STUDIES ON THE EQUATIONS OF INCE’S TABLE

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Abstract. We study the phase space of the equations of Ince’s table from the viewpoint of its accessible singularities and local index.

1. Introduction

In 1979, K. Okamoto constructed the spaces of initial conditions of Painlevé equations, which can be considered as the parametrized spaces of all solutions, including the meromorphic solutions (see [13]).

In this paper, we study the phase space of the equations of Ince’s table (see [4]) from the viewpoint of its accessible singularities and local index.

2. Accessible singularity and local index

Let us review the notion of accessible singularity. Let $B$ be a connected open domain in $\mathbb{C}$ and $\pi : W \rightarrow B$ a smooth proper holomorphic map. We assume that $\mathcal{H} \subset W$ is a normal crossing divisor which is flat over $B$. Let us consider a rational vector field $\tilde{v}$ on $W$ satisfying the condition

$$\tilde{v} \in H^0(W, \Theta_W(-\log \mathcal{H})(\mathcal{H})).$$

Fixing $t_0 \in B$ and $P \in W_{t_0}$, we can take a local coordinate system $(x_1, \ldots, x_n)$ of $W_{t_0}$ centered at $P$ such that $\mathcal{H}_{\text{smooth}}$ can be defined by the local equation $x_1 = 0$. Since $\tilde{v} \in H^0(W, \Theta_W(-\log \mathcal{H})(\mathcal{H}))$, we can write down the vector field $\tilde{v}$ near $P = (0, \ldots, 0, t_0)$ as follows:

$$\tilde{v} = \frac{\partial}{\partial t} + g_1 \frac{\partial}{\partial x_1} + \frac{g_2}{x_1} \frac{\partial}{\partial x_2} + \cdots + \frac{g_n}{x_1} \frac{\partial}{\partial x_n}.$$

This vector field defines the following system of differential equations

$$\frac{dx_1}{dt} = g_1(x_1, \ldots, x_n, t), \quad \frac{dx_2}{dt} = \frac{g_2(x_1, \ldots, x_n, t)}{x_1}, \ldots, \quad \frac{dx_n}{dt} = \frac{g_n(x_1, \ldots, x_n, t)}{x_1}.$$

Here $g_i(x_1, \ldots, x_n, t), \ i = 1, 2, \ldots, n$, are holomorphic functions defined near $P = (0, \ldots, 0, t_0)$.

Definition 2.1. With the above notation, assume that the rational vector field $\tilde{v}$ on $W$ satisfies the condition

$$\text{(A)} \quad \tilde{v} \in H^0(W, \Theta_W(-\log \mathcal{H})(\mathcal{H})).$$

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We say that \( \tilde{\nu} \) has an \textit{accessible singularity} at \( P = (0, \ldots, 0, t_0) \) if
\[
x_1 = 0 \text{ and } g_i(0, \ldots, 0, t_0) = 0 \text{ for every } i, \ 2 \leq i \leq n.
\]

If \( P \in \mathcal{H}_{\text{smooth}} \) is not an accessible singularity, all solutions of the ordinary differential equation passing through \( P \) are vertical solutions, that is, the solutions are contained in the fiber \( \mathcal{W}_0 \) over \( t = t_0 \). If \( P \in \mathcal{H}_{\text{smooth}} \) is an accessible singularity, there may be a solution of (1) which passes through \( P \) and goes into the interior \( \mathcal{W} - \mathcal{H} \) of \( \mathcal{W} \).

Here we review the notion of \textit{local index}. Let \( \nu \) be an algebraic vector field with an accessible singular point \( \tilde{p} = (0, \ldots, 0) \) and \( (x_1, \ldots, x_n) \) be a coordinate system in a neighborhood centered at \( \tilde{p} \). Assume that the system associated with \( \nu \) near \( \tilde{p} \) can be written as
\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n
\end{pmatrix} &= \frac{1}{x_1} \begin{pmatrix}
a_{11} & 0 & 0 & \cdots & 0 \\
a_{21} & a_{22} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} & 0 \\
a_{n1} & a_{n2} & \cdots & a_{n(n-1)} & a_{nn}
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n
\end{pmatrix} \\
&\quad + \begin{pmatrix}
h_1(x_1, \ldots, x_n, t) \\
h_2(x_1, \ldots, x_n, t) \\
\vdots \\
h_{n-1}(x_1, \ldots, x_n, t) \\
h_n(x_1, \ldots, x_n, t)
\end{pmatrix},
\end{align*}
\]
where \( h_1 \) is a polynomial which vanishes at \( \tilde{p} \) and \( h_i, i = 2, 3, \ldots, n \) are polynomials of order at least 2 in \( x_1, x_2, \ldots, x_n \). We call ordered set of the eigenvalues \( (a_{11}, a_{22}, \cdots, a_{nn}) \) \textit{local index} at \( \tilde{p} \).

We are interested in the case with local index
\[
(1, a_{22}/a_{11}, \ldots, a_{nn}/a_{11}) \in \mathbb{Z}^n.
\]
These properties suggest the possibilities that \( a_1 \) is the residue of the formal Laurent series:
\[
y_1(t) = \frac{a_{11}}{t - t_0} + b_1 + b_2(t - t_0) + \cdots + b_n(t - t_0)^{n-1} + \cdots \quad (b_i \in \mathbb{C}),
\]
and the ratio \( (1, a_{22}/a_{11}, \ldots, a_{nn}/a_{11}) \) is resonance data of the formal Laurent series of each \( y_i(t) \) \( (i = 2, \ldots, n) \), where \( (y_1, \ldots, y_n) \) is original coordinate system satisfying \( (x_1, \ldots, x_n) = (f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n)), f_i(y_1, \ldots, y_n) \in \mathbb{C}(t)(y_1, \ldots, y_n) \).

If each component of \( (1, a_{22}/a_{11}, \ldots, a_{nn}/a_{11}) \) has the same sign, we may resolve the accessible singularity by blowing-up finitely many times. However, when different signs appear, we may need to both blow up and blow down.

The \( \alpha \)-test,
\[
t = t_0 + \alpha T, \quad x_i = \alpha X_i, \quad \alpha \to 0,
\]
yields the following reduced system:

\[ \frac{d}{dT} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} = \frac{1}{X_1} \begin{pmatrix} a_{11}(t_0) & 0 & 0 & \ldots & 0 \\ a_{21}(t_0) & a_{22}(t_0) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{(n-1)1}(t_0) & a_{(n-1)2}(t_0) & \ldots & a_{(n-1)(n-1)}(t_0) & 0 \\ a_{n1}(t_0) & a_{n2}(t_0) & \ldots & a_{n(n-1)}(t_0) & a_{nn}(t_0) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix}, \]

where \( a_{ij}(t_0) \in \mathbb{C} \). Fixing \( t = t_0 \), this system is the system of the first order ordinary differential equation with constant coefficient. Let us solve this system. At first, we solve the first equation:

\[ X_1(T) = a_{11}(t_0)T + C_1 \quad (C_1 \in \mathbb{C}). \]

Substituting this into the second equation in \((6)\), we can obtain the first order linear ordinary differential equation:

\[ \frac{dX_2}{dT} = \frac{a_{22}(t_0)X_2}{a_{11}(t_0)T + C_1} + a_{21}(t_0). \]

By variation of constant, in the case of \( a_{11}(t_0) \neq a_{22}(t_0) \) we can solve explicitly:

\[ X_2(T) = C_2(a_{11}(t_0)T + C_1) + \frac{a_{22}(t_0)(a_{11}(t_0)T + C_1)}{a_{11}(t_0) - a_{22}(t_0)} \quad (C_2 \in \mathbb{C}). \]

This solution is a single-valued solution if and only if

\[ \frac{a_{22}(t_0)}{a_{11}(t_0)} \in \mathbb{Z}. \]

In the case of \( a_{11}(t_0) = a_{22}(t_0) \) we can solve explicitly:

\[ X_2(T) = C_2(a_{11}(t_0)T + C_1) + \frac{a_{21}(t_0)(a_{11}(t_0)T + C_1)\text{Log}(a_{11}(t_0)T + C_1)}{a_{11}(t_0)} \quad (C_2 \in \mathbb{C}). \]

This solution is a single-valued solution if and only if

\[ a_{21}(t_0) = 0. \]

Of course, \( \frac{a_{22}(t_0)}{a_{11}(t_0)} = 1 \in \mathbb{Z} \). In the same way, we can obtain the solutions for each variables \((X_3, \ldots, X_n)\). The conditions \( \frac{a_{ij}(t)}{a_{11}(t)} \in \mathbb{Z}, \quad (j = 2, 3, \ldots, n) \) are necessary condition in order to have the Painlevé property.

In the next section, in order to consider the phase spaces for each system, let us take the compactification \([z_0 : z_1 : z_2] \in \mathbb{P}^2 \) of \((x, y) \in \mathbb{C}^2 \) with the natural embedding

\[ (x, y) = (z_1/z_0, z_2/z_0). \]
Moreover, we denote the boundary divisor in \( \mathbb{P}^2 \) by \( \mathcal{H} \). Extend the regular vector field on \( \mathbb{C}^2 \) to a rational vector field \( \tilde{v} \) on \( \mathbb{P}^2 \). It is easy to see that \( \mathbb{P}^2 \) is covered by three copies of \( \mathbb{C}^2 \):

\[
U_0 = \mathbb{C}^2 \ni (x, y),
U_j = \mathbb{C}^2 \ni (X_j, Y_j) \ (j = 1, 2),
\]

via the following rational transformations

\[
X_1 = 1/x, \quad Y_1 = y/x,
X_2 = x/y, \quad Y_2 = 1/y,
\]

### 3. Canonical equation of type I

\[
\begin{align*}
I : \frac{d^2 u}{dt^2} &= 0, \\
II : \frac{d^2 u}{dt^2} &= 6u^2, \quad III : \frac{d^2 u}{dt^2} = 6u^2 + \frac{1}{2}, \quad IV : \frac{d^2 u}{dt^2} = 6u^2 + t, \\
V : \frac{d^2 u}{dt^2} &= -2u \frac{du}{dt} + q(t) \frac{du}{dt} + q'(t)u, \\
VI : \frac{d^2 u}{dt^2} &= -3u \frac{du}{dt} - u^3 + q(t) \left\{ \frac{du}{dt} + u^2 \right\}, \\
VII : \frac{d^2 u}{dt^2} &= 2u^3, \quad VIII : \frac{d^2 u}{dt^2} = 2u^3 + \beta u + \gamma, \\
IX : \frac{d^2 u}{dt^2} &= 2u^3 + tu + \gamma, \\
X : \frac{d^2 u}{dt^2} &= -u \frac{du}{dt} + u^3 - 12q(t)u + 12q'(t),
\end{align*}
\]

where \( ' := \frac{d}{dt} \).

### 4. Ince-V equation

The Ince-V equation is explicitly given by

\[
\frac{d^2 u}{dt^2} = -2u \frac{du}{dt} + q(t) \frac{du}{dt} + q'(t)u.
\]

Here \( u \) denotes unknown complex variable.

**Proposition 4.1.** The canonical transformation

\[
\begin{cases}
x = \frac{1}{u}, \\
y = \frac{du}{dt} + u^2 - q(t)u
\end{cases}
\]
takes the equation (12) to the system

\[
\begin{aligned}
\frac{dx}{dt} &= -x^2y - q(t)x + 1, \\
\frac{dy}{dt} &= 0.
\end{aligned}
\]

Here \(x, y\) denote unknown complex variables.

5. Ince-VI Equation

Ince-VI equation is explicitly given by

\[
\frac{d^2 u}{dt^2} = -3u \frac{du}{dt} - u^3 + q(t) \left\{ \frac{du}{dt} + u^2 \right\}.
\]

**Proposition 5.1.** The canonical transformation

\[
\begin{aligned}
x &= \frac{1}{u}, \\
y &= \frac{du}{dt} \frac{1}{u} + u
\end{aligned}
\]

takes the equation (14) to the system

\[
\begin{aligned}
\frac{dx}{dt} &= 1 - xy, \\
\frac{dy}{dt} &= -y^2 + q(t)y.
\end{aligned}
\]

This system is a Riccati extension of the Riccati equation:

\[
\frac{dy}{dt} = -y^2 + q(t)y.
\]

6. Ince-VII Equation

Ince-VII equation is explicitly given by

\[
\frac{d^2 u}{dt^2} = 2u^3.
\]

**Proposition 6.1.** The canonical transformation

\[
\begin{aligned}
x &= u, \\
y &= \frac{du}{dt} \\
\end{aligned}
\]

takes the equation (17) to the system

\[
\begin{aligned}
\frac{dx}{dt} &= xy, \\
\frac{dy}{dt} &= 2x^2 - y^2.
\end{aligned}
\]
Theorem 6.2. After a series of explicit blowing-ups at ten points in $\mathbb{P}^2$, we obtain the rational surface $\mathcal{X}$ of type $E_7^{(1)}$ (see Figure 1). The phase space $\mathcal{X}$ for the system (18) is obtained by gluing four copies of $\mathbb{C}^2 \times \mathbb{C}$:

$$U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \quad j = 0, 1, 2$$

via the following birational transformations:

0) $x_0 = x, \quad y_0 = y,$

1) $x_1 = xy, \quad y_1 = \frac{1}{y},$

2) $x_2 = xy^3 - y^4, \quad y_2 = \frac{1}{y},$

3) $x_3 = xy^3 + y^4, \quad y_3 = \frac{1}{y}.$

7. Ince-X equation

Ince-X equation is explicitly given by

$$\frac{d^2u}{dt^2} = -u \frac{du}{dt} + u^3 - 12q(t)u + 12q'(t).$$

Proposition 7.1. The canonical transformation

$$\begin{cases}
    x = u, \\
    y = \frac{du}{dt}
\end{cases}$$

takes the equation (19) to the system

$$\begin{cases}
    \frac{dx}{dt} = y, \\
    \frac{dy}{dt} = -xy + x^3 - 12q(t)x + 12q'(t).
\end{cases}$$
Theorem 7.2. After a series of explicit blowing-ups at eleven points in \( \mathbb{P}^2 \), we obtain the rational surface \( X \) of type \( E_8^{(1)} \) (see Figure 2). The phase space \( X \) for the system \([20]\) is obtained by gluing three copies of \( \mathbb{C}^2 \times \mathbb{C} \):

\[
U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \quad j = 0, 1, 2
\]

via the following birational transformations:

0) \( x_0 = x, \quad y_0 = y, \)

1) \( x_1 = \frac{1}{x}, \quad y_1 = (y + x^2 - 12q(t))x, \)

2) \( x_2 = \frac{1}{x}, \quad y_2 = \{(y - x^2/2 + 6q(t))x - 12q'(t)x + 12(6q^2(t) - C_1t - C_2)x - 144q(t)q'(t)\}x. \)

Lemma 7.3. The rational vector field \( \tilde{v} \) has one accessible singular point:

\[
P = \{(X_2, Y_2)|X_2 = Y_2 = 0\},
\]

where \( P \) has multiplicity of order 2.

Figure 2. Each bold line denotes \((-2)\)-curve. The leftarrow denotes blowing-ups and the rightarrow denotes blowing-downs. The symbol \((\ast)\) denotes intersection number of \( \mathbb{P}^1 \). The phase space \( \mathcal{X} \) is the rational surface of type \( E_8^{(1)} \).
8. Proof of Theorems 7.2

Now we are ready to prove Theorem 7.2.

By the following steps, we can resolve the accessible singular point $P$.

**Step 1:** We blow up at the point $P$:

$$x^{(1)} = X_2, \quad y^{(1)} = \frac{Y_2}{X_2}.$$ 

**Step 2:** We blow up at the point $\{(x^{(1)}, y^{(1)}) | x^{(1)} = y^{(1)} = 0\}$:

$$x^{(2)} = \frac{x^{(1)}}{y^{(1)}}, \quad y^{(2)} = y^{(1)}.$$ 

**Step 3:** We make a change of variables:

$$x^{(3)} = \frac{1}{x^{(2)}}, \quad y^{(3)} = y^{(2)}.$$ 

In the coordinate system $(x^{(3)}, y^{(3)})$, the rational vector field $\tilde{v}$ has two accessible singular points:

$$P^{(1)} = \{(x^{(3)}, y^{(3)}) | x^{(3)} = \frac{1}{2}, y^{(3)} = 0\},$$

$$P^{(2)} = \{(x^{(3)}, y^{(3)}) | x^{(3)} = -1, y^{(3)} = 0\}.$$ 

Next let us calculate its local index at each point.

| Singular point | Type of local index | Condition (3) |
|---------------|---------------------|---------------|
| $P^{(1)}$     | $(-3, -\frac{1}{2})$ | $-\frac{3}{2} = 6$ |
| $P^{(2)}$     | $(3, 1)$            | $\frac{3}{2} = 3$ |

This property suggests that we will blow up six times to the direction $x^{(3)}$ on the resolution process of $P^{(1)}$ and three times to the direction $x^{(3)}$ on the resolution process of $P^{(2)}$.

At first, we will resolve the accessible singular point $P^{(1)}$.

**Step 4:** We take the coordinate system centered at $P^{(1)}$:

$$x^{(4)} = x^{(3)} - \frac{1}{2}, \quad y^{(4)} = y^{(3)},$$

and the system (20) is rewritten as follows:

$$\frac{d}{dt} \begin{pmatrix} x^{(4)} \\ y^{(4)} \end{pmatrix} = \frac{1}{y^{(4)}} \begin{pmatrix} -3 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x^{(4)} \\ y^{(4)} \end{pmatrix} + \cdots$$

satisfying (2).

**Step 5:** We blow up at the point $\{(x^{(4)}, y^{(4)}) | x^{(4)} = y^{(4)} = 0\}$:

$$x^{(5)} = \frac{x^{(4)}}{y^{(4)}}, \quad y^{(5)} = y^{(4)}.$$
Step 6: We blow up at the point \( \{(x(5), y(5))|x(5) = y(5) = 0\} \):
\[
\begin{align*}
x(6) &= \frac{x(5)}{y(5)}, & y(6) &= y(5).
\end{align*}
\]

Step 7: We blow up at the point \( \{(x(6), y(6))|x(6) = -6q(t), y(6) = 0\} \):
\[
\begin{align*}
x(7) &= \frac{x(6) + 6q(t)}{y(6)}, & y(7) &= y(6).
\end{align*}
\]

Step 8: We blow up at the point \( \{(x(7), y(7))|x(7) = 12q'(t), y(7) = 0\} \):
\[
\begin{align*}
x(8) &= \frac{x(7) - 12q'(t)}{y(7)}, & y(8) &= y(7).
\end{align*}
\]

Step 9: We blow up at the point \( \{(x(8), y(8))|x(8) = -12(6q^2(t) - C_1t - C_2), y(8) = 0\} \):
\[
\begin{align*}
x(9) &= \frac{x(8) + 12(6q^2(t) - C_1t - C_2)}{y(8)}, & y(9) &= y(8).
\end{align*}
\]

Step 10: We blow up at the point \( \{(x(9), y(9))|x(9) = 144q(t)q'(t), y(9) = 0\} \):
\[
\begin{align*}
x(10) &= \frac{x(9) - 144q(t)q'(t)}{y(9)}, & y(10) &= y(9),
\end{align*}
\]

\[
\begin{align*}
dx(10)/dt &= \frac{24\{12(q'(t))^2 + 12q(t)q''(t) - q'''(t)\}}{y(10)} + g_1(x(10), y(10), t), \\
dy(10)/dt &= g_2(x(10), y(10), t),
\end{align*}
\]

where \( g_i(x(10), y(10), t) (i = 1, 2) \) are polynomials in \( x(10), y(10) \). Each right-hand side of the system (21) is a polynomial if and only if
\[
(22) \quad q'''(t) = 12(q'(t))^2 + 12q(t)q''(t).
\]

This condition is reduced as follows:
\[
(23) \quad q''(t) = 6q^2(t) - C_1t - C_2 \quad (C_1, C_2 \in \mathbb{C}).
\]

We remark that the coordinate system \( (x(10), y(10)) \) corresponds to \( (y_2, x_2) \) given in Theorem 7.2.

Next, we will resolve the accessible singular point \( P^{(2)} \).

Step 11: We take the coordinate system centered at \( P^{(2)} \):
\[
\begin{align*}
x(11) &= x(3) + 1, & y(11) &= y(3),
\end{align*}
\]

and the system (20) is rewritten as follows:
\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} x(11) \\ y(11) \end{pmatrix} &= \frac{1}{y(11)} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(11) \\ y(11) \end{pmatrix} + \cdots
\end{align*}
\]
satisfying (2).
Step 12: We blow up at the point \(((x^{(11)}, y^{(11)}))|x^{(11)} = y^{(11)} = 0\):

\[x^{(12)} = \frac{x^{(11)}}{y^{(11)}}, \quad y^{(12)} = y^{(11)}.\]

Step 13: We blow up at the point \(((x^{(12)}, y^{(12)}))|x^{(12)} = 12q(t), \ y^{(12)} = 0\):

\[x^{(13)} = \frac{x^{(12)} - 12q(t)}{y^{(12)}}, \quad y^{(13)} = y^{(12)}.\]

Thus we have resolved the accessible singular point \(P^{(2)}\). The coordinate system \((x^{(13)}, y^{(13)})\) corresponds to \((y_1, x_1)\) given in Theorem 7.2.

Thus, we have completed the proof of Theorem 7.2.

9. Canonical equation of type II

\(XI: \frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2,\)

\(XII: \frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 + \alpha u^3 + \beta u^2 + \gamma + \frac{\delta}{u},\)

\[XIII: \frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 - \frac{1}{t} \frac{du}{dt} + \frac{1}{t} (\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u},\]

\(XIV: \frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 + \left\{ q(t)u + \frac{r(t)}{u} \right\} + q'(t)u^2 - r'(t),\)

\(XV: \frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 + \frac{1}{u} \frac{du}{dt} + r(t)u^2 - u \frac{d}{dt} \left\{ \frac{r(t)}{r(t)} \right\},\)

\(XVI: \frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 - q(t) \frac{1}{u} \frac{du}{dt} + u^3 - q(t)u^2 + q''(t),\)

where \(\prime := \frac{d}{dt}.\)

10. Ince-XI equation

Ince-XI equation is explicitly given by

\(\frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2.\)

Here \(u\) denotes unknown complex variable.

**Proposition 10.1.** The canonical transformation

\[
\begin{cases}
  x = u, \\
  y = \frac{du}{dt}
\end{cases}
\]
takes the equation (25) to the system
\[
\begin{aligned}
\frac{dx}{dt} &= xy, \\
\frac{dy}{dt} &= 0.
\end{aligned}
\]
Here \(x, y\) denote unknown complex variables. This system can be solved by
\[
(x, y) = (c_2 e^{c_1 t}, c_1) \quad (c_1, c_2 \in \mathbb{C}).
\]

11. Ince-XII equation

Ince-XII equation is explicitly given by
\[
\frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 + \alpha u^3 + \beta u^2 + \gamma + \frac{\delta}{u}.
\]

**Proposition 11.1.** The canonical transformation
\[
\begin{aligned}
x &= u, \\
y &= \left( \frac{du}{dt} - \frac{\delta}{u} + \frac{\gamma}{\delta} \right) / u
\end{aligned}
\]
takes the equation (28) to the system
\[
\begin{aligned}
\frac{dx}{dt} &= \frac{\partial H}{\partial y} = x^2 y - \frac{\gamma}{\delta} x + \delta, \\
\frac{dy}{dt} &= -\frac{\partial H}{\partial x} = -xy^2 + \alpha x + \frac{\gamma}{\delta} y + \beta
\end{aligned}
\]
with the polynomial Hamiltonian
\[
H = \frac{x^2 y^2}{2} - \frac{\alpha}{2} x^2 - \frac{\gamma}{\delta} xy + \frac{\delta}{\delta} y - \beta x.
\]

We remark that the system (29) has the Hamiltonian \(H\) as its first integral.

**Theorem 11.2.** After a series of explicit blowing-ups in \(\mathbb{P}^2\), we obtain the rational surface \(X\) of type \(D^0_6\). The phase space \(X\) for the system (29) can be obtained by gluing four copies of \(\mathbb{C}^2 \times \mathbb{C}\):
\[
U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \quad j = 0, 1, 2, 3
\]
via the following birational and symplectic transformations:

0) \(x_0 = x, \quad y_0 = y,\)
1) \(x_1 = \frac{1}{x}, \quad y_1 = -\left( y - \sqrt{\alpha} x - \frac{\sqrt{\alpha \gamma + \beta \delta}}{\sqrt{\alpha \delta}} \right) x,\)
2) \(x_2 = \frac{1}{x}, \quad y_2 = -\left( y + \sqrt{\alpha} x - \frac{\sqrt{\alpha \gamma - \beta \delta}}{\sqrt{\alpha \delta}} \right) x,\)
3) \(x_3 = x, \quad y_3 = y - \frac{2\gamma}{\delta} + \frac{2\delta}{x^2}.\)
Theorem 11.3. The system (29) admits the affine Weyl group symmetry of type $C_2^{(1)}$ as the group of its Bäcklund transformations, whose generators $s_i$ are explicitly given as follows: with the notation $(\star) := (x, y, t; \alpha, \beta, \gamma, \delta)$,

\begin{align*}
  s_0 : (x, y; \alpha, \beta, \gamma, \delta) &\to \left( x - \frac{\sqrt{\alpha \gamma + \beta \delta}}{\sqrt{\alpha \delta}}, y; \alpha, -\frac{\gamma}{\delta} \sqrt{-\alpha}, -\frac{\beta \delta}{\sqrt{\alpha \delta}} \right), \\
  s_1 : (x, y; \alpha, \beta, \gamma, \delta) &\to \left( x - \frac{\sqrt{\alpha \gamma - \beta \delta}}{\sqrt{\alpha \delta}}, y; \alpha, \frac{\gamma}{\delta} \sqrt{-\alpha}, \frac{\beta \delta}{\sqrt{\alpha \delta}} \right), \\
  s_2 : (x, y; \alpha, \beta, \gamma, \delta) &\to \left( x - 2 \frac{\gamma}{\delta} x + 2 \frac{\delta}{x^2}; \alpha, \beta, \gamma, -\delta \right).
\end{align*}

12. INCE-XIII EQUATION

Ince-XIII equation is explicitly given by

\begin{equation}
  \frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 - \frac{1}{t} \frac{du}{dt} + \frac{1}{t} (\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}.
\end{equation}

This equation coincides with the Painlevé III equation.

13. MODIFIED INCE-XIV EQUATION

Modified Ince-XIV equation is explicitly given by

\begin{equation}
  \frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 + \frac{1}{2} \frac{d}{dt} \left( \frac{r(t)}{r(t)} \right) u + \frac{r(t)}{u} + \frac{c_2 e^{c_1 t}}{r(t)} u^2 + c_1 \sqrt{r(t)}.
\end{equation}

Proposition 13.1. The canonical transformation

\begin{equation}
  \left\{ \begin{array}{l}
  x = u, \\
  y = \left( \frac{du}{dt} - \frac{\sqrt{-r(t)}}{u} - \frac{2 \sqrt{1} c_1 r(t) + r'(t)}{2 r(t)} \right) / u
  \end{array} \right.
\end{equation}

takes the equation (32) to the system

\begin{equation}
  \left\{ \begin{array}{l}
  \frac{dx}{dt} = \frac{\partial H}{\partial y} = x^2 y + \left( \sqrt{-1} c_1 + \frac{r'(t)}{2 r(t)} \right) x + \sqrt{-r(t)}, \\
  \frac{dy}{dt} = -\frac{\partial H}{\partial x} = -x y^2 - \left( \sqrt{-1} c_1 + \frac{r'(t)}{2 r(t)} \right) y + \frac{c_2 e^{c_1 t}}{\sqrt{r(t)}}
  \end{array} \right.
\end{equation}

with the polynomial Hamiltonian

\begin{equation}
  H = \frac{x^2 y^2}{2} + \left( \sqrt{-1} c_1 + \frac{r'(t)}{2 r(t)} \right) x y + \sqrt{-r(t)} y - \frac{c_2 e^{c_1 t}}{\sqrt{r(t)}} x.
\end{equation}

Theorem 13.2. After a series of explicit blowing-ups and blowing-downs in $\mathbb{P}^2$, we obtain the rational surface $X$ of type $D_7^{(1)}$. The phase space $X$ for the system (33) can be obtained by gluing three copies of $\mathbb{C}^2 \times \mathbb{C}$.
\[ U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \ j = 0, 1, 2 \]

*via the following birational and symplectic transformations:*

0) \( x_0 = x, \ y_0 = y, \)

1) \( x_1 = x + \frac{2(1 + \sqrt{-1})c_1}{y} - \frac{2c_2 e^{c_1 t}}{r(t) y^2}, \ y_1 = y, \)

2) \( x_2 = x, \ y_2 = y + \frac{2\sqrt{-1}c_1}{x} + \frac{2\sqrt{-r(t)}}{x^2}. \)

14. **Ince-XV equation**

Ince-XV equation is explicitly given by

\[ \frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 - \frac{1}{u} \frac{du}{dt} + r(t) u^2 - g(t) u. \]  \hspace{1cm} (35)

**Proposition 14.1.** The canonical transformation

\[
\begin{cases}
  x = u, \\
  y = \left( \frac{du}{dt} - 1 \right) / u
\end{cases}
\]

takes the equation (35) to the system

\[
\begin{cases}
  \frac{dx}{dt} = xy + 1, \\
  \frac{dy}{dt} = r(t)x - g(t),
\end{cases}
\]  \hspace{1cm} (36)

where \( r(t) \) and \( g(t) \) satisfy the relation:

\[ g(t) = \frac{r(t) r(t) - (r'(t))^2}{r^2(t)} = \frac{d}{dt} \left\{ \frac{r'(t)}{r(t)} \right\}. \]  \hspace{1cm} (37)

**Lemma 14.2.** The rational vector field \( \tilde{v} \) has one accessible singular point:

\[ P = \{(X_1, Y_1)|X_1 = Y_1 = 0\}. \]

By the following steps, we can resolve the accessible singular point \( P \).

**Step 1:** We blow up at the point \( P \):

\[ x^{(1)} = \frac{X_1}{Y_1}, \ y^{(1)} = Y_1. \]

**Step 2:** We blow up at the point \( \{(x^{(1)}, y^{(1)})|x^{(1)} = y^{(1)} = 0\} \):

\[ x^{(2)} = x^{(1)}, \ y^{(2)} = \frac{y^{(1)}}{x^{(1)}}. \]

**Step 3:** We make a change of variables:

\[ x^{(3)} = x^{(2)}, \ y^{(3)} = \frac{1}{y^{(2)}}. \]
In the coordinate system \((x^{(3)}, y^{(3)})\), the rational vector field \(\tilde{v}\) has one accessible singular point:

\[
P^{(1)} = \{(x^{(3)}, y^{(3)}) | x^{(3)} = \frac{1}{2r(t)}, y^{(3)} = 0\}.
\]

Next let us calculate its local index at \(P^{(1)}\).

| Singular point | Type of local index | Condition \((3)\) |
|----------------|---------------------|------------------|
| \(P^{(1)}\)    | \((-1, -\frac{1}{2})\) | \(-\frac{1}{4} = 2\) |

This property suggests that we will blow up second times to the direction \(x^{(3)}\) on the resolution process of \(P^{(1)}\).

**Step 4:** We take the coordinate system centered at \(P^{(1)}\):

\[
x^{(4)} = x^{(3)} - \frac{1}{2r(t)}, \quad y^{(4)} = y^{(3)}.
\]

**Step 5:** We blow up at the point \(\{(x^{(4)}, y^{(4)}) | x^{(4)} = y^{(4)} = 0\}:

\[
x^{(5)} = \frac{x^{(4)}}{y^{(4)}}, \quad y^{(5)} = y^{(4)}.
\]

**Step 6:** We blow up at the point \(\{(x^{(5)}, y^{(5)}) | x^{(5)} = \frac{r'(t)}{r^2(t)}, y^{(5)} = 0\}:

\[
x^{(6)} = \frac{x^{(5)} - \frac{r'(t)}{r^2(t)}}{y^{(5)}}, \quad y^{(6)} = y^{(5)}.
\]

\[
\begin{cases}
\frac{dx^{(5)}}{dt} = \left(g(t) - \frac{r''(t)r(t) - (r'(t))^2}{r^2(t)}\right) / y^{(5)} + g_1(x^{(5)}, y^{(5)}, t), \\
\frac{dy^{(5)}}{dt} = g_2(x^{(5)}, y^{(5)}, t),
\end{cases}
\]

where \(g_i(x^{(5)}, y^{(5)}, t) \ (i = 1, 2)\) are polynomials in \(x^{(5)}, y^{(5)}\). Each right-hand side of the system (38) is a polynomial if and only if

\[
g(t) = \frac{r''(t)r(t) - (r'(t))^2}{r^2(t)} = \frac{d}{dt} \left\{ \frac{r'(t)}{r(t)} \right\}.
\]

Thus, we have resolved the accessible singular point \(P^{(1)}\).

15. **Ince-XVI equation**

Ince-XVI equation is explicitly given by

\[
\frac{d^2u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 - \frac{q(t)}{u} \frac{du}{dt} + u^3 - q(t)u^2 + q''(t).
\]
Figure 3. Each bold line denotes \((-2)\)-curve. The leftarrow denotes blowing-ups and the rightarrow denotes blowing-downs. The symbol \((\ast)\) denotes intersection number of \(\mathbb{P}^1\). The phase space \(\mathcal{X}\) is the rational surface of type \(D_4\).

**Proposition 15.1.** The canonical transformation

\[
\begin{align*}
x &= u, \\
y &= \frac{du}{dt} - q'(t) \\
u &= \frac{d^2u}{dt^2}
\end{align*}
\]

takes the equation (40) to the system

\[
\begin{align*}
\frac{dx}{dt} &= xy + q'(t), \\
\frac{dy}{dt} &= x^2 - x(t)x.
\end{align*}
\]

**Theorem 15.2.** After a series of explicit blowing-ups and blowing-downs in \(\mathbb{P}^2\), we obtain the rational surface \(\mathcal{X}\) of type \(D_4\) (see Figure 3). The phase space \(\mathcal{X}\) for the system (41) can be obtained by gluing three copies of \(\mathbb{C}^2 \times \mathbb{C}\):

\[
U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{ (x_j, y_j, t) \}, \quad j = 0, 1, 2
\]

via the following birational transformations:

0) \(x_0 = x, \quad y_0 = y\),
1) \(x_1 = \frac{1}{x}, \quad y_1 = xy - x^2 + q(t)x\),
2) \(x_2 = \frac{1}{x}, \quad y_2 = xy + x^2 - q(t)x\).
16. Canonical equation of type III

\[ XVII \frac{d^2 u}{dt^2} = \frac{m - 1}{mu} \left( \frac{du}{dt} \right)^2, \]

\[ XVIII \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + 4u^2, \]

\[ XIX \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + 4u^2 + 2u, \]

\[ XX \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + 4u^2 + 2tu, \]

\[ XXI \frac{d^2 u}{dt^2} = \frac{3}{4u} \left( \frac{du}{dt} \right)^2 + 3u^2, \]

\[ XXII \frac{d^2 u}{dt^2} = \frac{3}{4u} \left( \frac{du}{dt} \right)^2 - 1, \]

\[ XXIII \frac{d^2 u}{dt^2} = \frac{3}{4u} \left( \frac{du}{dt} \right)^2 + 3u^2 + \alpha u + \beta, \]

\[ XXIV \frac{d^2 u}{dt^2} = \frac{m - 1}{mu} \left( \frac{du}{dt} \right)^2 + q(t)u \frac{du}{dt} - \frac{mq^2(t)}{(m + 2)^2} u^3 + \frac{mq'(t)}{m + 2} u^2, \]

\[ XXV \frac{d^2 u}{dt^2} = \frac{3}{4u} \left( \frac{du}{dt} \right)^2 - \frac{3u du}{2u} \frac{du}{dt} - \frac{u^3}{4} + \frac{q'(t)}{2q(t)} \left( u^2 + \frac{du}{dt} \right) + r(t)u + q(t), \]

\[ XXVI \frac{d^2 u}{dt^2} = \frac{3}{4u} \left( \frac{du}{dt} \right)^2 + \frac{6q'(t) du}{u} \frac{du}{dt} + 3u^2 + 12q(t)u - 12q''(t) - \frac{36q'(t)^2}{u}, \]

\[ XXVII \frac{d^2 u}{dt^2} = \frac{m - 1}{mu} \left( \frac{du}{dt} \right)^2 + \left( f(t)u + \phi(t) - \frac{m - 2}{mu} \right) \frac{du}{dt} - \frac{m f(t)^2}{(m + 2)^2} u^3 + \frac{m (f'(t) - f(t) \phi(t))}{m + 2} u^2 + \psi(t)u - \phi(t) - \frac{1}{mu}, \]

\[ XXVIII \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 - (u - q(t)) \frac{du}{dt} + \frac{u^3}{2} - 2q(t)u^2 + 3 \left( q'(t) + \frac{1}{2} q(t)^2 \right) u - \frac{72r(t)^2}{u}, \]

\[ XXIX \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + \frac{3}{2} u^3, \]

\[ XXX \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + \frac{3}{2} u^3 + 4\alpha u^2 + 2\beta u - \frac{\gamma^2}{2u}, \]

\[ XXXI \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + \frac{3}{2} u^3 + 4tu^2 + 2(t^2 - \alpha)u - \frac{\beta^2}{2u}, \]

\[ XXXII \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 - \frac{1}{2u}, \]
XXXIII: $\frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + 4u^2 + \alpha u - \frac{1}{2u}$,

XXXIV: $\frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + 4\alpha u^2 - tu - \frac{1}{2u}$,

XXXV: $\frac{d^2 u}{dt^2} = \frac{2}{3u} \left( \frac{du}{dt} \right)^2 - \left( \frac{2}{3}u - \frac{2}{3}q(t) - \frac{r(t)}{u} \right) \frac{du}{dt} + \frac{2}{3}u^3 - \frac{10}{3}q(t)u^2$

$+ \left( 4q'(t) + r(t) + \frac{8}{3}q(t)^2 \right) u - 2q(t)r(t) - 3r'(t) - \frac{3r(t)^2}{u}$,

XXXVI: $\frac{d^2 u}{dt^2} = \frac{4}{5u} \left( \frac{du}{dt} \right)^2 - \left( \frac{2}{5}u + \frac{1}{5}q(t) - \frac{r(t)}{u} \right) \frac{du}{dt} + \frac{4}{5}u^3 + \frac{14}{5}q(t)u^2$

$+ \left( r(t) - 3q'(t) + \frac{6}{5}q(t)^2 \right) u - \frac{1}{3}(q(t)r(t) + 5r'(t)) - \frac{5r(t)^2}{9u}$,

where $' := \frac{d}{dt}$.

17. **INCE-XVII EQUATION**

Ince-XVII equation is explicitly given by

(43) \[ \frac{d^2 u}{dt^2} = m - \frac{1}{5u} \left( \frac{du}{dt} \right)^2. \]

Here $u$ denotes unknown complex variable.

**Proposition 17.1.** The canonical transformation

\[ \begin{cases} x = u, \\ y = \frac{du}{dt} \end{cases} \]

takes the equation (43) to the system

(44) \[ \begin{cases} \frac{dx}{dt} = xy, \\ \frac{dy}{dt} = -\frac{y^2}{m}. \end{cases} \]

Here $x, y$ denote unknown complex variables.

18. **INCE-XVIII EQUATION**

Ince-XVIII equation is explicitly given by

(45) \[ \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + 4u^2. \]
Proposition 18.1. The canonical transformation
\[
\begin{align*}
\begin{cases}
x = u, \\
y = \frac{du}{dt}
\end{cases}
\end{align*}
\]
takes the equation (45) to the system
\[
\begin{align*}
\begin{cases}
\frac{dx}{dt} = \frac{\partial H}{\partial y} = xy, \\
\frac{dy}{dt} = -\frac{\partial H}{\partial x} = -\frac{y^2}{2} + 4x
\end{cases}
\end{align*}
\]
with the polynomial Hamiltonian
\[
H = \frac{xy^2}{2} - 2x^2.
\]

We remark that the system (46) has the Hamiltonian \(H\) as its first integral.

Theorem 18.2. After a series of explicit blowing-ups in \(\mathbb{P}^2\), we obtain the rational surface \(X\) of type \(E_7^{(1)}\). The phase space \(X\) for the system (46) can be obtained by gluing three copies of \(\mathbb{C}^2 \times \mathbb{C}\):
\[
U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni (x_j, y_j, t), \quad j = 0, 1, 2
\]
via the following birational and symplectic transformations:

0) \(x_0 = x\), \(y_0 = y\),
1) \(x_1 = -\left(x - \frac{y^2}{4}\right)y^2\), \(y_1 = \frac{1}{y}\),
2) \(x_2 = -xy^2\), \(y_2 = \frac{1}{y}\).

19. Ince-XIX equation

Ince-XIX equation is explicitly given by
\[
(47) \quad \frac{d^2u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + 4u^2 + 2u.
\]

Proposition 19.1. The canonical transformation
\[
\begin{align*}
\begin{cases}
x = u, \\
y = \frac{du}{dt}
\end{cases}
\end{align*}
\]
takes the equation (47) to the system
\[
\begin{align*}
\begin{cases}
\frac{dx}{dt} = \frac{\partial H}{\partial y} = xy, \\
\frac{dy}{dt} = -\frac{\partial H}{\partial x} = -\frac{y^2}{2} + 4x + 2
\end{cases}
\end{align*}
\]
with the polynomial Hamiltonian

\[ H = \frac{xy^2}{2} - 2x^2 - 2x. \]

We remark that the system (48) has the Hamiltonian \( H \) as its first integral.

**Theorem 19.2.** After a series of explicit blowing-ups in \( \mathbb{P}^2 \), we obtain the rational surface \( \mathcal{X} \) of type \( E_7^{(1)} \). The phase space \( \mathcal{X} \) for the system (48) can be obtained by gluing three copies of \( \mathbb{C}^2 \times \mathbb{C} \):

\[ U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \quad j = 0, 1, 2 \]

via the following birational and symplectic transformations:

0) \( x_0 = x, \quad y_0 = y \),

1) \( x_1 = -\left( x - \frac{y^2}{4} + 1 \right) y^2, \quad y_1 = \frac{1}{y} \),

2) \( x_2 = -xy^2, \quad y_2 = \frac{1}{y} \).

20. Ince-XXI equation

Ince-XXI equation is explicitly given by

\[ \frac{d^2 u}{dt^2} = \frac{3}{4u} \left( \frac{du}{dt} \right)^2 + 3u^2. \]

**Proposition 20.1.** The canonical transformation

\[
\begin{cases}
  x = u, \\
  y = \frac{du}{dt} 
\end{cases}
\]

takes the equation (49) to the system

\[
\begin{cases}
  \frac{dx}{dt} = xy, \\
  \frac{dy}{dt} = -\frac{y^2}{4} + 3x.
\end{cases}
\]

This system has its first integral:

\[ I := x(4x - y^2)^2. \]

**Theorem 20.2.** The phase space \( \mathcal{X} \) (see Figure 4) for the system (50) can be obtained by gluing three copies of \( \mathbb{C}^2 \times \mathbb{C} \):

\[ U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \quad j = 0, 1, 2 \]
Figure 4. Each bold line denotes (-2)-curve. The leftarrow denotes blowing-ups.

via the following birational transformations:

0) \( x_0 = x, \quad y_0 = y, \)

1) \( x_1 = \left( x - \frac{y^2}{4} \right) y, \quad y_1 = \frac{1}{y}, \)

2) \( x_2 = xy^4, \quad y_2 = \frac{1}{y}. \)

21. Ince-XXII equation

Ince-XXI equation is explicitly given by

\[
\frac{d^2 u}{dt^2} = \frac{3}{4u} \left( \frac{du}{dt} \right)^2 - 1. \tag{52}
\]

**Proposition 21.1.** The canonical transformation

\[
\begin{align*}
&x = \frac{u}{\frac{du}{dt}}, \\
y = \frac{4u - \left( \frac{du}{dt} \right)^2}{4u \frac{du}{dt}}
\end{align*}
\]

takes the equation (52) to the system

\[
\begin{align*}
\frac{dx}{dt} &= xy + \frac{1}{2}, \\
\frac{dy}{dt} &= y^2.
\end{align*} \tag{53}
\]

This system can be solved by

\[
(x(t), y(t)) = \left( \frac{t^2/4 + c_1 t/2 + c_2}{t + c_1}, -\frac{1}{t + c_1} \right) \quad (c_1, c_2 \in \mathbb{C}). \tag{54}
\]
22. INCE-XXIII EQUATION

Ince-XXIII equation is explicitly given by

\[ \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + \frac{3}{2} u^3 + \frac{\alpha}{2} u + \frac{\beta}{2u}. \]  

**Proposition 22.1.** The canonical transformation

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
x = u, \\
y = \frac{du}{dt} - \sqrt{-\beta} \frac{u}{u}
\end{array}
\right.
\end{aligned}
\]

takes the equation (55) to the system

\[ \begin{aligned}
\frac{dx}{dt} & = \frac{\partial H}{\partial y} = xy + \sqrt{-\beta}, \\
\frac{dy}{dt} & = -\frac{\partial H}{\partial x} = \frac{3}{2} x^2 - \frac{1}{2} y^2 + \frac{\alpha}{2}
\end{aligned} \]

with the polynomial Hamiltonian

\[ H = \frac{1}{2} xy^2 - \frac{1}{2} x^3 - \frac{\alpha}{2} x + \sqrt{-\beta} y. \]

We remark that the system (56) has the Hamiltonian \( H \) as its first integral.

**Theorem 22.2.** After a series of explicit blowing-ups in \( \mathbb{P}^2 \), we obtain the rational surface \( \mathcal{X} \) of type \( E_6^{(1)} \). The phase space \( \mathcal{X} \) for the system (56) can be obtained by gluing four copies of \( \mathbb{C}^2 \times \mathbb{C} \):

\[ U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{ (x_j, y_j, t) \}, \quad j = 0, 1, 2, 3 \]

via the following birational and symplectic transformations:

0) \( x_0 = x, \quad y_0 = y \),

1) \( x_1 = \frac{1}{x}, \quad y_1 = -x + \frac{\alpha - 2\sqrt{-\beta}}{2} x \),

2) \( x_2 = \frac{1}{x}, \quad y_2 = -x + \frac{\alpha + 2\sqrt{-\beta}}{2} x \),

3) \( x_3 = -(xy + 2\sqrt{-\beta})y, \quad y_3 = \frac{1}{y} \).

**Theorem 22.3.** The system (56) is invariant under the following transformations:

\[
\begin{aligned}
s_0 : (x, y; \alpha, \beta) & \rightarrow \left( x - \frac{\alpha - 2\sqrt{-\beta}}{y - x}, \quad y - \frac{\alpha - 2\sqrt{-\beta}}{y - x}; \quad -\alpha + 6\sqrt{-\beta}, \quad \frac{-\alpha}{2} + 4\alpha\sqrt{-\beta} + \frac{4\beta}{16} \right), \\
s_1 : (x, y; \alpha, \beta) & \rightarrow \left( x + \frac{\alpha + 2\sqrt{-\beta}}{y + x}, \quad y - \frac{\alpha + 2\sqrt{-\beta}}{y + x}; \quad -\alpha - 6\sqrt{-\beta}, \quad \frac{-\alpha}{2} + 4\alpha\sqrt{-\beta} + \frac{4\beta}{16} \right), \\
s_2 : (x, y; \alpha, \beta) & \rightarrow \left( x, y + \frac{2\sqrt{-\beta}}{x}; \alpha, \beta \right).
\end{aligned}
\]
23. Ince-XXIV Equation

Ince-XXIV equation is explicitly given by

\[(57) \quad \frac{d^2 u}{dt^2} = \frac{m-1}{m u} \left( \frac{du}{dt} \right)^2 + q(t) u \frac{du}{dt} - \frac{mq^2(t)}{(m+2)^2} u^3 + \frac{mq'(t)}{m+2} u^2. \]

**Proposition 23.1.** The canonical transformation

\[
\begin{align*}
\begin{cases}
    x = \frac{(m+1)(m+2)}{-m q(t) u + (m+1)(m+2) \frac{du}{dt}}, \\
y = \frac{(m+1)\{m q(t) u - (m+2) \frac{du}{dt}\}}{m^2 q(t) u}
\end{cases}
\end{align*}
\]

takes the equation \[(57)\] to the system

\[(58) \quad \begin{align*}
\frac{dx}{dt} &= -\frac{m^2 q(t)}{(m+1)(m+2)} x^2 y - \frac{mq(t)}{m+1} x y - \frac{q'(t)}{q(t)} x - \frac{m+1}{m}, \\
\frac{dy}{dt} &= \frac{mq(t)}{(m+1)(m+2)} y^2 - \frac{q'(t)}{q(t)} y.
\end{align*}
\]

This system is a Riccati extension of the Riccati equation:

\[(59) \quad \frac{dy}{dt} = \frac{m q(t)}{(m+1)(m+2)} y^2 - \frac{q'(t)}{q(t)} y.\]

24. Ince-XXV Equation

Ince-XXV equation is explicitly given by

\[(60) \quad \frac{d^2 u}{dt^2} = \frac{3}{4u} \left( \frac{du}{dt} \right)^2 - \frac{3}{2} \frac{du}{dt} - \frac{1}{4} u^3 + \frac{q'(t)}{2q(t)} \left( u^2 + \frac{du}{dt} \right) + r(t) u + q(t).\]

**Proposition 24.1.** The canonical transformation

\[
\begin{align*}
\begin{cases}
    x = \frac{du}{dt} + u^2 + \frac{1}{4q(t)u} + \frac{du}{dt} + u^2, \\
y = \frac{u}{du/dt + u^2}
\end{cases}
\end{align*}
\]

takes the equation \[(60)\] to the system

\[(61) \quad \begin{align*}
\frac{dx}{dt} &= -q(t) x^2 - r(t) xy - \frac{q'(t)}{2q(t)} x + y + \frac{r(t)}{2q(t)}, \\
\frac{dy}{dt} &= -r(t) y^2 - q(t) xy - \frac{q'(t)}{2q(t)} y + \frac{1}{2}.
\end{align*}
\]

**Proposition 24.2.** The phase space \(\mathcal{X}\) for the system \[(61)\] is the 2-dimensional projective space \((x,y) \in \mathbb{C}^2 \subset \mathbb{P}^2\).
Ince-XXVI equation is explicitly given by

\[
\frac{d^2 u}{dt^2} = \frac{3}{4u} \left( \frac{du}{dt} \right)^2 + \frac{6q'(t) du}{u} + 3u^2 + 12q(t)u - 12q''(t) - \frac{36(q'(t))^2}{u}.
\]

**Proposition 25.1.** The canonical transformation

\[
\begin{align*}
x &= u, \\
y &= \frac{du}{dt} + 12q'(t)
\end{align*}
\]

takes the equation (62) to the system

\[
\begin{align*}
\frac{dx}{dt} &= xy - 12q'(t), \\
\frac{dy}{dt} &= -\frac{1}{4} y^2 + 3x + 12q(t).
\end{align*}
\]

**Theorem 25.2.** After a series of explicit blowing-ups in \( \mathbb{P}^2 \), we obtain the rational surface \( X \) of type \( E_8^{(1)} \). The phase space \( \mathcal{X} \) for the system (63) can be obtained by gluing three copies of \( \mathbb{C}^2 \times \mathbb{C} \):

\[
U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \quad j = 0, 1, 2
\]

via the following birational transformations:

0) \( x_0 = x, \quad y_0 = y \),

1) \( x_1 = \left( x - \frac{y^2}{4} + 12q(t) \right) y, \quad y_1 = \frac{1}{y} \),

2) \( x_2 = \left( (xy - 16q'(t))y - 32q''(t) \right)y + 768q(t)q'(t) - 128q''(t) \), \( y_2 = \frac{1}{y} \).

**Lemma 25.3.** The rational vector field \( \tilde{v} \) has two accessible singular points:

\[
P_1 = \{(X_1, Y_1)|X_1 = Y_1 = 0\}, \\
P_2 = \{(X_2, Y_2)|X_2 = Y_2 = 0\},
\]

where \( P_1 \) is multiple point of order 2.

By the following steps, we can resolve the accessible singular point \( P_2 \).

Next let us calculate its local index at \( P_2 \).

| Singular point | Type of local index | Condition (3) |
|----------------|---------------------|----------------|
| \( P_2 \)      | \((\frac{\pi}{4}, \frac{1}{2})\) | \( \frac{\pi}{4} = 5 \) |

This property suggests that we will blow up five times to the direction \( X_1 \) on the resolution process of \( P_2 \).

**Step 1:** We blow up at the point \( P_2 \):

\[
x^{(1)} = \frac{X_1}{Y_1}, \quad y^{(1)} = Y_1.
\]
Step 2: We blow up at the point \( \{(x^{(1)}, y^{(1)}) | x^{(1)} = y^{(1)} = 0\} \):

\[
x^{(2)} = \frac{x^{(1)}}{y^{(1)}}, \quad y^{(2)} = y^{(1)}.
\]

Step 3: We blow up at the point \( \{(x^{(2)}, y^{(2)}) | x^{(2)} = 16q'(t), y^{(2)} = 0\} \):

\[
x^{(3)} = \frac{x^{(2)} - 16q'(t)}{y^{(2)}}, \quad y^{(3)} = \frac{1}{y^{(2)}}.
\]

Step 4: We blow up at the point \( \{(x^{(3)}, y^{(3)}) | x^{(3)} = 32q''(t), y^{(3)} = 0\} \):

\[
x^{(4)} = \frac{x^{(3)} - 32q''(t)}{y^{(3)}}, \quad y^{(4)} = y^{(3)}.
\]

Step 5: We blow up at the point \( \{(x^{(4)}, y^{(4)}) | x^{(4)} = -768q(t)q'(t) + 128q'''(t), y^{(4)} = 0\} \):

\[
x^{(5)} = \frac{x^{(4)} + 768q(t)q'(t) - 128q'''(t)}{y^{(4)}}, \quad y^{(5)} = y^{(4)},
\]

\[
\begin{align*}
\frac{dx^{(5)}}{dt} &= \frac{128\{12((q'(t))^2 + 12q(t)q''(t) - q'''(t))\}}{y^{(5)}} + g_1(x^{(5)}, y^{(5)}, t), \\
\frac{dy^{(5)}}{dt} &= g_2(x^{(5)}, y^{(5)}, t),
\end{align*}
\] (64)

where \( g_i(x^{(5)}, y^{(5)}, t) (i = 1, 2) \) are polynomials in \( x^{(5)}, y^{(5)} \). Each right-hand side of the system (64) is a polynomial if and only if

\[
q'''(t) = 12((q'(t))^2 + 12q(t)q''(t)).
\] (65)

This equation reduces as follows:

\[
q''(t) = 6q(t)^2 + C_1 t + C_2 \quad (C_1, C_2 \in \mathbb{C}).
\] (66)

Thus, we have completed the proof of Theorem 25.2.

26. Ince-XXVII equation

Ince-XXVII equation is explicitly given by

\[
\frac{d^2 u}{dt^2} = \frac{m - 1}{mu}\left(\frac{du}{dt}\right)^2 + \left(fu + \phi - \frac{m - 2}{mu}\right)\frac{du}{dt} - \frac{mf^2}{(m + 2)^2}u^3 + \frac{m(f' - f\phi)}{m + 2}u^2 + \psi u - \phi - \frac{1}{mu}.
\] (67)

Proposition 26.1. The canonical transformation

\[
\begin{align*}
x &= \frac{1}{u}, \\
y &= \frac{du}{dt} - \frac{1}{u} - \frac{m}{m + 2}f(t)u
\end{align*}
\]
takes the equation (67) to the system
\[
\begin{aligned}
\frac{dx}{dt} & = -x^2 - xy - \frac{m}{m+2} f(t), \\
\frac{dy}{dt} & = -\frac{y^2}{m} + \phi(t)y + f(t) + \psi(t) - \frac{m}{m+2} f(t).
\end{aligned}
\]  

This system is a Riccati extension of the Riccati equation:
\[
\frac{dy}{dt} = -\frac{y^2}{m} + \phi(t)y + f(t) + \psi(t) - \frac{m}{m+2} f(t).
\]

27. **Ince-XXX equation**

Ince-XXX equation is explicitly given by
\[
\frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 + \frac{3}{2} u^3 + 4\alpha u^2 + 2\beta u - \frac{\gamma^2}{2u}.
\]

**Proposition 27.1.** The canonical transformation
\[
\begin{aligned}
x = u, \\
y = \frac{du}{dt} - \gamma
\end{aligned}
\]

takes the equation (70) to the system
\[
\begin{aligned}
\frac{dx}{dt} & = \frac{\partial H}{\partial y} = xy + \gamma, \\
\frac{dy}{dt} & = -\frac{\partial H}{\partial x} = \frac{3}{2} x^2 - \frac{1}{2} y^2 + 4\alpha x + 2\beta
\end{aligned}
\]

with the polynomial Hamiltonian
\[
H = \frac{1}{2} xy^2 - \frac{1}{2} x^3 - 2\alpha x^2 - 2\beta x + \gamma y.
\]

We remark that the system (71) has the Hamiltonian $H$ as its first integral.

**Theorem 27.2.** After a series of explicit blowing-ups in $\mathbb{P}^2$, we obtain the rational surface $X$ of type $E_6^{(1)}$. The phase space $X$ for the system (71) can be obtained by gluing four copies of $\mathbb{C}^2 \times \mathbb{C}$:

\[U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \quad j = 0, 1, 2, 3\]

via the following birational and symplectic transformations:

0) $x_0 = x$, $y_0 = y$,

1) $x_1 = \frac{1}{x}$, $y_1 = -\left( (y - x - 2\alpha)x + 2\alpha^2 - 2\beta + \gamma \right) x$,

2) $x_2 = \frac{1}{x}$, $y_2 = -\left( (y + x + 2\alpha)x - 2\alpha^2 + 2\beta + \gamma \right) x$,

3) $x_3 = -(xy + 2\gamma)y$, $y_3 = \frac{1}{y}$.
Theorem 27.3. The system (71) is invariant under the following transformations:

\[ s_0 : (x, y; \alpha, \beta, \gamma) \to \left( x + \frac{2\alpha^2 - 2\beta + \gamma}{y - x - 2\alpha}, y + \frac{2\alpha^2 - 2\beta + \gamma}{y - x - 2\alpha}; \alpha, \frac{2\alpha^2 + \gamma}{2}, \frac{-4\alpha^2 + 4\beta + \gamma}{3} \right), \]

\[ s_1 : (x, y; \alpha, \beta, \gamma) \to \left( x + \frac{-2\alpha^2 + 2\beta + \gamma}{y + x + 2\alpha}, y - \frac{-2\alpha^2 + 2\beta + \gamma}{y + x + 2\alpha}; \alpha, \frac{6\alpha^2 - 2\beta - 3\gamma}{4}, \frac{2\alpha^2 - 2\beta + \gamma}{2}, \right), \]

\[ s_2 : (x, y; \alpha, \beta, \gamma) \to \left( x, y + \frac{2\gamma}{x}; \alpha, \beta, -\gamma \right). \]

28. Ince-XXXII equation

Ince-XXXII equation is explicitly given by

\[ \frac{d^2 u}{dt^2} = \frac{1}{2u} \left( \frac{du}{dt} \right)^2 - \frac{1}{2u}. \] (72)

Proposition 28.1. The canonical transformation

\[ \begin{align*}
    &x = u, \\
    &y = \frac{du}{dt} - \frac{1}{u}
\end{align*} \]

takes the equation (72) to the system

\[ \begin{align*}
    &\frac{dx}{dt} = xy + 1, \\
    &\frac{dy}{dt} = -\frac{1}{2}y^2.
\end{align*} \] (73)

This system can be solved by

\[ (x(t), y(t)) = \left( -t + 2c_1 + c_2(t - 2c_1)^2, \frac{2}{t - 2c_1} \right) \quad (c_1, c_2 \in \mathbb{C}). \] (74)

29. Ince-XXXV equation

Ince-XXXV equation is explicitly given by

\[ \frac{d^2 u}{dt^2} = \frac{2}{3u} \left( \frac{du}{dt} \right)^2 - \left( \frac{2}{3} u - \frac{2}{3} q(t) - \frac{r(t)}{u} \right) \frac{du}{dt} + \frac{2}{3} u^3 \]

\[ - \frac{10}{3} q(t) u^2 + \left( 4q'(t) + r(t) + \frac{8}{3} q^2(t) \right) u + 2q(t)r(t) - 3r'(t) - \frac{3r^2(t)}{u}. \] (75)

Proposition 29.1. The canonical transformation

\[ \begin{align*}
    &x = u, \\
    &y = \frac{du}{dt} + 3r(t)
\end{align*} \]

takes the equation (75) to the system

\[
\begin{align*}
\frac{dx}{dt} &= xy - 3r(t), \\
\frac{dy}{dt} &= \frac{2}{3}x^2 - \frac{1}{3}y^2 - \frac{2}{3}xy - \frac{10}{3}q(t)x + \frac{2}{3}q(t)y + \frac{8}{3}q^2(t) + 3r(t) + 4q'(t).
\end{align*}
\]

**Theorem 29.2.** After a series of explicit blowing-ups in \(\mathbb{P}^2\), we obtain the rational surface \(X\) of type \(E_7^{(1)}\). The phase space \(X\) for the system (76) can be obtained by gluing four copies of \(\mathbb{C}^2 \times \mathbb{C}\):

\[
U_j \times \mathbb{C} = \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \quad j = 0, 1, 2, 3
\]

via the following birational transformations:

0) \(x_0 = x, \quad y_0 = y\),

1) \(x_1 = \frac{1}{x}, \quad y_1 = (y + x - 4q(t))x\),

2) \(x_2 = \frac{1}{x}, \quad y_2 = -((y - x/2 + 2q(t))x - 9r(t)/2 - 6q'(t))x + 9r'(t) + 12q''(t))x\),

3) \(x_3 = -\left((xy - 9q(t)/2)y + \frac{18q(t)r(t) - 27r(t)}{2}\right)y, \quad y_3 = \frac{1}{y}\).

In the process of making its phase space, we can obtain the system in dimension three. Setting

\[
x = q(t), \quad y = q'(t), \quad z = r(t),
\]

we obtain the system

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= -2xy - 3xz - \frac{3c_1 + c_2}{4}, \\
\frac{dz}{dt} &= 2xz + c_1.
\end{align*}
\]

Here \(x, y, z\) denote unknown complex variables and \(c_1, c_2\) are constant parameters.

**Theorem 29.3.** The system (77) is invariant under the following transformations:

\[
\begin{align*}
s_0 : (x, y, z; c_1, c_2) &\rightarrow \left(x + \frac{c_1}{z}, y - \frac{2c_1x}{z} - \frac{c_1^2}{z^2}, z; -c_1, -6c_1 + c_2\right), \\
s_1 : (x, y, z; c_1, c_2) &\rightarrow \left(\frac{4xy + 3xz + c_2}{4y + 3z}, \frac{16y^3 + 24y^2z + 9yz^2 + 8c_2xy + 6c_2xz + c_2^2}{(4y + 3z)^2}, \frac{27z^3 + 48y^2z + 72yz^2 - 32c_2xy - 24c_2xz - 4c_2^2}{3(4y + 3z)^2}; c_1 - 2c_2/3, -c_2\right).
\end{align*}
\]

Here, the transformations \(s_0, s_1\) satisfy the relations \(s_0^2 = s_1^2 = 1\).
Theorem 29.4. The phase space $X$ for the system (77) can be obtained by gluing three copies of $\mathbb{C}^3 \times \mathbb{C}$:

$$U_j \times \mathbb{C} = \mathbb{C}^3 \times \mathbb{C} \ni \{(x_j, y_j, z_j, t)\}, \ j = 0, 1, 2$$

via the following birational transformations:

0) $x_0 = x, \ y_0 = y, \ z_0 = z,$

1) $x_1 = \frac{1}{x}, \ y_1 = y + x^2, \ z_1 = -(xz + c_1)x,$

2) $x_2 = \frac{1}{x}, \ y_2 = \frac{1}{3}x(4xy + 3xz + c_2), \ z_2 = \frac{2}{3}(2x^2 + 2y + 3z).$

In the coordinate system $(x_2, y_2, z_2)$, we obtain the system

\[ \begin{cases} 
\frac{dx_2}{dt} = \frac{3}{2}x_2^4y_2 + \frac{c_2}{2}x_2^{-2} + \frac{3}{4}((c_1 - c_2/3)t + c_3)x_2^2 - 1, \\
\frac{dy_2}{dt} = -3x_2^2y_2 + \frac{3c_2}{2}x_2^{-2}y_2 - \frac{3}{2}((c_1 - c_2/3)t + c_3)x_2y_2 - \frac{c_2^2x_2}{6} - \frac{c_2}{4}((c_1 - c_2/3)t + c_3), \\
\frac{dz_2}{dt} = c_1 - \frac{1}{3}c_2.
\end{cases} \]

We see that this system has its first integral:

\[ \frac{dz_2}{dt} = c_1 - \frac{1}{3}c_2, \]

and we can solve explicitly as follows:

\[ z_2(t) = \left(c_1 - \frac{1}{3}c_2\right)t + c_3, \]

where $c_3$ is its integral constant.

Setting

\[ X = \frac{1}{x_2}, \ Y = (x_2y_2 + c_2/3)x_2, \]

we obtain the system

\[ \begin{cases} 
\frac{dX}{dt} = \frac{\partial H}{\partial Y} = X^2 - \frac{3}{2}Y - \frac{1}{4}(3c_1 - c_2)t - \frac{3}{4}c_3, \\
\frac{dY}{dt} = -\frac{\partial H}{\partial X} = -2XY + \frac{c_2}{3}
\end{cases} \]

with the polynomial Hamiltonian

\[ H = X^2Y - \frac{3}{4}Y^2 - \frac{1}{4}(3c_1 - c_2)tY - \frac{3}{4}c_3Y - \frac{c_2}{3}X. \]

This system is the second Painlevé system.
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