On estimating the constant of simultaneous Diophantine approximation

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Abstract

The paper is devoted to the problem of estimating the constant of the best Diophantine approximations. The estimates of lower bound \( C_n \) for \( n = 5 \) and \( n = 6 \) was improved.

The first chapter gives an overview of the history of estimates of the constant of the best Diophantine approximations. In the second chapter sets out the methods and results of numerical experiments leading to estimates of \( C_n \). The third chapter is devoted to estimating some functions by means of which in the fourth chapter we will proof the estimates of \( C_n \) for \( 3 \leq n \leq 6 \).

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Introduction

Let us first formulate the problem of best simultaneous Diophantine approximations in the multidimensional case. Let

$$\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

be an arbitrary vector of real numbers. We will be interested in the approximations $\vec{\alpha}$ by rational fractions

$$\vec{p}/q = \left(\frac{p_1}{q}, \frac{p_2}{q}, \ldots, \frac{p_n}{q}\right).$$
By the Dirichlet theorem, there are infinitely many rational vectors $\vec{p}/q$ such that
\[
|\alpha_i - \frac{p_i}{q}| < q^{-\frac{n+1}{n}}, \quad i = 1, \ldots, n.
\]

As an approximation quality measure we will use the value
\[
\max_{i=1,n} q|q\alpha_i - p_i|^n
\]

**Definition 1.** The measure of the quality of the simultaneous Diophantine approximations of the first kind of the vector $\vec{\alpha}$ by the rational vector $\vec{p}/q$ is the quantity
\[
D(\vec{\alpha}, \vec{p}/q) = \max_{i=1,n} q|q\alpha_i - p_i|^n \tag{1}
\]

Then it follows from the Dirichlet theorem that there exist numbers $C$ such that the inequality
\[
\max_{i=1,n} q|q\alpha_i - p_i|^n < C \tag{2}
\]
has an infinite number of solutions in integers $q > 0, p_1, \ldots, p_n$.

**Definition 2.** The constant of the best Diophantine approximations $C(\vec{x})$ for the vector $\vec{x}$ is the exact lower bound of the quantity $C$ for which there is an infinite number of rational vectors $\vec{p}/q$ satisfying the inequality
\[
D(\vec{x}, \vec{p}/q) < C \tag{3}
\]

Thus for any positive constant $C < C(\vec{x})$ the inequality
\[
D(\vec{x}, \vec{p}/q) < C
\]
has a finite number of solutions with rational vector $\vec{p}/q$ for $C > C(\vec{x})$ — infinite number of solutions and for $C(\vec{x})$ the question of the number of solutions remains open.

From the Dirichlet theorem it immediately follows that for any vector $\vec{x}$ the constant of the best Diophantine approximations $C(\vec{x}) \leq 1$.

**Definition 3.** The constant of the best Diophantine approximations $C_n$ is the exact upper bound of the number $C(\vec{x})$ for all vectors $\vec{x}$ dimension $n$:
\[
C_n = \sup_{\vec{x} \in \mathbb{R}^n} C(\vec{x})
\]

That is $C(\vec{x})$ is the smallest positive number at which the inequality has an infinite number of solutions for all $C = C_n + \varepsilon$ ($\varepsilon > 0$) and any $\vec{x}$ . Logically the question of estimating the value $C_n$ states. Logically this problem has a rich history (see 1) which revealed a significant connection between the theory of Diophantine approximations and various sections of mathematics (for example the geometry of numbers).

There is another special important case of the best Diophantine approximations for the algebraic vector.

**Definition 4.** The constant of the best Diophantine approximations $C_n^*$ of algebraic numbers are called the exact upper bound of the number $C(\vec{x})$ for all vectors $\vec{x}$ such that together with 1 they form the basis of a totally real field of algebraic numbers of degree $n + 1$.

Such vectors $\vec{x}$ are called algebraic $\vec{x}$. 
1 History

The problem of estimating the constant of the best Diophantine approximations has a rich history. An interesting feature of this problem is the variety of methods from various sections of mathematics with which help the results on this problem were obtained – continued fractions [17], linear algebra [15], geometry of numbers [6, 8, 10].

1.1 Estimates for \( n = 1 \) and \( n = 2 \)

First results on the estimating the constant of the best Diophantine approximations were obtained in the nineteenth century. First of all this is the result obtained by Dirichlet in 1842 [11].

**Theorem 1.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and \( Q \) be arbitrary real numbers and \( Q > 1 \). Then there is an integer \( q \) such that \( 1 \leq q < Q^n \) and

\[
\max_i (\|q\alpha_i\|) \leq \frac{1}{Q}
\]

Or equivalently: there are integers \( p_1, \ldots, p_n \) and \( q \) such that \( 1 \leq q < Q^n \) and

\[
\max_i \left( \left| \alpha_i - \frac{p_i}{q} \right| \right) \leq \frac{1}{Qq} < \frac{1}{q^{1 + \frac{1}{n}}}
\]

**Proof.** See [34]. □

From this theorem it follows immediately that \( C_n \geq 1 \).

In 1891 Hurwitz [17] using the theory of continued fractions and quadratic irrationalities proved the following theorem

**Theorem 2.** The following statements are true

- For any irrational number \( \alpha \) there are an infinite number of different rational numbers \( p/q \) satisfying the inequality

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}
\]

- The statement above becomes incorrect if we replace \( \sqrt{5} \) with any number \( A > \sqrt{5} \).

**Proof.** See [34]. □

This statement leads to the first and only exact value \( C_n \). It is \( C_1 = \frac{1}{\sqrt{5}} \). This equality is achieved for numbers from a quadratic field \( \mathbb{Q}(\sqrt{5}) \).

In the future a lot of research has been done on the subject of the continued fractions multiplication by the multidimensional in particular two-dimensional case.

The first one was considered by Euler in 1775 [12]. At the end of the nineteenth century Jacobi [18] developed the first algorithm allowing to decompose an arbitrary vector into a vector continued fraction. Perron in 1907 in his masters thesis investigated this algorithm [33]. In their honor he was called the Jacobi-Perron algorithm. This algorithm was investigated well enough [4, 29, 23].

In their honor he was called the Jacobi-Perron algorithm.
Later other algorithms for cheating continued fractions [38, 39, 40] were received later. However none of received algorithms allowed us to obtain estimates for $C_2$.

For $C_2$ significant results were obtained in the mid-twentieth century. In 1927 Furtwangler [15] after receiving estimates of some determinants (view 1.2) showed that $C_2 \geq \frac{1}{\sqrt{23}}$. Later Cassels [6] using the results of Davenport [10], received an estimate $C_2 \geq \frac{2}{7}$.

Further studies on this issue were often devoted to determining the classes of numbers at which the score is reached $C_2(\alpha_1, \alpha_2) = \frac{2}{7}$. This is due to the fact that the estimation of Cassels in this matter is not constructive. For case $n = 2$ significant results were obtained in this direction. The question of the estimation $C^*_2 = C(\alpha, \beta)$, where $\alpha$ and $\beta$ cubic irrationalities [2, 3, 9, 37] was thoroughly investigated. The result of these studies was the estimation $C^*_2 = 2/7$ which is achieved for algebraic integers from a totally cubic field $\mathbb{Q}(2 \cos \frac{2\pi}{7})$ [9].

1.2 Estimation of Furtwängler

In 1927 Furtwängler [15] proved the following statement.

Theorem 3. Let $k$ be a positive number less than $1/\sqrt{|\Delta|}$ where $\Delta$ is the smallest modulo discriminant of an algebraic field of degree $n + 1$. Then for any real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ inequalities

$$q \left| q\alpha_i - p_i \right|^n < k, \quad i = 1, n$$

have an infinite number of solutions in integer numbers $p_1, p_2, \ldots, p_n, q$.

Proof. See [15]. □

From this statement it immediately follows that

$$C_n \geq 1/\sqrt{|\Delta|}. \quad (4)$$

For example, for $n = 2$ the best estimate is achieved with $\Delta = -23$ [1] (the discriminant of the cubic field generated by the equation $x^3 - x^2 - 1 = 0$) and the corresponding estimate is $C_2 \geq 1/\sqrt{23}$. For $n = 3$ the smallest modulo discriminant is 117 [1] (the discriminant of the field is generated by the equation $x^4 - x^3 - x^2 + x + 1 = 0$) and the corresponding estimate is $C_3 \geq 1/\sqrt{117}$.

1.3 Estimations of Davenport and Cassels

Consider in more detail the assessments of Davenport and Cassels.

1.3.1 Preliminary concepts from the geometry of numbers

We recall some concepts from the geometry of numbers [7].

Definition 1. Function $F(\mathbf{x})$ where $\mathbf{x} = (x_1, \ldots, x_n)$ is called it radial, if

- $F(\mathbf{x})$ is non-negative, that is $F(\mathbf{x}) \geq 0$;
- $F(\mathbf{x})$ is continuous;
• $F(\tau)$ homogeneous, that is for any $t \geq 0$, $F(t\tau) = tF(\tau)$.

**Definition 2.** Let $a_1, \ldots, a_n$ be linearly independent points of a real Euclidean space. The set of all points

$$x = u_1a_1 + \ldots + u_na_n$$

with integer coefficients $u_1, \ldots, u_n$ is called the lattice. The quantity

$$d(\Lambda) = |\det(a_1, \ldots, a_n)|$$

is called the determinant of lattice $\Lambda$.

**Definition 3.** Let $F$ - point body. If lattice $\Lambda$ does not have in $F$ points other than $\emptyset$ ($\emptyset \in F$) than $\Lambda$ is admissible for $F$ or $F$-admissible. The exact lower bound of

$$\Delta(F) = \inf d(\Lambda)$$

the determinants $d(\Lambda)$ of all $F$-admissible lattices $\Lambda$ is called the critical determinant of the set $F$. If there is no $F$-admissible lattices, then $F$ is a set of infinite type and $\Delta(F) = \infty$.

**Definition 4.** Star body is the set with the following properties

• there is a point called ”start”, which is an internal point of the set;

• any ray emerging from the ”beginning” either does not intersect the boundary of the set or has only one common point with it.

### 1.3.2 Minkowskis theorem on linear forms

The following theorem is true

**Theorem 4.** Let $\alpha_{ij}(1 \leq i, j \leq n)$ be real number with a determinant equal to $\pm 1$. Let numbers $A_1, \ldots, A_n$ be positive and $A_1A_2\ldots A_n = 1$. Then there exists an integer point $\overline{\tau} = (x_1, \ldots, x_n) \neq 0$ such that

$$|\alpha_{11}x_1 + \ldots + \alpha_{in}x_n| < A_i, \quad (1 \leq i \leq n - 1),$$

$$|\alpha_{n1}x_1 + \ldots + \alpha_{in}x_n| \leq A_n$$

**Proof.** See [7].

Directly from this theorem follows that for arbitrary real numbers $\alpha_1, \ldots, \alpha_n$ and an integer $Q$ there are $n + 1$ integers $u_0, \ldots, u_n$ simultaneously not equal to zero, that

$$|u_0\alpha_j - u_j| < Q^{-1/n}, \quad 1 \leq j \leq n,$$

$$|u_0| \leq Q$$

Or else

$$\left| \frac{\alpha_j - u_j}{u_0} \right| < \frac{1}{u_0Q^{1/n}}, \quad 1 \leq j \leq n,$$

On the other hand

$$u_0 \left( \max_{1 \leq j \leq n} |u_0\alpha_j - u_j| \right)^n < 1 \quad (5)$$

In fact, this inequality has infinitely many solutions $u_0 > 0, u_1, \ldots, u_n$. We first of all wonder what constant can replace 1 on the right side (5). Essentially, this is the constant $C_n$ [2] described above

$$\max_{i=1,n} q|q\alpha_i - p_i|^n < C_n$$
1.3.3 Preliminary reasoning

Note, that instead of minimizing expression \( \max_{1 \leq j \leq n} |u_0 \alpha_j - u_j| \) we can minimize the expression \( \sum_{j=1}^{n} (u_0 \alpha_j - u_j)^2 \) or \( \prod_{j=1}^{n} |u_0 \alpha_j - u_j| \). All these tasks can be combined in the following general problem.

**Problem.** Let \( \Phi(x_1, \ldots, x_n) \) be the radial function of \( n \) variables. What is the smallest value
\[
u_0 \Phi^n(u_0 \alpha_1 - u_1, \ldots, u_0 \alpha_n - u_n)
\]
for the different sets \( u_0 > 0 \) and \( u_1, \ldots, u_n \)?

Let
\[
D(\Phi, \alpha_1, \ldots, \alpha_n) = \lim_{u_0 \to \infty} \inf u_0 \Phi^n(u_0 \alpha_1 - u_1, \ldots, u_0 \alpha_n - u_n)
\]
and
\[
D(\Phi) = \sup_{\alpha_1, \ldots, \alpha_n} D(\Phi, \alpha_1, \ldots, \alpha_n).
\]

Our task is to estimate from above \( D(\Phi) \).

Consider the radial function
\[
F(x_0, \ldots, x_n) = (|x_0| \Phi^n(x_1 \text{ sign } x_0, \ldots, x_n \text{ sign } x_0))^{\frac{1}{n+1}}.
\]

Let
\[
\delta(F) = \sup_{\Lambda} \frac{F^{n+1}(\Lambda)}{d(\Lambda)},
\]
where the exact upper bound is taken over all \((n + 1)\)-dimensional lattices. Then \[7\]
\[
\delta(F) = \Delta \mathbb{F}^{-1},
\]
where \( \mathbb{F} \) is \((n + 1)\)-dimensional star body
\[
\mathbb{F} : F(x_0, \ldots, x_n) < 1,
\]
and \( \Delta \mathbb{F} \) its critical determinant.

Then the following theorem is true

**Theorem 5.** Let \( \Phi \) and \( F \) connected as described above. Then
\[
D(\Phi) \leq \delta(F).
\]

**Proof.** See \[7\]. □
1.3.4 Estimation of Davenport

Let
\[ \Phi(x_1, \ldots, x_n) = \max_{1 \leq i \leq n} |x_i| \]
and
\[ F^{n+1}(x_0, \ldots, x_n) = |x_0| \max_{1 \leq i \leq n} |x_i|^n. \]

Then according to the theorem 5
\[ \sup_{\alpha_1, \ldots, \alpha_n} q |q\alpha_i - p_i|^n \leq \delta(F). \]

From where
\[ C_n \geq \delta(F) = \frac{1}{\Delta F} \]
where \( F \) is \((n+1)\)-dimensional star body
\[ \mathbb{F} : F(x_0, \ldots, x_n) < 1, \]
or
\[ |x_0| \max_{1 \leq i \leq n} |x_i|^n < 1 \] (6)
and \( \Delta F \) – its critical determinant (after we will denote it \( D_n \)). This leads us to

**Theorem 6.**
\[ C_n \geq \frac{1}{D_n}. \] (7)

**Proof.** See before. \( \square \)

1.3.5 Estimation of Cassels

However, the calculation \( D_n \) in practice proved to be very difficult. Instead of directly calculating \( D_n \) Cassels\[8\] could appreciate \( D_n \) using discriminant \( d \) of the arbitrary algebraic field \( F \) of degree \( n \) extent and thus estimate lower bound of \( C_n \).

**Theorem 7.** Let
\[ f_{n,s} = \frac{1}{2^s} \prod_{i=1}^s |x_i^2 + x_{s+i}^2| \prod_{i=2s+1}^n |x_i| \] (8)

and \( 2^n V_{n,s} \) is the maximum volume of parallelepiped centered at the origin, contained inside the shape
\[ f_{n,s} \leq 1 \] (9)

Let \( \Delta_{n,s} \) the smallest absolute value of the discriminant of the real field of degree \( n + 1 \) which has \( s \) pair of complex conjugate algebraic number (i.e. \( 2s \leq n + 1 \))/ Then
\[ D_n \leq \sqrt{\Delta_{n,s}/V_{n,s}}, \] (10)
or
\[ C_n \geq V_{n,s}/\sqrt{\Delta_{n,s}}. \] (11)
Proof.  
Let $M_1, \ldots, M_{n+1}$ are $n+1$ linear forms from $n$ variables such that coefficients of $M_1$ are form a basis and the coefficients of the remaining forms are their conjugate. Let $r+2s = (n+1)$ and $M_j$ and $M_{s+j}$ $(1 \leq j \leq s)$ are complex conjugate forms and $M_{2s+1}, \ldots, M_{n+1}$ are totally real forms. Then

$$|M_1 \cdot \ldots \cdot M_{n+1}| \geq 1$$

(12)

for arbitrary real numbers not all zero. The integer lattice

$$x_{j-1} + ix_{s+j-1} = \sqrt{2}M_j \quad (1 \leq j \leq s) \quad x_{j-1} = M_j \quad (2s + 1 \leq j \leq n + 1)$$

(13)

has a determinant $d^{1/2}$ and does not contain non-zero integer points inside (6). Since

$$x_{2j-1} + x_{2s+1} = 2M_jM_{s+j} = 2|M_j|^2$$

then

$$\max(|x_j|, |x_{s+j}|) \geq |M_j| = |M_{s+j}|.$$  

From here directly follows that $D_n \leq |d|^{1/2}$ which gives an estimate of Furtwängler (4).

We can strengthen this estimate (8), using the inequality

$$2^{-s}(x_0^2 + x_s^2) \cdot \ldots \cdot (x_{2s-1}^2 + x_{2s}^2)|x_{2s} \cdot \ldots \cdot x_n| \geq 1,$$

(14)

which is obtained from (12) and (13), for any $x_0, \ldots, x_n$ not all zero. Let $2^nV$ be the largest volume of $n$-dimensional parallelepiped $|y_1| \leq 1, \ldots, |y_n| \leq 1$ lying inside the figure

$$2^{-s}(x_0^2 + x_s^2) \cdot \ldots \cdot (x_{2s-1}^2 + x_{2s-1}^2)|x_{2s} \cdot \ldots \cdot x_{n-1}| \leq 1,$$

(15)

where $y_1, \ldots, y_n$ are linear forms from $x_0, \ldots, x_{n-1}$. The determinant of lattice generated by these forms is equal to $V^{-1}$ and using homogenity the left side of (15) is always $\leq \left(\max_{1 \leq i \leq n} |y_i|\right)^n$.

From here, using (14) we get that the lattice $y_1, \ldots, y_n, x_n$ is admissible for the region (6) whence

$$D_n \leq V^{-1}|d|^{1/2}.$$  

This leads us to estimate (11)

$$C_n \geq \frac{V}{|d|^{1/2}}.$$  

The theorem is proved. □

1.4 Known results

It is easy to show that $V_{2,0} = 2$ and $V_{2,1} = 1$. And since the smallest discriminant of a purely real cubic field is 49 (for the case $x^3 + 2x^2 - x - 1 = 0$) we get the estimate $C_2 \geq \frac{4}{7}(> \frac{1}{\sqrt{23}})$. This result belongs Cassels (6).

Estimation in the case $n = 3$ belongs to Cusick (9). He showed that

$$V_{3,1} = 2; \quad V_{3,0} = \frac{3^{3/2}}{2}.$$
Since \( \Delta_{3,1} = 275, \Delta_{3,0} = 725 \) then
\[
C_3 \geq \frac{2}{\sqrt{275}} > \frac{3^{3/2}}{2\sqrt{725}}.
\]

Krass \([21, 22]\) had shown that
\[
V_{4,2} \geq \frac{16}{9}; \quad V_{4,1} \geq 2; \quad V_{4,0} \geq 4
\]

Since \([16]\) \( \Delta_{4,2} = 1609, \Delta_{4,1} = 4511, \Delta_{4,0} = 14641 \) then
\[
C_4 \geq \frac{16}{9\sqrt{1609}} > \frac{4}{\sqrt{14641}} > \frac{2}{\sqrt{4511}}.
\]

He also owns more general estimates
\[
V_{n,[n/2]} \leq V_{n,[n/2]-1} \leq \ldots \leq V_{n,0}
\]
\[
V_{n,s}V_{n',s'} \leq V_{n+n',s+s'}
\]
whence (since \( V_{4,2} \geq \frac{16}{9} \)) the Furtwängler score can be improved \(\text{(14)}\) for \( C_n(n \geq 4) \)
\[
C_n \geq \frac{V_{n,[n/2]}}{\sqrt{\Delta_{n+1}}} \geq \frac{(16/9)^{[n/4]}}{\sqrt{\Delta_{n+1}}} > \frac{1}{\sqrt{\Delta_{n+1}}}.
\]

Also Crass \([22]\) had a numerical estimate
\[
V_{5,2} \geq 2.3932 \ldots
\]

### 1.5 Other estimates of the constant of the best Diophantine approximations

Among other significant results in estimating the constant of the best Diophantine approximations we should mention the Szekers result \([36]\)
\[
C^*_n \leq C_n,
\]
where \( C^*_n \) is the constant of approximations of algebraic numbers \(\text{(1)}\). We note that the inequality holds when the poorly approximated vector \( \vec{x} \) is not algebraic. For the case \( n = 1 \), \( C^*_1 = C_1 \). This equality is achieved for numbers from a quadratic field \( \mathbb{Q}(\sqrt{5}) \).

Prior to this we considered the estimates of \( C_n \) from below. Consider as well then known results on the estimation of the constant \( C_n \) from above \(\text{(11)}\).

In 1896 Minkowski \([26]\) received an assessment
\[
C_n \leq \left(1 - \frac{1}{n}\right)^n \sim \frac{1}{e}, \quad n \to \infty.
\]
In 1914 Blichfeldt \[5\] received an improvement in Minkowskis result

\[ C_n \leq \left( 1 - \frac{1}{n} \right)^n \cdot \frac{1}{1 + \left( \frac{n-1}{n+1} \right)^{n-1}} \sim \frac{1}{e+\frac{1}{e}}, \quad n \to \infty. \]

In 1948 Mullender \[28\] developing the method of Blichfeldt and generalizing the constructions of Mordell \[27\] and Koksma-Melenbeld \[20\] received the estimate

\[ C_n \leq \frac{1}{\beta}, \quad \beta = \frac{25}{8} + o(1), \quad n \to \infty \]

The next result belongs to Spohn (1967) \[35\]; he proved that

\[ C_n \leq \frac{1}{\beta_n}, \quad \beta_n \geq n \cdot 2^{n+1} \int_0^1 \frac{u^{n-1}du}{(1+u)^n + (1+u^n)} \sim \pi, \quad n \to \infty \]

In his work Spohn \[41\] assumes that this result is the best estimate which can be obtained with the theorem of the Minkowski convex body and the Blichfeldt approach.

Later Novak \[31\] proposed a design that improves the result Spohn and add the positive value \( \epsilon_n \) on the right-hand side

\[ C_n \leq \frac{1}{\beta_n}, \quad \beta_n \geq n \cdot 2^{n+1} \int_0^1 \frac{u^{n-1}du}{(1+u)^n + (1+u^n)} + \epsilon_n \]

Estimates for \( \epsilon_n \) are obtained for example by Moshevitin \[41\].

In the case of small dimensions there are more accurate estimates. For example it is known that

\[ C_2 \leq \left( \frac{8}{13} \right)^2 \]

### 2 Preliminary estimates of the maximal parallelepipeds

#### 2.1 Preliminary reasoning

Consider the \( n \)-dimensional matrix

\[ A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad (18) \]

and

\[ A_n^{-1} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \]
Let $E$ is $n$-dimensional unit cube consisting of points
\[ \vec{e} = (e_1, e_2, \ldots, e_n), \quad 0 \leq e_i \leq 1, \quad i = 1, n. \]

Matrix $A$ transforms it into $n$-dimensional parallelepiped
\[ A : \vec{a} = A \cdot \vec{e} \] (19)

Note that in this way each $n$-dimensional parallelepiped is uniquely determined by the matrix $A$. The volume of this parallelepiped is $2^n \det A$.

Let $F_{n,s}$ is $n$-dimensional star body
\[ F_{n,s} : f_{n,s} \leq 1, \]
where $f_{n,s}$ is (8).

We are interested in whether there is some parallelepiped $A$ inside the star body $F_{n,s}$. It is possible suggest the following method of verifying this statement. We will compose the optimization problem
\[
\begin{align*}
    f_{n,s} &\to \max, \\
    |b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n| &\leq 1, \\
    |b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n| &\leq 1, \\
    \ldots \\
    |b_{n1}x_1 + b_{n2}x_2 + \ldots + b_{nn}x_n| &\leq 1.
\end{align*}
\] (20)

If the solution of the problem $\leq 1$ then the parallelepiped $A$ lies entirely inside the star body $F_{n,s}$ otherwise the part of it is outside the star body.

Thus if the parallelepiped $A$ lies inside the star body $F_{n,s}$ then score are true
\[ V_{n,s} \geq \det A. \] (21)

In further our goal is to build a matrix $A$ such that problem (20) had a solution $\max f_{n,s} \leq 1$. We will only be interested in $f_{n,\lfloor n/2 \rfloor}$ (thats why $V_{n,\lfloor n/2 \rfloor}$ appears in the assessment in estimate (17)). In what follows we will call parallelepipeds $A$ for which $\det A$ is ”large” maximal. The matrix corresponding to it will also be called maximal.

2.2 Numerical experiments

At the first stage of the investigation it was decided to conduct computational experiments on the numerical determination of the largest values of $V_{n,s}$. As a research tool was chosen a mathematical package Wolfram Mathematica.

The idea of the experiment was as follows. We will perform a directed search of the matrices $A$ (18), with the aim of finding a matrix with the largest $\det A$ satisfying the condition (20). The essence of the search is the following – to build a ”grid” of coefficients of the matrix and gradually narrow it down to the side corresponding to large values of $\det A$. Note firstly that this approach does not guarantee finding the absolute minimum (however like many numerical optimization algorithms) and secondly with the help of such an approach we can not find the absolute greatest value of $V_{n,s}$ for some specific dimension. On the other hand this approach allows us to verify whether the matrix $A$ specific
structure be interesting for further research on whether it corresponds to the largest parallelepiped $A$.

A separate issue is the verification of the admissibility of a particular parallelepiped $A$ (is $A$ inside the star body $F_{n,s}$). As mentioned above it is sufficient to solve the problem (20). In practice it turned out that the mathematical package Wolfram Mathematica not always correctly can solve this optimization problem (in our case it sometimes underestimated the value of $\max f_{n,s}$). To partially correct this situation it was decided to perform an additional check of the values of $f_{n,s}$ in the tops, on the edges and diagonals. The resulting program is given in [6].

As a result of experiments for the dimensions 3 and 4 it was found that there are a set of maximal matrices (and the corresponding parallelepipeds) with the same $\det A$. Therefore a study was conducted to get the maximal matrix $A$ with the simplest structure. It turned out that one can find the maximal matrix $A$ of the following type (for examples of such matrices see [7])

$$
A = \begin{pmatrix}
  a & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
  0 & a & \cdots & 0 & 0 & \cdots & 0 & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & \cdots & a & 0 & \cdots & 0 & 0 \\
  0 & 0 & \cdots & 0 & a_1 & a_1 & \cdots & 0 \\
  0 & 0 & \cdots & 0 & -a_1 & a_1 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & \cdots & 0 & 0 & \cdots & a_k & a_k \\
  0 & 0 & \cdots & 0 & 0 & \cdots & -a_k & a_k 
\end{pmatrix}.
$$

For us the analytical values of these matrices are more interesting than numerical ones. To find them we can proceed as follows. We can try to determine the points at which the largest parallelepiped $V_{n,[n/2]}$ concerns the star body $F_{n,[n/2]}$ write out the boundary conditions at these points and on their basis get parallelepiped parameters. For example consider the case $n = 3$. Consider the matrix

$$
A^*_3 = \begin{pmatrix}
  a & 0 & 0 \\
  0 & b & b \\
  0 & -b & b 
\end{pmatrix}.
$$

We construct the inverse problem. We choose a set of points in which $f_{3,1}$ must be $\leq 1$ (if we choose as it all points $A^*_3$ (set $\delta_{\max}$) then the matrix is guaranteed to satisfy the problem (20)). For a fixed set of points $\delta_0$ we will maximizes the value of $\det A^*_3$. If $\det A^*_3$ coincides with $\det A_3$ of the largest matrix for $n = 3$ this will mean that in verification (20) we can go from the set $\delta_{\max}$ to the set $\delta_0$. Narrowing the set $\delta_0$ to a minimum we get the boundary points in which $f_{3,1} = 1$.

Carrying out numerical experiments and starting from points with coordinates $-1, 0, 1$ we came to a set consisting of a unique point $(1, 1, 0)$ (on the unit cube, the unit cube with the help of transformation (19) is reduced to $A^*_3$). This point by applying (19) is transformed into a point $(a, b, b)$ which leads us to the problem

$$
\begin{align*}
2ab^2 & \rightarrow \max, \\
\frac{1}{2} (a^2 + b^2) b &= 1.
\end{align*}
$$

(22)
Solving this problem we get the exact value $A^*_3$.
For $n = 4$ taking as the matrix

$$A^*_4 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & b \\ 0 & 0 & -b & b \end{pmatrix}$$

we find out that it is sufficient to take two points $(1, 1, 1, 0)$ and $(1, 1, 1, 1)$. This leads us to the task

$$2a^2b^2 \to \text{max},$$
$$\frac{1}{4} (a^2 + b^2)^2 = 1,$$
$$\frac{1}{4} a^2 (a^2 + 4b^2) = 1.$$  \hspace{1cm} (23)

For $n = 5$ we take the matrix

$$A^*_5 = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & 0 \\ 0 & -c & c & 0 & 0 \\ 0 & 0 & 0 & b & b \\ 0 & 0 & 0 & -b & b \end{pmatrix}.$$

In this case more complex boundary points are obtained: $(1, 1, 1, -1, 1)$, $(1, 1, -1, 1, 1)$ and $(1, 1, 2\varphi - 1, -1, 1)$, where $\varphi = \frac{\sqrt{5} - 1}{2}$ – inverse of the golden section. Note that

$$\varphi^2 = 1 - \varphi,$$
$$\varphi^3 = 2\varphi - 1,$$
$$\varphi^4 = 2 - 3\varphi,$$
$$\varphi^5 = 5\varphi - 3,$$
$$\varphi^6 = 5 - 8\varphi,$$
$$\varphi^7 = 13\varphi - 8,$$
$$\varphi^8 = 13 - 21\varphi,$$
$$\varphi^9 = 34\varphi - 21,$$
$$\varphi^{10} = 34 - 55\varphi.$$  \hspace{1cm} (24)

The corresponding problem has the form

$$4ab^2c^2 \to \text{max},$$
$$2a^2b^2c = 1,$$
$$\frac{8}{27} (a^2 + 4b^2) c^3 = 1,$$
$$2\varphi^2b^2 (a^2 + 4\varphi^4b^2) c = 1.$$  \hspace{1cm} (25)
For $n = 6$ the matrix has the form

$$A_6^* = \begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & b & b & 0 & 0 \\
0 & 0 & -b & b & 0 & 0 \\
0 & 0 & 0 & 0 & b & b \\
0 & 0 & 0 & 0 & -b & b \\
\end{pmatrix}.$$ 

In this case we again obtain two boundary points: $(1, 1, 1, 1, 1, 1)$ and $(1, 1, 2\varphi - 1, 1, 1, 1)$. The corresponding problem has the form

$$4a^2b^4 \to \text{max},$$

$$\frac{1}{2} a^2 b^2 (a^2 + 4b^2) = 1,$$

$$\frac{1}{2} \varphi^2 b^2 (a^2 + 4b^2) (a^2 + 4\varphi^4 b^2) = 1. \quad (26)$$

### 2.3 Derivation of estimates for the maximal parallelepipeds

Let solve the problems \((22), (23), (25), (26)\) and get the exact value for the maximal matrices are $A_3, A_4, A_5, A_6$. Later (in the chapter 4) we show that the matrices found do satisfy the problem \((20)\) and lead us to the estimates of $V_{n, [n/2]}$.

#### 2.3.1 Derivation of estimates for $n = 3$

Let solve \((22)\). We have

$$h_3(a, b) = 2ab^2 \to \text{max},$$

$$\frac{1}{2} (a^2 + b^2) b = 1.$$ 

Hence

$$\frac{1}{2} (a^2 + b^2) b = 1,$$

$$a^2 + b^2 = \frac{2}{b},$$

$$a^2 = \frac{2 - b^3}{b}.$$ 

We will take into account only positive values of $a, b$ (we will do the same in other cases). This will not lead to loss of generality as we are looking for the maximum.

$$a = \sqrt{\frac{2 - b^3}{b}},$$

$$h_3(b) = 2b^2 \sqrt{\frac{2 - b^3}{b}} \to \text{max}.$$
Since $h_3(b)$ is not negative, we can go to
\[
h_3^2(b) = 4b^3 \left(2 - b^3\right) = 8b^3 - 4b^6 \to \max
\]
\[
\frac{dh_3^2(b)}{db} = 24b^2 - 24b^5 = 0,
\]
\[
b^2 \left(1 - b^3\right) = 0,
\]
\[
b = 0 \quad \text{or} \quad b = 1,
\]
\[
h_3(0) = 0 \quad \text{or} \quad h_3(1) = 2.
\]
So
\[
a = b = 1.
\]
\[
\det A_3^* = 2ab^2 = 2.
\]

2.3.2 Derivation of estimates for $n = 4$

Let solve (23). We have
\[
h_4(a, b) = 2a^2b^2 \to \max,
\]
\[
\frac{1}{4} \left(a^2 + b^2\right)^2 = 1,
\]
\[
\frac{1}{4}a^2 \left(a^2 + 4b^2\right) = 1.
\]

From 1st restriction
\[
\frac{1}{4} \left(a^2 + b^2\right)^2 = 1,
\]
\[
a^2 + b^2 = 2,
\]
\[
b = \sqrt{2 - a^2},
\]

From 2nd restriction
\[
\frac{1}{4} a^2 \left(a^2 + 4 \left(2 - a^2\right)\right) = 1,
\]
\[
a^2 \left(a^2 + 8 - 4a^2\right) = 4,
\]
\[
3a^4 - 8a^2 + 4 = 0,
\]
\[
a^2 = \frac{8 \pm 4\sqrt{3}}{6},
\]
\[
a = \sqrt{\frac{2}{3}}, \quad b = \sqrt{\frac{4}{3}} \quad \text{or} \quad a = \sqrt{2}, \quad b = 0,
\]
\[
h_4 \left(\sqrt{\frac{2}{3}}, \sqrt{\frac{4}{3}}\right) = 0 \quad \text{or} \quad h_4 \left(\sqrt{2}, 0\right) = 0.
\]
So
\[
a = \sqrt{\frac{2}{3}}, \quad b = \sqrt{\frac{4}{3}}.
\]
\[
\det A_4^* = 2a^2b^2 = 2 \cdot \frac{2}{3} \cdot \frac{4}{3} = \frac{16}{9} \approx 1.77777...
2.3.3 Derivation of estimates for \( n = 5 \)

Let solve (25). We have

\[
4ab^2c^2 ightarrow \text{max},
\]

\[2a^2b^2c = 1,\]

\[
\frac{8}{27} (a^2 + 4b^2) c^3 = 1,
\]

\[2\varphi^2 b^2 (a^2 + 4\varphi^4 b^2) c = 1.\]

From 1st restriction

\[c = \frac{1}{2a^2b^2}.\]

Substituting in the 3rd restriction

\[2\varphi^2 b^2 (a^2 + 4\varphi^4 b^2) \cdot \frac{1}{2a^2b^2} = 1,\]

\[
\varphi^2 (a^2 + 4\varphi^4 b^2) = a^2,
\]

\[4\varphi^6 b^2 = (1 - \varphi^2) a^2;\]

\[4\varphi^6 b^2 = \varphi a^2, \text{ by (24)}\]

\[b^2 = \frac{a^2}{4\varphi^5}.\]

Now solve the 2nd constraint

\[
\frac{8}{27} \left( a^2 + 4 \cdot \frac{a^2}{4\varphi^5} \right) \cdot \frac{64\varphi^{15}}{8a^6 \cdot a^6} = 1,
\]

\[64 (1 + \varphi^5) a^2 \delta^{15} = 27 \cdot \delta^5 a^{12},\]

\[a^{10} = \delta^{10} \cdot \frac{64 (1 + \varphi^5)}{27}.\]

Finally

\[a = \sqrt[10]{\frac{64\varphi^{10} (1 + \varphi^5)}{27}}, \quad b = \sqrt[10]{\frac{(1 + \varphi^5)}{432\varphi^{15}}}, \quad c = \sqrt[5]{\frac{729\varphi^5}{128 (1 + \varphi^5)^2}}.\]

\[\det A_5 = 4ab^2c^2 = 4^{10} \sqrt[10]{\frac{26\varphi^{10} (1 + \varphi^5) \cdot (1 + \varphi^5)^2 \cdot 3^{24}\varphi^{20}}{3^3 \cdot 3^628\varphi^{30} \cdot 2^{28} (1 + \varphi^5)^8}} =
\]

\[= 4^{10} \sqrt[10]{\frac{3^{15}}{2^{30} (1 + \varphi^5)^8}} = \sqrt[10]{\frac{27}{4 (1 + \varphi^5)}} \approx 2.48831...\]
2.3.4 Derivation of estimates for $n = 6$

Let solve (26). We have

$$4a^2b^4 \rightarrow \text{max},$$

$$\frac{1}{2} a^2b^2 (a^2 + 4b^2) = 1,$$

$$\frac{1}{2} \varphi^2 b^2 (a^2 + 4b^2) (a^2 + 4\varphi^4b^2) = 1.$$ 

From 1st restriction

$$\frac{1}{2} a^2b^2 (a^2 + 4b^2) = 1,$$

$$4a^2b^4 + a^4b^2 - 2 = 0,$$

$$b^2 = \frac{-a^4 \pm \sqrt{a^8 + 32a^2}}{8a^2},$$

$$b = \sqrt{\frac{\sqrt{32 + a^6} - a^3}{8a}}.$$ 

since we only consider positive roots.

Substitute the 2nd restriction

$$\frac{\sqrt{32 + a^6} - a^3}{8a} \cdot \varphi^2 \cdot \left( a^2 + 4 \cdot \frac{\sqrt{32 + a^6} - a^3}{8a} \right) \left( a^2 + 4\varphi^4 \cdot \frac{\sqrt{32 + a^6} - a^3}{8a} \right) = 2,$$

$$\left( \sqrt{32 + a^6} - a^3 \right) \varphi^2 \left( 4a^3 + 4\sqrt{32 + a^6} \right) \left( 8a^3 + 4\varphi^4 \left( \sqrt{32 + a^6} - a^3 \right) \right) = 1024a^3,$$

$$\varphi^2 \left( 32 + a^6 - a^6 \right) \left( 2a^3 + \varphi^4 \sqrt{32 + a^6} - \varphi^4 a^3 \right) = 64a^3,$$

$$\varphi^2 \left( \varphi^4 \sqrt{32 + a^6} + 3\varphi a^3 \right) = 2a^3,$$  by (24)

$$\varphi^6 \sqrt{32 + a^6} = 2a^3 - 3\varphi^3 a^3,$$

$$\varphi^6 \sqrt{32 + a^6} = (2 - 3\varphi^3) a^3,$$

$$\varphi^{12} (32 + a^6) = (4 - 12\varphi^3 + 9\varphi^6) a^6,$$

$$32\varphi^{12} = (-\varphi^{12} + 9\varphi^6 - 12\varphi^3 + 4) a^6,$$

$$a^6 = \frac{32\varphi^{12}}{-\varphi^{12} + 9\varphi^6 - 12\varphi^3 + 4} = \frac{32\varphi^{12}}{-\varphi^{12} + 9\varphi^6 - 12\varphi^3 + 4\varphi^2 + 4\varphi (\varphi^2 + \varphi)} = \frac{32\varphi^{10}}{8 - 8\varphi + 9 (2 - 3\varphi) - (34 - 55\varphi)} = \frac{32\varphi^{10}}{20\varphi - 8} = \frac{8\varphi^{10}}{8\varphi - 3 + 1} = \frac{8\varphi^{10}}{1 + \varphi^5},$$  by (24)

$$a = \sqrt[6]{\frac{8\varphi^{10}}{1 + \varphi^5}}.$$  

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We introduce the following functions

\[
\frac{b^2}{a^2} = \frac{\sqrt{32 + a^6} - a^3}{8a^3} = \frac{\sqrt{32 + \frac{8\varphi^{10}}{1 + \varphi^5}} - \sqrt{\frac{8\varphi^{10}}{1 + \varphi^5}}}{8\sqrt{\varphi^{10}}}
\]

\[
= \frac{\sqrt{32 + 32\varphi^5 + 8\varphi^{10}} - \sqrt{8\varphi^{10}}}{8\sqrt{8\varphi^{10}}} = \frac{\varphi^5 + 2 - \varphi^5}{8\varphi^5} = \frac{1}{4\varphi^5}
\]

and

\[
b^6 = a^6, \quad \frac{1}{64\varphi^{15}} = \frac{8\varphi^{10}}{64\varphi^{15}(1 + \varphi^5)} = \frac{1}{8\varphi^5(1 + \varphi^5)}
\]

Finally

\[
a = \sqrt[6]{\frac{8\varphi^{10}}{1 + \varphi^5}}, \quad b = \sqrt[6]{\frac{1}{8\varphi^5(1 + \varphi^5)}}.
\]

\[
det A_6^* = 4a^2b^4 = 4\sqrt[6]{\frac{2^6\varphi^{20}}{(1 + \varphi^5)^2}. 2^{12}\varphi^{20}(1 + \varphi^5)^2} = \frac{2}{1 + \varphi^5} \approx 1.83458...
\]

3 Estimates of some functions

We introduce the following functions

\[
F_0 = \left( \frac{1}{2} + x^2 \right) \left( \frac{1}{2} + y^2 \right),
\]

\[F_1 = (1 + x^2)|y|,
\]

\[F_2 = (t_1 + y^2)(t_2x^2 + z^2)|w|, \quad \text{where} \quad t_1 = 10\sqrt{5} - 22, \quad \text{and} \quad t_2 = \frac{26 + 10\sqrt{5}}{27}
\]

\[F_3 = (t + x^2)(t + z^2)(y^2 + w^2), \quad \text{where} \quad t = 10\sqrt{5} - 22
\]

We will subsequently need several auxiliary theorems containing estimates of these functions. In the process of their proof, we will use the following statement.

**Theorem 8** (Newton, Sylvester). Let \( f(x) \) – polynomial of degree \( n \) without multiple roots. Consider the sequence \( f_0(x), f_1(x), \ldots, f_n(x) \) where

\[f_i(x) = \frac{(n - i)!}{n!} f^{(i)}(x),\]

and consider another sequence \( F_0(x), F_1(x), \ldots, F_n(x) \) where \( F_0(x) = f(x), \ F_n(x) = f_n^2(x) \) and

\[F_i(x) = f_i^2(x) - f_{i-1}(x)f_{i+1}(x), \quad i = 1, n - 1.
\]

We will only consider the pairs \( f_i(x), f_{i+1}(x) \) such that \( \text{sign} \ F_i(x) = \text{sign} \ F_{i+1}(x) \). Let \( N_+(x) \) is the number of pairs for which \( \text{sign} \ f_i(x) = \text{sign} \ f_{i+1}(x) \) and \( N_-(x) \) is the number of pairs for which \( \text{sign} \ f_i(x) = - \text{sign} \ f_{i+1}(x) \).

Then the number of roots between \( a \) and \( b \) where \( a < b \) and \( f(a)f(b) \neq 0 \) does not exceed both \( N_+(b) - N_+(a) \) and \( N_-(b) - N_-(a) \).

**Proof.** See [42] □
### 3.1 Estimate for $F_1$

**Theorem 9.**

\[
\max F_1(x, y) = (1 + x^2)|y| = 2,
\]

\[-2 \leq x + y \leq 2, \quad -2 \leq x - y \leq 2.
\]

**Proof.**

1. Note that $F_1(x, y) = F_1(x, -y) = F_1(-x, y) = F_1(-x, -y)$. So it is nessesary to consider only the values $x \geq 0, y \geq 0$. Therefore our task will take the form

\[
F_1^* = (1 + x^2)y \to \max,
\]

\[x \geq 0, \quad y \geq 0, \quad x + y \leq 2.\]

2. Lets find unconditional extremums

\[
\frac{\partial F_1^*}{\partial x} = 2xy = 0,
\]

\[
\frac{\partial F_1^*}{\partial y} = 1 + x^2 = 0.
\]

Therefore, there are no unconditional extremums.

3. Checking the values at the borders

\[
F_1^*(0, 2) = 2, \quad F_1^*(2, 0) = 0.
\]

4. Let $x + y = 2$. Then $y = 2 - x$. Hence

\[
F_1^* = (1 + x^2)(2 - x) = -x^3 + 2x^2 - x + 2,
\]

\[
\frac{\partial F_1^*}{\partial x} = -3x^2 + 4x - 1 = 0,
\]

\[
x = 1 \quad \text{or} \quad x = \frac{1}{3},
\]

\[
F_1^*(1) = 2 \quad \text{or} \quad F_1^* \left( \frac{1}{3} \right) = \frac{50}{27} < 2.
\]

So $\max F_1^* = 2$. \[\square\]
3.2 Estimate for $F_0$

**Theorem 10.**

$$\max F_0(x, y) = \left( \frac{1}{2} + x^2 \right) \left( \frac{1}{2} + y^2 \right) = \left( \frac{3}{2} \right)^2,$$

$$-2 \leq x + y \leq 2, \quad -2 \leq x - y \leq 2.$$ 

**Proof.**

1. Let $k = \frac{1}{2}$. Then

$$F_0 = \left( \frac{1}{2} + x^2 \right) \left( \frac{1}{2} + y^2 \right) = (k + x^2) (k + y^2).$$

2. Similar to the theorem (9) we notice that $F_0(x, y) = F_0(x, -y) = F_0(-x, y) = F_0(-x, -y)$. We come to the task

$$F_0 \rightarrow \max, \quad x \geq 0, \quad y \geq 0, \quad x + y \leq 2.$$ 

3. Let's find unconditional extremums

$$\begin{align*}
\frac{\partial F_0}{\partial x} &= 2x(k + y^2) = 0, \\
\frac{\partial F_0}{\partial y} &= 2y(k + x^2) = 0
\end{align*}$$

$$\Rightarrow x = y = 0.$$

Therefore, we get the global minimum $F_0(0, 0) = k^2$.

4. Check the values at the borders

$$F_0(0, 2) = F_0(2, 0) = k(k + 4).$$

5. Let $x + y = 2$. Then $y = 2 - x$. Hence

$$F_0 = (k + x^2)(k + (2 - x)^2),$$

$$\frac{\partial F_0}{\partial x} = (2x(k + (2 - x)^2) - 2(2 - x)(k + x^2)) = 0,$$

$$xk + x(2 - x)^2 - (2 - x)k - (2 - x)x^2 = 0,$$

$$xk + 4x - 4x^2 + x^3 - 2k + xk - 2x^2 + x^3 = 0,$$

$$2x^3 - 6x^2 + (4 + 2k)x - 2k = 0,$$

$$2(x - 1)(x^2 - 2x + k) = 0,$$

$$x = 1, \quad y = 1 \quad \text{or} \quad x = 1 \pm \sqrt{1 - k}, \quad y = 1 \mp \sqrt{1 - k}.$$ 

Compute the values at these points

$$F_0(1) = (k + 1)^2.$$
\[ F_0(1 \pm \sqrt{1 - k}) = (k + (1 + \sqrt{1 - k})^2)(k + (1 + \sqrt{1 - k})^2) = \\
= (k + 1 + 1 - k - 2\sqrt{1 - k})(k + 1 + 1 - k - 2\sqrt{1 + k}) = \\
= 4(1 - \sqrt{1 - k})(1 + \sqrt{1 - k}) = 4(1 - (1 - k)) = 4k. \]

6. Combining the results obtained, we obtain
\[ \max F_0 = \max \{(k + 1)^2, k(k + 4), 4k\} = \\
= \max \left\{ \left( \frac{3}{2} \right)^2, \frac{1}{2} \left( 4 + \frac{1}{2} \right), 4 \cdot \frac{1}{2} \right\} = \left( \frac{3}{2} \right)^2 \]. \]

### 3.3 Estimate for \( F_3 \)

**Theorem 11.** \[ \max F_3(x, y, z, w) = (t + x^2)(t + z^2)(y^2 + w^2) = 64(56 - 25\sqrt{5}), \]
where \( t = 10\sqrt{5} - 22 \) and
\[-2 \leq x + y \leq 2, \quad -2 \leq x - y \leq 2, \]
\[-2 \leq z + w \leq 2, \quad -2 \leq z - w \leq 2. \]

**Proof.**
Similar to theorem 9, we notice that
\[ F_3(x, y, z, w) = F_3(x, -y, z, w) = F_3(-x, y, z, w) = F_3(-x, -y, z, w), \]
\[ F_3(x, y, z, w) = F_3(x, y, z, -w) = F_3(x, y, -z, w) = F_3(x, y, -z, -w). \]

We come to task
\[ F_3(x, y, z, w) \to \max, \]
\[ x + y \leq 2, \quad z + w \leq 2 \]
\[ x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad w \geq 0. \]

The last conditions are not boundary and necessary only for clipping points.

We will find preliminary unconditional extremums
\[ \begin{align*}
\frac{\partial F_3}{\partial x} &= 2x(t + z^2)(y^2 + w^2) = 0, \\
\frac{\partial F_3}{\partial y} &= 2y(t + x^2)(t + z^2) = 0, \\
\frac{\partial F_3}{\partial z} &= 2z(t + z^2)(y^2 + w^2) = 0, \\
\frac{\partial F_3}{\partial w} &= 2w(t + x^2)(t + z^2) = 0
\end{align*} \]

\( \Rightarrow x = y = z = w = 0 \)

Hence, we obtain a global minimum of \( F_3(0, 0, 0, 0) = 0. \)
3.3.1 Boundary $x = 0, y = 2$

Check the values on the boundary $x = 0, y = 2$. Then

$$F_3^*(z, w) = t(t + z^2)(w^2 + 4) \to \text{max}$$

under the condition

$$z \geq 0, \quad w \geq 0, \quad z + w \leq 2.$$

1. Unconditional extremum

$$\begin{cases}
\frac{\partial F_3^*}{\partial z} = 2zt(4 + w^2) = 0, \\
\frac{\partial F_3^*}{\partial w} = 2wt(t + z^2) = 0
\end{cases} \Rightarrow z = w = 0.$$

Therefore we get the global minimum of $F_3^*(0, 0) = 4t^2$.

2. Check the values at the boundaries

$$F_3^*(0, 2) = 8t^2, \quad F_3^*(2, 0) = 4t(t + 4).$$

3. Let $z + w = 2$. Then $w = 2 - z$. Hence

$$F_3^* = t(t + z^2)((2 - z)^2 + 4) = t(t + z^2)(z^2 - 4z + 8),$$

$$\frac{\partial F_3^*}{\partial z} = t(2z(z^2 - 4z + 8) + (2z - 4)(t + z^2)) = 0,$$

$$2z^3 - 8z^2 + 16z + 2tz + 2z^3 - 4t - 4z^2 = 0,$$

$$2z^3 - 6z^2 + (8 + t)z - 2t = 0.$$

This equation has a root in the interval $(0, 2)$. However the second derivative

$$\frac{\partial^2 F_3^*}{\partial x^2} = t(6z^2 - 16z + 16 + 2t + 6z^2 - 8z) = 2t(6z^2 - 12z + 8 + t) = 2t(6(z - 1)^2 + 2 + t)$$

is strictly positive. Therefore this is the local minimum point.

As a result

$$\max F_3(0, 2, z, w) = \max \left\{ \begin{array}{ll} 8t^2, \\ 4t(t + 4) \end{array} \right. = \max \left\{ \begin{array}{ll} 8(10\sqrt{5} - 22)^2, \\ 4(10\sqrt{5} - 22)(10\sqrt{5} - 18) \end{array} \right.$$

$$= 4 \left( 10\sqrt{5} - 22 \right) \left( 10\sqrt{5} - 18 \right) = 16 \left( 5\sqrt{5} - 11 \right) \left( 5\sqrt{5} - 9 \right) =$$

$$= 16 \left( 5\sqrt{5} - 11 \right) \left( 5\sqrt{5} - 9 \right) = 16 \left( 125 - 55\sqrt{5} - 45\sqrt{5} + 99 \right) =$$

$$= 16 \left( 224 - 100\sqrt{5} \right) = 64 \left( 56 - 25\sqrt{5} \right).$$

(27)

As

$$22 < 10\sqrt{5} < 23$$

and

$$2 \left( 10\sqrt{5} - 22 \right) \leq \left( 10\sqrt{5} - 18 \right).$$
### 3.3.2 Boundary \( x = 2, y = 0 \)

Check the values on the boundary \( x = 2, y = 0 \). Then

\[
F_3^*(z, w) = (t + 4)(t + z^2)w^2 \rightarrow \max
\]

under the condition

\[
z \geq 0, \quad w \geq 0, \quad z + w \leq 2.
\]

1. Unconditional extremum

\[
\left\{ \begin{aligned}
\frac{\partial F_3^*}{\partial z} &= 2(t + 4)zw^2 = 0, \\
\frac{\partial F_3^*}{\partial w} &= 2(t + 4)w(t + z^2) = 0
\end{aligned} \right. \quad \Rightarrow w = 0.
\]

Hence we get global minimum \( F_3^*(z, 0) = 0 \).

2. Check values at boundaries

\[
F_3^*(0, 2) = 4t(t + 4), \quad F_3^*(2, 0) = 0.
\]

3. Let \( z + w = 2 \). Then \( z = 2 - w \). Hence

\[
F_3^* = (t + 4)(t + (2 - w)^2)w^2,
\]

\[
\frac{\partial F_3^*}{\partial w} = (t + 4)(2w(t + (2 - w)^2) - 2(2 - w)w^2) = (t + 4)2w(t + (2 - w)^2 - (2 - w)w) = 0,
\]

\[
w(t + (2 - w)^2 - (2 - w)w) = 0.
\]

If \( w = 0 \) then \( F_3^*(z, 0) = 0 \). Otherwise

\[
t + (2 - w)^2 - (2 - w)w = 0,
\]

\[
t + 4 - 4w + w^2 - 2w + w^2 = 0,
\]

\[
2w^2 - 6w + (t + 4) = 0,
\]

\[
w = \frac{3 \pm \sqrt{1-2t}}{2}.
\]

Then

\[
F_3^* \left( \frac{3 + \sqrt{1-2t}}{2} \right) = (t + 4) \left( t + \left( \frac{1 - \sqrt{1-2t}}{2} \right)^2 \right) \left( \frac{3 + \sqrt{1-2t}}{2} \right)^2 = \quad (28)
\]

\[
= \frac{(t + 4)}{16} \cdot (4t + 1 + 1 - 2t - 2\sqrt{1-2t}) \left( 3 + \sqrt{1-2t} \right)^2 =
\]
\[ F_3^* \left( \frac{3 - \sqrt{1 - 2t}}{2} \right) = (t + 4) \left( t + \left( \frac{1 + \sqrt{1 - 2t}}{2} \right)^2 \right) \left( \frac{3 - \sqrt{1 - 2t}}{2} \right)^2 = \]
\[ = \frac{(t + 4)}{16} \cdot (4t + 1 + 1 - 2t + 2\sqrt{1 - 2t}) (3 - \sqrt{1 - 2t})^2 = \]
\[ = \frac{(t + 4)}{8} \cdot (t + 1 + \sqrt{1 - 2t}) (3 - \sqrt{1 - 2t})^2 \]

As a result

\[ \max F_3(2, 0, z, w) = (t + 4) \max \left\{ \frac{4t}{8} (t + 1 + \sqrt{1 - 2t}) (3 - \sqrt{1 - 2t})^2, \frac{1}{8} (t + 1 - \sqrt{1 - 2t}) (3 + \sqrt{1 - 2t})^2 \right\}, = \]
\[ = \left( 10\sqrt{5} - 18 \right) \max \left\{ \frac{4}{8} \left( 10\sqrt{5} - 22 \right), \frac{1}{8} \left( 10\sqrt{5} - 21 + \sqrt{45 - 20\sqrt{5}} \right) (3 - \sqrt{45 - 20\sqrt{5}})^2, \frac{1}{8} \left( 10\sqrt{5} - 21 - \sqrt{45 - 20\sqrt{5}} \right) (3 + \sqrt{45 - 20\sqrt{5}})^2 \right\}, = \]
\[ = \left( 5\sqrt{5} - 9 \right) \max \left\{ \frac{16}{8} \left( 5\sqrt{5} - 11 \right), \frac{1}{4} \left( 10\sqrt{5} - 21 + 5 - 2\sqrt{5} \right) (3 - 5 + 2\sqrt{5})^2, \frac{1}{4} \left( 10\sqrt{5} - 21 - 5 + 2\sqrt{5} \right) (3 + 5 - 2\sqrt{5})^2 \right\}, = \]
\[ = \left( 5\sqrt{5} - 9 \right) \max \left\{ \frac{16}{8} \left( 5\sqrt{5} - 11 \right), \frac{1}{4} \left( 8\sqrt{5} - 16 \right) 4 (\sqrt{5} - 1)^2, \frac{1}{4} \left( 12\sqrt{5} - 26 \right) 4 (4 - \sqrt{5})^2 \right\}, = \]
\[ = 2 \left( 5\sqrt{5} - 9 \right) \max \left\{ \frac{8}{4} \left( 5\sqrt{5} - 11 \right), \frac{4 (\sqrt{5} - 2)}{6} (6 - 2\sqrt{5}), (6\sqrt{5} - 13) (21 - 8\sqrt{5}) \right\}, = \]
\[ = \left( 5\sqrt{5} - 9 \right) \max \left\{ \frac{8}{4} \left( 5\sqrt{5} - 11 \right), \frac{8 \left( 3\sqrt{5} - 6 - 5 + 2\sqrt{5} \right)}{126\sqrt{5} - 273 - 240 + 104\sqrt{5}} \right\}, = \]
\[ = 2 \left( 5\sqrt{5} - 9 \right) \max \left\{ \frac{8}{230\sqrt{5} - 513} \left( 5\sqrt{5} - 11 \right), \frac{8 \left( 5\sqrt{5} - 11 \right)}{230\sqrt{5} - 513}, \frac{16 \left( 5\sqrt{5} - 11 \right) \left( 5\sqrt{5} - 9 \right)}{64 \left( 56 - 25\sqrt{5} \right)} \right\}, = \]

because
\[ 230\sqrt{5} - 513 = 40\sqrt{5} - 88 + 5 \left( 38\sqrt{5} - 85 \right) \leq 40\sqrt{5} - 88 = 8 \left( 5\sqrt{5} - 11 \right), \]
because
\[ 38^2 \cdot 5 = 7220 < 7225 = 85^2, \]
\[ 38\sqrt{5} < 85. \]

The result is the same as \([27]\).
3.3.3 Boundary $x + y = 2$

Let $x + y = 2$. Then $x = 2 - y$. Hence

$$F_3^*(y, z, w) = (t + (2 - y)^2)(t + z^2)(y^2 + w^2) \rightarrow \max$$

with condition

$$0 \leq y \leq 2, \quad z \geq 0, \quad w \geq 0, \quad z + w \leq 2.$$

1. Unconditional extremum

$$\begin{align*}
\frac{\partial F_3^*}{\partial y} &= (t + z^2)[2y(t + (2 - y)^2) - 2(2 - y)(y^2 + w^2)] = 0, \\
\frac{\partial F_3^*}{\partial z} &= 2z(t + (2 - y)^2)(y^2 + w^2) = 0, \\
\frac{\partial F_3^*}{\partial w} &= 2w(t + (2 - y)^2)(t + z^2) = 0
\end{align*}$$

$$\Rightarrow \begin{cases}
w = 0, \\
2y(t + (2 - y)^2) - 2(2 - y)y^2 = 0, \quad \Rightarrow \\
2z(t + (2 - y)^2)y^2 = 0
\end{cases}$$

$$\Rightarrow \begin{cases}
w = 0, \\
\begin{cases}
y = 0, \\
z \in \mathbb{R}, \\
2(t + (2 - y)^2) - 2(2 - y)y = 0,
\end{cases} \quad z = 0
\end{cases}$$

If $y = 0$ then $F_3^*(0, z, 0) = (t + 4)(t + z^2) \cdot 0 = 0$. Otherwise

$$2(t + (2 - y)^2) - 2(2 - y)y = 0,$$

$$2t + 8 - 8y + 2y^2 - 4y + 2y^2 = 0,$$

$$2y^2 - 6y + (t + 4) = 0,$$

$$y = \frac{3 \pm \sqrt{1 - 2t}}{2}.$$

Hence (see 28)

$$F_3^* \left( \frac{3 + \sqrt{1 - 2t}}{2} \right) = t \left( t + \left( \frac{1 - \sqrt{1 - 2t}}{2} \right)^2 \right) \left( \frac{3 + \sqrt{1 - 2t}}{2} \right)^2 \leq F_3(2, 0, z, w), \quad (29)$$

and

$$F_3^* \left( \frac{3 - \sqrt{1 - 2t}}{2} \right) = t \left( t + \left( \frac{1 + \sqrt{1 - 2t}}{2} \right)^2 \right) \left( \frac{3 - \sqrt{1 - 2t}}{2} \right)^2 \leq F_3(2, 0, z, w).$$

So there are no new possible absolute maximum values.
2. We check the values on the boundary $z = 0, w = 2$ then

$$F_3^* = (t + (2 - y)^2)t(y^2 + 4) = t(y^2 + 4)(y^2 - 4y + t + 4) \to \max$$

with the condition

$$0 \leq y \leq 2.$$

We have

$$\frac{\partial F_3^*}{\partial y} = t \ [2y(y^2 - 4y + t + 4) - 2(2 - y)(y^2 + 4)] = 0,$$

$$y(t + (2 - y)^2) - (2 - y)(y^2 + 4) = 0,$$

$$ty + 4y - 4y^2 + y^3 - 2y^2 - 8 + y^3 + 4y = 0,$$

$$2y^3 - 6y^2 + (8 + t)y - 8 = 0.$$

This equation has a root in the interval $(0, 2)$ However, the second derivative

$$\frac{\partial^2 F_3^*}{\partial y^2} = t(6y^2 - 16y + 2t + 8 - 8y + 6y^2 + 8) = 2t(6y^2 - 12y + 8 + t) = 2t(6(y - 1)^2 + 2 + t)$$

is strictly positive. Therefore it is a local minimum point.

3. We check the values on the border $z = 2, w = 0$ Then

$$F_3^* = (t + (2 - y)^2)(t + 4)y^2 = (t + 4)(y^4 - 4y^3 + (t + 4)y^2) \to \max$$

with the condition

$$0 \leq y \leq 2,$$

$$0 \leq w \leq 2.$$

We have

$$\frac{\partial F_3^*}{\partial y} = (t + 4)(4y^3 - 12y^2 + 2(t + 4)y) = 0,$$

$$4y^3 - 12y^2 + 2(t + 4)y,$$

$$2y^2 - 6y + (t + 4) = 0$$

which leads us to the result $[27]$.

4. Let $z + w = 2$. Then $z = 2 - w$. Therefore

$$F_3^* = (t + (2 - y)^2)(t + (2 - w)^2)(y^2 + w^2) \to \max$$

under the condition

$$0 \leq y \leq 2, 0 \leq w \leq 2.$$

We equate to zero partial derivatives

$$\begin{cases}
\frac{\partial F_3^*}{\partial y} = (t + (2 - w)^2) [-2(2 - y)(y^2 + w^2) + 2y (t + (2 - y)^2)] = 0, \\
\frac{\partial F_3^*}{\partial w} = (t + (2 - y)^2) [-2(2 - w)(y^2 + w^2) + 2w (t + (2 - w)^2)] = 0
\end{cases} \Rightarrow$$

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\[
\begin{align*}
\begin{cases}
(2-y)(y^2+w^2) - y(t+(2-y)^2) &= 0, \\
(2-w)(y^2+w^2) - w(t+(2-w)^2) &= 0
\end{cases}
\Rightarrow
\end{align*}
\]

We express from the second equation
\[
y^2 + w^2 = \frac{y(t+(2-y)^2)}{2-y} = \frac{w(t+(2-w)^2)}{2-w},
\]
so
\[
w^2 = \Delta = y \left(2 - 2y + \frac{t}{2-y}\right) = y \left(\frac{2(1-y)(2-y) + t}{2-y}\right).
\]

Then
\[
(2-w) \cdot \frac{y(t+(2-y)^2)}{2-y} - w(t+(2-w)^2) = 0,
\]
\[
w \cdot \left[ -\frac{y(t^2+(2-y)^2)}{2-y} - (t+4) - y \left(2 - 2y + \frac{t}{2-y}\right) \right] = \frac{2y(t+(2-y)^2)}{2-y} - 4y \left(2 - 2y + \frac{t}{2-y}\right),
\]
so
\[
w = \frac{2y(t+(2-y)^2) + 4y(2(1-y)(2-y) + t)}{y(t+(2-y)^2) + (2-y)(t+4) + y(2(1-y)(2-y) + t)} =\]
\[
= 2y \cdot \frac{t + (2-y)^2 + 4(1-y)(2-y) + 2t}{(2-y)(y(2-y) + t + 4 + 2y(1-y)) + 2yt} =\]
\[
= 2y \cdot \frac{(2-y)(2y - y^2 + 4 + 2y - 2y^2) + (y+2)t}{(2-y)(2y - y^2 + 4 + 2y - 2y^2) + (y+2)t} =\]
\[
= 2y \cdot \frac{(2-y)(6-5y) + 3t}{(2-y)(4+4y-3y^2) + (y+2)t} =\]
\[
= 2y \cdot \frac{5y^2 - 16y + 12 + 3t}{3y^3 - 10y^2 + 4y + 8 + (y+2)t}.
\]

Let \(T = t + 4\) then
\[
\sqrt{\Delta} = 2y \cdot \frac{y(5y - 16) + 3T}{y^2(3y - 10) + (y+2)T}.
\]

So
\[
\frac{y(2y^2 - 6y + T)}{2-y} = \frac{4y^2(5y - 16) + 3T}{(y^2(3y - 10) + (y+2)T)^2},
\]
\[
(2y^2 - 6y + T) \left(y^2(3y - 10) + (y+2)T\right)^2 = 4y(2-y) \left(y(5y - 16) + 3T\right)^2,
\]
\[
\frac{y(2y^2 - 6y + T)}{2-y} = \frac{4y^2(5y - 16) + 3T}{(y^2(3y - 10) + (y+2)T)^2}.
\]
\[(2y^2 - 6y + T) (y^4(9y^2 - 60y + 100) + 2y^2(3y^3 - 10y)(y + 2) + (y^2 + 4y + 4)T^2) = \\
= 4y(2 - y) \left( y^2(25y^2 - 160y + 256) + 6y(5y - 16)T + 9T^2 \right), \]

\[(2y^2 - 6y + T) (9y^6 - 60y^5 + (100 + 6T)y^4 - 8Ty^3 + (T^2 - 40T)y^2 + 4T^2y + 4T^2) = \\
= 4y(2 - y) \left( 25y^4 - 160y^3 + (256 + 30T)y^2 - 96Ty + 9T^2 \right), \]

\[18y^8 - 174y^7 + (200 + 12T + 360 + 9T)y^6 - (16T + 600 + 36T + 60T)y^5 + \\
+ (2T^2 - 80T + 48T + 100T + 6T^2)y^4 - (-8T^2 + 6T^2 - 240T + 8T^2)y^3 + \\
+ (8T^2 - 24T^2 + T^3 - 40T^2)y^2 - (4T^3 - 24T^2)y + 4T^3 = \\
= -100y^6 + 840y^5 - (1024 + 120T + 1280)y^4 + \\
+ (2048 + 240T + 384T)y^3 - (768T + 36T^2)y^2 + 72T^2y, \]

\[18y^8 - 174y^7 + (21T + 660)y^6 - (112T + 1440)y^5 + (8T^2 + 188T + 2304)y^4 - \\
- (6T^2 + 384T + 2048)y^3 + (T^3 - 20T^2 + 768T)y^2 + (4T^3 - 96T^2)y + 4T^3 = 0, \]

\[18y^8 - (48 + 126)y^7 + (6T + 336 + (15T + 324))y^6 - (42T + (40T + 864) + (30T + 576))y^5 + \\
+ ((5T^2 + 108T) + (80T + 1536) + (3T^2 + 768))y^4 - ((10T^2 + 192T) + (8T^2 + 2048) + (-12T^2 + 192T))y^3 + \\
+ ((T^3 + 256T) + (-32T^2 + 512T) + 12T^2)y^2 + ((4T^3 - 64T^2) - 32T^2)y + 4T^3 = 0, \]

\[(3y^2 - 8y + T)(6y^6 - 42y^5 + (5T + 108)y^4 - (10T + 192)y^3 + (T^2 + 256)y^2 + (4T^2 - 64T)y + 4T^2) = 0 \]

I. e.

\[6y^6 - 42y^5 + (5T + 108)y^4 - (10T + 192)y^3 + (T^2 + 256)y^2 + (4T^2 - 64T)y + 4T^2 = 0, \]

or

\[3y^2 - 8y + T = 0 \implies y = \frac{8 \pm \sqrt{64 - 12T}}{6} = \frac{4 \pm \sqrt{16 - 3T}}{3} = \frac{4 \pm \sqrt{4 - 3T}}{3}. \]

Consider both cases separately.
3.3.4 Case 1

In the second case

\[6y^6 - 42y^5 + (5T + 108)y^4 - (10T + 192)y^3 + (T^2 + 256)y^2 + (4T^2 - 64T)y + 4T^2 = 0 \quad (30)\]

Let us prove that this equation has no solutions on the interval \((0, 2)\) (The figure shows the plot of the function on the left side of the equation on the interval \((0, 2)\)).

To prove this statement we use the theorem.

We will take into account that \(T = t + 4 = 10\sqrt{5} - 18\).

Write out the coefficients of the polynomial

\[a_0 = 6,\]
\[a_1 = -42,\]
\[a_2 = 5T + 108 = 50\sqrt{5} + 18 = 2 \left(25\sqrt{5} + 9\right),\]
\[a_3 = -(10T + 192) = - \left(100\sqrt{5} + 12\right) = -4 \left(25\sqrt{5} + 3\right),\]
\[a_4 = T^2 + 256 = \left(10\sqrt{5} - 18\right)^2 + 256 = 500 - 360\sqrt{5} + 324 + 256 = 1080 - 360\sqrt{5} = 360 \left(3 - \sqrt{5}\right),\]
\[a_5 = 4T^2 - 64T = 4 \left(10\sqrt{5} - 18\right)^2 - 64 \left(10\sqrt{5} - 18\right) = 2000 - 1440\sqrt{5} + 1296 - 640\sqrt{5} + 1152 = 4448 - 2080\sqrt{5} = 32 \left(139 - 65\sqrt{5}\right),\]
\[a_6 = 4T^2 = 4 \left(10\sqrt{5} - 18\right)^2 = 2000 - 1440\sqrt{5} + 1296 = 3296 - 1440\sqrt{5} = 32 \left(103 - 45\sqrt{5}\right)\]
So we can proceed to the investigation of the zeros of the function

\[ f(x) = 6x^6 - 42x^5 + 2\left(25\sqrt{5} + 9\right)x^4 - 4\left(25\sqrt{5} + 3\right)x^3 + \\
+ 360\left(3 - \sqrt{5}\right)x^2 + 32\left(139 - 65\sqrt{5}\right)x + 32\left(103 - 45\sqrt{5}\right) \]

Calculating the auxiliary functions \( f_i(x) \). We have

\[ f_0(x) = f(x), \]

\[ f_1(x) = 6x^5 - 35x^4 + \frac{4}{3}\left(25\sqrt{5} + 9\right)x^3 - 2\left(25\sqrt{5} + 3\right)x^2 + 120\left(3 - \sqrt{5}\right)x + \frac{16}{3}\left(139 - 65\sqrt{5}\right), \]

\[ f_2(x) = 6x^4 - 28x^3 + \frac{4}{5}\left(25\sqrt{5} + 9\right)x^2 - \frac{4}{5}\left(25\sqrt{5} + 3\right)x + 24\left(3 - \sqrt{5}\right), \]

\[ f_3(x) = 6x^3 - 21x^2 + \frac{2}{5}\left(25\sqrt{5} + 9\right)x - \frac{1}{5}\left(25\sqrt{5} + 3\right), \]

\[ f_4(x) = 6x^2 - 14x + \frac{2}{15}\left(25\sqrt{5} + 9\right), \]

\[ f_5(x) = 6x - 7, \]

\[ f_6(x) = 6. \]

We compute the next series of auxiliary functions

\[ F_0(x) = f(x), \]

\[ F_1(x) = \ldots \ldots \ldots = \]

\[ = \frac{209 - 100\sqrt{5}}{5}x^8 + \frac{16\left(25\sqrt{5} - 18\right)}{15}x^7 + \frac{4\left(9045\sqrt{5} - 22937\right)}{45}x^6 + \frac{32\left(2565\sqrt{5} - 5348\right)}{15}x^5 + \\
+ \frac{8\left(106781 - 49255\sqrt{5}\right)}{15}x^4 + \frac{128\left(63856 - 28165\sqrt{5}\right)}{45}x^3 + \frac{256\left(1969 - 892\sqrt{5}\right)}{3}x^2 + \\
+ \frac{512\left(2381 - 1060\sqrt{5}\right)}{5}x + \frac{1024\left(6507 - 2911\sqrt{5}\right)}{9}. \]

\[ F_2(x) = f_2^2(x) - f_1(x)f_3(x) = \]

\[ = \left(6x^4 - 28x^3 + \frac{4}{5}\left(25\sqrt{5} + 9\right)x^2 - \frac{4}{5}\left(25\sqrt{5} + 3\right)x + 24\left(3 - \sqrt{5}\right)\right)^2 - \\
- \left(6x^3 - 21x^2 + \frac{2}{5}\left(25\sqrt{5} + 9\right)x - \frac{1}{5}\left(25\sqrt{5} + 3\right)\right) \cdot \left(6x^5 - 35x^4 + \frac{4}{3}\left(25\sqrt{5} + 9\right)x^3 - 2\left(25\sqrt{5} + 3\right)x^2 + 120\left(3 - \sqrt{5}\right)x + \frac{16}{3}\left(139 - 65\sqrt{5}\right)\right) = \]

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\[
\begin{align*}
= & \left[ 36x^8 + 784x^6 + \frac{16}{25} (25\sqrt{5} + 9)^2 x^4 + \frac{16}{25} (25\sqrt{5} + 3)^2 x^2 + 576 \left(3 - \sqrt{5}\right)^2 - 336x^7 + \\
& \quad + \frac{48}{5} (25\sqrt{5} + 9) x^6 - \left( \frac{224}{5} (25\sqrt{5} + 9) + \frac{48}{5} (25\sqrt{5} + 3) \right) x^5 + \\
& \quad + \left( 288 (3 - \sqrt{5}) + \frac{224}{5} (25\sqrt{5} + 3) \right) x^4 - \left( \frac{32}{25} (25\sqrt{5} + 9) (25\sqrt{5} + 3) + 1344 (3 - \sqrt{5}) \right) x^3 + \\
& \quad + \frac{192}{5} (25\sqrt{5} + 9) (3 - \sqrt{5}) x^2 - \frac{192}{5} (25\sqrt{5} + 3) (3 - \sqrt{5}) x \right] - \\
& \left[ 36x^8 - 336x^7 + \left( \frac{52}{5} (25\sqrt{5} + 9) + 735 \right) x^6 - \left( \frac{66}{5} (25\sqrt{5} + 3) + 42 (25\sqrt{5} + 9) \right) x^5 + \\
& \quad + \left( 720 (3 - \sqrt{5}) + 49 (25\sqrt{5} + 3) + \frac{8}{15} (25\sqrt{5} + 9)^2 \right) x^4 - \\
& \quad - \left( \frac{32}{15} (139 - 65\sqrt{5}) + 2520 (3 - \sqrt{5}) + \frac{16}{15} (25\sqrt{5} + 3) (25\sqrt{5} + 9) \right) x^3 + \\
& \quad + \left( \frac{2}{5} (25\sqrt{5} + 3)^2 + 48 (3 - \sqrt{5}) (25\sqrt{5} + 9) - 112 (139 - 65\sqrt{5}) \right) x^2 - \\
& \quad + \left( \frac{32}{15} (139 - 65\sqrt{5}) (25\sqrt{5} + 9) - 24 (3 - \sqrt{5}) (25\sqrt{5} + 3) \right) x - \frac{16}{15} (139 - 65\sqrt{5}) (25\sqrt{5} + 3) \right] = \\
\quad = \left( 49 - \frac{4}{5} (25\sqrt{5} + 9) \right) x^6 + \left( \frac{18}{5} (25\sqrt{5} + 3) - \frac{14}{5} (25\sqrt{5} + 9) \right) x^5 + \\
& \quad + \left( \frac{8}{75} (3206 + 450\sqrt{5}) - 432 (3 - \sqrt{5}) - \frac{21}{5} (25\sqrt{5} + 3) \right) x^4 + \\
& \quad + \left( 1176 (3 - \sqrt{5}) - \frac{16}{75} (3152 + 300\sqrt{5}) - 32 (139 - 65\sqrt{5}) \right) x^3 + \\
& \quad + \left( \frac{6}{25} (3134 + 150\sqrt{5}) - \frac{48}{5} (66\sqrt{5} - 98) + 112 (139 - 65\sqrt{5}) \right) x^2 + \\
& \quad + \left( -\frac{32}{15} (2890\sqrt{5} - 6874) + \frac{72}{5} (72\sqrt{5} - 116) \right) x + \frac{16}{15} (3280\sqrt{5} - 7708) + 576 (14 - 6\sqrt{5}) = \\
\quad = \frac{209 - 100\sqrt{5}}{5} x^6 + \frac{4 (25\sqrt{5} - 18)}{5} x^5 + \frac{28125\sqrt{5} - 72497}{75} x^4 + \frac{8 (7875\sqrt{5} - 14929)}{75} x^3 + \\
& \quad + \frac{4 (107881 - 49235\sqrt{5})}{25} x^2 + \frac{32 (7657 - 3376\sqrt{5})}{15} x + \frac{64 (10\sqrt{5} - 37)}{15},
\end{align*}
\]

\[F_3(x) = f_3^2(x) - f_2(x)f_4(x) = \left( 6x^3 - 21x^2 + \frac{2}{5} (25\sqrt{5} + 9) x - \frac{1}{5} (25\sqrt{5} + 3) \right)^2 - \left( 6x^2 - 14x + \frac{2}{15} (25\sqrt{5} + 9) \right) \left( 6x^4 - 28x^3 + \frac{4}{5} (25\sqrt{5} + 9) x^2 - \frac{4}{5} (25\sqrt{5} + 3) x + \right.\]

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\[+24 \left(3 - \sqrt{5}\right) = \left[36x^6 + 441x^4 + \frac{4}{25} \left(25\sqrt{5} + 9\right)^2 x^2 - \frac{1}{25} \left(25\sqrt{5} + 3\right)^2 - 252x^5
\]
\[+ \frac{24}{5} \left(25\sqrt{5} + 9\right)x^4 - \frac{12}{5} \left(25\sqrt{5} + 3\right)x^3 - \frac{84}{5} \left(25\sqrt{5} + 9\right)x^3 + \frac{42}{5} \left(25\sqrt{5} + 3\right)x^2
\]
\[- \frac{4}{25} \left(25\sqrt{5} + 9\right) \left(25\sqrt{5} + 3\right)x - \left[36x^6 - 252x^5 + \left(\frac{28}{5} \left(25\sqrt{5} + 9\right) + 392\right)x^4 -
\]
\[- \left(\frac{24}{5} \left(25\sqrt{5} + 3\right) + \frac{224}{15} \left(25\sqrt{5} + 9\right)\right)x^3 +
\]
\[+ \left(144 \left(3 - \sqrt{5}\right) + \frac{56}{5} \left(25\sqrt{5} + 3\right) + \frac{8}{75} \left(25\sqrt{5} + 9\right)^2\right)x^2 -
\]
\[- \left(336 \left(3 - \sqrt{5}\right) + \frac{8}{75} \left(25\sqrt{5} + 3\right) \left(25\sqrt{5} + 9\right)\right)x + \frac{16}{5} \left(3 - \sqrt{5}\right) \left(25\sqrt{5} + 9\right)\right] =
\]
\[= \left(49 - \frac{4}{5} \left(25\sqrt{5} + 9\right)\right)x^4 + \left(\frac{12}{5} \left(25\sqrt{5} + 3\right) - \frac{28}{15} \left(25\sqrt{5} + 9\right)\right)x^3 +
\]
\[+ \left(\frac{4}{75} \left(3206 + 450\sqrt{5}\right) - \frac{14}{5} \left(25\sqrt{5} + 3\right) - 144 \left(3 - \sqrt{5}\right)\right)x^2 +
\]
\[+ \left(336 \left(3 - \sqrt{5}\right) + \frac{4}{75} \left(3152 + 300\sqrt{5}\right)\right)x + \left(\frac{1}{25} \left(3134 + 150\sqrt{5}\right) - \frac{16}{5} \left(66\sqrt{5} - 98\right)\right) =
\]
\[= \frac{209 - 100\sqrt{5}}{5} x^4 + \frac{8 \left(25\sqrt{5} - 18\right)}{15} x^3 + \frac{2 \left(3675\sqrt{5} - 10103\right)}{75} x^2 +
\]
\[+ \frac{16 \left(3937 - 1650\sqrt{5}\right)}{75} x + \frac{6 \left(1829 - 855\sqrt{5}\right)}{25},
\]

\[F_4(x) = f_4^2(x) - f_3^2(x)f_5(x) = \left(6x^2 - 14x + \frac{2}{15} \left(25\sqrt{5} + 9\right)\right)^2 -
\]
\[- \left(6x - 7\right) \left(6x^2 - 21x^2 + \frac{2}{5} \left(25\sqrt{5} + 9\right)x - \frac{1}{5} \left(25\sqrt{5} + 3\right)\right) =
\]
\[= \left(36x^4 + 196x^2 + \frac{4 \left(3206 + 450\sqrt{5}\right)}{225}\right) - 168x^3 + \frac{8}{5} \left(25\sqrt{5} + 9\right)x^2 - \frac{56}{15} \left(25\sqrt{5} + 9\right)x -
\]
\[- \left(36x^4 - 168x^3 + \left(147 + \frac{12}{5} \left(25\sqrt{5} + 9\right)\right)x^2 - \left(\frac{14}{5} \left(25\sqrt{5} + 9\right) + \frac{6}{5} \left(25\sqrt{5} + 3\right)\right)x
\]
\[+ \frac{7}{5} \left(25\sqrt{5} + 3\right)\right) = \frac{245 - 200\sqrt{5} - 36}{5} x^2 + \frac{18 \left(25\sqrt{5} + 3\right) - 14 \left(25\sqrt{5} + 9\right)}{15} x +
\]
\[4 \left(3206 + 450\sqrt{5}\right) - 105 \left(25\sqrt{5} + 3\right) =
\]
\[
F_5(x) = f_5^2(x) - f_4(x)f_6(x) = (6x - 7)^2 - \frac{12}{15} \left(45x^2 - 105x + \left(25\sqrt{5} + 9\right)\right) = \\
= \frac{209 - 100\sqrt{5}}{5}x^2 + \frac{4(25\sqrt{5} - 18)}{15}x + \frac{11879 - 6075\sqrt{5}}{225},
\]

Now we calculate the required signs of the functions \(f_i(x)\).

\[
f_1(0) = \frac{16}{3} \left(139 - 65\sqrt{5}\right) < 0, \text{ because } 139^2 = 19321 < 21125 = 65^2 \cdot 5,
\]

\[
f_2(0) = 24 \left(3 - \sqrt{5}\right) > 0, \text{ because } 3^2 = 9 > 5,
\]

\[
f_3(0) = \frac{1}{5} \left(25\sqrt{5} + 3\right) < 0,
\]

\[
f_4(0) = \frac{2}{15} \left(25\sqrt{5} + 9\right) > 0,
\]

\[
f_5(0) = -7 < 0,
\]

So \(N_+(0) = 0\), \(N_-(0) = 4\).

Similarly we define the signs of functions at \(F_i(x)\) the point 2.

\[
F_6(2) = 384 - 1344 + 32 \left(25\sqrt{5} + 9\right) - 32 \left(25\sqrt{5} + 3\right) + 1440 \left(3 - \sqrt{5}\right) + 64 \left(139 - 65\sqrt{5}\right) + \\
+32(103 - 45\sqrt{5}) = 128(123 - 55x^2\sqrt{5}) > 0, \text{ because } 123^2 = 15129 > 15125 = 55^2 \cdot 5,
\]
\[ F_1(2) = \frac{256 (209 - 100\sqrt{5})}{5} + \frac{2048 (25\sqrt{5} - 18)}{15} + \frac{256 (9045\sqrt{5} - 22937)}{45} + \frac{1024 (2565\sqrt{5} - 5348)}{15} + \]
\[ + \frac{128 (106781 - 49255\sqrt{5})}{15} + \frac{1024 (63856 - 28165\sqrt{5})}{45} + \frac{1024 (1969 - 892\sqrt{5})}{3} + \]
\[ + \frac{1024 (2381 - 1060\sqrt{5})}{5} + \frac{1024 (6507 - 2911\sqrt{5})}{9} = \frac{128}{45} \left( 3762 - 1800\sqrt{5} + 1200\sqrt{5} - 864 \right) + \]
\[ + \frac{128}{45} \left( 18090\sqrt{5} - 45874 + 61560\sqrt{5} - 128352 + 320343 - 147765\sqrt{5} + 510848 - 225320\sqrt{5} \right) + \]
\[ + \frac{128}{45} \left( 236280 - 107040\sqrt{5} + 171432 - 76320\sqrt{5} + 260280 - 116440\sqrt{5} \right) = \]
\[ = \frac{128}{9} \left( 265571 - 118767\sqrt{5} \right) < 0, \]

because \(265571^2 = 70 527 956 041 < 70 528 001 445 = 118767^2 \cdot 5,\)

\[ F_2(2) = \frac{64 (209 - 100\sqrt{5})}{5} + \frac{128 (25\sqrt{5} - 18)}{5} + \frac{16 (28125\sqrt{5} - 72497)}{75} + \]
\[ + \frac{64 (7875\sqrt{5} - 14929)}{75} + \frac{16 (107881 - 49235\sqrt{5})}{25} + \frac{64 (7657 - 3376\sqrt{5})}{15} + \frac{64 (10\sqrt{5} - 37)}{15} = \]
\[ = \frac{16}{75} \left( 12540 - 6000\sqrt{5} + 3000\sqrt{5} - 2160 + 28125\sqrt{5} - 72497 + 31500\sqrt{5} - 59716 \right) + \]
\[ + \frac{16}{45} \left( 323643 - 147705\sqrt{5} + 153140 - 67520\sqrt{5} + 200\sqrt{5} - 740 \right) = \]
\[ = \frac{32}{5} \left( 11807 - 5280\sqrt{5} \right) > 0, \] because \(11807^2 = 139 405 249 > 139 392 000 = 5280^2 \cdot 5,\)

\[ F_3(2) = \frac{16 (209 - 100\sqrt{5})}{5} + \frac{64 (25\sqrt{5} - 18)}{15} + \frac{8 (3675\sqrt{5} - 10103)}{75} + \]
\[ + \frac{32 (3937 - 1650\sqrt{5})}{75} + \frac{6 (1829 - 855\sqrt{5})}{25} = \frac{2}{75} \left( 25080 - 12000\sqrt{5} \right) + \]
\[ + \frac{2}{75} \left( 4000\sqrt{5} - 2880 + 14700\sqrt{5} - 40412 + 62992 - 26400\sqrt{5} + 16461 - 7695\sqrt{5} \right) = \]
\[ = \frac{2}{75} \left( 61241 - 27395\sqrt{5} \right) < 0, \]

because \(61241^2 = 3 750 460 081 < 3 752 430 125 = 27395^2 \cdot 5,\)

\[ F_4(2) = \frac{4 (209 - 100\sqrt{5})}{5} + \frac{8 (25\sqrt{5} - 18)}{15} + \frac{11879 - 6075\sqrt{5}}{225} = \]
\[ = \frac{37620 - 18000\sqrt{5} + 3000\sqrt{5} - 2160 + 11879 - 6075\sqrt{5}}{225} = \frac{47339 - 21075\sqrt{5}}{225} > 0, \]
because $47339^2 = 2240980921 > 2220778125 = 21075^2 \cdot 5$,

$$F_5(0) = \frac{209 - 100\sqrt{5}}{5} < 0,$$

$$F_6(0) = 36 > 0.$$  

That is $N_+ (2) = N_- (2) = 0$.

So the equation $\text{(30)}$ does not have roots in the interval $[0, 2)$ as required to prove.

### 3.3.5 Case 2

Let’s explore the points $y = \frac{4 \pm \sqrt{4 - 3t}}{3}$

1. Consider the point

$$y = \frac{4 + \sqrt{4 - 3t}}{3} = \frac{4 + \sqrt{70 - 30\sqrt{5}}}{3} = \frac{4 + 3\sqrt{5} - 5}{3} = \frac{3\sqrt{5} - 1}{3}.$$  

This is the local minimum point (See figure [2]).

![Figure 2: Local minimum point](image)

Instead of examining this point to the maximum-minimum calculate the value of the function at this point and show that it does not exceed $\text{(27)}$.

$$w = \sqrt{y \left( \frac{2(1 - y)(2 - y) + t}{2 - y} \right)} =$$
\[
\sqrt{\frac{3\sqrt{5} - 1}{3} \cdot 3 \left( \frac{2 \cdot \frac{4 - 3\sqrt{5}}{3} \cdot \frac{7 - 3\sqrt{5}}{3} + 10\sqrt{5} - 22}{7 - 3\sqrt{5}} \right) = \\
\sqrt{\frac{(3\sqrt{5} - 1) \left( 2 \left( 4 - 3\sqrt{5} \right) \left( 7 - 3\sqrt{5} \right) + 90\sqrt{5} - 198 \right)}{9 \left( 7 - 3\sqrt{5} \right)}} = \\
\sqrt{\frac{2 \left( 3\sqrt{5} - 1 \right) \left( 28 - 21\sqrt{5} - 12\sqrt{5} + 45 + 45\sqrt{5} - 99 \right)}{9 \left( 7 - 3\sqrt{5} \right)}} = \\
\sqrt{\frac{2 \left( 3\sqrt{5} - 1 \right) \left( 12\sqrt{5} - 26 \right)}{9 \left( 7 - 3\sqrt{5} \right)}} = \\
\sqrt{\frac{(3\sqrt{5} - 1) \left( 21\sqrt{5} - 7 + 45 + 3\sqrt{5} \right)}{9 \left( 7 - 3\sqrt{5} \right)}} = \\
\sqrt{\frac{(3\sqrt{5} - 1) \left( 3\sqrt{5} - 1 \right) \left( 7 - 3\sqrt{5} \right)}{9 \left( 7 - 3\sqrt{5} \right)}} = \\
\frac{3\sqrt{5} - 1}{3}
\]

So \( y = w \).

Note that
\[
3^2 \cdot 5 = 45 < 49 = 7^2, \\
3\sqrt{5} < 7, \\
6\sqrt{5} < 14, \\
6\sqrt{5} - 13 < 1.
\]

Hence
\[
F_3^* = (t + (2 - y)^2)(t + (2 - y)^2)(y^2 + y^2) = 2y^2(t + (2 - y)^2)^2 = \\
= 2 \left( \frac{3\sqrt{5} - 1}{3} \right)^2 \left( 10\sqrt{5} - 22 + \left( \frac{7 - 3\sqrt{5}}{3} \right)^2 \right)^2 = \\
= \frac{2}{729} \left( 46 - 6\sqrt{5} \right) \left( 90\sqrt{5} - 198 + 94 - 42\sqrt{5} \right)^2 = \\
= \frac{4}{729} \left( 23 - 3\sqrt{5} \right) \left( 48\sqrt{5} - 104 \right)^2 = \frac{256}{729} \left( 23 - 3\sqrt{5} \right) \left( 6\sqrt{5} - 13 \right)^2 < \frac{256}{729} \left( 23 - 3\sqrt{5} \right).
\]

Finish our assessment
\[
18 \cdot 213^2 \cdot 5 = 1658566845 < 1659095824 = 40732^2, \\
18 \cdot 213\sqrt{5} < 40732, \\
92 - 12\sqrt{5} < 40824 - 18225\sqrt{5}, \\
4 \left( 23 - 3\sqrt{5} \right) < 729 \left( 56 - 25\sqrt{5} \right), \\
\frac{256}{729} \left( 23 - 3\sqrt{5} \right) < 64 \left( 56 - 25\sqrt{5} \right).
\]

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2. Analogically consider the point

\[ y = \frac{4 - \sqrt{4 - 3t}}{3} = \frac{4 - \sqrt{70 - 30\sqrt{5}}}{3} = \frac{4 - 3\sqrt{5} + 5}{3} = 3 - \sqrt{5}. \]

This is the saddle point (See figure 3).

![Figure 3: Saddle point](image)

Again calculate

\[
w = \sqrt{y \left( \frac{2(1 - y)(2 - y) + t}{2 - y} \right)} = \sqrt{\left(3 - \sqrt{5}\right) \frac{(2 - \sqrt{5} - 2)(\sqrt{5} - 1) + 10\sqrt{5} - 22}{\sqrt{5} - 1}} = \sqrt{\frac{(3 - \sqrt{5})(4\sqrt{5} - 8)}{\sqrt{5} - 1}} = \sqrt{\frac{(3 - \sqrt{5})(3\sqrt{5} - 3 - 5 + \sqrt{5})}{\sqrt{5} - 1}} = 3 - \sqrt{5}.
\]

So \( y = w \).

Hence

\[ F_3^* = (t + (2 - y)^2)(t + (2 - y)^2)(y^2 + y^2) = 2y^2(t + (2 - y)^2)^2 = \]
\[= 2 \left(3 - \sqrt{5}\right)^2 \left(10\sqrt{5} - 22 + \left(\sqrt{5} - 1\right)^2\right)^2 = 2 \left(14 - 6\sqrt{5}\right) \left(10\sqrt{5} - 22 + 6 - 2\sqrt{5}\right)^2 =
\]
\[= 4 \left(7 - 3\sqrt{5}\right) \left(8\sqrt{5} - 16\right)^2 = 256 \left(7 - 3\sqrt{5}\right) \left(\sqrt{5} - 2\right)^2 = 256 \left(7 - 3\sqrt{5}\right) \left(9 - 4\sqrt{5}\right) =
\]
\[= 256 \left(63 - 28\sqrt{5} - 27\sqrt{5} + 60\right) = 256 \left(123 - 55\sqrt{5}\right).
\]

Finish our assessment
\[436^2 = 190,096 < 190,125 = 195^2 \cdot 5,
\]
\[436 < 195\sqrt{5},
\]
\[492 - 220\sqrt{5} < 56 - 25\sqrt{5},
\]
\[4 \left(123 - 55\sqrt{5}\right) < 56 - 25\sqrt{5}.
\]

As a result, we find that in the case 3.3.3 there are no local minima.

### 3.3.6 Final estimation

Combining the results obtained above we get that
\[
\max F_3 = 64 \left(56 - 25\sqrt{5}\right). \quad \Box
\]

### 3.4 Estimate for \(F_2\)

**Theorem 12.**

\[
\max F_2(x, y, z, w) = (t_1 + y^2)(t_2 x^2 + z^2)|w| = \frac{64 \left(5\sqrt{5} - 9\right)}{27},
\]

where \(t_1 = 10\sqrt{5} - 22\) and \(t_2 = \frac{26 + 10\sqrt{5}}{27}\) under the condition
\[
-2 \leq x + y \leq 2, \quad -2 \leq x - y \leq 2,
\]
\[
-2 \leq z + w \leq 2, \quad -2 \leq z - w \leq 2.
\]

**Proof.**

Similar to theorem 9, we notice that
\[
F_2(x, y, z, w) = F_2(x, -y, z, w) = F_2(-x, y, z, w) = F_2(-x, -y, z, w),
\]
\[
F_2(x, y, z, w) = F_2(x, y, z, -w) = F_2(x, y, -z, w) = F_2(x, y, -z, -w).
\]

We came to the task
\[
F^*_2(x, y, z, w) = (t_1 + y^2)(t_2 x^2 + z^2)w \to \max,
\]
\[
x + y \leq 2, \quad z + w \leq 2
\]
\[
x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad w \geq 0.
\]
The last conditions are not boundary and necessary only for clipping points.

We find unconditional extremums
\[
\begin{align*}
\frac{\partial F^*_2}{\partial x} &= 2xw(t_1 + y^2) = 0, \\
\frac{\partial F^*_2}{\partial y} &= 2yw(t_2x^2 + z^2) = 0, \\
\frac{\partial F^*_2}{\partial z} &= 2zw(t_1 + y^2) = 0, \\
\frac{\partial F^*_2}{\partial w} &= (t_1 + y^2)(t_2x^2 + z^2) = 0
\end{align*}
\]

\[\Rightarrow x = z = 0.\]

Therefore we get a global minimum of \(F^*_3(0, y, 0, w) = 0.\)

3.4.1 Boundary \(x = 0, y = 2\)

Check the values on the boundary \(x = 0, y = 2.\) Then
\[
F^*_2(z, w) = (t_1 + 4)z^2w \rightarrow \max
\]
under the condition
\[
z \geq 0, \quad w \geq 0, \quad z + w \leq 2.
\]

1. Unconditional extremum
\[
\begin{align*}
\frac{\partial F^*_2}{\partial z} &= 2zw(t_1 + 4) = 0, \\
\frac{\partial F^*_2}{\partial w} &= z^2(t_1 + 4) = 0
\end{align*}
\]

\[\Rightarrow z = 0.\]

Therefore we get the global minimum \(F^*_3(0, w) = 0.\)

2. Check the values at the borders
\[
F^*_3(0, 2) = 0, \quad F^*_3(2, 0) = 0.
\]

3. Let \(z + w = 2.\) Then \(w = 2 - z.\) Hence
\[
F^*_2 = (t_1 + 4)z^2(2 - z) = (t_1 + 4)(2z^2 - z^3),
\]
\[
\frac{\partial F^*_3}{\partial z} = (t_1 + 4)(4z - 3z^2) = 0,
\]
\[
4z - 3z^2 = 0,
\]
\[
z = 0 \quad \text{or} \quad z = \frac{4}{3},
\]

\[
F^*_2(0) = 0 \quad \text{or} \quad F^*_2\left(\frac{4}{3}\right) = \frac{32(t_1 + 4)}{27}.
\]

As a result
\[
\max F_2(0, 2, z, w) = \frac{32(t_1 + 4)}{27} = \frac{32(10\sqrt{5} - 18)}{27} = \frac{64(5\sqrt{5} - 9)}{27}. \quad (31)
\]
3.4.2 Boundary $x = 2, y = 0$

Check the values on the boundary $x = 2, y = 0$. Hence

$$F_2^* = t_1(4t_2 + z^2)w \rightarrow \max$$

under condition

$$z \geq 0, \quad w \geq 0, \quad z + w \leq 2.$$

1. Unconditional extremum

$$\begin{cases}
\frac{\partial F_2^*}{\partial z} = 2t_1zw = 0, \\
\frac{\partial F_2^*}{\partial w} = t_1(4t_2 + z^2) = 0
\end{cases} \Rightarrow \emptyset$$

Therefore we do not get local extremums.

2. Check the values at the borders

$$F_2^*(0, 2) = 8t_1t_2, \quad F_3^*(2, 0) = 0.$$ 

3. Let $z + w = 2$. Then $w = 2 - z$. Hence

$$F_2^* = t_1(4t_2 + z^2)(2 - z),$$

$$\frac{\partial F_2^*}{\partial x} = t_1 \left[-(4t_2 + z^2) + 2z(2 - z)\right] = 0,$$

$$-4t_2z + z^2 + 2z(2 - z) = 0,$$

$$z^2 + 4t_2 - 4z + 2z^2 = 0,$$

$$3z^2 - 4z + 4t_2 = 0,$$

$$D = 1 - 3t_2 = 1 - \frac{26 + 10\sqrt{5}}{9} = -\frac{17 + 10\sqrt{5}}{9} < 0.$$

So there are no roots in this case.

As a result

$$\max F_3(2, 0, z, w) = 8t_1t_2 = 827 \left(10\sqrt{5} - 22\right) \left(26 + 10\sqrt{5}\right) =$$

$$= \frac{32}{27} \left(65\sqrt{5} + 125 - 143 - 55\sqrt{5}\right) = \frac{64}{27} \left(5\sqrt{5} - 9\right).$$

The result obtained coincides with (31).
3.4.3 Boundary $x + y = 2$

Let $x + y = 2$. Then $y = 2 - x$. Hence

$$F_2^* = (t_1 + (2 - x)^2)(t_2x^2 + z^2)w \rightarrow \max$$

under condition

$$0 \leq x \leq 2, \quad z \geq 0, \quad w \geq 0, \quad z + w \leq 2.$$

1. Unconditional extremum

\[
\begin{align*}
\frac{\partial F_2^*}{\partial x} &= w \left[2x(t_1 + (2 - x)^2) - 2(2 - x)(t_2x^2 + z^2)\right] = 0, \\
\frac{\partial F_2^*}{\partial z} &= 2z(t_1 + (2 - x)^2)w = 0, \quad \Rightarrow x = z = 0, \\
\frac{\partial F_2^*}{\partial w} &= (t_1 + (2 - x)^2)(t_2x^2 + z^2) = 0
\end{align*}
\]

So $F_2^* = 0$.

2. Check values at the border $z = 0, w = 2$. Hence

$$F_2^* = 2(t_1 + (2 - x)^2)t_2x^2 \rightarrow \max$$

under condition

$$0 \leq x \leq 2$$

We have

$$\frac{\partial F_2^*}{\partial x} = 2t_2 \left[2x(t_1 + (2 - x)^2) - 2(2 - x)x^2\right] = 0,$$

$$x(t_1 + (2 - x)^2) - (2 - x)x^2 = 0.$$

If $x = 0$ then $F_2^* = 0$. Otherwise

$$t_1 + (2 - x)^2 - (2 - x)x = 0,$$

$$t_1 + 4 - 4x + x^2 - 2x + x^2 = 0,$$

$$2x^2 - 6x + (t_1 + 4) = 0,$$

$$x = \frac{3 \pm \sqrt{1 - 2t_1}}{2}.$$

Hence

$$F_2^* \left(\frac{3 + \sqrt{1 - 2t_1}}{2}\right) = 2t_2 \left(\frac{3 + \sqrt{1 - 2t_1}}{2}\right)^2 \cdot \left(t_1 + \left(\frac{1 - \sqrt{1 - 2t_1}}{2}\right)^2\right) =$$

$$= \frac{t_2}{8} \left(9 + 1 - 2t_1 + 6\sqrt{1 - 2t_1}\right) \left(4t_1 + (1 + 1 - 2t_1 - 2\sqrt{1 - 2t_1})\right) =$$

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\[
= \frac{t_2}{2} (5 - t_1 + 3\sqrt{1 - 2t_1}) (1 + t_1 - \sqrt{1 - 2t_1})
\]

and
\[
F_2^* \left( \frac{3 - \sqrt{1 - 2t_1}}{2} \right) = 2t_2 \left( \frac{3 - \sqrt{1 - 2t_1}}{2} \right)^2 \cdot \left( 1 + \frac{1 + \sqrt{1 - 2t_1}}{2} \right)^2 = \\
= \frac{t_2}{8} (9 + 1 - 2t_1 - 6\sqrt{1 - 2t_1}) (4t_1 + (1 + 1 - 2t_1 + 2\sqrt{1 - 2t_1})) = \\
= \frac{t_2}{2} (5 - t_1 - 3\sqrt{1 - 2t_1}) (1 + t_1 + \sqrt{1 - 2t_1}).
\]

So we get
\[
\max \left\{ F_2^* \left( \frac{3 + \sqrt{1 - 2t_1}}{2} \right), F_2^* \left( \frac{3 - \sqrt{1 - 2t_1}}{2} \right) \right\} = \\
= \max \left\{ \frac{t_2}{2} (5 - t_1 + 3\sqrt{1 - 2t_1}) (1 + t_1 - \sqrt{1 - 3t_1}), \frac{t_2}{2} (5 - t_1 - 3\sqrt{1 - 2t_1}) (1 + t_1 + \sqrt{1 - 3t_1}) \right\} = \\
= \max \left\{ \frac{13 + 5\sqrt{5}}{27} (5 - 10\sqrt{5} + 22 + 3\sqrt{45 - 20\sqrt{5}}) (1 + 10\sqrt{5} - 22 - \sqrt{45 - 20\sqrt{5}}), \frac{13 + 5\sqrt{5}}{27} (5 - 10\sqrt{5} + 22 - 3\sqrt{45 - 20\sqrt{5}}) (1 + 10\sqrt{5} - 22 + \sqrt{45 - 20\sqrt{5}}) \right\} = \\
= \max \left\{ \frac{1}{27} (13 + 5\sqrt{5}) (27 - 10\sqrt{5} + 6\sqrt{5} - 15) (10\sqrt{5} - 21 - 2\sqrt{5}) + 5), \frac{1}{27} (13 + 5\sqrt{5}) (27 - 10\sqrt{5} - 6\sqrt{5} + 15) (10\sqrt{5} - 21 + 2\sqrt{5}) - 5) \right\} = \\
= \max \left\{ \frac{1}{27} (13 + 5\sqrt{5}) (12 - 4\sqrt{5}) (8\sqrt{5} - 16), \frac{1}{27} (13 + 5\sqrt{5}) (42 - 16\sqrt{5}) (12\sqrt{5} - 26) \right\} = \\
= \max \left\{ \frac{32}{27} (13 + 5\sqrt{5}) (8 - \sqrt{5} - 6 - 2\sqrt{5}), \frac{32}{27} (13 + 5\sqrt{5}) (21 - 8\sqrt{5}) (6\sqrt{5} - 5) \right\} = \\
= \max \left\{ \frac{32}{27} (13 + 5\sqrt{5}) (3\sqrt{5} - 6 - 5 + 2\sqrt{5}), \frac{32}{27} (13 + 5\sqrt{5}) (126\sqrt{5} - 273 - 240 + 104\sqrt{5}) \right\} = \\
= \max \left\{ \frac{32}{27} (13 + 5\sqrt{5}) (5\sqrt{5} - 11), \frac{32}{27} (13 + 5\sqrt{5}) (230\sqrt{5} - 513) \right\} = \max \left\{ \frac{32}{27} (65\sqrt{5} - 143 + 125 - 55\sqrt{5}), \frac{32}{27} (2990\sqrt{5} - 6669 + 5750 - 2565\sqrt{5}) \right\} = \\
= \max \left\{ \frac{64}{27} (5\sqrt{5} - 9), \frac{64}{27} (425\sqrt{5} - 919) \right\} = \frac{64 (5\sqrt{5} - 9)}{27},
\]

because \(16 (5\sqrt{5} - 9) = 80\sqrt{5} - 144 > 425\sqrt{5} - 919\). Really
\[
24025 = 155^2 > 69^2 \cdot 5 = 23805,
\]
\[
155 > 69\sqrt{5},
\]
\[
775 > 345\sqrt{5},
\]
\[
80\sqrt{5} - 144 > 425\sqrt{5} - 919.
\]

This is the same as \(\text{[31]}\).
3. We check the values on the border $z = 2, w = 0$. Here $F^*_2 = 0$.

4. Let $z + w = 2$. Then $w = 2 - z$. Hence

$$F^*_2 = (t_1 + (2 - x)^2)(t_2x^2 + z^2)(2 - z) \rightarrow \max$$

under condition

$$0 \leq x \leq 2, 0 \leq z \leq 2.$$ 

We equate to zero the partial derivatives

$$\begin{align*}
\frac{\partial F^*_2}{\partial x} &= (2 - z) \left[ -2(2 - x)(t_2x^2 + z^2) + 2t_2x(t_1 + (2 - x)^2) \right] = 0, \\
\frac{\partial F^*_2}{\partial z} &= (t_1 + (2 - x)^2) \left[ -(t_2x^2 + z^2) + 2z(2 - z) \right] = 0
\end{align*}$$

We express from the second equation

$$t_2x^2 = 2z(2 - z) - z^2, \quad x^2 = \frac{4z - 3z^2}{t_2} = \frac{z(4 - 3z)}{t_2}, \quad (32)$$

We simplify the first equation

$$xt_1t_2 + 4xt_2 - 4x^2t_2 + x^3t_2 - 2x^2t_2 + x^3t_2 - 2z^2 + x^2 = 0,$$

$$x(t_1t_2 + 4t_2 + 2x^2t_2 + z^2) = 6x^2t_2 + 2z^2.$$ 

Therefore

$$x = \frac{6z(4 - 3z) + 2z^2}{t_1t_2 + 4t_2 + 2z(4 - 3z) + z^2} = \frac{24z - 18z^2 + 2z^2}{t_1t_2 + 4t_2 + 8z - 5z^2} = \frac{8z(3 - 2z)}{t_2(t_1 + 4) + z(8 - 5z)}$$

Combining this with (32) we get

$$\frac{z(4 - 3z)}{t_2} = \frac{64z^2(3 - 2z)^2}{(t_2(t_1 + 4) + z(8 - 5z))^2},$$

$$z(4 - 3z)(t_2(t_1 + 4) + z(8 - 5z))^2 = 64z^2t_2(3 - 2z)^2.$$ 

Let $T_1 = t_1 + 4$ then

$$(4 - 3z) \left( t_2^2T_1^2 + 2t_2T_1z(8 - 5z) + z^2(8 - 5z)^2 \right) = 64zt_2 \left( 9 - 12z + 4z^2 \right),$$

$$4t_2^2T_1^2 + 8t_2T_1z(8 - 5z) + 4z^2 \left( 64 - 80z + 25z^2 \right) - 3zt_2^2T_1^2 - 6t_2T_1z^2(8 - 5z)$$
3z^3(64 - 80z + 25z^2) = 576z_t - 768z^2t_2 + 256z^3t_2,

75z^5 - (100 + 240)z^4 + (320 - 30t_2T_1 + 192 + 256t_2)z^3 -
-(-40t_2T_1 + 256 - 48t_2T_1 + 768t_2)z^2 + (-64t_2T_1 + 3t_2T_1^2 + 576t_2)z - 4t_2^2T_1^2 = 0,

75z^5 - 340z^4 + (512 - 30t_2T_1 + 256t_2)z^3 - (256 + 768t_2 - 88t_2T_1)z^2 +
+(3t_2^2T_1^2 - 64t_2T_1 + 576t_2)z - 4t_2^2T_1^2 = 0. \tag{33}

Let us study this equation.

3.4.4 The study of equation (33)

Equation (33) has a single root in the interval (0, 2). (See 4).

![Graph of function g(z)](image)

Figure 4: The plot of the function $g(z)$

To verify this, let us investigate the function

$$g(z) = 75z^5 - 340z^4 + (512 - 30t_2T_1 + 256t_2)z^3 - (256 + 768t_2 - 88t_2T_1)z^2 +
+(3t_2^2T_1^2 - 64t_2T_1 + 576t_2)z - 4t_2^2T_1^2$$

Write down the coefficients of the polynomial, taking into account that $T_1 = t_1 + 4 = 10\sqrt{5} - 18$
and $t_2 = \frac{26 + 10\sqrt{5}}{27}$

$$a_0 = 75,$$
$$a_1 = -340,$$
\[ a_2 = 512 - 30t_2T_1 + 256t_2 = \]
\[ = \frac{512 \cdot 27 + (26 + 10\sqrt{5}) (256 - 30 (10\sqrt{5} - 18))}{27} = \frac{8 (1728 + (13 + 5\sqrt{5}) (199 - 75\sqrt{5}))}{27} = \frac{160 (122 + \sqrt{5})}{27}, \]
\[ a_3 = -(256 + 768t_2 - 88t_2T_1) = \]
\[ = \frac{-6912 + (26 + 10\sqrt{5}) (768 - 88 (10\sqrt{5} - 18))}{27} = \frac{-32 (216 + (13 + 5\sqrt{5}) (147 - 55\sqrt{5}))}{27} = \frac{-128 (188 + 5\sqrt{5})}{27}, \]
\[ a_4 = 3t_2^2T_1^2 - 64t_2T_1 + 576t_2 = \]
\[ = \frac{(26 + 10\sqrt{5}) \left( 27 (576 - 64 (10\sqrt{5} - 18)) + 3 (26 + 10\sqrt{5}) (10\sqrt{5} - 18)^2 \right)}{729} = \frac{16 (13 + 5\sqrt{5}) (9 (216 - 80\sqrt{5}) + (13 + 5\sqrt{5}) (206 - 90\sqrt{5}))}{729} = \frac{32 (13 + 5\sqrt{5}) (972 - 360\sqrt{5} + (1339 - 585\sqrt{5} + 515\sqrt{5} - 1125))}{729} = \frac{64 (13 + 5\sqrt{5}) (593 - 215\sqrt{5})}{729} = \frac{64 (7709 + 2965\sqrt{5} - 2795\sqrt{5} - 5375)}{729} = \frac{128 (1167 + 85\sqrt{5})}{243}, \]
\[ a_5 = -4t_2^2T_1^2 = -\frac{4 (26 + 10\sqrt{5})^2 (10\sqrt{5} - 18)^2}{729} = -\frac{164 (125 - 117 + 65\sqrt{5} - 45\sqrt{5})^2}{729} = -\frac{1024 (129 + 20\sqrt{5})}{729}. \]

So
\[ g(z) = 75z^5 - 340z^4 + \frac{160 (122 + \sqrt{5})}{27}z^3 - \frac{128 (188 + 5\sqrt{5})}{27}z^2 + \frac{128 (1167 + 85\sqrt{5})}{243}z - \frac{1024 (129 + 20\sqrt{5})}{729}. \]

We will prove that the derivative \( g'(z) \) is strictly positive on the interval \((0, 2)\).

\[ f(z) = g'(z) = 375z^4 - 1360z^3 + \frac{160 (122 + \sqrt{5})}{9}z^2 - \frac{256 (188 + 5\sqrt{5})}{27}z + \frac{128 (1167 + 85\sqrt{5})}{243}. \]
\[ g'(0) = \frac{128 (1167 + 85\sqrt{5})}{243} > 0 \]
Let us prove now that the equation

\[ f(z) = 0 \]  \hspace{1cm} (34)

has no roots. To do this we again use the theorem VIII.

Calculate the auxiliary functions \( f_i(z) \):

\[
\begin{align*}
  f_0(z) &= f(z), \\
  f_1(z) &= 375z^3 - 1020z^2 + \frac{80}{9} \left( 122 + \sqrt{5} \right) z - \frac{64}{27} \left( 188 + 5\sqrt{5} \right), \\
  f_2(z) &= 375z^2 - 680z + \frac{80}{27} \left( 122 + \sqrt{5} \right), \\
  f_3(z) &= 375z - 340, \\
  f_4(z) &= 375.
\end{align*}
\]

Now calculate \( F_i(z) \):

\[
\begin{align*}
  F_0(z) &= f(z), \\
  F_1(z) &= f_1^2(z) - f_0(z)f_2(z) = \ldots = \frac{400 \left( 449 + 25\sqrt{5} \right)}{9} z^4 + \frac{6400 \left( 373 + 29\sqrt{5} \right)}{27} z^3 - \frac{640 \left( 53691 + 6995\sqrt{5} \right)}{243} z^2 + \\
  &+ \frac{5120 \left( 13595 + 2739\sqrt{5} \right)}{729} z - \frac{1024 \left( 75553 + 23845\sqrt{5} \right)}{6561}, \\
  F_2(z) &= f_2^2(z) - f_1(z)f_3(z) = \ldots = -\frac{400 \left( 449 + 25\sqrt{5} \right)}{9} z^2 + \frac{3200 \left( 373 + 29\sqrt{5} \right)}{27} z - \frac{1280 \left( 11847 + 1075\sqrt{5} \right)}{729}, \\
  F_3(z) &= f_3^2(z) - f_2(z)f_4(z) = (375z - 340)^2 - 375 \left( 375z^2 - 680z + \frac{80}{27} \left( 122 + \sqrt{5} \right) \right) = \\
  &= 25 \left( 5625z^2 - 10200z + 4624 \right) = \ldots = -\frac{400 \left( 449 + 25\sqrt{5} \right)}{9}, \\
  F_4(z) &= f_4^2(z) = 140625
\end{align*}
\]

We determine the signs of the functions \( F_i(z) \) at the point 0.

\[
\begin{align*}
  F_0(0) &= \frac{128 \left( 1167 + 85\sqrt{5} \right)}{243} > 0, \\
  F_1(0) &= -\frac{1024 \left( 75553 + 23845\sqrt{5} \right)}{6561} < 0,
\end{align*}
\]
\[ F_2(0) = -\frac{1280 \left(11847 + 1075\sqrt{5}\right)}{729} < 0, \]
\[ F_3(0) = -\frac{400 \left(449 + 25\sqrt{5}\right)}{9} < 0, \]
\[ F_4(0) = 140625 > 0. \]

Then we calculate the required signs of the functions \( f_i(z) \)
\[ f_1(0) = -\frac{64}{27} \left(188 + 5\sqrt{5}\right) < 0, \]
\[ f_2(0) = \frac{80}{27} \left(122 + \sqrt{5}\right) > 0, \]
\[ f_3(0) = -340 < 0. \]

So \( N_+(0) = 0, N_-(0) = 2. \)

Similarly we determine the signs of the functions \( F_i(z) \) at the point 1.
\[ F_0(1) = \ldots = \frac{3909 + 3680\sqrt{5}}{243} > 0, \]
\[ F_1(1) = \ldots = -\frac{16 \left(430889 + 355265\sqrt{5}\right)}{6561} < 0, \]
\[ F_2(1) = \ldots = \frac{3760 \left(669 + 85\sqrt{5}\right)}{729} > 0, \]
\[ F_3(1) = -\frac{400 \left(449 + 25\sqrt{5}\right)}{9} < 0, \]
\[ F_4(1) = 140625 > 0. \]

So \( N_+(1) = N_-(1) = 0. \)

Now we determine the signs of functions \( F_i(z) \) at point 2.
\[ F_0(2) = \ldots = \frac{16 \left(12837 + 320\sqrt{5}\right)}{243} > 0, \]
\[ F_1(2) = \ldots = -\frac{16 \left(262019 + 139595\sqrt{5}\right)}{6561} < 0, \]
\[ F_2(2) = \ldots = -\frac{320 \left(27813 - 1235\sqrt{5}\right)}{729} < 0, \text{ because } \sqrt{5} < 3 \]
\[ F_3(2) = -\frac{400 \left(449 + 25\sqrt{5}\right)}{9} < 0, \]
\[ F_4(2) = 140625 > 0. \]

Then we calculate the required signs of the functions \( f_i(z) \).
\[ f_1(0) = \ldots = \frac{8}{27} \left(2171 + 20\sqrt{5}\right) > 0, \]
\[ f_2(0) = \ldots = \frac{20}{27} (677 + 4\sqrt{5}) > 0, \]
\[ f_3(0) = 750 - 340 = 410 > 0, \]

So \( N_+(2) = 2 \), \( N_-(2) = 0 \).

This means that equation (34) does not have roots both on the interval (0, 1) and on the interval (1, 2). Since
\[ f(1) = F_0(1) > 0 \]
it does not have any roots in the interval (0, 2) which we had to prove.

Since
\[ g(0) = -\frac{1024}{729} (129 + 20\sqrt{5}) < 0, \]
\[ g(2) = \ldots = \frac{32}{729} (5169 + 320\sqrt{5}) > 0, \]
equation \( g(z) = 0 \) has exactly 1 root on the interval (0, 2). Lets call it \( z_0 \). It will be important for us that
\[ 1 = z_1 < z_0 < z_2 = \frac{10}{9}. \quad (35) \]

Really
\[ g(1) = \ldots = \frac{1}{729} (159 - 800\sqrt{5}) < 0, \text{ because } \sqrt{5} < 3, \]
\[ g \left( \frac{10}{9} \right) = \ldots = \frac{32}{19683} (1239 + 320\sqrt{5}) > 0. \]

3.4.5 Investigation \( F_2^* \) at the point \( z_0 \)

Let us investigate the point \( z_0 \). We first obtain an estimate for the value of the variable \( x \) at this point. From (32)
\[ x = \sqrt{\frac{4z - 3z^2}{t_2}} = \sqrt{\frac{4}{3} - 3 \left( \frac{1 - \frac{2}{3}}{3} \right)^2} = 3 \sqrt{\frac{4 - (3z - 2)^2}{26 + 10\sqrt{5}}}, \]
hence
\[ 3 \sqrt{\frac{4 - (\frac{10}{3} - 2)^2}{26 + 10\sqrt{5}}} \leq x_0 \leq 3 \sqrt{\frac{4 - (3 - 2)^2}{26 + 10\sqrt{5}}}, \]
\[ \sqrt{\frac{20}{26 + 10\sqrt{5}}} \leq x_0 \leq \sqrt{\frac{27}{26 + 10\sqrt{5}}}. \]

Lets estimate the left part
\[ 3969 = 63^2 > 25^2 \cdot 5 = 3125, \]
\[ 63 > 25\sqrt{5}, \]
\[ 128 > 65 + 25\sqrt{5}, \]
\[ \frac{2}{13 + 5\sqrt{5}} > \frac{5}{64}, \]

49
\[
\frac{20}{26 + 10\sqrt{5}} > \frac{25}{64},
\]
so
\[
\sqrt{\frac{20}{26 + 10\sqrt{5}}} > \sqrt{\frac{25}{64}} = \frac{5}{8}.
\]

Now the right part
\[
121 = 11^2 < 5^2 \cdot 5 = 125,
\]
\[
11 < 5\sqrt{5},
\]
\[
24 < 13 + 5\sqrt{5},
\]
\[
\frac{3}{13 + 5\sqrt{5}} < \frac{1}{8},
\]
\[
\frac{27}{26 + 10\sqrt{5}} < \frac{9}{16},
\]
so
\[
\sqrt{\frac{20}{26 + 10\sqrt{5}}} < \sqrt{\frac{9}{16}} = \frac{3}{4}.
\]

As result
\[
\frac{5}{8} = x_1 < x_0 < x_2 = \frac{3}{4}.
\]

Note that \((x_0, z_0)\) is the saddle point (See figure [5]). For us it is important that there is no global maximum at this point. To prove this we will estimate \(F_2^*(x_0, z_0)\). For this we use the following estimate
\[
F_2^*(x_0, z_0) \leq F_2^*(x_1, z_1) + \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2} \cdot \max_{x_1 \leq x \leq x_2, z_1 \leq z \leq z_2} \text{grad} F_2 \leq
\]
\[
\leq F_2^*(x_1, z_1) + \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2} \cdot \max_{x_1 \leq x \leq x_2, z_1 \leq z \leq z_2} \sqrt{(\frac{\partial F_2^*}{\partial x})^2 + (\frac{\partial F_2^*}{\partial z})^2}.
\]

**Estimates of partial derivatives.** Let's estimate the partial derivatives \(F_2^*\) in the rectangle \((x_1, x_2) \times (z_1, z_2)\).

We will estimate the individual factors included in the derivative.

\[
\frac{8}{9} = 2 - \frac{10}{9} = 2 - z_2 \leq 2 - z \leq 2 - z_1 = 2 - 1 = 1.
\]

\[
\frac{5}{4} = 2 - \frac{3}{4} = 2 - x_2 \leq 2 - x \leq 2 - x_1 = 2 - \frac{5}{8} = \frac{11}{8}.
\]

\[
t_1 + (2 - x)^2 \leq t_1 + (2 - x_1)^2 = t_1 + \left(2 - \frac{5}{8}\right)^2 = t_1 + \frac{121}{64} < 10\sqrt{5} - 22 + 2 < \frac{5}{2}.
\]

\[
t_1 + (2 - x)^2 \geq t_1 + (2 - x_2)^2 = t_1 + \left(2 - \frac{3}{4}\right)^2 = t_1 + \frac{25}{16} > 10\sqrt{5} - 22 + \frac{3}{2} > \frac{3}{2},
\]

(36)
because $44^2 < 20^2 \cdot 5 < 45^2$ and $22 < 10\sqrt{5} < \frac{45}{2}$. So

$$\frac{3}{2} < t_1 + (2 - x)^2 < \frac{5}{2}.$$  

Then

$$t_2x^2 + z^2 \geq t_2x_1^2 + z_1^2 = \frac{26 + 10\sqrt{5}}{27} \cdot \frac{25}{64} + 1 > \frac{13 + 5\sqrt{5}}{27} \cdot \frac{24}{32} + 1 = \frac{13 + 5\sqrt{5}}{36} + 1 = \frac{13 + 5\sqrt{5} + 36}{36} = \frac{49 + 5\sqrt{5}}{36} > \frac{49 + 11}{36} > \frac{5}{3},$$

$$t_2x^2 + z^2 \leq t_2x_2^2 + z_2^2 = \frac{26 + 10\sqrt{5}}{27} \cdot \frac{9}{16} + \frac{100}{81} < \frac{13 + 5\sqrt{5}}{24} + \frac{30}{24} = \frac{13 + 5\sqrt{5} + 30}{24} = \frac{43 + 5\sqrt{5}}{24} < \frac{43 + 12}{24} = \frac{55}{24} = \frac{7}{3},$$

because $11^2 < 5^2 \cdot 5 < 12^2$ and $11 < 5\sqrt{5} < 12$. So

$$\frac{5}{3} < t_2x^2 + z^2 < \frac{7}{3}.$$
First estimate \( \frac{\partial F^*_z}{\partial x} \)

\[
\frac{\partial F^*_z}{\partial x} = 2(2 - z) \left[-(2 - x)(t_2x^2 + z^2) + t_2x(t_1 + (2 - x)^2)\right],
\]

Estimate the third cofactor

\[
-(2 - x)(t_2x^2 + z^2) + 2t_2x(t_1 + (2 - x)^2) \geq -\frac{11}{8} \cdot \frac{7}{3} + \frac{26 + 10\sqrt{5}}{27} \cdot \frac{5}{8} \cdot \frac{3}{2} = \frac{-77}{24} + \frac{65 + 25\sqrt{5}}{72} > \frac{-166}{24} + 26 + 10\sqrt{5} > -\frac{77}{24} + \frac{65 + 25\sqrt{5}}{72},
\]

because \( 5^2 \cdot 5 = 125 > 121 = 11^2 \) and \( 5\sqrt{5} > 11 \).

\[
-(2 - x)(t_2x^2 + z^2) + 2t_2x(t_1 + (2 - x)^2) \leq -\frac{1}{3} \cdot \frac{5}{3} + \frac{26 + 10\sqrt{5}}{27} \cdot \frac{3}{4} \cdot \frac{5}{2} = \frac{-5}{3} + \frac{65 + 25\sqrt{5}}{36} < \frac{5 + 60}{72} < 1,
\]

because \( 5^2 \cdot 5 = 125 < 144 = 12^2 \) and \( 5\sqrt{5} < 12 \).

So

\[
\left|-(2 - x)(t_2x^2 + z^2) + 2t_2x(t_1 + (2 - x)^2)\right| < \frac{25}{16}.
\]

In the end

\[
\left| \frac{\partial F^*_z}{\partial x} \right| \leq 2 \cdot |2 - z| \cdot \left|-(2 - x)(t_2x^2 + z^2) + 2t_2x(t_1 + (2 - x)^2)\right| \leq 2 \cdot 1 \cdot \frac{25}{16} = \frac{25}{8} < \frac{10}{3}.
\]

Now we estimate \( \frac{\partial F^*_z}{\partial z} \)

\[
\frac{\partial F^*_z}{\partial z} = (t_1 + (2 - x)^2) \left[-(t_2x^2 + z^2) + 2z(2 - z)\right],
\]

We will estimate the second factor

\[
-(t_2x^2 + z^2) + 2z(2 - z) \geq -\frac{7}{3} + 2 \cdot \frac{1}{9} = -\frac{21 + 16}{9} = -\frac{5}{9},
\]

\[
-(t_2x^2 + z^2) + 2z(2 - z) \leq -\frac{5}{3} + 2 \cdot \frac{10}{9} \cdot 1 = -\frac{15 + 20}{9} = \frac{5}{9},
\]

So

\[
\left|-(t_2x^2 + z^2) + 2z(2 - z)\right| < \frac{5}{9}.
\]

Hence

\[
\left| \frac{\partial F^*_z}{\partial z} \right| \leq \left|t_1 + (2 - x)^2\right| \cdot \left|-(t_2x^2 + z^2) + 2z(2 - z)\right| = \frac{7}{3} \cdot \frac{5}{9} = \frac{35}{27} < \frac{4}{3}.
\]
Estimation $F^*_2$ at the point $(x_0, z_0)$. We will estimate first

$$F^*_2(x_1, z_1) = (t_1 + (2 - x_1)^2)(t_2x_1^2 + z_1^2)(2 - z_1) =$$

$$= \left(10\sqrt{5} - 22 + \frac{121}{64}\right) \cdot \left(\frac{26 + 10\sqrt{5}}{27} \cdot \frac{25}{64} + 1\right) \cdot 1 <$$

$$< \left(\frac{179}{8} - 22 + \frac{61}{32}\right) \cdot \left(\frac{13 + 5\sqrt{5}}{32} + 1\right) < \frac{12 + 61}{32} \cdot \frac{45 + 5\sqrt{5}}{32} <$$

$$< \frac{73}{32} \cdot \frac{45 + 12}{32} = \frac{4161}{1024} < \frac{4224}{1024} = \frac{33}{8},$$

because

$$80^2 \cdot 5 = 32000 < 32041 = 179^2,$$

$$10\sqrt{5} < \frac{179}{8},$$

$$5^2 \cdot 5 = 125 < 144 = 12^2,$$

$$5\sqrt{5} < 12.$$

Then

$$(x_2 - x_1)^2 + (z_2 - z_1)^2 = \frac{1}{64} + 1 = \frac{64 + 81}{64 \cdot 81} = \frac{145}{64 \cdot 81} < \frac{196}{64 \cdot 81} = \left(\frac{14}{36}\right)^2 = \left(\frac{7}{36}\right)^2.$$ Now

$$\left(\frac{\partial F^*_2}{\partial x}\right)^2 + \left(\frac{\partial F^*_2}{\partial z}\right)^2 < \frac{100}{9} + \frac{16}{9} < \frac{121}{9} = \left(\frac{11}{3}\right)^2.$$ Combining the results obtained

$$F^*_2(x_0, z_0) \leq F^*_2(x_1, z_1) + \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2} \cdot \max_{x_1 \leq x \leq x_2, z_1 \leq z \leq z_2} \sqrt{\left(\frac{\partial F^*_2}{\partial x}\right)^2 + \left(\frac{\partial F^*_2}{\partial z}\right)^2} \leq$$

$$\leq \frac{33}{8} + \frac{7}{36} = \frac{33}{8} + \frac{77}{108} < \frac{33}{8} + \frac{81}{108} = \frac{33}{8} + \frac{3}{4} < 5.$$ On the other hand we can estimate (31)

$$\frac{64}{27} \left(5\sqrt{5} - 9\right) = \frac{1}{27} \left(320\sqrt{5} - 576\right) > \frac{715}{27} = \frac{139}{27} > \frac{138}{27} = 5,$$

because $320^2 \cdot 5 = 512000 > 511225 = 715^2$ and $320\sqrt{5} > 715$.

So the value of $F^*_2$ at the point $(x_0, z_0)$ does not exceed (31) which was to be proved.

### 3.4.6 Final score

Combining the results obtained above we see that

$$\max F_2 = \frac{64 \left(5\sqrt{5} - 9\right)}{27}. \Box$$
4 Proof of the estimates for the critical parallelepiped

As noted above it is known that
\[ V_{3,1} \geq 2, \quad [9] \]  
\[ V_{4,2} \geq \frac{16}{9}, \quad [21, 22] \]  
(37)  
(38)

We will prove these and two other results. Note that the proof procedure will differ from [9, 21, 22].

4.1 The idea of proof

We return to the proof of the estimates obtained in [2,3].

We will consider the matrices of the following kind

\[
A_* = \begin{pmatrix}
a & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & a & \cdots & 0 & 0 & \cdots & 0 & 0 \\
& & \cdots & & & & & \\
0 & 0 & \cdots & a & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & a_1 & a_1 & \cdots & 0 & 0 \\
& & \cdots & & & & & \\
0 & 0 & \cdots & 0 & -a_1 & a_1 & \cdots & 0 & 0 \\
& & \cdots & & & & & \\
0 & 0 & \cdots & 0 & 0 & \cdots & a_k & a_k \\
0 & 0 & \cdots & 0 & 0 & \cdots & -a_k & a_k \\
\end{pmatrix}
\]

Then

\[
A_*^{-1} = \begin{pmatrix}
\frac{1}{a} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{a} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
& & \cdots & & & & & \\
0 & 0 & \cdots & \frac{1}{a} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \frac{1}{2a_1} & \frac{1}{2a_1} & \cdots & 0 & 0 \\
& & \cdots & & & & & \\
0 & 0 & \cdots & 0 & -\frac{1}{2a_1} & \frac{1}{2a_1} & \cdots & 0 & 0 \\
& & \cdots & & & & & \\
0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{2a_k} & \frac{1}{2a_k} \\
0 & 0 & \cdots & 0 & 0 & \cdots & -\frac{1}{2a_k} & \frac{1}{2a_k} \\
\end{pmatrix}
\]
The task (20) takes the form

\[ f_{n,[n/2]} = \frac{1}{2^{n/2}} \prod_{i=1}^{[n/2]} |x_1^2 + x_{[n/2]+1}^2| \prod_{i=2}^{n} |x_i| \rightarrow \max, \]

\[ \frac{x_1}{a} \leq 1, \quad \cdots \quad \frac{x_n-2k}{a} \leq 1, \]

\[ \frac{x_{n-2k+1}}{2a_1} + \frac{x_{n-2k+2}}{2a_1} \leq 1, \quad \frac{x_{n-2k+1} - x_{n-2k+2}}{2a_1} \leq 1, \]

\[ \frac{x_{n-1}}{2a_k} + \frac{x_n}{2a_k} \leq 1, \quad \frac{x_{n-1} - x_n}{2a_k} \leq 1 \]

Making the replacement

\[ x_i = ay_i, \quad i = 1, n-2k \]

\[ x_{n-2(k-i)-1} = a_i y_{n-2(k-i)-1}, \quad x_{n-2(k-i)} = a_i y_{n-2(k-i)}, \quad i = 1, k \] (39)

the task takes the form

\[ f_{n,[n/2]} = \frac{1}{2^{n/2}} \prod_{i=1}^{[n/2]} |x_1^2 + x_{[n/2]+1}^2| \prod_{i=2}^{n} |x_i| \rightarrow \max, \]

\[ |y_1| \leq 1, \quad \cdots \quad |y_{n-2k}| \leq 1, \]

\[ |y_{n-2k+1} + y_{n-2k+2}| \leq 2, \quad |y_{n-2k+1} - y_{n-2k+2}| \leq 2, \] (40)

\[ |y_{n-1} + y_n| \leq 2, \quad |y_{n-1} - y_n| \leq 2 \]

In this task the restrictions do not depend on the original matrix \( A \). This property we will use later.

### 4.2 Estimate for \( V_{3,1} \)

**Theorem 13.**

\[ V_{3,1} \geq 2 \] (41)

**Proof.**

To prove this statement let's return to the task (40). As a matrix \( A_n \) we will consider

\[ A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}. \]

so

\[ V_{3,1} \geq \det A_3 = 2. \]

Hence the task (40) will take the form (by (39))

\[ f_{3,1} = \frac{1}{2} \left( x_1^2 + x_2^2 \right) |x_3| = \frac{1}{2} \left( y_1^2 + y_2^2 \right) |y_3| \rightarrow \max, \]
Let us prove that $\max f_{3,1} \leq 1$.
Note that the greatest value is achieved with $|y_1| = 1$. Indeed let there be a maximum such that
$\max f_{3,1} = f_{3,1}(\delta, y_2, y_3)$ where $|\delta| < 1$. Then
\[
f_{3,1}(\delta, y_2, y_3) = \frac{1}{2} (\delta^2 + y_2^2) |y_3| \leq \frac{1}{2} (1 + y_2^2) |y_3| = f_{3,1}(1, y_2, y_3).
\]
Contradiction. So $|y_1| = 1$.
Thus it is enough to prove that $\max f^*_{3,1} \leq 1$ under the condition
\[
|y_2 + y_3| \leq 2, \quad |y_2 - y_3| \leq 2
\]
where
\[
f^*_{3,1} = \frac{1}{2} (1 + y_2^2) |y_3|.
\]
We have
\[
f^*_{3,1} = \frac{1}{2} (1 + y_2^2) |y_3| = \frac{1}{2} F_0(y_2, y_3),
\]
where
\[
F_0(a, b) = (1 + a^2) |b|.
\]
By the theorem 10
\[
\max F_0(a, b) = 2
\]
under constraints (42). Hence $f^*_{3,1} \leq 1$. The theorem is proved. \(\square\)

4.3 Estimate for $V_{4,2}$

Theorem 14.

\[
V_{4,2} \geq \frac{16}{9}
\]

Proof.

The proof will be carried out similarly to the theorem 13. As a matrix $A_n$ we will consider
\[
A_4 = \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & \sqrt{2\alpha} & \sqrt{2\alpha} \\
0 & 0 & -\sqrt{2\alpha} & \sqrt{2\alpha}
\end{pmatrix}.
\]
where
\[
\alpha = \sqrt{\frac{2}{3}}
\]
so
\[
V_{4,2} \geq \det A_2 = \alpha^2 \cdot 4\alpha^2 = 4 \cdot \alpha^4 = 4 \cdot \frac{4}{9} = \frac{16}{9}.
\]
Then the task (40) will take the form
\[
f_{4,2} = \frac{1}{4} \left( x_1^2 + x_3^2 \right) \left( x_2^2 + x_4^2 \right) = \frac{1}{4} \left( \alpha^2 y_1^2 + 2 \alpha^2 y_3^2 \right) \left( \alpha^2 y_2^2 + 2 \alpha^2 y_4^2 \right) \to \max,
\]
\[
|y_1| \leq 1, \quad |y_2| \leq 1, \\
|y_3 + y_4| \leq 2, \quad |y_3 - y_4| \leq 2.
\]
Note that
\[
f_{4,2} = \frac{1}{4} \left( \alpha^2 y_1^2 + 2 \alpha^2 y_3^2 \right) \left( \alpha^2 y_2^2 + 2 \alpha^2 y_4^2 \right) = \alpha^4 \cdot \left( \frac{y_1^2}{2} + y_3^2 \right) \left( y_2^2 + y_4^2 \right).
\]
We will prove that \( \max f_{4,2} \leq 1 \).
Analogously to the proof of the theorem 13 we note that \( |y_1| = |y_2| = 1 \) and we come to the restrictions
\[
|y_3 + y_4| \leq 2, \quad |y_3 - y_4| \leq 2. \tag{44}
\]
So
\[
f^*_{4,2} = \alpha^4 \cdot \left( \frac{1}{2} + y_3^2 \right) \left( \frac{1}{2} + y_4^2 \right) = \alpha^4 \cdot F_1(y_3, y_4),
\]
where
\[
F_1(a, b) = \left( \frac{1}{2} + a^2 \right) \left( \frac{1}{2} + b^2 \right).
\]
By the theorem 9
\[
\max F_1(a, b) = \left( \frac{3}{2} \right)^2
\]
under restrictions (44). Then
\[
f^*_{4,2} \leq \alpha^4 \cdot \left( \frac{3}{2} \right)^2 = \left( \frac{2}{3} \right)^2 \cdot \left( \frac{3}{2} \right)^2 = 1.
\]
The theorem is proved. \( \square \)

4.4 Estimate for \( V_{5,2} \)

**Theorem 15.**
\[
V_{5,2} \geq \sqrt{\frac{27 \left( 9 + 5\sqrt{5} \right)}{88}} \approx 2.48831 \tag{45}
\]

**Proof.**
The proof will be carried out similarly to the theorem 13. As a matrix \( A_n \) we will consider
\[
A_5 = \begin{pmatrix}
\alpha & 0 & 0 & 0 & 0 \\
0 & \alpha \beta & \alpha \beta & 0 & 0 \\
0 & -\alpha \beta & \alpha \beta & 0 & 0 \\
0 & 0 & 0 & \alpha \beta \gamma & \alpha \beta \gamma \\
0 & 0 & 0 & -\alpha \beta \gamma & \alpha \beta \gamma
\end{pmatrix}
\]

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\[ \alpha = \sqrt{7 - 3\sqrt{5}} \cdot 10^{\frac{134 + 60\sqrt{5}}{27}}, \quad \beta = \sqrt{\frac{1}{10\sqrt{5} - 22}}, \quad \gamma = \sqrt{\frac{27}{26 + 10\sqrt{5}}} \]

so

\[ V_{5,2} \geq \det A_5 = \alpha \cdot 2\alpha^2\beta^2 \cdot 2\alpha^2\beta^2\gamma^2 = 4 \cdot \alpha^5 \cdot \beta^4 \cdot \gamma^2 = \]

\[= 4 \cdot (7 - 3\sqrt{5})^2 \sqrt{\frac{(7 - 3\sqrt{5}) (134 + 60\sqrt{5})}{27}} \cdot \frac{1}{(10\sqrt{5} - 22)^2} \cdot \frac{27}{26 + 10\sqrt{5}} = \]

\[= \frac{4(49 - 42\sqrt{5} + 45)}{(500 - 440\sqrt{5} + 484)(26 + 10\sqrt{5})} \cdot \sqrt{27 (938 + 420\sqrt{5} - 402\sqrt{5} - 900)} = \]

\[= \frac{94 - 42\sqrt{5}}{4(123 - 55\sqrt{5})(13 + 5\sqrt{5})} \cdot \sqrt{54 (19 + 9\sqrt{5})} = \]

\[= \frac{47 - 21\sqrt{5}}{2(1599 - 715\sqrt{5} + 615\sqrt{5} - 1375)} \cdot \sqrt{54 (19 + 9\sqrt{5})} = \]

\[= \frac{47 - 21\sqrt{5}}{2(224 - 100\sqrt{5})} \cdot \sqrt{54 (19 + 9\sqrt{5})} = \sqrt{\frac{54 (19 + 9\sqrt{5}) (47 - 21\sqrt{5})^2}{64 (56 - 25\sqrt{5})^2}} = \]

\[= \sqrt{\frac{27 (19 + 9\sqrt{5}) (2209 - 1974\sqrt{5} + 2205)}{32 (3136 - 2800\sqrt{5} + 3125)}} = \]

\[= \sqrt{\frac{27 (19 + 9\sqrt{5}) (2207 - 987\sqrt{5}) (6261 + 2800\sqrt{5})}{16 (6261 - 2800\sqrt{5}) (6261 + 2800\sqrt{5})}} = \]

\[= \sqrt{\frac{27 (19 + 9\sqrt{5}) ((2207 \cdot 6261 - 987 \cdot 2800 \cdot 5) + (-987 \cdot 6261 + 2207 \cdot 2800)\sqrt{5})}{16 \cdot (39200121 - 39200000)}} = \]

\[= \sqrt{\frac{27 (19 + 9\sqrt{5}) (27 - 7\sqrt{5})}{16 \cdot 121}} = \sqrt{\frac{27 (513 - 133\sqrt{5} + 243\sqrt{5} - 315)}{16 \cdot 121}} = \]

\[= \sqrt{\frac{27 (198 + 110\sqrt{5})}{16 \cdot 121}} = \sqrt{\frac{27 (9 + 5\sqrt{5})}{88}}. \]

Then the task (40) will take the form

\[ f_{5,2} = \frac{1}{4} (x_1^2 + x_3^2) (x_2^2 + x_4^2) |x_5| = \]

\[= \frac{1}{4} (\alpha^2 y_1^2 + \alpha^2 \beta^2 y_3^2) \cdot (\alpha^2 \beta^2 y_2^2 + \alpha^2 \beta^2 \gamma^2 y_4^2) \cdot |\alpha \beta \gamma y_5| \rightarrow \text{max}, \]

\[ |y_1| \leq 1, \]

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\[ |y_2 + y_3| \leq 2, \quad |y_2 - y_3| \leq 2, \]
\[ |y_1 + y_5| \leq 2, \quad |y_4 - y_5| \leq 2. \]

Note that
\[ f_{5,2} = \frac{1}{4} (\alpha^2 y_2^2 + \alpha^2 \beta^2 y_3^2) \cdot (\alpha^2 \beta^2 y_2^2 + \alpha^2 \beta^2 \gamma^2 y^2_4) \cdot |\alpha \beta \gamma y_5| = \]
\[ = \frac{\alpha^5 \beta^3 \gamma^3}{4} \cdot (y_1^2 + \beta^2 y_4^2) (y_2^2 + \gamma^2 y_4^2) |y_5| = \]
\[ = \frac{\alpha^5 \beta^5 \gamma^3}{4} \cdot \left( \frac{y_1^2}{\beta^2} + y_6^2 \right) \left( \frac{y_2^2}{\gamma^2} + y_4^2 \right) |y_5|. \]

We will prove that \( \max f_{5,2} \leq 1 \).
Analogously to the proof of the theorem 13, we note that \( |y_1| = 1 \) and we come to the restrictions
\[ |y_2 + y_3| \leq 2, \quad |y_2 - y_3| \leq 2, \quad |y_4 + y_5| \leq 2, \quad |y_4 - y_5| \leq 2. \quad (46) \]

So
\[ f_{5,2}^* = \frac{\alpha^5 \beta^5 \gamma^3}{4} \cdot \left( \frac{1}{\beta^2} + y_6^2 \right) \left( \frac{y_2^2}{\gamma^2} + y_4^2 \right) |y_5| = \frac{\alpha^5 \beta^5 \gamma^3}{4} \cdot F_2(y_2, y_3, y_4, y_5), \]
where
\[ F_2(a, b, c, d) = \left( \frac{1}{\beta^2} + b^2 \right) \left( \frac{a^2}{\gamma^2} + c^2 \right) |d|. \]

By the theorem 12
\[ \max F_2(a, b, c, d) = \frac{64 (5\sqrt{5} - 9)}{27}, \]
under restrictions (46). Then
\[ f_{5,2}^* \leq \frac{\alpha^5 \beta^5 \gamma^3}{4} \cdot \frac{64 (5\sqrt{5} - 9)}{27} = \frac{(7 - 3\sqrt{5})^2}{4} \sqrt{\frac{(7 - 3\sqrt{5}) (134 + 60\sqrt{5})}{27}}. \]

\[ = \frac{1}{(10\sqrt{5} - 22)^2 \sqrt{10\sqrt{5} - 22}} \cdot \frac{27}{26 + 10\sqrt{5}} \sqrt{\frac{27}{26 + 10\sqrt{5}} \cdot \frac{64 (5\sqrt{5} - 9)}{27} = \]
\[ = \frac{16 (49 - 42\sqrt{5} + 45) (5\sqrt{5} - 9)}{(500 - 440\sqrt{5} + 484) (26 + 10\sqrt{5})} \cdot \frac{(7 - 3\sqrt{5}) (134 + 60\sqrt{5})}{(10\sqrt{5} - 22) (26 + 10\sqrt{5})} = \]
\[ = \frac{32 (47 - 21\sqrt{5}) (5\sqrt{5} - 9)}{16 (123 - 55\sqrt{5}) (13 + 5\sqrt{5})} \cdot \frac{938 + 420\sqrt{5} - 402\sqrt{5} - 900}{4 (5\sqrt{5} - 11) (13 + 5\sqrt{5})} = \]
\[ = \frac{2 (235\sqrt{5} - 525 - 423 + 189\sqrt{5})}{1599 - 715\sqrt{5} + 615\sqrt{5} - 1375} \cdot \frac{2 (19 + 9\sqrt{5})}{4 (65\sqrt{5} + 125 - 143 + 55\sqrt{5})} = \]
\[ = \frac{2 (424\sqrt{5} - 948)}{224 - 100\sqrt{5}} \cdot \frac{2 (19 + 9\sqrt{5})}{4 (10\sqrt{5} - 18)} = \frac{2 (106\sqrt{5} - 237)}{56 - 25\sqrt{5}} \cdot \frac{\sqrt{19 + 9\sqrt{5}}}{4 (5\sqrt{5} - 9)}. \]
\[
\sqrt{(106\sqrt{5} - 237)^2 (19 + 9\sqrt{5}) (5\sqrt{5} + 9) / \sqrt{(56 - 25\sqrt{5})^2 (5\sqrt{5} - 9) (5\sqrt{5} + 9)}} =
\]
\[
\sqrt{(56180 - 50244\sqrt{5} + 56169) (95\sqrt{5} + 171 + 225 + 81\sqrt{5}) / (3136 - 2800\sqrt{5} + 3125) \cdot 44} =
\]
\[
\sqrt{(112349 - 50244\sqrt{5}) (396 + 176\sqrt{5}) / 44 (6261 - 2800\sqrt{5})} =
\]
\[
\sqrt{(112349 - 50244\sqrt{5}) (9 + 4\sqrt{5}) (6261 + 2800\sqrt{5}) / (6261 - 2800\sqrt{5}) (6261 + 2800\sqrt{5})} =
\]
\[
\sqrt{(112349 - 50244\sqrt{5}) (56349 + 25044\sqrt{5} + 25200\sqrt{5} + 56000) / 39200121 - 39200000} =
\]
\[
\sqrt{(112349 - 50244\sqrt{5}) (112349 + 50244\sqrt{5}) / 121} =
\]
\[
\sqrt{12622297801 - 12622297680 / 121} = \sqrt{121 / 121} = 1.
\]

The theorem is proved. □

4.5 Estimate for \( V_{6,3} \)

**Theorem 16.**

\[ V_{6,3} \geq \frac{9 + 5\sqrt{5}}{11} \approx 1.83458 \]  \hspace{1cm} (47)

**Proof.**

The proof will be carried out similarly to the theorem 13. As a matrix \( A_n \) we will consider

\[
A_6 = \begin{pmatrix}
\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha\beta & \alpha\beta & 0 & 0 \\
0 & 0 & -\alpha\beta & \alpha\beta & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha\beta & \alpha\beta \\
0 & 0 & 0 & 0 & -\alpha\beta & \alpha\beta
\end{pmatrix}
\]

where

\[
\alpha = \sqrt{\frac{8 (30\sqrt{5} - 67)}{11}}, \quad \beta = \frac{1}{\sqrt{10\sqrt{5} - 22}}
\]

so

\[ V_{6,3} \geq \det A_6 = \alpha^2 \cdot 2\alpha^2\beta^2 \cdot 2\alpha^2\beta^2 = 4 \cdot \alpha^6 \cdot \beta^4 = \]
Then the task (40) will take the form

\[
f_{6,3} = \frac{1}{8} (\alpha^2 y_1^2 + 2\beta^2 y_4^2) \cdot (\alpha^2 y_2^2 + 2\beta^2 y_5^2) \cdot (\alpha^2 y_3^2 + 2\beta^2 y_6^2) =
\]

\[
= \frac{\alpha^6}{8} \cdot (y_1^2 + \beta^2 y_4^2) (y_2^2 + \beta^2 y_5^2) (y_3^2 + \beta^2 y_6^2)
\]

\[
= \frac{\alpha^6 \beta^6}{8} \cdot \left( \frac{y_1^2}{\beta^2} + y_4^2 \right) \left( \frac{y_2^2}{\beta^2} + y_5^2 \right) \left( \frac{y_3^2}{\beta^2} + y_6^2 \right).
\]

We will prove that \( \max f_{6,3} \leq 1 \).

Analogously to the proof of the theorem 13 we note that \( |y_1| = |y_2| = 1 \) and we come to the restrictions

\[
|y_3 + y_4| \leq 2, \quad |y_3 - y_4| \leq 2, \quad |y_5 + y_6| \leq 2, \quad |y_5 - y_6| \leq 2.
\]

(48)

So

\[
f_{6,3} = \frac{\alpha^6 \beta^6}{8} \cdot \left( \frac{1}{\beta^2} + y_4^2 \right) \left( \frac{1}{\beta^2} + y_5^2 \right) \left( \frac{y_3^2}{\beta^2} + y_6^2 \right) = \frac{\alpha^6 \beta^6}{8} \cdot F_3(y_3, y_4, y_5, y_6),
\]

where

\[
F_3(a, b, c, d) = \left( \frac{1}{\beta^2} + a^2 \right) \left( \frac{1}{\beta^2} + c^2 \right) (b^2 + d^2).
\]

By the theorem 11

\[
\max F_3(a, b, c, d) = 64(56 - 25\sqrt{5})
\]

under restrictions (48). Then

\[
f_{6,3}^{\ast} \leq \frac{\alpha^6 \beta^6}{8} \cdot 64(56 - 25\sqrt{5}) = \frac{8(30\sqrt{5} - 67)}{11 \cdot 8 \cdot (10\sqrt{5} - 22)^3} \cdot 64(56 - 25\sqrt{5}) =
\]

\[
= \frac{64(30\sqrt{5} - 67)(56 - 25\sqrt{5})}{11(10\sqrt{5} - 22)^3} = \frac{64(1680\sqrt{5} - 3750 - 3752 + 1675\sqrt{5})}{11(5000\sqrt{5} - 33000 + 14520\sqrt{5} - 10648)} =
\]

\[
= \frac{214720\sqrt{5} - 480128}{214720\sqrt{5} - 480128} = 1.
\]

The theorem is proved. □
4.6 Estimate for a critical parallelepiped of arbitrary dimension

Consider the following general estimate for $V_{n,[n/2]}$.

**Theorem 17.** There is an estimate of

$$V_{n,[n/2]} \geq T_n \cdot \left(\frac{4}{3}\right)^{2[(n-3)/4]}$$

where

$$T_n = \begin{cases} 
2, & \text{if } n \equiv 3 \pmod{4}, \\
\frac{16}{9} \approx 1.7777..., & \text{if } n \equiv 0 \pmod{4}, \\
\sqrt{\frac{27(9+5\sqrt{5})}{88}} \approx 2.4883..., & \text{if } n \equiv 1 \pmod{4}, \\
\frac{9+5\sqrt{5}}{11} \approx 1.8345..., & \text{if } n \equiv 2 \pmod{4},
\end{cases}$$

**Proof.**

1. We will prove each estimate separately.

2. When $n \equiv 0 \pmod{4}$ then from the inequality \((16)\) follows that

$$V_{4k,2k} \geq (V_{4,2})^k = \left(\frac{16}{9}\right)^k = \left(\frac{4}{3}\right)^{2k}.$$ 

3. When $n \equiv 3 \pmod{4}$ we similarly have

$$V_{4k-1,2k-1} \geq V_{3,1} \cdot (V_{4,2})^{k-1} = 2 \cdot \left(\frac{16}{9}\right)^{k-1} = 2 \cdot \left(\frac{4}{3}\right)^{2(k-1)}.$$ 

4. When $n \equiv 1 \pmod{4}$ we similarly have

$$V_{4k+1,2k} \geq V_{5,2} \cdot (V_{4,2})^{k-1} = \sqrt{\frac{27(9+5\sqrt{5})}{88}} \cdot \left(\frac{16}{9}\right)^{k-1} = \sqrt{\frac{27(9+5\sqrt{5})}{88}} \cdot \left(\frac{4}{3}\right)^{2(k-1)}.$$ 

5. When $n \equiv 2 \pmod{4}$ we similarly have

$$V_{4k+2,2k+1} \geq V_{6,3} \cdot (V_{4,2})^{k-1} = \frac{9+5\sqrt{5}}{11} \cdot \left(\frac{16}{9}\right)^{k-1} = \frac{9+5\sqrt{5}}{11} \cdot \left(\frac{4}{3}\right)^{2(k-1)}.$$ 

The theorem is proved. □
Note 1. In the case of $n \equiv 0 \pmod{4}$ the estimate coincides with the Krass estimate \cite{17}.

In the case of $n \equiv 3 \pmod{4}$ the Krass estimate has the form

$$V_{4k-1,2k-1} > (16/9)^{[(4k-1)/4]} = (16/9)^{k-1}.$$  

So the result estimate is doubled improves the Krass estimate.

In the case of $n \equiv 1 \pmod{4}$ the Krass estimate has the form

$$V_{4k+1,2k} > (16/9)^{[(4k+1)/4]} = (16/9)^{k}.$$  

The obtained result somewhat improves the Krass estimate, since

$$\sqrt{\frac{27(9 + 5\sqrt{5})}{88}} \approx 2.48831... > 1.77777... \approx \frac{16}{9}.$$  

In the case of $n \equiv 2 \pmod{4}$ the Krass estimate has the form

$$V_{4k+2,2k+2} > (16/9)^{[(4k+2)/4]} = (16/9)^{k}.$$  

The obtained result somewhat improves the Krass estimate, since

$$\frac{9 + 5\sqrt{5}}{11} \approx 1.83458... > 1.77777... \approx \frac{16}{9}.$$  

5 Results

5.1 Minimal discriminants of some algebraic fields

In addition to $V_{n,s}$ the value $\Delta_{n,s}$ is included in the estimate \cite{11}. There are a lot of values $\Delta_{n,s}$ \cite{11} known however the calculation of this quantity is rather complex. The foundations of the methods of calculation $\Delta_{n,s}$ were laid by Mayer \cite{25} and Günter \cite{16}. Extensive results was made by Odlyzko \cite{32}. Now a lot of work in this direction is carried out by Klüner and Malle \cite{19,1}. They have built a large database of algebraic fields up to 19.

We give some values of $\Delta_{n,[n/2]}$ (with a sign) \cite{25,16,32,1} which will interest us for further estimates $C_n$.

| Field Degree | $\Delta_{n,[n/2]}$ | Decomposition $\Delta_{n,[n/2]}$ | Polynomial generating field with discriminant $n + 1$ |
|--------------|------------------|-------------------------------|--------------------------------------------------|
| 4            | -275             | $-5^2 \cdot 11$              | $x^4 - 2x^3 + x - 1$                             |
| 5            | 1609             | 1609                          | $x^5 - x^4 - x^3 + x^2 - 1$                       |
| 6            | 28037            | $23^2 \cdot 53$              | $x^6 + 3x^5 + x^4 - 2x^3 - x - 1$                 |
| 7            | -184607          | -184607                       | $x^7 - x^6 - x^5 + x^3 + x^2 - x - 1$             |
| n  | $a_n$       | $b_n$                | $c_n$                                                                 |
|----|------------|----------------------|----------------------------------------------------------------------|
| 8  | -4286875   | -54·193              | $x^8 - x^7 + x^5 - 2x^4 - x^3 + 2x^2 + 2x - 1$                        |
| 9  | 29510281   | 101·292181           | $x^9 - 3x^8 + 6x^7 - 8x^6 + 7x^5 - 3x^4 + 2x^2 - 2x + 1$             |
| 10 | -209352647 | -72·23·4312          | $x^{10} - 2x^9 + 3x^8 - 5x^7 + 9x^6 - 12x^5 + 13x^4 - 11x^3 + 7x^2 - 3x + 1$ |
| 11 | -5939843699| -12917·459847        | $x^{11} + x^9 - 2x^8 - 2x^7 - x^6 + 3x^4 + x^3 + x^2 - 1$              |

### 5.2 Estimates of the constant of the best Diophantine approximations

The results described above lead us to the following estimates $C_{n,s}$

\[
C_3 \geq \frac{2}{5\sqrt{11\cdot 16}} \approx 0.120605...
\]

\[
C_4 \geq \frac{9\sqrt{1609}}{\sqrt{1166}} \approx 0.044320...
\]

\[
C_5 \geq \frac{3}{46\sqrt{3\cdot (9 + 5\sqrt{5})}} \approx 0.014860...
\]

\[
C_6 \geq \frac{9 + 5\sqrt{5}}{11\sqrt{184607}} \approx 0.004269...
\]

\[
C_7 \geq \frac{4275\sqrt{19}}{256} \approx 0.001717...
\]

\[
C_8 \geq \frac{81\sqrt{29510281}}{256} \approx 0.000581...
\]

\[
C_9 \geq \frac{6}{9051\sqrt{3\cdot (9 + 5\sqrt{5})}} \approx 0.000229...
\]

\[
C_{10} \geq \frac{16}{99\sqrt{3\cdot 5939843699}} \approx 0.000042...
\]

For $n \geq 5$ these values improve the estimates given in [13].

### References

[1] A Database for Number Fields. [http://galoisdb.math.upb.de/](http://galoisdb.math.upb.de/)

[2] Adams W. W. Simultaneous Diophantine approximations and cubic irrationals. *Pacific J. Math.* 30 (1969) 1-14.

[3] Adams W. W. The best two-dimensional diophanite approximation constant for cubic irrationals. *Pacific J. Math.* 91 (1980) 29-30.
[4] Bernstein L. A 3-Dimensional Periodic Jacobi-Perron Algorithm of Period Length 8. *J. Number Theory* 4 (1972) 48-69.

[5] Blichfeldt H. A new principle in the geometry of numbers, with some applications. *Trans. Amer. Math. Soc.* 15 (1914) 227–235.

[6] Cassels J. W. S. Simultaneous Diophantine approximation. *J. London Math. Soc.* 30 (1955) 119-121.

[7] Cassels J. W. S. An Introduction to the Geometry of Numbers. Springer-Verlag (1959).

[8] Cusick J. W. Estimates for Diophantine approximation constants. *J. Number Theory* (1980) 543-556.

[9] Cusick J. W. The two dimensional diophantine approximation constant. *Pacific J. Math.* 105 (1983) 53-67.

[10] Davenport. H. On a theorem of Furtwängler. *J. London Math. Soc.* 30 (1955) 186-195.

[11] Dirichlet L. G. P. Verallgemeinerung eines Satzes aus der Lehre von den Kettenbruchenzahlen nebst einigen Anwendungen auf die Theorie der Zahlen. *S. B. Preuss. Akad. Wiss.* (1842) 93—95.

[12] Euler L. De relatione inter ternas plurisve quantitates instituenda // Petersburger Akademie Notiz. Exhib. August 14, 1775 // Commentationes arithmeticae collectae. V. II. St. Petersburg. (1849) 99-104.

[13] Finch S.R. Mathematical Constants. (2003) – (Encyclopedia of Mathematics and its Applications, Book 94).

[14] Fujita H. The minimum discriminant of totally real algebraic fields of degree 9 with cubic subfields. *Mathematics of Computation* 60 (1993) 801-810.

[15] Furtwängler H. Über die simultane Approximation von Irrationalzahlen *Math. Ann.* 96 (1927) 169-175.

[16] Hunter J. The minimum discriminant of quintic fields. *Proc. Glasgow Math. Assoc.* 3 (1957) 57-67.

[17] Hurwitz A. Über die angenaherte Darstellung der Irrationalzahlen durch rationale Böcke. *Math. Ann.* 39 (1891) 279—284.

[18] Jacobi C. G. J. Allgemeine Theorie der Kettenbruchanderlichen Algorithmen, in welchenjede Zahl aus drei vorhergehenden gebildet wird // J. Reine Angew. Math., 1868. V. 69. P. 29-64. // Gesammelte Werke, Bd. IV. Berlin: Reimer, 1891. S. 385-426

[19] Klüners J., Malle, G. A Database for Field Extensions of the Rationals. *LMS Journal of Computation and Mathematics* 4 (2001) 182-196.

[20] Koksma J., Meulenbeld B. Sur le theoreme de Minkowski, concernant un systeme de formes lineaires reelles. I, II, III, IV. *Kon. Nederl. Akad. Wetensch. Proc. Sect. Sci.* 45 (1942) 256–262, 354–359, 471–478, 578–584.
[21] Krass S. Estimates for $n$ -dimensional Diophantine approximation constants for $n \geq 4$. J. Number Theory (1985) 172-176.

[22] Krass S. The $N$ -dimensional diophantine approximation constants. J. Austral. Math. Soc. 32 (1985) 313-316.

[23] Lanker M., Petek P., Rugeji M. S. The continued fractions ladder of specific pairs of irrationals. https://arxiv.org/abs/1108.0087

[24] Mack J.M. Simultaneous Diophantine approximation. J. Austral. Math. Soc. 24 (1977) 266–285.

[25] Mayer J. Die absolut-kleinsten Diskriminanten der biquadratischen Zahlkorper. S.-B. Akad. Wiss. Wien Abt. Ila. 138 (1929) 733-742.

[26] Minkovski H. Geometrie der Zahlen. Berlin: Teubner (1896).

[27] Mordell L. Lattice points in some n-dimensional non-convex regions. I, II . Kon. Nederl. Akad. Wetensch Proc. Sect. Sci. 49 (1946) 773–781, 782–792.

[28] Mullender P. Lattice points in non-convex regions. Kon. Nederl. Akad. Wetensch. Proc. Sect. Sci. 51 (1948) 874–884.

[29] Murru N. On the Hermite problem for cubic irrationalit. https://arxiv.org/abs/1305.3285

[30] Nowak W. G. A note on simultaneous Diophantine approximation. Manuscr. Math. 36 (1981) 33-46.

[31] Nowak W. G. A remark concerning the $s$-dimensional simultaneous Diophantine approximation constants. Graz. Math. Ber. 318 (1993) 105–110.

[32] Odlyzko A. M. Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions : a survey of recent results. Journal de Théorie des Nombres de Bordeaux 2 (1990) 119-14.

[33] Perron O. Grundlagen fur eine Theorie des Jacobischen Ketten-bruchalgorithmus. Math. Ann. 64 (1907) 1-76.

[34] Schmidt. W. M. Diophantine approximations. Springer-Verlag (1980).

[35] Spohn W.G. Blichfeldt’s theorem and simultaneous Diophantine approximation. Amer. J. Math. 90 (1968) 885–894.

[36] Szekers G. The $n$ -dimensional approximation constant. Bull. Austral. Math. Soc. 29 (1984) 119-125.

[37] Woods A. C. The asymmetric product of three homogenous linear forms. Pacific J. Math. 93 (1981) 237–250.

[38] Bruno A. D. Algorithm of the generalized continued fraction. Preprint IAM of Keldysh 45 (2004).
6 Appendix 1

A program for finding the largest values of $V_{n,s}$ on a mathematical package Wolfram Mathematica.

```mathematica
logMessage = Function[{logFile, params, console},
   s = StringJoin[Map[Function[s, If[StringQ[s], s, ToString[s, InputForm]]],
     Prepend[params, DateString[] <> ' - ']]];
   w = OpenAppend[logFile, PageWidth -> 1000];
   Write[w, s];
   Close[w];
   If[console, Print[s]];
];

isCubeVerticesInsidedF = Function[{vertices2, f},
   inside = True;
   Do[
     inside = If[inside,
       w = Apply[f, x];
       Abs[w] <= 1,
       False];
     , {x, vertices2};
   inside]

isCubeDiagonalsInsidedF = Function[{vertices2, f},
   stepT = 0.3;
   verticesCount = Length[vertices2];
   inside = True;
   m = 0;
   Do[
     Do[
       v1 = vertices2[[i]];
       v2 = vertices2[[j]];
       d = v2 - v1;
       Do[
         x = v1 + d*t;
         inside = If[inside,
           w = Apply[f, x];
       x = v1 + d*t;
       inside = If[inside,
         w = Apply[f, x];
```
Abs[w] <= 1,
    False];
    , {t, 0, 1, stepT}];
    , {j, i + 1, verticesCount}];
    , {i, 1, verticesCount}];

inside
]

getMaxF = Function[{transform2, f, xParameter},
  n = Length[xParameter];
  m = transform2.xParameter;
  a = {Apply[f, xParameter]};
  Do[AppendTo[a, -1 <= m[[i]] <= 1], {i, Range[1, n]}];
  res1 = Check[
    NMaximize[a, xParameter, Method -> Automatic, AccuracyGoal -> 5, PrecisionGoal -> 5],
    NMaximize[a, xParameter, Method -> "DifferentialEvolution",
    AccuracyGoal -> 5, PrecisionGoal -> 5]]; 
  res2 = Check[
    NMinimize[a, xParameter, Method -> Automatic, AccuracyGoal -> 5, PrecisionGoal -> 5],
    NMinimize[a, xParameter, Method -> "DifferentialEvolution",
    AccuracyGoal -> 5, PrecisionGoal -> 5];
  res = Max[res1[[1]], -res2[[1]]];
  res
];

isCubeInsideF = Function[{transform, f, compiledF, xParameter, cubeVertices},
  transform2 = Inverse[transform];
  vertices2 = Map[Function[x, transform.x], cubeVertices];
  inside = isCubeVerticesInsideF[vertices2, compiledF];
  inside = If[inside, isCubeDiagonalsInsideF[vertices2, compiledF], False];
  inside = If[inside,
    fMax = getMaxF[transform2, f, xParameter];
    fMax <= 1,
    False];
  inside
];

iteration = Function[{f, compiledF, minVolume, xParameter, getTransformMatrix,
  a, b, intervals, vars, logFile},
  n = Length[xParameter];
  h = Map[Function[i, (b[[i]] - a[[i]]) / (intervals - 1.0)], Range[1, vars]]; 
  range = intervals`vars;
  degreeOfParallelizm = If[range > 100000, $ProcessorCount - 1, 1];
  cubeVertices = Tuples[{-1, 1}, n];

  coordsTransform = Function[point,
    Map[Function[i, a[[i]] + h[[i]]*point[[i]]], Range[1, vars]] ];

  getPoint = Function[i,
    t = i;
    point = {};
    Do[
AppendTo[point, Mod[t, intervals]];  
    t = Quotient[t, intervals];  
    , {j, vars}];  
    point
];  

partialRes = ParallelTable[
    maxVolume = minVolume;  
    mPoint = {};  
    prevPercent = 0;  
    j = 0;  
    prevJ = 0;  
    tt1 = AbsoluteTime[];
    Do[
        point = getPoint[i];  
        coords = coordsTransform[point];  
        transform = getTransformMatrix[coords];  
        det = Det[transform];  
        If[det > maxVolume,
            inside = isCubeInsideF[transform, f, compiledF, xParameter, cubeVertices];  
            volume = If[inside, det, 0];  
            If[volume > 0,
                maxVolume = volume;
                mPoint = volume, point, coords;
                logMessage[logFile, {'Thread ', thread, ' cube volume=', volume,
                    ' point=', point, ' transform=', transform}, False];]
        ];
    If[range > 1000000,
        curPercent = Floor[j++ * 100 / range];  
        If[curPercent > prevPercent,
            prevPercent = curPercent;
            tt2 = AbsoluteTime[];
            logMessage[logFile, {'Thread ', thread, ' Progress ',
                'curPercent', '% Performance ', Round[(j - prevJ) / (tt2 - tt1)],
                'FLOPS'}, False];
            prevJ = j;
            tt1 = tt2;];];
    , {i, thread - 1, range, degreeOfParallelizm}];
    coords = mPoint[[3]];
    {maxVolume, coords - h, coords + h}, {thread, degreeOfParallelizm}];
    res = partialRes[[1]];  
    Do[
        If[partialRes[[i]][[1]] > res[[1]], res = partialRes[[i]]];
    , {i, 2, degreeOfParallelizm}];
    res
];

solve = Function[{f, compiledF, xParameter, getTransformMatrix, a, b,
intervals, vars, iterations, logFile};

n = Length[xParameter];
intervals2 = intervals + 1;

a2 = ConstantArray[a, vars];
b2 = ConstantArray[b, vars];
prevVolume = 1;
curVolume = 1;

Do[
    t1 = AbsoluteTime[];
    res = iteration[f, compiledF, Min[prevVolume, curVolume] - 0.1, xParameter,
        getTransformMatrix, a2, b2, intervals2, vars, logFile];
    t2 = AbsoluteTime[];

    logMessage[logFile, {"iteration ", it + 1, " t=", N[t2 - t1], " volume= ",
        res[[1]], " a=", res[[2]], " b=", res[[3]]}, True];

    a2 = res[[2]];
b2 = res[[3]];
    prevVolume = curVolume;
curVolume = res[[1]];
intervals2 = 4
    , {it, 0, iterations}]

c = (a2 + b2) / 2;
cubeVertices = Tuples[-1,1, n];
transform = getTransformMatrix[c];
inside = isCubeInsideF[transform, f, compiledF, xParameter, cubeVertices];
volume = Det[transform];
transform2 = Inverse[transform];
logMessage[logFile, {"solve volume=", volume, " coords=", c, " inside=", inside,
    " transform=", transform, " restrict=", transform2}, True];

volume
];

SetDirectory[NotebookDirectory[]];
Import["core.m"];

vars = 3;
xParameter = {x1, x2, x3, x4, x5};

getTransformMatrix = Compile[{{coords, _Real, 1}},
    {{coords[[1]], 0, 0, 0, 0},
    {0, coords[[2]], coords[[2]], 0, 0},
    {0, -coords[[2]], coords[[2]], 0, 0},
    {0, 0, 0, coords[[3]], coords[[3]]},
    {0, 0, 0, -coords[[3]], coords[[3]]}}
];

f52 = Function[{x1, x2, x3, x4, x5}, (x1^2 + x3^2)*(x2^2 + x4^2)*x5/4.0];
compiledF52 = Compile[{x1, x2, x3, x4, x5}, (x1^2 + x3^2)*(x2^2 + x4^2)*x5/4.0];
solve[f52, compiledF52, xParameter, getTransformMatrix, 0.0, 2.0, 10, vars, 20, ‘’logV5s.txt’’];

7 Appendix 2

Numerical values of the largest matrices.

$$A_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & -1 & 1
\end{pmatrix}$$

$$\det A_3 = 2$$

$$A_4 \approx \begin{pmatrix}
0.81649 & 0 & 0 & 0 \\
0 & 0.81649 & 0 & 0 \\
0 & 0 & 1.15469 & 1.15469 \\
0 & 0 & -1.15469 & 1.15469
\end{pmatrix}$$

$$\det A_4 \approx 1.77777$$

$$A_5 \approx \begin{pmatrix}
0.67958 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.13157 & 1.13157 & 0 & 0 & 0 \\
0 & -1.13157 & 1.13157 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.84550 & 0.84550 & 0 \\
0 & 0 & 0 & -0.84550 & 0.84550 & 0
\end{pmatrix}$$

$$\det A_5 \approx 2.48831$$

$$A_6 \approx \begin{pmatrix}
0.62510 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.62510 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.04085 & 1.04085 & 0 & 0 & 0 \\
0 & 0 & -1.04085 & 1.04085 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.04085 & 1.04085 & 0 \\
0 & 0 & 0 & 0 & -1.04085 & 1.04085 & 0
\end{pmatrix}$$

$$\det A_6 \approx 1.83456$$