R-CLOSED HOMEOMORPHISMS ON SURFACES

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ABSTRACT. Let \( f \) be an \( R \)-closed homeomorphism on a connected orientable closed surface \( M \). In this paper, we show that if \( M \) has genus more than one, then each minimal set is either a periodic orbit or an extension of a Cantor set. If \( M = \mathbb{T}^2 \) and \( f \) is neither minimal nor periodic, then either each minimal set is a periodic orbit or an extension of a Cantor set. If \( M = S^2 \) and \( f \) is not periodic but orientation-preserving (resp. reversing), then the minimal sets of \( f \) (resp. \( f^2 \)) are exactly two fixed points and other circloids and \( S^2/\tilde{f} \cong [0, 1] \).

1. INTRODUCTION

In \cite{Ma}, it has shown that if \( f \) is orientation-preserving \( R \)-closed and non-periodic homeomorphism on \( S^2 \), then \( f \) has exactly two fixed points and every non-degenerate orbit closure is a homology 1-sphere. In this paper, we consider minimal sets of \( R \)-closed homeomorphisms on closed surfaces. Precisely, let \( f \) be an \( R \)-closed homeomorphism on a connected orientable closed surface \( M \). Then we show that if \( M \) has genus more than one, then each minimal set is either a periodic orbit or an extension of a Cantor set. If \( M = \mathbb{T}^2 \) and \( f \) is neither minimal nor periodic, then either the orbit class space \( \mathbb{T}^2/\tilde{f} \) is a 1-manifold and each minimal set is a finite disjoint union of essential circloids, or there is a minimal set which is an extension of a Cantor set. If \( M = S^2 \) and \( f \) is not periodic but orientation-preserving (resp. reversing), then the minimal sets of \( f \) (resp. \( f^2 \)) are exactly two fixed points and other circloids and \( S^2/\tilde{f} \cong [0, 1] \). Finally we state the applications for codimension two foliations.

2. PRELIMINARIES

By a flow, we mean a continuous action of a topological group \( G \) on a topological space \( X \). We call that \( G \) is \( R \)-closed if \( R := \{(x, y) \mid y \in \overline{G(x)}\} \) is closed. Recall that a subset \( S \) of \( G \) is is said to be (left) syndetic if there is a compact set \( K \) of \( G \) with \( KS = G \). For a point \( x \in X \) and an open \( U \) of \( X \), let \( N(x, U) = \{g \in G \mid gx \in U\} \). We say that \( x \) is an almost periodic point if \( N(x, U) \) is syndetic for every neighborhood \( U \) of \( x \). A flow \( G \) is pointwise almost periodic if every point \( x \in X \) is almost periodic. When \( X \) is a compact metrizable (i.e. compact Hausdorff) space, they are known that if \( f \) is \( R \)-closed, then \( f \) is pointwise almost periodic, and that \( f \) is pointwise almost periodic if and only if \( \overline{G(x)} \mid x \in X \) \( \) is a decomposition of \( X \). When \( f \) is pointwise almost periodic, write \( \tilde{F} := \{\overline{G(x)} \mid x \in X \} \) the decomposition of \( X \). Note that \( X/\tilde{F} \) is called an orbit class space and is also denoted by \( X/G \).
pointwise almost periodic flow $G$ is weakly almost periodic in the sense of Gottschalk \[\text{if the saturation of orbit closures for any closed subset of } X \text{ is closed (i.e. the quotient map } X \to X/G \text{ is closed). By Theorem 5 [Ma] and Proposition 1.2 [Y], the following are equivalent for a pointwise almost periodic flow } f \text{ on a compact metrizable space: } 1) \hat{F} \text{ is } R\text{-closed, } 2) \hat{F} \text{ is weakly almost periodic, } 3) \hat{F} \text{ is upper semi-continuous, } 4) X/\hat{F} \text{ is Hausdorff.}

By a continuum we mean a compact connected metrizable space which is not a singleton. A continuum $A \subset X$ is said to be annular if it has a neighbourhood $U \subset X$ homeomorphic to an open annulus such that $U - A$ has exactly two components each of which is homeomorphic to an annulus. We call any such $U$ an annular neighbourhood of $A$. We say a subset $C \subset X$ is a cirlcoid if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset. For a subset $A$ of $X$ and a decomposition $\hat{F}$, the saturation $\text{Sat}(A)$ of $A$ is the union $\bigcup\{L \in \mathcal{F} \mid A \cap L \neq \emptyset\}$ of elements of $\hat{F}$ intersecting $A$.

**Lemma 2.1.** Let $X$ be a sequentially compact space and $(C_n)$ a sequence of connected subsets of $X$. Suppose that there are disjoint open subsets $U, V$ of $X$ and sequences $(x_n)$ (resp. $(y_n)$) converging to $x \in U$ (resp. $y \in V$) with $x_n, y_n \in C_n$. Then there is an element $z \in (\bigcap_{n>0} \bigcup_{k>n} C_k) \setminus U \cup V$.

**Proof.** Let $F = X - U \cup V$ be a closed subset. Since $C_n$ is connected, each $C_n$ intersects $F$. Choose $z_n \in C_n \cap F$. Since $X$ is sequentially compact, we have $F$ is also sequentially compact. Hence there is a convergent subsequence of $z_n$ and so the limit $z \in F$ is desired. $\square$

We show that connected closures for an $R$-closed flow must converge to a connected closure.

**Lemma 2.2.** Let $G$ be an $R$-closed flow on a sequentially compact space $X$ and let $(w_n)$ be a convergent sequence to a point $w \in M$. If each $G(w_n)$ is connected, then the closure $\overline{G(w)}$ is connected.

**Proof.** Put $C_n := \overline{G(w_n)}$. Suppose that $\overline{G(w)}$ is disconnected. Then there are disjoint open subsets $U, W$ of $M$ such that $\overline{G(w)} \subseteq U \cup W$, $\overline{G(w)} \cap U \neq \emptyset$, and $\overline{G(w)} \cap W \neq \emptyset$. Then $G(w) \cap U \neq \emptyset$ and $G(w) \cap W \neq \emptyset$. Since $w_n$ converges to $w$, the continuity of $G$ implies that there are sequences $(x_n)$ (resp. $(y_n)$) converging $x \in U$ (resp. $y \in W$) such that $x_n, y_n \in C_n$. By Lemma 2.1, there is an element $z \in (\bigcap_{n>0} \bigcup_{k>n} C_k) \setminus U \cup W$. Then there is a convergent sequence $(z_n \in C_n)$ to $z$. Since $z_n \in C_n = G(w_n)$ and $z \notin \overline{G(w)}$, we obtain $(w_n, z_n) \in R$ and $(w, z) \notin R$. This contradicts the $R$-closedness. Therefore $C$ is connected. $\square$

Let $f$ be a pointwise almost periodic homeomorphism on an orientable connected closed surface $M$. Recall $\hat{F} = \mathcal{F}_f = \overline{\{O_f(x) \mid x \in M\}}$. Write $V = V_f := \{x \in M - \text{Fix}(f) \mid O(x) \text{ is connected }\} = \cup\{L \in \mathcal{F} : \text{connected}\} - \text{Fix}(f)$.

**Lemma 2.3.** If $f$ is not minimal, then $V$ consists of cirlcoids.

**Proof.** Since $f$ is pointwise almost periodic, we have that the non-wandering set of $f$ is $M$. By Theorem 1.1.1 [K], each element $C$ in $V$ of $\hat{F}$ is annular. Let $U$ be a sufficiently small annular neighbourhood of $C$ such that $U - C$ is a disjoint union of two open annuli $A_1, A_2$. Since $C$ is $f$-invariant and minimal, we have that $C = \partial A_1 \cap \partial A_2$. Suppose that there is an annular continuum $C' \subseteq C$. Then there
is an annular neighbourhood $U'$ of $C'$ such that $U' \subset U$. Embedding $U$ into $S^2$, we may assume that $U$ is a subset of $S^2$. Then $S^2 - C$ is a disjoint union of two open disks $D_1, D_2$ and $S^2 - C'$ is a disjoint union of two open disks $D'_1, D'_2$. Since $S^2 - C' \supseteq S^2 - C$, we have $D_1 \cup D_2 \subseteq D'_1 \cup D'_2$. Since $D_1 \cup D_2 \cup \{x\}$ for any element $x \in C$ is connected, we obtain $D'_1 \cup D'_2$ is connected. This contradicts to disconnectivity. Thus $C$ is a circloid. □

Note that a point $x$ is almost periodic if and only if for every open neighborhood $U$ of $x$, there is $K \in \mathbb{Z}_{\geq 0}$ such that $\mathbb{Z} = \{n, n + 1, \ldots, n + K \mid n \in N(x, U)\}$. The above lemmas implies the following statement.

**Corollary 2.4.** Suppose $f$ is not minimal but $R$-closed. Each point of $\overline{V} - V$ is a fixed point.

Taking the iteration, we obtain the following corollary.

**Corollary 2.5.** Suppose $f$ is not minimal. For any $x \in M$, if $\overline{O(x)}$ is not periodic but consists of finitely many connected components, then $\overline{O(x)}$ consists of circloids.

**Proof.** Let $k$ be the number of the connected components of $\overline{O(x)}$. By Theorem 1 [ES], we have $f^k$ is also pointwise almost periodic and so $\overline{O(x)}$ is connected. By Lemma 2.3 $O_{f^k}(x)$ is a circloid. Since each connected component of $\overline{O(x)}$ is homeomorphic to each other, the assertion holds. □

This corollary can sharpen Theorem 6 [Ma] into the following statement.

**Corollary 2.6.** Let $f$ be a non-periodic $R$-closed orientation-preserving (resp. reversing) homeomorphism on $S^2$. Then $S^2/f$ is a closed interval and $\hat{F}_f$ (resp. $\hat{F}_{f^2}$) consists of two fixed points and other circloids.

**Proof.** Suppose that $f$ is orientation-preserving. By Theorem 3 and 6 [Ma], there are exactly two fixed points and all other orbit closures of $f$ are connected. By Lemma 2.3 they are circloids. We show that $M/\hat{F}$ is a closed interval. Indeed, let $A$ be the sphere minus two fixed points. Suppose that there is a circloid $L$ which is null homotopic in $A$. Let $D$ be a disk bounded by $L$ in $A$. Since $M$ consists of non-wandering points, the Brouwer’s non-wandering Theorem [B] to $D$ implies that $f|_D$ has a fixed point. This contradicts to the non-existence of fixed points in $A$. On the orientation reversing case, since $f^2$ is orientation-preserving, the assertion holds. □

Note that if there is a dense orbit and $\hat{F}$ is pointwise almost periodic, then $\hat{F}$ is minimal and $V = T^2 = M$. Now we proof a key lemma.

**Lemma 2.7.** Suppose that $f$ is an orientation-preserving (resp. reversing) $R$-closed homeomorphism on an orientable connected closed surface $M$. If there is a minimal set which is a circloid, then $M/\hat{F} = V/\hat{F}$ is a closed interval or a circle. Moreover either $M \cong S^2$ and $\hat{F}_f$ (resp. $\hat{F}_{f^2}$) consists of exactly two fixed points and other circloids, or $M \cong \mathbb{T}^2$ and $\hat{F}_f$ (resp. $\hat{F}_{f^2}$) consists of essential circloids.

**Proof.** Fix a metric compatible to the topology of $M$. First, suppose that $f$ is orientation-preserving. First we show that $V$ is open. Let $L$ be a circloid of $\hat{F}$ with a sufficiently small annular neighbourhood $A$. Since $\hat{F}$ is $R$-closed, Lemma 1.6 [Y] implies that the quotient map $M \to M/\hat{F}$ is closed and so the saturation
Let \( f \) be a \( R \)-closed homeomorphism on a closed surface with genus more than one. Then each non-periodic minimal set of \( f \) has infinitely many connected components.

**Proof.** Suppose that there is a non-periodic minimal set \( M \) of \( f \) with at most finitely many connected components. Let \( k \) be the number of connected components of \( M \). Then each connected component \( M' \) of \( M \) is a minimal set of \( f^k \). By Lemma 2.3, we obtain that \( M' \) is a circloid. By Corollary 2.5, we have \( M \) is \( S^2 \) or \( T^2 \). This contradicts to the hypothesis. \( \square \)

### 3. Main Results and Their Proofs

We say that a minimal set \( M \) on a surface homeomorphism \( f : S \to S \) is an extension of a Cantor set (resp. a periodic orbit) if there are a surface homeomorphism \( \tilde{f} : S \to S \) and a surjective continuous map \( p : S^2 \to S^2 \) which is homotopic
to the identity such that $p \circ f = \tilde{f} \circ p$ and $p(M)$ is a Cantor set (resp. a periodic orbit) which is a minimal set of $\tilde{f}$. Now we state main results.

**Theorem 3.1.** Let $M$ be a connected orientable closed surface with genus more than one. Each minimal set of an $R$-closed homeomorphism on $M$ is either a periodic orbit or an extension of a Cantor set.

**Proof.** Let $M$ be a minimal set. By Lemma 2.7, $M$ is not a finite disjoint union of circloids. By Theorem [PX], we have that $M$ is an extension of either a periodic orbit or a Cantor set. We may assume that $M$ is an extension of a periodic orbit. By the proof of Addendum 3.17 [JKP] and Proposition 5.1 [PX], we obtain that $M$ has at most finitely many connected components. By Corollary 2.8, this minimal set $M$ is a periodic orbit.

In the toral case, we obtain the following statement.

**Theorem 3.2.** Each $R$-closed toral homeomorphism $f$ satisfies one of the following:
1. $f$ is minimal.
2. $f$ is periodic.
3. Each minimal set is a finite disjoint union of essential circloids.
4. There is a minimal set which is an extension of a Cantor set.

**Proof.** Suppose that $f$ is neither minimal nor periodic and there are no minimal sets which are extensions of Cantor sets. Since $f$ is not periodic, by Theorem 4 [JKP], there is a minimal set $M$ which is a finite disjoint union of circloids. Let $k$ be the number of the connected components of $M$. By Theorem 1.1 [Y2], the iteration $f^k$ is also $R$-closed. Applying Lemma 2.7 to $f^k$, we have that each minimal set of $f$ is a finite disjoint union of essential circloids.

Recall that $f$ is aperiodic if $f$ has no periodic orbits. By Theorem D [J], we obtain the following corollary.

**Corollary 3.3.** Each orbits closure of a non-minimal aperiodic $R$-closed toral homeomorphism isotopic to identity is a circloid.

4. Applications to codimension two foliations

In [Y], it show that a foliated space on a compact metrizable space which is minimal or “compact and without infinite holonomy”, is $R$-closed. Since each compact codimension two foliation on a compact manifold has finite holonomy [Ep] [Y], we have that the set of minimal or compact codimension two foliations is contained in the set of codimension two $R$-closed foliations. The following examples are codimension two $R$-closed foliations which are neither minimal nor compact. Considering an axisymmetric embedding of $T^2$ (resp. $S^2$) into $R^3$, any irrational rotation on it around the axis is a non-periodic $R$-closed homeomorphism. Taking a suspension on $T^2$ (resp. $S^2$), we obtain the following statement by Theorem 3.2 (resp. Corollary 2.10).

**Corollary 4.1.** Each suspension of a $R$-closed homeomorphism on $T^2$ or $S^2$ which is neither minimal nor periodic induces a codimension two $R$-closed foliation which is neither minimal nor compact. Moreover there are such homeomorphisms on $T^2$ and $S^2$. 
References

[B] L. Brouwer, Beweis des ebenen Translationssatzes Math. Ann. 72 (1912) 37–54.

[Ep] D.B.A. Epstein. Periodic flows on three-manifolds Ann. of Math., 95, 66–82, 1972.

[ES] Erdös, P., Stone, A. H., Some remarks on almost periodic transformations Bull. Amer. Math. Soc. 51, (1945). 126–130.

[G] Gottschalk, W. H., Almost period points with respect to transformation semi-groups Ann. of Math. (2) 47, (1946). 762–766.

[J] Jäger, T., Linearization of conservative toral homeomorphisms Invent. math. 176, 60–616 (2009).

[JKP] Jäger T., Kwakkel F., Passeggi A. A Classification of Minimal Sets of Torus Homeomorphisms Math. Z. (2012) to appear.

[K] Koropecki, A., Aperiodic invariant continua for surface homeomorphisms Math. Z. 266 (2010), no. 1, 229–236.

[Ma] Mason, W. K., Weakly almost periodic homeomorphisms of the two sphere Pacific J. Math. 48 (1973), 185–196.

[PX] A. Passeggi, J. Xavier, A classification of minimal sets for surface homeomorphisms [arXiv:1208.1650]

[S] Swanson, R., Periodic orbits and the continuity of rotation numbers Proc. Amer. Math. Soc. 117 (1993), no. 1, 269–273.

[V] Vogt, E., Foliations with few non-compact leaves Algebr. Geom. Topol. 2 (2002), 257–284.

[Y] Yokoyama, T., Recurrence, almost periodicity and orbit closure relation for topological dynamics preprint.

[Y2] Yokoyama, T., Toral or non locally connected minimal sets for R-closed surface homeomorphisms preprint.

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