Tangential real hypersurfaces on Hermite-like manifolds
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Abstract. Non-degenerate real hypersurfaces of almost Hermite-like manifolds are examined. Tangential real hypersurfaces are introduced and the main identities of such hypersurfaces are obtained. With the help of these identities, contact metric structures of certain kinds, namely $K$-contact and cosymplectic cases, are discussed.

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1 Introduction

The theory of real hypersurfaces in Hermitian manifolds is one of the most interesting topics in differential geometry thanks to their features that these hypersurfaces contain almost contact structures induced from almost complex structures. There exist various kinds of real hypersurfaces in Hermitian manifolds, namely Hopf hypersurfaces, ruled real hypersurfaces, totally real hypersurfaces, homogenous hypersurfaces, pseudo-Einstein real hypersurfaces, etc. For these kinds of hypersurfaces and their applications, we refer to [7, 9, 16, 18, 21, 22, 25], etc.

Besides Hermitian geometry, the theory of statistical manifolds is a new attractive topic nowadays, since this concept has important physical and geometrical aspects with two torsion-free, non-metric connections. These manifolds have many application in areas such as neural networks, machine learning, artificial intelligence, black holes [2, 8, 11, 26, 27]. In the field of differential geometry, there exist qualifying papers dealing with statistical manifolds and their submanifolds admitting various differentiable structures. Statistical manifolds were initially defined by S. Amari [1] in his book. Later, hypersurfaces of statistical manifolds were presented by H. Furuhata in [12, 13]. These manifolds admitting contact structures or complex structures and their submanifolds were investigated in [3-5, 10, 14, 15, 19, 20, 23, 24].

In [23], K. Takano developed a new perspective on statistical complex manifolds as follows:

Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold with almost complex structures $J$ and $J^*$ of types $(1, 1)$ satisfying the condition

$$\tilde{g}(JX, Y) = -\tilde{g}(X, J^*Y)$$

(1.1)
for any tangent vector fields $X, Y \in \Gamma(TM)$. Then $(\tilde{M}, \tilde{g}, J, J^*)$ is called a Hermite-like manifold. We note that a Hermite-like manifold becomes a Hermitian manifold when $J = J^*$.

Considering how complex and contact manifolds have been widely studied by many mathematicians and physicists so far, we think that the geometric properties of Hermite-like manifolds and contact manifolds with certain kinds will be studied by many authors in the coming years and this topic will be a very interesting subject soon.

Motivated by these facts, we aim to investigate real hypersurfaces of Hermite-like manifolds. With the help of this investigation, we shall present the main properties of contact-like manifolds namely $K$-contact-like and cosymplectic-like manifolds in this paper.

2 Preliminaries

Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold with a semi-Riemannian metric $\tilde{g}$ of constant index. Then there exists a unique affine connection $\tilde{\nabla}^0$ such that

i. $\tilde{\nabla}^0$ is torsion free,

ii. $\tilde{\nabla}^0$ preserves the metric, i.e., $\tilde{\nabla}^0 \tilde{g} = 0$.

This unique affine connection $\tilde{\nabla}^0$ is called the Levi-Civita connection of $(\tilde{M}, \tilde{g})$.

Let $\tilde{\nabla}$ be a torsion-free affine connection on $(\tilde{M}, \tilde{g})$. The triplet $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is called a statistical manifold if the following relation is satisfied for any tangent vector fields $X, Y$ and $Z$ on $\tilde{M}$

$$\tilde{g} (\tilde{\nabla} Z X, Y) = Z \tilde{g} (X, Y) - \tilde{g} (X, \tilde{\nabla}^*_Z Y),$$

(2.1)

where

$$\tilde{\nabla}^0 = \frac{1}{2} \left( \tilde{\nabla} + \tilde{\nabla}^* \right).$$

(2.2)

Here, $\tilde{\nabla}^*$ is called the dual connection of $\tilde{\nabla}$.

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(2.1)

where

$$\tilde{\nabla}^0 = \frac{1}{2} \left( \tilde{\nabla} + \tilde{\nabla}^* \right).$$

(2.2)

Here, $\tilde{\nabla}^*$ is called the dual connection of $\tilde{\nabla}$.
If \( c = 0 \) then the pair \((\tilde{\nabla}, \tilde{\nabla}^\ast)\) is called a Hessian structure on \( \tilde{M} \). From (2.3) and (2.4) it is clear that \((\tilde{M}, \tilde{g})\) is of constant curvature with respect to \( \tilde{R} \) then it is also constant with respect to \( \tilde{R}^\ast \) [12].

Now let \((\tilde{M}, \tilde{g}, J, J^\ast)\) be a Hermite-like manifold with almost complex structures \( J \) and \( J^\ast \). From (1.1) it is clear that \((\tilde{M}, \tilde{g})\) is of constant curvature with respect to \( \tilde{R} \) then it is also constant with respect to \( \tilde{R}^\ast \) [12].

Let \((\tilde{M}, \tilde{g}, J, J^\ast, \tilde{\nabla})\) be a Hermite-like statistical manifold. From (1.1) and (2.1), we see that

\[
\tilde{g}(\tilde{\nabla}_X J Y, Z) = -\tilde{g}(Y, (\tilde{\nabla}^\ast_X J^\ast) Z),
\]

which shows that \( J \) is parallel with respect to \( \tilde{\nabla} \) if and only if \( J^\ast \) is also parallel with respect to \( \tilde{\nabla}^\ast \). A Hermite-like manifold is called a Kaehler-like statistical manifold if \( J \) is parallel with respect to \( \tilde{\nabla} \).

A Kaehler-like statistical manifold \((\tilde{M}, \tilde{g}, J, J^\ast, \tilde{\nabla})\) is called space of constant holomorphic sectional curvature \( c \) if the Riemannian curvature tensor is satisfied the following relation for any \( X, Y, Z \in \Gamma(T\tilde{M}) \):

\[
\tilde{R}(X, Y) Z = \frac{c}{4} \{ \tilde{g}(Y, Z) X - \tilde{g}(X, Z) Y - \tilde{g}(Y, J Z) J X + \tilde{g}(X, J Z) J Y + \tilde{g}(X, J Y) J Z - \tilde{g}(J X, Y) J Z \},
\]

where \( c \) is constant, [12]. A Kaehler-like statistical manifold with constant holomorphic sectional curvature is usually denoted by \( \tilde{M}(c) \).

**Definition 2.1.** (see [23]) A semi-Riemannian manifold \((\tilde{M}, \tilde{g})\) with an almost contact structure \((\varphi, \xi, \eta)\) and another tensor field \( \varphi^\ast \) is called an almost contact metric manifold of certain kind if the following relation holds

\[
\tilde{g}(\varphi X, Y) = -\tilde{g}(X, \varphi^\ast Y)
\]

for any \( X, Y \in \Gamma(T\tilde{M}) \). In this case, one can find that

\[
(\varphi^\ast)^2 X = -X + \eta(X) \xi
\]

and

\[
\tilde{g}(\varphi X, \varphi^\ast Y) = \tilde{g}(X, Y) - \eta(X) \eta(Y),
\]

where \( \eta(X) = \tilde{g}(X, \xi) \).

For more details on almost contact metric manifolds of certain kinds, we refer to [23].
3 Tangential real hypersurfaces

In this section, we shall introduce tangential real hypersurfaces of Hermite-like manifolds and present the basic identities of these hypersurfaces.

Let \((\tilde{M}, \tilde{g}, J, J^*)\) be a Hermite-like manifold and \((M, g)\) be a non-degenerate real hypersurface of \((\tilde{M}, \tilde{g}, J, J^*)\). Suppose that \(N\) is the unit normal vector \(M\). From (1.1) we can write

\[ JN = \xi + \lambda N, \]
\[ J^*N = \xi^* - \lambda N, \]

where \(\lambda\) is a smooth function on \(\tilde{M}\), \(\xi\) and \(\xi^*\) are the tangential parts of \(JN\) and \(J^*N\), respectively.

**Definition 3.1.** Let \((M, g)\) be a non-degenerate real hypersurface of \((\tilde{M}, \tilde{g}, J, J^*)\). If \(JN\) and \(J^*N\) lie on \(\Gamma(TM)\) then \((M, g)\) is called a tangential real hypersurface of \((\tilde{M}, \tilde{g}, J, J^*)\).

**Remark 3.2.** We note that since \((\tilde{M}, \tilde{g})\) is a semi-Riemannian manifold, the induced metric \(g\) from \(\tilde{g}\) might be degenerate or non-degenerate. We shall study real hypersurfaces whose metrics are non-degenerate throughout this paper.

Let \((M, g)\) be a tangential real hypersurface of \((\tilde{M}, \tilde{g}, J, J^*)\) and suppose that \(JN = -\xi\) and \(J^*N = -\xi^*\). From (2.3), we get

\[ \tilde{g}(JN, J^*N) = \tilde{g}(N, N) = \varepsilon, \]

which is equivalent to

\[ \tilde{g}(\xi, \xi^*) = \varepsilon. \]

This shows that \(\xi\) and \(\xi^*\) can not be perpendicular with each other.

For any \(X \in \Gamma(TM)\), we can write

\[ JX = \varphi X + \mu_1(X)N \]

and

\[ J^*X = \varphi^* X + \mu_2(X)N, \]

where \(\varphi X, \varphi^* X \in \Gamma(TM)\) and \(\mu_1, \mu_2\) are real functions on \(M\). Then we have

\[ \tilde{g}(JX, N) = \tilde{g}(\varphi X + \mu_1(X)N, N) = \varepsilon \mu_1(X) \]
and
\[ g(J^*X, N) = \tilde{g}(\varphi^*X + \mu_2(X)N, N) = \varepsilon \mu_2(X). \]

In view of (1.1) and the above equations, we get
\[ \varepsilon \mu_1(X) = -\tilde{g}(X, J^*N) = g(X, \xi^*) \]
and
\[ \varepsilon \mu_2(X) = -\tilde{g}(X, JN) = g(X, \xi). \]
Therefore, we can write
\[ \mu_1(X) = \varepsilon \eta^*(X) \quad \text{and} \quad \mu_2(X) = \varepsilon \eta(X), \]
where \( \eta \) and \( \eta^* \) are 1-forms defined by
\[ \eta(X) = g(X, \xi) \quad \text{and} \quad \eta^*(X) = g(X, \xi^*) \]
for any \( X \in \Gamma(TM) \). Therefore, we can write \( JX \) and \( J^*X \) as
\[ JX = \varphi X + \varepsilon \eta^*(X)N \quad (3.1) \]
and
\[ J^*X = \varphi^*X + \varepsilon \eta(X)N, \quad (3.2) \]
respectively.

Now we shall give some examples of tangential real hypersurfaces as follows:

**Example 3.3.** Let us consider \( \mathbb{R}^4_2 \) whose line element in coordinates \((x, y, z, t)\) defined by
\[ ds^2 = 2dx^2 + 2dy^2 - dz^2 - dt^2. \]

In this case, the semi-Riemannian metric of \( \mathbb{R}^4_2 \) is defined by
\[ \tilde{g}(X, Y) = 2x_1y_1 + 2x_2y_2 - x_3y_3 - x_4y_4 \]
for any tangent vector fields \( X = (x_1, x_2, x_3, x_4) \) and \( Y = (y_1, y_2, y_3, y_4) \). Let us define
\[ J(x_1, x_2, x_3, x_4) = (x_3, x_4, -x_1, -x_2) \]
and
\[ J^*(x_1, x_2, x_3, x_4) = \frac{1}{2}(-x_3, -x_4, 4x_1, 4x_2). \]
Then, we see that \((\mathbb{R}^4, \tilde{g}, J, J^*)\) becomes a Hermite-like manifold.

Now let us consider a real hypersurface \(M\) defined by
\[
\{(x, y, z, 0) : \forall x, y, z \in \mathbb{R}\}
\]
in \((\mathbb{R}^4, \tilde{g}, J, J^*)\). By a direct computation, we get the tangent and normal spaces as follows:
\[
\Gamma(TM) = \text{Span}\{e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0)\}
\]
and
\[
\Gamma(TM^\perp) = \text{Span}\{N = (0, 0, 0, 1)\}
\]
respectively. Since \(JN = e_2\) and \(J^*N = -\frac{1}{2}e_2\) which show that \(JN, J^*N \in \Gamma(TM)\).
Therefore, \(M\) is a tangential real hypersurface of \((\mathbb{R}^4, \tilde{g}, J, J^*)\).

Example 3.4. Consider the 4-dimensional generalized Robertson-Walker spacetime (GRW) \(L^4_1(0, f) = (I \times_f \mathbb{E}^3, \tilde{g})\) whose line element in coordinates \((t, x, y, z)\) defined by
\[
ds^2 = -dt^2 + f^2(t)(dx^2 + dy^2 + dz^2),
\]
where \(f\) is a smooth function on an interval \(I \subset \mathbb{R}\) and \(\mathbb{E}^3\) denotes the Euclidean 3-space. Let us define almost complex structures \(J\) and \(J^*\) as follows:
\[
J(x_1, x_2, x_3, x_4) = (-x_2, x_1, x_4, -x_3)
\]
and
\[
J^*(x_1, x_2, x_3, x_4) = (-f^2(t)x_2, \frac{1}{f^2(t)}x_1, x_4, -x_3).
\]
Then it is clear that the equation (1.1) is satisfied and thus, \((L^4_1(0, f), J, J^*)\) is a Hermite-like manifold.

Now let us consider a real hypersurface \(M\) of \((L^4_1(0, f), J, J^*)\) defined by
\[
M = \{(t, \cos x \cos y, \cos x \sin y, \sin x) : t \in I, x \in [0, \frac{\pi}{2}), y \in [0, \frac{\pi}{2}]\}.
\]
By a direct computation, we obtain
\[
\Gamma(TM) = \text{Span}\{e_1 = (1, 0, 0, 0), e_2 = (0, -\sin x \cos y, -\sin x \sin y, \cos x),
\]
\[e_3 = (0, -\cos x \sin y, \cos x \cos y, 0)\}.
\]
and
\[
\Gamma(TM^\perp) = \text{Span}\{N = (0, \cos x \cos y, \cos x \sin y, \sin x)\}.
\]
Since \(JN\) and \(J^*N\) are perpendicular to \(N\), we see that \(M\) is a tangential real hypersurface of \((L^4_1(0, f), J, J^*)\).
Further examples of tangential real hypersurfaces could be given.

**Lemma 3.5.** Let $(M, g)$ be a tangential real hypersurface of $(\tilde{M}, \tilde{g}, J, J^*)$. Then the following identities satisfy for any $X, Y \in \Gamma(TM)$:

\[
g(\varphi X, Y) = -g(X, \varphi^* Y), \quad (3.3)
\]

\[
g(\varphi X, \varphi^* Y) = g(X, Y) - \varepsilon \eta^*(X) \eta(Y), \quad (3.4)
\]

or, equivalently,

\[
g(\varphi^* X, \varphi Y) = g(X, Y) - \varepsilon \eta(X) \eta^*(Y). \quad (3.5)
\]

**Proof.** Putting (3.1) in (3.1), we get immediately (3.3). Furthermore, (3.4) and (3.5) can be obtained from (2.5), (3.1) and, (3.2) simultaneously.

**Lemma 3.6.** Let $(M, g)$ be a tangential real hypersurface of $(\tilde{M}, \tilde{g}, J, J^*)$. Then the following identities satisfy for any $X \in \Gamma(TM)$:

\[
\varphi^2 X = -X + \varepsilon \eta^*(X)\xi, \quad \eta^*(\varphi X) = 0 \quad (3.6)
\]

and

\[
(\varphi^*)^2 X = -X + \varepsilon \eta(X)\xi^*, \quad \eta(\varphi^* X) = 0. \quad (3.7)
\]

**Proof.** Using the fact that $J^2 X = -X$ and (3.1), we obtain

\[
-X = \varphi^2 X + \varepsilon \eta^*(\varphi X)N - \varepsilon \eta^*(X)\xi.
\]

Considering the tangential and normal parts of the last equation, we obtain (3.6).

With similar arguments as in the proof of (3.6), one can obtain (3.7).

**Lemma 3.7.** For any tangential real hypersurface of $(\tilde{M}, \tilde{g}, J, J^*)$ we have

\[
\varphi \xi = 0 \quad \text{and} \quad \varphi^* \xi^* = 0. \quad (3.8)
\]

**Proof.** Using the fact that $J\xi = N$ and $J^* \xi^* = N$, we have

\[
J\xi = \varphi \xi + \varepsilon \eta^*(\xi)N = N
\]

and

\[
J^* \xi^* = \varphi^* \xi^* + \varepsilon \eta(\xi^*)N = N.
\]

Considering the tangential and normal parts of above equations, we get (3.8).
In view of Lemma 3.5, Lemma 3.6 and Lemma 3.7 we get the following theorem:

**Theorem 3.8.** Any tangential real hypersurface of $(\tilde{M}, \tilde{g}, J, J^*)$ is an $\varepsilon$-almost contact metric manifold of certain kind.

**Remark 3.9.** We note that if the unit normal vector field of any tangential real hypersurface is a space-like vector, then this hypersurface becomes an almost contact metric manifold of certain kind.

Now we shall state the Gauss and Weingarten formulas for these hypersurfaces following H. Furuhata [12]:

Let $(M, g)$ be a tangential real hypersurface of an Hermite-like statistical manifold $(\tilde{M}, \tilde{g}, J, J^*, \tilde{\nabla})$. Using the facts that $\tilde{g}(N, N) = \varepsilon$, the co-dimension of $(M, g)$ is equal to 1 and the equation (2.1), the Gauss and Weingarten formulas for tangential real hypersurfaces can be represented by

\[
\tilde{\nabla}_XY = \nabla_XY + \varepsilon g(A^*_N X, Y) N, \quad (3.9)
\]

\[
\tilde{\nabla}_X N = -A_N X + \varepsilon \kappa(X) N, \quad (3.10)
\]

\[
\tilde{\nabla}^*_X Y = \nabla^*_X Y + \varepsilon g(A^*_N X, Y) N, \quad (3.11)
\]

\[
\tilde{\nabla}^*_X N = -A^*_N X - \varepsilon \kappa(X) N, \quad (3.12)
\]

where $\nabla$ and $\nabla^*$ are the induced linear connections, $A_N$ and $A^*_N$ are the shape operators with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively and $\kappa$ is a 1-form.

A non-degenerate real hypersurface $(M, g, \nabla)$ is called totally geodesic with respect to $\tilde{\nabla}$ (resp. $\tilde{\nabla}^*$) if $A_N = 0$ (resp. $A^*_N = 0$) and it is called totally umbilical with respect to $\tilde{\nabla}$ (resp. $\tilde{\nabla}^*$) if there exists a smooth function $\lambda$ on $M$ such that $A_N X = \lambda X$ (resp. $A^*_N X = \lambda X$) for any $X \in \Gamma(TM)$.

In summary, the main formulas obtained in this section are given as follows:

**Corollary 3.10.** Let $(M, g)$ be a tangential real hypersurface of $(\tilde{M}, \tilde{g}, J, J^*)$. Then we have the following table:

| First part | Second part |
|------------|-------------|
| $JN = -\xi$ | $J^*N = -\xi^*$ |
| $J\xi = N$ | $J^*\xi^* = N$ |
| $JX = \phi X + \varepsilon \eta^*(X) N$ | $J^*X = \phi^* X + \varepsilon \eta(X) N$ |
| $\eta(X) = g(X, \xi)$ | $\eta^*(X) = g(X, \xi^*)$ |
| $g(\phi X, \phi^* Y) = g(X, Y) - \varepsilon \eta^*(X) \eta(Y)$ | $g(\phi^* X, \phi Y) = g(X, Y) - \varepsilon \eta(X) \eta^*(Y)$ |
| $g(\phi X, Y) = -g(X, \phi^* Y)$ | $g(\phi^* X, Y) = -g(X, \phi Y)$ |
| $\eta(\phi^* X) = 0$ | $\eta^*(\phi X) = 0$ |
| $\phi^* = 0$ | $\phi^* \xi^* = 0$ |
| $\phi^* \xi = -X + \varepsilon \eta^*(X) \xi$ | $(\phi^*)^2 X = -X + \varepsilon \eta(X) \xi^*$ |
| $\tilde{\nabla}_X Y = \nabla_X Y + \varepsilon g(A^*_N X, Y) N$ | $\tilde{\nabla}^*_X Y = \nabla^*_X Y + \varepsilon g(A^*_N X, Y) N$ |
| $\tilde{\nabla}_X N = -A_N X + \varepsilon \kappa(X) N$ | $\tilde{\nabla}^*_X N = -A^*_N X - \varepsilon \kappa(X) N$ |
4 Real hypersurfaces of Kaehler-like statistical manifolds

A Hermite-like statistical manifold $(\tilde{M}, \tilde{g}, J, J^*, \tilde{\nabla})$ is called a Kaehler-like statistical manifold if $\tilde{\nabla}J = 0$. It is known from [23] that $\tilde{\nabla}J = 0 \iff \tilde{\nabla}^*J^* = 0$.

**Proposition 4.1.** Let $(M, g)$ be a tangential real hypersurface of a Kaehler-like statistical manifold $(\tilde{M}, \tilde{g}, J, J^*, \tilde{\nabla})$. Then the following identities are satisfied for all $X \in \Gamma(TM)$:

\[
\nabla_X \xi = \varphi A_N X + \varepsilon \eta^*(\nabla_X \xi) \xi, \quad (4.1)
\]

\[
\nabla_X^* \xi^* = \varphi^* A_N^* X + \varepsilon \eta(\nabla_X^* \xi^*) \xi^*, \quad (4.2)
\]

\[
\eta^*(\nabla_X \xi) = -\eta(\nabla_X^* \xi^*) = \kappa(X). \quad (4.3)
\]

**Proof.** Since $J\xi = N$ and $\tilde{\nabla}_X N = -A_N X + \varepsilon \kappa(X) N$, we can write

\[
\tilde{\nabla}_X N = \tilde{\nabla}_X J\xi = -A_N X + \varepsilon \kappa(X) N.
\]

Using the fact that the ambient manifold $\tilde{M}$ is a Kaehler-like statistical manifold and the Gauss and Weingarten formulas, we obtain

\[
J(\nabla_X \xi + \varepsilon g(A_N^* X, \xi) N) = -A_N X + \varepsilon \kappa(X) N,
\]

which is equivalent to

\[
\varphi \nabla_X \xi + \varepsilon \eta^*(\nabla_X \xi) N - \varepsilon g(A_N^* X, \xi) \xi = -A_N X + \varepsilon \kappa(X) N.
\]

Considering the tangential and normal parts of the last equation, we obtain

\[
\varphi \nabla_X \xi = -A_N X + \varepsilon g(A_N^* X, \xi) \xi \quad (4.4)
\]

and

\[
\eta^*(\nabla_X \xi) = \kappa(X). \quad (4.5)
\]

From (4.4), we easily have

\[
\varphi^2 \nabla_X \xi = -\varphi A_N X. \quad (4.6)
\]

Using Lemma 3.6 in (4.6), we obtain (4.1).

With a similar proof way of (4.1), one can obtain the equation (4.2) and $\eta(\nabla_X^* \xi^*) = -\kappa(X)$. Therefore the proof of Proposition 4.1 is completed. \qed
Now we shall recall the following definition of K. Takano [23]:

**Definition 4.2.** Let \((\overline{M}, \overline{g}, \nabla, \varphi, \xi)\) be an almost contact metric manifold of certain kind.

1. \((\overline{M}, \overline{g})\) is called a \(K\)-contact-like manifold if \(\nabla_X \xi = \varepsilon \varphi X\) for any \(X \in \Gamma (T \overline{M})\).
2. \((\overline{M}, \overline{g})\) is called a cosymplectic-like manifold if \(\nabla_X \xi = 0\) for any \(X \in \Gamma (T \overline{M})\).

**Example 4.3.** Let us consider the hypersurface \(M\) given in Example [3.3]. For this hypersurface, we have
\[
\Gamma (TM) = \text{Span} \{e_1 = (1, 0, 0, 0), e_2 = (0, 0, 1, 0), \xi = (0, -1, 0, 0)\}.
\]

Let us suppose \(\nabla\) and \(\nabla^*\) satisfy the following relations:
\[
\begin{align*}
\nabla_{e_1} \xi &= -\varphi e_1 = e_2, \quad \nabla^*_{e_1} \xi = -e_2, \\
\nabla_{e_2} \xi &= -\varphi e_2 = -e_1, \quad \nabla^*_{e_2} \xi = e_1, \\
\nabla \xi \xi &= -\varphi \xi = 0, \quad \nabla^* \xi \xi = 0.
\end{align*}
\]

Then we see that \((M, g, \nabla, \nabla^*)\) becomes a \(K\)-contact-like manifold. If we write
\[
\nabla_{e_1} \xi = \nabla^*_{e_1} \xi = \nabla_{e_2} \xi = \nabla^*_{e_2} \xi = \nabla \xi \xi = \nabla^* \xi \xi = 0,
\]
then we see that \((M, g, \nabla, \nabla^*)\) becomes a cosymplectic-like manifold.

Further examples could be derived.

Considering the definition of totally \(\eta\)-umbilical real hypersurfaces of almost complex manifolds (cf. [25]), we shall state the following definition:

**Definition 4.4.** Any tangential real hypersurface \((M, g)\) is called totally \(\eta^*\)-umbilical if
\[
A_N X = a X + b \eta^*(X) \xi
\]
and it is called totally \(\eta\)-umbilical if
\[
A^*_N X = c X + d \eta(X) \xi
\]
for any \(X \in \Gamma (TM)\), where \(a, b, c, d\) are smooth functions on \(M\).

**Theorem 4.5.** Every \(K\)-contact-like tangential real hypersurface \((M, g)\) of a Kaehler-like statistical manifold \((\overline{M}, \overline{g}, J, J^*, \nabla)\) is totally \(\eta^*\)-umbilical.

**Proof.** From (4.4), we have
\[
A_N X = \varepsilon X - \eta^*(X) \xi + \varepsilon g(A^*_N X, \xi)\xi.
\]
If we put \(b \eta^*(X) = -\eta^*(X) + \varepsilon g(A^*_N X, \xi)\) for a function \(b\) on \(M\), then we get \(b = -1 + \eta(A^*_N \xi)\) from \(\eta^*(\xi) = \varepsilon\). Thus we find
\[
A_N X = \varepsilon X + \{-1 + \eta(A^*_N \xi)\}\eta^*(X) \xi,
\]
which shows that \((M, g)\) is totally \(\eta^*\)-umbilical. \(\Box\)
**Proposition 4.6.** Let \((M, g)\) be a cosymplectic-like tangential real hypersurface of a Kaehler-like statistical manifold \((\widetilde{M}, \widetilde{g}, J, J^*, \widetilde{\nabla})\). Then we have
\[
A_N X = \varepsilon \eta^*(A_N X) \xi \tag{4.9}
\]
for any \(X \in \Gamma(TM)\).

**Proof.** Using the fact that \(\nabla_X \xi = 0\) for any \(X \in \Gamma(TM)\) in (4.1), we get \(\varphi(X\xi) = 0\). Thus the proof is straightforward from \(\varphi \xi = 0\). \(\square\)

As a result of Proposition 4.6, we have the following theorem:

**Theorem 4.7.** There does not exist any totally umbilical cosymplectic-like tangential real hypersurface of \((\widetilde{M}, \widetilde{g}, J, J^*, \widetilde{\nabla})\).

**Proposition 4.8.** Let \((M, g)\) be a tangential real hypersurface of Kaehler-like statistical manifold \((\widetilde{M}, \widetilde{g}, J, J^*, \widetilde{\nabla})\). Then we have the following relations for any \(X, Y \in \Gamma(TM)\):
\[
(\widetilde{\nabla}_X \varphi)Y = -\varepsilon g(A_N X, Y)\xi + \varepsilon \eta^*(Y)A_N X - \varepsilon g(Y, \nabla^*_X \xi^*)N - \eta^*(Y)\kappa(X)N, \tag{4.10}
\]
\[
(\widetilde{\nabla}^*_X \varphi^*)Y = -\varepsilon g(A_N X, Y)\xi^* + \eta \eta^*(Y)A_N X - \varepsilon g(Y, \nabla_X \xi)N + \eta(Y)\kappa(X)N. \tag{4.11}
\]

**Proof.** From (3.1), we can write \(\varphi Y = JY - \varepsilon \eta^*(Y)N\) for any \(Y \in \Gamma(TM)\). Thus we have
\[
\widetilde{\nabla}_X (\varphi Y) = \tilde{\nabla}_X (JY - \varepsilon \eta^*(Y)N) = J\tilde{\nabla}_X Y - \varepsilon \tilde{\nabla}_X (\eta^*(Y))N - \varepsilon \eta^*(Y)\tilde{\nabla}_X N.
\]
Using the Gauss and Weingarten formulas in the last equation, we get
\[
\tilde{\nabla}_X (\varphi Y) = J\tilde{\nabla}_X Y + \varepsilon g(A_N X, Y)JN - \varepsilon X g(Y, \xi^*)N + \varepsilon \eta^*(Y)A_N X - \eta^*(Y)\kappa(X)N
\]
\[
= \varphi \nabla_X Y + \varepsilon \eta^*(\nabla_X Y)N - \varepsilon g(A_N X, Y)\xi - \varepsilon \tilde{g}(\tilde{\nabla}_X Y, \xi^*)N
\]
\[
- \varepsilon \tilde{g}(Y, \tilde{\nabla}^*_X \xi^*)N + \varepsilon \eta^*(Y)A_N X - \eta^*(Y)\kappa(X)N.
\]
Using \(\tilde{\nabla}_X (\varphi Y) = (\tilde{\nabla}_X \varphi)Y + \varphi \tilde{\nabla}_X Y\) in the last equation, we obtain (4.10).

Putting \(\varphi^* Y = J^* Y - \varepsilon \eta^*(Y)N\) and following a similar computation technique of (4.10), one can obtain (4.11). \(\square\)

**Proposition 4.9.** For a tangential real hypersurface \((M, g)\) of Kaehler-like statistical manifold \((\widetilde{M}, \widetilde{g}, J, J^*, \widetilde{\nabla})\), we have also the following relations:
\[
(\tilde{\nabla}_X \varphi)Y = (\nabla_X \varphi)Y + \varepsilon g(A_N X, \varphi Y)N, \tag{4.12}
\]
\[
(\tilde{\nabla}^*_X \varphi^*)Y = (\nabla^*_X \varphi^*)Y + \varepsilon g(A_N X, \varphi^* Y)N. \tag{4.13}
\]
Proof. For any $X, Y \in \Gamma(TM)$, we write
\[
(\bar{\nabla}_X \varphi) Y = \bar{\nabla}_X \varphi Y - \varphi \bar{\nabla}_X Y \\
= \nabla_X \varphi Y + \varepsilon g(A^*_N X, \varphi Y) Y - \varphi (\nabla_X Y + \varepsilon g(A^*_N X, Y) N) \\
= \nabla_X \varphi Y - \varphi \nabla_X Y + \varepsilon g(A^*_N X, \varphi Y) N,
\]
which is equivalent to (4.12).

With a similar way, one can obtain (4.13).

Taking into account of (4.10) and (4.12), we obtain the following proposition:

Proposition 4.10. For any tangential real hypersurface of a Kaehler-like statistical manifold $(\tilde{M}, \tilde{g}, J, J^*, \tilde{\nabla})$, we have
\[
(\nabla_X \varphi) Y = \nabla_X \varphi Y - \varphi \nabla_X Y \\
+ \varepsilon g(A^*_N X, \varphi Y) N,
\]
which is equivalent to (4.12).

Proposition 4.11. For any tangential real hypersurface of a Kaehler-like statistical manifold $(\tilde{M}, \tilde{g}, J, J^*, \tilde{\nabla})$, we have
\[
(\nabla^*_X \varphi^*) Y = \nabla^*_X \varphi^* Y - \varphi^* \nabla_X Y \\
+ \varepsilon g(A^*_N X, \varphi^* Y) N,
\]
which is equivalent to (4.12).

In the theory of contact metric manifolds, it is well known that if $(M, g)$ is a cosymplectic manifold, then
\[
\nabla^0_X \xi = 0
\]
for any $X \in \Gamma(TM)$. Here $\nabla^0$ denotes the Levi-Civita connection on $M$, [6]. However, this condition does not satisfy for any tangential real hypersurface, that is, the condition $\nabla_X \varphi = 0$ does not require the condition $\nabla_X \xi = 0$ for any $X \in \Gamma(TM)$.

We will show the accuracy of the above mentioned fact in two cases:

Case 1. Suppose that $(M, g)$ is a tangential real hypersurface of a Kaehler-like statistical manifold $(\tilde{M}, \tilde{g}, J, J^*, \tilde{\nabla})$ such that $\nabla_X \xi = \nabla^*_X \xi^* = 0$ for any $X \in \Gamma(TM)$. From (4.1) and (4.2), we get
\[
\varphi A_N X = \varphi^* A^*_N X = 0.
\]
Thus we write

\[ A_N X = \varepsilon \eta^* (A_N X) \xi \quad \text{and} \quad A_N^* X = \varepsilon \eta (A_N^* X) \xi^*. \]

If we consider these facts on (4.14) and (4.16), it follows that

\[ (\nabla_X \varphi) Y = [\eta^* (A_N X) - \eta (A_N^* X)] \eta (Y) \xi \]

and

\[ (\nabla_X^* \varphi^*) Y = [\eta^* (A_N X) - \eta (A_N^* X)] \eta (Y) \xi^* \]

for any \( X, Y \in \Gamma (TM) \). The above facts show that the condition \( \nabla_X \varphi = \nabla_X^* \varphi^* = 0 \) does not satisfy.

**Case 2.** Suppose that \((M, g)\) is a tangential real hypersurface of \((\tilde{M}, \tilde{g}, J, J^\ast, \tilde{\nabla})\) such that \( \nabla_X \varphi = \nabla_X^* \varphi^* = 0 \) for any \( X \in \Gamma (TM) \). Then, one can prove that the condition \( \nabla_X \xi = \nabla_X^* \xi^* = 0 \) also does not satisfy.

In the remaining part of this section, we shall express the Riemannian curvature tensor field for tangential real hypersurfaces.

Let \( R \) and \( R^* \) denote the Riemannian curvature tensor field of the induced connections \( \nabla \) and \( \nabla^* \) respectively. Then we have the following lemma:

**Lemma 4.12.** If \((M, g)\) is a tangential real hypersurface of the Kaehler-like statistical manifold, then we have for tangent vector fields \( X, Y, Z \) and the normal vector field \( N \) on \( M \):

\[
\tilde{R}(X, Y) Z = R(X, Y) Z - \varepsilon g(A_N^* Y, Z) A_N X + \varepsilon g(A_N^* X, Z) A_N Y \\
+ \varepsilon \left\{ g((\nabla^*_X A^*)_N Y, Z) - g((\nabla^*_Y A^*)_N X, Z) \right\} N,
\]

\[
\tilde{R}(X, Y) N = -(\nabla_X A)_N Y + (\nabla_Y A)_N X + \varepsilon \left\{ g(A_N X, A_N^* Y) - g(A_N Y, A_N^* X) \right\} N,
\]

\[
\tilde{R}^*(X, Y) Z = R^*(X, Y) Z - \varepsilon g(A_N Y, Z) A_N^* X + \varepsilon g(A_N X, Z) A_N^* Y \\
+ \varepsilon \left\{ g((\nabla^*_X A^*)_N Y, Z) - g((\nabla^*_Y A^*)_N X, Z) \right\} N,
\]

\[
\tilde{R}^*(X, Y) N = -(\nabla^*_X A^*)_N Y + (\nabla^*_Y A^*)_N X - \varepsilon \left\{ g(A_N X, A_N^* Y) - g(A_N Y, A_N^* X) \right\} N,
\]

where we put

\[
(\nabla^*_X A^*)_N Y = \nabla^*_X (A_N^* Y) + \varepsilon \kappa(X) A_N^* Y - A_N^* (\nabla^*_X Y),
\]

\[
(\nabla_X A)_N Y = \nabla_X (A_N Y) - \varepsilon \kappa(X) A_N Y - A_N (\nabla_X Y),
\]

\[
(\nabla_X \kappa)(Y) = X \{ \kappa(Y) \} - \kappa(\nabla_X Y).
\]
Proposition 4.13. Let $\hat{M}(c)$ be a Kaehler-like statistical manifold with constant holomorphic sectional curvature $c$ and $(M, g)$ be a tangential real hypersurface of $\hat{M}(c)$. Then we have

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \]

for any $X, Y, Z \in \Gamma(TM)$. Using (3.9) and (3.10), we obtain

\[ \tilde{R}(X, Y)Z = R(X, Y)Z + \varepsilon g(A^*_N X, \nabla_Y Z)N + \varepsilon \nabla_X (g(A^*_N Y, Z))N \]
\[ - \varepsilon g(A^*_N Y, Z)A_N X + g(A^*_N Y, Z)\kappa(X)N - \varepsilon g(A^*_N Y, \nabla_X Z)N \]
\[ - \varepsilon \nabla_Y (g(A^*_N X, Z))N + \varepsilon (A^*_N X, Z)A_N Y - \kappa(Y) g(A^*_N X, Z)N \]
\[ - \varepsilon g(A^*_N [X, Y], Z)N, \]

which is equivalent to (4.19).

With similar arguments as in the proof of (4.19), the equations (4.20), (4.21) and (4.22) could be proved. \qed

Proposition 4.13. Let $\hat{M}(c)$ be a Kaehler-like statistical manifold with constant holomorphic sectional curvature $c$ and $(M, g)$ be a tangential real hypersurface of $\hat{M}(c)$. Then we have

\[ R(X, Y)Z = \frac{c}{4} \left\{ g(Y, Z)X - g(X, Z)Y - g(Y, \varphi Z)\varphi X + g(X, \varphi Z)\varphi Y + g(X, \varphi Y)\varphi Z - g(\varphi X, Y)\varphi Z \right\} + \varepsilon g(A^*_N Y, Z)A_N X \]
\[ - \varepsilon g(A^*_N X, Z)A_N Y, \quad (4.23) \]

\[ R^*(X, Y)Z = \frac{c}{4} \left\{ g(Y, Z)X - g(X, Z)Y - g(Y, \varphi^* Z)\varphi^* X + g(X, \varphi^* Z)\varphi^* Y + g(X, \varphi^* Y)\varphi^* Z - g(\varphi^* X, Y)\varphi^* Z \right\} + \varepsilon g(A_N Y, Z)A^*_N X \]
\[ - \varepsilon g(A_N X, Z)A^*_N Y, \quad (4.24) \]

and

\[ (\nabla_X A^*)_N Y - (\nabla_Y A^*)_N X = \frac{c}{4} \left\{ \eta^*(X) \varphi^* Y - \eta^*(Y) \varphi^* X + g(X, \varphi Y)\xi^* - g(\varphi X, Y)\xi^* \right\}, \quad (4.25) \]

\[ (\nabla_X A)_N Y - (\nabla_Y A)_N X = \frac{c}{4} \left\{ \eta(X) \varphi Y - \eta(Y) \varphi X + g(X, \varphi Y)\xi - g(\varphi X, Y)\xi \right\}, \quad (4.26) \]

\[ g(A_N X, A^*_N Y) - g(A_N Y, A^*_N X) = \frac{c}{4} \left\{ \eta(Y) \eta^*(X) - \eta(X) \eta^*(Y) \right\} - (\nabla_X \kappa)(Y) + (\nabla_Y \kappa)(X). \quad (4.27) \]
Proof. If we put (2.9) and (3.1) in (4.19) then we get

\[ R(X, Y) Z = \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y - g(Y, \varphi Z) \varphi X + g(X, \varphi Z) \varphi Y \\
+ g(X, \varphi Y) \varphi Z - g(\varphi X, Y) \varphi Z \} + \varepsilon g(A^*_N Y, Z) A_N X \\
- \varepsilon g(A^*_N X, Z) A_N Y + \varepsilon \left\{ -\frac{c}{4} g(Y, \varphi Z) \eta^*(X) + \frac{c}{4} g(X, \varphi Z) \eta^*(Y) \\
+ \frac{c}{4} g(X, \varphi Y) \eta^*(Z) - \frac{c}{4} g(\varphi X, Y) \eta^*(Z) \\
- g((\nabla_X A^*)_N Y, Z) + g((\nabla_Y A^*)_N X, Z) \right\} N \]

for any \( X, Y, Z \in \Gamma(TM) \). Considering the tangential and normal parts of the last equation, the proofs of (4.23) and (4.25) are straightforward.

With similar arguments as in the proofs of (4.23) and (4.25), the equations (4.24), (4.26) and (4.27) could be proved. \qed

From Theorem 4.5 we have

**Proposition 4.14.** Let \( (M, g) \) be a tangential real hypersurface of \( \tilde{M}(c) \). If \( (M, g) \) is a K-contact-like then we have

\[ R(X, Y) Z = \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y - g(Y, \varphi Z) \varphi X + g(X, \varphi Z) \varphi Y \\
+ g(X, \varphi Y) \varphi Z - g(\varphi X, Y) \varphi Z \} + g(A^*_N Y, Z) X \\
- g(A^*_N X, Z) Y + \varepsilon \left\{ \eta(A^*_N \xi) - 1 \right\} \{ \eta^*(X) g(A^*_N Y, Z) \\
- \eta^*(Y) g(A^*_N X, Z) \} \xi. \]

From Proposition 4.6 we have

**Proposition 4.15.** Let \( (M, g) \) be a tangential real hypersurface of \( \tilde{M}(c) \). If \( (M, g) \) is a cosymplectic-like then we have

\[ R(X, Y) Z = \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y - g(Y, \varphi Z) \varphi X + g(X, \varphi Z) \varphi Y \\
+ g(X, \varphi Y) \varphi Z - g(\varphi X, Y) \varphi Z \} + \{ g(A^*_N Y, Z) \eta^*(A_N X) \\
- g(A^*_N X, Z) \eta^*(A_N Y) \} \xi. \]

5 Conclusions and future works

Contact structures have a significant role in the fields of differential geometry and physics. Theoretical backgrounds and their applications have been investigated by many authors. An important feature of contact manifolds is that basic properties can be constructed by the fact that real hypersurfaces of each almost complex manifold admit a contact structure which induced from the complex structure. Based on this fact, we
present the basic properties of real hypersurfaces of Hermite-like manifolds, which is a new version of Hermitian manifolds in this study. With these reviews, we have obtained the basic properties of contact-like manifolds. We note that contact-like manifolds namely contact manifolds with certain kinds were firstly defined by K. Takano in [23]. In this paper, by examining the geometric properties of the real hypersurfaces of Hermite-like manifolds, we find the possibility to add and update some concepts on K. Takano’s definitions.

In line with the results obtained in this study, we express the definitions of contact-like manifolds as follows:

**Definition 5.1.** An odd-dimensional smooth manifold \((M, g)\) is called an \(\varepsilon\)-almost contact-like manifold or almost contact manifold of certain kind if there exists an almost contact structure of certain kind \((\varphi, \varphi^*, \xi, \xi^*, \eta, \eta^*)\) consisting tensor fields \(\varphi\) and \(\varphi^*\) of types \((1, 1)\), vector fields \(\xi\) and \(\xi^*\), 1-forms \(\eta\) and \(\eta^*\) satisfying

\[
\varphi^2 = -I + \eta^* \otimes \xi, \quad \varphi \xi = 0, \quad \eta(\xi) = \varepsilon = \mp 1, \quad \eta \circ \varphi^* = 0,
\]

\[
(\varphi^*)^2 = -I + \eta \otimes \xi^*, \quad \varphi^* \xi^* = 0, \quad \eta^*(\xi) = \varepsilon = \mp 1 \quad \eta^* \circ \varphi = 0,
\]

where \(I\) denotes the identity map.

**Definition 5.2.** An \(\varepsilon\)-almost contact-like manifold \((M, g)\) with a semi-Riemannian metric \(g\) is called a \(\varepsilon\)-contact-like metric manifold if the following relation is satisfied for any \(X, Y \in \Gamma(TM)\).

\[
g(\varphi X, \varphi^* Y) = g(X, Y) - \varepsilon \eta^*(X) \eta(Y).
\]

Considering these definitions, the problem of examining the geometric and physical properties naturally arises in contact-like manifolds and their submanifolds.

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