BOREL-WEIL THEOREM FOR QUASIREDUCTIVE SUPergroupS

TAIKI SHIBATA

Abstract. Quasireductive supergroups $\mathcal{G}$ form a wide class of algebraic supergroups, which includes general linear supergroups, queer supergroups, and all Chevalley supergroups of classical type due to R. Fioresi and F. Gavarini. We study structure of $\mathcal{G}$ and give a systematic construction of all simple supermodules of $\mathcal{G}$ over a field of arbitrary characteristic. Especially when $\mathcal{G}$ has a distinguished parabolic super-subgroup, we classify simple supermodules of $\mathcal{G}$, and prove a super-analogue of the Kempf vanishing theorem and the Weyl character formula for $\mathcal{G}$.

Key Words: Borel-Weil theorem, quasireductive supergroup, Chevalley supergroup, algebraic supergroup, Hopf superalgebra.

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1. Introduction

This paper is to settle a framework of characteristic-free study of representation theory of quasireductive supergroups, and to prove some new results including a super-analogue of the Borel-Weil Theorem. An algebraic supergroup is a representable group-valued functor $\mathcal{G}$ defined on the category of commutative superalgebras, such that the representing (necessarily, Hopf) superalgebra $\mathcal{O}(\mathcal{G})$ is finitely generated. Here, the word “super” is a synonym of “graded by the group $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ of order two”. There exists the largest ordinary (= non-super) subgroup $\mathcal{G}_{ev}$ of $\mathcal{G}$, called the even part of $\mathcal{G}$. Since the notion of algebraic supergroups is a generalization of algebraic groups, it is natural to ask what is a good/nice generalization of split reductive groups to our super setting. In 2011, V. Serganova [27] answered this question, introducing the notion of quasireductive supergroups; it is an algebraic supergroup whose even-part $\mathcal{G}_{ev}$ is a split reductive group. There are plenty of examples of quasireductive supergroups: general linear supergroups $\mathfrak{GL}(m|n)$, queer supergroups $\mathfrak{Q}(n)$ (whose Lie superalgebra is the queer Lie superalgebra $\mathfrak{q}(n)$), and all Chevalley supergroups of classical type (which is a super-analogue of the Chevalley-Demazure groups, including special linear supergroups $\mathfrak{SL}(m|n)$ and ortho-symplectic supergroups $\mathfrak{sp}\mathcal{O}(m|n)$) due to R. Fioresi and F. Gavarini [7, 8] in 2012.

Over a field of characteristic zero, the study of quasireductive supergroups $\mathcal{G}$ is essentially the same as its (necessarily, finite-dimensional) Lie superalgebra $\text{Lie}(\mathcal{G})$ of $\mathcal{G}$. Ever since V. Kac [11] classified finite-dimensional simple Lie superalgebras over $\mathbb{C}$, representations of $\text{Lie}(\mathcal{G})$ has been well-studied, see Kac [12], I. Penkov and Serganova [26], and A. Sergeev [28] for example. See also I. Musson [23], S.-J. Cheng and W. Wang [5]. Recently, Serganova...
gave a systematic construction of simple supermodules for quasireductive supergroups over an algebraically closed field of characteristic zero, in terms of their Lie superalgebras.

On the other hand, the study of quasireductive supergroups over a field of positive characteristic has just started. A. Zubkov and F. Marko [14, 15, 32, 33] have obtained many results for representations of $\mathfrak{gl}(m|n)$. J. Brundan and A. Kleshchev [3] studied representations of $Q(n)$ which have a close relationship to modular representations of spin symmetric groups. By using representations of $\mathfrak{gl}(m|n)$, Mullineux conjecture (now is Mullineux theorem) was re-proven by Brundan and J. Kujawa [3]. B. Shu and Wang [29] classified simple supermodules of $\mathfrak{spO}(m|n)$ whose parameter set is described by combinatorics, which is related to Mullineux bijection.

This paper proposes a framework of characteristic-free study of representation theory of quasireductive supergroups $\mathfrak{g}$. The first purpose of the paper is to establish the Borel-Weil theorem for $\mathfrak{g}$. Namely, we give a systematic construction of all simple $\mathfrak{g}$-supermodules which extends Serganova’s construction to arbitrary characteristic. In the following, we explain the details. First, using a concept of the theory of Harish-Chandra pairs due to A. Masuoka [18], we are able to find a super-torus $T$ and a Borel super-subgroup $B$ of $\mathfrak{g}$. Next, we construct all simple $T$-supermodules $\{u(\lambda)\}_{\lambda \in \Lambda}$ by the theory of Clifford algebras, where $\Lambda$ is the character group of $T$. For the quotient superscheme $G/B$ (due to Masuoka and Zubkov [21]), we may consider cohomology groups $H^\alpha(\lambda) := H^\alpha(G/B, \mathcal{L}(u(\lambda)))$, where $\lambda \in \Lambda$, $n \geq 0$ and $\mathcal{L}(u(\lambda))$ is the associated sheaf to $u(\lambda)$ on $G/B$. Set $\Lambda^0 := \{\lambda \in \Lambda \mid H^0(\lambda) \neq 0\}$. Our main result is the following.

**Theorem (Theorem 4.12).** For $\lambda \in \Lambda^0$, the $\mathfrak{g}$-socle $L(\lambda)$ of $H^0(\lambda)$ is simple. The map $\lambda \mapsto L(\lambda)$ gives a one-to-one correspondence between $\Lambda^0$ and the isomorphism classes of all simple $\mathfrak{g}$-supermodules up to parity change.

In the analytic situation, Borel-Weil-Bott theorem for classical Lie supergroups over $\mathbb{C}$, was established by Penkov [25] in 1990.

The second purpose of the paper is to study quasireductive supergroups $\mathfrak{g}$ which include a distinguished “parabolic” super-subgroup $P$ such that $P_{ev} = \mathfrak{g}_{ev}$ and $\text{Lie}(P)_{1} = \text{Lie}(B)_{1}$. Then we can describe $H^\alpha(\lambda)$ in terms of the non-super cohomology group $H^\alpha_{ev}(\lambda) := H^\alpha(\mathfrak{g}_{ev}/B_{ev}, \mathcal{L}(k^\lambda))$ as $\mathcal{T}_{ev}$-supermodules, where $k^\lambda$ is the one-dimensional $\mathcal{T}_{ev}$-module of weight $\lambda$. This is a generalization of Zubkov’s result [32, Proposition 5.2 and Lemma 5.1] for $\mathfrak{g} = \mathfrak{gl}(m|n)$, see also Marko [14, §1.2]. As a corollary, we have the following results.

**Theorem (Section 5.2).** If $\mathfrak{g}$ has such a parabolic super-subgroup $P$, then

1. $\Lambda^0$ coincides with the set of all dominant weights $\Lambda^+$ for $\mathfrak{g}_{ev}$,
2. $H^\alpha(\lambda) = 0$ for $\lambda \in \Lambda^+$, $n \geq 1$, and
3. the formal character of $H^0(\lambda)$ is given by $\text{ch}(H^0_{ev}(\lambda)) \prod_{\delta \in \Delta^+_1} (1 + e^{-\delta})$ for $\lambda \in \Lambda^+$, where $\Delta^+_1$ is the set of all positive odd roots of $\text{Lie}(\mathfrak{g})$.

The above (2) (resp. (3)) is a super-analogue of the Kempf vanishing theorem (resp. the Weyl character formula).
We study structure of quasireductive supergroups over sub-pairs (Definition 2.4). In Section 3, we give a definition of (PID). As a generalization of [19, Definition 6.3], we re-define the notion of $\lambda \in \Lambda$ as a generalization of Serganova's one. We study structure of quasireductive supergroups $\mathcal{G}$ over a PID; super-tori $\mathcal{T}$, unipotent super-subgroups $\mathcal{U}, \mathcal{U}^+$ and Borel super-subgroups $\mathcal{B}, \mathcal{B}^+$ are given in §3.2 and §3.3. In particular, we give a PBW basis of Kostant's $\mathbb{Z}$-form for $\mathcal{G}$ (Theorem 3.1). As a consequence, we prove an algebraic interpretation of the “density” of the big cell $\mathcal{B} \times \mathcal{B}^+$ in $\mathcal{G}$ (Corollary 3.15).

This is needed to prove the Borel-Weil theorem for our $\mathcal{G}$. In the non-super situation, see B. Parshall and J. Wang [24], and J. Bichon and S. Riche [1]. In Section 4, we work over a field $k$ of characteristic not equal to 2. First, in §4.1, we construct all simples $\{u(\lambda)\}_{\lambda \in \Lambda}$ of a super-torus $\mathcal{T}$, using the theory of Clifford algebras (Appendix B). Here, $\Lambda$ is the character group $X(\mathcal{T}_{ev})$ of $\mathcal{T}_{ev}$. When $k$ is algebraically closed, we show that our simples are the same as Serganova’s (Remark 4.7). Next two subsections are devoted to cohomology groups $H^n(\lambda)$ and determination of all simple $\mathcal{B}$-supermodules. In §4.4, we prove our first main result, that is, the $\mathcal{G}$-socle $L(\lambda)$ of $H^0(\lambda) \neq 0$ is simple, and this gives a one-to-one correspondence between the set $\Lambda^+ := \{\lambda \in \Lambda \mid H^0(\lambda) \neq 0\}$ and the set of all isomorphism classes Simple$_\mathcal{G}(\mathcal{G})$ of simple $\mathcal{G}$-supermodules up to parity change. We also give a necessary and sufficient condition for $L(\lambda)$ to be of type $\mathcal{Q}$, that is, $L(\lambda) \cong \Pi L(\lambda)$ as $\mathcal{G}$-supermodules, using the language of Clifford algebras (Theorem 4.12). Here, $\Pi$ is the parity change functor. Some further properties of the induced supermodule $H^0(\lambda)$ are discussed in §4.5.

In the final Section 5, we still work over a field $k$ of characteristic not equal to 2. Zubkov [32] establish various results on $\mathcal{G}L(m|n)$, using its parabolic super-subgroups. In this section, we assume that our quasireductive supergroup $\mathcal{G}$ having a “parabolic” super-subgroup $\mathcal{P}$ such that $\mathcal{P}_{ev} = \mathcal{G}_{ev}$ and $\text{Lie}(\mathcal{P}) = \text{Lie}(\mathcal{B})$. In §5.1 we generalize some of results, stated in [32] §5. In §5.2 we prove that for all $\lambda \in \Lambda^0$ and $n \geq 0$, $H^n(\lambda)$ is isomorphic to $H^n(\lambda) \otimes \text{Lie}(\mathcal{U})^+_1$ as $\mathcal{T}_{ev}$-supermodules (Theorem 5.3), where $H^n(\lambda)$ is the cohomology group for $\mathcal{G}_{ev}$ and $\text{Lie}(\mathcal{U})^+_1$ is the dual space of the odd part of $\text{Lie}(\mathcal{U})^+$. As a consequence, we get the following three results: (1) $\Lambda^0$ coincides with the set of all dominant weights $\Lambda^+$ for $\mathcal{G}_{ev}$, (2) the Kempf vanishing theorem $H^n(\lambda) = 0$ for $\lambda \in \Lambda^+$, $n \geq 1$ (Corollary 5.6), and (3) the Weyl character formula, that is, an explicit description of the formal character of $H^0(\lambda)$ for $\lambda \in \Lambda^+$ (Corollary 5.8). In §5.3 we see some examples.

2. Preliminaries

In this section, the base ring $k$ supposed to be a principal ideal domain (PID) of characteristic different from 2. The unadorned $\otimes$ denotes the tensor product over $k$. Note that, a $k$-module is projective if and only if it is free. A $k$-module is said to be finite (resp. $k$-free/$k$-flat) if it is finitely generated (resp. free/flat). In particular, we say that a $k$-module is finite free if it is finite and $k$-free. All $k$-modules form a symmetric tensor category.
Mod with the trivial symmetry $V \otimes W \to W \otimes V$; $v \otimes w \mapsto w \otimes v$, where $V, W \in \text{Mod}$.

2.1. Superspaces. The group algebra of the group $Z_2 = \{0, 1\}$ of order two over $k$ is denoted by $kZ_2$ which has a unique Hopf algebra structure over $k$. We let $C := (\text{Mod}^{kZ_2}, \otimes, k)$ denote the tensor category of right $kZ_2$-comodules, that is, $Z_2$-graded modules over $k$. Here, $k$ is regarded as a purely even object; $k = k \oplus 0$. In what follows, for $V = V_0 \oplus V_1 \in C$, an element $v$ in $V$ is always regarded as a homogeneous element of $V$. We denote the parity of $v$ by $|v|$, that is, $|v| = \epsilon$ if $v \in V_\epsilon$ ($\epsilon \in Z_2$). For each $V, W \in C$, the following is the so-called supersymmetry.

$$c_{V,W} : V \otimes W \to W \otimes V; \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$ 

In this way, $(\text{Mod}^{kZ_2}, \otimes, k, c)$ forms a symmetric tensor category which we denote by $\text{SMod}$. An object in $\text{SMod}$ is called a superspace. We let $\text{SMod}(V, W)$ denote the set of all morphisms from $V$ to $W$ in $\text{SMod}$.

For $V \in \text{SMod}$, we define an object $\Pi V$ of $\text{SMod}$ so that $(\Pi V)_\epsilon = V_{\epsilon+1}$ for each $\epsilon \in Z_2$. For a morphism $f : V \to W$ in $\text{SMod}$, we define a morphism $\Pi f : \Pi V \to \Pi W$ in $\text{SMod}$ so that $\Pi f = f$. In this way, we get a functor $\Pi : \text{SMod} \to \text{SMod}$, called the parity change functor on $\text{SMod}$. For $V, W \in \text{SMod}$, we define an object $\text{SMod}(V, W)$ of $\text{SMod}$ as follows.

$$\text{SMod}(V, W)_0 := \text{SMod}(V, W), \quad \text{SMod}(V, W)_1 := \text{SMod}(\Pi V, W).$$

Given an object $V$ of $\text{SMod}$, we define a superspace $V^* := \text{SMod}(V, k)$, called the dual of $V$.

A superalgebra (resp. supercoalgebra/Hopf superalgebra/Lie superalgebra) is defined to be an algebra (resp. coalgebra/Hopf algebra/Lie algebra) object in $\text{SMod}$. A superalgebra $A$ is said to be commutative if $ab = (-1)^{|a||b|}ba$ for all $a, b \in A$. For a supercoalgebra $C$ with the comultiplication $\Delta_C : C \to C \otimes C$, we use the Hyneman-Sweedler notation $\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}$, where $c \in C$.

For a superalgebra $A$, we let $A\text{SMod}$ denote the category of all left $A$-supermodules. Similarly, we let $\text{SMod}^C$ denote the category of all right $C$-supercomodules for a supercoalgebra $C$. We also define a superspace $\text{SMod}^C(V, W) := \text{SMod}^C(V, W) \oplus \text{SMod}^C(\Pi V, W)$ for $V, W \in \text{SMod}^C$. The definition of $\text{SMod}^A(M, N)$ for $M, N \in \text{SMod}^A$ will be clear.

2.2. Algebraic Supergroups. An affine supergroup scheme (supergroup, for short) over $k$ is a representable functor $\mathcal{G}$ from the category of commutative superalgebras to the category of groups. We denote the representing object of $\mathcal{G}$ by $\mathcal{O}(\mathcal{G})$ which forms a commutative Hopf superalgebra. A supergroup $\mathcal{G}$ is said to be algebraic (resp. flat) if $A := \mathcal{O}(\mathcal{G})$ is finitely generated as a superalgebra (resp. $k$-flat). A closed super-subgroup of a supergroup $\mathcal{G}$ is a supergroup which is represented by a quotient Hopf superalgebra of $A$. For a supergroup $\mathcal{G}$, we define its even part $\mathcal{G}_{ev}$ as the restricted functor of $\mathcal{G}$ form the category of commutative algebras to the category of groups. This $\mathcal{G}_{ev}$ is an ordinary affine group scheme represented by the quotient (ordinary) Hopf algebra $\overline{A} := A/I_A$, where $I_A$ is the superideal of $A$ generated by the odd part $A_1$ of $A$. We denote the quotient map

$$\overline{A} := A/I_A.$$
by $A \rightarrow \overline{\tau}$, $a \mapsto \pi$. If $G$ is algebraic, then so is $G_{\text{ev}}$. An algebraic supergroup is said to be connected if its even part is connected, see [18, Definition 8].

**Example 2.1.** For a superspace $V$ over $k$, we define a group functor $GL(V)$ so that

$$GL(V)(R) := \text{Aut}_R(V \otimes R),$$

where $R$ is a commutative superalgebra and $\text{Aut}_R(V \otimes R)$ is the set of all bijective maps in $S\text{Mod}_R(V \otimes R, V \otimes R)$. Suppose that $V$ is finite free with $m = \text{rank}(V_0)$ and $n = \text{rank}(V_1)$. Then $GL(V)$ becomes an algebraic supergroup, called a general linear supergroup, and is denoted by $GL(m|n)$. The even part $GL(m|n)_{\text{ev}}$ is isomorphic to $GL_m \times GL_n$. For the explicit form of the Hopf algebra structure of $O(GL(m|n))$, see [11, 32] for example. $\square$

Let $G$ be a supergroup with $A := O(G)$. A superspace $M$ is said to be a left $G$-supermodule if there is a group functor morphism from $G$ to $GL(M)$. We let $gS\text{Mod}$ denote the tensor category of left $G$-supermodules. One easily sees that $M$ forms a right $A$-supercomodule. Conversely, any right $A$-supercomodule can be regarded as a left $G$-supermodule. In this way, we may identify $gS\text{Mod}$ and $S\text{Mod}^A$. Let $H$ be a closed super-subgroup of $G$ with $B := O(H)$. For a left $G$-supermodule $M$ and a left $H$-supermodule $V$, we may consider its restricted $H$-supermodule and induced $G$-supermodule, respectively:

$$\text{res}^G_H(M) := \text{res}^A_B(M) \quad \text{and} \quad \text{ind}^G_H(V) := \text{ind}^A_B(V),$$

via the quotient Hopf superalgebra map $A \rightarrow B$, see Appendix A.3. Suppose that $G$ and $H$ are flat. We also obtain the following Frobenius reciprocity, see [A.4].

$$S\text{Mod}^B(\text{res}^A_B(M), V) \xrightarrow{\cong} S\text{Mod}^A(M, \text{ind}^A_B(V)).$$

(2.1)

A. Zubkov [32, Proposition 3.1] showed that the category $H\text{SM}\text{od}$ of all left $H$-supermodules has enough injectives. Since the induction functor $\text{ind}^G_H(-)$ is left exact, we get its right derived functor $R^n\text{ind}^G_H(-)$ form $H\text{SM}\text{od}$ to $gS\text{Mod}$ ($n = 0, 1, 2, \ldots$).

A non-zero left $G$-supermodule $L$ is said to be simple (or irreducible) if $L$ is simple as a right $A$-supercomodule, that is, $L$ has no non-trivial $A$-super-subcomodule. If $L$ is simple, then so is $\Pi L$. As in [3], we shall use the following terminology.

**Definition 2.2.** A simple left $G$-supermodule $L$ is said to be of type $Q$ if $L$ is isomorphic to $\Pi L$ as left $G$-supermodules, and type $M$ otherwise.

Let $\text{Simple}(G)$ denote the set of isomorphism classes of simple left $G$-supermodules. The functor $\Pi$ naturally acts on $\text{Simple}(G)$ as a permutation of order 2. Let $\text{Simple}_\Pi(G)$ denote the set of $(\Pi)$-orbits in $\text{Simple}(G)$. In other words, two elements, $L, L'$ in $\text{Simple}(G)$ coincide in $\text{Simple}_\Pi(G)$ if and only if $L \cong L'$ or $\Pi L'$ as left $G$-supermodules.
2.3. Lie superalgebras and Super-hyperalgebras. In the following, we let $\mathfrak{g}$ be an algebraic supergroup with $A := \mathcal{O}(\mathfrak{g})$. Set $A^+ := \text{Ker}(\varepsilon_{\mathfrak{g}})$, where $\varepsilon_{\mathfrak{g}} : A \rightarrow k$ is the counit of $A$. As a $k$-super-submodule of $A^*$, we let

$$\text{Lie}(\mathfrak{g}) := (A^+/(A^+)^2)^*.$$ 

This naturally forms a Lie superalgebra (see [19, Proposition 4.2]), which we call the Lie superalgebra of $\mathfrak{g}$. Note that, $\text{Lie}(\mathfrak{g})$ is finite free and $\text{Lie}(\mathfrak{g})_0$ can be identified with the (ordinary) Lie algebra $\text{Lie}(\mathfrak{g}_{ev})$ of $\mathfrak{g}_{ev}$.

For any $n \geq 1$, we may regard $(A/(A^+)^n)^*$ as a $k$-super-submodule of $A^*$ through the dual of the canonical quotient map $A \rightarrow A/(A^+)^n$. Set

$$\text{hy}(\mathfrak{g}) := \bigcup_{n \geq 1} (A/(A^+)^n)^*.$$ 

This $\text{hy}(\mathfrak{g})$ forms a super-subalgebra of $A^*$. We call it the super-hyperalgebra of $\mathfrak{g}$. It is sometimes called the super-distribution algebra $\text{Dist}(\mathfrak{g})$ of $\mathfrak{g}$. Suppose that $\mathfrak{g}$ is infinitesimally flat, that is, $A/(A^+)^n$ is finitely presented and flat as $k$-module (or equivalently, finite free) for any $n \geq 1$. Then one sees that $\text{hy}(\mathfrak{g})$ has a structure of a cocommutative Hopf superalgebra such that the restriction

$$\langle \cdot, \cdot \rangle : \text{hy}(\mathfrak{g}) \times A \rightarrow k$$

of the canonical pairing $A^* \times A \rightarrow k$ is a Hopf pairing, see [19, Lemma 5.1]. One sees $\text{Lie}(\mathfrak{g})$ coincides with the set of all primitive elements $P(\text{hy}(\mathfrak{g})) := \{u \in \text{hy}(\mathfrak{g}) \mid \sum_u u(1) \otimes u(2) = u \otimes 1 + 1 \otimes u\}$ in $\text{hy}(\mathfrak{g})$.

**Remark 2.3** (See [18, Remark 2]). Suppose that $k$ is a field of characteristic zero. Then it is known that $\text{hy}(\mathfrak{g})$ coincides with the universal enveloping superalgebra $\mathcal{U}(\text{Lie}(\mathfrak{g}))$ of the Lie superalgebra $\text{Lie}(\mathfrak{g})$ of $\mathfrak{g}$. $\square$

For a left $\mathfrak{g}$-supermodule $V$, we regard $V$ as a left $\text{hy}(\mathfrak{g})$-supermodule by letting

$$u.v := \sum_v (-1)^{|v(0)||u|} v(0) \langle u, v(1) \rangle,$$

where $u \in \text{hy}(\mathfrak{g})$, $v \in V$ and $v \mapsto \sum_v u(0) \otimes v(1)$ is the right $A$-supercomodule structure of $V$. Suppose that $V$ is finite free. Then the dual superspace $V^*$ of $V$ forms a right $A$-supercomodule by using the antipode $S : A \rightarrow A$ of $A$. The induced left $\text{hy}(\mathfrak{g})$-supermodule structure, which we shall denote by $\rightarrow$, satisfies the following equation.

$$\langle u \rightarrow v^*, v \rangle = \langle v^*, S^*(u).v \rangle,$$

where $v \in V$, $v^* \in V^*$, $u \in \text{hy}(\mathfrak{g})$ and $S^*$ is the antipode of $\text{hy}(\mathfrak{g})$.

2.4. Sub-pairs. Let $\mathfrak{g}$ be an algebraic supergroup. It is known that the Lie superalgebra $\text{Lie}(\mathfrak{g})$ of $\mathfrak{g}$ is 2-divisible, that is, for any $v \in \text{Lie}(\mathfrak{g})_1$, there exists $x \in \text{Lie}(\mathfrak{g})_1$ such that $[v, v] = 2x$, see [19, Proposition 4.2]. We sometimes denote $x$ by $\frac{1}{2}[v, v]$. Set $A := \mathcal{O}(\mathfrak{g})$ and

$$W^A := (A^+/(A^+)^2)_1.$$ 

It is easy to see that the right coaction on $A$

$$\text{coad} : A \rightarrow A \otimes A : a \mapsto \sum_a (-1)^{|a(1)||a(2)|} a(2) \otimes S(a(1))a(3)$$

should be a group action of $\text{g}$ on $A$. We let $G := \text{Aut}(\mathfrak{g}) \otimes A$ act on $A$ by

$$a \mapsto \sum_a (-1)^{|a(1)||a(2)|} a(2) \otimes S(a(1))a(3),$$

where $a \in A$. Then $G$ is a group of automorphisms of $A$. Set $V := \text{Lie}(\mathfrak{g})_1$. We have a left $G$-supermodule $V$.

The set $\text{Lie}(\mathfrak{g})_1$ of all odd elements of $\text{Lie}(\mathfrak{g})$ forms a Lie superalgebra. We call it the super-hyperalgebra of $\mathfrak{g}$.
induces on $W^A$ a right $A$-comodule (or equivalently, left $G_{ev}$-module) structure, where $S$ is the antipode of $A$. Thus, if $W^A$ is finite free, then $\text{Lie}(G)_1$ can be regarded as a left $G_{ev}$-module.

Suppose that $G$ satisfies the following three conditions.

(E0) $G_{ev}$ is flat,
(E1) $W^A$ is finite free, and
(E2) $\overline{A^+/(A^+)^2}$ is finite free.

The labels (E0)–(E2) above are taken from [20, §5.2]. In other words, we suppose that $G_{ev}$ is an object of the category $(\text{gss-fsgroups})'_k$, see [20, Remark 5.8] for the notation. In this case, $G_{ev}$ is $\otimes$-split, in the sense that there exists a counit-preserving isomorphism

\[
A \rightarrow \overline{A} \otimes (W^A)
\]

of left $A$-comodule superalgebras, where $\wedge(W^A)$ is the exterior superalgebra on $W^A$. Here, we naturally regard $A$ as a left $A$-comodule superalgebra via $A \rightarrow \overline{A} \otimes A; a \mapsto \sum a(1) \otimes a(2)$.

Note that, a $k$-(super-)submodule of a finite free $k$-(super)module is again finite free, since $k$ is a PID. The following is a generalization of [20, Definition 6.3].

**Definition 2.4.** Let $G$ be an algebraic supergroup. For a closed subgroup $K$ of $G_{ev}$, and a Lie super-subalgebra $k$ of $\text{Lie}(G)$ with $\text{Lie}(K) = k_{\bar{0}}$, we say that the pair $(K,k)$ is a sub-pair of the pair $(G_{ev},\text{Lie}(G))$, if

(S0) $K$ is flat,
(S1) $k_{\bar{1}}$ is $K$-stable in $\text{Lie}(G)_1$, and
(S2) $k$ is 2-divisible.

**Remark 2.5.** In general, a Lie super-subalgebra $k$ of a 2-divisible Lie superalgebra $g$ is not always 2-divisible. For example, as super-subspaces of $\text{Mat}_{1|1}(Z)$ (see Example A.1 for the notation), we take

\[
k = \left\{ \begin{pmatrix} 2a & 3b \\ 3b & 2a \end{pmatrix} \mid a, b, c \in Z \right\} \subseteq g = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a, b, c \in Z \right\}.
\]

The following calculation shows that this $k$ is not 2-divisible.

\[
\left[ \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \right] = 2 \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}.
\]

Working over a PID, it is easy to see that a Lie super-subalgebra of an admissible Lie superalgebra (for the definition, see [19, Definition 3.1]) is admissible if and only if it is 2-divisible.

**Proposition 2.6.** Let $G$ be an algebraic supergroup satisfying (E0)–(E2). For a sub-pair $(K,k)$ of the pair $(G_{ev},\text{Lie}(G))$, there exists a closed subgroup $K$ of $G_{ev}$ which satisfies $K_{ev} = K$ and $\text{Lie}(K) = k$. Moreover, $K$ is $\otimes$-split.
\(O(\mathcal{G}_{\text{ev}})\) and \(\mathcal{G}_{\text{ev}}\) satisfies the condition (E2), one sees that \(O(K)\) satisfies the condition (F2). Since \(\mathcal{G} \in \text{(gss-fsgroups)}'_k\), the pair \((\mathcal{G}_{\text{ev}}, \text{Lie}(\mathcal{G}))\) is an object of the category \((\text{sHCP})'_k\). Then by the assumption (S1), we see that \((K, \mathfrak{k})\) satisfies the conditions (F3) and (F4). By the assumption (S2), we can define a 2-operation \((-\langle 2 \rangle : \mathfrak{k}_1 \to \mathfrak{k}_0\) by letting \(v\langle 2 \rangle := \frac{1}{2}[v, v]\). This is obviously \(K\)-stable, which shows that the condition (F5) is satisfied. □

3. Structures of Quasireductive Supergroups over a PID

It is known that the category of all representations of a split reductive groups over a field of characteristic zero is semisimple, that is, any object is decomposed into a direct sum of simple modules (i.e., irreducible representations). However, algebraic supergroups \(\mathcal{G}\) whose representation category is semisimple are rather restricted. Over an algebraically closed field of characteristic zero, R. Weissauer showed that such \(\mathcal{G}\) (which are not ordinary algebraic groups) are essentially exhausted by ortho-symplectic supergroups \(\mathfrak{sp}\mathcal{O}(2n|1)\), see [31, Theorem 6]. On the other hand, over a field of positive characteristic, A. Masuoka showed that such \(\mathcal{G}\) are necessarily purely even, that is, \(\mathcal{G} = \mathcal{G}_{\text{ev}}\), see [18, Theorem 45].

Hence, it is natural to ask what is a nice/good generalization of the notion of split reductive groups. In 2011, V. Serganova [27] answered this problem. She introduced the notion of “quasireductive supergroups” over a field.

In this section, we still suppose that \(k\) is a PID of characteristic not equal to 2, and we generalize her definition to our \(k\) and study its structure.

3.1. Quasireductive Supergroups. A split and connected reductive \(Z\)-group \(G_Z\) is a certain connected algebraic group over \(Z\) having a split maximal torus \(T_Z\) such that the pair \((G_Z, T_Z)\) corresponds to some root datum, see [9, Part II, Chap. 1] (see also [19, §5.2]). It is known that \(O(G_Z)\) is free as a \(Z\)-module and \(G_Z\) is infinitesimally flat.

Definition 3.1. An algebraic supergroup \(\mathcal{G}_Z\) over \(Z\) is said to be quasireductive if its even part \((\mathcal{G}_Z)_{\text{ev}}\) is a split and connected reductive group over \(Z\) and \(W^{O(\mathcal{G}_Z)}\) is finitely generated and free as a \(Z\)-module.

There are plenty of examples of quasireductive supergroups.

Examples 3.2. The followings are quasireductive supergroups over \(Z\).

(1) General linear supergroups \(GL(m|n)\).

(2) Queer supergroups \(Q(n)\):

\[Q(n)(R) := \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \in \mathfrak{gl}(n|n)(R) \right\},\]

where \(R\) is a commutative superalgebra over \(Z\). The even part is \(Q(n)_{\text{ev}} = GL_n\). For the Hopf superalgebra structures of \(Q(n)\), see J. Brundan and A. Kleshchev [3]. The Lie superalgebra \(q(n)\) of \(Q(n)\) is the so-called queer Lie superalgebra.

(3) Chevalley supergroups of classical type defined by R. Fioresi and F. Gavarini [7, 8], see also [19, §6]. For example, special linear supergroups \(\mathfrak{sl}(m|n)\) (consists of elements in \(\mathfrak{gl}(m|n)\) whose Berezinian determinants are trivial), ortho-symplectic supergroups \(\mathfrak{sp}\mathcal{O}(m|n)\) (see B. Shu and W. Wang [29] for details), etc.
In what follows, we fix a quasireductive algebraic supergroup $G_Z$ over $\mathbb{Z}$ and a split maximal torus $T_Z$ of $(G_Z)_{ev}$. Set $G_Z := (G_Z)_{ev}$ and $\Lambda := X(T_Z)$. Note that, the character group $X(T_Z)$ of $T_Z$ can be identified with the set of all group-like elements of the Hopf algebra $\mathcal{O}(T_Z)$. Let $\mathfrak{g}$ (resp. $G$, $T$) denote the base change of $G_Z$ (resp. $G_Z$, $T_Z$) to our base ring $k$, that is, $\mathcal{O}(G) := \mathcal{O}(G_Z) \otimes_k k$. Since $k$ is an integral domain, we can identify $\Lambda$ with $X(T) \cong \mathbb{Z}^\ell$, where $\ell$ is the rank of $G$.

By definition, we see that the algebraic supergroup $\mathfrak{g}$ satisfies the conditions (E0)–(E2). Therefore, $\mathfrak{g}$ is $\otimes$-split:

\begin{equation}
\psi : \mathcal{O}(\mathfrak{g}) \overset{\cong}{\rightarrow} \mathcal{O}(G) \otimes \Lambda(\mathcal{O}(\mathfrak{g})).
\end{equation}

In particular, we have the following.

**Proposition 3.3.** $\mathcal{O}(\mathfrak{g})$ is $k$-free and $\mathfrak{g}$ is infinitesimally flat.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the Lie superalgebra $\text{Lie}(\mathfrak{g})$ of $\mathfrak{g}$. The left adjoint action of $T$ on $\mathfrak{g}$ induces an action of $T$ on $\mathfrak{g}$ which preserves the parity of $\mathfrak{g}$. Since $T$ is a diagonalizable group scheme, the left $T$-supermodule $\mathfrak{g}$ decomposes into weight superspaces as follows.

\[ \mathfrak{g} = \bigoplus_{\alpha \in \Lambda} \mathfrak{g}^\alpha = \left( \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_0^\alpha \right) \oplus \left( \bigoplus_{\delta \in \Delta} \mathfrak{g}_1^\delta \right), \]

where $\mathfrak{g}^\alpha$ is the $\alpha$-weight super-subspace of $\mathfrak{g}$. It is known that

\[ \mathfrak{g}^0 = \{ X \in \mathfrak{g} | u \rightarrow X = \langle u, \alpha \rangle X \text{ for all } u \in \mathfrak{h}(T) \}, \]

see [9] Part I, 7.14. Here, we regard $\Lambda$ as a subset of $\mathcal{O}(T)$. Let $\mathfrak{h} := \mathfrak{g}^0 = 0$ be the $\alpha = 0$ weight super-subspace of $\mathfrak{g}$ which is a Lie super-subealgebra of $\mathfrak{g}$. Note that, the even part $\mathfrak{g}_0$ of $\mathfrak{g}$ coincides with the Lie algebra $\text{Lie}(G)$ of $G = G_{ev}$. By definition, we see that $\mathfrak{h}_0 = \text{Lie}(T)$. For $\epsilon \in \mathbb{Z}_2$, we set

\[ \Delta_\epsilon := \{ \alpha \in \Lambda | \mathfrak{g}^\alpha \neq 0 \} \setminus \{ 0 \} \]

and

\[ \Delta := \begin{cases} \Delta_0 \cup \Delta_1 & \text{if } \mathfrak{g}_1^{\delta = 0} = 0, \\ \Delta_0 \cup \Delta_1 \cup \{ 0 \} & \text{otherwise}. \end{cases} \]

We shall call $\Delta$ the root system of $\mathfrak{g}$ with respect to $T$. Note that, $\Delta_0$ is the root system of $G$ with respect to $T$.

By taking the linear dual of (3.1), we see that $\mathfrak{h}(\mathfrak{g})$ is $\otimes$-split, that is, there exists a (unique) unit-preserving isomorphism

\begin{equation}
\phi : \mathfrak{h}(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}_1) \overset{\cong}{\rightarrow} \mathfrak{h}(\mathfrak{g})
\end{equation}

of left $\mathfrak{h}(\mathfrak{g})$-module supercoalgebras satisfying

\begin{equation}
\langle \phi(z), a \rangle = \langle z, \psi(a) \rangle, \quad a \in \mathcal{O}(\mathfrak{g}), \quad z \in \mathfrak{h}(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}_1).
\end{equation}

For details, see [19] Lemma 5.2. The dual $\mathfrak{g}(\mathfrak{g})^*$ of the cocommutative Hopf superalgebra $\mathfrak{h}(\mathfrak{g})$ naturally has a structure of a commutative superalgebra. Since our quasireductive supergroup $\mathfrak{g}$ is connected, the canonical Hopf paring (2.2) induces an injection $\mathcal{O}(\mathfrak{g}) \hookrightarrow \mathfrak{g}(\mathfrak{g})^*$ of superalgebras, see [19] Lemma 5.3.
A left hy($\mathfrak{G}$)-supermodule $M$ is called a left hy($\mathfrak{G}$)-$T$-supermodule if the restricted left hy($T$)-module structure on $M$ arises from some left $T$-module structure on $M$. In this case, $M$ is said to be locally finite if $M$ is so as a left hy($\mathfrak{G}$)-supermodule. Then by [19] Theorem 5.8, we have the following.

**Theorem 3.4.** For a left $\mathfrak{G}$-supermodule, the induced left hy($\mathfrak{G}$)-supermodule is a locally finite left hy($\mathfrak{G}$)-$T$-supermodule. This gives an equivalence of categories between $k$-free, left $\mathfrak{G}$-supermodules and $k$-free, locally finite left hy($\mathfrak{G}$)-$T$-supermodules.

### 3.2. Super-tori and Unipotent Super-subgroups.

Recall that, $G = \mathfrak{G}_{ev}$, Lie($T$) = $\mathfrak{h}_0$ and $\mathcal{O}(T)$ is a free $\mathbb{Z}$-module.

**Lemma 3.5.** The pair $(T, h)$ forms a sub-pair of $(G, g)$.

**Proof.** The condition (S0) is obvious. By definition, we see that $\mathfrak{h}$ is hy($T$)-stable, and hence (S1) is clear. To see (S2), we take $K \in \mathfrak{h}_1$ and $X \in \mathfrak{g}$ so that $[K, K] = 2X$. For all $u \in \text{hy}(T)$ with $u \notin \mathfrak{k}$, we see that $2(u \rightarrow X) = u \rightarrow [K, K] = 0$. Since char($\mathfrak{k}$) $\neq 2$, we conclude that $X \in \mathfrak{h}_0$. □

Then by Proposition [2.9] we get a closed super-subgroup $\mathcal{T}$ of $\mathfrak{G}$ satisfying

\[
\mathcal{T}_{ev} = T, \quad \text{Lie}(\mathcal{T}) = \mathfrak{h}.
\]

We shall call $\mathcal{T}$ a super-torus of $\mathfrak{G}$.

Let $Z\Delta$ be the abelian subgroup of $X(T)$ generated by $\Delta$. In what follows, we fix a homomorphism $\gamma : Z\Delta \rightarrow \mathbb{R}$ of abelian groups such that $\gamma(\alpha) \neq 0$ for any $0 \neq \alpha \in \Delta$. As in [27] §9.3, we set

$$\Delta^\pm := \{ \alpha \in \Delta \setminus \{0\} \mid \pm \gamma(\alpha) > 0 \}, \quad \Delta_\epsilon^\pm := \Delta_\epsilon \cap \Delta^\pm$$

for $\epsilon \in \mathbb{Z}_2$. Set

$$n^\pm := \bigoplus_{\alpha \in \Delta^\pm} a^\alpha, \quad b^\pm := n^\pm \oplus \mathfrak{h}.$$

One easily sees that, these are Lie super-subalgebras of $\mathfrak{g}$.

As in [9] Part II, 1.8], there is a connected and unipotent subgroup $U$ of $G$ such that

$$U \cong \prod_{\alpha \in \Delta^-_0} G_{\text{add}}, \quad \text{Lie}(U) = n^-_0,$$

where $G_{\text{add}}$ is the one-dimensional additive group scheme over $k$. In the product above, we have fixed an (arbitrary) ordering of $\Delta^-_0$.

**Lemma 3.6.** The pair $(U, n^-)$ forms a sub-pair of the pair $(G, g)$.

**Proof.** The condition (S0) is clear. Take $X \in \mathfrak{g}^0$ ($\delta \in \Delta^-_0$) and $Y \in \mathfrak{g}$ so that $[X, X] = 2Y$. For all $u \in \text{hy}(T)$, we see that $2(u \rightarrow Y) = 2(u, \delta)[X, X]$. Since $2\delta \in \Delta^-_0$ and char($\mathfrak{k}$) $\neq 2$, we conclude that $Y \in n^-_0$, and hence $n^-$ is 2-divisible (S2). It remains to prove (S1), that is, $n^-_1$ is $U$-stable in $\mathfrak{g}_1$. Since $n^-_1$ is obviously $G$-stable in $\mathfrak{g}_1$, we see that it is a right hy($G$)-module via the induced action.

First, we show that $n^-_1$ is hy($U$)-stable in $\mathfrak{g}_1$. It is known that we can choose an element $X_\alpha \in \mathfrak{g}_0^\alpha$ for each $\alpha \in \Delta^-_0$ so that the set

$$\{ \prod_{\alpha \in \Delta^-_0} X_\alpha^{(n_\alpha)} \mid n_\alpha = 0, 1, 2, 3, \ldots \}$$

forms a sub-pair of the pair $(G, g)$. □
forms a k-basis of hy(U). Here, $X^{(n_\alpha)}_\alpha$ denotes the divided power, see [20, Part II, 1.12] (see also 3.4 below). We take $X_\beta \in \mathfrak{n}_1^\pm$ and $X^{(n_\alpha)}_\alpha \in \mathfrak{hy}(U)$. It is easy to see that the $T$-weight of $X^{(n_\alpha)}_\alpha \to X_\beta (\in g)$ is given by $\beta + n_\alpha\alpha$. Since $\beta \in \Delta^-_1$ and $\alpha \in \Delta^-_0$, we see $\gamma(\beta + n_\alpha\alpha) = \gamma(\beta) + n_\alpha\gamma(\alpha) \leq 0$, and hence $\beta + n_\alpha\alpha$ is in $\Delta^-_1$. Thus, $X^{(n_\alpha)}_\alpha \to X_\beta$ is an element of $\mathfrak{n}_1^-$. This implies that $\mathfrak{n}_1^-$ is $\mathfrak{hy}(U)$-stable in $\mathfrak{g}_1$.

Next, we show that $\mathfrak{n}_1^-$ is indeed $U$-stable. By construction, the corresponding Hopf algebra $\mathcal{O}(U)$ of $U$ is isomorphic to the polynomial algebra $k[T_\alpha; \alpha \in \Delta^-_0]$ in $\#(\Delta^-_0)$-variables $T_\alpha$ over $k$ with each $T_\alpha$ is primitive. Since $\mathfrak{n}_1^-$ is finite free, the right $\mathfrak{hy}(U)$-module structure on $\mathfrak{n}_1^-$ induces the following (well-defined) left $\mathcal{O}(U)$-comodule structure on $\mathfrak{n}_1^-$. 

$$\mathfrak{n}_1^- \to \mathcal{O}(U) \otimes \mathfrak{n}_1^-; \quad Y \mapsto \sum_{\alpha \in \Delta^-_0} \sum_{n_\alpha \geq 0} T^{n_\alpha}_\alpha \otimes (X^{(n_\alpha)}_\alpha \to Y),$$

see [9, Part II, 1.20] for detail. Thus, we conclude that $\mathfrak{n}_1^-$ is $U$-stable. \hfill \Box

By Proposition 2.6, we obtain a closed super-subgroup $\mathbb{U}$ of $\mathcal{G}$ which satisfies

$$U_{ev} = U, \quad \text{Lie}(\mathbb{U}) = \mathfrak{n}^-.$$ 

The supergroup $\mathbb{U}$ is connected, and hence the canonical Hopf paring induces an injection $\mathcal{O}(\mathbb{U}) \hookrightarrow \mathfrak{hy}(\mathbb{U})^*$ of superalgebras, as before. By [18, Theorem 41], we get the following.

**Proposition 3.7.** Suppose that $k$ is a field. Then the supergroup $\mathbb{U}$ is unipotent, that is, the corresponding Hopf superalgebra $\mathcal{O}(\mathbb{U})$ is irreducible.

Similarly, we get a closed super-subgroup $\mathbb{U}^+$ of $\mathcal{G}$ such that $\mathbb{U}_{ev}^+ = U^+$, $\text{Lie}(\mathbb{U}^+) = \mathfrak{n}^+$, where $U^+$ is an unipotent subgroup of $G$ corresponds to $\Delta^+_0$.

### 3.3. Borel Super-subgroups.

Our next aim is to construct “Borel super-subgroups” of $\mathcal{G}$.

**Lemma 3.8.** $\mathcal{T}$ normalizes $\mathbb{U}$.

**Proof.** First, we work over $Q := \text{Frac}(k)$ and show that $\mathcal{T}_Q := \mathcal{T} \times_k Q$ is included in the normalizer $\mathcal{N}_{\mathcal{G}_Q}(\mathbb{U}_Q)$ of $\mathbb{U}_Q$ in $\mathcal{G}_Q$. In the following, for simplicity, we shall drop the subscript $Q$. By [20, Thorem 6.6(1)], to show $\mathcal{T} \subseteq \mathcal{N}_{\mathcal{G}}(\mathbb{U})$, it is enough to check the following conditions (i) and (ii):

(i) $T$ is contained in $\mathcal{N}_{\mathcal{G}}(U) \cap \text{Stab}_{\mathcal{G}}(\mathfrak{n}_1^-)$;

(ii) $\mathfrak{h}_1$ is contained in $\text{inv}_{U}(\mathfrak{g}_1/\mathfrak{n}_1^-) \cap (\mathfrak{n}_0^- : \mathfrak{n}_1^-)$.

Here, $\text{Stab}_{\mathcal{G}}(\mathfrak{n}_1^-) = \text{the stabilizer of } \mathfrak{n}_1^- \text{ in } G$, $\text{inv}_{U}(\mathfrak{g}_1/\mathfrak{n}_1^-)$ is the largest $U$-submodule of $\mathfrak{g}_1$ including $\mathfrak{n}_1^-$ whose quotient $U$-module by $\mathfrak{n}_1^-$ is trivial, and $(\mathfrak{n}_0^- : \mathfrak{n}_1^-) := \{ X \in \mathfrak{g}_1 \mid [X, \mathfrak{n}_1^-] \subseteq \mathfrak{n}_0^- \}$.

By the definition of root spaces, $\mathfrak{n}_1^-$ is trivially $T$-stable. Since $T \subseteq \mathcal{N}_{\mathcal{G}}(U)$ is obvious, the condition (i) is clear. It is easy to see that $[\mathfrak{h}_1, \mathfrak{n}_1^-] \subseteq \mathfrak{n}_0^-$. Thus, we have $\mathfrak{h}_1 \subseteq (\mathfrak{n}_0^- : \mathfrak{n}_1^-)$. By using the Hopf pairing $\mathfrak{hy}(U) \otimes \mathcal{O}(U) \to Q$, one sees that

$$\text{inv}_{U}(\mathfrak{g}_1/\mathfrak{n}_1^-) = \{ X \in \mathfrak{g}_1 \mid (u \to X) - \varepsilon^+_U(u)X \in \mathfrak{n}_1^- \text{ for all } u \in \mathfrak{hy}(U) \},$$

where $\varepsilon^+_U(u)$ is the $+$-stabilizer of $\mathfrak{n}_1^-$ in $G$. By the definition of root spaces, $\mathfrak{n}_1^-$ is trivially $T$-stable. Since $T \subseteq \mathcal{N}_{\mathcal{G}}(U)$ is obvious, the condition (i) is clear. It is easy to see that $[\mathfrak{h}_1, \mathfrak{n}_1^-] \subseteq \mathfrak{n}_0^-$. Thus, we have $\mathfrak{h}_1 \subseteq (\mathfrak{n}_0^- : \mathfrak{n}_1^-)$. By using the Hopf pairing $\mathfrak{hy}(U) \otimes \mathcal{O}(U) \to Q$, one sees that

$$\text{inv}_{U}(\mathfrak{g}_1/\mathfrak{n}_1^-) = \{ X \in \mathfrak{g}_1 \mid (u \to X) - \varepsilon^+_U(u)X \in \mathfrak{n}_1^- \text{ for all } u \in \mathfrak{hy}(U) \},$$
and the group multiplication is given by \( \varepsilon_U^\gamma \) is the counit of \( \text{hy}(U) \). For each \( X \in \mathfrak{h}_1 \) and \( u = X_{\alpha}^{(n)} \in \text{hy}(U) \) \((\alpha \in \Delta^-, n \in \mathbb{Z}_{\geq 1})\), the weight of \( u \to X \) is \( na \in \Delta^- \). Since \( \varepsilon_U^\gamma(u) = 0 \), we have \((u \to X) - \varepsilon_U^\gamma(u)X \in \mathfrak{n}_-^-\). If \( u \in Q \), then \((u \to X) - \varepsilon_U^\gamma(u)X = 0 \). This shows \( \mathfrak{h}_1 \subseteq \text{ln}_{U}(\mathfrak{g}_1/\mathfrak{n}_1^-) \), and hence the condition (ii) follows. Thus, we get \( T \subseteq N_G(U) \) over \( k \).

Next, we prove that \( T \subseteq N_G(U) \) over \( k \). Since \( O(T) \) is \( k \)-free, the canonical map \( O(T) \to O(T_Q) \) is injective. Then one easily sees that the canonical quotient map \( O(N_{G_Q}(U_Q)) \to O(T_Q) \) induces the following diagonal arrow:

\[
\begin{array}{ccc}
O(\mathcal{G}) & \to & O(N_G(U)) \\
\downarrow & & \downarrow \kappa \\
O(T) & \to & O(T_Q)
\end{array}
\]

Here, the horizontal and vertical arrows are the canonical quotient maps. Thus, we are done. \( \square \)

Let \( T \times U \) be the crossed product supergroup scheme of \( T \) and \( U \). Namely, for any superalgebra \( R \), the group \((T \times U)(R)\) is just \( T(R) \times U(R) \) as a set, and the group multiplication is given by

\[
(t, u)(s, v) = (ts, (-1)^{|s||t|}ts^{-1}usv), \quad s, t \in T(R), \quad u, v \in U(R).
\]

We define a closed super-subgroup \( B \) of \( G \) as the image of the multiplication map \( T \times B \to G \). Since \( T \cap U \) and \( \mathfrak{h} \cap \mathfrak{n}^- \) is trivial, the intersection \( T \cap U \) is also trivial. Thus, the multiplication map of \( G \) induces an isomorphism \( T \times U \cong B \) of algebraic supergroups. By dualizing, we have an isomorphism of Hopf superalgebras

\[
O(B) \cong O(T \times U) = O(T) \bowtie O(U),
\]

where \( O(T) \bowtie O(U) \) denotes the Hopf-crossed product of \( O(U) \) and \( O(T) \) in \( \text{SMod} \). This implies that

\[
(3.7) \quad B_{ev} = T \times U, \quad \text{Lie}(B) = b^-.
\]

Thus, \( B_{ev} \) is a Borel subgroup of \( G \). In this sense, we shall call \( B \) a Borel super-subgroup of \( G \). The supergroup \( B \) is connected, and hence the canonical Hopf paring induces an injection \( O(B) \hookrightarrow \text{hy}(B)^* \) of superalgebras, as before. Similarly, we define \( B^+ := \text{Im}(T \times U^+ \to G) \cong T \times U^+ \).

**Remark 3.9.** Note that, the groups \( U, U^+, B \) and \( B^+ \) are depending on the choice of the homomorphism \( \gamma : Z\Delta \to \mathbb{R} \), defined in \( \{8,2\} \). In addition, all (possible) Borel super-subgroups are not conjugate. \( \square \)

Since \( T \subseteq B \) and \( \mathfrak{h} \subseteq b^- \), we see that \( T \) is a closed super-subgroup of \( B \). Then the inclusion map \( T \hookrightarrow B \) induces a Hopf superalgebra surjection \( O(B) \twoheadrightarrow O(T) \).

**Proposition 3.10.** The morphism \( O(B) \twoheadrightarrow O(T) \) is split epic.

**Proof.** We consider the following Hopf superalgebra map

\[
(3.8) \quad s_B : O(T) \xrightarrow{\text{id}} O(T) \bowtie O(U) \xrightarrow{\delta} O(B),
\]
where the first arrow is given by \( x \mapsto 1 \otimes x \) and the second arrow is the inverse map of \( \delta \). Then one easily sees that this \( \delta \) gives a section of \( \mathcal{O}(B) \to \mathcal{O}(T) \).

\[ \square \]

### 3.4. Kostant’s Z-Form and PBW Theorem

As a superspace, we have

\[ \mathfrak{g} = (\mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_0^\alpha) \oplus (\mathfrak{h}_1 \oplus \bigoplus_{\delta \in \Delta_1} \mathfrak{g}_1^\delta). \]

Note that, for \( \alpha \in \Delta_0 \), the rank of \( \mathfrak{g}_0^\alpha \) is always 1. However, for \( \delta \in \Delta_1 \), the rank of \( \mathfrak{g}_1^\delta \) may be greater than 1, see [22, Lemma 9.6]. For \( \epsilon \in \mathbb{Z}_2 \), we set

\[ \ell_\epsilon := \text{rank}(\mathfrak{h}_\epsilon). \]

Set \( \mathfrak{g}_Z := \text{Lie}(\mathfrak{G}_Z) \) and \( \mathfrak{h}_Z := \mathfrak{h} \cap \mathfrak{g}_Z \). For \( X \in (\mathfrak{g}_Z)_0 \setminus (\mathfrak{h}_Z)_0 \) and \( H \in (\mathfrak{h}_Z)_0 \), as elements in \( \text{hy}(\mathfrak{G}_Z) \otimes_{\mathbb{Q}} \mathbb{Q} \), we set

\[ X^{(n)} := X^n \otimes_{\mathbb{Z}} 1/m, \quad H^{(m)} := (\prod_{j=0}^{m-1} (H - j)) \otimes_{\mathbb{Z}} 1/m, \]

where \( n, m \in \mathbb{Z}_{\geq 0} \). As in [9] Part II, 1.11], we choose a \( \mathbb{Z} \)-free basis of \( (\mathfrak{g}_Z)_0 \)

\[ \mathcal{B}_0 := \{ X_\alpha \in (\mathfrak{g}_Z)_0 \mid \alpha \in \Delta_0 \} \cup \{ H_i \in (\mathfrak{h}_Z)_0 \mid 1 \leq i \leq \ell_0 \}, \]

so that the set of all products of factors of type \( X_\alpha^{(n_\alpha)} \) and \( H_i^{(m_i)} \) \((n_\alpha, m_i \in \mathbb{Z}_{\geq 0}, \alpha \in \Delta_0, 1 \leq i \leq \ell_0, \) taken in \( \text{hy}(\mathfrak{G}_Z) \) with respect to any order on \( \mathcal{B}_0 \), forms a \( \mathbb{Z} \)-basis of \( \text{hy}(\mathfrak{G}_Z) \).

Since \( (\mathfrak{g}_Z)_1 \) is also \( \mathbb{Z} \)-free, we take a \( \mathbb{Z} \)-free basis of \( \mathfrak{g}_Z \) as follows.

\[ \mathcal{B}_1 := \{ Y_{(\delta, p)} \in (\mathfrak{g}_Z)_1 \mid \delta \in \Delta_1, 1 \leq p \leq \text{rank}(\mathfrak{g}_1^\delta) \} \cup \{ K_t \in (\mathfrak{h}_Z)_1 \mid 1 \leq t \leq \ell_1 \}. \]

We see that \( \text{hy}(\mathfrak{G}) \) can be identified with \( \text{hy}(\mathfrak{G}_Z) \otimes_{\mathbb{Z}} \mathbb{K} \), since \( \mathfrak{G}_Z \) is infinitesimally flat (Proposition [8]). As elements of \( \text{hy}(\mathfrak{G}) = \text{hy}(\mathfrak{G}_Z) \otimes_{\mathbb{Z}} \mathbb{K} \), we set

\[ X_{\alpha,n} := X_\alpha^{(n)} \otimes_{\mathbb{Z}} 1, \quad H_{i,m} := H_i^{(m)} \otimes_{\mathbb{Z}} 1, \]

\[ Y_{(\delta, p),\epsilon} := Y_{(\delta, p)} \otimes_{\mathbb{Z}} 1, \quad K_{t,\epsilon} := K_t \otimes_{\mathbb{Z}} 1, \]

where \( \epsilon = 0 \) or 1. Then the supercoalgebra isomorphism \( \text{hy}(\mathfrak{G}) \cong \text{hy}(\mathfrak{G}_e) \otimes \bigwedge(\mathfrak{g}_1) \) given in [3,2] implies the following PBW theorem for \( \text{hy}(\mathfrak{G}) \).

**Theorem 3.11.** For any total order on the set \( \mathcal{B}_0 \cup \mathcal{B}_1 \), the set of all products of factors of type

\[ H_{i,m}, \quad X_{\alpha,n(\alpha)}, \quad K_{t,\epsilon(\ell)}, \quad Y_{(\delta, p),\epsilon(\delta, p)} \]

\((n(\alpha), m(i) \in \mathbb{Z}_{\geq 0}, \alpha \in \Delta_0, 1 \leq i \leq \ell_0, \delta \in \Delta_1, 1 \leq p \leq \text{rank}(\mathfrak{g}_1^\delta), 1 \leq t \leq \ell_1 \) and \( \epsilon(\ell), \epsilon(\delta, p) \in \{ 0, 1 \} \), taken in \( \text{hy}(\mathfrak{G}) \) with respect to the order, forms a \( \mathbb{K} \)-basis of \( \text{hy}(\mathfrak{G}) \).

**Remark 3.12.** To construct Chevalley supergroups over \( \mathbb{Z} \), it is necessary to prove that (1) any finite-dimensional simple Lie superalgebra \( \mathfrak{s} \) over \( \mathbb{C} \) has a Chevalley basis; (2) the Kostant’s \( \mathbb{Z} \)-form of \( \mathcal{U}(\mathfrak{s}) \) has a PBW basis, like above. These were done by Fioresi and Gavarini [7, Theorems 3.7, 4.7].

Since the \( \mathbb{K} \)-valued points of \( \mathfrak{G} \) is just \( \mathfrak{G}(\mathbb{K}) = \mathfrak{G}_e(\mathbb{K}) \), it is hard to consider a geometrical “denseness” of \( \mathbb{B} \times \mathbb{B}^+ \) in \( \mathfrak{G} \) (or, a “big cell” as in [9, Part II, 1.9]) directly. The following is an algebraic interpretation of the denseness, see B. Parshall and J. Wang [24, and J. Bichon and S. Riche [1].
Corollary 3.13. The following superalgebra map is injective.

\[
\mathcal{O}(\mathfrak{G}) \rightarrow \mathcal{O}(\mathfrak{G}) \otimes \mathcal{O}(\mathfrak{G}) \rightarrow \mathcal{O}(\mathfrak{B}) \otimes \mathcal{O}(\mathfrak{B}^+),
\]

where the first arrow is the comultiplication map of \( \mathcal{O}(\mathfrak{G}) \) and the second arrow is the tensor product of the canonical quotient maps \( \mathcal{O}(\mathfrak{G}) \rightarrow \mathcal{O}(\mathfrak{B}) \) and \( \mathcal{O}(\mathfrak{G}) \rightarrow \mathcal{O}(\mathfrak{B}^+) \).

Proof. As in the proof Proposition 3.10 we also get an injection \( \mathcal{O}(\mathfrak{U}) \rightarrow \mathcal{O}(\mathfrak{B}) \). Thus, to prove the claim, it is enough to see that the map \( \mathcal{O}(\mathfrak{G}) \rightarrow \mathcal{O}(\mathfrak{U}) \otimes \mathcal{O}(\mathfrak{B}^+) \) is injective.

The multiplication map \( \mu : \mathfrak{U} \times \mathfrak{B}^+ \rightarrow \mathfrak{G} \) induces a morphism \( \mathrm{hy}(\mu) : \mathrm{hy}(\mathfrak{U}) \otimes \mathrm{hy}(\mathfrak{B}^+) \rightarrow \mathrm{hy}(\mathfrak{G}) \) of supercoalgebras which is indeed bijective, by \( n^- \oplus b^+ = \mathfrak{g} \) and Theorem 3.11. Then the \( k \)-linear dual \( \mathrm{hy}(\mu)^* \) is an isomorphism of superalgebras. Since \( \mathrm{hy}(\mathfrak{U}) \) and \( \mathrm{hy}(\mathfrak{B}^+) \) are both \( k \)-free, the canonical map \( \mathrm{hy}(\mathfrak{U})^* \otimes \mathrm{hy}(\mathfrak{B}^+)^* \rightarrow (\mathrm{hy}(\mathfrak{U}) \otimes \mathrm{hy}(\mathfrak{B}^+))^* \) is injective. Therefore, we get the following commutative diagram of superalgebras:

\[
\begin{array}{ccc}
\mathrm{hy}(\mathfrak{G})^* & \overset{\mathrm{hy}(\mu)^*}{\twoheadrightarrow} & (\mathrm{hy}(\mathfrak{U}) \otimes \mathrm{hy}(\mathfrak{B}^+))^* \\
\downarrow & & \downarrow \\
\mathcal{O}(\mathfrak{G}) & \otimes & \mathcal{O}(\mathfrak{U}) \otimes \mathcal{O}(\mathfrak{B}^+).
\end{array}
\]

The vertical canonical maps are injection, since \( \mathfrak{G}, \mathfrak{U} \) and \( \mathfrak{B}^+ \) are connected. Therefore, the lower horizontal arrow is also injective. \( \square \)

4. Borel-Weil Theorem for \( \mathfrak{G} \)

Throughout the rest of the paper, we assume that \( k \) is a field of characteristic different from 2. All supergroups \( \mathfrak{G}, \mathfrak{B}, \mathfrak{U}, \mathfrak{T} \), etc. are defined over the field \( k \). In this section, we will construct all simple \( \mathfrak{G} \)-supermodules, which extends Serganova’s construction [27, §9] to arbitrary characteristic. The main idea is based on Brundan and Kleshchev’s argument [3, §6], see also Parshall and Wang [24], Bichon and Riche [11] for the non-super situation.

4.1. Simple \( \mathfrak{T} \)-Supermodules. We will construct all simple supermodules of the super-torus \( \mathfrak{T} \) of \( \mathfrak{G} \). In the following, we freely use the notations used in Appendix B.

Recall that, \( \ell_{\epsilon} \) denotes the dimension of \( \mathfrak{h}_\epsilon \) for \( \epsilon \in \mathbb{Z}_2 \). By Theorem 3.11 the super-hyperalgebra \( \mathrm{hy}(\mathfrak{T}) \) of \( \mathfrak{T} \) has a \( k \)-basis

\[
\{ \prod_{i=1}^{\ell_0} H_{i,m(i)} \prod_{t=1}^{\ell_1} K_{t,\epsilon(t)} | m(i) \in \mathbb{Z}_{\geq 0}, \epsilon(t) \in \{0, 1\} \}.
\]

In the following, we fix \( \lambda \in \Lambda \) and write \( \lambda(H) := \langle H, \lambda \rangle \) for \( H \in \mathfrak{h}_0 (\subseteq \mathrm{hy}(\mathfrak{T})) \). We let \( \mathrm{hy}(\mathfrak{T})^\lambda \) denote the quotient superalgebra of the superhyperalgebra \( \mathrm{hy}(\mathfrak{T}) \) of \( \mathfrak{T} \) by the two-sided super-ideal of \( \mathrm{hy}(\mathfrak{T}) \) generated by all

\[
H_{i,m} - \frac{1}{m!} \lambda(H_i)(\lambda(H_i) - 1) \cdots (\lambda(H_i) - m + 1),
\]

where \( 1 \leq i \leq \ell_0 \) and \( m \geq 0 \). Then we get the following.

\[
\dim \mathrm{hy}(\mathfrak{T})^\lambda = 2^{\ell_1}.
\]
For \( x, y \in \mathfrak{h}_1 \), we see that \([x, y] \in \mathfrak{h}_0\), and hence we can define a bilinear map \( b^\lambda : \mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathbb{k}\) as follows.

\[
b^\lambda(x, y) := \lambda([x, y]),
\]

where \( x, y \in \mathfrak{h}_1 \). It is easy to see that \((\mathfrak{h}_1, b^\lambda)\) forms a quadratic space over \(\mathbb{k}\). Thus, we get the Clifford superalgebra \(\text{Cl}(\mathfrak{h}_1, b^\lambda) := T(\mathfrak{h}_1)/I(\mathfrak{h}_1, b^\lambda)\) over \(\mathbb{k}\). We may regard \(\mathfrak{h}_1\) as a subspace of \(\mathfrak{h}(\mathfrak{T})\), since \(\mathfrak{h}\) coincides with the set of all primitive elements in \(\mathfrak{h}(\mathfrak{T})\). Then we have the following map.

\[
(4.2) \quad T(\mathfrak{h}_1) \to \mathfrak{h}(\mathfrak{T}) \to \mathfrak{h}(\mathfrak{T})^\lambda,
\]

where \(T(\mathfrak{h}_1)\) is the tensor algebra of \(\mathfrak{h}_1\).

**Lemma 4.1.** The map above induces an isomorphism \(\text{Cl}(\mathfrak{h}_1, b^\lambda) \cong \mathfrak{h}(\mathfrak{T})^\lambda\) of superalgebras.

**Proof.** It is easy to see that \((4.2)\) is surjective and the kernel contains the ideal \(I(\mathfrak{h}_1, b^\lambda)\) of \(T(\mathfrak{h}_1)\). Thus, \((4.2)\) induces a surjection \(\text{Cl}(\mathfrak{h}_1, b^\lambda) \to \mathfrak{h}(\mathfrak{T})^\lambda\) of superalgebras. On the other hand, it is known that the dimension of \(\text{Cl}(\mathfrak{h}_1, b^\lambda)\) is \(2^d\). Thus, we are done. \(\square\)

Take a non-degenerate subspace \((\mathfrak{h}_1)_s\) of \(\mathfrak{h}_1\) so that \(\mathfrak{h}_1 = \text{rad}(b^\lambda) \perp (\mathfrak{h}_1)_s\), and set

\[
(4.3) \quad d_\lambda := \ell_\mathfrak{T} - \dim \text{rad}(b^\lambda).
\]

We choose an orthogonal basis \(\{x_1, \ldots, x_d\}\) of \((\mathfrak{h}_1)_s\) and we let

\[
(4.4) \quad \delta_\lambda := (-1)^{d_\lambda(d_\lambda+1)/2} \lambda([x_1, x_1]) \cdots \lambda([x_{d_\lambda}, x_{d_\lambda}]),
\]

the signed determinant of \((\mathfrak{h}_1)_s\), see \((3.4)\). For simplicity, we let \(\delta_\lambda = 0\) if \(b^\lambda = 0\). By Lemma 4.1 and Proposition B.3, we have the following.

**Proposition 4.2.** The superalgebra \(\mathfrak{h}(\mathfrak{T})^\lambda\) has a unique simple supermodule \(u(\lambda)\) up to isomorphism and parity change. If \((1)\) \(\delta_\lambda = 0\) or \((2)\) \(d_\lambda\) is even and \(\delta_\lambda \in (\mathbb{k}^\times)^2\), then \(\Pi u(\lambda) \neq u(\lambda)\). Otherwise, \(\Pi u(\lambda) = u(\lambda)\).

For a left \(\mathfrak{h}(\mathfrak{T})^\lambda\)-supermodule \(M\), we may regard \(M\) as a left \(\mathfrak{h}(\mathfrak{T})\)-supermodule via the canonical quotient map \(\mathfrak{h}(\mathfrak{T}) \to \mathfrak{h}(\mathfrak{T})^\lambda\). It is easy to see that \(M\) is a left \(\mathfrak{h}(\mathfrak{T})\)-\(T\)-supermodule. Since \(u(\lambda)\) is finite-dimensional, it is obviously locally finite as a left \(\mathfrak{h}(\mathfrak{T})\)-\(T\)-supermodule.

**Lemma 4.3.** Let \(L\) be a locally finite left \(\mathfrak{h}(\mathfrak{T})\)-\(T\)-supermodule. If \(L\) is simple, then there exists \(\lambda \in \Lambda\) such that \(L\) is isomorphic to \(u(\lambda)\) or \(\Pi u(\lambda)\).

**Proof.** Since \(L\) is non-zero, there exists \(\lambda \in \Lambda\) such that the \(\lambda\)-weight space \(L^\lambda\) of \(L\) is non-zero. By definition, \(L^\lambda\) is a \(\mathfrak{h}(\mathfrak{T})^\lambda\)-supermodule. Hence by Proposition 4.2, \(L^\lambda\) contains \(u(\lambda)\) or \(\Pi u(\lambda)\), say \(u(\lambda)\). The simplicity assumption on \(L\) implies \(L = u(\lambda)\). \(\square\)

By Theorem 3.4 for \(\mathfrak{T}\), we see that the category of locally finite left \(\mathfrak{h}(\mathfrak{T})\)-\(T\)-supermodules is equivalent to the category of \(\mathfrak{T}\)-supermodules. Thus, we may regard \(u(\lambda)\) as a \(\mathfrak{T}\)-supermodule, and hence we get the following map.

\[
\Lambda = \mathcal{X}(T) \to \text{Simple}_{\mathfrak{T}}(\mathfrak{T}) : \lambda \mapsto u(\lambda).
\]

**Proposition 4.4.** The above map is bijective. Moreover, \(u(\lambda)\) is of type \(M\) if and only if \((1)\) \(\delta_\lambda = 0\) or \((2)\) \(d_\lambda\) is even and \(\delta_\lambda \in (\mathbb{k}^\times)^2\).
Remark 4.5. As a $\mathbb{T}$-supermodule, this $u(\lambda)$ is isomorphic to some copies of $k^\lambda$ up to parity change, where $k^\lambda$ is the one-dimensional purely even left $\text{hy}(T)$-supermodule of weight $\lambda$. If $\mathfrak{g} = \mathfrak{g}_0$, then we have $\text{hy}(\mathbb{T})^\lambda = k$, and hence $u(\lambda)$ is just $k^\lambda$ or $\Pi k^\lambda$.

By Proposition 4.4. Then by (B.6), we conclude that $\text{coind}(\mathfrak{a}) = 0$. Assume that our base field $k$ is algebraically closed (not necessarily $\text{char}(k) = 0$). In the following, we shall generalize her construction (in the supergroup level) to our $k$, and show that our $u(\lambda)$ is isomorphic to her $C_\lambda$ up to parity change, when $\text{char}(k) = 0$.

Fix a maximal totally isotropic subspace $\mathfrak{h}_1^\perp$ of $(\mathfrak{h}_1, b^\lambda)$. We set

$$b^\perp := \mathfrak{h}_0 \oplus \mathfrak{h}_1^\perp.$$ 

This forms a Lie super-subalgebra of $\mathfrak{h}$. It is obvious that the pair $(T, b^\perp)$ is a sub-pair of the pair $(T, \mathfrak{h})$, and hence we get a closed super-subgroup $\mathbb{T}^\perp$ of $\mathbb{T}$. In particular, $\text{hy}(\mathbb{T}^\perp)$ is isomorphic to $\text{hy}(T) \otimes \wedge(b^\perp)$ as supercoalgebras.

We regard the one-dimensional purely even left $\text{hy}(T)$-supermodule $k^\lambda$ as a left $\text{hy}(\mathbb{T}^\perp)$-supermodule by letting $K.k^\lambda = 0$ for any $K \in \mathfrak{h}_1^\perp$. Set

$$\text{coind}_{\mathbb{T}^\perp}^\mathbb{T}(\lambda) := \text{hy}(\mathbb{T}) \otimes_{\text{hy}(\mathbb{T}^\perp)} k^\lambda.$$ 

By (4.1), we see that $\dim \text{coind}_{\mathbb{T}^\perp}^\mathbb{T}(\lambda) = 2^\ell_1 - \dim(b^\perp)$. Since the radical $\text{rad}(b^\lambda)$ is contained in $\mathfrak{h}_1^\perp$, one sees that $\text{coind}_{\mathbb{T}^\perp}^\mathbb{T}(\lambda)$ is a $\text{hy}(\mathbb{T})$-$T$-supermodule, and hence it is a $\mathbb{T}$-supermodule. Thus, $\text{coind}_{\mathbb{T}^\perp}^\mathbb{T}(\lambda)$ must contain $u(\lambda)$ or $\Pi u(\lambda)$, by Proposition 4.4. Then by (B.6), we conclude that $\text{coind}_{\mathbb{T}^\perp}^\mathbb{T}(\lambda)$ is isomorphic to $u(\lambda)$ up to parity change. If $\text{char}(k) = 0$, then our $\text{coind}_{\mathbb{T}^\perp}^\mathbb{T}(\lambda)$ coincides with Serganova’s $C_\lambda$ by Remark 2.3.

4.2. Cohomology Groups and Induced Supermodules. In this section, we shall prepare some notations. Let $\mathcal{H}$ be a closed super-subgroup of $\mathcal{G}$. We consider the functor $(\mathcal{G}/\mathcal{H})_{(n)} : R \mapsto \mathcal{G}(R)/\mathcal{H}(R)$ from the category of superalgebras to the category of sets, which is called the naive quotient of $\mathcal{G}$ over $\mathcal{H}$. Here, $\mathcal{G}(R)/\mathcal{H}(R)$ is the set of right cosets of $\mathcal{H}(R)$ in $\mathcal{G}(R)$. Then by Masuoka and Zubkov [21, Theorem 0.1], the sheafification $\mathcal{G}/\mathcal{H}$ of the native quotient $(\mathcal{G}/\mathcal{H})_{(n)}$ becomes a noetherian superscheme endowed with a morphism $\pi_{\mathcal{G}/\mathcal{H}} : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$ satisfying the conditions (Q1)–(Q3) in [21, §2]. See also, Masuoka and Y. Takahashi [10]. Let $\mathcal{L} = \mathcal{L}_{\mathcal{G}/\mathcal{H}}$ be a functor from $\mathcal{H} \text{SMOD}$ to the category of quasi-coherent $\mathcal{O}_{\mathcal{G}/\mathcal{H}}$-$\mathcal{G}$-supermodules satisfying

$$\mathcal{L}(V)(\mathcal{U}) = V \square_{\mathcal{O}(\mathcal{H})} \mathcal{O}(\pi_{\mathcal{G}/\mathcal{H}}^{-1}(\mathcal{U}))$$

for $V \in \mathcal{H} \text{SMOD}$ and an open super-subfunctor $\mathcal{U} \subseteq \mathcal{G}/\mathcal{H}$ with $\pi_{\mathcal{G}/\mathcal{H}}^{-1}(\mathcal{U})$ is affine. This $\mathcal{L}(V)$ is the so-called associated sheaf to $V$ on $\mathcal{G}/\mathcal{H}$. 
Now, let us return to our situation. Recall that, $u(\lambda)$ is a simple $T$-supermodule for $\lambda \in \Lambda$. For simplicity, we shall denote the cohomology group $H^0(\mathcal{G}/B, \mathcal{L}(u(\lambda)))$ by $H^0(\lambda)$ for $\lambda \in \Lambda$ and $n \in \mathbb{Z}_{\geq 0}$. Note that, $H^0(\lambda)$ does depend on the choice of the morphism $\gamma : \mathbb{Z}\Delta \to \mathbb{R}$ defined in [3,2] Brundan [2, Corollary 2.4] showed that for each $n \geq 0$, we have an isomorphism

$$H^0(\lambda) \cong R^n\text{ind}^\mathcal{G}_B(\text{res}^\mathcal{G}_B(u(\lambda))), \quad n = 0, 1, 2, \ldots$$

of $G$-supermodules. This is a super-analogue result of [9, Part I, Proposition 5.12]. In the following, we shall use this identification. By construction, $\text{res}^\mathcal{G}_B(u(\lambda))$ is isomorphic to some copies of $k^\lambda$ as $B$-supermodules. Thus, for $c \otimes a \in u(\lambda) \otimes O(G)$, we have

$$c \otimes a \in H^0(\lambda) \iff c \otimes \lambda \otimes a = c \otimes \sum a_i \otimes a_{i+1},$$

where $a_i$ is the canonical image of $a \in O(G)$ in $O(B)$.

4.3. Simple $B$-Supermodules. The inclusion $U \subseteq B$ makes $O(B)$ into a right $O(U)$-supermodule. We regard $O(T) \otimes O(U)$ as a right $O(U)$-supermodule via $\text{id} \otimes \Delta_{O(U)}$. Then one sees that the isomorphism $O(B) \cong O(T) \otimes O(U)$ given in (4.10) is right $O(U)$-colinear. By taking the functor $(- \square_{O(U)} k)$ to both sides, we get

$$O(B)^U \cong O(T),$$

where $O(B)^U$ is the $U$-invariant super-subspace of $O(B)$, in other words, the $O(U)$-coinvariant super-subspace $O(B)^{\square_{O(U)}}$ of $O(B)$, see Appendix A.2. It is easy to see that this isomorphism is left $O(B)$- right $O(T)$-colinear.

In the following, for a left $B$-supermodule $V$, we consider $\text{res}^B_B(V)$ and $\text{ind}^B_B(V)$ via the quotient maps $O(B) \to O(T)$ and $O(B) \to O(U)$ respectively.

**Lemma 4.8.** For a left $B$-supermodule $V$, $\text{res}^B_U(V)^U$ is isomorphic to $\text{ind}^B_U(V)$ as left $T$-supermodules.

**Proof.** It is easy to see that the canonical isomorphism $M \square_{O(B)} O(B) \cong V$ is left $O(U)$-colinear. Then by taking the functor $(- \square_{O(U)} k)$ to both sides, we get

$$V \square_{O(B)} O(B) \square_{O(U)} k \cong \text{res}^B_U(V)^U.$$

The left hand side is isomorphic to $V \square_{O(B)} (O(B)^U)$. Thus, by combining this with (4.6), we are done.

For a left $T$-supermodule $N$, we consider $\text{res}^T_B(N)$ via the Hopf superalgebra splitting $s_B : O(T) \hookrightarrow O(B)$ given in (4.8). Since the composition the splitting $s_B$ with the quotient $O(B) \to O(T)$ is identity, $\text{res}^T_B(\text{res}^B_B(N))$ is just the original $N$.

**Proposition 4.9.** For $\lambda \in \Lambda$, the left $B$-supermodule $\text{res}^T_B(u(\lambda))$ is simple. Moreover, this gives a one-to-one correspondence between $\Lambda$ and $\text{Simple}_T(B)$. 
Proposition 4.11. Thus, we conclude that $\mu$ exists. By Frobenius reciprocity, we have $\Pi\text{res}^\lambda(L)$ contains $u(\lambda)$ or $\Pi u(\lambda)$. For simplicity, we shall concentrate on the case when $u(\lambda) \subseteq \text{ind}^\lambda L(L)$. Then by Frobenius reciprocity (2.1), we get

$$\text{BSmod}(\text{res}^\lambda(u(\lambda)), L) \cong \text{tSmod}(u(\lambda), \text{ind}^\lambda L) \neq 0.$$  

Hence, we get a $\mathcal{B}$-supermodule surjection $\text{res}^\lambda(u(\lambda)) \twoheadrightarrow L$. By applying the functor $\text{res}^\lambda(-)$ to both sides, we have $u(\lambda) \twoheadrightarrow \text{res}^\lambda L(L)$. Since this is indeed a bijection, we are done.

4.4. Simple $\mathcal{G}$-Supermodules. Set

$$\Lambda^\circ := \{\lambda \in \Lambda \mid H^0(\lambda) \neq 0\}, \quad L(\lambda) := \text{soc}_{\mathcal{G}}(H^0(\lambda)).$$  

Note that, for $\lambda \in \Lambda^\circ$, the left $\mathcal{G}$-supermodule $L(\lambda)$ is non-zero by Lemma A.3.

Let $N$ be a left $\mathcal{T}$-supermodule. By taking the functor $\text{res}^\lambda B(N) \otimes_{\mathcal{T}} (\mathcal{B})$ to the superalgebra inclusion $O(\mathcal{G}) \rightarrow O(\mathcal{B}) \otimes O(\mathcal{B}^+)$ given in (3.9), we get

$$\text{res}^\lambda B(N) \otimes_{\mathcal{T}} (\mathcal{B}) \rightarrow \text{res}^\lambda B(N) \otimes_{\mathcal{T}} (O(\mathcal{B}) \otimes O(\mathcal{B}^+)).$$

Since the right hand side is equal to $N \otimes O(\mathcal{B}^+)$, we have $\text{ind}^\lambda_{\mathcal{B}^+} (\text{res}^\lambda B(N)) \rightarrow N \otimes O(\mathcal{B}^+)$. It is easy to see that the image of the above map lies in $N \otimes O(\mathcal{T}) O(\mathcal{B}^+)$. Thus, we get an inclusion

$$(4.7) \quad \text{res}^\lambda_{\mathcal{B}^+} (\text{ind}^\lambda_{\mathcal{B}^+} (\text{res}^\lambda B(N))) \rightarrow \text{ind}^\lambda_{\mathcal{B}^+} (N)$$

of right $O(\mathcal{B}^+)$-supercomodules, or equivalently left $\mathcal{B}^+$-supermodules.

For simplicity, we write $\text{res}^\lambda_{\mathcal{B}^+} (H^0(\lambda))$ as $H^0(\lambda)$ for $\lambda \in \Lambda$. Then by (1.7), $H^0(\lambda)$ can be regarded as a $\mathcal{B}^+$-super-submodule of $\text{ind}^\lambda_{\mathcal{B}^+} (u(\lambda))$.

Lemma 4.10. For $\lambda \in \Lambda^\circ$, we have $\text{soc}^\lambda_{\mathcal{B}^+} (H^0(\lambda)) = \text{res}^\lambda_{\mathcal{B}^+} (u(\lambda))$.

Proof. We see that $\text{soc}^\lambda_{\mathcal{B}^+} (H^0(\lambda))$ is contained in $\text{soc}^\lambda_{\mathcal{B}^+} (\text{ind}^\lambda_{\mathcal{B}^+} (u(\lambda)))$. To prove the converse, it is enough to show that

$$(4.8) \quad \text{soc}^\lambda_{\mathcal{B}^+} (\text{ind}^\lambda_{\mathcal{B}^+} (u(\lambda))) = \text{res}^\lambda_{\mathcal{B}^+} (u(\lambda)).$$

By Proposition 4.9 any simple $\mathcal{B}^+$-super-submodule of $\text{ind}^\lambda_{\mathcal{B}^+} (u(\lambda))$ is either

(i) $\text{res}^\lambda_{\mathcal{B}^+} (\mu(u))$ or

(ii) $\Pi \text{res}^\lambda_{\mathcal{B}^+} (\mu(u))$ for some $\mu \in \Lambda$. First, we consider the case (i). In this case, we have

$$0 \neq \mathcal{B}^+ \text{SMod}(\text{res}^\lambda_{\mathcal{B}^+} (\mu(u)), \text{ind}^\lambda_{\mathcal{B}^+} (u(\lambda))) \cong \text{tSMod}(\mu(u), u(\lambda)),$$

by Frobenius reciprocity (2.1) and $\text{ind}^\lambda_{\mathcal{B}^+} (\text{ind}^\lambda_{\mathcal{B}^+} (u(\lambda))) = u(\lambda)$. Thus, we conclude that $\lambda = \mu$, and hence the equation (4.8) holds. Next, we consider the case (ii). Similarly, by Frobenius reciprocity, we have $\Pi u(\mu) \cong u(\lambda)$. Thus, we conclude that $\mu = \lambda$ and $u(\lambda)$ is of type $\mathfrak{Q}$, and hence the equation (4.8) also holds.

Proposition 4.11. For $\lambda \in \Lambda^\circ$, the left $\mathcal{G}$-supermodule $L(\lambda)$ is a unique simple super-submodule of $H^0(\lambda)$. 

Proof. Suppose that $L, L'$ are two simple $G$-super-submodules of $H^0(\lambda)$. Then by Lemma 4.10, $\text{soc}_B^+(\text{res}_B^G(L))$ and $\text{soc}_B^+(\text{res}_B^G(L'))$ should coincide with $\text{res}_B^+(u(\lambda))$. Thus, $u(\lambda)$ is included in $L \cap L'$, and hence $L = L'$. □

By Proposition 4.11, we get the following map.

$$\Lambda^\flat \longrightarrow \text{Simple}_\Pi(G); \quad \lambda \longmapsto L(\lambda).$$

The next is our main theorem, which is a generalization of Serganova’s result [27, Theorem 9.9].

**Theorem 4.12.** The map above is bijective. Moreover, if (1) $\delta_\lambda = 0$ or (2) $d_\lambda$ is even and $\delta_\lambda \in (\mathbb{k}^\times)^2$, then $L(\lambda)$ is of type $M$. Otherwise, $L(\lambda)$ is of type $Q$.

**Proof.** Let $L$ be a simple $G$-supermodule. We see that $\text{soc}_B(\text{res}_B^G(L^*)) \neq 0$ by Lemma A.3. Then $\text{res}_B^G(L^*)$ includes $\text{res}_B^+(u(\mu))$ or $\text{Irres}_B^+(u(\mu))$ for some $\mu \in \Lambda$. Taking the linear dual $(-)^*$, we get a morphism from $\text{res}_B^G(L)$ to $\text{res}_B^+(u(\mu))^*$ or $\text{Irres}_B^+(u(\mu))^*$. Here, we used $L = L^**$. Since $u(\mu)^*$ is simple, there exists $\lambda \in \Lambda$ such that $u(\mu)^* = u(\lambda)$ by Proposition 4.9. Thus, we get a non trivial morphism from $\text{res}_B^G(L)$ to $\text{res}_B^+(u(\lambda))$ or (ii) $\text{Irres}_B^+(u(\lambda))$.

First, we consider the case (i). By Frobenius reciprocity (2.1), we get

$$0 \neq _B\text{SMod}(\text{res}_B^G(L), \text{res}_B^+(u(\lambda))) \cong _\text{T}\text{SMod}(L, H^0(\lambda)).$$

Thus, we conclude that $L \hookrightarrow H^0(\lambda)$, and hence $L = L(\lambda)$. Next, we consider the case (ii). In this case, a similar argument ensures that $L \hookrightarrow \Pi H^0(\lambda)$, and hence $L = \Pi L(\lambda)$.

Since $\Pi L(\lambda) = \text{soc}_G(\Pi H^0(\lambda))$, we conclude that $L(\lambda)$ is of type $M$ (resp. $Q$) if and only if $u(\lambda)$ is of type $M$ (resp. $Q$). Thus, the last statement directly follows from Proposition 4.11.

If $h_1 = 0$, then $b^\flat = 0$ by definition. Hence, in this case, $L(\lambda)$ is always of type $M$, that is, $L(\lambda)$ is not isomorphic to $\Pi L(\lambda)$ as $G$-supermodules.

**Remark 4.13.** For the queer supergroup $G = Q(n)$, the result above is a part of Brundan and Kleshchev’s result [3, Theorem 6.11] (where the base field $k$ is assumed to be algebraically closed). Indeed, one easily sees that their $h_{\nu'}(\lambda)$ coincides with our $d_\lambda$, where $p := \text{char}(k)$ and $h_{\nu'}(\lambda) := \#\{i \in \{1, \ldots, n\} \mid p \mid d_i\}$ for $\lambda = d_1\lambda_1 + \cdots + d_n\lambda_n \in \Lambda \cong \bigoplus_{i=1}^n \mathbb{Z}\lambda_i$. □

4.5. **Some Properties of Induced Supermodules.** In this section, we study some properties of $H^0(\lambda)$ for $\lambda \in \Lambda^\flat$.

**Lemma 4.14.** For $\lambda \in \Lambda^\flat$, we have $H^0(\lambda)^{T^\perp} \cong u(\lambda)$ as left $T$-supermodules.

**Proof.** By taking the functor $\text{ind}_{B^\perp}^T(\cdot)$ to the inclusion $H^0(\lambda) \hookrightarrow \text{ind}_{B^\perp}^T(u(\lambda))$, we get an inclusion $\text{ind}_{B^\perp}^T(H^0(\lambda)) \hookrightarrow u(\lambda)$ of $T$-supermodules, and hence this is bijective. Then by a $B^\perp$-version of Lemma 4.13, we are done. □

For $\lambda \in \Lambda^\flat$, we regard $H^0(\lambda)$ as a left $T$-supermodule via the left adjoint action, as usual. Then for $\mu \in \Lambda \subseteq \text{O}(T)$, the $\mu$-weight superspace of $H^0(\lambda)$ is described as follows.

$$H^0(\lambda)^\mu = \{\xi \in H^0(\lambda) \mid u \rightarrow \xi = \langle u, \mu \rangle \xi \text{ for } u \in \text{hy}(T)\}.$$
We define a partial order $\leq$ on $\Lambda$ as follows.

\begin{equation}
\mu \leq \lambda : \iff \lambda - \mu \in \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \alpha.
\end{equation}

The following proof is based on the proof of [9, Part II, Proposition 2.2(a)].

**Proposition 4.15.** For $\lambda \in \Lambda^\circ$, $\lambda$ is a maximal $T$-weight of $H^0(\lambda)$ with respect to $\leq$ and $H^0(\lambda)^{\lambda} \cong \mu(\lambda)$ as left $\mathbb{T}$-supermodules.

**Proof.** Suppose that $\mu \in \Lambda$ is a maximal $T$-weight of $H^0(\lambda)$. Then by definition, the $\mu$ weight superspace $H^0(\lambda)^{\mu}$ is included in the $\mathbb{U}^+$-invariant super-subspace $H^0(\lambda)^{\mathbb{U}^+}$ of $H^0(\lambda)$.

On the other hand, since $u(\lambda)$ is isomorphic to some copies of $k^\lambda$ as $T$-modules, we see that $H^0(\lambda)^{\lambda} \neq 0$ by Lemma 4.14. We fix a non-zero element $x \in H^0(\lambda)^{\lambda}$ and consider the following map.

$$H^0(\lambda) \rightarrow H^0(\lambda)^{\lambda}; \quad c \otimes a \mapsto c \otimes \varepsilon_{\mathcal{G}}(a)x,$$

where $c \otimes a \in u(\lambda) \otimes \mathcal{O}(\mathcal{G})$ and $\varepsilon_{\mathcal{G}} : \mathcal{O}(\mathcal{G}) \rightarrow k$ is the counit of $\mathcal{O}(\mathcal{G})$. We will show that this is injective. For simplicity, we assume that $u(\lambda) = k^\lambda$.

For $c \otimes a \in H^0(\lambda)^{\mathbb{U}^+}$ with $c \neq 0$ and $\varepsilon_{\mathcal{G}}(a) = 0$, it is easy to see that the canonical images of $a \in \mathcal{O}(\mathcal{G})$ in $\mathcal{O}(\mathbb{U}^+)$ and $\mathcal{O}(\mathbb{B})$ are both zero. As in the proof of Corollary 3.13 we can also prove that the composition $\mathcal{O}(\mathcal{G}) \rightarrow \mathcal{O}(\mathcal{G}) \otimes \mathcal{O}(\mathcal{G}) \rightarrow \mathcal{O}(\mathbb{U}^+) \otimes \mathcal{O}(\mathbb{B})$ is injective. Thus, we can conclude that $a = 0$, and hence we may regard $H^0(\lambda)^{\mathbb{U}^+}$ as a $\mathbb{T}$-super-submodule of $H^0(\lambda)^{\lambda}$.

Combined with the above, we get $\mu = \lambda$ and $H^0(\lambda)^{\mathbb{U}^+} = H^0(\lambda)^{\lambda}$. \qed

Set $A := \mathcal{O}(\mathcal{G})$. Recall that, $I_A$ is the super-ideal of $A$ generated by the odd part $A_1$, see [2, 3]. Then the (finite) descending chain $I^0_A := A \supseteq I_A \supseteq I^2_A \supseteq \cdots$ defines the graded (ordinary) algebra

$$\text{gr}(A) := \bigoplus_{n \geq 0} I^0_A/I_{n+1}$$

of $A$. On the other hand, we regard $A$ as a right $\mathcal{A}$-comodule via the right coadjoint action [2, 3]. This coaction makes $\wedge(W^A)$ into a right $\mathcal{A}$-comodule Hopf algebra, and hence we can construct the cosmash product $\mathcal{A} \bowtie \wedge(W^A)$. Then we have an isomorphism $\text{gr}(A) \cong \mathcal{A} \bowtie \wedge(W^A)$ of graded (ordinary) Hopf algebras, see [17, Proposition 4.9(2)]. Recall that, $T$ is a split maximal torus of $G$. Since $\mathcal{O}(T)$ is a cosemisimple Hopf algebra, we see that the graded algebra $\text{gr}(A)$ is isomorphic to $A$ as right $\mathcal{O}(T)$-comodules via the adjoint action of $T$. Hence, we get an isomorphism

\begin{equation}
A \cong \mathcal{A} \bowtie \wedge(W^A)
\end{equation}

of left $\mathcal{A}$-right $\mathcal{O}(T)$-comodules (or equivalently, right $G$-left $T$-modules). Here, the left $\mathcal{A}$-comodule structure of right hand side is $\Delta_T \otimes \text{id}$.

**Lemma 4.16.** For a left $\mathbb{B}$-supermodule $V$, there is an inclusion

$$\text{ind}_{\mathbb{B}}^\mathcal{G}(V) \hookrightarrow \text{ind}_{\mathbb{B}}^\mathcal{G}(\text{res}_\mathbb{B}^\mathcal{G}(V)) \otimes \wedge(W^\mathcal{G}(\mathcal{G}))$$

of left $T$-modules.
Proof. Taking the functor $V \square_{O(B)} \text{res}^G_B(\cdot)$ to both sides in (4.10), we get $V \square_{O(B)} A \overset{\sim}{\to} (V \square_{O(B)} A) \otimes \Lambda(W^A)$. By the definition of the cotensor, we see that $V \square_{O(B)} A$ is contained in $V \square_{O(B)} A$. This completes the proof. □

Proposition 4.17. For a finite-dimensional left $B$-supermodule $V$, the induced supermodule $\text{ind}_B^G(V)$ is finite-dimensional. In particular, $H^0(\lambda)$ is finite for any $\lambda \in \Lambda$.

Proof. Since $O(B)$ is a finitely generated superalgebra, $W^{O(B)}$ is finite-dimensional by [17, Proposition 4.4]. On the other hand, it is known that $\text{ind}_B^G(V)$ is finite dimensional, see [9, Part I, 5.12(c)] for example. This claim follows immediately from Lemma 4.16. □

As a non-super version of $H^0(\lambda)$, we set $H^0_{ev}(\lambda) := \text{ind}_B^G(k\lambda)$ and $\Lambda^+ := \{\lambda \in \Lambda \mid H^0_{ev}(\lambda) \neq 0\}$. for $\lambda \in \Lambda$. It is known that $\Lambda^+$ consists of all dominant weights of $T$ with respect to $\Delta^+_0$, that is, $\Lambda^+ := \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta^+_0\}$, see [9, Part II, 2.6] for example. Then by Lemma 4.16, we get the following Proposition.

Proposition 4.18. For $\lambda \in \Lambda$, there is an inclusion

$$H^0(\lambda) \hookrightarrow H^0_{ev}(\lambda) \oplus n_\lambda \otimes \Lambda(W^{O(B)})$$

of left $T$-modules, where $n_\lambda := \dim(u(\lambda))$.

In particular, we see that the parameter set $\Lambda^\circ$ is included in $\Lambda^+$.

Remark 4.19. Assume that $k$ is algebraically closed. Brundan and Kleshchev [13, Theorem 6.11] determined $\Lambda^\circ$ explicitly for queer supergroups $Q(n)$; $\Lambda^\circ = X_\sigma^+(n)$ in their notation. Shu and Wang [29, Theorem 5.3] describe $\Lambda^\circ$ combinatorially for ortho-symplectic supergroups $SO(2m|n)$; $\Lambda^\circ = \mathcal{X}^\circ(T)$ in their notation. □

A unified description of the parameter set $\Lambda^\circ$ for all quasireductive supergroups, as in the case of (non-super) split reductive groups, is not known. This is an open problem.

5. Quasireductive Supergroups admitting Distinguished Parabolic Super-subgroups

Let $G$ be a quasireductive supergroup over a field $k$, as before. Recall that, $G = G_{ev}$ and $B$ is the Borel super-subgroup of $G$ satisfying $b^- = \text{Lie}(B)$. In this section, we assume that $G$ contains a closed super-subgroup $P$ such that

$$P_{ev} = G \quad \text{and} \quad \text{Lie}(P)_{ev} = b_-,$$

which we shall call a parabolic super-subgroup of $G$.

Examples 5.1. The followings satisfy the assumptions above.

(1) General linear supergroups $GL(m|n)$.
(2) Chevalley supergroups $G$ of classical type whose Lie superalgebra $g = \text{Lie}(G)$ is of type $I$.
(3) Periplectic supergroups $P(n)$, see Example 5.10.
However, queer supergroups $Q(n)$ do not satisfy the assumption. □

For simplicity, we also assume that $h_1 = 0$, or equivalently, $T = T$.

5.1. **A $\otimes$-Splitting Property.** First, we fix a totally ordered $k$-basis $G = ((X_i)_i, \leq)$ of $g_1$ and define the following unit-preserving supercoalgebra map

$$\iota_G : \Lambda(g_1) \rightarrow \text{hy}(G); \quad X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_r} \mapsto X_{i_1} X_{i_2} \cdots X_{i_r},$$

where $X_{i_1} < X_{i_2} < \cdots < X_{i_r}$ in $G$. The left $T$-supermodule structure on $g_1$ naturally induces a left $T$-supermodule structure on the exterior superalgebra $\Lambda(g_1)$. It is easy to see that $\iota_G$ is left $T$-linear. Then one can retake the isomorphism (3.2) so that

$$\phi_{G,G} : \text{hy}(G) \otimes \Lambda(g_1) \rightarrow \text{hy}(G); \quad u \otimes \xi \mapsto u \cdot \iota_G(\xi).$$

Moreover, by dualizing $\iota_G$, we get a counit-preserving left $T$-module superalgebra map $\pi_G : \Omega(G) \rightarrow \Lambda(W^O(G))$ satisfying

$$\langle \iota_G(\xi), \eta \rangle = \langle \xi, \pi_G(\eta) \rangle, \quad \xi \in \Lambda(g_1), \eta \in \Lambda(W^O(G))$$

and the isomorphism (3.1) can be chosen as follows.

$$\psi_{G, G} : \Omega(G) \rightarrow \Omega(G) \otimes \Lambda(W^O(G)); \quad a \mapsto \sum_a a_{(1)} \otimes \pi_G(a_{(2)}),$$

see [18, Proposition 22]. It is easy to see that this $\psi_{G, G}$ is also left $T$-linear.

Take $B, U^+ \subseteq G$ so that $G = B \sqcup U^+$ (disjoint union) and $B$ (resp. $U^+$) forms a $k$-basis of $b^+_1$ (resp. $u^+_1$). Then we also get $\pi_B : \Omega(B) \rightarrow \Lambda(W^O(B))$ and $\pi_{U^+} : \Omega(U^+) \rightarrow \Lambda(W^O(U^+))$. The canonical identification $g = b^- \oplus u^+$ induces an isomorphism $\Lambda(W^O(G)) \cong \Lambda(W^O(B)) \otimes \Lambda(W^O(U^+))$. One sees that the following diagram commutes.

$$\begin{array}{c}
\Omega(G) \\
\downarrow \pi_G \\
\Lambda(W^O(G))
\end{array} \cong \begin{array}{c}
\Omega(B) \otimes \Omega(U^+) \\
\circ \downarrow \pi_B \otimes \pi_{U^+} \\
\Lambda(W^O(B)) \otimes \Lambda(W^O(U^+))
\end{array}$$

(5.1)

where the upper arrow is given by (3.9).

In the following, we regard $\Omega(G)$ as a right $P$-supermodule via the canonical quotient map $\varpi_P : \Omega(G) \rightarrow \Omega(P)$, and regard $\Omega(P)$ and $\Lambda(W^O(U^+))$ as left $T$-supermodules, as before. Then we have the following.

**Proposition 5.2** ([32] Proposition 5.1). $\Omega(G)$ is isomorphic to $\Omega(P) \otimes \Lambda(W^O(U^+))$ as right $P$-left $T$-supermodules.

**Proof.** By the definition of $P$, we also have a counit-preserving left $\Omega(G)$-right $\Omega(T)$-comodule superalgebra isomorphism $\psi_{P, B} : \Omega(P) \rightarrow \Omega(G) \otimes \Lambda(W^O(B))$ which satisfies

$$\psi_{P, B}(\varpi_P(a)) = \sum_a a_{(1)} \otimes \pi_B(\varpi_B(a_{(2)})), \quad a \in \Omega(G),$$

for simplicity, we also assume that $h_1 = 0$, or equivalently, $T = T$. A $\otimes$-Splitting Property. First, we fix a totally ordered $k$-basis $G = ((X_i)_i, \leq)$ of $g_1$ and define the following unit-preserving supercoalgebra map

$$\iota_G : \Lambda(g_1) \rightarrow \text{hy}(G); \quad X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_r} \mapsto X_{i_1} X_{i_2} \cdots X_{i_r},$$

where $X_{i_1} < X_{i_2} < \cdots < X_{i_r}$ in $G$. The left $T$-supermodule structure on $g_1$ naturally induces a left $T$-supermodule structure on the exterior superalgebra $\Lambda(g_1)$. It is easy to see that $\iota_G$ is left $T$-linear. Then one can retake the isomorphism (3.2) so that

$$\phi_{G,G} : \text{hy}(G) \otimes \Lambda(g_1) \rightarrow \text{hy}(G); \quad u \otimes \xi \mapsto u \cdot \iota_G(\xi).$$

Moreover, by dualizing $\iota_G$, we get a counit-preserving left $T$-module superalgebra map $\pi_G : \Omega(G) \rightarrow \Lambda(W^O(G))$ satisfying

$$\langle \iota_G(\xi), \eta \rangle = \langle \xi, \pi_G(\eta) \rangle, \quad \xi \in \Lambda(g_1), \eta \in \Lambda(W^O(G))$$

and the isomorphism (3.1) can be chosen as follows.

$$\psi_{G, G} : \Omega(G) \rightarrow \Omega(G) \otimes \Lambda(W^O(G)); \quad a \mapsto \sum_a a_{(1)} \otimes \pi_G(a_{(2)}),$$

see [18, Proposition 22]. It is easy to see that this $\psi_{G, G}$ is also left $T$-linear.

Take $B, U^+ \subseteq G$ so that $G = B \sqcup U^+$ (disjoint union) and $B$ (resp. $U^+$) forms a $k$-basis of $b^+_1$ (resp. $u^+_1$). Then we also get $\pi_B : \Omega(B) \rightarrow \Lambda(W^O(B))$ and $\pi_{U^+} : \Omega(U^+) \rightarrow \Lambda(W^O(U^+))$. The canonical identification $g = b^- \oplus u^+$ induces an isomorphism $\Lambda(W^O(G)) \cong \Lambda(W^O(B)) \otimes \Lambda(W^O(U^+))$. One sees that the following diagram commutes.

$$\begin{array}{c}
\Omega(G) \\
\downarrow \pi_G \\
\Lambda(W^O(G))
\end{array} \cong \begin{array}{c}
\Omega(B) \otimes \Omega(U^+) \\
\circ \downarrow \pi_B \otimes \pi_{U^+} \\
\Lambda(W^O(B)) \otimes \Lambda(W^O(U^+))
\end{array}$$

(5.1)

where the upper arrow is given by (3.9).

In the following, we regard $\Omega(G)$ as a right $P$-supermodule via the canonical quotient map $\varpi_P : \Omega(G) \rightarrow \Omega(P)$, and regard $\Omega(P)$ and $\Lambda(W^O(U^+))$ as left $T$-supermodules, as before. Then we have the following.

**Proposition 5.2** ([32] Proposition 5.1). $\Omega(G)$ is isomorphic to $\Omega(P) \otimes \Lambda(W^O(U^+))$ as right $P$-left $T$-supermodules.

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$$\psi_{P, B}(\varpi_P(a)) = \sum_a a_{(1)} \otimes \pi_B(\varpi_B(a_{(2)})), \quad a \in \Omega(G),$$
where \( \varpi_B: \mathcal{O}(G) \to \mathcal{O}(B) \) is the canonical quotient map. Then by \( \text{[3.1]} \), we get the following commutative diagram in the category of left \( T \)-supermodules.

\[
\begin{array}{ccc}
\mathcal{O}(G) & \xrightarrow{\psi_{g,B}} & \mathcal{O}(G) \otimes (W^\mathcal{O}(G)) \\
\mathcal{O}(P) \otimes (W^\mathcal{O}(U^+)) & \xrightarrow{\psi_{P,B} \otimes \text{id}} & \mathcal{O}(G) \otimes (W^\mathcal{O}(B)) \otimes (W^\mathcal{O}(U^+)).
\end{array}
\]

Thus, we are done. \( \square \)

Recall that, \( G = \mathfrak{g}_{ev} \) and \( B = \mathfrak{b}_{ev} \). For a left \( B \)-supermodule \( V \), it is easy to see that the map \( \text{id} \otimes \varepsilon_P: V \boxtimes_{\mathcal{O}(B)} \mathcal{O}(P) \to V \) is left \( B \)-linear, where \( \varepsilon_P: \mathcal{O}(P) \to \mathbb{k} \) is the counit of \( \mathcal{O}(P) \). Then by Frobenius reciprocity \( \text{[2.1]} \), we have the following left \( G \)-module map

\[
\mathcal{N}_V: V \boxtimes_{\mathcal{O}(B)} \mathcal{O}(P) \to V \boxtimes_{\mathcal{O}(B)} \mathcal{O}(G); \quad v \otimes p \mapsto v \otimes \overline{p},
\]

where \( \overline{p} \) is the canonical image of \( p \in \mathcal{O}(P) \) in \( \mathcal{O}(G) \).

**Lemma 5.3.** The above \( \mathcal{N}: \text{res}^\mathcal{P}_{B} \text{ind}^B_B(-) \to \text{ind}^G_B \text{res}^B_B(-) \) is a natural equivalence.

**Proof.** The proof is essentially the same as Zubkov’s \( \text{[32, Proposition 5.2]} \). For simplicity, we write \( R := \mathcal{O}(B), \overline{R} := \mathcal{O}(B), P := \mathcal{O}(P) \). Note that, \( \overline{A} = \mathcal{O}(G) = \mathcal{O}(P_{ev}) \). First, we prove that \( \mathcal{N}_R \) is an isomorphism. By the right version of \( \text{[3.1]} \), we see that \( R \) is isomorphic to \( \wedge(W^R) \otimes \overline{A} \) as a right \( \overline{R} \)-supercomodule. Thus, we have

\[
\mathcal{N}_R: R \boxtimes_R P \to R \boxtimes_R \overline{A} \cong \text{co}^{\overline{R}}R \otimes \overline{A}.
\]

Here, \( \text{co}^{\overline{R}}R \) is the right \( \overline{R} \)-coinvariant subspace of \( R \), that is, \( \text{co}^{\overline{R}}R = \mathbb{k} \boxtimes_R \overline{R} \), where \( \mathbb{k} \) is regarded as the trivial one-dimensional right \( \overline{R} \)-comodule. It is known that \( \text{co}^{\overline{R}}R \) is isomorphic to \( \wedge(W^R) \), see \( \text{[17, Theorem 4.5]} \). Then one sees that this map coincides with the tensor decomposition \( P \cong \wedge(W^R) \otimes \overline{A} \) of \( P \), which is the right version of \( \psi_{P,B} \), and hence \( \mathcal{N}_R \) is an isomorphism. We see that \( \mathcal{N}_{\Pi B} \) is also an isomorphism.

Next, we prove that \( \mathcal{N}_V \) is an isomorphism for all injective object \( V \) in \( \text{SMod}^B_G \). By \( \text{[32, Proposition 3.1]} \), it is shown that \( V \) is a direct summand of a direct sum of some copies of \( B \) and \( \Pi B \). Thus, \( \mathcal{N}_V \) is an isomorphism. Then by \( \text{[24, Lemma 8.4.5]} \), we are done. \( \square \)

The following is a generalization of Zubkov’s result \( \text{[32, Proposition 5.2]} \).

**Proposition 5.4.** For a left \( B \)-supermodule \( V \), there is an isomorphism

\[
R^n\text{ind}^B_B(V) \cong R^n\text{ind}^G_B(\text{res}^B_B(V)) \otimes (W^\mathcal{O}(U^+))
\]

of left \( T \)-supermodules.

**Proof.** Since \( \text{res}^B_B(-) \) is exact, we get \( \text{res}^B_B \circ R^n\text{ind}^B_B(-) \cong R^n(\text{res}^B_B \text{ind}^B_B(-)) \), see \( \text{[9, Part I, 4.1(2)]} \). Then by Lemma \( \text{[5.3]} \) we see that \( \text{res}^B_B \circ R^n\text{ind}^B_B(-) \cong R^n(\text{ind}^G_B \text{res}^B_B(-)) \). By Proposition \( \text{[5.2]} \), we see that \( \mathcal{O}(G) \) is an injective object in the category of left \( \mathcal{O}(P) \)-supermodules, and hence the functor \( \text{ind}^B_B(-) \) is exact. Thus, again by \( \text{[9, Part I, 4.1(2)]} \), we get \( \text{ind}^B_B \circ R^n(\text{ind}^B_B(-)) \cong
Thus, in this case, we see that $\dim(Y) = 2$. Here, we put $p$.

Remark 5.7. For the case of a character of $W$, we have $\Lambda = \Lambda^+$. Some Applications.

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5.2. Some Applications. As before, we set $H^n_{\text{ev}}(\lambda) := R^0 \text{ind}_{\text{B}}^G(\text{res}_{\text{B}}^G(k^\lambda))$ for $\lambda \in \Lambda$ and $n \in \mathbb{Z}_{\geq 0}$. By Proposition 5.4, we get the following theorem.

Theorem 5.5. For $\lambda \in \Lambda$ and $n \geq 0$, there is an isomorphism

$$H^n(\lambda) \cong H^n_{\text{ev}}(\lambda) \otimes (W^O(U^+))$$

of left $T$-supermodules. In particular, we have $\Lambda^0 = \Lambda^+$.

This seems to be a certain dual version of the notion of Kac modules (cf. [12], §2.2 (b)). As a corollary, we get a super-analogue of the Kempf vanishing theorem [9, Part II, Proposition 4.5].

Corollary 5.6. For $\lambda \in \Lambda^+$ and $n \geq 1$, we have $H^n(\lambda) = 0$.

Set $\rho_0 := \frac{1}{2} \sum_{\alpha \in \Delta^+_0} \alpha$ in $\Lambda \otimes \mathbb{Q}$, and

$$\Lambda^+_p := \left\{ \lambda \in \Lambda^+ \mid 0 \leq \langle \lambda + \rho_0, \beta^\vee \rangle \leq p, \forall \beta \in \Delta^+_0 \right\} \quad \text{if } p = 0,$$

$$\Lambda^+_p := \left\{ \lambda \in \Lambda^+ \mid 0 \leq \langle \lambda + \rho_0, \beta^\vee \rangle \leq p, \forall \beta \in \Delta^+_0 \right\} \quad \text{if } p > 2.$$

Here, we put $p := \text{char}(k)$. Then for $\lambda \in \Lambda^+_p$, it is known that $G$-module $H^n_{\text{ev}}(\lambda)$ is simple, that is, $H^n_{\text{ev}}(\lambda) = L_{\text{ev}}(\lambda) := \text{soc}_G(H^n_{\text{ev}}(\lambda))$. See [9] Part II, 5.5. Thus, in this case, we see that $H^0(\lambda)$ is isomorphic to the direct sum of $2^{\dim \mathfrak{g}}$-copies of $L_{\text{ev}}(\lambda)$ as $T$-modules.

Remark 5.7. For the case of $G = GL(m|n)$ over an algebraically closed field of positive characteristic, F. Marko [14] gave a necessary and sufficient condition for the induced $G$-supermodule $H^0(\lambda)$ to be simple.

Let $ZA$ be the group algebra of $\Lambda$ over $\mathbb{Z}$, and let $\{e^\lambda \mid \lambda \in \Lambda\}$ be the standard basis of $ZA$. For a finite-dimensional $T$-supermodule $M$, we let $\text{ch}(M)$ denote the formal character of $M$, that is, $\text{ch}(M) = \sum_{\lambda \in \Lambda} \dim(M^\lambda) e^\lambda$. Set

$$A(\mu) := \sum_{w \in W(G_{\text{ev}}, T)} (-1)^{\ell(w)} e^{w\mu},$$

where $\mu \in \Lambda \otimes \mathbb{Q}$ and $W(G_{\text{ev}}, T)$ is the Weyl group of $G = G_{\text{ev}}$ with respect to $T$. The following is a super-analogue of the Weyl character formula.

Corollary 5.8. For $\lambda \in \Lambda^+$, we have

$$\text{ch}(H^0(\lambda)) = \frac{A(\lambda + \rho_0)}{A(\rho_0)} \prod_{\delta \in \Delta^+_1} (1 + e^{-\delta}).$$

Proof. By the Weyl character formula for $G_{\text{ev}}$, it is known that the formal character of $H^0_{\text{ev}}(\lambda)$ is given by $A(\lambda + \rho_0)/A(\rho_0)$, see [9] Part II, 5.10. On the other hand, by definition, $W^O(U^+)$ coincides with the dual $(\mathfrak{n}_1^+)^*$ of $\mathfrak{n}_1^+$ as right $T$-modules. The formal character of $\wedge(\mathfrak{n}_1^+)^*$ is given by $\prod_{\delta \in \Delta^+_1} (1 + e^{-\delta})$. Thus, we are done. □
Set $\rho_1 := \frac{1}{2} \sum_{\delta \in \Delta_1} \delta$ and $\rho := \rho_0 - \rho_1$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Assume that $w\rho_1 = \rho_1$ for all $w \in \mathcal{W}(\mathfrak{g}_{ev}, T)$. Then for each $\lambda \in \Lambda^+$, we have

$$\text{ch}(H^0(\lambda)) = A(\lambda + \rho) \cdot \frac{\prod_{\delta \in \Delta_1^+}(e^{\delta/2} + e^{-\delta/2})}{A(\rho_0)}.$$  

This assumption is satisfied if $\mathfrak{g} = \text{Lie}(\mathcal{G})$ is a simple Lie superalgebra of type I or $\mathfrak{g}(m|n)$.

### 5.3. Some Examples

In this section, we shall see some examples.

**Example 5.9.** Let us consider the case of $\mathfrak{gl}(m|n)$. In our setting, we shall see Corollary 5.4 in [32]. As usual, we choose a split maximal torus $T$ of $\mathfrak{g}_{ev} \cong \mathfrak{gl}_m \times \mathfrak{gl}_n$ so that

$$T(R) = \{ (\text{diag}(x_1, \ldots, x_m), \text{diag}(y_1, \ldots, y_n)) \in \mathfrak{gl}_m(R) \times \mathfrak{gl}_n(R) \},$$

where $R$ is a commutative algebra. We may identify $\Lambda = \bigoplus_{i=1}^{m+n} \mathbb{Z}\lambda_i$. Then we have $\text{Lie}(\mathfrak{gl}(m|n)) = \mathfrak{gl}(m|n) = \text{Mat}_{m|n}(k)$ and $\mathcal{W}(\mathfrak{gl}(m|n), T) \cong \mathfrak{S}_m \times \mathfrak{S}_n$, where $\mathfrak{S}_m$ is the symmetric group on $m$ letters. The root system of $\mathfrak{gl}(m|n)$ with respect to $T$ is $\Delta = \{ \lambda_i - \lambda_j | 1 \leq i \neq j \leq m + n \}$ and $\Delta_0 = \{ \lambda_i - \lambda_j | 1 \leq i \neq j \leq m \text{ or } m + 1 \leq i \neq j \leq m + n \}$.

Note that, $\Delta = \Delta_0 \cup \Delta_1$ (disjoint union). Define $\gamma : \mathbb{Z}\Delta \to \mathbb{R}$ so that $\gamma(\lambda_i) := -i$ for each $1 \leq i \leq m + n$. Then one sees that $\Delta^\pm = \{ \pm(\lambda_i - \lambda_j) | 1 \leq i < j \leq m + n \}$ and the Borel super-subgroup $\mathcal{B}$ (resp. $\mathcal{B}^+$) of $\mathfrak{gl}(m|n)$ consists of lower (resp. upper) triangular matrices. In this case, the parabolic super-subgroup of $\mathfrak{gl}(m|n)$ is given as

$$\mathcal{P}(R) = \{ \begin{pmatrix} X & Y \\ 0 & W \end{pmatrix} \in \mathfrak{gl}(m|n)(R) \}$$

where $R$ is a commutative superalgebra. By Theorem 5.8, we see that

$$\Lambda^0 = \Lambda^+ = \{ \sum_{i=1}^{m+n} d_i \lambda_i \in \Lambda \mid d_1 \geq \cdots \geq d_m \text{ and } d_{m+1} \geq \cdots \geq d_{m+n} \}.$$  

For $\lambda = \sum_{i=1}^{m+n} d_i \lambda_i \in \Lambda^0$, we shall write $\lambda_+ := \sum_{i=1}^{m} d_i \lambda_i$ and $\lambda_- := \sum_{i=m+1}^{m+n} d_i \lambda_i$. Then we have $H^0_{\mathfrak{ev}}(\lambda) \cong H^0_{\mathfrak{gl}_m}(\lambda_+) \otimes H^0_{\mathfrak{gl}_n}(\lambda_-)$, see [32], Part I, Lemma 3.8]. Here, $H^0_{\mathfrak{gl}_m}(\lambda_+)$ is the induced $\mathfrak{gl}_m$-module for $k^{\lambda_+}$. Let $t_1, \ldots, t_{m+n}$ be indeterminates. As an element of $\mathbb{Z}[t_1^\pm, \ldots, t_{m+n}^\pm]$, 

$$s_{\lambda_+}(t_1, \ldots, t_m) := \det(t_i^{t_j+m-j})_{1 \leq i, j \leq m} \cdot \prod_{1 \leq i < j \leq m} (t_i - t_j)$$

is the so-called Schur polynomial. It is well-known that the formal character of $H^0_{\mathfrak{gl}_m}(\lambda_+)$ is given by $A(\lambda_+ + \rho_0)/A(\rho_0) = s_{\lambda_+}(e^{t_1}, \ldots, e^{t_m})$. Then by Corollary 5.8, we see that the formal character of $H^0(\lambda)$ is given as

$$s_{\lambda_+}(t_1, \ldots, t_m) \cdot s_{\lambda_-}(t_{m+1}, \ldots, t_{m+n}) \cdot \prod_{1 \leq i \leq m, 1 \leq j \leq m+n} (1 + \frac{t_j}{t_i}).$$

Here, we have replaced each $e^{t_i}$ with $t_i$. \hfill \Box
For a superalgebra $R$, the super-transpose of $A \in \text{Mat}_{n|n}(R)$ is given as 
\[ stA := \begin{pmatrix} X & tY \\ -tY & Z \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}. \]
Here, $tX$ denotes the ordinary transpose of the matrix $X$.

**Example 5.10.** For $n \geq 2$, we consider the following closed super-subgroup $P(n)$ of $GL(n|n)$. For a commutative superalgebra $R$,
\[ P(n)(R) := \{ g \in GL(n|n)(R) \mid stg J_n g = J_n \}, \quad \text{where} \quad J_n := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}. \]
Here, $I_n$ is the identity matrix of size $n$. This is a quasisubreductive supergroup whose even part is $P(n)_{ev} \cong GL_n$.

We fix a split maximal torus $T$ of $P(n)_{ev} \cong GL_n$ as the subgroup of all diagonal matrices. We may identify $\Lambda = \bigoplus_{i=1}^{n} \mathbb{Z} \lambda_i$ and $W(P(n)_{ev}, T) = S_n$. It is easy to see that the Lie superalgebra of $P(n)$ is given as 
\[ \text{Lie}(P(n)) = \{ A \in \text{Mat}_{n|n}(\mathbb{k}) \mid stA J_n + J_n A = 0 \}. \]
This is the so-called periplectic Lie superalgebra, see [11, §2.1.3] and [5, §1.1.5]. For this reason, we shall call $P(n)$ the periplectic supergroup.

Then the root system of $P(n)$ with respect to $T$ is given as follows.
\[ \Delta = \{ \pm (\lambda_i - \lambda_j), \pm (\lambda_i + \lambda_j), 2\lambda_p \mid 1 \leq i < j \leq n, 1 \leq p \leq n \}. \]
Define $\gamma : \mathbb{Z} \Delta \to \mathbb{R}$ so that $\gamma(\lambda_i) := -i$ for each $1 \leq i \leq n$. Then we see that $\Delta^+ = \{ \lambda_i - \lambda_j, -\lambda_i + \lambda_j \mid 1 \leq i < j \leq n \}$ and $\Delta^- = \{ -\lambda_i - \lambda_j, \lambda_i + \lambda_j, 2\lambda_p \mid 1 \leq i < j \leq n, 1 \leq p \leq n \}$. In particular, $-\Delta^+ \neq \Delta^-$. In this case, the Borel subalgebra $B$ (resp. $B^+$) of $P(n)_{ev} \cong GL_n$ consists of all lower (resp. upper) triangular matrices. As a closed super-subgroup of $P(n)(R)$ (where $R$ is a commutative superalgebra), one sees that 
\[ B^+(R) = \{ \begin{pmatrix} X & 0 \\ Z & -tX \end{pmatrix} \mid X \in B^+(R_0), \quad t^TX = -tX \}, \]
\[ B(R) = \{ \begin{pmatrix} X & Y \\ 0 & -tX \end{pmatrix} \mid X \in B(R_0), \quad t^{-1}t^TY = YX^{-1} \}, \]
\[ P(R) = \{ \begin{pmatrix} X & Y \\ 0 & -tX \end{pmatrix} \mid X \in GL_n(R_0), \quad t^{-1}t^TY = YX^{-1} \}. \]

By Theorem [5.5] we see that $\Lambda^{b} = \Lambda^+ = \{ \sum_{i=1}^{n} d_i \lambda_i \in \Lambda \mid d_1 \geq \cdots \geq d_n \}$. For $\lambda = \sum_{i=1}^{n} d_i \lambda_i \in \Lambda^b$, the formal character of $H^0(\lambda)$ is given as follows.
\[ s_\lambda(t_1, \ldots, t_n) = \prod_{1 \leq i < j \leq n} (1 + \frac{1}{t_it_j}). \]
Here, we have replaced each $e^{\lambda_i}$ with $t_i$. \hfill \Box

**Remark 5.11.** As we mentioned before, the Borel super-subgroup $B$, the induced supermodule $H^0(\lambda)$, the order $\leq$ on $\Lambda$ given in [11,9], and the notion of maximal $T$-weight (see Proposition [11,15]) do depend on the choice of the homomorphism $\gamma : \mathbb{Z} \Delta \to \mathbb{R}$, defined in [3,2]. \hfill \Box
Appendix A. General Property

In this appendix, $k$ is a field of characteristic not equal to 2.

A.1. Supermodules. Let $A$ be a superalgebra. We consider the (ordinary) cross product algebra $A \times k\mathbb{Z}_2$ via the non-trivial action of the algebra $k\mathbb{Z}_2$ on $A$. By definition, the multiplication of $A \times k\mathbb{Z}_2$ is given as follows.

$$(a \times \epsilon)(b \times \eta) = a(b_0 + (-1)^s b_1) \times (\epsilon + \eta),$$

where $\epsilon, \eta \in \mathbb{Z}_2$, $a, b = b_0 + b_1 \in A$ with $b_0 \in A_0$, $b_1 \in A_1$.

The category of all left $A$-supermodules $A\text{SMod}$ is, by definition, the category of left $A$-right $k\mathbb{Z}_2$-Hopf modules $A\text{Mod}^{k\mathbb{Z}_2}$. Since the characteristic of $k$ is not 2, there is a unique Hopf algebra isomorphism $k\mathbb{Z}_2 \cong (k\mathbb{Z}_2)^\ast$. By using the isomorphism, we get the following equivalence of tensor categories.

$$(A.1) \quad A\text{SMod} \xrightarrow{\cong} A\times k\mathbb{Z}_2\text{Mod}; \quad M \mapsto M.$$

Here, the left $A \times k\mathbb{Z}_2$-module structure on $M$ is given by

$$(a \times \epsilon).m = a.(m_0 + (-1)^s m_1),$$

where $a \in A$, $\epsilon \in \mathbb{Z}_2$ and $m = m_0 + m_1 \in M$ with $m_0 \in M_0$, $m_1 \in M_1$.

A non-zero superalgebra is said to be simple if it has no non-trivial two-sided super-ideal. Here, the notion of a two-sided super-ideal is defined in an obvious way. Simple superalgebras over an algebraically closed field are well-studied, see T. Józefiak [10].

Example A.1. For natural numbers $m, n$ and an algebra $R$ (not necessarily commutative), we let $\text{Mat}_{m|n}(R)$ denote the set of all $(m + n) \times (m + n)$ matrices over $R$. This naturally forms a superalgebra whose grading is given as follows.

$$\text{Mat}_{m|n}(R)_0 = \{(X | 0 \atop 0 | W) \mid X \in \text{Mat}_m(R), \ W \in \text{Mat}_n(R)\},$$

$$\text{Mat}_{m|n}(R)_1 = \{(0 | Y \atop Z | 0) \mid Y \in \text{Mat}_{m,n}(R), \ Z \in \text{Mat}_{m,n}(R)\},$$

where $\text{Mat}_m(R)$ (resp. $\text{Mat}_{m,n}(R)$) is the set of all $m \times m$ (resp. $m \times n$) matrices over $R$. For simplicity, we let $\text{Mat}_{m|0}(R)$ denote the ordinary matrix algebra $\text{Mat}_m(R)$.

Proposition A.2. Let $A$ be a simple superalgebra. If $A_1 \neq 0$, then the algebra $A_0$ is Morita equivalent to $A \times k\mathbb{Z}_2$.

Proof. Since $A_1 A_1 \oplus A_1$ is a non-zero super-ideal of $A$, the $\mathbb{Z}_2$-grading of $A$ is strongly graded, that is, $A_1 A_1 = A_0$. This means that $A$ over $A_0$ is a right $k\mathbb{Z}_2$-Galois, see [22, §8]. Then by Hopf-Galois theory, we get the following equivalence.

$$(A.2) \quad A_0\text{Mod} \xrightarrow{\cong} A\text{Mod}^{k\mathbb{Z}_2} (= A\text{SMod}); \quad V \mapsto A \otimes_{A_0} V.$$

By combining the equivalence above with (A.1), the claim follows. \qed

In the situation as above, the parity change functor $\Pi : A\text{SMod} \rightarrow A\text{SMod}$ can be translated into the following functor.

$$A_0\text{Mod} \rightarrow A_0\text{Mod}; \quad V \mapsto A_1 \otimes_{A_0} V.$$
A.2. Supercomodules. Let $C$ be a supercoalgebra. As a dual of $A \rtimes k\mathbb{Z}_2$, we can consider the (ordinary) cosmash product coalgebra $k\mathbb{Z}_2 \ltimes C$ of $k\mathbb{Z}_2$ and $C$ which is $k\mathbb{Z}_2 \otimes C$ as a vector space. The comultiplication and the counit of $k\mathbb{Z}_2 \ltimes C$ are respectively given as follows.

$$\epsilon \triangleright c \mapsto \sum c \epsilon(c),$$

where $\epsilon \in \mathbb{Z}_2$ and $c \in C_0 \cup C_1$. Here, $\varepsilon_C : C \rightarrow k$ is the counit of $C$. Then as a dual of $(A.1)$, we get

$$(A.3) \quad \text{SMod}^C \cong \text{Mod}^{k\mathbb{Z}_2 \ltimes C}; \quad V \mapsto V.$$

Here, the right $k\mathbb{Z}_2 \ltimes C$-comodule structure on $V$ is given by

$$v \mapsto \sum v \otimes (|v| \triangleright v_{(1)}),$$

where $v \in V$ and $v \mapsto \sum v_{(0)} \otimes v_{(1)}$ is the given right $C$-supermodule structure on $V$.

For a right $C$-supermodule $V$, we let $\text{soc}_C(V)$ denote the sum of all simple $C$-super-subcomodules of $V$. In particular, $\text{corad}(C) := \text{soc}_C(C)$ is called the coradical of $C$. The identification $(A.3)$ ensures that any simple supercomodules are finite dimensional, for example. Moreover, we get the following lemma.

**Lemma A.3.** Suppose that $C$ is a Hopf superalgebra. Then for a right $C$-supermodule $V$, $V \neq 0$ if and only if $\text{soc}_C(V) \neq 0$.

A Hopf superalgebra $U$ is said to be irreducible if $\text{corad}(U)$ is trivial, or equivalently, the simple $U$-supercomodules are exhausted by the purely even or odd trivial $U$-comodule $k$, see [18, Definition 2]. For a right $U$-supercomodule $V$,

$$V^{\text{co}U} := \{ v \in V \mid \sum v_{(0)} \otimes v_{(1)} = v \otimes 1 \}$$

is called the $U$-coinvariant super-subspace of $V$.

**Lemma A.4.** Let $V$ be a right $U$-supercomodule. Then $V^{\text{co}U} = \text{soc}_U(V)$. In particular, $V \neq 0$ if and only if $V^{\text{co}U} \neq 0$.

A.3. Cotensors. Let $C, D$ be supercoalgebras and let $f : C \rightarrow D$ be a supercoalgebra map. For a right supercomodule $V$ with $\rho : V \rightarrow V \otimes C$ its structure map, the map $\rho|_D : V \rightarrow V \otimes C$ becomes $V \otimes D$ makes $V$ into a right $D$-supercomodule. We shall denote it by $\text{res}_D^C(V)$. Then $\text{res}_D^C(-)$ becomes a functor from $\text{SMod}^C$ to $\text{SMod}^D$. For a left $C$-supercomodule $V'$ with $\rho' : V' \rightarrow C \otimes V'$ its structure map, the cotensor product $V \bowtie_C V'$ of $V$ and $V'$ is given by

$$V \bowtie_C V' := \text{Ker}(V \otimes V' \overset{\rho \otimes \text{id} - \text{id} \otimes \rho'}{\longrightarrow} V \otimes C \otimes V').$$

This is a super-subspace of $V \otimes V'$. It is easy to see that $V \bowtie_C C \cong V$ and $C \bowtie_C V' \cong V'$ as superspaces. For a Hopf superalgebra $H$, the $H$-coinvariant superspace $V^{\text{co}H}$ of a right supercomodule $V$ is nothing but $V \bowtie_H k$, where $k$ is regarded as the trivial right $H$-supercomodule.
We regard $C$ as a left $D$-supercomodule whose structure map is given by $(f \otimes \text{id}) \circ \Delta_C$, where $\Delta_C : C \rightarrow C \otimes C$ is the comultiplication of $C$. For a right $D$-supercomodule $W$, one easily sees $\text{ind}^D_C(W) := W \otimes_D C$ forms a right $C$-supercomodule whose structure map is given by $id \otimes \Delta_C$. In this way, we get a (left exact) functor $\text{ind}^D_C(-)$ from $\text{SMod}^D$ to $\text{SMod}^C$.

In the super-situation, the following Frobenius reciprocity also holds: For a right $D$-supercomodule $W$ and a left $C$-supercomodule $V$, we have an isomorphism

$$\text{SMod}^D\left(\text{res}^D_C(V), W\right) \cong \text{SMod}^C\left(V, \text{ind}^C_D(W)\right)$$

of superspaces, which is given by $f \mapsto (f \otimes \text{id}_C) \circ \rho$, where $\rho : V \rightarrow V \otimes C$ is the $C$-supercomodule structure of $V$. Since $\text{ind}^C_D(-)$ is right adjoint to $\text{res}^D_C(-)$, we see that the functor $\text{ind}^C_D(-)$ preserves injective objects.

**APPENDIX B. Clifford Superalgebras**

In this Appendix, we review some basic facts and notations of Clifford superalgebras over a field $k$ of characteristic not equal to 2. Set $k^\times := k \setminus \{0\}$. For details, see C. T. C. Wall [30], I. Musson [23, A.3.2] and T. Y. Lam [13].

**B.1. Structures of Clifford Superalgebras.** Let $V$ be a finite-dimensional vector space. Set $r := \dim V$. For a symmetric bilinear form $b : V \times V \rightarrow k$ on $V$, the pair $(V,b)$ is called a quadratic space. Let $I(V,b)$ be the two-sided ideal of the tensor algebra $T(V)$ generated by all $xy + yx - b(x,y)$, where $x,y \in V$. Set

$$\text{Cl}(V,b) := T(V)/I(V,b).$$

For simplicity, we sometimes write $\text{Cl}(V)$ instead of $\text{Cl}(V,b)$. This forms a superalgebra over $k$ with each element in $V$ regards as odd, and is called the Clifford superalgebra for $(V,b)$. Since the characteristic of $k$ is not 2, we can choose an orthogonal basis $\{x_1, \ldots, x_r\}$ of $V$ with respect to $b$, that is, a basis of $V$ satisfying $b(x_i,x_j) = 0$ if $i \neq j$. For $1 \leq i \leq r$, set

$$(B.1) \quad \delta_i := b(x_i, x_i).$$

Then one sees that the superalgebra $\text{Cl}(V,b)$ is generated by the odd elements $x_1, \ldots, x_r$ with the relations

$$x_i^2 - \delta_i; \quad x_j x_k + x_k x_j,$$

where $1 \leq i \leq r$ and $1 \leq j < k \leq r$. One sees that $\text{Cl}(V,b)_e$ is a $2^{r-1}$-dimensional space for each $e \in \mathbb{Z}/2$, and hence $\text{Cl}(V,b)$ is $2^r$-dimensional. If $b$ is trivial (i.e., $b = 0$), then $\text{Cl}(V,b)$ is the so-called exterior superalgebra $\wedge(V)$ on $V$.

The orthogonal sum $(V,b) \perp (V',b')$ (or $V \perp V'$, for short) of two quadratic spaces $(V,b)$ and $(V',b')$ is a quadratic space whose underlying space is the direct sum $V \oplus V'$ and the bilinear form is given as follows.

$$(B.2) \quad \text{Cl}(V \perp V') \xrightarrow{\cong} \text{Cl}(V) \otimes \text{Cl}(V')$$

given by $(v,v') \mapsto v \otimes 1 + 1 \otimes v'$, where $v \in V, \ v' \in V'$. 
The radical of the quadratic space \((V, b)\) is defined as follows.

\[
\text{rad}(b) := \{ v \in V \mid b(v, w) = 0 \text{ for all } w \in V \}.
\]

We say that a quadratic space \((V, b)\) is non-degenerate if \(\text{rad}(b) = 0\).

Suppose that \((V, b)\) is non-degenerate. We define the signed determinant of \(V\) as follows.

\[
\delta := (-1)^{r(r-1)/2} \delta_1 \delta_2 \cdots \delta_r.
\]

It is known that the Clifford superalgebra \(\text{Cl}(V, b)\) is central simple over \(k\) in the category of superalgebras. In addition, if \(r = \dim V\) is even, then \(\text{Cl}(V, b)\) is still central simple over \(k\) in the category of algebras. Moreover, in this case (i.e., \(r\) is even), the following structure theorem is known. For more details, see [38 Chapter V, §2].

- If \(\delta \notin (k^\times)^2\), then the algebra \(\text{Cl}(V, b)_0\) is central simple over \(k(\sqrt{\delta})\).
- If \(\delta \in (k^\times)^2\), then the superalgebra \(\text{Cl}(V, b)\) is isomorphic to \(\text{Mat}_{n|n}(D)\), where \(n \in \mathbb{N}\) and \(D\) is a central division algebra over \(k\).

Here, \(k(\sqrt{\delta})\) is the quadratic field extension of \(k\), and

\[
(k^\times)^2 := \{ a \in k \mid a = b^2 \text{ for some } b \in k^\times \}.
\]

### B.2. Simples of Clifford Superalgebras

In the following, we let \((V, b)\) be a quadratic space (not necessarily non-degenerate). Then there is a non-degenerate subspace \(V_s\) of \(V\) such that \(V = \text{rad}(b) \perp V_s\). Let \(\delta\) be the sign determinant of \(V_s\). For simplicity, if \(b = 0\), then we let \(\delta := 0\). The following fact is well-known, see C. Chevalley [6 Chapter II, 2.7] for example. For convenience of the reader, we shall give a proof here.

**Lemma B.1.** The (ordinary) Jacobson radical \(J\) of the algebra \(\text{Cl}(V)\) coincides with the two-sided ideal of \(\text{Cl}(V)\) generated by \(\text{rad}(b)\), and the quotient (ordinary) algebra \(\text{Cl}(V)/J\) is isomorphic to \(\text{Cl}(V_s)\).

**Proof.** For simplicity, we let \(\mathfrak{r}\) be the two-sided ideal of \(\text{Cl}(V)\) generated by \(\text{rad}(b)\). We shall identify \(\text{Cl}(V_s)\) as a subalgebra of \(\text{Cl}(V)\) via the isomorphism \(\text{Cl}(V) \cong \text{Cl}(\text{rad}(b)) \otimes \text{Cl}(V_s)\). Then \(\mathfrak{r} + \text{Cl}(V_s)\) forms a subalgebra of \(\text{Cl}(V)\). Since \(V = \text{rad}(b) \perp V_s\), we see that \(\mathfrak{r} + \text{Cl}(V_s) = \text{Cl}(V)\), and hence we have

\[
\text{Cl}(V)/\mathfrak{r} \cong \text{Cl}(V_s)/(\mathfrak{r} \cap \text{Cl}(V_s)).
\]

Fix \(x \in \text{rad}(b)\). For each \(u = u_0 + u_1 \in \text{Cl}(V)\) with \(u_0 \in \text{Cl}(V)_0\) and \(u_1 \in \text{Cl}(V)_1\), we see that \((xu)^2 = xuxu = x^2(u_0 - u_1)u = 0\), since \(x^2 = 0\). Thus, we get that \(1 - xu\) is invertible in \(\text{Cl}(V)\), and hence \(\text{rad}(b) \subseteq J\). In particular, we have \(\mathfrak{r} \subseteq J\). Thus, \(\mathfrak{r} \cap \text{Cl}(V_s)\) is a nil-ideal of \(\text{Cl}(V_s)\). Since the Jacobson radical of \(\text{Cl}(V_s)\) is trivial, we get \(\text{Cl}(V)/\mathfrak{r} \cong \text{Cl}(V_s)\) and \(\mathfrak{r} = J\). \(\Box\)

To find simple supermodules, the following lemma is useful.

**Lemma B.2.** There exists a \((1 + \dim V)\)-dimensional quadratic space \((V', b')\) such that the category of \(\text{Cl}(V)\)-supermodules is equivalent to the category of \(\text{Cl}(V')\)-modules.
Proof. For simplicity, we set $A := \Cl(V, b)$. Let $kx$ be a one-dimensional vector space over $k$ with a base $x$, and let $b_x : kx \times kx \to k$ be a symmetric bilinear form defined by $b_x(x, x) = 1$. Then we have the following isomorphism of (ordinary) algebras.

$$A \times kZ_2 \cong A \otimes \Cl(kx, b_x); \quad a \times \epsilon \mapsto a \otimes x^\epsilon,$$

where $a \in A$ and $\epsilon \in Z_2$. Here, the tensor product $A \otimes \Cl(kx, b_x)$ is $Z_2$-graded, as before. Set $(V', b') := (V, b) \perp (kx, b_x)$. Then we have an isomorphism of (ordinary) algebras $A \times kZ_2 \cong \Cl(V', b')$. Hence, by the equivalence (A.1), the category of all left $A$-supermodules can be identified with the category of all left $\Cl(V', b')$-modules.

**Proposition B.3.** There is a unique simple left $\Cl(V, b)$-supermodule $u$ up to isomorphism and parity change. If (1) $\delta = 0$ or (2) $\dim V$ is even and $\delta \in (k^\times)^2$, then $u \neq \Pi u$. Otherwise, $u = \Pi u$.

Proof. First, suppose that $\delta = 0$. Then $\Cl(V, b)$ coincides with the exterior superalgebra $\wedge(V)$ on $V$. It is easy to see that $\wedge(V)$ has a unique one-dimensional simple supermodule $u$ up to isomorphism which satisfies $u \neq \Pi u$.

Next, suppose that $\delta \in k^\times$. For simplicity, we set $A := \Cl(V, b)$ and $r := \dim V$. By Lemma [B.1], it is enough to consider the case when $(V, b)$ is non-degenerate. (i) Assume that $r$ is odd. We use the notation in Lemma [B.2]. Note that, $V'$ is also non-degenerate. Since $\dim V' = r + 1$ is even, we see that $\Cl(V')$ is (Artinian) simple as an ordinary algebra. Thus, $A$ has a unique simple supermodule $u$ up to isomorphism which satisfies $\Pi u = u$. (ii) Assume that $r$ is even and $\delta \not\in (k^\times)^2$. In this case, $A_0$ is (Artinian) simple. By Proposition [A.2], $A \times kZ_2$ is Morita equivalent to $A_0$. Thus, $A$ has a unique simple supermodule $u$ up to isomorphism which satisfies $\Pi u = u$, by using the equivalence (A.1). (iii) Assume that $r$ is even and $\delta \in (k^\times)^2$. Then the superalgebra $A$ can be identified with $\Mat_{n | n}(D)$ for some $n \in \mathbb{N}$ and some central division algebra $D$ over $k$. Thus, in the following, we identify $A_0$ with

$$\Mat_{n | n}(D)_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix} \mid X, W \in \Mat_n(D) \right\}.$$ 

As column vectors of length $2n$ with entries in $D$, we set

$$e := \ell((1, \ldots, 1, 0, \ldots, 0)), \quad e' := \ell((0, \ldots, 0, 1, \ldots, 1)).$$

Then $M := A_0 e$ and $M' := A_0 e'$ are the distinct simple left $A_0$-modules. By Proposition [A.2] the corresponding simple left $A$-supermodules are given by $u := A \otimes_{A_0} M$ and $u' := A \otimes_{A_0} M'$. One easily sees that $\Pi u = u'$, and hence $u \neq \Pi u$. □

**B.3. The Algebraically Closed Case.** Assume that the base field $k$ is algebraically closed. In this section, we shall calculate the dimensions of simple supermodules of Clifford superalgebras.

Let $(V, b)$ be a quadratic space over $k$, in general. The quotient space $\nabla := V/\rad(b)$ is non-degenerate. Set $d := \dim \nabla$. By Proposition [B.3], $\Cl(V, b)$ has a unique simple supermodule $u$ up to isomorphism and parity change.
Proposition B.4. The dimension of the vector space $u$ is given by $2^\lfloor (d+1)/2 \rfloor$, where $\lfloor (d+1)/2 \rfloor$ is the largest integer greater than $(d+1)/2$.

Proof. First, we suppose that the signed determinant $\delta$ of $V$, defined in (B.4), is zero. In this case, $\dim u = 1$ and $d = 0$, and hence the claim follows. Next, we suppose that $\delta \neq 0$ and $d$ is odd. In this case, the algebra $\text{Cl}(V, b) \rtimes k\mathbb{Z}_2$ is central simple over $k$. Since $k$ is algebraically closed, one sees that the algebra $\text{Cl}(V, b) \rtimes k\mathbb{Z}_2$ is isomorphic to $\text{Mat}_m(k)$, where $m = 2^{(d+1)/2}$. Hence, we are done. Finally, we suppose that $\delta \neq 0$ and $d$ is even. Note that, $\lfloor (d+1)/2 \rfloor = d/2$. Since $k$ is algebraically closed, the element $\delta (\neq 0)$ is always in $(k^\times)^2 = k^\times$. Thus, the algebra $\text{Cl}(V, b)_0$ is isomorphic to $\text{Mat}_n(k) \times \text{Mat}_n(k)$, where $n = 2^{d/2}$. Hence, we have $\dim u = n = 2^{d/2}$. □

A maximal totally isotropic subspace $V^\dagger$ of $(V, b)$ is a maximal subspace $W$ of $V$ satisfying $b(W, W) = 0$. It is known that the Witt index $\text{ind}(V) := \lfloor d/2 \rfloor$ of $(V, b)$ coincides with the dimension of the quotient space $V^\dagger/\text{rad}(b)$, see [13, Chapter I, Corollary 4.4] for example. Therefore, we obtain $\dim V - \dim V^\dagger = \lfloor (d+1)/2 \rfloor$. Here, we used $d = \lfloor d/2 \rfloor + \lfloor (d+1)/2 \rfloor$. Then by Proposition B.4, we get

$$\dim u = 2^{\dim V - \dim V^\dagger}.$$  

This is another description of the dimension of $u$.

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References

[1] Julien Bichon and Simon Riche, *Hopf algebras having a dense big cell*, Trans. Amer. Math. Soc. 368 (2016), no. 1, 515–538.
[2] Jonathan Brundan, *Modular representations of the supergroup $Q(n)$, II*, Pacific J. Math. 224 (2006), no. 1, 65–90.
[3] Jonathan Brundan and Alexander Kleshchev, *Modular representations of the supergroup $Q(n)$, I*, J. Algebra 260 (2003), no. 1, 64–98, Special issue celebrating the 80th birthday of Robert Steinberg.
[4] Jonathan Brundan and Jonathan Kujawa, *A new proof of the Mullineux conjecture*, J. Algebraic Combin. 18 (2003), no. 1, 13–39.
[5] Shun-Jen Cheng and Weiqiang Wang, *Dualities and representations of Lie superalgebras*, Graduate Studies in Mathematics, vol. 144, American Mathematical Society, Providence, RI, 2012. MR 3012224
[6] Claude Chevalley, *The algebraic theory of spinors and Clifford algebras*, Collected works. Vol. 2, Springer-Verlag, Berlin, 1997.
[7] Rita Fioresi and Fabio Gavarini, *Chevalley supergroups*, Mem. Amer. Math. Soc. 215 (2012), no. 1014, vi+64.
[8] ______, *Algebraic supergroups with Lie superalgebras of classical type*, J. Lie Theory 23 (2013), no. 1, 143–158. MR 3060770
[9] Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003.
[10] Tadeusz Józefiak, *Semisimple superalgebras*, Lecture Notes in Math., vol. 1352, Springer, Berlin, 1988. MR 981821
[11] Victor G. Kac, *Lie superalgebras*, Advances in Math. 26 (1977), no. 1, 8–96.
[12] _, *Representations of classical Lie superalgebras*, Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977), Lecture Notes in Math., vol. 676, Springer, Berlin, 1978, pp. 597–626. MR 519631
[13] Tsit Yuen Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005.
[14] František Marko, *Irreducibility of induced supermodules for general linear supergroups*, J. Algebra 494 (2018), 92–110. MR 3723172
[15] František Marko and Alexandr N. Zubkov, *Blocks for the general linear supergroup GL(m|n)*, Transform. Groups 23 (2018), no. 1, 185–215. MR 3763946
[16] A. Masuoka and Y. Takahashi, *Geometric construction of quotients G/H in supersymmetry*, preprint arXiv: 1808.05753 (2018), [math.AG].
[17] Akira Masuoka, *The fundamental correspondences in super affine groups and super formal groups*, J. Pure Appl. Algebra 202 (2005), 284–312.
[18] ___, *Harish-Chandra pairs for algebraic affine supergroup schemes over an arbitrary field*, Transform. Groups 17 (2012), no. 4, 1085–1121.
[19] Akira Masuoka and Taiki Shibata, *Algebraic supergroups and Harish-Chandra pairs over a commutative ring*, Trans. Amer. Math. Soc. (2017), no. 369, 3443–3481.
[20] ___, *On functor points of affine supergroups*, J. Algebra (2018), no. 503, 534–572.
[21] ___, *Quotient sheaves of algebraic supergroups are superschemes*, J. Algebra 348 (2011), 135–170.
[22] Susan Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.
[23] Ian M. Musson, *Lie superalgebras and enveloping algebras*, Graduate Studies in Mathematics, vol. 131, American Mathematical Society, Providence, RI, 2012.
[24] Brian Parshall and Jian Pan Wang, *Quantum linear groups*, Mem. Amer. Math. Soc., 89 (1991), no. 4, vi+157.
[25] I. B. Penkov, *Borel-Weil-Bott theory for classical Lie supergroups*, Current problems in mathematics. Newest results, Vol. 32, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988, Translated in J. Soviet Math. 51 (1990), no. 1, 2108–2140, pp. 71–124. MR 957752
[26] Ivan Penkov and Vera Serganova, *Representations of classical Lie superalgebras of type I*, Indag. Math. (N.S.) 3 (1992), no. 4, 419–466. MR 1201236
[27] Vera Serganova, *Quasireductive supergroups*, New developments in Lie theory and its applications, Contemp. Math., vol. 544, Amer. Math. Soc., Providence, RI, 2011, pp. 141–159.
[28] Alexander Sergeev, *An analog of the classical invariant theory for Lie superalgebras. I, II*, Michigan Math. J. 49 (2001), no. 1, 113–146, 147–168. MR 1827078
[29] Bin Shu and Weiqiang Wang, *Modular representations of the ortho-symplectic supergroups*, Proc. Lond. Math. Soc. (3) 96 (2008), no. 1, 251–271.
[30] Charles Terence Clegg Wall, *Graded Brauer groups*, J. Reine Angew. Math. 213 (1963/1964), 187–199.
[31] Rainer Weissauer, *Semisimple algebraic tensor categories*, preprint arXiv: 0909.1793 (2009), [math.CT].
[32] Alexandr N. Zubkov, *Some properties of general linear supergroups and of Schur superalgebras*, Algebra Logika 45 (2006), no. 3, 257–299, 375.
[33] ___, *Some homological properties of GL(m|n) in arbitrary characteristic*, J. Algebra Appl. 15 (2016), no. 7, 1650119, 26. MR 3528547

Taiki Shibata: Department of Applied Mathematics, Okayama University of Science, 1-1 Ridai-cho Kita-ku Okayama-shi 700-0005, Japan
E-mail address: shibata@math.ous.ac.jp