Invariance of simultaneous similarity and equivalence of matrices under extension of the ground field

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Abstract

We give a new and elementary proof that simultaneous similarity and simultaneous equivalence of families of matrices are invariant under extension of the ground field, a result which is non-trivial for finite fields and first appeared in a paper of Klinger and Levy ([2]).

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1 Introduction

In this article, we let $K$ denote a field, $L$ a field extension of $K$, and $n$ and $p$ two positive integers.

Definition 1. Two families $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ of matrices of $M_n(K)$ indexed over the same set $I$ are said to be simultaneously similar when there exists $P \in \text{GL}_n(K)$ such that

$$\forall i \in I, \quad P A_i P^{-1} = B_i$$

(such a matrix $P$ will then be called a base change matrix with respect to the two families).

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Two families \((A_i)_{i \in I} \) and \((B_i)_{i \in I} \) of matrices of \(M_{n,p}(\mathbb{K})\) indexed over the same set \(I\) are said to be \textit{simultaneously equivalent} when there exists a pair \((P, Q) \in \text{GL}_n(\mathbb{K}) \times \text{GL}_p(\mathbb{K})\) such that
\[
\forall i \in I, \ P A_i Q = B_i.
\]
Of course, those relations extend the familiar relations of similarity and equivalence respectively on \(M_n(\mathbb{K})\) dans \(M_{n,p}(\mathbb{K})\), and they are equivalence relations respectively on \(M_n(\mathbb{K})\) dans \(M_{n,p}(\mathbb{K})\).

The simultaneous similarity of matrices is generally regarded upon as a “wild problem” where finding a useful characterisation by invariants seems out of reach. See [1] for an account of the problem and an algorithmic approach to its solution (for that last matter, also see [2]).

In this respect, our very limited goal here is to establish the following two results:

**Theorem 1.** Let \(\mathbb{K} - L\) be a field extension and \(I\) be a set. Let \((A_i)_{i \in I}\) and \((B_i)_{i \in I}\) be two families of matrices of \(M_n(\mathbb{K})\). Then \((A_i)_{i \in I}\) and \((B_i)_{i \in I}\) are simultaneously similar in \(M_n(\mathbb{K})\) if and only if they are simultaneously similar in \(M_n(L)\).

**Theorem 2.** Let \(\mathbb{K} - L\) be a field extension and \(I\) be a set. Let \((A_i)_{i \in I}\) and \((B_i)_{i \in I}\) be two families of matrices of \(M_{n,p}(\mathbb{K})\). Then \((A_i)_{i \in I}\) and \((B_i)_{i \in I}\) are simultaneously equivalent in \(M_{n,p}(\mathbb{K})\) if and only if they are simultaneously equivalent in \(M_{n,p}(L)\).

**Remarks.**

(i) In both theorems, the “only if” part is trivial.

(ii) It is an easy exercise to derive theorem 1 from theorem 2. However, we will do precisely the opposite!

## 2 A proof for simultaneous similarity

### 2.1 A reduction to special cases

In order to prove theorem 2 we will not, contra [2], try to give a canonical form for simultaneous similarity. Instead, we will focus on base change matrices and prove directly that if one exists in \(M_n(L)\), then another (possibly the same), also exists in \(M_n(\mathbb{K})\). To achieve this, we will prove the theorem in the two following special cases:

(i) \(\mathbb{K}\) has at least \(n\) elements;

(ii) \(\mathbb{K} - L\) is a separable quadratic extension.
Assuming these cases have been solved, let us immediately prove the general case. Case (i) handles the situation where $K$ is infinite. Assume now that $K$ is finite, and choose a positive integer $N$ such that $(\#K)^{2^N} \geq n$. Since $K$ is finite, there exists (see section V.4 of [3]) a tower of $N$ quadratic separable extensions

$$K \subset K_1 \subset K_2 \subset \cdots \subset K_N.$$ 

We let $M$ denote a compositum extension of $K_N$ and $L$ (as extensions of $K$):

$$\begin{array}{c}
  K \\
  | \\
  L \\
  | \\
  M.
\end{array}$$

Assume the families $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ of matrices of $M_n(K)$ are simultaneously similar in $M_n(L)$. Then they are also simultaneously similar in $M_n(M)$. However, $\#K_N = (\#K)^{2^N} \geq n$, so this simultaneous similarity also holds in $M_n(K_N)$. Using case (ii) by induction, when then obtain that that $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ are simultaneously similar in $M_n(K)$.

2.2 The case $\#K \geq n$

The line of reasoning here is folklore, but we reproduce the proof for sake of completeness. Let then $P \in \text{GL}_n(L)$ be such that

$$\forall i \in I, P A_i P^{-1} = B_i,$$

so

$$\forall i \in I, P A_i = B_i P.$$

Let $V$ denote the $K$-vector subspace of $L$ generated by the coefficients of $P$, and choose a basis $(x_1, \ldots, x_N)$ of $V$. Decompose then

$$P = x_1 P_1 + \cdots + x_N P_N$$

with $P_1, \ldots, P_N$ in $M_n(K)$, and let $W$ be the $K$-vector subspace of $M_n(K)$ generated by the $N$-tuple $(P_1, \ldots, P_N)$. Since the $A_i$’s and the $B_i$’s have all their coefficients in $K$, the previous relations give:

$$\forall i \in I, \forall k \in \{1, N\}, P_k A_i = B_i P_k$$

hence

$$\forall i \in I, \forall Q \in W, Q A_i = B_i Q.$$
It thus suffices to prove that $W$ contains a non-singular matrix. However, the polynomial $\det(Y_1 P_1 + \cdots + Y_N P_N) \in K[Y_1, \ldots, Y_N]$ is homogeneous of total degree $n$ and is not the zero polynomial because 
\[
\det(x_1 P_1 + \cdots + x_N P_N) = \det(P) \neq 0.
\]
Since $n \leq \# K$, we conclude that the map $Q \mapsto \det Q$ does not totally vanish on $W$, which proves that $W \cap \text{GL}_n(K)$ is non-empty, QED.

### 2.3 The case $L$ is a separable quadratic extension of $K$

We choose an arbitrary element $\varepsilon \in L \setminus K$ and let $\sigma$ denote the non-identity automorphism of the $K$-algebra $L$. Assume $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ are simultaneously similar in $M_n(L)$, and let $P \in \text{GL}_n(L)$ be such that 
\[
\forall i \in I, \quad P A_i P^{-1} = B_i.
\]
We first point out that the problem is essentially unchanged should $P$ be replaced with a $K$-equivalent matrix of $\text{GL}_n(L)$.

Indeed, let $(P_1, P_2) \in \text{GL}_n(K)^2$, and set $P' := P_1 P P_2^{-1} \in \text{GL}_n(L)$, and $A'_i := P_2 A_i (P_2)^{-1}$ and $B'_i := P_1 B_i (P_1)^{-1}$ for all $i \in I$. Then:
\[
\forall i \in I, \quad P' A'_i (P')^{-1} = B'_i.
\]

Since it follows directly from definition that $(A_i)_{i \in I}$ and $(A'_i)_{i \in I}$ are simultaneously similar in $M_n(K)$, and that it is also true of $(B_i)_{i \in I}$ and $(B'_i)_{i \in I}$, it will suffice to show that $(A'_i)_{i \in I}$ and $(B'_i)_{i \in I}$ are simultaneously similar in $M_n(K)$, knowing that they are simultaneously similar in $M_n(L)$.

Returning to $P$, we split it as $P = Q + \varepsilon R$ with $(Q, R) \in M_n(K)^2$.

The previous remark then reduces the proof to the case where the pair $(Q, R)$ is canonical in terms of Kronecker reduction (see chapter XII of [4] and our section 4). More roughly, when can assume, since $P$ is non-singular, that, for some $q \in [0, n]$
\[
Q = \begin{bmatrix} M & 0 \\ 0 & I_{n-q} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix}
\]
where $M \in M_q(K)$, $N$ is a nilpotent matrix of $M_{n-q}(K)$, and we have let $I_k$ denote the unit matrix of $M_k(K)$.

Let $i \in I$. Applying $\sigma$ coefficient-wise to $P A_i P^{-1} = B_i$, we get:
\[
\sigma(P) A_i \sigma(P)^{-1} = B_i = P A_i P^{-1},
\]

hence $A_i$ commutes with $\sigma(P)^{-1} P$. We now claim the following result:
Lemma 3. Under the preceding assumptions, any matrix of $M_n(K)$ that commutes with $\sigma(P)^{-1} P$ also commutes with $P$.

Assuming this lemma holds, we deduce that $\forall i \in I, P A_i P^{-1} = A_i$, hence $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ are equal, thus simultaneously similar in $M_n(K)$, which finishes our proof.

Proof of lemma 3. Let $A \in M_n(K)$ which commutes with $\sigma(P)^{-1} P$. Applying $\sigma$, we deduce that $A$ also commutes with $P^{-1} \sigma(P)$, hence with $I_n + (\sigma(\varepsilon) - \varepsilon) P^{-1} R$, hence with $P^{-1} R$ since $\sigma(\varepsilon) \neq \varepsilon$.

Notice then that $P^{-1} R = \begin{bmatrix} (M + \varepsilon I_q)^{-1} & 0 \\ 0 & (I_{n-q} + \varepsilon N)^{-1} N \end{bmatrix}$ with $(M + \varepsilon I_q)^{-1}$ non-singular and $(I_{n} + \varepsilon N)^{-1} N$ nilpotent, so $A$, which stabilizes both $\text{Im}(P^{-1} R)^n$ and $\text{Ker}(P^{-1} R)^n$, must be of the form $A = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ for some $(C, D) \in M_q(K) \times M_{n-q}(K)$.

Commutation of $A$ with $P^{-1} R$ ensures that $C$ commutes with $(M + \varepsilon I_q)^{-1}$, whereas $D$ commutes with $(I_{n-q} + \varepsilon N)^{-1} \neq (I_{n-q} + \varepsilon N)^{-1}$ hence with $(I_{n-q} + \varepsilon N)^{-1}$. It follows that $A$ commutes with $P^{-1}$, hence with $P$.

3 A proof for simultaneous equivalence

We will now derive theorem 2 from theorem 1. Under the assumptions of theorem 2, we choose an arbitrary object $a$ that does not belong to $I$, and define $C_a = D_a := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+p}(K)$

and, for $i \in I$,

$C_i = \begin{bmatrix} 0 & A_i \\ 0 & 0 \end{bmatrix}$ and $D_i = \begin{bmatrix} 0 & B_i \\ 0 & 0 \end{bmatrix}$ in $M_{n+p}(K)$.

The following two conditions are then equivalent :
(i) $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ are simultaneously equivalent ;
(ii) $(C_i)_{i \in I \cup \{a\}}$ and $(D_i)_{i \in I \cup \{a\}}$ are simultaneously similar.
Indeed, if condition (i) holds, then we choose \((P, Q) \in \text{GL}_n(\mathbb{K}) \times \text{GL}_p(\mathbb{K})\) such that \(\forall i \in I, PA_i Q = B_i\), set \(R := \begin{bmatrix} P & 0 \\ 0 & Q^{-1} \end{bmatrix}\), and remark that \(R \in \text{GL}_{n+p}(\mathbb{K})\) and \(\forall i \in I \cup \{a\}, RC_i R^{-1} = D_i\).

Conversely, assume condition (ii) holds, and choose \(R \in \text{GL}_{n+p}(\mathbb{K})\) such that \(\forall i \in I \cup \{a\}, RC_i R^{-1} = D_i\).

Equality \(RC_a R^{-1} = C_a\) then entails that \(R\) is of the form \(R = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\) for some \((P, Q) \in \text{GL}_n(\mathbb{K}) \times \text{GL}_p(\mathbb{K})\), and the other relations then imply that \(\forall i \in I, PA_i Q^{-1} = B_i\).

Using equivalence of (i) and (ii) with both fields \(\mathbb{K}\) and \(\mathbb{L}\), theorem 2 follows easily from theorem 1.

4 Appendix: on the Kronecker reduction of matrix pencils

Attention was brought to me that, in [4], the proof that every pencil of matrix is equivalent to a canonical one fails for finite fields. We will give a correct proof here in the case of a “weak” canonical form (that is all we need here, and reducing further to a true canonical form is not hard from there using the theory of elementary divisors).

Notation 2. For \(n \in \mathbb{N}\), set \(L_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & \ddots \\ 0 & 1 \end{bmatrix}\) \(\in M_{n,n+1}(\mathbb{K})\) and \(K_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & \ddots \\ 0 & 1 \end{bmatrix}\) \(\in M_{n,n+1}(\mathbb{K})\); and, for arbitrary objects \(a\) and \(b\), define the Jordan matrix:

\[ J_n(a, b) = \begin{bmatrix} a & b & 0 \\ 0 & a & b \\ \ddots & \ddots & \ddots \end{bmatrix} \in M_n(\{0, a, b\}). \]
Lemma 5. From there, the proof has two independent major steps:

- \( P + X I_r \) for some non-singular \( P \in GL_r(\mathbb{K}) \);
- \( J_r(1,X); J_r(X,1); L_r + X K_r; \) \((L_r + X K_r)^t\).

This decomposition is unique up to permutation of blocks and up to similarity on the non-singular \( P \).

We will only prove here that such a decomposition exists. Uniqueness is not needed here so we will leave it as an exercise for the reader.

We will consider \( A \) and \( B \) as linear maps from \( E = \mathbb{K}^p \) to \( F = \mathbb{K}^n \). Without loss of generality, we may assume \( \text{Ker} \, A \cap \text{Ker} \, B = \{0\} \) and \( \text{Im} \, A + \text{Im} \, B = F \). We define inductively two towers \((E_k)_{k \in \mathbb{N}}\) and \((F_k)_{k \in \mathbb{N}}\) of linear subspaces of \( E \) and \( F \) by:

(a) \( E_0 = \{0\} \); \( F_0 = A(\{0\}) = \{0\} \);

(b) \( \forall k \in \mathbb{N}, \, E_{k+1} = B^{-1}(F_k) \) and \( F_{k+1} = A(E_{k+1}) \).

Notice that \( E_1 = \text{Ker} \, B \). The sequences \((E_k)_{n \geq 0}\) and \((F_k)_{n \geq 0}\) are clearly non-decreasing so we can find a smallest integer \( N \) such that \( E_N = E_k \) for every \( k \geq N \). Hence \( F_N = F_k \) for every \( k \geq N \), and \( E_N = g^{-1}(F_N) \). It follows that \( A(E_N) = F_N \) and \( B(E_N) \subset F_N \). We now let \( f \) and \( g \) denote the linear maps from \( E_N \) to \( F_N \) induced by \( A \) and \( B \).

From there, the proof has two independent major steps:

Lemma 5. There are basis \( B \) and \( C \) respectively of \( E_N \) and \( F_N \) such that \( M_B C(f) + X M_B C(g) \) is block-diagonal with all non-zero blocks having one of the forms \( J_r(1,X) \) or \( L_s + X K_s \).

Lemma 6. There are splittings \( E = E_N \oplus E' \) and \( F = F_N \oplus F' \) such that \( A(E') \subset F' \) and \( B(E') \subset F' \).

Assuming those lemmas are proven, let us see how we can easily conclude:

- We deduce from the two previous lemmas that \( A + X B \) is \( \mathbb{K} \)-equivalent to some \( \begin{bmatrix} A' + X B' & 0 \\ 0 & C(X) \end{bmatrix} \) where \( C(X) \) is block-diagonal with all non-zero blocks of the form \( J_r(1,X) \) or \( L_s + X K_s \), and \( A' \) and \( B' \) have coefficients in \( \mathbb{K} \), with \( \text{Ker} \, B' = \{0\} \); it will thus suffice to prove the existence of a canonical form for the pair \( (A',B') \);
- applying the first step of the proof to the matrices \((A')^t \) and \((B')^t \), we find that \( A' + X B' \) is \( \mathbb{K} \)-equivalent to some \( \begin{bmatrix} A'' + X B'' & 0 \\ 0 & D(X) \end{bmatrix} \).
where $D(X)$ is block-diagonal with all non-zero blocks of the form $J_r(1,X)'$ (which is $\mathbb{K}$-similar to $J_r(1,X)$) or $(L_{s} + X K)_{s}'$, and $A''$ and $B''$ have coefficients in $\mathbb{K}$, with $\ker B'' = \{0\}$ and $\text{coker} B'' = \{0\}$. It follows that $B''$ is non-singular.

- Finally, $(B'')^{-1}(A'' + X B'') = (B'')^{-1}A'' + X I_k$ for some integer $k$, and the pair $(A'', B'')$ can thus be reduced by using the Fitting decomposition of $(B'')^{-1}A''$ combined with a Jordan reduction of its nilpotent part: this yields a block-diagonal matrix $\mathbb{K}$-equivalent to $A'' + X B''$ with all diagonal blocks of the form $J_r(X, 1)$ or $P + X I_s$ for some non-singular $P$. This completes the proof of existence.

**Proof of lemma**

We proceed by induction.

Assume, for some $k \in [1, N]$, that there are splittings $E = E_N \oplus E' \oplus E''$ and $F = F_N \oplus F'$ such that $A(E') \subset F' \oplus F_k$ and $B(E') \subset F' \oplus F_k$. Since $B^{-1}(F_N) = E_N$, the subspaces $F_N$ and $B(E')$ are independent. We can therefore find some $F''$ such that $F' \oplus F_k = F'' \oplus F_k$, $F_N \oplus F'' = F$ and $B(E') \subset F''$. Choose then a basis $(e_1, \ldots, e_p)$ of $E'$, and decompose $A(e_i) = f_i + f'_i$ for all $i \in [1, p]$, with $f_i \in F''$ and $f'_i \in F_k$. For $i \in [1, p]$, we have $f'_i = A(g_i)$ for some $g_i \in E_k$. Then $(e_1 - g_1, \ldots, e_p - g_p)$ still generates a supplementary subspace $E''$ of $E_N$ in $E$, and we now have $A(e_i - g_i) \in F''$ and $B(e_i - g_i) \in F'' \oplus F_{k-1}$ for all $i \in [1, p]$. Hence $E = E_N \oplus E''$ and $F = F_N \oplus F''$, now with $A(E'') \subset F'' \oplus F_{k-1}$ and $B(E'') \subset F'' \oplus F_{k-1}$. The condition is thus proven at the integer $k - 1$. By downward induction, we find that it holds for $k = 0$, QED.

**Proof of lemma**

The argument is similar to the standard proof of the Jordan reduction theorem.

- Split $F_N = F_{N-1} \oplus W_{N,N}$ and $E_N = E_{N-1} \oplus V_{N,N} \oplus V'_{N,N}$ such that $E_{N-1} \oplus V'_{N,N} = E_{N-1} + (E_N \cap \ker f)$, $V_{N,N} \subset \ker f$ and $f(V_{N,N}) = W_{N,N}$ (so $f$ induces an isomorphism from $V_{N,N}$ to $W_{N,N}$).

Set $W_{N,N-1} = g(V_{N,N})$ and $W'_{N,N-1} = g(V'_{N,N})$. Remark that $F_{N-2} \oplus W_{N,N-1} \oplus W'_{N,N-1} \subset F_{N-1}$, and split $F_{N-1} = F_{N-2} \oplus W_{N,N-1} \oplus W'_{N,N-1} \oplus W_{N-1,N-1}$.

- We then proceed by downward induction to define four families of linear subspaces $(V_{\ell,k})_{1 \leq k \leq \ell \leq N}$, $(V'_{\ell,k})_{1 \leq k \leq \ell \leq N}$ $(W_{\ell,k})_{1 \leq k \leq \ell \leq N-1}$ and $(W'_{\ell,k})_{1 \leq k \leq \ell - 1 \leq N-1}$ such that:

(i) for every $k \in [1, N]$, $E_k = E_{k-1} \oplus V_{k,k} \oplus V_{k+1,k} \oplus \cdots \oplus V_{N,k} \oplus V'_{k,k} \oplus V'_{k+1,k} \oplus \cdots \oplus V'_{N,k}$;

(ii) for every $k \in [1, N]$, $F_k = F_{k-1} \oplus W_{k,k} \oplus W_{k+1,k} \oplus \cdots \oplus W_{N,k} \oplus W'_{k,k} \oplus W'_{k+1,k} \oplus \cdots \oplus W'_{N,k}$;

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(iii) for every $k \in [1, N]$, $E_{k-1} + (E_k \cap \text{Ker} f) = E_{k-1} \oplus V'_{k,k}$ and $V'_{k,k} \subset \text{Ker} f$;
(iv) for every $\ell \in [1, N]$ and $k \in [2, \ell]$, $g$ induces an isomorphism $g'_{\ell,k} : V_{\ell,k} \xrightarrow{\sim} W_{\ell,k-1}$ and an isomorphism $g_{\ell,k} : V'_{\ell,k} \xrightarrow{\sim} W'_{\ell,k-1}$;
(v) for every $\ell \in [1, N]$ and $k \in [1, \ell]$, $f$ induces an isomorphism $f_{\ell,k} : V_{\ell,k} \xrightarrow{\sim} W_{\ell,k}$ and, if $k < \ell$, an isomorphism $f'_{\ell,k} : V'_{\ell,k} \xrightarrow{\sim} W'_{\ell,k-1}$.

\[ \begin{array}{ccccc}
& V_{\ell,1} & \cdots & V_{\ell,\ell} & \\
\{0\} & \downarrow g & \cdots & \downarrow g & \{0\} \\
& W_{\ell,1} & \cdots & W_{\ell,\ell} & \\
& \{0\} & \downarrow g & \cdots & \downarrow g & \{0\}. \\
\end{array} \]

- Set $\ell \in [1, N]$. Define $G_{\ell} = V_{\ell,1} \oplus \cdots \oplus V_{\ell,\ell}$, $G'_{\ell} = V'_{\ell,1} \oplus \cdots \oplus V'_{\ell,\ell}$, $H_{\ell} = W_{\ell,1} \oplus \cdots \oplus W_{\ell,\ell}$ and $H'_{\ell} = W'_{\ell,1} \oplus \cdots \oplus W'_{\ell,\ell-1}$.

Notice that:

\[ f(G_{\ell}) = H_{\ell}, \quad g(G_{\ell}) \oplus W_{\ell,\ell} = H_{\ell}, \quad f(G'_{\ell}) = H'_{\ell} \quad \text{and} \quad g(G'_{\ell}) = H'_{\ell}. \]

From there, it is easy to conclude.

- Let $n_{\ell} = \dim W_{\ell,\ell}$. Remark that $\dim V'_{\ell,k} = \dim W_{\ell,k} = n_{\ell}$ for every $1 \in [1, \ell]$ and choose a basis $C_{\ell,\ell}$ of $W_{\ell,\ell}$. Define $B_{\ell,\ell} = f_{\ell,\ell}^{-1}(C_{\ell,\ell})$, $C_{\ell,\ell-1} := g_{\ell,\ell}(B_{\ell,\ell})$ and proceed by induction to recover a basis for $V_{\ell,k}$ and $W_{\ell,k}$ for every suitable $k$: by gluing together those bases, we recover respective basis $(B_{\ell,1}, \ldots, B_{\ell,\ell})$ and $(C_{\ell,1}, \ldots, C_{\ell,\ell})$ of $G_{\ell}$ and $H_{\ell}$ and remark that $f$ and $g$ induce linear maps from $G_{\ell}$ to $H_{\ell}$ with respective matrices $L_{\ell} \otimes I_{n_{\ell}}$ and $K_{\ell} \otimes I_{n_{\ell}}$ in those basis (remember that $E_1 = \text{Ker} g$). A simple permutation of basis shows that those linear maps can be represented by $I_{n_{\ell}} \otimes L_{\ell}$ and $I_{n_{\ell}} \otimes K_{\ell}$ in a suitable common pair of basis.
• Proceeding similarly for $G'_\ell$ and $H'_\ell$, but starting from a basis of $V'_\ell$, we obtain that $f$ and $g$ induce linear maps from $G'_\ell$ to $H'_\ell$ and there is a suitable choice of basis so that their matrices are respectively $I_s \otimes I_\ell$ and $I_s \otimes J_{\ell}(0,1)$ for some integer $s$.

• Notice that we have defined splittings

$$E_N = G_1 \oplus G'_1 \oplus G_2 \oplus G'_2 \oplus \cdots \oplus G_N \oplus G'_N$$

and

$$F_N = H_1 \oplus H'_1 \oplus H_2 \oplus H'_2 \oplus \cdots \oplus H'_{N-1} \oplus H_N,$$

therefore lemma 5 is proven by glueing together the various basis built here.

\[\square\]

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