TORUS ACTIONS AND COMBINATORICS OF POLYTOPES

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Abstract. An n-dimensional polytope \( P^n \) is called simple if exactly n codimension-one faces meet at each vertex. The lattice of faces of a simple polytope \( P^n \) with \( m \) codimension-one faces defines an arrangement of even-dimensional planes in \( \mathbb{R}^{2m} \). We construct a free action of the group \( \mathbb{R}^{m-n} \) on the complement of this arrangement. The corresponding quotient is a smooth manifold \( \mathcal{Z}_P \) invested with a canonical action of the compact torus \( T^m \) with the orbit space \( P^n \). For each smooth projective toric variety \( M^{2n} \) defined by a simple polytope \( P^n \) with the given lattice of faces there exists a subgroup \( T^{m-n} \subset T^m \) acting freely on \( \mathcal{Z}_P \) such that \( \mathcal{Z}_P/T^{m-n} = M^{2n} \). We calculate the cohomology ring of \( \mathcal{Z}_P \) and show that it is isomorphic to the cohomology ring of the face ring of \( P^n \) regarded as a module over the polynomial ring. In this way the cohomology of \( \mathcal{Z}_P \) acquires a bigraded algebra structure, and the additional grading allows to catch the combinatorial invariants of the polytope. At the same time this gives an example of explicit calculation of the cohomology ring for the complement of an arrangement of planes, which is of independent interest.

Introduction

In this paper we study relations between the algebraic topology of manifolds and the combinatorics of polytopes. Originally, this research was inspired by the results of the toric variety theory. The main object of our study is the smooth manifold defined by the combinatorial structure of a simple polytope. This manifold is equipped with a natural action of the compact torus \( T^m \).

We define an \( n \)-dimensional convex polytope as a bounded set in \( \mathbb{R}^n \) that is obtained as the intersection of a finite number of half-spaces. So, any convex polytope is bounded by a finite number of hyperplanes. A convex \( n \)-dimensional polytope is called simple if there exactly \( n \) codimension-one faces (or facets) meet at each vertex. The bounding hyperplanes of a simple polytope are in general position at each vertex. A convex polytope can be also defined as the convex hull of a set of points in \( \mathbb{R}^n \). If these points are in general position, the resulting polytope is called simplicial, since all its faces are simplices. For each simple polytope there is defined the dual (or polar) simplicial polytope (see Definition 1.3). The boundary of a simplicial polytope defines a simplicial subdivision (triangulation) of a sphere.

We associate to each simple polytope \( P^n \) with \( m \) facets a smooth \( (m+n) \)-dimensional manifold \( \mathcal{Z}_P \) with a canonical action of the compact torus \( T^m \). A number of manifolds that play the important role in different aspects of topology, algebraic and symplectic geometry appear as special cases of the above manifolds \( \mathcal{Z}_P \) or as the quotients \( \mathcal{Z}_P/T^k \) for torus subgroups \( T^k \subset T^m \) acting on \( \mathcal{Z}_P \) freely.

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It turns out that no torus subgroup of the rank \( m > n \) can act on \( Z_P \) freely. We call the quotients of \( Z_P \) by tori of the maximal possible rank \( m - n \) quasitoric manifolds. The name refers to the fact that the important class of algebraic varieties known to algebraic geometers as toric manifolds fits the above picture. More precisely, one can use the above construction (i.e. the quotient of \( Z_P \) by a torus subgroup) to produce all smooth projective toric varieties (see [Da]), which we refer to as toric manifolds. The action of \( T^m \) on \( Z_P \) induces an action of \( T^n \) on the (quasi)toric manifold \( M^{2n} := Z_P/T^{m-n} \) which the same orbit space \( P^n \). However, there are combinatorial types of simple polytopes that can not be realized as the orbit space for a quasitoric (and also toric) manifold. This means exactly that for such combinatorial type \( P \) it is impossible to find a torus subgroup \( T^{m-n} \subset T^m \) of rank \( m - n \) that acts on the corresponding manifold \( Z_P \) freely. If the manifold \( Z_P \) defined by a combinatorial simple polytope \( P \) admits a free action of a torus subgroup of rank \( m - n \), then different such subgroups may produce different quasitoric manifolds over \( P^n \), and some of them may turn out to be toric manifolds. Our quasitoric manifolds originally appeared under the name “toric manifolds” in [DJ], where the authors developed combinatorial and topological methods for studying the corresponding torus actions. We use some results of [DJ] in our paper.

Our approach to constructing manifolds \( Z_P \) defined by simple polytopes is based on one construction from algebraic geometry, which was used in [Ba] for studying toric varieties. Namely, the lattice of faces of a simple polytope \( P^n \) defines a certain affine algebraic set \( U(P^n) \subset \mathbb{C}^m \) with an action of the algebraic torus \( (\mathbb{C}^*)^m \). This set \( U(P^n) \) is the complement of a certain arrangement of planes in \( \mathbb{C}^m \) defined by the combinatorics of \( P^n \). Toric manifolds arise when one can find a subgroup \( C \subset (\mathbb{C}^*)^m \) isomorphic to \( (\mathbb{C}^*)^{m-n} \) that acts on \( U(P^n) \) freely. The crucial fact in our approach is that it is always possible to find a subgroup \( R \subset (\mathbb{C}^*)^m \) isomorphic to \( \mathbb{R}^{m-n} \) and acting freely on \( U(P^n) \). In this case the quotient manifold is defined, which we refer to as the manifold defined by the simple polytope \( P^n \). There is a canonical action of the torus \( T^m \) on this manifold, namely, that induced by the standard action of \( T^m \) on \( \mathbb{C}^m \) by diagonal matrices. The another approach to construct \( Z_P \) was proposed in [DJ], where this manifold was defined as the quotient space \( Z_P = T^m \times P^n / \sim \) for a certain equivalence relation \( \sim \). We construct an equivariant embedding \( i_e \) of this manifold into \( U(P^n) \subset \mathbb{C}^m \) and show that for the above subgroup \( R \cong \mathbb{R}^{m-n} \) the composite map \( Z_P \to U(P^n) \to U(P^n)/R \) of the embedding and the orbit map is a homeomorphism. Hence, from the topological viewpoint, both approaches produce the same manifold. This is what we refer to as the manifold defined by simple polytope \( P^n \) and denote by \( Z_P \) throughout our paper.

The analysis of the above constructions shows that we can replace the \( m \)-dimensional complex space \( \mathbb{C}^m \cong (\mathbb{R}^2)^m \) by a space \( (\mathbb{R}^k)^m \) with arbitrary \( k \). Indeed, we may construct the open subset \( U(k)(P^n) \subset (\mathbb{R}^k)^m \) determined by the lattice of faces of \( P^n \) in the same way as in the case of \( \mathbb{C}^m \) (i.e. \( U(k)(P^n) \) is the complement of a certain set of planes, see Definition 2.8). The multiplicative group \( (\mathbb{R}_>)^m \cong \mathbb{R}^m \) acts on \( (\mathbb{R}^k)^m \) diagonally (i.e. as the product of \( m \) standard actions of \( \mathbb{R}_> \) on \( \mathbb{R}^k \) by dilations). For this action it is also possible to find a subgroup \( R \subset (\mathbb{R}_>)^m \) isomorphic to \( \mathbb{R}^{m-n} \) that acts on \( U(k)(P^n) \) freely. The corresponding quotient \( U(k)(P^n)/R \) is now of dimension \( (k-1)m + n \) and is invested with an action of the group \( O(k)^m \) (the product of \( m \) copies of the orthogonal group). This action is induced by the diagonal action of \( O(k)^m \) on \( (\mathbb{R}^k)^m \). In the case \( k = 2 \) the above considered action
of the torus $T^n$ on $U(P^n)/R$ is just the action of $SO(2)^n \subset O(2)^n$. In the case $k = 1$ we obtain for any simple polytope $P^n$ a smooth $n$-dimensional manifold $Z^n$ with an action of the group $(\mathbb{Z}/2)^n$ whose orbit space is $P^n$. This manifold is known as the universal Abelian cover of $P^n$ regarded as a right-angled Coxeter orbifold (or manifold with corners). The analogues of quasitoric manifolds in the case $k = 1$ are the so-called small covers. A small cover is a $n$-dimensional manifold $M^n$ with an action of $(\mathbb{Z}/2)^n$ whose orbit space is $P^n$. The name refers to the fact that any cover of $P^n$ by a manifold must have at least $2^n$ sheets. The case $k = 1$ was detailedly treated in [DJ], along with quasitoric manifolds. The other case of particular interest is $k = 4$, since $\mathbb{R}^4$ can be viewed as a one-dimensional quaternionic space. In this paper we study the case $k = 2$, which all the constructions below refer to.

One of our main goals here is to study relations between the combinatorics of simple polytopes and the topology of the above described manifolds. There is a well-known important algebraic invariant of a simple polytope: a graded ring $k(P)$ (here $k$ is any field), called the face ring (or the Stanley–Reisner ring), see [St]. This is the quotient of the polynomial ring $k[v_1, \ldots, v_m]$ by a homogeneous ideal determined by the lattice of faces of a polytope (see Definition 1.1). The cohomology modules $Tor_{k[v_1, \ldots, v_m]}^{-i}(k(P), k)$, $i > 0$, are of great interest to algebraic combinatorists. Some results on the corresponding Betti numbers $\beta^i(k(P)) = \dim_k Tor_{k[v_1, \ldots, v_m]}^{-i}(k(P), k)$ can be found in [St]. We show that the bigraded $k$-module $Tor_{k[v_1, \ldots, v_m]}(k(P), k)$ can be endowed with a bigraded $k$-algebra structure and its totalized graded algebra is isomorphic to the cohomology algebra of $Z_P$. Therefore, the cohomology of $Z_P$ possesses a canonical bigraded algebra structure. The proof of these facts uses the Eilenberg–Moore spectral sequence. This spectral sequence is usually applied in algebraic topology as a powerful tool for calculating the cohomology of homogeneous spaces for Lie group actions (see e.g., [Sm]). So, it was interesting for us to discover a quite different application of this spectral sequence. In our situation the $E_2$ term of the spectral sequence is exactly $Tor_{k[v_1, \ldots, v_m]}(k(P), k)$, and the spectral sequence collapses in the $E_2$ term. Using the Koszul complex as a resolution while calculating the $E_2$ term, we show that the above bigraded algebra is the cohomology algebra of a certain bigraded complex defined in purely combinatorial terms of the polytope $P^n$ (see Theorem 4.6). Therefore, our bigraded cohomology algebra of $Z_P$ contains all the combinatorial data of $P^n$. In particular, it turns out that the well-known Dehn–Sommerville equations for a simple polytope $P^n$ follow directly from the bigraded Poincaré duality for $Z_P$. Given the corresponding bigraded Betti numbers one can compute the numbers of faces of $P$ of fixed dimension (the so-called $f$-vector of the polytope). Many combinatorial results, such as the Upper Bound for the number of faces of a simple polytope, can be interpreted nicely in terms of the cohomology of the manifold $Z_P$.

Moreover, since the homotopy equivalence $Z_P \cong U(P^n)$ holds, our calculation of the cohomology is also applicable to the set $U(P^n)$. As it was mentioned above, $U(P^n)$ is the complement of an arrangement of planes in $\mathbb{C}^m$ defined by the combinatorics of $P^n$. Hence, here we have a special case of the well-known general problem of calculating the cohomology of the complement of an arrangement of planes. In [GM, part III] the corresponding Betti numbers were calculated in terms of the cohomology of a certain simplicial complex. In our case special properties
of the arrangement defined by a simple polytope allow to calculate the cohomology ring of the corresponding complement much more explicitly.

Problems considered here were discussed in the first author’s talk on the conference “Solitons, Geometry and Topology” devoted to the jubilee of our Teacher Sergey Novikov. A part of the results of this paper were announced in [BP].

1. Main constructions and definitions

1.1. Simple polytopes and their face rings. Let \( P^n \) be a simple polytope and let \( f_i \) be its number of codimension \((i+1)\) faces, \( 0 \leq i \leq n-1 \). We refer to the integer vector \((f_0, \ldots, f_{n-1})\) as the \( f \)-vector of \( P^n \). It is convenient to put \( f_{-1} = 1 \). Along with the \( f \)-vector we also consider the \( h \)-vector \((h_0, \ldots, h_n)\) whose components \( h_i \) are defined from the equation

\[
h_i = h_0 t^i + \ldots + h_{n-i} t^i + h_n = (t - 1)^n + f_0 (t - 1)^{n-1} + \ldots + f_{n-1}.
\]

Therefore, we have

\[
h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{k-i} f_{i-1}.
\]

We fix a commutative ring \( k \), which we refer to as the ground ring. A graded ring called face ring is associated to \( P^n \). More precisely, let \( \mathcal{F} = (F_1, \ldots, F_m) \) be the set of codimension-one faces of \( P^n \), \( m = f_0 \). Form the polynomial ring \( k[v_1, \ldots, v_m] \), where the \( v_i \) are regarded as indeterminates corresponding to the facets \( F_i \).

**Definition 1.1.** The face ring \( k(P) \) of a simple polytope \( P \) is defined to be the ring \( k[v_1, \ldots, v_m]/I \), where

\[
I = (v_{i_1} \ldots v_{i_s} : i_1 < i_2 < \ldots < i_s, F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_s} = \emptyset).
\]

Note that the face ring is determined only by a combinatorial type of \( P^n \) (i.e. by its lattice of faces).

In the literature (see [St]), the face ring is usually defined for simplicial complexes, as follows. Let \( K \) be a finite simplicial complex with the vertex set \( \{v_1, \ldots, v_m\} \). Form a polynomial ring \( k[v_1, \ldots, v_m] \) where the \( v_i \) are regarded as indeterminates.

**Definition 1.2.** The face ring \( k(K) \) of a simplicial complex \( K \) is the quotient ring \( k[v_1, \ldots, v_m]/I \), where

\[
I = (v_{i_1} \ldots v_{i_s} : i_1 < i_2 < \ldots < i_s, \{v_{i_1}, \ldots, v_{i_s}\} \text{ does not span a simplex in } K).
\]

We regard the indeterminates \( v_i \) in \( k[v_1, \ldots, v_m] \) as being of degree two; in this way \( k(K) \), as well as \( k(P) \), becomes a graded ring.

**Definition 1.3.** Given a convex polytope \( P^n \subset \mathbb{R}^n \), the dual (or polar) polytope \((P^n)^* \subset (\mathbb{R}^n)^*\) is defined as follows

\[
(P^n)^* = \{x' \in (\mathbb{R}^n)^* : \langle x', x \rangle \leq 1 \text{ for all } x \in P^n\}.
\]

It can be shown (see [Br]) that the above set is indeed a convex polytope. In the case of simple \( P^n \) the dual polytope \((P^n)^* \) is simplicial and its \( i \)-dimensional faces (simplices) are in one-to-one correspondence with the faces of \( P^n \) of codimension \( i+1 \). The boundary complex of \((P^n)^* \) defines a simplicial subdivision (triangulation) of a \((n-1)\)-dimensional sphere \( S^{n-1} \), which we denote \( K_P \). In this situation both Definitions 1.1 and 1.2 provide the same ring: \( k(P) = k(K_P) \). The face rings of
simple polytopes have very specific algebraic properties. To describe them we need some commutative algebra.

Now suppose that $k$ is a field and let $R$ be a graded algebra over $k$. Let $n$ be the maximal number of algebraically independent elements of $R$ (this number is known as the \textit{Krull dimension} of $R$, denoted $\text{Krull} R$). A sequence $(\lambda_1, \ldots, \lambda_k)$ of homogeneous elements of $R$ is called a \textit{regular sequence}, if $\lambda_{i+1}$ is not a zero divisor in $R/(\lambda_1, \ldots, \lambda_i)$ for each $i$ (in the other words, the multiplication by $\lambda_{i+1}$ is a monomorphism of $R/(\lambda_1, \ldots, \lambda_i)$ into itself). It can be proved that $(\lambda_1, \ldots, \lambda_k)$ is a regular sequence if and only if $\lambda_1, \ldots, \lambda_k$ are algebraically independent and $R$ is a free $k[\lambda_1, \ldots, \lambda_n]$-module. The notion of regular sequence is of great importance for algebraic topologists (see, for instance, [La], [Sm]). A sequence $(\lambda_1, \ldots, \lambda_n)$ of homogeneous elements of $R$ is called a \textit{homogeneous system of parameters} (hsop), if the Krull dimension of $R/(\lambda_1, \ldots, \lambda_n)$ is zero. The $k$-algebra $R$ is \textit{Cohen–Macaulay} if it admits a regular sequence $(\lambda_1, \ldots, \lambda_n)$ of $n = \text{Krull} R$ elements (which is then automatically a hsop). It follows from the above that $R$ is Cohen–Macaulay if and only if there exists a sequence $(\lambda_1, \ldots, \lambda_n)$ of algebraically independent homogeneous elements of $R$ such that $R$ is a finite-dimensional free $k[\lambda_1, \ldots, \lambda_n]$-module.

In our case the following statement holds (see [St]).

\textbf{Proposition 1.4.} The face ring $k(P^n)$ of a simple polytope $P^n$ is a Cohen–Macaulay ring. $\Box$

In what follows we need two successive generalizations of a simple polytope. As it was mentioned in the introduction, the bounding hyperplanes of a simple polytope are in general position at each vertex. First, we define a \textit{simple polyhedron} as a convex set in $\mathbb{R}^n$ (not necessarily bounded) obtained as the intersection of a finite number of half-spaces with the additional condition that no more than $(n+1)$ hyperplanes intersect in one point. The faces of a simple polyhedron are defined obviously; all of them are simple polyhedra as well. It is also possible to define the $(n-1)$-dimensional simplicial complex $K_P$ dual to the boundary of a simple polyhedron $P^n$. (And again the $i$-dimensional simplices of $K_P$ are in one-to-one correspondence with the faces of $P^n$ of codimension $i+1$.) However, the simplicial complex $K_P$ obtained in such way not necessarily defines a triangulation of a $(n-1)$-dimensional sphere $S^{n-1}$.

\textbf{Example 1.5.} The simple polyhedron

$$
\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0\},
$$

bounded by $n$ coordinate hyperplanes will appear many times throughout our paper. The simplicial complex dual to its boundary is a $(n-1)$-dimensional simplex $\Delta^{n-1}$.

We note, however, that not any $(n-1)$-dimensional simplicial complex can be obtained as the dual to a $n$-dimensional simple polyhedron. Because of this, we still need to generalize the notion of a simple polytope (and simple polyhedron). In this way we come to the notion of a \textit{simple polyhedral complex}. Informally, a simple polyhedral complex of dimension $n$ is “the dual to a general $(n-1)$-dimensional simplicial complex”. We take its construction from [DJ]. Let $K$ be a simplicial complex of dimension $n-1$ and let $K'$ be its barycentric subdivision. Hence, the vertices of $K'$ are simplices $\Delta$ of the complex $K$, and the simplices of $K'$ are sets $(\Delta_1, \Delta_2, \ldots, \Delta_k)$, $\Delta_i \in K$, such that $\Delta_1 \subset \Delta_2 \subset \ldots \subset \Delta_k$. For each simplex
Definition 1.8. Following Definition 1.8, the subcomplex of $K'$ consisting of all simplices of $K'$ of the form $\Delta = \Delta_0 \subset \Delta_1 \subset \ldots \subset \Delta_k$. If $\Delta$ is a $(k - 1)$-dimensional simplex, then we refer to $F_\Delta$ as a face of codimension $k$. Let $P_k$ be the cone over $K$. Then this $P_k$ together with its decomposition into “faces” $\{F_\Delta\}_{\Delta \in K}$ is said to be a simple polyhedral complex. Any simple polytope $P^n$ (as well as a simple polyhedron) can be obtained by applying this construction to the simplicial complex $K^{n-1}$ dual to the boundary $\partial P^n$.

1.2. The topological spaces defined by simple polytopes. Following [DJ], in this subsection we associate two topological spaces, $Z_P$ and $BT_P$, to any simple polyhedral complex $P$ (in particular, to any simple polytope).

Let $T^m = S^1 \times \ldots \times S^1$ be the $m$-dimensional compact torus. Let $F = (F_1, \ldots, F_m)$ denote, as before, the set of codimension-one faces of $P^n$ (or the vertex set of the dual simplicial complex $K^{n-1}$). We consider the lattice $\mathbb{Z}^m$ of one-parameter subgroups of $T^m$ and fix a one-to-one correspondence between the facets of $P^n$ and the elements of a basis $\{e_1, \ldots, e_m\}$ in $\mathbb{Z}^m$. Now we can define the canonical coordinate subgroups $T_{i_1, \ldots, i_k} \subset T^m$ as the tori corresponding to the coordinate subgroups of $\mathbb{Z}^m$ (i.e. to the subgroups spanned by basis vectors $e_{i_1}, \ldots, e_{i_k}$).

Definition 1.6. The equivalence relation $\sim$ on $T^m \times P^n$ is defined as follows

$$(g, p) \sim (h, q) \text{ iff } p = q \text{ and } g^{-1}h \in T_{i_1, \ldots, i_k},$$

where $p$ lies in the relative interior of the face $F_{i_1} \cap \ldots \cap F_{i_k}$.

We associate to any simple polytope $P^n$ a topological space $Z_P = (T^m \times P^n)/\sim$

As it follows from the definition, $\dim Z_P = m + n$ and the action of the torus $T^m$ on $T^m \times P^n$ descends to an action of $T^m$ on $Z_P$. In the case of simple polytopes, the orbit space for this action is a $n$-dimensional ball invested with the combinatorial structure of the polytope $P^n$ as described by the following proposition.

Proposition 1.7. Suppose that $P^n$ is a simple polytope. Then the action of $T^m$ on $Z_P$ has the following properties:

1. The isotropy subgroup of any point of $Z_P$ is a coordinate subgroup of $T^m$ of dimension $\leq n$.
2. The isotropy subgroups define the combinatorial structure of the polytope $P^n$ on the orbit space. More precisely, the interior of a codimension-$k$ face consists of orbits with the same $k$-dimensional isotropy subgroup. In particular, the action is free over the interior of the polytope.

Proof. This follows easily from the definition of $Z_P$. $\square$

Now we return to the general case of a simple polyhedral complex $P^n$. Let $ET^m$ be the contractible space of the universal principal $T^m$-bundle over $BT^m = (\mathbb{C}P^\infty)^m$. Applying the Borel construction to the $T^m$-space $Z_P$, we come to the following definition.

Definition 1.8. The space $BT_P$ is defined as

$$(3) \quad BT_P = ET^m \times_{T^m} Z_P.$$  

Hence, the $BT_P$ is the total space of the bundle (with the fibre $Z_P$) associated to the universal bundle via the action of $T^m$ on the $Z_P$. As it follows from the definition, the homotopy type of $BT_P$ is determined by a simple polyhedral complex $P^n$. 

1.3. Toric and quasitoric manifolds. In the previous subsection we defined for any simple polytope $P^n$ a space $Z_P$ with an action of $T^n$ and the combinatorial structure of $P^n$ in the orbit space (Proposition 1.7). As we shall see in the next section, this $Z_P$ turns out to be a smooth manifold. Another class of manifolds possessing the above properties is well known in algebraic geometry as toric manifolds (or non-singular projective toric varieties). Below we give a brief review of them. The detailed background material on this subject can be found in [Da], [Fu].

**Definition 1.9.** A toric variety is a normal algebraic variety $M$ containing the $n$-dimensional algebraic torus $(\mathbb{C}^*)^n$ as a Zariski open subvariety, with the additional condition that the diagonal action of $(\mathbb{C}^*)^n$ on itself extends to an action on the whole $M$ (so, the torus $(\mathbb{C}^*)^n$ is contained in $M$ as a dense orbit).

On any non-singular projective toric variety there exists a very ample line bundle whose zero cohomology (the space of global sections) is generated by the sections corresponding to the points with integer coordinates inside a certain simple polytope with vertices in the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Conversely, there is an algebraic construction which allows to produce a projective toric variety $M^{2n}$ of real dimension $2n$ starting from a simple polytope $P^n$ with vertices in $\mathbb{Z}^n$ (see, e.g., [Fu]). However, the resulting variety $M^{2n}$ is not necessarily non-singular. Namely, this construction gives us a non-singular variety if for each vertex of $P^n$ the normal covectors of $n$ facets meeting at this vertex form a basis of the dual lattice $(\mathbb{Z}^n)^*$. A toric variety is not uniquely determined by the combinatorial type of a polytope: it depends also on integer coordinates of vertices. Thus, a given combinatorial type of a simple polytope (i.e. a lattice of faces) gives rise to a number of toric varieties, one for each geometrical realization with integer vertices. Some of these toric varieties may be non-singular. However, there are combinatorial simple polytopes that produce only singular toric varieties. The corresponding examples will be discussed below.

The algebraic torus contains the compact torus $T^n \subset (\mathbb{C}^*)^n$, which acts on a toric manifold as well. It can be proved that all isotropy subgroups for this action are tori $T^k \subset T^n$ and the orbit space has the combinatorial structure of $P^n$, as described in the second part of Proposition 1.7 (here $P^n$ is the polytope defined by the toric manifold as described above). The action of $T^n$ on $M^{2n}$ is locally equivalent to the standard action of $T^n$ on $\mathbb{C}^n$ (by the diagonal matrices) in the following sense: every point $x \in M^{2n}$ lies in some $T^n$-invariant neighbourhood $U \subset M^{2n}$ which is $T^n$-equivariantly homeomorphic to a certain $(T^n$-invariant) open subset $V \subset \mathbb{C}^n$. Furthermore, there exists an explicit map $M^{2n} \to \mathbb{R}^n$ (the moment map), with image $P^n$ and $T^n$-orbits as fibres (see [Fu]). A toric manifold $M^{2n}$ (regarded as a smooth manifold) can be obtained as the quotient space $T^n \times P^n / \sim$ for some equivalence relation $\sim$ (see [DJ]; compare this with Definition 1.6 of the space $Z_P$).

Now, if we are interested only in topological and combinatorial properties, then we should not restrict ourselves to algebraic varieties; in this way, forgetting all the algebraic geometry of $M^{2n}$ and the action of the algebraic torus $(\mathbb{C}^*)^n$, we come to the following definition.

**Definition 1.10.** A quasitoric (or topologically toric) manifold over a simple polytope $P^n$ is a real orientable 2n-dimensional manifold $M^{2n}$ with an action of the compact torus $T^n$ that is locally isomorphic to the standard action of $T^n$ on $\mathbb{C}^n$ and whose orbit space has the combinatorial structure of $P^n$ (in the sense of the second part of Proposition 1.7).
Quasitoric manifolds were firstly introduced in [DJ] under the name “toric manifolds”. As it follows from the above discussion, all algebraic non-singular toric varieties are quasitoric manifolds as well. The converse is not true: the corresponding examples can be found in [DJ]. One of the most important result on quasitoric manifolds obtained there is the description of their cohomology rings. This result generalizes the well-known Danilov–Jurkiewicz theorem, which describes the cohomology ring of a non-singular projective toric variety (see [Da]). In section 3 we give a new proof of the result of [DJ] by means of the Eilenberg–Moore spectral sequence (see Theorem 3.3). In the rest of this subsection we describe briefly main constructions with quasitoric manifolds. The proofs can be found in [DJ].

Suppose $M^{2n}$ is a quasitoric manifold over a simple polytope $P^n$, and $\pi : M^{2n} \rightarrow P^n$ is the orbit map. Let $F_{n-1}$ be a codimension-one face of $P^n$; then for any $x \in \pi^{-1}(\text{int} F_{n-1})$ the isotropy subgroup at $x$ is an independent of the choice of $x$ rank-one subgroup $G_F \in T^n$. This subgroup is determined by a primitive vector $v \in \mathbb{Z}^n$. In this way we construct a function $\lambda$ from the set $F$ of codimension-one faces of $P^n$ to primitive vectors in $\mathbb{Z}^n$.

**Definition 1.11.** The defined above function $\lambda : F \rightarrow \mathbb{Z}^n$ is called the characteristic function of $M^{2n}$.

The characteristic function can be also viewed as a homomorphism $\lambda : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$, where $m = \#F = f_0$ and $\mathbb{Z}^m$ is the free $\mathbb{Z}$-module spanned by the elements of $F$.

It follows from the local equivalence of the torus action to the standard one that the characteristic function has the following property: if $F_{i_1}, \ldots, F_{i_n}$ are the codimension-one faces meeting at the same vertex, then $\lambda(F_{i_1}), \ldots, \lambda(F_{i_n})$ form an integer basis of $\mathbb{Z}^n$. For any function $\lambda : F \rightarrow \mathbb{Z}^n$ satisfying this condition there exists a quasitoric manifold $M^{2n}(\lambda)$ over $P^n$ with characteristic function $\lambda$, and $M^{2n}$ is determined by its characteristic function up to an equivariant homeomorphism. However, there are simple polytopes that do not admit any characteristic function. One of such examples is the dual to the so-called cyclic polytope $C_k^n$ for $k \geq 2^n$ (see [DJ]). This polytope can not be realized as the orbit space for a quasitoric (or toric) manifold.

Quasitoric manifolds over a simple polytope $P^n$ are connected with the spaces $Z_P$ and $B_T P$ (see the previous subsection) as follows. Since a quasitoric manifold $M^{2n}$ is a $T^n$-space, we can take the Borel construction $ET^n \times_{T^n} M^{2n}$. It turns out that all these spaces for the given combinatorial type $P^n$ are independent of $M^{2n}$ and have the homotopy type $B_T P$:

$$B_T P \simeq ET^n \times_{T^n} M^{2n}. \quad (4)$$

The relationship between the $T^m$-space $Z_P$ and quasitoric manifolds over $P^n$ is described by the following property: for each toric manifold $M^{2n}$ over $P^n$ the orbit map $Z_P \rightarrow P^n$ is decomposed as $Z_P \rightarrow M^{2n} \overset{\pi}{\longrightarrow} P^n$, where $Z_P \rightarrow M^{2n}$ is a principal $T^{m-n}$-bundle and $M^{2n} \overset{\pi}{\longrightarrow} P^n$ is the orbit map for $M^{2n}$. Therefore, quasitoric manifolds over the polytope $P^n$ correspond to rank $m-n$ subgroups of $T^m$ that act freely on $Z_P$; and each subgroup of this type produces a quasitoric manifold. As it follows from Proposition 1.7, subgroups of rank $\geq m-n$ can not act on $Z_P$ freely, so the action of $T^m$ on $Z_P$ is maximally free exactly when there is at least one quasitoric manifold over $P^n$. We will discuss this question in more details later (see subsection 4.3).
Theorem 1.12. Let $P$ be a simple polyhedral complex with $m$ codimension-one faces. The map $p^* : H^*(BT^m) \to H^*(BT^P)$ is an epimorphism, and after the identification $H^*(BT^m) \cong k[v_1, \ldots, v_m]$ it becomes the quotient epimorphism $k[v_1, \ldots, v_m] \to k(P)$, where $k(P)$ is the face ring. In particular, $H^*(BT^P) \cong k(P)$. □

Now let $M^{2n}$ be a quasitoric manifold over a simple polytope $P^n$ with the characteristic function $\lambda$. The characteristic function is obviously extended to a linear map $k^m \to k^n$. Consider the bundle $p_0 : BT^P \to BT^n$ with the fibre $M^{2n}$ (see (4)).

Theorem 1.13. The map $p_0^* : H^*(BT^n) \to H^*(BT^P)$ is a monomorphism and $p^* : H^2(BT^n) \to H^2(BT^P)$ coincides with $\lambda^* : k^n \to k^m$. Furthermore, after the identification $H^*(BT^n) \cong k[t_1, \ldots, t_n]$, the elements $\lambda_i = p_0^*(t_i) \in H^*(BT^P) \cong k(P)$ form a regular sequence of degree-two elements of $k(P)$. □

In particular, all the above constructions hold for (algebraic) toric manifolds. Toric manifolds are defined by simple polytopes $P^n \subset \mathbb{R}^n$ whose vertices have integer coordinates. As it follows from the above arguments, the value of the corresponding characteristic function on the facet $F_{n-1} \in F$ is its minimal integer normal (co)vector. All characteristic functions corresponding to (algebraic) toric manifolds can be obtained by this method.

2. Geometrical and homotopical properties of $Z_P$ and $BT^P$

2.1. A cubical subdivision of a simple polytope. In this subsection we suppose that $P^n$ is a simple $n$-dimensional polytope. Let $I^q$ denote the standard unit cube in $\mathbb{R}^q$:

$$I^q = \{(y_1, \ldots, y_q) \in \mathbb{R}^q : 0 \leq y_i \leq 1, i = 1, \ldots, q\}.$$

Definition 2.1. A $q$-dimensional cubical complex is a topological space $X$ presented as the union of homeomorphic images of $I^q$ (called cubes) in such a way that the intersection of any two cubes is a face of each.

Theorem 2.2. A simple polytope $P^n$ with $m = f_0$ facets is naturally a $n$-dimensional cubical complex $\mathcal{C}$ with $r = f_{n-1}$ cubes $I^m_v$ corresponding to the vertices $v \in P^n$. Furthermore, there is an embedding $i_P$ of $\mathcal{C}$ into the boundary of the standard $m$-dimensional cube $I^m$ that takes cubes of $\mathcal{C}$ to $n$-faces of $I^m$.

Proof. Let us choose a point in the relative interior of each face of $P^n$ (we also take all vertices and a point in the interior of the polytope). The resulting set $S$ of $1 + f_0 + f_1 + \ldots + f_{n-1}$ points will be the vertex set of the cubical complex $\mathcal{C}$. Since the polytope $P^n$ is simple, the number of $k$-faces meeting at each vertex is $\binom{n}{k}$, $0 \leq k \leq n$. Hence, for each vertex $v$ there is defined a $2^n$-element subset $S_v$ of $S$ consisting of the points chosen in the interiors of faces containing $v$ (including $v$ itself and the point in the interior of $P^n$). This set $S_v$ is said to be the vertex set of the cube $I^m_v \subset \mathcal{C}$ corresponding to $v$. The faces of $I^m_v$ are defined as follows. We take any two faces $F^k_1$ and $F^l_2$ of $P^n$ such that $v \in F^k_1 \subset F^l_2$, $0 \leq k = \dim F^k \leq l = \dim F^l \leq n$. Then there are $\binom{l-k}{i}$ faces $F^{k+i}$ of dimension $k+i$ such that $v \in F^k_i \subset F^{k+i} \subset F^l_2$, $0 \leq i \leq l-k$. Hence, there are $2^{l-k}$ faces “between” $F^k_i$ and $F^l_2$. The points inside these faces define a $2^{l-k}$-element subset of $S_v$, which is
said to be a vertex set of a \((l − k)\)-face \(I_{i_1, i_2}^{l-k}\) of the cube \(I^n\). Now, to finish the
definition of the cubical complex \(C\) we need only to check that the intersection of
any two cubes \(I^n_v, I^n_w\) is a face of each. To do this we take the minimal-dimension
face \(F^p \subset P^n\) that contains both vertices \(v\) and \(v'\) (clearly, there is only one such
face). Then it can be easily seen that \(I^n_v \cap I^n_w = I^n_{P^0, P^n}\) is the face of \(I^n_v\) and \(I^n_w\).

Now let us construct an embedding \(\mathcal{C} \hookrightarrow I^m\). First, we define the images of the
vertices of \(\mathcal{C}\), i.e. the images of the points of \(S\). To do this, we fix the numeration
of facets: \(F_1^{n-1}, \ldots, F_m^{n-1}\). Now, if a point of \(S\) lies inside the facet \(F_i^{n-1}\), then
we map it to the vertex \((1, \ldots, 1, 0, 1, \ldots, 1)\) of the cube \(I^n\), where 0 stands on
the \(i\)th place. If a point of \(S\) lies inside a face \(F^{n-k}\) of codimension \(k\), then we
write \(F^{n-k} = F_1^{n-1} \cap \ldots \cap F_{i_k}^{n-1}\), and map this point to the vertex of \(I^n\) whose
coordinates are zero and all other coordinates are 1. The point of \(S\) in
the interior of \(P^n\) maps to the vertex of \(I^m\) with coordinates \((1, \ldots, 1)\). Hence, we
constructed the map from the set \(S\) to the vertex set of \(I^m\). This map obviously
extends to a map from the cubical subdivision \(\mathcal{C}\) of \(P^n\) to the standard cubical
subdivision of \(I^m\). One of the ways to do this is as follows. Take a simplicial
subdivision \(K\) of \(P^n\) with vertex set \(S\) such that for each vertex \(v \in P^n\) there exists
a simplicial subcomplex \(K_v \subset K\) with vertex set \(S_v\) that subdivides the cube \(I^n_v\).
The simplest way to construct such a simplicial complex is to view \(P^n\) as the cone
over the barycentric subdivision of the complex \(K_{P-1}^{n-1}\) dual to the boundary \(\partial P\).
Then the subcomplexes \(K_v\) are just the cones over the barycentric subdivisions of the
\((n-1)\)-simplices of \(K_{P-1}^{n-1}\). Now we can extend the map \(S \hookrightarrow I^m\) linearly on
each simplex of the triangulation \(K\) to the embedding \(i_P : P^n \hookrightarrow I^m\) (which is
therefore a piecewise linear map). Figure 1 illustrates this embedding for \(n = 2, m = 3\).

In short, the above constructed embedding \(i_P : P^n \hookrightarrow I^m\) is determined by the
following property:

The cube \(I^n_v \subset P^n\) corresponding to a vertex \(v = F_1^{n-1} \cap \ldots \cap F_{i_1}^{n-1}\) is
\(5\) mapped onto the \(n\)-face of \(I^m\) determined by \(m - n\) equations \(y_j = 1, j \notin \{i_1, \ldots, i_n\}\).
Thus, all cubes of $C$ map to faces of $I^n$, which proves the assertion.

Lemma 2.3. The number $c_k$ of $k$-cubes in the cubical subdivision $C$ of a simple polytope $P^n$ is given by the formula

$$c_k = \sum_{i=0}^{n-k} f_{n-i-1} \binom{n-i}{k} = f_{n-1} \binom{n}{k} + f_{n-2} \binom{n-1}{k} + \ldots + f_{k-1},$$

where $(f_0, \ldots, f_{n-1})$ is the $f$-vector of $P^n$ and $f_{-1} = 1$.

Proof. This follows from the fact that $k$-cubes of $C$ are in one-to-one correspondence with pairs $(F_i, F_{i+k})$ of faces of $P^n$ such that $F_i \subset F_{i+k}$ (see the proof of Theorem 2.2).

2.2. $Z_P$ as a smooth manifold and an equivariant embedding of $Z_P$ in $C^n$.

Let us consider the standard polydisc $(D^2)^m \subset \mathbb{C}^m$:

$$(D^2)^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| \leq 1\}.$$ 

This $(D^2)^m$ is a $T^m$-stable subset of $\mathbb{C}^m$ (with respect to the standard action of $T^m$ on $\mathbb{C}^m$ by diagonal matrices). The corresponding orbit space is $I^m$. The main result of this subsection is the following theorem.

Theorem 2.4. For any simple polytope $P^n$ with $m$ facets the space $Z_P$ has the canonical structure of a smooth $(m+n)$-dimensional manifold for which the $T^m$-action is smooth. Furthermore, there exists a $T^m$-equivariant embedding $i_e : Z_P \hookrightarrow (D^2)^m \subset \mathbb{C}^m$.

Proof. Theorem 2.2 shows that $P^n$ is presented as the union of $n$-cubes $I^n_i$ indexed by the vertices of $P^n$. Let $\rho : Z_P \to P^n$ be the orbit map. It follows easily from the definition of $Z_P$ that for each cube $I^n_i \subset Z_P$ we have $\rho^{-1}(I^n_i) \cong (D^2)^n \times T^{m-n}$, where $(D^2)^n$ is the polydisc in $\mathbb{C}^n$ with the diagonal action of $T^n$. Hence, $Z_P$ is presented as the union of “blocks” of the form $B_e \cong (D^2)^n \times T^{m-n}$. Gluing these “blocks” together smoothly along their boundaries we obtain a smooth structure on $Z_P$. Since each $B_e$ is $T^m$-invariant, the $T^m$-action on $Z_P$ is also smooth.

Now, let us prove the second part of the theorem. Recall our numeration of codimension-one faces of $P^n$: $F_1^{n-1}, \ldots, F_m^{n-1}$. Take the block

$$B_e \cong (D^2)^n \times T^{m-n} = D^2 \times \ldots \times D^2 \times S^1 \times \ldots \times S^1$$

corresponding to a vertex $v \in P^n$. Each factor $D^2$ and $S^1$ above corresponds to a codimension-one face of $P^n$ and therefore acquires a number (index) $i$, $1 \leq i \leq m$.

Note that $n$ factors $D^2$ acquire the indices of those facets containing $v$, while other indices are assigned to $m-n$ factors $S^1$. Now we numerate the factors $D^2 \subset (D^2)^m$ of the polydisc in any way and embed each block $B_e \subset Z_P$ into $(D^2)^m$ according to the indices of its factors. It can be easily seen the set of embeddings $B_e \hookrightarrow (D^2)^m$ define an equivariant embedding $Z_P \hookrightarrow (D^2)^m$.

Example 2.5. If $P^n = \Delta^1$ is a 1-dimensional simplex (a segment), then $B_v = D^2 \times S^1$ for each of the two vertices, and we obtain the well-known decomposition $Z_{\Delta^1} \cong S^3 = D^2 \times S^1 \cup D^2 \times S^1$. If $P^n = \Delta^n$ is an $n$-dimensional simplex, we obtain the similar decomposition of a $(2n+1)$-sphere into $n + 1$ “blocks” $(D^2)^n \times S^1$. 


Lemma 2.6. The equivariant embedding \( i_e : Z_P \hookrightarrow (D^2)^m \subset \mathbb{C}^m \) (see Theorem 2.4) covers the embedding \( i_P : P^n \hookrightarrow I^m \) (see Theorem 2.2) as described by the commutative diagram

\[
\begin{array}{ccc}
Z_P & \xrightarrow{i_e} & (D^2)^m \\
\downarrow & & \downarrow \\
P^n & \xrightarrow{i_P} & I^m
\end{array}
\]

where the vertical arrows denote the orbit maps for the corresponding \( T^m \)-actions.

Proof. It can be easily seen that an embedding of a face \( I^n \subset I^m \) defined by \( m - n \) equations of the type \( y_i = 1 \) (as in (5)) induces an equivariant embedding of \( (D^2)^n \times T^{m-n} \) into \( (D^2)^m \). Then our assertion follows from the representation of \( Z_P \) as the union of blocks \( B_e \cong (D^2)^n \times T^{m-n} \) and from property (5).

The above constructed embedding \( i_e : Z_P \hookrightarrow (D^2)^m \subset \mathbb{C}^m \) allows as to relate the manifold \( Z_P \) with one construction from the theory of toric varieties. Below we describe this construction, following [Ba].

We introduce the complex \( m \)-dimensional space \( \mathbb{C}^m \) with coordinates \( z_1, \ldots, z_m \).

Definition 2.7. A subset of facets \( \mathcal{P} = \{F_i_1, \ldots, F_i_p\} \subset F \) is called a \textit{primitive collection} if \( F_i_1 \cap \ldots \cap F_i_p = \emptyset \), while any proper subset of \( \mathcal{P} \) has non-empty intersection. In terms of the simplicial complex \( K_P \) dual to the boundary of \( P^n \), the vertex subset \( \mathcal{P} = \{v_i_1, \ldots, v_i_p\} \) is called a primitive collection if \( \{v_i_1, \ldots, v_i_p\} \) does not span a simplex, while any proper subset of \( \mathcal{P} \) spans a simplex of \( K_P \).

Now let \( \mathcal{P} = \{F_i_1, \ldots, F_i_p\} \) be a primitive collection of facets of \( P^n \). Denote by \( A(\mathcal{P}) \) the \( (m - p) \)-dimensional affine subspace in \( \mathbb{C}^m \) defined by the equations

\[ z_{i_1} = \ldots = z_{i_p} = 0. \]

Since every primitive collection has at least two elements, the codimension of \( A(\mathcal{P}) \) is at least 2.

Definition 2.8. The set of planes (or the arrangement of planes) \( A(P^n) \subset \mathbb{C}^m \) defined by the lattice of faces of a simple polytope \( P^n \) is

\[ A(P^n) = \bigcup_{\mathcal{P}} A(\mathcal{P}), \]

where the union is taken over all primitive collections of facets of \( P^n \). Put

\[ U(P^n) = \mathbb{C}^m \setminus A(P^n). \]

Note that we may define \( U(P^n) \) without using primitive collections: the same set is obtained if we take the complement in \( \mathbb{C}^m \) of the union of all planes (6) such that the facets \( F_i_1, \ldots, F_i_p \) have empty intersection. However, we shall use the notion of a primitive collection in the next sections. We note also that the open set \( U(P^n) \subset \mathbb{C}^m \) is invariant with respect to the action of \( (\mathbb{C}^*)^m \) on \( \mathbb{C}^m \).

It follows from property (5) that the image of \( Z_P \) under the embedding \( i_e : Z_P \hookrightarrow \mathbb{C}^m \) (see Theorem 2.4) does not intersect \( A(P^n) \), and therefore, \( i_e(Z_P) \subset U(P^n) \).

We put

\[ \mathbb{R}^m_+ = \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m : \alpha_i > 0\}. \]

This is a group with respect to multiplication, which acts by dilations on \( \mathbb{R}^m \) and \( \mathbb{C}^m \) (an element \( (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \) takes \( (y_1, \ldots, y_m) \in \mathbb{R}^m \) to \( (\alpha_1 y_1, \ldots, \alpha_m y_m) \)).
There is the isomorphism \( \exp : \mathbb{R}^m \to \mathbb{R}^m \) between the additive and the multiplicative group taking \((t_1, \ldots, t_m) \in \mathbb{R}^m \) to \((e^{t_1}, \ldots, e^{t_m}) \in \mathbb{R}^m \).

Now, by definition, the polytope \( P^n \) is a set of points \( x \in \mathbb{R}^n \) satisfying \( m \) linear inequalities:
\[
P^n = \{ x \in \mathbb{R}^n : (l_i, x) \geq -a_i, \ i = 1, \ldots, m \},
\]
where \( l_i \in (\mathbb{R}^n)^* \) are normal (co)vectors of facets. The set of \((\mu_1, \ldots, \mu_m) \in \mathbb{R}^m \) such that \( \mu_1 l_1 + \cdots + \mu_m l_m = 0 \) forms an \((m-n)\)-dimensional subspace in \( \mathbb{R}^m \). We choose a basis \( \{ w_i = (w_{i1}, \ldots, w_{im})^T \}, 1 \leq i \leq m-n \), in this subspace and form the \( m \times (m-n) \)-matrix
\[
W = \begin{pmatrix}
w_{11} & \cdots & w_{1,m-n} \\
\vdots & \ddots & \vdots \\
w_{m1} & \cdots & w_{m,m-n}
\end{pmatrix}
\]
of maximal rank \( m - n \). This matrix satisfies the following property.

**Proposition 2.9.** Suppose that \( n \) facets \( F_{i1}^{n-1}, \ldots, F_{in}^{n-1} \) of \( P^n \) meet at the same vertex \( v \); \( F_{i1}^{n-1} \cap \cdots \cap F_{in}^{n-1} = \emptyset \). Then the minor \((m-n) \times (m-n)\)-matrix \( W_{i1,\ldots,i_n} \) obtained from \( W \) by deleting \( n \) rows \( i_1, \ldots, i_n \) is non-degenerate: \( \det W_{i1,\ldots,i_n} \neq 0 \).

**Proof.** If \( \det W_{i1,\ldots,i_n} = 0 \), then one can find a zero non-trivial linear combination of vectors \( l_{i_1}, \ldots, l_{i_n} \). But this is impossible: since \( P^n \) is simple, the set of normal vectors of facets meeting at the same vertex constitute a basis of \( \mathbb{R}^n \).

The matrix \( W \) defines the subgroup
\[
R_W = \{ (e^{w_{i1} \tau_1 + \cdots + w_{1,m-n} \tau_{m-n}}, \ldots, e^{w_{im} \tau_1 + \cdots + w_{m,m-n} \tau_{m-n}}) \in \mathbb{R}^m_+ \} \subset \mathbb{R}^m_+,
\]
where \((\tau_1, \ldots, \tau_{m-n})\) runs over \( \mathbb{R}^{m-n} \). This subgroup is isomorphic to \( \mathbb{R}^{m-n}_+ \). Since \( U(P^n) \subset \mathbb{C}^m \) (see Definition 2.8) is invariant with respect to the action of \( \mathbb{R}^m_+ \subset (\mathbb{C}^*)^m \) on \( \mathbb{C}^m \), the subgroup \( R_W \subset \mathbb{R}^m_+ \) also acts on \( U(P^n) \).

**Theorem 2.10.** The subgroup \( R_W \subset \mathbb{R}^m_+ \) acts on \( U(P^n) \subset \mathbb{C}^m \) freely. The composite map \( \mathbb{Z}_P = U(P^n) \to U(P^n) / R_W \) of the embedding \( i_e \) and the orbit map is a homeomorphism.

**Proof.** A point from \( \mathbb{C}^m \) may have the non-trivial isotropy subgroup with respect to the action of \( \mathbb{R}^m_+ \) on \( \mathbb{C}^m \) only if at least one of its coordinate vanishes. As it follows from Definition 2.8, if a point \( x \in U(P^n) \) has some zero coordinates, then all of them correspond to facets of \( P^n \) having at least one common vertex \( v \in P^n \). Let \( v = F_{i1}^{n-1} \cap \cdots \cap F_{in}^{n-1} \). The point \( x \) has non-trivial isotropy subgroup with respect to the action of \( R_W \) only if some linear combination of vectors \( w_{i1}, \ldots, w_{m-n} \) lies in the coordinate subspace spanned by \( e_{i_1}, \ldots, e_{i_n} \). But this means that \( \det W_{i1,\ldots,i_n} = 0 \), which contradicts Proposition 2.9. Thus, \( R_W \) acts on \( U(P^n) \) freely.

Now, let us prove the second part of the theorem. Here we use both embeddings \( i_e : \mathbb{Z}_P \to (D^m)^m \subset \mathbb{C}^m \) from Theorem 2.4 and \( i_P : P^n \to I^m \subset \mathbb{R}^m \) from Theorem 2.2. It is sufficient to prove that each orbit of the action of \( R_W \) on \( U(P^n) \subset \mathbb{C}^m \) intersects the image \( i_e(\mathbb{Z}_P) \) in a single point. Since the embedding \( i_e \) is equivariant, instead of this we may prove that each orbit of the action of \( R_W \) on the real part \( U_{\mathbb{R}}(P^n) = U(P^n) \cap \mathbb{R}^m_+ \) intersects the image \( i_P(P^n) \) in a single point. Let \( y \in i_P(P^n) \subset \mathbb{R}^m \). Then \( y = (y_1, \ldots, y_m) \) lies in some \( n \)-face \( I^n_+ \) of the unit cube \( I^m \subset \mathbb{R}^m \) as described in (5). We need to show that the \((m-n)\)-dimensional subspace spanned by the vectors...
(w_{11}y_1, \ldots, w_m y_m)^\top, \ldots, (w_{1,m-n}y_1, \ldots, w_m, m-n y_m)^\top$ is in general position with the $n$-face $I_n^p$. But this follows directly from (5) and Proposition 2.9.

The above theorem gives a new proof of the fact that $Z_P$ is a smooth manifold, which is embedded in $\mathbb{C}^m \cong \mathbb{R}^{2m}$ with trivial normal bundle.

**Example 2.11.** Let $P^n = \Delta^n$ ($n$-simplex). Then $m = n + 1, U(P^n) = \mathbb{C}^{n+1} \setminus \{0\}$, $R^m_{<n}$ is $\mathbb{R}^n_>$, and $\alpha \in \mathbb{R}^n_>$ takes $z \in \mathbb{C}^{n+1}$ to $az$. Thus, we have $Z_P = S^{2n+1}$ (this could be also deduced from Definition 1.6; see also Example 2.5).

Now, suppose that all vertices of $P^n$ belong to the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Such integer simple polytope $P^n$ defines a projective toric variety $M_P$ (see [Fu]). Normal (co)\textit{vectors} $l_i$ of facets of $P^n$ (see (7)) can be taken integer and primitive. The toric variety $M_P$ defined by $P^n$ is smooth if for each vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ the vectors $l_{i_1}, \ldots, l_{i_n}$ constitute an integer basis of $\mathbb{Z}^n$. As before, we may construct the matrix $W$ (see (8)) and then define the subgroup

$$C_W = \{(e^{w_{11}n_1 + \ldots + w_{1,m-n}n_{m-n}}, \ldots, e^{w_{m1n_1 + \ldots + w_{m,m-n}n_{m-n}}}) \} \subset (\mathbb{C}^*)^m,$$

where $(n_1, \ldots, n_{m-n})$ runs over $\mathbb{C}^{m-n}$. This subgroup is isomorphic to $(\mathbb{C}^*)^{m-n}$. It can be shown (see [Ba]) that $C_W$ acts freely on $U(P^n)$ and the toric manifold $M_P$ is identified with the orbit space $U(P^n)/C_W$. Thus, we have the commutative diagram

$$\begin{array}{ccc}
U(P^n) & \xrightarrow{R_W \cong \mathbb{R}^{m-n}_>^m} & Z_P \\
\downarrow_{C_W \cong (\mathbb{C}^*)^{m-n}} & & \downarrow_{T^{m-n}} \\
M^{2n} & \cong & M^{2n}.
\end{array}$$

Since $Z_P$ can be viewed as the orbit space of $U(P^n)$ with respect to an action of $R_W \cong \mathbb{R}^{m-n}_>$, the manifold $Z_P$ and the complement of an arrangement of planes $U(P^n)$ have the same homotopy type. Hence, all results on the cohomology of $Z_P$ obtained in section 4 remain true if we substitute $U(P^n)$ for $Z_P$.

### 2.3. Homotopical properties of $Z_P$ and $BT_P$.

**Lemma 2.12.** Suppose that $P^n$ is the product of two simple polytopes, $P^n = P_{1}^{1} \times P_{2}^{2}$. Then $Z_P = Z_{P_{1}} \times Z_{P_{2}}$.

**Proof.** This follows directly from the definition of $Z_P$:

$$Z_P = (T^n \times P^n)/\sim = ((T^{m_1} \times P^{n_1})/\sim) \times ((T^{m_2} \times P^{n_2})/\sim) = Z_{P_{1}} \times Z_{P_{2}}.$$ 

The next lemma also follows easily from the construction of $Z_P$.

**Lemma 2.13.** If $P_{1}^{n_1} \subset P^n$ is a face of a simple polytope $P^n$, then $Z_{P_{1}}$ is a submanifold of $Z_P$. 

Below we invest the space $BT_P$ defined by a simple polyhedral complex with a canonical cell structure.

We use the standard cell decomposition of $BT_m = (\mathbb{C}P^\infty)^m$ (each $\mathbb{C}P^\infty$ has one cell in every even dimension). The corresponding cell cochain algebra is $C^*(BT_m) = H^*(BT_m) = k[v_1, \ldots, v_m].$
Theorem 2.14. The space $B_T P = ET^m \times_{T^m} \mathbb{Z}_P$ defined by a simple polyhedral complex $P$ can be viewed as a cell subcomplex of $B^m$. This subcomplex is the union of subcomplexes $B_T^k \times_{T^k} \mathbb{Z}_P$ over all simplices $\Delta = (i_1, \ldots, i_k)$ of the simplicial complex $K_P$ dual to the boundary of $\partial P$. In this realization we have $C^i(B_T P) = H^*(B_T P) = k(P)$, and the inclusion $i : B_T P \hookrightarrow B^m$ induces the quotient epimorphism $C^i(B_T^m) = k[v_1, \ldots, v_m] \to k(P) = C^i(B_T P)$ (here $k(P)$ is the face ring of $P$).

Proof. A simple polyhedral complex $P$ is defined as the cone over the barycentric subdivision of a simplicial complex $K$ with $m$ vertices. We construct a cell embedding $i : B_T P \hookrightarrow B^m$ by induction on the dimension of $K$. If $\dim K = 0$, then $K$ is a disjoint union of vertices $v_1, \ldots, v_m$ and $P$ is the cone on $K$. In this case $B_T P$ is a bouquet of $m$ copies of $\mathbb{C}P^\infty$ and we have the obvious inclusion $i : B_T P \to B^m = (\mathbb{C}P^\infty)^m$. In degree zero $C^i(B_T P)$ is just $k$, while in degrees $\geq 1$ it is isomorphic to $k[v_1] \oplus \cdots \oplus k[v_m]$. Therefore, $C^i(B_T P) = k[v_1, \ldots, v_m]/I$, where $I$ is the ideal generated by all square free monomials of degree $\geq 2$, and $i^*$ is the projection onto the quotient ring. Thus, the theorem holds for $\dim K = 0$.

Now let $\dim K = k - 1$. By the inductive hypothesis, the theorem holds for the simple polyhedral complex $P'$ corresponding to the $(k-2)$-skeleton $K'$ of $K$, i.e. $i^*C^i(B_T^m) = C^i(B_T P') = k(K') = k[v_1, \ldots, v_m]/I'$. We add $(k-1)$-simplices one at a time. Adding the simplex $\Delta_{k-1}$ on vertices $v_{i_1}, \ldots, v_{i_k}$ results in adding all elements of the subcomplex $B_T^{k-1} \times \cdots \times B_T^{k-1} = B_T^1 \times \cdots \times B_T^1 \subset B_T^m$ to $B_T P' \subset B^m$. Then $C^i(B_T P' \cup B_T^k) = k(K' \cup \Delta_{k-1}) = k[v_1, \ldots, v_m]/I$, where $I$ is generated by $I'$ and $v_{i_1} v_{i_2} \cdots v_{i_k}$. It is also clear that a map of the cochain (or cohomology) algebras induced by $i : B_T P \to B^m$ is the projection onto the quotient ring. $\square$

In particular, we see that for $K_P = \Delta^m$ (i.e. $P = \mathbb{R}_m^+$) one has $B_T P = B^m$.

Below we apply the above constructed cell decomposition of $B_T P$ for calculating some homotopy groups of $B_T P$ and $Z_P$.

A simple polytope (or a simple polyhedral complex) $P^m$ with $m$ codimension-one faces is called $q$-neighbourly [Br] if the $(q-1)$-skeleton of the dual simplicial complex $K_{P}^{m-1}$ coincides with the $(q-1)$-skeleton of a $(m-1)$-simplex (this just means that any $q$ codimension-one faces of $P^m$ have non-empty intersection). Note that any simple polytope is 1-neighbourly.

Theorem 2.15. For any simple polyhedral complex $P^m$ with $m$ codimension-one faces we have:

1. $\pi_1(Z_P) = \pi_1(B_T P) = 0$;
2. $\pi_2(Z_P) = 0$, $\pi_2(B_T P) = \mathbb{Z}_m$;
3. $\pi_q(Z_P) = \pi_q(B_T P)$ for $q \geq 3$;
4. If $P^m$ is $q$-neighbourly, then $\pi_i(Z_P) = 0$ for $i < 2q + 1$, and $\pi_{2q+1}(Z_P)$ is a free Abelian group with generators corresponding to square-free monomials $v_{i_1} \cdots v_{i_{q+1}} \in I$ (see Definition 1.1; these monomials correspond to primitive collections of $q+1$ facets).

Proof. The identities $\pi_1(B_T P) = 0$ and $\pi_2(B_T P) = \mathbb{Z}_m$ follow from the cell decomposition of $B_T P$ described in the previous theorem. In order to calculate $\pi_q(Z_P)$ and $\pi_2(Z_P)$ we consider the following fragment of the exact homotopy sequence for
the bundle $p : B_T P \to BT^m$ with the fibre $Z_P$:

$$
\begin{array}{ccccccccc}
\pi_3(BT^m) & \to & \pi_2(Z_P) & \to & \pi_2(B_T P) & \xrightarrow{p_*} & \pi_2(BT^m) & \to & \pi_1(Z_P) & \to & \pi_1(B_T P) \\
\| & & \| & & \| & & \| & & \| & & \|
\end{array}
$$

It follows from Theorem 2.14 that $p_*$ above is an isomorphism, and hence, $\pi_1(Z_P) = \pi_2(Z_P) = 0$. The third assertion of the theorem follows from the fragment

$$
\begin{array}{ccccccccc}
\pi_{q+1}(BT^m) & \to & \pi_{q}(Z_P) & \to & \pi_{q}(B_T P) & \to & \pi_{q}(BT^m),
\end{array}
$$

in which $\pi_{q}(BT^m) = \pi_{q+1}(BT^m) = 0$ for $q \geq 3$. Finally, the cell structure of $B_T P$ shows that if $P^m$ is $q$-neighbourly, then the $(2q + 1)$-skeleton of $B_T P$ coincides with the $(2q + 1)$-skeleton of $BT^m$. Thus, $\pi_k(B_T P) = \pi_k(BT^m)$ for $k < 2q + 1$. Now, the last assertion of the theorem follows from the third one and from Theorem 2.14.

The form of the homotopy groups of $Z_P$ and $B_T P$ enables us to make a hypothesis that $Z_P$ is a first killing space for $B_T P$, i.e. $Z_P = B_T P |_3$. In order to see this, let us consider the following commutative diagram of bundles:

$$
\begin{array}{ccccccccc}
Z_P \times ET^m & \longrightarrow & ET^m \\
\| & & \| & & \| & & \| & & \| & & \| & & \|
\end{array}
$$

(9)

Since $ET^m$ is contractible, $Z_P \times ET^m$ is homotopically equivalent to $Z_P$. On the other hand, since $BT^m = K(Z^m, 2)$ and $\pi_2(B_T P) = Z^m$, we see that $Z_P \times ET^m$ is a first killing space for $B_T P$ by definition. Thus, $Z_P$ has homotopy type of a first killing space $B_T P |_3$ for $B_T P$.

3. The Eilenberg–Moore spectral sequence.

In [EM] Eilenberg and Moore developed a spectral sequence, which turns out to be of great use in our considerations. In the description of this spectral sequence we follow [Sm].

Suppose that $\xi_0 = (E_0, p_0, B_0, F)$ is a Serre fibre bundle, $B_0$ is simply connected, and $f : B \to B_0$ is a continuous map. Then we can form the diagram

$$
\begin{array}{ccccccccc}
F & & F \\
\| & & \| & & \| & & \| & & \| & & \|
\end{array}
$$

(10)

$$
\begin{array}{ccccccccc}
E & \longrightarrow & E_0 \\
\| & & \| & & \| & & \| & & \| & & \|
\end{array}
$$

where $\xi = (E, p, B, F)$ is the induced fibre bundle. Under these assumptions the following theorem holds

**Theorem 3.1** (Eilenberg–Moore). There exists a spectral sequence of commutative algebras $\{E_r, d_r\}$ with

1. $E_1 \Rightarrow H^*(E)$ (the spectral sequence converges to the cohomology of $E$),
2. $E_2 = \text{Tor}_{H^*(B_0)}(H^*(B), H^*(E_0))$. □
The Eilenberg–Moore spectral sequence lives in the second quadrant and the differential $d_r$ has bidegree $(r, 1 - r)$. In the special case when $B = *$ is a point (hence, $E = F$ is the fibre of $\xi$) we have

**Corollary 3.2.** Let $F \hookrightarrow E \rightarrow B$ be a fibration over the simply connected space $B$. There exists a spectral sequence of commutative algebras $\{E_r, d_r\}$ with

1. $E_r \Rightarrow H^*(E)$,
2. $E_2 = \text{Tor}_{H^*(B)}(H^*(E), k)$. \hfill $\Box$

As the first application of the Eilenberg–Moore spectral sequence we calculate the cohomology ring of a quasitoric manifold $M^{2n}$ over a simple polytope $P^n$ (this was already done in [DJ] by means of other methods). Along with the ideal $I$ (see Definition 1.1) we define an ideal $J \subset k(P)$ as $J = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i$ are the elements of the face ring $k(P)$ defined by the characteristic function $\lambda$ of the manifold $M^{2n}$ (see Theorem 1.13). As it follows from Theorem 1.13, $\lambda_i = \lambda_{i1} v_1 + \lambda_{i2} v_2 + \ldots + \lambda_{im} v_m$ are algebraically independent elements of degree 2 in $k(P)$, and $k(P)$ is a finite-dimensional free $k[\lambda_1, \ldots, \lambda_n]$-module. The inverse image of the ideal $J$ under the projection $k[v_1, \ldots, v_m] \rightarrow k(P)$ is the ideal generated by $\lambda_i = \lambda_{i1} v_1 + \ldots + \lambda_{im} v_m$ regarded as elements of $k[v_1, \ldots, v_m]$. This inverse image will be also denoted by $J$.

**Theorem 3.3.** The following isomorphism of rings holds for any quasitoric manifold $M^{2n}$:

$$H^*(M^{2n}) \cong k(P)/J = k[v_1, \ldots, v_m]/I + J.$$  

**Proof.** Consider the Eilenberg–Moore spectral sequence of the fibration

$$\begin{array}{ccc}
M^{2n} & \longrightarrow & B_T P \\
\downarrow & & \downarrow p_0 \\
* & \longrightarrow & BT^n
\end{array}$$

Theorem 1.13 gives the monomorphism

$$H^*(B_T^n) = k[t_1, \ldots, t_n] \xrightarrow{p_0^*} H^*(B_T P) = k(P),$$

such that $\text{Im} p_0^* = k[\lambda_1, \ldots, \lambda_n] \subset k(P)$. The $E_2$ term of the Eilenberg–Moore spectral sequence is

$$E_2^{*, *} = \text{Tor}_{H^*(B_T^n)}^*(H^*(B_T P), k) = \text{Tor}_{k[\lambda_1, \ldots, \lambda_n]}^*(k(P), k).$$

The right-hand side above is a bigraded $k$-module (see [Ma], [Sm]). The first (“external”) grading arises from a projective resolution of $H^*(B_T P)$ as a $H^*(B_T^n)$-module used in the definition of the functor Tor. The second (“internal”) grading arises from the gradings of $H^*(B_T^n)$-modules which enter the resolution; we assume that non-zero elements appear only in even internal degrees (remember that $\deg \lambda_i = 2$). Since $k(P)$ is a free $k[\lambda_1, \ldots, \lambda_n]$-module, we have

$$\text{Tor}_{k[\lambda_1, \ldots, \lambda_n]}^*(k(P), k) = \text{Tor}_{k[\lambda_1, \ldots, \lambda_n]}^0(k(P), k) = k(P) \otimes_{k[\lambda_1, \ldots, \lambda_n]} k = k(P)/J.$$  

Therefore, $E_2^{*, *} = k(P)/J$ and $E_2^{p, *} = 0$ for $p > 0$. Thus, $E_2 = E_{\infty}$ and $H^*(M^{2n}) = k(P)/J$. \hfill $\Box$
As we have already mentioned, this theorem generalizes the well-known Danilov–Jurkiewicz theorem for the cohomology ring of a non-singular projective toric variety.

**Corollary 3.4.** \(H^*(M^{2n}) = \text{Tor}_k[\lambda_1, \ldots, \lambda_n](k(P), k).\) \(\square\)

4. **Calculation of the cohomology of \(Z_P\)**

In this section we use the Eilenberg–Moore spectral sequence for describing the cohomology ring of \(Z_P\) in terms of the face ring \(k(P)\). We also obtain some additional results about this cohomology in the case when at least one quasitoric manifold exists over the polytope \(P\). Throughout this section we assume that \(k\) is a field.

4.1. **Additive structure of the cohomology of \(Z_P\).** Here we consider the Eilenberg–Moore spectral sequence of an arbitrary commutative square (10) one has \(F^p F^q = \text{Tor}_k[\lambda_1, \ldots, \lambda_n](k(P), k)\), which we denote elements) of \(k(P)\).

\[E_2^{p,n} = F^{-p}H^n(Z_P)/F^{-p+1}H^n(Z_P).\]

**Proposition 4.1.** \(F^0 H^*(Z_P) = H^0(Z_P) = k\) (here \(k\) is the ground field).

**Proof.** It follows from [Sm, Proposition 4.2] that for the Eilenberg–Moore spectral sequence of an arbitrary commutative square (10) one has \(F^0 H^*(E) = \text{Im}\{H^*(B) \otimes H^*(E_0) \to H^*(E)\}\). In our case this gives \(F^0 H^*(Z_P) = \text{Im}\{H^*(B_T P) \to H^*(Z_P)\}\). Now, the proposition follows from the fact that the map \(p^* : H^*(BT^m) \to H^*(B_T P)\) is an epimorphism (see Theorem 1.12). \(\square\)

The \(E_2\) term of the Eilenberg–Moore spectral sequence of the bundle \(p : B_T P \to BT^m\) is \(E_2 = \text{Tor}_k[v_1, \ldots, v_m](k(P), k)\). Let us consider a free resolution of \(k(P)\) as a \(k[v_1, \ldots, v_m]\)-module:

\[0 \longrightarrow R^{-h} \overset{d_{-h}}{\longrightarrow} R^{-h+1} \overset{d_{-h+1}}{\longrightarrow} \cdots \longrightarrow R^{-1} \overset{d_{-1}}{\longrightarrow} R^0 \overset{d_0}{\longrightarrow} k(P) \longrightarrow 0.\]

It is convenient for our purposes to assume that \(R^i\) are numbered by non-positive integers, i.e. \(h > 0\) above.

The minimal number \(h\) for which a free resolution of the form (11) exists is called the *homological dimension* of \(k(P)\) and is denoted by \(\text{hd}_k[v_1, \ldots, v_m](k(P))\). By the Hilbert syzygy theorem, \(\text{hd}_k[v_1, \ldots, v_m](k(P)) \leq m\). At the same time, since \(k(P)\) is a Cohen–Macaulay ring, it is known [Se, Chapter IV] that

\[\text{hd}_k[v_1, \ldots, v_m](k(P)) = m - n,\]

where \(n\) is the Krull dimension (the maximal number of algebraically independent elements) of \(k(P)\). In our case \(n = \dim P\).

We shall use a special free resolution (11) known as the *minimal* resolution (see [Ad]), which is defined in the following way. Let \(A\) be a graded connected commutative algebra, and let \(R, R'\) be modules over \(A\). Set \(I(A) = \sum_{q > 0} A_q = \{a \in A : \deg a \neq 0\}\) and \(J(R) = I(A) \cdot R\). The map \(f : R \to R'\) is called *minimal*, if \(\text{Ker} f \subset J(R)\). The resolution (11) is called *minimal*, if all \(d_i\) are minimal. For constructing a minimal resolution we use a notion of a *minimal set of generators*. A minimal set of generators for a \(A\)-module \(R\) can be chosen by means of the following procedure. Let \(k_1\) be the lowest degree in which \(R\) is non-zero. Choose
a vector space basis in \((R)^k\), say \(x_1, \ldots, x_p\). Now let \(R_1 = (x_1, \ldots, x_p) \subset R\) be the submodule generated by \(x_1, \ldots, x_p\). If \(R = R_1\) then we are done. Otherwise, consider the first degree \(k_2\) in which \(R \neq R_1\); then in this degree we can choose a direct sum decomposition \(R = R_1 \oplus \hat{R}_1\). Now choose in \(\hat{R}_1\) a vector space basis \(x_{p_1+p+1}, \ldots, x_p\) and set \(R_2 = (x_1, \ldots, x_{p_1})\). If \(R = R_2\) we are done, if not just continue to repeat the above process until we obtain a minimal set of generators for \(R\). A minimal set of generators has the following property: no element \(x_k\) can be decomposed as \(x_k = \sum a_i x_i\) with \(a_i \in A\), \(\deg a_i \neq 0\). Now, we construct a minimal resolution (11) as follows. Take a minimal set of generators for \(k(P)\) and span by them a free \(k[v_1, \ldots, v_m]\)-module \(R^0\). Then take a minimal set of generators for \(\ker d^0\) and span by them a free module \(R^{-1}\), and so on. On the \(i\)th step we take a minimal set of homogeneous generators for \(\ker d^{-i+1}\) as the basis for \(R^{-i}\). A minimal resolution is unique up to an isomorphism.

Now let (11) be a minimal resolution of \(k(P)\) as a \(k[v_1, \ldots, v_m]\)-module. Then we have \(h = m - n\), and \(R^0\) is a free \(k[v_1, \ldots, v_m]\)-module with one generator of degree 0. The generator set of \(R^i\) consists of elements \(v_{i_1}, \ldots, v_{i_k}\) of degree \(2k\) such that \(\{v_{i_1}, \ldots, v_{i_k}\}\) does not span a simplex in \(K\), while any proper subset of \(\{v_{i_1}, \ldots, v_{i_k}\}\) spans a simplex in \(K\). This means exactly that the set \(\{v_{i_1}, \ldots, v_{i_k}\}\) is a primitive collection in the sense of Definition 2.7 (here the \(v_i\) are regarded as the vertices of the simplicial complex \(K^{n-1}\) dual to \(\partial P\)).

Now let (11) be a minimal resolution of \(k(P)\) as a \(k[v_1, \ldots, v_m]\)-module. Then we have \(h = m - n\), and \(R^0\) is a free \(k[v_1, \ldots, v_m]\)-module with one generator of degree 0. The generator set of \(R^i\) consists of elements \(v_{i_1}, \ldots, v_{i_k}\) of degree \(2k\) such that \(\{v_{i_1}, \ldots, v_{i_k}\}\) does not span a simplex in \(K\), while any proper subset of \(\{v_{i_1}, \ldots, v_{i_k}\}\) spans a simplex in \(K\). This means exactly that the set \(\{v_{i_1}, \ldots, v_{i_k}\}\) is a primitive collection in the sense of Definition 2.7 (here the \(v_i\) are regarded as the vertices of the simplicial complex \(K^{n-1}\) dual to \(\partial P\)).

Note that the \(k[v_1, \ldots, v_m]\)-module structure in \(k\) is defined by the homomorphism \(k[v_1, \ldots, v_m] \to k, v_i \to 0\). Since the resolution (11) is minimal, all the differentials \(d^i\) in the complex

\[
(12)\quad 0 \longrightarrow R^{-(m-n)} \otimes_{k[v_1, \ldots, v_m]} k \longrightarrow R^{-(m-n)-1} \otimes_{k[v_1, \ldots, v_m]} k \longrightarrow \cdots \longrightarrow R^{-1} \otimes_{k[v_1, \ldots, v_m]} k \longrightarrow R^0 \otimes_{k[v_1, \ldots, v_m]} k \longrightarrow 0
\]

are trivial. The module \(R^i \otimes_{k[v_1, \ldots, v_m]} k\) is a finite-dimensional vector space over \(k\); its dimension is equal to the dimension of \(R^i\) as a free \(k[v_1, \ldots, v_m]\)-module:

\[
\dim_k R^i \otimes_{k[v_1, \ldots, v_m]} k = \dim_{k[v_1, \ldots, v_m]} R^i.
\]

Therefore, since all the differentials in the complex (12) are trivial, the following equality holds for the minimal resolution (11):

\[
(13)\quad \dim_k \text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k) = \sum_{i=0}^{m-n} \dim_{k[v_1, \ldots, v_m]} R^{-i}.
\]

Now we are ready to describe the additive structure of the cohomology of \(\mathbb{Z}_p\).

**Theorem 4.2.** The following isomorphism of graded \(k\)-modules holds:

\[
H^*(\mathbb{Z}_p) \cong \text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k),
\]

where the right-hand side is regarded as the totalized one-graded module. More precisely, there is a filtration \(\{F^{-p}H^*(\mathbb{Z}_p)\}\) in \(H^*(\mathbb{Z}_p)\) such that

\[
F^{-p}H^*(\mathbb{Z}_p)/F^{-p+1}H^*(\mathbb{Z}_p) = \text{Tor}_{k[v_1, \ldots, v_m]}^{-p}(k(P), k).
\]

**Proof.** First, we show that

\[
(14)\quad \dim_k H^*(\mathbb{Z}_p) \geq \dim_k \text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k).
\]
To prove this inequality we construct an injective map from the union of generator sets of the free \( k[v_1, \ldots, v_m] \)-modules \( R^i \) (see (11)) to a generator set of \( H^*(Z_P) \) (over \( k \)).

Consider the Leray–Serre spectral sequence of the bundle \( p : B_T P \to B^m T \) with the fibre \( Z_P \). The first column of the \( E_2 \) term of this spectral sequence is the cohomology of the fibre: \( H^*(Z_P) = E_2^{0,*} \). By Theorem 1.12, non-zero elements can appear in the \( E_\infty \) term only in the bottom line; this bottom line is the ring \( k(P) = H^*(B_T P) \):

\[
E_\infty^{*,p} = 0, \quad p > 0; \quad E_\infty^{*,0} = k(P).
\]

Therefore, all elements from the kernel of the map \( d^0 : R^0 = k[v_1, \ldots, v_m] = E_2^{0,*} \to E_\infty^{0,*} = k(P) \) (see (11)) must be killed by the differentials of the spectral sequence. This kernel is just our ideal \( I \). Let \( (x_1, \ldots, x_p) \) be a minimal generator set of \( I \). Then we claim that the elements \( x_i \) can be killed only by the transgression (i.e. by differentials from the first column). Indeed, suppose that the converse is true, so that \( x \in \{x_1, \ldots, x_p\} \) is killed by a non-transgressive differential: \( x = d_k y \) for some \( k \), where \( y \) is not from the first column. Then \( y \) is sent to zero by all differentials up to \( d_{k-1} \). This \( y \) arises from some element \( \sum_i l_i a_i \) in the \( E_2 \) term, \( l_i \in E_2^{i,*}, a_i \in E_2^{0,*} = k[v_1, \ldots, v_m] \). Suppose that all the elements \( l_i \) are transgressive (i.e. \( d_j(l_i) = 0 \) for \( j < k \)), and let \( d_k(l_i) = m_i, m_i \in E_k^{0,*} \). Since all \( m_i \) are killed by differentials, their inverse images in \( E_2 \) belong to \( I \). Hence, we have \( x = d_k y = \sum_i m_i a_i, m_i \in I \), which contradicts the minimality of the basis \( (x_1, \ldots, x_p) \). Therefore, there are some non-transgressive elements among \( l_i \), i.e. there exists \( p < k \) and \( i \) such that \( d_p(l_i) = m_i \neq 0 \) (see Figure 2). Then \( m_i \) survives in \( E_p \) and we have \( d_p(y) = m_i a_i + \ldots \neq 0 \) — contradiction. This means that all the minimal generators of \( I \) are killed by the transgression, i.e. they correspond to some (different) generators \( l_i^{(1)} \in H^*(Z) \).

Since \( E_2 = H^*(Z_P) \otimes k[v_1, \ldots, v_m] \), a free \( k[v_1, \ldots, v_m] \)-module generated by the elements \( l_i^{(1)} \) is included into the \( E_2 \) term as a submodule. Therefore, we have \( R^{-1} \subset E_2 \) and the map \( d^{-1} : R^{-1} \to R^0 = k[v_1, \ldots, v_m] \) is defined by the differentials of the spectral sequence. The kernel of this map, \( \text{Ker} d^{-1} \), can not be killed by the already constructed differentials. Using the previous argument, we deduce that the elements of a minimal generator set for \( \text{Ker} d^{-1} \subset R^{-1} \) can be killed only by some elements from the first column, say \( l_i^{(2)}, \ldots, l_i^{(q)} \). Therefore, a free \( k[v_1, \ldots, v_m] \)-module generated by the elements \( l_i^{(2)} \) is also included into the
$E_2$ term as a submodule, i.e. $R^{-2} \subset E_2$. Proceeding with this procedure, at the end we obtain $\sum_{i=0}^{n-3} \dim_{k[v_1,\ldots,v_m]} R^{-1}$ generators in the first column of the $E_2$ term. Using (13), we deduce the required inequality (14).

Now let us consider the Eilenberg–Moore spectral sequence of the bundle $p : B_P \to BT^m$ with the fibre $Z_P$. This spectral sequence has $E_2 = \text{Tor}_{k[v_1,\ldots,v_m]}(k(P), k)$, and $E_\infty \Rightarrow \mathcal{H}^*(Z)$. It follows from inequality (14) that $E_2 = E_\infty$, which concludes the proof of the theorem.

Let us turn again to the Eilenberg–Moore filtration $\{F^-P H^*(Z_P)\}$ in $H^*(Z_P)$. The Poncaré duality defines a filtration $\{F^-P H_*(Z_P)\}$ in the homology of $Z_P$. It turns out that elements from $F^{-1}H_*(Z_P)$ have very transparent geometric realization. Namely, the following statement holds:

**Theorem 4.3.** Elements of $\{F^{-1}H_*(Z_P)\}$ can be realized as embedded submanifolds of $Z_P$. These submanifolds are spheres of odd dimensions for generators of $\{F^{-1}H_*(Z_P,Z)\}$

**Proof.** It follows from Theorem 4.2 that

$$F^{-1}H^*(Z_P) / F^0H^*(Z_P) = \text{Tor}^{-1}_{k[v_1,\ldots,v_m]}(k(P), k).$$

By Proposition 4.1, $F^0H^*(Z_P) = H^0(Z_P)$. Take a basis of $\text{Tor}^{-1}_{k[v_1,\ldots,v_m]}(k(P), k)$ consisting of elements $v_{i_1,\ldots,i_p}$ of internal degree $2p$ such that $v_{i_1,\ldots,i_p}$ are primitive collections of the vertices of $K_P$ (see the proof of Theorem 4.2). If $v_{i_1,\ldots,i_p}$ is a primitive collection, then the subcomplex of $K_P$ consisting of all simplices with vertices among $v_{i_1,\ldots,i_p}$ is a simplicial complex consisting of all faces of a simplex except one of the highest dimension (i.e. the boundary of a simplex). In terms of the simple polytope $P$ the element $v_{i_1,\ldots,i_p}$ corresponds to the set $\{F_{i_1},\ldots,F_{i_p}\}$ of codimension-one faces such that $F_{i_1} \cap \cdots \cap F_{i_p} = \emptyset$, though any proper subset of $\{F_{i_1},\ldots,F_{i_p}\}$ has a non-empty intersection. Note that the element $v_{i_1,\ldots,i_p} \in \text{Tor}^{-1}_{k[v_1,\ldots,v_m]}(k(P), k)$ defines, by means of the isomorphism from Theorem 4.2, an element of $H^*(Z_P)$ of dimension $2p - 1$. Now, we take one point inside each face $F_{i_1} \cap \cdots \cap F_{i_r} \cap \cdots \cap F_{i_p}$, $1 \leq r \leq p$, ($F_{i_r}$ is dropped); then we can embed the simplex $\Delta^{p-1}$ on these points into the polytope $P$ in such a way that the boundary $\partial \Delta^{p-1}$ embeds into $\partial P$. (Compare this with the construction of the cubical decomposition of $P$ in Theorem 2.2.) Let $\rho : Z_P = (T^m \times P^n)/\sim \to P^n$ be the projection onto the orbit space; then it can be easily seen that $\rho^{-1}(\Delta^{p-1}) = (T^p \times \Delta^{p-1})/\sim \times T^{m-p} = S^{2p-1} \times T^{m-p}$. In this way we obtain an embedding $S^{2p-1} \hookrightarrow Z_P$ that realize the element of $H_*(Z_P)$ dual to $v_{i_1,\ldots,i_p}$.

**4.2. Multiplicative structure of the cohomology of $Z_P$.** Here we describe the ring $H^*(Z_P)$.

The bigraded $k$-module $\text{Tor}_{k[v_1,\ldots,v_m]}(k(P), k)$ can be calculated either by means of a resolution of the face ring $k(P)$ or by means of a resolution of $k$. In the previous subsection we studied the minimal resolution of $k(P)$ as a $k[v_1,\ldots,v_m]$-module. Here we use another approach based on the Koszul resolution of $k$ as a $k[v_1,\ldots,v_m]$-module. This allows to invest the bigraded $k$-module $\text{Tor}_{k[v_1,\ldots,v_m]}(k(P), k)$ with a bigraded $k$-algebra structure. We show that the corresponding total graded $k$-algebra is isomorphic to the algebra $H^*(Z_P)$. This approach also gives us the description of $H^*(Z_P)$ as a cohomology algebra of some differential (bi)graded algebra.
Let $\Gamma = k[y_1, \ldots, y_n]$, $\deg y_i = 2$, be a graded polynomial algebra over $k$, and let $\Lambda[u_1, \ldots, u_n]$ denote an exterior algebra over $k$ on generators $u_1, \ldots, u_n$. Consider the bigraded differential algebra
\[ \mathcal{E} = \Gamma \otimes \Lambda[u_1, \ldots, u_n], \]
whose gradings and differential are defined by
\[
\begin{aligned}
\text{bideg}(y_i \otimes 1) &= (0, 2), & d(y_i \otimes 1) &= 0; \\
\text{bideg}(1 \otimes u_i) &= (-1, 2), & d(1 \otimes u_i) &= y_i \otimes 1,
\end{aligned}
\]
and requiring that $d$ be a derivation of algebras. The differential adds $(1, 0)$ to bidegree, hence, the components $\mathcal{E}^{-s, \cdot}$ form a cochain complex. This complex will be also denoted by $\mathcal{E}$. It is well known that this complex defines a $\Gamma$-free resolution of $k$ (regarded as a $\Gamma$-module) called the Koszul resolution (see [Ma]).

**Proposition 4.4.** Let $\Gamma = k[y_1, \ldots, y_n]$, and let $A$ be a $\Gamma$-module, then
\[
\text{Tor}_r(\Gamma, k) = H[A \otimes \Lambda[u_1, \ldots, u_n], d],
\]
where $d$ is defined as $d(a \otimes u_i) = (y_i \cdot a) \otimes 1$ for any $a \in A$.

**Proof.** Let us consider the introduced above $\Gamma$-free Koszul resolution $\mathcal{E} = \Gamma \otimes \Lambda[u_1, \ldots, u_n]$ of $k$. Then
\[
\text{Tor}_r(\Gamma, k) = H[A \otimes _{\Gamma} \Gamma \otimes \Lambda[u_1, \ldots, u_n], d] = H[A \otimes \Lambda[u_1, \ldots, u_n], d].
\]

Now let us consider the principal $T^m$-bundle $Z_P \times ET^m \to BT^m$ pulled back from the universal $T^m$-bundle by the map $p : BT^m \to BT^m$ (see (9)). The following lemma holds.

**Lemma 4.5.** The following isomorphism describes the $E_3^{(s)}$ term of the Leray–Serre spectral sequence $\{E_r^{(s)}, d_r\}$ of the bundle $Z_P \times ET^m \to BT^m$:
\[ E_3^{(s)} \cong \text{Tor}_{k[y_1, \ldots, y_m]}(k(P), k). \]

**Proof.** First, consider the $E_2^{(s)}$ term of the spectral sequence. Since $H^*(T^m) = \Lambda[u_1, \ldots, u_m]$, $H^*(BT^m) = k(P) = k[v_1, \ldots, v_m]/I$, we have
\[ E_2^{(s)} = k(P) \otimes \Lambda[u_1, \ldots, u_m]. \]

It can be easily seen that the differential $d_2^{(s)}$ acts as follows:
\[ d_2^{(s)}(1 \otimes u_i) = v_i \otimes 1, \quad d_2^{(s)}(v_i \otimes 1) = 0 \]
(see Figure 3). Now, since $E_3^{(s)} = H[E_2^{(s)}, d_2^{(s)}]$, our assertion follows from Proposition 4.4 by putting $\Gamma = k[v_1, \ldots, v_m]$, $A = k(P)$.

Now we are ready to prove our main result on the cohomology of $Z_P$.

**Theorem 4.6.** The following isomorphism of graded algebras holds:
\[ H^*(Z_P) \cong H[k(P) \otimes \Lambda[u_1, \ldots, u_m], d], \]
\[
\begin{aligned}
\text{bideg} v_i &= (0, 2), & \text{bideg} u_i &= (-1, 2), \\
\text{bideg} (1 \otimes u_i) &= v_i \otimes 1, & d(v_i \otimes 1) &= 0.
\end{aligned}
\]

Hence, the Leray–Serre spectral sequence of the $T^m$-bundle $Z_P \times ET^m \to BT^m$ collapses in the $E_3$ term.
Proof. Let us consider the bundle $p : BT \rightarrow BT^m$ with the fibre $Z_P$. It follows from Theorem 2.14 that the correspondent cochain algebras are $C^*(BT^m) = k[v_1, \ldots, v_m]$ and $C^*(BT) = k(P)$, and the action of $C^*(BT^m)$ on $C^*(BT)$ is defined by the quotient projection. It was shown in [Sm, Proposition 3.4] that there is an isomorphism of algebras

$$\theta^* : \text{Tor}_{C^*(BT^m)}(C^*(BT), k) \rightarrow H^*(Z_P).$$

Now, it follows from above arguments and Proposition 4.4 that

$$\text{Tor}_{C^*(BT^m)}(C^*(BT), k) \cong H[k(P) \otimes \Lambda[u_1, \ldots, u_m], d],$$

which concludes the proof. \hfill \Box

4.3. Cohomology of $Z_P$ and torus actions. First, we consider the case where the simple polytope $P^n$ can be realized as the orbit space for some quasitoric manifold (see subsection 1.3). We show that the existence of a quasitoric manifold enables us to reduce calculating of the cohomology of $Z_P$ to calculating the cohomology of an algebra, which is much smaller than that from Theorem 4.6.

As it was already discussed in section 1.3, a quasitoric manifold $M^{2n}$ over $P^n$ defines a principal $T^{m-n}$-bundle $Z_P \rightarrow M^{2n}$. This bundle is induced from the universal $T^{m-n}$-bundle by a certain map $f : M^{2n} \rightarrow BT^{m-n}$.

**Theorem 4.7.** Suppose $M^{2n}$ is a quasitoric manifold over a simple polytope $P^n$; then the Eilenberg–Moore spectral sequences of the commutative squares

$$
\begin{array}{ccc}
Z_P \times ET^m & \rightarrow & ET^m \\
\downarrow & & \downarrow \\
BT \rightarrow & & BT^m \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
Z_P & \rightarrow & ET^{m-n} \\
\downarrow & & \downarrow \\
M^{2n} \rightarrow & & BT^{m-n} \\
\end{array}
$$

are isomorphic.

Proof. Let $\{E_r, d_r\}$ be the Eilenberg–Moore spectral sequence of the first commutative square, and let $\{E_r, d_r\}$ be that of the second one. Then, as it follows from the results of [EM], [Sm], the inclusions $BT^{m-n} \rightarrow BT^m$, $ET^{m-n} \rightarrow ET^m$, $M^{2n} \rightarrow BT \rightarrow BT^m$, and $Z_P \rightarrow Z_P \times ET^m$ define a homomorphism of spectral sequences: $g : \{E_r, d_r\} \rightarrow \{E_r, d_r\}$. First, we prove that $g_2 : E_2 \rightarrow E_2$ is an isomorphism.

The map $H^*(BT^m) \rightarrow H^*(BT^{m-n})$ is the quotient projection $k[v_1, \ldots, v_m] \rightarrow k[w_1, \ldots, w_{m-n}]$ with the kernel $J = (\lambda_1, \ldots, \lambda_n)$. By Theorem 3.3, we have $H^*(M^{2n}) = k[v_1, \ldots, v_m] / I + J$. Hence, $f^* : H^*(BT^{m-n}) \rightarrow H^*(M^{2n}) / I + J = H^*(M^{2n})$ is the quotient epimorphism.

The $E_2$-terms of our spectral sequences are $E_2 = \text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k)$ and $E_2 = \text{Tor}_{k[w_1, \ldots, w_{m-n}]}(k(P), k)$.
To proceed further we need the following result.

**Proposition 4.8.** Let $\Lambda$ be an algebra and $\Gamma$ a subalgebra and set $\Omega = \Lambda/\Gamma$. Suppose that $\Lambda$ is a free $\Gamma$-module and we are given a right $\Omega$-module $A$ and a left $\Lambda$-module $C$. Then there exists a spectral sequence $\{E_r, d_r\}$ with

$$E_r \Rightarrow \text{Tor}_\Lambda(A, C), \quad E_2^{p,q} = \text{Tor}_\Omega^{p+q}(A, \text{Tor}_\Gamma^q(C, k)).$$

Proof. See [CE, p.349].

The next proposition is a modification of one assertion from [Sm].

**Proposition 4.9.** Suppose $f : k[v_1, \ldots, v_m] \to A$ is an epimorphism of graded algebras, deg $v_i = 2$, and $J \subset A$ is an ideal generated by a length $n$ regular sequence of degree-two elements of $A$. Then the following isomorphism holds:

$$\text{Tor}_k[v_1, \ldots, v_m](A, k) = \text{Tor}_k[w_1, \ldots, w_{m-n}](A/J, k).$$

Proof. Let $J = (\lambda_1, \ldots, \lambda_n)$, deg $\lambda_i = 2$, and $\{\lambda_1, \ldots, \lambda_n\}$ is a regular sequence. Let $\hat{\lambda}_i$, $1 \leq i \leq n$, be degree-two elements of $k[v_1, \ldots, v_m]$ such that $f(\hat{\lambda}_i) = \lambda_i$. Hence, $\hat{\lambda}_i = \lambda_{i1}v_1 + \cdots + \lambda_{im}v_m$ and $\text{rk}(\lambda_{ij}) = n$. Let us take elements $w_1, \ldots, w_{m-n}$ of degree two such that

$$k[v_1, \ldots, v_m] = k[\hat{\lambda}_1, \ldots, \hat{\lambda}_n, w_1, \ldots, w_{m-n}],$$

and put $\Gamma = k[\hat{\lambda}_1, \ldots, \hat{\lambda}_n]$. Then $k[v_1, \ldots, v_m]$ is a free $\Gamma$-module, and therefore, by Proposition 4.8, we have a spectral sequence

$$E_r \Rightarrow \text{Tor}_k[v_1, \ldots, v_m](A, k), \quad E_2 = \text{Tor}_\Omega(\text{Tor}_\Gamma(A, k), k),$$

where $\Omega = k[v_1, \ldots, v_m]/\Gamma = k[w_1, \ldots, w_{m-n}]$.

Since $\lambda_1, \ldots, \lambda_n$ is a regular sequence, $A$ is a free $\Gamma$-module. Therefore,

$$\text{Tor}_\Gamma(A, k) = A \otimes_k k = A/J \quad \text{and} \quad \text{Tor}_\Gamma^q(A, k) = 0 \quad \text{for} \quad q \neq 0,$$

$$\Rightarrow \quad E_2^{p,q} = 0 \quad \text{for} \quad q \neq 0,$$

$$\Rightarrow \quad \text{Tor}_k[v_1, \ldots, v_m](A, k) = \text{Tor}_k[w_1, \ldots, w_{m-n}](A/J, k),$$

which concludes the proof of the proposition.

Now, we return to the proof of Theorem 4.7. Setting $A = k(P)$ in Proposition 4.9 we deduce that $g_2 : E_2 \to \hat{E}_2$ is an isomorphism. The $E_2$ terms of both spectral sequences contain only finite number of non-zero modules. In this situation a homomorphism $g$ that defines an isomorphism in the $E_2$ terms is an isomorphism of the spectral sequences (see [Ma, XI, Theorem 1.1]). Thus, Theorem 4.7 is proved.

**Corollary 4.10.** Suppose that $M^{2n}$ is a quasitoric manifold over a simple polytope $P^n$. Then the cohomology of $Z_P$ can be calculated as

$$H^*(Z_P) = \text{Tor}_{k[w_1, \ldots, w_{m-n}]}(H^*(M^{2n}), k).$$

Proof. By Theorem 4.6, $H^*(Z_P) = \text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k)$. Hence, our assertion follows from the isomorphism between the $E_2$ terms of the spectral sequences from Theorem 4.7.

Let us turn again to the principal $T^{m-n}$-bundle $Z_P \to M^{2n}$ defined by a quasitoric manifold $M^{2n}$. The following lemma is analogous to Lemma 4.5 (and is proved similarly).
Lemma 4.11. The following isomorphism holds for the Leray–Serre spectral sequence of the bundle $Z_P \to M^{2n}$:

$$E_3^{(s)} \cong \Tor_k[w_1, \ldots, w_{m-n}](H^*(M^{2n}), k) = \Tor_k[w_1, \ldots, w_{m-n}](k(P)/J, k),$$

where $E_3^{(s)}$ is the $E_3$ term of the Leray–Serre spectral sequence, and $H^*(M^{2n}) \cong k(P)/J$ is invested with a $k[w_1, \ldots, w_{m-n}]$-module structure by means of the map

$$k[w_1, \ldots, w_{m-n}] = k[v_1, \ldots, v_m]/J \to k[v_1, \ldots, v_m]/I+J = H^*(M^{2n}). \qed$$

Theorem 4.12. Suppose $M^{2n}$ is a quasitoric manifold over $P^n$. Then the Leray–Serre spectral sequence of the principle $T^{m-n}$-bundle $Z_P \to M^{2n}$ collapses in the $E_3$ term, i.e. $E_3 = E_\infty$. Furthermore, the following isomorphism of algebras holds

$$H^*(Z_P) = H[(k(P)/J) \otimes \Lambda[u_1, \ldots, u_{m-n}], d],$$

where $a \in k(P)/J = k[w_1, \ldots, w_{m-n}]/I$ and $\Lambda[u_1, \ldots, u_{m-n}]$ is an exterior algebra.

Proof. The cohomology algebra $H[(k(P)/J) \otimes \Lambda[u_1, \ldots, u_{m-n}], d]$ is exactly the $E_3$ term of the Leray–Serre spectral sequence for the bundle $Z_P \to M^{2n}$. At the same time, it follows from Proposition 4.4 that this cohomology algebra is isomorphic to $\Tor_k[w_1, \ldots, w_{m-n}](H^*(M^{2n}), k)$. Corollary 4.10 shows that this is exactly $H^*(Z_P)$. Since the Leray–Serre spectral sequence converges to $H^*(Z_P)$, it follows that it collapses in the $E_3$ term. \qed

The algebra $(k(P)/J) \otimes \Lambda[u_1, \ldots, u_{m-n}]$ from Theorem 4.12 is significantly smaller than the algebra $k(P) \otimes \Lambda[u_1, \ldots, u_m]$ from general Theorem 4.6. This enables to calculate the cohomology of $Z_P$ more efficiently.

A rank $m - n$ torus subgroup of $T^m$ that acts freely on $Z_P$ gives rise to a quasitoric manifold $M^{2n} = Z_P/T^{m-n}$ with orbit space $P^n$. In the general case, such a subgroup may fail to exist; however, one still may be able to find a subgroup of dimension less than $m - n$ that acts freely on $Z_P$. So, suppose that a subgroup $H \cong T^r$ acts on $Z_P$ freely. Then the inclusion $s : H \hookrightarrow T^m$ is defined by an integer $(m \times r)$-matrix $S = (s_{ij})$ such that the $\Z$-module spanned by its columns $s_j = (s_{1j}, \ldots, s_{mj})^T$, $j = 1, \ldots, r$ is a direct summand in $\Z^m$. Choose any basis $t_i = (t_{i1}, \ldots, t_{im})$, $i = 1, \ldots, m - r$ in the kernel of the dual map $s^* : (\Z^m)^* \to (\Z^n)^*$. Then the cohomology ring of the quotient manifold $Y_{(r)} = Z_P/H$ is described by the following theorem, which generalize both Corollary 3.4 and Theorem 4.2.

Theorem 4.13. The following isomorphism of algebras holds:

$$H^*(Y_{(r)}) \cong \Tor_k[t_1, \ldots, t_{m-r}](k(P), k),$$

where the action of $k[t_1, \ldots, t_{m-r}]$ on $k(P) = k[v_1, \ldots, v_m]/I$ is defined by the map

$$k[t_1, \ldots, t_{m-r}] \to k[v_1, \ldots, v_m], \quad t_i \to t_{i1}v_1 + \ldots + t_{im}v_m.$$ 

Remark. Corollary 3.4 corresponds to the value $r = m - n$, while Theorem 4.2 corresponds to the value $r = 0$. 

Proof. The inclusion of the subgroup $H \cong T^r \to T^m$ defines a map of classifying spaces $h : BT^r \to BT^m$. Let us consider the bundle pulled back by this map from the bundle $p : BTP \to BT^m$ with the fibre $Z_P$. It follows directly from the construction of $BTP$ (see subsection 1.2) that the total space of this bundle has homotopy type $Y(r)$ (more precisely, it is homeomorphic to $Y(r) \times ET^r$). Hence, we have the commutative square

$$
\begin{array}{ccc}
Y(r) & \longrightarrow & BTP \\
\downarrow & & \downarrow \\
BT^r & \longrightarrow & BT^m.
\end{array}
$$

The corresponding Eilenberg–Moore spectral sequence converges to the cohomology of $Y(r)$ and has the following $E_2$ term:

$$E_2 = \text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k[w_1, \ldots, w_r]),$$

where the action of $k[v_1, \ldots, v_m]$ on $k[w_1, \ldots, w_r]$ is defined by the map $s^*$, i.e. $v_i \mapsto s_i w_1 + \ldots + s_r w_r$. Using [Sm, Proposition 3.4] in the similar way as in the proof of Theorem 4.6, we show that the spectral sequence collapses in the $E_2$ term and the following isomorphism of algebras holds:

$$H^*(Y(r)) = \text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k[w_1, \ldots, w_r]).$$

Now put $\Lambda = k[v_1, \ldots, v_m], \Gamma = k[t_1, \ldots, t_{m-r}], A = k[w_1, \ldots, w_r]$, and $C = k(P)$ in Proposition 4.8. Since $\Lambda$ here is a free $\Gamma$-module and $\Omega = \Lambda/\Gamma = k[w_1, \ldots, w_r]$, a spectral sequence $\{E_s, d_s\}$ arises. Its $E_2$ term is

$$E_2^{p,q} = \text{Tor}_{k[w_1, \ldots, w_r]}^p(A, \text{Tor}_{k[t_1, \ldots, t_{m-r}]}^q(k(P), k)),
$$

and it converges to $\text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k[w_1, \ldots, w_r])$. Since $A$ is a free module over $k[w_1, \ldots, w_r]$ with one generator 1, we have

$$E_2^{p,q} = 0 \text{ for } p \neq 0, \quad E_2^{0,q} = \text{Tor}_{k[t_1, \ldots, t_{m-r}]}(k(P), k).$$

Thus, the spectral sequence collapses in the $E_2$ term, and we have the isomorphism of algebras:

$$\text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k[w_1, \ldots, w_r]) \cong \text{Tor}_{k[t_1, \ldots, t_{m-r}]}(k(P), k),$$

which together with the isomorphism (15) proves the theorem.

Below we characterize subgroups $H \subset T^m$ that act on $Z_P$ freely.

Let us consider again the integer $(m \times r)$-matrix $S$ defining the subgroup $H \subset T^m$ of rank $r$. For each vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$ of the polytope $P^n$ denote by $S_{i_1, \ldots, i_n}$ the $(m - n) \times r$-submatrix of $S$ that is obtained by deleting the rows $i_1, \ldots, i_n$. In this way we construct $f_{n-1}$ submatrices of the size $(m - n) \times r$. Then the following criterion for the freeness of the action of $H$ on $Z_P$ holds.

**Lemma 4.14.** The action of the subgroup $H \subset T^m$ defined by an integer $(m \times r)$-matrix $S$ on the manifold $Z_P$ is free if and only if for any vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$ of $P^n$ the corresponding $(m - n) \times r$-submatrix $S_{i_1, \ldots, i_n}$ defines a direct summand $\mathbb{Z}^r \subset \mathbb{Z}^{m-n}$.

**Proof.** It follows from Definition 1.6 that the orbits of the action of $T^m$ on $Z_P$ corresponding to the vertices $v = F_{i_1} \cap \cdots \cap F_{i_n}$ of $P^n$ have maximal (rank $n$) isotropy subgroups. These isotropy subgroups are the coordinate subgroups $T_{i_1, \ldots, i_n}^n \subset T^m$. A subgroup $H$ acts freely on $Z_P$ if and only if it has only unit in the intersection
with each isotropy subgroup. This means that the \( m \times (r + n) \)-matrix obtained by adding \( n \) columns \( (0, \ldots, 0, 1, 0, \ldots, 0) ^\top \) (1 stands on the place \( i_j, j = 1, \ldots, n \) to \( S \) defines a direct summand \( \mathbb{Z}^{k+n} \subset \mathbb{Z}^m \). (This matrix corresponds to the subgroup \( H \times T_{i_1, \ldots, i_n}^n \subset T^m \).) Obviously, this is equivalent to the requirements of the lemma.

In particular, for subgroups of rank \( m - n \) we obtain

**Corollary 4.15.** The action of the rank \( m - n \) subgroup \( H \subset T^m \) defined by an integer \( m \times (m - n) \)-matrix \( S \) on the manifold \( \mathbb{Z}_P \) is free if and only if for any vertex \( v = F_{i_1} \cap \ldots \cap F_{i_n} \) of \( P^n \) the minor \( (m - n) \times (m - n) \)-matrix \( S_{i_1, \ldots, i_n} \) has det \( S_{i_1, \ldots, i_n} = \pm 1 \). □

**Remark.** Compare this with Proposition 2.9 and Theorem 2.10. Note that unlike the situation of Theorem 2.10, the subgroup \( H \cong T^{m-n} \) satisfying the condition of Corollary 4.15 may fail to exist.

The inclusion \( s: \mathbb{Z}^{m-n} \to \mathbb{Z}^m \) defines the short exact sequence

\[
0 \longrightarrow \mathbb{Z}^{m-n} \xrightarrow{s} \mathbb{Z}^m \longrightarrow \mathbb{Z}^n \longrightarrow 0.
\]

It can be easily seen that the condition from Corollary 4.15 is equivalent to the following: the map \( \mathbb{Z}^m \to \mathbb{Z}^n \) above is a characteristic function in the sense of Definition 1.11. Thus, we have obtained the another interpretation of the fact that quasitoric manifolds exist over \( P^n \) if and only if it is possible to find a subgroup \( H \cong T^{m-n} \) that acts on \( \mathbb{Z}_P \) freely.

As it follows from Lemma 4.14, the one-dimensional subgroup \( H \cong T^1 \) defined to the diagonal inclusion \( T^1 \subset T^m \) always acts on \( \mathbb{Z}_P \) freely. Indeed, in this situation the matrix \( S \) is a column of \( m \) units and the condition from Lemma 4.14 is obviously satisfied. Theorem 4.13 gives the following formula for the cohomology of the corresponding quotient manifold \( \mathcal{Y}_{(1)} = \mathbb{Z}_P/H \):

\[
(16) \quad H^*(\mathcal{Y}_{(1)}) \cong \text{Tor}_{k[t_1, \ldots, t_{m-1}]}(k(P), k),
\]

where the action of \( k[t_1, \ldots, t_{m-1}] \) on \( k(P) = k[v_1, \ldots, v_m] / I \) is defined by the homomorphism

\[
k[t_1, \ldots, t_{m-1}] \to k[v_1, \ldots, v_m],
\]

\[
t_i \to v_i - v_m.
\]

The principal \( T^1 \)-bundle \( \mathbb{Z}_P \to \mathcal{Y}_{(1)} \) is pulled back from the universal \( T^1 \)-bundle by a certain map \( c: \mathcal{Y}_{(1)} \to BT^1 = CP^\infty \). Since \( H^*(CP^\infty) = k[v], v \in H^2(CP^\infty) \), the element \( c^*(v) \in H^2(\mathcal{Y}_{(1)}) \) is defined. Then, the following statement holds.

**Lemma 4.16.** A polytope \( P^n \) is \( q \)-neighbourly if and only if \( (c^*(v))^q \neq 0 \).

**Proof.** The map \( c^* \) takes the cohomology ring \( k[v] \) of \( CP^\infty \) to the subring \( k(P) \otimes_{k[t_1, \ldots, t_{m-1}]} k = \text{Tor}_{k[t_1, \ldots, t_{m-1}]}^0(k(P), k) \) of the cohomology ring of \( \mathcal{Y}_{(1)} \) (see (16)). This subring is isomorphic to the quotient ring \( k(P)/(v_1 = \ldots = v_m) \). Now, the assertion follows from the fact that a polytope \( P^n \) is \( q \)-neighbourly if and only if the ideal \( I \) (see Definition 1.1) does not contain monomials of degree less than \( q + 1 \). □
Now we return to the general case of a subgroup $H \cong T^r$ acting on $Z_P$ freely. For such a subgroup we have

$$B_T P = Z_P \times_{T^m} ET^m = ((Z_P/T^r) \times_{T^{m-r}} ET^{m-r}) \times ET^r = (Y_r \times T^{m-r}) \times ET^r.$$  

Hence, there is defined a principal $T^{m-r}$-bundle $Y_r \times ET^m \to B_T P$.

**Theorem 4.17.** The Leray–Serre spectral sequence of the $T^{m-r}$-bundle $Y_r \times ET^m \to B_T P$ collapses in the $E_3$ term, i.e. $E_3 = E_\infty$. Furthermore,

$$H^*(Y_r) = H[k(P) \otimes \Lambda[u_1, \ldots, u_{m-r}], d],$$

$$d(1 \otimes u_i) = (t_{i1}v_1 + \ldots + t_{im}v_m) \otimes 1, \quad d(a \otimes 1) = 0;$$

$$\text{bideg } a = (0, \deg a), \quad \text{bideg } u_i = (-1, 2),$$

where $a \in k(P) = k[v_1, \ldots, v_m]/I$ and $\Lambda[u_1, \ldots, u_{m-r}]$ is an exterior algebra.

**Proof.** In the similar way as in Lemma 4.5 we show that the $E_3$ term of the spectral sequence is

$$E_3 = H[k(P) \otimes \Lambda[u_1, \ldots, u_{m-k}], d] = \text{Tor}_{k[t_1, \ldots, t_{m-r}]}(k(P), k).$$

Theorem 4.13 shows that this is exactly $H^*(Y_r)$. □

**Remark.** Theorem 4.6 and Corollary 3.4 can be obtained from this theorem by setting $r = 0$ and $r = m - n$ respectively.

### 4.4. Explicit calculation of $H^*(Z_P)$ for some particular polytopes.

1. Our first example demonstrates how the above methods work in the simple case when $P$ is a product of simplices. So, let $P^n = \Delta^i \times \Delta^{i2} \times \ldots \times \Delta^{ik}$, where $\Delta^i$ is an $i$-simplex and $\sum_k i_k = n$. This $P^n$ has $n+k$ facets, i.e. $m = n+k$. Lemma 2.12 shows that $Z_P = Z_{\Delta^i} \times \ldots \times Z_{\Delta^{ik}}$.

The minimal resolution (11) of $k(P_i)$ in the case $P_i = \Delta^i$ is as follows

$$0 \to R^{-1} \xrightarrow{d^{-1}} R^0 \xrightarrow{d^0} k(P_i) \to 0,$$

where $R^0, R^{-1}$ are free one-dimensional $k[v_1, \ldots, v_{i+1}]$-modules and $d^{-1}$ is the multiplication by $v_1 \cdots v_{i+1}$. Hence, we have the isomorphism of algebras

$$\text{Tor}_{k[v_1, \ldots, v_{i+1}]}(k(P_i), k) = \Lambda[a], \quad \text{bideg } a = (-1, 2i + 2),$$

where $\Lambda[a]$ is an exterior $k$-algebra on one generator $a$. Now, Theorem 4.6 shows that

$$H^*(Z_{\Delta^i}) = \Lambda[a], \quad \deg a = 2i + 1,$$

Thus, the cohomology of $Z_P = Z_{\Delta^i} \times \ldots \times Z_{\Delta^{ik}}$ is

$$H^*(Z_P) = \Lambda[a_1, \ldots, a_k], \quad \deg a_l = 2i_l + 1.$$  

Actually, Example 2.11 shows that our $Z_P$ is the product of spheres: $Z_P = S^{2i_1+1} \times \ldots \times S^{2i_k+1}$. However, our calculation of the cohomology does not use the geometrical constructions from section 2.

2. In our next example we consider plane polygons, i.e. the case $n = 2$. Let $P^2$ be a convex $m$-gon. Then the corresponding manifold $Z_P$ is of dimension $m + 2$. First, we compute the Betti numbers of these manifolds.

It can be easily seen that there is at least one quasitoric manifold $M^4$ over $P^2$. Let us consider the $E_2$ term of the Leray–Serre spectral sequence for the
bundle $q: \mathcal{Z}_P \to M^4$ with the fibre $T^{m-2}$. Theorem 3.3 shows that $H^2(M^4)$ has rank $m - 2$ and the ring $H^*(M^4)$ is multiplicatively generated by elements of degree 2. The ring $H^*(T^{m-2})$ is an exterior algebra. We choose bases $w_1, \ldots, w_{m-2}$ in $H^2(M^4)$ and $u_1, \ldots, u_{m-2}$ in $H^2(T^{m-2})$ such that the second differential of the spectral sequence takes $u_i$ to $w_i$ (more precisely, $d^2(u_i \otimes 1) = 1 \otimes w_i$, see Figure 4). Furthermore, the map $q^*: H^*(M^4) \to H^*(\mathcal{Z}_P)$ is zero homomorphism in degrees $\geq 0$. This follows from the fact that the map $f^*: H^*(BT^{m-2}) \to H^*(M^4)$ is epimorphic (see the proof of Theorem 4.7) and from the commutative diagram

$$
\begin{array}{ccc}
\mathcal{Z}_P & \longrightarrow & ET^{m-2} \\
q & \downarrow & \downarrow \\
M^4 & \longrightarrow & BT^{m-2}.
\end{array}
$$

Using all these facts and Corollary 4.12 (which gives $E_3 = E_\infty$), we deduce that all differentials $d^0_2$ are monomorphisms, and all differentials $d^2_2$ are epimorphisms.

Now, using Theorem 4.12 we obtain by easy calculations the following formulae for the Betti numbers $b^i(\mathcal{Z}_P)$:

$$b^0(\mathcal{Z}_P) = b^{m+2}(\mathcal{Z}_P) = 1,$$

$$b^1(\mathcal{Z}_P) = b^2(\mathcal{Z}_P) = b^m(\mathcal{Z}_P) = b^{m+1}(\mathcal{Z}_P) = 0,$$

$$b^k(\mathcal{Z}_P) = (m-2) \binom{m-2}{k-2} - \binom{m-2}{k-1} - \binom{m-2}{k-3}$$

$$= \binom{m-2}{k-3} \frac{m(m-k)}{k-1}, \quad 3 \leq k \leq m-1. \quad (17)$$

For small $m$ the above formulae give us the following:

- $m = 3$: $b^0(\mathcal{Z}_5) = b^2(\mathcal{Z}_5) = 1$;
- $m = 4$: $b^0(\mathcal{Z}_6) = b^6(\mathcal{Z}_6) = 1$, $b^3(\mathcal{Z}_6) = 2$,

(all other Betti numbers are zero). Both cases are covered by the previous example. Indeed, for $m = 3$ we have $P^2 = \Delta^2$, and for $m = 4$ we have $P^2 = \Delta^1 \times \Delta^1$. As it
was pointed out above, in this cases $Z_p^7 = S^5$, $Z_p^6 = S^3 \times S^3$. Further, 
\[ m = 5 : \ b^0(Z^7) = b^7(Z^7) = 1, \ b^3(Z^7) = b^4(Z^7) = 5; \]
\[ m = 6 : \ b^0(Z^8) = b^8(Z^8) = 1, \ b^3(Z^8) = b^5(Z^8) = 9, \ b^6(Z^8) = 16, \]
(all other Betti numbers are zero), and so on.

Now we want to describe the ring structure in the cohomology. Theorem 4.6 gives us the isomorphism of algebras
\[ H^*(Z_{P}^{m+2}) \cong \text{Tor}_{k[v_1, \ldots, v_m]}(k(P^2), k) = H[k(P^2) \otimes \Lambda[u_1, \ldots, u_m], d]. \]

If $m = 3$, then $k(P) = k[v_1, v_2, v_3]/v_1v_2v_3$; if $m > 3$ we have $k(P) = k[v_1, \ldots, v_m]/I$, where $I$ is generated by monomials $v_iv_j$ such that $i \neq j \pm 1$. (Here we use the agreement $v_{m+i} = v_i$ and $v_{i-m} = v_i$.) Below we give the complete description of the multiplication in the case $m = 5$. The general case is similar but more involved.

It is easy to check that five generators of $H^3(Z_P)$ are represented by the cocycles $v_i \otimes u_{i+2} \in k(P^2) \otimes \Lambda[u_1, \ldots, u_m]$, $i = 1, \ldots, 5$, while five generators of $H^1(Z_P)$ are represented by the cocycles $v_j \otimes u_{j+2}u_{j+3}$, $j = 1, \ldots, 5$. The product of cocycles $v_i \otimes u_{i+2}$ and $v_j \otimes u_{j+2}u_{j+3}$ represents a non-trivial cohomology class in $H^7(Z_P)$ if and only if the set $\{i, i+2, j, j+2, j+3\}$ is the whole index set $\{1, 2, 3, 4, 5\}$. Hence, for each cohomology class $[v_i \otimes u_{i+2}]$ there is a unique (Poincaré dual) cohomology class $[v_j \otimes u_{j+2}u_{j+3}]$ such that the product $[v_i \otimes u_{i+2}] \cdot [v_j \otimes u_{j+2}u_{j+3}]$ is non-trivial. This product defines a fundamental cohomology class of $Z_P$ (for example, it is represented by the cocycle $v_1v_2 \otimes u_3u_4u_5$). In the next section we prove the similar statement in the general case. All other products in the cohomology algebra $H^*(Z_P)$ are trivial.

5. COHOMOLOGY OF $Z_P$ AND COMBINATORICS OF SIMPLE POLYTOPES

Theorem 4.6 shows that the cohomology of $Z_P$ is naturally a bigraded algebra. The Poincaré duality in $H^*(Z_P)$ regards this bigraded structure. More precisely, the Poincaré duality has the following combinatorial interpretation.

**Lemma 5.1.** In the bigraded differential algebra $[k(P) \otimes \Lambda[u_1, \ldots, u_m], d]$ from Theorem 4.6

1. For each vertex $v = F_{i_1}^{m-1} \cap \cdots \cap F_{i_n}^{m-1}$ of the polytope $P^n$ the element $v_{i_1} \cdots v_{i_n} \otimes u_{i_1} \cdots u_{i_m}$, where $j_1 < \cdots < j_{m-n}$, $\{i_1, \ldots, i_n, j_1, \ldots, j_{m-n}\} = \{1, \ldots, m\}$, represents the fundamental class of $Z_P$.
2. Two cocycles $v_{i_1} \cdots v_{i_p} \otimes u_{j_1} \cdots u_{j_r}$ and $v_{k_1} \cdots v_{k_s} \otimes u_{l_1} \cdots u_{l_t}$ represent Poincaré dual cohomology classes in $H^*(Z_P)$ if and only if $p + s = n$, $r + t = m - n$, $\{i_1, \ldots, i_p, k_1, \ldots, k_s, j_1, \ldots, j_r, l_1, \ldots, l_t\} = \{1, \ldots, m\}$.

**Proof.** The first assertion follows from the fact that the cohomology class under consideration is a generator of the module $\text{Tor}_{k[v_1, \ldots, v_m]}^{(m-n), 2m}(k(P), k) \cong H^{m+n}(Z_P^{m+2})$ (see Theorem 4.6). The second assertion holds since two cohomology classes are Poincaré dual if and only if their product is the fundamental cohomology class. \[ \square \]

In what follows we use the following notations: $T^i = \text{Tor}_{k[v_1, \ldots, v_m]}^{-i}(k(P), k)$ and $T^{-i,2j} = \text{Tor}_{k[v_1, \ldots, v_m]}^{-i,2j}(k(P), k)$. We define the bigraded Betti numbers of $Z_P$ as
\[ b^{-i,2j}(Z_P) = \dim_k T^{-i,2j}(k(P), k). \]
Then Theorem 4.2 can be reformulated as $b^k(\mathcal{Z}_P) = \sum_{2j-i=k} b^{-i,2j}(\mathcal{Z}_P)$. The second part of Lemma 5.1 shows that $b^{-i,2j}(\mathcal{Z}_P) = b^{-(m-n-i),2(m-j)}(\mathcal{Z}_P)$ for all $i,j$. These equalities can be written as the following identities for the Poincaré series $F(T^i, t) = \sum_{r=0}^m b^{-i,2r} t^r$ of $T^i$:

\begin{equation}
F(T^i, t) = t^{2m} F(T^{m-n-i}, \frac{1}{t}), \quad i = 1, \ldots, m-n.
\end{equation}

It is well known in commutative algebra that the above identities hold for the so-called Gorenstein rings (see [St]). The face ring of a simplicial subdivision of a simple polytope $P^k$ is Gorenstein (see [St]). The face ring of a simplicial subdivision of a simple polytope $P^k$ is Gorenstein (see [St]).

A simple combinatorial argument (see [St, part II, §1]) shows that for any $(n-1)$-dimensional simplicial complex $K$ the Poincaré series $F(k(K), t)$ can be written as follows

\begin{equation}
F(k(K), t) = 1 + \sum_{i=0}^{n-1} f_i t^{2i+1},
\end{equation}

where $(f_0, \ldots, f_{n-1})$ is the $f$-vector of $K$. This series can be also expressed in terms of the $h$-vector $(h_0, \ldots, h_n)$ (see (1)) as

\begin{equation}
F(k(K), t) = \frac{h_0 + h_1 t^2 + \cdots + h_n t^{2n}}{(1 - t^2)^n}.
\end{equation}

On the other hand, the Poincaré series of the $k[v_1, \ldots, v_m]$-module $k(P)$ (or $k(K)$) can be calculated from any free resolution of $k(P)$. More precisely, the following general theorem holds (see e.g., [St]).

**Theorem 5.2.** Let $M$ be a finitely generated graded $k[v_1, \ldots, v_m]$-module, deg $v_i = 2$, and there is given a finite free resolution of $M$:

\[ 0 \rightarrow R^{-h} \xrightarrow{d^{-h}} R^{-h+1} \xrightarrow{d^{-h+1}} \cdots \xrightarrow{d^{-1}} R^0 \xrightarrow{d^0} M \rightarrow 0. \]

Suppose that the free $k[v_1, \ldots, v_m]$-modules $R^{-i}$ have their generators in dimensions $d_{i1}, \ldots, d_{iq_i}$, where $q_i = \dim_k k[v_1, \ldots, v_m] R^{-i}$. Then the Poincaré series of $M$ can be calculated by the following formula:

\[ F(M, t) = \sum_{i=0}^{-h} (-1)^i t^{d_{i1} + \cdots + d_{iq_i}} (1 - t^2)^n. \] □

Now let us apply this theorem to the minimal resolution (11) of $k(P) = k[v_1, \ldots, v_m]/I$. Since all differentials of the complex (12) are trivial, we obtain

\begin{equation}
F(k(P), t) = (1 - t^2)^{-m} \sum_{i=0}^{m-n} (-1)^i F(T^i, t).
\end{equation}

Combining this with (20), we get

\[ F(k(P), t) = (1 - t^2)^{-m} \sum_{i=0}^{m-n} (-1)^i t^{2m} F(T^{m-n-i}, \frac{1}{t}) =
\]

\[ = (1 - \left(\frac{1}{t}\right)^2)^{-m} \cdot (-1)^m \sum_{j=0}^{m-n} (-1)^{m-n-j} F(T^j, \frac{1}{t}) = (-1)^n F(k(P), \frac{1}{t}). \]
Substituting here the expressions from the right-hand side of (21) for $F(k(P), t)$ and $F(k(P), 1)$, we finally deduce

\[(23) \quad h_i = h_{n-i}.\]

These are the well-known Dehn–Sommerville equations [Br] for simple (or simplicial) polytopes.

Thus, we see that the algebraic duality (20) and the combinatorial Dehn–Sommerville equations (23) follow from the Poincaré duality for the manifold $\mathcal{Z}_P$. Furthermore, combining (21) and (22) we obtain

\[(24) \quad \sum_{i=0}^{m-n} (-1)^i F(T^i, t) = (1 - t^2)^{m-n} h(t^2),\]

where $h(t) = \sum_{i=0}^n h_i t^i$.

We define the subcomplex $\mathcal{A}$ of the cochain complex $[k(P) \otimes \Lambda[u_1, \ldots, u_m], d]$ from Theorem 4.6 as follows. The $k$-module $\mathcal{A}$ is generated by monomials $v_{i_1} \cdots v_{i_p} \otimes u_{j_1} \cdots u_{j_q}$ and $1 \otimes u_{j_1} \cdots u_{j_q}$ such that $\{v_{i_1}, \ldots, v_{i_p}\}$ spans a simplex in $K_P$ and $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} = \emptyset$. It can be easily checked that $d(\mathcal{A}) \subset \mathcal{A}$ and, therefore, $\mathcal{A}$ is a cochain subcomplex. Moreover, $\mathcal{A}$ inherits the bigraded module structure from $k(P) \otimes \Lambda[u_1, \ldots, u_m]$ with differential $d$ adding $(1,0)$ to bidegree.

**Lemma 5.3.** The cochain complexes $[k(P) \otimes \Lambda[u_1, \ldots, u_m], d]$ and $[\mathcal{A}, d]$ have same cohomologies. Hence, the following isomorphism of $k$-modules holds:

\[H[\mathcal{A}, d] \cong \text{Tor}_{k[v_1, \ldots, v_m]}(k(P), k).\]

**Proof.** It is sufficient to prove that any cocycle $\omega = v_{i_1}^{\alpha_1} \cdots v_{i_p}^{\alpha_p} \otimes u_{j_1} \cdots u_{j_q}$ from $k(P) \otimes \Lambda[u_1, \ldots, u_m]$ that does not lie in $\mathcal{A}$ is a coboundary. To do this we note that if there is $i_k \in \{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\}$, then $d\omega$ contains the summand $v_{i_1}^{\alpha_1} \cdots u_{i_k}^{\alpha_k-1} \cdots v_{i_p}^{\alpha_p} \otimes u_{j_1} \cdots u_{j_q}$, hence, $\omega \neq 0$ — a contradiction. Therefore, $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} = \emptyset$. If $\omega$ contains at least one $v_k$ with degree $\alpha_k > 1$, then since $d\omega = 0$, we have $\omega = \pm d(v_{i_1}^{\alpha_1} \cdots v_{i_k}^{\alpha_k-1} \cdots v_{i_p}^{\alpha_p} \otimes u_{j_1} \cdots u_{j_q})$. Now, our assertion follows from the fact that all non-zero elements of $k(P)$ of the type $v_{i_1} \cdots v_{i_p}$ correspond to simplices of $K_P$. \(\square\)

Now let us introduce the submodules $\mathcal{A}^{*,2r} \subset \mathcal{A}$, $r = 0, \ldots, m$, generated by monomials $v_{i_1} \cdots v_{i_p} \otimes u_{j_1} \cdots u_{j_q} \in \mathcal{A}$ such that $p + q = r$. Hence, $\mathcal{A}^{*,2r}$ is the submodule in $\mathcal{A}$ consisting of all elements of internal degree $2r$ (i.e. for any $\omega \in \mathcal{A}^{*,2r}$ one has $\text{deg} \omega = (r, 2r)$; remember that the internal degree corresponds to the second grading). It is clear that $\sum_{r=0}^{2m} \mathcal{A}^{*,2r} = \mathcal{A}$. Since the differential $d$ does not change the internal degree, all $\mathcal{A}^{*,2r}$ are subcomplexes of $\mathcal{A}$. The cohomology modules of these complexes are exactly $T^{*,2r}$ and their dimensions are the bigraded Betti numbers $b^{-i,2r}(\mathcal{Z}_P)$. Let us consider the Euler characteristics of these subcomplexes:

\[\chi_r := \chi(\mathcal{A}^{*,2r}) = \sum_{q=0}^{m} (-1)^q \dim_k A^{q,2r} = \sum_{q=0}^{m} (-1)^q b^{-q,2r}(\mathcal{Z}_P),\]
Lemma 5.4. The Poincaré series $F(A^{*,*}, \tau, t) = \sum_{r,q} \dim_k A^{-q,2r} \tau^{-q} t^{2r}$ of the bigraded module $A^{*,*}$ is as follows

$$F(A^{*,*}, \tau, t) = \sum_j f_{j-1} \left(1 + \frac{t^2}{\tau}\right)^{m-j} t^{2j},$$

and define

$$\chi(t) = \sum_{r=0}^{m} \chi_r t^{2r}.$$  

Then it follows from Lemma 5.3 that

$$\chi(t) = \sum_{r=0}^{m} \sum_{q=0}^{m} (-1)^q \dim_k A^{-q,2r} t^{2r} = \sum_{q=0}^{m} (-1)^q \sum_{r=0}^{m} \dim_k H^{-q}[A^{*,2r}] t^{2r}$$

$$= \sum_{q=0}^{m} (-1)^q \sum_{r=0}^{m} \dim_k T^{q,2r} t^{2r} = \sum_{q=0}^{m} (-1)^q F(T^q,t),$$

where $T^{q,2r} = H^{-q,2r} [k(P) \otimes A[u_1, \ldots, u_m], d] = \text{Tor}_{k[u_1, \ldots, u_m]} (k(P), k)$. Combining this with formula (24), we get

$$\chi(t) = (1 - t^2)^{m-n} h(t^2).$$

This formula can be also obtained directly from the definition of $\chi_r$. Indeed, it can be easily seen that

$$(27) \quad \dim_k A^{-q,2r} = f_{r-q-1} \left(\frac{m-r+q}{q}\right), \quad \chi_r = \sum_{j=0}^{m} (-1)^{r-j} f_{j-1} \left(\frac{m-j}{r-j}\right),$$

(here we set $\binom{j}{k} = 0$ if $k < 0$). Then

$$(28) \quad \chi(t) = \sum_{r=0}^{m} \chi_r t^{2r} = \sum_{r=0}^{m} \sum_{j=0}^{m} t^{2j} t^{2(r-j)} (-1)^{r-j} f_{j-1} \left(\frac{m-j}{r-j}\right)$$

$$= \sum_{j=0}^{m} f_{j-1} t^{2j} (1 - t^2)^{m-j} = (1 - t^2)^m \sum_{j=0}^{n} f_{j-1} (t^{-2} - 1)^{-j}.$$ 

Further, it follows from (1) that

$$t^n h(t^{-1}) = (t - 1)^n \sum_{i=0}^{n} f_{i-1} (t - 1)^{-i}.$$ 

Substituting here $t^{-2}$ for $t$ and taking into account (28), we finally obtain

$$\frac{\chi(t)}{(1 - t^2)^m} = \frac{t^{-2n} h(t^2)}{(t^{-2} - 1)^n} = \frac{h(t^2)}{(1 - t^2)^n},$$

which is equivalent to (26).

Formula (26) allows to express the $h$-vector of a simple polytope $P^n$ in terms of the bigraded Betti numbers $h^{-q,2r}(Z_P)$ of the corresponding manifold $Z_P$.

**Lemma 5.4.** The Poincaré series $F(A^{*,*}, \tau, t) = \sum_{r,q} \dim_k A^{-q,2r} \tau^{-q} t^{2r}$ of the bigraded module $A^{*,*}$ is as follows

$$F(A^{*,*}, \tau, t) = \sum_j f_{j-1} \left(1 + \frac{t^2}{\tau}\right)^{m-j} t^{2j}.$$
Proof. Using formula (27), we calculate
\[
\sum_{r,q} \dim_k A^{-q,2r} \tau^{-q} t^{2r} = \sum_{r,q} f_{r-q-1} \left( \frac{m-r+q}{q} \right) \tau^{-q} t^{2r} = \sum_{j} f_{j-1} \left( \frac{1}{r-j} \right) \tau^{-j} t^{2j}.
\]

The bigraded Betti numbers \( b^{-i,2j}(Z_P) \) can be calculated either by means of Theorem 4.6 and the results of subsection 4.3 (as we did before) or by means of the following theorem, which reduces their calculation to calculating the cohomology of certain subcomplexes of the simplicial complex \( K^{n-1} \) dual to \( \partial P^n \).

**Theorem 5.5** (Hochster, see [Ho], [St]). Let \( K \) be a simplicial complex on the vertex set \( V = \{v_1, \ldots, v_m\} \), and let \( k(K) \) be its face ring. Then the Poincaré series of \( T^i = \text{Tor}^{-i}_k(k(P), k) \) is calculated as follows
\[
F(T^i, t) = \sum_{W \subseteq V} \left( \dim_k \tilde{H}_{|W|-i-1}(K_W) \right) t^{2|W|},
\]
where \( K_W \) is the subcomplex of \( K \) consisting of all simplices with vertices in \( W \).

However, easy examples show that the calculation based on the above theorem becomes very involved even for small complexes \( K \). It can be shown also that applying the discussed above result of [GM] (see subsection 2.2) to \( U(P^n) \) gives the same description of \( H^*(U(P^n)) \) as that of \( H^*(Z_P) \) given by the Hochster theorem. This, of course, conforms with our results from subsection 2.2.

**Lemma 5.6.** For any simple polytope \( P \) holds
\[
\text{Tor}^{-q,2r}_k(k(P), k) = 0 \quad \text{for} \quad 0 < r \leq q.
\]
Proof. This can be seen either directly from the construction of the minimal resolution (11), or from Theorem 5.5.

**Theorem 5.7.** We have
1. \( H^1(Z_P) = H^2(Z_P) = 0 \).
2. The rank of the third cohomology group of \( Z_P \) (i.e. the third Betti number \( b^3(Z_P) \)) equals the number of pairs of vertices of the simplicial complex \( K^{n-1} \) that are not connected by an edge. Hence, if \( f_0 = m \) is the number of vertices of \( K \) and \( f_1 \) is the number of edges, then
\[
b^3(Z_P) = \frac{m(m-1)}{2} - f_1.
\]
Proof. It follows from Theorem 4.2 and Lemma 5.6 that
\[
H^3(Z) = \text{Tor}^{-1,4}_k(k(P), k) = T^{1,4}.
\]
By Theorem 5.5,
\[
b^{-1,4}(Z_P) = \dim_k T^{1,4} = \sum_{W \subseteq V, |W| = 2} \dim_k \tilde{H}_0(K_W).
\]
Now the theorem follows from the fact that \( \dim_k \tilde{H}_0(K_W) = 0 \) if \( K_W \) is a 1-simplex, and \( \dim_k \tilde{H}_0(K_W) = 1 \) if \( K_W \) is a pair of disjoint vertices.

\[\square\]
Remark. Combining Theorems 4.2, 5.5 and Lemma 5.6 we can also obtain that
\[ b^4(Z) = \dim_k T^{2,6} = \sum_{W \subseteq V, |W| = 3} \dim_k H_0(K_W). \]

Manifolds \( Z_P \) allow to give a nice interpretation not only to the Dehn–Sommerville equations (23) but also to a number of other combinatorial properties of simple polytopes. In particular, using formula (26) one can express the well-known MacMullen inequalities, the Upper and the Lower Bound Conjectures (see. [Br]) in terms of the cohomology of \( Z_P \). We review here only two examples.

The first non-trivial MacMullen inequality for a simple polytope \( P \) can be written as \( h_1 \leq h_2 \) for \( n \geq 3 \). In terms of the \( f \)-vector this means that \( f_1 \geq mn - \binom{n+1}{2} \). Theorem 5.7 shows that \( b^3(Z_P) = \binom{m}{2} - f_1 \). Hence, we have the following upper bound for \( b^3(Z_P) \):

\[ b^3(Z_P) \leq \binom{m-n}{2} \]

if \( n \geq 3 \).

The Upper Bound Conjecture for the number of faces of a simple polytope can be formulated in terms of the \( h \)-vector as
\[ h_i \leq \binom{m-n+i-1}{i}. \]

Using the decomposition
\[ \left( \frac{1}{1-t^2} \right)^{m-n} = \sum_{i=0}^{\infty} \binom{m-n+i-1}{i} t^{2i}, \]
we deduce from (26) and (30) that
\[ \chi(t) \leq 1, \quad 0 \leq t < 1. \]

It would be interesting to obtain a purely topological proof of inequalities (29) and (31).
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