Complete set of unitary irreps of Discrete Heisenberg Group $\mathcal{HW}_{2^s}$.

E. Floratos* and I. Tsohantjis†

National and Kapodistrian University of Athens, Faculty of Physics, Department of Nuclear and Particle Physics, University Campus, Ilisia, 15771 Athens, Greece

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Abstract

Following the method of induced group representations of Wigner-Mackay, the explicit construction of all the unitary irreducible representations of the discrete finite Heisenberg-Weyl group $\mathcal{HW}_{2^s}$ over the discrete phase space lattice $\mathbb{Z}_{2^s} \otimes \mathbb{Z}_{2^s}$ is presented. We explicitly determine their characters and their fusion rules. We discuss possible physical applications for finite quantum mechanics and quantum computation.

*Email address: mflorato@phys.uoa.gr
†Email address: itsochantzis@phys.uoa.gr
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1 Introduction

Quantum computation and quantum information are fast advancing areas of theoretical and experimental studies of a variety of physical systems, ranging from integrable lattice models, many body systems, anyonic systems to black holes.

Technological advances in the control and study of few atoms quantum systems require the isolation of these systems from the environment, restricting and isolating to transitions between a fixed finite number of energy levels reduces the quantum mechanical problem to finite dimensional Hilbert spaces.

The usual theoretical approach is the description of the quantum systems with a finite number of spins and the corresponding models are various forms of one or two dimensional Heisenberg spin chains. On the other hand in this picture the role of the discrete Heisenberg-Weyl group which is the mathematical expression of wave-particle duality is not obvious. The Heisenberg-Weyl group plays an important role also in quantum information on phase space qubits [1], [2] as well as in quantum error correcting codes [3].

In another area for the fundamental questions about the nature of gravity and spacetime and the great difficulties encountered in the unification of gravity with quantum mechanics it is becoming obvious that due to the assignment of finite entropy to every local causal region connecting two different observers, the corresponding Hilbert space of states must be finite dimensional [4], [5], [6] and references therein.

In the case where the dimension of the Hilbert space is \(2^s\) which relevant for quantum computing and more generally for Heisenberg spin chains, typically one starts with a quantum system of \(s\) qubits. Information processing on such systems is carried out through quantum circuits which represent the action of unitary operators, \(2^s \times 2^s\) matrices acting on \(H_2^s\). These operators are themselves constructed as a tensor product of a finite number of two or three qubit unitary operators, the universal quantum gates, and describe the quantum mechanical evolution of the \(s\)-qubit system from an initial to a final state.

In this standard picture of quantum information processing the fundamental character of quantum mechanics namely the wave-particle duality is
only implicit and appears to play only a minor role if not at all. This situation is completely different to that of the continuous quantum mechanical systems where wave-particle duality and Heisenberg uncertainty principle play a central role in the description of the evolution of a quantum system. To this end one should focus in the inherently discrete character of information processing of $s$-qubits quantum systems where the wave-particle duality is expressed through the fundamental Heisenberg Weyl uncertainty relation at the group level: $QP = \omega PQ$, where $\omega = e^{2\pi i/2^s}$, $Q$ is the position operator $Q = \text{diag}[1, \omega, \omega^2, ..., \omega^{2^s-1}]$ for a fictitious particle hoping (anticlockwise) on the vertices of a canonical $s$-agon situated on the unit circle, and $P$ is the discrete momentum operator performing this (anticlockwise) motion by an $2\pi/2^s$-angle shift on the circle.

In this picture the wave-particle duality is transparent. The wave functions which are the eigenstates of $P$ are the discretization of the continuous planewaves. These eigenstates are the columns of a diagonalization matrix $F$ of $P$, with matrix elements $F_{kj} = 1/\sqrt{2^s} \omega^{kj}$, satisfying the relations $QF = FP$, $F^4 = I$. $F$ is thus a Finite Fourier transform between discrete position and momentum of the fictitious particle’s phase space $\mathbb{Z}_{2^s} \times \mathbb{Z}_{2^s}$.

The general framework of Finite Quantum Mechanics (FQM), is quantum mechanics with a finite dimensional Hilbert space. The reason for introducing this framework has been exemplified and discussed in detail by [7], [8], [9], [10], [11]. We should stress though that the previous references concern with the fundamentals of quantum mechanics for physical systems and there is no discussion about applications of this framework in quantum information.

In the present work we shall connect the area of quantum information with that of FQM. To this end, in this article, we first determine all the inequivalent irreducible unitary representations of the discrete Heisenberg-Weyl group $HW_{2^s}$ as well as their properties. This is the necessary first step to define FQM on the discrete phase space $\mathbb{Z}_{2^s} \times \mathbb{Z}_{2^s}$. The unitary evolution operators of FQM will be obtained from the unitary metaplectic representations of $Sp(2, \mathbb{Z}_{2^s})$ which is the automorphisms group of $HW_{2^s}$ [12].

The plan of the paper is as follows:

In section 2 the structure of $HW_{2^s}$ together with the determination of its conjugacy classes is presented.

In section 3 we derive up to equivalence all the inequivalent unitary irreducible representations of the discrete Heisenberg group $HW_{2^s}$, using Wigner’s and MacKey’s little group method on induced representations for the case of a semi-direct product group. Note though that we do not restrict
ourselves to only primitive roots of unity $\omega$ but also for any other admissible even or odd power of $\omega$. This corresponds to quantum mechanics with various Planck constants $h = 2\pi k/2^s$, where $k = 1, ..., 2^s - 1$. In this way we obtain not only the faithfull representations but also non-faithful ones \cite{13}, \cite{14}. It seems that the non faithful ones have not attracted much attention until now in the literature.

In section 4 we obtain the explicit matrix form of constructed representations which is then used to determine Finite Fourier Transforms and their properties corresponding both for the case of the faithful and non-faithful representations.

In section 5, the characters of the above representations are constructed as well as the decomposition of direct product of any two inequivalent representations is carried out in detail, thus establishing the set of fusion rules for $H_{W_2}$. We present examples of fusion rules for $H_{W_2}$ and $H_{W_4}$.

In section 6 we conclude and we discuss open issues and possible applications.

Finally in the appendix section 7, we give all the details for the determination of the distinct orbits of the little group methods and the proof of the main theorem 2 (see section 3).
2 Structure of the discrete and finite Heisenberg-Weyl group $HW_N$

In this section we define the discrete and finite Heisenberg-Weyl group $HW_N$ and determine its conjugacy classes.

$HW_N$ is the set of elements $z^m x^n y^l$ where $m, n, l = 0, 1, \ldots, N - 1$ and $x, y, z$ are subject to the relations

\begin{align*}
  x^N &= y^N = z^N = e, \quad xy = zy, \\
  zx &= xz, \quad zy = yz
\end{align*}

$e$ is the group identity element and the group has order $|HW_N| = N^3$.

$HW_N$ has the semi-direct product group structure

$$HW_N = H \circledast B$$

where the abelian subgroups $H$ and $B$ are given by

\begin{align*}
  H &= \{z^m x^n \mid m, n = 0, 1 \ldots N - 1\} \\
  B &= \{y^l \mid m, n = 0, 1 \ldots N - 1\} \approx \mathbb{Z}_N.
\end{align*}

It can be shown that if the prime decomposition of $N$ is

$$N = p_1^{r_1} p_2^{r_2} \ldots p_M^{r_M},$$

where $p_1, p_2, \ldots, p_M$ distinct primes and $r_1, r_2, \ldots, r_M$ positive integers, then $HW_N$ admits the tensor product structure

$$HW_N = \bigotimes_{i=1}^M HW_{p_i^{r_i}}.$$  

Due to the structure (7), it only suffices to find all the unitary inequivalent irreps of $HW_N$ for $N = p^s, p$ prime and $s$ positive integer.

In this article we shall focus on the specific case where $N = 2^s$. The general case for $N = p^s, p$ prime will be dealt elsewhere [15].

We proceed to the determination of the conjugacy classes of $HW_{2^s}$. Let $g = z^p x^n y^l$ be an arbitrary element of the group. We shall discuss with increasing complexity the conjugacy class of $g$. First we consider the case
\( n = l = 0 \). It is obvious that for every value of \( \beta = 0, 1, \ldots, 2^s - 1 \) the corresponding conjugacy class has one element, \( g \) and so in total we have for \( 2^s \) conjugacy classes.

Secondly we consider the case \( n \neq 0, l = 0 \), that is elements of the form \( g = z^\beta x^n \). In this case the conjugacy class corresponding to conjugation with an element \( h = z^\alpha x^k y^m \) contains elements of the form \( z^{\alpha + \beta} x^n \) where \( \alpha = -(mn) \mod 2^s \) and \( m = 0, 1, 2, \ldots, 2^s - 1 \). However not all of these elements are different. To count the distinct ones we consider the problem of finding the number of distinct points on the discrete line of the \((j, m)\)-plane, \( j = (-mn + \beta) \mod 2^s \), as \( m \) takes the values 0, 1, 2, \ldots, \( 2^s - 1 \).

To solve this problem we extract the maximum power of 2 from \( n = 2^t u_t \), \( u_t = 1, 3, 5, \ldots, 2^{s-t} - 1 \). For each value of \( t \), \( m \) will run from 0, 1, 2, \ldots, \( 2^s - t - 1 \) with \( u_t \) fixed. Similar arguments hold for the cases \( n = 0, l \neq 0 \) and \( n \neq 0, l \neq 0 \). The conjugacy classes of the elements \( g = z^\beta x^n y^l \), \( C_g \), of \( \text{HW}_{2^s} \), are of the form

\[
C_g = \{ z^\alpha g, \alpha = 0, 2^k, 2 \times 2^k, 3 \times 2^k, \ldots, (2^s-k-1)2^k \} \tag{8}
\]

where \( n, l \in \{ 0, 1, \ldots, 2^s - 1 \}, n = 2^t u_t, l = 2^{t'} u_{t'}, t, t' \in \{ 0, 1, 2, \ldots, s-1 \}, u_t = 1, 3, 5, \ldots, 2^{s-t} - 1, u_{t'} = 1, 3, 5, \ldots, 2^{s-t'} - 1 \). The index \( k \) above is \( k = \min(t, t') \) and \( \beta \) must be restricted to the range \( \{ 0, 1, \ldots, 2^k - 1 \} \) in order to avoid double counting. The order of each \( C_{z^\beta x^n y^l} \) is \( |C_{z^\beta x^n y^l}| = 2^{s-k} \). Counting arguments show that the total number of conjugacy classes \( N_C \) is given by

\[
N_C = \sum_{t,t'=0}^{s-1} (2^{2s-t-t'-2\min(t,t')}) + 2^s (s + 1) \tag{9}
\]

In what follows we shall derive the complete set of unitary irreducible matrix representations of \( \text{HW}_{2^s} \) using the method of induced representations as applied to finite groups and their number must be equal to \( N_C \) above.
3 Construction of the complete set of unitary irreducible matrix representations of $HW_{2s}$.

In this section we will present Induced representations are provided by the following theorem (c.f. [16], [17]). Bellow we will merely follow the reference [17].

**Theorem 1** Let $G$ be a finite group and $K$ a subgroup of $G$, of respective orders $|G|$ and $|K|$. Let $\Delta$ be a $d$-dimensional representation of $K$. Let $g_i$, $i = 1, 2, ..., \frac{|G|}{|K|}$ be the coset representatives for the decomposition of $G$, in to right cosets wrt $K$. Then the set of $d \times d$ matrices $\Gamma(g)$ defined by

$$\Gamma(g)_{im,jn} = \Delta(g_i gg_i^{-1})_{mn} \quad \text{for all} \quad g_i gg_i^{-1} \in K$$

$$= 0 \quad \text{otherwise} \quad (10)$$

for all $g \in G$, where $i, j = 1, 2, ..., \frac{|G|}{|K|}$ and $m, n = 1, ..., d$, provide a $d \times d$-dimensional representation of $G$ induced from the $d$-dimensional representation $\Delta$ of $K$, $\Gamma = \Delta(K)^{\uparrow} G$. Moreover the characters $\chi$ of the representation $\Gamma$ are given by

$$\chi(g) = \sum_{i=1}^{\frac{|G|}{|K|}} \chi_{\Delta}(g_i gg_i^{-1}) \quad (11)$$

where $\chi_{\Delta}$ is the character of the representation $\Delta$.

A very important fact is that in the case of semi-direct product groups $G = A \rtimes B$ with the $A$ component being an abelian invariant subgroup, this method when used with Wigner’s and Mackey little group and orbit techniques, provides with the complete set of inequivalent unitary irreps of the group.(c.f. [17]) faithful or non faithful. In using these techniques one determines all the non isomorphic semi-direct product subgroups $K$ of $G$, whose unitary inequivalent irreps will be used to induce the corresponding irreps of $G$ according to the above theorem.

In particular, consider the vector space of characters of $A$ with basis the irreducible characters $\chi^j_A$. The action of $B$ on the space of characters given by $g_B \chi^j_A(a) = \chi^j_A(g_B ag_B^{-1}) = \chi^{j'}_A(a')$ where $g_B \in B$, $a, a' \in A$, provide in general a reducible representation of $B$. To each $\chi^j_A$ the set of elements of $B$ that leave $\chi^j_A$ invariant form a subgroup of $B$ called the little group $B_j$ and the action of the coset $B / B_j$ on $\chi^j_A$ defined the orbit $Orb(\chi^j_A)$. Varying $\chi^j_A$ be obtain all the orbits which in general are not distinct. In what will
follows below we shall determine all the distinct orbits of $B$. In order to apply the above theorem 1 we choose the subgroups of $G$ which have the form $K_j = A \circ B_j$ where $B_i$ are subgroup of $B$, whose elements leave each character $\chi^j_A$, representative of the distinct orbits of $A$ invariant. Finally we construct the irreps of each $K_j$ which will be used to induce the irreps of $G$.

Turning to our case where $G = HW_{2s} = H \circ B$ where $H$ and $B$ are as in (4) (5) respectively.

The space of the unitary inequivalent irreps of the invariant abelian subgroup $H \approx \mathbb{Z}_{2s} \times \mathbb{Z}_{2s}$ is given by the set $\Omega = \{\chi^{(p,q)}_H, p, q = 0, 1, \ldots, 2^s - 1\}$ of characters, where

$$\chi^{(p,q)}_H(z^m x^n) = e^{\frac{2\pi i}{2s}(mp+nq)},$$

for all $m$, $n$, $p$, $q = 0, 1, \ldots, 2^s - 1$.

The action of $B$ on the characters results as follows,

$$y \chi^{(p,q)}_H(z^m x^n) = \chi^{(p,q)}_H(yz^m x^n y^{-1}) = \chi^{(p,p+q)}_H(z^m x^n)$$

for every $y \in B$. This action can be visualized as the geometrical action of the group $B$ on the two dimensional discrete torus $T_2(2^s) = \mathbb{Z}_{2s} \times \mathbb{Z}_{2s}$. Every point on this torus $(m, n)$ is mapped to point $(m + 1, n)$. That is the action of $B$ is an elementary translation across the first component. In the dual torus $(p, q)$ which is the space of the characters $\chi^{(p,q)}_H$, this action induces the mapping $(p, q) \rightarrow (p, p + q)$.

The little groups $B^{(p,q)}$ are defined as the subgroups of $B$ that leave invariant the point of the dual torus $(p, q)$. This implies that $B^{(p,q)} = y^t \in B : lp = 0 \text{ (mod} 2^s)$. In order to determine the elements of $B^{(p,q)}$ we distinguish two cases: for the case $p \neq 0$ and $p = 0$. In the case $p \neq 0$ we extract the maximum power of 2, $p = 2^t u_t (\text{mod} 2^s)$, introducing the parameters $t$ and $u_t$. In this case we obtain

$$B^{(p,q)} = \{(y^{2^s t})^v, v = 0, 1, \ldots, 2^t - 1\} \approx \mathbb{Z}_{2^t}$$

where $u_t = 1, 3, 5,\ldots, 2^s - 1$, and $t = 0, 1, \ldots, s - 1$. In the second case $p = 0$, we easily obtain $B^{(0,q)} = B$.

We observe that for fixed $t$, $B^{(p,q)}$ is independent of the values of $u_t$ and $q$. Thus there are $2^s - 1$ identical little groups as in (14). Similarly for $p = 0$, there are $2^s$ identical ones.

The inequivalent irreps, $\Gamma^{r}_B$, of $B^{(p,q)}$ labeled by $r$, which will be needed in what follows, are all one-dimensional. As $B^{(p,q)}$ are cyclic, these
are given by

\[
\Gamma^{r}_{B^{2^{-t}}}(y^{2^{s-1}v}) = e^{2\pi irv}, \quad r, v = 0, 1, \ldots, 2^t - 1, \quad p \neq 0
\]  \quad (15)

\[
\Gamma^{r}_{B^{0}}(y^{l}) = e^{2\pi i rl}, \quad r, l = 0, 1, \ldots, 2^s - 1, \quad p = 0
\]

Turning now to the structure of the orbit of \(\chi^{(p,q)}_{H}\), \(\text{Orb}(\chi^{(p,q)}_{H})\), under \(B\), notice that since the action of the elements of \(B^{(p,q)}\) leave \(\chi^{(p,q)}_{H}\) invariant, \(\text{Orb}(\chi^{(p,q)}_{H})\) consists of the irreducible characters of \(B\) obtained by the action of the factor group \(B/B^{(p,q)} = \{y^k, k = 0, 1, \ldots, 2^{s-t} - 1\}\) on \(\chi^{(p,q)}_{H}\). Using (13) and (12) \(\text{Orb}(\chi^{(p,q)}_{H})\) are given by

\[
\text{Orb}(\chi^{(p,q)}_{H}) = \{\chi^{(p,kp+q)}_{H}, \quad k = 0, 1, \ldots, 2^{s-t} - 1\}, \quad p \neq 0
\]  \quad (16)

\[
\text{Orb}(\chi^{(0,q)}_{H}) = \{\chi^{(0,q)}_{H}\}, \quad p = 0
\]

In appendix A we show that all the distinct orbits we are interested in, are produced by constraining the label \(q\) in the range \(q = 0, 1, \ldots, 2^t - 1\), when \(p = 2^t u_t (mod 2^s)\), while when \(p = 0\), \(q = 0, 1, \ldots, 2^s - 1\). Counting arguments show that there are \(2^s\) orbits of the form \(\text{Orb}(\chi^{(0,q)}_{H})\) and for each value of \(t\), \(2^{s-t-1} \times 2^t\) orbits of the form \(\text{Orb}(\chi^{(2^t u_t, q)}_{H})\), the total number of distinct orbits being \(\sum_{t=0}^{s-1}(2^{s-t-1} 2^t) + 2^s = 2^{s-t+1} 2^s + 2^s\) and as expected their union contains the \(\sum_{t=0}^{s-1}(2^{s-t-1} 2^t 2^{s-t}) + 2^s = 2^{2s}\) elements of the character set of \(H\). For each distinct orbit we choose its representative to be \(\chi^{(p,q)}_{H}\) for \(p \neq 0\) or \(\chi^{(0, q)}_{H}\).

Now for each \((p,q)\) labeling the representative of a distinct orbit, we construct the invariant subgroup of \(\text{HW}_{2^s}\), \(K^{(p,q)} = H \otimes B^{(p,q)}\) which is given by

\[
K^{(p,q)} = \{z^m x^n y^{2^{s-t}v}, \quad m, n = 0, 1, \ldots, 2^s - 1, \quad v = 0, 1, \ldots, 2^t - 1\}, \quad (17)
\]

has order \(|K^{(p,q)}| = 2^{2s+t}\) and the coset space \(\text{HW}_{2^s}/K^{(p,q)} = B/B^{(p,q)}\).

The next step in the construction is the determination of the one dimensional unitary representations of \(K^{(p,q)}\), \(\Gamma^{(p,q)}_{K^{(p,q)}}\), obtained by tensoring the unitary irreps of \(B^{(p,q)}\) and \(H\). For each unitary irrep of \(B^{(p,q)}\), \(\Gamma^{r}_{B^{(p,q)}}\), given by (15) and each distinct orbit representative \(\chi^{(p,q)}_{H}\), we find that

\[
\Gamma^{(p,q)}_{K^{(p,q)}}(z^m x^n y^{2^{s-t}v}) = \chi^{(p,q)}_{H}(z^m x^n) \Gamma^{r}_{B^{(p,q)}}(y^{2^{s-t}v}), \quad p \neq 0
\]  \quad (18)

\[
\Gamma^{(0,q)}_{K^{(p,q)}}(z^m x^n y^l) = \chi^{(0,q)}_{H}(z^m x^n) \Gamma^{r}_{B^{(0,q)}}(y^l), \quad p = 0
\]  \quad (19)
where \( m, n, l = 0, 1, \ldots, 2^s - 1 \). More explicitly,

\[
\begin{align*}
\Gamma^{(p,q),r}_{K(p,q)}(z^m x^n y^{2^t-v}) &= e^{2\pi i (mp+rq)/2} e^{2\pi i rv}, \quad v = 0, 1, \ldots, 2^t - 1 \\
\Gamma^{(0,q),r}_{K(0,q)}(z^m x^n y^l) &= e^{2\pi i (nq+rl)}
\end{align*}
\]  

(20)

We are ready now to apply theorem 1, by making the identifications \( G \equiv HW_2 \), \( K \equiv K(p,q) \) and \( \Delta \equiv \Gamma^{(p,q),r}_{K(p,q)} \). In this way we get the complete set of unitary irreps of \( HW_2 \), which are presented by the following theorem (c.f. [17]):

Theorem 2. The matrix representations \( \Gamma^{(p,q),r}_{K(p,q)} \) of \( HW_2 \), induced by the those of \( K(p,q) \), i.e. \( \Gamma^{(p,q),r}_{K(p,q)} \xrightarrow{} \Gamma^{(p,q),r}_{K(p,q)} \uparrow HW_2 \), provide with the complete set of inequivalent unitary irreducible matrix representations of \( HW_2 \), and they are given by

\[
\Gamma^{(p,q),r}_{K(p,q)}(z^m x^n y^l)_{kj} = \begin{cases} \\
\Gamma^{(p,q),r}_{K(p,q)}(y^k z^m x^n y^{l-j}) & \text{for all } y^k y^{j} y^{-j} \in B(p,q) \\
0 & \text{otherwise} 
\end{cases}
\]  

(21)

where \( k, j = 0, 1, \ldots, 2^s - t - 1 \) and for all \( m, n, l = 0, 1, \ldots, 2^s - 1 \). (for proof see appendix B).

Explicitely, upon using (12) and (20), (21) we find that

\[
\begin{align*}
\Gamma^{(p,q),r}_{K(p,q)}(z^m x^n y^l)_{kj} &= \sum_{v=0}^{2^t-1} \chi^r_{B(p,q)}(y^k z^m x^n) \chi^r_{B(p,q)}(y^{2^t-v}) \delta_{k+l-j,2^t-v} \\
&= \sum_{v=0}^{2^t-1} e^{2\pi i pm} e^{2\pi i (kp+q)n} e^{2\pi i rv} \delta_{k+l-j,2^t-v}
\end{align*}
\]  

(22)

and the corresponding characters are given by

\[
\chi^{(p,q),r}_{K(p,q)}(z^m x^n y^l) = \sum_{k=0}^{2^s-t-1} \sum_{l=0}^{2^t-1} \chi^{(p,pk+q)}_{K(p,q)}(z^m x^n) \chi^r_{B(p,q)}(y^l) \delta_{k+l-j,2^t-v}
\]  

(23)

for all \( m, n, l = 0, 1, \ldots, 2^s - 1 \) and where \( \delta_{k+l-j,2^t-v} = 1 \) iff \( k + l - j = 2^s - t \) (mod \( 2^s \)).

For \( t = 0 \), the matrix irreps given by \( \Gamma^{(u_0,0),0}_{K(u_0,0)} \), \( u_0 = 1, 3, \ldots, 2^s - 1 \), are faithful \( 2^s \)-dimensional, for \( t = 1, \ldots, s - 1 \), the \( \Gamma^{(2^t u, q),r}_{K(u,q)} \), \( u_t = 1, 3, \ldots, 2^s-t - 1 \), \( q, r = 0, 1, \ldots, 2^t - 1 \), irreps are non-faithful \( 2^s-t \)-dimensional and for \( p = 0 \), the \( \Gamma^{(0,q),r}_{K(p,q)} \), \( q, r = 0, 1, \ldots, 2^s - 1 \) irreps are 1-dim. Counting arguments show that the number \( N_s \) of inequivalent irreps of \( HW_2 \) is given by
\[ N_s = \sum_{t=0}^{s-1} (2^{s-t-1}2^t) + 2^{2s} = 2^{s-1}(3 \times 2^s - 1) \]
\[ = \sum_{t=0}^{s-1} N_s(t) + 2^{2s} \]

where \( N_s(t) \) is the number of inequivalent irreps of dimension \( d = 2^{s-t} \) and the number of inequivalent 1-dimensional irreps is equal to \( 2^{2s} \). As expected \( N_s \) gives precisely the same result with the number of conjugacy classes \( N_C \) in (9). In the table below, the numbers \( N_s \) of irreps (and conjugacy classes) of \( HW_{2^s} \) for various values of \( s \) are shown:

| \( s \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-------|----|----|----|----|----|----|----|----|----|----|
| \( N_s \) | 5  | 22 | 92 | 376| 1520| 6112| 24512| 98176| 392960| 1572352|

(24)

In closing this section we present a simplification of notation to be used in the rest of the article. A general element of the group will be denoted by \( J_{mnl} \equiv z^{m,x^ny^l} \), a unitary matrix irrep. \( \Gamma^{(p,q),r} \), by \( D \equiv [(p,q),r] \) so that \( J_{mnl}^D \equiv \Gamma^{(p,q),r}(J_{mnl}) \). Thus, for the non trivial irreps (i.e. \( p \neq 0 \)) the commutator of two elements \( J_{mnl}^D, J_{m'n'l'}^D \) is given by:

\[ [J_{mnl}^D, J_{m'n'l'}^D] = (\omega_{s-t}^{m+n'l'} - \omega_{s-t}^{m'n'l'})J_{m+m',n+n',l+l'}^D \]  

where \( \omega_{s-t} = e^{2\pi i s-t} \) \( m,n,l = 0,1,...,2^s - 1 \).
4 Generalized Finite Fourier Transforms

In this section we investigate the existence of a transformation between \( x \) and \( y \) and the conditions under which this transformation can be a Finite Fourier Transform. In Finite quantum mechanics \( x \) represents the position operator in the diagonal form and \( y \) the shift operator of a fictitious particle moving on a discrete circle with \( 2^{s-t} \) equidistant points. The corresponding phase space is the discrete torus \( T_2[2^{s-t}] = \mathbb{Z}_2 \times \mathbb{Z}_2 \).

To obtain the matrix form of such a Finite Fourier Transform we shall first present the matrix form of the generators \( z, x, \) and \( y \) of \( HW_{2^s} \). This matrix forms are important for explicit calculations in problems of finite quantum mechanics.

In the representations constructed above, it is understood that the carrier space of the irreps \( \Gamma^{(p,q),r} \) can be taken to be
\[
V^D \equiv V^{(p,q),r} = \text{span}\{ |j> , j = 0, 1, ..., 2^{s-t} - 1 \} \tag{26}
\]
where we use the canonical basis \( |j> = (e_j)_k = \delta_{kj} \) and \( <k|j> = \delta_{kj} \).

Following (22), the action of \( z, x, y \) elements on \( V^D \) is given by
\[
z_D|j> = \omega^{u_s}_{s-t}\delta_{k,j}|k>, x_D|j> = \omega^q_{s-t}\omega^{u_k}_{s-t}\delta_{k,j}|k>, \]
\[
y_D|0> = \omega^r_{t}|2^{s-t} - 1>, y_D|j> = \delta_{k+1,j}|k>, j = 1, ..., 2^{s-t} - 1. \tag{27}
\]

Thus the \( 2^{s-t} \times 2^{s-t} \) matrix form of \( z_D, x_D, \) and \( y_D \) are given by
\[
z_D = \omega^{u_t}_{s-t}I_{2^{s-t}}, x_D = \omega^q_{s-t} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^{u_t}_{s-t} & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & 0 & \omega^{u_t}_{(2^{s-t}-1)} \end{bmatrix} \tag{28}
\]
\[
y_D = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \omega^r_{t} & 0 & 0 & 0 & 0 \end{bmatrix}
\]

Moreover using (28) it is shown that
\[
y_Dx_D = z_Dx_Dy_D \tag{29}
\]
which can be used to show that
\[
x_D^{2^{s-t}} = \omega^q_{t}I_{2^{s-t}}, y_D^{2^{s-t}} = \omega^r_{t}I_{2^{s-t}}, x_D^{2^s} = y_D^{2^s} = z_D^{2^s} = I_{2^s}. \tag{30}
\]
Then the diagonalizing matrix \( F \) product form \( r \) where the matrix elements of \( \Omega \) powers of \( \omega \) around the \( x \) and \( y \) cycles of phase space torus \( T_s[2^{s-t}] \), characterized by powers of \( \omega_t \).

From the explicit matrix form of \( x \) and \( y \) (28), we observe that the fictitious particle is described by quantum mechanical states carrying winding number \( u_t \) and satisfying the twisted boundary conditions with phase \( \omega_t^k \) for the rotation around the \( x \)-axis and \( \omega_t^r \) for the \( y \) axis. It follows that the corresponding Fourier Transform is expected to be modified with respect to the standard one.

Using (28), the eigenvalues \( \lambda \) of \( y_D \), solutions of \( \lambda^{2^{s-t}} = \omega_t^k \), the corresponding normalized eigenvectors are respectively given by

\[
|\psi_k> = \frac{1}{\sqrt{2^{s-t}}} \begin{bmatrix} 1 & \omega_s^k & \omega_s^2 & \cdots & \omega_s^{(2^{s-t}-1)r} \end{bmatrix}^T
\]

\[
\lambda_k = \omega_s^k, \quad k = 0, \ldots, 2^{s-t} - 1,
\]

Then the diagonalizing matrix \( F_D \) of \( y_D \), can be easily deduced to have the product form

\[
F_D = \Omega_r F_{s-t}
\]

where the matrix elements of \( \Omega_r \), \( F_{s-t} \) and \( F_D \) are respectively given by

\[
(\Omega_r)_{kj} = \omega_s^{kj} \delta_{kj},
\]

\[
(F_{s-t})_{kj} = \frac{1}{\sqrt{2^{s-t}}} \omega_s^{kj},
\]

\[
(F_D)_{kj} = \omega_s^{rk} \omega_s^{kj}
\]

with \( k, j = 0, \ldots, 2^{s-t} - 1 \). In particular the matrices \( \Omega_r \) and \( F_{s-t} \) have the matrix form

\[
\Omega_r = \text{diag} \left[ 1, \omega_s^r, \omega_s^{2r}, \ldots, \omega_s^{(2^{s-t}-1)r} \right],
\]

\[
F_{s-t} = \frac{1}{\sqrt{2^{s-t}}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & \omega_{s-t} & \omega_s^{2} & \cdots & \omega_s^{(2^{s-t}-1)} \\ 1 & \omega_s^{2} & \omega_s^{4} & \cdots & \omega_s^{2(2^{s-t}-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_s^{(2^{s-t}-1)} & \omega_s^{2(2^{s-t}-1)} & \cdots & \omega_s^{(2^{s-t}-1)(2^{s-t}-1)} \end{bmatrix}
\]

and \( F_{s-t} \) is the standard Finite Fourier Transform for \( s - t \) qubits. In the discrete phase space \( F_{s-t} \) represents the rotation by \( \pi/2 \) degrees i.e. \( F_{s-t}^1 = I_D \). Moreover \( (F_D y D F_D^{-1})^t u_t = \omega_s^{ru_t} \omega_s^{-q x_D^{-1}} \) and thus

\[
F_D y D F_D^{-1} = \omega_s^{ru_t-q x_D^{-1}}
\]
where \( u_t = 1, 3, \ldots 2^{s-t} - 1 \) and \( r, q = 0, \ldots, 2^t - 1 \).

5 The irreducible characters and fusion rules of \( HW_{2^s} \) irreps.

Up on using (12), (15), (22) and (28), the characters \( \chi^D(J_{mn}) = \text{tr} J^D_{mn} \) of the \( 2^{s-t} \)-dim. representations are given by

\[
\chi^D(J_{mn}) = \sum_{k=0}^{2^{s-t}-1} \chi^D_{(p,pk+q)}(J_{mn0}) \chi^r_{B(p,q)}(J_{00l}) \delta_{l,2^{s-t}v} \tag{37}
\]

for all \( m,n,l = 0, 1, \ldots, 2^s - 1 \). In particular, the non-zero values of characters are given by

\[
\chi^D(J_{mn,2^{s-t}v_1,2^{s-t}v_2}) = 2^{s-t} \omega_s^{um_v_1 q + v_2 r} \tag{38}
\]

where \( m = 0, 1, \ldots, 2^s - 1 \), and \( v_1, v_2 = 0, \ldots, 2^t - 1 \). For example in the case of characters of \( HW_4 \) i.e. \( s = 2 \), for we have 22 inequivalent irreps: (i) for \( t = 0 \), the two 4-dim faithful irreps \( \Gamma^{(1,0)_0}, \Gamma^{(3,0)_0} \), (ii) for \( t = 1 \), the four 2-dim non-faithful, \( \Gamma^{(2,q)_r}, q, r = 0, 1 \) and (iii) for \( t = 2 \), the sixteen 1-dim, \( \Gamma^{(0,q)_r}, q, r = 0, 1, 2, 3 \).

In the following we shall establish the fusion algebra of the unitary irreps of \( HW_{2^s} \) as constructed above. Given any two such irreps, \( D_i, D_j \), the fusion algebra is defined by

\[
D_i \otimes D_j = \sum_k N_{jk}^i D_k
\]

where \( N_{jk}^i \) is the multiplicity of the \( D_k \) irrep appearing in the decomposition of the above tensor product. Since these multiplicities completely specifies the fusion algebra we shall provide with the explicit relation which computes them. For simplicity of exposition, let \( D_1, D_2 \) be any two given unitary irreps of \( HW_{2^s} \) and let \( D_3 \) be some irrep that would appear in the decomposition of \( D_1 \otimes D_2 \). Implementing the well known relation of the multiplicities in the case of the tensor product of two finite group irreps (c.f. [17]) we obtain

\[
N_{12}^3 = \frac{1}{2^{3s}} \sum_{m,n,l=0}^{2^{s-1}} \chi^{D_3}(J_{m,n,l}) \chi^{D_2}(J_{m,n,l}) \chi^{D_1}(J_{m,n,l})^* \tag{39}
\]
where the $\chi^D_i$, $i=1,2,3$ are given by (37). Using (37) and (38), relation (39) becomes

$$N_{12}^3 = 2^{s-t_2+t_1-t_3} \delta_{p_1+p_2, p_3(\mod 2^t_1)} \delta_{q_1+q_2, q_3(\mod 2^t_1)} \delta_{r_1+r_2, r_3(\mod 2^t_1)}$$

(40)

where $p_1 = 2^{t_1} u_1$, $p_2 = 2^{t_2} u_2$, $p_3 = 2^{t_3} u_3$ and without loss of generality it has been assumed that $t_1 \leq t_2$. Consequently the fusion algebra of $HW_2$, is completely specified by

$$[(p_1, q_1), r_1] \otimes [(p_2, q_2), r_2] \approx \bigoplus_{q_3(\mod 2^t_1)=q_1+q_2(\mod 2^t_1)} 2^{s-t_2+t_1-t_3} [(p_1 + p_2, q_3), r_3]$$

(41)

It should be understood that in the above multiplicity and fusion formula the values of $q_3$ and $r_3$ are in the range $\{0, 1, ..., 2^{t_3} - 1\}$ such that $q_3(\mod 2^t_1) = (q_1 + q_2)(\mod 2^t_1)$ and $r_3(\mod 2^t_1) = (r_1 + r_2)(\mod 2^t_1)$ and where $t_3$ is obtained by $p_3 = (p_1 + p_2)(\mod 2^t) = 2^{t_3} u_3$.

In particular, in the case of the fusion of two identical (up to equivalence) irreps $[(p, q), r] \otimes [(p, q), r]$ where $p = 2^t u_t$ we obtain that

$$[(p, q), r] \otimes [(p, q), r] \approx \bigoplus_{q_3(\mod 2^t)=2q(\mod 2^t)} r_3(\mod 2^t) = 2r(\mod 2^t) 2^{s-t-1} [(2^{t+1} u_{t+1}, q_3), r_3]$$

(42)

where with $t \neq 0$, we have set $p_3 = (2p)(\mod 2^t) = (2^{t+1} u_{t+1})(\mod 2^t) = 2^{t+1} u_{t+1}$ and $q_3, r_3 \in \{0, 1, ..., 2^{t+1} - 1\}$ such that $q_3 = 2q(\mod 2^t)$, $r_3 = 2r(\mod 2^t)$.

In the case of the fusion of two faithfull irreps $[(p_1, 0), 0] \otimes [(p_2, 0), 0]$ where $p_1, p_2$ are odd we obtain that

$$[(p_1, 0), 0] \otimes [(p_2, 0), 0] \approx \bigoplus_{q_3(\mod 1)=0} 2^{s-t_3} [(p_3, q_3), r_3]$$

(43)

with $p_3 = (p_1 + p_2)(\mod 2^t) = 2^{t_3} u_3$ and all $q_3, r_3 \in \{0, 1, ..., 2^{t_3} - 1\}$ from which its is apparent that only non-faithful irreps participate in the decomposition.

5.1 Example 1. Fusion rules of $HW_2$

According to the analysis of the previous sections, the inequivalent irreps of $HW_2$ are: (i) for $t=0$, the one 2-dim faithful irreps $\Gamma^{(1,0)}$, (ii) for $t=1$, the four 1-dim non-faithful, $\Gamma^{(0,q)}$, $q, r = 0, 1$. Using (41), the set of fusion

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rules among the irreps are given by

\[
\begin{align*}
\Gamma^{(1,0),0} \otimes \Gamma^{(1,0),0} & \approx \Gamma^{(0,0),0} \oplus \Gamma^{(0,0),1} \oplus \Gamma^{(0,1),0} \oplus \Gamma^{(0,1),1} \\
\Gamma^{(1,0),0} \otimes \Gamma^{(0,q),r} & \approx \Gamma^{(1,0),0} \quad \text{for all } q, r = 0, 1 \\
\Gamma^{(0,0),0} \otimes \Gamma^{(0,q),r} & \approx \Gamma^{(0,q),r} \quad \text{for all } q, r = 0, 1 \\
\Gamma^{(0,0),1} \otimes \Gamma^{(0,0),1} & \approx \Gamma^{(0,0),0} \oplus \Gamma^{(0,1),0} \approx \Gamma^{(0,1),1} \\
\Gamma^{(0,0),1} \otimes \Gamma^{(0,1),1} & \approx \Gamma^{(0,1),0} \oplus \Gamma^{(0,1),0} \approx \Gamma^{(0,0),0} \\
\Gamma^{(0,1),0} \otimes \Gamma^{(0,1),1} & \approx \Gamma^{(0,0),1} \oplus \Gamma^{(1,0),1} \approx \Gamma^{(0,0),0}
\end{align*}
\]

5.2 Example 2. Fusion rules of $HW_{2^2}$

According to the analysis of the previous sections, the inequivalent irreps of $HW_{2^2}$ are: (i) for $t = 0$, the two 4-dim faithful irreps $\Gamma^{(1,0),0}$, $\Gamma^{(3,0),0}$, (ii) for $t = 1$, the four 2-dim non-faithful, $\Gamma^{(2,q),r}$, $q, r = 0, 1$, and (iii) for $p = 0$ (or equivalently for $t = 2$), the sixteen 1-dim, $\Gamma^{(0,q),r}$, $q, r = 0, 1, 2, 3$. Using [11], the set of fusion rules among the non-trivial irreps are given by

\[
\begin{align*}
\Gamma^{(1,0),0} \otimes \Gamma^{(1,0),0} & \approx \Gamma^{(3,0),0} \otimes \Gamma^{(3,0),0} \\
\Gamma^{(1,0),0} \otimes \Gamma^{(3,0),0} & \approx \Gamma^{(2,0),0} \otimes \Gamma^{(2,0),1} \oplus 2\Gamma^{(2,0),0} \oplus 2\Gamma^{(2,1),0} \oplus 2\Gamma^{(2,1),1} \\
\Gamma^{(1,0),0} \otimes \Gamma^{(2,q),r} & \approx \Gamma^{(2,0),0} \quad \text{for all } q, r = 0, 1 \\
\Gamma^{(3,0),0} \otimes \Gamma^{(2,q),r} & \approx \Gamma^{(2,0),1} \otimes \Gamma^{(2,0),1} \approx \Gamma^{(2,1),0} \otimes \Gamma^{(2,1),0} \\
\Gamma^{(2,0),0} \otimes \Gamma^{(2,0),1} & \approx \Gamma^{(2,1),0} \otimes \Gamma^{(2,1),1} \approx \Gamma^{(2,1),0} \oplus \Gamma^{(2,1),1} \\
\Gamma^{(2,0),0} \otimes \Gamma^{(2,1),0} & \approx \Gamma^{(0,0),1} \oplus \Gamma^{(0,0),3} \oplus \Gamma^{(0,2),1} \oplus \Gamma^{(0,2),3} \\
\Gamma^{(2,0),0} \otimes \Gamma^{(2,1),0} & \approx \Gamma^{(0,1),0} \oplus \Gamma^{(0,1),2} \oplus \Gamma^{(0,3),0} \oplus \Gamma^{(0,3),2} \\
\Gamma^{(2,0),0} \otimes \Gamma^{(2,1),1} & \approx \Gamma^{(2,0),1} \otimes \Gamma^{(2,1),0} \approx \Gamma^{(2,0),0} \oplus \Gamma^{(2,0),1} \oplus \Gamma^{(2,0),3} \oplus \Gamma^{(2,1),0} \oplus \Gamma^{(2,1),1} \oplus \Gamma^{(2,1),3}
\end{align*}
\]
6 Conclusion

We conclude this work by a summary of our results. We have introduced a definition of the Heisenberg Weyl group $HW_2$, using only a presentation of the relations for its generators. This presentation allows a generalization of the standard discrete $HW_2$ group where only the primitive root of unity $\omega = e^{2\pi i}$ appears as the generator of its center. Our generalization allows all the possible roots of unity to appear with the crucial result the appearance of a rich spectrum of non faithful representations. We have thus to apply the Wigner-Mackey method of orbits and little groups to this extended group. Indeed we have found that all the representations are classified by a triplet of integers $(p, q, r)$, taking appropriate values in the range from 0 to $2^s - 1$. The dimensions of these representations are $2^{s-t}$ where $t$ runs from 0 to $s$. When $t = 0$ the representations are faithful while for $t$ from 1 to $s$ the representations are non-faithful. For each of these representations we have determined explicitly their matrix form as well as their characters and their fusion rules. In particular the rich structure of the fusion rules provide decompositions with non trivial multiplicities which may be of interest in anyonic systems. We have examined also the relation between the ”position” and ”momentum” operators for any irrep with the result the existence of generalized finite Fourier transforms.

To discuss possible applications and future work we note that for applications in quantum circuits it is necessary to construct the corresponding quantum circuit for the unitary matrices of any representation and any group element. To this end, it is enough to construct the quantum circuits for the ”momentum” and ”position” matrices $P$ and $Q$.

For applications in the area of quantum Hall effect, we should consider an important class of group elements of $HW_2$, the so called magnetic translations which represent the motion of an electron in two dimensional toroidal lattice $\mathbb{Z}_{2^s} \times \mathbb{Z}_{2^s}$ in the presence of a transverse magnetic flux $2\pi/2^s$ per lattice plaquete, in units of quantum of magnetic flux [13, 15] and references therein. The problem which appears for this value of flux is that the magnetic translations are ill defined and one has to go to a new phase space lattice $\mathbb{Z}_{2^{2s}} \times \mathbb{Z}_{2^{2s}}$ and the extension of the $HW_2$ to the tensor product $HW_2 \otimes HW_2$. The solution to this problem will appear elsewhere [15].

In this area considerations on the construction of models of topological quantum computations lead to the study of new excitations of many electron systems the so called anyonic excitations of exotic fractional statistics [19]. Here the representations of the finite Heisenberg group $HW_2$, as well as its automorphism group play an important role in the study of so called
metaplectic anyons. The Heisenberg group for a system of N electrons moving on the above toroidal lattice is the N-fold tensor product of $HW_{2^s}$ and its automorphism group is the finite symplectic group $Sp_{2N}(Z_{2^s})$ so we have to construct quantum circuits for the elements of this group. The anyonic excitations are eigenstates of the corresponding unitary operators because of the relation of the braid group $B_{2N+1}$ and the symplectic group. So in future work we shall study the metaplectic representation of $Sp_{2N}(Z_{2^s})$ [20].
7 Appendix

7.1 Determination of distinct orbits

Consider an irreducible character $\chi^{(p, q)}_H$ of $B$ and the corresponding little group $B^{(p, q)}$, where the values of $p$ are given either by $p = 2^t u_t$, for each $t = 0, 1, ..., s - 1$ and each $u_t = 1, 3, ..., 2^{s-t} - 1$ or by $p = 0$ and in both cases the values of $q = 0, 1, ..., 2^s - 1$. Since the action of the elements of $B^{(p, q)}$ leave $\chi^{(p, q)}_H$ invariant, the orbit of $\chi^{(p, q)}_H$, $Orb(\chi^{(p, q)}_H)$, under $B$ consists of the irreducible characters of $B$ obtained by the action of the factor group $B/B^{(p, q)} = \{y^k, k = 0, 1, ..., 2^{s-t} - 1\}$ on $\chi^{(p, q)}_H$ given by

$$y^k \chi^{(p, q)}_H(z^m x^n) = \chi^{(p, q)}_H(y^k z^m x^n y^{-k}) = \chi^{(p, kp+q)}_H(z^m x^n)$$

where (12) and (13) have been used. That is

$$Orb(\chi^{(p, q)}_H) = \{\chi^{(p, kp+q)}_H, k = 0, 1, ..., 2^{s-t} - 1\}, \quad p \neq 0$$
$$Orb(\chi^{(0, q)}_H) = \{\chi^{(0, q)}_H\}, \quad p = 0 \quad (44)$$

It is obvious that the sets $Orb(\chi^{(0, q)}_H)$ for $q \in \{0, 1, ..., 2^s - 1\}$, have all one element and are all distinct. However this is not the case for $Orb(\chi^{(p, q)}_H)$, $p \neq 0$ since for fixed $p$, $kp+q$ is evaluated mod$2^s$. For fixed $p = 2^t u_t$ the sets $K_q = \{kp+q, k = 0, 1, ..., 2^{s-t} - 1\}$ take the form $K_q = \{q, 2^t u_t + q, 2 \times 2^t u_t + q, ..., (2^{s-t} - 1) 2^t u_t + q\}$ from which it is obvious that changing the values of $u_t \in \{1, 3, ..., 2^{s-t} - 1\}$ leads to a mere rearrangement of the elements of the set. Thus it suffices to find the distinct sets $K_q$ when $u_t = 1$, given by

$$K_q = \{q, 2^t + q, 2 \times 2^t + q, ..., (2^{s-t} - 2) 2^t + q, (2^{s-t} - 1) 2^t + q\} \quad (45)$$

when $q$ takes the values from 0 to $2^s - 1$. For $q \in \{0, 1, ..., 2^t - 1\}$ it is evident that the sets $K_q$ are all distinct since for the maximum value $q = 2^t - 1$, the maximum value in the set is $2^t - 1$. For $q = 2^t$, $2^t + 1, ..., 2 \times 2^t - 1$ direct evaluation mod$2^s$ of the elements in the (45) shows that $K_{2^t} = K_0$, $K_{2^t+1} = K_1$, $K_{2^t+2} = K_2,..., K_{2 \times 2^t-1} = K_{2^t-1}$. Similarly for $q = 2 \times 2^t$, $2 \times 2^t + 1, ..., 3 \times 2^t - 1$ it is shown that $K_{2 \times 2^t} = K_0$, $K_{2 \times 2^t+1} = K_1$, $K_{2 \times 2^t+2} = K_2,..., K_{3 \times 2^t-1} = K_{2^t-1}$. Continuing this evaluation up to the values of $q = (2^{s-t} - 1) \times 2^t$, $(2^{s-t} - 1) \times 2^t + 1, (2^{s-t} - 1) \times 2^t + 2, ..., (2^{s-t} - 1) \times 2^t + 2^t - 1$ we finally obtain that for each fixed pair $s, t$, $K_0, K_1, ..., K_{2^t-1}$ are all distinct while for the rest values of $q$ from $2^t$ to $2^s - 1$, it is deduced that
\[ K_{j+q} = K_q, \text{ for each } q = 0, 1, ..., 2^t - 1 \]  
and all \( j = 2^t, 2 \times 2^t, 3 \times 2^t, ..., (2^{s-t} - 1)2^t \)

Consequently the distinct orbits are those for which either \( p = 0 \) and \( q = 0, 1, ..., 2^s - 1 \) or \( p = 2^t u_t \) and \( q = 0, 1, ..., 2^t - 1 \). These participate in the construction of the complete set of inequivalent irreps of \( HW_{2s} \). Counting arguments show that the number of distinct orbits \( N_o = \sum_{t=0}^{s-t-1} (2^s-t-12^t) + 2^s = 2^{s-1}(s+2) \).

Closing this appendix we can illustrate the determination of distinct orbits with the example of \( HW_{2s} \). In this case \( s = 2 \) and \( t = 0, 1 \) and \( q = 0, 1, 2, 3 \). For \( p = 0 \) we have 4 distinct orbits \( Orb(\chi_H^{(0,0)}), Orb(\chi_H^{(0,1)}), Orb(\chi_H^{(0,2)}), Orb(\chi_H^{(0,3)}) \) of one element each. For \( t = 0, u_0 = 1, 3, p = u_0 = 1, 3 \) and for \( q=0,1,2,3 \) we obtain

\[
\begin{align*}
Orb(\chi_H^{(1,0)}) &= \{ \chi_H^{(1,0)}, \chi_H^{(1,1)}, \chi_H^{(1,2)}, \chi_H^{(1,3)} \} = Orb(\chi_H^{(1,1)}) = Orb(\chi_H^{(1,2)}) = Orb(\chi_H^{(1,3)}) \\
Orb(\chi_H^{(3,0)}) &= \{ \chi_H^{(3,0)}, \chi_H^{(3,1)}, \chi_H^{(3,2)}, \chi_H^{(3,3)} \} = Orb(\chi_H^{(3,1)}) = Orb(\chi_H^{(3,2)}) = Orb(\chi_H^{(3,3)}) 
\end{align*}
\]

so the distinct orbits can be taken to be \( Orb(\chi_H^{(1,0)}) \) and \( Orb(\chi_H^{(3,0)}) \), that is for \( q = 0 \). For \( t=1, u_1=1, so p = 2u_1 = 2, and for q=0,1,2,3 \) we obtain

\[
\begin{align*}
Orb(\chi_H^{(2,0)}) &= \{ \chi_H^{(2,0)}, \chi_H^{(2,1)} \} = Orb(\chi_H^{(2,2)}) \\
Orb(\chi_H^{(2,1)}) &= \{ \chi_H^{(2,1)}, \chi_H^{(2,3)} \} = Orb(\chi_H^{(2,3)}) 
\end{align*}
\]

so the distinct orbits can be taken to be \( Orb(\chi_H^{(2,0)}) \) and \( Orb(\chi_H^{(2,1)}) \), that is for \( q = 0, 1 \).
7.2 Proof of theorem 2

Unitarity of $\Gamma^{(p,q),r}$ stems from the unitarity of $\Gamma^{(p,q),r}_{K^{(p,q)}}$. Irreducibility is easily proven by checking that $\sum_{m,n,l=0}^{2s-1} |\chi^{(p,q),r}_{\delta}(z^m x^n y^l)|^2$ equals the order of $HW_{2s}$. Indeed, using (23) we have that

$$\sum_{m,n,l=0}^{2s-1} |\chi^{(p,q),r}_{\delta}(z^m x^n y^l)|^2 = \sum_{m,n,l=0}^{2s-1} \sum_{k,k'=0}^{2s-1} \sum_{v,v'=0}^{2s-1} e^{\frac{2\pi i}{2s}(k-k')pm} e^{\frac{2\pi i}{2s}(v-v')\delta_{l,2s-t_v} \delta_{l,2s-t_{v'}}}$$

$$= 2^{2s} \sum_{l=0}^{2s-1} \sum_{v,v'=0}^{2s-1} e^{\frac{2\pi i}{2s}(v-v')\delta_{l,2s-t_v} \delta_{l,2s-t_{v'}}}$$

$$= 2^{2s} \sum_{v,v'=0}^{2s-1} e^{\frac{2\pi i}{2s}(v-v')\delta_{v,v'}} = 2^{3s}$$

To show the completeness of the set of the unitary irreps thus constructed we have first to demonstrate if we chose any other character $\chi$ than the representative $\chi^{(p,q)}_{\delta}$ from a distinct orbit we obtain equivalent representations and secondly that the same is true if we chose any other character from other non distinct orbits as classified above in appendix 1. These will be investigated by using the well known fact that for two representations $\Gamma^{(p,q),r}$ and $\Gamma^{(p',q'),r'}$ to be equivalent a necessary and sufficient condition is the equality of their character systems. The demonstration of the completeness is finalized by showing that the sum of the square of the dimensions of all the constructed irreps equals $2^{3s}$, the order of group.

To this end, let $\chi^{(p,q),r} = \chi^{(p,q)}_{\delta} \chi^{r}_{B(p,q)}$ be the character of an irrep constructed above where $\chi^{(p,q)}_{\delta}$ is the character representative of the corresponding distinct orbit and let $\chi^{(p,kp+q),r'} = \chi^{(p,kp+q)}_{\delta} \chi^{r'}_{B(p,kp+q)}$ be a character where $\chi^{(p,kp+q)}_{\delta}$ is in the orbit of $\chi^{(p,q)}_{\delta}$ for some fixed $k = 1, 2, ..., 2s-t - 1$. Since by construction $B^{(p,q)} = B^{(p,kp+q)}$, taking $r = r'$ and using (23), with $p = 2^t u_t$, $\chi^{(p,q),r}$ is given by:

$$\chi^{(p,q),r}_{\delta}(z^m x^n y^l) = \sum_{k'=0}^{2s-1} \sum_{v=0}^{2s-1} e^{\frac{2\pi i}{2s}(k'-q'p)n} e^{\frac{2\pi i}{2s}(r-v)\delta_{l,2s-t_v}}$$

$$= 2^{s-t} \sum_{v=0}^{2s-1} e^{\frac{2\pi i}{2s}qu} e^{\frac{2\pi i}{2s}(u_t n)\delta_{l,2s-t_v}}$$

while for $\chi^{(p,kp+q),r'}$
for all m, n, l = 0, 1, ..., 2^s - 1. Since \( \delta_{u,n,0} \) is evaluated mod\( 2^{s-t} \), for \( m = l = 0 \), the non zero characters are those for which \( n = 0, 2^{s-t}, 2 \times 2^{s-t}, 3 \times 2^{s-t}, ... \). Thus comparison of (47) with (48) shows that for each value of \( t \) with \( q = 0 \) the non zero characters are those for which \( s \) is a multiple of 2 and \( \Gamma^{(p,q)}, \Gamma^{(p,kp+q)}, r \) are equivalent.

Turning now to the case of two representations \( \Gamma^{(p,q)}, \Gamma^{(p,q+j)}, r' \) where \( q \) takes a fixed value between 0 to \( 2^l - 1 \) and \( j \) is in [16]. The character \( \chi^{(p,q+j), r'} = \chi^{(p,q)} H(\chi^{(p,q+j)}) \) is such that \( \chi^{(p,q+j)} \) is taken to be an orbit representative of the non distinct orbit \( Orb(\chi^{(p,q+j)}) \). Again since by construction \( B^{(p,q)} = B^{(p,q+j)} \), taking \( r = r' \) and using (23), with \( p = 2^l u_1 \), \( \chi^{(p,q)}, r \) is given by (47) while \( \chi^{(p,q+j)}, r \) is given by:

\[
\chi^{(p,q+j), r}(z^m, x^n y^l) = \sum_{k=0}^{2^{s-t}-1} \sum_{v=0}^{2^{l}-1} e^{2\pi i (kp+q+j)n} e^{2\pi i (rv) t} \delta_{l,2^{s-t}}
\]

where again for \( m = l = 0 \), the non zero characters are those for which \( n \) is a multiple of \( 2^{s-t} \). Since \( j \) is a multiple of \( 2^l \) the term \( e^{2\pi i j n} = 1 \) which implies the equality of (47) and (49) and thus the equivalence of \( \Gamma^{(p,q)}, r \) and \( \Gamma^{(p,q+j)}, r \).

Finally for the unitary irreps thus constructed, observe that for each value of \( t = 0, 1, ..., s - 1 \), there correspond \( 2^{s-t-1} \times 2^{2t} \) irreps \( \Gamma^{(2^t u_1, q)}, \Gamma^{(2^t u_1, q)}, r \), all of the same dimensionality \( 2^{s-t} \) and for \( p = 0 \), there correspond \( 2^{2s} 1 \)-dimensional irreps \( \Gamma^{(0, q)}, r \). Thus the sum of the square of the dimensions of all these irreps is \( \sum_{t=0}^{s-1} (2^{s-t-1}2^{2t}) 2^{2s-2t} + 2^{2s} = 2^{3s} \) as is the order of \( HW_{2^s} \).
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