New series with Cauchy and Stirling numbers, 
Part 2

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Abstract

We evaluate in closed form several series involving products of Cauchy numbers with other special numbers (harmonic, skew-harmonic, hyperharmonic, and central binomial). Similar results are obtained with series involving Stirling numbers of the first kind. We focus on several particular cases which give new closed forms for Euler sums of hyperharmonic numbers and products of hyperharmonic and harmonic numbers.

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1 Introduction

The Cauchy numbers $c_n$ are defined by the generating function

\[
\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n \quad (|x| < 1)
\]

(see [6, 9, 16]). They are called Cauchy numbers of the first kind by Comtet [9]. The numbers $c_n/n!$ are also known as the Bernoulli numbers of the second kind (see the comments in [6]). The Cauchy numbers have the important representation

\[
c_n = \int_{0}^{1} z(z-1)\cdots(z-n+1) \, dz \quad (1)
\]

The Stirling numbers of the first kind $s(n,k)$ are defined by the ordinary generating function

\[
z(z-1)\cdots(z-n+1) = \sum_{k=0}^{n} s(n,k) \, z^k
\]
or, equivalently
\[
n! \binom{z}{n} = \sum_{k=0}^{n} s(n,k) z^k
\] (2)

and together with the Cauchy numbers play a major role in this paper. Integrating equation (1) we see that the numbers \(c_n\) can be expressed in terms of \(s(n,k)\) in the following way
\[
c_n = \sum_{k=0}^{n} s(n,k) \frac{k}{k+1}.
\] (3)

The Stirling numbers of the first kind are very popular numbers in mathematics and have various important applications (see the comments and references in [6, 9]). In the recent paper [6] the first author stated the following two propositions:

**Proposition A:** Let \(f(z)\) be a function analytic in a region of the form \(\text{Re}(z) > \lambda\) for some \(\lambda < 0\) and with moderate growth in that region. Then we have the representation
\[
\frac{1}{0} f(x)dx = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) \right\}.
\] (4)

**Proposition B:** Under the same assumptions on the function \(f(z)\), for every \(m \geq 0\) we have the representation
\[
\frac{f^{(m)}(0)}{m!} = \sum_{n=0}^{\infty} \frac{(-1)^n s(n,m)}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) \right\}
\] (5)

and in particular,
\[
f'(0) = \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} f(k) \right\}.
\] (6)

(The summation in (5) de facto starts from \(n = m\) since \(s(n,m) = 0\) for \(n < m\).)

For details see [6]. In that paper various series identities were proved based on these two propositions by applying them to appropriate functions. The purpose of the present paper is to continue this project and present further results in this direction.

In the next section we prove new series identities involving Cauchy numbers and binomial coefficients. The short Section 3 deals with skew-harmonic numbers, while Section 4 is dedicated to series with hyperharmonic numbers. Our results are presented in several examples and propositions.

## 2 Series with Cauchy numbers and binomial coefficients

**Example 1:** In this example we construct the generating functions for the numbers \(c_n \binom{n}{q}\) and \(s(n,m) \binom{n}{q}\) for any integer \(q \geq 0\).
Theorem 1. For any non-negative integer $q$ and every $|z| < 1$ we have the representation

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \binom{n}{q} z^n = (-1)^q \left( \frac{z}{1-z} \right)^q \int_0^1 \left( \frac{x}{q} \right) (1-z)^x \, dx$$  

where

$$\int_0^1 \left( \frac{x}{q} \right) (1-z)^x \, dx = \frac{1}{q!} \sum_{k=0}^{q} s(q, k) A_k$$

with

$$A_k = k! \left( \frac{1}{(-\ln(1-z))^{k+1}} - (1-z) \sum_{j=0}^{k} \frac{1}{j!(-\ln(1-z))^{k-j+1}} \right).$$

In particular, with $z = 1/2$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \binom{n}{q} z^n = \frac{(-1)^q}{q!} \sum_{k=0}^{q} s(q, k) A_k$$

with

$$A_k = \int_0^1 x^k 2^{-x} \, dx = k! \left( \frac{1}{(\ln 2)^{k+1}} - \frac{1}{2} \sum_{j=0}^{k} \frac{1}{j!(\ln 2)^{k-j+1}} \right).$$

Proof. For the proof we use the binomial formula (see [4, Eq. (10.25)])

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{k}{q} \alpha^k = (-\alpha)^q (1-\alpha)^{n-q} \binom{n}{q}$$

where $0 \leq \alpha \leq 1$. We take $f(x) = \binom{x}{q} \alpha^x$ to get from [4]

$$\int_0^1 \left( \frac{x}{q} \right) \alpha^x \, dx = \frac{(-\alpha)^q}{1-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n c_n (1-\alpha)^n}{n!} \binom{n}{q}.$$  

When $\alpha = 1$ this becomes the familiar

$$c_q = q! \int_0^1 \left( \frac{x}{q} \right) \, dx.$$
Now equation \(7\) follows from \(10\) with \(z = 1 - \alpha\). For the evaluation of the integral we use equation \(2\) and entry 3.352(1) from [14].

Next, applying Proposition B to the same function \(f(x) = (\frac{x}{q})^{\alpha x}\) and using again the binomial identity \(9\) we come to the following result.

**Proposition 2** For every non-negative integer \(q\) and every \(0 \leq z < 1\) we have for \(m = 1, 2, \ldots\)

\[
\sum_{n=0}^{\infty} \frac{(-1)^n s(n, m)z^n}{n!} \binom{n}{q} = \frac{(-1)^q}{m!} \left( \frac{z}{1-z} \right)^q \left( \frac{d}{dx} \right)^m (1-z)^x \bigg|_{x=0}.
\]

(11)

For \(m = 1\) using the formula

\[
\frac{d}{dx} \left( \frac{x}{q} \right) = \frac{x}{q} \sum_{j=0}^{q-1} \frac{1}{x-j} = \frac{x}{q} \frac{x}{x} + \left( \frac{x}{q} \right) \sum_{j=1}^{q-1} \frac{1}{x-j}
\]

we compute

\[
\lim_{x \to 0} \left\{ \frac{d}{dx} \left( \frac{x}{q} \right) \right\} = \frac{(-1)^{q-1}}{q}.
\]

Also \(s(n, 1) = (-1)^{n-1}(n-1)!\) so that (11) takes the form

\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} \binom{n}{q} = \frac{1}{q} \left( \frac{z}{1-z} \right)^q
\]

which is equivalent to the well-known expansion

\[
\sum_{n=0}^{\infty} \binom{n}{q} z^n = \frac{z^q}{(1-z)^{q+1}}.
\]

**Example 2:** In this example we use the central binomial coefficients \(\binom{2n}{n} \).

We start with the binomial formula [4, Eq. (10.35a)]

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(2k-1)^n}{4^n} = \binom{2n}{n} \frac{1}{4^n}.
\]

(12)

So we apply \(4\) to the function

\[
f(x) = \left( \frac{2x}{x} \right) \frac{1}{4^x} = \frac{\Gamma(2x+1)}{\Gamma^2(x+1)4^x}
\]

to get from \(4\) and \(6\)

**Proposition 3**

\[
\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n! 4^n} \binom{2n}{n} = \int_0^{1} \left( \frac{2x}{x} \right) \frac{1}{4^x} dx \approx 0.6703837612
\]

(13)
and also for \( m = 1, 2, \ldots \)

\[
\frac{1}{m!} \left( \frac{d}{dx} \right)^m \left( \frac{2x}{x} \right)^{1/4} \bigg|_{x=0} = \sum_{n=1}^{\infty} \frac{(-1)^n s(n, m)}{n!4^n} \binom{2n}{n}. \quad (14)
\]

For \( m = 1, 2, 3 \) we have correspondingly

\[
\sum_{n=1}^{\infty} \frac{1}{n4^n} \binom{2n}{n} = \ln 4 \quad (15)
\]

\[
\sum_{n=1}^{\infty} \frac{H_{n-1}}{n4^n} \binom{2n}{n} = \frac{\pi^2}{6} + 2 \ln^2 2 \quad (16)
\]

\[
\sum_{n=1}^{\infty} \frac{(H_{n-1}^2 - H_{n-1}^{(2)})}{n4^n} \binom{2n}{n} = 4\zeta(3) + \frac{8}{3} \ln^3 2 + \frac{2\pi^2}{3} \ln 2 \quad (17)
\]

as \( s(n, 2) = (-1)^{n-2}(n-1)!H_{n-1} \) and \( s(n, 3) = (-1)^{n-3}(n-1)! \frac{1}{2}(H_{n-1}^2 - H_{n-1}^{(2)}) \).

Here

\[
H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \quad (k \geq 1), \quad H_0 = 0
\]

and

\[
H_k^{(2)} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \quad (k \geq 1), \quad H_0^{(2)} = 0.
\]

The first series (15) is known (see [19, Eq.(6)]). All these series are very slowly convergent.

**Example 3:** The starting point is the binomial identity ([4, Eq.(10.9)])

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{p+k}{k} = (-1)^n \binom{p}{n},
\]

where \( p \geq 0 \) is an integer. With the function \( f(x) = \binom{p+x}{p} \) we find from [4] the identity

\[
\int_0^1 \binom{p+x}{p} \ dx = \sum_{n=0}^{p} \frac{cn}{n!} \binom{p}{n}. \quad (18)
\]

We have

\[
\binom{p+x}{p} = \frac{(p+x)(p+x-1) \cdots (x+1)}{p!}
\]

\[
= (-1)^{p+1} \frac{(-x)(-x-1) \cdots (-x-(p+1)+1)}{p!}
\]

\[
= (-1)^{p+1} \sum_{k=0}^{p+1} (-1)^{k-1} s(p+1, k) x^{k-1}.
\]

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Then (after changing the index \( k = j + 1 \)) we get
\[
\int_0^1 \left( \frac{p + x}{p} \right) dx = (-1)^{p+1} \sum_{k=1}^{p+1} \frac{(-1)^{k-1} s(p + 1, k)}{k} \]
\[
= (-1)^{p+1} \sum_{j=0}^{p} \frac{(-1)^{j} s(p + 1, j + 1)}{j + 1}.
\]

Using the explicit expression of the Cauchy polynomials of the second kind \( \hat{c}_n(x) \) \[18, Theorem 2\]
\[
\hat{c}_n(-r) = \sum_{k=0}^{n} s_r(n,k) \frac{(-1)^k}{k + 1},
\]
where \( s_r(n,k) \) is the \( r \)-Stirling numbers of the first kind \[7\], and \( s_1(n,k) = s(n + 1, k + 1) \), we obtain
\[
\sum_{n=0}^{P} \frac{c_n}{n!} \binom{p}{n} = \frac{(-1)^P}{p!} \hat{c}_p(-1). \tag{19}
\]

With the function \( f(x) = \left( \frac{p + x}{p} \right) \), \[6\] implies
\[
\frac{1}{m!} \left( \frac{d}{dx} \right)^m \left( \binom{p + x}{p} \right) \bigg|_{x=0} = \sum_{n=m}^{P} \frac{s(n,m)}{n!} \binom{p}{n}.
\]

Moreover, it is known that \[20, Theorem 2\] \((n \geq m > 0)\)
\[
\left( \frac{d}{dx} \right)^m \left( \binom{n + x}{r} \right) \bigg|_{x=0} = (-1)^m \binom{n}{r} \ Y_m \left(-0! (H_n - H_{n-r}), \ldots, -(m-1)! (H^{(m)}_n - H^{(m)}_{n-r})\right),
\]
where \( H^{(m)}_n \) is the \( n \)th generalized harmonic number defined by
\[
H^{(m)}_n = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m} \quad (m \geq 1), \quad H^{(m)}_0 = 0
\]
and \( Y_i(t_1, t_2, \ldots, t_i) \) is the exponential complete Bell polynomial \[9, Sect. 3.3\].

Thus, we have
\[
\sum_{n=m}^{P} s(n,m) \binom{p}{n} = \frac{(-1)^m}{m!} \ Y_m \left(-0! H_p, \ldots, -(m-1)! H^{(m)}_p\right). \tag{20}
\]

The following proposition summarizes these results

**Proposition 4** For every non-negative integers \( m \) and \( p \), \[19\] and \[20\] are true.
For $m = 1$, (20) reduces to [4, Eq. (9.2)]

$$\sum_{n=1}^{p} \binom{p}{n} \frac{(-1)^{n+1}}{n} = H_p.$$  

For $m = 2$ and $m = 3$, we find correspondingly

$$\sum_{n=1}^{p} (-1)^{n+1} \binom{p+1}{n+1} \frac{H_n}{n+1} = \frac{H_{p+1}^2 - H_{p+1}^{(2)}}{2}$$

$$\sum_{n=1}^{p} (-1)^{n-1} \binom{p+2}{n+2} \frac{H_{n+1}^2 - H_{n+1}^{(2)}}{n+2} = \frac{H_{p+2}^3 - 3H_{p+2}H_{p+2}^{(2)} + H_{p+2}^{(3)}}{3}.$$  

**Example 4:** Here we give a new explicit formula, extending (3) and [16, Eq.(17)]. If $q > n$, $\binom{n}{q} = 0$.

Then (7) can be rewritten as

$$\int_{0}^{1} \left( \frac{x}{q} \right) (1 - z)^q \, dx = \sum_{n=q}^{\infty} \frac{(-1)^{n-q} c_n}{(n-q)!} z^n (1 - z)^q$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n!} z^n (1 - z)^q$$

Setting $z \to 1 - e^{-t}$ in the above, we have

$$\sum_{n=0}^{\infty} c_{n+q} \frac{(e^{-t} - 1)^n}{n!} e^{-tq} = q! \int_{0}^{1} \left( \frac{x}{q} \right) e^{-xt} \, dx.$$  

Using the generating function of the $r$-Stirling numbers of the second kind [7]

$$\frac{(e^t - 1)^n}{n!} e^{tr} = \sum_{k=n}^{\infty} S_r(k, n) \frac{t^k}{k!}$$

gives

$$\sum_{n=0}^{\infty} c_{n+q} \frac{(e^{-t} - 1)^n}{n!} e^{-tq} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \sum_{n=0}^{k} S_q(k, n) c_{n+q}.$$  

On the other hand, using (2) and Maclaurin’s expansion of $e^x$, we have

$$\int_{0}^{1} \left( \frac{x}{q} \right) e^{-xt} \, dx = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \sum_{m=0}^{q} \frac{s(q, m)}{k + m + 1}.$$  

Then comparing the coefficients of $\frac{(-t)^k}{k!}$ yields

$$\sum_{n=0}^{k} S_q(k, n) c_{n+q} = \sum_{m=0}^{q} \frac{s(q, m)}{k + m + 1}.$$  

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Finally, utilizing the $r$-Stirling transform, given by
\[ a_n = \sum_{k=0}^{n} S_r(n,k) b_k \ (n \geq 0) \] if and only if \[ b_n = \sum_{k=0}^{n} s_r(n,k) a_k \ (n \geq 0), \]
we obtain the following:

**Proposition 5** For any non-negative integers $k$ and $q$

\[ c_{k+q} = \sum_{n=0}^{k} \sum_{m=0}^{q} s_q(k,n) s(q,m). \]

**Example 5:** Now we evaluate an infinite series involving Cauchy numbers with shifted indices.

**Proposition 6** For every non-negative integer $q$

\[ \sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n!(n+q+1)\cdots(n+2q+1)} = \sum_{j=0}^{q} (q+j) \frac{1}{j!} \prod_{i=1}^{j} (1+q+i) \ln\left(\frac{j+1}{j+q+1}\right). \]

**Proof.** From (7), we have

\[ z^q \int_{0}^{1} \frac{x^q}{(1-z)^{q+1}} \ dx = \sum_{n=q}^{\infty} \frac{(-1)^{n-q} c_n}{(n-q)!} z^n (1-z)^q \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n! q!} z^{n+q} (1-z)^q. \]

Integrating both sides of the above with respect to $z$ from 0 to 1 and using well-known identity

\[ B(p,q) = \int_{0}^{1} z^{p-1} (1-z)^{q-1} \ dz = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)}, \]

we have

\[ \sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n!(n+q+1)\cdots(n+2q+1)} = q! \int_{0}^{1} \frac{x^q}{(1-q)^{x+q+2}} \ dx. \]

Utilizing (21), the properties $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(n+1) = n! \ (n \in \mathbb{N})$ and

\[ \frac{1}{(x+1)\cdots(x+q+1)} = \frac{1}{(q+1)!} \sum_{j=1}^{q+1} (-1)^{j-1} \binom{q+1}{j} \frac{j}{x+j}, \quad (21) \]
we obtain
\[
\sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n! (n + q + 1) \cdots (n + 2q + 1)} = \frac{1}{(q + 1)!} \sum_{j=1}^{q+1} \sum_{k=0}^{q} (-1)^{j-1} \binom{q+1}{j} s(q,k) j \int_0^1 \frac{x^k}{x+j} \, dx.
\]

One can see that
\[
\int_0^1 \frac{x^k}{x+j} \, dx = (-1)^k j^k \ln \left( \frac{j+1}{j} \right) + \sum_{m=1}^{k} \binom{k}{m} (-j)^{k-m} \ln \frac{j+1}{j+m} \left( (j+1)^m - j^m \right).
\]

Then we have
\[
\sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n! (n + q + 1) \cdots (n + 2q + 1)}
\]
\[
= \frac{1}{(q + 1)!} \sum_{j=1}^{q+1} (-1)^{j+k-1} \binom{q+1}{j} \ln \left( \frac{j+1}{j} \right) \sum_{k=0}^{q} (-1)^{q-k} s(q,k) j^{k+1}
\]
\[
+ \frac{1}{(q + 1)!} \sum_{j=1}^{q+1} \sum_{k=0}^{q} \sum_{m=1}^{k} (-1)^{j+m-1} \binom{q+1}{j} s(q,k) \binom{k}{m} \frac{j^{k-m+1}}{m} \left( (j+1)^m - j^m \right).
\]

The second sum of the right-hand side is zero since
\[
\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^j = 0 \text{ for } j < n.
\]

Moreover, using
\[
\sum_{k=0}^{q} (-1)^{q-k} s(q,k) j^k = j (j+1) \cdots (j+q-1),
\]
we come to the desired result. 

Example 6: In this example we use the reciprocal binomial coefficients \( \binom{n+l}{l}^{-1} \), where \( l \) is any non-negative integer. We take \( f(x) = \frac{1}{(x+l+1)\cdots(x+l+r)} \) and use [17, Proposition 2],
\[
\frac{1}{(r-1)! (n+r) \binom{n+l}{l}^{-1}} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+l+1) \cdots (k+l+r)} \quad (22)
\]
and [21]. Thus (4) and (5) imply that

**Proposition 7** For any integer \( r \geq 1 \)
\[
\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n! (n + r) \binom{n+l}{l}^{-1}} = \sum_{j=1}^{r} (-1)^{j-1} \binom{r-1}{j-1} \ln \frac{l+j+1}{l+j},
\]
\[
\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n,m)}{n! (n + r) \binom{n+l}{l}^{-1}} = \sum_{j=1}^{r} (-1)^{j-1} \binom{r-1}{j-1} \frac{1}{(l+j)^{m+1}}. \quad (23)
\]
When \( r = l + 1 \), these sums become

\[
\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{(n + l + 1)!} = \frac{1}{l!} \sum_{j=0}^{l} (-1)^j \binom{l}{j} \ln \left( \frac{l+j+2}{l+j+1} \right),
\]

(24)

\[
\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n, m)}{(n + l + 1)!} = \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} \frac{(-1)^j}{(l+j+1)^{m+1}}.
\]

(25)

Setting \( l = 0 \) in (23), we reach that

\[
\sum_{n=m}^{\infty} \frac{(-1)^{n-m}s(n, m)}{n!(n + r)} = (-1)^{r+1} (r-1)! S(-m, r),
\]

(26)

where \( S(-n, r) \) is the Stirling numbers of the second kind with negative integral values, defined by [2]

\[
\frac{(-1)^r}{r!} \sum_{j=1}^{r} \binom{r}{j} \frac{(-1)^j}{j^n} = S(-n, r).
\]

(27)

With the use of [20, Theorem 4], we can list some special cases as follows:

\[
S(0, r) = \frac{(-1)^{r+1}}{r!}, \quad S(-1, r) = \frac{(-1)^{r+1}}{r!} H_r, \quad S(-2, r) = \frac{(-1)^{r+1}}{2r!} \left( H_r^2 + H_r^{(2)} \right).
\]

It is good to note that (29) is slightly different from [23, Corollary 2.4]. (See [1, 15, 21, 22] for more examples of series involving Stirling numbers of the first kind.)

### 3 Series with skew-harmonic numbers.

We work here with the skew-harmonic numbers

\[
H_n^- = 1 - \frac{1}{2} + \frac{1}{3} + \ldots + \frac{(-1)^{n-1}}{n} \quad (n \geq 1), \quad H_0^- = 0.
\]

**Example 7:** Applying the binomial formula [4 Eq.(9.21)]

\[
\sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{1-2^k}{k} = H_n^-
\]

(28)

we use the (entire) function

\[
f(x) = \frac{1 - 2^x}{x} = - \sum_{n=0}^{\infty} \frac{(\ln 2)^{n+1} x^n}{(n+1)!}
\]
where \( f(0) = -\ln 2 \). With summation from \( k = 0 \) the binomial formula \([28]\) takes the form
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = -\ln 2 + H_n^-.
\]
Proposition A implies
\[
\int_{0}^{1} \frac{1 - 2^x}{x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \left\{ -\ln 2 + H_n^- \right\}.
\]
The integral can be computed this way: with \( t = x \ln 2 \)
\[
\int_{0}^{1} \frac{2^x - 1}{x} \, dx = \sum_{n=1}^{\infty} \frac{(\ln 2)^n}{n^n} = -Ein(-\ln 2)
\]
where
\[
Ein(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n! n}
\]
is the entire exponential integral function. This gives the evaluation

**Proposition 8**
\[
\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \left\{ H_n^- - \ln 2 \right\} = Ein(-\ln 2). \tag{29}
\]

Note that
\[
\lim_{n \to \infty} \left\{ H_n^- - \ln 2 \right\} = 0
\]
since \( \lim_{n \to \infty} H_n^- = \ln 2 \).

### 4 Series with hyperharmonic numbers

In this section, we work with the hyperharmonic numbers which are defined by the equation \([10]\)
\[
h_n^{(r)} = \binom{n + r - 1}{r - 1} \left( H_{n+r-1} - H_{r-1} \right). \tag{30}
\]
Please see \([3, 8, 10, 12, 13]\) for more detail on hyperharmonic numbers.

**Example 8:** Let \( r \) be an integer \( \geq 1 \). We use the function \( f(x) = \frac{1}{(x+1)^2(x+2)\cdots(x+r)} \)
together with the binomial identity \([17]\) Theorem 3

\[
\frac{h_n^{(r+1)}}{(n+1)\cdots(n+r)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^2(k+2)\cdots(k+r)} \tag{31}
\]
in Proposition A and Proposition B to obtain the following proposition.
Proposition 9

\[ \sum_{n=0}^{\infty} \frac{(-1)^n c_n h_{n+1}^{(r)}}{(n+r)!} = \frac{1}{2} \frac{r-1}{r!} \ln (2) + \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} \frac{(-1)^{j+1}}{j-1} \ln \left( \frac{2j^j}{(j+1)^j} \right) \]  

and

\[ \sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n,m) h_{n+1}^{(r)}}{(n+r)!} = (-1)^{r+1} \sum_{k=0}^{m} S(-k,r). \]  

Proof. One have

\[ \frac{1}{(x+1)^{2}(x+2) \cdots (x+r)} = \frac{1}{(r-1)! (x+1)^2} - \frac{1}{r!} \frac{1}{x+1} + \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} \frac{(-1)^{j+1}}{j-1} \left( \frac{1}{x+1} - \frac{j}{x+j} \right). \]  

Using Proposition A in the above equation yields (32).

For the proof of (33), we first use

\[ \frac{1}{(1+x)^{m}} = \sum_{n=0}^{\infty} \frac{(-1)^m (\alpha + m - 1)_n x^n}{m!} \]  

in (34) to obtain

\[ \sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n,m) h_{n+1}^{(r)}}{(n+r)!} = \frac{rm+1}{r!} + \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} \frac{(-1)^{j+1}}{j-1} \left( 1 - \frac{1}{j^m} \right) \]  

Then utilizing \( \sum_{k=1}^{n} x^k = (1-x^n)/(1-x) \) in the above equation we have

\[ \sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n,m) h_{n+1}^{(r)}}{(n+r)!} = \frac{rm+1}{r!} + \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} \frac{(-1)^{j+1}}{j^m} \sum_{k=0}^{m-1} j^k \]  

Then utilizing \( \sum_{k=0}^{m} \sum_{j=1}^{r} \binom{r}{j} \frac{(-1)^{j+1}}{j^{m-k}} \) in the above equation we have

With the use of (27), we come to the desired result. \( \blacksquare \)

It is good to note that the case \( r = 1 \) of (32) and (33) are given in [6]. Moreover, setting \( r = 2 \) in (32), \( m = 1 \) and \( m = 2 \) in (33) give

\[ \sum_{n=0}^{\infty} \frac{(-1)^n c_n H_{n+2}}{(n+1)!} = \ln 3 - \ln 2 + \frac{1}{2}, \]
\[
\sum_{n=1}^{\infty} \frac{h_{n+1}^{(r)}}{n^{(n+r)}} = 1 + H_r, \quad \sum_{n=2}^{\infty} \frac{H_{n-1} h_{n+1}^{(r)}}{n^{(n+r)}} = 1 + H_r + \frac{H_r^2 + H_r^{(2)}}{2},
\]
respectively.

**Example 9:** Here we exploit the binomial formula \[17, \text{Theorem } 9\]
\[
\frac{-h_n^{(r)}}{(n+1)\cdots(n+r)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)\cdots(k+r)} H_k,
\]
and the function \(f(x) = \frac{\psi(x+1)+\gamma}{(x+1)\cdots(x+r)}\). With the use of \((21)\) and
\[
\psi(x+1) + \gamma = \sum_{n=1}^{\infty} (-1)^{n+1} \zeta(n+1) x^n, \quad |x| < 1.
\]
we have
\[
\frac{\psi(x+1) + \gamma}{(x+1)\cdots(x+r)} = \frac{1}{r!} \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} \sum_{n=1}^{\infty} (-1)^{n+1} x^n \left( \sum_{k=1}^{n} \frac{\zeta(k+1)}{j^{m-k}} \right).
\]

Then using \((27)\) and \((5)\) give the following evaluation.

**Proposition 10**
\[
\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n, m) h_n^{(r)}}{(n+r)!} = (-1)^{r+1} \sum_{k=1}^{m} S(k-m, r) \zeta(k+1).
\]

The case \(r = 1\) was discussed in \[5\]. Setting \(m = 1\) in \((37)\) gives \[11, \text{Eq.}(27)\]. Moreover, \(m = 2\) and \(m = 3\) in \((37)\) yield
\[
\sum_{n=2}^{\infty} \frac{H_{n-1} h_n^{(r)}}{n^{(n+r)}} = H_r \frac{\pi^2}{6} + \zeta(3),
\]
\[
\sum_{n=3}^{\infty} \frac{(H_{n-1}^2 - H_{n-1}^{(2)}) h_n^{(r)}}{n^{(n+r)}} = \left( H_r^2 + H_r^{(2)} \right) \frac{\pi^2}{6} + 2H_r \zeta(3) + \frac{\pi^4}{45}
\]
respectively.

**Example 10:** In this example we take \(f(x) = \frac{\psi(x+1)+\gamma}{(x+1)\cdots(x+r)}\) and use the identity \[17, \text{Theorem } 11\]
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{(k+1)(k+1)^{(r)}} H_k = \frac{1}{2(n+1)(r-1)!} \left( (H_{n+r} - H_{r-1})^2 - \left( H_{n+r}^{(2)} - H_{r-1}^{(2)} \right) \right).
\]
Using (34) and (36), we have
\[
\psi (x + 1) + \gamma \frac{1}{(x + 1)^{2}} (x + 2) \cdots (x + r) \\
= \frac{1}{(r - 1)!} \sum_{m=1}^{\infty} (-1)^{m-1} x^{m} \sum_{k=1}^{m} (m - k + 1) \zeta (k + 1) \\
\left[ \sum_{m=1}^{\infty} (-1)^{m-1} x^{m} \sum_{k=1}^{m} \zeta (k + 1) \right] \\
+ \frac{1 - r}{r!} \sum_{m=1}^{\infty} (-1)^{m-1} x^{m} \sum_{k=1}^{m} \zeta (k + 1) \\
+ \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} \frac{(-1)^{j+1}}{j - 1} \left[ \sum_{m=1}^{\infty} (-1)^{m-1} x^{m} \sum_{k=1}^{m} \zeta (k + 1) \right] \\
+ \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} \frac{(-1)^{j}}{(j - 1)} \left[ \sum_{m=1}^{\infty} (-1)^{m-1} x^{m} \sum_{k=1}^{m} \frac{\zeta (k + 1)}{j^{m-k}} \right].
\]

From (5) and some arrangements we obtain
\[
\sum_{n=m}^{\infty} \left( (-1)^{n-m+1} s (n, m) \left[ (H_{n+r} - H_{r-1})^2 - (H_{n+r}^{(2)} - H_{r-1}^{(2)}) \right] \right) \\
\frac{2 (n + 1)!}{2 (n + 1)!} = \frac{1}{r} \sum_{k=1}^{m} \zeta (k + 1) + \frac{1}{r} \sum_{k=1}^{m} \zeta (k + 1) \sum_{j=1}^{r} \binom{r}{j} \frac{(-1)^{j+1}}{j^{m-k}} \left( 1 + j + \cdots + j^{m-k-1} \right).
\]

Then using (27) yields the following:

**Proposition 11**
\[
\sum_{n=m}^{\infty} \left( (-1)^{n-m+1} s (n, m) \left[ (H_{n+r} - H_{r-1})^2 - (H_{n+r}^{(2)} - H_{r-1}^{(2)}) \right] \right) \\
\frac{2 (n + 1)!}{2 (n + 1)!} = \frac{1}{r} \sum_{k=0}^{m-1} A_k (r) \zeta (m - k + 1) \tag{38}
\]

where
\[
A_k (r) = 1 + (-1)^{r+1} r! \sum_{l=1}^{k} S (-l, r).
\]

Since \( S (-l, 1) = 1 \), for \( r = 1 \) in (38), we have
\[
\sum_{n=m}^{\infty} \left( (-1)^{n-m} s (n, m) \left[ H_{n+1}^{(2)} - H_{n+1}^{2} \right] \right) \\
\frac{2 (n + 1)!}{2 (n + 1)!} = \sum_{k=1}^{m} (m - k + 1) \zeta (k + 1) \tag{39}
\]

On the other hand, we have \( H_{n+1}^{2} + H_{n+1}^{(2)} \)
\[
\sum_{n=m}^{\infty} \left( (-1)^{n-m} s (n, m) \left[ H_{n+1}^{2} + H_{n+1}^{(2)} \right] \right) \\
\frac{2 (n + 1)!}{2 (n + 1)!} = (m + 1) (m + 2).
\]
Combining this result with \(39\), we reach that
\[
\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n, m) H_{n+1}^{(2)}}{2(n+1)!} = (m+1)(m+2) + \sum_{k=1}^{m} (m-k+1) \zeta(k+1),
\]
\[
\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n, m) H_{n+1}^{2}}{2(n+1)!} = (m+1)(m+2) - \sum_{k=1}^{m} (m-k+1) \zeta(k+1)
\]
Setting \(m = 1, 2, 3\) in the above give
\[
\sum_{n=1}^{\infty} \frac{H_{n+1}^{(2)}}{n(n+1)} = 12 + \frac{\pi^2}{3}, \quad \sum_{n=1}^{\infty} \frac{H_{n+1}^{2}}{n(n+1)} = 12 - \frac{\pi^2}{3},
\]
\[
\sum_{n=2}^{\infty} \frac{H_{n-1}H_{n+1}^{(2)}}{n(n+1)} = 24 + \frac{2\pi^2}{3} + 2\zeta(3),
\]
\[
\sum_{n=2}^{\infty} \frac{H_{n-1}H_{n+1}^{2}}{n(n+1)} = 24 - \frac{2\pi^2}{3} - 2\zeta(3)
\]
\[
\sum_{n=3}^{\infty} \frac{(H_{n-1}^{2} - H_{n-1}^{(2)})H_{n+1}^{(2)}}{n(n+1)} = 80 + 2\pi^2 + 8\zeta(3) + \frac{2\pi^4}{45},
\]
\[
\sum_{n=3}^{\infty} \frac{(H_{n-1}^{2} - H_{n-1}^{(2)})H_{n+1}^{2}}{n(n+1)} = 80 - 2\pi^2 - 8\zeta(3) - \frac{2\pi^4}{45}.
\]

**Example 11.** Let \(r\) be an integer > 1. Then we use the identity \[11\]
\[
\frac{-h_{n}^{(r)}}{(n+r-1)!} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k}{(k+r-1)!}
\]
together with the function \(f(x) = \frac{x}{(x+r-1)^2}\). For \(|x| < |r-1|\) we have the Taylor series
\[
f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (m+1)}{(r-1)^{m+2}} x^{m+1}.
\]
From \[4\] and \[5\] we find the representations below:

**Proposition 12** Let \(r\) be an integer > 1. Then we have
\[
\sum_{n=0}^{\infty} \frac{(-1)^{n+1} c_n h_{n}^{(r)}}{n!(n+r-1)!^2} = \ln\left(\frac{r}{r-1}\right) - \frac{1}{r}, \quad (40)
\]
\[
\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n, m) h_{n}^{(r)}}{n!(n+r-1)!^2} = \frac{(m+1)}{(r-1)^{m+2}}, \quad (41)
\]
For \(m = 1\) and \(m = 2\) in \[41\] we find that
\[
\sum_{n=1}^{\infty} \frac{h_{n}^{(r)}}{n!(n+r-1)!^2} = \frac{2}{(r-1)^3} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{H_{n-1}h_{n}^{(r)}}{n!(n+r-1)!^2} = \frac{3}{(r-1)^4}.
\]
At the end of this section we want to note that if we apply (4) to the functions and binomial identities given in Examples 9 and 10, we come to very challenging integrals.

\[
\sum_{n=0}^{\infty} \frac{(-1)^{n+1} c_{n} H_{n}^{(r)}}{(n+r)!} = \int_{0}^{1} \frac{\psi(x+1) + \gamma}{(x+1) \cdots (x+r)} dx, \\
\]

and

\[
\frac{1}{(r-1)!} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} c_{n} \left\{ (H_{n+r} - H_{r-1})^2 - \left( H_{n+r}^{(2)} - H_{r-1}^{(2)} \right) \right\}}{(n+1)! (n+r)!} = \int_{0}^{1} \frac{\psi(x+1) + \gamma}{(x+1)^2 (x+2) \cdots (x+r)} dx.
\]

The first integral when \( r = 1 \) is

\[
\int_{0}^{1} \frac{\psi(x+1) + \gamma}{x+1} dx.
\]

Multiplying (39) by the geometric series \( (1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n \), we come to the expansion

\[
\frac{\psi(x+1) + \gamma}{x+1} = \sum_{n=1}^{\infty} x^n \left\{ (-1)^{n-1} \sum_{k=1}^{n} \zeta (k+1) \right\}.
\]

So we have

\[
\int_{0}^{1} \frac{\psi(x+1) + \gamma}{x+1} dx = \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n-1}}{n+1} \sum_{k=1}^{n} \zeta (k+1) \right\} \approx 0.3606201929.
\]

(This complements the results in [5].)

References

[1] Victor Adamchik, On Stirling numbers and Euler sums, J. Comput. Appl. Math. 79(1) (1997) 119–130.

[2] David Branson, An extension of Stirling numbers. Fibonacci Quart., 34: 1996, 213-222.

[3] Benjamin A.T., Gaebler D., Gaebler R.: A combinatorial approach to hyperharmonic numbers. Integers 3, #A15 (2003)

[4] Khristo N. Boyadzhiev, Notes on the Binomial Transform, Theory and Table, World Scientific, 2018.
[5] Khristo N. Boyadzhiev, A special constant and series with zeta values and harmonic numbers, Gazeta Matematica, Seria A, vol. 115 (3-4) (2018), 1-16.

[6] Khristo N. Boyadzhiev, New series identities with Cauchy, Stirling, and harmonic numbers, and Laguerre polynomials, J. Integer Seq., 23 (2020), Article 20.11.7.

[7] Andrei Zary Broder, The $r$-Stirling numbers, Discrete Math. 49 (1984) 241-259.

[8] Mümün Can, Muhammet C. Dağlı, Extended Bernoulli and Stirling matrices and related combinatorial identities. Linear Algebra Appl. 444, 114–131 (2014)

[9] Louis Comtet, Advanced Combinatorics, Kluwer, 1974.

[10] John H. Conway and Richard Guy, The book of numbers, Springer, 1996.

[11] Ayhan Dil and Khristo Boyadzhiev, Euler sums of hyperharmonic numbers. J. Number Theory 147 (2015) 490–498.

[12] Ayhan Dil and Istvan Mező, A symmetric algorithm for hyperharmonic and Fibonacci numbers. Appl. Math. Comp. 206, 942–951 (2008)

[13] Ayhan Dil and Erkan Muniroğlu, Applications of derivative and difference operators on some sequences, Appl. Anal. Discrete Math., 14 (2020), 406–430.

[14] Izrail S. Gradshteyn and Iosif M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, 1980.

[15] Charles Jordan, Calculus of finite differences, Chelsea, New York, 1950 (First edition: Budapest 1939).

[16] Levent Kargin. On Cauchy Numbers and Their Generalizations, Gazi University Journal of Science, 33(2) (2020), 456-474.

[17] Levent Kargin and Mümün Can, Harmonic number identities via polynomials with $r$-Lah coefficients, Comptes Rendus. Mathématique, 358(5), 535-550 (2020).

[18] Takao Komatsu and Istvan Mező. Several explicit formulae of Cauchy polynomials in terms of $r$-Stirling numbers, Acta Math. Hungar., 148.2 (2016), 522-529.

[19] Derrick Henry Lehmer, Interesting series involving the central binomial coefficient, Amer. Math. Monthly, 92 (1985), 449–457.

[20] Weiping Wang, Cangzhi Jia, Harmonic number identities via the Newton–Andrews method, Ramanujan J. 35 (2014), 263–285.
[21] Weiping Wang, and Lyu Yanhong, Euler sums and Stirling sums, J. Number Theory 185 (2018), 160-193.

[22] Ce Xu, Yan Yuhuan and Shi Zhijuan, Euler sums and integrals of polylogarithm functions, J. Number Theory 165 (2016) 84–108.

[23] Ce Xu, Mingyu Zhang, Weixia Zhu, Some evaluation of harmonic number sums, Integral Transforms Spec. Funct., 27(12) (2016) 937–955.