Neural Networks with Quantum Gated Nodes

Fariel Shafee
Department of Physics
Princeton University
Princeton, NJ 08540
USA.

We study a quantum neural network with superposed qubits replacing classical neurons with deterministic states, and also with quantum gate operators in place of the classical action potentials observed in biological contexts. With our choice of logic gates interconnecting the neural lattice, we find that the state of the system behaves in ways reflecting both the strength of coupling between neurons as well as the initial conditions, and depending on whether there is a threshold for emission from excited to ground state, the system shows either chaotic oscillations or coherent ones with periodicity that depends on the strength of coupling in a unique way. The spatial pattern of the initial input affects the subsequent dynamic behavior of the system in an interesting unambiguous way, which indicates that it can serve as a dynamic memory system analogous to biological ones, but with an unlimited lifetime.

PACS numbers: 03.67.Lx, 07.05.Mh, 84.35.+i
Keywords: neural network; quantum computing

I. INTRODUCTION

Quantum computers have recently become a topic of elaborate research. It has been shown [1] that a quantum computer can compute some NP hard problem much faster than a classical computer because the memory elements can simultaneously hold multiple information and the processor can operate in parallel on many qubits. Quantum entanglement between the elements permits the use of different kinds of algorithms [2, 3] for such quantum computers as well, which too are now being explored extensively.

Closely related to computing is the process of memory and pattern recognition. Biologically, the human brain has been modeled as a network of neurons firing signals with different time sequence patterns corresponding to different input signals. Hebbian learning is achieved by taking into account the plasticity of the weights with which the neurons are connected to each other. A dynamic type of memory system in a classical network with integrate-and-fire neurons was first presented by Hopfield and Herz [4] which was later extended to the more complicated case of a still classical model with nonzero widths of the action potentials [5] with some features similar to the zero-width case and other novel features related to the finite width.

An effort to enter the quantum realm with such dynamic networks was made in the semiclassical version reported in [6]. In all these works the periodicity and the pattern of collective and firings of the dynamical neural networks acted on by the environment and also with nearest neighbor action between the neighboring neurons were studied in detail. However, in the previous semiclassical neural network model the analogies retained with the classical case took it somewhat away from the current focus in quantum computing, where quantum gates replace arbitrary time-dependent potentials so that there is a well-defined unitary operator representing the transitions at the nodes of the network.

In the present work, we investigate a quantum machine with such easily referable and repeatable quantum gates, which at the same time also most closely resembles our previously studied models, so that we can understand aspects of the differences between a classical or a semiclassical model having given potentials between the nodes, with a quantum computer-like gated system, where the building block is the quantum transition operation representing a logical operation, not a general potential.

Whereas a quantum computer would be useful for calculating quantities, with an input operated on by the processor and producing a quantitative output, a quantum neural network may possibly also be used for the purpose of enriched learning of a variety different from the Hebbian learning of classical networks. This possibility has not yet been fully explored, though Altaisky [7] has made some preliminary investigations into a single quantum perceptron. Learning is of course an irreversible process, and the unitarity of the operators involved in a net does not permit an irreversible change of the system in the ordinary sense. There have been many attempts [8] to explain such a change using decoherence at the output with reversibility of the intermediate processes in
a quantum computer. This approach has mostly been suggested for quantum computers. For quantum neural networks usually the transition to a certain eigenstate is inserted in an ad hoc manner, as in the work of Altisky and also that of Zak et al.

In this work we shall not go into the details of the complications of the separation of the quantum processing followed by a classical output. We shall instead try to mimic as nearly as possible our previously biologically motivated model of an integrate-and-fire neural network, because biological or quasi-biological models are always a fascinating benchmark for the comparison of any system that aims at intelligence-like functions. However, in this case we have, in place of deterministic neurons, qubits being interconnected to nearest neighbors by quantum gates with well-defined operatorial roles.

II. THE QUANTUM NEURAL NETWORK MODEL

In the integrate-and-fire model a neuron receives a current from the fired neighbors, and when its own potential exceeds the threshold it too fires, feeding its own neighbors. We have studied the effect of the finite duration of the signal from a neuron to the neighbor. In a quantum process all transitions of the neurons must be designated by unitary operators. So in place of the firing of a neuron we have a less spectacular unitary transformation that simply performs a sort of rotation of the state vector or the qubit.

In principle this operation should involve time too, and we should write:

$$|t⟩ = U(t, t_0)|0⟩$$

(1)

to indicate the transformation of a neuron from time

to indicate the transformation of a neuron from time

$$U(t + dt, t) = U(t, t) + idtH$$

(2)

So that

$$d|t⟩ = idtH|t⟩$$

(3)

in the lowest order, with a hermitian operator $H$, usually the Hamiltonian.

In quantum computing it has been shown that a complete set of unitary operators exist to express the classical logical operations such as NOT, AND or XOR. These may make use of Hadamard gates, phase-change gates or controlled-NOT (c-NOT) gates. These gates may be combined to give entanglement between different nodes, e.g. the c-NOT or the Toffoli gate, which is a kind of adder.

It is not necessary for the whole network to be completely entangled by the basic operators of the net to form a useful network. It is known that we can have pair-wise entanglements at the lowest nontrivial level. However, even if we entangle only nearest neighbors, the entanglement may spread throughout the net after successive operations. The process is similar to obtaining a dense matrix from the multiplication of a large number of sparse matrices with nonzero elements at different positions.

We postulate the following physical model:

1. the neurons represent qubits;
2. an excited neuron $|1⟩$ will turn on a neighbor in a ground state $|0⟩$;
3. an excited state will make an excited neighbor ‘fire’ and go down to the ground state (induced emission);
4. the excited state itself will go down to the ground state after exciting the neighbors;
5. an unexcited neuron does nothing to itself or any neighbor.

Postulates 2, 3 and 5 can be satisfied by a c-NOT gate, with the first neuron serving as the controller, and operating on its neighbor. With a square lattice we consider for simplicity there are four neighbors for each neuron, so that in place of c-NOT gates we shall need $c – NOT^4$ gates, i.e. one controller flipping all four neighbors if it is in state $|1⟩$ and doing nothing if it is in state $|0⟩$.

Postulate 4 can be satisfied by using an AND gate connection every neuron with a common $|0⟩$ state after the $c – NOT^4$ gate.

For any particular neighbor the c-NOT gate can be represented by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

(4)

where $\sigma_1$ is the flipping Pauli matrix. Eq. represents a hermitian operator. We, however, connect the c-NOT matrices using a weight factor, $\epsilon$ to represent the strength with which a neuron can affect its neighbor. This weight factor is not the same as the weight factors in a classical neural network, which must sum up to one, but just a measure of the strength with which the neurons are able to affect the neighbors.

The $AND.|0⟩$ can be represented by another hermitian operator in the qubit space:

$$U' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(5)

where the sequence of states in the rows and columns are, as usual $|1⟩|1⟩$, $|1⟩|0⟩$, $|0⟩|1⟩$ and $|0⟩|0⟩$, the first being the controlling state.

Although hermitian, the AND operator is obviously not unitary. The nonunitarity of the AND operator is responsible for the collapse of the state to the ground state after it has reached a threshold. This situation is not the standard quantum computer situation, where all gates are required to be strictly unitary, but is a mix between unitary connections among neurons and a nonunitary collapse. The introduction of unitary rotations of
the neurons in the qubit space makes our device fast and
efficient, and the collapse to the ground state by the AND
operator lets us mimic the classical neural network model
as closely as possible. However, The collapse of the state
to the ground state simply restarts the rotational activity
of a neuron when connected to an excited neighbor. So
this situation is not quite the same as decoherence when
in contact with nature, and states are collapsed prob-
abilistically to one of the superposed states and delete
all memory. Rather, this collapse is a conditional col-
lapse to a specified state, and the timing of the collapse
holds information about when that specific threshold was
reached to enable a time sequence pattern.

So we get at each node the controlling qubit remaining
unchanged due to its own action:

\[
\begin{pmatrix}
  c \\
  s
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  c \\
  s
\end{pmatrix}
\] (6)

and the following change for the neighbors receiving
signal from it, i.e. operated on by the \( c - NOT \) gates:

\[
\begin{pmatrix}
  c' \\
  s'
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  c' \\
  s'
\end{pmatrix} + \epsilon \begin{pmatrix}
  -s'.c \\
  c'.c
\end{pmatrix}
\] (7)

Since the small \( \epsilon \) approximation of the unitary opera-
tors are not themselves unitary, in simulation it is neces-
sary to renormalize each qubit at each step.

III. RESULTS OF SIMULATION

As in the classical cases in previous works, for ease of
comparison we constructed a 40X40 network of qubits
with periodic boundary conditions, so that it behaves
in some ways as a much larger lattice. In the quantum
case of course in reality a large lattice would very dif-
ficult to realize in the laboratory at the present stage of
technology, but for such a theoretical study it makes no
difference.

We put the input data on qubits in the periphery and
made the inside neurons either all random or all zero
\([0, 1]\). Then we updated the qubits according to Eq.6
and 7

A large number (40,000) of time steps were chosen and
various \( \epsilon \) values. This parameter may be interpreted as
the strength of coupling of the neurons, but as it occurs
together with \( dt \), it may also indicate the width of the
pulse at each time step.

In our first model we did not use a threshold for the
firing of the neurons, so that each neuron was allowed to
reach to enable a time sequence pattern.

In our first model we did not use a threshold for the
firing of the neurons, so that each neuron was allowed to
go up to its full qubit value of \((1, 0)\) with \( c = +1 \) or \(-1\). It
is interesting to observe that in this case there is no well-
defined periodicity, either for any single neuron or for the
average neuron (i.e. the sum) in the network (Figs. 1,
2). Though all neurons do indeed go through the \((1, 0)\)
to \((0, 1)\) cycles, the oscillations are aperiodic. This may
be because the exact equation of motion coupling the
nearest neighbor neurons becomes insoluble in terms of
periodic functions.

We also experimented with a slightly different version
of the model of the network more akin to the classical one.
Here we introduced a threshold for the excited part of the
neuron, which when crossed, causes the qubit to jump to
the ground state \((0, 1)\), i.e. \( c > c_{\text{thres}} \), any \((c, s)\) makes
a transition to the \((0, 1)\) state. This may be considered
as due to emission of energy by an excited element
on reaching a threshold. So, in this case the \( AND, |0\)\) opera-
tion works on reaching the threshold. A notable difference between the two models is that whereas in the
first model, the AND gate brought back the qubit slowly
to the ground state, in the second model, the qubit was
collapsed to the ground state immediately.

We found more interesting results with this model. In
this case we get periodic oscillations of the system, with
all neurons almost in the same phase. We put the thresh-
hold at 0.7 which is just below \( 1/\sqrt{2} \), because this seems
to be critical threshold that gives regular oscillations.

\[\text{FIG. 1: Oscillations of the c part of a qubit for a non-cutoff model with } \epsilon = .01. \text{ There is no fixed periodicity.}\]

\[\text{FIG. 2: Correlation between two qubits for no threshold case with } \epsilon = 0.01, (10, 10|20, 21), \text{ where the qubits are located by their } (x, y) \text{ coordinates.}\]
We believe this happens because here the cut-off effectively serves to truncate the complicated coupled behavior of the system, reducing it to a simpler periodic system, just as the truncation of a transcendental function by a polynomial with a finite number of terms provides it with a simpler behavior. For example, the function:

\[ F(t) = \cos(\theta + \epsilon \sin \theta) \]  

with \( \theta = \omega t \) assumes the periodic form

\[ F(t) = \cos[(1 + \epsilon)\omega t] \]  

for small enough \( \epsilon \) only, but has a more complicated behavior at large \( \epsilon \). Obviously this requires more thorough and careful study.

However, the other reason for this behavior might be that the rate at which the AND operator acted was different from the rate at which the c-NOT operator acted on the system, and the coupling of the two incongruous frequencies yielded a complex pattern.

We note that these oscillations are seen in the behavior of a single neuron (Fig. 3), or the sum of all neurons of the system, or even in the correlation \( \langle i|j \rangle \) between neuron \( |i \rangle \) and neuron \( |j \rangle \) (Fig. 4).

Most interestingly, it appears that for large \( \epsilon (> 0.7) \), if we put the initial signal only at two parallel sides of the square, there is no oscillation (Fig. 5), but a static asymptotic state, which is quickly reached, but if we put the signal on all four sides, then periodic oscillations continue with changed frequency.

It is possible that the lack of signal in the orthogonal direction allows the neurons in the net the extra freedom to adjust themselves to fixed static states in the direction of the signal. This is reminiscent of the one-dimensional Ising model having a trivial phase transition. When signals arrive from both \( x \) and \( y \)-directions, presumably the attractor for the system becomes dynamic, as it tries to adjust in both directions, but cannot find a static equilibrium state.

IV. CONCLUSIONS

We have shown that a quantum neural network similar to the integrate-and-fire neuron network in some ways can be constructed with quantum elements, consisting of qubit nodes connected by c-NOT and AND gates. However, the quantum gated system also has significant differences.

We have found the interesting property that, with no threshold the system converges to a dynamic state with no fixed period and no phase locking, similar to a chaotic system, but with an average behavior which is not entirely chaotic.

With an assigned threshold that takes an excited qubit to the zero state, we see dynamic oscillations of the system. The period is almost inversely proportional to the coupling strength (Fig. 6), but shows nonlinearity for strong coupling. For quite strong coupling, if there are initial excitations only in one direction, the system seems to converge rapidly to a static attractor, but with excitations from both directions of the square lattice, dynamic
oscillations continue. This is a very significant difference in the mapping of the initial spatial pattern into dynamic patterns of the system that promises to make such a system a useful device for conversion of static patterns into dynamic memory.

In the case of fixed period oscillations, the correlation between neurons as measured by the overlap of the two qubits, $\langle i|j \rangle$ also shows periodic time dependence. Interestingly, we have found that despite the simplicity of the model of this quantum neural network with c-NOT gates, it can hold dynamic memories of the input indefinitely.

Obviously, with complex phases in the coupling the pattern generated may be more interesting, also if there is a dynamic external agent affecting the peripheral neurons, and not just an initial input. We can also introduce delay lines between the neurons to introduce a time scale.

[1] M.A. Nielsen and M. Chuang, Quantum computation and quantum information (Cambridge U.P., NY, 2000)
[2] P.W. Shor, SIAM J. Comp 26, 1448 (1997)
[3] J. Grover, Phys. Rev. Lett. 79, 325 (1997)
[4] J.J. Hopfield and A.V.M. Herz, Proc. Natl. Acad. Sci. 92, 6655 (1995)
[5] F. Shafee, cond-mat/0111151 (2001)
[6] F. Shafee, quant-ph/0202015 (2002)
[7] M.V. Altaisky, quant-ph/0107012 (2001)
[8] A. Eckert et al, University of Oxford preprint (2000) and other reviews at arxiv.org
[9] M. Zak et al, JPL-Caltech preprint 97-1153(1997)