Searching for MOND in scalar–tensor theories of gravity

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Received 16 May 2018, revised 12 July 2018
Accepted for publication 16 July 2018
Published 1 August 2018

Abstract
In this paper, I study spherically symmetric solutions in a simple class of geometric sigma models of the Universe. This class of models is a subclass of the wider class of scalar–tensor theories of gravity. The purpose of this work is to examine how the additional scalar degree of freedom modifies Newtonian gravitational force. The general solution for spherically symmetric metric far from the point source is obtained in a weak field approximation. As it turns out, it is parametrized by the mass of the source, and an additional function of time. One particular model is examined as an example. It is shown that there are solutions that accommodate MOND regime at some distances from the source. Unfortunately, the obtained interval of distances turns out to be smaller than it is needed. An additional analysis shows that genuine MOND, that explains galactic curves of all nearby galaxies, cannot be obtained.

Keywords: modified gravity, sigma model, MOND

1. Introduction
With this work, I begin the search for a plausible explanation of the observed galactic curves. While Newtonian gravity successfully explains trajectories of planets in the Solar system, and trajectories of stars close to the galaxy center, it encounters problems when trying to explain trajectories of more distant stars. Indeed, the observed trajectories of stars turn out to require more galactic mass than our telescopes can detect [1–8]. The missing mass is commonly referred to as dark matter. With the presence of dark matter, the explanation of flat galactic curves is quite simple. Still, there are two problems that cannot be ignored. The first is that distribution of dark matter in every galaxy must be fine tuned to produce the observed galactic curves. Then, one needs an explanation of how this specific distribution of mass has been formed in the first place. The second is the very nature of dark matter. We do not know what sort of exotic matter it is, and why it is not seen. Another explanation of flat galactic curves is that Newtonian gravity is somehow modified at large distances. One of the most successful
phenomenological models is Milgrom’s modified Newtonian dynamics, commonly referred to as MOND [9–16]. This model does not require dark matter, but it suffers from the nonexistence of a satisfactory underlying theory.

There have been various attempts in literature to construct a relativistic MOND theory. In [14, 17–21], the authors examined scalar–vector-tensor theories for the presence of MOND. In [22], Bekenstein’s TeVeS theory has been extended by the addition of a Galileon type term. A class of bimetric theories of gravity has been examined in [23]. The other proposals include non-local theories [24], extended metric theories with torsion [25], and many more. Notice, however, that all these proposals exclusively deal with the galactic scale dynamics. As a consequence, the expansion of the Universe is neglected, and the cosmological background is replaced by the Minkowski metric. In this work, I shall examine if a class of scalar–tensor theories, known for its success in explaining the early Universe, can be as successful in describing present epoch at small scales. To this end, I shall search for MOND in a given class of scalar–tensor theories.

The motivation for considering this particular class of theories is that scalar–tensor theories are widely excepted in literature because of their success in explaining the early Universe. In particular, they predict inflation with all its attractive consequences. Thus, the primary reason for popularity of this class of theories has nothing to do with MOND. My primary motivation is to find out if these theories can be as successful in explaining the late time behavior of the Universe as they are successful in explaining the early Universe. In particular, I want to calculate the modified gravitational force and see if it can fit the observation. I emphasize that my goal is not to construct a relativistic MOND theory, but rather to check if a given class of scalar–tensor theories can have MOND-like behavior at the galactic scale.

In what follows, I shall examine a class of geometric sigma models with one scalar field. These are a subclass of scalar–tensor theories of gravity, and differ from ordinary sigma models in two respects. First, their scalar fields can all be gauged away, leaving us with the metric alone. Second, they are constructed in an unconventional way. One first chooses metric one would like to be the vacuum of the model, and then builds a theory that has this metric as its solution. These models have first been proposed in [26] in the context of fermionic excitations of flat geometry. In [27] and [28], they are used for the construction of various inflationary and bouncing cosmologies. The purpose of this work is to examine how the additional scalar degree of freedom modifies Newtonian gravitational force. To this end, I shall consider a pointlike source, and calculate spherically symmetric metric far from it.

The results of the paper are summarized as follows. First, I obtained general spherically symmetric solution to the whole class of considered geometric sigma models. The solution is obtained in a weak field approximation, and in the form of a power series. As it turns out, this general solution has two free parameters. The first is a constant with the dimension of mass, and the second is an arbitrary function of time. The presence of a free function of time in the general solution is caused by the additional scalar degree of freedom. The second result concerns the comparison of the obtained solution with MOND. A specific example has been analyzed. I have shown that there exists a particular class of solutions that mimic MOND in a finite interval of distances, and a wide range of source masses. Sadly, the obtained interval of distances turns out to be much smaller than the original MOND suggests. To check if the second result is a general property of the model, I have done an alternative analysis of field equations. The result is that exact MOND can always be obtained at a fixed moment of time, such as the present epoch, but its time evolution is so rapid that MOND regime is almost instantly lost. In particular, it is seen that even the nearest galaxies are devoid of the exact MOND.
The layout of the paper is as follows. In section 2, a precise definition of the class of models to be considered is given. The very construction of geometric sigma models is only briefly recapitulated. In section 3, spherically symmetric ansatz is applied to field equations. The general solution is obtained in a weak field approximation, and in the form of a power series. Two free parameters parametrize the solution, and one of them is a free function of time. In section 4, a particular geometric sigma model is examined for a class of spherically symmetric solutions. This class is defined by the simplest nontrivial choice of the time dependent parameter. MOND behavior is found in a finite interval of distances, and for a wide range of source masses. In section 5, an alternative analysis of field equations is done. It is shown that, although MOND can always be obtained at a fixed moment of time, its time evolution is so rapid that it almost instantly disappears. Section 6 is devoted to concluding remarks.

My conventions are as follows. Indexes \( \mu, \nu, \ldots \) and \( i, j, \ldots \) from the middle of alphabet take values \( 0, 1, 2, 3 \). Indexes \( \alpha, \beta, \ldots \) and \( a, b, \ldots \) from the beginning of alphabet take values \( 1, 2, 3 \). Spacetime coordinates are denoted by \( x^\mu \), ordinary differentiation uses comma (\( X_{, \mu} \equiv \partial_\mu X \)), and covariant differentiation uses semicolon (\( X_{; \mu} \equiv \nabla_\mu X \)). Repeated indexes denote summation: \( X_{\alpha \beta} \equiv X_{11} + X_{22} + X_{33} \). Signature of the 4-metric \( g_{\mu \nu} \) is \((- , + , + , + )\), and curvature tensor is defined as \( R^{\mu \nu \rho \sigma} \equiv \partial_\rho \Gamma^{\mu \nu \lambda} - \partial_\nu \Gamma^{\mu \rho \lambda} + \Gamma^{\mu \sigma \lambda} \Gamma^{\rho \nu \lambda} - \Gamma^{\mu \rho \lambda} \Gamma^{\nu \sigma \lambda} \). Throughout the paper, the natural units \( c = \hbar = 1 \) are used.

2. Model

The model considered in this paper belongs to the class of geometric sigma models, originally defined in [26]. The main feature of every geometric sigma model is that it is defined by associating action functional with a fixed freely chosen metric \( g^{(o)}_{\mu \nu}(x) \). The action has the form

\[
I_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ R - F_{ij}(\phi) \phi^i \phi^j - V(\phi) \right],
\]

where \( F_{ij}(\phi) \) and \( V(\phi) \) are target metric and potential of four scalar fields \( \phi^i(x) \). The constant \( \kappa \equiv 8\pi G \) stands for the gravitational coupling constant. The target metric \( F_{ij}(\phi) \) is constructed by replacing \( x^i \) with \( \phi^i \) in the expression

\[
F_{ij}(x) \equiv R_{ij}^{(o)}(x) - \frac{1}{2} V(x) g_{ij}^{(o)}(x),
\]

where \( R_{ij}^{(o)}(x) \) is Ricci tensor for the metric \( g_{ij}^{(o)}(x) \). The same replacement in an arbitrary function \( V(x) \) defines the potential \( V(\phi) \). This construction guarantees that

\[
\phi^i = x^i, \quad g_{\mu \nu} = g_{\mu \nu}^{(o)}
\]

is a solution of the field equations defined by equation (1). In what follows, the solution equation (3) will be referred to as vacuum. It is seen that physics of small perturbations of this vacuum allows the gauge condition \( \phi^i(x) = x^i \). Gauge fixed field equations depend on the metric alone.

In cosmology, the vacuum metric \( g_{\mu \nu}^{(o)} \) is chosen to be the background metric

\[
d s^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right).
\]

The scale factor \( a(t) \) is an unspecified function of time, and it will remain unspecified throughout most of the paper. In other words, I am looking for the general spherically symmetric
solution in an arbitrary background. The model is defined by determining $V(x)$ and $F_{ij}(x)$. While $V(x)$ is kept arbitrary, $F_{ij}(x)$ is determined by equation (2). One obtains

$$F_{00} = W - 2\dot{H}, \quad F_{0b} = 0, \quad F_{ab} = -a^2 W \delta_{ab},$$

(5)

where $H \equiv \dot{a}/a$ is the Hubble parameter, and $W$ is defined by

$$V \equiv 2 \left( W + \dot{H} + 3H^2 \right).$$

(6)

The ‘dot’ denotes time derivative. The target metric $F_{ij}(\phi)$ and the potential $V(\phi)$ are obtained by the substitution $x^i \rightarrow \phi^i$ in $F_{ij}(x)$ and $V(x)$.

In what follows, I shall consider the simplest case $W = 0$. Then, the target metric becomes degenerate, as $F_{00}$ remains the only nonzero component. Owing to its independence of spatial coordinates, the resulting action depends on one scalar field only. Precisely,

$$I_g = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \left[ R - F(\phi)\phi_\mu\phi^\mu - V(\phi) \right],$$

(7)

where $F \equiv F_{00}$ and $\phi \equiv \phi^0$. The action equation (7) defines a class of scalar–tensor theories parametrized by the scale factor $a(t)$. The target metric and potential are defined by the substitution $t \rightarrow \phi$ in

$$F(t) \equiv -2\dot{H}, \quad V(t) \equiv 2 \left( \dot{H} + 3H^2 \right).$$

(8)

More detailed construction of geometric sigma models with one scalar field can be found in [27].

3. Field equations

The complete action needed for considerations of this paper must include matter fields. It has the form

$$I = I_g + I_m,$$

(9)

where $I_g$ is geometric action equation (7), and $I_m$ stands for the action of matter fields. The matter Lagrangian is assumed to be that of the standard model of elementary particles minimally coupled to gravity. This means that direct coupling of matter to the scalar field is absent, which leaves us with the following field equations. First, equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^\phi_{\mu\nu} + \kappa T^m_{\mu\nu},$$

(10)

is obtained by varying the action equation (9) with respect to the metric. Tensors on the right-hand side of equation (10) stand for the stress–energy of the scalar field and matter fields, respectively. Specifically,

$$T^m_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta I_m}{\delta g^{\mu\nu}},$$

(11)

$$T^\phi_{\mu\nu} = G_{\mu\nu} - \frac{1}{2} \left( G^\rho_\rho + V \right) g_{\mu\nu},$$

(12)

where

$$G_{\mu\nu} \equiv F(\phi) \phi_\mu\phi_\nu.$$
The scalar field equation is obtained by varying the action equation (9) with respect to $\phi$. Owing to the absence of direct coupling of matter to the scalar $\phi$, the same equation is obtained by varying geometric action equation (7). Thus obtained scalar equation implies covariant conservation of the stress–energy tensor of the scalar field,

$$\nabla^{\mu} T_{\mu \nu}^\phi = 0. \tag{13}$$

In fact, the scalar equation is equivalent to equation (13), as only one out of these four equations is truly independent. Finally, matter field equations are obtained by varying the action equation (9) with respect to matter fields. Specific form of these equations is not known unless $I_m$ is specified. However, some information on the dynamics of matter fields can be obtained directly from equations (10) and (13). Indeed, the Bianchi identities imply covariant conservation of the right-hand side of equation (10). With the help of equation (13), one then obtains

$$\nabla^{\mu} T_{\mu \nu}^m = 0. \tag{14}$$

Thus, the two stress–energy tensors, $T_{\mu \nu}^\phi$ and $T_{\mu \nu}^m$, are independently covariantly conserved.

In what follows, matter fields are assumed to be localized in a point, which I choose to be $\vec{x} = 0$. Then, the field equations in the region $\vec{x} \neq 0$ reduce to those obtained from the geometric action equation (7). It is easily checked that they possess the vacuum solution

$$\phi = t, \quad g_{\mu \nu} = g_{\mu \nu}^{(0)},$$

where $g_{\mu \nu}^{(0)}$ is defined by equation (4). In this work, I shall consider spherically symmetric deviations from this vacuum, caused by the presence of a massive particle in $\vec{x} = 0$. As these deviations are expected to be small far from $\vec{x} = 0$, one is allowed to fix the gauge $\phi = t$. In this gauge, the scalar equation is identically satisfied, and we are left with equation (10). Using the definition equation (2), it is rewritten as

$$R_{\mu \nu} - R_{\mu \nu}^{(0)} - \frac{V}{2} \left( g_{\mu \nu} - g_{\mu \nu}^{(0)} \right) = 0. \tag{15}$$

This noncovariant equation carries the full content of the model in the gauge $\phi = t$. The residual coordinate transformations are

$$x^\alpha \rightarrow x^\alpha + \xi^\alpha(x), \quad t \rightarrow t.$$

They allow for an additional gauge fixing, which I choose to be $g_{0 \alpha} = 0$. The adopted gauge fixing conditions

$$\phi = t, \quad g_{0 \alpha} = 0 \tag{16}$$

leave us with time independent parameters $\xi^\alpha = \xi^\alpha(x)$.

In what follows, I shall search for spherically symmetric solutions of equation (15). The most general spherically symmetric metric in the gauge equation (16) has the form

$$g_{00} = \mu, \quad g_{0 \alpha} = 0, \quad g_{\alpha \beta} = \nu P_{\alpha \beta}^\parallel + \rho P_{\alpha \beta}^\perp, \tag{17}$$

where $P_{\alpha \beta}^\parallel$ and $P_{\alpha \beta}^\perp$ are parallel and orthogonal projectors on $\vec{x}$,

$$P_{\alpha \beta}^\parallel \equiv \frac{x^\alpha x^\beta}{r^2}, \quad P_{\alpha \beta}^\perp \equiv \delta_{\alpha \beta} - \frac{x^\alpha x^\beta}{r^2}. \tag{18}$$

and $\mu, \nu, \rho$ are functions of $r$ and $t$, only. The radius $r$ is defined by $r^2 = x^2 + y^2 + z^2$. Now, the field equations (15) are straightforwardly expressed in terms of $\mu, \nu, \rho$ and the scale factor $a$. For this, one uses equation (17) to calculate $R_{\mu \nu}$, and the vacuum metric equation (4) to
determine $R_{\mu\nu}^{(\omega)}$. The potential $V$ is defined by equation (8). Skipping the details of cumbersome calculations, here I display the final result. The ‘00’ component of equation (15) takes the form

$$
\frac{1}{4} \left[ \frac{\ddot{\mu}}{\mu} \left( \frac{\dot{\nu}}{\nu} + 2 \frac{\ddot{\nu}}{\rho} \right) - \left( \frac{\nu^2}{\nu^2} + 2 \frac{\rho^2}{\rho^2} \right) - 2 \left( \frac{\nu}{\nu} + 2 \frac{\rho}{\rho} \right)_0 \right] \\
+ \frac{\nu'}{\nu} \left( \frac{\mu'}{\mu} - \frac{\nu'}{\nu} - 2 \frac{\rho'}{\rho} \right) \right] - \frac{1}{2} \left[ \left( \frac{\mu'}{\mu} \right)' + 2 \frac{\rho'}{\rho} \right] \\
- \left( H + 3H^2 \right) \mu + 2 \dot{H} = 0,
$$

(19a)

and the three ‘0α’ components are all equivalent to the equation

$$
\frac{1}{2} \left( \frac{\dot{\mu}}{\mu} - \frac{\dot{\rho}}{\rho} \right) \left( \frac{\rho'}{\rho} + \frac{2}{\nu} \right) + \frac{1}{2} \frac{\mu'}{\mu} \left( \frac{\rho'}{\rho} - \frac{2}{\nu} \right)' = 0.
$$

(19b)

The ‘αβ’ components of equation (15) are shown to have the form $AP_{\alpha\beta}^{||} + BP_{\alpha\beta}^{⊥} = 0$. They are equivalent to two equations

$$
\frac{1}{2} \left( \frac{\dot{\mu}}{\mu} \right)_0 + \frac{\dot{\nu}}{\mu} \left( \frac{\mu}{\nu} - \nu + 2 \frac{\rho}{\rho} \right) + \frac{1}{2} \left( \frac{\mu'}{\mu} + 2 \frac{\rho'}{\rho} \right)' \\
- \frac{1}{4} \frac{\nu'}{\nu} \left( \frac{\mu'}{\mu} + 2 \rho' \right) + \frac{1}{2} \left[ \left( \frac{\mu'}{\mu} \right)^2 + 2 \left( \frac{\rho'}{\rho} \right)^2 \right] \\
- \frac{1}{r} \left( \frac{\nu'}{\nu} - 2 \frac{\rho'}{\rho} \right) + \left( H + 3H^2 \right) \rho = 0,
$$

(19c)

$$
\frac{1}{2} \left( \frac{\dot{\mu}}{\mu} \right)_0 + \frac{\dot{\rho}}{\mu} \left( \frac{\mu}{\nu} + \frac{\ddot{\nu}}{\nu} \right) - \frac{1}{2} \frac{2}{r} \left( 1 - \frac{\rho}{\nu} \right) - \frac{\rho'}{\rho} \left( \frac{\nu'}{\nu} - 1 \right) \left( \frac{\rho'}{\rho} + \frac{2}{\nu} \right) \\
- \frac{1}{r} \left[ \frac{2}{r} \left( 1 - \frac{\rho}{\nu} \right) - \frac{\rho'}{\rho} \right] + \frac{1}{4} \left( \frac{\mu'}{\mu} + \frac{\nu'}{\nu} \right) \left( \frac{\rho'}{\rho} + \frac{2}{\nu} \right) \\
+ \left( H + 3H^2 \right) \rho = 0.
$$

(19d)

To check if equation (19) are correctly derived, I have solved these equations in three simplest cases. First, it is easily verified that the vacuum $\mu = -1$, $\nu = \rho = a^2$ is a solution of equation (19) for every $a(t)$. Second, the Schwarzschild metric

$$\rho = 1, \quad \mu = -\frac{1}{\nu} = - \left( 1 - \frac{\ell}{r} \right)$$

is obtained as the general spherically symmetric solution in the flat background $a(t) = 1$. Finally, I examined de Sitter background $a(t) = e^{\omega t}$. As expected, the Schwarzschild-de Sitter solution

$$\rho = 1, \quad \mu = -\frac{1}{\nu} = - \left( 1 - \frac{\ell}{r} - \omega^2 r^2 \right)$$

is obtained. The integration constant $\ell$ stands for the Schwarzschild radius of the point source.

The field equation (19) remain unchanged by the action of the residual coordinate transformation $r \to r + \xi(r)$. With respect to this, the variables of the theory transform as
\[\begin{align*}
\delta_0 \mu &= -\xi \mu', \\
\delta_0 \nu &= -\xi \nu' - 2\xi' \nu, \\
\delta_0 \rho &= -\xi \rho' - \frac{2}{r}\xi \rho. 
\end{align*}\]

These transformations define the residual gauge symmetry of equation (19). In what follows, I shall search for the general solution of equation (19), in which \(a(t)\) remains unspecified. To this end, I shall consider the region far from the source, where weak field approximation can be used. This is because the unperturbed equation (19) are too complicated to be solved without any approximation. Thus, I define

\[\begin{align*}
\mu &\equiv -1 + \mu_1, \\
\nu &\equiv a^2 (1 + \nu_1), \\
\rho &\equiv a^2 (1 + \rho_1).
\end{align*}\]

The new fields \(\mu_1, \nu_1, \rho_1\) are assumed to be small, so that quadratic and higher order terms can be neglected. After a lengthy calculation, the linearized field equations are brought to the form

\[\begin{align*}
\frac{1}{r} \left[\nu_1 - (r\rho_1)'\right]' - H \mu_1' &= \mathcal{O}_2, \\
2a^2 \left[ H(\dot{\nu}_1 + 2\dot{\rho}_1) + (\dot{H} + 3H^2) \mu_1 \right] + \frac{2}{r} \left[ \nu_1 - (r\rho_1)' \right]' + \frac{2}{r^2} \left[ \nu_1 - (r\rho_1)' \right] &= \mathcal{O}_2, \\
a^2 \left[ (\nu_1 - \rho_1)_{,00} + 3H(\nu_1 - \rho_1)_{,0} \right] + \mu_1'' - \frac{1}{r} \left[ \mu_1 - \nu_1 + (r\rho_1)' \right]' - \frac{2}{r^2} \left[ \nu_1 - (r\rho_1)' \right] &= \mathcal{O}_2,
\end{align*}\]

\[\begin{align*}
\dot{\nu}_1 + 2\dot{\rho}_1 + \mu_1 + \left( 6H + \frac{\ddot{H}}{H} \right) \mu_1 &= \mathcal{O}_2,
\end{align*}\]

where \(\mathcal{O}_2\) stands for quadratic and higher order terms in \(\mu_1, \nu_1, \rho_1\). The residual gauge symmetry of the linearized theory is the linearized version of equation (20). One finds

\[\begin{align*}
\delta_0 \mu_1 &= \mathcal{O}_2, \\
\delta_0 \nu_1 &= -2\xi' + \mathcal{O}_2, \\
\delta_0 \rho_1 &= -\frac{2}{r} \xi + \mathcal{O}_2.
\end{align*}\]

In the next section, I shall search for the general solution of equation (22).

### 4. Solution

The solution of equation (22) is searched for in the form of a power series. Specifically, I use the decomposition
\[ 
\mu_1 = \sum_{n=0}^{\infty} \alpha_n t^{n-1}, \\
\nu_1 = \sum_{n=0}^{\infty} \beta_n t^{n-1}, \\
\rho_1 = \sum_{n=0}^{\infty} \gamma_n t^{n-1}, 
\]

where \( \alpha_n(t), \beta_n(t), \gamma_n(t) \) are time dependent coefficients. The substitution of equations (24) into (22) yields the following set of ordinary differential equations. Equation (22a) becomes

\[ 
A_n \equiv \dot{\beta}_n - n \dot{\gamma}_n - (n - 1) H \alpha_n = 0 \tag{25} 
\]

for all \( n \geq 0 \). Equation (22b) leads to

\[ 
\beta_1 - \gamma_1 = 0, \tag{26a} 
\]

\[ 
B_n \equiv a^2 H \left[ \dot{\beta}_n + 2 \dot{\gamma}_n + \left( 3H + \frac{\dot{H}}{H} \right) \alpha_n \right] + (n + 2) \left[ \dot{\beta}_{n+2} - (n + 2) \dot{\gamma}_{n+2} \right] = 0. \tag{26b} 
\]

From equation (22c), one finds

\[ 
\alpha_0 - \beta_0 = 0, \tag{27a} 
\]

\[ 
C_n \equiv a^2 \left[ \dot{\beta}_n - \dot{\gamma}_n + 3H \left( \dot{\beta}_n - \dot{\gamma}_n \right) \right] + (n - 1) \left[ \dot{\beta}_{n+2} - (n + 2) \dot{\gamma}_{n+2} \right] + (n - 1)(n + 1) \alpha_{n+2} = 0, \tag{27b} 
\]

and from equation (22d)

\[ 
D_n \equiv \ddot{\beta}_n + 2 \ddot{\gamma}_n + \dddot{\alpha}_n + \left( 6H + \frac{\dot{H}}{H} \right) \alpha_n = 0. \tag{28} 
\]

The obtained ordinary differential equations are not all mutually independent. Indeed, a straightforward calculation shows that the identity

\[ 
(n - 1) \left[ (n + 2) a \alpha_{n+2} - (a \beta_n)_0 + a^3 \dot{H} D_n \right] + (n + 2) a \beta_n C_n - 3H (a^2 A_n)_0 \equiv 0 \tag{29} 
\]

holds true. It implies that equation (27b) should be abandoned, as it follows from equations (25), (26b) and (28). The remaining independent equations are then readily solved. In the first step, we consider the case \( n = 0 \). It leads to

\[ 
\alpha_0 = \beta_0 = \frac{\ell}{a}, \tag{30} 
\]

where \( \ell \) is a constant with the dimension of length. In the second step, the remaining equations are rewritten in the form of recurrent relations. They read

\[ 
\dot{\beta}_n + 2 \dot{\gamma}_n = - \left[ \dot{\alpha}_n + \left( 6H + \frac{\dot{H}}{H} \right) \alpha_n \right], \tag{31a} 
\]

\[ 
\dot{\beta}_1 - \dot{\gamma}_1 = 0. \tag{31b} 
\]

\[ 
\dot{\beta}_n - \dot{\gamma}_n = - \left[ \dddot{\alpha}_n + \left( 6H + \frac{\dot{H}}{H} \right) \alpha_n \right], \tag{31c} 
\]

\[ 
\dot{\beta}_1 = 0. \tag{31d} 
\]
\[
\beta_{n+2} = (n+2)^2 \gamma_{n+2} = \frac{a^2}{n+2} \left[ \dot{\alpha}_n + \left( 3H - \frac{\ddot{H}}{H} + \frac{\dot{H}}{H} \right) \alpha_n \right],
\]
(31b)

\[
\alpha_{n+2} = \frac{1}{n+1} \frac{1}{H} \left\{ \frac{a^2}{n+2} \left[ \dot{\alpha}_n + \left( 3H - \frac{\ddot{H}}{H} + \frac{\dot{H}}{H} \right) \alpha_n \right] \right\},
\]
(31c)

and

\[
\gamma_1 = \beta_1.
\]
(32)

These equations determine the coefficients \(\alpha_n\), \(\beta_n\) and \(\gamma_n\). The only undetermined coefficient is \(\alpha_1\). The constant \(\ell\) and the function \(\alpha_1(t)\) are the unique free parameters of the theory. This is because the free integration constants of equation (31a) turn out to be pure gauge. The proof goes as follows. First, the power expansion

\[
\xi(r) \equiv \sum_{n=0}^{\infty} \xi_n r^n
\]
is used to bring the residual gauge transformations equation (23) to the form

\[
\delta_0 \alpha_n = 0, \quad \delta_0 \beta_n = -2n \xi_n, \quad \delta_0 \gamma_n = -2 \xi_n.
\]
(33)

Second, equation (31a) is solved for \(\beta_n + 2 \gamma_n\). One obtains

\[
\beta_n + 2 \gamma_n = - \int_0^t \left[ \dot{\alpha}_n + \left( 6H - \frac{\ddot{H}}{H} \right) \alpha_n \right] dt + c_n,
\]
(34)

where \(c_n\) are free integration constants. Finally, the transformation law

\[
\delta_0 c_n = -2 (n+2) \xi_n
\]
is derived by applying the gauge transformations equation (33) to the general solution equation (34). It is seen that the integration constants \(c_n\) can all be gauged away, leaving us with no residual gauge symmetry. The unique free parameters of the gauge fixed theory are \(\ell\) and \(\alpha_1\).

Physical meaning of the parameters \(\ell\) and \(\alpha_1\) will be addressed in the next section, where their influence on the gravitational force is examined. Here, we anticipate that \(\ell\) measures the Schwarzschild radius, or equivalently, mass of the gravitational source, whereas \(\alpha_1\) stems from the additional scalar degree of freedom. Indeed, \(\ell = 0\) implies the absence of \(1/r\) terms in the expansion equation (24), which leaves us with the gravitational force without the familiar Newtonian contribution. Nevertheless, the gravitational force does not vanish. Instead, it is proportional to the arbitrary function \(\alpha_1(t)\), which indicates the presence of an additional degree of freedom. In the next section, I shall examine how the choice of \(\alpha_1\) influences geometry of the point source.

Before we go on, let me derive the formula for gravitational acceleration. One starts with the geodesic equation

\[
\frac{d\upsilon^\mu}{ds} + \Gamma_{\nu\rho}^\mu u^\nu u^\rho = 0,
\]
where \(u^\mu \equiv dx^\mu/ds\). The goal is to calculate trajectories of stars in galaxies. For this purpose, the non-relativistic approximation is known to work well. Thus, the geodesic equation for the metric equation (17) is brought to the form

\[
\frac{dr^\alpha}{dt} = \frac{\mu'_1 \chi^\alpha}{2a^2 \bar{r}} + O(v),
\]
(35)
where $v^α \equiv dx^α/dt$. What remains to be done is to rewrite this equation in terms of physical distance and velocity. In the approximation we work with, the physical distance is defined by the background metric equation (4). It tells us that

$$dr_{\text{phys}} = a(t)dr,$$

which should be integrated out to give the global physical distance $r_{\text{phys}}$. As meaningful notion of global distance is known to require static geometry, we shall restrict to small time intervals in which $a(t)$ remains practically unchanged. Then, one finds

$$r_{\text{phys}} \approx a(t_*) r$$

for all $t$ in the vicinity of $t_*$. One can think of $t_*$ as the epoch the observed galaxy lives in. For closest galaxies, it is the present epoch $t_0$. The time $|t - t_*|$, on the other hand, is related to how long it takes the light to travel across the galaxy. Typically, the expansion of the Universe during this time is negligible. The physical velocity $\vec{v}_{\text{phys}} \equiv dr_{\text{phys}}/dt$ is derived straightforwardly. One finds

$$\vec{v}_{\text{phys}} = a(t_*) \vec{v}.$$

The time $t_*$ in the above formulas is a fixed time. It should be emphasized, however, that $t_*$ is allowed to have different values, depending on what specific galaxy is considered. In what follows, I shall replace $t_*$ with more common $t$. One should only be careful not to do this during actual calculations. The replacement $t_* \rightarrow t$ is reserved for final expressions.

5. Example

Let me consider one specific example. For this purpose, I choose one of the geometric sigma models defined in section 2. As these are parametrized by their background geometries, the choice is made by specifying the scale factor $a(t)$. In this section, I choose

$$a = \ln \left[2\cosh(e^{\omega t}) - 1\right].$$

The graph of this scale factor is shown in figure 1. It represents a Universe that expands from a higher curvature de Sitter space at $t \rightarrow -\infty$ to a lower curvature de Sitter space at $t \rightarrow \infty$. The constant $\omega$ is a free parameter with the dimension of mass. The scale factor equation (37) is a solution of the sigma model equation (7) in which the target metric $F(\phi)$ and the potential $V(\phi)$ are defined by the replacement $t \rightarrow \phi$ in the expressions equation (8). As their explicit form is not needed for the forthcoming analysis, I choose not to display them here.
5.1. Gravitational acceleration

Let me now solve the recurrent relations equation (31c). To make it easier, I adopt two additional simplifications. First, I restrict my considerations to late times, where $a(t)$ is well approximated by the exponential function. This is the late de Sitter phase of the cosmological evolution, where our current epoch belongs to. Second, the parameter $\alpha_1(t)$ is chosen to be

$$\alpha_1 \equiv q,$$

where $q$ is unspecified dimensionless constant. With these assumptions, the coefficients $\alpha_n$ are found to take the following approximate values:

$$\begin{align*}
\alpha_0 &\approx \frac{\ell}{a}, \\
\alpha_1 &\equiv q, \\
\alpha_2 &\approx -\ell \omega^2 a^2, \\
\alpha_3 &\approx -\frac{1}{2} q \omega^2 a^3, \\
\alpha_4 &\approx \frac{5}{12} \ell \omega^4 a^4, \\
\alpha_5 &\approx \frac{1}{30} q \omega^4 a^5, \\
\alpha_6 &\approx -\frac{1}{2} \ell \omega^6 a^6, \\
\alpha_7 &\approx -\frac{9}{280} q \omega^6 a^9 \\
\end{align*}$$

and so on. This pattern gives us two separate formulas for even and odd coefficients. The even coefficients are collected in

$$\begin{align*}
\alpha_0 &\approx \frac{\ell}{a}, \\
\alpha_{2n} &\approx \ell \omega^{2n} a^{3n-1} c_n \\
\end{align*}$$

for all $n \geq 1$. The numerical coefficients $c_n$ have the values

$$c_n \equiv \frac{(-1)^n}{(2n)!} \prod_{k=1}^{n} (3k - 1).$$

The odd coefficients are given by

$$\alpha_{2n+1} \approx q \omega^{2n} a^{3n} d_n$$

with

$$d_n \equiv \frac{(-3)^n}{n!} \frac{n!}{(2n + 1)!}.$$
These hold true for all $n \geq 0$. With the known $\alpha_n$ coefficients, it is straightforward to calculate the $\mu$ component of the metric, and subsequently, the gravitational acceleration equation (36). The result is most conveniently expressed in terms of

$$x \equiv \omega r a \sqrt{a},$$

which serves as a dimensionless measure of spatial distance. Then, the gravitational acceleration becomes

$$g = \frac{\ell}{2} \omega^2 a \left( - \frac{1}{x^2} + J_1(x) \right) + q \omega \sqrt{a} J_2(x),$$

where

$$J_1 \equiv \sum_{n=1}^{\infty} (2n - 1) c_n x^{2n-2}, \quad J_2 \equiv \sum_{n=1}^{\infty} n d_n x^{2n-1}. \quad (43)$$

The leading term in equation (42) is recognized as the Newtonian gravitational acceleration. Indeed, the latter is given by

$$g_N \equiv - \frac{GM}{r_{\text{phys}}},$$

where $G$ is the gravitational constant and $M$ denotes the source mass. It is straightforward to verify that $g_N = -\ell \omega^2 a / 2x^2$, once the parameter $\ell$ is identified with the Schwarzschild radius. In terms of the source mass,

$$\ell \equiv 2GM. \quad (45)$$

Thus, the gravitational acceleration is a sum

$$g = g_N + \Delta g,$$

where

$$\Delta g \equiv \frac{\ell}{2} \omega^2 a J_1(x) + q \omega \sqrt{a} J_2(x).$$

The first term in $\Delta g$ is a universal modification that does not depend on the scalar degree of freedom. It remains the same irrespective of the choice of the free parameter $\alpha_1(t)$. The second, on the other hand, changes whenever the scalar field initial conditions are changed. The graphs of $J_1$ and $J_2$ are displayed in figure 2. It is seen that their contribution to the gravitational acceleration equation (42) can be both, positive and negative. At small distances, they increase the attractive Newtonian force whenever the free parameter $q$ is positive. At large distances, however, the gravitational force can become repulsive, as expected in the expanding Universe.

5.2. Comparison with MOND

In what follows, I shall examine how close to MOND the modified gravitational force can be. Let me start with the MOND formula

$$(\Delta g)_{\text{MOND}} = - \frac{\sqrt{GMg_0}}{r_{\text{phys}}},$$

(46)
where $g_0$ is a constant with the dimension of acceleration [9–11]. The astronomical observations indicate that the value of $g_0$ is close to $H_0/2\pi$, where $H_0$ is the Hubble parameter of the present epoch [29–36]. In our example, $H_0$ can be identified with the parameter $\omega$. Indeed, the late time behavior $a \sim e^{\omega t}$ indicates that $H \sim \omega$. This is the vacuum value of $H$, as no ordinary matter is considered. The inclusion of ordinary matter, however, has little influence on the measured value of the Hubble parameter. Because of this, I adopt the approximation

$$g_0 \approx \frac{\omega}{2\pi}. \quad (47)$$

Let me now examine the equality

$$\Delta g = (\Delta g)_{\text{MOND}}.$$  

With the adopted approximations, it reduces to

$$\sqrt{\pi \ell \omega a} = -\frac{1}{(J_1 + pJ_2)x}, \quad (48)$$

where

$$p \equiv \frac{2q}{\ell \omega \sqrt{a}}. \quad (49)$$

The fact that $t$ in $a(t)$ is fixed by the choice of the observed galaxy makes the parameter $p$ a substitute for the free constant $q$. In what follows, equation (48) will be solved for a variety of numerical values of $p$. Precisely, I search for the range of distances for which $J_1 + pJ_2$ behaves as $1/x$. This is done by a graphical method. In the first step, the graphs of the functions $(J_1 + pJ_2)x$ are drawn for different values of $p$. Then, flat portions of these curves are identified. The outcome of this analysis is shown in table 1. In the first column, the value of the free parameter $p$ defines which function $(J_1 + pJ_2)x$ is examined. As it turns out, each of these functions can be approximated by a constant in some interval of distances. These intervals are shown in the second column. The corresponding constant values of the functions $(J_1 + pJ_2)x$ are displayed in the third column. The last two columns are reserved for quantities that are straightforwardly derived from equations (48) and (49). The error that appears when portions of curves are approximated by constants is kept lower than $2\%$. Considering the typical precision of astronomical measurements, this is a very good approximation.

In what follows, I shall inspect the collected data more closely. Let me start with the last column of table 1. It tells us that the expression $q/\sqrt{\ell \omega}$ is practically independent of $\ell$. Precisely,
for all \( \ell \) that satisfy \( \ell \omega a < 0.1 \). Only then, the gravitational acceleration equation (42) has regions with MOND-like behavior. The range of \( \ell \) for which \( q(\ell) \) has the form equation (50) depends on time. Indeed, the time \( t \) in \( a = a(t) \) varies from one observed galaxy to another. For closest galaxies, \( t \) takes the present value \( t_0 \). It is determined by the current values of the Hubble and deceleration parameters, and so is the parameter \( \omega \). Instead of repeating these well known calculations, let me summarize the results. First, the parameter \( \omega \) is identified with the current value of the Hubble parameter. This is because the late time approximation I use in this paper implies the exponential law \( a \sim e^{\omega t} \). A rough estimation is that \( \omega \approx 10^{-10} \text{ yr}^{-1} \).

Second, the present time \( t_0 \), apart from belonging to the late de Sitter phase of the cosmological evolution, is freely chosen. This is possible because the relevant formulas have one free integration constant. This integration constant is related to the freedom of defining the origin of time. (The known result \( t_0 \approx 13 \text{ Gyr} \) obtained when the origin \( t = 0 \) is associated with the initial singularity.) In accordance with the assumed late time approximation \( (\omega t \gg 1) \), the present time \( t_0 \) is chosen to be

\[
\omega t_0 = 23.
\]

It implies \( a_0 \approx 10^{10} \), so that the inequality \( \ell \omega a < 0.1 \) turns into

\[
\ell < 10^{-1} \text{ ly}
\]  

(51)

at present time. The Schwarzschild radius of \( 10^{-1} \text{ ly} \) is known to correspond to the largest galaxies in the observed Universe. Thus, the restriction equation (51) does not rule out any of the observationally interesting astronomical objects in our vicinity.

Further inspection of data in table 1 reveals that the obtained MOND behavior is not as universal as the original MOND suggests. Indeed, it is only a small interval of distances far from the point source where the modified gravitational law equation (42) reduces to MOND. These distances are shown in the second column of table 1. For not too large source masses, \( x \in (1.15, 1.55) \), which corresponds to

\[
1.15 \times 10^5 \text{ ly} < r_{\text{phys}} < 1.55 \times 10^5 \text{ ly}.
\]

(52)

For comparison, the largest galaxies in the observable Universe are about \( 10^5 \text{ ly} \) in diameter. Thus, the stars in small galaxies \((10^3-10^4 \text{ ly})\) do not feel MOND regime. As an illustration, the gravitational acceleration equation (42) for \( \ell \approx 10^{-6} \text{ ly} \) is depicted in figure 3. This value

\[
q \approx 0.9 \sqrt{\ell \omega}
\]

(50)
of $\ell$ corresponds to the Schwarzschild radius of the black hole in the center of Milky Way. (The unit value of $g$ in the graph is $10^{-16} \text{ yr}^{-1}$.) It is seen that modified gravitational acceleration $g$ generally differs from the Newtonian acceleration $g_N$. This is certainly true in the interval $x \in (0.5, 10)$, where the magnitude of $g$ considerably exceeds that of $g_N$. It should be noted, however, that this modification has little to do with MOND. Indeed, MOND behavior is found only in the small interval $x \in (1.15, 1.55)$. For comparison, the diameter of Milky Way is $x \sim 1$. At small distances, $x \ll 1$, the modified gravitational force takes the familiar Newtonian form. As for $x \gg 1$, the gravitational acceleration becomes positive. It corresponds to the repulsive gravitational force that drives the expansion of the Universe.

In conclusion, I demonstrated that scalar–tensor theories may indeed have solutions that mimic MOND. The problem is that these solutions do not cover all the distances that the original MOND suggests. Of course, this is not a generic feature of the class of models considered in this paper. Hopefully, a different geometric sigma model could have a better behaving solution. Another possibility to get closer to MOND is to make a different choice of the free parameter $\alpha_1(t)$. Instead of the simplest choice $\alpha_1 = \text{const.}$, one may try $\alpha_1$ with a nontrivial time dependence. The problem is that one has no clue how to choose $\alpha_1(t)$ to obtain the desired result. In the next section, I shall do an alternative analysis of the field equations, with the idea to replace the free parameter $\alpha_1(t)$ with some standard initial conditions. Some of these initial conditions will be shown to allow for exact MOND at a fixed moment of time.

6. Alternative approach

Let me consider equation (22) once again. It is seen that equations (22a) and (22b) can be expressed in terms of $\mu_1$ and $\nu_1 - (r \rho_1)'$, only. Thus, I define the new variables

$$u \equiv rH\mu_1, \quad v \equiv r \left[ \nu_1 - (r \rho_1)' \right].$$

In terms of these new variables, equations (22a) and (22b) become

$$\ddot{v} = ru' - u, \quad \dot{u} = \frac{1}{ra^2} v' - Qu,$$

where

$$Q \equiv 3H - 2 \frac{\dot{H}}{H} + \frac{\ddot{H}}{H}.$$
This is a system of two first order partial differential equations, solved for \( \dot{u} \) and \( \dot{v} \). Hence, the solutions are naturally parametrized by the initial conditions \( u(t_0, r) = u_0(r) \) and \( v(t_0, r) = v_0(r) \). The problem is that \( u_0(r) \) and \( v_0(r) \) cannot be arbitrarily chosen. This is because the variables \( u \) and \( \dot{v} \) are subject to more than just two equation (54). As a consequence, their initial values are partially restricted. It is seen from the result of section 4 that only odd coefficients in the power expansion equation (24) carry some arbitrariness. Indeed, these are proportional to the arbitrary function \( \alpha_1(t) \) and its derivatives, whereas even coefficients are not. In what follows, I shall consider strictly odd functions of \( r \). Then, the allowed initial conditions have the form

\[
\begin{align*}
    u_0 &= \sum_{n=0}^{\infty} a_n r^{2n+1}, \quad \dot{u}_0 = \sum_{n=0}^{\infty} b_n r^{2n+3}, \\
    v_0 &= \sum_{n=0}^{\infty} b_n r^{2n+3},
\end{align*}
\]

where \( a_n \) and \( b_n \) take arbitrary values. (The absence of linear term in the power expansion of \( v_0 \) is a consequence of equation (32).) The arbitrariness of \( a_n \) and \( b_n \), on the other hand, implies some freedom in choosing the initial values of \( u \) and \( \dot{u} \). Precisely, the odd parts of the functions \( u(t_0, r) \) and \( \dot{u}(t_0, r) \) can be freely chosen. This can be utilized in the second order differential equation for the variable \( u \), which is easily derived from equation (54). One finds

\[
\ddot{u} + (Q + 2H)\dot{u} + (\dot{Q} + 2HQ)u = \frac{1}{a^2}u''.
\]

The partial differential equation (56) is too complicated to be solved analytically. Numerical calculations, on the other hand, require additional adaptations. First, all the variables and coefficients should be made dimensionless. The easiest way to achieve this is to fix the system of units. Let me choose \( \omega = 1 \).

In this system of units, the unit distance is the Hubble distance \( r_H \equiv 1/\omega \). Its approximate value is \( 10^{10} \) ly. In accordance with the adopted natural units \( c = \hbar = 1 \), the corresponding approximate value of the unit time is \( 10^{10} \) years. Second, the initial conditions should be specified. In this section, I choose the present epoch \( t_0 \) to play the role of the initial time. The ideal choice of the initial value of \( u \) would be

\[
u_0 = \frac{\ell}{a_0} - \sqrt{\frac{\ell}{\pi}} \ln r + \text{const.},
\]

as it implies exact MOND at \( t = t_0 \). Unfortunately, the choice equation (58) is not possible. This is because the even part of equation (58) does not agree with the result of section 4. The odd part of \( u_0 \), however, can be freely chosen. In accordance with this, I adopt

\[
u_0 = r \ln r, \quad \dot{u}_0 = 0.
\]

The choice equation (59) differs from the ideal choice equation (58) in two respects. First, the Newtonian term \( \ell/a_0 \) in equation (58) is missing in equation (59). However, this term has negligible influence on the flat galactic curves we are interested in. Indeed, if we restrict to regions far from the gravitational source, the initial conditions equation (59) are practically indistinguishable from equation (58). The second difference is that \( u_0 \) of equation (59) lacks the constant \(-\sqrt{\ell/\pi}\). This constant, however, has been deliberately omitted for simplicity. It can be restored later because the general solution of equation (56) is determined only up to a multiplicative constant. Finally, let me explain why the function \( r \ln r \) is considered odd. The reason is the regularization scheme that I use to replace the singular function \( \ln r \) with
everywhere regular, even function \( \frac{1}{2} \ln (r^2 + \varepsilon^2) \). At large distances, or small values of \( \varepsilon \), the latter is well approximated by \( \ln r \).

To recapitulate, I have demonstrated that it is always possible to have MOND at a fixed moment of time. It is achieved by imposing proper initial conditions, such as equation (59). What one should check is how this result evolves with time. Let me solve equation (56) numerically. In the first step, one specifies the support of the function \( u(t, r) \). This is done as follows. First, one determines the time interval in which physically relevant observations are made. It is well known that flat galactic curves are detected in most of the surrounding galaxies, including very distant ones. I am talking about distances up to \( 10^8 \) ly. Thus, the time coordinate \( t \) should go at least \( 10^8 \) years to the past. At the same time, the closest of the observed flat galactic curves is about \( 10^5 \) ly far from us. In the adopted system of units, this implies

\[
-10^{-2} < t - t_0 < -10^{-5}.
\]  

(60a)

The range of \( r \) coordinate is related to the radii of the observed galactic curves. In terms of physical distance, this range is defined by \( r_{\text{phys}} \in (10^2, 10^5) \) ly, which leads to

\[
10^{-8} < r a_0 < 10^{-5}
\]  

(60b)

in the system of units \( \omega = 1 \). The exact numerical values are obtained when the present time \( t_0 \), and the scale factor \( a(t) \) are specified. As an example, I shall consider the scale factor

\[
a = \frac{e^t}{1 + e^{-16t}}.
\]  

(61)

Its graph is shown in figure 4. The present time \( t_0 \) must belong to the late de-Sitter phase of the given geometry. This leads me to choose

\[ t_0 = 1. \]

The support of the function \( u(t, r) \) is then defined by

\[ t \in (0.9, 1), \quad r \in (10^{-8}, 10^{-5}). \]

This region of \( t-r \) plane does not exactly coincide with equation (60), but is more suitable for numerical calculations.

The solution of equation (56) with the initial conditions (59) is easily seen to have the structure

\[ u = f_0 r \ln r + \sum_{n=1}^{\infty} \frac{f_n}{r^{2n-1}}, \]

where \( f_n = f_n(t) \) for all \( n \geq 0 \). As a consequence, the partial differential equation (56) is rewritten as the system of ordinary differential equations

\[
\ddot{f}_0 + (Q + 2H) \dot{f}_0 + (\dot{Q} + 2QH) f_0 = 0,
\]

\[
\ddot{f}_1 + (Q + 2H) \dot{f}_1 + (\dot{Q} + 2QH) f_1 = \frac{1}{a^2} f_0,
\]

\[
\ddot{f}_n + (Q + 2H) \dot{f}_n + (\dot{Q} + 2QH) f_n = \frac{(2n-3)(2n-2)}{a^2} f_{n-1}
\]

for all \( n \geq 2 \). The corresponding initial conditions are
\[
\begin{align*}
f_0(1) &= 1, \quad \dot{f}_0(1) = 0, \\
\dot{f}_n(1) &= 0, \quad \ddot{f}_n(1) = 0
\end{align*}
\]
for all \( n \geq 1 \). The numerical solution for the first several functions \( f_n(t) \) is obtained straightforwardly. As it turns out, \( f_n(t) \) are well approximated by

\[
f_n \approx \kappa_n (1 - t)^{2n}
\]

in the interval \( t \in (0.9, 1) \). The constants \( \kappa_n \) take values

\[
\begin{align*}
\kappa_0 &= 1, \\
\kappa_1 &= 6 \times 10^{-2}, \\
\kappa_2 &= 1.4 \times 10^{-3}, \\
\kappa_3 &= 7.5 \times 10^{-5}, \\
\kappa_4 &= 3.7 \times 10^{-6}, \\
\kappa_5 &= 5 \times 10^{-7}, \\
\kappa_6 &= 5 \times 10^{-8}, \\
\kappa_7 &= 5 \times 10^{-9}, \\
&\ldots
\end{align*}
\]

With these results, the function \( u(t, r) \) can be rewritten as

\[
u \approx r \ln r + \frac{1 - t}{100} \sum_{n=0}^{\infty} \frac{\tilde{\kappa}_n}{(2n + 1)!} \left( \frac{1 - t}{r} \right)^{2n+1},
\]

where \( \tilde{\kappa}_n \) take values

\[
\tilde{\kappa}_n = 6, 1, 1, 2, 20, 200, 3000, \ldots
\]

Finally, one can calculate the gravitational acceleration \( g \). As seen from equations (36) and (53), the gravitational acceleration is proportional to \( (u/r)' \), so that one obtains

\[
g \propto \frac{1}{r} + \frac{t - 1}{(10r)^2} \sum_{n=0}^{\infty} \frac{\tilde{\kappa}_n}{(2n + 1)!} \left( \frac{1 - t}{r} \right)^{2n+1},
\]

where \( \tilde{\kappa}_n \) take values

\[
\tilde{\kappa}_n = 12, 4, 6, 16, 200, 2400, 42000, \ldots
\]

To estimate the value of \( g \), note that \( \tilde{\kappa}_n > 1 \) implies
It can be checked that

\[
\frac{1 - t}{r} > 10
\]

for all the observed galaxies in our neighborhood. Owing to this,

\[
g \propto \frac{1}{r} F \left( \frac{1 - t}{r} \right),
\]

where

\[
F \left( \frac{1 - t}{r} \right) > \frac{1 - t}{r} \sinh \frac{1 - t}{r}.
\]

It is seen that the gravitational law rapidly changes with time. In fact, the change is so rapid that the earlier found possibility to have exact MOND at the initial moment of time is completely disqualified. Indeed, even the nearest galaxies are time shifted with respect to the present time. If MOND held true in our Milky Way, it would definitely be lost in the surrounding galaxies. In fact, a quick look at equation (62) tells us that MOND can be found only in the region $\frac{1 - t}{r} \sim 1$. This rules out all but our own galaxy.

### 7. Concluding remarks

I have shown in this paper that geometric sigma models with one scalar field can indeed have MOND-like solutions. Unfortunately, the range of distances where the gravitational law reduces to MOND turns out to be much smaller than needed. This is an outcome of the example considered in section 5. Of course, one can repeat the analysis with another geometric sigma model, or with another choice of the free parameter $\alpha_1(t)$. While this may result in a better behaved gravitational law, it is not clear how exactly one can achieve it. Indeed, one can not possibly know which $\alpha_1(t)$ leads to the desired result.

Further development has been made in section 6, where $\alpha_1(t)$ has been replaced with some standard initial conditions. Precisely, I have demonstrated that initial conditions for even part of the metric component $g_{00}$ can be freely chosen. As a consequence, MOND behavior can always be achieved at a particular instance of time. Unfortunately, its time evolution turns out to be so rapid that exact MOND is quickly lost. In particular, the desired MOND behavior is preserved only in our own galaxy.

To summarize, the class of geometric sigma models with one scalar field can hardly be responsible for the appearance of MOND. Even if it can, there remains an important question of how fine tuned initial conditions can be avoided. As I have already explained, the possibility to obtain MOND relies a great deal on the extra scalar degree of freedom. This is because, unlike vector and tensor modes, the scalar mode is not frozen by the spherically symmetric ansatz. Obviously, this helps the search for MOND, but it leaves us with the problem of fine tuned initial conditions. In this respect, I believe that scalar-vector-tensor theories are worthy of investigation. A nice class of such models is the class of geometric sigma models with four scalar fields [28]. The particle spectrum of these models has been proven to consist of 2 scalar, 2 vector and 2 tensor degrees of freedom, rather than 2 tensor and 4 scalar ones. If vector
degrees of freedom happen to be responsible for MOND, no fine tuning problem would be left behind. The analysis of these models will be presented elsewhere.

Acknowledgments

This work is supported by the Serbian Ministry of Education, Science and Technological Development, under Contract No. 171031.

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