On Non-Cooperative Perfect Information Semi-Markov Games

K. G. Bakshi\textsuperscript{a} \textsuperscript{*}, S. Sinha\textsuperscript{b}
\textsuperscript{a,b}Jadavpur University, Department Of Mathematics, Kolkata, 700 032, India.
\textsuperscript{a}Email: kushalguhabakshi@gmail.com
\textsuperscript{b}Email: sagnik62@gmail.com

Abstract

We show that an $N$-person non-cooperative semi-Markov game under limiting ratio average pay-off has a pure semi-stationary Nash equilibrium. In an earlier paper, the zero-sum two person case has been dealt with. The proof follows by reducing such perfect information games to an associated semi-Markov decision process (SMDP) and then using existence results from the theory of SMDP. Exploiting this reduction procedure, one gets simple proofs of the following:

(a) zero-sum two person perfect information stochastic (Markov) games have a value and pure stationary optimal strategies for both the players under discounted as well as undiscounted pay-off criterion.
(b) Similar conclusions hold for $N$-person non-cooperative perfect information stochastic games as well.

All such games can be solved using any efficient algorithm for the reduced SMDP (MDP for the case of Stochastic games).

In this paper, we have implemented Mondal’s algorithm \cite{14} to solve an SMDP under limiting ratio average pay-off criterion.

Keywords: Semi-Markov games, Perfect information, Pure semi-stationary strategies, Nash equilibrium, Linear programming.

AMS subject classifications: 90C40, 91A05, 90C05, 91A10

1 Introduction

A semi-Markov game(SMG) is a generalisation of a stochastic game(\cite{20}), where the sojourn time depends not only on the present state and actions chosen but also on the state at the next decision epoch. The theory of semi-Markov games finds applications
in dynamic overlapping generations models ([18]) and dynamic oligopoly models ([9]). We study two-person non-cooperative semi-Markov games considered under limiting ratio average pay-off criterion. For other applications refer to Bhattacharya (1989) ([1]), Kurano (1985) ([10]) and Sennott (1994) ([19]). Lal and Sinha ([11]) studied two-person zero-sum semi-Markov games and established the existence of value and stationary optimal strategies for discounted pay-off and limiting average pay-off criterion under various ergodicity conditions. Single player semi-Markov games are called semi-Markov decision processes (SMDPs) which were introduced by Jewell ([7]) and Howard ([5]). A perfect information semi-Markov game (PISMG) is a natural extension of perfect information stochastic games (PISGs) ([4], [22], [13]), where at each state at most one player has more than one action available to him (i.e., all but one are dummies). Various types of recurrence like conditions are assumed to prove the existence of Nash equilibrium strategy in the literature of SMGs under limiting ratio average pay-offs. For example ([3], [11], [17]) considered ergodicity conditions whereas ([6], [23]) used some variants of a Lyapunov like conditions which yields the so-called weighted geometric ergodicity property. In this paper we establish the existence of pure semi-stationary Nash equilibria in non-cooperative perfect information semi-Markov games for the limiting ratio average pay-off criterion. We prove this result by using an existence result by Sinha and Mondal(2017) [21] and then using a strategic equivalence between an NCPISMG and an undiscounted(limiting ratio average) SMDP. The paper is organized as follows. Section 2 contains definitions and properties of an $N$-person non-cooperative semi-Markov games and semi-Markov decision processes considered under limiting ratio average pay-off. Section 3 contains main result of this paper. In section 4 we propose a linear programming algorithm to compute an optimal semi-stationary strategy pair for the players of such perfect information undiscounted semi-Markov games. Section 5 contains some numerical examples illustrating our theorem and proposed algorithm. Section 6 is reserved for the conclusion.

2 Preliminaries

2.1 Finite $N$-person non-cooperative perfect information semi-Markov games (NCPISMGs)

An $N$-person (finite) non-cooperative perfect information semi-markov game (NCPI SMG) $\Gamma$ is described by a collection of five objects $\Gamma = \{S, q, P, \{r_i : i \in \{1, 2, \ldots, N\}\}\}$, $S = \bigcup_{i=1}^{N} S_i = \{1, 2, \ldots, N\}$ is a (finite non-empty) state space and $S_i$ is the states where all players except player $i$ are dummies (i.e., they have exactly one action available in states belonging to $S_i$). Thus $\{S_1, S_2, \ldots, S_N\}$ constitutes a partition of $S$. $A_i(s) = \{1, 2, \ldots, m_i(s)\}$ is the non-empty set of admissible pure actions of the player $i$ respectively in the states of $S_i$, for $i = 1, 2, \ldots, N$ (indeed $A_i(s) = \{1\}$ if $s \in S - S_i$). Let us denote by $K = \{(s, a_1, a_2, \ldots, a_N) : s \in S, a_i \in A_i(s), i = 1, 2, \ldots, N\}$ the set of all possible state-action tuples. For each $(s, a_1, a_2, \ldots, a_N)$, we denote by $q(. | s, a_1, a_2, \ldots, a_N)$,
the corresponding transition law of the game $\Gamma$ and $P_{ss'}(\cdot | a_1, a_2, \cdots, a_N)$ is a distribution function on $[0, \infty]$ given $K \times S$, which is called the conditional transition (sojourn) time distribution. Finally $r_i(\cdot | a_1, a_2, \cdots, a_N)$ is the real valued functions on $K$, which represents the immediate reward for player $i$ for $i = 1, 2, \cdots, N$. The semi-Markov game over infinite time is played as follows: at the 1st decision epoch, the game starts at $s_1 \in S$ and the player $i$ chooses actions $a_i^1 \in A_i(s_1)$ independently of the actions chosen by other players ($\neq i$). Consequently player $i$ gets immediate reward $r_i(s_1, a_i^1, a_2^1, \cdots, a_N^1)$ ($i \in \{1, 2, \cdots, N\}$) and on the next decision epoch game moves to the state $s_2$ with probability $q(s_2 | s_1, a_1^1, a_2^1, \cdots, a_N^1)$ on the next decision epoch. The sojourn (transition) time to move from state $s_1$ to the state $s_2$ is determined by the distribution function $P_{s_1s_2}(\cdot | a_1^1, a_2^1, \cdots, a_N^1)$. After reaching the state $s_2$ on the next decision epoch, the game is repeated over and over again with state $s_1$ replaced by $s_2$. Thus the game proceeds over the infinite future.

By a behavioural strategy $\pi_i$ of the player $i$ (for $i \in \{1, 2, \cdots, N\}$), we mean a sequence \{(\pi_i)_n(. | hist_n)\}_{n=1}^\infty$, where $(\pi_i)_n$ specifies the action to be chosen by player-$i$ on the $n$-th decision epoch by associating with each history $hist_n$ of the system up to $n$th decision epoch (where $hist_n = (s_1, a_1^1, a_2^1, \cdots, a_N^1, s_2, a_1^2, a_2^2, \cdots, a_N^2, \cdots, s_{n-1}, a_1^{n-1}, a_2^{n-1}, \cdots, a_N^{n-1}, s_n)$ for $n \geq 2$ and $hist_1 = (s_1)$) and $(s_k, a_1^k, a_2^k, \cdots, a_N^k) \in K$ are respectively the state and actions of the players at the $k$-th decision epoch) a probability distribution $(\pi_i)_n(. | hist_n)$ on $A_i(s_n)$ (for $i \in \{1, 2, \cdots, N\}$). We denote by $\Pi_i$ the set of strategy (behavioural) spaces of the players $i$ (for $i \in \{1, 2, \cdots, N\}$). A strategy $\sigma_i = \{(\sigma_i)_n\}_{n=1}^\infty$ for the player $i$ is called semi-Markov if for each $n$, $(\sigma_i)_n$ depends on $s_1, s_n$ and the decision epoch number $n$ (for $i \in \{1, 2, \cdots, N\}$). A stationary strategy is a strategy that depends only on the current state. A stationary strategy for player $i$ is defined as an $N$ tuple $f_i = (f_i(1), f_i(2), \cdots, f_i(N))$, where each $f_i(s)$ is a probability distribution on $A_i(s)$ given by $f_i(s) = (f_i(s, 1), f_i(s, 2), \cdots, f_i(s, a_i))$ where $f_i(s, a)$ denotes the probability of choosing action $a$ in the state $s$ (for $i \in \{1, 2, \cdots, N\}$). Let us denote by $F_i$ the set of stationary strategies for player $i$ (for $i \in \{1, 2, \cdots, N\}$). A stationary strategy is called pure if any player selects a particular action with probability 1 while in a state $s$. We denote by $F_i^p$ to be the set of pure stationary strategies of $i$-th player (for $i \in \{1, 2, \cdots, N\}$). A semi-stationary strategy is a semi-Markov strategy which is independent of the decision epoch $n$, i.e., for an initial state $s_1$ and present state $s_n$, if a semi-Markov strategy $f(s_1, s_n, n)$ is independent of $n$, then we call it a semi-stationary strategy. Let $\xi_i$ and $\xi_i^p$ denote the set of semi-stationary and pure semi-stationary strategies for the $i$-th player (for $i\{1, 2, \cdots, N\}$).

Let $(X_1, A_1^1, A_2^1, \cdots, A_N^1, X_2, A_1^2, A_2^2, \cdots, A_N^2, \cdots)$ be the co-ordinate sequence in $S \times (A_1 \times A_2 \times \cdots \times A_N) \times S)^\infty$. Given a behavioural strategy tuple $(\pi_1, \pi_2, \cdots, \pi_N) \in \Pi_1 \times \Pi_2 \cdots \Pi_N$, and an initial state $s \in S$, there exists a unique probability measure $P_{\pi_1, \pi_2, \cdots, \pi_N}(\cdot | X_1 = s)$ (hence an expectation operator $E_{\pi_1, \pi_2, \cdots, \pi_N}(\cdot | X_1 = s)$) on the product $\sigma$-field of $K^\infty \times S$ by Kolmogorov’s extension theorem.

**Definition 1** For a strategy tuple $(\pi_1, \pi_2, \cdots, \pi_N) \in \Pi_1 \times \Pi_2 \cdots \times \Pi_N$ for the players $1, 2, \cdots, N$ respectively, the limiting ratio average (undiscounted) pay-off for
player \(i\), starting from a state \(s \in S\) is defined by:

\[
\phi_i(s, \pi_1, \pi_2, \ldots, \pi_N) = \lim_{n \to \infty} \frac{E_{\pi_1, \pi_2, \ldots, \pi_N} \sum_{m=1}^{n} \left[ r_i(X_m, A_{1}^m, A_{2}^m, \ldots, A_{N}^m) \mid X_1 = s \right]}{E_{\pi_1, \pi_2, \ldots, \pi_N} \sum_{m=1}^{n} \left[ \tau(X_m, A_{1}^m, A_{2}^m, \ldots, A_{N}^m) \mid X_1 = s \right]}.
\]

(2.1)

Here \(\tau(s, a_1, a_2, \ldots, a_N) = \sum_{s' \in S} q(s' \mid s, a_1, a_2, \ldots, a_N) \int_{0}^{\infty} tdP_{ss'}(t \mid a_1, a_2, \ldots, a_N)\) is the expected sojourn time in the state \(s\) for an action tuple \((a_1, a_2, \ldots, a_N) \in A_1(s) \times A_2(s) \times \cdots \times A_N(s)\).

For each stationary strategy tuple \((f_1, f_2, \ldots, f_N) \in F_1 \times F_2 \times \cdots \times F_N\) we define the one-step transition probability matrix as \(Q(f_1, f_2, \ldots, f_N) = [q(s' \mid s, f_1, f_2, \ldots, f_N)]_{s \times s}\), where \(q(s' \mid s, f_1, f_2, \ldots, f_N) = \sum_{a_1 \in A_1(s)} \sum_{a_2 \in A_2(s)} \sum_{a_N \in A_N(s)} q(s' \mid s, a_1, a_2, \ldots, a_N)f_1(s, a_1)f_2(s, a_2) \cdots f_N(s, a_N)\) is the probability that starting from the state \(s\), next state is \(s'\) when the players choose stationary strategies \(f_1, f_2, \ldots, f_N\) respectively. (For a stationary strategy \(f\), \(f(s, a)\) denotes the probability of choosing action \(a\) in the state \(s\)).

For any stationary strategy tuple \((f_1, f_2, \ldots, f_N) \in F_1 \times F_2 \times \cdots \times F_N\) of player I and II, since the limit in (2.1) exists we write the undiscounted pay-off for player \(i\) as

\[
\phi_i(s, f_1, f_2, \ldots, f_N) = \lim_{n \to \infty} \frac{\sum_{m=1}^{n} r_i^m(s, f_1, f_2, \ldots, f_N)}{\sum_{m=1}^{n} \tau^m(s, f_1, f_2, \ldots, f_N)} \text{ for all } s \in S.
\]

Where \(r_i^m(s, f_1, f_2, \ldots, f_N)\) and \(\tau^m(s, f_1, f_2, \ldots, f_N)\) are respectively the expected reward and expected sojourn time for player \(i\) at the \(m\) th decision epoch, when player \(i\) chooses \(f_i\) and the initial state is \(s\) (for \(i \in \{1, 2, \ldots, N\}\)). We define \(r_i(f_1, f_2, \ldots, f_N) = [r_i(s, f_1, f_2, \ldots, f_N)]_{s \times 1}, \tau(f_1, f_2, \ldots, f_N) = [\tau(s, f_1, f_2, \ldots, f_N)]_{s \times 1}\) as expected reward, expected sojourn time and for a stationary strategy tuple \((f_1, f_2, \ldots, f_N) \in F_1 \times F_2 \times \cdots \times F_N\). Now

\[
r_i^m(s, f_1, f_2, \ldots, f_N) = \sum_{s' \in S} P_{f_1f_2\cdots f_N}(X_m = s' \mid X_1 = s)r_i(s', f_1, f_2, \ldots, f_N) = \sum_{s' \in S} q^{m-1}(s' \mid s, f_1, f_2, \ldots, f_N)r_i(s', f_1, f_2, \ldots, f_N) = [Q^{m-1}(f_1, f_2, \ldots, f_N)r_i(f_1, f_2, \ldots, f_N)](s)
\]

and

\[
\tau^m(s, f_1, f_2, \ldots, f_N) = \sum_{s' \in S} P_{f_1f_2\cdots f_N}(X_m = s' \mid X_1 = s)\tau(s', f_1, f_2, \ldots, f_N) = \sum_{s' \in S} q^{m-1}(s' \mid s, f_1, f_2, \ldots, f_N)\tau(s', f_1, f_2, \ldots, f_N) = [Q^{m-1}(f_1, f_2, \ldots, f_N)\tau(f_1, f_2, \ldots, f_N)](s)
\]

Since \(Q(f_1, f_2, \ldots, f_N)\) is a Markov matrix, then for each \(i \in \{1, 2, \ldots, N\}\), we have by Kemeny et al.,

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{m=0}^{n} Q^m(f_1, f_2, \ldots, f_N) \text{ exists and equals to } Q^*(f_1, f_2, \ldots, f_N).
\]

Thus, it is obvious that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} r_i^m(f_1, f_2, \ldots, f_N) = [Q^*(f_1, f_2, \ldots, f_N)r_i(f_1, f_2, \ldots, f_N)] \text{ (equality is valid co-ordinate wise)}
\]
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \bar{\pi}^m (f_1, f_2, \ldots, f_N) = [Q^* (f_1, f_2, \ldots, f_N) \bar{\pi} (f_1, f_2, \ldots, f_N)] \text{ (equality is valid co-ordinate wise).} \]

Thus we have for any stationary strategy tuple \((f_1, f_2, \ldots, f_N) \in F_1 \times F_2 \times \cdots \times F_N, \)
\[ \phi_i (s, f_1, f_2, \ldots, f_N) = \frac{[Q^* (f_1, f_2, \ldots, f_N) \bar{\pi} (f_1, f_2, \ldots, f_N)] (s)}{[Q^* (f_1, f_2, \ldots, f_N) \bar{\pi} (f_1, f_2, \ldots, f_N)] (s)} \text{ for all } s \in S, i = 1, 2, \ldots, N. \]

where \(Q^* (f_1, f_2, \ldots, f_N)\) is the Cesaro limit matrix of \(Q(f_1, f_2, \ldots, f_N).\)

**Assumption 1** There exist \(\epsilon > 0\) and \(M > 0\) with \(\epsilon < M\) such that
\[ \epsilon \leq \bar{\pi} (s, a_1, a_2, \ldots, a_N) \leq M, \forall (s, a_1, a_2, \ldots, a_N) \in K. \]

**Definition 3** A strategy tuple \((\pi_1^*, \pi_2^*, \ldots, \pi_N^*) \in \Pi_1 \times \Pi_2 \times \cdots \times \Pi_N\) is called a Nash equilibrium strategy tuple if
\[ \phi_i (s, \pi_1^*, \pi_2^*, \ldots, \pi_N^*) \geq \phi_i (s, \pi_1^*, \pi_2^*, \ldots, \pi_i^{*+1}, \pi_i, \pi_{i+1}^*, \ldots, \pi_N^*) \forall s \in S, \text{ any } \pi_i \in \Pi_i \]
for \(i = 1, 2, \ldots, N.\)

### 2.2 Semi-Markov decision processes (SMDPs)

An SMDP is an NCPISMG for which \(N = 1\), formally we define a finite semi-Markov decision process by a collection of five objects \(\Gamma = < S, \hat{A} = \{A(s) : s \in S\}, \hat{q}, \hat{P}, \hat{r} >, \)
where \(S = \{1, 2, \ldots, z\}\) is a (finite) state space, \(\hat{A}\) is a (finite) action space, in which \(A(s)\) is the finite set of admissible actions in the state \(s\) of the controller. \(\hat{q}(s' | s, a)\) is the transition probability (i.e., \(\hat{q}(s' | s, a) \geq 0 \text{ and } \sum_{s' \in S} \hat{q}(s' | s, a) = 1\)) that the next state is \(s'\), where \(s\) is the initial state and the decision maker chooses action \(a \in \hat{A}(s)\) in the state \(s\). \(\hat{P}_{ss'}(s' | a)\) is a distribution function on \([0, \infty)\) called the transition (sojourn) time distribution function and \(\hat{r}\) is the immediate reward function. The decision process proceeds over infinite time just as in a semi-Markov game, where instead of \(N\) players we consider a single decision maker. The definition of various strategy spaces for the decision maker is same as in a semi-Markov game. Let us denote by \(\Pi, F, F_s, \xi\) and \(\xi^p\) as the set of behavioural, stationary, pure-stationary, semi-stationary and pure semi-stationary strategies respectively of the decision maker.

Let \((X_1, A_1, X_2, A_2, \cdots)\) be a coordinate sequence in \(S \times (\hat{A} \times S)^\infty \) in \(\Gamma\). Given a behavioural strategy \(\pi \in \Pi\), an initial state \(s \in S\), there exists a unique measure \(\mu_{\pi}(\cdot | X_1 = s)\) (hence an expectation operator \(E_{\pi}(\cdot | X_1 = s)\) on the product \(\sigma\)-field of \(S \times (\hat{A} \times S)^\infty\) by Kolmogorov’s extension theorem.

For a behavioural strategy \(\pi \in \Pi\), the expected limiting ratio average pay-off (in short undiscounted pay-off) is defined by
\[ \hat{\phi}(s, \pi) = \lim \inf_{n \to \infty} \frac{E_{\pi} \sum_{m=1}^{n} [\hat{r}(X_m, A_m) | X_1 = s]}{E_{\pi} \sum_{m=1}^{n} [\hat{r}(X_m, A_m) | X_1 = s]} \text{ for all } s \in S. \]
Where $\tau(s,a) = \sum_{s' \in S} \hat{q}(s' | s,a) \int_0^\infty td\hat{P}_{ss'}(t | a)$ is the expected sojourn time in the state $s$ when decision maker chooses the action $a \in \hat{A}(s)$.

**Theorem 1** (21) For a finite semi-Markov decision process with scalar reward considered under limiting ratio average pay-off, there exists a pure semi-stationary strategy $f^* = \{f^*_s : s \in S\} \in \xi^p$ which is optimal for the decision maker.

### 3 Main result

**Theorem 2** In a non-cooperative $N$-person perfect information semi-Markov game (NCPISMG) under limiting ratio average pay-off criterion, there exists a pure semi-stationary Nash equilibrium for the players.

**Proof.** To avoid notational complexity, we prove the theorem for $N = 2$. The proof follows by a strategic equivalence between an NCPISMG and an undiscounted (limiting ratio average) semi-Markov decision process (SMDP). Let $\Gamma = < S = S_1 \cup S_2, A = \{A(s) : s \in S_1\}, B = \{B(s) : s \in S_2\}, q, P, r_1, r_2 >$ be a non-cooperative two person perfect information semi-Markov game under limiting ratio average pay-off, where in the first $|S_1|$ number of states, player-II is a dummy and in the last $\{\{S_1\} + 1, \cdots, |S_1| + |S_2|\}$ states player-I is a dummy. We construct an undiscounted SMDP $\hat{\Gamma}$ from $\Gamma$ above, which is defined as $\hat{\Gamma} = < S = S_1 \cup S_2, \hat{A} = \{\hat{A}(s) = A(s) \forall s \in S_1\}$ and $\{\hat{A}(s) = B(s) \forall s \in S_2\}, \hat{q} = q, \hat{P} = P, \hat{r} >$, where $\hat{r}(s,.)$ is defined as:

$$\hat{r}(s,i) = \begin{cases} r_1(s,i) & s \in S_1, i \in \hat{A}(s) = A(s) \\ r_2(s,j) & s \in S_2, j \in \hat{A}(s) = B(s) \end{cases}$$

By theorem 1 in section 2, $\hat{\Gamma}$ admits an optimal pure semi-stationary strategy $\hat{f}^*$. Let $s_1$ be a fixed but arbitrary initial state, thus $\hat{f}^* = \{\hat{f}^*(s_1, s) : \hat{f}^*(s_1, s) \in \mathbb{P}_{\hat{A}(s)} \forall s_1, s \in S\}$, where $\mathbb{P}_{\hat{A}(s)}$ is the set of probability distributions over the action space $\hat{A}(s)$. We extract a pure semi-stationary strategy pair $(f^*_1, f^*_2)$ in $\Gamma$ from $\hat{f}^*$ as follows:

$$f^*_1(s_1, s) = \begin{cases} \hat{f}^*(s_1, s) & s \in S_1 \\ 1 & s \in S_2 \end{cases}$$

$$f^*_2(s_1, s) = \begin{cases} 1 & s \in S_1 \\ \hat{f}^*(s_1, s) & s \in S_2 \end{cases}$$

We denote by $\phi_1, \phi_2$ and $\hat{\phi}$ the undiscounted pay-off functions for the players-I and II in the NCPISMG $\Gamma$ and the SMDP $\hat{\Gamma}$ respectively. Let $\xi^p_1$ and $\xi^p_2$ be the set of pure semi-stationary strategies for player-I and player-II respectively in the NCPISMG $\Gamma$. By corollary 1 of [10], we know that if there exists a pure semi-stationary Nash equilibrium strategy pair $(f^*_1, f^*_2)$ among the class $(\xi^p_1, \xi^p_2)$ of pure semi stationary strategy pair, then $(f^*_1, f^*_2)$ is a pair of pure semi-stationary Nash equilibrium of the game among the set of all behavioural strategy pair $(\Pi_1, \Pi_2)$ of the game. Thus, we can concentrate only on the set of pure semi-stationary strategies instead of behavioural
strategies. Now, if we fix the strategy $f_2^*$ for the states $\{|S_1| + 1, \cdots, |S_1| + |S_2|\}$ in the NCPISMG $\Gamma$ then it reduces to a decision process $\hat{\Gamma}_{12}$. Similarly, we can derive another SMDP model $\hat{\Gamma}_{21}$ by fixing the strategy $f_1^*$ for the states $\{|1, 2, \cdots, |S_1|\}$ in the NCPISMG $\Gamma$. If we can prove that $f_1^*$ is an optimal pure semi-stationary strategy in the reduced SMDP $\hat{\Gamma}_{12}$ and $f_2^*$ is an optimal pure semi-stationary strategy in the reduced SMDP $\hat{\Gamma}_{21}$ then eventually $(f_1^*, f_2^*)$ becomes a pure semi-stationary Nash equilibrium pair for both the players in the NCPISMG $\Gamma$. We now prove the following lemma.

**Lemma 1** $f_1^*$ is an optimal pure semi-stationary strategy for the player I in the reduced SMDP $\hat{\Gamma}_{12}$.

**Proof.** We assume by contradiction, that $f_1^*$ is not an optimal pure semi-stationary optimal strategy in $\hat{\Gamma}_{12}$. Let $\hat{\phi}_1$ be the undiscounted pay-off function of the SMDP $\hat{\Gamma}_{12}$. Suppose, $f_1'$ is an optimal pure semi-stationary optimal strategy in $\hat{\Gamma}_{12}$. Thus we have $\hat{\phi}_1(s, f_1') \geq \hat{\phi}_1(s, f_1^*) \forall s \in S_1 \cup S_2$, with strict inequality for at least one state in $S_1$. Suppose for $s_1 \in S$, $\phi_1(s_1, f_1') > \phi_1(s_1, f_1^*)$ holds. This implies, $\phi_1(s_1, f_1', f_2^*) > \phi_1(s_1, f_1^*, f_2^*)$ in the NCPISMG $\Gamma$. Let us construct a pure semi-stationary strategy $\hat{f}_1^*$ in the SMDP $\hat{\Gamma}$, which consists of the strategy pair $(f_1', f_2^*)$, such that it coincides with $f_1'$ in $S_1$ and in $S_2$ it coincides with $f_2^*$. i.e.,

$$\hat{f}_1^*(s_1, s) = \begin{cases} f_1'(s_1, s) & s \in S_1 \\ f_2^*(s_1, s) & s \in S_2 \end{cases}$$

Thus we get the inequality $\hat{\phi}_1(s_1, \hat{f}_1^*) > \hat{\phi}_1(s_1, f_1^*)$, which contradicts that $\hat{f}_1^*$ is an optimal pure semi-stationary strategy in the SMDP $\hat{\Gamma}$. $\square$

Similarly we can prove the following lemma:

**Lemma 2** $f_2^*$ is an optimal pure semi-stationary strategy in the reduced SMDP $\hat{\Gamma}_{21}$.

**Proof.** Proof is similar to the proof of lemma 1. $\square$

Thus in view of lemma 1 and lemma 2, the proof of theorem 2 follows and $(f_1^*, f_2^*)$ is a pure semi-stationary Nash equilibrium strategy in the NCPISMG $\Gamma$. $\square$

### 4 The Algorithm

We implement Mondal’s algorithm [14] for solving an undiscounted SMDP to solve such games. In what follows we assume $s_1$ to be a fixed but arbitrary initial state. Consider the following linear programming problem in the variables $v(s_1)$, $g = (g_s : s \in S)$ and $h = (h_s : s \in S)$ as:

$$LP : \min v(s_1) \quad \text{subject to}$$
\[ g_s \geq \sum_{s' \in S} \hat{q}(s' \mid s, a)g_{s'}, \forall s \in S, a \in \hat{A}(s). \tag{4.1} \]

\[ g_s + h_s \geq \hat{r}(s, a) - v(s_1)\hat{r}(s, a) + \sum_{s' \in S} \hat{q}(s' \mid s, a)h_{s'}, \forall s \in S, a \in \hat{A}(s). \tag{4.2} \]

\[ g_{s_1} \leq 0. \tag{4.3} \]

The variables \( v(s_1), (g_s : s \in S, s \neq s_1) \) and \( (h_s : s \in S, s = s_1) \) are unrestricted in sign. The dual linear programming problem of this primal for the variables \( x = (x_{sa} : s \in S, a \in \hat{A}(s)) \) and \( y = (y_{sa} : s \in S, a \in \hat{A}(s)) \) and \( t \) is given by

\[ \text{DLP : max } R_s, \text{ where } R_s = \sum_{s \in S} \sum_{a \in \hat{A}(s)} \hat{r}(s, a)x_{sa} \]

subject to

\[ \sum_{s \in S} \sum_{a \in \hat{A}(s)} \{\delta_{ss'} - \hat{q}(s' \mid s, a)\}x_{sa} = 0 \forall s' \in S. \tag{4.4} \]

\[ \sum_{a \in \hat{A}(s')} x_{s'a} + \sum_{s \in S} \sum_{a \in \hat{A}(s)} \{\delta_{ss'} - \hat{q}(s' \mid s, a)\}y_{sa} = 0 \forall s' \in S - \{s_1\}. \tag{4.5} \]

\[ \sum_{a \in \hat{A}(s_1)} x_{s_1a} + \sum_{s \in S} \sum_{a \in \hat{A}(s)} \{\delta_{ss_1} - \hat{q}(s_1 \mid s, a)\}y_{sa} - t = 0 \forall s' \in S - \{s_1\}. \tag{4.6} \]

\[ \sum_{s \in S} \sum_{a \in \hat{A}(s)} \hat{r}(s, a)x_{sa} = 1. \tag{4.7} \]

\[ x_{sa}, y_{sa}, t \geq 0 \forall s \in S, a \in \hat{A}(s), t \geq 0. \tag{4.8} \]

where \( \delta_{ss'} \) is the Kronecker’s delta function. For a feasible solution \((x, y, t)\) of the DLP, we define the following sets:

\[ \hat{S}_x = \{s \in S : \sum_{a \in \hat{A}(s)} x_{sa} > 0\} \]
\[ \hat{S}_y = \{s \in S : \sum_{a \in \hat{A}(s)} x_{sa} = 0 \text{ and } \sum_{a \in \hat{A}(s)} y_{sa} > 0\} \]
\[ \hat{S}_{xy} = \{s \in S : \sum_{a \in \hat{A}(s)} x_{sa} = 0 \text{ and } \sum_{a \in \hat{A}(s)} y_{sa} = 0\}. \]

Thus \( \hat{S} = \hat{S}_x \cup \hat{S}_y \cup \hat{S}_{xy} \), where \( \hat{S}_x, \hat{S}_y \) and \( \hat{S}_{xy} \) are pairwise disjoint sets. A pure stationary strategy corresponding to the feasible solution \((x, y, t)\) of the DLP is defined by \( f_{xyt}^{ps_1} \), where \( s_1 \) is the fixed but arbitrary initial state \( f_{xyt}^{ps_1}(s) = a_s, s \in S \) such that:

\[ a_s = \begin{cases} a & \text{if } s \in \hat{S}_x \text{ and } x_{sa} > 0 \\ a' & \text{if } s \in \hat{S}_y \text{ and } y_{sa'} > 0 \\ \text{arbitrary} & \text{if } s \in \hat{S}_{xy} \end{cases} \]

Now by [15], we have the following theorem.

**Theorem 3** Suppose \( s_1 \) be an arbitrary but fixed initial state. Let \((x^*, y^*, t^*)\) be an
optimal solution of the DLP. Then \( {f^*_{x^*y^*}} \) is a pure stationary optimal strategy of the SMDP for the initial state \( s_1 \).

From the above theorem, we conclude that an optimal pure semi-stationary strategy of an SMDP can be found from an optimal solution of the DLP above. Thus this algorithm is useful to obtain an optimal pure semi-stationary Nash equilibrium in the non-cooperative perfect information semi-Markov game. Now for a fixed initial state \( s_1 \), we have \( {f^*_{x^*y^*}} \) as an optimal pure stationary strategy, where \( {f^*_{x^*y^*}}(s) = a_s \) and \( a_s \) is defined above. So, for each initial state, we have an optimal pure stationary strategy of the decision maker in the undiscounted SMDP. By corollary 1 of [15], we obtain an optimal pure semi-stationary strategy \( \hat{f}^* \) of the decision maker in the SMDP \( \hat{\Gamma} \), where \( \hat{f}^* = (\hat{f}_1^*, \hat{f}_2^*, \cdots, \hat{f}_s^*, \cdots, \hat{f}_N^*) \) and \( \hat{f}_s^* = {f^*_{x^*y^*}} \) is an optimal pure stationary strategy of the SMDP with initial state \( s_1 \). Now as discussed in section 3, we obtain an optimal pure semi-stationary Nash equilibrium strategy pair \( (f_1^*, f_2^*) \) for the players I and II in the NCPISMG \( \Gamma \) given by

\[
\begin{align*}
f_1^*(s_1, s) &= \begin{cases} \hat{f}^*(s_1, s) & s \in S_1 \\ 1 & s \in S_2 \end{cases} \\
f_2^*(s_1, s) &= \begin{cases} 1 & s \in S_1 \\ \hat{f}^*(s_1, s) & s \in S_2 \end{cases}
\end{align*}
\]

4.1 Verification through Complete Enumeration

To verify that the pair \((f_1^*, f_2^*)\) above is indeed a Nash Equilibrium, we calculate the undiscounted value of the SMDP for a pure stationary strategy \( f \) as:

\[
\hat{\phi}(s, f) = \frac{[Q^*(f)\hat{r}(f)](s)}{[Q^*(f)\hat{\tau}(f)](s)} \quad \text{for all } s \in S
\]

where \( \hat{r}(f) \) is the reward vector and \( \hat{\tau}(f) \) is the expected sojourn time vector for a pure stationary strategy \( f \). The algorithm [12] to compute Cesaro limiting matrix for a SMDP with \( n \) number of states is as follows:

**Input:** The transition matrix \( Q \in M_n(\mathbb{R}) \) (where \( M_n(\mathbb{R}) \) is the set of \( n \times n \) matrices over real numbers).

**Output:** The Cesaro limiting matrix \( Q^* \in M_n(\mathbb{R}) \).

**Step 1:** Determine the characteristic polynomial \( C_Q(z) = |Q - zI_n| \).

**Step 2:** Divide the polynomial \( C_Q(z) \) by \( (z - 1)^{m(1)} \) (where \( m(1) \) is the algebraic multiplicity of the eigenvalue \( z_0 = 1 \)) and call it quotient \( T(z) \).

**Step 3:** Compute the quotient matrix \( W = T(Q) \).

**Step 4:** Determine the limiting matrix \( Q^* \) by dividing the matrix \( W \) by the sum of its elements of any arbitrary row.

We observe that the LP algorithm is much faster than the complete enumeration method. We elaborate this fact through the following examples.
5 Numerical examples

Example 1: Consider an NCPISMG $\Gamma$ with five states $S = \{1, 2, 3, 4, 5\}$, $A(1) = \{1, 2\} = A(2)$, $B(1) = B(2) = \{1\} = B(5)$, $B(3) = B(4) = \{1, 2\}$, $A(3) = A(4) = \{1\}$. Player II is the dummy player in the state 1 and 2 and 5 and player I is the dummy player for the states 3 and 4. Rewards, transition probabilities and expected sojourn times for the players are given below:

State-1:

| (13, 2)       | 4 |
| (1, 0, 0, 0, 0) |   |
| (9, 1)        | 2 |
| (0, 0, 0, 1, 0) |   |

State-2:

| (4, 2)       | 2 |
| (0, 1, 0, 0, 0) |   |
| (3, 1)        | 1.6 |
| (0, 1, 0, 0, 0) |   |

State-3:

| (4, 7)       | 2 |
| (1, 3)       | 1.5 |
| (1 3, 2, 0, 0) |   |
| (0, 0, 0, 1) | 1 |

State-4:

| (6, 15)      | 5 |
| (1, 0, 0, 0, 0) |   |
| (7, 0)       | 1 |
| (0, 0, 0, 0, 1) |   |

State-5:

| (5, 2)       | 3 |
| (0, 0, 1, 0, 0) |   |

where a cell $\begin{bmatrix} (r_1, r_2) \\ (q_1, q_2, q_3, q_4, q_5) \end{bmatrix}$ represents that $r_1$ and $r_2$ are the immediate rewards of the players I and II respectively, $q_1, q_2, q_3, q_4, q_5$ represents that the next states are 1, 2, 3, 4 and 5 respectively and $\bar{\tau}$ is the expected sojourn time if this cell is chosen at present. The undiscounted SMDP $\hat{\Gamma}$ extracted from this NCPISMG is given below:

State-1:

| 13 |
| (1, 0, 0, 0, 0) |
| 9 |
| (0, 0, 0, 1, 0) |
| 2 |

State-2:

| 4 |
| (0, 1, 0, 0, 0) |
| 3 |
| (0, 1, 0, 0, 0) |
| 1.6 |

State-3:

| 7 |
| (1 3, 2, 0, 0) |
| 2 |
| (0, 0, 0, 1) |
| 1.5 |

State-4:

| 15 |
| (1, 0, 0, 0, 0) |
| 5 |
| (0, 0, 0, 0, 1) |
| 1 |

State-5:

| 5 |
| (0, 0, 1, 0, 0) |
| 3 |

Next we implement Mondal's [15] LP algorithm to solve this SMDP and ob-
tain optimal pure semi-stationary strategy of the decision maker. For a fixed ini-
tial state \( s_1 \), the DLP in the variables \( x = (x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}, x_{51}) \),
\( y = (y_{11}, y_{12}, y_{21}, y_{22}, y_{31}, y_{32}, y_{41}, y_{42}, y_{51}) \) and \( t \) can be written as

\[
\max R_{s_1} = 13x_{11} + 9x_{12} + 4x_{21} + 3x_{22} + 7x_{31} + 3x_{32} + 15x_{41} + 5x_{51}
\]
subject to

\[
\begin{align*}
3x_{12} - x_{31} - 3x_{41} &= 0 \quad (5.1) \\
x_{31} &= 0 \quad (5.2) \\
x_{31} + x_{32} - x_{51} &= 0 \quad (5.3) \\
-x_{12} + x_{41} &= 0 \quad (5.4) \\
-x_{32} - y_{42} + x_{51} &= 0 \quad (5.5) \\
3x_{11} + 3x_{12} + 3y_{12} - y_{31} - 3y_{41} - 3\delta_{s_1} t &= 0 \quad (5.6) \\
3x_{21} + 6x_{22} - 2y_{31} - 3\delta_{s_2} t &= 0 \quad (5.7) \\
x_{31} + x_{32} + y_{31} + y_{32} - y_{51} - \delta_{s_3} t &= 0 \quad (5.8) \\
x_{41} - y_{12} + y_{41} - \delta_{s_4} t &= 0 \quad (5.9) \\
x_{51} - y_{32} - y_{42} + y_{51} - \delta_{s_5} t &= 0 \quad (5.10) \\
4x_{11} + 2x_{12} + 2x_{21} + 1.6x_{22} + 2x_{31} + 1.5x_{32} + 5x_{41} + x_{42} + 3x_{51} &= 1 \quad (5.11) \\
x, y, t &\geq 0. \quad (5.12)
\end{align*}
\]

The solution of the above linear programming problem by dual-simplex method for different initial states are given by:

(i) For \( s_1 = 1 \): \( \max R_1 = 3.4286 \), \( x = (0, 0.1429, 0, 0, 0, 0, 0, 0, 1429, 0, 0) \),
\( y = (0, 0.1429, 0, 0, 0, 0, 0, 0, 0, 0) \), \( t = 0.285714 \).
(ii) For \( s_1 = 2 \): \( \max R_2 = 2, \ x = (0, 0.0.5, 0, 0, 0, 0, 0, 0, 0) \),
\( y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \), \( t = 0.5 \).
(iii) For \( s_1 = 3 \): \( \max R_3 = 2.6866 \), \( x = (0, 0.0.74629, 0, 0, 0, 0, 0, 74629, 0, 0) \),
\( y = (0, 0.0, 0.0, 0.4478, 0, 0, 0, 0, 0) \), \( t = 0.44761 \).
(iv) For \( s_1 = 4 \): \( \max R_4 = 3.4286 \), \( x = (0, 0, 0, 0.1429, 0, 0, 0, 1429, 0, 0) \),
\( y = (0, 0, 0, 0, 0, 0, 0, 0, 1429, 0) \), \( t = 0.28571 \).
(v) For \( s_1 = 5 \): \( \max R_5 = 2.6866 \), \( x = (0, 0, 0, 0, 0.74629, 0, 0, 0, 0, 74629, 0, 0) \),
\( y = (0, 0, 0, 0.746269, 0, 0, 0.4478, 0, 0, 0, 0, 0, 0) \), \( t = 0.5 \).

By Sinha et al., ([21]), we find SMDP \( \hat{\Gamma} \) has the value vector \( \hat{\phi} = (3.4286, 2, 2.6866, 3.4286, 2.6866) \) and an optimal pure semi-stationary strategy is given by \( \hat{f}^* = (f_1^*, f_2^*, f_3^*, f_4^*, f_5^*) \), where
\( f_1^* = f_3^* = f_4^* = f_5^* = (2, 2, 1, 1, 1) \) and \( f_2^* = (1, 1, 1, 1, 1) \) which we can calculate from the definition of \( f_{xy}^{\phi^*} \), in section 4. Now, we calculate the undiscouned value vector of the SMDP \( \hat{\Gamma} \) by complete enumeration method. There are 16 pure stationary strategies in the SMDP model given as \( f_1 = (1, 1, 1, 1, 1) \), \( f_2 = (1, 1, 1, 2, 1) \), \( f_3 = (1, 1, 2, 1, 1) \), \( f_4 = (1, 1, 2, 2, 1) \), \( f_5 = (1, 2, 1, 1, 1) \), \( f_6 = (1, 2, 2, 1, 1) \), \( f_7 = (1, 2, 1, 2, 1) \), \( f_8 = (1, 2, 2, 2, 1) \), \( f_9 = (2, 1, 1, 1, 1) \), \( f_{10} = (2, 1, 1, 2, 1) \), \( f_{11} = (2, 1, 2, 1, 1) \), \( f_{12} = (2, 1, 2, 2, 1) \), \( f_{13} = (2, 2, 1, 1, 1) \), \( f_{14} = (2, 2, 2, 1, 1) \), \( f_{15} = (2, 2, 1, 2, 1) \), \( f_{16} = (2, 2, 2, 2, 1) \).
The Cesaro limiting matrices for all pure stationary strategies are calculated as
in section 4.1. Thus we get, \[ P^\ast(f_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = P^\ast(f_5), \ P^\ast(f_2) = \]

\[ P^\ast(f_3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = P^\ast(f_4) = P^\ast(f_6), \ P^\ast(f_7) = \]

\[ P^\ast(f_8) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \ P^\ast(f_9) = P^\ast(f_{10}) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \ P^\ast(f_{11}) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = P^\ast(f_{14}), \]

\[ P^\ast(f_{12}) = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = P^\ast(f_{16}). \] Thus we get \( \hat{\phi}(f_1) = (3.25, 2, 2.625, 3.25, 2.625) \)

\( \hat{\phi}(f_2) = (3.25, 2, 2.625, 2.625, 2.625), \hat{\phi}(f_3) = (3.25, 2, 1.778, 3.25, 1.778), \hat{\phi}(f_4) = (3.25, 2, 1.778, 1.778, 1.778), \hat{\phi}(f_5) = (3.25, 1.875, 2.6389, 3.25, 2.6389), \hat{\phi}(f_6) = (3.25, 1.875, 2.639, 3.25, 2.639), \hat{\phi}(f_7) = (3.25, 1.875, 2.6389, 2.6389, 2.6389), \hat{\phi}(f_8) = (3.25, 2, 2.6389, 2.6389, 2.6389), \hat{\phi}(f_9) = (3.429, 2, 2.667, 3.429, 2.667), \hat{\phi}(f_{10}) = (2, 2, 2, 2, 2), \hat{\phi}(f_{11}) = (3.429, 1.875, 2.687, 3.429, 2.687), \hat{\phi}(f_{12}) = (2.556, 2, 1.809, 1.3684, 1.75), \hat{\phi}(f_{13}) = (3.429, 1.875, 2.687, 3.429, 2.687), \hat{\phi}(f_{14}) = (3.429, 1.875, 1.778, 3.429, 1.778), \hat{\phi}(f_{15}) = (1.875, 1.875, 1.875, 1.875, 1.875), \hat{\phi}(f_{16}) = (2.556, 1.875, 1.8095, 1.3684, 1.75). \) By Sinha et al.,(2017) [21], we have \( (3.429, 2, 2.687, 3.429, 2.687) \) as the value vector and the optimal pure semi-stationary strategy for \( \hat{\Gamma} \) is \( f_c = (f_{13}, f_1, f_{13}, f_{13}, f_{13}) \), where \( f_{13} = (2, 2, 1, 1, 1) \) and \( f_1 = (1, 1, 1, 1, 1) \). Note that this optimal pure semi-stationary strategy matches with the optimal pure semi-stationary strategy we obtained by LP algorithm. Player-I’s optimal semi-stationary strategy is given by \( f^\ast(1, 1) = f^\ast(1, 2) = f^\ast(3, 1) = f^\ast(3, 2) = f^\ast(5, 1) = f^\ast(5, 2) = 2 \) and \( f^\ast(1, 5) = f^\ast(2, 1) = f^\ast(2, 2) = f^\ast(2, 5) = f^\ast(3, 5) = f^\ast(4, 5) = f^\ast(5, 5) = 1. \) Player-II’s optimal strategy is given by \( g^\ast(1, 3) = g^\ast(1, 4) = g^\ast(2, 3) = g^\ast(2, 4) = g^\ast(3, 3) = g^\ast(3, 4) = g^\ast(4, 3) = g^\ast(4, 4) = g^\ast(5, 3) = g^\ast(5, 4) = 1. \) So, player-II has an optimal pure stationary strategy denoted by \( g^\ast(3) = g^\ast(4) = 1. \) Thus the Nash pay-off of the game is \( (3.429, 2, 2.687, 3.429, 2.687) \). Lastly, we calculate the total number of iterations to calculate the optimal pure semi-stationary strategy of the SMDP \( \Gamma \) by LP algorithm.
The following table gives the number of iterations against each initial state:

| Initial state | Number of iterations needed |
|---------------|----------------------------|
| \(s_1 = 1\)   | 2                          |
| \(s_1 = 2\)   | 2                          |
| \(s_1 = 3\)   | 1                          |
| \(s_1 = 4\)   | 3                          |
| \(s_1 = 5\)   | 2                          |

From the above table we see that the total number of iterations to solve the LP in a coding software is 10. So, from the above examples, we conclude that our algorithm is much faster than the complete enumeration method.

**Example 2** Consider an NCPISMG \(\Gamma\) with four states \(S = \{1, 2, 3, 4\}\), \(A(1) = \{1, 2\} = A(2)\), \(B(1) = B(2) = \{1\}\), \(B(3) = B(4) = \{1, 2\}\), \(A(3) = A(4) = \{1\}\). Player II is the dummy player in the state 1 and 2 and player I is the dummy player for the states 3 and 4. Rewards, transition probabilities and expected sojourn times for the players are given below:

State-1:

\[
\begin{array}{cccc}
(5,2) & (1,1) & (3,2) & (5,2) \\
(1,2,0,0) & (3,2,0,0) & (5,2,0,0) & (1,2,0,0) \\
1.1 & 1.1 & 1.1 & 1.1 \\
\end{array}
\]

State-2:

\[
\begin{array}{cccc}
(2,3) & (4,2) & (2,2) \\
(0,1,0,0) & (2,1,0,0) & (0,0,0,1) \\
1 & 1 & 1 \\
\end{array}
\]

State-3:

\[
\begin{array}{cccc}
(2) & (4,5.8) \\
(0,0,1,0) & (0,0,1,0) \\
0.9 & 1 \\
\end{array}
\]

State-4:

\[
\begin{array}{cccc}
(3,4) & (2,2) \\
(1,2,0,0) & (1,0,0,1) \\
1 & 1 \\
\end{array}
\]

The undiscounted SMDP \(\hat{\Gamma}\) extracted from this NCPISMG is given below:

State-1:

\[
\begin{array}{cccc}
(5) & (2) & (3) & (4) \\
(1,2,0,0) & (0,1,0,0) & (0,0,1,0) & (1,2,0,0) \\
1.1 & 1.1 & 0.9 & 1 \\
\end{array}
\]

State-2:

\[
\begin{array}{cccc}
(2) & (3) \\
(0,1,0,0) & (0,0,1,0) \\
1 & 1 \\
\end{array}
\]

State-3:

\[
\begin{array}{cccc}
(3) & (4) \\
(0,0,1,0) & (0,0,1,0) \\
0.9 & 1 \\
\end{array}
\]

State-4:

\[
\begin{array}{cccc}
(4) \\
(1,2,0,0) & (0,0,0,1) \\
1 & 1 \\
\end{array}
\]

We implement our LP algorithm to solve this SMDP and obtain a pure semi-stationary strategy of the decision maker. For a fixed initial state \(s_0\), the \(DLP\) in the variables \(x = (x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42})\), \(y = (y_{11}, y_{12}, y_{21}, y_{22}, y_{31}, y_{32}, y_{41}, y_{42})\) and \(t\) can be written as

\[
\max R_{s_0} = 5x_{11} + x_{12} + 2x_{21} + 4x_{22} + 3x_{31} + 5.8x_{32} + 4x_{41} + 2x_{42}
\]
subject to

\begin{align}
3x_{12} + 4x_{12} - 4x_{22} - 3x_{41} &= 0 \quad (5.13) \\
-3x_{11} - 4x_{11} + 4x_{22} &= 0 \quad (5.14) \\
x_{41} &= 0 \quad (5.15) \\
6x_{11} + 6x_{12} + 3y_{11} + 4y_{12} - 4y_{22} - 3y_{41} - 6\delta_{s_0}t &= 0 \quad (5.16) \\
6x_{12} + 6x_{22} - 3y_{11} - 4y_{12} + 4y_{22} - 6\delta_{s_0}t &= 0 \quad (5.17) \\
x_{31} + x_{32} - \delta_{s_0}t &= 0 \quad (5.18) \\
x_{41} + x_{42} + y_{41} - \delta_{s_0}t &= 0 \quad (5.19) \\
1.1x_{11} + 1.1x_{12} + x_{21} + 1.1x_{22} + 0.9x_{31} + x_{32} + x_{41} + x_{42} &= 1 \quad (5.20) \\
x, y, t &\geq 0. \quad (5.21)
\end{align}

The solution of the above linear programming problem by dual-simplex method for different initial states are given by:

(i) For $s_1 = 1$: \(\max R_1 = 4.16, x = (0.52, 0, 0, 0.39, 0, 0, 0, 0), \)
\(y = (0, 0.5844, 0, 0, 0, 0, 0, 0), t = 0.9091. \)

(ii) For $s_1 = 2$: \(\max R_2 = 4.16, x = (0.52, 0, 0, 0.39, 0, 0, 0, 0), \)
\(y = (0, 0.5844, 0, 0, 0, 0, 0, 0), t = 0.9091. \)

(iii) For $s_1 = 3$: \(\max R_3 = 5.8, x = (0, 0, 0, 0, 0, 1, 0, 0), \)
\(y = (0, 0, 0, 0, 0, 0, 0, 0), t = 1. \)

(iv) For $s_1 = 4$: \(\max R_4 = 4.16, x = (0.52, 0, 0, 0.39, 0, 0, 0, 0), \)
\(y = (0, 0.5844, 0, 0, 0, 0, 0, 0), t = 0.9091. \)

To verify that the SMDP \(\hat{\Gamma}\) has value vector \(\hat{\phi} = (4.16, 4.16, 5.8, 4.16)\) and an optimal pure semi-stationary strategy is given by \(f^* = (f_1^*, f_2^*, f_4^*, f_9^*)\), where \(f_1^* = (1, 2, 2, 2)\) and \(f_2^* = (1, 2, 2, 1)\), which we calculate from the definition of \(f_{xy^i}^{ps}\), in section 4.

Now, we calculate the undiscounted value by complete enumeration method. Observe that there are 16 pure stationary strategies available in the SMDP model \(\Gamma\), which are given by \(f_1 = (1, 1, 1, 1), f_2 = (1, 1, 1, 2), f_3 = (1, 1, 2, 1), f_4 = (1, 1, 2, 2), f_5 = (1, 2, 1, 1), f_6 = (1, 2, 2, 1), f_7 = (1, 2, 1, 2), f_8 = (1, 2, 2, 2), f_9 = (2, 1, 1, 1), f_{10} = (2, 1, 1, 2), f_{11} = (2, 1, 2, 1), f_{12} = (2, 1, 2, 2), f_{13} = (2, 2, 1, 1), f_{14} = (2, 2, 2, 1), f_{15} = (2, 2, 1, 2), f_{16} = (2, 2, 2, 2)\). As before, we get the Cesaro limiting matrices

\[
Q^*(f_1) = Q^*(f_3) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad Q^*(f_2) = Q^*(f_4) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
Q^*(f_5) = Q^*(f_6) = \begin{bmatrix}
0.57 & 0.43 & 0 & 0 \\
0.57 & 0.43 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0.5646 & 0.4214 & 0 & 0.0140
\end{bmatrix}, \quad Q^*(f_7) = Q^*(f_8) = \begin{bmatrix}
0.57 & 0.43 & 0 & 0 \\
0.57 & 0.43 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]
Following table gives us number of iterations against each initial state:

| Initial state | Number of iterations needed |
|---------------|-----------------------------|
| $s_1 = 1$     | 2                           |
| $s_1 = 2$     | 2                           |
| $s_1 = 3$     | 1                           |
| $s_1 = 4$     | 3                           |

From the above table we see that the total number of iterations to solve the LP in a coding software is 8. So, from the above examples, we conclude that our algorithm is faster than complete enumeration method to calculate the Nash pay-off and optimal pure semi-stationary Nash equilibrium strategies for the players in the NCPISMG.

6 Conclusion

We conclude this paper by observing that the reduced semi-Markov decision process is strategically equivalent to the original NCPISMG. The finite LP algorithm to solve such SMDP is also helpful to find a pure semi-stationary Nash equilibrium in the NCPISMG. A special case of our result needs a special mention. If $n = 2$ and $r_1 = -r_2$, $\bar{r} = 1$, a NCPISMG reduces to a zero-sum two person Stochastic game with perfect information. In the Stochastic game literature this result is well-known (Gillette(1957) [4], Liggett-Lippman(1969) [13]). But their proof is complex and uses
proof of discounted pay-off and then a version of Hardy-Littlewood theorem. But our proof is much simpler, direct and uses only Derman’s ([2]) result on the existence of a pure optimal strategy(policy) for an undiscounted Markov decision process.

7 Acknowledgement

The first author is thankful to the Department of Science and Technology, Govt. of India, INSPIRE Fellowship Scheme for financial support.

References

[1] Rabi N Bhattacharya and Mukul Majumdar. Controlled semi-markov models-the discounted case. *Journal of Statistical Planning and Inference*, 21(3):365–381, 1989.

[2] Cyrus Derman. Denumerable state markovian decision processes-average cost criterion. *The Annals of Mathematical Statistics*, 37(6):1545–1553, 1966.

[3] Mrinal K Ghosh and Sagnik Sinha. Non-cooperative n-person semi-markov games on a denumerable state space. *Comp. Appl. Math*, 21:833–847, 2002.

[4] Dean Gillette. Stochastic games with zero stop probabilities. *Contributions to the Theory of Games III*, 39:179–187, 1957.

[5] Ronald A Howard. Semi-markovian control systems. Technical report, MASSACHUSETTS INST OF TECH CAMBRIDGE, 1963.

[6] Anna Jaskiewicz. Zero-sum semi-markov games. *SIAM journal on control and optimization*, 41(3):723–739, 2002.

[7] William S Jewell. Markov-renewal programming i: Formulation, finite return models. *Operations Research*, 11(6):938–948, 1963.

[8] John G Kemeny and J Laurie Snell. Finite continuous time markov chains. *Theory of Probability & Its Applications*, 6(1):101–105, 1961.

[9] Alan P Kirman and Matthew J Sobel. Dynamic oligopoly with inventories. *Econometrica: Journal of the Econometric Society*, pages 279–287, 1974.

[10] Masami Kurano. Semi-markov decision processes and their applications in replacement models. *Journal of the Operations Research Society of Japan*, 28(1):18–30, 1985.

[11] Arbind K Lal and Sagnik Sinha. Zero-sum two-person semi-markov games. *Journal of applied probability*, 29(1):56–72, 1992.
[12] Alexandru Lazari and Dmitrii Lozovanu. New algorithms for finding the limiting and differential matrices in markov chains. *Buletinul Academiei de Științe a Moldovei. Matematica*, 92(1):75–88, 2020.

[13] Thomas M Liggett and Steven A Lippman. Stochastic games with perfect information and time average payoff. *Siam Review*, 11(4):604–607, 1969.

[14] Prasenjit Mondal. On zero-sum two-person undiscounted semi-markov games with a multichain structure. *Advances in Applied Probability*, 49(3):826–849, 2017.

[15] Prasenjit Mondal. Computing semi-stationary optimal policies for multichain semi-markov decision processes. *Annals of Operations Research*, 287(2):843–865, 2020.

[16] Prasenjit Mondal and Sagnik Sinha. Semi-stationary equilibrium strategies in non-cooperative n-person semi-markov games. In *International workshop of Mathematical Analysis and Applications in Modeling*, pages 331–343. Springer, 2018.

[17] Wojciech Polowczuk. Nonzero-sum semi-markov games with countable state spaces. *Applicationes Mathematicae*, 27:395–402, 2000.

[18] Lakshmi K Raut. Two-sided altruism, lindahl equilibrium, and pareto optimality in overlapping generations models. *Economic Theory*, 27(3):729–736, 2006.

[19] Linn I Sennott. Nonzero-sum stochastic games with unbounded costs: discounted and average cost cases. *Zeitschrift für Operations Research*, 40(2):145–162, 1994.

[20] Lloyd S Shapley. Stochastic games. *Proceedings of the national academy of sciences*, 39(10):1095–1100, 1953.

[21] Sagnik Sinha and Prasenjit Mondal. Semi-markov decision processes with limiting ratio average rewards. *Journal of Mathematical Analysis and Applications*, 455(1):864–871, 2017.

[22] Frank Thuijsman and Thirukkannamangai ES Raghavan. Perfect information stochastic games and related classes. *International Journal of Game Theory*, 26(3):403–408, 1997.

[23] Oscar Vega-Amaya. Zero-sum average semi-markov games: fixed-point solutions of the shapley equation. *SIAM journal on control and optimization*, 42(5):1876–1894, 2003.