Global well-posedness for the Phan-Thein-Tanner model in critical Besov spaces without damping

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Abstract

In this paper, we mainly investigate the Cauchy problem for the Phan-Thein-Tanner (PTT) model. The PTT model can be viewed as a Navier-Stokes equations couple with a nonlinear transport system. This model is derived from network theory for the polymeric fluid. We study about the global well posedness of the PTT model in critical Besov spaces. When the initial data is a small perturbation over around the equilibrium, we prove that the strong solution in critical Besov spaces exists globally.

2010 AMS Classification: 35A01, 35B45, 35Q35, 76A05, 76D03.

Keywords: The Phan-Thein-Tanner Model; Critical Besov Spaces; Global Existence.

1. Introduction

In this paper, we consider the initial value problem for the following incompressible Phan-Thein-Tanner (PTT) model\cite{27, 26}:

\[
\begin{aligned}
&u_t + u \cdot \nabla u - \mu \Delta u + \nabla p = \mu_1 \text{div} \tau, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
&\tau_t + u \cdot \nabla \tau + (a + b \text{tr} \tau) \tau + Q(\tau, \nabla u) = \mu_2 D(u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
&\text{div} u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
&u|_{t=0} = u_0(x), \quad \tau|_{t=0} = \tau_0(x), \quad x \in \mathbb{R}^3.
\end{aligned}
\]

(1.1)

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Here $u$ stands for the velocity and $p$ is the scalar pressure of fluid, $\tau$ is the stress tensor. $D(u)$ is the symmetric part of $\nabla u$, that is

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

$Q(\tau, \nabla u)$ is a given bilinear form

$$Q(\tau, \nabla u) = \tau \Omega(u) - \Omega(u)\tau + \lambda(D(u)\tau + \tau D(u)),$$

where $\Omega(u)$ is the skew-symmetric part of $\nabla u$, namely

$$\Omega(u) = \frac{1}{2}(\nabla u - (\nabla u)^T).$$

$\mu > 0$ is the viscosity coefficient and $\mu_1$ is the elastic coefficient. $a$ and $\mu_2$ are associated to the Deborah number $De = \frac{\mu_2}{a}$, which indicates the relation between the characteristic flow time and elastic time. $\lambda \in [-1, 1]$ is a physical parameter. In particular, we call the system co-rotational case when $\lambda = 0$. $b \geq 0$ is a constant relate to the rate of creation or destruction for the polymeric network junctions.

If $b = 0$, the system reduce to the famous Oldroyd-B model (See [24]) which has been studied widely. Let us review some mathematical results for the related Oldroyd type model. C. Guillopé and J.C. Saut [16, 17] proved the existence of local strong solutions and the global existence of one dimensional shear flows. In [14], E. Fernández-Cara, F. Guillén and R. Ortega studied the local well-posedness in Sobolev spaces. J. Chemin and N. Masmoudi [4] proved the local well-posedness in critical Besov spaces and give a low bound for the lifespan. In the co-rotational case, P. L. Lions and N. Masmoudi [21] proved the global existence of weak solutions. In [20], F. Lin, C. Liu and P. Zhang proved that if the initial data is a small perturbation around equilibrium, then the strong solution is global in time. The similar results were obtained in several papers by virtue of different methods, see Z. Lei and Y. Zhou [19], Z. Lei, C. Liu and Y. Zhou [18], T. Zhang and D. Fang [29], Y. Zhu [30], D. Fang, M. Hieber and R. Zi proved the global existence of strong solutions with a class of large data [12, 13]. Recently, Q. Chen and X. Hao [5] and X. Zhai [28] study about the global well-posedness in the critical Besov spaces respectively. For the Oldroyd-B model, the global existence of strong solutions in two dimension without small conditions is still an open problem.
In this paper, we suppose that \( b = \mu = \mu_1 = \mu_2 = 1 \) and \( a = \lambda = 0 \) in the PTT model. To our knowledge, there are a lot of numerical results about the PTT model (See, \([25, 23, 22, 2, 15]\)). However, there is no any well-posedness results about the PTT model. The nonlinear term \((\text{tr} \tau)\tau\) in the PTT model will leads to some interesting phenomenon that is quiet different between the Oldroyd-B model. By virtue of the characteristic method, we prove that the strong solution of \((\text{PTT})\) will blow up in finite time when the initial data \(\text{tr} \tau_0 < 0\). This is a new phenomenon can not be founded in other viscoelastic model.

On the other hand, when \(\text{tr} \tau_0\) has a positive low bound \(c_0\), we can prove the global existence of strong solution with small initial data. The idea is inspired by the method applied in \([6, 28]\). The main different is to deal with the nonlinear term \((\text{tr} \tau)\tau\). In \([6, 28]\), the authors study about the following mixed linear system

\[
\begin{aligned}
    u_t - \Delta u - \Lambda(\Lambda^{-1}P \text{div} \tau) &= PE, \\
    (\Lambda^{-1}P \text{div} \tau)_t + \Lambda u &= \Lambda^{-1}P \text{div} F,
\end{aligned}
\]

where \(P\) is the Leray projection operator and \(\Lambda^s = (-\triangle)^{\frac{s}{2}}\). Based on the above dissipative structure of \(u\) and \(\Lambda^{-1}P \text{div} \tau\), the authors in \([5, 28]\) prove the global existence of the strong solution for the Oldroyd-B model with small initial data in the critical Besov spaces. However, from the linearized system, we can not obtain any dissipation for \(\tau\). Thus we can’t control the nonlinear term \((\text{tr} \tau)\tau\) in large time even tough the initial data is small. In order to deal with this difficult term, we need to change the original system into a new form. Note that \(u = 0\) and

\[
\bar{\tau} = \frac{1}{3} \frac{1}{c_0 + t} I = \begin{pmatrix}
\frac{1}{3} \frac{1}{c_0 + t} & 0 & 0 \\
0 & \frac{1}{3} \frac{1}{c_0 + t} & 0 \\
0 & 0 & \frac{1}{3} \frac{1}{c_0 + t}
\end{pmatrix},
\]

is a special solution of \((1.1)\). Let \(\sigma = \tau - \bar{\tau}\), we rewrite \((1.1)\) in the perturbation form as:

\[
\begin{aligned}
    u_t + u \cdot \nabla u - \Delta u + \nabla p &= \text{div} \sigma, \quad \text{div} u = 0, \\
    \sigma_t + u \cdot \nabla \sigma + \frac{1}{c_0 + t} (\sigma + \frac{1}{3} (\text{tr} \sigma) I) + (\text{tr} \sigma) \sigma + Q(\sigma, \nabla u) &= D(u), \quad \text{(PTT)} \\
    u|_{t=0} = u_0(x), \quad \sigma|_{t=0} = \tau_0(x) - \frac{c_0}{3} I.
\end{aligned}
\]
If $c_0 > 0$, we see that the linear term $\frac{1}{c_0} \sigma$ will lead to some dissipation information for $\sigma$. Specifically, we will define the following basic energy in the low and high frequencies:

\[
E_1(t) = \|u^l\|_{L^\infty_t(B^\frac{1}{2}_{2,1})} + \|\sigma^l\|_{L^\infty_t(B^\frac{1}{2}_{2,1})},
\]

\[
E_2(t) = \|u^l\|_{L^1_t(B^\frac{3}{2}_{2,1})} + \|\Lambda^{-1}\mathbb{P} \text{div} \sigma^l\|_{L^1_t(B^\frac{3}{2}_{2,1})},
\]

\[
E_3(t) = \|u^h\|_{L^\infty_t(B^\frac{-1}{2}_{p,1})} + \|\sigma^h\|_{L^\infty_t(B^\frac{1}{2}_{p,1})} + \|u^h\|_{L^1_t(B^\frac{-p}{p+1}_{p,1})} + \|\Lambda^{-1}\mathbb{P} \text{div} \sigma^h\|_{L^1_t(B^\frac{3}{2}_{p,1})},
\]

\[
E_4(t) = \|\text{tr} \sigma^l\|_{L^1_t(B^\frac{3}{2}_{2,1})} + \|\text{tr} \sigma^h\|_{L^1_t(B^\frac{3}{2}_{p,1})},
\]

and then estimate the above terms one by one. By virtue of the Littlewood-Paley theory, we deduce that

\[
E(t) = E_1(t) + E_2(t) + E_3(t) + E_4(t)
\]

\[
\leq C^* \left[ 1 + \exp \left( E_2(t) + E_3(t) + E_4(t) \right) \right] E(0) + \left[ E_1^2(t) + E_2^2(t) + E_3^2(t) + E_4^2(t) \right].
\]

From the above estimate we can obtain the global existence by a standard continuous argument under the condition $u_0$ and $\sigma_0$ is small enough.

A functional space is called critical if the associated norm is invariant under the scaling transformation. Although the system (1.1) does not have any scaling invariance, one may find that $(u, \tau)$ is a solution of (1.2), then

\[
(u_\lambda, \tau_\lambda, p_\lambda) = (\lambda u(\lambda^2 t, \lambda x), \tau(\lambda^2 t, \lambda x), \lambda^2 p(\lambda^2 t, \lambda x))
\]

is also a solution of (1.2). Thus, we can use the linearized system to define the critical spaces. The reason to consider the well-posedness in critical spaces has been fully explained in [4].

There is a lot of papers study about the well-posedness in critical Besov spaces, one can refer to [6, 7, 8, 9, 10, 11] and references therein.

**Notation.** Since all function spaces in throughout the paper are over $\mathbb{R}^3$, for simplicity, we drop $\mathbb{R}^3$ in the notation of function spaces if there is no ambiguity. $A \lesssim B$ stands for $A \leq C B$ for some constant $C > 0$ independent of $A$ and $B$. $S'(\mathbb{R}^3)$ denotes the set of tempered distributions. For any $z \in S'(\mathbb{R}^3)$, the lower and higher oscillation parts can be expressed as

\[
z^l = \sum_{j \leq N} \hat{\Delta}_j z, \quad z^h = \sum_{j > N} \hat{\Delta}_j z,
\]
where $\hat{\Delta}_j$ are the Littlewood-Paley dyadic blocks and $N$ is a large but fixed integer.

Our main result can be stated as follow:

**Theorem 1.1.** Let $0 < c_0 < +\infty$. Suppose that $\text{div} u_0 = 0$, $(\sigma_0)_{ij} = (\sigma_0)_{ji}$, and the initial data $(u_0^f, \sigma_0^f) \in \dot{B}_{2,1}^{3/2}$, $u_0^h \in \dot{B}_{p,1}^{3/p-1}$, $\sigma_0^h \in \dot{B}_{p,1}^{3/p}$ with $p \in [2, 4]$. There exists a $\epsilon_0$ such that if

$$0 < \delta_0 := \|(u_0^f, \sigma_0^f)\|_{\dot{B}_{2,1}^{3/2}} + \|u_0^h\|_{\dot{B}_{p,1}^{3/p-1}} + \|\sigma_0^h\|_{\dot{B}_{p,1}^{3/p}} \leq \epsilon_0,$$

then the problem (PTT) admits a unique global solution $(u(t), \sigma(t))$ satisfying that for all $t \geq 0$:

$$\|u\|_{L_t^\infty(B_{2,1}^{3/2})} + \|\sigma\|_{L_t^\infty(B_{p,1}^{3/p})} + \|u^h\|_{L_t^\infty(B_{p,1}^{3/p-1})} + \|\sigma^h\|_{L_t^\infty(B_{p,1}^{3/p})}$$

$$+ \|u_f\|_{L_t^1(B_{2,1}^{3/2})} + \|\Lambda^{-1} \text{div} \sigma_f\|_{L_t^1(B_{2,1}^{3/2})} + \|u^h\|_{L_t^1(B_{p,1}^{3/p-1})} + \|\Lambda^{-1} \text{div} \sigma^h\|_{L_t^1(B_{p,1}^{3/p})} \leq C(c_0)\delta_0,$$

where $C > 0$ is a positive constant independent of $t$.

The remainder of the paper is organized as follows. In Section 2 we review some basic proposition about the homogeneous Besov space. In Section 3 devote to study about the global existence of the strong solution with small initial data.

## 2. Preliminary

In this section, we first give the definition of the homogeneous Besov space. (See [1] for more details) Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^3 | \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. There exists radial function $\varphi$, valued in the interval $[0, 1]$, such that

$$\forall \xi \in \mathbb{R}^3 \setminus \{0\}, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad (2.1)$$

$$|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j}\xi) \cap \text{Supp } \varphi(2^{-j'}\xi) = \emptyset. \quad (2.2)$$

The homogeneous dyadic blocks $\hat{\Delta}_j$ are defined by

$$\hat{\Delta}_j u = \varphi(2^{-j}D)u = \int_{\mathbb{R}^3} h(2^j y)u(x - y)dy, \quad (2.3)$$
\[ \dot{S}_j u = \chi(2^{-j}D)u = \int_{\mathbb{R}^3} \tilde{h}(2^j y)u(x-y)dy, \tag{2.4} \]

where
\[ h(x) = \mathcal{F}^{-1}(\varphi)(x), \quad \tilde{h}(x) = \mathcal{F}^{-1}(\chi)(x), \quad \chi(\xi) = 1 - \sum_{j \geq 0} \varphi(2^{-j} \xi). \]

We denote by \( S'_h(\mathbb{R}^3) \) the space of tempered distributions \( u \) such that
\[ \lim_{\lambda \to \infty} \|\theta(\lambda D)u\|_{L^\infty} = 0, \quad \forall \theta \in \mathcal{D}(\mathbb{R}^3). \]

The homogeneous Besov space is denoted by \( \dot{B}^s_{p,r} \), that is
\[ \dot{B}^s_{p,r} = \left\{ u \in S'_h \left| \| u \|_{\dot{B}^s_{p,r}} = \| 2^{js}\|\Delta_j u\|_{L^p} \|_r < \infty \right. \right\}, \]

where \( s \in \mathbb{R} \) and \( p, r \in [1, +\infty) \). One can easily check that
\[ \|u^\ell\|_{\dot{B}^s_{p,1}} \approx \sum_{j \leq N} 2^{js}\|\|\Delta_j u\|_{L^p}, \quad \text{and} \quad \|u^h\|_{\dot{B}^s_{p,1}} \approx \sum_{j > N} 2^{js}\|\|\Delta_j u\|_{L^p}. \]

In order to study the product acts on Besov spaces, we need to use the Bony decomposition.

**Definition 2.1.** For functions \( u \) and \( v \), Bony’s decomposition in the homogeneous context is defined by
\[ uv = \dot{T}_u v + \dot{R}(u, v) + \dot{T}_v u, \]

where
\[ \dot{T}_u v \triangleq \sum_j \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u, v) \triangleq \sum_{|k-j| \leq 1} \dot{\Delta}_k u \dot{\Delta}_j v. \]

The following two lemmas will be used in the sequel.

**Lemma 2.2.** Let \( p \in [2, \infty) \), for any \( v^\ell \in B^{\frac{1}{2}, 1}_{2,1}, \) \( v^h \in B^{\frac{3}{2}-1}_{2,1}, \) \( w^\ell \in B^{\frac{3}{2}}_{2,1}, \) \( w^h \in B^{\frac{3}{2}}_{2,1}, \) and \( \nabla u^\ell \in B^{\frac{3}{2}}_{2,1}, \) \( \nabla u^h \in B^{\frac{3}{2}}_{2,1}, \) then we have
\[
\begin{align*}
\sum_{j \leq N} 2^{\frac{3}{2}j} \|[u \cdot \nabla, \dot{\Delta}_j] v\|_{L^2} & \lesssim \|v^\ell\|_{B^{\frac{1}{2}}_{2,1}} \|\nabla u\|_{B^{\frac{3}{2}}_{2,1}} + \|v\|_{B^{\frac{1}{2}}_{2,1}} \|\nabla u^\ell\|_{B^{\frac{3}{2}}_{2,1}}, \\
\sum_{j > N} 2^{j(\frac{3}{2}-1)} \|[u \cdot \nabla, \dot{\Delta}_j] v\|_{L^p} & \lesssim \|v^h\|_{B^{\frac{3}{2}}_{2,1}} \|\nabla u\|_{B^{\frac{3}{2}}_{2,1}} + \|v\|_{B^{\frac{3}{2}}_{2,1}} \|\nabla u^h\|_{B^{\frac{3}{2}}_{2,1}}, \\
\sum_{j \leq N} 2^{\frac{3}{2}j} \|[u \cdot \nabla, \dot{\Delta}_j] w\|_{L^2} & \lesssim \|w^\ell\|_{B^{\frac{3}{2}}_{2,1}} \|\nabla u\|_{B^{\frac{3}{2}}_{2,1}} + \|w\|_{B^{\frac{3}{2}}_{2,1}} \|\nabla u^\ell\|_{B^{\frac{3}{2}}_{2,1}}, \\
\sum_{j > N} 2^{\frac{3}{2}j} \|[u \cdot \nabla, \dot{\Delta}_j] w\|_{L^p} & \lesssim \|w^h\|_{B^{\frac{3}{2}}_{2,1}} \|\nabla u\|_{B^{\frac{3}{2}}_{2,1}} + \|w\|_{B^{\frac{3}{2}}_{2,1}} \|\nabla u^h\|_{B^{\frac{3}{2}}_{2,1}}.
\end{align*}
\]
Proof. We only deal with the first inequality. Using Bony’s decomposition for \([u \cdot \nabla, \hat{\Delta}_j]v\), then we have

\[
[u \cdot \nabla, \hat{\Delta}_j]v = \sum_{|i-j| \leq 4} [\hat{\Delta}_i u \cdot \nabla, \hat{\Delta}_j] \hat{\Delta}_i v + \sum_{i \geq j-3} [\hat{\Delta}_i u \cdot \nabla, \hat{\Delta}_j] \hat{\Delta}_i v + \sum_{i \geq j+3} [\hat{\Delta}_i u \cdot \nabla, \hat{\Delta}_j] \hat{\Delta}_i v = I_1 + I_2 + I_3,
\]

where \(\hat{\Delta}_i = \hat{\Delta}_{i-1} \pm \hat{\Delta}_i \pm \hat{\Delta}_{i+1}\). By the Hölder inequality, we get

\[
\| I_1 \|_{L^2} + \| I_3 \|_{L^2} \lesssim \sum_{|i-j| \leq 4} 2^{-j} (\| \nabla \hat{\Delta}_i u \|_{L^\infty} \| \nabla \hat{\Delta}_j v \|_{L^2} + \| \nabla \hat{\Delta}_i u \|_{L^\infty} \| \nabla \hat{\Delta}_j v \|_{L^2})
\]

which give rise to

\[
\sum_{j \leq N} 2^{\frac{j}{2}} (\| I_1 \|_{L^2} + \| I_3 \|_{L^2}) \lesssim \| \nabla u \|_{B^\frac{3}{p},1} \| v^f \|_{B^\frac{1}{p},1}.
\]

As for \(I_2\), we split it into two terms

\[
I_2 = \sum_{|i-j| \leq 4} [\hat{\Delta}_i u \cdot \nabla, \hat{\Delta}_j] \hat{\Delta}_i v + \sum_{i \geq j+3} \hat{\Delta}_i u \cdot \nabla (\hat{\Delta}_j \hat{\Delta}_i v) = I_{21} + I_{22},
\]

by the Hölder inequality, we get

\[
\| I_2 \|_{L^2} \lesssim \sum_{|i-j| \leq 4} 2^{-j} (\| \nabla \hat{\Delta}_i u \|_{L^2} \| \nabla \hat{\Delta}_j v \|_{L^\infty} + \sum_{i \geq j+3} \| \hat{\Delta}_i u \cdot \nabla (\hat{\Delta}_j \hat{\Delta}_i v) \|_{L^2})
\]

which gives rise to

\[
\sum_{j \leq N} 2^{\frac{j}{2}} \| I_2 \|_{L^2} \lesssim \sum_{j \leq N} 2^{\frac{j}{2}} \| \nabla \hat{\Delta}_j u \|_{L^2} \| \Lambda^{-1} v \|_{L^\infty} + \sum_{j \leq N} 2^{\frac{j}{2}} \| \nabla u \|_{L^\infty} \| \hat{\Delta}_j \hat{\Delta}_i v \|_{L^2}
\]

Together with the above estimates, then we prove the Lemma.

\[\square\]

**Lemma 2.3.** Let \(p \in [2, 4]\), for any \(v^f \in \dot{B}^\frac{3}{p},1, w^h \in \dot{B}^\frac{3}{p-1}, w^d \in \dot{B}^\frac{3}{p}, w^h \in \dot{B}^\frac{3}{p}, w^d \in \dot{B}^\frac{3}{p},
\]
Taking advantage of the Bony decomposition again, we obtain

for the last term, we have

\[
\begin{align*}

\| (vu)^\ell \|_{B^\frac{2}{2}, 1} & \lesssim \left( \| v^\ell \|_{B^\frac{4}{2}, 1} + \| v^h \|_{B^\frac{4}{2}, 1} \right) \| u \|_{B^\frac{2}{2}, 1}, \\
\| (vu)^h \|_{B^\frac{2}{p}, 1} & \lesssim \left( \| v^\ell \|_{B^\frac{4}{2}, 1} + \| v^h \|_{B^\frac{4}{2}, 1} \right) \| u \|_{B^\frac{2}{2}, 1}, \\
\| (wu)^\ell \|_{B^\frac{2}{2}, 1} & \lesssim \left( \| w^\ell \|_{B^\frac{4}{2}, 1} + \| w^h \|_{B^\frac{4}{2}, 1} \right) \left( \| u^\ell \|_{B^\frac{4}{2}, 1} + \| u^h \|_{B^\frac{4}{2}, 1} \right), \\
\| (wu)^h \|_{B^\frac{2}{p}, 1} & \lesssim \| w \|_{B^\frac{2}{p}, 1} \| u \|_{B^\frac{2}{2}, 1}.
\end{align*}
\]

Proof. Using the Bony decomposition, then we get

\[
\dot{S}_{N+1}(vu) = \dot{S}_{N+1}(\dot{T}v + \dot{R}(v, u)) + \dot{T}_u \dot{S}_{N+1}v + [\dot{S}_{N+1}, \dot{T}_u]v.
\]

Let \( \frac{1}{q} = \frac{1}{2} - \frac{1}{p} \), \( q \geq p \), one can check that \( \frac{3}{q} - 1 < 0 \). By the definition of \( \dot{T}_v u \) and \( \dot{R}(v, u) \), we have

\[
\begin{align*}
\| \dot{S}_{N+1}(\dot{T}v + \dot{R}(v, u)) \|_{B^\frac{4}{2}, 1} & \lesssim \sum_{k \geq j-2} 2^{\frac{3}{2}j} \| \dot{\Delta}_j (\dot{S}_{k+2}v \dot{\Delta}_k u) \|_{L^2} \\
& \lesssim \sum_{k \geq j-2} 2^{(\frac{3}{2} + \frac{3}{2} - 1)j} \| \dot{S}_{k+2}v \|_{L^q} \| \dot{\Delta}_k u \|_{L^p} \\
& \lesssim \| v \|_{B^\frac{3}{2}, 1} \| u \|_{B^\frac{3}{2}, 1} \lesssim \| v \|_{B^\frac{3}{2}, 1} \| u \|_{B^\frac{3}{2}, 1},
\end{align*}
\]

and

\[
\begin{align*}
\| \dot{T}_u \dot{S}_{N+1}v \|_{B^\frac{4}{2}, 1} & \lesssim \| \dot{S}_{N+1}v \|_{B^\frac{4}{2}, 1} \| u \|_{L^\infty} \lesssim \| v^\ell \|_{B^\frac{4}{2}, 1} \| u \|_{B^\frac{4}{2}, 1},
\end{align*}
\]

for the last term, we have

\[
\begin{align*}
\| [\dot{S}_{N+1}, \dot{T}_u]v \|_{B^\frac{4}{2}, 1} & \lesssim \| v \|_{B^\frac{3}{2}, 1} \| \nabla u \|_{B^\frac{3}{2}, 1} \lesssim \| v \|_{B^\frac{3}{2}, 1} \| u \|_{B^\frac{3}{2}, 1}.
\end{align*}
\]

Taking advantage of the Bony decomposition again, we obtain

\[
(I - \dot{S}_{N+1})(vu) = (I - \dot{S}_{N+1})(\dot{T}v + \dot{R}(v, u)) + (I - \dot{S}_{N+1})\dot{T}_uv.
\]
Notice that $\frac{3}{q} - 1 < 0$. Let $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, then we have

$$\|(I - \dot{S}_{N+1})(\dot{T}_u + \dot{R}(v, u))\|_{B^{\frac{3}{q}}_{p, 1}}$$

$$\lesssim \sum_{k \geq j - 2} 2^{\left(\frac{3}{q} - 1\right)j} \|\dot{\Delta}_j (\dot{S}_{k+2} v \dot{\Delta}_k u)\|_{L^p} + \sum_{k \geq j - 2} 2^{\left(\frac{3}{q} - 1\right)j} \|\dot{\Delta}_j (\dot{\Delta}_k v \dot{S}_{k-1} u)\|_{L^p}$$

$$\lesssim \sum_{k \geq j - 2} 2^{\left(\frac{3}{q} + \frac{3}{q} - 1\right)j} \|\dot{S}_{k+2} v\|_{L^p} \||\dot{\Delta}_k u\|_{L^p} + \sum_{k \geq j - 2} 2^{\left(\frac{3}{q} - 1\right)j} \|\dot{\Delta}_k v\|_{L^p} \|\dot{S}_{k-1} u\|_{L^\infty}$$

$$\lesssim \|v\|_{B^{\frac{3}{q}}_{q, 1}} \|u\|_{B^{\frac{3}{q}}_{q, 1}} + \|v^h\|_{B^{\frac{3}{q}}_{q, 1}} \|u\|_{L^\infty}$$

$$\lesssim (\|v^f\|_{B^{\frac{3}{q}}_{2, 1}} + \|v^h\|_{B^{\frac{3}{q}}_{q, 1}}) \|u\|_{B^{\frac{3}{q}}_{q, 1}}$$

and

$$\|(I - \dot{S}_{N+1})\dot{T}_u\|_{B^{\frac{3}{q}}_{p, 1}} \lesssim \|v\|_{B^{\frac{3}{q}}_{p, 1}} \|u\|_{L^\infty} \lesssim \|v\|_{B^{\frac{3}{q}}_{p, 1}} \|u\|_{B^{\frac{3}{q}}_{p, 1}}.$$  

Let $\dot{T}_u = \dot{T}_u^* + \dot{R}(v, u)$. By a similar computation, we have

$$\|\dot{S}_{N+1}(w_u)\|_{B^{\frac{3}{q}}_{2, 1}} \lesssim \|\dot{S}_{N+1}(\dot{T}_u)\|_{B^{\frac{3}{q}}_{2, 1}} + \|\dot{S}_{N+1}(\dot{T}_u^*)\|_{B^{\frac{3}{q}}_{2, 1}}$$

$$\lesssim \|\dot{S}_{N+1, \dot{T}_u}^*\|_{B^{\frac{3}{q}}_{2, 1}} + \|\dot{T}_u^*(\dot{S}_{N+1} w)\|_{B^{\frac{3}{q}}_{2, 1}} + \|\dot{T}_u(\dot{S}_{N+1} w)\|_{B^{\frac{3}{q}}_{2, 1}}$$

$$\lesssim \|w\|_{B^{\frac{3}{q}}_{p, 1}} \|\nabla u\|_{B^{\frac{3}{q}}_{q, 1}} + \|w\|_{L^\infty} \|u^f\|_{B^{\frac{3}{q}}_{2, 1}} + \|u\|_{B^{\frac{3}{q}}_{p, 1}} \|\nabla w\|_{B^{\frac{3}{q}}_{q, 1}} + \|u\|_{L^\infty} \|u^f\|_{B^{\frac{3}{q}}_{2, 1}}$$

$$\lesssim (\|w^f\|_{B^{\frac{3}{q}}_{2, 1}} + \|w^h\|_{B^{\frac{3}{q}}_{q, 1}}) (\|u^f\|_{B^{\frac{3}{q}}_{2, 1}} + \|u^h\|_{B^{\frac{3}{q}}_{q, 1}}),$$

and

$$\|(I - \dot{S}_{N+1})(w_u)\|_{B^{\frac{3}{q}}_{p, 1}} \lesssim \|w\|_{B^{\frac{3}{q}}_{p, 1}} \|u\|_{L^\infty} + \|u\|_{B^{\frac{3}{q}}_{q, 1}} \|w\|_{L^\infty} \lesssim \|w\|_{B^{\frac{3}{q}}_{p, 1}} \|u\|_{B^{\frac{3}{q}}_{p, 1}}.$$  

Hence we prove the Lemma.

3. Global existence

In this section, we are going to prove our main result. There is no derivative in the additional term in (PPT), the proof of local well-posedness for (PPT) is similar to the Oldroyd-B model (See [14, 20, 28]) and we omit the detail here. In order to prove the global existence of strong solutions, we give the some basic energies as follows,

$$\mathcal{E}(0) = \|u^f_0\|_{B^{\frac{3}{q}}_{2, 1}} + \|u^h_0\|_{B^{\frac{3}{q}}_{q, 1}} + \|u^h_0\|_{B^{\frac{3}{q}}_{q, 1}} + \|\sigma^h_0\|_{B^{\frac{3}{q}}_{q, 1}}$$

(3.1)
\[ \mathcal{E}_1(t) = \| u^t \|_{L_\infty^2(B^1_{2,1})} + \| \sigma^t \|_{L_\infty^2(B^1_{2,1})}, \]  
\[ \mathcal{E}_2(t) = \| u^t \|_{L_1^3(B^3_{2,1})} + \| \Lambda^{-1} \mathbb{P} \text{ div } \sigma^t \|_{L_1^3(B^3_{2,1})}, \]  
\[ \mathcal{E}_3(t) = \| u^h \|_{L_\infty^2(B^3_{p,1})} + \| \sigma^h \|_{L_\infty^2(B^3_{p,1})} + \| u^h \|_{L_1^3(B^3_{p,1})} + \| \Lambda^{-1} \mathbb{P} \text{ div } \sigma^h \|_{L_1^3(B^3_{p,1})}, \]  
\[ \mathcal{E}_4(t) = \| \text{tr } \sigma^t \|_{L_1^3(B^3_{p,1})} + \| \text{tr } \sigma^h \|_{L_1^3(B^3_{p,1})}, \]  

where \( \mathbb{P} = \mathbb{I} - \Delta^{-1} \nabla \text{ div} \) is the Leray projection operator. We shall derive the a priori estimates of \( \mathcal{E}_1(t), \mathcal{E}_2(t), \mathcal{E}_3(t) \) and \( \mathcal{E}_4(t) \) respectively.

### 3.1. The estimates of \( \mathcal{E}_1(t) \)

Applying the operator \( \hat{\Delta}_j \mathbb{P} \) to the first equation of (PTT) and \( \hat{\Delta}_j \) to the second equation of (PTT), we have

\[
\begin{aligned}
(\hat{\Delta}_j u)_t + u \cdot \nabla \hat{\Delta}_j u - \Delta \hat{\Delta}_j u &= \hat{\Delta}_j \mathbb{P} \text{ div } \sigma + [u \cdot \nabla, \hat{\Delta}_j \mathbb{P}] u, \\
(\hat{\Delta}_j \sigma)_t + u \cdot \nabla \hat{\Delta}_j \sigma + \frac{1}{c_0 + t} (\hat{\Delta}_j \sigma + \frac{1}{3} (\text{tr } \hat{\Delta}_j \sigma) I) &= \hat{\Delta}_j D(u) - \hat{\Delta}_j ((\text{tr } \sigma) \sigma + Q(\sigma, \nabla u)) + [u \cdot \nabla, \hat{\Delta}_j \sigma].
\end{aligned}
\]

Notice that \( \text{div } u = 0 \). Taking the \( L^2 \) scalar product of the first equation of (3.6) with \( \hat{\Delta}_j u \) and the second equation of (3.6) with \( \hat{\Delta}_j \sigma \), then we obtain that

\[
\frac{1}{2} \frac{d}{dt} (\| \hat{\Delta}_j u \|^2_{L^2} + \| \hat{\Delta}_j \sigma \|^2_{L^2}) + \| \nabla \hat{\Delta}_j u \|^2_{L^2} + \frac{1}{2} \frac{1}{c_0 + t} \| \hat{\Delta}_j \sigma \|^2_{L^2} \leq \int_{\mathbb{R}^3} (\hat{\Delta}_j \mathbb{P} \text{ div } \sigma \cdot \hat{\Delta}_j u + \hat{\Delta}_j D(u) \cdot \hat{\Delta}_j \sigma) dx
\]

\[
+ \int_{\mathbb{R}^3} ([u \cdot \nabla, \hat{\Delta}_j \mathbb{P}] u \cdot \hat{\Delta}_j u + [u \cdot \nabla, \hat{\Delta}_j \sigma] \cdot \hat{\Delta}_j \sigma) dx
\]

\[
- \int_{\mathbb{R}^3} \hat{\Delta}_j ((\text{tr } \sigma) \sigma + Q(\sigma, \nabla u)) \cdot \hat{\Delta}_j \sigma dx.
\]

Since \( \sigma_{ij} = \sigma_{ji} \), it follows that

\[
\int_{\mathbb{R}^3} (\hat{\Delta}_j \mathbb{P} \text{ div } \sigma \cdot \hat{\Delta}_j u + \hat{\Delta}_j D(u) \cdot \hat{\Delta}_j \sigma) dx = 0.
\]

Integrating in time and multiplying both sides of (3.7) by \( 2^{j} \), summing up about \( j \leq N \), then we obtain that,

\[
\| u^t \|_{L_\infty^2(B^1_{2,1})} + \| \sigma^t \|_{L_\infty^2(B^1_{2,1})} \lesssim \| u_0^t \|_{B^1_{2,1}} + \| \sigma_0^t \|_{B^1_{2,1}} + \sum_{j \leq N} 2^{\frac{j}{2}} \| [u \cdot \nabla, \hat{\Delta}_j \mathbb{P}] u \|_{L_1^2(L^2)}
\]

\[
+ \sum_{j \leq N} 2^{\frac{j}{2}} \| [u \cdot \nabla, \hat{\Delta}_j] \sigma \|_{L_1^2(L^2)} + \sum_{j \leq N} 2^{\frac{j}{2}} \| \hat{\Delta}_j ((\text{tr } \sigma) \sigma + Q(\sigma, \nabla u)) \|_{L_1^2(L^2)}.
\]
Applying Lemma 2.2 yields that
\[
\sum_{j \leq N} 2^j \left( \| u \cdot \nabla, \hat{\Delta}_j \|_{L^1_t(L^2)} + \| [u \cdot \nabla, \hat{\Delta}_j] \sigma \|_{L^1_t(L^2)} \right)
\leq \int_0^t \left( \| u^f \|_{B^\frac{1}{2}} + \| u^h \|_{B^\frac{3}{2}} + \| \sigma^f \|_{B^\frac{1}{2}} + \| \sigma^h \|_{B^\frac{3}{2}} \right) \left( \| \nabla u^f \|_{B^\frac{1}{2}} + \| \nabla u^h \|_{B^\frac{3}{2}} \right) ds.
\]
Taking advantage of Lemma 2.3 we have
\[
\sum_{j \leq N} 2^j \| \hat{\Delta}_j (\text{tr } \sigma) + Q(\sigma, \nabla u) \|_{L^1_t(L^2)}
\leq \int_0^t \left( \| \sigma^f \|_{B^\frac{1}{2}} + \| \sigma^h \|_{B^\frac{3}{2}} \right) \left( \| \text{tr } \sigma^f \|_{B^\frac{3}{2}} + \| \text{tr } \sigma^h \|_{B^\frac{3}{2}} + \| \nabla u^f \|_{B^\frac{1}{2}} + \| \nabla u^h \|_{B^\frac{3}{2}} \right) ds.
\]
According to above estimates, we deduce that
\[
\| u^f \|_{L^\infty_t(B^\frac{1}{2})} + \| \sigma^f \|_{L^\infty_t(B^\frac{1}{2})}
\leq \mathcal{E}(0) + \int_0^t \left( \| u^f \|_{B^\frac{1}{2}} + \| u^h \|_{B^\frac{3}{2}} + \| \sigma^f \|_{B^\frac{1}{2}} + \| \sigma^h \|_{B^\frac{3}{2}} \right) \times \left( \| \text{tr } \sigma^f \|_{B^\frac{3}{2}} + \| \text{tr } \sigma^h \|_{B^\frac{3}{2}} + \| \nabla u^f \|_{B^\frac{1}{2}} + \| \nabla u^h \|_{B^\frac{1}{2}} \right) ds,
\]
which implies
\[
\mathcal{E}_1(t) \lesssim \mathcal{E}(0) + (\mathcal{E}_1(t) + \mathcal{E}_3(t)) (\mathcal{E}_2(t) + \mathcal{E}_3(t) + \mathcal{E}_4(t)).
\]

3.2. The estimates of $\mathcal{E}_2(t)$

Applying the operator $\mathbb{P}$ to the first equation of (PTT) and $\Lambda^{-1} \mathbb{P} \text{ div}$ to the second equation of (PTT), we have
\[
\begin{cases}
  u_t + \mathbb{P}(u \cdot \nabla u) - \Delta u = \mathbb{P} \text{ div } \sigma, \\
  (\Lambda^{-1} \mathbb{P} \text{ div } \sigma)_t + \Lambda^{-1} \mathbb{P} \text{ div } (u \cdot \nabla \sigma) + \frac{1}{\alpha_0 + t} \Lambda^{-1} \mathbb{P} \text{ div } \sigma + \frac{1}{2} \Lambda u
  \end{cases}
\]
\[
= -\Lambda^{-1} \mathbb{P} \text{ div } ((\text{tr } \sigma) \sigma + Q(\tau, \nabla u)).
\]

Applying $\hat{\Delta}_j$ to the system (3.10), then we obtain the following system:
\[
\begin{cases}
  \hat{\Delta}_j u_t + u \cdot \nabla \hat{\Delta}_j u - \Delta \hat{\Delta}_j u - \Lambda \hat{\Delta}_j \psi = f_j, \\
  \hat{\Delta}_j \psi_t + u \cdot \nabla \hat{\Delta}_j \psi + \frac{1}{\alpha_0 + t} \hat{\Delta}_j \psi + \frac{1}{2} \Lambda \hat{\Delta}_j u = g_j,
\end{cases}
\]
where
\[
\psi = \Lambda^{-1} \mathbb{P} \div \sigma, \quad f_j = [u \cdot \nabla, \hat{\Delta} \Lambda^{-1} \mathbb{P} \div] \sigma - \hat{\Delta} \Lambda^{-1} \mathbb{P} \div ((\operatorname{tr} \sigma) \sigma + Q(\sigma, \nabla u)).
\]

Let \(0 < \eta < 1\) be a small constant which will be determined later on. Taking inner product with \((1 - \eta)\hat{\Delta} j u\) for the first equation of \((3.11)\), and \(\hat{\Delta} j \psi\) for the second equation of \((3.11)\), and then we have
\[
\frac{1}{2} \frac{d}{dt} ((1 - \eta)\|\hat{\Delta} j u\|_{L^2}^2 + \|\hat{\Delta} \psi\|_{L^2}^2) + (1 - \eta)\|\nabla \hat{\Delta} j u\|_{L^2}^2 + \eta \int_{\mathbb{R}^3} \Lambda \hat{\Delta} j u \cdot \hat{\Delta} \psi dx
\leq \|f_j\|_{L^2} \|\hat{\Delta} j u\|_{L^2} + \|g_j\|_{L^2} \|\hat{\Delta} \psi\|_{L^2}.
\]
Denote \(\varphi = 2 \Lambda \psi - u\), we have the following equation:
\[
\hat{\Delta} j \varphi + u \cdot \nabla \hat{\Delta} j \varphi + (1 + \frac{2}{c_0 + t}) \Lambda \hat{\Delta} j \psi = 2 \Lambda g_j - f_j + 2[u \cdot \nabla, \Lambda] \hat{\Delta} j \psi,
\]
Taking \(L^2\) inner product of \(\hat{\Delta} j \varphi\), then we have
\[
\frac{1}{2} \frac{d}{dt} \|\hat{\Delta} j \varphi\|_{L^2}^2 + 2(1 + \frac{2}{c_0 + t}) \Lambda \|\hat{\Delta} \psi\|_{L^2}^2 - (1 + \frac{2}{c_0 + t}) \int_{\mathbb{R}^3} \Lambda \hat{\Delta} \psi \cdot \hat{\Delta} j u dx
\leq \|(f_j)\|_{L^2} + 2 \|g_j\|_{L^2} + \|\varphi\|_{L^2} \|(\Lambda \hat{\Delta} \psi)\|_{L^2} + \|\hat{\Delta} j u\|_{L^2}.
\]
Together with the above inequalities and using the fact that \(1 \leq (1 + \frac{2}{c_0 + t}) \leq 1 + 2c_0\), we deduce that
\[
\frac{1}{2} \frac{d}{dt} ((1 - \eta)\|\hat{\Delta} j u\|_{L^2}^2 + \|\hat{\Delta} \psi\|_{L^2}^2 + \eta \|\hat{\Delta} j \varphi\|_{L^2}^2) + (1 - \eta)\|\nabla \hat{\Delta} j u\|_{L^2}^2 + \eta \|\Lambda \hat{\Delta} \psi\|_{L^2}^2
\leq \|(f_j)\|_{L^2} + (1 + 2^j) \|g_j\|_{L^2} + \|\varphi\|_{L^2} \|(\Lambda \hat{\Delta} \psi)\|_{L^2} (\|\hat{\Delta} j u\|_{L^2} + (1 + 2^j) \|\hat{\Delta} \psi\|_{L^2}).
\]
For any \(j \leq N\), we can find a \(\eta = \eta(N) > 0\) small enough such that
\[
(1 - \eta)\|\hat{\Delta} j u\|_{L^2}^2 + \|\hat{\Delta} \psi\|_{L^2}^2 + \eta \|\hat{\Delta} j \varphi\|_{L^2}^2 \geq C_N (\|\hat{\Delta} j u\|_{L^2}^2 + \|\hat{\Delta} \psi\|_{L^2}^2).
\]
From the above inequality and using Bernsteins’s lemma, we verify that
\[
\frac{d}{dt} (\|\hat{\Delta} j u\|_{L^2} + \|\hat{\Delta} j \psi\|_{L^2} + 2^j \|\hat{\Delta} \varphi\|_{L^2} + 2^j \|\hat{\Delta} \psi\|_{L^2})
\leq (1 + 2^j) (\|f_j\|_{L^2} + (1 + 2^j) \|g_j\|_{L^2} + \|\varphi\|_{L^2} \|(\Lambda \hat{\Delta} \psi)\|_{L^2}).
\]
Integrating in time and multiplying both sides of the above inequality by \(2^\frac{j}{2}\), and summing up about \(j \leq N\), we have
\[
\|u^0\|_{L^t \infty (B^{\frac{1}{2},1})} + \|\psi^0\|_{L^t \infty (B^{\frac{1}{2},1})} + \|\varphi^0\|_{L^t \infty (B^{\frac{1}{2},1})} + \|u^0\|_{L^{1} (B^{\frac{1}{2},1})} + \|\psi^0\|_{L^{1} (B^{\frac{1}{2},1})}
\leq \|u^0\|_{B^{\frac{1}{2},1}} + \|\psi^0\|_{B^{\frac{1}{2},1}} + \|\varphi^0\|_{B^{\frac{1}{2},1}} + \int_0^t \sum_{j \leq N} 2^\frac{j}{2} (\|f_j\|_{L^2} + \|g_j\|_{L^2} + \|\varphi\|_{L^2} \|(\Lambda \hat{\Delta} \psi)\|_{L^2}) ds.
\]
Thanks to Lemma 2.2 and Lemma 2.3 we verify that

\[
\sum_{j \leq N} 2^{\frac{j}{2}} \| f_j \|_{L^2} \lesssim \left( \| u^f \|_{B^{\frac{3}{2}}_{2,1}} + \| u^h \|_{B^{\frac{5}{2}}_{p,1}} \right) \left( \| \nabla u^f \|_{B^{\frac{3}{2}}_{2,1}} + \| \nabla u^h \|_{B^{\frac{5}{2}}_{p,1}} \right);
\]

\[
\sum_{j \leq N} 2^{\frac{j}{2}} \| g_j \|_{L^2} \lesssim \left( \| \sigma^f \|_{B^{\frac{5}{2}}_{p,1}} + \| \sigma^h \|_{B^{\frac{5}{2}}_{p,1}} \right) \left( \| \nabla u^f \|_{B^{\frac{3}{2}}_{2,1}} + \| \nabla u^h \|_{B^{\frac{5}{2}}_{p,1}} + \| \text{tr} \sigma^f \|_{B^{\frac{5}{2}}_{p,1}} + \| \text{tr} \sigma^h \|_{B^{\frac{5}{2}}_{p,1}} \right);
\]

\[
\sum_{j \leq N} 2^{\frac{j}{2}} \| [u \cdot \nabla, \Lambda] \dot{\Delta}_j \psi \|_{L^2} \lesssim \sum_{j \leq N} 2^{\frac{j}{2}} \| \nabla u \|_{L^\infty} \| \nabla \dot{\Delta}_j \psi \|_{L^2}
\lesssim \| \psi^f \|_{B^{\frac{3}{2}}_{2,1}} \| \nabla u \|_{B^{\frac{5}{2}}_{p,1}} \lesssim \| \sigma^f \|_{B^{\frac{5}{2}}_{p,1}} \left( \| \nabla u^f \|_{B^{\frac{3}{2}}_{2,1}} + \| \nabla u^h \|_{B^{\frac{5}{2}}_{p,1}} \right).
\]

Combining the above estimates yields that

\[
\| u^f \|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + \| u^h \|_{L^\infty_t(B^{\frac{5}{2}}_{p,1})} + \| \sigma^f \|_{L^1_t(B^{\frac{5}{2}}_{p,1})} + \| \sigma^h \|_{L^1_t(B^{\frac{5}{2}}_{p,1})} \lesssim \mathcal{E}(0) + \int_0^t \left( \| u^f \|_{B^{\frac{3}{2}}_{2,1}} + \| u^h \|_{B^{\frac{5}{2}}_{p,1}} + \| \sigma^f \|_{B^{\frac{5}{2}}_{p,1}} + \| \sigma^h \|_{B^{\frac{5}{2}}_{p,1}} \right) \times \left( \| \nabla u^f \|_{B^{\frac{3}{2}}_{2,1}} + \| \nabla u^h \|_{B^{\frac{5}{2}}_{p,1}} + \| \text{tr} \sigma^f \|_{B^{\frac{5}{2}}_{p,1}} + \| \text{tr} \sigma^h \|_{B^{\frac{5}{2}}_{p,1}} \right) ds,
\]

which implies

\[
\mathcal{E}_2(t) \lesssim \mathcal{E}(0) + \left( \mathcal{E}_1(t) + \mathcal{E}_3(t) \right) \left( \mathcal{E}_2(t) + \mathcal{E}_3(t) + \mathcal{E}_4(t) \right).
\]

3.3. The estimates of \( \mathcal{E}_3(t) \)

Denote \( \Gamma = u - \Lambda^{-1} \psi \), we can get from (3.11) that

\[
\dot{\Delta}_j \Gamma + u \cdot \nabla \dot{\Delta}_j \Gamma + \frac{1}{\psi} + t \dot{\Delta}_j \Gamma - \Delta \dot{\Delta}_j \Gamma = \left( \frac{1}{\psi} - \frac{1}{2} \right) \dot{\Delta}_j u + [u \cdot \nabla, \Lambda^{-1}] \dot{\Delta}_j \psi + f_j - \Lambda^{-1} g_j.
\]

By the standard \( L^p \) estimate, we get

\[
\frac{d}{dt} \| \dot{\Delta}_j \Gamma \|_{L^p} + 2^{3j} \| \dot{\Delta}_j \Gamma \|_{L^p} \lesssim \| \dot{\Delta}_j u \|_{L^p} + \| [u \cdot \nabla, \Lambda^{-1}] \dot{\Delta}_j \psi \|_{L^p} + \| f_j \|_{L^p} + \| \Lambda^{-1} g_j \|_{L^p}.
\]

Integrating in time and multiplying both sides of the above inequality by \( 2^{\frac{3}{2} - 1} j \), summing up about \( j > N \), then we obtain that

\[
\| \Gamma^h \|_{L^\infty_t(B^{\frac{3}{2}}_{\frac{p-1}{(p-1)^2}})} + \| \Gamma^h \|_{L^1_t(B^{\frac{5}{2}}_{p,1})} \lesssim \| \Gamma^h \|_{B^{\frac{5}{2}}_{p,1}} + 2^{-N} \left( \| \Gamma^h \|_{L^1_t(B^{\frac{5}{2}}_{p,1})} + \| \psi^h \|_{L^1_t(B^{\frac{5}{2}}_{p,1})} \right) + \int_0^t \sum_{j > N} 2^{\frac{3}{2} - 1} j \left( \| [u \cdot \nabla, \Lambda^{-1}] \dot{\Delta}_j \psi \|_{L^p} + \| f_j \|_{L^p} + \| \Lambda^{-1} g_j \|_{L^p} \right) ds.
\]
Lemma 2.2 and Lemma 2.3 ensure that
\[
\sum_{j>N} 2^{\frac{p}{p-1}j} \| [u \cdot \nabla, \Lambda^{-1}] \hat{\Delta}_j \psi \|_{L^p} \lesssim \sum_{j>N} 2^{\frac{p}{p-1}j} \| \Lambda^{-1} u \|_{L^\infty} \| \hat{\Delta}_j \psi \|_{L^p} 
\]
\[
\lesssim \| u \|_{B^{\frac{4}{p}-1}_{p,1}} \| \psi^h \|_{B^{\frac{4}{p}}_{p,1}} \lesssim (\| u^f \|_{B^{\frac{4}{p}}_{p,1}} + \| u^h \|_{B^{\frac{4}{p}-1}_{p,1}}) \| \psi^h \|_{B^{\frac{4}{p}}_{p,1}},
\]
\[
\sum_{j>N} 2^{\frac{p}{p-1}j} \| f_j \|_{L^p} \lesssim (\| u^f \|_{B^{\frac{4}{p}}_{p,1}} + \| u^h \|_{B^{\frac{4}{p}-1}_{p,1}})(\| \nabla u^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \nabla u^h \|_{B^{\frac{4}{p}}_{p,1}}),
\]
\[
\sum_{j>N} 2^{\frac{p}{p-1}j} \| \Lambda^{-1} g_j \|_{L^p} \lesssim (\| \sigma^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \sigma^h \|_{B^{\frac{4}{p}-1}_{p,1}})(\| \nabla u^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \nabla u^h \|_{B^{\frac{4}{p}}_{p,1}} + \| \text{tr} \sigma^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \text{tr} \sigma^h \|_{B^{\frac{4}{p}}_{p,1}}).
\]

Together with the above estimates, we deduce that
\[
\| \Gamma^h \|_{L^\infty(B^{\frac{4}{p}-1}_{p,1})} + \| \Gamma^h \|_{L^1(B^{\frac{4}{p}+1}_{p,1})} 
\]
\[
\lesssim \| \Gamma^h \|_{B^{\frac{4}{p}}_{p,1}} + 2^{-2N}(\| \Gamma^h \|_{L^1(B^{\frac{4}{p}+1}_{p,1})} + \| \psi^h \|_{L^1(B^{\frac{4}{p}}_{p,1})}) 
\]
\[
+ \int_0^t (\| u^f \|_{B^{\frac{4}{p}}_{p,1}} + \| u^h \|_{B^{\frac{4}{p}-1}_{p,1}} + \| \sigma^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \sigma^h \|_{B^{\frac{4}{p}}_{p,1}}) 
\]
\[
\times (\| \psi^h \|_{B^{\frac{4}{p}}_{p,1}} + \| \nabla u^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \nabla u^h \|_{B^{\frac{4}{p}}_{p,1}} + \| \text{tr} \sigma^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \text{tr} \sigma^h \|_{B^{\frac{4}{p}}_{p,1}}) ds.
\]

We rewrite the second equation of (3.11) as follows
\[
\hat{\Delta}_j \psi_t + u \cdot \nabla \hat{\Delta}_j \psi + \left( \frac{1}{2} + \frac{1}{c_0 + \ell} \right) \hat{\Delta}_j \psi = g_j - \frac{1}{2} \Lambda \hat{\Delta}_j \Gamma.
\]

By virtue of the standard $L^p$ estimate, we get
\[
\frac{d}{dt} \| \hat{\Delta}_j \psi \|_{L^p} + \| \hat{\Delta}_j \psi \|_{L^p} \lesssim \| g_j \|_{L^p} + \| \Lambda \Delta_j \Gamma \|_{L^p},
\]
which leads to
\[
\| \psi^h \|_{L^\infty(B^{\frac{4}{p}}_{p,1})} + \| \psi^h \|_{L^1(B^{\frac{4}{p}+1}_{p,1})} 
\]
\[
\lesssim \| \psi^h \|_{B^{\frac{4}{p}}_{p,1}} + \int_0^t \sum_{j>N} 2^{\frac{p}{p-1}j}(\| g_j \|_{L^p} + \| \Lambda \Delta_j \Gamma \|_{L^p}) ds 
\]
\[
\lesssim \| \psi^h \|_{B^{\frac{4}{p}}_{p,1}} + \| \Gamma^h \|_{L^1(B^{\frac{4}{p}+1}_{p,1})} + \int_0^t \sum_{j>N} 2^{\frac{p}{p-1}j} \| g_j \|_{L^p} ds 
\]
\[
\lesssim \| \psi^h \|_{B^{\frac{4}{p}}_{p,1}} + \| \Gamma^h \|_{L^1(B^{\frac{4}{p}+1}_{p,1})} + \int_0^t (\| \sigma^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \sigma^h \|_{B^{\frac{4}{p}}_{p,1}}) 
\]
\[
\times (\| \nabla u^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \nabla u^h \|_{B^{\frac{4}{p}}_{p,1}} + \| \text{tr} \sigma^f \|_{B^{\frac{4}{p}}_{p,1}} + \| \text{tr} \sigma^h \|_{B^{\frac{4}{p}}_{p,1}}) ds.
\]
Taking $N \in \mathbb{N}^+$ is large enough, together with (3.15) and (3.17), we verify that
\[
\|\Gamma^h\|_{L^\infty_t(B^\frac{1}{p-1}_{p,1})} + \|\psi^h\|_{L^\infty_t(B^\frac{p}{p-1}_{p,1})} + \|\Gamma^h\|_{L^1_t(B^\frac{p}{p-1}_{p,1})} + \|\psi^h\|_{L^1_t(B^\frac{p}{p-1}_{p,1})} \\
\lesssim \mathcal{E}(0) + \int_0^t \left( \|u^f\|_{B^\frac{1}{p}_{p,1}} + \|u^h\|_{B^\frac{p}{p-1}_{p,1}} + \|\sigma^f\|_{B^\frac{1}{p}_{p,1}} + \|\sigma^h\|_{B^\frac{1}{p}_{p,1}} \right) \times \left( \|\psi^h\|_{B^\frac{p}{p-1}_{p,1}} + \|\nabla u^f\|_{B^\frac{p}{p-1}_{p,1}} + \|\nabla u^h\|_{B^\frac{p}{p-1}_{p,1}} + \|\nabla \sigma^f\|_{B^\frac{p}{p-1}_{p,1}} + \|\nabla \sigma^h\|_{B^\frac{p}{p-1}_{p,1}} \right) ds.
\]

We can deduce from $u = \Gamma + \Lambda^{-1}\psi$ that
\[
\sum_{j} \left[ \|u^h\|_{L^\infty_t(B^\frac{1}{p-1}_{p,1})} + \|u^h\|_{L^\infty_t(B^\frac{p}{p-1}_{p,1})} + \|u^h\|_{L^1_t(B^\frac{p}{p-1}_{p,1})} + \|\psi^h\|_{L^1_t(B^\frac{p}{p-1}_{p,1})} \right] \\
\lesssim \sum_{j} \left[ \|\Gamma^h\|_{L^\infty_t(B^\frac{1}{p-1}_{p,1})} + \|\psi^h\|_{L^\infty_t(B^\frac{p}{p-1}_{p,1})} + \|\Gamma^h\|_{L^1_t(B^\frac{p}{p-1}_{p,1})} + \|\psi^h\|_{L^1_t(B^\frac{p}{p-1}_{p,1})} \right],
\]
which implies
\[
\mathcal{E}_3(t) \lesssim \mathcal{E}(0) + (\mathcal{E}_1(t) + \mathcal{E}_3(t))(\mathcal{E}_2(t) + \mathcal{E}_3(t) + \mathcal{E}_4(t)).
\]  

3.4. The estimates of $\mathcal{E}_4(t)$

Applying $\text{tr}$ to the second equation of (3.17), yields that
\[
(tr \sigma)_t + u \cdot \nabla \text{tr} \sigma + (tr \sigma)^2 + \frac{2}{c_0 + t} \text{tr} \sigma = 0,
\]
which leads to
\[
\left[ \left( \frac{1}{c_0} + t \right)^2 \text{tr} \sigma \right]_t + u \cdot \nabla \left[ \left( \frac{1}{c_0} + t \right)^2 \text{tr} \sigma \right] \leq -\left( \frac{1}{c_0} + t \right)^2 (\text{tr} \sigma)^2.
\]

Applying the operator $\hat{\Delta}_j$ to above equation, we get
\[
\left[ \left( \frac{1}{c_0} + t \right)^2 \hat{\Delta}_j \text{tr} \sigma \right]_t + u \cdot \nabla \left[ \left( \frac{1}{c_0} + t \right)^2 \hat{\Delta}_j \text{tr} \sigma \right] \leq -\left( \frac{1}{c_0} + t \right)^2 (\hat{\Delta}_j(\text{tr} \sigma)^2 - [u \cdot \nabla, \hat{\Delta}_j] \text{tr} \sigma).
\]

By virtue of the standard $L^q$ estimate, we obtain
\[
\frac{d}{dt} \left[ \left( \frac{1}{c_0} + t \right)^2 \|\hat{\Delta}_j \text{tr} \sigma\|_{L^q} \right] \lesssim \left( \frac{1}{c_0} + t \right)^2 (\|\hat{\Delta}_j(\text{tr} \sigma)^2\|_{L^q} + \|[u \cdot \nabla, \hat{\Delta}_j] \text{tr} \sigma\|_{L^q}).
\]

Choose $q = 2$ and $q = p$ with respectively. Multiplying both sides of the above inequality by $2^j$ and $2^j$, and then summing up about $j \leq N$ and $j > N$ with respectively, then we obtain that
\[
\frac{d}{dt} \left[ \left( \frac{1}{c_0} + t \right)^2 (\|\text{tr} \sigma^f\|_{B^\frac{3}{2}_{2,1}} + \|\text{tr} \sigma^h\|_{B^\frac{3}{2}_{p,1}}) \right] \\
\lesssim \left[ \left( \frac{1}{c_0} + t \right)^2 (\|\text{tr} \sigma^f\|_{B^\frac{3}{2}_{2,1}} + \|\text{tr} \sigma^h\|_{B^\frac{3}{2}_{p,1}}) \right] (\|\text{tr} \sigma^f\|_{B^\frac{3}{2}_{2,1}} + \|\text{tr} \sigma^h\|_{B^\frac{3}{2}_{p,1}} + \|\nabla u^f\|_{B^\frac{3}{2}_{2,1}} + \|\nabla u^h\|_{B^\frac{3}{2}_{p,1}}) .
\]
By Gronwall’s inequality, then we have
\[
(\frac{1}{c_0} + t)^2 (\| \sigma_0 \|_{B_{2,1}^\frac{3}{2}} + \| \sigma_0 \|_{B_{p,1}^\frac{3}{2}})
\geq \frac{1}{c_0} (\| \sigma_0 \|_{B_{2,1}^\frac{3}{2}} + \| \sigma_0 \|_{B_{p,1}^\frac{3}{2}}) \exp \left\{ \int_0^t (\| \sigma_0 \|_{B_{2,1}^\frac{3}{2}} + \| \sigma_0 \|_{B_{p,1}^\frac{3}{2}} + \| \nabla u \|_{B_{2,1}^\frac{3}{2}} + \| \nabla u \|_{B_{p,1}^\frac{3}{2}}) ds \right\},
\]
which leads to
\[
\int_0^t (\| \sigma_0 \|_{B_{2,1}^\frac{3}{2}} + \| \sigma_0 \|_{B_{p,1}^\frac{3}{2}}) ds
\geq \frac{1}{c_0} (\| \sigma_0 \|_{B_{2,1}^\frac{3}{2}} + \| \sigma_0 \|_{B_{p,1}^\frac{3}{2}}) \exp \left\{ \int_0^t (\| \sigma_0 \|_{B_{2,1}^\frac{3}{2}} + \| \sigma_0 \|_{B_{p,1}^\frac{3}{2}} + \| \nabla u \|_{B_{2,1}^\frac{3}{2}} + \| \nabla u \|_{B_{p,1}^\frac{3}{2}}) ds \right\},
\]
that is
\[
E_4(t) \leq E(0) \exp (E_2(t) + E_3(t) + E_4(t)).
\]

3.5. Proof of the Theorem 1.1

**Proof.** In this subsection, we will combine the above a priori estimates of $E_1(t)$, $E_2(t)$, $E_3(t)$ and $E_4(t)$ together and give the proof of the Theorem 1.1; then exists for any $t \in [0, T]$, we have
\[
E(t) = E_1(t) + E_2(t) + E_3(t) + E_4(t)
\]
\[
\leq C^* \left[ 1 + \exp \left( E_2(t) + E_3(t) + E_4(t) \right) \right] E(0) + \left[ E_1^2(t) + E_2^2(t) + E_3^2(t) + E_4^2(t) \right].
\]
Due to the local existence theory, there exists a positive time $T$ such that
\[
E(t) \leq 3C^* \delta_0, \quad \forall t \in [0, T].
\]
Let $T^*$ be the largest possible time of $T$ for what (3.23) holds. Under the setting of initial data, there exists a small enough number $\epsilon_0$ such that $E(0) \leq \delta_0 \leq \epsilon_0$. By virtue of (3.22) and the smallness assumption on $\delta_0$, we get that
\[
E(t) \leq 2C^* \delta_0 + C \delta_0^2 < 3C^* \delta_0.
\]
By standard continuity argument and total energy (3.22), we can show that $T^* = \infty$ provided that $\delta_0$ is small enough. Hence, we finish the proof of the Theorem 1.1. \qed
Remark 3.1. As $\frac{1}{\varepsilon_0 + t}$ does not belong to any Besov spaces, we can’t say that $\tau$ belongs to any Besov spaces. However, we have

$$
\|\sigma\|_{L_t^\infty L_x^\infty} \lesssim \|\sigma^f\|_{L_t^\infty L_x^\infty} + \|\sigma^h\|_{L_t^\infty L_x^\infty} \lesssim \|\sigma^f\|_{L_t^\infty (B_{2,1}^3)} + \|\sigma^h\|_{L_t^\infty (B_{p,1}^3)},
$$

which implies that $\tau \in L_t^\infty L_x^\infty$. Moreover, one can check that $\tau \in C(\mathbb{R}^+ \times \mathbb{R}^3)$

Remark 3.2. The initial condition $c_0 > 0$ and $\epsilon_0$ is small enough implies that $\inf \text{tr} \tau_0 > 0$. On the other hand, if there exists a $x_0 \in \mathbb{R}^3$ such that $\text{tr} \tau_0(x_0) < 0$, we can deduce from the second equation of (1.1) that

$$
\text{tr} \tau + u \cdot \nabla \text{tr} \tau + (\text{tr} \tau)^2 = 0.
$$

(3.24)

Consider the trajectory equation

$$
\frac{d}{dt} q(t,x) = u(t,q(t,x)), \quad q(0,x) = x.
$$

It is easy to see that

$$
\text{tr} \tau(t,q(t,x_0)) = \frac{\text{tr} \tau_0(x_0)}{1 + \text{tr} \tau_0(x_0)t}, \quad \forall t \in [0,T],
$$

(3.25)

which leads to $\text{tr} \tau(t,q(t,x_0))$ blows up in finite time. Thus, the condition $\inf \text{tr} \tau_0 \geq 0$ is a necessary condition to ensure that the strong solution exists globally.

Acknowledgements. Wei Luo is partially supported by NSF of China under Grant 11701586 and 11671407. Xiaoping Zhai is partially supported by NSF of China under Grant 11601533.

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