CONNORMAL VARIETIES OF COVEXILLARY SCHUBERT VARIETIES

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ABSTRACT. A permutation is called covexillary if it avoids the pattern 3412. We construct an open embedding of a covexillary matrix Schubert variety into a Grassmannian Schubert variety. As applications of this embedding, we show that the characteristic cycles of covexillary Schubert varieties are irreducible, and provide a new proof of Lascoux’s model computing Kazhdan-Lusztig polynomials of vexillary permutations. Combining the above embedding with earlier work of the author on the conormal varieties of Grassmannian Schubert varieties, we develop an algebraic criterion identifying the conormal varieties of covexillary Schubert and matrix Schubert varieties as subvarieties of the respective cotangent bundles.

The conormal varieties of Schubert varieties play a central role in the Springer representations of Weyl groups, see [Spr78, Spr82, CG97]. The work of Aluffi, Mihailea, Shürmann, and Su [AMSS17] sheds light on the relation between the torus-equivariant (co)-homology classes of conormal varieties and the “stable basis” of Maulik and Okounkov [MO19]. The conormal variety is defined as the closure (in the cotangent bundle) of the conormal bundle of the smooth locus of a variety. While the conormal bundle of the smooth locus often admits simple descriptions, it is in general a difficult problem to describe the boundary, singularities, cohomology classes etc. of its closure. In this paper, we study covexillary Schubert varieties and their conormal varieties.

Recall that a partial permutation is called covexillary if its essential set goes from top-right to bottom-left, see Section 1.2 for details. Following [Ful92], a permutation, i.e., a full-rank partial permutation matrix, is covexillary if it avoids the pattern 3412. Lascoux and Schützenberger [LS82] (see also Fulton [Ful92]) showed that the cohomology class of a covexillary Schubert variety is a multi-Schur function, generalizing Giambelli’s formula for the cohomology class of a Grassmannian Schubert variety. More recently, Anderson, Ikeda, Jeon, and Kawago [AIMR21] have shown that covexillary Schubert varieties have the same singularities as Schubert subvarieties of Grassmannians. A priori, it is not clear why covexillary Schubert varieties behave like Grassmannian Schubert varieties for both these properties (cohomology classes and singularity types). In this paper, we provide a construction (see Theorem 0.1 below) that simultaneously explains both these properties.

Let $B$ denote the set of upper triangular invertible $n \times n$ matrices, $g$ the set of $n \times n$ matrices, $w$ a partial permutation matrix, and $g_w = BwB$ the closure (in $g$) of the $B \times B$-orbit of $w$, i.e., a matrix Schubert variety. Our first result (Theorem 1.6) is the construction, for covexillary $w$, of an open embedding of $g_w$ in some Schubert subvariety of the Grassmannian $Gr(n, 2n)$.

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1 We use the ‘homological’ indexing, corresponding to $\dim(F_{\ell(w)}) = \ell(w)$, where $\ell(w)$ is the length of the word $w$. In particular, the Schubert varieties that we call covexillary are called vexillary in [LS82, Ful92, Las95] etc.
Theorem 0.1. Let \( w \) be a covexillary permutation. Let \( h : g \to Gr(n, 2n) \) be the graph embedding, i.e., the map given by \( x \mapsto \text{Col} \left( \begin{array}{c} I(n) \\ x \end{array} \right) \). There exists a permutation \( \tau \in S_{2n} \) and a Schubert variety \( Gr_{w} \subset Gr(n, 2n) \) such that the composite map

\[
\tau \circ h : g \to Gr(n, 2n), \quad x \mapsto \tau \text{Col} \left( \begin{array}{c} I(n) \\ x \end{array} \right)
\]

induces an open immersion \( g_{w} \hookrightarrow Gr_{w} \).

Let us explain how Theorem 0.1 relates the singularities and cohomology classes of covexillary Schubert varieties with those of Grassmannian Schubert varieties. Let \( E_{\bullet} = (E_{1} \subset \cdots \subset E_{n-1}) \) denote the standard flag, \( Fl \) the flag variety, and \( \pi : G \to Fl \) the map \( g \mapsto gE_{\bullet} \). Let \( Fl_{w} = \overline{BwE_{\bullet}} \) denote the Schubert variety corresponding to a permutation matrix \( w \), and \( G_{w} := \pi^{-1}(Fl_{w}) \) the pull-back of \( Fl_{w} \) along \( \pi \). Observe that since \( \pi \) is a locally trivial fibration with smooth fibres, \( Fl_{w} \) has the same singularities as \( G_{w} \). Further, \( G_{w} \) is an open subset of \( g_{w} \). Now, if \( w \) is covexillary, it follows from Theorem 0.1 that \( g_{w} \) (and hence also \( G_{w} \)) can be viewed as an open subset of \( Gr_{w} \) and hence has the same singularities as \( Gr_{w} \).

Let \( w_{0} \) be the longest permutation in \( S_{n} \), and let \( T \subset B \) be the set of diagonal matrices in \( B \). Recall that the double Schubert polynomial \( S_{\nu_{w}w} \), which computes the \( T \)-equivariant cohomology class of \( Fl_{w} \), is precisely the \( T \times T \) multidegree of \( g_{w} \) in \( g \), i.e., the localization of the \( T \times T \)-equivariant cohomology class of \( g_{w} \) at \( 0 \). Let \( T_{2n} \) denote the set of diagonal matrices in \( GL_{2n} \), acting on \( Gr(n, 2n) \) by left multiplication. The embedding \( g_{w} \to Gr_{w} \) sends \( 0 \in g_{w} \) to \( \tau \in Gr(n, 2n) \) and is equivariant for an appropriate identification \( T \times T \cong T_{2n} \) (the precise identification depends on the word \( w \)). This relates the \( T \times T \) multidegree of \( g_{w} \) in \( g \), i.e., the double Schubert polynomial \( S_{\nu_{w}w} \), with the \( T_{2n} \)-equivariant localization of \( Gr_{w} \) at the point \( \tau \), which is a multi-Schur function.

The fact that covexillary Schubert varieties have the same singularities as Grassmannian Schubert varieties yields some immediate consequences. The characteristic cycle of the IC sheaf of a Schubert variety \( Fl_{w} \) is irreducible if and only if for each \( T \)-fixed point \( v \), the local Euler obstruction of \( Fl_{w} \) at \( v \) equals the evaluation of the Kazhdan-Lusztig polynomial \[KL80a, KL80b\] evaluated at \( 1 \). Both the Kazhdan-Lusztig polynomials and the Euler obstructions are local invariants, and hence we obtain the following result (Theorem 1.10).

Theorem 0.2. The characteristic cycle of the IC sheaf of a covexillary Schubert variety is irreducible.

In [LS81], Lascoux and Schützenberger gave a combinatorial model to compute the Kazhdan-Lusztig polynomial of a Grassmannian Schubert variety. Later, Lascoux [Las95] extended this to covexillary Schubert varieties, showing that the Kazhdan-Lusztig polynomials of a covexillary Schubert variety are the same as certain Kazhdan-Lusztig polynomials of some Grassmannian Schubert variety. Since the Kazhdan-Lusztig polynomials are local invariants, Theorem 0.1 gives an alternate proof of Lascoux’s result. Further, by Theorem 0.2, the Lascoux-Schützenberger model is also an effective algorithm for computing the Euler obstructions of covexillary Schubert varieties.
Finally, we turn our attention to the conormal variety \( N^*\mathcal{F}l_w \) of \( \mathcal{F}l_w \) in \( \mathcal{F}l \), which we relate to the conormal variety \( N^*Gr_w \) of \( Gr_w \) in \( Gr(n, 2n) \) via the commutative diagram in Equation (1). Here \( f_{\#} : T^*M \to T^*N \) denotes the open immersion induced on cotangent bundles from an open immersion \( f : M \to N \) of smooth varieties. The existence of a map \( \pi' : N_*G_w \to N_*\mathcal{F}l_w \) is a consequence of the smoothness of the map \( \pi : G \to \mathcal{F}l \), see [HIT08].

\[
\begin{align*}
N^*\mathcal{F}l_w & \leftarrow N_*G_w \xrightarrow{\pi_{\#}} N_*g_w \xrightarrow{\tau_{\#} \circ h_{\#}} N_*Gr_w \\
\mathcal{F}l_w & \leftarrow G_w \xrightarrow{\pi} g_w \xrightarrow{\tau_{\#}} Gr_w \\
\mathcal{F}l & \leftarrow G \xrightarrow{\pi} g \xrightarrow{\tau_{\#}} Gr(n, 2n)
\end{align*}
\]

In [Sim21], algebraic conditions, equivalently a (possibly non-reduced) system of equations, was developed identifying the conormal variety of a Grassmannian Schubert variety as a closed subvariety of the cotangent bundle of the Grassmannian. Following Equation (1), the conormal variety \( N_*g_w \) of the matrix Schubert variety \( g_w \) is simply the pull-back of the conormal variety \( N_*Gr_w \) along the map \( \tau_{\#} \circ h_{\#} : T_*g_w \to T_*Gr(n, 2n) \), or equivalently, \( N_*g_w = N_*Gr_w g_w \). Using this characterization of \( N_*g_w \), along with the results of [Sim21], we develop a description of \( N_*g_w \) as a subvariety of \( T^*g \). (see Theorem 2.4 for a precise statement).

Using the trace form, we make the identifications \( g^* = g \) and \( T^*g = g \times g \).

**Theorem 0.3.** Suppose \( w \) is covexillary. A point \( (x, y) \in g \times g = T^*g \) belongs to the conormal variety \( N_*g_w \) of \( g_w \) if and only if \( \) certain rank conditions are satisfied on certain sub-matrices of the \( 2n \times 2n \) matrix \( \begin{pmatrix} yx & y \\ xy & xy \end{pmatrix} \).

Consider the pull-back \( \pi^*T^*\mathcal{F}l \) of \( T^*\mathcal{F}l \) along \( \pi : G \to \mathcal{F}l \), and the induced map \( \pi' : \pi^*T^*\mathcal{F}l \to T^*\mathcal{F}l \). A point \( p \in T^*\mathcal{F}l \) belongs to the conormal variety \( N^*\mathcal{F}l_w \) if and only if \( \pi'\tau_{\#}(x) \in N^*G_w \). Moreover, the latter is precisely the restriction of \( N_*g_w \) along the inclusion \( G_w \to g_w \). This allows us to develop a characterization of \( N^*\mathcal{F}l_w \).

**Theorem 0.4.** Suppose \( w \) is covexillary, and let \( 1 \leq p_1 \leq \cdots \leq p_m \leq n \), \( 1 \leq q_1 \leq \cdots \leq q_m < n \) and \( r_1, \cdots, r_m \) be integers such that

\[
\dim(F_{q_i}/E_{p_i}) \leq r_i, \quad \forall 1 \leq i \leq m
\]

are the minimal conditions describing \( \mathcal{F}l_w \) as a subvariety of \( \mathcal{F}l \) (see Lemma 1.2). Consider the cotangent bundle,

\[
T^*\mathcal{F}l = \{(F_*, z) \in \mathcal{F}l \times g | zF_i \subset F_{i-1} \forall i\}.
\]

Then \( (F_*, z) \in N^*\mathcal{F}l_w \) if and only if \( F_* \in \mathcal{F}l_w \), and

\[
\dim(z(F_{q_i} + E_{p_i})/(F_{q_j} \cap E_{p_j})) \leq \begin{cases} (q_{i-1} - r_{i-1}) - (q_j - r_j), & \forall 1 \leq j < i \leq n \\ (p_i + r_i) - (p_{j+1} + r_{j+1}), & \forall 1 \leq j \leq n \end{cases}
\]
1. Matrix Schubert Varieties and the Graph Embedding

Let $E_n$ be a $n$-dimensional vector space with standard basis $e_1, \cdots, e_n$. For linear subspaces $V, V' \subseteq E_n$, we will denote by $V/V'$ the image of $V$ under the projection $E_n \to E_n/V'$.

Let $g$ be the set of $n \times n$ matrices, $G = GL_n$ the set of invertible matrices in $g$, and $B$ the set of upper triangular matrices in $G$. The algebra $g$ (resp. the group $G$) acts on $E_n$ by left multiplication with respect to the ordered basis $e_1, \cdots, e_n$.

1.1. Partial Permutations and the rank matrix. A partial permutation (of size $n$) is a $\{0,1\}$-matrix (of size $n \times n$) such that each row and each column contains at most one non-zero entry. A full-rank partial permutation matrix contains exactly one non-zero entry in each row and column, and hence corresponds to a permutation of the set $\{1, \cdots, n\}$. (The permutation $w \in S_n$ corresponds to the $n \times n$ matrix with 1s in the positions $\{(w(i), i) | 1 \leq i \leq n\}$, and 0s elsewhere).

Given a partial permutation $w$ of size $n \times n$, we define a rank matrix $r^w$ given by

$$r^w_{ij} = \dim(wE_j/E_{i-1}).$$

Equivalently, $r^w_{ij}$ is the rank of the bottom-left sub-matrix of $w$ spanned by the rows $i, i + 1, \cdots, n$, and the columns $1, \cdots, j$.

Example 1.1. Consider the permutation $w \in S_6$ given in one-line notation by $w = [351642]$. The corresponding permutation and rank matrices are given by

$$w = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad r^w = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 2 & 3 & 3 & 4 \\
1 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 2 & 2 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}.$$

1.2. Diagram and Essential Set. Consider a partial permutation $w$ of size $n$. For each non-zero entry in $w$, we shade every box that appears above it and every box that appears to its right. The set of unshaded boxes is called the diagram $D(w)$ of $w$. The set of northeast corners of $D(w)$ is called the essential set $\mathcal{E}ss(w)$ of $w$,

$$\mathcal{E}ss(w) = \{(p, q) \in D(w) | (p - 1, q), (p, q + 1), (p - 1, q + 1) \notin D(w)\}.$$ 

It is clear that the essential set does not contain any boxes in the top row or the last column.

Given a partial permutation $w$, let $\hat{w}$ be the unique $2n \times 2n$ permutation matrix whose bottom-left $n \times n$ submatrix is $w$, and the entries in the top $n$ rows and in the last $n$ columns go from bottom-left to top-right. Then the essential set of $\hat{w}$ is contained in the bottom-left $n \times n$ submatrix, and agrees with the essential set of $w$. The significance of essential sets can be gleaned from the following result.
Lemma 1.2. (cf. [Ful92]) The partial permutation matrix \( w \) is determined by the pair \((\mathcal{E}ss(w), r_w|\mathcal{E}ss(w))\), i.e., the essential set of \( w \), and the restriction of the rank matrix to this essential set.

An algorithm to recover \( w \) from the pair \((\mathcal{E}ss(w), r_w|\mathcal{E}ss(w))\) was developed by Eriksson and Linusson in [EL96].

Definition 1.3. We say that a partial permutation \( w \) is covexillary if its essential set goes from the top-left to the bottom-right, i.e., there exist integers \( 1 \leq p_1 \leq \cdots \leq p_{m-1} < n \) and \( 1 \leq q_1 \leq \cdots \leq q_{m-1} < n \) such that

\[
\mathcal{E}ss(w) = \{(p_1 + 1, q_1), \cdots , (p_{m-1} + 1, q_{m-1})\}.
\]

Lemma 1.4. (cf. [Ful92]) A permutation \( w \in S_n \) is covexillary if and only if it avoids the pattern 3412, i.e., there does not exist any quadruple \( 1 \leq i < j < k < l \leq n \) satisfying \( w(k) < w(l) < w(i) < w(j) \).

1.3. Schubert varieties in the Flag Manifold. Let \( E_i = \langle e_1, \cdots , e_n \rangle \), let \( E_\bullet = (E_1 \subset \cdots \subset E_{n-1}) \) denote the standard flag in \( E_n \), and let \( \mathcal{F}l \) denote the variety of flags in \( E_n \), i.e.,

\[
\mathcal{F}l = \{(F_1 \subset \cdots \subset F_{n-1}) \mid \dim F_i = i\}.
\]

For \( w \in S_n \), the \( B \)-orbit \( \mathcal{F}l_w^0 \) of the permutation flag \( wE_\bullet \) in \( \mathcal{F}l \) is called a Schubert cell, and its closure \( \mathcal{F}l_w \) a Schubert variety. We have

\[
\mathcal{F}l_w^0 = \{F_\bullet \in \mathcal{F}l \mid \dim(F_j/E_{i-1}) = r_w(i, j) \ \forall \ 1 \leq i, j \leq n\},
\]

\[
\mathcal{F}l_w = \{F_\bullet \in \mathcal{F}l \mid \dim(F_j/E_{i-1}) \leq r_w(i, j) \ \forall \ 1 \leq i, j \leq n\}.
\]

Given \( u, w \in S_n \), we have \( \mathcal{F}l_u \subset \mathcal{F}l_w \) if and only if

\[
r_u(i, j) \leq r_w(i, j) \ \forall \ 1 \leq i, j \leq n.
\]

The partial order \( u \leq w \iff \mathcal{F}l_u \subset \mathcal{F}l_w \) is called the Bruhat order on \( S_n \).

Remark 1.5. A permutation which avoids the pattern 2143 is called vexillary. Let \( w_0 \) denote the longest permutation in \( S_n \). Then \( w \in S_n \) is covexillary if and only if \( w_0w \) is vexillary. There are two common conventions for indexing Schubert varieties by permutations: one corresponds to \( \dim(\mathcal{F}l^w) = \ell(w_0) - \ell(w) \), the other to \( \dim(\mathcal{F}l_w) = \ell(w) \), where \( \ell(w) \) is the length of \( w \). The Schubert varieties indexed by vexillary permutations in the former convention correspond to the ones indexed by covexillary permutations in the latter.
1.4. Grassmannian Schubert varieties. Let $Gr(d, n)$ denote the Grassmannian variety of $d$-dimensional subspaces of $E_n$. The $B$-orbit closures $Gr_{w} \subset Gr(d, n)$ (called Grassmannian Schubert varieties) are indexed by increasing $d$-sequences $w$ taking values in $\{1, \cdots, n\}$. Given such a sequence $w = (u_1, \cdots, u_d)$, we have

$$Gr_{w} = \{ V \subset E_n \mid \dim (V + E_{u_i}) \leq d + u_i - i \forall 1 \leq i \leq d \}.$$  

We have $Gr_{w} \subset Gr_{w'}$ if and only if $u_i \leq v_i$ for all $1 \leq i \leq d$.

Note the redundancy inherent in the conditions in Equation (3). Given a sequence $w$ as above, let $S_w = \{ i \in [d] \mid u_{i+1} \neq u_i + 1 \}$. Then

$$Gr_{w} = \{ V \subset E_n \mid \dim (V + E_{u_i}) \leq d + u_i - i \forall i \in S_w \}.$$  

1.5. Matrix Schubert varieties. Consider the $B \times B$-action on $g$ given by

$$(b_1, b_2) \cdot x = b_1 xb_2^{-1}. $$

The orbits $g_w \subset g$ of this action are indexed by $n \times n$ partial permutations. Here $g_w$ denotes the $B \times B$-orbit in $g$ of the partial permutation matrix $w$. We call $g_w$ a matrix Schubert cell, and its closure $\overline{g}_w := \overline{g}_w^o$ a matrix Schubert variety. We have

$$\overline{g}_w^o = \{ x \in g \mid \dim (xE_j/E_{i-1}) = r(w)(i, j) \forall 1 \leq i, j \leq n \},$$

$$g_w = \{ x \in g \mid \dim (xE_j/E_{i-1}) \leq r(w)(i, j) \forall 1 \leq i, j \leq n \}.$$  

Given partial permutations $u, w$, we have $g_u \subset g_w$ if and only if

$$r_u(i, j) \leq r_w(i, j) \quad \forall i, j.$$  

The partial order $u \leq w \iff g_u \subset g_w$ (also called the Bruhat order) extends the Bruhat order on permutations.

The set $G \subset g$ is a $B \times B$-stable open subvariety of $g$. We have $g_w^o \subset G$ if and only if $w$ is a permutation matrix, i.e., if and only if $rk(w) = n$. In this case, we set

$$G_w = \overline{g}_w \cap G = \bigcup_{w \in S_n \atop w \leq w} g_w^o.$$  

Consider the map $\pi : G \rightarrow Fl$, given by $g \mapsto gE_\bullet$. Comparing Equations (2) and (4), we see that $G_w = \pi^{-1}(Fl_w)$.

1.6. Open immersion for coexillary matrix Schubert varieties. Let $w$ be a coexillary partial permutation, so that

$$E_{ss}(w) = \{ (p_1 + 1, q_1), \cdots, (p_{m-1} + 1, q_{m-1}) \}$$

for some $0 \leq p_1 \leq \cdots \leq p_{m-1} \leq n - 1$ and $1 \leq q_1 \leq \cdots \leq q_{m-1} \leq n$. Let $r_w$ be the rank matrix of $w$, and set $r_i = r_w^{p_i+1, q_i}$. For convenience, we set $p_m = q_m = n$, and $t_i = p_i + q_i$. Let $\tau$ be the $2n \times 2n$ permutation matrix given by

$$\tau = \begin{pmatrix}
I(q_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I(p_1) & 0 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & I(q_m - q_{m-1}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I(p_m - p_{m-1}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I(p_{m-1} - p_{m-2}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} ,$$

where $I(k)$ denotes the identity matrix of size $k$, and the 0s are rectangular zero matrices of appropriate sizes. By an abuse of notation, we will also denote by
\( \tau : Gr(n, 2n) \to Gr(n, 2n) \) the automorphism of \( Gr(n, 2n) \) induced by the left multiplication action of \( \tau \) on \( E_n \).

Let \( \text{Col}(\ ) \) denote the column span of a matrix, and consider the map

\[
(6) \quad h : g \to Gr(n, 2n), \quad h(x) = \text{Col} \left( \begin{array}{c} I_n \\ x \end{array} \right).
\]

If we view \( x \in g \) as a map \( x : E_n \to E_n \), then \( h(x) \) can be identified as the graph of this map. Hence we call \( h : g \to Gr(n, 2n) \) the graph embedding.

**Theorem 1.6.** Let \( w \in S_n \) and \( \tau \in S_{2n} \) be as in Equation (5). Let \( Gr_w \) be the Schubert subvariety of \( Gr(n, 2n) \) given by

\[
Gr_w = \{ V \in Gr(n, 2n) \mid \dim(V + E_t_i) \leq n + p_i + r_i \}.
\]

The composite map \( \tau \circ h : g \to Gr(n, 2n) \) induces an open immersion \( g_w \hookrightarrow Gr_w \).

The proof of Theorem 1.6 depends on the following lemma.

**Lemma 1.7.** Let \( V = (e_1, \cdots, e_{n+1}, \cdots, e_{n+p}) \). For \( x \in g \), we have

\[
\dim(xE_q/E_p) \leq r \iff \dim(h(x) + V) \leq n + p + r.
\]

**Proof.** Let us write \( x \in g \) in block-matrix form,

\[
x = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

with \( a \) a \( p \times q \) matrix, \( b \) a \( p \times (n-q) \) matrix, \( c \) a \((n-p) \times q \) matrix, and \( d \) a \((n-p) \times (n-q) \) matrix. The result follows from the observations \( \dim(xE_q/E_p) = \text{rk}(c) \), and

\[
\dim(h(x) + V) = \text{rk} \begin{pmatrix} I(q) & 0 & I(q) & 0 \\ 0 & I(p) & 0 & 0 \\ a & b & 0 & I(p) \\ c & d & 0 & 0 \end{pmatrix} = \text{rk} \begin{pmatrix} 0 & 0 & I(q) & 0 \\ 0 & I(p) & 0 & 0 \\ 0 & 0 & 0 & I(p) \\ c & 0 & 0 & 0 \end{pmatrix} = n + p + \text{rk}(c).
\]

\( \square \)

**Proof of Theorem 1.6** Since \( h : g \hookrightarrow Gr(n, 2n) \) is an open immersion, and \( V \hookrightarrow \tau V \) is an automorphism of \( Gr(n, 2n) \), the composite map \( \tau \circ h : g \to Gr(n, 2n) \) is an open immersion. Hence, it suffices to show that for \( x \in g \), we have \( \tau h(x) \in Gr_w \) if and only if \( x \in g_w \).

Observe that \( \tau h(x) \in Gr_w \) if and only if

\[
\dim(\tau h(x) + E_t_i) \leq n + p_i + r_i \quad \forall 1 \leq i \leq m - 1
\]

\( \iff \dim(h(x) + \tau^{-1}E_t_i) \leq n + p_i + r_i \quad \forall 1 \leq i \leq m - 1. \)

Following Equation (5), we see that

\[
(7) \quad \tau^{-1}E_t_i = (e_1, \cdots, e_{q_i}, e_{n+1}, \cdots, e_{n+p_i}).
\]

Following Lemma 1.7, we have

\[
\dim(h(x) + \tau^{-1}E_t_i) \leq n + p_i + r_i \quad \forall 1 \leq i \leq m - 1
\]

\( \iff \dim(xE_{q_i}/E_{p_i}) \leq r_i \quad \forall 1 \leq i \leq m - 1. \)
Comparing with Equation (4), we see that $\tau h(x) \in Gr_v$ if and only if $x \in g_w$. □

**Corollary 1.8.** Consider the open immersion $\iota : G \rightarrow g$. If $w \in S_n$ is covexillary permutation, we have an open immersion $\tau \circ h \circ \iota : G_w \rightarrow Gr_v$.

**1.7. Equivariance of the Embedding.** The $B \times B$-action on $g$ restricts to a $T \times T$-action on $g$. Let $T_{2n}$ be the group of invertible $2n \times 2n$ diagonal matrices, acting on $Gr(n, 2n)$ via left-multiplication. Let $\tau : T_{2n} \rightarrow T_{2n}$ denote conjugation by the permutation $\tau$. Making the identification $T \times T \cong T_{2n}$, we have a commutative diagram of actions,

$$
\begin{array}{ccc}
T \times T & \sim & T_{2n} \\
\downarrow & & \downarrow \\
g & \rightarrow & Gr(n, 2n)
\end{array}
$$

1.8. Characteristic Cycles and Euler Obstructions. The local Euler obstruction is a certain topological invariant of algebraic varieties. It was first introduced by MacPherson in [Mac74], and has seen many alternate definitions and interpretations, see [BSS09, Ch. 8] for a discussion. Combinatorial models computing the local Euler obstructions of Grassmannian Schubert varieties were developed by [BF97], see also [BFL90, BK81].

A long-standing problem in geometric representation theory is determining when the characteristic cycle of the IC sheaf of a Schubert variety $Fl_w$ is irreducible, and more generally, computing the characteristic cycle as a linear combination of conormal cycles. Euler obstruction computations are useful in the study of the irreducibility of the characteristic cycle of the IC sheaf of a Schubert variety. Precisely, the characteristic cycle is irreducible if and only if the Euler obstruction of $Fl_w$ at a point $v$ equals the value of the Kazhdan-Lusztig polynomial $[KL80a, KL80b]$ evaluated at 1. Bressler, Finkelberg, and Lunts [BFL90] used Zelevinsky’s small resolutions to show that the characteristic cycle of a Grassmannian Schubert variety is irreducible. Later, Boe and Fu [BF97] gave a different proof, showing that for Grassmannian Schubert varieties, Lascoux and Schützenberger’s combinatorial model [Las95] for Kazhdan-Lusztig polynomials also computes the Euler obstructions.
Since both the Kazhdan-Lusztig polynomials and the Euler obstructions are local invariants, the following is an immediate consequence of Theorem 1.6.

**Theorem 1.10.** The characteristic cycle of the IC sheaf a covexillary Schubert variety is irreducible.

2. **The Conormal Variety of a Covexillary Matrix Schubert Variety**

In this section, we give a (possibly non-reduced) system of equations identifying the conormal variety $N^*g_w$ of $g_w$ in $g$ as a subvariety of the cotangent bundle $T^*g$. To this end, we recall standard descriptions of the cotangent bundles $T^*g$ and $T^*Gr(n, 2n)$, and give a formula in Lemma 2.1 for the induced map on cotangent bundles, $\tau_\# \circ h_\# : T^*g \to T^*Gr(n, 2n)$, compatible with these descriptions.

2.1. **Conormal Varieties.** Let $M$ be a smooth variety, and $X \subset M$ a closed subvariety. We recall the definition of the conormal variety $N^*X$ (or $N^*(X, M)$ when it is necessary to emphasize the ambient variety $M$) of $X$ in $M$. Let $X^{sm}$ be the smooth locus of $X$. The conormal bundle $N^*X^{sm} \to X^{sm}$ is the vector bundle given by

$$N^*_xX^{sm} = \{ \alpha \in T^*_xM \mid \alpha(v) = 0 \forall v \in T_xM \},$$

i.e., it is the vector bundle whose fibre at a point $x \in X^{sm}$ is the annihilator (in $T^*_xM$) of the tangent subspace $T_xX^{sm}$. The closure of $N^*_xX^{sm}$ in $T^*_xM$ is called the conormal variety $N^*X$ of $X$ in $M$.

2.2. **Cotangent bundles and pull-backs.** Consider a smooth map $\gamma : D \to M$ to $M$ from some (Euclidean) neighborhood $D$ of $0 \in \mathbb{C}$. Then $\gamma$ corresponds to a tangent vector at the point $\gamma(0) \in M$, and every tangent vector can obtained in this manner. We will call $\gamma$ a local curve in $M$. Given a smooth map $f : M \to N$, we have an induced map $f_\# : TM \to TN$ of tangent bundles. If $v$ is the tangent vector corresponding to a curve $\gamma : D \to M$, then $f_\#(v)$ is the tangent vector corresponding to the curve $f \circ \gamma : D \to N$.

For $x \in M$, let $f_{x, x} : T_xM \to T_{f(x)}N$ denote the restriction of $f_\#$ to the tangent space $T_xM$. Let $f' : f^*T^*N \to T^*N$ be the base change of the map $f : M \to N$ along the map $T^*N \to N$. For $x \in M$, the fibre $(f^*T^*N)_x$ is precisely the cotangent space $T_{f(x)}N$. Dual to the map $f_{x, x}$, we have a map $f^*_x : (f^*T^*N)_x \to T^*_xM$, and hence a map $f^* : f^*T^*N \to T^*M$.

$$
\begin{array}{ccc}
T^*M & \xleftarrow{f^*} & f^*T^*N \\
\downarrow & & \downarrow \\
M & \times & N.
\end{array}
$$

If $f : M \to N$ is an open immersion, the map $f^*$ is an isomorphism, and we have an open immersion $f_\# : T^*M \to T^*N$ given by $f_\# = f' \circ (f^*)^{-1}$.

$$
\begin{array}{ccc}
T^*M & \xrightarrow{(f^*)^{-1}} & f^*T^*M \\
\downarrow & & \downarrow \\
M & \times & N.
\end{array}
$$

Equivalently, $T^*M$ is simply the restriction of $T^*N$ to $M$, and $f_\# : T^*M \to T^*N$ is the inclusion map.
2.3. Tangent and cotangent bundle of the Grassmannian. Since \( g \) is a vector space, we have natural identifications

\[
Tg = g \times g, \quad T^*g = g \times g^*.
\]

The point \((x, y) \in g \times g = Tg\) corresponds to the local curve \( t \mapsto x + ty\).

The trace form on \( g \) is a non-degenerate bilinear form, and hence induces an isomorphism \( \alpha : g \to g^* \). For \( y \in g \), we write \( \alpha_y \in g^* \) for the linear functional given by \( \alpha_y(x) = \text{tr}(yx) \).

Let \( E_{2n} \) be a \( 2n \)-dimensional vector space with standard basis \( e_1, \ldots, e_{2n} \), and let \( P \subset GL_{2n} \) denote the stabilizer of the subspace \( E_n \subset E_{2n} \). Let \( g_{2n} = \text{Lie}(GL_{2n}) \), \( p = \text{Lie}(P) \), and let \( \theta : g \to g_{2n}/p \) denote the linear isomorphism

\[
\theta(y) = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \pmod{p},
\]

where the matrix on the right is in block form with blocks of size \( n \times n \). Recall the isomorphism \( GL_{2n}/P = Gr(n, 2n) \) given by \( gP \mapsto gE_n \). We have

\[
T^*Gr(n, 2n) = GL_{2n} \times^P (g_{2n}/p),
\]

with \((g, \theta(y)) \in GL_{2n} \times^P (g_{2n}/p)\) corresponding to the curve \( t \mapsto g \text{Col}(I(n)_{ty}) \).

Let \( u_P \) be the unipotent radical of \( p \). The trace form on \( g_{2n} \) induces a non-degenerate bilinear pairing \( \text{tr} : g_{2n}/p \times u_P \to \mathbb{C} \). Let \( \theta^* : g^* \to u_P \) denote the isomorphism

\[
\theta^*(\alpha_y) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix},
\]

where the matrices are in block form with blocks of size \( n \times n \). We have a commutative diagram,

\[
g \times g^* \xrightarrow{\theta \times \theta^*} g_{2n}/p \times u_P \xrightarrow{\text{tr}} \mathbb{C}
\]

where the map \( g \times g^* \to \mathbb{C} \) is simply the evaluation map \((x, \alpha_y) \mapsto \alpha_y(x) = \text{tr}(yx)\).

Dual to Equation (9), the cotangent bundle of \( Gr(n, 2n) \) is given by

\[
T^*Gr(n, 2n) = GL_{2n} \times^P u_P.
\]

Lemma 2.1. Let \( h_1 : g \to GL_{2n} \) be the map given by

\[
h_1(x) = \begin{pmatrix} I(n) \\ x \end{pmatrix}.
\]

Recall from Equation (5) the graph embedding \( h : g \to Gr(n, 2n) \). The map \( h_# : T^*g \to T^*Gr(n, 2n) \) is given by

\[
h_#(x, \alpha_y) = (h_1(x), \theta^*(\alpha_y)) \in GL_{2n} \times^P u_P.
\]

Proof. Consider the local curve \( \gamma \) in \( g \) given by \( \gamma(t) = x + ty \). We have

\[
h \circ \gamma(t) = \text{Col}(I(n)_{x + ty}) = h_1(x) \text{Col}(I(n)_{ty}),
\]
We see that \( h_\ast(x, y) = (h_1(x), \theta(y)) \in GL_{2n} \times^P (g_{2n}/p) \), and hence
\[
  h_\#(x, \alpha_g) = (h_1(x), \theta^*(\alpha_g)) \in GL_{2n} \times^P u_P.
\]

**Lemma 2.2.** Recall \( \tau \in S_{2n} \) from Equation (5), and the associated map \( \tau : Gr(n, 2n) \to Gr(n, 2n) \) given by \( V \mapsto \tau V \). We have \( \tau_\#(g, x) = (\tau g, x) \in GL_{2n} \times^P u_P \).

**Proof.** At any point \( gE_n \in Gr(n, 2n) \), the pairing
\[
  T_{gE_n} Gr(n, 2n) \times T_{gE_n} Gr(n, 2n) \to \mathbb{C}
\]
is given by \( ((g, y), (g, x)) \mapsto \text{tr}(yx) \). The point \( (g, x) \in G \times^P (g/p) \) corresponds to the local curve \( \gamma \) given by \( \gamma(t) = gCol \begin{pmatrix} I(t) \\ ty \end{pmatrix} \). We have \( \tau \circ \gamma(t) = \tau gCol \begin{pmatrix} I(t) \\ ty \end{pmatrix} \), and hence \( \tau_\#(g, y) = (\tau g, y) \). It follows that \( \tau_\#(g, x) = (\tau g, x) \). \( \blacksquare \)

**2.4. The Springer map.** The Springer map \( \mu_{2n} : T^*Gr(n, 2n) \to g_{2n} \), given by
\[
  \mu_{2n}(g, x) = Ad(g)x = gxg^{-1}, \quad (g, x) \in GL_{2n} \times^P u_P,
\]
yields a closed immersion,
\[
  j : GL_{2n} \times^P u_P \to g_{2n}, \quad (g, x) \mapsto (gE_n, gxg^{-1}),
\]
(see also [CG97]). We have
\[
  j(T^*Gr(n, 2n)) = \{ (V, x) \in Gr(n, 2n) \times g_{2n} \mid \text{Im}(x) \subset V \subset \ker(x) \}.
\]

A system of defining equations for the conormal variety of any Grassmannian Schubert variety was developed in [Sin21]. We present the result in a language conducive to our needs.

**Proposition 2.3.** (cf. [Sin21] Thm B) Let \( Gr_w \subset Gr(d, N) \) be a Grassmannian Schubert variety given by the conditions
\[
  Gr_w = \{ V \in Gr \mid \dim(V + E_i) \leq d + c_i \forall i \}.
\]
for some integers \( 1 \leq t'_1 \leq \cdots \leq t'_k \leq N - 1 \) and \( 0 \leq c_1 \leq \cdots \leq c_k \leq N - d \). A point \( (V, x) \in j(T^*Gr(d, N)) \) is in \( j(N^*Gr_w) \) if and only if \( V \in Gr_w \) and
\[
  \dim(xE_i/E_{ij}) \leq \begin{cases} (t'_i - c_{i-1}) - (t'_j - c_j), & \forall i. \end{cases}
\]

**2.5. The conormal variety of a covexillary matrix Schubert variety.** Let \( w \) be a covexillary partial permutation, with essential set
\[
  \mathcal{E}ss(w) = \{ (p_1 + 1, q_1), \ldots, (p_{m-1} + 1, q_{m-1}) \}
\]
for some \( 0 \leq p_1 \leq \cdots \leq p_{m-1} \leq n - 1 \) and \( 1 \leq q_1 \leq \cdots \leq q_{m-1} \leq n \). Let \( r_t = r^{w}_{p_t - 1, q_t} \)
\[
  Gr_w = \{ V \in Gr(n, 2n) \mid \dim(V + E_t) \leq n + p_t + r_t \},
\]
and let \( \tau \in S_{2n} \) and \( \tau \circ h : g_{w} \to Gr_w \) be as in Theorem [1.6]. Applying Proposition 2.3 to \( Gr_w \subset Gr(n, 2n) \), we see that a point \( (V, x) \in j(T^*Gr(n, 2n)) \) belongs to \( j(N^*Gr_w) \) if and only if \( V \in Gr_w \) and
\[
  xE_{i}/E_{ij} \leq \begin{cases} (q_{i-1} - r_{i-1}) - (q_j - r_j), & \forall 1 \leq j < i \leq m. \end{cases}
\]
**Theorem 2.4.** Consider the $2n \times 2n$ matrix $M$ given in block form by

$$M = \begin{pmatrix} yx & y \\ xy & xy \end{pmatrix},$$

and let $M(i,j)$ be the submatrix of $M$ spanned by the rows $\{q_j + 1, \cdots, n, n + p_j + 1, \cdots, 2n\}$ and the columns $\{1, \cdots, q_i, n + 1, \cdots n + p_i\}$. We have $(x, \alpha_y) \in N^*g_w$ if and only if $x \in g_w$ and

$$\text{rk}(M(i,j)) \leq \begin{cases} (q_i-1-r_i-1) - (q_j - r_j), \\ (p_i + r_i) - (p_j+1 + r_j+1). \end{cases} \quad \forall 1 < j \leq i < m.$$

**Figure 2.** The matrix $M(i,j)$ is the shaded portion of the block matrix $M = \begin{pmatrix} yx & y \\ xy & xy \end{pmatrix}$.

**Proof.** The map $\tau \circ h : g_w \to Gr_w$ is an open immersion, hence we have the following Cartesian square,

$$\begin{array}{ccc} N^*g_w & \xrightarrow{\tau \circ h_w} & N^*Gr_w \\ \downarrow & & \downarrow \\ g_w & \xrightarrow{\tau \circ h} & Gr_w. \end{array}$$

It follows that for $(x, \alpha_y) \in T^*g$, we have

$$(x, \alpha_y) \in N^*g_w \iff \tau \circ h(x,\alpha_y) \in N^*Gr_w \iff j \circ \tau \circ h(x,\alpha_y) \in j(N^*Gr_w).$$
Following Lemmas 2.1 and 3.1 we see that
\[ j \circ \tau \circ h(x, \alpha y) = j(\tau h_1(x), \theta^*(y)) \]
\[ = j(\tau \left( \begin{array}{c} I(n) \\ x \end{array} \right), \left( \begin{array}{cc} 0 & y \\ 0 & 0 \end{array} \right)) \]
\[ = \left( \tau \text{Col} \left( \begin{array}{c} I(n) \\ x \end{array} \right), \tau \left( \begin{array}{cc} -yx & y \\ -xy & xy \end{array} \right) \tau^{-1} \right) \]

Following Equation (13), we have
\[ j \circ \tau \circ h(x, \alpha y) \in j(N^* Gr_w) \]
if and only if
\[ \dim \left( \tau \left( \begin{array}{cc} -yx & y \\ -xy & xy \end{array} \right) \tau^{-1} E_{i_j} / E_{i_j} \right) \leq \left\{ \begin{array}{c} (q_{i-1} - r_{i-1}) - (q_j - r_j), \\ (p_i + r_i) - (p_{j+1} + r_{j+1}) \end{array} \right\} \]

(14)

Next, observe that
\[ \dim \left( \tau \left( \begin{array}{cc} -yx & y \\ -xy & xy \end{array} \right) \tau^{-1} E_{i_j} / E_{i_j} \right) = \dim \left( \left( \begin{array}{cc} -yx & y \\ -xy & xy \end{array} \right) \tau^{-1} E_{i_j} / \tau^{-1} E_{i_j} \right) \]
\[ = \dim (M \tau^{-1} E_{i_j} / \tau^{-1} E_{i_j}) \]

Recall from Equation (7) that \( \tau^{-1} E_{i_j} = \langle e_1, \ldots, e_{q_i}, e_{n+1}, \ldots, e_{n+p_i} \rangle \). We deduce that the columns of the matrix \( M(i, j) \) form a basis of \( M \tau^{-1} E_{i_j} / \tau^{-1} E_{i_j} \), and hence the result follows from Equation (14).

3. The conormal variety of a covexillary Schubert variety

Let \( \iota : G \rightarrow \mathfrak{g} \) denote the inclusion of \( GL_n \) into the set of \( n \times n \) matrices. Recall the map \( \pi : G \rightarrow Fl \) given by \( g \mapsto gE \), and the corresponding Cartesian square,

\[ \pi^* T^* Fl \rightarrow T^* Fl \]
\[ \downarrow \quad \downarrow \]
\[ G \quad \pi \rightarrow Fl. \]

Since \( \pi : G \rightarrow Fl \) is a smooth morphism, for any subvariety \( X \subset Fl \), we have \( \pi^{-1}(N^* X) = N^*(\pi^{-1}(X)) \), see [HTT08, p. 65]. Applied to \( X = Fl_w \), this yields the following diagram of Cartesian squares,

\[ N^* Fl_w \leftarrow \pi^* N^* G_w \rightarrow N^* \mathfrak{g}_w \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ Fl_w \leftarrow \pi G_w \rightarrow \mathfrak{g}_w \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ Fl \leftarrow \pi G \rightarrow \mathfrak{g}. \]

In this section, we use Theorem 2.4 and Equation (16) to give a characterization of \( N^* Fl_w \) when \( w \) is a covexillary permutation.

Let \( w \in S_n \) be a covexillary permutation, and let \( \tau \in S_{2n} \) be as in Equation (5). Recall that \( \mathfrak{g} \) is naturally identified with the tangent space of \( G \) at identity; the point \( x \in \mathfrak{g} \) corresponds to the local curve \( t \mapsto 1+tx \). By \( G \)-equivariance, we have \( TG = G \times \mathfrak{g} \), with the point \( (g, x) \in G \times \mathfrak{g} \) corresponding to the curve \( t \mapsto g(1+tx) \).
Dually, we have $T^*G = g \times g^*$. Using the trace form on $g$, we have an isomorphism $\alpha : g \to g^*$, given by $\alpha_y = (x \mapsto \text{tr}(yx))$, see Section 2.3.

**Lemma 3.1.** Consider the open immersion $\iota : G \to g$. We have $\iota_\# (g, \alpha_y) = (g, \alpha_{y^{-1}})$.

*Proof.* Recall from Section 2.2 the induced map $\iota_* : TG \to Tg$ on tangent bundles. Consider the local curve $\gamma$ given by $\gamma(t) = g(1 + tx)$; we have $\iota \circ \gamma(t) = g + tx$, and hence $\iota_*(g, x) = (g, gx)$.

Since $\alpha_y(gx) = \text{tr}(ygx) = \alpha_{y^{-1}}(x)$, the dual map $\iota_*^* : T^*_g g \to T^*_g G$ is given by $\alpha_y \mapsto \alpha_{y^{-1}}$. Following Section 2.2, $\iota_\# : T^*_g G \to T^*_g g$ is the inverse of the isomorphism $\iota_*^*$, and hence is given by $\iota_\#(\alpha_y) = \alpha_{y^{-1}}$. It follows that $\iota_\#(g, \alpha_y) = (g, \alpha_{y^{-1}})$. □

Let $\mathfrak{b}$ be the Lie algebra of $B$, and let $u_B$ be the unipotent radical of $\mathfrak{b}$. Following the isomorphism $G/B \simeq Fl$ given by $gB \mapsto gE_\bullet$, we see that $T_{E_\bullet} Fl = \mathfrak{g}/\mathfrak{b}$, and hence by $G$-equivariance, $T^* Fl = G \times \mathfrak{g}/\mathfrak{b}$. The trace form on $g$ induces a non-degenerate bilinear form $\mathfrak{g}/\mathfrak{b} \times u_B \to \mathbb{C}$, and hence we have the identifications $u = (\mathfrak{g}/\mathfrak{b})^*$ and $T^* Fl = G \times \mathfrak{g}/\mathfrak{b}$.

Recall the Springer map $\mu : T^* G/B \to g$, given by $(g, y) \mapsto g y^{-1}$. Following [Spr69] (see also [CG97]), we have an embedding
\[
(17) \quad j : T^* Fl \hookrightarrow Fl \times g, \quad (g, y) \mapsto (g E_{\bullet}, g y^{-1}).
\]

The image of $T^* Fl$ under this map has the following description:
\[
j(T^* Fl) = \{ (F_{\bullet}, z) \in Fl \times g \mid zF_i \subset F_{i-1} \forall i \}.
\]

**Lemma 3.2.** We have an identification $\pi^* T^* Fl = G \times u_B$, with the inclusion $\pi^* T^* Fl \hookrightarrow T^* G$ given by $(g, y) \mapsto (g, \alpha_y)$. Moreover, the map $\pi' : \pi^* T^* Fl \to T^* Fl$ (see Equation (15)) is the quotient map
\[
G \times u_B \to G \times^B u_B, \quad (g, y) \mapsto (g, y)(\text{mod } B).
\]

*Proof.* It is clear that we have Cartesian squares,
\[
\begin{array}{ccc}
G \times u_B & \longrightarrow & G \times^B u_B \\
\downarrow & & \downarrow \\
G & \longrightarrow & Fl,
\end{array}
\quad
\begin{array}{ccc}
\pi^* T^* Fl & \longrightarrow & G \times^B u_B \\
\downarrow & & \downarrow \\
G & \longrightarrow & Fl.
\end{array}
\]

By the uniqueness of fibre products, we deduce an isomorphism identifying the pull-back $\pi^* T^* Fl$ with $G \times u_B$. Further, the map $(g, y) \mapsto (g, \alpha_y)$ sends $g \times u_B$ to the annihilator of $T_g (gB)$ (in $T^*_g G$), and hence can be identified with the inclusion $\pi^* T^* Fl \hookrightarrow T^* G$. □

**Theorem 3.3.** Consider $(F_{\bullet}, z) \in j(T^* Fl)$. We have $(F_{\bullet}, z) \in j(N^* Fl_w)$ if and only if $F_{\bullet} \in Fl_w$ and
\[
\dim (z(F_{q_i} + E_{p_i})/(F_{q_i} \cap E_{p_i})) \leq \begin{cases} 
\frac{q_i-1 - r_{i-1}}{q_j - r_j} - (p_i + r_i) \\
(p_i + r_i) - (p_{i+1} + r_{j+1})
\end{cases}
\]
for all $1 \leq i < j < m$.

*Proof.* Consider the commutative diagram (see Equation (16)),
\[
\[
\begin{array}{ccc}
F_{q_i} + E_{p_i} & \longrightarrow & F_{q_i} \\
\downarrow & & \downarrow \\
F_{q_i} \cap E_{p_i} & \longrightarrow & (F_{q_i} \cap E_{p_i}).
\end{array}
\]
Since the induced map \( \pi' : N^*G_w \to N^*\mathcal{F}_w \) is surjective, we have \((F_\bullet, z) \in j(N^*\mathcal{F}_w)\) if and only if there exists \((g, y) \in N^*G_w\) such that
\[
j \circ \pi'(g, y) = (gE_\bullet, gyg^{-1}) = (F_\bullet, z).
\]

Following Lemma 3.1 and Equation (16), we see that
\[
\begin{align*}
M &= \begin{pmatrix}
y & yg^{-1} \\
g & gyg^{-1}
\end{pmatrix} = \begin{pmatrix}g^{-1}zg & g^{-1}z \\
zg & z
\end{pmatrix},
\end{align*}
\]
and let \(M(i, j)\) be the submatrix of \(M\) spanned by the rows \(\{q_j + 1, \cdots, n, n+p_j + 1, \cdots, 2n\}\) and the columns \(\{1, \cdots, q_i, n + 1, \cdots, n + p_i\}\). Following Theorem 2.4, we have \((g, \alpha_{yg^{-1}}) \in N^*G_w\) if and only if \(g \in G_w\) and

\[
\text{rk}(M(i, j)) \leq \begin{cases}
(q_{i-1} - r_{i-1}) - (q_j - r_j) & \forall 1 \leq i < j < m.
\end{cases}
\]

We finish the proof by showing that Equations (18) and (19) are equivalent.

Consider the automorphism \(\tilde{g}\) of \(E_n \oplus E_n\) given by \((v, w) \mapsto (gv, w)\). Observe that \(\tilde{g}(E_{q_j} \oplus E_{p_j}) = F_{q_j} \oplus E_{p_j}\), and hence \(\tilde{g}\) descends to a map \(\tilde{g}_0\) satisfying the commutative diagram,
\[
\begin{array}{ccc}
E_n \oplus E_n & \xrightarrow{\tilde{g}} & E_n \oplus E_n \\
\downarrow \eta & & \downarrow \eta' \\
(E_n \oplus E_n)/(E_{q_j} \oplus E_{p_j}) & \xrightarrow{\tilde{g}_0} & (E_n \oplus E_n)/(F_{q_j} \oplus E_{p_j}),
\end{array}
\]
where \(\eta, \eta'\) are the quotient maps.

Let \(\kappa : E_n \to E_n \oplus E_n\) denote the diagonal embedding \(v \mapsto (v, v)\). Observe that
\[
\begin{pmatrix}g^{-1}z \\ z\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}g^{-1}zg \\ zg\end{pmatrix}
\]
are precisely the matrices of the maps \(\tilde{g}\kappa z\) and \(\tilde{g}\kappa zg\) respectively (with respect to the ordered bases \((e_1, \cdots, e_n)\) of \(E_n\) and \(((e_1, 0), \cdots, (e_n, 0), (0, e_1), \cdots, (0, e_n))\) of \(E_n \oplus E_n\)). Moreover, removing the columns indexed \(\{1, \cdots, q_j, n + 1, \cdots, n + p_j\}\) from \(M\) corresponds to quotienting \(E_n \oplus E_n\) by the subspace \(E_{q_j} \oplus E_{p_j}\).

It follows that first \(q_i\) columns of \(M(i, j)\) span the subspace
\[
\eta \tilde{g}^{-1}\kappa zgE_{q_i} = \eta \tilde{g}^{-1}\kappa zF_{q_i} = \tilde{g}_0^{-1}\eta' \kappa zF_{q_i},
\]
and the last \(p_i\) columns of \(M(i, j)\) span the subspace
\[
\eta \tilde{g}^{-1}\kappa zE_{p_i} = \eta \tilde{g}^{-1}\kappa zE_{p_i} = \tilde{g}_0^{-1}\eta' \kappa zE_{p_i}.
\]
Consequently, we have
\[ \text{rk}(M(i, j)) = \dim(\tilde{g}_0^{-1} \eta' \kappa z(F_{q_i} + E_{p_i})) = \dim(\eta' \kappa z(F_{q_i} + E_{p_i})) = \dim(\kappa z(F_{q_i} + E_{p_i})/(F_{q_j} \oplus E_{p_j})). \]

Finally, since \( \kappa : E_n \to E_n \oplus E_n \) is the diagonal embedding, we have
\[ \text{Im}(\kappa) \cap (F_{q_j} \oplus E_{p_i}) = \kappa(F_{q_j} \cap E_{p_i}), \]
and hence
\[ \kappa z(F_{q_i} + E_{p_i}) \cap (F_{q_j} \oplus E_{p_i}) = \kappa(z(F_{q_i} + E_{p_i}) \cap F_{q_j} \cap E_{p_i}). \]

It follows that
\[ \dim(z(F_{q_i} + E_{p_i})/(F_{q_j} \cap E_{p_i})) = \dim(\kappa z(F_{q_i} + E_{p_i})/(F_{q_j} \oplus E_{p_i})) = \text{rk}(M(i, j)), \]
and hence Equation (18) is equivalent to Equation (19). \( \square \)

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