In this work we study the many-body localization (MBL) transition and relate it to the eigenstate structure in the Fock space. Besides the standard entanglement and multifractal probes, we introduce the radial probability distribution of eigenstate coefficients with respect to the Hamming distance in the Fock space and relate the cumulants of this distribution to the properties of the quasi-local integrals of motion in the MBL phase. We demonstrate non-self-averaging property of the many-body fractal dimension \( D_q \) and directly relate it to the jump of \( D_q \) as well as of the localization length of the integrals of motion at the MBL transition. We provide an example of the continuous many-body transition confirming the above relation via the self-averaging of \( D_q \) in the whole range of parameters. Introducing a simple toy-model, which hosts ergodic thermal bubbles, we give analytical evidences both in standard probes and in terms of newly introduced radial probability distribution that the MBL transition in the Fock space is consistent with the avalanche mechanism for delocalization, i.e., the Kosterlitz-Thouless scenario. Thus, we show that the MBL transition can been seen as a transition between ergodic states to non-ergodic extended states and put the upper bound for the disorder scaling for the genuine Anderson localization transition with respect to the non-interacting case.

I. INTRODUCTION

Understanding the emergence of ergodicity in closed quantum many-body systems is an active front of research [1–6]. Generic interacting systems are expected to thermalize under their own quantum dynamics. Nevertheless, thermalization may fail if the system is subjected to strongly quenched disorder, giving arise to a new phase of matter dubbed as many-body localized (MBL) [7–13].

The MBL phase is best understood in terms of an emergent form of integrability, which is characterized by the existence of an extensive set of quasi-local conserved quantities, which strongly hinder thermalization in the system [14–16]. As a consequence, the system has Poison level statistic, area-law entanglement, and the partial local structure of the initial state is maintained under the evolution [9, 10, 17]. Instead at weak disorder, the system is in an ergodic phase, meaning that Eigenstate Thermalization Hypothesis [1, 2, 4, 6] (ETH) holds, and therefore local observable thermalize. This implies that the system is fully described in terms of few macroscopic conserved quantities, i.e., energy and/or particles number.

A quantum phase transition, referred to as MBL transition [9], is believed to separate an ergodic phase from an MBL one. The MBL transition is a dynamical phase transition, meaning that it occurs at the level of individual eigenstates even at high energy density. In the last decade an enormous effort, both numerically and theoretically [7, 17–25], has been made to understand the nature of this transition. Nevertheless, only little is known about the MBL transition. Numerically, the critical exponents associated with a putative second-order type of the transition, are in disagreement with generic bounds [17, 26]. This incongruence could be due to the fact that the system sizes analyzed using exact diagonalization techniques
are too small to capture the true asymptotic behavior.

Very recently, several theoretical works have doubted the underline assumptions that the transition is of the second order. Phenomenological renormalization group studies suggest that the MBL transition could be of a Kosterlitz-Thouless (KT) type [27–33]. This possible scenario is properly predicted by a possible delocalization mechanism called avalanche theory, which takes into account non-perturbative effects possibly destabilizing the MBL phase [27, 28]. Strictly speaking, the theory states that the presence of thermal bubbles in the system due to unavoidable entropic arguments is enough to destabilize the MBL phase if the localization length $\xi_{loc}$ settled by the disorder strength exceeds a finite critical length. An immediate consequence of this mechanism is that the MBL is characterized not by the divergence of the correlation length, as one expects from the ordinary second-order transition, but by a finite jump of the inverse localization length across the transition [28, 30].

A complementary interesting perspective is to characterize MBL systems in the Fock space, see Fig. 1 (a) for a pictorial representation. This paradigm is based on the original idea of mapping a disordered quantum dot to a localization problem in the Fock space [34] which has been developed further recently [35]. This has been used to provide evidence of the existence of an MBL transition by Basko, Aleiner, and Altshuler in their seminal work [7]. Ergodicity is then defined through the fractal dimensions $D_q$, which quantify the spread of a state in the Fock space [36]. Ergodic states at infinite temperatures are believed to behave like random vectors [37, 38], therefore they are spread homogeneously over the entire Fock space and $D_q = 1$. Instead, non-ergodic states cover only a vanishing fraction of the Fock space, $0 \leq D_q < 1$. A genuine localization in the Fock space requires $D_q = 0$, though due to the many-body nature of the problem is never reached at finite disorder [17, 39–42]. Thus, the MBL transition can be seen as an ergodic to non-ergodic transition in the Fock space, with $D_q = 1$ in the ETH phase and $D_q < 1$ in the MBL phase.

In a recent work [40], the behavior of $D_q$ for a certain MBL model has been inspected using extensive numerical calculations. The MBL transition was found to be characterized by a jump in the fractal dimensions $D_q$ at the critical point. The aforementioned investigations lead to the indication of the existence of an MBL transition. However, a clear connection between the above two viewpoints is still missing.

In this work, we focus on the existence of the above jump in the fractal dimensions and on its connection to the avalanche theory, i.e., to the KT-type transition from another perspective. Based on the breakdown of self-averaging for $D_q$ at the transition and on the recently developed relation of $D_q$ to the entanglement entropy [43], we show that the MBL transition is consistent with the jump of $D_q$ from $D_q = 1$ to $D_q < 1/2$. In addition, we focus on the radial distribution of a many-body eigenstate in the Fock space around its maximum, relate it to the behavior of local integrals of motion [44], and, thus, confirm the consistency of the KT-scenario [28, 30] for the MBL transition, see Fig. 1 for an overall picture.

This paper is organized as follows. In Sec. II we introduce the model and the indicators that we inspected numerically. In particular, we study the inverse participation ratio of eigenstate coefficients in the Fock space, from which we extract the fractal dimensions and the radial probability of eigenstate coefficients. Section III represents the numerical results concerning the fractal dimensions and the radial probability distribution. In Sec. IV we show our analytical considerations which underlie the connection between the avalanche theory and the observed jump in the fractal dimensions.

In Sec. V, we provide an example of a non-interacting model with many-body filling, which is known to have a delocalization-localization transition and characterized by a diverging localization length at this transition. We show the main difference of this model from the MBL transition which is believed to have a discontinuity in the inverse localization length. Finally, we draw our conclusions and outlooks in Sec. VI.

II. MODEL AND METHODS

We study the random quantum Ising model [44] with the Hamiltonian

$$\hat{H}_{MBL} = \sum_i \hat{\sigma}^+_i \sum_i h_i \hat{\sigma}^+_i + V \sum_i J_i \hat{\sigma}^+_i \hat{\sigma}^-_{i+1},$$

(1)

of a spin chain of the length $L$ with periodic boundary conditions and the Pauli operators at site $i$ given by $\sigma^\alpha_i$ for $\alpha \in \{x, y, z\}$, $h_i$ and $J_i$ are independent random variables uniformly distributed in $[-W, W]$ and $[0.8, 1.2]$, respectively. $W$ is the disorder and $V = 1$ is the interaction strength.

In [44] under mild assumptions of the absence on energy level attraction the existence of the MBL phase has been established for sufficiently large, but finite $W$. Moreover, numerically the critical disorder strength of the MBL transition has been identified as $W_c \approx 3.5$ [45].

In the non-interacting limit of $V = 0$ the Hamiltonian, Eq. (1), represents a system of uncoupled spins, which is trivially localized in the sense that all eigenstates are product states. In this limiting case, the Hamiltonian can be expressed, $\hat{H} = \sum_i \epsilon_i \hat{\tau}^z_i$, through its integrals of motion $\hat{\tau}^z_i = \hat{U}_i \hat{\sigma}^z_i \hat{U}^+_i$ obtained from the original spins by a single-spin rotation

$$\hat{U}_i = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix},$$

(2)

with $\sin \theta_i = \frac{1}{\sqrt{1 + h^2_i}}$ and the single-spin energies $\epsilon_i = \sqrt{1 + h^2_i}$. This example directly shows that the eigenstates $\{|\uparrow\rangle \}$ of $\hat{H}_{MBL}$ with $V = 0$ are adiabatically connected to the $\sigma^-_i$-basis product states $|\sigma^-_i \rangle = \otimes_i |\sigma^-_i \rangle$ with
where \( |\sigma^z_i\rangle \) stands for the configuration given by the vector \( \sigma^z \) of \( L \) values \( \sigma^z_i = \{+1, -1\} \), while \( \sigma^z \sim \sigma^z \) means that the corresponding vectors differ by a single spin flip. In this representation the first sum in \( \hat{H}_{\text{MBL}} \) can be understood as the Laplace operator on the hypercube as it connects spins configurations, which differ by one spin flip, and \( E_{\sigma^z} = \sum_i h_i \sigma^z_i + \sum_i J_{ij} \sigma^z_i \sigma^z_j \) are the diagonal energies. It is important to note that the \( 2^L \) diagonal entries \( \{E_{\sigma^z}\} \) are strongly correlated random variables since they are constructed only from \( 2L \) random variables \( \{h_i\} \) and \( \{J_{ij}\} \). Indeed, even though typical fluctuations of the entries scales as \( \sqrt{L} \), their level spacings \( E_{\sigma^z} - E_{\sigma^z} \) are \( O(1) \) if \( \sigma^z \sim \sigma^z \).

This model should be distinguished from the Quantum Random Energy Model (QREM) different from its classical counterpart \([46–48]\) for which the diagonal entries \( \{E_{\sigma^z}\} \) are replaced by independent identically distributed Gaussian random variables \( N(0, W^2 L) \) by a transverse field term \( \sum^L_i \tilde{\sigma}_i^z \). The QREM has an Anderson transition at \( W_c \sim \sqrt{L} \log L \) (cf. Eq. (10.15) \([49]\)).

Ergodic properties of an eigenstate \( |E\rangle \) of \( \hat{H}_{\text{MBL}} \) in Eq. (3) can be quantified using the generalized inverse participation ratio (IPR\( _q \))

\[
\text{IPR}_q = \sum_{\sigma^z} |\langle \sigma^z | E \rangle|^{2q},
\]

which quantifies the spread of \( |E\rangle \) over the Fock space. Through the IPR\( _q \) the fractal dimensions are defined as

\[
D_q = \frac{\log \text{IPR}_q}{(1-q) L \log 2}.
\]

Ergodic states at infinite temperature are characterized by \( D_q = 1 \) since they extend over the entire Fock space \( |\langle \sigma^z | E \rangle|^2 \sim 1/2^L \). In general, \( 0 < D_q < 1 \) corresponds to the non-ergodic (or multifractal) states, while the extreme case \( D_q = 0 \) refers to the localized ones.

For a model similar to \( \hat{H}_{\text{MBL}} \) considered in \([40]\) it has been shown that in the ergodic phase \( (W < W_c) \) mid-spectrum eigenstates show \( D_q = 1 \). Instead, in the MBL phase \( D_q < 1 \) and the fractal dimensions experience a jump at the critical point. It is important to point out that due to the many-body nature of the wave-functions \( D_q > 0 \) \([17, 39, 40, 42]\) for any finite values of \( W \), even deeply in the MBL phase.

The last observation is closely related to the tensor product structure of the Fock space and the Hamiltonian's local structure. Indeed, in the non-interacting limit \( (V = 0) \) spins are decoupled, i.e., \( \text{IPR}_q^{V=0} = \prod^L \text{IPR}_{\sigma^z}^{(i)} \) and the one-site IPR\( _q \) is smaller than one

\[
\text{IPR}_q = \sum_{\sigma^z \in \{+1, -1\}} |\langle \sigma^z | E \rangle|^2 < 1.
\]

As a consequence, \( \text{IPR}_q^{V=0} \sim 2^{-(q-1)D_q} L + O(\sqrt{L}) \) decays exponentially with the strictly positive exponent

\[
D_q = \frac{\log \text{IPR}_q}{(1-q) L \log 2} > 0.
\]

This fractal exponent is self-averaging as it is the sum of independent random variables and has fluctuations \( O(1/\sqrt{L}) \) shown above in the exponent of the IPR \([50]\).

At this point, it is important to appreciate the difference between the MBL phase of \( \hat{H}_{\text{MBL}} \) and the localized phase for the QREM. The first one is characterized by a strictly positive fractal dimension, while in the second one \( D_q = 0 \).

For a better understanding of the ergodicity properties from the Fock-space point of view, we define the radial probability distribution \( \Pi(x) \) \([51]\) of an eigenstate \( |E\rangle \) as

\[
\Pi(x) = \sum_{d(\sigma^z, \sigma^z_0) = x} |\langle \sigma^z | E \rangle|^2,
\]

where the sum runs over the \( \{x\} \) spin states \( \{\sigma^z\} \) which differ by \( x \) flips \((i.e., \text{at the Hamming distance} d(\sigma^z, \sigma^z_0) = x)\) from \( \{\sigma^z_0\} \), which corresponds to the maximal eigenstate coefficient \( \max_{\sigma^z} |\langle \sigma^z | E \rangle|^2 = |\langle \sigma^z_0 | E \rangle|^2 \). The overbar indicates the average over disorder and a few mid-spectrum eigenstates.

Compared to the IPR\( _q \), \( \Pi(x) \) gives more information and is a good probe of the eigenstate’s local structure in the Fock space. In particular, we can study the spread of \( \Pi(x) \) by defining the moments

\[
\overline{X^n} = \sum_x x^n \Pi(x),
\]

and the mean-square displacement of it

\[
\Delta X^2 = \overline{X^2} - \overline{X}^2.
\]

In the ergodic phase, the infinite-temperature wave function is spread homogeneously on the Fock space and \( \Pi^{\text{MBL}}(x) \) is given by a binomial distribution,

\[
\Pi_p(L, x) = \binom{L}{x} (1-p)^{L-x} p^x,
\]
with \( p = 1/2 \) and therefore \( \bar{X} = L/2 \) and \( \Delta X^2 = L/4 \). In the opposite limit of a strongly localized system, which can be approximated by the non-interacting case (\( V = 0 \)), \( \Pi(x) \) is still given by the binomial probability distribution Eq. (11), however the value of \( p = \sin^2(\theta_i/2) \) now strongly depends on the disorder \( W \) as

\[
p = \frac{1}{2} - \frac{|h_i|}{2\sqrt{1 + h_i^2}} = \frac{1}{2} \sqrt{\frac{W^2 + 1}{2W} - 1}.
\]  

(12)

As expected, \( p \approx 1/2 - W/4 \to 1/2 \) as \( W \to 0 \) like in the ergodic phase, but for \( V = 0 \) it happens due to the system localization in \( \sigma^x \)-basis, and \( p \approx 1/(2W) \to 0 \) in the opposite limit of \( W \to \infty \).

Note that the binomial probability distribution in Eq. (11), can be approximated by a Gaussian distribution with mean and the variance given by

\[
\bar{X} = pL, \quad \Delta X^2 = p(1-p)L.
\]  

(13)

III. RESULTS

In order to relate the ergodicity properties of the considered system with the local structure of its eigenstates in the Fock space, we focus on the behavior of the radial probability distribution \( \Pi(x) \) of mid-spectrum eigenstates of \( \hat{H}_{\text{MBL}} \). However, for sake of completeness we start our analysis by investigating some standard MBL indicators which quantify ergodicity in the real and Fock space, such as bipartite entanglement entropy and IPR, and compare their properties.

The entanglement entropy has been found to be a surrounding resource to test and quantify ergodicity in a system.

Figure 2(a) shows the half-chain bipartite entanglement entropy \( S = -\text{Tr}[\rho_{L/2} \log \rho_{L/2}] \) (blue lines) of the reduced density matrix of mixed eigenstates of \( \hat{H}_{\text{MBL}} \). As expected, at weak disorder \( S \) shows the volume law, \( S \sim L \), and flows towards the Page value \( S_{\text{Page}} = L/2 \log 2 - 1/2 \) [37], which is the value for states randomly drawn in Fock space.

Instead, at strong disorder \( S \) has an area law scaling, \( S \sim O(1) \), and thus \( S/S_{\text{Page}} \sim 1/L \) tends to zero. The crossover between the two behaviors occurring at \( W_c \approx 3.5 \) indicates the MBL transition.

The averaged fractal dimension \( D_2 \) in Eq. (5) is also shown in Fig. 2(a) (orange lines) and its behavior is consistent with the one of \( S \). In the ergodic phase \( D_2 \approx 1 \), while for \( W > W_c \), the fractal dimension converges with \( L \) to a value which is strictly smaller than one \( (D_2 < 1) \).

A few comments are in order: the finite-size flow of \( S/S_{\text{Page}} \) and \( D_2 \) to unity with the increasing system size within the ergodic phase and the stability in the localized phase. The discrete derivatives \( \partial D = 2(S(L) - S(L'))/(L - L') \log 2 \) [52] and \( \partial D = -\log (\text{IPR}_2(L)/\text{IPR}_2(L'))/(L - L') \log 2 \) in the inset of Fig. 2(a) tend to converge to discontinuous functions of \( W \) with increasing \( L_c \), with zero value of \( \partial D \) and strictly positive value of \( \partial D \) in the MBL phase. In addition, as clearly seen from Fig. 2(a), both \( S/S_{\text{Page}} \) and \( D_2 \) deviate from their ergodic values at the same disorder amplitude.

At the critical point the variance of the entanglement entropy \( \Delta S^2 \), which has been shown to be a useful probe for the transition [17, 19], diverges with \( L \) (blue lines in Fig. 2(b)). In analogy with \( S \), we also inspect the variance of \( -\log \text{IPR}_2 = LD_2 \log 2 \), which also diverges around the critical point (orange lines in Fig. 2(b)).

The above observations give an indication of the following scenario. The upper bound for the unaveraged \( S \leq D_1 L \log 2 \) derived in [43] and the deviation of \( S \) from its ergodic Page-value limit \( \bar{S} = S_{\text{Page}} \) at the same \( W \) as \( D_2 \), is consistent with the jump in \( D_q \) across the MBL transition from unity in the ergodic phase to a certain positive value \( D_q < 1/2 \) in the MBL phase. Moreover, the divergent fluctuations of \( D_q \) might lead to the satu-
of $D_2$ at the critical point. Summarizing, at the MBL transition $D_q$ might be characterized by a jump and it is not self-averaging. This non-self-averaging and the relation between $D_q$ and $S$ [43] might drive the simultaneous jump-like transitions in $S$ and $\overrightarrow{D_q}$.

It is important to point out once again that $D_q$ is strictly positive due to the many-body nature of the problem. As a consequence, the MBL transition cannot been considered as an Anderson-localization transition in the Fock space with $D_q = 0$ in the localized phase, but rather as a transition between an ergodic ($D_q = 1$) and a non-ergodic extended ($0 < D_q < 1$) phase. In order to reach a genuine localization transition in the Fock space one needs to rescale the disorder strength with $L$. Naively, as a first approximation, one might replace the Fock space on-site energies with independent distributed random variables with typical fluctuation $\sim \sqrt{L}$ as is done for the QREM [53–57]. If this would be the case, the Anderson transition would occur at $W_{AT} \sim \sqrt{L} \log L$ also for the quantum Ising model Eq. (1). Nevertheless, the Fock space on-site energies are strongly correlated and they cannot be approximated as independent random variables. Due to the presence of these correlations (similar to [58]) we expect that stronger disorder is needed to localize many-body states in the Fock space.

In order to understand the correct scaling of $W_{AT}$ with $L$, we rely on the exactly solvable non-interacting case ($cV = 0$) providing the lower bound for $D_q$ and then check numerically if the same scaling works as well at strong disorder in the interacting case. This comparison is motivated by the belief [44] that interacting eigenstates in the MBL phase are adiabatically connected to the ones of a non-interacting problem. Consequently, we expect to rescale $W$ with $L$ in the same way as in the non-interacting problem in order to have a genuine localization in the Fock space. Straightforward calculations of the single-spin $\text{IPR}_{\psi_i}^{(1)} = \sin^2 q + \cos^2 q$ in the non-interacting model show that at large disorder $W$

$$\overrightarrow{D_q} \sim \begin{cases} W^{-2q} & q < 1/2, \\ \log W & q = 1/2, \\ W^{-1} & q > 1/2. \end{cases}$$

Thus rescaling $W \sim L$, we have $\text{IPR}_q \sim O(1)$ for $q > 1/2$ and the system exhibits localization in the Fock space. In Fig. 2(c) we plot $-\log \text{IPR}_q$ for $V = 1$ which is collapsed after the rescaling $W/L$ in agreement with the prediction of the non-interacting calculation.

The last observation should be compared to the scaling of $W$ for the QREM, where $W_{\text{QREM}} \sim \sqrt{L} \log L \ll L$ and the random many-body energies are uncorrelated. However, due to the limitation of system sizes achievable with exact diagonalization techniques, it is important to notice that a reasonable collapse of the curves in Fig 2(c) can also be obtained by rescaling $W/(\sqrt{L} \log L)$. Thus, in order to draw a distinction between the two scalings, we consider another moment $q = 1/2$ for $\text{IPR}_q$. The critical value for the QREM is independent of $q$, while

![FIG. 3. Probability distribution function $\rho$ of the fractal dimension $D_2$ for different disorder strength (a) $W = 1$, (b) $W = 2.5$, (c) $W = 3.25$, (d) $W = 6$. Different colors represent system sizes $L$ shown in the legend of panel (b). The inset shows the variance of the fractal dimension versus $W$ establishing the non-self averaging property close to the MBL transition ($\Delta D_2^2 \sim O(L^2)$). The vertical black dashed line in (c) and (d) indicate the value $D_2 = 1/2$.](image)
for the MBL model we expect a different scaling with $L$ for $q \leq 1/2$, as shown in Eq. (15). Figure 2(d) clearly demonstrates the collapse of $\log \text{IPR}_{1/2}$ for several $L$, by rescaling $W$ with $L \log L$ in complete agreement with the prediction of Eq. (15).

Recently, some works [59, 60] claim that the critical point for the MBL transition is $L$-dependent and shifts as $W_c \sim L$. These claims are in conflict with our results for which the system is already Anderson-localized in the Fock space ($D_q = 0$) provided $W \sim L$. Localization in the Fock space is a much stronger breaking of ergodicity than the one defined using local observables (ETH). Since we expect to have $D_q = 1$ in a putative ergodic phase at infinite temperature, it implies that the critical point of the MBL transition, if it scales with system size, would have to scale slower than $L \left( W_c/L \rightarrow 0 \right)$, which rules out the prediction in Ref [59, 60].

Furthermore, a more refined analysis puts even stronger constraint on the possible behavior of $W_c$ with $L$ (if any). It is natural to expect that the stronger the interaction is, the stronger is the disorder needed to break the ergodicity and reach MBL. From this perspective, the uncorrelated on-site disorder in QREM can be expressed in terms of strings of $\hat{\sigma}^z$ operators

$$\sum_{z}^{2L} E_{\sigma^z} \langle \sigma^z \rangle = \sum_{k}^{L} \sum_{i_1, \ldots, i_k} j_{i_1, \ldots, i_k} \hat{\sigma}^z_{i_1} \cdots \hat{\sigma}^z_{i_k}, \quad (16)$$

with $j_{i_1, \ldots, i_k} = \frac{1}{2^k} \sum_{z}^{2L} E_{\sigma^z} \sigma^z_{i_1} \cdots \sigma^z_{i_k}$. As compared with Hamiltonian $\hat{H}_{\text{MBL}}$, the QREM contains stronger interactions, and, for the system Eq. (1), we expect

$$W_c \leq W_{\text{QREM}} \sim \sqrt{L \log L} < W_{\text{AT}} \sim L. \quad (17)$$

This imposes an even stronger upper bound on the possible scaling of the MBL transition with $L$, which is not consistent with [59, 60].

**A. Radial probability distribution $\Pi(x)$**

In this section, we show that the radial probability distribution $\Pi(x)$ in Eq. (8) of mid-spectrum eigenstate coefficients of $\hat{H}_{\text{MBL}}$ in the Fock space gives more detailed information on the wave function’s local structure in the Fock space compared to IPR$_q$. In addition, it can be related to the local integrals of motion of $\hat{H}_{\text{MBL}}$ [14–16, 44].

We recall that at strong disorder, deep in the MBL phase, the eigenstates $\{|E\rangle\}$ of $\hat{H}_{\text{MBL}}$ are believed to be adiabatically connected to the non-interacting ones $\{|\sigma^z\rangle\}$ through a sequence of quasi-local unitary operators,

$$|E\rangle = \hat{U} |\sigma^z\rangle, \quad (18)$$

defining the integrals of motions $\tilde{\tau}^z_i = \hat{U} \tilde{\sigma}^z_i \hat{U}^\dagger$ for which $[\hat{H}, \tilde{\tau}^z_i] = 0$.

![FIG. 4. (a) Radial probability distribution $\Pi(x)$ of the eigenstate coefficients in the Fock space, Eq. (8), for a fixed system size $L = 16$ and several disorder strengths $W$, ranging from the ergodic to the MBL phase (see legend). The blue dashed line shows the ergodic prediction for $\Pi_{\text{MBL}}(x) = \frac{L}{W} \left( \frac{L}{W} \right)$. (b) Mean $\overline{X}/W$, Eq. (9), and (c) variance $\Delta X^2/W$, Eq. (10), of $\Pi(x)$ as functions of $W$ for several $L$. Symbol and color code is shown in legend. The blue and green dashed lines correspond to the ergodic and the non-interacting ($V = 0$) cases, respectively.](image)
Averaging over disorder and energies $E$, we obtain
\[ \tau^2 = \frac{L}{2} - \frac{1}{2} \sum_i (E|\hat{\tau}^2_i|E) \]  

(22)

Similar calculations show that the variance in Eq. (10) of $\Pi(x, E)$, $\Delta X^2(E) = \overline{X^2}(E) - \overline{X}(E)^2$, is given by
\[ \Delta X^2(E) = \frac{1}{4} \sum_{i,j} \left( (E|\hat{\tau}^2_i \hat{\tau}^2_j|E) - (E|\hat{\tau}_i^2|E)(E|\hat{\tau}_j^2|E) \right) \]  

(23)

Thus, both first cumulants, $\overline{X}$ and $\Delta X^2$, provide the measures of the distance between the local integrals of motion ($\{\hat{\tau}^2_i\}$) and the undressed operators ($\{\hat{\tau}_i\}$). In particular, in the MBL phase we expect a perturbative expansion $\tau^2 = a^2_i \hat{\tau}^2_i / \xi_{\text{loc}} + b^2_i \hat{\tau}^2_i \hat{\tau}^2_j + \cdots$ [61], where $a^2_i \sim e^{-|i|/\xi_{\text{loc}}}$, $b^2_i \sim e^{-|i|/\xi_{\text{loc}}}$, and Latin letters run over site indices and Greek ones are in $(x, y, z)$. The coefficient $1/\sqrt{\xi_{\text{loc}}}$ with $a^2_i \sim O(1)$, provides the normalization of the operator ($\tau^2_i = 1$). Thus, $\overline{X}$ gives the direct estimate for the localization length
\[ \overline{X} \sim \frac{L}{2} \left( 1 - \frac{\xi_{\text{min}}}{\xi_{\text{loc}}} \right), \]  

(24)

where $\xi_{\text{min}}$ stands for some typical value of $(a^2_i)^2$.

Having elucidated the relation between the integrals of motion and the radial probability distribution, we now present the numerical results for $\Pi(x)$, $\overline{X}$ and $\Delta X^2$. Figure 4(a) shows $\Pi(x)$ for fixed system size $L = 16$ and several disorder strengths. As expected, at weak disorder, $\Pi(x)$ is centered in the middle of the chain, $\overline{X} = L/2$, with fluctuations $\Delta X^2 = L/4$ as shown in Fig. 4(b-c). In this case the shape of $\Pi(x)$ is close to the one $\Pi_{\text{Erg}}(x)$ of an ergodic system, Eq. (11) with $p = 1/2$. To quantify better the deviations of $\Pi(x)$ from $\Pi_{\text{Erg}}(x)$, we consider the Hellinger distance
\[ H_q = \frac{1}{\sqrt{2}} \sqrt{\sum_x (\sqrt{\Pi(x)} - \sqrt{\Pi_{\text{Erg}}(x)})^2}, \]  

(25)

which quantifies the distance between two probability distributions. The inset in Fig. 4(a) shows $H_q$ as a function of $W$. In agreement with the IPR$q$-analysis, at weak disorder $H_q$ tends to zero exponentially fast with $L$, since the system is ergodic in the Hilbert space. At strong disorder in the MBL phase, $\Pi(x)$ has a non-ergodic shape and it is skewed to the left with respect to its maximum, meaning that $\overline{X} < L/2$, Fig. 4(a), and $H_q$ flows slowly with $L$ to its maximal value $H_q = 1$.

Both mean $\overline{X}/L$ and variance $\Delta X^2/L$ normalized by $L$ in Fig. 4(b-c), demonstrate similar behavior to the fractal dimension, $D_f$, and the normalized entanglement entropy, $\overline{S}/L$ and its fluctuations shown in Fig. 2(a-b). The quantity $\overline{X}/L$ decreases monotonically with $W$ from its ergodic value (blue dashed line) toward the non-interacting one (green dashed curve). At $W \lesssim W_\ast$, $\overline{X}/L$ flows with $L$ towards the ergodic value $\overline{X}/L = 1/2$, consistent with $\xi_{\text{loc}}^{-1} = 0$ in Eq. (24). It shows the saturation with the system size in the MBL phase at the strictly positive value ($\xi_{\text{loc}} > \xi_{\text{min}}$). This behavior is consistent with the finite jump in $\xi_{\text{loc}}^{-1}$ at the MBL transition. The deviations of $\overline{X}/L$ and $\Delta X^2/L$ in the MBL phase from their non-interacting limits and the saturation of these deviations with the system size are possibly related to the partial melting of the ergodic bubbles in the avalanche scenario [28].

Close to the transition $\Delta X^2/L$ exhibits a peak which diverges with the system size, $\Delta X^2 \sim L^\alpha$, with $\alpha > 1$. The latter evidences a rather broad profile of $\Pi(x)$ going beyond the binomial approximation, Eq. (11). The inset to Fig. 4(c) emphasizes this by showing that the ratio $\Gamma_0 = L\Delta X^2/\overline{X}(L - \overline{X})$ is strictly larger than its binomial unit value and demonstrates the same divergence with $L$ as the variance $\Delta X^2/L$.

To sum up in this section we have studied the radial probability distribution $\Pi(x)$, which is directly related to the local integrals of motions. In particular, the mean $\overline{X}$ of $\Pi(x)$ can be used to define a localization length $\xi_{\text{loc}}$, Eq. (24). We have shown that at weak disorder $\overline{X} \to L/2$ corresponding to $\xi_{\text{loc}} \to \infty$, while in the localized phase $0 < \overline{X} < L/2$ giving finite $\xi_{\text{loc}} > \xi_{\text{min}}$. At the transition the fluctuations $\Delta X^2$ diverge which gives an evidence of a finite jump in $\overline{X}/L$ and therefore in $\xi_{\text{loc}}^{-1}$ in the thermodynamic limit $L \to \infty$. The above finite jump in $\xi_{\text{loc}}^{-1}$ is consistent with the avalanche theory of many-body delocalization [28] and therefore with the KT-type scaling of the transition [29, 30].

In the next section, we consider a simple toy model of dilute ergodic “bubbles”, which explains the observed phenomenon of the absence of $D_f$ self-averaging, and helps us to bridge the gap between Fock space structure of mid-spectrum eigenstates, the avalanche theory, and the recent studies of phenomenological renormalization groups of MBL transition.

IV. DILUTE ERGODIC BUBBLES APPROXIMATION

In this section we discuss random block Hamiltonian consisting of non-interacting pieces which is able to reproduce the numerical results described in the previous section. Ref. 28 proposes that non-perturbative effects, such as rare thermal inclusions, which are unavoidable due to entropic arguments, can destabilize the MBL phase provided their density exceeds a certain critical value. As a consequence, an abrupt finite jump of the localization length is expected at the transition. Moreover, RG-studies [29, 30] have shown that this avalanche theory should lead to a KT-like scaling behavior and the
probability distribution $P(\ell)$ of the size $\ell$ of the largest thermal bubble has a power-law fat tail leading to a diverging variance.

With the aim to keep the toy model as simple as possible and avoid introducing further “fitting” parameters, we consider bimodal approximation of $\rho L$ active ergodic spins [62] and $(1 - \rho)L$ non-interacting (frozen) spins. As $\Pi(x)$ is not sensitive to the phases of the eigenstate’s coefficients, and both ergodic and non-interacting limits of it are given by the binomial distribution in Eq. (11), we approximate the contributions of ergodic and frozen spins to the many-body wave function as the non-interacting ones (Eq. (2)), with $\sin^2 \frac{\theta_i}{2} = \frac{1}{2}$ for $\rho L$ active spins and $\sin \theta_i = 1/\sqrt{1 + \nu^2}$ for the remaining $(1 - \rho)L$ frozen ones, as they are subject to strong fields $h_i$. Due to self-averaging of the non-interacting contribution (14) we describe it by the mean value of $p$, Eq. (12). In this model, $\rho$ plays the role of the density of ergodic bubbles.

Within the above approximation, one can easily find $\Pi(x)$ as follows. At a spin-flip distance $x$ from the eigenstate maximum we take $k$ spin flips of ergodic type and $r - k$ of frozen type. The number of such paths from the maximal configuration point in the Fock space is given by

$$N_{k,x} = \left( \frac{\rho L}{k} \right) \left( \frac{(1 - \rho)L}{x - k} \right),$$

(26)

where the first combinatorial factor counts the number of possible selection of $k$ indistinguishable spin-flips of the ergodic type out of $\rho L$ available ones, the second one counts the number of possible $r - k$ indistinguishable spin-flips of the frozen type out of $(1 - \rho)L$ available ones. The corresponding wave function intensity averaged over $h_i$ will be

$$|\psi_{k,x}|^2 = \left( \frac{1}{2} \right)^\rho L p^{x-k}(1-p)^{(1-\rho)L-x+k}.$$  

(27)

The total number of selections for any $k$ is given by

$$N_x = \sum_{k=0}^{x} N_{k,x} = \left( \frac{L}{x} \right).$$

(28)

As a result, one can obtain the probability distribution $\Pi(x, \rho)$ at the fixed ergodic spin density $\rho$

$$\Pi(x, \rho) = \sum_{k=0}^{x} N_{k,x} |\psi_{k,x}|^2$$

$$= \sum_{k=0}^{x} \Pi_{1/2} (\rho L, k) \Pi_{p} ((1 - \rho)L, x - k)$$

(29)

which is simply given by the convolution of two binomial distributions

$$\Pi_p(L', x') = \left( \frac{L'}{x'} \right) p^{x'}(1-p')^{L'-x'},$$

(30)

one describing the ergodic bubble ($p' = 1/2$) and the other the remaining frozen spins with $p' = p < 1/2$.

Due to the simple nature of the model, the $\text{IPR}_q$ is the product of the $\text{IPR}_q$’s for the $\rho L$ ergodic spins and the $(1 - \rho)L$ frozen ones:

$$\text{IPR}_q = \sum_{x,k} N_{k,x} |\psi_{k,x}|^{2q} =$$

$$= \left( \frac{1}{2} \right)^{(q-1)\rho L} \left[ p^q + (1-p)^q \right]^{(1-\rho)L},$$

(31)

and thus the fractal dimensions, Eq. (5), take the form

$$D_q(\rho) = \rho + (1 - \rho)\overline{D_q},$$

(32)

with $\overline{D_q}$ the fractal dimension of the non-interacting frozen spins, which is a continuous function of $W$ with Eqs. (7), (15). In this approximation any jump in the fractal dimension $D_q(\rho)$ is directly related to the jump in the bubble density $\rho$.

By using the Gaussian approximation Eq. (13) for the binomial distribution Eq. (11), we can estimate the first and second moment of $\Pi(x, \rho)$ in Eq. (29)

$$\overline{X}_\rho = \frac{L}{2} - \nu \varepsilon L,$$

(33)

$$\overline{X}_\rho^2 = \left[ \frac{L}{2} - \nu \varepsilon L \right]^2 + L \left[ \frac{1}{4} - \nu \varepsilon^2 \right]$$

(34)

and

$$\Delta X^2_\rho = \overline{X}_\rho^2 - \overline{X}_\rho^2 = L \left[ \frac{1}{4} - \nu \varepsilon^2 \right],$$

(35)

where $\varepsilon = 1/2 - p$ and $\nu = 1 - \rho \leq 1$. These results hold for fixed density $\rho$. For $\rho = 0$ and $p$ given by (12), they coincide with the non-interacting results shown in Fig. 4 (b-c) as green dashed lines, which show plots of Eqs. (33) and Eq. (35) for these parameters.

We emphasize that the ratio $\Gamma_b(\rho) = L \Delta X_\rho^2/\overline{X}_\rho (L - \overline{X}_\rho)$ is strictly smaller than its binomial value $\Gamma_b$ in case $0 < \nu < 1, \varepsilon \neq 0,$ and

$$\overline{X}_\rho \nu (L - \overline{X}_\rho) = L \left[ \frac{1}{4} - \nu \varepsilon^2 \right],$$

(36)

which contradicts the numerics in Fig. 4. This observation underlines the importance of the fluctuations of $\rho$ and the corresponding disorder average. Moreover, even the scaling of the maximum of $\Delta X^2 \sim L^\alpha$, $\alpha > 1$, from the exact numerics with $L$ contradicts (35) with fixed $\rho$.

In order to recover the observed behavior of $\Gamma_b > 1$ in Fig. 4, it is crucial to take into account the non-trivial distribution $P(\rho)$ of the density of ergodic spins and perform the average over it. After averaging Eqs. (32), (33), and (34) and abbreviating

$$\int_{0}^{1} \rho P(\rho) d\rho = \bar{\rho} \equiv 1 - \nu, \quad \int_{0}^{1} \rho^2 P(\rho) d\rho - (\bar{\rho})^2 = \sigma^2_\rho,$$

(37)
one straightforwardly obtains
\begin{align}
D_q &= 1 - \nu(1 - D^0_q), \\
X &= \frac{L}{\nuL} - \nu\langle L \rangle \\
\Delta X^2 &= \frac{\sigma^2_{\rho > \nu} L^2 + L}{4} \left(1 - \nu\langle L \rangle^2\right).
\end{align}

The properties of the probability distribution $P(\rho)$ of the density of ergodic spins can be extracted from the works [27–30]. According to these works the distribution $P(\ell_i)$ of the length $\ell_i$ of a single ergodic bubble is (stretch)-exponential, $P(\ell_i) \sim e^{-\ell_i/\rho}$, or power-law, $P(\ell_i) \sim \ell_i^{-\alpha}$, $\alpha > 2$ in the MBL phase depending on its type. In the thermal phase, $P(\ell_i)$ is sharply concentrated on the length scale $L$ of the system size and tends to a delta-function for $L \to \infty$, meaning the entire system is ergodic. At the transition $P(\ell_i)$ is a power-law distributed $\sim \ell_i^{-\alpha c}$ with the critical exponent $2 \leq \alpha c \leq 3$ [28], in order to have finite mean $\ell \sim L^0$ and diverging variance as $\Delta^2 \sim L^{3-\alpha c}$.

In our consideration we focus on the power-law probability distribution and determine the power $\alpha$ independently. Due to entropic reasons, the mean density $\bar{\rho}$ of the ergodic bubbles has to be finite even in the MBL phase. This density can be approximated by the sum of individual bubbles of lengths $\ell_i$ normalized by the system size
\[\rho \approx \sum_{i=1}^{K} \frac{\ell_i}{L},\]
where $K$ is a cut-off ensuring finite total length of order $L$ of the chain. In the simplest approximation of independent identically distributed random $\ell_i$ one can apply the central limit theorem to $\rho$.

In the MBL phase where the mean and the variance of the length of a single ergodic bubble are finite, $\bar{\rho} \sim L^0$, $\bar{\rho}^2 \sim L^0$, i.e., $\alpha > 3$, and the number of bubbles scales as $K \sim L$ due to the boundness of $\bar{\rho} = K\bar{\rho}/L \sim L^0$, one obtains the Gaussian distribution of $\rho$ with the variance decaying as
\[\sigma^2_{\rho}(W > W_c) = \bar{\rho}^2 - \bar{\rho}^2 \sim \frac{1}{L},\]
consistent with self-averaging. This also immediately confirms the self-averaging of $D_q$ via Eq. (32). In an ergodic phase, the variance $\sigma_{\rho}(W < W_c)$ is exponentially small in $L$ as $\bar{\rho} \to 1$. At the MBL transition self-averaging of $\rho$ breaks down in this model only if $\bar{\rho}^2 \to L^{3-\alpha c}$ diverges in the thermodynamic limit, i.e. $\alpha c < 3$. Due to the finite mean density
\[\bar{\rho} = \sum_{i=1}^{K} \frac{\ell_i}{L} = \frac{K}{L} \max(L^0, L^{2-\alpha c}) \sim L^0,\]
the number of ergodic bubbles scales as $K \sim \min(L, L^{\alpha c-1})$, which puts the lower bound on $\alpha c > 1$.

The corresponding scaling of the variance takes the form
\[\sigma^2_{\rho}(W \simeq W_c) \sim \min(L^0, L^{2-\alpha c}), 1 < \alpha c < 3.\]

The condition of $\Delta X^2$ scaling faster than $L$ is in agreement with the observations in Fig. 4(c), and requires $\alpha c < 3$, which ultimately leads to the decay of the variance with $L$, but slower than $L^{-1}$. As a consequence, both in the MBL phase and at the transition the most significant contribution to $\Delta X^2$ in (40) is given by the large fluctuations in $\rho$, cf. Eqs. (42), (44).

The lack of self-averaging of $D_q$ in Eq. (32) implies the more restrictive condition $\alpha c \leq 2$. This is consistent with the finite mean $\ell \sim L^0$ of a single bubble, $\alpha c \geq 2$, only at the value $\alpha c = 2$.

To summarize, in the ergodic phase $\rho = 1 - \nu$ goes to unity exponentially $\sim e^{-\eta L}$ and its variance has to decay also at least exponentially and therefore $\Delta X^2(W \ll W_c) \sim L/4$, as expected in the ergodic case. In the MBL phase $\bar{\rho}$ is finite and the variance decays as $1/L$ (cf. (42)), which yields a linear behavior of the mean $\bar{\rho} \sim L$ (cf. Eq. (39)), and the variance $\Delta X^2 \sim L$ (cf. Eq. (40)). At the transition, $W = W_c$, according to Eq. (44), the variance of $\rho$ is large compared to $L^{-1}$ and, thus, it provides the main contribution to $\Delta X^2 \sim L^{4-\alpha c} \gg L$, with $2 \leq \alpha c \leq 3$, i.e.
\[\frac{\Delta X^2}{L} \sim \begin{cases}
\frac{1}{4} & W < W_c, \\
L^{3-\alpha c} & W = W_c, \\
< 1/4 & W > W_c.
\end{cases}\]

Finally, the constraint $\alpha c = 2$ implies also that $D_q$ is not self-averaging at the transition.

Thus, we have shown that our numerical results can be explained by considering a simple toy model of dilute ergodic bubbles embedded into a sea of frozen clusters. In particular, the divergence in $\Delta X^2/L$ requires to describe the probability distribution of the length of a typical ergodic bubble by a fat-tailed power-law distribution which has diverge fluctuations at the transition,
\[\Delta D_q^2 \sim \sigma^2_{\rho} \sim \begin{cases}
\nu e^{-\eta L} & W < W_c, \\
L^{2-\alpha c} & W = W_c, \\
L^{-1} & W > W_c.
\end{cases}\]

V. CONTINUITY OF THE TRANSITION FOR NON-INTERACTING SYSTEMS

In the previous sections, we have considered numerically the MBL transition from a Fock-space perspective. We have shown that it is could be characterized by a finite jump in the fractal dimensions $D_q$, which are not self-averaging at the critical point. This scenario is consistent with the avalanche theory which predict a KT-type scaling at the transition and therefore a finite jump on the inverse localization length $\xi^{-1}$. For a contrast, we now investigate the case in which the system undergoes
a delocalization-localization transition with continuous \( \xi^{-1}_{\text{loc}} \), corresponding to the divergence of the localization length at the transition, \( \lim_{W \to \infty} \xi_{\text{loc}} \to \infty \). Concretely, we consider the non-interacting Aubrey-André model

\[
\hat{H}_{\text{AA}} = -\frac{1}{2} \sum_i \hat{c}_i^\dagger \hat{c}_{i+1} + h.c. + W \sum_i h_i \hat{c}_i \hat{c}_{i+1}^\dagger,
\]

where \( \hat{c}_i^\dagger (\hat{c}_i) \) is the fermionic creation (annihilation) operator at site \( i \), \( h_i = \cos(2\pi \beta i + \delta) \) is a quasi-periodic potential with \( \beta = (\sqrt{5} + 1)/2 \), and \( \delta \) is a random phase uniformly distributed in the interval \([0, 2\pi)\). This single-particle model is known to have an Anderson transition at \( W_c = 2 \) between extended (\( W < 2 \)) and localized (\( W > 2 \)) phases [63]. The single-particle localization length at transition diverges as \( \xi_{\text{loc}} \sim 1/\log W/2 \).

We consider non-interacting \( \hat{H}_{\text{AA}} \) with the many-body fermionic filling and focus on multifractal structure of the Slater-determinant many-body wave functions in the Fock space. For this reason, we take the model on \( L \) sites in the chain at half-filling \( N = L/2 \), where \( N \) is the number of particles, maximizing the Hilbert space dimension \( (\frac{L}{2})^L \sim 2^L/L \). Fixing the basis of bare fermions \( |\psi\rangle = \prod_{n_i} (\hat{c}_i^\dagger)^{n_i} |0\rangle \) as the computational basis, with \( n_i \in \{0, 1\} \), we rewrite the model \( \hat{H}_{\text{AA}} \) in a similar form as Eq. (3) and investigate the ergodic properties of its eigenstates in the Fock space.

Figure 5(a) shows the fractal dimension \( \overline{D}_2 \) for the eigenstates of \( \hat{H}_{\text{AA}} \) constructed as a Slater determinant of taken at random \( N \) single-particle eigenstates. In the extended phase, \( W < 2 \), the fractal dimension tends to unity, meaning that the typical eigenstate covers homogeneously the available Hilbert space. Instead, in the single-particle localized phase, \( W > 2 \), \( \overline{D}_2 \) converges to a strictly positive value smaller than one. In fact, we expect that \( IP \overline{D}_2 \sim (\xi_{\text{loc}}^{-1})^\nu \) similarly to a non-interacting spin chain, Eq. (7), and as a consequence \( D_2 \sim \log \xi_{\text{loc}}^{-1} \) close to the transition. Inset of Fig. 5(a) shows \( D_2 \) as function \( \log \xi_{\text{loc}}^{-1} \) giving an evidence of our prediction and therefore the many-body \( D_2 \) does not experience a jump across the transition. Another important consequence is that \( D_2 \) is self-averaging as shown in Fig. 5(b), \( \Delta D_2^2 \sim 1/L \), everywhere including the transition. This self-averaging property should be compared to the one observed in the MBL model, for which the fractal dimensions are not self-averaging close to the MBL transition.

VI. CONCLUSION

In this work we have studied the MBL transition of a chain of interacting spins from a Fock-space point of view. In addition to the standard diagnostic tools for ergodicity, such as fractal dimensions and entanglement entropy, we consider the radial probability distribution of eigenstate coefficients with respect to the Hamming distance in the Fock space from the wave function maximum. We show that this radial probability distribution gives important insights about the integrals of motion of the problem and allows to extract the localization length from the cumulants of the distribution.

Numerically we have found that both the fractal dimensions and the radial probability distribution have strong fluctuations at the critical point. The fractal dimensions at the transition are not self-averaging and the probability distribution is extremely broad. This divergence provides a rather strong evidence of the existence of a possible jump of the fractal dimensions as well as of the localization length across the MBL transition.

Inspired by recent studies of renormalization group and avalanche MBL theory, we explain our findings by introducing a simple spin toy model. This model hosts a finite density of ergodic/thermal bubbles as well as frozen/localized spins. The MBL transition occurs by tuning the density of the ergodic bubbles. At the transition the probability distribution of the bubble density is fat-tailed in agreement with the RG studies. Using this simple model, we are able to explain our numerical findings, and thus to bridge the gap between recent studies of the nature of the MBL transition in the “real space” and in the Fock space perspectives.

As a result, we show that the MBL transition can be seen as a transition between ergodic states to non-ergodic extended states and we put an upper bound on the disorder scaling for a genuine Anderson localized regime with respect to the non-interacting case.

Finally, we provide an example of the many-body (non-interacting) model with a continuous localization transition showing the self-averaging of fractal dimensions in the whole range of parameters. This model confirms the conjectured relation between the non-self-averaging property of the fractal dimensions and their finite jump at the localization transition.
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