SINGULARITIES OF AXISYMMETRIC FREE SURFACE FLOWS
WITH GRAVITY

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Abstract. We consider a steady axisymmetric solution of the Euler equations for a fluid (incompressible and with zero vorticity) with a free surface, acted on only by gravity. We analyze stagnation points as well as points on the axis of symmetry. At points on the axis of symmetry which are not stagnation points, constant velocity motion is the only blow-up profile consistent with the invariant scaling of the equation. This suggests the presence of downward pointing cusps at those points.

At stagnation points on the axis of symmetry, the unique blow-up profile consistent with the invariant scaling of the equation is Garabedian’s pointed bubble solution with water above air. Thus at stagnation points on the axis of symmetry with no water above the stagnation point, the invariant scaling of the equation cannot be the right scaling. A fine analysis of the blow-up velocity yields that in the case that the surface is described by an injective curve, the velocity scales almost like \( \sqrt{x^2 + y^2 + z^2} \) and is asymptotically given by the velocity field

\[
V(\sqrt{x^2 + y^2}, Z) = c(-\sqrt{x^2 + y^2}, 2Z)
\]

with a nonzero constant \( c \).

The last result relies on a frequency formula in combination with a concentration compactness result for the axially symmetric Euler equations by J.-M. Delort. While the concentration compactness result alone does not lead to strong convergence in general, we prove the convergence to be strong in our application.

1. Introduction

Consider the steady axisymmetric Euler equations for a fluid (incompressible and with zero vorticity) with a free surface acted on only by gravity. Using cylindrical coordinates and the Stokes stream function \( \psi \) (see for example Exercise 4.18 (ii)), we obtain the free boundary problem

\[
\begin{align*}
\text{div} \left( \frac{1}{x_1} \nabla \psi(x_1, x_2) \right) &= 0 \text{ in the water phase } \{ \psi > 0 \} \quad (1.1) \\
\frac{1}{x_1^2} |\nabla \psi(x_1, x_2)|^2 &= -x_2 \text{ on the free surface } \partial \{ \psi > 0 \};
\end{align*}
\]
here the original velocity field

\[ V(X, Y, Z) = \left(-\frac{1}{x_1} \partial_2 \psi \cos \vartheta, -\frac{1}{x_1} \partial_2 \psi \sin \vartheta, \frac{1}{x_1} \partial_1 \psi \right), \]

where \((X, Y, Z) = (x_1 \cos \vartheta, x_1 \sin \vartheta, x_2).\)

Observe that the positive sign of \(\psi\) is chosen just for convenience and that replacing \(\psi\) by \(-\psi\) our analysis covers the case of negative \(\psi\) as well.

Note also that the equations above describe apart from a model, where the fluid is pumped in or sucked out at a fixed boundary, also the case of a traveling wave traveling in the direction of the axis of symmetry; here the equations describe the steady flow in the moving frame, so that the original velocity field is

\[ V(X, Y, Z, t) = \tilde{V}(X, Y, Z - c_0 t) + (0, 0, c_0), \]

where \(c_0\) is the speed of the traveling wave and

\[ \tilde{V}(X, Y, Z) = \left(-\frac{1}{x_1} \partial_2 \psi \cos \vartheta, -\frac{1}{x_1} \partial_2 \psi \sin \vartheta, \frac{1}{x_1} \partial_1 \psi \right). \]

[17] and [19], [18] are excellent reviews on two-dimensional water waves.

The free boundary problem \([1.1]\) has been studied in [2] where regularity away from the degenerate sets \(\{x_1 = 0\}\) (the axis of symmetry) and \(\{x_2 = 0\}\) (containing all stagnation points) has been shown for minimizers of a certain energy.

In the present paper we will focus on precisely those two sets and analyze the profile of the velocity vector field close to points in those sets.

Due to the degeneracy of the free boundary condition \(|\nabla \psi(x_1, x_2)|^2 = x_1^2 x_2\) at points \(x^0 = (x^0_1, x^0_2)\) with \(x^0_1 x^0_2 = 0\), we obtain four invariant scalings

- \(\frac{\psi(x^0 + r x)}{r} \) in the case \(x^0_1 \neq 0\) and \(x^0_2 \neq 0\),
- \(\frac{\psi(x^0 + r x)}{r^2} \) in the case \(x^0_1 \neq 0\) and \(x^0_2 = 0\),
- \(\frac{\psi(x^0 + r x)}{r^2} \) in the case \(x^0_1 = 0\) and \(x^0_2 \neq 0\),
- \(\frac{\psi(x^0 + r x)}{r^2} \) in the case \(x^0_1 = x^0_2 = 0\).

Note that the velocity (in the moving frame) would scale like \(1, |x|^\frac{1}{2}, 1, |x|^\frac{1}{2}\) in the respective cases.

In a first main result we determine the profile of the scaled solution as \(r \to 0\) (Proposition 3.10). In the case \(x^0_1 \neq 0\) and \(x^0_2 \neq 0\) the only asymptotics possible is constant velocity flow parallel to the free surface. In the case \(x^0_1 \neq 0\) and \(x^0_2 = 0\) the only asymptotics possible is the well-known Stokes corner flow (see [4], [15], [16], [21]). Due to the perturbed equation the situation is actually not unlike the two-dimensional problem in the presence of vorticity (see [20], [5], [6], [7] for two-dimensional results in the presence of vorticity). In the case \(x^0_1 = 0\) and \(x^0_2 \neq 0\) the
only asymptotics possible is constant velocity flow in the gravity direction. This suggests the possibility of air cusps pointing in the gravity direction (Figure 1).

Figure 1. Dynamics suggested by our analysis

In the case $x_1^0 = x_2^0 = 0$ the only asymptotics possible is the Garabedian pointed bubble solution with water above air (cf. [10], Figure 2). This comes at first as a surprise as it means that there is no nontrivial asymptotic profile at all with air above water and with the invariant scaling. However there remains at this stage the possibility that the solution has a higher growth than that suggested by the invariant scaling.

In Theorem 3.12 we first analyze the possible shapes of the surface close to stagnation points and close to points on the axis of symmetry. Assuming that the surface is given by an injective curve and assuming also a strict Bernstein inequality (corresponding to a Rayleigh-Taylor condition) we obtain the following result:

In the case $x_1^0 \neq 0$ and $x_2^0 = 0$ the only asymptotics possible are the well-known Stokes corner (an angle of opening $120^\circ$ in the direction of the axis of symmetry), and a horizontal point.

In the case $x_1^0 = 0$ and $x_2^0 < 0$ the only asymptotics possible are cusps in the direction of the axis of symmetry.

In the case $x_1^0 = x_2^0 = 0$ the only asymptotics possible are the Garabedian pointed bubble asymptotics (an angle of opening $\approx 114.799^\circ$ with water above air), and a horizontal point.

A fine analysis of the velocity profile in the last case ($x_1^0 = x_2^0 = 0$ and a horizontal point) is no mean feat, and we confine ourselves to the case of air above water. Here we prove (Theorem 7.1) that the velocity scales almost like $\sqrt{X^2 + Y^2 + Z^2}$ and is
Garabedian pointed bubble asymptotics asymptotically given by the velocity field

\[ V(\sqrt{X^2 + Y^2}, Z) = c(-\sqrt{X^2 + Y^2}, 2Z), \]

where \( c \) is a nonzero constant (Figure 3).
The proofs rely on a monotonicity formula as well as a frequency formula for the axisymmetric problem; as remarked in [21], it is for certain semilinear problems possible to derive on the set of highest density not a perturbation of Almgren’s frequency formula (see [1], [13], [12], [11]), but a true nonlinear frequency formula. Here we extend the formula of [21] to the axisymmetric case. In combination with a concentration compactness result for the axially symmetric Euler equations by J.-M. Delort [8], this leads to the already mentioned profile for the velocity vector field. Note that while the concentration compactness result alone does not lead to strong convergence in general, we prove the convergence to the limiting velocity vector field to be strong in our application.

2. Notation

We will use coordinates \((X, Y, Z)\) in the physical space \(\mathbb{R}^3\) together with partial derivatives \(\partial_X, \partial_Y, \partial_Z\) as well as two-dimensional coordinates \(x = (x_1, x_2)\) together with partial derivatives \(\partial_1, \partial_2\). Sometimes we are going to use cylindrical coordinates \((X, Y, Z) = (x_1 \cos \vartheta, x_1 \sin \vartheta, x_2)\). We denote by \(x \cdot y\) the Euclidean inner product in \(\mathbb{R}^n \times \mathbb{R}^n\), by \(|x|\) the Euclidean norm in \(\mathbb{R}^n\), by \(B_r(x^0) := \{x \in \mathbb{R}^n : |x - x^0| < r\}\) the ball of center \(x^0\) and radius \(r\), by \(B^+_r(x^0) := \{x \in \mathbb{R}^n : x_1 > 0\}\) and \(|x - x^0| < r\}, by \(\partial B^+_r(x^0) := \{x \in \mathbb{R}^n : x_1 > 0\}\) and \(|x - x^0| = r\}\) and \(\mathbb{R}^+ := \{(x_1, \ldots, x_n) : x_1 > 0\}\) the positive parts. Note that \(\partial B^+_r(x^0)\) is not the topological boundary of \(B^+_r(x^0)\) and that \(B^+_r(x^0)\) is not necessarily a half ball.

We will use the notation \(B_r\) for \(B_r(0)\) as well as \(B^+_r\) for \(B^+_r(0)\), and denote by \(\omega_2\) the 2-dimensional volume of \(B_1\).

We will use the weighted \(L^p\) space

\[
L^p_w(\mathbb{R}^2_+) := \{v \text{ measurable : } \int_{\mathbb{R}^2_+} \frac{1}{x_1} |v|^p \, dx < +\infty\}
\]

with norm \(\|f\|_{L^p_w(\mathbb{R}^2_+)} = \left(\int_{\mathbb{R}^2_+} \frac{1}{x_1} |v|^p \, dx\right)^{\frac{1}{p}}\), the weighted Sobolev space

\[
W^{1,p}_w(\mathbb{R}^2_+) := \{v \in L^p_w(\mathbb{R}^2_+) : \text{ all weak partial derivatives of } v \text{ are contained in } L^p_w(\mathbb{R}^2_+)\}
\]

as well as the local spaces

\[
L^p_{w,loc}(\mathbb{R}^2_+) := \{v \text{ measurable : } \int_{B^+_R} \frac{1}{x_1} |v|^p \, dx < +\infty \text{ for each } R \in (0, +\infty)\}
\]

and

\[
W^{1,p}_{w,loc}(\mathbb{R}^2_+) := \{v \text{ measurable : } v \in L^p_{w,loc}(\mathbb{R}^2_+) \text{ and all weak partial derivatives of } v \text{ are contained in } L^p_w(B^+_R) \text{ for each } R \in (0, +\infty)\}.
\]

We denote by \(\chi_A\) the characteristic function of a set \(A\). For any real number \(a\), the notation \(a^+\) stands for \(\max(a, 0)\) and \(a^-\) stands for \(\min(a, 0)\). Also, \(\mathcal{L}^n\) shall
denote the \( n \)-dimensional Lebesgue measure and \( \mathcal{H}^s \) the \( s \)-dimensional Hausdorff measure. By \( \nu \) we will always refer to the outer normal on a given surface. We will use functions of bounded variation \( BV(U) \), i.e. functions \( f \in L^1(U) \) for which the distributional derivative is a vector-valued Radon measure. Here \( |\nabla f| \) denotes the total variation measure. Note that for a smooth open set \( E \subset \mathbb{R}^2 \), \( |\nabla \chi_E| \) coincides with the surface measure on \( \partial E \). We will also use the reduced boundary \( \partial_{\text{red}} E \).

3. Notion of solution and monotonicity formula

Let \( \Omega \) be a bounded domain contained in \( \{(x_1, x_2) : x_1 \geq 0\} \), in which to consider the combined problem for fluid and air. We study solutions \( u \), in a sense to be specified, of the problem

\[
\text{div} \left( \frac{1}{x_1} \nabla u \right) = \frac{\partial}{\partial x_1} \left( \frac{1}{x_1} \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \frac{1}{x_1} \frac{\partial u}{\partial x_2} \right) = 0 \quad \text{in} \quad \Omega \cap \{u > 0\},
\]

\[
\frac{1}{x_1^2} |\nabla u|^2 = x_2 \quad \text{on} \quad \Omega \cap \partial \{u > 0\}.
\]

Note that, compared to the Introduction, we have switched notation from \( \psi \) to \( u \), and we have “reflected” the problem at the hyperplane \( \{x_2 = 0\} \). Since our results are completely local, we do not specify boundary conditions on \( \partial \Omega \).

We begin by introducing our notion of a variational solution of problem (3.1).

**Definition 3.1** (Variational Solution). We define \( u \in W^{1,2}_{\text{loc}}(\Omega) \) to be a variational solution of (3.1) if

\[
u \in C^0(\Omega) \cap C^2(\Omega \cap \{u > 0\}),
\]

\[
\frac{1}{x_1} \nabla u \in C^1(\Omega \cap \{u > 0\}),
\]

\[
u(x) = 0 \quad \text{on} \quad \{x_1 = 0\} \quad \text{(motivated by the fact that the velocity on the axis orthogonal to the axis direction should be zero)},
\]

\[
u \geq 0 \quad \text{in} \quad \Omega,
\]

and the first variation with respect to domain variations of the functional

\[
J(v) := \int_{\Omega} \left( \frac{1}{x_1} |\nabla v|^2 + x_1 x_2 \chi_{\{v > 0\}} \right) \, dx
\]

vanishes at \( v = u \), i.e.

\[
0 = -\frac{d}{d\epsilon} J(u(x + \epsilon \phi(x)))|_{\epsilon=0}
\]

\[
= \int_{\Omega} \left[ \left( \frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u > 0\}} \right) \text{div} \phi - 2 \frac{1}{x_1} \nabla u \cdot \nabla \phi + \left( - \frac{1}{x_1^2} |\nabla u|^2 + x_2 \chi_{\{u > 0\}} \right) \phi_1 + x_1 \chi_{\{u > 0\}} \phi_2 \right] \, dx
\]

for any \( \phi = (\phi_1, \phi_2) \in C^1_0(\Omega; \mathbb{R}^2) \) such that \( \phi_1 = 0 \) on \( \{x_1 = 0\} \).

A proof of the just mentioned first variation formula can be found in [14, Section 3.2]. An integration by parts shows that \( u \) satisfies on smooth parts of free boundary \( \partial \{u > 0\} \) in \( \{x_1 x_2 \neq 0\} \) the free boundary condition

\[
\frac{1}{x_1} |\nabla u|^2 = x_1 x_2.
\]
Theorem 3.2 (Monotonicity Formula). Let $u$ be a variational solution of (3.1), let $x^0 \in \Omega$ and let $\delta := \text{dist}(x^0, \partial\Omega)/2$. Let, for any $r \in (0, \delta)$,

$$I(r) = \int_{B^+_1(x^0)} \left( \frac{1}{x_1} \nabla u \cdot \nu + x_1 x_2 \chi_{\{u>0\}} \right) \, dx, \quad (3.2)$$

$$J(r) = \int_{\partial B^+_1(x^0)} \frac{1}{x_1} u^2 \, dH^1, \quad (3.3)$$

$$M^{\text{int}}(r) = r^{-2} I(r) - r^{-3} J(r), \quad (3.4)$$

$$M^{x^2}(r) = r^{-3} I(r) - \frac{3}{2} r^{-4} J(r), \quad (3.5)$$

$$M^{x^1}(r) = r^{-3} I(r) - 2 r^{-4} J(r), \quad (3.6)$$

$$M^{x^1, x^2}(r) = r^{-4} I(r) - \frac{5}{2} r^{-5} J(r). \quad (3.7)$$

Then, for a.e. $r \in (0, \delta)$,

$$(M^{\text{int}}(r))' = 2 r^{-2} \int_{\partial B^+_1(x^0)} \frac{1}{x_1} \left( \nabla u \cdot \nu - \frac{u}{r} \right)^2 \, dH^1 \quad (3.8)$$

$$+ r^{-3} \int_{B^+_1(x^0)} \left( - \frac{x_1 - x^0_1}{x_1} |\nabla u|^2 + \left( (x_1 - x^0_1) x_2 + (x_2 - x^0_2) x_1 \right) \chi_{\{u>0\}} \right) \, dx$$

$$+ r^{-4} \int_{\partial B^+_1(x^0)} \frac{x_1 - x^0_1}{(x_1)^2} u^2 \, dH^1. \quad (3.9)$$

In the case $x^2_0 = 0$,

$$(M^{x^2}(r))' = 2 r^{-3} \int_{\partial B^+_1(x^0)} \frac{1}{x_1} \left( \nabla u \cdot \nu - \frac{3}{2} \frac{u}{r} \right)^2 \, dH^1$$

$$+ r^{-4} \int_{B^+_1(x^0)} \left( - \frac{x_1 - x^0_1}{x_1} |\nabla u|^2 + (x_1 - x^0_1) x_2 \chi_{\{u>0\}} \right) \, dx$$

$$+ \frac{3}{2} r^{-5} \int_{\partial B^+_1(x^0)} \frac{x_1 - x^0_1}{(x_1)^2} u^2 \, dH^1. \quad (3.10)$$

In the case $x^1_0 = 0$,

$$(M^{x^1}(r))' = 2 r^{-3} \int_{\partial B^+_1(x^0)} \frac{1}{x_1} \left( \nabla u \cdot \nu - 2 \frac{u}{r} \right)^2 \, dH^1 \quad (3.11)$$

$$+ r^{-4} \int_{B^+_1(x^0)} (x_2 - x^0_2) x_1 \chi_{\{u>0\}} \, dx.$$ \( Last, \ in \ the \ case \ x_0^1 = x_2^0 = 0, \)

$$(M^{x^1, x^2}(r))' = 2 r^{-4} \int_{\partial B^+_1(x^0)} \frac{1}{x_1} \left( \nabla u \cdot \nu - \frac{5}{2} \frac{u}{r} \right)^2 \, dH^1. \quad (3.11)$$

Remark 3.3. (i) The integrand in the first integral on the right-hand side of (3.8) is a scalar multiple of $(\nabla u(x) \cdot (x - x^0) - u(x))^2$, and therefore vanishes if and only if $u$ is a homogeneous function of degree 1 with respect to $x^0$. 

(ii) The integrand in the first integral on the right-hand side of (3.9) is a scalar multiple of \((\nabla u(x) \cdot (x - x^0) - \frac{3}{2} u(x))^2\), and therefore vanishes if and only if \(u\) is a homogeneous function of degree 3/2 with respect to \(x^0\).

(iii) The integrand in the first integral on the right-hand side of (3.9) is a scalar multiple of \((\nabla u(x) \cdot (x - x^0) - 2u(x))^2\), and therefore vanishes if and only if \(u\) is a homogeneous function of degree 2 with respect to \(x^0\).

(iv) The integrand in the first integral on the right-hand side of (3.9) is a scalar multiple of \((\nabla u(x) \cdot x - \frac{5}{2} u(x))^2\), and therefore vanishes if and only if \(u\) is a homogeneous function of degree 5/2.

**Proof.** First, for each \(u \in W^{1, \frac{2}{\delta}}_{w, loc}(\mathbb{R}^2)\), each \(\alpha \in \mathbb{R}\) and a.e. \(r \in (0, \delta)\) we obtain, setting \(w_r(x) := u(x^0 + rx)\),

\[
\frac{d}{dr} \left( r^\alpha \int_{\partial B^+_{1} (x^0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1 \right) = \frac{d}{dr} \left( r^{\alpha + n - 1} \int_{\partial B^+_{1} (x^0)} \frac{1}{x_1 + rx_1} w_r^2 \, d\mathcal{H}^1 \right) \tag{3.12}
\]

\[
= (\alpha + n - 1)r^{\alpha - 1} \int_{\partial B^+_{1} (x^0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1 - r^{\alpha + n - 1} \int_{\partial B^+_{1} (x^0)} \frac{x_1}{(x_1^2 + rx_1)^2} w_r^2 \, d\mathcal{H}^1 \\
+ r^{\alpha + n - 1} \int_{\partial B^+_{1} (x^0)} \frac{2}{x_1 + rx_1} w_r \nabla u(x^0 + rx) \cdot \nabla \mathcal{H}^1 \\
= (\alpha + n - 1)r^{\alpha - 1} \int_{\partial B^+_{1} (x^0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1 - r^{\alpha - 1} \int_{\partial B^+_{1} (x^0)} \frac{x_1 - x_1^0}{(x_1^2)^2} u^2 \, d\mathcal{H}^1 \\
+ r^{\alpha} \int_{\partial B^+_{1} (x^0)} \frac{2}{x_1} u \nabla u \cdot \nabla \mathcal{H}^1.
\]

Suppose now that \(u\) is a variational solution of (3.1). For small positive \(\kappa\) and \(\eta_\kappa(t) := \max(0, \min(1, \frac{t}{\kappa}))\), we take after approximation \(\phi_\kappa := \eta_\kappa(|x - x^0|)(x - x^0)\) as a test function in the definition of a variational solution, obtaining

\[
0 = \int_{\Omega} \left( \frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u > 0\}} \right) \left( 2\eta_\kappa(|x - x^0|) + \eta'_\kappa(|x - x^0|)|x - x^0| \right) \, dx \\
- \int_{\Omega} \frac{2}{x_1} \left( \langle \partial_1 u \rangle^2 \left( \eta_\kappa(|x - x^0|) + \eta'_\kappa(|x - x^0|) \frac{(x_1 - x_1^0)^2}{|x - x^0|} \right) \\
+ \langle \partial_2 u \rangle^2 \left( \eta_\kappa(|x - x^0|) + \eta'_\kappa(|x - x^0|) \frac{(x_2 - x_2^0)^2}{|x - x^0|} \right) \right) \, dx \\
+ \int_{\Omega} \left( - \frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 \eta_\kappa(|x - x^0|) + [(x_1 - x_1^0)x_2 \\
+ (x_2 - x_2^0)x_1] \chi_{\{u > 0\}} \eta_\kappa(|x - x^0|) \right) \, dx.
\]
Passing to the limit as \( \kappa \to 0 \), we obtain for a.e. \( r \in (0, \delta) \),
\[
0 = 2 \int_{B^+_r(x^0)} x_1 x_2 \chi_{\{u > 0\}} \, dx - r \int_{\partial B^+_r(x^0)} \left( \frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u > 0\}} \right) \, d\mathcal{H}^1 \tag{3.13}
\]
\[
+ 2r \int_{\partial B^+_r(x^0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 \, d\mathcal{H}^1
\]
\[
+ \int_{B^+_r(x^0)} \left( - \frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 + [(x_1 - x_1^0)x_2 + (x_2 - x_2^0)x_1] \chi_{\{u > 0\}} \right) \, dx.
\]

Observe that letting \( \epsilon \to 0 \) in
\[
\int_{B^+_r(x^0)} \frac{1}{x_1} \nabla u \cdot \nabla \max(u - \epsilon, 0)^{1+\epsilon} \, dx = \int_{\partial B^+_r(x^0)} \frac{1}{x_1} \max(u - \epsilon, 0)^{1+\epsilon} \nabla u \cdot \nu \, d\mathcal{H}^1
\]
for a.e. \( r \in (0, \delta) \), we obtain the integration by parts formula
\[
\int_{B^+_r(x^0)} \frac{1}{x_1} |\nabla u|^2 \, dx = \int_{\partial B^+_r(x^0)} \frac{1}{x_1} u \nabla u \cdot \nu \, d\mathcal{H}^1 \tag{3.14}
\]
for a.e. \( r \in (0, \delta) \).

Note that
\[
(r^{-2} I(r))' = -2r^{-3} \int_{B^+_r(x^0)} \left( \frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u > 0\}} \right) \, dx
\]
\[
+ r^{-2} \int_{\partial B^+_r(x^0)} \left( \frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u > 0\}} \right) \, d\mathcal{H}^1,
\]
so that by (3.13) and (3.14),
\[
(r^{-2} I(r))' = r^{-3} \left( 2r \int_{\partial B^+_r(x^0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 \, d\mathcal{H}^1 - 2 \int_{\partial B^+_r(x^0)} \frac{1}{x_1} u \nabla u \cdot \nu \, d\mathcal{H}^1 \right.
\]
\[
+ \left. \int_{B^+_r(x^0)} \left( - \frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 + [(x_1 - x_1^0)x_2 + (x_2 - x_2^0)x_1] \chi_{\{u > 0\}} \right) \, dx \right),
\]
Combining (3.15) and (3.12) with \( \alpha = -3 \) yields (3.8).

Moreover,
\[
(r^{-3} I(r))' = -3r^{-4} \int_{B^+_r(x^0)} \left( \frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u > 0\}} \right) \, dx \tag{3.16}
\]
\[
+ r^{-3} \int_{\partial B^+_r(x^0)} \left( \frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u > 0\}} \right) \, d\mathcal{H}^1.
\]

In the case \( x_2^0 = 0 \) we obtain from (3.16) using (3.13) and (3.14), that
\[
(r^{-3} I(r))' = -r^{-4} \left( 2r \int_{\partial B^+_r(x^0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 \, d\mathcal{H}^1 - 3 \int_{\partial B^+_r(x^0)} \frac{1}{x_1} u \nabla u \cdot \nu \, d\mathcal{H}^1 \right.
\]
\[
+ \left. \int_{B^+_r(x^0)} \left( - \frac{x_1 - x_1^0}{x_1^2} |\nabla u|^2 + (x_1 - x_1^0)x_2 \chi_{\{u > 0\}} \right) \, dx \right),
\]

Combining (3.17) and (3.12) with \( \alpha = -4 \) yields (3.9). On the other hand, in the case \( x_1^0 = 0 \) we obtain from (3.16), using (3.13) and (3.14), that

\[
(r^{-3}I(r))' = r^{-4} \left( 2r \int_{\partial B_+^x(x^0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 \, dH^1 - 4 \int_{\partial B_+^x(x^0)} \frac{1}{x_1} u \nabla u \cdot \nu \, dH^1 \right) 
+ \int_{B_+^x(x^0)} (x_2 - x_2^0)x_1 \chi_{\{u > 0\}} \, dx,
\]

(3.18)

Combining (3.18) and (3.12) with \( \alpha = -4 \) yields (3.10).

Last, in the case \( x_1^0 = x_2^0 = 0 \), since

\[
(r^{-4}I(r))' = -4r^{-5} \int_{B_+^x(0)} \left( \frac{1}{x_1} |\nabla u|^2 + x_1x_2 \chi\{u > 0\} \right) \, dx 
+ r^{-4} \int_{\partial B_+^x(0)} \left( \frac{1}{x_1} |\nabla u|^2 + x_1x_2 \chi\{u > 0\} \right) \, dH^1,
\]

we obtain from (3.13) and (3.14) that

\[
(r^{-4}I(r))' = -5 \left( 2r \int_{\partial B_+^x(0)} \frac{1}{x_1} (\nabla u \cdot \nu)^2 \, dH^1 - 5 \int_{\partial B_+^x(0)} \frac{1}{x_1} u \nabla u \cdot \nu \, dH^1 \right).
\]

(3.19)

Combining (3.19) and (3.12) with \( \alpha = -5 \) yields (3.11).

**Lemma 3.4** (Bernstein estimate). *In \( \{u > 0\} \), the solution satisfies

\[
\Delta \left( \frac{|\nabla u|^2}{x_1} - x_1x_2 \right) = 2 \sum_{i,j=1}^2 \frac{(\partial_{ij} u)^2}{x_1}.
\]

**Proof.** Direct calculation. \( \square \)

**Remark 3.5.** Constructing barrier solutions it is therefore possible to verify \( \frac{|\nabla u|^2}{x_1} - x_1x_2 \leq 0 \) for certain domains \( \subset \{x_2 > 0\} \), certain Dirichlet boundary data and the minimal solution \( u \) (cf. [22]).

**Definition 3.6** (Weak Solution). We define \( u \in W^{1,2}_{u,loc}(\Omega) \) to be a weak solution of (3.1) if the following are satisfied: \( u \) is a variational solution of (3.1) and the topological free boundary \( \partial\{u > 0\} \cap \Omega^2 \cap \{x_2 \neq 0\} \) is locally a \( C^{2,\alpha} \)-surface.

**Remark 3.7.** (i) It follows that in \( \Omega^2 \cap \{x_2 \neq 0\} \) the solution is a classical solution of (3.1). It follows also that \( \partial\{u > 0\} \subset \{x_2 \geq 0\} \).

(ii) For any weak solution \( u \) of (3.1) such that

\[
\frac{|\nabla u|^2}{x_1} \leq Cx_1|x_2| \quad \text{locally in } \Omega,
\]
Lemma 3.8. Let $u$ be a variational solution of (3.1) and suppose that

$$\frac{|\nabla u|^2}{x_1} \leq C x_2$$

locally in $\Omega$.

Then:

(i) The limit $M^{int}(0^+) = \lim_{r \to 0^+} M^{int}(r)$ exists and is finite. If $x_2^0 = 0$, then the limit $M^{z_2}(0^+) = \lim_{r \to 0^+} M^{z_2}(r)$ exists and is finite. If $x_1^0 = 0$, then the limit $M^{z_1}(0^+) = \lim_{r \to 0^+} M^{z_1}(r)$ exists and is finite. If $x_1^0 = x_2^0 = 0$, then the limit $M^{z_1,z_2}(0^+) = \lim_{r \to 0^+} M^{z_1,z_2}(r)$ exists and is finite.

(iii) Let $x_1^0 > 0$, $x_2^0 > 0$ and $0 < r_m \to 0^+$ as $m \to \infty$ be a sequence such that the blow-up sequence

$$u_m(x) := u(x^0 + r_m x)/r_m$$

converges weakly in $W^{1,2}_{loc}(\mathbb{R}^2)$ to a blow-up limit $u_0$. Then $u_0$ is a homogeneous function of degree 1, i.e. $u_0(\lambda x) = \lambda u_0(x)$.

Let $x_2^0 = 0$ and let $0 < r_m \to 0^+$ as $m \to \infty$ be a sequence such that the blow-up sequence

$$u_m(x) := u(x^0 + r_m x)/r_m^{1/2}$$

converges weakly in $W^{1,2}_{loc}(\mathbb{R}^2)$ to a blow-up limit $u_0$. Then $u_0$ is a homogeneous function of degree 3/2.

Let $x_1^0 = 0$ and let $0 < r_m \to 0^+$ as $m \to \infty$ be a sequence such that the blow-up sequence

$$u_m(x) := u(x^0 + r_m x)/r_m^{2}$$

converges weakly in $W^{1,2}_{\omega,loc}(\mathbb{R}^2_+)$ to a blow-up limit $u_0$. Then $u_0$ is a homogeneous function of degree 2.
Let \( x_1^0 = x_2^0 = 0 \) and let \( 0 < r_m \to 0^+ \) as \( m \to \infty \) be a sequence such that the blow-up sequence

\[ u_m(x) := u(x^0 + r_m x)/r_m^{\frac{\sigma}{2}} \]  

(3.23)

converges weakly in \( W^{1,2}_{w,loc}(\mathbb{R}^2) \) to a blow-up limit \( u_0 \). Then \( u_0 \) is a homogeneous function of degree \( 5/2 \).

(iii) Let \( u_m \) be one of the converging sequences in (ii). Then \( u_m \) converges strongly in \( W^{1,2}_{w,loc}(\mathbb{R}^2) \) (strongly in \( W^{1,2}_{loc}(\mathbb{R}^2) \) in the cases where \( x_1^0 > 0 \)).

(iv) If \( x_1^0 > 0 \) and \( x_2^0 \neq 0 \), then

\[ M^{int}(0+) = x_1^0 x_2^0 \lim_{r \to 0^+} r^{-3} \int_{B_r^+(x^0)} x_2 \chi_{\{u > 0\}} dx. \]

Moreover, \( M^{int}(0+) = 0 \) implies that \( u_0 = 0 \) in \( \mathbb{R}^2 \) for each blow-up limit \( u_0 \) of \( u_m(x) = u(x^0 + r_m x)/r_m \).

If \( x_1^0 > 0 \) and \( x_2^0 = 0 \), then

\[ M^{x_2}(0+) = x_1^0 \lim_{r \to 0^+} r^{-3} \int_{B_r^+(x^0)} x_2 \chi_{\{u > 0\}} dx. \]

If \( x_1^0 = 0 \) and \( x_2^0 \neq 0 \), then

\[ M^{x_1}(0+) = x_2^0 \lim_{r \to 0^+} r^{-3} \int_{B_r^+(x^0)} x_1 \chi_{\{u > 0\}} dx. \]

Moreover, \( M^{x_1}(0+) = 0 \) implies that \( u_0 = 0 \) in \( \mathbb{R}^2_+ \) for each blow-up limit \( u_0 \) of \( u_m(x) = u(x^0 + r_m x)/r_m^2 \).

If \( x_1^0 = x_2^0 = 0 \), then

\[ M^{x_1,x_2}(0+) = \lim_{r \to 0^+} r^{-4} \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u > 0\}} dx. \]

Proof. (i) follows from the assumption

\[ \frac{|\nabla u|^2}{x_1} \leq C x_1 |x_2| \text{ locally in } \Omega \]

together with Theorem 3.2

(ii): For each \( 0 < \sigma < \infty \) the sequence \( u_m \) is in each case by assumption bounded in \( C^{0,1}(B^+_r) \) (bounded in \( C^{0,1}(B_r) \) in the case that \( x_1^0 > 0 \)). For any \( 0 < \tau < \sigma < \infty \), we write the identities (3.8), (3.9), (3.8), (3.11) in integral form as

\[ 2 \int_{\tau}^{-2} \int_{\partial B_{\tau}^+(x^0)} \frac{1}{x_1} \left( \nabla u \cdot \nu - \frac{u}{r} \right)^2 dH^1 dr \]

\[ = M^{int}(\sigma) - M^{int}(\tau) - \int_{\tau}^{\sigma} K^{int}(r) dr \text{ in the case } x_1^0 > 0 \text{ and } x_2^0 > 0, \]  

(3.24)

\[ 2 \int_{\tau}^{-3} \int_{\partial B_{\tau}^+(x^0)} \frac{1}{x_1} \left( \nabla u \cdot \nu - \frac{3 u}{2 r} \right)^2 dH^1 dr \]

\[ = M^{x_2}(\sigma) - M^{x_2}(\tau) - \int_{\tau}^{\sigma} K^{x_2}(r) dr \text{ in the case } x_1^0 > 0 \text{ and } x_2^0 = 0, \]  

(3.25)
2 \int_{\partial B_r^+ (x_0)} x_1 \left( \nabla u \cdot \nu - 2u \right)^2 \, dH^1 \, dr

= M^{x_1}(\sigma) - M^{x_1}(\tau) - \int_\tau^\sigma K^{x_1}(r) \, dr \text{ in the case } x_0^1 = 0 \text{ and } x_0^2 > 0, \quad (3.26)

2 \int_{\partial B_r^+ (x_0)} x_1 \left( \nabla u \cdot \nu - \frac{5}{2} u \right)^2 \, dH^1 \, dr

= M^{x_1 x_2}(\sigma) - M^{x_1 x_2}(\tau) \text{ in the case } x_0^1 = x_0^2 = 0; \quad (3.27)

Here \(K^{int}, K^{x_2}\) and \(K^{x_1}\) are defined by (3.24), (3.25) and (3.26), and they are all integrable.

It follows by rescaling in (3.24)-(3.27) that

\[
2 \int \left| x \right|^{-3} \frac{1}{x_1} \left( \nabla u_m (x) \cdot x - u_m (x) \right)^2 \, dx \\
\leq M^{int}(r_m \sigma) - M^{int}(r_m \tau) + \int_{r_m \tau}^{r_m \sigma} |K^{int}(r)| \, dr \to 0 \quad \text{as } m \to \infty,
\]

in the case \(x_0^1 > 0\) and \(x_0^2 > 0\),

\[
2 \int \left| x \right|^{-5} \frac{1}{x_1} \left( \nabla u_m (x) \cdot x - \frac{3}{2} u_m (x) \right)^2 \, dx \\
\leq M^{x_2}(r_m \sigma) - M^{x_2}(r_m \tau) + \int_{r_m \tau}^{r_m \sigma} |K^{x_2}(r)| \, dr \to 0 \quad \text{as } m \to \infty,
\]

in the case \(x_0^1 > 0\) and \(x_0^2 = 0\),

\[
2 \int \left| x \right|^{-5} \frac{1}{x_1} \left( \nabla u_m (x) \cdot x - 2 u_m (x) \right)^2 \, dx \\
\leq M^{x_1}(r_m \sigma) - M^{x_1}(r_m \tau) + \int_{r_m \tau}^{r_m \sigma} |K^{x_1}(r)| \, dr \to 0 \quad \text{as } m \to \infty,
\]

in the case \(x_0^1 = 0\) and \(x_0^2 > 0\),

\[
2 \int \left| x \right|^{-6} \frac{1}{x_1} \left( \nabla u_m (x) \cdot x - \frac{5}{2} u_m (x) \right)^2 \, dx \\
\leq M^{x_1 x_2}(r_m \sigma) - M^{x_1 x_2}(r_m \tau) \to 0 \quad \text{as } m \to \infty
\]

in the case \(x_0^1 = x_0^2 = 0\),

which yields the desired homogeneity of \(u_0\).

(iii): In order to show strong convergence of \(\nabla u_m\), it is in view of the weak \(L^2_w\)-convergence of \(\nabla u_m\) sufficient to prove convergence of the \(L^2_w\)-norm.

Let \(\delta := \text{dist}(x_0, \partial \Omega)/2\). Then, for each \(m, u_m\) is a variational solution of

\[
\text{div} \left( \frac{\nabla u_m (x)}{(x_0 + m x_1)} \right) = 0 \quad \text{in} \ B_{\delta/m} \cap \{u_m > 0\} \text{ in the case } x_0^1 > 0,
\]

\[
\text{in} \ B_{\delta/m}^+ \cap \{u_m > 0\} \text{ in the case } x_0^1 = 0. \quad (3.28)
\]
Since $u_m$ converges to $u_0$ locally uniformly, it follows from (3.28) that $u_0$ is harmonic in $\{u_0 > 0\}$ in the case $x_1^0 > 0$ and a solution of the equation

$$\text{div} \left( \frac{1}{x_1} \nabla u_0 \right) = 0$$

in the case $x_1^0 = 0$. Also, using the uniform convergence, the continuity of $u_0$ and its solution property in $\{u_0 > 0\}$ we obtain as in the proof of (3.14) that

$$o(1) + \int_{\mathbb{R}^2} \frac{1}{x_1^0} |\nabla u_m|_2^2 \eta \, dx$$

$$= \int_{\mathbb{R}^2} \frac{1}{x_1^0 + r_m x_1^0} |\nabla u_m|_2^2 \eta \, dx = - \int_{\mathbb{R}^2} u_m \frac{1}{x_1^0 + r_m x_1^0} \nabla u_m \cdot \nabla \eta \, dx$$

$$\to - \int_{\mathbb{R}^2} u_0 \frac{1}{x_1^0} \nabla u_0 \cdot \nabla \eta \, dx = \frac{1}{x_1^0} \int_{\mathbb{R}^2} |\nabla u_0|_2^2 \eta \, dx$$

in the case $x_1^0 > 0$ and that

$$\int_{\mathbb{R}^2} \frac{1}{x_1^0} |\nabla u_m|_2^2 \eta \, dx = - \int_{\mathbb{R}^2} u_m \frac{1}{x_1^0} \nabla u_m \cdot \nabla \eta \, dx$$

$$\to - \int_{\mathbb{R}^2} u_0 \frac{1}{x_1^0} \nabla u_0 \cdot \nabla \eta \, dx = \int_{\mathbb{R}^2} \frac{1}{x_1^0} |\nabla u_0|_2^2 \eta \, dx$$

as $m \to \infty$. It therefore follows that $\nabla u_m$ converges strongly in $L^2_w$ (and in $L^2$ if $x_1^0 > 0$) to $\nabla u_0$ as $m \to \infty$.

(iv): Let us take a sequence $r_m \to 0^+$ such that $u_m$ defined in (3.20)-(3.23) converges weakly in $W^{1,2}_{w,\text{loc}}(\mathbb{R}^2_+)$ (weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ in the case $x_1^0 > 0$) to a function $u_0$. Using (iii) and the homogeneity of $u_0$, we obtain that

$$\lim_{m \to \infty} M^{\text{int}}(r_m) = \frac{1}{x_1^0} \left( \int_{B_1^+} |\nabla u_0|^2 \, dx - \int_{\partial B_1^+} u_0^2 \, d\mathcal{H}^1 \right)$$

$$+ \lim_{r \to 0^+} r^{-2} \int_{B_1^+(x_0)} x_1 x_2 \chi_{\{u_0 > 0\}} \, dx$$

$$= x_1^0 x_2^0 \lim_{r \to 0^+} r^{-2} \int_{B_1^+(x_0)} \chi_{\{u_0 > 0\}} \, dx,$$

$$\lim_{m \to \infty} M^{\text{f2}}(r_m) = \frac{1}{x_1^0} \left( \int_{B_1^+} |\nabla u_0|^2 \, dx - \frac{3}{2} \int_{\partial B_1^+} u_0^2 \, d\mathcal{H}^1 \right)$$

$$+ \lim_{r \to 0^+} r^{-3} \int_{B_1^+(x_0)} x_1 x_2 \chi_{\{u_0 > 0\}} \, dx$$

$$= x_1^0 \lim_{r \to 0^+} r^{-3} \int_{B_1^+(x_0)} x_2 \chi_{\{u_0 > 0\}} \, dx,$$

$$\lim_{m \to \infty} M^{\text{f1}}(r_m) = \int_{B_1^+} \frac{1}{x_1^0} |\nabla u_0|^2 \, dx - 2 \int_{\partial B_1^+} \frac{1}{x_1^0} u_0^2 \, d\mathcal{H}^1$$

$$+ \lim_{r \to 0^+} r^{-3} \int_{B_1^+(x_0)} x_1 x_2 \chi_{\{u_0 > 0\}} \, dx$$
\[ x^2 \lim_{r \to 0^+} r^{-3} \int_{B^+_r(x^0)} x_1 \chi_{\{u > 0\}} \, dx, \]

\[ \lim_{m \to \infty} M^{x_1 x_2}(r_m) = \int_{B^+_1(x)} \frac{1}{x_1} |\nabla u_0|^2 \, dx - \frac{5}{2} \int_{\partial B^+_1(x)} \frac{1}{x_1} u_0^2 \, d\mathcal{H}^1 \]

\[ + \lim_{r \to 0^+} r^{-4} \int_{B^+_r(x^0)} x_1 x_2 \chi_{\{u > 0\}} \, dx \]

\[ = \lim_{r \to 0^+} r^{-4} \int_{B^+_r(x^0)} x_1 x_2 \chi_{\{u > 0\}} \, dx. \]

In the case \( x^2_0 > 0 \), \( M^{int}(0+) \geq 0 \) and \( M^{x_1}(0+) \geq 0 \), and equality implies that \( u_m \) converges to 0 in measure in \( \mathbb{R}^2_+ \). \( \square \)

The next lemma will be useful in the characterization of blow-up limits in Proposition 3.10.

**Lemma 3.9.** The Legendre function \( y = P_{3/2} \) satisfies

\[ x \mapsto \frac{y'(x)}{y'(-x)} \text{ is strictly increasing on } (-1, 1). \]

**Proof.** It suffices to prove that

\[ y''(x)y'(-x) + y''(-x)y'(x) > 0 \text{ in } (-1, 1). \]

Using the differential equation

\[ (1 - x^2)y''(x) - 2xy'(x) + \frac{3}{2} y(x) = 0, \]

we obtain

\[ y''(x)y'(-x) + y''(-x)y'(x) = -\frac{15}{4} \frac{1}{1 - x^2} (y(x)y'(-x) + y(-x)y'(x)). \]

Therefore it is sufficient to prove that \( f(x) = y(x)y'(-x) + y(-x)y'(x) < 0 \) in \((-1, 1)\). As \( f(x) \to -\infty \) for \( |x| \to 1 \), must have a maximum point in \((-1, 1)\). At the maximum point,

\[ 0 = f'(x) = \frac{2x}{1 - x^2} (y(x)y'(-x) + y(-x)y'(x)), \]

implying that \( x = 0 \) and that

\[ \max f = 2y(0)y'(0) = 3 \frac{\sqrt{\pi}}{\Gamma(-\frac{1}{4})\Gamma(\frac{7}{4})} P_{3/2}(0) \]

\[ = 3 \frac{\sqrt{\pi}}{\Gamma(-\frac{1}{4})\Gamma(\frac{7}{4})} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{4})\Gamma(\frac{7}{4})} < 0 \]

(see [http://functions.wolfram.com/07.07.20.0006.01](http://functions.wolfram.com/07.07.20.0006.01), [http://functions.wolfram.com/07.07.03.0001.01](http://functions.wolfram.com/07.07.03.0001.01)). \( \square \)
Proposition 3.10 (Characterization of blow-up Limits). Let \( u \) be a variational solution of (3.1), and suppose that
\[
\frac{\nabla u^2}{x_1} \leq C x_1 |x_2| \quad \text{locally in } \Omega,
\]
and that
\[
\int_{B^+_1(x_0)} \sqrt{x_1^2 + d|\nabla \chi_{\{u>0\}}|} \leq C_1 \left\{ \begin{array}{ll}
r^3, & x_2^0 = 0 \\
r \sqrt{x_1^2,} & x_2^0 > 0
\end{array} \right.
\]
for all sufficiently small \( r > 0 \).
Then the following hold:

(i) In the case \( x_1^0 > 0 \) and \( x_2^0 > 0 \), the only possible blow-up limits of \( u_m(x) = u(x^0 + r_m x)/r_m \) are
\[
u_0(x) = x_1^0 \sqrt{x_2^0} \max(x \cdot e, 0) \quad \text{and} \quad u_0(x) = \gamma |x \cdot e|,
\]
where \( e \) is a unit vector and \( \gamma \) is a nonnegative constant. In the case \( u_0(x) = x_1^0 \sqrt{x_2^0} \max(x \cdot e, 0) \), the corresponding density is \( M^{int}(0+) = x_1^0 x_2^0 \omega_2/2 \), in the case \( u_0(x) = \gamma |x \cdot e| \) with \( \gamma > 0 \) the density is \( M^{int}(0+) = x_1^0 x_2^0 \omega_2 \), while in the case \( u_0 = 0 \) the density has possible values \( M^{int}(0+) \in \{0, x_1^0 x_2^0 \omega_2\} \).

(ii) In the case \( x_1^0 > 0 \) and \( x_2^0 = 0 \), the only possible blow-up limits are
\[
u_0(\rho \sin \theta, \rho \cos \theta) = \sqrt{2x_1^0 \frac{3}{2}} \rho^{3/2} \cos\left(\frac{3}{2} \theta\right) \chi\{(\rho \sin \theta, \rho \cos \theta) : -\pi/3 < \theta < \pi/3\},
\]
with corresponding density
\[
M^{x_2}(0+) = x_1^0 \int_{B_1} x_2 \chi\{(\rho \sin \theta, \rho \cos \theta) : -\pi/3 < \theta < \pi/3\} \, dx,
\]
and \( u_0(x) = 0 \), with possible values of the density
\[
M^{x_2}(0+) \in \left\{x_1^0 \int_{B_1} x_2^+ \, dx, x_1^0 \int_{B_1} x_2^- \, dx, 0\right\}.
\]

(iii) In the case \( x_1^0 = 0 \) and \( x_2^0 > 0 \), the only possible blow-up limits are
\[
u_0(x) = \gamma x_1^2
\]
with \( \gamma \) a nonnegative constant and corresponding density
\[
M^{x_1}(0+) = x_2^0 \int_{B_1^+} x_1 \, dx,
\]
and \( u_0(x) = 0 \), with possible values of the density
\[
M^{x_1}(0+) \in \left\{x_2^0 \int_{B_1^+} x_1 \, dx, 0\right\}.
\]

(iv) In the case \( x_1^0 = x_2^0 = 0 \), the only possible blow-up limits are
\[
u_0(\rho \sin \theta, \rho \cos \theta) = r^{\frac{3}{2}} U_r(\theta)
\]
with corresponding density
\[ M^{x_1x_2}(0+) = \int_{B_1^+ \cap \{(\rho \sin \theta, \rho \cos \theta) : P_{3/2}(-\cos \theta) < 0\}} x_1 x_2 \, dx, \]

where \( P_{3/2} \) is the Legendre function and \( U_t \) is a unique function which is positive in \( B_1^+ \cap \{ P_{3/2}(-\cos \theta) < 0 \} \) (an angle of \( \approx 114.799^\circ \) in the positive \( x_2 \)-direction) and zero else, and \( u_0(x) = 0 \), with possible values of the density
\[ M^{x_1x_2}(0+) = \left\{ \int_{B_1^+} x_1 x_2^+ \, dx, \int_{B_1^+} x_1 x_2^- \, dx, 0 \right\}. \]

For \( U_t \) we have the relations
\[ \frac{5}{2} U_t(\theta) = c_0 \sin^2 \theta P_{3/2}(-\cos \theta), \quad U_t'(\theta) = \frac{3}{2} \sin \theta P_{3/2}(\cos \theta) \]
with a unique positive constant \( c_0 \).

**Proof.** Consider a blow-up sequence \( u_m \) as in Lemma 3.8 where \( r_m \to 0^+ \), with blow-up limit \( u_0 \). Because of the strong convergence of \( \nabla u_m \) to \( \nabla u_0 \) in \( L^2 \) and the compact embedding from \( BV \) into \( L^1 \), \( u_0 \) is a homogeneous solution of
\[ 0 = \int_{\mathbf{R}^2} \frac{1}{x_1} \left( |\nabla u_0|^2 \text{div} \phi - 2 \nabla u_0 D\phi \nabla u_0 \right) \, dx + x_1^0 x_2^0 \int_{\mathbf{R}^2} \chi_0 \text{div} \phi \, dx \quad (3.29) \]
in the case \( x_1^0 > 0 \) and \( x_2^0 > 0 \),
\[ 0 = \int_{\mathbf{R}^2} \frac{1}{x_1} \left( |\nabla u_0|^2 \text{div} \phi - 2 \nabla u_0 D\phi \nabla u_0 \right) \, dx + \int_{\mathbf{R}^2} \left( x_1^0 x_2 \chi_0 \text{div} \phi + x_1^0 \chi_0 \phi_2 \right) \, dx \quad (3.30) \]
in the case \( x_1^0 > 0 \) and \( x_2^0 = 0 \),
\[ 0 = \int_{\mathbf{R}^2} \frac{1}{x_1} \left( |\nabla u_0|^2 \text{div} \phi - \frac{1}{x_1} |\nabla u_0|^2 \phi_1 - 2 \nabla u_0 D\phi \nabla u_0 \right) \, dx \quad (3.31) \]
\[ + \int_{\mathbf{R}^2} \left( x_1 x_2 \chi_0 \text{div} \phi + x_2 \chi_0 \phi_1 \right) \, dx \]
in the case \( x_1^0 = 0 \) and \( x_2^0 > 0 \),
\[ 0 = \int_{\mathbf{R}^2} \frac{1}{x_1} \left( |\nabla u_0|^2 \text{div} \phi - \frac{1}{x_1} |\nabla u_0|^2 \phi_1 - 2 \nabla u_0 D\phi \nabla u_0 \right) \, dx \quad (3.32) \]
\[ + \int_{\mathbf{R}^2} \left( x_1 x_2 \chi_0 \text{div} \phi + x_2 \chi_0 \phi_1 + x_1 \chi_0 \phi_2 \right) \, dx \]
in the case \( x_1^0 = x_2^0 = 0 \);

the formulas are valid for every \( \phi = (\phi_1, \phi_2) \in C_0^1(\mathbf{R}^2; \mathbf{R}^2) \) in the case \( x_1^0 > 0 \) and for every \( \phi = (\phi_1, \phi_2) \in C_0^1(\mathbf{R}^2; \mathbf{R}^2) \) such that \( \phi_1 = 0 \) on \( \{ x_1 = 0 \} \) in the case \( x_1^0 = 0 \). Moreover \( \chi_0 \) is the strong \( L^1_{\text{loc}} \)-limit of \( \chi_{\{u_m > 0\}} \) along a subsequence. The values of the function \( \chi_0 \) are almost everywhere in \( \{ 0, 1 \} \), and the locally uniform convergence of \( u_m \) to \( u_0 \) implies that \( \chi_0 = 1 \) in \( \{ u_0 > 0 \} \). Moreover \( \chi_0 \) is constant...
in each connected component of \( \{ u_0 = 0 \}^\circ \setminus \{ x_2 = 0 \} \). In the case \( u_0 = 0 \), \((3.32)\) show that \( \chi_0 \) is constant in \( \{ x_2 \neq 0 \} \) in the cases \((3.30)\) and \((3.32)\) and that \( \chi_0 \) is constant in the cases \((3.29)\) and \((3.31)\). Its value may be either 0 or 1.

Let \( z \) be an arbitrary point in \( \partial \{ u_0 = 0 \} \setminus \{ 0 \} \). Consider first the case when \( B_\delta(z) \cap \{ u_0 > 0 \} \) has exactly one connected component. Note that the normal to \( \partial \{ u_0 = 0 \} \) has the constant value \( \nu(z) \) in \( B_\delta(z) \) for some \( \delta > 0 \). Plugging in \( \phi(x) := \eta(x) \nu(z) \) into \((3.29)-(3.32)\), where \( \eta \in C_1^1(B_\delta(z)) \) is arbitrary, and integrating by parts, it follows that

\[
0 = \int_{\partial \{ u_0 > 0 \}} \left( -\frac{1}{x_1} |\nabla u_0|^2 + x_1^0 x_2^0 (1 - \bar{\chi}_0) \right) \eta \, dH^1
\]

in the case \( x_1^0 > 0 \) and \( x_2^0 > 0 \),

\[
0 = \int_{\partial \{ u_0 > 0 \}} \left( -\frac{1}{x_1} |\nabla u_0|^2 + x_1^0 x_2^0 (1 - \bar{\chi}_0) \right) \eta \, dH^1
\]

in the case \( x_1^0 > 0 \) and \( x_2^0 = 0 \),

\[
0 = \int_{\partial \{ u_0 > 0 \}} \left( -\frac{1}{x_1} |\nabla u_0|^2 + x_1 x_2^0 (1 - \bar{\chi}_0) \right) \eta \, dH^1
\]

in the case \( x_1^0 = 0 \) and \( x_2^0 > 0 \),

\[
0 = \int_{\partial \{ u_0 > 0 \}} \left( -\frac{1}{x_1} |\nabla u_0|^2 + x_1 x_2 (1 - \bar{\chi}_0) \right) \eta \, dH^1
\]

in the case \( x_1^0 = x_2^0 = 0 \).

Here \( \bar{\chi}_0 \) denotes the constant value of \( \chi_0 \) in \( \{ u_0 = 0 \}^\circ \). Note that by Hopf’s principle, \( \nabla u_0 \cdot \nu \neq 0 \) on \( B_\delta(z) \cap \partial \{ u_0 > 0 \} \). In all cases it follows therefore that \( \bar{\chi}_0 \neq 1 \), and hence necessarily \( \bar{\chi}_0 = 0 \). We deduce from \((3.33)-(3.36)\) that

\[
|\nabla u_0|^2 = (x_1^0)^2 x_2^0 \text{ on } \partial \{ u_0 > 0 \}
\]

in the case \( x_1^0 > 0 \) and \( x_2^0 > 0 \),

\[
|\nabla u_0|^2 = (x_1^0)^2 x_2 \text{ on } \partial \{ u_0 > 0 \}
\]

in the case \( x_1^0 > 0 \) and \( x_2^0 = 0 \),

\[
|\nabla u_0|^2 = x_1 x_2^0 \text{ on } \partial \{ u_0 > 0 \}
\]

in the case \( x_1^0 = 0 \) and \( x_2^0 > 0 \),

\[
|\nabla u_0|^2 = x_1^2 x_2 \text{ on } \partial \{ u_0 > 0 \}
\]

in the case \( x_1^0 = x_2^0 = 0 \).

Next, let us try to compute \( u_0 \): In the cases where \( x_1^0 > 0 \), the homogeneity of \( u_0 \) and its harmonicity in \( \{ u_0 > 0 \} \) imply the following: if \( x_2^0 > 0 \), then each connected component of \( \{ u_0 > 0 \} \) is a half-plane passing through the origin. If \( x_2^0 = 0 \), then the fact that \( u_0 \) must be harmonic in \( \{ x_2 < 0 \} \), implies that \( \{ u_0 > 0 \} \) is a cone
with vertex at the origin and of opening angle 120° symmetric with respect to and containing \( \{(0, t) : t > 0\} \).

In the cases where \( x_1^0 = 0 \), solving the resulting ODE leads to hypergeometric functions and is slightly awkward, so we will instead use, in each section of the unit disk where \( u_0 > 0 \), the velocity potential \( \phi \) defined by

\[
\partial_1 \phi = \frac{1}{x_1} \partial_2 u, \quad \partial_2 \phi = -\frac{1}{x_1} \partial_1 u.
\]

In the case \( x_2^0 > 0 \) we obtain that \( \phi(\rho \sin \theta, \rho \cos \theta) \) is homogeneous of degree 1 and is on the unit circle given by a linear combination of \( P_1(\cos \theta) \) and \( \Re(Q_1(\cos \theta)) \), where \( P_1 \) and \( Q_1 \) are the Legendre functions. Now \( P_1(x) = x \) and \( \Re Q_1 \) is a strictly convex function with singularities at \(-1\) and 1, so that it is not possible that

\[
\alpha P_1'(x) + (\Re Q_1)'(x) = \alpha P_1'(y) + (\Re Q_1)'(y) \text{ for } x \neq y \in (-1, 1).
\]

It follows that there can be at most one free surface point of the solution \( \alpha P_1(\cos \theta) + \Re Q_1(\cos \theta) \) in \((0, \pi)\), but then the solution would have at least one singularity in the interval \([0, \pi]\). Thus the only solution possible is \( \sigma P_1(\cos \theta) = \sigma \cos \theta \), so that \( \phi(x) = \sigma x_2 \) and \( u_0(x) = cx_1^2 \), where \( c \) and \( \sigma \) are non-negative constants. The statement about the density follows as \( \chi_0 = 1 \) in \( \{u_0 > 0\} \).

In the case \( x_2^0 = 0 \) we obtain that \( \phi(\rho \sin \theta, \rho \cos \theta) \) is homogeneous of degree \( 3/2 \) and is on the unit circle given by a linear combination of \( P_{3/2}(\cos \theta) \) and \( P_{3/2}(-\cos \theta) \), where \( P_{3/2} \) is the Legendre function. It is well known that \( P_{3/2} \) has only one singularity at \(-1\) and that \( P_{3/2}' \) has in \((-1, 1)\) a unique zero \( z_0 \in (-1, 0) \).

By Lemma 3.9 we obtain as in the last case that \( \alpha P_{3/2}(\cos \theta) + \beta P_{3/2}(-\cos \theta) \) can have at most one free surface point in \((0, \pi)\). But then the solution would have at least one singularity in the interval \([0, \pi]\) unless \( \beta = 0 \). The fact that the singularity and the unique zero are both contained in \([-1, 0)\) implies therefore that either

\[
\phi(\rho \sin \theta, \rho \cos \theta) = \sigma \rho^{3/2} P_{3/2}(\cos \theta) \text{ in } \{0 < \theta < \arccos(z_0)\}
\]

or

\[
\phi(\rho \sin \theta, \rho \cos \theta) = \sigma \rho^{3/2} P_{3/2}(-\cos \theta) \text{ in } \{\arccos(-z_0) < \theta < \pi\}.
\]

However the free surface must not intersect \( \{x_2 < 0\} \), so that we obtain that the only admissible solution is

\[
\phi(\rho \sin \theta, \rho \cos \theta) = \sigma \rho^{3/2} P_{3/2}(-\cos \theta) \text{ in } \{\arccos(-z_0) < \theta < \pi\}
\]

for some nonzero constant \( \sigma \). Switching from the velocity potential back to \( u_0 \) we obtain the statement about \( u_0 \) as well as the density.

Last, consider the situation when the set \( B_3(z) \cap \{u_0 > 0\} \) has two connected components. The computations of \( u_0 \) in the respective cases show that this is only possible for \( x_1^0 > 0 \) and \( x_2^0 > 0 \). The argument for \( 3.33 \) yields in this case that
the constant values of $|\nabla u_0|^2$ on either side of $\partial\{u_0 > 0\}$ are equal. This concludes the proof. □

Lemma 3.11. Let $u$ be a weak solution of (3.1) such that $u = 0$ in $\{x_2 \leq 0\}$ and suppose that

$$\frac{|\nabla u|^2}{x_1} \leq x_1x_2^+ \text{ in } \Omega.$$ 

Then $x_2^0 = 0$, $x_1^0 > 0$ and $M^{x_2}(0+) = 0$ imply that $u \equiv 0$ in some open 2-dimensional ball containing $x^0$, while $x_1^0 = x_2^0 = M^{x_1x_2}(0+) = 0$ implies that $u \equiv 0$ in $B_\delta^+$ for some $\delta > 0$.

Proof. Suppose towards a contradiction that $x^0 \in \partial\{u > 0\}$, and let us take a blow-up sequence

$$u_m(x) := u(x^0 + r_m x)/r_m^{3/2}$$

converging weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ to a blow-up limit $u_0$ in the case that $x_1^0 > 0$, and a blow-up sequence

$$u_m(x) := u(x^0 + r_m x)/r_m^{5/2}$$

converging weakly in $W^{1,2}_{\text{w,loc}}(\mathbb{R}^2)$ to a blow-up limit $u_0$ in the case that $x_1^0 = 0$. Proposition 3.10 shows that $u_0 = 0$ in $\mathbb{R}^2$. Consequently,

$$0 \leftarrow \text{div} \left( \frac{1}{x_1^0 + r_m x_1} \nabla u_m \right)(B_2) \geq \int_{B_2 \cap \partial \text{red}\{u_m > 0\}} \sqrt{x_2} \, d\mathcal{H}^1 \text{ in the case } x_1^0 > 0,$$

$$0 \leftarrow \text{div} \left( \frac{1}{x_1^0} \nabla u_m \right)(B_2^+) \geq \int_{B_2^+ \cap \partial \text{red}\{u_m > 0\}} \sqrt{x_2} \, d\mathcal{H}^1 \text{ in the case } x_1^0 = 0,$$

as $m \to \infty$. (Recall that $\text{div} \left( \frac{1}{x_1} \nabla u \right)$ is a nonnegative Radon measure in $\Omega$.) On the other hand, there is at least one connected component $V_m$ of $\{u_m > 0\}$ touching the origin and containing by the maximum principle a point $x^m \in \partial A$, where $A = (-1,1) \times (0,1)$ in the case $x_1^0 > 0$ and $A = (0,1) \times (0,1)$ in the case $x_1^0 = 0$. If $\max\{x_2 : x \in V_m \cap \partial A\} \neq 0$ as $m \to \infty$, we immediately obtain a contradiction to (3.37). If $\max\{x_2 : x \in V_m \cap \partial A\} \to 0$, we use the free-boundary condition as well as $|\nabla u|^2/x_1^0 \leq x_2^+$ to obtain

$$0 = \text{div} \left( \frac{1}{x_1^0 + r_m x_1} \nabla u_m \right)(V_m \cap A) \leq \int_{V_m \cap \partial A} \sqrt{x_2} \, d\mathcal{H}^1 - \int_{A \cap \partial \text{red}\{V_m\}} \sqrt{x_2} \, d\mathcal{H}^1$$

in the case $x_1^0 > 0$,

$$0 = \text{div} \left( \frac{1}{x_1^0} \nabla u_m \right)(V_m \cap A) \leq \int_{V_m \cap \partial A} \sqrt{x_2} \, d\mathcal{H}^1 - \int_{A \cap \partial \text{red}\{V_m\}} \sqrt{x_2} \, d\mathcal{H}^1$$

in the case $x_1^0 = 0$.

However $\int_{V_m \cap \partial A} \sqrt{x_2} \, d\mathcal{H}^1$ is the unique minimiser of $\int_{\partial D} \sqrt{x_2} \, d\mathcal{H}^1$ with respect to all open sets $D$ with $D = V_m$ on $\partial A$. So $V_m$ cannot touch the origin, a contradiction. □
**Theorem 3.12** (Curve Case). Let $u$ be a weak solution of (3.1) satisfying

$$\frac{|\nabla u|^2}{x_1} \leq Cx_1|x_2| \text{ locally in } \Omega,$$

and let $x^0 \in \Omega$ be such that $x_1^0x_2^0 = 0$. Suppose in addition that $\partial \{u > 0\} \cap B^+_1(x^0)$ is in a neighborhood of $x^0$ a continuous injective curve $\sigma : I \to \mathbb{R}^2$ such that $\sigma = (\sigma_1, \sigma_2)$ and $\sigma(0) = x^0$. Then the following hold:

(i) Stokes corner: If $x_1^0 > 0$, $x_2^0 = 0$ and

$$M^{x_2}(0^+) = x_1^0 \int_{B^+_1} x_2 \chi_{\{\rho \sin \theta, \rho \cos \theta) : -\pi/3 < \theta < \pi/3\}} dx,$$

then (cf. Figure 4) $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$ and, depending on the parametrization, either

$$\lim_{t \to 0^+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}} \text{ and } \lim_{t \to 0^-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}};$$

or

$$\lim_{t \to 0^+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}} \text{ and } \lim_{t \to 0^-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}}.$$
Figure 5. Horizontal point \((x_0^1 > 0, x_0^2 = 0)\)

(iii) In the case \(x_0^1 > 0, x_0^2 = 0\) and \(M^{x_1}(0^+) = 0\) — which is according to Lemma 3.11 not possible at all provided that \(u = 0\) in \(\{x_2 \leq 0\}\) and the sharp Bernstein inequality holds —, then \(\sigma_1(t) \neq x_0^1\) in \((-t_1, t_1) \setminus \{0\}\), \(\sigma_1 - x_0^1\) does not change its sign at \(t = 0\), and

\[
\lim_{t \to 0^+} \frac{\sigma_2(t)}{\sigma_1(t) - x_0^1} = 0.
\]

(ii1) If \(x_0^2 > 0, x_0^1 = 0\) and \(M^{x_2}(0^+) = x_0^2 \int_{B_1^+} x_1 \, dx\), then (cf. Figures 6–8) \(\sigma_2(t) \neq x_0^2\) in \((0, t_1)\) and

\[
\lim_{t \to 0^+} \frac{\sigma_1(t)}{\sigma_2(t) - x_0^2} = 0,
\]

or \(\sigma_2(t) \neq x_0^2\) in \((-t_1, t_1) \setminus \{0\}\), \(\sigma_2 - x_0^2\) changes sign at \(t = 0\) and

\[
\lim_{t \to 0^+} \frac{\sigma_1(t)}{\sigma_2(t) - x_0^2} = 0.
\]

(ii2) The case \(x_0^2 > 0, x_0^1 = 0\) and \(M^{x_1}(0^+) = 0\) is not possible.

(iii1) Garabedian corner: If \(x_0^1 = x_0^2 = 0\) and

\[
M^{x_1 x_2}(0^+) = \int_{B_1^+ \cap \{(x_1, x_2) : x_1 < 0\}} x_1 x_2 \, dx,
\]

then (cf. Figure 9) \(\sigma_1(t) \neq 0\) in \((0, t_1)\) and,

\[
\lim_{t \to 0^+} \frac{\sigma_2(t)}{\sigma_1(t)} = \tan(\pi/2 - \arccos(-z_0)).
\]

(iii2) If \(x_0^1 = x_0^2 = 0\) and

\[
M^{x_1 x_2}(0^+) = \int_{B_1^+} x_1 x_2^+ \, dx \text{ or } M^{x_1 x_2}(0^+) = \int_{B_1^+} x_1 x_2^- \, dx,
\]
then (cf. Figure 10) $\sigma_1(t) \neq 0$ in $(0, t_1)$ and

$$\lim_{t \to 0} \frac{\sigma_2(t)}{\sigma_1(t)} = 0.$$
In the subsequent sections of the present paper we will analyze the precise asymptotics of the velocity field in the case $M^{x_1 x_2}(0^+) = \int_{B_1^+} x_1 x_2^+ dx$. 

(iii) If $x_1^0 = x_2^0 = 0$ and $M^{x_1 x_2}(0^+) = 0$ —which is according to Lemma 3.11 not possible at all provided that $u = 0$ in $\{x_2 \leq 0\}$ and the sharp Bernstein
inequality holds—, then $\sigma_1(t) \neq 0$ in $(-t_1, t_1) \setminus \{0\}$, and

$$\lim_{t \to 0} \frac{\sigma_2(t)}{\sigma_1(t)} = 0.$$  

**Remark 3.13.** Although we omit a proof in the present paper, a perturbation of the frequency formula in [21] (see [20]) can be used to prove that, if $x_0^1 > x_0^2 = 0$, then case $M^{x^2}(0+) = x_1^0 \int_{B_1^+} x_2^+ dx$ is not possible. Case $(i_1)$ seems possible as we have a nontrivial homogeneous solution. We do at present not have an existence proof for the cusps suggested here.

**Proof.** We prove the claimed results only case (iii), when $x_0^1 = x_0^2 = 0$, the analysis in the other cases being similar. For each $y = (y_1, y_2) \in \mathbb{R}^2$ with $y_1 \geq 0$ and $(y_1, y_2) \neq (0, 0)$, we define $\theta(y) \in [0, \pi]$ by the relation

$$(y_1, y_2) = (\rho(y) \sin \theta(y), \rho(y) \cos \theta(y)).$$

We now consider the set

$$\mathcal{L} = \{ \theta_0 \in [0, \pi] : \text{there is } t_m \to 0 \text{ such that } \theta(\sigma(t_m)) \to \theta_0 \text{ as } m \to \infty \}.$$  

Note that in fact $\mathcal{L} \subset [0, \pi/2]$, since the free boundary $\partial\{u > 0\}$ is contained in $\{x_2 \geq 0\}$.

We now claim that: The set $\mathcal{L}$ is a subset of $\{0, \theta^*, \pi/2\}$, where $\theta^* = \arccos(-z_0)$ is the angle corresponding to the Garabedian cone.

Indeed, suppose towards a contradiction that a sequence $0 \neq t_m \to 0+$, $m \to \infty$
On the other hand, exists such that \( \theta(\sigma(t_m)) \to \theta_0 \in \mathcal{L} \setminus \{0, \theta^*, \pi/2\} \), let \( r_m := |\sigma(t_m)| \) and let
\[
u_m(x) := \frac{u(r_m x)}{r_m^{5/2}}.
\]

For each \( \rho > 0 \) such that \( \hat{B} := B_\rho(\sin \theta_0, \cos \theta_0) \) satisfies \( \emptyset = \hat{B} \cap ((\alpha, 0) : \alpha \in \mathbb{R}_+) \cup \{(0, \alpha) : \alpha \in \mathbb{R}_+\} \cup \{(\alpha \sin \theta^*, \alpha \cos \theta^*) : \alpha \in \mathbb{R}_+\} \), we infer from the formula for the unique blow-up limit \( u_0 \) (see Theorem 3.10) that the convergence of measures
\[
(\text{div} \frac{1}{x_1} \nabla u_m)(\hat{B}) = (\text{div} \frac{1}{x_1} \nabla u_0)(\hat{B}) = 0 \text{ as } m \to \infty.
\]

On the other hand,
\[
\text{div} \frac{1}{x_1} \nabla u_m = \sqrt{x_2} \mathcal{H}^{1}|\partial\{u_m > 0\},
\]
which implies, since \( \hat{B} \cap \partial\{u_m > 0\} \) contains a curve of length at least \( 2\rho - o(1) \), that
\[
0 \leftarrow (\text{div} \frac{1}{x_1} \nabla u_m)(\hat{B}) \geq c(\theta_0, \rho) \text{ as } m \to \infty,
\]
where \( c(\theta_0, \rho) > 0 \), a contradiction. This proves the property claimed.

Now, a continuity argument yields that \( \mathcal{L} \) is a connected set. Consequently the limit
\[
\ell = \lim_{t \to 0^+} \theta(\sigma(t))
\]
exists and is contained in the set \( \{0, \theta^*, \pi/2\} \). In what follows, we identify the value of \( \ell \) in terms of the value of \( M^{x_1 x_2}(0^+) \).

Suppose first that \( M^{x_1 x_2}(0^+) = \int_{B_1^+ \cap \{P_{1/2}(-\cos \theta) < 0\}} x_1 x_2 \, dx \). Then, by Proposition 3.10, the blow-up limit is
\[
u_0(\rho \sin \theta, \rho \cos \theta) = r^\frac{5}{2} U_\ell(\theta).
\]
Since \( (\text{div} \frac{1}{x_1} \nabla u_0)(B_1/100(\sin \theta^*, \cos \theta^*)) > 0 \), it follows that we cannot have \( \ell \in \{0, \pi/2\} \), and therefore we must have \( \ell = \theta^* \). This proves case (iii) of the Theorem.

Suppose now that \( M^{x_1 x_2}(0^+) \in \left\{ \int_{B_1^+} x_1 x_2^+ \, dx, \int_{B_1^+} x_1 x_2^- \, dx, 0 \right\} \). Then the blow-up limit is \( u_0(x) = 0 \). The same argument given earlier in the proof shows that \( \ell \neq \theta^* \), so that necessarily \( \ell \in \{0, \pi/2\} \). But then the formula in Lemma 3.8 that
\[
M^{x_1 x_2}(0^+) = \lim_{r \to 0^+} r^{-4} \int_{B_r^+(x^0)} x_1 x_2 (\chi_{\{u > 0\}}) \, dx,
\]
shows that \( \ell = 0 \) implies \( M^{x_1 x_2}(0^+) = 0 \), while \( \ell = \pi/2 \) implies that \( M^{x_1 x_2}(0^+) \in \left\{ \int_{B_1^+} x_1 x_2^+ \, dx, \int_{B_1^+} x_1 x_2^- \, dx, 0 \right\} \). However, the possibility that \( M^{x_1 x_2}(0^+) = 0 \) and \( \ell = 0 \) is ruled out by the argument in the proof of Lemma 3.11 even in the absence of the strict Bernstein condition. This proves the cases (iii) and (iii) of the Theorem.

\[\Box\]
4. Frequency formula

From now on we will focus on the case \( x_1^0 = x_2^0 = 0 \), \( u = 0 \) in \( \{x_2 \leq 0\} \) and \( M^{x_1 x_2}(0+) = \int_{B^+_1} x_1 x_2^+ \, dx \), in which we will derive a precise asymptotic profile of the velocity.

**Theorem 4.1 (Frequency Formula).** Let \( u \) be a variational solution of (3.1), and let \( \delta := \text{dist}(0, \partial \Omega) / 2 \). Let, for any \( r \in (0, \delta) \),

\[
D(r) = \frac{r \int_{B^+_1(0)} \frac{1}{x_1^2} |\nabla u|^2 \, dx}{\int_{\partial B^+_1(0)} \frac{1}{x_1^2} u^2 \, d\mathcal{H}^1}
\]

and

\[
V(r) = \frac{r \int_{B^+_1(0)} x_1 x_2 (1 - \chi_{\{u > 0\}}) \, dx}{\int_{\partial B^+_1(0)} \frac{1}{x_1^2} u^2 \, d\mathcal{H}^1}.
\]

Then the “frequency”

\[
H(r) = D(r) - V(r)
\]

satisfies for a.e. \( r \in (0, \delta) \) the identities

\[
H'(r) = 2 \frac{r}{r \int_{\partial B^+_1(0)} \frac{1}{x_1^2} \, d\mathcal{H}^1} \left[ \left( \frac{r (\nabla u \cdot \nu)}{\left( \int_{\partial B^+_1(0)} \frac{1}{x_1^2} u^2 \, d\mathcal{H}^1 \right)^{1/2}} - D(r) \right) \frac{u}{\left( \int_{\partial B^+_1(0)} \frac{1}{x_1^2} u^2 \, d\mathcal{H}^1 \right)^{1/2}} \right]^2 \, d\mathcal{H}^1
\]

and

\[
H'(r) = 2 \frac{r}{r \int_{\partial B^+_1(0)} \frac{1}{x_1^2} \, d\mathcal{H}^1} \left[ \left( \frac{r (\nabla u \cdot \nu)}{\left( \int_{\partial B^+_1(0)} \frac{1}{x_1^2} u^2 \, d\mathcal{H}^1 \right)^{1/2}} - H(r) \right) \frac{u}{\left( \int_{\partial B^+_1(0)} \frac{1}{x_1^2} u^2 \, d\mathcal{H}^1 \right)^{1/2}} \right]^2 \, d\mathcal{H}^1
\]

\[
+ \frac{2}{r} V^2(r) + \frac{2}{r} V(r) \left( H(r) - \frac{5}{2} \right).
\]

**Proof.** Note that, for all \( r \in (0, \delta) \),

\[
H(r) = \frac{r^{-4} I(r) - \int_{B^+_1} x_1 x_2 \, dx}{r^{-5} J(r)}.
\]

Hence, for a.e. \( r \in (0, \delta) \),

\[
H'(r) = \frac{(r^{-4} I(r))'}{r^{-5} J(r)} - \frac{(r^{-4} I(r) - \int_{B^+_1} x_1 x_2 \, dx)}{r^{-5} J(r)} \frac{(r^{-5} J(r))'}{r^{-5} J(r)}.
\]
ensures that, for all $r \in (0, \delta)$,

$$H'(r) = \frac{2r \int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}}(\nabla u \cdot \nu)^{2} \, dH^{1} - 5 \int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}}u(\nabla u \cdot \nu) \, dH^{1}}{\int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}}u^{2} \, dH^{1}}$$

\[-(D(r) - V(r)) \frac{1}{r} \left( 2r \int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}}u(\nabla u \cdot \nu) \, dH^{1} - 5 \int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}}u^{2} \, dH^{1} \right)\]

\[= \frac{2}{r} \left( r^{2} \int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}}(\nabla u \cdot \nu)^{2} \, dH^{1} \frac{1}{x_{1}}u^{2} \, dH^{1} - \frac{5}{2} \right) \]

where we have also used the fact, which follows from \[\text{[3.14]}, \]

\[D(r) = \frac{r \int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}}u(\nabla u \cdot \nu) \, dH^{1}}{\int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}}u^{2} \, dH^{1}}. \quad (4.5)\]

Identity \[\text{[4.1]}\] now follows by merely rearranging \[\text{[4.4]}, \]

and the fact that $D(r) = V(r) + H(r)$.

Since \[\text{[4.1]}\] holds, it follows by inspection that \[\text{[4.2]}\] holds if and only if

\[\int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}} \left[ r(\nabla u \cdot \nu) - D(r)u \right]^{2} \, dH^{1} + V^{2}(r) \int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}}u^{2} \, dH^{1} = \int_{\partial B^{+}_{r}(0)} \frac{1}{x_{1}} \left[ r(\nabla u \cdot \nu) - H(r)u \right]^{2} \, dH^{1}. \quad (4.6)\]

However, \[\text{[4.6]}\] is easily verified as a consequence of \[\text{[4.5]}\] and the fact that $D(r) = H(r) + V(r)$. In conclusion, identity \[\text{[4.2]}\] also holds.  

\[\square\]

**Theorem 4.2.** Let $u$ be a variational solution of \[\text{[3.1]}\] such that $u = 0$ in $\{x_{2} \leq 0\}$, let $x^{0} = 0$, suppose that $M^{x_{1}x_{2}}(0+) = \int_{B^{+}_{1}} x_{1}x_{2}^{+} \, dx$, and let $\delta := \text{dist}(0, \partial \Omega)/2$. Then the following hold:

1. $H(r) \geq \frac{5}{2}$ for all $r \in (0, \delta)$.

2. The function $r \mapsto r^{-5}J(r)$ is nondecreasing on $(0, \delta)$.

3. The function $H$ is nondecreasing on $(0, \delta)$, and has a right limit $H(0+)$, where $H(0+) \geq 5/2$.

4. $r \mapsto \frac{1}{r}V^{2}(r) \in L^{1}(0, \delta)$.

**Proof.** (i) The monotonicity, which follows from Theorem 3.2 of the function $M^{x_{1}x_{2}}$ ensures that, for all $r \in (0, \delta)$,

\[r^{-4}I(r) - \frac{5}{2}r^{-5}J(r) \geq \int_{B^{+}_{r}} x_{1}x_{2}^{+} \, dx. \quad (4.7)\]

Using \[\text{[4.3]}, \]

the above inequality may be rearranged in the form of the claimed result.
(ii) Plugging \( \alpha = -5 \) into (3.12), using also (3.14), and then (4.7), we obtain, for a.e. \( r \in (0, \delta) \),

\[
(r^{-5} J(r))' = \frac{2}{r} \left( r^{-4} \int_{B_1^+(0)} \frac{1}{x_1} |\nabla u|^2 \, dx - \frac{5}{2} r^{-5} \int_{\partial B_1^+(0)} \frac{1}{x_1} u^2 \, dH^1 \right)
\geq 2r^{-5} \int_{B_1^+(0)} x_1 x_2 (1 - \chi_{\{u > 0\}}) \, dx \geq 0,
\]

which implies the claimed result.

(iii) The monotonicity of \( H \) on \( (0, \delta) \) is a consequence of (4.1) and (i). The remaining part of the claim is immediate.

(iv) The claimed result follows from (4.1) and (iii).

\[ \square \]

5. Blow-up limits

The Frequency Formula allows passing to blow-up limits.

Proposition 5.1. Let \( u \) be a variational solution of (3.1) such that \( u = 0 \) in \( \{x_2 \leq 0\} \), let \( x^0 = 0 \), and suppose that \( M^{x_1,x_2}(0+) = \int_{B_1^+} x_1 x_2 \, dx \). Then:

(i) There exist \( \lim_{r \to 0^+} V(r) = 0 \) and \( \lim_{r \to 0^+} D(r) = H(0+) \).

(ii) For any sequence \( r_m \to 0^+ \) as \( m \to \infty \), the sequence

\[ v_m(x) := \frac{u(r_m x)}{\sqrt{r_m^{-1} \int_{\partial B_1^+} \frac{1}{x_1} u^2 \, dH^1}} \quad (5.1) \]

is bounded in \( W^{1,2}_{w}(B_1^+) \).

(iii) For any sequence \( r_m \to 0^+ \) as \( m \to \infty \) such that the sequence \( v_m \) in (5.1) converges weakly in \( W^{1,2}_{w}(B_1^+) \) to a blow-up limit \( v_0 \), the function \( v_0 \) is homogeneous of degree \( H(0+) \) in \( B_1^+ \), and satisfies

\[ v_0 \geq 0 \text{ in } B_1, \quad v_0 \equiv 0 \text{ in } B_1^+ \cap \{x_2 \leq 0\} \text{ and } \int_{\partial B_1^+} \frac{1}{x_1} v_0^2 \, dH^1 = 1. \]

Proof. We first prove that, for any sequence \( r_m \to 0^+ \), the sequence \( v_m \) defined in (5.1) satisfies, for every \( 0 < \tau < \sigma < 1 \),

\[
\int_{B_1^+ \setminus B_{\tau}^+} \frac{1}{x_1} \left| x \right|^{-5} \left| \nabla v_m(x) \cdot x - H(0+)v_m(x) \right|^2 \, dx \to 0 \quad \text{as } m \to \infty. \quad (5.2)
\]

Indeed, for any such \( \tau \) and \( \sigma \), it follows by scaling from (4.2) that, for every \( m \) such that \( r_m < \delta \),

\[
\int_{\tau}^{\sigma} \frac{2}{r} \int_{\partial B_r^+} \frac{1}{x_1} \left( \frac{r \nabla v_m \cdot v}{\left( \int_{\partial B_r^+} \frac{1}{x_1} v_m^2 \, dH^1 \right)^{1/2}} - H(r_m r) \frac{v_m}{\left( \int_{\partial B_r^+} \frac{1}{x_1} v_m^2 \, dH^1 \right)^{1/2}} \right)^2 \, dH^1 \, dr \leq H(r_m \sigma) - H(r_m \tau) \to 0 \quad \text{as } m \to \infty,
\]

\[ \square \]
as a consequence of Theorem 4.2 (iii). The above implies that

\[
\int_\tau^\sigma \frac{2}{r} \int_{\partial B^+_r} \frac{1}{x_1} \left[ \frac{r(\nabla v_m \cdot \nu)}{\left( \int_{\partial B^+_r} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 \right)^{1/2}} - H(0+) \frac{v_m}{\left( \int_{\partial B^+_r} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 \right)^{1/2}} \right]^2 d\mathcal{H}^1 \, dr \to 0 \quad \text{as} \quad m \to \infty.
\]

(5.3)

Now note that, for every \( r \in (\tau, \sigma) \subset (0,1) \) and all \( m \) as before, it follows by using Theorem 4.2 (ii), that

\[
\int_{\partial B^+_r} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 = \frac{\int_{\partial B^+_r} \frac{1}{x_1} u^2 d\mathcal{H}^1}{\int_{\partial B^+_r} \frac{1}{x_1} u^2 d\mathcal{H}^1} \leq r^5 \leq 1.
\]

Therefore (5.2) follows from (5.3), which proves our claim. Let us also recall (4.5).

We can now prove all parts of the Proposition.

(i) Suppose towards a contradiction that (i) is not true. Let \( s_m \to 0 \) be such that the sequence \( V(s_m) \) is bounded away from 0. It is a consequence of Theorem 4.2 (iv) that

\[
\min_{r \in \{s_m, 2s_m\}} V(r) \to 0 \quad \text{as} \quad m \to \infty.
\]

Let \( t_m \in [s_m, 2s_m] \) be such that \( V(t_m) \to 0 \) as \( m \to \infty \). For the choice \( r_m := t_m \) for every \( m \), the sequence \( v_m \) given by (5.1) satisfies (5.2). The fact that \( V(r_m) \to 0 \) implies that \( D(r_m) \) is bounded, and hence that \( v_m \) is bounded in \( W^{1,2}_w(B^+_1) \). Let \( v_0 \) be any weak limit of \( v_m \) along a subsequence. Note that by the compact embedding \( W^{1,2}_w(B^+_1) \hookrightarrow L^2(\partial B^+_1), \) \( v_0 \) has norm 1 on \( L^2_w(\partial B^+_1), \) since this is true for \( v_m \) for all \( m \). It follows from (5.2) that \( v_0 \) is homogeneous of degree \( H(0+) \). Note that, by using Theorem 4.2 (ii),

\[
V(s_m) = \frac{s_m^4 \int_{\partial B^+_m} x_1 x_2 (1 - \chi_{\{u > 0\}}) \, dx}{s_m^5 \int_{\partial B^+_m} \frac{1}{x_1} u^2 d\mathcal{H}^1} \leq \frac{s_m^4 \int_{\partial B^+_m} x_1 x_2 (1 - \chi_{\{u > 0\}}) \, dx}{(r_m/2)^5 \int_{\partial B^+_m} \frac{1}{x_1} u^2 d\mathcal{H}^1} \leq \frac{1}{2} \int_{\partial B^+_{r_m/2}} \frac{1}{x_1} u^2 d\mathcal{H}^1 \, V(r_m) \leq \frac{1}{2} \int_{\partial B^+_{r_m/2}} \frac{1}{x_1} u^2 d\mathcal{H}^1 \, V(r_m).
\]

(5.4)

Since, at least along a subsequence,

\[
\int_{\partial B^+_{r_m/2}} \frac{1}{x_1} v_m^2 d\mathcal{H}^1 \to \int_{\partial B^+_{r_m/2}} \frac{1}{x_1} v_0^2 d\mathcal{H}^1 > 0,
\]

(5.4) leads to a contradiction. It follows that indeed \( V(r) \to 0 \) as \( r \to 0^+ \). This implies that \( D(r) \to H(0+) \).
Let us now consider the velocity field in three dimensions

\[ \phi \]

and excludes concentration. In order to do so we combine the concentration compactness result of J.-M. Delort [8] with information gained by our Frequency Formula. In addition, we obtain strong convergence of our blow-up sequence which is necessary in order to prove our main theorems.

6. Concentration compactness

In the present section we will prove a concentration compactness result which allows us to preserve variational solutions in the blow-up limit at degenerate points and excludes concentration. In order to do so we combine the concentration compactness result of J.-M. Delort [8] with information gained by our Frequency Formula. In addition, we obtain strong convergence of our blow-up sequence which is necessary in order to prove our main theorems.

**Theorem 6.1.** Let \( u \) be a variational solution of (3.1) such that \( u = 0 \) in \( x_2 \leq 0 \) and \( M^{x_1,x_2}(0+) = \int_{B_1^+} x_1 x_2^+ \, dx \). Let \( r_m \to 0+ \) be such that the sequence \( v_m \) given by (5.1) converges weakly to \( v_0 \) in \( W^{1,2}(B_1^+) \). Then \( v_m \) converges to \( v_0 \) strongly in \( W^{1,2}_{w,\text{loc}}(B_1^+ \setminus \{0\}) \), \( v_0 \) is continuous on \( B_1^+ \) and \( \text{div} \left( \frac{1}{x_1} \nabla v_0 \right) \) is a nonnegative Radon measure satisfying \( v_0 \text{div} \left( \frac{1}{x_1} \nabla v_0 \right) = 0 \) in the sense of Radon measures in \( B_1^+ \).

**Proof.** Note first that the homogeneity of \( v_0 \) given by Proposition 5.1, together with the fact that \( v_0 \) belongs to \( W^{1,2}(B_1^+) \), imply that \( v_0 \) is continuous.

Let \( \sigma \) and \( \tau \) with \( 0 < \tau < \sigma < 1 \) be arbitrary. We know that \( \text{div} \left( \frac{1}{x_1} \nabla v_m \right) \geq 0 \) and \( \text{div} \left( \frac{1}{x_1} \nabla v_m \right)(B_{(\sigma+1)/2}^+) \leq C_1 \) for all \( m \). We regularize each \( v_m \) by

\[ \tilde{v}_m := v_m * \phi_m \in C^\infty(B_1^+) \]

where \( \phi_m \) is a standard mollifier such that

\[ \text{div} \left( \frac{1}{x_1} \nabla \tilde{v}_m \right) \geq 0, \int_{B_1^+} \text{div} \left( \frac{1}{x_1} \nabla \tilde{v}_m \right) \leq C_2 < +\infty \quad \text{for all } m, \]

and

\[ \left\| \frac{1}{x_1} (\nabla v_m - \nabla \tilde{v}_m) \right\|_{L^2(B_1^+)} + \| v_m - \tilde{v}_m \|_{L^2(B_1^+)} \to 0 \quad \text{as } m \to \infty. \]

Let us now consider the velocity field in three dimensions

\[ V^m(X,Y,Z) := \left( -\frac{1}{x_1} \partial_2 \tilde{v}_m \cos \theta, -\frac{1}{x_1} \partial_1 \tilde{v}_m \sin \theta, \frac{1}{x_1} \partial_1 \tilde{v}_m \right). \]
where \((X, Y, Z) = (x_1 \cos \vartheta, x_1 \sin \vartheta, x_2)\), as well as their weak limit
\[
V(X, Y, Z) := \left(- \frac{1}{x_1} \partial_2 \tilde{v} \cos \vartheta, - \frac{1}{x_1} \partial_2 \tilde{v} \sin \vartheta, \frac{1}{x_1} \partial_1 \tilde{v}\right).
\]
We have that \(V^m\) is divergence free and satisfies
\[
\text{curl } V^m = \omega^m = (-\sin \vartheta, \cos \vartheta, 0) \alpha^m \text{ in } B_2(0)
\]
with a non-negative function \(\alpha^m\) that is bounded in \(L^1(B_\sigma)\). It follows that
\[
V^m_1 = \Delta^{-1}_{m_1} \partial_2 \omega^m_2,
V^m_2 = -\Delta^{-1}_{m_2} \partial_2 \omega^m_1,
V^m_3 = \Delta^{-1}_{m_3} (\partial_1 \omega^m_1 - \partial_2 \omega^m_2),
\]
where \(\Delta^{-1}_{m_i}\) is the inverse of the three dimensional Laplace operator with averaged Dirichlet boundary data \(V^m_i\), more precisely
\[
\Delta^{-1}_{m_i} f = \frac{2}{1 - \sigma} \int_{B_R} \int_{B_R} G_R f \, dx \, dR + \frac{2}{1 - \sigma} \int_{\partial B_R} V^m_i \nabla G_R \cdot \nu \, dH^2,
\]
where \(G_R\) is Green’s function with respect to the Laplace operator in \(B_R\). From the proof of [S Proposition 3.2], where [S (3.6)] holds with \(v^\varepsilon_i\) replaced by \(V^m_i\) and \(\omega^\varepsilon_i\) replaced by \(\omega^m_i\) but the remainder terms \(w^\varepsilon_i\) given by Greens formula in \(B_\sigma\), we infer that
\[
V^m_1 V^m_3 \rightharpoonup V_1 V_3 \text{ weakly in } L^2_{\text{loc}}(B_\sigma),
V^m_2 V^m_3 \rightharpoonup V_2 V_3 \text{ weakly in } L^2_{\text{loc}}(B_\sigma),
(V^m_1)^2 + (V^m_2)^2 - (V^m_3)^2 \rightharpoonup (V_1)^2 + (V_2)^2 - (V_3)^2 \text{ weakly in } L^2_{\text{loc}}(B_\sigma);
\]
note that as in [S] the remainder terms converge strongly in \(L^2_{\text{loc}}(B_\sigma)\).

It follows that
\[
\frac{1}{x_1} \partial_1 v_m \partial_2 v_m \to \frac{1}{x_1} \partial_1 v_0 \partial_2 v_0 \tag{6.1}
\]
and
\[
\frac{1}{x_1} ((\partial_1 v_m)^2 - (\partial_2 v_m)^2) \to \frac{1}{x_1} ((\partial_1 v_0)^2 - (\partial_2 v_0)^2)
\]
in the sense of distributions on \(B^+_\sigma\) as \(m \to \infty\). Let us remark that in contrast to the true two-dimensional problem, this alone would not allow us to pass to the limit in the domain variation formula for \(v_m!\)

Observe now that (5.2) shows that
\[
\nabla v_m(x) \cdot x - H(0+)v_m(x) \to 0
\]
strongly in \(L^2_\omega(B^+_\sigma \setminus B^+_\tau)\) as \(m \to \infty\). It follows that
\[
\partial_1 v_m x_1 + \partial_2 v_m x_2 \to \partial_1 v_0 x_1 + \partial_2 v_0 x_2
\]
strongly in \( L^2_w(B_1^+ \setminus B_1^+) \) as \( m \to \infty \). But then
\[
\int_{B_1^+ \setminus B_0^+} \frac{1}{x_1} (\partial_1 v_m \partial_1 v_m x_1 + \partial_1 v_m \partial_2 v_m x_2) \eta \, dx \\
\to \int_{B_1^+ \setminus B_0^+} \frac{1}{x_1} (\partial_1 v_0 \partial_1 v_0 x_1 + \partial_1 v_0 \partial_2 v_0 x_2) \eta \, dx
\]
for each \( \eta \in C^0_0(B_1^+ \setminus B_1^+) \) as \( m \to \infty \). Using (6.1), we obtain that
\[
\int_{B_1^+ \setminus B_0^+} (\partial_1 v_m)^2 \eta \, dx \to \int_{B_1^+ \setminus B_0^+} (\partial_1 v_0)^2 \eta \, dx
\]
for each \( 0 \leq \eta \in C^0_0(B_1^+ \setminus B_1^+) \) as \( m \to \infty \). Using once more (6.1) yields that \( \nabla v_m \) converges strongly in \( L^2_w(B_1^+ \setminus B_1^+) \). Since \( \sigma \) and \( \tau \) with \( 0 < \tau < \sigma < 1 \) were arbitrary, it follows that \( \nabla v_m \) converges to \( \nabla v_0 \) strongly in \( L^2_w,loc(B_1^+ \setminus \{0\}) \).

As a consequence of the strong convergence, we see that
\[
\int_{B_1^+} \frac{1}{x_1} \nabla (\eta v_0) \cdot \nabla v_0 = 0 \quad \text{for all} \quad \eta \in C^1_0(B_1^+ \setminus \{0\}).
\]
Combined with the fact that \( v_0 = 0 \) in \( B_1^+ \cap \{ x_2 \leq 0 \} \), this proves that \( v_0 \Delta v_0 = 0 \) in the sense of Radon measures on \( B_1^+ \).

\[ \square \]

7. Degenerate points

**Theorem 7.1.** Let \( u \) be a weak solution of (3.1) such that \( u = 0 \) in \( x_2 \leq 0 \) and \( M^{x_2 x_2}(0+) = \int_{B_1^+} x_1 x_2^2 \, dx \), let the free boundary \( \partial \{ u > 0 \} \cap B_1^+ \) be a continuous injective curve \( \sigma = (\sigma_1, \sigma_2) \) such that \( \sigma(0) = 0 \). Then \( \sigma_1(t) \neq 0 \) in \( [0, t_1) \setminus \{0\} \),
\[
\lim_{t \to 0^+} \frac{\sigma_2(t)}{\sigma_1(t)} = 0
\]
and
\[
\frac{u(rx)}{\sqrt{r^{-1} \int_{\partial B_1^+(0)} u^2 \, dH^1}} \to \frac{x_1^2 x_2}{\sqrt{\int_{\partial B_1^+(0)} x_1^2 x_2^2 \, dH^1}} \quad \text{as} \quad r \to 0^+,
\]
strongly in \( W^{1,2}_{w,loc}(B_1^+ \setminus \{0\}) \) and weakly in \( W^{1,2}(B_1^+) \). Moreover,
\[
\frac{u(rx)}{r^\alpha} \to 0 \quad \text{in} \quad L^2_w(B_1^+) \quad \text{for} \quad \alpha \in (0, 2) \quad \text{and}
\]
\[
\frac{u(rx)}{r^\alpha} \quad \text{is unbounded in} \quad L^2_w(B_1^+) \quad \text{for} \quad \alpha > 2.
\]

**Proof.** Let \( r_m \to 0^+ \) be an arbitrary sequence such that the sequence \( v_m \) given by (5.1) converges weakly in \( W^{1,2}(B_1^+) \) to a limit \( v_0 \). By Proposition 5.1 (iii) and Theorem 6.1, \( v_0 \neq 0 \), \( v_0 \) is homogeneous of degree \( H(0+) \geq 5/2 \), \( v_0 \) is continuous, \( v_0 \geq 0 \) and \( v_0 \equiv 0 \) on \( \{ x_1 = 0 \} \) and in \( \{ x_2 \leq 0 \} \), \( v_0 \text{div} \left( \frac{x_1}{x_1^2} \nabla v_0 \right) = 0 \) in \( B_1^+ \) as a Radon measure, and the convergence of \( v_m \) to \( v_0 \) is strong in \( W^{1,2}_{w,loc}(B_1^+ \setminus \{0\}) \).
Moreover, the strong convergence of $v_m$ and the fact proved in Proposition 5.1 (i) that $V(r_m) \to 0$ as $m \to \infty$ imply that

$$0 = \int_{\mathbb{R}^2} \left( \frac{1}{x_1^2} |\nabla v_0|^2 \text{div} \, \phi - 2\nabla u_0 D\phi \nabla v_0 \right)$$

for every $\phi \in C^1_0(\{x_1 > 0\} \cap \{x_2 > 0\}; \mathbb{R}^2)$, so that even an analysis in the case of $\{u = 0\}$ consisting of infinitely many disconnected components (similar to that in [21]) would be possible in principle. However the structure here is more complicated. For that reason we confine ourselves to the assumed injective curve case.

As in the proof of Proposition 3.10 we will use in each section of the unit disk where $v_0 > 0$ the velocity potential $\phi$ defined by

$$\partial_1 \phi = \frac{1}{x_1} \partial_2 v_0, \quad \partial_2 \phi = -\frac{1}{x_1} \partial_1 v_0.$$

We obtain that $\phi(\rho \sin \theta, \rho \cos \theta)$ is homogeneous of degree $m = H(0+) \geq 5/2$ and is on the unit circle given by a linear combination $f(\cos \theta) = \alpha P_m(\cos \theta) + \beta P_m(-\cos \theta)$, in the case that the Legendre function $P_m$ and the function $P_m(-x)$ are linearly independent, and $f(\cos \theta) = \alpha P_m(\cos \theta) + \beta \Re(Q_m(\cos \theta))$ in the case the Legendre function $P_m$ and the function $P_m(-x)$ are linearly dependent. Moreover $(1,0)$ is a free boundary point of $v_0$ so that $f'(0) = 0$, which implies $\alpha = \beta$ in the case of linear independence.

On the other hand, Theorem 3.12 (ii) implies that for any ball $\tilde{B} \subset B_1^+ \cap \{x_2 > 0\}$, $v_r = \frac{u(x)}{\sqrt{1-\int_{\partial B_1^+} u^2 dH}} > 0$ in $\tilde{B}$. Consequently $\text{div} \left( \frac{1}{x_1} \nabla v_0 \right) = 0$ in $\{x_1 > 0\} \cap \{x_2 > 0\}$. However, if there is a free boundary point $x$ in $(0,1) \times (0,1)$ then by homogeneity the half line connecting that point to the origin consists of free boundary points, so that $(\text{div} \left( \frac{1}{x_1} \nabla v_0 \right))(B_\delta(x)) > 0$ for each $\delta > 0$, a contradiction. Thus $\alpha P_m' + \beta Q_m' \not\equiv 0$ must be either strictly positive or strictly negative in $(0,1)$.

In the case $f(\cos \theta) = \alpha(P_m(\cos \theta) + P_m(-\cos \theta))$ we obtain now a contradiction to the fact that $P_m$ is bounded at 1 and has a singularity at $-1$.

In the case that $P_m$ is an even function, we obtain from $P_m''(0) = mP_{m-1}(0) = \frac{m\pi}{(\frac{m-1}{2})!} \frac{1}{(\frac{m+1}{2})!}$ and $Q_m'(0) = mQ_{m-1}(0) = -\frac{m2^{3/2} \tan(\pi (m-1)/2)}{(m-1)!} \frac{1}{(\frac{m-1}{2})!} \frac{1}{(\frac{m+1}{2})!}$

(see [http://functions.wolfram.com/07.07.0001.01](http://functions.wolfram.com/07.07.0001.01), [http://functions.wolfram.com/07.07.0003.01](http://functions.wolfram.com/07.07.0003.01), [http://functions.wolfram.com/07.07.0001.01](http://functions.wolfram.com/07.07.0001.01)) that $m$ is an even integer $\geq 2$ and that $\beta = 0$ so that $f$ is up to a nonzero multiplicative constant the Legendre polynomial $P_m$. But, using [3 Corollary on p. 114] there is only one even integer $\geq 2$ such that $P_m$ has no critical point in $(0,1)$, namely $m = 2$. We obtain $f(x) = c_2 P_2(x) = c_2 \frac{1}{2}(3x^2 - 1)$. 

In order to obtain the claimed growth we calculate for \( u_r(x) = u(rx)/r^\alpha \) and a.e. \( r \in (0, \delta) \), using (3.14),

\[
\left( \int_{\partial B_r^+(0)} \frac{1}{x_1} u_r^2 \, dH^1 \right)' = \frac{2}{r} \left( \int_{B_r^+(0)} \frac{1}{x_1} \|\nabla u_r\|^2 \, dx - \alpha \int_{\partial B_r^+(0)} \frac{1}{x_1} u_r^2 \, dH^1 \right)
\]

\[
\begin{align*}
\geq & \frac{\kappa}{r} \int_{\partial B_r^+(0)} \frac{1}{x_1} u_r^2 \, dH^1, \quad \alpha \in (0, 2), \\
\leq & -\frac{\kappa}{r} \int_{\partial B_r^+(0)} \frac{1}{x_1} u_r^2 \, dH^1, \quad \alpha > 2.
\end{align*}
\]

Integrating we obtain the result. \( \square \)

**References**

[1] F. J. Almgren, Jr. *Almgren’s big regularity paper*, volume 1 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000. Q-valued functions minimizing Dirichlet’s integral and the regularity of area-minimizing rectifiable currents up to codimension 2, With a preface by J. E. Taylor and V. Scheffer.

[2] Hans Wilhelm Alt, Luis A. Caffarelli, and Avner Friedman. *Jet flows with gravity*. J. Reine Angew. Math., 331:58–103, 1982.

[3] V. I. Arnold. *Lectures on Partial Differential Equations*. Springer, Berlin Heidelberg New York, 2004.

[4] C. J. Amick, L. E. Fraenkel, and J. F. Toland. On the Stokes conjecture for the wave of extreme form. *Acta Math.*, 148:193–214, 1982.

[5] Adrian Constantin and Walter Strauss. Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.*, 57(4):481–527, 2004.

[6] Adrian Constantin and Walter Strauss. Rotational steady water waves near stagnation. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 365(1858):2227–2239, 2007.

[7] Adrian Constantin and Walter Strauss. Periodic traveling gravity water waves with discontinuous vorticity. *Arch. Ration. Mech. Anal.*, 202:133–175, 2011.

[8] Jean-Marc Delort. Une remarque sur le problème des nappes de tourbillon axisymétriques sur \( \mathbb{R}^3 \). *J. Funct. Anal.*, 108(2):274–295, 1992.

[9] L. E. Fraenkel. *An introduction to maximum principles and symmetry in elliptic problems*, volume 128 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000.

[10] P. R. Garabedian. A remark about pointed bubbles. *Comm. Pure Appl. Math.*, 38(5):609–612, 1985.

[11] N. Garofalo and A. Petrosyan. Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem. *Invent. Math.*, 177:415–461, 2009.
[12] Nicola Garofalo and Fang-Hua Lin. Monotonicity properties of variational integrals, $A_p$ weights and unique continuation. *Indiana Univ. Math. J.*, 35(2):245–268, 1986.

[13] Nicola Garofalo and Fang-Hua Lin. Unique continuation for elliptic operators: a geometric-variational approach. *Comm. Pure Appl. Math.*, 40(3):347–366, 1987.

[14] M. Giaquinta and S. Hildebrandt, *Calculus of Variations I*, Springer-Verlag, Berlin, 1996.

[15] P. I. Plotnikov. Justification of the Stokes conjecture in the theory of surface waves. *Dinamika Sploshn. Sredy*, (57):41–76, 1982.

[16] P. I. Plotnikov. Proof of the Stokes conjecture in the theory of surface waves. *Stud. Appl. Math.*, 108(2):217–244, 2002. Translated from Dinamika Sploshn. Sredy No. 57 (1982), 41–76 [MR0752600 (85f:76036)].

[17] Walter A. Strauss. Steady water waves. *Bull. Amer. Math. Soc. (N.S.)*, 47(4):671–694, 2010.

[18] J. F. Toland. Errata to: “Stokes waves” [Topol. Methods Nonlinear Anal. 7 (1996), no. 1, 1–48; MR1422004 (97j:35130)]. *Topol. Methods Nonlinear Anal.*, 8(2):413–414 (1997), 1996.

[19] J. F. Toland. Stokes waves. *Topol. Methods Nonlinear Anal.*, 7(1):1–48, 1996.

[20] E. Varvaruca and G. S. Weiss. The Stokes conjecture for waves with vorticity. *Accepted for publication in Annales de l’Institut Henri Poincare. Analyse Non Lineaire*.

[21] E. Varvaruca and G. S. Weiss. A geometric approach to generalized Stokes conjectures. *Acta Math.*, 206:363–403, 2011.

[22] Georg S. Weiss. Boundary monotonicity formulae and applications to free boundary problems. I. The elliptic case. *Electron. J. Differential Equations*, pages No. 44, 12 pp. (electronic), 2004.

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