AF-algebras and topology of 3-manifolds

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Abstract

We construct a functor which maps conjugate pseudo-Anosov automorphisms of a surface to the so-called stably isomorphic stationary AF-algebras; the functor gives new topological invariants of three dimensional manifolds coming from the known invariants of the AF-algebras. The main invariant is a triple $(\Lambda, [I], K)$, where $\Lambda$ is an integral order in the real number field $K$ and $[I]$ the equivalence class of ideals in $\Lambda$.

Key words and phrases: AF-algebras, 3-dimensional manifolds

AMS (MOS) Subj. Class.: 19K, 46L, 57M.

Introduction

A. Three dimensional manifolds. Let $X$ be an orientable surface of genus $g$. We shall denote by $\text{Mod} (X)$ the mapping class group, i.e. a group of the orientation-preserving automorphisms of $X$ modulo the normal subgroup of trivial automorphisms. Let $\phi \in \text{Mod} (X)$ and consider a mapping torus $M_\phi = \{ X \times [0,1] \mid (x, 0) \mapsto (\phi(x), 1), \ x \in X \}$. The $M_\phi$ is a 3-dimensional manifold and $M_\phi \cong M_{\phi'}$ are homotopy equivalent if and only if $\phi' = \psi \circ \phi \circ \psi^{-1}$ are conjugate by an automorphism $\psi \in \text{Mod} (X)$ [7]. Equivalently, $M_\phi$ is a surface bundle over the circle given by the monodromy $\phi$; such bundles make by far the most interesting, the most complex and the most useful part of the 3-dimensional topology [9], p.358. Each $\phi \in \text{Mod} (X)$ is isotopic to

*Partially supported by NSERC.
an automorphism $\phi'$, such that either (i) $\phi'$ has a finite order, or (ii) $\phi'$ is a pseudo-Anosov automorphism, or else (iii) $\phi'$ is reducible by a system of curves to either type (i) or (ii) [10]. Recall *ibid.* that $\phi$ is pseudo-Anosov if there exist a pair of the stable $F_s$ and unstable $F_u$ mutually orthogonal measured foliations of the surface $X$, such that $\phi(F_s) = \lambda_\phi F_s$ and $\phi(F_u) = \lambda_\phi F_u$, where $\lambda_\phi > 1$ is a dilatation of $\phi$. The foliations $F_s$ and $F_u$ are minimal, uniquely ergodic and describe the automorphism $\phi$ up to a power. In the sequel, we shall classify surface bundles $M_{\phi}$, where $\phi$ is the pseudo-Anosov automorphisms of a surface $X$.

**B. The $AF$-algebras ([5]).** The $C^*$-algebra $A$ is an algebra over complex numbers endowed with the norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$, $a \in A$, such that $A$ is complete with respect to the norm, and such that $||ab|| \leq ||a|| \ ||b||$ and $||a^*a|| = ||a||^2$ for every $a, b \in A$. Any commutative algebra $A$ is isomorphic to the $C^*$-algebra $C_0(X)$ of continuous complex-valued functions on a locally compact Hausdorff space $X$; the algebras which are not commutative are deemed as noncommutative topological spaces. A *stable isomorphism* $A \rightarrow A'$ is defined as the (usual) isomorphism $A \otimes K \rightarrow A' \otimes K$, where $K$ is the $C^*$-algebra of compact operators on a Hilbert space; such an isomorphism corresponds to a homeomorphism between the noncommutative spaces $A$ and $A'$. The matrix algebra $M_n(\mathbb{C})$ is an example of noncommutative finite-dimensional $C^*$-algebra; a natural generalization are approximately finite-dimensional ($AF$-) algebras, which are given by an ascending sequence $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots$ of finite-dimensional semi-simple $C^*$-algebras $M_i = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$ and homomorphisms $\varphi_i$ arranged into an infinite graph as follows. The two sets of vertices $V_i, \ldots, V_k$ and $V'_i, \ldots, V'_k$ are joined by the $b_{rs}$ edges, whenever the summand $M_i$ contains $b_{rs}$ copies of the summand $M_i'$ under the embedding $\varphi_i$; as $i \rightarrow \infty$, one gets a *Bratteli diagram* of the $AF$-algebra. Such a diagram is defined by an infinite sequence of incidence matrices $B_i = (b_{ij}^{(s)})$. If the homomorphisms $\varphi_1 = \varphi_2 = \ldots = Const$, the $AF$-algebra is called *stationary*; its Bratteli diagram looks like an infinite graph with the incidence matrix $B = (b_{rs})$ repeated over and over again.

**C. The functoriality problem.** Let $\phi \in Mod(X)$ be a pseudo-Anosov automorphism; the problem we seek a solution is as follows. Given $\phi$ one assigns to it an $AF$-algebra, $A_{\phi}$, such that for every automorphism $h \in Mod(X)$ the following diagram commutes:
In words, if \( \phi \) and \( \phi' \) are the conjugate pseudo-Anosov automorphisms, then the corresponding \( AF \)-algebras \( A_\phi \) and \( A_{\phi'} \) are stably isomorphic; the following simple example indicates, that the functoriality problem has a solution.

**D. A model example (case \( g = 1 \)).** Let \( T^2 \) be two-dimensional torus and \( \phi \in Mod ( T^2 ) \) an automorphism given by non-negative hyperbolic matrix \( A_\phi \in SL_2 ( \mathbb{Z} ) \). Consider a stationary \( AF \)-algebra \( A_\phi \) given by the infinite Bratteli diagram with the constant incidence matrix \( B = A_\phi \); it is verified directly, that \( F : \phi \mapsto A_\phi \) is a correctly defined map on the set of hyperbolic matrices with non-negative entries. Let us show that if \( \phi, \phi' \in Mod ( T^2 ) \) are conjugate automorphisms, then \( A_\phi \) is stably isomorphic to \( A_{\phi'} \). Indeed, let \( \phi' = h \circ \phi \circ h^{-1} \) for an \( h \in Mod ( X ) \); then \( A_{\phi'} = T A_\phi T^{-1} \) for a matrix \( T \in SL_2 ( \mathbb{Z} ) \) and \( ( A_{\phi'} )^n = ( T A_\phi T^{-1} )^n = T A_\phi^n T^{-1} \) for any \( n \in \mathbb{N} \). Recall that the \( AF \)-algebras are stably isomorphic if and only if their Bratteli diagrams contain a common block of an arbitrary length; this claim follows from [5], Theorem 2.3, where the order-isomorphism is replaced by an equivalent condition of Bratteli diagrams having the same infinite tail. Consider the Bratteli diagrams \( A_\phi = \lim_{n \to \infty} A_\phi^n \) and \( A_{\phi'} = \lim_{n \to \infty} T A_{\phi'}^n T^{-1} \); the latter have a common block of arbitrary length. Thus, \( A_\phi \otimes \mathbb{K} \cong A_{\phi'} \otimes \mathbb{K} \), which gives a solution to the functoriality problem in the case \( g = 1 \).

**E. The \( AF \)-algebra \( A_\phi \) (case \( g \geq 1 \)).** Denote by \( \mathcal{F}_\phi \) the unstable foliation of a pseudo-Anosov automorphism \( \phi \in Mod ( X ) \). For brevity, we assume that \( \mathcal{F}_\phi \) is an oriented foliation given by the trajectories of a closed 1-form \( \omega \in H^1 ( X ; \mathbb{R} ) \); if \( \mathcal{F}_\phi \) is not oriented, the standard double cover construction brings it to the oriented case [8]. Let \( v^{(i)} = \int_{\gamma_i} \omega \), where \( \{ \gamma_1, \ldots, \gamma_n \} \) be a basis in the relative homology \( H_1 ( X, Sing \mathcal{F}_\phi ; \mathbb{Z} ) \), such that \( \theta = ( \theta_1, \ldots, \theta_{n-1} ) \) is a vector with positive coordinates \( \theta_i = v^{(i+1)}/v^{(1)} \); while each \( \theta_i \) depends on a basis in the homology group, the \( \mathbb{Z} \)-module generated by \( \theta_i \) does not
(Lemma 1). Consider the Jacobi-Perron continued fraction of vector $\theta$ ([2]):

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix},$$

where $b_k = (b_1^{(i)}, \ldots, b_{n-1}^{(i)})^T$ is a vector of the non-negative integers, $I$ the unit matrix and $I = (0, \ldots, 0, 1)^T$. We shall denote by $A_\phi$ an (isomorphism class of) AF-algebra given by the Bratteli diagram, whose incidence matrices coincide with $B_k = \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}$ for $k = 1, \ldots, \infty$. Notice that such a definition coincides with the one given in the model example (case $g = 1$). The basic lemma says that $A_\phi$ is a stationary AF-algebra (Lemma 4).

**F. The result.** Let $B$ be the incidence matrix of the stationary algebra $A_\phi$; let $\lambda_B$ be the Perron-Frobenius eigenvalue of $B$ and $(v_B^{(1)}, \ldots, v_B^{(n)})$ the corresponding normalized eigenvector with $v_B^{(i)} \in K = \mathbb{Q}(\lambda_B)$. The endomorphism ring of the module $m = \mathbb{Z}v_B^{(1)} + \ldots + \mathbb{Z}v_B^{(n)}$ will be denoted by $\Lambda$. The equivalence class of ideals in the ring $\Lambda$ generated by the ideal $m$, we shall write as $[I]$. Finally, let $\Phi$ be a category of all pseudo-Anosov automorphisms of the surface of genus $g \geq 1$; the morphisms are the conjugations between the automorphisms. Likewise, let $A$ be a category of all stationary AF-algebras $A_\phi$, where $\phi$ runs the set $\Phi$; the morphisms of $A$ are the stable isomorphisms among the $A_\phi$’s. Our main result can be expressed as follows.

**Theorem 1** Let $F : \Phi \to A$ be a map given by the formula $\phi \mapsto A_\phi$. Then:

(i) $F$ is a covariant functor, which maps the conjugate pseudo-Anosov automorphisms to the stably isomorphic AF-algebras;

(ii) $F^{-1}(A_\phi) = [\phi]$, where $[\phi] = \{ \phi' \in \Phi \mid (\phi')^m = \phi^n, \ m, n \in \mathbb{N} \}$ is the commensurability class of the pseudo-Anosov automorphism $\phi$.

**Corollary 1** The triple $(\Lambda, [I], K)$ is a homotopy invariant of the manifold $M_\phi$.

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1 Preliminaries
1.1 Jacobian of measured foliation

Let $\mathcal{F}$ be a measured foliation on a compact surface $X$ [10]. For the sake of brevity, we shall assume that $\mathcal{F}$ is an oriented foliation, i.e. given by the trajectories of a closed 1-form $\omega$ on $X$; each non oriented foliation is covered by an oriented one on a surface $\tilde{X}$, which is a double cover of $X$ ramified at the singular points of the half-integer index of the non-oriented foliation [8]. Let $\{\gamma_1, \ldots, \gamma_n\}$ be a basis in the relative homology group $H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z})$, where $\text{Sing } \mathcal{F}$ is the set of singular points of the foliation $\mathcal{F}$. It is well known that $n = 2g + m - 1$, where $g$ is the genus of $X$ and $m = |\text{Sing } (\mathcal{F})|$. The periods of $\omega$ in the above basis we shall write as $\lambda_i = \int_{\gamma_i} \omega$. By a Jacobian $\text{Jac}(\mathcal{F})$ of $\mathcal{F}$, we understand a $\mathbb{Z}$-module $m = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ regarded as a subset of the real line $\mathbb{R}$.

**Lemma 1** The $\mathbb{Z}$-module $m$ is independent of the choice of a basis in $H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z})$.

**Proof.** Indeed, let $A = (a_{ij}) \in GL_n(\mathbb{Z})$ and let $\gamma'_i = \sum_{j=1}^n a_{ij} \gamma_j$ be a new basis in $H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z})$. Then using the integration rules: $\lambda'_i = \int_{\gamma'_i} \omega = \int_{\sum_{j=1}^n a_{ij} \gamma_j} \omega = \sum_{j=1}^n \int_{\gamma_j} \omega = \sum_{j=1}^n a_{ij} \lambda_j$. To prove that $m = m'$, consider the following equations: $m' = \sum_{i=1}^n z\lambda'_i = \sum_{i=1}^n z \sum_{j=1}^n a_{ij} \lambda_j = \sum_{j=1}^n (\sum_{i=1}^n a_{ij} z) \lambda_j \subseteq m$. Let $A^{-1} = (b_{ij}) \in GL_n(\mathbb{Z})$ be an inverse to the matrix $A$. Then $\lambda_i = \sum_{j=1}^n b_{ij} \lambda'_j$ and $m = \sum_{i=1}^n z\lambda_i = \sum_{i=1}^n z \sum_{j=1}^n b_{ij} \lambda'_j = \sum_{j=1}^n (\sum_{i=1}^n b_{ij} z) \lambda'_j \subseteq m'$. Since both $m' \subseteq m$ and $m \subseteq m'$, we conclude that $m' = m$. Lemma 1 follows. □
Recall that the measured foliations $\mathcal{F}$ and $\mathcal{F}'$ are said to be *topologically conjugate*, if there exists an automorphism $h \in \text{Mod} (X)$, which sends the leaves of the foliation $\mathcal{F}$ to the leaves of the foliation $\mathcal{F}'$. Note that such an equivalence deals with the topological foliations (i.e. the projective classes of measured foliations [10]) and does not preserve transversal measure.

**Lemma 2** Let $\mathcal{F}$ and $\mathcal{F}'$ be topologically conjugate measured foliations on a surface $X$. Then $\text{Jac} (\mathcal{F}') = \mu \text{Jac} (\mathcal{F})$, where $\mu > 0$ is a real number.

*Proof.* Let $h : X \to X$ be an automorphism of the surface $X$. Denote by $h_*$ its action on $H_1(X, \text{Sing} (\mathcal{F}); \mathbb{Z})$ and by $h^*$ on $H^1(X; \mathbb{R})$ connected by the formula: $\int_{h_*(\gamma)} \omega = \int_\gamma h^*(\omega)$, $\forall \gamma \in H_1(X, \text{Sing} (\mathcal{F}); \mathbb{Z})$, $\forall \omega \in H^1(X; \mathbb{R})$.

Let $\omega, \omega' \in H^1(X; \mathbb{R})$ be the closed 1-forms whose trajectories define the foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. Since $\mathcal{F}, \mathcal{F}'$ are topologically conjugate, $\omega' = \mu h^*(\omega)$ for a $\mu > 0$. Let $\text{Jac} (\mathcal{F}) = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $\text{Jac} (\mathcal{F}') = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$. Then $\gamma_i = \int_{\gamma_i} \omega' = \mu \int_{\gamma_i} h^*(\omega) = \mu \int_{h_*(\gamma_i)} \omega$, $1 \leq i \leq n$.

By lemma 1, it holds: $\text{Jac} (\mathcal{F}) = \sum_{i=1}^n \mathbb{Z} \int_{\gamma_i} \omega = \sum_{i=1}^n \mathbb{Z} \int_{h_*(\gamma_i)} \omega$. Therefore $\text{Jac} (\mathcal{F}') = \sum_{i=1}^n \mathbb{Z} \int_{\gamma_i} \omega' = \mu \sum_{i=1}^n \mathbb{Z} \int_{h_*(\gamma_i)} \omega = \mu \text{Jac} (\mathcal{F})$. Lemma 2 follows. \(\square\)

### 1.2 Functoriality for measured foliations

Let $\mathcal{F}$ be a foliation of surface $X$ endowed with the unique ergodic measure; suppose that $\mathcal{F}$ is given by the trajectories of a closed 1-form $\omega \in H^1(X; \mathbb{R})$.

Let $v^{(i)} = \int_{\gamma_i} \omega$, where $\{\gamma_1, \ldots, \gamma_n\}$ be a basis in the relative homology $H_1(X, \text{Sing} \mathcal{F}_0; \mathbb{Z})$, such that $\theta = (\theta_1, \ldots, \theta_{n-1})$ is a vector with the positive coordinates $\theta_i = v^{(i+1)}/v^{(1)}$. Consider the Jacobi-Perron continued fraction of $\theta$ ([2]):

$$\left(\begin{array}{c} 1 \\ \theta \end{array}\right) = \lim_{k \to \infty} \left(\begin{array}{cc} 0 & 1 \\ I & b_1 \end{array}\right) \cdots \left(\begin{array}{cc} 0 & 1 \\ I & b_k \end{array}\right) \left(\begin{array}{c} 1 \\ 1 \end{array}\right),$$

where $b_i = (b_1^{(i)}, \ldots, b_{n-1}^{(i)})^T$ is a vector of the non-negative integers, $I$ the unit matrix and $I = (0, \ldots, 0, 1)^T$. Let $\mathcal{A}_\mathcal{F}$ be an (isomorphism class of) $\mathcal{A}$-algebra given by the Bratteli diagram, whose incidence matrices coincide with $B_k = \left(\begin{array}{cc} 0 & 1 \\ I & b_k \end{array}\right)$ for all $k = 1, \ldots, \infty$; notice that $\mathcal{A}_\mathcal{F}$ is correctly defined, since the Jacobi-Perron fraction of uniquely ergodic measured foliation is convergent [1]. The following lemma establishes functoriality of the algebras $\mathcal{A}_\mathcal{F}$ with respect to the topological conjugacy.
Lemma 3  If \( \mathcal{F} \) and \( \mathcal{F}' \) are topologically conjugate foliations, then \( \mathcal{A}_\mathcal{F} \) and \( \mathcal{A}_\mathcal{F}' \) are stably isomorphic AF-algebras.

Proof. (i) First, let us show that if \( m = \mathbb{Z} \lambda_1 + \ldots + \mathbb{Z} \lambda_n \) and \( m' = \mathbb{Z} \lambda'_1 + \ldots + \mathbb{Z} \lambda'_n \) are two \( \mathbb{Z} \)-modules, such that \( m' = \mu m \) for a \( \mu > 0 \), then the Jacobi-Perron continued fractions of the vectors \( \lambda \) and \( \lambda' \) coincide except, may be, a finite number of terms. Indeed, let \( m = \mathbb{Z} \lambda_1 + \ldots + \mathbb{Z} \lambda_n \) and \( m' = \mathbb{Z} \lambda'_1 + \ldots + \mathbb{Z} \lambda'_n \). Since \( m' = \mu m \), where \( \mu \) is a positive real, one gets the following identity of the \( \mathbb{Z} \)-modules:

\[
\mathbb{Z} \lambda'_1 + \ldots + \mathbb{Z} \lambda'_n = \mathbb{Z} (\mu \lambda_1) + \ldots + \mathbb{Z} (\mu \lambda_n).
\]

One can always assume that \( \lambda_i \) and \( \lambda'_i \) are positive reals; there exists a basis \( \{ \lambda''_1, \ldots, \lambda''_n \} \) of the module \( m' \), such that:

\[
\begin{cases}
\lambda''_i = A(\mu \lambda) \\
\lambda''_i = A' \lambda',
\end{cases}
\]

where \( A, A' \in GL_n^+(\mathbb{Z}) \) are the matrices, whose entries are non-negative integers. In view of the Proposition 3 of [1]:

\[
\begin{cases}
A = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}, \\
A' = \begin{pmatrix} 0 & 1 \\ I & b'_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b'_l \end{pmatrix},
\end{cases}
\]

where \( b_i, b'_i \) are non-negative integer vectors. Since the continued fraction for the vectors \( \lambda \) and \( \mu \lambda \) coincide for any \( \mu > 0 \) [2], we conclude that:

\[
\begin{cases}
\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \cdots \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
\begin{pmatrix} 1 \\ \theta' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ I & b'_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b'_l \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{cases}
\]

where

\[
\begin{pmatrix} 1 \\ \theta'' \end{pmatrix} = \lim_{i \to \infty} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & a_i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

In other words, the continued fractions of the vectors \( \lambda \) and \( \lambda' \) coincide, except a finite number of terms.

(ii) By lemma 2 topologically conjugate foliations \( \mathcal{F} \) and \( \mathcal{F}' \) have proportional Jacobians, i.e. \( m' = \mu m \) for a \( \mu > 0 \). Thus, the continued fraction expansion of the basis vectors of the proportional Jacobians must coincide,
except a finite number of terms; the $AF$-algebras $A_F$ and $A_{F'}$ are given by the Bratteli diagrams, which are identical, except a finite part of the diagram. It is well known ([5], Theorem 2.3) that the $AF$-algebras, which have such a property, are stably isomorphic. □

1.3 Basic lemma

There exists a countable family of measured foliations, which come from the pseudo-Anosov automorphisms of surfaces; we shall restrict our attention to this class of foliations. Let $\phi \in \text{Mod} (X)$ be a pseudo-Anosov automorphism of the surface; then there exist a stable $F_s$ and unstable $F_u$ mutually orthogonal measured foliations on $X$, such that $\phi(F_s) = \frac{1}{\lambda_\phi} F_s$ and $\phi(F_u) = \lambda_\phi F_u$, where $\lambda_\phi > 1$ is called a dilatation of $\phi$. The foliations $F_s, F_u$ are minimal, uniquely ergodic and describe the automorphism $\phi$ up to a power; we shall understand by $F_\phi$ the unstable foliation of $\phi$. Let $\mathbb{A}_\phi := \mathbb{A}_{F_\phi}$ be the $AF$-algebra of the measured foliation $F_\phi$; the following lemma describes the basic property of such an algebra (to be proved in the next section).

**Lemma 4** $\mathbb{A}_\phi$ is stably isomorphic to a stationary $AF$-algebra.

Recall that any stationary $AF$-algebra is given by a positive integer matrix $A$; the similarity class of the matrix corresponds to the stable isomorphism class of the $AF$-algebra $\mathbb{A}_\phi$ [5].

2 Proofs

2.1 Proof of basic lemma

Let $\phi \in \text{Mod} (X)$ be a pseudo-Anosov automorphism of the surface $X$; we proceed by showing, that invariant foliation $F_\phi$ is given by form $\omega \in H^1(X; \mathbb{R})$, which is an eigenvector of the linear map $[\phi] : H^1(X; \mathbb{R}) \to H^1(X; \mathbb{R})$ induced by $\phi$. Indeed, let $\lambda_\phi$ be a dilatation of $\phi$ and $\Omega$ the corresponding volume element; by definition, $\phi(\Omega) = \lambda_\phi \Omega$. Note, that $\Omega$ is given by restriction of form $\omega$ to a 1-dimensional manifold, transverse to the leaves of $F_\phi$. The leaves of $F_\phi$ are fixed by $\phi$ and, therefore, $\phi(\Omega)$ is given by a multiple $\lambda_\phi \omega$ of form $\omega$. Since $\omega \in H^1(X; \mathbb{R})$ is a vector, whose coordinates
define $F_\phi$ up to a scalar, we conclude, that $[\phi](\omega) = \lambda_\phi \omega$, i.e. $\omega$ is an eigenvector of the linear map $[\phi]$. Let $(\lambda_1, \ldots, \lambda_n)$ be a basis of the Jacobian of $F_\phi$, such that $\lambda_i > 0$. Notice, that $\phi$ acts on $\lambda_i$ as multiplication by constant $\lambda_\phi$; indeed, since $\lambda_i = \int_{\gamma_i} \omega$, we have:

$$
\lambda'_i = \int_{\gamma_i} [\phi](\omega) = \int_{\gamma_i} \lambda_\phi \omega = \lambda_\phi \lambda_i,
$$

(1)

where $\{\gamma_i\}$ is a basis in $H_1(X, \text{Sing } F_\phi; \mathbb{Z})$. Since $\phi$ preserves the leaves of $F_\phi$, one concludes that $\lambda'_i \in \text{Jac } (F_\phi)$; therefore, $\lambda'_i = \sum b_{ij} \lambda_i$ for a non-negative integer matrix $B = (b_{ij})$. According to [1], matrix $B$ can be written as a finite product:

$$
B = \begin{pmatrix} 0 & 1 \\ I & b_1 \\ & \ddots \\ & & 0 & 1 \\ & & I & b_p \end{pmatrix} = B_1 \cdots B_p,
$$

(2)

where $b_i = (b_i^{(1)}, \ldots, b_i^{(n-1)})^T$ is a vector of non-negative integers and $I$ the unit matrix. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$. Consider a purely periodic Jacobi-Perron continued fraction:

$$
\lim_{i \to \infty} B_1 \cdots B_p \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},
$$

(3)

where $\mathbb{I} = (0, \ldots, 0, 1)^T$; by a basic property of such fractions, it converges to an eigenvector $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ of matrix $B_1 \cdots B_p$ [3], Ch.3. But $B_1 \cdots B_p = B$ and $\lambda$ is an eigenvector of matrix $B$; therefore, vectors $\lambda$ and $\lambda'$ are collinear. The collinear vectors are known to have the same continued fractions; thus, we have

$$
\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{i \to \infty} \overline{B_1 \cdots B_p \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}},
$$

(4)

where $\theta = (\theta_1, \ldots, \theta_{n-1})$ and $\theta_i = \lambda_{i+1}/\lambda_1$. Since vector $(1, \theta)$ unfolds into a periodic Jacobi-Perron continued fraction, we conclude, that the $AF$-algebra $A_{\phi}$ is stationary. Lemma 4 is proved. □

2.2 Proof of theorem 1

(i) Let us prove the first statement; we start with the following lemma.

**Lemma 5** Let $\phi$ and $\phi'$ be the conjugate pseudo-Anosov automorphisms of a surface $X$. Then the invariant foliations $F_\phi$ and $F_{\phi'}$ are topologically conjugate.

\[9\]
Proof. Let $\phi, \phi' \in \text{Mod}(X)$ be conjugate, i.e. $\phi' = \psi \circ \phi \circ \psi^{-1}$ for an automorphism $\psi \in \text{Mod}(X)$. Since $\phi$ is the pseudo-Anosov automorphism, there exists a measured foliation $F_\phi$, such that $\phi(F_\phi) = \lambda_\phi F_\phi$. Let us evaluate the automorphism $\phi'$ on the foliation $\psi(F_\phi)$:

$$\phi'(\psi(F_\phi)) = \psi \circ \phi \circ \psi^{-1}(\psi(F_\phi)) = \psi \phi(F_\phi) = \psi \lambda_\phi F_\phi = \lambda_\psi(\psi(F_\phi)).$$

(5)

Thus, $F_{\phi'} = \psi(F_\phi)$ is the invariant foliation for the pseudo-Anosov automorphism $\phi'$ and foliations $F_\phi$ and $F_{\phi'}$ are topologically conjugate. Note also, that the pseudo-Anosov automorphisms $\phi$ and $\phi'$ have the same dilatation. □

One can prove claim (i) of theorem 1; let $\phi$ and $\phi'$ be conjugate pseudo-Anosov automorphisms. Functor $F$ acts by the formulas $\phi \mapsto \mathcal{A}_\phi$ and $\phi' \mapsto \mathcal{A}_{\phi'}$, where $\mathcal{A}_\phi$ and $\mathcal{A}_{\phi'}$ are the AF-algebras of the invariant foliations $F_\phi$ and $F_{\phi'}$. In view of lemma 5, foliations $F_\phi$ and $F_{\phi'}$ are topologically conjugate. By lemma 3, the AF-algebras $\mathcal{A}_\phi$ and $\mathcal{A}_{\phi'}$ are stably isomorphic; claim (i) is proved.

(ii) Let $\phi \in \text{Mod}(X)$ be a pseudo-Anosov automorphism. Then there exists a unique measured foliation $F_\phi$, such that $\phi(F_\phi) = \lambda_\phi F_\phi$, where $\lambda_\phi > 1$; let us evaluate the automorphism $\phi^2 \in \text{Mod}(X)$ on the foliation $F_\phi$:

$$\phi^2(F_\phi) = \phi(\phi(F_\phi)) = \phi(\lambda_\phi F_\phi) = \lambda_\phi \phi F_\phi = \lambda_\phi^2 F_\phi,$$

(6)

where $\lambda_{\phi^2} := \lambda_\phi^2$. Thus, the foliation $F_\phi$ is an invariant foliation for the automorphism $\phi^2$ as well; by induction, we conclude that $F_\phi$ is an invariant foliation for the automorphism $\phi^n$ for any $n \geq 1$. Denote by $[\phi]$ the set of all pseudo-Anosov automorphisms $\psi$ of $X$, such that $\psi^m = \phi^n$ for some positive integers $m$ and $n$.

Lemma 6 The foliation $F_\phi$ is an invariant foliation for every automorphism $\psi \in [\phi]$.

Proof. Suppose that $\psi \in \text{Mod}(X)$ is a pseudo-Anosov automorphism, such that $\psi^m = \phi^n$ for some $m \geq 1$ and $\psi \neq \phi$; then $F_\phi$ is an invariant foliation for the automorphism $\psi$. Indeed, $F_\phi$ is an invariant foliation for the
automorphism $\psi^m$. If there exists $\mathcal{F}' \neq \mathcal{F}_\phi$, such that the foliation $\mathcal{F}'$ is an invariant foliation of $\psi$, then the foliation $\mathcal{F}'$ is an invariant foliation of the pseudo-Anosov automorphism $\psi^m$. Thus, by the uniqueness of invariant foliations, $\mathcal{F}' = \mathcal{F}_\phi$. □

In view of lemma 6, one arrives at the following identities among the $AF$-algebras:

$$\mathbb{A}_\phi = \mathbb{A}_{\phi^2} = \ldots = \mathbb{A}_{\psi^n} = \ldots = \mathbb{A}_{\psi^2} = \mathbb{A}_\psi.$$  (7)

Thus, the functor $F : \Phi \to \mathbb{A}$ is not injective, since the preimage $F^{-1}$ of the $AF$-algebra $\mathbb{A}_\phi$ is a countable set of pseudo-Anosov automorphisms $\psi \in [\phi]$ commensurable with the automorphism $\phi$.

Theorem 1 is proved. □

2.3 Proof of corollary 1

Lemma 4 says that $\mathbb{A}_\phi$ is a stationary $AF$-algebra given by a positive integer matrix $B$. By the Perron-Frobenius theory, matrix $B$ has a real eigenvalue $\lambda_B > 1$, which exceeds the absolute values of all other roots of the characteristic polynomial of $B$; note that $\lambda_B$ is an algebraic number. Consider a real algebraic number field $K = \mathbb{Q}(\lambda_B)$ obtained as an extension of the field of the rational numbers by $\lambda_B$. Let $(v^{(1)}_B, \ldots, v^{(n)}_B)$ be the eigenvector corresponding to the eigenvalue $\lambda_B$; one can normalize the eigenvector so that $v^{(i)}_B \in K$. Consider the $\mathbb{Z}$-module $\mathfrak{m} = \mathbb{Z}v^{(1)}_B + \ldots + \mathbb{Z}v^{(n)}_B$; denote by $\Lambda$ the endomorphism ring of $\mathfrak{m}$ and by $I$ an ideal in the ring $\Lambda$ generated by $\mathfrak{m}$. The ring $\Lambda$ is an order in the algebraic number field $K$ and therefore $I$ belongs to an ideal class in $\Lambda$; the ideal class of $I$ is denoted by $[I]$. The triple $(\Lambda, [I], K)$ is an invariant of the stable isomorphism class of the stationary $AF$-algebra $\mathbb{A}_\phi$ (Handelman [6], §5). By theorem 1, $(\Lambda, [I], K)$ is an invariant of the conjugacy class of $\phi$ and by Hemion [7] of the homotopy class of manifold $M_\phi$. □
3 Numerical invariants

3.1 Determinant and signature

One can derive numerical invariants of the stable isomorphism classes of stationary $AF$-algebras from the triple $(\Lambda, [I], K)$; one such invariant is associated with the trace function on the algebraic number field $K$. Recall that $Tr : K \to \mathbb{Q}$ is a linear function on $K$ such that $Tr (\alpha + \beta) = Tr (\alpha) + Tr (\beta)$ and $Tr (a\alpha) = a\ Tr (\alpha)$ for $\forall \alpha, \beta \in K$ and $\forall a \in \mathbb{Q}$. Let $m$ be a full $\mathbb{Z}$-module in the field $K$. The trace function defines a symmetric bilinear form $q(x, y) : m \times m \to \mathbb{Q}$ by the formula $(x, y) \mapsto Tr (xy)$, $\forall x, y \in m$. The form

$$q(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{n} s_{ij} x_i y_j, \quad \text{where} \quad s_{ij} = Tr (\lambda_i \lambda_j); \quad (8)$$

depends on the basis $\{\lambda_1, \ldots, \lambda_n\}$ in the module $m$; however, certain numerical quantities will not depend on the basis. Namely, consider a symmetric matrix $S = (s_{ij})$ corresponding to the bilinear form $q(x, y)$. In a new basis matrix $S$ will take the form $S' = U^T SU$, where $U \in GL_n(\mathbb{Z})$; thus $\det (S') = \det (U^T SU) = \det (U^T) \det (S) \det (U) = \det (S)$. Therefore, the rational integer

$$\Delta = \det (Tr (\lambda_i \lambda_j)), \quad (9)$$

does not depend on the choice of the basis $\{\lambda_1, \ldots, \lambda_n\}$ in the module $m$; it is called a determinant of the bilinear form $q(x, y)$. Clearly, $\Delta$ discerns the modules $m$ and $m'$.

Finally, recall that the form $q(x, y)$ can be brought by the integer linear substitutions to the diagonal form:

$$s_1 x_1^2 + s_2 x_2^2 + \ldots + s_n x_n^2, \quad (10)$$

where $s_i \in \mathbb{Z} - \{0\}$. We let $s_i^+$ be the positive and $s_i^-$ the negative entries in the diagonal form. In view of the law of inertia for the bilinear forms, the integer number $\Sigma = (#s_i^+) - (#s_i^-)$ does not depend on the choice of basis in the module $m$; it is called a signature of the form. Thus, $\Sigma$ discerns the modules $m$ and $m'$. 

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3.2 Numerical invariants of Anosov automorphisms

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension of the field of rational numbers $\mathbb{Q}$. Further we suppose that $d$ is a positive square free integer. Let

$$\omega = \begin{cases} 
\frac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4, \\
\sqrt{d} & \text{if } d \equiv 2, 3 \mod 4.
\end{cases} \quad (11)$$

**Proposition 1** Let $f$ be a positive integer. Every order in $K$ has form $\Lambda_f = \mathbb{Z} + (f\omega)\mathbb{Z}$, where $f$ is the conductor of $\Lambda_f$.

*Proof.* See [4] pp. 130-132. □

The proposition 1 allows to classify the similarity classes of the full modules in the field $K$. Indeed, there exists a finite number of $m_f(1), \ldots, m_f(s)$ of the non-similar full modules in the field $K$, whose coefficient ring is the order $\Lambda_f$, cf Theorem 3, Ch 2.7 of [4]. Thus, proposition 1 gives a finite-to-one classification of the similarity classes of full modules in the field $K$.

Let $\Lambda_f$ be an order in $K$ with the conductor $f$. Under the addition operation, the order $\Lambda_f$ is a full module, which we denote by $m_f$. Let us evaluate the invariants $q(x,y)$, $\Delta$ and $\Sigma$ on the module $m_f$. To calculate $(s_{ij}) = Tr(\lambda_i\lambda_j)$, we let $\lambda_1 = 1, \lambda_2 = f\omega$. Then:

$$s_{11} = 2, \quad s_{12} = a_{21} = f, \quad s_{22} = \frac{1}{2} f^2(d + 1) \quad \text{if } d \equiv 1 \mod 4$$

$$s_{11} = 2, \quad s_{12} = s_{21} = 0, \quad s_{22} = 2f^2d \quad \text{if } d \equiv 2, 3 \mod 4, \quad (12)$$

and

$$q(x,y) = 2x^2 + 2fxy + \frac{1}{2} f^2(d + 1)y^2 \quad \text{if } d \equiv 1 \mod 4$$

$$q(x,y) = 2x^2 + 2f^2dy^2 \quad \text{if } d \equiv 2, 3 \mod 4. \quad (13)$$

Therefore

$$\Delta = \begin{cases} 
2f^2d & \text{if } d \equiv 1 \mod 4, \\
4f^2d & \text{if } d \equiv 2, 3 \mod 4,
\end{cases} \quad (14)$$

and $\Sigma = +2$ in the both cases, where $\Sigma = \#(\text{positive}) - \#(\text{negative})$ entries in the diagonal normal form of $q(x,y)$.
3.3 Example

Let us consider a numerical example, which illustrates an advantage of the above invariants in comparison to the classical Alexander polynomials. Denote by $M_\phi$ and $M_{\phi'}$ the torus bundles given by the monodromy

$$B = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix},$$

respectively. The Alexander polynomial of three dimensional manifolds $M_\phi$ and $M_{\phi'}$ are identical $\Delta(t) = \Delta'(t) = t^2 - 6t + 1$. However, the bundles $M_\phi$ and $M_{\phi'}$ are not homotopy equivalent.

Indeed, the Perron-Frobenius eigenvector of matrix $B$ is $v_B = (1, \sqrt{2} - 1)$ while of the matrix $B'$ is $v_{B'} = (1, 2\sqrt{2} - 2)$. The bilinear forms for the modules $m_B = \mathbb{Z} + (\sqrt{2} - 1)\mathbb{Z}$ and $m_{B'} = \mathbb{Z} + (2\sqrt{2} - 2)\mathbb{Z}$ can be written as

$$q_B(x, y) = 2x^2 - 4xy + 6y^2, \quad q_{B'}(x, y) = 2x^2 - 8xy + 24y^2,$$

respectively. The modules $m_B$ and $m_{B'}$ are not similar in the number field $K = \mathbb{Q}(\sqrt{2})$, since their determinants $\Delta(m_B) = 8$ and $\Delta(m_{B'}) = 32$ are not equal. Therefore the matrices $B$ and $B'$ are not similar $^1$ in $SL(2, \mathbb{Z})$.

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$^1$The reader may verify this fact using the method of periods, which dates back to Gauss. First we have to find the fixed points $Bx = x$ and $B'x = x$, which gives us $x_B = 1 + \sqrt{2}$ and $x_{B'} = \frac{1 + \sqrt{2}}{2}$, respectively. Then one unfolds the fixed points into a periodic continued fraction, which gives us $x_B = [2, 2, 2, \ldots]$ and $x_{B'} = [1, 4, 1, 4, \ldots]$. Since the period (2) of $x_B$ differs from the period (1, 4) of $B'$, the matrices $B$ and $B'$ are not similar in $SL(2, \mathbb{Z})$. 

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