Entropic regularization of Wasserstein distance between infinite-dimensional Gaussian measures and Gaussian processes

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Abstract This work studies the entropic regularization formulation of the 2-Wasserstein distance on an infinite-dimensional Hilbert space, in particular for the Gaussian setting. We first present the Minimum Mutual Information property, namely the joint measures of two Gaussian measures on Hilbert space with the smallest mutual information are joint Gaussian measures. This is the infinite-dimensional generalization of the Maximum Entropy property of Gaussian densities on Euclidean space. We then give closed form formulas for the optimal entropic transport plan, 2-Wasserstein distance, and Sinkhorn divergence between two Gaussian measures on a Hilbert space, along with the fixed point equations for the barycenter of a set of Gaussian measures. Our formulations fully exploit the regularization aspect of the entropic formulation and are valid both in singular and nonsingular settings. In the infinite-dimensional setting, both the entropic 2-Wasserstein distance and Sinkhorn divergence are Fréchet differentiable, in contrast to the exact 2-Wasserstein distance, which is not differentiable. Our Sinkhorn barycenter equation is new and always has a unique solution. In contrast, the finite-dimensional barycenter equation for the entropic 2-Wasserstein distance fails to generalize to the Hilbert space setting. In the setting of reproducing kernel Hilbert spaces (RKHS), our distance formulas are given explicitly in terms of the corresponding kernel Gram matrices, providing an interpolation between the kernel Maximum Mean Discrepancy (MMD) and the kernel 2-Wasserstein distance.

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1 Introduction

In this work, we study the entropic regularization formulation of the 2-Wasserstein distance in the Hilbert space setting, with a particular focus on Gaussian measures and covariance operators on Hilbert space. This is the infinite-dimensional generalization of recent work on the entropic 2-Wasserstein distance between Gaussian measures on $\mathbb{R}^n$, as reported in [55,45,24]. Our work is along the direction of entropic regularization in optimal transport, which has recently attracted much attention in various fields, in particular machine learning and statistics [20,83,30,35,36,56,75,77], with applications in computer vision, density functional theory, and inverse problems (e.g. [36,37,53,69]). This direction of research is also closely connected with the Schrödinger bridge problem [81], which has been studied extensively [11,18,21,31,78,91,33,52,79,80].

Our focus in the Gaussian setting stems not only from its use in elucidating various aspects of the abstract theory, since many quantities of interest admit closed form formulas, but also from numerous applications utilizing Gaussian measures and covariance matrices/operators. These include brain imaging [3,27], computer vision [88,89,87], and brain computer interfaces [16]. Many distances/divergences have been studied and employed in practice, including the affine-invariant Riemannian metric [71], corresponding to the Fisher-Rao distance between centered Gaussians, the Alpha Log-Determinant divergences [13], corresponding to Rényi divergences between centered Gaussians, the Log-Euclidean metric [4], and recent work attempting to unify them [2,15,86,63].

Infinite-dimensional setting. The generalization of distances/divergences for Gaussian measures and covariance matrices on $\mathbb{R}^n$ to the infinite-dimensional setting of Gaussian measures and covariance operators on Hilbert spaces has been carried out by various authors. In general, the infinite-dimensional formulations are substantially more complex than the finite-dimensional ones and regularization is often necessary. This is the case for the affine-invariant Riemannian distance [51], the Log-Hilbert-Schmidt metric [60], the Alpha and Alpha-Beta Log-Determinant divergences [59,61,58,64,65]. The settings for these distances/divergences are the sets of positive definite unitized trace class/Hilbert-Schmidt operators, which are positive trace class/Hilbert-Schmidt operators plus a positive scalar multiple of the identity operator so that operations such as inversion, logarithm, and determinant, are well-defined. A particular advantage of the 2-Wasserstein distance compared to the above distances/divergences is that the finite and infinite-dimensional distance formulas [34,19] are the same and no regularization is necessary.

Reproducing kernel Hilbert space (RKHS) setting. From the computational and practical viewpoint, this setting is particularly interesting since many quantities of interest admit closed forms via kernel Gram matrices which can be efficiently computed. Examples include the kernel Maximum Mean Discrepancy (MMD) [41] and the RKHS covariance operators, the latter resulting in powerful nonlinear algorithms with substantial improvements over
finite-dimensional covariance matrices, see e.g. [93,43,60,66,92] for examples of applications in computer vision.

**Contributions of this work.**

1. We generalize the Maximum Entropy property of Gaussian densities in $\mathbb{R}^n$ to the Hilbert space setting, namely the Minimum Mutual Information of joint measures of two Gaussian measures on Hilbert space.

2. For two Gaussian measures on a Hilbert space $H$, we provide closed form formulas for the optimal entropic transport plan, the entropic $2$-Wasserstein distance and the Sinkhorn divergence, generalizing results in [55,45,24].

3. For a set of Gaussian measures, we show a new Sinkhorn barycenter equation, with always a unique non-trivial solution. In contrast, we show that the finite-dimensional barycenter equation for the entropic $2$-Wasserstein distance fails to generalize to the Hilbert space setting.

4. In the RKHS setting, we present closed form formulas for the distances via the finite kernel Gram matrices, providing an interpolation between the kernel MMD [41] and kernel Wasserstein distance [92,62].

5. Our proofs and results fully exploit the regularization aspect of the entropic formulation and are valid both in singular and non-singular settings. This is novel also in the finite-dimensional setting (compared to [55,45,24]).

6. As we discuss in detail, many properties of the Gaussian case have not been proved in the general theory, due to (i) $\dim(H) = \infty$, (ii) $c(x,y) = ||x-y||^2$ is unbounded on $H$, (iii) the support of Gaussian measures is unbounded.

**2 Background and finite-dimensional results**

Let $(X,d)$ be a complete separable metric space equipped with a lower semi-continuous cost function $c : X \times X \rightarrow \mathbb{R}_{\geq 0}$. Let $\mathcal{P}(X)$ denote the set of all probability measures on $X$. The optimal transport (OT) problem between two probability measures $\nu_0, \nu_1 \in \mathcal{P}(X)$ is (see e.g. [90])

$$OT(\nu_0, \nu_1) = \min_{\gamma \in \text{Joint}(\nu_0, \nu_1)} \mathbb{E}_\gamma[c] = \min_{\gamma \in \text{Joint}(\nu_0, \nu_1)} \int_{X \times X} c(x,y)d\gamma(x,y)$$

where $\text{Joint}(\nu_0, \nu_1)$ is the set of joint probabilities with marginals $\nu_0$ and $\nu_1$. For $1 \leq p < \infty$, let $\mathcal{P}_p(X)$ denote the set of all probability measures $\mu$ on $X$ of finite moment of order $p$, i.e. $\int_X d^p(x_0, x)d\mu(x) < \infty$ for some (and hence any) $x_0 \in X$. The $p$-Wasserstein distance $W_p$ between $\nu_0$ and $\nu_1$ is defined as

$$W_p(\nu_0, \nu_1) = OT_{d^p}(\nu_0, \nu_1)^{\frac{1}{p}}.$$  

This distance defines a metric on $\mathcal{P}_p(X)$ (Theorem 7.3, [90]). For two multivariate Gaussian distributions $\nu_i = \mathcal{N}(m_i, C_i)$, $i = 0, 1$, on $\mathbb{R}^n$, $W_2(\nu_0, \nu_1)$
admits the following closed form [40, 26, 67, 49]

\[ W_2^2(\nu_0, \nu_1) = \|m_0 - m_1\|^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - 2\text{Tr} \left( C_1^T C_0 C_1^T \right)^{1/2}. \] (3)

**Entropic regularization and Sinkhorn divergence.** The exact OT problem (1) is often computationally challenging and it is more numerically efficient to solve the following regularized optimization problem, for a given \( \epsilon > 0 \),

\[ \text{OT}_\epsilon^0(\mu, \nu) = \min_{\gamma \in \text{Joint}(\mu, \nu)} \left\{ \mathbb{E}_{\gamma}[c] + \epsilon \text{KL}(\gamma||\mu \otimes \nu) \right\}, \] (4)

where \( \text{KL}(\nu||\mu) \) denotes the Kullback-Leibler divergence between \( \nu \) and \( \mu \). The KL in (4) acts as a bias [30], with the consequence that in general \( \text{OT}_\epsilon^0(\mu, \mu) \neq 0 \). The following \( \epsilon \)-Sinkhorn divergence [30] removes this bias

\[ S_\epsilon^0(\mu, \nu) = \text{OT}_\epsilon^0(\mu, \nu) - \frac{1}{2} (\text{OT}_\epsilon^0(\mu, \mu) + \text{OT}_\epsilon^0(\nu, \nu)). \] (5)

For \( \nu_i = N(m_i, C_i) \), \( i = 0, 1 \), both \( \text{OT}_\epsilon^0(\nu_0, \nu_1) \) and \( S_\epsilon^0(\nu_0, \nu_1) \) admit closed form formulas. Let \( N_{ij}^\epsilon = I + \left( I + \frac{16}{\epsilon^2} C_i^T C_j C_i^T \right)^{1/2} \), \( i, j = 0, 1 \), then [55, 45, 24]

\[ \text{OT}_\epsilon^0(\nu_0, \nu_1) = \|m_0 - m_1\|^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2} \log \det(N_{01}^\epsilon) - \log \det(N_{10}^\epsilon) + n \log 2 - 2n, \] (6)

\[ S_\epsilon^0(\nu_0, \nu_1) = \|m_0 - m_1\|^2 + \frac{\epsilon}{4} \left( \text{Tr}(N_{00}^\epsilon - 2N_{01}^\epsilon + N_{11}^\epsilon) + \log \left( \frac{\det(N_{00}^\epsilon)}{\det(N_{01}^\epsilon \det(N_{11}^\epsilon))} \right) \right). \] (7)

In this case, the unique minimizer \( \gamma \) in (4) is a joint Gaussian measure of \( \nu_0 \) and \( \nu_1 \), a direct consequence of the Maximum Entropy of Gaussian densities (see below). In particular, \( \lim_{\epsilon \to 0} \text{OT}_\epsilon^0(\nu_0, \nu_1) = \lim_{\epsilon \to 0} S_\epsilon^0(\nu_0, \nu_1) = W_2^2(\nu_0, \nu_1) \) and \( \lim_{\epsilon \to \infty} S_\epsilon^0(\nu_0, \nu_1) = \|m_0 - m_1\|^2 \). For related work, see also [50, 14].

### 3 From finite to infinite-dimensional settings

In the current work, we generalize the results in [55, 45, 24] to the Hilbert space setting. Throughout the following, let \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) be a real, separable Hilbert space, with \( \dim(\mathcal{H}) = \infty \) unless explicitly stated otherwise. For two separable Hilbert spaces \( (\mathcal{H}_i, \langle \cdot, \cdot \rangle_i), i = 1, 2 \), let \( \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) denote the Banach space of bounded linear operators from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \), with operator norm \( \|A\| = \sup_{\|x\|_i \leq 1} \|Ax\|_2 \). For \( \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H} \), we use the notation \( \mathcal{L}(\mathcal{H}) \).

Let \( \text{Sym}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}) \) be the set of bounded, self-adjoint linear operators on \( \mathcal{H} \). Let \( \text{Sym}^+(\mathcal{H}) \subset \text{Sym}(\mathcal{H}) \) be the set of self-adjoint, positive operators on \( \mathcal{H} \), i.e. \( A \in \text{Sym}^+(\mathcal{H}) \iff \langle Ax, x \rangle \geq 0 \forall x \in \mathcal{H} \). Let \( \text{Sym}^{++}(\mathcal{H}) \subset \text{Sym}^+(\mathcal{H}) \) be
the set of self-adjoint, strictly positive operator on \( \mathcal{H} \), i.e \( A \in \text{Sym}^+ (\mathcal{H}) \iff (x, Ax) > 0 \ \forall x \in \mathcal{H}, x \neq 0 \). We write \( A \geq 0 \) for \( A \in \text{Sym}^+ (\mathcal{H}) \) and \( A > 0 \) for \( A \in \text{Sym}^{++} (\mathcal{H}) \). If \( I + A > 0 \), where \( I \) is the identity operator, \( \gamma \in \mathbb{R}, \gamma > 0 \), then \( \gamma I + A \) is also invertible, in which case it is called positive definite.

The Banach space \( \text{Tr}(\mathcal{H}) \) of trace class operators on \( \mathcal{H} \) is defined by (see e.g. [76]) \( \text{Tr}(\mathcal{H}) = \{ A \in \mathcal{L}(\mathcal{H}) : \| A \|_{\text{tr}} = \sum_{k=1}^{\infty} \langle e_k, (A^* A)^{1/2} e_k \rangle < \infty \} \), for any orthonormal basis \( \{ e_k \}_{k \in \mathbb{N}} \in \mathcal{H} \). For \( A \in \text{Tr}(\mathcal{H}) \), its trace is defined by \( \text{Tr}(A) = \sum_{k=1}^{\infty} \langle e_k, Ae_k \rangle \), which is independent of choice of \( \{ e_k \}_{k \in \mathbb{N}} \).

The Hilbert space \( \text{HS}(\mathcal{H}_1, \mathcal{H}_2) \) of Hilbert–Schmidt operators from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) is defined by (see e.g. [47]) \( \text{HS}(\mathcal{H}_1, \mathcal{H}_2) = \{ A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) : \| A \|_{\text{HS}}^2 = \text{Tr}(A^* A) = \sum_{k=1}^{\infty} \| Ae_k \|^2 < \infty \} \), for any orthonormal basis \( \{ e_k \}_{k \in \mathbb{N}} \) in \( \mathcal{H}_1 \), with inner product \( \langle A, B \rangle_{\text{HS}} = \text{Tr}(A^* B) \). For \( \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H} \), we write \( \text{HS}(\mathcal{H}) \). We have \( \text{Tr}(\mathcal{H}) \subseteq \text{HS}(\mathcal{H}) \) when \( \text{dim}(\mathcal{H}) = \infty \), with \( \| A \|_{\text{HS}} \leq \| A \|_{\text{tr}} \).

Some key differences between the finite and infinite-dimensional settings are

1. On \( \mathbb{R}^n \), for two random variables \( X, Y \) with joint and marginal measures \( \mu_{XY}, \mu_X, \mu_Y \), having densities \( f(x, y), f_X(x), f_Y(y) \), respectively, with respect to the Lebesgue measure, their mutual information is defined as [17]

   \[
   I(X; Y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \log \left( \frac{f(x, y)}{f_X(x) f_Y(y)} \right) f(x, y) dxdy = H(X) + H(Y) - H(X, Y)
   \]

   \[
   = \text{KL}(\mu_{XY} \| \mu_X \otimes \mu_Y),
   \]

   \[
   (8)
   \]

   where \( H(X) = -\int_{\mathbb{R}^n} \log[f_X(x)] f_X(x) dx \) is the differential entropy of \( X \). The classical Maximum Entropy of Gaussian densities property ([43], Theorem 9.6.5) states that if \( X \) has mean zero and covariance matrix \( C \), then

   \[
   H(X) \leq \frac{1}{2} \log(2\pi e)^n \det(C), \text{ with equality if and only if } X \sim \mathcal{N}(0, C).
   \]

   Thus if both \( X \) and \( Y \) have Gaussian densities, then \( I(X; Y) \) is minimum if and only if their joint density is Gaussian, so that for \( c(x, y) = ||x - y||^2 \), and \( \nu_0, \nu_1 \) being Gaussian, a minimizing \( \gamma \) in (4) is necessarily a joint Gaussian measure of \( \nu_0, \nu_1 \). When \( C \) is a covariance operator on \( \mathcal{H} \) with \( \text{dim}(\mathcal{H}) = \infty \), the quantity \( \det(C) \) is no longer well-defined. However, \( I(X; Y) = \text{KL}(\mu_{XY} \| \mu_X \otimes \mu_Y) \) is well-defined and finite whenever \( \mu_{XY} \) is absolutely continuous with respect to \( \mu_X \otimes \mu_Y \). In the following, we show that the above Minimum Mutual Information property of joint Gaussian measures generalizes to the infinite-dimensional setting.

2. If \( A \) is a strictly compact operator on \( \mathcal{H} \), then \( A^{-1} \) is unbounded when \( \text{dim}(\mathcal{H}) = \infty \). Thus finite-dimensional methods that utilize matrix inversion extensively, e.g. in [55, 45, 24], are not applicable when \( \text{dim}(\mathcal{H}) = \infty \). Instead, we fully exploit the regularization aspect of problem (4) and invert operators of the form \( \gamma I + A > 0 \), thus our proofs fully resolve this issue and are valid in the general setting when \( A \) can be singular.
3. The identity operator $I$ is not trace class when $\dim(H) = \infty$. This leads to the breakdown in the entropic barycenter problem (Theorem 12) and has consequences for the analysis of the existence of solutions of the barycenter equations (detail given in Section 11).

4 Main Results

We first state the following generalization of the Maximum Entropy of Gaussian densities in $\mathbb{R}^n$. To the best of our knowledge, this property has not been explicitly and rigorously presented in the literature in the infinite-dimensional setting. It states that among all joint measures, with the same covariance operators, of two Gaussian measures $\mu_X, \mu_Y$ on two separable Hilbert spaces $H_1, H_2$, the ones with the minimum Mutual Information are precisely the joint Gaussian measures on $H_1 \times H_2$. In the following, $\text{Gauss}(\mathcal{H})$ denotes the set of all Gaussian measures on $\mathcal{H}$ and $\text{Gauss}(\mu_X, \mu_Y)$ denotes the set of joint Gaussian measures having marginals $\mu_X$ and $\mu_Y$.

**Theorem 1 (Minimum Mutual Information of Joint Gaussian Measures)** Let $H_1, H_2$ be two separable Hilbert spaces. Let $\mu_X = \mathcal{N}(m_X, C_X) \in \text{Gauss}(H_1)$, $\mu_Y = \mathcal{N}(m_Y, C_Y) \in \text{Gauss}(H_2)$, $\ker(C_X) = \ker(C_Y) = \{0\}$. Let $\gamma \in \text{Joint}(\mu_X, \mu_Y), \gamma_0 \in \text{Gauss}(\mu_X, \mu_Y)$, $\gamma_0$ is equivalent to $\mu_X \otimes \mu_Y$. Assume that $\gamma$ and $\gamma_0$ have the same covariance operator $\Gamma$ and that $\mu_X \otimes \mu_Y$ has covariance operator $\Gamma_0$. Then

$$\text{KL}(\gamma || \mu_X \otimes \mu_Y) \geq \text{KL}(\gamma_0 || \mu_X \otimes \mu_Y) = -\frac{1}{2} \log \det(I - V^*V). \quad (9)$$

Equality happens if and only if $\gamma = \gamma_0$. Here $V$ is the unique bounded linear operator satisfying $V \in \text{HS}(H_2, H_1), ||V|| < 1$, such that $\Gamma = \Gamma_0^{1/2} \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} \Gamma_0^{-1/2}$.

The operator $V$ in Theorem 1 is defined in Section 7. It links the covariance operators $C_X$ and $C_Y$ with the cross-covariance operator $C_{XY}$ via the relation $C_{XY} = C_X^{1/2} V C_Y^{1/2}$. In Eq.(9), $\det$ refers to the Fredholm determinant (see e.g., [82]). Let $A \in \text{Tr}(\mathcal{H})$, then the Fredholm determinant of $I + A$ is given by $\det(I + A) = \prod_{j=1}^{\infty} (1 + \lambda_j)$, where $\{\lambda_j\}_{j \in \mathbb{N}}$ are the eigenvalues of $A$.

Theorem 1 in turn follows from the following more general result on the KL divergence on Hilbert space. It states in particular that if $\mu$ is a Gaussian measure on $\mathcal{H}$, then among all probability measures with the same mean and covariance operator, $\text{KL}(\gamma || \mu)$ is minimum if and only if $\gamma$ is Gaussian.

**Theorem 2** Let $\mu = \mathcal{N}(m_1, Q)$, $\ker(Q) = \{0\}$. Let $\nu = \mathcal{N}(m_2, R_0)$ be equivalent to $\mu$. Let $S \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ be such that $R_0 = Q^{1/2} (I - S) Q^{1/2}$. Let $\gamma \in P_2(\mathcal{H})$ be absolutely continuous with respect to $\mu$, with mean $m_3, m_3 - m_1 \in \text{Im}(Q^{1/2})$, and covariance operator $R_1 = Q^{1/2} A Q^{1/2}, A \in \text{Sym}^+(\mathcal{H})$. Assume further that one of the following (non-mutually exclusive) conditions hold
1. $S \in \text{Tr}(\mathcal{H})$.

2. $I - A \in \text{HS}(\mathcal{H})$.

Then the following decomposition holds

$$
\text{KL}(\gamma||\mu) = \text{KL}(\gamma||\nu) - \frac{1}{2} ||(I - S)^{-1/2}Q^{-1/2}(m_2 - m_1)||^2
$$

$$
- \frac{1}{2}[(I - S)^{-1}Q^{-1/2}(m_3 - m_1), Q^{-1/2}(m_3 - m_1)]
$$

$$
+ \langle (I - S)^{-1}Q^{-1/2}(m_2 - m_1), Q^{-1/2}(m_3 - m_1) \rangle
$$

$$
+ \frac{1}{2} \text{Tr}[S(I - (I - S)^{-1}A)] - \frac{1}{2} \log \det_2(I - S). \quad (10)
$$

In particular, for $m_3 = m_2$ and $A = I - S$, i.e. $R_\gamma = R_\nu$,

$$
\text{KL}(\gamma||\mu) = \text{KL}(\gamma||\nu) + \frac{1}{2} ||Q^{-1/2}(m_2 - m_1)||^2 - \frac{1}{2} \log \det_2(I - S) \quad (11)
$$

$$
= \text{KL}(\gamma||\nu) + \text{KL}(\nu||\mu). \quad (12)
$$

In this case $\text{KL}(\gamma||\mu) \geq \text{KL}(\nu||\mu)$, with equality if and only if $\gamma = \nu$, i.e. if and only if $\gamma$ is Gaussian.

In Theorem 2, $\det_2$ refers to the Hilbert-Carleman determinant (see e.g. [82]). For $A \in \text{HS}(\mathcal{H})$, the Hilbert-Carleman determinant of $I + A$ is defined by

$$
\det_2(I + A) = \det[(I + A) \exp(-A)],
$$

with $\det$ being the Fredholm determinant. The different conditions in Theorem 2 can be satisfied simultaneously. In particular, they are both automatically satisfied in the case $\text{dim}(\mathcal{H}) < \infty$.

**Entropic 2-Wasserstein distance between Gaussian measures.** In the following, let $\mu_i = \mathcal{N}(m_i, C_i), i = 0, 1$, be two Gaussian measures on $\mathcal{H}$, where $m_i \in \mathcal{H}, C_i \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$. Consider the cost function $c(x, y) = ||x - y||^2$ on $\mathcal{H} \times \mathcal{H}$ and the corresponding optimization problem

$$
\text{OT}_\epsilon^{2} (\mu_0, \mu_1) = \min_{\gamma \in \text{Joint}(\mu_0, \mu_1)} \mathbb{E}_{\gamma} ||x - y||^2 + \epsilon \text{KL}(\gamma||\mu_0 \otimes \mu_1). \quad (13)
$$

A direct consequence of Theorem 1 is that if $C_0, C_1$ are nonsingular, then a minimizer of problem (13) is necessarily a joint Gaussian measure of $\mu_0$ and $\mu_1$. We show that this holds in the general setting, i.e. in both nonsingular and singular cases. The following result gives the explicit formula for this minimizer, which is unique for any $\epsilon > 0$. It is proved in Section 8, using two different methods: (i) by directly solving the optimization (13), and (ii) by solving the corresponding Schödinger system.

**Theorem 3 (Optimal entropic transport plan)** Let $\mu_0 = \mathcal{N}(m_0, C_0)$, $\mu_1 = \mathcal{N}(m_1, C_1)$. For each fixed $\epsilon > 0$, problem (13) has a unique minimizer.
\ \gamma^\epsilon, \text{ which is the Gaussian measure}
\gamma^\epsilon = \mathcal{N}\left(\begin{pmatrix} m_0 \\ m_1 \end{pmatrix}, \begin{pmatrix} C_0 & C_{XY} \\ C_{XY} & C_1 \end{pmatrix}\right), \quad \text{(14)}

where \( C_{XY} = \frac{2}{\epsilon} C_0^{1/2} \left( I + \frac{1}{2} M^\epsilon_{01} \right)^{-1} C_0^{1/2} C_1 \). \quad \text{(15)}

The Radon-Nikodym derivative of \( \gamma^\epsilon \) with respect to \( \mu_0 \otimes \mu_1 \) is given by
\[ \frac{d\gamma^\epsilon}{d(\mu_0 \otimes \mu_1)}(x,y) = \alpha^\epsilon(x)\beta^\epsilon(y) \exp\left(-\frac{||x-y||^2}{\epsilon}\right), \quad \text{(16)} \]

where the functions \( \alpha^\epsilon : \mathbb{R} \to \mathbb{R} \) and \( \beta^\epsilon : \mathbb{R} \to \mathbb{R} \) take the form
\[ \alpha^\epsilon(x) = \exp\left( \langle x-m_0, A(x-m_0) \rangle + \frac{2}{\epsilon} \langle x-m_0, m_0 - m_1 \rangle + a \right), \quad \text{(17)} \]
\[ \beta^\epsilon(y) = \exp\left( \langle y-m_1, B(y-m_1) \rangle + \frac{2}{\epsilon} \langle y-m_1, m_1 - m_0 \rangle + b \right). \quad \text{(18)} \]

The constants \( a, b \in \mathbb{R} \) and the operators \( A, B : \mathcal{H} \to \mathcal{H} \) are given by
\[ A = \frac{1}{\epsilon} I - \frac{2}{\epsilon^2} C_0^{1/2} \left[ I + \frac{1}{2} M^\epsilon_{01} \right]^{-1} C_0^{1/2}, \]
\[ B = \frac{1}{\epsilon} I - \frac{2}{\epsilon^2} C_0^{1/2} \left[ I + \frac{1}{2} M^\epsilon_{01} \right]^{-1} C_0^{1/2}, \quad \text{(19)} \]
\[ \exp(a+b) = \exp\left( \frac{||m_0-m_1||^2}{\epsilon} \right) \sqrt{\det \left( I + \frac{1}{2} M^\epsilon_{01} \right)}. \]

Here \( \det \) is the Fredholm determinant and \( M^\epsilon_{ij} : \mathcal{H} \to \mathcal{H} \), are defined by
\[ M^\epsilon_{ij} = -I + \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_i C_j C_0^{1/2} \right)^{1/2}, \quad i, j = 0, 1. \quad \text{(20)} \]

Remark 1 The operator \( M^\epsilon_{ij} \) as defined in Eq.(20) can be rewritten as
\[ M^\epsilon_{ij} = \frac{16}{\epsilon^2} C_i^{1/2} C_j C_i^{1/2} \left[ I + \left( I + \frac{16}{\epsilon^2} C_i^{1/2} C_j C_i^{1/2} \right)^{1/2} \right]^{-1}, \quad \text{(21)} \]
from which it follows that \( M^\epsilon_{ij} \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \). Thus for the operator
\[ I + \frac{1}{2} M^\epsilon_{ij} = \frac{1}{2} I + \frac{1}{4} \left( I + \frac{16}{\epsilon^2} C_i^{1/2} C_j C_i^{1/2} \right)^{1/2}, \quad \text{(22)} \]
the Fredholm determinant \( \det \left( I + \frac{1}{2} M^\epsilon_{ij} \right) \) is well-defined and positive.
Finite-dimensional case. For $\mathcal{H} = \mathbb{R}^n$ and $C_0, C_1 \in \text{Sym}^{+}(n)$, the cross-covariance operator $C_{XY}$ in Theorem 3 has the same expression as that given in Theorem 1 in [45], namely $C_{XY} = \frac{1}{2}[-I + C_0^{1/2}(I + C_0)^{1/2}C_1C_0^{1/2})C_0^{-1/2}]$.

Theorem 4 (Entropic 2-Wasserstein distance between Gaussian measures on Hilbert space) Let $\mu_0 = \mathcal{N}(m_0, C_0)$ and $\mu_1 = \mathcal{N}(m_1, C_1)$. For each fixed $\epsilon > 0$,

$$OT^\epsilon_{d^2}(\mu_0, \mu_1) = \|m_0 - m_1\|^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2} \text{Tr}(M^\epsilon_{01})$$

$$+ \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2} M^\epsilon_{01} \right).$$

(23)

Remark 2 While we use the term entropic distance, $OT^\epsilon_{d^2}$ is neither a distance nor a divergence, since generally $OT^\epsilon_{d^2}(\mu, \mu) \neq 0$, as noted before.

Connection with the entropic Kantorovich duality formulation. Following [25], let $(X, \mu)$ be a Polish space. The class of Entropy-Kantorovich potentials is defined by the set of measurable functions $\varphi$ on $X$ satisfying

$$L^\text{exp}_\epsilon(X, \mu) = \left\{ \varphi : X \to [-\infty, \infty] : 0 < E_\mu \left[ \exp \left( \frac{1}{\epsilon} \varphi \right) \right] < \infty \right\}.\quad (24)$$

The dual Kantorovich functional is defined to be

$$D(\varphi, \psi) = E_{\mu_0}[\varphi] + E_{\mu_1}[\psi] - \epsilon \left( E_{\mu_0 \otimes \mu_1} \left[ \frac{(\varphi \oplus \psi) - d^2}{\epsilon} \right] - 1 \right),$$

where $(\varphi \oplus \psi)(x, y) = \varphi(x) + \psi(y)$. For $X = \mathcal{H}$, $c(x, y) = \|x - y\|^2$, the entropic Kantorovich dual formulation of $OT^\epsilon_{d^2}$ is given by [25,30,36,39,52]

$$OT^\epsilon_{d^2}(\mu_0, \mu_1) = \sup_{\varphi \in L^\text{exp}_\epsilon(\mathcal{H}, \mu_0), \psi \in L^\text{exp}_\epsilon(\mathcal{H}, \mu_1)} D(\varphi, \psi).$$

(26)

The following shows that in our setting, the supremum in (26) is attained.

Corollary 1 Let $\mu_i = \mathcal{N}(m_i, C_i), i = 0, 1$. Let $\varphi^\epsilon = \epsilon \log \alpha^\epsilon$, $\psi = \epsilon \log \beta^\epsilon$, with $\alpha^\epsilon$, $\beta^\epsilon$ as defined in Theorem 3. Then $\varphi^\epsilon \in L^\text{exp}_\epsilon(\mathcal{H}, \mu_0)$, $\psi^\epsilon \in L^\text{exp}_\epsilon(\mathcal{H}, \mu_1)$, and

$$OT^\epsilon_{d^2}(\mu_0, \mu_1) = D(\varphi^\epsilon, \psi^\epsilon).$$

(27)

We remark that in the case the cost function $c(x, y)$ is bounded, much more can be said about the duality formulation, see [25].

Theorem 5 (Convexity) Let $\mu_0 = \mathcal{N}(m_0, C_0)$, $\mu_1 = \mathcal{N}(m, X)$, then $OT^\epsilon_{d^2}(\mu_0, \mu_1)$ is convex in each argument. In particular, let $C_0$ be fixed, then the function $X \to FE(X) = OT^\epsilon_{d^2}(\mathcal{N}(0, C_0), \mathcal{N}(0, X))$ is convex in $X \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$. Furthermore, it is strictly convex if $C_0$ is strictly positive, i.e. $\text{ker}(C_0) = \{0\}$. 


Theorem 6 (Lower bound for Entropic 2-Wasserstein distance between general probability measures on Hilbert space) Let $\mu_0, \mu_1$ be two probability measures on $\mathcal{H}$, with means $m_0, m_1$ and covariance operators $C_0, C_1$, respectively. Then

$$
\text{OT}_2^\epsilon(\mu_0, \mu_1) \geq ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2} \text{Tr}(M_{01}')
+ \frac{\epsilon}{2} \log \det \left(I + \frac{1}{2} M_{01}' \right)
.$$  

(28)

Equality happens if and only if $\mu_0, \mu_1$ are Gaussian measures.

Theorem 7 (Sinkhorn divergence between Gaussian measures on Hilbert space) Let $\mu_0 = \mathcal{N}(m_0, C_0)$, $\mu_1 = \mathcal{N}(m_1, C_1)$. Then

$$
S_{d_2}^\epsilon(\mu_0, \mu_1) = ||m_0 - m_1||^2 + \frac{\epsilon}{4} \text{Tr} \left[ M_{00}' - 2 M_{01}' + M_{11}' \right]
+ \frac{\epsilon}{4} \log \det \left[ \frac{(I + \frac{1}{2} M_{01}')^2}{(I + \frac{1}{2} M_{00}')(I + \frac{1}{2} M_{11}')} \right].
$$  

(29)

Finite-dimensional case. For $C_0, C_1 \in \text{Sym}^+(n)$, one verifies directly that Eqs.(23) and (29) reduce to Eqs.(6) and (7), respectively.

In [30], the Sinkhorn divergence was proved to be convex in each variable for either a compact metric space $X$ or for measures with bounded support on $\mathbb{R}^n$, with cost function $c(x, y) = ||x - y||^p$, $p = 1, 2$. In [44], this was shown for sub-Gaussian measures on $\mathbb{R}^n$, $c(x, y) = ||x - y||^2$. The following shows strict convexity for Gaussian measures on $\mathcal{H}$ with $c(x, y) = ||x - y||^2$. This property is crucial for guaranteeing the uniqueness of the barycenter problem below.

Theorem 8 (Strict convexity of Sinkhorn divergence) Let $\mu_0 = \mathcal{N}(m_0, C_0)$, $\mu_1 = \mathcal{N}(m, X)$. Then $S_{d_2}^\epsilon(\mu_0, \mu_1)$ is strictly convex in each argument. In particular, let $C_0$ be fixed, then the function $X \to F_S(X) = S_{d_2}^\epsilon[\mathcal{N}(0, C_0), \mathcal{N}(0, X)]$ is strictly convex in $X \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$.

In [30], positivity of the Sinkhorn divergence was proved for either a compact metric space $X$ or for measures with bounded support on $\mathbb{R}^n$, with cost function $c(x, y) = ||x - y||^p$, $p = 1, 2$. We now show that in the Gaussian case, with $c(x, y) = ||x - y||^2$, this holds in the much more general Hilbert space setting.

Theorem 9 (Positivity of Sinkhorn divergence) The function $S_{d_2}^\epsilon : \text{Gauss}(\mathcal{H}) \times \text{Gauss}(\mathcal{H}) \to \mathbb{R}_{\geq 0}$ satisfies

$$
S_{d_2}^\epsilon(\mu_0, \mu_1) \geq 0, \quad \forall \mu_0, \mu_1 \in \text{Gauss}(\mathcal{H}),
$$  

(30)

$$
S_{d_2}^\epsilon(\mu_0, \mu_1) = 0 \iff \mu_0 = \mu_1.
$$  

(31)
Differentiability. In the finite-dimensional setting, \( \text{Sym}^+(n) \) is an open subset in the vector space \( \text{Sym}(n) \) and Fréchet derivatives can be properly defined on this set. In contrast, \( \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \) is not an open subset of \( \text{Sym}(\mathcal{H}) \) when \( \dim(\mathcal{H}) = \infty \). In particular, for the exact 2-Wasserstein distance, the function \( X \to W_2^2(\mathcal{N}(0,C_0),\mathcal{N}(0,X)) = \text{Tr}(C_0) + \text{Tr}(X) - 2\text{Tr}[(C_0^{1/2}X C_0^{1/2})^{1/2}] \) is not Fréchet differentiable on \( \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \).

To discuss Fréchet differentiability of both \( \text{OT}^\epsilon_2 \) and \( S^\epsilon_2 \) in the covariance operator component, we can extend their definition, thanks to the regularization effect, to a larger, open set, containing \( \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \), as follows.

**Theorem 10 (Differentiability of entropic Wasserstein distance and Sinkhorn divergence)** Let \( C_0 \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \) be fixed. Both functions \( F_E \) in Theorem 5 and \( F_S \) in Theorem 8 are well-defined and twice Fréchet differentiable on the open, convex set \( \Omega = \{ X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) : I + c^2_0 X C_0^{1/2} X C_0^{1/2} > 0 \} \supset \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}), \ c_0 = \frac{1}{4} \). Furthermore, \( F_S \) is strictly convex and \( F_E \) is convex on \( \Omega \), with strict convexity if \( C_0 \) is strictly positive.

The expressions for \( \text{OT}^\epsilon_2 \) and \( S^\epsilon_2 \) in Theorems 4 and 7 are not intuitively close to the exact OT formula, which is the same as in the finite-dimensional setting [34,19]. Theorem 11 below gives equivalent formulas that better express the connections between the exact and regularized settings.

**Theorem 11 (Equivalent expressions for entropic 2-Wasserstein distance and Sinkhorn divergence)** Let \( \mu_0 = \mathcal{N}(m_0, C_0), \mu_1 = \mathcal{N}(m_1, C_1) \). Define \( L^\epsilon_{ij} = \frac{\epsilon}{8} (-I + (I + \frac{\epsilon}{8} C_0^{1/2} C_1^{1/2} C_0^{1/2})^{1/2}) = \frac{\epsilon}{8} M^\epsilon_{ij}, \ i,j = 0, 1 \). Then

\[
\text{OT}^\epsilon_2(\mu_0, \mu_1) = \|m_0 - m_1\|_2^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - 2\text{Tr}[C_0^{1/2} C_1^{1/2} - L^\epsilon_{01}]^{1/2} + \frac{\epsilon}{2} \log \text{det} \frac{C_0^{1/2} C_1^{1/2}}{L^\epsilon_{01}}.
\]

(32)

\[
S^\epsilon_2(\mu_0, \mu_1) = \|m_0 - m_1\|_2^2 + \text{Tr}[(C_0^2 - L^\epsilon_{00})^{1/2} - 2(C_0^{1/2} C_1^{1/2} - L^\epsilon_{01})^{1/2} + (C_1^2 - L^\epsilon_{11})^{1/2}] + \frac{\epsilon}{4} \log \text{det} \frac{L^\epsilon_{00} L^\epsilon_{11}}{(L^\epsilon_{01})^2}.
\]

(33)

Furthermore, we verify directly that

\[
\lim_{\epsilon \to 0} \text{OT}^\epsilon_2(\mu_0, \mu_1) = \|m_0 - m_1\|_2^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - 2\text{Tr}[C_0^{1/2} C_1^{1/2}]^{1/2} = W_2^2(\mu_0, \mu_1),
\]

(34)

\[
\lim_{\epsilon \to \infty} \text{OT}^\epsilon_2(\mu_0, \mu_1) = \|m_0 - m_1\|_2^2 + \text{Tr}(C_0) + \text{Tr}(C_1).
\]

(35)
\[
\lim_{\epsilon \to 0} S^\epsilon_{d^2}(\mu_0, \mu_1) = ||m_0 - m_1||^2_2 + \text{Tr}(C_0) + \text{Tr}(C_1) - 2\text{Tr}[C_0^{1/2}C_1C_0^{1/2}]^{1/2}
\]
\[
= W_2^2(\mu_0, \mu_1),
\]
\[
\lim_{\epsilon \to \infty} S^\epsilon_{d^2}(\mu_0, \mu_1) = ||m_0 - m_1||^2_2.
\]

**Entropic 2-Wasserstein barycenter of Gaussian measures.**

Given \( N \) probability measures \( \mu_i \in P(\mathcal{H}), i = 1, 2, ..., N \), the entropic barycenter \( \bar{\mu} \) with weights \( w_i > 0, \sum_{i=1}^{N} w_i = 1 \), is defined as the Fréchet mean

\[
\bar{\mu} := \arg \min_{\mu \in P(\mathcal{H})} \sum_{i=1}^{N} w_i d_{d^2}^\epsilon(\mu, \mu_i), \quad w_i > 0, \quad \sum_{i=1}^{N} w_i = 1.
\]

The entropic barycenter problem illustrates clearly the bias effect of the entropic regularization term, as analyzed in the finite-dimensional case, e.g. [44, 45]. This bias effect is sharp when \( \dim(\mathcal{H}) = \infty \), with the finite-dimensional barycenter equation failing to generalize to this case. In the following, we call a barycenter trivial if it is a Dirac delta measure in \( \mathcal{H} \).

**Theorem 12 (Entropic Barycenter of Gaussians)** Let \( \mu_i = N(m_i, C_i), i = 1, 2, ..., N \) be a set of Gaussian measures on \( \mathcal{H} \). Then a minimizer of problem (38) in \( P_2(\mathcal{H}) \) is necessarily Gaussian. Assume at least one of the \( C_i \)'s is strictly positive. Then on \( \text{Gauss}(\mathcal{H}) \), problem (38) is strictly convex and the first order minimality condition is

\[
\sum_{i=1}^{N} w_i \left[ C_i^{1/2} \left( I + \left( I + \frac{16}{\epsilon^2} C_i^{1/2} X C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} \right] = \frac{\epsilon}{4} I.
\]

1. If \( \epsilon I \geq 2 \sum_{i=1}^{N} w_i C_i \), the unique barycenter is the Dirac delta measure centered at \( \bar{m} = \sum_{i=1}^{N} w_i m_i \).

2. If \( \dim(\mathcal{H}) < \infty \), a necessary condition for the existence of a non-trivial barycenter is

\[
0 < \epsilon I < 2 \sum_{i=1}^{N} w_i C_i.
\]

A sufficient condition for the existence of a non-trivial barycenter is

\[
\exists \alpha \in \mathbb{R}, \alpha > 0 \text{ such that } C_i \geq \alpha I, 1 \leq i \leq N, \text{ and } 0 < \epsilon < 2\alpha.
\]

In this case, the barycenter is unique and is the Gaussian measure \( N(\bar{m}, \bar{C}) \), where \( \bar{C} > 0 \) is the unique solution of Eq.(39). Equivalently, \( \bar{C} \) is the unique strictly positive solution of the following equation

\[
X = \frac{\epsilon}{4} \sum_{i=1}^{N} w_i \left[ -I + \left( I + \frac{16}{\epsilon^2} X \bar{C} X \right)^{1/2} \right].
\]
3. If \( \dim(H) = \infty \), then Eq.(39) has no solution in \( \text{Sym}^+(H) \). If \( \epsilon \not\geq 2 \sum_{i=1}^{N} w_i C_i \), then the barycenter, if it exists, is not the Dirac measure centered at \( \tilde{m} \).

**Discussion of results.** Eq.(39) is the first order optimality condition for the strictly convex problem (38). It has sharply different behavior when \( \dim(H) = \infty \) compared with the case \( \dim(H) < \infty \), as we stated.

Conditions (40) and (41) show that, even in the finite-dimensional setting, a non-trivial entropic barycenter of Gaussian measures exists if and only if \( \epsilon \) is sufficiently small. Condition (40), namely \( 0 < \epsilon I < 2 \sum_{i=1}^{N} w_i C_i \), cannot be satisfied in the case \( \dim(H) = \infty \) since \( I \) is not trace class.

Under condition (41), Eq.(39) has a unique solution, which is strictly positive. Then Eq.(42) also has a unique strictly positive solution, but it also has the trivial solution \( X_0 = 0 \) and uncountably infinitely many positive solutions, which are singular (see Theorem 14 and Proposition 7).

**Remark 3** In general, for \( \epsilon > 0 \), the entropic barycenter problem (38) does not make much sense. Consider the following one-dimensional scenario, where \( C_1 = \cdots = C_N = \sigma^2 > 0 \). The unique solution of (39) is \( \bar{C} = \sigma^2 - \frac{\epsilon}{2} > 0 \iff \sigma^2 > \frac{\epsilon}{2} \). Thus if \( \epsilon \geq 2\sigma^2 \), then Equation (39) has no positive solution. The above solution \( \bar{C} \) is also obtained for the case \( N = 1, w_1 = 1, C_1 = \sigma^2 > 0 \), in which case a sensible solution should be \( \bar{C} = C_1 \), i.e. the barycenter of a set of one point should the point itself. This seemingly pathological behavior is not necessarily surprising, since OT\(_{\epsilon}^d\) is neither a distance nor a divergence.

**Sinkhorn barycenter of Gaussian measures.** We now consider the barycenter problem with respect to the Sinkhorn divergence. For a set of probability measures \( \{\mu_i\}_{i=1}^{N} \) on \( H \) and a set of weights \( \sum_{i=1}^{N} w_i = 1, w_i > 0, 1 \leq i \leq N \), their Sinkhorn barycenter is defined to be

\[
\bar{\mu} = \arg \min_{\mu \in \mathcal{P}(H)} \sum_{i=1}^{N} w_i S_{\epsilon}^d(\mu, \mu_i), \quad \sum_{i=1}^{N} w_i = 1, w_i > 0, 1 \leq i \leq N. \tag{43}
\]

In the current work, we consider barycenter of the \( N \) Gaussian measures \( \{\mu_i\}_{i=1}^{N} \) in the set of all Gaussian measures on \( H \)

\[
\bar{\mu} = \arg \min_{\mu \in \text{Gauss}(H)} \sum_{i=1}^{N} w_i S_{\epsilon}^d(\mu, \mu_i), \quad \sum_{i=1}^{N} w_i = 1, w_i > 0, 1 \leq i \leq N. \tag{44}
\]

In contrast to the entropic barycenter problem, the debiased Sinkhorn barycenter problem has a consistent generalization to the infinite-dimensional setting, with a unique solution that is valid in both singular and nonsingular cases.

**Theorem 13 (Sinkhorn barycenter of Gaussian measures)** Consider the set of Gaussian measures \( \{N(m_i, C_i)\}_{i=1}^{N} \) on \( H \), with \( m_i \in \mathcal{H} \) and \( C_i \in \mathcal{H} \).
Sym$^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$. Their Sinkhorn barycenter in Gauss($\mathcal{H}$), as defined in Eq.(44), is the unique Gaussian measure $\bar{\mu} = N(\bar{m}, \bar{C})$, where $\bar{m} = \sum_{i=1}^{N} w_i m_i$ and $\bar{C}$ is the unique solution of the following equation, with $c_\epsilon = \frac{4}{\epsilon}$,

$$X = \left(I + (I + c_\epsilon^2 X^2)^{1/2}\right)^{1/2} \sum_{i=1}^{N} w_i \left[C_i^{1/2} \left(I + \left(I + c_\epsilon^2 X^1/2 C_i^{1/2} X^{1/2}\right)^{1/2}\right)^{-1} C_i^{1/2}\right]$$

$$\times \left(I + (I + c_\epsilon^2 X^2)^{1/2}\right)^{1/2}. \quad (45)$$

Furthermore, $\bar{C}$ is strictly positive if and only if

$$\sum_{i=1}^{N} w_i C_i > 0. \quad (46)$$

Under the additional hypothesis that $\bar{C} > 0$, $\bar{C}$ is equivalently the unique strictly positive solution of the following equation

$$X = \frac{1}{c_\epsilon} \left[-I + \left(\sum_{i=1}^{N} w_i \left(I + c_\epsilon^2 X^{1/2} C_i^{1/2} X^{1/2}\right)^{1/2}\right)^2\right]^{1/2}. \quad (47)$$

Define the following map $\mathcal{F} : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H})$ by

$$\mathcal{F}(X) = \left(I + (I + c_\epsilon^2 X^2)^{1/2}\right)^{1/2} \sum_{i=1}^{N} w_i \left[C_i^{1/2} \left(I + \left(I + c_\epsilon^2 C_i^{1/2} X^{1/2}\right)^{1/2}\right)^{-1} C_i^{1/2}\right]$$

$$\times \left(I + (I + c_\epsilon^2 X^2)^{1/2}\right)^{1/2}. \quad (48)$$

Then the unique solution of Eq.(45) is the unique fixed point of $\mathcal{F}$.

**Limiting cases.** When $\epsilon \to 0$, both Eqs. (42) and (47) become

$$X = \sum_{i=1}^{N} w_i (X^{1/2} C_i^{1/2} X^{1/2})^{1/2}. \quad (49)$$

In the finite-dimensional setting, this is the barycenter equation for the exact 2-Wasserstein distance [1], assuming that $\bar{C} > 0$. As of the current writing, to the best of our knowledge, a rigorous proof for the infinite-dimensional case has not yet been established. We note that the proof given in [54], which uses the transport map in [19] to compute gradients, is only applicable in the case $\dim(\mathcal{H}) < \infty$, since the transport map is generally unbounded when $\dim(\mathcal{H}) = \infty$, see also the discussion in [57].

**Theorem 14 (Singular solutions of fixed point equations)** Let $\dim(\mathcal{H}) \geq 2$. The following equations have uncountably infinitely many positive, singular solutions, apart from the trivial solution $X_0 = 0$. Here $c_\epsilon = \frac{4}{\epsilon}$. 

1. **Exact 2-Wasserstein barycenter**, $\sum_{i=1}^{N} w_{i}C_{i} > 0$,

\[
X = \sum_{i=1}^{N} w_{i}(X_{i}^{1/2}C_{i}X_{i}^{1/2})^{1/2}.
\]  

(50)

Without the condition $\sum_{i=1}^{N} w_{i}C_{i} > 0$, this equation always has at least one positive, nonzero singular solution.

2. **Entropic Wasserstein barycenter**, $2 \leq \dim(\mathcal{H}) < \infty$, $0 < \epsilon I < 2 \sum_{i=1}^{N} w_{i}C_{i}$,

\[
X = \frac{1}{\epsilon c} \sum_{i=1}^{N} w_{i} \left[ -I + \left( I + \epsilon^{2}X_{i}^{1/2}C_{i}X_{i}^{1/2} \right)^{1/2} \right].
\]  

(51)

3. **Sinkhorn barycenter (second version)**, $\sum_{i=1}^{N} w_{i}C_{i} > 0$,

\[
X = \frac{1}{\epsilon c} \left[ -I + \left( \sum_{i=1}^{N} w_{i} \left( I + \epsilon^{2}X_{i}^{1/2}C_{i}X_{i}^{1/2} \right)^{1/2} \right)^{2} \right]^{1/2}.
\]  

(52)

Without the condition $\sum_{i=1}^{N} w_{i}C_{i} > 0$, this equation always has at least one positive, nonzero singular solution.

**Comparison of Eqs. (45) and (47).** Eq.(45) is general and is always valid whether the covariance operators $C_{i}$’s and the barycenter $C$ are singular or nonsingular. This equation always has a unique solution, which can be positive and singular or strictly positive. Furthermore, this solution is strictly positive if and only if $\sum_{i=1}^{N} w_{i}C_{i} > 0$.

Eq.(47) has the same form as the finite-dimensional version reported in [55] and [45]. It is, however, only applicable for finding the barycenter in the case it is strictly positive, since it is derived under this explicit assumption. If the solution of Eq.(45) is strictly positive, then it is also the unique strictly positive solution of Eq.(47). Eq.(47), however, always has the trivial solution $X = 0$. Furthermore, if $\dim(\mathcal{H}) \geq 2$ and at least one of the $C_{i}$’s is positive, then it has uncountably infinitely many positive solutions, which are singular (Proposition 14). The same phenomenon happens for the barycenter equation (50) in the exact, unregularized setting [1], i.e. when $\epsilon = 0$.

As we discuss in detail in Section 11, it is not straightforward to extend the approach in [1] for Eq.(50) and [45] for Eq.(47) in the finite-dimensional setting, which requires all $C_{i}$’s to be strictly positive for the existence of $\bar{C} > 0$, to the infinite-dimensional setting. This is because it is no longer possible to uniformly lower bound the $C_{i}$’s by $\alpha I$ for some $\alpha > 0$ and it is not clear whether this lower bound can be replaced by another strictly positive operator.

We also remark on our condition $\sum_{i=1}^{N} w_{i}C_{i} > 0$ for the strict positivity of $\bar{C}$, which is more general than requiring all $C_{i}$’s to be strictly positive (e.g. [45]).
In fact, we can have $\sum_{i=1}^{N} w_i C_i > 0$, guaranteeing $\bar{C} > 0$, with all $C_i$‘s being singular (see Section 10 for an example).

**The RKHS setting.** We now apply the abstract Hilbert space setting above to the reproducing kernel Hilbert space (RKHS) setting. In this case, we obtain an interpolation between Kernel Maximum Mean Discrepancy (MMD) and Kernelized $L^2$-Wasserstein Distance. The RKHS formulas are expressed explicitly in terms of the kernel Gram matrices, which are readily computable.

Let $\mathcal{X}$ be a complete separable metric space. Let $K$ be a continuous positive definite kernel on $\mathcal{X} \times \mathcal{X}$. Then the reproducing kernel Hilbert space (RKHS) $\mathcal{H}_K$ induced by $K$ is separable ([84], Lemma 4.33). Let $\Phi : \mathcal{X} \to \mathcal{H}_K$ be the corresponding canonical feature map, so that $K(x,y) = \langle \Phi(x), \Phi(y) \rangle_{\mathcal{H}_K}$ $\forall (x,y) \in \mathcal{X} \times \mathcal{X}$. Let $\rho$ be a Borel probability measure on $\mathcal{X}$ such that $\int_{\mathcal{X}} ||\Phi(x)||_{\mathcal{H}_K}^2 d\rho(x) = \int_{\mathcal{X}} K(x,x) d\rho(x) < \infty$. (53)

Then the RKHS mean vector $\mu_\Phi \in \mathcal{H}_K$ and covariance operator $C_\Phi : \mathcal{H}_K \to \mathcal{H}_K$ induced by the feature map $\Phi$ are both well-defined and are given by

$$\mu_\Phi = \int_{\mathcal{X}} \Phi(x) d\rho(x) \in \mathcal{H}_K,$$

$$C_\Phi = \int_{\mathcal{X}} (\Phi(x) - \mu_\Phi) \otimes (\Phi(x) - \mu_\Phi) d\rho(x).$$

(55)

Here the rank-one operator $u \otimes v$ is defined by $(u \otimes v)w = \langle v, w \rangle_{\mathcal{H}_K} u$, $u, v, w \in \mathcal{H}_K$. Then $C_\Phi$ is a positive trace class operator on $\mathcal{H}_K$ (see e.g. [66]).

Let $X = [x_1, \ldots, x_m], m \in \mathbb{N}$, be a data matrix randomly sampled from $\mathcal{X}$ according to a Borel probability distribution $\rho$, where $m \in \mathbb{N}$ is the number of observations. The feature map $\Phi$ on $\mathcal{X}$ defines the bounded linear operator $\Phi(X) : \mathbb{R}^m \to \mathcal{H}_K, \Phi(X)b = \sum_{j=1}^{m} b_j \Phi(x_j), b \in \mathbb{R}^m$. The corresponding empirical mean vector and covariance operator for $\Phi(X)$ are defined to be

$$\mu_{\Phi(X)} = \frac{1}{m} \sum_{j=1}^{m} \Phi(x_j) = \frac{1}{m} \Phi(X)1_m,$$

$$C_{\Phi(X)} = \frac{1}{m} \Phi(X)J_m \Phi(X)^* : \mathcal{H}_K \to \mathcal{H}_K,$$

(56) (57)

where $J_m = I_m - \frac{1}{m} 1_m 1_m^T$, $1_m = (1, \ldots, 1)^T \in \mathbb{R}^m$, is the centering matrix, with $J_m^2 = J_m$ and $AJ_m$ is the matrix obtained from the (possibly infinite) matrix $A$ by subtracting the mean column.

Let $X = [x_i]_{i=1}^{m}$, $Y = [y_i]_{i=1}^{m}$, be two random data matrices sampled from $\mathcal{X}$ according to two Borel probability distributions $\rho_0$ and $\rho_1$ on $\mathcal{X}$. Let $\mu_X, \mu_Y$ and $C_{\Phi(X)}, C_{\Phi(Y)}$ be the corresponding mean vectors and covariance operators induced by $K$, respectively. Let us derive the explicit expression for $\text{OT}^*_{\text{KL}}(\mu_0, \mu_1)$.
and $S_{d^2}(\mu_0, \mu_1)$ when $\mu_0 \sim \mathcal{N}(\mu_{\Phi_X}, C_{\Phi(X)})$, $\mu_1 \sim \mathcal{N}(\mu_{\Phi_Y}, C_{\Phi(Y)})$. Define the following $m \times m$ Gram matrices

$$K[X] = \Phi(X)^*\Phi(X), \quad K[Y] = \Phi(Y)^*\Phi(Y), \quad K[X, Y] = \Phi(X)^*\Phi(Y). \quad (58)$$

**Theorem 15** Let $\epsilon > 0$ be fixed. For $\mu_0 = \mathcal{N}(\mu_{\Phi_X}, C_{\Phi(X)}), \mu_1 = \mathcal{N}(\mu_{\Phi_Y}, C_{\Phi(Y)})$,

\[
\begin{align*}
OT_{d^2}(\mu_0, \mu_1) &= \frac{1}{m^2} \text{Tr}(K[X] + K[Y] - 2K[X, Y])1_m \\
&+ \frac{1}{m} \text{Tr}(K[X]J_m) + \frac{1}{m} \text{Tr}(K[Y]J_m) \\
&- \frac{\epsilon}{2} \text{Tr} \left[ -I + \left( I + \frac{16}{\epsilon^2 m^2} J_m K[X, Y] J_m K[Y, X] J_m \right)^{1/2} \right] \\
&+ \frac{\epsilon}{2} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2 m^2} J_m K[X, Y] J_m K[Y, X] J_m \right)^{1/2} \right). \\
\end{align*}
\]

\[
\begin{align*}
S_{d^2}(\mu_0, \mu_1) &= \frac{1}{m^2} \text{Tr}(K[X] + K[Y] - 2K[X, Y])1_m \\
&+ \frac{\epsilon}{4} \text{Tr} \left[ -I + \left( I + \frac{16}{\epsilon^2 m^2} (J_m K[X] J_m)^2 \right)^{1/2} \right] \\
&+ \frac{\epsilon}{4} \text{Tr} \left[ -I + \left( I + \frac{16}{\epsilon^2 m^2} (J_m K[Y] J_m)^2 \right)^{1/2} \right] \\
&- \frac{\epsilon}{2} \text{Tr} \left[ -I + \left( I + \frac{16}{\epsilon^2 m^2} J_m K[X, Y] J_m K[Y, X] J_m \right)^{1/2} \right] \\
&+ \frac{\epsilon}{2} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2 m^2} J_m K[X, Y] J_m K[Y, X] J_m \right)^{1/2} \right) \\
&- \frac{\epsilon}{4} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2 m^2} (J_m K[X] J_m)^2 \right)^{1/2} \right) \\
&- \frac{\epsilon}{4} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2 m^2} (J_m K[Y] J_m)^2 \right)^{1/2} \right). \\
\end{align*}
\]

In particular, as $\epsilon \to \infty$,

\[
\begin{align*}
\lim_{\epsilon \to \infty} S_{d^2}(\mu_0, \mu_1) &= ||\mu_{\Phi_X} - \mu_{\Phi_Y}||^2_{H_K} \\
&= \frac{1}{m^2} \text{Tr}(K[X] + K[Y] - 2K[X, Y])1_m. \\
\end{align*}
\]
This is the empirical squared Kernel MMD distance. As \( \epsilon \to 0 \),
\[
\lim_{\epsilon \to 0} S^2_\epsilon(\mu_0, \mu_1) = \frac{1}{m^2} \sum_{i=1}^{m} [K[X] + K[Y] - 2K[X, Y]]m \\
+ \frac{1}{m} \text{Tr}(K[X]J_m) + \frac{1}{m} \text{Tr}(K[Y]J_m) \\
- \frac{2}{m} \text{Tr}[J_m K[X, Y]J_m K[Y, X]J_m]^{1/2}.
\] (62)

This is the Kernelized Wasserstein Distance \([92, 62]\).

Remark 4 To keep our expressions simple, we have assumed that the number of data points in \( X = [x_i]_{i=1}^{m} \) and \( Y = [y_i]_{i=1}^{n} \) are the same, i.e. \( m = n \). The extension to the case \( m \neq n \) is straightforward.

5 From Gaussian measures to Gaussian processes

Let us discuss the translation of the results for Gaussian measures on an abstract Hilbert space \( H \) into the setting of Gaussian processes, see also \([68, 32, 73, 54, 57]\). Consider the following correspondence between Gaussian measures and Gaussian processes with paths in a Hilbert space, as established in \([74]\).

Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a probability space. Let \( T \) be an index set. Let \((T, \mathcal{A}, \nu)\) be a measurable space, \( \nu \) nonnegative, \( \sigma \)-finite, such that \( L^2(T, \mathcal{A}, \nu) \) is separable (e.g. \( T \subset \mathbb{R}^n \) measurable, \( \mathcal{A} = \mathcal{B}(T), \nu \) is the Lebesgue measure).

Let \( \xi = (\xi_t)_{t \in T} = (\xi(t, \omega))_{t \in T} \) be a real \( \mathcal{F}/\mathcal{A} \)-measurable Gaussian process on \((\Omega, \mathcal{F}, \mathcal{P})\), with mean \( m(t) \) and covariance function \( K(s, t) \), denoted by \( \text{GP}(m, K) \). The sample paths \( \xi(\cdot, \omega) \in H = L^2(T, \nu) \) almost \( \mathcal{P} \)-surely, i.e.
\[
\int_T \xi^2(t, \omega) d\nu(t) < \infty, \quad \text{almost } \mathcal{P} \text{-surely,}
\] (63)
if and only if (\([74]\), Theorem 2 and Corollary 1)
\[
\int_T m^2(t) d\nu(t) < \infty, \quad \int_T K(t, t) d\nu(t) < \infty.
\] (64)

Then \( \xi \) induces the following Gaussian measure \( \mathcal{P}_\xi \) on \((H, \mathcal{B}(H))\)
\[
\mathcal{P}_\xi(B) = \mathcal{P}\{\omega \in \Omega : \xi(\cdot, \omega) \in B\}, \quad B \in \mathcal{B}(H),
\] (65)
with mean \( m \in H \) and covariance operator \( \mathcal{C}_K : H \to H \), defined by
\[
(C_K f)(s) = \int_T K(s, t) f(t) d\nu(t), \quad f \in H.
\] (66)

Conversely, let \( \mu \) be a Gaussian measure on \((H = L^2(T, \mathcal{A}, \nu), \mathcal{B}(H))\). Then there is an \( \mathcal{F}/\mathcal{A} \)-measurable Gaussian process \( \xi = (\xi_t)_{t \in T} \) on \((\Omega, \mathcal{F}, \mathcal{P})\) with sample paths in \( H \), such that the induced probability measure is \( \mathcal{P}_\xi = \mu \).
Correspondence between covariance function and covariance operator via Mercer Theorem. Covariance functions, being positive definite kernels, can be fully expressed via their induced covariance operators, as follows. In the following, let $T$ be a $\sigma$-compact metric space, that is $T = \bigcup_{i=1}^{\infty} T_i$, where $T_1 \subset T_2 \subset \cdots$, with each $T_i$ compact. Let $\nu$ be a positive, non-degenerate Borel measure on $T$, i.e. $\nu(B) > 0$ for any open $U \subset T$, with $\nu(T_i) < \infty \forall i \in \mathbb{N}$.

**Theorem 16 (Mercer Theorem - version in [85])** Let $T$ be a $\sigma$-compact metric space and $\nu$ a positive, non-degenerate Borel measure on $T$. Let $K : T \times T \to \mathbb{R}$ be continuous, positive definite. Assume furthermore that \[
\int_T K(s,t)^2 d\nu(t) < \infty, \quad \forall s \in T. \tag{67}
\]
\[
\int_{T \times T} K(s,t)^2 d\nu(s)d\nu(t) < \infty. \tag{68}
\]
Then $C_K$ is Hilbert-Schmidt, self-adjoint, positive. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenvalues of $C_K$, with corresponding orthonormal eigenvectors $\{\phi_k\}_{k=1}^{\infty}$. Then
\[
K(s,t) = \sum_{k=1}^{\infty} \lambda_k \phi_k(s) \phi_k(t), \tag{69}
\]
where the series converges absolutely for each pair $(s,t) \in T \times T$ and uniformly on each compact subset of $T \times T$.

Mercer Theorem thus describes the covariance function $K(s,t)$ fully and explicitly via its covariance operator $C_K$. Since $K$ is positive definite, $K(s,t)^2 \leq K(s,s)K(t,t) \forall s,t \in T \times T$. Thus the condition $\int_T K(t,t)d\nu(t) < \infty$ in (64) implies both conditions (67) and (68) in Mercer Theorem and from (69)
\[
\text{Tr}(C_K) = \sum_{k=1}^{\infty} \lambda_k = \int_T K(t,t)d\nu(t) < \infty. \tag{70}
\]

We now generalize ideas in [68,32,73,54,57], using the fact that Gaussian processes are fully determined by their mean and covariance functions. Most importantly, the following incorporates Mercer Theorem to quantify the correspondence $\text{GP}(m,K) \iff \mathcal{N}(m,C_K)$.

**Definition 1 (Divergence between Gaussian processes)** Let $T$ be a $\sigma$-compact metric space, $\nu$ a positive, non-degenerate Borel measure on $T$. Let $H = \mathcal{L}^2(T,\mathcal{B}(T),\nu)$. Let $\xi^i = \text{GP}(m_i,K_i)$, $i = 1,2$, be two Gaussian processes with mean $m_i \in H$, covariance function $K_i$ continuous, and $\int_T K_i(t,t)d\nu(t) < \infty$. Let $D$ be a divergence function on $\text{Gauss}(H) \times \text{Gauss}(H)$. The corresponding divergence $D_{\text{GP}}$ between $\xi^1$ and $\xi^2$ is defined to be
\[
D_{\text{GP}}(\xi^1||\xi^2) = D(\mathcal{N}(m_1,C_{K_1})||\mathcal{N}(m_2,C_{K_2})). \tag{71}
\]
Mercer Theorem immediately implies the following.

**Theorem 17** Assume the hypothesis in Definition 1. Then

\[ D_{GP}(\xi^1||\xi^2) \geq 0, \quad (72) \]

\[ D_{GP}(\xi^1||\xi^2) = 0 \iff \xi^1 = \xi^2. \quad (73) \]

**Remark 5** In our current context, we can immediately apply Definition 1 and Theorem 17 to the exact Wasserstein distances and Sinkhorn divergences. Definition 1 also can also be extended to cover the entropic OT distances, however since they are not metrics/divergences, Theorem 17 no longer holds.

6 Kullback-Leibler divergence between Gaussian measures

In this section, we briefly review the KL divergence between Gaussian measures on Hilbert space and prove Theorem 2. For a separable Hilbert space \( \mathcal{H} \), consider the set \( \mathcal{P}(\mathcal{H}) \) of probability measures on \( (\mathcal{H}, \mathcal{B}(\mathcal{H})) \), where \( \mathcal{B}(\mathcal{H}) \) denotes the Borel \( \sigma \)-algebra on \( \mathcal{H} \). We focus on the subset \( \mathcal{P}_2(\mathcal{H}) \) defined by

\[ \mathcal{P}_2(\mathcal{H}) = \{ \mu \in \mathcal{P}(\mathcal{H}) : \int_{\mathcal{H}} ||x||^2 d\mu(x) < \infty \}. \quad (74) \]

For \( \mu \in \mathcal{P}_2(\mathcal{H}) \), its mean vector \( m \in \mathcal{H} \) and covariance operator \( C : \mathcal{H} \to \mathcal{H} \) are well-defined and are given by

\[ \langle m, u \rangle = \int_{\mathcal{H}} \langle x, u \rangle d\mu(x), \quad u \in \mathcal{H}, \quad (75) \]

\[ \langle Cu, v \rangle = \int_{\mathcal{H}} \langle x - m, u \rangle \langle x - m, v \rangle d\mu(x), \quad u, v \in \mathcal{H}. \quad (76) \]

In particular \( C \) is a self-adjoint, positive, and trace class operator on \( \mathcal{H} \).

We recall that for two measures \( \mu \) and \( \nu \) on a measure space \( (\Omega, \mathcal{F}) \), with \( \mu \) \( \sigma \)-finite, \( \nu \) is said to be absolutely continuous with respect to \( \mu \), denoted by \( \nu << \mu \), if for any \( A \in \mathcal{F}, \mu(A) = 0 \iff \nu(A) = 0 \). In this case, the Radon-Nikodym derivative \( \frac{d\nu}{d\mu} \in \mathcal{L}^1(\mu) \) is well-defined. The Kullback-Leibler (KL) divergence between \( \nu \) and \( \mu \) is defined by

\[ \text{KL}(\nu||\mu) = \begin{cases} \int_{\mathcal{H}} \log \left( \frac{d\nu}{d\mu}(x) \right) d\nu(x) & \text{if } \nu << \mu, \\ \infty & \text{otherwise}. \end{cases} \quad (77) \]

If \( \mu << \nu \) and \( \nu << \mu \), then we say that \( \mu \) and \( \nu \) are equivalent, denoted by \( \mu \sim \nu \). We say that \( \mu \) and \( \nu \) are mutually singular, denoted by \( \mu \perp \nu \), if there exist \( A, B \in \mathcal{F} \) such that \( \mu(A) = \nu(B) = 1 \) and \( A \cap B = \emptyset \).

**Equivalence of Gaussian measures.** Let \( Q, R \) be two self-adjoint, positive trace class operators on \( \mathcal{H} \) such that \( \ker(Q) = \ker(R) = \{0\} \). Let \( m_1, m_2 \in \mathcal{H} \). A fundamental result in the theory of Gaussian measures is the Feldman-Hajek
Theorem [29], [42], which states that two Gaussian measures \( \mu = \mathcal{N}(m_1, Q) \) and \( \nu = \mathcal{N}(m_2, R) \) are either mutually singular or equivalent. The necessary and sufficient conditions for the equivalence of the two Gaussian measures \( \nu \) and \( \mu \) are given by the following.

**Theorem 18** ([9], Corollary 6.4.11, [23], Theorems 1.3.9 and 1.3.10)

Let \( \mathcal{H} \) be a separable Hilbert space. Consider two Gaussian measures \( \mu = \mathcal{N}(m_1, Q) \) and \( \nu = \mathcal{N}(m_2, R) \) on \( \mathcal{H} \). Then \( \mu \) and \( \nu \) are equivalent if and only if the following hold

1. \( m_2 - m_1 \in \text{Im}(Q^{1/2}) \).
2. There exists \( S \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \), without the eigenvalue 1, such that \( R = Q^{1/2}(I - S)Q^{1/2} \).

We now recall results on Kullback-Leibler divergences between two Gaussian measures \( \mu = \mathcal{N}(m_1, Q) \) and \( \nu = \mathcal{N}(m_2, R) \) on \( \mathcal{H} \). If \( \mu \perp \nu \), then \( \text{KL}(\nu||\mu) = \infty \). If \( \mu \sim \nu \), then we have the following result.

**Theorem 19** ([65]) Let \( \mu = \mathcal{N}(m_1, Q), \ \nu = \mathcal{N}(m_2, R), \) with \( \ker(Q) = \ker R = \{0\} \), and \( \mu \sim \nu \). Let \( S \in \text{HS}(\mathcal{H}) \cap \text{Sym}(\mathcal{H}) \), \( I - S > 0 \), be such that \( R = Q^{1/2}(I - S)Q^{1/2} \), then

\[
\text{KL}(\nu||\mu) = \frac{1}{2} \left|\left|Q^{-1/2}(m_2 - m_1)\right|\right|^2 - \frac{1}{2} \log \det_2(I - S). \tag{78}
\]

For two equivalent Gaussian measures \( \mu, \nu \) on \( \mathcal{H} \), the Radon-Nikodym derivative involves only the means and covariance operators ([65], Theorem 11). This motivates Theorem 2, which seeks to extend the validity of Eq.(78) by generalizing the expression \( \int_{\mathcal{H}} \log \left( \frac{d\nu}{d\mu} \right) d\nu \) to \( \int_{\mathcal{H}} \log \left( \frac{d\nu}{d\gamma} \right) d\gamma \) where \( \gamma \in \mathcal{P}_2(\mathcal{H}) \) is any probability measure with the same mean and covariance operator as \( \nu \).

To prove Theorem 2, in the following we utilize the concept of white noise mapping, see e.g. [22, 23]. For \( \mu = \mathcal{N}(m, Q), \ \ker(Q) = \{0\}, \) we define \( \mathcal{L}^2(\mathcal{H}, \mu) = \mathcal{L}^2(\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu) = \mathcal{L}^2(\mathcal{H}, \mathcal{B}(\mathcal{H}), \mathcal{N}(m, Q)) \). Consider the following mapping

\[
W : Q^{1/2}(\mathcal{H}) \subseteq \mathcal{H} \to \mathcal{L}^2(\mathcal{H}, \mu), \ \ z \in Q^{1/2}(\mathcal{H}) \to W_z \in \mathcal{L}^2(\mathcal{H}, \mu), \tag{79}
\]

\[
W_z(x) = \langle x - m, Q^{-1/2}z \rangle, \ \ z \in Q^{1/2}(\mathcal{H}), x \in \mathcal{H}. \tag{80}
\]

For any pair \( z_1, z_2 \in Q^{1/2}(\mathcal{H}) \), we have by definition of the covariance operator

\[
\langle W_{z_1}, W_{z_2} \rangle_{\mathcal{L}^2(\mathcal{H}, \mu)} = \int_{\mathcal{H}} \langle x - m, Q^{-1/2}z_1 \rangle \langle x - m, Q^{-1/2}z_2 \rangle N(m, Q)(dx) = \langle Q(Q^{-1/2}z_1), Q^{-1/2}z_2 \rangle = \langle z_1, z_2 \rangle_{\mathcal{H}}. \tag{81}
\]

Thus the map \( W : Q^{1/2}(\mathcal{H}) \to \mathcal{L}^2(\mathcal{H}, \mu) \) is an isometry, that is

\[
\|W_z\|_{\mathcal{L}^2(\mathcal{H}, \mu)} = \|z\|_{\mathcal{H}}, \ \ z \in Q^{1/2}(\mathcal{H}). \tag{82}
\]
Since $\ker(Q) = \{0\}$, the subspace $Q^{1/2}(\mathcal{H})$ is dense in $\mathcal{H}$ and the map $W$ can be uniquely extended to all of $\mathcal{H}$, as follows. For any $z \in \mathcal{H}$, let $\{z_n\}_{n \in \mathbb{N}}$ be a sequence in $Q^{1/2}(\mathcal{H})$ with $\lim_{n \to \infty} ||z_n - z||_{\mathcal{H}} = 0$. Then $\{z_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, so that by isometry, $\{W_{z_n}\}_{n \in \mathbb{N}}$ is also a Cauchy sequence in $\mathcal{L}^2(\mathcal{H}, \mu)$, thus converging to a unique element in $\mathcal{L}^2(\mathcal{H}, \mu)$. Thus we can define

$$W : \mathcal{H} \to \mathcal{L}^2(\mathcal{H}, \mu), \quad z \in \mathcal{H} \to W_z$$

by the following unique limit in $\mathcal{L}^2(\mathcal{H}, \mu)$

$$W_z(x) = \lim_{n \to \infty} W_{z_n}(x) = \lim_{n \to \infty} \langle x - m, Q^{-1/2}z_n \rangle.$$  

(83)

The map $W : \mathcal{H} \to \mathcal{L}^2(\mathcal{H}, \mu)$ is called the *white noise mapping* associated with the measure $\mu = \mathcal{N}(m, Q)$. $W_z$ can be expressed explicitly in terms of the finite-rank orthogonal projections $P_N = \sum_{k=1}^N e_k \otimes e_k$ onto the $N$-dimensional subspace of $\mathcal{H}$ spanned by $\{e_k\}_{k=1}^N$, $N \in \mathbb{N}$, where $\{e_k\}_{k \in \mathbb{N}}$ are the orthonormal eigenvectors of $Q$. For any $z \in \mathcal{H}$, we have

$$P_N z = \sum_{k=1}^N \langle z, e_k \rangle e_k \Rightarrow Q^{-1/2} P_N z = \sum_{k=1}^N \frac{1}{\lambda_k} \langle z, e_k \rangle e_k.$$  

(85)

Thus $Q^{-1/2} P_N z$ is always well-defined $\forall z \in \mathcal{H}$. Furthermore, for all $x, y \in \mathcal{H}$,

$$\langle Q^{-1/2} P_N x, y \rangle = \sum_{j=1}^N \frac{1}{\lambda_j} \langle x, e_j \rangle \langle y, e_j \rangle = \langle x, Q^{-1/2} P_N y \rangle.$$  

(86)

The operator $Q^{-1/2} P_N : \mathcal{H} \to \mathcal{H}$ is bounded and self-adjoint $\forall N \in \mathbb{N}$. Since the sequence $\{P_N z\}_{N \in \mathbb{N}}$ converges to $z$ in $\mathcal{H}$, we have, in the $\mathcal{L}^2(\mathcal{H}, \mu)$ sense,

$$W_z(x) = \lim_{N \to \infty} W_{P_N z}(x) = \lim_{N \to \infty} \langle x - m, Q^{-1/2} P_N z \rangle.$$  

(87)

The Radon-Nikodym derivative between two equivalent Gaussian measures on $\mathcal{H}$ is expressed explicitly via the white noise mapping, as follows.

**Theorem 20 ([65], Theorem 11)** Let $\mu = \mathcal{N}(m_1, Q)$, $\nu = \mathcal{N}(m_2, R)$, $\ker(Q) = \ker(R) = 0$ be equivalent, that is $m_2 - m_1 \in \text{Im}(Q^{1/2})$, $R = Q^{1/2}(I - S)Q^{1/2}$ for $S \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$. Let $\{\alpha_k\}_{k \in \mathbb{N}}$ be the eigenvalues of $S$, with corresponding orthonormal eigenvectors $\{\phi_k\}_{k \in \mathbb{N}}$. Let $W$ be the white noise mapping induced by $\mu$. The Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is given by

$$\frac{d\nu}{d\mu}(x) = \exp \left[-\frac{1}{2} \sum_{k=1}^{\infty} \Phi_k(x) \right] \exp \left[-\frac{1}{2} ||(I - S)^{-1/2} Q^{-1/2}(m_2 - m_1)||_2 \right],$$  

(88)

where for each $k \in \mathbb{N}$,

$$\Phi_k = \frac{\alpha_k}{1 - \alpha_k} W_{\phi_k}^2 - \frac{2}{1 - \alpha_k} \langle Q^{-1/2}(m_2 - m_1), \phi_k \rangle W_{\phi_k} + \log(1 - \alpha_k).$$  

(89)

The series $\sum_{k=1}^{\infty} \Phi_k$ converges in $\mathcal{L}^1(\mathcal{H}, \mu)$ and $\mathcal{L}^2(\mathcal{H}, \mu)$. 
Lemma 1 Let $\mu = N(m_1, Q)$, ker $Q = \{0\}$. Let $W$ be its induced white noise mapping. Let $\nu \in P_2(\mathcal{H})$, $\nu << \mu$, with mean $m_2$, where $m_2 - m_1 \in \text{Im}(Q^{1/2})$, and covariance operator $R = Q^{1/2}AQ^{1/2}$, where $A \in \text{Sym}^+(\mathcal{H})$. Then

$$\int_{\mathcal{H}} W_z(x) d\nu(x) = (Q^{-1/2}(m_2 - m_1), z). \tag{90}$$

$$\langle W_{z_1}, W_{z_2}\rangle_{L^2(\mathcal{H}, \nu)} = \langle Az_1, z_2 \rangle, \quad z_1, z_2 \in \mathcal{H},$$

$$+ \langle Q^{-1/2}(m_2 - m_1), z_1 \rangle \langle Q^{-1/2}(m_2 - m_1), z_2 \rangle. \tag{91}$$

$$\|W_z\|_{L^2(\mathcal{H}, \nu)}^2 = \langle Az, z \rangle + \langle Q^{-1/2}(m_2 - m_1), z \rangle^2, \quad z \in \mathcal{H}. \tag{92}$$

Proof For $z \in \text{Im}(Q^{1/2})$, $W_z(x) = (x - m_1, Q^{-1/2}z)$. Thus for $z_1, z_2 \in \text{Im}(Q^{1/2})$,

$$\langle W_{z_1}, W_{z_2}\rangle_{L^2(\mathcal{H}, \nu)} = \int_{\mathcal{H}} W_{z_1}(x) W_{z_2}(x) d\nu(x)$$

$$= \int_{\mathcal{H}} \langle x - m_2 + m_2 - m_1, Q^{-1/2}z_1 \rangle \langle x - m_2 + m_2 - m_1, Q^{-1/2}z_2 \rangle d\nu(x)$$

$$= \int_{\mathcal{H}} \langle x - m_2, Q^{-1/2}z_1 \rangle \langle x - m_2, Q^{-1/2}z_2 \rangle d\nu(x) + \langle m_2 - m_1, Q^{-1/2}z_1 \rangle \langle m_2 - m_1, Q^{-1/2}z_2 \rangle$$

$$= (RQ^{-1/2}z_1, Q^{-1/2}z_2) + \langle Q^{-1/2}(m_2 - m_1), z_1 \rangle \langle Q^{-1/2}(m_2 - m_1), z_2 \rangle$$

$$= \langle Az_1, z_2 \rangle + \langle Q^{-1/2}(m_2 - m_1), z_1 \rangle \langle Q^{-1/2}(m_2 - m_1), z_2 \rangle,$$

since $R = Q^{1/2}AQ^{1/2}$. In particular, for $z \in \text{Im}(Q^{1/2})$,

$$\|W_z\|_{L^2(\mathcal{H}, \nu)}^2 = \int_{\mathcal{H}} W_z^2(x) d\nu(x) = \langle Az, z \rangle + \langle Q^{-1/2}(m_2 - m_1), z \rangle^2$$

$$\leq \|A\| + \|Q^{-1/2}(m_2 - m_1)\|^2 \|z\|^2 = \|A\| + \|Q^{-1/2}(m_2 - m_1)\|^2 \|W_z\|_{L^2(\mathcal{H}, \nu)}^2.$$
For the first expression, for any \( z \in \mathcal{H}, N \in \mathbb{N} \),
\[
\int_{\mathcal{H}} W_{P_N z}(x) d\nu(x) = \int_{\mathcal{H}} \langle x - m_1, Q^{-1/2} P_N z \rangle d\nu(x)
\]
\[
= \int_{\mathcal{H}} \langle x - m_2 + m_2 - m_1, Q^{-1/2} P_N z \rangle d\nu(x) = \langle m_2 - m_1, Q^{-1/2} P_N z \rangle
\]
\[
= \langle Q^{-1/2}(m_2 - m_1), P_N z \rangle.
\]

Since \( \nu \) is a probability measure, we also have \( \lim_{N \to \infty} ||W_{P_N z} - W_z||_{L^2(\mathcal{H}, \nu)} = 0 \) by Hölder Inequality. Thus
\[
\int_{\mathcal{H}} W_z(x) d\nu(x) = \lim_{N \to \infty} \int_{\mathcal{H}} W_{P_N z}(x) d\nu(x) = \langle Q^{-1/2}(m_2 - m_1), z \rangle.
\]
This completes the proof. \( \square \)

**Proposition 1.** Let \((X, \Sigma, \mu)\) be a measurable space. Let \( \Phi = (\phi_k)_{k=1}^{\infty} \) be an orthonormal sequence in \( L^2(X, \mu) = L^2(X, \Sigma, \mu) \). Let \( f : X \to \mathbb{R} \) be a measurable function such that \( \phi_k f \in L^1(X, \mu) \ \forall k \in \mathbb{N} \) and \( b_k = \int_X \phi_k(x) f(x) d\mu(x) \), \( k \in \mathbb{N} \) satisfy \((b_k)_{k \in \mathbb{N}} \in l^2 \). Then for any \( \sum_{k=1}^{\infty} a_k \phi_k \in L^2(X, \mu) \), the following integral is well-defined and finite
\[
\int_X \left( \sum_{k=1}^{\infty} a_k \phi_k(x) \right) f(x) d\mu(x) = \int_X \left( \sum_{k=1}^{\infty} a_k \phi_k(x) \right) \left( \sum_{k=1}^{\infty} b_k \phi_k(x) \right) d\mu(x)
\]
\[
= \sum_{k=1}^{\infty} a_k b_k.
\]
(93)

We note that if \( \Phi \) is an orthonormal basis for \( L^2(X, \mu) \), then the hypothesis of Proposition 1 becomes \( f \in L^2(X, \mu) \) and the conclusion is immediate.

**Proof.** Let \( S_\Phi = \text{span}\{\phi_k\}_{k \in \mathbb{N}} \) be the closed Hilbert subspace of \( L^2(X, \mu) \) with orthonormal basis \( \Phi \). We show that the following linear functional
\[
A_f : S_\Phi \to \mathbb{R}, \quad A_f(g) = \int_X g(x) f(x) d\mu(x),
\]
is well-defined and bounded. First, by assumption, \( \phi_k f \in L^1(X, \mu) \ \forall k \in \mathbb{N} \), so that \( A_f(\phi_k) \) is well-defined \( \forall k \in \mathbb{N} \), with
\[
A_f(\phi_k) = \int_X \phi_k(x) f(x) d\mu(x) = b_k.
\]
It follows that \( A_f \) is well-defined on the dense subspace \( \text{span}\{\phi_k\}_{k \in \mathbb{N}} \), with
\[
A_f(g_N) = \int_X \left( \sum_{k=1}^{N} a_k \phi_k(x) \right) f(x) d\mu(x) = \sum_{k=1}^{N} a_k b_k \quad \text{for any } g_N = \sum_{k=1}^{N} a_k \phi_k, \ N \in \mathbb{N}.
\]
Let now \( g = \sum_{k=1}^{\infty} a_k \phi_k \in L^2(X, \mu) \), \((a_k)_{k \in \mathbb{N}} \in \ell^2\), then by linearity

\[
A_f(g) = \sum_{k=1}^{\infty} a_k b_k \quad \text{with} \quad |A_f g| \leq (\sum_{k=1}^{\infty} a_k^2)^{1/2}(\sum_{k=1}^{\infty} b_k^2)^{1/2} < \infty.
\]

Furthermore, for \( g_N = \sum_{k=1}^{N} a_k \phi_k, \ N \in \mathbb{N}\),

\[
|A_f(g_N) - A_f(g)| = \sum_{k=N+1}^{\infty} a_k b_k \leq (\sum_{k=N+1}^{\infty} a_k^2)^{1/2}(\sum_{k=N+1}^{\infty} b_k^2)^{1/2} = \|g_N - g\|_{L^2(X, \mu)}(\sum_{k=N+1}^{\infty} b_k^2)^{1/2} \to 0
\]
as \( N \to \infty \). Thus \( A_f \) is well-defined and bounded on \( S_\Phi \), with \( \|A_f\| \leq (\sum_{k=1}^{\infty} b_k^2)^{1/2} \). By the Riesz Representation Theorem, there exists a unique element \( h \in S_\Phi \) such that \( A_f(g) = (g, h)_{L^2(X, \mu)} \). It is clear that this element \( h \) is given by \( h = \sum_{k=1}^{\infty} b_k \phi_k \in L^2(X, \mu) \). \( \square \)

**Lemma 2** Assume the hypothesis of Theorem 2. Let \( \{\alpha\}_{k \in \mathbb{N}} \) be the eigenvalues of \( S \), with corresponding orthonormal eigenvectors \( \{\phi_k\}_{k \in \mathbb{N}} \). Let \( g = \sum_{k=1}^{\infty} \frac{1}{1-\alpha_k}(Q^{-1/2}(m_2 - m_1), \phi_k)W_{\phi_k} \). Then \( g \in L^2(H, \gamma) \), \( g \in L^1(H, \gamma) \), and

\[
\int_{H} g(x)d\gamma(x) = \langle (I - S)^{-1}Q^{-1/2}(m_2 - m_1), Q^{-1/2}(m_3 - m_1) \rangle.
\]

**Proof** Let \( a = Q^{-1/2}(m_2 - m_1), b = Q^{-1/2}(m_3 - m_1) \). By Lemma 1,

\[
\int_{H} W_{\phi_k}(x)d\gamma(x) = \langle Q^{-1/2}(m_3 - m_1), \phi_k \rangle = \langle b, \phi_k \rangle, \ \forall k \in \mathbb{N}.
\]

Using the expression for \( \langle W_{\phi_j}, W_{\phi_k} \rangle_{L^2(H, \nu)} \) from Lemma 1, we have

\[
\|g\|_{L^2(H, \gamma)}^2 = \left\| \sum_{k=1}^{\infty} \frac{a_k}{1-\alpha_k} W_{\phi_k} \right\|_{L^2(H, \gamma)}^2 = \sum_{k,j=1}^{\infty} \frac{\langle a, \phi_k \rangle \langle a, \phi_j \rangle}{(1-\alpha_k)(1-\alpha_j)} \langle W_{\phi_k}, W_{\phi_j} \rangle_{L^2(H, \gamma)}
\]

\[
= \sum_{k,j=1}^{\infty} \frac{\langle a, \phi_k \rangle \langle a, \phi_j \rangle}{(1-\alpha_k)(1-\alpha_j)} \langle A\phi_k, \phi_j \rangle + \sum_{k,j=1}^{\infty} \frac{\langle a, \phi_k \rangle \langle a, \phi_j \rangle}{(1-\alpha_k)(1-\alpha_j)} \langle b, \phi_k \rangle \langle b, \phi_j \rangle
\]

\[
= \sum_{k=1}^{\infty} \frac{\langle a, \phi_k \rangle}{1-\alpha_k} ((I - S)^{-1}a, A\phi_k) + \left( \sum_{k=1}^{\infty} \frac{\langle a, \phi_k \rangle \langle b, \phi_k \rangle}{1-\alpha_k} \right)^2
\]

\[
= \langle ((I - S)^{-1}a, A(I - S)^{-1}a) + ((I - S)^{-1}a, b) \rangle^2
\]

\[
= \|A^{1/2}(I - S)^{-1}a\|^2 + ((I - S)^{-1}a, b)^2 < \infty.
\]
Thus \( g \in \mathcal{L}^2(\mathcal{H}, \gamma) \). Furthermore, for any \( N \in \mathbb{N} \) and \( T_N = \sum_{k=1}^{N} \phi_k \otimes \phi_k \),

\[
\sum_{k,j=N+1}^{\infty} \frac{\langle a, \phi_k \rangle \langle a, \phi_j \rangle}{(1 - \alpha_k)(1 - \alpha_j)} (A\phi_k, \phi_j) = \sum_{k=N+1}^{\infty} \frac{\langle a, \phi_k \rangle}{1 - \alpha_k} (I - T_N)(I - S)^{-1} a, A\phi_k
\]

\[
= \langle (I - T_N)(I - S)^{-1} a, A(I - T_N)(I - S)^{-1} a \rangle = \|A^{1/2}(I - T_N)(I - S)^{-1} a\|^2,
\]

\[
\left( \sum_{k=N+1}^{\infty} \frac{\langle a, \phi_k \rangle \langle b, \phi_k \rangle}{1 - \alpha_k} \right)^2 = \langle (I - T_N)(I - S)^{-1} a, b \rangle^2.
\]

Let \( g_N = \sum_{k=1}^{N} \frac{1}{1 - \alpha_k} \langle Q^{-1/2}(m_2 - m_1), \phi_k \rangle W_\phi_k \), then

\[
\|g_N - g\|_{\mathcal{L}^2(\mathcal{H}, \gamma)}^2 = \|A^{1/2}(I - T_N)(I - S)^{-1} a\|^2 + \langle (I - T_N)(I - S)^{-1} a, b \rangle^2
\]

\[
\leq \|A\| + \|b\|^2 \| (I - T_N)(I - S)^{-1} a \|^2 \to 0 \text{ as } N \to \infty.
\]

Since \( \gamma \) is a probability measure, by Hölder Inequality, we have \( g \in \mathcal{L}^1(\mathcal{H}, \gamma) \) and \( \lim_{N \to \infty} \|g_N - g\|_{\mathcal{L}^1(\mathcal{H}, \gamma)} = 0 \). It follows that

\[
\int_\mathcal{H} g(x) d\gamma(x) = \lim_{N \to \infty} \int_\mathcal{H} g_N(x) d\gamma(x) = \sum_{k=1}^{\infty} \frac{\langle a, \phi_k \rangle \langle b, \phi_k \rangle}{1 - \alpha_k} = \langle (I - S)^{-1} a, b \rangle.
\]

Letting \( a = Q^{-1/2}(m_2 - m_1), b = Q^{-1/2}(m_2 - m_1) \) gives the final answer. \( \Box \)

We are now ready to prove Theorem 2.

**Proof (of Theorem 2)** Since \( \nu \sim \mu \) and \( \gamma \ll \mu \), the Radon-Nikodym derivatives \( \frac{d\nu}{d\mu}, \frac{d\gamma}{d\nu} \) are both well-defined. By the chain rule,

\[
\text{KL}(\gamma||\mu) = \int_\mathcal{H} \log \left( \frac{d\gamma}{d\mu} \right) d\gamma = \int_\mathcal{H} \log \left( \frac{d\gamma}{d\nu} \right) d\nu + \int_\mathcal{H} \log \left( \frac{d\nu}{d\mu} \right) d\nu
\]

\[
= \text{KL}(\gamma||\nu) + \int_\mathcal{H} \log \left( \frac{d\nu}{d\mu} \right) d\gamma.
\]

Let us evaluate the second term. Let \( \{\alpha_k\}_{k=1}^{\infty} \) be the eigenvalues of \( S \), with corresponding orthonormal eigenvectors \( \{\phi_k\} \), which forms an orthonormal basis in \( \mathcal{H} \). By the assumption \( S \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \), we have \( \{\alpha_k\}_{k \in \mathbb{N}} \in \ell^2 \), \( \alpha_k \in \mathbb{R} \ \forall k \in \mathbb{N} \) and the following quantity is finite

\[
\log \det_2 (I - S) = \log \det[(I - S) \exp(S)] = \sum_{k=1}^{\infty} [\alpha_k + \log(1 - \alpha_k)]. \quad (95)
\]

By Theorem 20,

\[
\log \left( \frac{d\nu}{d\mu} (x) \right) = -\frac{1}{2} \|(I - S)^{-1/2}Q^{-1/2}(m_2 - m_1)\|^2 - \frac{1}{2} \sum_{k=1}^{\infty} \phi_k (x), \quad (96)
\]
where for each $k \in \mathbb{N}$,
\[
\phi_k = \frac{\alpha_k}{1-\alpha_k} W_{\phi_k}^2 + \frac{2}{1-\alpha_k} (Q^{-1/2}(m_2-m_1), \phi_k) W_{\phi_k} + \log(1-\alpha_k).
\]
By Lemma 1,
\[
\int_{\mathcal{H}} W_{\phi_k}^2(x) d\gamma(x) = \langle A\phi_k, \phi_k \rangle + \|Q^{-1/2}(m_3-m_1), \phi_k \|^2.
\]

(i) $S \in \text{Tr}(\mathcal{H})$. Since $I - S > 0$, we have $\log(I - S) \in \text{Tr}(\mathcal{H})$ and $\sum_{k=1}^{\infty} \log(1-\alpha_k) = \log \det(I - S)$ is finite. By Tonelli Theorem,
\[
\int_{\mathcal{H}} \sum_{k=1}^{\infty} \frac{\alpha_k}{1-\alpha_k} W_{\phi_k}^2 d\gamma = \int_{\mathcal{H}} \sum_{k=1}^{\infty} \frac{\alpha_k}{1-\alpha_k} W_{\phi_k}^2 d\gamma = \sum_{k=1}^{\infty} \frac{\alpha_k}{1-\alpha_k} \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \frac{\alpha_k}{1-\alpha_k} [\langle A\phi_k, \phi_k \rangle + \|Q^{-1/2}(m_3-m_1), \phi_k \|^2]
\]
\[= \text{Tr}[S(I-S)^{-1}A] + \langle S(I-S)^{-1}Q^{-1/2}(m_3-m_1), Q^{-1/2}(m_3-m_1) \rangle.
\]

Combining this with Eq.(96) and Lemma 2, we obtain
\[
\int_{\mathcal{H}} \log \left( \frac{d\nu}{d\mu}(x) \right) d\gamma(x) = -\frac{1}{2} \| (I - S)^{-1/2} Q^{-1/2}(m_2-m_1) \|^2
\]
\[\quad - \frac{1}{2} \text{Tr}[S(I-S)^{-1}A] + \langle S(I-S)^{-1}Q^{-1/2}(m_3-m_1), Q^{-1/2}(m_3-m_1) \rangle
\]
\[\quad - \frac{1}{2} \log \det(I - S) + \langle (I-S)^{-1}Q^{-1/2}(m_2-m_1), Q^{-1/2}(m_3-m_1) \rangle.
\] (97)

(ii) $S \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ and $I - A \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$. In this case we have
\[
\frac{\alpha_k}{1-\alpha_k} W_{\phi_k}^2 + \log(1-\alpha_k) = \frac{\alpha_k}{1-\alpha_k} [W_{\phi_k}^2 - 1] + \left[ \frac{\alpha_k}{1-\alpha_k} + \log(1-\alpha_k) \right].
\]
The second term gives the series of constants
\[
\sum_{k=1}^{\infty} \left[ \frac{\alpha_k}{1-\alpha_k} + \log(1-\alpha_k) \right] = \sum_{k=1}^{\infty} \frac{\alpha_k}{1-\alpha_k} + \sum_{k=1}^{\infty} [\alpha_k + \log(1-\alpha_k)]
\]
\[= \text{Tr}[S^2(I-S)^{-1}] + \log \det_2(I - S).
\] (98)
The functions \( \{\psi_k = \frac{1}{\sqrt{2}} W_{\phi_k}^2 \} \) form an orthonormal sequence in \( L^2(\mathcal{H}, \mu) \) ([23], Proposition 1.2.6). Furthermore, \( \psi_k \frac{d\gamma}{d\mu} \in L^1(\mathcal{H}, \mu) \) $\forall k \in \mathbb{N}$, with
\[
b_k = \int_{\mathcal{H}} \left( \psi_k \frac{d\gamma}{d\mu}(x) \right) d\mu(x) = \int_{\mathcal{H}} \psi_k(x) d\gamma(x)
\]
\[= \frac{1}{\sqrt{2}} [\langle A\phi_k, \phi_k \rangle + \|Q^{-1/2}(m_3-m_1), \phi_k \|^2 - 1]
\]
\[= \frac{1}{\sqrt{2}} [\langle (I-A)\phi_k, \phi_k \rangle + \|Q^{-1/2}(m_3-m_1), \phi_k \|^2].
\]
The sequence \((b_k)_{k \in \mathbb{N}}\) satisfies
\[
\sum_{k=1}^{\infty} b_k^2 = \frac{1}{2} \sum_{k=1}^{\infty} \left[ - \langle (I - A) \phi_k, \phi_k \rangle + \| (Q^{-1/2}(m_3 - m_1), \phi_k) \|^2 \right]^2
\leq \sum_{k=1}^{\infty} \| (I - A) \phi_k, \phi_k \|^2 + \sum_{k=1}^{\infty} \| (Q^{-1/2}(m_3 - m_1), \phi_k) \|^4
\leq \sum_{k=1}^{\infty} \| (I - A) \phi_k \|^2 + \sum_{k=1}^{\infty} \| (Q^{-1/2}(m_3 - m_1), \phi_k) \|^2 \|^2
= \| (I - A) \|_{HS}^2 + \| (Q^{-1/2}(m_3 - m_1)) \|^4 < \infty.
\]
Since \((\alpha_k)_{k \in \mathbb{N}} \in \ell^2\), we apply Proposition 1 to \((\psi_k)_{k \in \mathbb{N}}\) and \(f = \frac{d\gamma}{d\mu}\) to obtain
\[
\int_{\mathcal{H}} \sum_{k=1}^{\infty} \frac{\alpha_k}{1 - \alpha_k} (W^2 - 1) d\gamma(x) = \int_{\mathcal{H}} \sum_{k=1}^{\infty} \frac{\sqrt{2} \alpha_k}{1 - \alpha_k} \sqrt{2} (W^2 - 1) d\gamma(x) d\mu(x)
= \sum_{k=1}^{\infty} \frac{\sqrt{2} \alpha_k b_k}{1 - \alpha_k} = \sum_{k=1}^{\infty} \frac{\alpha_k}{1 - \alpha_k} \left[ - \langle (I - A) \phi_k, \phi_k \rangle + \| (Q^{-1/2}(m_3 - m_1), \phi_k) \|^2 \right]
= -\text{Tr} [S(I - S)^{-1}(I - A)] + \langle S(I - S)^{-1}Q^{-1/2}(m_3 - m_1), Q^{-1/2}(m_3 - m_1) \rangle.
\]
Combining this with Eqs. (96), (98), and Lemma 2, we obtain
\[
\int_{\mathcal{H}} \log \left( \frac{d\gamma(x)}{d\mu(x)} \right) d\gamma(x) = -\frac{1}{2} \| (I - S)^{-1/2}Q^{-1/2}(m_2 - m_1) \|^2
- \frac{1}{2} \langle S(I - S)^{-1}Q^{-1/2}(m_3 - m_1), Q^{-1/2}(m_3 - m_1) \rangle
+ \frac{1}{2} \text{Tr} [S(I - S)^{-1}A] - \frac{1}{2} \log \det_2 (I - S)
+ \langle (I - S)^{-1}Q^{-1/2}(m_2 - m_1), Q^{-1/2}(m_3 - m_1) \rangle.
\]
(99)
For \(S \in \text{Tr}(\mathcal{H})\), we have \(\det_2 (I - S) = \text{Tr}(S) + \log \det (I - S)\), so that (99) reduces to (97). For \(m_3 = m_2\) and \(A = I - S\), (99) simplifies to
\[
\int_{\mathcal{H}} \log \left( \frac{d\gamma(x)}{d\mu(x)} \right) d\gamma(x) = \frac{1}{2} \| Q^{-1/2}(m_2 - m_1) \|^2 - \frac{1}{2} \log \det_2 (I - S)
= \text{KL}(\gamma \| \mu).
\]
Thus in the case we have \(\text{KL}(\gamma \| \mu) = \text{KL}(\gamma \| \nu) + \text{KL}(\nu \| \mu)\) and hence \(\text{KL}(\gamma \| \mu) \geq \text{KL}(\nu \| \mu)\), with equality if and only \(\gamma = \nu\).

### 7 Mutual information of Gaussian measures on Hilbert space

In this section, we prove Theorem 1. For completeness, we give a new, shorter proof of the mutual information between two Gaussian measures [7]. We start by reviewing joint measures and cross-covariance operators on Hilbert space.
**Joint measures.** Following [6], let \((\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)\) and \((\mathcal{H}_2, \langle \cdot, \cdot \rangle_2)\) be two real separable Hilbert spaces with corresponding Borel \(\sigma\)-algebras \(\mathcal{B}(\mathcal{H}_1)\) and \(\mathcal{B}(\mathcal{H}_2)\). Let \(\mathcal{H}_1 \times \mathcal{H}_2\) be the Hilbert space with inner product defined by \(\langle (u_1, u_2), (v_1, v_2) \rangle_{12} = \langle u_1, v_1 \rangle_1 + \langle u_2, v_2 \rangle_2\) and Hilbert norm \(\| (u, v) \|_2^2 = \| u \|_1^2 + \| v \|_2^2\). A **joint measure** \(\mu_{XY}\) is a probability measure defined on \((\mathcal{H}_1 \times \mathcal{H}_2, \mathcal{B}(\mathcal{H}_1) \times \mathcal{B}(\mathcal{H}_2))\). Let \(\mu_X\) and \(\mu_Y\) be its marginal probability measures defined on \((\mathcal{H}_1, \mathcal{B}(\mathcal{H}_1))\) and \((\mathcal{H}_2, \mathcal{B}(\mathcal{H}_2))\), respectively. Assume further that \(\mu_{XY} \in \mathcal{P}_2(\mathcal{H}_1 \times \mathcal{H}_2)\). Under this assumption, the mean vector \(m_{XY} \in \mathcal{H}_1 \times \mathcal{H}_2\) and covariance operator \(\Gamma_{XY} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \times \mathcal{H}_2\) are well-defined. Furthermore, since

\[
\int_{\mathcal{H}_1 \times \mathcal{H}_2} \| (u, v) \|_2^2 \, d\mu_{XY}(u, v) = \int_{\mathcal{H}_1} \| u \|_1^2 \, d\mu_X(u) + \int_{\mathcal{H}_2} \| v \|_2^2 \, d\mu_Y(v),
\]

it follows that \(\mu_{XY} \in \mathcal{P}_2(\mathcal{H}_1 \times \mathcal{H}_2)\) if and only \(\mu_X \in \mathcal{P}_2(\mathcal{H}_1)\) and \(\mu_Y \in \mathcal{P}_2(\mathcal{H}_2)\). Subsequently, throughout the paper, we assume that \(\mu_{XY} \in \mathcal{P}_2(\mathcal{H}_1 \times \mathcal{H}_2)\) for all joint measures \(\mu_{XY}\) on \((\mathcal{H}_1 \times \mathcal{H}_2, \mathcal{B}(\mathcal{H}_1) \times \mathcal{B}(\mathcal{H}_2))\).

**Cross-covariance operators.** Let \((m_X, C_X)\) and \((m_Y, C_Y)\) be the means and covariance operators of \(\mu_X\) and \(\mu_Y\), respectively. Since \(\mu_{XY} \in \mathcal{P}_2(\mathcal{H}_1 \times \mathcal{H}_2)\), the following linear functional \(G : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{R}\) is well-defined and bounded,

\[
G(u, v) = \int_{\mathcal{H}_1 \times \mathcal{H}_2} \langle x - m_X, u \rangle_1 \langle y - m_Y, v \rangle_2 \, d\mu_{XY}(x, y).
\]

By the Riesz Representation Theorem, there exist single-valued, bounded, linear maps \(C_{XY} : \mathcal{H}_2 \to \mathcal{H}_1, C_{YX} : \mathcal{H}_1 \to \mathcal{H}_2\) such that

\[
\langle u, C_{XY} v \rangle_1 = G(u, v), \quad \langle C_{YX} u, v \rangle_2 = G(u, v), \quad u \in \mathcal{H}_1, v \in \mathcal{H}_2.
\]

Clearly, \(C_{XY} = C_{YX}^*\). In operator tensor product notation,

\[
C_{XY} = \int_{\mathcal{H}_1 \times \mathcal{H}_2} (x - m_X) \otimes (y - m_Y) \, d\mu_{XY}(x, y).
\]

\(C_{XY}\) is called the **cross-covariance operator** of \(\mu_{XY}\). It is closely related to the covariance operators of the marginals \(\mu_X\) and \(\mu_Y\) via the following.

**Theorem 21 ([6])** Consider the projection operators \(P_X : \mathcal{H}_1 \to \text{Im}(C_X), P_Y : \mathcal{H}_2 \to \text{Im}(C_Y)\). Then \(C_{XY}\) admits the following representation

\[
C_{XY} = C_X^{1/2} V C_Y^{1/2}
\]

where \(V : \mathcal{H}_2 \to \mathcal{H}_1\) is a unique bounded linear operator such that \(\| V \| \leq 1\) and \(V = P_X V P_Y\).

In column vector notation, the mean vector of the joint measure \(\mu_{XY}\) is given by \(m_{XY} = \begin{pmatrix} m_X \\ m_Y \end{pmatrix} \in \mathcal{H}_1 \times \mathcal{H}_2\). In operator-valued matrix notation, the covariance operator \(\Gamma_{XY} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \times \mathcal{H}_2\) of \(\mu_{XY}\) is given by

\[
\Gamma_{XY} = \begin{pmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{pmatrix}, \Gamma_{XY} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} C_X u + C_{XY} v \\ C_{YX} u + C_Y v \end{pmatrix}, u \in \mathcal{H}_1, v \in \mathcal{H}_2.
\]
Mutual information of Gaussian measures. We now consider joint Gaussian measures. The covariance operator for $\mu_X \otimes \mu_Y$ is $\Gamma_0 = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$. Let $V \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ be as in Theorem 21. The covariance operator for $\mu_{XY}$ is

$$
\Gamma = \begin{pmatrix} C_1 & C_1^{1/2} V C_2^{1/2} \\ C_1^{1/2} V C_2^{1/2} & C_2 \end{pmatrix} = \Gamma_0^{1/2} \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} \Gamma_0^{1/2}.
$$

(106)

**Lemma 3** Let $V \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. Then the operator $\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \times \mathcal{H}_2$ is Hilbert-Schmidt on $\mathcal{H}_1 \times \mathcal{H}_2$ if and only if $V \in \text{HS}(\mathcal{H}_2, \mathcal{H}_1)$.

**Proof** Let $\{e_j^i\}_{j \in \mathbb{N}}$ be an orthonormal basis in $\mathcal{H}_i$, $i = 1, 2$. Then $\{\begin{pmatrix} e_j^1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_j^2 \end{pmatrix}\}_{j \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_1 \times \mathcal{H}_2$ and

$$
\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \begin{pmatrix} e_j^1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ V^* e_j^1 \end{pmatrix}, \quad \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e_j^2 \end{pmatrix} = \begin{pmatrix} V e_j^2 \\ 0 \end{pmatrix}.
$$

Thus $\left\| \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \right\|_{\text{HS}}^2 = \sum_{j=1}^{\infty} \left( \|V^* e_j^1\|_2^2 + \|V e_j^2\|_2^2 \right) = \|V^*\|_2^2 + \|V\|_2^2 = \text{Tr}(VV^*) + \text{Tr}(V^*V)$, which is finite if and only if both $\text{Tr}(VV^*)$ and $\text{Tr}(V^*V)$ are finite, in which case they are both equal to $\|V\|_2^2$. \hfill \Box

**Lemma 4** Let $V \in \text{HS}(\mathcal{H}_2, \mathcal{H}_1)$. Then $\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$ has nonzero eigenvalues $\{\pm \sqrt{\gamma_j}\}_{j \in \mathbb{N}}$, with $\{\gamma_k\}_{k \in \mathbb{N}}$ being the nonzero eigenvalues of $VV^* : \mathcal{H}_1 \to \mathcal{H}_1$. Moreover,

$$
\left\| \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \right\| < 1 \iff \|V\| < 1,
$$

in which case the following quantity is finite

$$
\log \det_2 \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} = \log \det(I - V^*V).
$$

(107)

Furthermore, $\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \in \text{Tr}(\mathcal{H}_1 \times \mathcal{H}_2) \iff (VV^*)^{1/2} \in \text{Tr}(\mathcal{H}_1)$. In this case

$$
\det_2 \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} = \det \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} = \det(I - V^*V).
$$

**Proof** By Lemma 3, the operator $\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$ is self-adjoint and Hilbert-Schmidt on $\mathcal{H}_1 \times \mathcal{H}_2$, thus admits a countable sequence of eigenvalues, with corresponding eigenvectors forming an orthonormal basis for $\mathcal{H}_1 \times \mathcal{H}_2$. Consider the square

$$
\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}^2 = \begin{pmatrix} VV^* & 0 \\ 0 & V^*V \end{pmatrix}.
$$

The nonzero eigenvalues of the last operator are precisely the nonzero eigenvalues of $VV^* : \mathcal{H}_1 \to \mathcal{H}_1$ and $V^*V : \mathcal{H}_2 \to \mathcal{H}_2$, which are identical. Consider the following eigenvalue equation, where $\lambda \neq 0$,

$$
\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} Vb \\ V^*a \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{pmatrix} Vb = \lambda a \\ V^*a = \lambda b \end{pmatrix}, \quad a \in \mathcal{H}_1, b \in \mathcal{H}_2.$$
Combining these two equations, we obtain $VV^*a = \lambda^2 a$ and $V^*Vb = \lambda^2 b$. Thus $a$ and $b$ are necessarily eigenvectors of $VV^*$ and $V^*V$, respectively, under the same eigenvalue $\lambda^2$. Due to the square factor, it follows that $-\lambda$ is an eigenvalue of \( \begin{pmatrix} V^* & V \\ \end{pmatrix} \) corresponding to eigenvector \( \begin{pmatrix} a \\ -b \\ \end{pmatrix} \). Let \( \{\gamma_k\}_{k=1}^\infty \) be the nonzero eigenvalues of $VV^*$, then the nonzero eigenvalues of \( \begin{pmatrix} 0 & V \\ V^* & 0 \\ \end{pmatrix} \) are \( \{\pm \sqrt{\gamma_k}\}_{k=1}^\infty \).

Thus \( \left\| \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \right\| < 1 \iff \|VV^*\| = \|V\|^2 < 1 \iff \|V\| < 1. \)

Let \( \{e_j^\pm\}_{j=1}^\infty \) denote the corresponding orthonormal eigenvectors. Then
\[
\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} = \sum_{j=1}^\infty [\sqrt{\gamma} e_j^+ \otimes e_j^- - \sqrt{\gamma} e_j^- \otimes e_j^+].
\]

By definition of the Hilbert-Carleman determinant,
\[
\det_2 \left( \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} \right) = \det \left[ \begin{pmatrix} I & V \\ V^* & 0 \end{pmatrix} \right] \exp \left( -\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \right) \]
\[
= \prod_{j=1}^\infty (1 + \sqrt{\gamma_j})(1 - \sqrt{\gamma_j}) e^{\sqrt{\gamma_j} e^{-\sqrt{\gamma_j}}} = \prod_{j=1}^\infty (1 - \gamma_j) = \det(I - V^*V),
\]
giving the log det formula. Since \( \left( \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \right) = \begin{pmatrix} (VV^*)^{1/2} & 0 \\ 0 & (V^*V)^{1/2} \end{pmatrix} \), \( \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \) belongs to \( \text{Tr}(\mathcal{H}_1 \times \mathcal{H}_2) \iff (VV^*)^{1/2} \in \text{Tr}(\mathcal{H}_1) \iff (V^*V)^{1/2} \in \text{Tr}(\mathcal{H}_2). \)

In this case, \( \det(I - V^*V) = \prod_{j=1}^\infty (1 - \gamma_j) = \det_2 \left( \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} \right) = \det \left( \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} \right). \)

By applying Formula (78) in Theorem 19 and Lemma 4 to our setting, we obtain the expression for $\text{KL}(\mu_X | | \mu_Y)$, first proved in [7] (Proposition 2) by a direct approach. The case $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ was proved in [5]. As we now show, by employing the more general result on the KL divergence between Gaussian measures in Theorem 19, this result is obtained immediately.

**Theorem 22** Let $\mathcal{H}_1, \mathcal{H}_2$ be two separable Hilbert spaces. Let $\mu_X = \mathcal{N}(\mu_X, \Sigma_X) \in \text{Gauss}(\mathcal{H}_1)$, $\mu_Y = \mathcal{N}(\mu_Y, \Sigma_Y) \in \text{Gauss}(\mathcal{H}_2)$, ker($\Sigma_X$) = \{0\}, ker($\Sigma_Y$) = \{0\}. Assume that $\mu_{XY} \in \text{Gauss}(\mu_X, \mu_Y)$. Then $\mu_{XY} \sim \mu_X \otimes \mu_Y \iff \|V\| < 1, V \in \text{HS}(\mathcal{H}_2, \mathcal{H}_1)$, where $\Sigma_{XY} = \Sigma_X^1/2 \Sigma_Y^1/2$. Furthermore,
\[
\text{KL}(\mu_{XY} | | \mu_X \otimes \mu_Y) = -\frac{1}{2} \log \det(I - V^*V) \text{ if } \mu_X \sim \mu_X \otimes \mu_Y,
\]
\[
\infty \text{ if } \mu_{XY} \perp \mu_X \otimes \mu_Y.
\]

**Proof (of Theorem 22)** The condition for $\mu_{XY} \sim \mu_X \otimes \mu_Y$ follows from Lemmas 3, 4 and Theorem 18. Assume that $\mu_{XY} \sim \mu_X \otimes \mu_Y$. Since the mean
vector for $\mu_{XY}$ and $\mu_X \otimes \mu_Y$ is the same, namely $\left( \frac{m_X}{m_Y} \right)$, the first term in Eq.(78) is equal to zero. From Eqs.(106) and (78), we have

$$\text{KL}(\mu_{XY} \| \mu_X \otimes \mu_Y) = -\frac{1}{2} \log \det_2 \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} = -\frac{1}{2} \log \det(I - V^*V),$$

where the last equality follows from Lemma 4.

We are now ready to prove Theorem 1, which we restate here for clarity.

**Theorem 23 (Minimum Mutual Information of Joint Gaussian Measures)** Let $H_1, H_2$ be two separable Hilbert spaces. Let $\mu_X = \mathcal{N}(m_X, C_X) \in \text{Gauss}(H_1)$, $\mu_Y = \mathcal{N}(m_Y, C_Y) \in \text{Gauss}(H_2)$, $\ker(C_X) = \ker(C_Y) = \{0\}$. Let $\gamma \in \text{Joint}(\mu_X, \mu_Y), \gamma_0 \in \text{Gauss}(\mu_X, \mu_Y)$, $\gamma_0 \sim \mu_X \otimes \mu_Y$. Assume that $\gamma$ and $\gamma_0$ have the same covariance operator $\Gamma$ and that $\mu_X \otimes \mu_Y$ has covariance operator $\Gamma_0$. Then

$$\text{KL}(\gamma \| \mu_X \otimes \mu_Y) \geq \text{KL}(\gamma_0 \| \mu_X \otimes \mu_Y) = -\frac{1}{2} \log \det(I - V^*V). \tag{109}$$

Equality happens if and only if $\gamma = \gamma_0$. Here $V$ is the unique bounded linear operator satisfying $V \in \text{HS}(H_2, H_1), ||V|| < 1$, such that $\Gamma = \Gamma_0^{1/2} \left( \begin{pmatrix} I & V \\ V^* & I \end{pmatrix} \right) \Gamma_0^{1/2}$.

**Proof (of Theorem 23)** We can assume that $\gamma \ll \mu_X \otimes \mu_Y$, since otherwise $\text{KL}(\gamma \| \mu_X \otimes \mu_Y) = \infty$ and the inequality is obviously true. Since $\gamma_0 \sim \mu_X \otimes \mu_Y$, we also have $\gamma \ll \gamma_0$ and the Radon-Nikodym derivatives $\frac{d\gamma}{d\mu_X \otimes \mu_Y}$ and $\frac{d\gamma_0}{d\mu_X \otimes \mu_Y}$ are both well-defined. Since $\gamma, \gamma_0$, and $\mu_X \otimes \mu_Y$ all have the same mean, namely $\left( \frac{m_X}{m_Y} \right)$, and $\gamma, \gamma_0$ have the same covariance operators, we have by Theorem 2,

$$\text{KL}(\gamma \| \mu_X \otimes \mu_Y) = \text{KL}(\gamma \| \gamma_0) + \text{KL}(\gamma_0 \| \mu_X \otimes \mu_Y)$$

$$\geq \text{KL}(\gamma_0 \| \mu_X \otimes \mu_Y) = -\frac{1}{2} \log \det(I - V^*V) \text{ by Theorem 22}.$$ 

Thus $\text{KL}(\gamma \| \mu_X \otimes \mu_Y) \geq \text{KL}(\gamma_0 \| \mu_X \otimes \mu_Y)$, with equality if and only if $\gamma = \gamma_0$, that is if and only if $\gamma$ is Gaussian. \qed

8 Entropic regularized 2-Wasserstein distance between Gaussian measures on Hilbert space

Consider the entropic OT problem (4), which we restate here

$$\text{OT}_e^c(\mu_0, \mu_1) = \min_{\gamma \in \text{Joint}(\mu_0, \mu_1)} \left\{ \mathbb{E}_{\gamma} c(x,y) + c\text{KL}(\gamma \| \mu_0 \otimes \mu_1) \right\}. \tag{110}$$
The nonsingular case. We first solve (110) for \( c(x, y) = \|x - y\|^2 \) under the nonsingular Gaussian setting. Let \( \mu_X, \mu_Y \in \mathcal{P}_2(\mathcal{H}) \), with means \( m_X, m_Y \) and covariance operators \( C_X, C_Y \), respectively, then \( C_X, C_Y \in \text{Tr}(\mathcal{H}) \) and

\[
E_{\mu_X} \|x - m_X\|^2 = \text{Tr}(C_X), \quad E_{\mu_Y} \|y - m_Y\|^2 = \text{Tr}(C_Y). \tag{111}
\]

**Lemma 5** Let \( \mu_X, \mu_Y \in \mathcal{P}_2(\mathcal{H}), \mu_{XY} \in \text{Joint}(\mu_X, \mu_Y) \). Then \( C_{XY} \in \text{Tr}(\mathcal{H}) \) and for any \( A, B \in \mathcal{L}(\mathcal{H}) \),

\[
E_{\mu_{XY}}(A(x - m_X), B(y - m_Y)) = \text{Tr}(AC_{XY}B^*). \tag{112}
\]

**Proof** By Theorem 21, \( C_{XY} = C_X^{1/2}VC_Y^{1/2} \). Since \( C_X \) and \( C_Y \) are both trace class, \( C_X^{1/2} \) and \( C_Y^{1/2} \) are both Hilbert-Schmidt. Thus it follows that \( C_{XY} \) is trace class. Recall that \( C_{XY} : \mathcal{H} \rightarrow \mathcal{H} \) is given by

\[
C_{XY} = \int_{\mathcal{H} \times \mathcal{H}} (x - m_X) \otimes (y - m_Y) d\mu_{XY}(x, y).
\]

It suffices to consider \( m_X = m_Y = 0 \). Let \( \{e_j\}_{j=1}^{\infty} \) be an orthonormal basis on \( \mathcal{H} \), then for each \( j \in \mathbb{N} \),

\[
\int_{\mathcal{H} \times \mathcal{H}} \langle Ax, e_j \rangle \langle By, e_j \rangle d\mu_{XY}(x, y) = \int_{\mathcal{H} \times \mathcal{H}} \langle x, A^*e_j \rangle \langle y, B^*e_j \rangle d\mu_{XY}(x, y) = \langle A^*e_j, C_{XY}B^*e_j \rangle.
\]

For each \( N \in \mathbb{N} \),

\[
\sum_{j=1}^{N} \left\| \langle Ax, e_j \rangle \langle By, e_j \rangle \right\| \leq \sum_{j=1}^{N} \left[ \sum_{j=1}^{N} \left\| \langle Ax, e_j \rangle \right\|^2 \right]^{1/2} \left[ \sum_{j=1}^{N} \left\| \langle By, e_j \rangle \right\|^2 \right]^{1/2} \leq \|Ax\| \|By\| \left[ \sum_{j=1}^{N} \left\| \langle Ax, e_j \rangle \right\|^2 \right]^{1/2} \left[ \sum_{j=1}^{N} \left\| \langle By, e_j \rangle \right\|^2 \right]^{1/2}.
\]

Thus

\[
\int_{\mathcal{H} \times \mathcal{H}} \|x\|^2 + \|y\|^2 d\mu_{XY}(x, y) = \int_{\mathcal{H}} \|x\|^2 d\mu_X(x) + \int_{\mathcal{H}} \|y\|^2 d\mu_Y(y) = \text{Tr}(C_X) + \text{Tr}(C_Y) < \infty.
\]

By Lebesgue Dominated Convergence Theorem,

\[
E_{\mu_{XY}}(Ax, By) = \int_{\mathcal{H} \times \mathcal{H}} \langle Ax, By \rangle d\mu_{XY}(x, y) = \int_{\mathcal{H} \times \mathcal{H}} \sum_{j=1}^{\infty} \langle Ax, e_j \rangle \langle By, e_j \rangle d\mu_{XY}(x, y) = \sum_{j=1}^{\infty} \int_{\mathcal{H} \times \mathcal{H}} \langle x, A^*e_j \rangle \langle y, B^*e_j \rangle d\mu_{XY}(x, y) = \sum_{j=1}^{\infty} \langle A^*e_j, C_{XY}B^*e_j \rangle = \text{Tr}(AC_{XY}B^*). \tag*{Q.E.D.}
\]

**Corollary 2** Let \( \mu_X, \mu_Y \in \mathcal{P}_2(\mathcal{H}), \mu_{XY} \in \text{Joint}(\mu_X, \mu_Y) \). Then

\[
E_{\mu_{XY}} \|x - y\|^2 = \|m_X - m_Y\|^2 + \text{Tr}(C_X) + \text{Tr}(C_Y) - 2\text{Tr}(C_{XY}) \quad \tag{113}
\]

\[
= \|m_X - m_Y\|^2 + \text{Tr}(C_X) + \text{Tr}(C_Y) - 2\text{Tr}(C_X^{1/2}VC_Y^{1/2}).
\]
Proposition 2 Let \( \mu_0 = \mathcal{N}(m_0, C_0), \mu_1 = \mathcal{N}(m_1, C_1) \), with \( \ker(C_0) = \ker(C_1) = \{0\} \). If \( \gamma^\epsilon \) is the minimizer of problem (13), then necessarily \( \gamma^\epsilon \in \text{Gauss}(\mu_0, \mu_1) \) and \( \gamma^\epsilon \sim \mu_0 \otimes \mu_1 \).

Proof By Corollary 2, for any \( \gamma \in \text{Joint}(\mu_0, \mu_1) \),

\[
I_\epsilon(\gamma) = \mathbb{E}_\gamma ||x - y||^2 + \epsilon \text{KL}(\gamma||\mu_0 \otimes \mu_1)
= ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - 2\text{Tr}(C_0^{1/2}VC_1^{1/2}) + \epsilon \text{KL}(\gamma||\mu_0 \otimes \mu_1),
\]

with the first terms depending solely on the means and covariance operators. First, we must have \( \gamma << \mu_0 \otimes \mu_1 \), since otherwise \( \text{KL}(\gamma||\mu_0 \otimes \mu_1) = \infty \) and such a \( \gamma \) cannot be a minimizer. By Theorem 1, if \( \gamma << \mu_0 \otimes \mu_1 \) and \( \gamma_0 \in \text{Gauss}(\mu_0, \mu_1) \), \( \gamma_0 \sim \mu_0 \otimes \mu_1 \), has the same covariance operator as \( \gamma \), then

\[
\text{KL}(\gamma||\mu_0 \otimes \mu_1) \geq \text{KL}(\gamma_0||\mu_0 \otimes \mu_1),
\]

with equality if and only if \( \gamma = \gamma_0 \). Thus we must have \( \gamma^\epsilon \in \text{Gauss}(\mu_0, \mu_1) \) and \( \gamma^\epsilon \sim \mu_0 \otimes \mu_1 \). \( \square \)

The following technical lemmas are used extensively throughout the paper.

Lemma 6 Let \( A : \mathcal{H} \rightarrow \mathcal{H} \) be a compact operator. Then

\[
A(I + A^*A)^{1/2} = (I + AA^*)^{1/2}A. \tag{114}
\]

Proof This is a special case of Lemma 10 and Corollary 2 in [61].

The following result is then immediate.

Lemma 7 Let \( A : \mathcal{H} \rightarrow \mathcal{H} \) be a compact operator. Then

\[
[I + (I + AA^*)^{1/2}]A = A[I + (I + A^*A)^{1/2}], \tag{115}
\]

\[
A[I + (I + A^*A)^{1/2}]^{-1} = [I + (I + AA^*)^{1/2}]^{-1}A. \tag{116}
\]

Lemma 8 Let \( C \in \text{Sym}^+(\mathcal{H}), X \in \text{Sym}^+(\mathcal{H}) \) be fixed. For any constant \( a \in \mathbb{R}, a \neq 0 \),

\[
X^{1/2}C^{1/2} \left( I + \left( I + a^2C^{1/2}XC^{1/2} \right)^{1/2} \right)^{-1}C^{1/2}X^{1/2}
= -\frac{1}{a^2}I + \frac{1}{a^2} \left( I + a^2X^{1/2}CX^{1/2} \right)^{1/2}. \tag{117}
\]
Proof The desired equality is

\[ X^{1/2}C^{1/2} \left( I + \left( I + a^2 C^{1/2} X C^{1/2} \right)^{1/2} \right)^{-1} C^{1/2} X^{1/2} \]

\[ = \frac{1}{a^2 I + \frac{1}{a^2} \left( I + a^2 X^{1/2} C X^{1/2} \right)^{1/2}} \]

\[ \iff a^2 X^{1/2} C^{1/2} \left( I + \left( I + a^2 C^{1/2} X C^{1/2} \right)^{1/2} \right)^{-1} C^{1/2} X^{1/2} \]

\[ = -I + \left( I + a^2 X^{1/2} C X^{1/2} \right)^{1/2} \]

\[ \iff a^2 X^{1/2} C^{1/2} \left( I + \left( I + a^2 C^{1/2} X C^{1/2} \right)^{1/2} \right)^{-1} C^{1/2} X^{1/2} \]

\[ = a^2 X^{1/2} C X^{1/2} \left( I + \left( I + a^2 X^{1/2} C X^{1/2} \right)^{1/2} \right)^{-1} \]

\[ \iff a^2 X^{1/2} C^{1/2} \left( I + \left( I + a^2 C^{1/2} X C^{1/2} \right)^{1/2} \right)^{-1} \]

\[ \times C^{1/2} X^{1/2} \left( I + \left( I + a^2 X^{1/2} C X^{1/2} \right)^{1/2} \right)^{-1} = a^2 X^{1/2} C X^{1/2}. \]

This last equality is valid as a consequence of Lemma 6, which gives

\[ C^{1/2} X^{1/2} \left( I + \left( I + a^2 X^{1/2} C X^{1/2} \right)^{1/2} \right) \]

\[ = C^{1/2} X^{1/2} + C^{1/2} X^{1/2} \left( I + a^2 X^{1/2} C X^{1/2} \right)^{1/2} \]

\[ = C^{1/2} X^{1/2} \left( I + a^2 C^{1/2} X C^{1/2} \right)^{1/2} C^{1/2} X^{1/2} \]

\[ = \left( I + \left( I + a^2 C^{1/2} X C^{1/2} \right)^{1/2} \right) C^{1/2} X^{1/2}. \]

Together with the left hand side of the previous expression, this gives the desired equality. \(\Box\)

We apply the following result on the log-concavity of the Fredholm determinant from [59], which is a generalization of Ky Fan’s inequality for the log-concavity of the determinant on the set of symmetric positive definite matrices [28].

Proposition 3 (Proposition 7 in [59]) Let \( A, B \in \text{Sym}(H) \cap \text{Tr}(H) \) be such that \( I + A > 0, I + B > 0 \). Then for any fixed \( 0 < \alpha < 1 \),

\[ \det(I + \alpha A + (1 - \alpha)B) \geq \det(I + A)^\alpha \det(I + B)^{1-\alpha}, \quad (118) \]

with equality if and only if \( A = B \).
Lemma 9 Let $\Omega = \{X \in \text{Tr}(\mathcal{H}), \|X\| < 1\}$. The function $f: \Omega \to \mathbb{R}$ defined by $f(X) = \log \det(I - X^*X)$ is strictly concave, i.e. for $0 < \alpha < 1$ fixed,
\[
\log \det[I - (\alpha A + (1 - \alpha)B)^*(\alpha A + (1 - \alpha)B)] \\
\geq \alpha \log \det(I - A^*A) + (1 - \alpha) \log \det(I - B^*B),
\]
for all $A, B \in \Omega$. Equality happens if and only if $A = B$.

Proof By Lemma 4, for $X \in \text{Tr}(\mathcal{H}), \|X\| < 1$, $\|\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}\| < 1$ and thus $\left(I - X^*X\right) > 0$, with $\det(I - X^*X) = \det\left(I - \begin{pmatrix} 0 & X \\ X^* & 1 \end{pmatrix}\right)$. For any $A, B \in \Omega$ and a fixed $0 < \alpha < 1$, we have $\alpha A + (1 - \alpha)B \in \Omega$. Thus by Proposition 3,
\[
\det[I - (\alpha A + (1 - \alpha)B)^*(\alpha A + (1 - \alpha)B)] \\
= \det\left(\begin{pmatrix} I & \alpha A + (1 - \alpha)B \\ \alpha A + (1 - \alpha)B^* & I \end{pmatrix}\right)^{-1} \\
= \det(I + \alpha A + (1 - \alpha)B)^* \det(I - B^*B)^{1 - \alpha},
\]
with equality if and only $A = B$, from which the desired result follows. \hfill \square

Theorem 24 (Optimal entropic transport plan and entropic Wasserstein distance - the nonsingular case) Let $\mu_X = N(m_0, C_0)$, $\mu_Y = N(m_1, C_1)$, with $\ker(C_0) = \ker(C_1) = \{0\}$. Then
\[
\min_{\gamma \in \text{Joint}(\mu_X, \mu_Y)} \left\{ \mathbb{E}_{\gamma}\|x - y\|^2 + \epsilon \text{KL}(\gamma || \mu_X \otimes \mu_Y) \right\} \\
= \|m_0 - m_1\|^2 + \text{Tr}(C_0) + \text{Tr}(C_1) \\
- \max_{\mathcal{V} \in \text{HS}(\mathcal{H}), \|V\| < 1} \left\{ 2\text{Tr}(V C_1^{1/2} C_0^{1/2}) + \frac{\epsilon}{2} \log \det(I - V^*V) \right\} \\
= \|m_0 - m_1\|^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2} \text{Tr}(M_0) + \frac{\epsilon}{2} \log \det\left(I + \frac{1}{2} M_0\right).
\]
The unique minimizer is $V = \frac{4}{\epsilon} \left(I + (I + \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2}) C_1^{1/2})^{-1} C_0^{1/2} C_1^{1/2}.}$

This corresponds to the unique minimizing Gaussian measure
\[
\gamma^{\epsilon} = N\left(\begin{pmatrix} m_0 \\ m_1 \end{pmatrix}, \begin{pmatrix} C_0 & C_{XY} \\ C_{XY} & C_1 \end{pmatrix}\right),
\]
where $C_{XY} = \frac{4}{\epsilon} C_0^{1/2} \left(I + (I + \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2})^{1/2})^{-1} C_0^{1/2} C_1.}$


Proof (of Theorem 24) By Proposition 2, a minimizer of Eq. (120) must necessarily be Gaussian and satisfy $\gamma \sim \mu_X \otimes \mu_Y$. By Theorem 22,

\[
E_\gamma||x - y||^2 + \epsilon \text{KL}(\gamma || \mu_X \otimes \mu_Y), \quad \text{with } \gamma \in \text{Gauss}(\mu_X, \mu_Y)
\]

\[
= ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - 2\text{Tr}(C_0^{1/2}VC_1^{1/2}) - \frac{\epsilon}{2} \log \det(I - V^*V)
\]

\[
= ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - 2\text{Tr}(VC_1^{1/2}C_0^{1/2}) - \frac{\epsilon}{2} \log \det(I - V^*V),
\]

where $V \in \text{HS}(\mathcal{H}), ||V|| < 1$. It follows that

\[
\min_{\gamma \in \text{Gauss}(\mu_X, \mu_Y)} \{E_\gamma||x - y||^2 + \epsilon \text{KL}(\gamma || \mu_X \otimes \mu_Y)\}
\]

\[
= ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1)
\]

\[
- \max_{V \in \text{HS}(\mathcal{H}), ||V|| < 1} \left\{2\text{Tr}(VC_1^{1/2}C_0^{1/2}) + \frac{\epsilon}{2} \log \det(I - V^*V)\right\}.
\]

Let $g : \mathcal{L}(\mathcal{H}) \rightarrow \text{Sym}(\mathcal{H})$ be defined by $g(X) = X^*X$, then

\[
Dg(X_0)(X) = X_0^*X + X^*X_0, \quad X_0, X \in \mathcal{L}(\mathcal{H}). \quad (124)
\]

Let $\Omega = \{X \in \text{HS}(\mathcal{H}), ||X|| < 1\}$. Let $f : \Omega \rightarrow \mathbb{R}$ be defined by

\[
f(X) = 2\text{Tr}(XC_1^{1/2}C_0^{1/2}) + \frac{2}{\epsilon} \log \det(I - X^*X). \quad (125)
\]

By the chain rule, we have for $X_0 \in \Omega, X \in \text{HS}(\mathcal{H})$

\[
Df(X_0)(X) = 2\text{Tr}(XC_1^{1/2}C_0^{1/2}) - \frac{\epsilon}{2} \text{Tr}[(I - X_0^*X_0)^{-1}(X_0^*X + X^*X_0)]
\]

\[
= 2\text{Tr}(XC_1^{1/2}C_0^{1/2}) - \epsilon \text{Tr}[(I - X_0^*X_0)^{-1}X_0^*X].
\]

Thus $Df(X_0)(X) = 0 \forall X \in \text{HS}(\mathcal{H})$ if and only if

\[
(I - X_0^*X_0)^{-1}X_0^* = \frac{2}{\epsilon} C_1^{1/2}C_0^{1/2}. \quad (126)
\]

Since the right hand side of Eq.(127) is trace class, any solution $X_0$ must necessarily satisfy $X_0 \in \text{Tr}(\mathcal{H})$. By Lemma 9, the function $f$, as defined in Eq.(125), is strictly concave in the set $\Omega_2 = \{X \in \text{Tr}(\mathcal{H}), ||X|| < 1\}$, since the first term is linear in $X$. Thus any solution of Eq.(127) is necessarily unique and is the unique maximizer of $f$ on the larger set $\Omega = \{X \in \text{HS}(\mathcal{H}), ||X|| < 1\}$, corresponding therefore to the unique minimizer of Eq.(120).

We claim that, with $c_\epsilon = \frac{1}{\epsilon}$, the unique solution of Eq.(127) in $\Omega_2$ is

\[
X_0 = c_\epsilon (I + (I + c_\epsilon^2 C_0^{1/2}C_1^{1/2})^{1/2})^{-1} C_0^{1/2}C_1^{1/2}. \quad (128)
\]

Clearly $X_0 \in \text{Tr}(\mathcal{H})$. By Lemma 7, we also have

\[
X_0 = c_\epsilon C_0^{1/2} C_1^{1/2} (I + (I + c_\epsilon^2 C_0^{1/2}C_1^{1/2})^{1/2})^{-1}. \quad (129)
\]
We first show that with the above expression for $X_0$,
\[
I - X_0^*X_0 = \left( \frac{1}{2}I + \frac{1}{2}(I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right)^{-1}.
\] (130)

This gives $||X_0||^2 = ||X_0^*X_0|| < 1$. Eq.(130) is equivalent to

\[
X_0^*X_0 = I - \left( \frac{1}{2}I + \frac{1}{2}(I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right)^{-1}
= \left( -I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right) \left( I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right)^{-1}
= c_2 C_{1/2}^1 C_0 C_{1/2}^1 \left( I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right)^{-2}
= c_2 C_{1/2}^1 C_0 C_{1/2}^1 \left( I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right)^{-1} C_{1/2}^0 C_{1/2}^1 \left( I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right)^{-1}.
\]

The last equality is valid by Eqs.(128) and (129). It follows from Eq.(130), by invoking Lemma 7, that

\[
(I - X_0^*X_0)^{-1}X_0^* = \frac{4}{\epsilon} \left( \frac{1}{2}I + \frac{1}{2}(I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right) C_{1/2}^1 C_0^{1/2}
\times \left( I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right)^{-1}
= \frac{2}{\epsilon} C_{1/2}^1 C_0^{1/2} \left[ I + c_2 (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right] \left( I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right)^{-1}
= \frac{2}{\epsilon} C_{1/2}^1 C_0^{1/2},
\]

which is (127).

With $X_0$ as given in Eq.(128), we obtain the unique minimizing Gaussian measure $\gamma^*$ of Eq.(120), with $C_{XY} = C_{1/2}^0 X_0 C_{1/2}^1$. Furthermore,

\[
\text{Tr}[X_0 C_{1/2}^1 C_0^{1/2}] = \text{Tr} \left[ c_2 (I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2})^{-1} C_{1/2}^0 C_0^{1/2} \right]
= \frac{1}{c_2} \text{Tr} \left[ -I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2} \right] = \frac{\epsilon}{4} \text{Tr}[M_{01}],
\]

\[
\log \det(I - X_0^*X_0) = - \log \det(I + (I + c_2 C_{1/2}^1 C_0 C_{1/2}^1)^{1/2})
= - \log \det(I + \frac{1}{2} M_{10}^*) = - \log \det(I + \frac{1}{2} M_{01}).
\]

Combining the last two expressions gives the entropic distance formula. \( \square \)

Let us now compute the Radon-Nikodym density of the optimal entropic transport plan $\gamma^*$ in Theorem 24 with respect to $\mu_0 \otimes \mu_1$. We note that the optimal $V$ in Theorem 24 is trace class. Consequently, we can apply the following result (see Proposition 1.3.11 in [23] or Corollary 2 in [65]).
Proposition 4  Let $\mu = \mathcal{N}(m, Q)$, $\nu = \mathcal{N}(m, R)$, with $\ker Q = \{0\}$ and $\mu \sim \nu$. Assume that $R = Q^{1/2}(I - S)Q^{1/2}$ with $S \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$. Then

$$\frac{d\nu}{d\mu}(x) = \det[(I - S)^{-1/2}] \exp \left\{ -\frac{1}{2}(Q^{-1/2}(x - m), S(I - S)^{-1}Q^{-1/2}(x - m)) \right\}$$

(131)

where, in the $\mathcal{L}^1(\mathcal{H}, \mu)$ sense,

$$\langle Q^{-1/2}(x - m), S(I - S)^{-1}Q^{-1/2}(x - m) \rangle \approx \lim_{N \to \infty} \langle Q^{-1/2}P_N(x - m), S(I - S)^{-1}Q^{-1/2}P_N(x - m) \rangle.$$  

(132)

The following result can be obtained by direct verification.

Lemma 10 Let $B, C \in \mathcal{L}(\mathcal{H})$ be such that $(I - BC)$ is invertible, then the block operator $\left( \begin{array}{cc} I & B \\ C & I \end{array} \right) : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ is invertible, with

$$\left( \begin{array}{cc} I & B \\ C & I \end{array} \right)^{-1} = \left( \begin{array}{cc} (I - BC)^{-1} & -(I - BC)^{-1}B \\ -C(I - BC)^{-1} & I + C(I - BC)^{-1}B \end{array} \right) \in \mathcal{L}(\mathcal{H} \times \mathcal{H}).$$  

(133)

Proposition 5  Assume the hypothesis of Theorem 24. The Radon-Nikodym density of the optimal entropic transport plan $\gamma^\epsilon$ with respect to $\mu_0 \otimes \mu_1$ is

$$\frac{d\gamma^\epsilon}{d(\mu_0 \otimes \mu_1)}(x, y) = \sqrt{\det \left( I + \frac{1}{2} M_{01}^\epsilon \right)} \exp(\langle x - m_0, A_\epsilon(x - m_0) \rangle) \times \exp(\langle y - m_1, B_\epsilon(y - m_1) \rangle) \exp \left( \frac{\epsilon}{2}(x - m_0, y - m_1) \right).$$  

(134)

Here $A_\epsilon = -\frac{2}{\epsilon^2} C_1^{1/2} (I + \frac{1}{2} M_{01}^\epsilon)^{-1} C_1^{1/2}$, $B_\epsilon = -\frac{2}{\epsilon^2} C_0^{1/2} (I + \frac{1}{2} M_{01}^\epsilon)^{-1} C_0^{1/2}$.

Eq.(134) is equivalent to Eq.(16) in Theorem 3 via the identity

$$||x - y||^2 = ||x - m_0||^2 + ||y - m_1||^2 + ||m_0 - m_1||^2 - 2(x - m_0, y - m_1) + 2(x - m_0, m_0 - m_1) - 2(y - m_1, m_0 - m_1).$$  

(135)

Proof  For $||V|| < 1$, $(I - VV^*)$ is invertible and by Lemma 10,

$$\left( \begin{array}{cc} I & V \\ V^* & I \end{array} \right)^{-1} = \left( \begin{array}{cc} (I - VV^*)^{-1} & -(I - VV^*)^{-1}V \\ -V^*(I - VV^*)^{-1} & I + V^*(I - VV^*)^{-1}V \end{array} \right).$$

By the identity $V^*(I - VV^*)^{-1} = (I - V^*V)^{-1}V^*$,

$$\left( \begin{array}{cc} 0 & V \\ V^* & 0 \end{array} \right) \left( \begin{array}{cc} I & V \\ V^* & I \end{array} \right)^{-1} = \left( \begin{array}{cc} -V^*(I - VV^*)^{-1} & (I - VV^*)^{-1}V \\ V^*(I - VV^*)^{-1} & -V^*(I - VV^*)^{-1}V \end{array} \right) \left( \begin{array}{cc} -(I - VV^*)^{-1}V^* & (I - VV^*)^{-1}V \\ (I - V^*V)^{-1}V^* & -(I - V^*V)^{-1}V^* \end{array} \right).$$
For $V = c_c(I + c_c^2 C_0^1 C_1^0 C_1^0 C_1^0)^{1/2} - 1 C_0^1 C_1^0 = \frac{c_c}{2}(I + \frac{1}{2} M_{01}^0)^{-1} C_0^1 C_1^0$, where $c_c = \frac{\epsilon}{2}$, we have $V \in \text{Tr}(H)$. Similar to the proof of Theorem 24,

$$I - V^* V = \left(\frac{1}{2} I + \frac{1}{2} I + c_c^2 C_0^1 C_0^1 C_1^0 C_1^0 \right)^{-1} = \left(I + \frac{1}{2} M_{10}^0 \right)^{-1}$$

$$I - V V^* = \left(\frac{1}{2} I + \frac{1}{2} I + c_c^2 C_0^1 C_0^1 C_1^0 C_1^0 \right)^{-1} = \left(I + \frac{1}{2} M_{01}^0 \right)^{-1}$$

$$(I - V^* V)^{-1} V = \frac{c_c}{2} C_1^0 C_0^1 C_0^1, \quad (I - V V^*)^{-1} V = \frac{c_c}{2} C_0^1 C_1^0 C_1^0$$

$$(I - V^* V)^{-1} V^* = \frac{1}{4} c_c^2 C_0^1 C_1^0 C_1^0 C_1^0 \left(I + \frac{1}{2} M_{10}^0 \right)^{-1} \left(I - V V^* \right)^{-1} C_1^0 C_0^1 C_0^1$$

$$(I - V^* V)^{-1} V^* = \frac{1}{4} c_c^2 C_1^0 C_0^1 C_0^1 \left(I + \frac{1}{2} M_{01}^0 \right)^{-1} C_0^1 C_1^0$$

Since $V \in \text{Tr}(H), S = -\left(\begin{array}{c} 0 & V \\ V^* & 0 \end{array}\right) \in \text{Tr}(H \times H)$ by Lemma 4, and

$$\det(I - S) = \det(I - V^* V) = \left[\det(I + \frac{1}{2} M_{10}^0)\right]^{-1} = \left[\det(I + \frac{1}{2} M_{01}^0)\right]^{-1}$$

since $C_0^1 C_1^0 C_1^0$ and $C_1^0 C_0^1 C_1^0$ have the same eigenvalues. Let $Q = \left(\begin{array}{cc} C_0^1 & 0 \\ 0 & C_1 \end{array}\right)$, $A_c = -\frac{c_c}{2} C_0^1 C_1^0 C_1^0$, $B_c = -\frac{c_c}{2} C_0^1 C_1^0 C_1^0$, then

$$S(I - S)^{-1} = -Q^{1/2} \left(\begin{array}{cc} 2A_c & \frac{1}{2} c_c I \\ \frac{1}{2} c_c I & 2B_c \end{array}\right) Q^{1/2}$$

Thus for any $x, y \in H$ and $N \in \mathbb{N}$,

$$\lim_{N \to \infty} \left\{ Q^{-1/2} P_N \left(\begin{array}{c} x - m_0 \\ y - m_1 \end{array}\right), S(I - S)^{-1} Q^{-1/2} P_N \left(\begin{array}{c} x - m_0 \\ y - m_1 \end{array}\right) \right\}$$

$$= \left\{ \left(\begin{array}{c} x - m_0 \\ y - m_1 \end{array}\right), \frac{2A_c}{2B_c} \left(\begin{array}{c} x - m_0 \\ y - m_1 \end{array}\right) \right\}$$

$$= -2(x - m_0, A_c(x - m_0) - c_c(x - m_0, y - m_1) - 2(y - m_1, B_c(y - m_1)).$$

Combining this with det$(I - S)$ and Proposition 4 gives the desired result. $\square$

**The general case.** We note that while Theorem 24 and Proposition 5 are proved under the hypothesis that $\ker(C_0) = \ker(C_1) = \{0\}$, the optimal solution obtained is clearly mathematically valid without this assumption. We now confirm that this is indeed the case via a different approach.

Let $k(x, y) = \exp\left(-\frac{c(x, y)}{\epsilon}\right), \epsilon > 0$. It has been shown that, see e.g. [11,18, 25,38,78], problem (110) has a unique minimizer $\gamma^\epsilon$ if and only if there exist functions $\alpha^\epsilon, \beta^\epsilon$ satisfying the *Schrödinger system*

$$\alpha^\epsilon(x) \mathbb{E}_{\nu_\epsilon}[\beta^\epsilon(y) k(x, y)] = 1,$$

$$\beta^\epsilon(y) \mathbb{E}_{\nu_\epsilon}[\alpha^\epsilon(x) k(x, y)] = 1.$$ (136)
In this case, the unique minimizer $\gamma^\epsilon$ is the probability measure whose Radon-Nikodym derivative with respect to $\mu_0 \otimes \mu_1$ is given by

$$\frac{d\gamma^\epsilon}{d(\mu_0 \otimes \mu_1)}(x,y) = \alpha^\epsilon(x)\beta(y)k(x,y). \quad (137)$$

Motivated by Proposition 5, we now solve the Schrödinger system (136) when $c(x,y) = ||x - y||^2$ on $\mathcal{H}$, leading to another proof of Theorem 3, which is valid in the general setting, where $C_0$ and $C_1$ can be singular.

We make use of the following results on Gaussian integrals on Hilbert spaces.

**Theorem 25 ([23], Proposition 1.2.8)** Consider the Gaussian measure $\mathcal{N}(0, C)$ on $\mathcal{H}$. Assume that $M$ is a self-adjoint operator on $\mathcal{H}$ such that $(C^{1/2}MC^{1/2}x, x) < ||x||^2 \forall x \in \mathcal{H}, x \neq 0$. Let $b \in \mathcal{H}$. Then

$$\int_{\mathcal{H}} \exp \left( \frac{1}{2} (My,y) + \langle b, y \rangle \right) d\mathcal{N}(0, C)(y) = [\det(I - C^{1/2}MC^{1/2})]^{-1/2} \exp \left( \frac{1}{2} ||(I - C^{1/2}MC^{1/2})^{-1/2}C^{1/2}b||^2 \right). \quad (138)$$

The following result then follows immediately

**Corollary 3** Consider the Gaussian measure $\mathcal{N}(m, C)$ on $\mathcal{H}$. Assume that $M$ is a self-adjoint operator on $\mathcal{H}$ such that $(C^{1/2}MC^{1/2}x, x) < ||x||^2 \forall x \in \mathcal{H}, x \neq 0$. Let $b \in \mathcal{H}$. Then

$$\int_{\mathcal{H}} \exp \left( \frac{1}{2} (M(y-m), (y-m)) + \langle b, y-m \rangle \right) d\mathcal{N}(m, C)(y) = [\det(I - C^{1/2}MC^{1/2})]^{-1/2} \exp \left( \frac{1}{2} ||(I - C^{1/2}MC^{1/2})^{-1/2}C^{1/2}b||^2 \right). \quad (139)$$

**Proof (of Theorem 3: optimal entropic transport plan - the general case)** We first have

$$||x - y||^2 = ||(x-m_0) - (y-m_1) + (m_0 - m_1)||^2$$

$$= ||x-m_0||^2 + ||y-m_1||^2 + ||m_0 - m_1||^2 - 2\langle x-m_0, y-m_1 \rangle$$

$$+ 2\langle x-m_0, m_0 - m_1 \rangle - 2\langle y-m_1, m_0 - m_1 \rangle.$$

Expanding $\alpha^\epsilon(x)\beta^\epsilon(y) \exp \left( -\frac{||x-y||^2}{\epsilon} \right)$ under the assumptions

$$\alpha^\epsilon(x) = \exp \left( (x-m_0, A(x-m_0)) + \frac{2}{\epsilon} (x-m_0, m_0 - m_1) + a \right),$$

$$\beta^\epsilon(y) = \exp \left( (y-m_1, B(y-m_1)) + \frac{2}{\epsilon} (y-m_1, m_1 - m_0) + b \right),$$

Such that...
we obtain

\[ \alpha'(x) \beta'(y) \exp \left( -\frac{||x - y||^2}{\epsilon} \right) = \exp(a + b) \exp \left( -\frac{||m_0 - m_1||^2}{\epsilon} \right) \times \exp \left( \left( x - m_0, A - \frac{1}{\epsilon} I \right) (x - m_0) \right) \times \exp \left( \left( y - m_1, B - \frac{1}{\epsilon} I \right) (y - m_1) \right) \times \exp \left( \frac{2}{\epsilon} (x - m_0, y - m_1) \right). \]

The Schrödinger system (136) then becomes

\[ 1 = \exp(a + b) \exp \left( -\frac{||m_0 - m_1||^2}{\epsilon} \right) \times \int_{\mathcal{H}} \exp \left( \left( x - m_0, A - \frac{1}{\epsilon} I \right) (x - m_0) \right) \times \exp \left( \left( y - m_1, B - \frac{1}{\epsilon} I \right) (y - m_1) \right) \times \exp \left( \frac{2}{\epsilon} (x - m_0, y - m_1) \right) d\mu_1(y), \]

\[ 1 = \exp(a + b) \exp \left( -\frac{||m_0 - m_1||^2}{\epsilon} \right) \times \int_{\mathcal{H}} \exp \left( \left( x - m_0, A - \frac{1}{\epsilon} I \right) (x - m_0) \right) \times \exp \left( \left( y - m_1, B - \frac{1}{\epsilon} I \right) (y - m_1) \right) \times \exp \left( \frac{2}{\epsilon} (x - m_0, y - m_1) \right) d\mu_0(x). \]

Let \( A_\epsilon = A - \frac{1}{\epsilon} I \), \( B_\epsilon = B - \frac{1}{\epsilon} I \), then by Corollary 3,

\[ \int_{\mathcal{H}} \exp \left( \left( x - m_0, A_\epsilon(x - m_0) \right) + \frac{2}{\epsilon} (x - m_0, y - m_1) \right) d\mu_0(x) = [\det(I - 2C_0^{1/2} A_\epsilon C_0^{1/2})]^{-1/2} \exp \left( \frac{2}{\epsilon^2} ||(I - 2C_0^{1/2} A_\epsilon C_0^{1/2})^{-1/2}C_0^{1/2}(y - m_1)||^2 \right), \]

\[ \int_{\mathcal{H}} \exp \left( \left( y - m_1, B_\epsilon(y - m_1) \right) + \frac{2}{\epsilon} (x - m_0, y - m_1) \right) d\mu_1(y) = [\det(I - 2C_1^{1/2} B_\epsilon C_1^{1/2})]^{-1/2} \exp \left( \frac{2}{\epsilon^2} ||(I - 2C_1^{1/2} B_\epsilon C_1^{1/2})^{-1/2}C_1^{1/2}(x - m_0)||^2 \right). \]

Thus for the system of equations (140) to hold \( \forall x, y \in \mathcal{H} \), we must have

\[ B_\epsilon = -\frac{2}{\epsilon^2} C_0^{1/2}(I - 2C_0^{1/2} A_\epsilon C_0^{1/2})^{-1}C_0^{1/2}, \]

\[ A_\epsilon = -\frac{2}{\epsilon^2} C_1^{1/2}(I - 2C_1^{1/2} B_\epsilon C_1^{1/2})^{-1}C_1^{1/2}, \]

\[ \exp(a + b) = \exp \left( -\frac{||m_0 - m_1||^2}{\epsilon} \right) [\det(I - 2C_0^{1/2} A_\epsilon C_0^{1/2})]^{1/2}, \]  

\[ \exp(a + b) = \exp \left( -\frac{||m_0 - m_1||^2}{\epsilon} \right) [\det(I - 2C_1^{1/2} B_\epsilon C_1^{1/2})]^{1/2}. \]
We claim that the following $A_\epsilon$ and $B_\epsilon$ solve the system of equations (141)

\[
A_\epsilon = -\frac{2}{\epsilon^2}C_1^{1/2} \left[ \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_1^{1/2} C_0 C_1^{1/2} \right)^{1/2} \right]^{-1} C_1^{1/2},
\]

\[
B_\epsilon = -\frac{2}{\epsilon^2}C_0^{1/2} \left[ \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2} \right)^{1/2} \right]^{-1} C_0^{1/2}.
\]

Let us verify the first equation in (141) (the second one is analogous). We have

\[
C_0^{1/2} A_\epsilon C_0^{1/2} = -\frac{2}{\epsilon^2} C_0^{1/2} C_1^{1/2} \left[ \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_0 C_1^{1/2} \right)^{1/2} \right]^{-1} C_1^{1/2} C_0^{1/2}, \tag{142}
\]

\[
I - 2C_0^{1/2} A_\epsilon C_0^{1/2} = I + \frac{4}{\epsilon^2} C_0^{1/2} C_1^{1/2} \left[ \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_1^{1/2} C_0 C_1^{1/2} \right)^{1/2} \right]^{-1} C_1^{1/2} C_0^{1/2}. \tag{143}
\]

Thus in order to have $B_\epsilon = -\frac{2}{\epsilon^2} C_0^{1/2} (I - 2C_0^{1/2} A_\epsilon C_0^{1/2})^{-1} C_0^{1/2}$, we need

\[
\frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2} \right)^{1/2} = I - 2C_0^{1/2} A_\epsilon C_0^{1/2} \tag{144}
\]

Replacing the right hand side with the expression in Eq.(143), this is

\[
\frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2} \right)^{1/2}
\]

\[
= I + \frac{4}{\epsilon^2} C_0^{1/2} C_1^{1/2} \left[ \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_0 C_1^{1/2} \right)^{1/2} \right]^{-1} C_1^{1/2} C_0^{1/2}
\]

\[
\Leftrightarrow -I + \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2} \right)^{1/2}
\]

\[
= \frac{16}{\epsilon^2} C_0^{1/2} C_1^{1/2} \left[ I + \left( I + \frac{16}{\epsilon^2} C_1^{1/2} C_0 C_1^{1/2} \right)^{1/2} \right]^{-1} C_1^{1/2} C_0^{1/2}.
\]

This is precisely Lemma 8 with $a = \frac{4}{\epsilon}$. Similarly, we have

\[
\frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_1^{1/2} C_0 C_1^{1/2} \right) = I - 2C_1^{1/2} B_\epsilon C_1^{1/2}. \tag{145}
\]

Finally, for the expression of $\exp(a + b)$, we note that it is clear that from Eq.(142) that $C_0^{1/2} A_\epsilon C_0^{1/2} \in \operatorname{Tr}(\mathcal{H})$ and from Eqs. (143) and (144) that $I -
$2C_0^{1/2}A_0C_0^{1/2} > 0$. Thus the Fredholm determinant of the latter expression is well-defined and positive. From Eq. (144),

$$
\det(I - 2C_0^{1/2}A_0C_0^{1/2}) = \det \left( \frac{1}{2}I + 2 \left( I + \frac{16}{e^2}C_0^{1/2}C_0^{1/2} \right)^{1/2} \right)
$$

$$
= \det \left( \frac{1}{2}I + 2 \left( I + \frac{16}{e^2}C_1^{1/2}C_0^{1/2} \right)^{1/2} \right) = \det(I - 2C_1^{1/2}B_0C_1^{1/2}).
$$

Here we have used the fact that the eigenvalues of $C_0^{1/2}C_1C_0^{1/2}$ are the same as those of $C_1^{1/2}C_0C_1^{1/2}$. \(\square\)

**Lemma 11** Let $\mu \in \mathcal{P}_2(\mathcal{H})$ with mean $m$ and covariance operator $C$. Let $A \in \mathcal{L}(\mathcal{H})$. Then

$$
\int_{\mathcal{H}} \langle x - m, A(x - m) \rangle d\mu(x) = \text{Tr}(CA). \quad (146)
$$

**Proof** It suffices to consider the case $m = 0$.

(i) Suppose $A \in \text{Sym}^+(\mathcal{H})$. Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis in $\mathcal{H}$. Then

$$
\int_{\mathcal{H}} \langle x, Ax \rangle d\mu(x) = \int_{\mathcal{H}} \|A^{1/2}x\|^2 d\mu(x) = \int_{\mathcal{H}} \sum_{k=1}^\infty \langle A^{1/2}x, e_k \rangle^2 d\mu(x)
$$

$$
= \int_{\mathcal{H}} \sum_{k=1}^\infty \langle x, A^{1/2}e_k \rangle^2 d\mu(x) = \sum_{k=1}^\infty \int_{\mathcal{H}} \langle x, A^{1/2}e_k \rangle^2 d\mu(x)
$$

by Lebesgue Monotone Convergence Theorem

$$
= \sum_{k=1}^\infty \langle CA^{1/2}e_k, A^{1/2}e_k \rangle = \sum_{k=1}^\infty \langle A^{1/2}CA^{1/2}e_k, e_k \rangle = \text{Tr}(A^{1/2}CA^{1/2}) = \text{Tr}(CA).
$$

(ii) Suppose now $A \in \text{Sym}(\mathcal{H})$, then $A = A_1 - A_2$, where $A_1 = \frac{1}{2}(|A| + A) \in \text{Sym}^+(\mathcal{H}), A_2 = \frac{1}{2}(|A| - A) \in \text{Sym}^+(\mathcal{H})$. Thus this case reduces to case (i).

(iii) For any $A \in \mathcal{L}(\mathcal{H}), \int_{\mathcal{H}} \langle x, Ax \rangle d\mu(x) = \frac{1}{2} \int_{\mathcal{H}} \langle x, (A + A^*)x \rangle d\mu(x)$. Using the fact $\text{Tr}(CA^*) = \text{Tr}(AC) = \text{Tr}(CA)$, this case reduces to case (ii). \(\square\)

**Corollary 4** (Entropic Wasserstein distance between Gaussian measure on Hilbert space - the general case) Let $\mu_0 = \mathcal{N}(m_0, C_0)$ and $\mu_1 = \mathcal{N}(m_1, C_1)$. For each fixed $\epsilon > 0$,

$$
\text{OT}_{\epsilon}(\mu_0, \mu_1) = ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2} \text{Tr}(M_{01})
$$

$$
+ \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2}M_{01} \right). \quad (147)
$$
Proof For the optimal \(d_{\alpha^*(\mu_0, \mu_1)}(x, y) = \alpha^*(x) \beta^*(y) \exp(-\frac{||x-y||^2}{\epsilon})\),

\[
\text{OT}_{\alpha^*(\mu_0, \mu_1)} = E_{\gamma^*} ||x - y||^2 + \epsilon \int_{\mathcal{H} \times \mathcal{H}} \log \left( \frac{d\gamma^*}{d(\mu_0 \otimes \mu_1)}(x, y) \right) d\gamma^*(x, y)
\]

\[
= \epsilon \int_{\mathcal{H} \times \mathcal{H}} \log \alpha^*(x) d\gamma^*(x, y) + \epsilon \int_{\mathcal{H} \times \mathcal{H}} \log \beta^*(y) d\gamma^*(x, y)
\]

Recall that

\[
\alpha^*(x) = \exp \left( (x - m_0, A(x - m_0)) + \frac{2}{\epsilon} (x - m_0, m_0 - m_1) + a \right),
\]

\[
\beta^*(y) = \exp \left( (y - m_1, B(y - m_1)) + \frac{2}{\epsilon} (y - m_1, m_1 - m_0) + b \right),
\]

\[
A = \frac{1}{\epsilon} I - \frac{2}{\epsilon^2} C_1^{1/2} \left[ \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_1^{1/2} C_0 C_1^{1/2} \right)^{1/2} \right]^{-1} C_1^{1/2},
\]

\[
B = \frac{1}{\epsilon} I - \frac{2}{\epsilon^2} C_0^{1/2} \left[ \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2} \right)^{1/2} \right]^{-1} C_0^{1/2},
\]

\[
M_{01}^* = -I + \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2} \right)^{1/2} = \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2} \left[ I + \left( I + \frac{16}{\epsilon^2} C_0^{1/2} C_1 C_0^{1/2} \right)^{1/2} \right]^{-1},
\]

\[
M_{10}^* = -I + \left( I + \frac{16}{\epsilon^2} C_1^{1/2} C_0 C_1^{1/2} \right)^{1/2} = \frac{16}{\epsilon^2} C_1^{1/2} C_0 C_1^{1/2} \left[ I + \left( I + \frac{16}{\epsilon^2} C_1^{1/2} C_0 C_1^{1/2} \right)^{1/2} \right]^{-1},
\]

\[
\exp(a + b) = \exp \left( \frac{||m_0 - m_1||^2}{\epsilon} \right) \sqrt{\det \left( I + \frac{1}{2} M_{01}^* \right)}
\]

\[
\Longleftrightarrow (a + b) = \frac{||m_0 - m_1||^2}{\epsilon} + \frac{1}{2} \log \det \left( I + \frac{1}{2} M_{01}^* \right).
\]

It follows that

\[
\text{OT}_{\alpha^*(\mu_0, \mu_1)} = \epsilon \left[ (a + b) + c E_{X \sim \mu_0} \left[ (X - m_0, A(X - m_0)) + \frac{2}{\epsilon} (X - m_0, m_0 - m_1) \right] \right]
\]

\[
+ c E_{Y \sim \mu_1} \left[ (Y - m_1, B(Y - m_1)) + \frac{2}{\epsilon} (Y - m_1, m_1 - m_0) \right]
\]

\[
= \epsilon (a + b) + \epsilon \left( \text{Tr} [C_0 A] + \text{Tr} [C_1 B] \right).
\]

Here we have invoked Lemma 11. For the first trace term, we have

\[
\text{Tr}[C_0 A] = \frac{1}{\epsilon} \text{Tr}(C_0) - \frac{2}{\epsilon^2} \text{Tr} \left[ C_1^{1/2} C_0 C_1^{1/2} \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_1^{1/2} C_0 C_1^{1/2} \right)^{1/2} \right)^{1/2} \right]^{-1}
\]

\[
= \frac{1}{\epsilon} \text{Tr}(C_0) - \frac{1}{4} \text{Tr}(M_{10}^*). \]
Similarly, the second term is
\[ \text{Tr}(C_1 B) = \frac{1}{\epsilon} \text{Tr}(C_1) - \frac{1}{4} \text{Tr}(M_{01}^*). \]

Combining all the previous expressions, we obtain
\[
\text{OT}_{d^2}(\mu_0, \mu_1) = ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{4} \text{Tr}(M_{01}^*) - \frac{\epsilon}{4} \text{Tr}(M_{10}^*) + \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2} M_{01}^* \right) \\
= ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2} \text{Tr}(M_{01}^*) + \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2} M_{01}^* \right).
\]

Here we have used the fact that the nonzero eigenvalues of \( C_0^{1/2} C_1 C_0^{1/2} \) and \( C_1^{1/2} C_0 C_1^{1/2} \) are the same, so that \( \text{Tr}(M_{01}^*) = \text{Tr}(M_{10}^*). \)

\[ \square \]

Proof (of Corollary 1 - dual formulation) With \( \varphi = \epsilon \log \alpha, \psi = \epsilon \log \beta, \)
\[
E_{\mu_0} \left[ \exp \left( \frac{1}{\epsilon} \varphi \right) \varphi \right] = E_{\mu_0}[\alpha^*] = \int_{\mathcal{H}} \alpha^*(x) d\mu_0(x) \\
= \int_{\mathcal{H}} \exp \left( (x - m_0, A(x - m_0)) + \frac{2}{\epsilon} (x - m_0, m_0 - m_1) + a \right) d\mu_0(x) \\
= \exp(a) \text{Tr}(I - 2C_0^{1/2} A C_0^{1/2})^{-1/2} \exp \left( \frac{2}{\epsilon^2} \|I - 2C_0^{1/2} A C_0^{1/2}\|^{-1/2}(m_0 - m_1)^2 \right)
\]
by Corollary 3. Clearly \( 0 < E_{\mu_0} \left[ \frac{1}{\epsilon} \varphi \right] < \infty \) and thus \( \varphi \in L_{\text{exp}}^\epsilon(\mathcal{H}, \mu_0). \) Similarly \( \psi \in L_{\text{exp}}^\epsilon(\mathcal{H}, \mu_1). \) Let us now compute \( D(\varphi, \psi). \) We have
\[
E_{\mu_0} [\varphi] = \epsilon E_{\mu_0}[\log \alpha], \quad E_{\mu_1} [\psi] = \epsilon E_{\mu_1}[\log \beta], \\
\int_{\mathcal{H} \times \mathcal{H}} \left[ \exp \left( \frac{\varphi(x) + \psi(y) - d^2(x, y)}{\epsilon} \right) - 1 \right] d(\mu_0 \otimes \mu_1)(x, y) \\
= -1 + \int_{\mathcal{H} \times \mathcal{H}} \alpha^*(x) \beta^*(y) \exp \left( - \frac{||x - y||^2}{\epsilon} \right) d(\mu_0 \otimes \mu_1)(x, y) \\
= -1 + \int_{\mathcal{H} \times \mathcal{H}} \frac{d\gamma^*}{d(\mu_0 \otimes \mu_1)}(x, y) d(\mu_0 \otimes \mu_1)(x, y) = -1 + \int_{\mathcal{H} \times \mathcal{H}} d\gamma^*(x, y) = 0.
\]
It follows that \( D(\varphi, \psi) = \epsilon E_{\mu_0}[\log \alpha^*] + \epsilon E_{\mu_1}[\log \beta^*] = \text{OT}_{d^2}(\mu_0, \mu_1), \) as in the proof of Corollary 4.

\[ \square \]

Corollary 5 (Sinkhorn divergence between Gaussian measures on Hilbert space) Let \( \mu_0 = \mathcal{N}(m_0, C_0), \mu_1 = \mathcal{N}(m_1, C_1). \) Then
\[
S_{d^2}(\mu_0, \mu_1) = ||m_0 - m_1||^2 + \frac{\epsilon}{4} \text{Tr}(M_{01}^* - 2M_{01}^* M_{11}^* + M_{11}^*) \\
+ \frac{\epsilon}{4} \log \det \left[ \frac{(I + \frac{1}{2} M_{00}^*)^2}{(I + \frac{1}{2} M_{00}^*) (I + \frac{1}{2} M_{11}^*)} \right].
\]

(148)
Proof By definition of the Sinkhorn divergence and Theorem 4,

\[
S_{\epsilon d}^T(\mu_0, \mu_1) = OT_{\epsilon d}^T(\mu_0, \mu_1) - \frac{1}{2} OT_{\epsilon d}(\mu_0, \mu_0) - \frac{1}{2} OT_{\epsilon d}^T(\mu_1, \mu_1)
\]

\[
= ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2} \text{Tr}(M_0^\epsilon) + \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2} M_0^\epsilon \right)
\]

\[
- \frac{1}{2} \left[ 2\text{Tr}(C_0) - \frac{\epsilon}{2} \text{Tr}(M_0^\epsilon) + \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2} M_0^\epsilon \right) \right]
\]

\[
- \frac{1}{2} \left[ 2\text{Tr}(C_1) - \frac{\epsilon}{2} \text{Tr}(M_1^\epsilon) + \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2} M_1^\epsilon \right) \right]
\]

\[
= ||m_0 - m_1||^2 + \frac{\epsilon}{4} \text{Tr} [M_0^\epsilon - 2M_0^\epsilon + M_1^\epsilon] + \frac{\epsilon}{4} \log \det \left[ \left( I + \frac{1}{2} M_0^\epsilon \right) \left( I + \frac{1}{2} M_1^\epsilon \right) \right].
\]

This completes the proof. \(\square\)

Proof of Theorem 11 - Equivalent expressions. Let \(c_\epsilon = \frac{\epsilon}{2}\). Then

\[
C_i^{1/2} C_j C_i^{1/2} - L_i^{\epsilon} = c_\epsilon^2 (C_i^{1/2} C_j C_i^{1/2} - L_i^{\epsilon})^2 \left( I + (I + c_\epsilon^2 C_i^{1/2} C_j C_i^{1/2})^{1/2} \right)^2
\]

\[
(C_i^{1/2} C_j C_i^{1/2} - L_i^{\epsilon})^{1/2} = c_\epsilon C_i^{1/2} C_j C_i^{1/2} \left( I + (I + c_\epsilon^2 C_i^{1/2} C_j C_i^{1/2})^{1/2} \right)^{-1}
\]

\[
= \frac{1}{c_\epsilon} \left( -I + (I + c_\epsilon^2 C_i^{1/2} C_j C_i^{1/2})^{-1} \right) = \frac{\epsilon}{4} M_{ij}^\epsilon
\]

\[
= \frac{-\epsilon}{4} + \left( \frac{\epsilon^2}{16} I + C_i^{1/2} C_j C_i^{1/2} \right)^{1/2},
\]

\[
\frac{C_i^{1/2} C_j C_i^{1/2}}{L_i^{\epsilon}} = \frac{1}{2} I + \frac{1}{2} (I + c_\epsilon^2 C_i^{1/2} C_j C_i^{1/2})^{-1} = I + \frac{1}{2} M_{ij}^\epsilon,
\]

\[
\frac{L_0^\epsilon L_1^{\epsilon}}{(L_0^{\epsilon})^2} = \frac{(I + \frac{1}{2} M_{00}^\epsilon)^2}{(I + \frac{1}{2} M_{00}^\epsilon)(I + \frac{1}{2} M_{11}^\epsilon)}.
\]

Using these expressions, we see that the formulas for \(OT_{\epsilon d}^T\) and \(S_{\epsilon d}^T\) coincide with those in Theorems 4 and 7. Furthermore, it is immediately clear that

\[
\lim_{\epsilon \to 0} \text{Tr}(C_i^{1/2} C_j C_i^{1/2} - L_i^{\epsilon})^{1/2} = \text{Tr}(C_i^{1/2} C_j C_i^{1/2})^{1/2},
\]

\[
\lim_{\epsilon \to \infty} \text{Tr}(C_i^{1/2} C_j C_i^{1/2} - L_i^{\epsilon})^{1/2} = 0.
\]

By L’Hopital’s rule, we have \(\forall x \geq 0, \lim_{\epsilon \to 0} \epsilon \log\left( \frac{1}{2} + \frac{1}{2} (1 + \frac{16}{3} x)^{1/2} \right) = \lim_{\epsilon \to \infty} \epsilon \log\left( \frac{1}{2} + \frac{1}{2} (1 + \frac{16}{3} x)^{1/2} \right) = 0\). It follows that

\[
\lim_{\epsilon \to 0} \epsilon \log \det(I + \frac{1}{2} M_{ij}^\epsilon) = \lim_{\epsilon \to \infty} \epsilon \log \det(I + \frac{1}{2} M_{ij}^\epsilon) = 0.
\]

Combining these limits give the limiting behavior of \(OT_{\epsilon d}^T\) and \(S_{\epsilon d}^T\). \(\square\)
Theorem 26 (Lower bound for entropic 2-Wasserstein distances between general probability measures) Let \( \mu, \mu_1 \in \mathcal{P}_2(\mathcal{H}) \), with means \( m_0, m_1 \) and covariance operators \( C_0, C_1 \), respectively. Then

\[
\text{OT}^2_{\mu} (\mu_0, \mu_1) \geq \|m_0 - m_1\|^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2} \text{Tr}(M_{\mu_0}^0) \\
+ \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2} M_{\mu_0}^0 \right).
\]

Equality happens if and only if \( \mu_0, \mu_1 \) are Gaussian measures.
Lemma 12 If \( \gamma^c \in \text{Joint}(\mu_0, \mu_1) \) is a minimizer of the entropic regularization problem (110), then necessarily \( \gamma^c \sim \mu_0 \otimes \mu_1 \).

Proof The Schrödinger system (136) implies that \( \alpha^c(x) \neq 0 \), \( \beta^c(y) \neq 0 \) \( \forall x, y \in \mathcal{H} \). It thus follows that \( \frac{d\gamma}{d(\mu_0 \otimes \mu_1)}(x, y) > 0 \) \( \forall x, y \in \mathcal{H} \). Hence \( \gamma^c \sim \mu_0 \otimes \mu_1 \). \( \square \)

Proof (of Theorem 26) By Lemma 12, it is sufficient to consider the set of all probability measures on \( (\mathcal{H} \times \mathcal{H}, \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H})) \) that are equivalent to \( \mu_0 \otimes \mu_1 \). Consider the probability measure \( \gamma_0 \) on \( (\mathcal{H} \times \mathcal{H}, \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H})) \) whose Radon-Nikodym derivative with respect to \( \mu_0 \otimes \mu_1 \) is given by

\[
\frac{d\gamma_0}{d(\mu_0 \otimes \mu_1)}(x, y) = \alpha^c(x)\beta^c(y) \exp\left(-\frac{||x-y||^2}{\epsilon}\right),
\]

where the functions \( \alpha^c \) and \( \beta^c \) take the form

\[
\alpha^c(x) = \exp\left((x - m_0, A(x - m_0)) + \frac{2}{\epsilon}(x - m_0, m_0 - m_1) + a\right),
\]

\[
\beta^c(y) = \exp\left((y - m_1, B(y - m_1)) + \frac{2}{\epsilon}(y - m_1, m_1 - m_0) + b\right),
\]

with \( A, B, a, b \) as defined in Eq.(19). Then \( \gamma_0 \sim \mu_0 \otimes \mu_1 \).

Let \( \gamma \in \text{Joint}(\mu_0, \mu_1) \) such that \( \gamma \sim \mu_0 \otimes \mu_1 \). Then \( \gamma \sim \gamma_0 \) and we have

\[
I_c(\gamma) = \mathbb{E}_\gamma||x-y||^2 + \epsilon \text{KL}(\gamma||\mu_0 \otimes \mu_1)
\]

\[
= \int_{\mathcal{H} \times \mathcal{H}} ||x-y||^2 \gamma(x, y) + \epsilon \int_{\mathcal{H} \times \mathcal{H}} \log\left(\frac{d\gamma}{d(\mu_0 \otimes \mu_1)}(x, y)\right) \gamma(x, y)
\]

\[
= \int_{\mathcal{H} \times \mathcal{H}} ||x-y||^2 \gamma(x, y) + \epsilon \int_{\mathcal{H} \times \mathcal{H}} \log\left(\frac{d\gamma_0}{d(\mu_0 \otimes \mu_1)}(x, y)\right) \gamma_0(x, y)
\]

\[
\quad + \epsilon \int_{\mathcal{H}} \log\left(\frac{d\gamma_0}{d(\mu_0 \otimes \mu_1)}(x, y)\right) \gamma_0(x, y)
\]

\[
= \epsilon \text{KL}(\gamma||\gamma_0) + \epsilon \int_{\mathcal{H} \times \mathcal{H}} \log(\alpha^c(x)) \gamma(x, y) + \epsilon \int_{\mathcal{H} \times \mathcal{H}} \log(\beta^c(y)) \gamma(x, y).
\]

By Lemma 11,

\[
\int_{\mathcal{H}} \log(\alpha^c(x)) \gamma(x, y)
\]

\[
= a + \int_{\mathcal{H} \times \mathcal{H}} \left((x - m_0, A(x - m_0)) + \frac{2}{\epsilon}(x - m_0, m_0 - m_1)\right) \gamma(x, y)
\]

\[
= a + \text{Tr}(C_0 A).
\]

\[
\int_{\mathcal{H}} \log(\beta^c(y)) \gamma(x, y)
\]

\[
= b + \int_{\mathcal{H} \times \mathcal{H}} \left((y - m_1, B(y - m_1)) + \frac{2}{\epsilon}(y - m_1, m_1 - m_0) + b\right) \gamma(x, y)
\]

\[
= b + \text{Tr}(C_1 B).
\]
Thus it follows that
\[ I_\epsilon(\gamma) = \epsilon\text{KL}(\gamma||\gamma_0) + \epsilon(a + b) + \epsilon\text{Tr}(C_0 A) + \epsilon\text{Tr}(C_1 B) \]
\[ = \epsilon\text{KL}(\gamma||\gamma_0) + ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2}\text{Tr}(M_{01}') \]
\[ + \frac{\epsilon}{2}\log\det\left(I + \frac{1}{2}M_{01}'\right), \text{ as in the proof of Theorem } 4 \]
\[ \ge ||m_0 - m_1||^2 + \text{Tr}(C_0) + \text{Tr}(C_1) - \frac{\epsilon}{2}\text{Tr}(M_{01}') + \frac{\epsilon}{2}\log\det\left(I + \frac{1}{2}M_{01}'\right). \]
Equality happens if and only if \( \text{KL}(\gamma||\gamma_0) = 0 \iff \gamma = \gamma_0 \). If \( \mu_0', \mu_1' \) are Gaussians, \( \mu_0' = N(m_0, C_0), \mu_1' = N(m_1, C_1) \), then we show in Theorems 3 and 4 that \( \gamma_0 \in \text{Joint}(\mu_0', \mu_1') \), \( \gamma_0 \sim \mu_0' \otimes \mu_1' \) and that \( I_\epsilon(\gamma_0) = \text{OT}_{d_2}'(\mu_0', \mu_1') \).
Thus \( \mu_0' \otimes \mu_1' \sim \mu_0 \otimes \mu_1 \) and
\[ \frac{d(\mu_0' \otimes \mu_1')}{d(\mu_0 \otimes \mu_1)} = \frac{d(\mu_0' \otimes \mu_1')}{d\gamma_0} \frac{d\gamma_0}{d(\mu_0 \otimes \mu_1)} = 1 \iff \mu_0' \otimes \mu_1' = \mu_0 \otimes \mu_1 \]
\[ \iff \mu_0' = \mu_0, \mu_1' = \mu_1. \]
Hence equality happens if and only if \( \mu_0, \mu_1 \) are Gaussians. \( \square \)

9 Entropic 2-Wasserstein barycenter

In this section, we prove Theorem 5 on the convexity of \( \text{OT}_{d_2}' \) and Theorem 12 on the \( \text{OT}_{d_2}' \)-based barycenter. We first recall the concept of the Fréchet derivative on Banach spaces (see e.g. [46]). Let \( V, W \) be Banach spaces and \( \mathcal{L}(V, W) \) be the Banach space of bounded linear maps between \( V \) and \( W \). Assume that \( f : \Omega \to W \) is well-defined, where \( \Omega \) is an open subset of \( V \). Then the map \( f \) is said to be Fréchet differentiable at \( x_0 \in \Omega \) if there exists a bounded linear map \( Df(x_0) : V \to W \) such that
\[ \lim_{h \to 0} \frac{||f(x_0 + h) - f(x_0) - Df(x_0)(h)||_W}{||h||_V} = 0. \]
The map \( Df(x_0) \) is called the Fréchet derivative of \( f \) at \( x_0 \). Let now \( W = \mathbb{R} \). If \( x_0 \) is a local minimizer for \( f \), then necessarily (see e.g. Theorem 1.33 in [72])
\[ Df(x_0) = 0. \] (150)
Furthermore, if \( f \) is Fréchet differentiable, then (Proposition 3.11 in [72]) for \( \Omega \subset V \) open and convex,
\[ f \text{ is strictly convex } \iff f(y) > f(x) + Df(x)(y - x), \forall x, y \in \Omega. \] (151)
Thus \( Df(x_0) = 0 \) implies that \( x_0 \) is the unique global minimizer for \( f \).
If the map \( Df : \Omega \to \mathcal{L}(V, W) \) is differentiable at \( x_0 \), then its Fréchet derivative at \( x_0 \), denoted by \( D^2f(x_0) : V \to \mathcal{L}(V, W) \), is called the second order derivative.
of $f$ at $x_0$. The bounded linear map $D^2 f(x_0) \in \mathcal{L}(\mathcal{L}(V, V), W))$, can be identified with a bounded bilinear map from $V \times V \to W$, via

$$D^2 f(x_0)(x, y) = (D^2 f(x_0)(x))(y), \quad x, y \in V.$$  

Under this identification, $D^2 f(x_0)$ is a symmetric, continuous bilinear map from $V \times V \to W$, so that $D^2 f(x_0)(x, y) = D^2 f(x_0)(y, x)$ $\forall x, y \in V$. If $f$ is twice differentiable on $\Omega$, then in addition to (151),

$$f \text{ is convex } \iff D^2 f(x_0)(x, x) \geq 0 \forall x_0 \in \Omega, \forall x \in V; \quad (152)$$

$$D^2 f(x_0)(x, x) > 0 \forall x_0 \in \Omega, \forall x \in V \Rightarrow f \text{ is strictly convex}. \quad (153)$$

In the following, we focus on the Banach space $\mathcal{L}(\mathcal{H})$ of bounded operators, the subspace $\text{Sym}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ of self-adjoint bounded operators, and the Banach space $\text{Tr}(\mathcal{H})$ of trace class operators on $\mathcal{H}$, respectively.

The following two results are straightforward.

**Lemma 13** Let $A, B \in \mathcal{L}(\mathcal{H})$ be fixed and consider the function $f : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ defined by $f(X) = AXB$. Then

$$Df(X_0)(X) = AXB, \quad X_0, X \in \mathcal{L}(\mathcal{H}). \quad (154)$$

**Lemma 14** For the function $\text{Tr} : \text{Tr}(\mathcal{H}) \to \mathbb{R}$,

$$D\text{Tr}(X_0)(X) = \text{Tr}(X), \quad X_0, X \in \text{Tr}(\mathcal{H}). \quad (155)$$

The following are special cases of Lemmas 3 and 4 in [58], respectively.

**Lemma 15** Let $\Omega = \{A \in \text{Tr}(\mathcal{H}) : I + A \text{ is invertible}\}$. Let $f : \Omega \to \mathbb{R}$ be defined by $f(X) = \det(I + X)$. Then $Df(X_0) : \text{Tr}(\mathcal{H}) \to \mathbb{R}$, $X_0 \in \Omega$, is given by

$$Df(X_0)(X) = \det(I + X_0)^{-1} \text{Tr}[(I + X_0)^{-1} X], \quad X \in \text{Tr}(\mathcal{H}). \quad (156)$$

**Lemma 16** Let $\Omega = \{A \in \text{Tr}(\mathcal{H}) : I + A > 0\}$. Let $f : \Omega \to \mathbb{R}$ be defined by $f(X) = \log \det(I + X)$. Then $Df(X_0) : \text{Tr}(\mathcal{H}) \to \mathbb{R}$, $X_0 \in \Omega$, is given by

$$Df(X_0)(X) = \text{Tr}[(I + X_0)^{-1} X], \quad X \in \text{Tr}(\mathcal{H}). \quad (157)$$

**Lemma 17** Let $\text{sqrt} : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H})$ be defined by $\text{sqrt}(X) = X^{1/2}$. Let $\Omega \subset \text{Sym}^+(\mathcal{H})$ be an open subset. The derivative $D\text{sqrt}(X_0) : \text{Sym}(\mathcal{H}) \to \text{Sym}(\mathcal{H})$, $X_0 \in \Omega \subset \text{Sym}^+(\mathcal{H})$, is given by

$$(X_0)^{1/2} D\text{sqrt}(X_0)(X) + D\text{sqrt}(X_0)(X)(X_0)^{1/2} = X, \quad X \in \text{Sym}(\mathcal{H}). \quad (158)$$

If, furthermore, $X_0$ is invertible and $X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$, then

$$\text{Tr}[D\text{sqrt}(X_0)(X)] = \frac{1}{2} \text{Tr}[(X_0)^{-1/2} X]. \quad (159)$$
In general, let $f: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be such that $f(X_0)$ and $X_0$ commute, then
\[
\text{Tr}[f(X_0)D\sqrt{(X_0)}(X)] = \frac{1}{2}\text{Tr}[(X_0)^{-1/2}f(X_0)X]. \tag{160}
\]

Proof i) For the function $sq(X) = X^2$, we have
\[
Dsq(X_0)(X) = X_0X + XX_0. \tag{161}
\]
Let $f(X) = X = sq(\sqrt{X})$, then by the chain rule
\[
Df(X_0)(X) = X = Dsq(\sqrt{X_0}) \circ D\sqrt{X_0}(X)
= (X_0)^{1/2}D\sqrt{X_0}(X) + D\sqrt{X_0}(X)(X_0)^{1/2}.
\]
This gives us the first identity.

ii) For the second identity, we note that
\[
(X_0)^{1/2}D\sqrt{X_0}(X) + D\sqrt{X_0}(X)(X_0)^{1/2} = X
\Leftrightarrow D\sqrt{X_0}(X) + (X_0)^{-1/2}D\sqrt{X_0}(X)(X_0)^{1/2} = (X_0)^{-1/2}X.
\]
Taking the trace operation on both sides gives
\[
2\text{Tr}D\sqrt{X_0}(X) = \text{Tr}[(X_0)^{-1/2}X]
\]
which gives us the desired expression.

iii) For the third expression,
\[
f(X_0)(X_0)^{1/2}D\sqrt{X_0}(X) + f(X_0)D\sqrt{X_0}(X)(X_0)^{1/2} = f(X_0)X.
\]
Since $f(X_0)$ and $X_0^{1/2}$ commute, this is the same as
\[
(X_0)^{1/2}f(X_0)D\sqrt{X_0}(X) + f(X_0)D\sqrt{X_0}(X)(X_0)^{1/2} = f(X_0)X
\Leftrightarrow f(X_0)D\sqrt{X_0}(X) + (X_0)^{-1/2}f(X_0)D\sqrt{X_0}(X)(X_0)^{1/2} = (X_0)^{-1/2}f(X_0)X.
\]
Taking trace on both sides gives
\[
\text{Tr}[f(X_0)D\sqrt{X_0}(X)] = \frac{1}{2}\text{Tr}[(X_0)^{-1/2}f(X_0)X].
\]

Lemma 18 Let $C \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$ be fixed. Let $\Omega = \{X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) : I + c^2C^{1/2}XC^{1/2} > 0\}$, $c \in \mathbb{R}$. Let $f : \Omega \to \mathbb{R}$ be defined by $f(X) = \text{Tr}\left[-I + (I + c^2C^{1/2}XC^{1/2})^{1/2}\right]$. Then $Df(X_0) : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$, $X_0 \in \text{Sym}^+ \cap \text{Tr}(\mathcal{H})$, is given by
\[
Df(X_0)(X) = \frac{c^2}{2}\text{Tr}\left[C^{1/2}\left(I + c^2C^{1/2}X_0C^{1/2}\right)^{-1/2}C^{1/2}X\right]. \tag{162}
\]
Proof Let $g(X) = (I + c^2C^{1/2}XC^{1/2})^{1/2}$. $f(X) = \text{Tr}[-I + g(X)]$, then

$$Df(X_0)(X) = Df(g(X_0)) \circ Dg(X_0)(X) = \text{Tr}[Dg(X_0)(X)]$$

$$= c^2 \text{Tr} \left[ D\sqrt{\text{tr}}(I + c^2C^{1/2}X_0C^{1/2})(C^{1/2}XC^{1/2}) \right].$$

By Lemma 17,

$$\text{Tr} \left[ D\sqrt{\text{tr}}(I + c^2C^{1/2}X_0C^{1/2})(C^{1/2}XC^{1/2}) \right] = \frac{1}{2} \text{Tr} \left[ (I + c^2C^{1/2}X_0C^{1/2})^{-1/2} C^{1/2}XC^{1/2} \right].$$

It thus follows that

$$Df(X_0)(X) = \frac{c^2}{2} \text{Tr} \left[ \left( I + c^2C^{1/2}X_0C^{1/2} \right)^{-1/2} (C^{1/2}XC^{1/2}) \right]$$

$$= \frac{c^2}{2} \text{Tr} \left[ C^{1/2} \left( I + c^2C^{1/2}X_0C^{1/2} \right)^{-1/2} C^{1/2}X \right].$$

Lemma 19 Let $C \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$ be fixed. Let $\Omega = \{X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) : I + c^2C^{1/2}XC^{1/2} > 0\}$, $c \in \mathbb{R}$. Let $f: \Omega \rightarrow \mathbb{R}$ be defined by $f(X) = \log \det \left[ \frac{1}{2}I + \frac{1}{2} (I + c^2C^{1/2}XC^{1/2})^{1/2} \right]$. Then $Df(X_0) : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \rightarrow \mathbb{R}$, $X_0 \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$, is given by

$$Df(X_0)(X) = \frac{c^2}{2} \text{Tr} \left[ C^{1/2} \left( I + c^2C^{1/2}X_0C^{1/2} \right)^{1/2} + \left( I + c^2C^{1/2}X_0C^{1/2} \right)^{-1} C^{1/2}X \right].$$

(163)

Proof Let $g(X) = -\frac{1}{2}I + \frac{1}{2} (I + c^2C^{1/2}XC^{1/2})^{1/2}$, then $f(X) = \log \det [I + g(X)]$. By Lemma 16,

$$Df(X_0)(X) = Df(g(X_0)) \circ Dg(X_0)(X) = \text{Tr}[(I + g(X_0))^{-1} Dg(X_0)(X)].$$

As in the proof of Lemma 18,

$$Dg(X_0)(X) = \frac{c^2}{2} D\sqrt{\text{tr}} \left( I + c^2C^{1/2}X_0C^{1/2} \right) (C^{1/2}XC^{1/2}).$$

Thus it follows from Lemma 17 that

$$Df(X_0)(X) = \frac{c^2}{2} \text{Tr} \left[ \left( I + c^2C^{1/2}X_0C^{1/2} \right)^{1/2} \right]^{-1} D\sqrt{\text{tr}} \left( I + c^2C^{1/2}X_0C^{1/2} \right) (C^{1/2}XC^{1/2})$$

$$= \frac{c^2}{4} \text{Tr} \left[ \left( I + c^2C^{1/2}X_0C^{1/2} \right)^{-1/2} \left( I + c^2C^{1/2}X_0C^{1/2} \right)^{1/2} \right]^{-1} (C^{1/2}XC^{1/2})$$

$$= \frac{c^2}{2} \text{Tr} \left[ C^{1/2} \left( I + c^2C^{1/2}X_0C^{1/2} \right)^{1/2} + \left( I + c^2C^{1/2}X_0C^{1/2} \right)^{-1} C^{1/2}X \right].$$
Lemma 20 Let $A \in \mathcal{L}(\mathcal{H})$. Then

$$\text{Tr}[AX] = 0 \quad \forall X \in \text{Tr}(\mathcal{H}) \iff A = 0. \quad (164)$$

If, furthermore, $A \in \text{Sym}(\mathcal{H})$, then

$$\text{Tr}[AX] = 0 \quad \forall X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \iff A = 0. \quad (165)$$

Proof The first statement follows from the fact that $\mathcal{L}(\mathcal{H}) = [\text{Tr}(\mathcal{H})]^*$, that is the map $A \to \text{Tr}(A)$ is an isometric isomorphism between $\mathcal{L}(\mathcal{H})$ and the dual space $[\text{Tr}(\mathcal{H})]^*$ of $\text{Tr}(\mathcal{H})$ (see e.g. Theorem VI.26 in [76]).

For the second statement, let $Y \in \text{Tr}(\mathcal{H})$ be arbitrary, then

$$\text{Tr}[AY] = \frac{1}{2}[\text{Tr}(AY) + \text{Tr}((AY)^*)] = \frac{1}{2}[\text{Tr}(AY) + \text{Tr}(Y^*A)] = \frac{1}{2}[\text{Tr}(AY) + \text{Tr}(AY^*)] = \frac{1}{2}\text{Tr}[A + Y^*] = 0.$$

Since $Y$ is arbitrary, this implies $A = 0$ by the first statement.

Lemma 21 Let $\Omega = \{X \in \mathcal{L}(\mathcal{H}) : I + X \text{ invertible}\}$. For the map $f : \Omega \to \Omega$ defined by $f(X) = (I + X)^{-1}$, $DF(X_0) : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is given by

$$DF(X_0)(X) = -(I + X_0)^{-1}X(I + X_0)^{-1}, \quad X_0 \in \Omega, X \in \mathcal{L}(\mathcal{H}). \quad (166)$$

Proof Using the identity $A^{-1} - B^{-1} = -A^{-1}(A - B)B^{-1}$, we have

$$(I + X_0 + tX)^{-1} - (I + X_0)^{-1} = -t(I + X_0 + tX)^{-1}X(I + X_0)^{-1},$$

$$(I + X_0 + tX)^{-1} - (I + X_0)^{-1} + t(I + X_0)^{-1}X(I + X_0)^{-1}$$

$$= -t[(I + X_0 + tX)^{-1} - (I + X_0)^{-1}]X(I + X_0)^{-1}$$

$$= t^2(I + X_0 + tX)^{-1}X(I + X_0)^{-1}X(I + X_0)^{-1}.$$ 

Thus $\lim_{t \to 0} \frac{||f(X_0 + tX) - f(X_0) - Df(X_0)(tX)||}{|t||X||} = 0.$

Lemma 22 For the map $f : \text{Tr}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \mathbb{R}$ defined by $f(X)(Y) = \text{Tr}[XY]$, the Fréchet derivative $DF(X_0) : \text{Tr}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \mathbb{R}$ is given by

$$[DF(X_0)(X)](Y) = \text{Tr}[XY], \quad X \in \text{Tr}(\mathcal{H}), Y \in \mathcal{L}(\mathcal{H}). \quad (167)$$

Proof For $g \in W = \mathcal{L}(\mathcal{H}), \mathbb{R}$, $g : \mathcal{L}(\mathcal{H}) \to \mathbb{R}$, we have $||g||_W = \sup_{||X|| \leq 1} |g(X)|$. Let $V = \text{Tr}(\mathcal{H})$, then with $f : V \to W$,

$$\lim_{t \to 0} \frac{||f(X_0 + tX) - f(X_0) - Df(X_0)(tX)||}{|t||X||} = \lim_{t \to 0} \sup_{Y \in \mathcal{L}(\mathcal{H}), ||Y|| \leq 1} \frac{|f(X_0 + tX)(Y) - f(X_0)(Y) - Df(X_0)(tX)(Y)|}{|t||X||}$$

$$= \lim_{t \to 0} \sup_{Y \in \mathcal{L}(\mathcal{H}), ||Y|| \leq 1} \frac{|\text{Tr}(X_0 + tX)Y - \text{Tr}(X_0Y) - t\text{Tr}(XY)|}{|t||X||} = 0.$$
We remark that the condition that $B$ is generally false even if $B, C$ commute.

Assume further that $B$ and $C$ commute. Then

\[ \text{Tr}[C(AB)^2] = ||C^{1/2}B^{1/2}AB^{1/2}||^2_{\text{HS}} \geq 0. \]  
(168)

If in addition $B, C$ are invertible, then equality happens if and only if $A = 0$.

We remark that the condition that $B$ and $C$ commute is crucial in Lemma 23. It can be verified numerically that, without this condition, the stated inequality is generally false even if $A$ is also positive.

**Proof** By the assumption that $B$ and $C$ commute,

\[ \text{Tr}[C(AB)^2] = \text{Tr}[CABAB] = \text{Tr}[C^{1/2}AB^{1/2}B^{1/2}AB^{1/2}C^{1/2}B^{1/2}] \]
\[ = \text{Tr}[C^{1/2}B^{1/2}AB^{1/2}B^{1/2}AB^{1/2}C^{1/2}] = ||C^{1/2}B^{1/2}AB^{1/2}||^2_{\text{HS}} \geq 0. \]

Equality happens if and only if $C^{1/2}B^{1/2}AB^{1/2} = 0$. If $B$ and $C$ are invertible, then this happens if and only if $A = 0$. \(\Box\)

**Lemma 24** Let $Y_0, Z_0 \in \text{Sym}^+(\mathcal{H})$. Assume that $Y_0$ and $Z_0$ commute, then $\forall X \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$,

\[ \text{Tr}[\sqrt{Z_0}(X)Y_0XY_0] = 2||Z_0^{1/4}Y_0^{1/2}\sqrt{\text{Dsqrt}(Z_0)(X)}Y_0^{1/2}||^2_{\text{HS}} \geq 0. \]  
(169)

If in addition $Y_0, Z_0$ are invertible, then equality happens if and only if $X = 0$.

**Proof** By Lemma 17, for any $X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$,

\[ Z_0^{1/2}\sqrt{\text{Dsqrt}(Z_0)(X)}Y_0^{1/2} = X. \]  
(170)

Pre- and post-multiplying both sides by $\sqrt{\text{Dsqrt}(Z_0)(X)}Y_0$ and $Y_0$, respectively,

\[ \sqrt{\text{Dsqrt}(Z_0)(X)}Y_0Z_0^{1/2}\sqrt{\text{Dsqrt}(Z_0)(X)}Y_0 + D\sqrt{\text{Dsqrt}(Z_0)(X)}Y_0^2D\sqrt{\text{Dsqrt}(Z_0)(X)}Z_0^{1/2}Y_0 = D\sqrt{\text{Dsqrt}(Z_0)(X)}Y_0XY_0. \]

Taking trace on both sides and applying Lemma 23 gives

\[ \text{Tr}[\sqrt{\text{Dsqrt}(Z_0)(X)}Y_0XY_0] = \text{Tr} \left( Z_0^{1/2} \text{Dsqrt}(Z_0)(X)Y_0^2 + Z_0^{1/2} [\text{Dsqrt}(Z_0)(X)]^2 \right) \]
\[ = 2\text{Tr} \left( Z_0^{1/2} \text{Dsqrt}(Z_0)(X)Y_0^2 \right) = 2||Z_0^{1/4}Y_0^{1/2}\sqrt{\text{Dsqrt}(Z_0)(X)}Y_0^{1/2}||^2_{\text{HS}} \geq 0. \]

If $Y_0, Z_0$ are invertible, by Lemma 23, the zero equality happens if and only if $\text{Dsqrt}(Z_0)(X) = 0$, which is equivalent to $X = 0$ by Eq.(170) \(\Box\)
Lemma 25. Let $C \in \text{Sym}^{+}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$. Let $\Omega = \{X \in \text{Sym}(\mathcal{H}) : I + c^2C^{1/2}XC^{1/2} > 0\}$, $c \in \mathbb{R}$, $c \neq 0$. Define the map $f : \Omega \to \mathcal{L}(\mathcal{H}, \mathbb{R})$, by
\[
f(X)(Y) = \text{Tr} \left[ C^{1/2} \left( I + (I + c^2C^{1/2}XC^{1/2})^{1/2} \right)^{-1} C^{1/2}Y \right].
\] (171)

The Fréchet derivative $Df(X_0) : \text{Sym}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}, \mathbb{R})$ is given by
\[
[Df(X_0)](Y) = \text{Tr}[\text{Dsqr}(Z_0)(C^{1/2}XC^{1/2})(I + Z_0^{1/2})^{-1}C^{1/2}YC^{1/2}(I + Z_0^{1/2})^{-1}],
\] (172)
where $Z_0 = I + c^2C^{1/2}X_0C^{1/2}$. In particular, for $Y = X$,
\[
[Df(X_0)](X) = -2c^2 \|Z_0^{1/4}(I + Z_0^{1/2})^{-1/2}\|_{\text{HS}} \|Z_0^{1/2}\|_{\text{HS}} \leq 0.
\] (173)

For $c \neq 0$, equality happens if and only $C^{1/2}XC^{1/2} = 0$. If $C$ is strictly positive, then equality happens if and only if $X = 0$.

Proof. Let $h(X) = (I + h(X))^{-1}$, and $f(X)(Y) = \text{Tr}[C^{1/2}g(X)C^{1/2}Y]$. By the chain rule and Lemma 21,
\[
Dg(X_0)(X) = [Dg(h(X_0)) \circ Dh(X_0)](X)
= -c^2(I + h(X_0))^{-1}\text{Dsqr}(I + c^2C^{1/2}X_0C^{1/2})(I + h(X_0))^{-1}.
\]

Let $Z_0 = I + c^2C^{1/2}X_0C^{1/2}$ and $h(X) = Z_0^{1/2}$. By Lemma 22,
\[
[Df(X_0)](Y) = [Df(g(X_0)) \circ Dg(X_0)](X)(Y) = \text{Tr}[C^{1/2}Dg(X_0)(X)C^{1/2}Y]
= -c^2\text{Tr}[C^{1/2}(I + h(X_0))^{-1}\text{Dsqr}(I + c^2C^{1/2}X_0C^{1/2})(I + h(X_0))^{-1}C^{1/2}Y]
= -c^2\text{Tr}[\text{Dsqr}(Z_0)(C^{1/2}XC^{1/2})(I + Z_0^{1/2})^{-1}C^{1/2}YC^{1/2}(I + Z_0^{1/2})^{-1}].
\]

In particular, for $Y = X$, by Lemma 24,
\[
[Df(X_0)](X) = -c^2\text{Tr}[\text{Dsqr}(Z_0)(C^{1/2}XC^{1/2})(I + Z_0^{1/2})^{-1}C^{1/2}XC^{1/2}(I + Z_0^{1/2})^{-1}]
= -2c^2 \|Z_0^{1/4}(I + Z_0^{1/2})^{-1/2}\|_{\text{HS}} \|Z_0^{1/2}\|_{\text{HS}} \leq 0.
\]

Since $Z_0$ and $(I + Z_0^{1/2})$ are invertible, equality happens if and only if $C^{1/2}XC^{1/2} = 0$. If $C$ is strictly positive, then $C^{1/2}XC^{1/2} = 0 \iff X = 0$.

\[\square\]

Lemma 26. Let $A, B \in \mathcal{L}(\mathcal{H})$. Assume that $B$ is compact, self-adjoint, and $B > 0$. Then
\[
AB = 0 \iff A = 0,
BA = 0 \iff A = 0.
\] (174)
Proof Let \( \{\lambda_k\}_{k \in \mathbb{N}} \) be the eigenvalues of \( B \), \( \lambda_k > 0 \forall k \in \mathbb{N} \), with corresponding orthonormal eigenvectors \( \{e_k\}_{k \in \mathbb{N}} \), forming an orthonormal basis in \( \mathcal{H} \). Then
\[
0 = ABCe_k = \lambda_k Ae_k = 0 \quad \forall k \in \mathcal{N} \Rightarrow Ax = 0 \quad \forall x \in \mathcal{H} \Rightarrow A = 0.
\]

The second expression then follows by via the adjoint operation. \( \square \)

**Proposition 6** Let \( C \in \text{Sym}^+ \cap \text{Tr}(\mathcal{H}) \) be fixed. Let \( \Omega = \{X \in \text{Sym}(\mathcal{H}) : I + c^2C^{1/2}XC^{1/2} > 0\}, \ c \in \mathbb{R}, \ c \neq 0 \). Let \( f : \Omega \to \mathbb{R} \) be defined by
\[
f(X) = \log \det \left( \frac{1}{2} I + \frac{1}{2} (I + c^2C^{1/2}XC^{1/2})^{1/2} \right) - \text{Tr} \left[ -I + (I + c^2C^{1/2}XC^{1/2})^{1/2} \right]. \tag{175}
\]

Then \( f \) is convex on \( \Omega \). Furthermore, \( f \) is strictly convex if \( C \) is strictly positive.

Proof For any \( X_0 \in \Omega, \ X \in \text{Sym(\mathcal{H})} \), by Lemmas 19 and 18,
\[
Df(X_0)(X) = \frac{c^2}{2} \text{Tr} \left[ C^{1/2} \left( (I + c^2C^{1/2}X_0C^{1/2})^{1/2} + (I + c^2C^{1/2}X_0C^{1/2})^{-1}C^{1/2} \right) \right] \\
- \frac{c^2}{2} \text{Tr} \left[ C^{1/2} \left( (I + c^2C^{1/2}X_0C^{1/2})^{-1/2}C^{1/2} \right) \right] = -\frac{c^2}{2} \text{Tr} \left[ C^{1/2} \left( (I + c^2C^{1/2}X_0C^{1/2})^{1/2} \right) C^{1/2} \right].
\]

Thus we have the map \( Df : \Omega \to \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathbb{R}) \), with
\[
Df(X)(Y) = -\frac{c^2}{2} \text{Tr} \left[ C^{1/2} \left( (I + c^2C^{1/2}X^{1/2})^{1/2} \right)^{-1} C^{1/2} Y \right].
\]

Differentiating this map gives the second-order Fréchet derivative
\[
[D^2f(X_0)](X,Y) = [D^2f(X_0)](Y) \quad \text{and} \quad [D^2f(X_0)](X,Y) = [D^2f(X_0)](Y)\]
\[
= \frac{c^4}{4} \text{Tr} \left[ \text{Tr}(Z_0)(C^{1/2}X^{1/2})(I + Z_0^{1/2})^{-1}C^{1/2}Y C^{1/2}(I + Z_0^{1/2})^{-1} \right],
\]
where \( Z_0 = I + c^2C^{1/2}X_0C^{1/2} \), by Lemma 25. In particular, for \( Y = X \),
\[
[D^2f(X_0)](X,X) = [Df(X_0)](X) \quad \text{and} \quad [D^2f(X_0)](X,X) = [Df(X_0)](X)\]
\[
= c^4 \| Z_0^{1/4} \| (I + Z_0^{1/2})^{-1/2} \text{Tr}(Z_0)(C^{1/2}X^{1/2})(I + Z_0^{1/2})^{-1/2} \|_{\text{HS}}^2 \geq 0.
\]

Equality happens if and only if \( C^{1/2}X^{1/2} = 0 \). If \( C \) is strictly positive, then equality happens if and only if \( X = 0 \) by Lemma 26. Thus \( f \) is convex on \( \Omega \) and furthermore, it is strictly convex if \( C \) is strictly positive. \( \square \)
Proof (of Theorem 5 - Convexity of entropic Wasserstein distance)

By the strict convexity of the square Hilbert norm \(|x|^2\), the function \(m \mapsto |m - m_0|^2\) is strictly convex in \(m\). For the covariance part, by Theorem 4,

\[
F(X) = \text{OT}_{\text{de}}(\mathcal{N}(0, C_0), \mathcal{N}(0, X)) = \text{Tr}(X) + \text{Tr}(C_0) - \frac{\epsilon}{2} \text{Tr} \left[ -I + \left( I + \frac{16}{\epsilon^2} C_0^{1/2} X C_0^{1/2} \right)^{1/2} \right] + \frac{\epsilon}{2} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} X C_0^{1/2} \right)^{1/2} \right).
\]

Since \(\text{Tr}(X)\) is linear in \(X\) and the remaining part is convex in \(\text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})\) by Proposition 6, \(F\) is convex in \(X \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})\), with strict convexity if \(C_0\) is strictly positive.

\[\square\]

Lemma 27 Assume that \(\sum_{i=1}^N w_i \sigma_i^2 > 0\). For a fixed \(\epsilon > 0\), define the function \(f : [0, \infty) \to \mathbb{R}\) by \(f(x) = \sum_{i=1}^N w_i f_i(x)\), where for \(1 \leq i \leq N\), \(c_i = \frac{\epsilon}{4}\),

\[
f_i(x) = \sigma_i^2 + x - \frac{\epsilon}{2} \left[ -1 + \left( 1 + c_i^2 \sigma_i^2 x \right)^{1/2} \right] + \frac{\epsilon}{2} \log \left[ 1 + \frac{1}{2} \left( 1 + c_i^2 \sigma_i^2 x \right)^{1/2} \right],
\]

(176)

Then \(f\) is strictly convex. For the minimum of \(f\), there are two scenarios

1. \(\epsilon \geq 2 \sum_{i=1}^N w_i \sigma_i^2\): in this case \(\min f = f(0) = \sum_{i=1}^N w_i \sigma_i^2\).

2. \(0 < \epsilon < 2 \sum_{i=1}^N w_i \sigma_i^2\): in this case \(\min f = f(x^*)\), where \(x^* > 0\) is the unique solution of the following equation

\[
\sum_{i=1}^N w_i \sigma_i^2 \left[ 1 + \left( 1 + c_i^2 \sigma_i^2 x \right)^{1/2} \right]^{-1} = \frac{\epsilon}{4}.
\]

(177)

Equivalently, \(x^*\) is the unique positive solution of the equation

\[
x = \frac{\epsilon}{4} \sum_{i=1}^N w_i \left[ -1 + \left( 1 + c_i^2 \sigma_i^2 x \right)^{1/2} \right],
\]

(178)

which also has the solution \(x_0 = 0\).

Proof We have \(f'(x) = 1 - \frac{\epsilon}{4} \sum_{i=1}^N w_i \sigma_i^2 \left[ 1 + \left( 1 + c_i^2 \sigma_i^2 x \right)^{1/2} \right]^{-1}\), \(f''(x) = \frac{\epsilon}{16} \sum_{i=1}^N w_i \sigma_i^4 \left[ 1 + \left( 1 + c_i^2 \sigma_i^2 x \right)^{1/2} \right]^{-2} \left( 1 + c_i^2 \sigma_i^2 x \right)^{-1/2} > 0 \forall x \geq 0\). Thus \(f'(x)\) is a strictly increasing function on \([0, \infty)\). Furthermore, \(f'(0) = 1 - \frac{\epsilon}{4} \sum_{i=1}^N w_i \sigma_i^2\), \(\lim_{x \to \infty} f'(x) = 1\). We have the following three scenarios

1. \(\epsilon > 2 \sum_{i=1}^N w_i \sigma_i^2\). In this case \(f'(0) > 0\) and thus \(f'(x) > 0 \forall x > 0\) and thus the minimum for \(f\) on \([0, \infty)\) is \(f(0) = \sum_{i=1}^N w_i \sigma_i^2\).
2. \( \epsilon = 2 \sum_{i=1}^{N} w_i \sigma_i^2 \). In this case \( f'(0) = 0, f'(x) > 0 \forall x > 0 \) and thus the minimum for \( f \) on \([0, \infty)\) is \( f(0) = \sum_{i=1}^{N} w_i \sigma_i^2 \).

3. \( 0 < \epsilon < 2 \sum_{i=1}^{N} w_i \sigma_i^2 \). In this case \( f'(0) < 0 \) and thus there exists a unique \( x^* > 0 \) such that \( f'(x^*) = 0 \). This is the unique global minimizer of \( f \) and

\[
\begin{align*}
\frac{d}{dx} f(x^*) &= 0 \iff \sum_{i=1}^{N} w_i \sigma_i^2 \left[ 1 + (1 + c_i^2 \sigma_i^2 x^*)^{1/2} \right]^{-2} c_i x^* \\
&= \epsilon \left[ 1 + (1 + c_i^2 \sigma_i^2 x^*)^{1/2} \right]^{-1} = \epsilon \left( 1 + (1 + c_i^2 \sigma_i^2 x^*)^{1/2} \right)
\end{align*}
\]

which can be verified via the identity \(-1 + (1 + a^2)^{1/2} = a[1 + (1 + a^2)^{1/2}]^{-1}\).

We note the last equation also has the solution \( x = 0 \). \( \square \)

**Proof (of Theorem 12 - Infinite-dimensional setting)** As a direct consequence of Theorem 6, the entropic barycenter of a set of Gaussian measures in \( \mathcal{P}_2(\mathcal{H}) \), if it exists, must be another Gaussian measure on \( \mathcal{H} \). By Theorem 4, \( \text{OT}_{\mathcal{H}}^\varepsilon(\mathcal{N}(m_0, C_0), \mathcal{N}(m_1, C_1)) \) decomposes into the squared Euclidean distance \( \|m_0 - m_1\|^2 \) and the distance \( \text{OT}_{\mathcal{H}}^\varepsilon(\mathcal{N}(0, C_0), \mathcal{N}(0, C_1)) \). It follows that we can compute the barycentric mean and covariance operator separately.

The barycentric mean is obviously that the Euclidean mean \( \bar{m} = \sum_{i=1}^{N} w_i m_i \).

Consider now the centered Gaussian measures \( \{ \mathcal{N}(0, C_i) \}_{i=1}^{N} \). We define the following functions \( F, F_i : \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}, 1 \leq i \leq N \),

\[
F(C) = \sum_{i=1}^{N} w_i F_i(C),
\]

\[
F_i(C) = \text{OT}_{\mathcal{H}}^\varepsilon(\mathcal{N}(0, C), \mathcal{N}(0, C_i))
\]

\[
= \text{Tr}(C) + \text{Tr}(C_i) - \frac{\epsilon}{2} \text{Tr}(M_{01}) + \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2} M_{01} \right)
\]

\[
= \text{Tr}(C) + \text{Tr}(C_i) - \frac{\epsilon}{2} \text{Tr} \left( -I + \left( I + \frac{16}{\epsilon^2} C_i^{1/2} C_i^{1/2} \right)^{1/2} \right)
\]

\[
+ \frac{\epsilon}{2} \log \det \left( I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_i^{1/2} C_i^{1/2} \right)^{1/2} \right).
\]

Each function \( F_i \) is well-defined on the larger, open, convex set \( \Omega_i = \{ X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) : I + c_i^2 C_i^{1/2} X C_i^{1/2} > 0 \} \), \( c_i = \frac{\epsilon}{2} \). If \( C_i \) is strictly positive, then \( F_i \) is strictly convex by Theorem 5. Then \( F \) is well-defined on the open, convex set \( \Omega = \cap_{i=1}^{N} \Omega_i = \{ X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) : I + c_i^2 C_i^{1/2} X C_i^{1/2} > 0, i = 1, \ldots, N \} \). On \( \Omega, F \) is Fréchet differentiable. Since we assume that at least one of the \( C_i \)'s is strictly positive, \( F \) is strictly convex. Thus a minimizer \( X_0 \in \Omega \) of \( F \) must necessarily be unique and satisfy \( DF(X_0) = 0 \).
Combining Lemmas 14, 18, and 19, we obtain the Fréchet derivative $DF_i(X_0) : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$, $X_0 \in \Omega$, $X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$, as follows. With $c_i = \frac{4}{\epsilon}$,

$$
\begin{align*}
DF_i(X_0)(X) &= \text{Tr}(X) - \frac{4}{\epsilon} \text{Tr} \left[ C_i^{1/2} \left( I + c_i^2 C_i^{1/2} X_0 C_i^{1/2} \right)^{-1} C_i^{1/2} X \right] \\
&\quad + \frac{4}{\epsilon} \text{Tr} \left[ C_i^{1/2} \left( \left( I + c_i^2 C_i^{1/2} X_0 C_i^{1/2} \right)^{1/2} + \left( I + c_i^2 C_i^{1/2} X_0 C_i^{1/2} \right) \right)^{-1} C_i^{1/2} X \right] \\
&= \text{Tr}(X) - \frac{4}{\epsilon} \text{Tr} \left[ C_i^{1/2} \left( I + \left( I + c_i^2 C_i^{1/2} X_0 C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} X \right].
\end{align*}
$$

Summing over $i, i = 1, \ldots, N$, we obtain

$$
DF(X_0)(X) = \sum_{i=1}^{N} w_i DF_i(X_0)(X) = \sum_{i=1}^{N} w_i \text{Tr} \left[ \left( I - \frac{4}{\epsilon} \left[ C_i^{1/2} \left( I + \left( I + c_i^2 C_i^{1/2} X_0 C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} \right] \right) X \right].
$$

By Lemma 20, $DF(X_0)(X) = 0 \ \forall X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$ if and only if

$$
\sum_{i=1}^{N} w_i \left[ I - \frac{4}{\epsilon} \left[ C_i^{1/2} \left( I + \left( I + c_i^2 C_i^{1/2} X_0 C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} \right] \right] = 0.
$$

\begin{equation}
\sum_{i=1}^{N} w_i \left[ C_i^{1/2} \left( I + \left( I + c_i^2 C_i^{1/2} X_0 C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} \right] = \frac{4}{\epsilon} I. \tag{180}
\end{equation}

When $\text{dim}(\mathcal{H}) = \infty$, this identity is impossible, since the left hand side is a trace class operator, whereas the identity operator is not trace class. Thus the function $F$ does not have a global minimum on the open set $\Omega$.

Consider the possible global minima of $F$ on $\text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \subseteq \Omega$.

(i) Consider the case $\epsilon \geq 2 \sum_{i=1}^{N} w_i C_i$. For any $X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$,

$$
DF(0)(X) = \sum_{i=1}^{N} w_i \text{Tr} \left[ \left( I - \frac{2}{\epsilon} C_i \right) X \right] = \text{Tr} \left[ \left( I - \frac{2}{\epsilon} \sum_{i=1}^{N} w_i C_i \right) X \right].
$$

By Lemma 41, we have

$$
A \succeq 0, B \succeq 0 \implies A^{1/2} BA^{1/2} \succeq 0 \implies \text{Tr}(AB) = \text{Tr}(A^{1/2} BA^{1/2}) \geq 0.
$$

Then $\forall Y \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$, we have by Eq.(151)

$$
F(Y) > F(0) + DF(0)(Y) = F(0) + \text{Tr} \left[ \left( I - \frac{2}{\epsilon} \sum_{i=1}^{N} w_i C_i \right) Y \right] \geq F(0).
$$
Thus $X_0 = 0$ is the unique global minimizer of $F$ in $\text{Sym}^+ (\mathcal{H}) \cap \text{Tr}(\mathcal{H})$.

(ii) Assume now that $\epsilon I \not\preceq 2 \sum_{i=1}^{N} w_i C_i$. We show that $X_0 = 0$ is not a global minimum of $F$ in $\text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$. For any $X_0 \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$,

$$DF(X_0)(X_0) = \sum_{i=1}^{N} w_i \text{Tr} \left[ X_0 - \frac{4}{\epsilon} \left( X_0^{1/2} C_i^{1/2} \left( I + \left( I + c_i^2 X_0^{1/2} C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} X_0^{1/2} \right) \right]$$

$$= \sum_{i=1}^{N} w_i \text{Tr} \left[ X_0 + \frac{\epsilon}{4} I - \frac{\epsilon}{4} \left( I + c_i^2 X_0^{1/2} C_i X_0^{1/2} \right)^{1/2} \right].$$

The assumption $\epsilon I \not\preceq 2 \sum_{i=1}^{N} w_i C_i$ means that $\exists u \in \mathcal{H}, ||u|| = 1$, such that $0 < \epsilon = \epsilon ||u||^2 < 2 \sum_{i=1}^{N} w_i \langle u, C_i u \rangle$. Consider $X_u = x(u \otimes u)$. By Lemma 40,

$$DF(X_u)(X_u) = \sum_{i=1}^{N} w_i \text{Tr} \left[ \left( x + \frac{\epsilon}{4} - \frac{\epsilon}{4} (1 + c_i^2 \langle u, C_i u \rangle)^{1/2} \right) (u \otimes u) \right]$$

$$= x - \frac{\epsilon}{4} \sum_{i=1}^{N} w_i \left[ -1 + (1 + c_i^2 \langle u, C_i u \rangle)^{1/2} \right].$$

According to Lemma 27, the property $0 < \epsilon < 2 \sum_{i=1}^{N} w_i \langle u, C_i u \rangle$ implies that there exists a unique $x^* > 0$ such that

$$x^* = \frac{\epsilon}{4} \sum_{i=1}^{N} w_i \left[ -1 + (1 + c_i^2 x^* \langle u, C_i u \rangle)^{1/2} \right].$$

Thus with $X_u^* = x^* (u \otimes u)$, we have $DF(X_u^*)(X_u^*) = 0$. By Eq.(151),

$$F(0) > F(X_u^*) - DF(X_u^*)(X_u^*) = F(X_u^*).$$

Thus $X_0 = 0$ is not a global minimum of $F$ in $\text{Sym}^+ \cap \text{Tr}(\mathcal{H})$. \hfill $\square$

**Proof (of Theorem 12 - Finite-dimensional setting)** When $\text{dim}(\mathcal{H}) < \infty$, if we impose the additional condition that $\ker(X_0) = \{0\}$, that is $X_0$ is invertible, then the identity (180) is equivalent to

$$\sum_{i=1}^{N} w_i \left[ X_0^{1/2} C_i^{1/2} \left( I + \left( I + c_i^2 X_0^{1/2} C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} X_0^{1/2} \right] = \frac{\epsilon}{4} X_0.$$ 

By Lemma 8, on the left hand side,

$$X_0^{1/2} C_i^{1/2} \left( I + \left( I + c_i^2 X_0^{1/2} C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} X_0^{1/2}$$

$$= -\frac{1}{c_i^2} I + \frac{1}{c_i^2} \left( I + c_i^2 X_0^{1/2} C_i X_0^{1/2} \right)^{1/2}.$$
Substituting into the previous equation, we obtain
\[ X_0 = \frac{\epsilon}{4} \sum_{i=1}^{N} w_i \left[ -I + \left( I + \epsilon^2 X_0^{1/2} C_i X_0^{1/2} \right)^{1/2} \right]. \]

This gives Eq.(42). Define the following map \( \mathcal{F} : \text{Sym}^+(\mathcal{H}) \rightarrow \text{Sym}^+(\mathcal{H}) \) by
\[
\mathcal{F}(X) = \sum_{i=1}^{N} w_i \left[ C_i^{1/2} \left( I + \left( I + c_i^2 C_i^{1/2} X C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} \right],
\]
then we have by the monotonicity of the square root function
\[
\mathcal{F}(0) = \frac{1}{2} \sum_{i=1}^{N} w_i C_i, \quad \mathcal{F}(X) \leq \frac{1}{2} \sum_{i=1}^{N} w_i C_i, \quad \forall X \geq 0.
\]

Thus it is clear that
(i) If \( \epsilon I > 2 \sum_{i=1}^{N} w_i C_i \), then Eq.(39) has no solution \( X_0 \geq 0 \).
(ii) If \( \epsilon I = 2 \sum_{i=1}^{N} w_i C_i \), then Eq.(39) has the solution \( X_0 = 0 \), which is necessarily unique due to the strict convexity of the entropic OT distance.
(iii) The condition \( 0 < \epsilon I < 2 \sum_{i=1}^{N} w_i C_i \) is thus necessary if Eq.(39) is to have a solution \( X_0 \geq 0, X \neq 0 \). By Proposition 9, if \( C_i \geq \alpha I, 1 \leq i \leq N \), and \( 0 < \epsilon < 2\alpha \), then Eq.(39) has a strictly positive solution, which is necessarily unique due to the strict convexity of the entropic OT distance. \( \square \)

**Proposition 7** Define the following map \( \mathcal{G} : \text{Sym}^+(n) \rightarrow \text{Sym}^+(n) \), \( c_i = \frac{1}{c} \),
\[
\mathcal{G}(X) = \frac{\epsilon}{4} \sum_{i=1}^{N} w_i \left[ -I + \left( I + c_i^2 X^{1/2} C_i C_i^{1/2} \right)^{1/2} \right].
\]

1. Suppose \( \exists \alpha \in \mathbb{R}, \alpha > 0 \) such that \( C_i \geq \alpha I, 1 \leq i \leq N \) and \( 0 < \epsilon < 2\alpha \). Then \( \mathcal{G} \) has a strictly positive fixed point, that is Eq.(42) has a strictly positive solution.
2. Let \( u \in \mathbb{R}^n, \|u\| = 1 \). If \( 0 < \epsilon < 2 \sum_{i=1}^{N} w_i \langle u, C_i u \rangle \), then \( X_u = x_u (u \otimes u) = x_u u u^T \) is a fixed point of \( \mathcal{G} \), where \( x_u \) is the unique positive solution of the one-dimensional fixed point equation
\[
x = \frac{\epsilon}{4} \sum_{i=1}^{N} w_i \left[ -1 + \left( 1 + c_i^2 x \langle u, C_i u \rangle \right)^{1/2} \right].
\]

If \( 0 < \epsilon < 2 \sum_{i=1}^{N} w_i C_i \), then for \( \forall u \in \mathcal{H}, \|u\| = 1 \), \( 0 < \epsilon = \epsilon \|u\|^2 < 2 \sum_{i=1}^{N} w_i \langle u, C_i u \rangle \). In particular, if \( C_i \geq \alpha I, 1 \leq i \leq N \), and \( 0 < \epsilon < 2\alpha \), then \( 0 < \epsilon < 2 \sum_{i=1}^{N} w_i \langle u, C_i u \rangle \). Hence under these assumptions, the map \( \mathcal{G} \) as defined in Eq.(183), has uncountably infinitely many fixed points, which are positive but singular.
Proof (of Proposition 7) For the first part, let $\gamma \in \mathbb{R}, \gamma > 0$ be such that $C_i \leq \gamma I, 1 \leq i \leq N$. Let $\beta = \alpha - \frac{\epsilon^2}{2} > 0$ and consider the following set

$$K_\epsilon = \{X \in \text{Sym}^+(n) : \beta I \leq X \leq \gamma I\}. \quad (185)$$

This is a compact, convex set in $\text{Sym}(n)$. By Lemma 41, the operator monotonicity of the square root function, and the inequality $(1 + a^2)^{1/2} \leq 1 + a, a \in \mathbb{R}, a \geq 0$, we have with $c_\epsilon = 4\epsilon$,

$$0 \leq X \leq \gamma I \Rightarrow \mathcal{G}(X) \leq \frac{\epsilon}{4} \left[-I + \left(I + c^2_\epsilon \gamma^2 I\right)^{1/2}\right] \leq \gamma I,$$

$$X \geq \beta I \Rightarrow \mathcal{G}(X) \geq \frac{\epsilon}{4} \left[-I + \left(I + c^2_\epsilon \alpha \left(\alpha - \frac{\epsilon^2}{2}\right) I\right)^{1/2}\right] = \left(\alpha - \frac{\epsilon^2}{2}\right) I = \beta I.$$

Thus the continuous map $\mathcal{G}$ maps the compact convex set $K_\epsilon$ into itself. By Brouwer Fixed Point Theorem, $\mathcal{G}$ has at least a fixed point in $K_\epsilon$. For the second part, by Lemma 40,

$$(u \otimes u)^{1/2} C_i (u \otimes u)^{1/2} = (u \otimes u) C_i (u \otimes u) = \langle u, C_i u \rangle (u \otimes u).$$

Therefore, for $X_u = x(u \otimes u)$,

$$I + c^2_\epsilon X_u^{1/2} C_i X_u^{1/2} = (I - u \otimes u) + \left(1 + c^2_\epsilon x(u, C_i u)\right) (u \otimes u).$$

Since $(I - u \otimes u)^2 = (I - u \otimes u)$ and $(I - u \otimes u)(u \otimes u) = 0$, by Lemma 40,

$$\left(I + c^2_\epsilon X_u^{1/2} C_i X_u^{1/2}\right)^{1/2} = (I - u \otimes u) + \left(1 + c^2_\epsilon x(u, C_i u)\right)^{1/2} (u \otimes u).$$

It follows that

$$\mathcal{G}(X_u) = \frac{\epsilon}{4} \sum_{i=1}^N w_i \left[-1 + \left(1 + c^2_\epsilon x(u, C_i u)\right)^{1/2}\right] (u \otimes u).$$

Thus the fixed point equation $X_u = \mathcal{G}(X_u)$ is equivalent to

$$x = \frac{\epsilon}{4} \sum_{i=1}^N w_i \left[-1 + \left(1 + c^2_\epsilon x(u, C_i u)\right)^{1/2}\right].$$

By the assumption $0 < \epsilon < 2 \sum_{i=1}^N w_i \langle u, C_i u \rangle$, this equation has a unique positive solution by Lemma 27. \qed
10 Sinkhorn barycenter of Gaussian measures

In this section, we prove Theorem 8 on the strict convexity of the Sinkhorn divergence and Theorem 13 on the barycenter of a set of Gaussian measures on $\mathcal{H}$ under the Sinkhorn divergence.

**Strict convexity of Sinkhorn divergence.** We need the following results.

**Lemma 28** Let $f : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$ be defined by $f(X) = \text{Tr} \left[ -I + \left( I + c^2 X^2 \right)^{1/2} \right]$, $c \in \mathbb{R}$. $Df(X_0) : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$, $X_0 \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$, is given by

$$Df(X_0)(X) = c^2 \text{Tr} \left[ \left( I + c^2 X_0^2 \right)^{-1/2} X_0 X \right], \quad X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}).$$

**Proof** Let $g(X) = -I + \left( I + c^2 X^2 \right)^{1/2}$, then $f(X) = \text{Tr} [g(X)]$ and

$$Df(X_0)(X) = Df(g(X_0)) \circ Dg(X_0)(X) = \text{Tr} [Dg(X_0)(X)]$$

by Lemma 17

$$= c^2 \text{Tr} \left[ \text{Dsqrt} \left( I + c^2 X_0^2 \right) (X_0 X + X X_0) \right]$$

$$= \frac{1}{2} c^2 \text{Tr} \left[ \left( I + c^2 X_0^2 \right)^{-1/2} (X_0 X + X X_0) \right],$$

$$= c^2 \text{Tr} \left[ \left( I + c^2 X_0^2 \right)^{-1/2} X_0 X \right].$$

**Lemma 29** Let $f : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$ be defined by $f(X) = \log \det \left[ \frac{1}{2} I + \frac{1}{2} \left( I + c^2 X^2 \right)^{1/2} \right]$. Then $Df(X_0) : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$, $X_0 \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$, is defined by

$$Df(X_0)(X) = c^2 \text{Tr} \left[ \left( I + c^2 X_0^2 \right)^{1/2} + \left( I + c^2 X^2 \right)^{1/2} \right]^{-1} X_0 X.$$  

**Proof** Let $g(X) = -I + \left( I + c^2 X^2 \right)^{1/2}$, then $f(X) = \log \det [I + \frac{1}{2} g(X)]$. By Lemmas 16 and 17,

$$Df(X_0)(X) = \text{Tr} \left[ \left( I + \frac{1}{2} g(X_0) \right)^{-1} Dg(X_0)(X) \right]$$

$$= \frac{1}{2} c^2 \text{Tr} \left[ \left( \frac{1}{2} I + \frac{1}{2} \left( I + c^2 X_0^2 \right)^{1/2} \right)^{-1} \text{Dsqrt} \left( I + c^2 X_0^2 \right) (X_0 X + X X_0) \right]$$

$$= \frac{1}{4} c^2 \text{Tr} \left[ \left( I + c^2 X_0^2 \right)^{-1/2} \left( \frac{1}{2} I + \frac{1}{2} \left( I + c^2 X_0^2 \right)^{1/2} \right)^{-1} (X_0 X + X X_0) \right]$$

$$= c^2 \text{Tr} \left[ \left( I + c^2 X_0^2 \right)^{1/2} + \left( I + c^2 X^2 \right)^{1/2} \right]^{-1} X_0 X.$$  

**Lemma 30** Let $W$ be a Banach algebra and $\Omega \subset W$ be an open subset. Let $g : \Omega \to W$ be Fréchet differentiable at $X_0$. Let $f(X) = g(X) X$ and $h(X) = X g(X)$. Then $f$ and $h$ are Fréchet differentiable at $X_0$, with $Df(X_0)(X) = Dg(X_0)(X) X_0 + g(X_0) X$ and $Dh(X_0)(X) = X_0 Dg(X_0)(X) + X g(X_0)$. 

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Proof By assumption, \( \lim_{t \to 0} \frac{|g(X_0 + tX) - g(X_0) - tDg(X_0)(X)|_W}{|t| |X|_W} = 0. \) Thus

\[
\lim_{t \to 0} \frac{||f(X_0 + tX) - f(X_0) - Df(X_0)(tX)||_W}{|t| |X|_W} = \lim_{t \to 0} \frac{||g(X_0 + tX)(X_0 + tX) - g(X_0)X_0 - tDg(X_0)(X)X_0 - t\theta(X_0)X||}{|t| |X|_W}
\]

\[
\leq \lim_{t \to 0} \frac{||g(X_0 + tX) - g(X_0) - tDg(X_0)(X)||_W}{|t| |X|_W} + \lim_{t \to 0} ||g(X_0 + tX) - g(X_0)||_W = 0.
\]

This proves the formula for \( Df(X_0) \). Here we use the fact the Fréchet Differentiability implies continuity. The proof for \( Dh(X_0) \) is entirely similar. \( \square \)

**Lemma 31** Define the function \( f : \text{Sym}(\mathcal{H}) \to \text{Sym}(\mathcal{H}) \) by \( f(X) = (I + (I + c^2 X^2)^{1/2})^{-1} X \). Then \( Df(X_0) : \text{Sym}(\mathcal{H}) \to \text{Sym}(\mathcal{H}) \) is given by

\[
Df(X_0)(X) = -c^2 (I + h(X_0))^{-1} D\sqrt{(I + c^2 X^2)}(X_0 X + X X_0)(I + h(X_0))^{-1} X_0
\]

\[
+ (I + h(X_0))^{-1} X
\]

\[
= -c^2 X_0 (I + h(X_0))^{-1} D\sqrt{(I + c^2 X^2)}(X_0 X + X X_0)(I + h(X_0))^{-1}
\]

\[
+ X (I + h(X_0))^{-1}.
\]

Here \( X, X_0 \in \text{Sym}(\mathcal{H}) \) and \( h(X_0) = (I + c^2 X^2_0) \).

**Proof** Let \( h(X) = (I + c^2 X^2)^{1/2} \) and \( g(X) = (I + h(X))^{-1} \). By the chain rule,

\[
Dg(X_0)(X) = [Dg(h(X_0)) \circ Dh(X_0)](X)
\]

\[
= - (I + h(X_0))^{-1} Dh(X_0)(X)(I + h(X_0))^{-1}
\]

\[
= -c^2 (I + h(X_0))^{-1} D\sqrt{(I + c^2 X^2)}(X_0 X + X X_0)(I + h(X_0))^{-1}.
\]

By Lemma 30, with \( f(X) = g(X)X \),

\[
Df(X_0)(X) = Dg(X_0)(X)X_0 + g(X_0)X
\]

\[
= -c^2 (I + h(X_0))^{-1} D\sqrt{(I + c^2 X^2)}(X_0 X + X X_0)(I + h(X_0))^{-1} X_0
\]

\[
+ (I + h(X_0))^{-1} X.
\]

Since \( g(X) \) and \( X \) commute, we also have \( f(X) = X g(X) \) and thus

\[
Df(X_0)(X) = X_0 Dg(X_0)(X) + X g(X_0)
\]

\[
= -c^2 X_0 (I + h(X_0))^{-1} D\sqrt{(I + c^2 X^2)}(X_0 X + X X_0)(I + h(X_0))^{-1}
\]

\[
+ X (I + h(X_0))^{-1}.
\]

This gives the second, equivalent, expression for \( Df(X_0)(X) \). \( \square \)
Lemma 32 Let \( f : \text{Sym}(H) \cap \text{Tr}(H) \to \mathcal{L}(\mathcal{L}(H), \mathbb{R}) \) be defined by
\[
f(X)(Y) = \text{Tr}[(I + (I + c^2 X^2)^{1/2})^{-1} X Y], \quad Y \in \mathcal{L}(H). \tag{190}
\]
Then \( Df(X_0) : \text{Sym}(H) \cap \text{Tr}(H) \to \mathcal{L}(\mathcal{L}(H), \mathbb{R}) \) is given by
\[
[Df(X_0)](Y) = \frac{1}{2} \text{Tr}[(I + Z_0^{1/2})^{-1}(XY + YX)]
- \frac{c^2}{2} \text{Tr}[(I + Z_0^{1/2})^{-1} D\text{sqrt}(Z_0)(X_0 X + X X_0)(I + Z_0^{1/2})^{-1}(X_0 Y + Y X_0)].
\tag{191}
\]
In particular, for \( Y = X \),
\[
[Df(X_0)](X) = \text{Tr}[(I + Z_0^{1/2})^{-1} X^2]
- \frac{c^2}{2} \text{Tr}[(I + Z_0^{1/2})^{-1} D\text{sqrt}(Z_0)(X_0 X + X X_0)(I + Z_0^{1/2})^{-1}(X_0 X + X X_0)].
\tag{192}
\]
Here \( X, X_0 \in \text{Sym}(H) \cap \text{Tr}(H) \) and \( Z_0 = I + c^2 X_0^2 \).

Proof Let \( g(X) = (I + (I + c^2 X^2)^{1/2})^{-1}X \), \( h(X) = (I + c^2 X^2)^{1/2} \), then \( f(X)(Y) = \text{Tr}[g(X)Y] \). By Lemmas 25 and 31, combining Eqs. (188) and (189), we obtain, with \( Z_0 = I + c^2 X_0^2 \),
\[
[Df(X_0)](Y) = [Df(g(X_0)) \circ Dg(X_0)](Y) = \text{Tr}[Dg(X_0)g(X_0)Y]
= \frac{c^2}{2} \text{Tr}[(I + h(X_0))^{-1} D\text{sqrt}(I + c^2 X_0^2)(X_0 X + X X_0)(I + h(X_0))^{-1}(X_0 Y + Y X_0)]
+ \frac{1}{2} \text{Tr}[(I + h(X_0))^{-1}(XY + YX)]
= \frac{c^2}{2} \text{Tr}[(I + Z_0^{1/2})^{-1} D\text{sqrt}(Z_0)(X_0 X + X X_0)(I + Z_0^{1/2})^{-1}(X_0 Y + Y X_0)]
+ \frac{1}{2} \text{Tr}[(I + Z_0^{1/2})^{-1}(XY + YX)].
\]
In particular, for \( Y = X \),
\[
[Df(X_0)](X) = \text{Tr}[(I + Z_0^{1/2})^{-1} X^2]
- \frac{c^2}{2} \text{Tr}[(I + Z_0^{1/2})^{-1} D\text{sqrt}(Z_0)(X_0 X + X X_0)(I + Z_0^{1/2})^{-1}(X_0 X + X X_0)].
\]

Lemma 33 Let \( X_0 \in \text{Sym}(H) \) be a fixed compact operator. Let \( Z_0 = I + c^2 X_0^2 \), \( c \in \mathbb{R} \). Then \( \forall X \in \text{Sym}(H) \cap \text{HS}(H), X \neq 0, \)
\[
\frac{c^2}{2} \text{Tr}[(I + Z_0^{1/2})^{-1} D\text{sqrt}(Z_0)(X_0 X + X X_0)(I + Z_0^{1/2})^{-1}(X_0 X + X X_0)]
< \text{Tr}[(I + Z_0^{1/2})^{-1} X^2]. \tag{193}
\]
Proof Let $Y_0 = (I + Z_0^{1/2})^{-1} = (I + (I + c^2X_0^2)^{1/2})^{-1}$. By Lemma 24, since $Y_0$ and $Z_0$ commute, we have

$$\text{Tr}[\text{Dsqrt}(Z_0)(X)Y_0XY_0] = 2||Z_0^{1/4}Y_0^{1/2}\text{Dsqrt}(Z_0)(X)Y_0^{1/2}||_{\text{HS}}.$$ 

Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be the eigenvalues of $X_0$, with corresponding orthonormal eigenvectors $\{e_k\}_{k \in \mathbb{N}}$ forming an orthonormal basis in $\mathcal{H}$. Then $Z_0$ and $Y_0$ have eigenvalues $\{z_k = (1 + c^2\lambda_k^2)\}_{k \in \mathbb{N}}$ and $\{y_k = (1 + z_k^{1/2})^{-1}\}_{k \in \mathbb{N}}$, respectively, with the same eigenvalues. We then have $\forall X \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$,

$$\frac{1}{2}\text{Tr}[\text{Dsqrt}(Z_0)(X)Y_0XY_0] = ||Z_0^{1/4}Y_0^{1/2}\text{Dsqrt}(Z_0)(X)Y_0^{1/2}||_{\text{HS}}$$

$$= \sum_{k=1}^{\infty} ||Z_0^{1/4}Y_0^{1/2}\text{Dsqrt}(Z_0)(X)Y_0^{1/2}e_k||^2 = \sum_{k=1}^{\infty} y_k ||Z_0^{1/4}Y_0^{1/2}\text{Dsqrt}(Z_0)(X)e_k||^2$$

$$= \sum_{k=1}^{\infty} y_k \sum_{j=1}^{\infty} \langle Z_0^{1/4}Y_0^{1/2}\text{Dsqrt}(Z_0)(X)e_k, e_j \rangle^2$$

$$= \sum_{j,k=1}^{\infty} y_k y_j z_j^{1/2} \langle e_k, \text{Dsqrt}(Z_0)(X)e_j \rangle^2. \quad (194)$$

We recall the following identity from Lemma 17

$$Z_0^{1/2}\text{Dsqrt}(Z_0)(X) + \text{Dsqrt}(Z_0)(X)Z_0^{1/2} = X.$$ 

Applying the inner product with $e_k$ and $e_j$ on both sides, taking into account that $\text{Dsqrt}(Z_0)(X) \in \text{Sym}(\mathcal{H})$ and $z_k > 0\forall k \in \mathbb{N}$, gives

$$\langle e_k, \text{Dsqrt}(Z_0)(X)e_j \rangle = \frac{1}{z_k^{1/2} + z_j^{1/2}} \langle e_k, Xe_j \rangle.$$ 

Since $X, X_0 \in \text{Sym}(\mathcal{H})$, $\langle e_k, (XX_0 + X_0X)e_j \rangle = (\lambda_k + \lambda_j)\langle e_k, Xe_j \rangle$ and thus

$$\langle e_k, \text{Dsqrt}(Z_0)(XX_0 + X_0X)e_j \rangle = \frac{\lambda_k + \lambda_j}{z_k^{1/2} + z_j^{1/2}}\langle e_k, Xe_j \rangle$$

Substituting this into Eq.(194) gives

$$\frac{1}{2}\text{Tr}[\text{Dsqrt}(Z_0)(XX_0 + X_0X)Y_0(XX_0 + X_0X)Y_0]$$

$$= ||Z_0^{1/4}Y_0^{1/2}\text{Dsqrt}(Z_0)(XX_0 + X_0X)Y_0^{1/2}||_{\text{HS}}$$

$$= \sum_{k,j=1}^{\infty} y_k y_j z_j^{1/2} \left( \frac{\lambda_k + \lambda_j}{z_k^{1/2} + z_j^{1/2}} \right)^2 \langle e_k, Xe_j \rangle^2.$$ 

With $z_k = 1 + c^2X_k^2$, $y_k = (1 + z_k^{1/2})^{-1}$, we have $z_k^{1/2}y_k < 1\forall k \in \mathbb{N}$ and

$$c^2 \left( \frac{\lambda_k + \lambda_j}{z_k^{1/2} + z_j^{1/2}} \right)^2 = \left( \frac{c\lambda_k + e\lambda_j}{(1 + c^2\lambda_k^2)^{1/2} + (1 + c^2\lambda_j^2)^{1/2}} \right)^2 < 1, \forall k, j \in \mathbb{N}.$$
It follows that $\forall X \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H}), X \neq 0$,

$$
c^2 ||Z_0^{1/4}Y_0^{-1/2}D\sqrt{\text{tr}}(Z_0)(XX_0 + X_0X)Y_0^{-1/2}||_{\text{HS}}^2 < \sum_{k,j=1}^\infty y_k(Xe_k, e_j)^2
$$

$$
= \sum_{k=1}^\infty ||Xe_k||^2 = \sum_{k=1}^\infty ||XY_0^{-1/2}e_k||^2 = ||XY_0^{-1/2}||_{\text{HS}}^2 = \text{Tr}(Y_0X^2).
$$

Substituting $Y_0 = (I + Z_0^{1/2})^{-1}$ gives the desired result. \hfill \Box

**Proposition 8** Let $f : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$ be defined by

$$
f(X) = \text{Tr} \left[ -I + (I + c^2X^2)^{1/2} \right] - \log \det \left( \frac{1}{2} I + \frac{1}{2}(I + c^2X^2)^{1/2} \right),
$$

where $c \in \mathbb{R}, c \neq 0$. Then $f$ is at least twice Fréchet differentiable and strictly convex on $\text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$.

**Proof** By Lemmas 28, 29, $Df(X_0) : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$ is given by, $\forall X_0, X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}),$

$$
Df(X_0)(X) = c^2\text{Tr} \left[ (I + c^2X_0^2)^{-1/2}X_0X \right]
- c^2\text{Tr} \left[ \left( (I + c^2X_0^2)^{1/2} + (I + c^2X_0^2) \right)^{-1}X_0X \right]
= c^2\text{Tr} \left[ (I + (I + c^2X_0^2)^{1/2})^{-1}X_0X \right].
$$

Thus we have the map $Df : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathbb{R})$, with

$$
Df(X)(Y) = c^2\text{Tr} \left[ (I + (I + c^2X^2)^{1/2})^{-1}XY \right].
$$

Differentiating this map gives the 2nd-order Fréchet derivative, by Lemma 32,

$$
[D^2f(X_0)](X, Y) = [D^2f(X_0)(X)](Y) = \frac{c^4}{2}\text{Tr}[(I + Z_0^{1/2})^{-1}(XY + YX)]
- \frac{c^2}{2}\text{Tr}[(I + Z_0^{1/2})^{-1}\text{D}\sqrt{\text{tr}}(Z_0)(XX_0 + X_0X)(I + Z_0^{1/2})^{-1}(X_0Y + YX_0)].
$$

In particular, for $Y = X$,

$$
[D^2f(X_0)](X, X) = c^2\text{Tr}[(I + Z_0^{1/2})^{-1}X^2]
- \frac{c^2}{2}\text{Tr}[(I + Z_0^{1/2})^{-1}\text{D}\sqrt{\text{tr}}(Z_0)(XX_0X + XX_0)(I + Z_0^{1/2})^{-1}(X_0X + XX_0)] > 0,
$$

with the strict inequality being valid $\forall X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}), X \neq 0$ by Lemma 33. Thus $f$ is strictly convex on $\text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$. \hfill \Box
**Proof (of Theorem 8 - Strict convexity of Sinkhorn divergence)** By the strict convexity of the square Hilbert norm \( \| \|_2 \), the function \( m \to \| m - m_0 \|_2 \) is strictly convex in \( m \). For the covariance part, by Theorem 7,

\[
F(X) = S_{\alpha}(\mathcal{N}(0, C_0), \mathcal{N}(0, X)) = \frac{\epsilon}{4} \text{Tr} \left[ \left( I + \frac{16}{\epsilon^2} C_0^{1/2} \right)^{1/2} - 2 \left( I + \frac{16}{\epsilon^2} C_0^{1/2} X C_0^{1/2} + \frac{16}{\epsilon^2} X^2 \right)^{1/2} \right] \\
+ \frac{\epsilon}{2} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} X C_0^{1/2} \right)^{1/2} \right) \\
- \frac{\epsilon}{4} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} \right)^{1/2} \right) - \frac{\epsilon}{4} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} C_0^{1/2} \right)^{1/2} \right).
\]

By Proposition 6, the function \( f(X) = \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + c^2 C_0^{1/2} X C_0^{1/2} \right)^{1/2} \right) - \text{Tr} \left[ \left( I + c^2 C_0^{1/2} X C_0^{1/2} \right)^{1/2} \right] \) is convex on \( \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \). By Proposition 8, \( g(X) = \text{Tr} \left[ \left( I + \left( I + \frac{16}{\epsilon^2} X^2 \right)^{1/2} \right)^{1/2} \right] + \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{\epsilon^2} X^2 \right)^{1/2} \right) \) is strictly convex in \( \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \). Thus \( F \) is strictly convex on \( \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \).

**Proof (of Theorem 10 - Differentiability)**. The differentiability and convexity \( F_E \) follows from Lemmas 18, 19, and Proposition 6. The differentiability and strict convexity of \( F_S \) follows from Lemmas 18, 19, 28, 29, and Proposition 6 and 8.

**Proof (of Theorem 9 - Positivity)** Let \( \mu_0 = (m_0, C_0) \) and \( \mu_1 = (m, X) \). It suffices to prove for the case \( m_0 = m = 0 \). Let \( C_0 \) be fixed. By Theorem 10, the function \( F_S : X \to S_{\alpha}(\mathcal{N}(0, C_0), \mathcal{N}(0, X)) \) is well-defined, twice Fréchet differentiable, and strictly convex on the open, convex set \( \Omega = \{ X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) : I + c^2 C_0^{1/2} X C_0^{1/2} > 0 \} \supset \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \). Thus a minimizer of \( F_S \) in \( \Omega \) is necessarily unique. Proceeding as in Proposition 9, the Fréchet derivative for \( F_S \) is given by, for \( X_0 \in \Omega, X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \),

\[
DF_S(X_0)(X) = -c \text{Tr} \left[ C_0^{1/2} \left( I + \left( I + c^2 C_0^{1/2} X_0 C_0^{1/2} \right)^{1/2} \right)^{-1} C_0^{1/2} X \right] + c \text{Tr} \left[ \left( I + \left( I + c^2 X_0^2 \right)^{1/2} \right)^{-1} X_0 X \right]. \tag{196}
\]

By Lemma 20, \( DF_S(X_0)(X) = 0 \) \( \forall X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \) if and only if

\[
\left( I + \left( I + c^2 X_0^2 \right)^{1/2} \right)^{-1} X_0 = C_0^{1/2} \left( I + \left( I + c^2 C_0^{1/2} X_0 C_0^{1/2} \right)^{1/2} \right)^{-1} C_0^{1/2}
\]

\( \iff X_0^{1/2} \left( I + \left( I + c^2 X_0^2 \right)^{1/2} \right)^{-1} X_0^{1/2} = C_0^{1/2} \left( I + \left( I + c^2 C_0^{1/2} X_0 C_0^{1/2} \right)^{1/2} \right)^{-1} C_0^{1/2}. \)
This equation obviously has solution \( X_0 = C_0 \), which must be unique since \( F_S \) is strictly convex in \( \Omega \). Thus the unique global minimum of \( F_S \) in \( \Omega \), and hence in \( \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \), is \( F_S(C_0) = 0 \). Hence \( F_S(X) \geq 0 \), \( \forall X \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \), with \( F_S(X) = 0 \iff X = C_0 \). \( \square \)

**Derivation of the barycenter equations.** We start by deriving Eqs. (45) and (47) for the barycenter of Gaussian measures, which we restate here.

**Proposition 9** Consider the Gaussian measures \( \mathcal{N}(0, C), \mathcal{N}(0, C_i), 1 \leq i \leq N \). Define the following function \( F : \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}, 1 \leq i \leq N, \) by

\[
F(C) = \sum_{i=1}^{N} w_i S_{\epsilon^2}(\mathcal{N}(0, C), \mathcal{N}(0, C_i)).
\]  

(197)

Then \( F \) is well-defined on the larger, open set \( \Omega = \{ X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) : I + \frac{16}{C_i^2} X C_i^2 > 0, i = 1, \ldots, N \} \). \( F \) is Fréchet differentiable on \( \Omega \) and the condition \( DF(X_0) = 0 \) is equivalent to

\[
X_0 = \left( I + \left( I + \frac{16}{C_i^2} X_0^2 \right)^{1/2} \right)^{1/2} \sum_{i=1}^{N} w_i \left[ C_i^{1/2} \left( I + \left( I + \frac{16}{C_i^2} X_0 C_i 1/2 \right)^{1/2} \right)^{-1} C_i^{1/2} \right] \times \left( I + \left( I + \frac{16}{C_i^2} X_0^2 \right)^{1/2} \right)^{1/2}.
\]  

(198)

A solution \( X_0 \) of Eq. (198) must necessarily satisfy \( X_0 \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \). Under the additional hypothesis that \( X_0 > 0 \), Eq. (198) is equivalent to

\[
X_0 = \frac{\epsilon}{4} \left[ -I + \left( \sum_{i=1}^{N} w_i \left( I + \frac{16}{C_i^2} X_0^{1/2} C_i X_0^{1/2} \right)^{1/2} \right)^{2} \right]^{1/2}.
\]  

(199)

**Proof** Let \( F_1(C) = S_{\epsilon^2}(\mathcal{N}(0, C), \mathcal{N}(0, C_i)) \). By Theorem 7,

\[
F_1(C) = S_{\epsilon^2}(\mathcal{N}(0, C), \mathcal{N}(0, C_i))
\]

\[
= \frac{\epsilon}{4} \text{Tr} \left[ M_{00}^2 - 2 M_{01} + M_{11} \right] + \frac{\epsilon}{4} \log \det \left[ \left( I + \frac{1}{2} M_{00}^2 \left( I + \frac{1}{2} M_{11} \right) \right) \right]
\]

\[
= \frac{\epsilon}{4} \text{Tr} \left[ \left( I + \frac{16}{C_i^2} C_i \right)^{1/2} - 2 \left( I + \frac{16}{C_i^2} C_i 1/2 C_i 1/2 \right)^{1/2} \right] + \frac{\epsilon}{2} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{C_i^2} C_i 1/2 \right)^{1/2} \right)
\]

\[- \frac{\epsilon}{4} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{C_i^2} C_i \right)^{1/2} \right) - \frac{\epsilon}{4} \log \det \left( \frac{1}{2} I + \frac{1}{2} \left( I + \frac{16}{C_i^2} C_i \right)^{1/2} \right).\]
Clearly, $F_i$ is well-defined on the larger, open set $\Omega = \{X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) : I + \frac{4c_i}{i} X C_{i1/2}^{1/2} > 0\}$ and hence $F$ is well-defined on $\Omega = \cap_{i=1}^N \Omega_i$. The Fréchet derivative of $F_i$ at each $X_0 \in \Omega_i$ is a linear map $DF_i(X_0) : \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$. Combining Lemmas 18, 19, 28, 29, we obtain, $\forall X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$, with $c_i = \frac{4}{i}$,

$$DF_i(X_0)(X) = -c_i\text{Tr}\left[C_i^{1/2} \left(I + c_i^2 C_i^{1/2} X_0 C_i^{1/2}\right)^{-1/2} C_i^{1/2} X\right]$$

$$+ c_i\text{Tr}\left[(I + c_i^2 X_0^2)^{-1/2} X_0 X\right]$$

$$+ c_i\text{Tr}\left[C_i^{1/2} \left(\left(I + c_i^2 C_i^{1/2} X_0 C_i^{1/2}\right)^{1/2} + \left(I + c_i^2 C_i^{1/2} X_0 C_i^{1/2}\right)^{-1}\right) C_i^{1/2} X\right]$$

$$- c_i\text{Tr}\left[(I + c_i^2 X_0^2)^{1/2} + (I + c_i^2 X_0^2)^{-1}\right] X_0 X$$

$$= -c_i\text{Tr}\left[C_i^{1/2} \left(I + \left(I + c_i^2 C_i^{1/2} X_0 C_i^{1/2}\right)^{1/2}\right)^{-1} C_i^{1/2} X\right]$$

$$+ c_i\text{Tr}\left[(I + (I + c_i^2 X_0^2)^{1/2})^{-1} X_0 X\right].$$

Summing over $i$, $1 \leq i \leq N$, we obtain the Fréchet derivative for $F$, namely

$$DF(X_0)(X) = -c_i \sum_{i=1}^N w_i \text{Tr}\left[C_i^{1/2} \left(I + \left(I + c_i^2 C_i^{1/2} X_0 C_i^{1/2}\right)^{1/2}\right)^{-1} C_i^{1/2} X\right]$$

$$+ c_i\text{Tr}\left[(I + (I + c_i^2 X_0^2)^{1/2})^{-1} X_0 X\right]. \quad (200)$$

By Lemma 20, the first order optimality condition $DF(X_0)(X) = 0 \forall X \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$ is equivalent to $DF(X_0) = 0$, which in turn is equivalent to

$$\left(I + (I + c_i^2 X_0^2)^{1/2}\right)^{-1} X_0 = \sum_{i=1}^N w_i \left[C_i^{1/2} \left(I + \left(I + c_i^2 C_i^{1/2} X_0 C_i^{1/2}\right)^{1/2}\right)^{-1} C_i^{1/2}\right]$$

$$\iff \left(I + (I + c_i^2 X_0^2)^{1/2}\right)^{-1/2} X_0 \left(I + (I + c_i^2 X_0^2)^{1/2}\right)^{-1/2}$$

$$= \sum_{i=1}^N w_i \left[C_i^{1/2} \left(I + \left(I + c_i^2 C_i^{1/2} X_0 C_i^{1/2}\right)^{1/2}\right)^{-1} C_i^{1/2}\right]$$

$$\iff X_0 = \left(I + (I + c_i^2 X_0^2)^{1/2}\right)^{1/2} \sum_{i=1}^N w_i \left[C_i^{1/2} \left(I + \left(I + c_i^2 C_i^{1/2} X_0 C_i^{1/2}\right)^{1/2}\right)^{-1} C_i^{1/2}\right]$$

$$\times \left(I + (I + c_i^2 X_0^2)^{1/2}\right)^{1/2}.$$
Assume now $X_0 > 0$. We rewrite $DF(X_0) = 0$ as

$$
(1 + (1 + c^2 X_0^{1/2})^{-1/2})^{-1} X_0 = \sum_{i=1}^{N} w_i \left[ C_i^{1/2} \left( 1 + (1 + c^2 C_i^{1/2} X_0 C_i^{1/2})^{1/2} \right)^{-1} C_i^{1/2} \right]
$$

which under the condition $X_0 > 0$, by Lemma 8, pre- and post-multiplying $X_0^{1/2}$ on both sides gives the equivalent expression

$$
X_0 \left( 1 + (1 + c^2 X_0^{2})^{1/2} \right)^{-1} X_0
$$

$$
= \sum_{i=1}^{N} w_i \left( X_0^{1/2} C_i^{1/2} \left( 1 + (1 + c^2 C_i^{1/2} X_0 C_i^{1/2})^{1/2} \right)^{-1} C_i^{1/2} X_0^{1/2} \right)
$$

$$
\iff \frac{1}{c^2} \left[ -1 + (1 + c^2 X_0^{1/2})^{1/2} \right] = \frac{1}{c^2} \sum_{i=1}^{N} w_i \left[ -I + (1 + c^2 X_0^{1/2} C_i X_0^{1/2})^{1/2} \right],
$$

where the right hand side follows from Lemma 8. This in turn is

$$
(1 + c^2 X_0^{1/2}) = \sum_{i=1}^{N} w_i \left( 1 + c^2 X_0^{1/2} C_i X_0^{1/2} \right)^{1/2}
$$

$$
\iff X_0^2 = \frac{1}{c^2} \left[ -1 + \left( \sum_{i=1}^{N} w_i \left( 1 + c^2 X_0^{1/2} C_i X_0^{1/2} \right)^{1/2} \right)^2 \right]
$$

$$
\iff X_0 = \frac{1}{c^2} \left[ -1 + \left( \sum_{i=1}^{N} w_i \left( 1 + c^2 X_0^{1/2} C_i X_0^{1/2} \right)^{1/2} \right)^2 \right]^{1/2}.
$$

This completes the proof. 

**Existence of the Fixed Point.** It is clear from Eq.(45) that if it has a solution $X_0$, then necessarily $X_0 \geq 0$. We now prove that Eq.(45) has at least one solution $X_0$, which is then necessarily unique by the strict convexity of the Sinkhorn divergence. This is done via the Schauder Fixed Point Theorem. Let $E$ be a Banach space and $M \subset E$. We recall that a mapping $f : M \rightarrow E$ is said to be compact if it is continuous and maps bounded subsets into relatively compact subsets of $E$, that is subsets whose closures are compact.

**Theorem 27 (Schauder Fixed Point Theorem)** Let $M$ be a bounded closed convex subset of a Banach space $E$. Assume that $f : M \rightarrow M$ is a compact mapping. Then $f$ has at least one fixed point in $M$. 

Consider the map $F : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H})$ as defined in Eq. (48). The proof of the existence of a fixed point of $F$ consists of two steps.

1. We show that $F$ is compact.

2. Let $\gamma \in \mathbb{R}, \gamma > 0$ be such that $C_i \leq \gamma I, 1 \leq i \leq N$, and consider the set

$$K = \{X \in \text{Sym}^+(\mathcal{H}) : 0 \leq X \leq \gamma I\}. \quad (201)$$

We show that $F$ maps $K$ into itself. We can then apply Schauder Fixed Point Theorem to obtain the existence of a fixed point of $F$ in $K$.

**Lemma 34** Let $B \in \mathcal{L}(\mathcal{H}) \cap \text{Sym}(\mathcal{H})$. Let $A \in \mathcal{L}(\mathcal{H})$ with $\ker(A^*) = \{0\}$. Then

$$A^*BA \geq 0 \iff B \geq 0. \quad (202)$$

**Proof** If $B \geq 0$, then by Lemma 41, we have $A^*BA \geq 0$. Assume now that $A^*BA \geq 0$, which means that $\langle x, A^*BAx \rangle = \langle Ax, BAx \rangle \geq 0 \forall x \in \mathcal{H}$. In particular, for $y = Ax \in \text{Im}(A)$, we have $(y, BY) \geq 0$.

Since $\ker(A^*) = \{0\}$, we have $\text{Im}(A) = \ker(A^*) = \mathcal{H}$, hence $\text{Im}(A)$ is dense in $\mathcal{H}$. Thus for each $y \in \mathcal{H}$, there is a sequence $\{y_n\}_{n \in \mathbb{N}}$ in $\text{Im}(A)$ such that $\lim_{n \to \infty} ||y_n - y|| = 0$. Then

$$|(y, BY) - (y_n, BY_n)| = |(y - y_n, BY) + (y_n, B(y - y_n))| \leq ||y_n - y|| ||B|| (||y|| + ||y_n||) \to 0 \text{ as } n \to \infty.$$  

It follows that $(y, BY) = \lim_{n \to \infty} \langle y_n, BY_n \rangle \geq 0$. Since this holds for all $y \in \mathcal{H}$, we have $B \geq 0$.  

**Remark 7** Lemma 34 is generally not true without the condition $\ker(A^*) = \{0\}$. As an example, consider the case $\mathcal{H} = \mathbb{R}^2$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = (b_{ij})_{i,j=1,2}$.

Then $ABA = \begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix} \geq 0 \iff b_{11} \geq 0$.

**Lemma 35 (Corollary 3.2 in [48])** For any two positive operators $A, B$ on $\mathcal{H}$ such that $A \geq cI > 0, B \geq cI > 0$, for any bounded operator $X$ on $\mathcal{H}$,

$$||A^rX - XB^r||_p \leq r c^{r-1} ||AX - XB||_p, 0 < r \leq 1, 1 \leq p \leq \infty. \quad (203)$$

The following result is then immediate.

**Corollary 6** Let $\mathcal{C}_p(\mathcal{H})$ denote the set of $p$th Schatten class operators on $\mathcal{H}$, $1 \leq p \leq \infty$. For two operators $A, B \in \text{Sym}^+(\mathcal{H}) \cap \mathcal{C}_p(\mathcal{H})$,

$$||(I + A)^r - (I + B)^r||_p \leq r ||A - B||_p, 0 \leq r \leq 1. \quad (204)$$
**Theorem 28** (Theorem 2.3 in [48]) Let $A, B$ be two positive operators on $\mathcal{H}$ and $f$ any operator monotone function with $f(0) = 0$. Then

$$ ||f(A) - f(B)|| \leq f(||A - B||). \quad (205) $$

The following result is then immediate.

**Corollary 7** Let $A, B$ be two positive operators on $\mathcal{H}$. Then

$$ ||A^r - B^r|| \leq ||A - B||^r, \quad 0 < r \leq 1. \quad (206) $$

**Proposition 10** Let $C \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$. The following maps are compact

1. $F_1 : \text{Sym}^+(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ defined by $F_1(X) = X^{1/2}C^{1/2}$.
2. $F_2 : \text{Sym}^+(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ defined by $F_2(X) = C^{1/2}X^{1/2}$.
3. $F_3 : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H})$ defined by $F_3(X) = X^{1/2}CX^{1/2}$.
4. $F_4 : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H})$ defined by $F_4(X) = C^{1/2}XC^{1/2}$

**Proof** (i) By Corollary 7, we have

$$ ||F_1(X) - F_1(Y)|| \leq ||X^{1/2} - Y^{1/2}|| \cdot ||C^{1/2}|| \leq ||C||^{1/2}||X - Y||^{1/2}. $$

Thus the map $F_1$ is continuous on $\text{Sym}^+(\mathcal{H})$.

Since $C^{1/2}$ is a compact operator on $\mathcal{H}$, it maps bounded subsets of $\mathcal{H}$ into relatively compact subsets of $\mathcal{H}$. Consider the set $\mathcal{Y} = \{C^{1/2}x : x \in \mathcal{H}, ||x|| \leq 1\} \subset \mathcal{H}$, then $\mathcal{Y}$ being relatively compact means that every sequence $\{y_n = C^{1/2}x_n\}_{n \in \mathbb{N}}$ in $\mathcal{Y}$ contains a subsequence $\{y_{n_k} = C^{1/2}x_{n_k}\}_{k \in \mathbb{N}}$ that converges in $\mathcal{H}$, that is $\exists y \in \mathcal{H}$ such that $\lim_{k \to \infty} ||C^{1/2}x_{n_k} - y|| = 0$.

Since $C \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$, we have $C^{1/2} \in \text{HS}(\mathcal{H})$ and $X^{1/2}C^{1/2} \in \text{HS}(\mathcal{H}) \forall X \in \text{Sym}^+(\mathcal{H})$. In particular $X^{1/2}C^{1/2}$ is a compact operator.

Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be the eigenvalues of $C$, arranged in decreasing order, with corresponding orthonormal eigenvectors $\{e_k\}_{k \in \mathbb{N}}$. Let $N \in \mathbb{N}$ be fixed and consider the finite-rank operator $C_N^{1/2} = \sum_{k=1}^{N} \sqrt{\lambda_k}(e_k \otimes e_k)$. Then for any $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that

$$ ||C_N^{1/2} - C^{1/2}||_{\text{HS}} = \left( \sum_{k=N+1}^{\infty} \lambda_k \right)^{1/2} < \epsilon, \quad \forall N > N(\epsilon). $$

Consequently, for $||X|| \leq 1$, $N > N(\epsilon)$,

$$ ||X^{1/2}C_N^{1/2} - X^{1/2}C^{1/2}||_{\text{HS}} \leq ||X^{1/2}|| \cdot ||C_N^{1/2} - C^{1/2}|| < \epsilon. $$
For a fixed \( N \in \mathbb{N} \), consider the set \( Z_N = \{ X^{1/2}C_N^{1/2} : \|X\| \leq 1 \} \subset \text{HS}(\mathcal{H}) \) and a sequence \( \{ X_N^{1/2}C_N^{1/2} \}_{n \in \mathbb{N}} \subset Z_N \). We now show that this sequence has a convergent subsequence in \( \text{HS}(\mathcal{H}) \). We have

\[
\|X_n^{1/2}C_N^{1/2}\|_{\text{HS}}^2 = \sum_{j=1}^{\infty} \|X_n^{1/2}C_N^{1/2}e_j\|^2 \leq \|C_N^{1/2}\|_{\text{HS}}^2 \leq \text{Tr}(C) < \infty.
\]

The sequence \( \{\|X_n^{1/2}C_N^{1/2}\|_{\text{HS}}\}_{n \in \mathbb{N}} \) is a bounded sequence of non-negative numbers and thus, by the Bolzano-Weierstrass Theorem, has a convergent subsequence \( \{\|X_{n_0}^{1/2}C_N^{1/2}\|_{\text{HS}}\}_{n_0 \in \mathbb{N}} \), with

\[
\lim_{n_0 \to \infty} \|X_{n_0}^{1/2}C_N^{1/2}\|_{\text{HS}} = B, \quad \text{for some constant } B \geq 0.
\]

For \( j = 1 \), the sequence \( \{ X_n^{1/2}C_N^{1/2}e_1 \}_{n_0 \in \mathbb{N}} \), belonging to a relatively compact set in \( \mathcal{H} \), contains a convergent subsequence \( \{ X_{n_1}^{1/2}C_N^{1/2}e_1 \} \), i.e.

\[
\lim_{n_1 \to \infty} \|X_{n_1}^{1/2}C_N^{1/2}e_1 - y_1\| = 0, \quad \text{for some } y_1 \in \mathcal{H}.
\]

Similarly, for \( j = 2 \), the sequence \( \{ X_n^{1/2}C_N^{1/2}e_2 \}_{n \in \mathbb{N}} \) contains a convergent subsequence \( \{ X_{n_2}^{1/2}C_N^{1/2}e_2 \} \), i.e. \( \exists y_2 \in \mathcal{H} \) such that

\[
\lim_{n_2 \to \infty} \|X_{n_2}^{1/2}C_N^{1/2}e_2 - y_2\| = 0 \quad \text{and the same time} \quad \lim_{n_2 \to \infty} \|X_{n_2}^{1/2}C_N^{1/2}e_1 - y_1\| = 0.
\]

Carrying out this procedure iteratively, we obtain a subsequence \( \{ X_{n_j}^{1/2}C_N^{1/2} \} \) in \( \text{HS}(\mathcal{H}) \) and \( (y_j)_{j=1}^{N}, y_j \in \mathcal{H} \), such that

\[
\lim_{n_j \to \infty} \|X_{n_j}^{1/2}C_N^{1/2}e_j - y_j\| = 0, \quad 1 \leq j \leq N.
\]

Furthermore,

\[
\sum_{j=1}^{N} \|y_j\|^2 = \sum_{j=1}^{N} \lim_{n_j \to \infty} \|X_{n_j}^{1/2}C_N^{1/2}e_j\|^2 = \lim_{n_j \to \infty} \sum_{j=1}^{N} \|X_{n_j}^{1/2}C_N^{1/2}e_j\|^2 = \lim_{n_j \to \infty} \|X_{n_j}^{1/2}C_N^{1/2}\|_{\text{HS}}^2 = B^2 < \infty.
\]

Define the following finite-rank operator \( Y_N \in \mathcal{L}(\mathcal{H}) \) by

\[
Y_N e_j = \begin{cases} y_j & \text{for } 1 \leq j \leq N, \\ 0 & \text{else.} \end{cases}
\]

Then \( \|Y_N\|_{\text{HS}}^2 = \sum_{j=1}^{\infty} \|Y_N e_j\|^2 = \sum_{j=1}^{N} \|y_j\|^2 = B^2 < \infty \) and

\[
\lim_{n_N \to \infty} \|X_{n_N}^{1/2}C_N^{1/2} - Y_N\|_{\text{HS}}^2 = \lim_{n_N \to \infty} \sum_{j=1}^{N} \|(X_{n_N}^{1/2}C_N^{1/2} - Y_N)e_j\|^2 = \sum_{j=1}^{N} \lim_{n_N \to \infty} \|X_{n_N}^{1/2}C_N^{1/2}e_j - y_j\|^2 = 0.
\]
Thus \( \{X^{1/2}C_N^{1/2}\} \) is the desired convergent subsequence, with limit \( Y_N \in \text{HS}(\mathcal{H}) \). This shows that the set \( Z_N = \{X^{1/2}C_N : ||X|| \leq 1\} \) is relatively compact \( \forall N \in \mathbb{N} \) in \( \text{HS}(\mathcal{H}) \), so that \( \forall \epsilon > 0 \), there is a finite \( \epsilon \)-net \( \{Z_i\}_{i=1}^{N_2(\epsilon)} \) in \( \text{HS}(\mathcal{H}) \) such that

\[
\{X^{1/2}C_N^{1/2} : ||X|| \leq 1\} \subset \bigcup_{i=1}^{N_2(\epsilon)} B_{\text{HS}(\mathcal{H})}(Z_i, \epsilon).
\]

Consequently, for \( N > N(\epsilon) \),

\[
\{X^{1/2}C_N^{1/2} : ||X|| \leq 1\} \subset \bigcup_{i=1}^{N_2(\epsilon)} B_{\text{HS}(\mathcal{H})}(Z_i, 2\epsilon).
\]

This shows that the set \( \{X^{1/2}C_N^{1/2} : ||X|| \leq 1\} \) is relatively compact in \( \text{HS}(\mathcal{H}) \), hence in \( \mathcal{L}(\mathcal{H}) \) and thus \( F_1 \) is a compact map on \( \text{Sym}^+(\mathcal{H}) \). Moreover, each sequence \( \{X_n^{1/2}C_n^{1/2} : ||X|| \leq 1\} \) contains a convergent subsequence \( \{X_{k(n)}^{1/2}C_{k(n)}^{1/2}\}_{n \in \mathbb{N}} \) in \( \text{HS}(\mathcal{H}) \), i.e. \( \exists Y \in \text{HS}(\mathcal{H}) \) such that

\[
\lim_{k(n) \to \infty} ||X_{k(n)}^{1/2}C_{k(n)}^{1/2} - Y||_{\text{HS}} = 0.
\]

(ii) Similarly, \( F_2 \) is a continuous map on \( \text{Sym}^+(\mathcal{H}) \) and each sequence \( \{C_N^{1/2}X_n^{1/2} : ||X_n|| \leq 1\}_{n \in \mathbb{N}} \) contains a convergent subsequence \( \{C_{k(n)}^{1/2}X_{k(n)}^{1/2}\}_{n \in \mathbb{N}} \), with

\[
\lim_{k(n) \to \infty} ||C_{k(n)}^{1/2}X_{k(n)}^{1/2} - Y^*||_{\text{HS}} = 0,
\]

where \( Y \in \text{HS}(\mathcal{H}) \) is as defined in Part (i). Thus \( F_2 \) is a compact map on \( \text{Sym}^+(\mathcal{H}) \).

(iii) Since \( F_1(X) = F_1(X)F_2(X) \), \( F_3 \) is continuous on \( \text{Sym}^+(\mathcal{H}) \). Furthermore, each sequence \( \{X_n^{1/2}CX_n^{1/2} : ||X_n|| \leq 1\}_{n \in \mathbb{N}} \) contains a convergent subsequence \( \{X_{k(n)}^{1/2}CX_{k(n)}^{1/2}\}_{n \in \mathbb{N}} \), with

\[
\lim_{k(n) \to \infty} ||X_{k(n)}^{1/2}CX_{k(n)}^{1/2} - YY^*||_{\text{HS}} = 0,
\]

where \( Y \in \text{HS}(\mathcal{H}) \) is as defined in Part (i). Thus \( F_2 \) is a compact map on \( \text{Sym}^+(\mathcal{H}) \).

(iv) Entirely analogous to \( F_3 \), the map \( F_4 \) is compact on \( \text{Sym}^+(\mathcal{H}) \). \( \Box 

**Corollary 8** Let \( A, B \in \text{Sym}^+(\mathcal{H}) \) be given. Let \( 1 \leq p \leq \infty \). Then

\[
||(I + (I + A)^{1/2})^{-1} - (I + (I + B)^{1/2})^{-1}||_p \leq \frac{1}{8}||A - B||_p.
\]

In particular, let \( A \in \text{Sym}^+(\mathcal{H}) \), \( \{A_n\}_{n \in \mathbb{N}} \), \( A_n \in \text{Sym}^+(\mathcal{H}) \) \( \forall n \in \mathbb{N} \) be such that \( \lim_{n \to \infty} ||A_n - A||_p = 0 \). Then

\[
\lim_{n \to \infty} ||(I + (I + A_n)^{1/2})^{-1} - (I + (I + A)^{1/2})^{-1}||_p = 0.
\]
Proof By Corollary 6,
\[
||((I + (I + A)^{1/2})^{-1} - (I + (I + B)^{1/2})^{-1})||_p
\]
\[
= ||((I + (I + A)^{1/2})^{-1}[(I + (I + A)^{1/2}) - (I + (I + B)^{1/2})](I + (I + B)^{1/2})^{-1}||_p
\]
\[
\leq ||(I + (I + A)^{1/2})^{-1}|| ||(I + (I + A)^{1/2} - (I + B)^{1/2})||_p ||(I + (I + B)^{1/2})^{-1}||
\]
\[
\leq \frac{1}{8}||A - B||_p.
\]
The second result is then immediate. □

Corollary 9 Let \( C \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \). Let \( a \in \mathbb{R}, a \neq 0 \). Consider the map \( F : \text{Sym}^+(\mathcal{H}) \rightarrow \text{Sym}^+(\mathcal{H}) \) defined by
\[
F(X) = \left(I + \left(I + a^2C^{1/2}XC^{1/2}\right)^{1/2}\right)^{-1}.
\]
Then \( F \) is a compact map on \( \text{Sym}^+(\mathcal{H}) \).

Proof By Corollary 8,
\[
||F(X) - F(Y)|| \leq \frac{a^2}{8}||C^{1/2}XC^{1/2} - C^{1/2}YC^{1/2}|| \leq \frac{a^2}{8}||C^{1/2}||^2||X - Y||.
\]
Thus \( F(X) \) is a continuous map on \( \text{Sym}^+(\mathcal{H}) \). By Proposition 10, the map \( g : \text{Sym}^+(\mathcal{H}) \rightarrow \text{Sym}^+(\mathcal{H}) \) defined by \( g(X) = C^{1/2}XC^{1/2} \) is compact, so that each sequence \( \{C^{1/2}X_nC^{1/2}, ||X_n|| \leq 1\}_{n \in \mathbb{N}} \) contains a convergent subsequence \( \{C^{1/2}X_{k(n)}C^{1/2}\} \) with
\[
\lim_{k(n) \rightarrow \infty} ||C^{1/2}X_{k(n)}C^{1/2} - Y^*Y||_{\text{HS}} = 0,
\]
where \( Y \in \text{HS}(\mathcal{H}) \) is as defined in the proof of Proposition 10. By Corollary 8,
\[
||F(X_{k(n)}) - (I + (I + a^2Y^*Y)^{1/2})^{-1}|| \leq \frac{a^2}{8}||C^{1/2}X_{k(n)}C^{1/2} - Y^*Y||
\]
\[
\leq \frac{a^2}{8}||C^{1/2}X_{k(n)}C^{1/2} - Y^*Y||_{\text{HS}} \rightarrow 0
\]
as \( k(n) \rightarrow \infty \). Thus the set \( \{F(X), ||X|| \leq 1\} \subset \text{Sym}^+(\mathcal{H}) \) is relatively compact, showing that \( F \) is compact. □

Lemma 36 Let \( C \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \). Let \( a \in \mathbb{R}, a \neq 0 \). The following map
\( F : \text{Sym}^+(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}) \) is compact
\[
F(X) = (I + (I + a^2X^2)^{1/2})^{1/2}C^{1/2}.
\]
(212)
Proof Define the map \( g : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H}) \) by \( g(X) = (1 + (1 + a^2X^2)^{1/2})^{1/2} \). By the inequality \((1 + a^2)^{1/2} \leq 1 + a\) for \( a \in \mathbb{R}, a \geq 0 \),

\[
||g(X)|| \leq 1 + (1 + a^2)||X||^2^{1/4} \leq 1 + (1 + a||X||)^{1/2} \leq 2 + \sqrt{a}||X||^{1/2}.
\]

Thus the set \( \{ g(X) : ||X|| \leq 1 \} \) is bounded, with max\(_{||X|| \leq 1} ||g(X)|| \leq 2 + \sqrt{a} \). Applying Corollary 6 twice, we obtain

\[
||g(X) - g(Y)|| = ||(I + (I + a^2X^2)^{1/2})^{1/2} - (I + (I + a^2Y^2)^{1/2})^{1/2}||
\leq \frac{1}{2}||(I + a^2X^2)^{1/2} - (I + a^2Y^2)^{1/2}||
\leq \frac{a^2}{4}||X^2 - Y^2|| \leq \frac{a^2}{4}(||X|| + ||Y||)||X - Y||.
\]

This shows that \( g \) is continuous on \( \text{Sym}^+(\mathcal{H}) \). Hence \( F \) is continuous on \( \text{Sym}^+(\mathcal{H}) \). As in the proof of Proposition 10, for each sequence \( \{ F(X_n) = g(X_n)C^{1/2}, ||X_n|| \leq 1 \} \), there exists a subsequence \( \{ F(X_{k(n)}) \} \) and an operator \( Z \in \text{HS}(\mathcal{H}) \) such that \( \lim_{k(n) \to \infty} ||F(X_{k(n)}) - Z||_{\text{HS}} = 0 \). Thus the set \( \{ F(X) : ||X|| \leq 1 \} \) is relatively compact, proving that \( F \) is compact.

Lemma 37 Let \( E \) be a Banach algebra and \( M \subset E \). Let \( f, g : M \to E \) be compact. Then the sum and product maps \( h_1, h_2 : M \to E \) defined by \( h_1(X) = f(X) + g(X) \) and \( h_2(X) = f(X)g(X) \) are compact.

Proof Let us show that the product map is compact. Let \( M_B \) be any bounded, non-empty subset of \( M \). By assumption of compactness, each sequence \( \{ f(X_n), X_n \in M_B \}_{n \in \mathbb{N}} \) contains a convergent subsequence \( \{ f(X_{n_k}) \} \) with limit \( Y_1 \) in \( E \). Next, the sequence \( \{ g(X_{n_k}) \} \) contains a convergent subsequence \( \{ g(X_{n_{k_2}}) \} \) with limit \( Y_2 \) in \( E \). We then have

\[
||f(X_{n_2})g(X_{n_2}) - Y_1Y_2|| \leq ||(f(X_{n_2}) - Y_1)g(X_{n_2})|| + ||Y_1(g(X_{n_2}) - Y_2)||
\leq ||f(X_{n_2}) - Y_1|| ||g(X_{n_2})|| + ||Y_1|| ||g(X_{n_2}) - Y_2|| \to 0
\]

as \( n_2 \to \infty \). Thus the set \( \{ h_2(X) = f(X)g(X) : X \in M_B \} \) is relatively compact. An analogous argument shows that \( h_2 \) is continuous, hence \( h_2 \) is a compact map on \( M \).

Proposition 11 Let \( C_i \geq 0, 1 \leq i \leq N, \) be fixed. The map \( \mathcal{F} : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H}) \) as defined in Eq. (48) is continuous in the operator \( || \cdot || \) norm. In particular, if \( 0 \leq C_i \leq \gamma I, i = 1, \ldots, N, \) and \( 0 \leq X, Y \leq \gamma I \), then

\[
||\mathcal{F}(X) - \mathcal{F}(Y)|| \leq \frac{8\gamma^2}{\epsilon^2} \left( 1 + \frac{\sqrt{2}}{\epsilon} \right) \left( 3 + \frac{\sqrt{2}}{\epsilon} \right) ||X - Y||. \quad (213)
\]
Proof Let \( c_i = \frac{4}{\epsilon} \) and \( g(X) = \left( I + (I + c_i^2 X^2)^{1/2} \right)^{1/2} \) and for \( 1 \leq i \leq N \), define \( h_i(X) = C_i^{1/2} \left( I + (I + c_i^2 C_i^{1/2} X C_i^{1/2})^{1/2} \right)^{-1} C_i^{1/2} \), then

\[
\|F(X) - F(Y)\| = \|g(X)\| \sum_{i=1}^{N} w_i h_i(X) g(X) - g(Y) \sum_{i=1}^{N} w_i h_i(Y) g(Y)\|
\]

\[
\leq \|g(X)\| \sum_{i=1}^{N} w_i h_i(X) \|g(X) - g(Y)\| + \|g(X)\| \sum_{i=1}^{N} w_i \|h_i(X) - h_i(Y)\| \|g(Y)\| + \|g(X) - g(Y)\| \sum_{i=1}^{N} w_i h_i(Y) g(Y)\|.
\]

(214)

Using the inequality \( (1 + a^2)^{1/2} \leq 1 + a \) \( \forall a \geq 0 \), we obtain

\[
\|g(X)\| \leq 2 + \frac{2}{\epsilon^2} \|X\|^{1/2}, \quad \|g(Y)\| \leq 2 + \frac{2}{\epsilon^2} \|Y\|^{1/2}.
\]

(215)

Applying Corollary 6 twice gives

\[
\|g(X) - g(Y)\| \leq \frac{4}{\epsilon^2} (\|X\| + \|Y\|) \|X - Y\|.
\]

(216)

For \( \sum_{i=1}^{N} w_i h_i(X) \),

\[
\left\| \sum_{i=1}^{N} w_i h_i(X) \right\| = \left\| \sum_{i=1}^{N} w_i \left[ C_i^{1/2} \left( I + (I + c_i^2 C_i^{1/2} X C_i^{1/2})^{1/2} \right)^{-1} C_i^{1/2} \right] \right\|
\]

\[
\leq \frac{1}{2} \sum_{i=1}^{N} w_i \|C_i\|.
\]

(217)

By Corollary 7,

\[
\|h_i(X) - h_i(Y)\|
\]

\[
= \left\| C_i^{1/2} \left[ (I + (I + c_i^2 C_i^{1/2} X C_i^{1/2})^{1/2})^{-1} - (I + (I + c_i^2 C_i^{1/2} Y C_i^{1/2})^{1/2})^{-1} \right] C_i^{1/2} \right\|
\]

\[
\leq \frac{1}{8} c_i^2 \|C_i^{1/2}\| \|C_i^{1/2} X C_i^{1/2} - C_i^{1/2} Y C_i^{1/2}\| \leq \frac{2}{\epsilon^2} \|C_i\|^2 \|X - Y\|.
\]

(218)
Combining Eqs. (214), (215), (217), (216), and (218) gives

\[
\|F(X) - F(Y)\| \leq \left(2 + \frac{2}{\sqrt{\epsilon}}\right) \left(\frac{1}{2} \sum_{i=1}^{N} w_i \|C_i\| \right) \frac{4}{\epsilon^2} \|X\| \|Y\| \|X - Y\| \\
+ \left(2 + \frac{2}{\sqrt{\epsilon}}\right) \left(\frac{1}{2} \sum_{i=1}^{N} w_i \|C_i\| \right) \frac{4}{\epsilon^2} \|X\| \|Y\| \|X - Y\| \\
+ \left(2 + \frac{2}{\sqrt{\epsilon}}\right) \left(\frac{1}{2} \sum_{i=1}^{N} w_i \|C_i\| \right) \frac{4}{\epsilon^2} \|X\| \|Y\| \|X - Y\| \\
= \frac{4}{\epsilon^2} \left(\sum_{i=1}^{N} w_i \|C_i\| \right) \left(2 + \frac{2}{\sqrt{\epsilon}}\right) \left(\frac{1}{2} \sum_{i=1}^{N} w_i \|C_i\| \right) \frac{4}{\epsilon^2} \|X\| \|Y\| \|X - Y\| \\
+ \frac{8}{\epsilon^2} \left(\sum_{i=1}^{N} w_i \|C_i\| \right) \left(1 + \frac{1}{\sqrt{\epsilon}}\right) \left(1 + \frac{1}{\sqrt{\epsilon}}\right) \|X - Y\|.
\]

This shows that \(F\) is continuous in the operator norm \(\|\cdot\|\). In particular, for \(0 \leq X \leq \gamma I\), \(0 \leq Y \leq \gamma I\), and \(0 \leq C_i \leq \gamma I\), \(1 \leq i \leq N\),

\[
\|F(X) - F(Y)\| \leq \frac{16\gamma^2}{\epsilon^2} \left(1 + \sqrt{\frac{2}{\epsilon}}\right) \|X - Y\| + \frac{8\gamma^2}{\epsilon^2} \left(1 + \sqrt{\frac{2}{\epsilon}}\right)^2 \|X - Y\| \\
= \frac{8\gamma^2}{\epsilon^2} \left(1 + \sqrt{\frac{2}{\epsilon}}\right) \left(3 + \sqrt{\frac{2}{\epsilon}}\right) \|X - Y\|.
\]

This completes the proof. \(\Box\)

**Proposition 12** Let \(C_i \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H})\), \(1 \leq i \leq N\). Consider the map \(F : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H})\) as defined in Eq. (48). Then \(F\) is compact.

**Proof (of Proposition 12)** By Proposition 11, \(F\) is continuous in the operator norm \(\|\cdot\|\). Let \(g_i(X) = \left(I + (I + \frac{16}{\sqrt{\epsilon}} X) \frac{1}{\sqrt{\epsilon}}\right)^{1/2} C_i^{1/2}\), \(1 \leq i \leq N\), then \(g_i\) is compact by Lemma 36. Let \(h_i(X) = \left(I + \left(I + \frac{16}{\sqrt{\epsilon}} C_i^{1/2} X C_i^{1/2}\right)^{1/2}\right)^{-1}\), \(1 \leq i \leq N\), then \(h_i\) is compact by Corollary 9. Then we have the summation \(F(X) = \sum_{i=1}^{N} w_i g_i(X) h_i(X) g_i(X)\), which is compact by Lemma 37. \(\Box\)

**Lemma 38** Let \(\gamma \in \mathbb{R}, \gamma > 0\) be fixed. Assume that \(0 \leq C_i \leq \gamma I\), \(1 \leq i \leq N\). Consider the map \(F : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H})\) as defined in Eq. (48). Then

\[
0 \leq X \leq \gamma I \Rightarrow 0 \leq X F(X) X \leq \gamma X^2.
\] (219)
Proof Since $0 \leq X \leq \gamma I$, by Lemma 41,
\[ 0 \leq X \leq \gamma I \Rightarrow X^{1/2}(X)X^{1/2} \leq \gamma X \Rightarrow c_i^2 C_i^{1/2} X^2 C_i^{1/2} \leq \gamma c_i^2 C_i^{1/2} X C_i^{1/2} \]
\[ \iff \gamma I + c_i^2 C_i^{1/2} X^2 C_i^{1/2} \leq \gamma \left( I + c_i^2 C_i^{1/2} X C_i^{1/2} \right) \]
\[ \Rightarrow \left( I + \frac{1}{\gamma} c_i^2 C_i^{1/2} X^2 C_i^{1/2} \right)^{1/2} \leq \left( I + c_i^2 C_i^{1/2} X C_i^{1/2} \right)^{1/2} \]
\[ \iff I + \left( I + \frac{1}{\gamma} c_i^2 C_i^{1/2} X^2 C_i^{1/2} \right)^{1/2} \leq I + \left( I + c_i^2 C_i^{1/2} X C_i^{1/2} \right)^{1/2} \]
\[ \iff \left( I + \left( I + c_i^2 C_i^{1/2} X C_i^{1/2} \right)^{1/2} \right)^{-1} \leq \left( I + \left( I + \frac{1}{\gamma} c_i^2 C_i^{1/2} X^2 C_i^{1/2} \right)^{1/2} \right)^{-1}. \]

By Lemma 41, pre- and post-multiplying by $C_i^{1/2}$ gives
\[ C_i^{1/2} \left( I + \left( I + c_i^2 C_i^{1/2} X C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} \leq C_i^{1/2} \left( I + \left( I + \frac{c_i^2}{\gamma} C_i^{1/2} X^2 C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2}. \]

Once again applying Lemma 41, pre- and post-multiplying by $X$ gives
\[ XC_i^{1/2} \left( I + \left( I + c_i^2 C_i^{1/2} X C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} X \]
\[ \leq XC_i^{1/2} \left( I + \left( I + \frac{c_i^2}{\gamma} C_i^{1/2} X^2 C_i^{1/2} \right)^{1/2} \right)^{-1} \]
\[ \leq XC_i^{1/2} \left( \frac{\gamma}{c_i^2} \left[ -I + \left( I + \frac{c_i^2}{\gamma} X C_i X \right) \right] \right) \]
\[ \text{where the last expression follows from Lemma 8. Since } 0 \leq C_i \leq \gamma I, \text{ we have} \]
\[ XC_i X \leq X(\gamma I)X = \gamma X^2 \text{ by Lemma 41. Thus,} \]
\[ XC_i^{1/2} \left( I + \left( I + c_i^2 C_i^{1/2} X C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} X \leq \frac{\gamma}{c_i^2} \left[ -I + \left( I + c_i^2 X^2 \right)^{1/2} \right] \]
\[ = \gamma X^2 \left( I + \left( I + c_i^2 X^2 \right)^{1/2} \right)^{-1}. \]

By Lemma 41,
\[ X F(X)X = \left( I + \left( I + c_i^2 X^2 \right)^{1/2} \right)^{1/2} \]
\[ \times \sum_{i=1}^{N} w_i \left[ XC_i^{1/2} \left( I + \left( I + c_i^2 C_i^{1/2} X C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} X \right] \left( I + \left( I + c_i^2 X^2 \right)^{1/2} \right)^{1/2} \]
\[ \leq \left( I + \left( I + c_i^2 X^2 \right)^{1/2} \right)^{1/2} \sum_{i=1}^{N} w_i \gamma X^2 \left[ I + \left( I + c_i^2 X^2 \right)^{1/2} \right]^{-1} \left( I + \left( I + c_i^2 X^2 \right)^{1/2} \right)^{1/2} \]
\[ = \gamma X^2. \]

This completes the proof. \( \square \)
Proposition 13 Let $\gamma \in \mathbb{R}, \gamma > 0$ be fixed. Assume that $0 \leq C_i \leq \gamma I$, $1 \leq i \leq N$. Consider the map $F : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H})$ as defined in Eq.(48). Then under either one of the following two additional conditions

1. $X$ is strictly positive,

2. $C_i, 1 \leq i \leq N$, and $X$ are compact, not necessarily strictly positive,

the following holds

$$0 \leq X \leq \gamma I \Rightarrow 0 \leq F(X) \leq \gamma I.$$  \hfill (220)

Proof (of Proposition 13) It is clear that $F(X) \geq 0$. For $X = 0$, we have

$$F(0) = \sum_{i=1}^{N} w_i C_i \leq \gamma I \text{ since } C_i \leq \gamma I \forall i = 1, \ldots, N.$$  

Assume now that $X \neq 0$. By Lemma 38, we have

$$X F(X) X \leq \gamma X^2 \iff X[\gamma I - F(X)]X \geq 0.$$  

If $X$ is strictly positive, that is $\ker(X) = \{0\}$, then by Lemma 34, the previous inequality implies $\gamma I - F(X) \geq 0 \iff F(X) \leq \gamma I$.

Assume now that $X$ is compact and singular, $X \neq 0$. Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be the set of eigenvalues of $X$, $\lambda_k \geq 0$, arranged in decreasing order, with corresponding orthonormal eigenvectors $\{e_k\}_{k \in \mathbb{N}}$ forming an orthonormal basis in $\mathcal{H}$. Then

$$X = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k \text{ and } X \leq \gamma I \iff \lambda_1 \leq \gamma.$$  

For any $0 < \delta \leq \lambda_1$, define the following operator

$$X' = \sum_{k=1}^{\infty} \lambda'_k e_k \otimes e_k, \text{ where } \lambda'_k = \begin{cases} \lambda_k & \text{if } \lambda_k > 0, \\ \delta & \text{if } \lambda_k = 0. \end{cases}$$  

Then $X'$ is compact, strictly positive, with $X' \leq \gamma I$ and $\|X - X'\| < \delta$. Thus $F(X') \leq \gamma I$ by the first part of the proposition. By Proposition 11,

$$\|F(X) - F(X')\| \leq \frac{8\gamma^2}{\varepsilon^2} \left(1 + \sqrt{\frac{\gamma}{\varepsilon}}\right) \left(3 + \sqrt{\frac{\gamma}{\varepsilon}}\right) \|X - X'\|.$$  

This implies that

$$\|F(X)\| \leq \|F(X')\| + \frac{8\gamma^2}{\varepsilon^2} \left(1 + \sqrt{\frac{\gamma}{\varepsilon}}\right) \left(3 + \sqrt{\frac{\gamma}{\varepsilon}}\right) \|X - X'\|$$

$$\leq \gamma + \frac{8\gamma^2}{\varepsilon^2} \left(1 + \sqrt{\frac{\gamma}{\varepsilon}}\right) \left(3 + \sqrt{\frac{\gamma}{\varepsilon}}\right) \delta.$$  

Since $\delta$ can be arbitrarily close to zero, this implies that $\|F(X)\| \leq \gamma$. With the additional conditions that $C_i, 1 \leq i \leq N$ are compact, the operator $F(X)$ is self-adjoint, compact, positive, and thus $0 \leq F(X) \leq \gamma I$. \hfill \Box
Proof (of Theorem 13) As with the entropic 2-Wasserstein distance, the Sinkhorn divergence $S_\epsilon^{d_2}(N(m_0, C_0), N(m_1, C_1))$ is the sum of the squared Euclidean distance $||m_0 - m_1||^2$ and the Sinkhorn divergence $S_\epsilon^{d_2}(N(0, C_0), N(0, C_1))$. We can thus consider the means and covariance operators separately.

The barycentric mean is obviously the Euclidean mean $\bar{m} = \sum_{i=1}^{N} w_im_i$.

Consider now the centered Gaussian measures $N(0, C_0)$, $N(0, C_i)$, $1 \leq i \leq N$.

Define the function $F : \text{Sym}^+(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \to \mathbb{R}$, $1 \leq i \leq N$, by

$$F(C) = \sum_{i=1}^{N} w_i S_\epsilon^{d_2}(N(0, C_0), N(0, C_i)).$$

Then $F$ is strictly convex, since $S_\epsilon^{d_2}$ is strictly convex, thus its minimum, if it exists, is unique. By Proposition 9,

$$DF(X_0) = 0 \iff X_0 = F(X_0)$$

where $F : \text{Sym}^+(\mathcal{H}) \to \text{Sym}^+(\mathcal{H})$ is the map defined by Eq.(48). By Proposition 12, $F$ is a compact map on $\text{Sym}^+(\mathcal{H})$. Let $\gamma \in \mathbb{R}$, $\gamma > 0$ be such that $C_i \leq \gamma I$, $1 \leq i \leq N$ and consider the set

$$\mathcal{K} = \{X \in \text{Sym}^+(\mathcal{H}) : 0 \leq X \leq \gamma I\}. \quad (221)$$

Then $\mathcal{K}$ is a closed, bounded, convex subset of $\mathcal{L}(\mathcal{H})$. By Proposition 13, $X \in \mathcal{K} \Rightarrow F(X) \in \mathcal{K}$. Thus by Schauder Fixed Point Theorem, there exists $X_0 \in \mathcal{K}$ such that $X_0 = F(X_0)$, which must be the unique global minimizer of the strictly convex function $F$. Clearly, with $c_\epsilon = \frac{4}{\epsilon}$,

$$X_0 > 0 \iff \sum_{i=1}^{N} w_i C_i^{1/2} \left( I + \left( I + c_\epsilon^{2} C_i^{1/2} X_0 C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} > 0.$$  

Since $0 \leq X_0 \leq \gamma I$, $0 \leq C_i \leq \gamma I$, $1 \leq i \leq N$,

$$\left( 1 + \left( 1 + c_\epsilon^{2} \gamma^2 \right)^{1/2} \right)^{-1} \sum_{i=1}^{N} w_i C_i^{1/2} \leq \sum_{i=1}^{N} w_i C_i^{1/2} \left( I + \left( I + c_\epsilon^{2} C_i^{1/2} X_0 C_i^{1/2} \right)^{1/2} \right)^{-1} C_i^{1/2} \leq \frac{1}{2} \sum_{i=1}^{N} w_i C_i,$$

Thus it follows that $X_0 > 0 \iff \sum_{i=1}^{N} w_i C_i > 0.$ \qed
11 Comparison of barycenter fixed point equations

We now show that for \( \dim(\mathcal{H}) \geq 2 \), \( \sum_{i=1}^{N} w_{i} C_{i} > 0 \), Eq.(47) has uncountably infinitely many positive, singular solutions.

Consider first the case \( \mathcal{H} = \mathbb{R} \) and \( C_{i} = \sigma_{i}^{2} \), \( i = 1, \ldots, N \).

**Lemma 39** Assume that \( \sum_{i=1}^{N} w_{i} \sigma_{i}^{2} > 0 \). The function

\[
F(x) = \left( 1 + (1 + c_{x}^{2} x^{2})^{1/2} \right) \sum_{i=1}^{N} w_{i} \sigma_{i}^{2} \left( 1 + (1 + c_{x}^{2} x^{2})^{1/2} \right)^{-1}, \quad x \geq 0, \tag{222}
\]

has a unique fixed point \( x^{*} \), which satisfies \( x^{*} > 0 \). The function

\[
G(x) = \frac{\epsilon}{4} \left( -1 + \left( \sum_{i=1}^{N} w_{i} \left( 1 + c_{x}^{2} \sigma_{i}^{2} x^{2} \right)^{1/2} \right)^{2} \right)^{1/2}, \quad x \geq 0, \tag{223}
\]

has two fixed points, namely \( x^{*} \) and \( x_{0} = 0 \).

*Proof* The fixed point equation \( x = F(x) \) is equivalent to

\[
\left[ 1 + (1 + c_{x}^{2} x^{2})^{1/2} \right]^{-1} x - \sum_{i=1}^{N} w_{i} \sigma_{i}^{2} \left( 1 + (1 + c_{x}^{2} x^{2})^{1/2} \right)^{-1} = 0.
\]

Consider the left hand side, which is \( f(x) = \left( 1 + (1 + c_{x}^{2} x^{2})^{1/2} \right)^{-1} [x - F(x)] \), with \( f(0) = -\frac{\epsilon}{4} \sum_{i=1}^{N} w_{i} \sigma_{i}^{2} < 0 \), \( \lim_{x \to \infty} f(x) = \frac{\epsilon}{4} > 0 \), and

\[
f'(x) = \left[ 1 + (1 + c_{x}^{2} x^{2})^{1/2} \right]^{-1} \left( 1 + c_{x}^{2} x^{2} \right) + \left( 1 + c_{x}^{2} x^{2} \right)^{1/2} \cdot
\]

\[
+ \frac{8}{\epsilon} \sum_{i=1}^{N} w_{i} \sigma_{i}^{4} \left( 1 + (1 + c_{x}^{2} \sigma_{i}^{2} x^{2})^{1/2} \right)^{-2} \left( 1 + c_{x}^{2} \sigma_{i}^{2} x^{2} \right)^{-1/2} \geq 0 \quad \forall x \geq 0.
\]

Thus \( f(x) \) is strictly increasing on \([0, \infty)\) and hence there must exist a unique \( x^{*} > 0 \) at which \( f(x^{*}) = 0 \iff x^{*} = F(x^{*}) \). Since \( x^{*} > 0 \), by Proposition 9,

\[
x^{*} = F(x^{*}) \iff x^{*} = G(x^{*}) = \frac{\epsilon}{4} \left( -1 + \left( \sum_{i=1}^{N} w_{i} \left( 1 + c_{x}^{2} \sigma_{i}^{2} x^{2} \right)^{1/2} \right)^{2} \right)^{1/2}.
\]

It is obvious that \( G \) has another fixed point \( x_{0} = 0 \). \( \square \)

**Lemma 40** Let \( u \in \mathcal{H} \), \( \|u\| = 1 \). Consider the rank-one operator \( u \otimes u : \mathcal{H} \to \mathcal{H} \) defined by \( (u \otimes u)x = \langle u, x \rangle u \). Then for any \( A \in \mathcal{L}(\mathcal{H}) \) and any \( k \in \mathbb{N} \),

\[
[(u \otimes u)A(u \otimes u)]^{k} = \langle u, Au \rangle^{k} (u \otimes u). \tag{224}
\]

If \( A \) is self-adjoint, positive, then

\[
[(u \otimes u)A(u \otimes u)]^{1/k} = \langle u, Au \rangle^{1/k} (u \otimes u). \tag{225}
\]
Since \((u \otimes u)A(u \otimes u)x = (u \otimes u)A(u, x)u = \langle u, x \rangle (u, Au)u = (u, Au)(u \otimes u)x\),

Since \((u \otimes u)^2 = (u \otimes u)\), we have

\[
[(u \otimes u)A(u \otimes u)]^2 x = \langle u, Au \rangle (u \otimes u)A(u \otimes u)x = (u, Au)^2(u \otimes u)x.
\]

For the first expression, the general case then follows by induction.

Consider the cases for the solution of the following one-dimensional fixed point equation since otherwise \(\sum_{i=1}^\infty u_i C_i u_1 \neq 0\) is not satisfied for at least one \(u_i \in \mathcal{H}, u_i \neq 0\), since otherwise \(C_1 = \cdots = C_N = 0\). If \(\sum_{i=1}^N w_i C_i > 0\), then \(\sum_{i=1}^N w_i (u_i C_i u) > 0\) for \(u_i \in \mathcal{H}, u_i \neq 0\). Thus under this assumption, for \(\dim(\mathcal{H}) \geq 2\), \(G\) has uncountably infinitely many fixed points of the form \(X_u = x_u(u \otimes u)\).

\[X_0 = 0\] is a fixed point of \(G\). Let \(u \in \mathcal{H}, ||u|| = 1\) be such that \(\sum_{i=1}^N w_i \langle u, C_i u \rangle > 0\). Then \(X_u = x_u(u \otimes u)\) is a fixed point of \(G\), where \(x_u\) is the unique positive solution of the following one-dimensional fixed point equation

\[
x = \frac{\epsilon}{4} \left[ -1 + \left( \sum_{i=1}^N w_i \left( 1 + c_i^2 x(u, C_i u) \right)^{1/2} \right)^2 \right]^{1/2}.
\]

The condition \(\sum_{i=1}^N w_i \langle u, C_i u \rangle > 0\) is satisfied for at least one \(u_i \in \mathcal{H}, u_i \neq 0\), since otherwise \(C_1 = \cdots = C_N = 0\). If \(\sum_{i=1}^N w_i C_i > 0\), then \(\sum_{i=1}^N w_i (u_i C_i u) > 0\) for \(u_i \in \mathcal{H}, u_i \neq 0\). Thus under this assumption, for \(\dim(\mathcal{H}) \geq 2\), \(G\) has uncountably infinitely many fixed points of the form \(X_u = x_u(u \otimes u)\).

\[X_0 = 0\] is always a fixed point of \(G\). Consider the rank-one operator \(u \otimes u, ||u|| = 1\), with eigenvalue 1 and eigenvector \(u\), we have \((u \otimes u)^{1/2} = u \otimes u\). By Lemma 40,

\[
(u \otimes u)^{1/2} C_i (u \otimes u)^{1/2} = (u \otimes u)C_i (u \otimes u) = \langle u, C_i u \rangle (u \otimes u).
\]

Therefore, for \(X_u = x(u \otimes u)\),

\[
I + c_i^2 X_u^{1/2} C_i X_u^{1/2} = (I - u \otimes u) + (1 + c_i^2 x(u, C_i u)) (u \otimes u).
\]

Since \((I - u \otimes u)^2 = (I - u \otimes u)\) and \((I - u \otimes u)(u \otimes u) = 0\), by Lemma 40,

\[
\left( I + c_i^2 X_u^{1/2} C_i X_u^{1/2} \right)^{1/2} = (I - u \otimes u) + (1 + c_i^2 x(u, C_i u))^{1/2} (u \otimes u).
\]
Applying the same argument and using the fact that $\sum_{i=1}^{N} w_i = 1$, we have

$$G(X_u) = \frac{\epsilon}{4} \left[ -1 + \left( \sum_{i=1}^{N} w_i \left( 1 + c_i^2 x(u, C_i u) \right)^{1/2} \right)^2 \right]^{1/2} (u \otimes u).$$

Thus the fixed point equation $X_u = G(X_u)$ becomes

$$x = \frac{\epsilon}{4} \left[ -1 + \left( \sum_{i=1}^{N} w_i \left( 1 + c_i^2 x(u, C_i u) \right)^{1/2} \right)^2 \right]^{1/2}.$$

As shown in Lemma 39, under the condition $\sum_{i=1}^{N} w_i \langle u, C_i u \rangle > 0$, this one-dimensional fixed point equation has a unique positive solution $x_u^*$. Thus $X_u = x_u^*(u \otimes u)$ is a fixed point of $G$. \hfill \Box

When $\epsilon = 0$, the fixed points $X_u$ of $G$ in Proposition 14 admit a closed form.

**Proposition 15** Let $C_i \in \text{Sym}^+(H)$, $1 \leq i \leq N$ be fixed. Consider the following map $G : \text{Sym}^+(H) \to \text{Sym}^+(H)$, defined by

$$G(X) = \sum_{i=1}^{N} w_i (X^{1/2} C_i X^{1/2})^{1/2}. \quad (228)$$

Then $X_0 = 0$ is a fixed point of $G$. Let $u \in H, ||u|| = 1$ be such that $\sum_{i=1}^{N} w_i \langle u, C_i u \rangle^{1/2} > 0$, then the following is a fixed point of $G$

$$X_u = \left( \sum_{i=1}^{N} w_i \langle u, C_i u \rangle^{1/2} \right)^2 (u \otimes u). \quad (229)$$

The condition $\sum_{i=1}^{N} w_i \langle u, C_i u \rangle^{1/2} > 0$ is satisfied for at least one $u \in H, u \neq 0$, since otherwise $C_1 = \cdots = C_N = 0$. If $\sum_{i=1}^{N} w_i C_i > 0$, then $\sum_{i=1}^{N} w_i \langle u, C_i u \rangle > 0 \forall u \in H, u \neq 0$. Since $w_i > 0$, we have at least one $i$ for which $\langle u, C_i u \rangle > 0$. This implies that under this assumption, $\sum_{i=1}^{N} w_i \langle u, C_i u \rangle^{1/2} > 0$ is satisfied for all $u \in H, u \neq 0$. In this case, when $\dim(H) \geq 2$, $G$ has uncountably infinitely many fixed points of the form $X_u$.

**Example.** Consider the simplest setting $C_1 = \cdots = C_N = C > 0$, then in both Propositions 14 and 15,

$$X = G(X) \iff X = (X^{1/2} C X^{1/2})^{1/2}. \quad (230)$$

One can immediately see that some of the solutions of the above equation include $X = 0$, $X = C$, $X_u = \langle u, C u \rangle (u \otimes u)$, for any $u \in H, ||u|| = 1$, including $X_k = \lambda_k (c_k \otimes c_k)$, $k \in \mathbb{N}$, where $\lambda_k$ are the eigenvalues of $C$, with corresponding orthonormal eigenvectors $\{c_k\}_{k \in \mathbb{N}}$. 
Proof (of Proposition 15) By Lemma 40, $X_u = x(u \otimes u)$, $x > 0$, we have

$$G(X_u) = \sqrt{x} \left( \sum_{i=1}^{N} w_i \langle u, C_i u \rangle^{1/2} \right) (u \otimes u).$$

Thus the fixed point equation $X_u = G(X_u)$ becomes

$$x = \left( \sum_{i=1}^{N} w_i \langle u, C_i u \rangle^{1/2} \right)^2 > 0$$

by the assumption $\sum_{i=1}^{N} w_i \langle u, C_i u \rangle^{1/2} > 0$. Hence $\left( \sum_{i=1}^{N} w_i \langle u, C_i u \rangle^{1/2} \right)^2 (u \otimes u)$ is a fixed point of $G \forall u \in H$, $\|u\| = 1$. □

Proof (of Theorem 14 - Singular solutions of fixed point equations) This is the combination of Propositions 7, 14, and 15. □

Comparison with the finite-dimensional setting. In the case $H = \mathbb{R}^n$, the existence of the strictly positive solution of Eq.(47) is proved via the Brouwer Fixed Point Theorem (see e.g. [10]) as follows. This is the technique employed by [1] for the case $\epsilon = 0$ and [45] for the case $\epsilon > 0$.

Theorem 29 (Brouwer Fixed Point Theorem) Let $M \subset \mathbb{R}^n$ be a compact convex subset and $f : M \to M$ be continuous. Then $f$ has a fixed point in $M$.

Remark 8 Unlike the Banach Fixed Point Theorem, both Brouwer and Schauder Fixed Point Theorems guarantee the existence of one fixed point but not its uniqueness. This needs to be proved via other means, e.g. strict convexity.

Assume that there exist $\alpha, \beta \in \mathbb{R}$, $\alpha > 0, \beta > 0$ such that $\alpha I \leq C_i \leq \beta I$, $1 \leq i \leq N$. Consider the set $\mathcal{K}_2 = \{ X \in \text{Sym}^{++}(n) : \alpha I \leq X \leq \beta I \}$. Then by Lemma 41

$$\alpha^2 I \leq \alpha X = X^{1/2} \alpha X^{1/2} \leq X^{1/2} C_i X^{1/2} \leq X^{1/2} \beta X^{1/2} = \beta X \leq \beta^2 I.$$  

It follows that

$$\alpha I \leq G(X) = \frac{\epsilon}{4} \left[ -I + \left( \sum_{i=1}^{N} w_i \left( I + \epsilon_2^2 X^{1/2} C_i X^{1/2} \right)^{1/2} \right)^2 \right]^{1/2} \leq \beta I. \quad (231)$$

Thus the continuous map $G$ maps the compact convex set $\mathcal{K}_2$ into itself, thus $G$ has a fixed point in $\mathcal{K}_2$. This strictly positive solution of Eq.(47) is then precisely the unique solution of Eq.(45).

It is not clear, however, whether this proof strategy can be extended to the infinite-dimensional setting. This is because one can no longer assume that there is a uniform lower bound of the form $\alpha I$ for the $C_i$’s and $X$ as above.
One might assume instead that there is an operator \( C > 0 \) such that \( C_i \geq C \), \( 1 \leq i \leq N \) and consider the set \( K_3 = \{ X \in \text{Sym}^+(H) : C \leq X \leq \beta I \} \). This does not help, however, since the condition \( X \geq C > 0 \) does not imply that \( X^{1/2}CX^{1/2} \geq C^2 \). The following is a counterexample:

\[
X = \begin{pmatrix}
1.6254 & -0.6825 & -1.2503 \\
-0.6825 & 1.9105 & 0.0516 \\
-1.2503 & 0.0516 & 2.2376 \\
\end{pmatrix},
\quad C = \begin{pmatrix}
0.2867 & -0.3297 & 0.1976 \\
-0.3297 & 0.6925 & -0.2484 \\
0.1976 & -0.2484 & 0.1392 \\
\end{pmatrix}.
\]

(232)

It can be verified numerically that \( X \geq C \) but \( X^{1/2}CX^{1/2} \nless C^2 \).

We note also that the condition \( 0 < \alpha I \leq C_i \leq \beta I, 1 \leq i \leq N \), above is more restrictive than the condition \( \sum_{i=1}^N w_i C_i > 0 \) stated in Theorem 13, which guarantees the existence of a strictly positive solution of Eq.(47). As an example, let \( H = \mathbb{R}^N \) and \( \{ e_i \}_{i=1}^N \) be an orthonormal basis in \( H \). Define

\[
C_i = e_i \otimes e_i \quad \text{then} \quad \sum_{i=1}^N w_i C_i = \sum_{i=1}^N w_i (e_i \otimes e_i) > 0,
\]

(233)

guaranteeing that \( \tilde{C} > 0 \), even though all the \( C_i \)'s are singular.

12 Miscellaneous Technical Results

**Operator monotone functions.** We recall the concept of operator monotone functions. Let \( A, B \) be two self-adjoint bounded operators on \( H \), then we say \( A \leq B \) if \( B - A \geq 0 \). Let \( I \subseteq \mathbb{R} \) be an interval. A function \( f : I \rightarrow \mathbb{R} \) is said to be **operator monotone** if \( A \leq B \Rightarrow f(A) \leq f(B) \). Some of the well-known operator monotone functions are the following.

**Proposition 16 (see e.g. [70])** The function \( f(t) = t^r \) on \([0, \infty)\) is operator monotone if and only if \( 0 \leq r \leq 1 \).

In particular \( f(t) = t^{1/2} \) on \([0, \infty)\) is operator monotone. Thus \( 0 \leq A \leq B \Rightarrow 0 \leq A^{1/2} \leq B^{1/2} \). The function \( f(t) = t^2 \) on \([0, \infty)\), on the other hand, is **not** operator monotone. However, we still have \( 0 \leq A \leq \lambda I \Rightarrow 0 \leq \lambda A \leq \lambda^2 I \) and \( A \geq \lambda I \Rightarrow A^2 \geq \lambda^2 I, \lambda > 0 \).

The following is the generalization of Proposition V.1.6 in [8] to the infinite-dimensional setting, with the additional assumption that \( A, B \) be invertible, since if \( \dim(H) = \infty \), then \( A > 0 \) does not imply that \( A \) is invertible.

**Proposition 17** The function \( f(t) = -\frac{1}{t} \) is operator monotone on \((0, \infty)\).

Thus if \( A, B \in \mathcal{L}(H) \) are invertible then \( 0 < A \leq B \Rightarrow A^{-1} \geq B^{-1} \).

**Lemma 41** (see [8], Lemma V.1.5)

\[
A \leq B \Rightarrow X^*AX \leq X^*BX \quad \forall X \in \mathcal{L}(H).
\]

(234)
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