A Characterization of Minimal Surfaces in $S^5$ with Parallel Normal Vector Field

Rodrigo Ristow Montes *

Departamento de Matemática ,
Universidade Federal da Paraíba,
BR– 58.051-900 João Pessoa, P.B., Brazil

Abstract

In this paper we proof that the Holomorphic angle for compact minimal surfaces in the sphere $S^5$ with constant Contact angle and with a parallel normal vector field must be constant.

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1 Introduction

The notion of Kähler angle was introduced by Chern and Wolfson in [3] and [12]; it is a fundamental invariant for minimal surfaces in complex manifolds. Using the technique of moving frames, Wolfson obtained equations for the Laplacian and Gaussian curvature for an immersed minimal surface in $\mathbb{C}P^n$. Later, Kenmotsu in [7], Ohnita in [10] and Ogata in [11] classified minimal surfaces with constant Gaussian curvature and constant Kähler angle.

A few years ago, Li in [14] gave a counterexample to the conjecture of Bolton, Jensen and Rigoli (see [2]), according to which a minimal immersion (non-holomorphic, non anti-holomorphic, non totally real) of a two-sphere in $\mathbb{C}P^n$ with constant Kähler angle would have constant Gaussian curvature.

*ristow@mat.ufpb.br and ristow@math.wustl.edu
In [8] we introduced the notion of contact angle, that can be considered as a new geometric invariant useful to investigate the geometry of immersed surfaces in $S^3$. Geometrically, the contact angle ($\beta$) is the complementary angle between the contact distribution and the tangent space of the surface. Also in [8], we deduced formulas for the Gaussian curvature and the Laplacian of an immersed minimal surface in $S^3$, and we gave a characterization of the Clifford Torus as the only minimal surface in $S^3$ with constant contact angle.

We define $\alpha$ to be the angle given by $\cos \alpha = \langle ie_1, v \rangle$, where $e_1$ and $v$ are defined in section 2. The holomorphic angle $\alpha$ is the analogue of the Kähler angle introduced by Chern and Wolfson in [3].

Recently, in [9], we construct a family of minimal tori in $S^5$ with constant contact and holomorphic angle. These tori are parametrized by the following circle equation

$$a^2 + \left( b - \frac{\cos \beta}{1 + \sin^2 \beta} \right)^2 = 2 \frac{\sin^4 \beta}{(1 + \sin^2 \beta)^2};$$

where $a$ and $b$ are given in Section 3 (equation (9)). In particular, when $a = 0$ in (7), we recover the examples found by Kenmotsu, in [6]. These examples are defined for $0 < \beta < \frac{\pi}{2}$. Also, when $b = 0$ in (9), we find a new family of minimal tori in $S^5$, and these tori are defined for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$. Also, in [9], when $\beta = \frac{\pi}{2}$, we give an alternative proof of this classification of a Theorem from Blair in [1], and Yamaguchi, Kon and Miyahara in [13] for Legendrian minimal surfaces in $S^5$ with constant Gaussian curvature.

In this paper, we classify minimal surfaces in $S^5$ with constant Contact angle and with a parallel normal vector field. We suppose that $e_3$ (in equation (3)) is a parallel normal vector field, and we get the following

**Theorem 1.** The holomorphic angle ($0 < \alpha < \frac{\pi}{2}$) is constant for compact minimal surfaces in $S^5$ with constant Contact angle $\beta$ and null principal curvatures $a, b$.

## 2 Contact Angle for Immersed Surfaces in $S^{2n+1}$

Consider in $\mathbb{C}^{n+1}$ the following objects:

- the Hermitian product: $(z, w) = \sum_{j=0}^{n} z^j \bar{w}^j$;
- the inner product: $\langle z, w \rangle = Re(z, w)$;
- the unit sphere: $S^{2n+1} = \{z \in \mathbb{C}^{n+1} | (z, z) = 1\}$;
- the Reeb vector field in $S^{2n+1}$, given by: $\xi(z) = iz$;
- the contact distribution in $S^{2n+1}$, which is orthogonal to $\xi$:

$$\Delta_z = \{v \in T_zS^{2n+1} | \langle \xi, v \rangle = 0 \}.$$
We observe that $\Delta$ is invariant by the complex structure of $\mathbb{C}^{n+1}$.

Let now $S$ be an immersed orientable surface in $S^{2n+1}$.

**Definition 1.** The contact angle $\beta$ is the complementary angle between the contact distribution $\Delta$ and the tangent space $TS$ of the surface.

Let $(e_1, e_2)$ be a local frame of $TS$, where $e_1 \in TS \cap \Delta$. Then $\cos \beta = \langle \xi, e_2 \rangle$. Finally, let $v$ be the unit vector in the direction of the orthogonal projection of $e_2$ on $\Delta$, defined by the following relation

$$e_2 = \sin \beta v + \cos \beta \xi.$$  \hspace{1cm} (2)

### 3 Equations for Gaussian curvature and Laplacian of a minimal surface in $S^5$

In this section, we deduce the equations for the Gaussian curvature and for the Laplacian of a minimal surface in $S^5$ in terms of the contact angle and the holomorphic angle. Consider the normal vector fields

$$\begin{align*}
e_3 &= i \csc \alpha e_1 - \cot \alpha v \\
e_4 &= \cot \alpha e_1 + i \csc \alpha v \\
e_5 &= \csc \beta \xi - \cot \beta e_2
\end{align*}$$  \hspace{1cm} (3)

where $\beta \neq 0, \pi$ and $\alpha \neq 0, \pi$. We will call $(e_j)_{1 \leq j \leq 5}$ an adapted frame.

Using (2) and (3), we get

$$\begin{align*}
v &= \sin \beta e_2 - \cos \beta e_5, \quad iv &= \sin \alpha e_4 - \cos \alpha e_1 \\
\xi &= \cos \beta e_2 + \sin \beta e_5
\end{align*}$$  \hspace{1cm} (4)

It follows from (3) and (4) that

$$\begin{align*}
ie_1 &= \cos \alpha \sin \beta e_2 + \sin \alpha e_3 - \cos \alpha \cos \beta e_5 \\
ie_2 &= -\cos \beta z - \cos \alpha \sin \beta e_1 + \sin \alpha \sin \beta e_4
\end{align*}$$  \hspace{1cm} (5)

Consider now the dual basis $(\theta^i)$ of $(e_j)$. The connection forms $(\theta^i_j)$ are given by

$$De_j = \theta^i_j e_k,$$

and the second fundamental form with respect to this frame are given by

$$II^j = \theta^j_1 \theta^1 + \theta^j_2 \theta^2; \quad j = 3, ..., 5.$$
Using (5) and differentiating \(v\) and \(\xi\) on the surface \(S\), we get

\[
D\xi = - \cos\alpha \sin\beta \theta^2 e_1 + \cos\alpha \sin\beta \theta^1 e_2 + \sin\alpha \theta^1 e_3 + \sin\alpha \sin\beta \theta^2 e_4 - \cos\alpha \cos\beta \theta^1 e_5, \tag{6}
\]

\[
Dv = (\sin\beta \theta_2^1 - \cos\beta \theta_2^0) e_1 + \cos\beta (d\beta - \theta_3^1) e_2 + (\sin\beta \theta_3^2 - \cos\beta \theta_3^1) e_3 + (\sin\beta \theta_4^2 - \cos\beta \theta_4^1) e_4 + \sin\beta (d\beta + \theta_5^2) e_5.
\]

Differentiating \(e_3\), \(e_4\) and \(e_5\), we have

\[
\begin{align*}
\theta_3^1 &= -\theta_1^3 \\
\theta_3^2 &= \sin\beta (d\alpha + \theta_1^1) - \cos\beta \sin\alpha \theta^1 \\
\theta_3^4 &= \csc\beta \theta_2^1 - \cot\alpha (\theta_3^1 + \csc\beta \theta_2^1) \\
\theta_3^5 &= \cot\beta \theta_2^1 - \csc\beta \sin\alpha \theta^1 \\
\theta_4^1 &= -d\alpha - \csc\beta \theta_2^3 + \sin\alpha \cot\beta \theta^1 \\
\theta_4^2 &= -\theta_4^1 \\
\theta_4^3 &= \csc\beta \theta_2^1 + \cot\alpha (\theta_3^1 + \csc\beta \theta_2^1) \\
\theta_4^5 &= \cot\beta \theta_2^4 - \sin\alpha \theta^2 \\
\theta_5^1 &= -\cos\alpha \theta^2 - \cot\beta \theta_2^1 \\
\theta_5^2 &= d\beta + \cos\alpha \theta^1 \\
\theta_5^3 &= -\cot\beta \theta_2^3 + \csc\beta \sin\alpha \theta^1 \\
\theta_5^4 &= -\cot\beta \theta_2^4 + \sin\alpha \theta^2
\end{align*}
\tag{7}
\]

The conditions of minimality and of symmetry are equivalent to the following equations:

\[
\theta_1^3 \wedge \theta^1 + \theta_2^3 \wedge \theta^2 = 0 = \theta_1^4 \wedge \theta^2 - \theta_2^4 \wedge \theta^1. \tag{8}
\]

On the surface \(S\), we consider

\[
\theta_1^3 = a\theta^1 + b\theta^2
\]

It follows from (8) that

\[
\begin{align*}
\theta_1^3 &= a\theta^1 + b\theta^2 \\
\theta_2^3 &= b\theta^1 - a\theta^2 \\
\theta_1^4 &= d\alpha + (b \csc\beta - \sin\alpha \cot\beta)\theta^1 - a \csc\beta \theta^2 \\
\theta_2^4 &= d\alpha \circ J - a \csc\beta \theta^1 - (b \csc\beta - \sin\alpha \cot\beta)\theta^2 \\
\theta_1^5 &= d\beta \circ J - \cos\alpha \theta^2 \\
\theta_2^5 &= -d\beta - \cos\alpha \theta^1
\end{align*}
\tag{9}
\]

where \(J\) is the complex structure of \(S\) is given by \(Je_1 = e_2\) and \(Je_2 = -e_1\). Moreover,
the normal connection forms are given by:

\[
\begin{align*}
\theta^4_3 &= -\sec \beta d\alpha \circ J - \cot \alpha \csc \beta d\alpha \circ J + a \cot \alpha \cot^2 \beta \theta^1 \\
&\quad + (b \cot \beta - \cos \alpha \cot \beta \csc \beta + 2 \sec \beta \cos \alpha) \theta^2 \\
\theta^5_3 &= (b \cot \beta - \csc \beta \sin \alpha) \theta^1 - a \cot \beta^2 \\
\theta^5_4 &= \cot \beta (d\alpha \circ J) - a \cot \beta \csc \beta \theta^1 + (-b \csc \beta \cot \beta + \sin \alpha (\cot^2 \beta - 1)) \theta^2,
\end{align*}
\]

while the Gauss equation is equivalent to the equation:

\[
d\theta^1 + \theta^k_k \wedge \theta^2 = \theta^1 \wedge \theta^2.
\]

Therefore, using equations (9) and (11), we have

\[
K = 1 - |\nabla \beta|^2 - 2 \cos \alpha \beta_1 - \cos^2 \alpha - (1 + \csc^2 \beta)(a^2 + b^2) \\
+ 2b \sin \alpha \csc \beta \cot \beta + 2 \sin \alpha \cot \beta \alpha_1 - |\nabla \alpha|^2 \\
+ 2a \csc \beta \alpha_2 - 2b \csc \beta \alpha_1 - \sin^2 \alpha \cot^2 \beta \\
= 1 - (1 + \csc^2 \beta)(a^2 + b^2) - 2b \csc \beta (\alpha_1 - \sin \alpha \cot \beta) + 2a \csc \beta \alpha_2 \\
- |\nabla \beta + \cos \alpha e_1|^2 - |\nabla \alpha - \sin \alpha \cot \beta e_1|^2
\]

Using (11) and the complex structure of \( S \), we get

\[
\theta^2_1 = \tan \beta (d\beta \circ J - 2 \cos \alpha \theta^2)
\]

Differentiating (13), we conclude that

\[
d\theta^1_2 = -(1 + \tan^2 \beta)|\nabla \beta|^2 - \tan \beta \Delta \beta - 2 \cos \alpha (1 + 2 \tan^2 \beta) \beta_1 \\
+ 2 \tan \beta \sin \alpha \alpha_1 - 4 \tan^2 \beta \cos^2 \alpha \theta^1 \wedge \theta^2
\]

where \( \Delta = \text{tr} \nabla^2 \) is the Laplacian of \( S \). The Gaussian curvature is therefore given by:

\[
K = -(1 + \tan^2 \beta)|\nabla \beta|^2 - \tan \beta \Delta \beta - 2 \cos \alpha (1 + 2 \tan^2 \beta) \beta_1 \\
+ 2 \tan \beta \sin \alpha \alpha_1 - 4 \tan^2 \beta \cos^2 \alpha.
\]

From (12) and (14), we obtain the following formula for the Laplacian of \( S \):

\[
\tan \beta \Delta \beta = (1 + \csc^2 \beta)(a^2 + b^2) + 2b \csc \beta (\alpha_1 - \sin \alpha \cot \beta) - 2a \csc \beta \alpha_2 \\
- \tan^2 \beta (|\nabla \beta + \cos \alpha e_1|^2 - |\cot \beta \nabla \alpha + \sin \alpha (1 - \cot^2 \beta) e_1|^2) \\
+ \sin^2 \alpha (1 - \tan^2 \beta)
\]

4 Gauss-Codazzi-Ricci equations for a minimal surface in \( S^5 \) with constant Contact angle \( \beta \)

In this section, we will compute Gauss-Codazzi-Ricci equations for a minimal surface in \( S^5 \) with constant Contact angle \( \beta \).

Using the connection form (9) and (10) in the Codazzi-Ricci equations, we have

\[
d\theta^3_1 + \theta^3_2 \wedge \theta^2_1 + \theta^3_1 \wedge \theta^1_4 + \theta^5_3 \wedge \theta^5_1 = 0
\]
This implies that

\((b_1 - a_2) + (a^2 + b^2) \cot \alpha \csc \beta \cot^2 \beta - a \cot \alpha (\csc^2 \beta + \cot^2 \beta) \alpha_2\) \hspace{1cm} (16)

\(+b(cot\alpha(\csc^2 \beta + \cot^2 \beta)\alpha_1 - \cos \alpha \cot \beta(csc^2 \beta + \cot^2 \beta - 3 \sec^2 \beta(1 + \sin^2 \beta))) - \cos \alpha \csc \beta(2(\cot \beta - \tan \beta)\alpha_1 - \sin \alpha(\cot^2 \beta - 3)) + \cot \alpha \csc \beta |\nabla \alpha|^2 = 0\)

Replacing the following (10) in the Codazzi-Ricci equations

\[d\theta_2^3 + \theta_3^1 \wedge \theta_2^1 + \theta_3^2 \wedge \theta_2^1 = 0\]
\[d\theta_1^3 + \theta_3^1 \wedge \theta_1^2 + \theta_3^2 \wedge \theta_1^2 = 0\]
\[d\theta_3^5 + \theta_3^1 \wedge \theta_3^5 + \theta_3^2 \wedge \theta_3^5 = 0\]

We get

\[(a_1 + b_2) + b \cot \alpha \alpha_2 + a(\cot \alpha \alpha_1 + 6 \tan \beta \cos \alpha) - 2 \sec \beta \cos \alpha \alpha_2 = 0\] \hspace{1cm} (17)

Using the connection form (10) in the Codazzi-Ricci equations

\[d\theta_2^3 + \theta_3^1 \wedge \theta_2^1 + \theta_3^2 \wedge \theta_2^1 = 0\]
\[d\theta_1^3 + \theta_3^1 \wedge \theta_1^2 + \theta_3^2 \wedge \theta_1^2 = 0\]
\[d\theta_3^5 + \theta_3^1 \wedge \theta_3^5 + \theta_3^2 \wedge \theta_3^5 = 0\]

We have

\[(a_2 - b_1) - (a^2 + b^2) \cot \alpha \sin \beta \cot^2 \beta + a \cot \alpha \alpha_2\] \hspace{1cm} (18)
\[+b(-cot\alpha\alpha_1 + 2 \cos \alpha(cot \beta - 3 \tan \beta)) + 2 \cos \alpha \sin \beta(cot \beta - \tan \beta)\alpha_1\]
\[+ \sin \alpha \cos \alpha \sin \beta(5 - cot^2 \beta) + \sin \beta \Delta \alpha = 0\]

Codazzi-Ricci equations

\[d\theta_1^3 + \theta_3^1 \wedge \theta_1^3 + \theta_3^2 \wedge \theta_1^3 + \theta_3^5 \wedge \theta_1^5 = 0\]
\[d\theta_1^4 + \theta_3^1 \wedge \theta_1^4 + \theta_3^2 \wedge \theta_1^4 + \theta_3^5 \wedge \theta_1^4 = 0\]

give the following equation

\[(a^2 + b^2)(1 + \csc^2 \beta) + 2b \csc \beta(\alpha_1 - \cot \beta \sin \alpha) - 2a \csc \beta \alpha_2 \]
\[+|\nabla \alpha|^2 + 2 \sin \alpha(\tan \beta - \cot \beta)\alpha_1 - 4 \tan^2 \beta \cos^2 \alpha\]
\[- \sin^2 \alpha(1 - \cot^2 \beta) = 0\] \hspace{1cm} (19)

The following Codazzi equation is automatically verified

\[d\theta_2^5 + \theta_1^3 \wedge \theta_2^5 + \theta_2^3 \wedge \theta_2^3 \wedge \theta_4^1 \wedge \theta_2^4 = 0\]
5 Applications

5.1 Minimal Surfaces in $S^5$ with Constant Contact Angle $\beta$ and parallel normal vector field

In this section, we will give Gauss-Codazzi-Ricci equations for a minimal surface in $S^5$ with constant contact angle and null principal curvatures $a, b$.

Suppose that $a, b$ are nulls and the Contact angle $\beta$ is constant, then using the Codazzi equation (16), we have

$$\cos \alpha (2(\cot \beta - \tan \beta)\alpha_1 - \sin \alpha (\cot^2 \beta - 3)) - \cot \alpha |\nabla \alpha|^2 = 0$$

On the other hand, Codazzi equation (18) with $a, b$ nulls and constant contact angle implies

$$2\cos \alpha (\cot \beta - \tan \beta)\alpha_1 + \sin \alpha \cos \alpha (5 - \cot^2 \beta) + \Delta \alpha = 0$$

Using equations (20) and (21), we obtain a new Laplacian equation of $\alpha$

$$\Delta \alpha = -\sin(2\alpha) - \cot \alpha |\nabla \alpha|^2$$

Now using the Hopf’s Lemma, we get the Theorem 1.

Remark 1. The Theorem 1 implies a general classification in [9] that gives a family of minimal flat tori in $S^5$ with constant Contact angle and constant Holomorphic angle.

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