Lyapunov criteria for robust forward completeness of distributed parameter systems

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Abstract

We show that the robust forward completeness for distributed parameter systems is equivalent to the existence of a corresponding Lyapunov function that increases at most exponentially along the trajectories.

Keywords: Nonlinear systems, infinite-dimensional systems, forward completeness, reachability sets, Lyapunov methods

1. Introduction

A control system is called forward complete if for any initial condition \( x \), and any input \( u \), the corresponding trajectory \( \phi(\cdot, x, u) \) is well-defined on the whole nonnegative time axis. If additionally, for any magnitude \( R > 0 \) and any time \( \tau > 0 \)

\[
\sup\limits_{\|x\| \leq R, \ u \in D, \ t \in [0, \tau]} \|\phi(t, x, u)\| < +\infty,
\]

where \( D \) is the space of admissible inputs, then a control system is said to be robustly forward complete (the concept is coined by [7]).

Robust forward completeness (RFC), as well as a related concept of boundedness of reachability sets, are important in many contexts. They were instrumental for the derivation of converse Lyapunov theorems for global asymptotic stability [11]. Uniform global asymptotic stability for infinite-dimensional systems has been characterized in terms of uniform weak attractivity, local stability, and RFC property in [13]. Criteria for input-to-state stability in terms of uniform limit property, local stability, and boundedness of reachability sets were proved for general nonlinear control systems in [14]. These characterizations, in turn, paved the way for the development of non-coercive Lyapunov methods [15, 14, 5], characterization of global asymptotic stability for retarded systems [3], to name a few.

Sufficient conditions for the global existence of solutions for ordinary differential equations (ODEs) and other classes of control systems are a classic subject [20, 2, 6, 13, 4]. For instance, Wintner’s theorem [20] shows that an ODE

\[
\dot{x} = f(x)
\]

with locally Lipschitz \( f \) has unique global solutions provided that \( |f(x)| \leq L(|x|) \) with \( L \) satisfying

\[
\int_c^\infty \frac{1}{L(s)} ds = +\infty \quad \forall c > 0.
\]

In particular, if \( f \) is globally Lipschitz continuous or linearly bounded, the solutions for the above ODE exist globally, and the reachability sets are bounded (i.e., the system is RFC). This result can be extended to evolution equations in Banach spaces and other system classes, e.g., [16, Theorem 3.3, p. 199].

The analysis of necessary conditions for forward completeness is more recent. Necessary and sufficient conditions of Lyapunov type for forward completeness of ODEs without inputs have been proposed in [10]. However, Lyapunov functions constructed in [10] are time-variant even for time-invariant ODEs.

In [11, 1] for systems

\[
\dot{x} = f(x, u),
\]

with Lipschitz continuous (in both arguments) \( f \), it was shown that: forward completeness, boundedness of reachability sets for ODEs with inputs, and the existence of a Lyapunov function that increases at most exponentially, are equivalent properties.

For distributed parameter systems, the situation is more complex. Linear forward complete infinite-dimensional systems have always bounded reachability sets [19, Proposition 2.5]. However, nonlinear forward complete infinite-dimensional systems with Lipschitz continuous right-hand sides do not necessarily have bounded reachability sets, even for systems without inputs, as demonstrated in [14, Example 2]. This fact indicates that the RFC property (establishing uniform bounds for solutions on finite time intervals) is a bridge between the pure well-posedness theory (that studies existence and uniqueness but does not
care much about the bounds for solutions) and the stability theory (which is interested in establishing certain bounds for solutions for all nonnegative times, as well as their convergence).

In this work, we consider a broad class of control systems satisfying the so-called boundedness-implies-continuation property and having flows that are Lipschitz continuous on compact intervals. We show that for this class of systems, robust forward completeness is equivalent to the existence of a Lyapunov function that increases at most exponentially along the trajectories.

Our proof is different from that of [1], where a closely related result was shown for ODE systems. Namely, for ODEs with Lipschitz right-hand sides, local solutions exist not only in a positive direction but also in a negative direction. This fact was used for the construction of “RFC Lyapunov functions” in [1]. At the same time, for the class of systems that we consider, the solutions backward in time do not necessarily exist, and if they do, then they do not need to be unique. To overcome this challenge, we propose a different proof scheme motivated by the converse Lyapunov results for the UGAS property, e.g., [2, Theorem 4.2.1].

Notation. We write \( \mathbb{N}, \mathbb{R}, \) and \( \mathbb{R}_+ \) for the sets of positive integers, real numbers, and nonnegative real numbers, respectively. We say that \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) belongs to the class \( K \) if \( \gamma \) is continuous, \( \gamma(0) = 0 \), and \( \gamma \) is strictly increasing. \( \gamma \in K_\infty \) if \( \gamma \in K \) and it is unbounded.

For a normed vector space \( S \) we denote the open ball of radius \( r \) around \( 0 \in S \) by \( B_r(S) := \{u \in S : \|u\|_S < r\} \). If \( S \) is the state space \( X \), then we denote for short \( B_r := B_r(X) \).

2. General class of systems

We start with a general definition of a control system.

**Definition 2.1.** Consider the triple \( \Sigma = (X, \mathcal{U}, \phi) \) consisting of

(i) A normed vector space \( (X, \| \cdot \|_X) \), called the state space, endowed with the norm \( \| \cdot \|_X \).

(ii) A normed vector space of inputs \( \mathcal{U} \subset \{u : \mathbb{R}_+ \to U\} \) endowed with a norm \( \| \cdot \|_U \), where \( U \) is a normed vector space of input values. We assume that the following axiom holds:

The axiom of shift invariance: for all \( u \in \mathcal{U} \) and all \( \tau \geq 0 \) the time shift \( u(\cdot + \tau) \) belongs to \( \mathcal{U} \) with \( \|u\|_U \geq \|u(\cdot + \tau)\|_U \).

(iii) A map \( \phi : D_\phi \to X \), \( D_\phi \subseteq \mathbb{R}_+ \times X \times \mathcal{U} \) (called transition map), such that for all \( (x, u) \in X \times \mathcal{U} \) it holds that \( D_\phi \cap (\mathbb{R}_+ \times \{(x, u)\}) = [0, t_m(x, u)) \times \{(x, u)\} \subset D_\phi \), for a certain \( t_m = t_m(x, u) \in (0, +\infty) \).

The corresponding interval \([0, t_m)\) is called the maximal domain of definition of \( t \to \phi(t, x, u) \).

The triple \( \Sigma \) is called a (control) system, if the following properties hold:

(S1) The identity property: for every \( (x, u) \in X \times \mathcal{U} \) it holds that \( \phi(0, x, u) = x \).

(S2) Causality: for every \( (t, x, u) \in D_\phi \), for every \( \tilde{u} \in \mathcal{U} \), such that \( u(s) = \tilde{u}(s) \) for all \( s \in [0, t] \) it holds that \([0, t] \times \{(x, \tilde{u})\} \subset D_\phi \) and \( \phi(t, x, u) = \phi(t, x, \tilde{u}) \).

(S3) Continuity: for each \( (x, u) \in X \times \mathcal{U} \) the map \( t \to \phi(t, x, u) \) is continuous on its maximal domain of definition.

(S4) The cocycle property: for all \( x \in X, u \in \mathcal{U} \), for all \( t, h \geq 0 \) so that \([0, t + h] \times \{(x, u)\} \subset D_\phi \), we have \( \phi(h, \phi(t, x, u), u(t + h)) = \phi(t + h, x, u) \).

Definition 2.1 can be viewed as a direct generalization and a unification of the concepts of strongly continuous nonlinear semigroups with abstract linear control systems [19]. This class of systems encompasses control systems generated by ODEs, switched systems, time-delay systems, evolution partial differential equations, differential equations in Banach spaces and many others [8, Chapter 1].

For a wide class of control systems, the boundedness of a solution implies the possibility of prolonging it to a larger interval, see [8, Chapter 1]. Next, we formulate this property for abstract systems:

**Definition 2.2.** We say that a system \( \Sigma = (X, \mathcal{U}, \phi) \) satisfies the boundedness-implies-continuation (BIC) property if for each \( (x, u) \in X \times \mathcal{U} \) such that the maximal existence time \( t_m(x, u) \) is finite, and for all \( M > 0 \), there exists \( t \in [0, t_m(x, u)) \) with \( \|\phi(t, x, u)\|_X > M \).

Take any \( R \in \mathbb{R}_+ \cup \{\infty\} \), and assume that the inputs are restricted to the set

\[ \mathcal{D} := \{u \in \mathcal{U} : \|u\|_U \leq R\} \] (1)

**Definition 2.3.** We say that a control system \( \Sigma = (X, \mathcal{U}, \phi) \) is forward complete (for inputs in \( \mathcal{D} \)), if \( \mathbb{R}_+ \times X \times \mathcal{D} \subset D_\phi \), that is for every \( (x, u) \in X \times \mathcal{D} \) and for all \( t \geq 0 \) the value \( \phi(t, x, u) \in X \) is well-defined.

Forward completeness alone does not imply, in general, the existence of any uniform bounds on the trajectories emanating from bounded balls, even in the absence of inputs [14, Example 2]. If the system does exhibit such a bound, it deserves a special name (the term was first introduced in [5, Definition 2.2], though implicitly it was used before, e.g., in [11, Corollary 2.4]).

**Definition 2.4.** Consider a control system \( \Sigma = (X, \mathcal{U}, \phi) \), and let \( \mathcal{D} \) be the set of disturbances as defined by (1). We say that \( \Sigma \) is robustly forward complete (RFC) if \( \Sigma \) is forward complete for inputs in \( \mathcal{D} \), and for any \( C > 0 \) and any \( \tau > 0 \), it holds that

\[ \sup \{\|\phi(t, x, u)\|_X : \|x\|_X \leq C, u \in \mathcal{D}, t \in [0, \tau]\} < \infty. \]
For ODEs with Lipschitz continuous right-hand side, forward completeness is always robust provided that \( R < \infty \), as shown in \([11]\) Proposition 2.5. However, if \( \mathcal{D} = \mathcal{U} \), robust forward completeness is, in general, essentially stronger than forward completeness, even for scalar systems.

In particular, the scalar system \( \dot{x} = xu \) with \( X = \mathbb{R} \) and \( \mathcal{D} = \mathcal{U} := L^{\infty}(\mathbb{R}_+, \mathbb{R}) \) is forward complete, but it is not RFC. A simple example of a scalar RFC system is given by the following scalar system with the same \( X, \mathcal{U}, \mathcal{D} \):

\[
\dot{x} = \frac{1}{1 + |u(t)|^x}.
\]

3. Criteria for robust forward completeness

In this section, we derive Lyapunov criteria for RFC.

3.1. Technical lemmas

We need the following two simple lemmas

**Lemma 3.1.** Let \( f, g : \mathcal{D} \to \mathbb{R}_+ \) be any functions for which \( \sup_{d \in \mathcal{D}} f(d) \) is finite. Then

\[
\sup_{d \in \mathcal{D}} f(d) - \sup_{d \in \mathcal{D}} g(d) \leq \sup_{d \in \mathcal{D}} (f(d) - g(d)).
\]

**Lemma 3.2.** For any \( k \in \mathbb{N} \), consider the function

\[
G_k : r \mapsto \max\left\{ r - \frac{1}{k}, 0 \right\}.
\]

Then

(i) \( G_k \) is Lipschitz continuous with a unit Lipschitz constant, i.e., for all \( r_1, r_2 \geq 0 \) it holds that

\[
|G_k(r_1) - G_k(r_2)| \leq |r_1 - r_2|.
\]

(ii) For any \( a \geq 1 \) and any \( k \in \mathbb{N} \), it holds that

\[
G_k(ar) \leq aG_k(r) + \frac{a - 1}{k}, \quad r \geq 0.
\]

**Proof.** (i) holds as each \( G_k \) is the maximum of two Lipschitz continuous functions with Lipschitz constant at most 1.

(ii). For \( a \geq 1 \) we have that

\[
G_k(ar) = \max\left\{ ar - \frac{1}{k}, 0 \right\} = \max\left\{ ar - a\frac{1}{k} + \frac{a - 1}{k}, 0 \right\} \leq \max\left\{ ar - a\frac{1}{k}, 0 \right\} + \frac{a - 1}{k} = aG_k(r) + \frac{a - 1}{k}.
\]

The following lemma is taken from \([8]\) p.130] (where it was stated informally); see also \([12]\) Lemma A.18, where a more general result is stated.

**Lemma 3.3.** For any \( a \in \mathbb{K}_\infty \), there exists \( \rho \in \mathbb{K}_\infty \) so that \( \rho(s) \leq a(s) \) for all \( s \in \mathbb{R}_+ \) and \( \rho \) is globally Lipschitz with a unit Lipschitz constant, i.e. for any \( s_1, s_2 \geq 0 \) it holds that

\[
|\rho(s_1) - \rho(s_2)| \leq |s_1 - s_2|.
\]

For a continuous function \( y : \mathbb{R} \to \mathbb{R} \), let the right upper Dini derivative be defined by \( D^+y(t) := \lim_{h \to +0} \frac{y(t+h)-y(t)}{h} \).

**Proposition 3.4 (Comparison principle).** For any \( g \in C(\mathbb{R}_+, \mathbb{R}_+) \) satisfying the differential inequality

\[
D^+y(t) \leq ay(t) + M \quad \forall t > 0,
\]

with certain \( a, M > 0 \), it holds that

\[
y(t) \leq y(0)e^{at} + \frac{M}{a}(e^{at} - 1) \quad \forall t \geq 0.
\]

**Proof.** Define \( z(t) := y(t) - \frac{M}{a}(e^{at} - 1), \quad t \geq 0 \). Then

\[
D^+z(t) = D^+y(t) - Me^{at}
\]

\[
\leq a\left(z(t) + \frac{M}{a}(e^{at} - 1)\right) + M - Me^{at} = az(t).
\]

Arguing as in \([13]\) the proof of Lemma 3.2, p. 464], we obtain the counterpart of the estimate \([13]\) eq. (41) for \( z \):

\[
z(t) \leq \eta^{-1}(\eta(z(0)) + at),
\]

with \( h(s) = \ln(s) \), which implies that \( z(t) \leq z(0)e^{at} \), and thus \([8]\) holds.

3.2. Lyapunov characterization of RFC

We call a function \( h : \mathbb{R}_+^3 \to \mathbb{R}_+ \) increasing, if \((r_1, r_2, r_3) \leq (R_1, R_2, R_3) \) implies that \( h(r_1, r_2, r_3) \leq h(R_1, R_2, R_3) \), where we use the component-wise partial order on \( \mathbb{R}_+^3 \).

The regularity of Lyapunov functions, constructed via converse Lyapunov techniques, depends on the regularity of the flow map.

**Definition 3.5.** Let \( \mathcal{D} \) be the set of disturbances as defined by \([11]\). We say that the flow of a control system \( \Sigma = (X, \mathcal{U}, \phi) \) is Lipschitz continuous on compact intervals uniformly in inputs from \( \mathcal{D} \), if for any \( \tau > 0 \) and any \( r > 0 \), there exists \( L = L(\tau, r) > 0 \) so that

\[
x, y \in B_r, \quad t \in [0, \tau], \quad u \in \mathcal{D}
\]

\[
\Rightarrow \quad ||\phi(t, x, u) - \phi(t, y, u)||_X \leq L||x - y||_X.
\]

We assume that the axiom of concatenation is valid for the inputs in \( \mathcal{D} \).

**Assumption 3.1 (The axiom of concatenation).** We suppose that for all \( u_1, u_2 \in \mathcal{D} \) and for all \( t > 0 \) the concatenation of \( u_1 \) and \( u_2 \) at time \( t \), defined by

\[
u_1 \dot{\diamond} u_2(t) := \begin{cases} u_1(t), & \text{if } t \in [0, t], \\ u_2(t - t), & \text{otherwise,} \end{cases}
\]

belongs to \( \mathcal{D} \).
Remark 3.6. If $\mathcal{D} = \mathcal{U}$, Assumption 3.1 is satisfied for most of the standard input spaces. If $\mathcal{D}$ is a bounded ball (i.e., $R$ in (1) is finite), Assumption 3.1 becomes more restrictive as then the norm of the concatenation of two inputs from $\mathcal{D}$ cannot exceed the maximum of the norm of the concatenated inputs. In particular, Assumption 3.1 is valid with $R < \infty$ if $U$ is a Banach space, and $\mathcal{U} = L^p(\mathbb{R}_+, U)$ (the space of essentially bounded strongly measurable $U$-valued functions), if $\mathcal{U} = C_b(\mathbb{R}_+, U)$ (the space of continuous bounded $U$-valued functions), or if $\mathcal{U} = PC_b(\mathbb{R}_+, U)$ (the space of piecewise continuous bounded $U$-valued functions). At the same time, Assumption 3.1 is not valid with finite $R$ for $\mathcal{U} = L^p(\mathbb{R}_+, U)$, $p \in [1, +\infty)$ (the space of strongly measurable functions $u : \mathbb{R}_+ \rightarrow U$ such that $s \mapsto \|u(s)\|_U^p$ is Lebesgue integrable).

Next, we characterize the RFC property in Lyapunov terms

**Theorem 3.7 (Criteria for RFC property).**
Consider a control system $\Sigma = (X, \mathcal{U}, \phi)$. Let $\mathcal{D}$ be the set of disturbances as defined by (1) satisfying Assumption 3.1. Let $\Sigma$ satisfy the BIC property and have a flow that is Lipschitz continuous on compact intervals uniformly in inputs from $\mathcal{D}$.

The following statements are equivalent:

(i) $\Sigma$ is robustly forward complete.

(ii) There exists a continuous, increasing function $\mu : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, such that for all $x \in X$, $u \in \mathcal{D}$ and all $t \geq 0$ we have

$$\|\phi(t, x, u)\|_X \leq \mu(\|x\|_X, t). \tag{11}$$

(iii) There exists a continuous function $\mu : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ such that for all $x \in X$, $u \in \mathcal{D}$ and all $t \geq 0$ the inequality (11) holds.

(iv) There are $\xi \in K_\infty$ and $c > 0$, such that for all $t \geq 0$, $x \in X$, $u \in \mathcal{D}$

$$\|\phi(t, x, u)\|_X \leq \xi(\|x\|_X) + \xi(t) + c. \tag{12}$$

(v) There are Lipschitz continuous functions $V : X \rightarrow \mathbb{R}_+$, maps $\psi_1, \psi_2 \in K_\infty$, and $C > 0$ such that

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X) + C, \quad x \in X, \tag{13}$$

and there are $a, M > 0$, such that for all $x \in X$ and $u \in \mathcal{D}$, the following holds:

$$\hat{V}_a(x) \leq aV(x) + M, \tag{14}$$

where $\hat{V}_a(x)$ denotes the right upper Dini derivative of the map $t \mapsto V(\phi(t, x, u))$ at $t = 0$, i.e.,

$$\hat{V}_a(x) := \lim_{t \rightarrow 0^+} \frac{1}{t} (V(\phi(t, x, u)) - V(x)). \tag{15}$$

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). This was shown in Lemma 2.12.

(ii) $\Rightarrow$ (iv). Define $\zeta(r) := \mu(r, r) + r, r \geq 0$. As $\mu$ is increasing and continuous, $\zeta$ is strictly increasing and continuous. Now, for all $x, u, t$, we have

$$\mu(\|x\|_X, t) \leq \zeta(\|x\|_X + \zeta(t)).$$

Define $\xi(r) := \zeta(r) - \lim_{r \rightarrow 0^+} \zeta(r)$. Then $\xi \in K_\infty$, and (12) holds with this $\xi$ and $c := 2 \lim_{r \rightarrow 0^+} \zeta(r)$.

(v) $\Rightarrow$ (i). Pick any $x \in X$ and any $u \in \mathcal{D}$. As $\Sigma$ is a well-defined control system, there is a maximal time $t_m(x, u)$ such that $\phi(\cdot, x, u)$ is well-defined on $[0, t_m(x, u))$. By the axiom of shift invariance, $u(\cdot + r) \in \mathcal{U}$ and $\|u\|_U \geq \|u(\cdot + r)\|_U$ for any $r \geq 0$. This shows that also $u(\cdot + r) \in \mathcal{D}$, and thus the set $\mathcal{D}$ is invariant w.r.t. time-shift of the input signal as well. Thus, by (14), for all $t \in [0, t_m(x, u))$ we have that

$$D^+V(\phi(t, x, u)) = \hat{V}_a(x) \leq aV(\phi(t, x, u)) + M.$$

Employing Proposition 3.4 for the continuous map

$$y(t) := V(\phi(t, x, u)), \quad t \in [0, t_m(x, u)),$$

we obtain:

$$V(\phi(t, x, u)) \leq e^{at}V(x) + \frac{M}{a}(e^{at} - 1), \quad t \in [0, t_m(x, u)).$$

Thanks to the “sandwich inequality” (13), we have for all $t \in [0, t_m(x, u))$ that

$$\psi_1(\|\phi(t, x, u)\|_X) \leq e^{at}(\psi_2(\|x\|_X) + C) + \frac{M}{a}(e^{at} - 1),$$

and thus

$$\|\phi(t, x, u)\|_X \leq \psi_1^{-1}(e^{at}(\psi_2(\|x\|_X) + C) + \frac{M}{a}(e^{at} - 1)). \tag{16}$$

Now, if $t_m(x, u)$ is finite, the trajectory $\phi(\cdot, x, u)$ is uniformly bounded on $[0, t_m(x, u))$, and we obtain a contradiction to the BIC property. Hence, $t_m(x, u) = +\infty$, and (16) shows the robust forward completeness.

(v) $\Rightarrow$ (v). This implication (converse Lyapunov result) will be proved in several steps.

**Construction of “pre-Lyapunov functions” $V_k$.** Let $\xi \in K_\infty$ be as in (iv). Pick $\rho \in K_\infty$ such that $\rho \leq \xi^{-1}$ pointwise and $\rho$ is globally Lipschitz continuous with a unit Lipschitz constant. Such $\rho$ exists in view of Lemma 3.3.

For any $k \in \mathbb{N}$, consider $V_k : X \rightarrow \mathbb{R}_+$, defined for all $x \in X$ as follows:

$$V_k(x) := \sup_{u \in \mathcal{D}} \sup_{t \geq 0} G_k\left(e^{-t}\rho\left(\frac{1}{3}\|\phi(t, x, u)\|_X\right)\right), \tag{17}$$

where $G_k$ are given by $3$. To upperestimate for $V_k$, recall that for all $\alpha \in K_\infty$ and all $a, b, c \in \mathbb{R}_+$ it holds that

$$\alpha(a + b + c) \leq \alpha(3a) + \alpha(3b) + \alpha(3c). \tag{18}$$
Take any $x \in X$. Using in \ref{17} the estimate \ref{12}, and the fact that $\rho \leq \xi^{-1}$ and $G_k(r) \leq r$ pointwise, we have:

$$V_k(x) \leq \sup_{u \in D, t \geq 0} e^{-t} \xi^{-1} \left( \frac{1}{3} \left( \xi \|x\|_X + \xi(t) + e \right) \right).$$

Applying \ref{13} with $\alpha := \xi^{-1}$, we obtain that

$$V_k(x) \leq \sup_{t \geq 0} e^{-t} \left( \|x\|_X + t + e^{-1}(c) \right) \leq \|x\|_X + C, \quad (19)$$

for a certain constant $C > 0$ and all $x \in X$.

**Proving the growth estimate** (14) for $V_k$ with $a = 1$. Now take any $v \in D$, and any $h > 0$. By the cocycle property, we obtain that

$$V_k(\phi(h, x, v)) = \sup_{u \in D, t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t, h, x, u)\|_X \right) \right) = \sup_{u \in D, t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t + h, x, u)\|_X \right) \right).$$

Here, the concatenation $v \diamond u$ was defined in \ref{10}.

Assumption 3.1 ensures that $v \diamond u \in D$. Thus, we only increase the right-hand side by taking the supremum over a larger space of inputs:

$$V_k(\phi(h, x, v)) \leq \sup_{u \in D, t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t + h, x, u)\|_X \right) \right) = \sup_{u \in D, t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t + h, x, u)\|_X \right) \right).$$

Applying Lemma 3.2(ii), we proceed to

$$V_k(\phi(h, x, v)) \leq e^{h} \sup_{u \in D, t \geq 0} G_k \left( e^{-t(1+h)} \rho \left( \frac{1}{3} \|\phi(t + h, x, u)\|_X \right) \right) + \frac{e^h - 1}{k}.$$  

Thus, for all $v \in D$, we have

$$V_k(\phi(h, x, v)) = \lim_{h \to 0} \sup_{u \in D, t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t + h, x, u)\|_X \right) \right) + \frac{e^h - 1}{k}.$$  

**Lipschitz continuity for $V_k$ on bounded balls.** Take any $R > 0$. Arguing as in \ref{14}, we see that for all $x \in B_R$, all $u \in D$, and all $t \geq 0$, it holds that

$$e^{-t} \rho \left( \frac{1}{3} \|\phi(t, x, u)\|_X \right) \leq e^{-t} (R + \xi^{-1}(c)).$$

Hence for any $k \in \mathbb{N}$, there is a time $T = T(R, k)$, such that

$$t \geq T(R, k) \Rightarrow e^{-t} \rho \left( \frac{1}{3} \|\phi(t, x, u)\|_X \right) \leq \frac{1}{k}, \quad x \in B_R, \; u \in D.$$

Thus, the domain of maximization in the definition of $V_k$ has a finite length. That is, for all $R > 0$ and all $x \in B_R$, the function $V_k$ can be equivalently defined by

$$V_k(x) = \sup_{u \in D, t \in [0, T(R, k)]} G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t, x, u)\|_X \right) \right).$$

Now pick any $x, y \in B_R$, and consider

$$|V_k(x) - V_k(y)| = \left| \sup_{u \in D, t \in [0, T(R, k)]} G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t, x, u)\|_X \right) \right) \right| - \sup_{u \in D, t \in [0, T(R, k)]} G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t, y, u)\|_X \right) \right).$$

Using Lemma 3.1 we proceed to

$$|V_k(x) - V_k(y)| \leq \sup_{u \in D, t \in [0, T(R, k)]} \left| G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t, x, u)\|_X \right) \right) \right| - \left| G_k \left( e^{-t} \rho \left( \frac{1}{3} \|\phi(t, y, u)\|_X \right) \right) \right|.$$  

As $G_k$ is globally Lipschitz with unit Lipschitz constant, we continue the estimates as follows:

$$|V_k(x) - V_k(y)| \leq \sup_{u \in D, t \in [0, T(R, k)]} \left| \rho \left( \frac{1}{3} \|\phi(t, x, u)\|_X \right) \right| - \left| \rho \left( \frac{1}{3} \|\phi(t, y, u)\|_X \right) \right|.$$  

As $\rho$ is also globally Lipschitz with unit Lipschitz constant, we proceed to

$$|V_k(x) - V_k(y)| \leq \frac{1}{3} \sup_{u \in D, t \in [0, T(R, k)]} \left| \phi(t, x, u) - \phi(t, y, u) \right|_X.$$  

Since $\phi$ is Lipschitz continuous on compact intervals uniformly in inputs in $D$, there is some $M = M(R, k)$, which we assume without loss of generality to be increasing with respect to both arguments, such that:

$$|V_k(x) - V_k(y)| \leq M(R, k) \|x - y\|_X, \quad x, y \in B_R.$$

**Defining “RFC Lyapunov function”**. Setting in \ref{17} $t := 0$, and using the identity axiom of $\Sigma$, we estimate $V_k$ from below as

$$V_k(x) \geq G_k \left( \rho \left( \frac{1}{3} \|\phi(0, x, 0)\|_X \right) \right) = G_k \circ \rho \left( \frac{1}{3} \|x\|_X \right).$$
Thus, 
\[ \rho\left(\frac{1}{3}\|x\|_X\right) > \frac{1}{k} \Rightarrow V_k(x) > 0. \]  
(21)

At the same time, if \( \rho\left(\frac{1}{3}\|x\|_X\right) < \frac{1}{k} \), we do not have a coercive estimate from below for \( V_k \). Hence (motivated by \[8, p. 133\]), we define a Lyapunov function candidate
\[ \psi \]
by coercive estimate from below for \( V \).

Clearly, \( \psi(0) = 0 \). Since for each \( x \neq 0 \) there is some \( k \in \mathbb{N} \) such that \( \rho\left(\frac{1}{3}\|x\|_X\right) > \frac{1}{k} \), the condition (21) ensures that \( \psi(r) \) is proper for \( r > 0 \). Furthermore, for any \( r, s \geq 0 \) we have
\[ |\psi(r) - \psi(s)| \]
\[ \leq \sum_{k=1}^{\infty} \frac{1}{1+M(k,k)} G_k \circ \rho\left(\frac{1}{3}\|x\|_X\right) \]
\[ \leq \sum_{k=1}^{\infty} \frac{1}{1+M(k,k)} |r - s|. \]

As both \( G_k \), \( k \in \mathbb{N} \), and \( \rho \) are globally Lipschitz with unit Lipschitz constant, we proceed to
\[ |\psi(r) - \psi(s)| \leq \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{1+M(k,k)} |r - s| \leq \frac{1}{3} |r - s|, \]
which shows the global Lipschitz continuity of \( \psi \). Finally, as \( \rho \) is increasing to infinity, \( \psi \) shares this property. Overall, \( \psi \in \mathcal{K}_{\infty} \).

Differentiating \( W \) along the trajectory, we obtain:
\[ \dot{W}(x) \leq \sum_{k=1}^{\infty} \frac{1}{1+M(k,k)} V_k(x) \]
\[ \leq \sum_{k=1}^{\infty} \frac{2^{-k} M(k,k)}{1+M(k,k)} \left( V_k(x) + \frac{1}{k} \right) = W(x) + C_2, \]
for a certain \( C_2 > 0 \).

Now pick any \( R > 0 \) and any \( x, y \in B_R \). Exploiting (20), we have
\[ |W(x) - W(y)| \leq \sum_{k=1}^{\infty} \frac{2^{-k}}{1+M(k,k)} \left( V_k(x) - V_k(y) \right) \]
\[ \leq \sum_{k=1}^{\infty} 2^{-k} M(k,k) \frac{1}{1+M(k,k)} \|x - y\|_X \]
\[ \leq \left( 1 + \sum_{k=1}^{\infty} \frac{2^{-k} M(k,k)}{1+M(k,k)} \right) \|x - y\|_X. \]

This shows that \( W \) is a Lyapunov function for \( \Sigma \) in the sense of (v), which is Lipschitz continuous on bounded balls. \( \square \)

4. Discussion

4.1. Relation to finite-dimensional results

Having proved a characterization of robust forward completeness for a general class of infinite-dimensional control systems, it is of virtue to see how much it can tell us in the special case of ODE systems, and in particular, how Theorem 3.7 relates to Lyapunov characterization of forward completeness derived in [1, Theorem 2].

Let \( \Sigma \) be an ODE system
\[ \dot{x} = f(x,u), \]
(23)
where \( x(t) \in X := \mathbb{R}^n, u \in U := L^\infty(\mathbb{R}_+, \mathbb{R}^m) \), and the nonlinearity is as follows:

**Assumption 4.1.** \( f \) is continuous on \( \mathbb{R}^n \times \mathbb{R}^m \) and is Lipschitz continuous in \( x \) on bounded sets.

This assumption ensures that for any initial condition and any input, the corresponding mild solution (in the sense of Carathéodory) of (23) exists and is unique on a certain finite interval. Furthermore, the flow \( \phi \) is well-defined, and \( \Sigma := (X, U, \phi) \) is a well-defined control system with the BIC property, see [1, Theorem 1.16, Proposition 1.20].

Let \( R \) be finite, and let \( \mathcal{D} \) be the set of disturbances as defined by (4).

Recall that a map \( f : \mathbb{R}^n \to \mathbb{R}_+ \) is called proper if the preimage of any compact subset of \( \mathbb{R}_+ \) is compact in \( \mathbb{R}^n \).

For systems (23), Theorem 3.7 takes the form

**Proposition 4.1.** Let Assumption 4.1 hold. System (23) is robustly forward complete if and only if there exists a proper and Lipschitz continuous function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) and \( a, M > 0 \) such that the following exponential growth condition holds:
\[ \dot{V}_a(x) \leq aV(x) + M, \quad x \in \mathbb{R}^n, \quad u \in \mathcal{D}. \]
(24)

**Proof.** By [12, Corollary A.11], \( V \in C(\mathbb{R}^n, \mathbb{R}_+) \) is proper if and only if there is a \( \psi \in \mathcal{K}_{\infty} \), such that \( V(x) \geq \psi_1(|x|) \) for all \( x \in \mathbb{R}^n \). Furthermore, \( V(x) \leq \omega(|x|) \), where \( \omega : r \mapsto \sup_{|y| \leq r} V(y) \) is a continuous nondecreasing function. Setting \( \psi_2(r) := r + \omega(r) - \lim_{s \to +0} \omega(s) \), we obtain \( V(x) \leq \psi_2(|x|) + \lim_{s \to +0} \omega(s) \). Thus, \( V \) is proper if and only if the "sandwich bounds" (13) hold.

"\( \Leftarrow \)." Follows from the above argument and Theorem 3.7.

"\( \Rightarrow \)." As (23) is robustly forward complete and Assumption 4.1 holds, [13, Lemma 4.6] ensures that the flow \( \phi \) is Lipschitz continuous on compact intervals uniformly in inputs from \( \mathcal{D} \). The rest follows from Theorem 3.7. \( \square \)

Proposition 4.1 is a version of [1, Theorem 2]. Both results guarantee for a robustly forward complete ODE
system the existence of a Lyapunov function with at most exponential growth rate. However, the Lyapunov function constructed in [1] Theorem 2] satisfies (24) with \( M = 0 \), while in our construction \( M > 0 \). Another difference is that in Proposition [1.1] our Lyapunov function \( V \) is Lipschitz continuous, while in [1] Theorem 2) the existence of an infinitely differentiable Lyapunov function with the same properties is shown.

Basically, the authors in [1, Theorem 2] construct first a Lipschitz continuous Lyapunov functional (using a procedure different from that in this paper) and afterward apply the smoothing procedure based on results in [1], which is developed specifically for ODE systems. At the same time, in Proposition 4.1 it is not required that \( f \) is Lipschitz continuous with respect to inputs, which is assumed in [1.1].

4.2. Remarks on boundedness of reachability sets

The following property is closely related to robust forward completeness and is frequently used in control theory:

**Definition 4.2.** We say that \( \Sigma = (X, \mathcal{U}, \phi) \) has bounded reachability sets (BRS) if it is forward complete and for any \( C > 0 \) and any \( \tau > 0 \), it holds that

\[
\sup \{ \| \phi(t, x, u) \|_X : \| x \|_X \leq C, \| u \|_U \leq C, \; t \in [0, \tau] \} < \infty.
\]

It is not hard to see that a control system \( \Sigma \) has BRS if and only if it is robustly forward complete with respect to \( \mathcal{D} \) defined in (1) for all \( R < \infty \). Thus, the BRS property is (in general) stronger than RFC with respect to \( \mathcal{D} \) with a fixed finite \( R \). At the same time, BRS is generally weaker than RFC with \( \mathcal{D} = \mathcal{U} \).

It is reasonable to ask whether one can obtain a Lyapunov characterization of the BRS property as well. The following result proposes a natural candidate for a “BRS Lyapunov function”:

**Proposition 4.3.** Consider a control system \( \Sigma = (X, \mathcal{U}, \phi) \) satisfying the BIC property.

Let there exist a Lipschitz continuous map \( V : X \to \mathbb{R}_+ \), maps \( \psi_1, \psi_2 \in K_\infty \), and \( C > 0 \) such that

\[
\psi_1(\| x \|_X) \leq V(x) \leq \psi_2(\| x \|_X) + C, \; x \in X,
\]

and there are \( a > 0 \) and \( \gamma \in K_\infty \), such that for all \( x \in X \) and \( u \in \mathcal{U} \) the following holds:

\[
\| x \|_X \geq \gamma(\| u \|_U) \Rightarrow \dot{V}_u(x) \leq aV(x).
\]

Then \( \Sigma \) has bounded reachability sets.

**Proof.** Pick any \( x \in X \) and any \( u \in \mathcal{U} \). As \( \Sigma \) is a well-defined control system, there is a maximal time \( t_m(x, u) \) such that \( \phi(\cdot, x, u) \) is well-defined on \([0, t_m(x, u))\).

Take any finite \( \tau \leq t_m(x, u) \), and define

\[
P := \{ t \in (0, \tau) : \| \phi(t, x, u) \|_X > \gamma(\| u \|_U) \}.
\]

For \( t \in [0, \tau) \setminus P \) we have

\[
\| \phi(t, x, u) \|_X \leq \max\{ \| x \|_X, \gamma(\| u \|_U) \}.
\]

If \( P \neq \emptyset \), take any \( t \in P \), and consider the maximal (w.r.t. the set inclusion) open interval \( I = (t-, t_+) \subset P \), such that \( t \in I \).

By continuity of \( \phi(\cdot, x, u) \), such an interval is well-defined, and either \( \| \phi([t-, t_+], x, u) \|_X = \gamma(\| u \|_U) \), or \( t_+ = \infty \), and by the identity axiom \( \| \phi(t, x, u) \|_X = \| x \|_X \).

By the axiom of shift invariance, \( \| u \|_U \geq \| u(t + \cdot) \|_U \) for any \( r \geq 0 \), and we have that

\[
\| \phi(t, x, u) \|_X > \gamma(\| u(t + \cdot) \|_U), \; t \in I.
\]

By (26), for all \( t \in [t-, t_+] \) we have that

\[
D^+ V(\phi(t, x, u)) = \dot{V}_u(t+, t) \leq aV(\phi(t, x, u)).
\]

Employing Proposition 3.3 for the continuous map

\[
g(t) := V(\phi(t, x, u)), \; t \in [t-, t_+],
\]

we obtain:

\[
V(\phi(t, x, u)) \leq e^{a(t-t_-)}V(\phi(t-, x, u)), \; t \in [t-, t_+).
\]

Thanks to the “sandwich inequality” (25), we have for all \( t \in [t-, t_+] \) that

\[
\psi_1(\| \phi(t, x, u) \|_X) \leq e^{a(t-t_-)}(\psi_2(\| \phi(t-, x, u) \|_X) + C).
\]

Thus, for all \( t \in P \)

\[
\| \phi(t, x, u) \|_X \leq \psi_1^{-1}\left( e^{a\tau}(\psi_2\left( \max\{ \| x \|_X, \gamma(\| u \|_U) \} + C \right) \right).
\]

Together with (27), this shows that the trajectory \( \phi(\cdot, x, u) \) is uniformly bounded on \([0, \tau] \). If \( t_m(x, u) \) is finite, taking \( \tau := t_m(x, u) \), we obtain a contradiction to the BIC property. Hence, \( t_m(x, u) = +\infty \), and (27) and (28) show the BRS property.

\( \square \)

A possible approach to obtaining the converse Lyapunov theorem for the BRS property is to transform the control system \( \Sigma \) into an RFC auxiliary system \( \Sigma \) by using the state feedback \( u(x) = d(t)k(x) \), where \( d \) is understood as a disturbance belonging to the bounded closed ball of a fixed radius, and \( k \) is a carefully chosen feedback law. Using RFC characterization (Theorem 3.7) shown in this work, one obtains the RFC Lyapunov function for the modified system \( \Sigma \). It could be used to obtain a BRS Lyapunov function for the original system \( \Sigma \). The modification method was successfully employed in [17] for the characterization of the ISS property for ODE systems and in [1] for the characterization of the so-called unboundedness observability property for ODE systems with outputs, as well as for the BRS property for ODE systems.

Employing this method for the systems considered in this work raises several challenges. One of them is that
the abstract systems used in this work are defined in terms of the flow map. Thus, there is no trivial way to define explicitly the modified system that will be obtained after adding feedback. Infinite-dimensionality adds additional complexities as a question appears, whether such a feedback makes an auxiliary closed-loop system well-posed. These interesting problems are left for future research.

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