Learning Distributed Quantum State Discrimination with Noisy Classical Communications

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Abstract—Consider a distributed quantum sensing system in which Alice and Bob are tasked with detecting the state of a quantum system that is observed partly at Alice and partly at Bob via local operations and classical communication (LOCC). Prior work introduced LOCCNet, a distributed protocol that optimizes the local operations via parameterized quantum circuits (PQCs) at Alice and Bob. This paper presents Noise Aware-LOCCNet (NA-LOCCNet) for distributed quantum state discrimination in the presence of noisy classical communication. We propose specific ansatzes for the case of two observed qubit pairs, and we describe a noise-aware training design criterion. Through experiments, we observe that quantum, entanglement-breaking noise on the observed quantum system can be useful in improving the detection capacity of the system when classical communication is noisy.

Index Terms—Quantum machine learning, state discrimination, parameterized quantum circuits, entanglement-breaking channel

I. INTRODUCTION

Quantum state discrimination is central to many applications of quantum sensing, communication, networking, and computing [1]–[3]. Of particular interest for quantum sensing are settings in which distributed nodes have access to correlated quantum subsystems, and they are tasked with discriminating between two possible joint states of the overall system [4]. As illustrated in Fig. 1, in such a situation, the nodes may be able to implement local operations (LOs) on their shares of the quantum system, as well as to communicate using classical communication (CC) links. This paper introduces a novel protocol that leverages variational quantum algorithms [5, 6] for the design of the LOs, while accounting for noisy CC.

Traditionally, quantum state discrimination protocols based on LO and CC, also known as LOCC protocols, have been designed by hand by focusing on the discrimination of specific pairs of states. Specific examples include the discrimination of orthogonal pure states [7] and the discrimination of maximally entangled states [8]. Assuming the presence of two nodes, Alice and Bob, these methods select the unitary at Bob as a function of the output of measurements made by Alice and shared on a noiseless CC link with Bob. Recently, a flexible framework for the design of LOCC protocols via variational quantum algorithms was introduced in [9]. The approach, termed LOCCNet, prescribes the use of parameterized quantum circuits (PQCs) for the local unitaries applied by Alice and Bob.

PQCs have been widely investigated in recent years as means to program small-scale noisy quantum computers via classical optimization, with applications ranging from combinatorial optimization to generative modelling [5]. A PQC typically consists of a sequence of one- and two-qubit rotations, whose parameters can be optimized, as well as of fixed entangling gates. The design of LOCCNet in [9] considers the problem of distinguishing two orthogonal maximally entangled Bell states, where one of the Bell state is corrupted by an entanglement-breaking quantum channel. The design assumes ideal, noiseless, classical communications, and it operates on a single pair of qubits.

While the reviewed existing solutions assume ideal classical communications, this paper studies the case, illustrated in Fig. 1, in which the CC link between Alice and Bob is noisy. To address this more challenging scenario, we introduce ansatzes for the PQCs to be applied by Alice and Bob that allow the joint processing of two qubit pairs. The proposed
design targets the optimization of the probability of successful detection while accounting for the presence of CC noise. The introduced approach, termed Noise Aware-LOCCNet (NA-LOCCNet), is shown via experiments to have significant advantages over existing protocols designed for noiseless communications. Furthermore, we observe that, depending on the level of classical noise, a larger level of entanglement-breaking noise can be advantageous to facilitate successful decentralized discrimination.

In the related work [10], we have addressed the distinct problem of distributed entanglement distillation, and proposed a noise aware protocol that operates in the presence of noisy CC. Problem setting, ansatzes, and main results in [10] have no bearing on the contributions of this paper.

The rest of the paper is organized as follows. In Section II we formulate the problem setting of quantum state discrimination in the presence of noisy CC, and define the performance metrics of interest. In Section III we review the LOCCNet protocol [9], and in Section IV we present the proposed NA-LOCCNet protocol. Experimental results are presented in Section V and Section VI concludes the paper.

**Notations and Definitions:** For any non-negative integer $K$, $[K]$ represents the set $\{0, 1, \ldots, K\}$. Given a discrete set $A$ and positive integer $S$, $A^S$ represents the set of strings of length $S$ from the alphabet $A$. The Kronecker product is denoted as $\otimes$; $I$ represents the identity matrix; $M^\dagger$ represents the complex conjugate transpose of the matrix $M$; and $tr(M)$ represents trace of the matrix $M$. We adopt standard notations for quantum states, computational basis, and quantum gates [11]. Let $A$ and $B$ be two Hilbert spaces of dimensions $d_A$ and $d_B$, with computational basis $\{|i\rangle\}_{i=0}^{d_A-1}$ and $\{|j\rangle\}_{j=0}^{d_B-1}$, respectively. Any $2^{d_A+d_B} \times 2^{d_A+d_B}$ complex matrix $M$ on the Hilbert space $A \otimes B$, can be written as $M = \sum_{i,j,k,l} M_{ijkl} |i\rangle \langle j| \otimes |k\rangle \langle l|$, where $M_{ijkl}$ are complex numbers and the sums range over the sets $i, j \in [2^{d_A-1}]$ and $k, l \in [2^{d_B-1}]$. The partial transpose operator of $M$ with respect to $B$ is defined as $M^{T_B} = \sum_{i,j,k,l} M_{ijkl} |i\rangle \langle j| \otimes |k\rangle \langle l|$. The partial trace of $M$ with respect to $A$ is defined as $tr_A(M) = \sum_{i, j, k, l} M_{ijkl} |i\rangle \otimes |j\rangle (|l\rangle \otimes |k\rangle)$, where $|\rangle \otimes |\rangle$ is the $2^{d_A} \times 2^{d_A}$ identity matrix.

II. SETTING AND PERFORMANCE METRICS

As in [9], we study the distributed quantum state discrimination problem illustrated in Fig. 1. In it, two agents, Alice and Bob, observe pairs of entangled qubits, and are tasked with detecting the joint quantum state of the qubit pairs. To this end, Alice and Bob can carry out local operations (LOs), as well as classical communication (CC) from Alice to Bob, i.e., they can implement an LOCC protocol. Unlike [9], we assume that the CC link between Alice and Bob is noisy. Applications of this setting include quantum sensor networks, as well as diagnostic functionalities for entanglement testing in the quantum internet [14] [12].

A. Setting

Alice and Bob share $S$ qubit pairs $(A_s, B_s)$ with $s \in [S-1]$, where each qubit $A_s$ is at Alice and each qubit $B_s$ is at Bob. Each qubit pair $(A_s, B_s)$ is entangled in one of two possible ways: The joint state of each pair $(A_s, B_s)$ is either given by the density matrix $\rho_0 = \frac{1}{2} (\ket{\Phi^+} \bra{\Phi^+} + \ket{\Phi^-} \bra{\Phi^-})$, with maximally entangled Bell state $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle) / \sqrt{2}$; or it is in state $\rho_1 = N(|\Phi^-\rangle \bra{\Phi^-})$, where $N(\cdot)$ is an amplitude damping (AD) channel and $|\Phi^-\rangle = (|00\rangle - |11\rangle) / \sqrt{2}$ is a maximally entangled Bell state orthogonal to $|\Phi^\pm\rangle$. The AD channel applies separately to the two qubits, and is expressed as

$$N(\rho) = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} E_{ij} \rho E_{ij}^\dagger,$$

where $E_{ij} = E_i \otimes E_j$ with Kraus matrices

$$E_0 = |0\rangle \langle 0| + \sqrt{1-\gamma}|1\rangle \langle 1|$$

and

$$E_1 = \sqrt{\gamma}|0\rangle \langle 1|,$$

where $0 \leq \gamma \leq 1$ represents the noise parameter of the AD channel. For $\gamma = 0$ the AD channel does not alter the input Bell state $|\Phi^-\rangle$, whereas for $\gamma = 1$ the AD channel breaks the entanglement of the Bell state $|\Phi^-\rangle$, converting it to the product state $|00\rangle$. We note that results in this paper apply at a qualitative level to any other entanglement-breaking channel [13].

As seen in Fig. 1, Alice applies a parameterized quantum circuit (PQC) to the $S$ qubits $A_0, A_1, \ldots, A_{S-1}$ in her possession; then, it measures the $S$ qubits, and sends the $S$ classical bits obtained from the measurements to Bob. The PQC applied by Alice implements a $2^S \times 2^S$ unitary matrix $U^{\Theta}(\theta^A)$ that is parameterized by vector $\theta^A$. Given that the input state for each qubit pair is $\rho_i$, with $i \in \{0, 1\}$, the corresponding output state for the $2^S$ qubits $A_0, A_1, \ldots, A_{S-1}$ and $B_0, B_1, \ldots, B_{S-1}$ can be written as

$$\rho^{AB}_i = (U^A(\theta^A) \otimes I^B)\rho_i^{\otimes S}(U^A(\theta^A) \otimes I^B)^\dagger,$$

where $I^B$ is a $2^S \times 2^S$ identity matrix. The notation $\rho^{AB}_i$ represents the state of the $S$ qubit pairs, with qubits ordered so that Alice qubits $A_0, A_1, \ldots, A_{S-1}$ are listed prior to Bob's qubits $B_0, B_1, \ldots, B_{S-1}$.

Furthermore, Alice measures her qubits $A_0, A_1, \ldots, A_{S-1}$ using the $2^S$ projection matrices $\Pi^A_a = |a\rangle \langle a| \otimes I$ with $a \in \{0, 1\}^S$, where $|a\rangle$ is the computational basis vector corresponding to bit string $a$. The measurement returns the output $a \in \{0, 1\}^S$ with probability given by the Born rule, i.e.,

$$P^A_{a|\theta} = tr(\Pi^A_a \rho^{AB}_i).$$

Note that the probability $P^A_{a|\theta}$ is conditioned on the true initial state $\rho_i$ of the qubit pairs, Alice communicates the $S$ classical bits $a \in \{0, 1\}^S$ obtained from the measurement to Bob through a memoryless binary symmetric channel with bit-flip probability $p$.

As a result, Bob receives a message $\hat{a} \in \{0, 1\}^S$ with probability $P^{AB}_{\hat{a}|a} = p^{d_{\hat{a}, a}}(1-p)^{S-d_{\hat{a}, a}}$, where $d_{\hat{a}, a}$ is the Hamming distance between the bit strings $a$ and $\hat{a}$. We note that this model can also account for measurement noise at Alice [14] [15]. Therefore, the probability of receiving message
at Bob, when the qubit pairs initial state is $\rho_1$, is given by
\begin{equation}
P_{\hat{a}|i}^B = \sum_{a \in \{0,1\}^S} P_{\hat{a}|B} P_{A \rightarrow B} P_{A|a}.
\end{equation}
and the corresponding $2^S \times 2^S$ post-measurement density state of the $S$ qubits at Bob is
\begin{equation}
P_{\hat{a}|i}^B = \sum_{a \in \{0,1\}^S} P_{\hat{a}|B} P_{A \rightarrow B} P_{A|a} (|a \rangle \otimes I) P_{\hat{a}|B} P_{A \rightarrow B} P_{A|a} (|a \rangle \otimes I).
\end{equation}
Depending on the message $\hat{a} \in \{0,1\}^S$ received at Bob, Bob performs a local operation given by the unitary $U^B(\theta^B_{\hat{a}})$, leaving the $S$ qubits in his possession in the density state
\begin{equation}
P_{\hat{a}|B}^B = \sum_{a \in \{0,1\}^S} P_{\hat{a}|B} P_{A \rightarrow B} (\theta^B_{\hat{a}}) P_{\hat{a}|B} P_{A \rightarrow B} (\theta^B_{\hat{a}}) I.
\end{equation}
Finally, Bob applies a parity projective measurement on the $S$ qubits, using the projection matrices $\Pi_{\hat{a}|B} = \sum_{b \in \{0,1\}^S} |b \rangle \langle b|$ and $\Pi_{\hat{a}|B} = \sum_{\text{odd } b} |b \rangle \langle b|$, where “even” and “odd” refer to the number of 1’s in the bit string $b$, with $b \in \{0,1\}^S$. This produces the output $i \in \{0,1\}$ with probability
\begin{equation}
P_{\hat{a}|B}^B = \text{tr}(\Pi_{\hat{a}}^{i} P_{\hat{a}|B}^B).
\end{equation}

The goal of this work is to design the PQC parameters $\theta^A$ and $\theta^B = \{\theta^B_{\hat{a}}\}_{\hat{a} \in \{0,1\}^S}$ at Alice and Bob such that the estimated state index $\hat{i} \in \{0,1\}$ at Bob equals the true state index $i$ with high probability. We specifically focus on protocols with single qubit pair, i.e., $S = 1$ as studied in [9] in Section III and with two qubit pairs, i.e., $S = 2$, in Section IV.

B. Performance Metrics

Assuming that the two states $\rho_0$ and $\rho_1$ are selected a priori with equal probability, the average success probability is computed as
\begin{equation}
P_{\text{succ}}(\theta^A, \theta^B) = \frac{1}{2} \sum_{i=0}^{1} P_{\hat{a}|B}^i.
\end{equation}

This probability is a function of the PQC parameters $\theta^A$ and $\theta^B$ at Alice and Bob respectively. We are interested in the problem of maximizing the average success probability
\begin{equation}
\max_{\theta^A, \theta^B} P_{\text{succ}}(\theta^A, \theta^B).
\end{equation}

Problem [11] requires a search over the space of $|\theta^A| + \sum_{\hat{a} \in \{0,1\}^S} |\theta^B_{\hat{a}}|$ PQC parameters, where $|\theta|$ represents the size of the vector $\theta$. This search can be carried out using standard optimization techniques, such as gradient descent.

We now discuss two upper bounds on the average success probability [10], namely the Helstrom bound and the PPT bound.

**Helstrom Bound:** Assume that all $S$ qubit pairs were available at a central node that could perform global measurements on all qubits. The maximum probability of successful detection in this system provides an upper bound on the probability of success for the decentralized system under study. Allowing for a general positive operator valued measure (POVM), this approach yields the Helstrom bound [16] [17]
\begin{equation}
P_{\text{succ}} \leq \frac{1}{2} + \frac{1}{4} \|\rho_0 \circ S - \rho_1 \circ S\|_1,
\end{equation}
where $\|H\|_1$ represents the $l_1$-norm of the Hermitian matrix $H$, which is defined as the sum of the absolute values of the eigenvalues of matrix $H$.

**Positive Partial Transpose (PPT) Bound:** A tighter bound is obtained by restricting the type of measurements that are allowed at the central node having access to all $S$ qubit pairs. In particular, such restriction can be defined as to include as a special case LOCC operations [18]. The resulting PPT bound is obtained as the maximum value of the objective function of the semidefinite program (SDP) [19]
\begin{equation}
\max_{M_0, M_1} \frac{1}{2} \text{tr}(M_0 \rho_0 \circ S + M_1 \rho_1 \circ S)
\end{equation}
subject to
\begin{equation}
\begin{cases}
{M_1}_i \geq 0 & \text{for } i = 0,
M_1 \geq 0 & \text{for } i = 1,
M_0 + M_1 = I,
\end{cases}
\end{equation}
where $M_1 = T_n$ represents the partial transpose of the operator $M_i$ [9] [20] with respect to the Hilbert space of Bob’s qubits.

III. LOCCNET

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{LOCCNet.png}
\caption{Illustration of the LOCCNet protocol [9], which operates on a single pair of qubits ($S = 1$).}
\end{figure}

In this section, we review the LOCCNet protocol introduced in [9], which applies separately to each pair of qubits, i.e., $S = 1$. As illustrated in Fig. 2 in LOCCNet the PQCs at Alice and Bob consists of Pauli $Y$-rotation gates, where the one-qubit Pauli $Y$-rotation gate is defined as [11]
\begin{equation}
R_Y(\theta) = \begin{bmatrix}
\cos(\theta/2) & -\sin(\theta/2) \\
\sin(\theta/2) & \cos(\theta/2)
\end{bmatrix}.
\end{equation}

LOCCNet assumes a noiseless CC from Alice to Bob, and hence it addressed the special case of the optimization problem in [11] with $p = 0$. The optimized rotation angles are given as $\theta^A = \pi/2$ and $\theta^B = (-1)^k(\pi - \arctan(\alpha))$, with $\alpha \in \{0,1\}$ and $\alpha = (2 - \gamma)/2$, where $\gamma \in [0,1]$ is the noise parameter of the AD channel.

In Section V, we will also evaluate the performance of the system illustrated in Fig. 2 when the PQC parameters $\theta^A$ and $\theta^B$ are optimized by addressing problem [11] with the correct value of the channel bit flip probability $p$. 

IV. NOISE AWARE-LOCCNET

In this section, we introduce the NA-LOCCNet protocol, which operates on $S = 2$ qubit pairs. There are two main innovations as compared to the LOCCNet protocol: (i) We introduce an ansatz for the PQCs at Alice and Bob based on two-qubit rotation gates that can outperform the separate application of the LOCCNet protocol in Fig. 2 to the two qubit pairs; (ii) We propose the direct optimization of the noise-aware performance objective (11), which is capable of adapting to the current classical noise level $p$, as well as to the quantum noise level $\gamma$.

For the PQCs, we adopt the architecture shown in Fig. 3 where the two qubit Pauli $ZY$-rotation gate is defined as

$$R_{ZY}(\theta) = \exp \left( -i \frac{\theta}{2} (Z \otimes Y) \right).$$ (15)

Note that the Pauli $ZY$-rotation gates are followed at Alice, and preceded at Bob, by a controlled NOT (CNOT) gate. This ansatz has been selected through an empirical search.

For every value of the noise level $\gamma$ and bit flip probability $p$ we propose to optimize the average success probability in (10) over the rotation angles $\theta_A^{\hat{a}}$ and $\theta_B^{\hat{a}}$ where $\hat{a} \in \{0, 1\}^2$. 

For the experiments, we evaluate the performance of the proposed NA-LOCCNet protocols in the presence of a noisy CC link from Alice to Bob. We assume the availability of $S$ qubit pairs, and we consider LOCCNet, reviewed in Section III, as the benchmark protocol. As discussed in Section IV, LOCCNet applies separately to the two qubit pairs, while the proposed NA-LOCCNet operates jointly on the two qubit pairs. LOCCNet is designed as in [9] by setting $p = 0$ in the optimization problem (11), and we also evaluate the performance of the LOCCNet architecture in Fig. 2 when the optimization is done by accounting for the actual value of $p$. We label this scheme as NA-LOCCNet ($S = 1$), since the design is noise aware. Optimization is done using Adam gradient descent optimizer [22], with 0.01 learning rate and 1000 iterations. As performance bounds, we show the PPT bounds described in Section II-B, which are tighter than Helstrom bounds, for both the cases $S = 1$ and $S = 2$.

We simulated all the experiments on a laptop with i7 processor and 16 GB RAM. The PyTorch and MATLAB code for regenerating the results of this paper is available at https://github.com/kclip/Noise-Aware-LOCCNet.
Fig. 5] which demonstrates the gains of NA-LOCCNet at all values of the noise parameter of the AD channel $\gamma$. Interestingly, the probability of success first decreases and then increases as a function of the noise strength $\gamma$. To explain this behavior, consider the case $p = 0.5$ of a fully noisy CC link and assume that Alice does not perform any operation on her qubits. In this case, Bob needs to distinguish $\rho_0^{\otimes 2}$ and $\rho_1^{\otimes 2}$ based solely on the local states $\text{tr}_A(\rho_0^{\otimes 2})$ and $\text{tr}_A(\rho_1^{\otimes 2})$, where $\text{tr}_A(\cdot)$ represents the partial trace operation with respect to the qubits at Alice. The maximal probability of success for detection at Bob is given by the Helstrom bound (12) as

$$P_{\text{succ}} = \frac{1}{2} + \frac{1}{4} \left| \text{tr}_A(\rho_0^{\otimes 2}) - \text{tr}_A(\rho_1^{\otimes 2}) \right|_1. \quad (16)$$

The probability of success (16) takes the minimal value 0.5 when there is no AD quantum noise, i.e., when $\gamma = 0$, since in this case we have $\text{tr}_A(\rho_0^{\otimes 2}) = \text{tr}_A(\rho_1^{\otimes 2}) = 0.5I$. In contrast, at the other extreme, when $\gamma = 1$, we have $\text{tr}_A(\rho_0^{\otimes 2}) = 0.5I$ and $\text{tr}_A(\rho_1^{\otimes 2}) = \langle 0 | 0 \rangle$, and hence the probability of success (16) is given by $P_{\text{succ}} = 0.75 > 0.5$. This argument suggests that, when the CC noise level $p$ is sufficiently large, the presence of an entanglement-breaking channel can be instrumental in improving the detection performance achievable via LOCC.

VI. CONCLUSION

In this paper, we have studied the problem of quantum state discrimination in the presence of noisy classical communication. The states to be discriminated involve orthogonal Bell states, where the qubits of one of the Bell states are corrupted by quantum noise from an entanglement-breaking channel. We have proposed an LOCC protocol based on a variational quantum algorithm that targets the maximization of the average success probability. Unlike existing protocols, which assume ideal noiseless classical communication, the proposed noise-aware-LOCCNet (NA-LOCCNet) directly accounts for the presence of noisy classical communication. Simulation results have confirmed the advantages of the proposed solution. It is observed that quantum entanglement-breaking noise on the observed system can be advantageous to improve the detection capacity when classical communication is noisy.

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