Wave-packet trains of a time-dependent harmonic oscillator

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By using a test-function method, we construct \(n\) exact solutions of a quantum harmonic oscillator with a time-dependent "spring constant". Any \(n\)-th solution describes a wave-packet train consisting of \(n+1\) packets. Its center oscillates like the classically harmonic oscillator with variable frequency, and width and highness of each packet change simultaneously. When the deformation is small, it behaves like a soliton train, and the large deformation is identified with collapse and revival of the wave-packet train.

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Classical and quantum motions of the time-dependent harmonic oscillator are practically important, because of their physical realization in a Paul trap\textsuperscript{1},\textsuperscript{2}. For some periodic and time-dependent "spring constants", the classical motions are governed by the well-known Mathieu equation\textsuperscript{3}. Most of works for the system are of classical dynamics, such as the stability analysis\textsuperscript{4}, order and chaos\textsuperscript{5,6}, and so on. In the ultracold ion case, quantum motion becomes important for practical applications\textsuperscript{7,8}. For convenience' sake, many works based on quantum mechanics, say, the quantum computations\textsuperscript{11,12} and state preparations\textsuperscript{13,14}, were performed by adopting the pseudopotential model\textsuperscript{8} with constant trapped frequencies. There exist a few of works on the quantum motions in a Paul trap with periodically variable trapped frequencies. Cook and coworkers proximately treated the time-dependent quantum oscillator by perturbing about a harmonic oscillator solution for a time-independent, effective potential\textsuperscript{15}. Combescure, Brown and Feng\textsuperscript{16,17} gave similar exact solutions of the time-dependent system by employing different methods respectively. Their results showed that quantum-mechanical solutions of the system depend on the corresponding classical ones.

Very recently, Moya-Cessa and Guasti reported a coherent state solution of the time-dependent quantum harmonic oscillator\textsuperscript{19} that may be associated with summing Brown's \(n\) exact solutions. For a time-independent trapped frequency, we suggest a test-function method to get \(n\) exact solutions of the system\textsuperscript{20}. In the present paper, we shall adopt this method to find \(n\) exact solutions of the time-dependent quantum harmonic oscillator. These solutions describe \(n\) wave-packet trains that propagate and breathe simultaneously. By selecting initial conditions to fix centers of the trains, the result is reduced to Brown's one\textsuperscript{17}. And the ground state with \(n = 0\) may be related to Cessa's coherent state\textsuperscript{19}.

We consider a single ion confined in a Paul trap that provides an oscillating quadrupole potential with the resulting quantum motion governed by the time-dependent Schrödinger equation

\[
\frac{\partial \psi}{\partial t} = \frac{1}{2} \frac{d^2 \psi}{dx^2} + \frac{1}{2} k(t) x^2 \psi, \quad (1)
\]

where the "spring constant" is a periodic function of time, \(k(t) = U^2 + V \cos \omega t\) with \(\omega\) being the rf-driven frequency and \(U^2, V\) the trap parameters. We have adopted the unit with \(\hbar = 1\) and will normalize time in unit \(2/\omega\) so that \(k(t) = U^2 + V \cos 2t\). The spatial coordinate \(x\) and probability density \(|\psi|^2\) are normalized by the harmonic oscillator length \(l_h = \sqrt{\hbar/(m\omega)}\) and inverse one \(l_h^{-1}\) respectively. Consider the test function as a solution of Eq. (1) in the form

\[
\psi_n = a_n(t) H_n(\xi) \exp\left[\frac{b(t)x^2 - f(t)}{2}\right], \quad \xi = e(t)x - f(t). \quad (2)
\]

Here \(a_n(t), b(t), c(t)\) are the complex functions of time and \(e(t), f(t)\) the real functions. Applying Eq. (2) to Eq. (1), we arrive at the equation

\[
e^{2i}\frac{\partial^2 H_n}{\partial \xi^2} + 2(be - i f + i e x - 2cex) \frac{\partial H_n}{\partial \xi} + 2\left[ i \frac{\partial}{\partial a_n} - if \frac{b^2}{2} - c + (ib - 2bc)e \right] x + \left( 2c^2 - i c - \frac{1}{2} k(t) \right) x^2) H_n = 0. \quad (3)
\]

Noticing the Hermitian equation \(\partial^2 H_n/\partial \xi^2 - 2\xi \partial H_n/\partial \xi + 2nH_n = 0\), Eq. (3) implies

\[
i e = 2c^2 - k(t)/2, \quad i b = 2bc, \quad i e = 2c^2 - e^3, \quad i f = be - e^2 f, \quad i a_n/a_n = if - b^2/2 + c + ne^2. \quad (4)
\]
The first of Eq. (4) is a complex Riccati equation, which can be changed into a complex equation of a classical harmonic oscillator
\[
\ddot{c} = -k(t)c = -(U^2 + V \cos 2t)c,
\]
through the function transformation \( c = \varphi / (2i\varphi) \). This equation is just the Mathieu’s one whose exact solution can be taken as the form of infinite trigonometrical series
\[
\varphi_1 = A \cos(Ut + \alpha) + CV^{-1} \left( \sin Ut \int_0^t \cos Ut \cos 2t \varphi_1 dt - \cos Ut \int_0^t \sin Ut \cos 2t \varphi_1 dt \right),
\]
\[
\varphi_2 = B \cos(Ut + \beta) + CV^{-1} \left( \sin Ut \int_0^t \cos Ut \cos 2t \varphi_2 dt - \cos Ut \int_0^t \sin Ut \cos 2t \varphi_2 dt \right),
\]
where \( A, B, \alpha \) and \( \beta \) are some real constants associated with initial conditions of the classical oscillator. In quantum-mechanical treatment, there exist many possibilities on these constants that correspond to different states \( \psi \). The two equations in Eq. (6) can be directly proved by inserting them into Eq. (5). Applying Eq. (6), we construct the solution of Eq. (5) as
\[
\varphi = \varphi_1 + i\varphi_2 = \rho(t)e^{i\theta(t)},
\]
\[
\rho(t) = \sqrt{\varphi_1^2 + \varphi_2^2}, \quad \theta(t) = \arctan \frac{\varphi_2}{\varphi_1}.
\]
The real functions \( \rho(t) \) and \( \theta(t) \) will be used as the known functions in follows. So far, most of experimental works and theoretical analyses on the Paul-trapped ions are focused on the first stability region \([22]\), where the trap parameters \( U^2 \) and \( V \) are small, \( U^2 < 1, V < 1 \) and \( V \sim U^2 < U \). In such case, we can treat the terms proportional to \( V \) of exact solution (12) for different quantum number \( n \) as perturbations and adopt the iteration method to produce the perturbed solution.

Returning to the transformation between \( \varphi \) and \( c \) yields
\[
c = \frac{\varphi}{2i\varphi} = \frac{1}{2} \dot{\theta} - i \frac{\dot{\varphi}}{2 \varphi},
\]
Substitution of Eq. (7) into Eq. (5) yields equations of the phase \( \theta \) and module \( \rho \) as
\[
\ddot{\theta} = -2\dot{\varphi} / \rho, \quad \ddot{\varphi} = \dot{\rho}^2 - k(t) \rho
\]
with the first integration
\[
c_0 = \rho^2 \ddot{\theta} = \varphi_1 \ddot{\varphi}_2 - \varphi_2 \ddot{\varphi}_1.
\]
Combining Eqs. (8) with Eq. (4) and applying the relation (10), we easily obtain
\[
b = b_0 \exp \left( \frac{-i\theta}{\rho} \right), \quad e = \sqrt{\frac{c_0}{\rho}} = \sqrt{\theta}, \quad f = \frac{b_0}{\sqrt{c_0}} \cos \theta,
\]
\[
a_n = \frac{A_0}{\sqrt{\rho}} \exp \left\{ -i \left( \frac{1}{2} + n \right) \theta - \frac{b_0^2}{4c_0} \sin 2\theta \right\},
\]
where \( b_0 \) and \( A_0 \) are arbitrary constants. Inserting these into Eq. (2) leads to the exact solution \( \psi_n(x, t) \), and the normalization condition \( \int |\psi_n|^2 dx = A_0^2 \sqrt{\rho^2} \int H_n^2(\xi) \exp(-\xi^2) d\xi = A_0^2 \sqrt{\rho^2} \sqrt{2\pi} n! \) gives the constant \( A_0 = \left[ \sqrt{\rho^2} / \sqrt{2\pi} n! \right]^{1/2} \) such that from Eqs. (2), (8) and (11) we have the normalized wavefunction
\[
\psi_n = R_n(x, t) \exp[i\Theta_n(x, t)], \quad n = 0, 1, 2, \ldots,
\]
\[
R_n = \left[ \frac{\sqrt{c_0}}{\sqrt{\pi^2 n! \rho(t)}} \right]^{1/2} H_n(\xi) \exp \left( -\frac{1}{2} \xi^2 \right), \quad \xi = \frac{\sqrt{c_0} x}{\rho(t)} - \frac{b_0}{\sqrt{c_0}} \cos \theta(t),
\]
\[
\Theta_n = \frac{\dot{\rho}^2 x^2}{2 \rho(t)} - \frac{b_0 x}{\rho(t)} \sin \theta(t) + \frac{b_0^2}{4c_0} \sin[2\theta(t)] - \left( \frac{1}{2} + n \right) \theta(t).
\]
The module \( R_n \) of exact solution (12) for different quantum number \( n \) describes the wave-packet trains con-
sisting of \( n+1 \) packets. Orbit of center of the wave-packet trains \( x_c(t) \) is given by \( \xi = 0 \). Applying Eq. (7), from \( \xi = 0 \) we have the orbit

\[
x_c = \frac{b_0}{c_0} \rho(t) \cos(\theta(t) = \frac{b_0}{c_0} \varphi_1,
\]

which is proportional to real part of the complex solution (7), namely the orbit of a time-dependent classical oscillator. Comparing our Eq. (12) with Eq. (24) of Brown’s paper \[15\], we find that if one let the arbitrary constant \( b_0 \) be zero, our solution agrees with Brown one. The orbit equation (13) indicates difference between behaviors of the two solution: our wave-packet trains oscillate their centers, but the centers of Brown’s wave-packets are rested.

The function \( \rho(t) \) appearing in \( \xi \) describes widths of the wave-packet trains and each packet. For any train the average width of the packets is \( \rho(t)/\sqrt{\alpha} \). Same function appearing in radical of Eq. (12) governs highnesses of every packets. So we call \( \rho(t) \) the function of width and highness. When the changes of the widths and highnesses are small, behavior of the wave-packets seems to be that of the soliton trains. The greatly variable widths and highnesses show collapse and revival of the wave-packet trains, like behaviors of multiple breathers. The normalization condition implies that the broader wave-packet train is associated with smaller mean highness and the narrower wave-packet train corresponds to larger mean highness.

Phase of the exact solution (12) contains the term \((1/2 + n)\theta(t)\) that infers average energy \( E_n(t) = \langle \psi_n | i \partial / \partial t | \psi \rangle \) be proportional to \((1/2 + n)\). In the processes of motions, the wave-packet trains may spontaneously transit from states of higher average energies to that of lower ones. Some perturbations also could cause transitions between the states of different quantum numbers. The ground state \( \psi_0 \) of Eq. (12) may be similar to Cessa’s coherent state \[19\]. We shall numerically illustrate the analytical results for the system parameters \( U = 0.5, V = 0.05 \) as follows.

**The solitonlike trains** To show the solitonlike behavior, we require larger amplitude of Eq. (13) and keep small change of module \( \rho \) such that the wave-packet trains described by Eq. (12) can propagate through greater distance with small deformation. For simplicity, we take the parameter set \( \alpha = 0, \beta = -\pi/2, A = B = c_0 = 1, b_0 = -10 \) and make iteration from Eq. (6) only to the first order. That is, let \( \varphi_1^{(0)} \) and \( \varphi_2^{(0)} \) be \( \cos Ut \) and \( \sin Ut \) respectively, and use them instead of \( \varphi_1 \) and \( \varphi_2 \) in the corresponding integrations of Eq. (6), obtaining the approximate solutions

\[
\begin{align*}
\varphi_1 & \approx \cos 0.5t + 0.05 \sin 0.5t \sin 2t \\
& + 0.025 \cos 0.5t (1 - \cos 2t), \\
\varphi_2 & \approx \sin 0.5t + 0.025 \sin 0.5t (1 - \cos 2t).
\end{align*}
\]

Applying the first of Eq. (14) to Eq. (13) yields the orbit of wave-packets’ center. Obviously, the center of wave-packet train oscillates in the spatial region \(-10 \leq x_c \leq 10 \) with period \( 4\pi \). Combining Eq. (14) with Eq. (7), we obtain the function of width and highness \( \rho(t) \) whose time evolution is plotted as Fig. 1. In the considered case the module oscillates with very small amplitude that implies width and highness of the wave-packet train being changed also quite small.

Substituting Eq. (14) into Eq. (7) and further into Eq. (12) leads to a determined form of the module \( R_n(x,t) \). Using this form we numerically plot the probability density \( R_n^2 \) versus \( x \) for \( n = 8 \) at the times \( t = 0, \pi/2, 2\pi \) respectively as Fig. 2a, 2b, and 2c. These plots show that the solitonlike train propagates from \( x_c = -10 \) to \( x_c = 10 \) and approximately keeps its shape. In the next half period, \( 2\pi \leq t \leq 4\pi \), an inverse process will occur.

**Collapse and revival of the wave-packets** From Eq. (13) we know that when small constants \( A \) and \( b_0/c_0 \) are applied to Eq. (12), the wave-packet trains will approximately fix their centers. Further we let constant \( B \) be much greater than \( A \). Eqs. (6) and (7) could lead to large change of the function \( \rho(t) \) for \( \alpha = \beta = \pi/2 \). This will cause consequently large changes of the widths and highnesses of the wave-packet trains described by Eq. (12). The large deformations can be identified with the so-called collapse and revival of the wave-packets \[23\]. As an example, we take the parameter set \( \alpha = 0, \beta = -\pi/2, A = b_0 = 0.02, B = 10, c_0 = 1 \), from Eqs. (6), (7) and (13) to solve for \( \rho(t) \) and \( x(t) \). The center of the wave-packet train could oscillate with amplitude only in \( Ab_0 = 10^{-4} \) order. The function of width and highness \( \rho(t) \) periodically changes with amplitude 10 and minimum value 0.02. The large amplitude and small minimum value of \( \rho(t) \) mean periodical collapse and revival of the wave-packet train. Using these parameters, from Eq. (12) we can plot spatial evolution of the probability density with \( n = 4 \) for several different times as in Fig 3. This figure displays that the initially higher and narrower wave-packet train is collapsed to very low and width at \( t = \pi \) and revived to initial state at \( t = 2\pi \), as in Fig. 3b and 3c respectively.

In conclusion, we have investigated the quantum and classical motions of a time-dependent harmonic oscillator, which is associated with a Paul-trapped ion system. A new kind of exact solutions of the quantum-
mechanical Schrödinger equation is constructed by using the corresponding exact solutions of the classical-mechanical Mathieu equation, which describes propagations and deformations of the wave-packet trains. When the deformations are small, they behave like some soliton trains. The large deformations are identified with collapse and revival of the wave-packet trains. If we select the initial conditions to fix center of the wave-packet trains, the result is reduced to Brown’s quantum motion [17]. Our ground state solution with $n = 0$ seems to be Cessa’s coherent state [19]. The state $\psi_1$ of two wave-packets is similar to the Schrödinger’s cat state [13]. We desire the new exact solutions to play an important role in treating various harmonically confined systems, e.g. the Bose-Einstein condensate held in a magnetic well.

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