Models of universe with a polytropic equation of state: III. The phantom universe

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We construct models of universe with a generalized equation of state $p = (\alpha \rho + k \rho^{1+1/n})c^2$ having a linear component and a polytropic component. The linear equation of state $p = \alpha \rho c^2$ with $-1 \leq \alpha \leq 1$ describes radiation ($\alpha = 1/3$), pressureless matter ($\alpha = 0$), stiff matter ($\alpha = 1$), and vacuum energy ($\alpha = -1$). The polytropic equation of state $p = k \rho^{1+1/n}c^2$ may be due to Bose-Einstein condensates with repulsive ($k > 0$) or attractive ($k < 0$) self-interaction, or have another origin. In this paper, we consider the case where the density decreases as the universe expands. This corresponds to a “phantom universe” for which $w = p/\rho c^2 < -1$ (this requires $k < 0$). We complete previous investigations on this problem and analyze in detail the different possibilities. We describe the singularities using the classification of [S. Nojiri, S. D. Odintsov, S. Tsujikawa, Phys. Rev. D 71, 063004 (2005)]. We show that for $\alpha > -1$ there is no Big Rip singularity although $w \leq -1$. For $n = -1$, we provide an analytical model of phantom bouncing universe “disappearing” at $t = 0$. We also determine the potential of the phantom scalar field and phantom tachyon field corresponding to the generalized equation of state $p = (\alpha \rho + k \rho^{1+1/n})c^2$.

I. INTRODUCTION

In previous papers of this series, we have constructed models of universe with a generalized equation of state

\[ p = (\alpha \rho + k \rho^{1+1/n})c^2, \]

having a linear component and a polytropic component. In Papers I and II, we have assumed $\alpha + 1 + k \rho^{1/n} \geq 0$ corresponding to $w = p/\rho c^2 \geq -1$. In that case, the density decreases as the universe expands. For $n > 0$, the polytropic component dominates the linear component when the density is high: This describes the early universe (Paper I). For $n < 0$, the polytropic component dominates the linear component when the density is low: This describes the late universe (Paper II). When the polytropic pressure is positive ($k > 0$), the solutions of the Friedmann equations exhibit past or future singularities (or peculiarities). When the polytropic pressure is negative ($k < 0$), there is no singularity. Furthermore, the polytropic equation of state implies the existence of an upper bound $\rho_{\text{max}}$ (in the past) and a lower bound $\rho_{\text{min}}$ (in the future) for the density. It makes sense to identify the maximum density to the Planck density $\rho_P = 5.16 \times 10^{98} \text{g/m}^3$ and the minimum density to the cosmological density $\rho_\Lambda = 7.02 \times 10^{-24} \text{g/m}^3$. These constant densities imply in turn the existence of two phases of exponential inflation, one in the early universe and one in the late universe. During the inflation, the universe is accelerating. The early inflation is necessary to solve notorious difficulties such as the singularity problem, the flatness problem, and the horizon problem [1, 2]. The late inflation is necessary to account for the observed accelerating expansion of our universe [3] driven by dark energy [4]. In that context, the equation of state [4] with $k < 0$ and $n < 0$ corresponds to the generalized Chaplygin gas [3] that has been proposed as a model for dark energy. From the generalized polytropic equation of state [11], we have obtained a model of universe without singularity that possesses striking “symmetries” between the past and the future (aioniotic universe). This model, which could have been obtained from a principle of “simplicity” without making any observation, turns out to be strikingly consistent with what we know of the real universe. It is consistent with the standard model [4, 7] but refines it by removing the primordial singularity (Big Bang). In this model, the Planck density and the cosmological density are interpreted as two fundamental bounds for the density determined by the Planck constant $\hbar$ (microphysics) and the cosmological constant $\Lambda$ (cosmophysics), respectively. These bounds differ by 122 orders of magnitude, a difference that appears to be quite natural instead of representing a “problem” [8].

In this paper, we consider a case that has not been treated in our previous papers. This is the case where the density increases as the universe expands. Since the nature of dark energy is unknown, this situation cannot be rejected a priori. It corresponds to an equation of state parameter $w$ less than $-1$ which violates the null dominant energy condition. This is referred to as a “phantom universe” [9] because when the equation of state with $w < -1$ is constructed in terms of a scalar field, the corresponding kinetic term has the wrong sign (negative kinetic energy). It represents therefore a phantom (ghost) scalar field (see reviews [4, 10]).

Actually, there is a rich recent literature on this situation (more than one thousand papers are related to phantom dark energy) since observations do not exclude the possibility that we live in a phantom universe. Indeed, observational data indicate that the equation of state parameter $w$ lies in a narrow strip around $w = -1$ possibly being below this value [11]. The models based on phantom dark energy usually predict a future singularity in which the scale factor, the energy density, and the pressure of the universe become infinite in a finite time. This would lead to the death of the universe in a singularity called “Big Smash” [12], “Big Rip” or “Cos-
mic Doomsday" [13]. Contrary to the “Big Crunch”, the
universe is destroyed not by excessive contraction but
rather by excessive expansion. In phantom cosmology,
every gravitationally bound system (e.g. the solar sys-
tem, the Milky Way, the local group, galaxy clusters) is
dissociated before the singularity [13, 14], and the black
holes gradually lose their mass and finally vanish [15, 16].
This scenario allows the explicit calculation of the rest of
the lifetime of our universe. Actually, as we approach the
singularity, the energy scale may grow up to the Planck
one, giving rise to a second quantum gravity era. Eventu-
ally, quantum effects may moderate or even prevent the
singularity [17]. Other aspects of phantom cosmology
have been studied in [18].

There are many interesting recent works on the study
of singularities. In particular, Nojiri et al. [19] consid-
ered an equation of state of the form \( p = -\rho - f(\rho) \) and
obtained a classification of finite-time future singularities
(see supplements in [20]). They are of four types:

- **Type 0** (Big Bang or Big Crunch): For \( t \to t_s \), \( a \to 0 \),
  \( \rho \to +\infty \), and \( |p| \to +\infty \).

- **Type I** (Big Rip): For \( t \to t_s \), \( a \to +\infty \), \( \rho \to +\infty \),
  and \( |p| \to +\infty \).

- **Type II** (sudden singularity): For \( t \to t_s \), \( a \to a_s \),
  \( \rho \to \rho_s \), and \( |p| \to +\infty \).

- **Type III** (Big Freeze): For \( t \to t_s \), \( a \to a_s \), \( \rho \to +\infty \),
  and \( |p| \to +\infty \).

- **Type IV** (generalized sudden singularity): For \( t \to t_s \),
  \( a \to a_s \), \( \rho \to \rho_s \), \( |p| \to p_s \), and higher derivatives of \( H \)
diverge.

In this classification, \( t_s \), \( a_s \), \( \rho_s \), and \( p_s \) are all finite
constants (\( a_s \neq 0 \)). Type 0 is the standard Big Bang
or Big Crunch singularity arising in the original Fried-
mann models [6]. Type I is the Big Rip singularity which
emerges from the phantom equation of state \( p = \alpha \rho c^2 \)
with constant \( \alpha < -1 \) [8, 13], and from the equation of
state \( p = \rho c^2 \) with \( \alpha = -1 \) [13]. Type II corre-
sponds to the sudden future singularity found by
Barrow [22] at which \( \rho \) and \( p \) are finite but \( p \) diverges.
Type III, arising in the equation of state \( p = \rho c^2 \) [23]
differs from the sudden future singularity in the
sense that \( \rho \) diverges. Type IV appears in the model
described in [19].

It is important to stress that the phantom models with
\( w < -1 \) do not necessarily lead to future singularities.
For example, the equation of state \( p = \rho c^2 \) with \( \alpha = -1 \)
and \( -2 \leq n < 0 \) does not present future singularity [21].
However, since the scale factor and the density increase
indefinitely, this has been called “Little Rip” [24].

On the other hand, the models with \( w > -1 \) may lead
to past or future singularities. For example, the new
form of primordial singularity (for \( n > 0 \) and \( k > 0 \))
described in Secs. IV C, VI and in Appendix A of Paper I
corresponds to a past singularity of type III: The universe
starts at \( t = 0 \) with a finite scale factor and an infinite
density. On the other hand, the future singularity (for
\( -1 < n < 0 \) and \( k > 0 \)) described in Sec. IV C and in
Appendix B of Paper II corresponds to a singularity of
type II: At a finite time \( t_s \), the universe reaches a point
at which the scale factor is finite, the density vanishes
and the pressure is infinite.

In paper II, we have also introduced a notion of “pecu-
liarity”. This is when the density vanishes \( \rho = 0 \) while
the scale factor has a finite value \( a_s \) (when \( a_s = 0 \) we shall
call it generalized peculiarity). In that case, the universe
is empty (in other works, it “disappears”). Although there
is no singularity, this situation is very peculiar. However,
since the nature of dark energy is unknown, all possibil-
ities should be contemplated.

In this paper, we provide an exhaustive study of the
equation of state \( \rho = \rho(t) c^2 \) in the case \( w \leq -1 \) (requiring \( k < 0 \))
for arbitrary \( -1 \leq \alpha \leq 1 \) and \( n \). This is a natural
continuation of our previous works which assumed \( w = -1 \)
(Papers I and II). Our paper also completes previous
studies of the case \( w \geq -1 \) that considered \( \alpha = -1 \)
[21] or \( \alpha = 0 \) [23]. An interesting result of our study
is that the equation of state \( \rho = \rho(t) c^2 \) with \( \alpha < -1 \)
[13] or the equation of state \( \rho = \alpha \rho c^2 \) with \( \alpha = -1 \)
[13] or \( \alpha = 0 \) [23]. Another interesting result of our study is the
construction of a bouncing phantom universe for \( -2 < n < 0 \).
For \( -1 < n < 0 \), this bouncing universe presents
a past singularity of type II: At \( t = 0 \), the the pressure is
infinite while the scale factor has a finite value and the
density vanishes. For \( \alpha > -1 \) and \( n = -1 \), corresponding
to a constant negative pressure, the bouncing phantom
universe admits a simple analytical expression.

The paper is organized as follows. In Sec. III we recall
the basic equations of cosmology. In Secs. III and IV
we study the generalized equation of state \( \rho = \rho(t) c^2 \) for
any value of the parameters \( -1 < \alpha \leq 1 \), \( k < 0 \) and \( n \).
Assuming \( w = -1 \) (phantom cosmology). In Sec. VII we
determine the potential of the phantom scalar field and
the potential of the phantom tachyon field corresponding
to the generalized equation of state \( \rho = \rho(t) c^2 \). In Appendix A
we treat the case \( \alpha = -1 \). In Appendix D we summarize
all the results obtained in our series of papers and analyze
the different singularities in terms of the classification of
[19].

II. BASIC EQUATIONS OF COSMOLOGY

We assume that the universe is isotropic and homo-
genous at large scales and contains a uniform perfect
fluid of energy density \( \rho(t) c^2 \) and pressure \( p(t) \).
We also assume that the universe is flat in agreement
with observations of the cosmic microwave background
(CMB) [7]. Finally, in this paper, we ignore the cos-
omical constant (\( \Lambda = 0 \)). It that case, the Einstein

\[ \frac{\left( a^2 \right)^{\frac{\alpha}{2}}}{|\dot{a}|} \left( 1 - \alpha \right) \frac{\ddot{a}}{a} = 0 \]

1 We shall not consider this type of singularities in this paper.
equations reduce to
\[ \frac{d\rho}{dt} + 3\frac{\dot{a}}{a} (\rho + \frac{p}{c^2}) = 0, \]
(2)
\[ \frac{\dot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right), \]
(3)
\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho, \]
(4)
where \( a(t) \) is the scale factor (“radius” of the universe) and \( H = \frac{\dot{a}}{a} \) is the Hubble parameter. These are the well-known Friedmann equations describing a non-static universe [1]. The first equation can be viewed as an “equation of continuity”. For a given barotropic equation of state \( p = \rho(w) \), it determines the relation between the density and the scale factor. Then, the temporal evolution of the scale factor is given by Eq. (4). Introducing the density and the scale factor. Then, the temporal evolution of state is given by Eq. (4). Introducing the density parameter \( a(t) \) that the universe is decelerating if \( w > -1/3 \) (strong energy condition) and accelerating if \( w < -1/3 \). On the other hand, according to Eq. (9), the density decreases with the scale factor if \( w > -1 \) (null dominant energy condition) and increases with the scale factor if \( w < -1 \). In this last case, we are dealing with a “phantom universe”.

We will also need the thermodynamical equation
\[ \frac{dp}{dT} = \frac{1}{T}(\rho c^2 + p), \]
which can be derived from the first principle of thermodynamics [8]. For a given barotropic equation of state \( p = \rho(w) \), this equation can be integrated to obtain the relation \( T = T(\rho) \) between the temperature and the density. It can be shown [3] that the Friedmann equations conserve the entropy of the universe
\[ S = \frac{a^3}{T} (p + \rho c^2). \]
If we impose that the entropy is positive, we conclude from Eq. (6) that the temperature is positive when \( w > -1 \) while it is negative when \( w < -1 \). The fact that a phantom universe has a negative temperature was mentioned in [16]. Negative temperatures arise in other domains of physics such as 2D turbulence [25].

The simplest model of phantom universe corresponds to the linear equation of state \( p = \alpha \rho c^2 \) with \( \alpha < -1 \) [9]. The continuity equation (2) can be integrated into
\[ \rho \propto a^{3[1+\alpha]}. \]
(7)
Substituting Eq. (7) in Eq. (4) and solving the resulting equation for \( a(t) \), we find that the scale factor, the Hubble parameter and the density increase in time as [9]:
\[ a \propto (t_s - t)^{-2/(3[1+\alpha])}, \]
(8)
They all diverge at a finite time \( t = t_s \). This is the “Big Rip” singularity [13], which is a singularity of type I [19]. We also find from Eq. (6) that the temperature behaves as
\[ T \propto -a^{\alpha/(\alpha+1)} \propto -a^{3[\alpha]} \propto -(t_s - t)^{-2\alpha/(1+\alpha)}. \]
(11)
The temperature becomes more and more negative as the universe expands, and it diverges when \( t \rightarrow t_s \).

III. GENERALIZED EQUATION OF STATE
WITH \( w < -1 \)

We consider a generalized equation of state of the form
\[ p = (\alpha \rho + k \rho^{1+1/n})c^2. \]
(12)
This is the sum of a standard linear equation of state \( p = \alpha \rho c^2 \) and a polytropic equation of state \( p = k \rho^{n} \), where \( k \) is the polytropic constant and \( \gamma = 1 + 1/n \) is the polytropic index. Concerning the linear equation of state, we assume \( -1 \leq \alpha \leq 1 \) (the case \( \alpha = -1 \) is treated specifically in Appendix A). Concerning the polytropic equation of state, we remain very general, so that \( k \) and \( n \) can take arbitrary values. In papers I and II, we assumed that \( \alpha + 1 + k \rho^{1/n} \geq 0 \), so that the density decreases with the scale factor (\( w \geq -1 \)). In the present paper, we assume that \( \alpha + 1 + k \rho^{1/n} \leq 0 \) (a necessary condition is \( k < 0 \)) so that the density increases with the scale factor (\( w \leq -1 \)). This corresponds to a “phantom universe”.

A. The density

For the equation of state (12), the Friedmann equation (2) becomes
\[ \frac{d\rho}{dt} + 3\frac{\dot{a}}{a} (1 + \alpha + k \rho^{1/n}) = 0. \]
(13)
Assuming \( \alpha + 1 + k \rho^{1/n} \leq 0 \), this equation can be integrated into
\[ \rho = \frac{\rho_*/[1 - (a_{*}/a)^{3(1+\alpha)/n}]}{1}. \]
(14)
where \( \rho_* = [(\alpha + 1)/|k|]^{n} \) and \( a_* \) is a constant of integration.

For \( n > 0 \), the density is defined only when \( a < a_* \). When \( a \rightarrow 0 \), \( \rho \rightarrow \rho_* \) and \( p \rightarrow -\rho_* c^2 \). When \( a \rightarrow a_* \),
\[ \frac{\rho}{\rho_*} \sim \left[ \frac{n}{3(1+\alpha)} \right]^{n} \frac{1}{(1 - a/a_*)^{n}} \rightarrow +\infty, \]
(15)
and \( p \to -\infty \).

For \( n < 0 \), the density is defined only when \( a > a_* \).

When \( a \to a_* \),

\[
\frac{\rho}{\rho_*} \approx \frac{3(1+\alpha)}{|n|} |\alpha n|^{n} \to 0. \tag{16}
\]

In the same limit, \( p \to -\infty \) for \( n > -1 \), \( p \) tends to a finite value for \( n = -1 \), and \( p \to 0 \) for \( n < -1 \). On the other hand, when \( a \to +\infty \), \( \rho \to \rho_* \) and \( p \to -\rho_* c^2 \).

Some curves giving the evolution of the density \( \rho \) as a function of the scale factor \( a \) are plotted in Fig. 1 for \( n > 0 \) and \( n < 0 \).

### B. The temperature

For the equation of state \([12]\), the thermodynamical equation \([5]\) can be integrated into

\[
T = -T_* \left[ (\rho/\rho_*)^{1/n} - 1 \right]^{(\alpha n + 1)/(\alpha + 1)} \left( \rho/\rho_* \right)^{\alpha/(\alpha + 1)}, \tag{17}
\]

where \( T_* > 0 \) is a constant of integration. Combined with Eq. \([14]\), we obtain

\[
T = -T_* \left( \frac{\alpha n}{1 - (\alpha n)} \right)^{3(\alpha n + 1)/n}, \tag{18}
\]

We have to consider different cases.

We first assume \( n > 0 \). When \( a \to 0 \), \( T \to 0 \); when \( a \to a_* \), \( T \to -\infty \).

We now assume \( n < 0 \). When \( a \to a_* \), \( T \to 0 \) for \( n < -1 \) and \( T \to -\infty \) for \( n = -1 \). When \( a \to +\infty \), \( T \to 0 \) for \( n + \alpha + 1 > 0 \) and \( T \to -\infty \) for \( n + \alpha + 1 < 0 \).

The extremum of temperature (when it exists) is located at

\[
\frac{\rho c}{\rho_*} = \left[ \frac{\alpha n}{(1 + \alpha)(n + 1)} \right]^n. \tag{19}
\]
D. The velocity of sound

For the equation of state \([12]\), the velocity of sound is given by

\[
c_s^2 = p'(\rho) = \left[ \alpha - (\alpha + 1) \frac{n + 1}{n} \left( \frac{\rho}{\rho_*} \right)^{1/n} \right] \left( \frac{\rho}{\rho_*} \right). \tag{24}
\]

For \(n < 0\), the velocity of sound vanishes at the point \([19, 21]\) where the temperature is extremum. At that point, the pressure is maximum with value

\[
\frac{p_*}{\rho_*} = \frac{\alpha}{n + 1} \left[ \frac{an}{(1 + \alpha)(n + 1)} \right]^n. \tag{25}
\]

The case \(c_s^2 < 0\) corresponds to an imaginary velocity of sound. We also define

\[
\frac{\rho_s}{\rho_*} = \left[ \frac{(1 - \alpha)n}{(1 + \alpha)(n + 1)} \right]^n. \tag{26}
\]

\[
\frac{a_s}{a_*} = \left[ \frac{\alpha + 2n + 1}{n(1 - \alpha)} \right]^{n/[3(1 + \alpha)]}, \tag{27}
\]

corresponding to a possible point where the velocity of sound is equal to the speed of light. Different cases have to be considered.

We first assume \(n > 0\). When \(a \to 0\), \((c_s/c)^2 \to -(\alpha + n + 1)/n\); when \(a \to a_*\), \((c_s/c)^2 \to -\infty\). The velocity of sound is always imaginary.

We now assume \(n < 0\). When \(a \to a_*\), \((c_s/c)^2 \to +\infty\) for \(n > -1\) and \((c_s/c)^2 \to -\infty\) for \(n < -1\); when \(a \to +\infty\), \((c_s/c)^2 \to -(\alpha + n + 1)/n\). For \(n > -1\) and \(\alpha + n + 1 > 0\), \(c_s^2\) in always positive. For \(n > -1\) and \(\alpha + n + 1 < 0\), it is positive for \(a < a_e\) and negative for \(a > a_e\). For \(n < -1\) and \(\alpha + n + 1 < 0\), \(c_s^2\) is always negative. For \(n < -1\) and \(\alpha + n + 1 > 0\), it is negative for \(a < a_e\) and positive for \(a > a_e\). For \(n > -1\) and \(\alpha + 2n + 1 > 0\), the velocity of sound is always larger than the speed of light. For \(n > -1\) and \(\alpha + 2n + 1 < 0\), the velocity of sound is larger than the speed of light for \(a < a_s\) and smaller for \(a > a_s\). For \(n < -1\) and \(\alpha + 2n + 1 < 0\), velocity of sound is always smaller than the speed of light. For \(n < -1\) and \(\alpha + 2n + 1 > 0\), the velocity of sound is smaller than the speed of light for \(a < a_s\) and larger for \(a > a_s\).

Some curves giving the evolution of \((c_s/c)^2\) as a function of the scale factor \(a\) are plotted in Figs. 3 and 4 for \(n > 0\) and \(n < 0\).

IV. EVOLUTION OF THE SCALE FACTOR

A. The deceleration parameter

The deceleration parameter is defined by Eqs. (I-77) and (I-78). A phantom universe is always accelerating since \(q \leq -1 < 0\). For the equation of state \([12]\), using Eq. (24), we get

\[
q(t) = \frac{1 + 3\alpha}{2} - \frac{3}{2}(\alpha + 1) \left( \frac{\rho}{\rho_*} \right)^{1/n}. \tag{28}
\]

For \(n > 0\), \(q \to -1\) when \(a \to 0\) and \(q \to -\infty\) when \(a \to a_*\).

For \(n < 0\), \(q \to -\infty\) when \(a \to a_*\) and \(q \to -1\) when \(a \to +\infty\).

Some curves giving the evolution of \(q\) as a function of the scale factor \(a\) are plotted in Figs. 3 and 4 for \(n > 0\) and \(n < 0\).

B. The differential equation

The temporal evolution of the scale factor \(a(t)\) is determined by the Friedmann equation \([11]\). Introducing the normalized radius \(R = a/a_*\), the density \([12]\) can be written

\[
\rho = \frac{\rho_*}{[1 - R^{3(1+\alpha)/n}]}. \tag{29}
\]
Substituting this expression in Eq. (39), we obtain the differential equation

$$\dot{R} = \frac{\epsilon KR}{[1 - R^{3(1+\alpha)/n}]^{n/2}};$$  \hspace{1cm} (30)

where $K = (8\pi G\rho_*/3)^{1/2}$ and $\epsilon = \pm 1$. In general, we shall select the sign $\epsilon = +1$ corresponding to an expanding universe ($\dot{R} > 0$), except in the case of a bouncing universe where both signs of $\epsilon$ must be considered. The solution can be written as

$$\epsilon Kt = \int \left[ 1 - R^{3(1+\alpha)/n} \right]^{n/2} \frac{dR}{R},$$  \hspace{1cm} (31)

or, after a change of variables $x = R^{3(1+\alpha)/n}$, as

$$\frac{3(\alpha + 1)}{n} \epsilon Kt = \int_{1}^{R^{3(1+\alpha)/n}} (1 - x)^{n/2} \frac{dx}{x}. \hspace{1cm} (32)$$

The integral can be expressed in terms of hypergeometric functions. Some simple analytical expressions can be obtained for specific values of $n$. Actually, we can have a good idea of the behavior of the solution of Eq. (30) by considering asymptotic limits (see below). The complete solution is represented in the figures by solving Eq. (30) numerically.

C. The case $n > 0$

The universe starts from $t \to -\infty$ with a vanishing radius $R = 0$, a finite density $\rho = \rho_*$, and a finite pressure $p = \rho c^2$. When $t \to -\infty$,

$$R \sim Ae^{Kt}.$$  \hspace{1cm} (33)

This corresponds to an exponential expansion (early inflation) due to the fact that the density is approximately constant. Then, the universe undergoes a finite time singularity at a time $t_*$. When $t \to t_*$, the radius tends to its maximum value $R = 1$ while the density tends to $+\infty$ and the pressure to $-\infty$. This is a future singularity of type III. Close to the singularity, we have

$$1 - R \sim \left\{ \frac{2 + n}{2} \left[ \frac{n}{3(\alpha + 1)} \right]^{n/2} K(t_* - t) \right\}^{2/(2+n)}.$$  \hspace{1cm} (34)

and

$$\frac{\rho}{\rho_*} \sim \left[ \frac{3}{2} (1 + \alpha) \frac{2 + n}{n} K(t_* - t) \right]^{-2n/(2+n)}.$$  \hspace{1cm} (35)

The evolution of the scale factor is represented in Fig. 5. Some simple analytical results can be obtained in particular cases.

For $n = 1$, using the identity

$$\int \sqrt{1 - x} \frac{dx}{x}$$

and for $n = 2$, using the identity

$$\int (1 - x) \frac{dx}{x} = -x + \ln x,$$

we can obtain $t(R)$ from Eq. (32).

D. The case $n < 0$

The early evolution of the universe depends on the value of $n$. Different cases must be considered.

(i) For $n < -2$, the universe starts from $t \to -\infty$ with a finite radius $R = 1$, a vanishing density $\rho = 0$, and a vanishing pressure $p = 0$ (past peculiarity). When $t \to -\infty$,

$$R - 1 \sim \left\{ \frac{2}{|n| - 2} \left[ \frac{|n|}{3(\alpha + 1)} \right]^{n/2} \frac{1}{(-Kt)} \right\}^{2/(|n|-2)}, \hspace{1cm} (38)$$

and

$$\frac{\rho}{\rho_*} \sim \left[ \frac{2}{3(\alpha + 1)} \frac{|n|}{|n| - 2} \frac{1}{(-Kt)} \right]^{2n/(|n|-2)}.$$  \hspace{1cm} (39)

The density tends to zero algebraically rapidly.

(ii) For $n = -2$, the universe starts from $t \to -\infty$ with a finite radius $R = 1$, a vanishing density $\rho = 0$, and a vanishing pressure $p = 0$ (past peculiarity). When $t \to -\infty$,

$$R - 1 \sim Ae^{\frac{3(n+1)}{2} Kt},$$  \hspace{1cm} (40)

and

$$\frac{\rho}{\rho_*} \sim \frac{9}{4} (1 + \alpha)^2 A^2 e^{3(1+\alpha)Kt}. \hspace{1cm} (41)$$

The density tends to zero exponentially rapidly.

(iii) For $-2 < n < 0$, the universe starts at $t = 0$ with a finite radius $R = 1$ and a vanishing density $\rho = 0$ (past peculiarity). When $t \to 0$,

$$R - 1 \sim \left\{ \frac{2 - |n|}{2} \left[ \frac{|n|}{3(\alpha + 1)} \right]^{n/2} Kt \right\}^{2/(2-|n|)},$$  \hspace{1cm} (42)

and

$$\frac{\rho}{\rho_*} \sim \left[ \frac{3}{2} (1 + \alpha) \frac{2 - |n|}{|n|} Kt \right]^{2n/(2-|n|)}.$$  \hspace{1cm} (43)
At $t = 0$, the pressure vanishes for $-2 < n < -1$, is finite for $n = -1$ and tends to $-\infty$ for $n > -1$. In this last case, there is a past singularity of type II. Actually, we can extend the solution to $t < 0$ (except, maybe, in the case $n > -1$ where the pressure diverges). This describes a phase of contraction of the universe, corresponding to the solution of Eq. (21) with $\epsilon = -1$. This leads to a model of bouncing phantom universe that collapses for $t < 0$ (with decreasing density), disappears at $t = 0$ (the density vanishes), and expands for $t > 0$ (with increasing density).

On the other hand, for $t \to +\infty$, the density tends to a constant $\rho_*$, implying an exponential growth of the scale factor as

$$R \sim A' e^{Kt}. \quad (44)$$

This corresponds to a phase of late inflation. The pressure $p \to -\rho_* c^2$.

![FIG. 6. Evolution of the radius $R$ as a function of time for $n < 0$. Some simple analytical results can be obtained in particular cases.](image)

For $n = -2$, using the identity

$$\int \frac{1}{\sqrt{1-x}} \frac{dx}{x} = \ln \left( \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} \right), \quad (45)$$

we obtain

$$R = \cosh^{2/[(3+1)\alpha]} \left[ \frac{3}{2} (1+\alpha) Kt \right], \quad (46)$$

$$\frac{\rho}{\rho_*} = \tanh^2 \left[ \frac{3}{2} (1+\alpha) Kt \right]. \quad (47)$$

This provides an analytical solution of a bouncing phantom universe (see Fig. 7). We can explicitly check that Eq. (10) has the asymptotic forms (12) and (14) with $A' = 2^{-2/[3(1+\alpha)]}$. For $\alpha = 0$, this model has a constant negative pressure $p = -|k| c^2$. It belongs therefore to the same “class” as the $\Lambda$CDM model (Paper I) and the anti-$\Lambda$CDM model (Paper II) that also have a constant pressure. These three models admit simple analytical expressions.

![FIG. 7. Analytical model of a bouncing phantom universe corresponding to $n = -1$. We have taken $\alpha = 0$. The universe starts from $t = -\infty$ with a maximum density $\rho_*$. It first experiences a phase of contraction during which its density decreases. At $t = 0$, it reaches its minimum radius $R = 1$ and its density vanishes (the universe “disappears”). For $t > 0$, the universe expands and its density increases up to the maximum value $\rho_*$.](image)

For $n = -2$, using the identity

$$\int \frac{1}{1-x} \frac{dx}{x} = \ln \left( \frac{x}{1-x} \right), \quad (48)$$

we obtain

$$R = \left[ 1 + e^{2(1+\alpha)Kt} \right]^{2/[3(1+\alpha)]}, \quad (49)$$

$$\frac{\rho}{\rho_*} = \frac{1}{\left[ 1 + e^{-2(1+\alpha)Kt} \right]^2}. \quad (50)$$

This provides an analytical solution of a phantom universe exhibiting a past peculiarity and a late inflation. Equation (49) has the asymptotic forms (40) and (41) with $A' = 1$ and $A = 2/[3(1+\alpha)]$.

For $n = -1/2$, we have the identity

$$\int \frac{1}{(1-x)^{1/4}} \frac{dx}{x} = 2 \arctan \left( (1-x)^{1/4} \right) + \ln \left[ 1 + (1-x)^{1/4} \right], \quad (51)$$

which determines $t(R)$ using Eq. (32). For $\alpha = 0$, this solution corresponds to the phantom Chaplygin gas.

Remark: We note that the solution with $n \leq -2$ looks similar to the Eddington-Lemaître model (see Fig. 1 in
Paper I) since the universe is “static” in the past with a finite radius \(R = 1\) and expands exponentially rapidly in the future. However, in the Eddington-Lemaître model, the density decreases with time, while, in the present (phantom) model, it increases with time. In addition, in the Eddington-Lemaître model, the density tends to a finite value when \(t \to -\infty\) while in the present model, it tends to zero. Therefore, these models are physically very different.

V. SCALAR FIELD MODELS

In this section, we introduce a representation of the phantom universe in terms of scalar field models. We determine the potential of the scalar field corresponding to the equation of state \(\alpha\) using the general methodology exposed in [4]. We consider a normal scalar field and a tachyon field. Although totally equivalent to fluid equations, this scalar field representation may be useful in order to make the link with more fundamental theories, like those arising in particle physics and string theory [10].

A. Phantom scalar field

A fluid with an equation of state parameter satisfying \(w > -1\) can be described in terms of an ordinary scalar field minimally coupled to gravity called a quintessence field [26]. A fluid with an equation of state parameter satisfying \(w < -1\) can be described in terms of a phantom scalar field. The phantom field can be obtained from the quintessence field by making the transformation \(\phi \to i\phi\). As a result, the phantom scalar field evolves according to the equation

\[
\ddot{\phi} + 3H \dot{\phi} - \frac{dV}{d\phi} = 0,
\]

where \(V(\phi)\) is the potential of the scalar field. The density and the pressure are given by

\[
p = -\frac{1}{2} \dot{\phi}^2 - V(\phi),
\]

with \(\dot{\phi} = (d\phi/da)Ha\), and the Friedmann equation [11] valid for a flat universe, we obtain

\[
\frac{d\phi}{da} = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\frac{\sqrt{|1 + w|}}{a},
\]

For the equation of state \(\alpha\), using Eqs. [14] and [23], and setting \(R = a/a_\star\), we can rewrite Eq. [55] in the form

\[
\frac{d\phi}{dR} = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\frac{\sqrt{\alpha + 1}}{R}\frac{R^{3(1 + \alpha)/2n}}{\sqrt{1 - R^{3(1 + \alpha)/2n}}}.
\]

With the change of variables

\[
x = R^{3(\alpha + 1)/2n}, \quad \psi = \left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{3\sqrt{\alpha + 1}}{2n}\phi,
\]

we find that

\[
\psi = \int \frac{dx}{\sqrt{1 - x^2}} = \text{Arcsin}(x).
\]

On the other hand, according to Eq. [53], we have

\[
V = \frac{1}{2}(1 - w)pc^2.
\]

For the equation of state \(\alpha\), using Eqs. [14] and [23], we obtain

\[
V = \frac{1}{2}\rho_\star c^2(2 - (1 - \alpha)x^2)\frac{(1 - x^2)^{n+1}}{1 + n x^2}\frac{1}{\phi^{2n}}.
\]

Since \(x = \sin\psi\), the scalar field potential is explicitly given by

\[
V(\psi) = \frac{1}{2}\rho_\star c^2(1 - \alpha)\cos^2\psi + \alpha + 1\frac{1}{\cos^2(\frac{\pi}{2} - \psi)}.
\]

and \(R^{3(\alpha + 1)/2n} = \sin\psi\). In these models, \(0 \leq \psi \leq \pi/2\).

The case \(\alpha = -1\) and \(k < 0\) (see Appendix A) must be treated specifically. Repeating the preceding procedure, we find that the potential of the scalar field is

\[
V(\phi) = \rho_\star c^2\left(\frac{|h|c^2}{2\pi G}\right)^n(1 + \frac{n^2c^2}{12\pi G\phi^2})\frac{1}{\phi^{2n}},
\]

\[-\ln R = \frac{2\pi G}{ac^2}\phi^2.
\]

In these models \(\phi \geq 0\).

Finally, for a linear equation of state \(p = \alpha pc^2\) with \(\alpha < -1\), writing the relation between the density and the scale factor as \(\rho/\rho_\star = (a/a_\star)^{3|\alpha| + 1}\), we obtain [4]:

\[
V(\phi) = \frac{1}{2}\rho_\star c^2(1 - \alpha)e^{\sqrt{\alpha + 1}(\frac{\pi c}{2\pi})^2/\phi},
\]

\[
\phi = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{|1 + \alpha|}\ln R,
\]

where \(R = a/a_\star\) and \(\phi \leq 0\). Since \(R \propto (t_s - t)^{-2/(3|1 + \alpha|)}\), the scalar field evolves in time as \(\phi = -(c^2/6\pi G)[1 + \alpha]^{1/2}\ln(t_s - t)\).
B. Phantom tachyon field

Performing the transformation $\phi \rightarrow i\phi$ in the equations of an ordinary tachyon field [27], we find that a phantom tachyon field evolves according to the equation

$$\frac{\dot{\phi}}{1 + \phi^2} + 3H\phi - \frac{1}{V} \frac{dV}{d\phi} = 0. \quad (66)$$

The density and the pressure are given by

$$\rho c^2 = \frac{V(\phi)}{\sqrt{1 + \phi^2}}, \quad p = -V(\phi)\sqrt{1 + \phi^2}. \quad (67)$$

From these equations, we obtain

$$\dot{\phi}^2 = |1 + w|, \quad (68)$$

where we have written $p = w\rho c^2$. Using $\phi = (d\phi/da)H a$, and the Friedmann equation (11), we get

$$\frac{d\phi}{da} = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \sqrt{1 + w} \frac{\sqrt{1 + \phi^2}}{\rho c^2} \quad (69).$$

For the equation of state (12), using Eqs. (14) and (23), we can rewrite Eq. (69) in the form

$$\frac{d\phi}{dR} = \frac{1}{\sqrt{\rho c^2}} \left(\frac{3c^2}{8\pi G}\right)^{1/2} \sqrt{\frac{1 + \phi^2}{\rho c^2}} \frac{R^{3(1+\alpha)/2n}}{\rho^{3(1+\alpha)/2n}} \times \left[1 - R^{3(1+\alpha)/n}\right]^{(n-1)/2}. \quad (70)$$

With the change of variables

$$x = R^{3(\alpha+1)/2n}, \quad \psi = \sqrt{\rho c^2} \left(\frac{8\pi G}{3c^2}\right)^{1/2} \frac{3\sqrt{1 + \alpha}}{2n} \phi, \quad (71)$$

we find that

$$\psi = \int (1 - x^2)^{(n-1)/2} dx. \quad (72)$$

On the other hand, from Eq. (67), we have

$$V^2 = -wp^2c^4. \quad (73)$$

For the equation of state (12), using Eqs. (14) and (23), we obtain

$$V^2 = \rho c^4 \frac{\alpha x^2 + 1}{(1 - x^2)^{2n+1}}. \quad (74)$$

Therefore, the scalar field potential $V(\psi)$ is given in parametric form by Eqs. (72) and (74). Let us consider particular cases.

(i) For $n = 1$, we find that $x = \psi$. Therefore, we obtain

$$V^2 = \rho c^4 \frac{\alpha x^2 + 1}{(1 - x^2)^2}, \quad (75)$$

and $R^{3(\alpha+1)/2n} = \psi$ with $0 \leq \psi \leq 1$.

(ii) For $n = -1$, we find that $x = \tanh \psi$. Therefore, we obtain

$$V^2 = \rho c^4 \frac{\alpha \tanh^2 \psi + 1}{\cosh^2 \psi}, \quad (76)$$

and $R^{3(\alpha+1)/2n} = 1/\tanh \psi$ with $\psi \geq 0$.

(iii) For $n = -2$, we find that $x = \psi/\sqrt{1 + \psi^2}$. Therefore, we obtain

$$V^2 = \rho c^4 \frac{(\alpha + 1)\psi^2 + 1}{(1 + \psi^2)^2}, \quad (77)$$

and $R^{3(\alpha+1)/2n} = \sqrt{1 + \psi^2}/\psi$ with $\psi \geq 0$.

(iv) For $n = -1/2$ and $\alpha = 0$ (phantom Chaplygin gas), we find that $V(\phi) = \rho c^2$ is constant.

The case $\alpha = -1$ and $k > 0$ (see Appendix A) must be treated specifically. Repeating the preceding procedure, we find that the potential of the scalar field is

$$V(\phi)^2 = \rho c^4 \left[\frac{|n|}{2\pi G\rho c(n + 1)^2}\right]^{2n/(n+1)} \frac{1}{\phi^{4n/(n+1)}} \times \left[1 + \frac{|n|}{2\pi G\rho c(n + 1)^2}\right]^{1/(n+1)} \frac{1}{\phi^{2/(n+1)}}. \quad (78)$$

$$-\ln R = \text{sgn}(n) \left[\frac{2\pi G\rho c(n + 1)^2}{|n|}\right]^{1/(n+1)} \phi^{2/(n+1)}. \quad (79)$$

In these models $\phi \geq 0$.

Finally, for a linear equation of state $p = \alpha \rho c^2$ with $\alpha < -1$, writing the relation between the density and the scalar factor as $\rho/\rho_* = (a/a_*)^{3|1+\alpha|}$, we obtain [4]:

$$V(\phi) = \sqrt{-\alpha} \frac{c^2}{|1 + \alpha|} \frac{1}{6\pi G \phi^2}. \quad (80)$$

$$\phi = -\frac{2}{3} \frac{1}{\rho c^2} \left(\frac{3c^2}{8\pi G}\right)^{1/2} \frac{1}{\sqrt{|1 + \alpha|}} R^{-3|1+\alpha|/2}, \quad (81)$$

where $R = a/a_*$ and $\phi \leq 0$. Since $\rho = \rho_* R^{-3|1+\alpha|} = 1/[6\pi G(1+\alpha)^2(t_* - t)^2]$, the scalar field evolves in time as $\phi = -\sqrt{|1 + \alpha|}(t_* - t)$.

VI. CONCLUSION

In this paper, we have performed an exhaustive study of the generalized equation of state (12) in the case where the pressure increases with the scale factor. This corresponds to the so-called phantom cosmology [4].

The case $\alpha = -1$ was previously treated in [21]. For $n < -2$, the universe experiences a future singularity of type I (Big Rip): The scale factor and the density diverge...
at a finite time. For \(-2 \leq n < 0\), the scale factor and the density diverge in infinite time (Little Rip). For \(n > 0\), the universe experiences a future singularity of type III: The density diverges at a finite time while the scale factor tends to a constant.

We have found that when \(\alpha > -1\), the universe does not experience a Big Rip singularity. For \(n < 0\), there is a phase of late inflation and, for \(n > 0\), the universe experiences a future singularity of type III. The past evolution of the density is very different (it starts from zero in the infinite past and increases as the universe expands). For \(-2 < n < 0\), we have obtained a model of bouncing universe which also possesses peculiar features (the density decreases in the past, vanishes at \(t = 0\), and increases in the future). For \(-1 < n < 0\), this bouncing universe presents a past singularity of type II since the pressure at \(t = 0\) is infinite while the scale factor is finite and the density vanishes. For \(n = -1\), corresponding to a constant negative pressure, the bouncing phantom universe admits a simple analytical expression.

Of course, most of these models are academic, and do not correspond to the true evolution of our universe. However, we believe that it is important to study the equation of state \( (12) \) in full generality. On the other hand, there are indications \([11]\) that the equation of state parameter \(w\) of our universe may become less than \(-1\) in the close future (or even at present), so the late evolution of the phantom models described in this paper may be physically relevant.

A drawback of the simple form of phantom cosmology considered in this paper is that it does not connect smoothly to the matter era (which has \(w = 0\)). Therefore, we cannot realistically extend the phantom models to the past and obtain unified models of dust matter and phantom dark energy (\(w < -1\)), contrary to the unified models of dust matter and quintessence dark energy (\(w > -1\)) based on the generalized Chaplygin gas considered in Paper II. A unification of dust matter and phantom dark energy can be achieved in more general models allowing to cross the phantom divide \([28]\). This generalization assumes an interaction between dark matter and dark energy. These models are very interesting because they may provide a solution to the “cosmic coincidence problem” (the fact that the ratio of dark matter and dark energy is of order one).

The phantom cosmology is also interesting for its connection to Hoyle’s version of the steady state theory \([29]\), for its connection to wormholes \([30]\), and for its very strange thermodynamics allowing for the existence of negative temperatures \([18]\) like in 2D turbulence \([22]\).

However, we may recall that there is no firm evidence that we live in a phantom universe. The model of Paper II, corresponding to the standard \(\Lambda\)CDM model with the primordial singularity removed, may correctly describe the whole evolution of our universe. Therefore, a more precise determination of the equation of state parameter \(w\) will help discriminate between these different models.

### Appendix A: Equation of state \( p = ( - \rho + k \rho^\gamma ) c^2 \) with \( k < 0 \)

In this Appendix, we specifically study the equation of state \([12]\) with \(\alpha = -1\) and \(k < 0\), namely

\[ p = ( - \rho - |k| \rho^\gamma ) c^2. \]  

(A1)

Since \( w < -1 \), this equation of state describes a phantom universe. This equation of state was introduced by Nojiri & Odintsov \([17]\) and studied by Stefancic \([21]\). Nojiri et al. \([19]\) used it to illustrate their classification of future finite time singularities. For the completeness of our study, we shall re-derive their results in a more compact form (with our notations) and give a few complements. We follow the same presentation as in Papers I and II.

#### 1. The case \( n > 0 \)

The equation of continuity \([2]\) can be integrated into

\[ \rho = \frac{\rho_*}{\ln( a_*/a)^n}, \]  

(A2)

where \(\rho_* = (n/3|k|)^n\) and \(a_*\) is a constant of integration. The density is defined for \(n \leq a_*\). When \(a \to 0\), \(\rho \to 0\) and \(p \to 0\); when \(a \to a_*\), \(\rho \to +\infty\) and \(p \to -\infty\).

![FIG. 8. Evolution of the density and temperature as a function of the scale factor. We have taken \(n = 1\).](image)

The thermodynamical equation \([14]\) can be integrated into

\[ T = -T_* ( \frac{\rho}{\rho_*} )^{(n+1)/n} e^{-3(\rho/\rho_*)^{1/n}}, \]  

(A3)
where $T_\ast > 0$ is a constant of integration. Combined with Eq. (A2), we obtain

$$T = -\frac{T_\ast}{\ln(a_\ast/a)^{n+1}} \left( \frac{a}{a_\ast} \right)^3. \quad (A4)$$

When $a \to a_\ast$, $T \to 0$; when $a \to a_\ast^-$, $T \to -\infty$. The evolution of the density and temperature as a function of the scale factor is represented in Fig. 8.

The equation of state can be written as $p = w\rho c^2$ with

$$w = -1 - \frac{n}{3} \left( \frac{\rho}{\rho_\ast} \right)^{1/n}. \quad (A5)$$

When $a \to 0$, $w \to -1$; when $a \to a_\ast$, $w \to -\infty$.

The deceleration parameter is given by Eqs. (I-77) and (I-78). Together with Eq. (A6), we obtain

$$q = -1 - \frac{n}{2} \left( \frac{\rho}{\rho_\ast} \right)^{1/n}. \quad (A6)$$

When $a \to 0$, $q \to -1$; when $a \to a_\ast$, $q \to -\infty$.

The velocity of sound is given by

$$\frac{c_s^2}{c^2} = -1 - \frac{n+1}{3} \left( \frac{\rho}{\rho_\ast} \right)^{1/n}. \quad (A7)$$

When $a \to 0$, $(c_s/c)^2 \to -1$; when $a \to a_\ast$, $(c_s/c)^2 \to -\infty$. The velocity of sound is always imaginary.

The evolution of $w$, $q$, and $(c_s/c)^2$ as a function of the scale factor $a$. We have taken $n = 1$.

Setting $R = a/a_\ast$, the Friedmann equation (4) can be written

$$\dot{R} = \frac{KR}{(-\ln R)^{n/2}}, \quad (A8)$$

where $K = (8\pi G\rho_\ast/3)^{1/2}$. Its solution is

$$R(t) = e^{\left[2\ln^2 K(t_\ast - t)\right]^{2/(2+n)}}, \quad (A9)$$

$$\frac{\rho(t)}{\rho_\ast} = \left[\frac{2+n}{2}K(t_\ast - t)\right]^{-2n/(2+n)} \quad (A10)$$

The universe starts from $t = -\infty$ with a vanishing radius $R = 0$, a vanishing density $\rho = 0$ and a vanishing pressure $p = 0$. This corresponds to a generalized past peculiarity. It also undergoes a future singularity of type III: At $t = t_\ast$, the density tends to $+\infty$ and the pressure tends to $-\infty$ while the radius reaches its maximum value $R = 1$.

The evolution of the radius with time for $n > 0$ (specifically $n = 1$). We have arbitrarily taken $t_\ast = 1$.

The evolution of the scale factor is represented in Fig. 10.

The equation of continuity (2) can be integrated into

$$\rho = \frac{\rho_\ast}{\ln(a/a_\ast)^n}, \quad (A11)$$

where $\rho_\ast = ([n]/3k)^n$ and $a_\ast$ is a constant of integration. The density is defined for $a \geq a_\ast$. When $a \to a_\ast$, $\rho \to 0$.

In the same limit, $p \to 0$ for $n < -1$, $p$ tends to a finite value for $n = -1$, and $p \to -\infty$ for $n > -1$. When $a \to +\infty$, $\rho \to +\infty$, and $p \to -\infty$.

The thermodynamical equation (5) can be integrated into

$$T = -T_\ast \left( \frac{\rho}{\rho_\ast} \right)^{(n+1)/n} e^{3(\rho_\ast/\rho)^{1/n}}, \quad (A12)$$

where $T_\ast > 0$ is a constant of integration. Combined with Eq. (A11), we obtain

$$T = -\frac{T_\ast}{\ln(a/a_\ast)^{n+1}} \left( \frac{a}{a_\ast} \right)^3 \quad (A13)$$

When $a \to a_\ast$, $T \to 0$ for $n < -1$ and $T \to -\infty$ for $n > -1$. When $a \to +\infty$, $T \to -\infty$. For $n < -1$, the temperature reaches its maximum at

$$\frac{\rho_e}{\rho_\ast} = \left( \frac{3}{n+1} \right)^n, \quad \frac{a_e}{a_\ast} = e^{(n+1)/3}, \quad (A14)$$
The equation of state can be written as $p = w \rho c^2$ with

$$w = -1 + \frac{n}{3} \left( \frac{\rho}{\rho_*} \right)^{1/n}.$$  \hspace{1cm} (A16)

When $a \to a_*$, $w \to -\infty$; when $a \to +\infty$, $w \to -1$.

The deceleration parameter is given by Eqs. (I-77) and (I-78). Together with Eq. (A16), we obtain

$$q = -1 + \frac{n}{2} \left( \frac{\rho}{\rho_*} \right)^{1/n}.$$ \hspace{1cm} (A17)

When $a \to a_*$, $q \to -\infty$; when $a \to +\infty$, $q \to -1$.

The velocity of sound is given by

$$\frac{c_s^2}{c^2} = -1 + \frac{n + 1}{3} \left( \frac{\rho}{\rho_*} \right)^{1/n}.$$ \hspace{1cm} (A18)

We have to distinguish several cases. We first assume $n < -1$. When $a \to a_*$, $(c_s/c)^2 \to -\infty$; when $a \to +\infty$, $(c_s/c)^2 \to -1$. The velocity of sound is always imaginary. We now assume $n > -1$. When $a \to a_*$, $(c_s/c)^2 \to +\infty$; when $a \to +\infty$, $(c_s/c)^2 \to -1$. The velocity of sound vanishes at the point (A14) at which the temperature is maximum. At that point, the pressure is maximum with value

$$\frac{p}{\rho_* c^2} = -\frac{3^n}{(n+1)^{n+1}}.$$ \hspace{1cm} (A19)

The velocity of sound is real for $a < a_*$ and imaginary for $a > a_*$. On the other hand, the velocity of sound is equal to the speed of light at

$$\frac{\rho_s}{\rho_*} = \left( \frac{6}{n+1} \right)^n, \quad \frac{a_s}{a_*} = e^{(n+1)/6}. \hspace{1cm} (A20)$$

The velocity of sound is larger than the speed of light when $a < a_*$ and smaller when $a > a_*$. The evolution of $w$, $q$, and $(c_s/c)^2$ as a function of the scale factor $a$ is represented in Fig. [12].

Setting $R = a/a_*$, the Friedmann equation [4] can be written

$$\dot{R} = \frac{\epsilon K R}{(\ln R)^{n/2}},$$ \hspace{1cm} (A21)

where $K = (8\pi G \rho_*/3)^{1/2}$ and $\epsilon = \pm 1$. We must distinguish three cases.

(i) For $n < -2$,

$$R(t) = e^{\left( \frac{|n|}{2} K(t_* - t) \right)^{-2/|n|}}, \hspace{1cm} (A22)$$

$$\frac{\rho(t)}{\rho_*} = \left[ \frac{|n|}{2} K(t_* - t) \right]^{-2|n|/|n|-2}.$$ \hspace{1cm} (A23)

The universe starts from $t = -\infty$ with a finite radius $R = 1$, a vanishing density, and a vanishing pressure (past
We have obtained the following results: 

(i) For $n > 0$ and $k < 0$, the universe undergoes an early inflation. It starts from $t = -\infty$ with a vanishing radius $a = 0$ and a finite density $\rho_*$. Its radius increases indefinitely in time while its density decreases. There is no singularity.

(ii) For $n > 0$ and $k > 0$, the universe exhibits a past singularity of type III. It starts at $t = 0$ with a finite radius $a_*$ and an infinite density $\rho = +\infty$. For $t > 0$, its radius increases indefinitely in time while its density decreases. There is no future singularity.

(iii) For $n < 0$ and $k > 0$, the universe exhibits a Big Bang singularity. It starts at $t = 0$ with a vanishing radius $a = 0$ and an infinite density $\rho = +\infty$. The universe also undergoes a late inflation. Its radius increases to $+\infty$ as $t \to +\infty$ while its density decreases to a finite value $\rho_*$. There is no future singularity.

(iv) For $n < 0$ and $k > 0$, the universe exhibits a Big Bang singularity. It starts at $t = 0$ with a vanishing radius $a = 0$ and an infinite density $\rho = +\infty$. The universe also exhibits a future peculiarity. Its radius increases to a finite value $a_*$ while its density decreases to zero $\rho = 0$ (the universe “disappears”). For $n \leq -2$, this peculiarity is reached in infinite time. For $n > -2$, this peculiarity is reached in a finite time $t_*$ (for $n < -1$, there is a future singularity of type II because the pressure diverges at $t = t_*$. For $t_* < t < 2t_*$, the radius decreases to zero while the density increases to $+\infty$. This leads to a Big Crunch singularity. These phases of expansion and contraction continue periodically (cyclic universe).

- In this paper, we have studied the case $w < -1$ (requiring $k > 0$) corresponding to a phantom universe. We have obtained the following results:

  (i) For $n > 0$, the universe undergoes an early inflation. It starts from $t = -\infty$ with a vanishing radius $a = 0$ and a finite density $\rho_*$. The universe also undergoes a future singularity of type III: At a finite time $t_*$, its radius tends to $+\infty$ while its density diverges $\rho \to +\infty$.

  (ii) For $n < 0$, the universe exhibits a past peculiarity. Its radius starts from a finite value $a_*$ while its density vanishes $\rho = 0$. For $n \leq -2$, this peculiarity occurs in the infinite past. For $n > -2$, this peculiarity occurs at $t = 0$ (for $n < -1$, there is a past singularity of type II because the pressure diverges at $t = 0$). The universe also undergoes a late inflation. Its radius increases to $+\infty$ as $t \to +\infty$ while its density increases to a finite value $\rho_*$. There is no future singularity. Actually, the solutions with $n > -2$ can be continued symmetrically for $t < 0$ leading to models of bouncing phantom universe.

Appendix B: Summary of all the possible cases

The study of the polytropic equation of state in cosmology is very rich. This is also the case for the study of polytropic distributions in stellar structure and in other areas of physics and biology. In this Appendix, we summarize all the results obtained in our series of papers and analyze the different singularities in terms of the classification of.

1. The case $-1 < \alpha \leq 1$

- In papers I and II, we have studied the case $w \geq -1$. We have obtained the following results:

  (i) For $n > 0$ and $k < 0$, the universe undergoes an early inflation. It starts from $t = -\infty$ with a vanishing radius $a = 0$ and a finite density $\rho_*$. Its radius increases indefinitely in time while its density decreases. There is no singularity.

  (ii) For $n > 0$ and $k > 0$, the universe exhibits a past singularity of type III. It starts at $t = 0$ with a finite radius $a_*$ and an infinite density $\rho = +\infty$. For $t > 0$, its radius increases indefinitely in time while its density decreases. There is no future singularity.

  (iii) For $n < 0$ and $k > 0$, the universe exhibits a Big Bang singularity. It starts at $t = 0$ with a vanishing radius $a = 0$ and an infinite density $\rho = +\infty$. The universe also undergoes a late inflation. Its radius increases to $+\infty$ as $t \to +\infty$ while its density decreases to a finite value $\rho_*$. There is no future singularity.

  (iv) For $n < 0$ and $k > 0$, the universe exhibits a Big Bang singularity. It starts at $t = 0$ with a vanishing radius $a = 0$ and an infinite density $\rho = +\infty$. The universe also exhibits a future peculiarity. Its radius increases to a finite value $a_*$ while its density decreases to zero $\rho = 0$ (the universe “disappears”). For $n \leq -2$, this peculiarity is reached in infinite time. For $n > -2$, this peculiarity is reached in a finite time $t_*$ (for $n < -1$, there is a future singularity of type II because the pressure diverges at $t = t_*$. For $t_* < t < 2t_*$, the radius decreases to zero while the density increases to $+\infty$. This leads to a Big Crunch singularity. These phases of expansion and contraction continue periodically (cyclic universe).

2. The case $\alpha = -1$

- In papers I and II we have studied the case $w > -1$ (requiring $k > 0$). We have obtained the following results:

  (i) For $n > 0$, the universe undergoes an early inflation. It starts from $t = -\infty$ with a vanishing radius $a = 0$ and a finite density $\rho_*$. The universe also undergoes a future singularity of type III: At a finite time $t_*$, its radius tends to $+\infty$ while its density diverges $\rho \to +\infty$.

  (ii) For $n < 0$, the universe exhibits a past peculiarity. Its radius starts from a finite value $a_*$ while its density vanishes $\rho = 0$. For $n \leq -2$, this peculiarity occurs in the infinite past. For $n > -2$, this peculiarity occurs at $t = 0$ (for $n < -1$, there is a past singularity of type II because the pressure diverges at $t = 0$). The universe also undergoes a late inflation. Its radius increases to $+\infty$ as $t \to +\infty$ while its density increases to a finite value $\rho_*$. There is no future singularity. Actually, the solutions with $n > -2$ can be continued symmetrically for $t < 0$ leading to models of bouncing phantom universe.
indefinitely in time while its density decreases. There is no future singularity.

(ii) For \( n < -2 \), the universe exhibits a Big Bang singularity. It starts at \( t = 0 \) with a vanishing radius \( a = 0 \) and an infinite density \( \rho = +\infty \). The universe also exhibits a future peculiarity. Its radius increases to a finite value \( a_* \) while its density decreases to zero \( \rho = 0 \). This peculiarity is reached algebraically rapidly in infinite time.

(iii) For \( n = -2 \), the universe starts from \( t = -\infty \) with a vanishing radius \( a = 0 \) and an infinite density \( \rho = +\infty \). The universe exhibits a future peculiarity. Its radius increases to a finite value \( a_* \) while its density decreases to zero \( \rho = 0 \). This peculiarity is reached exponentially rapidly in infinite time.

(iv) For \( -2 < n < 0 \), the universe starts from \( t = -\infty \) with a vanishing radius \( a = 0 \) and an infinite density \( \rho = +\infty \). The universe exhibits a future peculiarity. Its radius increases to a finite value \( a_* \) while its density decreases to zero \( \rho = 0 \) (the universe “disappears”). This peculiarity is reached in a finite time \( t_* \) (for \( -1 < n < 0 \), there is a future singularity of type II because the pressure diverges at \( t = t_* \)). For \( t > t_* \), the radius decreases to zero while the density increases to \( +\infty \). This takes place in infinite time.

• In this paper, we have studied the case \( w < -1 \) (requiring \( k < 0 \)) corresponding to a phantom universe. We have obtained the following results (see also [19, 21]):

(i) For \( n > 0 \) the universe starts from \( t = -\infty \) with a vanishing radius \( a = 0 \) and a vanishing density \( \rho = 0 \). This corresponds to a generalized past peculiarity. The universe also exhibits a future singularity of type III: At a finite time \( t_* \), its radius tends to a finite value \( a_* \) while its density diverges \( \rho \to +\infty \).

(ii) For \( n < -2 \), the universe exhibits a past peculiarity. It starts from \( t = -\infty \) with a finite radius \( a_* \) and a vanishing density \( \rho = 0 \). It also exhibits a future singularity of type I (Big Rip): At a finite time \( t_* \), its radius and density are infinite. This singularity is reached algebraically rapidly.

(iii) For \( n = -2 \), the universe exhibits a past peculiarity. It starts from \( t = -\infty \) with a finite radius \( a_* \) and a vanishing density \( \rho = 0 \). Then, the radius and the density increase indefinitely (Little Rip).

(iv) For \( -2 < n < 0 \), the universe exhibits a past peculiarity: It starts at \( t = 0 \) with a finite radius \( a_* \) and a vanishing density \( \rho = 0 \) (for \( -1 < n < 0 \), there is a past singularity of type II because the pressure diverges at \( t = 0 \)). Then, the radius and the density increase indefinitely (Little Rip). Actually, the solution can be continued symmetrically for \( t < 0 \) This leads to a model of bouncing phantom universe.

[1] A.H. Guth, Phys. Rev. D 23, 347 (1981); A.D. Linde, Phys. Lett. B 108, 389 (1982); A. Albrecht, P.J. Steinhardt, M.S. Turner, F. Wilczek, Phys. Rev. Lett. 48, 1437 (1982)
[2] A. Linde, Particle Physics and Inflationary Cosmology (Harwood, Chur, Switzerland, 1990)
[3] A.G. Riess et al., Astron. J. 116, 1009 (1998); S. Perlmutter et al., ApJ 517, 565 (1999); P. de Bernardis et al., Nature 404, 955 (2000); S. Hanany et al., ApJ 545, L5 (2000)
[4] E.J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006)
[5] A. Kamenshchik, U. Moschella, V. Pasquier, Phys. Lett. B 511, 265 (2001); N. Bilic, G.B. Tupper, R. Voilier, Phys. Lett. B 535, 17 (2002); J.S. Fabris, S.V. Goncalves, P.E. de Souza, Gen. Relativ. Gravit. 34, 53 (2002); M.C. Bento, O. Bertolami, A.A. Sen, Phys. Rev. D 66, 043507 (2002); V. Gorini, A. Kamenshchik, U. Moschella, Phys. Rev. D 67, 063509 (2003); M.C. Bento, O. Bertolami, A.A. Sen, Phys. Rev. D 70, 083519 (2004)
[6] S. Weinberg, Gravitation and Cosmology (John Wiley & Sons, 1972)
[7] J. Binney, S. Tremaine, Galactic Dynamics (Princeton University Press, 2008)
[8] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989)
[9] R.R. Caldwell, Phys. Lett. B 545, 23 (2002)
[10] Y.F. Cai, E.N. Saridakis, M.R. Setare, J.Q. Xia, Phys. Rept. 493, 1 (2010)
[11] U. Alam, V. Sahni, T. Deep Saini, A.A. Starobinsky, Mon. Not. Roy. Astron. Soc. 354, 603512 (2004); P.S. Corasaniti, M. Kunz, D. Parkinson, E.J. Copeland, B.A. Bassett, Phys. Rev. D 70, 083006 (2004); B. Novosyadlyj, O. Sergijenko, R. Durrer, V. Perykh, arXiv:1206.5194
(1998); Ph. Brax, J. Martin, Phys. Lett. B 468, 40 (1999); A. Albrecht, C. Skordis, Phys. Rev. Lett. 84, 2076 (2000); T. Barreiro, E.J. Copeland, N.J. Nunes, Phys. Rev. D 61, 127301 (2000); L.A. Ureña-López, T. Matos, Phys. Rev. D 62, 081302(R) (2000); P. Brax, J. Martin, Phys. Rev. D 61, 103502 (2000); T.D. Saini, S. Raychaudhury, V. Sahni, A.A. Starobinsky, Phys. Rev. Lett. 85, 1162 (2000); V. Sahni, A.A. Starobinsky, Int. J. Mod. Phys. D 9, 373 (2000); V. Sahni, Class. Quantum Grav. 19, 3435 (2002); M. Pavlov, C. Rubano, M. Sazhin, P. Scudellaro, Astrophys. J. 566, 619 (2002); V. Sahni, T.D. Saini, A.A. Starobinsky, U. Alam, JETP Lett. 77, 201 (2003)

[27] A. Sen, JHEP 0204, 008 (1999); JHEP 0207, 065 (2002); G.W. Gibbons, Phys. Lett. B 537, 1 (2002); T. Padmanabhan, Phys. Rev. D 66, 021301(R) (2002); A. Frolov, L. Kofman, A. Starobinsky, Phys. Lett. B 545, 8 (2002)

[28] H. García-Compeán, G. García-Jimeńez, O. Obregón, C. Ramírez, JCAP 7, 16 (2008)

[29] F. Hoyle, Monthly Not. Roy. Astron. Soc. 108, 372 (1948)

[30] M. Visser, S. Kar, N. Dadhich, Phys. Rev. Lett. 90, 201102 (2003)

[31] S. Chandrasekhar, An Introduction to the Study of Stellar Structure (Dover, 1958)

[32] P.H. Chavanis, C. Sire, Phys. Rev. E 69, 016116 (2004); Phys. Rev. E 78, 061111 (2008)

[33] P.H. Chavanis, A. Campa, Eur. Phys. J. B 76, 581 (2010)

[34] P.H. Chavanis, Eur. Phys. J. B 62, 179 (2008)