Matrix Weighted Kolmogorov–Riesz’s Compactness Theorem

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Abstract In this paper, several versions of the Kolmogorov–Riesz compactness theorem in weighted Lebesgue spaces with matrix weights are obtained. In particular, when the matrix weight $W$ is in the known $A_p$ class, a characterization of totally bounded subsets in $L^p(W)$ with $p \in (1, \infty)$ is established.

Keywords Kolmogorov–Riesz theorem, matrix weight, totally bounded, metric measure space, variable exponent Lebesgue space

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1 Introduction

In this paper, we investigate totally bounded sets in matrix weighted Lebesgue spaces, from which one can obtain corresponding compactness criteria via the Hausdorff criterion for compactness, that is, a set is compact if and only if it is complete and totally bounded.

In classical Lebesgue spaces $L^p$, the characterization of precompact sets was given by the celebrated Kolmogorov–Riesz theorem (see [22]) which was first discovered by Kolmogorov [26] in $L^p([0, 1])$ for $p \in (1, \infty)$. Subsequently, Tamarkin [32] extended the result to the case in which the underlying space can be unbounded, with an additional condition related to the behaviour at infinity. Tulajkov [35] showed that Tamarkin’s result was also true when $p = 1$. At the same time, Riesz [30] independently proved a similar result. In [34], Tsuji extended Kolmogorov–Riesz’s result to the case $p \in (0, 1)$. Since then, compactness criteria of subsets in Lebesgue spaces have been studied in various settings, e.g., see [1–3,19,20,29] for a series of works on compactness criteria in variable exponent function spaces.

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Meanwhile, Kolmogorov–Riesz’s theorem has abroad applications in the field of harmonic analysis. For instance, the usual approach to verify the compactness of certain bounded operators depends on it, e.g., see [38] for characterizations of weighted compactness of certain commutators. In addition, very recently, Cao, Olivo and Yabuta [7] established several “compact versions” of Rubio de Francia’s weighted extrapolation theorem for multilinear operators and further improved previous results on the compactness of commutators for several kinds of multilinear operators, whose method relies on developing scalar weighted versions of the Kolmogorov–Riesz theorem.

Recently, Hanche-Olsen and Holden [22] seminally showed that both the Arzelá–Ascoli theorem (see [17]) and Kolmogorov–Riesz’s theorem are consequences of a simple lemma on compactness in metric spaces via a finite dimension argument. Inspired by the method used in [22], Clop and Cruz [9] first gave a compactness criterion in scalar weighted Lebesgue spaces \( L^p(\omega) \) for \( p \in (1, \infty) \) with a scalar weight \( \omega \in A_p \). Their result was then improved by Guo and Zhao in [21, Theorem 3.3, Corollary 3.8], in which they gave the following compactness criterion in \( L^p(\omega) \), where \( 1 \leq p < \infty \) and \( \omega \in L^1_{\text{loc}}(\mathbb{R}^n) \) is a nonnegative function, or \( 0 < p < \infty \) and \( \omega \) is a scalar weight, a nonnegative locally integrable function on \( \mathbb{R}^n \) that is positive almost everywhere.

**Theorem 1.1.** Let \( 1 \leq p < \infty \) and \( \omega \in L^1_{\text{loc}}(\mathbb{R}^n) \) be a nonnegative function, or \( 0 < p < \infty \) and \( \omega \) be a scalar weight. A subset \( \mathcal{F} \) of \( L^p(\omega) \) is totally bounded if the following conditions are valid:

(a) \( \mathcal{F} \) is bounded, i.e., \( \sup_{f \in \mathcal{F}} \|f\|_{L^p(\omega)} < \infty \);

(b) \( \mathcal{F} \) uniformly vanishes at infinity, that is,

\[
\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \|f \chi_{(B(0,R))^c}\|_{L^p(\omega)} = 0;
\]

(c) \( \mathcal{F} \) is equicontinuous, that is,

\[
\lim_{r \to 0} \sup_{f \in \mathcal{F}} \sup_{y \in B(0,r)} \|\tau_y f - f\|_{L^p(\omega)} = 0.
\]

Here, \( \tau_y \) denotes the translation operator: \( \tau_y f(x) := f(x - y) \).

There is a natural question whether Theorem 1.1 can be extended to the setting of matrix weights. As a natural vector-valued generalization of scalar Muckenhoupt \( A_p \) weights, the theory of matrix weights was first introduced by Bloom [4,5] in 1981. Then the theory was pushed forward through the seminal work of Nazarov, Treil and Volberg [28, 33, 36], Christ and Goldberg [8, 18], and Frazier and Roudenko [15, 31] in the late 1990s, that arose from problems in the theory of stationary processes, the theory of Toeplitz operators and
multivariable elliptic PDEs. From then on, harmonic analysis with matrix weights has been considered by many authors in various directions. For the references, we refer to [11, 12] for recent developments on matrix weights, [24, 25, 27] for the matrix $A_2$ conjecture related to the sharp norm estimates for singular operators, and [13, 14] for some applications of matrix weights.

Matrix weights share many properties with scalar weights, for example, the definition of matrix $A_p$ weights due to Frazier and Roudenko [15, 31] seems to be an intuitive extension of the definition of scalar $A_p$ weights and the Hilbert transform is bounded on $L^p(W)$ if and only if $W \in A_p$. However, due to the non-commutativity in the matricial setting, many techniques of the classical harmonic analysis fail to generalize to the case of vector-valued functions with matrix weights, so the vector-valued case cannot be easily reduced to the scalar case. For example, in the setting of matrix weights, a suitable theory of weak-type spaces $L^{p,\infty}(W)$ is unknown.

In this paper, we mainly obtain several generalizations of Theorem 1.1 in weighted Lebesgue spaces with matrix weights. To be precise, we first obtain a Kolmogorov–Riesz theorem in the weighted variable Lebesgue space $L^p(\cdot)(\rho)$ on $\mathbb{R}^n$, where $\rho = \{\rho_x\}_{x \in \mathbb{R}^n}$ is a family of norms on $\mathbb{C}^d$. We then establish another version of compactness criterion in $L^p(W)$ when $p \in [1, \infty)$ and $W$ is a matrix weight on $\mathbb{R}^n$. We further extend the above version to the case in which $p \in (0, \infty)$ and $W$ is an invertible matrix weight on $\mathbb{R}^n$. We also obtain a Kolmogorov–Riesz theorem in $L^p(\rho, \mu)$ for $p \in [1, \infty)$ on metric measure spaces $(X, d, \mu)$, and obtain an equivalent characterization of precompact subsets in $L^p(W)$ for $p \in (1, \infty)$ and $W$ in the $A_p$ class on $\mathbb{R}^n$ as an application.

We would like to emphasize that due to the special structure of matrix weighted Lebesgue spaces, neither matrix weighted Lebesgue spaces $L^p(W)$ nor weighted Lebesgue spaces $L^p(\rho)$ are in the framework of (quasi-)Banach function spaces in [6, 20, 21] or Lebesgue–Bochner spaces in [17]. For example, in the case of vector-valued functions with matrix weights, much of the ability to compare objects and dominate one by another is lost. Moreover, for a scalar weight $\omega$, we have the fact that a function $f \in L^p(\omega)$ if and only if $|f| \in L^p(\omega)$. Unfortunately, it is not true for matrix weights.

Based on these facts, different from the classical case, the available methods to prove Kolmogorov–Riesz’s theorem seem to be not applicable in the setting of matrix weights. And we use a finite dimension argument without using the key lemma in [22] to obtain compactness criteria in matrix weighted Lebesgue spaces on $(\mathbb{R}^n, |\cdot|, m)$, where $m$ denotes the Lebesgue measure.

The paper is organized as follows. In Section 2, we recall some basic notations and facts related to matrix weights. In particular, we recall the so-called John ellipsoid theorem which shows the existence of a positive-definite self-adjoint matrix for a given norm $\rho$ on $\mathbb{C}^d$; see Lemma 2.2 below.

Section 3 is devoted to the study of totally bounded sets in matrix weighted Lebesgue spaces on $(\mathbb{R}^n, |\cdot|, m)$. For a given family of norms $\rho = \{\rho_x\}_{x \in \mathbb{R}^n}$ on
we first apply the John ellipsoid theorem and establish a Kolmogorov–Riesz theorem in $L^p(\cdot)(\cdot)$ for an exponent function $p(\cdot)$. When $p \in [1, \infty)$ and $W$ is a matrix weight on $\mathbb{R}^n$ which is not necessarily invertible, we also obtain another version of compactness criterion in $L^p(W)$ by following some idea from [13]. By using Tsuji’s method in [34], we further extend the above result to the case in which $p \in (0, \infty)$ and $W$ is a matrix weight on $\mathbb{R}^n$. As an application, a compactness criterion in degenerate Sobolev spaces with matrix weights is given.

In Section 4, let $(X, d, \mu)$ be a proper metric measure space such that $\mu$ is continuous with respect to the metric $d$ and $\rho = \{\rho_x\}_{x \in X}$ be a family of norms on $\mathbb{C}^d$. We present a Kolmogorov–Riesz theorem in weighted Lebesgue spaces $L^p(\rho, \mu)$ for $p \in [1, \infty)$ on $(X, d, \mu)$ in terms of the average operator, and apply to $L^p(W, \mu)$ when $W$ is an invertible matrix weight on $\mathbb{R}^n$ and $\mu$ is continuous with respect to $|\cdot|$. We would like to mention that our method to prove Theorem 3.1 (see also [22, Theorem 5] and [21, Theorem 3.1]) relies on the translation invariance of the Euclidean metric and the Lebesgue measure, which fails on metric measure spaces.

In Section 5, based on the result obtained in Section 4, when the matrix weight $W$ is in the known $A_p$ class on $\mathbb{R}^n$ in [31], we further obtain an equivalent characterization of compact subsets in $L^p(W)$ for $p \in (1, \infty)$.

Throughout this paper, we will use the following notations. We always use $\omega(\cdot)$ to denote a scalar weight while $W(\cdot)$ to denote a matrix weight. Given two non-negative $A$ and $B$, we will write $A \lesssim B$ if there exists a positive constant $c$, independent of appropriate quantities involved in $A$ and $B$, such that $A \leq cB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. We will use $p'$ to denote the conjugate exponent of $p$ when $p \in (1, \infty)$. For a given set $E \subset \mathbb{R}^n$ (or $X$), $\chi_E$ means the characteristic functions of $E$ and $E^c := \mathbb{R}^n \setminus E$ (or $X \setminus E$). Additionally, unless otherwise noted, $(\mathbb{R}^n, |\cdot|, m)$ is the underlying measure space.

## 2 Preliminaries

In this section, we recall some basic notations and facts about matrix weights; see [13, 18] and the references therein.

Let $\mathcal{M}_d$ denote the set of all complex-valued, $d \times d$ matrices. A matrix function on $\mathbb{R}^n$ is a map $W : \mathbb{R}^n \to \mathcal{M}_d$. We say that it is measurable if each component of $W$ is a measurable function, and invertible if $\det W(x) \neq 0$ a.e. $x \in \mathbb{R}^n$ and so $W^{-1}$ exists. Let $\mathcal{S}_d$ be the set of all those $A \in \mathcal{M}_d$ that are self-adjoint and non-negative-definite. For each $A \in \mathcal{S}_d$, $A$ has $d$ non-negative real-valued eigenvalues $\lambda_i$, $1 \leq i \leq d$, and the norm of $A$ is defined as the operator norm

$$\|A\|_{\text{op}} := \sup_{v \in \mathbb{C}^d, |v|=1} |Av| = \max_i \lambda_i.$$ 

Moreover, there exists a unitary matrix $U$ such that $U^H A U$ is diagonal, where
$U^H$ denotes the conjugate transpose matrix of $U$. We denote a diagonal matrix by $D(\lambda_1, \ldots, \lambda_d) = D(\lambda_i)$. For every $s > 0$, we define $A^s := UD(\lambda_i^s)U^H$. Furthermore, if $A$ is positive-definite, we set $A^{-s} := UD(\lambda_i^{-s})U^H$.

The following technical lemma is from [13, Lemma 3.1].

**Lemma 2.1.** Given a measurable matrix function $W : \mathbb{R}^n \to \mathcal{S}_d$, there exists a $d \times d$ measurable matrix function $U$ defined on $\mathbb{R}^n$ such that $U^H(x)W(x)U(x)$ is diagonal, and $U(x)$ is unitary for every $x \in \mathbb{R}^n$.

Based on Lemma 2.1, it is easy to see that for any measurable matrix function $W : \mathbb{R}^n \to \mathcal{S}_d$, $W^s$ is a measurable matrix function satisfying that, for any $x \in \mathbb{R}^n$,

$$\|W^s(x)\|_{\text{op}} = \max_i \lambda_i^s(x). \quad (2.1)$$

If $W$ is invertible, $W^{-s}$ is a measurable matrix function satisfying that, for any $x \in \mathbb{R}^n$,

$$\|W^{-s}(x)\|_{\text{op}}^{-1} = \min_i \lambda_i^s(x). \quad (2.2)$$

By a matrix weight on $\mathbb{R}^n$ we mean a measurable matrix function $W : \mathbb{R}^n \to \mathcal{S}_d$ such that $\|W\|_{\text{op}} \in L^1_{\text{loc}}(\mathbb{R}^n)$. Equivalently, each eigenvalue function $\lambda_i \in L^1_{\text{loc}}(\mathbb{R}^n)$, $1 \leq i \leq d$. Define the matrix weighted Lebesgue space $L^p(W)$ for $p \in (0, \infty)$ to be the set of all measurable vector-valued functions $f := (f_1, \ldots, f_d)^T : \mathbb{R}^n \to \mathbb{C}^d$ such that

$$\|f\|_{L^p(W)}^p := \int_{\mathbb{R}^n} |W^{1/p}(x)f(x)|^p dx < \infty.$$ 

In many cases, it is more convenient to characterize the matrix weighted Lebesgue space in the following language; see [36]. Let $\rho := \{\rho_x\}_{x \in \mathbb{R}^n}$ be a family of norms on $\mathbb{C}^d$, where for each $x \in \mathbb{R}^n$, $\rho_x : \mathbb{C}^d \to \mathbb{R}^+ := [0, \infty)$. Define the weighted Lebesgue space $L^p(\rho)$ for $p \in (0, \infty)$ to be the set of all measurable vector-valued functions $f : \mathbb{R}^n \to \mathbb{C}^d$ such that

$$\|f\|_{L^p(\rho)}^p := \int_{\mathbb{R}^n} [\rho_x(f(x))]^p dx < \infty,$$

where we always assume that $\rho_x(f(x))$ is a measurable function on $\mathbb{R}^n$ for any measurable vector-valued function $f$.

For any given invertible matrix weight $W$, one can reduce $L^p(\rho)$ to $L^p(W)$ by setting $\rho_x(\cdot) := |W^{1/p}(x)\cdot|$. The following so-called John ellipsoid theorem (see [18, Proposition 1.2]) shows that the two matrix weighted Lebesgue spaces above actually coincide.
Lemma 2.2. Given a norm $\rho$ on $\mathbb{C}^d$, there exists a positive-definite self-adjoint matrix $W$ such that

$$\rho(v) \leq |W(v)| \leq d^{\frac{1}{2}}\rho(v), \quad \forall v \in \mathbb{C}^d. \quad (2.3)$$

We now recall the definition of matrix $A_p$ weight in [15, 31] of Frazier and Roudenko; see also [28] for another definition of matrix $A_p$ weights when $p > 1$.

Definition 2.1. Let $W$ be an invertible matrix weight.

(i) When $p \in (1, \infty)$, we say $W \in A_p$ if $\|W^{-1}\|_{op}^{p'} \in L^1_{loc}(\mathbb{R}^n)$ and

$$[W]_{A_p} := \sup_Q \frac{1}{|Q|} \int_Q \left( \frac{1}{|Q|} \int_Q \|W^\frac{1}{p}(x)W^{-\frac{1}{p}}(y)\|_{op}^{p'} dy \right)^\frac{p}{p'} dx < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

(ii) When $p \in (0, 1]$, we say $W \in A_p$ if $\|W^{-1}\|_{op} \in L^1_{loc}(\mathbb{R}^n)$ and

$$[W]_{A_p} := \supesssup_Q \frac{1}{|Q|} \int_Q \|W^\frac{1}{p}(y)W^{-\frac{1}{p}}(x)\|_{op}^{p} dy < \infty,$$

where the first supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

We would like to mention that when $p \geq 1$, $d = 1$ and $W(x) = \omega(x)$ is a scalar weight, the matrix $A_p$ condition is the Muckenhoupt $A_p$ condition. Moreover, we have the following lemma due to [13, Lemma 4.5].

Lemma 2.3. Let $1 < p < \infty$. If $W \in A_p$, then $\|W\|_{op}$ and $\|W^{-1}\|_{op}^{-1}$ are scalar $A_p$ weights.

Next we recall a variant of the maximal operator introduced by Christ and Goldberg in [8, 18]. The Christ–Goldberg maximal operator $M_W$ is defined as

$$M_W f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |W^\frac{1}{p}(x)W^{-\frac{1}{p}}(y)f(y)| dy,$$

where the supremum is taken over all balls in $\mathbb{R}^n$ containing $x$. They obtained the following strong-type estimate for the Christ–Goldberg maximal operator.

Lemma 2.4. Let $1 < p < \infty$. If $W \in A_p$, then there exists $\delta > 0$ such that when $q \in \{q > 1 : |p - q| < \delta\}$,

$$\|M_W f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n, \mathbb{C}^d)}, \quad \forall f \in L^q(\mathbb{R}^n, \mathbb{C}^d),$$

where the implicit constant depends only on $q$ and

$$\|f\|_{L^q(\mathbb{R}^n, \mathbb{C}^d)} := \left( \int_{\mathbb{R}^n} |f(x)|^q dx \right)^{\frac{1}{q}}.$$
3 Compactness Criteria on $\mathbb{R}^n$

This section is devoted to the study of Kolmogorov–Riesz’s theorem in matrix weighted Lebesgue spaces on $\mathbb{R}^n$. In [19, Theorem 5], Görka and Macios established a compactness criterion in variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. The first main result of this section is to obtain a generalization of [19, Theorem 5] to $L^{p(\cdot)}(\rho)$. Before that, we recall some basic notations and results about variable exponent Lebesgue spaces; see [10].

Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot): \mathbb{R}^n \to [1, \infty]$. The elements of $\mathcal{P}(\mathbb{R}^n)$ are called exponent functions. Given an exponent function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, we put

$$p_+ := \text{ess sup}_{x \in \mathbb{R}^n} p(x), \quad p_- := \text{ess inf}_{x \in \mathbb{R}^n} p(x).$$

We assume that exponent functions $p(\cdot)$ are bounded, i.e., $p_+ < \infty$. Define the weighted variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ to be the set of all measurable vector-valued functions $f: \mathbb{R}^n \to \mathbb{C}^d$ such that the modular

$$\int_{\mathbb{R}^n} [\rho_x(f(x))]^{p(x)} dx < \infty,$$

equipped with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\rho)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left[ \frac{\rho_x(f(x))}{\lambda} \right]^{p(x)} dx \leq 1 \right\}.$$

According to the above definition and the convexity of the modular, we obtain the following useful results on the relationship between the modular and the norm.

**Lemma 3.1.** Let $p_+ < \infty$, $0 < \lambda \leq 1$, and $f \in L^{p(\cdot)}(\rho)$. Then the following statements are true:

(a) If $\|f\|_{L^{p(\cdot)}(\rho)} \leq 1$, then $\int_{\mathbb{R}^n} [\rho_x(f(x))]^{p(x)} dx \leq \|f\|_{L^{p(\cdot)}(\rho)}$;

(b) If $\|f\|_{L^{p(\cdot)}(\rho)} > 1$, then $\int_{\mathbb{R}^n} [\rho_x(f(x))]^{p(x)} dx \geq \|f\|_{L^{p(\cdot)}(\rho)}$;

(c) $\|f\|_{L^{p(\cdot)}(\rho)} \leq \int_{\mathbb{R}^n} [\rho_x(f(x))]^{p(x)} dx + 1$;

(d) If $\int_{\mathbb{R}^n} [\rho_x(f(x))]^{p(x)} dx \leq \lambda^{p_+}$, then $\|f\|_{L^{p(\cdot)}(\rho)} \leq \lambda$.

**Proof.** (a)–(c) hold by adapting the arguments in [10, Corollary 2.22]. To prove (d), noting that when $p_+ < \infty$ and $0 < \lambda \leq 1$, we have

$$\int_{\mathbb{R}^n} \left[ \frac{\rho_x(f(x))}{\lambda} \right]^{p(x)} dx \leq \int_{\mathbb{R}^n} [\rho_x(f(x))]^{p(x)} \lambda^{-p_+} dx \leq 1,$$

which implies $\|f\|_{L^{p(\cdot)}(\rho)} \leq \lambda$. \qed
Based on Lemma 3.1, we now present a Kolmogorov–Riesz theorem in $L^p(\cdot)(\rho)$.

**Theorem 3.1.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $p_+ < \infty$ and $\rho := \{\rho_x\}_{x \in \mathbb{R}^n}$ be a family of norms on $\mathbb{C}^d$ such that $W_x$ is an invertible matrix weight satisfying (2.3) for every $x \in \mathbb{R}^n$ and $\|W_x\|^p_{\text{op}} \in L^1_{\text{loc}}(\mathbb{R}^n)$. A subset $\mathcal{F} \subset L^p(\cdot)(\rho)$ is totally bounded if the following conditions are valid:

(a) $\mathcal{F}$ is bounded in the sense of the modular, that is,

$$\sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} [\rho_x(f(x))]^{p(x)} dx < \infty;$$

(b) $\mathcal{F}$ uniformly vanishes at infinity, that is,

$$\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \int_{(B(0,R))^c} [\rho_x(f(x))]^{p(x)} dx = 0;$$

(c) $\mathcal{F}$ is equicontinuous, that is,

$$\lim_{r \to 0} \sup_{f \in \mathcal{F}} \sup_{y \in B(0,r)} \int_{\mathbb{R}^n} [\rho_x(\tau_y f(x) - f(x))]^{p(x)} dx = 0.$$

**Proof.** Assume that $\mathcal{F} \subset L^p(\cdot)(\rho)$ satisfies (a)–(c). Given $\epsilon > 0$ small enough, to prove the total boundedness of $\mathcal{F}$, it suffices to find a finite $\epsilon$-net of $\mathcal{F}$. Denote by $R_i := [-2^i, 2^i]^n$ for $i \in \mathbb{Z}$. Then by condition (b), there exists a positive integer $m$ large enough such that

$$\sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} [\rho_x(f(x) - f(x)\chi_{R_m}(x))]^{p(x)} dx < \epsilon. \quad (3.1)$$

Moreover, by condition (c), there exists an integer $t$ such that

$$\sup_{f \in \mathcal{F}} \sup_{y \in R_t} \int_{\mathbb{R}^n} [\rho_x(f(x - y) - f(x))]^{p(x)} dx < \epsilon. \quad (3.2)$$

Let $\mathcal{D}_i, i \in \mathbb{Z}$, be the family of dyadic cubes in $\mathbb{R}^n$, open on the right, whose vertices are adjacent points of the lattice $(2^i \mathbb{Z})^n$. For each $i \in \mathbb{Z}$, the cubes in $\mathcal{D}_i$ are either disjoint or coincide. Thus there exists a sequence $\{Q_j\}_{j=1}^N$ of disjoint cubes in $\mathcal{D}_t$ such that $R_m = \bigcup_{j=1}^N Q_j$, where $N = 2^{(m+1-t)n}$ is a positive integer.

For any $f \in \mathcal{F}$ and $x \in \mathbb{R}^n$, define

$$\Phi(f)(x) := \begin{cases} f_{Q_j} := \frac{1}{|Q_j|} \int_{Q_j} f(y) dy, & x \in Q_j, \ j = 1, \ldots, N, \\ 0, & \text{otherwise.} \end{cases}$$
Then for every fixed \( x \in \mathbb{R}^n \), by Lemma 2.2, there exists a positive-definite self-adjoint matrix \( W_x \) such that for each \( j \),

\[
\rho_x((f(x) - f_{Q_j})(x)) \leq |W_x(f(x) - f_{Q_j})(x)|
\]

\[
\lesssim \frac{1}{|Q_j|} \int_{Q_j} |W_x(f(x) - f(y))| dy \chi_{Q_j}(x)
\]

\[
\lesssim \frac{1}{|Q_j|} \int_{Q_j} \rho_x(f(x) - f(y)) dy \chi_{Q_j}(x),
\]

where the implicit constants only depend on \( d \). Then from the choices of \( \{Q_j\}_{j=1}^N \), the Jensen inequality, the Fubini theorem and (3.2), it follows that

\[
\int_{\mathbb{R}^n} \left[ \rho_x(f(x)\chi_{R_m}(x) - \Phi(f)(x)) \right]^{p(x)} dx
\]

\[
\lesssim \sum_{j=1}^{N} \int_{Q_j} \left[ \frac{1}{|Q_j|} \int_{Q_j} \rho_x(f(x) - f(y)) dy \right]^{p(x)} dx
\]

\[
\lesssim \sum_{j=1}^{N} \frac{1}{|Q_j|} \int_{Q_j} \int_{Q_j} [\rho_x(f(x) - f(y))]^{p(x)} dx dy
\]

\[
\approx 2^{-nt} \sum_{j=1}^{N} \int_{Q_j} \int_{Q_j} [\rho_x(f(x) - f(x - y))]^{p(x)} dy dx
\]

\[
\lesssim 2^{-nt} \int_{\mathbb{R}^n} \int_{R_t} [\rho_x(f(x) - f(x - y))]^{p(x)} dy dx
\]

\[
\approx 2^{-nt} \int_{R_t} \int_{\mathbb{R}^n} [\rho_x(f(x) - f(x - y))]^{p(x)} dx dy
\]

\[
\lesssim 2^{-nt} |R_t| \sup_{y \in R_t} \int_{\mathbb{R}^n} [\rho_x(f(x) - f(x - y))]^{p(x)} dx \lesssim 2^n \epsilon, \quad (3.3)
\]

where we use the fact that \( x - Q_j := \{x - y : y \in Q_j\} \subset R_t \) when \( x \in Q_j \). Note that

\[
\int_{\mathbb{R}^n} \left[ \rho_x(f(x) - \Phi(f)(x)) \right]^{p(x)} dx
\]

\[
= \left[ \int_{\mathbb{R}^n \setminus R_m} + \int_{R_m} \right] [\rho_x(f(x) - \Phi(f)(x))]^{p(x)} dx
\]

\[
= \int_{\mathbb{R}^n} [\rho_x(f(x) - f(x)\chi_{R_m}(x))]^{p(x)} dx
\]

\[
+ \int_{\mathbb{R}^n} [\rho_x(f(x)\chi_{R_m}(x) - \Phi(f)(x))]^{p(x)} dx.
\]

This via (3.1) and (3.3) implies that

\[
\sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} [\rho_x(f(x) - \Phi(f)(x))]^{p(x)} dx \lesssim \epsilon, \quad (3.4)
\]
where the implicit constant depends only on \( n \) and \( d \). Since \( p_+ < \infty \), then from (3.4) and Lemma 3.1, it suffices to show that \( \Phi(\mathcal{F}) \) is totally bounded in \( L^{p(\cdot)}(\rho) \).

Note that by condition (a), we have

\[
\sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} \left[ \rho_x(\Phi(f)(x)) \right]^{p(x)} dx \leq 2^{p-1} \left( \sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} \left[ \rho_x(f(x) - \Phi(f)(x)) \right]^{p(x)} dx \right) + \sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} \left[ \rho_x(f(x)) \right]^{p(x)} dx < \infty.
\]

Then by [10, Remark 2.10], it follows that for any \( f \in \mathcal{F} \),

\[
|W_x \Phi(f)(x)| \leq d^\frac{1}{2} \rho_x(\Phi(f)(x)) < \infty \quad \text{a.e. } x \in \mathbb{R}^n.
\]

Since \( W_x \) is positive-definite for every \( x \in \mathbb{R}^n \), we obtain

\[
|\Phi(f)(x)| < \infty \quad \text{a.e. } x \in \mathbb{R}^n,
\]

which implies \( |f_{Q_j}| < \infty, j = 1, \ldots, N \). From this and \( \|W_x\|_{op}^{p_+} \in L^1_{\text{loc}}(\mathbb{R}^n) \), we see that \( \Phi \) is a map from \( \mathcal{F} \) to \( \mathcal{B} \), a finite dimensional Banach subspace of \( L^{p(\cdot)}(\rho) \). Notice that \( \Phi(\mathcal{F}) \subset \mathcal{B} \) is bounded, and hence is totally bounded. The proof of Theorem 3.1 is complete. \( \square \)

As a corollary of Theorem 3.1, by taking \( p(\cdot) \equiv p \in [1, \infty) \), we have the following Kolmogorov–Riesz theorem in \( L^p(\rho) \) defined in Section 2.

**Corollary 3.1.** Let \( 1 \leq p < \infty, \rho := \{\rho_x\}_{x \in \mathbb{R}^n} \) be a family of norms on \( \mathbb{C}^d \) such that \( W_x \) is an invertible matrix weight satisfying (2.3) for every \( x \in \mathbb{R}^n \) and \( \|W_x\|_{op}^{p_+} \in L^1_{\text{loc}}(\mathbb{R}^n) \). A subset \( \mathcal{F} \) of \( L^p(\rho) \) is totally bounded if the following conditions hold:

(a) \( \mathcal{F} \) is bounded, i.e., \( \sup_{f \in \mathcal{F}} \|f\|_{L^p(\rho)} < \infty \);

(b) \( \mathcal{F} \) uniformly vanishes at infinity, that is,

\[
\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \|f \chi_{(B(0,R))^c}\|_{L^p(\rho)} = 0;
\]

(c) \( \mathcal{F} \) is equicontinuous, that is,

\[
\lim_{r \to 0} \sup_{f \in \mathcal{F}} \sup_{y \in B(0,r)} \|\tau_y f - f\|_{L^p(\rho)} = 0.
\]

As an application of Corollary 3.1, by setting \( \rho_x(\cdot) := |W_1^{-\frac{1}{2}}(x) \cdot | \) and (2.1), we have the following compactness criterion in \( L^p(W) \) for \( p \in [1, \infty) \).
Corollary 3.2. Let $1 \leq p < \infty$, $W$ be an invertible matrix weight. A subset $\mathcal{F}$ of $L^p(W)$ is totally bounded if the following conditions hold:

(a) $\mathcal{F}$ is bounded, i.e., $\sup_{f \in \mathcal{F}} \|f\|_{L^p(W)} < \infty$;

(b) $\mathcal{F}$ uniformly vanishes at infinity, that is,
$$\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \|f \chi_{B^c(0,R)}\|_{L^p(W)} = 0;$$

(c) $\mathcal{F}$ is equicontinuous, that is,
$$\limsup_{r \to 0} \sup_{f \in \mathcal{F}, y \in B(0,r)} \|\tau_y f - f\|_{L^p(W)} = 0.$$

Remark 3.1. It is worth noting that compared with the compactness criteria in [7, 9, 39], there is no additional assumption on $W$. Based on this, from the fact that matrix weighted Lebesgue spaces are not translation invariant, on which the translation operator $\tau_y$ is not continuous, it follows that Corollary 3.2 is a strong sufficient condition for precompactness in $L^p(W)$.

When $p \in [1, \infty)$ and $W$ is a matrix weight which is not necessarily invertible, we also have the following characterization of totally bounded sets in $L^p(W)$, which extends [21, Theorem 3.3] to the setting of matrix weights.

Theorem 3.2. Let $1 \leq p < \infty$, $W$ be a matrix weight. A subset $\mathcal{F}$ of $L^p(W)$ is totally bounded if it satisfies the conditions (a), (b) in Corollary 3.2 and

(c*) $\mathcal{F}$ is equicontinuous in the sense that
$$\limsup_{r \to 0} \sup_{f \in \mathcal{F}, y \in B(0,r)} \left( \int_{\mathbb{R}^n} |D^1 \tilde{\mathcal{F}}(x)(U^H(x-y)f(x-y)-U^H(x)f(x))|^p dx \right)^{\frac{1}{p}} = 0.$$

Here, $D^1 \tilde{\mathcal{F}} := U^H W^\frac{1}{p} U$ is a matrix weight by Lemma 2.1.

Proof. We use some ideas from [13, Proposition 3.6]. Assume that $\mathcal{F} \subset L^p(W)$ satisfies (a)–(c*). For any $f \in L^p(W)$, denote by $\tilde{f} := U^H f$ and $\tilde{\mathcal{F}} := \{\tilde{f} \}_{f \in \mathcal{F}}$. Then condition (c*) is equivalent to the equicontinuity of $\tilde{\mathcal{F}} \subset L^p(D)$, that is,
$$\limsup_{r \to 0} \sup_{f \in \mathcal{F}, y \in B(0,r)} \|\tau_y \tilde{f} - \tilde{f}\|_{L^p(D)} = 0.$$

Moreover, from the orthogonality of $U$, it follows that
$$|D^\frac{1}{p} \tilde{f}| = |U^H W^\frac{1}{p} UU^H f| = |W^\frac{1}{p} f|.$$

This via (a) and (b) for $\mathcal{F}$ shows that $\tilde{\mathcal{F}} \subset L^p(D)$ is also bounded and uniformly vanishes at infinity. Observe that if $\tilde{\mathcal{F}} \subset L^p(D)$ is totally bounded, so is
\( \mathcal{F} \subset L^p(W) \) (indeed, the converse is also true). It suffices to verify the total boundedness of \( \tilde{\mathcal{F}} \subset L^p(D) \), that is, to find a finite \( \epsilon \)-net of \( \tilde{\mathcal{F}} \) for each fixed \( \epsilon > 0 \).

Note that for every \( 1 \leq i \leq d \), we have
\[
|D^{1/p} \tilde{f}_i| = |D(\lambda_i^{1/p}) \tilde{f}_i| \geq \lambda_i^{1/p} |\tilde{f}_i|,
\]
which implies that \( \{\tilde{f}_i\}_{f \in \mathcal{F}} \subset L^p(\lambda_i) \) satisfies conditions (a)–(c) of Theorem 1.1 with \( \omega \) replaced by \( \lambda_i \), where \( \lambda_i \in L^1_{\text{loc}}(\mathbb{R}^n) \) is a nonnegative eigenvalue function and \( \tilde{f} := (\tilde{f}_1, \ldots, \tilde{f}_d)^T \). Then we obtain that \( \{\tilde{f}_i\}_{f \in \mathcal{F}} \subset L^p(\lambda_i) \) is totally bounded by Theorem 1.1.

Hence, given \( \epsilon > 0 \), for every \( f \in \mathcal{F} \) and \( 1 \leq i \leq d \), there exists \( g_i \in L^p(\lambda_i) \) such that \( \|\tilde{f}_i - g_i\|_{L^p(\lambda_i)} < \epsilon \). Let \( g := (g_1, \ldots, g_d)^T \). Then by the equivalence of norms on \( \mathbb{C}^d \) and our choice of the \( g_i \)'s, we have
\[
\|\tilde{f} - g\|_{L^p(D)} \approx \sum_{i=1}^d \|\tilde{f}_i - g_i\|_{L^p(\lambda_i)} \lesssim \epsilon,
\]
where implicit constants depend only on \( p \) and \( d \). Finally, by the total boundedness of \( \{\tilde{f}_i\}_{f \in \mathcal{F}} \), we conclude that \( \{g\} \subset L^p(D) \) is a finite \( \epsilon \)-net of \( \tilde{\mathcal{F}} \). This completes the proof of Theorem 3.2.

**Remark 3.2.** When \( d = 1 \) and \( W(x) = \omega(x) \), we have \( D^{1/p}(x) = \omega^{1/p}(x) \) and \( U(x) = 1 \). In this case, both Theorem 3.2 (c*) and Corollary 3.2 (c) become Theorem 1.1 (c).

Based on the method to prove Theorem 3.2, we immediately obtain the following theorem which extends [21, Corollary 3.8] to the setting of matrix weights.

**Theorem 3.3.** Let \( 0 < p < \infty \), \( W \) be an invertible matrix weight. A subset \( \mathcal{F} \) of \( L^p(W) \) is totally bounded if it satisfies the conditions (a)–(c*) in Theorem 3.2.

Finally, we end this section with an application in degenerate Sobolev spaces with matrix weights. Let \( W \) be an invertible matrix weight and set \( v := \|W\|_{\text{op}} \).

For \( p \in [1, \infty) \), the degenerate Sobolev space \( \mathcal{W}^{1,p}_W(\mathbb{R}^n) \) due to Cruz-Uribe, Moen and Rodney [13] is defined as the set of all \( f \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}^n) \) such that
\[
\|f\|_{\mathcal{W}^{1,p}_W(\mathbb{R}^n)} := \|f\|_{L^p(v)} + \|\nabla f\|_{L^p(W)} < \infty,
\]
where \( \nabla f \) is the gradient of \( f \). The degenerate Sobolev space \( \mathcal{W}^{1,p}_W(\mathbb{R}^n) \) is an extension of scalar weighted Sobolev spaces. From Theorem 1.1 and Corollary 3.2, we have the following compactness criterion in \( \mathcal{W}^{1,p}_W(\mathbb{R}^n) \), which is an extension of [22, Corollary 9] and [1, Theorem 12].

**Corollary 3.3.** A subset \( \mathcal{F} \subset \mathcal{W}^{1,p}_W(\mathbb{R}^n) \) is totally bounded if the following conditions hold:
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(a) \( \mathcal{F} \) is bounded, i.e., \( \sup_{f \in \mathcal{F}} \| f \|_{W^{1,p}(\mathbb{R}^n)} < \infty \);

(b) \( \mathcal{F} \) uniformly vanishes at infinity, that is,

\[
\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \| f \chi_{B(0,R)c} \|_{W^{1,p}(\mathbb{R}^n)} = 0;
\]

(c) \( \mathcal{F} \) is equicontinuous, that is,

\[
\lim_{r \to 0} \sup_{f \in \mathcal{F}} \sup_{y \in B(0,r)} \| \tau_y f - f \|_{W^{1,p}(\mathbb{R}^n)} = 0.
\]

\textbf{Proof.} First, note that \( \mathcal{F} \subset W^{1,p}(\mathbb{R}^n) \) satisfies (a)–(c) if and only if \( \mathcal{F} \subset L^p(v) \) satisfies (a)–(c) of Theorem 1.1 and \( \nabla(\mathcal{F}) := \{ \nabla f \}_{f \in \mathcal{F}} \subset L^p(W) \) satisfies (a)–(c) of Corollary 3.2. Then since \( W \) is an invertible matrix weight, by Theorem 1.1 and Corollary 3.2, we obtain that both \( \mathcal{F} \subset L^p(v) \) and \( \nabla(\mathcal{F}) \subset L^p(W) \) are totally bounded. Corollary 3.3 then follows from the fact that a set in metric spaces is totally bounded if and only if it is Cauchy-precompact, that is, every sequence admits a Cauchy subsequence; see [37, Lemma 39.8].

\[\Box\]

\section{Compactness Criteria on Metric Measure Spaces}

This section is devoted to the study of totally bounded sets in matrix weighted Lebesgue spaces on metric measure spaces. We begin with some basic facts about metric measure spaces in [16,20].

Let \((X,d,\mu)\) be a metric measure space equipped with a metric \(d\) and a positive Borel regular measure \(\mu\). Let

\[
B(x,r) := \{ y \in X : d(x,y) < r \}
\]

be the ball of the radius \(r > 0\) with center \(x \in X\). We assume that the measure of every open nonempty set is strictly positive, and that the measure of every bounded set is finite.

\textbf{Definition 4.1.} A metric space is proper if every closed bounded set is compact.

We remark that since every bounded set in a geometrically doubling metric space is totally bounded (see [23, Lemma 2.3]), a geometrically doubling metric space is proper if and only if it is complete via the Hausdorff criterion.

\textbf{Definition 4.2.} Let \((X,d,\mu)\) be a metric measure space. The measure \(\mu\) is said to be continuous with respect to the metric \(d\) if for any \(x \in X\) and \(r > 0\) the following condition is valid:

\[
\lim_{d(x,y) \to 0} \mu[B(x,r) \Delta B(y,r)] = 0,
\]

where \(A \Delta B\) stands for the symmetric difference of sets \(A\) and \(B\). We call such a measure metrically continuous for short, when no confusions arise.
From Definition 4.2, we have the following lemma.

**Lemma 4.1.** If \( \mu \) is metrically continuous, then for every compact set \( K \subset X \) and \( r > 0 \),
\[
\inf_{x \in K} \mu[B(x,r)] > 0.
\]

**Proof.** First, since \( \mu \) is metrically continuous, then from Definition 4.2, we deduce that for any \( x, y \in X \) and \( r > 0 \),
\[
|\mu[B(x,r)] - \mu[B(y,r)]| \leq \mu[B(x,r) \Delta B(y,r)],
\]
which implies that for any given \( r > 0 \), the map \( x \mapsto \mu[B(x,r)] \) is continuous. Lemma 4.1 then follows from the extreme value theorem. \( \square \)

We now recall a vector-valued version of the classical Arzelà–Ascoli theorem; see [17, Lemma 2.1] and [22, Theorem 2].

**Lemma 4.2.** Let \( K \) be a compact topological space and \( C(K, \mathbb{C}^d) \) be the space of \( \mathbb{C}^d \)-valued continuous functions on \( K \) with the topology of uniform convergence. A subset \( \mathcal{F} \) of \( C(K, \mathbb{C}^d) \) is totally bounded if and only if the following conditions are valid:

(a) \( \mathcal{F} \) is pointwise bounded, i.e., \( \sup_{f \in \mathcal{F}} |f(x)| < \infty \), \( \forall x \in K \);

(b) \( \mathcal{F} \) is equicontinuous, that is, for every \( x \in K \) and \( \epsilon > 0 \), there is a neighborhood \( U \) of \( x \) such that
\[
|f(x) - f(y)| < \epsilon, \quad \forall y \in U, \ f \in \mathcal{F}.
\]

Now we give some necessary definitions and notations of matrix weights on metric measure spaces. A matrix function on \( X \) is a map \( W : X \to \mathbb{M}_d \). We say that it is \( \mu \)-measurable if each component of \( W \) is a \( \mu \)-measurable function on \( X \), and invertible if \( \det W(x) \neq 0 \) \( \mu \)-a.e. and so \( W^{-1} \) exists.

By a matrix weight on \( X \) we mean a \( \mu \)-measurable matrix function \( W : X \to \mathbb{M}_d \) such that \( \|W\|_{\text{op}} \in L^{1}_{\text{loc}}(X,d,\mu) \). Equivalently, each eigenvalue function \( \lambda_i \in L^{1}_{\text{loc}}(X,d,\mu) \), \( 1 \leq i \leq d \). Let \( \rho := \{\rho_x\}_{x \in X} \) be a family of norms on \( \mathbb{C}^d \), where for each \( x \in X \), \( \rho_x : \mathbb{C}^d \to \mathbb{R}^+ \). Define the weighted Lebesgue space \( L^p(\rho,\mu) \) for \( p \in (0,\infty) \) to be the class of all \( \mu \)-measurable vector-valued functions \( f : X \to \mathbb{C}^d \) such that
\[
\|f\|_{L^p(\rho,\mu)}^p := \int_X |\rho_x(f(x))|^p d\mu(x) < \infty,
\]
where we always assume that \( \rho_x(f(x)) \) is a \( \mu \)-measurable function on \( X \) for any \( \mu \)-measurable vector-valued function \( f \).

Now we present our main result in this section as follows, in which we replace the translation operator by the average operator and apply Lemma 4.2, inspired by [20, Theorem 3.1].
**Theorem 4.1.** Assume that \((X, d, \mu)\) is a proper metric measure space with a metrically continuous measure \(\mu\). Let \(1 < p < \infty\), \(\rho := \{\rho_x\}_{x \in X}\) be a family of norms on \(\mathbb{C}^d\) such that \(W_x\) is an invertible matrix weight satisfying (2.3) for every \(x \in X\) and \(\|W_x\|_{\text{op}}, \|W_x^{-1}\|_{\text{op}} \in L^1_{\text{loc}}(X, d, \mu)\). A subset \(\mathcal{F}\) of \(L^p(\rho, \mu)\) is totally bounded if the following conditions hold:

(a) \(\mathcal{F}\) is bounded, i.e., \(\sup_{f \in \mathcal{F}} \|f\|_{L^p(\rho, \mu)} < \infty\);

(b) \(\mathcal{F}\) uniformly vanishes at infinity, that is, for some \(x_0 \in X\),

\[\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \|f\chi_{X \setminus B(x_0, R)}\|_{L^p(\rho, \mu)} = 0;\]

(c) \(\mathcal{F}\) is equicontinuous, that is,

\[\lim_{r \to 0} \sup_{f \in \mathcal{F}} \|S_rf - f\|_{L^p(\rho, \mu)} = 0.\]

Here, \(S_r\) denotes the average operator:

\[S_rf(x) := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f(y) \, d\mu(y).\]

**Proof.** Assume that \(\mathcal{F} \subset L^p(\rho, \mu)\) satisfies (a)–(c). Given \(\epsilon > 0\), to prove the total boundedness of \(\mathcal{F}\), it suffices to find a finite \(\epsilon\)-net of \(\mathcal{F}\). By condition (b), there exists \(R > 0\) such that

\[\sup_{f \in \mathcal{F}} \|f - f\chi_{B(x_0, R)}\|_{L^p(\rho, \mu)} < \frac{\epsilon}{3}. \tag{4.1}\]

Moreover, by condition (c), there exists \(r \in (0, R)\) such that

\[\sup_{f \in \mathcal{F}} \|S_rf - f\|_{L^p(\rho, \mu)} < \frac{\epsilon}{3}. \tag{4.2}\]

Then by (4.1) and (4.2), we have

\[\sup_{f \in \mathcal{F}} \|(S_rf)\chi_{B(x_0, R)} - f\|_{L^p(\rho, \mu)} \leq \sup_{f \in \mathcal{F}} \|(S_rf)\chi_{B(x_0, R)} - f\chi_{B(x_0, R)}\|_{L^p(\rho, \mu)} + \sup_{f \in \mathcal{F}} \|f - f\chi_{B(x_0, R)}\|_{L^p(\rho, \mu)} < \frac{2\epsilon}{3}.\]

So we only need to show that \(\{(S_rf)\chi_{B(x_0, R)}\}_{f \in \mathcal{F}}\) has a finite \(\frac{\epsilon}{3}\)-net. Next, we turn to verify that \(\{S_rf\}_{f \in \mathcal{F}}\) is pointwise bounded and equicontinuous on \(\bar{B}(x_0, R)\), where

\(\bar{B}(x_0, R) := \{y \in X : d(x_0, y) \leq R\}\).
is a closed bounded subset of $X$, and hence is compact by Definition 4.1. From Lemma 2.2, we have the following estimate for any $f \in L^p(\rho, \mu)$,

$$
\int_{B(x_0,2R)} |f(y)|d\mu(y) = \int_{B(x_0,2R)} |W_{y}^{-1}W_{y} f(y)|d\mu(y) \leq \int_{B(x_0,2R)} \|W_{y}^{-1}\|_{op} |W_{y} f(y)|d\mu(y)
$$

$$
\leq \left( \int_{B(x_0,2R)} \|W_{y}^{-1}\|_{op}^{p}d\mu(y) \right)^{\frac{1}{p}} \left( \int_{B(x_0,2R)} |W_{y} f(y)|^{p}d\mu(y) \right)^{\frac{1}{p}}
$$

$$
\approx \left( \int_{B(x_0,2R)} \|W_{y}^{-1}\|_{op}^{p}d\mu(y) \right)^{\frac{1}{p}} \left( \int_{B(x_0,2R)} \rho_{y}(f(y))^{p}d\mu(y) \right)^{\frac{1}{p}}
$$

$$
\leq \left( \int_{B(x_0,2R)} \|W_{y}^{-1}\|_{op}^{p}d\mu(y) \right)^{\frac{1}{p}} \|f\|_{L^{p}(\rho, \mu)}. \tag{4.3}
$$

It follows that for any $f \in \mathcal{F}$ and any fixed $x \in \bar{B}(x_0, R)$,

$$
|S_{r}f(x)| \lesssim \frac{1}{\mu[B(x, r)]} \int_{B(x, r)} |f(y)|d\mu(y)
$$

$$
\approx \frac{1}{\mu[B(x, r)]} \int_{B(x_0,2R)} |f(y)|d\mu(y)
$$

$$
\approx \sup_{f \in \mathcal{F}} \frac{\|f\|_{L^{p}(\rho, \mu)}}{\mu[B(x, r)]} \left( \int_{B(x_0,2R)} \|W_{y}^{-1}\|_{op}^{p}d\mu(y) \right)^{\frac{1}{p}}, \tag{4.4}
$$

where we use the fact that $B(x, r) \subset B(x_0, 2R)$ when $x \in \bar{B}(x_0, R)$. Since $\|W_{y}^{-1}\|_{op}^{p} \in L^{1}_{loc}(X, d, \mu)$, then by condition (a), we obtain that $\{S_{r}f\}_{f \in \mathcal{F}}$ is pointwise bounded on $\bar{B}(x_0, R)$.

Furthermore, for any $f \in \mathcal{F}$ and any fixed $x \in B(x_0, R)$, by (4.4), we have the following estimate that for any $y \in B(x_0, R)$,

$$
|S_{r}f(y) - S_{r}f(x)|
$$

$$
\leq \left| \frac{1}{\mu[B(y, r)]} \int_{B(y, r)} f(z)d\mu(z) - \frac{1}{\mu[B(x, r)]} \int_{B(x, r)} f(z)d\mu(z) \right|
$$

$$
+ \left| \frac{1}{\mu[B(y, r)]} \int_{B(y, r)} f(z)d\mu(z) - \frac{1}{\mu[B(x, r)]} \int_{B(x, r)} f(z)d\mu(z) \right|
$$

$$
\lesssim \frac{\mu[B(x, r)\Delta B(y, r)]}{\mu[B(x, r)]\mu[B(y, r)]} \int_{B(y, r)} |f(z)|d\mu(z)
$$

$$
+ \frac{1}{\mu[B(x, r)]} \int_{B(x, r)\Delta B(y, r)} |f(z)|d\mu(z)
$$

$$
\lesssim \frac{\mu[B(x, r)\Delta B(y, r)]}{\mu[B(x, r)]\mu[B(y, r)]} \int_{B(x_0,2R)} |f(z)|d\mu(z)
$$

$$
\lesssim \frac{\mu[B(x, r)\Delta B(y, r)]}{\mu[B(x, r)]\mu[B(y, r)]} \int_{B(x_0,2R)} |f(z)|d\mu(z)
$$
which finishes the proof of Theorem 4.1.

\[
+ \frac{1}{\mu[B(x,r)]} \int_{B(x,r) \Delta B(y,r)} |f(z)|d\mu(z)
\]
\[
\leq \frac{\mu[B(x,r) \Delta B(y,r)]}{\mu[B(x,r) \mu[B(y,r)]} \sup_{f \in \mathcal{F}} \|f\|_{L^p(\rho, \mu)} \left( \int_{B(x,2R)} |W_{\chi}^{-1}|_{op}^p d\mu(z) \right)^{\frac{1}{p}}
\]
\[
+ \frac{1}{\mu[B(x,r)]} \int_{B(x,r) \Delta B(y,r)} |f(z)|d\mu(z).
\]

(4.5)

Since \( \mu \) is metrically continuous, then by (4.3), (4.4) and Lemma 4.1, for any \( 0 < h \leq r, x, y \in B(x_0, R) \) and \( f \in \mathcal{F} \), a direct calculation yields that

\[
\int_{B(x,r) \Delta B(y,r)} |f(z)|d\mu(z)
\]
\[
\leq \int_{B(x,r) \Delta B(y,r)} |f(z) - S_h f(z)|d\mu(z) + \int_{B(x,r) \Delta B(y,r)} |S_h f(z)|d\mu(z)
\]
\[
\leq \int_{B(x_0,2R)} |f(z) - S_h f(z)|d\mu(z) + \int_{B(x,r) \Delta B(y,r)} |S_h f(z)|d\mu(z)
\]
\[
\leq \left( \int_{B(x_0,2R)} |W_{\chi}^{-1}|_{op}^p d\mu(z) \right)^{\frac{1}{p}} \sup_{f \in \mathcal{F}} \|S_h f - f\|_{L^p(\rho, \mu)}
\]
\[
+ \frac{\mu[B(x,r) \Delta B(y,r)]}{\inf_{z \in B(x_0,2R) \mu[B(z,h)]} \left( \int_{B(x_0,3R)} |W_{\chi}^{-1}|_{op}^p d\mu(z) \right)^{\frac{1}{p}} \sup_{f \in \mathcal{F}} \|f\|_{L^p(\rho, \mu)},
\]

(4.6)

where we use the fact that \( B(z,h) \subset B(x_0,3R) \) when \( z \in B(x_0,2R) \), and implicit constants depend only on \( d \). Then by the arbitrariness of \( h \), condition (c), (4.5) and (4.6), we obtain that \( \{S_{\rho}f\}_{f \in \mathcal{F}} \) is equicontinuous on \( B(x_0, R) \).

So from Lemma 4.2, we further conclude that \( \{S_{\rho}f\}_{f \in \mathcal{F}} \) is totally bounded in \( C(B(x_0,r), \mathbb{C}^d) \). It follows that there exists \( \{f_k\}_{k=1}^N \subset \mathcal{F} \) such that \( \{S_{\rho}f_k\}_{k=1}^N \) is an \( \epsilon \)-net of \( \{S_{\rho}f\}_{f \in \mathcal{F}} \) for given \( \epsilon \), where \( A := 3\left( \int_{B(x_0,R)} |W_{\chi}^{-1}|_{op}^p d\mu(x) \right)^{\frac{1}{p}} \).

Hereafter, we shall show that \( \{(S_{\rho}f_k)\chi_{B(x_0,R)}\}_{k=1}^N \) is exactly a finite \( \frac{\epsilon}{3} \)-net of \( \{(S_{\rho}f)\chi_{B(x_0,R)}\}_{f \in \mathcal{F}} \) in \( L^p(\rho, \mu) \). Note that by Lemma 2.2,

\[
\|(S_{\rho}f)\chi_{B(x_0,R)} - (S_{\rho}f_k)\chi_{B(x_0,R)}\|_{L^p(\rho, \mu)}
\]
\[
\leq \left( \int_{B(x_0,R)} |W_x(S_{\rho}f(x) - S_{\rho}f_k(x))|^p d\mu(x) \right)^{\frac{1}{p}}
\]
\[
\leq \left( \int_{B(x_0,R)} |W_x|_{op}^p |S_{\rho}f(x) - S_{\rho}f_k(x)|^p d\mu(x) \right)^{\frac{1}{p}}
\]
\[
\leq \left( \int_{B(x_0,R)} |W_x|_{op}^p d\mu(x) \right)^{\frac{1}{p}} \sup_{x \in B(x_0,R)} |S_{\rho}f(x) - S_{\rho}f_k(x)| < \frac{\epsilon}{3},
\]

which finishes the proof of Theorem 4.1. \( \square \)
Remark 4.1. (i) By using the same argument as in Theorem 4.1 with some
minor changes, one can prove that Theorem 4.1 also holds for \( p = 1 \) under the
additional assumptions that \( W_x \) is an invertible matrix weight satisfying (2.3)
for every \( x \in X \) and \( \| W_x^{-1} \|_{op} \in L^\infty_{loc}(X, d, \mu) \).

(ii) The assumption on \( \rho_x \) in Theorem 4.1 is necessary for our method,
which relies on the structure of Banach function spaces (see [20, Definition 2.1]).
Moreover, Tsuji’s method is invalid here to relax the range of the exponent
\( p \in (0, \infty) \).

As an application, we obtain the following compactness criterion in matrix
weighted Lebesgue spaces on \((\mathbb{R}^n, | \cdot |, \mu)\) with a metrically continuous
measure \( \mu \), by applying Theorem 4.1 with \( \rho_x(\cdot) := |W_{\frac{1}{p}}(x) \cdot | \). This extends the
corresponding results of [9, Theorem 5], [39, Lemma 4.1] and [7, Proposition
2.9].

Corollary 4.1. Let \( 1 \leq p < \infty \), \((\mathbb{R}^n, | \cdot |, \mu)\) be the Euclidean metric measure
space with a metrically continuous measure \( \mu \). Assume that \( W \) is an invertible
matrix weight satisfying

\[
\text{(i) } \| W^{-1} \|_{op} \in L^\infty_{loc}(\mathbb{R}^n, | \cdot |, \mu) \text{ when } p = 1;
\]
\[
\text{(ii) } \| W^{-1} \|_{op} \in L^1_{loc}(\mathbb{R}^n, | \cdot |, \mu) \text{ when } p \in (1, \infty).
\]

Define

\[
\| f \|^p_{L^p(W, \mu)} := \int_{\mathbb{R}^n} |W_{\frac{1}{p}}(x)f(x)|^p d\mu(x).
\]

A subset \( \mathcal{F} \) of \( L^p(W, \mu) \) is totally bounded if the following conditions are valid:

(a) \( \mathcal{F} \) is bounded, i.e., \( \sup_{f \in \mathcal{F}} \| f \|^p_{L^p(W, \mu)} < \infty \);

(b) \( \mathcal{F} \) uniformly vanishes at infinity, that is,

\[
\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \| f \chi(B(0,R)^c) \|^p_{L^p(W, \mu)} = 0;
\]

(c) \( \mathcal{F} \) is equicontinuous, that is,

\[
\lim_{r \to 0} \sup_{f \in \mathcal{F}} \| S_rf - f \|^p_{L^p(W, \mu)} = 0.
\]

Clearly, by Definition 2.1, if \( W \) is a matrix \( A_p \) weight for \( p \in (1, \infty) \), then
\( W \) satisfies the assumption in Corollary 4.1 on \((\mathbb{R}^n, | \cdot |, m)\).
5 A Characterization of Compactness on $\mathbb{R}^n$

In this section, we give a necessary and sufficient condition for total boundedness of subsets in matrix weighted Lebesgue spaces. Before that, we need some lemmas.

For any $0 < r < \infty$, let $S_r$ be the average operator on $\mathbb{R}^n$ defined by

$$S_r f(x) := \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy, \quad \forall f \in L^p(W).$$

Then we have the following useful lemma.

**Lemma 5.1.** Let $1 < p < \infty$. If $W \in A_p$, then

$$\|S_r f\|_{L^p(W)} \lesssim \|f\|_{L^p(W)}, \quad \forall f \in L^p(W),$$

where the implicit constant depends only on $p$ and $d$.

**Proof.** Note that for any $f \in L^p(W)$ and $0 < r < \infty$,

$$\left| \frac{W^\frac{1}{p}}{|B(x, r)|} \int_{B(x, r)} f(y) dy \right| \lesssim \frac{1}{|B(x, r)|} \int_{B(x, r)} \left| W^\frac{1}{p}(x) f(y) \right| dy$$

$$\approx \frac{1}{|B(x, r)|} \int_{B(x, r)} \left| W^\frac{1}{p}(x) W^{-\frac{1}{p}}(y) W^\frac{1}{p}(y) f(y) \right| dy$$

$$\lesssim M_W (W^\frac{1}{p} f)(x). \quad (5.1)$$

Then from (5.1) and Lemma 2.4, it follows that

$$\|S_r f\|_{L^p(W)} \lesssim M_W \left( W^\frac{1}{p} f \right)_{L^p(\mathbb{R}^n)} \lesssim \|W^\frac{1}{p} f\|_{L^p(\mathbb{R}^n, \mathbb{C}^d)} \approx \|f\|_{L^p(W)}.$$ 

This completes the proof of Lemma 5.1. \qed

The following is a vector-valued extension of the Lebesgue differentiation theorem in the setting of matrix weights.

**Lemma 5.2.** Let $1 < p < \infty$. If $W \in A_p$, then for any $f \in L^p(W)$,

$$\lim_{r \to 0} |S_r f(x) - f(x)| = 0 \quad \text{a.e. } x \in \mathbb{R}^n.$$

**Proof.** First, for any $f := (f_1, \ldots, f_d)^T \in L^p(W)$, it suffices to show that $f_i \in L^1_{\text{loc}}(\mathbb{R}^n)$ for each $1 \leq i \leq d$. Since $W \in A_p$, then by (2.2),

$$|f|^p = |W^{-\frac{1}{p}} W^\frac{1}{p} f|^p \leq \|W^{-\frac{1}{p}}\|_{\text{op}}^p |W^\frac{1}{p} f|^p = \|W^{-1}\|_{\text{op}} |W^\frac{1}{p} f|^p.$$ 

It follows that

$$|f|^p \|W^{-1}\|_{\text{op}}^{-1} \leq |W^\frac{1}{p} f|^p.$$
From Lemma 2.3, we conclude that $\|W^{-1}\|_{op}^{-1}$ is a scalar $A_p$ weight, and hence $|f| \in L^1_{loc}(\mathbb{R}^n)$, which implies that $f_i \in L^1_{loc}(\mathbb{R}^n)$ for each $1 \leq i \leq d$.

Hence, by the classical Lebesgue differentiation theorem, for each $1 \leq i \leq d$, we have

$$\lim_{r \to 0} |S_rf_i(x) - f_i(x)| = 0 \quad \text{a.e. } x \in \mathbb{R}^n.$$ 

Lemma 5.2 then follows from the fact that for any $x \in \mathbb{R}^n$,

$$|S_r f(x) - f(x)| \leq d^{\frac{1}{2}} \max_{1 \leq i \leq d} |S_rf_i(x) - f_i(x)|.$$

We now present a characterization for total boundedness of subsets in $L^p(W)$ when $W \in A_p$.

**Theorem 5.1.** Let $1 < p < \infty$, $W \in A_p$. A subset $\mathcal{F}$ of $L^p(W)$ is totally bounded if and only if the following conditions hold:

1. $\mathcal{F}$ is bounded, i.e., $\sup_{f \in \mathcal{F}} \|f\|_{L^p(W)} < \infty$;
2. $\mathcal{F}$ uniformly vanishes at infinity, that is,

$$\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \|f\chi_{(B(0,R))}^\epsilon\|_{L^p(W)} = 0;$$
3. $\mathcal{F}$ is equicontinuous, that is,

$$\lim_{r \to 0} \sup_{f \in \mathcal{F}} \|S_rf - f\|_{L^p(W)} = 0.$$

**Proof.** The sufficiency is due to Corollary 4.1. We now prove the necessity. Assume that $\mathcal{F} \subset L^p(W)$ is totally bounded. Then for any given $\epsilon > 0$, there exists $\{f_k\}_{k=1}^N \subset \mathcal{F}$ such that $\{f_k\}_{k=1}^N$ is an $\epsilon$-net of $\mathcal{F}$, which implies that for any $f \in \mathcal{F}$, there exists $f_k$ such that $\|f - f_k\|_{L^p(W)} < \epsilon$.

Clearly, (a) is true. As for (b), for each $1 \leq k \leq N$, since $f_k \in L^p(W)$, by the monotone convergence theorem, there exists $R_k > 0$ such that

$$\|f_k\chi_{(B(0,R_k))}^\epsilon\|_{L^p(W)} < \epsilon.$$

Set $R := \max\{R_k : 1 \leq k \leq N\}$. It follows that for given $f \in \mathcal{F}$,

$$\|f\chi_{(B(0,R))}^\epsilon\|_{L^p(W)} \leq \|f - f_k\|_{L^p(W)} + \|f_k\chi_{(B(0,R))}^\epsilon\|_{L^p(W)} < 2\epsilon,$$

which implies (b).

As for (c), for each $1 \leq k \leq N$, by $W \in A_p$, Lemma 2.4 and (5.1),

$$|W^{\frac{1}{p}}(S_rf_k(x) - f_k(x))| \leq |W^{\frac{1}{p}}(x)S_rf_k(x)| + |W^{\frac{1}{p}}(x)f_k(x)|$$

$$\leq M_W(W^{\frac{1}{p}}f_k(x) + |W^{\frac{1}{p}}(x)f_k(x)| \in L^p(\mathbb{R}^n).$$
Then by Lemma 5.2 and Lebesgue’s dominated convergence theorem, there exists \( r > 0 \) such that for any \( h \leq r \),
\[
\max_{1 \leq k \leq N} \| S_h f_k - f_k \|_{L^p(W)} < \epsilon.
\]
From Lemma 5.1, it follows that
\[
\| S_h f - f \|_{L^p(W)} \leq \| S_h f - S_h f_k \|_{L^p(W)} + \| S_h f_k - f_k \|_{L^p(W)} + \| f_k - f \|_{L^p(W)}
\approx \| f_k - f \|_{L^p(W)} + \| S_h f_k - f_k \|_{L^p(W)} \lesssim \epsilon,
\]
where implicit constants depend only on \( p \) and \( d \). This implies (c) and completes the proof of Theorem 5.1.

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