A characterization of the ellipsoid by planar grazes

I. Gonzalez-García¹, J. Jerónimo-Castro², D. J. Ver dusco-Hernández³, E. Morales-Amaya⁴

¹,²Facultad de Ingeniería
Universidad Autónoma de Querétaro, México
³,⁴Facultad de Matemáticas-Acapulco,
Universidad Autónoma de Guerrero, México

Dedicated to David George Larman

Abstract

In this paper we proved the following: \( K, L \subset \mathbb{R}^3 \) be two \( \mathcal{O} \)-symmetric convex bodies with \( L \subset \text{int}K \) strictly convex. Suppose that from every \( x \) in \( \text{bd}K \) the graze \( \Sigma(L, x) \) is a planar curve and \( K \) is almost free with respect to \( L \). Then \( L \) is an ellipsoid.

1 Introduction

A very important problem in Geometric Tomography is to establish properties of a given convex body, i.e., a compact and convex set with non-empty interior, if we know some properties over its sections or its projections. An interesting result was given by C. A. Rogers [12]:

\( \text{Let } K \subset \mathbb{R}^n \text{ be a convex body, } n \geq 3, \text{ and } p \text{ be a point in } \mathbb{R}^n. \text{ If all the 2-dimensional sections of } K \text{ through } p \text{ have a centre of symmetry, then } K \text{ also has a centre of symmetry.} \)

In the same article, Rogers also conjectured that \( K \) must be an ellipsoid if we also have the condition that \( p \) is not the centre of symmetry of \( K \). This conjecture was first
proved by P. W. Aitchison, C. M. Petty, and C. A. Rogers \[1\] and is known as the False Centre Theorem. Later, L. Montejano and E. Morales-Amaya gave a simpler proof of the False Centre Theorem in \[10\]. In the following theorem, due to S. Olovjanishnikov \[11\], the considered sections of \(K\) are not concurrent:

A convex body \(K \subset \mathbb{R}^n, n \geq 3\), is an ellipsoid provided all hyperplanar sections of \(K\) that divide the volume of \(K\) in a given ratio \(\lambda \neq 1\) have a centre of symmetry.

In \[2\], J. A. Barker, and D. G. Larman conjectured the following, which is a variant of Olovjanishnikov’s theorem: Suppose \(K, L \subset \mathbb{R}^n\) are convex bodies, with \(n \geq 3\) and \(L \subset \text{int} K\). Suppose that whenever \(H\) is a hyperplane supporting \(L\) the section \(H \cap K\) of \(K\) is centrally symmetric. Then \(K\) is an ellipsoid.

Another variant of Olovjanishnikov’s theorem would be the following conjecture by G. Bianchi, and P. M. Gruber \[3\]: Let \(K\) be a convex body in \(\mathbb{R}^n, n \geq 3\), and let \(\delta\) be a continuous real function on \(S^{n-1}\) such that for each vector \(u \in S^{n-1}\) the hyperplane \(\{x : \langle x, u \rangle = \delta(u)\}\) intersects the interior of \(K\). If any such intersection is centrally symmetric and (with, possibly, a few exceptions) does not contain a possibly existing centre of \(K\), then \(K\) is an ellipsoid.

There is another kind of characterizations of the ellipsoid considering properties of the intersections of cones or cylinders with the body. Given a point \(x \in \mathbb{R}^n \setminus K\) we denote the cone generated by \(K\) with apex \(x\) by \(C(K, x)\), i.e., \(C(K, x) := \{x + \lambda(y - x) : y \in K, \lambda \geq 0\}\). The boundary of \(C(K, x)\) is denoted by \(S(K, x)\), in other words, \(S(K, x)\) is the support cone of \(K\) from the point \(x\). We denote the graze of \(K\) from \(x\) by \(\Sigma(K, x)\), i.e., \(\Sigma(K, x) := S(K, x) \cap \partial K\). Using this notion of grazes, A. Marchaud proved the following in \[9\]:

Let \(K \subset \mathbb{R}^3\) be a convex body and \(H\) be a plane which is either disjoint from \(K\) or meets \(K\) at a single point. Then \(K\) is an ellipsoid if for every point \(x \in H \setminus K\), the graze \(\Sigma(K, x)\) contains a planar convex curve \(\gamma\) such that \(\text{conv} \gamma \cap \text{int} K \neq \emptyset\).

However, if we change the plane \(H\) for a surface \(\Gamma\) which encloses \(K\) it is not known whether \(K\) is an ellipsoid or not. Marchaud’s theorem is also true if the apexes of the cones are points at infinity. In this case the grazes are called shadow boundaries and are obtained by intersections of \(\partial K\) with circumscribed cylinders. The first proof that a convex body is an ellipsoid if and only if every shadow boundary lies in a hyperplane is due to H. Blaschke \[4\]. We suspect the following is true.

**Conjecture 1.** Let \(L \subset \text{int} K \subset \mathbb{R}^n\) be convex bodies such that for every point \(x \in \partial K\) it holds that \(\Sigma(L, x)\) lies in a hyperplane. Then \(L\) is an ellipsoid.

With the additional condition that the grazes are ellipses, it was proved in \[5\] that \(L\) is
an ellipsoid. In this work we give another progress in order to prove this conjecture, and
was very unexpected that under the hypotheses of Theorem 1 the grazes of the body $L$
result to be centrally symmetric (Lemma 1). Before we give the statement of the main
result in this work, we give some more definitions and notation. Let $L$ and $K$ be two
$O$-symmetric convex bodies in $\mathbb{R}^3$, with $L \subset \text{int} K$. We say that the points $x, y \in \text{bd} K$
are free with respect to $L$ if the line through $x$ and $y$, $\ell(x, y)$, does not meet $L$. Suppose
that for every point $x \in \text{bd} K$, the graze $\Sigma(L, x)$ is a planar curve and denote the plane
where it is contained by $\Delta_x$. The body $K$ is said to be almost free with respect to $L$ if
for each $z \in \text{bd} K$ and $w \in \Pi_z \cap \text{bd} K$, where $\Pi_z$ is a plane through $O$ parallel to $\Delta_z$,
the points $z$ and $w$ are free with respect to $L$ (see Fig. 1).

![Figure 1: $K$ is almost free with respect to $L$](image)

The main result of this article is the following.

**Theorem 1.** Let $K, L \subset \mathbb{R}^3$ be two $O$-symmetric convex bodies with $L \subset \text{int} K$ strictly
convex. Suppose that from every $x$ in $\text{bd} K$ the graze $\Sigma(L, x)$ is a planar curve and $K$
is almost free with respect to $L$. Then $L$ is an ellipsoid.

It is worth to notice that Theorem 1 is not only evidence for the veracity of Conjecture 1. Once we have the conclusion of Lemma 1 we arrive to a particular cases of, for
one hand, the Bianchi and Gruber’s conjecture and, on the other hand, the Baker and
Larman’s conjecture mentioned before, and Theorem 1 gives a positive answers to this
cases.
2 Main result

We first prove two lemmas.

**Lemma 1.** For each \( x \in \text{bd}K \) the graze \( \Sigma(L, x) \) is centrally symmetric with centre at the point \( O_x := \ell(x, -x) \cap \Delta_x \).

**Proof.** Let \( \Omega_x := S(L, x) \cap S(L, -x) \). By Lemma 2.2 in [8] we have that \( \Omega_x \) is a simple and closed curve, moreover, since \( L \) is centred at \( O \) we have that \( \Omega_x \) is centrally symmetric with centre at \( O \). Let \( a \in \Sigma(L, x) \) be any point and let \( z \) be the point where the line \( \ell(x, a) \) intersects \( \Omega_x \) (see Fig. 2). Consider the point \( b \in \Sigma(L, x) \) such that \([a, b]\) is an affine diameter of \( \Sigma(L, x) \). Suppose that \( O_x \not\in [a, b] \) and let \( a' \) be the point where \([-z, x]\) intersects \( \Sigma(L, x) \). Let \( H_a \) be a support plane of \( L \) through the points \( x \) and \( z \), and let \( \ell_a := \Delta_x \cap H_a \), and let \( \ell_z \) be the support lines of \( \Sigma(L, x) \) and \( \Omega_x \), through \( a \) and \( z \), respectively, with \( \ell_z \) parallel to \( \ell_a \). Let \( \ell_b \) be the support line of \( \Sigma(L, x) \) through \( b \) and parallel to \( \ell_a \) (this line exist since \([a, b]\) is an affine diameter of \( \Sigma(L, x) \)), and let \( H_b \) be the support plane of \( L \) through \( \ell_b \) and \( x \).

![Figure 2: \( \Omega_x \) is a planar and closed curve](image)
Now, since $\Omega_x$ is an $O$-symmetric set, the line $-\ell_z$ is a support line of $\Omega_x$ through $-z$. The plane $H_{a'}$ through $-\ell_z$ and $x$ contains the line $\ell(x, -z)$, hence, the line $\ell_{a'} := \Delta_x \cap H_{a'}$ is a support line of $\Sigma(L, x)$, parallel to $\ell_b$. This can happen only if $a' = b$, otherwise we obtain that one of $H_{a'}$ or $H_b$ is not a support plane of $L$. We have proved that the affine diameter $[a, b]$ passes through $O_x$, and the same happens for any other affine diameter of $\Sigma(L, x)$, by a theorem of P. C. Hammer [6] we have that $\Sigma(L, x)$ has centre of symmetry at $O_x$.

Lemma 2. For every $x \in \text{bd}K$ we have that $\Omega_x$ is a planar curve parallel to $\Sigma(L, x)$.

Proof. We use the notation of Lemma 1 and Fig. 2. Since the points $x, b, -z, z, a$, are coplanar and $O_x$ and $O$ are the midpoints of the segments $[b, a]$ and $[-z, z]$, respectively, by elementary Geometry we have that $[-z, z]$ must be parallel to $[b, a]$. It follows that $\Omega_x$ is parallel to $\Sigma(L, x)$, indeed, they are homothetic with centre of homothety at $x$.

Remark 1. If $K$ is a Euclidean ball then we can prove at this point that $L$ is also a Euclidean ball. It is not difficult to prove that for every $x \in \text{bd}K$, by Lemma 2, the intersection $S(L, x) \cap S(L, -x)$ is a planar curve contained in $x^\perp$. It was proved in [2] (Theorem 2) that under this condition the body $L$ is a Euclidean ball.

Lemma 3. For every $u \in S^2$ there exists $v(u) \in S^2$ such that for every $x \in u^\perp \cap \text{bd}K$, the plane $\Delta_x$ is parallel to $v(u)$.

Proof. Consider a point $x \in u^\perp \cap \text{bd}K$, and let $y, -y \in \text{bd}K$ be such that the planes $\Delta_y$, and $\Delta_{-y}$ are parallel to $u^\perp$. We claim that $\Delta_x$ is parallel to $\ell(y, -y)$. Suppose to the contrary that $\Delta_x$ is not parallel to $\ell(y, -y)$. By virtue of Lemma 1 we have that

$$\Sigma(L, -y) = \Sigma(L, y) - 2 \cdot O_y. \quad (1)$$

Let $\mu := \text{conv}(\Sigma(L, x) \cap \Sigma(L, y))$ be the chord with extreme points in the intersection $\Sigma(L, x) \cap \Sigma(L, y)$. Notice that the chord $\mu$ is well defined since $K$ is almost free with respect to $L$. By (1), the chord $\mu - 2 \cdot O_y$ belongs to $\Delta_{-y}$ but, since $\Delta_x$ is not parallel to $\ell(y, -y)$, this chord is not contained in $\Delta_x$. On the other hand, by virtue that $\Delta_y$ and $\Delta_{-y}$ are at the same distance from $u^\perp$ and given that $\Sigma(L, x)$ has centre at $O_x \in u^\perp$, the image $\ell$ of $\mu$, under the central symmetry with respect to $O_x$ restricted to the plane $\Delta_x$, is in $\Delta_{-y}$ (see Fig. 3). Applying the same argument for $-x$, it follows that $\Sigma(L, -y)$ has four parallel chords of the same length which contradicts the strict convexity of $L$. Thus $\Delta_x$ is parallel to $\ell(y, -y)$. Finally we define $v(u)$ as the unit vector parallel to $\ell(y, -y)$.
Figure 3: $\Delta_x$ is parallel to $\ell(y, -y)$

**Proof of Theorem 1.** Let $x$ be a point in $\mathrm{bd}K$ and we define the unit vector $z = \frac{x}{|x|}$. Let $u \in z^\perp \cap S^2$. We are going to prove that $u^\perp \cap \Delta_x$ is a line of affine symmetry of $\Sigma(L, x)$. In order to show this, we are going to prove that through the extreme points of the chords of $\Sigma(L, x)$ parallel to $v(u)$, $v(u)$ given by Lemma 3 there passes support lines of $\Sigma(L, x)$ that intersect each other in a point in $u^\perp \cap \Delta_x$. By this property and since one of the chords pass through the centre of $\Sigma(L, x)$, we have by Lemma 3 in [7] that $u^\perp \cap \Delta_x$ is a line of affine symmetry for $\Sigma(L, x)$.

Let $y \in \mathrm{bd}(u^\perp \cap K)$ be a point such that line $\ell(x, y)$ is contained in $u^\perp \setminus C(L, x)$ (see Fig. 4). We denote by $\Gamma_1, \Gamma_2$ the supporting planes of $L$ containing the line $\ell(x, y)$, by $a, b$ the common points between $\mathrm{bd}L$ and $\Gamma_1, \Gamma_2$, respectively. Since $\Gamma_1, \Gamma_2$ are supporting planes of $C(L, x)$ it follows $a, b \in \Sigma(L, x)$. Analogously we conclude that $a, b \in \Sigma(L, y)$. Hence $\ell(a, b) = \Delta_x \cap \Delta_y$. By Lemma 3 $\Delta_x$ and $\Delta_y$ are parallel to $v(u)$. Thus the line $\Delta_x \cap \Delta_y$ is parallel to $v(u)$. We denote by $L_1, L_2$ the lines $\Gamma_1 \cap \Delta_x, \Gamma_2 \cap \Delta_x$, respectively, and by $c$ the point $\ell(x, y) \cap \Delta_x$. It is clear that $c \in u^\perp \cap \Delta_x$ and that $L_1, L_2$ are supporting lines of $\Sigma(L, x)$ passing through $c$ and $a$ and $c$ and $b$, respectively. Varying $y \in \mathrm{bd}(u^\perp \cap K)$, such that line $\ell(x, y)$ is contained in $u^\perp \setminus C(L, x)$ it follows that $\Sigma(L, x)$ satisfies the required property and, consequently, $u^\perp \cap \Delta_x$ is a line of affine symmetry of $\Sigma(L, x)$. Finally varying $u \in (z^\perp \cap S^2)$ we conclude that $\Sigma(L, x)$ is an ellipse. Finally,
we conclude, using Theorem 5 in [5] that $L$ is an ellipsoid.

$\square$

Figure 4: $u^\perp \cap \Delta_x$ is a line of affine symmetry for $\Sigma(L, x)$

References

[1] P. W. Aitchison, C. M. Petty, and C. A. Rogers, *A convex body with a false centre is an ellipsoid*. Mathematika **18** (1971), 50 – 59.

[2] J. A. Barker, and D. G. Larman, *Determination of convex bodies by certain sets of sectional volumes*. Discrete Math. **241** (2001), 79 – 96.

[3] G. Bianchi, and P. M. Gruber, *Characterizations of ellipsoids*. Arch. Math. (Basel) **49** (1987), no. 4, 344 – 350.

[4] W. Blaschke, *Kreis und Kugel*. Göschen, Leipzig, (1916).

[5] I. González-García, J. Jerónimo-Castro, E. Morales-Amaya, and D. J. Ver dusco-Hernández, *Sections and projections of nested convex bodies*. Aequat. Math. **96** (2022), 885 – 900.
[6] P. C. Hammer, *Diameters of convex bodies*. Proc. Amer. Math. Soc. 5 (1954), 304 – 306.

[7] J. Jerónimo-Castro, *A characterization of the ellipse related to illumination bodies*. Elem. Math. 70 (2015), 95 – 102.

[8] J. Jerónimo-Castro, and T. B. McAllister, *Two characterizations of ellipsoidal cones*. J. Convex Anal. 20 (2013), 1181 – 1187.

[9] A. Marchaud, *Un théoreme sur les corps convexes*. Ann. Scient. École Norm. Supér. 76 (1959), 283 – 304.

[10] L. Montejano, and E. Morales-Amaya, *Variations of classic characterizations of ellipsoids and a short proof of the false centre theorem*. Mathematika 54 (2007), no. 1-2, 35 – 40.

[11] S. P. Olovjanishnikov, *On a characterization of the ellipsoid*. Ucen. Zap. Leningrad. State Univ. Ser. Mat. 83 (1941), 114 – 128.

[12] C. A. Rogers, *Sections and projections of convex bodies*. Port. Math. 24 (1965), 99 – 103.