Parity Transformation in the Front Form *

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Abstract

By considering the parity-transformation properties of the $(1/2, 0)$ and $(0, 1/2)$ fields in the front form we find ourselves forced to study the front-form evolution both along $x^+$ and $x^-$ directions. As a by product, we find that half of the dynamical degrees of freedom of a full theory live on the $x^+ = 0$ surface and the other half on the $x^- = 0$ surface. Elsewhere, Jacob shows how these results are required to build a satisfactory, and internally consistent, front-form quantum field theory.

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In a recent series of papers [1–10], the \((j, 0) \oplus (0, j)\) representation space has been investigated in some detail and some unexpected results have been discovered. For instance, in Ref. [1] it was found that the \((1, 0) \oplus (0, 1)\) representation space supports a Bargmann-Wightman-Wigner-type quantum field theory in which a boson and its antiparticle have opposite relative intrinsic parity. In Ref. [4], we applied the approach used previously for the instant-form formalism [3] to the front-form case and obtained the \((j, 0) \oplus (0, j)\) spinors and generalized Melosh transformations for any spin. The front-form formalism was seen to be endowed with several advantages. The work of Ref. [4], apart from the indicated generalization, reproduced some of the well-known results of Melosh [11], Lepage and Brodsky [12], and Dziembowski [13] for spin-\(1/2\). Due to certain magic of Wigner’s time-reversal operator in Ref. [9], we were able to present a Majorana-like construct in the \((j, 0) \oplus (0, j)\) representation space. Even though much further work is needed to exploit the Majorana-like construct, we suspect that for spin-1 it may have a deep connection with \(\mathcal{P}\) and \(\mathcal{C}\mathcal{P}\) violation [10] in nature.

During the above-indicated investigations, we have come across a problem in the front form of field theory. The problem deals with the operation of parity. It is the purpose of this paper to bring attention to this problem and present its solution. We note that it is not the first time that the problem of parity in the front form has been considered. What we present is a new and significantly more critical analysis of the subject. For example, one of the earliest considerations of this problem appears in the 1971 thesis of Soper [14]. More recently, Jacob [15] has considered the question of parity while considering the quantization of the scalar field, and he arrived at similar conclusions to those presented in this paper but with a very different perspective. We look at the transformation properties of the \((1/2, 0)\) and \((0, 1/2)\) fields in the front form and find that the consideration of parity-covariance forces us to consider evolution not only along \(x^+\) (or \(x^-\)) but simultaneously along \(x^+\) and \(x^-\). Our considerations are applicable to massive as well as massless particles. McCartor [16], while investigating the quantization of massless fields in the front form concluded that a spin-\(1/2\) system must be specified on both \(x^+\) and \(x^-\) surfaces. Later, McCartor and Robertson [17] argued that \(x^+\) and \(x^-\) should be considered symmetrically. The arguments that we construct below lie at the heart of space-time symmetries (and are independent of any specific
Lagrangian), and the reader should find that these arguments convincingly establish that parity-covariance requires studying the evolution of a system (in the front form of field theory) both along $x^+$ and $x^-$ directions. The arguments we present are true for all $(j, 0)$ and $(0, j)$ fields. However, for conceptual clarity and general familiarity, we choose spin-$\frac{1}{2}$ as an example case.

To define the problem, we recall that in the instant form $(1/2, 0)$ and $(0, 1/2)$ fields transform as 

\begin{equation}
\begin{align*}
(1/2, 0) : & \quad \phi_R(p^\mu) = \exp\left[ + \varphi \cdot \frac{\sigma}{2} \right] \phi_R(\tilde{p}^\mu) , \\
(0, 1/2) : & \quad \phi_L(p^\mu) = \exp\left[ - \varphi \cdot \frac{\sigma}{2} \right] \phi_L(\tilde{p}^\mu) .
\end{align*}
\end{equation}

In the above equations, $p^\mu$ represents the four-momentum of the particle and $\tilde{p}^\mu$ corresponds to the particle at rest. The boost operator $\varphi$ that appears in Eq. (1) is defined as

\begin{equation}
\begin{align*}
cosh(\varphi) &= \gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{E}{m} , \\
\sinh(\varphi) &= v = \frac{|p|}{m} , \\
\varphi &= \frac{p}{|p|} , \\
\varphi &= |\varphi| ,
\end{align*}
\end{equation}

with $p$ the three-momentum of the particle. It is immediately obvious from Eqs. (1) and (2) that under the operation parity, $P$, $(1/2, 0)$ and $(0, 1/2)$ representation spaces get interchanged,

\begin{equation}
P : (1/2, 0) \leftrightarrow (0, 1/2) .
\end{equation}

It is because of the result (3) that any parity-conserving interaction must involve both the $(1/2, 0)$ and $(0, 1/2)$ fields. One then introduces the $(1/2, 0) \oplus (0, 1/2)$ Dirac spinor, which in the chiral representation (indicated by the use of curly brackets enclosing the argument $p^\mu$) reads

\begin{equation}
\psi\{p^\mu\} = \begin{bmatrix}
\phi_R(p^\mu) \\
\phi_L(p^\mu)
\end{bmatrix} .
\end{equation}

These results are well known and can indeed be found in any modern textbook on quantum field theory [18–20]. For the sake of later reference, let’s note that the more familiar [21] canonical representation is defined as (argument $p^\mu$ of spinors in the canonical representation would be enclosed in square brackets)

\begin{equation}
\psi[p^\mu] = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & -\mathbb{I} \end{pmatrix} \psi\{p^\mu\} ,
\end{equation}
where $\mathbb{1}$ is a $2 \times 2$ identity matrix. Under the operation of parity operator $S(P) = \gamma^0$ in the $(1/2, 0) \oplus (0, 1/2)$ representation space, the particle and antiparticle spinors, in the usual notation of Refs. [3,21, transform as
\begin{align*}
u_\sigma[p_\mu] &= +\gamma^0 u_\sigma[p_\mu] , \\
\nu_\sigma[p_\mu] &= -\gamma^0 v_\sigma[p_\mu] .
\end{align*}

The $p'_\mu$ is the parity-transformed $p_\mu$.

The front-form counterpart of the simple and important instant-form relation (3), and other equations such as (6), is a little subtle. To see this note, that counterpart of transformation properties of the right- and left-handed fields in the front-form of evolution associated with $x^+ = x^0 + x^3$ reads (as was recently shown in Ref. [4])
\begin{align*}(1/2, 0)^{[x^+]} : \quad \phi_\kappa^{[x^+]}(p_\mu) &= \exp \left[ +\beta \cdot \frac{\boldsymbol{\sigma}}{2} \right] \phi_\kappa^{[x^+]}(\tilde{p}_\mu) , \\
(0, 1/2)^{[x^+]} : \quad \phi_\lambda^{[x^+]}(p_\mu) &= \exp \left[ -\beta^* \cdot \frac{\boldsymbol{\sigma}}{2} \right] \phi_\lambda^{[x^+]}(\tilde{p}_\mu) .
\end{align*}

The superscript $[x^+]$ in the above equations serves the purpose of reminding that these relations hold true for the evolution along $x^+$. The $\sigma$ are the standard Pauli matrices. The boost parameter $\beta$ that appears in Eq. (7) is defined [4] as:
\begin{align*} \beta &= \eta \left( \alpha v^r , -i \alpha v^r , 1 \right) ,
\end{align*}

where $\alpha = [1 - \exp(-\eta)]^{-1}$, $v^r = v_x + i v_y$ (and $v^\ell = v_x - i v_y$). In terms of the front-form variable $p^+ \equiv E + p_z$, one can show that
\begin{align*}\cosh(\eta/2) &= \Omega (p^+ + m) , \quad \sinh(\eta/2) = \Omega (p^+ - m) ,
\end{align*}

with $\Omega = [1/(2m)] \sqrt{m/p^+}$. Under the operation of parity, $P$, an inspection of Eqs. (9), indicates that $(1/2, 0)$ and $(0, 1/2)$ representation spaces do not get interchanged;
\begin{align*} P : \quad (1/2, 0) \not\leftrightarrow (0, 1/2) , \quad \text{for evolution along } x^+ = x^0 + x^3 .
\end{align*}

Therefore, it is obvious that the standard front-form formulation (which expands field operators in terms of the spinors associated only with the evolution along the $x^+ = x^0 + x^3$, or $x^- = x^0 - x^3$
directions) of field theories is not covariant under parity. Added to this observation is the fact that only *half* of the degrees of freedom associated with such a field are dynamical [22]. We claim that there is in fact no loss/reduction of degrees of freedom in the front form of field theories and no violation of parity covariance *if* one is a little careful. We now elaborate.

One must note that under the operation of parity, \( P \), unlike the instant-form case, the direction of evolution (say) \( x^+ \) gets interchanged with \( x^- \). So, we suspect that one must obtain counterparts of transformation properties (7) for the evolution along the parity-transformed \( x^+ \), that is \( x^- \), and see how the fields transform. Algebraically, this exercise is a little involved but follows parallel to our previous analysis of Ref. [4]. Here, we just quote the result of our calculations. We find that the right- and left-handed fields in the front form of evolution associated with \( x^- \) direction transform as follows

\[
\begin{align*}
(1/2, 0)^{[x^-]} & : \quad \phi^{[x^-]}_R(p^\mu) = \exp \left[ -\beta^* \cdot \frac{\sigma}{2} \right] \phi^{[x^-]}_R(\hat{p}^\mu) , \\
(0, 1/2)^{[x^-]} & : \quad \phi^{[x^-]}_L(p^\mu) = \exp \left[ +\beta \cdot \frac{\sigma}{2} \right] \phi^{[x^-]}_L(\hat{p}^\mu) .
\end{align*}
\]

(11)

The superscript \( [x^-] \) in the above equations serves the purpose of reminding that these relations hold true for the evolution along \( x^- \).

Comparison of transformation properties (7) and (11) yields the front-form counterpart of the instant-form relation (3),

\[
\begin{array}{c}
\text{Comparison of transformation properties (7) and (11) yields the front-form counterpart of the instant-form relation (3),}
\\
P : \\
(1/2, 0)^{[x^+]} \leftrightarrow (0, 1/2)^{[x^-]} \\
(0, 1/2)^{[x^+]} \leftrightarrow (1/2, 0)^{[x^-]}
\end{array}
\]

(12)

Thus, under the operation of parity, the representation space \( (1/2, 0)^{[x^+]} \oplus (0, 1/2)^{[x^+]} \) maps one-to-one onto \( (1/2, 0)^{[x^-]} \oplus (0, 1/2)^{[x^-]} \). To be more explicit, one may carry out an exercise similar to the one presented in our recent work [4] and obtain the \( u^{[x^-]}_\mu[p^\mu] \) and \( v^{[x^-]}_\mu[p^\mu] \) spinors in the \( (1/2, 0)^{[x^-]} \oplus (0, 1/2)^{[x^-]} \) representation space. We already know the \( (1/2, 0)^{[x^+]} \oplus (0, 1/2)^{[x^+]} \)-spins, \( u^{[x^+]}_\mu[p^\mu] \) and \( v^{[x^+]}_\mu[p^\mu] \), from Ref. [4]. The parity operation \( S(P) \) still remains \( \gamma^0 \) because Melosh transformation, as was explicitly proved in [4], does not mix particle and antiparticle spinors, and the front-form \( (1/2, 0) \oplus (0, 1/2) \) spinors turn out to be the superposition of the instant-
form spinors with $p^\mu$-dependent coefficients contained in the Melosh matrix for spin-$\frac{1}{2}$. The above indicated exercise yields the front-form counterpart of the identities (3),

$$u^{[x^-]}_\mu [p'^\mu] = + \gamma^0 u^{[x^+]}_h [p^\mu] ,$$
$$v^{[x^-]}_\mu [p'^\mu] = - \gamma^0 v^{[x^+]}_h [p^\mu] .$$

The $h$ and $\mu$ in the above expressions correspond to the helicity degrees of freedom associated with the front-form helicity operators (respectively associated with evolution along $x^+$ and $x^-$):

$$J_{3}^{[x^+]} \equiv J_3 + \frac{1}{P_-} (G_1 P_2 - G_2 P_1) ,$$
$$J_{3}^{[x^-]} \equiv J_3 + \frac{1}{P_+} (D_1 P_2 - D_2 P_1) .$$

For dynamical significance and the definition of various generators involved in the above expressions, we refer the reader to Sec. II of Ref. [4]. The non-trivial space-time structure of the results we obtain, such as Eqs. (13), is intuitively satisfactory. The reader should carefully examine all the associated super- and sub-scripts!

To conclude, we note that evolution along the $x^+$ direction contains two dynamically independent spinorial degrees of freedom [22]. It can now be verified, following arguments similar to [22], that the other two dynamically independent degrees of freedom are contained in the field operator constructed for the evolution along $x^-$. A parity-covariant front-form field theory requires specification of a system on both $x^+ = 0$ and $x^- = 0$ surfaces and carries four independent degrees of freedom (as in the instant-form) — two on each surface. Jacob [23], in a recent preprint, shows that the result (13) is essential for front-form quantization, which involves specification of a system on both $x^+ = 0$ and $x^- = 0$ surfaces.

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