On Hypersurface of a Finsler space subjected to $h$-Matsumoto change

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Abstract

Recently, we have studied the Finsler space with $h$-Matsumoto change and found Cartan connection for the transformed space $[2]$. In this paper, we have discussed certain geometrical properties of the hypersurface of a Finsler space for the $h$-Matsumoto change.

Keywords: Finsler space, hypersurface, Matsumoto change, $h$-vector.

1 Introduction

In 1984, C. Shibata [11] introduced a change of Finsler metric called $\beta$-change. An important class of $\beta$-change is Matsumoto change which is given by $\overline{L}(x, y) = \frac{L^2(x, y)}{L_{\beta}}$, where $\beta(x, y) = b_i(x)y^i$ is one form on $M^n$. The concept of an $h$-vector $b_i(x, y)$ has been introduced by H. Izumi [7], which is $v$-covariant constant and satisfies $LC^h_{ij}b_h = \rho_{hij}$, where $\rho$ is a non-zero scalar function. He showed that the scalar $\rho$ is independent of directional arguments. M. K. Gupta and P. N. Pandey [3] proved that the scalar $\rho$ is constant if the $h$-vector $b_i$ is gradient. B. N. Prasad [8] discussed the Cartan connection of Finsler space whose metric is given by $h$-Randers change of a Finsler metric. In 2016, M. K. Gupta and A. K. Gupta [4] studied Finsler space subjected to $h$-exponential change. Recently, we have obtained [2] the relation between the Cartan connection of Finsler space $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$, where $\overline{L}(x, y)$ is obtained by the transformation

$$\overline{L}(x, y) = \frac{L^2(x, y)}{L(x, y) - b_i(x, y)y^i}, \quad (1.1)$$

and $b_i$ is an $h$-vector in $(M^n, L)$.

“A Hypersurface is a generalization of the concept of hyperplane”. The theory of hypersurface in a Finsler space has been first considered by E. Cartan [1] from two points of view, i.e.
(i) A hypersurface as the whole of tangent line-elements and then it is also a Finsler space.

(ii) A hypersurface as the whole of normal line-elements and then it is a Riemannian space.

Three kinds of hyperplane has been introduced by A Rapcsák [12] while M. Matsumoto [10] has classified them and developed a systematic theory of Finslerian hypersurfaces. M. K. Gupta and P. N. Pandey [6] discussed the hypersurfaces of a Finsler space whose metric is given by Kropina change with an $h$-vector. The hypersurface of a Finsler space subjected to an $h$-exponential change of metric has been studied by Gupta and Gupta [5].

In the present paper, we discuss the geometrical properties of hypersurface of a Finsler space subjected to the $h$-Matsumoto change given by (1.1).

The terminologies and notations are referred to Matsumoto [9].

2 Preliminaries

Let $F^n = (M^n, L)$ be an $n$-dimensional Finsler space equipped with the fundamental function $L(x, y)$, satisfying the requisite conditions [9]. The normalized supporting element, angular metric tensor, metric tensor and Cartan tensor are defined by $l_i = \dot{\partial}_i L$, $h_{ij} = L\dot{\partial}_i \dot{\partial}_j L$, $g_{ij} = \frac{1}{2} \partial_i \partial_j L^2$ and $C_{ijk} = \frac{1}{2} \partial_k g_{ij}$ respectively. The Cartan connection in $F^n$ is given as $C\Gamma = (F^{ij}, C^i, C^i_{jk})$.

A hypersurface $M^{n-1}$ of the underlying smooth manifold $M^n$ may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where $u^\alpha$ are Gaussian coordinates on $M^{n-1}$ and Greek indices run from 1 to $n-1$. We assume that the matrix of projection factors $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is of rank $n-1$. If the supporting element $y^i$ at a point $u = (u^\alpha)$ of $M^{n-1}$ is assumed to be tangent to $M^{n-1}$, we may then write $y^i = B^i_\alpha(u) v^\alpha$, so that $v = (v^\alpha)$ is thought of as the supporting element of $M^{n-1}$ at the point $u^\alpha$. Since the function $L(u, v) = L\left(x(u), y(u, v)\right)$ gives rise to a Finsler metric on $M^{n-1}$, we get an $(n-1)$-dimensional Finsler space $F^{n-1} = (M^{n-1}, L(u, v))$.

A unit normal vector $B^i(u, v)$ at each point $u^\alpha$ of $F^{n-1}$ is defined by [10],

$$g_{ij} B^i_\alpha B^j = 0, \quad g_{ij} B^i B^j = 1.$$  \hfill (2.1)

If the inverse projection of $B^i_\alpha$ is $B^\alpha_i(u, v)$, then we have

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta,$$  \hfill (2.2)

where $g^{\alpha\beta}$ is the inverse of the metric tensor $g_{\alpha\beta}$ of $F^{n-1}$.

In view of equation (2.1) and (2.2), we get

$$B^i_\alpha B^j_\beta = \delta^j_\beta, \quad B^i_\alpha B_j = 0, \quad B^i B^\alpha_i = 0, \quad B^i B_i = 1,$$  \hfill (2.3)

and further

$$B^i_\alpha B^\alpha_j + B^j B_j = \delta^i_j.$$  \hfill (2.4)
The second fundamental $h$-vector $H_{\alpha\beta}$ and the normal curvature vector $H_{\alpha}$ for the induced Cartan connection $IC\Gamma=(F^\alpha_{\beta\gamma}, C^\alpha_{\beta\gamma})$ on $F^{n-1}$ are given by

$$H_{\alpha\beta} = B_i(B^i_{\alpha\beta} + F^i_{jk}B^k_{\alpha}B^j_{\beta}) + M_{\alpha}H_{\beta}, \quad (2.5)$$

and

$$H_{\alpha} = B_i(B^i_{\alpha0} + G^i_{j}B^j_{\alpha}), \quad (2.6)$$

where

$$M_{\alpha} = C_{ijk}B^i_{\alpha}B^j_{\beta}B^k_{\alpha}, \quad B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad B^i_{\alpha0} = B^i_{\beta\alpha}v^{\beta}. \quad (2.7)$$

The second fundamental $v$-tensor $M_{\alpha\beta}$ is defined as

$$M_{\alpha\beta} = C_{ijk}B^i_{\alpha}B^j_{\beta}B^k_{\alpha}. \quad (2.8)$$

The relative $h$- and $v$-covariant derivative of $B^i_{\alpha}$ and $B^i$ are given by

$$B^i_{\alpha\beta} = H_{\alpha\beta}B^i, \quad B^i_{\alpha\beta} = M_{\alpha\beta}B^i, \quad B^i_{\beta} = -H_{\alpha\beta}B^\alpha_{ij}g^{ij}, \quad B^i_{\beta} = -M_{\alpha\beta}B^\alpha_{ij}g^{ij}. \quad (2.9)$$

The relative $h$- and $v$-covariant derivatives of a covariant vector field $X_i$ are given by,

$$X_{i\beta} = X_{ij}B^j_{\beta} + X_{i}B^j_{\beta}H_{\beta}, \quad X_{i\beta} = X_{i}B^j_{\beta}. \quad (2.10)$$

Different kinds of hyperplane and their characteristic conditions are classified by Matsumoto [10] which are given in the following Lemmas

**Lemma 1** A hypersurface $F^{n-1}$ is a hyperplane of the first kind if and only if $H_{\alpha} = 0$ or equivalently $H_{\alpha0} = 0$.

**Lemma 2** A hypersurface $F^{n-1}$ is a hyperplane of the second kind if and only if $H_{\alpha\beta} = 0$.

**Lemma 3** A hypersurface $F^{n-1}$ is a hyperplane of the third kind if and only if $H_{\alpha\beta} = 0 = M_{\alpha\beta}$.

### 3 The Finsler space $\overline{F}^{\mu} = (M^n, \overline{L})$

Let $F^n = (M^n, L)$ be the Finsler space equipped with the fundamental function $L(x, y)$ given by (1.1), where $\beta = b_i(x, y)y^i$, $b_i$ is $h$-vector defined by

(i) $b_i|_k = 0$, \hspace{1cm} (ii) $L C^h_{ij} b_h = \rho b_{ij}$, \hspace{1cm} $\rho \neq 0$. \quad (3.1)

Thus from the above definition, we have

$$L \dot{b}_i = \rho b_{ij}. \quad (3.2)$$

In this paper, the geometric objects corresponding to $\overline{F}^{\mu}$ is denoted by bar over the quantity.
We have obtained the normalized supporting element and the angular metric tensor of $\overline{F}^{\alpha\beta}$ as [2]

\[
\overline{t}_i = \frac{\tau}{(\tau - 1)}l_i + \frac{\tau^2}{(\tau - 1)^2}m_i,
\]

and

\[
\overline{t}_{ij} = \frac{\tau^2(\tau + \rho \tau - 2)}{(\tau - 1)^3}h_{ij} + \frac{2\tau^4}{(\tau - 1)^3}m_im_j,
\]

where $\tau = \frac{4}{\beta}$, $m_i = b_i - \frac{1}{\tau}l_i$.

The metric tensor and Cartan tensor of the transformed space are derived as follows [2]

\[
\overline{g}_{ij} = pg_{ij} + p_1l_il_j + p_2(m_il_j + m_jl_i) + p_3m_im_j,
\]

and

\[
\overline{C}_{ijk} = pC_{ijk} + V_{ijk},
\]

where

\[
p = \frac{\tau^2(\tau + \rho \tau - 2)}{(\tau - 1)^3}, \quad p_1 = \frac{\tau^2(1 - \rho \tau)}{(\tau - 1)^3}, \quad p_2 = \frac{\tau^3}{(\tau - 1)^3}, \quad p_3 = \frac{3\tau^4}{(\tau - 1)^4},
\]

\[
V_{ijk} = K_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + K_2m_im_jm_k,
\]

\[
K_1 = \frac{\tau^3(\tau + 3\rho \tau - 4)}{2L(\tau - 1)^4}, \quad K_2 = \frac{6\tau^4}{\beta(\tau - 1)^5}.
\]

**Remark 1** $V_{ijk}$ is an indicatory tensor and satisfies;

(i) $V_{ijk}m^i = (2K_1 + K_2m^2)m_im_k + K_1m^2h_{jk},$

(ii) $V_{ijk}g^{ir} = K_1(h_{jk}m^r + h_{kr}m_j + h_{jr}m_k) + K_2m_jm_km^r,$

where $h_{kr} = h_{rk}g^{ir}$.

The inverse metric tensor of $\overline{F}^{\alpha\beta}$ is obtained as follows [2]

\[
\overline{g}^{ij} = qg^{ij} + q_1l^il^j + q_2(l^im^j + l^jm^i) + q_3m^im^j,
\]

where

\[
q = \frac{1}{p}, \quad q_1 = \frac{-1}{2}\left[\frac{p_1p_3 - p_2^2}{(p_1 + p)p_3 - p_2^2} + \frac{2p_2^2p_3^2}{(3p + 2p_3m^2)(p_1 + p)p_3 - p_2^2}\right],
\]

\[
q_2 = \frac{-2p_3p_2}{(3p + 2p_3m^2)(p_1 + p)p_3 - p_2^2}, \quad q_3 = \frac{-2p_3}{p(3p + 2p_3m^2)},
\]

and $m$ is the magnitude of the vector $m^i = g^{ij}m_j$.

We have obtained the relation between the Cartan connection coefficients $F_{jk}^i$ and $\overline{F}_{jk}^i$ as [2]

\[
\overline{F}_{jk}^i = F_{jk}^i + D_{jk}^i.
\]
The difference tensor $D^i_{jk}$ is given by

\[
D^i_{jk} = g^{is} \left\{ Q_j F_{sk} + Q_s E_{kj} + Q_k F_{js} + p \left( C_{jkm} D^m_s - C_{skm} D^m_j - C_{jsm} D^m_k \right) + V_{jkm} D^m_s - V_{skm} D^m_j - V_{jsm} D^m_k + B_{js} \beta_k - B_{jk} \beta_s + B_{sk} \beta_j \right\}, \tag{3.9}
\]

where

\[
D^i_j = g^{is} \left\{ -2 D^m (p C_{mrj} + V_{mrj}) + Q_r E_{j0} + E_{00} B_{rj} + p_2 L F_{rj} + Q_j F_{r0} + \frac{1}{2} p_2 \rho_k h_{rj} y_k^k \right\}, \tag{3.10}
\]

and

\[
Q_r = (p_2 l_r + p_3 m_r), \quad B_{rj} = K_1 h_{rj} + K_2 m_r m_j
\]

\[
2 E_{ij} = b_{ij} + b_{ji}, \quad 2 F_{ij} = b_{ij} - b_{ji}, \tag{3.12}
\]

\[
\beta_j = \delta_{ij}, \quad \rho_k = \rho_{jk} = \partial_k \rho.
\]

The zero ‘0’ in subscript is denoted for the contraction by $y^i$, for example, $F_{ij} y^j = F_i$. If the $h$-vector $b_i$ is parallel, i.e., $b_{ij} = 0$, then the Cartan connection coefficient for both spaces are equivalent. Moreover then the Berwald connection coefficient for both the spaces are also identical [2].

4 Hypersurface $\tilde{F}^{n-1}$ of the space $\tilde{F}^n$

Let $F^{n-1} = (M^{n-1}, L(u, v))$ be a Finslerian hypersurface of the space $F^n$. The functions $B^i_\alpha (u)$ may be considered as the components of $(n - 1)$ linearly independent vectors tangent to $F^{n-1}$. Let $B^i$ be the unit normal vector at a point of $F^{n-1}$. Then the unit normal vector $\overline{B}^i (u, v)$ of $\tilde{F}^{n-1}$ is uniquely determined by

\[
\overline{g}_{ij} B^i_\alpha \overline{B}^j_\beta = 0, \quad \overline{g}_{ij} \overline{B}^i \overline{B}^j = 1. \tag{4.1}
\]

The inverse projection factors $\overline{B}^i_\alpha$ are uniquely defined along $\tilde{F}^{n-1}$ by

\[
\overline{g}^{i\beta} \overline{B}^i_\alpha = 0, \quad \overline{g}^{ij} \overline{B}^i \overline{B}^j = 1. \tag{4.2}
\]

where $\overline{g}^{i\beta}$ is the inverse of the metric tensor $\overline{g}_{\alpha\beta}$ of $\tilde{F}^{n-1}$.

From (4.2), it follow that

\[
B^i_\alpha \overline{B}^i_\beta = \delta^i_\alpha, \quad B^i_\alpha \overline{B}^i_\alpha = 0, \quad \overline{B}^i \overline{B}^i = 0, \quad \overline{B}^i \overline{B}^i = 1. \tag{4.3}
\]

Transvecting equation (2.1) by $v^\alpha$ and using $B^i_\alpha v^\alpha = y^i$, we get

\[
y_{ij} B^j = 0. \tag{4.4}
\]
Equation (3.5) is contracting by $B^i B^j$ and using (4.1) and (4.4) we have,
\[ \overline{g}_{ij} B^i B^j = p + p_3(m_i B^i)^2, \]  
which shows that $B^i / \sqrt{p + p_3(m_i B^i)^2}$ is a unit normal vector. Again contracting (3.5) by $B^i\alpha B^j$ and using (2.1), (4.4), we obtain
\[ \overline{g}_{ij} B^i\alpha B^j = (p_2 l_i + p_3 m_i) B^i\alpha (B^j m_j). \]  
The above equation shows that the vector $B^j$ is normal to $\overline{F}^{n-1}$ if and only if
\[ (p_2 l_i + p_3 m_i) B^i\alpha (B^j m_j) = 0. \]  
This implies at least one of the following holds
\[ (i) \quad (p_2 l_i + p_3 m_i) B^i\alpha = 0 \quad (ii) \quad B^j m_j = 0. \]  
(i) on transvecting by $v^\alpha$ gives $L = 0$, which is not possible. Therefore (ii) holds, i.e.
\[ B^j m_j = 0, \]  
which, in view of (4.4), can be equivalently written as
\[ B^j b_j = 0. \]  
This shows that the vector $B^j$ is normal to $\overline{F}^{n-1}$ if and only if $b_j$ is tangent to $\overline{F}^{n-1}$. In view of equation (4.5), (4.6) and (4.8) we can say that $B^i / \sqrt{p}$ is a unit normal vector of $\overline{F}^{n-1}$ i.e.
\[ \overline{B}^i = \frac{B^i}{\sqrt{p}}, \]  
which gives
\[ \overline{B}_i = \overline{g}_{ij} \overline{B}^j = \sqrt{p} B_i. \]  
Thus, we have

**Theorem 4.1** Let $\overline{F}^n$ be the Finsler space obtained from $F^n$ by the h-Matsumoto change (1.1). If $\overline{F}^{n-1}$ are the hypersurface of these spaces then the vector $b_i$ is tangential to the hypersurface $F^{n-1}$ if and only if every vector normal to $F^{n-1}$ is also normal to $\overline{F}^{n-1}$.

As $h_{ij} = g_{ij} - l_i l_j$, in view of equation (2.1) and (4.4), we get
\[ h_{ij} B^i\alpha B^j = 0, \quad h_{ij} B^i = B_j. \]  
Then the tensors $B_{ij}$ and $Q_j$ given by (3.12), satisfy the relations
\[ B_{ij} B^i B^j\alpha = 0, \quad B_{ij} B^i = B_j, \quad Q_j B^j = 0. \]  
Transvecting (3.7) by $B_i$ and using $l^i B_i = 0 = m^i B_i$, we get
\[ \overline{g}^{ij} B_i = q B^s. \]
From (2.6), (3.10) and (4.10), we get
$$\overline{H}_\alpha = \sqrt{p}(H_\alpha + B_j B^j_\alpha).$$
Contracting the above equation by $v^\alpha$ and using $v^\alpha B^k_\alpha = y^k$, we get
$$\overline{H}_0 = \sqrt{p}(H_0 + B_i D^i).$$

Equation (3.11) can be rewritten as
$$D^i = \frac{1}{2} \left\{ (q + q_1)p_2 + q_3p_3m^2 \right\} E_{00} + 2q_3p_2 L F_{30} \right\} l^i + \frac{1}{2} \left\{ \mu E_{00} + 2q_3p_2 L F_{30} \right\} m^i + qp_2 L F_{30}.$$  

Transvecting the above equation by $B_i$ and using $m_i B^i = 0$ and $l_i B^i = 0$, we get
$$D^i B_i = qp_2 L B_i F_{30}.$$  

M. K. Gupta and P. N. Pandey [3] proved the following Lemma,

**Lemma 4** If the $h$-vector $b_i$ is gradient then the scalar $\rho$ is constant.

From the above Lemma we get
$$\rho_i = 0.$$  

In view of (4.17), the equation (4.16) becomes
$$D^i B_i = 0.$$  

and then equation (4.14) reduces to
$$\overline{H}_0 = \sqrt{p} H_0.$$  

Thus in view of Lemma (4.2), we have

**Theorem 4.2** For the $h$-Matsumoto change, let the $h$-vector $b_i(x, y)$ be gradient and tangent to the hypersurface $F^{n-1}$. Then the hypersurface $F^{n-1}$ is a hyperplane of the first kind if and only if hypersurface $\overline{F}^{n-1}$ is a hyperplane of the first kind.

The second fundamental $h$-tensor $\overline{H}_{\alpha\beta}$ for hyperplane $\overline{F}^{n-1}$ is given by
$$\overline{H}_{\alpha\beta} = \overline{B}_i (B^i_{\alpha\beta} + F^i_{jk} B^j_\alpha B^k_\beta) + \overline{M}_\alpha \overline{H}_\beta.$$  

In view of equation (3.8) and (4.10), above equation gives
$$\overline{H}_{\alpha\beta} - \overline{M}_\alpha \overline{H}_\beta = \sqrt{p}(H_{\alpha\beta} + D^i_{jk} B_i B^j_\alpha B^k_\beta) - \sqrt{p} M_\alpha H_\beta.$$  

Using (4.17) and (4.18), the equation (3.9) reduces to
$$D^i_{jk} = \frac{1}{2} \left\{ Q_s E_{kj} + pC_{jkm} D^m_s + V_{jkm} D^m_s - pC_{skm} D^m_j - V_{skm} D^m_j ight\}$$
$$- pC_{jsm} D^m_k - V_{jsm} D^m_k + B_j \beta_k + B_k \beta_j - B_{jk} \beta_s.$$  

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Transvecting equation (4.22) by $B_iB_j^\alpha B_k^\beta $ and using $\overline{\gamma}_j B_j = qB_i$, $B^sQ_s = 0$, $B_{sk}B^sB_k^\beta = 0$, we get
\[ D^i_{jk}B_iB_j^\alpha B_k^\beta = qB^sB_j^\alpha B_k^\beta \left\{ pC_{jkm}D^m_s - V_{jkm}D^m_s - pC_{skm}D^m_j - V_{skm}D^m_j 
- pC_{jsm}D^m_k - V_{jsm}D^m_k - B_{jk}\beta \right\}. \] (4.23)

In view of equation (4.15), and using the indicatory property of $C_{ijk}$, $V_{ijk}$, $h_{ij}$, $m_i$, the equation (3.10) can be rewritten as
\[ D^m_s = \overline{\gamma}^{mr}\left\{ \lambda h_{sr} + \phi m_s m_r + Q_rE_{s0} \right\}, \] (4.24)
where
\[ \lambda = \left[ -\mu \left( \frac{\rho \rho}{L} + m^2 K_1 \right) + K_1 \right] E_{00} \quad \text{and} \quad \phi = \left[ -\mu \left( 2K_1 + K_2m^2 \right) + K_2 \right] E_{00}. \] (4.25)

Transvecting equation (4.24) by $C_{jkm}$ we get
\[ C_{jkm}D^m_s = C_{jkm} \left\{ qg^{mr} + q_1 l^m m^r + q_2 (l^m m^r + l^r m^m) + q_3 m^m m^r \right\} \left\{ \lambda h_{sr} + \phi m_s m_r + Q_rE_{s0} \right\}, \] (4.26)
which can be simplified as
\[ C_{jkm}D^m_s = q\lambda C_{jsk} + \left[ (q + q_3 m^2) \phi + q_3 \lambda \right] \frac{\rho}{L} h_{jkm} + \frac{\rho}{L} \mu h_{jkm} E_{s0}. \] (4.27)

Similarly we can write the expressions for $C_{skm}D^m_j$ and $C_{jsm}D^m_k$ as
\[ C_{skm}D^m_j = q\lambda C_{jsk} + \left[ (q + q_3 m^2) \phi + q_3 \lambda \right] \frac{\rho}{L} h_{skm} + \frac{\rho}{L} \mu h_{skm} E_{j0}, \] (4.28)
and
\[ C_{jsm}D^m_k = q\lambda C_{jsk} + \left[ (q + q_3 m^2) \phi + q_3 \lambda \right] \frac{\rho}{L} h_{skm} + \frac{\rho}{L} \mu h_{skm} E_{k0}. \] (4.29)

Transvecting equations (4.27), (4.28) and (4.29) by $B^sB_j^\alpha B_k^\beta $, and using (2.8) and (11.1), we get respectively
\[ B^sC_{jkm}D^m_s B^\mu B^\nu_k = q\lambda M_{\alpha\beta} + \frac{\rho}{L} \mu h_{jkm} B^\mu B^\nu_k B^sE_{s0}, \] (4.30)
\[ B^sC_{skm}D^m_j B^\mu B^\nu_k = q\lambda M_{\alpha\beta}, \] (4.31)
\[ B^sC_{jsm}D^m_k B^\mu B^\nu_k = q\lambda M_{\alpha\beta}. \] (4.32)

Again, transvecting Equation (4.24) by $V_{jkm}$, we get
\[ V_{jkm}D^m_s = V_{jkm} \left\{ qg^{mr} + (q_2 l^r + q_3 m^r) m^m \right\} \left\{ \lambda h_{sr} + \phi m_s m_r + Q_rE_{s0} \right\}. \] (4.33)

By using Remark 1, the above equation can be rewritten as
\[ V_{jkm}D^m_s = q \left\{ K_1[h_{jkm} r + h_r m_k + h_m k m_j] + K_2m_j m_k m_r \right\} \left\{ \lambda h_{sr} + \phi m_s m_r + Q_rE_{s0} \right\}, \] (4.34)
which can be simplified as

\[
V_{jkm}D^m_s = \left\{\psi_1K_1h_{jk} + (\psi_1K_2 + 2\psi_2K_1)m_jm_k \right\}m_s + q\lambda K_1 (h_{js}m_k + h_{sk}m_j) \\
+ \left\{ \mu [m^2K_1h_{sk} + (2K_1 + K_2m^2)m_jm_k] \right\} E_{s0},
\]

(4.34)

where

\[
\psi_1 = (\lambda + \phi m^2) (q + q_3m^2) \quad \text{and} \quad \psi_2 = (\lambda + \phi m^2) q_3 + q\phi.
\]

Similarly we can write the expression for \(V_{skm}D^m_j\) and \(V_{jsm}D^m_k\) as

\[
V_{skm}D^m_j = \left\{\psi_1K_1h_{sk} + (\psi_1K_2 + 2\psi_2K_1)m_jm_k \right\}m_j + q\lambda K_1 (h_{js}m_k + h_{jk}m_s) \\
+ \left\{ \mu [m^2K_1h_{sk} + (2K_1 + K_2m^2)m_jm_k] \right\} E_{j0},
\]

(4.35)

and

\[
V_{jsm}D^m_k = \left\{\psi_1K_1h_{sj} + (\psi_1K_2 + 2\psi_2K_1)m_jm_s \right\}m_k + q\lambda K_1 (h_{sk}m_j + h_{jk}m_s) \\
+ \left\{ \mu [m^2K_1h_{sj} + (2K_1 + K_2m^2)m_jm_k] \right\} E_{k0}.
\]

(4.36)

Contracting equation (4.34), (4.35), (4.36) by \(B^sB^j\alpha B^k\beta\) and using \(B^i m_i = 0 = h_{ij}B^i\alpha\), we get respectively

\[
B^sV_{jkm}D^m_s B^j\alpha B^k\beta = \mu \left\{K_1m^2h_{jk} + (2K_1 + K_2m^2)m_jm_k \right\} B^j\alpha B^k\beta E_{s0},
\]

(4.37)

\[
B^sV_{skm}D^m_j B^j\alpha B^k\beta = 0,
\]

(4.38)

\[
B^sV_{jsm}D^m_k B^j\alpha B^k\beta = 0.
\]

(4.39)

Putting the value of equation (4.30), (4.31), (4.32), (4.37), (4.38) and (4.39) in equation (4.23), we get

\[
D^i_j B_i\alpha B^j\beta = \left[ \frac{\mu\rho}{L} - q \left\{ [2K_1 + K_2m^2]m_jm_k \right\} + K_1m^2h_{jk} \right\} B^j\alpha B^k\beta E_{s0} - q\lambda M_{\alpha\beta}.
\]

(4.40)

Now taking the relative \(h\)-covariant differentiation of \(b_iB^i = 0\) with respect to the Cartan connection of \(E^{m-1}\), we get

\[
b_{ij}B^i + b_iB^i_{\beta} = 0.
\]

In view of equation (2.9) and (2.10), above equation becomes

\[
\left(b_{ij}B^j H_{\beta} + b_{ij}B^j_{\beta}\right) B^i - b_iH_{\alpha\beta} B^\alpha g^{ij} = 0,
\]

which on contraction by \(v^\beta\) and using (2.7), gives

\[
b_{i0}B^i = (H_{\alpha} + M_{\alpha}H_0) B^\alpha b^i - b_{ij}H_0 B^j B^i.
\]

In view of Lemma 1, if the hypersurface to be first kind then \(H_0 = 0 = H_{\alpha}\). Thus the above equation reduces to \(b_{i0}B^i = 0\). The vector \(b_i\) is gradient, \(i.e.\ b_{ij} = b_{jii}\), then we get

\[
E_{i0}B^i = b_{i0}B^i = 0.
\]

(4.41)
Therefore equation (4.40) reduces to
\[ D^i_{jk}B^j_iB^k_\alpha = -q\lambda M_{\alpha\beta}. \] (4.42)

In view of the above equation and (4.21), we get
\[ \overline{H}_{\alpha\beta} - \overline{M}_\alpha \overline{H}_\beta = \sqrt{p} (H_{\alpha\beta} - q\lambda M_{\alpha\beta}) - \sqrt{p} M_\alpha H_\beta. \] (4.43)

Now transvecting (3.6) by \( B^i_\alpha B^j_\beta B^k \) and in view of equation (4.8) and (4.11), we obtain
\[ \overline{C}_{ijk}B^i_\alpha B^j_\beta B^k = pC_{ijk}B^i_\alpha B^j_\beta B^k. \] (4.44)

From (2.8) and (4.9), equation (4.44) may be written as
\[ \overline{M}_{\alpha\beta} = \sqrt{p} M_{\alpha\beta}. \] (4.45)

Thus from (4.43) and (4.45), we have

**Theorem 4.3** For the \( h \)-Matsumoto change, let the \( h \)-vector \( b_i \) be a gradient and tangential to hypersurface \( F^{n-1} \) and satisfies condition (4.41). Then

1. \( F^{n-1} \) is a hyperplane of second kind if \( F^{n-1} \) is hyperplane of second kind and \( M_{\alpha\beta} = 0 \).
2. \( F^{n-1} \) is a hyperplane of third kind if \( F^{n-1} \) is hyperplane of third kind.

For the \( h \)-Matsumoto change, let the vector \( b_i \) be parallel with respect to the Cartan connection of \( F^n \). Then Cartan connection coefficient and Berwald connection coefficient for both the spaces are identical [2], i.e.
\[ \overline{F}^i_{jk} = F^i_{jk}, \] (4.46)
and
\[ \overline{C}^i_{jk} = C^i_{jk}. \] (4.47)

Thus we have

**Theorem 4.4** For the \( h \)-Matsumoto change, let the vector \( b_i(x,y) \) be parallel with respect to the Cartan connection of \( F^n \) and tangent to the hypersurface \( F^{n-1} \). Then \( F^{n-1} \) is a hyperplane of the second(third) kind if and only if \( F^{n-1} \) is also a hyperplane of the second(third) kind.

The \((v)hv\)-torsion tensor for the hyperplane of first kind is given by [10]
\[ P^i_{\beta\gamma} = B^i_\alpha K^\alpha_{\beta\gamma}, \] (4.48)
where \( K^i_{\beta\gamma} = P^j_{jk}B^j_iB^k_\beta \). Now contracting equation (2.4) by \( K^i_{\beta\gamma} \) and using the above equation, we get
\[ K^i_{\beta\gamma} = B^i_\delta P^\delta_{\beta\gamma} + B^i_\lambda B^\lambda_\delta K^\delta_{\beta\gamma}. \] (4.49)

Since the \((v)hv\)-torsion tensor \( P^i_{jk} \) is given by
\[ P^i_{jk} = G^i_{jk} - F^i_{jk}, \] (4.50)
In view of (4.46) and (4.47), the above equation gives
\[ \overline{P}_{jk} = P_{jk}. \] (4.51)

Thus in view of (4.51) and \( K^i_{\beta\gamma} = P^i_{jk} B^j_{\alpha} B^k_{\beta} \), we get
\[ \overline{K}_{\beta\gamma} = K^i_{\beta\gamma}. \]

From the above relation and using equation (4.49) we get
\[ \overline{P}_{\beta\gamma}^i = \overline{B}_{\beta\gamma}^i \left[ P^\alpha_{\beta\gamma} B^i_{\alpha} + K^j_{\beta\gamma} B^i_{\beta} B^j_{\gamma} \right]. \] (4.52)

In view of equation (2.3) and (4.48), the above equation gives us
\[ \overline{P}_{\beta\gamma}^i = P^\alpha_{\beta\gamma}. \] (4.53)

We know that if the \((v)hv\)-torsion tensor \( P^i_{jk} \) vanishes then a Finsler space \( F^n \) is called Landsberg space. Thus we have

**Theorem 4.5** For the \( h \)-Matsumoto change, let the \( h \)-vector \( b_i(x, y) \) be parallel with respect to the Cartan connection of \( F^n \) and tangent to the hypersurface \( F^{n-1} \). Then a hyperplane \( F^{n-1} \) of first kind is Landsberg space if and only if the hyperplane \( \overline{F}^{n-1} \) of first kind is a Landsberg space.

For the hyperplane of first kind, the Berwald connection coefficients \( G^\alpha_{\beta\gamma} \) are given by [10]
\[ G^\alpha_{\beta\gamma} = B^i_{\alpha} A^i_{\beta\gamma}, \] (4.54)

where \( A^i_{\beta\gamma} = B^i_{\beta\gamma} + G^i_{jk} B^j_{\beta} B^k_{\gamma} \). Now contracting equation (2.4) by \( A^i_{\beta\gamma} \) and using the above equation, we get
\[ A^i_{\beta\gamma} = B^i_{\beta\gamma} G^\delta_{\beta\gamma} + B^i B^h A^h_{\beta\gamma}. \] (4.55)

Thus in view of equation (4.46), (4.47), and from the above equation we get
\[ \overline{A}_{\beta\gamma}^i = A_{\beta\gamma}^i. \]

By using this relation and equation (4.54) and (4.55), we obtain
\[ \overline{G}_{\beta\gamma}^i = \overline{B}_{\beta\gamma}^i \left( B^j_{\beta\gamma} G^\delta_{\beta\gamma} + B^j A^h_{\beta\gamma} \right), \] (4.56)

In view of (2.3), the above equation gives as
\[ \overline{G}_{\beta\gamma}^\alpha = G^\alpha_{\beta\gamma}. \] (4.57)

If the Berwald connection coefficients \( G^i_{jk} \) are function of position only then a Finsler space \( F^n \) is called Berwald space. Thus from (4.57), we obtain

**Theorem 4.6** For the \( h \)-Matsumoto change, let the \( h \)-vector \( b_i(x, y) \) be parallel with respect to the Cartan connection of \( F^n \) and tangent to the hypersurface \( F^{n-1} \). Then a hyperplane \( F^{n-1} \) of first kind is Berwald space if and only if the hyperplane \( \overline{F}^{n-1} \) of first kind is a Berwald space.
Discussion

“${\mathcal{F}}^{n-1}$ is a hyperplane of third kind if $F^{n-1}$ is hyperplane of third kind.”

This result has been proved by Gupta and Gupta [5] for $h$-exponential change (which is infinite in nature) with the $h$-vector $b_i$ be gradient and tangential to hypersurface $F^{n-1}$ and satisfies the condition
\[ \beta_i C^j_{ij} = 0. \] (4.58)

While, Gupta and Pandey [6] have also obtained the same result with same condition for Kropina change (which is finite in nature) with an $h$-vector.

In the present paper we have proved the same result for $h$-Matsumoto change (which is also infinite in nature) without using the condition (4.58).

Notice that the above result holds

(i) For both the changes i.e. $h$-exponential change [5] (infinite nature) and Kropina change [6] (finite nature) with $h$-vector by using condition (4.58).

(ii) For $h$-Matsumoto change (infinite nature) without using (4.58).

The question is that, Is there any specific change with $h$-vector (without using the condition (4.58)) for which $F^{n-1}$ is a hyperplane of third kind if $F^{n-1}$ is hyperplane of third kind?

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