Mediated tunable coupling of flux qubits

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Abstract. It is sketched how a monostable rf- or dc-SQUID can mediate an inductive coupling between two adjacent flux qubits. The non-trivial dependence of the SQUID’s susceptibility on external flux makes it possible to continuously tune the induced coupling from antiferromagnetic (AF) to ferromagnetic (FM). In particular, for suitable parameters, the induced FM coupling can be sufficiently large to overcome any possible direct AF inductive coupling between the qubits.

The main features follow from a classical analysis of the multi-qubit potential. A fully quantum treatment yields similar results, but with a modified expression for the SQUID susceptibility.

Since the latter is exact, it can also be used to evaluate the susceptibility—or, equivalently, energy-level curvature—of an isolated rf-SQUID for larger shielding and at degenerate flux bias, i.e., a (bistable) qubit. The result is compared to the standard two-level (pseudospin) treatment of the anticrossing, and the ensuing conclusions are verified numerically.

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1. Introduction

Superconducting devices made of mesoscopic Josephson junctions are one of the few remaining viable candidates for the realization of quantum bits (qubits). The major types encode quantum information in the charge, phase, or flux degree of freedom [1]. Quantum coherence in such devices was conclusively demonstrated at the turn of the century. While much remains to be done in improving the coherence times, control and readout of individual qubits, another key challenge on the road towards quantum-register prototypes is the coupling of several qubits. In the past years, coupling was achieved for each of the mentioned qubit types [2, 3].

In all published experiments, the coupling strength is a constant determined at the time of fabrication. For each model of quantum computing (gate-operations, adiabatic/ground-state, etc), however, it would be very desirable to have the coupling tunable in situ, even if this requirement can sometimes be circumvented in ‘software’ [4]. In particular, whenever the coupling can be continuously tuned between two values with opposite signs, it can be switched off naturally.

For charge qubits, a design fulfilling these requirements was presented in [5]. The coupling between the two qubits is mediated via a third device, the bias parameters of which can be used to tune the coupling constant. While the coupling element thus is non-trivial, it is important that it be passive, i.e., that it does not get entangled with the qubits so that the system’s total effective Hilbert space remains four-dimensional. In practice, this means that the coupler must stay in its ground state, adiabatically following the qubit dynamics.

Mediated coupling can be adapted to flux qubits [6]. This paper’s purpose is to contribute to its analytical theory. The main results can be obtained from a purely classical analysis of the system’s potential, which may have some independent pedagogical interest. This is carried out in section 2 for two choices of the coupler, both SQUID devices. A full quantum analysis is presented in section 3; the main result for the tunable coupling constant is highly analogous to its classical counterpart, but some additional effects are uncovered concerning a renormalization of the qubit parameters. In the formula for the coupling constant, the non-trivial factor is an exact expression for the coupler’s ground-state magnetic susceptibility. In section 4, this is applied to a different regime, where the coupler behaves like a qubit itself, allowing for a
Figure 1. A system of three rf-SQUIDs, of which the two outer ones function as qubits and the inner one as a coupler.

comparison with the predictions of the two-state model. Some concluding remarks are made in section 5.

2. Classical analysis

2.1. rf-SQUID coupler

Consider three rf-SQUIDs $a$ through $c$ in a row (figure 1). The $a$- and $c$-devices are supposed to work as (effectively) degenerately biased flux qubits, while the $b$-SQUID is a coupling element, tunable by an external flux bias. To derive the indirect $a$–$c$ coupling, it suffices to consider the potential, consisting of both Josephson and magnetic terms ($\hbar = 1$ throughout),

$$U = -E_a \cos \phi_a - E_b \cos \phi_b - E_c \cos \phi_c + \frac{1}{8e^2} (\phi - \phi^x)^\top L^{-1} (\phi - \phi^x),\quad (1)$$

with the inductance matrix

$$L = \begin{pmatrix}
L_a & -M_{ab} & 0 \\
-M_{ab} & L_b & -M_{bc} \\
0 & -M_{bc} & L_c
\end{pmatrix},\quad (2)$$

i.e., the direct AF inductive $a$–$c$ coupling is ignored. In principle, this can always be achieved through a gradiometric layout. However, as long as $M_{ac}$ is small, one can probably just as well simply add an interaction $M_{ac} I_a I_c$ to the final result (9) below. In accordance with the roles played by the various loops, the bias fluxes (in phase units) are chosen as

$$\phi^x = \begin{pmatrix}
\pi + (M_{ab}/L_b)(\phi_b^{(0)} - \phi_b^x) \\
\phi_b^x \\
\pi + (M_{bc}/L_b)(\phi_b^{(0)} - \phi_b^x)
\end{pmatrix}.\quad (3)$$

Thus, $\phi^x_{a,c}$ compensate externally for the shielding flux which the $b$-loop couples to in the $a$- and $c$-SQUIDs; see (6) for the precise definition of $\phi_b^{(0)}$.

The $b$-SQUID should act as a passive coupler without introducing additional degrees of freedom, so bistability must be avoided. This can be achieved either by biasing it with a flux close to an integer number of quanta (small $\phi_b^0$), or for any flux bias by taking the shielding
coefficient $\beta_b \equiv 4e^2 L_b E_b < 1$ (in SI units, $\beta = 2\pi L I_c / \Phi_0$). The calculation that follows is valid in either case, and their relative merits will be discussed afterwards.

We proceed by expansion in $M$. Without suggesting that the regime of $M/L \ll 1$ (say, distant loops) is the most practical, this leads to a transparent result, which can guide numerical studies in the general case. The $b$-mediated $a$–$c$ coupling is expected to be $\mathcal{O}(M^2)$, so the junction phases will be written as

$$\phi_j = \phi_j^{(0)} + \phi_j^{(1)} + \phi_j^{(2)} + \mathcal{O}(M^3);$$

these can be determined by solving the equilibrium condition $\nabla \phi U = 0$. In leading order, the $\phi_j^{(0)}$ are just the stationary phases of an isolated rf-SQUID,

$$\beta_j \sin \phi_j^{(0)} + \phi_j^{(0)} - \pi = 0 \quad (j = a, c),$$

$$\beta_b \sin \phi_b^{(0)} + \phi_b^{(0)} - \phi_b^a = 0,$$

where $\phi_{a,c}^{(0)} - \pi$ can have either sign. To the same order as (4), one has

$$L^{-1} = \begin{pmatrix}
\frac{1}{L_a} + \frac{M_{ab}^2}{L_a L_b} & \frac{M_{ab}}{L_a L_b} & \frac{M_{ab} M_{bc}}{L_a L_b L_c} \\
\frac{M_{ab}}{L_a L_b} & \frac{1}{L_b} + \frac{M_{ab}^2}{L_a L_b^2} & \frac{M_{ab}^2}{L_b L_c} \\
\frac{M_{ab} M_{bc}}{L_a L_b L_c} & \frac{M_{bc}}{L_b L_c} & \frac{1}{L_c} + \frac{M_{bc}^2}{L_b L_c^2}
\end{pmatrix} + \mathcal{O}(M^3).$$

While (7) may look tedious, using it consistently actually leads to significant cancellation below.

In $\mathcal{O}(M)$, one finds that $\phi_j^{(1)} = 0$ due to the special choice of $\phi^x$ in (3), while the $b$-loop picks up the shielding fluxes of the neighbouring ones, with a prefactor reflecting its susceptibility:

$$\phi_b^{(1)} = \left[ 1 + \beta_b \cos \phi_b^{(0)} \right]^{-1} \left[ \frac{M_{ab}}{L_a} (\pi - \phi_a^{(0)}) + \frac{M_{bc}}{L_c} (\pi - \phi_c^{(0)}) \right].$$

Calculating the $\phi_j^{(2)}$ in $\mathcal{O}(M^2)$ is straightforward but unnecessary: since one expands around a minimum of $U$, the $\phi_j^{(2)}$ do not contribute to the relevant order.

All that remains is to substitute (3)–(8) into $U$ in (1). Since, e.g., $(\phi_a^{(0)} - \pi)^2$ does not depend on the qubit state, one is left with

$$U = \text{const} + \frac{M_{ab} M_{bc}}{4e^2 L_a L_b L_c} \frac{\beta_b \cos \phi_b^{(0)}}{1 + \beta_b \cos \phi_b^{(0)}} (\phi_a^{(0)} - \pi)(\phi_c^{(0)} - \pi)$$

$$= \text{const} + \frac{M_{ab} M_{bc} \beta_b \cos \phi_b^{(0)}}{L_b (1 + \beta_b \cos \phi_b^{(0)})} I_a I_c.$$  

1 A $\beta_b < 1$ can clearly be achieved by reducing $E_b$ while keeping the area of the $b$-loop, and hence the inductive coupling, appreciable. If required, a shunting capacitance could be placed across the $b$-junction to keep the $b$-SQUID in the flux regime, where it has the largest response. Also, even a SQUID with a unique potential minimum will have excited states, corresponding to plasma oscillations $\omega_p$. While these are outside the present scope, the corresponding excitation energies should exceed any transition in the $a$- and $c$-devices if the $b$-SQUID is to remain passive. There is an apparent conflict, since the shunting capacitance contemplated above would lower $\omega_p$; however, the relevant energy scale, given by the tunnel splittings in qubits $a$ and $c$, is so small that all requirements can be met simultaneously in practice. Section 3 has a few more details on the system's quantum aspects.
The product of mutual inductances is expected geometrically, while $I_a I_c \propto \sigma_a^z \sigma_c^z$ in qubit parlance.

The fraction is worth a closer look:

$$\frac{\beta_b \cos \phi_b^{(0)}}{1 + \beta_b \cos \phi_b^{(0)}} \approx \begin{cases} \beta_b \cos \phi_b^x, & \beta_b \ll 1, \\ 1, & \beta_b \gg 1. \end{cases} \quad (10)$$

Thus, for small $\phi_b^x$, the coupling is AF, but it changes sign to FM as $\phi_b^x \to \pi$ (only attainable for $\beta_b < 1$).

This behaviour, and the coupling mechanism in general, have a transparent interpretation. Consider the self-flux curve $\phi_b^{(0)}(\phi_b^x) - \phi_b^x$ for the free $b$-SQUID (figure 2), which up to constants simply is the shielding current versus applied flux. If the $a$-qubit flips from $\uparrow$ to $\downarrow$, then due to the mutual inductance $-M_{ab} < 0$ this effectively increases $\phi_b^x$. For small $\phi_b^x$, the curve has a slope $< 0$, so that this increase generates a self-flux in the $\downarrow$-direction. In turn, the latter, through $-M_{bc} < 0$, acts as an $\uparrow$-bias for the $c$-loop, favouring the $\uparrow$-state there, i.e., opposite to the state of the $a$-qubit. This also explains the bound of unity exemplified by the second line of (10), since the maximum AF response is perfect shielding for $\beta_b \to \infty$ (an uninterrupted superconducting loop).

However, near $\phi_b^x = \pi$ the argument works in the opposite direction. There, the self-flux curve has a slope $> 0$ so that, differentially, the self-flux does not shield at all but actually cooperates with the external flux increase. Thus, the coupling changes sign to FM. Moreover, as $\beta_b \uparrow 1$, the slope of this curve increases without bound (a precursor to bistability), so that the FM coupling can actually be large; this corresponds to the zero in the denominator of (10). On the one hand, the divergence is never realized in practice, since for any finite $M$’s one deals with finite differences, not slopes, on this curve (a nice counterpart to the discussion in [5] for
charge devices\textsuperscript{2\textsuperscript{1}}). On the other, this potentially large FM coupling makes it possible to overcome any residual direct AF coupling through $M_{ab}$. Since (10) shows that the large-$\beta_b$ regime is quite inflexible, and that for $\beta_b \ll 1$ the coupling strength is limited by the small shielding flux, this case of $\beta_b \lesssim 1$ is thought to be the preferred one.

The above is mostly a straightforward derivation from (1), so very few specifics of the circuit are involved. For instance, the $a$- and $c$-SQUIDs could also be placed inside a large $b$-loop, which changes the signs of both $M_{ab}$ and $M_{bc}$, so that (9) is invariant. This design modification makes it clearer that the $b$-loop is mostly a flux transformer [7], with a weak link providing the tunability. Moreover, the final result (9) depends only trivially on the properties of the $a$- and $c$-devices—through the flux which these couple into the $b$-SQUID. Hence, the generalization to other types of flux qubits should be obvious.

2.2. dc-SQUID coupler

The above indicates that the simple rf-SQUID should be a satisfactory coupling element. Still, it is worthwhile to also examine a dc-SQUID [6], which provides additional flexibility since it can be both flux- and current-biased. In particular, if only a current bias $I_b^x$ would turn out to suffice for coupling tunability, that could be preferable in situations where a flux bias is problematic due to stray fields, etc. The $b$-device now has separate left and right arms, designated as $b_1$ and $b_2$, together carrying $I_b^x$ (figure 3). Again ignoring the direct $a\cdots c$ mutual inductance (not that this makes much difference anywhere), we thus take the coil-to-coil inductance matrix as

\[
\begin{pmatrix}
  L_a & -M_{ab1} & -M_{ab2} & 0 \\
- M_{ab1} & L_{b1} & M_{b12} & -M_{b1c} \\
- M_{ab2} & M_{b12} & L_{b2} & -M_{b2c} \\
0 & -M_{b1c} & -M_{b2c} & L_c
\end{pmatrix}.
\]

However, always $I_{b2} = I_{b1} + I_b^x$, so the flux–current equation relates three-vectors pertaining to the loops:

\[
\Phi - \tilde{\Phi} = \mathbb{I} I,
\]

\textsuperscript{2\textsuperscript{1}} Indeed, since the shielding current is the derivative of the energy with respect to the external flux, the induced coupling strength is proportional to the second derivative or finite difference of the SQUID band structure, in complete analogy with [5].

Figure 3. A dc-SQUID mediating a coupling between two rf-SQUID qubits.
with
$$\Phi = (\Phi_a, \Phi_b, \Phi_c)^\top,$$
(13)

$$\tilde{\Phi}^x = \Phi^x + (-M_{ab2}, \tilde{L}_{b2}, -M_{bc2})^\top \tilde{I}_b^x,$$
(14)

$$I = (I_a, I_{b1}, I_c)^\top,$$
(15)

$$\mathbb{L} = \begin{pmatrix} L_a & -M_{ab} & 0 \\ -M_{ab} & L_b & -M_{bc} \\ 0 & -M_{bc} & L_c \end{pmatrix},$$
(16)

$$M_{ab} = M_{ab1} + M_{ab2}, \quad M_{bc} = M_{b1c} + M_{b2c},$$
(17)

$$L_b = \tilde{L}_{b1} + \tilde{L}_{b2}, \quad \tilde{L}_{b1} = L_{b1} + M_{b12}, \quad \tilde{L}_{b2} = L_{b2} + M_{b12}.$$  
(18)

On the other hand, the four junction phases are all independent dynamical variables. They are related to the fluxes by

$$\langle \Phi_1, \Phi_1', \Phi_1'' \rangle = 2e \Phi.$$  

One should first of all determine the Hamiltonian. Of the Kirchhoff (circuit) equations, the first four are simply the Josephson relations

$$\dot{\phi}_a = 2e Q_a/C_a \text{ etc, with } Q_a \text{ the junction charge and } C_a \text{ its capacitance.}$$

The other four express current conservation:

$$\dot{Q}_a = -2e E_a \sin \phi_a - I_a = -2e E_a \sin \phi_a - [\mathbb{L}^{-1}(\Phi - \tilde{\Phi}^x)]_a,$$
(19)

$$\dot{Q}_{b1} = -2e E_{b1} \sin \phi_{b1} - [\mathbb{L}^{-1}(\Phi - \tilde{\Phi}^x)]_b,$$
(20)

$$\dot{Q}_{b2} = -2e E_{b2} \sin \phi_{b2} - I_{b1} - I_b^x = -2e E_{b2} \sin \phi_{b2} - [\mathbb{L}^{-1}(\Phi - \tilde{\Phi}^x)]_b - I_b^x,$$
(21)

$$\dot{Q}_c = -2e E_c \sin \phi_c - [\mathbb{L}^{-1}(\Phi - \tilde{\Phi}^x)]_c.$$  
(22)

It is readily seen that these are the Hamiltonian equations

$$\dot{\phi}_j = 2e \partial H/\partial Q_j \text{ and } \dot{Q}_j = -2e \partial H/\partial \phi_j,$$

for

$$H = \sum_j \left[ \frac{Q_j^2}{2C_j} - E_j \cos \phi_j \right] + \frac{1}{2} (\Phi - \tilde{\Phi}^x)^\top \mathbb{L}^{-1}(\Phi - \tilde{\Phi}^x) + \frac{I_b^x}{2e} \phi_{b2},$$
(23)

with $j \in \{a, b1, b2, c\}$. It is less apparent that $H$ is actually symmetric in the two SQUID arms, as it should be: $H$ is a function of the junction phases and charges only, and the asymmetric choice of $I_{b1}$ as the loop current was merely an interim convention. That is, designating the last two terms of (23) as ‘magnetic’ and ‘bias’ energy, respectively, is a bit arbitrary. Working out $\mathbb{L}^{-1}$, one sees that up to a constant, one can rewrite

$$H = \sum_j \left[ \frac{Q_j^2}{2C_j} - E_j \cos \phi_j \right] + \frac{1}{2} (\Phi - \tilde{\Phi}^x)^\top \mathbb{L}^{-1}(\Phi - \tilde{\Phi}^x) + H_{bias},$$
(24)
where

$$\frac{2e \det \mathbb{L}}{I_b^c} H_{\text{bias}} = [ (M_{ab} L_{b1} - M_{ab} L_{b2}) L_c + (M_{ab} M_{bc} - M_{ab} M_{bc}) M_{bc}] \phi_a$$

$$- [M_{ab} M_{bc} L_c + L_a M_{bc} M_{bc} - L_a L_{b2} L_c] \phi_1$$

$$+ [ - M_{ab} M_{bc} L_c - L_a M_{bc} M_{bc} + L_a L_{b1} L_c] \phi_2$$

$$+ [ L_a (\tilde{L}_{b1} M_{bc} - \tilde{L}_{b2} M_{bc}) + M_{ab} (M_{ab} M_{bc} - M_{ab} M_{bc})] \phi_c,$$

(25)

with \( \det \mathbb{L} = L_a L_b L_c - L_a M_{bc}^2 - M_{bc}^2 L_c \). It would be instructive to see if the above, especially the form (25) for \( H_{\text{bias}} \), can be reproduced (presumably almost mechanically) by the ‘node formalism’ of circuit analysis [8, 9].

From here on, the analysis is a straightforward generalization of the rf-SQUID case, again writing the phases as in (4), and expanding \( H_{\text{bias}} \) in (25) to the same order. We again want the \( a \)- and \( c \)-qubits to be effectively at degeneracy, which presently means that their own external flux should compensate also for \( I_b^c \):

$$\phi^c = \left( \begin{array}{c} \pi + L_b^{-1} [M_{ab} (\phi^{(0)}_b + \phi^{(0)}_b - \phi^c) + 2e I_b^c (M_{ab} \tilde{L}_{b1} - M_{ab} \tilde{L}_{b2})] \\ \pi + L_b^{-1} [M_{bc} (\phi^{(0)}_b + \phi^{(0)}_b - \phi^c) + 2e I_b^c (\tilde{L}_{b1} M_{bc} - \tilde{L}_{b2} M_{bc})] \end{array} \right).$$

(26)

In \( \mathcal{O} (M^0) \), the stationary phases obey the standard equations for the isolated devices:

$$\beta_j \sin \phi^{(0)}_j + \phi^{(0)}_j - \pi = 0, \quad (j = a, c),$$

(27)

$$\beta_{b1} \sin \phi^{(0)}_{b1} + \phi^{(0)}_{b1} + \phi^{(0)}_{b2} - \phi^c - 2e I_b^c \tilde{L}_{b2} = 0,$$

(28)

$$\beta_{b2} \sin \phi^{(0)}_{b2} + \phi^{(0)}_{b1} + \phi^{(0)}_{b2} - \phi^c + 2e I_b^c \tilde{L}_{b1} = 0.$$

(29)

Here, \( \beta_{a(c)} = 4e^2 L_{a(c)} E_{a(c)} \) is standard, but note that the definitions \( \beta_{bj} = 4e^2 L_b E_{bj} \) \( (j = 1, 2) \) involve the full loop inductance of the \( b \)-SQUID, not the individual arm inductances.

In \( \mathcal{O} (M) \), our choice (26) for \( \phi^c \) again ensures that \( \phi^{(1)}_a = \phi^{(1)}_c = 0 \), while the equations for \( \phi^{(1)}_{b1(2)} \) are coupled, as were (28) and (29) for \( \phi^{(0)}_{b1(2)} \) (although, unlike the latter, they are of course linear):

$$\begin{pmatrix} \phi^{(1)}_{b1} \\ \phi^{(1)}_{b2} \end{pmatrix} = \begin{pmatrix} 1 + \beta_{b1} \cos \phi^{(0)}_{b1} & 1 \\ 1 & 1 + \beta_{b2} \cos \phi^{(0)}_{b2} \end{pmatrix}^{-1} \begin{pmatrix} M_{ab} \frac{\pi - \phi^{(0)}_a}{L_a} + M_{bc} \frac{\pi - \phi^{(0)}_c}{L_c} \\ 1 \end{pmatrix}.$$
As for the rf-SQUID, it is best to leave the $\phi_j^{(2)}$ unevaulated in the $M$-expansion of $U$, and one sees that their contribution cancels (to this order), since one expands around a potential minimum. Taking advantage of some further cancellation and dropping all terms which do not depend on the qubit state, one arrives at

$$U = \text{const} + M_{ab} M_{bc} \frac{\beta_{b1} \beta_{b2} \cos \phi_{b1}^{(0)} \cos \phi_{b2}^{(0)}}{L_b (\beta_{b1} \cos \phi_{b1}^{(0)} + \beta_{b2} \cos \phi_{b2}^{(0)} + \beta_{b1} \beta_{b2} \cos \phi_{b1}^{(0)} \cos \phi_{b2}^{(0)})} I_a I_c, \quad (31)$$

in which all tunability thus comes via the dependence of $\phi_{b1(2)}^{(0)}$ on $\phi_b^x$, $I_b^x$ as given in (28) and (29). For a simple consistency check, let us take $I_b^x \to 0$ and $E_{b2} \to \infty$. Then, (29) shows that $\phi_{b2}^{(0)} = 0$, while (28) reduces to $\beta_{b1} \sin \phi_{b1}^{(0)} + \phi_{b1}^{(0)} - \phi_b^x = 0$, and the large fraction in (31) becomes $\beta_{b1} \cos \phi_{b1}^{(0)}/(1 + \beta_{b1} \cos \phi_{b1}^{(0)})$; in view of the remark below (29), this is exactly the result for rf-SQUID coupling.

Another well-known case is the symmetric dc-SQUID with negligible shielding, for which (28) and (29) reduce to $\phi_{b1}^{(0)} + \phi_{b2}^{(0)} = \phi_b^x$ and $2I_b \cos(\phi_b^x/2) \sin(\phi_{b1}^{(0)}-\phi_b^x/2) = I_b^x$. Even in this simple case, one can have FM coupling, for $\cos^2(\phi_b^x/2) < |I_b^x|/2I_c < |\cos(\phi_b^x/2)|$. While one thus needs a nonzero flux bias, tunability of $I_b^x$ suffices for changing the sign of the coupling.

One notes a few features, and it should be investigated whether these carry over to more generic devices described by our equations: (i) to achieve FM coupling, one needs nonzero flux and current bias; (ii) the denominator in (31) is always positive; (iii) $\cos \phi_{b2}^{(0)}$ never become negative simultaneously. Need for current bias: for a symmetric SQUID this is easy to prove, since for $I_b^x = 0$ one then has $\phi_{b1}^{(0)} = \phi_{b2}^{(0)}$, and for any $\phi_b^x$ there is a stationary state with $|\phi_{b1}^{(0)}| \leq \pi/2$. This cannot be generalized further, since we have already seen that the (asymmetric) dc-SQUID contains the rf-SQUID as a special case, and the latter can mediate FM coupling without any current bias. Need for flux bias: again easily proved for symmetric devices with arbitrary shielding, which have $\phi_{b1}^{(0)} = -\phi_{b2}^{(0)}$, etc for $\phi_b^x = 0$; in general, however, (28) and (29) show that an inductance imbalance between the two SQUID arms can play the same role as a nonzero $\phi_b^x$. Finally, note that the rhs in (30) features the second-derivative matrix for the potential of the free $b$-SQUID (characterizing the device’s susceptibility). For a stable minimum, this matrix should be positive-definite, from which (ii) and (iii) immediately follow in general. (As remarked above, the second derivatives keep appearing because of the $M$-expansion. For finite $M$, one has second differences instead [5].)

One of the possible uses of a dc-SQUID coupler could be to moonlight as a readout device through switching-current measurement, so let us treat that in the present notation. Rather than thinking of the maximum bias for which (28) and (29) have solutions, it is convenient to fix $\phi_b^x$ and consider $I_b^x$ as a function of the junction phases, which are still related among themselves by one non-trivial constraint. Critical bias now corresponds to $dI_b^x/d\phi = 0$; working things out readily yields

$$\beta_{b1} \cos \phi_{b1}^{(0)} + \beta_{b2} \cos \phi_{b2}^{(0)} + \beta_{b1} \beta_{b2} \cos \phi_{b1}^{(0)} \cos \phi_{b2}^{(0)} = 0 \quad (32)$$
as the relevant condition\(^3\). To calculate switching-current sensitivity, we now vary \(\phi^b\) and see which variation in \(I^b\) conserves the above relation. One finds

\[
\frac{dI_{b,x}^{\text{max}}}{d\phi^b} = \frac{\beta b_1 \cos \phi^{(0)}_{b_1}}{2e[L_b + \tilde{L}_{b_1} \beta_{b_1} \cos \phi^{(0)}_{b_1}]},
\]

using (32), one again finds that this is more symmetric in the SQUID arms than it looks. So the sensitivity vanishes iff, at critical bias, both junctions become critical individually. This happens for

\[
\phi^b = \pm \frac{\tilde{L}_{b_1} \beta_{b_1} - \tilde{L}_{b_2} \beta_{b_2}}{L_b}, \quad I^b = \pm \frac{\beta_{b_1} + \beta_{b_2}}{2eL_b}.
\]

(34)

As long as the ‘critical flux’ product \(\tilde{L}_{b_j} I_{c,b_j}\) is different for the two arms, one has a finite sensitivity at \(\phi^b = 0\) and in fact for any \(\phi^b\), since one can ramp \(I^b\) with either sign. For the sake of completeness we also give the zero-sensitivity points

\[
\phi^b = \pm \left[\pi - \frac{\tilde{L}_{b_1} \beta_{b_1} + \tilde{L}_{b_2} \beta_{b_2}}{L_b}\right], \quad I^b = \pm \frac{\beta_{b_2} - \beta_{b_1}}{2eL_b},
\]

(35)

which correspond to minima in the switching current.

Finally, the above analysis only shows that the potential part of (24) and (25) enables tunable coupling in the classical case. That is, there could be devices outside its regime of validity which are nonetheless suitable couplers, as long as the transformer remains passive, i.e., confined to its lowest energy state/band [5]. For a quantum analysis, the interacting, biased Hamiltonian in (24) and (25) can be used as a starting point. The method of \(M\)-expansion should again reduce the problem to one for the uncoupled, biased, dc-SQUID. Below, however, the quantum case is taken up for rf-SQUIDs only.

### 3. Quantum analysis

Given that quantum tunnelling is crucial for the qubits \(a\) and \(c\), a consistent quantum treatment of the whole system is desirable. In the quantum case, the \(M\)-expansion works as above. The other crucial step is an adiabatic approximation for the \(b\)-coupler (cf [5]), replacing operators with their ground-state expectation. This very language is only meaningful if the rf-SQUID \(b\) is a well-defined separate entity, i.e., for weak coupling. Specifically, the coupling energies must be much smaller than the first excitation energy of the \(b\)-device, but this does not restrict their size compared to the level splittings of qubits \(a\) and \(c\).

\(^3\) Compare (31) and the end of the previous paragraph: one can equivalently characterize critical bias as the current for which the potential becomes marginally unstable. We note in passing that, for near-critical bias, the induced coupling is always FM, except for the doubly non-generic points given by (34) and (35).
Thus, let us collect all terms in the Hamiltonian \( H = \sum_{j=a,b,c} Q_j^2/2C_j - U \) (with \( U \) as in (1)) involving the coupler. To the relevant order in \( M \) one has

\[
H_b \equiv \frac{Q_b^2}{2C_b} - E_b \cos \phi_b + \frac{1}{8e^2 L_b} \left[ \left( 1 + \frac{M_{ab}^2}{L_a L_b} + \frac{M_{bc}^2}{L_b L_c} \right) (\phi_b - \phi_c^x)^2 
+ 2 \frac{M_{ab}}{L_a} (\phi_a - \phi_a^x) (\phi_b - \phi_c^x) 
+ 2 \frac{M_{bc}}{L_c} (\phi_b - \phi_c^x) (\phi_c - \phi_c^x) \right] 
+ \frac{Q_b^2}{2C_b} - E_b \cos \phi_b + \frac{(\phi_b - \phi_b^{x,\text{eff}})^2 - (\delta \phi_b^x)^2}{8e^2 L_b^{\text{eff}}},
\]

with

\[
L_b^{\text{eff}} = L_b \left( 1 - \frac{M_{ab}^2}{L_a L_b} - \frac{M_{bc}^2}{L_b L_c} \right),
\]

\[
\phi_b^{x,\text{eff}} = \phi_b^x + \delta \phi_b^x = \phi_b^x + \frac{M_{ab}}{L_a} (\phi_a^x - \phi_a) + \frac{M_{bc}}{L_c} (\phi_c^x - \phi_c).
\]

The adiabatic step amounts to the replacement

\[
H_b \mapsto \mathcal{E}_b^{(0)}(\phi_b^{x,\text{eff}}) - \frac{(\delta \phi_b^x)^2}{8e^2 L_b^{\text{eff}}},
\]

and for small coupling, one can further write

\[
\mathcal{E}_b^{(0)}(\phi_b^{x,\text{eff}}) \approx \mathcal{E}_b^{(0)}(\phi_b^x) - \frac{I_b^{(0)}}{2e} \delta \phi_b^x + \frac{\chi_b^{(0)}}{8e^2} (\delta \phi_b^x)^2.
\]

Here, \( I_b^{(0)} \) is the ground-state expectation of the loop current; the susceptibility is defined as

\[
\chi_b^{(0)} = \frac{- \mathrm{d} I_b^{(0)}}{\mathrm{d} \phi_b^x} = \frac{1}{L_b} \left( 1 - \frac{\mathrm{d} (\phi_b^x)_0}{\mathrm{d} \phi_b^x} \right),
\]

which gives the bound \( \chi_b^{(0)} < 1/L_b \) (no absolute value on the lhs)—the maximum shielding is perfect diamagnetism for an uninterrupted loop (cf below (10)).

All that remains is to substitute (40) and (41) back into the full \( H \); after some cancellation, one finds

\[
H = \frac{Q_a^2}{2C_a} - E_a \cos \phi_a + \frac{(\phi_a - \phi_a^{x,\text{eff}})^2}{8e^2 L_a^{\text{eff}}} + (a \leftrightarrow c) + H_{\text{int}},
\]

\[
\phi_a^{x,\text{eff}} = \phi_a^x - 2e I_b^{(0)} M_{ab},
\]

\[
L_a^{\text{eff}} = L_a \left( 1 - \frac{M_{ab}^2}{L_a \chi_b^{(0)}} \right),
\]

\[
H_{\text{int}} = M_{ab} M_{bc} \chi_b^{(0)} I_a I_c.
\]
This does not involve assumptions on the states of the $a$- and $c$-qubits, or on their flux biases. Note that (46) involves current operators for the $a$- and $c$-devices [10], in contrast to, e.g., (42) and (44). Equations (44) and (45) imply that the qubits’ effective parameters depend subtly on the coupler’s working point, presenting a possible experimental challenge.

As in the classical case, the perturbative step has reduced the problem to that of the uncoupled $b$-SQUID, and we henceforth omit its index as well as the zero superscript indicating free devices. Evaluating (42) in the potential minimum, one verifies that (46) reduces to (9) in the classical limit. In the quantum case, numerical differentiation in (42) presents no problems, but it will be instructive, and useful for section 4, to express $\chi$ in terms of quantities at a single bias point. To this end, (42) is evaluated in terms of $\xi_0(\phi) \equiv \partial_{\phi}^x \psi_0(\phi)$, where normalization implies that $\langle \psi_0 | \xi_0 \rangle = 0$. In its turn, $\xi_0$ is solved from the $\phi^x$-differentiated Schrödinger equation

$$ (H - \mathcal{E}_0) \xi_0 + \left( \frac{\phi^x - \phi}{4e^2L} - \mathcal{E}_0^\prime \right) \psi_0 = 0, \quad (47) $$

in which $\psi_0$ figures as an inhomogeneous term, and where $\mathcal{E}_0^\prime \equiv \partial_{\phi} \mathcal{E}_0 = (\phi^x - \phi)/4e^2L$. Using a variation-of-constant solution to (47), one then derives

$$ \chi = \frac{1}{L} \left\{ 1 - \frac{C}{4e^2L} \int_{-\infty}^\infty \frac{d\phi}{\psi_0^2(\phi)} \left[ \int_{-\infty}^{\phi} d\phi_1 (\phi_1 - \langle \phi_1 \rangle_0) \psi_0^2(\phi_1) \right]^2 \right\}. \quad (48) $$

This already expresses considerable simplification, in that two of the integrals in a triple integral were written as the square of

$$ k(\phi) = \int_{-\infty}^{\phi} d\phi_1 (\phi_1 - \langle \phi_1 \rangle_0) \psi_0^2(\phi_1) = -\int_{\phi}^{\infty} d\phi_1 (\phi_1 - \langle \phi_1 \rangle_0) \psi_0^2(\phi_1). $$

However, $k(\phi) \to 0$ exponentially for $\phi \to \infty$ ($\phi \to -\infty$) will not be realized numerically for the former (latter) representation of $k$. Hence, we have to use both. Putting

$$ l_1(\zeta) \equiv \int_{-\zeta}^{\zeta} d\phi k^2(\phi)/\psi_0^2(\phi), \quad l_2(\zeta) \equiv \int_{\zeta}^{\infty} d\phi k^2(\phi)/\psi_0^2(\phi), $$

one has $\chi = L^{-1} \{ 1 - (C/4e^4L)[l_1(0) + l_2(0)] \}$. Now, $\chi$ can be found by a single integration of the ODE systems

$$ \begin{cases} 
 l_1' = k^2/\psi_0^2, \\
 k'(\phi) = (\phi - \langle \phi \rangle_0) \psi_0^2(\phi), \\
 l_1(-\infty) = k(-\infty) = 0,
 \end{cases} \quad \begin{cases} 
 l_2' = -k^2/\psi_0^2, \\
 k'(\phi) = (\phi - \langle \phi \rangle_0) \psi_0^2(\phi), \\
 l_2(\infty) = k(\infty) = 0.
 \end{cases} \quad (49) $$

A different derivation uses that, for stationary states, the shielding current equals the Josephson current (readily proven formally):

$$ \chi = 4e^2E \frac{d(sin \phi)_0}{d\phi^x}. \quad (50) $$

---

4 All wavefunctions are taken real w.l.o.g.
It is correct to evaluate (50) with the aid of (47), but this yields unwieldy asymmetric expressions. Instead, we observe that a change in flux bias is a uniform translation if there is no Josephson potential, i.e., we set

$\xi_0 \equiv \eta_0 - \psi'_0$ with $\langle \psi_0 | \eta_0 \rangle = 0$ and

$$\left( H - E_0 \right) \eta_0 + \left( E \sin \phi - E'_0 \right) \psi_0 = 0,$$

using the matching expression $E'_0 = E \langle \sin \phi \rangle_0$. Combination leads to

$$\chi = 4e^2 E \langle \cos \phi \rangle_0 - 4CE^2 \int_{-\infty}^{\infty} \frac{d\phi}{\psi_0^2(\phi)} \left[ \int_{-\infty}^{\phi} d\phi_1 \left( \sin \phi_1 - \langle \sin \phi_1 \rangle_0 \right) \psi_0^2(\phi_1) \right]^2. \tag{52}$$

The translation of (52) to ODE form is wholly analogous to (49) and will not be repeated.

For small $\beta$, (52) is clearly preferable to (48), since the double integral yields only a small correction to the first term. Compare (50), readily yielding an estimate of $\chi$ for weakly shielded SQUIDs, to (42), where in this case one has to evaluate the difference of two nearly equal terms. Conversely, for large $\beta$, $\langle \phi \rangle_0 \approx 2\pi n$ is approximately constant over most of the bias range, giving a clear advantage to (42) and (48), which capture this behaviour in their first terms. The equivalence of the four expressions for $\chi$ has been thoroughly tested numerically.

Results are shown in figure 4. It is seen that the finite spread of the ground-state wavefunction smooths out the classical susceptibility, in particular near $\phi^x = \pi$ where the FM response is maximal. The finite quantum tunnelling amplitude also means that there is no longer a sharp transition at $\beta = 1$, although the small tunnel splitting for larger $\beta$ is likely to violate the adiabaticity requirement.

4. Application: SQUID susceptibility near an anticrossing

4.1. Analytics

While the above was derived with monostable coupling devices in mind, it should be general, hence equally applicable to rf-SQUID qubits close to the anticrossing, where it can be compared

Figure 4. The reduced SQUID susceptibility $L \chi$ versus flux bias $\Phi^x$ for $\beta = 0.95$. (a) Classical response. (b) Close-up of the classical (———-) and quantum response (- - - - - , for $g = 5 \times 10^3$) for biases near $\frac{1}{2} \Phi_0$; the two curves are indistinguishable elsewhere. (c) The quantum susceptibility at $\Phi^x = \frac{1}{2} \Phi_0$ versus capacitance, showing the approach to the classical limit.
to the predictions of the two-state model

$$\hat{E}_0^\phi(\delta \Phi^x) = \text{const} - \sqrt{\Delta(\delta \Phi^x)^2 + \epsilon(\delta \Phi^x)^2},$$

(53)

with $\delta \Phi^x \equiv \Phi^x - \frac{1}{2} \Phi_0$. The bias $\epsilon$ is defined as half the difference between the ‘local well energies’, which for definiteness are taken as the ground-state energies with an artificial nodal condition imposed at the potential maximum. If $\beta - 1$ is taken fixed, then the two-state model should be valid asymptotically in $g \equiv 2CE/\epsilon^2$. Note that it incorporates several quantum effects short of tunnelling: at finite bias the curvatures of the two wells change in opposite directions, affecting the zero-point energies (note 15 in [11]), and the anharmonicity of the wells is also accounted for. The ‘tunnelling amplitude’ $\Delta$ is primarily defined through (53), without reliance on WKB estimates. Bearing in mind that $\Delta$[deltaPhi^x] [epsilon(deltaPhi^x)] is even [odd], one finds $\chi^0(0) = \hat{E}_0^0(0)^n = -\Delta'(0) - I_p/\Delta$ for the two-state approximation to $\chi$, with $\Delta \equiv \Delta(0)$ and $I_p \equiv -\epsilon(0)$.

These predictions make it clear that $\chi(0)$ is exponentially large in $g$, so that in the evaluation of (48) and (52), their first terms can be safely ignored. Further, $(\phi)_0 = \pi$ and $(\sin \phi)_0 = 0$ by symmetry for $\phi^x = \pi$, which will be assumed henceforth. In either formula, the increase in $\chi$ can only come from the factor $\psi_0^{-2}$, which is large for $|\phi| \rightarrow \infty$ and near the barrier. However, in both formulas and both for $\phi \rightarrow \infty$ and $\phi \rightarrow -\infty$, this is overcome by $[\cdots]^2$ decaying as $\psi_0^4$. Therefore, the only relevant contribution comes from the barrier region, where to within exponentially small corrections the factors $[\cdots]^2$ are constant, and readily related to the loop current. Thus, in the regime of interest, both (48) and (52) evaluate to

$$\chi(0) \approx \frac{-CI_p^2}{4e^2} \int_b^d \frac{d\phi}{\psi_0^2(\phi)},$$

(54)

where $\int_b^c$ means an integral over the barrier region, which for definiteness can be taken as everything between the two well minima (so as not to rely on ‘classical turning points’). To relate the integral in (54) to $\Delta$, define $f(\phi, \mathcal{E})$ as the solution to the Schrödinger equation at (continuously tunable) energy $\mathcal{E}$ satisfying the left boundary condition. Of course, this function will diverge for $\phi \rightarrow \infty$ unless $\mathcal{E}$ is an eigenenergy. Its energy derivative satisfies a Schrödinger equation in which $f$ figures as an inhomogeneous term (cf (47) for the bias derivative), which we solve by the variation-of-constant ansatz $\partial_c f = fh$. Assuming that the $\mathcal{E}$-dependence of $f$ can be linearized for $\mathcal{E}_0 \leq \mathcal{E} \leq \mathcal{E}_1$, the significance of these functions is that $\partial_c f(\pi, \mathcal{E}) \approx [0 - \psi_0(\pi)]/2\Delta \Rightarrow h(\pi, \mathcal{E}_0) \approx -1/2\Delta$. Solving for $h$, one finds

$$h'(\phi) = -\frac{C}{2e^2} \int_{-\infty}^{\phi} d\phi' f^2(\phi_1) f^2(\phi);$$

(55)

this defines $h$ only up to a constant, since $f$ was defined only up to a, possibly $\mathcal{E}$-dependent, normalization. We now fix $\int_{-\infty}^{\pi} d\phi f^2(\phi) = \frac{1}{2}$, so that $h'(\phi) \approx -C/4e^2 f^2(\phi)$ throughout the barrier region. Further, with $\phi_{\min}$ the left potential minimum, $h(\phi_{\min}, \mathcal{E})$ is of order one, hence negligible, and one finds $h(\pi, \mathcal{E}_0) \approx -(C/4e^2) \int_{\phi_{\min}}^{\pi} d\phi \psi_0^{-2}(\phi) = -(C/8e^2) \int_b^d d\phi \psi_0^{-2}(\phi)$.

Combination shows that $\Delta^{-1} = (C/4e^2) \int_b^d d\phi \psi_0^{-2}(\phi)$, and substitution in (54) yields

$$\chi(0) \approx \frac{-I_p^2}{\Delta}.$$
Comparing with what was found below (53) for $\chi_{qb}(0)$ in the two-state approximation, this leads one to conclude that actually $\Delta''(0) = 0$. More precisely, comparing the orders of magnitude, it indicates that $\Delta$ does not have relative variations of order one in the interval $|\epsilon| \sim \Delta$, which is all one cares about. This flatness of $\Delta$ in the anticrossing region is often assumed without much discussion, let alone proof; even the comparatively detailed WKB treatment in [12] is complete only for highly excited states in the local wells. Yet, the standard WKB expression for the tunnelling amplitude is very sensitive to the barrier action integral, suggesting that there could be an equally strong sensitivity to bias variations.

All in all, the above paints a positive picture of the standard two-level approximation. Its regime of validity should follow from $|\epsilon(\delta/\Phi_1)| \ll E_2 - E_1$, where the rhs is of the order of the plasma frequency. Hence, the range of allowed $\delta\Phi^x$ is actually algebraically decreasing in $g$ so that asymptotically, variations in loop current etc. become negligible within this range. At the same time, the range of permissible $\epsilon/\Delta$ increases exponentially, so that wide sweeps in $\epsilon$, as e.g. studied in the context of Landau–Zener transitions [13], can be performed without leaving this range.

4.2. Numerics

If one numerically finds the two lowest eigenfunctions of a near-degenerate bistable rf-SQUID, one can \textit{uniquely} determine the parameters of the corresponding two-level low-energy Hamiltonian

$$H_{qb} = -\epsilon \sigma^z - \Delta \sigma^x.$$  \hfill (57)

Namely, from the numerics one extracts the gap $\mathcal{E}_{10} \equiv \mathcal{E}_1 - \mathcal{E}_0$, plus the well distribution of the ground-state wavefunction, $p^2 = \int_{-\infty}^{\phi_b} d\phi \psi_0^2(\phi)$ and $q^2 = \int_{\phi_b}^{\infty} d\phi \psi_0^2(\phi)$, where the barrier point $\phi_b$ denotes the location of the potential maximum. The average energy is an irrelevant constant (assumed zero in the traceless (57)), $p$ and $q$ are constrained by normalization, and the well distribution of the first excited state is fixed by orthogonality, so that one has two pieces of data. Comparing these to the predictions of (57), one can solve for the two unknowns as

$$\epsilon = \mathcal{E}_{10} \frac{q^2 - p^2}{2}, \quad \Delta = \mathcal{E}_{10} pq.$$  \hfill (58)

Thus, the coefficients of the two-state model are extracted without any handwaving, even without the artifice of an extra boundary condition stipulated below (53); the only ingredient is a physically transparent coarse-graining of the wavefunctions. Note how $\epsilon$, $\Delta$ attain the expected limits at degeneracy $p = q$, and $\epsilon$ also for large bias $pq \rightarrow 0$.

Another way to arrive at this result is easier to generalize to multiple qubits. It involves using the transition matrix \textsuperscript{6} $U = \begin{pmatrix} q & p \\ p & q \end{pmatrix}$ (i.e., having the eigenstates for its columns) to rotate

\textsuperscript{5} We have used similar wordings to describe the procedure in [3, 11], but the concepts are different. There, the impedance response as a function of flux bias was used to fit a single set of qubit-Hamiltonian coefficients. Here, we have numerical access to the wavefunctions, and can therefore determine these coefficients independently at each point in parameter space.

\textsuperscript{6} Sign convention: it is logical to have $\delta\Phi^x < 0$ correspond to $\epsilon < 0$. The former favours smaller $\phi$, i.e., the left well; by (57), the latter favours $\sigma^z = -1$, so that the bottom row in $U$ corresponds to the left well. Further, the columns in $U$ and those in $H_\epsilon$ of course have to be ordered consistently.
For consistency, the columns of $\phi$ of last entries in table 1 for each small compared to the depopulated well is so small that contributions near contradict the analytical conclusions. The breakdown of (57) occurs when the wavefunction in $\tilde{\psi}$ should thus consider there is no natural counterpart to this shortcut, and the $4 \times 4$ in terms of not the end of section 4.1 for increasing a coarse-grained description. Even then, the linearity of $\text{unitarity on } U$ has circumvented coarse-graining the excited state $\psi_1$. Already for two qubits, there is no natural counterpart to this shortcut, and the $4 \times 4$ matrix $U$ having coarse-grained eigenstates for its columns is not exactly unitary, introducing a small uncertainty into the above procedure. The root cause of these subtleties is the problematic nature of the ubiquitous term ‘flux basis’, which strictly speaking should consist of $\delta$-functions in the coordinate representation. Pending further research, a completely satisfactory resolution may be as follows: calculate the matrix elements of $\phi$ in the low-energy subspace. This Hermitian matrix has an eigenbasis, and the eigenvectors can be taken as columns of a transition matrix $V$.\textsuperscript{7} Recognizing that $V$ consists of $\phi$-eigenvectors in the energy basis whereas $U$ consists of $\varepsilon$-eigenvectors in the flux basis, one should thus consider $H_{\text{qb}} \equiv V^\dagger H_{E} V$ as an alternative to (59).

\textsuperscript{7} For consistency, the columns of $V$ should be ordered according to decreasing eigenvalues. For, say, two qubits, these columns will be the simultaneous eigenvectors of the commuting operators $\phi_a, \phi_b$ in the low-energy subspace.

| $\beta$ | $\delta \Phi^x/\Phi_0$ | $\epsilon$ | $\Delta$ |
|---|---|---|---|
| 2 | $5 \times 10^{-6}$ | $2.63 \times 10^{-5}$ | $5.22120 \times 10^{-3}$ |
| 2 | $5 \times 10^{-5}$ | $2.63 \times 10^{-4}$ | $5.22134 \times 10^{-3}$ |
| 2 | $5 \times 10^{-4}$ | $2.63 \times 10^{-3}$ | $5.23581 \times 10^{-3}$ |
| 2 | $5 \times 10^{-3}$ | $2.64 \times 10^{-2}$ | $6.09330 \times 10^{-3}$ |
| 3 | $5 \times 10^{-6}$ | $2.28 \times 10^{-5}$ | $3.00032 \times 10^{-4}$ |
| 3 | $5 \times 10^{-5}$ | $2.28 \times 10^{-4}$ | $3.00082 \times 10^{-4}$ |
| 3 | $5 \times 10^{-4}$ | $2.28 \times 10^{-3}$ | $3.05012 \times 10^{-4}$ |
| 3 | $5 \times 10^{-3}$ | $2.28 \times 10^{-2}$ | $5.61648 \times 10^{-4}$ |
| 6 | 0 | 0 | $5.64441 \times 10^{-6}$ |
| 6 | $5 \times 10^{-7}$ | $1.38 \times 10^{-6}$ | $5.64441 \times 10^{-6}$ |
| 6 | $5 \times 10^{-6}$ | $1.38 \times 10^{-5}$ | $5.64450 \times 10^{-6}$ |
| 6 | $5 \times 10^{-5}$ | $1.38 \times 10^{-4}$ | $5.65360 \times 10^{-6}$ |
| 6 | $5 \times 10^{-4}$ | $1.38 \times 10^{-3}$ | $6.50793 \times 10^{-6}$ |

Results are summarized in table 1. With increasing $\beta$, one sees the same trends predicted at the end of section 4.1 for increasing $g$: the range of validity of (57) with constant $\Delta$ increases, in terms of not $\delta \Phi^x$ but $\epsilon/\Delta$. Remaining deviations $\Delta(\delta \Phi^x) - \Delta(0) \sim (\delta \Phi^x)^2$ are too small to contradict the analytical conclusions. The breakdown of (57) occurs when the wavefunction in the depopulated well is so small that contributions near $\phi_0$ can no longer be neglected, precluding a coarse-grained description. Even then, the linearity of $\epsilon(\delta \Phi^x)$ is still excellent, as shown by the last entries in table 1 for each $\beta$, and the breakdown is of little practical consequence as $\Delta$ is too small compared to $|\epsilon|$ to have an appreciable effect on the spectrum of (57).

The above situation for a single qubit is uncharacteristically fortunate, in that the requirement of unitarity on $U$ has circumvented coarse-graining the excited state $\psi_1$. Already for two qubits, there is no natural counterpart to this shortcut, and the $4 \times 4$ matrix $U$ having coarse-grained eigenstates for its columns is not exactly unitary, introducing a small uncertainty into the above procedure. The root cause of these subtleties is the problematic nature of the ubiquitous term ‘flux basis’, which strictly speaking should consist of $\delta$-functions in the coordinate representation. Pending further research, a completely satisfactory resolution may be as follows: calculate the matrix elements of $\phi$ in the low-energy subspace. This Hermitian matrix has an eigenbasis, and the eigenvectors can be taken as columns of a transition matrix $V$. Recognizing that $V$ consists of $\phi$-eigenvectors in the energy basis whereas $U$ consists of $\varepsilon$-eigenvectors in the flux basis, one should thus consider $H_{\text{qb}} \equiv V^\dagger H_{E} V$ as an alternative to (59).
5. Conclusion

A different sign-tunable flux transformer, relying on the gradiometric nature of the employed qubit, was previously considered in [14]. The transformer itself is also gradiometric, and tunability is achieved by incorporating compound junctions with variable couplings in either arm. In another type of gradiometric flux transformer, each arm of the gradiometer couples to one adjacent device, and the tunable element is a single compound junction in the central leg [15]. This does not give sign-tunability, though: compare (4) in [15] to (10) above. For the present type of mediated coupling, the prediction of sign-tunability is extremely generic: for any SQUID-type device, the $I(\Phi)$ curve is smooth and periodic, and therefore will have pieces with both positive and negative slope in any $\Phi_0$-interval. Comparing the two implementations, the rf-SQUID (section 2.1) does not have any galvanic coupling to external circuitry, which should limit decoherence. On the other hand, it was already mentioned that a dc-SQUID coupler (section 2.2) [6] could be useful for readout. Either seems promising, and we look forward to experimental results on tunable coupling, which may well be the next big step forward in superconducting-qubit technology.

The analysis in section 4 is a prelude to the case of multiple interacting qubits, where the precise form of the low-energy Hamiltonian $H_{qb}$ is still open to discussion. Namely, there could be multi-qubit tunnelling, say, along the diagonals in a two-qubit potential, which would lead to $\sigma^x-\sigma^y$ and $\sigma^y-\sigma^x$ interactions ([10], note in proof). There is little doubt that such terms would be small, but this does not automatically imply that their effect is negligible. Namely, they could, say, cause a gap at an $|\uparrow\downarrow\rangle \leftrightarrow |\downarrow\uparrow\rangle$ anticrossing (for AF-coupled qubits) in the first order, while conventional tunnelling terms $\sim \sigma^+$ contribute to this gap only in the second order. Further, there seems to have been little investigation of the possibility that (single-qubit) tunnelling in qubit $a$ might depend on the state of qubit $b$ and vice versa, which would lead to $\sigma^+_a \sigma^-_b$ etc terms. To the extent that, for $a$-tunnelling, the $b$-qubit can be regarded as an external flux bias, the flatness of $\Delta$ observed in section 4 indicates that such terms should be very small. Finally, for more than two qubits, one can anticipate three-qubit etc interactions—particularly if, for stronger coupling, one qubit’s persistent current can depend on the state of the others. In any case, rotation methods such as in (59) should enable one to settle all these questions conclusively.

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