Abstract
The present paper tries to answer the question: Can a de Sitter phase in presence of radiation be a competitor of the standard inflationary paradigm for the early universe? This kind of a de Sitter phase can exist in cosmological models where gravitational particle production takes place. To address the issue the metric perturbations in the de Sitter phase in presence of radiation must be known. The evolution of metric perturbations are explicitly calculated in the paper. It is seen that the evolution of scalar and vector perturbations are considerably different from standard inflationary models. These differences arise due to the particle production mechanism. The scalar perturbation power spectrum grows exponentially at small length scales. However, one cannot uniquely specify the scale at which this exponential growth starts because of the dependence of the power spectrum on the initial perturbation value. The time slice on which the initial perturbation starts is arbitrary. The arbitrariness of the initial time slice is related to the problem of setting the initial condition on the perturbations in the present model. The paper briefly opines on vector and tensor perturbations in the de Sitter space filled with radiation. It is seen that quantization of the perturbation modes in the present model is considerably difficult. One has to modify the present model to provide a consistent scheme of quantization of the perturbation modes.

Keywords de Sitter space · Cosmological perturbations · Particle production

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1 Introduction

In warm inflation models [1–4] the universe enters a de Sitter phase in presence of radiation. The radiation in the de Sitter phase (strictly speaking a quasi de Sitter phase) arises from the decay of the scalar inflaton field. The inflaton has thermal fluctuations and these thermal fluctuations act as source to the density fluctuations in the warm inflationary scenario. In this paper we revisit the idea of a de Sitter space in presence of radiation, but this radiation does not arise from the decay of any inflaton field. We assume that the very early universe was filled with radiation, as conventional cosmological models assume. A part of this radiation also arises from gravitational particle production in the de Sitter phase. Parker’s original work [5] showed that, in the cosmological backdrop, it is impossible to produce massless particles in a radiation dominated universe where the scale-factor of the universe $a(t) \propto t^{1/2}$. In the case of super cool inflation, Ford has also calculated the energy density due to the created particles [6]. Later it has been shown that one can think of gravitational production of massless particles, in a universe filled with radiation, if $a(t)$ is not proportional to $t^{1/2}$. If the universe, in the presence of radiation, is in a de Sitter phase then one can actually think of gravitational production of massless particles [7]. Authors have previously worked on de Sitter phase of the universe in presence of gravitational particle production [8, 9]. In this paper we present the cosmological metric perturbation of the universe passing through a de Sitter phase in presence of radiation following the phenomenological model of particle creation as proposed by Prigogine, Geheniau, Gunzig and Nardone in Ref. [10]. Gravitational particle creation in open systems was studied in detail in Ref. [10].

As particle creation can happen in a dynamical spacetime as the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, the produced particles can slowdown the rate at which energy density gets diluted in an expanding universe. For a long time people have been working phenomenologically on big bang models of the universe with particle creation [11, 12]. Previous workers have assumed various forms of the particle creation rate as in Ref. [13, 14]. The slowdown of the rate of the energy density of particles can be taken care of by a negative pressure component arising purely due
to the creation of particles. Thermodynamics of such a cosmological model has been studied in detail in Ref. [15]. Various models of cosmological particle creation have also been used by many authors to model the late development of the universe [16–18].

The negative pressure arising in models with gravitational particle production can produce a de Sitter space in the early universe if the rate of particle production in the early universe was high. If the universe is filled with radiation then one can have a de Sitter phase in presence of radiation. The cosmological models for such interesting spaces were discussed in detail by previous authors in Ref. [7]. Can this kind of a de Sitter space in presence of radiation, in the early phase of the universe, be a good competitor for the traditional super cool inflationary models where the de Sitter space originates due to the potential energy of one scalar field [19–21]? We have tried to address this important question in the present article by studying the dynamics of metric perturbations in de Sitter space in presence of radiation where gravitational particle production takes place. In the traditional models of inflation the inflaton field produces a de Sitter expansion and the perturbation of the inflaton field in conjunction with other metric perturbations produce the inhomogeneities present in the early universe. In the present case although we have a de Sitter phase in the early universe, the scenario lacks the inflaton perturbations. Only metric perturbations and the accompanying perturbations in the fluid parameters are present in our present model of accelerated expansion of the early Universe. The scalar and vector power spectrum in the present model are very different from corresponding power spectra in the standard inflationary models. The tensor perturbation result remains identical to the standard inflationary result at linear order where scalar and tensor perturbations decouple. However in higher orders the tensor perturbation power spectrum is affected by the modified scalar perturbation results and consequently the tensor perturbation power spectrum, in general, will also be different from the form it has in standard inflationary theories.

In Ref. [22] the authors tried to formulate a similar problem but with a different aim. In this work the authors used a different strategy to analyze the gauge invariant cosmological dynamics in presence of gravitational particle production. In the above reference the authors do not define any slow-roll parameters and consequently do not interpret the results in the light of a slow-roll process. The authors of Ref. [22] work out their result using adiabatic perturbation on comoving hypersurfaces by choosing the gauge where the fluid velocity potential vanishes. In the present paper we give a complete description of the initial de Sitter phase arising out of a universe filled with radiation and show how this de Sitter space dynamics can be modelled as a slow-roll process. In contrast to the above referred work the techniques used to analyze cosmological perturbations in the present paper are new and we think these techniques can now be refined in the near future to generalize our method of solving cosmological dynamics in open systems. We have presented a method by which one can compare the nature of perturbations in de Sitter phase in presence of radiation with simple inflationary model predictions.

One of the important observations of the present work is related to the seed perturbations or the initial value of the perturbations. Treated as a purely classical theory one has to assume the presence of the seed perturbations without any proper justification for their presence. On the other hand if one wants to quantize the scalar perturbations then the quantization procedure meets stiff challenge because of the ultraviolet
instability of the theory. In presence of ultraviolet instability (related to small scale perturbation modes) it becomes difficult to define a Bunch-Davies kind of vacuum. In the standard Bunch-Davies vacua all the modes of perturbation remains stable at an earlier (conformal) time, and the initial conditions are specified in that early phase. In the absence of such a scheme the issue about the initial conditions become problematic as the low (length) scale or high $k$ modes turns out to be non-perturbative. As a result of this the power spectrum in the subhorizon limit starts to grow steeply and the exact value of $k$ for which the power spectrum starts to become steep depends upon the initial conditions applied on an arbitrary time slice. In standard inflationary theories this problem does not arise. One probable way to address this issue is related to the assumption of a pre-de Sitter phase where the perturbation modes are well behaved at all length scales. In this paper we discuss the problem related to the initial conditions briefly in the concluding section. The proper solution of the initial value problem for the perturbations require more focussed research in this area in the future.

The material in the paper is organized in the following way. In the next section we present the relevant calculations and results required to understand the background (unperturbed) cosmology accompanied by gravitational particle production. In Sect. 3 we present the details about the de Sitter phase in presence of radiation. Section 4 presents the results related to scalar cosmological perturbations in the de Sitter model in presence of radiation. In Sect. 5 we briefly present the results related to vector and tensor perturbations in such models. The next section concludes the paper by summarizing the results obtained in it.

2 The cosmological model with gravitational particle production

In the present case we work with spatially flat FLRW metric

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + r^2 d\theta^2 + \sin^2\theta d\phi^2\right],$$

(1)

where $a(t)$ is the dimensionless scale-factor. The universe is pervaded by a fluid whose elements move with 4-velocity $u^\mu$ and the 4-velocity is normalized as $u^\mu u_\mu = 1$. For the background evolution $\bar{u}^\mu = (1, 0)$. In our notation all the barred quantities will stand for background values. The fluid 4-velocity is $u^\mu = \bar{u}^\mu + \delta u^\mu$, where $\delta u^\mu$ is the fluid 4-velocity perturbation. Unbarred quantities include the background value and the perturbative fluctuations. Here we are talking about metric perturbations. The energy-momentum tensor of the fluid is

$$T^\mu_\nu = (\rho + P)u^\mu u^\nu - P g^{\mu\nu},$$

(2)

where $\rho$ and $P$ are the energy density and pressure of the fluid. Here we assume the fluid to be a barotropic one with an equation of state

$$P = \omega \rho,$$

(3)
where \( \omega \) is a constant. The fluid is made up of particles whose number may increase with time. The particle flow vector is defined as:

\[
N^{\mu} = nu^{\mu},
\]

(4)

where \( n \) is the particle number density and is given by \( n = N/V \) where the number of particles present in a physical volume \( V = a^3 \) is \( N \). Due to matter creation there arises a pressure which gives rise to a new energy-momentum tensor as \([10, 23, 24]\)

\[
T_{c}^{\mu\nu} = P_{c}(u^{\mu}u^{\nu} - g^{\mu\nu}),
\]

(5)

where \( P_{c} \) is called the particle creation pressure. In the present paper the cosmological system is assumed to be an open thermodynamic system. The particle creation process makes the system open. This is a well known approach which started with the work of Prigogine in 1989 \([10]\). Later on many authors have used this approach. In this approach the cosmological system is receiving particles from some source. It is assumed that the source is generating the particles gravitationally. The exact method in which the source creates the particles is not so important for the analysis. For the above mentioned reason the cosmological system is assumed to be an open thermodynamic system which is continuously fed with new particles. The exact purpose for introducing the creation pressure is presented in Appendix A. The Einstein equation in presence of particle creation is then given by

\[
G^{\mu\nu} = \kappa (T_{f}^{\mu\nu} + T_{c}^{\mu\nu}),
\]

(6)

where \( \kappa = 8\pi G \). Since the two fluids exchange energy-momentum we have

\[
D_{\mu}(T_{f}^{\mu\nu} + T_{c}^{\mu\nu}) = 0,
\]

(7)

where the result of the action of the covariant derivative on a second rank contravariant tensor is \( D_{\mu}T^{\mu\nu} = \partial_{\mu}T^{\mu\nu} + \Gamma_{\mu\alpha}^{\mu}T^{\alpha\nu} + \Gamma_{\mu\alpha}^{\nu}T^{\mu\alpha} \), and in our convention

\[
\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2}g_{\mu\sigma}(\partial_{\beta}g_{\alpha\sigma} + \partial_{\alpha}g_{\beta\sigma} - \partial_{\sigma}g_{\alpha\beta}).
\]

In the present case the energy-momentum conservation equation for the unperturbed fluid is

\[
\dot{\rho} + 3H(\bar{\rho} + \bar{P} + \bar{P}_{c}) = 0.
\]

(8)

Here \( H \) is the Hubble parameter defined as \( H \equiv \dot{a}/a \). As particles are being produced due to cosmological expansion, we must have a conservation equation as \([24]\)

\[
D_{\mu}N^{\mu} = n\Gamma
\]

(9)

where \( \Gamma \) stands for the rate of change of particle number in a physical volume \( V \) and \( n \) is the number of particles per unit physical volume. The above relation is assumed
to be true for cosmological models with particle production although we do not know the value of $\Gamma$ precisely. We assume throughout our work that $\Gamma \geq 0$.

We will see later that the creation rate can be estimated by using a basic principle and phenomenological methods. As because expansion of the universe is at the root of gravitational particle production we assume that the particle creation rate is dependent on some parameter which specifies the expansion of the universe. The only parameter in spatially flat cosmologies which carries the information about expansion of the universe is the Hubble parameter. We assume $\Gamma = \Gamma(H)$ such that it vanishes when expansion of the universe stops. Under a metric perturbation the expansion rate of the universe remains the same as that of the background FLRW spacetime and consequently metric perturbations do not affect the value of $\Gamma$. In our model the fluctuations in metric variables, energy density and number density all become locally spacetime dependent but the particle production rate remains homogeneous and isotropic. As we assume that the particle production rate $\Gamma$ depends only on the property of the background evolution we have

$$\delta \Gamma = 0,$$

where $\delta \Gamma$ arises due to metric perturbations. Henceforth we will only work with an unbarred $\Gamma$ (as $H$ or $a$) whereas the other variables will have their metric fluctuations.

From the form of Eq. (9) one can write

$$D_\mu \tilde{N}^\mu = \dot{n} + \tilde{n} \Gamma^\mu \mu_0 = \dot{n} + 3H \tilde{n} = \tilde{n} \Gamma.$$

From the above equation we can write

$$\frac{\dot{n}}{n} = \Gamma - 3H.$$

These equations hold true for the unperturbed variables. Next we write down the general thermodynamic relation in the fluid.

The energy conservation equation can be written as

$$T ds = d\left(\frac{\rho}{n}\right) + P d\left(\frac{1}{n}\right),$$

where $T$ is the temperature of the fluid and $s$ is the entropy per particle, or the specific entropy. Assuming adiabatic particle creation where specific entropy is conserved, i.e. when $\dot{s} = 0$, the unperturbed variables satisfy

$$0 = \dot{\tilde{\rho}} - \frac{\dot{n}}{n}(\tilde{\rho} + \tilde{P}) = \dot{\tilde{\rho}} + 3H \left(1 - \frac{\Gamma}{3H}\right)(\tilde{\rho} + \tilde{P}).$$
Using the above result and the relation in Eq. (8) we get the expression for the unperturbed creation pressure as

\[ \bar{P}_c = -\frac{\Gamma}{3H} (\bar{\rho} + \bar{P}) = -\frac{\Gamma}{3H} (1 + \omega)\bar{\rho} . \]  

(14)

As long as \( \omega > -1, \bar{\rho} > 0 \) and \( \Gamma > 0 \) we will have \( \bar{P}_c < 0 \). The above equation is like an equation of state (EOS) connecting \( \bar{P}_c \) and \( \bar{\rho} \) and \( \Gamma \). As in cosmology the EOS of barotropic fluid does not depend upon orders of perturbation we assume the above EOS to be valid in general to all orders of metric perturbation and write

\[ P_c = -\frac{\Gamma}{3H} (\rho + P) = -\frac{\Gamma}{3H} (1 + \omega)\rho , \]

(15)

where up to first order \( P_c = P_c + \delta P_c \) and \( \rho = \bar{\rho} + \delta \rho \). We will work with \( \delta P_c \) later when we introduce metric perturbation. This calculation shows that the unperturbed pressure of the fluid due to particle creation is negative although \( \delta P_c \) may not be smaller than zero.

From Eq. (13) it is seen that \( \dot{\bar{\rho}} + 3H\bar{\rho}(1 + \omega) = \Gamma\bar{\rho}(1 + \omega) \) producing

\[ \frac{\dot{\bar{\rho}}}{\bar{\rho}} = (1 + \omega)(\Gamma - 3H) \]

Comparing this result with the expression of \( \dot{n}/n \) we can write

\[ (1 + \omega) \frac{\dot{n}}{n} = \frac{\dot{\bar{\rho}}}{\bar{\rho}} , \]

(16)

which has a solution as [25]

\[ \bar{n} = n_0 \left( \frac{\bar{\rho}}{\rho_0} \right)^{1/(1+\omega)} , \]

(17)

where \( n_0 \) and \( \rho_0 \) are the values of the variables at some given instant of cosmic time.

The temperature evolution due to cosmological expansion is affected by the particle production mechanism [23, 26]. Assuming \( \bar{T}, \bar{n} \) to be the basic thermodynamic variables and taking \( \bar{\rho} = \bar{\rho}(\bar{n}, \bar{T}) \) one can show

\[ \frac{\dot{\bar{T}}}{\bar{T}} = \left( \frac{\partial \bar{P}}{\partial \bar{\rho}} \right)_n \frac{\dot{\bar{n}}}{\bar{n}} , \]

(18)

giving us the time dependence of temperature. A derivation of the above important relation is given in Appendix B. Using the equation of state in Eq. (3) we can write the above equation as

\[ \frac{\dot{\bar{T}}}{\bar{T}} = \omega \frac{\dot{\bar{n}}}{\bar{n}} , \]

(19)
whose solution is \[ \bar{n}(\bar{T}) = n_0 \left( \frac{\bar{T}}{T_0} \right)^{1/\omega}. \] (20)

Using the result from Eq. (16) in Eq. (19) we immediately get

\[ \bar{\rho}(\bar{T}) = \rho_0 \left( \frac{\bar{T}}{T_0} \right)^{(1+\omega)/\omega}, \] (21)

where in the above equations \( n_0, T_0 \) and \( \rho_0 \) are the values of the respective variables at some specific time. The above relations show that in presence of radiation fluid when \( \omega = 1/3 \) we have \( \bar{\rho}_r \propto \bar{T}^4 \) as in standard cosmological models without particle creation.

One can write down the Einstein equations from Eq. (6) as:

\[ H^2 = \frac{\kappa}{3} \bar{\rho}, \quad 2\dot{H} + 3H^2 = -\kappa (\bar{P} + \bar{P}_c). \] (22)

The above two equations can be combined and written as

\[ \dot{H} = -\frac{3}{2} (1 + \omega) H^2 \left[ 1 - \frac{\Gamma}{3H} \right]. \] (23)

The above equation shows that as long as \( \Gamma \ll 3H \) one recovers the standard FLRW solutions in GR. If \( \Gamma = 3H \), irrespective of the value of \( \omega \) the model produces a de Sitter phase where \( \dot{H} = 0 \). In this paper we will focus on the de Sitter phase in presence of radiation. In general \( \Gamma \) will be a function of time and the de Sitter phase can be obtained for a short time interval and after that interval \( \Gamma < 3H \) and the system will start to evolve towards standard radiation dominated cosmological evolution if \( \omega = 1/3 \).

### 3 de Sitter phase in presence of radiation

In the present case one can indeed have a de Sitter phase in presence of radiation as discussed in Ref. [7]. In the above reference the authors briefly mention about such a phase which can exist in the very early universe. The previous work only gave the background cosmological solution, the question related to the metric perturbations on the background solutions was not addressed. In the present work we will address the issue related to cosmological metric perturbations in de Sitter space in presence of radiation. Let us write down the specific equations relevant for studying this phase when \( \omega = 1/3 \) and the energy density is \( \bar{\rho}_r \) due to radiation. The equations are

\[ H^2 = \frac{\kappa}{3} \bar{\rho}_r, \] (24)
and

\[
\dot{H} = -2H^2 \left[ 1 - \frac{\Gamma}{3H} \right],
\]

where \( \Gamma \) is the particle creation rate, in the de Sitter phase, in the presence of radiation. In this phase we have

\[
\bar{\rho}_r = A\bar{T}^4,
\]

where \( A = \frac{\pi^2}{30} g_* \), \( g_* \) being the effective relativistic internal degrees of freedom.

To proceed further we require a concrete form of the particle production rate. As discussed in the introduction, there are various ways in which people have tried to write down a mathematical form of \( \Gamma \). From our discussion in the last section we have assumed \( \Gamma = \Gamma(H) \) such that it vanishes when expansion of the universe stops [7, 13] i.e., \( H = 0 \) for all time. In general one can propose that the particle production rate is given by [27]

\[
\Gamma = \Gamma_0 + lH^2 + mH + \frac{p}{H}, \quad (H > 0)
\]

where \( \Gamma_0, l, m, p \) are constants. We assumed that \( \Gamma_0 \) and \( \frac{p}{H} \) exists for a non-vanishing Hubble parameter. Here except \( m \) the other constants are dimensional. If only the first term \( \Gamma_0 \) is present then it will be able to model an emergent universe scenario where the initial singularity is avoided. Cosmological models where \( \Gamma \) is proportional to \( H^2 \) can be applied to the early period of radiation domination. The universe filled with radiation can proceed towards a de Sitter phase as we will see in this paper. The other terms in the above general expression of \( \Gamma \) like the term proportional to \( H \) or the one proportional to \( 1/H \) can give the particle production rates in the early decelerated phase of expansion and later accelerated phase of expansion, respectively. In general one can combine all of them together, as done above, to write down the general phenomenological expression of particle production rate. In this paper we will follow the form of \( \Gamma \) as discussed in Ref. [7]. In this model \( \Gamma \) is proportional to \( H^2 \) as we are interested in the very early universe in the presence of radiation. If we model the system with particle creation rate as [7]

\[
\Gamma = \frac{3H^2}{H_I},
\]

where \( H_I \) is the value of the Hubble parameter when \( \dot{H} \) is exactly zero during the initial phase of expansion. In reality \( H \) will be near \( H_I \) during the de Sitter phase and as long as \( H \sim H_I \) the quasi de Sitter phase will exist. As \( \dot{H} \sim 0 \) in the de Sitter phase \( H \) is a slowly varying function and consequently from Eq. (24) we see that \( \bar{\rho}_r \) is also a slowly varying function during this phase. From Eq. (26) it becomes clear that during the quasi de Sitter expansion the temperature of the universe will approximately remain constant. As \( \dot{H} \sim 0 \) during the de Sitter phase we must have
during the quasi de Sitter phase. Here $a_i$ is the scale-factor at the initial moment of de Sitter phase. To define the quasi de Sitter phase one must have some variables which mimic the “slow-roll parameter” of standard inflationary scenario. In our case we can define one parameter as

$$\epsilon \equiv -\frac{\dot{H}}{H^2}.$$  

(30)

and assume that quasi de Sitter phase exists as long as

$$\epsilon \ll 1.$$  

(31)

From Eq. (25) we can immediately write

$$\epsilon = 2\left[1 - \frac{\Gamma}{3H}\right],$$  

(32)

which yields

$$H = \left(1 - \frac{\epsilon}{2}\right)H_i.$$  

(33)

In the absence of any potential function for the inflaton field only one slow-roll parameter $\epsilon$ is meaningful. Figs. 1 and 2 show the variation of particle number density and temperature during radiation dominated de Sitter phase in the cosmological model with particle creation where the particle creation rate is given in Eq. (28). In producing the plots we have used the initial values of the energy density and number density of particles as

$$\rho_0 = \frac{\pi^2}{30} g_* T_0^4, \quad n_0 = \frac{\zeta(3)}{\pi^2} g_* T_0^4$$  

(34)

where $T_0 = .008$ and $g_* \sim 100$. Here $\zeta(3)$ is the Riemann-Zeta function of 3. All the quantities are in Planck units. To get the value of the variables in natural units where all the quantities are expressed in GeV or its inverse powers one has to multiply each quantity with a power of Planck mass where the power is specified by the mass dimension of each quantity. The figures above show that during radiation dominated de Sitter phase the number density and temperature essentially remains constant. This fact is in stern contrast with standard cold inflation scenarios where temperature and number density of particles rapidly vanish to zero. On the other hand these features may be a bit (superficially) similar to warm inflation predictions [1, 2].

Before we proceed to the topic of cosmological perturbations in the present model we want to specify an important point regarding the form of the particle production rate $\Gamma$. In Eq. (28) we proposed phenomenologically a form of $\Gamma$ as given in Ref. [7]
and used it inside the Einstein equation and predicted the background cosmological development. One may invert the process and write down an equivalent form of the particle production rate from the Einstein equation. In this case one does not require to assume any ad hoc form of $\Gamma$ to begin with. From Eq. (25) and the definition of the slow-roll parameter $\epsilon$ in Eq. (30) one can easily write

$$
\Gamma = 3H \left( 1 - \frac{\epsilon}{2} \right) = \frac{3H}{a} \left( 1 - \frac{\epsilon}{2} \right).
$$

(35)
Here $\mathcal{H}$ is the Hubble parameter expressed in conformal time. In this case we have used the Einstein equation and the slow-roll parameter to define the particle production rate. One can easily verify that in the slow-roll approximation the above form of $\Gamma$ is the same as given by the expression in Eq. (28).

### 4 Scalar cosmological perturbations

Following the conventions set in Ref. [28] we can write the perturbed line element as

$$ds^2 = a^2(\eta) \left\{ (1 + 2\phi) d\eta^2 - 2\partial_i B dx^i d\eta - \left[ (1 - 2\psi) \delta_{ij} + 2\partial_i \partial_j E \right] dx^i dx^j \right\}. \quad (36)$$

where $\phi, \psi, B$ and $E$ are scalar functions of space and time. Here $d\eta = dt/a$ where $\eta$ is the conformal time. In conformal time

$$\mathcal{H} = \frac{a'}{a},$$

where $a' = da/d\eta$. In this section we will use conformal time exclusively. Henceforth we work in the longitudinal gauge where $B = E = 0$.

As in the present case the spatial part of the energy-momentum tensor is diagonal we have $\phi = \psi$. The particle production mechanism particularly affects the scalar perturbations. To see how it affects various sectors of the theory let us first see how the perturbed continuity equation gets affected in cosmological models with particle production.

To find out how particle production affects the perturbed continuity equation we calculate

$$\delta(D_\mu T^\mu_{f0}) = \partial_\mu \delta T^\mu_{f0} + (\delta \Gamma^\mu_{\mu\alpha}) \tilde{T}^\alpha_{f0} + \tilde{T}^\mu_{f0} \delta T^\alpha_{f0} - (\delta \Gamma^\alpha_{\mu\nu}) \tilde{T}^\mu_{f0} - \tilde{T}^\alpha_{f0} \delta T^\mu_{f0}. \quad (37)$$

Using the expressions of the perturbed affine connections in conformal time we first obtain

$$\delta(D_\mu T^\mu_{f0}) = \delta \rho' + 3\mathcal{H}(\delta \rho + \delta P) + (\bar{\rho} + \bar{P}) aV^2 - 3\phi'(\bar{\rho} + \bar{P}). \quad (38)$$

In our notation

$$\tilde{T}^0_{f0} = \bar{\rho}, \quad \tilde{T}^i_{fj} = -P \delta^i_j, \quad \tilde{T}^i_{c j} = -P_c \delta^i_j,$$

and

$$\delta T^0_{f0} = \delta \rho, \quad \delta T^i_{f0} = -(\bar{\rho} + \bar{P}) a \delta^i V, \quad \delta T^0_{fi} = -(\bar{\rho} + \bar{P}) a^{-1} \partial_i V, \quad \delta T^i_{fj} = -\delta^i_j \delta P,$$

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whereas
\[ \delta T^0_{c0} = 0, \quad \delta T^i_{c0} = -\bar{P}_c \delta^i J V, \quad \delta T^0_{ci} = -\bar{P}_c a^{-1} \partial_i V, \quad \delta T^i_{cj} = -\delta^i_j \delta P_c. \]

In the above calculations we use the scalar part of velocity perturbation given by \( \delta u^i = -\partial^i V \) where \( V \) is the fluid velocity potential. Using the above results we can write
\[ \delta(D \mu T^\mu_{c0}) = a \bar{P}_c \nabla^2 V - 3 \phi' \bar{P}_c + 3 \mathcal{H} \delta P_c. \]  
(39)

As \( \delta[(D \mu T^\mu_i) + (D \mu T^\mu_{ci})] = 0 \) we have from Eq. (39) and Eq. (38)
\[ \delta \rho' + 3 \mathcal{H} (\delta \rho + \delta \mathcal{P}) + (\bar{\rho} + \bar{P}) a \nabla^2 V - 3 \phi' (\bar{\rho} + \bar{P}) + \bar{P}_c (a \nabla^2 V - 3 \phi') + 3 \mathcal{H} \delta P_c = 0, \]
(40)
giving us the continuity equation in a cosmological model with particle creation. One can also find out the Euler equation by working out \( \delta[(D \mu T^\mu_i) + (D \mu T^\mu_{ci})] = 0. \) Here \( \delta P_c \) is obtained from Eq. (15) as
\[ \delta P_c = -\frac{a}{3 \mathcal{H}} (1 + \omega) \Gamma \delta \rho. \]
(41)

From the above expression we see that \( \delta P_c \) vanishes when \( \Gamma \) vanishes. When \( \Gamma = 0 \) the above continuity equation reduces to the standard continuity equation in cosmology.

### 4.1 The other relevant scalar perturbations

Let us particularly focus on perturbation of \( N^\mu \) which gives \( \delta N^\mu = \bar{u}^\mu \delta n + \bar{n} \delta u^\mu \) where a bar over the variables specify the unperturbed values. For the scalar part \( \delta u^i = -\partial^i V \) where \( V \) is the fluid velocity potential. This gives us
\[ \delta N^0 = \bar{u}^0 \delta n, \quad \delta N^i = -\bar{n} \partial^i V. \]

First we perturb the number conservation condition in Eq. (9) and get
\[ \delta(D \mu N^\mu) = \Gamma \delta n. \]

Using the fact \( \partial_i \bar{n} = 0 \) the above equation becomes
\[ (\delta n)' + 3 \mathcal{H} \delta n + \bar{n} \delta \Gamma^\mu_{\mu_0} + a \bar{n} \nabla^2 V = a \Gamma \delta n, \]
where \( \nabla^2 \equiv -\partial_i \partial^i \). The above equation can also be written as
\[ (\delta n)' + (3 \mathcal{H} - a \Gamma) \delta n + \bar{n} \delta \Gamma^\mu_{\mu_0} + a \bar{n} \nabla V = 0. \]
We can express Eq. (9) in conformal time as

\[
\frac{\delta n'}{\bar{n}} + \frac{\delta n}{\bar{n}} + \bar{n}\delta \Gamma^\mu_{\mu 0} + a\bar{n} \nabla^2 V = 0.
\]

As \(\delta \Gamma^\mu_{\mu 0} = -2\phi'\), in the longitudinal gauge, we can write the final expression of the perturbed particle number conservation as:

\[
(\delta n') - \frac{\delta n' \bar{n}}{\bar{n}} - 2\phi' \bar{n} + a\bar{n} \nabla^2 V = 0.
\]  (42)

This equation specifies how \(\delta n\) evolves with time. This is a coupled partial differential equation and can only be solved if we know the evolution equation for \(\phi\) and the form of \(\delta \Gamma\).

In the present case the energy momentum tensor is \(T^{\mu\nu} = T^{\mu\nu}_f + T^{\mu\nu}_c\) as given in Eq. (6) and consequently we have

\[
T^0_0 = \rho, \quad T^i_j = -(P + P_c)\delta^i_j.
\]  (43)

Following the method of gauge invariant scalar cosmological perturbations in Ref. [28] one can write the scalar perturbation equations in the longitudinal gauge as:

\[
\nabla^2 \phi - 3\mathcal{H}\phi' - 3\mathcal{H}^2\phi = \frac{k}{2}a^2\delta \rho,
\]  (44)

\[
D_i (a\phi)' = \frac{k}{2}(\bar{\rho} + \bar{P} + \bar{P}_c)a^2\delta u_i,
\]  (45)

\[
\phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = \frac{k}{2}a^2(\delta P + \delta P_c).
\]  (46)

The mathematical form of \(\delta P_c\) is given in Eq. (41). Using the fact that \(\delta P = \omega \delta \rho\) one can simplify the above equations as

\[
\phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = \frac{k}{2}a^2\omega \delta \rho + \frac{k}{2}a^2\delta P_c.
\]

For the sake of generality we will try to keep the general equation of state explicit in the equations written below. Only when we state our results we will use \(\omega = 1/3\) as we are working primarily in a universe filled with radiation. One can use the form of \((k/2)a^2\delta \rho\) from Eq. (44) in the first term on the right hand side of the above equation and obtain:

\[
\phi'' - \omega \nabla^2 \phi + 3\mathcal{H}(1 + \omega)\phi' + [2\mathcal{H}' + \mathcal{H}^2(1 + 3\omega)]\phi = \frac{k}{2}a^2\delta P_c.
\]  (47)

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Using the expression of $\delta P_c$ one can write the above expression as

$$
\phi'' - \left[ \omega - \frac{a(1 + \omega)\Gamma}{3\mathcal{H}} \right] \nabla^2 \phi + (3\mathcal{H} - a\Gamma) (1 + \omega)\phi' + [2\mathcal{H}' + \mathcal{H}^2 (1 + 3\omega) - a(1 + \omega)\Gamma \mathcal{H}] \phi = 0 .
$$

This equation specifies the growth of $\phi$ in presence of gravitational particle production. From Eq. (45) we can write

$$
\nabla^2 V = -\frac{2}{\kappa a^2} \frac{a' \nabla^2 \phi + a \nabla^2 \phi'}{\bar{\rho} + \bar{P} + \bar{P}_c}.
$$

Using this equation we can write Eq. (42) as

$$
\delta n' - \frac{\bar{n}'}{n} \delta n - \left[ 2\phi' + \frac{2(\mathcal{H} \nabla^2 \phi + \nabla^2 \phi')}{\kappa(\bar{\rho} + \bar{P} + \bar{P}_c)} \right] \bar{n} = 0 .
$$

These above equations specify the scalar perturbation equations of the cosmological model with particle creation. These equations can be applied to any cosmological model where gravitational particle creation takes place.

### 4.2 Evolution of perturbations

Before we write down the evolution equation of the perturbations using the above fact we want to opine on the temperature growth of the background and some properties of temperature fluctuations. Using Eq. (19) and the expression of $\dot{n}/n$ we can write

$$
\frac{\bar{T}'}{\bar{T}} = \omega \frac{\bar{n}'}{\bar{n}} = \omega (a\Gamma - 3\mathcal{H}) ,
$$

where primes naturally denote differentiation with respect to conformal time $\eta$. The temperature fluctuations can be approximately figured out by assuming that the metric perturbations do not take the system out of local thermal equilibrium. Assuming the perturbed system to be near local thermodynamic equilibrium one can still assume

$$
\rho = \bar{\rho} + \delta \rho \sim \frac{\pi^2}{30g_\ast} T^4
$$

where $T = \bar{T} + \delta T$. Up to first order the relation gives

$$
\delta T \sim \frac{15}{2\pi^2 g_\ast} \frac{\delta \rho}{T^3} .
$$

We will see later that metric perturbations may induce fluid flow in the initial stages of the de Sitter phase and consequently local thermodynamic equilibrium can only hold approximately. The above simple estimation can give us the temperature
fluctuations induced by metric perturbations assuming the system is near about local thermal equilibrium. In this paper we have assumed that radiation fluid is the main constituent of the universe, where $\omega = 1/3$, so that Eq. (51) can easily be integrated to see how the background temperature evolves when one has

$$\Gamma = \frac{3\mathcal{H}^2}{a^2 H_I},$$

(54)

which can be obtained from Eq. (28). For plane wave like perturbations where the perturbation modes are proportional to $e^{i k \cdot x}$ we will have the basic perturbation equations as:

$$\phi_k'' + k^2 \left[ \omega - \frac{a(1 + \omega)\Gamma}{3\mathcal{H}} \right] \phi_k + (3\mathcal{H} - a\Gamma)(1 + \omega)\phi_k' + [2\mathcal{H}' + \mathcal{H}^2(1 + 3\omega)

-a(1 + \omega)\Gamma H] \phi_k = 0,$$

(55)

$$\frac{\delta n_k'}{\bar{n}} \frac{\delta n_k}{2} \left[ \phi_k' - \frac{k^2(\mathcal{H}\phi_k + \phi_k')}{\kappa(\bar{\rho} + \bar{P} + \bar{P}_c)} \right] = 0,$$

(56)

$$V_k + \frac{2}{\kappa a^2} \frac{a'\phi_k + a\phi_k'}{\bar{\rho} + \bar{P} + \bar{P}_c} = 0,$$

(57)

where $\phi_k(\eta)$, $\delta n_k(\eta)$ and $V_k(\eta)$ are the $k$th Fourier modes of the respective perturbations. Using the following expression

$$\bar{\rho} + \bar{P} + \bar{P}_c = (1 + \omega)\bar{\rho} \left[ 1 - \frac{a\Gamma}{3\mathcal{H}} \right] = \frac{3\mathcal{H}^2}{\kappa a^2 (1 + \omega)} \left[ 1 - \frac{\mathcal{H}}{a H_I} \right],$$

we can write Eq. (56) and Eq. (57) as

$$\frac{\delta n_k'}{\bar{n}} \frac{\delta n_k}{2} \left[ \phi_k' - \frac{a^2 k^2(\mathcal{H}\phi_k + \phi_k')}{3\mathcal{H}^2(1 + \omega)(1 - \frac{\mathcal{H}}{a H_I})} \right] = 0,$$

(58)

$$V_k + \frac{2}{3\mathcal{H}^2} \frac{a'\phi_k + a\phi_k'}{(1 + \omega)(1 - \frac{\mathcal{H}}{a H_I})} = 0.$$

(59)

In the present case one can write $\mathcal{H}'$ and $\mathcal{H}$ as functions of the slow-roll parameter as

$$\mathcal{H}' = (1 - \epsilon)\mathcal{H}^2, \quad \frac{\mathcal{H}}{a H_I} = 1 - \frac{\epsilon}{2}.$$

(60)

The differential equations show the nature of the scalar perturbation for $k < \mathcal{H}$, which corresponds to the superhorizon limit and $k > \mathcal{H}$, corresponding to the subhorizon limit. The perturbation equations can be compactly written down using the slow-roll parameter as:
Fig. 3 Evolution of $\phi_k$ with $\eta$ in the radiation dominated de Sitter phase for $\delta \Gamma = 0$ and $k = 10^{-5}$, more details given in text.

\begin{align}
\phi_k'' + \frac{3}{2}(1 + \omega)\mathcal{H}\epsilon\phi_k' + \left[\frac{\epsilon}{2}k^2(1 + \omega) - k^2 + 2\mathcal{H}' + \mathcal{H}^2(1 + 3\omega)ight] \phi_k &= 0, \\
-3(1 + \omega)\mathcal{H}^2(1 - \frac{\epsilon}{2}) \phi_k &= 0,
\end{align}

\begin{align}
\delta n_k' + \frac{3}{2}\epsilon\mathcal{H}\delta n_k - 2\left[\phi_k' - \frac{2a^2k^2(\mathcal{H}\phi_k + \phi_k')}{3\mathcal{H}^2(1 + \omega)\epsilon}\right] \bar{n} &= 0, \\
V_k + \frac{4}{3\mathcal{H}^2} \frac{a'\phi_k + a\phi_k'}{(1 + \omega)\epsilon} &= 0.
\end{align}

It must be noted that $\delta n_k \ll \bar{n}$ but in an absolute sense $\delta n_k$ need not be smaller than one as the fluctuation in the number of particles in unit volume cannot be less than one. The above gravitational and hydrodynamic perturbation equations are distinctly different from standard perturbation equations in radiation domination or matter domination or from the perturbation equations in warm inflation or super cold inflation. In standard perturbation equations in presence of matter the concept of slow-roll never arises and in inflationary models the concept of slow-roll arises from the nature of inflaton potential. In the present case we have a slow-roll mechanism in presence of hydrodynamic fluid where the slowness of the process is related to the slow decrement of the particle creation rate which sustained a near de Sitter phase initially.

The plots of the perturbations are shown in Figs. 3, 4 and 5. In all of these plots we have taken $k = 10^{-5}$ and $\omega = 1/3$. We have used Planck units as described earlier.
Fig. 4 Evolution of $\delta n_k/n$ with $\eta$ in the radiation dominated de Sitter phase for $\delta \Gamma = 0$ and $k = 10^{-5}$, details given text

Fig. 5 Evolution of $V_k$ with $\eta$ in the radiation dominated de Sitter phase for $\delta \Gamma = 0$ and $k = 10^{-5}$

While plotting the above figures we used $T_0 = 0.008$ and $g_\ast \sim 100$ and $H_0 = 0.001$. The zero subscript specify initial value of any variable. The scale-factor initially is taken to be unity. For the perturbations we have used $\phi_k(\eta_0) = 0.01$, $\phi'_k(\eta_0) = 0.0001$, and $\delta n_k(\eta_0) = 0.001 \times n_0$ where $n_0$ is specified in Eq. (34). In all of the plots $\eta_0 = -10^3$. 

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The plots show well behaved perturbations of the relevant quantities in the radiation dominated de Sitter phase.

The above perturbation equations may produce unstable behavior when \( k > \mathcal{H} \). The perturbations do not turn out to be singular in this regime but tend to be unstable and may slowly attain value near to or greater than one. From Eq. (61) it is seen that when \( k^2 > \mathcal{H}^2 \), \( \phi_k \) can become unstable.\(^1\) To see this explicitly we write Eq. (61) when \( \epsilon \ll 1 \) as

\[
\phi_k'' \sim k^2 \phi_k , \quad \text{when } k > \mathcal{H}
\]

showing that \( \phi_k \) can increase without bound in this limit. The condition \( k^2 > \mathcal{H}^2 \) will not stay forever and consequently \( \phi_k \) will not become singular. It is seen that \( \phi_k \) can indeed become non-perturbative, particularly if the initial value of it is not too small, when \( k^2 > \mathcal{H}^2 \). On the other hand when \( k^2 < \mathcal{H}^2 \) and \( \epsilon \ll 1 \), Eq. (61) becomes \( \phi_k'' \sim 2(\mathcal{H}^2 - \mathcal{H}'')\phi_k = \epsilon \mathcal{H}^2 \phi_k \), and in our approximation scheme we can write the last condition as

\[
\phi_k'' \sim 0 , \quad \text{when } k < \mathcal{H}
\]

showing that \( \phi_k \) will be approximately a linear function of conformal time in this limit. In particular if \( \phi_k' \sim 0 \) then the scalar perturbation will be a constant in this limit. As \( \epsilon \to 0 \) when \( \bar{\Gamma} \to 3\mathcal{H}/a \) we see Eq. (62) can also be written as

\[
\left( \frac{\delta n_k}{n} \right)' \sim 2 \left[ \phi_k' - \frac{2a^2 k^2 (\mathcal{H} \phi_k + \phi_k')}{3\mathcal{H}^2 (1 + \omega) \epsilon} \right]. \quad (66)
\]

From the above expression we see the fractional number density can build up if \( \phi_k \) increases in the limit \( k > \mathcal{H} \). Depending upon the sign of \( \phi_k' \) and \( \phi_k \) we see that \( \delta n_k/\bar{n} \) can be positive or negative implying that at some length scales there will be more fractional particle and at some length scales the particle fraction is less. These results show that one can have inhomogeneities in the radiation filled universe during the de Sitter phase.

Before we end the discussion on scalar cosmological perturbations in de Sitter space in presence of radiation we want to comment on the nature of the gravitational potential power spectrum. We know that the simplest inflationary models predicts nearly scale invariant power spectrum. In the present case we have seen that perturbation with high \( k \) tend to become nonlinear and we expect the power spectrum to be not nearly scale invariant. In the present case we plot \((k^3/2\pi^2)|\phi_k|^2\) with \( k \) in Fig. 6. In Fig. 6 we have assumed the slow-roll parameter \( \epsilon = .001 \) and have plotted the perturbations calculated at \( \eta = -500 \). The initial Hubble parameter used to plot the perturbations is \( \mathcal{H} = .0001 \). The initial values of the perturbation and its time derivative remain the same as stated before. Both \( \eta \) and Hubble parameter are expressed in Planckian units. We see that the nature of the curve does not portray a nearly scale invariant power spectrum. In the high \( k \) region the curve is approximately proportional to \( k^3 \).

\(^1\) In this limit we must also have \( k^2 > \mathcal{H}' \) as \( \mathcal{H}' = (1 - \epsilon)\mathcal{H}^2 \) and \( \epsilon \ll 1 \).
From this figure it is apparent that the present model of de Sitter space in presence of radiation, accompanied by gravitational particle production, does not reproduce the simplest features seen in the simple inflationary models.

5 Brief comment on the other kind of metric perturbations

The de Sitter phase in a radiation filled universe can produce interesting features in pure vector perturbation sector. The tensor perturbations on the other hand do not produce any novel features. The pure vector fluid velocity perturbation can get amplified due to the particle production process inducing radiation fluid movement in the shorter length scales whereas tensor perturbations in the present case do not get amplified and its properties remain similar to the corresponding result in standard super cool inflationary scenario. In this section we will discuss briefly about the vector and tensor perturbations in cosmology with particle creation. The basic results related to the metric perturbations quoted in the present section follows the conventions set in Ref. [28].

5.1 Vector perturbations

When only vector perturbations are excited the metric is given as

$$g_{\mu\nu} = a^2(\eta) \left[ \frac{1}{S_i} \left| -\delta_{ij} + \partial_i F_j + \partial_j F_i \right| \right],$$

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where $S^i$ and $F^i$ are 3-vectors satisfying the constraint $\partial_t S^i = \partial_t F^i = 0$. In the Newtonian gauge where $F^i = 0$, the standard theory of vector perturbation gives

$$v^i_k = \frac{k^2}{2a^2 \kappa (\bar{\rho} + \bar{P} + \bar{P}_c)} S^i_k = \frac{k^2}{a^2 \kappa (1 + \omega) \bar{\rho} \epsilon} S^i_k = \frac{k^2}{3H^2 (1 + \omega) \epsilon} S^i_k,$$

and

$$S^i_k(\eta) = \frac{C^i_k}{a(\eta)^2},$$

where $C^i_k$ is a constant 3-vector. Here $v^i_k$ is the $k$th Fourier component of the pure vector part of the fluid velocity 3-vector $v^i$. Here $S^i_k$ designates the $k$th Fourier mode of the metric vector perturbation. For all these variables $k \cdot v^i_k = k \cdot S^i_k = k \cdot C^i_k = 0$. In this case we see that the metric perturbation $S^i_k$ becomes negligible as the universe expands, but from Eq. (67) it is seen that the initial values of $v^i_k$ may not be negligible when $\epsilon$ is small and $k > H$. The pure vector part of fluid velocity perturbations cannot be neglected in this case and this may have important cosmological consequences. In pure radiation domination the pure vector part of fluid velocity perturbations remains constant, and if they were negligible initially they remain so throughout the radiation dominated phase. In the present case the pure vector part of fluid velocity perturbations cannot be neglected in the de Sitter phase.

5.2 Tensor perturbations

In order to tackle tensor perturbations

$$ds^2 = a^2(\eta) \left[ d\eta^2 - (\delta_{ij} - h_{ij})dx^i dx^j \right],$$

where $h_{ij}$ is a symmetric, traceless tensor with $\partial_i h_{ij} = 0$ one can rescale $h_{ij}$ and write

$$h_{ij} = \frac{v}{a} e_{ij},$$

where $v$ is a function of $\eta$, $x$ and $e_{ij}$ is the time-independent polarization tensor. Expanding $v$ in plane waves we get,

$$v''_k + \left( k^2 - \frac{a''}{a} \right) v_k = 0.$$

In our case during the initial stages of quasi-de Sitter expansion $v_k$ has a solution which is similar to the solution of the analogous problem in de Sitter space in the absence of any radiation. The tensor perturbations do not carry any special signature from the particle production model up to linear order of perturbations. For non-linear orders the
modified scalar perturbations can act as source for the tensor modes and consequently, in general, the tensor power spectrum will also differ from the standard inflationary result. In the present case once the particle production rate decreases considerably, normal radiation domination returns when $a(\eta) \propto \eta$ and then $v_k \propto \exp(\pm i k \eta)$.

The presence of the source term $\Gamma$, where the form of $\Gamma$ as given in Eq. (28), in the perturbation equations modify the behavior of the evolution of cosmological perturbations from pure radiation-dominated universe. Cosmological particle production in the slow-roll regime introduces finite instabilities in the short wavelength limit of the perturbations. The nature of the instabilities greatly depend upon the form of $\Gamma$ chosen for the background particle production rate. The physical origin of the instabilities can be intuitively understood. In de Sitter space, filled with radiation fluid, one has a casual Hubble patch. The radius of the Hubble sphere defines the superhorizon and subhorizon limits. Events inside the Hubble patch are understood to be casually interrelated. In general it is very difficult for a hydrodynamic fluid to exist for some time in a de Sitter space as due to exponential increase of the scale-factor, the fluid density will rapidly fade. In our model the radiation bath persists as there is a constant rate of gravitational particle production per unit spatial volume. In absence of fluid velocity fluctuations there is no instability of the model and as a result the background model is perfectly stable. As soon metric perturbations are switched on there appears (scalar) fluid velocity fluctuations which can deplete some regions of the universe and overpopulate some causally connected region of the universe. These overpopulated regions have larger gravitational effect and attract more fluid from the neighborhood, where particle creation is happening in an unperturbed manner, and as a result we have unstable perturbations in the causally connected subhorizon limit. Even the pure vector velocity perturbation can become appreciably big inside the subhorizon region. In our model the particle production rate is not perturbed, remains practically constant during the slow-roll phase, and as a result some regions can continuously supply particles to other dense regions. The instabilities persist in the subhorizon scale because depletion and overpopulation are causally connected and primarily affects the system in the subhorizon scales. This intuitive interpretation of the model show that the instabilities we see in our model are a natural consequence of the assumptions we have made to define our cosmological model.

6 Discussion and Conclusion

In this article we have tried to formulate metric perturbations in a cosmological model where gravitational particle production is taking place. The formulation requires various new assumptions about the particle production process in cosmology. Previous authors have presented the cosmological background (unperturbed) calculations of radiation filled universe in a de Sitter phase where the de Sitter negative pressure is supplied by the particle creation pressure. If the particle creation rate attains a specific value then a radiation filled universe may enter a de Sitter phase where one may try to see whether such a phase can be modelled by a slow-roll process. The expression of the creation pressure turns out to be proportional to the particle production rate and energy density and always remains negative. We have used the expression connecting
the creation pressure and energy density like a new equation of state from which one can get the perturbed creation pressure. In the initial part of the present paper we have presented the basic equations governing the background cosmological evolution in a cosmological model with particle production. It has been pointed out how such a cosmological model can enter a de Sitter phase in presence of radiation. In the model of particle production, as studied in the paper, specific entropy does not increase and the created radiation can locally equilibrate with the existing radiation, although the whole system is undergoing an irreversible process where total entropy increases due to the enlargement of the phase space. As long as the radiation filled universe remains quasi de Sitter, massless particle production can go on, and as slowly the system comes out of the de Sitter phase massless particle production goes down as predicted by Leonard Parker [5]. Following previous works on the cosmology of de Sitter space in presence of radiation we have assumed the particle production rate is proportional to the square of the Hubble parameter.

The main results of the paper are related to the perturbation of the cosmological model with gravitational particle creation. Initially the scalar cosmological perturbation calculations are presented. We have consistently used the longitudinal gauge in our calculations for the scalar perturbations. While working out the scalar perturbations we have set the perturbation of the particle production rate to be zero. This assumption is based on the fact that in the gravitational particle production models the particle production rate is primarily a function of the expansion rate of the universe which is given by the Hubble parameter. As the expansion rate is not affected by the metric perturbations it is natural to assume that the perturbation of the particle production rate due to metric fluctuations is zero. With this assumption and the equation of state connecting creation pressure and energy density one can write down the scalar perturbation equations in a cosmological model accompanied by gravitational particle production. When the equations are solved for the de Sitter universe, filled with radiation, various interesting results come out. The perturbation equations show that all the relevant Fourier modes of the perturbations in longitudinal gauge show instability in the subhorizon limit where as on the superhorizon case the Fourier mode of the metric perturbation in the longitudinal gauge linearly increases with time or becomes constant. In general for a radiation dominated universe the short wavelength perturbations oscillate in a stable fashion. The effect of the de Sitter phase is to make the small scale perturbations unstable which may cause inhomogeneities in the very early universe. We have shown that the scalar perturbation power spectrum steeply rises with the comoving wave number which predicts there is more power for high values of \( k \). The exact value of \( k \) after which the steepness increases cannot be ascertained as this value depends on the initial condition of the perturbations set on an arbitrary time slice. This feature distinguishes our model from simple inflationary models.

In the previous section we have briefly discussed the fate of pure vector and tensor perturbations in cosmology where gravitational particle production takes place. The vector metric perturbation dies down quickly in an exponentially expanding spacetime where as the pure vector velocity perturbations may not die down so easily. It has been shown that in the de Sitter phase the fluid pure vector perturbation may be amplified in the subhorizon limit indicating directed flow of radiation from one region to other. These strong local flows can produce interesting effects in the de Sitter model.
studied. Further investigation is required to unravel the effects of such flow in the universe pervaded by radiation fluid. The tensor part of the perturbation sector does not carry any special signature of gravitational particle production process, there are no amplification or instability in this sector. The tensor perturbations in the present model are similar to the tensor perturbations in standard inflationary theory.

A serious point concerning the present model of cosmology is related to the ultra-violet instability of the scalar cosmological perturbations. The small scale or high $k$ modes of the scalar perturbations become perturbatively unstable. This may mean production of primordial black holes or scalar induced gravitational waves [29]. As the present theory does not require any specific inflaton like scalar field to produce the de Sitter phase so one naturally does not explicitly think about the quantum nature of the scalar fluctuations. The whole theory can be thought of as a classical theory. The difficulty of this classical interpretation is related to the cause of the seed fluctuations. In standard inflationary theory the seed fluctuations are inherent quantum fluctuations which are unavoidable in any realistic physical system. In the present case one faces considerable difficulty if the scalar cosmological perturbations are assumed to be quantized. The quantum nature of the perturbations can naturally answer the question about origin of the seed perturbations but the quantization scheme becomes obscure in the present model. Any quantization scheme of scalar fields will require a vacuum to be defined. In standard inflationary theories the vacuum is called the Bunch-Davies vacuum. This vacuum is defined in such a way that on the ultraviolet (sub-Hubble) scales the perturbation modes become plane-wave modes. Any similar method of choosing a vacuum for the scalar perturbations in the present model faces a problem because in the ultraviolet limit the theory does not remain perturbative.

One possible way to evade the above problem is to assume a pre-de Sitter phase where one can quantize the perturbations in a manageable way. In the present model it is seen that if one keeps the particle production rate as given in Eq. (28) then at times when $H > H_I$ (assuming an expanding universe where $H > 0$) one obtains the phantom phase in presence of radiation. This fact can be verified from Eq. (25) which shows that $\dot{H} > 0$ for $H > H_I$. Whether the perturbation modes in that kind of an exotic phantom phase remains perturbative, in the ultraviolet limit, is an open question. It is difficult to guess an answer here because there are no works related to quantum fluctuations in a phantom phase in presence of radiation. In Ref. [30] the authors have presented a field theoretic interpretation of cosmological perturbations in the phantom phase in presence of a scalar field. In this reference it is shown that the perturbation theory of the scalar field, which is responsible for the phantom phase, requires an ultraviolet cutoff. In our case the phantom phase is not produced by any scalar field and consequently one has to formulate the perturbation theory in a new context. Here we do not present any details about this reformulation as it is out of scope of the present paper. The present paper shows that the construction of a cosmological model based on a de Sitter phase in presence of radiation is incomplete as the question related to the initial conditions of the perturbations remains unanswered. A focussed research in this field in the future can resolve this issue.

In conclusion we can say that the present work shows that although the particle production model in de Sitter phase can produce all the kinds of metric perturbations produced in inflationary models but it cannot reproduce all the features of the simplest
inflationary paradigm. One has to modify the present model to make the scalar power spectrum nearly scale invariant and make the theory quantum mechanically complete.

**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

### A Appendix I

In an expanding universe the energy density naturally decreases with time. In presence of particle production the decrement in energy density, with time, slows down (compared to the corresponding rate in standard cosmological models) due to the production of new particles which contributes to the energy density. This decrement in the rate of energy density, in models with particle production, can be explained if we take into account the action of the creation pressure $P_c$. To understand this point we must note that the effective pressure of the system is the sum of the fluid pressure and the creation pressure,

$$P_T = P + P_c,$$

where $P = \omega \rho$ is the fluid pressure. The total pressure appears in the total energy momentum tensor appearing in Eq. (6). As total energy momentum tensor is conserved, we must have

$$\frac{d(\rho \Delta V)}{dt} = -P_T \frac{d(\Delta V)}{dt}.$$  

Assuming $d(\Delta V)/dt > 0$ for an expanding universe, we can decrease the time rate of change of the energy (compared to standard cosmological models where there is no particle production and $P_T = P$) by making $P_T < P$. For positive values of $P_T$ and $P$ one can get $P_T < P$ when the creation pressure $P_c$ is negative. As because by introducing the negative creation pressure we can decrease the rate of decrement of the of energy (compared to the case where no particles are created) we do not again require to alter the energy density. This is the reason why we do not alter the energy density for the cosmological scenarios where there is particle production.

### B Appendix II

Assuming $\bar{T}, \bar{n}$ to be the basic thermodynamic variables we have $\dot{\rho} = \dot{\rho}(\bar{n}, \bar{T})$. As a consequence

$$\dot{\rho} = \left( \frac{\partial \rho}{\partial \bar{n}} \right)_T \dot{\bar{n}} + \left( \frac{\partial \rho}{\partial \bar{T}} \right)_n \dot{\bar{T}}.$$
Using the above relation in \( \dot{\bar{\rho}} + 3H(\bar{\rho} + \bar{P}) = \Gamma(\bar{\rho} + \bar{P}) \), which comes from Eq. (13), one gets
\[
\left( \frac{\partial \bar{\rho}}{\partial n} \right)_T \dot{n} + \left( \frac{\partial \bar{\rho}}{\partial T} \right)_n \dot{T} = -3H(\bar{\rho} + \bar{P}) - 3H \bar{P}_c.
\]

Using the relation of \( \bar{\dot{n}}/\bar{n} = \Gamma - 3H \), the above equation can also be written as
\[
\left( \frac{\partial \bar{\rho}}{\partial T} \right)_n \dot{T} = -3H(\bar{\rho} + \bar{P}) - 3H \bar{P}_c - \left( \frac{\partial \bar{\rho}}{\partial n} \right)_T \left( \bar{n} \Gamma - 3H \bar{n} \right).
\]

The above equation can also be written as:
\[
\dot{T} = -\frac{1}{(\bar{\rho} + \bar{P})_n} \left\{ \left( \frac{\partial \bar{\rho}}{\partial n} \right)_T - \bar{n} \left( \frac{\partial \bar{\rho}}{\partial T} \right)_n \right\} 3H + 3H \bar{P}_c + \left( \frac{\partial \bar{\rho}}{\partial n} \right)_T \bar{n} \Gamma. \tag{72}
\]

To proceed further we have to spend some time on thermodynamic relations. From Eq. (12) (for the background cosmological model) we can write
\[
d\bar{s} = \frac{1}{\bar{n}T} \left[ d\bar{\rho} - \left( \frac{\bar{\rho} + \bar{P}}{\bar{n}} \right) d\bar{n} \right],
\]
which gives,
\[
\left( \frac{\partial \bar{s}}{\partial n} \right)_T = \frac{1}{\bar{n}T} \left[ \left( \frac{\partial \bar{\rho}}{\partial n} \right)_T - \left( \frac{\bar{\rho} + \bar{P}}{\bar{n}} \right) \right], \tag{73}
\]
\[
\left( \frac{\partial \bar{s}}{\partial T} \right)_n = \frac{1}{\bar{n}T} \left( \frac{\partial \bar{\rho}}{\partial T} \right)_n. \tag{74}
\]

For partial derivatives we know
\[
\frac{\partial}{\partial T} \left\{ (\partial \bar{s}/\partial n)_T \right\}_n = \frac{\partial}{\partial n} \left\{ (\partial \bar{s}/\partial T)_n \right\}_T,
\]
which gives
\[
\frac{\partial}{\partial T} \left\{ \frac{1}{\bar{n}T} \left[ \left( \frac{\partial \bar{\rho}}{\partial n} \right)_T - \left( \frac{\bar{\rho} + \bar{P}}{\bar{n}} \right) \right] \right\}_n = \frac{\partial}{\partial n} \left\{ \frac{1}{\bar{n}T} \left( \frac{\partial \bar{\rho}}{\partial T} \right)_n \right\}_T.
\]

Calculating the derivatives in both sides and cancelling appropriate terms the above equation yields [26]
\[
\bar{T} \left( \frac{\partial \bar{P}}{\partial T} \right)_n = (\bar{\rho} + \bar{P}) - \bar{n} \left( \frac{\partial \bar{\rho}}{\partial n} \right)_T. \tag{75}
\]
Using this relation in Eq. (72) we get

\[ \dot{T} = -\frac{1}{(\partial \tilde{\rho} / \partial T)_n} \left\{ 3H \tilde{T} \left( \frac{\partial \tilde{P}}{\partial T} \right)_n + 3H \tilde{P}_c + \left( \frac{\partial \tilde{\rho}}{\partial n} \right)_T \tilde{n} \Gamma \right\}, \]

which can also be written as

\[ \dot{T} = -3H \left( \frac{\partial \tilde{P}}{\partial \tilde{\rho}} \right)_n - \frac{3H \tilde{P}_c + (\partial \tilde{\rho} / \partial n)_T \tilde{n} \Gamma}{\tilde{T} (\partial \tilde{\rho} / \partial T)_n}, \quad (76) \]

where we have used

\[ \frac{(\partial \tilde{P} / \partial \tilde{T})_n}{(\partial \tilde{\rho} / \partial \tilde{T})_n} = \left( \frac{\partial \tilde{P}}{\partial \tilde{\rho}} \right)_n. \]

If specific entropy is conserved then the expression of \( \tilde{P}_c \) is given as in Eq. (14) and we know the expression of \( \tilde{n}(\partial \tilde{\rho} / \partial \tilde{n})_T \) from Eq. (75). Using these expressions we can write

\[ 3H \tilde{P}_c + \tilde{n} \left( \frac{\partial \tilde{\rho}}{\partial \tilde{n}} \right)_T \Gamma = -\Gamma (\tilde{\rho} + \tilde{P}) + \left[ (\tilde{\rho} + \tilde{P}) - \tilde{T} \left( \frac{\partial \tilde{P}}{\partial \tilde{T}} \right)_n \right] \Gamma = -\tilde{T} \Gamma \left( \frac{\partial \tilde{P}}{\partial \tilde{T}} \right)_n. \]

Using this result in Eq. (76) for cosmological evolution conserving specific entropy we get

\[ \dot{T} = -3H \left( \frac{\partial \tilde{P}}{\partial \tilde{\rho}} \right)_n + \Gamma \left( \frac{\partial \tilde{P}}{\partial \tilde{T}} \right)_n, \]

and using the expression of \( \dot{n} / n \) the above equation becomes

\[ \dot{T} = \left( \frac{\partial \tilde{P}}{\partial \tilde{\rho}} \right)_n \frac{\dot{n}}{n}. \quad (77) \]

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