Self-Dual Soliton Solution in a Generalized Jackiw-Pi Model

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Abstract

We consider a generalization of Jackiw-Pi model by introducing a nonstandard kinetic term. We present a Bogomolnyi framework for this theory and as a particular case we show that the Bogomolnyi equations of Chern-Simons Higgs theory can be obtained. Finally, the dimensionally reduced theory is analyzed and novel solitonic equations emerge.

Keywords: Chern-Simons gauge theory, Topological solitons

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1 Introduction

The two dimensional matter field interacting with gauge fields whose dynamics is governed by a Chern-Simons term support soliton solutions[1], [2]. These models have the particularity to became auto-dual when the self-interactions are suitably chosen[3, 4]. When this occur the model presents particular mathematical and physics properties, such as the supersymmetric extension of the model[5], and the reduction of the motion equation to first order derivative equation[6]. The Chern-Simons gauge field inherits its dynamics from the matter fields to which it is coupled, so it may be either relativistic[3] or non-relativistic[4]. In addition the soliton solutions are of topological and non-topological nature[7].

In the recent years, theories with nonstandard kinetic term, named $k$-field models, have received much attention. The $k$-field models are mainly in connection with effective cosmological models[8, 9] as well as the strong interaction physics, strong gravitational waves[10] and dark matter[11]. One interesting aspect to analyze in these
models concern to its topological structure. In this context several studies have been conducted, showing that the $k$-theories can support topological soliton solutions both in models of matter as in gauged models[12, 13]. These solitons have certain features such as their characteristic size, which are not necessarily those of the standard models[14]. Other interesting aspects are that they do not interact at large distances and they are, in general, not self-dual.

In this paper we are interested in studying the Jackiw-Pi model with a generalized dynamics. The so call Jackiw-Pi model is a nonrelativistic and Galilean invariant model which is also self-dual[4]. Here, we will show that introducing a nonstandard dynamics in the Jackiw-Pi Lagrangian, via a function $\omega$ depending on the Higgs field, we can obtain self-dual or Bogomolnyi equations by minimizing the energy functional of the model. As particular case, we will show that choosing a suitable $\omega$, the Bogomolnyi equations of Chern-Simons Higgs theory can be obtained. Finally, we will study the the dimensional reduction of the model to $(1 + 1)$-dimensions and we will arrive to the existence of interesting soliton solutions in the system.

2 The Model

Let us start by considering the model proposed by Jackiw and Pi[4]

$$S = \int d^3x \left( \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + i \phi^* D_0 \phi - \frac{1}{2m} |D_i \phi|^2 + \lambda |\phi|^4 \right)$$  \hspace{1cm} (1)

This is a nonrelativistic model where the gauge fields dynamics is dictated by a Chern-Simons term and matter is represented by scalar field $\phi(x)$. The covariant derivative is defined as $D_\mu = \partial_\mu + ieA_\mu$ ($\mu = 0, 1, 2$). The metric tensor is $g^{\mu\nu} = (1, -1, -1)$ and $\epsilon^{\mu\nu\lambda}$ is the totally antisymmetric tensor such that $\epsilon^{012} = 1$.

The field equations corresponding to this action are

$$iD_0 \phi = -\frac{1}{2m} D_i^2 \phi - 2\lambda |\phi|^2 \phi$$
$$B = \frac{e}{\kappa} \rho$$
$$E^i = -\frac{1}{\kappa} \epsilon^{ij} j_j$$  \hspace{1cm} (2)

where $\rho = |\phi|^2$ and $j^i = -\frac{i}{2m} \left( \phi^* D_i \phi - (D_i \phi)^* \phi \right)$. The second of this equations is the Chern-Simons Gauss law, which can be integrated, over the entire plane, to obtain the important consequence that any object with charge $Q = e \int d^2x \rho$ also carries magnetic flux $\Phi = \int Bd^2x$ [15]:

$$\Phi = \frac{1}{\kappa} Q$$  \hspace{1cm} (3)
Here, we are interested in time-independent soliton solutions that ensure the finiteness of the action (1). These are the stationary points of the energy which for the static field configuration reads

$$E = \int d^2 x \left( \frac{1}{2m} |D_i \phi|^2 - \lambda |\phi|^4 \right)$$

(4)

In order to find the minimum of the energy, the expression (4) can be rewritten as

$$E = \int d^3 x \left( \frac{1}{2m} |D_\pm \phi|^2 + (-\lambda \mp \frac{e^2}{2m\kappa}) |\phi|^4 \right)$$

(5)

where we have used the Chern-Simons Gauss law and the identity

$$|D_i \phi|^2 = |(D_1 \pm iD_2) \phi|^2 \mp eB|\phi|^2 \pm m \epsilon^{ij} \partial_i J_j$$

(6)

Thus, with the self-dual coupling

$$\lambda = \mp \frac{e^2}{2m\kappa}$$

(7)

and sufficiently well behaved fields so that the integral over all space of $\epsilon^{ij} \partial_i J_j$ vanishes, the energy becomes

$$E = \int d^3 x \frac{1}{2m} |D_\pm \phi|^2$$

(8)

Thus, the energy is bounded below by zero, and this lower bound is saturated by fields obeying the first order self-duality equations

$$(D_1 \pm iD_2) \phi = 0$$

$$B = \frac{e}{\kappa} \rho$$

(9)

Following the same idea of the works cited in Ref.[12], we will consider, here, a generalization of the Jackiw-Pi model described by the action

$$S = \int d^3 x \left( \frac{2}{\kappa} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + i \omega(\rho) \phi^* D_0 \phi - \frac{1}{2m} |D_\rho \phi|^2 - V(\rho) \right)$$

(10)

where we have replaced the usual kinetic term $i\phi^* D_0 \phi$ by a more generalized term $i\omega(\rho) \phi^* D_0 \phi$. Here, $\omega(\rho)$ is, in principle, an arbitrary function of the complex scalar field $\phi$ and $V(\rho)$ is the scalar field potential to be determined below.

The equations of motion for this system are given by

$$i \left( \frac{\partial \omega(\rho)}{\partial \phi^*} \phi^* D_0 \phi + \omega(\rho) D_0 \phi \right) = -\frac{1}{2m} D_i^2 \phi + \frac{\partial V(\rho)}{\partial \phi^*}$$

$$B = \frac{e}{\kappa} \omega(\rho) \rho$$

$$E^i = -\frac{1}{\kappa} \epsilon^{ij} j_j$$

(11)
where the two first equations differ from those present in Eq.(2) by the presence of the function \( \omega(\rho) \).

The theory may be described in terms of the Hamiltonian formulation as

\[
H = \int d^2x \left( \frac{1}{2m} |D_i \phi|^2 + V(\rho) \right)
\]

(12)

which may be rewritten using the Gauss Law and the identity (6) in the form

\[
E = \int d^3x \left( \frac{1}{2m} |D_\pm \phi|^2 \mp \frac{e^2}{2m\kappa} \omega(\rho)^2 + V(\rho) \right)
\]

(13)

In order to relate the solutions in this theory with those present in the Chern-Simons Higgs theory, we may choose, as a particular case, the following \( \omega(\rho) \) function

\[
\omega(\rho) = 2m \frac{e^2}{\kappa} (\rho - 1)
\]

(14)

Then, the energy functional (13) is written as

\[
E = \int d^3x \left( \frac{1}{2m} |D_\pm \phi|^2 \mp \frac{e^4}{\kappa^2} \rho^2 (\rho - 1)^2 + V(\rho) \right)
\]

(15)

and the Gauss law of the equation (11) takes the form

\[
B = 2m \frac{e^3}{\kappa^2} (\rho - 1)\rho
\]

(16)

The form of the potential \( V(\rho) \) that we choose is motivated by the desire to find self-dual soliton solution. Thus, if we choose the potential as

\[
V(\rho) = \pm \frac{e^4}{\kappa^2} \rho (\rho - 1)^2
\]

(17)

and replace it in the expression (15) we arrive to following expression of energy functional

\[
E = \int d^3x \left( \frac{1}{2m} |D_\pm \phi|^2 \mp \frac{e^2}{2m} B \right)
\]

(18)

which is bounded below by a multiple of the magnitude of the magnetic flux (for positive flux we choose the lower signs, and for negative flux we choose the upper signs):

\[
E \geq \frac{e}{2m} |\Phi|
\]

(19)

Here, the magnetic flux is determined by the requirement of finite energy. This implies that the covariant derivative must vanish asymptotically, which fixes the behavior of the gauge field \( A_i \). Then we have

\[
\Phi = \int d^2x B = \oint_{|x| = \infty} A_i dx^i = 2\pi N
\]

(20)
where $N$ is a topological invariant which takes only integer values. It is interesting to remark, here, the existence of the topological bound which is not present in the Jackiw-Pi model. So, this a non-relativistic model with a topological bound and therefore we shall expect to find topological solitons.

This bound is saturated by fields satisfying the first-order self-duality equations

$$D_{\pm} \phi = (D_1 \pm i D_2) \phi = 0$$

$$B = 2m e^3 \kappa^2 (\rho - 1) \rho$$

These equations may be compared with the self-duality equations of the Chern-Simons Higgs theory. We can note that if we fix $m = 1$ and choose the plus sign in the potential expression we arrive to the Chern-Simons Higgs self-duality equations

$$D_1 \phi = 0$$

$$B = 2m e^3 \kappa^2 (\rho - 1) \rho$$

On the other hand the anti-self-duality equations may be obtained by choosing the function $\omega(\rho)$ as

$$\omega(\rho) = -2m e^2 \kappa^2 (\rho - 1)$$

In that case, if we desire to arrive to the expression, we must choose the following potential term

$$V(\rho) = \pm e^4 \kappa^2 \rho(\rho - 1)^2$$

Then, the Bogomolny equations becomes

$$D_{\pm} \phi = (D_1 \pm i D_2) \phi = 0$$

$$B = -2m e^3 \kappa^2 (\rho - 1) \rho$$

Choosing $V(\rho) = e^4 \kappa^2 \rho(\rho - 1)^2$ we obtain the anti-self-duality equations

$$D_- \phi = (D_1 - i D_2) \phi = 0$$

$$B = -2m e^3 \kappa^2 (\rho - 1) \rho$$

Thus, we can obtain the same Bogomolny equations as those present in the Chern-Simons Higgs model. The difference lies in the fact that in our case, we have dealing with a non-relativistic model and we have imposed the constraint $m = 1$. Another interesting fact is that, here, we expect to find both topological and nontopological soliton solutions, just as in Chern-Simons Higgs theory. Also it is worth noting that the Bogomolny equations in Chern-Simons Higgs theory are neither solvable nor integrable. However, numerical solutions can be found using a radial vortex-like ansatz.

An important comment is that the generalized Jackiw-Pi model, studied here, is a self-dual model. This is important because the generalized pure Chern-Simons system previously explored are not self-dual (For instance see Ref. [12]).
3 Dimensional reduction and the solitonic solution

In this section we are interested in analyzing the dimensional reduction of the model (10) as well as in studying the soliton solution in the dimensionally reduced model. In order to analyze the lineal problem [16, 17], it is natural to consider a dimensional reduction of the action (10) by suppressing dependence on the second spacial coordinate, renaming $A_y$ as $B$. Then, the action (10) becomes

$$S = \int d^2x \left( \kappa (A_0 \partial_x B + B \partial_0 A_1) + i \omega(\rho) \phi^* D_0 \phi - \frac{1}{2m} |D_x \phi|^2 - \frac{e^2}{2m} B^2 \rho - V(\rho) \right)$$

(31)

Notice that the Gauss law constraint for this action is

$$\partial_x B = \frac{e}{\kappa} \omega(\rho) \rho,$$

(32)

which can be solved as

$$B(x) = \frac{e}{2\kappa} \int dz \epsilon(x - z) \omega(\rho(z)) \rho(z)$$

(33)

where $\epsilon(x) = \theta(x) - \theta(-x)$ is the odd step function.

By using the Gauss law and the explicit form of $B$ given by equation (33), the action can be written simply as

$$S = \int d^2x \left[ \left( \frac{e}{2} \int dz \epsilon(x - z) \omega(\rho(z)) \rho(z) \right) \partial_0 A_1 + i \omega(\rho) \phi^* \partial_0 \phi - \frac{1}{2m} |D_x \phi|^2 - \frac{e^2}{2m} B^2 \rho - V(\rho) \right]$$

(34)

Following the Ref. [17], the gauge field $A_x$ may be eliminate from the action (34) by a gauge transformation. Indeed, after transforming the matter field as

$$\phi(x) \rightarrow e^{-i\alpha(x)} \phi(x)$$

(35)

with

$$\alpha(x) = \frac{e}{2} \int dz \epsilon(x - z) A_x(z)$$

(36)

we arrive to the following action

$$S = \int d^2x \left( i \omega(\rho) \phi^* \partial_0 \phi - \frac{1}{2m} |\partial_x \phi|^2 - \frac{e^2}{2m} B^2 \rho - V(\rho) \right)$$

(37)

Consider, now, the derivation of the Bogomolnyi equations in the reduced model (37). As discussed in Ref. [18, 19] the field $B$ plays an important role in the derivation of the
self-dual equations. Indeed the expression (33) of \( B \), involves the existence of a novel soliton solution. Using the relation
\[
\int d^2x \left( (\partial_x + \gamma eB)^2 + \gamma e\partial_x B \rho \right) = \int d^2x \left( |\partial_x \phi|^2 + e^2 B^2 \rho \right)
\] (38)
we can rewrite the action (37) as
\[
S = \int d^2x \left( i\omega(\rho)\phi^* \partial_0 \phi - \frac{1}{2m}|(\partial_x + \gamma eB)\phi|^2 - \frac{e^2}{2m}\partial_x B \rho - V(\rho) \right)
\] (39)
Here, \( \gamma = \pm 1 \). The Gauss law (32) may be used to replace the derivative of the field \( B \) in the action. Then we have, in the static field configuration, that the Hamiltonian associated to action is written as
\[
H = \int dx \left( \frac{1}{2m}|(\partial_x + \gamma eB)\phi|^2 + \frac{e^2}{2m}\omega(\rho)\rho^2 + V(\rho) \right)
\] (40)
As in the \((2 + 1)\)-dimensional case, we can choose \( \omega(\rho) = 2m\frac{e^2}{\kappa}(\rho - 1) \) and \( V(\rho) = \mp \frac{e^4}{4\kappa}\rho(\rho - 1)^2 \) to obtain
\[
H = \int dx \left( \frac{1}{2m}|(\partial_x + \gamma eB)\phi|^2 + \frac{e^2}{2m}\partial_x B \right)
\] (41)
Since the field \( B \) must be zero in the boundary the last term in the Hamiltonian vanish and we have
\[
H = \int dx \frac{1}{2m}|(\partial_x + \gamma eB)\phi|^2
\] (42)
This is non-negative and therefore takes the minimum when the \( \phi \) satisfies
\[
(\partial_x + \gamma eB)\phi = 0
\] (43)
We can write this equation in a more explicitly form by using the Eq. (33)
\[
\partial_x \phi(x) + \frac{\gamma e}{2\kappa} \int dz \epsilon(x - z)\omega(\rho(z))\rho(z)\phi(x) = 0
\] (44)
To solve (44), we can proceed as in Ref. [19]. Thus, we assume that \( \phi \) may be written as \( \phi = \sqrt{\rho} \), which leads to
\[
\frac{1}{2} \partial_x (\log \rho(x)) + \frac{e\gamma}{2\kappa} \int dz \epsilon(x - z)\omega(\rho(z))\rho(z) = 0
\] (45)
Differentiating the above equation with respect to \( x \), we arrive to the following 1-dimensional Liouville type equation
\[
\frac{1}{2} \partial_x^2 (\log \rho(x)) + \frac{e\gamma}{\kappa} \omega(\rho(x))\rho(x) = 0
\] (46)
This is the general Bogomolnyi equation corresponding to dimensionally reduced model. In particular we are analyzing the case $\omega(\rho) = 2m^2(\rho - 1)$, so that the previous equation becomes

$$\frac{1}{2}\partial_x^2(\log \rho(x)) + \frac{2e^3\gamma}{\kappa^2}(\rho - 1)\rho(x) = 0 \quad (47)$$

This equation presents two types of solution, one derived from the topological solution and the other from nontopological solution present in the two dimensional Chern-Simons Higgs theory. Let us start by considering the solution derived from the topological case. For this case we propose as a solution the following series

$$\rho = 1 + \sum_{n=1}^{\infty} a_n \text{sech}^n(bx) \quad (48)$$

where $a_n$ are the real coefficients of series and $b$ is a real constant. To check that this is really a solution, we rewrite Eq. (47) as

$$-(\partial_x \rho)^2 + (\partial_x^2 \rho)\rho + v\rho^3(\rho - 1) = 0 \quad (49)$$

with $v = \frac{4e^3\gamma}{\kappa^2}$. Inserting the series (48) into equation (49), we obtain

$$\sum_{n,m=1}^{\infty} \left[ -a_n a_m mnb^2 + n^2 a_n a_m b^2 + 3va_n a_m \right] \text{sech}^{n+m}(bx) +$$

$$\sum_{n,m=1}^{\infty} \left[ a_n a_m mnb^2 - n^2 a_n a_m b^2 - na_n a_m b^2 \right] \text{sech}^{n+m+2}(bx) -$$

$$\sum_{n=1}^{\infty} (n^2 + n)a_n b^2 \text{sech}^{2+n}(bx) + \sum_{n=1}^{\infty} \left[ n^2 a_n b^2 + va_n \right] \text{sech}^n(bx) +$$

$$3v \sum_{n,m,i=1}^{\infty} a_n a_m a_i \text{sech}^{n+m+i}(bx) + v \sum_{n,m,i,j=1}^{\infty} a_n a_m a_i a_j \text{sech}^{n+m+i+j}(bx) = 0 \quad (50)$$

where, in this last equation, we have used the relation

$$\tanh^2(bx) = 1 - \text{sech}^2(bx) \quad (51)$$

Since the expression (50) is an expansion of powers of $\text{sech}(bx)$ each coefficient of different powers must vanish separately. This implies that for the coefficient of $\text{sech}(bx)$, we have following relation

$$b^2 = -v \quad (52)$$

whereas that for the coefficients of $\text{sech}^2(bx)$ and $\text{sech}^3(bx)$, we deduce

$$a_2 = 2a_1^2 \quad (53)$$
and

$$a_3 = \frac{13}{8} a_1^3 - \frac{1}{2} a_1$$

(54)

This procedure may be repeated in order to determine the successive coefficients.

There is another type of solution, which satisfies the boundary condition \( \rho \to 0 \) as \( |x| \to \infty \). This case is more complicated and we are not able to propose an analytic expression for the solution. However, it is not difficult to obtain approximate asymptotic solution for large \(|x|\). In this limit the equation (47) may be approximated by

$$\partial_x^2 (\log \rho(x)) + v(\rho - 1)\rho(x) = 0$$

(55)

As in the previous case the solution exists for \( v < 0 \);

$$\rho = a \sech^2(bx)$$

(56)

where the \( a \) and \( b \) are related by \( b^2 = -\frac{a^2}{2} \). Thus if the solution exist, should approach to zero exponentially.

4 Conclusion

We have studied a generalized Jackiw-Pi model by introducing a non-standard dynamics \( \omega(\rho) \) in the original Jackiw-Pi Lagrangian. It was shown that this model support Bogomolnyi equations and soliton solutions therein, which represent an important fact because the other soliton solutions found in generalized pure Chern-Simons models are not self-dual. In particular have shown that choosing a suitable function \( \omega(\rho) \) and a sixth-order self-dual potential, the energy of the model is bounded below by a topological number. In addition, the resulting Bogomolnyi equations are those of the Chern-Simons Higgs theory.

The introduction of the function \( \omega(\rho) \) has also consequences at quantum level. Although the model may be viewed as the second quantized \( N \)-particle system interacting with a Chern-Simons gauge field, its dynamics is modified by \( \omega(\rho) \), and therefore differs from dynamics of the Jackiw-Pi model. In this case the quantum field equation is

$$i\partial_t \Phi(x) = [\Phi(x), H]$$

(57)

and the gauge field should be subject to the constraint

$$B = e \kappa \omega(\rho) \rho$$

(58)

Finally, it is interesting to mention that the \((1 + 1)\)-dimensional model, obtained by dimensional reduction of the generalized Jackiw-Pi, presents novel solitons solutions. One type of these solitons has a topological origin and have been able to find the analytical expression, the other has non-topological origin and we have found its asymptotic behavior.
References

[1] S. K. Paul and A. Khare, Phys. Lett. B 174, 420 (1986) [Erratum-ibid. 177B, 453 (1986)].
H. J. de Vega and F. A. Schaposnik, Phys. Rev. D 34, 3206 (1986).

[2] for review see:
Peter A. Horvathy and Pengming Zhang, Phys.Rept. 481, 83 (2009).
Peter A. Horvathy, [arXiv:hep-th/0704.3220].
F. A. Schaposnik, [arXiv:hep-th/0611028].
Gerald V. Dunne, [arXiv:hep-th/9902115].

[3] R. Jackiw and E. Weinberg, Phys. Rev. Lett. 64 2334 (1990).
J. Hong, Y. Kim and P-Y. Pac, Phys. Rev. Lett. 64 2330 (1990).

[4] R. Jackiw and S. Y. Pi, Phys. Rev. Lett. 64, 2969 (1990).
R. Jackiw and S. Y. Pi, Phys. Rev. D 42, 3500 (1990). [Erratum-ibid. D 48, 3929 (1993)].

[5] C. Lee, K. Lee and E.J. Weinberg, Phys. Lett. B 253, 105 (1990).

[6] E. Bogomolyi, Sov. J. Nucl. Phys 24, 449 (1976).
H. de Vega and F. A. Schaposnik, Phys. Rev. D 14, 1100 (1976).

[7] R. Jackiw, Ki-Myeong Lee and E. Weinberg, Phys.Rev. D 42, 3488 (1990).

[8] E. Babichev, V. Mukhanov and A. Vikman, J. High Energy Phys. 02, 101 (2008).

[9] C. Armendariz-Picon, T. Damour and V. Mukhanov, Phys. Lett. B 458, 209 (1999).

[10] V. Mukhanov and A. Vikman, J. Cosmol. Astropart. Phys. 02, 004 (2005).

[11] C. Armendariz-Picon and E. A. Lim, J. Cosmol. Astropart. Phys. 08, 007 (2005).

[12] D. Bazeia, E. da Hora, C. dos Santos and R. Menezes, Phys. Rev. D 81, 125014 (2010).
D. Bazeia, E. da Hora, R. Menezes, H. P. de Oliveira and C. dos Santos, Phys. Rev. D 81, 125016 (2010).
C. dos Santos and E. da Hora, Eur. Phys. J. C 70, 1145 (2010); Eur. Phys. J. C 71, 1519 (2011).
C. dos Santos, Phys. Rev. D 82, 125009 (2010).
D. Bazeia, E. da Hora, C. dos Santos, R. Menezes, Eur. Phys. J. C 71, 1833 (2011).
D. Bazeia, J. D. Dantas, Phys. Rev. D 85, 067303 (2012).

[13] C. Adam, N. Grandi, J. Sanchez-Guillen and A. Wereszczynski, J. Phys. A 41, 212004 (2008). Erratum- ibid. A 42, 159801 (2009).
C. Adam, J. Sanchez-Guillen and A. Wereszczynski, J. Phys. A 40, 13625 (2007). Erratum-ibid. A 42, 089801 (2009).
C. Adam, N. Grandi, P. Klimas, J. Sanchez-Guillen and A. Wereszczynski, J. Phys. A 41, 375401 (2008).
C. Adam, P. Klimas, J. Sanchez-Guillen and A. Wereszczynski, J. Phys. A 42, 135401 (2009).

[14] E. Babichev, Phys. Rev. D 74, 085004 (2006).
D. Bazeia, L. Losano, R. Menezes and J. C. R. E. Oliveira, Eur. Phys. J. C 51, 953 (2007).
X. Jin, X. Li. and D. Liu, Classical Quantum Gravity 24, 2773 (2007).

[15] J. Schonfeld, Nucl. Phys. B 185, 157 (1981).
S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. 48, 975 (1982).
S. Deser, R. Jackiw, and S. Templeton, Ann. Phys.(N.Y.) 140, 372 (1982).

[16] Gustavo S. Lozano, Oliver Piguet, Fidel A. Schaposnik and Lucas Sourrouille, Phys. Rev. A 75, 023608 (2007). [arXiv:cond-mat/0609553].

[17] U. Aglietti, L. Griguolo, R. Jackiw, S. Y. Pi and D. Seminara, Phys. Rev. Lett. 77, 4406 (1997) [arXiv:hep-th/9606141]; see also
R. Jackiw, Journal of Nonlinear Physics, 4 (1997) 241, [arXiv:hep-th/9611185];
R. Jackiw and S. Y. Pi, [arXiv:hep-th/9808036]; H. c. Kao, K. M. Lee and T. Lee, Phys. Rev. D 55, 6447 (1997) [arXiv:hep-th/9612183].

[18] P. Oh and C. Rim, Phys. Lett. B 404, 89 (1997) [arXiv:hep-th/9612028].

[19] Lucas Sourrouille, Physical Review D 83:045016,(2011) [arXiv:hep-th/1010.6277].