Conformal Equitorsion and Concircular Transformations in a Generalized Riemannian Space

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Abstract: In the beginning, the basic facts about a conformal transformations are exposed and then equitorsion conformal transformations are defined. For every five independent curvature tensors in Generalized Riemannian space, the above cited transformations are investigated and corresponding invariants-5 concircular tensors of concircular transformations are found.

Keywords: generalized Riemannian space; conformal; equitorsion and concircular transformations

1. Introduction

In the sense of Eisenhart’s definition [1], a generalized Riemannian space \((GR_N)\) is a differentiable \(N\)-dimensional manifold that is endowed with basic non-symmetric tensor \((g_{ij} \neq g_{ji})\), where \(\det g_{ij} \neq 0\).

The symmetric part of \(g_{ij}\) is noted with \(g^{\underline{ij}}\) and antisymmetric one with \(g^{ij}_V\). The lowering and rising of indices in \(GR_N\) is defined by \(g^{\underline{ij}}\) and \(g^{ij}_V\), respectively, where \(g^{\underline{ij}}g^{\underline{jk}} = \delta^k_j\) \((\det g_{ij} \neq 0)\). The Christoffel symbols in \(GR_N\) are given in the next manner:

\[
a) \Gamma^i_{jk} = \frac{1}{2} (g^{ji}_{jk} - g^{jk}_{ji} + g^{kj}_{ji}), \quad b) \Gamma^i_{jk} = \frac{1}{2} g^{ip}_{jk} \Gamma^p_{jk} = \frac{1}{2} g^{ip}_{jk} (g^{ip}_{jk} - g^{jk}_{ip} + g^{pk}_{ij}),
\]

where, e.g., \(g^{ij}_{jk} = \partial g^{ij}/\partial x^k\).

Because of non-symmetry of the affine connection coefficients \(\Gamma^i_{jk}\) by indices \(j\) and \(k\), there are four kinds of covariant differentiation in the space \(GR_N\). Namely, for a tensor \(a^i_j\), these covariant derivatives are defined as:

\[
a^i_{|m} = a^i_{,m} + \Gamma^i_{pm} a^p_j - \Gamma^p_{jm} a^i_j.
\]

Yano in [2] investigates a conformal and concircular transformations in the \(R_N\). In that case, of course, he considers one that is Riemannian curvature tensor. De and Mandal in [3] studied concircular curvature tensors as important tensors from the differential geometric point of view. In [4–11], Mikeš et al. have studied special kinds of transformations in Riemannian space.

Mincić, in his doctoral dissertation (Novi Sad, 1975), obtained 12 curvature tensors, using non-symmetric connection. Among these 12 tensors, five of them are independent (see also [12–17]) and they are noted \(R_1, \ldots, R_5\).
In [18], another combination of five independent curvature tensors is obtained, and they are denoted by $K_1 \ldots K_5$.

For five independent tensors $K_\theta$ in [19], the invariants $Z_\theta$ were found, which are different from the invariants $\tilde{Z}_\theta$ in the present paper (see Remark 3.1, at the end). Compare e.g., $Z_1$ from the present paper and $Z_1$ from [19], where $R = K_1$.

Investigation of various kinds of mappings in the settings of generalized Riemannian spaces is an active research topic, numerous results were obtained in the recent years; see, for instance [20–22]. Very recently, conformal and concircular diffeomorphisms of generalized Riemannian spaces have been studied by M. Z Petrović, M. S. Stanković and P. Peška [23].

2. Equitorsion Conformal Transformation in Generalized Riemannian Space

Consider a special transformation of the objects in $GR_N$.

**Definition 1.** Conformal transformation is that one under which the basic tensor is changed according to the law

$$\bar{g}_{ij}(x) = \rho^2(x)g_{ij}(x), \quad (g_{ij} \neq g_{ji}),$$

(3)

where $\rho(x) = \rho(x^1, \ldots, x^N)$ is some differentiable function of coordinates in $GR_N$.

We see that $g$ and $\bar{g}$ are considered in the common system of coordinates. The same is valid for the other geometric objects.

Furthermore, we have:

$$ds^2 = g_{ij}dx^idx^j, \quad d\bar{s}^2 = \bar{g}_{ij}dx^idx^j = \rho^2g_{ij}dx^idx^j,$$

$$d\bar{s}^2 = \rho^2ds^2 \Leftrightarrow d\bar{s}/ds = \rho.$$  

(4)

If the transformation (3) is effected, for the Christoffel symbols, it is obtained

$$\bar{\Gamma}_{i,jk} = \frac{1}{2}(\bar{g}_{ji,k} - \bar{g}_{jk,i} + \bar{g}_{ik,j})$$

$$= \rho^2 \left[ \rho_{,k}g_{ji} - \rho_{,j}g_{ik} + \rho_{,i}g_{jk} + \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \right].$$

(6)

Denoting

$$(\ln \rho)_i = \frac{\partial(\ln \rho)}{\partial x^i} = \frac{1}{\rho} \rho_{,i} = \rho_{,i},$$

(5)

the previous equation gives

$$\bar{\Gamma}_{i,jk} = \rho^2(\Gamma_{i,jk} + g_{ji}\rho_k - g_{jk}\rho_i + g_{ik}\rho_j).$$

(6)

For $\Gamma^i_{jk}$, according to (1), we get

$$\bar{\Gamma}^i_{jk} = \frac{1}{2}g^{ip}(\bar{g}_{jp,k} - \bar{g}_{jk,p} + \bar{g}_{pk,j}).$$

(7)

Because the inverse matrix for $(g_{ij})$ is the matrix $(g^{ij})$, we get

$$\bar{g}^{ij}(x) = [\rho(x)]^{-2}g^{ij}(x)$$

(8)
and, based on (1), (6), (8),
\[ \Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{jk} + \xi^i_{jk}, \] (9)
where
\[ \xi^i_{jk} = g^{ip} (g_{jp} \rho_k - g_{kp} \rho_j + g_{pk} \rho_j). \] (10)

Denote
\[ \rho^i = g^{ip} \rho_p = (5) g^{ip} (ln \rho). \] (11)

From (9), it is obtained: for the symmetric part of the connection
\[ \Gamma^i_{jk} = \Gamma^i_{kj} + \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{jk}, \] (12)
and for the torsion tensor (double skewsymmetric part of the connection)
\[ \bar{T}^i_{jk} = 2 \Gamma^i_{jk} = 2 \rho^i g^{ip} (g_{jp} \rho_k - g_{kp} \rho_j + g_{pk} \rho_j) = T^i_{jk} + 2 \xi^i_{jk}. \] (13)

**Definition 2.** An equitorsion conformal transformation of the connection in GR\(_N\) is that conformal transformation (3) on the base of which the torsion is not changed, i.e.,
\[ T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} = \Gamma^i_{jk} - \Gamma^i_{kj} = T^i_{jk}. \] (14)

From (13), we conclude that

**Theorem 1.** Necessary and sufficient condition for a conformal transformation of the connection to be equitorsion is
\[ \xi^i_{jk} = g^{ip} (g_{jp} \rho_k - g_{kp} \rho_j + g_{pk} \rho_j) = 0. \] (15)

3. Curvature Tensors in Equitorsion Conformal and Concircular Transformation in Generalized Riemannian Space

3.1. The First Curvature Tensor

The 1\(^{st}\) from the cited curvature tensors in GR\(_N\) is [12,13]
\[ R^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm} \Gamma^i_{pn} - \Gamma^p_{jn} \Gamma^i_{pm}. \] (16)

Based on (15), (9), we obtain
\[ \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{jk}. \] (17)

If by the transformation of the connection \( \Gamma \) into \( \bar{\Gamma} \) we write
\[ a) \ \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \rho^i_{jk}, \quad b) \rho^i_{jk} = \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{jk} = P^i_{jk}, \] (18)
we can consider how e.g., some curvature tensors from the above mentioned independent ones are transformed.
With respect to (18), for $\tilde{R}_i^i_{1jmn}$, one obtains

$$\tilde{R}_i^i_{1jmn} = \tilde{\Gamma}_{jmn}^i + \tilde{\Gamma}_{jm}^i_{1n} + \tilde{\Gamma}_{jn}^i_{1m} = R_{jmn}^i + R_{jm}^i_{1n} + R_{jn}^i_{1m}$$

(19)

and substituting $P$ from (18b):

$$\tilde{R}_i^i_{1jmn} = R_{jmn}^i + \delta_m^i(\rho_{m|n} - \rho_{n|m} + T_{mn}^p \rho_p) + \delta_m^i(\rho_{j|m} + \rho_{j|n})$$

$$- \delta_m^i(\rho_{j|m} - \rho_{j|n}) + (\rho_{j|m} - \rho_{j|n} ) g_{jm} - (\rho_{j|m} - \rho_{j|n} ) g_{jm}$$

$$+ \rho_p \rho_p (\delta_m^i g_{jm} - \delta_m^i g_{jm}) + T_{mn}^p \rho - T_{j, mn} \rho$$

(20)

where $|m$ denotes covariant derivative of the first kind on $x^m$. Because

$$\rho_{m|n} - \rho_{n|m} = -T_{mn}^p \rho_p$$

the 2nd addend on the right side in (20) is 0. Introducing the notation

$$\rho_{ij} = \rho_{i|j} - \rho_{j|i} + \frac{1}{2} g^{ij} \rho_p s g_{ij} = \rho_{i|j} - \rho_{j|i} + \frac{1}{2} \rho_p \rho^p g_{ij}$$

(22)

we obtain

$$\rho_{mn} - \rho_{nm} = \rho_{m|n} - \rho_{n|m} = -T_{mn}^p \rho_p$$

(23)

and, for $R_{jmn}^i$

$$\tilde{R}_i^i_{1jmn} = R_{jmn}^i + \delta_m^i(\rho_{m|n} - \rho_{n|m} + \frac{1}{2} g^{ij} \rho_p s g_{ij}) - \delta_m^i(\rho_{j|m} - \rho_{j|n})$$

$$+ \frac{1}{2} g^{ij} \rho_p s g_{ij}$$

(24)

is obtained, where

$$A_{jmn}^i = T_{mn}^i \rho - T_{j,mn} \rho^j$$

Furthermore,

$$\tilde{R}_i^i_{1jmn} = R_{jmn}^i + \delta_m^i(\rho_{m|n} - \rho_{n|m} + \frac{1}{2} g^{ij} \rho_p s g_{ij}) - \delta_m^i(\rho_{j|m} - \rho_{j|n})$$

$$+ \frac{1}{2} g^{ij} \rho_p s g_{ij}$$

(25)

from where

$$\tilde{R}_i^i_{1jmn} = R_{jmn}^i + \delta_m^i \rho_{m|n} - \delta_m^i \rho_{j|m} - \delta_m^i \rho_{n|m} \rho_p \rho^p + \delta_m^i \rho_{m|n} \rho_p \rho^p$$

$$+ \rho_p \rho_p (\delta_m^i g_{jm} - \delta_m^i g_{jm}) + A_{jmn}^i$$
and putting in order:

\[
\bar{R}^i_{jmn} = R^i_{jmn} + \delta^i_m \rho^j_{ pn} - \delta^i_n \rho^j_{ pm} + \rho^i_m \bar{g}^j_{ pn} - \rho^i_n \bar{g}^j_{ pm} + A^i_{jmn},
\]

(26)

where \(A^i_{jmn}\) is given in (24). We are using the next definition from [2]

**Definition 3.** If a conformal transformation in a Riemannian space \(R^N\):

\[
\bar{g}^{ij} = \rho^2 g^{ij}, \quad (g^{ij} = g_{ji})
\]

transforms every geodesic circle into geodesic circle, the function \(\rho(x)\) satisfies the partial differential equation

\[
\rho_{ij} = \Phi(x) g^{ij}(x), \quad (g^{ij} = g_{ji}),
\]

(27)

where

\[
\rho_{ij} = \rho_{ij} - \rho_i \rho_j + \frac{1}{2} \rho_p \rho^p g_{ij}, \quad (g^{ij} = g_{ji}).
\]

(28)

Such a transformation is called a **concircular transformation** in \(R^N\), and **concircular geometry** is geometry that treats the concircular transformations and the spaces that allow such kinds of transformations.

In the \(GR_N\), we consider transformations

\[
\bar{g}^{ij} = \rho^2 g^{ij}, \quad (g^{ij} \neq g_{ji})
\]

(29)

where, based on (22), \(\rho_{ij} \neq \rho_{ji}\) in \(GR_N\). Now, we take

\[
\rho_{ij} = \Phi(x) g^{ij}(x), \quad (g^{ij} \neq g_{ji}),
\]

(30)

and such a transformation we name a **concircular transformation of the first kind** in \(GR_N\).

We have to find the function \(\Phi\). Substituting \(\rho^1\) from (30) into (26), we get:

\[
R^i_{jmn} = R^i_{jmn} + 2 \Phi(\delta^i_m \bar{g}^j_{ pn} - \delta^i_n \bar{g}^j_{ pm}) + A^i_{jmn}.
\]

(31)

If we effect the contraction with \(i = n\), it follows that

\[
\bar{R}^i_{jm} = R^i_{jm} + 2 \Phi(\delta^i_n \bar{g}^j_{ ji} - \delta^i_j \bar{g}^j_{ ni}),
\]

where \(\bar{R}^i_{jm} = R^i_{jm}\), and so on, and we get:

\[
\bar{R}^i_{jm} = R^i_{jm} + 2(1 - N) \Phi \bar{g}^i_{jm} + A_{jm}.
\]

By multiplying the corresponding sides of previous equation and the equation

\[
\rho^2 \bar{g}^i_{jm} = \bar{g}^i_{jm},
\]

we obtain

\[
\rho^2 \bar{R}^i_{jm} = \bar{g}^i_{jm}\{R^i_{jm} + 2(1 - N) \Phi \bar{g}^i_{jm} + A_{jm}\}.
\]
where \( \bar{R}_{1}^{im}G_{1}^{im} = \bar{R} \) and so on, while

\[
A_{jmn}G_{im}^{im} = A_{jmn}^{i}G_{1}^{im} = (T_{jmn}^i - T^{i}_{jmn})G_{im}^{im} = 0,
\]

and we get

\[
\rho^2 \frac{\bar{R}}{1} = R + \Phi[-2(N - 1)N],
\]

wherefrom it follows that

\[
\Phi(x) = -\frac{\rho^2 \bar{R} - R}{2(N - 1)N}.
\]

Substituting \( \Phi \) into (31), we get

\[
R_{1}^{i} = R_{1}^{i} - \frac{\rho^2 \bar{R} - R}{(N - 1)N} (\bar{g}_{1}^{i} - \bar{g}_{1}^{i}) + A_{jmn}^{i}
\]

and from here

\[
\bar{R}_{1}^{i} = \frac{\rho^2 \bar{R}}{(N - 1)N} - \frac{\rho^2 \bar{R}}{(N - 1)N} (\bar{g}_{1}^{i} - \bar{g}_{1}^{i}) + A_{jmn}^{i},
\]

Taking into consideration that

\[
\rho_i = \frac{1}{2N} [(\ln \bar{g})_i - (\ln g)_i] = \frac{1}{2N} (\bar{g}_i - g_i), \quad g = det(g_{ij}),
\]

with respect to (24) and (35)

\[
A_{jmn}^{i} = T_{jmn}^i - T^{i}_{jmn} = T_{jmn}^i - T^{i}_{jmn}G_{ip}p = \frac{1}{2N} [(T_{jmn}^i - T^{i}_{jmn})g_{ip}p - T_{jmn}^i G_{ip}p - T^{i}_{jmn} G_{ip}p]
\]

where \( T_{jmn}^i = T^{i}_{jmn} \) (for the first addend) and \( G_{ip}p = G_{ip}p \) (for the third addend). By substituting from (36) into (34) and because of

\[
\bar{g}_{1}^{i} = \bar{g}_{1}^{i}, \quad \rho^2 \bar{g}_{1}^{i} = \bar{g}_{1}^{i}, \quad \bar{g}_{1}^{i} = \bar{g}_{1}^{i},
\]

we obtain

\[
\bar{R}_{1}^{i} = \frac{\bar{R}(\bar{g}_{1}^{i} G_{ip}p - \bar{g}_{1}^{i} G_{ip}p)}{(N - 1)N} + \frac{1}{2N} (T_{jmn}^i G_{ip}p - T^{i}_{jmn} G_{ip}p),
\]

In that manner, we conclude that the following theorem is valid:
Theorem 2. The tensor

\[
\tilde{Z}^i_{1jmn} = \bar{Z}^i_{1jmn} + \frac{R(\delta^i_m g_{jn} - \delta^i_n g_{jm})}{(N - 1)N} + \frac{1}{2N} (T_{jmn} g^{ip} g_{p} - T_{i}^{i}_{mn} g_{j}) \tag{39}
\]

is an invariant in the space GR\(_N\), by an equitorsion concircular transformation i.e., according to (38):

\[
\tilde{Z}^i_{1jmn} = \bar{Z}^i_{1jmn}, \tag{40}
\]

where e.g., \(g_j = (\ln g)_j = \frac{\partial (\ln g)}{\partial x^j}\) and \(\bar{Z}^i_{1jmn}\) is given by (39).

The tensor \(\tilde{Z}^i_{1jmn}\) is an equitorsion concircular tensor of the first kind in GR\(_N\).

3.2. The Second Curvature Tensor

The tensor \(R^i_{2jmn}\) in GR\(_N\) is [12,17]

\[
R^i_{2jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{mp}, \tag{41}
\]

and, for \(\bar{R}^i_{2jmn}\), by virtue of (18), it follows that

\[
\bar{R}^i_{2jmn} = R^i_{2jmn} + \rho^i_{m|j} - \rho^i_{n|j} + \rho^i_{m} g_{jn} - \rho^i_{n} g_{jm} - \Lambda^i_{jmn}. \tag{42}
\]

Substituting from (18) into the previous equation and arranging, one obtains

\[
\bar{R}^i_{2jmn} = R^i_{2jmn} + \delta^i_{m|j} (\rho^i_{m|j} - \rho^i_{n|j} - T^p_{mn} \rho_p) + \delta^i_{n|j} (\rho^i_{m|j} - \rho^i_{n|j} - T^p_{nm} \rho_p) + \delta^i_{m|n} (\rho^i_{m|n} - \rho^i_{n|m})

+ (\rho^i_{m|n} - \rho^i_{n|m}) g_{jn} - (\rho^i_{m|n} - \rho^i_{n|m}) g_{jm} + \rho^p (\delta^i_{m} g_{jn} - \delta^i_{n} g_{jm}) - T^p_{mn} \rho_j + T_{jmn} \rho^i. \tag{43}
\]

The term in the 1st bracket on the right side is 0 because of

\[
\rho^i_{m|n} - \rho^i_{n|m} = T^p_{mn} \rho_p. \tag{44}
\]

If we introduce the denotation

\[
\rho^i_{2|j} = \rho^i_{|j} - \rho^i j + \frac{1}{2} \rho^i p \rho^j g_{ij}, \tag{45}
\]

we have

\[
\rho^i_{2mn} - \rho^i_{2nm} = \rho^i_{m|n} - \rho^i_{n|m} = T^p_{mn} \rho_p
\]

and, for \(\bar{R}^i_{2jmn}\) from (43)–(45), it follows that

\[
\bar{R}^i_{2jmn} = R^i_{2jmn} + \delta^i_{m|j} (\rho^i_{m|j} - \rho^i_{n|j} - T^p_{mn} \rho_p) + \delta^i_{n|j} (\rho^i_{m|j} - \rho^i_{n|j} - T^p_{nm} \rho_p) + \delta^i_{m|n} (\rho^i_{m|n} - \rho^i_{n|m})

+ (\rho^i_{m|n} - \rho^i_{n|m}) g_{jn} - (\rho^i_{m|n} - \rho^i_{n|m}) g_{jm} + \rho^p (\delta^i_{m} g_{jn} - \delta^i_{n} g_{jm}) - T^p_{mn} \rho_j + T_{jmn} \rho^i. \tag{46}
\]
where $A_{jmn}^i$ is given (24). Furthermore, we use the concircular transformation for $R$

$$\rho_2 = \Phi(x) g_{ij}(x).$$

(47)

By substitution of $\rho_2$ into (46), by procedure as for $R$, we obtain

$$\Phi = -\frac{\rho_2 R - R}{2(N - 1)N}$$

(48)

and at the end:

$$\bar{R}^i_{jmn} = R^i_{jmn} + \frac{\rho_2 R - R}{2(N - 1)N} (\delta^i_m g_{jn} - \delta^i_n g_{jm}) - A_{jmn}^i$$

(49)

where $A_{jmn}^i$ is given in (24).

Thus, we conclude that the next theorem is valid.

**Theorem 3.** The tensor

$$\tilde{Z}^i_{jmn} = R^i_{jmn} + \frac{\rho_2 R - R}{2(N - 1)N} (\delta^i_m g_{jn} - \delta^i_n g_{jm})$$

(50)

is an invariant in $GR_N$ with respect to an equitorsion concircular transformation, i.e., in force is

$$\bar{Z}^i_{jmn} = Z^i_{jmn}.$$  

(51)

The tensor $\tilde{Z}$ is an equitorsion concircular tensor of the 2nd kind at $GR_N$ and e.g., $g_j = (\ln g)_{j} = \frac{\partial}{\partial x^j}$.

3.3. The Third Curvature Tensor

The tensor $R$ in $GR_N$ [12,14,17] is

$$R^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm} r^i_{np} - \Gamma^p_{nj} r^i_{pm} + \Gamma^i_{pm} T^i_{pj},$$

(52)

where $T^i_{pj}$ is torsion tensor in local coordinates. For $R^i_{jmn}$ on the base of (18), it is obtained

$$\bar{R}^i_{jmn} = R^i_{jmn} + P^i_{jmn} - P^i_{jmn} - P^i_{jmn} P^i_{jmn} + P^i_{jmn} P^i_{jmn} + T^i_{pj} T^i_{pj},$$

(53)

where we take into consideration that $P^i_{jp}$ is symmetric, with respect to (18).

By substituting from (18) into the previous equation and arranging, one obtains

$$\bar{R}^i_{jmn} = R^i_{jmn} + \frac{\delta^i_m (\rho_{jn} - \frac{1}{2} \rho_{pm} \rho^p g_{jm} - \delta^i_n (\rho_{jm} - \frac{1}{2} \rho_{pm} \rho^p g_{jm})}
+ \frac{1}{2} \rho_{pm} \rho^p g_{jm} - \delta^i_n (\rho_{pm} - \frac{1}{2} \rho_{pm} \rho^p g_{jm})}
+ \rho_{pmp} \delta^i_{jm} - \rho_{pm} \delta^i_{jm} + D^i_{jmn}.$$  

(54)
where

\[ D^i_{jmn} = T^i_{jm} \rho_n + T^i_{nj} \rho_m + g^{ps} g_{mn} T^i_{jp} \rho_s. \]  

(55)

From (55), it is obtained that

\[ \bar{R}^i_{jmn} = R^i_{jmn} + \delta^i_{m} \rho_j - \delta^i_{n} \rho_m + \rho^i_{m} \rho_n + \rho^i_{n} \rho_m - \frac{1}{2} \rho^i_{mn} \rho + D^i_{jmn}. \]  

(56)

Consider, further, the concircular transformation for the tensor \( \bar{R}^i_{jmn} \) in the following manner. Taking

\[ \rho_{ij} = \frac{\Phi(x) g_{ij}(x)}{3}, \quad \theta = 1, 2, \]  

(57)

we obtain from (56)

\[ \bar{R}^i_{jmn} = R^i_{jmn} + 2 \Phi \frac{\delta^i_{m} \rho_j - \delta^i_{n} \rho_m + \rho^i_{m} \rho_n + \rho^i_{n} \rho_m - \frac{1}{2} \rho^i_{mn} \rho}{3} + D^i_{jmn}. \]  

(58)

Putting \( i = n \), we get

\[ \bar{R}^i_{jmn} = R^i_{jmn} + 2 \Phi \frac{\delta^i_{m} \rho_j - \delta^i_{n} \rho_m + \rho^i_{m} \rho_n + \rho^i_{n} \rho_m - \frac{1}{2} \rho^i_{mn} \rho}{3} + D^i_{jmn}, \]  

(59)

and contracting with \( \rho^2 \bar{g}^{im} = g^{im} \) on the left and the right sides correspondingly in (59), we get

\[ \rho^2 \bar{R} = R + 2 \Phi (1 - N) N \]  

(60)

because

\[ D = D_{jm} g^{im} = D^i_{jm} g^{im} = (T^i_{jm} \rho_i + T^i_{ij} \rho_m + g^{ps} g_{mn} T^i_{jp} \rho_s) g^{im} \]

\[ = T^i_{jm} \rho_i \rho_j + 0 + g^{ps} T^i_{jp} \rho_s = T^i_{jm} (\rho^i_{m} \rho_j - \delta^i_{n} \rho_m) \]  

\[ = T^i_{jm} \rho^i_{m} \rho_j - \frac{1}{2} \rho^i_{mn} \rho = - \frac{1}{2} \rho^i_{mn} \rho = 0. \]  

(61)

By the further procedure as in the case of \( R \), we obtain

\[ \Phi(x) = - \frac{\rho^2 \bar{R} - R}{3 (N - 1) N}. \]  

(62)

Consider, further, the tensor \( D^i_{jmn} \). By virtue of (35), one gets

\[ D^i_{jmn} = \frac{1}{2N} \left( T^i_{jm} (\bar{g}_n - g_n) + T^i_{nj} (\bar{g}_m - g_m) + g^{ps} g_{mn} T^i_{jp} (\bar{g}_s - g_s) \right), \]  

(63)

where the equitorsiion is taken into consideration.

Substituting from (62), (63) into (58), it follows that

\[ \bar{R}^i_{jmn} = R^i_{jmn} - \frac{\rho^2 \bar{R} - \frac{R}{3}}{(N - 1) N} \left( \delta^i_{m} g_{jn} - \delta^i_{n} g_{jm} \right) \]

\[ + \frac{1}{2N} \left[ T^i_{jm} (\bar{g}_n - g_n) + T^i_{nj} (\bar{g}_m - g_m) + g^{ps} g_{mn} T^i_{jp} (\bar{g}_s - g_s) \right]. \]  

(64)

from where we conclude that the next theorem is valid.
Theorem 4. The tensor
\[
\tilde{Z}_{ijmn} = R_{ijmn}^i + \frac{R(\delta^i_m g_{jn} - \delta^i_n g_{jm})}{(N - 1)N} \\
- \frac{1}{2N} (T_{jm,n}^i g_n + T_{nj,m}^i g_m + \gamma^p_{m n} T_{ijp} g_s)
\] (65)
is an invariant in $GR_N$ with respect to an equitorsion concircular transformation, i.e., it is
\[
\tilde{Z}_{ijmn} = \tilde{Z}_{ijmn}^i.
\] (66)
The tensor $\tilde{Z}$ is an equitorsion concircular tensor of the 3rd kind at $GR_N$.

3.4. The Fourth Curvature Tensor

For the tensor $R_{ijmn}^4$ in $GR_N$, we have [13,14,17]
\[
R_{ijmn}^4 = \Gamma_{jm,n}^i - \Gamma_{nj,m}^i + \Gamma_{jm}^p \Gamma_{np}^i - \Gamma_{nj}^p \Gamma_{pm}^i + \Gamma_{mn}^p T_{ipj}^i
\] (67)
where $T_{ipj}^i$ is torsion tensor in local coordinates. For $R_{ijmn}^4$ on the base of (18), it is obtained
\[
\tilde{R}_{ijmn}^4 = R_{ijmn}^4 + P_{|j|n}^i - P_{|i|n}^j + P_{jm}^p P_{|p|n}^i - P_{pm}^i P_{|p|n}^j + T_{ipj}^i p_{pm}^i
\] (68)
From (53), (68), it follows that
\[
\tilde{R}_{ijmn}^4 - R_{ijmn}^4 = R_{ijmn}^4 - R_{ijmn}^3 + 2P_{|j|n}^i T_{ip}^i p_{pm}^i = R_{ijmn}^4 - R_{ijmn}^3
\]
because $P_{|j|n}^i = 0$. Thus, we have
\[
\tilde{R}_{ijmn}^4 - R_{ijmn}^4 = \tilde{R}_{ijmn}^3 - R_{ijmn}^3 \\
= \frac{\delta^i_m p_{jn}^j - \delta^i_n p_{jm}^i + p_{jm}^i \gamma^j_{mn}^i - \delta^i_n \gamma^j_{jm}^i + D_{ijmn}^i}{2}
\] (69)
where $D_{ijmn}^i$ is given in (55). For the concircular transformation for the tensor $R_{ijmn}^4$, we put
\[
\rho_{ij} = \Phi(x) g_{ij}(x), \quad \theta = 1, 2,
\]
and, by the same procedure as in the previous case, the next theorem is obtained.

Theorem 5. The tensor
\[
\tilde{Z}_{ijmn}^4 = R_{ijmn}^4 + \frac{R(\delta^i_m g_{jn} - \delta^i_n g_{jm})}{(N - 1)N} \\
- \frac{1}{2N} (T_{jm,n}^i g_n + T_{nj,m}^i g_m + \gamma^p_{m n} T_{ijp} g_s)
\] (70)
is an invariant in $GR_N$ with respect to an equitorsion concircular transformation, i.e., in force is
\[
\tilde{Z}_{ijmn}^4 = \tilde{Z}_{ijmn}^4.
\] (71)
The tensor $\tilde{Z}$ is an equitorsion concircular tensor of the 4th kind at $GR_N$.

3.5. The Fifth Curvature Tensor

Finally, consider the 5th curvature tensor $R^i_{\bar{5}jmn}$ in $GR_N$ (in [12] $R$ is denoted with $\bar{R}$). We have according to [12,17]

$$R^i_{\bar{5}jmn} = \frac{1}{2} (\Gamma^j_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m}) + \Gamma^p_{jm} \Gamma^i_{nm} + \Gamma^p_{nj} \Gamma^i_{mn} - \Gamma^p_{jn} \Gamma^i_{mp} - \Gamma^p_{nj} \Gamma^i_{pm}),$$  \hspace{1cm} (72)

which can be written in the form [17]:

$$\tilde{R}^i_{\bar{5}jmn} = \frac{1}{2} (P^i_{jm|n} - P^i_{jn|m} + P^i_{mj|n} - P^i_{nj|m}) + P^p_{jm} P^i_{nm} - P^p_{jn} P^i_{mp} + P^p_{nj} P^i_{np} - P^p_{mj} P^i_{np}),$$  \hspace{1cm} (73)

where $P^i_{jk}$ is given in (18). With substitution of $P$ from (18) into (73), one obtains

$$\tilde{R}^i_{\bar{5}jmn} = R^i_{\bar{5}jmn} + \frac{1}{2} \{ \delta^j_{\bar{5}} (\rho_{\bar{5}m|n} - \rho_{\bar{5}n|m} + \rho_{\bar{5}m|n} - \rho_{\bar{5}n|m}) + \delta^i_{\bar{5}} (\rho_{\bar{5}j|m} + \rho_{\bar{5}j|n} - 2 \rho_{\bar{5}j} \rho_{\bar{5}m} + \rho_{\bar{5}j} \rho_{\bar{5}n} - \rho_{\bar{5}j} \rho_{\bar{5}n}) - \delta^i_{\bar{5}} (\rho_{\bar{5}j|m} + \rho_{\bar{5}j|n} - 2 \rho_{\bar{5}j} \rho_{\bar{5}m} + \rho_{\bar{5}j} \rho_{\bar{5}n} - \rho_{\bar{5}j} \rho_{\bar{5}n})\}.$$  \hspace{1cm} (74)

Using (23) and (44) and introducing the denotation

$$\rho_{\bar{5}}_{ij} = \frac{1}{2} (\rho_{\bar{5}ij} - \rho_{\bar{5}ji} + 2 \rho_{\bar{5}ij} + \rho_{\bar{5}j} \rho_{\bar{5}i} \rho_{\bar{5}j} \rho_{\bar{5}i}) = \frac{1}{2} (\rho_{\bar{5}ij} + \rho_{\bar{5}ji} = \rho_{\bar{5}ij}),$$  \hspace{1cm} (75)

Equation (74) obtains the form

$$\tilde{R}^i_{\bar{5}jmn} = R^i_{\bar{5}jmn} + \delta^i_{\bar{5}} \rho_{\bar{5}jm} - \delta^i_{\bar{5}} \rho_{\bar{5}jm} + \rho_{\bar{5}j} \rho_{\bar{5}i} \rho_{\bar{5}j} \rho_{\bar{5}i} - \rho_{\bar{5}j} \rho_{\bar{5}i} \rho_{\bar{5}j} \rho_{\bar{5}i}) + \delta^i_{\bar{5}} \rho_{\bar{5}jm} - \rho_{\bar{5}j} \rho_{\bar{5}i} \rho_{\bar{5}j} \rho_{\bar{5}i} - \rho_{\bar{5}j} \rho_{\bar{5}i} \rho_{\bar{5}j} \rho_{\bar{5}i} - \rho_{\bar{5}j} \rho_{\bar{5}i} \rho_{\bar{5}j} \rho_{\bar{5}i}).$$  \hspace{1cm} (76)

Let us apply a concircular transformation for the tensor $R^i_{\bar{5}jmn}$. By virtue of (75), we put

$$\rho_{\bar{5}}_{ij} = \Phi(x) g_{ij}(x) = \rho_{\bar{5}}_{ij},$$  \hspace{1cm} (77)

into (76) and we get

$$\tilde{R}^i_{\bar{5}jmn} = R^i_{\bar{5}jmn} + 2 \Phi(x) g_{ij}(x) = R^i_{\bar{5}jmn}.$$  \hspace{1cm} (78)
Contracting by \( i = n \), we obtain
\[
\hat{R}_{jm} = R_{jm} + (1 - N)g_{jm}(2\Phi - \rho \rho').
\]
Multiplying this equation with \( \rho^2 \hat{g}_{jm} = \hat{g}_{jm} \), it follows that
\[
2\Phi = \frac{\rho^2 \hat{R} - \hat{R}}{N - 1} + \rho \rho'.
\]
and substituting this value into (78), one gets that the following theorem is valid.

**Theorem 6.** The tensor
\[
\hat{Z}^i_{jmn} = R^i_{jmn} + \frac{R}{N - 1} \left( \delta^i_{m} \delta^j_{n} - \delta^i_{n} \delta^j_{m} \right)
\]
(79)
is an invariant in \( GR_N \) with respect to an equitorsion concircular transformation, i.e., in force is
\[
\hat{Z}^i_{jmn} = \hat{Z}^i_{jmn}.
\]
(80)

The tensor \( \hat{Z} \) is an equitorsion concircular tensor of the 5th kind at \( GR_N \).

**Remark 1.** In [19], is \( K^i_{jmn} = R^i_{1jmn} \), \( K^i_{2jmn} = K^i_{3jmn} \), while \( R^i_{\theta jmn} \notin \{ K^i_{1jmn}, K^i_{2jmn}, K^i_{3jmn} \} \), \( \theta = 2, 4, 5 \). However, because of different procedures, it is \( \hat{Z}^i_{jmn} \notin \{ Z^i_{1jmn}, \ldots, Z^i_{5jmn} \} \), \( \theta = 1, \ldots, 5 \), where \( Z^i_{\theta jmn} \) are from [19]. Thus, \( \hat{Z}^i_{jmn} \) are new invariants of the considered transformations.

**Remark 2.** In the case of \( R_N(\delta_{ij} = \delta_{ji}, T^i_{jk} = 0) \), each of the obtained tensors \( \hat{Z}^i_{\theta jmn} \) reduces to a known concircular tensor [2] \( Z^i_{jmn} = R^i_{jmn} + \frac{R(\delta^i_{m} \delta^j_{n} - \delta^i_{n} \delta^j_{m})}{N - 1} \).

4. Conclusions

Conformal equitorsion concircular transformations are investigated and corresponding invariants-5 concircular tensors of concircular transformations are found.

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