Improving coherence with nested environments

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For many experimental quantum information devices, the central system is well isolated against simple decoherence processes such as spontaneous emission or coupling to a heat bath, but it is subject to some imperfections in the apparatus. To describe this situation, we insert a near environment which provides the residual coupling between central system and heat bath, and neglect any direct couplings. In the framework of random matrix theory we calculate the decoherence of the central system, using a linear response expansion for the coupling to the far environment and a dephasing coupling between central system and near environment. We find that the increase of the coupling to the far environment will slow down the decoherence in the central system, and thus improve the quality of the device. We extend this result to stronger couplings by numerical calculations in a modified Caldeira-Leggett model.

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In many quantum optics experiments and quantum information devices we find the following situation: The central system, well protected from simple decoherence processes such as spontaneous emission or direct coupling to a structureless heat bath, still suffers some decoherence from the coupling to or instabilities of the quantum part of the apparatus. We will call the former the far environment and the latter the near environment. An example results from the celebrated Haroche experiment [1, 2], if we interpret the two-level atom as the central system, the cavity as the near environment and its leaks and absorption as coupling to a far environment. In such situations, different strategies have been developed in order to deal with the residual decoherence process, such as using particular spectral density functions, describing a structured heat bath, non-Markovian quantum master equations [3], or incorporating some degrees of freedom of the environment into the central system Ref. [4–6].

Assuming a tripartite system without direct coupling between central system and far environment, we find that increasing the coupling of the near to the far environment can protect the central system against decoherence. This fact seems somewhat anti-intuitive but in the setting of the Haroche experiment the late M. C. Nemes discussed this possibility with one of the authors [7] twelve years ago. More recently, additional numerical evidence in various settings has appeared [8], some of which were master thesis related to the present work [9–11]. Finally, it was shown in a different setting setting that in a strong coupling limit a protected subspace can appear [12].

To obtain results with some claim of universality we use random matrix theory (RMT) of decoherence [13–15]. We simplify the picture by limiting the coupling between central system and near environment to dephasing, and assume the coupling between near and far environment to be separable. In this setup, many analytical expressions exist [16–20] in the absence of the far environment, we can take the advantage of these expressions, if we treat the far environment within a linear response calculation. We thus obtain analytic expressions for weak couplings between near and far environment, note that this range of coupling strengths is exactly opposite to that treated in Ref. [12]. To study the effect of the far environment beyond the linear response approximation, we perform numerical simulations, using a modified Caldeira-Leggett master equation [21], whose equivalence to RMT models has first been discussed in Ref. [22].

Model: The full system consists of three parts, the central system, the near environment and the far environment with Hilbert spaces \( \mathcal{H}_c, \mathcal{H}_e \) and \( \mathcal{H}_f \). The unitary evolution of the entire system is given by the Hamiltonian

\[
H_{\text{tot}} = H_0 + \nu_c \otimes \mathbb{1}_f + \mathbb{1}_c \otimes V_e \otimes \mathbb{1}_f + \mathbb{1}_c \otimes V'_e \otimes \mathbb{1}_f \quad (1)
\]

where \( H_0 = h_c \otimes \mathbb{1}_e \otimes \mathbb{1}_f + \mathbb{1}_c \otimes H_e \otimes \mathbb{1}_f + \mathbb{1}_c \otimes V_{ef} \otimes \mathbb{1}_f \). Tracing out both environments leads to the reduced dynamics of the central system \( \varrho_c(t) = \text{tr}_{e,f}[\varrho_{\text{tot}}(t)] \), with

\[
\varrho_{\text{tot}}(t) = \exp(-iH_{\text{tot}}t/\hbar) \varrho_c \otimes \varrho_{c,f} \exp(iH_{\text{tot}}t/\hbar) \quad (2)
\]

where \( \varrho_c \) represents the initial state of the central system (typically assumed to be pure), and \( \varrho_{c,f} \) represents the initial states of the environment. The separable couplings are given by \( \nu_c \otimes V_e \) (between central system and near environment) and \( \gamma V'_e \otimes V_f \) (between near and far environment). The former will be chosen as dephasing, such that \( [h_c, \nu_c] = 0 \). Such couplings are frequently used for open systems [3], as they simplify calculations and maintain many essential properties.

Dynamics: We write the Hamiltonian as \( H_{\text{tot}} = \sum_{j} |j\rangle\langle j| \otimes H_{c,f}^{(j)} \), with

\[
H_{c,f}^{(j)} = (\bar{\varepsilon}_j \mathbb{1}_c + \nu_j V_e) \otimes \mathbb{1}_f + \mathbb{1}_c \otimes H_f + \gamma V'_e \otimes V_f \quad (3)
\]
where the set of states $\{|j\rangle\}$ is a common eigenbasis of $\hat{h}_c$ and $\nu_e$, while $\varepsilon_j$ and $\nu_j$ are the corresponding eigenvalues. The evolution of the whole system can be written as $\rho_{\text{tot}}(t) = \sum_{jk} \rho_{jk}(0) |j\rangle \langle k| \otimes \rho^{(j,k)}_{\text{eff}}(t)$, where $\rho_{\text{tot}} = \sum_{jk} \rho_{jk}(0) |j\rangle \langle k|$ is the initial state of the central system, and

$$\rho^{(j,k)}_{\text{eff}}(t) = \exp \left( -iH_{c,\text{eff}} t / \hbar \right) \rho_{\text{eff}} \exp \left( iH_{c,\text{eff}} t / \hbar \right).$$

We find for the matrix elements of the reduced state of the central system: $\rho_{jk}(t) = \rho_{jk}(0) \text{tr}_{c,i}[\rho^{(j,k)}_{\text{eff}}(t)]$. Since $\text{tr}_{c,i}[\rho^{(j,j)}_{\text{eff}}(t)] = 1$, the diagonal elements are constant in time, while the off-diagonal ones (i.e. the coherences) are given as expectation values of generalized echo operators in the composite environment (see Eq. (7) and Ref [23].)

In other words focusing on an individual matrix element, $\rho_{jk}(t)$, we may introduce

$$H_\lambda = H_0 + \lambda V_{\text{eff}} = H_c + \nu_j V_c, \quad H_0 = H_c + \nu_k V_c,$$

such that $\lambda V_{\text{eff}} = (\nu_j - \nu_k) V_c$. This allows to connect the coherences for vanishing coupling ($\gamma \to 0$) to the far environment, with fidelity amplitudes [16, 17]. Introducing the relative coherences

$$f_{\lambda,i}(t) = \frac{\rho_{jk}(t)}{\rho_{jk}(0)} = \text{tr}_{c,i} \left[ e^{-iH_{c,\text{eff}} t / \hbar} \rho_{\text{eff}} e^{iH_{c,\text{eff}} t / \hbar} \right],$$

where $H_{\lambda,\gamma} = H_0 \otimes \mathbb{1}_c + \mathbb{1}_e \otimes H_{\text{eff}} + \gamma V_c^\dagger \otimes V_c$, we find that $f_{\lambda,0}(t) = f_{\lambda}(t)$ with

$$f_{\lambda}(t) = \text{tr}_c [M_{\lambda}(t) \text{tr}_{c,i}(\rho_{\text{eff}})], \quad M_{\lambda}(t) = e^{iH_0 t / \hbar} e^{-iH_{\lambda,\gamma} t / \hbar}.$$

Hence, $f_{\lambda,0}(t)$ becomes the fidelity amplitude for perturbing the Hamiltonian $H_0$ by $\lambda V_{\text{eff}}$, given the initial state $\text{tr}_{c,i}(\rho_{\text{eff}})$ in the near environment.

**Perturbative calculation:** Applying the linear response approximation in the Fermi golden rule regime to the coupling between near and far environment, one arrives after some lengthy but straightforward algebra [24] at

$$f_{\lambda,\Gamma}(t) \sim (1 - \Gamma t) f_{\lambda}(t) + \Gamma \int_0^t d\tau \ f_{\lambda}(\tau) f_{\lambda}(t - \tau),$$

with the transition rate $\Gamma = \gamma^2 \tau_N / \hbar^2 \ (\tau_N$ is the Heisenberg time in the far environment, and $N_c$ is the dimension of the near environment). Here and below, the symbol $\sim$ means equal up to $O(\Gamma^2)$, and we will replace $\gamma$ by the physically more meaningful transition rate $\Gamma$. As we will see below, Eq. (8) is valid as long as $\Gamma t \ll 1$.

Analyzing this result for different functional forms for $f_{\lambda}(t)$, we find that the coupling to the far environment is indeed slowing down the decoherence in the central system. However, the effect can be more or less pronounced. For generic systems, one often finds that $f_{\lambda}(t)$ changes from an exponential decay in the Fermi golden rule regime to a Gaussian decay in the perturbative regime [18, 25].

In the Fermi golden rule regime the effect is zero, which can also be understood in physical terms. In that regime the temporal correlations $(V_{\text{eff}}(t) V_{\text{eff}}(t'))$ of the perturbation in the interaction picture decay very fast – on a time scale $t_{\text{corr}} \ll t_{\text{dec}(ce)}$, the decoherence time in the central system. Therefore, even if the decoherence time in the near environment $t_{\text{dec}(ef)}$ (due to the far environment) is smaller than $t_{\text{dec}(ce)}$, as long as $t_{\text{dec}(ef)} > t_{\text{corr}}$, the far environment will not have any effect on the decoherence in the central system.

In the perturbative regime by contrast, $f_{\lambda}(t) \sim e^{-\lambda^2 t^2}$, such that $f_{\lambda,\Gamma}(t) \sim g_{\lambda}(\lambda t)$ with

$$g_{\lambda}(x) = (1 - \alpha x) e^{-x^2} + \alpha \sqrt{\pi / 2} e^{-x^2 / 2} \text{erf}(x / \sqrt{2}).$$

For simplicity, we will compare our results to the exponentiated linear response (ELR) expression for the fidelity amplitude $f_{\lambda}(t)$ from Ref. [18], though note that an exact analytical result is also available [19, 20].

**Caldeira-Leggett master equation:** For full-fledged random matrix calculations, we would need to work in the Hilbert space of near and far environment. For the far environment, we would need a smaller mean level spacing in combination with a larger spectral span, as compared to the near environment. Still, in order to justify the use of RMT, we would need as many levels as possible also in the near environment. Such random matrix calculations are not viable, due to the dimension of the Hamiltonian matrices involved.

We will therefore use an approach which allows to work in the Hilbert space of the near environment alone, taking the effect of the far environment into account via a quantum master equation. For convenience we choose the Caldeira-Leggett master equation [21], where we replace the diagonal matrix representation of the harmonic oscillator Hamiltonian with a random matrix, defined as in Eq. (5). We choose both matrices $H_0$ and $V_{\text{eff}}$ from the Gaussian orthogonal ensemble (GOE). We scale $H_0$ in such a way that the mean level spacing becomes one in the center of the spectrum. The matrix elements of $V_{\text{eff}}$ are chosen to have the variances $(V_{\text{eff}}^2)_{ij} = 1 + \delta_{ij}$. In that way, the strength of the perturbation (implied by the dephasing coupling to the central system), measured in units of the mean level spacing $d_0$, is given by $\lambda$. In the following figures we scale time by the Heisenberg time $t_H = 2\tau_N / d_0$. From earlier studies of similar models [9, 10], we know that the effect of the heat bath is quite compatible with the coupling to a further RMT environment in the Fermi golden rule regime, as long as the coupling to the heat bath, determined by $\Gamma$, is not
we consider the case of the statistical uncertainty of the results. This gives us an idea about the quality of our analytical result from Eq. (8), and the nearest thin dotted lines show three statistically independent ensemble averages for each case. (b) The same quantity as in panel (a), but for the theoretical curves we choose best fit values for the coupling to the heat bath: $\Gamma/\lambda = 0.097$ (circles), 0.44 (triangles), and 0.77 (shaped crosses), obtained for the region $0 < t < 15$. We can clearly see that the curves which correspond to $\Gamma = 0$ are different from zero, due to the reasons discussed above. Finally, we show three additional cases with increasing coupling to the heat bath. For those cases, the relative coupling strength $\alpha = \Gamma/\lambda$ is 0.1 (circles), 0.5 (triangles), and 1.0 (shaped crosses). We can observe that the theory agrees with the simulations, only in the case of smallest coupling, for stronger coupling the effect is systematically overestimated.

In Fig. 2(b) we intend to adjust the value for $\Gamma$ (now denoted by $\Gamma_0$) such that the theory agrees best with the numerical simulations. A good agreement between theory and simulations could not be achieved for all times,
we repeat the comparison for the fitted values \( \Gamma \).

In particular not for moderate and strong coupling. However, a best fit for \( \Gamma_{\text{fit}} \) restricted to times up to the maximum which is always located at approximately \( t_{\text{max}} \approx 15 \), yields quite satisfactory results.

In Fig. 3 we repeat the comparison for \( \lambda = 0.02 \), where the coupling between central system and RMT environment is close to the perturbative regime. We use the same fitting procedure as in Fig. 2(b). However, in this case we consider a broader range of different values for \( \alpha = \Gamma/\lambda \) which goes from \( \alpha = 0.1 \) until \( \alpha = 10.0 \). Essentially, we observe a similar behavior as in the case of \( \lambda = 0.1 \). However, it becomes also clear that a very large ratio \( \Gamma/\lambda \) yields stronger deviations between simulations and theory, even with fitted values for \( \Gamma \) and restricting ourselves to small times (here, \( t \lesssim 60 \)). Nevertheless, the slowing down of decoherence in the central system due to the increasing coupling to the far environment, occurs just as before.

Finally, we compare in Fig. 4 the fitted values \( \Gamma_{\text{fit}} \) for the coupling to the far environment, with the nominal ones, by plotting \( \Gamma_{\text{fit}}/\lambda \) versus \( \Gamma/\lambda \). This is done for different dimensions of the near environment, for different coupling strengths between central system and near environment, and different couplings to the far environment. The derivation of our theoretical result within linear response theory showed that the deviation from the exact result should be quadratic in \( \Gamma \). Hence, for sufficiently small values of \( \Gamma \) one would expect that the points in Fig. 4 would come close to the line \( \Gamma_{\text{fit}} = \Gamma \). For larger values of \( \Gamma \), we find that the effect of the far environment is smaller than predicted by our theory, as the fitted values \( \Gamma_{\text{fit}} \) are smaller than the nominal ones. To guide the eye we plotted the straight line \( \Gamma_{\text{fit}}/\lambda = \alpha \) and \( b \alpha/(b+\alpha) \) with \( b = 3.77 \), which describes the overall behavior of the points quite well.

Summarizing, we have been able to obtain an analytic expression confirming that nested environments can improve coherence of a central system as the coupling between near and far environment increases, as long as this coupling is small. We also extended previous limited numerical evidence for large coupling using a Caldeira-Leggett master equation which has been derived from RMT considerations in previous work [10]. This confirms that the effect subsists at large couplings between near and far environment, but subsides if the central system is strongly coupled to the near environment. An explanation on the basis of the quantum Zeno effect is tempting but problematic, at least in as far as we consider weak couplings between near and far environment.

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Derivation of the main result (Eq. 8)

**Dephasing coupling**

Under dephasing coupling, the nondiagonal element of the qubit reduced state is just the fidelity amplitude of the RMT-environment with respect to the perturbation induced by the coupling between central system and near RMT environment. For an initial state \(\Omega_0\), and with the abbreviation \(H_\lambda = H_e + \nu_t V_e\),

\[
f_\lambda(t) = \text{tr} \left[ \Omega_0 e^{i(H_0+H_t+\gamma V_{e,c})t/\hbar} e^{-i(H_\lambda+H_t+\gamma V_{e,c})t/\hbar} \right] = \text{tr} \left\{ e^{i(H_\lambda+H_t)t/\hbar} e^{-i(H_\lambda+H_t)t/\hbar} \Omega_0 e^{i(H_0+H_t+\gamma V_{e,c})t/\hbar} e^{-i(H_0+H_t)t/\hbar} e^{i(H_0+H_t)t/\hbar} e^{-i(H_\lambda+H_t)t/\hbar} \right\}.
\]

The later two evolution operators are separable and therefore simplify as follows:

\[
e^{i(H_0+H_t)t/\hbar} e^{-i(H_\lambda+H_t)t/\hbar} = e^{iH_0 t/\hbar} \otimes e^{iH_t t/\hbar} e^{-iH_\lambda t/\hbar} \otimes e^{-iH_t t/\hbar} = M_\lambda(t) \otimes \mathbb{I}_f, \quad M_\lambda(t) = e^{iH_0 t/\hbar} e^{-iH_\lambda t/\hbar}.
\]

Since \(\text{tr}_f [AM \otimes \mathbb{I}_f] = A_{ij,kl} M_{km} \delta_{ij} = \text{tr}_f(A) M\) then

\[
f_s(t) = \text{tr}_e \left[ \hat{g}_{e,f}(t) M_\lambda(t) \right], \quad \hat{g}_{e,f}(t) = \text{tr}_f \left[ e^{i(H_\lambda+H_t)t/\hbar} e^{-i(H_\lambda+H_t+\gamma V_{e,c})t/\hbar} \Omega_0 e^{i(H_0+H_t+\gamma V_{e,c})t/\hbar} e^{-i(H_0+H_t)t/\hbar} \right].
\]

Linear response approximation for the coupling to the far environment

The trace over the far environment in Eq. (12) is almost exactly of the form as the reduced density matrix (in the interaction picture) treated in the reference [14], namely with \(M_\gamma(t) = e^{i(H_\lambda+H_t)t/\hbar} e^{-i(H_\lambda+H_t+\gamma V_{e,c})t/\hbar}\), we may write

\[
f_s(t) = \text{tr}_e \left[ \hat{g}_c(t) M_\lambda(t) \right], \quad \hat{g}_c(t) = \text{tr}_f \left[ M_\gamma(t) \Omega_0 M_\gamma(t)^\dagger \right].
\]
However, in order to apply the formalism of the reference [14], we should assume the coupling $V_{e,f}$ and the initial state $\Omega_0$ to be separable:
\[
\Omega_0 = \varrho_e \otimes \varrho_f , \quad V_{e,f} = \varrho_e \otimes V_f , \quad \tilde{V}_{e,f}(t) = e^{i(H_\lambda + H_t) t/\hbar} \varrho_e \otimes V_f e^{-i(H_\lambda + H_t) t/\hbar},
\]
where we have already defined the representation $\tilde{V}_{e,f}(t)$ of the coupling operator to the far environment in the interaction picture. Of course there remains the very important difference, that here we have different echo operators on the left and the right side of the initial state. Nevertheless, following carefully the calculation made in [14], after the developed the echo operators into its Dyson series we find:
\[
\varrho_{e,f}(t) = \varrho_e - \frac{\gamma^2}{\hbar^2}(A_J - A_I) , \quad A_J = \text{tr}_i[ J(t) \Omega_0 + \Omega_0 J(t)^\dagger ] , \quad A_I = \text{tr}_i [ I(t) \Omega_0 I(t) ]
\]
\[
J_\lambda(t) = \int_0^t d\tau \int_0^\tau d\tau' \tilde{V}_{e,f}(\tau) \tilde{V}_{e,f}(\tau'), \quad I_\lambda(t) = \int_0^t d\tau \tilde{V}_{e,f}(\tau).
\]
(14)

Now, note that $\tilde{V}_{e,f}(\tau)$ is separable, such that
\[
\tilde{V}_{e,f}(\tau) = \tilde{\varrho}_\lambda(\tau) \otimes \tilde{V}_f(\tau) , \quad \tilde{\varrho}_\lambda(\tau) = e^{iH_\lambda t/\hbar} \varrho_e e^{-iH_\lambda t/\hbar},
\]
and similarly for $\tilde{V}_f(\tau)$.

The calculation for the average over $J_\pm(t)$ with respect to the random matrix $V_f$:
\[
\langle J_\lambda(t) \rangle = \int_0^t d\tau \int_0^\tau d\tau' c(\tau - \tau') \tilde{\varrho}_\lambda(\tau) \tilde{\varrho}_\lambda(\tau') \otimes 1_f ,
\]
(15)

where $c(\tau)$ describes the spectral correlations of $H_f$ and $\beta$ is the Dyson parameter, such that for a GUE ($\beta = 2$) or a GOE ($\beta = 1$) with Heisenberg time $\tau_H = 2\pi \hbar/\delta$: $c(\tau) = 3 - \beta + \delta(\tau/\tau_H) - \beta(\tau/\tau_H)$. Similarly for $A_I$:
\[
\langle A_I \rangle = \int_0^t d\tau \int_0^\tau d\tau' c(\tau - \tau') \tilde{\varrho}_\lambda(\tau) \varrho_e \tilde{\varrho}_\lambda(\tau') .
\]
(16)

Finally, we obtain
\[
\langle A_J - A_I \rangle = \int_0^t d\tau \int_0^\tau d\tau' c(\tau - \tau') \{ \tilde{\varrho}_\lambda(\tau) [ \varrho_e - \varrho_e \tilde{\varrho}_\lambda(\tau') ] - [ \varrho_e \tilde{\varrho}_\lambda(\tau') - \varrho_e \tilde{\varrho}_\lambda(\tau') ] \tilde{\varrho}_\lambda(\tau) \}
\]
(17)

Fermi golden rule regime and master equation

If we assume that the Heisenberg time of the far environment $\tau_H$ is very large, and that we are in the Fermi golden rule regime for the coupling to the far environment, then from the correlation function $c(\tau)$, we only need to take the delta function into account. That reduces Eq. (17) to
\[
\langle A_J - A_I \rangle = \frac{\tau_H}{2} \int_0^t d\tau \{ \tilde{\varrho}_\lambda(\tau)[ \varrho_e - 2 \tilde{\varrho}_\lambda(\tau) \varrho_e \tilde{\varrho}_\lambda(\tau') + \varrho_e \tilde{\varrho}_\lambda(\tau) \tilde{\varrho}_\lambda(\tau') ] \}
\]
(18)

Next, we will average that expression over the coupling matrix $\varrho_e$, which is the near environment part of the coupling between near and far environment. Since this matrix is assumed to be an element of the GUE, we find:
\[
\langle \varrho^2_{e,ik} \rangle = \sum_j \langle \varrho_{ij} \varrho_{jk} \rangle = N_e \delta_{ik} \quad \Rightarrow \quad \langle \varrho^2_e \rangle = \langle \tilde{\varrho}_\lambda(\tau) \tilde{\varrho}_\lambda(\tau) \rangle = N_e \mathbb{I}_e
\]
(19)

On the other hand, we find
\[
\langle \varrho_e \tilde{\varrho}_\lambda(\tau) \tilde{\varrho}_\lambda(\tau) \rangle_{ij} = (u^\dagger)_i j (\varrho_e)_{jk} (u)_{kl} (\varrho_e)_{lm} (u^\dagger)_{mn} (\varrho_e)_{np} (u)_{pq} = (u^\dagger)_i j \delta_{kn} \delta_{jp} (u)_{kl} (\varrho_e)_{lm} (u^\dagger)_{mn} (u^-)_{pq} = (u^\dagger)_i j (u)_{kl} (\varrho_e)_{lm} (u^\dagger)_{mk} (u)_{jq}
\]
(20)

This can be written as
\[
\langle \varrho_e \tilde{\varrho}_\lambda(\tau) \rangle = [M(\tau)]_{ij} \text{tr}[ \varrho_e M(\tau) ] \quad \text{since} \quad M(\tau) = u^\dagger u , \quad M(\tau)^\dagger = u^\dagger u .
\]
(21)
Therefore, we obtain for $\rho_{e,f}(t)$:

$$
\rho_{e,f}(t) = \rho_e - \gamma^2 \frac{\tau H}{2\hbar^2} \left( 2 N_e t \rho_e - 2 \int_0^t d\tau \text{tr}[\rho_e M(\tau)] M(\tau)^\dagger \right)
$$

(22)

Let us denote $\Gamma = \gamma^2 \tau H N_e / \hbar^2$. Then we obtain for the fidelity amplitude:

$$
f_{\lambda,\Gamma}(t) = \text{tr} \left[ (1 - \Gamma t) \rho_e M(t) + \frac{\Gamma}{N_e} \int_0^t d\tau \text{tr}[\rho_e M(\tau)] M(\tau)^\dagger M(t) \right]
$$

(23)

thus

$$
f_{\lambda,\Gamma}(t) \sim (1 - \Gamma t) f_\lambda(t) + \Gamma \int_0^t d\tau f_\lambda(\tau) f_\lambda(t - \tau)
$$

(24)

where we have used that $f_\lambda(t) = \text{tr}[\rho_e M(\tau)]$ and $N_e f_\lambda(t - \tau) = \text{tr}[M(t - \tau)]$. So $f_\lambda(t)$ denotes the fidelity amplitude in the near environment, if there is no coupling to the far environment ($\gamma = 0$). The first line is exact (in the limit $\Gamma t \ll 1$), assuming that no ensemble averaging has been applied with respect to $H_e$ and $V_e$. The second line assumes self averaging for the quantities $\text{tr}[\rho_e M(\tau)]$ and $\text{tr}[M(t - \tau)]$ which will probably hold for generic initial states $\rho_e$ and sufficiently large near environment ($N_e \gg 1$).