Regular frames and particle’s rotation near a black hole

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We consider a particle moving towards a rotating black hole. The number of revolution \( n \) is studied using the coordinate frame regular near the horizon. It turns out that \( n \) coincides with that obtained in our previous work by the subtraction procedure \( n_1 - n_2 \) where 1 is an original particle and 2 is some reference one. Meanwhile, the present results for \( n \) can differ from those obtained with the help of other subtraction procedures (such as measurements with respect to the horizon of a rotating black hole or motion of a non-geodesic particle). For a nonextremal black hole, regularity of a coordinate frame leads to finiteness of a number of revolutions around a black hole. The regular frames we use can be considered as generalization of the Kerr coordinates known for the Kerr metric. The transformation under discussion ensure simultaneously the regularity of the metric and finiteness of \( n \) for particles (apart from some exceptional cases).

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I. INTRODUCTION

It is well known that if a particle falls towards a black hole, there is a sharp contrast between the time intervals measured by an observer at infinity \( \tau \) and an observer comoving with the particle (the proper time \( t \)). To reach the event horizon, a particle needs an infinite \( t \). Meanwhile, \( \tau \) (except from some special situations) is finite. The similar relation exists between the number of revolutions \( n \) around a rotating black hole. For a remote observer, it is always infinite. For a comoving observer it can be finite or infinite.

In the previous work [1] we showed that the relationship between these two characteristics can be quite nontrivial. In particular, it can depend not only on the type of black hole and a trajectory but also on a way the measurement are performed. We considered (i) the measurement for a free-falling particle with respect to another free-falling one, (ii) measurements with respect to a particle that is not in free fall, (iii) with respect to a black hole itself. In all three cases it was impossible to view the effect using one fixed coordinate frame.

The divergences of \( n \) are connected with the fact that an original coordinates (like the Boyer-Lindquist ones for the Kerr metric) fail near the horizon, so both \( t \) and the polar angle variable \( \phi \) diverge. Meanwhile, in recent years, coordinate frames regular near the rotating black holes were constructed [2–3] that enabled one to trace rather subtle details of particle behavior near the horizon [2, 3] or give physical interpretation of metrics in terms of rotating fluid [3]. The aim of the present work is to consider the properties of revolution of a particle around a black hole using this type of angle variable. Then, there is no need to develop special schemes for subtraction of the contribution to \( n \) from the reference frame since this subtraction is already implicitly contained in the definition of a “good” angle variable. We will see that this simple and direct procedure agrees with the aforementioned method (i) that, however, can be different from (ii) and (iii).

It turned out that overlap between different but related issues (description of geometry near the horizon and description of particle revolution around it) lead us to construction of the rotational analogue of the contracting/expanding Eddington-Finkelstein (or Lemaître) frames for black/white holes. This generalizes corresponding construction, known for the Kerr or Kerr-Newman metric.

II. METRIC AND EQUATIONS OF MOTION

Let us consider the metric

\[
ds^2 = -N^2 dt^2 + g_\phi(d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_{\theta\theta}d\theta^2.
\]

Here, the event horizon is described by \( N = 0 \). We assume that the metric coefficients do not depend on \( t \) and \( \phi \), so that the energy \( E \) and angular momentum \( L \) of a particle are conserved. We also assume

\[
N^2 = \alpha \Delta, \quad A = \frac{\Delta}{\rho^2},
\]

where \( \Delta = \Delta(r) \), \( \alpha > 0 \) and \( \rho > 0 \) depend on both \( r \) and \( \theta \). The maximum zero \( r = r_+ \) of \( \Delta \) corresponds to
the event horizon, \( \alpha \) and \( \rho \) are supposed to remain finite and nonzero at \( r = r_+ \).

In what follows, we use notation

\[
X = E - \omega L.
\]

(3)

We restrict ourselves by motion in the equatorial plane \( \theta = \pi/2 \) only.

Equations of geodesic motion read

\[
m \frac{dt}{d\tau} = X \frac{N^2}{N^2},
\]

(4)

\[
m \frac{d\phi}{d\tau} = \frac{L}{g_\phi} + \frac{\omega X}{N^2},
\]

(5)

\[
m \rho \sqrt{\alpha} \frac{dr}{d\tau} = -Z, \quad Z = \sqrt{X^2 - N^2 \left( \frac{L^2}{g_\phi} + m^2 \right)},
\]

(6)

where we assumed that a particle moves towards a black hole, so \( dr/d\tau < 0 \). Here, \( m \) is a particle’s mass, \( \tau \) being a proper time. It follows from (5), (6) that

\[
\frac{d\phi}{dr} = -\frac{\rho \sqrt{\alpha}}{Z} \left( \frac{L}{g_\phi} + \frac{\omega X}{N^2} \right),
\]

(7)

\[
t = -\int^r X \rho \frac{dr'}{\sqrt{\alpha} \Delta Z}.
\]

(8)

We imply that the so-called forward-in-time condition \( dt/d\tau > 0 \) is satisfied because of which \( X \geq 0 \).

### III. COORDINATE TRANSFORMATION TO REGULAR FRAMES

In [5], a general approach was suggested that enables one to build frames regular near the horizon. This includes previous known coordinate frames, in particular, Painlevé-Gulstrand ones for the Kerr and Kerr-Newman metrics [2]–[4]. For the metric (1), transformations within the equatorial plane have the form

\[
dt = d\bar{t} + \frac{z(r)dr}{\Delta},
\]

(9)

\[
d\phi = d\bar{\phi} + \frac{\xi dr}{\Delta} + \delta d\theta.
\]

(10)

We restrict ourselves by motion of particles in the equatorial plane \( \theta = \text{const} = \pi/2 \), so the last term in (10) is irrelevant. The corresponding functions obey the relations

\[
h \Delta = \xi - \omega z,
\]

(11)

\[
\mu \Delta = \rho^2 - z^2 \alpha,
\]

(12)

where \( h \) and \( \mu \) are finite on the horizon. This enables us to kill the divergences in the metric coefficient \( g_{rr} \). If we specify some functions \( z(r) \) and \( h(r, \theta) \) that are finite on the horizon, it follows from (11), (12) that

\[
(\xi - \omega z)_{r=r_+} = 0,
\]

(13)

\[
(z^2 \alpha - \rho^2)_{r=r_+} = 0.
\]

(14)

Then,

\[
\frac{d\bar{\phi}}{dr} = -\frac{\xi}{\Delta} - \frac{\rho \sqrt{\alpha}}{Z} \left( \frac{L}{g_\phi} + \frac{\omega X}{N^2} \right) = -\frac{\omega}{\Delta} \left( z + \frac{X \rho}{Z \sqrt{\alpha}} \right),
\]

(15)

The number of revolutions that particle experiences during travel between points 1 and 2 is equal to

\[
n = \frac{\Delta \bar{\phi}}{2\pi}, \quad \Delta \bar{\phi} = \bar{\phi}_2 - \bar{\phi}_1.
\]

(16)

It is worth paying attention to the choice

\[
z = -\frac{\rho \sqrt{1 - \frac{\alpha \Delta}{\alpha}}}{\alpha},
\]

(17)

\[
h = \frac{\omega \rho^2}{z}, \quad \mu = \alpha \rho^2,
\]

(18)

\[
\xi = \frac{\omega \rho^2}{z \alpha}.
\]

(19)

Then, for a particle moving with \( L = 0, E = m \) we see the angle \( \bar{\phi} = \text{const} \). This is the generalization of the corresponding property [6] inherent to the Kerr metric in coordinates of Ref. [2]. One reservation is order. The aforementioned choice implies that the combination (17) does not contain the angle \( \theta \). This is valid for the Kerr metric but not necessarily in a general case. However, for motion in the plane \( \theta = \pi/2 = \text{const} \) the ansatz (17) is sufficient.

Let us consider now different types of black holes and of particles separately.

### IV. NONEXTREMAL BLACK HOLE

Now, near the horizon the expansion of \( \omega \) takes the form [9]

\[
\omega = \omega_H - B_1 N^2 + O(N^3).
\]

(20)

Hereafter, subscript “H” means that a corresponding quantity is calculated on the horizon. For the Kerr-Newman metric, \( B_1 > 0 \).

Then, it follows from (13) and (14) that near the horizon

\[
X = X_H + B_1 N^2 L + O(N^3),
\]

(21)
\[ Z \approx X - \frac{N^2}{2X_H} \left( \frac{L^2}{g_H} + m^2 \right). \]

We have from \((14)\) that near the horizon either \(z + \rho/\sqrt{\alpha} \approx 0\) or \(z - \rho/\sqrt{\alpha} \approx 0\). Choosing the first option, we have near the horizon

\[ z + \frac{\rho}{\sqrt{\alpha}} \approx -b\Delta, \]

where \(g_H = g_\alpha\) \((N = 0)\) and \(b\) is the model-dependent coefficient. Expansion \((23)\) agrees with \((14)\). Then,

\[
\left( \frac{d\bar{\phi}}{dr} \right)_H = \omega_H b - h_H - \frac{L}{g_H X_H} - \frac{1}{2} \left[ X_H + 2B_1 L + \frac{\rho H \sqrt{\alpha} H}{X_H} \left( \frac{L^2}{g_H} + m^2 \right) \right].
\]

It is finite. It follows from the above equations that the quantity \(\bar{\phi}\) obtained by the integration of the right hand side of \((15)\) is also finite. Thus the angle changes at a finite value during infall of a particle to a black hole.

**V. EXTREMAL BLACK HOLE**

Now

\[ N^2 = D(r - r_+)^2 + O((r - r_+)^3), \]

where \(D > 0\) is some constant. The expansion for \(\omega\) takes the form \((8)\)

\[ \omega = \omega_H - B_1 N + O(N^2). \]

Below, we use classification of particles according to which a particle with \(X_H > 0\) is called usual and that with \(X_H = 0\) is called critical.

**A. Usual particles**

Then, one can see that the expression for \((24)\) on the horizon retains its validity, so \(\bar{\phi}\) is finite.

**B. Critical particles**

Using \((26)\), one obtains

\[ X = B_1 L N + O(N^2), \]

\[ Z = Z_1 N + O(N^2), \quad Z_1 = \sqrt{\frac{E^2}{\omega_H^2} \left( B_1^2 - \frac{1}{g_H} \right) - m^2}. \]

As a result,

\[ n \approx \frac{\rho^2 H \alpha H \omega H (B_1 \bar{E} / \omega H - Z_1)}{2\pi Z_1 D(r - r_+)}, \]

so \((29)\) diverges. Eq. \((29)\) corresponds to eq. (48) of \([1]\).

**VI. TWO TYPES OF FRAMES: ROTATIONAL ANALOGUES OF EXPANDING AND CONTRACTING FRAMES**

We saw that, after introducing a regular frame, the number of revolutions becomes a finite, except a rather special case of the critical particle moving around the extremal horizon. Meanwhile, one can notice here a rather interesting peculiarity. Near the horizon, eq. \((14)\) admits two branches for \(z\). Regularization of \(n\) leads to a finite result for a quite definite choice of a sign of \(z\) near the horizon. However, if instead of \((23)\) we take \(z \approx +\rho/\sqrt{\alpha}\), the quantity \(n\) remains infinite, although the metric itself looks regular since \((13)\) is satisfied. How can it happen?

To elucidate this issue, let us consider eq. \((30)\). It follows from it that

\[ \bar{t} = \bar{t} - \int_{r_1}^r \frac{z(r') dr'}{\Delta(r')}, \]

where \(r_1\) is some constant. Let a free particle falls towards a black hole. Using equation of motion \((5)\), we have, choosing the integration constant properly,

\[ \bar{t} = - \int_{r_1}^r \frac{dr'}{\Delta} \left( z \pm \frac{X \rho}{Z \sqrt{\alpha}} \right), \]

where \(\bar{t}(r_1) = 0\). A particle moves with decreasing \(r\), so further \(r < r_1, t > 0\).

For definiteness, let us consider a nonextremal black hole. Then, near the horizon, \(X/Z \approx 1\) as is seen \(r\). Correspondingly, it is the choice \((23)\) that enables us to obtain a finite value for \(\bar{t}\) in other words, a particle moving from the outside towards a horizon, reaches it for a finite interval of \(\bar{t}\), making a finite number of revolutions.

In a similar way, we can consider a particle that moves outward. Then, \(dr/dt > 0\) and, instead of \((31)\) we have

\[ \bar{t} = - \int_{r_1}^r \frac{dr'}{\Delta} \left( z \pm \frac{X \rho}{Z \sqrt{\alpha}} \right). \]

With the same choice \((23)\) we obtain divergent \(\bar{t}\). Also, the angle variable diverges if \(r \to r_+\).

This has a clear analogy with the contracting and expanding Eddington-Finkelstein or Lemaître frames (see, e.g. Sec. 33 in \([10]\), Sec. 2.4 and 2.5 of \([12]\)). The contracting frame describes properly a history of particles falling under the horizon (black hole) but is unable to describe the history of particle appearing from the inner region (white hole). For the expanding system, the situation is opposite. The aforementioned frames are suited for the description of radial motion. Meanwhile, now we dealt with a similar problem for a rotational motion.

**VII. DRAGGING EFFECT FOR PARTICLE SCATTERED BY A BLACK HOLE**

We want to stress the following interesting property of a particle that is scattered by a black hole. Let such a
The black hole angular velocity
\[ \Delta_1 \phi = + \int_{r_m}^{r_1} \eta(r') \, dr', \tag{33} \]
where \( \eta = \frac{\rho^2}{Z} \left( \frac{\partial}{\partial \phi} + \frac{\omega X}{\omega X} \right) \) according to [7]. From \( r_m \) to \( r_1 \) a particle moves with \( dr/dt > 0 \), so that if a particle returns to the same \( r_1 \), it receives an additional change expressed by the same formula [33]. As a result, the full change \( \Delta \phi = 2 \Delta_1 \phi \). If \( r_m \to r_+ \), \( \Delta \phi \to \infty \) due to the factor \( N^2 \) in the denominator of the second term in \( \eta \).

If, instead of \( \phi \), one uses \( \hat{\phi} \), one can easily see that in [15] two integrals containing \( \xi \) mutually cancel, and again we obtain \( \Delta \phi = 2 \Delta_1 \phi \), where \( \Delta_1 \phi \) is given by eq. [33]. In this sense, the coordinate transformation does not change this result. What is especially interesting is that the dragging effect persists and is described by the same formula \( \Delta \phi = 2 \Delta_1 \phi \) even if, by choosing appropriate functions \( z(r) \), \( h(r) \), we can achieve \( \phi = \text{const} \) for the falling in equatorial plane. (But, in the chosen coordinates, the angle variable will not remain constant during motion in the outward direction.)

VIII. KERR METRIC

Let us consider the Kerr metric as an explicit example. Then,
\[ \Delta = r^2 - 2Mr + a^2, \quad \alpha = \frac{1}{\Sigma}, \quad g_\phi = \Sigma, \quad g_\theta = \rho^2, \tag{34} \]
\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Sigma = r^2 + a^2 + \frac{2Mar^2}{\rho^2} \sin^2 \theta, \tag{35} \]
\[ \omega = \frac{2Mar}{\rho^2 \Sigma}, \quad \rho^2 \Sigma = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta. \tag{36} \]
The black hole angular velocity
\[ \omega_H = \omega|_{\Delta = 0} = \frac{a}{r^2 + a^2}. \tag{37} \]
The choice
\[ z = -\sqrt{2Mr(r^2 + a^2)}, \tag{38} \]
\[ h = \frac{\omega \rho^2}{z} = \frac{a\sqrt{2Mr}}{\Sigma \sqrt{r^2 + a^2}}, \tag{39} \]
\[ \xi = \frac{\sqrt{2Mar}}{\sqrt{r^2 + a^2}} \tag{40} \]
in which took into account [11] corresponds to Ref. [2].

FIG. 1. The angles of rotation for particles falling from the point \( r = 9M \) with \( E = m \), \( L = 2mM \) (red lines), \( L = 0 \) (green lines), and \( L = -2mM \) (blue lines) for the coordinates of Natário (the top lines in groups), Doran (the middle lines in groups), and Kerr (the lower lines in groups).

The choice
\[ h = 0, \quad z = -\sqrt{2Mr(r^2 + a^2)}, \]
\[ \xi = -\frac{2Mar\sqrt{2Mr(r^2 + a^2)}}{\rho^2 \Sigma}, \quad \mu = \frac{\rho^2}{\Sigma} \tag{41} \]
corresponds to the Natário’s frame [3]. In the plane \( \theta = \pi/2 \),
\[ \xi = -\frac{2Mar\sqrt{2Mr(r^2 + a^2)}}{r^2 + a^2 + \frac{2Mar^2}{r}} \tag{42} \]
The choice
\[ z = -(r^2 + a^2), \quad \xi = -a \tag{43} \]
corresponds to so-called Kerr coordinates — see Sec. 33.2 of [10] or pp. 163, 164 of [11]. It is also instructive to trace how changes the angle \( \phi \) during motion of a particle when it approaches the black hole horizon. See Fig. 1.

IX. DISCUSSION AND CONCLUSION

It is instructive to compare the above results with those from [1]. For nonextremal black holes, the situation unambiguously agrees with Sec. 5 of [1] and the corresponding line in Table 1 there. For extremal black holes, the situation is more subtle. If a particle is usual, it is shown in [1] that the result depends on a way the angle is measured (see line 3 in Table 1 there). Now, we saw that \( n \) is finite. For critical particles, \( n \) was found in [1] to be infinite in all cases. However, the asymptotic behavior is different depending on the procedure of measurement of the angle (either as \( (r - r_+)^{-1} \) or \( \ln(r - r_+) \)). Now, it behaves like \( (r - r_+)^{-1} \) that agrees with eq. (48) of [1].

Thus we see that the behavior of \( n \) [10] in terms of \( \phi \) coincides exactly with that obtained in [1] for measurement of relative angles \( \phi \) of two particles (this is column
$n_1 - n_2$ in Table 1 there). Now $\phi$ is a well-behaved single coordinate, so one does not need to use the subtraction procedure. Meanwhile, now disagreement is possible between [10] and other methods of computing $n$, as is seen from the same Table 1 [1]. In this sense, the procedure of subtraction used in the relative measurement in [1] is the most natural and is confirmed now by direct calculations of the coordinate behavior.

Also, even without referring to previous results [1], we can formulate an important observation. There is an ultimate connection between two properties: (i) the regularity of a coordinate system that can penetrate inside across the horizon, (ii) a number of revolutions performed by a free falling particle around the horizon. For a nonextremal black hole, a frame can be chosen in such a way that $n$ is always finite. It would be of interest to extend the approach and results of the present work to the case of nonequatorial orbits.

We want to stress that the main our result concerns not the properties of particle motion as such but has rather conceptual nature. We generalized so-called Kerr coordinates in that our approach is applicable to generic stationary axially symmetric black/white holes. There are two different entities: (i) the regularity of the metric near the horizon and (ii) the finiteness of a number of revolution during particle motion. We showed that both properties become regular (finite) simultaneously, if a coordinate transformation is chosen properly.

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