AN ARBITRARY ORDER SCHEME ON GENERIC MESHES FOR MISCIBLE DISPLACEMENTS IN POROUS MEDIA

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Abstract. We design, analyse and implement an arbitrary order scheme applicable to generic meshes for a coupled elliptic-parabolic PDE system describing miscible displacement in porous media. The discretisation is based on several adaptations of the Hybrid-High-Order (HHO) method due to Di Pietro et al. [Computational Methods in Applied Mathematics, 14(4), (2014)]. The equation governing the pressure is discretised using an adaptation of the HHO method for variable diffusion, while the discrete concentration equation is based on the HHO method for advection-diffusion-reaction problems combined with numerically stable flux reconstructions for the advective velocity that we have derived using the results of Cockburn et al. [ESAIM: Mathematical Modelling and Numerical Analysis, 50(3), (2016)]. We perform some rigorous analysis of the method to demonstrate its $L^2$ stability under the irregular data often presented by reservoir engineering problems and present several numerical tests to demonstrate the quality of the results that are produced by the proposed scheme.

Keywords: hybrid high-order methods, porous medium, miscible fluid flow, stability analysis, numerical tests.

1. Introduction

The single-phase flow of incompressible, miscible fluids in a porous medium, arising in the modelling of enhanced oil recovery is described by a coupled system of non-linear elliptic-parabolic equations on the pressure and concentration of invading solvent, sometimes referred to as the Peaceman model as derived by Peaceman in [34] (not to be confused with the Peaceman model of wells). Given its complexity, the behaviour of this system can only be fully understood through numerical approximations. These approximation must account for the specificities of flows in porous media: discontinuous data, non-cartesian (and possibly non-conforming) grids, etc.

In this paper, we design an arbitrary-order numerical scheme for the Peaceman model. The stability of the approximation is established, and numerical results are provided. Based on these, we provide advice on how to choose the various parameters of the scheme (spatial order of approximation, time-stepping method, etc.) to optimise the accuracy of the result and the overall computational cost.

Existence of a weak solution to this miscible flow model has been first established in [26], and then extended in [9] to include gravity effects and various boundary
conditions. In practice, the molecular diffusion is much smaller than the dispersion effect (driven by the Darcy viscosity), and often neglected in numerical simulation. In that case, the parabolic equation takes on a degenerate form, which makes the analysis of the model even more complex. [1] establishes the existence of a solution in the case of a vanishing molecular diffusion and regular source terms. Given the scale of the reservoir and the well bores, it is customary in simulations to consider wells concentrated on measures (Dirac measures in 2D, measures along lines in 3D). The corresponding mathematical analysis has been carried out in [20] for non-zero molecular diffusion and in [21] for vanishing molecular diffusion.

A variety of numerical schemes have been considered, with or without convergence analysis, for the Peaceman model. Finite-element based methods coupled with a modified method of characteristic methods for the advective part of the flow, are applied in [24] and analysed in [23]. Another method of characteristics, the Eulerian–Langrangian Adjoint Method, is coupled with $RT_k$ mixed finite element (for the pressure) and $Q_k$ finite elements (for the concentration) in [36, 37]. A method combining $RT_k$ finite elements and discontinuous Galerkin (dG) schemes is analysed in [2]. Optimal error estimates for conforming $P_k$ finite elements on triangles are established in [31], based on maximal regularity properties of the continuous model. Finite-element based methods are natural and well known, but suffer from restrictions on the mesh geometries, which must be conforming and essentially made of triangles or squares. Other schemes have been considered to deal with meshes with generic geometries, as encountered in geophysical applications. In the last few years, schemes have been developed to be applicable on such generic grids. They are mostly based on finite volume techniques, which have the advantage of providing conservative approximations of the Darcy fluxes [17], that can be used in the discretisation of the advective terms. In [6], the mixed finite volume of [18] is adapted to the Peaceman model, its convergence analysed and numerical tests are provided; as shown in [19], this method can be embedded in a larger family, the hybrid mimetic method (HMM) family, that also contains the SUSHI scheme of [25] and the mixed-hybrid mimetic finite difference methods of [5]. Discrete duality finite volume (DDFV) methods are considered in [7, 8]. HMM and DDFV are finite volume schemes with first-order approximation properties, and rely for the miscible displacement model on upwinding to stabilise the advective terms; this raises the concern of an over-diffusion of the transition layer between the invading solvent and the residing oil.

In this work, we develop an arbitrary-order numerical scheme for the Peaceman model, which is applicable on generic grids. The scheme is an adaptation of the hybrid high-order (HHO) method, initially developed for stationary diffusion PDEs in [15, 16] and then extended to stationary advection–diffusion–reaction models in [13]. The HHO method can be seen as a higher-order extension of the HMM method, and is very close to virtual element methods [3], to non-conforming mimetic finite difference methods [32] and to hybridizable dG methods [10]. The initial degrees of freedom of the HHO method are scalar valued polynomials of arbitrary order $k$ on the cells and faces of the mesh. The cell degrees of freedom can however be eliminated by a local static condensation procedure, and only the face degrees of freedom remain coupled, in a way that is however highly parallelisable. The HHO scheme is built on a collection of high-order local reconstruction operators that
Our executive summary is as follows:

- There is a real advantage in going for a higher order method. The choice \( k = 0 \) leads to strong grid effects, that are mostly eliminated by taking \( k = 1 \). This choice \( k = 1 \) seems to be optimal in terms of accuracy vs. computational cost; the choices \( k = 2, 3, \ldots \) increase the computational cost with only minor further improvements of the accuracy.
- A Crank-Nicolson or second order backward differentiation formula (BDF) time-stepping is sufficient to obtain good results; high-order BDF tend to become unstable unless the time step is reduced a lot, and do not lead to perceptible improvement (even considering higher order spatial approximation, that is \( k \geq 2 \)).
- The specific mesh geometry is mostly irrelevant to the quality of the numerical approximation, which mostly seems to only depend on the number of edges of the mesh (which is expected, the edge unknowns being the main unknowns in the HHO method).

Let us conclude by describing the organisation of the paper. In the next section, we describe the continuous miscible displacement model, both in strong and weak form. The scheme is described in Section 3, starting from the time stepping, designed in a classical way to decouple the pressure and concentration equations. As explained above, the HHO method is built on local polynomial spaces and reconstruction operators; these are respectively described in Sections 3.2 and 3.3. The discretised pressure equation is then presented. In order to discretise the convection terms appearing in the concentration equation, cell Darcy velocities and corresponding fluxes have to be designed from the numerical solution of the pressure equation. The reconstruction of this velocity and fluxes is described in Section 3.5. We note that this reconstruction has to be carefully performed to preserve the overall schemes’ stability; in particular, this implies discretising the pressure equation at an order twice the order chosen for the concentration equation. The numerical approximation of this latter is described in Section 3.6. Existence, uniqueness and stability results for our complete scheme are stated in Theorems 3.7 and 3.8 at the end of Section 3. Extensive numerical tests are provided in Section 4. We analyse in particular the effect (in terms of cost as well as efficiency) of varying the spatial degree of the method and of having to use distorted polygonal meshes. A short conclusion is given in Section 5, and the proof of the existence, uniqueness and stability result is given in the appendix, Section 6. These proofs show in particular the importance of a proper choice of the Darcy velocity and fluxes.

### 2. The continuous model

We introduce the following notation that will be used to describe the model, and then present the aforementioned system.
\( d \in \{2, 3\} \) the number of dimensions considered in the model,
\( \Omega \subset \mathbb{R}^d \) a bounded Lipschitz domain representing the reservoir,
\( (0, t_f) \subset \mathbb{R} \) the time interval on which we consider the problem,
\( p : (0, t_f) \times \Omega \to \mathbb{R} \) the pressure in the mixture,
\( U : (0, t_f) \times \Omega \to \mathbb{R}^d \) the Darcy velocity of the fluid,
\( c : (0, t_f) \times \Omega \to \mathbb{R} \) the concentration of the invading solvent in the reservoir,
\( \Phi : \Omega \to \mathbb{R} \) the porosity of the medium,
\( \mu : [0, 1] \to \mathbb{R} \) the viscosity of the fluid mixture at a given concentration,
\( K : \Omega \to \mathbb{R}^{d \times d} \) the absolute permeability tensor of the medium,
\( D : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d \times d} \) the diffusion-dispersion tensor of the medium,
\( q^+ : (0, t_f) \times \Omega \to \mathbb{R} \) the source term corresponding to the injection well,
\( q^- : (0, t_f) \times \Omega \to \mathbb{R} \) the source term corresponding to the production well.

Taking the effects of gravity to be negligible, the model reads:

\[
\begin{align*}
\text{div}(U) &= q^+ - q^- \quad \text{in } (0, t_f) \times \Omega, \\
U &= -\frac{K(x)}{\mu(c)} \nabla p \quad \text{in } (0, t_f) \times \Omega, \\
\Phi(x) \frac{\partial c}{\partial t} - \text{div}(D(x, U) \nabla c - cU) + q^- c &= q^+ \hat{c} \quad \text{in } (0, t_f) \times \Omega.
\end{align*}
\]

This system is comprised of two very natural parts. The pressure equation (2.1a) is an anisotropic diffusion equation with diffusivity \(-\frac{K(x)}{\mu(c)}\). For simplicity of notation, we may instead write

\[ \kappa = \frac{K(x)}{\mu(c)}. \]

The concentration equation (2.1b) describes the convection of the fluid mixture via an advection-diffusion-reaction equation with diffusivity \( D \) and an advective velocity corresponding to the Darcy velocity of the fluid mixture \( U \). We will now briefly summarise the models used for the data. Following [33], Peaceman derived the diffusion-dispersion tensor \( D \) to be

\[ D(x, U) = \Phi(x) \left( d_m I + |U| (d_l E(U) + d_t (I - E(U))) \right), \]

where \( d_m \) is the molecular diffusion coefficient, \( d_l \) and \( d_t \) are the longitudinal and transverse dispersion coefficients, and \( E(U) \) is an orthogonal projection in the direction of the Darcy velocity, given by the outer-product

\[ E(U) = \frac{U \otimes U}{|U|^2} = \left[ \frac{U_i U_j}{|U|^2} \right]_{1 \leq i, j \leq d}. \]

Physical experiments reveal that the longitudinal dispersion \( d_l \) is far stronger than the transverse dispersion \( d_t \), and that the molecular dispersion \( d_m \) is negligible in comparison.

The viscosity of the mixture is determined by the mixing rule given in [29]:

\[ \mu(c) = \mu(0) \left( 1 + (M^{\frac{1}{4}} - 1)c \right)^{-4} \quad c \in [0, 1], \]

where \( \mu(0) \) is the viscosity of the oil and \( M \) is the mobility ratio between the oil and the injected solvent, given by \( M = \frac{\mu(0)}{\mu(1)}. \)
In order to maintain a balance of mass in the domain, the boundary of the reservoir \( \partial \Omega \) is taken to be impermeable. Consequently, we include, denoting by \( n \), the unit exterior normal to \( \partial \Omega \), homogeneous no flow Neumann boundary conditions:

\[
\begin{aligned}
U \cdot n &= 0 & \text{on } (0, t_f) \times \partial \Omega, \\
D \nabla c \cdot n &= 0 & \text{on } (0, t_f) \times \partial \Omega.
\end{aligned}
\]  

(2.1g)

Additionally, in order to satisfy the no flow boundary conditions and maintain mass, we must further impose that our injection and production source terms are compatible:

\[
\int_{\Omega} q^+ (\cdot, x) dx = \int_{\Omega} q^- (\cdot, x) dx \quad \text{in } (0, t_f).
\]

(2.1h)

We must also prescribe an initial condition:

\[
c(0, x) = c_0(x).
\]

(2.1i)

In practice we take the initial concentration in the well to be 0 everywhere. It is also usual to take \( \hat{c} \), the injected concentration at the injection well to be 1. Lastly, since the pressure is only defined up to an arbitrary constant, we normalise \( p \) by the following condition:

\[
\int_{\Omega} p(\cdot, x) dx = 0 \quad \text{in } (0, t_f).
\]

(2.1j)

2.1. Weak formulation. Noting that the geometry of a typical oil reservoir will contain many geological layers of varying porosity and permeability, it is very important to take into account the fact that these quantities can not be assumed to be smooth and continuous everywhere. The following assumptions given in [6] are reasonable and allow us to devise both analytically sound and physically acceptable solutions.

\[
\Phi \in L^\infty(\Omega) \text{ such that there exists } \Phi_* > 0 \text{ satisfying } \Phi_* \leq \Phi \leq \Phi_*^{-1} \text{ almost everywhere in } \Omega,
\]

(2.2a)

\[
q^+, q^- \in L^\infty(0, t_f; L^2(\Omega)) \text{ are non-negative and compatible, i.e. } \int_{\Omega} q^+ (\cdot, x) dx = \int_{\Omega} q^- (\cdot, x) dx \text{ for almost every } t \in (0, t_f),
\]

(2.2b)

\[
K : \Omega \to M_d(\mathbb{R}) \text{ is a bounded matrix-valued function admitting symmetric, uniformly coercive values, that is, } \exists \alpha_K > 0 \text{ s.t. } K(x) \xi \cdot \xi \geq \alpha_K |\xi|^2 \text{ for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^d,
\]

(2.2c)

\[
\exists \Lambda_K > 0 \text{ s.t. } |K(x)| \leq \Lambda_K \text{ for almost every } x \in \Omega,
\]

(2.2d)

\[
D : \Omega \times \mathbb{R}^d \to M_d(\mathbb{R}) \text{ is given by } (2.1d) \text{ with } d_l, d_m, d_t > 0.
\]

(2.2e)

\[
\mu \in C(\mathbb{R}) \text{ is positively bounded, that is } 0 \leq a \leq \mu(c) \leq b \text{ for positive constants } a, b \text{ for all } c \in \mathbb{R}. \text{ This is clearly satisfied by } (2.1f),
\]

(2.2f)

\[
\hat{c} \in L^\infty((0, t_f) \times \Omega) \text{ satisfies } 0 \leq \hat{c} \leq 1 \text{ almost everywhere,}
\]

(2.2g)

\[
c_0 \in L^\infty(\Omega) \text{ satisfies } 0 \leq c_0 \leq 1 \text{ almost everywhere.}
\]

(2.2h)
Under the regularity assumptions (2.2), the existence of a weak solution to (2.1a)–(2.1b) is established in [9, 26]. The HHO scheme presented in Section 3 consists in discretising the following equations, satisfied by this weak solution \((p, U, c)\):

\[
\begin{align*}
\begin{cases}
p \in L^\infty(0, t_f; H^1(\Omega)), & U \in L^\infty(0, t_f; L^2(\Omega))^d, \\
c \in C([0, t_f]; L^2(\Omega)) \cap L^2(0, t_f; H^1(\Omega)) \text{ with } \\
\Phi \partial_t c \in L^2(0, t_f; (W^{1,4}(\Omega))') & \text{and } c(0) = c_0,
\end{cases}
\end{align*}
\]

(2.3)

For almost every \(t \in (0, t_f)\), for all \(\varphi \in H^1(\Omega)\),

\[
\begin{align*}
- \int_\Omega U(t, \cdot) \cdot \nabla \varphi &= \int_\Omega (q^+(t, \cdot) - q^-(t, \cdot)) \varphi
\end{align*}
\]

(2.4)

and

\[
\begin{align*}
\begin{cases}
\Phi \partial_t c, \varphi \rangle_{W^{1,4}, W^{1,4}} + \int_\Omega D(\cdot, U(t, \cdot)) \nabla c(t, \cdot) \cdot \nabla \varphi \\
- \int_\Omega c(t, \cdot) U(t, \cdot) \cdot \nabla \varphi + \int_\Omega q^-(t, \cdot) c(t, \cdot) \varphi = \int_\Omega q^+(t, \cdot) \hat{c} \varphi.
\end{cases}
\end{align*}
\]

(2.5)

3. Numerical scheme

The scheme for the pressure equation consists in the standard HHO method for variable diffusion problems [15], taking into account the coupling of the pressure with the concentration equation. Adapting [10], we derive conservative discrete estimates to the Darcy velocity and its fluxes that are required to discretise the advective term \(\text{div}(U c)\) in the concentration equation. The spatial terms in this equation are discretised by using the HHO method for linear advection–diffusion–reaction [13], incorporating the aforementioned discrete Darcy velocity and fluxes. The time stepping presented here is based the Crank-Nicolson method, due to its strong stability properties [30] (we also tested BDF time stepping).

3.1. Time stepping. We define our time-stepping as follows, let \(N \in \mathbb{N}\) be the number of time-steps to be taken and let

\[
\Delta t = \frac{t_f}{N}, \quad t^n = n \Delta t, \quad n = 0, 1, ..., N.
\]

(3.1)

Let us denote the pressure, concentration and Darcy velocity at the \(n\)’th time-step by \(p^n = p(t^n, \cdot),\ c^n = c(t^n, \cdot)\) and \(U^n = U(t^n, \cdot) = -\kappa(c^n, \cdot) \nabla p^n\). The Crank-Nicolson time stepping of (2.1b) consists in writing

\[
\Phi(x) \frac{c^{n+1} - c^n}{\Delta t} - \text{div}(D(x, U^{n+1/2}) \nabla c^{n+1/2} - c^{n+1/2} U^{n+1/2}) + q^- c^{n+1/2}
\]

(2.2)

\[
= q^+ \hat{c}(t^{n+1/2}),
\]

where the intermediate time-stepped values are defined by

\[
\xi^{n+1/2} = \frac{\xi^{n+1} + \xi^n}{2} \quad (\xi = t, c \text{ or } U).
\]

(3.3)
The problem data, e.g., the well terms $q^+$ and $q^-$ are evaluated at $t^{n+1/2}$. The formulation (3.2) leads to the following direct relation between $c^n$ and $c^{n+1/2}$:

$$\Phi(x)2\frac{(c^{n+1/2} - c^n)}{\Delta t} - \text{div}(D(x, U^{n+1/2})\nabla c^{n+1/2} - c^{n+1/2} U^{n+1/2})$$

$$+ q^- c^{n+1/2} = q^+ \hat{c}(t^{n+1/2}).$$

We note that this formulation is equivalent to performing a half time-step with an implicit Euler scheme to obtain $c^{n+1/2}$ and then linearly extrapolating $c^n$ with $c^{n+1/2}$ to obtain $c^{n+1}$ using (3.3).

3.2. Discretisation spaces. Let us briefly introduce the notion of a mesh and define the polynomial spaces central to the scheme.

**Definition 3.1.** A mesh $\mathcal{M}$ of $\Omega \subset \mathbb{R}^d$ is a collection of cells and faces $(T, F)$ where:

1. $T$ (The cells or control volumes) is a subdivision of $\Omega$ into a disjoint family of open, non-empty polygons (or polyhedra in higher dimensions). Formally,
   $$\bigcup_{T \in T} T = \overline{\Omega}. \tag{3.5}$$

2. $F$ (The faces or edges) is a disjoint family of non-empty affine subsets of $\Omega$, with positive $(d-1)$ dimensional measure such that for each cell $T \in T$, there exists $F_T \subset F$ where
   $$\bigcup_{F \in F_T} F = \partial T, \tag{3.6}$$

   such that each face $F \in F$ borders exactly one or two cells.

Additionally, we may employ the following notation

- $F_T \subset F$ the faces that border the cell $T$,
- $F_b \subset F$ the faces on the boundary of the domain $\Omega$,
- $|T|$ or $|F|$ the Lebesgue measure of the cell $T$ or face $F$,
- $\mathbf{x}_T$ the centre of mass of the cell $T$,
- $\mathbf{n}_{TF}$ a normal vector to the face $F$ facing outward from the cell $T$,
- $h_T$ or $h_F$ the diameter of a cell $T$ or a face $F$,
- $h = \max_{T \in T} h_T$ the maximum diameter of any cell.

Take $\mathcal{M}_h = (T_h, F_h)$ a generic polygonal/polytopal mesh of $\Omega$ as above, where $h$ is the mesh size (maximum of the diameters of the cells). The degrees of freedom of the scheme are scalar valued polynomials on the cells $T \in T_h$ and the edges $F \in F_h$. No continuity conditions between cells and faces, or between cells and cells, are imposed on the degrees of freedom. Selecting an integer $m \geq 0$, the following notation will help us describe the polynomial spaces:

- $\mathbb{P}^m(K)$ the space of $l$-variate polynomials of degree $\leq m$ on $K \subset \mathbb{R}^l$,
- $\nabla \mathbb{P}^m(K)$ the space of functions $\{ \nabla u : u \in \mathbb{P}^m(K) \}$,
- $\pi_K^m : L^2(K) \rightarrow \mathbb{P}^m(K)$ the $L^2$ orthogonal projection onto $\mathbb{P}^m(K)$,

where the $L^2$ orthogonal projector is given by:

$$\pi_K^m(u) = v \in \mathbb{P}^m(K) \text{ is such that, for all } w \in \mathbb{P}^m(K), \int_K (u - v)w = 0. \tag{3.7}$$
The spaces of degrees of freedom are then given as follows.

**Definition 3.2.** Let $\mathcal{M}_h$ be a mesh. For each cell $T \in \mathcal{T}_h$, the space of local degrees of freedom on $T$ is defined by

$$X^m_T = \mathbb{P}^m(T) \times \left\{ \bigotimes_{F \in \mathcal{F}_T} \mathbb{P}^m(F) \right\}. \quad (3.8)$$

The space of global degrees of freedom on the mesh is defined as

$$X^m_h = \left\{ \bigotimes_{T \in \mathcal{T}_h} \mathbb{P}^m(T) \right\} \times \left\{ \bigotimes_{F \in \mathcal{F}_h} \mathbb{P}^m(F) \right\}. \quad (3.9)$$

For a given $k \geq 0$, these spaces will be used to approximate both the pressure (with $m = 2k$) and the concentration (with $m = k$). An arbitrary set of global degrees of freedom is denoted by

$$u_h = (u_T)_{T \in \mathcal{T}_h}, (u_F)_{F \in \mathcal{F}_h} \in X^m_h, \quad (3.10)$$

and, similarly, a set of local degrees of freedom is

$$u_T = (u_T, (u_F)_{F \in \mathcal{F}_T}) \in X^m_T. \quad (3.11)$$

For a given $u_h \in X^m_h$, we define $u_h \in L^2(\Omega)$ as the piecewise polynomial function given by $(u_h)_T = u_T$ for all $T \in \mathcal{T}_h$. The space of degrees of freedom with zero average is then

$$X^m_{h,*} = \left\{ u_h \in X^m_h : \int_{\Omega} u_h = 0 \right\}. \quad (3.12)$$

3.3. **Reconstruction operators.** We now introduce the local reconstruction operators which are central to the scheme. The cornerstone of the HHO method is a high-order local gradient reconstruction operator based on the cell and face polynomial degrees of freedom.

**Definition 3.3.** Let $\Lambda$ be a bounded, real, symmetric, coercive tensor valued function on $\Omega$. Take a cell $T \in \mathcal{T}_h$ of $\mathcal{M}$. The local reconstruction operator $r^{m+1}_{T, \Lambda} : X^m_T \to \mathbb{P}^{m+1}(T)$ is defined such that, for a bundle of local degrees of freedom $u_T = (u_T, (u_F)_{F \in \mathcal{F}_T}) \in X^m_T$ and any test function $w \in \mathbb{P}^{m+1}(T)$,

$$\int_T \Lambda \nabla r^{m+1}_{T, \Lambda} u_T \cdot \nabla w = \int_T \Lambda \nabla u_T \cdot \nabla w + \sum_{F \in \mathcal{F}_T} \int_F (u_F - u_T) \nabla w \cdot (\Lambda n_f), \quad (3.13)$$

and

$$\int_T r^{m+1}_{T, \Lambda} u_T = \int_T u_T. \quad (3.14)$$

In addition to the local reconstruction operator, we define the local high-order correction operator.

**Definition 3.4.** The high-order correction operator $R^{m+1}_{T, \Lambda} : X^m_T \to \mathbb{P}^{m+1}(T)$ is defined such that

$$R^{m+1}_{T, \Lambda} u_T = u_T + (r^{m+1}_{T, \Lambda} u_T - r^{m+1}_{T, \Lambda} u_T). \quad (3.15)$$

We remark that the bracketed term of (3.15) is orthogonal to $\mathbb{P}^m(T)$ by construction, and hence this operator can be seen as adding a high-order orthogonal correction to the cell unknown $u_T \in \mathbb{P}^m(T)$. 
3.4. The pressure equation. Let us fix an integer $k \geq 0$. The HHO scheme for (2.1) consists in discretising the pressure in $X_h^{2k}$ and the concentration in $X_h^k$. The choice of an order $2k$, instead of $k$, for the pressure is driven by stability considerations, which is made clear in the proof of Theorem 3.8 (see Remark 6.3).

To write the scheme on the pressure, we introduce the following local bilinear forms: $a_{\Lambda,T} : X_T^{2k} \times X_T^{2k} \to \mathbb{R}$ and $s_{\Lambda,T} : X_T^{2k} \times X_T^{2k} \to \mathbb{R}$, defined by

$$a_{\Lambda,T}(u_T, w_T) = \int_T \Lambda \nabla r_{2k+1}^{2k+1} u_T \cdot \nabla r_{2k+1}^{2k+1} w_T + s_{\Lambda,T}(u_T, w_T),$$  \hspace{1cm} (3.16)

$$s_{\Lambda,T}(u_T, w_T) = \sum_{F \in F_T} \frac{\Lambda_F}{h_F} \int_F \pi_F^{2k}(u_F - R_{T,\Lambda}^{2k+1} u_T)\pi_F^{2k}(w_F - R_{T,\Lambda}^{2k+1} w_T),$$  \hspace{1cm} (3.17)

where

$$\Lambda_F = \|n_{TF} \cdot \mathbf{A} n_{TF}\|_{L^\infty(F)}$$  \hspace{1cm} (3.18)

is a controlling factor for the size of $\Lambda$ across the face $F$. The first term of $a_{\Lambda,T}$ can readily be recognised as a discrete analogue of the weak diffusive terms of the weak formulation. The function $s_{\Lambda,T}$ is the diffusive stabilisation term whose purpose is to enforce a least-squares penalty between face unknowns, and the projection of the high-order correction of the cell unknown. This is required to ensure that the cell and face unknowns are related, and to ensure that the global bilinear form defined in (3.19) below is symmetric positive definite on $X_h^{2k}$ (see Lemma 6.1).

Finally we can define the global pressure bilinear form that mimics the weak pressure equation (2.4). We denote by $a_{\Lambda,h} : X_h^{2k} \times X_h^{2k} \to \mathbb{R}$, the global bilinear form such that

$$a_{\Lambda,h}(u_h, w_h) = \sum_{T \in T_h} a_{\Lambda,T}(u_T, w_T).$$  \hspace{1cm} (3.19)

We also write the linear functional $l_{p,n+1/2}^h : X_h^{2k} \to \mathbb{R}$ such that

$$l_{p,n+1/2}^h(w_h) = \int_\Omega (q^+(t^{n+1/2}, \cdot) - q^-(t^{n+1/2}, \cdot))w_h,$$  \hspace{1cm} (3.20)

which mimics the right-hand side of the weak pressure equation.

Recalling that the pressure equation has diffusion tensor $\mathbf{A} = \kappa(c)$ given by (2.1c), the discrete pressure $p_{h,n+1/2}^n \in X_h^{2k}$ on $[t^n, t^{n+1})$ is computed by solving

$$a_{\kappa^{n+1/2},h}(p_{h,n+1/2}^n, w_h) = l_{p,n+1/2}^h(w_h) \quad \forall w_h \in X_h^{2k},$$  \hspace{1cm} (3.21)

where $\kappa^{n+1/2} = \kappa(\hat{c}^{n+1/2})$ with $\hat{c}^{n+1/2}$ the following interpolation of the concentration at time $t^{n+1/2}$:

$$\hat{c}^{n+1/2} = \frac{3}{2} c^n - \frac{1}{2} c^{n-1}$$  \hspace{1cm} (3.22)

(we take $c_{T}^{-1} = c_{T}^0 = \pi_T^k c_0$ for all $T \in T_h$). Note that this choice of interpolant decouples the pressure equation from the concentration equation (see Algorithm 1 below). Of course, (3.21) only defines the pressure up to an additive constant, so we normalise by imposing

$$\int_\Omega p_{h,n+1/2}^n = 0.$$  \hspace{1cm} (3.23)
3.5. The discrete fluxes. As can be readily seen from the concentration equation (2.1b), the distribution of the concentration depends heavily on the Darcy velocity of the invading fluid as defined by the pressure equation (2.1a). In the HHO framework, the discretisation of advective terms is done by using both the velocity in each cell and its fluxes through the faces [13]. This velocity and fluxes must be properly chosen to ensure the numerical stability of the discretised concentration – the fact that $\text{div}(\mathbf{U}) = q^+ - q^-$ at the continuous level is what ensures stability estimates on the continuous concentration, and this must be mimicked at the discrete level.

For simplicity of notation, we let here $p_h^k = p_h^{n+1/2}$ and $\kappa = \kappa^{n+1/2}$. The numerical flux $U_{TF}$, outward from cell $T$ through the face $F$, is defined by following [10]:

$$ U_{TF} = -\kappa \nabla r_T^{2k+1} p_T \cdot \mathbf{n}_{TF} + \frac{\kappa_F}{h_F} s_{\partial T}^{2k+1} \left( \pi_T^{2k+1} (R_T^{2k+1} p_T - p_F) \right), $$

(3.24)

where $\kappa_F$ is defined from $\kappa$ by (3.18) and $s_{\partial T}^{2k+1}$ is the adjoint, for the $L^2(\partial T)$ inner product, of the operator $s_{\partial T}^{2k} : X_h^{2k} \rightarrow X_h^{2k}$ defined by

$$ s_{\partial T}^{2k}(w_T) = (0, \left( \pi_T^{2k} (w_T - r_T^{2k+1} w_T + \pi_T^{2k} r_T^{2k+1} w_T) \right)_{F \in \mathcal{F}_T}). $$

(3.25)

The first term of (3.24) can be seen as the “naive” discrete flux that we would obtain if we simply substituted the discrete pressure into the definition of the continuous fluxes. The second term can thus be thought of as a discrete conservative correction to the “naive” flux.

The discrete Darcy velocity in cell $T$ is then given by

$$ U_T = -\kappa \nabla r_T^{2k+1} p_T. $$

(3.26)

Finally, it is important to note that the discrete fluxes and Darcy velocity satisfy the following local conservation condition from [10].

**Theorem 3.5.** Let $T \in \mathcal{T}_h$ and let $p_h$ be the solution to (3.21). Then for any $w_T \in X_h^{2k}$, the discrete Darcy velocity and fluxes (3.24) satisfy

$$ a_{\kappa,T}(p_{TF}, w_T) = - \int_T \mathbf{U}_T \cdot \nabla w_T + \sum_{F \in \mathcal{F}_T} \int_F U_{TF}(w_T - w_F). $$

(3.27)

3.6. Concentration equation. The discrete concentration equation is formulated in terms of a stationary advection–diffusion–reaction equation as in [13], with source and reaction terms incorporating the time-stepping. We recall that, for the concentration equation, the relevant discrete space is $X_h^{2k}$.

We consider $\mathbf{U} = ((U_T)_{T \in \mathcal{T}_h}, (U_{TF})_{T \in \mathcal{T}_h, F \in \mathcal{F}_T})$ the Darcy velocity and fluxes reconstructed above at time $t^{n+1/2}$. The following discrete advective derivative $G_{U,T}^{k} : X_h^{2k} \rightarrow P^k(T)$ is designed to be a high-order approximation to the continuous quantity $\mathbf{U} \cdot \nabla v$ on the cell $T$.

**Definition 3.6.** For all $T \in \mathcal{T}_h$, the discrete advective derivative $G_{U,T}^{k} : X_h^{2k} \rightarrow P^k(T)$ is such that, for any $v_T \in X_h^{2k}$ and any test function $w \in P^k(T)$,

$$ \int_T (G_{U,T}^{k} v_T) w = \int_T (\mathbf{U}_T \cdot \nabla v_T) w + \sum_{F \in \mathcal{F}_T} \int_F U_{TF}(v_F - v_T) w. $$

(3.28)

Recalling Definition (2.1d), we define the discrete reconstructed diffusion tensor $D$ for each cell $T \in \mathcal{T}_h$, for any $x \in T$ by

$$ D(x) = \Phi(x)(d_m I + |U_T(x)|(d_l E(U_T(x)) + d_r (I - E(U_T(x)))). $$

(3.29)
The local advection–reaction bilinear form \( a_{R,U,T} : X^k_h \times X^k_h \to \mathbb{R} \) is then defined as

\[
a_{R,U,T}(u_T, w_T) = -\int_T u_T(G^k_{U,T} w_T) + \int_T R u_T w_T + s_{U,T}(u_T, w_T),
\]

(3.30)

where \( R \) are the reaction terms, encompassing the time-stepping, given by

\[
R = \frac{2\Phi}{\Delta t} + q^- (t^{n+1/2}, \cdot).
\]

(3.31)

Furthermore, \( s_{U,T} : X^k_T \times X^k_T \to \mathbb{R} \) is the advective stabilisation term given by

\[
s_{U,T}(u_T, w_T) = \sum_{F \in T_T} \int_F [U_T F]^- (u_F - u_T)(w_F - w_T).
\]

(3.32)

The global advection-reaction bilinear form \( a_{R,U,h} : X^k_h \times X^k_h \to \mathbb{R} \) is defined such that

\[
a_{R,U,h}(u_h, w_h) = \sum_{T \in T_h} a_{R,U,T}(u_T, w_T).
\]

(3.33)

Combining the diffusion and advection–reaction bilinear forms, the complete bilinear form for advection–diffusion–reaction is \( a_{D,R,U,h} : X^k_h \times X^k_h \to \mathbb{R} \) such that

\[
a_{D,R,U,h}(u_h, w_h) = a_{D,h}(u_h, w_h) + a_{R,U,h}(u_h, w_h),
\]

(3.34)

and the global linear functional \( l^c_{h,n+1/2} : X^k_h \to \mathbb{R} \) is

\[
l^c_{h,n+1/2}(w_h) = \int_\Omega \left( q^+ (t^{n+1/2}, \cdot) \tilde{c} (t^{n+1/2}, \cdot) + \frac{2\Phi}{\Delta t} c_h^n \right) w_h,
\]

(3.35)

The discrete concentration scheme then consists in seeking \( \phi_{h,n+1/2} \in X^k_h \) such that

\[
a_{D,R,U,h}(\phi_{h,n+1/2}, w_h) = l^c_{h,n+1/2}(w_h), \quad \forall w_h \in X^k_h,
\]

(3.36)

then extrapolating to obtain \( \phi_{n+1} \). We recall that \( D \) and \( U \) are computed from \( p_{n+1/2} \), and therefore depend on \( n \).

The full scheme is summarised in Algorithm 1.

**Algorithm 1** Complete scheme for the pressure–concentration

1. Set \( c_{T}^{-1} = c_{T}^{0} = \pi_T^k c_0 \) for all \( T \in T_h \)
2. for \( n = 0 \) to \( N - 1 \) do
3. Compute \( p_{n+1/2} \) by (3.21)–(3.23), from \( c_{h,n-1} \) and \( c_{h,n} \).
4. Compute \( U \) by (3.24) and (3.26) with \( p_h = p_{h}^{n+1/2} \).
5. Compute \( \phi_{h,n+1/2} \) by (3.36) and set \( \phi_{h,n+1} = 2\phi_{h,n+1/2} - \phi_{h,n} \).
6. end for

At each iteration, \( (p_{h}^{n+1/2}, \phi_{h}^{n+1/2}) \) are computed by solving two decoupled linear equations, one corresponding to an HHO scheme for a pure diffusion equation, the other one to an HHO scheme for a diffusion–advection–reaction equation. By [13, 16], each of these equations has a unique solution. The existence and uniqueness of a solution to the coupled scheme therefore follows immediately.
**Theorem 3.7** (Existence of a solution to the coupled scheme). Let $\mathcal{M} = (\mathcal{T}_h, \mathcal{F}_h)$ be a mesh of $\Omega$ and take $N \geq 1$. If $(q^+, q^-, K, D, \Phi, \hat{c}, c_0)$ are data satisfying (2.2), then there exists a unique solution $(\mathcal{P}_h, \mathcal{S}_h)$ to the iterative scheme described in Algorithm 1.

An essential question when designing a numerical scheme, beyond the existence of its solution, is its stability with respect to the considered grid (especially as the grid size goes to zero). The following theorem brings an answer to that, by providing a bound on the cell unknowns corresponding to the concentration. We note that this bound is, of course, uniform with respect to the grid size, but also with respect to other important parameters, in particular the molecular, longitudinal and transverse dispersion coefficients $d_m$, $d_l$ and $d_t$. A stability with respect to $d_m$ is all the more essential that this coefficient tends to be taken equal to 0 in numerical tests [6, 37].

**Theorem 3.8** (A priori estimates for the discrete solutions). Let $\mathcal{M} = (\mathcal{T}_h, \mathcal{F}_h)$ be a mesh of $\Omega$ and take $N \geq 1$. If $(q^+, q^-, K, D, \Phi, \hat{c}, c_0)$ are data satisfying (2.2) and $(\mathcal{P}_h, \mathcal{S}_h)$ is a solution to the iterative scheme described in Algorithm 1, then

$$
\|c_h^N\|_{L^2(\Omega)}^2 \leq \frac{(2 + 2e)}{\Phi^2} \left( t_f^2 \|q^+\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|c_0^h\|_{L^2(T)}^2 \right) .
$$

The proof of Theorem 3.8 is provided in the appendix.

4. Tests

In this section, we illustrate the results obtained from the HHO scheme for the simulation of miscible fluid displacement in an oil reservoir. Some of the following test cases first appeared in [35] for the ELLAM-MFEM method and have been subsequently applied in [6] for the MFV scheme. In every simulation here, we use the spatial domain $\Omega = (0, 1000)^2$ measured in ft$^2$ and consider the time period $[0, 3600]$ (approximately 10 years) measured in days. The injection and production source terms are Dirac masses, approximated as usual by a piecewise constant function on each mesh cell. The injection well is located at $(1000, 1000)$ with an injection rate of $30$ ft$^2$/day. The production well is correspondingly located at $(0, 0)$ with a production rate of $30$ ft$^2$/day. We always take the injected concentration $\hat{c} = 1$ with an initial condition given by $c_0(x) = 0$. The viscosity of the oil is given by $\mu(0) = 1.0$ cp and the mobility ratio is $M = 41$ (see (2.1f)). We assume that molecular diffusion is negligible, setting $d_m = 0.0$ ft$^2$/day contrasting with the dispersion effects $d_l = 50.0$ ft$^2$/day and $d_t = 5.0$ ft$^2$/day. The porosity of the medium is taken to be a constant $\Phi(x) = 0.1$. For each test we present surface and contour plots of the concentration $c$, the principle quantity of interest.

**Remark 4.1.** Taking $d_m > 0$ is required to meet the regularity assumptions (2.2) but physically unrealistic due to the fact that the magnitude of the molecular diffusion present in miscible fluid flow is negligibly small compared to the dispersive effects. We will present results taking $d_m = 0$ to demonstrate the suitability of the scheme to real-world parameters. We recall that the stability result of Theorem 3.8 is independent of $d_m$, and is therefore uniformly valid up to the limit $d_m \to 0$.

Table 1 gives the numbers of faces and sizes (maximum ratio of area to perimeter across all cells) for each mesh used in the following tests.
### Table 1. Mesh parameters.

|          | Triangular | Cartesian | Kershaw | Hexagonal |
|----------|------------|-----------|---------|-----------|
| Mesh 1   | 92         | 40        | 612     | 62        |
| Mesh 2   | 352        | 144       | 2380    | 220       |
| Mesh 3   | 1376       | 544       | 5304    | 824       |
| Mesh 4   | 5540       | 2112      | 9384    | 3184      |
| Mesh 5   | 21632      | 8320      | 14620   | 12512     |

|          | Triangular | Cartesian | Kershaw | Hexagonal |
|----------|------------|-----------|---------|-----------|
| Mesh 1   | 31.8       | 62.5      | 16.2    | 70.6      |
| Mesh 2   | 15.9       | 31.2      | 8.96    | 36.7      |
| Mesh 3   | 7.95       | 15.6      | 6.12    | 18.5      |
| Mesh 4   | 3.98       | 7.81      | 4.64    | 9.26      |
| Mesh 5   | 1.99       | 3.91      | 3.73    | 4.63      |

4.1. **Numerical results.** In these first two tests, we use a polynomial degree $k = 1$ and a Cartesian mesh for the spatial discretisation. The time-step is $\Delta t = 18$ days ($N \approx 200$ time-steps over 10 years).

**Test 4.1.1.** We take a homogeneous permeability tensor $K = 80I$ uniformly over a $32 \times 32$ grid (Mesh 4 in Table 1). The mobility within the solvent saturated regions caused by the large adverse mobility ratio $M$ and the lack of molecular diffusion should result in the front of the injected fluid progressing most rapidly along the diagonal between the injection and production wells. These effects are seen on the surface and contour plots in Figure 1 and imply that the flow is indeed strongest along the diagonal direction as expected. This effect is well studied in the literature and is referred to as the macroscopic fingering phenomenon [22]. Notably, our results at $t = 3$ with $k \geq 1$ are far more realistic than those given in [6], which suffer from fluid progressing much too rapidly along the boundary of the domain. The scheme in [6] corresponds to a variant of the HHO method with $k = 0$; our own tests with $k = 0$ reproduced similar results as in this reference (see, e.g., Figures 5(A) and 6(A)).

**Test 4.1.2.** For this test, we take a discontinuous permeability tensor $K = 80I$ except on the four subdomains $(200, 400) \times (200, 400)$, $(600, 800) \times (200, 400)$, $(200, 400) \times (600, 800)$ and $(600, 800) \times (600, 800)$ where instead $K = 20I$. The grid is now $40 \times 40$ in order to ensure that the regions of discontinuity are aligned with the mesh. The results shown in Figure 2 are of great interest to us as they depict very different behaviour to those presented in [6]. Notably in [6], by $t = 10$ years, the invading fluid has yet to subsume the two low permeability regions along the main diagonal. However, the HHO scheme depicts both blocks almost entirely saturated by $t = 10$ years. This implies the presence of a significantly higher amount of dispersion in the solution compared to that produced by the MFV scheme, which suggests that low order schemes may underestimate the amount of diffusion described by the model. Additionally, we note that the region saturated with solvent
4.2. **Comparison of higher order parameters and mesh varieties.** Although the scheme is arbitrary order in space and can easily be extended to arbitrary order in time (by substituting the Crank-Nicholson time-stepping for a higher-order scheme such as backward differentiation), we argue that the $k = 1$ scheme provides the best balance of accuracy and computational efficiency. We will also show that the scheme produces consistent results on a variety of different meshes.

In order to demonstrate that this is the case, we first present the following tests which compare the solutions produced by the schemes with high-order parameters. To contrast the different solutions, we compare the principal quantity of interest to us, which is the total volume of oil recovered after ten years, as measured by the integral

$$
\int_{\Omega} \Phi(x)c_N^h(x).
$$
(A) Surface plot at $t = 3$ years  
(B) Contour plot at $t = 3$ years  
(C) Surface plot at $t = 10$ years  
(D) Contour plot at $t = 10$ years  

**Figure 2.** Concentration of the invading solvent in Test 4.1.2 with $k = 1$ and $\Delta t = 18$ on a $40 \times 40$ Cartesian mesh with a discontinuous permeability.

For the $k = 1$ scheme as presented in Test 4.1.1, the total of volume of oil recovered can be measured to be 65.798% of the total volume of the reservoir. For the high-order tests, we range $k$ from 0 to 3, and replace the Crank-Nicholson time-stepping scheme with a high order backward differentiation formula of order 4 in order to minimise the contribution of the temporal error. Additionally, in order to mitigate the majority of the extrapolation error produced by the pressure estimate when extrapolating $c_h$ (Equation (3.22)), we take a reduced time-step of $\Delta t = 7.2$ days (approximately $N = 500$ steps.)

The four mesh varieties are shown in Figure 3. The triangular, Cartesian and Ker-shaw meshes were first introduced in the FVCA5 benchmark [27] as mesh families 1, 2 and 4.1 respectively. The hexagonal-dominant mesh was used in [4, 11].

Figure 4 depicts the total recovery volume on each family of meshes for decreasing mesh sizes $h$ and various polynomial degrees $k$. The $k = 0$ scheme is shown to perform quite poorly, producing results that are well out-of-line with the rest of the schemes. All of the other schemes however quickly converge to a similar estimate as the mesh size is refined, which is consistently within 1% of the estimate produced by Test 4.1.1 with only $k = 1$, a Crank-Nicholson time-stepping, and $\Delta t = 18$ days. We note that the scheme behaves well even on distorted meshes (the
Kershaw mesh) until $k = 3$, where the linear system becomes too difficult to solve (Figure 4 (c)). This solvability issue of HHO with high degrees (remember that for $k = 3$, the pressure equation is approximated with an order $2k = 6$) on severely distorted meshes has already been noticed for Navier–Stokes equations, and might be a consequence of rounding errors [12]. However, in our experiments, the quality of the results do not significantly improve when using orders higher than $k = 1$ or 2, and the computational cost increases drastically. This difficulty encountered by the highest order schemes on the distorted meshes therefore further justifies avoiding such higher order schemes in practice.

The quality of the numerical approximations for various values of $k$ can also be observed visually. Depicted in Figure 6 are the contour plots for the solution to Test 4.1.1 using various values of $k = 0, 1, 2, 3$. The low-order solution using $k = 0$ suffers from obvious grid effects in which the fluid mixture is progressing too rapidly along and clinging to the boundary of the domain. A similar effect is present in the results of the MFV scheme of [6], suggesting that this artefact is a result of the low order of the scheme. Moving to a higher order scheme, even just $k = 1$, remedies this effect and shows the solvent mixture progressing in a physically realistic pattern. The results of Figures 4, 5 and 6 also show that there is little advantage in selecting a spatial order $k \geq 2$, since the results for these higher order are similar to those obtained with $k = 1$.

4.3. Computational cost. Although the high-order schemes have been shown to produce very reliable results, their computational cost presents a major drawback from a practical point of view. The computational cost of the method increases quite rapidly with respect to $k$, which further supports our argument that the $k = 1$ scheme may be the best balance of accuracy and speed. On a given mesh $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ with polynomials of degree $m$, the number of degrees of freedom present in the discretisation is given by

$$
(m + d) |\mathcal{T}_h| + \left( \frac{m + d - 1}{d - 1} \right) |\mathcal{F}_h|.
$$

The cost of the scheme therefore scales exponentially with $m$, even more so for higher dimensional domains. We also recall that the pressure is solved at a degree $m = 2k$, making the higher order methods all the more expensive. The cost of the time discretisations can be considered independent since the cost is clearly linear in the number of time-steps $N$. Figure 7 illustrates the relative costs of the methods. Depicted are the average times taken per step using $N = 100$ time-steps for the data given in Test 4.1.1. We emphasise that our implementation is not optimised.

Figure 3. The four varieties of meshes. Figures obtained from [16].
for high performance, and that the tests are performed on a personal computer, so these measurements give only a very rough indication of the comparative running times of the various order schemes. As predicted, the running times begin to grow very rapidly for fine meshes with high-order $k$. Since the number of degrees of freedom of the scheme is directly tied to the number of faces in the mesh, the execution time can be seen to be larger for meshes with a greater number of faces (see Table 1).

5. Conclusion

We designed and implemented an arbitrary-order scheme for a miscible incompressible flow model used in tertiary oil recovery. The scheme was based on the Hybrid-High Order method, and is applicable on any kind of polygonal or polyhedral meshes. To ensure the stability of the numerical approximation, the pressure equation has to be approximated with an order twice as large as the order used for the concentration equation, and special care must be taken in reconstructing advective Darcy fluxes from the approximate pressure.
Figure 5. Comparison of the quality of the numerical approximation for $k = 0, 1, 2, 3$ on a Cartesian mesh at time $t = 3$ years. For $k = 0$, the fluid mixture clings to and overly rapidly progresses along the reservoir boundary. Moreover, the expected fingering effect is not visible. These issues are remedied by already selecting $k = 1$, and further increases of the order does not noticeably impact the solution.

We produced several numerical tests on classical 2D test cases encountered in the literature. These tests show that the best balanced of accuracy vs. computational cost is obtained by selecting a spatial order $k = 1$ and a Crank-Nicholson time stepping. The results are stable with respect to the geometry of the meshes, in the sense that the total recovered oil is similar for all meshes and $k \geq 1$. Selecting an order $k = 1$ fixes grid effects that are present with $k = 0$ (as in low-order finite volume methods).

6. Appendix: Stability and existence of a solution to the scheme

We prove here Theorems 3.8 and 3.7. Let $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ be a mesh. As in [14], we take a matching simplicial subdivision $\mathcal{I}_h$ of $\mathcal{M}_h$ and $\varrho > 0$ such that for any simplex $S \in \mathcal{I}_h$ of diameter $h_S$ and inradius $r_S$, $\varrho h_S \leq r_S$, and for all $T \in \mathcal{T}_h$ and all $S \in \mathcal{I}_h$ such that $S \subset T$, $\varrho h_T \leq h_S$. In the following, $A \lesssim B$ means that $A \leq CB$ for some $C$ depending only on $\Omega$ and $\varrho$ (not on $h$).
The following lemma states a stability result for the discrete elliptic bilinear form. A proof can be found for \( \Lambda = \text{Id} \) in [16], and a sketch for extending this to generic \( \Lambda \) is given in [15].

**Lemma 6.1.** Let \( \Lambda \) be a bounded, symmetric, uniformly coercive tensor valued function on \( \Omega \). Then for any \( \mathbf{w}_h \in X_h^k \), the discrete diffusive bilinear form \( a_{\Lambda,h} \) (3.19) satisfies

\[
a_{\Lambda,h}(\mathbf{w}_h,\mathbf{w}_h) \gtrsim \sum_{T \in T_h} \| \Lambda^{1/2} \nabla \mathbf{w}_T \|_{L^2(T)}^2 + \sum_{T \in T_h} \sum_{F \in F_T} \frac{\Lambda_F}{h_F} \| \mathbf{w}_F - \mathbf{w}_V \|_{L^2(F)}^2. \tag{6.1}
\]

The following lemma is the key ingredient in proving the stability of the solution to the scheme. It holds true because of the specific choice of reconstructed Darcy velocity, chosen to be compatible with the discretisation of the concentration equation.

**Lemma 6.2.** Let \( R \) be given by (3.31) and \((\mathbf{U}_T)_{T \in T_h}, (\mathbf{U}_{TF})_{T \in T_h, F \in F_T}\) be the reconstructed Darcy velocity (3.24)–(3.26) corresponding to a solution \( p_{n+1/2} \) of the

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**Figure 6.** Comparison of the quality of the numerical approximation for \( k = 0, 1, 2, 3 \) on a Cartesian mesh at time \( t = 10 \) years.
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Figure 7. The average execution time for one time-step of Test 4.1.1 against varying mesh sizes.

discrete pressure equation (3.21). Then for any \( w_h \in X_h^k \), the discrete advection–reaction bilinear form (3.33) satisfies

\[
a_{R,U,h}(w_h, w_h) \geq \int_{\Omega} \Delta t w_h^2 + \sum_{T \in T_h} \sum_{F \in F_T} \frac{1}{2} \int_F | U_{TF} | (w_T - w_F)^2.
\]  

Proof. Proof: Equation (3.30) and (3.33) give

\[
a_{R,U,h}(w_h, w_h) = \sum_{T \in T_h} \left\{ - \int_T w_T (G_{U,T}^k w_T) + \int_T R w_T^2 + s_{U,T}^k (w_T, w_T) \right\}. \]

Expanding via the definitions of the advective derivative \( G_{U,T}^k \) (Definition 3.6), the advection stabilisation \( s_{U,T} \) (3.32) and the reaction terms \( R \) (3.31) and collecting cell and face terms, we find

\[
a_{R,U,h}(w_h, w_h) = \sum_{T \in T_h} \left\{ \int_T \frac{2 \Phi}{\Delta t} w_T^2 - \int_T (U_T \cdot \nabla w_T) w_T + \int_T q^- (t^{n+1/2}, \cdot) w_T^2 \right\} \\
+ \sum_{T \in T_h} \sum_{F \in F_T} \left\{ - \int_F U_{TF} (w_F - w_T) w_T + \int_F \left| U_{TF} \right| (w_F - w_T)^2 \right\}.
\]  

(6.4)
Considering the second cell term of (6.4), we notice that

\[-(U_T \cdot \nabla w_T)w_T = -U_T \cdot \nabla \left( \frac{1}{2} w_T^2 \right). \tag{6.5}\]

Next we make use of the identity \((a - b)a = a^2 - ab = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a - b)^2\) to rewrite the first face term of (6.4) as

\[-\int_F (U_{TF}(w_T - w_F))w_T = \frac{1}{2} \int_F U_{TF}(w_T^2 - w_F^2) + \frac{1}{2} \int_F U_{TF}(w_T - w_F)^2. \tag{6.6}\]

Inspired by (6.5) and (6.6), we set for any cell \(T \in \mathcal{T}_h\),

\[\tilde{w}_T = \frac{1}{2}(w_T^2, (w_F^2)_{F \in \mathcal{F}_T}) \in \mathbb{R}^{2^k}, \tag{6.7}\]

and use the conservation of the fluxes (Theorem 3.5) with \(\tilde{w}_T\) instead of \(w_T\) to write

\[-\frac{1}{2} \int_T U_T \cdot \nabla (w_T^2) + \frac{1}{2} \sum_{F \in \mathcal{F}_T} \int_F U_{TF}(w_T^2 - w_F^2)\]

\[= -\int_T U_T \cdot \nabla \tilde{w}_T + \sum_{F \in \mathcal{F}_T} \int_F U_{TF}(\tilde{w}_T - \tilde{w}_F) = a_{\kappa_{n+1/2}, T}(p_{n+1/2}^h, \tilde{w}_T). \tag{6.8}\]

Summing over the cells and using the fact that \(p_{n+1/2}^h\) solves the discrete pressure equation (3.21), we deduce that

\[-\frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_T U_T \cdot \nabla (w_T^2) + \frac{1}{2} \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F U_{TF}(w_T^2 - w_F^2)\]

\[= a_{R,T,h}(\tilde{w}_h) = \sum_{T \in \mathcal{T}_h} \left\{ \frac{1}{2} \int_T (q^+(t^{n+1/2}, \cdot) - q^-(t^{n+1/2}, \cdot))w_T^2 + \int_T q^-(t^{n+1/2}, \cdot)w_T^2 \right\}. \tag{6.9}\]

Gathering the results from (6.6)–(6.9) and substituting into (6.4), we have

\[a_{R,T,h}(\tilde{w}_h, w_h)\]

\[= \sum_{T \in \mathcal{T}_h} \left\{ \int_T \frac{2\phi}{\Delta t} w_T^2 + \frac{1}{2} \int_T (q^+(t^{n+1/2}, \cdot) - q^-(t^{n+1/2}, \cdot))w_T^2 + \int_T q^-(t^{n+1/2}, \cdot)w_T^2 + \int_F U_{TF}^2 (w_T - w_F)^2 \right\}\]

\[+ \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \left\{ \frac{1}{2} \int_F U_{TF}(w_T - w_F)^2 + \int_F |U_{TF}|^2 (w_T - w_F)^2 \right\}. \tag{6.10}\]

Combining the second and third cell terms of (6.10), and using \(\frac{1}{2} U_{TF} + |U_{TF}| = \frac{1}{2}|U_{TF}|\) and the non-negativity of \(q^+\) and \(q^-\), we find

\[a_{R,T,h}(\tilde{w}_h, w_h) = \sum_{T \in \mathcal{T}_h} \left\{ \int_T \frac{2\phi}{\Delta t} w_T^2 + \frac{1}{2} \int_T (q^+(t^{n+1/2}, \cdot) + q^-(t^{n+1/2}, \cdot))w_T^2 \right\}\]

\[+ \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \frac{1}{2} \int_F |U_{TF}|^2 (w_T - w_F)^2\]

\[\geq \int_{\Omega} \frac{2\phi}{\Delta t} w_h^2 + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \frac{1}{2} \int_F |U_{TF}|^2 (w_T - w_F)^2. \]

The proof is complete. \(\square\)
 Remark 6.3 (Order 2k on the pressure, and choice of the Darcy fluxes). The reason for discretising the pressure equation with an HHO scheme of order 2k, instead of k, is found in (6.8). Obtaining this relation requires the usage of $\overline{w}_T$, defined by (6.7) and belonging to $X^k_T$, into (3.27).

Equation (6.8) is an essential component of the stability proof, and it also justifies our choice of Darcy flux and volumetric velocity (3.24) and (3.26).

The last ingredient for the proof of Theorem 3.8 is a Grönwall inequality, whose proof can be found in [28].

Lemma 6.4 (Discrete Grönwall Inequality). Let $(y_n)_n, (f_n)_n, (g_n)_n$ be non-negative sequences that satisfy

$$y_n \leq f_n + \sum_{k=0}^{n-1} g_k y_k, \quad n \geq 0.$$  \hspace{1cm} (6.11)

Then $(y_n)_n$ satisfies

$$y_n \leq f_n + \sum_{k=0}^{n-1} f_k g_k \exp \left( \sum_{j=k}^{n-1} g_j \right), \quad n \geq 0.$$  \hspace{1cm} (6.12)

We are now ready to prove our stability result.

Proof of Theorem 3.8. By (3.36), we have for all $\overline{w}_h \in X_h^k$

$$a_{\mathbf{D},R,U,H}(\zeta_h^{n+1/2}, \overline{w}_h) = \int_{\Omega} \zeta_h^{n+1/2}(\overline{w}_h).$$  \hspace{1cm} (6.13)

Select $\overline{w}_h = \zeta_h^{n+1/2}$ as the test function and expand by the definitions of the discrete linear forms to write

$$a_{\mathbf{D},h}(\zeta_h^{n+1/2}, \zeta_h^{n+1/2}) + a_{R,U,h}(\zeta_h^{n+1/2}, \zeta_h^{n+1/2}) = \int_{\Omega} \zeta_h^{n+1/2}(\zeta_h^{n+1/2}).$$

Using Lemma 6.1 (with $\mathbf{A} = \mathbf{D}$) and Lemma 6.2, the definition (3.35) of $\zeta_h^{n+1/2}$ then yields

$$\int_{\Omega} \frac{2\Phi(c_h^{n+1/2})^2}{\Delta t} \leq \int_{\Omega} \left( q^+(t^{n+1/2}, \cdot) \hat{c}(t^{n+1/2}, \cdot) + \frac{2\Phi}{\Delta t} c_h^{n+1/2} \right) c_h^{n+1/2}.\hspace{1cm} (6.14)$$

Gathering the time-stepping terms together, we then write

$$\int_{\Omega} \frac{2\Phi(c_h^{n+1/2} - c_h^n)^2}{\Delta t} \zeta_h^{n+1/2} \leq \int_{\Omega} \left( q^+(t^{n+1/2}, \cdot) \hat{c}(t^{n+1/2}, \cdot) \right) \zeta_h^{n+1/2}.\hspace{1cm} (6.15)$$

Recalling the definition of the half time-stepped concentration (3.3), we easily deduce

$$\int_{\Omega} \frac{2\Phi(c_h^{n+1/2} - c_h^n)^2}{\Delta t} \zeta_h^{n+1/2} = \int_{\Omega} \Phi \left( \frac{(c_h^{n+1/2})^2 - (c_h^n)^2}{2\Delta t} \right).$$

Hence, using the Cauchy–Schwarz and Young’s inequalities in the right-hand side of (6.15) yield, for any $\varepsilon > 0$,

$$\int_{\Omega} \Phi \left( \frac{(c_h^{n+1/2})^2 - (c_h^n)^2}{2\Delta t} \right) \leq \| q^+(t^{n+1/2}, \cdot) \hat{c}(t^{n+1/2}, \cdot) \|_{L^2(\Omega)} \| c_h^{n+1/2} \|_{L^2(\Omega)}$$

$$\leq \frac{1}{2} \| q^+(t^{n+1/2}, \cdot) \hat{c}(t^{n+1/2}, \cdot) \|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \| c_h^{n+1/2} \|^2_{L^2(\Omega)}.$$
and using the fact that $N|_{\text{side}}$. Using we infer

$$\sum_{n=0}^{N-1} \left( ||c_h^n||_{L^2(\Omega)}^2 + ||c_h^{n+1}||_{L^2(\Omega)}^2 \right)$$

we infer

$$\int_{\Omega} \Phi \left( \frac{(c_h^n)^2 - (c_h^0)^2}{2\Delta t} \right) \leq \frac{N}{2\varepsilon} \left( ||q^+||_{L^\infty(0,T;L^2(\Omega))}^2 \right) + \frac{\varepsilon}{2} \sum_{n=0}^{N-1} \left( ||c_h^n||_{L^2(\Omega)}^2 + ||c_h^{n+1}||_{L^2(\Omega)}^2 \right)$$

Applying the boundedness of $\Phi$ (2.2a),

$$\int_{\Omega} \Phi \left( \frac{(c_h^n)^2}{2\Delta t} \right) \leq \int_{\Omega} \Phi \left( \frac{(c_h^0)^2}{2\Delta t} \right) + \frac{N}{2\varepsilon} \left( ||q^+||_{L^\infty(0,T;L^2(\Omega))}^2 \right) + \frac{\varepsilon}{2} \sum_{n=0}^{N-1} \left( ||c_h^n||_{L^2(\Omega)}^2 + ||c_h^{n+1}||_{L^2(\Omega)}^2 \right)$$

Multiplying both sides by $2\Delta t/\Phi_*$ and recalling that $N\Delta t = t_f$ yields

$$\int_{\Omega} (c_h^n)^2 \leq \int_{\Omega} \left( \frac{(c_h^0)^2}{\Phi_*} + \frac{t_f}{\varepsilon \Phi_*} ||q^+||_{L^\infty(0,T;L^2(\Omega))}^2 \right) + \frac{\varepsilon t_f}{\Phi_* N} \sum_{n=0}^{N-1} \int_{\Omega} (c_h^n)^2.$$  \hspace{1cm} (6.16)

From here we wish to cancel the $N^{th}$ term in the rightmost sum of (6.16), so we choose $\varepsilon = \frac{\Phi_*}{2t_f}$. Cancelling and multiplying through what remains we obtain

$$\int_{\Omega} (c_h^n)^2 \leq \frac{2N}{2N-1} \left( \int_{\Omega} \left( \frac{(c_h^0)^2}{\Phi_*^2} + \frac{t_f^2}{\Phi_*^2} ||q^+||_{L^\infty(0,T;L^2(\Omega))}^2 \right) \right)$$

Defining the constant

$$C = \left( \int_{\Omega} \left( \frac{(c_h^0)^2}{\Phi_*^2} + \frac{t_f^2}{\Phi_*^2} ||q^+||_{L^\infty(0,T;L^2(\Omega))}^2 \right) \right),$$

and using the fact that $\frac{N}{2N-1} \leq 1$ for all $N \geq 1$, we are left with

$$\int_{\Omega} (c_h^n)^2 \leq 2C + \frac{1}{2N-1} \sum_{n=0}^{N-1} \int_{\Omega} (c_h^n)^2.$$ 

Applying the discrete Grönwall inequality (Lemma 6.4) to the sequence $y_n = \int_{\Omega} (c_h^n)^2$ then gives us the estimate

$$\int_{\Omega} (c_h^n)^2 \leq 2C \left( 1 + \frac{1}{2N-1} \sum_{n=0}^{N-1} \exp \left( \frac{N-1}{2N-1} \right) \right) \leq 2C(1 + \varepsilon).$$

which is the desired stability bound. \hfill \Box
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