Fermions in the Rindler spacetime

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Abstract

In this paper we study the Dirac equation in the Rindler spacetime. The solution of the wave equation in an accelerated reference frame is obtained. The differential equation associated to this wave equation is mapped into a Sturm-Liouville problem of a Schrödinger-like equation. We derive a compact expression for the energy spectrum associated with the Dirac equation in an accelerated reference. It is shown that the noninertial effect of the accelerated reference frame mimics an external potential in the Dirac equation and, moreover, allows the formation of bound states.

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I. INTRODUCTION

In recent years, the scientific interest in the study of noninertial effects on physical systems has been renewed and many systems have been studied \cite{1-5}. For instance, in \cite{1} it was shown that the noninertial effect breaks the symmetrical of the energy spectrum about \( E = 0 \). In this way, it was shown that the energy spectrum associated with scalar bosons in a rotating reference frame is different from the one obtained in a usual inertial reference frame, in other words, the energy levels are shifted by the effects of the rotating frame.

Another kind of noninertial system that may be investigated with this purpose are the uniformly accelerated observers in Minkowski spacetime, the so-called Rindler spacetime. In order to investigate the energy states of any quantized field in a accelerated reference frame it was suggested that a uniformly accelerated detector in vacuum measures blackbody radiation, i.e, an observer who undergoes a acceleration apparently sees a fixed surface radiate \cite{6-8}.

Parallel to this, gravitational effects on quantum mechanical systems have been studied intensively over the last few years \cite{9-19}. A fundamental question in physics is how quantum systems are affected by the structure of the spacetime, and if exists some significant effect. In order to study these gravitational effects, systematic studies are being carried out \cite{19, 20}. For example, in \cite{19} where the effects of very intense magnetic fields in the energy levels, as intense as the ones expected to be produced in ultra-relativistic heavy-ion collisions, are investigated, in \cite{20} where bosons inside cosmic strings are considered.

In this paper, a single particle solution of the Dirac equation in an accelerated reference frame is discussed. The motivation for this work besides the ones pointed out above is the understanding of the physical consequences of the Dirac equation in noninertial systems of reference that undergo translational acceleration.

The paper is organized as follows: section 2 contains a brief review about the wave equation for spin 1/2 particles in curved spacetimes where the basic formulation and the equations that will be needed in the next sections will be shown. In section 3, we will present an equation for spin 1/2 fermions in the Rindler spacetime, i.e., the Dirac equation in an accelerated reference frame. In section 4, we will see that the energy spectrum associated with the Dirac equation in Rindler space is discrete and depends on \( a \), the acceleration of the reference frame. Finally, section 5 presents our conclusions. In this work, we use natural
II. FERMIONS IN CURVED SPACETIMES

A essential characteristic of the Dirac operator in flat spacetimes is its invariance under Lorentz transformations, so when these particles are studied in curved spacetimes, it is necessary to preserve this aspect (locally). It can be written by using the tetrads $e^{(a)}_{\mu}$ that may be defined in order to satisfy the expression

$$g_{\eta\lambda} = e^{(a)}_{\eta} e^{(b)}_{\lambda} \eta_{(a)(b)},$$

where $\eta_{(a)(b)}$ is the Minkowski tensor, and $g_{\eta\lambda}$ the general metric tensor. We use $(a), (b), (c), \ldots$ to denote the local Lorentz spacetime and $\alpha, \beta, \gamma, \ldots$ to denote the general spacetime coordinate [21–23]. From eq. (1), we can see that the tetrads may be used in order to project vectors from the curved spacetime in the flat spacetime with the equation $A_{\mu} = e^{(a)}_{\mu} A_{(a)}$ that relates the form of a vector in different spacetimes.

Now we study the behavior of the elements of the Dirac equation under transformations that preserve the Lorentz symmetry. We note that a spinor transforms according to $\psi \rightarrow \rho(\Lambda) \psi$, with $\rho(\Lambda) = 1 + \frac{1}{2} i \epsilon^{(a)(b)} \Sigma_{(a)(b)}$, and $\Sigma_{(a)(b)}$ is the spinoral representation of the generators of the Lorentz transformation, written in terms of the $\gamma^{(c)}$ matrices, $\Sigma_{(a)(b)} \equiv \frac{1}{2} i [\gamma^{(a)}, \gamma^{(b)}]$ [24]. The first task is to construct a covariant derivative $\nabla_{(a)} \psi$ that is locally Lorentz invariant, for this purpose we impose the transformation condition

$$\nabla_{(a)} \psi \rightarrow \rho A_{(a)}^{(b)} \nabla_{(b)} \psi.$$

The common way to obtain the explicit form of the covariant derivative operator is by supposing the combination of terms

$$\nabla_{(a)} \psi = e^{(a)}_{(a)}^{\mu} (\partial_{\mu} + \Omega_{\mu}) \psi,$$

we note that the operator $\Omega_{\mu}$ transforms according

$$\Omega_{\mu} \rightarrow \rho \Omega_{\mu} \rho^{-1} + \partial_{\mu} \rho \rho^{-1}.$$

The next step is to find the explicit form of the operator $\Omega_{\mu}$. If we consider the combination of terms

$$\Omega_{\mu} = \frac{1}{2} i \Gamma_{(a)(b)} \Sigma^{(a)(b)} = \frac{1}{2} i \epsilon^{(a)}_{\nu} \nabla_{\mu} e^{(b)\nu} \Sigma_{(a)(b)},$$
the eq. (4) and (2) are satisfied, where the term $\Gamma_{(a)\mu(b)}$ is given by

$$\Gamma_{(a)\mu(b)} = e_{(a)\nu} \left( \partial_\mu e_{(b)}^{\nu} + \Gamma^\nu_{\mu\lambda} e_{(b)}^{\lambda} \right), \quad (6)$$

where $\Gamma^\nu_{\mu\lambda}$ are the Christoffel symbols. As a result, we obtain the final form of the covariant derivative operator

$$\nabla_{(c)} \psi = e_{(c)\mu} \left( \partial_\mu + \frac{1}{2} ie_{(a)\nu} \nabla_\mu e_{(b)}^{\nu} \Sigma_{(a)(b)} \right) \psi. \quad (7)$$

By replacing the conventional derivative operator of the Dirac equation in a flat spacetime by the one obtained in (7) we obtain the final form of the wave equation for fermions particles in a curved spacetime

$$ie_{(a)\mu} \gamma^{(a)} \left( \partial_\mu + \Omega_{\mu} + ieA_\mu \right) \psi - m\psi = 0. \quad (8)$$

It is common to define the term $\gamma^\mu = e_{(a)\mu} \gamma^{(a)}$ as a Dirac matrix in a given curved spacetime and it is easy to see that it satisfies the Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = g^{\mu\nu} 1. \quad (9)$$

If the spinor $\psi$ is coupled to the gauge field $A_\mu$, we may introduce this effect by a minimal coupling

$$ie_{(a)\mu} \gamma^{(a)} \left( \partial_\mu + \Omega_{\mu} + ieA_\mu \right) \psi - m\psi = 0. \quad (10)$$

**WAVE EQUATION IN THE RINDLER SPACETIME**

The Rindler metric represents an accelerated reference frame in the Minkowski spacetime where the line element may be written in the form

$$ds^2 = (1 + a\xi)^2 d\tau^2 - d\xi^2, \quad (11)$$

where $\xi$ and $\tau$ are the proper coordinates of the accelerated frame, and $a$ is the acceleration [25]. The coordinate of distance $\xi$ is constrained by $\xi > -\frac{1}{a}$ while the proper time $\tau$ varies in the interval $-\infty < \tau < \infty$. We shall now rewrite the metric (11) in a conformally flat form. The usual way to obtain the conformal form of eq. (11) is by making the transformation

$$\bar{\tau} = \tau,$$
$$x = \frac{1}{a} \ln \left(1 + a\xi\right). \quad (12)$$
In this way, we obtain the line element
\[ ds^2 = e^{2ax} \left( d\bar{\tau}^2 - dx^2 \right). \] (13)

Now the conformal coordinates \( x, \bar{\tau} \) vary in the interval \(-\infty < \bar{\tau} < \infty\), and \(-\infty < x < \infty\), that means an extension of the spacetime metric. From eq. (13) we can see that the metric is conformally flat because the conformal factor \( e^{2ax} \) is multiplied by the Minkowski line element. In addition, the spacetime metric (13) represents a flat spacetime since it is related to the Minkowski line element by a coordinate transformation.

Thus, the next step is to choose the tetrad basis for the line element (13). The diagonal form of (13) suggests the following tetrad basis
\[ e_{\nu}^{(a)} = \begin{pmatrix} e^{\frac{\sigma(x)}{2}} & 0 \\ 0 & e^{-\frac{\sigma(x)}{2}} \end{pmatrix}, \] (14)
\[ e_{\nu}^{(a)} = \begin{pmatrix} e^{-\frac{\sigma(x)}{2}} & 0 \\ 0 & e^{\frac{\sigma(x)}{2}} \end{pmatrix}, \] (15)
where \( \sigma(x) \equiv 2ax \). It is easy to verify that it satisfies the equation
\[ g_{\mu\nu} = e^{(a)}_{\mu} e^{(b)}_{\nu} \eta_{(a)(b)}. \]

Observing that the only non-zero term \( \Omega_0 \) in the wave equation, relative to the tetrad (14) is given by
\[ \Omega_0 = \frac{i\gamma^0 \cdot \gamma^1}{4} \frac{d\sigma}{dx}, \] (16)
equation (8) becomes
\[ \left[ i\gamma^{(0)} \frac{\partial}{\partial \bar{\tau}} + i\gamma^{(1)} \frac{\partial}{\partial x} + \frac{i\gamma^{(1)}}{4} \frac{d\sigma(x)}{dx} - me^{\frac{1}{2}\sigma(x)} \right] \psi(\bar{\tau}, x) = 0, \] (17)
where the usual representation for the gamma matrices in 1 + 1 dimensions is considered
\[ \gamma^{(0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{(1)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \] (18)

In eq. (17), we will suppose a solution of the form
\[ \psi(t, x) = e^{-ie\bar{\tau}} \begin{bmatrix} \bar{g}(x) \\ \bar{f}(x) \end{bmatrix}, \] (19)
where $\varepsilon$ is the energy of particle. So, equation (17) may be written in an explicit form

\[
\begin{bmatrix}
\frac{d}{dx} + \frac{1}{4} \frac{d\sigma (x)}{dx} + me^{\frac{1}{2}\sigma(x)} \\
-\frac{d}{dx} - \frac{1}{4} \frac{d\sigma (x)}{dx} + me^{\frac{1}{2}\sigma(x)}
\end{bmatrix}
\begin{bmatrix}
\bar{g} (x) \\
\bar{f} (x)
\end{bmatrix} = \varepsilon \begin{bmatrix}
\bar{f} (x) \\
\bar{g} (x)
\end{bmatrix},
\]

(20)

\[
\begin{bmatrix}
\frac{d}{dx} + \frac{1}{4} \frac{d\sigma (x)}{dx} + me^{\frac{1}{2}\sigma(x)} \\
-\frac{d}{dx} - \frac{1}{4} \frac{d\sigma (x)}{dx} + me^{\frac{1}{2}\sigma(x)}
\end{bmatrix}
\begin{bmatrix}
\bar{f} (x) \\
\bar{g} (x)
\end{bmatrix} = \varepsilon \begin{bmatrix}
\bar{g} (x) \\
\bar{f} (x)
\end{bmatrix}.
\]

(21)

Making a unitary transformation \[26\] in the system of equations

\[
U (x) = \begin{bmatrix}
e^{\frac{1}{4}\sigma(x)} & 0 \\
0 & e^{\frac{1}{4}\sigma(x)}
\end{bmatrix},
\]

(22)

we obtain a simplified form

\[
\begin{bmatrix}
\frac{d}{dx} + z (x) \\
-\frac{d}{dx} + z (x)
\end{bmatrix}
\begin{bmatrix}
g (x) \\
f (x)
\end{bmatrix} = \varepsilon \begin{bmatrix}
f (x) \\
g (x)
\end{bmatrix},
\]

(23)

(24)

where $z (x) = me^{\frac{1}{2}\sigma(x)}$ and

\[
\bar{\psi} = U \psi = U \begin{bmatrix}
\bar{g} (x) \\
\bar{f} (x)
\end{bmatrix} = \begin{bmatrix}
g (x) \\
f (x)
\end{bmatrix}.
\]

(25)

In order to investigate the solutions of the system of equations, we will consider that the set of equations (23) and (24) may be decoupled. The function $f$ can be isolated in eq. (23) and then, using (23) to eliminate the terms containing the $g$. The result is

\[
\frac{d^2 f}{dx^2} - \frac{dz}{dx} f - z^2 f + \varepsilon^2 f = 0,
\]

(26)

in a similar way, we derive the equation for $g$

\[
\frac{d^2 g}{dx^2} + \frac{dz}{dx} g - z^2 g + \varepsilon^2 g = 0,
\]

(27)

that may be resumed in the form

\[
\frac{d^2 F}{dx^2} - s \frac{dz}{dx} F - z^2 F + \varepsilon^2 F = 0,
\]

(28)

where $F = f$ when $s = 1$ and $F = g$ when $s = -1$. For the case of the Rindler metric, we have $\sigma = 2ax$ so that

\[
g_{\mu\nu} = \exp (2ax) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(29)
If we consider small values of the acceleration $a$, we may solve eq. (28) neglecting terms in higher orders, and then $z(x)$ can be expanded as

$$z(x) = m \left( 1 + \frac{ax}{2} + \ldots \right).$$

(30)

So, substituting this equation into Eq. (28) we obtain the following expression

$$\frac{d^2 F}{dx^2} - \frac{s m a}{2} F - \frac{a^2 m^2}{4} \left( \frac{2}{a} + x \right)^2 F + \varepsilon^2 F = 0.$$  

(31)

In order to investigate the solutions of the Eq. (31), we will consider the transformation

$$y = \sqrt{\frac{am}{2}} \left( x + \frac{2}{a} \right),$$

(32)

and as a result, the equation will take the form

$$\frac{d^2 F}{dy^2} + (\eta - V_{ef}) = 0,$$

(33)

here we have defined $\eta = \frac{2\varepsilon^2}{am}$ and $V_{ef} = y^2 + s$. In the next section we will obtain two classes of solutions of the Dirac equation in the Rindler spacetime. Indeed, eq. (33) may be mapped into a Sturm–Liouville problem of a Schrödinger-like equation.

### III. BOUND-STATE SOLUTIONS

As we may observe, equation (33) is similar to the Schrödinger equation, and the term $V_{ef} = y^2 + s$ may be identified as an effective potential and as we can see, the system has the form of a harmonic oscillator. In fact, the solution of the equation (33) may be mapped into a 2-dimensional harmonic oscillator-like, and then the solutions are given in terms of the Hermite polynomials

$$\bar{\psi} = \begin{pmatrix} B_1' \exp (-y^2) H_{n+1} (y) \\ B_2' \exp (-y^2) H_n (y) \end{pmatrix},$$

(34)

where $B_1'$ and $B_2'$ are normalization constants. The Hermite polynomials satisfy the recurrence relation

$$a_{j+2} = \frac{2j + 1 - \eta}{(j + 2)(j + 1)} a_j,$$

(35)
FIG. 1. The plots of the energy spectrum for \( a = 0.01, a = 0.02 \) and \( a = 0.03 \). The energy is symmetrical about \( \varepsilon = 0 \).

as the series must be finite in order for our solution to have physical meaning, we suppose that there exists some \( n \) such that when \( j = n \), the numerator is \( 2j + 1 - \eta = 0 \). In this way, we obtain the energy spectrum

\[
\varepsilon = \pm \sqrt{am \left( n + \frac{1 + s}{2} \right)}.
\]  \hspace{1cm} (36)

We can see that the energy spectrum associated with the Dirac equation in the Rindler space is discrete and depends on \( a \), the acceleration of the reference frame. This is an interesting feature of the system because the noninertial effect mimics an external potential in the Dirac equation. From Fig. 1, we can see that the discrete set of energies are symmetrical about \( \varepsilon = 0 \), that means the particle and antiparticle have the same energy. As it may be seen in Fig. 2, the solution \( g(x) \) decreases with the coordinate \( x \) and becomes negligible as \( x \to \pm \infty \).
FIG. 2. The lower spinor component $g(x)$ as the function of $x$ for three fixed values of $a$, Up: left and right the figures are plots for $n = 0$ and $n = 1$. Down: left and right the figures are plots for $n = 2$ and $n = 3$.

IV. CONCLUSIONS

In this work a brief review about the wave equation for spin 1/2 particles in curved spaces is done. We have determined a single particle solution of the Dirac equation in an accelerated reference frame, and as result, a compact expression for the energy spectrum associated with the Dirac equation in an accelerated reference frame has been obtained. It was shown that the noninertial effect mimics an external potential in the Dirac equation and, moreover, allows the formation of bound states. We also have shown that the energy spectrum associated with fermions in this kind spacetime is discrete. The solution is obtained by adopting the limit $a \ll 1$, that means a not so fast acceleration.

With these results it is possible to have an idea about the general aspects of the behavior of spin 1/2 particles in the Rindler space. Potential applications of our work include physical
systems with conformally flat metrics, i.e., invariant under conformal transformations. For instance, in applications of the AdS/CFT correspondence to the study of strongly coupled QCD, we look for solutions of the wave equations where the conformal symmetry plays a pivotal role.

It is interesting to observe that the results obtained above, in addition to previous ones, for example, show many important aspects of quantum systems studied in spacetimes with different structures. However, these results rather than being considered as final, may be considered as a motivation for future works in order to obtain a deeper understanding of this fundamental theme of physics.

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