Abstract

We consider the problem of learning how to optimally allocate treatments whose cost is uncertain and can vary with pre-treatment covariates. This setting may arise in medicine if we need to prioritize access to a scarce resource that different patients would use for different amounts of time, or in marketing if we want to target discounts whose cost to the company depends on how much the discounts are used. Here, we show that the optimal treatment allocation rule under budget constraints is a thresholding rule based on priority scores, and we propose a number of practical methods for learning these priority scores using data from a randomized trial. Our formal results leverage a statistical connection between our problem and that of learning heterogeneous treatment effects under endogeneity using an instrumental variable. We find our method to perform well in a number of empirical evaluations.

1 Introduction

Data-driven resource allocation is increasingly prevalent across a number of fields. One popular approach starts by modeling treatment heterogeneity. Given a treatment (or intervention) and an outcome of interest, we also collect a large number of (pre-treatment) covariates and seek to estimate how these covariates modulate the effect of the treatment on the outcome. We then allocate treatment to those individuals who are predicted to respond most strongly to it based on their covariates. As examples of this paradigm, in medicine, Basu, Sussman, and Hayward [2017] consider assigning more aggressive treatment to reduce blood pressure to cardiovascular disease patients who are estimated to benefit from it the most; in marketing, Ascarza [2018] and Lemmens and Gupta [2020] consider targeting retention offers to customers who are estimated to be most responsive to them; while in economics, Kitagawa and Tetenov [2018] discuss prioritizing eligibility to job training programs to those job applicants who are estimated to get the largest employment boost from the program.

One limitation of this line of work, however, is that existing methods for treatment personalization mostly do not consider the cost of assigning treatment. In all three cases considered above, this is not a problem: Here, treating any one specific person costs roughly the same as treating another, and so allocating treatment based on estimated outcomes alone is valid. However, in many problem settings the cost of treating different people is not the
same, and is unknown pre-treatment. In these settings, when there is a budget constraint limiting the total resources that we can spend on the treatment, then determining which individuals to prioritize depends requires learning the benefits as well as the costs of the treatment, and how they relate to the pre-treatment covariates.

**Example 1. Marketing incentives.** Suppose a gym wants to evaluate a campaign that gives a “first month free” offer to some potential customers, with the goal of enrolling more long-term members. Clearly, the treatment effect may vary across customers, as may the cost. Some recipients of the offer may visit the gym just a handful of times during their free month (low cost) and then upgrade to a regular membership at the end of the month (high reward), while others may use the gym’s facilities every day during their free month (high cost) but then fail to convert (low reward). A marketing campaign that allocates resources only based on rewards but not costs may not spend its budget optimally. We analyze a marketing experiment with this structure run by a sharing economy company in Section 6.2.

**Example 2. Targeting scarce healthcare resources in a crisis.** Consider a hospital that has insufficient intensive care beds to treat all incoming patients, and needs to choose whom to prioritize given available resources. Suppose, moreover, that the hospital only has two types of incoming patients. Patients of type A are responsive to treatment, and their chance of survival rises by 10% if admitted to intensive care; however, their recovery is slow, and they will spend 20 days in the unit if admitted. In contrast, Patients of type B get a 5% increase in chance of survival if admitted, but will only spend 5 days in the unit if admitted. Here, targeting based on treatment heterogeneity would prioritize patients of type A, but this is not the utility-maximizing prioritization rule: If the hospital only targets patients of type A, in the long run it can save 0.5 patients per day per 100 intensive care beds, whereas if it only targeted patients of type B it could double this number to 1 patient per day per 100 intensive care beds.

**Example 3. Insurance subsidies.** Suppose a philanthropic organization wants to offer a subsidized insurance product. The organization has a finite budget, and wants to design its program to maximize benefits (e.g., in the case of health insurance, to maximize the total improvement along a target health metric). In this setting, utility-maximization requires considering both how much a recipient would benefit from the insurance, and how many claims they might make (and thus how much of the total budget they would use up).

In this paper, we propose an approach to optimal treatment prioritization in a setting where we have a limited budget, and our treatment of interest has costs that are both variable and uncertain. We show that the optimal feasible treatment rule ranks units by a cost-aware priority score, formed as a ratio of conditional expected incremental benefits to conditional expected incremental costs, and then treats people ordered by this priority score until budget runs out (or the intervention is no longer beneficial).

The main learning problem in the paper is estimating the optimal priority scores; our proposed policies then involve targeting using the estimated priority score. We start by showing that in a semi-parametric setting—where the priority score is linear in the pre-treatment covariates—a moment-based estimator of the score function converges at a \(1/\sqrt{n}\)-rate and has an asymptotically normal sampling distribution. In the more general non-parametric setting, we show that the scores can be learned by using existing algorithms based on generalized random forests [Breiman, 2001, Athey, Tibshirani, and Wager, 2019].
We also provide a method for estimation and inference on the benefit of a priority-based rule for a certain budget.

We find our approach to perform well in a number of applications, and to enable meaningful gains relative to approaches that do not account for variable costs in targeting. We also contrast our proposed methods to existing approaches based on directly solving an empirical version of the cost-benefit optimization problem [Hoch et al., 2002, Xu et al., 2020, Huang and Xu, 2020, Sun, 2021, Wang et al., 2018]. In doing so, we argue that the fact that our approach is priority-based, i.e., that it first ranks units by priority and then allocated them to treatment until the budget has been spent, has some notable practical advantages: It ensures monotonicity in treatment assignment (i.e., the set of people treated at a higher budget level is a superset of people treated at a lower budget level), and enables us to more precisely enforce the budget constraint when deploying the policy to new data. Approaches based on direct optimization of an empirical objective are not priority based, and so do not generally have these advantages.

1.1 Related Work
The need to account for the costs of an intervention arises in a number of application areas. The effectiveness of an intervention across studies is often compared on the basis of cost-effectiveness, i.e., the positive effect for a dollar invested. Hendren and Sprung-Keyser [2020] perform a meta-study in which they compare a large number of experiments with public expenditures on the basis of cost-effectiveness, and also discuss a common and sensible way to construct the costs and the benefits variables across studies. Dhaliwal et al. [2013] do the same focusing on education. However, while such cost-effectiveness comparisons across interventions or treatments are ubiquitous in the literature, these papers do not generally consider the heterogeneity in the cost-effectiveness estimates within their study in a systematic way, or the potential for targeted treatments.

Our contribution fits broadly into a growing literature on treatment personalization, including Bertsimas, Dunn, and Mundru [2019], Hahn, Murray, and Carvalho [2020], Kallus and Zhou [2020], Kennedy [2020], Künzel et al. [2019], Nie and Wager [2021], Wager and Athey [2018], Zhao et al. [2012] and Zhou, Athey, and Wager [2023]. Most of this literature has focused on settings where cost of treatment is constant across units and so doesn’t enter into considerations about optimal targeting; however, there are a handful of recent exceptions, involving two general approaches to taking costs into account for treatment personalization. Each of these solve the same optimization problem of maximizing outcomes while constraining costs to meet a budget, but use algorithms that are not priority-based to do so.

The first approach, considered by Hoch, Briggs, and Willan [2002] and Xu et al. [2020], is to create a new outcome, called the net monetary benefit, which captures both the cost and benefit of treatment. Concretely, this approach specifies outcomes of the form “reward $-\nu \times$ cost” and then runs standard methods for personalization of these outcomes. This approach is helpful if we are able to pre-commit to a value of $\nu$ that brings costs and rewards to the same scale. However, enforcing a specific target budget exactly is not feasible. A value of $\nu$ can be chosen using a hold-out set of the data to meet a target budget in expectation, but when implementing the rule on a new sample in practice, will sometimes violate the budget.¹

¹Difficulties with the net monetary benefit may also arise in a setting like Example 2. Here, treatment doesn’t really have a clear monetary costs; rather, the only reason the hospital may fail to treat a patient
Another challenge with this approach is that it is less practical to consider interventions at multiple budget levels (e.g., in an advertising application, perhaps management would specify a target budget for a campaign, but also ask for estimates on what could be accomplished if this target budget were increased). Re-fitting a model using multiple different values of $\nu$ can lead to a number of practical difficulties: It increases computational requirements and, furthermore, finite-sample effects can lead to non-monotonicity in estimated treatment rules, whereby some units are moved from treatment to control even though we increase the overall budget. In contrast, our approach relies on ranking by priority scores, and the budget only impacts the cutoff above which individuals are treated. This means that the treatment can be rolled out sequentially until the deployment budget is exhausted, and the performance of the rule can be evaluated at multiple budget levels using a single estimate of the priority scores.

The second approach, considered in Huang and Xu [2020], Sun [2021] and Wang, Fu, and Zeng [2018], is to directly impose cost constraints into the outcome-weighted learning approach of Zhao et al. [2012]. This approach is conceptually direct and is amenable to extensions, such as multiple treatments, which are not straightforward using a priority-based rule. However, it relies on a non-trivial optimization problem that can be difficult to solve with many thousands of observations. The thresholding rule that we consider leads to fast, scalable, and simple estimation procedures, based on existing random forest-based algorithms. Furthermore, by solving the optimization problem directly, we again lose the advantages of a priority-based rule that separates the budget from the main learning problem. The direct optimization approach must be solved from scratch any time the budget changes, and the approach only meets the budget in expectation when it is implemented on a new data sample. Wang et al. [2018] also discuss another algorithm—they refer to it as the regression-model-based learning algorithm—which is distinct from the outcome-weighted learning approach otherwise analyzed in that paper and can be understood as a version of the direct ratio approach discussed in this paper.

Finally, Luedtke and van der Laan [2016] and Bhattacharya and Dupas [2012] discuss the role of budget when allocating treatments; however, they assume a constant cost of treatment (i.e., the budget determines the fraction of the population that may be treated). We also note work on “cost-sensitive” decision rules, including Greiner, Grove, and Roth [2002] and Lakkaraju and Rudin [2017], which considers the cost of covariate acquisition in defining a decision rule. In their setting, one may prefer a simple although slightly less accurate prioritization rule if we can save costs by not measuring some covariates; here, in contrast, the full covariate set is always available, but we do not know a priori how much it will cost to assign treatment to any given individual.

2 Optimal Allocation under Budget Constraints

Throughout this paper, we formalize causal effects using the potential outcomes framework [Imbens and Rubin, 2015]. We assume that we observe independent and identically distributed tuples $(X_i, W_i, Y_i, C_i)^i \sim \text{iid} P$ for $i = 1, \ldots, n$, where $X_i \in \mathcal{X}$ denotes pre-treatment covariates, $W_i \in \{0, 1\}$ denotes treatment assignment, $Y_i \in \mathbb{R}$ denotes the observed outcome, and $C_i \in \{0, 1\}$ denotes incurred cost. Here, both $Y_i$ and $C_i$ depend on the assigned treatment $W_i$, and we capture this relationship via potential outcomes: We posit pairs \{$(Y_i(0), Y_i(1))$ and $(C_i(0), C_i(1))$\} denoting the outcomes (and respectively costs) we would is if all intensive care beds are already full.
have observed for treatment assignments \( W_i = 0 \) and \( W_i = 1 \), such that we in fact observe \( Y_i = Y_i(W_i) \) and \( C_i = C_i(W_i) \) given the realized treatment \( W_i \). In many applications, we may know a priori that \( C_i(0) = 0 \) (i.e., there is no cost to not assigning treatment); for now, however, we also allow for the general case where \( C_i(0) \) may be non-zero. Throughout, we assume that treatment increases costs in the following sense.

**Assumption 1.** \( C_i(1) \geq C_i(0) \) almost surely, and \( \mathbb{E}[C_i(1) - C_i(0) | X_i = x] > a \) for any value of \( x \in X \) and some value of \( a \).

The goal is to use the sample of data \((X_i, W_i, Y_i, C_i)\) for \( i = 1, \ldots, n \) to estimate the optimal treatment allocation rule. The first step is to define the optimal treatment allocation rule in the population \( P \) under a budget constraint and variable costs. A treatment allocation rule (or policy) is a function \( \pi : X \to [0, 1] \) mapping pre-treatment covariates to an action, where prescriptions \( 0 < \pi(x) < 1 \) are interpreted as random actions (i.e., we randomly assign treatment with probability \( \pi(x) \)). The (incremental) value \( V \) of a policy \( \pi \) is the expected gain it achieves by treating the units it prescribes treatment to, \( V(\pi) = \mathbb{E}[\pi(X_i)(Y_i(1) - Y_i(0))] \), while the (incremental) cost \( G \) of \( \pi \) is \( G(\pi) = \mathbb{E}[\pi(X_i)(C_i(1) - C_i(0))] \). Given a budget constraint \( B \), the optimal policy \( \pi_B^* \) solves the following knapsack-type problem

\[
\pi_B^* := \arg \max \{ V(\pi) : G(\pi) \leq B \}.
\]

Recall that the knapsack problem involves selecting a set of items such as to maximize the aggregate “value” of the selected items subject to a constraint on the allowable “weight”; and, in our setting, the treatment effect \( Y_i(1) - Y_i(0) \) is the value we want to maximize while the incremental cost \( C_i(1) - C_i(0) \) acts as a weight. There is a key difference between our treatment allocation problem and the traditional knapsack problem. We do not know the distribution of the outcomes or costs, and need to learn them from data. Here, we momentarily abstract away from the learning problem and first write down the form of the optimal treatment assignment rule given the true data generating distribution; then, we will turn towards learning in the following sections.

In this setting, the form of the optimal treatment allocation rule (1) follows directly from the well known solution to the fractional knapsack problem given in Dantzig [1957]. The optimal policy involves first computing the following conditional cost-benefit ratio function,

\[
\rho(x) := \frac{\mathbb{E}[Y_i(1) - Y_i(0) | X_i = x]}{\mathbb{E}[C_i(1) - C_i(0) | X_i = x]},
\]

and then prioritizing treatment in decreasing order of \( \rho(x) \). The following result formalizes this statement. The proof of Theorem 1 given in the appendix generalizes an argument from Luedtke and van der Laan [2016] to the setting with variable costs. We also note that Wang et al. [2018] prove a special case of Theorem 1 that applies in the case of where \( \rho(X_i) \) has a continuous distribution (and so a non-randomized optimal policy exists).

**Theorem 1.** Under Assumption 1, the optimal (stochastic) policy \( \pi_B^* \) admits the following

\[\text{In the medical literature, this quantity is also known as the incremental cost-effectiveness ratio [Hoch, Briggs, and Willan, 2002]. We use the convention that } \alpha/0 \text{ is equal to } +\infty \text{ if } \alpha > 0, -\infty \text{ if } \alpha < 0, \text{ and } 0 \text{ if } \alpha = 0.\]
characterization: There are constants $\rho_B \in \mathbb{R}$ and $a_B \in [0, 1]$ such that

$$
\pi^*_B(x) = \begin{cases} 
0 & \text{if } \rho(x) < \rho_B, \\
 a_B & \text{if } \rho(x) = \rho_B, \\
 1 & \text{if } \rho(x) > \rho_B,
\end{cases}
$$

(3)

where either $\rho_B = a_B = 0$ (i.e., we have sufficient budget to treat everyone with a positive treatment effect), or $\rho_B > 0$ and the pair $(\rho_B, a_B)$ is the unique pair for which this policy has cost exactly $B$ in expectation. In the case where $\rho(X_i)$ has a bounded density, $\mathbb{P}[\rho(X_i) = \rho_B] = 0$, the policy $\pi^*_B$ is both deterministic and the unique optimal policy.

Remark 1. We emphasize that $\pi^*_B$ involves ranking units by the ratio of conditional expectations $\rho(x)$, rather than by the actual cost-benefit ratios $R_i = (Y_i(1) - Y_i(0)) / (C_i(1) - C_i(0))$ as one might expect in a classical deterministic knapsack specification. The issue here is that the policy $\pi^*_B$ must make decisions based only on knowledge of pre-treatment covariates $X_i$, and $R_i$ is not measurable in terms of pre-treatment covariates.

2.1 Identifying the Priority Score in Randomized Trials

To make use of Theorem 1 in practice, we need to make assumptions that let us identify the target $\rho(x)$ from observable data. The difficulty here is that $\rho(x)$ depends on all four potential outcomes $Y_i(0), Y_i(1), C_i(0)$ and $C_i(1)$, whereas we only get to observe the realized outcomes $Y_i = Y_i(W_i)$ and $C_i = C_i(W_i)$. Such difficulties are recurrent in the literature on treatment effect estimation, and arise from what Holland [1986] calls the fundamental problem of causal inference.

Here, we address this difficulty by assuming that we have access to data from a randomized controlled trial, i.e., where $W_i$ is determined by an exogenous random process; or, more generally, that we have data where the treatment assignment mechanism is unconfounded in the sense of Rosenbaum and Rubin [1983], i.e., that it is as good as random once we condition on pre-treatment covariates $X_i$. Randomized controlled trials are frequently used to guide treatment allocation decision in application areas where costs may matter [see, e.g., Banerjee and Duflo, 2011, Gupta et al., 2020, Kohavi et al., 2009], and unconfoundedness assumptions are widely used in the literature on treatment personalization [Künzel et al., 2019, Wager and Athey, 2018].

The following result shows how, under unconfoundedness, we can re-write $\rho(x)$ in terms of observable moments. Given this result, the problem of estimating $\rho(x)$ now reduces to a pure statistical problem of estimating a ratio of conditional covariances.

Proposition 2. In the setting of Theorem 1, suppose further more that the treatment assignment mechanism is unconfounded,

$$
\{Y_i(0), Y_i(1), C_i(0), C_i(1)\} \perp \perp W_i \mid X_i,
$$

(4)

and that it satisfies overlap, $0 < \mathbb{P}[W_i = 1 \mid X_i = x] < 1$. Then,

$$
\rho(x) = \frac{\text{Cov} [Y_i, W_i \mid X_i = x]}{\text{Cov} [C_i, W_i \mid X_i = x]}.
$$

(5)

At first glance, the problem of estimating a ratio of covariances as in (5) may seems like an explicit but potentially difficult statistical problem. However, there is a useful connection
between the statistical task of estimating (5), and that of estimating a (conditional) local average treatment effect using an instrumental variable [Angrist, Imbens, and Rubin, 1996, Durbin, 1954]. Specifically, suppose we have independent and identically distributed samples \((X_i, Y_i, T_i, Z_i)\) where the \(X_i\) are pre-treatment covariates, \(T_i\) is a (potentially endogenous) treatment, \(Y_i\) is an outcome, and \(Z_i\) is an (exogenous) instrument. In this setting and under further assumptions discussed in Imbens and Angrist [1994], the (conditional) local average treatment effect,

\[
\lambda(x) = \frac{\text{Cov}[Y_i, Z_i | X_i = x]}{\text{Cov}[T_i, Z_i | X_i = x]},
\]

is a natural measure of the causal effect of the endogenous treatment \(T_i\) on the outcome \(Y_i\).

Several authors, including Abadie [2003], Angrist and Pischke [2008], Chernozhukov et al. [2018], Athey, Tibshirani, and Wager [2019] and Wang, Li, and Hopp [2022], have then used this instrumental variables setting as motivation for developing methods that boil down to estimating a ratio of conditional covariances \(\lambda(x)\) as in (6).

The upshot is that, although our problem and that of treatment effect estimation with instruments are conceptually very different, they both reduce to statistically equivalent ratio estimation problems: Despite divergent derivations and motivations, there is no difference between the statistical targets (5) and (6). Thus, we can take any method for estimating \(\lambda(x)\) in (6), and turn it into an estimator for \(\rho(x)\) in (5) by simply plugging in our treatment \(W_i\) where the method expects an “instrument” \(Z_i\), and plugging in our cost \(C_i\) where it expects a “treatment” \(T_i\).

### 2.2 Is the Priority Score a Local Average Treatment Effect?

Given the statistical connection between the instrumental variables problem and our problem, it is natural to ask whether some deeper conceptual connection exists. In particular, is it possible to interpret the cost-adjusted benefit of a treatment rule as a type of local average treatment effect? One key assumption made by Imbens and Angrist [1994] to show that (6) identifies a local average treatment effect is that an “exclusion restriction” holds, i.e., that all effects of the instrument \(Z_i\) on the outcome \(Y_i\) are mediated by the treatment \(T_i\). In contrast, in our setting, we have made no such assumption; and furthermore a comparison between our setting and that of Imbens and Angrist [1994] reveals that this is the only material way in which our model differs from their abstract model. Thus, the answer to the above question is that our priority score can be interpreted as a local average treatment effect if and only if we add an exclusion restriction to our setting, i.e., we assume that all effects of the treatment \(W_i\) on the outcome \(Y_i\) are mediated by costs \(C_i\).

Whether or not such an exclusion restriction is credible will depend on the setting. As one example where the exclusion restriction may hold, consider a social policy that seeks to promote employment by offering a free 1-week interview preparation workshop for first-time job seekers. One could argue that this policy can only have an effect on employment via participation in the workshop. And then, if this exclusion restriction holds, the work of Imbens and Angrist [1994] would imply that the priority score arising from our setting corresponds to the local average treatment effect of the investment on employment (i.e., it measures the average benefit from a dollar of spending on the interview preparation workshop on those people who choose to make use of the workshop).

We emphasize, however, that in many cases of interest this type of exclusion restriction will not hold. In the setting of Example 1, one would hope that the effect of a first-month-free offer on future subscriptions would at least partially operate by creating goodwill and brand
visibility, and not be entirely mediated by the amount the company spends on honoring the offer. And, in the setting of Example 3, one might expect that access to reliable health insurance would, in addition to benefits mediated by spending on healthcare, also have direct benefits, e.g., by enabling access to preventive care and by reducing stress from exposure to uninsured health events. Thus, while our priority score statistically looks like a (conditional) local average treatment effect—and can sometimes be interpreted as one—our overall setting is more general, and our priority scores are also valid for optimal targeting in a number of important settings where they cannot be interpreted as local average treatment effects.

3 Learning Treatment Allocation Rules

The simple characterization of the optimal treatment rule $\pi^*_B$ given in Theorem 1 suggests the following simple algorithm for treatment prioritization. To keep the algorithm straightforward, we here focus on the case where $\rho(X_i)$ has a bounded density, so that the optimal policy is unique and deterministic.

1. Get an estimate $\hat{\rho}(x)$ of the ratio (2) on a training set where pre-treatment covariates $X_i$, treatment $W_i$, and realized costs and outcomes ($Y_i, C_i$) are observed.

2. On the test set, where only $X_i$ is observed, rank units $i$ in descending order of $\hat{\rho}(X_i)$, and treat those with estimated ratio above the estimated threshold $\hat{\rho}_B$: $\hat{\pi}(X_i) = 1(\hat{\rho}(X_i) > \hat{\rho}_B)$.

In other words, each individual is assigned a priority score, and the estimate of this priority score will not depend on the budget. Individuals are assigned to the treatment in order of their priority, up until a threshold, where the threshold ensures the budget constraint is respected.

As emphasized in the introduction, our use of a priority-based treatment assignment rule has a number of advantages. The expected performance of the treatment rule can be estimated for multiple different budgets, using a single estimate of the priority score $\hat{\rho}(x)$. Furthermore, the treatment assignment is monotonic in the budget. For $B' \geq B$, and any test set of individuals, any individual that is treated under budget $B$ is also treated under $B'$. This allows a budget for a campaign to be increased after the campaign has already started. Finally, on the test set, it is possible in many cases to satisfy the budget exactly, rather than in expectation (see Remark 2).

Now, to make use of this framework, it remains to develop estimators for $\rho(x)$. First, in Section 3.1, we consider a semi-parametric specification where $\rho(x)$ is assumed to be linear in $x$, but the conditional covariances $\text{Cov} \left[ Y_i, W_i \mid X_i = x \right]$ and $\text{Cov} \left[ C_i, W_i \mid X_i = x \right]$ themselves may have a complex dependence on $x$. In this setting, we develop a Neyman-orthogonal estimator for $\rho(x)$ that allows for $1/\sqrt{n}$ rates of convergence. Second, in Section 3.2, we propose a non-parametric estimator for $\rho(x)$ based on random forests. Finally, in Section 4, we discuss how to generate confidence intervals for the lift generated by estimated targeting rules.

Remark 2. One important setting where we can (nearly) exactly satisfy the budget constraint on the test set is when the control arm has no cost (i.e., $C_i(0) = 0$), we have an upper bound on the treatment costs, $C_i(1) \leq M$, and the treatment cost is immediately (or rapidly) revealed for units if they’re assigned to treatment. In this case, we can satisfy the budget to within tolerance $M/n$ by: treating units in descending order of $\hat{\rho}(X_i)$ and
keeping track of the accumulated costs from treated individuals; and then stopping when the accumulated cost of treatment is within $M/n_{\text{test}}$ of $B$.

In settings where there is a significant delay in observing realized costs after treatment, then this algorithm is not feasible and controlling the realized costs on the test set will in general not be possible. Instead, $\hat{\rho}_B$ can be estimated on the training set with $i = 1, \ldots, n_{\text{train}}$ observations, given $\hat{\rho}(x)$ and an estimate $\hat{\gamma}(x) = E[C_i(1) − C_i(0)|X_i = x]$, as:

$$\hat{\rho}_B = \min \left\{ p \in [0, \infty) : \frac{1}{n_{\text{train}}} \sum_{i=1}^{n_{\text{train}}} 1(\hat{\rho}(X_i) > p)\hat{\gamma}(X_i) \leq B \right\}.$$  

When the rule is then deployed on a test set, then the budget is met in expectation, but it is possible in finite samples that there are violations of the budget constraint.

### 3.1 Parametric Estimation of the Priority Score

If we assume that the benefit-cost ratio is linear in the covariates, so that $\rho(x) = x'\beta$, then we can define $\beta$ as the solution to an unconditional moment restriction. This representation leads to a method-of-moments type estimator that has the same form as the just-identified instrumental variables estimator and an asymptotic theory for the estimator. Although the linearity assumption is a strong assumption in many practical settings, understanding the performance of the estimator in the parametric setting is helpful before turning to the non-parametric setting. To do so, we follow the approach to instrumental variables estimation taken in, e.g., Chernozhukov et al. [2018], and start by defining a score function

$$e_i(\beta, h(X_i)) = (W_i - h_w(X_i))(Y_i - h_y(X_i)) - (C_i - h_c(X_i))X_i'\beta, \quad h_w(x) := E[W_i|X_i = x], \quad h_y(x) := E[Y_i|X_i = x], \quad h_c(x) := E[C_i|X_i = x].$$  

(7)

Note that, under the conditions of Proposition 2, the identification result (5) is equivalent to the score function being mean-zero at the true value of $\beta$ (the details are in Appendix B),

$$E[e_i(\beta, h(X_i))|X_i = x] = 0 \text{ for all } x \in \mathcal{X}. \quad (8)$$

The terms $h(x)$ in (7) are nuisance components, i.e., unknown functions that are not of direct interest, but are required to form the score functions. However, the construction (7) is Neyman orthogonal, i.e., the identifying result (8) is robust to small errors in the nuisance components: for any perturbation function $\delta(x)$,

$$\left[d \frac{d}{\varepsilon} E[e_i(\beta, h(x) + \varepsilon\delta(x))|X_i = x]\right]_{\varepsilon=0} = 0, \text{ for all } x \in \mathcal{X}; \quad (9)$$

see the proof of Theorem 3 for details. As argued in Chernozhukov et al. [2018], this Neyman-orthogonality property is crucial to estimators motivated by (8) enabling robust estimation of $\beta$.

Now, the identification result (8) implies a conditional moment restriction at each value $x \in \mathcal{X}$, and so may be difficult to work with in practice if the $X_i$ have continuous support or are high dimensional. However, (8) also implies that, given

$$\mathcal{B} = \{ \beta' : E[X_i e_i(\beta', h(X_i))] = 0 \}, \quad (10)$$
Theorem 3. Estimating \( \beta \) a semi-parametrically efficient estimator of \( \beta \). The covariates are bounded:

Denoting \( \rho \) with \( \beta \) we must have \( \beta \) holds. There exists a sequence \( \delta \) such that, when trained on \( n \) IID samples from our generative distribution \( P \), we obtain an estimator \( \hat{h} \) satisfying, with probability tending to 1 as \( n \) gets large,

\[
E_{X \sim P_X} \left[ (\hat{h}_y(X) - h_y(X))^2 \right] \leq \rho_y, \quad E_{X \sim P_X} \left[ (\hat{h}_c(X) - h_c(X))^2 \right] \leq \rho_c, \quad E_{X \sim P_X} \left[ (\hat{h}_w(X) - h_w(X))^2 \right] \leq \rho_w,
\]

with \( \rho_y, \rho_c, \rho_w, \leq \frac{\delta_y}{n^{1/2}}, \rho_y, \rho_c, \rho_w, \leq \frac{\delta_c}{n^{1/2}} \) and also \( \rho_y, \rho_c, \rho_w, < \delta_y, \rho_c, \rho_w, < \delta_c \), \( \rho_y, \rho_c, \rho_w, < \delta_n \).

Assumption 2. Assume that \( \mathcal{X} \subseteq \mathbb{R}^m \). We use estimators \( \hat{h} \) of \( h \) for which the following holds. There exists a sequence \( \delta_n \to 0 \) and constants \( a, A > 4 \) such that, when trained on \( n \) IID samples from our generative distribution \( P \), we obtain an estimator \( \hat{h} \) satisfying, with probability tending to 1 as \( n \) gets large,

\[
E_{X \sim P_X} \left[ (\hat{h}_y(X) - h_y(X))^2 \right] \leq A, \quad E_{X \sim P_X} \left[ (\hat{h}_c(X) - h_c(X))^2 \right] \leq A, \quad E_{X \sim P_X} \left[ (\hat{h}_w(X) - h_w(X))^2 \right] \leq A.
\]

We show below that this estimator achieves a parametric rate of convergence for \( \beta \) provided the nuisance components \( \hat{h} \) converge reasonably fast (but not necessarily at a parametric rate themselves), and the moment condition \( (10) \) is full rank. Our proof follows from general results developed in Chernozhukov et al. [2018].

Assumption 3. Denoting \( V_i = W_i - E[W_i | X_i], D_i = (C_i - E[C_i | X_i]) \) and \( U_i = Y_i - E[Y_i | X_i] - D_i X_i' \beta \). We assume that for some constants \( A, q > 4 \): \( E[X_i X_i'] \) is full rank, \( E[V_i^2 U_i^2 | X_i] \geq a, \ E[C_i^q]^{1/q} \leq A, \ E[Y_i^q]^{1/q} \leq A \) and \( E[V_i^2 | X_i] \leq A^2, \ E[U_i^2 | X_i] \leq A^2 \). The covariates are bounded: \( X_i \in [-A, A]^k \).

Theorem 3. Under the assumptions of Proposition 2, suppose furthermore that Assumption 2 and Assumption 3 hold. Then, our estimator \( \hat{\beta} \) described above satisfies

\[
\sqrt{n} \left( \hat{\beta} - \beta \right) \to N(0, V_\beta), \quad V_\beta = E[V_i D_i X_i X_i']^{-1} E[U_i^2 V_i^2 X_i X_i'] E[V_i D_i X_i X_i']^{-1}.
\]

\[3\] The construction \( (10) \) is not the only way to turn \( (7) \) into a practical, unconditional moment restriction. In fact, Chernozhukov et al. [2018] shows that, writing \( \sigma^2(x) = E[e_i(\beta, h(X_i))]^2 | X_i = x \) and \( R(x) = E \left[ \sigma^2(x) R(X_i)e_i(\beta, h(X_i)) | X_i = x \right] \), then the moment condition \( E \left[ \sigma^{-2}(X_i) R(X_i)e_i(\beta, h(X_i)) \right] = 0 \) leads to a semi-parametrically efficient estimator of \( \beta \), reaching the Chamberlain [1992] efficiency bound. However, estimating \( \sigma^2(x) \) and \( R(x) \) leads to additional complexity, and so we rely on the simple form \( (10) \) here.
The rank condition on $\mathbb{E}[X_iX'_i]$ ensures that the moment condition in Equation 10 has a unique solution. The key property of Theorem 3 is that we get $1/\sqrt{n}$-rate convergence for $\hat{\beta}$ even if the rest of the problem is not parametrically specified. In particular, the numerator and denominator used to define $\rho(x)$ in (5), i.e., $\text{Cov} \left[ Y_i, W_i \mid X_i = x \right]$ and $\text{Cov} \left[ C_i, W_i \mid X_i = x \right]$, need not admit a linear specification. Rather, it’s enough to be able to estimate relevant nuisance components at slower rates, e.g. $\hat{\rho}(X_i) - \rho(X_i) = o_p(n^{-1/4})$, and this can be done via flexible machine learning methods such as deep learning [Farrell, Liang, and Misra, 2021].

3.2 Non-Parametric Estimation of the Priority Score

If we’re willing to assume that $\rho(x)$ admits a linear form, then the estimator discussed above achieves excellent large-sample performance. However, in many applications, we may not be willing to assume a linear specification $\rho(x) = x'\beta$, and instead seek a non-parametric estimator for $\rho(x)$. In this case, one possible approach would be to first separately estimate the numerator and denominator in (5), $\text{Cov} \left[ Y_i, W_i \mid X_i = x \right]$ and $\text{Cov} \left[ C_i, W_i \mid X_i = x \right]$, and then form $\hat{\rho}(x) = \widehat{\text{Cov}} \left[ Y_i, W_i \mid X_i = x \right] / \widehat{\text{Cov}} \left[ C_i, W_i \mid X_i = x \right]$. This approach, however, is potentially suboptimal: If the numerator and denominator are more complex than $\rho(x)$, then the rates of convergence we could achieve via this approach would be slower than ones we could get via directly targeting $\rho(x)$ [Foster and Syrgkanis, 2019, Nie and Wager, 2021].

Here, we consider one particular solution to direct estimation of $\rho(x)$ based on the “generalized random forest” framework of Athey, Tibshirani, and Wager [2019]. Generalized random forests provide an approach to turn any conditional moment restriction for a target parameter, such as (7), into an estimator for the target parameter that adapts the popular random forest method of Breiman [2001]. The key idea of the algorithm is that it grows a forest specifically designed to express heterogeneity in $\rho(x)$, and this can be done via flexible machine learning methods such as deep learning [Farrell, Liang, and Misra, 2021].

Like random forests, the approach starts by growing a set of $B$ decision trees by recursive partitioning on the covariates $X_i$. For each tree indexed $b = 1, \ldots, B$ and a given test point $x$, let $L_b(x)$ denote the set of observations $i = 1, \ldots, n$ falling in the leaf-node containing $x$, and define forest weights

$$\alpha_i(x) = \frac{1}{B} \sum_{b=1}^{B} \frac{1 \left( \{i \in L_b(x)\} \right)}{\sum_{j=1}^{n} 1 \left( \{j \in L_b(x)\} \right)}.$$  (13)

Conceptually, the weights $\alpha_i(x)$ capture the relevance of each observation $i = 1, \ldots, n$ for estimation at $x$; formally, we note that the usual regression forest prediction at $x$ can be expressed as a weighted average of outcomes $Y_i$ with weights $\alpha_i(x)$. In our setting, generalized random forests estimate $\rho(x)$ by solving an empirical version of (8) with the forest weights $\alpha_i(x)$:

$$\hat{\rho}(x) = \frac{\sum_{i=1}^{n} \alpha_i(x) (Y_i - \bar{h}_y(X_i)) (W_i - \bar{h}_w(X_i))}{\sum_{i=1}^{n} \alpha_i(x) (C_i - \bar{h}_c(X_i)) (W_i - \bar{h}_w(X_i))},$$

$$\bar{h}_y(x) = \sum_{i=1}^{n} \alpha_i(x) Y_i, \quad \bar{h}_c(x) = \sum_{i=1}^{n} \alpha_i(x) C_i, \quad \bar{h}_w(x) = \sum_{i=1}^{n} \alpha_i(x) W_i.$$  (14)
As discussed in Athey, Tibshirani, and Wager [2019], it is helpful to consider compare (14) to a simpler $k$-nearest neighbors estimator that first discards all but the $k$ closest observations to $x$ in covariate space, and then estimates $\rho(x)$ by solving an unconditional version of (8) on those $k$ observations. From the perspective of this comparison, the advantage of generalized random forests is that the weights $\alpha_i(x)$ provide a well tuned, data-adaptive notion of neighbors relevant to estimating $\rho(x)$.

We refer to Athey, Tibshirani, and Wager [2019] and Athey and Wager [2019] for details, including a discussion of how the recursive partitioning used to grow the individual trees in the forest is run. At a high level, the trees are grown to greedily express as much heterogeneity as possible in $\rho(x)$. These papers also detail how subsampling and subsample splitting are used to stabilize the estimator. The formal results given in Athey, Tibshirani, and Wager [2019] apply directly to our setting, and ensure large-sample consistency of the learned $\hat{\rho}(x)$ under the conditions of Proposition 2.

Finally, from a practical perspective, we can again make use of the formal connection to instrumental variables estimation here. Although the specification above would be enough to build a generalized random forest for estimating $\rho(x)$, doing so would seem require a non-trivial amount of implementation work. However, it turns out that the calculations required to estimate $\rho(x)$ are exactly the same as are already performed in the “instrumental forest” method provided in the grf package of Athey, Tibshirani, and Wager [2019], and so we can re-purpose this function for our use case. Specifically, we use instrumental forests to estimate $\rho(x)$ by replacing the method’s inputs $Z_i$ and $T_i$ with $W_i$ and $C_i$ respectively (we pass covariates $X_i$ and the outcome $Y_i$ to the instrumental forest as usual).

## 4 Evaluating the Performance of a Targeting Rule

So far in this section, we have showed how to estimate $\rho(X_i)$, which allows us to implement the algorithm for treatment prioritization provided in Section 3. In deciding whether or not to implement a treatment prioritization rule, it is useful to characterize for a fixed budget how much the population is expected to benefit in expectation from prioritization compared to a uniform rule. In this section, we show how to estimate and perform inference on the lift of a prioritization rule that relies on a given estimator for $\rho(X_i)$ computed on a training set of data.

Let $S_i$ be the score that we assign to an individual and use for prioritization. For example, the score might be an estimate of the cost-benefit ratio $\tilde{\rho}(X_i)$, where $\tilde{\rho}(X_i)$ is computed on some training set of the data. Let $s \in [0, 1]$ be some score cutoff. If we treat individuals with a score above $s$, then we can define the budget as $B(s)$ and the reward as $R(s)$.

$$B(s) = \mathbb{E}[(C_i(1) - C_i(0))1\{S_i \geq s\}]$$
$$R(s) = \mathbb{E}[(Y_i(1) - Y_i(0))1\{S_i \geq s\}]$$

Under this definition, the expected budget spend for a sample of $n$ individuals is $B(s)n$ and the expected reward earned is $R(s)n$. We can define the reward for a given budget $b$ as $Q(b) = R(B^{-1}(b))$, where $b$ ranges from 0 to $B(0)$. $B(s)$ is strictly monotonic, since by Assumption 1 $\frac{\partial B(s)}{\partial s} = f(s)\mathbb{E}[C_i(1) - C_i(0)|S_i = s] > 0$, where $f(s)$ is the density function for scores. Thus, $B^{-1}(b)$ is a function and $Q(b)$ is well-defined. For a given spend $b$, we can
where \( \pi \) (\( \hat{R} \) and \( a \) uniformly randomized trial, \( \pi \) define the lift over the reward given by the random allocation as

\[
\Delta(b) = Q(b) - b \frac{R(0)}{B(0)}.
\]

Below, we focus on performing estimation and inference on \( \Delta(b) \), the lift at a single budget value, directly.

To this end, suppose that we have trained a scoring rule \( \hat{S} : \mathcal{X} \rightarrow [0, 1] \) on a training set, and have access to a test set with \( i = 1, \ldots, n_{\text{test}} \) units for evaluation. We then construct an adaptation of the QINI curve that shows the cost and benefit of treating different fractions of units, as prioritized by \( \hat{S} \). The QINI curve is a popular visualization that, for a family of thresholded scoring rules, plots the cost of treatment on the \( x \)-axis and the benefit of treatment on the \( y \)-axis [Ascarza, 2018, Imai and Li, 2019, Rzepakowski and Jaroszewicz, 2012, Yadlowsky et al., 2021]. Existing results on estimating QINI curves, however, assume that the cost of treating each unit is the same, and so the cost of treatment on the \( x \)-axis is equivalent to the number of units treated; however, in our setting, this equivalence no longer holds.

To address this challenge, we propose the following estimator for the QINI curve in a setting with uncertain costs. We first form inverse-propensity weighted estimators of \( B(s) \) and \( R(s) \) as follows,

\[
\hat{B}(s) = \frac{1}{n_{\text{test}}} \sum_{i=1}^{n_{\text{test}}} \left( \frac{W_i}{\pi(X_i)} - \frac{(1-W_i)}{1-\pi(X_i)} \right) C_i \mathbb{1}\{\hat{S}(X_i) \geq s\}
\]

\[
\hat{R}(s) = \frac{1}{n_{\text{test}}} \sum_{i=1}^{n_{\text{test}}} \left( \frac{W_i}{\pi(X_i)} - \frac{(1-W_i)}{1-\pi(X_i)} \right) Y_i \mathbb{1}\{\hat{S}(X_i) \geq s\},
\]

where \( \pi(X_i) = \mathbb{P}[W_i = 1 | X_i = x] \) is the treatment probability for units with \( X_i = x \) (in a uniformly randomized trial, \( \pi(x) = \pi \) would be constant); these are unbiased for \( B(s) \) and \( R(s) \) by the randomization of \( W_i \) [Imbens and Rubin, 2015]. We then plot the curve \((\hat{R}(S_{i_k}), \hat{B}(S_{i_k}))\) for \( k = 1, \ldots, n_{\text{test}} \), where \( S_{i_1} \leq \ldots \leq S_{i_{n_{\text{test}}}} \) are the ordered scores \( S(X_i) \) on the test set. Figures 1, 3 and 4 illustrate this approach in applications. The point at which this curve intersects the vertical line at \( x = b \) corresponds to an estimate of the lift that can be achieved with budget \( b \).

Finally, using the estimators for \( \hat{R}(s) \) and \( \hat{B}(s) \), we can also construct estimators for \( Q(b) \) and \( \Delta(b) \). Let \( s(b) = B^{-1}(b) \), and \( \hat{s}(b) \in \hat{B}^{-1}(b) \). Then, we can define estimators:

\[
\hat{Q}(b) = \hat{R}(\hat{s}(b)),
\]

\[
\hat{\Delta}(b) = \hat{Q}(b) - b \frac{\hat{R}(0)}{\hat{B}(0)}.
\]

In order to derive an inference strategy, our first result is that we can write \( \hat{Q}(b) - Q(b) \) in asymptotically linear form. Note that, given our assumption that scores lie between 0 and 1, \( B(0) \) and \( R(0) \) respectively denote the cost and benefit of treating everyone.

**Theorem 4.** *Under Assumption 1, if we have a scoring rule \( S : \mathcal{X} \rightarrow [0, 1] \) such that \( B(s) \) and \( R(s) \) are continuously differentiable in \( s \), and there is an approximate inverse in finite samples, i.e., \( \hat{B}(\hat{s}(b)) - b = o_p(n^{-0.5}) \), then*

\[
\hat{Q}(b) - Q(b) = \hat{R}(s(b)) - R(s(b)) - R'(s(b)) \left( \frac{\hat{B}(s(b))}{\hat{B}'(s(b))} - B(s(b)) - B(b) \right) + o_p\left(n^{-0.5}\right).
\]

(15)
Furthermore, \( \hat{Q}(b) \) and \( \hat{\Delta}(b) \) are asymptotically normal and have asymptotically linear representations:

\[
\sqrt{n} \left( \hat{Q}(b) - Q(b) \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi^q_i + o_p(1) \Rightarrow \mathcal{N}(0, \text{Var} \left[ \psi^q_i \right]),
\]

\[
\sqrt{n} \left( \hat{\Delta}(b) - \Delta(b) \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi^d_i + o_p(1) \Rightarrow \mathcal{N}(0, \text{Var} \left[ \psi^d_i \right]),
\]

where

\[
\psi^q_i = R_i(s(b)) - R(s(b)) - \frac{R'(s(b))}{B'(s(b))} (B_i(s(b)) - B(s(b))),
\]

\[
\psi^d_i = \psi^q_i - b \frac{R_i(0)}{B(0)} + b \frac{R(0)(B_i(0) - B(0))}{B(0)^2} + b \frac{R(0)}{B(0)}.
\]

\( R_i(s) = \left( \frac{W_i}{\pi} - \frac{1-W_i}{1-\pi} \right) C_i \mathbb{1}(S_i \geq s) \) and \( B_i(s) = \left( \frac{W_i}{\pi} - \frac{1-W_i}{1-\pi} \right) Y_i \mathbb{1}(S_i \geq s) \).

The asymptotic linear representation in Theorem 4 then implies that various resampling-based estimators [Efron, 1982] yield valid confidence intervals for \( \Delta(b) \) [Chung and Romano, 2013, Yadlowsky et al., 2021]. In particular, Lemma 12 of Yadlowsky et al. [2021] implies that the half-sample bootstrap will yield valid confidence intervals in this setting. We use this result to justify confidence intervals in our applications below. We emphasize that these confidence statements are conditional on the training set, i.e., we take the prioritization rules learned on the training set as given, and only quantify test set uncertainty in estimating the QINI curve. The continuous differentiability of \( B(s) \) and \( R(s) \) required for this result is satisfied in settings where the score \( S_i \) has a continuously differentiable distribution function and both \( \mathbb{E}[Y_i(1) - Y_i(0)|S_i = s] \) and \( \mathbb{E}[C_i(1) - C_i(0)|S_i = s] \) are continuous functions in \( s \).

Given a method for estimating the QINI curve in the setting with uncertain costs, we can also estimate the area under the QINI curve, known as the QINI coefficient. The QINI coefficient provides a single metric by which we can judge the performance of an allocation rule in budget-independent way. The QINI coefficient is the area between the estimated reward of the treatment allocation rule and the random treatment rule with the same cost, as the average budget ranges from 0 to the average cost of treating everyone in the sample,

\[
\text{QINI} = \int_{0}^{B(0)} \Delta(b)db
\]

The natural plug-in estimator for this quantity is \( \hat{\text{QINI}} = \int_{0}^{B^0} \hat{\Delta}(b)db \), where \( \hat{\Delta}(b) \) is as given above. We believe it plausible that the result from Theorem 4 can also be used extended to provide a central limit theorem for the QINI coefficient (see also the discussions in Yadlowsky et al. [2021]); however, we leave this question to further work.

5 Simulation Study

In order to understand numerical aspects of treatment allocation with uncertain costs, we conduct a simulation-based comparison of 7 methods for targeting. We consider 5 priority-based methods (i.e., that first rank units and then treat them according to the ranking until the budget runs out); for the first naïve method, this score is an estimate of the treatment
effect ignoring cost, whereas for the next 4 methods this score is an estimate of \( \rho(x) \) from (2) using different approaches detailed below. We also consider two direct optimization methods, proposed by Hoch et al. [2002] and Sun [2021], that are not priority based.

Details for each method are as follows. In all our experiments, there is no cost to withholding treatment (i.e., \( C_i(0) = 0 \)) and we have data from a randomized trial with \( \mathbb{P}[W_i = 1] = \pi \). All methods below will make use of these facts whenever appropriate; for example, causal forests allow the user to pass in values for conditional randomization probabilities (or propensity scores) \( \mathbb{P}[W_i = 1 \mid X_i] \), and in this case we pass the method the true randomization probabilities \( \pi \).

1) **Ignore Cost.** We ignore cost, and simply score observations using an estimate \( S_i = \hat{\tau}(x) \) of the treatment effect \( \tau(x) = \mathbb{E}[Y_i(1) - Y_i(0) \mid X_i = x] \). We estimate \( \hat{\tau}(x) \) using causal forests as implemented in the R-package `grf` [Athey, Tibshirani, and Wager, 2019, R Core Team, 2019].

2) **Direct Ratio Estimation \( \rho(x) \).** Our second baseline build on the characterization result from Theorem 1, but not on the connection to instrumental variables estimation from Proposition 2. We start by estimating \( \tau(x) \) using causal forests as above, and we also estimate the conditional cost function \( \gamma(x) = \mathbb{E}[C_i(1) - C_i(0) \mid X_i = x] = \mathbb{E}[C_i \mid X_i = x, W_i = 1] \) by using a regression forest from `grf` to predict \( C_i \) from \( X_i \) for treated units. Finally, we score observations using \( S_i = \rho_{dir}(X_i) = \hat{\tau}(X_i) / \hat{\gamma}(X_i) \).

3) **Direct Ratio Estimation with R-boost.** This method is analogous to the Direct Ratio method using `grf`. The difference is that we use `xgboost` instead to estimate \( \gamma(x) \) and \( \tau(x) \). To estimate the CATE model for \( \tau(x) \) using XGBoost we fit the XGBoost model on pseudo-outcomes \( \hat{Y}_i = \frac{Y_i - \hat{h}_y(X_i)}{W_i - \pi} \) with weights \( (W_i - \pi)^2 \), where \( \hat{h}_y(X_i) \) are out of bag predictions of `grf` model fitting \( \mathbb{E}[Y_i \mid X_i = x] \). This approach allows us to fit an R-learner loss with an XGBoost model, see Nie and Wager [2021] for a discussion of the R-learner approach. To estimate \( \gamma(x) = \mathbb{E}[C_i \mid X_i = x, W_i = 1] \) we use a regular XGBoost regression. For all XGBoost models we use a separately generated large sample to select the number of boosting steps.

4) **Proposed Method using Generalized Random Forests.** Our proposed method gets estimates \( S_i = \hat{\rho}(x) \) from an instrumental forest with “remapped” inputs. We call into the function instrumental forest in `grf`, except where the function expects an “instrument” we provide \( W_i \), and where the function expects a “treatment” we provide \( C_i \) (the covariates \( X_i \) and outcome \( Y_i \) are passed to the function as usual).

5) **Proposed Method using Linear IV.** We use Algorithm 1 to fit \( \hat{\beta} \), and we use \( \hat{\beta} \) to produce scores \( \hat{\rho}(X_i) = X_i'h \hat{\beta} \).

6) **Hoch et al. [2002].** The method predicts a linear combination of the reward and the cost \( m(x) = \mathbb{E}[Y_i(1) - Y_i(0) - \lambda(C_i(1) - C_i(0)) \mid X_i = x] \) for an appropriate choice of the coefficient \( \lambda \) to satisfy the budget constraint. An individual is treated whenever \( \hat{m}(x) > 0 \). We use a causal forest from the `grf` package to estimate \( m \). In practice, to meet a specific budget constraint, the \( \lambda \) parameter should be chosen by splitting the training dataset, which
can add additional noise to the estimates. For the simulation in Section 5, λ is chosen in advance to meet the budget constraint in expectation using a separate large sample of data from the data-generating process.

7) **Sun [2021]**. To address the budget violations of non-monotone decision rules, a researcher can choose to satisfy a more strict budget such that the probability of the violation of the total budget is small, which is proposed by Sun [2021]. We first estimate the standard deviation of the budget spent by Hoch et al. [2002] in simulation. Then we take a more strict budget threshold (1.96 standard deviations less) and run the method of Hoch et al. [2002] once again to select a different larger λ.

We first compare the above methods using a simple simulation study that highlights the behavior of the methods under consideration. For this experiment, we generate covariates and potential outcomes as follows with \( p = 12 \) (where left unspecified, variables are generated independently):

\[
\begin{align*}
X_{ij} &\sim \text{Unif} (-1, 1) \quad \text{for } j = 1, \ldots, p, \quad W_i \sim \text{Bern} (\pi), \quad \varepsilon_i \sim \mathcal{N} (0, 1), \\
Y_i (w) &= \max \{ X_{i1} + X_{i3}, 0 \} + \max \{ X_{i5} + X_{i6}, 0 \} + \omega e^{X_{i1} + X_{i2} + X_{i3} + X_{i4} + \varepsilon_i},
\end{align*}
\]

where Unif(\( a, b \)) is a uniform distribution on the interval \([ a, b ]\), \( \mathcal{N} (\mu, \sigma^2) \) is a Gaussian distribution with mean \( \mu \) and variance \( \sigma^2 \), and Bern(\( \pi \)) stands for the Bernoulli distribution with success probability \( \pi \). We also consider two settings for the cost \( C_i (1) \) of treating a unit: One baseline setting where cost is random but unpredictable, and another where cost can be anticipated in terms of covariates:

\[
\begin{align*}
\text{Unpredictable cost,} & \quad C_i (1) \mid X_i \sim \text{Pois}(1), \\
\text{Predictable cost,} & \quad C_i (1) \mid X_i \sim \text{Pois} \left( e^{X_{i2} + X_{i3} + X_{i4} + X_{i5}} \right),
\end{align*}
\]

where Pois(\( \mu \)) is a Poisson distribution with mean \( \mu \). We run both simulations on training sets of size \( n = 1,000 \) and with treatment randomization probability \( \pi = 0.5 \).

In order to evaluate the quality of these treatment rules, we consider results in terms of the QINI curve \( Q(b) \) described in Section 4 that maps different possible budget levels to the value we can get using the considered policy at this budget level. Figure 1 compares average test set performance of the different priority-based methods in terms of their QINI curves. In the left panel, with unpredictable costs, there is no visible difference between the four methods. This is as expected, as the optimal strategy is simply to prioritize units in decreasing order of \( \tau (x) = \mathbb{E} [ Y_i (1) - Y_i (0) \mid X_i = x] = e^{x_1 + x_2 + x_3 + x_4} \). In the second setting, however, there is a divergence between the treatment effect \( \tau (x) \) (which remains the same), and the cost-benefit ratio \( \rho (x) = e^{x_1 - x_5} \) we should use for prioritization, and this is reflected in the performance of different methods. Here, the “ignore cost” baseline is targeting the wrong objective, and so performs poorly. In the data generating process for this simulation, the priority score is not linear in the covariates, so the linear method does not perform very well, although it does slightly outperform the method that ignores costs entirely. The “direct ratio” baseline is targeting the correct objective and does better, but still does not match the performance of our proposed method which is designed to focus on \( \rho (x) \).

We note that, here, the function \( \tau (x) \) and \( \gamma (x) \) are somewhat aligned, and the induced cost-benefit ratio function \( \rho (x) = \tau (x) / \gamma (x) \) takes a simpler form than either \( \tau (x) \) or \( \gamma (x) \) on its own; specifically units with large values of \( x_2 \) or \( x_3 \) have large values of both \( \tau (x) \) and \( \gamma (x) \), and these effects cancel each other out. This type of structure may arise when there
Figure 1: QINI curves for the simulation settings (16) and (17), averaged over 100 simulation replicates. For each replicate, each method is trained on \( n = 1,000 \) samples. We generate the QINI curves using a shared test set of \( n_{\text{test}} = 10,000 \) samples that we do not regenerate across simulation replicates. For each test point \( i = 1, \ldots, n_{\text{test}} \) we use knowledge of the simulation design to compute the expected value of treating that sample \( \tau(X_i) = \mathbb{E} [Y_i(1) - Y_i(0) \mid X_i] \), and the expected cost \( \gamma(X_i) = \mathbb{E} [Y_i(1) \mid X_i] \).

Then, given any treatment rule derived from the training set, we rank the test set in decreasing order of the scores used by the treatment rule, and compute an estimate \( \bar{R}(S_{i_k}) = \frac{1}{n_{\text{test}}} \sum_{i=1}^{n_{\text{test}}} \tau(X_i) \mathbb{1}(S_i \geq S_{i_k}) \) and \( \bar{B}(S_{i_k}) = \frac{1}{n_{\text{test}}} \sum_{i=1}^{n_{\text{test}}} \gamma(X_i) \mathbb{1}(S_i \geq S_{i_k}) \) as cumulative sums along that ranking from \( i_1, \ldots, i_{n_{\text{test}}} \). The above displays are obtained by computing one such QINI curve for each simulation replicate, interpolating these QINI curves, and then (vertically) averaging the interpolated curves.

is some group of units that are overall just very responsive to treatment, in a sense where they both produce considerable value but also incur large costs; and instrumental forests are well positioned to take advantage of such structure as they can purely focus on fitting \( \rho(x) \). In other settings, where \( \tau(x) \) and \( \gamma(x) \) vary in more unrelated ways, the “direct ratio” baseline may also be a reasonable candidate for learning \( \rho(x) \).

Computing QINI curves for the methods from the related literature which are not priority-based is computationally difficult, since it requires resolving an optimization problem for each possible budget value in the curve. To compare the performance of the priority-based methods to those in the related literature, we describe results at a fixed budget constraint of 0.5 in Table 1. The instrumental forest has the highest lift at this budget level, while the method of Hoch et al. [2002] performs similarly well in terms of lift. The two direct ratio approaches perform slightly worse. However, since the method of Hoch et al. [2002] only meets the budget on the test set in expectation, it often violates the budget. The approach of Sun [2021] remedies this problem by ensuring the budget is met with high probability on the test set, rather than in expectation, but comes at the cost of performance. In contrast, the priority-based methods always spend the correct budget on the test set.
Table 1: The table shows the performance of different methods in the partially predictable costs simulation, under a budget constraint of 0.5. The table shows the estimated lift of the reward over the uniform allocation $\Delta$ from a sample of 1,000 individuals, averaged over 500 simulation replicates. The next column shows the half-sample bootstrapped (1,000 bootstrap samples) standard deviation of $\hat{\Delta}$ averaged across 500 simulation replications and the coverage of the $(\hat{\Delta} - 1.96 \text{se}(\hat{\Delta}), \hat{\Delta} + 1.96 \text{se}(\hat{\Delta}))$ confidence interval, where the ground truth was computed via simulation. Standard errors for direct optimization methods are not currently available in the literature. We also report the average budget spent and its standard deviation, as well as the percentage of simulation replicates for which the budget spent in the test set is higher than 0.5.

Furthermore, confidence intervals for the lift computed using the bootstrap have coverage close to 0.95 for the priority-based methods, as expected from the results in Section 4.

In practice, a budget for a campaign may be increased after an initial sample of individuals is already treated. Figure 2 shows that in finite samples the approach based on Hoch et al. [2002] is non-monotonic in the budget. Some individuals who are treated for lower levels of $\lambda$ are dropped at higher budget levels. This makes it infeasible to roll out the treatment by repeatedly applying the optimization procedure as the budget increases. In contrast, for a priority-based rule, an individual treated at budget $B$ is always treated at $B'$ for $B' \geq B$.

6 Example Applications

We further investigate our proposed random forest algorithm by deploying on two applications: the Oregon Health Insurance Experiment and a marketing problem. We also compare our approach to baselines #1 and #2 from the simulation study (ignore costs and direct ratio estimation). In the Oregon Health Insurance Experiment, the treatment propensity depends on the household size $H \in \mathbb{Z}$, $\mathbb{P}[W_i = 1 | H_i = h] = \pi(h)$, and so we also need to carry out the propensity estimation component to our algorithm. The marketing application is a randomized experiment so we can deploy all approaches like in the simulation study.

6.1 Oregon Health Insurance Experiment

In 2008, Oregon conducted a lottery for a limited number of spots in its Medicaid program [Finkelstein et al., 2018, 2012]. The authors enriched the data on lottery signups with surveys and administrative data and found positive effects of health insurance on self-reported health outcomes, health care utilization, and financial well-being. This dataset allows us to analyze how a government might optimize a self-reported health outcome under a constraint on Medicaid expenses, for example which depend on the utilization of health services.
For the purpose of our method, the target “reward” variable $Y_i$ is self-reported health, which we encode as a binary variable, where 1 maps to ‘good’, ‘very good’ or ‘excellent’ and 0 maps to ‘bad’ or ‘fair’. Meanwhile, we consider two possible “cost” variables $C_i$: the number of outpatient visits in the treatment group $C_i$, and the number of prescribed drugs in the treatment group. We consider the costs $C_i$ to be non-zero in the treatment group, since we consider our constraint to be on the resources used in the Medicaid expansion.

The baseline survey includes all of the lottery winners as well as an approximately equal amount of lottery losers, which amounts to an initial sample of 58,405 lottery subscribers. 23,777 subjects completed the endline survey in 12 months after the baseline, allowing us to measure the outcome variables. A few hundreds observations are also lost because of incomplete answers in the endline survey, leaving us with a sample of 18,062 when prescribed medications is the cost variable and 23,119 when outpatient visits is the cost variable. Finkelstein et al. [2012] check the balance of covariates in their paper and argue that the attrition is balanced across treatment groups and doesn’t invalidate the experiment. We split the sample equally into a training set and a testing set, stratifying the split on the number of household members and the assigned treatment.

Medicaid applies to all family members, while the lottery registrations are individual, therefore the chances of winning are confounded with the household size $H \in \mathbb{Z}$, i.e., members of larger households have a better chance of getting treated; so, we also estimate the propensity score $\pi(h) = \mathbb{P} [W_i = 1 \mid H_i = h]$. We use the short demographic characteristics from the registration form, emergency department visits history and the baseline survey data on demographics, employment, health conditions and past doctor visits to build the model $\hat{\rho}(X_i)$ of health improvement per resource usage. We drop some variables from the baseline survey, which could be affected (or are shown in the paper to be affected) by the treatment. The purpose of this example is to demonstrate the method, therefore we are...
Figure 3: QINI curves for the Oregon Health Insurance Experiment described in Section 6.1. The total sample size for the left figure is 18062 and 23119 for the right figure, split equally into the test and train samples.

using all of the available pretreatment information in learning $\hat{\rho}$. The full list of variables is included in the Appendix C.

We build the QINI curve $\hat{Q}(b)$ in the same way we did in the previous examples; however, to improve robustness due to using estimated propensity scores $\hat{\pi}(H_i)$, we use a doubly robust adaptation of $\hat{B}(s)$ following Yadlowsky et al. [2021]. Results are shown in Figure 3. In this application, both the instrumental forest and the direct ratio baseline have a comparable performance and both noticeably outperform the baseline “ignore costs” in the case we use the number of prescribed medications as a cost variable. This result is also robust to alternative choices of the reward variable, e.g., interpreting “fair” as a good health state.

Finally, we also present the estimated lifts $\Delta(1)$ for a chosen budget of 1 prescribed medication or 1 outpatient visit per person. We estimate standard errors using a bootstrap clustered at the household level. Results are presented in the Table 2. The instrumental forest and the direct ratio methods significantly outperform a random choice rule. Conversely, the baseline that ignores costs doesn’t give a statistically significant lift for this budget level. Quantitatively, if we have budget that allows us to prescribe on average 1 medication per patient among new medicaid enrollees, then targeting using instrumental forests lets us improve the % of healthy individuals from 2.3% to 3.4%. To summarize Figure 3 using a single metric, we also report the QINI coefficient, as defined in Section 3, in Table 4 of Appendix C. The metric shows that the instrumental forest performs roughly equivalent to the direct ratio method and vastly outperforms a treatment allocation policy that ignores costs.

4When deploying a method of this type in practice, one would need to audit the covariates used for equity, social and ethical concerns, as well as gameability; see Athey and Wager [2021] and Kitagawa and Tetenov [2018] for further discussion.
| Cost variable:       | Medications | Outpatient visits |
|---------------------|-------------|------------------|
|                     | (1)         | (2)              |
| IV                  | 0.0107**    | 0.0093***        |
|                     | (0.0052)    | (0.0044)         |
| Direct ratio        | 0.0110**    | 0.0068*          |
|                     | (0.0053)    | (0.0044)         |
| Ignore cost         | 0.0012      | 0.0035           |
|                     | (0.0048)    | (0.0045)         |
| \(\hat{Q}(1)\) under a uniform rule | 0.0232*** | 0.0253***        |
|                     | (0.0043)    | (0.0045)         |

Table 2: Additional lift \(\Delta(1)\) relative to random choice, for different prioritization rules and cost variables and bootstrapped standard deviations for them. We also include the \(\hat{Q}(1)\), i.e. the total reward under a budget constraint of 1 under the random choice rule, for reference. The standard deviations are in parentheses and are clustered at the household level. The stars denote confidence levels: *\(p<0.1\); **\(p<0.05\); ***\(p<0.01\).

6.2 Marketing Application

Finally, we turn to the problem of optimizing a user engagement campaign at a “sharing economy” company. For confidentiality reasons, we cannot describe the application in detail. At a high level, however, the campaign faced the same trade-offs as the ones described in Example 1 in the introduction. The dataset has \(p=39\) pre-treatment covariates that can be used for targeting, and treatment \(W_i\) was randomized with probability \(\pi = 0.5\).

For the purpose of our experiment, we randomly split our dataset into a training set of size \(n = 50,000\) and a test set of size \(n_{test} = 500,000\). We then trained all 3 methods under consideration on on the training set, and compare their QINI curves on the test set. Results shown in the left panel Figure 4 mirror those in our simulation study, except now our method outperforms the “direct ratio” baseline by a larger margin than before, while the “ignore cost” baseline results in slightly worse performance than random treatment choices. We also note that the difference in rewards attained by the three methods are statistically significant. Given a budget of \(B = 0.2\), the targeting rule learned with an instrumental forest achieves a value of 0.35 ± 0.03, with 95% confidence intervals obtained via the bootstrap. In other words we can expect to get roughly 35%±3% of the rewards from targeting everyone by only spending 20% of the budget needed to target everyone. In comparison, the 95% confidence interval of the \(B = 0.2\) value for the “direct ratio” baseline is 0.28±0.03, while for the “ignore cost” baseline it is 0.21 ± 0.04. Furthermore, a McNemar-type paired bootstrap yields a
QINI curve

cost vs. gain

Figure 4: QINI curves for the marketing application described in Section 6.2, with all methods trained on $n = 50,000$ samples. The left panel shows a QINI curve estimated via inverse-propensity weighting on a test set of size $n = 500,000$, as described in the text. The right panel shows a scatterplot of test set observations where, on the $x$-axis we show $\hat{\gamma}(X_i)$, while on the $y$-axis we show the reward implied by the instrumental forest method, i.e., $\hat{\tau}_{instr}(X_i) = \hat{\rho}(X_i)\hat{\gamma}(X_i)$. We re-scale cost and rewards so that $E[\gamma(X_i)] = E[\tau(X_i)] = 1$, i.e., the axes in both above displays are unit free.

95% confidence interval of $0.07 \pm 0.03$ for the value difference from using the treatment rule learned using instrumental forests versus the direct ratio baseline, and an associated $p$-value of $5 \times 10^{-6}$.

The right panel of Figure 4 provides further insight into the data-generating distribution. As in our simulation study, we see that there is considerable alignment between the estimated costs and rewards of treating any unit. Thus—assuming these estimates are accurate—a good treatment rule should prioritize units that are above the diagonal to those who are below it. The larger observed difference in performance between the “direct ratio” baseline and our proposed method relative to that seen in the simulation study may reflect the instrumental forest being able to better leverage a large sample size when dealing with a more complex statistical setting.

7 Discussion

In this paper, we considered the problem of optimally prioritizing (or targeting) a treatment under budget constraints, while allowing the cost of treating different people to be both variable and uncertain. Problems with this structure appear frequently in medicine, marketing, and other areas; however, with a handful of exceptions, this setting has not been a focus of the existing literature on data-driven decision making. Here, we derived the form of the optimal prioritization rule using the solution of Dantzig [1957] to the fractional knapsack problem, and established a statistical connection to the problem of heterogeneous treatment effect estimation with instrumental variables that allowed us to develop a number of estima-
tors for the optimal prioritization rule, including one that re-purposes off-the-shelf random forest software from Athey, Tibshirani, and Wager [2019]. In the simulation and empirical applications, the proposed approach shows considerable promise in helping us effectively learn whom to prioritize for treatment.

It is interesting to consider generalizations of our approach to a wider class of treatment targeting problems. In some applications, the decision maker may wish to add further constraints on how budget is allocated. For example, they may impose a constraint that the average per-person spending must be the same across a number of pre-specified protected groups. Extending our approach to allow for budget constraints across non-overlapping groups is straightforward; and one can derive the optimal treatment allocation by first computing the target budget level for each group, and then applying Theorem 1 separately to each group. On the other hand, our approach does not extend as directly to settings with budget constraints that apply to overlapping groups, and studying such settings would be an interesting topic for further work.

With multiple treatments, it is still possible to estimate incremental conditional benefit-cost ratios for each treatment and each individual in the sample. However, since there are multiple ratios for each individual, a priority-based approach no longer follows directly from the estimation of the ratios. Further work is needed to construct a priority-based approach that solves the multiple treatment problem with uncertain costs and benefits.

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A Targeting using a Subset of Confounders

The main analysis assumes that the set of pre-treatment covariates and set of confounders coincide. In this section we extend the analysis to the case, when the set of confounders \( \chi^{(\text{conf})} \) contains the set of pre-treatment covariates \( \chi \). This affects how the priority score is defined through observable moments and how it is estimated. We first derive an analog of the Proposition 2, expressing \( \rho(x) \) in terms of observable moments:

**Proposition 5.** In the setting of Theorem 1, suppose further more that the treatment assignment mechanism is unconfounded,

\[
\{Y_i(0), Y_i(1), C_i(0), C_i(1)\} \perp W_i \mid X_i^{(\text{conf})},
\]

(18)

also assume strict overlap, i.e. there exists \( \eta > 0 \), such that

\[
h_w(X_i^{(\text{conf})}) = \mathbb{E}\left[ W_i \mid X_i^{(\text{conf})} \right] \in (\eta, 1 - \eta)
\]

(19)

Then,

\[
\rho(x) = \frac{\mathbb{E}\left[ \text{Cov}\left[ Y_i, W_i \mid X_i^{(\text{conf})} \right] \mid X_i = x \right]}{\mathbb{E}\left[ h_w(X_i^{(\text{conf})}(1 - h_w(X_i^{(\text{conf})}))) \mid X_i = x \right]}
\]

(20)

Now, having a population analog of the priority score, we can estimate it either under a linearity assumption \( \rho(x) = x^T \beta \), or non-parametrically using generalized random forest. The Equation 20 implies that estimators will need to take the following two steps. First, fit nuisance parameters \( h \) using the whole set of confounders. Then, fit the model for scores using weights \( (h_w(X_i^{(\text{conf})})(1 - h_w(X_i^{(\text{conf})})))^{-1} \).

Consider first the non-parametric instrumental forest based estimation. To estimate 20 we need to adjust the instrumental forest based formula for \( \hat{\rho}(x) \).

\[
\hat{\rho}(x) = \frac{\sum_{i=1}^{n} \alpha_i(x)(h_w(X_i^{(\text{conf})})(1 - h_w(X_i^{(\text{conf})})))^{-1}\left( Y_i - Y_a(X_i^{(\text{conf})}) \right) \left( W_i - W_a(X_i^{(\text{conf})}) \right)}{\sum_{i=1}^{n} \alpha_i(x)(h_w(X_i^{(\text{conf})})(1 - h_w(X_i^{(\text{conf})})))^{-1}\left( C_i - C_a(X_i^{(\text{conf})}) \right) \left( W_i - W_a(X_i^{(\text{conf})}) \right)},
\]

\[
\hat{h}_y(x) = \sum_{i=1}^{n} \alpha_i(x)Y_i, \quad \hat{h}_c(x) = \sum_{i=1}^{n} \alpha_i(x)C_i, \quad \hat{h}_w(x) = \sum_{i=1}^{n} \alpha_i(x)W_i.
\]

(21)

It is possible to implement this estimator using a weighted instrumental forest from the \texttt{grf} package:

1. Estimate each of \( h_y, h_c, h_w \) with a regression forest using the whole set of confounders

2. Using an instrumental forest, pass \( \hat{h}_y \) as the expected outcome, \( \hat{h}_c \) as treatment propensities parameter, and \( \hat{h}_w \) as instrument propensities parameter, \( (\hat{h}_w(X_i^{(\text{conf})})(1 - \hat{h}_w(X_i^{(\text{conf})})))^{-1} \) as sample weights, and finally, \( Y \) as an outcome, \( C \) as a treatment variable and \( W \) as an instrument.
To estimate the score under linearity assumption $\rho(x) = x^\prime \beta$, we can rewrite the Equation 7 as:

$$e_i(\beta, h(X_i)) = w(X_i^{(conf)}) (W_i - h_w(X_i^{(conf)})) \left[(Y_i - h_y(X_i^{(conf)})) - (C_i - h_c(X_i^{(conf)})) \beta \right],$$

$$h_w(x) := E \left[ W_i | X_i^{(conf)} = x \right], \quad h_y(x) := E \left[ Y_i | X_i^{(conf)} = x \right], \quad h_c(x) := E \left[ C_i | X_i^{(conf)} = x \right]$$

$$w(x) = (1 - h_w(x)) h_w(x)^{-1}.$$  \hspace{1cm} (22)

We now once again define $\hat{\beta}$ as an output of the Algorithm 1 using the score function (22) (the 2SLS estimate is weighted with $w(X_i^{(conf)})$). This estimate is a consistent estimate of $\beta$ and it is asymptotically normal, which is stated in a Theorem analogous to the Theorem 3.

**Theorem 6.** Under the Assumptions of Proposition 5, suppose furthermore that Assumption 2 and Assumption 3 hold. Then, our estimator $\hat{\beta}$ described above satisfies

$$\sqrt{n} \left( \hat{\beta} - \beta \right) \Rightarrow N(0, V_\beta),$$

$$V_\beta = E[w(X_i^{(conf)})V_iD_iX_iX_i^\prime]^{-1} E[w^2(X_i^{(conf)})U^2V_iX_iX_i^\prime] E[w(X_i^{(conf)})V_iD_iX_iX_i^\prime]^{-1}.$$

Finally, note that when the set of confounders is a subset of the pre-treatment covariates used for targeting, no adjustments to the original method are needed. The Oregon Health Experiment, considered in the empirical part of the paper shows one example of it: the treatment is confounded only by the size of the household, while we use many more pre-treatment covariates for targeting.

**B Proofs**

**Proof of Theorem 1**

To ease the presentation, we first define the conditional average treatment effect function for both rewards and costs as

$$\delta_C(x) = E[C(1) - C(0) \mid X = x], \quad \delta_Y(x) = E[Y(1) - Y(0) \mid X = x].$$

Because $C_i(1) \geq C_i(0)$ almost surely, we see that $\beta(\rho) = E[I \{ \rho(X) > \rho \delta_C(X) \}$ is a non-increasing function of $\rho$. Let

$$\eta_B := \inf \{ \rho : \beta(\rho) \leq B \}, \quad \rho_B = \max \{ \eta_B, 0 \}.$$  \hspace{1cm} (23)

The claimed optimal (stochastic) decision rule in (3) can then be rewritten as

$$\pi^*_B(x) = \begin{cases} a_B & \text{if } \rho(x) = \rho_B, \\ 1 & \text{if } \rho(x) > \rho_B, \end{cases}$$

where

$$a_B = \begin{cases} 0 & \text{if } E[I \{ \rho(X) = \rho_B \} \delta_C(X)] = 0, \\ \min \left\{ \frac{B - E[I \{ \rho(x) > \rho_B \} \delta_C(x)]}{E[I \{ \rho(x) = \rho_B \} \delta_C(x)]}, 1 \right\} & \text{if } E[I \{ \rho(X) = \rho_B \} \delta_C(X)] > 0. \end{cases}$$

(24)
Note that $\pi^*_B(x)$ and $I\{\rho(x) > \rho_B\}$ are almost surely equal if $P[\rho(X_i) = \rho_B] = 0$ or if $\eta_B < 0$, and they should return the same decision in these settings. Moreover, $E[\pi^*_B(x)\delta_C(x)] = B$ if $\rho_B > 0$.

To verify that the above rule is in fact optimal, let $r(X)$ denote any other stochastic treatment rule which satisfies the budget constraint $B$. It remains to argue that

$$E[\delta_Y(X)\pi^*_B(X)] \geq E[\delta_Y(X)r(X)],$$

i.e., that $r(X)$ cannot achieve higher rewards than $\pi^*_B$ while respecting the budget. From now on, we assume that $\delta_C(X) > 0$ almost surely, i.e., that there are no units that are free to treat in expectation; because if there are units with $\delta_C(X) = 0$ then clearly one should just treat them according to the sign of $\delta_Y(X)$ (as is done by our policy), and this has no budget implications. Given this setting, we see that

$$E[\delta_Y(X)(\pi^*_B(X) - r(X))] = E[\rho(X)\delta_C(X)(\pi^*_B(X) - r(X))]$$

$$\geq \rho_B E[\delta_C(X)(\pi^*_B(X) - r(X))],$$

where the inequality follows by observing that, by definition of $\pi^*_B$, we must have $\pi^*_B(X) - r(X) \geq 0$ whenever $\rho(x) > \rho_B$ and $\pi^*_B(X) - r(X) \leq 0$ whenever $\rho(x) < \rho_B$.

We conclude by considering two cases: Either $\rho_B > 0$ or $\rho_B = 0$. In the first case, we know that $\pi^*_B$ spends the whole budget, i.e., $E[\delta_C(X)\pi^*_B(X)] = B$; thus, by the budget constraint on $r(X)$ (i.e., $E[\delta_C(X)r(X)] \leq B$), we see that $E[\delta_Y(X)(\pi^*_B(X) - r(X))] \geq 0$.

Meanwhile, in the second case, the lower bound in (25) is 0, and so our conclusion again holds. Finally, by an extension of the same argument, we see that when $P[\rho(X_i) = \rho_B] = 0$, our policy $\pi^*_B(x)$ is almost surely equivalent to $I\{\rho(x) > \rho_B\}$, and is both deterministic and the unique reward-maximizing decision rule that respects the budget constraint.

**Proof of Proposition 2**

In this section, we show the equation (5) in Proposition 2. Assume $W \in \{0, 1\}$ and let $e(x) := P[W = 1|X]$. Notice that

$$\text{Cov}[Y, W | X] = E[Y|W | X] - E[Y | X]E[W | X]$$

$$= E[Y(1)W | X] - E[Y | X]E[W | X]$$

$$= e(X)E[Y(1)|X] - e(X)^2E[Y(1)|X] + e(X)(1 - e(X))E[Y(0)|X]$$

$$= e(X)(1 - e(X))\{E[Y(1)|X] - E[Y(0)|X]\}$$

$$= e(X)(1 - e(X))\delta_Y(X),$$

where the second equality comes from the consistency assumption that $Y = WY(1) + (1 - W)Y(0)$ and the third equality comes from the unconfounded assumption (4). Similarly, we can show that

$$\text{Cov}[C, W | X] = e(X)(1 - e(X))\delta_C(X)$$
and thus
\[
\begin{align*}
\text{Cov} \left[ Y, W \mid X = x \right] &= e(x) \left\{ 1 - e(x) \right\} \delta_Y(x) \\
\text{Cov} \left[ C, W \mid X = x \right] &= e(x) \left\{ 1 - e(x) \right\} \delta_C(x) \\
\text{Cov} \left[ C, W \mid X = x \right] &= \frac{\delta_Y(x)}{\delta_C(x)} \\
&= \rho(x),
\end{align*}
\]

which completes the proof of the Proposition 2.

**Derivation of Equation 8**

When \( \rho(x) = x' \beta \), then Equation 5 is equivalent to
\[
\begin{align*}
x' \beta &= \text{Cov} \left[ Y_i, W_i \mid X_i = x \right] \\
&= \frac{\text{Cov} \left[ Y_i, W_i \mid X_i = x \right]}{\text{Cov} \left[ C_i, W_i \mid X_i = x \right]}
\end{align*}
\]

Using the definition of conditional covariance, and rearranging, we have that
\[
\begin{align*}
E[(W_i - h_w(X_i))(C_i - h_c(X_i))|X_i = x] &= E[(W_i - h_w(X_i))(Y_i - h_y(X_i))|X_i = x] \\
0 &= E[(W_i - h_w(X_i))(Y_i - h_y(X_i))|X_i = x] - E[(W_i - h_w(X_i))(C_i - h_c(X_i))X'_i |X_i = x] \\
0 &= E[(W_i - h_w(X_i))(Y_i - h_y(X_i)) - (C_i - h_c(X_i))X'_i |X_i = x]
\end{align*}
\]

This is equivalent to
\[
E[e_i(\beta, h(X_i))|X_i = x] = 0
\]

**Proof of Theorem 3**

We will use Theorem 3.1 and Theorem 3.2 of Chernozhukov et al. [2018], therefore we need to verify the Assumptions 3.1 and 3.2 from the paper, which will complete the proof.

From (10), we have that \( \beta \) satisfies the following unconditional moment restriction
\[
E[\psi_i(\beta, h(X_i))] = 0
\]

where the score function is
\[
\psi_i(\beta, h(X_i)) = U_i V_i = [(W_i - h_w(X_i))X_i][Y_i - h_y(X_i) - (C_i - h_c(X_i))X'_i |X_i = x] = \psi_i(0)(h(X_i)) + \psi_i(1)(h(X_i)) \beta.
\]

We have that the score function is linear in \( \beta \). This verifies Assumption 3.1b) of Chernozhukov et al. [2018]. To apply the Theorem, we must verify the remaining components of Assumption 3.1 and Assumption 3.2.

3.1a) is satisfied, since Equation 5 is equivalent to \( E[\psi_i(\beta, h(X_i))]|X_i = x] = 0 \) under the linearity assumption for \( \rho(x) \).

3.1c) is satisfied, since the score function is linear in both \( \beta \) and the nuisance parameters, it is twice differentiable in the nuisance parameters. For 3.1d), we show Neyman-Orthogonality by showing that the partial derivative, evaluated at zero, of the conditional
moment restriction with respect to each component of a perturbation of the nuisance function.

\[
\frac{\partial \mathbb{E}[\epsilon_i(\beta, h(x) + \epsilon\delta(x))|X_i = x]}{\partial \epsilon_y} \bigg|_{\epsilon=0} = -\delta_y(x)\mathbb{E}[W_i - \mathbb{E}[W_i|X_i = x]|X_i = x] = 0
\]

Similarly,

\[
\frac{\partial \mathbb{E}[\epsilon_i(\beta, h(x) + \epsilon\delta(x))|X_i = x]}{\partial \epsilon_c} \bigg|_{\epsilon=0} = X'\delta_c(x)\mathbb{E}[W_i - \mathbb{E}[W_i|X_i = x]|X_i = x] = 0.
\]

Lastly,

\[
\frac{\partial \mathbb{E}[\epsilon_i(\beta, h(x) + \epsilon\delta(x))|X_i = x]}{\partial \epsilon_w} = -\delta_w(x)\mathbb{E}[Y_i - \mathbb{E}[Y_i|X_i = x]|X_i = x] + X'\delta_w(x)\mathbb{E}[C_i - \mathbb{E}[C_i|X_i = x]|X_i = x] = 0.
\]

For 3.1e), we need \( \mathbb{E}[V_iD_iX_iX'_i] \) is invertible and

\[
\mathbb{E}[V_iD_iX_iX'_i] = \mathbb{E}[(W_i - \mathbb{E}[W_i|X_i])(C_i - \mathbb{E}[C_i|X_i])X_iX'_i]
\]

\[
= \mathbb{E}_x[\mathbb{E}[(W_i - \mathbb{E}[W_i|X_i])(C_i - \mathbb{E}[C_i|X_i])X_iX'_i|X_i]]
\]

\[
= \mathbb{E}_x[X_iX'_i\text{Cov}(C_i, W_i|X_i)].
\]

Since we are in the setting of Proposition 2 and unconfoundedness applies as well as the overlap condition \( 0 < \epsilon(x) < 1 \), we have that

\[
\text{Cov}(C_i, W_i|X_i) = \epsilon(x)(1 - \epsilon(x))\mathbb{E}[C_i(1) - C_i(0)|X_i = x] > 0,
\]

where the inequality is from Assumption 1. Then, \( \mathbb{E}[V_iD'_i] \) is invertible as long as \( \mathbb{E}[X_iX'_i] \)

is full rank, which is by assumption, further, the singular vectors of \( X_iX'_i \) are bounded from above, since \( X'_i \) are bounded.

We now verify the assumptions 3.2a) through c). The point is to show various bounds on \( \psi^{(1)} \) and \( \psi \) defined in (28) with constants \( a, A \) and a sequence \( \delta_n \), featuring in the Assumptions 1, 2, 3.

We will first introduce here additional notation: For a vector or a matrix \( ||A|| \) means some vector (matrix) norm. For a stochastic matrix or a vector \( ||A||_q \) means a \( q \) norm: \( \mathbb{E}[\sum_{i,j} A^{ij}_q]^{1/q} \). Also \( 1_m \) means a column vector of 1 of a size \( m \).

**Useful Inequalities.** Before turning to the verification of the assumptions, we will derive some useful bounds, which are used throughout the proof.

Throughout all of the derivations we will use the following inequalities (for any \( p < q \), which hold by Assumptions 2 and 3:

\[
||W - \hat{h}_w(X)||_p \leq ||W - \hat{h}_w(X)||_q \leq ||h_w(X) - \hat{h}_w(X)||_q + ||W||_q + ||h_w(X)||_q \leq 3A
\]

\[
||C - \hat{h}_c(X)||_p \leq ||C - \hat{h}_c(X)||_q \leq ||h_c(X) - \hat{h}_c(X)||_q + ||C||_q + ||h_c(X)||_q \leq 3A
\]

\[
||Y - \hat{h}_y(X)||_p \leq ||Y - \hat{h}_y(X)||_q \leq ||h_y(X) - \hat{h}_y(X)||_q + ||Y||_q + ||h_y(X)||_q \leq 3A.
\]
We can replace the RHS by $2A$ if we have a population version of $h(X)$ on the left hand side, which will be useful for derivation of a bound on $\beta$.

Now we also derive a bound on $\beta$. We make use of Assumption 1. Also, we use the assumption that the matrix $\mathbb{E}[XX']$ is invertible, therefore it’s singular values are bounded from below. Assume that the constant $a$ is low enough to be a valid bound for singular values. We also use the fact that $1_m^T1_m = m$, so $\mathbb{E}(XX')^{-1}1_m \leq 1_m a^{-2}m$:

$$\beta = \mathbb{E}(XX'(W - h_w(X))(C - h_c(X)))^{-1}\mathbb{E}(X'(W - h_w(X))(Y - h_y(X)))$$

$$\leq \mathbb{E}(XX'a)^{-1}mA\|W - h_w(X)\|_2\|Y - h_y(X)\|_2$$

$$\leq \mathbb{E}(XX'a)^{-1}mA\|W - h_w(X)\|_q\|Y - h_y(X)\|_q$$

$$\leq 1_ma^{-2}mA^4.$$  

(30)

We can also bound $X$. This will give me the following related bounds:

$$\|XX'|\|_\infty \leq m^2A^2$$

$$\|XX'\beta\|_\infty \leq 4a^{-2}m^3A^5$$

Since $X$ is bounded, for any scalar random variable $\xi$: $\|X\xi\|_p \leq mA\|\xi\|_p$

Verifying assumptions 3.2a) of Chernozhukov et al. [2018] Let the realization set $T_N$ be the set of estimates satisfying the conditions in the Assumption 2. Establishing the bounds bellow we will consider $\hat{h}$ functions from this realization set.

Verifying Assumptions 3.2b) of Chernozhukov et al. [2018] The goal is to establish an upper bound on $\langle E[\|\psi(\beta, \hat{h})\|^{q/2}]\rangle^{2/q}$ and $\langle E[\|\psi^{(1)}(\hat{h})\|^{q/2}]\rangle^{2/q}$.

We use the Hölder inequality, a bound on $\|XX'|\|_\infty$ and the previously derived bounds to to derive a bound on $\langle E[\|\psi^{(1)}(\hat{h})\|^{q/2}]\rangle^{2/q}$:

$$\left( E\left[\|\psi^{(1)}(\hat{h})\|^{q/2}\right]^2\right)^{2/q} = \|XX'(C - \hat{h}_c(X))(W - \hat{h}_w(X))\|_q^{2/q} \leq \|XX'|\|_\infty\|C - \hat{h}_c(X)\|_q\|W - \hat{h}_w(X)\|_q \leq m^2A^29A^2 = 9mA^4.$$  

Now we will re-use the bound above to verify the second equation of Assumption 3.2b) of Chernozhukov et al. [2018]. We also use the established bound on $\beta$ (30):

$$(E[\|\psi^{(2)}(\beta, \hat{h})\|^{r/2}]^{2/q} = \|\psi(\beta, \hat{h})\|_q^{r/2}$$

$$= \|X(W - \hat{h}_w(X))(Y - \hat{h}_y(X) - X'(C - \hat{h}_c(X))\beta)\|_q^{r/2}$$

$$\leq \|X(Y - \hat{h}_y(X))(W - \hat{h}_w(X))\|_q^{r/2} + \|XX'\beta(C - \hat{h}_c(X))(W - \hat{h}_w(X))\|_q^{r/2}$$

$$\leq mA\|Y - \hat{h}_y(X)\|_q\|W - \hat{h}_w(X)\|_q + 4a^{-2}m^3A^5\|(C - \hat{h}_c(X))(W - \hat{h}_w(X))\|_q^{r/2}$$

$$\leq 9mA^3 + 36a^{-2}m^3A^7.$$  

Therefore we established an upper bound on $\langle E[\|\psi(\beta, \hat{h})\|^{q/2}]\rangle^{2/q}$ and $\langle E[\|\psi^{(1)}(\hat{h})\|^{q/2}]\rangle^{2/q}$ as required by the assumption.
Verifying Assumptions 3.2c) of Chernozhukov et al. [2018]. Here we need to show the convergence to 0 of $\|E[\psi(\beta)\hat{h}] - E[\psi(\beta)h]\|$, $(E[\|\psi(\beta)\hat{h} - \psi(\beta)h\|^2])^{1/2}$ and $\sqrt{n}\|\partial^2 E[\psi(\beta, h + r(\hat{h} - h))]|$.

For the first equation we use boundedness of $X$, Assumption 2 and the bounds (29):

\[
\|E[\psi(\beta)\hat{h}] - E[\psi(\beta)h]\| = \|E[(W - \hat{h}_w(X))(C - \hat{h}_c(X))XX' - (W - h_w(X))(C - h_c(X))XX']\|
\leq m^2A^2\|(W - \hat{h}_w(X))(C - \hat{h}_c(X)) - (W - h_w(X))(C - h_c(X))\|_1
\leq m^2A^2\|(h_w(X) - \hat{h}_w(X))(C - h_c(X))\|_1
\leq m^2A^2\|(h_w(X) - \hat{h}_w(X))(h_c(X) - \hat{h}_c(X))\|_1
\leq m^2A^2\|(h_w(X) - \hat{h}_w(X))\|_2\|C - h_c(X)\|_2
\leq 4m^2A^3\delta_n + 2m^2A^3\delta_n/\sqrt{n}.
\]

Deriving the next inequality, we use the boundedness of conditional variance of $V$ and $U$, the fact that $\|\hat{h}_w(X) - h_w(X)\|_\infty$ is less than 1 (both $\hat{h}_w(X)$, $h_w(X)$ map into $[0,1]$), and the bounds on $X$ and $\beta$:

\[
(E[\|\psi(\beta, \hat{h}) - \psi(\beta, h)\|^2])^{1/2} = \|X(W - \hat{h}_w(X))(Y - \hat{h}_y(X) - (C - \hat{h}_c(X))X'\beta) - X(W - h_w(X))(Y - h_y(X) - (C - h_c(X))X'\beta)\|_2
\leq mA\|h_w(X) - \hat{h}_w(X)\|U\|_2 + mA\|h_y(X) - \hat{h}_y(X)\|V\|_2 + 4a^{-2}m^3A^5\|h_c(X) - \hat{h}_c(X)\|V\|_2
\leq A^2m\|h_w(X) - \hat{h}_w(X)\|_2 + 2A^2m\|h_y(X) - h_y(X)\|_2 + 4a^{-2}m^3A^5\|h_c(X) - \hat{h}_c(X)\|_2
\leq (2A^2m + 4a^{-2}m^3A^5 + mA + 4a^{-2}m^3A^5)\delta_n
\]

Finally, let

\[f(r) = E\left[X(U - r(\hat{h}_y(X) - h_y(X)) + r(\hat{h}_c(X) - h_c(X))X'\beta)(V - r(\hat{h}_w(X) - h_w(X))\right].\]

The derivative:

\[
\partial f(r) = E\left[X(\hat{h}_y(X) - h_y(X))(V - r(\hat{h}_w(X) - h_w(X))\right]
\leq E\left[X(\hat{h}_c(X) - h_c(X))X'\beta(V - r(\hat{h}_w(X) - h_w(X))\right]
- E\left[X(U - r(\hat{h}_y(X) - h_y(X)) + r(\hat{h}_c(X) - h_c(X))X'\beta)(\hat{h}_w(X) - h_w(X))\right]
\partial^2 f(r) = 2E\left[X((\hat{h}_y(X) - h_y(X)) - (\hat{h}_c(X) - h_c(X))X'\beta)\hat{h}_w(X) - h_w(X)\right]
\]

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We can bound
\[ |\partial^2 f(r)| \leq 2 \|X(\hat{h}_w(X) - h_p(X))(\hat{h}_w(X) - h_w(X))\| + 2 \|XX'\beta(\hat{h}_c(X) - h_c(X))(\hat{h}_w(X) - h_w(X))\| \leq 2mA \delta_n/\sqrt{n} + 8a^2m^3 \delta_n/\sqrt{n} \]

This establishes the convergence to 0 of \( \|E[\psi(1)(\hat{h})] - E[\psi(1)(h)]\|, (E[\|\psi(\beta, \hat{h}) - \psi(\beta, h)\|^2])^{1/2} \) and \( \sqrt{n}\|\partial^2 E[\psi(\beta, h + r(h - h))]\| \)

Assumption 3.2 d) also requires that the variance of the score \( E[V_i^2 U_i^2 X_i'] \) is non-degenerate. \( E[V_i^2 U_i^2 X_i'] = E[E[V_i^2 U_i^2 | X_i] X_i X_i'] \geq a E[X_i X_i'] \), which is full rank by assumption.

Given we have verified that Assumptions 3.1 and 3.2 hold, then the result of Theorem 3 comes directly from Theorem 3.1 of Chernozhukov et al. [2018].

**Proof of Theorem 4**

We first prove a couple of useful Lemmas.

**Lemma 1.** The estimated threshold converges to the true threshold \( \hat{s}(b) \rightarrow_p s(b) \) and has an asymptotically linear representation:
\[
\sqrt{n}(\hat{s}(b) - s(b)) = - \frac{1}{\sqrt{n}B'(s(b))} (\hat{B}(s(b)) - B(s(b))) + o_p(1).
\]

**Proof.** We can define \( \hat{s}(b) \) as a Z-estimator, where it is the possibly non-unique and approximate solution to
\[
\hat{B}(\hat{s}(b)) - b = 0.
\]

We can then use Theorem 5.9 of Vaart [1998] to prove that \( \hat{s}(b) \rightarrow_p s(b) \). Using this Lemma requires verifying two conditions:

First, the uniform convergence of \( \hat{B}(s) - b \rightarrow B(s) - b \) follows from Lemma 2.4 of Newey and McFadden [1994]. We have the weak continuity and boundedness of \( B_i(s) = \left( \frac{W_i}{n} - \frac{1-W_i}{n} \right) Y_i \text{1}(S_i \geq s) \) in \( s \) (given that \( S_i \) is continuously distributed) and that \( s \) comes from a compact space.

\[
\sup_{s \in [0,1]} \| \hat{B}(s) - B(s) \| \rightarrow_p 0.
\]

Next, we note that \( B(s) - b \) is continuous in \( s, s \in [0,1] \) which is a compact space, and \( B(s) - b \) has a unique zero at \( s(b) \) since \( B(s) \) is strictly monotonic, so has an inverse. This shows the second condition of Theorem 5.9 of Vaart [1998] (see Problem 5.27):
\[
\inf_{s: d(s,s(b)) \geq \epsilon} \| B(s) - b \| > 0 = \| B(s(b)) - b \|.
\]

We have now verified the conditions of Theorem 5.9 and shown that \( \hat{s}(b) \rightarrow_p s(b) \). \( \Box \)

**Lemma 2.** The following convergence in probability holds:

1. \( \sqrt{n}(\hat{R}(\hat{s}(b)) - R(\hat{s}(b)) - \sqrt{n}(\hat{R}(s(b)) - R(s(b))) \rightarrow_p 0 \)
2. \( \sqrt{n}(\hat{B}(\hat{s}(b)) - B(\hat{s}(b)) - \sqrt{n}(\hat{B}(s(b)) - B(s(b))) \rightarrow_p 0 \)

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Proof. We use Lemma 19.24 of Vaart [1998]. Given that we have shown in the previous Lemma that \( \hat{s}(b) \to_p s(b) \), then the convergence in probability that we require holds as long as the following two conditions hold:

1. Define the function classes

\[
\mathcal{F}^R = \left\{ \left( \frac{w}{\pi} - \frac{1-w}{1-\pi} \right) \mathbb{I}(q \geq s) : s \in [0,1] \right\},
\]
\[
\mathcal{F}^B = \left\{ \left( \frac{w}{\pi} - \frac{1-w}{1-\pi} \right) s \mathbb{I}(q \geq s) : s \in [0,1] \right\}.
\]

\( \mathcal{F}^R \) and \( \mathcal{F}^B \) are \( P \)-Donsker, where \( P \) defines the probability distribution of \( S_i, W_i, Y_i, C_i \).

2. \( E \left[ \left( R(\hat{s}(b)) - R(s(b)) \right)^2 \right] \to_p 0 \) and \( E \left[ \left( B(\hat{s}(b)) - B(s(b)) \right)^2 \right] \to_p 0 \).

**Showing Condition 1.**

Both \( \mathcal{F}^R \) and \( \mathcal{F}^B \) can be represented as \( \mathcal{F} = \{ c(x) \mathbb{I}(s \geq q) : s \in [0,1] \} \), where \( c(x) \) is a uniformly bounded function of the data. The function class represented by \( c(x) \) is of course \( P \)-Donsker, for any distribution over \( x \), since it does not depend on \( s \); for any \( s, s' \), then \( 0 = c(x, s) - c(x, s') \leq ||s - s'|| \), which by Example 19.7 of Vaart [1998] implies that it is a Donsker class. We have that a class of functions made of the product of two functions, each of which are in a Donsker class that is uniformly bounded, is also Donsker (see Example 19.20 of Vaart [1998]). So, to verify that \( \mathcal{F} \) is Donsker, we can verify that \( \mathcal{H} = \{ \mathbb{I}(q \geq s) : s \in [0,1] \} \) is Donsker. This in turn is equivalent to verifying that \( \mathcal{G} = \{ \mathbb{I}(1-q \leq s) : s \in [0,1] \} \) is Donsker, since each element of \( \mathcal{G} \) is just 1 minus each element of \( \mathcal{H} \). But a function in \( \mathcal{G} \) is an indicator function, the average of which is the empirical distribution function for a random variable on \([0,1] \subset \mathbb{R} \). Example 19.6 of Vaart [1998] verifies that this type of class is a Donsker class through a bracketing argument. We have thus verified that both \( \mathcal{F}^R \) and \( \mathcal{F}^B \) are \( P \)-Donsker.

**Showing Condition 2.**

We can write \( R(s) = E[Q_i^R \mathbb{I}(S_i \geq s)] \) and \( B(s) = E[Q_i^B \mathbb{I}(S_i \geq s)] \) where \( Q_i^B = \left( \frac{W_i}{\pi} - \frac{1-W_i}{1-\pi} \right) C_i \) and \( Q_i^R = \left( \frac{W_i}{\pi} - \frac{1-W_i}{1-\pi} \right) Y_i \) are both drawn i.i.d. and are bounded. We can then prove the required mean-squared convergence by proving it for \( E[Q_i \mathbb{I}(S_i \geq s)] \) for bounded \( Q_i \).

Let \( \gamma(t) = E[(Q_i[\mathbb{I}(S_i \geq t) - \mathbb{I}(S_i \geq s)])^2] \). The goal is to show that \( \gamma(t) \) is continuous in \( t \), so we can use the CMT for the required quadratic mean convergence, since we have proved already that \( \hat{s}(b) \to_p s(b) \) and the required mean squared convergence is equivalent to \( \gamma(\hat{s}(b)) \to_p \gamma(s(b)) \). If \( t \geq s(b) \), then we have that

\[
\gamma(t) = E[Q_i^2 \mathbb{I}(S_i \leq t)].
\]

If \( t \leq s(b) \), then we have that

\[
\gamma(t) = E[Q_i^2 \mathbb{I}(t \leq S_i \leq s(b))].
\]

\( S_i \) has bounded density, so we have that

\[
\lim_{\delta \to 0} \frac{Pr(t \leq S_i \leq t + \delta)}{\delta} = f(s).
\]
Since $Q_i$ is also bounded, then we can write that
\[ |\gamma(t + \delta) - \gamma(t)| \leq M^2 \delta f(s) \]
Then, we can write
\[
\lim_{\delta \to 0} |\gamma(t + \delta) - \gamma(t)| = 0,
\]
which implies for all $t \in [0, 1]$,
\[
\lim_{\delta \to 0} \gamma(t + \delta) = \gamma(t).
\]
This gives the continuity of $\gamma(t)$ in $t$, and we can now use the CMT to show the mean-squared convergence required by Condition 2. This concludes the proof of Lemma 2.

Next, for the asymptotically linear representation, we use Theorem 5.21 of Vaart [1998]. This requires verifying the following:

- Rather than relying on the Lipschitz condition from the original theorem, we can instead verify that the following condition holds from Lemma 19.24 of Vaart [1998]:
  \[
  \sqrt{n}(\hat{B}(\hat{s}(b)) - B(\hat{s}(b)) - \sqrt{n}(\hat{B}(s(b)) - B(s(b))) \to_p 0.
  \]
  This holds from Lemma 2.

- We need that $B(s) - b$ is differentiable at $s(b)$ with $B'(s(b)) \neq 0$. Since we have assumed that $B(s)$ is continuously differentiable and we know that $B(s)$ is strictly monotonic, then this holds.

- Since $B(s) - b$ is bounded, then its variance is bounded.

Now that we have verified these conditions, then the expansion from Theorem 5.21 holds:
\[
\sqrt{n}(\hat{s}(b) - s(b)) = \frac{1}{\sqrt{n}} \left( \hat{B}(s(b)) - B(s(b)) \right) + o_p(1).
\]

The following expansion holds for $\hat{Q}(b)$ under the Assumptions of Theorem 4
\[
\hat{Q}(b) - Q(b) = \hat{R}(\hat{s}(b)) - R(s(b))
= \hat{R}(\hat{s}(b)) - R(\hat{s}(b)) + R(\hat{s}(b)) - R(s(b))
= \hat{R}(s(b)) - R(s(b)) + R(\hat{s}(b)) - R(s(b)) + o_p(n^{-0.5})
\]
We use an expansion to get (32). Then, for (33), we applied Lemma 2 which indicates that $\hat{R}(\hat{s}(b)) - R(\hat{s}(b)) = \hat{R}(s(b)) - R(s(b)) + o_p(n^{-0.5})$. Next, since we have that $R(s)$ is differentiable in $s$, we can take a Taylor Expansion of $\hat{R}(\hat{s}(b))$ around $s(b)$:
\[
R(\hat{s}(b)) - R(s(b)) = R'(s)(\hat{s}(b) - s(b)) + o_p(n^{-0.5})
\]
Now we can use the expansion from Lemma 1 which indicates that :
\[
\hat{s}(b) - s(b) = -\frac{1}{B'(s(b))} \left( \hat{B}(s(b)) - B(s(b)) \right) + o_p(n^{-0.5}).
\]
This indicates
\[ R(\hat{s}(b)) - R(s(b)) = -\frac{R'(s)}{B'(s(b))}(\hat{B}(s(b)) - B(s(b))) + o_p(n^{-0.5}). \]
Plugging this back into Equation 33, we now have an expansion for \( \hat{\Delta}(b) \):
\[ \hat{\Delta}(b) - R(s(b)) = \hat{R}(s(b)) - R(s(b)) - \frac{R'(s(b))}{B'(s(b))}(\hat{B}(s(b)) - B(s(b))) + o_p(n^{-0.5}). \]

The RHS of the expression for \( \hat{Q}(b) \) is an i.i.d. average with finite variance so the central limit theorem applies and \( \hat{Q}(b) \) is asymptotically normal. We can write \( \sqrt{n}(\hat{Q}(b) - Q(b)) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_i^q + o_p(1) \) where \( \psi_i^q = R_i(s(b)) - R(s(b)) - \frac{R'(s(b))}{B'(s(b))}(B_i(s(b)) - B(s(b))) \) and so it has an asymptotically linear representation. \( R_i(s) = \left( \frac{w_s}{\pi} - \frac{(1-w_s)}{1-\pi} \right) C_i1(S_i \geq s) \) and \( B_i(s) = \left( \frac{w_s}{\pi} - \frac{(1-w_s)}{1-\pi} \right) Y_i1(S_i \geq s). \)

Next we convert \( \Delta \) to an asymptotically linear representation.
\[ \hat{\Delta}(b) = \hat{Q}(b) - b \frac{\hat{R}(0)}{B(0)}. \]
Let \( f(d) = b \frac{\hat{R}(0)}{d} \). Take a Taylor expansion of \( f(\hat{B}(0)) \) around \( B(0) \):
\[ f(\hat{B}(0)) = f(B(0)) + f'(B(0))(\hat{B}(0) - B(0)) + o_p(n^{-0.5}). \]
where \( f'(B(0)) = -b \frac{\hat{R}(0)}{B(0)} \). The small remainder term is because a CLT applies to \( \hat{B}(0) - B(0) \) given \( \hat{B}(0) \) is an i.i.d. average with bounded variance.

Plugging this into the expression for \( \hat{\Delta}(b) \), we have:
\[ \hat{\Delta}(b) = \hat{Q}(b) - b \frac{\hat{R}(0)}{B(0)} + b \frac{R(0)(\hat{B}(0) - B(0))}{B(0)^2} + o_p(n^{-0.5}) \]
Now we can take a Taylor expansion again of \( f(r) = r(\hat{B}(0) - B(0)) \).
\[ f(\hat{R}(0)) = f(R(0)) + (\hat{B}(0) - B(0))(\hat{R}(0) - R(0)) + o_p(n^{-0.5}) \]
\[ = f(R(0)) + o_p(n^{-0.5}) \]
where we can drop the first order term since both \( \hat{B}(0) - B(0) \) and \( \hat{R}(0) - R(0) \) converge at a \( \sqrt{n} \) rate.

This means that we can write
\[ \hat{\Delta}(b) = \hat{Q}(b) - b \frac{\hat{R}(0)}{B(0)} + b \frac{R(0)(\hat{B}(0) - B(0))}{B(0)^2} + o_p(n^{-0.5}). \]
This now gives an expression for \( \hat{\Delta}(b) \) in terms of an i.i.d. average which is asymptotically normal.
\[ \sqrt{n}(\hat{\Delta}(b) - \Delta(b)) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_i^d + o_p(1), \]
\[ \psi_i^d = \psi_i^q - b \frac{R_i(0)}{B(0)} + b \frac{R(0)(B_i(0) - B(0))}{B(0)^2} + b \frac{R(0)}{B(0)}. \]
Proof of Proposition 5

In this section, we show the equation (20) in Proposition 5. Assume $W \in \{0, 1\}$ and let $e(x) := \mathbb{P}[W = 1 \mid X^{(\text{conf})}]$. Notice that

\[
\text{Cov} \left[ Y, W \mid X^{(\text{conf})} \right] = E[YW \mid X^{(\text{conf})}] - E[Y \mid X^{(\text{conf})}]E[W \mid X^{(\text{conf})}]
\]

\[
= E[Y(1)W \mid X^{(\text{conf})}] - E[Y \mid X^{(\text{conf})}]E[W \mid X^{(\text{conf})}]
\]

\[
= e(X^{(\text{conf})})E[Y(1) \mid X^{(\text{conf})}] - e(X^{(\text{conf})})^2 E[Y(1) \mid X^{(\text{conf})}] + e(X)(1 - e(X))E[Y(0) \mid X^{(\text{conf})}]
\]

\[
= e(X^{(\text{conf})})\{1 - e(X^{(\text{conf})})\} \{E[Y(1) \mid X^{(\text{conf})}] - E[Y(0) \mid X^{(\text{conf})}]\}
\]

\[
(34)
\]

where the second equality comes from the consistency assumption that $Y = WY(1) + (1 - W)Y(0)$ and the third equality comes from the unconfounded assumption (4). Similarly, we can show that

\[
\text{Cov} \left[ C, W \mid X^{(\text{conf})} \right] = e(X^{(\text{conf})})\{1 - e(X^{(\text{conf})})\} \{E[C(1) \mid X^{(\text{conf})}] - E[C(0) \mid X^{(\text{conf})}]\}
\]

and thus

\[
\frac{E \left[ \frac{\text{Cov}[Y_i, W_i \mid X_i^{(\text{conf})}]}{e(X_i^{(\text{conf})})(1 - e(X_i^{(\text{conf})}))} \mid X_i = x \right]}{E \left[ \frac{\text{Cov}[C_i, W_i \mid X_i^{(\text{conf})}]}{e(X_i^{(\text{conf})})(1 - e(X_i^{(\text{conf})}))} \mid X_i = x \right]}
\]

\[
= \frac{E \left[ Y(1) \mid X^{(\text{conf})} \right] - E \left[ Y(0) \mid X^{(\text{conf})} \right] \mid X_i = x}{E \left[ C(1) \mid X^{(\text{conf})} \right] - E \left[ C(0) \mid X^{(\text{conf})} \right] \mid X_i = x}
\]

\[
= \frac{\delta_Y(x)}{\delta_C(x)}
\]

\[
= \rho(x),
\]

which completes the proof of the Proposition 5.

Proof of Theorem 6

We will use Theorem 3.1 of Chernozhukov et al. [2018]. From (10), we have that $\beta$ satisfies the following unconditional moment restriction, where the score function is

\[
\psi_i(\beta, h(X_i^{(\text{conf})})) = \frac{W_iX_i - E[W_i\mid X_i^{(\text{conf})}]X_i}{(W_i - E[W_i\mid X_i^{(\text{conf})}])E[W_i\mid X_i^{(\text{conf})}]}[Y_i - E[Y_i\mid X_i^{(\text{conf})}] - (C_i - E[C_i\mid X_i^{(\text{conf})}])X_i^T\beta]
\]

\[
E[\psi_i(\beta, h(X_i^{(\text{conf})}))] = 0
\]

We have that the score function is linear in $\beta$. This verifies Assumption 3.1b) of Chernozhukov et al. [2018]. To apply the Theorem, we must verify the remaining components of Assumption 3.1 and Assumption 3.2.
3.1a) is satisfied, since Equation 18 is equivalent to $E[\psi_i(\beta, h(X_i^{\text{conf}}))]=0$ under the linearity assumption for $\rho(x)$.

3.1c) is satisfied under the assumption of strict overlap, in case of propensity scores bounded away from zero the function is twice differentiable in the parameters. For 3.1d), we show Neyman-Orthogonality by showing that the partial derivative, evaluated at zero, of the conditional moment restriction with respect to each component of a perturbation of the nuisance functions is zero. Then, the Law of Iterated Expectations implies Neyman-Orthogonality for the unconditional score function.

$$\frac{\partial E[e_i(\beta, h(x) + \epsilon \delta(x)) \mid X_i^{\text{conf}} = x]}{\partial \epsilon_y} \bigg|_{\epsilon=0} = -\delta_y(x)E \left[ \frac{W_i - E[W_i | X_i^{\text{conf}} = x]}{E[W_i | X_i^{\text{conf}} = x]} \mid X_i^{\text{conf}} = x \right]$$

$$= 0$$

Similarly,

$$\frac{\partial E[e_i(\beta, h(x) + \epsilon \delta(x)) \mid X_i = x]}{\partial \epsilon_c} \bigg|_{\epsilon=0} = X' \delta_e(x)E \left[ \frac{W_i - E[W_i | X_i^{\text{conf}} = x]}{E[W_i | X_i^{\text{conf}} = x]} \mid X_i^{\text{conf}} = x \right]$$

$$= 0.$$}

Lastly,

$$\frac{\partial E[e_i(\beta, h(x) + \epsilon \delta(x)) \mid X_i = x]}{\partial \epsilon_w} = (1 - \delta_w(x))^{-2}E[Y_i - E[Y_i | X_i^{\text{conf}} = x] \mid X_i^{\text{conf}} = x] - X' \beta (1 - \delta_w(x))^{-2}E[C_i - E[C_i | X_i^{\text{conf}} = x] \mid X_i^{\text{conf}} = x] = 0$$

For 3.1d), we need that $E[A_i S'_i]$ is invertible.

$$E[A_i S'_i] = E \left[ \frac{(W_i X_i - E[W_i | X_i^{\text{conf}}] X_i)(C_i - E[C_i | X_i^{\text{conf}}] X_i)'}{E[W_i | X_i^{\text{conf}} = x]}(W_i - E[W_i | X_i^{\text{conf}} = x]) \right]$$

$$= E_x \left[ \frac{(W_i X_i - E[W_i | X_i^{\text{conf}}] X_i)(C_i - E[C_i | X_i^{\text{conf}}] X_i)'}{E[W_i | X_i^{\text{conf}} = x]}(W_i - E[W_i | X_i^{\text{conf}} = x]) \mid X_i \right]$$

$$= E_x[X_i X'_i \text{Cov}(C_i, W_i | X_i^{\text{conf}})]$$

Since we are in the setting of Proposition 5 and unconfoundedness applies as well as the strict overlap condition, we have that

$$\text{Cov}(C_i, W_i | X_i = e(x)(1 - e(x))E[C_i(1) - C_i(0)] X_i = x] > 0,$$

where the inequality is from Assumption 1. Then, $E[A_i S'_i]$ is invertible as long as $E[X_i X'_i]$ is full rank, which it is by assumption.

To verify the Assumption 3.2 we may represent

$$\psi_i(\beta, h(X_i^{\text{conf}})) = \frac{1}{(1 - h_w(X_i^{\text{conf}})) h_w(X_i^{\text{conf}}) \phi_i(\beta, h(X_i^{\text{conf}}))}$$

, where $\phi_i(\beta, h(X_i^{\text{conf}})) = X_i (W_i - h_w^{\text{conf}}(X_i))(Y_i - h_p(X_i^{\text{conf}}) - (C_i - h_c(X_i^{\text{conf}}))) X_i \beta$.

Since $|\psi_i(\beta, h(X_i^{\text{conf}}))| \leq a^{-2} |\phi_i(\beta, h(X_i^{\text{conf}}))|$, this part of the proof reduces to the respective part in the proof of Theorem 3.

Given we have verified that Assumptions 3.1 and 3.2 hold, then the result of Theorem 6 comes directly from Theorem 3.1 of Chernozhukov et al. [2018].
C Empirical Appendix

Pre-treatment variables for the Oregon Health Insurance Experiment
| Variable name                     | Variable description                                                                 |
|----------------------------------|--------------------------------------------------------------------------------------|
| numhh_list                       | Number of people in household on lottery list                                         |
| birthyear_list                   | Birth year: lottery list data                                                         |
| have_phone_list                  | Gave a phone number on lottery sign up: lottery list data                             |
| english_list                     | Individual requested english-language materials: lottery list data                    |
| female_list                      | Female: lottery list data                                                             |
| first_day_list                   | Signed up for lottery list on first day: lottery list data                            |
| last_day_list                    | Signed up for lottery list on last day: lottery list data                             |
| pobox_list                       | Gave a PO Box as an address: lottery list data                                         |
| self_list                        | Individual signed him or herself up for the lottery list                              |
| zip_msa_list                     | Zip code from lottery list is a metropolitan statistical area                         |
| snap_ever_presurvey12m           | Ever personally on SNAP, 6 month pretreatment                                         |
| snap_tot hh_presurvey12m         | Total household benefits from SNAP, 6 month pretreatment                              |
| tanf_ever_presurvey12m           | Ever personally on TANF, 6 month pretreatment                                         |
| tanf_tot hh_presurvey12m         | Total household benefits from TANF, 6 month pretreatment                              |
| any_visit_pre_ed                 | Any ED visit,                                                                          |
| any_hosp_pre_ed                  | Any ED visit resulting in a hospitalization                                           |
| any_out_pre_ed                   | Any Outpatient ED visit                                                               |
| any_on_pre_ed                    | Any weekday daytime ED visit                                                          |
| any_off_pre_ed                   | Any weekend or nighttime ED visits                                                   |
| num_edcnp_pre_ed                 | Number of emergent, non-preventable ED visits                                         |
| num_edcnpa_pre_ed                | Number of emergent, preventable ED visits                                            |
| num_epct_pre_ed                  | Number of primary care treatable ED visits                                           |
| num_ne_pre_ed                    | Number of non-emergent ED visits                                                     |
| num_unclas_pre_ed                | Number of of unclassified ED visits                                                  |
| any_acsc_pre_ed                  | Any ambulatory case sensitive ED visit                                               |
| any_chron_pre_ed                 | Any ED visit for chronic condition                                                   |
| any_inj_pre_ed                   | Any ED visit for injury                                                               |
| any_skin_pre_ed                  | Any ED visit for skin conditions                                                     |
| any_abdo_pre_ed                  | Any ED visit for abdominal pain                                                      |
| any_back_pre_ed                  | Any ED visit for back pain                                                           |
| any_heart_pre_ed                 | Any ED visit for chest pain                                                          |
| any_head_pre_ed                  | Any ED visit for headache                                                            |
| any_depres_pre_ed                | Any ED visit for mood disorders                                                      |
| any_psysub_pre_ed                | Any ED visit for psych conditions/substance abuse                                    |
| charg_tot_pre_ed                 | Sum of total charges                                                                 |
| ed_charg_tot_pre_ed              | Sum of total ED charges                                                               |
| any_hiun_pre_ed                  | Any ED visit to a high uninsured volume hospital                                      |
| any_lomun_pre_ed                 | Any ED visit to a low uninsured volume hospital                                       |
| need_med_0m                      | Survey data: Needed medical care in the last six months                               |
| need_rx_0m                       | Survey data: Needed prescription medications in the last six months                   |
| rx_num_mod_0m                    | Survey data: Number of prescription medications currently taking                      |
| rx_any_0m                        | Survey data: Currently taking any prescription medications                            |
| Variable name               | Variable description                                                                 |
|----------------------------|---------------------------------------------------------------------------------------|
| need_dent_0m               | Survey data: Needed dental care in the last six months                                 |
| doc_any_0m                 | Survey data: Any primary care visits                                                  |
| doc_num_mod_0m             | Survey data: Number of primary care visits, truncated                                   |
| er_any_0m                  | Survey data: Any ER visits                                                             |
| er_num_mod_0m              | Survey data: Number of ER visits, truncated                                            |
| er_noner_0m                | Survey data: Used emergency room for non-emergency care                                 |
| reason_er_need_0m         | Survey data: Went to ER (reason): needed emergency care                                 |
| reason_er_closed_0m       | Survey data: Went to ER (reason): clinics closed                                       |
| reason_er_apt_0m          | Survey data: Went to ER (reason): couldn’t get doctor’s appointment                    |
| reason_er_doc_0m          | Survey data: Went to ER (reason): didn’t have personal doctor                          |
| reason_er_copay_0m        | Survey data: Went to ER (reason): couldn’t afford copay to see a doctor                |
| reason_er_go_0m           | Survey data: Went to ER (reason): didn’t know where else to go                         |
| reason_er_other_0m        | Survey data: Went to ER (reason): other reason                                         |
| reason_er_rx_0m           | Survey data: Went to ER (reason): needed prescription drug                              |
| reason_er_dont_0m         | Survey data: Went to ER (reason): don’t know                                           |
| hosp_any_0m                | Survey data: Any hospital visits                                                      |
| hosp_num_mod_0m           | Survey data: Number hospital visits, truncated at 2*99th%ile                           |
| total_hosp_0m             | Survey data: Total days spent in hospital, last 6 months                               |
| dia_dx_0m                 | Survey data: Diagnosed diabetes                                                        |
| ast_dx_0m                 | Survey data: Diagnosed asthma                                                          |
| hbp_dx_0m                 | Survey data: Diagnosed high blood pressure                                             |
| emp_dx_0m                 | Survey data: Diagnosed COPD                                                            |
| chf_dx_0m                 | Survey data: Diagnosed congestive heart failure                                        |
| dep_dx_0m                 | Survey data: Diagnosed depression or anxiety                                           |
| female_0m                 | Survey data: Is female                                                                 |
| birthyear_0m              | Survey data: Birth year                                                                 |
| employ_0m                 | Survey data: Currently employed                                                        |
| employ_det_0m             | Survey data: Currently employed or self-employed                                       |
| hhinc_cat_0m              | Survey data: Household income category                                                 |
| employ_hrs_0m             | Survey data: Average hrs worked/week                                                   |
| edu_0m                    | Survey data: Highest level of education completed                                       |
| living_arrange_0m         | Survey data: Current living arrangement                                                 |
| hhsfize_0m                | Survey data: Household Size (adults and children)                                      |
| hhinc_pctfpl_0m           | Survey data: Household income as percent of federal poverty line                       |
| num19_0m                  | Survey data: Number of family members under 19 living in house                         |
| preperiod_any_visits      | Any ED visit (the date range is different from any_visit_pre_ed)                      |

Table 3: List of variables used as pre-treatment covariates in the Oregon Health Experiment application
Table 4: Area under the curve metric. It is calculated as the area between the uniform allocation line and the QINI curve of the respected metric.