Hidden symmetries, instabilities, and current suppression in Brownian ratchets

David Cubero1 and Ferruccio Renzoni2

1Departamento de Física Aplicada I, EUP, Universidad de Sevilla, Calle Virgen de África 7, 41011 Sevilla, Spain
2Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom

The operation of Brownian motors is usually described in terms of out-of-equilibrium and symmetry-breaking settings, with the relevant spatiotemporal symmetries identified from the analysis of the equations of motion for the system at hand. When the appropriate conditions are satisfied, symmetry-related trajectories with opposite current are thought to balance each other, yielding suppression of transport. The direction of the current can be precisely controlled around these symmetry points by finely tuning the driving parameters. Here we demonstrate, by studying a prototypical Brownian ratchet system, the existence of hidden symmetries, which escape the identification by the standard symmetry analysis, and require different theoretical tools for their revelation. Furthermore, we show that system instabilities may lead to spontaneous symmetry breaking with unexpected generation of directed transport.

Motion at the nanoscale presents features very different from those encountered in the macroscopic world. Noise is a dominant process at such a scale, and may contribute constructively to the dynamics rather than play the usual role of a disturbance. New mechanisms of transport emerge at the nanoscale, and in particular directed motion may occur in the absence of an applied bias force. Brownian ratchets [1–3, 24], the archetypal model system capturing the mechanisms behind such a transport process, represent a key to understand several biological processes [4, 5], as well as they inspired a plethora of new nanodevices displaying directed motion [6–21]. All these systems are usually described in terms of operation away from thermal equilibrium, with directed motion following from the breaking of certain spatiotemporal symmetries, identified from the analysis of the equations of motion from the system at hand. Here we prove the existence of hidden symmetries, which escape the identification by the standard symmetry analysis [2, 3, 22, 23], and require different theoretical tools for their revelation. The main assumption of the standard symmetry analysis, i.e. that two trajectories that are connected by a symmetry transformation carry the same statistical weight, a reasoning which can be traced back to Loschmidt’s paradox [24], yields incorrect predictions in these dissipative systems, failing to account for system instabilities that lead to spontaneous symmetry breaking.

Results — A large class of Brownian motor systems, from particles in solution [8] to vortices in superconductors [14] and atoms in dissipative optical lattices [21], corresponds to a Brownian particle diffusing in a periodic potential under the action of a driving force with zero-average. The particle’s motion is described by the following Langevin equation

\[ m\ddot{x} = -\gamma\dot{x} + \mathcal{F}(x, t) + \xi(t), \]

where \( \gamma \) is the friction coefficient, \( \mathcal{F}(x, t) \) a generic deterministic force, and \( \xi(t) \) a fluctuating force, modelled as a Gaussian white noise with autocorrelation \( \langle \xi(t)\xi(t') \rangle = 2\Gamma\delta(t - t') \), with the noise strength \( \Gamma \) related to the temperature \( T \) of the environment via the fluctuation-dissipation relation \( \Gamma = \gamma k_B T \). The directed current is defined as \( \langle v \rangle \equiv \lim_{t \to \infty} \langle x(t) \rangle / t \), where the angle brackets denote average over noise realizations. For fine noise strengths \( \Gamma > 0 \), ergodicity implies \( \langle v \rangle = \lim_{t \to \infty} x(t) / t \). In very small systems, from the nanoscale to the microscale, the Brownian dynamics of small particles is frequently in the overdamped regime, where inertia effects—

![FIG. 1. Shift-symmetric potentials act like spatially symmetric ones in one-dimensional overdamped systems. a) Ratchet potential \( U_{\text{rat}}(x) = -U_0[\sin(kx) + (1/4)\sin(2kx)] \) with period \( L = 2\pi/k \). b) Shift-symmetric potential defined from \( U_{\text{rat}}(x) = U_{\text{rat}}(x) \) in the first half-period, and \( U_{\text{rat}}(x) = -U_{\text{rat}}(x - L/2) \) in the second half-period. c) Directed current for a Brownian particle subject to the mixed potential \( U(x) = U_{\text{rat}}(x)(1 - a) + U_{\alpha}(x)a \) and to a shift-symmetric force defined by \( F(t) = g(t) \equiv A[\sin(\omega t) + (1/4)\sin(2\omega t)] \) in the first half-period, and \( F(t) = -g(t - \tau/2) \) in the second half, where \( \tau = 2\pi/\omega \). Reduced units are defined such as \( m = L = 10\omega = 1 \). Other parameters are \( A = 4, U_0 = 10/2\pi, \Gamma = 10 \). The directed current vanishes in the overdamped limit (large frictions \( \gamma \)) for the shift-symmetric potential (\( a = 1 \)) because of hidden symmetries.](image-url)
the term $m\ddot{x}$ in (1) can be neglected. This is the regime of interest here.

The standard symmetry analysis [2] [3] [22] [23] relies on the identification of transformations which leave the equation of motion (1) unchanged and reverse the sign of the particle momentum. Trajectories with opposite momentum are equivalent, with a net null contribution to the directed current, which thus turns out to be zero. We will show that this picture does not fully capture the basic principles behind the operation of Brownian ratchets. To do this, we consider a more general approach [25], and regard the directed current $\langle \psi \rangle$ as a generic functional of the driving force $F$, thus using the notation $\psi[F(x,t)]$. Several properties follow from symmetry considerations.

First, due to the vectorial nature of both the force $F$ and the current, the transformation $x \to -x$ yields the following property

$$v[-F(-x,t)] = -v[F(x,t)].$$  \hspace{1cm} (2)

Second, an arbitrary translation along the $x$ or $t$ axis does not alter the current, i.e.

$$v[F(x,t)] = v[F(x+x_0,t)] = v[F(x,t+t_0)].$$  \hspace{1cm} (3)

Let us consider now a forced ratchet, i.e., $F(x,t) = f(x) + F(t)$, where $f(x) = -\partial U(x)/\partial x$ is a conservative force and $F(t)$ a driving force.

If the system is spatially symmetric with respect a certain point $x_0$, then the potential satisfies $U(x+x_0) = U(-x + x_0)$. Without loss of generality, we choose the coordinate’s origin such that $x_0 = 0$. Then $-f(-x) = f(x)$, which together with (2) yields a characteristic property of spatially symmetric systems

$$v[f(x) - F(t)] = -v[f(x) + F(t)].$$  \hspace{1cm} (4)

A shift-symmetric force is defined as $F(t + t_0) = -F(t)$—for periodic drives $t_0 = \pi/2$, where $\pi$ is the period, $F(t + \pi) = F(t)$. The direct application of properties (4) and (3) yields no current for shift-symmetric forces in spatially symmetric systems,

$$v[f(x) - F(t)] = v[f(x) + F(t + t_0)] = v[f(x) + F(t)]$$
$$= -v[f(x) + F(t)].$$  \hspace{1cm} (5)

This is a well known result of spatially symmetric systems, already captured by the standard symmetry analysis [2] [3] [22] [23]. However, our current approach reveals two additional symmetries for overdamped one-dimensional systems, which are not captured by the standard approach. They are:

$$v[f(x) + F(t)] = v[f(x) + F(t)],$$  \hspace{1cm} (6)

$$v[f(x) + F(-t)] = v[f(x) + F(t)].$$  \hspace{1cm} (7)

A proof of (6), (7) based on the Smoluchowski equation is given in the Supplemental Material [26]. The symmetries (6) and (7), together with (3), yield the following property for shift-symmetric potentials,

$$v[f(x) + F(t)] = v[f(-x) + F(t)] = -v[-f(x) - F(t)]$$
$$= -v[f(x + L/2) - F(t)] = -v[f(x) + F(t)].$$  \hspace{1cm} (8)

This is the same property as Eq. (4), and proceeding as before it implies current suppression when combined with a shift-symmetric driving force. Therefore, quite counter-intuitively, in one-dimensional underdamped systems, the condition for current suppression for systems with shift-symmetric potentials—like the one shown in Fig. (1b)—is, despite being spatially asymmetric, the same as for spatially symmetric potentials. Figure (1c) confirms this unexpected behavior for large enough frictions. In the underdamped regime this property is not satisfied exactly. Nevertheless, even in this regime the overdamped symmetry (6) identified here has a lasting effect: The zero-current point determined by the overdamped symmetry is displaced to a lower value of the symmetry parameter $a$ in the underdamped regime. Thus, the overdamped symmetry (6) determines a current reversal in the underdamped regime.

The discovery of hidden symmetries reported above does not represent the only departure from the conclusions which can be drawn from the standard symmetry analysis. The presence of instabilities may also alter the picture, as trajectories which are solutions of the equation of motion with opposite momenta may have very different stability properties, and thus result into a total non-zero contribution to the system current. Such a scenario of spontaneous symmetry breaking is best illustrated via a specific case study.

In the overdamped regime, from every solution $x(t)$, the trajectory $\tilde{x}(t) = x(-t) + L/2$ is also a solution of (1) provided the potential is shift-symmetric. It corresponds to a transformed random force $\xi(t) = -\xi(-t)$, which is statistically equivalent to $\xi(t)$, and a driving force $\tilde{F}(t) = -F(-t)$. Following the standard symmetry analysis, no current is expected when anti-symmetric driving forces $F(t + t_0) = -F(-t + t_0)$ are applied [3] [22] [23] [42]—an appropriate choice of the time origin yields $t_0 = 0$. This prediction is correct in one-dimensional systems, as readily verified by numerical simulations. However the same reasoning predicts no current also in the case of higher dimensions, a result which is contradicted by our numerical simulations, as shown in Fig. 2 for a two-dimensional potential and an applied split biharmonic drive, as well as by independent results by Peter Reimann’s group (see Ref. [43], page 16). The presence of instabilities is the key to understand such an unexpected, spontaneous symmetry-breaking behaviour. The standard analysis fails to account for the actual instability of the transformed solutions $\tilde{x}(t)$, which makes them very unlikely. Even in the noiseless limit, the above transformation maps stable oscillations about the potential minima into highly unstable oscillations about
The potential maxima. We have verified via numerical simulations that, given a stable solution \( x(t) \), the transformed solution \( \tilde{x}(t) \) is unstable and thus quickly collapses onto \( x(t) \). This occurs both in one dimension, and in higher dimensional systems. Given that instabilities destroy the mechanisms of current suppression due to the contributions of a trajectory and the transformed one, the observed suppression of directed transport in one-dimensional systems must be associated to a different mechanism. This suppression under anti-symmetric forces is actually a consequence of the symmetry \( \Phi \), which yields no current for systems—which include spatially symmetric as well as spatially shift-symmetric systems of interest here—satisfying the property \( \Phi \):

\[
\psi[f(x) + F(t)] = -\psi[f(x) - F(t)] = -\psi[f(x) + F(-t)] = -\psi[f(x) + F(t)].
\]

A consequence of this analysis is that also truly spatially symmetric systems should exhibit no current in one-dimensional overdamped systems when anti-symmetric forces are driving the system. This phenomenon is illustrated in Fig. 3. It was already experimentally observed in Ref. [15], but it remained unexplained until the present work. These results are a confirmation of the validity of the approach based on a more general symmetry analysis, which does not rely on the direct analysis of the solutions of the equation of motion.

It is worth stressing that in the present discussion the dimensionality of the system corresponds to the number of particles per period, \( n \), and the potential depth \( U_0/E_0 \), with the left panels referring to the original one-particle potential \( U(x) \) (depicted on the upper panel), and the right panels to a shift-symmetric potential built from the former as \( U_{\text{sym}}(x) = U(x) - U(x + L/2) \). The interaction between the particles is accounted for by the pair potential \( V_{\text{int}}(r) = -E_0 \ln(r) \), with \( r \) the particle separation. The system is driven by a single-harmonic force acting on each particle, which is both shift-symmetric and anti-symmetric. The parameters are the same as in Fig. 2 of [14].
of spatial degrees of freedom taking part into the rectification mechanism, and not necessarily to the dimensionality of the potential landscape. The violation of the symmetries (6)–(7) in the above 2D overdamped setup is due to a rectification mechanism taking place in the two perpendicular directions. However, the symmetries (6)–(7) are not restricted to strictly one-dimensional systems, they are still present in higher-dimensions overdamped systems provided the rectification mechanism involves one spatial dimension only. For example, the dashed line in Fig. 3(c) shows the suppression of current for anti-symmetric driving for the same 2D system shown in Fig. 2 when the bi-harmonic driving force is applied in the y-direction only. Additional examples are shown in [20].

Discussion — The hidden symmetries identified in the present work are of relevance to current experiments, as well as they allow to recast known results within a more general theoretical framework. This is well exemplified by two specific case studies presented in the following.

The first case study correspond to the system of a.c.-driven vortices trapped in a superconductor experimentally studied in Ref. [13]. Here, inter-particle interactions provide an additional path to escape from the symmetries (6)–(7). Our results of Fig. 1 precisely refer to the one-dimensional system of interacting Brownian particles, successfully used in Ref. [13] to explain the multiple current reversals observed on a.c.-driven vortices trapped in a superconductor. Despite not being strictly satisfied, the influence of the symmetries (6)–(7) is quite noticeable, cancelling the ratchet effect and most of the current reversals in regions of the parameter space where the appearance of a current is not directly related to particle interactions.

As second case study, we refer to the celebrated flashing ratchet model [8–11, 46, 47], where the ratchet potential is periodically switched on and off in the absence of any additional additive driving \( F(t) \) — i.e., here \( F(x,t) = -\partial U(x,t)/\partial x \)— and more specifically to the known [48] result that a flashing shift-symmetric potential can not produce directed motion. The theoretical framework, and the related new symmetries, introduced here allows for a simple explanation of such a result.

Generally, in one-dimensional overdamped systems the following symmetry is satisfied [20]

\[
v[F(-x, -t)] = v[F(x, t)]. \tag{10}\]

In two-state systems that are periodically switched, reversing the direction of time has no effect, \( v[F(x, -t)] = v[F(x, t)] \), a fact which together with [20, 10] and [3] yield no current for shift-symmetric potentials,

\[
v[F(x, t)] = v[-F(-x, -t)] = -v[-F(x, -t)] = -v[-F(x, t)] = -v[F(x, t)]. \tag{11}\]

Therefore, a flashing ratchet with a shift-symmetric potential, regardless if it is spatially asymmetric or not, can not produce directed motion, thus showing that in overdamped systems shift-symmetric potentials behave like spatially symmetric ones.

Conclusions — The present work addresses the outstanding issue of providing a general theoretical framework for the identification of symmetries not captured by the standard symmetry analysis, examples of which were already given in previous works [41, 48] with ad-hoc treatments. We have proven the existence in a prototypical 1D overdamped system of hidden symmetries, which escape the identification by the standard symmetry analysis and require different theoretical tools for their revelation. Though rigorously not satisfied in higher-dimensional systems, their effects are shown to be still noticeable in them. Our results pave the way to new mechanisms of manipulating transport. The hidden symmetries determine in fact current reversals, which can be used to precisely control transport and implement mechanisms for particles separation. Specific realizations for optical tweezers and cold atom set-ups are discussed in the Supplemental Material [20].

Financial support from the Royal Society (IE130734) (DC and FR), and the Leverhulme Trust (Grant RPG 2012-809) (FR) is acknowledged.

\[ dcubero@us.es \]
\[ f.renzoni@ucl.ac.uk \]

†

[1] R.D. Astumian, “Thermodynamics and kinetics of a brownian motor,” Science 276, 917–922 (2000).
[2] P. Reimann, “Brownian motors: noisy transport far from equilibrium,” Phys. Rep. 361, 57 (2002).
[3] P. Hänggi and F. Marchesoni, “Artificial brownian motors: Controlling transport on the nanoscale,” Rev. Mod. Phys. 81, 387 (2009).
[4] L. Mahadevan and P. Matsudaira, “Motility powered by supramolecular springs and ratchets,” Science 288, 95–99 (2000).
[5] M. Schliwa and G. Wolhike, “Molecular motors,” Nature 422, 759–765 (2003).
[6] J.V. Hernández, E.R. Kay, and D.A. Leigh, “A reversible synthetic rotary molecular motor,” Science 306, 1532–1537 (2004).
[7] M.V. Costache and S.O. Valenzuela, “Experimental spin ratchet,” Science 330, 1645–1648 (2010).
[8] J. Rousselet, L. Salome, A. Ajdari, and J. Prost, “Directional motion of brownian particles induced by a periodic asymmetric potential,” Nature 370, 446 (1994).
[9] Joel S. Bader, Richard W. Hammond, Steven A. Henck, Michael W. Deem, Gregory A. McDermott, James M. Bustillo, John W. Simpson, Gregory T. Mulhern, and Jonathan M. Rothberg, “Dna transport by a micromachined brownian ratchet device,” Proc. Natl. Acad. Sci. U.S.A. 96, 13165–13169 (1999).
[10] C. F. Chou, O. Bakajin, S. W. P. Turner, T. A. J. Duke, S. S. Chan, E. C. Cox, H. G. Craighead, and R. Austin,
“Sorting by diffusion: An asymmetric obstacle course for continuous molecular separation,” Proc. Natl. Acad. Sci. U.S.A. 96, 13762–13765 (1999).

[11] A. van Oudenaarden and Steven G. Boxer, “Brownian ratchets: Molecular separations in lipid bilayers supported on patterned arrays,” Science 285, 1046 (1999).

[12] Sven Matthias and Frank Müller, “Asymmetric pores in a silicon membrane acting as massively parallel brownian ratchets,” Nature 424, 53 (2003).

[13] J.E. Villegas, S. Savel’ev, F. Nori, E.M. Gonzalez, J.V. Anguita, R. Garcia, and J.L. Vicent, “A superconducting reversible rectifier that controls the motion of magnetic flux quanta,” Science 302, 1188 (2003).

[14] C.C. de Souza Silva, J. Van de Vondel, M. Morelle, and V.V. Moshchalkov, “Controlled multiple reversals of a ratchet effect,” Nature 440, 651–654 (2006).

[15] H. Linke, T.E. Humphrey, A. Löfgren, A.O. Sushkov, R. Newbury, R.P. Taylor, and P. Omling, “Experimental tunneling ratchets,” Science 286, 2314 (1999).

[16] T. Salger, S. Kling, T. Hecking, C. Geckeler, L. Morales-Molina, and M. Weitz, “Directed transport of atoms in a hamiltonian quantum ratchet,” Science 326, 1241 (2009).

[17] C. Drexler, S.A. Tarasenko, P. Olbrich, J. Karch, M. Hirmer, F. Müller, M. Gmitra, J. Fabian, R. Yakhmova, S. Lara-Avila, S. Kubatkin, M. Wang, R. Vajtai, P. M. Ajayan, J. Kono, and S.D. Ganichev, “Magnetic quantum ratchet effect in graphene,” Nature Nanotech. 8, 104 (2013).

[18] V. Serreli, C.F. Lee, E.R. Kay, and D.A. Leigh, “A molecular information ratchet,” Nature 445, 523–527 (2007).

[19] T.R. Kelly, H. de Silva, and R.A. Silva, “Undirectional rotary motion in a molecular system,” Nature 401, 150–152 (1999).

[20] J. Siegel, “Inventing the nanomolecular wheel,” Science 310, 63–64 (2005).

[21] R. Gommers, S. Bergamini, and F. Renzoni, “Dissipation-induced symmetry breaking in a driven optical lattice,” Phys. Rev. Lett. 95, 073003 (2005).

[22] S. Flach, O. Yevtushenko, and Y. Zolotaryuk, “Directed current due to broken time-space symmetry,” Phys. Rev. Lett. 84, 2358 (2000).

[23] S. Denisov, S. Flach, and P. Hänggi, “Tunable transport with broken spacetime symmetries,” Phys. Rep. 538, 77–120 (2014).

[24] J. Loschmidt and J. Sitzungsber., J. Sitzungsber. der kais. Akad. d. W. Math. Naturw. II 73, 128 (1876).

[25] J. Casado-Pascual, J. A. Cuesta, and N. R. Quintero R. Alvarez-Nodarse, “General approach for dealing with dynamical systems with spatiotemporal periodicities,” Phys. Rev. E 91, 022905 (2015).

See Supplemental Material at [URL], which includes Refs. 27[10].

[27] H. Risken, The Fokker-Planck Equation (Springer, Berlin, 1984).

[28] L.P. Faucheux, L.S. Bourdieu, P.D. Kaplan, and A.J. Libchaber, “Optical thermal ratchet,” Phys. Rev. Lett. 74, 1504 (1995).

[29] Sang-Hyuk Lee, Kosta Ladavac, Marco Polin, and David G. Grier, “Observation of flux reversal in a symmetric optical thermal ratchet,” Phys. Rev. Lett. 94, 110601 (2005).

[30] A. V. Arzola, K. VoIko-Sepulveda, and J. L. Mateos, “Experimental control of transport and current reversals in a deterministic optical rocking ratchet,” Phys. Rev. Lett. 106, 168104 (2011).

[31] O.M. Marago, P.H. Jones, P.G. Gucciardi, G. Volpe, and A.C. Ferrari, “Optical trapping and manipulation of nanostructures,” Nature Nano. 8, 807 (2013).

[32] C.-S. Lee, B. Jankó, I. Derényi, and A.-L. Barabási, “Reducing vortex density in superconductors using the ‘ratchet effect’,” Nature 400, 337 (1999).

[33] D.E. Shálon and H. Pastoriza, “Vortex motion rectification in josephson junction arrays with a ratchet potential,” Phys. Rev. Lett. 94, 177001 (2005).

[34] J.F. Wambaugh, C. Reichhardt, C.J. Olson, F. Marchesoni, and F Nori, “Superconducting fluxon pumps and lenses,” Phys. Rev. Lett. 83, 5106 (1999).

[35] D. Cole, S. Bending, S. Savel’ev, A. Grigorenko, T. Tamegai, and F. Nori, “Ratchet without spatial asymmetry for controlling the motion of magnetic flux quanta using time-asymmetric drives,” Nature Mater. 5, 305–311 (2006).

[36] C. Mennerat-Robilliard, D. Lucas, S. Guibal, J. Tabosa, C. Jurczak, J.-Y. Courtois, and G. Gryenberg, “Ratchet for cold rubidium atoms: The asymmetric optical lattice,” Phys. Rev. Lett. 82, 851 (1999).

[37] M. Schiavoni, L. Sanchez-Palencia, F. Renzoni, and G. Gryenberg, “Phase control of directed diffusion in a symmetric optical lattice,” Phys. Rev. Lett. 90, 094101 (2003).

[38] R. Gommers, S. Denisov, and F. Renzoni, “Quasiperiodically driven ratchets for cold atoms,” Phys. Rev. Lett. 96, 240604 (2006).

[39] D. Cubero, V. Lebedev, and F. Renzoni, “Current reversals in a rocking ratchet: dynamical vs symmetry-breaking mechanisms,” Phys. Rev. E 82, 041116 (2010).

[40] A. Wickenbrock, P. C. Holz, N. A. Abdul Wahab, P. Phoonthong, D. Cubero, and F. Renzoni, “Vibrational mechanics in an optical lattice: Controlling transport via potential renormalization,” Phys. Rev. Lett. 108, 020603 (2012).

[41] S. Denisov, Y. Zolotaryuk, S. Flach, and O. Yevtushenko, “Vortex and translational currents due to broken time-space symmetries,” Phys. Rev. Lett. 100, 224102 (2008).

[42] P. Reimann, “Supersymmetric ratchets,” Phys. Rev. Lett. 86, 4992 (2001).

[43] D. Speer, Spontaneous Symmetry Breaking Transport, Ph.D. thesis, Universität Bielefeld (2011).

[44] S. Denisov, S. Flach, A. A. Ovchinikov, O. Yevtushenko, and Y. Zolotaryuk, “Broken space-time symmetries and mechanisms of rectification of ac fields by nonlinear (non)adiabatic response,” Phys. Rev. E 66, 041104 (2002).

[45] S. Ooi, S. Savel’ev, M. B. Gaifullin, T. Mochiku, K. Hirata, and F. Nori, “Nonlinear nanodevices using magnetic flux quanta,” Phys. Rev. Lett. 99, 207003 (2007).

[46] A. Ajdari and J. Prost, “Mouvement induit par un potentiel périodique de basse symétrie: diélectrophorèse pulsée,” C. R. Acad. Sci. Paris Sér. II 315, 1635 (1992).

[47] G.P. Harmer and D. Abbott, “Losing strategies can win by parrondo’s paradox,” Nature 402, 864 (1999).

[48] R. Kanada and K. Sasaki, “Thermal ratchets with symmetric potentials,” J. Phys. Soc. Jpn. 68, 3759–3762 (1999).