Co-evolution of dynamical states and interactions in dynamic networks

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Recent studies on the structure of social, technological and biological networks have shown that they share salient features which situate them far from being completely regular or random.1,2,3 Most of the models proposed to construct these networks are grounded in a graph-theoretical approach, i.e., algorithmic methods to build graphs formed by elements (the nodes) and links that evolve according to pre-specified rules. Despite the progress made, there are still several open questions.2 An important issue to be considered among these questions is that networks are dynamical entities4 that evolve and adapt driven by the actions of the elements that form a network.

The aim of this paper is to analyze a simple setting of such adaptive and evolving network, in which there is co-evolution of the state of the elements in the nodes of the network and the interaction links defining the network. Interactions among elements are modeled with a game theoretic approach, specially algorithmic models of evolutionary game theory,10 frequently applied in social, economic and biological situations. This mathematical theory models an interaction involving two (or more) elements, each with two or more ‘strategies’ or states, such that the outcome depends on the choices of all the interacting elements. The outcome is given in the form of a ‘utility’ or payoff given to each element according to the state of the site with the greatest payoff in its neighborhood4–6. The state of each element can be (1,0) or (0,1). In each step (generation), every i-th element interacts with all other elements inside its neighborhood Vi, and accumulates a payoff Πi = ∑j∈Vi xijx∗j, depending on the chosen states x∗ and payoff matrix J = (π00 π01 π10 π11). The i-th element compares its own payoff with all j ∈ Vi and changes its state to the state of the site with the greatest payoff in {i} ∪ Vi. The plasticity of the network is introduced here as network dynamics in which existing links can be severed and replaced by new ones. We make the assumption that whether an interaction link is severed depends on the joint payoff, i.e., the total payoff by the pair of interacting elements: the interactions giving the lowest benefit will be removed.

In the remainder, for the sake of concreteness, we will address the case of the Prisoner’s Dilemma (PD) game, which has been widely used as a model displaying complex behavior emerging from the competition between cooperative and selfish behavior.4 In its simplest form, two elements may either choose to cooperate (C, xC = (1,0)), or defect (D, xd = (0,1)). If both elements choose C, each gets a payoff π00; if one defects while the other cooperates, the former gets payoff π10 > π00, while the latter gets the ‘suckers’ payoff π11 < π00; if both defect, each gets π11. Under the standard restrictions π10 + π01 < 2π00, π10 > π00 > π11 > π01, defection is the best choice in a one-shot game resulting in a Nash equilibrium where both elements defect. Following previous studies, we consider a simplified version of the game given by the interaction matrix π00 = 1, π10 = b, π11 = ε, π01 = 0, in the limit ε = 0.2

In this context the dynamical rule proposed for local

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neighborhood adaptation, *plasticity*, is defined by analyzing the joint benefit obtained by each of the possible pairwise interactions: C–C, C–D, and D–D. Thus, according to the payoff obtained the worst interaction is clearly observed in a D–D situation where both elements will be better off by searching a new partner. Given this simplistic analysis, taking into account that we are considering undirected links, and assuming that the probability to rewire a C–D interaction is much smaller than to rewire a D–D interaction, our implementation of plasticity will allow Defectors to exchange (probabilistically) a D-neighbor by another randomly chosen element.

Thus, the game is divided in three stages. (i) Each element *i* plays the PD game with the same current state with all its neighbors, and collects an aggregate payoff \( \Pi_i \). (ii) Each element *i* updates its current state by comparing its payoff with its neighbors and *imitates* the state of the wealthiest element. An element is said *satisfied* if its own payoff is the highest among its neighbors (otherwise it is unsatisfied). (iii) Unsatisfied D-elements which imitate a Defector, replace this link with probability \( p \) by a new one pointing to a randomly chosen element.

The plasticity parameter \( p \) leads to a time evolution of the local connectivity of the network leaving the average degree \( \langle k_i \rangle \) constant. The parameter \( p \) sets a timescale for the evolution of the network with respect to the state update. In general we expect \( p \ll 1 \), so that the state update evolves in a much faster timescale than the network evolution, while \( p = 1 \) represents the limit of simultaneous update of interactions and states.

We have characterized numerically the model using \( N = 10,000 \) elements, averaged over 100 different random initial conditions, with an initial population of 0.6N Cooperators randomly distributed in the network. The initial network is generated by randomly distributing \( N\langle k_i \rangle/2 \) links. A prototype value of \( \langle k_i \rangle = 8 \) was chosen in order to secure an initial large connected component. The game is played *synchronously*, i.e., elements decide their state in advance and they all play at the same time.

**Stationary states.** To characterize the macroscopic behavior of the system we introduce the fraction of cooperators at a given time, \( \rho_C(t) \). We define the order parameter \( \rho_C \) as the average over realizations of the stationary cooperators density. In the case of random mixing, *i.e.*, in the absence of an interaction network, population dynamics gives \[ \dot{\rho}_C = \rho_C^2 (1 - \rho_C)(1 - b) \]. Thus, for \( b > 1 \) the only stable solution corresponds to a fully defecting population. For fixed networks (\( p = 0 \)), a typical time evolution shows in general that the order parameter fluctuates around an average value that decreases as the incentive to defect \( b \) increases (Fig. 1). At \( b \approx 2 \) the defectors dominate the network. For fixed networks, the precise value for this transition has been studied in detail. In contrast to random mixing, *context preservation* (fixed interactions) sustains partial cooperation.

This picture changes when the elements turn on their plasticity behavior (\( p > 0 \)) [see Fig. 1]. Extensive numerical simulations show that the system either reaches a stationary configuration with \( \rho_C > 0 \) (where the states and the network do not change in time), or an absorbing state with all elements being Defectors \( \rho_C = 0 \). The *cooperative phase*—the stationary states with a large value of \( \rho_C \) is formed by a set of solutions, corresponding to different network configurations and distribution of cooperators. In Fig. 1, we characterize these states showing that \( \rho_C > 0.8 \), a value always much larger than in the non-adaptive case. Slight variations exist for different \( \langle k \rangle \gtrsim 4 \). The crucial difference is the disappearance of the behavior observed in the case \( p = 0 \) in which, increasing \( b \), the large majority of the realizations reaches a configuration with a very low fraction of cooperators. The plasticity parameter \( p \) changes the time it takes to reach the stationary state: smaller \( p \) produce longer transients.

**Network structure.** In order to understand how such
a highly cooperative structure can be sustained, we analyze the implications of the proposed dynamical rules in the network structure. Consider that element $i$ updates its state imitating the state of element $j$; we define the correspondence $l: \mathcal{N} \rightarrow \mathcal{N}$ such that $l(i) = j$. Focusing only on those links, we identify the imitation network as the sub-network composed of directed links $i \rightarrow l(i)$ (Fig. 2). A necessary and sufficient condition for a stationary state ($\rho_C > 0, p \neq 0$) is (a) there are no links between two Defectors, and (b) each C-neighbor $i$ of a Defector $\gamma$, satisfies the payoff relation:

$$\Pi_j > \Pi_i > \Pi_\gamma, \quad j = l(i)$$  \hspace{1cm} (1)

In other words, in a stationary state all defectors become satisfied interacting only with cooperators, while cooperators can be unsatisfied while imitating from other cooperators. These steady state conditions naturally imply that the element with largest payoff in a stationary configuration is a satisfied cooperator. In Fig. 2 we show a partial view (the nodes in the lower level are not shown) of an imitation network, where the nodes in a layer imitate those elements in an upper layer indicated by the directed edges. At the top of the figure lie the nodes whose action is imitated by a chain of Cooperators.

A first characterization of how the structure of the cooperative stationary network configurations changes as a function of $b$ and $p$ is obtained by measuring the normalized degree variance $\sigma_n^2 = \langle (k_i^2) - \langle k_i \rangle^2 \rangle / \langle k_i \rangle$ (Fig. 3). We find that the degree distribution departs significantly from the initial Poisson distribution ($\sigma_n^2 = 1$) only for large values of the plasticity parameter $p$. For increasing $b$ the tail of the degree distribution expands and approaches an exponential form, indicating some elements become more connected than others (hubs).

We now address the question of whether the structure generated in our dynamical model has the characteristics of a small world network. The clustering coefficient $c$ measures the fraction of neighbors of a node that are connected among them, averaged over all the nodes in the network. In our simulations we find (Fig. 4) that the clustering coefficient increases very mildly with respect to a fixed random network $c_{\text{rand}} = \langle k_i \rangle / N$. Thus, even though the average path length is similar to a random network, in order to account for the high clustering we need to introduce “local” neighborhood selection. This mechanism is easily implemented introducing a parameter $q$ that modifies step (iii), so that with probability $q$ the new neighbor is selected among the neighbors of the neighbors; otherwise with probability $1 - q$ the random neighbor is chosen. We find that, while most of our results previously discussed are qualitatively independent of $q$, the clustering coefficient reaches a very large value even for a small value of $q$. For instance, a slightly 1% ($q = 0.01$) of local neighbor selection is enough to increase $c$ a hundred times, being the clustering largest for a slow evolution of the network ($p \ll 1$). In addition, the clustering coefficient decreases slightly with system size, an indication of a decay slower than the $N^{-1}$ decay expected for random graphs. All together our results indicate that local neighbor selection is needed in order to generate a small world network.

It is worth noting that an evolutionary model based on the PD game with a more complex strategy representation also shows, in the absence of local neighbor selection, that the increase of the clustering coefficient can be related to the change of the degree distribution. In contrast with Ref. [16], we don’t observe a power law degree distribution.

### Dynamics: global avalanches and network stability.

The hierarchical structure of the network is of fundamental importance to the dynamics on the system. A closer look at the the evolution towards a stationary state indicates the presence of avalanches (Fig. 5). The transient dynamics is characterized in general by large oscillations in $\rho_C(t)$. When the payoff of any D element increases above the upper limit of Eq. (1), an avalanche towards defection is triggered. This D element will be imitated by C neighbors, and each will initiate an avalanche of replication of D state through all those elements connected by the imitation network. During the avalanche, recovery of cooperation is possible through those satisfied Cooperators, which re-build the hierarchical topology.

The description in terms of the imitation network also indicates the vulnerability of the structure to stochastic
fluctuations. Figure 5 illustrates the sensitivity of the stationary network structure to perturbations acting on the highly connected nodes, which reflects their key role in sustaining cooperation. At time $t = 500$, the most connected node is externally forced to change state from C to D, triggering an avalanche. Notice the large oscillations in $\rho_C$, reproducing the transient dynamics in which the system searches for a new stationary globally cooperative structure.

**Conclusion.**—We have addressed the general question of network formation from the perspective of co-evolution between the dynamics of the elements’ state and the interactions network. Our model of cooperation with network plasticity leads to hierarchical topologies [18], the emergence of global cascades [19, 20] and vulnerability to attacks acting on specific targets [21]. The hierarchical interaction network is reached as a stationary network starting from a random network of interactions. The network appears structured from a few highly connected elements easily identified through the imitation network. Such network has the characteristics of a small world when a mechanism of local neighbor selection is introduced in the adaptive dynamics of the network. The hierarchical structure supports a stationary highly cooperative state for general situations in which for a fixed network the system would not settle in a stationary state and in which the cooperation level would be much smaller. The stability of the network is very sensitive to changes in state of the few highly connected nodes: external perturbations acting on these nodes, trigger global avalanches leading to transient dynamics in which network completely reorganizes itself searching for a new highly cooperative stationary state. Future work should explore the robustness of these results in slightly different settings. For instance we have checked that the same qualitative results are obtained with asynchronous update regarding Figure 1, and that adding continuous noise weakens the cooperative phase by the spontaneous occurrence of avalanches. Work along these lines is in progress.

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