A NOTE ON SINGULAR INTEGRAL

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Abstract. In this paper, we prove that for $\frac{2}{n} + \frac{1}{4} < \alpha \leq \frac{3}{4}$, the convolution operator

$$S_\alpha f(x) = \int_{|y| \geq 1} f(x-y) (|y|^2 - 1)^{-\alpha} \, dy$$

is bounded from $L^p$ to $L^q$ for certain values of $p$ and $q$.

1. Introduction

In this paper, we consider the $L^p - L^q$ estimates for the convolution operators having kernel with singularity along the sphere and at infinity which is non zero outside the neighbourhood of zero.

For convolutions on Euclidean space with homogeneous kernels, the fractional integration theorem and the Calderón-Zygmund inequalities give essentially the best possible results [4]. Such kind of kernels have singularities at the point 0 and $\infty$. It was Strichartz who first proved that other than that kind of singularity, the corresponding operator is still bounded from $L^p$ to other $L^q$. Such kind of operators naturally arise in the study of the wave equation and gives new information about solutions of this type of equation.

For $0 < \alpha < 1$, Strichartz considered the operator

$$T_\alpha f(x) = \int_{|y| \leq 1} f(x-y) \phi_\alpha(y) \, dy,$$

where the kernel is defined by

$$\phi_\alpha(y) = \begin{cases} (1 - |y|^2)^{-\alpha}, & \text{if } |y| < 1, \\ 0, & \text{if } |y| \geq 1. \end{cases} \quad (1.1)$$

The author investigated $L^p - L^q$ boundedness of such convolution operator with kernel having singularities on the sphere and at infinity. In general, from [2], if we consider the family of generalized functions $(ax^2 + bx + c)^\lambda_\pm$, where

$$(ax^2 + bx + c)^\lambda_+ = \begin{cases} (ax^2 + bx + c)^\lambda, & \text{if } ax^2 + bx + c > 0, \\ 0, & \text{if } ax^2 + bx + c \leq 0, \end{cases}$$

then it is always possible to perform a linear transformation on $x$ such that the generalized function $(ax^2 + bx + c)^\lambda_\pm$ is transformed to one of the forms $(1 - x^2)^\lambda_\pm$, $(1 + x^2)^\lambda_\pm$, $(x^2 - 1)^\lambda_\pm$, or $x^2_\pm$ respectively, to the four cases in which $ax^2 + bx + c > 0$. Here in this paper, we mainly interested in the case $(x^2 - 1)^\lambda_\pm$. Note that, all other cases have been treated already. In particular, Strichartz in [15] consider the first case $(1 - x^2)^\lambda_\pm$ and proved the following result.

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Theorem 1.1. [15]

(1) \( ||T_\alpha f||_q \leq A_\alpha ||f||_p \) provided \( 1 < p \leq 2 \leq q < \infty \) and \( \frac{1}{p} - \frac{1}{q} \leq \frac{(n+1-2\alpha)}{2n} \) for \( 0 < \alpha \leq \frac{(n+1)}{2} \).

(2) \( ||T_\alpha f||_{p'} \leq A_\alpha ||f||_p \) for \( p = \frac{(n+1)}{(n+1-\alpha)}, p' = \frac{(n+1)}{\alpha} \) and \( 0 < \alpha \leq \frac{(n+1)}{2} \).

(3) \( ||T_\alpha f||_q \leq A_\alpha ||f||_p \) for \( \frac{1}{2} \leq \alpha \leq \frac{(n+1)}{2} \) provided \( \frac{(n+1)}{(n+1-\alpha)} \leq p \leq 2 \) and \( \frac{n}{q} = \alpha - \frac{1}{p} \), or

\[ \frac{n}{(n+\frac{1}{2}-\alpha)} \leq p \leq \frac{(n+1)}{(n+1-\alpha)} \text{ and } \frac{1}{q} = \alpha - \frac{n}{p}. \]

Considerable attention has been devoted to discovering generalizations to new contexts for the boundedness of convolution operator by several researchers in different contexts. For example, \( L^p - L^q \) boundedness of convolution operator was studied by Amri and Gaidi in [1] in the Dunkel setting and investigated the solutions of wave equations associated to Dunkl Laplacian operator. Karapetyants studied \( L^p - L^q \) boundedness for convolutions operator with kernels having singularities on a sphere in [5]. F. Ricci and T. Giancarlo in [8] studied boundedness of convolution operators defined by singular measures. Further, \( \mathcal{L} \)-characteristic of some potential type convolution operator \( A_\alpha \) is bounded from \( L^p \) into \( L^q \) was given in the works of Karapetyants and Nogin [6]. There are several papers in the direction of bounds for potential-type convolution operators and close to them, we refer to [3,7,9–12] and references therein. Such operators naturally arise in applications to the theory of fractional powers of differential operators, in particular, classical operators of mathematical physics: the wave operator, the Klein-Gordon and Schrödinger operators, the telegraph operator.

The main aim of this paper is to prove \( L^p \) to \( L^q \) bounded of the convolution operator

\[ S_\alpha f(x) = \int_{|y| \geq 1} f(x-y) \xi_\alpha(y) \, dy \]  

(1.2)

for \( \alpha > 0 \), where the kernel is defined by

\[ \xi_\alpha(y) = \begin{cases} 
(y^2 - 1)^{-\alpha}, & \text{if } |y| > 1, \\
0, & \text{if } |y| \leq 1,
\end{cases} \]  

(1.3)

for certain \( p \) and \( q \). The kernel defined in (1.3) is different form the kernel (1.1), considered by Strichartz. The main difference in the kernel defined in (1.3) is that this kernel has singularity along the sphere and in infinity. Also it is non zero outside the neighbourhood of zero which is not consider by any author in past and which makes the problem rather slightly different. However, in the sense of distribution, the range of \( \alpha \) can be extended similarly as in Equation (3) of [15] by integration by parts.

Our main results in this article is the following \( L^p - L^q \) boundedness theorem of convolution operator \( S_\alpha \).

Theorem 1.2. Let \( \frac{n}{2} + \frac{1}{4} < \alpha \leq \frac{n+1}{2} \). Then for \( 1 < p \leq 2 \leq q < \infty \) and

\[ \frac{1}{p} - \frac{1}{q} \leq \frac{2\alpha - n - \frac{1}{2}}{n}, \]

the convolution operator

\[ S_\alpha f = \xi_\alpha * f, \quad f \in \mathcal{S}(\mathbb{R}^n) \]

is bounded from \( L^p \) to \( L^q \).
The proof of the theorem is given in Section 3. The main ingredient we use is the the Hardy-Littlewood multiplier theorem given in Theorem 1.11 of [4], to prove Theorem 1.2 using a similar technique given in [15].

The presentation of this manuscript is divided into three sections including the introduction. In section 2, we recall some basics properties and results from euclidean Fourier analysis which will be used in order to prove our main results. In Section 3, we prove our main results.

2. Preliminaries

In this section, we first recall some notation and basic properties of Fourier analysis on \( \mathbb{R}^n \). The Fourier transform \( \hat{f} \) of a function \( f \in L^1(\mathbb{R}^n) \) is defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^n.
\]

The Fourier transform can be extended to \( L^2(\mathbb{R}^n) \) using the standard density arguments. The inverse Fourier transform is given by

\[
f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-ix \cdot \xi} d\xi, \quad x \in \mathbb{R}^n,
\]

where \( f \) belongs to a suitable function space. The space \( S(\mathbb{R}^n) \) is the Schwartz class of rapidly decreasing smooth functions and \( S'(\mathbb{R}^n) \) is the space of tempered distributions. Let \( T \in S'(\mathbb{R}^n) \), its fourier transform \( \hat{T} \) defined in the sense of distribution by

\[
\hat{T}(\phi) = T(\hat{\phi}), \quad \phi \in S(\mathbb{R}^n).
\]

Moreover, \( \hat{T} \) is called multiplier if it is \( L^p - L^q \) bounded.

As usual, the space \( L_p^q \) denotes the Banach space of \( S'(\mathbb{R}^n) \)-distributions \( T \) such that the closure in \( L^p \) of the convolution \( T \ast u \) for \( u \in S(\mathbb{R}^n) \), is an \( L^p - L^q \) bounded translation invariant operator. The space \( L_p^q \) is thus isomorphic to a closed subspace of the Banach space of all bounded linear mappings of \( L^p \) into \( L^q \) and hence is also a Banach space. Also, \( M_p^q = L_p^q \) stands for the space of all \( L^p - L^q \) multipliers type \( (p,q) \) [4]. Moreover, we have

\[
L_p^\infty = L_1^p = L^p, \quad p < \infty \text{ and } L_\infty^\infty = L_1^1 = M.
\]

From [4], we have the following Hardy-Littlewood multiplier theorem.

**Theorem 2.1.** [4] Let \( 0 < a < n \) and \( m \) be a measurable function such that

\[
|m(\xi)| \leq \frac{c}{|\xi|^a}
\]

some constant \( c > 0 \). Then the operator \( T_m = F^{-1}(mF) \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \), provided

\[
1 < p \leq 2 \leq q < \infty \text{ and } \frac{1}{p} - \frac{1}{q} = \frac{a}{n}.
\]

By \( J_\alpha(s) \), we denotes the classical Bessel function of size \( \alpha \). From [14], we have the following estimate:

\[
|t^{-(a+ib)} J_{(a+ib)}(t)| \leq C_a e^{c|b|} (1 + t)^{-a - \frac{1}{2}}, \quad 0 < t < \infty.
\] (2.1)

For detailed study and several properties on Bessel functions, we refer to [14, 16].
3. $L^p$-$L^q$-Boundedness

This section is devoted to study our main results of this paper. The Hardy-Littlewood multiplier theorem plays a key role in the proof of the theorem.

3.1. Proof of Theorem 1.2.

Proof. For $\alpha > 0$, let us consider

$$\xi_\alpha(y) = \begin{cases} (|y|^2 - 1)^{-\alpha}, & \text{if } |y| > 1, \\ 0, & \text{if } |y| \leq 1. \end{cases}$$

Then the Fourier transform of $\xi_\alpha$ is not classically defined, as the function $\xi_\alpha$ is not in $L^1(\mathbb{R}^n)$. We define it by the sense of tempered distribution. For $\alpha \neq n, n \in \mathbb{N}$, we define

$$\hat{\xi}_\alpha(s) = \sqrt{\pi} \Gamma(1-\alpha) \left|\frac{s}{2}\right|^{\frac{\alpha - n}{2}} \cot \pi \left(\frac{n}{2} - \alpha\right) \left( J_{\alpha - \frac{n}{2}}(|s|) - J_{\frac{\alpha - n}{2}}(|s|) \right)$$

for $0 \leq \alpha < 1$, where $J_{\alpha}(s)$ denotes the classical Bessel function of size $\alpha$. This is well defined by the analytic continuation of $\Gamma$ almost everywhere. Since

$$S_\alpha f = \xi_\alpha * f, \quad f \in S(\mathbb{R}^n),$$

then the Fourier transform of $S_\alpha f$ is given by

$$\hat{S_\alpha f}(s) = \sqrt{\pi} \Gamma(1-\alpha) \left|\frac{s}{2}\right|^{\frac{\alpha - n}{2}} \cot \pi \left(\frac{n}{2} - \alpha\right) \left( J_{\alpha - \frac{n}{2}}(|s|) - J_{\frac{\alpha - n}{2}}(|s|) \right) \hat{f}(s).$$

The right-hand side of the above inequality is a single-valued analytic function of $\alpha$ in the complex plane except at the positive integers.

Let $\frac{n}{2} + \frac{1}{q} < \alpha \leq \frac{n+1}{2}$ and consider the function

$$m(s) = \sqrt{\pi} \Gamma(1-\alpha) \left|\frac{s}{2}\right|^{\frac{\alpha - n}{2}} \cot \pi \left(\frac{n}{2} - \alpha\right) \left( J_{\alpha - \frac{n}{2}}(|s|) - J_{\frac{\alpha - n}{2}}(|s|) \right).$$

Using the classical estimate (2.1) on the size of Bessel functions, we have

$$J_{\alpha - \frac{n}{2}}(|s|) \leq (1 + |s|)^{-\alpha + \frac{n}{2} - \frac{1}{2}}$$

and

$$J_{\frac{\alpha - n}{2}}(|s|) \leq (1 + |s|)^{-\alpha + \frac{n}{2} - \frac{1}{2}}.$$  \hspace{1cm} (3.4)

Using the estimates (3.3) and (3.4), from the relation (3.5), we get

$$|m(s)| \leq c|s|^{-\left(2\alpha - n - \frac{1}{2}\right)},$$

for some constant $c$. Thus by Theorem 2.1, $m$ is a $L^p - L^q$ multiplier provided

$$1 \leq p \leq 2 \leq q < \infty \quad \text{and} \quad \frac{1}{p} - \frac{1}{q} = \frac{2\alpha - n - \frac{1}{2}}{n}.$$

Thus, the operator $S_\alpha$ is bounded from $L^p$ to $L^q$ for $\frac{1}{p} - \frac{1}{q} \leq \frac{2\alpha - n - \frac{1}{2}}{n}$. This completes the proof of the theorem. \qed

In the next theorem we show that the convolution operator $S_\alpha$ is bounded from $L^p$ to $L^{p'}$ in one dimentional.

**Theorem 3.1.** Let $0 \leq \alpha \leq 1$. Then for $p = \frac{2}{(2 - \alpha)}$ and $p' = \frac{2}{\alpha}$, the convolution opeartor $S_\alpha$ is bounded from $L^p$ to $L^{p'}$. 

Proof. Let $S_{\alpha}$ is an family of operators defined in the strip $0 \leq \text{Re}(\alpha) \leq 1$ in the complex plane in the sense of Stein [13]. When $\text{Re}(\alpha) = 1$, using Plancherel theorem, (3.1), (3.5), and (3.5), we have

$$\|S_{\alpha}f\|_2 \leq c_1 e^{c\text{Im}(\alpha)} \|f\|_2.$$ 

Again on the other side, when $\text{Re}(\alpha) = 0$, using the fact that $|\xi_{\alpha}(y)| \leq 1$ and $|\Gamma(1+ib)| = \left(\frac{\pi b}{\sinh b}\right)^{\frac{1}{2}}$, from the relation (3.2), we have the estimate

$$\|S_{\alpha}f\|_{\infty} \leq c_1 e^{c\text{Im}(\alpha)} \|f\|_1.$$ 

Then by Stein’s interpolation theorem theorem [13], we have the result. \hfill $\square$

An immediate consequence of the above theorem is the following $L^p - L^q$ boundedness of the convolution operator $S_{\alpha}$.

**Theorem 3.2.** Let $\frac{1}{2} \leq \alpha \leq 1$. Then for

$$\frac{2}{(2-\alpha)} \leq p \leq 2 \quad \text{and} \quad \frac{1}{q} = \alpha - \frac{1}{p}$$

or

$$\frac{1}{(\frac{3}{2} - \alpha)} \leq p \leq \frac{2}{(2-\alpha)} \quad \text{and} \quad \frac{1}{q} = \alpha - \frac{1}{p'},$$

the convolution operator $S_{\alpha}$ is bounded from $L^p$ to $L^q$.

**Proof.** If $\alpha > \frac{1}{2}$, then the result follows from Theorem 1.2 and 3.1 by an application of Riesz interpolation theorem. Now let us consider $\alpha = \frac{1}{2}$. Since $\hat{\xi}_{\frac{1}{2}}$ is in weak $L^2$, then $T_{\frac{1}{2}}$ maps $L^1$ to weak $L^2$. Then the required result follows by applying the Marcinkiewicz interpolation theorem [17] and a duality argument [4]. \hfill $\square$

**Remark 3.3.** Note that, $\hat{\xi}_{\alpha}$ (defined in (1.3)) is a multiplier in terms of Bessel functions.

**Remark 3.4.** If we define kernels by replacing $|y|^2$ by any nondegenerate quadratic form using the computations of [2], Chapter III), then all the results are also in this case.

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