Remarks on weak-strong uniqueness for two-fluid model

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Abstract. This paper concerns with the compressible two-fluid model with algebraic pressure closure. We prove a conditional weak-strong uniqueness principle, meaning that a finite energy weak solution, with bounded densities, coincides with the classical solution on the lifespan of the latter emanating from the same initial data.

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1. Introduction

We consider the compressible two-fluid model with algebraic pressure closure in the three-dimensional torus $\mathbb{T}^3$:

$$
\begin{aligned}
\partial_t R + \text{div}_x(Ru) &= 0, \\
\partial_t Q + \text{div}_x(Qu) &= 0, \\
\partial_t [(R + Q)u] + \text{div}_x([(R + Q)u \otimes u] + \nabla_x p(Z)) &= \mu \Delta u + (\mu + \lambda)\nabla_x \text{div}_x u. 
\end{aligned}
$$

(1.1)

Here, $R$ and $Q$ are densities of two fluids; $u \in \mathbb{R}^3$ means the velocity field. $p = p(Z)$ stands for the scalar pressure, relating implicitly to $R, Q$ through

$$
\begin{aligned}
Q &= \left(1 - \frac{R}{Z}\right)Z^\gamma, \\
\gamma &= \gamma_+ / \gamma_-, \\
R &\leq Z.
\end{aligned}
$$

The model describes the motion of two immiscible compressible fluids sharing the same velocity field and obeying the algebraic pressure closure. The derivation of (1.1) may be found in Bresch et al. [1]. We refer to the monographs [2, 7] for more discussions on related models.

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Since the problem under consideration is evolutionary, we supplement (1.1) with the initial conditions:

\[(R, Q, u)|_{t=0} = (R_0, Q_0, u_0).\]  

(1.2)

We now introduce the concept of finite energy weak solution.

**DEFINITION 1.1.** \((R, Q, u)\) is said to be a finite energy weak solution to the problem (1.1)-(1.2) in \((0, T) \times \mathbb{T}^3\) provided that

- \((R, Z) \in L^\infty(0, T; L^{1+}(\mathbb{T}^3)), Q \in L^\infty(0, T; L^{1-}(\mathbb{T}^3)),\)
- \(\nabla_x u \in L^2(0, T; L^2(\mathbb{T}^3; \mathbb{R}^{3 \times 3}));\)

- the equation of continuity for \(R\)
  
  \[\int_0^T \int_{\mathbb{T}^3} (R \partial_t \phi + R u \cdot \nabla_x \phi) \, dx \, dt + \int_{\mathbb{T}^3} R_0 \phi(0, \cdot) \, dx = 0\]
  
  for any \(\phi \in C^\infty_c([0, T) \times \mathbb{T}^3);\)

- the equation of continuity for \(Q\)
  
  \[\int_0^T \int_{\mathbb{T}^3} (Q \partial_t \phi + Q u \cdot \nabla_x \phi) \, dx \, dt + \int_{\mathbb{T}^3} Q_0 \phi(0, \cdot) \, dx = 0\]
  
  for any \(\phi \in C^\infty_c([0, T) \times \mathbb{T}^3);\)

- the momentum equation
  
  \[\int_0^T \int_{\mathbb{T}^3} \left((R + Q) u \cdot \partial_t \varphi + (R + Q) u \otimes u : \nabla_x \varphi + p(Z) \text{div}_x \varphi \right) \, dx \, dt\]
  
  \[= \int_0^T \int_{\mathbb{T}^3} \left(\mu \nabla_x u : \nabla_x \varphi + (\mu + \lambda) \text{div}_x u \text{div}_x \varphi \right) \, dx \, dt - \int_{\mathbb{T}^3} (R_0 + Q_0) u_0 \cdot \varphi(0, \cdot) \, dx\]
  
  for any \(\varphi \in C^\infty_c([0, T) \times \mathbb{T}^3; \mathbb{R}^3);\)

- the energy inequality holds a.e. in \((0, T)\)
  
  \[\int_{\mathbb{T}^3} \left[\frac{1}{2} (R + Q) |u|^2 + \frac{1}{\gamma_+ - 1} \left(\frac{R}{\alpha}\right)^{\gamma_+} \alpha + \frac{1}{\gamma_- - 1} \left(\frac{Q}{1 - \alpha}\right)^{\gamma_-} (1 - \alpha)\right] (t, x) \, dx\]
  
  \[+ \int_0^t \int_{\mathbb{T}^3} \left[\mu |\nabla_x u|^2 + (\mu + \lambda) (\text{div}_x u)^2 \right] \, dx \, dt\]
  
  \[\leq \int_{\mathbb{T}^3} \left[\frac{1}{2} (R_0 + Q_0) |u_0|^2 + \frac{1}{\gamma_+ - 1} \left(\frac{R_0}{\alpha_0}\right)^{\gamma_+} \alpha_0 + \frac{1}{\gamma_- - 1} \left(\frac{Q_0}{1 - \alpha_0}\right)^{\gamma_-} (1 - \alpha_0)\right] \, dx\]
  
  where we set
  
  \[\alpha := \frac{R}{Z}.\]

The existence of finite energy weak solutions for system (1.1) was first proved by Bresch et al. [1] in the semi-stationary regime, which was later extended by Novotný et al. [10] in the general case. In a series of work, for instance [4, 6], the fundamental property of weak-strong uniqueness was verified for the compressible Navier-Stokes system in the framework of weak solutions. However, due to the complicated form of pressure, much less is known for the compressible two-fluid
models. Very recently, Jin et al. [8] proved the weak-strong uniqueness for the two-fluid model of Baer-Nunziato type. It should be noticed that the basic tool in their proof is the celebrated relative energy inequality.

As a consequence, it is natural to explore the property of weak-strong uniqueness for the two-fluid model (1.1). This is the motivation of the present note. However, due to the implicit form of pressure, it is not clear how to apply the well-developed method of relative entropy. Instead, we appeal to the Gronwall-type argument, inspired by Germain [6] and Desjardins [3]. As the expense, the boundedness of densities for finite energy weak solutions is imposed additionally. More precisely, we have the following result.

**Theorem 1.2.** Let \((R, Q, u)\) be a finite energy weak solution to (1.1)-(1.2) such that
\[
(R, Q) \in L^\infty(0, T; L^\infty(T^3)).
\]
Assume that \((\tilde{R}, \tilde{Q}, \tilde{u})\) is the classical solution to the same problem on \([0, T]\), starting from the same initial data. Then
\[
R = \tilde{R}, \quad Q = \tilde{Q}, \quad u = \tilde{u} \quad \text{in} \quad [0, T] \times T^3.
\]

Observe that the local existence and uniqueness of classical solutions as well as the global existence and uniqueness of classical solutions under smallness of initial data have recently been obtained by Piasecki and Zatorska [11] in a \(L^p - L^q\) maximal regularity setting. Our main theorem gives the stability of classical solutions within finite energy weak solutions by imposing the boundedness of densities additionally. The rest of this note is devoted to its proof.

2. Weak-strong uniqueness principle

In the sequel, we shall present the formal computations of the main steps without caring about the regularity issues of \((R, Q, u)\), in analogy with the classical literature [3, 6]. The rigorous proof is implemented with the standard regularization procedure, which is omitted here.

To simplify the notations, we denote by
\[
\mathfrak{R} = R - \tilde{R}, \quad Q = Q - \tilde{Q}, \quad U := u - \tilde{u},
\]
where \((R, Q, u)\) and \((\tilde{R}, \tilde{Q}, \tilde{u})\) satisfy the assumptions of Theorem 1.2. Moreover, due to the symmetry of the problem, we may assume, without loss of generality, that \(\gamma_+ \leq \gamma_-\), i.e., \(\gamma \leq 1\).

To begin with, we estimate the \(L^2(T^3)\)-norm of \((\mathfrak{R}, Q)\).

**Lemma 2.1.**
\[
\frac{d}{dt}\|\mathfrak{R}(t)\|_{L^2(T^3)} \leq C \left( (\|R\|_{L^\infty(T^3)} + \|\tilde{R}\|_{L^\infty(T^3)}) \|\nabla_x U\|_{L^2(T^3)} \right)
\]
\[
+ C \left( \|\nabla_x \tilde{u}\|_{L^\infty(T^3)} \|\mathfrak{R}\|_{L^2(T^3)} + \|\nabla_x \tilde{R}\|_{L^3(T^3)} \|U\|_{L^6(T^3)} \right); \quad (2.1)
\]
\[
\frac{d}{dt}\|Q(t)\|_{L^2(T^3)} \leq C \left( (\|Q\|_{L^\infty(T^3)} + \|\tilde{Q}\|_{L^\infty(T^3)}) \|\nabla_x U\|_{L^2(T^3)} \right)
\]
\[
+ C \left( \|\nabla_x \tilde{u}\|_{L^\infty(T^3)} \|Q\|_{L^2(T^3)} + \|\nabla_x \tilde{Q}\|_{L^3(T^3)} \|U\|_{L^6(T^3)} \right). \quad (2.2)
\]
Proof. It follows from the continuity equations of \((R, u)\) and \((\tilde{R}, \tilde{u})\) that

\[
\partial_t R + \text{div}_x \left( RU + R \tilde{u} \right) = 0.
\]

Equivalently,

\[
\partial_t R + R \text{div}_x U + U \cdot \nabla_x R + R \text{div}_x \tilde{u} + \tilde{u} \cdot \nabla_x R + U \cdot \nabla_x \tilde{R} = 0.
\]

Multiplying the above equation by it and integrating over \(\mathbb{T}^3\), with the help of Hölder’s inequality, gives (2.1); the verification of (2.2) follows exactly the same way.

Next, we give the estimate of \(U\).

**Lemma 2.2.** Suppose

\[
\|(R, Q, \tilde{R}, \tilde{Q})\|_{L^\infty(0,T;L^\infty(\mathbb{T}^3))} \leq M. \tag{2.3}
\]

Then there exists a positive constant \(C = C(M)\) such that

\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{R + Q} U \|_{L^2(\mathbb{T}^3)}^2 + \| \nabla_x U \|_{L^2(\mathbb{T}^3)}^2 \leq \left( \| R \|_{L^2(\mathbb{T}^3)} + \| Q \|_{L^2(\mathbb{T}^3)} \right) \| \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u} \|_{L^2(\mathbb{T}^3)} \| U \|_{L^2(\mathbb{T}^3)} + C \| \nabla_x U \|_{L^2(\mathbb{T}^3)} \left( \| R \|_{L^2(\mathbb{T}^3)} + \| Q \|_{L^2(\mathbb{T}^3)} + \| \sqrt{R + Q} U \|_{L^2(\mathbb{T}^3)} \| \nabla_x \tilde{u} \|_{L^\infty(\mathbb{T}^3)} \right). \tag{2.4}
\]

**Proof.** The momentum equations easily imply

\[
(R + Q) \partial_t U + (R + Q) u \cdot \nabla_x U + \nabla_x Z^{\gamma^+} - \nabla_x \tilde{Z}^{\gamma^+}
\]

\[
= \mu \Delta (u - \tilde{u}) + (\mu + \lambda) \nabla_x \text{div}_x (u - \tilde{u}) - (R + Q)(\partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u}) - (R + Q) U \cdot \nabla_x \tilde{u}, \tag{2.5}
\]

where \(\tilde{Z}\) is uniquely solved by

\[
\begin{cases}
\dot{Q} = \left(1 - \frac{\tilde{R}}{Z}\right) \tilde{Z}^\gamma, \\
\tilde{R} \leq \tilde{Z}.
\end{cases}
\]

Direct computations show

\[
\partial_R Z = \frac{Z^{\gamma-1}}{\gamma Z^{\gamma-1} - (\gamma - 1)RZ^{\gamma-2}}, \quad \partial_Q Z = \frac{1}{\gamma Z^{\gamma-1} - (\gamma - 1)RZ^{\gamma-2}},
\]

whence

\[
|\partial_R Z| \leq \frac{1}{\gamma}, \quad |\partial_Q Z| \leq \frac{Z^{1-\gamma}}{\gamma}. \tag{2.6}
\]

In addition, it is known that (see for instance Remark 1.1 in [9])

\[
Z \leq \max \left\{ 2 \| R \|_{L^\infty(0,T;L^\infty(\mathbb{T}^3))}, 2 \| Q \|_{L^\infty(0,T;L^\infty(\mathbb{T}^3))} \right\}^{1/\gamma};
\]

As a direct consequence of (2.3),

\[
\|(Z, \tilde{Z})\|_{L^\infty(0,T;L^\infty(\mathbb{T}^3))} \leq C(M). \tag{2.7}
\]

We then deduce from (2.6)-(2.7) that

\[
\begin{align*}
\left| \int_{\mathbb{T}^3} \left( \nabla_x Z^{\gamma^+} - \nabla_x \tilde{Z}^{\gamma^+} \right) \cdot U dx \right| \\
\leq \| Z^{\gamma^+} - \tilde{Z}^{\gamma^+} \|_{L^2(\mathbb{T}^3)} \| \nabla_x U \|_{L^2(\mathbb{T}^3)} \\
\leq C(M)(\| R \|_{L^2(\mathbb{T}^3)} + \| Q \|_{L^2(\mathbb{T}^3)}) \| \nabla_x U \|_{L^2(\mathbb{T}^3)}.
\end{align*}
\]

\[
\leq C(M)(\| R \|_{L^2(\mathbb{T}^3)} + \| Q \|_{L^2(\mathbb{T}^3)}) \| \nabla_x U \|_{L^2(\mathbb{T}^3)}. \tag{2.8}
\]
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Testing (2.5) by \( U \), with the help of (2.8) and Hölder’s inequality, and integrating by parts gives rise to (2.4) immediately.

\[ \square \]

In order to apply the generalized Poincaré inequality, we give the estimate on the mean value of \( U \).

**Lemma 2.3.**

\[
\left| \int_{T^3} U dx \right| \leq \frac{C}{\int_{T^3} (R_0 + Q_0) dx} \left[ \| R \|_{L^\infty(T^3)} + \| Q \|_{L^\infty(T^3)} \| \nabla_x U \|_{L^2(T^3)} + \| \nabla_x \tilde{u} \|_{L^2(T^3)} \left( \| R \|_{L^2(T^3)} + \| Q \|_{L^2(T^3)} \right) \right].
\] (2.9)

**Proof.** Obviously,

\[
\int_{T^3} (R + Q) U dx = \int_{T^3} \left[ (R + Q) \left( U - \int_{T^3} U dx \right) + (R + Q) \int_{T^3} U dx \right] dx
\]

\[
= \int_{T^3} (R + Q) \left( U - \int_{T^3} U dx \right) dx + \int_{T^3} U dx \int_{T^3} (R_0 + Q_0) dx,
\] (2.10)

since we know from the continuity equations that

\[
\int_{T^3} (R + Q) dx = \int_{T^3} (R_0 + Q_0) dx.
\]

Thus,

\[
\int_{T^3} U dx = -\frac{1}{\int_{T^3} (R_0 + Q_0) dx} \left[ \int_{T^3} (R + Q) \left( U - \int_{T^3} U dx \right) dx - \int_{T^3} (R + Q) U dx \right].
\] (2.11)

Observe next that

\[
\int_{T^3} (R + Q) U dx = \int_{T^3} (R + Q) (u - \tilde{u}) dx
\]

\[
= \int_{T^3} (R + Q) u dx - \int_{T^3} (R + Q) \tilde{u} dx
\]

\[
= \int_{T^3} (R_0 + Q_0) u_0 dx - \int_{T^3} (R + Q) \tilde{u} dx - \int_{T^3} (\tilde{R} + \tilde{Q}) \tilde{u} dx
\]

\[
= -\int_{T^3} (R + Q) \tilde{u} dx = -\int_{T^3} (R + Q) \left( \tilde{u} - \int_{T^3} \tilde{u} dx \right) dx
\] (2.12)

due to the fact that

\[
\int_{T^3} (R + Q) dx = 0.
\]

With the aid of Hölder and generalized Poincaré inequalities, (2.9) follows from (2.11)-(2.12) readily. The proof of Lemma 2.3 is thus finished.

\[ \square \]

With Lemmas 2.1-2.3 at hand, we are now ready to give the proof of weak-strong uniqueness principle.
Proof of Theorem 1.2. To begin with, we conclude from Lemma 2.3 and Sobolev’s inequality that
\[ \|U\|_{L^6(T^3)} \leq C \left( \| \nabla_x U \|_{L^2(T^3)} + \| \nabla_x U \|_{L^2(T^3)} \left( \| R \|_{L^2(T^3)} + \| Q \|_{L^2(T^3)} \right) \right). \]  
(2.13)
Combining (2.1), (2.2) and (2.13), it follows that
\[ \frac{d}{dt} \left( \| R(t) \|_{L^2(T^3)} + \| Q(t) \|_{L^2(T^3)} \right) \]
\[ \leq C \left( \| \nabla_x U \|_{L^2(T^3)} + \left( \| \nabla_x U \|_{L^2(T^3)} + \| \nabla_x U \|_{L^2(T^3)} \right) \left( \| R \|_{L^2(T^3)} + \| Q \|_{L^2(T^3)} \right) \right). \]  
(2.14)
Upon invoking the classical Gronwall’s inequality, the above differential inequality yields
\[ \| R(t) \|_{L^2(T^3)} + \| Q(t) \|_{L^2(T^3)} \leq C \int_0^t \| \nabla_x U(s) \|_{L^2(T^3)} ds. \]  
(2.15)
Consequently, the energy inequality (2.4) may be strengthened as, with the help of (2.13), (2.15) and Hölder’s inequality,
\[ \frac{1}{2} \frac{d}{dt} \| \sqrt{R + Q} U \|_{L^2(T^3)}^2 + \| \nabla_x U \|_{L^2(T^3)}^2 \]
\[ \leq C \left[ \| \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u} \|_{L^3(T^3)} \| \nabla_x U \|_{L^2(T^3)} \right] \int_0^t \| \nabla_x U(s) \|_{L^2(T^3)} ds \]
\[ + C \left[ \| \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u} \|_{L^3(T^3)} \| \nabla_x U \|_{L^2(T^3)} \left( \int_0^t \| \nabla_x U(s) \|_{L^2(T^3)} ds \right)^2 \right] \]
\[ + C \left[ \| \nabla_x U \|_{L^2(T^3)} \int_0^t \| \nabla_x U(s) \|_{L^2(T^3)} ds + \| \sqrt{R + Q} U \|_{L^2(T^3)} \| \nabla_x U \|_{L^2(T^3)} \right] \]
\[ \leq C \left[ \| \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u} \|_{L^3(T^3)} + 1 \right] \| \nabla_x U \|_{L^2(T^3)} \int_0^t \| \nabla_x U(s) \|_{L^2(T^3)} ds \]
\[ + C \left[ t \| \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u} \|_{L^3(T^3)} \| \nabla_x U \|_{L^2(T^3)} \int_0^t \| \nabla_x U(s) \|_{L^2(T^3)}^2 ds \right. \]
\[ + \| \sqrt{R + Q} U \|_{L^2(T^3)}^2 \| \nabla_x U \|_{L^2(T^3)} \]. \]  
(2.16)
By choosing
\[ f(t) = \frac{1}{2} \| \sqrt{R + Q} U(t) \|_{L^2(T^3)}^2 + \frac{1}{2} \int_0^t \| \nabla_x U(s) \|_{L^2(T^3)}^2 ds, \]
\[ g(t) = \int_0^t \| \nabla_x U(s) \|_{L^2(T^3)} ds, \]
\[ \alpha(t) = C \left( \int_0^t \| \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u})(t) \|_{L^3(T^3)} \| \nabla_x \tilde{u}(t) \|_{L^2(T^3)} + \| \nabla_x \tilde{u}(t) \|_{L^2(T^3)} \right), \]
\[ \beta(t) = C \left( \int_0^t \| \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u})(t) \|_{L^3(T^3)} + 1 \right), \]
in the generalized Gronwall’s inequality (recalled in Section 3), we conclude from (2.16) that
\[ U = 0, \ R = 0, \ Q = 0, \]
thus finishing the proof of Theorem 1.2.

Conflict of interest The authors declare that there is no conflict of interest.
3. An auxiliary lemma

For the convenience of the reader, we recall the following generalized Gronwall’s inequality proved in Lemma 2.2 of Ref. [5].

**Lemma 3.1.** Assume that 
\[ f' + (g')^2 \leq \alpha f + \beta gg', \]
where \( f, g', \alpha, \beta \) are positive functions with variable \( t \in (0, T) \) and \( f \in L^\infty((0, T)), g(0) = 0, g' \in L^2((0, T)), \alpha \in L^1((0, T)), \sqrt{\beta}(t) \in L^2((0, T)) \).

Then, for any \( t \in [0, T] \), it holds
\[
e^{-\int_0^t \alpha(\tau)d\tau} f(t) + \left( e^{-\int_0^t \alpha(\tau)d\tau} - \frac{1}{2} - \frac{1}{2} \int_0^t \tau \beta^2(\tau)d\tau \right) \int_0^t (g'(\tau))^2d\tau \leq f(0).
\]

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