Semi-Anchored Multi-Step Gradient Descent Ascent Method for Structured Nonconvex-Nonconcave Composite Minimax Problems

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Abstract
Minimax problems, such as generative adversarial network, adversarial training, and fair training, are widely solved by a multi-step gradient descent ascent (MGDA) method in practice. However, its convergence guarantee is limited. In this paper, inspired by the primal-dual hybrid gradient method, we propose a new semi-anchoring (SA) technique for the MGDA method. This makes the MGDA method find a stationary point of a structured nonconvex-nonconcave composite minimax problem; its saddle-subdifferential operator satisfies the weak Minty variational inequality condition. The resulting method, named SA-MGDA, is built upon a Bregman proximal point method. We further develop its backtracking line-search version, and its non-Euclidean version for smooth adaptable functions. Numerical experiments, including a fair classification training, are provided.

Key words. minimax problem, nonconvex-nonconcave problem, multi-step gradient descent ascent method, Bregman proximal point method

AMS subject classifications. 90C47, 90C26, 65K05, 47J25, 68T05

1 Introduction
Generative adversarial network [1, 33], adversarial training [46, 56] and fair training [57, 68] involve solving a minimax problem:

$$\min_{u \in U} \max_{v \in V} \phi(u, v), \quad (1)$$

where $U \subseteq \mathbb{R}^{d_u}$ and $V \subseteq \mathbb{R}^{d_v}$. To solve (1), variants of the multi-step gradient descent ascent (MGDA) method, which consists of one gradient descent update of $u$ and multiple gradient ascent updates of $v$, are widely used in practice (see e.g., [1, 14, 18, 33, 36, 42, 46, 55, 56]). However, the MGDA method is unstable, e.g., for some standard cases such as bilinear

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problems and smooth convex-concave problems \cite{53, 88}. Therefore, this paper develops a new semi-anchoring (SA) technique that makes the MGDA find a stationary point of a structured nonconvex-nonconcave composite problem; its saddle-subdifferential operator satisfies the weak Minty variational inequality (MVI) condition in \cite{25}. The weak MVI condition is weaker than the MVI condition that has received recent attention as one of standard nonconvex-nonconcave settings in the optimization community \cite{20, 50} and the machine learning community \cite{53, 79, 88}.

The proposed method, named SA-MGDA, is built upon the Bregman proximal point (BPP) method \cite{1, 12, 26} of a monotone operator, such as the saddle-subdifferential operator of a convex-concave function. Under a more general weak MVI condition \cite{25}, we show that the worst-case rate of the BPP method (with a strongly convex and smooth Legendre function) is $O(1/k)$, in terms of the Bregman distance between the successive iterates, where $k$ denotes the number of iterations. We also show that the BPP method has a linear rate under the strong MVI condition in \cite{79, 88}.

The BPP method is a nonlinear extension of a proximal point method \cite{52, 75} via a Bregman distance \cite{13}. A specific choice of the Bregman distance in this paper leads to the SA-MGDA method. Therefore, such conceptual SA-MGDA method, requiring an exact maximization oracle on $v$, consequently has the $O(1/k)$ worst-case rate, in terms of the associated Bregman distance (and the squared subgradient norm), for the structured nonconvex-nonconcave minimax problems. Then, we show that its practical version, performing one (proximal) gradient descent update of $u$ and a finite number of "anchored" (proximal) gradient ascent steps on $v$, requires total $O(\epsilon^{-1} \log \epsilon^{-1})$ gradient steps to find an $\epsilon$-stationary point. This matches the $O(\epsilon^{-1})$ complexity of the conceptual SA-MGDA up to a logarithmic factor.

The SA-MGDA is an implementation-wise simple modification of the MGDA with an improved convergence guarantee. This is inspired by and includes the primal-dual hybrid gradient (PDHG) method \cite{15, 28}. The PDHG is an instance of a preconditioned proximal point method \cite{39} for a bilinearly-coupled minimax problem, and our result is its nonlinear extension via the BPP. Similar but different extensions of the PDHG have been recently studied in \cite{38, 83, 86}.

Our main contributions are summarized as follows.

- We study the properties of the BPP method and analyze its worst-case rate (for a strongly convex and smooth Legendre function), in terms of the Bregman distance between two successive iterates, under the weak MVI and the strong MVI conditions, in Section 4.

- Built upon Section 4, we develop a new semi-anchoring (SA) approach for the MGDA, named SA-MGDA, and provide its worst-case rates for the structured nonconvex-nonconcave composite problems, in Section 5.

- We construct a backtracking line-search version of the SA-MGDA, in Section 6 for the case where the Lipschitz constant is unavailable. We also develop a non-Euclidean version of the SA-MGDA for smooth adaptable functions \cite{11}.
2 Related work

The MGDA can be viewed as solving the following equivalent minimization problem [2, 41, 68]:

\[
\min_{u \in \mathcal{U}} \left\{ \Psi(u) := \max_{v \in \mathcal{V}} \phi(u, v) \right\}
\] (2)

by a gradient descent method on \( u \). This requires the following gradient computation:

\[
\nabla \Psi(u) = \nabla_u \phi(u, v_*(u)),
\] (3)

for \( v_*(u) := \arg \max_{v \in \mathcal{V}} \phi(u, v) \), based on Danskin’s theorem [21] (under additional conditions for a differentiability of \( \Psi \) explained soon). This involves a maximization with respect to \( v \) in (3), e.g., many number of gradient ascent updates of \( v \) given \( u \). Unfortunately, even under the smoothness assumption on \( \phi \), the function \( \Psi \) is not differentiable in general. Therefore, the analysis in [41] relies on the theory of a subgradient method [24] on \( u \), resulting in a slow rate, under assumptions that \( \phi \) is smooth and also Lipschitz continuous.

To make a function \( \Psi \) differentiable, many existing literatures either introduce assumptions on \( \phi \) or apply a smoothing technique [64] to \( \phi \) [44, 78, 87], or consider both [2, 48, 68, 69, 81]. These all further assume that \( \mathcal{V} \) is a compact set. In specific, if a smooth function \( \phi \) is further assumed to be strongly concave on \( v \) with a convex compact set \( \mathcal{V} \), the function \( \Psi \) is differentiable and one can apply a gradient descent method on \( u \), using (3) [2, 68]. In addition, in [68], the Polyak-Łojasiewicz condition, which is weaker than the strong concavity, on \( v \) was assumed for a smooth function \( \phi \) to make \( \Psi \) smooth.

In many practical cases, desirable conditions such as a strong convexity are not given. However, a smoothing technique [64] can make the function \( \Psi \) smooth, while approximating the original problem. This usually involves adding an appropriate regularization term on \( v \), such as \(-\frac{\lambda}{2}||v - \tilde{v}||^2\), in (2), where \( \lambda \) is a tuning parameter and \( \tilde{v} \) is some point. For example, for a bilinearly-coupled problem of (2), i.e., \( u \) and \( v \) are coupled only through a bilinear term \( u^\top B v \), a smoothing technique in [64] makes \( \Psi \) smooth and enables us to use a gradient descent method and its accelerated variants [63, 64]. Such regularization has been further found to be useful practically, e.g., in machine learning applications [36, 54, 76, 78]. However, tuning a regularization parameter and being unable to find an exact solution of the original problem remain problematic.

The proposed SA-MGDA method does not require assumptions that \( \phi \) is strongly concave on \( v \) for a convergence guarantee. It also does not need smoothing the function \( \phi \) to make \( \Psi \) smooth, which involves a regularization parameter tuning. The proposed semi-anchoring technique incorporates an anchor point in the variable \( v \) (hence named semi-anchoring) at each iteration, which enforces the iterates to stay close to the anchor point for stability. This originates from the implicit regularization (smoothing) nature of the Bregman proximal point method. In particular, the anchoring chooses a convex combination of the anchor point and an updated point as a next iterate. This somewhat resembles Halpern iteration [37] that uses an initial point as an anchor point throughout the entire process. Inspired by Halpern iteration, the paper [77] proposes a method called anchoring, which is recently further studied in [47, 84]. The anchoring-type techniques also appear in a conditional gradient method [29] and a version of the celebrated Nesterov’s fast gradient method in [64].

On the other hand, single-step simultaneous gradient descent ascent methods are also gaining interest. The standard version converges under a restrictive partial cocoercivity condition for a Lipschitz continuous saddle-differential operator [72]. It also works when the function is strongly concave on \( v \) [17]. However, it diverges for a simple bilinear case [54, 85]
Table 1: Comparison of the problem settings for some representative single-step and multi-step methods for structured nonconvex-nonconcave minimax problems of this paper’s interest, under various assumptions on the Lipschitz continuous saddle-differential operator $M_{\theta} = (\nabla_\theta \phi, -\nabla_\phi \phi)$ of a minimax problem $f$. Some methods handle constrained problems $f$ or composite problems $g$.

| Type       | Method       | Paper       | Monotone $(\rho = 0)$ | MVI $(\rho > 0)$ | Weak MVI $(\rho > 0)$ | Const. Comp. |
|------------|--------------|-------------|-----------------------|------------------|------------------------|--------------|
| Single-step| EG, DE       | [45, 61, 65]| ✓                     | ✓                | ✓                      | ✓            |
|            | EG, OptDE    | [20, 53, 79]| ✓                     | ✓                | ✓                      | ✓            |
|            | GRAAL        | [50]        | ✓                     | ✓                | ✓                      | ✓            |
|            | EG+          | [25]        | ✓                     | ✓                | ✓                      | ✓            |
|            | CEG+         | [70]        | ✓                     | ✓                | ✓                      | ✓            |
| Multi-step | MGDA         | [2, 41, 68] | ✓                     | ✓                | ✓                      | ✓            |
|            | SA-MGDA      | This paper  | ✓                     | ✓                | ✓                      | ✓            |

Table 1 compares the convergence guarantees of the extragradient-type methods, the MGDA and the proposed SA-MGDA, under the problem settings of this paper’s interest; the monotonicity, the MVI, and the weak MVI. Table 1 illustrates that the SA-MGDA converges under settings that the extragradient-type methods work, unlike the MGDA, which is our main contribution. Note that our contribution is not meant to compete with the extragradient type methods in terms of how fast they find a stationary point with respect to the number of gradient computations. Under the nonconvex-nonconcave setting, a preliminary work [41] suggests that the MGDA type methods might find a stationary point that can be preferred over the one found by the simultaneous methods, making the MGDA worth studying further.

3 Preliminaries

3.1 Bregman distance

This paper uses a Legendre function [73] and its associated Bregman distance [13], defined below, as a non-Euclidean proximity measure [26, 80]. These help us to better handle the nonlinear geometry of a problem.

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1. [2, 48, 68, 81] analyze the complexity of the MGDA type methods for finding an $\epsilon$-stationary point of a (non)convex-concave problem (including the monotonicity case, but excluding the (weak) MVI case), by adding a regularization term. This, however, cannot find a stationary point exactly even with an exact maximization oracle, unlike the SA-MGDA.

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Definition 3.1. A function $h : \mathbb{R}^d \to (-\infty, \infty]$ which is closed proper strictly convex and essentially smooth\footnote{A closed convex proper function $h$ is called essentially smooth if it satisfies the following three conditions: (a) $\text{int dom } h$ is nonempty; (b) $h$ is differentiable throughout $\text{int dom } h$; (c) $\lim_{x \to \infty} ||\nabla h(x)|| = +\infty$ whenever $x_1, x_2, \ldots$, is a sequence in $\text{int dom } h$ converging to a boundary point $x$ of $\text{int dom } h$.} will be called a Legendre function.

Definition 3.2. Let $h : \mathbb{R}^d \to (-\infty, \infty]$ be a Legendre function. The Bregman distance associated to $h$, denoted by $D_h : \text{dom } h \times \text{int dom } h \to \mathbb{R}_+$ is defined by

$$D_h(x, y) := h(x) - h(y) - \langle \nabla h(y), x - y \rangle.$$ 

A Bregman distance $D_h$ reduces to the Euclidean distance for $h(x) = \frac{1}{2}||x||^2$. $D_h$ is not symmetric in general, except for the case $h(x) = \frac{1}{2}||x||^2$. In addition, $D_h(x, y) \geq 0$ for all $(x, y) \in \text{dom } h \times \text{int dom } h$, and $D_h(x, y) = 0$ if and only if $x = y$ due to the strict convexity of $h$. Popular examples of $h$ are $h(x) = \sum_{i=1}^d |x_i|^p$ for $p \geq 2$, a Shannon entropy $h(x) = \sum_{i=1}^d x_i \log x_i$, dom $h = [0, \infty)^d$, and a Burg entropy $h(x) = -\sum_{i=1}^d \log x_i$, dom $h = (0, \infty)^d$.

An appropriate choice of $h$ from the above (partial) list, especially that captures the geometry of the constraint sets of the problem, has been found useful in many applications (see, e.g., \cite{3, 9, 11, 49}). This paper, however, chooses $h$ not from the above standard list. Our choice of $h$ in Section \cite{5} is inspired by that of the PDHG \cite{15, 28, 89}.

### 3.2 Composite minimax problem and weakly monotone operator

We are interested in the minimax problem in a form:

$$\min_{u \in \mathbb{R}^{d_u}} \max_{v \in \mathbb{R}^{d_v}} \{ \Phi(u, v) := f(u) + \phi(u, v) - g(v) \},$$ \hspace{1cm} (4)

which satisfies the following assumption. Let $d := d_u + d_v$.

**Assumption 1.** $f : \mathbb{R}^{d_u} \to (-\infty, \infty]$ and $g : \mathbb{R}^{d_v} \to (-\infty, \infty]$ are closed, proper and convex functions. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable and has a Lipschitz continuous gradient, i.e., there exists $L_{uu} , L_{uv} , L_{vu} , L_{vv} > 0$ such that, for all $u, \bar{u} \in \mathbb{R}^{d_u}$ and $v, \bar{v} \in \mathbb{R}^{d_v}$,

$$||\nabla_u \phi(u, v) - \nabla_u \phi(\bar{u}, v)|| \leq L_{uu} ||u - \bar{u}|| + L_{uv} ||v - \bar{v}||,$$

$$||\nabla_v \phi(u, v) - \nabla_v \phi(u, \bar{v})|| \leq L_{vu} ||u - \bar{u}|| + L_{vv} ||v - \bar{v}||.$$  

Both the gradient $\nabla \phi$ and the saddle-differential of $\phi$, denoted by $M_\phi := (\nabla_u \phi, -\nabla_v \phi)$, are $L$-Lipschitz continuous (see Appendix \cite{A}), i.e., there exists $L > 0$ such that $||M_\phi x - M_\phi y|| \leq L ||x - y||$ for all $x, y \in \mathbb{R}^d$.

Finding a first-order stationary point $x_* := (u_*, v_*) \in \mathbb{R}^d$ of (4), is equivalent to finding a zero of the following set-valued valued saddle-subdifferential operator of $\Phi$:

$$M := (\nabla_u \phi + \partial f, -\nabla_v \phi + \partial g) : \mathbb{R}^d \rightrightarrows (-\infty, \infty]^d.$$

(5)

Let $X_*(M) := \{ x_* : 0 \in M x_* \}$ be a nonempty solution set. We consider the squared subgradient norm at $x$, $\min_{x \in X_* M} ||s||^2$, as an optimality criteria, which is standard in nonconvex-nonconcave minimax problems \cite{25, 47, 70}.

Under Assumption \cite{1}, the operator $M$ \cite{3} satisfies the following weak monotone condition with $\gamma = \max\{L_{uu}, L_{vv}\}$ (see Appendix \cite{B}) and is maximal.
**Assumption 2.** For some $\gamma \geq 0$, an operator $M$ is $\gamma$-weakly monotone, i.e.,
$$\langle x - y, w - z \rangle \geq -\gamma \|x - y\|^2, \quad \forall (x, w), (y, z) \in \text{gra} M,$$
where $\text{gra} M := \{(x, w) \in \mathbb{R}^d \times \mathbb{R}^d : w \in Mx\}$ denotes the graph of $M$. Also, it is maximal, i.e., there exists no $\gamma$-weakly monotone operator that its graph properly contains $\text{gra} M$.

### 3.3 Structured nonconvex-nonconcave problem

This section considers the structured nonconvex-nonconcave conditions; the weak Minty variational inequality (MVI) condition in [25] and the strong MVI condition in [79, 88].

The MVI problem is to find $x^*$ such that
$$\langle x - x^*, w \rangle \geq 0, \quad \forall (x, w) \in \text{gra} M.$$
For a continuous $M$, a solution set of the MVI problem is a subset of $X^*(M)$, and if $M$ is monotone, they are equivalent. The MVI condition, assuming that a solution of the MVI problem exists, is studied in [20, 50], which is also studied under the name, the coherence, in [53, 79, 88]. Recently, [25] introduced the following weaker condition, named weak MVI condition.

**Assumption 3 (Weak MVI).** For some $\rho \geq 0$, there exists a solution $x^* \in X^*(M)$ such that
$$\langle x - x^*, w \rangle \geq -\rho \|w\|^2, \quad \forall (x, w) \in \text{gra} M.$$

Let $X^\rho(M)$ be the associated solution set. Assumption 3 is implied by the $-\rho$-comonotonicity [6], or equivalently the $\rho$-cohypomonotonicity [19], i.e.,
$$\langle x - y, w - z \rangle \geq -\rho \|w - z\|^2, \quad \forall (x, w), (y, z) \in \text{gra} M.$$

The comonotonicity is also implied by the $\alpha \geq 0$-interaction dominant condition in [35] (see [47, Example 1]). [22, Proposition 2] consider a (constrained) two-agent zero-sum reinforcement learning problem, called the von Neumann ratio game, that satisfies the weak MVI condition, but neither the MVI condition nor the comonotonicity condition. [70] also provides examples satisfying the weak MVI condition.

We also consider the strong MVI condition in [79, 88]. This condition is also non-monotone, and includes the $\mu$-strong pseudomonotonicity [67] (see [67] for examples). Let $S^\mu(M)$ be the associated solution set.

**Assumption 4 (Strong MVI).** For some $\mu \geq 0$, there exists a solution $x^* \in X^*(M)$ such that
$$\langle x - x^*, w \rangle \geq \mu \|x - x^*\|^2, \quad \forall (x, w) \in \text{gra} M.$$

### 4 Bregman proximal point (BPP) method

#### 4.1 The $h$-resolvent

The $h$-resolvent of a monotone operator $M$ with respect to a Legendre function $h$ is defined as [26]
$$R^h_M := (\nabla h + M)^{-1} \nabla h,$$
where we omit $M$ and $h$ in $R^h_M$, for simplicity hereafter, unless necessary. This reduces to the standard resolvent operator $(I + M)^{-1}$ for $h = \frac{1}{2} \| \cdot \|^2$, where $I$ is an identity operator. The $h$-resolvent $R$ is single-valued on its domain for a monotone operator $M$ [4] Proposition 3.8, and we extend this for a weakly monotone operator.

**Lemma 4.1.** Let $M$ satisfy Assumption [3] for some $\gamma \geq 0$, and $h$ be a $\mu_h$-strongly convex Legendre function. Then, if $\mu_h > \gamma$, the $h$-resolvent $R$ is single-valued on $\text{int dom } h$.

*Proof.* Note that $\nabla h + M = (\nabla h - \gamma I) + (M + \gamma I)$. From the condition that $\mu_h > \gamma$ and $M$ is $\gamma$-weakly monotone, it is straightforward to show that $h - \frac{\gamma}{2} \| \cdot \|^2$ is a Legendre function and $M + \gamma I$ is maximally monotone. Then [7, Corollary 2.3] shows that $\text{ran}(\nabla h + M) = \mathbb{R}^d$. This implies that $Rx$ is nonempty for all $x \in \text{int dom } h$. Assume that $y, z \in Rx$. Since $\nabla h(x) - \nabla h(y) \in M y$ and $\nabla h(x) - \nabla h(z) \in M z$, we have the following inequality:

$$-\gamma \|y - z\|^2 \leq -\langle \nabla h(y) - \nabla h(z), y - z \rangle \leq -\mu_h \|y - z\|^2.$$  

So if $\mu_h > \gamma$, the inequality implies that $y = z$. $\Box$

### 4.2 BPP under weak MVI condition

The BPP method [26] iteratively applies the $h$-resolvent as, for $k = 0, 1, \ldots$,

$$x_{k+1} = R(x_k),$$  

which converges to a zero of a monotone operator $M$. We analyze the worst-case convergence behavior of the BPP method under Assumptions [2] and [3] (or [4]). We first state the Bregman nonexpansive property of $R$ below. For $\rho = 0$ and for any Legendre function $h$, this reduces to the quasi-Bregman firmly nonexpansive property $D_h(x, Rx) \leq D_h(x_r, x) - D_h(Rx, x)$ [12, 26].

**Lemma 4.2.** Let $M$ satisfy Assumption [3] for some $\rho \geq 0$, and $h$ be an $L_h$-smooth Legendre function. Then, if $Rx$ exists, for any $x_r \in X^*_r(M)$,

$$D_h(x_r, Rx) \leq D_h(x_r, x) - (1 - \rho L_h)D_h(Rx, x).$$  

*Proof.* By the definition of $Rx$, we have $\nabla h(x) - \nabla h(Rx) \in M Rx$. Then, Assumption [3] on $M$ implies that

$$0 \leq \langle \nabla h(x) - \nabla h(Rx), Rx - x \rangle + \frac{\rho}{2} \|\nabla h(x) - \nabla h(Rx)\|^2$$  

$$= \langle \nabla h(x), Rx - x \rangle - \langle \nabla h(x), x - x \rangle + \langle \nabla h(Rx), x - Rx \rangle$$  

$$+ \frac{\rho}{2} \|\nabla h(x) - \nabla h(Rx)\|^2$$  

$$= -D_h(Rx, x) + D_h(x, x) - D_h(x, Rx) + \frac{\rho}{2} \|\nabla h(x) - \nabla h(Rx)\|^2$$  

$$\leq -D_h(Rx, x) + D_h(x, x) - D_h(x, Rx) + \rho L_h D_h(Rx, x),$$

where the last inequality follows from the convex and $L_h$-smooth properties of $h$, i.e., $\frac{1}{2L_h} \|\nabla h(x) - \nabla h(y)\|^2 \leq D_h(x, y)$ for all $x, y$ [30, Theorem 2.1.5]. $\Box$
This presents that the condition $\rho L_h \leq 1$ guarantees the quasi-Bregman nonexpansiveness $D_h(x_*, R_x) \leq D_h(x_*, x)$. We then have the following worst-case rate in terms of the best Bregman distance between two successive iterates under the weak MVI condition, and the convergence property of the iterate sequences. These built upon [7] Theorem 1 of the BPP for a monotone operator.

**Theorem 4.3.** Let $M$ satisfy Assumptions [3] and [4] for some $\gamma, \rho \geq 0$, and $h$ be a $\mu_h$-strongly convex and $L_h$-smooth Legendre function with $\mu_h > \gamma$ and $\rho L_h < 1$, respectively. Then, the sequence $\{x_k\}$ of the BPP method (6) satisfies, for $k \geq 1$ and for any $x_* \in X'_s(M)$,

$$\min_{i=1, \ldots, k} D_h(x_i, x_{i-1}) \leq \frac{D_h(x_*, x_0)}{(1 - \rho L_h)k}.$$ 

Moreover, all limit points of the sequence $\{x_k\}$ of the BPP method are in $X_s(M)$, and if we further assume that $X'_s(M) = X_s(M)$, the sequence $\{x_k\}$ converges to a solution $x_* \in X'_s(M)$.

**Proof.** By Lemma [4.1], the condition $\mu_h > \gamma$ implies that $R_x$ exists for any $x$. By Lemma [4.2] we get

$$0 \leq D_h(x_*, x_i) - D_h(x_*, x_{i+1}) - (1 - \rho L_h)D_h(x_{i+1}, x_i)$$

(7)

for all $i \geq 0$. By summing over the above inequality, we get

$$\sum_{i=1}^{k} (1 - \rho L_h)D_h(x_i, x_{i-1}) \leq \sum_{i=1}^{k} (D_h(x_*, x_{i-1}) - D_h(x_*, x_i))$$

$$= D_h(x_*, x_0) - D_h(x_*, x_k)$$

$$\leq D_h(x_*, x_0).$$

Hence, by dividing the both sides of the inequality by $(1 - \rho L_h)k$, we get

$$\min_{i=1, \ldots, k} D_h(x_i, x_{i-1}) \leq \frac{1}{k} \sum_{i=1}^{k} D_h(x_i, x_{i-1}) \leq \frac{D_h(x_*, x_0)}{(1 - \rho L_h)k}.$$ 

Next, we prove that all limit point of the sequence $\{x_k\}$ are first-order stationary points. By [7], the sequence $\{D_h(x_*, x_k)\}$ is bounded above, which implies that $\{x_k\}$ is a bounded sequence due to the strong convexity of $h$. Let $x_\infty$ be any limit point of $\{x_k\}$, and let $\{x_{k(j)}\}$ be a subsequence that converges to $x_\infty$. Note that $D_h(x_{k(j)+1}, x_\infty) \to 0$ (and $\nabla h(x_{k(j)}) - \nabla h(x_{k(j)+1}) \to 0$ by the convex and $L_h$-smooth properties of $h$, i.e., $\frac{1}{\mu_h}||\nabla h(x) - \nabla h(y)||^2 \leq D_h(x, y)$ [6] Theorem 2.1.5) as $k \to \infty$, since $\sum_{i=1}^{k} D_h(x_i, x_{i-1}) \leq \frac{D_h(x_*, x_0)}{1 - \rho L_h}$.

Then $\{x_{k(j)+1}\}$ also converges to $x_\infty$ since

$$||x_{k(j)+1} - x_\infty||^2 \leq 2||x_{k(j)} - x_\infty||^2 + 2||x_{k(j)+1} - x_{k(j)||^2$$

$$\leq 2||x_{k(j)} - x_\infty||^2 + \frac{4}{\mu_h}D_h(x_{k(j)+1}, x_{k(j)}),$$

where the last inequality uses the strong convexity of $h$, i.e., $\frac{\mu_h}{2}||x_{k(j)+1} - x_{k(j)||^2 \leq D_h(x_{k(j)+1}, x_{k(j)}).$ Since $M + \gamma I$ is maximally monotone and satisfies $\nabla h(x_{k(j)}) - \nabla h(x_{k(j)+1}) + \gamma x_{k(j)+1} \in (M + \gamma I)(x_{k(j)+1}),$ we finally have $0 \in M x_\infty$ by [3] Proposition 20.32.

Lastly, assume that $X'_s(M) = X_s(M)$. Then $x_\infty$ is in $X'_s(M)$, and since $\lim_{j \to \infty} D_h(x_\infty, x_{k(j)}) = 0$ and $\{D_h(x_\infty, x_k)\}$ is a nonincreasing sequence by (6), we get $\lim_{k \to \infty} D_h(x_\infty, x_k) = 0$. Therefore, by the strong convexity of $h$, $\{x_k\}$ converges to $x_\infty \in X'_s(M)$. 

□
The BPP has a linear rate under the strong MVI condition.

**Theorem 4.4.** Let $M$ satisfy Assumptions 2 and 4 for some $\gamma, \mu \geq 0$, and $h$ be a $\mu_h$-strongly convex and $L_h$-smooth Legendre function with $\mu_h > \gamma$. Then, for $k \geq 1$ and for any $x_\ast \in S_\mu^\ast (M)$, the sequence $\{x_k\}$ of the BPP method (6) satisfies

$$D_h(x_\ast, x_k) \leq \left( \frac{2\mu}{L_h} + 1 \right)^{-k} D_h(x_\ast, x_0).$$

**Proof.** By Lemma 4.1, the condition $\mu_h > \gamma$ implies that $Rx$ exists for any $x$. By the definition of $Rx$, we have

$$\nabla h(x) - \nabla h(Rx) \in MRx.$$  

Then, Assumption 4 on $M$ implies that

$$\mu ||x_\ast - Rx||^2 \leq \langle \nabla h(x) - \nabla h(Rx), Rx - x_\ast \rangle = -D_h(Rx, x) + D_h(x_\ast, x) - D_h(x_\ast, Rx).$$

By letting $x = x_{i-1}$ and using the $L_h$-smoothness of $h$, i.e., $D_h(x_\ast, x_i) \leq \frac{L_h}{2} ||x_\ast - x_i||^2$, we have

$$\left( \frac{2\mu}{L_h} + 1 \right) D_h(x_\ast, x_i) \leq -D_h(x_i, x_{i-1}) + D_h(x_\ast, x_{i-1}) \leq D_h(x_\ast, x_{i-1}).$$

$\square$

### 4.3 Example: primal-dual hybrid gradient method

This section considers the following bilinearly-coupled convex-concave minimax problems

$$\min_{u \in \mathbb{R}^{d_u}} \max_{v \in \mathbb{R}^{d_v}} \{ f(u) + \langle u, Bv \rangle - g(v) \},$$

which is an instance of (4). The following choice of $h$ with $\tau \in (0, \frac{1}{||B||})$:

$$h(u, v) = \frac{1}{2\tau} \left( ||u||^2 + ||v||^2 \right) - \langle u, Bv \rangle$$

yields a (linear) preconditioner [39]

$$\nabla h(u, v) = P(u, v) := \begin{bmatrix} \frac{1}{\tau} I & -B \\ -B^\top & \frac{1}{\tau} I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$  

In [39], the resulting Bregman (preconditioned) proximal point method is shown to be equivalent to the PDHG [15, 23].

$$u_{k+1} = \text{prox}_{\tau f} [u_k - \tau Bv_k],$$  

$$v_{k+1} = \text{prox}_{\tau g} \left[ v_k + \tau B^\top (2u_{k+1} - u_k) \right],$$  

for $k = 0, 1, \ldots$, where the proximal operator is defined as $\text{prox}_{\psi} := (I + \partial \psi)^{-1}$.

The PDHG is similar to the alternating (proximal) gradient descent ascent method, except that the PDHG uses the term $B^\top (2u_{k+1} - u_k)$ in the $v_{k+1}$ update, instead of $B^\top u_{k+1}$. This simple modification improves convergence, which is a one-sided variant of the extragradient-type methods [40, 45, 50, 71].
By Theorem 4.3 and a stronger nonexpansive property of $R$ associated with the PDHG, i.e., $\langle Rx - Ry, P(Rx - Ry) \rangle \leq \langle x - y, P(x - y) \rangle$ for all $x, y \in \mathbb{R}^d$, the last iterate of the PDHG satisfies $D_h(x_k, x_{k-1}) = \frac{1}{2} \langle x_k - x_{k-1}, P(x_k - x_{k-1}) \rangle \leq O(1/k)$, where $x_k := (u_k, v_k)$. In [33], the rate is improved to a fast $O(1/k^2)$ rate by a new momentum technique. A more widely-known rate of PDHG for the averaged iterate $\bar{x}_k = \frac{1}{k} \sum_{i=1}^{k} x_i$, in terms of the primal-dual gap, is studied in [10].

When $B$ is a square matrix with full rank, the PDHG has a linear rate $1 - \tau^2 \sigma^2_{\min}(B)$, similar to its ODE analysis in [83].

**Proposition 4.5.** Assume that $f = g = 0$, and $B$ is a square matrix with full rank. Let $\{x_k\}$ be generated by the PDHG. Then, for any $\epsilon > 0$, there exists $K \geq 0$ such that, for all $k \geq K$,

$$||x_k - \hat{x}_k||^2 \leq \left(\sqrt{1 - \tau^2 \sigma^2_{\min}(B)} + \epsilon\right)^{2k} ||x_0 - \hat{x}_0||^2$$

where $\hat{x}_k := (x_k, x_{k-1})$ and $\hat{x}_s := (x_s, x_s) = 0$.

**Proof.** See Appendix C.

The standard proximal point method (the BPP method with $h(x) = \frac{1}{2}||x||^2$) has a linear rate $\frac{1}{1 + \tau^2 \sigma^2_{\min}(B)}$ for bilinear problems [74], which is slower than that of the PDHG. In [59], the extragradient type method has a rate similar but slower than that of the proximal point method, implying that there is potential in further studying the PDHG type method.

The PDHG has been extended to tackle nonlinear problems [1] in [33] by generalizing the term $B^T(2u_{k+1} - u_k)$. In particular, under the setting different from this paper, the prediction method in [33] uses the term $\nabla_v \phi(u_{k+1} - u_k, v_k)$, and papers [33] [86] consider $2\nabla_v \phi(u_k, v_k) - \nabla_v \phi(u_{k-1}, v_{k-1})$ with different update ordering. Next section proposes a different extension of the PDHG method by choosing a specific nonlinear function $h$ for the BPP method, leading to the semi-implicit gradient term $2\nabla_v \phi(u_{k+1}, v_{k+1}) - \nabla_v \phi(u_k, v_k)$.

5 Semi-anchored multi-step gradient descent ascent (SA-MGDA)

5.1 Constructing SA-MGDA from BPP

Inspired by $h$ [9] of the PDHG, this section considers the following:

$$h(u, v) = \frac{1}{2\tau} (||u||^2 + ||v||^2) - \phi(u, v)$$

(10)

under Assumption 1 which is $(\frac{1}{\tau} - L)$-strongly convex when $\frac{1}{\tau} > L$. This yields the Bregman distance

$$D_h(x, y) = \phi(y) + \langle \nabla \phi(y), x - y \rangle + \frac{1}{2\tau} ||x - y||^2 - \phi(x)$$

We have that $\langle \nabla h(x) - \nabla h(y), x - y \rangle = \langle \left(\frac{1}{\tau} I - \nabla \phi\right)x - \left(\frac{1}{\tau} I - \nabla \phi\right)y, x - y \rangle = \frac{1}{\tau}||x - y||^2 - \langle \nabla \phi(x) - \nabla \phi(y), x - y \rangle \geq \frac{1}{\tau}||x - y||^2 - ||\nabla \phi(x) - \nabla \phi(y)|| ||x - y|| \geq (\frac{1}{\tau} - L) ||x - y||^2$ for all $x, y \in \mathbb{R}^d$, where the last inequality uses the $L$-Lipschitz continuity of $\nabla \phi$. 

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which is a difference between the function $\phi$ and its quadratic upper bound at $y$. This is a non-standard optimality measure in minimax optimization, but due to the smoothness and convexity of $h$, we can show that this upper bounds the squared subgradient norm that is a standard optimality measure in nonconvex-nonconcave minimax optimization [25, 47, 70].

Since $M$ in (6) is $\hat{L} := \max\{L_{uu}, L_{vv}\}$-weakly monotone, the corresponding BPP update with $h = \frac{1}{\tau}I - \nabla \phi$ is well-defined for $\frac{1}{\tau} - \hat{L} > 0$, by Lemma 4.1. The BPP update with $h$ in (10) is

$$
\begin{bmatrix}
  u_{k+1} \\
v_{k+1}
\end{bmatrix} = \left(\frac{1}{\tau}I - \nabla \phi + M\right)^{-1} \left(\frac{1}{\tau}I - \nabla \phi\right) \begin{bmatrix}
u_k \\
v_k
\end{bmatrix}.
$$

This is equivalent to the followings:

$$
\begin{align*}
\frac{1}{\tau} u_{k+1} - \nabla u\phi(u_{k+1}, v_{k+1}) + \nabla v\phi(u_{k+1}, v_{k+1}) + \partial f(u_{k+1}) &\subseteq \frac{1}{\tau} u_k - \nabla u\phi(u_k, v_k), \\
\frac{1}{\tau} v_{k+1} - \nabla u\phi(u_{k+1}, v_{k+1}) - \nabla v\phi(u_{k+1}, v_{k+1}) + \partial g(v_{k+1}) &\subseteq \frac{1}{\tau} v_k - \nabla v\phi(u_k, v_k),
\end{align*}
$$

and rewriting the above in the minimization and maximization form respectively yields the followings.

$$
\begin{align*}
u_{k+1} &= \arg \min_{u \in \mathbb{R}^d_u} \left\{ \phi(u_k, v_k) + \langle \nabla u\phi(u_k, v_k), u - u_k \rangle + \frac{1}{2\tau} \| u - u_k \|^2 + f(u) \right\}, \\
v_{k+1} &= \arg \max_{v \in \mathbb{R}^d_v} \left\{ 2\phi(u_{k+1}, v) - \phi(u_k, v_k) - \langle \nabla v\phi(u_k, v_k), v - v_k \rangle - \frac{1}{2\tau} \| v - v_k \|^2 - g(v) \right\}.
\end{align*}
$$

The minimization in $u$ can be solved by one proximal gradient update, while the maximization in $v$ is equivalent to an implicit update

$$
v_{k+1} = \text{prox}_{\tau g}[v_k + \tau(2\nabla u\phi(u_{k+1}, v_{k+1}) - \nabla v\phi(u_k, v_k))].
$$

In other words, an iterative method is required for the maximization in $v$. This maximization problem consists of a $(\frac{1}{\tau} + 2L_{uu})$-smooth and $(\frac{1}{\tau} - 2L_{vv})$-strongly concave function, and a concave but possibly nonsmooth function $-g$. We thus use a proximal gradient method for total $J$ number of (inner) iterations for the (inexact) maximization in $v$. The corresponding proximal gradient ascent steps are, for $j = 0, 1, \ldots, J - 1,$

$$
v_{k,j+1} = \arg \max_{v \in \mathbb{R}^d_v} \left\{ 2\phi(u_{k+1}, v_{k,j}) + \langle \nabla v\phi(u_{k+1}, v_{k,j}), v - v_{k,j} \rangle - L_{vv} \| v - v_{k,j} \|^2 \\
- \frac{1}{2\tau} \| v - (v_k - \tau \nabla u\phi(u_k, v_k)) \|^2 - g(v) \right\}.
$$

Note that fast proximal gradient methods [10, 11, 63] can be used for acceleration. The resulting SA-MGDA method is illustrated in Algorithm 4 for $L_{uv} > 0$. The explicit update of $v_{k+1}$ in Algorithm 4 can be derived by examining the optimality condition of (12):

$$
2\nabla v\phi(u_{k+1}, v_{k,j}) - 2L_{uv}(v - v_{k,j}) - \frac{1}{\tau}(v - (v_k - \tau \nabla u\phi(u_k, v_k))) - \partial g(v) \subseteq 0.
$$
The explicit update of $v_{k,j+1}$ in Algorithm 1 involves a convex combination of an anchor point, $v_k - \tau \nabla_v \phi(u_k, v_k)$, that only depends on the previous point $(u_k, v_k)$ and a recent point, $v_{k,j} + \frac{1}{L_w} \nabla_v \phi(u_{k+1}, v_{k,j})$, that depends on $(u_{k+1}, v_{k,j})$. Since the anchoring appears only in the variable $v$, we name this approach as semi-anchoring.

**Algorithm 1 SA-MGDA for $\tau \in \left(\frac{\rho}{1-\rho L}, \frac{1}{L+\tau}\right)$**

\[
\text{for } k = 0, 1, \ldots \text{ do} \\
u_{k+1} = \text{prox}_{\tau f}[u_k - \tau \nabla_u \phi(u_k, v_k)], \quad v_{k,0} = v_k \\
\text{for } j = 0, \ldots, J - 1 \text{ do} \\
v_{k,j+1} = \text{prox}_{\eta g}^{\left[\frac{\eta}{\tau} \left(v_k - \tau \nabla_u \phi(u_k, v_k)\right) + 2\eta L_w \left(v_{k,j} + \frac{1}{L_w} \nabla_v \phi(u_{k+1}, v_{k,j})\right)\right]} \\
v_{k+1} = v_{k,j}, \quad x_{k+1} = (u_{k+1}, v_{k+1})
\]

**5.2 SA-MGDA with $J = \infty$**

The following theorems of the SA-MGDA method with $J = \infty$ (or equivalently, with an exact maximization oracle) are byproducts of Lemma 4.1 and Theorems 4.3 and 4.4 of the BPP method, for a specific $h$ in (10) that is $\mu_h$-strongly convex and $L_h$-smooth with $\mu_h = \frac{\tau}{\tau} - L$ and $L_h = \frac{\tau}{\tau} + L$.

**Theorem 5.1.** Let $M$ [5] of the composite problem [4] satisfy Assumption 3 for some $\rho \in \left[0, \frac{1}{2L+L}\right]$, and let $f, g$ and $\phi$ satisfy Assumption 7. Then, the sequence $\{x_k\}$ of the SA-MGDA (with $J = \infty$) satisfies, for $k \geq 1$, $\tau \in \left(\frac{\rho}{1-\rho L}, \frac{1}{L+\tau}\right)$ and for any $x_0 \in X^\rho_0(M)$,

\[
\min_{i=1, \ldots, k} \min_{s_i \in M x_i} \frac{||s_i||^2}{2 \left(\frac{1}{\tau} + L\right)} \leq \min_{i=1, \ldots, k} D_h(x_i, x_{i-1}) \leq \frac{D_h(x_0, x_0)}{1 - \rho \left(\frac{1}{\tau} + L\right)} \cdot k.
\]

Moreover, all limit points of the sequence $\{x_k\}$ of the SA-MGDA method are in $X_\star(M)$, and if we further assume that $X^\rho_\star(M) = X_\star(M)$, the sequence $\{x_k\}$ converges to a solution $x_\star \in X^\rho_\star(M)$.

**Proof.** The proof follows from Theorem 4.3 with constraints $\mu_h > \hat{L}$ and $\rho L_h < 1$, yielding $\tau < \frac{1}{\hat{L}+L}$ and $\tau > \frac{\rho}{1-\rho L}$. We also need $\rho < \frac{1}{2L+L}$, so that $\tau$ exists. The first inequality follows from $\nabla h(x_{i}) - \nabla h(x_{i-1}) \in M x_i$, and the convex and $L_h$-smooth properties of $h$, i.e., $\frac{1}{\tau} ||\nabla h(x) - \nabla h(y)||^2 \leq D_h(x, y)$ [60] Theorem 2.1.5.

**Remark 5.2.** A worst-case rate of the BPP method, in terms of the function value, is studied in [61]. Under the monotone condition on $M_\delta$ with $f(u) = \delta_U(u) := \begin{cases} 0, & \text{if } u \in U, \\ \infty, & \text{otherwise}, \end{cases}$ and $g(v) = \delta_V(v)$, where $U$ and $V$ are compact sets, the BPP method with a strongly convex $h$ (and thus the SA-MGDA) satisfies [61]:

\[
\max_{v \in V} \phi(\bar{u}_k, v) - \min_{u \in U} \phi(u, \bar{v}_k) \leq \frac{\max_{x \in U \times V} D_h(x, x_0)}{k}
\]

for the averaged iterates $\bar{u}_k = \frac{1}{k} \sum_{i=1}^k u_i$ and $\bar{v}_k = \frac{1}{k} \sum_{i=1}^k v_i$. This, however, intrinsically cannot be generalized to the nonconvex-nonconcave case.
Theorem 5.3. Let $M$ of the composite problem (4) satisfy Assumption 1 for $\mu \geq 0$, and let $f, g$ and $\phi$ satisfy Assumption 1. Then, for $k \geq 1$, $\tau \in (0, \frac{1}{L + \hat{L}})$ and for any $x_* \in S^\mu_\tau(M)$, the sequence of the SA-MGDA (with $J = \infty$) satisfies

$$D_h(x_*, x_k) \leq \left(1 + \frac{2\tau \mu}{1 + \tau L}\right)^{-k} D_h(x_*, x_0).$$

Proof. The proof follows from Theorem 4.4 with $\mu_h > \hat{L}$. \hfill \Box

5.3 SA-MGDA with a finite $J$

In most of the practical cases, an exact maximization oracle (or equivalently, considering $J = \infty$) is computationally intractable. This section thus studies the convergence behavior of the SA-MGDA with a finite $J$, under an additional bounded domain assumption that includes the case in Remark 5.2.

Assumption 5. For some $\Omega \geq 0$, $\|x - y\| \leq \Omega$ for all $x, y \in \text{dom} f \times \text{dom} g$.

The SA-MGDA with a finite $J$ can be viewed as an inexact variant of the BPP method [27], which generates a point $x_{k+1}$ different from $Rx_k$ at the $k$th iteration. Therefore, the proof of the following theorem for the SA-MGDA with a finite $J$ first extends Theorem 4.3 of the (exact) BPP to its inexact variant in Appendix D.1. Then, we consequently have the following result for the SA-MGDA.

Theorem 5.4. Let $M$ of the composite problem (4) satisfy Assumption 3 for some $\rho \in \left[0, \frac{1}{2L + \hat{L}}\right)$, and let $f, g$ and $\phi$ satisfy Assumptions 1 and 5 for some $\Omega \geq 0$. Then, the sequence $\{x_k\}$ of the SA-MGDA (with a finite $J$) satisfies, for $k \geq 1$, $\tau \in \left(\rho, \frac{1}{L + \hat{L}}\right)$ and for any $x_* \in X^\rho_\tau(M)$,

$$\min_{i=1,\ldots,k} \min_{s_i \in MRx_{i-1}} \|s_i\|^2 \leq \min_{i=1,\ldots,k} D_h(Rx_{i-1}, x_{i-1}) \leq \frac{3\Omega^2 (\frac{1}{\tau} + L)}{2 (1 - \rho (\frac{1}{\tau} + L))} \exp\left(-\frac{1}{\tau} - 2L_{vv}\right) \frac{1}{k}.$$ \hfill \Box

This inequality reduces to that of Theorem 5.1 as $J \to \infty$. By Theorem 5.4, the following corollary illustrates that one can find an $\epsilon$-stationary point using total $O(\epsilon^{-1} \log \epsilon^{-1})$ gradient computations in SA-MGDA.

Corollary 5.5. Under the conditions in Theorem 5.4, the SA-MGDA method finds an $\epsilon$-stationary point, i.e., a point $x$ satisfying $\min_{s \in MRx} \|s\|^2 \leq \epsilon$, with $k = O(\epsilon^{-1})$ number of outer iterations and $J = O(\log(\epsilon^{-1}))$ number of inner iterations, requiring total $O(\epsilon^{-1} \log \epsilon^{-1})$ gradient computations.

We similarly analyze the SA-MGDA with a finite $J$ under the strong MVI condition. Note that the following theorem reduces to Theorem 5.3 as $J \to \infty$. 13
Theorem 5.6. Let $M$ of the composite problem (4) satisfy Assumption 4 for $\mu \geq 0$, and let $f, g$ and $\phi$ satisfy Assumptions 1 and 3 for some $\Omega \geq 0$. Then, for $k \geq 1$, $\tau \in (0, \frac{1}{L+\hat{L}})$ and for any $x_\ast \in S_\mu(M)$, the sequence of the SA-MGDA (with finite $J$) satisfies

$$D_h(x_\ast, x_k) \leq \left(1 + \frac{2\tau\mu}{1 + \tau L}\right)^{-k} D_h(x_\ast, x_0) + \sum_{i=1}^{k} \left(1 + \frac{2\tau\mu}{1 + \tau L}\right)^{-i+1} \frac{3\Omega^2 (\frac{1}{\tau} + L)}{2} \exp \left(-\frac{1}{2} - \frac{2\tau}{\tau^2 + 2L_{uv}} J\right).$$

Proof. See Appendix D.2.

Corollary 5.7. Under the conditions in Theorem 5.6, the SA-MGDA method achieves $D_h(x_\ast, x_k) \leq \epsilon$ with $k = O(\log(\epsilon^{-1}))$ number of outer iterations and $J = O(\log(\epsilon^{-1}))$ number of inner iterations, requiring total $O(\log^2(\epsilon^{-1}))$ gradient computations.

5.4 Comparison to proximal point method

Lemma 4.1 and Theorems 4.3 and 4.4 also apply to the standard proximal point method (with $h(x) = \frac{1}{2\tau}||x||^2$), i.e., $x_{k+1} = (I + \tau M)^{-1}x_k$. Such method, however, requires solving a regularized minimax problem,

$$(u_{k+1}, v_{k+1}) = \arg \min_{u \in \mathbb{R}^d_u} \max_{v \in \mathbb{R}^d_v} \left\{f(u) + \phi(u, v) - g(v) + \frac{1}{2\tau}||u - u_k||^2 - \frac{1}{2\tau}||v - v_k||^2\right\},$$

at each iteration, while the SA-MGDA needs one gradient descent update and a maximization at each iteration. Both proximal point methods intrinsically have an implicit regularization (smoothing), and thus have a good convergence guarantee, while the latter is preferred in terms of the computational complexity.

6 Extensions of SA-MGDA

6.1 SA-MGDA with backtracking line-search

The SA-MGDA method requires the knowledge of the global Lipschitz constants of $\phi$, such as $L$, $\hat{L}$ and $L_{uv}$, which can be locally conservative. In addition, they are usually difficult to compute in practice. To deal with these two drawbacks, we adapt a backtracking line-search technique [10, 51, 60] in Algorithm 2, which adjusts (decreases) $\tau$ at each iteration, according to the local Lipschitz constant. Note that the existence of $R_M^h(x_k)$ is guaranteed if the maximization in $v$ [11]:

$$\arg \max_{v \in \mathbb{R}^d_v} \left\{2\phi(u_{k+1}, v) - \frac{1}{2\tau}||v - (v_k - \tau\nabla_v \phi(u_k, v_k))||^2 - g(v)\right\}$$

is nonempty, including a local maximum. Regarding the computation of $R_M^h$ in Algorithm 2, the standard SA-MGDA in Algorithm 4 uses a proximal gradient ascent method on $v$ with known $L_{uv}$. Here, one can apply its backtracking version in [10], without the knowledge of $L_{uv}$. 

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Algorithm 2 SA-MGDA with backtracking line-search for $\tau_0 \in \left[ \frac{\delta}{L+\hat{L}}, \infty \right)$, $\delta \in (2\rho(L+\hat{L}),1)$

for $k = 0,1, \ldots$ do

Find the smallest nonnegative integer $i_k$ such that $\bar{x} = R_M^{h_k}(x_k)$ exists and satisfies
\[
\frac{\bar{\tau}}{4} \|\nabla \bar{h}(\bar{x}) - \nabla h(x_k)\| \leq D_k(\bar{x}, x_k), \quad \text{where } \bar{\tau} = \delta^{i_k} \tau_k \quad \text{and} \quad \bar{h} = \frac{1}{2\bar{\tau}} \|\cdot\| - \phi.
\]

Let $\tau_{k+1} = \delta^{i_k} \tau_k$ and $h_{k+1} = \frac{1}{2 \tau_{k+1}} \|\cdot\| - \phi$.

$x_{k+1} = R_M^{h_{k+1}}(x_k)$

Under an additional assumption that the sequence $\{x_k\}$ of the SA-MGDA with backtracking line-search is bounded\(^4\), the following theorem shows that it has an $O(1/k)$ worst-case convergence rate, when the backtracking parameters $\tau_0$ and $\delta$ are appropriately chosen\(^5\).

**Theorem 6.1.** Let $M$ of the composite problem \(^1\) satisfy Assumption \(^3\) for some $\rho \in [0, \frac{1}{2(L+\hat{L})})$, and let $f, g$ and $\phi$ satisfy Assumption \(^4\). Additionally, assume that $\|x_k - x_*\| \leq C$ for all $x_k$ and for some $C \geq 0$. Then, the sequence $\{x_k\}$ of the SA-MGDA with backtracking line-search for some $\tau_0 \in \left[ \frac{\delta}{L+\hat{L}}, \infty \right)$ and $\delta \in (2\rho(L+\hat{L}),1)$ satisfies, for any $x_* \in X^*_c(M)$,

\[
\min_{i=1, \ldots, k} D_{h_i}(x_i, x_{i-1}) \leq \frac{D_{h_i}(x_*, x_0) + \frac{L}{2} C^2}{(1 - \rho L)k}, \quad \text{where } \hat{L} := \frac{2(L + \hat{L})}{\delta}.
\]

**Proof.** See Appendix \(^5\) \square

6.2 Non-Euclidean SA-MGDA for smooth adaptable problem

This section relaxes the smoothness condition on $\phi$ in Assumption \(^1\) to its non-Euclidean extension, called smooth adaptable condition \(^11\).

6.2.1 Smooth adaptable problem and BPP

We consider a function $\phi$ that is smooth with respect to some Legendre function $\psi$, i.e., both functions $L \psi - \phi$ and $L \psi + \phi$ are convex. Examples of $\psi$ are $\psi(x) = -\sum_{i=1}^d \log x_i$ \(^3\) and $\psi(x) = \frac{1}{2} \|x\|^4 + \frac{1}{2} \|x\|^2$ \(^11\).

**Assumption 6.** A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable and $L$-smooth with respect to $\psi$, i.e., there exists $L > 0$ such that, for all $x, y \in \text{int dom } \psi$,

\[
|\langle \nabla \phi(x) - \nabla \phi(y), x - y \rangle| \leq L (\nabla \psi(x) - \nabla \psi(y), x - y).
\]

Under Assumption \(^6\) we consider

\[
h(u, v) = \frac{1}{\bar{\tau}} \psi(u, v) - \phi(u, v),
\]

\(^4\)This holds, for example, under Assumption \(^3\) i.e., when the domain of the problem is bounded.

\(^5\)For instance, let $\tau_0$ be large enough, and $\delta = \frac{1}{2}$. Then by Theorem 6.1, the inequality $\min_{i=1, \ldots, k} D_{h_i}(x_i, x_{i-1}) \leq \frac{D_{h_i}(x_*, x_0) + 3L D^2}{(1 - \rho L)^2 k}$ holds, for the case $\rho \in [0, \frac{1}{L^2})$, and $L \in [0, \frac{1}{4})$. Note that the region of $\rho$ becomes small for an aggressive backtracking line-search, i.e., a small $\delta$.
which is \((\frac{1}{T} - L)\)-strongly convex\(^6\) with respect to \(\psi\), when \(\frac{1}{T} > L\). We further assume that \(M\) is weakly monotone with respect to \(\psi\).

**Assumption 7.** For some \(\gamma \geq 0\), an operator \(M\) is \(\gamma\)-weakly monotone with respect to \(\psi\), i.e.,
\[
(x - y, w - z) \geq -\gamma (\nabla \psi(x) - \nabla \psi(y), x - y),
\]
for all \(x, y \in \text{int dom } \psi\) and \((x, w), (y, z) \in \text{gra } M\). Also, it is maximal, i.e., there exists no \(\gamma\)-weakly monotone operator, with respect to \(\psi\), that its graph properly contains gra \(M\).

Then, under Assumption \(^7\) the BPP update \(^6\) with \(\nabla h = \frac{1}{T} \nabla \psi - \nabla \phi\) is well-defined for \(\frac{1}{T} - L > \gamma\).

**Lemma 6.2.** Let \(M\) satisfy Assumption \(^7\) for some \(\psi\) and \(\gamma \geq 0\), and \(h\) be a Legendre function and \(\mu_h\)-strongly convex with respect to \(\psi\), i.e.,
\[
(\nabla h(x) - \nabla h(y), x - y) \geq \mu_h (\nabla \psi(x) - \nabla \psi(y), x - y),
\]
for all \(x, y \in \text{int dom } \psi\). Then, if \(\mu_h > \gamma\), the \(h\)-resolvent \(R\) is single-valued on \(\text{int dom } h\).

**Proof.** Note that \(\nabla h + M = (\nabla h - \gamma \nabla \psi) + (M + \gamma \nabla \psi)\). From the condition that \(\mu_h > \gamma\) and \(M\) is \(\gamma\)-weakly monotone with respect to \(\psi\), it is straightforward to show that \(h - \gamma \psi\) is a Legendre function, and \(M + \gamma \nabla \psi\) is maximally monotone. Then Corollary 2.3 in \([7]\) shows that \(\text{ran}(\nabla h + M) = \mathbb{R}^d\). This implies that \(Rx\) is nonempty for all \(x \in \text{int dom } h\). Assume that \(y, z \in Rx\). Since \(\nabla h(x) - \nabla h(y) \in Mw\) and \(\nabla h(x) - \nabla h(z) \in Mz\), we have the following inequality:
\[
-\gamma (\nabla \psi(y) - \nabla \psi(z), y - z) \leq - (\nabla h(y) - \nabla h(z), y - z) \leq -\mu_h (\nabla \psi(y) - \nabla \psi(z), y - z).
\]
So if \(\mu_h > \gamma\), the inequality implies that \(y = z\). \(\square\)

### 6.2.2 Non-Euclidean SA-MGDA method from BPP

The resulting BPP method for smooth adaptable problems, named the non-Euclidean SA-MGDA, uses mirror descent steps \(^6\) with respect to \(\psi\) (see also \([8, 11]\) instead of the gradient steps in the standard SA-MGDA. This simplifies, for a separable \(\psi(u, v) = \psi_u(u) + \psi_v(v)\), as below, which reduces to the standard SA-MGDA \([11]\) when \(\psi(x) = \frac{1}{2}\|x\|^2\):

\[
\begin{align*}
\mathbf{u}_{k+1} &= \arg \min_{u \in \mathbb{R}^d_u} \left\{ \phi(u_k, v_k) + \langle \nabla_u \phi(u_k, v_k), u - u_k \rangle + \frac{1}{T} D_{\psi_u}(u, u_k) + f(u) \right\} \quad (13) \\
\mathbf{v}_{k+1} &= \arg \max_{v \in \mathbb{R}^d_v} \left\{ 2\phi(u_{k+1}, v) - \phi(u_k, v_k) - \langle \nabla_v \phi(u_k, v_k), v - v_k \rangle - \frac{1}{T} D_{\psi_v}(v, v_k) - g(v) \right\} \\
&= \arg \max_{v \in \mathbb{R}^d_v} \left\{ 2\phi(u_{k+1}, v) - \frac{1}{T} D_{\psi_v}(v, \tilde{v}_k) - g(v) \right\},
\end{align*}
\]

\(^6\)We have that \((\nabla h(x) - \nabla h(y), x - y) = (\frac{1}{T} \nabla \psi - \nabla \phi)(x - (\frac{1}{T} \nabla \psi - \nabla \phi)y, x - y) = \\frac{1}{T} (\nabla \psi(x) - \nabla \psi(y), x - y) - (\nabla \phi(x) - \nabla \phi(y), x - y) \geq \frac{1}{2} (\frac{1}{T} - L)(\nabla \psi(x) - \nabla \psi(y), x - y)\), for all \(x, y \in \mathbb{R}^d\), where the inequality uses the smooth adaptable condition on \(\phi\).
where \( \hat{v}_k := \nabla \psi^*_v(\nabla \psi_v(v_k) - \tau \nabla_x \phi(u_k, v_k)) \) is an anchoring point and \( \psi^*_v \) is the conjugate of \( \psi_v \). Note that the analogous standard non-Euclidean MGDA that considers \( v_k \), instead of \( \hat{v}_k \), in [13] does not have a convergence guarantee, unlike this method under the considered setting. The minimization in \( u \) can be solved by one proximal mirror descent update (see [3, 11]), while the maximization in \( v \) needs multiple steps of a proximal mirror descent update under our underlying assumption that \( \phi(u, v) \) is relative smooth in \( v \) with respect to \( \psi_v \).

This method has a rate \( O(1/k) \) under the MVI condition, \( i.e. \), Assumption [3] with \( \rho = 0 \).

**Theorem 6.3.** Let \( M \) [4] of the composite problem [4] satisfy Assumption [3] for \( \rho = 0 \) and Assumption [7] for some \( \gamma \geq 0 \), and let \( \phi \) satisfy Assumption [6]. Then, the sequence \( \{x_k\} \) of the non-Euclidean SA-MGDA (with \( J = \infty \)) satisfies, for \( k \geq 1 \), \( \tau \in (0, \frac{1}{L+\gamma}) \) and for any \( x^* \in X^0(M), \min_{i=1,...,k} D_h(x_i, x_{i-1}) \leq \frac{D_h(x^*, x_0)}{k} \).

**Proof.** By Lemma [6.2] \( \mu_h = \frac{1}{\tau} - L > \gamma \) implies that \( Rx \) exists for any \( x \). The proof of Lemma [6.2] also works for \( \rho = 0 \) and any Legendre function \( h \), \( i.e., 0 \leq D_h(x^*, x_i) - D_h(x^*, x_{i-1}) \leq \frac{D_h(x^*, x_0)}{k} \) for all \( i \geq 0 \). Then, based on the proof of Theorem [4.3], we have \( \min_{i=1,...,k} D_h(x_i, x_{i-1}) \leq \frac{1}{k} \sum_{i=1}^k D_h(x_i, x_{i-1}) \leq \frac{D_h(x^*, x_0)}{k} \). \( \square \)

### 7 Numerical results

This section compares our SA-MGDA method with the MGDA [68] on two toy experiments and one realistic fair training experiment [7].

#### 7.1 Toy examples

We first consider

\[
\phi(u, v) = -\frac{\rho L^2}{4} u^2 + L \sqrt{1 - \frac{\rho^2 L^2}{4} uv} + \frac{\rho L^2}{4} v^2,
\]

with \( f = g = 0 \). Its saddle-subdifferential \( M_\phi \) satisfies the \( L \)-Lipschitz continuity and the \( \rho \)-weak MVI condition, where \( L = 1 \) and \( \rho = \frac{1}{2} \). The problem is nonconcave on \( v \), so directly applying MGDA is not guaranteed to work. For a practical comparison, we apply the MGDA [68] to a regularized function \( \phi(u, v) - \frac{\lambda}{2}(v - v_0)^2 \) with \( \lambda \geq L_{uv} = \frac{\rho^2 L^2}{4} = \frac{1}{5} \), so that it becomes strongly concave on \( v \). (This stems from the strategy taken in [68] when the function is concave in \( v \).) Figure [1a] illustrates that SA-MGDA (with \( \lambda = 0 \)) outperforms the MGDA [68] with \( \lambda = 1, \frac{1}{2}, \frac{1}{4} \), for \((u_0, v_0) = (0,1)\). Here, the maximization on \( v \) is computed exactly for all methods.

Next, we consider another toy example in [70, Example 3]:

\[
M_\phi(u, v) = (\psi(u, v) - v, \psi(v, u) + u),
\]

where \( \psi(u, v) = \frac{1}{8}u(-1 + u^2 + v^2)(-1 + 4u^2 + 4v^2) \), with \( f(u) = \delta_{1_1}(u) \) and \( g(v) = \delta_{1_1}(v) \). The operator \( M_\phi \) satisfies \( L \)-Lipschitz continuity (within the domain) and \( \rho \)-weak MVI condition, where \( \rho < \frac{1}{12} \).

For a practical comparison, we apply the MGDA

\[L = \frac{\sqrt{5449798437} - 173756\sqrt{8067229}}{160000} \approx 1.6251 \text{ and } \rho = \frac{2004}{16000} \approx 0.1399.\]

[7] The code is available at [https://github.com/csfh1379/sa-mgda](https://github.com/csfh1379/sa-mgda).
7.2 Fair classification

To ensure that the trained model is fair to all categories, [57] considered a minimax problem that minimizes the maximum loss among the categories. We study such fair classification experiment in [68] on the Fashion MNIST data set [82]. Similar to [57, 68], we focus on the data labeled as T-shirt/top, Coat, and Shirt. The corresponding minimax problem is

$$\min_{u} \max_{v} \mathcal{L}_i(u),$$

where $u$ denotes the parameters of the neural network (see Appendix F for the details), and $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ denote the cross-entropy losses of the training data in each category, respectively. This is equivalent to the problem

$$\min_{u} \max_{v} \sum_{i=1}^{3} v_i \mathcal{L}_i(u),$$

where $\mathcal{V} = \{v \in \mathbb{R}^3_+ : \sum_{i=1}^{3} v_i = 1\}$, i.e., $\phi(u, v) = \sum_{i=1}^{3} v_i \mathcal{L}_i(u)$ with $f = 0$ and $g(v) = \delta(v)$. Since the problem is not strongly concave in $v$, [68] applied the MGDA to a regularized problem $\min_u \max_{v \in \mathcal{V}} \sum_{i=1}^{3} v_i \mathcal{L}_i(u) - \frac{\lambda}{2} \sum_{i=1}^{3} v_i^2$, where $\lambda$ is a positive regularization parameter.

We ran 8000 iterations of the MGDA and SA-MGDA methods with the learning rate $\tau = 0.01$, where the maximization on $v$ is computed exactly. For the MGDA, we considered various regularization parameters $\lambda = 0, 0.01, 0.1, 1$. We also ran a normal training, $\min_u \sum_{i=1}^{3} \frac{1}{3} \mathcal{L}_i(u)$, using a gradient descent (GD) method on $u$ for comparison. We performed 50 independent simulations for each case, and, in Table 2, we report the mean and standard deviation of the number of correctly classified test data (out of 1000) for each category and the worst category. Fig. 4 plots the results of the worst category versus iterations.

Table 2 presents that the model learned by the SA-MGDA method has the best performance, in terms of the worst category, yielding the most fairness, even without an explicit regularization and its parameter tuning. In addition, the learned model with the SA-MGDA method has the best accuracy for two categories and the second best for one category among fair trainings.

We present an additional numerical result with smaller learning rate, $\tau = 0.001$. The other settings of experiment are equivalent to the case $\tau = 0.01$, except that we ran 40000 iterations for each case.

---

9This consists of 28×28 grayscale cloth images of ten categories; 60000 data for training and 10000 for test.

10Suppose $i$ has $n$ data $\{x_{i,j}\}_{j=1}^{n}$. Then, the corresponding cross-entropy loss is defined as $\mathcal{L}_i(u) = -\frac{1}{n} \sum_{j=1}^{n} \log f_u^{(i)}(x_{i,j})$, where $f_u^{(i)}$ is the $i$-th entry of the neural network $f_u$ that learns the probability of the data to be in the class $i$.

11The problem is highly nonconvex and nonsmooth on $u$, while it is concave on $t$. Therefore, the purpose of this experiment is to investigate whether or not our theoretical understanding expands to real-world problems.

12The worst category denotes the smallest number of correctly classified test data among the three categories.
Figure 1: Toy examples

(a) First toy example

(b) Second toy example

Figure 2: Fair classification: the number of correctly classified test data for the worst category vs. iteration. (Left) $\tau = 0.01$, (Right) $\tau = 0.001$

Table 2: Fair classification: the mean and standard deviation of the number of correctly classified test data for normal and fair trainings ($\tau = 0.01$).

| Method | $\lambda$ | T-shirt/top | Coat | Shirt | Worst |
|--------|-----------|-------------|------|-------|-------|
|        |           | mean  std   | mean std | mean std | mean std |
| Normal | GD        | 852.5 12.3  | 855.5 18.4 | 678.6 26.3 | 678.6 26.3 |
| Fair   | MGDA 0.1  | 783.4 20.3  | 781.4 33.0 | 770.3 21.2 | 753.5 16.9 |
|        | MGDA 0.01 | 778.3 28.8  | 772.0 29.4 | 773.4 25.6 | 748.5 17.7 |
|        | MGDA 0 | 777.6 27.8  | 772.0 35.3 | 772.1 21.4 | 749.5 18.3 |
|        | MGDA 1    | 812.7 16.8  | 813.0 25.6 | 743.8 21.4 | 743.2 20.6 |
|        | MGDA 0.01 | 807.9 33.2  | 822.2 29.7 | 781.7 33.1 | 766.4 23.2 |

8 Conclusion

This paper proposed a semi-anchoring approach to a multi-step gradient descent ascent method for structured nonconvex-nonconcave smooth minimax problems. This is a new
Table 3: Fair classification: the mean and standard deviation of the number of correctly classified test data for normal and fair trainings. \((\tau = 0.001)\)

| Method | \(\lambda\) | T-shirt/top | Coat | Shirt | Worst |
|--------|-------------|-------------|------|-------|-------|
|        |             | mean std    | mean std | mean std | mean std |
| Normal | GD          | 849.2 9.8   | 843.2 17.8 | 658.6 23.5 | 658.6 23.5 |
| Fair   | MGDA 0.1    | 805.8 14.3  | 797.2 20.2 | 731.0 14.8 | 731.0 14.8 |
|        | 0.01        | 772.0 15.6  | 761.3 21.1 | 767.7 14.5 | 763.0 14.7 |
|        | 0           | 775.2 14.9  | 761.4 24.0 | 767.0 13.0 | 752.9 18.9 |

instance of the Bregman proximal point method for operators under the weak MVI condition. We showed that the proposed method guarantees convergence without regularization and its parameter tuning, unlike the smoothing technique. We further studied its backtracking line-search version, and its non-Euclidean version for smooth adaptable functions. Numerical experiments suggest that the proposed approach has potential to improve training dynamics of real-world minimax problems. We leave extending this work to a more general stochastic nonconvex-nonconcave setting as future work.

A Proof of Lipschitz continuity of \(\nabla \phi\) and \(M_\phi\)

Let \(x := (u, v)\) and \(y := (\bar{u}, \bar{v})\). Since \(\|\nabla \phi(u, v) - \nabla \phi(\bar{u}, \bar{v})\| = \|M_\phi x - M_\phi y\|\), it is enough to show that there exists a constant \(L > 0\) such that \(\|\nabla \phi(u, v) - \nabla \phi(\bar{u}, \bar{v})\| \leq L\|x - y\|\).

By Assumption 1 we have the following bounds

\[
\|\nabla \phi(u, v) - \nabla \phi(\bar{u}, \bar{v})\| \leq \sqrt{L_{uu}^2 + L_{uv}^2}\|u - \bar{u}\| + \sqrt{L_{uv}^2 + L_{vv}^2}\|v - \bar{v}\|
\]

Then, we can show the Lipschitz continuity as below.

\[
\|\nabla \phi(u, v) - \nabla \phi(\bar{u}, \bar{v})\| \leq \sqrt{L_{uu}^2 + L_{uv}^2}\|u - \bar{u}\| + \sqrt{L_{uv}^2 + L_{vv}^2}\|v - \bar{v}\|
\]

\[
\leq \sqrt{L_{uu}^2 + L_{uv}^2 + L_{uv}^2 + L_{vv}^2}\|u - \bar{u}\| + \|v - \bar{v}\|^2
\]

\((:: Cauchy-Schwarz inequality)\)

\[
= \sqrt{L_{uu}^2 + L_{uv}^2 + L_{uv}^2 + L_{vv}^2}\|x - y\|
\]

\(\square\)
B Proof of weak monotonicity of $M$

By Assumption 1, $\phi(\cdot, v)$ is $L_{uv}$-weakly convex for fixed $v$, and $-\phi(u, \cdot)$ is $L_{vu}$-weakly convex for fixed $u$. Then, using the weak convexity on $u$, we have

$$\phi(u, v) \geq \phi(u, v) + \langle \nabla_u \phi(u, v), u - u \rangle - \frac{L_{uu}}{2} ||u - u||^2,$$

$$-\phi(u, v) \geq -\phi(u, v) + \langle \nabla_v \phi(u, v), v - v \rangle - \frac{L_{vu}}{2} ||v - v||^2,$$

for all $u, u \in \mathbb{R}^{d_u}$ and $v, v \in \mathbb{R}^{d_v}$. Similarly, using the weak convexity on $v$, we have

$$-\phi(u, v) \geq -\phi(u, v) - \langle \nabla_u \phi(u, v), u - u \rangle - \frac{L_{uu}}{2} ||u - u||^2,$$

$$-\phi(u, v) \geq -\phi(u, v) - \langle \nabla_v \phi(u, v), v - v \rangle - \frac{L_{vu}}{2} ||v - v||^2.$$

Let $x = (u, v), y = (u, v), w = (\nabla_u \phi(u, v), -\nabla_v \phi(u, v)), \text{ and } z = (\nabla_u \phi(u, v), -\nabla_v \phi(u, v))$. Then, summing the above four inequalities yields

$$\langle x - y, w - z \rangle \geq \langle x - y, w - z \rangle \geq -L_{uu} ||u - u||^2 - L_{vu} ||v - v||^2 \geq - \max \{L_{uu}, L_{vu} \} ||x - y||^2,$$

for all $(x, w), (y, z) \in \text{gra } M$, where the first inequality uses the convexity of $f$ and $g$. \hfill $\Box$

C Proof of Proposition 4.5

When $f = g = 0$, the sequence $\{u_k, v_k\}$ generated by the PDHG has a relationship

$$u_{k+1} - 2u_k + u_{k-1} = -\tau B(v_k - v_{k-1}) = -\tau^2 BB^T (2u_k - u_{k-1}).$$

This can be written in a matrix form

$$\begin{bmatrix} u_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} 2(I - \tau^2 BB^T) & -(I - \tau^2 BB^T) \\ I & 0 \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix}.$$ 

The spectral radius of $T_u$, denoted by $\rho(T_u)$, determines the convergence rate of the PDHG. In specific, for any $\epsilon > 0$, there exists $K \geq 0$ such that $|\rho(T_u)|^k \leq ||T_u^k|| \leq |\rho(T_u) + \epsilon|^k$ for all $k \geq K$, and this yields

$$||\hat{u}_{k+1} - \hat{u}_s||^2 \leq (\rho(T_u) + \epsilon)^{2k} ||\hat{u}_1 - \hat{u}_s||^2,$$

where $\hat{u}_k := (u_k^T u_{k-1}^T)$ and $\hat{u}_s := (u_s^T u_s^T)^T = 0$.

Considering the SVD factorization of $B = U\Sigma V^T$, the spectral radius of $T_u$ can be rewritten as

$$\rho(T_u) = \max_{\sigma_{\min}(B) \leq \sigma \leq \sigma_{\max}(B)} \rho(T_{u, \sigma}), \text{ where }$$

$$T_{u, \sigma} := \begin{bmatrix} 2(1 - \tau^2 \sigma^2) & -(1 - \tau^2 \sigma^2) \\ 1 & 0 \end{bmatrix}.$$
The matrix $T_{u,\sigma}$ has two complex eigenvalues $1 - \tau^2\sigma^2 \pm i\sqrt{1 - \tau^2\sigma^2 - (1 - \tau^2\sigma^2)^2}$ with magnitude $\sqrt{1 - \tau^2\sigma^2}$. Therefore, we have $\rho(T_u) = \sqrt{1 - \tau^2\sigma^2_{\text{min}}(B)}$ and

$$
\|\hat{u}_{k+1} - \hat{u}_*\|^2 \leq \left(\sqrt{1 - \tau^2\sigma^2_{\text{min}}(B)} + \epsilon\right)^{2k} \|\hat{u}_1 - \hat{u}_*\|^2.
$$

Similarly, the sequence $\{v_k\}$ generated by the PDHG has a relationship

$$
\begin{bmatrix}
  v_{k+1} \\
  v_k
\end{bmatrix} =
\begin{bmatrix}
  2(I - \tau^2 B^\top B) & -(I - \tau^2 B^\top B) \\
  I & 0
\end{bmatrix}
\begin{bmatrix}
  v_k \\
  v_{k-1}
\end{bmatrix},
$$

which then yields

$$
\|\hat{v}_{k+1} - \hat{v}_*\|^2 \leq \left(\sqrt{1 - \tau^2\sigma^2_{\text{min}}(B)} + \epsilon\right)^{2k} \|\hat{v}_1 - \hat{v}_*\|^2,
$$

where $\hat{v}_k := (v_k^\top v_{k-1}^\top)^\top$ and $\hat{v}_* := (v_*^\top v_*^\top)^\top = 0$. Concatenating the results for $\hat{u}_k$ and $\hat{v}_k$ concludes the proof. \qed

## D Proofs and derivations for Section 5

### D.1 Proof of Theorem 5.4

We first extend Theorem 4.3 of the (exact) BPP method to its inexact variant that approximately computes the $h$-resolvent in BPP.

**Lemma D.1.** Let $\{x_k\}$ be generated by an inexact BPP, and $x_k^* := Rx_{k-1}$ be an exactly updated point from $x_{k-1}$, where $x_k \neq x_k^*$ in general. Then, under the conditions in Theorem 4.3, the sequence $\{x_k\}$ satisfies, for $k \geq 1$ and for any $x_* \in X^*(M)$,

$$
\min_{i=1, \ldots, k} D_h(x_i^*, x_{i-1}) \leq \frac{1}{(1 - \rho L_h)k} \left( D_h(x_*, x_0) + \sum_{i=1}^{k} \left( \frac{L_h}{2} \|x_i - x_*\|^2 + L_h \|x_i^* - x_i\| \cdot \|x_* - x_i^*\| \right) \right)
$$

**Proof.** Since $\nabla h(x_{i-1}) - \nabla h(x_i^*) \in Mx_i^*$, the weak MVI condition implies

$$
0 \leq \langle \nabla h(x_{i-1}) - \nabla h(x_i^*), x_i^* - x_* \rangle + \frac{\rho}{2} \|\nabla h(x_{i-1}) - \nabla h(x_i^*)\|^2
$$

$$
= D_h(x_*, x_{i-1}) - D_h(x_*, x_i^*) - D_h(x_i^*, x_{i-1}) + \frac{\rho}{2} \|\nabla h(x_{i-1}) - \nabla h(x_i^*)\|^2
$$

$$
\leq D_h(x_*, x_{i-1}) - D_h(x_*, x_i^*) - (1 - \rho L_h)D_h(x_i^*, x_{i-1})
$$

$$
= D_h(x_*, x_{i-1}) - D_h(x_*, x_i)
$$

$$
+ (D_h(x_*, x_*) - D_h(x_*, x_i^*)) - (1 - \rho L_h)D_h(x_i^*, x_{i-1}).
$$

The term $D_h(x_*, x_i) - D_h(x_*, x_i^*)$ can be further bounded as

$$
D_h(x_*, x_i) - D_h(x_*, x_i^*) = h(x_i^*) - h(x_i) - \langle \nabla h(x_i), x_i - x_i^* \rangle + \langle \nabla h(x_i^*), x_* - x_i^* \rangle
$$

$$
= h(x_i^*) - h(x_i) - \langle \nabla h(x_i), x_i^* - x_i \rangle
$$

$$
+ \langle \nabla h(x_i^*), x_* - x_i^* \rangle
$$

$$
\leq \frac{L_h}{2} \|x_i^* - x_i\|^2 + L_h \|x_i^* - x_i\| \cdot \|x_* - x_i^*\|.
$$

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Therefore, we get
\[(1 - \rho L_h)D_h(x^*_i, x_{i-1}) \leq D_h(x_*, x_{i-1}) - D_h(x_*, x_i)\]
\[+ \frac{L_h}{2} ||x^*_i - x_i||^2 + L_h ||x^*_i - x_i|| \cdot ||x_* - x^*_i||.\]

Then the result follows directly by summing over the inequalities for all \(i = 1, \ldots, k\) and dividing both sides by \((1 - \rho L_h)k\). □

This lemma reduces to Theorem 4.3 when \(x^*_i = x_k\) for all \(k\). Under the bounded domain assumption, the inequality in Lemma D.1 is further bounded as
\[
\min_{i=1, \ldots, k} D_h(x^*_i, x_{i-1}) \leq \frac{1}{(1 - \rho L_h)k} \left( D_h(x_*, x_0) + \sum_{i=1}^k \frac{3\Omega L_h}{2} ||x^*_i - x_i|| \right).
\]

The sequence \(\{(u_i, v_i)\}_{i \geq 0}\) of SA-MGDA (with a finite \(J\)), an instance of the inexact BPP, satisfies \(u_i = u^*_i\), so \(||x^*_i - x_i|| = ||v^*_i - v_i||\). Since the function except \(g\) in the maximization problem with respect to \(v\) is \((1/2 - 2L_{vv})\)-strongly concave and \((1/2 + 2L_{vv})\)-smooth, \(J\) number of (inner) proximal gradient ascent steps satisfy \(||v^*_i - v_i|| \leq \Omega \exp\left(-\frac{\rho - 2L_{vv}}{2(1/2 + 2L_{vv})} J\right)\)
(by Theorem 10.29 of [8]) and \(L_h = \frac{1}{\tau} + L\), which concludes the proof.

### D.2 Proof of Theorem 5.6

We first extend Theorem 4.4 of the (exact) BPP method to its inexact variant.

**Lemma D.2.** Let \(\{x_k\}\) be generated by an inexact BPP, and \(x^*_k := Rx_{k-1}\) be an exactly updated point from \(x_{k-1}\), where \(x_k \neq x^*_k\) in general. Then, under the conditions in Theorem 4.4, the sequence \(\{x_k\}\) satisfies, for \(k \geq 1\) and for any \(x_* \in S^iv(M)\),
\[
D_h(x_*, x_k) \leq \left(\frac{2\mu}{L_h} + 1\right)^{-k} D_h(x_*, x_0)
+ \sum_{i=1}^k \left(\frac{2\mu}{L_h} + 1\right)^{-i+1} \left(\frac{L_h}{2} ||x^*_i - x_i||^2 + L_h ||x^*_i - x_i|| \cdot ||x_* - x^*_i||\right).
\]

**Proof.** Since \(\nabla h(x_{i-1}) - \nabla h(x^*_i) \in Mx^*_i\), the strong MVI condition implies
\[
\mu ||x_* - x^*_i||^2 \leq \langle \nabla h(x_{i-1}) - \nabla h(x^*_i), x^*_i - x_* \rangle = D_h(x_*, x_{i-1}) - D_h(x_*, x^*_i) - D_h(x^*_i, x_{i-1}).
\]

Since \(D_h(x_*, x^*_i) \leq \frac{L_h}{2} ||x_* - x^*_i||^2\), we have
\[
\left(\frac{2\mu}{L_h} + 1\right) D_h(x_*, x^*_i) \leq D_h(x_*, x_{i-1}) - D_h(x^*_i, x_{i-1}) \leq D_h(x_*, x_{i-1}).
\]

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Therefore,
\[
D_h(x_s, x_i) \leq \left(\frac{2\mu}{L_h} + 1\right)^{-1} D_h(x_s, x_{i-1}) + (D_h(x_s, x_i) - D_h(x_s, x_i^*))
\]
\[
eq \left(\frac{2\mu}{L_h} + 1\right)^{-1} D_h(x_s, x_{i-1})
\]
\[
+ (D_h(x_s, x_i) - \langle \nabla h(x_i), x_i^* - x_i \rangle)
\]
\[
\leq \left(\frac{2\mu}{L_h} + 1\right)^{-1} D_h(x_s, x_{i-1})
\]
\[
+ \left(\frac{L_h}{2} \left\| x_i^* - x_i \right\|^2 + L_h \left\| x_i^* - x_i \right\| \cdot \left\| x_s - x_i^* \right\| \right).
\]

Then the result follows directly by recursively applying the inequalities for all \(i = 1, \ldots, k\). \(\square\)

The rest of the proof is similar to the proof of Theorem 5.4 in Appendix D.1.

**E Proof of Theorem 6.1**

We first show that \(\tau_k\) is lower bounded by \(\delta \frac{L + L}{L + L^2}\) for all \(k \geq 0\). Suppose that \(\tau_k \geq \delta \frac{L + L}{L + L^2}\) and \(\tau_{k+1} := \delta \tau_k < \frac{\delta}{L + L}\) for some \(k \geq 0\). Let \(\tau_{k+1} := \delta \tau_k = \frac{\tau_{k+1}}{\delta} < \frac{1}{L + L}\) and define a function \(h'_{k+1} := \frac{1}{\tau_{k+1}} \cdot \| \cdot \|^2 - \rho\), which is convex and \(\left(\frac{1}{\tau_{k+1}} + L\right)\)-smooth. Since \(\frac{1}{\tau_{k+1}} - L > \hat{L}, x'_{k+1} := R_{\hat{h}'_{k+1}}(x_k)\) exists, by Lemma 4.1 in addition, since \(\frac{1}{\tau_{k+1}} + L < \frac{2}{\tau_{k+1}}\), the inequality \(\frac{\tau_{k+1}}{\delta} \left\| \nabla h'_{k+1}(x'_{k+1}) - \nabla h'_{k+1}(x_k) \right\|^2 \leq D_{\hat{h}'_{k+1}}(x'_{k+1}, x_k)\) holds. This contradicts the definition of \(i_k\). Therefore, \(\tau_{k+1} \geq \frac{\delta}{L + L}\) for all \(k \geq 0\).

Let \(\hat{L} := \frac{2(L + L)}{\delta}\). By the definition of \(x_{i+1}\) and the proof of Lemma 4.2 we have
\[
0 \leq -D_{h_{i+1}}(x_{i+1}, x_i) + D_{h_{i+1}}(x_s, x_i) - D_{h_{i+1}}(x_s, x_{i+1})
\]
\[
+ \frac{\rho}{2} \left\| \nabla h_{i+1}(x_i) - \nabla h_{i+1}(x_{i+1}) \right\|^2
\]
\[
\leq D_{h_{i+1}}(x_s, x_i) - D_{h_{i+1}}(x_s, x_{i+1}) - (1 - \rho \hat{L}) D_{h_{i+1}}(x_{i+1}, x_i)
\]
for all \(i \geq 0\).

By summing over the above inequality, we get
\[
\sum_{i=1}^{k} (1 - \rho \hat{L}) D_{h_i}(x_i, x_{i-1})
\]
\[
\leq \sum_{i=1}^{k} (D_{h_i}(x_s, x_{i-1}) - D_{h_i}(x_s, x_i))
\]
\[
= D_{h_k}(x_s, x_k) - D_{h_k}(x_s, x_k) + \sum_{i=1}^{k-1} (D_{h_{i+1}}(x_s, x_i) - D_{h_i}(x_s, x_i))
\]
\[
= D_{h_k}(x_s, x_k) - D_{h_k}(x_s, x_k) + \sum_{i=1}^{k-1} \left( \frac{1}{2\tau_{i+1}} - \frac{1}{2\tau_i} \right) \left\| x_s - x_i \right\|^2
\]

24
\[
\leq D_{h_1}(x^*, x_0) - D_{h_k}(x^*, x_k) + \sum_{i=1}^{k-1} \left( \frac{1}{2\tau_{i+1}} - \frac{1}{2\tau_i} \right) C^2
\]
\[
\leq D_{h_1}(x^*, x_0) - D_{h_k}(x^*, x_k) + \frac{1}{2\tau_k} C^2
\]
\[
\leq D_{h_1}(x^*, x_0) - D_{h_k}(x^*, x_k) + \frac{\tilde{L}}{4} C^2
\]
\[
\leq D_{h_1}(x^*, x_0) + D_{\phi}(x^*, x_k) + \frac{\tilde{L}}{4} C^2
\]
\[
\leq D_{h_1}(x^*, x_0) + L C^2 + \frac{\tilde{L}}{4} C^2
\]
\[
\leq D_{h_1}(x^*, x_0) + \frac{\tilde{L}}{2} C^2,
\]

which uses \( D_{h_k}(x^*, x_k) = \frac{1}{2\tau_k} \| x^* - x_k \|^2 - D_{\phi}(x^*, x_k) \). Therefore, the result in Theorem 6.1 directly follows under the condition that \( \rho \tilde{L} < 1 \), which is equivalent to \( \delta > 2 \rho (L + \tilde{L}) \). We also need \( \rho < \frac{1}{2(L+\tilde{L})} \), so that \( \delta \in (2\rho(L + \tilde{L}), 1) \) exists.

\[\Box\]

\section{Details of the structure of the neural network}

| Layer Type       | Shape       |
|------------------|-------------|
| Convolution + tanh | 3 × 3 × 5  |
| Max Pooling     | 2 × 2       |
| Convolution     | 3 × 3 × 10  |
| Max Pooling     | 2 × 2       |
| Fully Connected + tanh | 250      |
| Fully Connected + tanh | 100      |
| Softmax         | 3           |

Table 4: Details of the Structure of the Neural Network

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