Dynamic similitude for small-scale re-entry experiment

V I Trushlyakov, V V Yudintsev
Omsk State Technical University, 11, Mira Ave., Omsk, 644050, Russian Federation

Abstract. The dynamic similitude criteria for detachable sections of rocket like fairing leaf are presented. The criteria ensure the similarity of the center of mass and attitude motion of the small-scale model to the full scale body. The application of the presented criteria is illustrated by a numerical experiment. These criteria can be used to conduct small-scale experiment for the investigation of the re-entry motion of launch vehicles and its parts.

1. Introduction
Not all of detachable sections of launch vehicles are burning up in the Earth’s atmosphere. It requires the allocation of significant drop zones that leads to economic, environmental and social problems [1, 2]. One of the possible solution to decrease the drop zones is using the dispersion of the falling bodies during atmospheric re-entry. This solution is described in [3, 4]. Pyrotechnic compositions can be used to increase the temperature of the body during entry heating to initiate its combustion. For example, pyrotechnic compositions can be integrated directly into the honeycomb sandwich structure of the fairing leaf that separated from the launch vehicle. Also the control system of the dispersion system should be installed to initiate the pyrotechnic compositions at the height about of 10 km where the burning process can be supported by the atmospheric oxygen.

To design the dispersion control system and its operation algorithm the motion of the body during re-entry should be investigated. Atmospheric re-entry is a complex dynamical process that can’t be investigated by using only mathematical models [5–7]. These models should be validated by experimental data.

Small-scale models of the launch vehicles parts can be used to investigate its re-entry dynamics to decrease the cost of the experiment. These small models have to preserve the thermomechanical properties of the original full-scale body and its motion should be investigated in the same conditions including the net force acting on the original body during the re-entry [8]. The small-scale models can be installed on launch vehicles as a piggyback payload and separated at the required height and speed.

The ignition time of the pyrotechnic compositions depends on the center of mass and attitude motion of the body. The dispersion process should be completed before the possible defragmentation of the body due to the action of the aerodynamic forces. The small-scale model has smaller mass and moments of inertia so its center of mass and attitude motion during the re-entry are not the same as the motions of the original full-scale body. So to conduct an adequate experiment not only the thermomechanical properties of the original body, but also the center of mass and attitude motion should be preserved.
In this paper we present dynamic similitude criteria that preserve the center of mass and attitude motion of the small-scale model. The proposed criteria demonstrated by the numerical experiment.

2. Dynamic similitude criteria

Let us consider the motion of the body in the Earth’s atmosphere. It is supposed that we know drag \( c_D \) and lift \( c_L \) coefficients of the body and its reference area \( S \). The motion of the body is considered on the interval of several minutes during re-entry, so the rotation of the Earth is not considered in the developed model. Also we suppose that the gravity field is homogeneous. The motion of the center of mass of the body is described by the following differential equations [9]

\[
\frac{dv}{dt} = \frac{D}{m} - g(r) \sin \vartheta = -\frac{\rho v^2 c_D S}{2m} - g(r) \sin \vartheta
\]

\[
\frac{vd\vartheta}{dt} = \frac{L}{m} - \left( g(r) - \frac{v^2}{r} \right) \cdot \cos \vartheta = -\frac{\rho v^2 c_L S}{2m} - \left( g(r) - \frac{v^2}{r} \right) \cdot \cos \vartheta
\]

\[
\frac{dr}{dt} = v \sin \vartheta
\]

where \( v \) is the speed of the body, \( \vartheta \) the flight path angle measured from the local horizontal (positively up), \( r \) the radial distance from the center of the Earth. The atmospheric density \( \rho \) and acceleration gravity \( g \) are functions of the distance \( r \).

![Figure 1. Geometry of the entry trajectory](image)

2.1. Dynamic similitude for center of mass motion

If the geometrical dimensions of the body are proportionally changed, its aerodynamic coefficients remain unchanged. The reference area \( S \) changes in proportion to the square of the variation in the dimensions (\( S \propto L^2 \)), and the mass of the body changes in proportion to the cube of the variation in the dimensions (\( S \propto L^3 \)), therefore, as the size of the body is reduced, a proportional increase in its center of mass acceleration is expected

\[
\frac{dv}{dt} \propto \frac{c_D S}{m} \propto \frac{c_D L^2}{L^3} = \frac{c_D}{L}
\]
To maintain the acceleration of the center of mass of the small-scale body at the same level which ensure that its will be close to the motion of the full-scale body, it is necessary to maintain the ratio of the reference area to the mass of the body for the small-scale model and full-scale body:

$$\frac{S}{m} = \frac{S'}{m'}$$  \hspace{1cm} (5)

where $S'$ and $m'$ are the reference area and mass of the small-scale model of the original body. To fulfil the condition (5) the mass of the small-scale model should be decreased in proportion to its characteristic area

$$m' = \frac{S'}{S} m = \frac{1}{k^2} m$$  \hspace{1cm} (6)

where $k = L/L'$ is the scale ratio (figure 2).

![Figure 2. Full-scale body and its small-scale model](image)

For a body in the form of a thin shell, like a fairing leaf, this condition is satisfied for the unchanged density of unit surface. Let us consider a half cylindrical shell with the radius $R$, height $H$ and the thickness $\delta$. The mass if this shell is

$$m = \rho \pi R H \delta$$  \hspace{1cm} (7)

If the size of this shell decreases in $k$ times the mass of the shell will decrease in $k^3$ times

$$m' = \rho \frac{R H \delta}{k \frac{k}{k}} = \frac{m}{k^3}$$  \hspace{1cm} (8)

To decrease the mass proportional to $k^2$ the mass of the unit area $m_s = \rho \delta$ should not be changed

$$m' = \frac{m}{k^2} \Rightarrow \frac{\rho}{k} \frac{\delta}{k} \frac{\pi R H}{k} = \frac{m}{k^3} \Rightarrow \rho' = k \rho$$  \hspace{1cm} (9)

The results of the analysis show that to preserve the motion of the center of mass of the small-scale model of the body during the re-entry it is necessary to reduce the mass of the model in proportion to the power of two of the change in the geometrical dimensions of the original body

$$L' = \frac{L}{k} \Rightarrow m' = \frac{m}{k^2}$$  \hspace{1cm} (10)

2.2. Attitude motion

The attitude motion of small-scale model of the body will also change with the change of the size of the body. In general, the equation of motion of the object around its center of mass has the form

$$J \dot{\omega} + \omega \times J \cdot \omega = M_a$$  \hspace{1cm} (11)
where \( J \) is the inertia tensor of the body, \( \omega \) the coordinate vector of the angular velocity of the body in the body frame, \( M_a \) the net torque of the aerodynamic forces acting on the object about its center of mass. Here we consider simplified attitude motion of the body in the plane of the angle of attack \( \alpha \). This equation has the form [9]

\[
\ddot{\alpha} = \frac{1}{J_z} \left[ m_z q S L + \frac{m_z^\alpha q S L^2}{v} \dot{\alpha} \right]
\]  

(12)

where \( J_z \) is the moment of inertia of the body relative to the transverse axis, \( L \) the reference length, \( m_z \) the aerodynamic moment coefficient, \( m_z^\alpha \) is the damping torque coefficient. In the absence of damping, the equation can be written as

\[
\ddot{\alpha} = m_z q \frac{S L}{J_z}
\]  

(13)

Assuming, as before, that the aerodynamic coefficient \( m_z \) remains constant during the proportional change of all geometrical dimensions of the body. To maintain the angular acceleration of the small-scale model equal to the angular acceleration of the full-scale body it is necessary to satisfy the following condition

\[
\frac{S L}{J_z} = \frac{S' L'}{J'_z}
\]  

(14)

where \( J_z = \rho^2 m \) is the moment of inertia of the body and \( J'_z = \rho'^2 m' \) is the moment of inertia of the small-scale body, \( \rho \) the radius of gyration of the body, \( \rho' \) the radius of gyration of the small-scale model.

The nominator of the expression \( S L/J_z \) changes proportional to the change in third power of the dimensions of the body and the denominator of the expression is proportional to the change in fifth power of the dimensions of the body

\[
\rho \propto L \quad m \propto L^3 \Rightarrow J_z \propto L^5
\]  

(15)

So it should be expected that the angular acceleration of the small-scale model \( \ddot{\alpha}' \) increases proportional to the power of two of the scale coefficient

\[
L' = \frac{L}{k} \Rightarrow \ddot{\alpha}' \propto \frac{S' L'}{J'_z} = \frac{S/k^2 L/k}{J_z/k^5} \propto \ddot{\alpha} k^2
\]  

(16)

To maintain the attitude motion of the small-scale model close to the attitude motion of the original body it is necessary to decrease the moments of inertia of the small-scale model in proportion to the third power of the decrease in size

\[
\frac{S L}{J_z} = \frac{S' L'}{J'_z} = \frac{(S/k^2)(L/k)}{J_z/k^5} \Rightarrow J'_z \propto \frac{J_z}{k^3}
\]  

(17)

According to (10)

\[
m' = \frac{m}{k^2} \Rightarrow \rho'^2 = \frac{J'_z}{m'} = \frac{(J_z/k^3)}{m/k^2} = \frac{\rho^2}{k} \Rightarrow \rho' = \frac{\rho}{\sqrt{k}}
\]  

(18)

In other words, to maintain the attitude motion of the small-scale model close to the attitude motion of the original body it is necessary to decrease the radius of gyration of the small-scale model in proportion to the square root of the decrease coefficient \( k \).
3. Numerical simulation

3.1. Parameters and initial conditions

In this section the motion of the fairing leaf and its small-scale model are compared. We consider the re-entry process of the fairing leaf from the initial height about of 100 km with the initial velocity about of 2.8 km/s. The parameters of the fairing leaf and its small-scale model are presented in Table 1. The small-scale model is three times smaller than the original body. The mass of the small-scale model, in accordance with condition (10), is reduced by a factor of nine.

Table 1. Parameters of the fairing leaf and its small-scale model

| Parameter               | Fairing leaf | Small-scale model |
|-------------------------|--------------|-------------------|
| Mass, kg                | 600          | 67                |
| L, m                    | 8            | 2.7               |
| S, m²                   | 10           | 3.3               |
| Moment of inertia, Jₓ, kg·m² | 900         | 33                |
| Moment of inertia, Jᵧ, kg·m² | 4000       | 148               |
| Moment of inertia, Jᶻ, kg·m² | 4000       | 148               |

3.2. Center of mass motion

Figure 3 shows full aerodynamic force-to-weight ratios of the fairing leaf and its small-scale model during the re-entry. Figure 3(a) shows the time history of the mean value of the full aerodynamic force-to-weight ratio and figure 3(b) the time history of the maximum of this value. The graphs are built on the interval from 80 to 10 km (up to the height of the supposed ignition of the dispersion system). The value of the force-to-weight ratio is averaged over an interval of 40 s.

![Figure 3](image-url)

**Figure 3.** Mean and maximum aerodynamic to weight ratio of the fairing leaf and its small-scale model as functions of the height

For the solution \( n(t₀), n(t₁), n(t₂), \ldots, n(tₙ) \) where \( \Delta t = t_{i+1} - t_i = \text{const} \) the mean value at the
time $t_i$ is calculated as
\[ n^{\text{mean}}(t_i) = \frac{\sum_{k=-r}^{r} n(t_i-k)}{2r+1} \]  
where $r = 200$ and $\Delta t = t_{i+1} - t_i = 0.1$. The time history of the maximum value is calculated as
\[ n^{\text{max}}(t_i) = \max(n(t_i-r), n(t_i-r+1), \ldots, n(t_i), n(t_i+1), \ldots, n(t_{i+r})) \]  
where $r = 100$.

The simulation results show that the patterns of the change in the mean and maximum aerodynamic force-to-weight ratios of the original fairing leaf and its small-scale model differs slightly. The maximum acceleration of the original fairing leaf and its model does not exceed $20g$ and these values are achieved at the height about of $40\ldots45$ km.

Figure 4 shows the time histories of the dynamic pressures of the fairing leaf and its small-scale model during the re-entry. We see that the time histories of the dynamic pressure mean value and its maximum value for the fairing leaf and its model coincide almost exactly. The mean an maximum values are calculated using the expressions (19) and (20). Variation in the average and maximum velocity show that the pattern of the motion of the center of mass of the original fairing leaf and its small-scale model, which determines the effect of aerodynamic forces, coincide.

3.3. Attitude motion
Let us consider the attitude motion of the fairing leaf and its small-scale model. The model is one third as large as fairing leaf and its moments of inertia are 27 times smaller.

Figure 5 shows the time histories of the mean values and maximum values of angle of attack of the fairing leaf and its small-scale model. We see that graphs are not coincide exactly but the patterns of change of the mean and maximum values of the angle of attack for the full scale and small-scale fairing are close. Original fairing leaf and its model pass max-q point (40\ldots45 km) with coincide mean angles of attack about of 50 degrees. Further, there is a difference in the start time of the increasing in the average angle of attack of the original fairing leaf and its model, however, the patterns of the change in the average angles of attack are close.
The maximum angle of attack of the original fairing leaf and its model at the max-q time are coincides. At the altitudes of 10 to 30 km the maximum values of the angles of attack attack are close and do not exceed 170 degrees (figure 5b).

Figure 6 shows the time history of the mean value and maximum value of the angular velocities of the fairing leaf and its small-scale model

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

(21)

The value of the angular velocity is averaged over an interval of 40 s. As before we see almost complete coincidence of the graphs to an altitude of 35 km. Further the patterns of change of two graphs are close to each other. The average angular velocity of the original fairing leaf and its small-scale model belongs to the interval from 2 to 4 rad/s.

4. Conclusion
In this paper we state that if the geometrical dimensions of the small-scale model are $k$ times smaller than the dimensions of the original object than the mass of the small-scale model should be $k^2$ times smaller than the mass of the original object. This condition ensure the similarity of the center of mass motion of the body and its small-scale model. The moments of inertia of the small-scale model should be decreased in proportion of $k^3$ to ensure that the attitude motion of the small-scale model will be closely related to the attitude motion of the original object.

Acknowledgments
This work is supported by Russian Science Foundation grant No. 16-19-10091.

References
[1] Klyushnikov V Y, Kuznetsov I I and Osadchenko A S 2014 Solar System Research 48 582-587 ISSN 0038-0946
[2] Perminov A N 2010 Otoronnopromyshlennyi kompleks Rossii (Russian Defense-Industrial Sector) 177–184
[3] Lempert D B, Trushlyakov V I and Zarko V E 2015 Combustion, Explosion, and Shock Waves 51 619–622 ISSN 0010-5082
Figure 6. Mean and maximum dynamic pressure of the fairing leaf and its small-scale model as functions of the height

[4] Trushlyakov V and Davydovich D 2017 The Use of Pyrotechnic Composition for Dispersing Fairings during Atmospheric Re-entry Procedia Engineering vol 174 pp 4–10 ISSN 18777058
[5] Pardini C and Anselmo L 2004 On the accuracy of satellite reentry predictions Advances in Space Research vol 34 pp 1038–1043 ISSN 02731177
[6] Frank M V, Weaver M A and Baker R L 2005 A probabilistic paradigm for spacecraft random reentry disassembly Reliability Engineering and System Safety vol 90 pp 148–161 ISSN 09518320
[7] Anselmo L and Pardini C 2005 Computational methods for reentry trajectories and risk assessment Advances in Space Research vol 35 pp 1343–1352 ISSN 02731177
[8] Tewari A 2009 Journal of Spacecraft and Rockets 46 299–306 ISSN 0022-4650
[9] Vinh N X, Busemann A and Culp R D 1980 Hypersonic and Planetary Entry Flight Mechanics (Rexdale: The University of Michigan Press) ISBN 9780472093045