Anomaly and Cobordism Constraints

Beyond Standard Model: Topological Force

Juven Wang

Center of Mathematical Sciences and Applications, Harvard University, Cambridge, MA 02138, USA

Abstract

Standard lore uses perturbative local anomalies to check the kinematic consistency of gauge theories coupled to chiral fermions, such as Standard Models (SM) of particle physics. In this work, based on a systematic cobordism classification [1, 2], we examine the constraints from invertible quantum anomalies (including all perturbative local anomalies and all nonperturbative global anomalies) for SM and chiral gauge theories. We also clarify the different uses of these anomalies: including (1) anomaly cancellations of dynamical gauge fields, (2) ’t Hooft anomaly matching conditions of background fields of global symmetries, and others. We find several new powerful 4d anomaly constraints of $\mathbb{Z}_{16}$, $\mathbb{Z}_4$, and $\mathbb{Z}_2$ classes, beyond the familiar perturbative $\mathbb{Z}$ class local anomalies of Feynman diagrams, and also beyond Witten SU(2) and the new SU(2) global anomalies. As an application, for $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ SM with 15n chiral Weyl fermions (such as the SU(5) Grand Unification) and a discrete $X^X = 5(B - L) - 4Y$ preserved, we discover a new hidden sector previously unknown to the Georgi-Glashow model — by appending to the known SM nearly gapless sector in a strongly-correlated manner, we propose a new gapped sector with a new Topological Mass mechanism: either (1) 4d non-invertible non-abelian topological quantum field theory at low energy but with heavy anyon excitations from 1d particle worldline and 2d string worldsheet coupled to gravity (i.e., intrinsic Topological Orders with long-range entanglement in 3+1 spacetime dimensions, which is strongly interacting accessible from neither naïve Dirac nor Majorana mass gap for the 16th Weyl fermions: 3 generations of right-handed neutrinos, but potentially from the dual fermionic vortex zero mode bound state condensation via topological quantum phase transition or 4d duality/“mirror symmetry”), or (2) 5d topological field theory (i.e., Symmetry-Protected/Enriched Topological states with short/long-range entanglement in extra dimensions) beyond Standard Models. Above a higher energy scale, the discrete $X$ becomes a dynamical gauge vector boson mediator $X_g$, the entangled Universe in 4d and 5d (i.e., $5d$ bulk topological quantum computer coupled to dynamical gravity is in a unified math and physics framework with SM, but hierarchically develop absolutely different new “chemistry and biology,” thus where hypothetical “God” or “foreign higher-beings,” formed by higher-dimensional extended operators, may exist) is mediated by Topological Force. Our model resolves puzzles, surmounting sterile neutrinos and dark matter, in fundamental physics.

Dedicated to James Clerk Maxwell and Mother on the eve of June 12/13, 2020

jw@cmsa.fas.harvard.edu

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“Der schwer gefaßte Entschluß. Muß es sein? Es muß sein!”
“The heavy decision. Must it be? It must be!”

String Quartet No. 16 in F major, op. 135

Ludwig van Beethoven in 1826
1 Introduction

The Universe where we reside, to our contemporary knowledge, is governed by the laws of quantum theory, the information and long-range entanglement, and gravity theory. Quantum field theory (QFT), especially gauge field theory, under the name of Gauge Principle following Maxwell [3], Hilbert, Weyl [4], Pauli [5], and many pioneers, forms a foundation of the fundamental physics. Yang-Mills (YM) gauge theory [6], generalizing the U(1) abelian gauge group to a non-abelian Lie group, has been given credits for theoretically and experimentally essential to describe the Standard Model (SM) physics [7–9].

The SM of particle physics is a chiral gauge theory in 4d\(^1\) encoding three of the four known fundamental forces or interactions (the electromagnetic, weak, and strong forces, but without gravity). The SM also classifies all experimentally known elementary particles so far: Fermions include three generations of quarks and leptons. (See Table 1 for \(15 + 1 = 16\) Weyl fermions in SM for each one of the three generation, where the additional 1 Weyl fermion is the sterile right-handed neutrino). Bosons include the vector gauge bosons: one electromagnetic force mediator photon \(\gamma\), the eight strong force mediator gluon \(g\) (denoted the gauge field \(A^a\)), the three weak force mediator \(W^\pm\) and \(Z^0\) gauge bosons; and the scalar Higgs particle \(\phi_H\). (See Table 2 for 12 gauge boson generators for SM.) While the spin-2 boson, the graviton, has not yet been experimentally verified, and it is not within SM. Physics experiments had confirmed that at a higher energy of SM, the electromagnetic and weak forces are unified into an electroweak interaction sector: Glashow-Salam-Weinberg (GSW) SM [7–9].

Grand Unifications and Grand Unified Theories (GUT) hypothesize that at a further higher energy, the strong and the electroweak interactions will be unified into an electroweak-nuclear GUT interaction sector. The GUT interaction is characterized by one larger gauge group and its force mediator gauge bosons with a single unified coupling constant [10,11]. \(^2\)

In this article, we revisit the Glashow-Salam-Weinberg SM [7–9], with four possible gauge groups:

\[
G_{SM_q} = \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_q}, \quad q = 1, 2, 3, 6. \tag{1.1}
\]

We revisit the embedding of this special \(q = 6\) SM (with a gauge group \(\frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_q}\)) into Georgi-Glashow SU(5) GUT with an SU(5) gauge group [10] (with 24 Lie algebra generators for SU(5) gauge bosons shown in Table 2), and Fritzsch-Minkowski SO(10) GUT with a Spin(10) gauge group [11]. Our main motivation to revisit these well-known SM and GUT models, following our prior work [1,2], is that the recent systematic cobordism classification of topological invariants [12] can be applied to classify all the invertible quantum anomalies, including

\(^1\)We denote \(dd\) for the \(d\) spacetime dimensions. The \(d + 1d\) means the \(d\) spatial and 1 time dimensions. The \(nD\) means the \(n\) space dimensions.

\(^2\)Unifying the fourth fundamental force, gravity, with the three fundamental forces from GUT interactions may give rise to a Theory of Everything (TOE). However, in our present work, we discover a possible new force, which we name Topological Force.

This Topological Force may be a possible Fifth force not included in the known four fundamental forces and not explored adequately in the prior particle physics literature. On the other hand, when we consider the constraints from the anomaly matching, the gravity mostly plays the role of the gravitational background probed fields (instead of dynamical gravity), such as in the gravitational anomaly or the mixed gauge-gravitational anomaly. We will mostly leave out dynamical gravity outside our model. The only exception that we discuss in the influences of dynamical gravity (or the hypothetical particle: graviton) such as in Quantum Gravity and Topological Gravity will be in Sec. 6.1.
• all perturbative local anomalies, and
• all nonperturbative global anomalies,

which can further mathematically and rigorously constrain SM and GUT models. In fact, many earlier works and recent works suggest the cobordism theory is the underlying math structure of invertible quantum anomalies, see a list of selective References [13–25] and an overview therein. By the completion of all invertible anomalies, we mean that it is subject to a given symmetry group

\[ G \equiv \left( \frac{G_{\text{spacetime}} \ltimes G_{\text{internal}}}{N_{\text{shared}}} \right), \]

(1.2)

where the \( G_{\text{spacetime}} \) is the spacetime symmetry,\(^3\) the \( G_{\text{internal}} \) is the internal symmetry,\(^4\) the \( \ltimes \) is a semi-direct product from a “twisted” extension,\(^5\) and the \( N_{\text{shared}} \) is the shared common normal subgroup symmetry between \( G_{\text{spacetime}} \) and \( G_{\text{internal}} \). Ref. [1,2,26] applies Thom-Madsen-Tillmann spectra [27,28], Adams spectral sequence (ASS) [29], and Freed-Hopkins theorem [12] in order to compute, relevant for SM and GUT, the bordism group

\[ \Omega^G_d, \]

(1.3)

and a specific version of cobordism group (firstly defined to classify Topological Phases \([\text{TP}]\) in [12])

\[ \Omega^d_G \equiv \Omega^d\left( \frac{G_{\text{spacetime}} \ltimes G_{\text{internal}}}{N_{\text{shared}}} \right) \equiv \text{TP}_d(G). \]

(1.4)

For a given \( G \), Ref. [1,2,26] find out corresponding all possible topological terms and all possible anomalies relevant for SM and GUT. See more mathematical definitions, details, and References therein our prior work [1,2].

Along this research direction, other closely related pioneer and beautiful works by Etxebarria-Montero [30] and Davighi-Gripaios-Lohitsiri [31] use a different mathematical tool, based on Atiyah-Hirzebruch spectral sequence (AHSS) [32] and Dai-Freed theorem [33], also compute the bordism groups \( \Omega^G_d \) and classify possible global anomalies in SM and GUT.\(^6\) As we will see, many SM and GUT with extra discrete symmetries require the twisted \( G \), whose \( \Omega^G_d \) is very difficult, if not simply impossible, to be determined via AHSS alone [30,31], but they can be more easily computed via ASS [1,36]. Therefore, we will focus on the results obtained in [1,2].

In this work, we have the plans and outline as follows, with topics by sections:

3For example, \( G_{\text{spacetime}} \) can be the \( \text{Spin}(d) \), the double cover of Euclidean rotational symmetry \( \text{SO}(d) \) for the QFT of a spacetime dimension \( d \).

4For example, \( G_{\text{internal}} \) can be the \( G_{\text{SM}_q} \equiv \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{Z_q} \), \( q = 1, 2, 3, 6 \).

5The “twisted” extension is due to the symmetry extension from \( G_{\text{internal}} \) by \( G_{\text{spacetime}} \). For a trivial extension, the semi-direct product \( \ltimes \) becomes a direct product \( \times \).

6A practical comment is that Adams spectral sequence (ASS) used in Ref. [1,2,12] turn out to be more powerful than the Atiyah-Hirzebruch spectral sequence (AHSS) used in [30,31].

• Ref. [1,2,34–36] based on Adams spectral sequence (ASS) and Freed-Hopkins theorem [12] includes the more refined data, containing both module and group structure, thus with the advantages of having less differentials. In addition, we can conveniently read and extract the topological terms and co/bordism invariants from the Adams chart and ASS [1].

• Ref. [30,31] is based on Atiyah-Hirzebruch spectral sequence (AHSS), which includes only the group structure, with the disadvantage of having more differentials and some undetermined extensions. It is also not straightforward to extract the topological terms and co/bordism invariants directly from the AHSS.
Sec. [2]. **Overview on SM, GUT and Anomalies** in Sec. 2:

We first overview the ingredients of various SM and GUT to set up the stage in Sec. 2.1. Then we comment on the different types of anomalies, and we clarify the different uses of these anomalies: including (1) anomaly cancellations of dynamical gauge fields, (2) ’t Hooft anomaly matching conditions of background fields of global symmetries, and others, in Sec. 2.2 and Fig. 1. We also describe the perturbative local anomalies vs nonperturbative global anomalies; also bosonic vs fermionic anomalies, etc. These are presented in Sec. 2.

Sec. [3]. **Dynamical Gauge Anomaly Cancellation** in Sec. 3:

We explicitly show the classification of all possible anomalies of SM by cobordism data in Table 3, and explicitly check all (invertible) anomaly cancellations for perturbative local anomalies vs nonperturbative global anomalies in Sec. 3.

Sec. [4]. **Anomaly Matching for SM and GUT with Extra Symmetries** in Sec. 4:

By including additional extra symmetries (such as $B - L$ baryon minus lepton numbers or $X$ symmetry [42], motivated in [30] and [2]) into SM and SU(5) GUT, we show the classification of all possible (invertible) anomalies of SM and GUT with extra symmetries by cobordism data in Table 4, and explicitly check whether all (invertible) anomaly cancellations for perturbative local anomalies vs nonperturbative global anomalies in Sec. 4. It turns out that a certain $Z_{16}$ anomaly may not be matched. The resolution, other than introducing a sterile right-handed neutrino, can also be that introducing a new gapped topological sector matching the same anomaly, but preserves the discrete $Z_{4,X}$ symmetry.

Sec. [5]. **Beyond Three “Fundamental” Forces: Hidden New Topological Force** in Sec. 5:

We recall that to match the ’t Hooft anomalies of global symmetries, there are several ways:

(a) Symmetry-breaking:
   - (say, discrete or continuous $G$-symmetry breaking. Explicitly breaking or spontaneously breaking [may give Nambu-Goldstone modes]).

(b) Symmetry-preserving:
   - Degenerate ground states (like the “Lieb-Schultz-Mattis theorem [43,44],” may host intrinsic topological orders),
   - Gapless, e.g., conformal field theory (CFT). There are also novel cases where the anomaly and symmetry together enforces the robustness of gapless ground states [49–51],
   - Symmetry-preserving topological quantum field theory (TQFT): symmetry-enriched anomalous topological orders.

(c) Symmetry-extension [52]: The symmetry-extension in any dimension is a helpful intermediate stepstone, to construct another earlier scenario: *symmetry-preserving TQFT*, via gauging the extended-symmetry [52].

We present the possible ways of matching of missing global anomalies of SM and GUT with extra symmetries, in Sec. 5. The condensed matter realization of ’t Hooft matching by a symmetry-preserving gapped anomalous TQFT is especially exotic and important to us. It is known as the surface topological order, pointed out firstly by Senthil-Vishwanath [53], and the developments are nicely reviewed [54]. The particular $2+1d$ boundary and $3+1d$ bulk results from [55,56] will be helpful for us finding out the $3+1d$ boundary and $4+1d$ bulk analogy.

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7The perturbative parts of calculations are standard on the QFT textbook [37–40]
8Topological order in the sense of Wen’s definition [45] and References therein. The gapped and gauged topological order in the colloquial sense can have fractionalized excitations such as anyons [46]. Some of the quantum vacua we look for in 4d and 5d may be regarded as a certain version of topological quantum computer [47, 48].
Sec. [6]. **Ultra Unification: Grand Unification + Topological Force/Matter** in Sec. 6:

In certain cases of anomaly matching, we require new hidden gapped topological sectors beyond SM and GUT. We may term the unification including SM, Grand Unification plus additional topological sectors with Topological Force and Matter as Ultra Unification. We then suggest possible resolutions to sterile neutrinos, neutrino oscillations, three generation mystery (or three family problem), and Dark Matter, in Sec. 6.

**Notations:** Throughout our work, we follow the same notations as [1, 2]. The imaginary number is $i \equiv \sqrt{-1}$. We use standard notation for characteristic classes [57]: $c_i$ for the Chern class, $e_n$ for the Euler class, $p_i$ for the Pontryagin class, and $w_i$ for the Stiefel-Whitney class. We abbreviate $c_i(G)$, $e_n(G)$, $p_i(G)$, and $w_i(G)$ for the characteristic classes of the associated vector bundle $V_G$ of the principal $G$ bundle (normally denoted as $c_i(V_G)$, $e_n(V_G)$, $p_i(V_G)$, and $w_i(V_G)$). For simplicity, we denote the Stiefel-Whitney class of the tangent bundle $TM$ as $w_j \equiv w_j(TM)$; if we do not specify $w_j$ with which bundle, then it is for $TM$. The PD is defined as the Poincaré dual. All the product notations between cohomology classes are cup product, such as $w_2w_3 \equiv w_2(TM)w_3(TM) = w_2(TM) \cup w_3(TM)$. All the product notations between a cohomology class $A$ and a $\eta$ (say, the eta invariant $\eta$ of Atiyah-Patodi-Singer [13–15], similarly for the $\tilde{\eta}$ as a mod 2 index of 1d Dirac operator, or Arf invariant [58], or Arf-Brown-Kervaire [ABK] invariant [59, 60], etc.), namely $A\eta$, are defined as the value of $\eta$ (or $\tilde{\eta}$, Arf, ABK, etc) on the submanifold of $M$ which represents the Poincaré dual of $A$. Notice that here for $A\eta$, it is crucial to have $A$ as a cohomology class, so we can define its Poincaré dual of $A$ as $\text{PD}(A)$. In other words, the $A\eta \equiv \eta(\text{PD}(A))$. For cobordism invariants, we may implicitly make a convention that cohomology classes are pulled back to the manifold $M$. For example, $A_{\mathbb{Z}_2} \in H^1(M, \mathbb{Z}_2)$ is the generator from $A_{\mathbb{Z}_2} \in H^1(B(\mathbb{Z}_4/\mathbb{Z}_2^F), \mathbb{Z}_2)$ of $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4$; the former is the pullback of the later to $M$. 


Table 1: We show the quantum numbers of $15 + 1 = 16$ left-handed Weyl fermion spinors in each of three generations of matter fields in SM. The 15 of 16 Weyl fermion are $\mathbf{5} \oplus \mathbf{10}$ of SU(5); namely, $(\mathbf{3}, 1, 1/3, 2/3, -1/3) \sim \mathbf{5}$ and $(\mathbf{3}, 2, 1/6, 1/3, 1/3) \sim \mathbf{10}$ of SU(5). The 1 of 16 is presented neither in the standard GSW SM nor in the SU(5) GUT, but it is within $\mathbf{16}$ of the SO(10) GUT. The numbers in the Table entries indicate the quantum numbers associated to the representation of the groups given in the top row. We show the first generation of SM fermion matter fields in Table 1. There are 3 generations, triplicating Table 1, in SM.

Table 2: We show quantum numbers of the electroweak Higgs boson $\phi_H$, and 24 gauge bosons corresponding to 24 Lie algebra $su(5)$ generators of Lie group SU(5). The readers should not be confused with the symmetry charge $X$ (written in the Italic form) and its gauge boson $X_g$, with the Georgi-Glashow (GG) model gauge boson $X$ (written in the text form). See also the caption in Table 1.
2 Overview on Standard Models, Grand Unifications, and Anomalies

2.1 SM and GUT: local Lie algebras to global Lie groups, and representation theory

We shall first overview the local Lie algebra the representation theory of matter field contents, and the global Lie groups of of SM and GUT. Then we will be able to be precise about the spacetime symmetry group $G_{\text{spacetime}}$ and internal symmetry group $G_{\text{internal}}$ relevant for SM and GUT physics,

[I]. The local gauge structure of Standard Model is the Lie algebra $u(1) \times su(2) \times su(3)$. This means that the Lie algebra valued 1-form gauge fields take values in the Lie algebra generators of $u(1) \times su(2) \times su(3)$. There are $1 + 3 + 8 = 12$ Lie algebra generators. The 1-form gauge fields are the 1-connections of the principals $G_{\text{internal}}$-bundles that we should determine.

II. Fermions are the spinor fields, as a sections of the spinor bundles. For the left-handed Weyl spinor $\Psi_L$, it is a doublet spin-1/2 representation of spacetime symmetry group $G_{\text{spacetime}}$ (Minkowski Spin$(3,1)$ or Euclidean Spin$(4)$), denoted as $\Psi_L \sim 2_L$ of $\text{Spin}(3,1)$, or $\Psi_L \sim 2_L$ of $\text{Spin}(4) = SU(2)_L \times SU(2)_R$. These Spin groups are the double-cover or universal-cover version of the Lorentz group $SO(3,1)_+$ or Euclidean rotation $SO(4)$, extended by the fermion parity $\mathbb{Z}_2$. In the first generation of SM, the matter fields as Weyl spinors $\Psi_L$ contain:

- The left-handed up and down quarks ($u$ and $d$) form a doublet $\begin{pmatrix} u \\ d \end{pmatrix}_L$ in 2 for the SU(2)$_{\text{weak}}$, and they are in 3 for the SU(3)$_{\text{strong}}$.
- The right-handed up and down quarks, each forms a singlet, $u_R$ and $d_R$, in 1 for the SU(2)$_{\text{weak}}$. They are in 3 for the SU(3)$_{\text{strong}}$.
- The left-handed electron and neutrino form a doublet $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ in 2 for the SU(2)$_{\text{weak}}$, and they are in 1 for the SU(3)$_{\text{strong}}$.
- The right-handed electron forms a singlet $e_R$ in 1 for the SU(2)$_{\text{weak}}$, and it is in 1 for the SU(3)$_{\text{strong}}$.

There are two more generations of quarks: charm and strange quarks ($c$ and $s$), and top and bottom quarks ($t$ and $b$). There are also two more generations of leptons: muon and its neutrino ($\mu$ and $\nu_{\mu}$), and tauon and its neutrino ($\tau$ and $\nu_{\tau}$). So there are three generations (i.e., families) of quarks and leptons:

\[
\begin{pmatrix}
(u) \\ (d)
\end{pmatrix}_L \times 3_{\text{color}}, \quad u_R \times 3_{\text{color}}, \quad d_R \times 3_{\text{color}}, \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad e_R \tag{2.2}
\]

\[
\begin{pmatrix}
(c) \\ (s)
\end{pmatrix}_L \times 3_{\text{color}}, \quad c_R \times 3_{\text{color}}, \quad s_R \times 3_{\text{color}}, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \mu_R \tag{2.3}
\]

\[
\begin{pmatrix}
(t) \\ (b)
\end{pmatrix}_L \times 3_{\text{color}}, \quad t_R \times 3_{\text{color}}, \quad b_R \times 3_{\text{color}}, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \quad \tau_R \tag{2.4}
\]

In short, for all of them as three generations, we can denote them as:

\[
\begin{pmatrix}
(u) \\ (d)
\end{pmatrix}_L \times 3_{\text{color}}, \quad u_R \times 3_{\text{color}}, \quad d_R \times 3_{\text{color}}, \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad e_R \times 3 \text{ generations.} \tag{2.5}
\]
In fact, all the following four kinds of
\[ G_{\text{internal}} = \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_q} \]  
with \( q = 1, 2, 3, 6 \) are compatible with the above representations of fermion fields. (See an excellent exposition in a recent work by Tong [61].) These 15 \times 3 Weyl spinors can be written in the following more succinct forms of representations for any of the internal symmetry group \( G_{\text{internal}} \) with \( q = 1, 2, 3, 6 \):

\[ \begin{pmatrix} (3, 2, 1/6)_L, (3, 1, 2/3)_R, (3, 1, -1/3)_R, (1, 2, -1/2)_L, (1, 1, 1/2)_R \end{pmatrix} \times 3 \text{ generations} \]

\[ \Rightarrow \begin{pmatrix} (3, 2, 1/6)_L, (3, 1, -2/3)_L, (3, 1, 1/3)_L, (1, 2, -1/2)_L, (1, 1, 1)_L \end{pmatrix} \times 3 \text{ generations}. \]  

The triplet given above is listed by their representations:

\[ (\text{SU}(3) \text{ representation}, \text{SU}(2) \text{ representation}, \text{hypercharge } Y). \]  

(2.8)

For example, \((3, 2, 1/6)\) means that 3 in SU(3), 2 in SU(2) and 1/6 for hypercharge. In the second line of (2.7), we transforms the right-handed Weyl spinor \( \Psi_R \sim 2_R \) of Spin(3,1) to its left-handed \( \Psi_L \sim 2_L \) of Spin(3,1), while we flip their representation (2.8) to its complex conjugation representations.\(^9\) If we include the right-handed neutrinos (say \( \nu_{eR}, \nu_{\mu R}, \) and \( \nu_{\tau R} \)), they are all in the representation

\[ (1, 1, 0)_R \]  

(2.9)

with no hypercharge. We can also represent a right-handed neutrino by the left-handed (complex) conjugation version

\[ (1, 1, 0)_L. \]  

(2.10)

Also the complex scalar Higgs field \( \phi_H \) is in a representation

\[ (1, 2, 1/2). \]  

(2.11)

In the Higgs condensed phase of SM, the conventional Higgs vacuum expectation value (vev) is chosen to be \( \langle \phi_H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \), which vev has \( Q_{\text{EM}} = 0 \).

Note that our hypercharge \( Y \) is given conventionally by the relation: \( Q_{\text{EM}} = T_3 + Y \) where \( Q_{\text{EM}} \) is the unbroken (not Higgsed) electromagnetic gauge charge and \( T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) is a generator of SU(2)\(_{\text{weak}}\). However, some other conventions are common, we list down three convention

\[ Q_{\text{EM}} = T_3 + Y = T_3 + \frac{1}{2} Y_W = T_3 + \frac{1}{6} \tilde{Y}. \]  

(2.12)

\(^9\)Note that 2 and \( \overline{2} \) are the same representation in SU(2), see, e.g., in the context of Yang-Mills gauge theories with discrete symmetries and cobordism [62].
In the $Q_{EM} = T_3 + \frac{1}{6} \tilde{Y}$ version, we have the integer quantized $\tilde{Y} = 6Y$. We can rewrite (2.7) as:

$$
\left( (3, 2, Y = 1/6)_L, (3, 1, Y = 2/3)_R, (3, 1, Y = -1/3)_R, (1, 2, Y = -1/2)_L, (1, 1, Y = -1)_L \right) \times 3 \text{ generations}
$$

$$
= \left( (3, 2, Y = 1/6)_L, (3, 1, Y = -2/3)_L, (\bar{3}, 1, Y = 1/3)_L, (1, 2, Y = -1/2)_L, (1, 1, Y = 1)_L \right) \times 3 \text{ generations}
$$

$$
= \left( (3, 2, Y_W = 1/3)_L, (3, 1, Y_W = 4/3)_R, (3, 1, Y_W = -2/3)_R, (1, 2, Y_W = -1)_L, (1, 1, Y_W = -2)_R \right) \times 3 \text{ generations}
$$

$$
= \left( (3, 2, Y_W = 1/3)_L, (3, 1, Y_W = -4/3)_L, (\bar{3}, 1, Y_W = 2/3)_L, (1, 2, Y_W = -1)_L, (1, 1, Y_W = 2)_L \right) \times 3 \text{ generations}
$$

$$
= \left( (3, 2, \tilde{Y} = 1)_L, (3, 1, \tilde{Y} = 4)_R, (3, 1, \tilde{Y} = -2)_R, (1, 2, \tilde{Y} = -3)_L, (1, 1, \tilde{Y} = -6)_R \right) \times 3 \text{ generations}
$$

$$
= \left( (3, 2, \tilde{Y} = 1)_L, (\bar{3}, 1, \tilde{Y} = -4)_L, (\bar{3}, 1, \tilde{Y} = 2)_L, (1, 2, \tilde{Y} = -3)_L, (1, 1, \tilde{Y} = 6)_L \right) \times 3 \text{ generations.}
$$

(2.13)

The right hand neutrino $\nu_R$ is in a representation

$$
(1, 1, Y = 0) = (1, 1, Y_W = 0) = (1, 1, \tilde{Y} = 0).
$$

(2.14)

times some generation number. Also the complex scalar Higgs field $\phi_H$ is in a representation

$$
(1, 2, Y = 1/2) = (1, 2, Y_W = 1) = (1, 2, \tilde{Y} = 3),
$$

(2.15)

where $\tilde{Y} = 3Y_W = 6Y$. In the Higgs condensed phase, the conventional Higgs vacuum expectation value (vev) is chosen to be $\langle \phi_H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu \end{pmatrix}$, which vev has $Q_{EM} = 0$. We organize the above data in Table 1.

[III]. If we include the $3 \times 2 + 3 + 3 + 2 + 1 = 15$ left-handed Weyl spinors from one single generation, we can combine them as a multiplet of $5$ and $10$ left-handed Weyl spinors of SU(5):  

$$
(\bar{3}, 1, 1/3)_L \oplus (1, 2, -1/2)_L \sim 5 \text{ of SU(5)},
$$

(2.16)

$$
(3, 2, 1/6)_L \oplus (\bar{3}, 1, -2/3)_L \oplus (1, 1, 1)_L \sim 10 \text{ of SU(5)}.
$$

(2.17)

Hence these are matter field representations of the SU(5) GUT with a SU(5) gauge group. Other than the electroweak Higgs $\phi_H$, we also need to introduce a different GUT Higgs field $\phi_{GG}$ to break down SU(5) to $SU(3) \times SU(2) \times U(1) / \mathbb{Z}_6$. The $\phi_{GG}$ is in the adjoint representation of SU(5) as

$$
24 = (8, 1, Y = 0) \oplus (1, 3, Y = 0) \oplus (1, 1, Y = 0) \oplus (3, 2, Y = -\frac{5}{6}) \oplus (3, 2, Y = \frac{5}{6}).
$$

(2.18)

[IV]. If we include the $3 \times 2 + 3 + 3 + 2 + 1 = 15$ left-handed Weyl spinors from one single generation, and also a right-handed neutrino, we can combine them as a multiplet of 16 left-handed Weyl spinors:

$$
\Psi_L \sim 16^+ \text{ of Spin}(10),
$$

(2.19)

which sits at the 16-dimensional irreducible spinor representation of Spin(10). (In fact, $16^+$ and $16^-$-dimensional irreducible spinor representations together form a 32-dimensional reducible spinor representation of Spin(10).) Namely, instead of an SO(10) gauge group, we should study the SO(10) GUT with a Spin(10) gauge group.
[V]. Lie algebra generators and gauge bosons: We can count the number of Lie algebra generators to represent the local 1-form gauge field of gauge bosons. For example, there are $1 + 3 + 8 = 12$ independent Lie algebra generators thus gauge bosons for the Lie algebra $u(1) \times su(2) \times su(3)$. There are $24$ independent Lie algebra generators thus gauge bosons for the Lie algebra $su(5)$, and $45$ Lie algebra generators thus gauge bosons for the Lie algebra $so(10)$. In this work, we focus on $SU(5)$ GUT thus also list down their quantum numbers in Table 2.

[VI]. $U(1)_{B-L}$ symmetry and $U(1)_X$ symmetry: For SM with $G_{\text{internal}} = G_{\text{SM}}$ of $q = 1, 2, 3, 6$, we have a $B-L$ (baryon minus lepton numbers) $U(1)_{B-L}$ global symmetry. For $SU(5)$ GUT with $G_{\text{internal}} = SU(5)$, we can have a $U(1)_X$ symmetry. Here we have used (2.12).

\[ X \equiv 5(B - L) - 4Y = 5(B - L) - 2Y_W, \]  
\[ \tilde{X} \equiv 3X = 5 \cdot 3(B - L) - 2 \cdot 3Y_W = 5 \cdot 3(B - L) - 2\tilde{Y} = 5(B - \tilde{L}) - 2\tilde{Y}, \]  
\[ \tilde{Y} = 3Y_W = 6Y. \]

Here we have used (2.12).

[VII]. $Z_{4,X}$ symmetry and discrete symmetries: We will learn it is natural to consider a $Z_{4,X}$ subgroup of $U(1)_X$, when we attempt to embed the $SU(5)$ GUT to SO(10) GUT. In fact, the $Z_{4,X}$ can be regarded as the $Z_4$ center of the Spin(10) gauge group as $Z(\text{Spin}(10)) = Z_4$. Since Spin(10) is fully dynamically gauged in SO(10) GUT, the $Z_{4,X}$ is also dynamically gauged. In summary, Remarks [VI] and [VII] show that

$U(1)_{B-L}$: global symmetry in the SMs for all $q = 1, 2, 3, 6$.  

$U(1)_X$: global symmetry in the $SU(5)$ GUT.  

$Z_{4,X}$: dynamical gauge symmetry in the SO(10) GUT.  

But $Z_{4,X}$ can be chosen to be a global (or gauge) symmetry for the SM and SU(5) GUT. (2.23)

Colloquially, we may call $Z_{4,X}$ a gauge symmetry but we should warn the readers that a gauge symmetry is not really a symmetry nor global symmetry, but only a gauge redundancy.

These extra symmetries are well-motivated in the earlier pioneer works [63–66] and References therein the recent work [30,67,68]. Extra discrete symmetries can be powerful give rise to new anomaly cancellation constraints. Part of the new ingredients we will survey are the new global anomalies for discrete symmetries of SMs, in Sec. 4.

Let us discuss how these extra groups can be embedded into the total groups in Remarks [VIII], [IX], [X], and [XI].

[VIII]. We find the Lie group embedding for the internal symmetry of GUTs and Standard Models [2,26]:

\[ \text{SO(10)} \supset \text{SU(5)} \supset \frac{U(1) \times SU(2) \times SU(3)}{Z_6}, \]  
\[ \text{Spin(10)} \supset \text{SU(5)} \supset \frac{U(1) \times SU(2) \times SU(3)}{Z_6}. \]

Only $q = 6$, but not other $q = 1, 2, 3$, for $G_{\text{SM}}$ can be embedded into Spin(10) nor SU(5). So from the GUT perspective, it is natural to consider the Standard Model gauge group $\frac{U(1) \times SU(2) \times SU(3)}{Z_6}$.  

[IX]. We also find the following group embedding for the spacetime and internal symmetries of GUTs and Standard Models (Ref. [2,26], see also [36] for the derivations):

\[ \frac{\text{Spin}(d) \times \text{Spin}(10)}{Z_2^F} \supset \text{Spin}(d) \times \text{SU(5)} \supset \frac{\text{Spin}(d) \times \text{SU}(3) \times SU(2) \times SU(1)}{Z_6}. \]

\[ \text{Spin}(d) \times \text{Spin}(10) \supset \text{Spin}(d) \times \text{SU}(5) \supset \text{Spin}(d) \times \frac{\text{SU}(3) \times SU(2) \times SU(1)}{Z_6}. \]

\[ ^{10}\text{In fact, the author believe and declare that } G_{\text{SM}} \equiv \frac{U(1) \times SU(2) \times SU(3)}{Z_6} \text{ is the correct and natural gauge group of SM.} \]
For an extra $U(1)_{B-L}$ or $U(1)_X$ symmetry, we need to consider $\text{Spin}^c \equiv \text{Spin}(d) \times \mathbb{Z}_2 \times U(1)$ structure. We find the embedding:

\[
\text{Spin}^c(d) \times \text{SU}(5) \supset \text{Spin}^c(d) \times \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6}.
\] (2.28)

For an extra $\mathbb{Z}_4, \mathbb{X}$ symmetry, we need to consider $\text{Spin}(d) \times \mathbb{Z}_2 \times \mathbb{Z}_4$ structure. In order to contain these groups embedded in $\frac{\text{Spin}(d) \times \text{Spin}(10)}{\mathbb{Z}_2}$, it is more naturally to consider:

\[
\frac{\text{Spin}(d) \times \text{Spin}(10)}{\mathbb{Z}_2} \supset \text{Spin}(d) \times \mathbb{Z}_2 \times \text{SU}(5) \supset \text{Spin}(d) \times \mathbb{Z}_2 \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \times \mathbb{Z}_6.
\] (2.29)

Ref. [2, 26, 30, 31] study the cobordism theory of some of these SM, BSM, and GUT groups. We shall particularly pay attention to $d = 5$ for $\Omega^{d=5}_G \equiv \Omega^{d=d}_{\text{spacetime} \times \text{internal}} \equiv \text{TP}_{d=5}(G)$, since the 5d cobordism invariants of $\text{TP}_{d=5}(G)$ can classify the 4d invertible anomalies of the total group $G$. We particularly apply the results in [2], for anomaly constraints and anomaly matchings of BSM physics, in the following Sec. 3.

### 2.2 Classifications of anomalies and their different uses

To start, let us mention different concepts of anomalies and their different uses. See for example Fig. 1 for perturbative local anomalies captured by Feynman-Dyson graphs. Generally, we also have nonperturbative global anomalies not captured by Feynman graphs, but their uses are similar as follows:

1. Dynamical gauge anomaly of $G_{\text{internal}}$ (e.g., Fig. 1 (1)): Anomaly matching cancellation must be zero for a dynamical gauge theory of its group $G_{\text{internal}}$.

2. ’t Hooft anomaly (e.g., Fig. 1 (2)): Anomaly matching of background symmetry of $G$ including $G_{\text{internal}}$ for their background fields, surprisingly, need not to be zero for a QFT:

   (• i). If the ’t Hooft anomaly of a QFT of any $G$ background fields turn out to be exactly zero, this means that these global symmetries of QFT can be realized local and onsite (or on an $n$-simplex for a higher generalized global $n$-symmetry [71]) on a lattice. This also means that we do not need to regularize the QFT living on the boundary of Symmetry-Protected/Enriched Topological states [26, 52, 72–74]. In fact, the QFT may be well-defined as the low energy theory of a quantum local lattice model in its own dimensions, as a high-energy ultraviolet (UV) completion — if it is free from all anomalies, via a cobordism argument [26].

   (• ii). If the ’t Hooft anomaly of a QFT of any $G$ background fields turn out to be not zero nor cancelled: Then this can be regarded as several meanings and implications:

     • **First**, the symmetries of QFT in fact are non-local or non-onsite (or non-on-$n$-simplex for a higher generalized global $n$-symmetry) on a lattice. This means the obstruction of gauging such non-local symmetries, thus this obstruction is equivalent with the definition of ’t Hooft anomaly [41].

     • **Second**, the QFT can have a hidden sector of the same dimension. We can match the ’t Hooft anomaly of QFT with additional sector $S'$ of the same dimension, but with the $S'$ sector with the
opposite ’t Hooft anomaly. So, the combined system, the QFT and $S'$, can have a no ’t Hooft anomaly at all.

- **Third,** the QFT can have a hidden sector of one higher dimension. In order to have the symmetries of QFT with ’t Hooft anomaly to be local or onsite instead, we need to append and regularize this $d$QFT living on the boundary of $(d + 1)d$ Symmetry-Protected/Enriched Topological states (SPTs/SETs) [26,52,72–74].\(^1\)

This is related to the Callan-Harvey anomaly inflow [80] of $(d + 1)d$ bulk and $dd$ boundary with their spacetime dimensions differed by one.

- **Fourth,** of course, we can also have a certain combination of the First, Second, and Third scenarios above, in order to make sense of QFT with ’t Hooft anomaly.

(3) Adler-Bell-Jackiw (ABJ [69,70]) type of anomalies (e.g., Fig. 1 (3)): The non-conservations of the global symmetry current $J$ is coupled to the background non-dynamical field $\mathcal{A}$ via the action term $\int \mathcal{A} \wedge \star \mathcal{J} \equiv$

\(^{1}\)Symmetry-Protected Topological states (SPTs) are the short-range entangled states as a generalization of the free non-interacting topological insulators and superconductors [75–79] with interactions. Symmetry-Enriched Topological states (SETs) are topological ordered states enriched by global symmetries. The readers with enthusiasms can overview the condensed matter terminology bridging to QFT in Ref. [62] for QFT theorists, or the excellent condensed matter reviews [45,54].
\[ \int d^d x (A_\mu J^\mu) \] with the Hodge dual star \(*\). The non-conservation of current is proportional to the anomaly factor in \(d\)d spacetime

\[ d(\star J) \propto (F_a)^{d/2} \propto F_a \wedge F_a \wedge \ldots. \]

Here \(a\) are dynamical gauge fields; we can also modify the equation appropriate for several different dynamical 1-form or higher-form gauge fields. For \(d = 4\), the ABJ anomaly formula is precisely captured by Fig. 1 (\(b\)).

(4). Anomaly that involves two background fields of global symmetries and one dynamical gauge field (e.g., Fig. 1 (4)).

Indeed it is obvious to observe that the anomalies (1) are tighten to anomalies (2).

(\(a\)). Anomalies from (1) can be related to anomalies from (2) via the gauging principle.

(\(b\)). Anomalies from (2) can be related to anomalies from (1) via the ungauging principle, or via gauging the higher symmetries [71].

Thus if we learn the gauge group of a gauge theory (e.g., SM, GUT or BSM), we may identify its ungauged global symmetry group as an internal symmetry group, say \(G_{\text{internal}}\) via ungauging.

Now let us discuss the classifications of anomalies. By “all invertible quantum anomalies” obtain from cobordism classification, we mean the inclusion of:

(i). **Perturbative local anomalies** captured by perturbative Feynman graph loop calculations, classified by the integer group \(\mathbb{Z}\) classes, or the free classes in mathematics. Some selective examples from QFT or gravity are:

(1): Perturbative fermionic anomalies from chiral fermions with \(U(1)\) symmetry, originated from Adler-Bell-Jackiw (ABJ) anomalies [69, 70] with \(\mathbb{Z}\) classes.

(2): Perturbative bosonic anomalies from bosonic systems with \(U(1)\) symmetry with \(\mathbb{Z}\) classes [1].

(3): Perturbative gravitational anomalies [81].

(ii). **Non-perturbative global anomalies**, classified by a product of finite groups such as \(\mathbb{Z}_n\), or the torsion classes in mathematics. Some selective examples from QFT or gravity are:

(1): An \(SU(2)\) anomaly of Witten in 4d or in 5d [82] with a \(Z_2\) class, which is a gauge anomaly.

(2): A new \(SU(2)\) anomaly in 4d or in 5d [83] with a different \(Z_2\) class, which is a mixed gauge-gravity anomaly.

(3): Higher ’t Hooft anomalies of \(Z_2\) class for a pure 4d \(SU(2)\) YM theory with a second-Chern-class topological term [84–86] (or the so-called \(SU(2)_{\theta=\pi}\) YM): The higher anomaly involves a discrete 0-form time-reversal symmetry and a 1-form center \(Z_2\)-symmetry. The first anomaly is discovered in [84]; later the anomaly is refined via a mathematical well-defined 5d bordism invariant as its topological term, with additional new \(Z_2\) class anomalies found for Lorentz symmetry-enriched four siblings of Yang-Mills gauge theories [85, 86].

---

\(^{12}\)By gauging or ungauging, also depending on the representation of the matter fields that couple to the gauge theory, we may gain or lose symmetries or higher symmetries [71]. It will soon become clear, for our purpose, we need to primarily focus on the ordinary (0-form) internal global symmetries and their anomalies.
(4): Global gravitational anomalies [87] detected by exotic spheres.

(5): Bosonic anomalies: Many types of bosonic anomalies in diverse dimensions are global anomalies [19, 88–91]. These bosonic anomalies only require bosonic degrees of freedom, but without the requirement of chiral fermions. Many such bosonic anomalies are related to group cohomology or generalized group cohomology theory, living on the boundary of SPTs [72], closely related to Dijkgraaf-Witten topological gauge theories [92].

Our present work explore global anomalies (generically not captured by Feynman graphs), in order to help readers to digest their physical meanings as in Fig. 2, we should imagine the computation of anomalies on generic curved manifolds in for the cobordism theory setting, with mixed gauge and gravitational background probes.

![Figure 2: General nonperturbative global anomalies not captured by Feynman graphs can still be characterized by generic curved manifolds with mixed gauge and gravitational background probes. The figure shows a bordism between manifolds. Here $M$ and $M'$ are two closed $d$-manifolds, $N$ is a compact $d + 1$-manifold whose boundary is the disjoint union of $M$ and $M'$, so $\partial N = M \sqcup M'$. If there are additional $G$-structures on these manifolds, then the $G$-structure on $N$ is required to be compatible with the $G$-structures on $M$ and $M'$. If there are additional maps from these manifolds to a fixed topological space, then the maps are also required to be compatible with each other. If these conditions are obeyed, then $M$ and $M'$ are called bordant equivalence and $N$ is called a bordism between $M$ and $M'$.](image-url)
3 Dynamical Gauge Anomaly Cancellation

In this section, we explicitly check the dynamical gauge anomaly cancellations of various SMs with four
gauge group \( G_{SM} \equiv (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_q \) with \( q = 1, 2, 3, 6 \). The cobordism classifications of
\( G_{SM} \)'s 4d anomalies are done in \([2,31]\). The 4d anomalies can be written as 5d cobordism invariants,
which are 5d invertible TQFT (iTQFT). These 5d cobordism invariants/iTQFT are derived in \([2]\), we
summarized the classifications and invariants in Table 3.

Moreover, as Ref. \([31]\) points out correctly already, we will show that dynamical gauge anomaly
cancellation indeed holds to be true for all \( G_{SM} \) with \( q = 1, 2, 3, 6 \). In Ref. \([31]\) has not written down
the 5d cobordism invariants nor explicit anomaly polynomials. However, by simply looking at the group
classifications of anomalies, Ref. \([31]\) argues that there is only the famous Witten SU(2) nonperturbative
global anomaly of \( \mathbb{Z}_2 \) class \([82]\), other anomalies all are perturbative local anomalies captured by
Feynman graphs. So why do we bother to do the calculations again, if Ref. \([31]\) has found all dynamical
gauge anomalies cancel for \( G_{SM} \)? There are multiple reasons:

- First, we will show that the 5d cobordism invariants obtained in \([2]\) indeed match with the anomaly
  polynomials. For perturbative local anomalies of \( \mathbb{Z} \) classes, see Fig. 4, we indeed can show that the
  5d cobordism invariants map to some one-loop Feynman graph calculations known in the standard
  QFT textbooks \([37–40]\). (In contrast, Ref. \([31]\) focus on global anomalies, and pays less attention
  on perturbative local anomalies of \( \mathbb{Z} \) classes. Follow Ref. \([2]\), we will fill in this gap by considering
  all local and all global anomalies.) We should work through all these correspondences carefully to
  gain a solid confidence for our understanding of cobordism classifications of anomalies.

- Second, we can learn how to translate cobordism data from math into the anomalies in physics.
  For example, in Table 3, we see that the 5d cobordism invariant \( c_2(SU(2))\tilde{\eta} \) in fact captures the 4d
  boundary theory has the Witten SU(2) nonperturbative global anomaly \([82]\).

- Third, the most important thing, we will discover new phenomena later when we include the
  additional discrete symmetries into SM and GUT in Sec. 4. In fact, we will discover entirely new
  physics that previously have never been figured out in the past.

Notations: Throughout our work, we write the three SU(2) Lie algebra generator \( \sigma^a \) of the rank-2
matrix of fundamental representation satisfying \( \text{Tr}[\sigma^a \sigma^b] = \frac{1}{2} \delta_{ab} \) with \( a, b \in \{1, 2, 3\} \). We write
the eight SU(3) Lie algebra generator \( \tau^a \) of the the rank-3 matrix of fundamental representation satisfying
\( \text{Tr}[\tau^a \tau^b] = \frac{1}{2} \delta_{ab} \) with \( a, b \in \{1, 2, \ldots, 8\} \).
### 3.1 Summary of anomalies from cobordism theory and Feynman diagrams

| Classes | Cobordism group $TP_d(G)$ with $G_{SM_q} \equiv (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_q$ with $q = 1, 2, 3, 6$ | Cobordism invariants |
|---------|-------------------------------------------------------------------------------------------------|---------------------|
| $5d$ $\mathbb{Z}^5 \times \mathbb{Z}_2$ | $\mu(\text{PD}(c_1(U(1))))$, $\frac{\text{CS}^{U(1)}_1 c_1(U(1))^2}{2}$, $\frac{\text{CS}^{U(1)}_1 c_2(SU(2))}{2}$, $\frac{\text{CS}^{SU(3)}_3}{2}$, $\frac{\text{CS}^{SU(3)}_3 c_2(SU(2))}{2}$ | $G = \text{Spin} \times G_{SM_1}$ |
| $5d$ $\mathbb{Z}^5$ | $\mu(\text{PD}(c_1(U(2))))$, $\frac{\text{CS}^{U(2)}_1 c_1(U(2))^2}{2}$, $\frac{\text{CS}^{U(2)}_1 c_2(SU(2))}{2}$, $\frac{\text{CS}^{SU(3)}_3}{2}$, $\frac{\text{CS}^{SU(3)}_3 c_2(SU(2))}{2}$ | $G = \text{Spin} \times G_{SM_2}$ |
| $5d$ $\mathbb{Z}^5 \times \mathbb{Z}_2$ | $\mu(\text{PD}(c_1(U(3))))$, $\frac{\text{CS}^{U(3)}_1 c_1(U(3))^2}{2}$, $\frac{\text{CS}^{U(3)}_1 c_2(SU(2))}{2}$, $\frac{\text{CS}^{SU(3)}_3}{2}$, $\frac{\text{CS}^{SU(3)}_3 c_2(SU(2))}{2}$ | $G = \text{Spin} \times G_{SM_3}$ |
| $5d$ $\mathbb{Z}^5$ | $\mu(\text{PD}(c_1(U(2))))$, $\frac{\text{CS}^{U(2)}_1 c_1(U(2))^2}{2}$, $\frac{\text{CS}^{U(2)}_1 c_2(SU(2))}{2}$, $\frac{\text{CS}^{SU(3)}_3}{2}$, $\frac{\text{CS}^{SU(3)}_3 c_2(SU(2))}{2}$ | $G = \text{Spin} \times G_{SM_6}$ |

Table 3: The 4d anomalies can be written as 5d cobordism invariants of $\Omega^{d=5}_{\text{SM}} \equiv TP_d(G)$, which are 5d iTQFTs. These 5d cobordism invariants/iTQFTs are derived in [2]. We summarized the group classifications of 4d anomalies and their 5d cobordism invariants. The anomaly classification of $\mathbb{Z}^5$ means that there are 5 perturbative local anomalies (of $\mathbb{Z}$ classes descended from the 6d bordism group $\Omega^{d=6}_G$), precisely match 5 perturbative one-loop triangle Feynman diagrams in Fig. 4. The anomaly classification of $\mathbb{Z}_2$ means that there is a 1 nonperturbative global anomaly, which turns out to be Witten SU(2) anomaly [82]. The $c_j(G)$ is the $j$th Chern class of the associated vector bundle of the principal $G$-bundle. The $\mu$ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = (\frac{2-\text{F-F}}{8})(M^4)$, thus $\mu(\text{PD}(c_1(U(1))))$ is related to $c_1(U(1))(\sigma-F-F)$. Here $\cdot$ is the intersection form of $M^4$. The $F$ is the characteristic 2-surface [93] in a 4-manifold $M^4$, it obeys the condition $F\cdot x = x\cdot x$ mod 2 for all $x \in H_2(M^4, \mathbb{Z})$. By the Freedman-Kirby theorem, we have $(\frac{2-\text{F-F}}{8})(M^4) = \text{Arf}(M^4, F)$ mod 2. The PD is defined as the Poincaré dual. The Arf is a 2d Arf invariant, whose condensed matter realization is the Kitaev fermionic chain [94]. The $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. In Sec. 3.7 and Ref. [2], we propose that the $\mathbb{Z}_2$ class 5d cobordism invariant $c_2(SU(2))\tilde{\eta}$ corresponds to the 4d Witten SU(2) anomaly [82]. See our notational conventions in Sec. 1 and Sec. 1.2.4 of Ref. [2]. The symbol “$\sim$” here means the equivalent rewriting of cobordism invariants on closed 5-manifolds.
Figure 3: Examples of dynamical gauge anomaly cancellations in SM. In fact, the 5 perturbative local anomalies from perturbative one-loop triangle Feynman diagrams precisely match anomaly classification of $Z_5^5$ obtained from the cobordism group calculations in Table 3 and Ref. [2].

### 3.2 $U(1)_Y^3$: 4d local anomaly from 5d $CS_{1}^{U(1)}c_1(U(1))^2$ and 6d $c_1(U(1))^3$

We read from Ref. [2] and Table 3 for the $Z$ class of the 5d cobordism invariants of the following:
5d $CS_{1}^{U(1)}c_1(U(1))^2$ for $G_{SM_1}$, 5d $CS_{1}^{U(2)}c_1(U(2))^2$ for $G_{SM_2}$ and $G_{SM_6}$, while 5d $CS_{1}^{U(3)}c_1(U(3))^2$ for $G_{SM_3}$ and $G_{SM_6}$. These 5d cobordism invariants correspond to the 4d perturbative local anomalies captured by the one-loop Feynman graph:

\[
\begin{array}{cc}
\text{U(1)$_Y$ gauge} & \text{U(1)$_Y$ gauge} \\
\text{U(1)$_Y$ gauge} & \text{U(1)$_Y$ gauge}
\end{array}
\]

Without losing generality, we focus on the 4d cubic anomaly $(U(1)_Y)^3$ from 5d $CS_{1}^{U(1)}c_1(U(1))^2$, which also descends from 6d $c_1(U(1))^3$ of bordism group $\Omega_6$ in Ref. [2]. Plug in data in Sec. 2.1, it is standard to check the anomaly (3.1) vanishes:\(^{13}\)

\[
\sum_{q} \text{Tr}[(\hat{Y}_q)^3] = \sum_{qL,qR} (Y_{qL})^3 - (Y_{qR})^3 = \frac{1}{2} \delta_{ab} N_{\text{generation}} \left( N_c \cdot \left( 2 \cdot (1/6)^3 + (-2/3)^3 + (1/3)^3 \right) + 2 \cdot (-1/2)^3 + (1)^3 + (0)^3 \right)
\]

\(^{13}\)When we switch between from $L$-chiral fermion to the $R$-chiral fermion (anti-chiral fermion), there could be an additional minus sign.
\[ N_{\text{generation}} \cdot (-N_c + 3) \cdot (1/4), \]  
\eqref{eq:generation}

which is 0 when \( N_c = 3 \) for 3 colors as we have. The \( N_{\text{generation}} \) (or \( N_{\text{family}} \)) counts the number of generations (same as families).

### 3.3 \( U(1)_Y \)-\( SU(2)^2 \): 4d local anomaly from 5d \( CS_1^{U(1)} c_2(SU(2)) \) and 6d \( c_1(U(1))c_2(SU(2)) \)

We read from Ref. [2] and Table 3 for a \( Z \) class of 5d cobordism invariants of the following:

- 5d \( CS_1^{U(1)} c_2(SU(2)) \) for \( G_{SM_1} \), 5d \( \frac{CS_1^{U(2)} c_2(U(2))}{2} \sim c_1(U(2))CS_2^{U(2)} \) for \( G_{SM_2} \),
- 5d \( CS_1^{U(3)} c_2(SU(2)) \sim c_1(U(3))CS_3^{SU(2)} \) for \( G_{SM_3} \), and 5d \( \frac{CS_1^{U(3)} c_2(U(2))}{2} \sim c_1(U(3))CS_3^{U(2)} \) for \( G_{SM_4} \). These 5d cobordism invariants correspond to the 4d perturbative local anomalies captured by the one-loop Feynman graph:

\[
\text{U(1)}_Y \text{ gauge} \quad \text{SU(2) gauge} \quad \text{SU(2) gauge}
\]

Without losing generality, we focus on the 4d anomaly \( U(1)_Y \)-\( SU(2)^2 \) from 5d \( CS_1^{U(1)} c_2(SU(2)) \), which also descends from 6d \( c_1(U(1))c_2(SU(2)) \) of bordism group \( \Omega_6 \) in Ref. [2]. Plug in data in Sec. 2.1, we check the anomaly \( \eqref{eq:3.3} \) vanishes,

\[
\sum_q \text{Tr}[\hat{Y}_q \sigma^a \sigma^b] = \frac{1}{2} \delta_{ab} \left( \sum_{q_L, q_R} (Y_{q_L}) - (Y_{q_R}) \right) = \frac{1}{2} \delta_{ab} N_{\text{generation}} \cdot \left( N_c \cdot \left( 2 \cdot \frac{1}{6} \right) + 2 \cdot \frac{1}{2} \right) = \frac{1}{2} \delta_{ab} N_{\text{generation}} \cdot (N_c/3 - 1),
\]

which is 0 when \( N_c = 3 \).

### 3.4 \( U(1)_Y \)-\( SU(3)^2 \)_c: 4d local anomaly from 5d \( CS_1^{U(1)} c_2(SU(3)) \) and 6d \( c_1(U(1))c_2(SU(3)) \)

We read from Ref. [2] and Table 3 for a \( Z \) class of 5d cobordism invariants of the following:

- 5d \( CS_1^{U(1)} c_2(SU(3)) \sim c_1(U(1))CS_3^{SU(3)} \) for \( G_{SM_1} \), 5d \( CS_1^{U(2)} c_2(SU(3)) \sim c_1(U(2))CS_3^{SU(3)} \) for \( G_{SM_2} \), 5d \( \frac{CS_1^{U(3)} c_2(U(3))}{2} + CS_5^{U(3)} \sim c_1(U(3))CS_3^{U(3)} + CS_5^{U(3)} \) for \( G_{SM_4} \), and 5d \( \frac{CS_1^{U(3)} c_2(U(3))}{2} + CS_5^{U(3)} \sim c_1(U(3))CS_3^{U(3)} + CS_5^{U(3)} \) for \( G_{SM_6} \). (Note that part of the additional contribution from \( CS_5^{U(3)} \) or \( CS_5^{SU(3)} \) will be separately discussed later in Sec. 3.6 and \( \eqref{eq:3.9} \).)

These 5d cobordism invariants correspond to the 4d perturbative local
anomalies captured by the one-loop Feynman graph:

\[ \begin{array}{c}
\text{U}(1)_Y \text{ gauge} \\
\downarrow \\
\text{SU(3)}_c \text{ gauge} \\
\end{array} \]

Without losing generality, we focus on the 4d anomaly \( \text{U}(1)_Y - \text{SU(3)}_c^2 \) from 5d \( \text{CS}^U_1 \text{c}_2(\text{SU}(3)) \sim c_1(\text{U}(1)) \text{CS}^U_3\text{SU}(3) \), which also descends from 6d \( c_1(\text{U}(1))c_2(\text{SU}(3)) \) of bordism group \( \Omega_6 \) in Ref. \[2\]. Plug in data in Sec. 2.1, we check the anomaly (3.6) vanishes,

\[
\sum_q \text{Tr}[\hat{Y}_q^a \tau^b] = \sum_{q_L, q_R} \text{Tr}[\hat{Y}_{q_L}^a \tau^b] - \text{Tr}[\hat{Y}_{q_R}^a \tau^b] = \frac{1}{2} \delta_{ab} \left( \sum_{q_L, q_R} (Y_{q_L}^a) - (Y_{q_R}^a) \right) = \frac{1}{2} \delta_{ab} N_{\text{generation}} \cdot \left( N_c \cdot \left( 2 \cdot \frac{1}{6} + (-2/3) + (1/3) \right) \right) = \frac{1}{2} \delta_{ab} N_{\text{generation}} \cdot N_c \cdot 0 = 0. \tag{3.6}
\]

### 3.5 \( \text{U}(1)_Y \)-\( (\text{gravity})^2 \): 4d local anomaly from 5d \( \mu(\text{PD}(c_1(\text{U}(1)))) \) and 6d \( \frac{c_1(\text{U}(1))(\sigma - F \cdot F)}{8} \)

We read from Ref. \[2\] and Table 3 for a \( Z \) class of 5d cobordism invariants of the following:

5d \( \mu(\text{PD}(c_1(\text{U}(1)))) \) for \( G_{\text{SM}_1} \), 5d \( \mu(\text{PD}(c_1(\text{U}(2)))) \) for \( G_{\text{SM}_2} \) and \( G_{\text{SM}_3} \), while 5d \( \mu(\text{PD}(c_1(\text{U}(3)))) \) for \( G_{\text{SM}_4} \) and \( G_{\text{SM}_5} \). These 5d cobordism invariants correspond to the 4d perturbative local anomalies captured by the one-loop Feynman graph:

\[ \begin{array}{c}
\text{U}(1)_Y \text{ gauge} \\
\downarrow \\
\text{gravity} \\
\end{array} \]

Without losing generality, we focus on the 4d anomaly \( \text{U}(1)_Y \)-\( (\text{gravity})^2 \) of \( \text{U}(1)_Y \)-gravitational anomaly from 5d \( \mu(\text{PD}(c_1(\text{U}(1)))) \), which also descends from 6d \( \frac{c_1(\text{U}(1))(\sigma - F \cdot F)}{8} \) of bordism group \( \Omega_6 \) in Ref. \[2\]. Plug in data in Sec. 2.1, we check the anomaly (3.7) vanishes,

\[
\sum_q \text{Tr}[\hat{Y}_q^a] = \sum_{q_L, q_R} (Y_{q_L}^a) - (Y_{q_R}^a) = N_{\text{generation}} \cdot \left( N_c \cdot \left( 2 \cdot \frac{1}{6} + (-2/3) + (1/3) \right) + 2 \cdot (-1/2) + (1) + (0) \right)
\]
\[ N_{\text{generation}} \cdot (0 \cdot N_c + 0) = 0. \] (3.8)

We remark that if we view the gravity as dynamical fields, then (3.7) checks the dynamical gauge anomaly cancellation of the Fig. 1 (I) and Remark (I); if we view the gravity as background probe fields, then (3.7) checks the anomaly cancellation of the type of Fig. 1 (4) and Remark (4).

3.6 SU(3)_c³: 4d local anomaly from 5d \( \frac{1}{2} \text{CS}^{SU(3)}_5 \) and 6d \( \frac{1}{2} c_3(SU(3)) \)

We read from Ref. [2] and Table 3 for a \( \mathbb{Z} \) class of 5d cobordism invariants of the following: 5d \( \frac{1}{2} \text{CS}^{SU(3)}_5 \) for \( G_{SM_1} \) and \( G_{SM_2} \), and 5d \( \text{CS}^{SU(3)}_5 \) for \( G_{SM_3} \) and \( G_{SM_6} \). (Note that part of the contributions from \( \text{CS}^{SU(3)}_5 \) also occur in Sec. 3.4.) These 5d cobordism invariants correspond to the 4d perturbative local anomalies captured by the one-loop Feynman graph:

\[
\text{SU(3)}_c \text{ gauge} \\
\text{SU(3)}_c \text{ gauge} \quad \text{SU(3)}_c \text{ gauge} \\
\text{SU(3)}_c \text{ gauge}
\]

Without losing generality, we focus on the 4d anomaly \( SU(3)_c^3 \) from 5d \( \frac{1}{2} \text{CS}^{SU(3)}_5 \), which also descends from 6d \( \frac{1}{2} c_3(SU(3)) \) of bordism group \( \Omega_6 \) in Ref. [2]. Plug in data in Sec. 2.1, we check the anomaly (3.9) vanishes. In fact, in the context of SM physics, even without checking explicitly, it is clear that this \( SU(3)_c^3 \) anomaly (3.9) must vanish, since the color \( SU(3)_c \) is vector gauge theory not chiral gauge theory respect to the color charge. We recall that only U(1)_Y and SU(2)_weak are chiral gauge theories in SM.

Readers may ask what happen to the \( SU(2)_c^3 \) anomaly by replacing the gauge fields in (3.9) to \( SU(2) \), since \( SU(2)_\text{weak} \) is chiral? The answer is that \( SU(2)_c^3 \) anomaly does not exist thus must vanish, because there is no such corresponding 5d cobordism invariant read from Ref. [2] and Table 3. In fact, for \( SU(2) \) and SO(\( N \)) group, all representations have zero 4d perturbative local anomalies, thus they must have none of \( \mathbb{Z} \) classes of 5d cobordism invariants, agreed with Ref. [2].

3.7 The old SU(2) anomaly (Witten \( \mathbb{Z}_2 \) global anomaly) from 5d \( c_2(SU(2))\tilde{\eta} \) and 6d \( c_2(SU(2))\text{Arf} \)

The old SU(2) anomaly of Witten in 4d [82] is summarized in [83] for the context we need. We read from Ref. [2] and Table 3 for a \( \mathbb{Z}_2 \) class of 5d cobordism invariant and suggest the 4d SU(2) anomaly corresponds to 5d \( c_2(SU(2))\tilde{\eta} \), and descends from 6d \( c_2(SU(2))\text{Arf} \) of bordism group \( \Omega_6 \) in Ref. [2].

1. 5d \( c_2(SU(2))\tilde{\eta} \) and 4d Witten SU(2) anomaly: This 4d anomaly is a mod 2 index of \( \mathbb{Z}_2 \) class counts the spin-\( 2r + 1/2 \) (or \( 4r + 2 \) in the dimension of representation) Weyl spinor as fermion doublet under \( SU(2) \) [83]. From Sec. 2.1, there are four of spin-\( 2r + 1/2 \) fermions from \((3,2,1/6)_L \) and
(1, 2, −1/2)_L, multiplied by \( N_{\text{generation}} \). So we check overall the Witten anomaly vanishes in SM:

\[
(\text{even number})|_{\text{of 2}} \mod 2 = 0.
\]

2. Naively, 5d \( c_2(\text{SU}(2))\eta \) only presents for \( G_{\text{SM}_1} \) and \( G_{\text{SM}_2} \), but not for \( G_{\text{SM}_3} \) and \( G_{\text{SM}_6} \). Readers may wonder how does Witten anomaly vanish for SM of \( q = 2, 6 \)?

- Ref. [95] explains nicely and accurately that the Witten anomaly mutates from a subclass of perturbative local \( Z \) class when we changes the SU(2) group to the U(2) group, namely for \( q = 2, 6 \).
- Ref. [2] gives a formal explanation as follows: The difference between \( k \) for \( G = \text{SU}(2) \) and \( Z \) for \( G = \text{U}(2) \) becomes \( 1 \) for \( G = \text{SU}(2) \) and \( 2 \) for \( G = \text{U}(2) \).

In summary of Sec. 3, by checking five \( Z \) classes of local anomalies and one \( Z \) class of Witten SU(2) global anomaly, we have shown that the for SM with \( G_{\text{SM}_q} \) of \( q = 1, 2, 3, 6 \) are indeed free from all dynamical gauge anomalies, thus dynamical gauge anomaly cancellation holds.

As we have checked, the \( G_{\text{SM}_q} \) is a healthy chiral gauge theory by its own with a dynamical gauge group \( G_{\text{SM}_q} \). However, what if we include additional global symmetries or gauge sectors? Such as the \( B - L \) or \( X \equiv 5(B - L) - 4Y \)? This motivates us to explore further in the next section Sec. 4.

4 Anomaly Matching for SM and GUT with Extra Symmetries

4.1 SM and GUT with extra continuous symmetries and cobordism theory

In Sec. 2.1 Remark [X], for SM and SU(5) GUT with an extra U(1)_B−L or U(1)_X symmetry, we need to consider \( \text{Spin}^c \equiv \text{Spin}(d) \times \mathbb{Z}_2 \) U(1) structure. We find the embedding:

\[
\text{Spin}^c(d) \times \text{SU}(5) \supset \text{Spin}^c(d) \times \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6}.
\]

\[\text{It has been long sought that } G_{\text{SM}_q} \text{ is a healthy chiral gauge theory by its own with a dynamical gauge group } G_{\text{SM}_q} \text{ free from all dynamical gauge anomalies. However, it is only until very recently by the systematic computations of cobordism groups in [30] (which checks } q = 1 \), [95] and [2] (which the two papers checks } q = 1, 2, 3, 6 \) completing the full checks on } G_{\text{SM}_q}. \text{ Without a cobordism classification of anomalies, previous literature either only check perturbative anomalies, or may still miss additional global anomaly constraints (as we shall see new anomaly constraints in Sec. 4).}
Figure 4: Examples of anomaly constraint for SM (or GUT) with extra symmetries such as $\mathbf{B} - \mathbf{L}$ or $\mathbf{X} \equiv 5(\mathbf{B} - \mathbf{L}) - 4Y$. We only show perturbative local anomalies from perturbative one-loop triangle Feynman diagrams discussed in (4.2). We will explore nonperturbative global anomalies (not captured by Feynman diagrams) in later sections. Assume the gravity contributes as background field:

- If $\mathbf{B} - \mathbf{L}$ or $\mathbf{X}$ is not gauged, (i), (ii), (iii), and (vi) are ABJ anomalies of Fig. 1 (3) and Remark (3); (iv) and (v) are ’t Hooft anomalies of Fig. 1 (2) and Remark (2).
- If $\mathbf{B} - \mathbf{L}$ or $\mathbf{X}$ is gauged, (i)-(iv), (vi) are dynamical gauge anomalies of Fig. 1 (1) and Remark (1); (v) is an anomaly of Fig. 1 (4) and Remark (4).

If these anomalies are not matched, we can still saturate the anomalies by proposing new sectors appending to the QFT; we will explore those new physics in Sec. 5.

Ref. [31] checks that the 5d bordism group:

$$\Omega^\text{Spin}^c_5(\mathbf{SU}(3) \times \mathbf{SU}(2) \times \mathbf{U}(1)/\mathbb{Z}_q) = 0,$$

which means no global anomalies. Ref. [36] computes the following cobordism groups $\text{TP}_5$ and bordism groups $\Omega_6$:

$$\text{TP}_5(\text{Spin}^c \times \mathbf{SU}(5)) = \mathbb{Z}^4, \quad \Omega^\text{Spin}^c_6(\mathbf{SU}(5)) = \mathbb{Z}^4.$$

$$\text{TP}_5(\text{Spin}^c \times (\mathbf{SU}(3) \times \mathbf{SU}(2) \times \mathbf{U}(1))/\mathbb{Z}_q) = \mathbb{Z}^{11}, \quad \Omega^\text{Spin}^c_6(\mathbf{SU}(3) \times \mathbf{SU}(2) \times \mathbf{U}(1)/\mathbb{Z}_q) = \mathbb{Z}^{11}.$$  \hspace{1cm} (4.2)

Ref. [36] finds that these $\text{TP}_5$ and $\Omega_6$ only contain $\mathbb{Z}$ classes, thus they only correspond to 4d local anomalies captured by the one-loop Feynman graph.

We emphasize that if $\mathbf{U}(1)_{\mathbf{B} - \mathbf{L}}$ or $\mathbf{U}(1)_{\mathbf{X}}$ is free of all anomalies, then we can dynamically gauge this symmetry. In that case, we should regard the corresponding gauge field as a Spin$^c$ connection instead of the familiar U(1) gauge fields, since the original theory requires to be defined on Spin$^c$ manifolds. (When we mention gauge fields for gauging $\mathbf{U}(1)_{\mathbf{B} - \mathbf{L}}$ or $\mathbf{U}(1)_{\mathbf{X}}$, what we really have in mind is the Spin$^c$ connection.) The $\mathbf{U}(1)_{\mathbf{B} - \mathbf{L}}$ or $\mathbf{U}(1)_{\mathbf{X}}$ is free of all anomalies if they are free from perturbative local anomalies, since they do not have global anomalies given by (4.2).
Are all the perturbative local anomalies canceled for these SM and GUT with extra continuous symmetries? The baryon and lepton local currents densities are

\[ j_B^\mu = \frac{1}{3}(\bar{q}_L \gamma^\mu q_L + \bar{u}_R \gamma^\mu u_R + \bar{d}_R \gamma^\mu d_R), \quad j_L^\mu = (\bar{L}_L \gamma^\mu L_L + \bar{e}_R \gamma^\mu e_R + N_{\nu_R} \bar{\nu}_R \gamma^\mu \nu_R) \] (4.3)

Here \( N_{\nu_R} \) is the number of right-handed neutrinos in one generation whose representation is given in (2.9):

- In the standard GSW SM, we have \( N_{\nu_R} = 0 \).
- In the SU(5) GUT, it is common to have \( N_{\nu_R} = 0 \).
- In the SO(10) GUT, we have \( N_{\nu_R} = 1 \).

We used to believe that there are no perturbative local anomalies for an additional U(1) if this U(1) is the \( U(1)_{B-L} \) or \( U(1)_X \). Let us check explicitly in the next subsections. We should pay attention on the anomaly cancellation and its dependence on \( N_{\nu_R} = 0 \) or \( 1 \). What we will check is the conservation of the \( U(1)_{B-L} \) current,

\[ d \star (j_B - j_L) = \partial_{\mu}(j_B^\mu - j_L^\mu) d^d x, \] (4.4)

by taking into account all possible anomaly contributions from cobordism considerations.

### 4.1.1 \((B - L)\)-U(1)_Y^2: 4d local anomaly

Plug in data from Table 1 to check 4d local anomaly of \((B - L)\)-U(1)_Y^2:

\[
\begin{array}{c}
\text{Global sym } B - L \\
(\text{Backgrd. field})
\end{array}
\begin{array}{c}
\text{U(1)}_Y \text{ gauge}
\end{array}
\begin{array}{c}
\text{U(1)}_Y \text{ gauge}
\end{array}
\]

we find the anomaly factor contributes to 15

\[
\begin{align*}
  j_B & : N_{\text{generation}} \cdot (N_c/3) \cdot (2 \cdot (1/6)^2 - (-2/3)^2 - (1/3)^2) = N_{\text{generation}} \cdot (N_c/3) \cdot (1/2), \\
  j_L & : N_{\text{generation}} \cdot (2 \cdot (-1/2)^2 + 1^2) = N_{\text{generation}} \cdot (1/2),
\end{align*}
\] (4.6)

such that \( d \star j_B \neq 0 \) and \( d \star j_L \neq 0 \), but \( d \star (j_B - j_L) = 0 \) if \( N_c = 3 \) as the color number it is.

15The minus sign can be interpreted either from the anti-quark of the \( R \)-chiral fermion (anti-chiral fermion).
4.1.2 \((B - L)\text{-SU}(2)^2\): 4d local anomaly

Plug in data from Table 1 to check 4d local anomaly of \((B - L)\text{-SU}(2)^2\):

\[
\begin{align*}
\text{Global sym } B - L \\
(\text{Backgrd. field})
\end{align*}
\]

we find the anomaly factor contributes to

\[
\begin{align*}
\mathbf{j}_B : \quad & N_{\text{generation}} \cdot \left( \frac{N_c}{3} \cdot \text{Tr} \left[ \frac{\sigma^a \sigma^b}{2} \right] \right) = N_{\text{generation}} \cdot \left( \frac{N_c}{3} \cdot \frac{\delta_{ab}}{2} \right), \\
\mathbf{j}_L : \quad & N_{\text{generation}} \cdot \left( \text{Tr} \left[ \frac{\sigma^a \sigma^b}{2} \right] \right) = N_{\text{generation}} \cdot \left( \frac{\delta_{ab}}{2} \right),
\end{align*}
\]

(4.8)

such that \(d \star \mathbf{j}_B \neq 0\) and \(d \star \mathbf{j}_L \neq 0\), but \(d \star (\mathbf{j}_B - \mathbf{j}_L) = 0\) if \(N_c = 3\) as the color number it is.

4.1.3 \((B - L)\text{-SU}(3)^2\): 4d local anomaly

Plug in data from Table 1 to check 4d local anomaly of \((B - L)\text{-SU}(3)^2\):

\[
\begin{align*}
\text{Global sym } B - L \\
(\text{Backgrd. field})
\end{align*}
\]

we find the anomaly factor contributes to

\[
\begin{align*}
\mathbf{j}_B : \quad & N_{\text{generation}} \cdot \left( (2 - 1 - 1) \text{Tr} \left[ \frac{\tau^a \tau^b}{2} \right] \right) = N_{\text{generation}} \cdot 0 \cdot \left( \frac{\delta_{ab}}{2} \right) = 0, \\
\mathbf{j}_L : \quad & N_{\text{generation}} \cdot 0 = 0,
\end{align*}
\]

(4.10)

such that \(d \star \mathbf{j}_B = d \star \mathbf{j}_L = d \star (\mathbf{j}_B - \mathbf{j}_L) = 0\).
4.1.4 \((B-L)\cdot(\text{gravity})^2\): 4d local anomaly

Plug in data from Table 1 to check 4d local anomaly of \((B-L)\cdot(\text{gravity})^2\):

we find the anomaly factor contributes to

\[
\begin{align*}
 j_B & : \quad N_{\text{generation}} \cdot (N_c/3) \cdot \left(2 - 1 - 1\right) = 0. \\
 j_L & : \quad N_{\text{generation}} \cdot \left(2 - 1 - N_{\nu_R}\right) = N_{\text{generation}} \cdot \left(1 - N_{\nu_R}\right).
\end{align*}
\]

It turns out that \(d \star j_B = 0\) but \(d \star j_L \neq 0\) unless \(N_{\nu_R} = 1\). Same for \(d \star (j_B - j_L) = 0\) only if \(N_{\nu_R} = 1\). Perturbative anomaly seems to suggest one right-handed neutrino \(N_{\nu_R} = 1\) to saturate the \((B-L)\) current non-conservation. Are there other ways to saturate this ABJ type anomaly? We will resolve the issue with other novel possibilities in Sec. 5.

4.1.5 \((B-L)^3\): 4d local anomaly

Plug in data from Table 1 to check 4d local anomaly of \((B-L)\cdot(\text{gravity})^2\):

we find the anomaly factor contributes to

\[
\begin{align*}
 j_B & : \quad N_{\text{generation}} \cdot N_c \cdot (1/3)^3 \cdot \left(2 - 1 - 1\right) = 0. \\
 j_L & : \quad N_{\text{generation}} \cdot (1)^3 \cdot \left(2 - 1 - N_{\nu_R}\right) = N_{\text{generation}} \cdot \left(1 - N_{\nu_R}\right).
\end{align*}
\]
It turns out that $d \star j_B = 0$ but $d \star j_L \neq 0$ unless $N_{\nu R} = 1$. Same for $d \star (j_B - j_L) = 0$ only if $N_{\nu R} = 1$. Perturbative anomaly seems to suggest one right-handed neutrino $N_{\nu R} = 1$ to saturate the $(B - L)$ current non-conservation. Are there other ways to saturate this ABJ type anomaly? We will resolve other novel possibilities in Sec. 5.

### 4.1.6 $X$-SU(5)$^2$: 4d local anomaly

Recall in Sec. 2.1, the $U(1)_{B-L}$ is not a proper symmetry of SU(5) GUT. The “baryon minus lepton number symmetry” of SU(5) GUT is $U(1)_X$. Plug in data from Table 1 to check 4d local anomaly of $X$-SU(5)$^2$:

\begin{equation}
\frac{\text{Global sym } X}{\text{(Backgrd. field)}}
\end{equation}

\begin{equation}
\text{SU(5) gauge}
\end{equation}

\begin{equation}
\text{SU(5) gauge}
\end{equation}

we find the anomaly factor contributed from the representation $R$ of fermions in SU(5) as the anti-fundamental $R = \overline{5}$ and anti-symmetric $R = 10$, from the 15 Weyl fermions $\overline{5} \oplus 10$ in one generation. Let us check the $X$ current conservation or violation by ABJ type anomaly:

\begin{equation}
d \star (j_X) \propto \sum_R X_R \cdot \text{Tr}[F_{SU(5)} \wedge F_{SU(5)}] \propto \sum_R X_R \cdot c_2(SU(5)).
\end{equation}

Here $c_2(SU(5))$ is the second Chern class of SU(5), which is also related to the 4d instanton number of SU(5) gauge bundle. For $\overline{5} \oplus 10$ with $N_{\text{generation}}$, from Table 1, we get the $U(1)_X$ charges for

\begin{equation}
X_{\overline{5}} = -3, \quad X_{10} = 1,
\end{equation}

so\footnote{To evaluate the $c_2$ or the instanton number in different representations, $R_1$ and $R_2$, we use the fact that
\begin{equation}
\text{Tr}[F \wedge F]/\text{Tr}[F \wedge F] = (C_2(R_1) d(R_1))/(C_2(R_2) d(R_2)),
\end{equation}
here $C_2(R)$ and $d(R)$ are respectively the quadratic Casimir and the dimension of an irreducible representation $R$. For the representation $R$ of SU($N$), we have
\begin{equation}
\begin{array}{c|c|c}
R & d & C_2 \\
\hline
\text{Fundamental} & N & N^2 - 1 \\
\text{Antisymmetric} & N(N-1)/2 & 2(N+1)(N-2)
\end{array}
\end{equation}
For SU(5) with $N = 5$, we get $\text{Tr}_{10}[F \wedge F] = (N - 2)\text{Tr}_{\overline{5}}[F \wedge F] = 3\text{Tr}_{\overline{5}}[F \wedge F]$.}

\begin{equation}
d \star (j_X) \propto N_{\text{generation}} \left( X_{\overline{5}} \text{Tr}_{\overline{5}}[F \wedge F] + X_{10} \text{Tr}_{10}[F \wedge F] \right) = N_{\text{generation}} \cdot 0 = 0
\end{equation}

vanishes. We confirm that the $U(1)_X$ symmetry is ABJ anomaly free at least perturbatively in SU(5) GUT.
4.2 SM and GUT with extra discrete symmetries and cobordism theory

In the subsection, we aim to digest better how robust is the anomalies from Sec. 4.1.4 and Sec. 4.1.5 that seems only to be matched with a right-handed neutrino (the 16th Weyl spinor) per generation. These anomalies are not dynamical gauge anomalies if (B – L) and X are only global symmetries but the theory is only suffered from ‘t Hooft anomaly which only results in nonlocal or nononsite (B – L); however they may have to be gauged in SO(10) GUT. If fact, at least for the discrete $Z_{4,X} \subset U(1)_X$ as the $Z_4$ center of Spin(10),

$$Z_{4,X} = Z(\text{Spin}(10)) \subset \text{Spin}(10)$$  \hspace{1cm} (4.22)

needs to be dynamically gauged in the SO(10) GUT. This fact motivates Ref. [30] to use the $\Omega_5^{\text{Spin} \times Z_2} Z_4 = Z_{16}$ to argue the 16 chiral fermions for the 4d GUT in one generation. This fact also motivates Ref. [2, 36] to check the following cobordism groups TP$_d(G)$ with $G \supset \text{Spin} \times Z_2^p \ Z_4$ summarized in Table 4.

| Cobordism group TP$_d(G)$ with $G_{SM_q} \equiv (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_q$ and $q = 1, 2, 3, 6$ |
|---|---|---|
| $dd$ | classes | cobordism invariants |
| | | |
| 5d $Z^5 \times Z_2 \times Z_4^2 \times Z_{16}$ | $\mu(\text{PD}(c_1(U(1))))$, $\text{CS}_1^{U(1)} c_1(U(1))^2$, $\text{CS}_1^{U(1)} c_2(SU(3)) \sim c_1(U(1)) \text{CS}_3^{SU(2)}$, $\text{CS}_1^{U(1)} c_2(SU(3)) \sim c_1(U(1))\text{CS}_3^{SU(3)}$, $\text{CS}_1^{U(2)} c_2(SU(3)) \sim c_1(U(1))\text{CS}_3^{SU(3)}$, $(A_{Z_2})^2 \text{CS}_1^{SU(3)} + \text{CS}_2^{SU(3)}$, $(A_{Z_2}) c_2(SU(3)) \eta'$, $c_1(U(1))^2 \eta'$, $\eta(\text{PD}(A_{Z_2}))$ |
| $G = \text{Spin} \times Z_2 \ Z_4 \times G_{SM_2}$ | |
| 5d $Z^5 \times Z_2^2 \times Z_4 \times Z_{16}$ | $\mu(\text{PD}(c_1(U(2))))$, $\text{CS}_1^{U(2)} c_1(U(2))^2$, $(A_{Z_2})^2 \text{CS}_1^{SU(3)} + \text{CS}_1^{SU(2)} c_2(U(2)) \sim (A_{Z_2})^2 \text{CS}_1^{SU(3)} + c_1(U(2))\text{CS}_3^{SU(3)}$, $\text{CS}_1^{U(2)} c_2(SU(3)) \sim c_1(U(2))\text{CS}_3^{SU(3)}$, $(A_{Z_2})^2 \text{CS}_3^{SU(3)} + \text{CS}_2^{SU(3)}$, $(A_{Z_2}) c_2(SU(3))$, $(A_{Z_2}) c_2(U(2))$, $c_1(U(2))^2 \eta'$, $\eta(\text{PD}(A_{Z_2}))$ |
| $G = \text{Spin} \times Z_2 \ Z_4 \times G_{SM_3}$ | |
| 5d $Z^5 \times Z_2 \times Z_4^2 \times Z_{16}$ | $\mu(\text{PD}(c_1(U(3))))$, $c_1(U(3))^2 \text{CS}_1^{U(3)}$, $\text{CS}_1^{U(3)} c_2(SU(2)) \sim c_1(U(3))\text{CS}_3^{SU(2)}$, $(A_{Z_2})^2 \text{CS}_1^{SU(3)} + c_1(U(3))\text{CS}_3^{U(3)} + \text{CS}_5^{U(3)}$, $(A_{Z_2}) c_2(U(3))$, $(A_{Z_2}) c_2(U(2))$, $c_1(U(3))^2 \eta'$, $\eta(\text{PD}(A_{Z_2}))$ |
| $G = \text{Spin} \times Z_2 \times Z_4 \times G_{SM_6}$ | |
| 5d $Z^5 \times Z_2^2 \times Z_4 \times Z_{16}$ | $\mu(\text{PD}(c_1(U(3))))$, $c_1(U(3))^2 \text{CS}_1^{U(3)}$, $(A_{Z_2})^2 \text{CS}_1^{SU(3)} + c_1(U(3))\text{CS}_3^{U(3)} + \text{CS}_5^{U(3)}$, $(A_{Z_2}) c_2(U(3))$, $(A_{Z_2}) c_2(U(2))$, $c_1(U(3))^2 \eta'$, $\eta(\text{PD}(A_{Z_2}))$ |
| $G = \text{Spin} \times Z_2 \ Z_4 \times SU(5)$ | |
| 5d $Z \times Z_2 \times Z_{16}$ | $(A_{Z_2})^2 \text{CS}_1^{SU(3)} + \text{CS}_2^{SU(3)}$, $(A_{Z_2}) c_2(SU(5))$, $\eta(\text{PD}(A_{Z_2}))$ |

Table 4: Our setup follows Table 3 and Ref. [2, 36]. The $A_{Z_2} \in H^1(M, Z_2)$ is the generator from $H^1(B(Z_4/Z_2^p), Z_2)$ of $\text{Spin} \times Z_2^p \ Z_4$. The $\eta'$ is a $Z_4$ valued 1d eta invariant which is the extension of a quotient $A_{Z_2}$ by the normal 1d $\eta$. So $(A_{Z_2}) \equiv (A_{Z_2})$ mod 2 is the quotient, while $Z_{4,X} \subset U(1)_X$. The $\eta(\text{PD}(A_{Z_2}))$ is the value of $\eta \in Z_{16}$ on the Poincaré dual (PD) submanifold of $A_{Z_2}$.
We aim to initiate a new approach on matching the nonperturbative global $Z_{16}$ anomaly and the perturbative local $Z$ anomalies from Sec. 4.1.4 and Sec. 4.1.5 for the missing neutrinos.

The $Z^5$ classes perturbative local anomalies are the same results parallel to Sec. 3.1 and Table 3. So in the following subsections, we only focus on checking global anomaly cancellations of $Z_n$ classes. We may not have Feynman diagrams to characterize global anomalies (so we do not present Feynman diagrams below), but we can characterize them by generic curved manifolds as in Fig. 2 with gauge, gravity, or mixed gauge-gravity background fields.

### 4.2.1 Witten anomaly $c_2(SU(2))\tilde{\eta}$ vs $c_2(SU(2))\eta'$: 4d $Z_2$ vs $Z_4$ global anomalies

The old SU(2) anomaly of Witten in 4d is a mod 2 class $Z_2$ global anomaly from 5d $c_2(SU(2))\tilde{\eta}$ in [2] and Table 4 and 6d $c_2(SU(2))\text{Arf}$. The $c_2(SU(2))\tilde{\eta}$ as Witten SU(2) anomaly counts the spin-$2r + 1/2$ Weyl spinor as fermion doublet under SU(2). There are four of spin-$2r + 1/2$ fermions from $(3,2,\tilde{Y} = 1)_L$ and $(1,1,\tilde{Y} = 6)_L$. The $c_2(SU(2))\tilde{\eta}$ counts the number of 4d Weyl spinors of SU(2) fundamental 2 mod 2.

From (2.13) for $N_{\text{generation}}$, we have:

$$N_{\text{generation}} \cdot (3 + 1) = 0 \mod 2. \quad (4.23)$$

There is an extended $Z_4$ global anomaly from the 5d cobordism invariant $c_2(SU(2))\eta'$ in [2] and Table 4 counting the number of 4d Weyl spinors of SU(2) fundamental 2 mod 4. From (2.13) for $N_{\text{generation}}$, we have:

$$N_{\text{generation}} \cdot (3 + 1) = 0 \mod 4. \quad (4.24)$$

Therefore, we have checked no global SU(2) anomaly of $Z_2$ or $Z_4$ classes for SM with Spin $\times Z_2^F Z_4$ structure.

### 4.2.2 $(A_{Z_2})c_2(SU(2))$: 4d $Z_2$ global anomaly

The 4d $Z_2$ global anomaly from the 5d cobordism invariant $(A_{Z_2})c_2(SU(2))$ in [2] and Table 4 counts the number of 4d left-handed Weyl spinors of SU(2) fundamental 2 mod 2. Here $(A_{Z_2}) \equiv (A_{Z_4}) \mod 2$, where $A_{Z_2} \in H^1(M,\mathbb{Z}_2)$ is the generator from $H^1(B(\mathbb{Z}_4/\mathbb{Z}_2^F),\mathbb{Z}_2)$ of Spin $\times Z_2^F \mathbb{Z}_4$. So $(A_{Z_2}) \equiv (A_{Z_4}) \mod 2$ is the quotient, while $Z_{4,X} \subset \text{U}(1)_X$. The $(A_{Z_2})c_2(SU(2))$ counts the number of 4d Weyl spinors of SU(2) fundamental 2 mod 2. From (2.13) for $N_{\text{generation}}$, we have:

$$N_{\text{generation}} \cdot (3 + 1) = 0 \mod 2. \quad (4.25)$$

Thus the anomaly vanishes. We have no obstruction to gauge the $Z_4$ by making $A_{Z_4}$ dynamical at least from this anomaly cancellation.

### 4.2.3 $(A_{Z_2})c_2(SU(3))$: 4d $Z_2$ global anomaly

The 4d $Z_2$ global anomaly from the 5d cobordism invariant $(A_{Z_2})c_2(SU(3))$ in [2] and Table 4 counts the number of 4d left-handed Weyl spinors of SU(3) fundamental 3 mod 2. From (2.13), we count $(3,2,\tilde{Y} = 1)_L$, $(\bar{3},1,\tilde{Y} = -4)_L$, and $(\bar{3},1,\tilde{Y} = 2)_L$ with 3 generations. For $N_{\text{generation}}$, we have:

$$N_{\text{generation}} \cdot (2 - 1 - 1) = 0 \mod 4. \quad (4.26)$$
Thus there is no anomaly. We have no obstruction to gauge the $Z_4$ by making $A_{Z_4}$ dynamical at least from this anomaly cancellation.

4.2.4 $c_1(U(1))^2\eta'$: 4d $Z_4$ global anomaly

The 4d $Z_4$ global anomaly from a 5d cobordism invariant $c_1(U(1))^2\eta'$ in [2] and Table 4 counts the number of 4d Weyl spinors of U(1) charge mod 4. Let us apply $\hat{Y}$ for U(1)$_Y$ charge from (2.13) with $N_{\text{generation}}$. For each generation with U(1) charge, we have:

$$3 \cdot 2 \cdot 1 + 3 \cdot 1 \cdot (-4) + 3 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot (-3) + 1 \cdot 1 \cdot 6 = 0 \mod 4.$$  \hfill (4.27)

The $c_1(U(1))^2$ also counts the U(1) instanton number up to a proportional factor. Thus there is no anomaly. We have no obstruction to gauge the $Z_4$ by making $A_{Z_4}$ dynamical at least from this anomaly cancellation.\footnote{In fact, the cancellation mod 4 also holds true for U(1)$^2$ charge too: From (2.13) with $N_{\text{generation}}$, for each generation with U(1)$^2$ charge, we have:}

$$3 \cdot 2 \cdot 1^2 + 3 \cdot 1 \cdot (-4)^2 + 3 \cdot 1 \cdot 2^2 + 1 \cdot 2 \cdot (-3)^2 + 1 \cdot 2 \cdot (6)^2 = 3 \cdot 4 \cdot 13 = 0 \mod 4.$$  \hfill (4.28)

4.2.5 $(A_{Z_2})c_2(SU(5))$: 4d $Z_2$ global anomaly

Similar to (4.18), we consider the discrete 4d $Z_2$ global anomaly from the 5d cobordism invariant $(A_{Z_2})c_2(SU(5))$ in [2, 36] and Table 4. Here $c_2(SU(5))$ is the second Chern class of SU(5), which is also related to the 4d instanton number of SU(5) gauge bundle. For $\bar{5} \oplus 10$ with $N_{\text{generation}}$, from Table 1, we get the $Z_4, X$ charges for $X_{\bar{5}} = -3 = 1 \mod 4$, $X_{10} = 1 \mod 4$.

By footnote 16, we compute the anomaly factor

$$N_{\text{generation}}((X_{\bar{5}} \mod 4)\text{Tr}_{\bar{5}}[F \wedge F] + (X_{10} \mod 4)\text{Tr}_{10}[F \wedge F])$$

$$= N_{\text{generation}} \cdot \left(1 \cdot 1 + 1 \cdot 3\right) = N_{\text{generation}} \cdot 4 = 0 \mod 4.$$  \hfill (4.29)

This certainly vanishes for the mod 2 anomaly for SU(5) GUT.

There is also another $Z$ class local anomalies for SU(5) GUT captured by 5d cobordism invariants $(A_{Z_2})^2 CS_{SU(3)}^3 + CS_{SU(3)}^3$, we can easily check that SU(5) GUT is free from any local anomaly [96].

4.2.6 $\eta(PD(A_{Z_2}))$: 4d $Z_{16}$ global anomaly

The 4d $Z_{16}$ global anomaly from a 5d cobordism invariant $\eta(PD(A_{Z_2}))$ in [2] and Table 4 counts the number mod 16 of 4d left-handed Weyl spinors ($\Psi_L \sim 2_L$ of Spin(3, 1) or $\Psi_L \sim 2_L$ of Spin(4) = SU(2)$_L \times$ SU(2)$_R$). From (2.13) with $N_{\text{generation}}$ (e.g., 3 generations), for each generation, we have:

$$3 \cdot 2 + 3 \cdot 1 + 3 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 = 15 = -1 \mod 16.$$  \hfill (4.30)
For 1 generation, we need to saturates the anomaly:

\[ \nu = -1 \mod 16. \]  \hspace{1cm} (4.31)

For 3 generations, we need

\[ 3 \left( 3 \cdot 2 + 3 \cdot 1 + 3 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 \right) = 45 = -3 \mod 16. \]  \hspace{1cm} (4.32)

Therefore we need to saturates the anomaly:

\[ \nu = -3 \mod 16. \]  \hspace{1cm} (4.33)

For \( N_{\text{generation}} \) generations, we need to saturates the anomaly:

\[ \nu = -N_{\text{generation}} \mod 16. \]  \hspace{1cm} (4.34)

This anomaly can be canceled by adding new degrees of freedom

\[ \nu = N_{\text{generation}} \cdot (N_{\nu_R} = 1) \mod 16. \]  \hspace{1cm} (4.35)

The anomaly matching in this Sec. 4.2.6 seems to be matched with a right-handed neutrino (the 16th Weyl spinor) per generation, similar to Sec. 4.1.4 and Sec. 4.1.5. This also shows the robustness of Sec. 4.1.4 and Sec. 4.1.5 even if we break down \( U(1)_{B-L} \) or \( U(1)_X \) down to \( Z_4_{B-L} \) or to \( Z_4_X \). Again this \( Z_4 \) as the center \( Z(\text{Spin}(10)) \) of Spin(10) is important for the SO(10) GUT.

Are there other ways to match the anomaly other than introducing the right-handed neutrino (the 16th Weyl spinor) per generation? Let us explore in the next subsection.

### 4.3 How to match the anomaly? Preserving or breaking \( Z_{4,X} \)?

Let us summarize what we learn from the anomaly computation and matching in previous sections. We have shown that all anomalies presented in Table 3 and Table 4 can be cancelled, except the additional anomaly constraint from Sec. 4.1.4, Sec. 4.1.5, and Sec. 4.2.6 may not matched unless we obey

\[ N_{\text{generation}} \cdot (1 - N_{\nu_R} + \text{hidden sector}) = 0, \quad \text{from Sec. 4.1.4 and Sec. 4.1.5.} \]  \hspace{1cm} (4.36)

\[ N_{\text{generation}} \cdot (1 - N_{\nu_R} + \text{hidden sector}) = 0 \mod 16, \quad \text{from Sec. 4.2.6.} \]

So the above naïvely suggests we need the right-handed neutrino (the 16th Weyl spinor) \( N_{\nu_R} = 1 \) per generation. However, we do have different ways to match the anomaly other than introducing the right-handed neutrino (the 16th Weyl spinor) \( N_{\nu_R} = 1 \). Can we match the anomaly by **additional new hidden sectors** not yet discovered in SM or in SU(5) Georgi-Glashow GUT? Let us focus on the \( Z_{4,X} \) symmetry instead of \( U(1)_X \) for the sake of thinking \( Z_{4,X} = Z(\text{Spin}(10)) \) in SO(10) GUT eventually, and let us enumerate the possibilities to match the anomaly (4.36):

1. **Anomaly matched by a right-handed neutrino (the 16th Weyl spinor) \( N_{\nu_R} = 1 \):**

   For the right-handed neutrino \( \nu_R \) to be massless (or gapless) while preserving the \( Z_{4,X} \) symmetry, we need to have \( \nu_R \) to be a complex Weyl spinor (with a \( Z_{4,X} \) charge \(-1 \mod 4\)), instead of a real Majorana spinor, in order to have the \( Z_{4,X} \) symmetry transformation manifest:

   \[ \nu_R \to \exp(-2\pi i/4) \nu_R = (-i) \nu_R. \]  \hspace{1cm} (4.37)
Since this is a sterile neutrino with a trivial representation of SM gauge group (2.9), (1,1,0)$_R$, we can rotate it to left-handed Weyl spinor $\bar{\nu}_R = \nu_L$ with (1,1,0)$_L$ and flips the $Z_{4,X}$ representation to its complex conjugation:

$$\nu_L \rightarrow \exp(2\pi i/4) \nu_L = (i) \nu_L.$$

What can the low energy dynamics of the $\nu_R$ be?

1. The $Z_{4,X}$ preserving: If the sterile neutrino $\nu_R$ remains gapless, we can match the anomaly (4.36) by a symmetric gapless low energy theory with an action on a 4d spacetime $M^4$:

$$\int_{M^4} \bar{\nu}_R (i \sigma^\mu \partial_\mu) \nu_R, \quad \text{or equivalently} \quad \int_{M^4} \bar{\nu}_L (i \bar{\sigma}^\mu \partial_\mu) \nu_L,$$

with $\bar{\sigma} \equiv (1, \vec{\sigma})$.

2. The $Z_{4,X}$ explicit breaking: If the sterile neutrino $\nu_R$ becomes gapped by Majorana mass $m_{\text{Maj}}$, the spacetime spinor becomes in a real Majorana representation, and the $Z_{4,X}$ symmetry in (4.37) and (4.38) is explicitly broken. We can still match the anomaly (4.36) by a $Z_{4,X}$-symmetry-breaking gapped theory with an action on a 4d spacetime $M^4$:

$$\int_{M^4=\partial M^5} \chi^T (i \bar{\sigma}^\mu \nabla_\mu) \chi + \frac{im_{\text{Maj}}}{2} (\chi^T \sigma^2 \chi + \chi^T \sigma^2 \chi^*),$$

where we have written the 4-component Majorana spinor as $\Psi_{\text{Maj}} = \left( \begin{array}{c} \chi \\ i \sigma^2 \chi^* \end{array} \right)$ with the transpose $T$, complex conjugate $*$, and complex conjugate transpose $\dagger$.

2. Anomaly matched by new additional or hidden sectors beyond SM: Let us hypothesize many scenarios with different low energy dynamics following Ref. [2]'s Sec. 8.2:

1. $Z_{4,X}$-symmetry-preserving anomalous gapless or interacting 4d CFT.
2. $Z_{4,X}$-symmetry-breaking gapless or interacting 4d CFT.
3. $Z_{4,X}$-symmetry-preserving anomalous gapped 4d TQFT.
4. $Z_{4,X}$-symmetry-breaking gapped 4d TQFT.
5. $Z_{4,X}$-Symmetry-Protected Topological state (SPTs) in 5d captured by 5d cobordism invariant

$$\eta(\text{PD}(A_{Z_2})).$$

(4.41)

(6). $Z_{4,X}$-gauged-(Symmetry)-Enriched Topological state (SETs) in 5d coupled to gravity.

We propose that all the above scenarios, conventional or exotic, if existing, can saturate the anomaly (4.36). Based on the contemporary knowledge of SM physics and experimental hints,

- Scenario (1) and (2) seem unpractical, because it is less likely to have any new gapless or interacting CFT that we do not observe below the TeV energy scale. Also if there is spontaneous symmetry-breaking (SSB), for $U(1)_X$ SSB, we expect to observe new Goldstone boson modes; for $Z_{4,X}$ SSB, we may observe different vacua or domain walls between different vacua. This shall be falsifiable in the experiments.

- Scenario (3) is exotic but very interesting, which we discover new insights into the neutrino physics and Dark Matter. Moreover, Scenario (3) can give rise to Scenario (4), if we construct such a $Z_{4,X}$-symmetry-preserving anomalous TQFT first, we can break some of the symmetry to obtain the symmetry-breaking gapped TQFT. So we should focus on Scenario (3) explored in Sec. 5.2.
• Scenario (5) implies that our 4d SM lives on the boundary of 5d $Z_{4,X}$-SPTs given by 5d $\eta(\text{PD}(A_{Z_2}))$. If Scenario (5) describes our universe, we discover at least an extra dimension from the 5d theory. We explore this 5d SPT or invertible TQFT theory in Sec. 5.1.

• Scenario (6) Above certain higher energy scale, the $Z_{4,X}$ may be dynamically gauged such as in $Z_{4,X} = Z(\text{Spin}(10))$ of the SO(10) GUT, then the 4d and 5d bulk are fully gauged and entangled together. The 5d bulk is in fact a 5d Symmetry-Enriched Topologically ordered state (SETs) coupled to dynamical gravity. We explore this 5d SET coupled to dynamical gravity theory in Sec. 6.1.

5 Beyond Three "Fundamental" Forces: Hidden New Topological Force

Follow Sec. 4.3, now we propose new Scenarios, beyond SM and beyond Georgi-Glashow (GG) SU(5) GUT, to match the anomalies of $Z_{16}$ from Sec. 4.2.6 (also match the constraint from Sec. 4.1.4 and Sec. 4.1.5). We focus on the Scenario (3) (thus also Scenario (4)) for a hidden 4d non-invertible TQFT in Sec. 5.2, and for a 5d SPTs or 5d SETs coupled to gravity in Scenario (5) and (6) in Sec. 6.1.

5.1 Hidden 5d invertible TQFT or 5d Symmetry-Protected Topological state (SPTs)

We can saturate the missing anomaly of $\nu = -N_{\text{generation}} \mod 16$ in (4.34) by a 5d invertible TQFT (or a 5d SPTs protected by $G = \text{Spin} \times Z_2 \times Z_4 \times G_{\text{internal}}$-symmetry in a condensed matter language) with a 5d partition function:

$$Z_{5d-iTQFT} = \exp \left( \frac{2\pi i}{16} \cdot (-N_{\text{generation}}) \cdot \eta(\text{PD}(A_{Z_2})) \right|_{M^5}. \right.$$  (5.1)

Here we define the eta invariant

$$\eta \in Z_{16}, \quad \text{and} \quad \eta(\text{PD}(A_{Z_2})) \in Z_{16}$$

slightly different by a proportional factor in the math literature. Let us overview quickly what we need about the APS eta invariant $\eta$ [13–15,22,97]. Let us consider the 4d $\eta$ invariant and then relate to our 4d $\eta^{18}$ and 5d $\eta(\text{PD}(A_{Z_2}))$ invariants.

On an Euclidean signature curved spacetime manifold $M^d$, the path integral of a massive fermion spinor with a mass $m$ coupled to a background gauge field $A$ or a background gravity of a metric $g_{\mu\nu}$, the Euclidean path integral is

$$\int [D\psi][D\bar{\psi}] \exp (-S_E) = \int [D\psi][D\bar{\psi}] \exp \left( -\int_{M^d} d^4x_E \sqrt{\det g(\bar{\psi}(D_A + m)\psi)} \right) = \det(\bar{\psi}(D_A + m)), \quad (5.2)$$

where locally $D_A \equiv e_{\mu}^\gamma \gamma^\mu (\partial_\mu + i\omega_\mu - iA_\mu)$, $e_{\mu}^\gamma$ is a vielbein, with a spin connection $\omega_\mu$, and a gauge connection $A_\mu$ for a gauge bundle of a group $G$. More explicitly, the components are $\omega_\mu = i\omega^\lambda_\mu [\gamma^\lambda, \gamma^\nu]/8$ and $A_\mu = A^a_{\mu} T^a$ with $T^a$ generators of the Lie algebra Lie($G$). We also specify the transition functions

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18The 4d $\eta$ invariant has condensed matter realizations. It is known as the class DIII topological superconductor [TSC] in the free fermion limit with a $Z$ classification [77–79]. The $\nu = 1$ of this TSC is the Balian-Werthamer (BW) state of the B phase of $^3$He liquid ($^3$He B) [98]. A 2+1d single Majorana fermion can live on the surface protected by time reversal symmetry in the non-interacting free system.
that relate fields $\psi$ and $\bar{\psi}$ (two independent Grassmann fields) and $A$ on different patches in order to 
\textit{globally} define the Dirac operator $D_A$ on $M^d$. The transition function also leaves the local expression 
of the partition function invariant. If we define $m \to \infty$ to be a trivial gapped vacua without any 
topological feature, then the $m \to -\infty$ gapped vacua can host a nontrivial iTQFT or SPTs in $d d$. Follow 
the notations in [62], the $d d$ partition function of nontrivial iTQFT or SPTs can be defined via a ratio:

\[ Z_{dd-iTQFT}[A] = \lim_{|m| \to \infty} \frac{\det(D_A - |m|)}{\det(D_A + |m|)} \equiv \lim_{|m| \to \infty} \prod_{\lambda} \frac{i \lambda - |m|}{i \lambda + |m|} = \frac{1}{2} (N_0 + \lim_{s \to 0^+} \sum_{\lambda \neq 0} \text{sign} \lambda \cdot |\lambda|^{-s}). \quad (5.3) \]

Here $D_A$ is anti-Hermitian, so $-i D_A$ is Hermitian. The $\lambda$ are eigenvalues of $-i D_A$ so $\lambda \in \mathbb{R}$ are real. 
$N_0$ are the number of the operator $-i D_A$’s zero modes. Depending on the underlying $G$-structure of manifolds and $-i D_A$ (such as $G = \text{Pin}^+$ [22], $G = \text{Pin}^c$ [97, 99], or $G = \text{Pin}^+ \times \mathbb{Z}_2 \text{SU}(2)$ [62]), the $4d$ 
iTQFT/SPT partition functions are respectively:

\[ \left( Z_{4d-iTQFT} \right)^{\nu} \lim_{|m| \to \infty} \left\{ \begin{array}{ll}
\exp\left(\frac{2\pi i}{16} \cdot \nu \cdot \eta_{\text{Pin}^+} \right), & \text{with } \eta_{\text{Pin}^+} \in \frac{1}{8} \mathbb{Z}, \quad \nu \in \mathbb{Z}_{16}.
\exp\left(\frac{2\pi i}{16} \cdot \nu \cdot \eta_{\text{Pin}^c} \right), & \text{with } \eta_{\text{Pin}^c} \in \frac{1}{8} \mathbb{Z}, \quad \nu \in \mathbb{Z}_8.
\end{array} \right. \quad (5.4) \]

However, in our work, we adjust above definitions to make $\eta_{\text{Pin}^+} \in \mathbb{Z}_{16}$ via $\eta_{\text{Pin}^+} = 8\eta_{\text{Pin}^+} \mod 16$, so:

\[ \left( Z_{4d-iTQFT} \right)^{\nu} \lim_{|m| \to \infty} \exp\left(\frac{2\pi i}{16} \cdot \nu \cdot \eta_{\text{Pin}^+} \right), \quad \text{with } \eta_{\text{Pin}^+} \in \mathbb{Z}_{16}, \quad \nu \in \mathbb{Z}_{16}. \quad (5.5) \]

Similarly, based on the Smith homomorphism [30, 67, 68, 100] we characterize the 5d iTQFT whose manifold generator for bordism group is $\mathbb{R}P^5$ at $\nu = 1$ below. The 
cobordism invariant for any $\nu$ corresponds to a 5d iTQFT partition function:

\[ \left( Z_{5d-iTQFT} \right)^{\nu} \lim_{|m| \to \infty} \exp\left(\frac{2\pi i}{16} \cdot \nu \cdot \eta(\text{PD}(A_{Z_2})) \right), \quad \text{with } \eta_{\text{Pin}^+} \in \mathbb{Z}_{16}, \quad \nu \in \mathbb{Z}_{16}. \quad (5.9) \]

\textbf{19} The $\det(-i D_A) = \prod_{\lambda} \lambda$. One can regularize it by $\det(-i D_A) = \prod_{\lambda} \frac{1}{\lambda + i M}$ via a Pauli-Villars regulator of mass $M$. 
The APS $\eta$-invariant [13–15] is a regularization of $\sum_{\lambda} \text{sign} \lambda$.

\textbf{20} In fact, there are more Smith homomorphisms in any dimension. Related results along these maps are abundant, 
see [100] and also [20, 36, 68]:

\begin{align*}
\Omega_5^{\text{Spin} \times \mathbb{Z}_2} Z_4 &= Z_{16} \quad \text{generated by } \mathbb{R}P^5, \\
\Omega_4^{\text{Pin}^+} &= Z_{16} \quad \text{generated by } \mathbb{R}P^5, \\
\Omega_4^{\text{Spin} \times \mathbb{Z}_2} Z_4 &= Z_{16} \quad \text{generated by } \mathbb{R}P^5.
\end{align*}

\textbf{5.6} Notice that $H^4(\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2$, for example by using $n = 5$, if we choose $A_{Z_2}$ for $H^4(\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2$, then $\mathbb{R}P^4$ and $\mathbb{R}P^5$ detect all $\nu \in \mathbb{Z}_{16}$:

\[ \exp\left(\frac{2\pi i}{16} \cdot \nu \right) = \exp\left(\frac{2\pi i}{16} \cdot \nu \right). \quad (5.7) \]
Given a $G \supseteq \text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4$ structure, the cohomology class $A_{\mathbb{Z}_2} \in H^1(M, \mathbb{Z}_2)$ is the generator from $H^1(B(\mathbb{Z}_4/\mathbb{Z}_2^F), \mathbb{Z}_2)$. So $\langle A_{\mathbb{Z}_2} \rangle \equiv \langle A_{\mathbb{Z}_4} \rangle \mod 2$ is the quotient for the $\mathbb{Z}_4$ gauge field $A_{\mathbb{Z}_4}$, which is the background field for the field $\mathbb{Z}_4, X \subset \text{U}(1)$. The $\eta(\text{PD}(A_{\mathbb{Z}_2}))$ is the value of $\eta \in \mathbb{Z}_{16}$ on the Poincaré dual (PD) submanifold of the cohomology class $A_{\mathbb{Z}_2}$. This PD is the precise meaning of (5.8) taking $\cap A_{\mathbb{Z}_2}$ from 5d to 4d, or the other way around.

Thus, to summarize, the full 5d iTQFT and the 4d SM or SU(5) GG GUT model action $S_{\text{SM or GUT}}$ together can make the anomaly (4.34) matched via the full 5d-4d coupled partition function

$$Z_{5d-4d} = \exp\left(\frac{2\pi i}{16} \cdot \eta(\text{PD}(A_{\mathbb{Z}_2}))|_{M^5}\right) \cdot \int [D\psi][D\bar{\psi}][DA] \cdots \exp (iS_{\text{SM or GUT}}|_{M^4}). \quad (5.10)$$

with a bulk-boundary correspondence $\partial M^5 = M^4$. The full $Z_{5d-4d}$ is gauge invariant, in particular also invariant under the background $\mathbb{Z}_4, X$ transformation (at least at the higher-energy of GUT scale, but it can be broken: SSB or explicitly at low energy). This concludes the 5d SPTs coupled to 4d SM or GUT in the Scenario (5).

### 5.2 Hidden 4d non-invertible and non-abelian TQFT or 4d topological order

Now we explore the Scenario (3) (thus also Scenario (4)) for a hidden 4d non-invertible TQFT in Sec. 5.2 to match the missing anomaly. In fact to construct such a symmetry-preserving anomalous 4d TQFT (here we preserve $\mathbb{Z}_4, X$ in Spin $\times \mathbb{Z}_2 \mathbb{Z}_4$ structure), we take the inspirations from the quantum condensed matter phenomenon in one lower dimension, known as the symmetry-enriched anomalous surface topological order in 3d (2+1d spacetime) living on the boundary of 4d SPTs (3+1d spacetime). In condensed matter thinking, it can be understood as saturating the ’t Hooft anomaly by gapped sector, without breaking any symmetry breaking and without gapless modes, via smearing the ’t Hooft anomaly and anomalous symmetry into the long-range entanglement. In the context of 3d boundary and 4d bulk, a novel surface topological order was firstly pointed out by Vishwanath-Senthil in an insightful work [53], later on many people follow up for the surface topological order constructions (see an excellent review in [45,54]). We need to generalize this condensed matter approach to find a symmetric anomalous 4d TQFT living on the boundary of 5d SPTs (5.9). In particular, we require following ingredients and insights from previous work.

#### 5.2.1 3d non-abelian Chern-Simons theory SU(2)$_6$ and SO(3)$_3$ as toy model

1) A 2+1d non-abelian topological order construction of Fidkowski-Chen-Vishwanath (FCV) [55] from Walker-Wang model [101]: Ref. [55] constructs on the boundary of a 3+1d $\mathbb{Z}_{16}$-class topological superconductor (TSC) with $\mathbb{Z}_4^{CT}$ time reversal symmetry ($T^2 = (-1)^F$ or 4d Pin$^+$-structure as in (5.6)) by an exactly solvable lattice Hamiltonian model. The 4d bulk $\nu \in \mathbb{Z}_{16}$ class of TSC is precisely described by the partition function (5.5). Ref. [55] construct the $\mathbb{Z}_{16}$-class 2+1d surface topological order closely related to Kitaev’s [48]’s 16-fold ways of 2+1d anyon models. Ref. [55] claims that for the odd $\nu$ mod 2 = 1, the 3d boundary must have a non-abelian topological order in order to be fully symmetric and saturate the full anomaly (from $T^2 = (-1)^F$ or 4d Pin$^+$’s $\eta$-invariant). This 3d candidate non-abelian topological order has a low energy TQFT given by a 3d SO(3)$_3$ Chern-Simons (CS) theory. The SO(3)$_3$ CS can be obtained from SU(2)$_6$ CS by doing any of the following:
• projecting out half-integer spin representation line operators of SU(2)$_6$ CS.
• gauging the 1-form $\mathbb{Z}_2$ electric symmetry (defined as $\mathbb{Z}_2^{\epsilon}$) of SU(2)$_6$ CS [71].
• anyon condensation via condensing the anyon (whose line operator is a $\mathbb{Z}_2^{\epsilon}$ symmetry generator) of SU(2)$_6$ CS. By anyons, we mean the fractionalized quasi-excitation living on the open ends of 1d line operators. In other words, the worldline of anyon corresponds to exactly the 1d line operator.

More importantly, we can precisely write down the data of SU(2)$_6$ CS and SO(3)$_3$ CS in Table 5.

| SU(2)$_6$ Chern-Simons TQFT in 3d |
|-----------------------------------|
| Rep.  | $n = 2j + 1$ | 0 | 1/2 | 1 | 3/2 | 2 | 5/2 | 3 |
| spin statistics. | $\frac{j(j+1)}{8}$ | 0 | $\frac{3}{32}$ | $\frac{1}{4}$ | $\frac{15}{32}$ | $\frac{3}{4}$ | $\frac{35}{32}$ | $\frac{3}{2}$ |
| $\mathcal{T}$-matrix | $e^{-\frac{i2\pi j(j+1)}{8}}$ | 1 | $e^{\frac{i2\pi}{16}}$ | $e^{\frac{i15\pi}{16}}$ | $\overline{-i}$ | $e^{\frac{i3\pi}{16}}$ | $-1$ |

| SO(3)$_3$ Chern-Simons TQFT in 3d |
|-----------------------------------|
| Rep.  | $n = 2j + 1$ | 0 | 1 | 2 | 3 | 5 | 7 |
| spin statistics. | $\frac{j(j+1)}{8}$ | 0 | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{3}{2}$ |
| $\mathcal{T}$-matrix | $e^{-\frac{i2\pi j(j+1)}{8}}$ | 1 | $e^{\frac{i2\pi}{16}}$ | $e^{\frac{i15\pi}{16}}$ | $\overline{-i}$ | $e^{\frac{i3\pi}{16}}$ | $-1$ |
| quantum dimension $d_Q$ | (abelian) | (non-abelian) | (abelian) |

Table 5: Data of 3d Chern-Simons (CS) TQFTs for SU(2)$_6$ CS and SO(3)$_3$ CS.

We present the complete 7 line operators (i.e., 7 anyons, or 7 superselection sectors) for 3d SU(2)$_6$ CS, labeled by integer or half-integer spin or $n = 2j + 1$-dimensional representations. We present the complete 4 line operators (i.e., 4 anyons, or 4 superselection sectors) for 3d SO(3)$_3$ CS, labeled by integer spin or odd $n$-dimensional representations. We present their anyonic spin-statistics (beyond bosons and fermions) and their modular $\mathcal{T}$-matrix representation from mapping class group of 2-torus $\text{MCG}(T^2) = \text{SL}(2,\mathbb{Z})$. We also show the quantum dimensions of 4 anyons of SO(3)$_3$ CS, two anyons are abelian (with quantum dimension $d_Q = 1$, including a trivial boson) and two anyons are non-abelian (with $d_Q > 1$ and with semion self-statistics: an adiabatic $2\pi$ self-rotation gains its wavefunction a $\pm$ sign, which is a half of fermion of a $-1$ sign). SO(3)$_3$ CS has nonabelian fusion rules (here the “×” means fusion, the “+” means the splitting into different fusion channels [thus nonabelian fusion statistics]): $f \times f = 1$. $S \times S = \tilde{S} \times \tilde{S} = 1 + S + \tilde{S}$. $S \times \tilde{S} = S + \tilde{S} + f$. $S \times f = \tilde{S}$. $\tilde{S} \times f = S$. Note that SU(2)$_6$ CS can be defined on non-spin manifold and known in the frame work of bosonic modular tensor category. But SO(3)$_3$ CS is a fermionic spin-TQFT (defined on spin manifolds), which is pre-modular in the category context.

The SU(N)$_k$ CS has a chiral central charge $c_-$ [102]

$$c_- \equiv c_L - c_R = \frac{k(N^2 - 1)}{k + N}. \quad (5.11)$$

SU(2)$_6$ CS and SO(3)$_3$ CS has the same chiral central charge $c_- := c_L - c_R = \frac{6(2^2 - 1)}{6 + 2} = \frac{9}{4}$, which naïvely seems to suggest that there are chiral edge modes and cannot be time-reversal ($CT$) invariant. However,
on the 3d boundary of 4d bulk $\nu \in \mathbb{Z}_{16}$ class of TSC, the SO(3)$_3$ CS can be made to be CT invariant. Meanwhile CT does not transform the trivial anyon 1 and another fermion $f$, so in an abbreviation, CT(1) = 1 and CT($f$) = $f$. The CT switches $S$ and $\tilde{S}$, so CT($S$) $\propto$ $\tilde{S}$ and CT($\tilde{S}$) $\propto$ $S$ up to a complex phase. There are in fact two versions of SO(3)$_3$ CS depending on the CT symmetry assignment [103]:

\[
\begin{align*}
\text{SO(3)$_3$}_+\text{CS} & : (CT)^2 = +i \text{ for } S, \quad (CT)^2 = -i \text{ for } \tilde{S}, \quad \text{match } \nu = 3 \in \mathbb{Z}_{16} \text{ anomaly.} \\
\text{SO(3)$_3$}_-\text{CS} & : (CT)^2 = -i \text{ for } S, \quad (CT)^2 = +i \text{ for } \tilde{S}, \quad \text{match } \nu = -3 \in \mathbb{Z}_{16} \text{ anomaly. (5.12)}
\end{align*}
\]

Since the anomaly we need to match by 4d TQFT is (4.34) as $\nu = -N_{\text{generation mod 16}}$ in particular $N_{\text{generation}} = 3$. Serendipitously we claim that the suitable 3d TQFT for our later 4d TQFT construction should be SO(3)$_3$+$\text{CS}$ for adding $\nu = +3 \in \mathbb{Z}_{16}$ anomaly that plays an important rule to match the missing 3 sterile right-handed neutrinos.

2). It is also useful to understand the phenomenon of symmetric gapped TQFTs saturating various anomalies in general grounds and in general dimensions. One approach is via so-called the symmetry-extension [52], which can be generalized to cases involving higher-symmetry extension [50]. In contrast to known gapped phase saturate anomaly via symmetry breaking (either global symmetry breaking or Anderson-Higgs mechanism for gauge theory), this approach is based on symmetry extension (thus beyond symmetry breaking and Anderson-Higgs mechanism). This approach applies to QFT or lattice models, as well as gauge theory [90] or mathematical ways of constructions [104].

5.2.2 Categorification: from 3d to 4d?

How do we go from 3d SO(3)$_3$+$\text{CS}$ matching anomaly for $\nu = \pm3 \in \mathbb{Z}_{16}$ of 4d $\eta$ invariant to a 4d TQFT matching anomaly for $\nu = \pm3 \in \mathbb{Z}_{16}$ of 5d $\eta(PD(\mathbb{A}_{2n}))$ invariant? One formal mathematical idea is called the categorification and decategorification. There is the categorification from math category theory perspective [105–106]. There is also the the categorification from QFT, TQFT and quantum topology perspective [107–109]. Categorification aims to replaces set-theoretic theorems by category analogues. Decategorification reverses the procedure of categorification. In fact, for bosonic and fermionic finite group gauge theories, there are explicit constructions to related 3d TQFT to 4d TQFT by dimensional extension (categorification) or dimensional reduction (decategorification) by compactification on a $S^1$ circle [68,110,111].

We may ask whether it make senses to categorify 3d SO(3)$_3$+$\text{CS}$? It turns out that categorifying SU(2)$_6$ or SO(3)$_3$ CS is challenging, since the partition functions on 3-manifolds and expectations values of links of Wilson loops are not integer valued, so they cannot represent dimensions of any spaces. There is an analytic continuation with respect to CS level $k$ and then consider expansion in $q = \exp(2\pi i/k)$, which does have integer $\mathbb{Z}$ coefficients (by fixing subtleties for non-trivial 3-manifolds), and then we can categorify these CS in terms of 4d theory with an extra U(1) symmetry for which $q$ becomes the fugacity. This categorification is given by Khovanov-Rozansky link cohomology In the case of links in $\mathbb{R}^3$ or $S^3$ [108,109]. This categorification is given by [112,113] in the case of partition functions on more generic 3-manifolds. Although these constructions seem mathematically appealing, they do have troubles to make connection to what we look for. This categorification has the CS level not fixed. Also the underlying 4d theory is non-unitary from the topological twisting of a supersymmetric QFT. Also the dimension of Hilbert space is generally not finite and not integer on a 4-manifold $M^3 \times S^1$.

However, we do require our desired 4d TQFT to be unitary, with a finite dimension of Hilbert space (by computing partition function $Z(M^3 \times S^1)$). We shall leave possible unitary 4d TQFT from a categorification of 3d CS for future work. We seek for other more physical routes next.
5.2.3 4d non-abelian TQFT from 3d non-abelian TQFT

The experience to construct a lower dimensional TQFT (in 3d) can help to construct a higher dimensional TQFT (in 4d), by putting a former theory on the symmetry-breaking domain wall. In fact, the $Z_{16}$ class of 5d iTQFT/SPTs $\eta(PD(A_{Z_{16}}))$ can be constructed from decorating the $Z_{4} = Z_{4,X}$-symmetry breaking domain wall with the $Z_{16}$ class of 4d iTQFT/SPTs $\eta$. Suppose we start with the 3d $SO(3)_3$ CS $Z_{SO(3)}^{CS} = \int [D\mathcal{A}] \exp(i k(2\pi) \int_{M^4} CS_{SO(3)}^{SO(3)}(\mathcal{A})) = \int [D\mathcal{A}] \exp(\frac{ik}{4\pi} \int_{M^4} Tr(\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3)),$ (5.13)

where the two versions of $SO(3)_{3,\pm}CS$ are differed only by assignment of the $Z_4$ symmetry in (5.12) (the $Z_4^{CT}$ in 3d, or the $Z_{4,X}$ in 4d later). We may shorthand and omit the obvious wedge product: $\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \equiv (\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \wedge \mathcal{A} \wedge \mathcal{A})$. In principle, we hope to extend to 4d but also keep (at least part of) the gauge invariance for some global $\mathcal{U}$ transformations in $SO(3)$:

$$\mathcal{A} \rightarrow \mathcal{U}(\mathcal{A} + \frac{i}{g} d\mathcal{A})^{-1}, \quad \mathcal{U} = \exp(i\theta^a T^a).$$

Motivated by the 4d BF theory [114] (see a systematic summary of continuum TQFT formulations [115] for twisted versions of unitary BF TQFTs), a naïve schematic 4d TQFT can contain the following term:

$$Z_{4dTQFT}(A_{Z_4}) = \int [DB][D\mathcal{A}] \exp(i \frac{k_{d4}}{2\pi} \int_{M^4} \sum I k_{d4,I} B_I \wedge d\mathcal{A} + i \frac{k_{d4}}{4\pi} \int_{M^4} A_{Z_4} \wedge Tr(\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3) + \ldots),$$ (5.14)

where we append the local 2-form $B$ gauge field to make the BF term $B \wedge d\mathcal{A}$. We also append the background gauge field $A_{Z_4}$ for $Z_{4}/Z_2^F$ to the 3d CS term $A_{Z_4} \wedge Tr(\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3)$. We have to fix the gauge invariance by adding additional terms and fields into the action. With the quantized levels $k_{d4,I}$ and $k_{d4}$ to be fixed later in this subsection.

How can we make sense such a partition function (5.14) physically? Certainly, there are some fair motivations that a certain modified version of (5.14) can make sense. For a TQFT with a finite gauge group in 4d, such as Dijkgraaf-Witten topological gauge theory [92], indeed Ref. [115–117] shows that such a continuum TQFT path integral can be defined and computed explicitly:

$$\int [DB_I][D\mathcal{A}_I] \exp(i \int_{M^4} \sum I N_I \mathcal{B}_I \wedge d\mathcal{A}_I + \frac{1}{2\pi} \sum I,J N_{IJ} p_{IJK} (\mathcal{A}_I \wedge \mathcal{A}_J \wedge \mathcal{A}_K),$$

$$\int [DB_I][D\mathcal{A}_I] \exp(i \int_{M^4} \sum I N_I \mathcal{B}_I \wedge d\mathcal{A}_I + \frac{1}{2\pi} \sum I,J,N \mathcal{N}_{IĴN} p_{I234} \mathcal{A}_I \wedge \mathcal{A}_J \wedge \mathcal{A}_K \wedge \mathcal{A}_L) \quad \text{for (5.15).}$$ (5.15)

They are invariant under appropriate gauge transformations defined in [115–118] (in fact, gauge variations are exactly cancelled globally to all order):

$$\mathcal{A}_I \rightarrow \mathcal{A}^I + dg^I, \quad B_I \rightarrow B^I + d\eta^I + \epsilon_{IJ} \frac{N_I p_{IJK}}{2\pi N_I} d\eta^J \wedge \mathcal{A} K \text{ for (5.15).}$$

$$\mathcal{A}^I \rightarrow \mathcal{A}^I + dg^I, \quad B^I \rightarrow B^I + d\eta^I - \frac{c_{IJ} K L}{N_I} \mathcal{A}^J \mathcal{A}^K g^L + \frac{c_{IJ} K L}{N_I} \mathcal{A}^J g^K d\eta^L - \frac{\pi c_{IJ} K L}{3} N_I g^J d\mathcal{A}^K d\mathcal{A}^L \text{ for (5.16),}$$

with $c_{IJ} K L \equiv \epsilon_{IJN} \frac{1}{2\pi} \frac{N_I N_J N_K N_L p_{I234}}{N_{1234}}$ the $\eta^I$ and $g^I$ are locally 1-form and 0-form gauge transformation parameters. We can also formulate a TQFT combination of two types: (5.15) and (5.16).

\footnote{For 3d Chern-Simons 3-form $CS_3^{SO(3)}(\mathcal{A})$ as a cobordism invariant, see Tables in Sec. 1.6 of arXiv version of [1].}
Their link invariants include:
(1) Aharanov-Bohm type of particle and string braiding statistics (such as the link of 1-worldline $S^1$ and 2-worldsheet $S^2$ linked in a spacetime $S^4$),
(2) Triple link invariants of three 2-surfaces [119] in 4d spacetime from three sets of gauge-invariant 2-surface operator $B_I + \ldots$ [115]. This is known as the multi-string 3-loop braiding in the Hamiltonian picture [110, 120, 121].
(3) Quadruple link invariants of four 2-surfaces in 4d spacetime from four sets of gauge-invariant 2-surface operator $B_I + \ldots$ [115]. This is known as the multi-string 4-loop braiding in the Hamiltonian picture [116, 117, 122].

* Their dimensions of Hilbert space on $M^3 \times S^1$ as the ground state degeneracy on a spatial $M^3$ are computed explicitly from $Z(M^3 \times S^1)$ by [111]. These types of continuum TQFT formulation work for bosonic non-spin TQFTs [115] as well as [68] for fermionic spin-TQFTs for finite group gauge theories.

How to sharpen the statement of (5.14) from the known formulations of (5.15) and (5.16)? We have to beware the former 4d TQFT is more challenging than the later 4d TQFTs. Let us further make sense of (5.14) by physical arguments:
(1) Equations (5.15) and (5.16) are gauge theories with continuous gauge transformations, with the 2-form or 2-cochain $B$ fields transform in a more non-abelian fashion. In fact, 4d TQFT (5.16) is intrinsically non-abelian, such that the string excitations from the 2-worldsheet of $B$ fields are nonabelian — (5.16) have nonabelian strings in its gapped spectrum. Nevertheless, (5.15) and (5.16) are still equivalent to some finite group gauge theories similar to 4d Dijkgraaf-Witten twisted topological gauge theory [92]. However, (5.14) contains a continuous Lie group $SO(3)$ gauge structure at least in 3d. To formulate a non-abelian 4d TQFT with a continuous Lie group structure (5.14) is fairly more challenging than formulating twisted abelian or non-abelian finite group TQFT [68, 115].
(2) The $k_{3d}$ level: The $SU(2)_6$ CS has an action $\frac{k_{3d}}{4\pi} \int_M 3 \text{Tr}(A \wedge dA + \frac{2}{3} A^3)$ with $k_{3d} = 6$; while gauging the 1-form $Z_2$ electric symmetry yields $SO(3)_3$ CS. We should regard $k_{3d} = 6$ from the $SU(2)_6$ CS perspective, or regard $k_{3d} = 3$ from the $SO(3)_3$ CS perspective.
(3) The $k_{4d}$ level: Naïvely, there is a BF term $\frac{k_{4d}}{2\pi} \int_M B \wedge dA$ to pair the $A$ with a dual $B$ field. We know the dimensions of Hilbert space for 3d TQFTs:
\[
\dim(H)_{SU(6)_6} \big|_{T^2} = Z(T^2 \times S^1)_{SU(2)_6} \text{CS} = 7, \quad \dim(H)_{SO(3)_3} \big|_{T^2} = Z(T^2 \times S^1)_{SO(3)_3} \text{CS} = 4. \quad \text{(5.17)}
\]
If we extend the theory to 4d TQFT and pair each $A$ with a dual $B$ field, we should have the $k_{4d} = 4$ to have 4 different dual $B$ fields; but this is only the case for a 4d TQFT of the similar types of BF or Dijkgraaf-Witten twisted theory. More generally, we may expect that an action $\frac{1}{2\pi} \int_M \sum_I k_{4d,I} B_I \wedge dA + \ldots$ that can still pair an $A$ with a dual $B$ such that the $\sum_I k_{4d,I} = 4$.
(4) The 4d TQFT is non-invertible and non-abelian:
• It is not-invertible in the sense its partition function $|Z_{4dTQFT}(A_Z)| \neq 1$, thus it does not make sense to discuss its generic inverted partition function $Z^{-1}_{4dTQFT}$. It is non-invertible also because the $Z_{4dTQFT}$ on $M^3 \times S^1$ has its dimensions of Hilbert space larger than 1, because it is already the case for its 3d reduction (5.17).
• It is non-abelian in the sense the quantum dimensions $d_Q > 1$ and fusion rules of $S$ and $\tilde{S}$ semion show its non-abelian nature in the dimensionally reduced 3d. The Wilson lines of $S$ and $\tilde{S}$ are 1-worldlines of nonabelian anyons. We expect that there are $B + \ldots$ as 2-worldsheet of nonabelian strings, which is found in the theory of (5.16) in [115]. We propose braiding statistics and link invariants for these anyonic particles and anyonic strings in Sec. 6.2.

Above we have suggested possible routes to attack this 4d TQFT. We will leave the mathematical...
rigorous formulation of such a 4d non-invertible non-abelian TQFT in the companion work [96].

5.3 Gapping Neutrinos: Majorana mass vs Dirac mass vs Topological mass

It should be clear that the 4d nonabelian TQFT in Sec. 5.2 with an index \( \nu = N_{\text{generation}} \mod 16 \) in (4.34) cannot be access from gapping the free \( \nu = N_{\text{generation}} = N_{\text{generation}} \mod 16 \) number of spacetime spinors of Weyl fermion or Majorana fermion. We will comment how to access the 4d nonabelian TQFT from the dual fermionic vortex zero mode bound state condensation via the topological quantum phase transition [123,124] or “4d mirror symmetry [125]” (i.e., a duality of QFTs in the version of 4d spacetime). We comment more about the 4d duality in Sec. 5.3.3. Before then, let us clarify the meanings of Mass or Energy Gap.

The masses for left-handed neutrinos (thus also right-handed antineutrinos) are experimentally estimated to be very small, nearly million times smaller than electron mass [126]

\[
m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau} < 1\text{eV} \ll m_e \simeq 0.51\text{MeV}.
\]

(5.18)

More precisely the flavor states \((\nu_e, \nu_\mu, \nu_\tau)\) are superpositions of the different mass eigenstates. So a flavor state should be weighed and averaged over different masses of the different mass eigenstates. The current experiments show that the sum of the masses of the three neutrinos should also be below about 1eV. These mass bounds hold for neutrinos regardless being Dirac fermion or Majorana fermion particles. These mass bounds only apply to left-handed neutrinos (thus also right-handed antineutrinos, or its own anti-particle if a neutrino is a Majorana fermion). But we do not yet know about the mass, or mass bound, or the existence of right-handed neutrinos (called sterile neutrinos because they are in the trivial representation \((1,1,0)_R\) in (2.9) and do not interact with any of three SM forces). They also can be regarded as the left-handed \((1,1,0)_L\) with complex conjugation on the [here trivial] complex representation.

Let us overview the two known mass generation mechanism and propose a new third mechanism (a topological mass or energy gap mechanism to gap the neutrinos) :\(^{22}\)

1. **Dirac mass** mechanism [127,128]: Dirac mass is believed to give other Standard Model particles their masses.

2. **Majorana mass** mechanism [129]: This requires that the neutrino and antineutrino to be the same particle. If a neutrino is indeed a Majorana fermion, then lepton-number violating processes such as neutrino-less double beta decay would be allowed. The neutrino-less double beta decay is not allowed if neutrinos are Dirac fermions.

3. **Topological mass** mechanism (or **Topological Energy Gap** mechanism from the gapped excitations of interacting systems, many-body quantum matter, invertible TQFT, non-invertible TQFT, or topological order): (or the energy gap induced by the interaction mechanism)

\(^{22}\)By gap, we mean giving a **mass gap** or an **energy gap** to the system’s energy spectrum, which is the eigenenergy values of the quantum Hamiltonian of the system. (E.g. Solving a big matrix eigenvalues in the linear algebra.) The mass gap usually already assumes the free particle descriptions exist. However, in the interacting systems such as CFT or strongly-correlated many-body quantum matter, we may not always be able to find a suitable free particle description. In the later case, we can still have an energy gap from the interacting or many-body physics. The energy gap can be a generalization of the particle mass gap for the interacting or many-body systems. After all, Einstein had told us long ago the energy is the mass: \(\Delta E \sim (\Delta m)c^2\).
Majorana spinor \((\chi\chi)\). We use the convention Lie algebras in the Lie algebra of the Lorentz group 16 sterile neutrinos or right-handed Weyl spinors. \(h.c.\)

Then the mass needs to be an SU(2) singlet. The neutrino from the SU(2) doublet \(\nu\) the neutrino and antineutrino to be the same particle (as its anti-particle). \(\bar{\nu}\) where \(N\) of 3 generations \(N\) right-handed spinor fields. We write the 4-component spinors (such as Dirac or Majorana) in terms of two 2-component Weyl spinors.\(^{23}\)

**Dirac spinor** \(\Psi_D \equiv \begin{pmatrix} \chi_0 \\ \zeta^\dagger \end{pmatrix} \). \(\bar{\Psi}_D \equiv (\zeta^\alpha, \chi_\alpha^\dagger)\).

**Dirac mass**: \(\bar{\Psi}_D \Psi_D = \zeta^\alpha \chi_\alpha + \chi_\alpha \zeta^{\dagger \dagger} = \zeta \chi + (\zeta \chi)^\dagger = \zeta \chi + \chi^\dagger \zeta^\dagger. \quad \text{(5.19)}\)

**Majorana spinor** \(\Psi_M \equiv \begin{pmatrix} \chi_0 \\ \chi_1^\dagger \end{pmatrix} \). \(\bar{\Psi}_M \equiv (\chi_\alpha^\dagger, \chi_\alpha)\). \(\text{Majorana mass}: \quad \bar{\Psi}_M \Psi_M = \chi \chi + h.c. = \chi \chi + (\chi \chi)^\dagger = \chi \chi + \chi^\dagger \chi^\dagger. \quad \text{(5.20)}\)

If the right-handed neutrino mass is generated by Dirac mass 1 like other Standard Model fermions, then the mass needs to be an SU(2) singlet. The neutrino from the SU(2) doublet \(U_{\nu}\) would have the Yukawa interactions with the neutral component of the SU(2) doublet Higgs \(\phi_H\) and an SU(2) singlet \(\chi_\nu\). Another mechanism is that neutrino mass can be generated by a Majorana mass 2, which would require the neutrino and antineutrino to be the same particle (as its anti-particle).

The anomaly \(\nu = -N_{\text{generation}} \mod 16\) in (4.34) dictates that in the free particle limit, we need to add \(\nu = N_{\text{generation}} = 3 \mod 16\) right-handed spinors.\(^{24}\) Denote the right-handed spinor of sterile neutrinos of 3 generations \(\nu_{\mu}, \nu_{\tau}, \nu_{\tau}\) as \((\chi_{\nu_e}, \chi_{\nu_{\mu}}, \chi_{\nu_{\tau}})\), we can write down the free quadratic non-interacting action where \(\tilde{\sigma} \equiv (1, \tilde{\sigma})\),

\[
\chi_{\nu_e}^\dagger \sigma^\mu \partial_\mu \chi_{\nu_e} + \chi_{\nu_{\mu}}^\dagger \sigma^\mu \partial_\mu \chi_{\nu_{\mu}} + \chi_{\nu_{\tau}}^\dagger \sigma^\mu \partial_\mu \chi_{\nu_{\tau}} + \frac{1}{2} \begin{pmatrix} \chi_{\nu_e}, \chi_{\nu_{\mu}}, \chi_{\nu_{\tau}} \end{pmatrix} \begin{pmatrix} M \end{pmatrix} \begin{pmatrix} \chi_{\nu_e} \\ \chi_{\nu_{\mu}} \\ \chi_{\nu_{\tau}} \end{pmatrix} + h.c. \quad \text{(5.21)}
\]

Along the rank-3 mass matrix \(M\), its diagonalized elements represent the Majorana mass (5.20).

\(^{23}\)Here we use the 2-component Weyl spinor notation. Not to confuse the “\(\mu\)” in muon neutrino index \(\nu_{\mu}\) with the spacetime index \(\mu\). For two general left-handed Weyl spinors, say \(\chi\) and \(\chi'\) (where each component is a Grassman number with anti-commutation properties \(\chi_{\alpha} \chi'_{\beta} = -\chi'_{\beta} \chi_{\alpha}\)), we have

\[
\chi = \chi_{\alpha} \chi'_{\beta} = \epsilon^{\alpha \beta} \chi_{\beta} \chi'_{\alpha} = -\epsilon^{\alpha \beta} \chi_{\beta} \chi'_{\alpha} = -\chi'_{\beta} \chi_{\alpha} = \epsilon^{\alpha \beta} \chi_{\alpha} \chi'_{\beta} = \chi_{\alpha} \chi'_{\beta} \equiv \chi'_{\beta} \chi_{\alpha},
\]

\((\chi \chi')^\dagger = (\chi^\prime)^\dagger (\chi^\alpha) = (\chi^\alpha)(\chi^\prime) = \chi^\prime \chi^\dagger = (\chi^\prime)^\dagger (\chi^\alpha) = \chi^\prime \chi^\dagger = (\chi \chi')^\dagger. \quad \text{(5.21)}\)

We use the convention \(\epsilon_{\alpha \beta} = \epsilon^{\beta \alpha} = -\epsilon_{\beta \alpha} = \epsilon^{\alpha \beta} = \epsilon^{\alpha \beta} = \epsilon_{\alpha \beta} = (i \epsilon^2)_{\alpha \beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), also \(\chi^\alpha \equiv \epsilon^{\alpha \beta} \chi_{\beta}\)

and \((\chi^\alpha)^\dagger = (\chi^\prime)^\dagger = (\chi^\prime)^\dagger (\chi^\alpha) = (\chi^\alpha)(\chi^\prime) = \chi^\prime \chi^\dagger = (\chi^\prime)^\dagger (\chi^\alpha) = \chi^\prime \chi^\dagger = (\chi \chi')^\dagger. \quad \text{(5.21)}\)

We use the convention \(\epsilon^{\alpha \beta} = \epsilon^{\beta \alpha} = -\epsilon_{\beta \alpha} = \epsilon^{\alpha \beta} = \epsilon^{\alpha \beta} = \epsilon_{\alpha \beta} = (i \epsilon^2)_{\alpha \beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), also \(\chi^\alpha \equiv \epsilon^{\alpha \beta} \chi_{\beta}\)

and \((\chi^\alpha)^\dagger = (\chi^\prime)^\dagger = (\chi^\prime)^\dagger (\chi^\alpha) = (\chi^\alpha)(\chi^\prime) = \chi^\prime \chi^\dagger = (\chi^\prime)^\dagger (\chi^\alpha) = \chi^\prime \chi^\dagger = (\chi \chi')^\dagger. \quad \text{(5.21)}\)

\(^{24}\)This anomaly \(\nu = -N_{\text{generation}} \mod 16\) in (4.34) also rules out many BSM introducing more than \(N_{\text{generation}} = 3 \mod 16\) sterile neutrinos or right-handed Weyl spinors.
More generally, we can pair $\nu = N_{\text{generation}} = 3$ right-handed Weyl spinors (sterile neutrinos) with each other and with the 3 left-handed Weyl spinors via another free non-interacting matrix term action:

$$\frac{1}{2} \sum_{i,j} \left( \begin{array}{c}
\langle \phi \rangle \\
\langle \bar{\phi} \rangle
\end{array} \right) \begin{pmatrix}
0 & M_{\text{Dirac}} \\
M_{\text{Dirac}} & 0
\end{pmatrix} \begin{pmatrix}
\chi_{\nu e}^\dagger \\
\chi_{\nu \mu}^\dagger \\
\chi_{\nu \tau}^\dagger
\end{pmatrix} \begin{pmatrix}
l_{L\nu e} \\
l_{L\nu \mu} \\
l_{L\nu \tau}
\end{pmatrix} + \text{h.c.} \right). \quad (5.22)
$$

There is a rank-6 mass matrix. Here the Dirac mass scale, for example, is given by Higgs vev $M_{\text{Dirac}} \sim |\langle \phi \rangle|$. We remind that the flavor states can be superpositions of the different mass eigenstates. In above, we rewrite the three generations of SU(2) doublet 2 of left-handed neutrino $(l_{L\nu e}, l_{L\nu \mu}, l_{L\nu \tau})$ pair with the SU(2) doublet 2 Higgs as $(l_{L\nu e}, l_{L\nu \mu}, l_{L\nu \tau})\langle \phi \rangle / |\phi \rangle$. The usual seesaw mechanism (e.g., [130]) sets the scale of $|M_S| \gg |M_{\text{Dirac}}|$. So the three mass eigenstates have mass $\sim |M_{\text{Dirac}}|^2 / |M_S| \ll |M_{\text{Dirac}}|$ (for the observed 3 neutrinos much smaller other Dirac mass of MeV or GeV scales), while the other three mass eigenstates have mass $\sim |M_S|$ which can set to be the GUT scale (thus too heavy yet to be detected).

We should emphasize that by having Majorana mass or Dirac mass to any of the Weyl fermion spinors, this quadratic mass would break the $Z_4$ symmetry, (4.37) and (4.38), assign to the complex Weyl fermions. Is it necessary to break $Z_4$ symmetry in order to saturate the anomaly by a gapped theory? No, we do not have to break the $Z_4$ symmetry if we introduce a Topological Mass/Energy Gap for Weyl fermion, see Sec. 5.3.2.

### 5.3.2 Topological Mass and Topological Energy Gap

Now we introduce Topological mass and Topological energy gap mechanism (the concept of free particle mass may not be appropriate for them), which we should digest them from the interacting theory viewpoint. We may colloquially call any of the following as topological energy gap:

1. **Interaction-induced mass or interaction-induced energy gap**: This idea has been used to classify the topological phases of interacting quantum matter. Given the symmetry $G$ (including the $G_{\text{spacetime}}$ and $G_{\text{internal}}$) of the system, are the two ground states (i.e., two vevs) deformable to each other by preserving the symmetry $G$? This is the key question for the community of Symmetry-Protected Topological state (SPTs) physics.

   - Fidkowski-Kitaev [131, 132] had shown that 1+1d $Z$ classification [77–79] of $T^2 = +1$ topological superconductor with $Z_2^T \times Z_2^F$ symmetry can be reduced to $Z_8$ clas. Fidkowski-Kitaev may be the first example of showing the interaction can produce the energy gap between 8 Majorana fermions in 1+1d without breaking the original symmetry. (See recent discussions along QFT reviewed in [133,134])

   - Kitaev and Fidkowski-Chen-Vishwanath (FCV) [55] suggested that 3+1d $Z$ classification of $T^2 = (-1)^F$ topological superconductor (TSC) with $Z_2^T \supset Z_2^F$ symmetry can be reduced to $Z_{16}$ class. This implies that the 16 number of 2+1d Majorana fermions on the boundary of 3+1d TSC can open up an energy gap without introducing any free quadratic mass: neither Majorana nor Dirac masses.
• Wen [74], and the author and Wen [26,135], suggest that the all G-anomaly-free gapless theory can be fully gapped without breaking G-symmetry. This idea includes introduce a random disorder new Higgs field [74]; or introduce direct non-perturbative interactions [26,135] (which are usually irrelevant or marginal operator deformations viewed from IR QFTs). Many of such examples are applied to construct chiral fermion or chiral gauge theories on the lattice. This includes You-BenTov-Xu [136], You-Xu [137]. BenTov and Zee [133,138] names this mechanism as Kitaev-Wen mechanism, or mass without mass. This deformation of G-anomaly-free theory is consistent with Seiberg’s conjecture on the deformation classes of QFTs constrained by symmetry and anomaly [139].

(2). Vortex condensation: This is an approach commonly used in condensed matter literature for 2+1d strongly-correlated systems. The idea is that the symmetry-breaking defects (such as vortices) may trap the zero modes and which carry nontrivial quantum number. The question is to find which number of vortex zero modes with what kind of symmetry assignment can be proliferated to restore the broken global symmetry — this would drive the quantum phase transition between the symmetry-breaking phase and the symmetry-restoring phase. This approach, called the vortex condensation, has been used to construct 2+1d surface topological orders, see the condensed matter review [54].

(3). Symmetry-extension and symmetry-preserving gapped topological order/TQFT: As mentioned briefly in Sec. 5.2.1 Symmetry-extension mechanism [52] and higher-symmetry extension generalization [50] are a rather exotic mechanism, which trivialize the anomaly not by breaking G to its subgroup $G_{sub} \subset G$, but extend it to a larger group $G_{total}$ which can be regarded as a fibration of the original group G as a quotient group. (See down-to-earth lattice constructions in any dimension [52] and in 1+1d bulk [140]) a useful intermediate stepstone, to construct another earlier scenario: symmetry-preserving TQFT, via gauging the extended-symmetry [52]. This approach is applicable to bosonic systems [52,111] and fermionic systems [68,141] in any dimensions. In contrast to known gapped phase saturate anomaly via symmetry breaking (either global symmetry breaking or Anderson-Higgs mechanism for gauge theory), this approach is based on symmetry extension (thus beyond symmetry breaking and Anderson-Higgs mechanism).

The Topological mass and Topological energy gap mechanism including (1), (2) and (3), in fact, is obviously beyond the familiar Dirac, Majorana mass, and seesaw mechanism [130]. In a colloquial sense, we do not have a Higgs field $\phi_H$ breaking the symmetry and gives vev $\langle \phi_H \rangle \neq 0$. In certain case, we can consider the disorder Higgs field such that [26,74]

$$\langle \phi_H \rangle = 0, \quad \langle |\phi_H|^2 \rangle \neq 0. \quad (5.23)$$

So Topological mass/energy gap is a quantum behavior beyond the mean field quadratic semiclassical theory, beyond Higgs, and beyond Landau-Ginzburg symmetry-breaking paradigm.

5.3.3 4d duality for an odd number of Majorana fermions and “mirror symmetry”

Before apply the Topological Mass from Sec. 5.3.2 to neutrino physics furthur, we like to introduce a potential helpful supersymmetry (SUSY) duality in 4d known as Seiberg duality [142] studied in $\mathcal{N} = 1$ theory and supersymmetric quantum chromodynamics (SQCD). Seiberg duality is an $\mathcal{N} = 1$ electric magnetic duality in SUSY non-abelian gauge theories with weak strong duality. On the left-hand side of the duality may have quarks and gluons, on the right-hand side dual theory’s they become the solitons (such as nonabelian magnetic monopoles) of the elementary fields. When the left-hand side theory is Higgsed by an expectation value of a squark, the right-hand side dual theory’s is confined. Massless
glueballs, baryons, and magnetic monopoles in the confined strongly coupled description in the left-hand side theory becomes some weakly coupled elementary quarks in the right-hand side dual Higgs description.

Schematically, there is an IR duality between left-hand side (LHS) and right-hand side (RHS) under renormalization group (RG) flow for $\mathcal{N} = 1$:

$$\text{SU}(N_c) \text{ gauge theory with } N_f \text{ chiral and antichiral multiplets } Q, \tilde{Q} \text{ in color fundamental } N_c, \tilde{N}_c \xleftarrow{\text{IR duality}} \text{SU}(N_f-N_c) \text{ gauge theory with } q \text{ and } \tilde{q} \text{ in color fundamental } N_f-N_c, \tilde{N}_f-N_c \text{ and meson } M.$$ (5.24)

Include the representations of chiral multiplet/superfields, we have the relations:

| 4d $\mathcal{N} = 1$ Seiberg duality | LHS: SQCD | RHS: dual theory |
|--------------------------------------|------------|-----------------|
| color gauge group                    | $N_c$      | $\text{SU}(N_f-N_c)$ |
| Same global internal symmetries     | $\text{SU}(N_f)_L \times \text{SU}(N_f)_R \times \text{U}(1)_B \times \text{U}(1)_R$ | $\text{SU}(N_f-N_c)$ |
| Chiral multiplet/superfields:        | $Q: (N_f,1,1,\frac{N_f-N_c}{N_f})$ | $q: (1,N_f,\frac{N_c}{N_f-N_c},\frac{N_c}{N_f})$ |
| Representation $\mathbf{R}$          | $\tilde{Q}: (1,\tilde{N}_f,-1,\frac{N_f-N_c}{N_f})$ | $\tilde{q}: (\tilde{N}_f,1,\frac{N_c}{N_f-N_c},\frac{N_c}{N_f})$ |
|                                       | $M: (N_f,\tilde{N}_f,0,\frac{2(N_f-N_c)}{N_f})$ | |

We are particularly interested in the case when the RHS flows to free Weyl spinors, which means that we can choose as simple as $N_c = 2$ and $N_f = 3$, and $\mathcal{N} = 1$:

$\text{SU}(N_c = 2)$ with $N_f = 3$ chiral and anti-chiral multiplets $Q$ and $\tilde{Q} \xleftarrow{\text{IR duality}} 15$ Weyl spinors (5.26)

For Weyl spinor counting we have $N_f = 3$ chiral multiplets and $\tilde{N}_f = 3$ anti-chiral multiplets, each is the fundamental or anti fundamental 2 or 2 of SU(2), thus they contribute $2 \cdot 2 \cdot 3 = 12$ Weyl spinors. There is also a vector multiplet which sits at the adjoint representation 3 of SU(2), this contributes another 3 Weyl spinors. So in total we have $2 \cdot 2 \cdot 3 + 3 = 15$ Weyl spinors. The $N_f = N_c + 1 = 3$ is interesting because it sits at the lower boundary $(3N_c/2 = N_f)$ of $3N_c/2 < N_f < 3N_c$, where the origin of the moduli space is an interacting CFT and non-abelian Coulomb phase. Also this case we have $N_f = N_c + 1$ thus two moduli spaces are identical but the interpretations of the singularity at the origin are different — massless particles can be regarded as, either strongly coupled mesons and baryons on LHS, or weakly interacting or free quarks on RHS.

This duality helps as $15 \mod 16 = -1 \mod 16$ so to cancel the anomaly (4.31) as $\nu = -N_{\nu_R}$ mod 16 = $-1 \mod 16$ in one generation of SM. One way to simplify 15 Weyl spinors to 1 Weyl spinor on RHS would be that adding 1 Weyl spinor on both sides in the trivial representation, and adding nonperturbative deformations to gap the RHS completely:

$$\text{SU}(N_c = 2) \text{ with } N_f = 3 \text{ chiral and anti-chiral multiplets } Q \text{ and } \tilde{Q} + \text{ deformations} \xleftarrow{\text{IR duality}} \text{ (gapping } 16 \text{ Weyl spinors via nonperturbative interacting deformations)} + (-1) \text{ Weyl spinors (in the complex conjugate representation).}$$ (5.27)

We leave details of this construction in an upcoming work [96]. In the following subsections, we can argue several phenomenon of gapping Weyl spinors based on this proposed duality (5.27). The hope is that we can access the 4d nonabelian TQFT from the dual fermionic vortex zero mode bound state condensation via the topological quantum phase transition [123,124] or “4d mirror symmetry [125] description” (i.e., a duality of QFTs in the version of 4d spacetime).
In a general colloquial sense, this duality is also related to the particle-vortex duality [143,144], the renowned supersymmetric version of duality in 3+1d includes, for example, \( \mathcal{N} = 2 \) Seiberg-Witten theory [145], and \( \mathcal{N} = 1 \) Seiberg duality [142], and many other theories (see a review [146]). A fermionic non-supersymmetric version of particle-vortex duality in 3+1d, a recent ongoing pursuit along this direction includes [50,147−149].

5.3.4 Gapping 3 Weyl spinors / sterile right-handed neutrinos

In Sec. 5.3.1, the anomaly \( \nu = -N_{\text{generation}} \mod 16 \) in (4.34) dictates that in the free particle limit, we need to add \( \nu = N_{\text{generation}} = 3 \mod 16 \) right-handed spinors. But as mentioned, in Sec. 5.2, we can propose a 4d noninvertible nonabelian TQFT to saturate the \( \nu = N_{\text{generation}} = 3 \mod 16 \) anomaly. In order to achieve this, we can go through a quantum phase transition by using 3 times of (5.27), and we can access the 4d noninvertible nonabelian TQFT from the LHS. On the RHS, on the other hand, we do not gain the expectation values for the quadratic mass:

\[
\begin{pmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{pmatrix}
\xrightarrow{\text{mean field vev}}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(5.28)

The energy gap for 3 sterile neutrinos are fully interacting and energetic from the LHS of 3 times of (5.27). We have a 4d extension of 3d SO(3)\(_+\) CS in Sec. 5.2.3 to saturate the \( \nu = \pm 3 \mod 16 \) anomaly [96].

5.3.5 Gapping 2 Weyl spinors / sterile right-handed neutrinos

Suppose we only play with gapping 2 Weyl spinors / sterile right-handed neutrinos by Topological Mass mechanism, then the other can still have either a Majorana mass or Dirac mass pairing with other left-handed spinor. We do not gain the expectation values (vev) for the quadratic mass of the 2 Weyl spinors, but only for the other one:

\[
\begin{pmatrix}
M_{11} & M_{12} & 0 \\
M_{21} & M_{22} & 0 \\
0 & 0 & M_{33}
\end{pmatrix}
\xrightarrow{\text{mean field vev}}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \langle \phi_H \rangle
\end{pmatrix}.
\]

(5.29)

Here for phenomenology purpose, the \( \langle \phi_H \rangle \) can be the same Higgs \( \phi_H \) of SM (as a doublet \( 2 \) of SU(2) for Dirac mass), or another new distinct Higgs (say a singlet \( 1 \) of SU(2)) for Majorana mass.

We can go through a quantum phase transition by using 2 times of (5.27), and we can access the 4d noninvertible abelian TQFT from the LHS [96]. We can relate the 3d Semion-Fermion\(_+\) TQFT [55,56,103] to its 4d extension TQFT [96]. The 3d CS theory for 3d Semion-Fermion\(_+\) TQFT can be as simple as a U(1)\(_2\) CS \( \times \{1, f\} \), or a U(1)\(_2\) CS \( + \) (trivial fermionic invertible spin TQFT) in 3d.

\[
\text{a U(1)\(_2\) CS} \times \{1, f\}, \text{ or a U(1)\(_2\) CS} + \text{ (trivial fermionic invertible spin TQFT) in 3d.}
\]

(5.30)

We can derive a 4d extension of 3d Semion-Fermion\(_+\) TQFT to saturate the \( \nu = \pm 2 \mod 16 \) anomaly [96].

\[\text{There were also a fermionic version in 2+1d of particle-vortex duality proposed in [150−152] and formalized in [153−155]. They have achieved great success on understanding condensed matter phenomena in 2+1d. They may provide insightful guidelines to the 3+1d, namely 4d duality construction.}\]
| Anyons      | 1 | s | f | sf |
|-------------|---|---|---|----|
| spin statistics | 0 | 1/4 | 1/2 | 3/4 |
| $\mathcal{T}$-matrix | 1 | i | -1 | -i |
| $(CT)^2$    | 1 | ±1 | -1 | ±1 |
| quantum dimension $d_Q$ | (abelian) | (abelian) | (abelian) | (abelian) |

Table 6: Two version of 3d Semion-Fermion± TQFT data, differed by their $CT$ assignment. Similar to Table 5’s setting. They are $\nu = \pm 2 \in \mathbb{Z}_{16}$ surface topological order in 3d for $(CT)^2 = (-1)^F$ TSC in 4d.

### 5.3.6 Gapping 1 Weyl spinor / sterile right-handed neutrino

Suppose we only play with gapping 2 Weyl spinors / sterile right-handed neutrinos by Topological Mass mechanism, then the other can still have either a Majorana mass or Dirac mass pairing with other left-handed spinor. We do not gain the expectation values (vev) for the quadratic mass of the 1 Weyl spinor, but only for the other twos:

$$\begin{pmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{pmatrix} \xrightarrow{\text{mean field vev}} \text{quadratic mass term vev} \propto \begin{pmatrix} 0 & 0 & 0 \\ 0 & \#\langle \phi_H^\nu \rangle & \#\langle \phi_H^\nu \rangle \\ 0 & \#\langle \phi_H^\nu \rangle & \#\langle \phi_H^\nu \rangle \end{pmatrix}.$$ (5.31)

The notations of $\phi_H^\nu$ is explained in Sec. 5.3.5. We can go through a quantum phase transition by using a single one of (5.27), and we can access the 4d noninvertible nonabelian TQFT from the LHS [96]. There are 3d version of $\nu = +1$ and $\nu = -1$ TQFT toy models that we can start with, which are

$$\text{SO(3)}_{3,+} \text{ CS + Semion-Fermion}_- \text{ TQFT, } \ \text{SO(3)}_{3,-} \text{ CS + Semion-Fermion}_+ \text{ TQFT.}$$ (5.32)

The 3d CS theory for 3d Semion-Fermion+ TQFT can be as simple as a $U(1)_2$ CS × $\{1, f\}$, or a $U(1)_2$ CS + a trivial fermionic invertible spin TQFT in 3d. We can derive a 4d extension TQFT of these 3d TQFTs to saturate the $\nu = \pm 1 \mod 16$ anomaly [96].
6 Ultra Unification: Grand Unification + Topological Force and Matter

In previous sections, we had shown that in order to match some nonperturbative global anomaly (Sec. 4.2) but still preserve $Z_{4,X}$ for the global symmetry of SM and SU(5) GUT Georgi-Glashow (GG) model, and for the gauge subgroup $Z_{4,X} = Z(\text{Spin}(10)) \subset \text{Spin}(10)$ of the SO(10) GUT, we can introduce a new hidden gapped sector appending to the SM and GUT:

- Topological Mass/Energy Gap (Sec. 5.2 and Sec. 5.3) to gap the 16th Weyl fermions (right-handed neutrinos). The outcome low energy gives rise to 4d non-invertible TQFT.

- 5d invertible TQFT (iTQFT) with one extra dimension (Sec. 5.1, known as 5d SPTs in quantum condensed matter analogy).

Overall, we can consider the combinations of two solutions. In either cases, we require new hidden gapped topological sectors beyond SM and GUT. We may term the unification including SM, Grand Unification plus additional topological sectors with Topological Force and Matter as Ultra Unification.

6.1 4d-5d Theory: Quantum Gravity and Topological Gravity coupled to TQFT

I]: The combinations of two solutions from adding 4d non-invertible TQFT and 5d invertible TQFT mean that we can propose a new schematic partition function / path integral, generalizing (5.10) and (5.14) to

$$Z_{4d\text{-}iTQFT} = \exp \left( \frac{2\pi i}{16} \cdot \nu_{5d} \cdot \eta(\text{PD}(A_{Z_2})) \right) \cdot \int [D\psi][D\bar{\psi}][DA][D\phi_H][D\mathcal{A}][D\mathcal{B}] \cdots

\exp(i S_{4d\text{-}SM/GUT}[\psi, \bar{\psi}, A, \phi_H, \ldots, A_{Z_4}]) \bigg|_{M^5} + i S_{4d\text{-}TQFT}[\mathcal{A}, \mathcal{B}, \ldots, A_{Z_2}] \bigg|_{M^4} \right) \text{ exp(} \nu_{5d} - \nu_{4d} = -N_{\text{generation}} \text{)}.

The anomaly (4.34) is now matched by:

$$\nu = \nu_{5d} - \nu_{4d} = -N_{\text{generation}} \mod 16.$$ (6.2)

The $S_{4d\text{-}SM/GUT}$ is the 4d SM or GUT action. The $\psi, \bar{\psi}, A, \phi_H$ are SM and GUT quantum fields, where $\psi, \bar{\psi}$ are the 15 or 16 Weyl spinor fermion fields, the $A$ are 12 or 24 gauge bosons given by gauge group Lie algebra generators, and $\phi_H$ is the electroweak Higgs (we can also add GUT Higgs).

The $S_{4d\text{-}iTQFT}$ is a 4d noninvertible TQFT outlined in Sec. 5.3 (for example, $\nu_{4d} = \pm 3, \pm 2, \pm 1$ in Sec. 5.3.4, Sec. 5.3.5, and Sec. 5.3.6). The $\mathcal{A}$ and $\mathcal{B}$ (and possibly others fields) are TQFT gauge fields (locally differential 1-form and 2-form anti-symmetric tensor gauge connections)

II]: Many different perspectives guide to an understanding that at high enough energy scale, every global symmetry should be either gauged or broken [156–159]. By every global symmetry, we include the internal symmetry $G_{\text{internal}}$, the spacetime $G_{\text{spacetime}}$, the fermion parity $Z_2^F$, the time reversal symmetry $Z_2^T$ or $Z_4^T \supset Z_2^T$ and so on. By gauging fermion parity $Z_2^F$, this means the UV completion at higher energy scale should be all bosonic, presumably from local tensor product

26By preserving $Z_{4,X}$, we mean the $Z_{4,X}$-symmetry is preserved at some higher energy scale above the electroweak scale. Of course, the energy scale lower such as the Higgs scale 125 GeV, the usual Dirac mass terms would break the $Z_{4,X}$.
Hilbert space — this is consistent with the “It from Qubit” of lattice model and quantum condensed matter view [26]. Therefore, it is natural for us to pursue a fully gauged version of 5d iTQFT such that the $\mathbb{Z}_{4X}$ is also gauged, at some GUT scale. By doing so, we need to sum over $A_{\mathbb{Z}_2} \in H^1(M,\mathbb{Z}_{4X}/\mathbb{Z}_F^2)$ for a given spacetime topology and geometry, and also sum over all spacetime $M \in \forall$ topology, $\forall$ geometry. Let us consider the case the spacetime is a closed $M = M^5$ without boundary. But summing over all spacetime $M$ certainly diverges, one needs to make sense of the partition function by regularization. To deal with such a path integral is a challenging problem of quantum gravity, we will not be able to solve it for now. In summary, a schematic path integral says:

$$\sum_{M \in \forall \text{ topology, } \forall \text{ geometry}} \sum_{A_{\mathbb{Z}_2} \in H^1(M,\mathbb{Z}_{4X}/\mathbb{Z}_F^2)} \exp\left(\frac{2\pi i}{16} \cdot \nu_{5d} \cdot \eta(\text{PD}(A_{\mathbb{Z}_2}))|_M\right) \quad (6.3)$$

A few thoughts can help to attack this challenging problem on regularization of summing over spacetimes.

- In 2d topological gravity, we can sum over conformal structures. This can be a finite dimensional integral for a fixed topology.
- One can simplify the problem to sum over different topologies given by the $5d \eta(\text{PD}(A_{\mathbb{Z}_2}))$ invariant with $M^5$ of Spin $\times \mathbb{Z}_2 \mathbb{Z}_4 = \text{Spin} \times \mathbb{Z}_F^2 \mathbb{Z}_4$ structure
- Recent work by Dijkgraaf and Witten on 2d topological gravity provides a guide to the analogous 2d partition function [160] related to the 2d Arf invariant. Our theory (6.3) may be regarded as a 5d gravity (dynamical, quantum and topological) related to the 5d $\eta(\text{PD}(A_{\mathbb{Z}_2}))$ invariant.

III: Suppose we find a way to regularize the path integral (such as on a computer or a certain lattice model), then we can consider the theory (6.3) with boundary, where we can place the 4d SM theory. We thus rewrite (6.1) as the fully gauged and coupled version

$$Z_{5d-TQFT,QG}/4d-QFT = \sum_{M \in \{\text{topo/geo}\}, \ A_{\mathbb{Z}_2} \in H^1(M,\mathbb{Z}_{4X}/\mathbb{Z}_F^2)} \exp\left(\frac{2\pi i}{16} \cdot \nu_{5d} \cdot \eta(\text{PD}(A_{\mathbb{Z}_2}))|_M\right) \cdot \int [\mathcal{D}A_{\mathbb{Z}_4}] [\mathcal{D}\psi] [\mathcal{D}\bar{\psi}] [\mathcal{D}A] [\mathcal{D}\phi_H] [\mathcal{D}\phi] [\mathcal{D}\mathcal{B}] \cdots$$

$$\exp\left(i \ S_{4d-SM/GUT}[\psi, \bar{\psi}, A, \phi_H, \ldots, A_{\mathbb{Z}_4}]|_M + i \ S_{4d-TQFT}[\phi, \mathcal{B}, \ldots, A_{\mathbb{Z}_2}]|_M\right) \bigg|_{\nu_{5d}-\nu_{4d}=-N_{\text{generation}}} \quad (6.4)$$

Recall we gauge $A_{\mathbb{Z}_4}$ as it is the $\mathbb{Z}_{4X}$ gauge field and $A_{\mathbb{Z}_2}$ mod 2 = $A_{\mathbb{Z}_2}$ This is a 4d and 5d coupled fully gauged quantum system, where 4d has SM, GUT and noninvertible TQFT, and 5d has a certain gravity (dynamical, quantum and topological) coupled to 5d TQFT. Part of the 5d TQFT is the gauged version of 5d iTQFT of $\eta(\text{PD}(A_{\mathbb{Z}_2}))$, and the 1-form gauge field gauging the 0-form $\mathbb{Z}_{4X}$ symmetry can mediate between 5d and 4d.
6.2 Braiding statistics and link invariants in 4d or 5d: Quantum communication with the “God” from our Standard Model physics?

Follow Sec. 6.1, above a higher energy scale (way above SM and above SU(5) GUT but around SO(10) GUT), the discrete \( X \) becomes dynamically gauged, with a dynamical gauge vector boson mediator \( X_g \). So the entangled Universe in 4d and 5d may be described by (6.4).

The 4d and 5d gauged TQFT sectors are noninvertible TQFT, which is the low energy of topological order under UV completion. In fact such a medium can be regarded as some version of topological quantum computer [47, 48]. The 4d version of TQFT with \( Z_{4,X} \) gauged, does not need to directly couple to dynamical gravity. But the 5d version of TQFT with \( Z_{4,X} \) gauged, does directly couple to dynamical gravity.

So in an appropriate analogy, the 5d bulk topological quantum computer coupled to dynamical gravity governed, which are in a unified math and physics framework with SM shown in (6.4). However, the 4d TQFT and 5d TQFT (including gravity) can hierarchically develop absolutely different new “chemistry and biology,” thus where hypothetical “God” or “foreign higher-beings,” formed by higher-dimensional extended operators, may exist. While our bodies and living creatures from 4d SM are made by the protons, neutrons and electrons (bound states from point particles \( u,d \) quarks and electron \( e \)), the “foreign higher-beings” can be hypothetically formed by higher-dimensional extended operators, such as the extended \( \mathcal{A} \) line and \( \mathcal{B} \) surface operators, and their other composite operators. We may simply call such “foreign higher-beings” as “God” for an abbreviation.

There is an immediate philosophical but now also scientific question: Do we have any way to communicate or interact with “God” or “foreign higher-beings” living in 4d or 5d TQFTs? Could we communicate with “God” within SM particles and forces that we human beings and creatures are made of? Another way to phrase this question is: Can we, the beings, sense the Topological Force?

Since the extended \( \mathcal{A} \) line and \( \mathcal{B} \) surface operators do not end on the SM particles and gauge forces, it seems that we cannot naïvely. But in fact, the answer is yes, we can communicate with “God” within SM particles and forces. The idea is that the TQFT (5.14) contains a term

\[
\int [D\mathcal{A}_{Z_4}] \int [D\mathcal{B}] [D\mathcal{A}] \exp(\cdots + \frac{ik_{3d}}{4\pi} \int_{M^4} \mathcal{A}_{Z_2} \wedge \text{Tr}(\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3) + \cdots),
\]

(6.5)

for the nonabelian 4d TQFT such as \( \nu = \pm 3, \pm 1 \) for Sec. 5.3.4 and Sec. 5.3.6. The abelian TQFT such as \( \nu = \pm 2 \) for Sec. 5.3.5 contains, for an abelian gauge field \( \mathcal{A} \),

\[
\int [D\mathcal{A}_{Z_4}] \int [D\mathcal{B}] [D\mathcal{A}] \exp(\cdots + \frac{ik_{3d}}{4\pi} \int_{M^4} \mathcal{A}_{Z_2} \wedge \mathcal{A} d\mathcal{A} + \cdots).
\]

(6.6)

We include the path integral \( \int [D\mathcal{A}_{Z_4}] \) when we gauge \( \mathcal{A}_{Z_4} \) as it is the \( Z_{4,X} \) gauge field and \( \mathcal{A}_{Z_4} \mod 2 = \mathcal{A}_{Z_2} \). So what are the experimental designs to interact or communicate with “God”?\textsuperscript{29}

\textsuperscript{27}Again this is a discrete \( Z_{4,X} \) or \( U(1)_X \) gauge theory’s gauge boson \( X_g \), different from the SU(5) GUT’s continuous nonabelian Lie group’s gauge boson X and Y. We have been denoted \( X_g \) as \( \mathcal{A}_{Z_4} \) for the case of \( Z_{4,X} \).

\textsuperscript{28}Of course, this is a scientific article. We do not have a claim to prove the existence of “foreign higher-beings” or “God.” One way to address this question is to have an algorithm to derive and calculate the absolutely different new “chemistry and biology” from the hidden topological world.

\textsuperscript{29}In fact, previous works on braiding statistics and link invariants, such as multi-string 3-loop braiding ([110, 120, 121] and [115–117, 122] and other exotic braiding process, provide helpful guidelines to these topological design, shown in Fig. 5 and Fig. 6.
The action term \( A_{Z_2} \wedge \mathcal{A} d\mathcal{A} \) in (6.6) prompts us to design a configuration in Fig. 5. Suppose there is such a Hopf link, the link of two \( S^1 \) circles from the loop operators of \( \mathcal{A} \) fields, drawn in blue and green circles. We claim that the link configuration (if measured in the long distance thus regarded as a small link) can carry nontrivial odd \( Z_{4,X} \) charge. Thus we can have a codimension 1 operator \( \star J_{Z_4} \) (for 4d, the \( \star J_{Z_4} \) is 3d) from

\[
\int A_{Z_4} \wedge \star J_{Z_4}
\]

wrapping around the small Hopf link of \( \mathcal{A} d\mathcal{A} \), such that the corresponding 0-form \( Z_{4,X} \)-symmetry charge is indeed odd.

(Extendable to a 5d bulk)

Figure 5: A nontrivial Hopf link between two line operators \( \mathcal{A} \) (shown as the blue and green \( S^1 \) loop operators) can carry a nontrivial odd \( Z_{4,X} \) charge (via \( A_{Z_2} \wedge \mathcal{A} d\mathcal{A} \)) measured by a codimension 1 operator \( \star J_{Z_4} \) (shown on the red \( S^3 \)) in \( A_{Z_4} \wedge \star J_{Z_4} \). The Wilson line (either dynamical gauge or background probe) with a line integral \( \exp(iqX \int A_{Z_4}) \) of \( Z_{4,X} \) along the gold color dashed line, can connect and mediate between the topological world and our Standard Model. A design to communicate and interact with the topological world of “God.”

The action term \( A_{Z_2} \wedge \text{Tr}(\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3) \) in (6.5) prompts us to design a configuration in Fig. 5 and Fig. 6, and their superpositions. Suppose there is such a Hopf link in Fig. 5 or a Borromean rings in Fig. 6, the link of two or three \( S^1 \) circles from the loop operators of \( \mathcal{A} \) fields, drawn in blue, green, and cyan circles. We claim that the link configuration (if measured in the long distance thus regarded as a small link) can carry nontrivial odd \( Z_{4,X} \) charge. Thus we can have a codimension 1 operator \( \star J_{Z_4} \) (for 4d, the \( \star J_{Z_4} \) is 3d, again from \( A_{Z_4} \wedge \star J_{Z_4} \)) wrapping around the small Hopf link or Borromean rings of \( \mathcal{A} d\mathcal{A} \) or \( \mathcal{A}^3 \), such that the corresponding 0-form \( Z_{4,X} \)-symmetry charge is indeed odd.\(^{30}\)

\(^{30}\)The similar phenomena in 3d, 4d TQFT and other dimensions are explored and summarized in [115].
4d spacetime $S^4$ or $\mathbb{R}^4$

$S^1 \times S^1 \times S^1 \star J_{AZ^2}$ on $\Sigma^3$ (e.g., $S^3$)

4d non-abelian TQFT line operators form Borromean rings carrying a charge of $X = 5(B - L) - 4Y$ (Extendable to a 5d bulk)

Figure 6: Similar to Fig. 5. Nontrivial Borromean rings between three line operators $\mathcal{A}$ (shown as the blue, green, and cyan $S^1$ loop operators) can carry a nontrivial odd $\mathbb{Z}_4$,X charge measured by a codimension 1 operator $\star J_{\mathcal{A}_4}$ (shown on the red $S^3$) in $\mathcal{A}_{\mathcal{Z}_4} \wedge \star J_{\mathcal{Z}_4}$. The Wilson line (either dynamical gauge or background probe) with a line integral $\exp(iq_X \int \mathcal{A}_4)$ of $\mathbb{Z}_4$,X along the gold color dashed line, can connect and mediate between the topological world and our Standard Model. A second design to communicate and interact with the topological world of “God.”

To be more practical, then, these Hopf links or Borromean rings cannot be designed by us (unless we can access the high enough energy to the anyonic excitation scales of TQFT and topological order), but by the “God”’s world. Suppose there are such Hopf links or Borromean rings formed in the “God”’s world, our next question is: How to communicate or interact with these topological links?

In fact, the odd $\mathbb{Z}_{4,X}$ charges of these topological links mean that the dynamically gauged Wilson line with a line integral $\exp(iq_X \int X_g) \propto \exp(iq_X \int \mathcal{A}_{\mathcal{Z}_4})$ of $\mathbb{Z}_{4,X}$ can mediate between these topological links and any other matters carry the nontrivial (especially the odd) $\mathbb{Z}_{4,X}$ charges.

So we only need to look for all SM and GUT particles carrying the $\mathbb{Z}_{4,X}$ charges (especially the odd $\mathbb{Z}_{4,X}$ charge). We now look at Table 1 and 2, all the left-handed Weyl spinors carry $\mathbb{Z}_{4,X}$ charge $+1$ (and the right-handed Weyl spinors carry a complex conjugate of $\mathbb{Z}_{4,X}$ charge $-1 = 3 \mod 4$). The electroweak Higgs $\phi_H$ carries an even $\mathbb{Z}_{4,X}$ charge $+2 = -2 \mod 4$. So, yes, in fact, all SM Weyl spinors carry odd $\mathbb{Z}_{4,X}$ charge, thus, all SM Weyl spinors can be the other end of the Wilson line with a line integral $\exp(iq_X \int \mathcal{A}_{\mathcal{Z}_4})$ of $\mathbb{Z}_{4,X}$. All the odd number of bound states (like proton and neutrons in our body, also electrons) presumably can carry the odd $\mathbb{Z}_{4,X}$. In summary, we have a schematic communication...
between SM particle $\psi_{SM}$ and topological link via a line operator

$$\exp(i \sum_I \int \mathscr{A}_I) \bigg|_{\text{links formed around } p_1} \cdot \exp(i q X \int_{p_1}^{p_2} A_{Z_4}) \cdot \psi_{SM}(p_2),$$

(6.7)

where $X_g$ is $A_{Z_4}$ for the discrete $Z_{4,X}$. Here $(\sum_I \int \mathscr{A}_I)$ are several sets of line operators forming the links as Fig. 5 and Fig. 6. So our SM world and the gapped topological sector can be mediated and communicated by Topological Force.

### 6.3 Topological Force as the Fourth or Fifth force?

We had mentioned that to complete the GUT, it is natural to include a nonperturbative topological sector of 4d TQFT, 5d iTQFT or 5d TQFT. The Topological Force derived here is not within three of Fundamental Forces: Electromagnetism, Weak, and Strong. So we may view the Topological Force as a new force. Should Topological Force be regarded as the Fourth force (related to gravity) or Fifth force? Our model in Sec. 6.1 suggests that perhaps the Topological Force is a Fifth force in the model (6.1). But the Topological Force may mix with the gravity (the Fourth force) in the model (6.4), when $Z_{4,X}$ is gauged and the geometry/topology are summed but regularized. It is not clear how the gravity in Sec. 6.1 has anything to do with Einstein gravity. It may be interesting to explore this direction in the future.

### 6.4 Neutrinos and Three Generation Mystery (or Three Family Problem)

The renown old puzzle about Three Generation Mystery (or Three Family Problem) [161] on $N_{\text{generation}} = 3$ may be now explained when $\nu = 3 = N_{\text{generation}} \mod 16$ anomaly turns out to be extremely special for gapped topological sector construct — a special way of gapping the missing sterile neutrinos described in Sec. 5. We had tried to indicate the 4d TQFT has a nice 3d analogous as $SO(3)_{3,\pm}$ Chern-Simons theory, which is indeed a very special construction to preserve $Z_4^T$ (or $Pin^+$ structure in math) in 3d and $Z_{4,X}$ (or $Spin \times_{Z_2} Z_4 = Spin \times_{Z_2} Z_{4,X}$ structure in math) in 4d.

Neutrino oscillations may also be explained if we consider the Majorana zero modes of the vortices in the 4d TQFT defects in Sec. 5. The left-handed neutrinos (observed in experiments) are nearly gapless/massless, when traveling through the 4d TQFT defects from gapping the right-handed sterile neutrinos (not yet observed in experiments) via Topological Mass / Energy Gap in Sec. 5.3, we may see nearly gapless left-handed neutrino flavor oscillations interfering with the Majorana zero modes of the vortices in the 4d TQFT defects.

**Right-handed sterile neutrinos are not sterile to $Z_{4,X}$ gauge field:** We should also point out that although the right-handed neutrinos are sterile to SM forces, but they are not sterile to the $Z_{4,X}$ gauge field $A_{Z_4}$, because they carry the odd $Z_{4,X}$ charge. So it is possible to regard the degrees of freedom of right-handed neutrinos encoding into the gapped 4d TQFT defects and zero modes. We can try to detect these degrees of freedom by the mediator Topological Force by the design in Sec. 6.2.
6.5 Dark Matter

Dark Matter: In the heavy sector of TQFTs that we described in Sec. 5, the line $\mathcal{A}$ is the worldline of heavy particle excitations, and the surface $\mathcal{B}$ is the worldsheet of heavy string excitations. Those are new objects very heavy maybe to the GUT scales. So these new heavy higher-dimensional extended objects may be a suitable candidate account for the Dark Matter.

Dark Energy: We have very little to say about the Dark Energy. Except that in the gravity version (sum over geometry and topology; dynamical, quantum, and topological gravity) of partition function in (6.4), it may be worthwhile to investigate the underlying energy stored in this partition function. But it seems that 4d and 5d topological sectors can be very heavy and store a much huge amount of energy than what we knew of from our Standard Model world.

7 Bibliography

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31 Based on the author’s conversations (i.e., my conversations) with eminent theoretical physicists on the related subjects of author’s present work and other related ideas, there are several conservative opinions. In 2016, Alexei Kitaev commented: “The idea may be correct mathematically. But our universe and physical nature would not do this.” Xiao-Gang Wen commented in 2018: “There may be no particle physics phenomenological consequences on this pursuit, thus experimentally not falsifiable nor verifiable. So your pursuit may not be productive. Our Universe may be just messy, not beautiful, like a junk.” Edward Witten originally thought the idea may be not feasible nor attractive (with sarcasm), especially about a non-perturbative lattice regularization, at least by the end of 2018; but he gradually became more supportive about my idea in the 2019 summer. In 2019 fall, Dan Freed was surprised when I said “the known Standard Model may have anomalies so we can use it to detect the hidden sector” — by that of course I meant the ’t Hooft anomaly, but not the dynamical gauge anomalies. But Dan Freed was still not convinced how the “the known Standard Model may have (’t Hooft) anomalies.” When I told David Huse and Bert Halperin these quantum condensed matter ideas should have a big impact revolutionizing physics beyond Standard Model, they became rather conservative. Erich Poppitz once commented that “It will be much nicer if your idea can really apply or happen in our universe... rather than a toy model...” I replied confidently and seriously, “It is. Of course. It indeed happened and is still happening in our universe, beyond Standard Model!”

Although many comments from these renowned physicists may sound discouraging (if not simply negative), the author had been believing in that the present new model with topological sector is indeed the correct path beyond Standard Model — we should look for high-energy physics phenomenological consequences — our universe indeed does this and speaks this language physically and naturally.

Instead of writing or drawing an image of the author’s mental feelings, a piece of Johannes Brahms’s music “The Variations and Fugue on a Theme by Handel, Op. 24 (1861)” may illuminate this well. In particular, listen to the Variation 11 and Variation 12 sing. Listen:
