CLASSICAL LIMIT OF THE TRAJECTORY REPRESENTATION OF QUANTUM MECHANICS, LOSS OF INFORMATION AND RESIDUAL INDETERMINACY

Edward R. Floyd
10 Jamaica Village Road, Coronado, CA 92118-3208, USA
floyd@crash.cts.com

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Abstract
The trajectory representation in the classical limit ($\hbar \to 0$) manifests a residual indeterminacy. We show that the trajectory representation in the classical limit goes to neither classical mechanics (Planck’s correspondence principle) nor statistical mechanics. This residual indeterminacy is contrasted to Heisenberg uncertainty. We discuss the relationship between residual indeterminacy and ’t Hooft’s information loss and equivalence classes.

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1 Introduction
The transition from quantum motion to classical motion has been a conceptually blurred area for the various representations of quantum mechanics. Recently, Faraggi and Matone have shown that, while quantum mechanics is compliant with the equivalence principle where all quantum systems can be connected by an equivalence coordinate transformation (trivializing map), classical mechanics is not. Faraggi and Matone through the equivalence principle have independently derived, free from axioms, the quantum Hamilton-Jacobi foundation for the trajectory representation of quantum mechanics. Herein, we theoretically test Planck’s correspondence principle by investigating the transition from quantum to classical mechanics for the trajectory representation, embedded in a non-relativistic quantum Hamilton-Jacobi theory, for the Planck constant going to zero in the limit.

The quantum stationary Hamilton-Jacobi equation (QSHJE) is a phenomenological equation just like the classical stationary Hamilton-Jacobi equation (CSHJE) and the Schrödinger equation. The QSHJE and CSHJE render generators of the motion that describe trajectories that differ. As we shall show herein, these trajectories still differ in the classical limit. The QSHJE in one dimension is given for non-relativistic quantum motion by

$$\frac{W^2}{2m} + V - E = -\frac{\hbar^2}{4m} \langle W; x \rangle$$

(1)
where $W$ is the reduced action (also known as Hamilton’s characteristic function), $W_x$ is the conjugate momentum, $V$ is the potential, $\hbar = h/(2\pi)$, $h$ is the Planck constant, $m$ is the mass, and $\langle W; x \rangle$ is the Schwarzian derivative that manifests the quantum effects. The Schwarzian derivative contains higher-order derivative terms given by

$$\langle W; x \rangle = \frac{W_{xxx}}{W_x} - \frac{3}{2} \left( \frac{W_x}{W_x} \right)^2.$$  

The complete solutions for the reduced action and conjugate momentum are well known and given for energy $E$ by

$$W = \hbar \arctan \left( \frac{b\theta/\phi + c/2}{(ab - c^2/4)^{1/2}} \right) + K, \quad x > 0. \tag{2}$$

and

$$W_x = (2m)^{1/2}(a\phi^2 + b\theta^2 + c\phi\theta)^{-1}. \tag{3}$$

where $K$ is an integration constant that we may arbitrarily set $K = 0$ for the rest of this investigation and where $(\phi, \theta)$ is the set of independent solutions to the associated stationary Schrödinger equation for energy $E$. The Wronskian $W(\phi, \theta)$ is normalized so that $W^2 = 2m/|\hbar^2(ab - c^2/4)|$. The set of coefficients $(a, b, c)$ for energy $E$ determines the particular solution for $W$ and $W_x$. The set $(a, b, c)$ is determined by a sufficient set of a combination of initial values or constants of the motion other than $E$ for the third-order QSHJE. This requires three independent values. For example, the set of coefficients $(a, b, c)$ can be specified by the initial values $[W_x(x_0), W_{xx}(x_0)]$ for the QSHJE plus the Wronskian, $W$, which is a constant for the Schrödinger equation. The particular solution was shown to specify the particular microstate which had not been detected in the Schrödinger representation.

The left side of the QSHJE, Eq. (2), is the CSHJE. If $\hbar$ were zero (as distinguished from the limit $\hbar$ going to zero), then Eq. (1) would trivially reduce to the classical time-independent Hamilton-Jacobi equation. Herein, we test Planck’s correspondence principle by investigating the proposition that quantum mechanics transitions to classical mechanics in the limit that $\hbar$ goes to zero. We specify the limit $\hbar \to 0$ to be the classical limit. (Another “classical limit”, neither used nor investigated herein, is Bohr’s correspondence principle where the principle quantum number increases without limit.)

The trajectory representation of quantum mechanics is well suited for testing Planck’s correspondence principle. While the CSHJE is a first-order nonlinear differential equation, the QSHJE is a third-order nonlinear differential equation whose second- and third-order terms contain the factor $\hbar^2$. This investigation must include, in the limit $\hbar \to 0$, the effects of annulling all higher than first order terms in a differential equation upon the particular solution and the set of necessary and sufficient initial values.

With the trajectory representation, we show that quantum mechanics in the classical limit does not generally reduce to classical mechanics. Nor does it reduce to statistical mechanics. A residual indeterminacy implies an uncertainty exists in general in the classical limit. We investigate this residual indeterminacy and contrast it to Heisenberg uncertainty. We also gain insight into the quantum term $\hbar^2 \langle W; x \rangle/(4m)$.

Recently, ’t Hooft proposed that underlying contemporary quantum mechanics there should exist a more fundamental, albeit still unknown, theory at the Planck level that would provide more information than the Schrödinger wave function. In ’t Hooft’s proposal, the additional information distinguishes primordial states at the Planck level, but this information is lost through dissipation as these states evolve into states forming an equivalence class. ’t Hooft suggested quantum gravity would dissipate information. States of an equivalence class, after a while, become indistinguishable from each other even though they have different pasts. ’t Hooft’s ideas bear upon this investigation by giving us insight into the relationship of the Copenhagen interpretation to the trajectory representation where information regarding microstates (primordial states) becomes “lost” and where the Schrödinger wave function becomes an equivalence class. We compare residual indeterminacy with ’t Hooft’s information loss and equivalence classes. This comparison is preliminary because the underlying fundamental theory of ’t Hooft’s proposal is still incomplete.
We examine the classical limit for three cases: a particle in the classically allowed region; a particle in the classically forbidden region (beyond the WKB turning point); and a particle in the vicinity of the WKB turning point. We choose potentials that are heuristic and whose trajectory representation can be presented by familiar functions. One dimension suffices for this investigation.

As noted in the opening paragraph, Faraggi and Matone in independently deriving the QSHJE, have shown that such an equivalence principal does not exist for classical mechanics (i.e., \( \bar{\hbar} = 0 \)). This begs the question “What happens when \( \bar{\hbar} \to 0 \)?” Faraggi and Matone essentially examined the \( \lim_{\bar{\hbar} \to 0} W \to W_{\text{classical}} \) in Ref. 4 (more on the differences between their and our limiting processes in Sec. 5). Still, Faraggi and Matone found that the equivalence principal in the classical limit is not valid.

Although the Bohmian school initially reported that the classical limit was specified by \( \lim_{\bar{\hbar} \to 0} \), it latter recanted and now opposes such a classification because \( \hbar \) is finite and a constant that cannot be varied. The differences between the trajectory representation and Bohmian mechanics has been discussed elsewhere. In fact, \( \hbar \) is finite but very small. For this exposition, we treat \( \hbar \) not as a physical constant but as an independent variable. This allows us to study the infinitesimal limit to show that even in the \( \lim_{\bar{\hbar} \to 0} \), quantum trajectories do not in general reduce to classical trajectories.

In Sec. 2, we investigate the classical limit of the trajectory representation of quantum mechanics for a particle in the classically allowed region. In Sec. 3, we examine a trajectory in the classical limit in the classical forbidden region. We investigate in Sec. 4 a trajectory as it transits between the classically allowed and forbidden regions across the WKB turning point. In Sec. 5, we discuss the impact of the classical limit upon the set of initial values necessary and sufficient to specify the trajectory. In Sec. 6, we present the relationship between the this investigation and 't Hooft’s information loss and equivalence classes. In Sec. 7, we contrast residual indeterminacy to Heisenberg uncertainty and distinguish the classical limit of the trajectory representation from classical and statistical mechanics.

2 The Classically Allowed Case

We examine here a particle in the classically allowed region, \( E > V(x) \), in the limit that \( \hbar \to 0 \). In the limiting process, the quantum motion does not approach the classical motion as \( \hbar \to 0 \) in spite of the \( \hbar^2 \) factor in the right side of the QSHJE. Quantum physics is more subtle.

Let us consider a heuristic example. We choose a free particle, \( V = 0 \), for finite \( \hbar \). An acceptable set of independent solutions to the Schrödinger equation for the free particle is given by

\[
\phi = [E(ab - c^2/4)]^{-1/4} \cos[(2mE)^{1/2}x/\hbar] \quad \text{and} \quad \theta = [E(ab - c^2/4)]^{-1/4} \sin[(2mE)^{1/2}x/\hbar].
\]  

(4)

The coefficients \( a, b, c \) specify the particular microstate for a specified energy \( E \) in accordance with a sufficient set of a combination of initial values and constants of the motion other than energy.

The reduced action, Eq. (2), may be expressed for \( V = 0 \) by

\[
W = \hbar \arctan \left( b \tan[(2mE)^{1/2}x/\hbar] + c/2 \right) / (ab - c^2/4)^{1/2}. \]

(5)

For \( a = b \) and \( c = 0 \), then \( W = (2mE)^{1/2}x \) which coincides with the classical reduced action. For \( a \neq b \) or \( c \neq 0 \), l’Hôpital’s rule is used to evaluate \( W \) in the classical limit rendering

\[
\lim_{\hbar \to 0} W = \frac{2(ab - c^2/4)^{1/2}(2mE)^{1/2}x}{a + b + [(a - b)^2 + c^2]^{1/2} \cos[2(2mE)^{1/2}x/\hbar] + \cos^{-1}[(c/(a - b))]}. \]

(6)

The \( 1/\hbar \) factor in the argument of the cosine term in the denominator on the right side of Eq. (6) induces in the classical limit an essential singularity in the cosine term. This essential singularity in turn induces an indeterminacy in the reduced action, \( W \), in the classical limit if \( a \neq b \) or \( c \neq 0 \). The magnitude of this indeterminacy is a function of the particular microstate as determined by the set of coefficients \( (a, b, c) \).
phase shift of this indeterminacy is a function of $c/(a - b)$. This indefiniteness in $\lim_{\hbar \to 0} W$ does not exist when $\hbar$ is finite. This is consistent with Faraggi’s and Matone’s findings that the equivalence principle was applicable to quantum mechanics but not classical mechanics ($\hbar$ must not be zero for equivalence). We shall return latter in this section after we have established the equations of motion to show that this indefiniteness can be removed and that another generator of the motion, Hamilton’s principal function, $S_{\text{classical}}$, for quantum motion in the classical limit goes to the Hamilton’s principal function, $S_{\text{classical}}$, of classical mechanics.

The quantum conjugate momentum for $V = 0$ can be expressed as

$$W_x = \frac{2(2mE)^{1/2}(ab - c^2/4)^{1/2}}{a + b + [(a-b)^2 + c^2]^{1/2} \cos \{[2(2mE)^{1/2}x/\hbar] + \cot^{-1}[c/(a-b)]\}}. \quad (7)$$

In the limit $\hbar \to 0$ the cosine term in the denominator in Eq. (7) fluctuates with an infinitesimally short wave length. This induces in $W_x$ a residual indeterminacy in the classical limit. In the trajectory theory, we know the microstates, but in classical mechanics, we do not know the microstate because classical mechanics operates with a reduced set of initial values insufficient to specify the microstate. (The Copenhagen interpretation by precept forbids knowledge of the microstate.) Since $\hbar$ is very small but finite, we must consider what happens in the limit that $\hbar \to 0$.

In the classical limit, the cosine term in Eq. (7) becomes indefinite for $a \neq b$ or $c \neq 0$ even when we know the microstate. Our inability to evaluate $W_x$ in its classical limit even with knowledge of the set of coefficients $(a, b, c)$ specifying the particular microstate is a residual indeterminacy that is hypothetical since in reality $\hbar$ is finite. For finite $\hbar$, $W_x$ is always specified by $(E, a, b, c, x)$ in the trajectory representation, but the Copenhagen interpretation denies knowledge of $(a, b, c)$ while championing Heisenberg uncertainty. Nevertheless, we can evaluate its average momentum by averaging $W_x$ over one cycle of the cosine term using standard integral tables. We shall use this averaging process to gain insight into the quantum term $\hbar^2 \langle W; x \rangle/(4m)$. Nothing herein implies that we are considering an ensemble of identical microstates rather than a solitary microstate. The averaging process leads to

$$\langle \lim_{\hbar \to 0} W_x \rangle_{\text{ave}} = \lim_{\hbar \to 0} \langle \lim_{\hbar \to 0} (2mE)^{1/2} \int_{-\hbar/2}^{\hbar/2} W_x(E, a, b, c, x + x') dx' = \frac{2(2mE)^{1/2}(ab - c^2/4)^{1/2}}{(a + b)[1 - \frac{(a-b)^2 - c^2}{(a+b)^2}]^{1/2}} = (2mE)^{1/2}. \quad (8)$$

[We have changed for this step the operational order of evaluating the classical limit and averaging on the right side of Eq. (8) because the averaging domain is dependent upon $\hbar$. We continue this practice throughout.] In the classical limit, the average conjugate momentum is the classical momentum and microstate information as specified by the set of coefficients $(a, b, c)$ is lost.

The average for $\lim_{\hbar \to 0} W_x^2$ is given with the aid of standard integral tables to be

$$\langle \lim_{\hbar \to 0} W_x^2 \rangle_{\text{ave}} = \langle \lim_{\hbar \to 0} W_x \rangle_{\text{ave}}^2 = mE \frac{a + b}{(ab - c^2/4)^{1/2}} \geq 2mE. \quad (9)$$

If we identify $W_x^2/(2m)$ as the effective kinetic energy, then the average of the classical limit of the effective kinetic energy for $V = 0$ is greater than $E$ for $a \neq b$ or $c \neq 0$ and is equal to $E$ for $a = b$ and $c = 0$.

Now let us examine the variance of $W_x$ in the classical limit. By Eqs. (8) and (9), we have

$$\langle \lim_{\hbar \to 0} W_x^2 \rangle_{\text{ave}} - \langle \lim_{\hbar \to 0} W_x \rangle_{\text{ave}}^2 = 2mE \frac{a + b - (4ab - c^2)^{1/2}}{(4ab - c^2)^{1/2}} \geq 0. \quad (10)$$

Even in the classical limit, the variance of the quantum conjugate momentum, $W_x$, still is a function of the coefficients $(a, b, c)$ that, in turn, manifest microstates. For $a = b$ and $c = 0$, then the variance of $W_x$ is zero. In this particular microstate, the quantum motion reduces to classical motion for any value of $\hbar$ because the additional necessary initial values of the QSHJE, $[W_x(x_o), W_x(x_o)]$ are both zero for a given energy $E$.

The average energy associated with the Schwarzian derivative term of the QSHJE in the classical limit is given from Eqs. (8) and (9) by
\[ \left\langle \lim_{\hbar \to 0} \frac{\hbar^2}{4m} \langle W; x \rangle \right\rangle_{\text{ave}} = E \left( 1 - \frac{(a + b)/2}{(ab - c^2/4)^{1/2}} \right) = -\frac{\text{variance of } \lim_{\hbar \to 0} W_x}{2m} \leq 0. \] (11)

So the average energy, in the classical limit, of the quantum term, \( \hbar^2 \langle W; x \rangle / (4m) \), which is also known for unbound states as Bohm’s quantum potential, \( Q \), is proportional to the negative of the variance of the classical limit of the conjugate momentum. The quantum potential is a function of the particular microstate and may be finite even in the classical limit as shown by Eq. (11). As such, this potential is not a function of spatial position alone but is path dependent and, thus, cannot be a conservative potential. Other objections to Bohm’s quantum potential have already been discussed elsewhere. Others have identified \( Q \) to be, even in a non-relativistic case, an internal motion, type of spin, or a Zitterbewegung. But Carroll has observed that this work assumes a particle velocity inconsistent with Jacobi’s theorem. Let me comment on this spin or Zitterbewegung. If the quantum term is associated with a spin or Zitterbewegung, this would imply that it should manifest a kind of kinetic energy. But such a kinetic energy, even for a free particle, cannot be positive on average by the Eq. (11). This undermines the concept of kinetic energy. For particles in classically forbidden regions, such spin or Zitterbewegung energy must be strongly negative. This questions attributing such motions to the Schwarzian derivative.

Let us now consider the equation of motion that is given by Jacobi’s theorem, \( t - t_o = W_E \). For the free particle with energy \( E \), the motion is given by

\[ t - t_o = \frac{(ab - c^2/4)^{1/2}(2m/E)^{1/2}x}{a + b + (a^2 - 2ab + b^2 + c^2)^{1/2} \cos \left(2(2mE)^{1/2}x/h + \cot^{-1}\left[c/(a - b)\right]\right)} \]. (12)

Let us now evaluate Hamilton’s principal function, \( S = W - Et \), in the classical limit. Since we have solved the equations of motion, we are able to show that \( S \) is the time integral of the Lagrangian of classical mechanics, \( L^{\text{classical}} \). For \( V = 0 \), the classical Lagrangian is a constant give by \( L^{\text{classical}} = E \). As the right sides of Eqs. (6) and (12) are dynamically similar with regard to \( x \), the indeterminacy on the right side of Eq. (12) can be removed and the reduced action in the classical limit may be expressed as a function of time by \( \lim_{\hbar \to 0} W = 2E(t - t_o) \). Also, the classical limit of the reduced action is independent of the set of coefficients \( (a, b, c) \). Subsequently, Hamilton’s principal function may be expressed by

\[ \lim_{\hbar \to 0} S = E(t - t_o) = \int_{t_o}^{t} L^{\text{classical}} \, dt = S^{\text{classical}}. \] (13)

Note that Eq. (13) holds regardless of the particular microstate. The right side of Eqs. (6) and (12) also have similar form regarding the set of coefficients \( (a, b, c) \). When we used the equation of motion, Eq. (12), to remove the indefiniteness on the right side of Eq. (6), we also annulled microstate specification.

Let us now return to the equation of motion. As before with Eq. (6), in the limit \( \hbar \to 0 \) the cosine term in the denominator above fluctuates with an infinitesimally short wavelength. This induces a residual indeterminacy in \( t(x) \). Again as before in the classical limit, we can evaluate the average (this time for \( t \) rather than for \( W_x \)) by averaging the right side of the preceding equation over one cycle of the cosine term in the denominator although here we have \( x \) as a factor in the numerator. We again use standard tables where the \( x \) factor in the numerator is fasted to a single value over the infinitesimally short wavelength of the cosine term. This leads to

\[ \left\langle \lim_{\hbar \to 0} (t - t_o) \right\rangle_{\text{ave}} = \frac{[(ab - c^2/4)(2m/E)^{1/2}x]}{(a + b)[1 - \frac{(a-b)^{1/2}x}{(a+b)^{1/2}}]^{1/2}} = \left( \frac{m}{2E} \right)^{1/2} x \] (14)

independent of the coefficients \( (a, b, c) \). While individual microstates have a residual indeterminacy in the classical limit in its motion in the \( [x, t] \) domain, this indeterminacy is centered about the classical motion in the \( [x, t] \) domain regardless of which particular microstate is specified. Nevertheless, the degree of indeterminacy is a function of the microstate as specified by \( (a, b, c) \).
3 Classically Forbidden Case

The QSHJE renders real solutions, including conjugate momentum, for the trajectory in the classically forbidden region. On the other hand, the CSHJE has a turning point at the WKB turning point. If classical trajectories were permitted beyond the turning point, the classical momentum would become imaginary. Let us now examine a particle in the classically forbidden region, \( E < V(x) \), in the classical limit, \( \hbar \to 0 \). Here, we choose

\[
V = \begin{cases} 
U > E, & x \geq 0 \\
0, & x < 0 
\end{cases}
\]

representing an infinite step barrier. In the classically forbidden region, \( x > 0 \), the set of independent solutions to the associated Schrödinger equation is given by

\[
\phi = \exp\left\{ -\frac{[2m(U - E)]^{1/2}x}{(U - E)(ab - c^2)]^{1/4}} \right\} \quad \text{and} \quad \theta = \exp\left\{ \frac{[2m(U - E)]^{1/2}x}{(U - E)(ab - c^2)]^{1/4}} \right\}.
\]

From Eq. (3), the conjugate momentum can be expressed by

\[
W_x = \frac{(ab - c^2/4)^{1/2}[8m(U - E)]^{1/2}}{a \exp\left\{[-2m(U - E)]^{1/2}x/\hbar\right\} + b \exp\{[2m(U - E)]^{1/2}x/\hbar\} + c}.
\]

As \( \hbar \to 0 \), the \( \exp\{[2m(U - E)]^{1/2}x/\hbar\} \) term in the denominator of the above increases without limit. Since its coefficient \( b \) is finite real, we have in the forbidden zone in the classical limit for the conjugate momentum that

\[
\lim_{\hbar \to 0} W_x = 0 \quad (15)
\]

regardless of either the particular microstate specified by \((a, b, c)\) or where the particle is in the forbidden region as long as it is in the forbidden region by a finite distance. However, the conjugate momentum is generally not the mechanical momentum, i.e. \( W_x \neq m\dot{x} \).

The reduced action in the forbidden region is given by

\[
W = \hbar \arctan\left( \frac{b \exp\{[2m(U - E)]^{1/2}x/\hbar\} + c/2}{(ab - c^2/4)^{1/2}} \right), \quad x > 0.
\]

The classical limit for reduced action in the forbidden region renders

\[
\lim_{\hbar \to 0} W \to \hbar/4, \quad x > 0
\]

independent of either the particular microstate or where the particle is as long as \( x \) is finite positive. As the reduced action in the forbidden region is a constant, we arbitrarily choose to use the principal value to evaluate \( \arctan(\infty) = \pi/2 \) in the above equation. This is consistent with the Maslov index [23] which becomes exact for a piecewise constant potential.

The equation of motion is given by Jacobi’s theorem, \( t - t_o = W_E \). For the particle with sub-barrier energy inside the step barrier, the motion is given by

\[
t - t_o = \frac{(ab - c^2/4)^{1/2}}{a \exp\{-2m(U - E)]^{1/2}x/\hbar\} + b \exp\{2m(U - E)]^{1/2}x/\hbar\} + c \left( \frac{2m}{U - E} \right)^{1/2}} x, \quad x > 0.
\]

In the limit \( \hbar \to 0 \), the equation of motion for a particle at a finite distance inside the step barrier is given by

\[
\lim_{\hbar \to 0} (t - t_o) = 0, \quad \text{for} \ x \ \text{finite positive}.
\]

\[15\]
Thus, the particle travels with infinite speed in the classical limit. This counterintuitive finding is consistent with the findings regarding tunnelling that showed that dwell time decreased with increasing $[2m(U - E)]^{1/2}/\hbar$. For completeness, some insight has been already gained on this counterintuitive phenomenon by examining a particle with oblique incidence to the barrier. Such a particle was shown to have a trajectory that is not normal to the iso-$W$ surface inside the barrier. In the classical limit, it was shown that the particle’s trajectory becomes imbedded in an iso-$W$ surface.

The quantum term, $\hbar^2 (W; x)/(4m)$, on the right side of Eq. (17) is given in the forbidden region inside the step barrier by

$$\frac{\hbar^2}{4m} (W; x) = (U - E) \left(-\frac{a \exp\{-2[2m(U - E)]^{1/2} x/\hbar\} + b \exp\{2[2m(U - E)]^{1/2} x/\hbar\}}{a \exp\{-2[2m(U - E)]^{1/2} x/\hbar\} + b \exp\{2[2m(U - E)]^{1/2} x/\hbar\} + c}\right)^2, \quad x > 0. $$

In the classical limit, the quantum term becomes

$$\lim_{\hbar \to 0} \left(\frac{\hbar^2}{4m} (W; x)\right) = E - U \leq 0, \quad x > 0. \quad (18)$$

Equations (17) and (18) balance the QSHJE, Eq. (11) in the classically forbidden region in the classical limit as expected.

## 4 WKB Turning Point

We now investigate the trajectory in the classical limit in the vicinity of the WKB turning point. We also examine the transition between the classically allowed and classically forbidden regions in the classical limit as the trajectory transits the WKB turning point. We choose the potential to be

$$V = fx, \quad (19)$$

which represents a constant force $f > 0$ acting on our particle. Any well-behaved one-dimensional potentials for which the force remains finite and continuous can always be approximated by a linear potential in a sufficiently small region containing the WKB turning point.

Let us digress briefly. In the previous two sections, we examined potentials that were at least piecewise constant. Even though the independent solution set, $(\phi, \theta)$, for a step potential is mathematical simpler, such a potential does not have a classical short-wave correspondence at the turning point for the relative change in the potential over a wavelength remains large there. The reflection coefficient is independent of $\hbar$. Even a sixteen-inch armor-piercing naval projectile with super barrier energy would still experience partial backscatter.

For a particle with energy $E$, the WKB turning point, $x_\ell$, is given by $x_\ell = E/f$. An acceptable set of independent solutions, $(\phi, \theta)$, to the Schrödinger equation is formed of Airy functions given by

$$\phi = \frac{(2m)^{1/12} \pi^{1/2} \text{Ai}[2mf/\hbar^2]^{1/3}(x - E/f)}{(\hbar f)^{1/6}(ab - c^2/4)^{1/2}} \quad \text{and} \quad \theta = \frac{(2m)^{1/12} \pi^{1/2} \text{Bi}[2mf/\hbar^2]^{1/3}(x - E/f)}{(\hbar f)^{1/6}(ab - c^2/4)^{1/2}}. \quad (20)$$

The reduced action for the linear potential specified by Eq. (19) is given by

$$W = \hbar \arctan \left(\frac{b \text{Bi}(\xi/\hbar^{2/3})/\text{Ai}(\xi/\hbar^{2/3}) + c/2}{(ab - c^2)^{1/2}}\right) \quad (21)$$
where \( \xi = (2mf)^{1/3}(x - E/f) \) for expository convenience. For \((fx - E) \) finite negative, the classical limit of the reduced action in the classically allowed region outside any infinitesimal neighborhood containing the WKB turning point is given by

\[
\lim_{\hbar \to 0} W = \frac{(4/3)(ab - c^2)^{1/2}(2m)^{1/2}}{a + b + [(a - b)^2 + c^2]^{1/2} \sin[\frac{\xi^{3/2}}{3 \hbar} + \cot^{-1}(\frac{c}{a-b})]} (E - fx)^{3/2}, \quad \text{for } (fx - E) \text{ finite negative.} \tag{22}
\]

If \( a = b \) and \( c = 0 \), then Eq. (22) simplifies to

\[
\lim_{\hbar \to 0} W \bigg|_{a=b,c=0} = \frac{[2m(E - fx)]^{3/2}}{3mf}, \quad \text{for } (fx - E) \text{ finite negative,}
\]

which is consistent with the reduced action for classical mechanics for the corresponding linear potential as expected. For \((fx - E) \) finite positive and for any microstate, then the classical limit of the reduced action in the classically forbidden region outside any infinitesimal neighborhood containing the WKB turning point is constant given by

\[
\lim_{\hbar \to 0} W = h/4, \quad \text{for } (fx - E) \text{ finite positive,}
\]

which is a constant consistent with the Maslov index.

The conjugate momentum for the linear potential specified by Eq. (19) is given by

\[
W_x = \frac{(2mf)^{1/3}(ab - c^2/4)^{1/2}}{\pi [aA^2(\xi/h^{2/3}) + bB^2(\xi/h^{2/3}) + cAi(\xi/h^{2/3})Bi(\xi/h^{2/3})]}. \tag{23}
\]

For \((fx - E) \) finite negative, the classical limit of the conjugate momentum in the classically allowed region outside any infinitesimal neighborhood containing the WKB turning point is given by

\[
\lim_{\hbar \to 0} W_x = \frac{2(ab - c^2/4)^{1/2}[2m(E - fx)]^{1/2}}{a + b + [(a - b)^2 + c^2]^{1/2} \sin[\frac{\xi^{3/2}}{3 \hbar} + \cot^{-1}(\frac{c}{a-b})]}, \quad \text{for } (fx - E) \text{ finite negative.} \tag{24}
\]

For \( a = b \) and \( c = 0 \), then \( \lim_{\hbar \to 0} W_x = [2m(E - fx)]^{1/2} \), which is consistent with the classical momentum in the allowed region. For \((fx - E) \) finite positive and for any microstate, the classical limit of the reduced action in the classically forbidden region outside any infinitesimal neighborhood containing the WKB turning point is zero as expected.

We now consider the quantum equation of motion that is given by Jacobi’s theorem. For the particle with energy \( E \) and subject to the linear potential, Eq. (19), the motion is given by

\[
t - t_o = \frac{h^{1/3}}{\pi} \frac{(ab - c^2/4)^{1/2}(2m/f^2)^{1/3}}{aA^2(\xi/h^{2/3}) + bB^2(\xi/h^{2/3}) + cAi(\xi/h^{2/3})Bi(\xi/h^{2/3})}. \tag{24}
\]

For \((fx - E) \) finite negative, the classical limit of the quantum equation of motion in the classically allowed region outside any infinitesimal neighborhood containing the WKB turning point is given by

\[
\lim_{\hbar \to 0} (t - t_o) = \frac{2(ab - c^2/4)^{1/2}[2m(E - fx)]^{1/2}/f}{a + b + [(a - b)^2 + c^2]^{1/2} \sin[\frac{\xi^{3/2}}{3 \hbar} + \cot^{-1}(\frac{c}{a-b})]}, \quad \text{for } (fx - E) \text{ finite negative.} \tag{25}
\]

For the microstate specified by \( a = b \) and \( c = 0 \), the equation of motion simplifies to

\[
\lim_{\hbar \to 0} (t - t_o) \bigg|_{a=b,c=0} = \frac{[2m(E - fx)]^{1/2}}{f}, \quad \text{for } (fx - E) \text{ finite negative,}
\]
which is consistent with the classical equation of motion for a linear potential. In the forbidden region
outside any infinitesimal region containing the WKB turning point, the equation of motion renders
\[
\lim_{h \to 0} (t - t_o) = 0, \quad \text{for} \ (fx - E) \ \text{finite positive},
\]
which is consistent with the classical forbidden region.

In summary, we see that in the classical limit there is a consistent transition across a WKB turning point
between the classically allowed and forbidden regions. Nevertheless, the motion remains “quantum” in an
infinitesimal region containing the WKB turning point. However, the measure of this region in the classical
limit reduces to zero.

5 Initial Values

The QSHJE is a third order nonlinear differential equation while the CSHJE is first order. The reduced
action, \( W \) or \( W^{\text{classical}} \), does not explicitly appear in either the quantum or classical equation respectively.
Then, a set of necessary and sufficient initial values at \( x_o \) needed to specify the quantum conjugate momentum
in addition to the constant of motion \( E \) is \([W_x(x_o), W_{xx}(x_o)]\) while just \( E \) is necessary and sufficient to
specify the classical conjugate momentum. Knowing \( W(x_o) \) specifies the integration constant \( K \) in Eq. (2). Subsequently, the quantum and classical reduced actions are known to within an arbitrary integration constant.

The initial values for the solution of the QSHJE specify the particular microstate. We note that, when \( a = b \) and \( c = 0 \), the quantum trajectory solution for \( V = 0 \) and for \( V = fx \) in the classical limit reduced to the
classical trajectory solution. The underlying physics is that the coefficients \( a = b \) and \( c = 0 \) for \( V = 0 \)
render by Eq. (5) that \( W_x \) be a constant, \((2mE)^{1/2}\), independent of \( \hbar \) consistent with the corresponding
\( W_x^{\text{classical}} \). Even for finite \( \hbar \), selecting \( a = b \) and \( c = 0 \) causes \( W_{xx} \) and \( W_{xxx} \) to be zero consistent with classical
mechanics for \( V = 0 \). If \( E \) is unknown, then \([W_x(x_o), W_{xx}(x_o), W_{xxx}(x_o)]\) forms a necessary and sufficient
set of initial values to specify \( W(x) \). For the linear potential \( V = fx \), selecting \( a = b \) and \( c = 0 \), we have
by Eq. (23) and its differentiation that \( W_x = [2m(E - fx)]^{1/2} \) and \( W_{xx} = f \) again consistent with classical
mechanics for a linear potential. Thus, choosing \( a = b \) and \( c = 0 \) tacitly induces the necessary initial values
\([W_x(x_o), W_{xx}(x_o)]\) for the QSHJE to correspond to the superfluous initial values \([W_x^{\text{classical}}(x_o), W_{xx}^{\text{classical}}(x_o)]\)
for the corresponding CSHJE. Another view is that classical mechanics inherently assumes that \( a = b \) and
\( c = 0 \) for our choices, Eqs. (1) and (20), for the set \((\phi, \theta)\) of independent solutions. For completeness, had we
chosen a different set of independent solutions than those specified by Eqs. (1) or (20) for \( V = 0 \) and \( V = fx \)
respectively, then, for that potential, we would have had to choose a different set of coefficients \((a, b, c)\) to
achieve Planck correspondence to classical mechanics.

Let us now consider a special class of microstates. For this class, we set \((a - b)^2 + c^2 = \eta(h, \chi)\) for the
sets of independent solutions given by Eqs. (1) and (21). While \( \hbar \) is treated as an independent variable
in this exposition, \( \chi \) is not but rather is a set of other physical constants. We use this notation to make
it explicit that \( \eta \) to be dimensionless, must be dependent upon other physical constants. Mathematically,
\( \eta \) is the square of the amplitude of the trigonometric term, whose argument has inverse \( \hbar \) dependence, in
the solutions for \( W_x \) as shown by Eqs. (1) and (24). Also let us set \( \lim_{\hbar \to 0} \eta = 0 \). Then the classical limit
of the trajectory representation for this special class of microstates would have Planck correspondence to
classical mechanics including correspondence to superfluous initial values \([W_x(x_o), W_{xx}(x_o)]\). For this class
of microstates, the limiting procedure with respect to \( \hbar \) would be analogous to the procedure used by
Faraggi and Matone in Ref. 4.

6 Loss of Information

Passing to the classical limit incurs a loss of information, associated with a set of necessary and sufficient
initial values, due to going from the third-order QSHJE to the first-order CSHJE. Likewise, even for finite \( \hbar \),
the Copenhagen interpretation loses the information inherent to microstates of the trajectory representation despite the Copenhagen school asserting that the Schrödinger wave function is exhaustive. For completeness, we study loss of information for both cases and compare our results with ’t Hooft. There are similarities and significant differences.

First, we shall examine the quantum level (ℏ finite). As already noted, the trajectory representation in its Hamilton-Jacobi formulation manifests microstates not discernible by the Schrödinger representation showing that the Schrödinger wave function cannot be the exhaustive description of nature. For a specified energy \( E \), each microstate, by itself, specifies the Schrödinger wave function. Yet, each microstate of energy eigenvalue \( E \) has a distinct trajectory specified by the set of initial values \( [W_x(x_o), W_{xx}(x_o)] \). Are these microstates primordial at the Planck level? Yes, we have already shown elsewhere that the trajectories are deterministic and that the trajectory representation has arbitrary initial conditions without concern for any nonequilibrium initial values that encumbered Bohm at the Planck level. As the Copenhagen school asserts that \( \psi \) should be the exhaustive description of natural phenomenon, the Copenhagen school denies that primordial microstates could exist. Viewed externally, the Copenhagen school unwittingly makes \( \psi \) to be a de facto equivalence class of any putative microstates. This is consistent with QSHJE being more fundamental than the stationary Schrödinger equation, in contrast to Messiah’s assertion because, as shown elsewhere, the bound-state boundary conditions of the QSHJE do not generate a unique solution but rather generate an infinite number of “primordial” microstates while the boundary conditions for the Schrödinger equation do generate a unique “equivalence-class” bound-state wave function. Let us make a few comparisons with a ’t Hooft process. The primordial microstates are deterministic trajectories of discrete energy \( E \) in contrast to the ’t Hooft primordial states that are of the continuum. Nevertheless, as all initial values are allowed for the microstates, the trajectories for any equivalence class manifested by \( \psi \) densely spans finite phase space. Also, the Copenhagen school loses information on primordial microstates by default and not through a ’t Hooft dissipative process.

Second, let us examine loss of information in the trajectory representation by executing the classical limit. In the trajectory representation, residual indeterminacy manifests loss of information. The residual indeterminacy for the solution, \( W_x \), of the QSHJE in the classical limit is given for \( V = 0 \) and \( V = f \) by the trigonometric terms

\[
[(a - b)^2 + c^2]^{1/2} \cos\left\{\frac{2mE}{\hbar}x/\hbar + \cot^{-1}\frac{c}{(a - b)}\right\}
\]

and

\[
[(a - b)^2 + c^2]^{1/2} \sin\left\{\frac{d}{3}c^{3/2}/\hbar + \cot^{-1}\frac{c}{(a - b)}\right\}
\]

in the denominators of Eqs. (7) and (23) respectively. Here, we set the square of the amplitude of these trigonometric terms to be \( A = (a - b)^2 + c^2 \) where \( A \) is dimensionless. Explicitly, \( A \) is specified to be independent of \( \hbar \) in contrast to \( \eta \) of Sec. 5. The phase shift of the argument of these trigonometric terms is manifested by \( \cot^{-1}\frac{c}{(a - b)} \). The factor \( \hbar^{-1} \) in the argument of these trigonometric terms induces an indeterminacy in the classical limit that makes the phase shift due to \( \cot^{-1}\frac{c}{(a - b)} \) irrelevant. This represents a loss of information. On the other hand, The phase shift, \( \cot^{-1}\frac{d/3}{(a - b)} \), is not redundant for specifying the set coefficients \( (a, b, c) \) from the set of necessary and sufficient initial values \( [W_x(x_o), W_{xx}(x_o)] \) for the QSHJE for a given \( E \). Hence, the irrelevance of the phase shift gives the set of coefficients \( (a, b, c) \) another degree of freedom that makes the set of coefficients underspecified in the classical limit. This underspecification of coefficients \( (a, b, c) \) permits the primordial microstates to form into equivalence classes where the primordial microstates establish the membership within any particular equivalence class and become identical with one another in the classical limit. This information loss differs with that of ’t Hooft. As before, the primordial microstates have discrete rather than continuum energies. Also again, no dissipation of information occurs in the trajectory representation when going to the classical limit, but rather this loss of information induces an indeterminacy.

We may generalize to say that as classical mechanics has a smaller set of necessary and sufficient initial values than the trajectory representation of quantum mechanics, then there is some loss of information...
and the formation of equivalence classes as we go to the classical limit. Also, the Copenhagen school by precept considers $\psi$ to be exhaustive and disregards any microstate information. In either case, this loss of information is not due to any dissipation as it is in 't Hooft’s proposal. Without dissipation, this loss of information may occur for stationarity of the quantum Hamilton-Jacobi equation.

7 Indereminarin

Let us begin by contrasting the residual indeterminacy of the trajectory representation to Heisenberg uncertainty. Residual indeterminacy and Heisenberg uncertainty manifest themselves in different regimes of $\hbar$. In the classical limit, $\hbar \to 0$, Heisenberg uncertainty goes to zero as the commutation relations are linear in $\hbar$. On the other hand, the residual uncertainty in the trajectory representation exists only for $\hbar \to 0$. We note that residual indeterminacy is consistent with the findings of Faraggi and Matone that the equivalence principle exists for quantum mechanics but not for classical mechanics. Otherwise, the trajectory representation remains causal and deterministic.

Heisenberg uncertainty exists in the $[x, p]$ domain (where $p$ is momentum) since the Hamiltonian operates in the $[x, p]$ domain. On the other hand, the trajectory representation through a canonical transformation to its Hamilton-Jacobi formulation operates in the $[x, t]$ domain. Residual indeterminacy of the trajectory representation is in the $[x, t]$ domain, cf. Eqs. (14) and (25).

Heisenberg uncertainty, which denies the existence of simultaneous knowledge of position and momentum, tacitly implies that the existence of such knowledge would suffice to render determinism even without knowing energy. Bohm, in his salad days as a Copenhagenist, cited the Heisenberg uncertainty principle for denying determinism. Bohm had also noted that knowledge of the exact position and velocity of an electron at some particular time and knowledge of the forces acting on the electron at all times would render the electron's classical trajectory and classical determinism. While the set $[p(t_0), x(t_0)]$ is sufficient to render $E$ and subsequently a unique solution (i.e., determinism) in classical mechanics, it is an insufficient set of initial values for specifying a unique solution for the QSHJE. The QSHJE, as a third-order differential equation, requires knowing the initial values of the higher order derivatives to specify a unique solution. As already noted herein, if $E$ is unknown, then the set of initial conditions $[W_x(x_0), W_{xx}(x_0), W_{xxx}(x_0)]$ for finite $x_0$ is necessary and sufficient to determine $E$ and a unique solution, $W_x(x)$, for the QSHJE. The Heisenberg uncertainty principle masks the fundamental cause of indeterminism in the Copenhagen interpretation. As long as the Copenhagen interpretation, even without Heisenberg uncertainty, assumes an insufficient set of initial values, tacitly or otherwise, to tackle the QSHJE or any other representation capable of rendering quantum motion, it has forfeited any chance to determine a unique solution. Without a sufficient set of initial values, it is premature to postulate any uncertainty that may exist within the insufficient set $[p(t_0), x(t_0)]$.

In closing, we remark on the impact of residual indeterminacy. In the classical limit, $\hbar \to 0$, the trajectory representation of quantum mechanics does not generally go to classical mechanics invalidating Planck’s correspondence principle. Nor does it go to statistical mechanics as the amplitude of the indeterminacy is given by $[(a - b)^2 + c^2]^{1/2}$ for the sets of independent solutions of the Schrödinger equation used herein, cf. Eqs. (6), (22), (23), and (25).

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