Unbounded Wiener-Hopf Operators and Isomorphic Singular Integral Operators

Domenico P.L. Castrigiano
Technischen Universität München, Fakultät für Mathematik, München, Germany
E-mail address: castrig@ma.tum.de

Some preliminaries and basic facts regarding unbounded Wiener-Hopf operators (WH) are provided. WH with rational symbols are studied in detail showing that they are densely defined closed and have finite dimensional kernels and deficiency spaces. The later spaces as well as the domains and ranges are explicitly determined. A further topic concerns semibounded WH. Expressing a semibounded WH by a product of a closable operator and its adjoint this representation allows for a natural self-adjoint extension. It is shown that it coincides with the Friedrichs extension. Polar decomposition gives rise to a Hilbert space isomorphism relating semibounded WH to singular integral operators of a well-studied type based on the Hilbert transformation.

Mathematics Subject Classification: 47G10, 47B35, 47A53, 47B25
Keywords: Unbounded Wiener-Hopf operators, singular integral operators of Hilbert transformation type, rational and semibounded symbols, Fredholm operators

1 Introduction

Results on Wiener-Hopf operators (WH) $W_{\kappa}$ with unbounded symbol $\kappa$ are scarce and probably in the literature there exists no introduction to this subject. So sec. 2 deals with preliminaries and basics regarding unbounded WH. In particular we are concerned with conditions on the symbol $\kappa$ ensuring that the domain of $W_{\kappa}$ is either the whole space or dense or trivial, and prove that $\text{dom} W_{\kappa}$ is either trivial or dense. A classical result on the eigenvalues of a WH is shown to remain valid in the unbounded case. It implies that non-trivial symmetric WH have no eigenvalues. A further result characterizes WH by their invariance under unilateral shifts. In sec. 3 WH with rational symbols are studied. They constitute a welcome source of densely defined closed operators with finite index. An explicit description of the domains, ranges, kernels, deficiency spaces, spectral and Fredholm points is given. The remainder of this article deals in sec. 4 and sec. 5 with densely defined semibounded WH. A semibounded operator $W_{\kappa}$ can be expressed by a product of a closable operator $A$ and its adjoint. Replacing $A$ by its closure one obtains quite naturally a self-adjoint extension $\tilde{W}_{\kappa}$. It is proven to coincide with the Friedrichs extension. Inverting the order of the factors one obtains a singular integral operator $L_{\phi}$ of type Hilbert transformation. As shown, for the operators of the mention type being not trivial there is a necessary condition analogous to that for WH. The self-adjoint extensions $\tilde{L}_{\phi}$ and $\tilde{W}_{\kappa}$ are isometric, which follows from the polar decomposition of $A$. Actually $\tilde{W}_{\kappa}$ is Hilbert space isomorphic to the reduction of $\tilde{L}_{\phi}$ on $\ker(\tilde{L}_{\phi})^\perp$, and the spectral representations of $\tilde{L}_{\phi}$ and $\tilde{W}_{\kappa}$ can be achieved in an explicit manner from each other. To conclude, this method is illustrated by a non-trivial example diagonalizing Lalescu’s operator and the isometrically related singular integral operator. In [22, sec. 3.3] the spectral representations of $W_{1[-1,1]}$ and the finite Hilbert transformation were related to each other by this method.
Notations. Let \( \mathcal{F} \) denote the Fourier transformation on \( L^2(\mathbb{R}) \). For measurable \( E \subset \mathbb{R} \) introduce the projection \( P_E : L^2(\mathbb{R}) \to L^2(E) \), \((P_E f)(x) := f(x)\). (For convenience define \( L^2(E) = \{0\} \), \( P_E = 0 \) if \( E = \emptyset \).) Its adjoint \( P_E^* \) is the injection \((P_E^* f)(x) = f(x) \) for \( x \in E \) and \( = 0 \) otherwise. Note
\[
P_E P_E^* = I_{L^2(E)}, \quad P_E^* P_E = M(1_E)
\]
with \( M(1_E) \) the multiplication by the indicator function \( 1_E \) for \( E \). We call \( E \) proper if neither \( E \) nor the complement \( \mathbb{R} \setminus E \) is a null set. Put \( \mathbb{R}_+ := [0, \infty] \) and \( P_+ := P_{\mathbb{R}_+} \). Analogously define \( P_- \).

Throughout let \( \kappa : \mathbb{R} \to \mathbb{C} \) denote a measurable function and \( M(\kappa) \) the multiplication operator by \( \kappa \) in \( L^2(\mathbb{R}) \) with dense domain \( \{ f \in L^2(\mathbb{R}) : \kappa f \in L^2(\mathbb{R}) \} \). \( M(\kappa) \) is normal satisfying \( M(\kappa)^* = M(\overline{\kappa}) \) and \( M(\kappa) M(\kappa) = M(|\kappa|^2) \). Moreover, \( M(\kappa) \) is self-adjoint if and only if \( \kappa \) is almost real, and \( M(\kappa) \) is nonnegative if and only if \( \kappa \geq 0 \) a.e.

**Definition.** The operator in \( L^2(\mathbb{R}_+) \)
\[
W_\kappa := P_+ \mathcal{F} M(\kappa) \mathcal{F}^{-1} P_+^*
\]
is called the Wiener-Hopf operator (WH) with symbol \( \kappa \). Occasionally we write \( W(\kappa) \) instead of \( W_\kappa \). Clearly, \( W_\kappa = W_{\kappa'} \) if \( \kappa = \kappa' \) a.e. Often we shall refer to this tacitly. The symbol \( \kappa \) is called proper if \( \text{dom} W_\kappa \neq \{0\} \).

The theory of WH with bounded symbol is well developed. We content ourselves to refer here to the book [1, Chapter 9] and to mention the origins [2]. Obviously in case of a bounded symbol the operators \( W_\kappa \) are bounded with \( \text{dom} W_\kappa = L^2(\mathbb{R}_+) \) and adjoint \( W_\kappa^* = W_\overline{\kappa} \). \( W_\kappa \) is the convolution on the real half line with kernel \( \kappa \), i.e.,
\[
(W_\kappa g)(x) = \int_0^\infty \kappa(x-y) g(y) \, dy
\]
if \( \kappa \in L^\infty \cap L^2 \) and \( k := (2\pi)^{-1/2} \mathcal{F} \kappa \), or if \( \kappa = \int e^{it\kappa} \kappa(t) \, dt \) for \( \kappa \in L^1(\mathbb{R}) \). For the case of integrable kernel there is the rather exhaustive theory by M. G. Krein [3]. Generally the tempered distribution \( k := (2\pi)^{-1/2} \mathcal{F} \kappa \), where \( \kappa \in L^\infty(\mathbb{R}) \) is considered as a regular tempered distribution, satisfies \( \mathcal{F} M(\kappa) \mathcal{F}^{-1} u = k \ast u \) for every Schwartz function \( u \) in the distributional sense (e.g. [4, Theorem IX.4]). For instance the kernel for \( W_{- \text{sgn}} \) is the tempered distribution \( k(x) = -\frac{1}{x^2} \) or that for \( W_{- \text{tanh}} \) equals \( k(x) = (21 \sinh(\pi x/2))^{-1} \). In the literature the generalizations of WH stay mostly within the realm of bounded operators. One deals with the traces (compressions) of bounded bijective operators in Banach space on a closed subspace [5]. The results concern the solvability of the associated Wiener-Hopf equations.

## 2 Unbounded Wiener-Hopf Operators

As put it by [6] results on unbounded WH are practically inexistent. Indeed they are scarce. See [7, 1.3] for some notes. An important result is due to M. Rosenblum [8], [9], obtained for Toeplitz operators and hence valid for the Hilbert space isomorphic \( \text{WH} W_\kappa \) (see (3)). So in the case that the symbol \( \kappa \) is real bounded below not almost constant and \((1+x^2)^{-1} \kappa \) is integrable, [9] furnishes the spectral representation of the extension \( \hat{W}_\kappa \), which is shown to be the Friedrichs extension [14], [15] and which by [9] is absolutely continuous. — There are investigations on unbounded general WH dealing with conditions for their invertibility [10]. — In [6] real bounded below Wiener-Hopf quadratic forms from distributional kernels
$k$ are considered, and it is shown that such a form determines a WH if and only if the form is closable or, equivalently, if and only if $\sqrt{2\pi k}$ is the Fourier transform of a locally integrable bounded below function $\kappa$ with integrable $(1 + x^2)^{-n}\kappa$ for some $n \in \mathbb{N}$. Clearly $\kappa$ is the symbol and $\text{dom} \ W_\kappa \supset C_c^\infty(\mathbb{R}_+)$ holds. See further [7]. Furthermore, the methods applied for the study of unbounded analytic Toeplitz operators [11] can also produce results on unbounded WH, as (4)(i), (13), (14)(c).

Starting the preliminary remarks note that $\text{dom} \ W_\kappa = \text{dom} \ M(\kappa)F^{-1}P_+$. Therefore $\kappa$ is proper if and only if $\kappa h$ is square-integrable for some Hardy function (2) $h \in \text{ran}(F^{-1}P_+)$. Clearly $\text{dom} \ W_\kappa \subset \text{dom} \ W_\kappa'$ if $|\kappa'| \leq |\kappa|$. Moreover, $\text{dom} \ W_\kappa = \text{dom} \ W_\kappa = \text{dom} \ W_\kappa(1 + |\kappa|)$. Note that $(g', W_\kappa g) = (W_\kappa g', g)$ for $g,g' \in \text{dom} \ W_\kappa$. Hence, if $W_\kappa$ is densely defined then $\text{dom} \ W_\kappa \subset \text{dom} \ W_\kappa'$, whence $W_\kappa$ is closable and $\text{dom} \ W_\kappa \subset \text{dom} \ W_\kappa'$. But unbounded WH may and may not be closed (8), (11). If $\text{dom} \ W_\kappa$ is densely defined then $W_\kappa$ is symmetric, i.e., $\text{dom} \ W_\kappa \subset \text{dom} \ W_\kappa^*$, if and only if $\kappa$ is almost real (4)(n). If $\text{dom} \ W_\kappa$ is densely defined symmetric, then $\text{dom} \ W_\kappa$ is bounded below if and only if $\kappa$ is essentially bounded below (4)(o). Recall that the numerical range $\{(g, W_\kappa g) : g \in \text{dom} \ W_\kappa, \|g\| = 1\}$ of $W_\kappa$ is convex. (Indeed, the numerical range of every operator in Hilbert space is convex [12].) It is determined in (4)(o), (p).

If the symbol $\kappa$ is unbounded, then $\text{dom} \ W_\kappa \neq L^2(\mathbb{R}_+)$. The alternative holds that either $\text{dom} \ W_\kappa$ is trivial or $\text{dom} \ W_\kappa$ is dense. In other words, as shown in (4)(i), if $\kappa$ is proper, then $\text{dom} \ W_\kappa$ is densely defined. In (4)(c) and (4)(g) explicit characterizations of proper symbols are given. There is also the useful criterion in (4)(b) for $\kappa$ to be proper. So proper symbols may have polynomial growth and countably many singularities with integrable logarithm like as $\exp |x|^\alpha, -1 < \alpha < 0$. It is easy to give examples of non-proper symbols (4)(h).

(2) Hardy spaces. Recall the Hardy spaces $H_{\pm} := \text{ran}(F^{-1}P_\pm^*)$. Obviously $L^2(\mathbb{R}) = H_+ \oplus H_-$ and $h \in H_+ \iff h \in H_-$ as well $h \in H_+ \iff \hat{h} \in H_-$. We tacitly refer to the well-known Paley-Wiener Theorem characterizing the Fourier transforms of $L^2$-functions vanishing on a half-axis, see e.g. [13, Theorem 95]. In particular $h \in H_+$ if and only if there is a $\phi$ holomorphic on the upper half-plane such that its partial maps $\phi_\xi(x) := \phi(x + iy)$ for $y > 0$ satisfies $\phi_\xi \in L^2(\mathbb{R})$, $\{\|\phi_\xi\| : y > 0\}$ bounded, and $\phi_{\xi} \rightarrow h$ for $y \rightarrow 0$ in the mean and pointwise a.e. Actually $\phi$ converges to $h$ non-tangentially a.e., and $\|\phi_\xi\|\|\xi\|$ for $y \downarrow 0$ (see e.g. [14, III 3.3, II 2.6]). Moreover, every $h \in H_{\pm} \setminus \{0\}$ vanishes only on a null set. Indeed, according to a Luzin-Privalov Theorem [15, IV 2.5] a meromorphic function on the upper or lower half-plane which takes non-tangential boundary values zero on a set of positive Lebesgue measure is zero. The former property is also an immediate consequence of the following result on the modulus of a Hardy function:

For $f \in L^2(\mathbb{R}) \setminus \{0\}$ there is $h \in H_+$ satisfying $|h| = |f|$ if and only if $\frac{\ln |f|}{1 + x^2} \in L^1(\mathbb{R})$.

One proves (2.1) using the outer function with prescribed modulus on the torus ([16, Chap. 3, Def. 1.1 and Prop. 3.2]) and the Hilbert space isomorphism $\Gamma$ in (3). Recall also [17, Theorem XII ] for $\ln |h|/(1 + x^2) \in L^1$ if $h \in H_+$.

Let $M_+(\kappa)$ denote the trace of $M(\kappa)$ on $H_+$, i.e.

$$M_+(\kappa) = P_{H_+} M(\kappa) P_{H_-}^*$$

with $P_{H_+} f$ the orthogonal projection of $f \in L^2(\mathbb{R})$ on $H_+$. Note that $P_{H_+} F^{-1} P_+^* : L^2(\mathbb{R}_+) \rightarrow H_+$ is a Hilbert space isomorphism with its inverse $P_+ F P_{H_+}^*$, by which $W_\kappa$ is Hilbert space isomorphic to $M_+(\kappa)$. Often it is convenient to deal with $M_+(\kappa)$ in place of $W_\kappa$.

Finally $H_+^\infty$ is the set of all measurable bounded $\alpha : \mathbb{R} \rightarrow \mathbb{C}$ such that there is a bounded holomorphic
A on the upper half-plane with the partial maps $A_y \to \alpha$ for $y \to 0$ pointwise a.e. Actually $A$ converges to $\alpha$ non-tangentially a.e. (see e.g. [14, III 3.3, II 2.6]).

(3) Remark on Toeplitz Operators. Let the torus $\mathbb{T}$ be endowed with the normalized Lebesgue measure. The Hardy space $H^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ with orthonormal basis $e_n(w) := w^n$, $n \in \mathbb{N}_0$. Let $\Gamma : H^2(\mathbb{T}) \to H_+,$ $\Gamma u)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} u(C(y)) \, dy$ be the Hilbert space isomorphism based on the Cayley transformation $C(x) := \frac{x-i}{x+i}$. Given a measurable $\omega : \mathbb{T} \to \mathbb{C}$ then

$$T_\omega = \Gamma^{-1} M_\alpha (\omega \circ C) \Gamma$$  \hfill (2.3)

is the Toeplitz operator with symbol $\omega$. Explicitly one has the formula $T_\omega = P_{H^2(\mathbb{T})} M(\omega) P_{H^2(\mathbb{T})}^*$ quite analogous to (2.2). Obviously by this relationship results and methods regarding Toeplitz operators may be transferred for the study of WH.

(4) Theorem. (a) $\text{dom} W_\kappa = L^2(\mathbb{R}_+)$ $\Rightarrow$ $\kappa$ is bounded $\Rightarrow \text{dom} W_\kappa = L^2(\mathbb{R}_+)$ and $W_\kappa$ is bounded.

(b) If $\kappa q s \in L^2(\mathbb{R})$, where $q$ is a polynomial and $s$ is the inverse Fourier transform of a Schwartz function with support in $[0, \infty]$, then $qs \in \text{dom} W_\kappa$ and $\kappa$ is proper. — Suppose that $\kappa q s \in L^2(\mathbb{R})$ for a polynomial $q$ with only real zeros and for every Schwartz function $s$. Then $\text{dom} W_\kappa \supset \{ q(1 + \frac{d}{d\omega}) \phi : \phi \in C_c^\infty(\mathbb{R}_+) \}$.

(c) $W_\alpha(\text{dom} W_\kappa) \subset \text{dom} W_\kappa \forall \alpha \in H^\infty_+$ and, equivalently, $ah \in \text{dom} M_+ (\kappa) \forall \alpha \in H^\infty_+$, $h \in \text{dom} M_+ (\kappa)$.

(d) For all $\alpha \in H^\infty_+$ one has (1) $W_\nu W_\alpha = W_{\alpha \nu}$ and (2) $W_\alpha^* W_\kappa \subset W_{\alpha \kappa}$.

For $b \geq 0$ let $S_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unilateral translation $S_b g(x) := g(x - b)$ if $x > b$ and $:= 0$ otherwise. Then one has the translational invariance

$$W_\kappa = S_b^* W_\kappa S_b$$

(e) Let $p \in [0, \infty]$. Let $j$ denote a real-valued function such that $\frac{1}{1 + x^p}$ is integrable. Then

$$\kappa \text{ proper } \iff \kappa x^j \in L^p(\mathbb{R}) \text{ for some } j$$

(f) $\kappa$ is proper if and only if $\kappa^2$ is proper. More generally, let $r > 0$ and let $\kappa_1$, $\kappa_2$ be two symbols satisfying $|\kappa_1| = |\kappa_2|^r$. Then $\kappa_1$ is proper if and only if $\kappa_2$ is proper. Finally, if $\kappa_1$ and $\kappa_2$ are proper symbols then so are $\kappa_1 \kappa_2$ and $\kappa_1 + \kappa_2$.

(g) $\kappa$ is proper if and only if $\frac{\ln(1 + |\kappa|)}{1 + x^2}$ is integrable.

(h) $\kappa$ is not proper if $|\kappa(x)|$ increases not less than exponentially for $x \to \infty$ or $x \to -\infty$, i.e., if there are positive constants $a$, $\delta$, $\lambda$ such that $|\kappa(x)| \geq \delta e^{\lambda |x|}$ for $x \geq a$ or $x \leq -a$.

(i) If $\kappa$ is proper, then $\text{dom} W_\kappa$ is dense. More precisely, if $h \in \text{dom} M_+ (\kappa) \setminus \{0\}$, then there is $h' \in H_+$ with $|h| = |h'|$ such that $H_+^{\infty} \frac{1}{x+1} h'$ is dense and contained in $\text{dom} M_+ (\kappa)$.

(j) Put $\rho := (1 + |\kappa|)^{-1}$. Then $\kappa$ is proper if and only if $\overline{M(\rho) H_- \cap M(\rho) H_+} = \{0\}$ and $\kappa$ is not proper if and only if $\overline{M(\rho) H_-} = L^2(\mathbb{R})$.

(k) Let $\kappa$ be not almost constant. If $\lambda \in \mathbb{C}$ is an eigenvalue of $W_\kappa$, then $\overline{\lambda}$ is not an eigenvalue of $\overline{W_\kappa}$. If $\kappa$ is almost real, then $W_\kappa$ has no eigenvalues and in particular $W_\kappa$ is injective.

(l) Let $\lambda \in \mathbb{C}$ such that $\kappa^{-1}(\{\lambda\})$ is proper. Then $\lambda$ is not an eigenvalue of $W_\kappa$. 

4
(m) Let $W_{\kappa_1}$ and $W_{\kappa_2}$ coincide on a dense set of $L^2(\mathbb{R}_+)$. Then $\kappa_1 = \kappa_2$ a.e.

(n) Let $W_\kappa$ be densely defined. Then $W_\kappa$ is symmetric if and only if $\kappa$ is almost real.

(o) Let $W_\kappa$ be densely defined symmetric. Then $W_\kappa$ is bounded below if and only if $\kappa$ is real essentially bounded below, and the maximal lower bound of $W_\kappa$ equals the maximal essential lower bound of $\kappa$. If $W_\kappa$ is bounded below but not bounded, then the numerical range $\{\langle g, W_\kappa g \rangle : g \in \text{dom} W_\kappa, ||g|| = 1\}$ equals $[\alpha, \infty]$ with $\alpha$ the maximal lower bound.

(p) Let $W_\kappa$ be bounded symmetric and not a multiple of $I$. Then the numerical range equals $|a, b|$ with $a$ and $b$ the minimum and maximum, respectively, of the essential range of $\kappa$.

Proof. (a) The second implication is obvious. As to the first one assume that $\kappa$ is not bounded. Then $f \in L^2(\mathbb{R})$ exists with $\kappa f \not\in L^2(\mathbb{R})$. Write $f = h_+ + h_-$ with $h_+ \in H_+$. As $dom M_+(\kappa) = L^2(\mathbb{R}_+)$ one has $\kappa h_+ \in L^2(\mathbb{R})$, and hence $\kappa h_- \not\in L^2(\mathbb{R})$. Similarly, $\kappa f \not\in L^2(\mathbb{R})$, $\bar{f} = \bar{h}_- + h_+$ with $\bar{h}_+ \in H_+$, and $\kappa \bar{h}_- \in L^2(\mathbb{R})$. This contradicts $\kappa h_- \not\in L^2(\mathbb{R})$.

(b) As to the first claim note that $s$ is a Schwartz function in $H_+$, whence $qs \in H_+$ by (7)(c).

Now let $D$ denote the differential operator $i \frac{d}{dx}$ and let $u$ be any Schwartz function with support in $\mathbb{R}_+$. Then $q(D)u$ is still such a function, and $F^{-1}q(D)u = qF^{-1}u$ are Schwartz functions in $H_+$. Hence it remains to show that $\{q(D)\phi : \phi \in D\}$ is dense in $L^2(\mathbb{R}_+)$, where $D$ denotes the space of test functions $C_c^\infty(\mathbb{R}_+)$. Assume $g \in L^2(\mathbb{R}_+)$ with $g \perp q(D)\phi$, i.e., $\int_{\mathbb{R}_+} \bar{g}(\phi) d x = 0$. The claim is $g = 0$.

Indeed, regarding $\phi$ as a regular distribution in $D'$, one has $\int (q(D)\phi) = q(-D)\bar{\phi} = 0$ for all test functions $\phi$, whence $q(-D)\bar{\phi} = 0$. Thus $\bar{\phi}$ is a solution of the differential equation $q(-D)F = 0$ for $F \in D'$. As known all its solutions are regular. Hence $\bar{\phi} \in T$, where $T$ denotes the space of linear combinations of functions on $\mathbb{R}_+$ of the kind $x \rightarrow x^k e^{i\lambda x}$, $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Then $G \in T$ for $G(x) := \int_0^x |g(t)|^2 dt$. One has $G(x) \rightarrow ||g||_2 < \infty$ for $x \rightarrow \infty$. Write $G = \sum_{k=0}^n x^k A_k$ with $A_k$ a linear combination of periodic functions $e^{i\lambda x}$. Then $A_n(x) \rightarrow 0$ for $x \rightarrow \infty$. Since $A_n$ is almost periodic this implies $A_n = 0$. The result follows.

(c) Let $h \in dom M_+(\kappa)$. The claim is $M_+(\alpha)h \in dom M_+(\kappa)$. Now $M_+(\alpha)h = ah$ since obviously $ah \in H_+$. Moreover $\kappa ah \in L^2(\mathbb{R})$, whence the claim.

(d) As to (1) note $h \in dom (M_+(\kappa)M_+(\alpha)) \Leftrightarrow h \in H_+, \quad P^-_{H_+}(ah) \in dom M_+(\kappa)$. Since $ah \in H_+$ the latter is equivalent to $ah \in dom M_+(\kappa) \Leftrightarrow \kappa ah \in L^2(\mathbb{R}) \Leftrightarrow h \in dom M_+(\alphah)$. — Regarding (2) note $M_+(\alpha) = M_+(\alpha)(M_+)(\kappa)$, dom $(M_+(\alpha)M_+(\kappa)) = dom M_+(\kappa)$, and $P^+_{H_+}P^-_{H_+} + P^-_{H_+}P^+_{H_+} = I - P^+_{H_+}P^-_{H_+} = P_{H_+} - I$. Hence for $h \in dom M_+(\kappa)$ one has $M_+(\alpha)h = M_+(\alpha)(M_+)(\kappa)h = M_+(\alpha)(\kappa)(h) = M_+(\alpha)(\kappa)(h) = P_{H_+}(\alpha)(\kappa)(h) - P^-_{H_+}(\alpha)(\kappa)(h) = M_+(\kappa)(h) - P^+_{H_+}(\alpha)(\kappa)(h) = M_+(\kappa)(h) - I$, since $\kappa H_+ = H_+$. — For the translational invariance check $S_\kappa = W(e^{i \kappa}h(\cdot))$, dom $W(e^{i \kappa}h(\cdot)) = dom W_\kappa$ and apply the foregoing results.

(e) Here we prove the case $p = 2$ and the implication $\Leftarrow$ for $p = \infty$ and $p = 1$. The remainder is shown in the proof of (f).

Let $h \in dom M_+(\kappa) \setminus \{0\}$. Then $\kappa h \in L^2(\mathbb{R})$, whence $\kappa e^{in|\kappa|} \in L^2(\mathbb{R})$. By (2.1), $\ln|\kappa|/(1 + x^2)$ is integrable. For the converse implication put $j := 1(|x| \leq 0)j - |x|^{1/2}$. Then $j'/(1 + x^2)$ is integrable, $e^{j'}$ is square-integrable. Moreover, $\kappa e^{j'}$ is square-integrable since $j' \leq j$. By (2.1) there is $h \in H_+$ with $|\kappa| = e^{j'}$. So $h \in dom M_+(\kappa) \setminus \{0\}$. — Next turn to $\Leftarrow$ for $p = \infty$. For $j'$ from above $|\kappa| e^{j'} \leq |\kappa| e^{j-|x|^{1/2}}$ is square-integrable. Hence $\kappa$ is proper by the case $p = 2$ just shown. — Now consider $\Leftarrow$ for $p = 1$. Due to the assumption $(1 + x^2)^{-1}(1 + |\kappa| e^{j'})$ is integrable, whence $j'(1 + x^2)^{-1}$ is integrable for $j' := \ln(1 + |\kappa| e^{j'})$. Therefore $(1 + |\kappa| e^{j'}) e^{-j'} = 1$ implying $|\kappa| e^{j-|x|^{1/2}} \leq 1$, whence the result by $\Leftarrow$ for $p = \infty$.

(f) Let $\kappa^2$ be proper. Then $1 + |\kappa|^2$ is proper and $|\kappa| \leq 1 + |\kappa|^2$, whence $\kappa$ is proper. — Now let $\kappa$
be proper. Then by (e) \((p = 2), \kappa e^j \in L^2(\mathbb{R})\) for some real-valued \(j\) with integrable \(j/(1 + x^2)\). Since \(\kappa := |\kappa|^2 e^{2\alpha}\) is integrable, \(\kappa\) is proper by (e) \((\Leftarrow \; \text{for } p = 1)\). Then (e) \((p = 2)\) yields \(\kappa e^j \in L^2(\mathbb{R})\) for some real-valued \(j\) with integrable \(j/(1 + x^2)\). Hence \(|\kappa|^2 e^{2\alpha+j} \in L^2(\mathbb{R})\), whence the claim by (e) \((p = 2)\).

The general case is easily reduced to the claim that, for \(r > 1\) and \(\kappa \geq 0, \kappa\) is proper if and only if \(\kappa^r\) is proper. So let \(\kappa\) be proper. Let \(n \in \mathbb{N}\) satisfy \(r \leq 2^n\). By the foregoing result \(\kappa^{2^n}\) is proper. Then \(1 + \kappa^{2^n}\) is proper and \(\kappa^\nu \leq 1 + \kappa^{2^n}\). Hence \(\kappa^\nu\) is proper. Conversely, if \(\kappa^\nu\) is proper, then \(1 + \kappa^{2^n}\) is proper and \(\kappa \leq 1 + \kappa^\nu\), whence \(\kappa\) is proper.

Now we complete the proof of (e). Consider first the case \(p \in [0, \infty]\). Let \(\kappa e^j \in L^p(\mathbb{R})\) for some \(j\). Then \(|\kappa|^{p/2} e^{j\nu} \in L^2(\mathbb{R})\). Hence \(|\kappa|^{p/2}\) is proper as shown in (e) \((p = 2)\). The foregoing result applies, whence \(\kappa\) is proper. The converse follows in the same way due to (e) \((\Rightarrow \; \text{for } p = 2)\). — Now let \(p = \infty\) and let \(\kappa\) be proper. As just shown, \(\kappa e^j\) is integrable for some real-valued \(j\) with integrable \((1 + x^2)^{-1}\).

Then \(|\kappa| e^j \leq 1\) for \(j' := j - \ln(1 + |\kappa| e^j)\) as shown in the proof of (e) \((\Leftarrow \; \text{for } p = 1)\).

Finally, let \(\kappa_1, \kappa_2\) be proper. Apply (e) for \(p = \infty\). So \(|\kappa_i| \leq e^j, i = 1, 2\). Then \(|\kappa_1\kappa_2| \leq e^{j_1+j_2}\). Assume without restriction \(j_1 \geq 0, i = 1, 2\). Then \(|\kappa_1 + \kappa_2| \leq 2e^{j_1+j_2}\).

\((g)\) Let \(\kappa\) be proper. Then \(|1 + |\kappa|\) is proper, and by (e) \((p = \infty)\) one has \(|1 + |\kappa|| \leq e^j\) for some real-valued \(j\) with integrable \(\frac{e^j}{1 + x^2}\). Then \(|1 + |\kappa|| \leq j\), whence the claim. Conversely let \(\kappa\) be integrable for \(j := \ln(1 + |\kappa|)\). Then \(|1 + |\kappa|| = e^j\), whence \(||\kappa\kappa| \leq e^j\) and \(\kappa\) is proper by (e) \((p = \infty)\).

\((h)\) \(\ln(1 + |\kappa(x)|) \geq \ln |\kappa(x)| \geq \ln(\delta) + \lambda|x|\) for all \(x \geq a\) or \(x \leq -a\). Hence \(\frac{\ln(1 + |\kappa(x)|)}{1 + x^2}\) is not integrable. Apply \((g)\).

\((i)\) Let \(h \in \text{dom } M_+(\kappa) \setminus \{0\}\). Then \(\kappa h \in L^2(\mathbb{R})\), whence \(\kappa e^j \in L^2(\mathbb{R})\) for \(j := \ln |h|\). By (2.1), \(j/(1 + x^2)\) is integrable.

Recall (3). Put \(c := C^{-1}\). Check that \(j \circ c \geq 0\) is integrable on the torus \(T\). Hence \(U(w) := \exp \left( \frac{1}{2\pi} \int_0^{2\pi} e^{e^j} d\tau \right)\) is an outer function on the disc \(D\) (see e.g. [16, Chapter 3]). So \(U\) converges non-tangentially a.e. to a function \(u\) on \(T\) satisfying \(|u| = e^{j\circ c} \in L^2(T)\), whence \(U \in H^2(D)\). Therefore, as known (see e.g. [19, sec. 3]), \(UH^\infty(D)\) is dense in \(H^2(D)\). This implies that \(uH^\infty(T)\) is dense in \(H^2(T)\).

Put \(h' := u \circ C\). Note \(|h'| = e^j = |h|\). The above result is transferred to \(H_+\) by \(\Gamma\) in (3). Accordingly, \(\frac{1}{1 + x^2}h' \in H_+\) and \(\frac{1}{1 + x^2}h' H^\infty_+\) is dense in \(H_+\). The latter is contained in \(\text{dom } M_+(\kappa)\) by \((c)\), since \(\kappa \frac{1}{1 + x^2}h'\) is square-integrable and hence \(\frac{1}{1 + x^2}h' \in \text{dom } M_+(\kappa)\). Finally \(h' \in H_+\) by \((7)\).(c).

\((j)\) Note \(0 < \rho \leq 1\). So \(h \in \text{dom } M_+(\kappa) = \text{dom } M_+(\rho^{-1}) \Rightarrow \exists f \in L^2(\mathbb{R})\) such that \(h = f \in H_+\). The latter means \(0 = (k, pf) = (pk, f)\) \(\forall k \in H_-\). Hence \(P_{H_+}^\perp \text{dom } M_+(\kappa) = M(\rho)(M(\rho)H_+)\perp\). — Now let \(h_0 \in H_+\) with \(h_0 \downarrow M(\rho)(M(\rho)H_+)\perp\). The latter means \(0 = \langle h_0, pf \rangle = \langle ph_0, f \rangle \forall f \in (M(\rho)H_+)\perp\) and hence equivalently \(\rho h_0 \in M(\rho)H_+\perp\). — Since by \((i)\) \(\text{dom } M_+(\kappa)\) is either dense if \(\kappa\) is proper or trivial if \(\kappa\) is not proper, the result follows.

\((k)\) Cf. the proof for bounded \(\kappa\) in [20, 2.8]. Keep \((2)\) in mind. Since \(W_\kappa - \lambda I = W(\kappa - \lambda 1_\mathbb{R})\) assume without restriction \(\lambda = 0\). Suppose \(W_{\kappa}u = 0\), \(W_{\kappa}v = 0\). Set \(h_+ := F^{-1}P_+\). Then \(h_+ \in H_+\) and \(h_- := k_+ + H_-\). Set \(k_- := F^{-1}P_+ v\). Then \(k_- \in H_-\) and \(k_- \in H_-\), whence \(k_+ := k_+ + H_+\).

Note that \(j := h_- \kappa k_+ = h_+ k_-\). Hence there is a holomorphic \(\chi : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}\) such that its partial maps satisfy \(\chi_y \in L^1(\mathbb{R})\), \(K := \sup\|\chi_y\|_{1; y \neq 0} < \infty, \chi_y \to j\) pointwise a.e. and \(\|\chi_y - j\|_1 \to 0\) for \(y \to 0\). By a standard argument (see also [21, Theorem II]) \(\chi\) extends to an entire function still called \(\chi\) with \(\chi |\kappa = j\) a.e. Fix \(z \in \mathbb{C}, |z| > 1\). We use the representation \(\pi(z) = \int_D \chi(z + w) d^2 w\) were \(D\) denotes the disc with center 0 and radius 1. Then \(\pi(\chi(z)) \leq \int_1^{\infty} \int_{-\infty}^{\infty} |\chi(x + u + i(y + v))| d u d v \leq 2K\) so that \(\chi\) is constant equal to 0, whence \(h_- \kappa k_+ = 0, h_+ k_+ = 0\). (An alternative argument implying this result uses [13, Theorem 76], by which \((2\pi)^{1/2} F_{\ell 1}(h_0 k_+) = F h_+ \ast F k_+\). Accordingly, \(F h_+ \ast F k_- = F h_+ \ast F k_+ = 0\),
since $Fh_-, Fk_-$ vanish on $[0, \infty[$ and $Fh_+, Fk_+$ vanish on $]-\infty, 0]$. The cases $k_- = 0$ or $h_+ = 0$ are trivial. Otherwise $h_+ \neq 0$ and $\diagup k_+ = h_- = 0$, whence $\kappa = 0$ a.e.

Now let $\kappa$ be real, let $\lambda \in \mathbb{C}$, and let $g \in \text{dom} W_\kappa$ satisfy $W_\kappa g = \lambda g$. Then $\langle g, g \rangle = (g, W_\kappa g) = \langle W_\kappa g, g \rangle = \overline{\langle g, g \rangle}$, hence $g = 0$, since otherwise $\lambda \in \mathbb{R}$ would contradict the foregoing result.

(l) Since $W_\kappa - A = W(\kappa - \lambda 1_{\mathbb{R}})$ and $\kappa^{-1}(\{\lambda\}) = (\kappa - \lambda 1_{\mathbb{R}})^{-1}(\{0\})$ assume without restriction $\lambda = 0$. Put $E := \kappa^{-1}(\{0\})$. Suppose $W_\kappa g = 0$. Then $h_+ := \mathcal{F}^{-1}1_{E^c}g \in H_+$ and $h_- := \kappa h_+ \in H_-$. Since $h_-$ vanishes on the non-null set $E$, $h_- = 0$ follows. Hence $h_+$ vanishes on the non-null set $\mathbb{R} \setminus E$ implying $h_+ = 0$ and hence $g = 0$.

(m) Put $\beta := \kappa_1 - \kappa_2$. Then $W_\beta |_D = 0$ for some dense $D \subseteq L^2(\mathbb{R}_+)$. Let $k \in \text{dom} W_\beta^*$. Then for all $g \in D$ one has $\langle k, W_\beta g \rangle = \langle W_\beta^*k, g \rangle$, whence $W_\beta^*k \neq 0$. Since $W_\beta^* \subseteq W_\beta^*$, it follows $W_\beta^* |_D = 0$. Then $\beta$ is almost constant by (k), whence $\beta = 0$ a.e.

(n) Suppose $W_\kappa \subseteq W_\kappa^*$. Since generally $W_\kappa \subseteq W_\kappa^*$ and dom $W_\kappa = \text{dom} W_\kappa$, it follows $W_\kappa = W_\kappa^*$, whence $\kappa = \overline{\kappa}$ a.e. by (m). The converse is obvious.

(o) By (n) $\kappa$ is real. First suppose that $\kappa$ is bounded below with maximal lower bound $a$. Put $\kappa := \kappa - a \geq 0$. Then $\langle h, M_+(\kappa)h \rangle = \int x[h]^2 \, dx \geq 0$, whence $\langle h, M_+(\kappa)h \rangle \geq a \|h\|^2 \forall h \in \text{dom} M_+(\kappa) = \text{dom} M_+(\kappa)$. So $a$ is a lower bound for $M_+(\kappa)$.

Now let $M_+(\kappa)$ be bounded below with maximal lower bound $a$. Put $\kappa := \kappa - a$. Then $\langle h, M_+(\kappa)h \rangle \geq 0 \forall h \in \text{dom} M_+(\kappa)$. If $\kappa = 0$ and $\|h\| = 1$ with $\langle \kappa h_1, M_+(\kappa)h_1 \rangle < \epsilon$. Put $A := \{x \in \mathbb{R} : \kappa(x) \geq 0\}$. Hence $\int 1_A \kappa |h_1|^2 \, dx < \epsilon$. Assume that $B := \mathbb{R} \setminus A$ is not null set. Let $h_2 \in \text{dom} M_+(1_B \kappa) \supseteq \text{dom} M_+(\kappa)$, $h_2 \neq 0$. Then $c := \int 1_B \kappa |h_2|^2 \, dx < 0$. Choose $\epsilon < |c|$. The function $f := 1_A |h_1| + 1_B |h_2|$ satisfies $\|f(x)\| \leq \|\kappa h_1(x)\| + \|\kappa h_2(x)\|$. Since $h_1, h_2 \in H_+$ it follows that $\int f(x)/(1 + x^2)^\gamma$ is integrable, whence there $h \in H_+$ with $\|h\| = f$. Note that $h \in \text{dom} M_+(\kappa)$, since $|\kappa h| = |1_A \kappa h_1| + |1_B \kappa h_2| \in L^2(\mathbb{R})$. It follows the contradiction $\langle h, M_+(\kappa)h \rangle = \int \kappa |h|^2 \, dx \leq c + \epsilon < 0$. Therefore $B$ is a null set, whence $\kappa \geq 0$ a.e. and $a$ is an essential lower bound of $\kappa$. This proves $a = \alpha$.

$p$ follows readily from (o).

In view of (4)(k) we recall that self-adjoint bounded WH, which are not a multiple of $I$, i.e., $\kappa$ real bounded not almost constant, are even absolutely continuous. Indeed, these operators are Hilbert space isomorphic to self-adjoint bounded Toeplitz operators (see (3), or e.g. [1, 9.5(c)], [22, 3.3.2(13)]), which by Rosenblum [9] are absolutely continuous. Actually, as already mentioned, it follows from [9] that for real bounded below not almost constant $\kappa$, for which $(1 + x^2)^{-1} \kappa$ is integrable, the Friedrichs extension $\hat{W}_\kappa$ (15) of $W_\kappa$ is absolutely continuous.

As mentioned section 4 is concerned with the case that $\kappa$ is proper real and semibounded. This is the general case that $W_\kappa$ is densely defined symmetric semibounded (4)(n),(o). The natural self-adjoint extension $\hat{W}_\kappa$ is studied in (14), (15).

If $\kappa$ is proper real and even (i.e. $\kappa(-x) = \kappa(x)$) then $W_\kappa$ is densely defined symmetric and has a self-adjoint extension. This holds true since $L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$, $g \rightarrow g$ is a conjugation, which leaves dom $W_\kappa$ invariant and satisfies $W_\kappa g = W_\kappa g$ (see [4, Theorem X.3]). If $\kappa$ is odd instead of even then in general $W_\kappa$ has no self-adjoint extension. Examples are furnished by real rational symbols as $\kappa(x) = x$. In (10) an explicit description of the deficiency spaces of $W_\kappa$ for real rational $\kappa$ are given yielding further examples of densely defined symmetric WH with self-adjoint extensions.
Concluding this section we deal with the unilateral translation invariance (4)(d) of WH. We are inspired by [6, sec. 2.3]) which treats the bounded case (6). Observe the easily verifiable relation

$$T_b^*P_\kappa^*W_\kappa P_+T_b = M(1_{[-b,\infty[})F \kappa \mathcal{F}^{-1}M(1_{[b,\infty[})$$  

(2.4)

where $T_b, b \in \mathbb{R}$ denotes the unitary one-parameter group of translations $T_b f(x) := f(x - b)$ on $L^2(\mathbb{R})$. It shows again the invariance

$$W_\kappa = S_b^\dagger W_\kappa S_b$$  

(2.5)

under the unilateral translations $S_b, b \geq 0$ (see (4)(d)), since $S_b = P_+ T_b P_+^*$. Recall $S_b = W(e^{ib(\cdot)})$. Moreover it implies that $T_b^* P_\kappa^* W_\kappa P_+ T_b P_+^*$ converges as $b \to \infty$ if $f \in T_a^* P_\kappa^* (\text{dom } W_\kappa)$ for some $a \geq 0$, yielding the limit $F \kappa \mathcal{F}^{-1} f$. In particular $\{ ||W_\kappa S_b g||; b \geq 0 \}$ is bounded for every $g \in \text{dom } W_\kappa$. If $W_\kappa$ is densely defined then also $\{ ||W_\kappa^* S_b g||; b \geq 0 \}$ is bounded as $W_\kappa^* |_{\text{dom } W_\kappa} = W_\kappa$.

(5) Theorem. Let $A$ be a densely defined operator in $L^2(\mathbb{R}^+)$ satisfying

$$A \subset S_b^\dagger A S_b \quad \forall b \geq 0$$  

(2.6)

If $\text{dom } A \subset \text{dom } A^*$ and $\{ ||A S_b g||; b \geq 0 \}$ and $\{ ||A^* S_b g||; b \geq 0 \}$ are bounded for every $g \in \text{dom } A$ then there is a WH $W_\kappa$ extending $A$.

Proof. (i) For $a \geq 0$ put $D_a := T_a^* P_+^*(\text{dom } A)$ and $D := \bigcup_{a \geq 0} D_a$. Then $D$ is a dense translation invariant subspace of $L^2(\mathbb{R})$.

Indeed, let $f \in L^2(\mathbb{R})$ and $\varepsilon > 0$. Since $f_a := 1_{[-a,\infty[} f \to f$ for $a \to \infty$ in the mean, one has $||f - f_a|| \leq \varepsilon/2$ for some $a \geq 0$. As $T_a f_a = P_+^* P_+ T_a f$, there is $g \in \text{dom } A$ with $||P_+ T_a f - g|| \leq \varepsilon/2$. Hence $||f - T_a P_+^* g|| \leq \varepsilon/2$. It follows that $D$ is dense.

Note that $S_b(\text{dom } A) \subset \text{dom } A$ for $b \geq 0$ due to $A \subset S_b^\dagger A S_b$. Then $D_a \subset D_b$ for $a \leq b$. Indeed, put $c := b - a \geq 0$. Then check $T_c^* P_+^* = T_b^* P_+^* T_a$, whence $D_a = T_a^* P_+^* S_c, (\text{dom } A) \subset T_b^* P_+^*(\text{dom } A) = D_b$. It follows that $D$ is a subspace of $L^2(\mathbb{R})$.

For the translation invariance of $D$ it suffices to show $T_c^* P_+^* g \in D$ for $c \in \mathbb{R}, g \in \text{dom } A$. If $c \geq 0$ this is obvious. Let $c < 0$. Then $T_c^* P_+^* g = T_{-c}^* P_+^* g = P_+^* P_+ T_{-c} P_+^* g = P_+^* S_{-c} g \in D$ as $S_{-c} g \in \text{dom } A$.

(ii) Let $f \in D$. Then $f = T_0^* P_+^* g$ for some $a \geq 0$, $g \in \text{dom } A$, whence $f_b := T_b^* P_+^* A P_+ T_b f = T_b^* P_+^* S_{-b} g$ is well-defined for all $b \geq a$. We are going to show that $\lim_{b \to \infty} f_b$ exists.

Indeed, let $c \geq b \geq a$. Then $||f_c - f_b||^2 = ||f_c||^2 + ||f_b||^2 - (f_c, f_b) = ||f_c - f_b||^2 = ||A P_+ T_c f||^2 = ||S_{-c} A S_{-b} P_+ T_b f||^2 = ||A P_+ T_b f||^2$ by (2.6) since $P_+ T_b f \in \text{dom } A$. Hence $||f_c - f_b||^2 = ||A P_+ T_c f||^2$ is increasing. Being bounded by the assumption it follows $||f_c - f_b||^2 \to 0$ for $c, \infty \to \infty$ so that $\lim_{b \to \infty} f_b$ exists.

(iii) Thus $C_1 f := \lim_{b \to \infty} f_b$ defines an operator on $D$. It is translation invariant since $T_c^* C_1 T_c f = \lim_{b \to \infty} T_c^* T_b^* P_+^* A P_+ T_b f = \lim_{b \to \infty} T_b^* P_+^* A P_+ T_b f = \lim_{b \to \infty} T_b^* P_+^* A P_+ T_b f = C_1 f$.

(iv) Also $C_2 f := \lim_{b \to \infty} T_b^* P_+^* A P_+ T_b f$ exists for $f \in D$ thus defining an operator $C_2$ on $D$. This result follows replacing $A$ in (ii) by $A^\# := A^\dagger |_{\text{dom } A}$. It remains to verify $A^\# \subset S_b^\dagger A^\# S_b, b \geq 0$. Indeed, for $g, g' \in \text{dom } A$ one has $\langle g, S_b^\dagger A^\# S_b g' \rangle = \langle S_b g, A^\# g' \rangle = \langle A S_b g, S_b g' \rangle$ since $S_b g \in \text{dom } A \subset \text{dom } A^\dagger = \text{dom } A^\#$. Hence $\langle g, S_b^\dagger A^\# S_b g' \rangle = \langle S_b A S_b g, g' \rangle = \langle A g, g' \rangle = \langle g, A^\# g' \rangle$ by (2.6) and $g, g' \in \text{dom } A$. This implies $S_b^\dagger A^\# S_b g = A^\# g'$, whence the claim.

(v) Obviously $C_2 \subset C_1^\dagger$. Hence $C_1^\dagger$ is densely defined and the closure $C := \overline{C_1}$ exists. Clearly translation invariance $C = T_b^* C T_b$ holds. Equivalently $\mathcal{F}^{-1} C \mathcal{F}$ commutes with $M(e^{ib(\cdot)}) = \mathcal{F}^{-1} T_b \mathcal{F}$ for
all \( b \in \mathbb{R} \). Thus \( \mathcal{F}^{-1}C \mathcal{F} = M(\kappa) \) for some measurable function \( \kappa \). Hence \( P_+CP_+^* = W_\kappa \). Finally, for \( g \in \text{dom } A \) one has \( P_+^*g \in D \) and \( W_\kappa g = P_+C_1P_+^*g = \lim_{b \to \infty} P_+T_b^*P_+^*AP_+T_bP_+^*g = \lim_{b \to \infty} S_b h g = \lim_{b \to \infty} A g = A g. \) □

(6) Corollary. Let \( A \) be a bounded operator on \( L^2(\mathbb{R}^+) \). Then \( A \) is a WH if and only if

\[ A = S_b^* A S_b \quad \forall \; b \geq 0 \]

Proof. It remains to observe that \( A \subseteq W_\kappa \) by (5) implies \( A = W_\kappa \). □

For (6) see also [6, (2.10)], where the existence of \( \lim_{b \to \infty} T_b^* P_+^* A P_+ T_b f \) (see (ii) of the proof of (5)) is not proven.

3 Rational Symbols

WH for rational symbols \( \kappa = \frac{P}{Q} |_{\mathbb{R}} \) with polynomials \( P \neq 0, Q \neq 0 \) permit some more general analysis. According to (4)(b) they are densely defined. In (8) we show that they are closed and we determine their domains, ranges, and kernels and deficiency spaces, which are finite dimensional, and their spectral and Fredholm points. In particular, in the symmetric case, i.e., for a real rational symbol the deficiency spaces and indices are explicitly available (10).

Mostly we will omit \( |_{\mathbb{R}} \) indicating the restriction on \( \mathbb{R} \). A polynomial with a negative degree is the null function.

(7) Lemma. Let \( P \neq 0 \) and \( Q \neq 0 \) be polynomials.

(a) Let \( P \) and \( Q \) have no common zeros. Then \( \frac{P}{Q} \in H_+ (\in H_-) \) if and only if \( \deg P < \deg Q \) and all zeros of \( Q \) are in the lower (upper) half-plane.

(b) Let \( h \in H_+ \setminus \{0\} \) such that \( \frac{P}{Q} h \in H_- \). Then there is a polynomial \( R \) with \( \deg R < \min\{\deg P, \deg Q\} \) such that all zeros of \( P \) in the closed upper half-plane as well as all zeros of \( Q \) in the closed lower half-plane are zeros of \( R \) and such that \( h \in L^2(\mathbb{R}) \). Conversely, it is obvious that \( h_+: \frac{P}{Q} \) and \( h_- : \frac{P}{Q} \) satisfy \( h_+ \in H_+ \) and \( \frac{P}{Q} h_+ = h_- \).

(c) Let \( Q \) have no zeros in the upper half-plane. If \( h \in H_+ \) and \( \frac{P}{Q} h \in L^2(\mathbb{R}) \), then \( \frac{P}{Q} h \in H_+ \).

(d) Let \( P \) and \( Q \) have no common zeros. Suppose that \( h \in H_+ \) and \( \frac{P}{Q} h \in H_+ \). Then \( \frac{Q}{P} h \in H_+ \).

Proof. (i) As to (a) put \( h := \frac{P}{Q} |_{\mathbb{R}} \). Obviously \( h \in L^2(\mathbb{R}) \) if and only if \( \deg P < \deg Q \) and \( Q \) has no real zeros. Moreover, if all zeros of \( Q \) are in the lower half-plane, then \( \psi := \frac{Q}{P} \) is holomorphic in the upper half-plane with bounded \( \|\psi\|_{2; y > 0} \), whence \( h \in H_+ \). Conversely, let \( \phi \) be the holomorphic function on the upper half-plane associated with \( h \in H_+ \). Then \( \phi \) converges to \( h \) non-tangentially a.e., and \( \psi \) is meromorphic without real poles, whence \( \psi(z) \to h(x) \) for \( z \to x \in \mathbb{R} \). Hence \( \phi \) and \( \psi \) coincide on the upper half-plane by [15, IV 2.5] so that \( \psi \) has no poles there.

(ii) As to (b) we prove \( h = R/P \) for some polynomial \( R \) supposing that \( Q \) has no real zeros and \( \deg Q \geq \deg P \).

Let \( \phi \) and \( \psi \) be the holomorphic functions on the upper half-plane and lower half-plane associated with \( h \) and \( \frac{P}{Q} h \), respectively. There is \( \delta > 0 \) such that \( B := \frac{P}{Q} \) is holomorphic and bounded in the strip.
\( z : -\delta < \text{Im} z < \delta \), and hence \( B \phi \) is holomorphic on \( \{ 0 < \text{Im} z < \delta \} \). Since \( B \) is bounded, one easily infers \( \int_J |(B \phi)(x + iy) - \psi(x - iy)| \, dx \to 0, 0 < y \to 0 \) for any bounded interval \( J \). Then by [21, Theorem II] there is a holomorphic function \( \chi \) on \( \{ z : \text{Im} z < \delta \} \) extending \( B \phi \) and \( \psi \). So \( Q \chi \) is still holomorphic on \( \{ z : \text{Im} z < \delta \} \) coinciding with \( P \phi \) on \( \{ z : 0 < \text{Im} z < \delta \} \) and with \( Q \psi \) on the lower half-plane. Hence there is an entire function \( R \) extending \( P \phi \) on the upper and \( Q \psi \) on the lower half-plane. Introduce \( S \) being equal to \( P \) on the upper half-plane and equal to \( Q \) on the lower half-plane. Analogously define \( \Sigma \) with respect to \( \phi \) and \( \psi \).

Fix \( z \in \mathbb{C}, |z| > 1 \). We use the representation \( R(z) = \frac{1}{\pi} \int_D R(z + w) \, d^2w \) were \( D \) denotes the disc with center 0 and radius 1. Then

\[
|R(z)|^2 \leq \pi^{-2} \int_D |S(z + w)|^2 \, d^2w \int_1^{\infty} \int_{-\infty}^{\infty} |\Sigma(x + u + i(y + v))|^2 \, dudv
\]

The first integral is easily estimated \( \leq \text{constant} \cdot |z|^{2n} \) with \( n := \deg Q \). The double integral is bounded independently of \( z \), since \( \int |\Sigma(u + iv)|^2 \, du \leq K \) for all \( v \neq 0 \) with some constant \( K < \infty \). Therefore \( R \) is a polynomial and \( \phi = \frac{R}{P} \).

(iii) Next we show \( P h \in H_+ \) for \( h \in H_+ \) if \( P h \in L^2(\mathbb{R}) \).

Let \( a \in \mathbb{C} \) be a zero of \( P \) and write \( P = (x-a)P' \). Note that \( (x-a)h \in L^2(\mathbb{R}) \) since \( |x-a| \leq c |P(x)| \), \( x \in \mathbb{R} \setminus J \) for some bounded interval \( J \) and constant \( c \). Hence it suffices to prove \( (x-a)h \in H_+ \) and proceed with \( P' \) in place of \( P \).

Let \( k \in H_- \) and \( \langle \frac{k}{x-1}, (x + i)h \rangle = 0 \). To conclude the proof obviously it suffices to show that \( \langle \frac{k}{x-1}, k \in H_- \rangle \) is dense in \( H_- \). So let \( k_0 \in H_- \) satisfy \( \langle k_0, \frac{k}{x-1} \rangle = 0 \) for all \( k \in H_- \). Then \( \langle \frac{k_0}{x-1}, k \rangle = 0 \) implying \( k_0 \frac{k}{x-1} \in H_+ \) or equivalently \( k_0 \frac{P}{P'} h \in H_- \), whence \( k_0 = 0 \) by (ii).

(iv) Let \( Q \) have no zeros in the upper half-plane. The claim is \( \frac{1}{Q} h \in H_+ \) for \( h \in H_+ \) if \( \frac{1}{Q} h \in L^2(\mathbb{R}) \).

Write \( Q = q Q_\epsilon \), where the zeros of \( q \) are exactly the real zeros of \( Q \). Let \( q_\epsilon(z) := q(z + i \epsilon) \) for \( \epsilon > 0 \) and put \( Q_\epsilon := q_\epsilon Q_\epsilon \). Note \( |q(x)/q_\epsilon(x)| < 1 \), \( x \in \mathbb{R} \). Hence \( h/Q_\epsilon \in L^2(\mathbb{R}) \). Since \( 1/Q_\epsilon \) is bounded on the upper half-plane, \( h/Q_\epsilon \in H_+ \). Moreover, \( h/Q_\epsilon \to h/Q \) for \( \epsilon \to 0 \) pointwise and in the mean, whence \( h/Q \in H_+ \).

(v) It follows the proof of (c). Without restriction let \( P \) and \( Q \) be without common zeros. Then \( \frac{1}{Q} h \in L^2(\mathbb{R}) \) implies \( \frac{1}{Q} h \in L^2(\mathbb{R}) \). Indeed, let \( K \) be a compact neighborhood of the real zeros of \( Q \) containing no real zero of \( P \). Then \( h/Q \) is bounded by \( C |P h/Q| \) on \( K \) and by \( c|h| \) on \( \mathbb{R} \setminus K \) for some finite constants \( C, c \), whence \( \frac{1}{Q} h \in L^2(\mathbb{R}) \). — Now \( \frac{1}{Q} h \in H_+ \) by (iv) and hence \( \frac{1}{Q} h \in H_+ \) by (iii).

(vi) We proceed with the proof of (b). Without restriction let \( P \) and \( Q \) be without common zeros. Write \( Q = q Q_0 \), where the zeros of \( q \) are exactly the real zeros of \( Q \). Then \( \frac{1}{Q} h \in L^2(\mathbb{R}) \) implies \( \frac{1}{Q} h \in L^2(\mathbb{R}) \), cf. (v). Hence \( h' := h/q \in H_+ \) by (iv). Next let \( P = p P_0 \) with \( \deg P_0 = \deg Q_0 \) if \( \deg P > \deg Q_0 \) and \( p = 1 \) otherwise. Then \( h'' := p h' \in L^2(\mathbb{R}) \), since \( h'' = \frac{Q_0}{P_0} \frac{P_0}{P} h \), where in the case \( p \neq 1 \) the factor \( \frac{Q_0}{P_0} \) is bounded outside a bounded interval. By (iii) this implies \( h'' \in H_+ \). By assumption \( \frac{P}{Q} h'' \in H_- \). Therefore \( h'' = R_0/P_0 \) by (ii) for some polynomial \( R_0 \). It follows \( h = R/P \in H_+ \) for \( R := q R_0 \) and hence \( R/Q \in H_- \). The proof is accomplished applying (a) proved in (i).

(vii) As to the proof of (d) assume first that \( Q \) has no real zeros. Let \( \phi \) and \( \psi \) be the holomorphic functions on the upper half-plane related to \( h \) and \( \frac{1}{Q} h \), respectively. Then \( \frac{P}{Q} \phi \) is meromorphic on the upper half-plane and converges non-tangentially to \( \frac{1}{Q} h \) a.e. Since \( \psi \) does the same, according to [15, IV 2.5], \( \psi = \frac{P}{Q} \phi \) holds. Hence \( \phi/Q \) is holomorphic on the upper half-plane with \( \langle \phi/Q, \psi \rangle \to h/Q \) for \( 0 < y \to 0 \) a.e. Let \( 0 < \delta < c \) such that \( C := [-c, c] \times i[\delta, c] \) is a neighborhood of the zeros of
$Q$ in the upper half-plane. Then $|1/Q|$ is bounded by some constant $L$ on $\{ z : \text{Im } z \geq 0 \} \setminus C$, and $|\phi/Q|$ is bounded on $C$ by some $M$. Recall that $\| \phi_y \|_2$ is bounded for $y > 0$ by some $K$. Then $\int |(\phi/Q)(x + iy)|^2 \, dx \leq L^2 \int |\phi(x + iy)|^2 \, dx + \int_0^\infty M^2 \, dx \leq L^2 K + 2cM^2$ for all $y > 0$. Finally, for $0 < y < \delta$, $\int |(\phi/Q)(x + iy) - (h/Q)(x)|^2 \, dx \leq L^2 \int |\psi(x + iy) - h)|^2 \, dx \to 0$ for $y \to 0$, whence $h/Q \in H_+$. Let $h \in L^2(\mathbb{R})$, cf. (v). Write $Q = qQ_0$, where the zeros of $q$ are exactly the real zeros of $Q$. Let $q_\epsilon(z) := q(z + i \epsilon)$ for $\epsilon > 0$ and put $Q_\epsilon := q_\epsilon Q_0$. Note $|q(z)/q_\epsilon(z)| < 1$ on the upper half-plane. Therefore $\frac{\partial}{\partial \epsilon} h = \frac{q_\epsilon}{q} h \in H_+$, where $Q_\epsilon$ has no real zeros. Moreover, for $\epsilon > 0$ small enough, $P$ and $Q_\epsilon$ have no common zeros. Hence the foregoing result applies so that $h/Q_\epsilon \in H_+$. Now $h/Q_\epsilon = \frac{\epsilon}{q} h/Q \to h/Q$ for $\epsilon \to 0$ in the mean implying $h/Q \in H_+$. \hfill $\square$

Recall that a densely defined closed operator between Banach spaces with finite dimensional kernel and cokernel is called a Fredholm operator if its range is closed (cf. [23]).

(8) Theorem. Let $\kappa = \frac{P}{Q}$ be a rational function, where the polynomials $P$ and $Q$ have no common zeros. Then $M_+(\kappa)$ is densely defined and closed and

\begin{enumerate}[(a)]
  \item $\text{dom } M_+(\kappa) = \frac{q}{(x+i)}H_+$ and $\text{ran } M_+(\kappa) = P_{H_+} \left( \frac{P}{Q}, \frac{Q}{Q(x+i)}; H_+ \right)$
  \item $\text{ker } M_+(\kappa) = \left\{ \frac{Q}{P} r : r \text{ polynomial with } \deg r < \min \{ \deg P_-, \deg Q_-, \deg P_+ \} \right\}$
  \item $\left( \text{ran } M_+(\kappa) \right)^\perp = \left\{ \frac{Q}{P} r : r \text{ polynomial with } \deg r < \deg P_+ - \deg Q_+ \right\}$
  \item the following statements are equivalent:
    \begin{enumerate}[(1)]
      \item $M_+(\kappa)$ is a Fredholm Operator
      \item $\text{ran } M_+(\kappa)$ closed
      \item $\deg Q \leq \deg P$ and $P$ without real zeros
      \item $0 \notin \kappa(\mathbb{R})$
    \end{enumerate}
\end{enumerate}

Here the zeros of the polynomial $q$ are the real zeros of $Q$, $\zeta := \max \{ \deg q, \deg P - \deg Q + \deg q \}$. Moreover $P = P_<P_>$, where the zeros of $P_<$ and $P_>$ are exactly the zeros of $P$ in the lower half-plane and in the closed upper half-plane, respectively. $P$ denotes the polynomial whose coefficients are the complex conjugates of $P$. Analogous notations concern $Q$.

Proof. For the closeness of $M_+(\kappa)$ write $\frac{P}{Q}$ in the form $\frac{P}{Q} = \frac{P_0}{Q_0} + \frac{P}{q}$ with polynomials $P_0, Q_0, p, q$ such that $Q = Q_0q$, $Q_0$ has no real zeros, $q$ has only real zeros, $\deg P_0 < \deg Q_0$, and $p$ and $q$ have no common zeros and satisfy $\zeta = \max \{ \deg p, \deg q \}$.

Since $\kappa_0 := \frac{P_0}{Q_0} |_{\mathbb{R}}$ is bounded, $M_+(\kappa_0)$ is bounded. It follows $M_+(\kappa) = M_+(\kappa_0) + M_+(\frac{Q}{q})$ and it remains to show that $M_+(\frac{Q}{q})$ is closed. Let $h_n \in \text{dom } M_+(\frac{Q}{q})$ such that $(h_n)$ converges to some $h \in H_+$ and $(M_+(\frac{Q}{q})h_n)$ converges to some $k \in H_+$. By (7)(c), $\frac{Q}{P} h_n \in H_+$. Hence one has $h_n \to h$ and $\frac{Q}{P} h_n \to k$ in $L^2(\mathbb{R})$. Since $M_+(\frac{Q}{q})$ is closed, $h \in \text{dom } M_+(\kappa)$ and $k = M_+(\frac{Q}{q})h$ follows.

(a) $\text{dom } M_+(\kappa)$ is dense by (4)(b). Arguing as above it remains to show $\text{dom } M_+(\frac{Q}{q}) = \frac{Q}{(x+i)}H_+$. Let $h \in H_+$. Then, by (7)(c), $\frac{Q}{(x+i)}h \in H_+$ and $\frac{Q}{q} (x+i)h \in H_+$ implying $\frac{Q}{(x+i)}H_+ \subset \text{dom } M_+(\frac{Q}{q})$. For the converse inclusion argue $g \in \text{dom } M_+(\frac{Q}{q}) \Rightarrow g \in H_+$, $\frac{Q}{q} g \in L^2 \Rightarrow h := \frac{Q}{q} g \in H_+$ by (7)(c). Hence
\[ g = \frac{q}{p}h, \text{ whence } \frac{1}{p}h \in H_+ \text{ by (7)(d). Since } \zeta = \max\{\deg p, \deg q\} \text{ one infers } k = \frac{(x+i)^{\zeta}}{p^2}h \in L^2, \text{ whence } k \in H_+ \text{ applying (7)(c) to } \frac{1}{p}h \in H_+. \text{ This shows } g = \frac{q}{(x+i)^{\zeta}}k \in \frac{q}{h}H_+. \text{ } \]

(b) Check the implications: \( h_0 \in \ker M_+(\kappa) \iff h_0 \in \text{dom} M_+(\kappa), M_+(\kappa) h_0 = 0 \iff h_0 \in H_+, \kappa h_0 \in L^2(\mathbb{R}), \text{ and } \kappa h_0 \in H_- \). According to (7)(b) this means \( h_0 = R/P, \) where \( R \) is a polynomial with \( \deg R < \min\{\deg P, \deg Q\} \) and \( R = P_\varnothing Q_\varnothing r, \) whence the claim.

(c) Using (a) one has \( h_0 \in (\text{ran } M_+(\kappa))^\perp \iff 0 = (h_0, \frac{P}{Q} Q_\varnothing(x+i)^{\zeta})h = (\frac{P}{Q} Q_\varnothing(x+i)^{\zeta})h_0, h) \forall h \in H_+ \iff \frac{P}{Q} Q_\varnothing(x+i)^{\zeta}h_0 \in H_- \). By (7)(b) this means \( h_0 = R/P, \) where \( R \) is a polynomial with \( \deg R < \min\{\deg P, \deg Q + \zeta\} = \deg P \) and \( R = P_\varnothing Q_\varnothing r = P_\varnothing Q_\varnothing r, \) whence the claim.

(d) By the foregoing results (1)\( \iff (2) \) holds. Moreover (3)\( \iff (4) \) is quite obvious. So we turn to (2)\( \iff (3). \)

Show first (3)\( \Rightarrow (2). \) Put \( R := (\frac{P}{Q} Q_\varnothing(x+i)^{\zeta})/R_\varnothing. \) Since by the assumptions nominator and denominator of \( R \) have equal degree and have no real zeros, \( R \) and \( R/\varnothing \) are bounded on \( \mathbb{R}. \) Hence \( M(R) \) is a homeomorphism on \( L^2(\mathbb{R}), \) whence \( RH_+ \) is closed. Since \( (Q_\varnothing)_+ = Q_\varnothing \) it follows by (7)(c) that \( RH_+ \subset H_+. \) Hence \( RH_+ = \text{ran } M_+(\kappa') \) for \( \kappa' = \frac{P}{Q'} \) with \( Q' := \hat{Q} Q_\varnothing q, \) whence \( \dim(RH_+)^\perp < \infty \) by (c). So it suffices to show that \( RH_+ \subset \text{ran } M_+(\kappa). \) By (7)(c) \( \frac{Q_\varnothing}{Q_\varnothing} H_+ \subset H_+. \) Hence, by (a), \( \text{ran } M_+(\kappa) \supset P_{R_\varnothing} H_+ = P_{R_\varnothing}(RH_+) = RH_+. \)

Now turn to (2)\( \Rightarrow (3). \) Consider first the case \( \deg P < \deg Q. \) Put \( R := Q_\varnothing Q_\varnothing(x+i)^{\zeta}. \) Note \( \deg R = \deg Q \) and \( \text{ran } M_+(\kappa) = P_{R_\varnothing} H_+ \) by (a). By (7)(c) and since \( M(R_{\varnothing}) \) is unitary on \( L^2(\mathbb{R}), \) \( \frac{R_{\varnothing}}{R_\varnothing} H_+ \) is a closed subspace of \( H_+. \) By (c) one has \( \left( \frac{R_{\varnothing}}{R_\varnothing} H_+ \right)^\perp = \left\{ \frac{r}{R_\varnothing} : \deg r < \deg R_\varnothing \right\} \) and hence

\[ \frac{P}{R} H_+ = \left\{ \frac{r}{R_\varnothing} : \deg r < \deg R_\varnothing \right\} + \frac{P}{R_\varnothing} H_+ \]

Applying (c) for \( \kappa = \frac{P}{R_\varnothing} \) and \( \kappa = \frac{P_\varnothing}{P_\varnothing}, \) check \( \left( \frac{P}{R_\varnothing} H_+ \right)^\perp = \left\{ \frac{r}{P_\varnothing} : \deg r < \deg P_\varnothing \right\}. \)

Hence \( \frac{P}{R_\varnothing} R_{\varnothing} H_+ \) is dense in \( \frac{P}{P_\varnothing} H_+, \) which is closed in \( H_+. \) Note that \( P_{R_\varnothing} H_+ = \left\{ \frac{r}{P_\varnothing} : \deg r < \deg R_\varnothing \right\} \subset V := \left\{ \frac{B}{P_\varnothing} : \deg B < \deg R_\varnothing \right\}. \)

Now assume that \( P_{R_\varnothing}(\frac{P}{P_\varnothing} H_+) \) is closed. Then the above considerations imply that \( \frac{P}{P_\varnothing} H_+ \subset V + \frac{P}{R_\varnothing} R_{\varnothing} H_+. \) Thus given \( h_0 \in H_+, \) there is \( h_0 \in H_+ \) and some polynomial \( B \) with \( \deg B < \deg R \) such that \( h = \frac{B}{P_\varnothing} R_{\varnothing} h_0 + \frac{B}{P_\varnothing}. \) Hence \( |h| \geq \left| \frac{B}{P_\varnothing} R_{\varnothing} h_0 \right| - \frac{B}{P_\varnothing} \) with \( m := \deg R - \deg P_\varnothing - \deg P_\varnothing \geq 1 \) and \( l := \deg B - \deg P \leq m - 1. \) Choose \( h_0 \in H_+ \) satisfying \( |h_0| = (1 + |x|)^{-3/4}. \) \( h_0 \) exists by (2.1) since square-integrable and \( \ln(1 + |x|) \leq \sqrt{2|x|}. \) Because of \( m - \frac{3}{4} - l > 0 \) the right side tends to \( \infty \) like \( |x|^{-m-3/4} \) for \( |x| \to \infty \) contradicting \( h \in L^2(\mathbb{R}). \)

To complete the proof of the implication (2)\( \Rightarrow (3) \) it remains to treat the case that \( \deg P \geq \deg Q \) and \( P \) has a real zero. Proceeding as in the foregoing case, here one has \( \deg R = \deg P. \) Hence, assuming that \( P_{R_\varnothing}(\frac{P}{P_\varnothing} H_+) \) is closed, one has \( m \geq 1 \) and \( l \leq -1 \) and the same contradiction follows. \( \blacksquare \)

From (8) one immediately obtains

\[ \text{(9) Corollary. Let } \lambda \in \mathbb{C} \text{ and put } P^\lambda := P + \lambda Q. \text{ Then referring to } M_+(\kappa), \lambda \text{ is} \]

- a Fredholm point (i.e. \( M_+(\kappa) - \lambda I \) is a Fredholm operator) iff \( \lambda \notin \mathbb{C}(\mathbb{R}); \) if \( \lambda \) is a Fredholm point, then \( \dim \ker(M_+(\kappa) - \lambda I) = \max\{0, \deg Q_\varnothing - \deg P_\varnothing\} \), \( \dim \text{ran}(M_+(\kappa) - \lambda I)^\perp = \max\{0, \deg P_\varnothing - \deg Q_\varnothing\}. \]
\[
\deg Q_\rangle, \text{ and } \text{ind}(M_+(\kappa) - \lambda I) = \deg Q_\rangle - \deg P_\lambda^\kappa
\]

- a regular value (i.e. \(M_+(\kappa) - \lambda I\) is continuously invertible) iff \(\lambda \not\in \kappa(\mathbb{R})\) and \(\deg Q_\rangle \leq \deg P_\lambda^\kappa\)

- in the resolvent set iff \(\lambda \not\in \kappa(\mathbb{R})\) and \(\deg Q_\rangle = \deg P_\lambda^\kappa\)

- a spectral value iff \(\lambda \in \kappa(\mathbb{R})\) or \(\deg Q_\rangle \neq \deg P_\lambda^\kappa\)

- in the point spectrum iff \(\deg Q_\leq < \deg P_\lambda^\kappa\) and \(\deg P_\lambda^\kappa < \deg Q_\rangle\)

- in the continuous spectrum (i.e. \(M_+(\kappa) - \lambda I\) is injective with dense not closed range) iff \(\lambda \in \kappa(\mathbb{R})\), \(\deg P_\lambda^\kappa \leq \deg Q_\rangle\) and either \(\deg P_\lambda^\kappa \leq \deg Q_\leq\) or \(\deg Q_\leq \leq \deg P_\lambda^\kappa\)

- in the residual spectrum (i.e. \(M_+(\kappa) - \lambda I\) is injective with not dense range) iff \(\deg Q_\rangle < \deg P_\lambda^\kappa\)

The characterization of the Fredholm points of \(M_+(\kappa)\) and the fact that at a Fredholm point either the kernel or the deficiency space is trivial, are familiar from Krein’s theory [3] for the case of integrable kernel. \(\lambda \mapsto \deg P_\lambda^\kappa\) is locally non-decreasing due to the continuity of the roots of a polynomial on its coefficients [24]. On \(\mathbb{C} \setminus \kappa(\mathbb{R})\) it is even locally constant, since there \(P_\lambda^\kappa = P_\lambda\). Hence, besides \(\text{ind}(M_+(\kappa) - \lambda I)\), also \(\text{dim ker}(M_+(\kappa) - \lambda I)\) and \(\text{dim ran}(M_+(\kappa) - \lambda I)^\perp\) are constant on the components of \(\mathbb{C} \setminus \kappa(\mathbb{R})\).

(10) Theorem. Let \(\kappa\) be a real rational function. Let \(p, q\) be real polynomials without common zeros such that \(\kappa = \frac{p}{q}\) and put

\[
p - i q = Q_+ Q_-
\]

where the zeros of the polynomials \(Q_+\) and \(Q_-\) are all in the upper and lower half-plane, respectively. Then \(M_+(\kappa)\) is closed symmetric with \(\text{dom } M_+(\kappa) = \frac{q}{(x+iy)^{p+1}} H_+\), where the zeros of \(q\) are the real zeros of \(q\) and \(\varsigma := \max\{\deg q', \deg p - \deg q + \deg q'\}\). The deficiency indices are

\[
n_\pm := \dim \text{ran } (M_+(\kappa) \mp i I)^\perp = \deg Q_\pm - \deg q_\leq
\]

and the deficiency spaces are

\[
\text{ran } (M_+(\kappa) - i I)^\perp = \left\{ \frac{q_\leq}{Q_+} r : r \text{ polynomial with } \deg r < \deg Q_+ - \deg q_\leq \right\}
\]

\[
\text{ran } (M_+(\kappa) + i I)^\perp = \left\{ \frac{q_\leq}{Q_-} r : r \text{ polynomial with } \deg r < \deg Q_- - \deg q_\leq \right\}
\]

Proof. For \(\text{dom } M_+(\kappa)\) see (8)(a). Adopting the notation of (8) one has \(M_+(\kappa) - i I = M_+(\frac{p}{q})\) with \(P := p - i q\) and \(Q = q\). Obviously \(P\) and \(Q\) have no common zeros. In view of (8)(c) note \(\tilde{Q}_\leq = q_\leq\), \(\tilde{P}_\leq = Q_+\), \(\deg P_\leq = \deg Q_+\), \(\deg Q_\leq = \deg q_\leq\). Hence the formula for \(\text{ran } (M_+(\kappa) - i I)^\perp\) holds, which implies \(n_\pm = \max\{0, \deg Q_+ - \deg q_\leq\}\). It remains to show \(\deg q_\leq \leq \deg Q_+.\) The assertions about the deficiency space for \(- i\) follow similarly.

Let \(m_t\) denote the number of zeros in the upper half-plane of \(p - i t q\) for \(t > 0\). \(p - i t q\) has no real zeros. So by the continuity of the roots of a polynomial, \(m_t\) is locally constant and hence constant \(= \deg Q_+.\) Obviously \(m_t\) is also the number of zeros in the upper half-plane of \(\frac{1}{t} p - i q\). Then by the same
continuity the zeros of \( q \) in the upper half-plane stay there for all \( t \) large enough, whence \( m_t \geq \deg q_\cdot \). This yields the result.

An interesting property of polynomials follows immediately from the foregoing considerations. Let \( R \) be a polynomial without real zeros, and let \( n_\cdot \) and \( n_\cdot \) denote the numbers of its zeros in the upper and lower half-plane, respectively. Write \( R = p + i q \) with real polynomials \( p \) and \( q \), and let \( n_p \) and \( n_q \) denote half the numbers of the non-real zeros of \( p \) and \( q \), respectively. Then \( \max\{n_p, n_q\} \leq \min\{n_\cdot, n_\cdot\} \). To illustrate this consider the case that all zeros of \( R \) lie in one half-plane. Then all zeros of \( p \) and \( q \) are real. However the converse does not hold as for instance \( R = z^2 + i \) shows.

\( W_\cdot \) for \( \kappa \) in (10) has unequal deficiency indices and hence no self-adjoint extension if \( \max\{\deg p, \deg q\} \) is odd. The deficiency indices \( (n_\cdot, n_\cdot) \) of \( W(x^l) \), \( l \in \mathbb{Z} \), are \( (\frac{\{0\}}{2}, \frac{\{0\}}{2}) \) if \( l \) is even and \( (\frac{\{0\}}{2}, \frac{\{0\}}{2}) \) otherwise. Other interesting examples are \( W(\zeta^2 + i) \) and \( W(\zeta^2 - i) \) with deficiency indices \( (1, 1) \) and \( (2, 0) \), respectively. Compare the later with \( W(\zeta^2 - i) \) having deficiency indices \( (1, 1) \).

This section is concluded by a much needed

(11) Example. \( W_{|\cdot|} \) is essentially self-adjoint but not closed. (The same holds true for \( W_{1/|\cdot|} \) ) So there are WH (also essentially self-adjoint semibounded ones), which are not closed.

As to the proof, by (10)(a), \( \text{dom} M_+(|\cdot|) = \frac{1}{x+i}H_+ \) and hence \( R_\pm := \text{ran}(M_+(|\cdot|) \mp i I) = \frac{|x| \mp i}{x+i}H_\cdot \). It suffices to show that \( R_\cdot \) is dense and \( \not= H_\cdot \).

The latter is easily inferred. Assume the contrary. Then there is \( h \in H_\cdot \) satisfying \( \frac{|x| \mp i}{x+i} h = \frac{1}{x+i} \). Hence \( h = \frac{1}{|x|-i} \in H_\cdot \) and \( h = \bar{h} \in H_\cdot \), whence the contradiction \( h = 0 \).

As to the former claim, let \( h_\pm \in R_\pm \). Then \( \langle h_\pm, \frac{|x| \mp i}{x+i} h \rangle = 0 \) for all \( h \in H_\cdot \), whence \( k_\pm := \frac{|x| \mp i}{x+i} h_\pm \in H_\cdot \). Let \( \phi_{\pm} \) be holomorphic on the upper half-plane with \( \phi_{\pm,y} \to h_{\pm} \) as \( y \to 0 \) pointwise a.e. and in the mean (cf. (2)). Similarly let \( \psi_{\pm} \) be holomorphic in the lower half-plane with \( \psi_{\pm,y} \to k_{\pm} \) as \( y \to 0 \). Consider \( \phi := (z+i)\varphi_\cdot \phi_+ \) and \( \psi := (z-i)\varphi_\cdot \psi_\cdot \) holomorphic on the upper and lower half-plane, respectively. Check that \( \phi_\cdot \) for \( 0 < y \to 0 \) and \( \psi_\cdot \) for \( 0 > y \to 0 \) converge pointwise a.e. and in \( L_\cdot_{\text{loc}} \), to \( f := (x+i)h_\cdot h_\cdot \). By [21, Theorem II] there is an entire function \( \chi \) extending \( \phi \) and \( \psi \). Let \( \chi_{\pm} \) be equal to \( \phi_{\pm} \) and \( \psi_{\pm} \) on the upper and lower half-plane, respectively. Let \( z = x + iy, |z| > 1 \). Then \( \pi |\chi(x)| \leq \int_{|w| \leq 1} \frac{|x(z + w)|^2}{2} \leq (|z| + 2) \left( \int_{|w| \leq 1} |x(z + w)|^2 \right)^{1/2} \left( \int_{|w| \leq 1} |x(z - w)|^2 \right)^{1/2} \). Now \( \int_{|w| \leq 1} \frac{|x(z + w)|^2}{2} \leq \int_{-\infty}^1 \frac{|x(z + (w + i(y + v))|^2}{2} \frac{d u}{d v} \leq \int_{-\infty}^1 C \frac{d u}{d v} = 2C \) for some finite constant \( C \). Hence \( |\chi(x)| \leq C |z| \) \( \forall |z| > 1 \). So \( \chi \) is a polynomial \( a + bz \). For \( y > 0, x \mapsto \phi_{\cdot,y}(x) \varphi_\cdot(x) = \frac{a + bz}{x+i} \) is integrable on \( \mathbb{R} \). This implies \( a = 0 \) and hence \( \chi = 0 \). Therefore either \( h_+ = 0 \) or \( h_- = 0 \). This means that one and hence both deficiency spaces are \( \{0\} \). So \( R_\cdot \) is dense.

4 Semibounded Wiener-Hopf operators

In (14), (15) a semibounded densely defined WH \( W_\kappa \) is expressed in a natural way by the product of a closable operator and its adjoint. Replacing the operator by its closure one obtains a self-adjoint extension \( W_\kappa \) of \( W_\kappa \), which is semibounded by the same bound. The bound is not an eigenvalue of the extension. \( W_\kappa \) is shown to be the Friedrichs extension of \( W_\kappa \).
(12) Lemma. Let $\gamma : \mathbb{R} \to \mathbb{C}$ be measurable. Put $E := \gamma^{-1}(\mathbb{C} \setminus \{0\})$ and let $A := P_{E}FM(\gamma)P_{E}$. Then $A$ is densely defined and $A^* = P_{E}M(\gamma)F^{-1}P_{E}^*$ holds, and dom $A^* = \text{dom} W_{\gamma}$ is either $\{0\}$ or dense. If $\gamma$ is not almost zero, then $A^*$ is injective. If $\gamma \neq 0$ a.e., then $A$ is injective. If $\gamma \neq 0$ a.e., then $\ker A = \{f \in L^2(\mathbb{R}) : \gamma f \in H_{-}\}$, which equals $\{0\}$ if and only if $\frac{1}{\gamma}$ is not proper. Finally $\ker A^* A = \ker A$, $W(|\gamma|^2) = AA^*$, and

$$
\gamma \text{ proper } \iff \text{dom } W(|\gamma|^2) \neq \{0\} \iff \text{dom } A^* \neq \{0\} \iff A \text{ closable}
$$

(\ast)

Proof. Note $M(\gamma)P_{E} = P_{E}M(\gamma)E_{E}$ and $P_{E}M(\gamma) = M(\gamma|_{E})P_{E}$, where $M(\gamma|_{E})$ denotes the multiplication operator in $L^2(E)$. Hence $A = P_{E}FM(\gamma|_{E})P_{E}$ is densely defined, and $A^* = P_{E}M(\gamma)F^{-1}P_{E}^*$ by [25, 13.2(2)]. Note that dom $A^* = \text{dom } M(\gamma)|F^{-1}P_{E}^*$ = dom $W_{\gamma}$, whence the claim on dom $A^*$ by (4)(i).

First let $E$ be proper. Then $A^* g = M(\gamma|_{E})(P_{E}F^{-1}P_{E}^*g)$ implies $P_{E}F^{-1}P_{E}^*g = 0$, whence $g = 0$. Similarly, $Ak = P_{E}F(P_{E}M(\gamma)g)$ means that $f_{-} := P_{E}M(\gamma)g \in H_{-}$ and vanishes on $\mathbb{R} \setminus E$, whence $f_{-} = 0$ and hence $k = 0$. Now assume at once $E = \mathbb{R}$. Then $A^* = M(\gamma)F^{-1}P_{E}^*$ is injective as $M(\gamma)$ is injective. Furthermore, $f \in \ker A \iff f, \gamma f \in L^2(\mathbb{R})$ with $P_{E}F(\gamma f) = 0 \iff f \in L^2(\mathbb{R})$, $\gamma f \in H_{-} \iff f \in L^2(\mathbb{R})$, $\gamma f \in \mathbb{H}_{+}$, and recall that $\frac{1}{\gamma}$ is proper $\iff \frac{1}{\gamma} h \in L^2(\mathbb{R})$ for some $h \in \mathbb{H}_{+} \setminus \{0\}$.

$\ker A^* A = \ker A$ is obvious since either $\gamma = 0$ a.e. or $A^*$ is injective. $M(\gamma)P_{E}P_{E}M(\gamma) = M(\gamma)(1_{E})M(\gamma) = M(\gamma)|F^{-1}P_{E}^*|$, whence $AA^* = W(|\gamma|^2)$.

Turn to the final claim (\ast). Recall that dom $A^* = \text{dom } W_{\gamma}$ is either trivial or dense. So the last equivalence is standard and the remaining equivalences hold by (4)(f).

In view of (12)(\ast) recall the results on the domain of a WH in (4). If $A$ is closable it need not be closed, even if $AA^*$ is closed. (Indeed, $W_{x_{2}} = AA^*$ is closed and $\neq W_{x_{2}} = AA^*$ by (10),(14).) Recall that ran $A^*$ is not dense as $A$ is not injective. $A$ is not injective for $\gamma \neq 0$ a.e. if for instance $\frac{1}{\gamma}F^{-1}s \in L^2(\mathbb{R})$ with $q$ a polynomial and $s$ a Schwartz function with support in $]-\infty, 0]$. (Indeed, $qF^{-1}s$ is a Schwartz function in $H_{-}$, whence $\frac{1}{\gamma}F^{-1}s \in \ker A$.) Recall that $A$ is injective if and only if $\gamma \neq 0$ not a.e. or $\gamma \neq 0$ a.e. and $\frac{1}{\gamma}$ non-proper.

(13) Lemma. Let $\gamma$ be proper. Suppose that $\gamma \neq 0$ not a.e. or that $\gamma \neq 0$ a.e. and $\frac{1}{\gamma}$ is not proper. Then $A$ is closable and $\overline{A}$ is injective.

Proof. $A$ is closable by (12)(\ast). According to (4)(e)(p = $\infty$) one has $|\gamma| e^{j} \leq 1$ for some $j$. Put $j' = 1_{\{j \leq 0\}}j - |x|^{1/2}$. Then $|\gamma| e^{j'} \leq 1$, and $e^{j'}$ and $|\gamma| e^{j'}$ are square integrable. By (2.1) there is $h \in H_{+}$ with $|h| = e^{j'}$. Hence $h \in \text{dom } M_{+}(\gamma) \setminus \{0\}$. Then $h D_{+} \subset \text{dom } M_{+}(\gamma)$ for $D := \frac{1}{1+1}H_{+}^\infty$. $D$ is dense in $H_{+}$ since $D = \Gamma(H_{+}^\infty)$ (see (3)). Now let $f \in \ker A = (\text{ran } A)^{\perp}$. For $d \in D$ one has $0 = \langle f, P_{E}M(\gamma)P_{E}^* (hd) \rangle = \langle P_{E}f, \gamma h d \rangle = \langle \gamma h P_{E} f, d \rangle$, whence $\gamma h P_{E} f \in H_{-}$. If $\gamma \neq 0$ not a.e., then it follows $\gamma P_{E} f = 0$, whence $f = 0$. If $\gamma \neq 0$ a.e., then $h \overline{f} = \frac{1}{\gamma} h'$ for some $h' \in H_{+}$. Assume $f \neq 0$. Then $h' \neq 0$ and $|\gamma|^{-1}|h'|$ is integrable. Hence $|\gamma|^{-1}$ is proper by (2.1) and (4)(e)(p = $\infty$), which however is excluded by the premise.

The foregoing lemma is needed only in sec.5. The main result of this section follows.

(14) Theorem. Let $\kappa \geq 0$. Put $E := \kappa^{-1}(\mathbb{R} \setminus \{0\})$ and $A := P_{E}FM(\sqrt{\kappa})P_{E}$. Then $W_{\kappa} = AA^*$.

Now let $\kappa$ be proper not almost zero. Put $W_{\kappa} := \overline{AA^*}$. Then
(a) $W_\kappa$ is densely defined symmetric nonnegative and $\tilde{W}_\kappa$ is an injective nonnegative self-adjoint extension of $W_\kappa$.

(b) $\text{dom} W_\kappa$ is a core of $A^*$ and $\text{dom} A^* \cap \text{ran}(I + W_\kappa)^\perp = \emptyset$ holds.

(c) $\tilde{W}_\kappa$ is the Friedrichs extension of $W_\kappa$.

Proof. (a) Apply (12) for $\gamma := \sqrt{\kappa}$. Accordingly, $W_\kappa = AA^*$ and, if $\kappa$ is proper, $W_\kappa$ is densely defined and symmetric nonnegative by (4)(n),(o), and $A^*$ is densely defined. Then $\overline{A} = A^{**}$ and by [25, 13.13(a)] $\overline{A}A^*$ is self-adjoint. Clearly $\overline{A}A^*$ is nonnegative. Check that $\overline{A}A^*$ is injective as $A^*$ is injective by (12).

(b), (c) According to [4, Theorem X.23], $\tilde{W}_\kappa$ is the Friedrichs extension only if $\text{dom} \tilde{W}_\kappa \subset H_{W_\kappa}$, where $H_{W_\kappa}$ is the completion of $\text{dom} W_\kappa$ with respect to the sesquilinear form $(f, g)_{W_\kappa} := (g, g) + (g, W_\kappa g)$.

Endow $\text{dom} A^*$ with the inner product $(g, g)_{A^*} := (g, g) + (A^*g, A^*g)$, by which $\text{dom} A^*$ becomes a Hilbert space $K$ since $A^*$ is closed. Then the subspace $\text{dom} \overline{A}A^*$ is dense in $K$ since $\text{dom} \overline{A}A^*$ is a core for $A^*$, see [25, 13.13(b)]. One easily checks that $H_{W_\kappa}$ is the closure of $\text{dom} W_\kappa$ in $K$. Therefore $\text{dom} \overline{A}A^* \subset H_{W_\kappa}$ if and only if $H_{W_\kappa} = K$, which means that $\text{dom} W_\kappa$ is a core of $A^*$. A short computation shows also that $H_{W_\kappa} = K$ is equivalent to $\text{dom} A^* \cap \text{ran}(I + W_\kappa)^\perp = \emptyset$.

Hence it remains to show $\text{dom} A^* \cap \text{ran}(I + W_\kappa)^\perp = \emptyset$. Explicitly this means that $h_0 = 0$ if $h_0 \in H_+$ satisfies $\sqrt{\kappa}h_0 \in L^2(\mathbb{R})$ and $(h_0, 1 + \kappa h) = 0 \forall h \in H_+$ with $\kappa h \in L^2(\mathbb{R})$ (*).

To this end a sequence $(\alpha_n)_n$ in $H^\infty$ will be constructed with $|\alpha_n| = \epsilon^{j_n}$ a.e., where $j_n(x) := 0$ if $1 + \kappa(x) < n$ and $j_n(x) := -\frac{1}{2} \ln(1 + \kappa(x))$ otherwise, and satisfying $\alpha_{n_k} \to 1$ pointwise a.e. for some subsequence $(n_k)$. Provided $(\alpha_n)_n$, set $h_n := \alpha_n h_0$. Then $h_n \in H_+$ and almost everywhere $(1 + \kappa(x))|h_n(x)|$ is less than $n|h_0(x)|$ if $1 + \kappa(x) \leq n$ and equals $\sqrt{1 + \kappa(x)}|h_0(x)|$ otherwise, which proves $(1 + \kappa)h_0 \in L^2(\mathbb{R})$. Moreover $\sqrt{1 + \kappa}h_n \leq \sqrt{1 + \kappa}h_0$ since $|\alpha_n| \leq 1$ and $\sqrt{1 + \kappa}h_n \to \sqrt{1 + \kappa}h_0$ pointwise a.e., whence $\sqrt{1 + \kappa}h_{n_k} \to \sqrt{1 + \kappa}h_0$ in $L^2(\mathbb{R})$ by dominated convergence. Thus (*) holds for $h = h_{n_k}$, whence $0 = (h_0, 1 + \kappa h_{n_k}) = (\sqrt{1 + \kappa}h_0, \sqrt{1 + \kappa}h_{n_k}) \to (\sqrt{1 + \kappa}h_0, \sqrt{1 + \kappa}h_0) = \|\sqrt{1 + \kappa}h_0\|^2$ implying $h_0 = 0$.

We turn to the construction of $(\alpha_n)_n$. By (4), $\frac{\ln(1 + \kappa)}{1 + \kappa}$ and hence all $\frac{j_n}{1 + \kappa}$ are integrable. For convenience we pass from $\mathbb{R}$ to the torus $\mathbb{T}$ by means of the Cayley transformation $C$ (see (3)). So let $\tilde{j}_n := j_n \circ C^{-1}$, which is integrable on $T$. Put

$$F_n(w) := \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} j_n(e^{it}) \, dt$$

for $w \in \mathbb{D}$. Then $\exp \circ F_n$ is an outer function. Let $\tilde{\alpha}_n$ denote its nontangential limit a.e. on $T$. It satisfies $|\tilde{\alpha}_n| = \epsilon^{j_n}$ a.e. Hence $\tilde{\alpha}_n \in H^\infty(\mathbb{T})$. It remains to show the existence of a subsequence $(n_k)$ satisfying $\tilde{\alpha}_{n_k} \to 1$ a.e. The formula $Tf(z) := \lim_{r \to 1} \frac{1}{r^2} \int_0^{2\pi} f(e^{it}) \, dt$, $z \in \mathbb{T}$ defines a bounded operator on $L^1(\mathbb{T})$ into weak-$L^1(\mathbb{T})$, whence $\{z \in \mathbb{T} : |Tf| > \delta\} \leq C \|f\|_1 / \delta$ for all $\delta > 0$ and $f \in L^1(\mathbb{T})$. Therefore, if $f_n \to 0$ in $L^1(\mathbb{T})$, then $Tf_n \to 0$ in probability, which implies $Tf_{n_k} \to 0$ a.e. for some subsequence $(n_k)$. This applies to $(j_n)_n$ yielding $\tilde{\alpha}_{n_k} = \exp \circ T\tilde{j}_{n_k} \to 1$ a.e. \(\square\)

For $\kappa \geq 0$ and $W_\kappa$ densely defined recall that the deficiency subspace $\text{ran}(I + W_\kappa)^\perp$ of $W_\kappa$ at $-1$ is trivial if and only if $W_\kappa$ is essentially self-adjoint.

(15) **Semibounded symbol.** Let $\kappa$ be real semibounded. Then there are $\alpha > 0$ and $\eta \in \{1, -1\}$ such that $\kappa := \alpha 1 + \eta \kappa \geq 0$. Clearly $W'_\kappa = \alpha I + \eta W_\kappa$. Let $W_\kappa$ be densely defined. Then so is $W_\kappa$, and
according to (14) there is the injective nonnegative self-adjoint extension \( \tilde{W}_\kappa \) of \( W_\kappa \). So
\[
\tilde{W}_\kappa := -\eta \alpha I + \eta \tilde{W}_\kappa
\] (4.1)
is a semibounded self-adjoint extension of \( W_\kappa \) with bound \(-\eta \alpha \), which is not an eigenvalue of \( \tilde{W}_\kappa \). It is the Friedrichs extension of \( W_\kappa \).

## 5 Isomorphic Singular Integral Operators

This section is concerned with the symmetric singular integral operator in \( L^2(E) \) for proper \( E \) or \( E = \mathbb{R} \)
\[
(L_\phi f)(x) := \frac{1}{2} \phi(x)f(x) + \frac{1}{2\pi i} \int_E \frac{\sqrt{\phi(x)}\sqrt{\phi(y)}}{y-x} f(y) \, dy
\] (5.1)
in the sense of the principal value at \( x \) where \( \phi : E \to \mathbb{R} \) is measurable positive. \( L_\phi \) turns out to be closely related to \( W_\kappa \), where \( \kappa \) extends \( \phi \) on \( \mathbb{R} \) by zero.

\( L_\phi \) belongs to the studied class of singular integral operators in \( L^2(E) \) of type Hilbert transformation
\[
(L(a,b)f)(x) := a(x)f(x) + \frac{1}{2\pi i} \int_E b(x)\overline{b(y)}(y-x)^{-1} f(y) \, dy
\] (5.2)
where \( a, b \) are measurable functions on \( E \) with \( a \) real and \( b \neq 0 \text{ a.e.} \). There is the obvious unitary equivalence
\[
U L(a,b)U^{-1} = L_\phi + M(\alpha)
\] (5.3)
for \( \phi = 2|b|^2 \) and \( \alpha = a - |b|^2 \), where \( U \) is the multiplication operator by \( \overline{b}/|b| \) and \( M(\alpha) \) the multiplication operator by \( \alpha \) in \( L^2(E) \). So we are concerned with the case \( a = |b|^2 \).

The operator \( L(a,b) \) for bounded \( b \) and bounded below \( a \) is treated by Rosenblum in [26]. It is shown to be self-adjoint on \( \text{dom} \ L(a,b) = \text{dom} \ M(\alpha) \), and is diagonalization is achieved. See also [27] and the literature cited in [26], [27]. The really unbounded case however is there when \( b \) is unbounded. [28] is concerned with this case replacing \( L(a,b) \) by the limit of truncated \( L(a_n,b_n) \) which are bounded. Our analysis of \( L_\phi \) will show (18) that \( L(a,b) \subset M(\alpha) \) if the extension of \( b \) on \( \mathbb{R} \) by zero is not proper. Hence for \( L(a,b) \) in (5.2) being not trivial it is necessary that the extension of \( b \) is proper. In this case \( L_\phi \) in (5.3) has a self-adjoint extension (20).

The Hilbert transformation \( H \) on \( L^2(\mathbb{R}) \) is defined by the singular integral
\[
Hf(x) = \frac{1}{i\pi} \int_{-\infty}^{\infty} f(y) \frac{1}{y-x} \, dy
\] (5.4)
Recall its representation
\[
H = \mathcal{F}^{-1}M(\text{sgn})\mathcal{F} = -\mathcal{F}M(\text{sgn})\mathcal{F}^{-1}
\] (5.5)
on \( L^2(\mathbb{R}) \) with \( \text{sgn} \) the signum function on \( \mathbb{R} \) (see e.g. [13, Theorems 91, 95] or [20, Lemma 1.35] or [14, Chapter II 4.3], and for more details [29, Teorema 1.1.1]). Let us introduce
\[
H_E := P_E H P_E^*
\] (5.6)
the trace on \( L^2(E) \) of the Hilbert transformation \( H \). Its spectrum is determined in [30]. For \( E = [a, b] \), \(-\infty \leq a < b \leq \infty \), \( H_E \) is called finite and semi-finite Hilbert transformation if \( E \) is bounded and semi-bounded, respectively. Its spectral representation is achieved in [31].
5.1 Isometry relating $\tilde{L}_\phi$ to $\tilde{W}_\kappa$

In what follows we use the polar decomposition $C = S|C|$ of a closed densely defined operator $C$ from a Hilbert space $\mathcal{H}$ into another $\mathcal{H}'$ (see e.g. [4, VIII.9]). $|C|$ denotes the square root of the self-adjoint nonnegative operator $C^*C$ in $\mathcal{H}$. One has $\text{dom}|C| = \text{dom} C$ and $\text{dom} C^*C$ is a core for $C$. $S$ is a partial isometry from $\mathcal{H}$ into $\mathcal{H}'$. Its initial space $(\ker S)^\perp$ equals $\overline{\text{ran } C^*} = \overline{\text{ran } |C|} = \overline{\text{ran } |C|^2}$. Similarly, its final space $\text{ran } S$ equals $\overline{\text{ran } C} = \overline{\text{ran } C^*} = \overline{\text{ran } C^*|^2}$. The partial isometry $S^*$ satisfies $(\ker S^*)^\perp = \text{ran } S$ and $\text{ran } S^* = (\ker S)^\perp$. Important is the relation

$$CC^* = SC^*CS^*$$

(5.7)

which means that the reductions of $CC^*$ and $C^*C$ on the orthogonal complements of their respective null spaces are Hilbert space isomorphic by the restriction of the partial isometry $S$ to its initial state $(\ker S)^\perp$ and its final state $\text{ran } S$.

Throughout this section $\kappa \geq 0$ and $\phi > 0$ are related to each other by $\phi = \kappa|E$ for $E = \kappa^{-1}(\mathbb{R}_+)$. 

(16) Lemma. Let $\kappa \geq 0$. Recall $A = P_\pm FM(\sqrt{\kappa})P^*_E$, $A^* = P_E M(\sqrt{\kappa}) F^{-1} P^*_+$. Then

(a) $W_\kappa = AA^*$

(b) $L_\phi = \frac{1}{2} M(\phi) + \frac{1}{2} M(\sqrt{\phi}) H_E M(\sqrt{\phi}) = P_E M(\sqrt{\kappa}) P^*_H P_H M(\sqrt{\kappa}) P^*_E = A^*A$

Now suppose that $\kappa$ is proper. Let $\overline{A} = T|\overline{A}|$ be the polar decomposition of $\overline{A}$. Then the partial isometry $T: L^2(E) \to L^2(\mathbb{R}_+)$ is surjective, its adjoint $T^*$ is injective, and

(c) $\bar{W}_\kappa := \overline{A} A^*$ is a self-adjoint extension of $W_\kappa$.

(d) $\bar{L}_\phi := A^* \overline{A}$ is a self-adjoint extension of $L_\phi$.

(e) $\ker \bar{L}_\phi = \ker \overline{A}$, $(\ker T)^\perp = (\ker \bar{L}_\phi)^\perp = \overline{\text{ran } A^*}$.

Proof. (a) See (14). (b) $A^*A = P_E M(\sqrt{\kappa}) F^{-1} P^*_E P_\pm FM(\sqrt{\kappa}) P^*_E P_E M(\sqrt{\kappa}) P^*_H P_H M(\sqrt{\kappa}) P^*_E = P_E M(\sqrt{\kappa}) \frac{1}{2} I + H). M(\sqrt{\kappa}) P^*_E$ by (5.5). Hence $A^*A = L_\phi$ as $P_E M(\sqrt{\kappa}) = M(\sqrt{\phi}) P_E$, $M(\sqrt{\kappa}) P^*_E = P^*_E M(\sqrt{\phi})$. Recall (5.6). (c) and (d) are obvious. Finally, $T$ is surjective as $\overline{\text{ran } T} = (\overline{\text{ran } A})^\perp = (\ker A^*)^\perp = L^2(\mathbb{R}_+)$ by (12), and (e) follows directly from polar decomposition. 

As a first result we note that $L_\phi$ in (5.1) can be trivial.

(17) Corollary. Let $\kappa \geq 0$ be not proper. Then $L_\phi \subset 0$. Moreover, dom $L_\phi = \{0\}$ if and only if either $\kappa \neq 0$ not a.e. or otherwise $\frac{1}{\kappa}$ is not proper. If $\kappa \neq 0$ a.e. and $\frac{1}{\kappa}$ is proper, then dom $L_\phi = \frac{1}{\sqrt{\kappa}} H_- \cap L^2(\mathbb{R})$.

Proof. dom $A^* = \{0\}$ by (12)(*), whence $L_\phi \subset 0$ and dom $L_\phi = \ker A$ because of (16)(b), (12). Recall (4)(f) by which $1/\kappa$ is not proper if and only if $1/\sqrt{\kappa}$ is not proper. The remainder is shown in (12).
For example \( \kappa = e^x \) is positive and, by (4)(h), \( \kappa \) and \( 1/\kappa \) are not proper. Hence \( \text{dom} \ L_{e^x} = \{0\} \).

Now (17) and (5.3) have the following important outcome on \( L(a,b) \). Accordingly, for \( L(a,b) \) in (5.2) being not trivial it is necessary that the extension of \( b \) on \( \mathbb{R} \) by zero be proper.

(18) **Theorem.** Let the extension of \( b \) on \( \mathbb{R} \) by zero be not proper. Then \( L(a,b) \subset M(a - |b|^2) \) with

(a) \( \text{dom} \ L(a,b) = \{ f \in L^2(\mathbb{R}) : |b| f \in H_-, (a - |b|^2) f \in L^2(\mathbb{R}) \} \) if \( E = \mathbb{R} \)

(b) \( \text{dom} \ L(a,b) = \{0\} \) if and only if \( E \) is proper, or if \( E = \mathbb{R} \) and \( \frac{1}{b} \) is not proper, or if \( E = \mathbb{R} \) and \( \frac{1}{b} \) is proper and \( \frac{a - |b|^2}{b} \) is not proper.

(19) **Corollary.** Let \( E = \mathbb{R} \) and suppose that \( \frac{1}{b} \) is bounded. Then \( L_\phi \) is densely defined and

\[
\text{dom} \ L_\phi = \left\{ \frac{1}{\sqrt{\phi}}(h + k) : h \in H_+, \sqrt{\phi}h \in L^2(\mathbb{R}), k \in H_- \right\}, \quad L_\phi \left( \frac{h + k}{\sqrt{\phi}} \right) = \sqrt{\phi}h \quad \text{(*)}
\]

with \( \ker L_\phi = \frac{1}{\sqrt{\phi}} H_- \) and \( (\ker L_\phi)^\perp = \{ \sqrt{\phi}h \in L^2(\mathbb{R}) : h \in H_+ \} \).

If \( \phi \) is proper then \( \tilde{L}_\phi = \ker L_\phi \neq L^2(\mathbb{R}) \). If \( \phi \) is not proper then \( \tilde{L}_\phi = 0 \).

**Proof.** Note \( \kappa = \phi, \ L_\phi = M(\sqrt{\phi})P_{H_+}P_{H_+}M(\sqrt{\phi}) \) by (16)(b). Then (\*) is easily verified and \( \ker L_\phi = \frac{1}{\sqrt{\phi}} H_- \) follows. Check \( f \in L^2(\mathbb{R}), f \perp \frac{1}{\sqrt{\phi}} H_- \iff f \in L^2(\mathbb{R}), 0 = \langle \sqrt{\phi} f, k \rangle = \langle \frac{1}{\sqrt{\phi}} f, k \rangle \forall k \in H_- \iff f \in L^2(\mathbb{R}), \frac{1}{\sqrt{\phi}} f \in H_+ \), proving the claim on (\( \ker L_\phi \))\(^\perp \).

Note \( D := \text{dom} A^* = \{ h \in H_+ : \sqrt{\phi}h \in L^2(\mathbb{R}) \} \), whence \( \text{ran} A^* = \{ \sqrt{\phi}h \in L^2(\mathbb{R}) : h \in H_+ \} \) which equals \( (\ker L_\phi)\perp \).

Now let \( \phi \) be proper. Then \( D \) is dense by (12) and \( A^* \neq \{0\} \). Hence by (16)(e) on has \( \ker \tilde{L}_\phi = (\text{ran} A^*)\perp = \ker L_\phi \neq L^2(\mathbb{R}) \). Moreover, \( \text{dom} L_\phi \) is dense, since \( f \in (\text{dom} L_\phi)\perp \Rightarrow 0 = \langle \sqrt{\phi}, (h + k) \rangle \forall h \in D, k \in H_- \Rightarrow f/\sqrt{\phi} \in H_+, \langle f/\sqrt{\phi}, h \rangle \forall h \in D \Rightarrow f/\sqrt{\phi} = 0 \Rightarrow f = 0 \). If \( \phi \) is not proper then \( D = \{0\} \) by (12), whence \( \text{dom} L_\phi = \frac{1}{\sqrt{\phi}} H_- \). The latter equals \( \ker L_\phi \).

Finally, \( (\ker L_\phi)\perp = \{ \sqrt{\phi}h \in L^2(\mathbb{R}) : h \in H_+ \} = \{0\} \) since \( D = \{0\} \).

For example, \( L_{e^{-|x|^2}} \subset 0 \) with dense \( \text{dom} L_{e^{-|x|^2}} = e^{-|x|^2}/2 \).

The main outcome of this section is

(20) **Theorem.** Let \( \kappa \geq 0 \) be proper. Then

\[
\tilde{L}_\phi = T^* \tilde{W}_\kappa T
\]

(5.8)

Let \( \tilde{L}_\phi \) and \( T^* \) denote the reduction of \( \tilde{L}_\phi \) and the restriction of \( T \) on the orthogonal complement of the null space of \( \tilde{L}_\phi \), respectively. Then \( T^* \) is a Hilbert space isomorphism onto \( L^2(\mathbb{R}_+) \) satisfying \( \tilde{L}_\phi = T^{d-1} \tilde{W}_\kappa T^* \).

**Proof.** (5.8) follows immediately from polar decomposition \( \tilde{A} = T|\tilde{A}| \) (cf. (5.7)). By (16) \( T \) is surjective. Hence \( T^* \) is a Hilbert space isomorphism by (16)(e). Finally recall (5.8).
Of course one has $\hat{W}_\kappa = T\hat{L}_\phi T^*$ as well. It allows to study $\hat{W}_\kappa$ starting from $\hat{L}_\phi$. — By (15) it is easy to extend (20) to semibounded symbols.

(21) Corollary. Let $\frac{c}{1+x^2}$ be integrable. Then $\hat{L}_\phi$ is absolutely continuous.

Proof. Recall that $\hat{T}_{\kappa\omega\gamma}$ and hence $\hat{W}_\kappa$ (see (3)) is absolutely continuous by [9]. Apply (20).

The special case of (21) that $\phi$ is bounded is treated in [26, sec. 3] and [27, Theorem]. — We remind that for $\hat{L}_\phi$ being injective it is necessary that $E$ is proper or that $1/\kappa$ is not proper.

(22) Corollary. Let $\kappa \geq 0$ be proper. Suppose that $E$ is proper or $1/\kappa$ is not proper. Then $\hat{L}_\phi$ is injective and $T : L^2(E) \to L^2(\mathbb{R}_+)$ is a Hilbert space isomorphism with

$$\hat{L}_\phi = T^{-1}\hat{W}_\kappa T$$

Proof. $\mathfrak{T}$ is injective by (13). Hence $\{0\} = \ker \hat{L}_\phi = \ker T$ and $T$ is an isomorphism. Apply (20).

So it is worth noting that $\hat{L}_\phi$ is absolutely continuous, if $\frac{c}{1+x^2}$ is integrable and if $E$ is proper or $1/\kappa$ is not proper.

Let $\phi$ be bounded. Then $L_\phi, W_\kappa$ are bounded and $\hat{L}_\phi = L_\phi, \hat{W}_\kappa = W_\kappa$, and $L_\phi = T^*W_\kappa T$. Moreover, if $E$ is proper or $1/\kappa$ is not proper, then even $\hat{L}_\phi = L_\phi$ by (22), whence

$$L_\phi = T^{-1}W_\kappa T$$

An example for the latter case is $L_{e^{-|x|}}$, which is injective. Generally, if $\kappa > 0$ does not decrease too rapidly (so that $1/\kappa$ is proper) the kernel of $L_\kappa$ is not trivial. For an instructive example see sec. 5.2. The trivial example here is $\kappa = 1\mathbb{R}$. Note that $W(1\mathbb{R}) = I_{L^2(\mathbb{R}_+)}$ and $L_{1\mathbb{R}}$ is the orthogonal projection on $H_+$, and $T' : H_+ \to L^2(\mathbb{R}_+), T'h := P_+\mathcal{F}h$. Examples for proper $E$ are the isomorphic pairs $W(1_E) \simeq \frac{1}{2}(I + H_E)$ with $H_E$ in (5.6), which we like to write as

$$H_E = T^{-1}W_{21_E-1}T$$

(5.9)

The case of the finite Hilbert transformation $H_{[-1,1]}$ is studied in detail in [22, sec. 3.3], [22, (3.20)].

In conclusion we make a remark on the spectral representations of $\hat{W}_\kappa$ and $\hat{L}_\phi$ in (20). By the spectral theorem in the multiplication operator version, $\hat{W}_\kappa$ is Hilbert space isomorphic to the multiplication operator $M(\varphi)$ on $L^2_\mu(\mathbb{R})$ for some Borel-measurable positive $\varphi : \mathbb{R} \to \mathbb{R}$ and finite Borel measure $\mu$. Let $V : L^2_\mu(\mathbb{R}) \to L^2(\mathbb{R}_+)$ be an isomorphism satisfying

$$\hat{W}_\kappa = V M(\varphi) V^{-1}$$

(5.10)

The spectral measure for $\hat{W}_\kappa$ is given by $E_{\hat{W}_\kappa}(\Delta) = VM(1_{\varphi^{-1}(\Delta)})V^{-1}$ for measurable $\Delta \subset \mathbb{R}$.

(23) Corollary. Suppose the representation (5.10) of $\hat{W}_\kappa$. Then $W_0 := A^*VM(\sqrt{\varphi})$ with $A^*$ in (16) is closable, its closure $W$ is the Hilbert space isometry $T^*V$ with $	ext{ran } W = (\ker \hat{L}_\phi)^\perp$, by which $\hat{L}_\phi =$
and that of the associated Hilbert space isomorphism
\[ A \]

For the last equality recall \( A^* = T^* |A^*|^{-1} \). Further note that \( \text{dom}(|A^*|^{-1}) = \text{ran}(|A^*|) \) is dense as \( |A^*| = W_\kappa^{-1/2} \) is self-adjoint injective. Since \( T^* \) is bounded, \( W_0 \) is closable and its closure \( W \) equals \( T^*V \). The remainder is obvious. \( \Box \)

5.2 Example: Lalescu's operator

Supposing \( \kappa > 0, \kappa \in L^2 \), and kernel \( k \in L^1 \), in [32] a spectral theory of \( W_\kappa \) is proposed by a reduction to a previously developed theory for singular integral operators. Its application in [32] to Lalescu’s operator \( W_\Lambda \) with symbol
\[ \lambda(x) := \frac{2}{1 + x^2} \]  

however does not produce the right normalization (5.13) of the generalized eigenfunctions (5.12). In establishing (5.13) we get also the result (25) on orthogonal polynomials.

The diagonalization of Lalescu’s operator \( W_\Lambda \) is achieved in sec.5.2.1 and that of the associated singular integral operator \( L_\Lambda \) is derived in sec.5.2.2. The generalized eigenfunctions (5.17) of the latter are no longer regular distributions.

5.2.1 Spectral representation of \( W_\Lambda \)

Clearly the spectrum of the WH \( W_\Lambda \) lies in \([0,2]\). The kernel for \( W_\Lambda \) is \( e^{-|x|} \). The obvious ansatz \( a e^{sx} + b e^{sy} \) for \( u(x), \ x > 0 \) in \( s u(x) - \int_0^\infty e^{-|x-y|} u(y) \, dy = 0 \) yields for every \( s \in [0,2] \) the generalized eigenfunction (cf. [2])
\[ q_s(x) = n(s) \left( (\tau - i) e^{\tau x} + (\tau + i) e^{-\tau x} \right), \quad \tau := \left( \frac{2}{\lambda} - 1 \right)^{1/2} \]  

The claim is that there is the unique positive normalization constant
\[ n(s) = (4\pi st)^{-1/2} \]  

such that \( q(s, x) := q_s(x) \) is the kernel for a Hilbert space isomorphism
\[ V : L^2(0, 2) \to L^2(\mathbb{R}_+), \quad Vh = \text{l.i.m.} \int_0^2 q(s, \cdot) h(s) \, ds \]  

Indeed, the assertion follows immediately from the following Lemma (24) as \( V = U^* \Gamma \) holds for the Hilbert space isomorphism
\[ \Gamma : L^2(0, 2) \to L^2(\mathbb{R}_+), \quad \Gamma h(y) := \frac{2\sqrt{\pi}}{1+y} h\left( \frac{2}{1+y} \right) \]  

due to the change of variable \( \gamma(y) := \frac{2}{1+y} \).

(24) Lemma. \( Ug(x) := \text{l.i.m.} \int_0^\infty u(x, y) g(y) \, dy \) with
\[ u(x, y) = (2\pi)^{-1/2} (1 + x^2)^{-1/2} \left( (x - i) e^{xy} + (x + i) e^{-iy} \right) \]
is unitary on $L^2(\mathbb{R}_+)$. 

Proof. Using the generating function $l_q(x) = (1 - q)^{-1} \exp \left( -\frac{1+i\varphi}{1-q} x \right)$, $|q| < 1$ of the Laguerre functions $l_n$, $n \geq 0$ one finds $\lambda_n(x) := U_n(x) = \sqrt{\frac{2}{\pi}} (1 + x^2)^{-1/2} \left( \frac{x+i}{1+ix} \gamma(x)^n + \frac{x-i}{1-ix} \gamma(x)^{-n} \right)$ with $\gamma(x) := -\frac{1 + i x}{1 + 2ix}$.

Note that $\int_{-\infty}^{\infty} (1 + x^2)^{-1} \left( \frac{x+i}{1+ix} \right)^2 \gamma(x)^k \, dx = 0$ for $k \in \mathbb{Z}$, since the integral is a parametrization of the complex integration of $\frac{1}{1+2ix} z^k$ along the unit circle. Using this it follows easily that $\lambda_n$, $n = 0, 1, 2, \ldots$ are orthonormal in $L^2(\mathbb{R}_+)$. It remains to show the completeness of $(\lambda_n)_n$.

Let $\varphi \in [0, \pi]$ satisfy $e^{i\varphi} = \gamma(x)$, $x > 0$. Put $\xi := \cos \varphi$. Then $\lambda_n(x) = \sqrt{\frac{18(1-\xi)}{\pi(5-3\xi)}} \sqrt{1-\xi^2} (\frac{\sin(n+1)\varphi}{\sin \varphi} - \frac{1}{3} \frac{\sin n \varphi}{\sin \varphi}) = \sqrt{\frac{18(1-\xi)}{\pi(5-3\xi)}} \sqrt{1-\xi^2} (U_n(\xi) - \frac{1}{3} U_{n-1}(\xi))$, where $U_n$ denotes the Chebyshev polynomial of the second kind of degree $n$ for $n = 0, 1, 2, \ldots$ and $U_{-1} := 0$ [33, 10.11(2)]. Therefore the Hilbert space isomorphism $\Gamma : L^2(\mathbb{R}_+) \to L^2(-1,1)$ due to the change of variable $\gamma(\xi) := \frac{1}{2} \sqrt{\frac{1+\xi}{1+\xi}}$, $\gamma'(\xi) = \frac{1}{2} \sqrt{\frac{1-\xi}{1+\xi}} (1-\xi)^{-2}$ maps $\lambda_n$ onto $\Gamma \lambda_n(\xi) = \sqrt{\frac{9}{\pi(5-3\xi)}} (1-\xi^2)^{1/4} Q_n(\xi)$, where $Q_n := U_n - \frac{1}{3} U_{n-1}$ is a polynomial of degree $n$ for $n = 0, 1, 2, \ldots$ Thus (25) is shown, which implies the assertion. 

(25) Corollary. Let $Q_n := U_n - \frac{1}{3} U_{n-1}$, $n = 0, 1, 2, \ldots$ with $U_n$ the Chebyshev polynomial of the second kind of degree $n$ and $U_{-1} := 0$. Then $Q_n$ is a sequence of orthonormal polynomials on $[-1,1]$ with respect to the weight function $\xi \mapsto \frac{2 \sqrt{1-\xi^2}}{\pi(5-3\xi)}$. It obeys the recurrence $Q_{n+1} = 2Q_n - Q_{n-1}$ for $Q_0 = 1, Q_1(\xi) = 2\xi - \frac{1}{2}$. 

(26) Theorem. The spectral representation $\mathcal{W}_\lambda = VM(\text{id}_{[0,2]})V^{-1}$ holds.

Proof. First note $|q(s,x)| \leq (s(2-s))^{-1/2}$. Let $h \in L^2(0,2)$ with support in $[\delta, 2-\delta]$ for some $0 < \delta < 1$. Then $(VM(\text{id}_{[0,2]})h)(x) = \int_{\delta}^{2-\delta} q(s,x) h(s) \, ds$. Substitute $s = q(s,x) = \int_{0}^{\infty} e^{-|x-y|} q(s,y) \, dy$. Obviously the integrations can be interchanged by Fubini’s theorem yielding $\int_{0}^{\infty} e^{-|x-y|} \left( \int_{\delta}^{2-\delta} q(s,y) h(s) \right) \, dy = \int_{0}^{\infty} e^{-|x-y|} V h(y) \, dy = W\lambda V h(y)$. This implies the result. 

5.2.2 Spectral representation of $L_\lambda$

Let us first establish the kernel of $L_\lambda$. By (16)(f) it equals that of $A = P_+ F M(\sqrt{\lambda})$, whence $\ker(L_\lambda) = \{ f \in L^2(\mathbb{R}) : \sqrt{f} \in H_- \}$ by (12). Put $u(x) := \frac{x-i}{\sqrt{1+2ix}}$, $x \in \mathbb{R}$. Note that the multiplication operator $M(u)$ is unitary. It follows

$$\ker(L_\lambda) = M(u) H_-, \quad \ker(L_\lambda) \perp = M(u) H_+ \quad (5.16)$$

Indeed, as to the less trivial implication let $f \in \ker(L_\lambda)$. Then $h := \sqrt{f} \in H_-$, whence $(x-i) h \in L^2(\mathbb{R})$. Therefore $k := 2^{-1/2}(x-i) h \in H_-$ using (7)(c), and $f = uk$ follows.

Hence the reduction $L'_\lambda$ of $L_\lambda$ on the orthogonal complement of its kernel is self-adjoint bounded on $M(u) H_+$ with spectrum $[0,2]$. The following computations are valid in a distributional sense. So generalized eigenfunctions $Q_s$ for $L_\lambda$ are given by $A^s Q_s = M(\sqrt{\lambda}) P_+ Q_s$ since $L_\lambda A^s Q_s = A^s A^s Q_s = \ldots$
$sA^*q_s$. Recall $\int_{0}^{\infty} e^{ixy} \, dy = \frac{1}{x} + \pi \delta(x)$, see e.g. [34, Table of Fourier Transforms 1.23]. Hence one yields the not regular distributions on $\mathbb{R}$

$$Q_s(x) = \frac{n(s)}{\sqrt{s}} \sqrt{\frac{2}{1 + x^2}} \frac{1}{2\pi} \left( (\tau - i) \left( \frac{i}{x + \tau} + \pi \delta(x + \tau) \right) + (\tau + i) \left( \frac{i}{x - \tau} + \pi \delta(x - \tau) \right) \right)$$  \tag{5.17}

The claim is that $n(s)\sqrt{s}$ is the unique positive normalization constant such that $Q(s, x) := Q_s(x)$ is the kernel for an isometry $W$ on $L^2(0, 2)$ in $L^2(\mathbb{R})$ satisfying

$$W h = \int_{0}^{2} Q(s, \cdot) h(s) \, ds$$  \tag{5.18}

for test functions $h$. The additional factor $\frac{1}{\sqrt{s}}$ regarding the normalization constant of $Q_s$ corresponds to the factor $\frac{1}{\sqrt{s}}$ for $W_0$ in (23) and is suggested heuristically by $\langle A^*q_s, A^*q_s \rangle = \langle q_s, AA^*q_s \rangle = s \langle q_s, q_s \rangle$. For the proof recall the isomorphism $\Gamma$ (5.15) and the Hilbert transformation $H$ (5.5). For $g : \mathbb{R}_+ \to \mathbb{C}$ let $g_{oe}$ denote the odd extension $g_{oe}(x) = -g(-x)$ for $x \leq 0$ of $g$.

**Theorem (27)**. The integral operator (5.18) determines a Hilbert space isometry $W : L^2(0, 2) \to L^2(\mathbb{R})$ with $\text{ran} W = M(u)H_+$. It satisfies $W h = M(u)\frac{1}{2}(I + H)(\Gamma h)_{oe}$ for $h \in L^2(0, 2)$ and yields the representation

$$L_\lambda = WM(\text{id}_{[0, 2]})W^*$$

**Proof.** By the change of variable $t = 2/(1 + s^2)$ for the integration in (5.18) one obtains $\frac{1}{2\pi} \int_{0}^{\infty} \left( (t - i) \left( \frac{i}{x + t} + \pi \delta(x + t) \right) + (t + i) \left( \frac{i}{x - t} + \pi \delta(x - t) \right) \right) \Gamma(h) \, dt$. The integral is easily done yielding $(\ast)$ $Wh = M(u)\frac{1}{2}(I + H)(\Gamma h)_{oe}$ for test functions $h$. Check $\langle H f, f \rangle = 0$ if $f \in L^2(\mathbb{R})$ is odd. Therefore $\|W h\|^2 = \frac{1}{4} \| (I + H)(\Gamma h)_{oe} \|^2 = \frac{1}{4} \| (I + H)(\Gamma h)_{oe} \|^2 = \frac{1}{4} \| (\Gamma h)_{oe} \|^2 = \|h\|^2$. So $W$ is an isometry, and $(\ast)$ extends to all $h \in L^2(0, 2)$. Obviously $Wh \in M(u)H_+$. Moreover, if $k \in H_+$ then $k = \frac{1}{2}(I + H)(k - k)$ as $k \in H_-$. Since $k - k$ is odd, this implies ran $W = M(u)H_+$. The representation of $L_\lambda$ follows from $WM(\text{id}_{[0, 2]})h = L_\lambda Wh$ for test functions $h$. The latter is shown along the lines of the proof of (26). We omit the technicalities. \hfill $\square$

**References**

[1] A. Böttcher, B. Silbermann: *Analysis of Toeplitz Operators*, Springer, Berlin 1990

[2] N. Wiener, E. Hopf: *Über eine Klasse singulärer Integralgleichungen*, Sitz. Berlin. Akad. Wiss. 696-706 (1931). See also [17, Chap. IV] and [13, 11.17].

[3] M. G. Krein: *Integral Equations on a Half-Line with Kernels Depending up on the Difference of the Arguments*, Amer. Math. Soc. Transl. (2) **22**, 163-288 (1962)

[4] M. Reed, B. Simon: *Methods of Modern Mathematical Physics*, Academic Press, New York 1975

[5] A. Devinatz, M. Shnibrot: *General Wiener-Hopf Operators*, Trans. AMS **145**, 467-494 (1969). F. O. Speck: *General Wiener-Hopf factorization methods*, Pitman, Boston 1985

23
[6] D.R. Yafaev: *On Semibounded Wiener-Hopf Operators*, J. London Math. Soc. **95**, 742-762 (2017) or arXiv:1606.01361v1[math.FA]

[7] D.R. Yafaev: *Toeplitz versus Hankel: Semibounded Operators*, Opuscula Math. **38** no. 4, 573590 (2018), https://doi.org/10.7494/OpMath.2018.38.4.573

[8] M. Rosenblum: *Self-adjoint Toeplitz Operators and Associated Orthonormal Functions*, Proceedings of the American Mathematical Society **13**, 590-595 (1962)

[9] M. Rosenblum: *A Concrete Spectral Theory for Self-adjoint Toeplitz Operators*, American Journal of Mathematics **87**, 709-718 (1965)

[10] V. J. Pellegrini: *Unbounded General Wiener-Hopf Operators*, Indiana University Mathematics J. **21**, 85-90 (1971)

[11] D. Sarason: *Unbounded Toeplitz Operators*, Integr. equ. oper. theory **61**, 281-298 (2008)

[12] K. Gustafson: *The Toeplitz-Hausdorff Theorem for Linear Operators*, Proceedings of the American Mathematical Society, Shorter Notes 203-204 (1970)

[13] C. Titchmarsh: *Introduction to the Theory of Fourier Integrals*, Oxford at the Clarendon Press, second edition 1948

[14] F. Ricci: *Hardy Spaces in One Complex Variable*, <http://homepage.sns.it/fricci/papers/hardy.pdf> (2005)

[15] I.I. Privalov: *Randeigenschaften analytischer Funktionen*, VEB Deutscher Verlag der Wissenschaften, Berlin 1965

[16] A. Aleman: *Introduction to Hardy spaces*, <www ctr.maths.lu.se hardy-spaces-lecture-notes 2> (2014)

[17] R.E.A.C. Paley, N. Wiener: *Fourier Transforms in the Complex Domain*, Amer. Math. Soc. Coll. Pub. **19** (1934).

[18] E. Hewitt, K. Stromberg: *Real and Abstract Analysis*, Springer, New York 1969

[19] D. Vukotić: *Analytic Toeplitz Operators on the Hardy Space H^p: a Survey*, Bull. Belg. Math. Soc. **10**, 101-113 (2003)

[20] R. Duduchava: *Integral Equations with Fixed Singularities*, Teubner, Leipzig 1979

[21] T. Carleman: *L’integrale de Fourier et questions qui s’y rattachent*, Almquist Wiksells, Uppsala 1944

[22] D.P.L. Castrigiano: *Spectral Representation of the Wiener-Hopf Operator for the sinc Kernel and some Related Operators*, preprint february 2020

[23] M. Schechter: *Basic Theory of Fredholm Operators*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome **21**, no 2, p. 261-280 (1967)

[24] F. Cucker, A.G. Corbalan: *An Alternate Proof of the Continuity of the Roots of a Polynomial*, Amer. Math. Monthly **96**, 342-345 (1989)

[25] W. Rudin: *Functional Analysis*, TATA McGraw-Hill TMH Edition, New Delhi 1974
[26] M. Rosenblum: *A Spectral Theory for Self-adjoint Singular Integral Operators*, Amer. J. Math. 88, 314-328 (1966)

[27] C.R. Putnam: *Absolute Continuity of Singular Integral Operators*, Amer. J. Math. 91, 453-462 (1969)

[28] J.D. Pincus: *Symmetric Singular Integral Operators*, Indiana Univ. Math. J. 23, 537-556 (1973)

[29] G. Talenti: *Sulle equazioni integrali di Wiener Hopf*, Boll. Un. Mat. Ital. (4) 7 suppl. fasc. 1, 18-118 (1973)

[30] H. Widom: *Singular Integral Equations in $L_p$*, Trans. Amer. Math. Soc. 97 (1960)

[31] W. Koppelman, J. D. Pincus: *Spectral Representations for Finite Hilbert Transformations*, Math. Zeitschrift 71, 399-407 (1959)

[32] J.D. Pincus: *The Spectral Theory of Self-Adjoint Wiener-Hopf Operators*, Bull. Am. Math. Soc. 72, 882-887 (1966)

[33] Erdelyi et. al.: *Higher Transcendental Functions Vol. 2*, McGraw-Hill, New York 1953

[34] I.M. Gel’fand, G.E. Shilov: *Generalized Functions Vol. I*, Academic Press, New York 1964