Fermions in loop quantum cosmology and the role of parity

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Abstract
Fermions play a special role in homogeneous models of quantum cosmology because the exclusion principle prevents them from forming sizable matter contributions. They can thus describe the matter ingredients only truly microscopically and it is not possible to avoid strong quantum regimes by positing a large matter content. Moreover, possible parity-violating effects are important especially in loop quantum cosmology whose basic object is a difference equation for the wavefunction of the universe defined on a discrete space of triads. The two orientations of a triad are interchanged by a parity transformation, which leaves the difference equation invariant for ordinary matter. Here, we revisit and extend loop quantum cosmology by introducing fermions and the gravitational torsion they imply, which renders the parity issue non-trivial. A treatable locally rotationally symmetric Bianchi model is introduced which clearly shows the role of parity. General wavefunctions cannot be parity-even or odd, and parity-violating effects in matter influence the microscopic big bang transition which replaces the classical singularity in loop quantum cosmology.

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1. Introduction

Most cosmological models—classical or quantum—introduce the matter ingredients of the universe as bosonic fields, in particular scalar ones. While this provides a good measure for the implications of matter energy on spacetime, some effects of realistic fermionic particles may be overlooked. Especially in homogeneous models of quantum cosmology there is an important difference between bosonic and fermionic models: the exclusion principle forbids large matter energies when symmetry reduction leaves only a few, finitely many fermionic degrees of freedom. A massive universe can then be obtained by only two possibilities: a
homogeneous description with many different fermionic species, or inhomogeneity with many local degrees of freedom of a few species (as in [1]).

Both options differ from what is modeled by large values of homogeneous bosonic fields which rather resemble a Bose–Einstein condensate of many identical excitations. In fact, fermion condensates have been suggested for such a purpose, with characteristic effects [2, 3]. This is an effective picture starting from an inhomogeneous perspective in which fermions condense under certain conditions, after which a symmetry reduction can be done. It differs from a fundamental description from fermions in quantum cosmology where constraints due to the exclusion principle cannot as easily be avoided. Potentially fundamental mechanisms which rely on a large amount of bosonic matter, such as bounce scenarios to avoid the big bang singularity, have to be reanalyzed if matter is fermionic. A truly microscopic description will then be achieved. Here, we perform an analysis of the role of fermions in loop quantum cosmology.

Loop quantum cosmology [4] provides a general mechanism for fundamental singularity resolution [5, 6]. Commonly in quantum cosmology, when volume is used as an intrinsic measure of time, evolution must stop at the classical singularity where the volume vanishes. In loop quantum cosmology, by contrast, the timeline is naturally extended, first at the kinematical quantum level, by including orientation into the basic variables: the (densitized) triad knows about the size as well as the orientation of the universe which make it take all real values, not just positive ones. Vanishing size is then no longer a boundary but an interior point of minisuperspace.

What is more, even dynamically the classical singularity is removed because the quantized Hamiltonian constraint equation uniquely extends any wavefunction defined on minisuperspace across the subset of vanishing sizes. Dynamics is dictated by a difference equation [7, 8] which remains regular where classical relativity and Wheeler–DeWitt quantum cosmology would reach their limits. Rather than being singular, the big bang transition then appears as a place where space flips its orientation—turning its inside out—while it changes from being contracting to being expanding3. Still, the region of vanishing volume does remain special in the underlying recurrence scheme. Some coefficients of the difference equation can vanish at labels corresponding to zero volume, which leads to consistency conditions implied by the dynamical law [11, 12]. This is welcome because, at least partially, it frees one from having to pose initial values for a wavefunction independently of the dynamics. The wavefunction of the universe is restricted by the theory alone, relaxing the need to pick one solution among many which could correspond to our universe.

While the set of configurations of vanishing volume is not a boundary within the theory, in the presence of ordinary matter one may choose to consider parity transformations as large gauge transformations which complete the gauge group of triad rotations to all orthogonal transformations. Then, one would restrict solutions to only those states which are either even or odd under parity reversal. This would essentially factor out the orientation degree of freedom introduced by the use of triad variables, and again demote the set of vanishing sizes to a boundary rather than an interior regime. This factoring has indeed been assumed in recent constructions of physical Hilbert spaces for specific isotropic models and the corresponding intuitive bounce pictures based on [13].

Sometimes it is suggested that this small-volume regime is avoided altogether because wave packets may turn around in a bounce at some minimal nonzero volume. This indeed happens for homogeneous models containing sufficiently much kinetic energy of matter [9, 10]. However, this does not appear as a general mechanism which would be valid in this form for generic quantum states or for inhomogeneous situations. Fundamental singularity resolution which deals with the wavefunction right at vanishing volume is thus required.
But if this is used crucially for the constructions, what happens if more realistic matter is included which, as we know from particle physics, cannot be parity invariant? Do properties of the specific solutions based on the assumption of reflection symmetry depend on the conservation of parity by matter, and if so, how reliable are the conclusions drawn from this assumption? Only the inclusion of parity-violating terms, at least as a possibility, can provide a sufficiently general mechanism of singularity resolution.

It may also give rise to new effects related to the role of parity violation in the big bang transition. If this were to happen, an intriguing new link between particle physics and quantum gravity would result. Seeing whether this is indeed the case requires the introduction of fermions, which is available in loop quantum gravity [14–16] (see also [17–19]). In general, however, the parity behavior of loop quantum gravity is highly non-trivial due to the fact that the basic variable conjugate to the densitized triad, namely the Ashtekar–Barbero connection, is the sum of a parity-even and a parity-odd term. It does not have a simple parity behavior and, moreover, it appears in quantized expressions only nonlinearly through holonomies. Even in vacuum, this makes a direct demonstration of parity invariance of loop quantum gravity—or the lack thereof—very complicated [16].

In this paper, we introduce a homogeneous model which allows one to analyze the parity behavior in a clear-cut way. At the same time, the model is amenable to the techniques which have been proven useful for explicit constructions of Hamiltonian constraint equations through the difference equations of loop quantum cosmology [7, 8, 20]. The microscopic nature of fermions due to the exclusion principle is explicitly realized. As we will see, quantization of this model does not introduce unexpected parity violations in the absence of classical parity violations. But the inclusion of parity-violating matter interactions is possible, which can be used to illustrate the role of parity for singularity removal. Then indeed, wavefunctions change under triad reflections. The big bang transition through vanishing sizes is a non-trivial event, which represents true local evolution in internal time rather than merely the application of a symmetry transformation.

2. Classical symmetry reduction

In this section, we provide the formulation of symmetry reduced cosmological models which may have torsion due to the presence of fermions. We follow the symmetry reduction of torsion-free Bianchi class A models [20–22], combined with the canonical formulation of gravity with fermions [14–16, 23]; the general formulation without symmetry is summarized in the appendix. Here, we combine these research lines and explore the symmetry reduction of gravity coupled to fermions in a first-order formalism, implying a theory with torsion. As we will see, there are non-trivial changes in the underlying equations, such that the analysis done here provides a crucial consistency test of the robustness of existing models. At the same time, we clarify the constructions of loop quantum cosmology [4] from the viewpoint of some recent developments.

2.1. Diagonalization

Bianchi class A models constitute all homogeneous models with a symmetry group $S$ acting freely on the space manifold $\Sigma \cong S$ and for which standard Hamiltonian formulations exist. The symmetry group is characterized by its structure constants $C^K_{IJ}$, which for class A models satisfy $C^K_{IJ} = 0$ [24] and can be parametrized as $C^K_{IJ} = \epsilon^K_{IJ}n^I$ with three coefficients $n^I$ which either vanish or take values $\pm 1$. Some of these models can be reduced further by imposing rotational symmetry with one axis (where $S$ has isotropy group $F = U(1)$) or even
isotropy \((F = SO(3))\). Later in this paper we will present a locally rotationally symmetric (LRS \([25]\)) model with torsion in detail.

The action of a symmetry group \(S\) on \(\Sigma\) provides invariant 1-forms \(\omega^I_a\) which are used for the reduction of Ashtekar–Barbero variables. For each \(s \in S\), they satisfy
\[
 s^I \omega^I_a = \text{Ad}(s)_I^J \omega^J_a \quad \text{or, in terms of the Lie-algebra valued 1-form} \quad \Omega_a := \omega^I_a T_I \quad \text{with generators} \quad T_I \text{ of} \quad S, \quad s^I \Omega_a = s^{-1} \Omega_a s.
\]

The left invariant 1-forms then yield the decomposition
\[
 A^I_a = V_0^{-1/3} \phi_I^a \quad \text{of an invariant connection with spatially constant coefficients} \quad \phi_I^a \quad (\text{see the appendix of} \quad [4] \quad \text{for more details on invariant connections}). \text{ Here, we have explicitly included a factor of} \quad V_0 = \int d^3x [\det(\omega^I_a)] \quad \text{of the spatial coordinate volume (or the volume of any finite region used to define the homogeneous variables) as it will be convenient later on. A corresponding decomposition of the densitized triad is given by} \quad E^a_I = V_0^{-2/3} p_I^a X_I^a \quad \text{with} \quad X_I^a \quad \text{being densitized left invariant vector fields dual to the 1-forms:} \quad \omega^I_a X_I^a = \delta^I_J [\det(\omega^K_b)]. \quad \text{The symplectic structure of the reduced model is given by}
\]
\[
 \{ \phi_I, p_J^a \} = \gamma I \delta^a_J \quad (1)
\]
as it follows from
\[
 (\gamma I \kappa)^{-1} \int d^3 x \ A^I_a E^a_E = (\gamma I \kappa)^{-1} \phi_I^a.
\]

For the purpose of loop quantization, it is useful to further reduce the number of independent components of the invariant connection and its conjugate momentum. In some cases, this will allow very explicit calculations of matrix elements of the Hamiltonian constraint and the difference equation it implies for physical states \([20]\). Both the connection and the densitized triad can be cast into diagonal form
\[
 A^I_a = V_0^{-1/3} c_{(K)} A^{K}_a \omega^K_a, \quad E^a_I = V_0^{-2/3} p^{(K)} A^I_K X^K_I \quad (2)
\]
with six spatially constant coefficients \(c_I\) and \(p^I\) which are considered as the only dynamical components while \(A \in SO(3)\) is fixed up to gauge transformations. Using the same \(A^I_a\) for \(A^I_a\) and \(E^a_I\) is consistent with the Gauss constraint for diagonal torsion-free Bianchi class \(A\) models which is then solved identically. From the diagonal densitized triad, moreover, we find the co-triad \(e^a_I = V_0^{-1/3} a_{(I)} A^K_a \omega^K_a\) with \(|a_I| = \sqrt{|p^2 p^3/p^1|}\) and cyclic. It determines the diagonal anisotropic spatial metric
\[
 q_{ab} = e^a_I e^b_I = V_0^{-2/3} a_{(I)} \delta_I^J a^J_b \omega^I_a = q_{IJ} a^I_a \omega^J_b
\]
with three independent scale factors \(V_0^{-1/3}|a_I|\).

By construction, \(c_I\), \(p^I\) and \(a_I\) are independent of coordinates as long as the diagonalized homogeneous form is respected. In particular in a Bianchi I model where \(\omega^I_a = \partial_a x^I = \delta^I_a\) in terms of Cartesian coordinates \(x^I\), spatial coordinates can be rescaled arbitrarily without affecting the basic variables. However, the specific values do depend on \(V_0\) and the choice of the integration volume. Obviously, the \(V_0\)-dependence is a consequence of the symmetry reduction to homogeneity, since \(V_0\) does not occur at all in an inhomogeneous framework. Thus, the dependence has to be interpreted with care especially after quantization where, fundamentally, the relation to coordinates is lost. As a consequence, the role of \(V_0\) cannot be properly understood if considerations are limited to purely homogeneous models because only the reduction from inhomogeneity shows how \(V_0\) enters; see \([26]\) for a discussion from the point of view of inhomogeneous states.

Note that \(p^I\) and \(a_K\) are allowed to take negative values to represent different triad orientations while the orientation of \(A \in SO(3)\) is fixed. A parity transformation then simply implies \(p^I \mapsto -p^I\) for the triad components (leaving coordinates unchanged), while the transformation of \(c_I\) is in general more complicated. In fact, we have
\[
 A^I_a = \hat{\Gamma}^I_a + \gamma K^I_a \quad \text{with the parity-even torsion-free spin connection}
\]
\[
 \hat{\Gamma}^I_a = \frac{1}{2} e^{IJK} \delta^a_J \left( 2 \partial_I e^I_b + e^I_b \partial_I e^I_b \right) \quad (3)
\]
and the odd extrinsic curvature $K^i_a = K_{ab}e^b_i$. In the torsion-free case, it follows from (3) that the homogeneous spin connection can be expressed as $\tilde{\Gamma}'_i = \tilde{\Gamma}'_{(k)} \omega_a^b_{(k)} \omega_a^b$ [22] with

$$\tilde{\Gamma}'_i = \frac{1}{2} \left( \frac{a_f}{a_k} n^f + \frac{a_k}{a_f} n^k - \frac{a_j}{a_f} a^b_k n^j - \frac{a_j}{a_k} a^b_f n^j \right)$$

for indices such that $\epsilon_{ijk} = 1$ (4)

and the same $\Lambda'^i_j$ as used for the densitized triad. Similarly $\tilde{K}^i_a = K^i_{(k)} \Lambda'^i_j \omega_a^b_{(k)}$ also with the same $\Lambda'_i$. Then, $c_I = \tilde{\Gamma}'_I + y K_I$ does not have a straightforward parity behavior unless $\tilde{\Gamma}'_I = 0$ (as in the Bianchi I model).

The diagonalization is sufficient to capture the crucial dynamical behavior of Bianchi models, such as the approach to a singularity. For the quantization, it has the advantage that it reduces $SU(2)$ to $U(1)^3$; holonomies of a homogeneous connection, computed along curves generated by the invariant vector fields $X^a_I$, take the form $h^\mu(\mu) = \exp(\mu \phi_I^a \tau_I)$ with a real number $\mu$ depending, e.g., on the coordinate length of a curve used to compute the holonomy. For $\phi_I = c_{(i)} \Lambda'_I$, we have

$$h^\mu(\mu) = \exp(\mu J^I c_{(i)} \Lambda'_I \tau_I) = \cos(\frac{1}{2} \mu J^I c_{(i)} \tau_I) + 2\Lambda'_I \tau_I \sin(\frac{1}{2} \mu J^I c_{(i)} \tau_I).$$

(5)

While any $SU(2)$-holonomy along $X^a_I$ can be written in this way, the diagonalization implies that $\Lambda'_I$ becomes a mere background quantity not subject to dynamics. Thus, it is sufficient to consider only the simple commuting exponentials $\exp(i \mu J^I c_{(i)})$ to separate diagonal connections. After a loop quantization, as we will see in detail below, this will have the implication that a triad representation exists, which simplifies the analysis of dynamics considerably. In fact, triad operators will simply be $\hat{p}^I = -i\hbar \ell^2 \partial / \partial c_I$, with the Planck length $\ell_P = \sqrt{\hbar}$, which form a complete commuting set. Their eigenstates

$$\langle c_1, c_2, c_3 | \mu_1, \mu_2, \mu_3 \rangle = \exp(\frac{1}{2} i (\mu_1 c_1 + \mu_2 c_2 + \mu_3 c_3))$$

(written here in the connection representation) form an orthonormal basis such that the coefficients in

$$|\psi\rangle = \sum_{\mu_1, \mu_2, \mu_3} s_{\mu_1, \mu_2, \mu_3} |\mu_1, \mu_2, \mu_3 \rangle$$

form the triad representation of arbitrary states. This explicit representation, which becomes available only after diagonalization [20], has been the basis of all investigations so far in homogeneous loop quantum cosmology. As we will see in this paper, arriving at such a representation is less trivial in the presence of torsion.

2.2. Torsion effects

This scheme of diagonalization of the basic torsion-free gravitational variables relies on the fact that both the connection and its conjugate momentum can be diagonalized with the same $\Lambda'_I$. In other words, the $su(2)$ valued connection and its conjugate momentum are parallel to each other in the tangent space of the internal symmetry group. This can be seen from the torsion-free Gauss constraint which expressed in terms of the diagonalized variables takes the form $p^{(i)} c_{(j)} e_{ijk} \Lambda'^i_j \Lambda'_I = 0$ and is identically satisfied. However, the presence of torsion via

4 General curves do not provide this simple form. For instance, along $X^a_I + X^a_2$ holonomies are not of the (almost) periodic form in $c_1$ or $c_2$ (but in $\sqrt{c_1^2 + c_2^2}$). If curves are considered which are not even straight with respect to the given symmetry, the behavior is more complicated due to path ordering and do not give rise to almost periodic functions [27]. However, such curves do not play a role in the kinematical symmetry reduction, which uses the given set of $X^a_I$ to introduce particular quantum geometries, just like classical symmetric metrics which are used in adapted coordinates but can look complicated in arbitrary coordinates.
the axial fermion current $J_i$, as summarized in the appendix, enters the Gauss constraint (A.7) implying that

$$\phi^i p^i \epsilon_{ijk} = \frac{i}{\sqrt{\det(p^i_j)}} J_i.$$  \hspace{1cm} (6)

(We only discuss the case where torsion is implied by the coupling to fermions. The implications of torsion on the diagonalizability of basic variables are, however, more general.) For $\phi^i = c_{ij} \Lambda^i_j$ and $p^i = p^{(ij)} \Lambda^i_j$ as above, this would only allow vanishing spatial components of the fermion current and severely restrict the allowed models. This situation becomes more obvious if we try to express the spin connection including its torsion contribution as $\Gamma^i_j = \Gamma(K) \Lambda^i_j \omega^K_a$ with the same $\Lambda^i_j$ as used for the triad: one can easily verify that the partial torsion contribution (A.6) to the connection cannot be expressed as $C^i_a = c_{ij} \Lambda^i_j \omega^K_a$ if $J^{i'} \neq 0$. Then also the Ashtekar–Barbero connection cannot be diagonal in the same basis. Therefore, our first result is that the presence of torsion does not allow us to diagonalize both canonical variables, i.e. the connection and the densitized triad, simultaneously.

Moreover, fermion terms require us to use a connection $A^i_a$ in (A.11) which carries an extra term compared to the Ashtekar–Barbero connection, depending on the fermion current. We then write the new diagonal variables as

$$A^i_a = V_0^{-1/3} c_{(K)} \Lambda^i_j \omega^K_a, \hspace{1cm} E^a_i = V_0^{-2/3} p^{(K)} T^i K X^a_K,$$  \hspace{1cm} (7)

where in general $T^i_j \neq \Lambda^i_j$. Not both $\Lambda^i_j$ and $T^i_j$ can be fixed because partially they are determined by dynamical fields as, e.g., per the Gauss constraint (6). This has an immediate implication for the symplectic structure because $c_{ij}$ and $p^{ij}$ will no longer be canonically conjugate

$$\int d^3 x E^a_i \mathcal{L}_i A^a_i = p^{(ij)} T^j_i \mathcal{L}_i (c_{(ij)} \Lambda^j_i) = p^{(ij)} \mathcal{L}_i (c_{(ij)} \Lambda^j_i T^j_i) - c_{(ij)} p^{(ij)} \Lambda^j_i T^j_i.$$  \hspace{1cm} (8)

Thus, it is not $c_{ij}$ which is conjugate to $p^{ij}$ but $c_{(ij)} \Lambda^j_i T^{(ij)}$. This is not a pure connection component but depends on the relative angles between the connection direction $\Lambda^i_j$ and the triad direction $T^i_j$ in internal space. (It is not possible to fix both $\Lambda^i_j$ and $T^i_j$ because this would require six parameters while the Gauss constraint allows one to fix only three.) Moreover, some of the angles enter the symplectic structure as independent variables. We can, for instance, (Euler) parametrize $T^i_j$ as the matrix $T(\phi^i) = \exp(\phi_3 T_3) \exp(\phi_2 T_2) \exp(\phi_1 T_1)$ using generators $T_i$ of $SO(3)$. Inserting this in (8) shows that the angles $\phi^i$ acquire canonical momenta given in terms of the angles in $\Lambda^i_j$, e.g. $\phi^i$ being conjugate to $-\text{tr}(c^2 \Lambda^i_j (p \cdot T(\phi^i + \pi/2, \phi_2, \phi_3)))$, where $c$ and $p$ denote the diagonal matrices with components $c_{ij}$ and $p^{ij}$, respectively. (Taking a derivative of $T(\phi^i)$ amounts to switching sines and cosines, which is the same as shifting an angle by $\pi/2$.)

The corresponding phase space and the constrained system defined on it is rather involved, and so we consider a more special case which still allows the non-trivial implications of torsion to be seen: we are interested in the case where the presence of a fermion current is the sole reason for anisotropy, while the two-dimensional space transversal to the spatial current is rotationally invariant. We can then assume that there are bases for $A^i_a$ and $E^a_i$, respectively, such that

$$\Lambda^i_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \rho & -\sin \rho \\ 0 & \sin \rho & \cos \rho \end{pmatrix}, \hspace{1cm} T^i_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix},$$  \hspace{1cm} (9)

where $\rho$ and $\phi$ are the only non-vanishing rotation angles. As we will demonstrate below, this allows non-trivial solutions where the fermion current is aligned in the 1-direction. The
Liouville term in the action can then be expressed as

\[
\frac{1}{\gamma \kappa} \int d^3 x \, E_i^a \mathcal{L}_i A^a_j = \frac{1}{\gamma \kappa} \delta^{(l)} \mathcal{L}_i \left( c_{(l)} \Lambda^l_j T^l_i \right) - c_{(l)} p^{(l)} L_i \mathcal{T}_T^i
\]

\[
= \frac{1}{\gamma \kappa} \left( \hat{c}_1 p^1 + \mathcal{L}_i (c_2 \cos(\rho - \phi)) p^2 + \mathcal{L}_i (c_3 \cos(\rho - \phi)) p^3 \right)
- \hat{\phi} (c_2 p^2 + c_3 p^3) \sin(\rho - \phi)

= \frac{1}{\gamma \kappa} \left( \hat{c}_1 p^1 + \hat{c}_2 p^2 + \hat{c}_3 p^3 + \hat{\phi} p_t \right),
\]

(10)

where we introduced

\[
\hat{c}_2 = c_2 \cos(\rho - \phi), \quad \hat{c}_3 = c_3 \cos(\rho - \phi), \quad p_t = -(c_2 p^2 + c_3 p^3) \sin(\rho - \phi).
\]

(11)

In these components, the symplectic structure is

\[
\{ \hat{c}_1, p^1 \} = \gamma \kappa, \quad \{ \hat{c}_2, p^2 \} = \gamma \kappa, \quad \{ \hat{c}_3, p^3 \} = \gamma \kappa, \quad \{ \phi, p_t \} = \gamma \kappa.
\]

(12)

Note that the presence of torsion at this stage introduces a new kinematical degree of freedom \( \phi \). It will be removed after solving the Gauss constraint (6), which is now non-trivial.

There is a useful interpretation of the canonical variables in the presence of torsion: we can write, e.g.,

\[
\hat{c}_2 = c_2 \cos(\rho - \phi) = c_2 \Lambda^l_j T^l_i = \phi_i^l T^l_i
\]

in terms of the general homogeneous coefficients \( \phi_i^l = c_{(l)} \Lambda^l_j \). Since \( T^l_i \) gives the direction of \( E_i^a \), we can interpret \( \hat{c}_2 \) as a component

\[
V_0^{-1/3} \hat{c}_2 = A_i^a E^b_i X_2 \omega_b^2 V_0^{2/3} p^2
\]

of the projection of \( A_i^a \) onto \( E_i^a \). In the absence of torsion, this would be a pure connection component because \( A_i^a \) and \( E_i^a \) would be parallel. With torsion, however, \( \hat{c}_2 \) is only part of an \( A_i^a \)-component: using the expression (A.13), the projection removes the term \( \epsilon_{ikl} e_i^a J^l \) perpendicular to \( E_i^a \) which happens to be the torsion contribution to extrinsic curvature. Moreover, the projection transversal to \( E_i^a \) is just (half of) the variable \( p_t \) due to the sine, which thus takes a value equal to the torsion contribution. This agrees with the solution of the Gauss constraint (14) below. Recall that the identification of the torsion contribution to extrinsic curvature used in (A.13) cannot be completed without partially solving equations of motion. In the projection defining \( \hat{c}_2 \) and \( \hat{c}_3 \), on the other hand, no equations of motion have been used. Thus, these canonical variables which we are naturally led to at the basic kinematical level present torsion-free contributions without explicitly splitting off torsion. (Something similar happens in inhomogeneous models such as spherical symmetry [28] or Gowdy models [29]. There it is spin connection contributions that are split off by a natural definition of canonical variables which then allows a manageable loop quantization.)

2.3. Reduced constraints

In terms of the diagonal variables the Gauss constraint (A.7) becomes

\[
G_i = \frac{1}{\gamma \kappa} \epsilon_{ijk} c_{(l)} p^{(l)} A^l_j T^l_k - \frac{1}{2} \sqrt{|p^1 p^2 p^3|} T^l_i J_l
\]

\[
= - \frac{\epsilon_{ijk}}{\gamma \kappa} (c_2 p^2 + c_3 p^3) \sin(\rho - \phi) - \frac{\sqrt{|p^1 p^2 p^3|}}{2} T^l_i J_l = 0.
\]

(13)
For $i = 2, 3$, it thus implies $J_2 = 0 = J_3$ while the remaining condition

$$
\epsilon_{123} p_\phi = \frac{\gamma k}{2} \sqrt{|p^1 p^2 p^3|} T_i J_i
$$

relates $J_1$ to $p_\phi$

$$
p_\phi = \frac{\gamma k}{2} \sqrt{|p^1 p^2 p^3|} J_1 =: \frac{1}{2} \gamma k J_1,
$$

where $J_i = \xi |\sigma| \xi + \chi |\sigma| \chi$ denotes the densitized axial fermion current (which is bilinear in half-densitized fermions $\xi$ and $\chi$). The form (14) of the Gauss constraint together with the expression for the densitized fermion current in terms of canonical fermion fields makes it clear that the new connection components remain gauge invariant: $c_1$ and $c_2$ commute with $p_\phi$ as well as with fermion fields. With the choice (9) of bases the fermion current $J_i$ is aligned along the first (fixed) internal direction: $J_2 = J_3 = 0$. This defines a specific class of models with a non-trivial spatial fermion current, as $J_1$ may be nonzero.

Similarly, the diffeomorphism constraint (A.8) can be written as

$$
D_\alpha N^\alpha = -c_{ij} p_i^\alpha N^j = N^1 (n^2 c_2 p^3 + n^3 c_3 p^3) \sin(\rho - \phi) = 0,
$$

where $N^\alpha = N^1 X_\alpha^i$ with $N^1$ constant and $C_{ij}^k = \epsilon^k_{ij} n^{(k)}$ to specify different Bianchi class A models are used. We have also imposed that the partial derivatives of spinor fields vanish in a homogeneous model, e.g. $\partial_\xi \psi = 0$. A conclusion to be drawn from (13) and (15) is that torsion is strongly restricted in Bianchi class A models with $n^2 + n^3 \neq 0$ since this implies that $p_\phi = \gamma k J_1 / 2 = 0$, and thus all spatial components of the axial vector current vanish.

Finally, the Hamiltonian constraint (A.9) is

$$
H_{\text{Bianchi}} = \frac{\kappa^{-1}}{\sqrt{|p^1 p^2 p^3|}} (n^1 c_1 p^2 p^3 + n^2 c_2 p^2 p^3 \cos(\rho - \phi) + n^3 c_3 p^2 p^3 \cos(\rho - \phi))
$$

$$
- \frac{\kappa^{-1} \gamma^{-2}}{\sqrt{|p^1 p^2 p^3|}} (c_1 p^1 c_2 p^2 \cos(\rho - \phi) + c_1 p^1 c_3 p^3 \cos(\rho - \phi) - c_2 c_3 p^3) + \frac{\kappa^{-1} \gamma^{-2} (1 + \gamma^2)}{\sqrt{|p^1 p^2 p^3|}} ((c_1 - \Gamma_1) p^1 (\Gamma_2 p^2 + \Gamma_3 p^3) \cos(\rho - \phi)
$$

$$
- (c_2 - \Gamma_2) p^2 \Gamma_3 p^3) + \frac{1}{2 \sqrt{|p^1 p^2 p^3|}} (\gamma (c_2 p^2 + c_3 p^3) \sin(\rho - \phi) J_1
$$

$$
+ \theta (\Gamma_1 p^1 + \Gamma_2 p^2 + \Gamma_3 p^3) \cos(\rho - \phi)) J^0
$$

$$
+ \frac{\gamma}{4 \alpha} \left( n^1 \left| \frac{p^2 p^3}{p^1} \right| + n^2 \left| \frac{p^1 p^3}{p^2} \right| + n^3 \left| \frac{p^2 p^1}{p^3} \right| \right) J^0
$$

$$
- \frac{3 \gamma \kappa \theta}{16 \sqrt{|p^1 p^2 p^3|}} \left( \frac{2}{\alpha} + \frac{\gamma \theta}{1 + \gamma^2} \right) J_0^2
$$

$$
+ \frac{\gamma}{16 \sqrt{|p^1 p^2 p^3|} (1 + \gamma^2)} \left( 2 \gamma \beta \left( 3 - \frac{\gamma}{\alpha} + 2 \gamma^2 \right) - \theta^2 \right) J_i^2
$$

where $\alpha$, $\beta$ and $\theta$ are defined in the appendix. It is important to emphasize that since $\Gamma_\mu^i$ is not diagonalized in either $N^j_\mu$ or $T^j_\mu$ in the presence of torsion, the Hamiltonian constraint in (A.9) expressed in terms of $\Gamma_\mu^i$ by splitting torsion from the spin connection is essential to obtain a controlled loop quantization as will be shown below.

### 2.4. The Bianchi I LRS model with torsion

If there is an isotropy group $F = U(1)$ for the action of the symmetry group $S$, one obtains locally rotationally symmetric (LRS) models. Therefore, two of the diagonal components of
the connection as well as of the triad, e.g. the second two for definitiveness, have to equal each other and only two degrees of freedom are left which we choose to be \((c_1, p^1)\) and \((\bar{c}_2, p^2)\) embedded into the general Bianchi model by

\[
(c_1, \bar{c}_2) \mapsto (c_1, \bar{c}_2, \bar{c}_3) = (c_1, \bar{c}_2, \bar{c}_2), \quad (p^1, p^2) \mapsto (p^1, p^2, p^3) = (p^1, p^2, p^2).
\]

The symplectic structure can be pulled back by this embedding providing Poisson brackets

\[
\{c_1, p^1\} = \gamma \kappa, \quad \{\bar{c}_2, p^2\} = \frac{1}{2} \gamma \kappa, \quad \{\phi, p_\phi\} = \gamma \kappa
\]

from (12), where \(p_\phi\) is now \(p_\phi := -2c_2 p^2 \sin(\rho - \phi)\). (Solutions of this symmetry type in the presence of torsion due to spin fluids have been studied in [30, 31].)

For the LRS model, the diffeomorphism and the Hamiltonian constraints, (15) and (16) respectively, further reduce to

\[
\mathcal{D}_u N^a = \frac{1}{2 \gamma \kappa} N^1 (n^2 + n^3) p_\phi = 0
\]

and

\[
H_{\text{LRS}} = \frac{\kappa^{-1} - 1}{|p^2|\sqrt{|p^1|}} \left( n^1 c_1 (p^2)^2 + n^2 \bar{c}_2 p^1 p^2 + n^3 \bar{c}_2 p^2 p^1 - \gamma^{-2} \left( 2c_1 p^1 \bar{c}_2 p^2 + (\bar{c}_2 p^2)^2 + \frac{1}{4} p_\phi^2 \right) \right)
\]

\[
+ \frac{\kappa^{-1} - 1 - \gamma^{-2}(1 + \gamma^2)}{|p^2|\sqrt{|p^1|}} \left( (c_1 - \bar{\Gamma}_1) p^1 (\bar{\Gamma}_2 p^2 + \bar{\Gamma}_3 p^3) \right. \left. \frac{2|\bar{c}_2 p^2|}{\sqrt{p_\phi^2 + 4(\bar{c}_2 p^2)^2}} \right)
\]

\[
- \frac{\text{sgn}(\bar{c}_2 p^2)^2}{2}\left( p_\phi^2 + 4(\bar{c}_2 p^2)^2 - \bar{\Gamma}_3 p^3 \right)
\]

\[
- \frac{1}{2|p^2|\sqrt{|p^1|}} \gamma p_\phi J_1 - \theta \left( \bar{\Gamma}_1 p^1 + (\bar{\Gamma}_2 p^2 + \bar{\Gamma}_3 p^3) \right. \left. \frac{2|\bar{c}_2 p^2|}{\sqrt{p_\phi^2 + 4(\bar{c}_2 p^2)^2}} \right) J_0^0
\]

\[
+ \frac{\gamma}{4\alpha} \left( n^1 (p^2)^2 |p^1| + (n^2 + n^3)|p^1| \right) J_0^0 - \frac{3 \gamma \kappa \theta}{16|p^2|\sqrt{|p^1|}} \left( \frac{2}{\alpha} + \frac{\gamma \theta}{1 + \gamma^2} \right) J_0^0
\]

\[
+ \frac{\kappa}{16|p^2|\sqrt{|p^1|}(1 + \gamma^2)} \left( 2\gamma^2 \beta \left( 3 - \frac{\gamma}{2} \right) - \theta^2 \right) J_0^2,
\]

where we have used the definitions of \(\bar{c}_2\) and \(p_\phi\) to write

\[
\cos(\rho - \phi) = \frac{2|\bar{c}_2 p^2|}{\sqrt{p_\phi^2 + 4(\bar{c}_2 p^2)^2}}.
\]

To allow a non-vanishing \(J_1\) and to be specific, we work from now on with the Bianchi I model. Here, the diffeomorphism constraint (18) vanishes identically and does not impose any restriction on \(p_\phi\). This has the additional advantage that the resulting Hamiltonian constraint will be free of terms such as \(\sqrt{p_\phi^2 + 4(\bar{c}_2 p^2)^2}\), which lack simple quantizations. (While there are well-defined operators with this classical limit, given that both \(p_\phi^2\) and \((\bar{c}_2 p^2)^2\) would be mutually commuting positive operators whose square root can be taken after summing them, not all the operators involved have discrete spectra. Thus, it would not be straightforward to compute explicit matrix elements of the square-root operator which would be required for the quantized Hamiltonian. Once the square root is quantized, its inverse in (20) could easily be obtained from \(2\gamma \kappa p^2 \cos(\rho - \phi) = \sqrt{p_\phi^2 + 4(\bar{c}_2 p^2)^2}\).)
For the Bianchi I LRS model, we then have $\tilde{\Gamma}_I = 0$ and thus the Hamiltonian constraint is finally given by

\[
H_{\text{ILRS}} = -\frac{\kappa^{-1} \gamma^{-2}}{|p^2|/\sqrt{|p^1|}} \left( 2c_1 p^1 \tilde{c}_2 p^2 + (\tilde{c}_2 p^3)^2 + \frac{1}{4} p^2_\varphi \right) \]

\[
- \frac{\gamma}{2 |p^2|/\sqrt{|p^1|}} p_\varphi J_1 - \frac{3\gamma \kappa \theta}{16 |p^2|/\sqrt{|p^1|}} \left( \frac{2}{\alpha} + \frac{\gamma \theta}{1 + \gamma^2} \right) J^2_0
\]

\[
+ \frac{\kappa}{16 |p^2|/\sqrt{|p^1|} (1 + \gamma^2)} \left( 2\gamma \beta \left( 3 - \frac{\gamma}{\alpha} + 2\gamma^2 \right) - \theta^2 \right) J^2_I.
\]

(21)

This concludes the classical symmetry reduction of canonical gravity non-minimally coupled to fermions.

2.5. Parity behavior

Because we are mainly concerned about the role of parity in loop quantum cosmology, we end this section on the classical equations with a discussion of parity invariance. As pointed out in [16], parity invariance in loop quantum gravity is not guaranteed. The Ashtekar connection is a sum of a parity-even and a parity-odd term and thus does not have a straightforward parity behavior. This already occurs in the absence of fermions and torsion, but is aggravated by the parity-mixing terms of torsion contributions due to a fermion current (see (A.6), noting that $J_1$ is even and $J_0$ is odd). Classically, one can explicitly split these contributions, which essentially amounts to replacing the Ashtekar connection with extrinsic curvature. However, a complete splitting requires equations of motion to be used, which will not be possible after quantization. It is then not guaranteed that quantum corrections due to the loop quantization will preserve parity even in vacuum or in the absence of parity-violating matter.

The model introduced here provides a clear view on parity in the classical theory as well as after quantization, as we will see below. One key property is that the canonical variables (11) we are led to do, in hindsight, perform the splitting into torsion-free and torsion components without using equations of motion. Thus, in the new variables every single term in the Hamiltonian constraint (21) has a clear and simple behavior under parity: among the gravitational variables, only $c_1$ and $p^1$ change sign under parity (reversing orientation) while the rest remains unchanged. (Since changing the sign of $p^2$ in an LRS model implies a reflection of both directions related by the rotational symmetry, it is equivalent to a triad rotation and thus mere gauge.) This is accompanied by the usual parity transformation of the fermions present, which implies that $J_1$ is parity invariant while $J_0$ changes sign as these are space and time components of an axial vector. In particular, it is immediately clear from (21) that the Hamiltonian constraint is parity invariant for free fermions. Parity violation will only result if suitable interactions are introduced to the model, which can easily be done by adding, e.g. $\sqrt{-\det g} \mathcal{V}_\mu J^\mu$ with the vector current $\mathcal{V}^\mu$ to the action. We will avail ourselves of this possibility in what follows to understand the role of parity in the loop quantized model.

3. Quantization of the Bianchi I LRS model

Loop quantum cosmology allows one to complete many of the constructions of full loop quantum gravity in simplified and explicit forms, which then provides indications toward the physical implications of the theory. In this section, we provide a self-contained description of anisotropic models with an emphasis on the effects of fermions, torsion and parity.
3.1. Quantum kinematics

We start with basic variables according to the Poisson structure (17). As in any loop quantization, states in the connection representation are constructed by taking exponentials

\[ \exp(\mu_1 c_1 \Lambda_1^\dagger \tau_i) \in SU(2), \quad \exp(\mu_2 \tilde{c}_2 \Lambda_1^\dagger \tau_i) \in SU(2), \]

\[ \exp(ik \phi) \in U(1) \quad \text{for all} \quad \mu_j \in \mathbb{R}, k \in \mathbb{Z}, \Lambda_1 \in SO(3) \]

as they arise in holonomies. Using holonomies in the general setting is important for a background independent basic algebra of variables. This crucial feature is then reflected also in symmetric models based on exponentials of connection components. The parameters \( \mu_j \) can take any real value, corresponding to evaluating holonomies along straight edges (tangential to \( X^j \)) of arbitrary length. The variable \( \phi \), on the other hand, was introduced as a periodic angle in (9), such that only strictly periodic functions \( \exp(ik \phi) \) with \( k \in \mathbb{Z} \) are allowed. This unphysical degree of freedom, which we were led to introduce due to the presence of torsion, will be removed after solving the Gauss constraint.

Matrix elements of the exponentials in (22) form a \( C^* \)-algebra of (almost) periodic functions, as seen from (5). Any function generated by this set can be written as

\[ g(c_1, \tilde{c}_2, \phi) = \sum_{\mu_1, \mu_2} \xi_{\mu_1, \mu_2} \exp\left(\frac{i}{2} i \mu_1 c_1 + \frac{i}{2} i \mu_2 \tilde{c}_2 + i k \phi\right). \]

with coefficients \( \xi_{\mu_1, \mu_2} \in \mathbb{C} \), where the sum is over finitely many \( \mu_1, \mu_2 \in \mathbb{R} \) and \( k \in \mathbb{Z} \). Note that while \( g(c_1, \tilde{c}_2, \phi) \) is almost periodic in \( c_1 \) and \( \tilde{c}_2 \), it is exactly periodic in \( \phi \). This provides a complete set of continuous functions on \( \mathbb{R}_{\text{Bohr}} \times \mathbb{R}_{\text{Bohr}} \times S^1 \), where \( \mathbb{R}_{\text{Bohr}} \) is the Bohr compactification of the real line. (By definition, \( \mathbb{R}_{\text{Bohr}} \) is the compactification of \( \mathbb{R} \) such that the set of all continuous functions on it is just the set of almost periodic functions. See, e.g. [32] for a recent discussion of further properties.) All spaces in the product are compact Abelian groups and carry a unique normalized Haar measure \( d\mu(c) \) in the case of \( \mathbb{R}_{\text{Bohr}} \), where

\[ \int f(c) \, d\mu(c) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(c) \, dc, \]

and \( d\phi \) for \( S^1 \).

By Cauchy completion, we obtain the Hilbert space as a tensor product \( \mathcal{H}_{\text{grav}} = \mathcal{H}_{\text{Bohr}}^{g_2} \otimes \mathcal{H}_{S^1} \) with the Hilbert spaces \( \mathcal{H}_{\text{Bohr}} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu(c)) \) and \( \mathcal{H}_{S^1} = L^2(S^1, d\phi) \) of square integrable functions on the Bohr compactification of the real line and the circle, respectively. Orthonormal bases for these spaces are given by \( \langle c | \mu \rangle = \exp(i c \mu / 2), \mu \in \mathbb{R}, \) and \( \langle \phi | k \rangle = \exp(ik \phi), k \in \mathbb{Z} \), respectively, with

\[ \langle \mu | \mu' \rangle = \delta_{\mu, \mu'}, \quad \langle k | k' \rangle = \delta_{k, k}. \]

The configuration variables act in the obvious manner: for all \( g_1 \) and \( g_2 \) of the form (23), we have

\[ \langle \hat{g}_1 g_2 \rangle (c_1, \tilde{c}_2, \phi) = g_1(c_1, \tilde{c}_2, \phi) g_2(c_1, \tilde{c}_2, \phi) \]

and the momentum operators are represented by

\[ \hat{p}^1 = -i \gamma \ell_2^2 \frac{\partial}{\partial c_1}, \quad \hat{p}^2 = -i \gamma \ell_2^2 \frac{\partial}{\partial \tilde{c}_2}, \quad \text{and} \quad \hat{p}_\phi = -i \gamma \ell_2^2 \frac{\partial}{\partial \phi}, \]

where \( \ell_2^2 = \kappa \hbar \). (The densitized triad in general is quantized via fluxes, i.e. two-dimensional integrations over surfaces. In a homogeneous context, however, this is not required and densitized triad components can directly be promoted to operators. This simple representation exists only due to our use of variables; had we used \( c_2 \) instead of \( \tilde{c}_2 \), the operator \( \hat{p}^2 \) and thus
the volume operator would have been more complicated.) Common eigenstates of all triad operators $\hat{p}^j$ are

$$|\mu_1, \mu_2, k\rangle := |\mu_1\rangle \otimes |\mu_2\rangle \otimes |k\rangle,$$

with

$$\hat{p}^1|\mu_1, \mu_2, k\rangle = \frac{\gamma^\ell f^j_{\mu\nu} \mu_1}{2} |\mu_1, \mu_2, k\rangle,$$

$$\hat{p}^2|\mu_1, \mu_2, k\rangle = \frac{\gamma^\ell f^j_{\mu\nu} \mu_2}{4} |\mu_1, \mu_2, k\rangle$$

and

$$\hat{p}_\Phi|\mu_1, \mu_2, k\rangle = \gamma^\ell f^j_{\mu\nu} k |\mu_1, \mu_2, k\rangle.$$

From triad operators we construct the volume operator

$$\hat{V}|\mu_1, \mu_2, k\rangle = |\hat{p}^2|\sqrt{\hat{p}^1}|\mu_1, \mu_2, k\rangle = \frac{\gamma^\ell f^j_{\mu\nu}}{4\sqrt{2}} |\mu_1\rangle |\mu_2\rangle |\mu_1, \mu_2, k\rangle.$$

The full Hilbert space is a further tensor product of $\mathcal{H}_{\text{grav}}$ with the fermionic Hilbert space $\mathcal{H}_{\text{fermion}}$. We represent the latter as the space of functions $f(\Theta_a)$ of four independent half-density Grassmann-valued variables $\Theta_a, \alpha = 1, \ldots, 4$, for the four components contained in the fermion fields $\xi$ and $\chi$ in this order. The fermionic momenta $p_\xi = -i\hat{\xi}$ and $p_\chi = -i\hat{\chi}$ then give rise to components $\Theta_a$ which are represented as $h \partial/\partial \Theta_a$. In particular, for the axial current components $J^0 = \xi^\dagger \chi - \chi^\dagger \xi$ and $J_1 = \xi^\dagger \sigma_1 \xi + \chi^\dagger \sigma_1 \chi$ we have operators

$$J^0 = h \frac{\partial}{\partial \Theta_1} + \tilde{h} \frac{\partial}{\partial \Theta_2} - h \frac{\partial}{\partial \Theta_3} - \tilde{h} \frac{\partial}{\partial \Theta_4},$$

$$J_1 = h \frac{\partial}{\partial \Theta_2} + \tilde{h} \frac{\partial}{\partial \Theta_1} + h \frac{\partial}{\partial \Theta_4} - \tilde{h} \frac{\partial}{\partial \Theta_3}.$$

(The component $\tilde{J}_0$ is subject to ordering ambiguities which we can ignore here.)

The currents are easy to diagonalize: each 2-spinor copy has eigenstates of $\partial/\partial \Theta_1 + \tilde{\partial}/\tilde{\partial} \Theta_2$ given by $f_0(\Theta) = 1$ and $f_2(\Theta) = \Theta_1 \Theta_2$ of eigenvalue zero, $f_1(\Theta) = \Theta_1 \pm \Theta_2$ of eigenvalue ±1. The tensor product of both 2-spinor copies $\xi$ and $\chi$ then gives eigenstates of eigenvalues zero, ±$\hbar$ and ±$\pm \hbar$ for $\tilde{J}_1$. The time component $J^0$ has the same eigenstates.

A general state in $\mathcal{H} = \mathcal{H}_{\text{grav}} \otimes \mathcal{H}_{\text{fermion}}$ can then be written in a form using fermion dependent coefficient functions in the triad eigenbasis (27)

$$|\rangle = \sum_{\mu_1, \mu_2, k} s_{\mu_1, \mu_2, k}(\Theta)|\mu_1, \mu_2, k\rangle.$$

One can define the coefficients $s_{\mu_1, \mu_2, k}(\Theta)$ for all values of $\mu_1, \mu_2 \in \mathbb{R}$ and $k \in \mathbb{Z}$ in this way. However, gauge invariance implies that the state must be invariant under changing the sign of $\mu_2$ because this corresponds to a triad rotation (without changing orientation). Thus, we require $s_{\mu_1, -\mu_2, k}(\Theta) = s_{\mu_1, \mu_2, k}(\Theta)$.

The remaining sign freedom, sgn $\mu_1$, is physical and crucial because it determines the relative orientation of the triad. Thus, we have a simple action

$$s_{\mu_1, \mu_2, k}(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \rightarrow \tilde{s}_{\mu_1, -\mu_2, k}(\Theta_3, \Theta_4, \Theta_1, \Theta_2)$$

of the parity operator $\tilde{P}$ on states. For the fermion dependence, we have represented the parity action $\tilde{P}\Psi = \gamma^0\Psi$ for Dirac spinors by switching the fermion values $\Theta_a$ corresponding to $\xi$ and $\chi$, respectively. This implies

$$\tilde{P}\tilde{J}_0\tilde{P}^{-1} = -\tilde{J}_0, \quad \tilde{P}\tilde{J}_1\tilde{P}^{-1} = \tilde{J}_1.$$

For gravitational operators, a direct calculation shows
\[ \hat{\Gamma} \hat{p} \hat{\Gamma}^{-1} = -\hat{p}, \quad \hat{\Gamma} \hat{p}^2 \hat{\Gamma}^{-1} = \hat{p}^2 \]  
(35)

\[ \hat{\Gamma} \exp(i\mu_1 \tilde{c}_1/2) \hat{\Gamma}^{-1} = \exp(-i\mu_1 c_1/2), \quad \hat{\Gamma} \exp(i\mu_2 \tilde{c}_2/2) \hat{\Gamma}^{-1} = \exp(i\mu_2 \tilde{c}_2/2) \]  
(36)

as required.

Finally, we can directly solve the Gauss constraint which requires \( \hat{\rho}_\phi = \frac{1}{2} \gamma \kappa \hat{J}_1 \) and thus allows us to eliminate \( k \) as an independent quantity. Using the spectra of the operators already determined, this provides solutions with either \( k = 0 \) or \( k = \pm 1 \). In the second case, there is a non-vanishing value of the spatial axial current \( J_1 \) of size \( \pm 2\hbar \). The values \( \pm \hbar \) for the fermion current, which do exist as eigenvalues, are ruled out because they do not correspond to integer \( k \). Both 2-spinors present must thus have the same or opposite \( \hat{J}_1 \)-eigenvalues, which allows them to be parity eigenstates. The parity behavior of the full state according to (33), however, is determined by the \( \mu_1 \)-dependence, which required the dynamics of quantum gravity coupling the triad to fermions.

The allowed values for the current are only microscopic and may not seem of interest to describe a macroscopic universe of large matter content; they all vanish in the classical limit \( \hbar \to 0 \). Nevertheless, this provides an interesting model where one can study the effects of fermions and parity in loop quantum gravity. Physically, it is also clear why the matter contribution can only be microscopic: as always in homogeneous quantum cosmological models, each field component is reduced to a single degree of freedom for all of space. For the fermion, this allows only one excitation per component due to Pauli’s principle. Unlike with scalar matter, one cannot simply make the matter content large by choosing a high ‘occupation’ such as a large momentum of the scalar. Significant fermionic matter can only be included by adding more independent spinor fields, or by introducing inhomogeneity which provides independent field values at different points (represented by fermions at different vertices of a spin network state in loop quantum gravity). Rather than being a limitation, we consider this as an important physical property of quantum cosmology in the presence of realistic fermionic matter.

3.2. Quantum dynamics: the Hamiltonian constraint

A useful feature of the torsion-free Bianchi I model is that the Lorentzian Hamiltonian constraint is related to the Euclidean part simply by \( H = -\gamma^{-2} H^{(E)} \) thanks to \( K_{ik} K_{jk} \propto \gamma_{ab} \epsilon_{ijk} \), making use of homogeneity as well as the fact that the spin connection vanishes. This has been used in almost all investigations of loop quantum cosmology so far. If this relation is not used, one can still quantize the Lorentzian constraint following techniques of the full theory [33]. This results in a more complicated constraint operator [8], but without crucial differences.

However, in the presence of torsion, such a simple relationship can be obtained only after splitting the torsion contribution from the spin connection as shown in (A.9), which is now to be quantized: even for the Bianchi I model, \( \Gamma^a_\theta \) is no longer zero due to torsion. Fortunately, torsion contributions to \( \Gamma^a_\theta \), namely \( C^a_\theta \) in (A.6), are completely determined by second class constraints. They can thus be split off and quantized separately together with the matter terms. For the Bianchi I LRS model, one can use a further key simplification which, as pointed out above, allows us to project out torsion contributions without directly computing them. All we need to do is use the new variable \( \tilde{c}_2 \) instead of \( c_2 \). The resulting contribution to the gravitational Hamiltonian constraint is the same as the torsion-free one and thus can be quantized in the same way.

Mimicking the steps done in the full theory [15, 33], one writes curvature components \( F^I_{ab} \) as a product of (point) holonomies \( h_I = \cos(\frac{1}{2} \delta_I c_I) + 2\Lambda_I^a \tau_i \sin(\frac{1}{2} \delta_I c_I) \) forming a closed loop,
whose 'edge lengths' are denoted as $\delta_1$ and $\delta_2$ for the two independent directions. Moreover, using

$$\frac{1}{2} e_{abc} \varepsilon^{ijk} \frac{E_i^b E_j^c}{\sqrt{|\det(E^a_i)|}} = c'_a = \frac{2}{\gamma^2} \left\{ A'_b(x), V \right\}$$

(37)

relevant products of triad components, including their inverse powers, are reduced to a Poisson bracket of the general form $h_I \{ h^{-1}_J, V \}$ where $V$ is the spatial volume and $h_J$ again a holonomy. This allows one to write an operator in compact form, which corresponds to a densely defined operator in the full theory

$$\hat{H}_G = \frac{4i \operatorname{sgn}(\hat{p}^1 \hat{p}^2 \hat{p}^3)}{\gamma^3 \kappa \ell^2_0 \delta_1 \delta_2} \sum_{IJK} e_{ijk} \operatorname{tr}(h_I h_J h_K^{-1} h_J^{-1} h_K^{-1} \{ h^{-1}_K, \hat{V} \}).$$

(38)

We can now compute the product of holonomies and take the trace explicitly, using the basic properties of Pauli matrices. We do this directly for LRS variables with only two independent holonomies such that $\delta_2 = \delta_3$. Moreover, the sign factor is now solely determined by $\operatorname{sgn} \hat{p}^1$ since $\hat{p}^2 \hat{p}^3$ cannot be negative. This results in [20]

$$\hat{H}_G = \frac{32i \operatorname{sgn}(\hat{p}^1)}{\gamma^3 \kappa \ell^2_0 \delta_1 \delta_2} \left( 2 \sin(\frac{1}{2} \delta_1 c_1) \cos(\frac{1}{2} \delta_1 c_1) \sin(\frac{1}{2} \delta_2 c_2) \cos(\frac{1}{2} \delta_2 c_2) \right. \times (\sin(\frac{1}{2} \delta_2 c_2) \hat{V} \cos(\frac{1}{2} \delta_2 c_2) - \cos(\frac{1}{2} \delta_2 c_2) \hat{V} \sin(\frac{1}{2} \delta_2 c_2)) + \sin^2(\frac{1}{2} \delta_2 c_2) \times \left. \cos^2(\frac{1}{2} \delta_2 c_2) (\sin(\frac{1}{2} \delta_1 c_1) \hat{V} \cos(\frac{1}{2} \delta_1 c_1) - \cos(\frac{1}{2} \delta_1 c_1) \hat{V} \sin(\frac{1}{2} \delta_1 c_1)) \right)$$

(39)

Because we have implicitly eliminated the torsion contributions from holonomies by our choice of basic variables, we can directly use this expression as it is known from torsion-free models. The torsion contribution will then be added to the constraint operator via the fermion current.

We emphasize that the meaning and form of the parameters $\delta_1$ and $\delta_2$ cannot be fully elucidated purely in homogeneous models. In the absence so far of a derivation from a full, inhomogeneous constraint (which itself is currently subject to changes in its general form depending on ongoing developments) it appears best to refrain from specific, heuristic arguments as to what values they may take. (For instance, there is currently no firm basis for a relation of those parameters to an eigenvalue of the area operator of the full theory, as initially proposed in [34].) We therefore follow a more general route which allows whole classes of these parameters, and confine attention to effects which are insensitive to the specific form. To us, this seems most advisable given that it is not just the numerical values of these parameters but even their possible functional dependence on basic variables which remains open; see section 3.3 for further discussions.

In order to quantize the matter Hamiltonian, we must in particular quantize the inverse volume $1/p^2 \sqrt{|p|^3}$. Here, we use the standard procedure [15], first writing

$$\frac{1}{\sqrt{|\det(E^a_i)|}} = \frac{\operatorname{sgn} \det(e^a_i)}{6 |\det(E^a_i)|} e_{abc} e_{ijk} e^i_a e^j_b e^k_c$$

$$= \frac{36}{\gamma^3 \kappa} \operatorname{sgn} \det(e^a_i) e_{abc} e_{ijk} \left\{ A'_a, V^{1/3} \right\} \left\{ A'_b, V^{1/3} \right\} \left\{ A'_c, V^{1/3} \right\}$$
based on (37), which is then quantized to

$$\frac{1}{V} = \frac{144i \text{sgn}(b^3)}{\gamma^1 \epsilon_0^3 \delta_1 \delta_3} \sum_{IJK} \epsilon^{IJK} \text{tr}(h_I [h_{J}^{-1}, \hat{V}^{1/3}] h_J [h_{K}^{-1}, \hat{V}^{1/3}])$$

$$= \frac{12^3 \text{sgn}(b^3)}{\gamma^1 \epsilon_0^3 \delta_1 \delta_2} \left( \sin\left(\frac{1}{2} \delta_1 c_1 \right) \hat{V}^{1/3} \cos\left(\frac{1}{2} \delta_1 c_1 \right) - \cos\left(\frac{1}{2} \delta_1 c_1 \right) \hat{V}^{1/3} \sin\left(\frac{1}{2} \delta_1 c_1 \right) \right) \times \left( \sin\left(\frac{1}{2} \delta_2 c_2 \right) \hat{V}^{1/3} \cos\left(\frac{1}{2} \delta_2 c_2 \right) - \cos\left(\frac{1}{2} \delta_2 c_2 \right) \hat{V}^{1/3} \sin\left(\frac{1}{2} \delta_2 c_2 \right) \right)^2. \quad (40)$$

The action of this operator as well as the Hamiltonian constraint is easily computed using the action of \(\sin\left(\frac{1}{2} \delta_1 c_1 \right)\) and \(\cos\left(\frac{1}{2} \delta_1 c_1 \right)\) on the triad eigenstates,

$$\cos\left(\frac{1}{2} \delta_1 c_1 \right) \left| \mu_1, \mu_2, k \right> = \frac{1}{2} \left( | \mu_1 + \delta_1, \mu_2, k > + | \mu_1 - \delta_1, \mu_2, k > \right)$$

$$\sin\left(\frac{1}{2} \delta_1 c_1 \right) \left| \mu_1, \mu_2, k \right> = -\frac{i}{2} \left( | \mu_1 + \delta_1, \mu_2, k > - | \mu_1 - \delta_1, \mu_2, k > \right)$$

and the volume operator (29). From matrix elements of the Hamiltonian constraint one can then write the constraint equation \(\{ \hat{H}_C + \hat{H}_{\text{matter}}, \gamma^I \} = 0\) as a difference equation for coefficients \(s_{\mu_1, \mu_2, k}(\Theta)\) of the state in the triad representation. We do this immediately on states solving the Gauss constraint which determines \(k\) in terms of the action of \(\hat{J}_I\). Dropping the label \(k\) on those states, we have

$$2(|\mu_2 + 3\delta_2| - |\mu_2 + \delta_2|) \left( |\mu_1 + 2\delta_1 |^{1/2} s_{\mu_1 + 2\delta_1, \mu_2 + 2\delta_2}(\Theta) - |\mu_1 - 2\delta_1 |^{1/2} s_{\mu_1 - 2\delta_1, \mu_2 + 2\delta_2}(\Theta) \right)$$

$$+ 2(|\mu_2 - \delta_2| - |\mu_2 - 3\delta_2|) \left( |\mu_1 + 2\delta_1 |^{1/2} s_{\mu_1 + 2\delta_1, \mu_2 - 2\delta_2}(\Theta) - |\mu_1 - 2\delta_1 |^{1/2} s_{\mu_1 - 2\delta_1, \mu_2 - 2\delta_2}(\Theta) \right)$$

$$- |\mu_1 + \delta_1 |^{1/2} s_{\mu_1 + \delta_1, \mu_2 - 2\delta_2}(\Theta) + |\mu_1 - \delta_1 |^{1/2} s_{\mu_1 - \delta_1, \mu_2 + 2\delta_2}(\Theta)$$

$$- 2 |\mu_1 |^{1/2} s_{\mu_1, \mu_2}(\Theta) + |\mu_2 - 3\delta_2 |^{1/2} s_{\mu_1, \mu_2 - 4\delta_2}(\Theta)$$

$$= -\frac{27}{8} |\mu_1 |^{1/3} |\mu_2 |^{1/3} \left( |\mu_1 + \delta_1 |^{1/6} - |\mu_1 - \delta_1 |^{1/6} \right) \left( |\mu_2 + \delta_2 |^{1/3} - |\mu_2 - \delta_2 |^{1/3} \right)$$

$$\times \left( \left( 1 + 4\gamma^2 - \frac{2\gamma \beta}{1 + \gamma^2} \left( 3 - \frac{\gamma}{\alpha} + 2\gamma^2 \right) - \frac{\theta^2}{1 + \gamma^2} \right) \frac{\gamma^2}{h^2} \right.$$

$$\left. + 3\gamma \theta \left( \frac{2}{\alpha} + \frac{\gamma \theta}{1 + \gamma^2} \right) \frac{\gamma^2}{h^2} \right) s_{\mu_1, \mu_2}(\Theta). \quad (42)$$

This equation is based on a non-symmetric constraint operator because in (39) we ordered all holonomy factors to the left and kept the commutator terms with the volume operator to the right. It is sometimes useful to have a symmetric ordering, although this is not strictly required for constraints. (But it is required by some methods to derive the physical Hilbert space.) There is only one way to order the constraint symmetrically, namely by introducing \(\frac{1}{2} (\hat{H} + \hat{H}^\dagger)\). Other possibilities have been suggested, such as splitting the sines and cosines and writing some to the left, others to the right of the commutator term. They are, for instance, useful to prove self-adjointness [35]. However, this corresponds to splitting the holonomy product \(h_I h_J h_{J}^{-1} h_{I}^{-1}\) into different factors, which cannot be done in a general setting where there would rather be a single holonomy \(h_a\) around a closed loop \(a\). The direct symmetrization, on the other hand, is always possible and in our case results in a difference equation

$$2 \left( (|\mu_2 + 3\delta_2| - |\mu_2 + \delta_2|)|\mu_1 + 2\delta_1 |^{1/2} + (|\mu_2 + \delta_2| - |\mu_2 - \delta_2|)|\mu_1 |^{1/2} \right) s_{\mu_1 + 2\delta_1, \mu_2 + 2\delta_2}(\Theta)$$

$$- 2 ((|\mu_2 + 3\delta_2| - |\mu_2 + \delta_2|)|\mu_1 - 2\delta_1 |^{1/2} + (|\mu_2 + \delta_2| - |\mu_2 - \delta_2|)|\mu_1 |^{1/2} \right) s_{\mu_1 - 2\delta_1, \mu_2 + 2\delta_2}(\Theta)$$

$$+ 2 ((|\mu_2 - \delta_2| - |\mu_2 - 3\delta_2|)|\mu_1 |^{1/2} - 2\delta_1 |^{1/2}$$

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3.3. Lattice refinement

So far, we have left the increments $\delta_1$ and $\delta_2$ unspecified. It is clear that as constants they would not influence the recurrence behavior of the difference equation, although specific solutions certainly depend on their values. However, in general $\delta_1$ and $\delta_2$ may not be constant but be functions of $\mu_1$ and $\mu_2$; this captures the way in which the discrete structure of a state underlying spatial expansion and contraction in loop quantum gravity is being refined dynamically [26, 36, 37]: at larger $\mu_1$, an increment of the total size by a Planck-scale amount has a weaker relative influence on the geometry. As a consequence, $\delta_I$ decrease with increasing spatial extensions. This can also be seen from more direct considerations of holonomies in inhomogeneous states and how they appear in Hamiltonian constraint operators. Since this involves the dynamical relation between models and a full non-symmetric theory, the precise behavior of lattice refinement has not been completely determined. (Since the parameters $\delta_I$ are related to edge lengths of spin network states, a derivation would require one to construct specific inhomogeneous states which correspond to an anisotropic geometry. Then, the action of a full Hamiltonian constraint operator would have to be projected to the homogeneous states; see [36] for further details.) However, consequences of different behaviors can be explored in several models. Sometimes, this is already quite restrictive even though it is impossible to derive a unique form of lattice refinement based solely on homogeneous models.

Non-trivial functions, such as power laws, have a much stronger influence than constants because they make the difference equation non-equidistant. Solutions are then more difficult to analyze and find, even numerically (but see [38, 39]). Only in the special cases where $\delta_1 \propto \mu_1^I$ and $\delta_2 \propto \mu_2^I$ can the equation be mapped to an equidistant one by a redefinition of independent variables. However, such cases have been ruled out [40] because they do not provide the correct semiclassical behavior near a horizon of Schwarzschild black holes, whose interior is treated as a homogeneous Kantowski–Sachs model. (The analysis in [40] uses corrections to classical equations due to the use of holonomies in the loop quantization, but ignores other effects such as quantum back-reaction [41, 42]. This type of phenomenological equations may not capture correctly the behavior of strong quantum regimes such as the black hole singularity. However, if these equations do not provide the correct semiclassical behavior in classical regimes, this cannot be corrected by the inclusion of quantum back-reaction. The fact that some refinement schemes are ruled out is thus a reliable feature.) In general, one has to expect functions of the form $\delta_1(\mu_1, \mu_2)$ and $\delta_2(\mu_1, \mu_2)$ with a non-trivial dependence on both arguments (which may not be of power-law form).

\[
\begin{align*}
+ (|\mu_2 + \delta_2| - |\mu_2 - \delta_2|)|\mu_1|^{1/2} s_{\mu_1 - 2\delta_1, \mu_2 - 2\delta_2}(\Theta)
- 2(|\mu_2 - \delta_2| - |\mu_2 - 3\delta_2|)|\mu_1 + 2\delta_1|^{1/2}
+ (|\mu_2 + \delta_2| - |\mu_2 - \delta_2|)|\mu_1|^{1/2} s_{\mu_1 + 2\delta_1, \mu_2 - 2\delta_2}(\Theta)
+ (|\mu_1 + \delta_1|^{1/2} - |\mu_1 - \delta_1|^{1/2})(|\mu_2| + |\mu_2 + 4\delta_1|) s_{\mu_1, \mu_2 + 4\delta_1}(\Theta)
- 4|\mu_2| s_{\mu_1, \mu_2}(\Theta) + (|\mu_2| + |\mu_2 - 4\delta_1|) s_{\mu_1, \mu_2 - 4\delta_1}(\Theta)
= - \frac{27}{4} |\mu_1|^{1/3} |\mu_2|^{1/3} (|\mu_1 + \delta_1|^{1/6} - |\mu_1 - \delta_1|^{1/6})(|\mu_2 + \delta_2|^{1/3} - |\mu_2 - \delta_2|^{1/3})^2
\times \left(1 + 4\gamma^2 - \frac{2\gamma\beta}{1 + \gamma^2} \left(3 - \frac{\gamma}{\alpha} + 2\gamma^2\right) - \frac{\theta^2}{1 + \gamma^2}\right) \frac{J_0^2}{\hbar^2}
+ 3\gamma \theta \left(\frac{2}{\alpha} + \frac{\gamma\theta}{1 + \gamma^2}\right) \frac{J_0^2}{\hbar^2} s_{\mu_1, \mu_2}(\Theta). \quad (43)
\end{align*}
\]
As we will see below, a discussion of fundamental singularity resolution only involves the recurrence near $\mu_1 = 0$. This is, fortunately, insensitive to the particular refinement scheme and thus presents a result of much wider generality than anything which applies at larger volume where the specific refinement can be crucial. The constructions and results of this paper are thus valid for any functional behavior of $\mu_1$ on phase space variables, even though finding explicit or numerical solutions to the difference equation would be more complicated in general.

4. Cosmological implications

It follows immediately from the difference equation (42) or (43) that it is parity invariant since all its terms change sign under (33). Thus, if $s_{\mu_1, \mu_2}(\Theta_1, \Theta_2, \Theta_3, \Theta_4)$ is a solution, so is $s_{\mu_1, \mu_2}(\Theta_3, \Theta_4, \Theta_1, \Theta_2)$. In particular, any solution can be written as a combination of even and odd solutions $s_{\mu_1, \mu_2}(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \pm s_{\mu_1, \mu_2}(\Theta_3, \Theta_4, \Theta_1, \Theta_2)$. This is no longer the case if we had matter interactions violating parity, such as a term proportional to $V_0 J_0^I$. In this case, no parity-even or odd solutions would exist. Wavefunctions for $\mu_1 > 0$ generically differ from their form for $\mu_1 < 0$, even though those values are deterministically related via the difference equation. At this stage, the precise form of parity violations in the matter system is crucial to determine the behavior of the wavefunction near the classical singularity at $\mu_1 = 0$.

To complete the construction, one would solve the difference equation and determine a physical inner product on the solution space. Ideally, one could then compute the behavior of observables of the system and derive detailed cosmological scenarios including the role of quantum effects. Unfortunately, such complete descriptions at an exact level are possible only in rare, specific models. While such models are instructive mathematically, conclusions drawn are difficult to interpret because one could not be certain about the robustness of results: if specific results are available only in a few special models where exact mathematical solutions in the physical Hilbert space can be found, there is no guarantee that they are not just the very result only of demanding this high mathematical control.

In this context, an aspect of particular interest is the fact that most models of loop quantum cosmology where physical Hilbert spaces have been constructed explicitly [13, 43, 44] specifically assume parity invariance in some form and make use of the corresponding restriction of states when parity is considered as a large gauge transformation. As we have seen here, physical states of quantum cosmology are neither even nor odd in triad reflections if parity-violating matter is present. It may thus be misleading to treat parity as a large gauge transformation even in cases where matter preserves parity. Results based on this assumption may be spurious, and one has to re-analyze the constructions of physical Hilbert spaces without the assumption of parity invariant states. Fortunately, the intuitive pictures of bounces which have sometimes been derived from physical observables are insensitive to the specific construction of the physical Hilbert space: they can be derived analytically in a representation independent formalism based on effective equations [10, 45]. Then, the assumption of parity as a large gauge transformation is not necessary, and it can be dropped without affecting the bounce result.

At a fundamental level, singularity resolution is also insensitive to the physical Hilbert space construction and can directly be determined using the difference equation (42) or (43). (Here, it is important that all solutions are non-singular, which then also include physical ones.) In general, coefficients of a difference equation of the type obtained in loop quantum cosmology may vanish and prevent certain values of $s_{\mu_1, \mu_2}$ from being determined in a recurrence starting from initial values. This happens for the non-symmetric equation (42) where none of the values $\psi_{0, \mu_2}$—right at the classical singularity—is determined by initial values because their
coefficients in the difference equation vanish. (The corresponding states $|0, \mu_2\rangle$ are mantic [63].) However, for the difference equations realized such undetermined values, if they arise, drop out completely of the recurrence. In particular, even though values for $\mu_1 = 0$ remain undetermined by initial values in the non-symmetrized version of the equation, coefficients at $\mu_1 < 0$ follow deterministically from coefficients at $\mu_1 > 0$.

In parity preserving models the wavefunction $s_{\mu_1, \mu_2}$ for $\mu_1 < 0$ could simply be the mirror image of its cousin at $\mu_1 > 0$, and it had to be symmetric if parity is considered a large gauge transformation. However, if there is parity violation, the transition through $\mu_1 = 0$ constitutes true evolution since values at $\mu_1 < 0$ must now differ from the mirror image at $\mu_1 > 0$. The wavefunction at $\mu_1 < 0$ cannot be determined simply by reflection, but it has to be derived by local evolution through all intermediate values of $\mu_1$. In this case, the region of $\mu_1 < 0$ can by no means be removed from considerations but must be considered as a physical domain on equal footing with that at $\mu_1 > 0$. In particular, the orientation-reversing big bang transition thus becomes physical and cannot be argued away as a large gauge transformation.

For both forms of difference equations derived here, there are consistency conditions arising due to vanishing coefficients around $\mu_1 = 0$, analogous to dynamical initial conditions [11, 12]. If we evaluate any of the difference equations at $\mu_1 = 0$, matter terms drop out and we obtain the universal relation

$$((|\mu_2 + 3\delta_2| - |\mu_2 + \delta_2|) s_{2\delta_0, \mu_2 + 2\delta_2} - (|\mu_2 - \delta_2| - |\mu_2 - 3\delta_2|) s_{2\delta_0, \mu_2 - 2\delta_2}$$

$$= (|\mu_2 + 3\delta_2| - |\mu_2 + \delta_2|) s_{-2\delta_0, \mu_2 + 2\delta_2} - (|\mu_2 - \delta_2| - |\mu_2 - 3\delta_2|) s_{-2\delta_0, \mu_2 - 2\delta_2}$$

(44)

valid for all $\mu_2$. In particular, at $\mu_2 = 2\delta_2$ we have $s_{2\delta_0, 4\delta_2} = s_{-2\delta_0, 4\delta_2}$. At odd integer multiples of $\mu_2 = 2\delta_2$, we obtain a recurrence relation which requires $s_{2\delta_0, 2(2n+1)\delta_2} = s_{-2\delta_0, 2(2n+1)\delta_2}$ for all integer $n$.

There are thus reflection symmetry conditions which directly follow from the dynamical law even in the presence of parity-violating terms. (This symmetry has been observed first in the vacuum case [46].) However, evolution away from $\mu_1 = \pm 1$ depends on whether $\mu_1$ is positive or negative if parity is not preserved: unlike equation (44), the matter Hamiltonian then enters the recurrence and for parity-violating matter the coefficients of the difference equation at negative and positive $\mu_1$ differ. Thus, the wavefunction is not mirror symmetric even though the dynamical initial condition closely ties the values $s_{\pm 2\delta_0, \mu_2}$ to each other.

5. Conclusions

We have introduced fermions into the framework of loop quantum cosmology which gave rise to several non-trivial changes due to the presence of torsion and potential parity non-invariance. We have observed several key features which have a bearing on cosmological scenarios and which do not arise for bosonic matter such as scalar fields as they are used commonly in cosmological models. First, the amount of matter is limited for each fermionic degree of freedom due to the exclusion principle. Thus, large matter contents as they are sometimes used to bring a quantum cosmological model into a semiclassical regime where it may bounce more easily cannot straightforwardly be achieved. The only possibilities are to allow many copies of independent fermions or inhomogeneity where fermionic components at different points will be independent. Physically, both possibilities are quite different from having a single bosonic field of high occupation. The methods used here may also be of interest for a supersymmetric version of loop quantum cosmology along, e.g., the lines of [47] (see also [48]). Fermions in quantum cosmology also play a role for decoherence [49].
This shows that it is crucial to consider the small-volume regime of a quantum cosmological model which cannot be avoided in the absence of much matter energy. Here, the recurrence scheme of an underlying difference equation of loop quantum cosmology becomes essential to determine whether the model is singular or not. As we showed, the singularity resolution mechanism of loop quantum cosmology [6] remains unchanged under the inclusion of fermionic matter even if it violates parity. At the same time, the model we used allows us to show that in its realm parity violations can only arise due to matter interactions, not due to pure gravity. In other models or the full theory, this situation may be different because the basic objects quantized, in particular holonomies, do not transform straightforwardly under parity. The model introduced here thus also serves the purpose of providing one example where parity invariance of pure gravity can be demonstrated after a loop quantization.

If one introduces parity-violating interactions, wavefunctions cannot be mirror symmetric. Then, the branches at the two opposite orientations of triads are independent of each other, and joined through degenerate geometries by the dynamics of loop quantum cosmology. The big bang transition now becomes a non-trivial event where space turned its inside out in a quantum process which in general cannot be described by an intuitive geometrical picture such as a simple bounce.

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Appendix A. The full constraints

To set the notations, the basic configuration variables in a Lagrangian formulation of fermionic field theory are the Dirac bi-spinor \( \Psi = (\psi, \eta)^T \) and its complex conjugate in \( \Psi = (\Psi^*)^T \gamma^0 \) with \( \gamma^a \) being the Minkowski signature Dirac matrices. We note that \( \psi \) and \( \eta \) transform with density weight zero and are spinors according to the fundamental representation of \( SL(2, \mathbb{C}) \).

Then the non-minimal coupling of gravity to fermions can be expressed by the total action composed of the gravitational contribution \( S_G \) and the matter contribution \( S_F \) resulting from the fermion field

\[
S[e, \omega, \Psi] = S_G[e, \omega] + S_F[e, \omega, \Psi] = \frac{1}{16\pi G} \int_M d^4x |e| e^\mu_I e^\nu_J P^{IJ}_{KL} F_{\mu
u}^{KL}(\omega) + \frac{1}{2} \int_M d^4x |e| \left( \nabla \gamma^I e^\mu_I \left( 1 - \frac{i}{\alpha} \gamma_5 \right) \nabla \Psi - \nabla \Psi \left( 1 - \frac{i}{\alpha} \gamma_5 \right) \gamma^I e^\mu_I \Psi \right),
\]

where \( I, J, \ldots = 0, 1, 2, 3 \) denote the internal Lorentz indices and \( \mu, \nu, \ldots = 0, 1, 2, 3 \) spacetime indices, and \( \alpha \in \mathbb{R} \) is the parameter for non-minimal coupling. (We keep \( \alpha \) free for generality. The value \( \alpha = \gamma \) corresponds to minimal coupling of fermions to Einstein–Cartan gravity [23], while \( \alpha \to \infty \) would be the minimal case from the point of view of the Holst action. The action is parity invariant for all real values of \( \alpha \), although torsion components have the expected parity behavior only if \( \alpha = \gamma \) [16, 23]. Results of this paper are insensitive to the specific coupling and are thus kept general.)

We have expressed the gravitational action in terms of the tetrad field \( e^\mu_I \), where \( e \) is its determinant and \( e^\mu_I \) the inverse, using the Holst action [50]. It presents a first-order formulation where the Lorentz connection, denoted by \( \omega^I_{\mu
u} \) and with curvature \( F^{KL}_{\mu
u}(\omega) = 2\partial_{[\mu} \omega^I_{\nu]} + \left[ \omega_{\mu\nu}, \omega_{\delta\rho} \right]^I_J \), is treated as a variable independent of the tetrad before equations of
motion are imposed. In the Holst action, we have
\[ P^{IJ}_{\ KL} = \frac{\delta^{[I}_{K} \delta^{J]}_{L}}{\gamma} - \frac{1}{\gamma} \frac{\epsilon^{IJ}_{\ KL}}{2} \]  
(A.2)
with inverse
\[ P^{-1}_{IJ}^{\ K\ L} = \frac{\gamma^{2}}{\gamma^{2} + 1} \left( \frac{\delta^{[I}_{J} \delta^{J]}_{K}}{\gamma} + \frac{1}{\gamma} \frac{\epsilon^{IJ}_{\ K\ L}}{2} \right) \]  
(A.3)
where \( \gamma \) is the Barbero–Immirzi parameter [51, 52]. The connection appears also in the matter part via the covariant derivative \( \nabla_{\mu} \) of Dirac spinors defined by
\[ \nabla_{\mu} \equiv \partial_{\mu} + \frac{1}{4} \omega_{\mu}^{IJ} \gamma_{IJK}, \quad [\nabla_{\mu}, \nabla_{\nu}] = \frac{1}{4} F^{IJ}_{\mu\nu} \gamma^{IJK} \]  
(A.4)
in terms of Dirac matrices \( \gamma_{I} \) (which will always carry an index such that no confusion with the Barbero–Immirzi parameter should arise).

For \( \alpha \to \infty \) we have minimal coupling from the viewpoint of the Holst action, while \( \alpha = \gamma \) corresponds to minimal coupling from the viewpoint of Einstein–Cartan theory [23]. In this paper, we allow all possible real values of \( \alpha \) as introduced in [23] (see also [53]). We emphasize that we have parity invariance for all real \( \alpha \), even though some torsion components such as those written below consist of contributions of different parity behavior.

A canonical analysis of the action yields first and second class constraints. To summarize the result of this analysis [16], we use \( \kappa = 8\pi G, \tau_{j} = -\frac{i}{2} \sigma_{j} \) in terms of Pauli matrices, the axial fermion current components
\[ J_{I} = \psi^{\dagger} \sigma^{I} \psi + \eta^{\dagger} \sigma^{I} \eta, \quad J^{0} = \psi^{\dagger} \psi - \eta^{\dagger} \eta \]  
(A.5)
and the parameters
\[ \theta = 1 - \frac{\gamma}{\alpha} \quad \text{and} \quad \beta = \gamma + \frac{1}{\alpha} \]  
(A.6)
which are useful as a shortcut. For a consistent loop quantization, the half-densitized \( \xi := \sqrt{q} \psi \) instead of \( \psi \) (and \( \chi := \sqrt{q} \eta \) instead of \( \eta \)) is required to be the classical canonical variable for fermions [14], and \( \pi_{\xi} = -i \xi^{\dagger} \) is the conjugate momentum for \( \xi \), using the spatial metric \( q_{ab} \) and its determinant \( q \).

Upon solving the second class constraints, which provides the expression
\[ C_{a}^{i} = \frac{\gamma^{2}}{4(1 + \gamma^{2})} \left( \theta \epsilon_{ij}^{[i} \epsilon_{a]j} - \beta \epsilon_{ij}^{[i} \right) \]  
(A.7)
for the torsion contribution to the spin connection \( \Gamma_{a}^{i} \), the usual first class constraints remain: the Gauss constraint
\[ G[\Lambda] := \int_{\Sigma} d^{3}x \ \Lambda^{i} \left( D_{b} P_{a}^{b} - \frac{1}{2} \sqrt{q} J_{i} \right) = \int_{\Sigma} d^{3}x \ \Lambda^{i} \left( D_{b} P_{a}^{b} - \pi_{\xi} \tau_{i} \xi - \pi_{\chi} \tau_{i} \chi \right) \]  
(A.8)
the diffeomorphism constraint
\[ D[N^{a}] := \int_{\Sigma} d^{3}x \ N^{a} \left( 2 \left( P_{a}^{b} \partial_{[a} A_{b]}^{i} - A_{i}^{a} \partial_{b} P_{a}^{b} \right) + \frac{1}{2} \left( \pi_{\xi} \partial_{a} \xi - (\partial_{a} \pi_{\xi}) \xi - \pi_{\chi} \partial_{a} \chi - (\partial_{a} \pi_{\chi}) \chi \right) \right) \]
and the Hamiltonian constraint (modulo Gauss constraint)
\[ H_{\text{total}}[N] = \int_{\Sigma} d^{3}x \ N \left( \frac{\gamma^{2} \kappa}{\sqrt{q}} \ P_{i}^{a} P_{j}^{b} \epsilon_{ij}^{[a} \mathcal{F}_{ab]}^{j} - 2(\gamma^{2} + 1) \mathcal{K}_{[a}^{i} \mathcal{K}_{b]}^{j} \right) + \frac{\gamma^{2} \kappa}{\sqrt{q}} P_{i}^{a} \Gamma_{a}^{i} \right) \]  
(A.9)
expressed in terms of the densitized axial fermion current components

\[ J^i = \xi^\dagger \sigma^i \xi + \chi^\dagger \sigma^i \chi, \quad J^0 = \xi^\dagger \xi - \chi^\dagger \chi. \]  

(A.10)

The Hamiltonian constraint has been written in a useful form after splitting the spin connection \( \Gamma^i_a = \tilde{\Gamma}^i_a + C^i_a \) with the torsion-free connection \( \tilde{\Gamma}^i_a \). As usually, \( N \) and \( N^a \) are, respectively, the lapse function and shift vector used to foliate the spacetime manifold \( M \).

As basic gravitational variables we use a canonical pair given by the densitized triad \( P^a_i = \frac{E^a_i}{\gamma \kappa} \) together with the connection \( A^i_a = \tilde{A}^i_a + \gamma \tilde{K}^i_a + \kappa \gamma \frac{\epsilon^i_{jk} e^j_a J^k}{4(1 + \gamma^2)} \) (A.11)

where

\[ C^i_a = \frac{\alpha}{4} \left( \frac{1}{\gamma} \epsilon^i_{jk} e^j_a J^k - e^i_a J^0 \right) \]  

(A.12)

whose curvature and covariant derivative we denote as \( \mathcal{F}^k_{ab} \) and \( D \), respectively. The \( J^0 \)-term in the connection \( A^i_a \), compared to the Ashtekar–Barbero connection, is required for a formulation in terms of half-densitized fermions. Solutions of the second class constraints tell us only what the torsion contribution \( C^i_a \) to the spin connection is. To know the torsion contribution to extrinsic curvature \( K^i_a \), and thus to the Ashtekar–Barbero connection, one has to partially solve equations of motion. Doing so [16], the \( J^0 \)-term is cancelled and we can write

\[ A^i_a = \tilde{\Gamma}^i_a + \gamma \tilde{K}^i_a + \frac{\kappa \gamma}{4} \epsilon^i_{jk} e^j_a J^k. \]  

(A.13)

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