EQUIVARIANT STABLE CATEGORIES FOR INCOMPLETE SYSTEMS OF TRANSFERS

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Abstract. In this paper, we construct incomplete versions of the equivariant stable category; i.e., equivariant stabilization of the category of $G$-spaces with respect to incomplete systems of transfers encoded by an $N_{\infty}$ operad $O$. These categories are built from the categories of $O$-algebras in $G$-spaces. Using this operadic formulation, we establish incomplete versions of the usual structural properties of the equivariant stable category, notably the tom Dieck splitting. Our work is motivated in part by the examples arising from the equivariant units and Picard space functors.

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1. Introduction

The foundation of stable homotopy theory is the study of the stable category. Boardman’s original construction of the stable category involved a process of formal stabilization of the category of finite CW complexes and then the addition of

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colimits (e.g., see [37]). Subsequently, the needs of calculation drove a lengthy period of focus on explicit models of the stable category as the homotopy category of various constructions of categories of spectra or (for the connective subcategory) as grouplike algebras in spaces over the $E_\infty$-operad. In the equivariant setting, until very recently essentially the only constructions of the equivariant stable category were given in terms of homotopical categories of equivariant spectra (e.g., see [24] and [26]).

Recently, there has been renewed interest in formal descriptions of the equivariant stable category in terms of structural properties. A substantial amount of this activity has been motivated by diagrammatic characterizations of the stable category that are suggested by the framework of $\infty$-categories; for example, regarding the equivariant spectral category as spectral presheaves on the Burnside category (e.g., see [17] and [5]) or suitable diagrams of geometric fixed-points glued via the Tate construction (e.g., see [16] and [1]). A significant aspect of establishing the correctness of these descriptions is grappling with the formal properties of the equivariant stable category, which are considerably more subtle than those of the stable category, especially when $G$ is a compact Lie group.

We take the perspective that the equivariant stable category should be thought of as characterized by equivalently

1. the existence of transfer maps,
2. the Wirthmuller isomorphisms,
3. or the tom Dieck splitting of suspension spectra.

For instance, as part of an effort to understand equivariant infinite loop space theory for infinite compact Lie groups, the first author gave a characterization of the equivariant stable category in terms of the existence of Wirthmuller isomorphisms [7].

These efforts to understand the structural properties of the equivariant stable category received a boost from the foundational work done as part of the Hill–Hopkins–Ravenel solution to the Kervaire invariant one problem [22]. Notably, the introduction of the multiplicative norm maps provided a new unifying language to think about the formal structure of the equivariant stable category. Previously, the authors studied the question of the way in which operadic algebras in spectra and spaces for various kinds of equivariant $E_\infty$ operads admit “norm maps”; in spectra, these norm maps are the multiplicative norms, and in spaces, they are transfers [8]. The result of that work was to introduce intermediate versions of equivariant commutative ring structures, structured by $N_\infty$ operads, interpolating between algebras over the naive $E_\infty$ operad regarded as $G$-trivial and the “genuine” $E_\infty$ $G$-operad that describes classical equivariant commutative ring spectra. In subsequent work, the authors gave an algebraic characterization of compatible systems of norms or transfers, called indexing systems [9].

The purpose of this paper is to construct and study the equivariant stable categories that are determined by the indexing systems specified by arbitrary $N_\infty$ operads $O$. Of course, when the indexing system comes from an equivariant little disks operad, there already exist constructions of incomplete equivariant stable categories described in terms of categories of orthogonal $G$-spectra specified by an incomplete universe $U$. However, a surprising result of [8] is that not all $N_\infty$ operads are equivalent to little disks operads; further examples of this phenomenon are given in [34]. Therefore, the work of this paper constructs new kinds of equivariant stable categories that do not seem to admit models using existing technology. Our
approach is to build a category of spectra completely removed from the universes, considering only the combinatorial data encoded by the $N_\infty$ operads. The way we do this has the interesting consequence that the structural properties of the stable category are directly exposed.

The starting point for our description of the $O$-stable category is the category $\text{Top}^G_{\ast}[\tilde{P}_O]$ of $O$-algebras in based $G$-spaces, where here $\tilde{P}_O$ denotes the monad on based spaces associated to $O$. We will generically refer to an object of $\text{Top}^G_{\ast}[\tilde{P}_O]$ as an $N_\infty$ space. Recall that an $O$-algebra $B$ is grouplike if $\pi^H_0(B)$ is a group for every closed subgroup $H \subset G$. A map $A \to B$ of $O$-spaces is a group completion if $B$ is grouplike and each induced map of fixed-point spaces $A^H \to B^H$ is a (non-equivariant) group completion.

We take the point of view that grouplike $N_\infty$-spaces model the connective part of the $O$-stable category, and we will produce the rest of the category by formal stabilization. To carry this out, we begin by constructing a model structure on the connective objects. The following theorem is standard.

**Theorem 1.1.** Let $\mathcal{O}$ be an $N_\infty$ operad. There is a cofibrantly generated model structure, the standard model structure, on $\text{Top}^G_{\ast}[\tilde{P}_O]$ where the weak equivalences and fibrations are determined by the forgetful functor $\text{Top}^G_{\ast}[\tilde{P}_O] \to \text{Top}^G_{\ast}$.

The first basic observation is that we can group-complete an $O$-algebra via the nonequivariant delooping. Specifically, we have the following result:

**Proposition 1.2.** Let $A$ be a cofibrant $O$-algebra. Then the natural map

$$A \to \Omega(S^1 \otimes A)$$

is a group completion, where $S^1 \otimes (-)$ denotes the based tensor and $\Omega(-)$ the based cotensor with $S^1$.

Here we are using the fact that the based space $S^1$ is a model of the bar construction. Following [35], we now localize this model structure:

**Definition 1.3.** Let $\mathcal{O}$ be a reduced $N_\infty$ operad. The $O$-stable model structure on the category $\text{Top}^G_{\ast}[\tilde{P}_O]$ has weak equivalences the maps $A \to B$ such that

$$\Omega(S^1 \otimes A) \to \Omega(S^1 \otimes B)$$

is a weak equivalence. The fibrant objects are the grouplike $O$-algebras.

We refer to $\text{Ho}(\text{Top}^G_{\ast}[\tilde{P}_O])$, formed with respect to the $O$-stable model structure, as the connective $O$-stable category. Note that the transfers that arise from the $N_\infty$ operad impose structure on the homotopy groups of an algebra.

**Proposition 1.4.** Let $\mathcal{O}$ be a reduced $N_\infty$ operad and let $B$ be a grouplike $O$-algebra. Then the homotopy groups $\{\pi^H_n(B)\}$ naturally extend to an $O$-Mackey functor.

We now turn to the formal addition of nonconnective objects to get a model for $O$-spectra. For any category $\mathcal{C}$ tensored and cotensored over based spaces, we can define the category $\mathcal{S}_p(\mathcal{C})$ of spectra in $\mathcal{C}$ as the collections of objects $\{B_i\}$ equipped with structure maps $S^1 \otimes B_n \to B_{n+1}$. This definition is motivated by the fact that when $\mathcal{C}$ is a model category, the based tensor of a cofibrant object with $S^1$ models the Quillen suspension on $\text{Ho}(\mathcal{C})$. 

Definition 1.5. Let $\mathcal{O}$ be a reduced $N_\infty$ operad. The category of $\mathcal{O}$-spectra $\text{Sp}_G^{\mathcal{O}}$ is the category $\text{Sp}_G^{\mathcal{O}}$ of spectrum objects in $\text{Top}_G^{\mathcal{O}}[\tilde{\mathcal{P}}]$.

Standard arguments show that this has a stable model structure lifted from the standard model structure on $\text{Top}_G^{\mathcal{O}}[\tilde{\mathcal{P}}]$, where the stable equivalences are detected by maps into $\Omega$-spectra or by stable homotopy groups.

Theorem 1.6. The category of $\mathcal{O}$-spectra has a cofibrantly generated model structure where the weak equivalences are detected by the stable equivalences. We refer to $\text{Ho}(\text{Sp}_G^{\mathcal{O}})$ as the $\mathcal{O}$-stable category.

The functor $\Omega(S^1 \otimes (-))$ induces a fully-faithful embedding of the connective $\mathcal{O}$-stable category into the $\mathcal{O}$-stable category. In addition, there is an adjoint pair of functors

$$\Sigma_\infty^O : \text{Top}_G^{\mathcal{O}} \rightleftarrows \text{Sp}_G^{\mathcal{O}} : \Omega_\infty^O,$$

where $\Sigma_\infty^O$ is the composite of the free $\mathcal{O}$-algebra functor and the suspension spectrum.

Notice that this is conceptually a different approach than the standard construction of equivariant spectra; we do not invert representation spheres, but just the trivial spheres in operadic algebras. The fact that this works is precisely because grouplike $N_\infty$-algebras in spaces are already stable objects, and we are simply formally adding shifts. As a consequence, an advantage of this approach to the construction of the $\mathcal{O}$-stable category is that basic results follow directly from analysis of the free $\mathcal{O}$-algebra in spaces. For example, we obtain the following incomplete version of the tom Dieck splitting by studying the fixed points of the free $\mathcal{O}$-algebra.

Theorem 1.7. Let $X$ be a $G$-CW complex. Then we have a natural equivalence

$$(\Sigma_\infty^O X_+)^G \simeq \bigvee_{G/H \in \pi_0 \text{O}(G)} EW_G(H)_+ \wedge_{W_G(H)} \Sigma_\infty^X H.$$

(Here $G/H \in \pi_0 \text{O}(G)$ denotes the isomorphism classes of admissible $G$-sets for the indexing system determined by $\mathcal{O}$.)

Equivalently, we also have an incomplete version of the Wirthmüller isomorphism.

Theorem 1.8. Let $G/H$ be an admissible set for $\mathcal{O}$. Then the functor $G_+ \wedge_H (-)$ is both the left and the right adjoint to the forgetful functor $i^*_H$ on the $\mathcal{O}$-stable category.

These results provide an axiomatic characterization of the $\mathcal{O}$-stable category. We also have the following consistency check. For the next result, recall that a map $\mathcal{O} \to \mathcal{O}'$ of $N_\infty$ operads is a weak equivalence when each constituent map $\mathcal{O}(n) \to \mathcal{O}'(n)$ is an equivalence of $G \times \Sigma_n$-spaces.

Proposition 1.9. Let $\mathcal{O}$ be weakly equivalent to a little disk operad $\mathcal{D}(U)$ for a $G$-universe $U$. Then the underlying $\infty$-category for the stable model structure on $\text{Sp}_G^{\mathcal{O}}$ is equivalent to the underlying $\infty$-category for the stable model category of orthogonal $G$-spectra on $U$. The homotopy category of orthogonal $G$-spectra for the universe $U$ is equivalent to the $\mathcal{O}$-stable category.
To explain the restriction to little disks, it is illuminating to indicate why a simple approach using $\Gamma_G$-spaces does not in general suffice to describe the $\mathcal{O}$-stable category. In order to work with $\Gamma_G$ spaces, we can consider analogues of Shimakawa’s special condition, restricting attention only to the maps coming from the admissible $G$-sets. That is, we would require the maps

$$X(T) \to F(T, X(1^+))$$

to be $G$-equivalences for admissible $G$-sets $T$. When all of the admissible sets for $\mathcal{O}$ are generated by the admissible sets for $G$ itself, this captures the correct homotopy theory. However, for certain other $N_\infty$ operads (e.g., the linear isometries operad considered below), all of the interesting structure in the associated indexing system takes place at proper subgroups of $G$.

Finally, we turn to explain in more detail our motivation for the work of this paper. As we mentioned above, our interest in $\mathcal{O}$-stable categories arose from the surprising fact that there exist exotic $\mathcal{O}$-stable categories that do not arise from categories of equivariant orthogonal spectra for any universe. In fact, there are interesting examples of this phenomenon that are simple enough to describe completely explicitly.

Consider the $C_4$-universe $U = \infty(\mathbb{R} \oplus \lambda)$, where $\lambda$ is the unique irreducible 2-dimensional real representation. We can consider $C_4$-spectra indexed on $U$, and the homotopy coefficient systems here naturally have transfer maps from the trivial group to $C_2$ and to $C_4$, but no transfer from $C_2$ to $C_4$. The category of $C_4$-spectra indexed on $U$ is a symmetric monoidal category, and commutative monoids are equivalent to operadic algebras for the linear isometries operad structured on $U$. A very curious fact which motivated the development of $N_\infty$ operads is that the little disks and linear isometries operads for $U$ are distinct and moreover, the linear isometries operad for $U$ is not equivalent to the little disks operad for any $C_4$ universe. If we try to form $GL_1$ or related constructions, then we still produce a group-like operadic algebra, but the only way to build an associated spectrum is to restrict the structure to little disks on a trivial $C_4$-universe, throwing away structure.

This example arises from geometric considerations. Specifically, Manolescu’s $Pin(2)$-equivariant approach to the $11/8$-conjecture considered $Pin(2)$-equivariant stable homotopy indexed on a particular incomplete universe [29] (see also [23]). When restricted to $C_4 \subset S^1 \subset Pin(2)$, this universe is exactly the universe $U$.

We should note that despite the obvious connections to equivariant infinite loop space theory, we do not construct a delooping machine for arbitrary $N_\infty$ spaces, although we think this is a very interesting topic of study. Also, a distinct disadvantage of the formalism we study is that the multiplicative structure is less explicit. There are models for the $\mathcal{O}$-stable category based on generalizations of the excisive functor [21] and equivariant symmetric spectra [28] approaches. We intend to return to these kinds of approaches in future work with Dotto and Mandell.

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2. $N_\infty$-OPERADS AND ALGEBRAS

In this section, we briefly review the notion of an $N_\infty$ operad and record some structural properties of categories of algebras over these operads. We refer the reader to [8, 9] for more detailed discussion. Throughout the following discussion, we as usual work in the category of compactly-generated weak Hausdorff spaces.

**Notation 2.1.** We write $\mathcal{O}p_{G}$ for the category of $G$-spaces and equivariant maps between them, viewed as a category enriched in spaces. We write $\mathcal{T}op_{G}$ for the category of $G$-spaces and all maps between them, viewed as a category enriched in $\mathcal{O}p_{G}$. In both cases, we denote the corresponding pointed categories with a subscript of an asterisk.

2.1. $N_\infty$-operads and indexing systems. Equivariant stable categories are controlled by certain kinds of $G$-operads. We will assume that the $G$-operads we consider have spaces which are of the homotopy type of $G$-CW complexes.

**Definition 2.2.** An $N_\infty$ operad is an operad $\mathcal{O}$ in $G$-spaces such that

1. The space $\mathcal{O}(0)$ is $G$-contractible,
2. The action of $\Sigma_n$ on $\mathcal{O}(n)$ is free, and
3. The space $\mathcal{O}(n)$ is a universal space for a family of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{1\}$ for $H \subseteq G$.

We say that an $N_\infty$-operad is reduced if $\mathcal{O}(0)$ is a point.

A central result about $N_\infty$ operads is that the homotopy category of $N_\infty$ operads is essentially algebraic.

**Definition 2.3.** A weak equivalence of $N_\infty$ operads is a map $\mathcal{O} \rightarrow \mathcal{O}'$ that induces weak equivalences $\mathcal{O}(n) \rightarrow \mathcal{O}'(n)$ of $G \times \Sigma_n$-spaces for each $n$.

Inverting these weak equivalences, the homotopy category of $N_\infty$ operads is equivalent to a poset [8, 20, 33, 10] described in terms of indexing systems.

**Definition 2.4.** A symmetric monoidal coefficient system is a contravariant functor from the orbit category of $G$ to the category of small symmetric monoidal categories and strong monoidal functors.

The canonical example of a symmetric monoidal coefficient system is $\mathcal{S}et$, which assigns to $G/H$ the category of finite $H$-sets.

**Definition 2.5.** An indexing system $\mathcal{C}$ is a symmetric monoidal full sub-coefficient system of $\mathcal{S}et$ that contains all trivial sets and is closed under

1. Passage to finite limits and
2. Self-induction, in the sense that if $H/K \in C(H)$ and $T \in C(K)$, then $H \times T \in C(H)$.

Indexing systems form a poset ordered by inclusion, and the theorem is that the homotopy category of $N_\infty$ operads is equivalent to the poset of indexing systems. Another interesting characterization of indexing systems and hence $N_\infty$ operads uses the theory of polynomials:

**Theorem 2.6.** There is an isomorphism of categories between the poset of indexing systems and the poset of wide, pullback stable, finite coproduct complete subcategories of the category of finite $G$-sets.
At the extremes, there are two distinguished $N_\infty$ operads, corresponding to the initial and terminal indexing systems: $O^{ir}$, the homotopy initial $N_\infty$ operad, which has the homotopy type of an ordinary, non-equivariant $E_\infty$-operad viewed as a $G$-operad with trivial $G$-action, and $O^{gen}$, the homotopy terminal $N_\infty$ operad, which is a $G-E_\infty$-operad in the sense of [24]. Since the $N_\infty$ operad $O^{ir}$ is initial in the homotopy category of $N_\infty$ operads, for any $O$ we have a canonical map in the homotopy category
\[ O^{ir} \to O, \]
which can be represented on the point-set level by the inclusion $O^G \to O$.

Remark 2.7. Any $N_\infty$ operad is weakly equivalent to one which is reduced, and hence the homotopy category of $N_\infty$ operads can be completely described in terms of reduced operads. In particular, the $N_\infty$ Barratt-Eccles operads described in [33] are reduced. Since it is technically convenient, we will assume throughout the remainder of the paper that our operads are reduced.

2.2. Homotopical categories of $O$-algebras.

2.2.1. Point-set categories. For an $N_\infty$ operad $O$, we are interested in the category of $O$-algebras in $G$-spaces and based $G$-spaces.

Definition 2.8. Let $P_O$ denote the monad on $\mathcal{T}op^G$ associated to the operad $O$; to be precise, this is specified as usual by the formula
\[ P_O X = \bigoplus_{i \geq 0} O(i) \times \Sigma_i X. \]

The reduced monad $\tilde{P}_O$ on $\mathcal{T}op^*_G$ is specified by defining $\tilde{P}_O X$ to be the quotient of $P_O X$ by basepoint identifications that ensure that the basepoint in $X$ coincides with the basepoint induced by $O(0)$. These identifications are specified as follows: there are maps $\sigma_i: O(n) \to O(n-1)$ that evaluate on the identity everywhere except at $i$ and maps $s_i: X^n \to X^{n+1}$ that insert $\ast$ at position $i$. We impose the equivalence relation $(o, s_i x) \sim (\sigma_i o, x)$ for $o \in O(n)$ and $x \in X^{n-1}$. (See for example [30, §4].)

Note that the categories of monadic algebras $\mathcal{T}op^G[\tilde{P}_O]$ and $\mathcal{T}op^*_G[\tilde{P}_O]$ are equivalent; this follows from the isomorphism of monads $\tilde{P}_O \circ (-)_+ \cong P_O$ on unbased spaces by [14] II.6.1. (See also [30, 4.4] for further discussion of this comparison.)

The usual arguments (e.g., see [14, I.7.2]) show that $P_O$ preserves reflexive coequalizers, and the category $\mathcal{T}op^G[\tilde{P}_O]$ is complete and cocomplete. Limits are created in $\mathcal{T}op^*_G$, as are some colimits.

Proposition 2.9. sifted colimits in $\mathcal{T}op^G[\tilde{P}_O]$ are created in $\mathcal{T}op^*_G$.

There is an enrichment of the category $\mathcal{T}op^*_G[\tilde{P}_O]$ in based $G$-spaces, where we consider all maps, not just equivariant ones.

Definition 2.10. If $A$ and $B$ are $O$-algebras, then let
\[ \mathcal{T}op^*_G[\tilde{P}_O](A, B) \]
be the $G$-space of all (not necessarily equivariant) maps
\[ A \to B \]
which commute with monadic structure maps. Let $\mathcal{T}op^*_G[\tilde{P}_O]$ be the category enriched over $\mathcal{T}op^*_G$ with objects $O$-algebras and these hom $G$-spaces.
The two enrichments are connected by passage to \(G\)-fixed points. Using the equivariant enrichment, we can talk about the standard tensoring and cotensoring over \(G\)-spaces and based \(G\)-spaces.

**Proposition 2.11.** The category \(\mathcal{T}op^G_{\ast}[\mathbb{P}_O]\) is tensored and cotensored over based \(G\)-spaces. These are defined via the adjunctions

\[
\mathcal{T}op_{\ast}[\mathbb{P}_O](X \otimes A, B) \cong \mathcal{T}op^G_{\ast}(X, \mathcal{T}op_{\ast}[\mathbb{P}_O](A, B)) \cong \mathcal{T}op_{\ast}[\mathbb{P}_O](A, F(X, B)).
\]

Finally, on free algebras, the tensoring operation is especially simple, while the underlying based \(G\)-space for the cotensoring is especially simple.

**Proposition 2.12.** Let \(X\) be a based \(G\)-space and \(A = \mathbb{P}_O(Y)\). Then

\[
X \otimes A \cong \mathbb{P}_O(X \wedge Y).
\]

For any object \(B\) in \(\mathcal{T}op^G_{\ast}[\mathbb{P}_O]\), the underlying \(G\)-space for \(F(X, B)\) is the ordinary \(G\)-space of maps \(\mathcal{T}op_{\ast}(X, B)\).

2.2.2. Model structures. We will use several model structures on \(\mathcal{T}op^G_{\ast}[\mathbb{P}_O]\), beginning with one which arises quite naturally from work of Schwede–Shipley [36]. Specifically, since all objects in \(\mathcal{T}op^G\) are fibrant, the following theorem follows easily.

**Theorem 2.13.** There is a compactly generated \(G\)-topological model structure on \(\mathcal{T}op^G_{\ast}[\mathbb{P}_O]\) in which the fibrations and weak equivalences are determined by the forgetful functor to \(\mathcal{T}op^G_{\ast}\). The cofibrant objects are retracts of filtered colimits of pushouts attaching free cells. We call this the standard model structure.

The assertion that the model structure is \(G\)-topological means that the enrichment and (co)tensor structure of \(\mathcal{T}op^G_{\ast}[\mathbb{P}_O]\) satisfies the analogue of Quillen’s SM7.

For a map of \(N_\infty\) operads \(\iota: \mathcal{O} \to \mathcal{O}'\), there are associated maps of monads \(\mathbb{P}_\mathcal{O} \to \mathbb{P}_{\mathcal{O}'}\) and \(\mathbb{P}^\wedge\mathcal{O} \to \mathbb{P}^\wedge\mathcal{O}'\). We denote by \((\iota_{\mathbb{P}^\wedge\mathcal{O}}, \iota_{\mathbb{P}^\wedge\mathcal{O}'})\) the adjunction between \(\mathcal{T}op^G_{\ast}[\mathbb{P}_\mathcal{O}]\) and \(\mathcal{T}op^G_{\ast}[\mathbb{P}_{\mathcal{O}'}]\). Since the right adjoint evidently preserves fibrations and weak equivalences, the following lemma is immediate.

**Lemma 2.14.** For a map \(\iota: \mathcal{O} \to \mathcal{O}'\) of \(N_\infty\) operads, the adjunction \((\iota_{\mathbb{P}^\wedge\mathcal{O}}, \iota_{\mathbb{P}^\wedge\mathcal{O}'}\)) is a Quillen adjunction for the standard model structures.

This allows us to describe a peculiar phenomenon for homotopy colimits: homotopy colimits in algebras over \(\mathcal{O}\) are always computed in the underlying category \(\mathcal{T}op^G_{\ast}[\mathbb{P}_{\mathcal{O}'\ast}]\), for any \(N_\infty\) operad \(\mathcal{O}\). In particular, for distinct \(N_\infty\) operads \(\mathcal{O}\) and \(\mathcal{O}'\), homotopy colimits in \(\mathcal{T}op^G_{\ast}[\mathbb{P}_\mathcal{O}]\) and \(\mathcal{T}op^G_{\ast}[\mathbb{P}_{\mathcal{O}'\ast}]\) coincide. We can immediately deduce this from Proposition 2.9 and the fact that finite coproducts and products coincide for \(\mathcal{O}\)-algebras; we prove this latter statement as a consequence of an analysis of the free \(\mathcal{O}\)-algebra below in Corollary 4.18.

**Lemma 2.15.** Let \(\mathcal{O}\) be an \(N_\infty\) operad. The forgetful functor \(\iota_{\mathcal{O}''\ast}: \mathcal{T}op^G_{\ast}[\mathbb{P}_\mathcal{O}] \to \mathcal{T}op^G_{\ast}[\mathbb{P}_{\mathcal{O}''\ast}]\) creates homotopy colimits.

2.2.3. Fixed points of \(\mathcal{O}\)-algebras. We close this section by recording some standard observations about the behavior of the categorical fixed-point functor on \(\mathcal{O}\)-algebras.

**Lemma 2.16.**
(1) If $B$ is an $O$-algebra, then $B^H$ is naturally an $O^{tr}$-algebra.

(2) If $q: G/H \to G/K$ is a $G$-map, then the induced restriction map

$$q^*: B^K \to B^H$$

is naturally a map of $O^{tr}$-algebras.

**Proof.** If $B$ is an $O$-algebra, then by restriction, $B$ is an $O^{tr}$-algebra. Since $O^{tr}$ has a trivial $G$-action, and since passage to fixed points is a strong symmetric monoidal on spaces, $B^G$ is automatically an $O^{tr}$-algebra.

For the second point, the map $q^*$ is the $G$-fixed points of the map

$$\text{Map}(G/K, B) \xrightarrow{q^*} \text{Map}(G/H, B),$$

which is given by cotensoring out of the map $G/H \to G/K$, and hence is a map of $O^{tr}$-algebras. \hfill \square

This lemma implies that an $O$-algebra specifies a diagram over the orbit category of algebras over $O^{tr}$.

**Corollary 2.17.** There is a contravariant functor

$$\mathcal{T}_G^G \left[ \mathbb{P}_O \right] \to \text{Fun}(\text{Orb}^G)^{op}, \mathcal{T}_G^G \left[ \mathbb{P}_{O^{tr}} \right]$$

specified by

$$B \mapsto (G/H \mapsto B^H)$$

and $G/H \to G/K$ goes to $q^*$.

A model for $O^{tr}$ is the little disks operad for a trivial universe $\mathbb{R}^\infty$, and so assuming that $B^H$ is grouplike for all $H$, using a functorial non-equivariant delooping machine we can associate to an object of $\text{Fun}(\text{Orb}^G)^{op}, \mathcal{T}_G^G \left[ \mathbb{P}_{O^{tr}} \right]$ a contravariant functor

$$\text{Orb}^G \to \mathcal{S}p.$$ This is the data of a naive $G$-spectrum, and passing to homotopy groups gives rise to a coefficient system of abelian groups. For general operads $O$, there is more structure on the homotopy groups:

**Theorem 2.18 ([8, 7.5]).** If $B$ is an $O$-algebra in spaces, then coefficient system

$$T \mapsto [T_+ \wedge S^k, B] =: \pi_k(B)(T)$$

extends naturally to an $O$-Mackey functor.

Finally, we record two results about the interaction of fixed points with the tensor and cotensor with (non-equivariant) spaces.

**Lemma 2.19.** Let $B$ be an $O$-algebra in based $G$-spaces. For any $H \subset G$ and based space $X$ that is the geometric realization of a levelwise-finite simplicial set, the natural map

$$X \otimes B^H \to (X \otimes B)^H$$

is a weak equivalence of $O^H$-algebras.

**Proof.** By Proposition 2.9, the tensor with $X$ can be computed as the geometric realization of the levelwise tensor. Since $X$ is levelwise-finite, these tensors are just coproducts. Therefore, we can reduce to showing that the natural map

$$B^H \vee_{O^H} B^H \to (B \vee O)B^H$$
induced by the two inclusions \( B \to B \vee B \) is an equivalence for all \( H \subseteq G \). This now follows from the commutative diagram

\[
\begin{array}{ccc}
B^H \vee_{O^H} B^H & \longrightarrow & (B \vee_{O} B)^H \\
\downarrow & & \downarrow \\
B^H \times B^H & \longrightarrow & (B \times B)^H,
\end{array}
\]

as the vertical maps are equivalences by Corollary 4.18 below, and the bottom horizontal map is an equivalence because fixed-points and products commute. □

The case of the cotensor is immediate, since cotensor in \( O^{tr} \) is formed in the underlying spaces.

**Lemma 2.20.** Let \( B \) be an \( O \)-algebra in spaces and \( X \) a based space. Then we have a natural equivalence of \( O^G \)-algebras

\[
(F(X,B))^G \simeq F(X,B^G).
\]

3. The \( O \)-Spanier-Whitehead category and the \( O \)-stable category

Our model of the equivariant \( O \)-stable category is motivated by very classical considerations. The original approach to the stable category regarded the objects of the Spanier-Whitehead category as fundamental and the other objects as determined by formal considerations.

In the Spanier-Whitehead category, the objects are finite based CW-complexes, and the morphisms from \( X \) to \( Y \) are the stable homotopy classes of maps from \( X \) to \( Y \), defined as

\[
\{X,Y\} := \text{colim}_n [\Sigma^n X, \Sigma^n Y] \cong [X, \text{colim} \Omega^n \Sigma^n Y].
\]

For connected \( X \) and \( Y \), we can interpret this description operadically; May’s approximation theorem [31, 2.7] implies that there is a natural zig-zag of equivalences of \( O \)-algebras

\[
\text{colim}_n \Omega^n \Sigma^n Y \simeq \tilde{\mathfrak{P}}_O(Y)
\]

for any \( E_\infty \) operad \( O \). The free-forgetful adjunction then provides a natural isomorphism

\[
\{X,Y\} \cong [\tilde{\mathfrak{P}}_O(X), \tilde{\mathfrak{P}}_O(Y)],
\]

where on the right we are working with homotopy classes of maps of \( O \)-algebras. One proceeds by showing that the suspension functor is fully-faithful and then formally inverting it to produce a category in which suspension is an auto-equivalence.

This definition applies equally well equivariantly. Here, the objects are finite based \( G \)-CW complexes, while the maps are defined as

\[
\{X,Y\}_U^G := \text{colim}_V [\Sigma^V X, \Sigma^V Y]^G \cong [X, \text{colim} \Omega^V \Sigma^V Y]^G,
\]

where the colimit is taken over all finite dimensional subspaces of some complete universe \( U \). The equivariant analogue of the approximation theorem (e.g., see [11, 1.18]) implies that for \( G \)-connected \( X \) and \( Y \) we have a description

\[
\{X,Y\}_U^G \cong [\tilde{\mathfrak{P}}_O(X), \tilde{\mathfrak{P}}_O(Y)]_{\text{top}}^G[\mathfrak{P}_O],
\]

where \( O \) is any genuine \( G \)-\( E_\infty \) operad. Here again, the suspension by any finite dimensional representation \( V \) is fully-faithful, so we formally invert these. In fact,
by a beautiful observation of Lewis, at this point it suffices to invert the trivial suspensions; all other are automatically inverted in this context.

The operadic description then generalizes immediately.

**Definition 3.1.** Let $G$ be a finite group and let $\mathcal{O}$ be an $\mathcal{N}_\infty$ operad. Define the $\mathcal{O}$-Spanier-Whitehead category $\text{SW}_\mathcal{O}^G$ by letting the objects be finite based $G$-CW complexes and the morphisms be

$$\text{SW}_\mathcal{O}^G(X, Y) = \{X, Y\}_G^G := \text{HoTop}_*^G(\tilde{\mathcal{P}}_\mathcal{O}([\tilde{\mathcal{P}}_\mathcal{O}(\Sigma X), \tilde{\mathcal{P}}_\mathcal{O}(\Sigma Y)])^G$$

$$\cong \text{HoTop}_*^G(X, \Omega \tilde{\mathcal{P}}_\mathcal{O}(\Sigma Y))^G.$$

Composition is defined via the category $\text{Top}_*^G[\tilde{\mathcal{P}}_\mathcal{O}]$.

As a consequence of Lemma [BL] which we prove below, we obtain the following basic result about the $\mathcal{O}$-Spanier-Whitehead category.

**Theorem 3.2.** If $\mathcal{O}$ is an indexing system for $G$, then the trivial suspension induces a fully-faithful self-embedding

$$\Sigma: \text{SW}_\mathcal{O}^G \to \text{SW}_\mathcal{O}^G.$$

We have an obvious “stabilization” functor taking a $G$-space to an $\mathcal{O}$-stable object.

**Notation 3.3.** Let $\mathcal{W}_G$ denote the full subcategory of $\text{Top}_*^G$ spanned by the finite $G$-CW complexes and $\mathcal{W}_G^\ast$ the full subcategory of $\text{Top}_*^G$ spanned by the based finite $G$-CW complexes.

**Definition 3.4.** Let

$$\Sigma^\infty: \mathcal{W}_G^\ast \to \text{SW}_\mathcal{O}^G$$

be the functor specified on objects by the assignment $X \mapsto X$ and which takes a morphism $f: X \to Y$ to the induced morphism

$$\tilde{\mathcal{P}}_\mathcal{O}(\Sigma X) \to \tilde{\mathcal{P}}_\mathcal{O}(\Sigma Y).$$

Let

$$\Sigma^\infty_+: \mathcal{W}_G \to \text{SW}_\mathcal{O}^G$$

be the composite of $\Sigma^\infty$ with the natural functor $\text{Top}_*^G \to \text{Top}_*^G$ adding a $G$-fixed disjoint basepoint.

This description suggests that we regard grouplike $\mathcal{O}$-algebras as stable objects. Just as classically, these will model connective spectra. The process of stabilization is then the formal process of inverting the suspension.

### 3.1. Equivariant group completions

A key step in stabilization is group completion. To study this process, we work at the level of coefficient systems.

**Definition 3.5.**

1. A $\mathcal{O}$-algebra $B$ is grouplike if $\pi_0(B)(T)$ is a group for every finite $G$-set $T$.
2. A map $A \to B$ of $\mathcal{O}$-spaces is an equivariant group completion if $B$ is grouplike and for every $H \subset G$ the induced map of fixed-point spaces $A^H \to B^H$ is a (non-equivariant) group completion.

Given a $\mathcal{O}$-algebra, we can form the group completion as follows.
Lemma 3.6. Let $B$ be a cofibrant $O$-algebra. Then the natural map

\[ B \to F(S^1, S^1 \otimes B) \]

is an equivariant group-completion.

Proof. For each subgroup $H \subset G$, passing to $H$-fixed points yields the map

\[ B^H \to F(S^1, S^1 \otimes B)^H. \]

Since $S^1$ has trivial $H$-action, we can factor this map as the composite

\[ B^H \to F(S^1, S^1 \otimes B^H) \to F(S^1, (S^1 \otimes B)^H) \cong F(S^1, S^1 \otimes B)^H, \]

and by Lemma 2.19 all maps except the first one are weak equivalences. Moreover, $B^H$ has an action of $O^H$, which by definition is a nonequivariant $E_\infty$ operad. We know that if $B$ is cofibrant then $B^H$ has the homotopy type of a cofibrant algebra over $O^H$, by Corollary 4.16 below. Since for any cofibrant algebra $A$ over an $E_\infty$ operad the natural map

\[ A \to F(S^1, S^1 \otimes A) \]

is a group-completion [6, 6.5], the result follows. \qed

To incorporate the group completion into the model category structure, we proceed as follows. We can take the map

\[ f: \tilde{P}O \to F(S^1, S^1 \otimes \tilde{P}O) \]

and factor into a cofibration $\tilde{f}$ followed by an acyclic fibration. Performing Bousfield localization with respect to $\tilde{f}$ then leads to the following model structure (Compare [35, 5.3]).

Theorem 3.7. There is a cofibrantly generated $G$-topological model structure, the group-completion model structure, on $\mathcal{T}_{op}^G[\tilde{P}O]$ in which

1. the cofibrations are the same as in the standard model structure
2. the weak equivalences are the maps which induce weak equivalences after cofibrant-replacement and group-completion, and
3. the fibrations are determined by the right lifting property with respect to the acyclic cofibrations.

The fibrant objects in this category provide our model for connective $O$-spectra: In this category, finite wedges are finite products and the hom sets in the homotopy category are all abelian groups. Moreover, the homotopy coefficient systems extend naturally to $O$-Mackey functors. This is an algebraic shadow of a more general Wirthmüller isomorphism we will prove below.

3.2. The category of $O$-spectra. The category $\mathcal{T}_{op}^G[\tilde{P}O]$, equipped with the group-completion model structure (or equivalently the full subcategory of group-complete objects in $\mathcal{T}_{op}^G[\tilde{P}O]$), provides a model of connective $O$-spectra. In order to obtain the nonconnective objects, we formally invert the suspension, working with the category of spectrum objects in $O$-spectra.

Definition 3.8. A spectrum object $E$ in $\mathcal{T}_{op}^G[\tilde{P}O]$ is a collection of $O$-algebras $\{E_i\}$ for $i \in \mathbb{N}$ together with natural maps of $O$-algebras

\[ S^1 \otimes E_i \to E_{i+1} \]

for each $i$. 
A morphism of spectra $E \to E'$ is a collection of maps of $O$-algebras

$$E_i \to E'_i$$

that commute with the structure maps.

We write $\mathcal{S}p^G_O$ to denote the category of spectrum objects and refer to the objects as $O$-spectra.

**Definition 3.9.** An $\Omega$-spectrum in $\mathcal{S}p^G_O$ is defined to be a spectrum $E$ such that each adjoint structure map

$$E_n \to \Omega E_{n+1}$$

is a weak equivalence of $O$-algebras.

In order to construct a model structure on spectrum objects, it is convenient to work with stable homotopy objects. Since the homotopy coefficient system of an $O$-algebra is naturally an $O$-Mackey functor, we have a similar extension.

**Definition 3.10.** Let $E$ be an $O$-spectrum. The homotopy $O$-Mackey functors of $E$ are the $O$-Mackey functors

$$\pi_n(E) := \text{colim}_k \pi_{n+k}(E_k),$$

where the transition maps in the direct system are given by applying homotopy groups to the adjoint structure maps in the spectrum.

**Definition 3.11.** We say a map $E \to E'$ in $\mathcal{S}p^G_O$ is a stable equivalence if the induced map of homotopy $O$-Mackey functors $\pi_n(E) \to \pi_n(E')$ is an isomorphism for each $n \in \mathbb{Z}$.

The following lemma is immediate.

**Lemma 3.12.** A map $f : E \to E'$ of $\Omega$-spectra in $\mathcal{S}p^G_O$ is a weak equivalence if and only if $E_i \to E'_i$ is a weak equivalence in $\text{Top}^G[\tilde{P}_O]$ for each $i \in \mathbb{N}$.

By the argument for [6, 3.1], we can conclude the following theorem.

**Theorem 3.13.** There is a model structure on $\mathcal{S}p^G_O$ where

1. the cofibrations are the maps $E \to E'$ such that

$$E_0 \to E'_0$$

is a cofibration and

$$(S^1 \otimes E'_n)_{(S^1 \otimes E_n)} \wedge E_{n+1} \to E'_{n+1}$$

is a cofibration for each $n$,

2. the fibrations are the maps $E \to E'$ such that

$$E_n \to E'_n$$

is a fibration and

$$E_n \to E'_n \times_{\Omega E'_n+1} \Omega E_{n+1}$$

is a fibration for each $n$, and

3. the weak equivalences are the stable equivalences.

(Here we are using the standard model structure on $\text{Top}^G[\tilde{P}_O]$, not the group-complete model structure.)
It is straightforward to check directly the following proposition.

**Proposition 3.14.** A fibrant object in the model structure on $Sp^G_O$ is an $\Omega$-spectrum.

It suffices to work with the standard model structure when studying the homotopy theory of $Sp^G_O$ because for an $\Omega$-spectrum, $E_0$ is grouplike. Conversely, given a $O$-algebra $B$, we can form the “suspension spectrum”

$$(\Sigma^\infty B)_n = (S^1 \wedge \ldots \wedge S^1) \otimes B,$$

where the structure maps are given by the canonical maps. When $B$ is cofibrant, the adjoint structure maps

$$(\Sigma^\infty B)_n \to \Omega(\Sigma^\infty B)_{n+1}$$

are weak equivalences for $n > 0$ and the group completion for $n = 0$. In particular, if $B$ is already grouplike, this is an $\Omega$-spectrum.

**Remark 3.15.** Given a cofibrant $O$-algebra $B$, we can also consider the stabilization obtained as follows. Let $\Gamma_G$ denote the category of finite based $G$-sets, and form the $\Gamma_G$-space specified by $T \mapsto X \otimes T$.

We can now describe the suspension spectrum functor from $G$-spaces

**Definition 3.16.** Let

$$\Sigma^\infty: \mathcal{Top}^G \to Sp^G_O$$

be the functor

$$X \mapsto \Sigma^\infty (\tilde{\mathcal{P}}_O(X)).$$

Let

$$\Sigma^\infty_+ : \mathcal{Top}^G \to Sp^G_O$$

be the composite of $\Sigma^\infty$ with the functor $(-)_+$ adding a disjoint basepoint.

We note that the suspension spectrum of a $G$-space is always an $\Omega$-prespectrum above level 0, by Lemma 3.6.

**Corollary 3.17.** Let $X$ be a based and connected $G$-CW complex. Then the suspension spectrum $\Sigma^\infty_+ X$ is an $\Omega$-spectrum except at level 0, where it is the group completion.

### 3.3. Structural properties of $O$-spectra.

We now turn to the structural properties of the categories $Sp^G_O$.

#### 3.3.1. Induction, coinduction, and Wirthmüller isomorphisms.

We begin by enumerating the construction of the usual change of group and fixed-point functors. For $H \subset G$, there are natural forgetful functors.

**Definition 3.18.** For each $H \subset G$, there is a forgetful functor

$$i^*_H : Sp^G_O \to Sp^H_{\mathcal{P}O}$$

specified by the levelwise application of the forgetful functor on $O$-algebras

$$i^*_H : \mathcal{Top}^G_\mathcal{P} \to \mathcal{Top}^{H_\mathcal{P}}_\mathcal{P}.$$
**Proposition 3.19** (See e.g., [8 §6.2.2]). If $B$ is an $i^*_H O$-algebra in based $H$-spaces, then $Map^H(G_+, B)$ is naturally an $O$-algebra in $G$-spaces and the coinduction functor $B \mapsto Map^H(G_+, B)$ is right adjoint to the forgetful functor $i^*_H$. The adjunction is a Quillen adjunction.

**Proof.** Let $B$ be an $i^*_H O$-algebra. We produce the action map $O(n) \times (Map^H(G_+, B))^\times n \to Map^H(G_+, B)$ as the adjoint of the composite

$$i^*_H(O(n) \times (Map^H(G_+, B))^\times n) \cong i^*_H O(n) \times (i^*_H Map^H(G_+, B))^\times n \to i^*_H O(n) \times B^\times n \to B,$$

where we are using the fact that $Map^H(G_+, -)$ is the adjoint to $i^*_H$ in spaces. It is straightforward to check that these action maps satisfy the required compatibilities, and the fact that this is the right adjoint in operadic algebras is a consequence of the construction. To see that this is a Quillen adjunction, observe that $Map^H(G_+, -)$ preserves fibrations since these are determined by the forgetful functor to $G$-spaces. □

We can extend the right adjoint to spectra by applying it levelwise, getting a functor

$$F^H(G_+, -): Sp^H_{i^*_H O} \to Sp^G_O.$$

**Corollary 3.20.** The functor

$$E \mapsto F^H(G_+, E)$$

is right adjoint to $i^*_H$ and this is a Quillen adjunction. If $E$ is an $\Omega i^*_H O$-spectrum, then $F^H(G_+, E)$ is naturally an $\Omega O$-spectrum.

The left adjoint is slightly harder to describe, since it is built out of the operadic coproduct, rather than the underlying one. Again, we begin in $O$-algebras in spaces.

**Proposition 3.21.** The forgetful functor $i^*_H$ has a left adjoint

$$G_+ \otimes_H (-): Top^H[i^*_H O] \to Top^G[O].$$

The construction is fairly straightforward. By the universal property, on free objects, we have

$$G_+ \otimes_H \tilde{P}_{i^*_H O}(X) \cong \tilde{P}_O(G_+ \wedge_H X).$$

Writing any general operadic algebra as a reflexive coequalizers (or more generally a sifted colimit like a geometric realization) yields the standard formula for the left adjoint in general.

Applying this levelwise gives us a functor

$$G_+ \otimes_H (-): Sp^H_{i^*_H O} \to Sp^G_O.$$

**Corollary 3.22.** The functor $G_+ \otimes_H (-)$ is left adjoint to $i^*_H$. The adjunction is a Quillen adjunction.

**Proof.** To see that this is a Quillen adjunction, observe that it is immediate that $i^*_H$ preserves weak equivalences and fibrations. □
A defining property of the genuine equivariant category is that induction and coinduction agree. In the setting of $\mathcal{O}$-spectra, this is true for admissible sets. There is a canonical natural transformation $H_+ \otimes_K (-) \Rightarrow F^K(H_+,-)$, and we prove the following comparison result below as Theorem 5.4.

**Theorem 3.23.** If $H/K$ is an admissible $H$-set for $\mathcal{O}$, then the natural transformation $H_+ \otimes_K (-) \Rightarrow F^K(H_+,-)$ induces a natural isomorphism in $\text{Ho}(\mathcal{S}_\mathcal{O}^H)^*_+\mathcal{G}/\mathcal{O})$.

These equivalences endow the set of homotopy classes of maps between $\mathcal{O}$-spectra with extra structure; they are automatically coefficient systems, and the Wirthmüller isomorphisms endows them with transfers.

**Corollary 3.24.** If $E$ and $E'$ are spectra, then the coefficient system $T \mapsto [T_+ \wedge E, E']$ extends naturally to an $\mathcal{O}$-Mackey functor.

All algebraic invariants represented by $\mathcal{O}$-spectra naturally then give $\mathcal{O}$-Mackey functors.

The question of which Wirthmüller isomorphisms we have is closely connected to various kinds of rigidity of compact generators in the homotopy category of $\mathcal{O}$-spectra. Since we can build cofibrant $\mathcal{O}$-spectra out of free $\mathcal{O}$-algebras on $G$-CW complexes, we immediately deduce that the compact generators are again suspension spectra of orbits.

**Proposition 3.25.** The category $\text{Ho}(\mathcal{S}_\mathcal{O}^G)$ has a set of compact generators given by the suspension spectra of orbits $\Sigma^\infty_{\mathcal{O}_+} G/H$.

**Remark 3.26.** Although we have not constructed the symmetric monoidal structure on the category $\text{Ho}(\mathcal{S}_\mathcal{O}^G)$, in principle the Wirthmüller isomorphism then tells us when the compact generators are dualizable. Specifically, we deduce from the Wirthmüller isomorphism applied to $E = S^0$ that if $H/K$ is an admissible $H$-set, then $H/K$ is dualizable (and its dual is itself). That other orbits are not self-dual is an easy consequence of the tom Dieck splitting established below. In particular, if we are not in the genuine case, then the associated tensor triangulated category is not rigid in the sense of [4].

### 3.4. Fixed and geometric fixed points

The usual fixed-point constructions are quite sensitive to the operad that shows up. We begin by discussing change of operad in the context of the $\mathcal{O}$-stable category; the results here are a straightforward extension of the results in the context of operadic algebras in spaces.

**Proposition 3.27.** Let $\mathcal{O}$ and $\mathcal{O}'$ be $N_\infty$ operads such that there is a map $\mathcal{O} \rightarrow \mathcal{O}'$. Then there are adjoint functors

$$i^*_\mathcal{O} : \mathcal{S}_\mathcal{O}^G \rightarrow \mathcal{S}_\mathcal{O}'^G$$

and

$$i_*^\mathcal{O} : \mathcal{S}_\mathcal{O}'^G \rightarrow \mathcal{S}_\mathcal{O}^G$$

that form a Quillen adjunction.

The $\mathcal{O}$-stable category is entirely determined by the indexing system of $\mathcal{O}$ in the following sense.
Corollary 3.28. If $\mathcal{O} \to \mathcal{O}'$ is a weak equivalence of $N_\infty$ operads, then the adjunction $(\iota_{\mathcal{O}}^\mathcal{O}', \iota_{\mathcal{O}}^\mathcal{O})$ is a Quillen equivalence.

Proof. This follows from the fact that we can compute $\iota_{\mathcal{O}}^\mathcal{O}_X$ using the bar construction $B(\mathcal{O}', \mathcal{O}, X)$; if $\mathcal{O} \to \mathcal{O}'$ is a weak equivalence, this bar construction is equivalent to $B(\mathcal{O}, \mathcal{O}, X)$ which is always homotopic to $X$. □

3.4.1. Categorical fixed points. There are categorical fixed points functors which are adjoint to the appropriate “trivial” spectra. Let $N$ be a normal subgroup of $G$. Then $O^N$ is an $N_\infty$-operad for both $G$ and for $Q = G/N$, and the obvious inclusion $O^N \hookrightarrow O$ is a map of $G$ $N_\infty$-operads. Since $N$ acts trivially on $O^N$, the inclusion of $G/N$-spaces into $G$-spaces gives an inclusion

$$q^* : Sp^G/G_{O^N} \to Sp^G_{O^N}.$$ 

Definition 3.29. The pushforward from $G/N$ $O^N$-spectra to $G$ $O$-spectra is the composite

$$\iota_{\mathcal{O}}^\mathcal{O}_N \circ q^* : Sp^G/G_{O^N} \to Sp^G_{O^N}.$$ 

Just as classically, the pushforward has a right adjoint.

Definition 3.30. The categorical $N$-fixed-point functor

$$(-)^N : Sp^G \to Sp^G_{O^N}$$ 

is specified by the space-level $N$-fixed points functor applied to the restriction to an $O^N$-spectrum.

Unraveling the adjunctions gives the following, just as classically.

Proposition 3.31. The $N$-fixed points functor is the right adjoint to $\iota_{\mathcal{O}}^\mathcal{O}_N \circ q^*$, and this is a Quillen adjunction.

There is a version of the tom Dieck splitting, allowing us to identify the fixed points of suspension spectra. Since $O^G$ is just an ordinary $E_\infty$ operad, a group-like $O^G$-algebra is a connective spectrum. Our analysis of the fixed points of free algebras in Section 4 below gives a proof of this $O$-algebra tom Dieck splitting.

Theorem 3.32 (O-Spectra tom Dieck). Let $X$ be a $G$-CW complex. Then we have a natural equivalence

$$(\Sigma^\infty X_+)^G \simeq \bigvee_{G/H \in \pi_0 O(G)} EW_G(H)_+ \wedge_{W_G(H)} \Sigma^\infty X^H,$$ 

where here the index for the wedge is all isomorphism classes of orbits $G/H$ which are admissible for $O$.

3.4.2. Geometric fixed points. For many purposes it is desirable to have a fixed-point functor which has the property that the value on $\Sigma^\infty X_+$ would be $\Sigma^\infty X^G$. This is accomplished by the geometric fixed points. Heuristically, the geometric fixed points remove the “algebraic” part of the fixed points arising from transfer maps, leaving only the last piece of the terms in the tom Dieck splitting.

Homotopically, this is achieved by the composite of two functors:
(1) the Bousfield localization which nullifies the localizing, triangulated subcategory generated by the induced cells $G/H_+$ for $H$ a proper subgroup, followed by

(2) the categorical fixed points, which induces an equivalence of categories between the local objects and the stable category.

We can carry this for $\mathcal{O}$-spectra in exactly the same way it is done classically.

**Definition 3.33.** The homotopical geometric fixed-point functor

$$\Phi^G: \mathcal{S}p^{\mathcal{O}} \rightarrow \mathcal{S}p$$

is specified by the formula

$$E \mapsto \((\tilde{E}\mathcal{P} \wedge E)^{fib}\)G.$$

By construction, $\Phi^G(\cdot)$ preserves weak equivalences between cofibrant objects. The “localization” part is given by tensoring with $\tilde{E}\mathcal{P}$. By definition, the restriction of this to any proper subgroup is equivariantly contractible, so the result is also true for the tensored $\tilde{E}\mathcal{P} \wedge E$. We then compose with the categorical fixed points.

There is always a canonical map

$$E^G \rightarrow \Phi^G E.$$  

Note that in contrast to the situation with traditional models of spectra, we do not have a good point-set construction of the geometric fixed points.

**Remark 3.34.** For a general $\mathcal{O}$, the canonical map

$$E^G \rightarrow \Phi^G(E)$$

can be closer to an equivalence than expected from the usual genuine case. Notably, if the only admissible $G$-sets are those with a trivial action, then this is an equivalence. This holds, for example, for $\mathcal{O}^{tr}$ (which is to say that the fixed points of naive $G$-spectra agree with the geometric fixed points), but it also holds for the $C_4$-linear isometries operads associated to $\infty(\mathbb{R} \oplus \lambda)$ described above.

We turn now to proving the key structural properties of $\mathcal{O}$-spectra, showing the tom Dieck splitting and the Wirthmüller isomorphism.

**4. The tom Dieck splitting of a free $\mathcal{O}$-algebras**

Since our model of the $\mathcal{O}$-stable category is built from the category of $\mathcal{O}$-algebras in $\text{Top}^G$, the core of our analysis of the category of $\mathcal{O}$-spectra will depend on describing the equivariant homotopy type of free $\mathcal{O}$-algebras. The purpose of this section is to calculate this homotopy type. In particular, we establish a splitting theorem that is a version of the tom Dieck splitting in the category of $\mathcal{O}$-algebras.

**4.1. Fixed points of free $\mathcal{O}$-algebras.** The proof of the following theorem is the goal of this subsection. In the statement, we write $\text{Map}^G(\cdot, \cdot)$ to denote the set of $G$-maps between $G$-spaces and $\text{Aut}_G(T)$ for the automorphisms of $T$ as a $G$-set. Note that since $\text{Aut}_G(T)$ is a group under composition (with unit the identity map), we can form the universal space $E\text{Aut}_G(T)$ via the two-sided bar construction

$$E\text{Aut}_G(T) = B(\text{Aut}_G(T), \text{Aut}_G(T), \ast).$$

For an $\infty$ operad $\mathcal{O}$, we write $\pi_0\mathcal{O}(G)$ to denote the set of isomorphism classes of $G$-sets which are admissible for $\mathcal{O}$. 

Theorem 4.1. Let $\mathcal{O}$ be an $N_\infty$-operad, and let $X$ be a $G$-CW complex. Then for all $n$ we have a natural weak equivalence
\[
(O_n \times X^n)^G \simeq \prod_{T \in \pi_0 \mathcal{O}(G), |T| = n} E \text{Aut}_G(T) \times \text{Map}^G(T, X).
\]
Moreover, if $\mathcal{O} \rightarrow \mathcal{O}'$ is a map of $N_\infty$ operads, then the natural map
\[
(O_n \times X^n)^G \rightarrow (O_n' \times X^n)^G
\]
is homotopic to the obvious inclusion of wedge summands induced by the inclusion $\pi_0 \mathcal{O}(G) \subset \pi_0 \mathcal{O}'(G)$.

As an immediate consequence, we get the following description of the fixed-points of the free $\mathcal{O}$-algebra on a space $X$.

**Corollary 4.2.** Let $\mathcal{O}$ be an $N_\infty$-operad, and let $X$ be a $G$-CW complex. Then we have a natural weak equivalence
\[
(P_{\mathcal{O}}(X))^G \simeq \prod_{T \in \pi_0 \mathcal{O}(G)} E \text{Aut}_G(T) \times \text{Map}^G(T, X).
\]

Throughout the course of this section, we will refine this corollary several times, deducing increasingly strong statements.

**Remark 4.3.** When $\mathcal{O}$ is the trivial $N_\infty$ operad, then there is a unique isomorphism class of admissible $G$-sets of cardinality $n$, namely the set of $n$ elements with the trivial action. In this case, we recover the classical results
\[
(O_n^{tr} \times X^n)^G \simeq E \Sigma_n \times (X^G)^n,
\]
and
\[
(P_{\mathcal{O}^{tr}}(X))^G \simeq \prod_{n \geq 0} E \Sigma_n \times (X^G)^n \simeq P_{\mathcal{O}^{tr}}(X^G).
\]
In words, the fixed points of the free $\mathcal{O}$-algebra over a trivial $N_\infty$ operad on $X$ is the free $E_\infty$ algebra on $X^G$.

Before we begin, we need a small lemma which helps in working with universal spaces and determining fixed points thereof. We first record a categorical observation about families of subgroups. Recall that a sieve in a category $C$ is a subcategory $S$ such that for any object $X$ in $C$, all morphisms $Z \rightarrow X$ are in $S$.

**Lemma 4.4** ((e.g., [8, 3.5], [18, 6.3])). Let $G$ be a finite group. Families of subgroups of $G$ are equivalent to sieves in the orbit category $\text{Orb}^G$. Because of this, we will tacitly elide the distinction in the following discussion, identifying a family $\mathcal{F}$ with the sieve in $\text{Orb}^G$.

We now develop some machinery in order to use the model of the classifying space of a family provided by Elmendorf’s theorem. Recall that an $\text{Orb}^G$-space $X$ is a functor $(\text{Orb}^G)^{op} \rightarrow \text{Top}$. Given a covariant functor $\mathcal{F} : \text{Orb}^G \rightarrow \text{Top}^G$, we can form the the two-sided bar construction $B(\mathcal{F})$, which is the simplicial object in $G$-spaces with $k$-simplices specified by the assignment
\[
[k] \mapsto \prod_{G/H_k \rightarrow \cdots \rightarrow G/H_0} X(G/H_0) \times \mathcal{F}(G/H_k),
\]
where $G$ acts via the action on $\mathcal{F}(G/H_k)$. The geometric realization then becomes a $G$-space, as the action is evidently compatible with the simplicial structure maps.

We now recall two particularly important examples of inputs for this two-sided bar construction.

**Definition 4.5** ([13]). Let $\mathcal{F}$ be family of subgroups of $G$. Let $\mathcal{F}^\text{op}: (\text{Orb}^G)^\text{op} \to \text{Top}$ be the functor defined by

$$\mathcal{F}^\text{op}(G/H) = \begin{cases} \ast & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F}. \end{cases}$$

Let $J: \text{Orb}^G \to \text{Top}^G$ be the functor which sends an orbit to itself (viewed now as a $G$-space).

The following lemma gives a convenient simplification of this two-sided bar construction when $X$ is the functor $\mathcal{F}^\text{op}$, using the characterization of Lemma 4.4.

**Lemma 4.6.** Let $\mathcal{F}$ be a family of subgroups of a group $G$, and let $\mathcal{C}$ be any covariant functor $\text{Orb}^G \to \text{Top}^G$. Then we have an equivariant isomorphism of simplicial $G$-spaces

$$B^\bullet(\mathcal{F}^\text{op}, \text{Orb}^G, \mathcal{C}) \cong B^\bullet(\ast, \mathcal{F}, \mathcal{C}|_{\mathcal{F}}),$$

where $\ast$ is the constant functor from $\mathcal{F}$ with value a point.

**Proof.** Recall that a point in $B^k(\mathcal{F}^\text{op}, \text{Orb}^G, \mathcal{C})$ is a $(k + 2)$-tuple

$$(x \in \mathcal{F}^\text{op}(G/H_0), G/H_k \to \cdots \to G/H_0, c \in \mathcal{C}(G/H_k)),$$

and $G$ acts on this via the action in the last coordinate. Since $\mathcal{F}^\text{op}$ is non-empty only on the sieve $\mathcal{F}$, any such point necessarily has $G/H_0 \in \mathcal{F}$, and since $\mathcal{F}$ is a sieve, this implies that all of $G/H_0, \ldots, G/H_k$ are in $\mathcal{F}$. That is, $x$ is in fact a point in $B^k(\ast, \mathcal{F}, \mathcal{C}|_{\mathcal{F}})$. This identification is clearly compatible with the simplicial structure maps. □

We are now ready to perform the basic calculation of the fixed-points of the extended powers for an $N_\infty$ operad. In the following statement, recall that an admissible set $T$ in an indexing system $\mathcal{O}$ corresponds to a homomorphism $\Gamma: H \to \Sigma_{|T|}$ such that the associated family contains the graph of $\Gamma$. We will refer to the subgroup associated to $T$ as $\Gamma_T$.

**Theorem 4.7.** Let $\mathcal{O}$ be an $N_\infty$ operad. Then for all $n$ and for all $G \times \Sigma_n$-CW complexes $Y$, we have a natural equivalence

$$(\mathcal{O}_n \times Y)^G \cong \bigotimes_{T \in \pi_0(\mathcal{O}(G)), \ |T|=n} E \text{Aut}_{G \times \Sigma_n}(G \times \Sigma_n/\Gamma_T) \times \prod_{\text{Aut}_{G \times \Sigma_n}(G \times \Sigma_n/\Gamma_T)} ((G \times \Sigma_n/\Gamma_T) \times Y)^G.$$

Moreover, if $\mathcal{O} \to \mathcal{O}'$ is a map of $N_\infty$ operads, then the natural map

$$(\mathcal{O}_n \times Y)^G \to (\mathcal{O}'_n \times Y)^G$$

is homotopic to the inclusion of summands induced by $\mathcal{O}(G) \subset \mathcal{O}'(G)$. 

Proof. By hypothesis, \( \mathcal{O}_n \) is a universal space for the family \( F_n \) of subgroups of \( G \times \Sigma_n \) of the form \( \Gamma_T \), where \( T \) ranges over all admissible sets of cardinality \( n \) for all subgroups. Moreover, \( \mathcal{O}_n \) is \( \Sigma_n \)-free and has the homotopy type of a CW-complex; we can replace \( \mathcal{O}_n \) up to weak equivalence, and therefore we have flexibility to use any point-set model we find convenient for this homotopy type. For our purposes, we will use Elmendorf’s bar construction [13] as simplified by Lemma [4.0].

In particular, we have for any \( G \times \Sigma_n \)-CW complex \( Y \) a canonical equivariant homeomorphism
\[
E \mathcal{F}_n \times Y \cong |\mathcal{B}_\bullet(\mathcal{F}_n^{op}, \text{Orb}^G, J)| \times Y \cong |\mathcal{B}_\bullet(\ast, \mathcal{F}_n, J|_{\mathcal{F}_n})| \times Y \\
\cong |\mathcal{B}_\bullet(\mathcal{F}_n^{op}, \mathcal{F}_n, J|_{\mathcal{F}_n} \times Y)|,
\]
where \( J|_{\mathcal{F}_n} \times Y \) is the function which sends an orbit \( G \times \Sigma_n / \Lambda \) to its product with \( Y \); here we are using the fact that finite products commute with geometric realization. Furthermore, we have a \( G \)-equivariant homeomorphism
\[
E \mathcal{F}_n \times Y_{\Sigma_n} \cong |\mathcal{B}_\bullet(\ast, \mathcal{F}_n, J|_{\mathcal{F}_n} \times Y)|_{\Sigma_n},
\]
since colimits commute. Using the fact that the geometric realization can be described as a sequential colimit of pushouts along closed inclusions, we can conclude that fixed points commute with geometric realization:
\[
(E \mathcal{F}_n \times Y)^G \cong |\mathcal{B}_\bullet(\ast, \mathcal{F}_n, (J|_{\mathcal{F}_n} \times Y)^G)|_{\Sigma_n}.
\]

If \( \Lambda \in \mathcal{F}_n \), then \( \Lambda \) is of the form \( \Gamma_T \) for some subgroup \( H \subset G \) and some admissible finite \( H \)-set of cardinality \( n \). If \( H \subset G \), then in fact \( \Gamma_T \subset H \times \Sigma_n \), and we have an isomorphism of \( G \times \Sigma_n \)-spaces
\[
G \times \Sigma_n / \Gamma_T \cong G \times H \left( H \times \Sigma_n / \Gamma_T \right).
\]
We therefore deduce that for any \( G \times \Sigma_n \)-space \( Y \), we have a \( G \)-equivariant homeomorphism
\[
(G \times \Sigma_n / \Gamma_T) \times Y \cong G \times H \left( (H \times \Sigma_n / \Gamma_T) \times i_{\Sigma_n} \right).
\]
In particular, if \( H \subset G \), then this has no \( G \)-fixed points.

Putting this together, we see that the fixed points \((E \mathcal{F}_n \times Y)^G\) can be computed as the geometric realization of a simplicial object with \( k \)-simplices:
\[
\left\{ (G \times \Sigma_n / \Gamma_{T_k}) \rightarrow \cdots \rightarrow (G \times \Sigma_n / \Gamma_{T_0}), y \in ((G \times \Sigma_n / \Gamma_{T_k}) \times Y)^G \right\}
\]
\[
| T_0, \ldots, T_k \text{ admissible for } \mathcal{O}, T_k \in \mathcal{O}(G) \}
\]
Now if \( T \) is a \( G \)-set, then \( \Gamma_T \) is a maximal element in the family of subgroups. In particular, all of the maps in the \( k \)-simplices above must be isomorphisms.

Thus, when we pass to fixed points, we are simply restricting to the maximal subgroupoid of \( \mathcal{F}_n \) containing the maximal subgroups:
\[
(E \mathcal{F}_n \times Y)^G \cong \\
\prod_{T \in \pi_n \mathcal{O}(G), |T| = n} E \text{Aut}_{G \times \Sigma_n}(G \times \Sigma_n / \Gamma_T) \times \text{Aut}_{G \times \Sigma_n(G \times \Sigma_n / \Gamma_T)} ((G \times \Sigma_n / \Gamma_T) \times Y)^G_{\Sigma_n}.
\]
For the second part of the statement, note that the map of $N_\infty$ operads $O \to O'$ induces a map on $n$th spaces. Since these are universal spaces, there is a unique homotopy class of such a map, so we can without loss of generality assume that the map is the one induced by the inclusion of the family of $O_n$ into that for $O'_n$. □

We can now restrict to the case that $Y = X^n$ for some $G$-space $X$ with the evident induced $G \times \Sigma_n$-action. Here the orbits in Theorem 4.7 can be put in a more illuminating form.

**Lemma 4.8** ([8, 6.2]). For all $G$-spaces $X$ and all finite $G$-sets $T$ of cardinality $n$, we have a natural $G$-equivariant homeomorphism

$$(G \times \Sigma_n/\Gamma_T) \times X^n \cong \text{Map}(T, X),$$

where the mapping space has the conjugation action.

Next, we can identify the Weyl group of $\Gamma_T$ as a subgroup of $G \times \Sigma_n$.

**Lemma 4.9.** Let $T$ be a finite $G$-set of cardinality $n$, and let $X$ be a $G$-space.

1. We have a natural isomorphism

$$\text{Aut}_{G \times \Sigma_n}(G \times \Sigma_n/\Gamma_T) = W_{G \times \Sigma_n}(\Gamma_T) \cong \text{Aut}_G(T).$$

2. The isomorphism of Lemma 4.8 is $\text{Aut}_G(T)$ equivariant, where the left-hand side has the evident action induced by the isomorphism above and $\text{Map}(T, X)$ is endowed with the obvious action of $\text{Aut}_G(T)$.

**Proof.** We begin by identifying the normalizer of $\Gamma_T$. Let $f$ be the homomorphism $G \to \Sigma_n$ defining $T$; the graph of $f$ is precisely $\Gamma_T$. Then the normalizer of $\Gamma_T$ in $G \times \Sigma_n$ is

$$N_{G \times \Sigma_n}(\Gamma_T) = \{ (g, \sigma) \mid \forall h \in G, (g, \sigma)(h, f(h))(g^{-1}, \sigma^{-1}) \in \Gamma_T \}.$$ 

Unpacking the condition, we see that $(g, \sigma) \in N_{G \times \Sigma_n}(\Gamma_T)$ if and only if for all $h \in G$,

$$\sigma f(h) \sigma^{-1} = f(g) f(h) f(g)^{-1}.$$ 

Therefore, we can define a natural homomorphism

$$\phi: \text{Aut}_G(T) \to N_{G \times \Sigma_n}(\Gamma_T) \to W_{G \times \Sigma_n}(\Gamma_T)$$

as the composite of the map which sends an automorphism $\sigma$ to the pair $(e, \sigma)$ followed by the projection.

We now deduce two consequences of equation (1):

- Taking $h = g$, we have

$$\sigma f(g) \sigma^{-1} = f(g) f(g) f(g)^{-1} = f(g),$$

i.e., $f(g)$ and $\sigma$ commute.

- Multiplying by $\sigma$ on the right and $f(g)^{-1}$ on the left, we obtain

$$f(g)^{-1} \sigma f(h) = f(h) f(g)^{-1} \sigma,$$

i.e., $f(g)^{-1} \sigma$ commutes with $f$ and thus specifies an automorphism of $T$.
We now use these facts to show that \( \phi \) is an isomorphism. Recall that for any \( g \in G \), by definition the element \( (g, f(g)) \in \Gamma_T \). The following equalities
\[
(g, \sigma) \simeq (g, \sigma)(g^{-1}, f(g)^{-1}) = (e, \sigma f(g)^{-1}) = (e, f(g)^{-1}) = \phi(f(g)^{-1})
\]
now show that any element \( (g, \sigma) \in N_{G \times \Sigma_n}(\Gamma_T) \) is equivalent modulo \( \Gamma_T \) to something in the image of \( \phi \). That is, \( \phi \) is surjective. Here the second equality uses equation (2) and the third equality uses equation (3). Similarly, if \( \phi(\sigma) = \phi(\sigma') \), then there is some \( g \in G \) such that
\[
(e, \sigma)(g, f(g)) = (e, \sigma')(g, f(g)).
\]
This forces \( g = e = f(g) \), and hence \( \sigma = \sigma' \); \( \phi \) is injective.

The second part is now obvious: passage to \( \Sigma_n \) orbits allows us to move a \( G \)-equivariant automorphism \( \sigma \) of \( T \) (viewed as a particular element in \( \Sigma_n \)) to the exponent of \( X^n \), which is precisely the definition of the action of \( \text{Aut}_G(T) \) on the set of \( n \) elements. \( \square \)

**Proof of Theorem 4.1.** Applying theorem 4.7 to \( Y = X^n \) yields the formula
\[
(O_n \times X^n)^G \cong \prod_{T \in \pi_0 \Omega(G), |T| = n} E \text{Aut}_G \times \Sigma_n(G \times \Sigma_n/\Gamma_T) \times \text{Aut}_G \times \Sigma_n(G \times \Sigma_n/\Gamma_T) \times (G \times \Sigma_n/\Gamma_T) \times X^n)^G.
\]
Using lemma 4.8, we identify the fixed points of the orbits in the summand associated to \( T \):
\[
((G \times \Sigma_n/\Gamma_T) \times X^n)^G \cong \text{Map}^G(T, X).
\]
Finally, Lemma 4.9 identifies the group \( \text{Aut}_G \times \Sigma_n(G \times \Sigma_n/\Gamma_T) \) and the action. \( \square \)

4.2. **The tom Dieck splitting for \( \mathcal{O} \)-algebras.** We now apply the identifications of the previous section to deduce a tom Dieck splitting for \( \mathcal{O} \)-algebras. Specifically, we will apply the identification of Corollary 4.2 and the naturality in the operad \( \mathcal{O} \).

4.2.1. The unbased case. We begin with some notation and an elementary observation.

**Notation 4.10.** Let \( (H)_G \) denote the set of \( G \)-conjugacy classes of \( H \). If \( G \) is clear from the context, we will also denote this simply as \( (H) \).

Since an automorphism preserves stabilizers of points, no automorphism can change orbit types:

**Proposition 4.11.** If
\[
T = \prod_{(H)} n_H \cdot G/H,
\]
where the coproduct ranges over conjugacy classes of subgroups, then we have an identification
\[
\text{Aut}_G(T) \cong \prod_{(H)} \text{Aut}_G(n_H \cdot G/H).
\]
Coupled with the universal property of the product, we have the following further refinement of Corollary 4.2.

**Proposition 4.12.** Let $\mathcal{O}$ be an $N_\infty$ operad, and let $X$ be a $G$-CW complex. Then we have an identification

$$\left(\mathbb{P}_\mathcal{O}(X)\right)^G \simeq \prod_{G/H \in \pi_0\mathcal{O}(G)} \prod_{n \geq 0} E \text{Aut}_G(n \cdot G/H) \times \text{Aut}_{\mathcal{O}(G)}(X^H)^n.$$

If $\mathcal{O} \subset \mathcal{O}'$, then the identification can be chosen as the inclusion of those factors corresponding to $G/H$ with $G/H \in \pi_0\mathcal{O}(G)$.

**Proof.** Since any finite $G$-set is a disjoint union of orbits and a finite set is admissible if and only if all of the orbits occurring in such a decomposition are admissible, the identification is essentially a rewriting of Corollary 4.2. The only further refinement is the application of the isomorphism

$$\text{Map}_G(n \cdot G/H, X) \cong \text{Map}_G(G/H, X^n) \cong (X^H)^n.$$

Next, we record the standard computation of the $G$-automorphisms of the $G$-set

$$n \cdot G/H = \coprod_n G/H.$$

**Proposition 4.13.** Let $H \subset G$. Then we have an isomorphism

$$\text{Aut}_G(n \cdot G/H) \cong \text{Aut}_G(G/H) \wr \Sigma_n \cong W_G(H) \wr \Sigma_n.$$

We are now ready to prove the main theorem of this subsection.

**Theorem 4.14.** Let $\mathcal{O}$ be an $N_\infty$ operad, and let $X$ be a $G$-CW complex. Let $\mathcal{E}$ denote any non-equivariant $E_\infty$ operad. Then we have a decomposition of $\mathcal{E}$-algebras

$$\left(\mathbb{P}_\mathcal{O}(X)\right)^G \simeq \prod_{G/H \in \pi_0\mathcal{O}(G)} \mathbb{P}_\mathcal{E}(EW_G(H) \times W_G(H)^n X^H).$$

**Proof.** Propositions 4.12 and 4.13 give the stated splitting in the category of spaces. We need to show that this is a splitting of $\mathcal{E}$-algebras; this is a consequence of naturality.

Recall from [8, 6.25] that the cofree $N_\infty$ operad $F(EG, \mathcal{O})$ is a genuine $G-E_\infty$ operad for any $G$-operad $\mathcal{O}$. Since the cofree functor is the right adjoint to the forgetful functor to trivial operads, the unit gives us a natural map of operads

$$\mathcal{O} \to F(EG, i_*^\mathcal{O})$$

which provides an explicit model for the terminal map in the homotopy category of $N_\infty$ operads. For convenience, we write $\mathcal{O}^{gen}$ for $F(EG, i_*^\mathcal{O})$ in what follows.

On the other hand, $\mathcal{O}^G$ is an $E_\infty$ operad [8, B.1], and so the inclusion gives rise to a map

$$\mathcal{O}^G \to \mathcal{O}$$

de $N_\infty$ operads where $\mathcal{O}^G$ is given the trivial $G$-action.

Putting this together, we see that the composite

$$\left(\mathbb{P}_\mathcal{O}(X)\right)^G \to \left(\mathbb{P}_{\mathcal{O}^{gen}}(X)\right)^G$$

...
is a map of $\mathcal{O}^G$-algebras. The equivariant Barratt-Priddy-Quillen theorem proved as [19, 6.12] shows that we have an equivalence of the desired form for $F(EG, i_*^e\mathcal{O})$. In particular, the projection maps

$$q_H : (\mathbb{P}_{\mathcal{O}^e}(X))^G \to \mathbb{P}_{\mathcal{E}}(EW_G(H) \times_{W_G(H)} X^H)$$

are all maps of $\mathcal{O}^G$-algebras. Composing these maps with the map in Equation 4 then shows that the projection maps

$$q_H : (\mathbb{P}_{\mathcal{O}}(X))^G \to \mathbb{P}_{\mathcal{E}}(EW_G(H) \times_{W_G(H)} X^H)$$

are all maps of $\mathcal{O}^G$-algebras, and hence the splitting from Proposition 4.12 is one of $\mathcal{O}^G$-algebras. Naturality in maps of operads and the usual product trick then allows us to convert this to a splitting of $\mathcal{E}$-algebras, as desired.

We will now use the fact that for the free $\mathcal{O}^G$-algebra functor $\mathbb{P}_{\mathcal{O}^G}$, the natural map

$$\mathbb{P}_{\mathcal{O}^G}(X \vee Y) \to \mathbb{P}_{\mathcal{O}^G} X \times \mathbb{P}_{\mathcal{O}^G} Y$$

is a weak equivalence [6, 6.9] and more generally for cofibrant $\mathcal{O}^G$-algebras, the natural map

$$X \vee_{\mathcal{O}^G} Y \to X \times Y$$

is a weak equivalence [6, 6.8].

In particular, we can immediately deduce the following result about naturality of the splitting in maps of operads.

**Corollary 4.15.** If $\mathcal{O} \to \mathcal{O}'$ is a map of $N_\infty$ operads, then for any $G$-CW complex $X$, the induced map

$$(\mathbb{P}_{\mathcal{O}}(X))^G \to (\mathbb{P}_{\mathcal{O}'}(X))^G$$

is the inclusion of an $E_\infty$ direct summand.

As another corollary of the proof, we can maintain homotopical control on the fixed points:

**Corollary 4.16.** For an $N_\infty$ operad $\mathcal{O}$, the fixed points

$$(\mathbb{P}_{\mathcal{O}}(X))^G \simeq \prod_{G/H \in \pi_0 \mathcal{O}(G)} \mathbb{P}_{\mathcal{E}}(EW_G(H) \times_{W_G(H)} X^H).$$

have the homotopy type of a cofibrant $\mathcal{O}^G$-algebra.

We can use these results to express the decomposition in terms of the free algebra on the summands, as follows. For $H \subseteq G$ such that $G/H$ is admissible for $\mathcal{O}$, let $\Gamma_{G/H}$ be the graph subgroup of $G \times \Sigma_{|G/H|}$ corresponding to an ordering of the points of $G/H$. Our description of the $G$-fixed points of $\mathbb{P}_{\mathcal{O}}(X)$ shows that there is an inclusion

$$EW_G(H) \times_{W_G(H)} X^H \xrightarrow{n_{G/H}} \mathbb{P}_{\mathcal{O}}(X)^G$$

corresponding to the map on homotopy colimits induced by the inclusion of the orbit $G \times \Sigma_{|G/H|}/\Gamma_{G/H}$ into the family $\mathcal{F}_{G/H}$. By the free-forget adjunction, this gives us a map of $\mathcal{O}^G$-algebras

$$\mathbb{P}_{\mathcal{O}^G}(EW_G(H) \times_{W_G(H)} X^H) \xrightarrow{n_{G/H}} \mathbb{P}_{\mathcal{O}}(X)^G.$$
The following proposition now follows immediately from Theorem 4.14 and equation (5).

**Proposition 4.17.** The map

\[ \prod_{n_{G/H}}: \mathcal{P}_\mathcal{O} \left( \prod_{G/H \in \pi_0 \mathcal{O}(G)} EW_G(H) \times X^H \right) \to \mathcal{P}_\mathcal{O}(X)^G \]

is a weak equivalence of \( E \)-algebras.

As a corollary, we deduce a first weak kind of stability for the category of \( N_\infty \)-spaces.

**Corollary 4.18.** Let \( X \) and \( Y \) be cofibrant \( \mathcal{O} \)-algebras in the standard model structure. Then the natural map

\[ X \vee \mathcal{O} Y \to X \times Y \]

is an equivalence of \( \mathcal{O} \)-algebras.

**Proof.** First, observe that using the bar construction equivalence \( B(\mathcal{P}_\mathcal{O}, \mathcal{P}_\mathcal{O}, X) \to X \) we can reduce as in the argument for [6, 6.8] to the case where we are considering the natural map

\[ \mathcal{P}_\mathcal{O} X \vee \mathcal{P}_\mathcal{O} Y \to \mathcal{P}_\mathcal{O} X \times \mathcal{P}_\mathcal{O}. \]

It suffices to consider the \( G \)-fixed points, and we can now apply Theorem 4.14 and use the fact that \( \mathcal{P}_\mathcal{O} X \vee \mathcal{P}_\mathcal{O} Y \simeq \mathcal{P}_\mathcal{O}(X \vee Y) \); we have that

\[ (\mathcal{P}_\mathcal{O}(X \vee Y))^G \simeq \prod_{G/H \in \pi_0 \mathcal{O}(G)} \mathcal{P}_E \left( EW_G(H) \times X^H \right). \]

Since coproducts commute with fixed-points and orbits, we can write the righthand side as

\[ \prod_{G/H \in \pi_0 \mathcal{O}(G)} \mathcal{P}_E \left( EW_G(H) \times X^H \right) \vee \prod_{G/H \in \pi_0 \mathcal{O}(G)} \mathcal{P}_E \left( EW_G(H) \times Y^H \right). \]

Using the non-equivariant comparison of coproduct and product for \( E \)-spaces (equation (5)), we can rewrite again as

\[ \prod_{G/H \in \pi_0 \mathcal{O}(G)} \mathcal{P}_E \left( EW_G(H) \times X^H \right) \times \prod_{G/H \in \pi_0 \mathcal{O}(G)} \mathcal{P}_E \left( EW_G(H) \times Y^H \right), \]

which by Theorem 4.14 is precisely

\[ \mathcal{P}_\mathcal{O}(X) \times \mathcal{P}_\mathcal{O}(Y). \]

\( \blacksquare \)

**4.2.2. The based case.** We can deduce the corresponding splitting in the based case from the unbased statement. Recall that we have a natural isomorphism of monads \( \mathcal{P}_\mathcal{O}(-)_+ \simeq \mathcal{P}_\mathcal{O}(-) \). Moreover, for any based \( G \)-space \( X \), there is a map \( S^0 \to X_+ \) specified by sending the non-basepoint of \( S^0 \) to the basepoint of \( X \). If \( X \) has a nondegenerate basepoint (which we can assume without loss of generality), then there is a cofiber sequence of based \( G \)-spaces \( S^0 \to X_+ \to X \). Since \( \mathcal{P}_\mathcal{O} \) preserves cofiber sequences and so does the righthand side of Theorem 4.14, we can conclude the following result in the based case.
Theorem 4.19. The norm maps \( n_{G/H} \) for \( G/H \) admissible induce a natural weak equivalence of \( \mathcal{O}^G \)-algebras

\[
(\tilde{P}_\mathcal{O}(X))^G \simeq \prod_{G/H \in \pi_0 \mathcal{O}(G)} \tilde{P}_\mathcal{O}(\text{EW}_G(H)_+ \wedge_{W_G(H)} X^H).
\]

Delooping this gives us the tom Dieck splitting.

Proof of Theorem 3.32. The definition of \( \mathcal{S}p_G^G \) implies that we can compute the fixed points of a suspension spectrum in terms of the zero-space. Specifically, Using Lemma 2.19, Corollary 3.17 implies that we can deduce the incomplete to m Dieck splitting from Theorem 4.14. □

5. Equivariant stability

5.1. The Wirthmüller Isomorphism in \( \mathcal{O} \)-spectra. The infinite loop space version of the Wirthmüller isomorphism is that for every subgroup \( H \subset G \) and for every \( \mathcal{O}^{gen} \)-algebra \( B \) in based \( H \)-spaces, we have a natural weak equivalence

\[
G_+ \otimes_H B \xrightarrow{\sim} \text{Map}_H(G_+, B).
\]

This is true much more generally.

Notation 5.1. Let

\[
\delta: i^*_H G_+ \to H_+
\]

be the based \( H \)-map

\[
\delta_{eH} (g) = \begin{cases} g & g \in H \\ + & g \notin H. \end{cases}
\]

We will also use \( \delta \) to denote the natural map of based \( H \)-spaces

\[
i^*_H(G_+ \wedge_H X) \to H_+ \wedge_H X \cong X.
\]

Theorem 5.2. Let \( H \subset G \) be such that \( G/H \) is an admissible \( \mathcal{O} \)-set. Then the natural map of \( \mathcal{O} \)-algebras

\[
\tilde{P}_\mathcal{O}(G_+ \wedge_H X) \xrightarrow{\omega_{\mathcal{O},H}} \text{Map}_H \left( G_+ \tilde{P}_{i^*_H \mathcal{O}}(X) \right)
\]

adjoint to

\[
i^*_H \tilde{P}_\mathcal{O}(G_+ \wedge_H X) \cong \tilde{P}_{i^*_H \mathcal{O}}(i^*_H G_+ \wedge_H X) \xrightarrow{\tilde{P}_\mathcal{O}(\delta)} \tilde{P}_\mathcal{O}(X)
\]

is a weak equivalence in the standard model structure.

The proof of Theorem 5.2 is a somewhat elaborate excursion through finite group theory, so we briefly postpone it to state and prove our desired result: a Wirthmüller isomorphism for the \( \mathcal{O} \)-Spanier-Whitehead category.

Theorem 5.3. For \( H \) an admissible \( \mathcal{O} \)-set, the functor

\[
X \mapsto G_+ \wedge_H X
\]

is both the left and right adjoint to the forgetful functor \( i^*_H \) on \( \text{SW}_\mathcal{O}^H \).
Proof. Since induction is the left adjoint in G-spaces, and since the mapping sets in \( \mathcal{SW}_p^G \) can be computed as just mapping sets in G-spaces, induction is the left adjoint in \( \mathcal{SW}_p^H \).

Since limits in \( \mathcal{O} \)-algebras are formed in the underlying category, we know that \( \text{Map}_H(G_+, -) \) is also the right adjoint to the forgetful functor \( i^*_H : \mathcal{O}-\text{Alg}^G \to \mathcal{O}-\text{Alg}^H \).

Theorem 5.2 gives us a natural equivariant weak equivalence for any \( H \)-space \( Y \)
\[
\Omega \tilde{\mathcal{P}}_\mathcal{O}(G_+ \wedge_H Y) \to \Omega \text{Map}_H \left( G_+, \tilde{\mathcal{P}}_{i^*_H \mathcal{O}}(Y) \right) \cong \text{Map}_H \left( G_+, \Omega \tilde{\mathcal{P}}_{i^*_H \mathcal{O}}(Y) \right).
\]

Thus for any finite \( G \)-CW complex \( X \), we have a natural isomorphism
\[
\{ X, G_+ \wedge_H Y \}^G_\mathcal{O} = \mathcal{O}-\text{Alg}^G \left( \Omega \tilde{\mathcal{P}}_\mathcal{O}(\Sigma X), \Omega \tilde{\mathcal{P}}_\mathcal{O}(G_+ \wedge_H Y) \right)
\cong \mathcal{O}-\text{Alg}^G \left( \Omega \tilde{\mathcal{P}}_\mathcal{O}(\Sigma X), \text{Map}_H \left( G_+, \Omega \tilde{\mathcal{P}}_{i^*_H \mathcal{O}}(Y) \right) \right)
\cong \mathcal{O}-\text{Alg}^H \left( i^*_H \Omega \tilde{\mathcal{P}}_\mathcal{O}(\Sigma X), \Omega \tilde{\mathcal{P}}_{i^*_H \mathcal{O}}(Y) \right)
\cong \{ i^*_H X, Y \}^H_\mathcal{O},
\]
where the last isomorphism is the obvious one that the free algebra in \( G \)-spaces on \( X \) restricts to the free algebra in \( H \)-spaces on \( i^*_H X \).

Finally, we can conclude the Wirthmuller isomorphism in \( \mathcal{SP}_p^G \); our argument is analogous in spirit to the formal criterion given in [15]. For any subgroup \( H \subset G \) and \( i^*_H \mathcal{O} \)-spectrum \( X \), there is a natural map
\[
G_+ \otimes_H X \to F_H(G, X)
\]
obtained as the adjoint of the natural map \( X \to i^*_H F_H(G, X) \) that takes \( x \) to the map \( \delta(-)x \).

Theorem 5.4. Let \( B \) be an \( i^*_H \mathcal{O} \)-spectrum. If \( G/H \) is an admissible \( G \)-set, then there is a natural isomorphism in \( \text{Ho}(\mathcal{SP}_p^G) \)
\[
G_+ \otimes_H B \cong F_H(G, B).
\]

Proof. Since both sides commute with sifted homotopy colimits, it suffices to show that the map is an isomorphism for the generating objects of \( \mathcal{SP}_p^G \); by Proposition 3.25 these are the suspension spectra of \( H/K \) where \( K \) ranges over the subgroups of \( H \). These are in particular suspension spectrum \( H \)-CW complexes, and the result now follows from Theorem 5.3. \( \square \)

5.2. Proof of Theorem 5.2. For self-containedness, we will prove in this subsection that the natural map
\[
\tilde{\mathcal{P}}_\mathcal{O}(G_+ \wedge_H X) \xrightarrow{\omega \circ_H} \text{Map}_H \left( G_+, \tilde{\mathcal{P}}_\mathcal{O}(X) \right)
\]
is a weak equivalence. If we restrict both sides to a proper subgroup \( K \subset G \), then we have the map
\[
\tilde{\mathcal{P}}_\mathcal{O}(i^*_K G_+ \wedge_H X) \xrightarrow{\omega \circ_H} \text{Map}_H \left( i^*_K G_+, \tilde{\mathcal{P}}_\mathcal{O}(X) \right),
\]
so by induction on the subgroup lattice, it will suffice to show that the map \( \tilde{\mathcal{P}}_\mathcal{O}(\delta) \) is an equivalence on \( G \)-fixed points. We will first analyze the fixed points, then use our usual comparison trick to deduce that the map is a weak equivalence.
The \( G \)-fixed points of the target is elementary, since we are mapping into something coinduced.

**Proposition 5.5.** We have a natural weak equivalence of \( \mathcal{O}^{tr} \)-algebras
\[
\text{Map}_H(G_+, \tilde{\mathbb{P}}_\mathcal{O}(X))^G \simeq \prod_{H/K \in \pi_0 \mathcal{O}(G)} \tilde{\mathbb{P}}_\mathcal{O}(EW_H(K)_+ \wedge X^K)
\]

**Proof.** For any \( \mathcal{O} \)-space, the diagonal map provides a natural weak equivalence of \( \mathcal{O}^{tr} \)-algebras
\[
\tilde{\mathbb{P}}_\mathcal{O}(X)^H \xrightarrow{\simeq} \text{Map}_H(G_+, \tilde{\mathbb{P}}_\mathcal{O}(X))^G.
\]
The result then follows from Theorem 4.19. \( \square \)

Theorem 4.19 also describes the fixed points of the source as an \( \mathcal{O}^{tr} \)-algebra.

**Proposition 5.6.** We have a natural weak equivalence of \( \mathcal{O}^{tr} \)-algebras
\[
\tilde{\mathbb{P}}_\mathcal{O}(G_+ \wedge_H X)^G \simeq \prod_{G/K \in \pi_0 \mathcal{O}(G)} \tilde{\mathbb{P}}_\mathcal{O}(EW_G(K)_+ \wedge W_G(K) (G_+ \wedge_H X)^K).
\]

At this point, it is not even clear that the two products we have to compare run over the same indexing set. We begin simplifying. Additionally, we must pass from the Weyl groups in \( G \) to those in \( H \). With a miraculous result about finite groups, all of this is possible.

We begin by analyzing the \( K \)-fixed points of \( G_+ \wedge_H X \). First note that the double coset decomposition of \( G \) gives an identification
\[
i_K^*(G_+ \wedge_H X) \cong \bigsqcup_{KgH \in K \backslash G/H} KgH_+ \wedge H X,
\]
where we use \( KgH \) both for the double coset as an equivalence class and as a \( K \)-\( H \)-biset. As a \( K \)-set, we have a natural (in \( X \)) \( K \)-equivariant homeomorphism
\[
KgH_+ \wedge_X H \cong K_+ \wedge_{(K \cap gHg^{-1})} c_g^* X,
\]
where \( c_g^* : \text{Top}^H \rightarrow \text{Top}^{gHg^{-1}} \) is the pull-back along the isomorphism \( gHg^{-1} \cong H \).

**Proposition 5.7.** We have a natural homeomorphism
\[
(G_+ \wedge_H X)^K \cong \bigsqcup_{KgH \in K \backslash G/H, \ \ g^{-1}Kg \subset H} X^{g^{-1}Kg}.
\]

We need this with the Weyl action of \( K \) in \( G \) to understand the homotopy orbits. Here it acts both on the indexing set and on the fixed points. In particular, we have an action on the double cosets.

**Proposition 5.8.** The normalizer of \( K \) in \( G \) acts on the set of double cosets by
\[
n \cdot Kg'H = nKg'H = Kn^g'H.
\]
This gives an obvious action of the Weyl group.

**Proof.** The \( H \) term is largely immaterial: since we are multiplying any double coset \( Kg'H \) on the left by \( n \) (either before or after the \( K \) ), replacing \( g' \) with \( g'h \) for any \( h \in H \) yields the same value. Similarly, since \( g \in N_G(K) \), we know that if we replace \( g' \) by \( kg' \) for some \( k \in K \), then
\[
Kn^g'H = Kk'ng'H = Kn^g'H,
\]
where \( k' = nkn^{-1} \in K \). This is therefore well defined. It is also obviously a left group action. Since \( kK = K \) for all \( k \in K \), we see that \( K \) acts trivially, giving the final part. \( \square \)

In general, the stabilizers could be a little ugly. However, we consider only those double cosets \( KgH \) for which \( g^{-1}Kg \subset H \).

**Definition 5.9.** Let \((K : G : H)\) denote the set of double cosets \( KgH \) such that \( g^{-1}Kg \subset H \).

**Proposition 5.10.** The subset \((K : G : H)\) is an \( \mathcal{W}_G(K) \)-equivariant subset of \( K \backslash G \backslash H \).

**Proof.** Let \( w \in \mathcal{W}_G(K) \) be represented by \( n \in \mathcal{N}_G(K) \), and let \( KgH \in (K : G : H) \). Then since \( n \) normalizes \( K \),

\[
(ng)^{-1}Kng = g^{-1}n^{-1}Kng = g^{-1}Kg.
\]

\( \square \)

**Proposition 5.11.** Let \( g \) be such that \( g^{-1}Kg \subset H \). Then the stabilizer in \( \mathcal{W}_G(K) \) of \( KgH \) is \( \mathcal{W}_{gHg^{-1}}(K) \) which under the conjugation isomorphism is \( \mathcal{W}_{Hg^{-1}}(g^{-1}Kg) \).

**Proof.** Let \( n \in \mathcal{N}_G(K) \) stabilize \( KgH \). Equivalently, there are \( k \in K \) and \( h \in H \) such that

\[
g^{-1}Kg = kgh.
\]

This means that \( ghg^{-1} \in \mathcal{N}_G(K) \), and as elements of the Weyl group,

\[
[n] = [ghg^{-1}].
\]

The second part is obvious. \( \square \)

**Corollary 5.12.** We have an \( \mathcal{W}_G(K) \)-equivariant homeomorphism

\[
(G^+_H X)^K \cong \bigvee_{[KgH] \in (K : G : H) / \mathcal{W}_G(K)} \mathcal{W}_{gHg^{-1}}(K) \ast_{g} X^{g^{-1}Kg}.
\]

Putting this together with Proposition 5.7 then gives the following.

**Corollary 5.13.** We have a natural weak equivalence of \( \mathcal{O}^\text{gr} \)-algebras

\[
\hat{P}_O(G^+_H X)^G \simeq \prod_{G/K \in \pi_0 \mathcal{O}(G), [KgH] \in (K : G : H) / \mathcal{W}_G(K)} \prod_{K < H} \hat{P}_O(EW_H(g^{-1}Kg)_{+} \ast_{g} X^{g^{-1}Kg}).
\]

Here \( K < H \) denotes the subconjugacy relation.

This actually brings us much closer to finishing our proof. The indexing set for the first product is the \( G \)-conjugacy classes of subgroups \( K \) such that \( G/K \) is admissible and \( K \) is subconjugate to \( H \). Since \( G/H \) is admissible, by assumption, this is the same as those \( G \)-conjugacy classes of subgroups \( K \) such that \( H/K \) is admissible.

**Proposition 5.14.** If \( G/H \) is admissible and if \( K \subset H \), then \( G/K \) is admissible if and only if \( H/g^{-1}Kg \) is admissible for all \( g^{-1}Kg \subset H \).
Proof. Since $G/H$ is admissible, and since admissible sets are closed under self-induction, if $H/K$ is admissible, then

$$G/K \cong G \times_{H} H/K$$

is admissible. For the other direction, if $G/K$ is admissible, then $G/g^{-1}Kg$ is admissible for all $g \in G$, since these are isomorphic $G$-sets. In particular, for all $g$ such that $g^{-1}Kg \subset H$, $G/g^{-1}Kg$ is an admissible $G$-set. Restricting to $H$, this gives

$$i_{H}^{}G/g^{-1}Kg = T_{g} H/g^{-1}Kg$$

for some finite $H$-set $T_{g}$. Since admissible sets are closed under finite limits, we conclude that $H/g^{-1}Kg$ is admissible. □

Corollary 5.15. The indexing set for the first product in Corollary 5.13 is the set of admissible $H$-orbits modulo conjugation in $G$.

This explains how the two sides of the map in Equation 6 for Theorem 5.2 are packaging the data: we take the conjugacy classes of subgroups of $H$ and group them according to conjugacy in $G$.

Definition 5.16. Let $H, K \subset G$. Let

$$(K)_{G:H} = \{ g^{-1}Kg \mid g \in G, g^{-1}Kg \subset H \}/H\text{-conjugacy}$$

be the set of $H$-conjugacy classes of $G$-conjugates of $K$ which sit in $H$.

In other words, $(K)_{G:H}$ is what we get when we take the conjugacy class of $K$, throw out all the terms which are not subgroups of $H$ and then work up to $H$-conjugacy.

Proposition 5.17. The map $\theta: (K:G:H) \rightarrow (K)_{G:H}$ defined by

$$\theta(KgH) = g^{-1}Kg$$

is Weyl equivariant (where the target has a trivial Weyl action) and induces a bijection

$$(K:G:H)/W_{G}(K) \cong (K)_{G:H}.$$ 

Proof. The map $\theta$ is obviously onto if it is well-defined. However, since we are considering only $H$-conjugacy classes of subgroups of $H$, replacing $g$ by $kg$ results in an $H$-conjugate subgroup, and the map is well-defined.

For the second part, if $n \in N_{G}(K)$, then as before

$$(ng)^{-1}Kng = g^{-1}Kg,$$

and hence $\theta$ factors through the Weyl orbits. On the other hand, if

$$\theta(KgH) = \theta(Kg'H),$$

then there is an $h \in H$ such that

$$g^{-1}Kg = h^{-1}g'^{-1}Kg'h = (g'h)^{-1}K(g'h).$$

Rearranging, this means that $n = g'hg^{-1} \in N_{G}(K)$, and by construction,

$$n \cdot KgH = K(g'hg^{-1})gH = Kg'hH = Kg'H.$$

□

Corollary 5.18. The left- and right-hand sides of the $G$-fixed points of Equation 6 are abstractly equivalent as $O^{tr}$-algebras.
We need slightly more, in that we need that the map $\omega_{G,H}$ from equation (6) is an equivalence of $\mathcal{O}^{tr}$-algebras. Here we again compare to a larger operad in which we know the answer. Recall from Corollary 4.15 that if $\mathcal{O}'$ is any larger $N_\infty$ operad, then the fixed points of the free $\mathcal{O}$-algebra sit as the obvious $\mathcal{O}^{tr}$-summand of the fixed points of the free $\mathcal{O}'$-algebra. The map in Equation (6) fits into a commutative square

\[
\begin{array}{c}
\mathbb{P}_\mathcal{O}(G_+ \wedge X) \\
\downarrow \\
\mathbb{P}_{\mathcal{O}'}(G_+ \wedge X)
\end{array}
\xrightarrow{\omega_{\mathcal{O},H}}
\begin{array}{c}
\text{Map}_H \left( G_+, \mathbb{P}_\mathcal{O}(X) \right) \\
\downarrow \\
\text{Map}_H \left( G_+, \mathbb{P}_{\mathcal{O}'}(X) \right)
\end{array}
\]

Taking fixed points of the bottom row and using that the vertical maps are the inclusion of summands, we see that the map in Equation (6) for $\mathcal{O}'$ can be written as

\[
(\mathbb{P}_\mathcal{O}(G_+ \wedge X))^G \vee_{\mathcal{O}'} E_s \xrightarrow{\omega_{\mathcal{O},H}} \left( \text{Map}_H \left( G_+, \mathbb{P}_\mathcal{O}(X) \right) \right)^G \vee_{\mathcal{O}'} E_t,
\]

where $E_s$ and $E_t$ are the complementary summands. Here $\omega_{\mathcal{O},H}$ is a “square matrix” in the sense that the source and target are abstractly isomorphic $\mathcal{O}^{tr}$-algebras.

Now consider any universe $U$ in which $G/H$ equivariantly embeds and for which $D(U)$ is bigger than $\mathcal{O}$ (at worst, one can take a complete universe). Then Adams’ original argument for the Wirthmuller isomorphism [7, §5] shows that the map

\[
\mathbb{P}_{D(U)}(G_+ \wedge X) \rightarrow \text{Map}_H \left( G_+, \mathbb{P}_{D(U)}(X) \right)
\]

is an equivariant weak equivalence. In particular, it is a weak equivalence on fixed points. The matrix form of this map in Equation (7) then shows that $\omega_{\mathcal{O},H}$ is a weak equivalence, as desired.

6. Comparisons

The purpose of this section is to compare the category of $\mathcal{O}$-spectra for certain $N_\infty G$-operads $\mathcal{O}$ to the usual categories of equivariant orthogonal $G$-spectra. Specifically, we explain the proof of the following theorem.

**Theorem 6.1.** Let $\mathcal{O}$ be an $N_\infty$ operad which is equivalent to the equivariant Steiner operad for some universe $U$. Then there is an equivalence of $\infty$-categories between $N(\mathcal{S}^G_{\mathcal{O}})[W^{-1}]$ and $N(\mathcal{S}^G)[W_U^{-1}]$; i.e., between the $\mathcal{O}$-stable category and the category of orthogonal $G$-spectra with the stable equivalences determined by $U$.

Here by the equivariant Steiner operad for a universe $U$, we mean the following. First, recall the definition of the Steiner operad $\mathcal{K}_V$ for a finite-dimensional real inner product $G$-space $V$. A Steiner path is a map $h : I \rightarrow \text{Map}(V, V)$ such that $I$ lands in the distance-reducing linear embeddings and $h(1)$ is the identity map. The $n$th space of the Steiner operad consists of tuples $(h_1, \ldots, h_n)$ of Steiner paths such that the embeddings $\{h_i(0)\}$ have disjoint image. The symmetric group acts by permuting the tuples, $G$ acts by conjugation on embeddings, and the composition is determined by the pointwise composition. The homotopy type of $\mathcal{K}_V$ can be described in terms of an equivariant configuration space.
For $V \subset W$, there is an evident inclusion $\mathcal{K}_V(n) \to \mathcal{K}_W(n)$ that is equivariant for the $G \times \Sigma_n$ action. Moreover, these inclusions assemble into a map of operads $\mathcal{K}_V \to \mathcal{K}_W$. Therefore, for a universe $U$, we define the equivariant Steiner operad for $U$ as

$$\mathcal{K}_U = \colim_{V \in U} \mathcal{K}_V,$$

the colimit over the collection of finite-dimensional subspaces of $U$ with morphisms inclusions.

The proof of Theorem 6.1 depends on the following result, which is essentially folklore but has not previously appeared in the literature.

**Theorem 6.2.** There is a zig-zag of connective Quillen equivalences between the category $\text{Top}^G[\mathcal{O}]$ with the grouplike model structure and the category $\text{Sp}_G$ of orthogonal $G$-spectra with the stable equivalences determined by $U$. 

Here recall from the discussion preceding [27, 0.10] that a connective Quillen equivalence is a Quillen equivalence which restricts to an equivalence on the full subcategories of the respective homotopy categories spanned by the connective objects.

We first explain the proof of Theorem 6.1 given Theorem 6.2. This is an immediate consequence of the following lemma. Here let $\text{Sp}_G^c$ denote the subcategory of connective objects in $\text{Sp}_G$ and denote by $\text{Sp}(\mathcal{C})$ the $\infty$-category of spectrum objects in an $\infty$-category $\mathcal{C}$ [25, §1.4.2].

**Lemma 6.3.** There is an equivalence of $\infty$-categories

$$\text{Sp}(N(\text{Sp}_G^c)[W_{-1}^{-1}]) \simeq N(\text{Sp}_G)[W_{-1}^{-1}].$$

**Proof.** The inclusion

$$N(\text{Sp}_G^c)[W_{-1}^{-1}] \to N(\text{Sp}_G)[W_{-1}^{-1}]$$

is exact and therefore induces a functor

$$\text{Sp}(N(\text{Sp}_G^c)[W_{-1}^{-1}]) \to \text{Sp}(N(\text{Sp}_G)[W_{-1}^{-1}]).$$

Computation of homotopy groups shows that this is an equivalence, and since the canonical suspension spectrum functor

$$N(\text{Sp}_G)[W_{-1}^{-1}] \to \text{Sp}(N(\text{Sp}_G)[W_{-1}^{-1}])$$

is an equivalence because $N(\text{Sp}_G)[W_{-1}^{-1}]$ is stable, the result follows. \(\square\)

We now discuss the proof of Theorem 6.2. Since $\mathcal{O}$ is equivalent to the equivariant Steiner operad on $U$, we can approach this using the basic strategy of the (equivariant analogue) of [2, 5.43]. Here we replace the specific discussion of the operadic bar construction with the work of [32, §6]. The outline for this kind of result is given in [30, §9], and we now explain the verification of the specific facts we need. Recall that $\mathcal{Q}$ denotes the monad associated to the construction $\colim_V \Omega^V \Sigma^V X$ as $V$ varies over the indexing spaces in the universe.

1. The map of monads from $\mathcal{K}_U \to \mathcal{Q}$ is an equivariant group completion. This follows from the work of Caruso-Waner [11], as reviewed and stated in [19, 1.11].
2. The geometric realization of proper simplicial $G$-spaces and orthogonal $G$-spectra preserves levelwise weak equivalences. This is standard (e.g., see [3, §4] for a discussion).
(3) Finally, we need to know that $\Omega^V|X\cdot| \cong |\Omega^V X\cdot|$ for suitably connected $X$; as reviewed in the proof of [19, 1.14], this is due to Hauschild and a published source is [12].

It is natural to wonder about the analogues of the comparison results in this section for arbitrary indexing systems. As we discussed briefly in the introduction, the subtleties of the situation are illuminated by consideration of the definition of very special $\Gamma$-$G$-spaces. Let $T$ be a based finite $G$-set, and $X$ a $\Gamma$-$G$-space; recall this is a functor from the category of based finite sets to $G$-spaces. Then $X$ is special if the natural map

$$X(T) \to \text{Map}(T, X(1^+))$$

is an equivalence of $G$-spaces, where $X(T)$ denotes the value of $X$ on the underlying set of $T$ and $1^+$ denotes the one-point set with a disjoint basepoint added. We can of course ask for the special condition to hold only for admissible based finite $G$-sets $T$; this corresponds to indexing systems which we denote “disk-like”, where the relevant data is determined by what happens at $G/G$. All little disks operads give rise to disk-like indexing systems, but in fact Rubin [34] shows that there exist disk-like indexing systems which do not come from any little disks operad.

There is a natural forgetful functor from $G$-$\Gamma$-spaces to $H$-$\Gamma$-spaces, and one can use this to define versions of special $\Gamma$-spaces in terms of conditions that hold for $H$-sets for proper subgroups of $G$. Generalizing this kind of condition, it is possible to use excisive functors, spectral presheaves, and equivariant symmetric spectra to model the $O$-stable category. We intend to return to these matters in future work.

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