Duality and syzygies for semimodules over numerical semigroups.

Julio José Moyano-Fernández

University Jaume I of Castellón

International meeting AMS-EMS-SPM — Porto 2015
June 11th, 2015
The talk is based on my joint work with Jan Uliczka

*Duality and syzygies for semimodules over numerical semigroups*

published “on-line first” in Semigroup Forum.
Our motivation was to gain a better understanding of certain semimodules over numerical semigroups with 2 generators appearing in previous investigations concerning *Hilbert depth*.
Our motivation was to gain a better understanding of certain semimodules over numerical semigroups with 2 generators appearing in previous investigations concerning *Hilbert depth*.

1. Lattice paths and \( \langle \alpha, \beta \rangle \)-lean sets

2. Syzygies of \( \langle \alpha, \beta \rangle \)-semimodules

3. Dual semimodules
Γ-lean sets and Γ-semimodules

Definition

Let Γ be a numerical semigroup. A set \( \{x_0 = 0, x_1, \ldots, x_n\} \subseteq \mathbb{N} \) is called Γ-lean if \( |x_i - x_j| \notin \Gamma \) for \( 0 \leq i < j \leq n \).

A key notion will be that of a module over a numerical semigroup Γ:

Definition

A Γ-semimodule \( \Delta \) is a non-empty subset of \( \mathbb{N} \) such that \( \Delta + \Gamma \subseteq \Delta \).

Every Γ-semimodule \( \Delta \) has a unique minimal system of generators.

The minimal system of generators of a normalized Γ-semimodule is Γ-lean, and conversely, every Γ-lean subset of \( \mathbb{N} \) generates minimally a normalized Γ-semimodule.
Gaps of \( \langle \alpha, \beta \rangle \) and lattice points

From now on we only consider semigroups \( \Gamma = \langle \alpha, \beta \rangle \) with \( \alpha < \beta \).

There is a map \( G \to \mathbb{N}^2, \alpha\beta - a\alpha - b\beta \mapsto (a, b) \) which identifies a gap with a lattice point. Since \( \alpha\beta - a\alpha - b\beta > 0 \), the point lies inside the triangle with corners \((0, 0), (\beta, 0), (0, \alpha)\).
\[ \langle \alpha, \beta \rangle \text{-lean sets and lattice paths} \]

\[ I = [0, 8, 6, 9] \text{ and } J = [15, 13, 16, 14]. \]
Gaps and ordering

For $\Gamma = \langle \alpha, \beta \rangle$ it holds

$$\ell \in \mathbb{N} \setminus \Gamma \iff \exists \ a, b \in \mathbb{N}_{>0} \text{ with } \ell = \alpha \beta - a\alpha - b\beta.$$ 

This means that, for gaps $i_k = \alpha \beta - a_k \alpha - b_k \beta$, $k = 1, 2$, we have that

$$|i_1 - i_2| \in \mathbb{N} \setminus \Gamma \iff (a_2 - a_1)(b_2 - b_1) < 0.$$ 

This allows us to introduce a partial ordering for the gaps:

$$i_1 < i_2 :\iff a_1 > a_2 \land b_1 < b_2.$$
Syzygies of $\langle \alpha, \beta \rangle$-semimodules

Next we explain the meaning of $J$ in terms of $\langle \alpha, \beta \rangle$-semimodules: Every $\langle \alpha, \beta \rangle$-semimodule $\Delta$ yields another $\langle \alpha, \beta \rangle$-semimodule $\text{Syz}(\Delta)$.

**Definition**

Let $I$ be an $\langle \alpha, \beta \rangle$-lean set, and let $\Delta$ be the $\langle \alpha, \beta \rangle$-semimodule generated by $I$. The syzygy of $\Delta$ is the $\langle \alpha, \beta \rangle$-semimodule

$$\text{Syz}(\Delta) := \bigcup_{i, i' \in I, \ i \neq i'} \left( (i + \langle \alpha, \beta \rangle) \cap (i' + \langle \alpha, \beta \rangle) \right).$$

The semimodule $\text{Syz}(\Delta)$ consists of those elements in $\Delta$ which admit more than one presentation of the form $i + x$ with $i \in I$, $x \in \langle \alpha, \beta \rangle$. 
Syz(Δ) can be also recognized in the lattice path corresponding to Δ:

**Theorem**

Let I, J sets of turning points as in the example. Let Δ be the ⟨α, β⟩-semimodule generated by the elements of I. Then

\[
\text{Syz}(\Delta) = \bigcup_{0 \leq k < m \leq n} (i_k + \langle \alpha, \beta \rangle \cap i_m + \langle \alpha, \beta \rangle) = \bigcup_{k=0}^{n} (j_k + \langle \alpha, \beta \rangle).
\]

i. e. , Syz(Δ) is generated by the elements of J.
Iterated syzygies and their orbits

The procedure of building a syzygy can be iterated; we set

\[ \text{Syz}^\ell(\Delta) := \text{Syz}(\text{Syz}^{\ell-1}(\Delta)), \quad \ell \geq 2. \]

Since all semimodules \( \text{Syz}^\ell(\Delta) \) share the same number of generators, it is clear that this sequence must be periodic up to isomorphism.

The set of isomorphism classes appearing in such a sequence of syzygies will be called an orbit.
It is easily seen that taking the syzygy cyclically permutates the top row of the matrix by one position to the left:

\[
\Delta \mapsto \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}
\]

\[
\text{Syz}(\Delta) \mapsto \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \end{pmatrix}
\]

\[
\text{Syz}^2(\Delta) \mapsto \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}
\]

\[
\text{Syz}^3(\Delta) \mapsto \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}
\]
Dual semimodules

For any $\Gamma$-semimodule $\Delta$ we set the *dual* of $\Delta$

$$\Delta^* := \text{Hom}_\Gamma(\Delta, \Gamma) \cong \{ c \in \mathbb{Z} \mid c + \Delta \subseteq \Gamma \} =: \Gamma - \Delta.$$ 

Dual semimodules behave as expected:

Let $\Delta, \Delta'$ be $\Gamma$-semimodules, and let $d \in \mathbb{Z}$. Then

(a) $(\Delta + d)^* = \Delta^* - d$.

(b) $(\Delta \cup \Delta')^* = \Delta^* \cap (\Delta')^*$.

(c) $\Gamma^* = \Gamma$. 

Duality and syzygies for semimodules over numerical semigroups. Julio José Moyano-Fernández
We found a describing formula:

**Theorem**

Let \( I = \{0, i_1, \ldots, i_n\} \) be a \( \Gamma \)-lean set with gaps

\[
i_k = \alpha \beta - a_k \alpha - b_k \beta
\]

which are ordered increasingly with respect to \( \prec \), and let

\[
\Delta_I = \bigcup_{i \in I} (\Gamma + i),
\]

then

\[
\Delta^*_I = (\Gamma + a_1 \alpha) \cup \bigcup_{k=1}^{n-1} (\Gamma + a_{k+1} \alpha + b_k \beta) \cup (\Gamma + b_n \beta).
\]

**Corollary**

\[
(\Delta^*_I)^* = \Delta_I.
\]
Let \( \mathbb{F} \) be a field. Consider \( \mathbb{F}[\Gamma] \), which may be identified with \( R = \mathbb{F}[t^\alpha, t^\beta] \).

The counterparts of \( \Gamma \)-semimodules are the graded \( R \)-submodules of \( \mathbb{F}[t] \).

Let \( I = \{0, i_1, \ldots, i_n\} \) be a \( \Gamma \)-lean set with \( i_k > 0 \), and let \( M_I = \sum_{i \in I} R t^i \).

Consider the first syzygy of \( M_I \), the kernel of the map

\[
\bigoplus_{i \in I} R(-i) \xrightarrow{\varphi^1} M_I
\]

\[
(f_0, \ldots, f_n) \mapsto \sum_{k=0}^{n} f_k t^{i_k}.
\]
By a result of Piontkowski this kernel is generated by \textit{bivectors}

\[(0, \ldots, 0, t^{\gamma_k}, 0, \ldots, 0, -t^{\gamma_m}, 0, \ldots, 0)\] with \(i_k + \gamma_k = i_m + \gamma_m\).

In fact \(n + 1\) special bivectors are sufficient, namely

\[
\begin{align*}
    f_0 &= (t^{(\beta-a_1)\alpha}, -t^{b_1\beta}, 0, \ldots, 0) \\
    f_k &= (0, \ldots, 0, t^{(a_k-a_{k+1})\alpha}, -t^{(b_{k+1}-b_k)\beta}, 0, \ldots, 0) \text{ for } k = 1, \ldots, n-1 \\
    f_n &= (-t^{(\alpha-b_n)\beta}, 0, \ldots, 0, t^{a_n\alpha}).
\end{align*}
\]

The degrees \(\text{deg } f_k = j_k\) are exactly the elements of the set \(J\).

Hence, the support of the syzygy \(\ker \varphi_1\) agrees with the object we called the syzygy of \(\Delta_I\).
The second step of the free resolution of $M_i$ is the map

$$
\bigoplus_{j \in J} R(-j) \xrightarrow{\varphi_2} \ker \varphi_1
$$

$$(g_0, \ldots, g_n) \mapsto \sum_{k=0}^{n} g_k f_k.$$

The condition $\varphi_2(g_0, \ldots, g_n) = 0$ yields the following system of equations:

\[
\begin{align*}
g_0 t^{(\beta-a_1)\alpha} & - g_n t^{(\alpha-b_n)\beta} = 0 \\
g_1 t^{(a_1-a_2)\alpha} & - g_0 t^{b_1\beta} = 0 \\
g_k t^{(a_k-a_{k+1})\alpha} & - g_{k-1} t^{(b_k-b_{k-1})\beta} = 0 \quad \text{for } k = 2, \ldots, n-1 \\
g_n t^{a_n\alpha} & - g_{n-1} t^{(b_n-b_{n-1})\beta} = 0
\end{align*}
\]
We can solve for $g_0$ and get

$$g_k = g_0 t^{b_k \beta - (a_1 - a_{k+1}) \alpha} \quad \text{for } k = 1, \ldots, n - 1$$

$$g_n = g_0 t^{b_n \beta - a_1 \alpha},$$

as one easily checks by induction on $k$.

Hence $g = (g_0, \ldots, g_n)$ is an element of $\ker \varphi_2$ if and only if it can be written in the form

$$g = g_0 \left(1, t^{b_1 \beta - (a_1 - a_2) \alpha}, \ldots, t^{b_{n-1} \beta - (a_1 - a_n) \alpha}, t^{b_n \beta - a_1 \alpha} \right)$$

with some $g_0 \in R$ such that all the entries are in $R$ as well.
In the language of $\Gamma$-semimodules this means that we are looking for the dual of the semimodule

$$\hat{\Delta}_I := \Gamma \cup \bigcup_{k=1}^{n-1} \left( \Gamma + (b_k \beta - (a_1 - a_{k+1}) \alpha) \right) \cup \left( \Gamma + b_n \beta - a_1 \alpha \right).$$

The Theorem above implies

$$\hat{\Delta}_I^* = a_1 \alpha + \Delta_I,$$

hence $\ker \varphi_2$ equals

$$\left\{ g_0 \left( 1, t^{b_1 \beta - (a_1 - a_2) \alpha}, \ldots, t^{b_{n-1} \beta - (a_1 - a_n) \alpha}, t^{b_n \beta - a_1 \alpha} \right) \mid g_0 \in M_I \cdot t^{a_1 \alpha} \right\},$$

$$\cong M_I.$$
Therefore we have shown:

**Theorem**

Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Let $I$ be a $\Gamma$-lean set, and let $M_I = \sum_{i \in I} R t^i$ with $R = \mathbb{F}[t^\alpha, t^\beta]$. Then the minimal graded free resolution of $M_I$ is —up to a shift— periodic of period 2.
Therefore we have shown:

**Theorem**

Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Let $I$ be a $\Gamma$-lean set, and let $M_I = \sum_{i \in I} R^i$ with $R = \mathbb{F}[t^\alpha, t^\beta]$. Then the minimal graded free resolution of $M_I$ is —up to a shift— periodic of period 2.

So we recover part of a result of Eisenbud (TAMS 1980):

**Theorem**

Let $A$ be a regular local ring, $x \in A$, and let $B = A/x$. If $F : \cdots \to F_1 \to F_0$ is the minimal $B$-free resolution of a finitely generated $B$-module $M$, then:

(i) $F$ becomes periodic of period 2 after $\dim A + 1$ steps;

(ii) $F$ is periodic (necessarily of period 2) iff $M$ is a maximal CM $B$-module with no free summand.