Zero-viscosity Limit for Boussinesq Equations with Vertical Viscosity and Navier Boundary in the Half Plane

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Abstract. In this paper we study the zero-viscosity limit of 2-D Boussinesq equations with vertical viscosity and zero diffusivity, which is a nonlinear system with partial dissipation arising in atmospheric sciences and oceanic circulation. The domain is taken as $\mathbb{R}^2_+$ with Navier-type boundary. We prove the nonlinear stability of the approximate solution constructed by boundary layer expansion in conormal Sobolev space. The expansion order and convergence rates for the inviscid limit are also identified in this paper. Our paper extends a partial zero-dissipation limit result of Boussinesq system with full dissipation by Chae D. [Adv. Math. 203 (2006), no.2, 497–513] in the whole space to the case with partial dissipation and Navier boundary in the half plane.

Keywords: Boussinesq equations; Anisotropic dissipation; Zero-viscosity limit; Navier boundary.

AMS subject classifications: 35B40, 35Q86, 76D10

1. Introduction

We consider the following 2-D Boussinesq equations with only vertical viscosity and zero diffusivity in $\mathbb{R}^2_+$:

$$
\begin{align*}
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon - \varepsilon^2 \partial_y^2 u^\varepsilon &= \varepsilon^2 e_2, \\
\partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon &= 0, \\
\nabla \cdot u^\varepsilon &= 0, \\
|u^\varepsilon|_{t=0} &= u^\varepsilon_0(x, y), \quad |\theta^\varepsilon|_{t=0} = \theta^\varepsilon_0(x, y),
\end{align*}
$$

where $t \geq 0$ and $(x, y) \in \mathbb{R} \times \mathbb{R}_+$ are time and space variables. Here $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2)$ and $p^\varepsilon$ are, respectively, the velocity and pressure of the fluid. The scalar function $\theta^\varepsilon$ denotes the temperature or the density. We use $e_2 = (0, 1)$ to represent the unit vector in the vertical direction, and $\varepsilon^2$ to denote the kinematic viscosity. The Boussinesq equations with

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anisotropic (full or partial) dissipation play an important role in the study of atmospheric and oceanographic flows and Raleigh-Bénard convection, mathematically and physically. For more backgrounds of the Boussinesq equations, one can refer to [14,46,49,60].

In this paper, the Navier-type (slip) boundary condition of the system (1.1) is given by

\[ u_2^\varepsilon = 0, \quad \partial_y u_1^\varepsilon = \alpha u_1^\varepsilon \quad \text{on} \quad \{y = 0\}, \quad (1.2) \]

where \( \alpha \in \mathbb{R} \) is used to characterize the tendency of the fluid to slip on the boundary. Here the initial data given in (1.1) and in the sequel should satisfy the compatibility conditions on the boundary and the divergence free condition.

Our purpose is to survey the zero-viscosity limit behaviour of the Boussinesq system with partial viscosity and zero diffusivity in a half space satisfying the Navier boundary condition. It is extremely challenging to deal with the loss of partial dissipation and the boundary layer effects, especially, for the bounded / unbounded domain with non-slip boundary. In this paper we set our problem in the half plane with the Navier boundary condition (1.2) for the first step. To our best knowledge, this paper is the first one to consider the strong zero-viscosity limit of Boussinesq equations with partial viscosity and boundaries, which is from physical consideration and involves layer effects.

Formally, letting \( \varepsilon \to 0 \), the partially viscous Boussinesq system (1.1) is then reduced to the following zero dissipation system in \( \mathbb{R}^2_+ \):

\[
\begin{aligned}
\partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 &= \theta^0 e_2, \\
\partial_t \theta^0 + u^0 \cdot \nabla \theta^0 &= 0, \\
\nabla \cdot u^0 &= 0, \\
u^0|_{t=0} &= u^0_0(x, y), \quad \theta^0|_{t=0} = \theta^0_0(x, y),
\end{aligned}
\]

which is complemented with the boundary condition:

\[ u^0_{2}|_{y=0} = 0. \quad (1.4) \]

We note that the initial data \((u^0_0, \theta^0_0)\) satisfy the compatibility condition

\[ \nabla \cdot u^0_0 = 0, \quad u^0_0 \cdot n = 0, \quad \text{with} \quad n = (0, -1). \]
Setting the zero-dissipation system (1.3) into the whole space \( \mathbb{R}^2 \) or a domain with boundary, the local in time existence theory of (1.3) has been studied in [7, 9] or [8, 10, 26, 29], respectively. From [8, 9], one can expect the local existence of (1.3) on a time interval \([0, T]\) for initial data satisfying \( \nabla \cdot u_0 = 0 \) and \((u_0^0, \theta_0^0) \in H^s(\mathbb{R}_+^2)\) for some \(s > 2\). In particular, the local in time existence of (1.3) in \( \mathbb{R}_+^2 \) holds for the \(C^\infty\)-smooth initial data [26] and for the \(C^{1,\alpha}(\mathbb{R}_+^2)\) initial data [10] with \(\alpha < \alpha_0\) for some \(\alpha_0 < 1\), both developing singularities in finite time. It is suggested that there are some strong analogues between the 2-D zero-dissipation system (1.3) and the 3-D Euler equations (see [15, 45]). Therefore, to understand the local existence theory of the 2-D Boussinesq system with no dissipation (1.3), one can also refer to the local existence theory and finite time blow-up criteria for 3-D incompressible Euler equations [3, 10–12, 17, 24, 25, 35, 43, 54].

The system (1.1) is a particular case of the 2-D Boussinesq system with anisotropic dissipation, which reads

\[
\begin{align*}
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon &= \nu_1 \partial_x^2 u^\varepsilon + \nu_2 \partial_y^2 u^\varepsilon + \theta^\varepsilon e_2, \\
\partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon &= \kappa_1 \partial_x^2 \theta^\varepsilon + \kappa_2 \partial_y^2 \theta^\varepsilon, \\
\nabla \cdot u^\varepsilon &= 0,
\end{align*}
\]  

(1.5)

where \(\nu_1, \nu_2\) and \(\kappa_1, \kappa_2\) are nonnegative constants to characterize viscosity and diffusivity, respectively. Recently, extensive progress has been made on the global existence theory of the Boussinesq system (1.5) with full dissipation or partial dissipation in \(\mathbb{R}^2\) (cf. [1, 2, 4, 5, 18, 19, 21, 23, 30, 31, 38, 40, 48, 62] and the references therein), or in an appropriate domain with boundary (cf. [22, 32, 37, 38, 53, 63] and the references therein). However, the global existence theories for the Boussinesq system with \(\nu_2 > 0\) and \(\nu_1 = \kappa_1 = \kappa_2 = 0\) (i.e. (1.1)), or \(\nu_1 = \nu_2 = \kappa_1 = \kappa_2 = 0\) (i.e. (1.3)), are still challenging open problems even in the whole space \(\mathbb{R}^2\). For the coupled advective scalar equation of \(\theta\) in (1.5) and the related quasigeostrophic equations, they are also attractive research topics in PDE theory [13–15, 17, 34, 50, 61].

In addition, the zero-dissipation limit behaviour of Boussinesq equations with boundary is a meaningful physical problem, which is associated with the boundary layer theory. Recently, Jiang et al [33] have studied the zero-diffusivity limit of the 2-D Boussinesq equations with full dissipation in the half plane. They have justified the zero-diffusivity limit of the velocity \(u\) and the temperature \(\theta\), respectively, in \(H^1\) and \(L^2\) norm uniformly on \([0, T]\) with convergence...
rate. In [57], Wang and Xie have investigated the $L^2$ zero-dissipation limit of Boussinesq equations in a bounded domain with Navier type boundary for the velocity and Neumann boundary for the temperature using boundary layer expansion. One can also consult [58] about the well-posedness of the boundary layer equation for a geophysical model or [27, 39] on the zero-diffusivity limit for the Boussinesq system in the weak sense.

The vanishing viscosity problem is an attractive topic in fluid dynamics, due to its physical importance. It is well known that the famous Prandtl boundary layer equations can be derived from the vanishing viscosity limit problem of classical incompressible Navier-Stokes equations with non-slip boundary. The Prandtl equations are regarded as a leading order of the inner layer approximation. Similarly, the Prandtl equations can also be derived from the inviscid limit problem of Boussinesq equations with non-slip boundary condition on velocity. In past decades, much attention have been paid to the well-posedness theory of Prandtl equations in different function spaces, mainly overcoming the difficulties from the lack of dissipation in horizontal direction. Based on the well-posedness theory of Prandtl equations, a natural question is that whether the Prandtl layer approximation is stable or not? In this direction, a lot of progresses have been made for Navier-Stokes equations ( For example, see [36, 44, 51, 52, 56] and the references therein). It is worthy to investigate the similar questions in ocean dynamic describing by anisotropic Boussinesq equations. Especially, when the horizontal viscosity is lacking, just like the model presented in this paper, the $L^\infty$ zero-viscosity limit problem is extremely challenging in a general domain with non-slip boundary, due to the strong Prandtl layer and the less dissipation. As a first step, in this paper we set our inviscid limit problem in the half plane with vertical viscosity and a (slip) Navier boundary. Meanwhile, our results can be extend to other anisotropic Boussinesq models with weak boundary layers in a half space. We hope to come back to the inviscid limit problem of anisotropic Boussinesq equations in a general domain in the future study.

In this paper we concentrate on investigating the strong $L^\infty$ zero-viscosity limit of the Boussinesq equations (1.1) with Navier boundary in $\mathbb{R}^2_+$. To begin with, we construct an approximate solution with outer layer profile away from the boundary and inner layer profile near the boundary, using boundary layer expansion method. Due to the structure of the
Navier boundary condition, we can derive an approximate solution with leading profile

$$(u^0, \theta^0) + \varepsilon(U, \Theta)(t, x, \frac{y}{\varepsilon}) + \cdots$$

to characterize the singularity near the boundary as $\varepsilon$ goes to zero. Here we can observe that the $H^2$ norm of $u^\varepsilon$ is not uniformly bounded. The study of zero-viscosity limit is then reduced to the stability analysis of the approximate solution and derive the uniform estimates on $[0, T]$ with $T$ independent of $\varepsilon$. If the expansion order is larger than or equal to two, we can prove the linear stability in the conormal Sobolev setting by taking advantage of the anisotropic Sobolev embedding inequality, since the first order derivatives of the inner layer profile are bounded. In the nonlinear stability analysis, we introduce an energy function for the perturbation equation around the approximate scheme, involving $L^\infty$-norm of $\theta$ and $u$ and their derivatives, which is inspired by the remarkable work [47]. This will overcome the difficulties from nonlinearity with the help of a precise $L^\infty$ estimates. More precisely, we deal with the $L^\infty$ estimates of the velocity $u$ in spirit of the $L^\infty$ estimates for incompressible Navier-Stokes equations in [47], taking advantage of estimates on Green’s function for the operator of an approximate equation. However, to close the essential nonlinear estimates, a higher order conormal energy estimate is needed and the order of singularity will increase, compared with the linear argument. Here we can establish a uniform convergence estimate through improving the order of boundary layer expansion. Notice that the lack of horizontal dissipation can be handled owing to the divergence free condition and the boundary setting. Throughout our stability analysis, we can construct an approximate solution near the boundary with high accuracy. Different from the inviscid limit problem of Navier-Stokes equations with Navier boundary considered in [47], the Boussinesq equations in this paper are coupling with an advective scalar equation on $\theta$. As for the $L^\infty$ estimates of the density (or temperature) $\theta$, we apply the maximum principle for a particular advective scalar equation (cf. [50]). Here we shall notice that in [47] the authors have proved the zero-viscosity limit of incompressible Navier-Stokes equations with Navier boundary using compactness argument, in which the convergence rate can not be deduced. Owing to the boundary layer expansion and stability analysis in this paper, we can deduce the convergence rate $O(\varepsilon)$ and identify the expansion order $K$ for the boundary layer expansion of the partially dissipative Boussinesq system.
Now we state our main result in the following theorem.

**Theorem 1.1.** Let \((u^0, \theta^0)\) be a solution to (1.3) (1.4) defined on \([0,T]\). Then there exists \((u^\varepsilon, \theta^\varepsilon)\) a solution to (1.1) (1.2) defined on \([0,T_1]\) for some \(T_1\) independent of \(\varepsilon\) and \(T_1 \leq T\) such that

\[
\sup_{[0,T_1]} (\|u^\varepsilon - u^0\|_{L^2(\mathbb{R}_+^2)} + \|\theta^\varepsilon - \theta^0\|_{L^2(\mathbb{R}_+^2)}) \to 0,
\]

(1.6)

\[
\sup_{[0,T_1]} (\|u^\varepsilon - u^0\|_{L^\infty(\mathbb{R}_+^2)} + \|\theta^\varepsilon - \theta^0\|_{L^\infty(\mathbb{R}_+^2)}) \to 0,
\]

(1.7)

as \(\varepsilon\) goes to 0. Furthermore, the rate of convergence is \(O(\varepsilon)\).

**Remark 1.2.** The convergence results in Theorem 1.1 also hold for Boussinesq system (1.5) in \(\mathbb{R}_+^2\) with either one of the following two conditions:

1. \(\nu_2 = \varepsilon^2, \nu_1 = 0\) or \(\varepsilon^2, \kappa_1 = 0\) or \(\varepsilon^2, \kappa_2 = 0\) and Navier boundary condition (1.2);
2. \(\nu_1 = \varepsilon^2, \kappa_1 = 0\) or \(\varepsilon^2, \nu_2 = \kappa_2 = 0\) and boundary condition \(u \cdot n = 0\) on \(\{y = 0\}\).

In addition, this result can be extended to the inviscid limit problem of the parallel three-dimensional case.

The arrangement of the remaining sections is as following. We devote Section 2 to constructing an approximate solution using boundary layer expansion. We will give the stability analysis of the linearized system around the approximate solution in conormal Sobolev space in Section 3. In Section 4, we will focus on dealing with the nonlinear terms and deriving the \(L^\infty\) estimates to close the uniform estimates.

### 2. Boundary Layer Expansion

We construct an approximate solution of system (1.1) in the following form

\[
(u_a, \theta_a, p_a) = \sum_{i=0}^{K} \varepsilon^i (u^i, \theta^i, p^i)(t, x, y) + \sum_{i=0}^{K} \varepsilon^i (U^i, \Theta^i, P^i) \left(t, x, \frac{y}{\varepsilon}\right),
\]

(2.1)

where \(K\) is an arbitrarily large integer, and \(U^i := (U^i_1, U^i_2)\). We use \((u^i, \theta^i, p^i)(t, x, y)\) in the first sum to approximate the outer layer, meanwhile, \((U^i, \Theta^i, P^i) \left(t, x, \frac{y}{\varepsilon}\right)\) to characterize the inner layer behaviour near the boundary in the second sum. In the following presentation, we denote the fast variable as \(z = y/\varepsilon\) for simplicity.
We shall impose the fast decay condition as following
\[
(U^i, \Theta^i, P^i)(t, x, z) \to 0, \quad \text{as} \quad z \to +\infty.
\] (2.2)

In addition, the matched boundary conditions for Navier boundary (1.2) read
\[
u_2^i(t, x, 0) + U_2^i(t, x, 0) = 0, \quad \text{for all} \quad i \geq 0,
\]
\[\partial_z U_1^0(t, x, 0) = 0,
\]
\[\partial_y u_1^i(t, x, 0) + \partial_z U_1^{i+1}(t, x, 0) = \alpha (u_1^i + U_1^i)(t, x, 0), \quad i \geq 0.
\] (2.3)

In the sequel, we denote \(\Gamma f\) as \(\Gamma f = f(t, x, y)|_{y=0}\).

We expect the leading order of the outer layer in the approximation (2.1) to be the solution of the zero viscosity Boussinesq system (1.3) (1.4), whose local well-posedness theory can be deduced from \([8, 9]\) for initial data satisfying \(\nabla \cdot u_0^0 = 0\) and \((u_0^0, \theta_0^0) \in H^s(\mathbb{R}^2_+))\) for some \(s > 2\). In fact, substituting the approximate scheme (2.1) into the viscous Boussinesq system (1.1) and collecting the \(O(1)\) order terms, consequently we can get the leading order terms for the outer layer satisfying (1.3) by taking \(z \to +\infty\).

In the inner zone, by collecting the \(O(\varepsilon^{-1})\) terms, one has
\[
(\Gamma u_2^0 + U_2^0) \partial_z U_1^0 = 0,
\]
\[(\Gamma u_2^0 + U_2^0) \partial_z U_2^0 + \partial_z P^0 = 0,
\]
\[\partial_z U_2^0 = 0,
\]
\[(\Gamma u_2^0 + U_2^0) \partial_z \Theta^0 = 0.
\]

Then we can deduce from (2.2) and (2.3) that
\[
U_2^0(t, x, z) = 0, \quad P^0(t, x, z) = 0.
\] (2.4)

In turn, \(\Gamma u_2^0 = 0\) holds.

Now, we collect the \(O(1)\) terms for the first equation of velocity in the inner zone to obtain
\[
\partial_t (\Gamma u_1^0 + U_1^0) + (\Gamma u_1^0 + U_1^0) \partial_x (\Gamma u_1^0 + U_1^0)
\]
\[+ (\Gamma u_2^1 + U_2^1 + z \Gamma \partial_y u_2^1) \partial_z (\Gamma u_1^0 + U_1^0) + \partial_z (\Gamma P^0) - \partial_{zz} (\Gamma u_1^0 + U_1^0) = 0,
\] (2.5)
Combining with the $O(1)$ terms from the divergence free condition in the inner layer,

$$\partial_x(\Gamma u_1^0 + U_1^0) + \partial_z(\Gamma u_1^1 + U_2^1 + z\Gamma \partial_y u_2^0) = 0,$$  \hspace{1cm} (2.6)

one has a closed system for $\Gamma U_0^0 + U_1^0$ and $\Gamma U_2^1 + U_2^1 + z\Gamma \partial_y u_2^0$, together with the boundary condition

$$\partial_z U_1^0(t,x,0) = 0.$$  \hspace{1cm} (2.7)

We notice that the equations (2.5) (2.6) and (2.7) have a trivial solution

$$U_1^0 = 0.$$  

In the following analysis, we shall assume that the approximate solutions (2.1) satisfy $U_1^0 = 0$.

Clearly, from the $O(1)$ order terms of the second velocity equation and the transport equation, respectively,

$$\partial_t(\Gamma u_2^0 + U_2^0) + (\Gamma u_1^0 + U_1^0)\partial_x(\Gamma u_2^0 + U_2^0)$$

$$+ (\Gamma u_2^0 + U_2^0)\partial_z(\Gamma u_1^1 + U_2^1 + z\Gamma \partial_y u_2^0) + \Gamma \partial_y p^0 + \partial_z P^1 = \Gamma \theta^0 + \Theta^0,$$

$$\partial_t(\Gamma \theta^0 + \Theta^0) + (\Gamma u_1^0 + U_1^0)\partial_x(\Gamma \theta^0 + \Theta^0) + (\Gamma u_2^1 + U_2^1 + z\Gamma \partial_y u_2^0)\partial_z \Theta^0 = 0,$$

the leading order for the temperature (or the density) in the inner zone has a trivial solution

$$\Theta^0 = 0.$$  \hspace{1cm} (2.7)

Consequently, we also have $P^1 = 0$.

**Remark 2.1.** *In the study of zero-viscosity limit for incompressible Navier-Stokes equations with non-slip boundary, one derive the system (2.5) (2.6) with boundary condition

$$\Gamma u_1^0 + U_1^0(t,x,0) = 0,$$  \hspace{1cm} (2.8)

which just forms the famous Prandtl system. For the local well-posedness theory of Prandtl system(2.5) (2.6) (2.8) or the inviscid limit of Navier-Stokes equations with non-slip boundary, one may refer to [20, 36, 41, 44, 51, 52, 56] and the references therein.*

Till now, we get the leading order for both inner zone and outer zone for the approximate solution. i.e. $(u^0, p^0, \theta^0)$ and $(U^0, P^0, \Theta^0)$ are solved. Next, following the standard method on constructing approximate solution by boundary layer expansion (see [28, 57, 59] and the
references therein for example), one can obtain the terms \((u^i, p^i, \theta^i)\) and \((U^i, P^i, \Theta^i)\) order by order. Indeed, we collect the \(O(\varepsilon^i), i \geq 1\) terms in the outer layer expansion to obtain

\[
\begin{align*}
\partial_t u^i + u^i \cdot \nabla u^0 + u^0 \cdot \nabla u^i + \nabla p^i &= \theta^i \varepsilon_2 + \partial_y^2 u^{i-2} + f_u^i, \\
\partial_t \theta^i + u^i \cdot \nabla \theta^0 + u^0 \cdot \nabla \theta^i &= f_{\theta}^i, \\
\nabla \cdot u^i &= 0,
\end{align*}
\]

where \(u^{-1} = 0\), and \(f_u^i, f_{\theta}^i\) are depending on \(u^j, \theta^j\) with \(j \leq i - 1\). This, together with the boundary condition \(u^j|_{y=0} = 0\), forms a linear closed system of \((u^i, \theta^i, p^i)\). Hence the outer layer terms have been determined. For the inner layer expansion in the order \(O(\varepsilon^i), i \geq 1\), we have that

\[
\begin{align*}
\partial_t (\Gamma u_1^i + U_1^i) + (\Gamma u_1^i + U_1^i) \partial_x (\Gamma u_0^i + U_0^i) + (\Gamma u_1^i + U_1^i) \partial_x (\Gamma u_0^i + U_1^i) \\
+ (\Gamma u_2^i + 1 + U_2^i + z \Gamma \partial_y u_2^i) \partial_x (\Gamma u_0^i + U_2^i) + \Gamma \partial_x p_i^i + \partial_x P_i^i = \partial_{zz} (\Gamma u_1^i + U_1^i) + F_{u_1}^i, \\
\partial_t (\Gamma u_2^i + U_2^i) + (\Gamma u_0^i + U_0^i) \partial_x (\Gamma u_2^i + U_2^i) + \Gamma \partial_y p_i^i + \partial_x P_i^{i+1} \\
+ (\Gamma u_2^i + 1 + z \Gamma \partial_y u_2^i + U_2^i) \partial_x (\Gamma u_2^i + U_2^i + z \Gamma \partial_y u_2^i) \\
+ (\Gamma u_2^i + U_2^i + z \Gamma \partial_y u_2^i - 1) \partial_x (\Gamma u_2^i + z \Gamma \partial_y u_2^i + U_2^i) \\
= \partial_{zz} (\Gamma u_2^i + U_2^i) + \Theta^i + \Gamma \theta^i + F_{u_2}^i, \\
\partial_t (\Gamma_0^i + \Theta^i) + (\Gamma U_1^i + U_1^i) \partial_x (\Gamma_0^i + \Theta^i) + (\Gamma U_1^i + U_1^i) \partial_x (\Gamma_0^i + \Theta^i) \\
+ (\Gamma U_1^i + U_1^i + z \Gamma \partial_y u_2^i) \partial_x (\Gamma_0^i + \Theta^i + z \Gamma \partial_y \theta^i - 1) \\
+ (\Gamma U_2^i + U_2^i + z \Gamma \partial_y u_2^i - 1) \partial_x (\Gamma_0^i + \Theta^i + z \Gamma \partial_y \theta^i) = F_{\theta}^i, \\
(\Gamma u_2^{i+1} + U_2^{i+1} + z \Gamma \partial_y u_2^i) &= - \int_0^z \partial_x (\Gamma u_1^i + U_1^i)(t, x, \tau) d\tau,
\end{align*}
\]

where \(F_{u_1}^i, F_{u_2}^i, F_{\theta}^i\) are depending on \(U^j, \Theta^i, u^j, \theta^j\) with \(j \leq i - 1\). Here the last equality is from the divergence free condition. Recall that \(P_1 = 0\) and the boundary conditions for the inner layer satisfy (2.3). Then, order by order, we obtain a linear closed differential and integral system for \((U^i, \Theta^i, P^{i+1})\), \(i \geq 1\). Now we conclude that the approximate solutions \((u_a, p_a, \theta_a)\) for (1.1) can be expressed in the form

\[
(u^0, p^0, \theta^0)(t, x, y) + \varepsilon \left( (u^1, p^1, \theta^1)(t, x, y) + (U^1, P^1, \Theta^1)(t, x, \frac{y}{\varepsilon}) \right) + \cdots.
\]
Let us denote
\[ \omega^0 = \partial_y u^0_1 - \partial_x u^0_2, \quad \text{and} \quad \omega^0_0 = \omega^0(t = 0, x, y). \]

Then we can gather our results on boundary layer approximations of the solutions for the system (1.1) and (1.2) in the following theorem.

**Theorem 2.2.** Let \( K \in \mathbb{N}_+ \). For any initial data \((\theta^0_0, u^0_0)\) satisfying \( \nabla \cdot u^0_0 = 0 \), \((u^0_0, \theta^0_0) \in H^s(\mathbb{R}^2)\) for some \( s > 2 \) and some compatibility conditions on \( \{y = 0\} \), then there exists \( T > 0 \) and a smooth approximate solution \((u_a, \theta_a)\) of (1.1) under the form (2.1) with order \( K \) such that

i) we have \((u^0, \theta^0) \in C^0([0, T], H^s(\mathbb{R}^2))\) as a solution of the inviscid system (1.3) with initial data \((u^0_0, \theta^0_0)\);

ii) for all \( 1 \leq i \leq K \), \((u^i, \theta^i) \in C^0([0, T], H^s(\mathbb{R}^2))\);

iii) for all \( 0 \leq i \leq K \), \((U^i, P^i, \Theta^i)\) are solved at least locally and satisfy the fast decay property (2.2) with respect to the last variable.

iv) we consider \((u^\varepsilon, \theta^\varepsilon)\) a solution to (1.1), and denote the error terms \((u, p, \theta)\) as following

\[ u = (u_1, u_2) = u^\varepsilon - u_a, \quad p = p^\varepsilon - p_a, \quad \theta = \theta^\varepsilon - \theta_a, \]

where \( u_a \) naturally satisfies

\[ \nabla \cdot u_a = 0, \quad u_a \cdot n|_{y=0} = 0, \]

with \( n = (0, -1) \).

Then \((u, \theta)\) satisfies the system of equations

\[ \begin{align*}
\partial_t u + u \cdot \nabla(u + u_a) + u_a \cdot \nabla u + \nabla p - \varepsilon^2 \partial_y^2 u &= \theta e_2 + \varepsilon^K R_u, \\
\partial_t \theta + u \cdot \nabla(\theta + \theta_a) + u_a \cdot \nabla \theta &= \varepsilon^K R_\theta, \\
\nabla \cdot u &= 0, \\
\partial_y u_1 &= \alpha u_1, \quad u_2 = 0, \quad \text{on} \quad \{y = 0\},
\end{align*} \]

where \( R_u := (R_{u1}, R_{u2}) \) and \( R_\theta \) are remainders satisfying

\[ \sup_{[0, T]} \|\nabla^\beta R_{u,\theta}\| \leq C_a \varepsilon^{-\beta_2}, \quad \forall \beta = (\beta_1, \beta_2) \in \mathbb{N}^2, \]
with $C_a > 0$ independent of $\varepsilon$. Here and thereafter we use the symbol $\| \cdot \|$ to represent the standard $L^2$ norm $\| \cdot \|_{L^2(\mathbb{R}^2)}$.

3. Linear Stability Estimates

This section is devoted to the stability analysis of the approximate solution constructed by boundary layer expansion in Theorem 2.2. Due to the essential challenge in nonlinear energy estimates caused by the boundary layer effects and the loss of dissipation in $x$-direction, we first consider the linear stability of the boundary layer approximation. In the following writing, we use the notation

$$\Omega := \mathbb{R}^2_+.$$

To begin with, we linearize the system (2.9) around the approximate solution to obtain that, in $\mathbb{R}^2_+$,

\begin{align*}
&\partial_t u + u \cdot \nabla u_a + u_a \cdot \nabla u + \nabla p - \varepsilon^2 \partial^2_y u = \theta e_2 + \varepsilon^K R_u, \quad (3.1a) \\
&\partial_t \theta + u \cdot \nabla \theta_a + u_a \cdot \nabla \theta = \varepsilon^K R_\theta, \quad (3.1b) \\
&\nabla \cdot u = 0, \quad (3.1c) \\
&\partial_y u_1 = \alpha u_1, \; u_2 = 0, \; \text{on} \; \{y = 0\}, \quad (3.1d)
\end{align*}

where $R_u := (R_{u_1}, R_{u_2})$ and $R_\theta$ are remainders satisfying

$$\sup_{[0,T]} \| \nabla^\beta R_{u,\theta} \| \leq C_a \varepsilon^{-\beta_2}, \; \forall \beta = (\beta_1, \beta_2) \in \mathbb{N}^2, \quad (3.2)$$

with $C_a > 0$ independent of $\varepsilon$. The initial data of the system (2.9) and (3.1) are given by

$$u|_{t=0} = \varepsilon^{K+1} u_0, \quad \theta|_{t=0} = \varepsilon^{K+1} \theta_0. \quad (3.3)$$

Here we can always write the initial data $(u_0^\varepsilon, \theta_0^\varepsilon)$ of the system (1.1) as

$$(u_0^\varepsilon, \theta_0^\varepsilon) = \left( \sum_{i=0}^{K} \varepsilon^i (u_0^i, \theta_0^i) \right) + \varepsilon^{K+1}(u_0, \theta_0),$$

where $(u_0^i, \theta_0^i)$ is taken as the initial data for each approximate profile in (2.1). Hence $\varepsilon^{K+1}(u_0, \theta_0)$ is regarded as the initial data of the perturbation equations (2.9) or (3.1).

In the remaining parts of this section, we will give the stability analysis of the linear system (3.1) for $\alpha \in \mathbb{R}$. 

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3.1. **Preliminaries.** Before we process the uniform estimates, some basic notations and useful inequalities will be given in this subsection. We will use \( \lesssim \) to denote \( \leq C(\cdot) \) or \( \leq C_a(\cdot) \) for a generic constant \( C \) or a constant \( C_a \) depending on the approximate solution, but both independent of \( \varepsilon \). The standard Sobolev norm is denoted by \( \| \cdot \|_s \) for \( \| \cdot \|_{H^s} \) with \( s \geq 0 \). In particular, \( \| \cdot \| \) is for \( \| \cdot \|_{L^2} \).

We introduce conormal operator

\[
Z^\beta = Z_1^\beta_1 Z_2^\beta_2
\]

with \( \beta = (\beta_1, \beta_2) \in \mathbb{N}^2 \), where

\[
Z_1 = \partial_x, \quad Z_2 = \varphi(y) \partial_y,
\]

and \( \varphi(y) \) is a smooth function satisfying \( \varphi(0) = 0 \) and \( \varphi'(0) > 0 \), such as

\[
\varphi(y) = \frac{y}{y + 1}.
\]

In the sequel, we denote

\[
Z^k = Z^\beta, \quad \text{with } |\beta| = k, \ k \in \mathbb{N}.
\]

The conormal Sobolev space \( H^s_{co} \) for \( s \in \mathbb{N} \) is defined by

\[
H^s_{co} := \left\{ u \in L^2_{x,y}(\mathbb{R}^2_+) : \| u \|_{H^s_{co}}^2 = \sum_{|\beta| \leq s} \| Z_1^{\beta_1} Z_2^{\beta_2} u \|_{L^2_{x,y}(\mathbb{R}^2_+)}^2 < \infty \right\}.
\]

In spirit of this setting, we denote the \( H^s_{co} \)-inner product by \( \langle \cdot, \cdot \rangle_{H^s_{co}} \). Set

\[
\| u \|_{k,\infty} = \sum_{|\beta| \leq k} \| Z^\beta u \|_{L^\infty},
\]

and we say that \( u \in W^k_{co,\infty} \) if \( \| u \|_{k,\infty} \) is finite.

Now we state the anisotropic Sobolev embedding inequality and some useful estimates (cf. [47]) in the following lemmas.

**Lemma 3.1.** For \( m_0 \geq 1, m_0 \in \mathbb{N} \), and conormal Sobolev norm defined in (3.6), we have

\[
\| u \|_{L^\infty}^2 \lesssim \| \partial_y u \|_{H^{m_0}_{co}}\| u \|_{H^{m_0}_{co}} + \| u \|_{H^{m_0}_{co}}^2.
\]

Let us denote the vorticity \( \omega \) as

\[
\omega = \text{curl } u = \partial_y u_1 - \partial_x u_2,
\]

and set

\[
\eta = \omega - \alpha u_1.
\]
Lemma 3.2 (cf. [47], Proposition 12). In \( \mathbb{R}^2_+ \), for \( m_0 \geq 1 \), we have
\[
\| u \|_{W^{1,\infty}} \lesssim \| u \|_{H^{m_0+2}_{loc}} + \| \eta \|_{H^{m_0+1}_{loc}} + \| \eta \|_{L^{\infty}},
\]  
(3.9)
\[
\| u \|_{2,\infty} \lesssim \| u \|_{H^{m_0+3}_{loc}} + \| \eta \|_{H^{m_0+2}_{loc}},
\]  
(3.10)
\[
\| \nabla u \|_{1,\infty} \lesssim \| u \|_{H^{m_0+3}_{loc}} + \| \eta \|_{H^{m_0+3}_{loc}} + \| \eta \|_{L^{\infty}}.
\]  
(3.11)

3.2. Uniform Estimates for \( \alpha \in \mathbb{R} \). In this subsection, we will derive the uniform estimates in conormal Sobolev space. To begin with, we state our main results for the linear system (3.1) with partial viscosity.

Theorem 3.3. Let \( m \geq 1 \), and \( (\theta^0_0, u^0_0) \in C^\infty(\mathbb{R}^2_+) \) be initial data for (1.3) satisfying \( \nabla \cdot u^0 \) and some compatibility conditions on \( \{ y = 0 \} \). Let \( K \in \mathbb{N}_+, K > m \) and \( (u_a, p_a, \theta_a) \) an approximate solution at order \( K \) given by Theorem 2.2. Then, for every \( \varepsilon \in (0, 1) \), there exists a \( T_0 > 0 \) such that the solution of (3.1)-(3.3) defined on \([0, T_0] \) satisfies the estimate
\[
\| (u, \theta) \|_{H^{m_0}} + \| \partial_y (u, \theta) \|_{H^{m_0-1}} \lesssim \varepsilon^{K-m}.
\]  
(3.12)

Therefore, it holds that
\[
\| (u, \theta) \|_{L^\infty(\mathbb{R}^2_+)} \lesssim \varepsilon^{K-1}.
\]  
(3.13)

The proof of the Theorem 3.3 can be deduced from the following \( L^2 \)-estimates, conormal energy estimates, normal derivatives estimates and pressure estimates.

\( L^2 \)-estimates.

Lemma 3.4. Let \( (u_a, \theta_a) \) be the approximate solution in Theorem 2.2. Given \( \alpha \in \mathbb{R} \), if \( (u, \theta) \) is the solution of the linear system (3.1) defined on \([0, T] \), then it holds that
\[
\| (u, \theta)(t,x,y) \|^2 + c_0 \varepsilon^2 \int_0^T \| \partial_y u(t,x,y) \|^2 \, dt \lesssim \varepsilon^{2K}
\]  
(3.14)
for some positive constant \( c_0 \).

Remark 3.5. The \( L^2 \) estimates (3.14) also holds for the nonlinear system (2.9), due to \( \nabla \cdot u = 0 \), and \( u \cdot n \big|_{y=0} = 0 \).
Proof. We multiply the equation (3.1a) and (3.1b), respectively, by \( u \) and \( \theta \). Then, adding the resulting equations together and integrating over \( \Omega \), together with integration by parts, we can obtain that

\[
\frac{d}{dt} \int_{\Omega} (u^2 + \theta^2) + \varepsilon^2 \int_{\Omega} |\partial_y u|^2 + \alpha \varepsilon^2 \int_{\partial \Omega} u_1^2 \\
= - \int_{\Omega} (u \cdot \nabla u_\alpha \cdot u + u \cdot \nabla \theta_\alpha \theta) + \int_{\Omega} \theta e_2 \cdot u + \varepsilon^K \int_{\Omega} (R_u u + R\theta \theta),
\]

(3.15)
since

\[
\int_{\Omega} \nabla p u = - \int_{\Omega} \nabla \cdot up + \int_{\partial \Omega} pu \cdot n = 0,
\]

\[
\int_{\Omega} u_\alpha \cdot \nabla uu = \int_{\Omega} u_\alpha \cdot \nabla u|^2 = - \int_{\Omega} \nabla \cdot u_\alpha |u|^2 + \int_{\partial \Omega} u_\alpha \cdot n |u|^2 = 0,
\]

\[
\int_{\Omega} \partial_y^2 uu = - \int_{\Omega} |\partial_y u|^2 - \int_{\partial \Omega} \partial_y u_1 u_1 = - \int_{\Omega} |\partial_y u|^2 - \alpha \int_{\partial \Omega} u_1^2,
\]

\[
\int_{\Omega} u_\alpha \cdot \nabla \theta \theta = - \int_{\Omega} \nabla \cdot u_\alpha \theta^2 + \int_{\partial \Omega} u_\alpha \cdot n \theta^2 = 0.
\]

Due to

\[
\int_{\Omega} (u \cdot \nabla u_\alpha \cdot u + u \cdot \nabla \theta_\alpha \theta) \leq \| \nabla u_\alpha \|_{L^\infty} \int_{\Omega} |u|^2 + \| \nabla \theta_\alpha \|_{L^\infty} \int_{\Omega} |\theta| \leq C_\alpha \int_{\Omega} (|u|^2 + |\theta|^2),
\]

and

\[
\int_{\Omega} \theta e_2 \cdot u \leq \int_{\Omega} (|u|^2 + |\theta|^2),
\]

then we have, for \( \alpha \geq 0 \),

\[
\frac{d}{dt} \int_{\Omega} (u^2 + \theta^2) + \varepsilon^2 \int_{\Omega} |\partial_y u|^2 + \alpha \varepsilon^2 \int_{\partial \Omega} u_1^2 \\
\lesssim \int_{\Omega} (u^2 + \theta^2) + \varepsilon^{2K},
\]

(3.16)

where the Cauchy inequality has been used. However, for \( \alpha < 0 \), the boundary term is not good in the energy estimates. Using trace theorem in \( \mathbb{R}^2_+ \), we have

\[
|\alpha| \varepsilon^2 \int_{\partial \Omega} u_1^2 \leq |\alpha| \varepsilon^2 \| \partial_y u_1 \| \| u_1 \|.
\]

Then, together with the Young inequality, one has

\[
\frac{d}{dt} \int_{\Omega} (u^2 + \theta^2) + c_0 \varepsilon^2 \int_{\Omega} |\partial_y u|^2 \lesssim \int_{\Omega} (u^2 + \theta^2) + \varepsilon^{2K}.
\]

(3.17)

Hence (3.14) follows from (3.17) and the Gronwall’s inequality. \( \square \)
Conormal energy estimates.

**Lemma 3.6.** Let \((u_a, \theta_a)\) be the approximate solution in Theorem 2.2. Given \(\alpha \in \mathbb{R}\), assume that \((u, \theta)\) is a solution of the linear system (3.1) defined on \([0,T]\), then, for \(m \geq 1\), we have

\[
\frac{d}{dt}(\|(u, \theta)\|_{H^m_{co}}^2) + c_0\epsilon^2 \|\partial_y u\|_{H^m_{co}}^2 \leq \|\partial_y u\|_{H^{m-1}_{co}}^2 + \|\theta\|_{H^m_{co}}^2 + \|u\|_{H^m_{co}}^2 + \|\nabla p\|_{H^{m-1}_{co}} \|u\|_{H^m_{co}} + \epsilon^{2K-2m},
\]

(3.18)

where \(c_0\) is some positive constant independent of \(\epsilon\).

**Proof.** Acting \(Z^\beta\) with \(|\beta| \leq m\) on the equations (3.1a) and (3.1b), we obtain that

\[
\begin{align*}
\partial_t Z^\beta u + u \cdot \nabla Z^\beta u_a + u_a \cdot \nabla Z^\beta u + \nabla Z^\beta p - \epsilon^2 \partial_y^2 Z^\beta u &= Z^\beta \theta e_2 + C_u + \epsilon^K Z^\beta R_u, \\
\partial_t Z^\beta \theta + u \cdot \nabla Z^\beta \theta_a + u_a \cdot \nabla Z^\beta \theta = C_\theta + \epsilon^K Z^\beta R_\theta,
\end{align*}
\]

(3.19) (3.20)

where the commutators \(C_u, C_\theta\) are defined as following

\[
C_u = -[Z^\beta, u \cdot \nabla]u_a - [Z^\beta, u_a \cdot \nabla]u - [Z^\beta, \nabla]p + \epsilon^2[Z^\beta, \partial_y^2]u \quad \text{:=} \quad C_1 + C_2 + C_3 + C_4,
\]

\[
C_\theta = -[Z^\beta, u \cdot \nabla]u_a - [Z^\beta, u_a \cdot \nabla]u \quad \text{:=} \quad C_5 + C_6.
\]

Now we calculate \((3.19) \times Z^\beta u + (3.20) \times Z^\beta \theta\) and then integrate the resulting equality over \(\Omega\) to get

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega (|Z^\beta u|^2 + |Z^\beta \theta|^2) + \epsilon^2 \int_\Omega |\partial_y Z^\beta u|^2 + \epsilon^2 \int_{\partial \Omega} \partial_y Z^\beta u_1 Z^\beta u_1 \\
= - \int_\Omega (u \cdot \nabla Z^\beta u_a Z^\beta u + u \cdot \nabla Z^\beta \theta_a Z^\beta \theta) - \int_\Omega \nabla Z^\beta p Z^\beta u + \int_\Omega Z^\beta \theta e_2 Z^\beta u \\
+ \int_\Omega C_u Z^\beta u + \int_\Omega C_\theta Z^\beta \theta + \epsilon^K \int_\Omega (Z^\beta R_u Z^\beta u + Z^\beta R_\theta Z^\beta \theta) := \sum_{j=1}^6 I_i.
\end{align*}
\]

We deduce from Navier boundary condition (1.2) that

\[
\int_{\partial \Omega} \partial_y Z^\beta u_1 Z^\beta u_1 = \int_{\partial \Omega} Z^\beta \partial_y u_1 Z^\beta u_1 + \int_{\partial \Omega} \partial_y [Z^\beta] u_1 Z^\beta u_1 \\
= \alpha \int_{\partial \Omega} |Z^\beta u_1|^2 = \alpha \int_{\partial \Omega} |\partial_y | u_1|^2 = \alpha \|u_1\|_{H^m_{co}(\partial \Omega)}^2,
\]

(3.21)

due to

\[
[\partial_y, Z_1] = 0,
\]

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\[ Z^\beta u_1|_{\partial \Omega} = 0, \text{ if } \beta_2 > 0. \]

Here we note that if the boundary \( \partial \Omega \) is not flat, \( Z_2 u|_{\partial \Omega} = 0 \) does not hold. Hence, by using trace theorem, we can derive that

\[ |\alpha| \int_{\partial \Omega} |Z^\beta u_1|^2 \leq C|\alpha||\partial_\alpha u_1||H^m_{\partial\Omega}||u_1||H^m_{\partial\Omega}. \]  

(3.22)

Then we estimate \( I_i \) \((1 \leq i \leq 6)\) one by one. For \( I_1 \), it holds that

\[ |I_1| \lesssim |\nabla Z^\beta u_a|_{L^\infty} \int_{\Omega} uZ^\beta u + |(Z^\beta \partial_x \theta_a, \partial_y Z^\beta \theta_a)|_{L^\infty} \int_{\Omega} (u_1 Z^\beta \theta + u_2 Z^\beta \theta) \]

\[ \lesssim (|Z^\beta \partial_y u_a|_{L^\infty} + |Z^\beta \partial_x u_a|_{L^\infty} + |(Z^\beta \partial_x \theta_a, \partial_y Z^\beta \theta_a)|_{L^\infty}) \int_{\Omega} (|u|^2 + |Z^\beta u|^2 + |Z^\beta \theta|^2) \]

\[ \lesssim \int_{\Omega} (|u|^2 + |Z^\beta u|^2 + |Z^\beta \theta|^2) \lesssim \|(u, \theta)\|^2_{H^m_{\Omega}}, \]  

(3.23)

due to the leading order term \((U^0_1, U^0_2)\) and \( \Theta^0 \) vanishing in the inner layer expansion. Note that

\[ I_2 = -\int_{\Omega} \nabla Z^\beta pZ^\beta u = \int_{\Omega} Z^\beta p \nabla \cdot Z^\beta u - \int_{\partial \Omega} Z^\beta pZ^\beta u \cdot n = -\int_{\Omega} Z^\beta p (|Z^\beta, \nabla| u), \]

and

\[ [Z_2, \nabla \cdot] u = -\varphi' \partial_y u_2 = \varphi' \partial_x u_1, \]

\[ [Z_1, \nabla \cdot] u = 0. \]

Hence we obtain

\[ |I_2| \lesssim \int_{\Omega} (|Z^\beta u| |Z^\beta p|) \lesssim \|u\|_{H^m_{\Omega}} \|\nabla p\|_{H^m_{\Omega}}. \]  

(3.24)

By virtue of the Cauchy inequality, we have

\[ |I_3| \lesssim \int_{\Omega} (|Z^\beta \theta|^2 + |Z^\beta u|^2) \lesssim \|u\|^2_{H^m_{\Omega}} + \|\theta\|^2_{H^m_{\Omega}}, \]  

(3.25)

\[ |I_6| \lesssim \int_{\Omega} (|Z^\beta \theta|^2 + |Z^\beta u|^2) + \varepsilon^2 K \lesssim \|u\|^2_{H^m_{\Omega}} + \|\theta\|^2_{H^m_{\Omega}} + \varepsilon^{2K - 2m}. \]  

(3.26)

As for the term \( I_4 \) containing commutators \( C_i \) \((1 \leq i \leq 4)\), we have

\[ C_1 = -[Z^\beta, u \cdot \nabla] u_a = - (Z^\beta (u \cdot \nabla u_a) - u \cdot \nabla Z^\beta u_a) \]

\[ = - \left( \sum_{\gamma + \zeta = \beta, \gamma \neq 0} c^{1}_{\gamma, \zeta} Z^\gamma u \cdot Z^\zeta \nabla u_a + u \cdot [Z^\beta, \nabla] u_a \right), \]  

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\[ C_2 = -[Z^\beta, u_a \cdot \nabla]u = -(Z^\beta(u_a \cdot \nabla u) - u_a \cdot \nabla Z^\beta u) \]
\[ = -\left(\sum_{\gamma + \zeta = \beta, \gamma \neq 0} c_{\gamma, \zeta}^1 Z^\gamma u_a \cdot Z^\zeta \nabla u + u_a \cdot [Z^\beta, \nabla]u\right), \]

where \( c_{\gamma, \zeta}^1, c_{\gamma, \zeta}^2 \) is, respectively, from the expansion of \( Z^\beta(u \cdot \nabla u) \) and \( Z^\beta(u_a \cdot \nabla u) \) by Leibniz formula. We obtain

\[
\int_{\Omega} C_1 Z^\beta u \lesssim \|u\|_{H^{m_0}},
\]
\[
\int_{\Omega} C_2 Z^\beta u \lesssim \|u\|_{H^{m_0}}^2 + \|\partial_y u\|_{H^{m_0-1}},
\]
due to

\[ |\nabla u_a|_{L^\infty} \leq C_a, \quad u_{a2}|_{\partial \Omega} = 0 \quad \text{(i.e. } \Gamma u_{a2} = 0), \]

and

\[ \|u_{a2}\partial_y Z^{\beta-1}u\| = \left\| \frac{u_{a2}}{\varphi(y)} \varphi(y) \partial_y Z^{\beta-1}u \right\| \lesssim |u_{a2}|_{W^{1,\infty}} \|Z^\beta u\|. \]

One can easily get

\[ \|C_3\| \lesssim \|Z^{\beta-1}\nabla p\| \lesssim \|\nabla p\|_{H^{m_0-1}}. \quad (3.27) \]

Together with

\[ [Z_1, \partial_y^2]u = 0, \quad [Z_2, \partial_y^2]u = -\varphi'' \partial_y u - 2\varphi' \partial_y^2 u, \quad Z_2 u|_{\partial \Omega} = 0, \]

we can derive the estimates for the most difficult case when

\[ C_4 = [Z_2^\beta, \partial_y^2]u = -\varepsilon^2 (\partial_y (\beta \varphi' \varphi^{\beta-1} \partial_y u) + \beta \varphi' \varphi^{\beta-1} \partial_y^2 u) \]

using integration by parts:

\[
\int_{\Omega} C_4 Z^\beta u - \varepsilon^2 \int_{\Omega} (\beta \varphi' Z^{\beta-1} \partial_y u)^2
\]
\[
= \varepsilon^2 \int_{\Omega} (\beta \varphi' Z^{\beta-1} \partial_y u) \cdot (Z^\beta \partial_y u) - \varepsilon^2 \int_{\Omega} (\beta \varphi' \varphi^{\beta-1} \partial_y^2 u) \cdot (Z^\beta u)
\]
\[
\lesssim \varepsilon^2 \|\partial_y u\|_{H^{m_0}} \|\partial_y u\|_{H^{m_0-1}}. \quad (3.28) \]
The other cases can be estimated similarly. Hence we can obtain the conormal estimates for $I_4$, using the Cauchy inequality,

$$|I_4| \lesssim \varepsilon^2 \|\partial_y u\|_{H^m_{co}} \|\partial_y u\|_{H^{m-1}_{co}} + \|\partial_y u\|^2_{H^m_{co}} + \|\nabla p\|_{H^{m-1}_{co}} \|u\|_{H^m_{co}}.$$  \hspace{1cm} (3.29)

Similarly, one can obtain that

$$|I_5| \lesssim \|\theta\|^2_{H^m_{co}} + \|u\|^2_{H^m_{co}}.$$  \hspace{1cm} (3.30)

Now one can deduce (3.18) from (3.21), (3.23), (3.24), (3.25), (3.26), (3.29), (3.30) and the Young inequality. The proof of Lemma 3.6 is completed. \hfill \Box

**Normal derivatives estimates.**

In view of the conormal energy estimates, we shall give the estimates of the normal derivatives $\|\partial_y u\|_{H^{m-1}_{co}}$. Due to the divergence free condition, we obtain that

$$\|\partial_y u_2\|_{H^{m-1}_{co}} \lesssim \|u_1\|_{H^m_{co}} \lesssim \|u\|_{H^m_{co}}.$$  

Hence it remains to estimate the normal derivatives of the tangential velocity $\|\partial_y u_1\|_{H^{m-1}_{co}}$.

Recall that the vorticity $\omega$ is defined as

$$\omega = \text{curl } u = \partial_y u_1 - \partial_x u_2.$$  

For the approximate solution, we denote $\omega_a$ as

$$\omega_a = \text{curl } u_a = \partial_y u_a - \partial_x u_a.$$  

Then we can derive from the velocity equation in (3.1) that

$$\partial_t \omega + u \cdot \nabla \omega_a + u_a \cdot \nabla u - \varepsilon^2 \partial_y^2 \omega = -\partial_x \theta + \varepsilon K \text{curl } R_u,$$  \hspace{1cm} (3.31)

where $\text{curl } R_u = (\partial_y R_{u_1} - \partial_x R_{u_2})$.

For the new variables

$$\eta = \omega - \alpha u_1, \quad \eta_a = \omega_a - \alpha u_{a1},$$

we have the advantages on the boundary that

$$\eta|_{\partial \Omega} = 0, \quad \eta_a|_{\partial \Omega} = 0.$$  \hspace{1cm} (3.32)
Moreover, it holds that
\[ \| \partial_y u_1 \|_{H^{m-1}_{c_0}} \lesssim \| \eta \|_{H^{m-1}_{c_0}} + \| u \|_{H^m_{c_0}}. \]

Combining with the first equation of the velocity in (3.1), we rewrite the vorticity equation (3.31) into the following form
\[ \partial_t \eta + u \cdot \nabla \eta_a + u_a \cdot \nabla \eta - \varepsilon^2 \partial_y^2 \eta = \alpha \partial_x p - \partial_x \theta - \varepsilon K R_{u_1} + \varepsilon K \text{curl} R_u. \] (3.33)

Taking normal derivatives on (3.1b), one has
\[ \partial_t \partial_y \theta + u \cdot \nabla \partial_y \theta_a + u_a \cdot \nabla \partial_y \theta = C' + \varepsilon K \partial_y R_{\theta}, \] (3.34)
where \( C' = -[\partial_y, u \cdot \nabla] \theta_a - [\partial_y, u_a \cdot \nabla] \theta. \)

Then we have the following normal derivatives estimates.

**Lemma 3.7.** For every \( m \geq 1 \) and every smooth solution of the system (3.1) with partial viscosity, we have
\[
\frac{d}{dt} \left( \| \eta \|_{H^{m-1}_{c_0}}^2 + \| \partial_y \theta \|_{H^{m-1}_{c_0}}^2 \right) + c_0 \varepsilon^2 \| \partial_y \eta \|_{H^{m-1}_{c_0}}^2 \\
\lesssim \| \nabla p \|_{H^{m-1}_{c_0}} \| \eta \|_{H^{m-1}_{c_0}} + \| \eta \|_{H^{m-1}_{c_0}}^2 + \| u \|_{H^m_{c_0}}^2 + \| \partial_y \theta \|_{H^{m-1}_{c_0}}^2 + \varepsilon^{2K-2m} (3.35)
\]
for some \( c_0 > 0 \).

**Proof.** We take \( L^2 \) inner product on equation (3.33) with \( \eta \) to get
\[
\frac{d}{dt} \int_\Omega \eta^2 + \varepsilon^2 \int_\Omega |\partial_y \eta|^2 = \int_\Omega [-u \cdot \nabla \eta_a \eta + \alpha \partial_x p \eta + \partial_x \eta \partial_y \theta - \varepsilon K R_{u_1} \eta + \varepsilon K \text{curl} R_u \eta] := RHS,
\]
where \( RHS \) satisfies
\[
RHS \lesssim \int_\Omega \left[ |u_1 \partial_x \eta_a \eta| + \left| \frac{u_2}{\varphi(y)} \varphi(y) \partial_y \eta_a \eta \right| + |\partial_x p \eta| + |\partial_y \theta \eta| + \varepsilon K |R_{u_1} \eta| + \varepsilon K |\text{curl} R_u \eta| \right] \\
\lesssim \| u \|^2 + \| \nabla p \| \| \eta \| + \| \eta \|^2 + \| Z(u, \theta) \|^2 + \varepsilon^{2K-2},
\]
with the aid of the Cauchy inequality and Hardy inequality. Then we have
\[
\frac{d}{dt} \| \eta \|^2 + \varepsilon^2 \| \partial_y \eta \|^2 \lesssim \| u \|_{H^m_{c_0}}^2 + \| \nabla p \| \| \eta \| + \| \eta \|^2 + \| Z \theta \|^2 + \varepsilon^{2K-2}. \] (3.36)

Similarly, we process \( L^2 \) inner product on the equation (3.34) with \( \partial_y \theta \) to get
\[
\frac{d}{dt} \| \partial_y \theta \|^2 \lesssim \| \eta, \partial_y \theta \|^2 + \| Z u \|^2 + \varepsilon^{2K-2}, \] (3.37)
using the Cauchy inequality and the Hardy inequality.

We take $H^{k-1}_{co}$—inner product on the equations (3.33) (3.34) with $\eta$ and $\partial_y \theta$, respectively. Then the desired results (3.35) can be verified by induction on $k \geq 1$. Here we omit the detail for simplicity. □

**Pressure estimates.**

In this part, we give the conormal estimates of the pressure. In view of the velocity equations in the partially viscous system (3.1a), we first consider the following system

\[
\begin{align*}
\partial_t u - \varepsilon^2 \partial_y^2 u + \nabla p &= F, \\ y > 0, \\
\nabla \cdot u &= 0, \\ y > 0, \\
u_2 &= 0, \\ \partial_y u_1 &= \alpha u_1, \\ y = 0.
\end{align*}
\]  

where $F$ represents the given source term. Here $F$ reads

\[
F = -(u_a \cdot \nabla u + u \cdot \nabla u_a) + \theta \varepsilon^2 + \varepsilon^K R_u
\]

in the linear system.

By taking divergence on the equations (3.38), using divergence free condition, we obtain the following elliptic equation of pressure

\[
\Delta p = \nabla \cdot F, \quad y > 0.
\]

One can deduce from (3.39) (3.40) that, the boundary condition for (3.41),

\[
\partial_y p(x, 0) = \varepsilon^2 \partial_y u_2(x, 0) - \partial_t u_2(x, 0) + F_2(x, 0) \\
= -\varepsilon^2 \partial_y \partial_x u_1(x, 0) + F_2(x, 0) \\
= -\varepsilon^2 \alpha \partial_x u_1(x, 0) + F_2(x, 0).
\]

We can express $p$ as following

\[
p = p_1 + p_2,
\]

where $p_1$ satisfies

\[
\Delta p_1 = \nabla \cdot F, \quad y > 0, \\ \partial_y p_1(x, 0) = F_2(x, 0),
\]

(3.42)
and \( p_2 \) solves

\[
\Delta p_2 = 0, \quad y > 0, \quad \partial_y p_2(x, 0) = -\alpha \varepsilon^2 \partial_x u_1(x, 0).
\]

(3.43)

Then the desired estimates for the pressure \( p \) can be achieved from the estimates of \( p_1 \) and \( p_2 \) by standard elliptic theory. We state the estimate of \( p \) in the following proposition.

**Proposition 3.8.** Considering the partially viscous system (3.38)-(3.40), for every \( m \geq 2 \), there exists \( C > 0 \) such that for every \( t \geq 0 \), we have the estimates

\[
\| \nabla p \|_{H^{m-1}_{\infty}} \leq C(\| F \|_{H^{m-1}_{\infty}} + \| \nabla \cdot F \|_{H^{m-2}_{\infty}} + \varepsilon^2 \| \nabla u \|_{H^{m-1}_{\infty}} + \| u \|_{H^{m}_{\infty}}).
\]

(3.44)

Taking the source term \( F \) in (3.44) as

\[
F = -(u_n \cdot \nabla u + u \cdot \nabla u_n) + \theta e_2 + \varepsilon K R_u,
\]

we can reach the desired estimates of the pressure \( p \) in the following lemma.

**Lemma 3.9.** For every \( m \geq 2 \), and every \( \varepsilon \in (0, 1] \), assume \((u, \theta)\) be a smooth solution of (3.1) on \([0, T]\). Then it holds that

\[
\| \nabla p \|_{H^{m-1}_{\infty}} \lesssim (1 + \varepsilon^2) \| u \|_{H^{m}_{\infty}} + \| (u, \theta) \|_{H^{m-1}_{\infty}} + \| \partial_y u \|_{H^{m-1}_{\infty}} + \| \partial_y \theta \|_{H^{m-2}_{\infty}} + \varepsilon^{K-m+1}.
\]

(3.45)

**Proof of the Theorem 3.3.** Combining the estimates (3.16) (3.18) (3.35) (3.45), we can obtain

\[
\frac{d}{dt} \left( \| (u, \theta) \|_{H^{m}_{\infty}}^2 + \| (\eta, \partial_y \theta) \|_{H^{m-1}_{\infty}}^2 \right) + c_0 \varepsilon^2 \| (\partial_y u, \partial_y \eta) \|_{H^{m-1}_{\infty}}^2 \lesssim \| (u, \theta) \|_{H^{m}_{\infty}}^2 + \| (\eta, \partial_y \theta) \|_{H^{m-1}_{\infty}}^2 + \varepsilon^{2K-2m}.
\]

(3.46)

Then (3.12) follows from (3.46) and the Gronwall inequality. Consequently, we have (3.13) by using (3.12) and the Lemma 3.1. The proof the Theorem 3.3 is completed.

\[ \square \]

**4. Nonlinear Stability Estimates**

In this section, we prove the nonlinear stability of the approximate solutions constructed in the Section 2 under the form (2.1). The zero-viscosity limit from the partially viscous system (1.1) to the inviscid system (1.3) will be verified with the detailed convergence rates.
In contrast with the linear stability estimates in the Section 3, we shall deal with the nonlinear terms $[Z^\beta, \mathbf{u} \cdot \nabla]\mathbf{u}$, $[Z^\beta, \mathbf{u} \cdot \nabla]\theta$, i.e.

$$
\sum_{\gamma+\zeta=\beta, \gamma \neq 0} c_{\gamma,\zeta} Z^\gamma \mathbf{u} \cdot Z^\zeta \nabla \mathbf{u} + \mathbf{u} \cdot [Z^\beta, \nabla]\mathbf{u}, \quad \sum_{\gamma+\zeta=\beta, \gamma \neq 0} d_{\gamma,\zeta} Z^\gamma \mathbf{u} \cdot Z^\zeta \nabla \theta + \mathbf{u} \cdot [Z^\beta, \nabla]\theta
$$

in the conormal energy estimates with $|\beta| \leq m$; and $[Z^\beta, \mathbf{u} \cdot \nabla]\eta$, $[Z^\beta, \mathbf{u} \cdot \nabla]\partial_y \theta$, i.e.

$$
\sum_{\gamma+\zeta=\beta, \gamma \neq 0} \tilde{c}_{\gamma,\zeta} Z^\gamma \mathbf{u} \cdot Z^\zeta \nabla \eta + \mathbf{u} \cdot [Z^\beta, \nabla]\eta, \quad \sum_{\gamma+\zeta=\beta, \gamma \neq 0} \tilde{d}_{\gamma,\zeta} Z^\gamma \mathbf{u} \cdot Z^\zeta \nabla \partial_y \theta + \mathbf{u} \cdot [Z^\beta, \nabla]\partial_y \theta
$$

in the normal derivatives estimates with $|\beta| \leq m - 1$, respectively. Here $c_{\gamma,\zeta}, d_{\gamma,\zeta}, \tilde{c}_{\gamma,\zeta}, \tilde{d}_{\gamma,\zeta}$ are positive constants depending on $\gamma$ and $\zeta$. The pressure estimates should involve the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ in the source term $F$. In the nonlinear analysis, the estimates of $\|\eta\|_{1,\infty}$ and $\|\partial_y \theta\|_{1,\infty}$ are crucial to the closure of energy estimates. We deal with the term $\|\eta\|_{1,\infty}$ in the spirit of the methods in [47] (using Maximum principle for transport-diffusion equation and precise estimates for the Green’s function of the operator for an approximate equation). As for $\|\partial_y \theta\|_{1,\infty}$, we take advantage of a maximum principle for a particular advective scalar equation (cf. [17, 50]).

Let us define

$$
E_m(t) = \|(u, \theta)\|^2_{H^m_{\text{co}}} + \|(\eta, \partial_y \theta)\|^2_{H^{m-1}_{\text{co}}} + \|(\eta, \partial_y \theta)\|^2_{1,\infty}.
$$

The proof of the Theorem 1.1 can be reduced to the justification of the following proposition, due to the Lemma 3.1. To close the nonlinear estimates, the minimum expansion order is $K = 6$.

**Proposition 4.1.** Assume $(u, \theta)$ be a solution of (2.9) (3.3) defined on $[0, T]$ with $T$ independent of $\varepsilon$. For $m \geq 5$, $K > m$, we have the a priori estimate

$$
E_m(t) \lesssim E_m(0) + (1 + t) \int_0^t (E_m^2(s) + E_m(s)) ds + \varepsilon^{2K-2m}.
$$

(4.1)

Note that

$$
\|u \cdot [Z^\beta, \nabla]\mathbf{u}\| \lesssim \sum_{|\gamma| \leq m-1} \|u_2 \partial_y Z^\gamma \mathbf{u}\| = \sum_{|\gamma| \leq m-1} \|\frac{u_2}{\varphi(y)} \varphi(y) \partial_y Z^\gamma \mathbf{u}\| \lesssim \|u_2\|_{W^{1,\infty}} \|\mathbf{u}\|_{H^{2m}_{\text{co}}}
$$

for $|\beta| \leq m$, due to $u_2|_{y=0} = 0$ and Hardy inequality. Similarly, we have

$$
\|u \cdot [Z^\beta, \nabla]\theta\| \lesssim \|u_2\|_{W^{1,\infty}} \|\theta\|_{H^{2m}_{\text{co}}}.
$$
By virtue of the fact that
\[ \|Z^{\beta_1}uZ^{\beta_2}v\| \lesssim \|u\|_{L^\infty}\|v\|_{H^k_{co}} + \|v\|_{L^\infty}\|u\|_{H^k_{co}}, \quad |\beta_1| + |\beta_2| = k, \] (4.2)
one can deduce that
\[ \|c_{\gamma,\zeta}Z^{\gamma}u \cdot Z^{\zeta}\nabla u\| \lesssim \|\nabla u\|_{L^\infty}(\|u\|_{H^m_{co}} + \|\partial_y u\|_{H^{m-1}_{co}}), \] (4.3)
\[ \|d_{\gamma,\zeta}Z^{\gamma}u \cdot Z^{\zeta}\nabla \theta\| \lesssim \|\nabla u\|_{L^\infty}(\|\theta\|_{H^m_{co}} + \|\partial_y \theta\|_{H^{m-1}_{co}}) + \|\nabla \theta\|_{L^\infty}\|u\|_{H^m_{co}}. \] (4.4)

Hence we can obtain the conormal energy estimates for the nonlinear system
\[
\begin{aligned}
\frac{d}{dt}(\|(u, \theta)\|^2_{H^m_{co}}) + c_0\varepsilon^2\|\partial_y u\|^2_{H^m_{co}} \\
\lesssim (1 + \|u\|_{W^{1,\infty}})(\|\theta, u\|^2_{H^m_{co}} + \|\partial_y (u, \theta)\|^2_{H^{m-1}_{co}}) + \|\nabla p\|_{H^{m-1}_{co}}\|u\|_{H^m_{co}} \\
+ \|\nabla \theta\|_{L^\infty}\|u\|^2_{H^m_{co}} + \varepsilon^2K.
\end{aligned}
\] (4.5)

In the estimates of normal derivatives \(\|\eta, \partial_y \eta\|_{H^{m-1}_{co}},\) to avoid the appearance of the terms like \(\|\partial_y \eta\|_{L^\infty}\) and \(\|\partial_y \eta\|_{H^m_{co}},\) we write, for \(\zeta + \gamma = \beta, \gamma \neq 0, |\beta| \leq m - 1,\)
\[
\begin{aligned}
\|c_{\gamma,\zeta}Z^{\gamma}u \cdot Z^{\zeta}\nabla \eta\| \\
\lesssim \|Z^{\gamma}u_1Z^{\zeta}\partial_x \eta\| + \|Z^{\gamma}u_2Z^{\zeta}\partial_y \eta\| \\
\lesssim \|\nabla u_1\|_{L^\infty}\|\eta\|_{H^{m-1}_{co}} + \|\eta\|_{L^\infty}\|Zu_1\|_{H^{m-1}_{co}} + \left\|\frac{1}{\varphi(y)}Z^{\gamma}u_2\varphi(y)Z^{\zeta}\partial_y \eta\right\| \\
\lesssim \|u\|_{W^{1,\infty}}(\|\eta\|_{H^{m-1}_{co}} + \|u\|_{H^m_{co}}) + (\|u\|_{2,\infty} + \|\eta\|_{L^\infty})(\|\eta\|_{H^{m-1}_{co}} + \|u\|_{H^m_{co}}) \\
\lesssim (\|u\|_{W^{1,\infty}} + \|u\|_{2,\infty} + \|\eta\|_{L^\infty})(\|\eta\|_{H^{m-1}_{co}} + \|u\|_{H^m_{co}}),
\end{aligned}
\]
where we have used (4.2) and divergence free condition. Indeed, for the estimate of \(\|Z^{\gamma}u_2Z^{\zeta}\partial_y \eta\|,\) we use the expansion of the form
\[
\frac{1}{\varphi(y)}Z^{\gamma}u_2\varphi(y)Z^{\zeta}\partial_y \eta = c_{\gamma,\zeta}Z^{\tilde{\gamma}}\left(\frac{1}{\varphi(y)}u_2\right)Z^{\tilde{\zeta}}(\varphi \partial_y \eta), \] (4.6)
where \(|\tilde{\gamma} + \tilde{\zeta}| \leq m - 1, |\tilde{\gamma}| \neq m - 1\) and \(c_{\gamma,\zeta}\) is some smooth bounded coefficient (For the justification of this expansion (4.6), one can refer to [47] for more details). Then, with the aid of (4.2) and Hardy inequality, we have
\[
\begin{aligned}
\left\|Z^{\tilde{\gamma}}\left(\frac{1}{\varphi(y)}u_2\right)Z^{\tilde{\zeta}}(\varphi \partial_y \eta)\right\| \lesssim \left\|Z^{\tilde{\gamma}}(\partial_y u_2)Z^{\tilde{\zeta}}(\varphi \partial_y \eta)\right\| \\
\lesssim \|\partial_y u_1\|_{H^{m-2}} + \|\partial_x u_1\|_{L^\infty}\|\eta\|_{H^{m-2}}.
\end{aligned}
\]
Together with
\[ m | \leq \| \nabla d \|_1 + \int_0^t (1 + \\cdots) \\cdots, \]

it follows that
\[ \| u \|_{W^{1,\infty}} \| \eta \|_{H^{m-1}_{00}} + \| Z \eta \|_{L^\infty} \| (\| u \|_{H^{m-1}_{00}} + \| \eta \|_{H^{m-1}_{00}} + \| u \|_{H^{m-1}_{00}} + \| Z \eta \|_{L^\infty}) \| (\| u \|_{H^{m-1}_{00}} + \| \eta \|_{H^{m-1}_{00}} + \| u \|_{H^{m-1}_{00}}) \]

for \(|\beta| \leq m - 1\), it follows that
\[
\frac{d}{dt} \left( \| u \|_{H^{m-1}_{00}}^{2} + \| \nabla u \|_{H^{m-1}_{00}}^{2} \right) + c_0 \varepsilon^2 \| \nabla u \|_{H^{m-1}_{00}}^{2} \\
\lesssim \| \nabla p \|_{H^{m-1}_{00}}^{2} + \| \nabla u \|_{H^{m-1}_{00}}^{2} + (1 + \| u \|_{W^{1,\infty}} + \| u \|_{H^{m-1}_{00}} + \| \eta \|_{H^{m-1}_{00}} + \| u \|_{H^{m-1}_{00}} + \| Z \eta \|_{L^\infty}) \| (\| u \|_{H^{m-1}_{00}} + \| \eta \|_{H^{m-1}_{00}} + \| u \|_{H^{m-1}_{00}}) \]

\[ + \| \nabla \eta \|_{1,\infty} \| u \|_{H^{m-1}_{00}}^{2} + (1 + \| u \|_{W^{1,\infty}} + \| u \|_{H^{m-1}_{00}}) \| \nabla \eta \|_{H^{m-1}_{00}}^{2} + \varepsilon^{2K-2}. \quad (4.8) \]

In the pressure estimates for the nonlinear system (2.9), we consider the source term as
\[ F = - (u_a \cdot \nabla + u \cdot \nabla u_a) + u \cdot \nabla u + \theta \varepsilon_2 + \varepsilon^K R_u. \]

Then it follows from (3.44) that
\[ \| \nabla p \|_{H^{m-1}_{00}} \lesssim (1 + \| u \|_{W^{1,\infty}}) \left( \| u \|_{H^{m-1}_{00}} + \| \nabla u \|_{H^{m-1}_{00}} \right) + \| (u, \theta) \|_{H^{m-1}_{00}} + \| \nabla \theta \|_{H^{m-1}_{00}} + \varepsilon^{K-1}, \quad (4.9) \]
due to
\[ \| u \cdot \nabla u \|_{H^{m-1}_{00}} \lesssim \| u \|_{W^{1,\infty}} \left( \| u \|_{H^{m-1}_{00}} + \| \nabla u \|_{H^{m-1}_{00}} \right), \]
\[ \| \nabla u \cdot \nabla u \|_{H^{m-1}_{00}} \lesssim \| \nabla u \|_{L^\infty} \| \nabla u \|_{H^{m-1}_{00}}. \]

Hence we can deduce from (4.5) (4.8) (4.9) that
\[
\frac{d}{dt} \left( \| (u, \theta) \|^{2}_{H^{m}_{00}} + \| (\eta, \nabla \theta) \|^{2}_{H^{m-1}_{00}} \right) + c_0 \varepsilon^2 \left( \| \nabla \eta \|_{H^{m}_{00}}^{2} + \| \nabla \eta \|_{H^{m-1}_{00}}^{2} \right) \\
\lesssim (1 + \| u \|_{W^{1,\infty}} + \| u \|_{2,\infty} + \| Z \eta \|_{L^\infty}) \left( \| (\theta, \eta) \|_{H^{m}_{00}}^{2} + \| (\theta, \eta) \|_{H^{m-1}_{00}}^{2} \right) + \| \nabla \theta \|_{L^\infty} + \| \nabla \theta \|_{1,\infty} \left( \| u \|_{H^{m}_{00}} + \varepsilon^{2K-2} \right). \quad (4.10) \]

\[ L^\infty \text{ estimates.} \]
It follows from (3.9) (3.10) (3.11) in the Lemma 3.2 and (3.7) that
\[
\|u\|_{W^{1,\infty}} \lesssim E_m^{1/2}(t), \quad \|u\|_{2,\infty} \lesssim E_m^{1/2}(t), \quad \|\nabla u\|_{1,\infty} \lesssim E_m^{1/2}(t), \quad m \geq m_0 + 3, \tag{4.11}
\]
\[
\|\theta\|_{1,\infty} \lesssim \|\theta\|_{m} + \|\partial_y \theta\|_{m-1} \lesssim E_m^{1/2}(t), \quad \|\theta\|_{2,\infty} \lesssim E_m^{1/2}(t), \quad m \geq m_0 + 3, \tag{4.12}
\]
for \(m_0 \geq 1\).

Combining the inequalities (4.10) (4.11) and (4.12), we still need to estimate
\[
\|\eta\|_{1,\infty}, \quad \|\partial_y \theta\|_{1,\infty}
\]
to close the energy estimates. Similar to the Proposition 13 in [47] on the estimates of \(\|\eta\|_{1,\infty}\), we can derive the following results:

**Lemma 4.2.** For \(m \geq 5\), we have the estimate
\[
\|\eta\|_{1,\infty}^2 \lesssim E_m(0) + (1 + t) \int_0^t (E_m^2(s) + E_m(s)) ds + \varepsilon^{2K-2m}. \tag{4.13}
\]

**Proof.** Recall that \(\eta\) satisfies the following nonlinear equations
\[
\partial_t \eta + (u_a + u) \cdot \nabla \eta - \varepsilon^2 \partial_y^2 \eta = \mathcal{R} := -u \cdot \nabla \eta_a + \alpha \partial_x p - \partial_x \theta - \varepsilon K R_{u_1} + \varepsilon^K \text{curl} R_u,
\]
\[
\eta = 0 \quad \text{on} \quad \{y = 0\}.
\]

Here \(\eta\) solves a transport-diffusion equation and satisfies a homogenous Dirichlet boundary condition. The estimates of \(\|\eta\|_{L^\infty}\) and \(\|Z_1 \eta\|_{L^\infty}\) can be derived by maximal principle for transport-diffusion equation. The estimate of \(\|Z_2 \eta\|_{L^\infty}\) would be the most difficult one, due to the commutators between \(Z_3\) and \(\partial_y^2\). Inspired by the idea in [47], one can similarly use the estimates on the Green’s function of an approximate equation for the above equation near the boundary, i.e.
\[
\partial_t \eta + (u_{1a}(t, x, 0) + u_1(t, x, 0)) \partial_x \eta + y \partial_y (u_{2a}(t, x, 0) + u_2(t, x, 0)) \partial_y \eta - \varepsilon^2 \partial_y^2 \eta
\]
\[
= \mathcal{R} - G, \tag{*}
\]
where \(G = [u_{1a} + u_1 - u_{1a}(1, x, 0) - u_1(t, x, 0)] \partial_x \eta + [u_{2a} + u_2 - y \partial_y (u_{2a}(t, x, 0) - u_2(t, x, 0)] \partial_y \eta.
\]
Notice that the solution of the equation (*) can be expressed by the generator \(S(t, \tau)\) of the operator in the left hand side of (*) through Duhamel formula. Namely, for \(\forall t \geq \tau\),
\[
\eta(t) = S(t, \tau) \eta_0 + \int_0^t S(t, \tau) (\mathcal{R} - G)(\tau) d\tau.
\]
This, together with
\[ \|y \partial_y S(t, \tau) \eta_0\|_{L^\infty} \lesssim \|\eta_0\|_{L^\infty} + \|y \partial_y \eta_0\|_{L^\infty}, \]
gives the estimate of \( \|Z_2\eta\|_{L^\infty} \),
\[ \|Z_2\eta\|_{L^\infty} \leq \left( \|\eta_0\|_{1, \infty} + \int_0^t \|\tilde{\mathcal{R}} - G\|_{1, \infty} \right). \]
Then (4.13) can be deduced from a closed conormal argument for \( \|\tilde{\mathcal{R}} - G\|_{1, \infty} \lesssim E_m^2(t) + E_m(t), m \geq 5 \). Here we omit the details for simplicity.
□

Therefore, it remains to estimate \( \|\partial_y \theta\|_{1, \infty} \), which can be derived from the maximum principle for an advective scalar equation (cf. [17, 50]).

**Lemma 4.3.** For \( m \geq 5 \), we have the estimate
\[ \|\partial_y \theta\|_{1, \infty}^2 \lesssim E_m(0) + \int_0^t E_m(s) ds + \varepsilon^{2K-4}. \] (4.14)

**Proof.** We rewrite (2.9b) in the following form
\[ \partial_t \theta + (u + u_a) \cdot \nabla \theta = f(t, x, y) := -u \cdot \nabla \theta_a + \varepsilon^K R_\theta, \quad y > 0, \] (4.15)
where \( u, u_a \) satisfy
\[ \nabla \cdot u = 0, \quad \nabla \cdot u_a = 0, \quad u_2(t, x, 0) = 0, \quad u_{a2}(t, x, 0) = 0. \]
We transform the problem into the whole space by defining \( \tilde{\theta}, \tilde{\theta}_a, \tilde{u}, \tilde{u}_a, \tilde{f} : \)
\[ (\tilde{\theta}, \tilde{\theta}_a, \tilde{u}, \tilde{u}_a) = (\theta, \theta_a, u, u_a)(t, x, y), \quad y > 0; \]
\[ (\tilde{\theta}, \tilde{\theta}_a, \tilde{u}, \tilde{u}_a) = -(\theta, \theta_a, (-u_1, u_2), (-u_{1a}, u_{2a}))(t, x, -y), \quad y < 0; \]
\[ \tilde{f} = -\tilde{u} \cdot \nabla \tilde{\theta}_a + \varepsilon^K \tilde{R}_\theta. \]
Then one has
\[ \partial_t \tilde{\theta} + (\tilde{u} + \tilde{u}_a) \cdot \nabla \tilde{\theta} = \tilde{f}, \quad \tilde{\theta}_0 = \tilde{\theta}|_{t=0} \] (4.16)
with \( \nabla \cdot \tilde{u} = 0 \) and \( \nabla \cdot \tilde{u}_a = 0 \). Applying maximum principle for advective scalar equations (see [50], 3.2 A priori Bounds. and [13, 17]), we obtain that, for \( t \in [0, T] \),
\[ \|\tilde{\theta}\|_{L^\infty} \lesssim \|\tilde{\theta}|_{t=0}\|_{L^\infty} + \int_0^t \|\tilde{f}(\tau)\|_{L^\infty} d\tau. \]
\[ \lesssim \varepsilon^{K+1} \| \theta_0 \|_{L^\infty} + \int_0^t \| u(\tau) \|_{L^\infty} d\tau + \varepsilon^K. \] (4.17)

Similarly, we can derive the \( L^\infty \)-estimates of \( \nabla \tilde{\theta} \) and \( Z \partial_y \tilde{\theta} \),

\[ \| \nabla \tilde{\theta} \|_{L^\infty} \lesssim \varepsilon^{K+1} \| \nabla \theta_0 \|_{L^\infty} + \int_0^t (\| u(\tau) \|_{L^\infty} + \| \nabla u(\tau) \|_{L^\infty}) d\tau + \varepsilon^{K-1}, \] (4.18)

\[ \| Z \partial_y \tilde{\theta} \|_{L^\infty} \lesssim \varepsilon^{K+1} \| Z \partial_y \theta_0 \|_{L^\infty} + \int_0^t (\| u(\tau) \|_{W^{1,\infty}} + \| u \|_{1,\infty} + \| Z \partial_y u(\tau) \|_{L^\infty}) d\tau + \varepsilon^{K-2}, \] (4.19)

due to

\[ \| u_2 \partial_y \partial_y \theta_a \|_{L^\infty} \leq \left\| \frac{u_2}{\varphi(y)} \varphi(y) \partial_y \partial_y \theta_a \right\|_{L^\infty} \lesssim \| u_2 \|_{W^{1,\infty}}. \] (4.20)

Hence the proof of (4.14) is completed with the aid of Lemma 3.2. \( \square \)

Therefore, the proof of Proposition 4.1 can be obtained by combining (4.10) (4.13) (4.14) and the standard continuous argument.

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