The Symmetry Preserving Removal Lemma

BALÁZS SZEGEDY

September 16, 2008

Abstract

In this note we observe that in the hyper-graph removal lemma the edge removal can be done in a way that the symmetries of the original hyper-graph remain preserved. As an application we prove the following generalization of Szemerédi’s Theorem on arithmetic progressions. If in an Abelian group $A$ there are sets $S_1, S_2, \ldots, S_t$ such that the number of arithmetic progressions $x_1, x_2, \ldots, x_t$ with $x_i \in S_i$ is $o(|A|^2)$ then we can shrink each $S_i$ by $o(|A|)$ elements such that the new sets don’t have such a diagonal arithmetic progression.

1 Introduction

A directed $k$-uniform hyper-graph $H$ on the vertex set $V$ is a subset of $V^k$ such that there is no repetition in the $k$ coordinates. A homomorphism between two directed $k$-uniform hyper-graphs $F$ and $H$ with vertex sets $V(F)$ and $V(H)$ is a map $f : V(F) \to V(H)$ such that $(f(a_1), f(a_2), \ldots, f(a_k))$ is in $H$ whenever $(a_1, a_2, \ldots, a_k)$ is in $F$. The automorphism group $\text{Aut}(H)$ is the group of bijective homomorphisms $\pi : V(H) \to V(H)$. The homomorphism density $t(F, G)$ of $F$ in $G$ is the probability that a random map $f : V(G) \to V(H)$ is a homomorphism.

The so-called hyper-graph removal lemma ([3],[4],[1],[2],[7])(in the directed setting) says the following

**Theorem 1 (Removal Lemma)** For every $k \in \mathbb{N}$, $\epsilon > 0$ and $k$-uniform directed hyper-graph $F$ there is a constant $\delta = \delta(k, \epsilon, F) > 0$ such that for every $k$-uniform directed hyper-graph $G$ with $t(F, G) \leq \delta$ there is a subset $S \subseteq G$ with $S \leq \epsilon|V(G)|^k$ such that $t(F, G \setminus S) = 0$.

Using this deep result we observe that the edge removal can be done in a way that the symmetries of $G$ remain preserved.
Theorem 2 (Symmetry Preserving Removal Lemma) For every $k \in \mathbb{N}$, $\epsilon > 0$ and $k$-uniform directed hyper-graph $F$ there is a constant $\delta_2 = \delta_2(k, \epsilon, F) > 0$ such that for every $k$-uniform directed hyper-graph $G$ with $t(F,G) \leq \delta_2$ there is a subset $S \subseteq G$ with $S \leq \epsilon |V(G)|^k$ such that $t(F,G \setminus S) = 0$ and furthermore $\text{Aut}(G) \subseteq \text{Aut}(G' \setminus S)$.

Proof. Let $V = V(G)$. Using the original removal lemma it remains to show that if $S \subseteq V^k$ satisfies $t(F,G \setminus S) = 0$ then there is $S' \subseteq V^k$ which is $\text{Aut}(G)$ invariant, $t(F,G \setminus S') = 0$ and $S' \leq |F||S|$. Such an $S'$ is the union of those orbits $O$ of $\text{Aut}(G)$ on $V^k$ for which $|O|/|O \cap S| \leq |F|$. Assume that $f : V(F) \rightarrow V$ is a homomorphism from $F$ to $G \setminus S'$. Then for every fixed $e \in F$ and for random element $\pi \in \text{Aut}(G)$ the probability that $\pi(f(e)) \in G \setminus S$ is less than $1/|F|$ and so there is some $\pi \in \text{Aut}(G)$ with $\pi(f(F)) \subseteq G \setminus S$ which is contradiction.

The argument given for the symmetry preserving removal is very general. It applies for various modified versions of the removal lemma. An important such version is the $t$-partite removal lemma where $t$ is a fixed natural number. The vertex set of a $t$-partite $k$-uniform hypergraph is a $t$ tuple $\{V_i\}_{i=1}^t$ of finite sets. An edge of a $t$-partite hypergraph is an element from $\prod_{i=1}^t V_a$ where $a_1, a_2, \ldots, a_k$ are $k$ distinct numbers between 1 and $t$. Let $G_1, G_2$ be two $t$ partite $k$-uniform hyper-graphs with vertex sets $\{V_i\}_{i=1}^t$ and $\{W_i\}_{i=1}^t$. A homomorphism from $G_1$ to $G_2$ is a $t$ tuple of maps $\{\phi_i : V_i \rightarrow W_i\}_{i=1}^t$ such that $(\phi_{a_1}(r_1), \phi_{a_2}(r_2), \ldots, \phi_{a_k}(r_k)) \in \prod_{i=1}^t W_a$ is an edge in $G_2$ whenever $(r_1, r_2, \ldots, r_k) \in \prod_{i=1}^t V_a$ is an edge in $G_1$. An automorphism is a bijective homomorphism from $G_1$ to $G_1$ and the homomorphism density $t(G_1, G_2)$ is the probability that a random $t$ tuple of maps $\{\phi_i : V_i \rightarrow W_i\}_{i=1}^t$ is a homomorphism.

We give an example for an application of the symmetry preserving removal lemma and then we generalize it in the next chapter.

Example 1.: Let $S$ be a subset of a group $T$. The Cayley graph $\text{Cy}(T, S) \subseteq G \times G$ is the collection of pairs $(a, b)$ with $ab^{-1} \in S$. The automorphism group of $\text{Cy}(T, S)$ contains $T$ with the action $(a, b) \mapsto (ag, bg)$. Clearly any subset of $T \times T$ invariant under this action of $G$ is a Cayley graph itself. This means that the $T$-orbit of edges in $\text{Cy}(T, S)$ correspond to elements of $S$. We apply the symmetry preserving removal lemma for $F = \{(1, 2), (1, 3), (2, 3)\}$ with $V(F) = \{1, 2, 3\}$ and for $G = \text{Cy}(T, S)$. A homomorphism from $F$ to $G$ is a map $f : \{1, 2, 3\} \rightarrow T$ such that $a = f(1)f(2)^{-1}$, $b = f(2)f(3)^{-1}$ and $c = f(1)f(3)^{-1}$ are all in $S$. Consequently, the number of such homomorphisms is the number is $|T||\{(a, b, c) : ab = c, a, b, c \in S\}$.
removal lemma yields that if \( ab = c \) has \( o(|T|^2) \) solutions in \( S \) then one can remove \( o(|T|) \) elements from \( S \) such that in the new set there is no solution of \( ab = c \). This was first proved by Ben Green \[9\] for Abelian groups and generalized for groups by Kral, Serra and Vena \[8\].

2 Cayley Hypergraphs

In this chapter we describe a potential way of generalizing Cayley graphs to the hypergraphs setting and then discuss the symmetry preserving removal lemma on such graphs.

**Definition 2.1** Let \( G_1, G_2, \ldots, G_t \) be \( t \) finite groups and let \( H \) be a subgroup of \( \prod_{i=1}^{t} G_i \). The group \( H \) is acting on each \( G_i \) by \( (h_1, h_2, \ldots, h_t)g = h_ig \) where \( (h_1, h_2, \ldots, h_t) \in H \) and \( g \in G_i \). An \( t \)-partite \( k \)-uniform hypergraph \( T \) on the vertex set \( \{ G_1 \}^t_{i=1} \) is called a Cayley hypergraph if its automorphism group contains \( H \) with the previous action.

This definition is very general so we will start to analyze a special setting. Assume that all the groups \( G_1, G_2, \ldots, G_t \) are isomorphic to an Abelian group \( A \). Furthermore, to get something interesting we want to assume that \( H \) is not too big and not too small. Let \( C = \{ C_1, C_2, \ldots, C_r \} \) be a collection of \( k \)-element subsets of \( \{1,2,\ldots,t\} \). Each set \( C_i \) defines a projection \( p_i : H \to A^k \) to the coordinates in \( C_i \). Assume that the factor group \( A^{C_i}/p_i(H) \cong A \) and let \( \psi_i : A^{C_i} \to A \) be a homomorphism with kernel \( p_i(H) \). Now we pick a subsets \( S_i \subseteq A \) for \( 1 \leq i \leq r \) and we define the graph \( H_{k,t}(A,\{S_i\},C) \) as

\[
\bigcup_{i=1}^{r} \psi_i^{-1}(S_i)
\]

where \( \psi_i^{-1}(S_i) \) is the union of cosets in \( A^{C_i} \) of \( p_i(H) \) representing an element in \( S_i \). Note that the way we produced \( H_{k,t}(A,\{S_i\},C) \) guarantees that its automorphism group contains \( H \) as a subgroup.

The symmetry preserving removal lemma for \( t \)-partite hypergraphs directly implies the following lemma:

**Lemma 2.1 (Cayley Hypergraph Removal Lemma)** For every \( k, t \) natural numbers and \( \epsilon > 0 \) there exists a constant \( \delta > 0 \) such that if

\[
t(F, H_{k,t}(A,\{S_i\},C)) \leq \delta
\]

for some \( t \) partite \( k \)-uniform hypergraph \( F \) then there are subsets \( S'_i \) in \( A \) of size at most \( \epsilon |A| \) such that \( t(F, H_{k,t}(A,\{S_i \setminus S_i'\},C)) = 0 \).
Example 2.: This example uses an idea by Solymosi [6] who showed that the Hypergraph Removal Lemma implies Szemeredi’s theorem on arithmetic progressions (even in a multi dimensional setting). Let $t$ be a natural number, $k = t - 1$ and $A$ be an Abelian group. We define $H$ to be the subgroup in $A^t$ of the elements $(a_1, a_2, \ldots, a_t)$ with $\sum_{i=1}^{t} a_i = 0$ and $\sum_{i=1}^{t} (i - 1) a_i = 0$. Now set

$$C = \{\{1, 2, \ldots, i - 1, i, \ldots, t\}\}_{i=1}^{t}$$

and

$$\psi_i(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_t) = \sum_{j=1}^{t} (j - i) a_i.$$ 

The functions $\psi_i$ are computed in a way that $\ker(\psi_i^{-1})$ is the projection of $H$ to the coordinates in $C_i$.

Let $F$ be the complete $t$ partite $t - 1$ uniform hypergraph on four point. Lemma 2.1 applied to $F$ and the above hypergraph $H_{t-1,t}(A, \{S_i\}, C)$ implies that if the system

$$x_i = \sum_{j=1}^{t} (j - i) a_i \in S_i$$

has $o(|A|^t)$ solutions then we can delete $o(|A|)$ elements from each $S_i$ such that the previous system has no solution. It is clear that $x_1, x_2, \ldots, x_t$ are forming a $t$ term arithmetic progression and in fact any such progression with $x_i \in S_i$ gives rise to $|A|^{t-2}$ solution of the previous system. Using this we obtain the following:

**Theorem 3 (Diagonal Szemeredi Theorem)** For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $A$ is an Abelian group, $S_1, S_2, \ldots, S_t$ are subsets in $A$ and there are at most $\delta|A|^2$ $t$-tuples $x_1, x_2, \ldots, x_t$ with $x_i \in S_i$ such that they are forming a $t$ term arithmetic progression then we can shrink each $S_i$ by at most $\epsilon|A|$ elements such that the new sets don’t have such a configuration.

This theorem implies Szemeredi’s theorem [10] if we apply it for $S = S_i$, $1 \leq i \leq t$ since $S$ contains the trivial progressions $a, a, a \ldots, a$ which are only removable if we delete the whole set $S$. 


References

[1] T. Gowers, Quasirandomness, counting and regularity for 3-uniform hypergraphs. Combin. Probab. Comput. 15 (2006), no. 1-2, 143–184.

[2] Y. Ishigami, A Simple Regularization of Hypergraphs

[3] B. Nagle, V. Rödl and M. Schacht, The counting lemma for regular k-uniform hypergraphs. Random Structures Algorithms 28 (2006), no. 2, 113–179.

[4] V. Rödl, M. Schacht, Regular partitions of hypergraphs: regularity lemmas. Combin. Probab. Comput. 16 (2007), no. 6, 833–885.

[5] V. Rödl, J. Skokan, Regularity lemma for k-uniform hypergraphs. Random Structures Algorithms 25 (2004), no. 1, 1–42.

[6] J. Solymosi, A note on a question of Erdős and Graham. Combin. Probab. Comput. 13 (2004), no. 2, 263–267.

[7] T. Tao, A variant of the hypergraph removal lemma. J. Combin. Theory Ser. A 113 (2006), no. 7, 1257–1280.

[8] D. Král, O. Serra, L. Vena A combinatorial proof of the Removal Lemma for Groups preprint

[9] B. Green, A szemerédi type regularity lemma in Abelian groups, with applications Geom. Funct. anal. 15(2005), no. 2, 340-376.

[10] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression Acta Arith. 27 (1975), 199-245