Fractal-based dynamic response of a pair of spur gears considering microscopic surface morphology

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Abstract
The meshing surfaces of a gear pair are rough from a microscopic perspective and the surface topography will affect the dynamic response. To study the influence of real surface topography on the gear system dynamic performance, this paper establishes a 3-degree of freedom transverse-torsional dynamic model with regard to the morphology of the interface. By fractal theory, the expression of backlash between gears is modified based on the height of asperities. The time-varying stiffness is calculated according to the fractal method rather than assuming a constant, which is more realistic. The dimensionless dynamic differential equations are established and solved with surface topography affected backlash function and time-varying stiffness. The dynamic response of the gear system with respect to fractal dimension and fractal roughness is analyzed.

Keywords
dynamic response, fractal theory, gear pair, surface topography

1 | INTRODUCTION

As a kind of widely used transmission mode, gears have been studied extensively. The dynamics of gear pair is the center and key content of gear research. Among the generally applied gear dynamic models, the gear pair is always simplified as two elastomeric contact bodies. In the process of meshing, the backlash and meshing stiffness are the main nonlinear dynamic parameters. Accordingly, their accurate modeling is the key to solve the dynamic response of gears.

Traditionally, the backlash has been treated mathematically as a constant or a steady random number conforming to the normal distribution.1–3 Chen et al.4 proposed a dynamic model of gear backlash combined with the offset of the gear center and obtained a dynamic expression of backlash in the transverse-torsional gear system. Lu et al.5 proposed a model of assembling backlash, based on which the complete gear dynamic model was established and the effects of backlash were analyzed. Walha et al.6 modeled the nonlinear backlash by the discontinuous function of elastic gear pair. Moradi and Salarieh7 proposed a nonlinear backlash model composed of a third-order polynomial function, which was applied to a single degree of freedom (DOF) dynamic model. Considering the radial backlash as a control parameter, Sheng et al.8 studied its effects on unstable gears and chaos of the system. Also, the wear of the surface may lead to the change of tooth thickness at the meshing region and finally the variation in backlash. In this respect, Onishchenko9 took instantaneous contact temperature into consideration and proposed a tooth wear model. Feng et al.10 established an updatable wear model in which system vibration was calculated considering the initial wear, which subsequently affects the current wear. The model formed a closed-loop update of wear. However, only setting an appropriate constant for the backlash from a mathematical point of view has high randomness, which cannot reflect the influences of surface topography in reality.

In terms of meshing stiffness, it has become a consensus to adopt time-varying meshing stiffness, which has different calculation methods.
Kaharman and Singh\textsuperscript{11} had earlier introduced time-varying stiffness and studied its influence on gear system dynamics. Wang et al.\textsuperscript{12,13} studied the time-varying meshing stiffness of a 5-DOF gear torsional model and analyzed the dynamic responses under various loads. Bai et al.\textsuperscript{13} established an electromechanical coupling model of the motor-gear system combining transient voltage. The results showed that voltage has obvious influences on dynamic response. Yu et al.\textsuperscript{14} proposed a new dynamic model of cylindrical gear pair with localized tooth spalling defects. Depending on the depth and extent of a spall, the resulting contribution of the spall to the gear dynamics can be classified in terms of mesh stiffness reduction and displacement excitation. Wang and Lim\textsuperscript{15} established a meshing stiffness model with asymmetry and accordingly solved the nonlinear gear dynamic model. Chen and Shao\textsuperscript{16} proposed a method to calculate meshing stiffness considering the effects of tooth profile and root crack based on the potential energy method and investigated the relation between tooth profile and stiffness. Liao et al.\textsuperscript{17} studied the meshing stiffness composed of stochastic components and expressed the uncertainty of stochastic by random variables. Among the above methods, the common method for calculating the contact stiffness is based on Hertz theory, which leads to an ideal solution that ignores the roughness of the tooth surface.

To sum up, the time-varying meshing stiffness and backlash are generally studied. Meanwhile, the effects of surface topography on lubrication\textsuperscript{18} and contact behavior\textsuperscript{19} in gear meshing have been studied. Nevertheless, very few studies in the literature have related the effects of the morphology on the backlash and stiffness in the gear system, especially on both of them simultaneously. Regarding any one of them from a purely mathematical point of view will result in incomplete consideration of surface morphology in the dynamic model. In fact, as displayed in Figure 1, the roughness of tooth surfaces exists certainly. The height of microscopic asperities will influence the backlash of meshing gears and the surface features will affect the normal contact stiffness of the interface, which ultimately influences the mechanics and dynamic behavior of gears. Therefore, the study of gear dynamics needs to consider the influence of meshing surface morphology to obtain more accurate results.

In this study, considering the microscopic topography of meshing teeth surfaces, the nonlinear dynamic response of spur gear based on a transverse-torsional dynamic model is studied. Both the backlash function and meshing stiffness of the gear pair are derived with regard to the influence of surface topography by the fractal method.

The system dynamic performance under various fractal parameters that represent the surface topography is analyzed and discussed. The results show that morphology has a significant effect on the dynamic response.

2 | MODELING OF A PAIR OF SPUR GEARS

2.1 | Transverse-torsional dynamic model of gear system

The simplified dynamic model of a gear pair is shown in Figure 2. For a pair of meshing spur gears, the transverse-torsional vibration model contains three degrees of freedom, which are the relative torsional motion of gears and the radial vibration of the pinion as well as the gear.

In Figure 2, $R_i (i = p, g)$ is the radius of pinion and gear, respectively; $K_i (i = p, g)$ is the bearing stiffness; $c_i (i = p, g)$ denotes the damping coefficient of the pinion bearing and the gear bearing, respectively; $\theta_i (i = p, g)$ is the rotational displacement, $T_i (i = p, g)$ is the external load torque; $F_i (i = p, g)$ is the external radial preloads applied on bearings; $K_{en} (t)$ is the time-varying meshing stiffness; and $c_{en}$ denotes the damping coefficient between meshing teeth.

In the process of gear meshing, the translational displacement contains the motion of the gear center along the $y$-direction and meshing point. Therefore, the total displacement of the meshing point is expressed as

\[ y_p = \bar{y}_p + R_p \theta_p, \]
\[ y_g = \bar{y}_g + R_g \theta_g, \]

where $\bar{y}_p$ is the displacement of the pinion center and $\bar{y}_g$ is the displacement of the gear center.

The torsional DOF is equivalent to the relative displacement of gears, that is

\[ x = R_p \theta_p - R_g \theta_g + y_p - y_g - e(t), \]

where $e(t)$ denotes the time-varying comprehensive error that originates from manufacturing,\textsuperscript{2} assembly and transmission\textsuperscript{10} error. The dynamic form is related to the periodic meshing of gears. According to

![Figure 1: Microscopic rough surfaces of meshing gears](image-url)
available literature, time-varying comprehensive error is always simplified as first harmonic function, that is, $e(t) = e_0 \cos(\omega_m t + \phi_m)$, $\omega_m$ is the meshing frequency of gear pair.

Considering Equations (1) and (2), the differential equations governing the coupled transverse-torsional motion are expressed as

$$\begin{align*}
\sum m_k \ddot{y}_k + c_m \dot{y}_m + \sum c_k \dot{x}_k + m_k \dot{x}_k + K_m(t)f(x) &= -F, \\
\sum m_k \ddot{y}_k + c_m \dot{y}_m + \sum c_k \dot{x}_k - c_m \dot{x}_m + K_m(t)f(x) &= F, \\
\sum m_k \ddot{x}_k + c_m \dot{x}_m + K_m(t)f(x) &= -F - m_c e(t)
\end{align*}$$

where $m_k$ is the equivalent mass of the gear pair decided by $m_k = 1/(R_p^2/l_p + R_g^2/l_g)$, $l(i = pg)$ is the rotational inertia of pinion and gear respectively, $c_m$ is the damping coefficient of meshing gears, that is

$$c_m = 2\xi \sqrt{\frac{K_m m_k R_p^4 + R_g^4}{2(m_p R_p^4 + m_g R_g^4)}}.$$  

$F$ is the external force that decided by $F = T_p/R_p = T_g/R_g$, $f(x)$ denotes the function of backlash

$$f(x) = \begin{cases} 
  x - b, & x > b \\
  0, & -b < x < b, \\
  x + b, & x < -b
\end{cases}$$

where $b$ is the backlash of the gear pair.

### 2.2 Classical expressions of nonlinear backlash and meshing stiffness

In Equation (3), the effects of time-varying meshing stiffness, backlash, damping, and comprehensive transmission error on the dynamic response are taken into consideration. The strong nonlinear factors are the meshing stiffness and backlash. The calculation of backlash and meshing stiffness is the first step to calculate the dynamic response.

The meshing stiffness of the gear pair is essentially composed of five parts: bending stiffness $K_b$, shear stiffness $K_s$, axial compression stiffness $K_a$, the fillet-foundation deformation stiffness $K_f$, and contact stiffness $K_c$. One of the commonly used methods to calculate stiffness is the potential energy method, which treats gear tooth as a cantilever beam. Based on the principle of virtual work and considering the tooth profile, the work borne by each meshing point under the meshing force was integrated to obtain the virtual work possessed by the whole tooth. Also, the potential energy of the beam consists of five components corresponding to the five parts of stiffness. Subsequently, stiffness is calculated by corresponding potential energy. To simplify the calculation, a common method is to fit the meshing stiffness by Fourier series

$$K_m(t) = K_{am}[1 + \varepsilon \cos(\omega_m t + \phi_m)].$$

where $K_{am}$ is the average meshing stiffness, which is always set as a suitable constant. $\cos(\omega_m t + \phi_m)$ is the first-order harmonic component of the stiffness and stands for the variant with time, $\phi_m$ is the initial phase.

Besides, the backlash in Equation (5) is always supposed as a fixed value or a steady random number that meets the normal distribution. But in fact, the backlash is related to the machining error, and the machining process also determines the surface topography, hence the gear backlash must also be affected by the surface morphology.

Thus, the above methods of calculating backlash and stiffness cannot reflect the influence of microscopic features between meshing interfaces. Since the contact stiffness of two interfaces is under the influence of microscopic features, the method to assume stiffness as a constant will ignore the effects, resulting in reduced accuracy and limited application scope. In this study, we will propose a modified expression of meshing stiffness and backlash to reflect the influence of surface topography.

### 3 THE EXPRESSION OF BACKLASH AND MESHING STIFFNESS UNDER SURFACE TOPOGRAPHY

#### 3.1 Effects of surface morphology on the backlash

The fractal theory is a way to describe a rough surface with fractal characteristics, which obeys the cross-scale self-affinity and self-similarity of the surface in reality. Different from the statistics method, the fractal method has the advantage of being independent of instrument resolution. Meanwhile, fractal dimension $D$ and fractal roughness $G$ are the main parameters to determine

![Simplified dynamic model of a gear pair](image-url)
surface topography in fractal theory. A smooth surface leads to a large fractal dimension $D$ and a smaller $G$. According to fractal theory, the rough meshing interfaces have fractal character and can be simulated by a series of microscopic asperities as shown in Figure 3.

Based on the W-M function, the expression of backlash is\(^{22,23}\)

$$z_i(t) = \sum_{n=0}^{\text{max}} \lambda_i^{D-2n} \sin(\lambda_i t), \quad (7)$$

where $z_i(t)$ is the time-varying backlash, $\lambda$ is the scale coefficient, $D$ is the fractal dimension ($1 < D < 2$), $n$ denotes the frequency index that determines the length scale of asperities. Equation (7) cannot reflect the effects of fractal roughness $G$. To take $G$ into consideration, Equation (7) can be modified as

$$z_i(t) = L(G/L)^{D-2} \sum_{n=0}^{\text{max}} \lambda_i^{D-2n} \sin(\lambda_i t), \quad (8)$$

where $L$ is the sampling length of the fractal surface. For meshing gears, $L$ means actual contact length and can be derived as

$$L = \left( \frac{3F_m R_m}{4E} \right)^{1/3}, \quad (9)$$

where $R_m = R_p R_g / (R_p + R_g)$ is the comprehensive curvature base radius of gears, $F_m$ is the meshing force decided by $F_m = F / \cos \alpha$, with $\alpha$ being the pressure angle.

Based on Equation (8), the backlash of gear pair considering surface topography is derived as

$$b_m(t) = b_0 - z_g(t) - z_p(t) = b_0 - \sum_{ij} L(G/L)^{D-2} \sum_{n=0}^{\text{max}} \lambda_i^{D-2n} \sin(\lambda_i t). \quad (10)$$

where $b_0$ is the initial backlash as an appropriate constant, $D_y, G_y$ are the fractal factors of the gear, $D_p, G_p$ are the fractal factors of the pinion.

Substituting Equation (10) into Equation (5), the modified backlash function is derived as

$$f'(x) = \begin{cases} x - b_0 + \sum_{ij} L(G/L)^{D-2} \sum_{n=0}^{\text{max}} \lambda_i^{D-2n} \sin(\lambda_i t), & x > b_m \\ 0, & -b_m < x < b_m \\ x + b_0 - \sum_{ij} L(G/L)^{D-2} \sum_{n=0}^{\text{max}} \lambda_i^{D-2n} \sin(\lambda_i t), & x < -b_m \end{cases} \quad (11)$$

Equation (11) indicates that the values of backlash are affected by surface topography directly. The details are illustrated in Figure 4. In Figure 4A, the backlash of a single tooth, i.e., the height of asperities, decreases with fractal dimension $D$. Since a higher fractal dimension means a smoother surface, the height of asperities will be less under a big value of $D$. The result in Figure 4A is consistent with reality. Nevertheless, in Figure 4B, as the backlash in the value of constant $b_0$ minus $z(t)$, for higher $D$, the backlash of a gear pair increases. Besides, as the backlash is a function of time, the influence of backlash becomes time-varying, which is a big difference from the traditional method.

### 3.2 Effects of surface morphology on meshing stiffness

Based on fractal theory, the stiffness of contact interfaces is the sum of all asperities at a microscopic scale. The stiffness of a single asperity in elastic deformation mode is

$$k_y = 2E(a/\pi)^{3/2}, \quad (12)$$

where $E$ is the equivalent elastic modulus, $a$ is the real contact area of a single asperity.

For a single asperity, the elastic deformation mode occurs at initial contact, and then the transition into plastic deformation can take place. The contact area and critical deformation to divide the two stages are given as\(^{24}\)

$$\omega_c = \sqrt{\frac{q_\sigma}{2E}} \left( \frac{\rho}{G^{D-1}} \right)^2, \quad (13)$$

respectively, where $q$ is the coefficient of Poisson’s ratio, $\alpha_y$ is the yield strength. The height of a single asperity is expressed as

$$\delta = G(D-1)\rho^{1-D}. \quad (14)$$

When the critical deformation exceeds the asperity height, the contact will be in pure elastic deformation mode. Combining Equations (13) and (14), the critical length scale $l_c$ can be derived as

$$l_c = G \left( \frac{2E}{q_\sigma \alpha_y} \right)^{1/2}. \quad (15)$$
The distribution function of contact area between a pair of engaged gear surfaces is¹⁵

\[ n(a) = \frac{\lambda_m D_a^2}{2} a_0^2 e^{-a^2/a_0^2} \]  

(16)

where \( a \) denotes the maximum contact area of a single asperity, \( a \) denotes the single asperity contact area, \( \lambda_m \) is the coefficient of the curved interface, that is

\[ \lambda_m = \frac{4L R_p R_g}{\pi (R_p + R_g)} \left( \frac{1}{R_p} + \frac{1}{R_g} \right)^2. \]  

(17)

Accordingly, combining Equations (12), (15), and (16), based on the coupling of contact area and critical length scale,²⁶ a double integration can be employed to calculate contact stiffness as

\[ K_n = \int_0^1 \int_0^a k_n n(a) da \]  

(18)

where \( D \) is equivalent fractal dimension of meshing surfaces decided by the fractal dimension of meshing surfaces as

\[ D = 2D_p D_g / (D_p + D_g), \]  

(19)

\( G \) is the equivalent fractal roughness of meshing interfaces by

\[ G = 2G_p G_g / (G_p + G_g), \]  

(20)

\( a_0 \) denotes the largest contact area of a single asperity, that is²⁷

\[ a_0 = \left( \frac{3E_p L_0^2}{4E_{hertz} G^0^{D-1}} \right)^{\frac{3}{2}}. \]  

(21)

Substituting Equation (21) into Equation (18), the contact stiffness is finally expressed as

\[ K_n = \frac{\left( \frac{3}{2} \right)^{D-3} \pi^{D-3} \beta G^0^{D-2+1} a_0^2 L_0^2}{(1 - D)(D - D^2 + 1)(q_r/2)^{D-1}} \left( L_0^{D-D^2+1} - L_0^{D-D^2+1} \right). \]  

(22)

It can be seen that in Equation (22), contact stiffness is decided by given surface topography. As the time variation of stiffness is caused by bending stiffness \( K_b \), shear stiffness \( K_s \), axial compression stiffness \( K_a \), and the fillet-foundation deformation stiffness \( K_f \), the contact stiffness can be applied as the average stiffness in Equation (6). And the modified meshing stiffness is

\[ K_{m, \text{eff}}(t) = K_n \left[ 1 + \cos(\omega_n t + \phi_n) \right]. \]  

(23)

Equation (23) shows that the time-varying stiffness is under the influence of surface topography. For various fractal dimensions and fractal roughness, i.e., the changeable morphology, the stiffness has obvious changes. The detailed change is displayed in Figure 5. As illustrated in Figure 5A, the meshing stiffness increases clearly and monotonically with \( D \). Since a high fractal dimension \( D \) represents a very smooth surface, the meshing stiffness at \( D = 1.9 \) is very close to the ideal smooth situation by Hertz theory. The Hertz theory can be well applied to an ideal smooth surface, which can be expressed as \( K_{\text{Hertz}} = E^{0.9} D^{0.8} F^{0.1}/1.275 \). This can also verify the correctness of
the proposed model. While as Figure 5B displays, the values of fractal roughness $G$ at $D = 1.2$ have gentle effects on stiffness. The curves of time-varying stiffness are very close. From above, the meshing stiffness presents variability for various values of $D$ and $G$, demonstrating the big effects of surface topography on the stiffness. Also, fractal dimension $D$ has greater influences than fractal roughness $G$.

### 3.3 Nondimensionalization of the dynamic model

Based on the modified expression of time-varying meshing stiffness and backlash, the dynamic equations can be nondimensionalized. According to Equation (3), the inherent frequency of the gear system is

$$\omega_n = \sqrt{\frac{K_m}{m_e}}. \quad (24)$$

To make the parameters in the dynamic equation dimensionless, let $t^* = t / \omega_n$, $x^* = x / b_1$, $y_0^* = y_0 / b_1$, $y_s^* = y_s / b_1$, $\omega_m^* = \omega_m / \omega_n$, where $b_1$ is the characteristic length. Then the dimensionless form of gear dynamic model is derived as

$$\begin{align*}
\dot{y}_0^* + \xi_1 y_0^* + \xi_{11} y_0^{*2} + \xi_{13} x^* + \xi_{13} f(x^*) &= -F_p^* \quad (25) \\
y_s^* + \xi_{22} y_s^* + \xi_{22} x^* - \xi_{23} x^{*2} - \xi_{23} f(x^*) &= F_s^* \\
x^* - \dot{y}_0^* + \dot{y}_s^* + \xi_{32} x + \xi_{33} f(x^*) &= F^* - F_p^* + F_s^*
\end{align*}$$

where $\xi_{11} = c_m / m_e \omega_n$, $\xi_{13} = c_m / m_e \omega_n$, $\xi_{22} = c_g / m_e \omega_n$, $\xi_{23} = c_g / m_e \omega_n$, $\xi_{32} = c_m / m_e \omega_n$, $\xi_{33} = c_g / m_e \omega_n$, $\xi_2 = K_p / m_p \omega_n$, $\xi_3 = K_m(t) / m_p \omega_n$, $\xi_5 = K_p / m_p \omega_n$, $\xi_6 = K_m(t) / m_p \omega_n$, $F_p^* = F / (m_e b_1 \omega_n)$, $F_s^* = e_1(t) / (m_e b_1 \omega_n)$. For a clearer expression, the above dimensionless dynamic equation of transverse-torsional gear system can be rewritten into a matrix form, that is

\[
\begin{bmatrix}
\ddot{y}_0^* \\
\ddot{y}_s^* \\
\dot{\xi}_n^* \\
\end{bmatrix}
= \begin{bmatrix}
\xi_{11} & \xi_{13} & \xi_{13} \\
\xi_{22} & \xi_{22} & \xi_{23} \\
0 & 0 & \xi_{32}
\end{bmatrix}
\begin{bmatrix}
y_0^* \\
y_s^* \\
x^*
\end{bmatrix}
+ \begin{bmatrix}
\xi_{11} & 0 & \xi_{13} \\
0 & \xi_{22} & -\xi_{23} \\
0 & 0 & \xi_{33}
\end{bmatrix}
\begin{bmatrix}
y_0^{*2} \\
y_s^{*2} \\
f(x^*)
\end{bmatrix}
+ \begin{bmatrix}
\xi_{11} & 0 & \xi_{13} \\
0 & \xi_{22} & -\xi_{23} \\
0 & 0 & \xi_{33}
\end{bmatrix}
\begin{bmatrix}
y_0^* \\
y_s^* \\
f(x^*)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= F_p^* + F_s^* + F^* - F_p^* + F_s^* \quad (26)
\]

### 4 SIMULATION AND DISCUSSION

To study the dynamic response under different surface topography, fractal parameters $D$ and $G$ are used to determine the microscopic morphology. The basic parameters of the gear pair for simulation are provided in Table 1. Meanwhile, the gear system is assumed to be subject to a light load to avoid the possible topography changes caused by overloading. The main dynamic parameters are set as follows,
\[ \xi_{11} = 1, \xi_{13} = 0.5, \xi_{22} = 1, \xi_{23} = 0.5, \xi_{33} = -0.1, \kappa_{11} = 0.1, \kappa_{13} = 0.1, \kappa_{22} = 0.1, \kappa_{23} = -0.1, \epsilon = 0.5, F_p^* = F_g^* = 2.5, F^* = 0.1, F_e^* = 0.05. \]

### 4.1 Dynamic response with respect to \( D \)

Fractal dimension \( D \) is the main factor to determine the surface topography. As illustrated from Figures 6 to 10, the dynamic response, including the time history chart, phase diagram, FFT spectrum, and Poincare section diagrams are analyzed. It can be seen that the dynamic response is strongly affected by fractal dimension \( D \) and shows no monotonic relation.

When \( D_p = D_g = 1.1 \), as shown in Figure 6, the meshing interfaces are very rough, and the displacement varies from about 0.8 to 1.5. In the phase diagram, there is a close but wide band. The vibration frequency has only one wave crest in the FFT spectrum which is around 1.5. The points distribution appears to be relatively less concentrated but still not divergent. From above, the gear system is in quasiperiodic stage for rough surfaces at \( D = 1.1 \).

From Figure 7, as \( D_p \) and \( D_g \) grow to 1.3, i.e., the surface gets smoother, the gear system translates into an evident chaotic region. Both the time history chart and phase diagram show intense irregularity. The phase diagram shows a lot of random curves. Another issue presented in the FFT spectrum is that there are two peak values, indicating multiple frequencies are coupled within the system motion. The points distribution shows stronger decentralization and has a larger range. It reveals that the system becomes very chaotic under \( D = 1.3 \).

However, as demonstrated in Figure 8, the tendency of becoming chaotic reverses for \( D_p = D_g = 1.5 \). It can be seen in the time history chart that the dynamic response apparently obeys periodic motion. The thin and close line in the phase diagram and the centralized points in the Poincare section diagrams prove the periodic situation as well. The system has two main frequencies in the FFT spectrum, but the lower frequency has a higher amplitude.

According to Figure 9, the periodic dynamic response does not change with \( D_p = D_g = 1.7 \). The time history and FFT spectrum show the system has only one frequency. There is only one thin cycle in the phase diagram, indicating the good periodic response of the gear system. Meanwhile, the points distribution has strong centralization.

As illustrated in Figure 10, the greatest difference of dynamic response under \( D = 1.9 \) is the coupling of motion frequency. From the time history chart, the curve of displacement is clearly composed of multiple various periods' waves. The several peaks in the FFT spectrum also prove the coupling of frequencies. Besides, as the phase diagram and Poincare section diagram illustrated, the gear system has periodic dynamic response characteristics.

From the above, the fractal \( D \) has direct and obvious effects on the dynamic response of the gear system, including vibration range, frequency, velocity, and chaos characteristics. The chaos characteristics of the gear system show nonmonotonic with fractal dimension \( D \), the system first comes into quasiperiodic stage at \( D = 1.1 \), but then becomes chaotic as \( D = 1.3 \). However, the system displays good periodicity under \( D = 1.5 \) and \( D = 1.7 \).
Dynamic response with respect to $G$

Fractal roughness $G$ is another main factor to decide the surface topography. To analyze the influence of $G$ on the dynamic response of the gear system, here we select $D = 1.3$, i.e., the most chaotic situation, and show the results in detail. Figures 11–15 present the dynamic response under $G = 3 \times 10^{-10}$ m, $5 \times 10^{-10}$ m, $7 \times 10^{-10}$ m, and $9 \times 10^{-10}$ m. As the time history chart illustrates, the dynamic response fluctuates gently with $G$ compared to fractal dimension $D$. The wave shape has a high degree of similarity as $G$ changes.
The main waveform form remains unchanged, but a clear trend exists that the fluctuation of the wave gradually decreases. Meanwhile, the displacement range is from 1 to 1.3 for all the values of $G$, in contrast to the large changes in amplitude when $D$ changes, it can be considered that the change in $G$ does not affect the range of vibrations.

With the change of fractal roughness $G$, another trend is shown in the FFT spectrum. The vibration frequency is...
affected by fractal roughness $G$. As $G$ grows, the two wave peaks get more clear, that is there are fewer small wave crests. This phenomenon may be due to the weakening of displacement coupling.

According to the phase diagram and Poincare section diagram, it can be seen that the system keeps in a chaotic stage no matter how $G$ changes when $D = 1.3$. Although the vibration range does not change, the velocity decreases with $G$. The largest velocity is 0.15

**FIGURE 11** Dynamic response under surface topography of $G = 5 \times 10^{-10}$ m. (A) Time history chart, (B) Phase diagram, (C) FFT spectrum, and (D) Poincare section diagrams.

**FIGURE 12** Dynamic response under surface topography of $G = 1 \times 10^{-9}$ m. (A) Time history chart, (B) Phase diagram, (C) FFT spectrum, and (D) Poincare section diagrams.
when $G = 1 \times 10^{-10}$ m, but becomes only about 0.015 when $G = 1 \times 10^{-8}$ m.

From above, the fractal roughness $G$ does influence the dynamic response of the gear system although the effect is slighter than fractal dimension $D$. The main influence of $G$ presents on the velocity that the range of velocity will decrease obviously with $G$. The wave frequency is also under the influence of $G$, indicating that the displacement coupling...
may be affected by surface topography, which is in our next research plan.

5 | CONCLUSIONS

Based on the fractal theory, the 3-DOF transverse-torsional dynamic equation considering the influences of surface topography is modified. The backlash and time-varying meshing stiffness are under direct effects of microscopic features simultaneously and the revised expression is derived. Ultimately, the dynamic response according to the various surface morphology of the gear system is solved and analyzed. The detailed conclusions are as follows:

(1) The two main nonlinear dynamic factors of gear system backlash and time-varying stiffness are all under clear and direct influences of surface topography. As fractal dimension \( D \) increases, the surface gets smoother and the backlash as well as time-varying stiffness grows. When fractal dimension \( D \) goes to 1.9, indicating a very smooth surface, the calculation result is very close to an ideally smooth surface.

(2) The dynamic response of the gear system is sensitive to fractal dimension \( D \). For various values of \( D \), the dynamic behavior presents different chaos characteristics. The relationship is not simply monotonic. At \( G = 1 \times 10^{-10} \) m, the system is first quasiperiodic for \( D = 1.1 \), and then shows the chaos at about \( D = 1.3 \) but is periodic at \( D = 1.5 \) and 1.7. The nonmonotonic reason may be related to the displacement coupling in the 3-DOF model. The vibration frequency and displacement are also affected by \( D \).

(3) Another fractal factor \( G \) has more gentle effects on the dynamic response. The main waveform of the vibration remains the same, but the amplitude of the lower order oscillation decreases gradually when \( G \) increases. The range of velocity decreases as \( G \) grows. But the chaos characteristic has little change.

In this paper, the coupling of meshing stiffness and backlash under microscopic roughness is not considered, which can be researched in depth. Besides, the influences of morphology on other contact behavior like lubrication, friction, and stress are still worth to be studied. On the other hand, it will be a good research direction to apply the three-dimensional fractal method to gears.

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CONFLICT OF INTEREST

The authors declare that there are no conflict of interest.

DATA AVAILABILITY STATEMENT

The datasets generated during the current study are available from the corresponding author on reasonable request.
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