left( H(t, s) - H_\infty(s) \right)^2 ds = \mathbb{E} \left[ \left( \int_0^t H(t, s) dB(s) - \int_0^\infty H_\infty(s) dB(s) \right)^2 \right] - \int_0^\infty H_\infty(s)^2 ds,\]
The therefore by the hypothesis of (B), we arrive at\[
abla \limsup_{t \to \infty} \int_0^t (H(t, s) - H_\infty(s))^2 ds = 0,\]
as required. \qquad \Box

**Proof of Proposition 1.** We prove that (A) implies (B) first. To prove (2.6), note for any $t \geq T$ we have the estimate\[
abla \int_T^t H^2(t, s) ds = \int_T^t (H(t, s) - H_\infty(s) + H_\infty(s))^2 ds \leq 2 \int_T^t (H(t, s) - H_\infty(s))^2 ds + 2 \int_T^t H_\infty^2(s) ds.\]
Since $H_\infty \in L^2([0, \infty); \mathbb{R})$ and (2.4) holds, we have\[
abla \limsup_{t \to \infty} \int_T^t H^2(t, s) ds \leq \int_T^\infty H_\infty^2(s) ds.\]
Since the lefthand side is monotone in $T$, we may take the limit as $T \to \infty$ on both sides, using the fact that $H_\infty \in L^2([0, \infty); \mathbb{R})$ to obtain the desired conclusion.

To show (2.7), let $T > 0$ be arbitrary. Then, for any $t \geq T$ we have\[
abla \int_0^t (H(t, s) - H_\infty(s))^2 ds \leq \int_0^t (H(t, s) - H_\infty(s))^2 ds,
\]
whence the result letting $t \to \infty$ and applying (2.4). Thus (A) implies (B).

To prove that (B) implies (A), we first must show that $H_\infty \in L^2([0, \infty); \mathbb{R})$. We start by observing that (2.6) is nothing other than\[
abla L(T) := \limsup_{t \to \infty} \int_T^t H(t, s)^2 ds.
\]
Since $L$ is non--increasing, for every $\epsilon > 0$ there exists $T_0(\epsilon) > 0$ such that $L(T) < \epsilon$ for all $T \geq T_0(\epsilon)$. Now, let $T \geq T_0$. Suppose also that $t \geq T$. Then\[
abla \int_T^{T_0} H_\infty^2(s) ds \leq 2 \int_T^{T_0} (H_\infty(s) - H(t, s))^2 ds + 2 \int_T^{T_0} H(t, s)^2 ds \leq 2 \int_0^{T_0} (H_\infty(s) - H(t, s))^2 ds + 2 \int_0^t H(t, s)^2 ds.
\]
So by (2.4) we have\[
abla \int_T^{T_0} H_\infty^2(s) ds \leq 2L(T_0),
\]
and since the righthand side is independent of $T$, it follows that $H_\infty \in L^2([0, \infty); \mathbb{R})$, which is one part of (2.4).
To prove the other part, let \( t \geq T > 0 \). Then we have the estimate
\[
\int_0^t (H(t, s) - H_\infty(s))^2 \, ds \\
= \int_0^T (H(t, s) - H_\infty(s))^2 \, ds + \int_T^t (H(t, s) - H_\infty(s))^2 \, ds \\
\leq \int_0^T (H(t, s) - H_\infty(s))^2 \, ds + 2 \int_T^t H(t, s)^2 \, ds + 2 \int_T^\infty H_\infty(s)^2 \, ds.
\]
Since \( H_\infty \in L^2([0, \infty); \mathbb{R}) \) and \( H \) obeys (2.7), we have
\[
\limsup_{t \to \infty} \int_0^t (H(t, s) - H_\infty(s))^2 \, ds \leq 2 \limsup_{t \to \infty} \int_T^t H(t, s)^2 \, ds + 2 \int_T^\infty H_\infty(s)^2 \, ds.
\]
Now letting \( T \to \infty \) on both sides of the inequality and using (2.6) proves the other part of (2.4). \( \square \)

**Proof of Theorem 4.** Suppose for a moment that \( f(s) = 1 \) for all \( s \geq 0 \). Then by (2.3) it follows that \( \int_0^\infty H_\infty(s) \, dB(s) \) exists. Therefore, by the martingale convergence theorem, we have that \( H_\infty \in L^2([0, \infty); \mathbb{R}) \), which is one part of (2.5). Therefore, for \( f \in BC([0, \infty); \mathbb{R}) \) the processes \( X_f \) and \( X_\infty^f \) in (2.2) and (2.3) are well-defined. Also, as \( H_\infty \in L^2([0, \infty); \mathbb{R}) \), we have that \( X_f^* \) is well-defined, and thus \( X_\infty^f(t) \to X_f^* \) as \( t \to \infty \) a.s., where
\[
X_f^*(t) = \int_0^\infty H_\infty(s) \, dB(s).
\]
Next, define \( Y_f(t) := X_f(t) - X_\infty^f(t) \) for \( t \geq 0 \). Evidently, we have that \( \mathbb{E}[Y_f(t)] = 0 \) for all \( t \geq 0 \). Also, we have from (2.3) that \( X_f(t) \to X_f^* \) as \( t \to \infty \) a.s. Therefore
\[
\lim_{t \to \infty} Y_f(t) = \lim_{t \to \infty} \{ X_f(t) - X_\infty^f(t) \} = \lim_{t \to \infty} \{ X_f(t) - X_f^* + X_f^* - X_\infty^f(t) \} = 0,
\]
amost surely. Notice also that \( (Y_f(t))_{t \geq 0} \) is a Gaussian process. Since it converges a.s., it does so to a Gaussian random variable which has zero mean and zero variance, and by the argument of pp304–305 in Shiryaev [14], we have that \( \text{Var}[Y_f(t)] \to 0 \) as \( t \to \infty \). Since \( \mathbb{E}[Y_f(t)] = 0 \), we also have \( \mathbb{E}[Y_f^2(t)] \to 0 \) as \( t \to \infty \). However, by Itô’s isometry,
\[
\mathbb{E}[Y_f^2(t)] = \mathbb{E}[(X_f(t) - X_\infty^f(t))^2] = \int_0^t (H(t, s) - H_\infty(s))^2 f^2(s) \, ds.
\]
Therefore we have
\[
\lim_{t \to \infty} \int_0^t (H(t, s) - H_\infty(s))^2 f^2(s) \, ds = 0,
\]
and choosing \( f(s) = 1 \) for all \( s \geq 0 \), we arrive at the rest of (2.4). Clearly (2.5) now holds by virtue of Theorem 3. \( \square \)

4. **Proof of Theorem 7**

Define \( \tilde{H} = H - H_\infty \). Notice that \( H_\infty \in L^2([0, \infty); \mathbb{R}) \) implies that
\[
\lim_{t \to \infty} \int_t^\infty H_\infty(s) f(s) \, dB(s) = 0, \quad \text{a.s.}
\]
so that proving
\[
\lim_{t \to \infty} \int_0^t \tilde{H}(t, s) f(s) \, dB(s) = 0, \quad \text{a.s.} \quad (4.1)
\]
is equivalent to establishing (2.8).
Since $H \in C^{1,0}$, we have $\tilde{H}_1 = H_1$. Therefore, we have
\[
\tilde{X}_f(t) := \int_0^t \tilde{H}(t, s) f(s) dB(s) = \int_0^t \left( \tilde{H}(s, s) f(s) + \int_0^s \tilde{H}_1(u, s) f(s) \, du \right) dB(s).
\]
By a stochastic Fubini theorem, we have
\[
\tilde{X}_f(t) = \int_0^t \tilde{H}(s, s) f(s) dB(s) + \int_0^t \left( \int_0^s \tilde{H}_1(u, s) f(s) \, dB(s) \right) \, du.
\]
Now, let $(t_n)_{n \geq 0}$ be an increasing sequence with $t_0 = 0$ and $t_n \to \infty$ as $n \to \infty$. In fact, choose
\[
t_n = n^\theta, \quad \text{for some } \theta \in (0, 1/(1+q) \wedge 1/(1+2q)) \subset (0, 1), \quad (4.2)
\]
where $q$ is the number in (2.14).

Therefore for $t \in [t_n, t_{n+1})$, we have
\[
\tilde{X}_f(t) = \tilde{X}_f(t_n) + \int_{t_n}^t H(s, s) f(s) dB(s) - \int_{t_n}^t H_\infty(s) f(s) dB(s) + \int_{t_n}^t \left( \int_0^s H_1(u, s) f(s) \, dB(s) \right) \, du.
\]
Hence
\[
\sup_{t_n \leq t \leq t_{n+1}} |\tilde{X}_f(t)| \leq |\tilde{X}_f(t_n)| + \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^t H_\infty(s) f(s) \, dB(s) \right| + \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^t \int_0^s H_1(u, s) f(s) \, dB(s) \right| \, du. \quad (4.3)
\]
We now show that each of the four terms on the righthand side of (4.3) tends to zero as $n \to \infty$ a.s.

**STEP 1: First term on the righthand side of (4.3).** First we prove that
\[
\lim_{n \to \infty} \tilde{X}_f(t_n) = 0, \quad \text{a.s.} \quad (4.4)
\]
Notice that $\tilde{X}_f(t_n)$ is normally distributed with mean zero and variance $v_n^2$ where
\[
v_n^2 := \int_0^{t_n} \tilde{H}^2(t_n, s) f^2(s) \, ds \leq \int_0^{t_n} \tilde{H}^2(t_n, s) \, ds \cdot \sup_{s \geq 0} f^2(s),
\]
Using (2.13) and the fact that $t_n \to \infty$ as $n \to \infty$, we have
\[
\lim_{n \to \infty} \int_0^{t_n} \tilde{H}(t_n, s)^2 \, ds \cdot \log t_n = 0.
\]
Therefore
\[
\limsup_{n \to \infty} v_n^2 \log t_n \leq \limsup_{n \to \infty} \int_0^{t_n} \tilde{H}(t_n, s)^2 \, ds \cdot \sup_{s \geq 0} f^2(s) \cdot \log t_n = 0. \quad (4.5)
\]
Since $X_n := \tilde{X}_f(t_n)/v_n$ is a standardised normal random variable, we have that
\[
\limsup_{n \to \infty} \frac{|\tilde{X}_f(t_n)|}{\sqrt{2v_n \log n}} = \limsup_{n \to \infty} \frac{|X_n|}{\sqrt{2 \log n}} \leq 1, \quad \text{a.s.,}
\]
the last inequality being a routine consequence of the Borel–Cantelli lemma. Thus
\[
\limsup_{n \to \infty} |\tilde{X}_f(t_n)| = \limsup_{n \to \infty} \frac{|\tilde{X}_f(t_n)|}{\sqrt{2v_n \log n}} \cdot \sqrt{2v_n (\log n)^{1/2}} \leq \sqrt{2} \limsup_{n \to \infty} v_n (\log t_n)^{1/2} \sqrt{\frac{\log n}{\log t_n}} = 0,
\]
due to (1.2) and (1.3), proving (4.1).

**STEP 2:** Second term on the righthand side of (4.3) Next we show that
\[
\lim_{n \to \infty} \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} H_\infty(s) f(s) dB(s) \right| = 0, \quad \text{a.s.} \quad (4.6)
\]
To do this, notice for every \( \epsilon > 0 \) by Chebyshev’s inequality and the Birkholder–Davis–Gundy inequality, c.f. e.g. [10, Theorem 1.3.8, Theorem 1.7.3] that
\[
P \left[ \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} H_\infty(s) f(s) dB(s) \right| > \epsilon \right] \leq \frac{1}{\epsilon^2} \mathbb{E} \left[ \left( \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} H_\infty(s) f(s) dB(s) \right| \right)^2 \right]
\leq \frac{4}{\epsilon^2} \mathbb{E} \left[ \left( \int_{t_n}^{t_{n+1}} H_\infty(s) f(s) dB(s) \right)^2 \right]
= \frac{4}{\epsilon^2} \int_{t_n}^{t_{n+1}} H_\infty^2(s) f^2(s) \, ds.
\]
Since \( H_\infty \in L^2([0, \infty); \mathbb{R}) \), and \( f \in BC([0, \infty); \mathbb{R}) \), we have
\[
\sum_{n=0}^{\infty} P \left[ \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} H_\infty(s) f(s) dB(s) \right| > \epsilon \right] \leq \frac{4}{\epsilon^2} \int_{0}^{\infty} H_\infty^2(s) f^2(s) \, ds.
\]
By the Borel–Cantelli Lemma, we have that (4.6) holds.

**STEP 3:** Third term on the righthand side of (4.3).
\[
\lim_{n \to \infty} U_n = 0, \quad \text{a.s.}
\]
where
\[
U_n = \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} H(s, s) f(s) dB(s) \right|.
\]
Note that \( (U_n)_{n \geq 0} \) is a sequence of independent random variables.

Notice that on the interval \([t_n, t_{n+1}]\), by the martingale time change theorem, there exists a Brownian motion \( \tilde{B} \) such that
\[
U_n = \sup_{t_n \leq t \leq t_{n+1}} \left| \tilde{B}_n \left( \int_{t_n}^{t} H^2(s, s) f^2(s) \, ds \right) \right|
= \sup_{0 \leq \tau \leq f_{t_n}^{t_{n+1}} H^2(s, s) f^2(s) \, ds} |\tilde{B}_n(\tau)|
\leq \sup_{0 \leq \tau \leq f_{t_n}^{t_{n+1}} H^2(s, s) \sup_{v \geq 0} f^2(v)} |\tilde{B}_n(\tau)|.
\]
Therefore, with \( w_n := \int_{t_n}^{t_{n+1}} H^2(s, s) \, ds \cdot \sup_{v \geq 0} f^2(v) \), we have for some Brownian motion \( W \) that
\[
P[U_n > \epsilon] \leq \mathbb{P}[\sup_{0 \leq \tau \leq w_n} |W(\tau)| > \epsilon].
\]
Using the symmetry of the distribution function leads to the estimate
\[
P[U_n > \epsilon] \leq 2P[|W(w_n)| > \epsilon] \leq 4P[W(w_n) > \epsilon] = 4P[Z > \epsilon/\sqrt{w_n}],
\]
where \( Z \) is a standard normal random variable, and we interpret the right hand side as zero if \( w_n = 0 \). Hence if \( \Phi \) is the distribution function of a standard normal
random variable and
\[
\sum_{n=0}^{\infty} \left\{ 1 - \Phi \left( \frac{\varepsilon}{\sqrt{\int_{t_n}^{t_{n+1}} H^2(s, s) \, ds}} \right) \right\} < +\infty, \quad \text{for all } \varepsilon > 0,
\]
we have that \( \lim_{n \to \infty} U_n = 0 \), a.s. The sum is finite provided
\[
\lim_{n \to \infty} \int_{t_n}^{t_{n+1}} H^2(s, s) \, ds \cdot \log n = 0.
\]
Since \( H(t, t)^2 \leq c_q(1 + t^{2q}) \), we have that
\[
\int_{t_n}^{t_{n+1}} H^2(s, s) \, ds \cdot \log n \leq c_q \int_{n^\theta}^{(n+1)^\theta} \{1 + s^{2q}\} \, ds \cdot \log n,
\]
so the right hand side is of the order \( n^{-1+\theta} n^{2q} \log n = n^{-1+(2q+1)\theta} \log n \to 0 \) as \( n \to \infty \), because \( \theta < 1/(1+2q) \).

**STEP 4: Fourth term on the righthand side of (4.3).** Finally, we show that
\[
\lim_{n \to \infty} Z_n = 0, \quad \text{a.s.} \tag{4.7}
\]
where
\[
Z_n := \int_{t_n}^{t_{n+1}} \left| \int_0^u H_1(u, s) f(s) \, dB(s) \right| \, du. \tag{4.8}
\]
By (2.14) there exists \( c_q > 0 \) such that
\[
\int_0^t H_1^2(t, s) \, ds \leq c_q(1 + t)^{2q}, \quad t \geq 0.
\]
By (4.2), \( q < 1/(1+q) \leq 1 \), so we can choose \( p \in \mathbb{N} \) so large that \( 2p[1-(1+q)\theta] > 1 \).
Clearly for such a \( p \in \mathbb{N} \) we have, via Jensen’s inequality
\[
Z_n^{2p} \leq (t_{n+1} - t_n)^{2p-1} \int_{t_n}^{t_{n+1}} \left( \int_0^u H_1(u, s) f(s) \, dB(s) \right)^{2p} \, du,
\]
so there exists \( C_p > 0 \) such that
\[
\mathbb{E}[Z_n^{2p}] \leq C_p (t_{n+1} - t_n)^{2p-1} \int_{t_n}^{t_{n+1}} \left( \int_0^u H_1^2(u, s) f^2(s) \, ds \right)^p \, du
\]
\[
\leq C_p \sup_{s \geq 0} f^{2p}(s) (t_{n+1} - t_n)^{2p-1} \int_{t_n}^{t_{n+1}} \left( \int_0^u H_1^2(u, s) \, ds \right)^p \, du.
\]
Then
\[
\mathbb{E}[Z_n^{2p}] \leq C_p \sup_{s \geq 0} f^{2p}(s) (t_{n+1} - t_n)^{2p-1} \int_{t_n}^{t_{n+1}} (c_q(1 + u)^{2q})^p \, du
\]
\[
\leq C_p c_q^{2p} \sup_{s \geq 0} f^{2p}(s) (t_{n+1} - t_n)^{2p-1} \int_{t_n}^{t_{n+1}} (1 + u)^{2p} \, du
\]
\[
\leq C_p c_q^{2p} \sup_{s \geq 0} f^{2p}(s) (t_{n+1} - t_n)^{2p} (1 + t_{n+1})^{2p}.
\]
Since \( t_n = n^\theta \), the right hand side is of the order \( [n^{\theta-1}]^{2p} n^{2p \theta} = n^{-2p[1-(1+q)\theta]} \) as \( n \to \infty \). By Chebyshev’s inequality, for any \( \epsilon > 0 \) we have
\[
\mathbb{P}[|Z_n| > \epsilon] \leq \frac{1}{\epsilon^{2p}} \mathbb{E}[Z_n^{2p}] \leq C c_p n^{-2p[1-(1+q)\theta]},
\]
and because \( 2p[1-(1+q)\theta] > 1 \), the righthand side is summable. Therefore, by the Borel–Cantelli lemma, we have (4.7).
4.1. Proof of Examples.

Proof of Example \[5\] Since $H_\infty$ is in $L^2([0, \infty); \mathbb{R})$, for each $f \in BC([0, \infty); \mathbb{R})$ we have that the integral $\int_0^t H_\infty(s) f(s) dB(s)$ tends almost surely as $t \to \infty$ to $\int_0^\infty H_\infty(s) f(s) dB(s)$. Thus

$$
\int_0^t H(t, s) f(s) dB(s) - \int_0^\infty H_\infty(s) f(s) dB(s) = H^2(t) \int_0^t f(s) dB(s) - \int_t^\infty H_\infty(s) f(s) dB(s). \quad (4.9)
$$

As $H^2$ obeys \[2.11\] we have $H^2(t) \to 0$ as $t \to \infty$. If $f \in L^2([0, \infty); \mathbb{R})$, then $\int_0^t f(s) dB(s)$ tends to a finite limit a.s., and therefore both terms on the righthand side of \[4.9\] tend to zero as $t \to \infty$ a.s., and \[2.8\] holds.

On the other hand, if $f \not\in L^2([0, \infty); \mathbb{R})$, then the martingale time change theorem and the Law of the Iterated Logarithm, c.f. e.g. \[12, Exercise 5.1.15\] as well as the boundedness of $f$ and \[2.11\] give that $H^2(t) \int_0^t f(s) dB(s) \to 0$ as $t \to \infty$ a.s. Hence we have that \[2.8\] holds.

We turn now to the question of relaxing \[2.11\] and still having \[2.8\] holding. By virtue of the fact that $H^2$ obeys \[2.10\], we have that \[2.5\] holds. By the Law of the Iterated Logarithm, we have that

$$
\limsup_{t \to \infty} |H^2(t) B(t)| < +\infty, \quad \text{a.s.}
$$

However,

$$
\limsup_{t \to \infty} |H^2(t) B(t)| \geq \limsup_{n \to \infty} |H^2(n) B(n)| = \sqrt{2} \limsup_{n \to \infty} \frac{|B(n)|}{\sqrt{2n \log \log(n + 2)}} = \sqrt{2},
$$

a.s., by the discrete version of the Law of the iterated logarithm, c.f. e.g. \[5\] Theorem 10.2.1. If $f(t) = 1$ for all $t \geq 0$ in \[4.9\], we have

$$
\int_0^t H(t, s) dB(s) - \int_0^\infty H_\infty(s) dB(s) = H^2(t) B(t) - \int_t^\infty H_\infty(s) dB(s).
$$

The second term on the righthand side has zero limit as $t \to \infty$ a.s., but by the above argument, the first term obeys

$$
0 < \limsup_{t \to \infty} |H^2(t) B(t)| < +\infty, \quad \text{a.s.}
$$

and therefore \[2.12\] holds, as claimed.

Proof of Example \[6\] To show that it is sufficient, suppose $f \in BC([0, \infty); \mathbb{R})$. Therefore, we have the identity

$$
\int_0^t H(t, s) f(s) dB(s) - \int_0^\infty H_\infty(s) f(s) dB(s) = (H^2(t) - 1) \int_0^t H_\infty(s) f(s) dB(s) - \int_t^\infty H_\infty(s) f(s) dB(s).
$$

Since $H^2(t) \to 1$ as $t \to \infty$, the limit as $t \to \infty$ of the righthand side is zero, and so we have \[2.8\]. Therefore, the condition $H^2(t) \to 1$ as $t \to \infty$ is necessary and sufficient both for \[2.8\] and for \[2.5\].
Proof of Example 3. Suppose that \( \lim_{t \to \infty} f^{t+1}_t \sigma(s)^2 \, ds = 0 \). Then clearly one has \( \int_{n}^{n+1} \sigma(s)^2 \, ds \to 0 \) as \( n \to \infty \). Now, for \( n \leq t < n + 1 \), where \( n = \lfloor t \rfloor \),

\[
E[Y(t)^2] = \sum_{j=1}^{n} \int_{j-1}^{j} e^{-2(t-s)} \sigma(s)^2 \, ds + \int_{n}^{t} e^{-2(t-s)} \sigma(s)^2 \, ds \\
\leq \sum_{j=1}^{n} e^{-2(n-j)} \int_{j-1}^{j} \sigma(s)^2 \, ds + \int_{n}^{n+1} \sigma(s)^2 \, ds.
\]

The first term on the right-hand side of the inequality is the convolution of a summable sequence with a sequence tending to zero and thus the convolution itself tends to zero as \( t \to \infty \). Thus we have \( \lim_{t \to \infty} E[Y(t)^2] = 0 \).

Conversely, suppose that \( \lim_{t \to \infty} E[Y(t)^2] = 0 \). Then defining

\[
y(t) = E[Y(t)^2] = \int_{0}^{t} e^{-2(t-s)} \sigma(s)^2 \, ds,
\]

one obtains the identity

\[
y(t + 1) - y(t) = -2 \int_{t}^{t+1} y(s) \, ds + \int_{t}^{t+1} \sigma(s)^2 \, ds.
\]

Thus, \( y(t) \to 0 \) as \( t \to \infty \) implies \( \lim_{t \to \infty} \int_{t}^{t+1} \sigma(s)^2 \, ds = 0 \). Hence \( \int_{t}^{t+1} \sigma(s)^2 \, ds \to 0 \) as \( t \to \infty \) completely characterises the mean square convergence of \( Y(t) \) to zero.

We now show that \( \liminf_{t \to \infty} \int_{t}^{t+1} \sigma(s)^2 \, ds \cdot \log t = L \in (0, \infty) \) implies that

\[
\lim_{t \to \infty} \int_{0}^{t} H(t, s)^2 \, ds \cdot \log t \in (0, \infty)
\]

(4.10)

Observe, for \( n \leq t < n + 1 \),

\[
\int_{0}^{t} e^{-2(t-s)} \sigma(s)^2 \, ds \cdot \log t = \left( \sum_{j=1}^{n} e^{-2t} \int_{j-1}^{j} e^{2s} \sigma(s)^2 \, ds + \int_{n}^{t} e^{-2(t-s)} \sigma(s)^2 \, ds \right) \log t \\
\geq e^{-2n} \sum_{j=1}^{n} \frac{e^{2j}}{\log(n+1)} \int_{j-1}^{j} \sigma(s)^2 \, ds \cdot \log(j + 1) \cdot \frac{1}{e^{2n} / \log n}.
\]

Standard application of Toeplitz’s Lemma, [14] Lemma IV.3.1, then gives (4.10).

We show now that \( \lim_{t \to \infty} \int_{t}^{t+1} \sigma(s)^2 \, ds \cdot \log t = 0 \) implies that (2.13) holds. Observe that for \( n \leq t < n + 1 \),

\[
\int_{0}^{t} H(t, s)^2 \, ds \cdot \log t = \sum_{j=1}^{n} \int_{j-1}^{j} e^{-2(t-s)} \sigma(s)^2 \, ds \log t + \int_{n}^{t} e^{-2(t-s)} \sigma(s)^2 \, ds \log t \\
\leq \sum_{j=1}^{n} \frac{2e^{2j}}{\log(n+1)} \int_{j-1}^{j} \sigma(s)^2 \, ds \log j + \int_{n}^{n+1} \sigma(s)^2 \, ds \log(n + 1) \cdot \frac{1}{e^{2n} / \log(n + 1)}.
\]

Again, standard application of [14] Lemma IV.3.1 gives the desired result.

Observe that \( H(t, s) = -t - H(t, s) \), thus with the supposition \( \int_{0}^{t} \sigma(s)^2 \, ds \to 0 \), this gives the first part of (2.13). The remainder of (2.13) is clearly satisfied if \( \lim_{t \to \infty} \sigma(s)^2 \log t = L \in (0, \infty) \). It is obvious that \( \lim_{t \to \infty} \sigma(s)^2 \log t = L \in (0, \infty) \) implies \( \lim_{t \to \infty} \int_{t}^{t+1} \sigma(s)^2 \, ds \cdot \log t = L \) which hence violates (2.13). □

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