MIXED BRUHAT OPERATORS AND
YANG-BAXTER EQUATIONS FOR WEYL GROUPS

FRANCESCO BRENTI, SERGEY FOMIN, AND ALEXANDER POSTNIKOV

1. Introduction

In this paper, we introduce and study a family of operators which act in the
span of a Weyl group $W$ and provide a multi-parameter solution to the quantum
Yang-Baxter equations of the corresponding type. These operators are then used
to obtain new results (as well as new proofs of the known facts) concerning the
Bruhat order of $W$.

Let us briefly review the contents of the paper. Section 2 is devoted to prelimi-
naries related to Coxeter groups and associated Yang-Baxter equations. The mixed
Bruhat operators $M_\tau$ are defined in Section 3 by the formula

$$M_\tau(w) = \begin{cases} 
p_\tau \tau w & \text{if } \ell(\tau w) > \ell(w); 
q_\tau \tau w & \text{if } \ell(\tau w) < \ell(w),
\end{cases}$$

where $p_\tau$ and $q_\tau$ are scalar parameters that depend on a reflection $\tau$. In Section 3, we
also state our main result, which describes a family of choices for the parameters $p_\tau$
and $q_\tau$ such that the associated operators $R_\tau = 1 + \varepsilon M_\tau$ satisfy the Yang-Baxter
equations. The well known $W$-analogue of the classical Yang’s solution can be
obtained from our solution as a particular degeneration.

In Section 4, we define the quantum Bruhat operators as certain limiting case
of the $M_\tau$. These operators play an important role (which we briefly explain)
in the explicit description of the multiplicative structure of the (small) quantum
cohomology ring of the flag manifold of the corresponding type.

Section 5 contains the proof of the main result. To make the presentation more
accessible, we first treat the case of the symmetric group, and then provide the proof
in complete generality. We also show how our family of solutions arises naturally
in an attempt to solve the Yang-Baxter equations within this class of operators.

Section 6 is devoted to combinatorial applications of our operators. For an
arbitrary element $u \in W$, we define a graded poset with unique minimal element $u$
that we call a tilted Bruhat order. The usual Bruhat order corresponds to the special
case $u = e$. We then prove that tilted Bruhat orders are lexicographically shella-
ble graded posets whose every interval is Eulerian. This generalizes well known results
of Verma, Björner, Wachs, and Dyer.

Date: April 8, 1998.

1991 Mathematics Subject Classification. Primary 05E15, Secondary 06A07, 14M15, 20F55.
Key words and phrases. Coxeter group, Weyl group, Bruhat order, Yang-Baxter equation,
quantum cohomology, Eulerian poset, Möbius function.

The authors were supported in part by MSRI (NSF grant #DMS-9022140). The second author
was also supported by NSF grant #DMS-9700927.
We first introduce some standard terminology and notation related to Coxeter groups and root systems. In what follows, $W$ is a Weyl group, and $S$ is the set of its simple reflections. In particular, $(W, S)$ is a Coxeter system, i.e., the finite (Coxeter) group $W$ is given by the presentation

$$(st)^{m(s, t)} = 1, \quad s, t \in S,$$

where the nonnegative integers $m(s, t)$ satisfy $m(s, s) = 1$ and $m(s, t) = m(t, s) > 1$ for $s \neq t$. The group $W$ is finite; let $w_0$ denote its longest element. Most constructions in this section can be extended naturally to arbitrary Coxeter groups.

For an element $w \in W$, an expansion $w = s_1 \cdots s_l$ of minimal possible length $l$ is called a reduced decomposition. The number $l = \ell(w)$ is the length of $w$. The elements of the set $T = \{wsu^{-1} : w \in W; s \in S\}$ are the reflections of $W$.

The Bruhat order on $W$ is defined as follows: $u \leq v$ if and only if there exist $t_1, \ldots, t_r \in T$ such that $t_r \cdots t_1 u = v$ and $\ell(t_i \cdots t_1 u) > \ell(t_{i-1} \cdots t_1 u)$ for $i = 1, \ldots, r$.

Geometrically, the group $W$ can be represented in terms of a root system $\Phi$. A subgroup $W'$ of $W$ generated by a subset $A \subseteq T$ is called a reflection subgroup; it corresponds to a root subsystem of $\Phi$. Hence $W'$ is again a Coxeter group, with the set of canonical (Coxeter) generators $S'$ corresponding to the simple roots of this subsystem. (This can be extended to any Coxeter group: see 11 or 16.2.) We will only be interested in the case where $W'$ is a dihedral reflection subgroup, i.e., $S'$ has two elements. A dihedral reflection subgroup is maximal if it is not contained in another such subgroup. Maximal dihedral subgroups correspond to two-dimensional root subsystems obtained by intersecting $\Phi$ with a plane spanned by a pair of positive roots.

Let $N = \ell(w_0)$. Following Dyer [12], we say that a bijection $\varphi : T \to \{1, \ldots, N\}$ is a (total) reflection ordering if, for any dihedral reflection subgroup $W'$ with canonical generators $a$ and $b$, the sequence $\varphi(a), \varphi(ab), \varphi(abab), \ldots, \varphi(babab), \varphi(bab), \varphi(b)$ is either increasing or decreasing. (It is enough to require this for every maximal dihedral subgroup.) Reflection orderings correspond (bijectively) to reduced decompositions of $w_0$ in the following standard way: $\varphi$ is a reflection ordering if and only if there exists a reduced decomposition $w_0 = s_1 \cdots s_N$ such that

$$(2.1) \quad \varphi^{-1}(j) = sNs_{N-1} \cdots s_{j+1}s_{j}s_{j+1} \cdots s_{N-1}s_N$$

for $j = 1, \ldots, N$.

**Definition 2.1.** A family $\{R_t\}_{t \in T}$ of elements of a monoid is called an (extensible) solution to the Yang-Baxter equations for $W$ if for any dihedral reflection subgroup $W'$ of $W$ with canonical generators $a$ and $b$, we have

$$(2.2) \quad R_aR_{aba}R_{abab} \cdots R_{bab}R_b = R_bR_{bab} \cdots R_{abab}R_{aba}R_a.$$

In particular, if $a, b \in T$ and $ab = ba$, then $R_aR_b = R_bR_a$. The collection $\{R_t\}_{t \in T}$ satisfying the Yang-Baxter equations (2.2) is frequently called an (extensible) $R$-matrix (of the corresponding type); we will not use this terminology here.

The definition above makes sense for any finite Coxeter group. In the case of a Weyl group, equations (2.2), stated case by case in terms of the root system for $W$,
were given by Cherednik (implicit in [6] and explicit in [7, Definition 2.1a]), along with a number of solutions.

**Remark 2.2.** The word “extensible” (which we will later omit; cf. [7, Definition 2.2]) indicates that we ask for (2.2) to be satisfied for all dihedral subgroups, not just for the maximal ones. (The distinction is only relevant in non-simply-laced cases.) This stronger condition, however not needed for the general “Yang-Baxter machinery” to work, will actually be satisfied by all solutions constructed in this paper, which explains our choice of definition.

For the type $A_{n-1}$, the Weyl group is the symmetric group $S_n$, the set $T$ consists of all transpositions $(ij) \in S_n$, and the equations (2.2) are the celebrated (quantum) Yang-Baxter equations (see, e.g., [17]). Let us explain. Let $R_{ij}$ be a shorthand for $R(ij)$, then (2.2) becomes

\[ R_{ij}R_{kl} = R_{kl}R_{ij} \text{ if } i, j, k, l \text{ are distinct}; \]

\[ R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij} \text{ if } i < j < k. \]

**Example 2.3.** The first solution to the Yang-Baxter equations was given by Yang in his pioneering paper [25], where he observed that the elements

\[ R_{ij} = 1 + \frac{(ij)}{x_j - x_i} \]

of the group algebra of the symmetric group $S_n$ satisfy (2.3)–(2.4), for any choice of distinct parameters $x_1, \ldots, x_n$. This generalizes to an arbitrary Weyl group as follows [6, 7]:

\[ R_\tau = 1 + \frac{\kappa_\tau}{\langle x, \alpha \rangle}, \]

where $\alpha$ is the positive root corresponding to $\tau$, and $\kappa_\tau$ is a scalar whose value only depends on whether the root $\alpha$ is short or long. (In other words, $\kappa_\tau = \kappa_\sigma$ if reflections $\tau$ and $\sigma$ are conjugate to each other.)

The fact that every two reduced decompositions of the element $w_0 \in W$ are related by a sequence of elementary Coxeter transformations (see, e.g., [16, Section 8.1]) translates (using (2.1)) into every two reflection orderings being related by a sequence of Yang-Baxter-type moves of the form

\[ \ldots, a, aba, ababa, \ldots, bab, b, \ldots, ababa, aba, a, \ldots, \]

where $a$ and $b$ are the canonical generators of the (maximal) dihedral subgroup they generate. This implies the following statement.

**Proposition 2.4.** Let $\{R_\tau\}$ be a solution of the Yang-Baxter equations for a finite Coxeter group $W$, and let $\varphi : T \to \{1, \ldots, N\}$ be a reflection ordering on $T$. Then the product

\[ \prod_{i=1}^{N} R_{\varphi^{-1}(i)} = R_{\varphi^{-1}(1)} \cdots R_{\varphi^{-1}(i)}(w_0) \]

does not depend on the choice of a reflection ordering $\varphi$. 

3. Mixed Bruhat operators

We will work over a ground field \( k \) of characteristic 0.

**Definition 3.1.** Let \( \{ p_\tau \} \) and \( \{ q_\tau \} \) be two families of scalar parameters indexed by reflections \( \tau \in T \). The mixed Bruhat operators \( M_\tau \) are linear operators acting in the \( k \)-span \( k[W] \) of the group \( W \) by

\[
M_\tau(w) = \begin{cases} 
   p_\tau \tau w & \text{if } \ell(\tau w) > \ell(w); \\
   q_\tau \tau w & \text{if } \ell(\tau w) < \ell(w).
\end{cases}
\]

Let \( \varepsilon \) be a formal variable with values in \( k \), and define the operators

\[
R_\tau = 1 + \varepsilon M_\tau .
\]

We will now describe a particular multi-parametric construction that allows to choose the \( p_\tau \) and \( q_\tau \) so that the operators \( R_\tau \) satisfy the Yang-Baxter equations (2.2).

**Definition 3.2.** A function \( \alpha \mapsto E(\alpha) \) defined on the set of positive roots is called multiplicative if, whenever \( \alpha, \beta, \) and \( \alpha + \beta \) are positive roots, we have

\[
E(\alpha + \beta) = E(\alpha) E(\beta).
\]

To construct such a function, simply assign arbitrary values to the simple roots, and then extend by multiplicativity. A typical example of a multiplicative function is given by

\[
E(\alpha) = e^{\langle \alpha, x \rangle},
\]

where \( x \) is an arbitrary vector. Notice, however, that (3.4) does not allow for \( E(\alpha) = 0 \), a possibility that we do not want to exclude.

**Theorem 3.3.** Let \( \alpha \mapsto E_1(\alpha) \) and \( \alpha \mapsto E_2(\alpha) \) be multiplicative functions on the set of positive roots such that \( E_1(\alpha) \neq E_2(\alpha) \) for every \( \alpha \). Let \( \kappa_\tau \) be a scalar whose value only depends on the length of the positive root \( \alpha \) corresponding to \( \tau \). Define parameters \( p_\tau, q_\tau, \) for \( \tau \in T \), by

\[
p_\tau = \frac{\kappa_\tau E_1(\alpha)}{E_1(\alpha) - E_2(\alpha)}
\]

and

\[
q_\tau = \frac{\kappa_\tau E_2(\alpha)}{E_1(\alpha) - E_2(\alpha)}.
\]

Then the operators \( R_\tau \) given by (3.1)–(3.2) satisfy the quantum Yang-Baxter equations (2.2).

**Example 3.4.** Consider the type \( A_{n-1} \) case where \( W \) is the symmetric group \( S_n \). For a reflection \( \tau = (ij) \), we will use the notation \( p_{ij}, q_{ij}, M_{ij}, \) and \( R_{ij} \) instead of \( p_\tau, q_\tau, M_\tau, \) and \( R_\tau \). Hence

\[
R_{ij}(w) = \begin{cases} 
   w + \varepsilon p_{ij} \tau w & \text{if } \ell(\tau w) > \ell(w); \\
   w + \varepsilon q_{ij} \tau w & \text{if } \ell(\tau w) < \ell(w).
\end{cases}
\]
The positive root corresponding to \( \tau = (ij) \) is \( \alpha = \alpha_i + \cdots + \alpha_{j-1} \), where \( \alpha_1, \ldots, \alpha_{n-1} \) are the simple roots, ordered in a standard way. Thus the multiplicative functions \( E_1 \) and \( E_2 \) are determined by the values \( p_i = E_1(\alpha_i) \) and \( q_i = E_2(\alpha_i) \), as follows: \( E_1(\alpha) = p_i \cdots p_{j-1} \), \( E_2(\alpha) = q_i \cdots q_{j-1} \). This leads to

\[
(3.8) \quad p_{ij} = \frac{\kappa p_i \cdots p_{j-1}}{p_i \cdots p_{j-1} - q_i \cdots q_{j-1}}
\]

and

\[
(3.9) \quad q_{ij} = \frac{\kappa q_i \cdots q_{j-1}}{p_i \cdots p_{j-1} - q_i \cdots q_{j-1}}.
\]

(Since all roots have the same length, we drop the subscript \( \tau \) in \( \kappa_\tau \).) Substituting this into (3.7), we obtain a family of solutions of the Yang-Baxter equations (of type \( A \)). In formulas (3.8)–(3.9), \( \kappa \) is an arbitrary scalar, while the parameters \( p_i \) and \( q_i \) should be chosen so that none of the denominators vanish. Notice that we do not use a single set of parameters \( t_i = q_i/p_i \) in order to, first, keep the symmetry between the \( p_i \) and the \( q_i \) and, second, allow for the possibility of \( p_i = 0 \).

**Remark 3.5.** The analogue (2.6) of Yang’s solution of the Yang-Baxter equation can be obtained from the solution given in Theorem 3.3 as a particular limiting case. Let \( \kappa_\tau = \delta \kappa_\tau \), where \( \delta \) is a scalar. Fix a vector \( x \), and set \( E_1(\alpha) = e^{\delta(\alpha,x)} \) and \( E_2(\alpha) = 1 \). Making these substitutions into (3.8)–(3.9) and taking the limit as \( \delta \to 0 \), we obtain \( p_\tau = q_\tau = \frac{\kappa_\tau}{\langle \alpha, x \rangle} \), which means that the operators \( \lim_{\delta \to 0} M_\tau \) act by left multiplication by \( \frac{\kappa_\tau}{\langle \alpha, x \rangle} \), as desired.

4. Rescaling. Quantum Bruhat operators

Rescaling is a very simple yet sometimes helpful way of producing new solutions to the Yang-Baxter equations from existing ones. In this section, we show how rescaling of the mixed Bruhat operators leads in the limiting case to the construction of “quantum Bruhat operators” for an arbitrary Weyl group \( W \). These operators, introduced in [14] for type \( A \), appear in the analogue of Monk’s formula for the (small) quantum cohomology ring of the flag manifold (see below). In this paper, we are mainly concerned with their combinatorial applications.

Suppose that \( \{ M_\tau \}_{\tau \in T} \) is a family of mixed Bruhat operators such that the corresponding operators \( R_\tau = 1 + \varepsilon M_\tau \) satisfy the Yang-Baxter equations (2.2). Let \( \{ \gamma_w : w \in W \} \) be a collection of nonzero scalars. Then the rescaled operators \( \tilde{M}_\tau \) defined by

\[
(4.1) \quad \tilde{M}_\tau(w) = \frac{\gamma_\tau w}{\gamma_w} M_\tau(w)
\]

are also a solution to (2.2). This follows from the fact that \( M_\tau(w) \) is always a scalar multiple of \( \tau w \), and therefore \( \tilde{M}_\tau = \Gamma M_\tau \Gamma^{-1} \), where \( \Gamma(v) = \gamma_v v \) for \( v \in W \).

Let \( \text{ht}(\alpha) \) denote the height of a positive root \( \alpha \), i.e., the sum of the coefficients in the expansion of \( \alpha \) in the basis of simple roots. Then for any scalar \( h \) and any multiplicative function \( \alpha \mapsto E(\alpha) \), the function \( \alpha \mapsto \gamma^{\text{ht}(\alpha)} E(\alpha) \) is also multiplicative.

Let \( \delta \neq 0 \) be a scalar parameter (eventually, we will take \( \delta \to 0 \)), and let \( \alpha \mapsto E(\alpha) \) be a multiplicative function. Let the parameters \( p_\tau \) and \( q_\tau \) of the mixed
Bruhat operators $M_\tau$ be given by (3.5)–(3.6) with
\[ \kappa_\tau = \delta^{-1}, \]
\[ E_1(\alpha) = 1, \]
\[ E_2(\alpha) = \delta^{2ht(\alpha)}E(\alpha). \]
Using notation $F \approx G$ for $\lim_{\delta \to 0} F/G = 1$, we then obtain:
\[ p_\tau \approx \delta^{-1}, \]
\[ q_\tau \approx \delta^{2ht(\alpha)-1}E(\alpha), \]
where, as before, $\alpha$ is the positive root corresponding to $\tau$. Now let the operators $\tilde{M}_\tau$ be given by (4.1) with $\gamma_w = \delta^{\ell(w)}$. Then
\[ \tilde{M}_\tau(w) = \delta^{\ell(\tau w)-\ell(w)}M_\tau(w). \]
Combining this with (4.3) and (3.1) yields
\[ \tilde{M}_\tau(w) \approx \begin{cases} \tau w & \text{if } \ell(\tau w) = \ell(w) + 1; \\ E(\alpha)\tau w & \text{if } \ell(\tau w) = \ell(w) - \ell(\tau) \text{ and } \ell(\tau) = 2ht(\alpha) - 1; \\ 0 & \text{otherwise}. \end{cases} \]
For the symmetric group, the requirement $\ell(\tau) = 2ht(\alpha) - 1$ in (4.4) is superfluous, and we recover the type $A$ quantum Bruhat operators $Q_\tau = \lim_{\delta \to 0} \tilde{M}_\tau$ given by
\[ Q_\tau(w) = \begin{cases} \tau w & \text{if } \ell(\tau w) = \ell(w) + 1; \\ E(\alpha)\tau w & \text{if } \ell(\tau w) = \ell(w) - \ell(\tau) \text{ and } \ell(\tau) = 2ht(\alpha) - 1; \\ 0 & \text{otherwise}. \end{cases} \]

**Corollary 4.1.** Let $\{Q_\tau\}_{\tau \in T}$ be the quantum Bruhat operators defined by (4.4). Then the operators $R_\tau = 1 + \varepsilon Q_\tau$ satisfy the Yang-Baxter equations (2.3)–(2.4).

We will now briefly explain the connection between our quantum Bruhat operators and the quantum cohomology of the generalized flag manifold $G/B$. Here $G$ is a semisimple connected complex Lie group associated with the dual root system $\Phi^\vee$, and $B$ is a Borel subgroup in $G$. Let us identify each element $w \in W$ with the Schubert class
\[ [w] = \sigma_{w^{-1}} = [(B^{w^{-1}}B)/B] \in H^{2\ell(w)}(G/B, \mathbb{Z}), \]
viewed as an element of the small quantum cohomology ring. (The reader is referred to [13, 12] and references therein for relevant background.) In particular, the generators $s \in S$ will correspond to special Schubert classes $[s]$. Extending the map $w \mapsto [w]$ to a linear isomorphism between $k[W]$ and the (quantum) cohomology ring
assigns obvious meaning to expressions of the form \([Q(w)]\), where \(Q\) is an operator acting in \(k[W]\).

Let the quantum Bruhat operators \(Q_\tau\) be given by (4.4), where the values of the multiplicative function \(\alpha \mapsto E(\alpha)\) at simple roots are set equal to the corresponding deformation parameters of the quantum cohomology ring. Then the (quantum Monk’s) formula for quantum multiplication of an arbitrary Schubert class \([w]\) by a special Schubert class \([s]\) can be written as follows:

\[
[w] \ast [s] = \sum_{\alpha > 0} \langle \omega, \alpha \rangle [Q_\tau(w)] + \sum_{\alpha > 0} \langle \omega, \alpha \rangle E(\alpha) [w_{\tau}],
\]

(4.5)

where, as before, the reflection \(\tau\) corresponds to the positive root \(\alpha \in \Phi\), and \(\omega\) denotes the fundamental weight corresponding to \(s\).

For the type \(A\) case, formula (4.5) was first stated and proved in [13]. For a general type, it was given by D. Peterson (reproduced in [5], without proof).

5. Motivation and proof

In this section, we prove Theorem 3.3 and also explain the origin of our solution (3.5)–(3.6).

Let us investigate the problem of choosing the parameters \(p_\tau\) and \(q_\tau\) so that the operators \(R_\tau\) given by (3.1)–(3.2) satisfy the Yang-Baxter equations (2.2). First of all, one easily checks that, for any choice of parameters, operators \(R_\tau\) and \(R_\sigma\) commute whenever \(\tau\) and \(\sigma\) do. Therefore we only need to take care of (2.2) in the cases where both sides involve at least three factors. In particular, for type \(A\) we only have to make sure that the operators \(R_{ij}\) satisfy the quantum Yang-Baxter equation (2.4).

5.1. Cosets modulo dihedral subgroups. Notice that each operator \(R_\tau\) stabilizes the span of every left coset \(W'w\) for any subgroup \(W'\) containing \(\tau\). Let \(W'\) be a dihedral reflection subgroup. (Thus \(W'\) is of type \(A_2\), \(B_2\), or \(G_2\).) Then the span of every left coset of \(W'\) is invariant under all operators appearing in the corresponding Yang-Baxter equation (2.2). Thus the operators \(R_\tau\) satisfy (2.2) if and only if so do the restrictions of these operators onto each space \(k[W'w]\) (which has dimension 6, 8, or 12). Our plan is to explicitly write down the matrices of these restrictions, plug them into the Yang-Baxter equation, and derive the complete set of equations for the parameters \(\{p_\tau\}\) and \(\{q_\tau\}\).

The first step is to understand the combinatorics of the coset \(W'w\) as a subposet of the Bruhat order. The following statement is known to hold for any Coxeter group (see Dyer [10]): in the special case of a Weyl group, it has a simple proof provided below.

**Lemma 5.1.** Let \(W'\) be a reflection subgroup of \(W\), and let \(S'\) be its set of canonical generators. Then the Bruhat order on \(W'\) (viewed as a Coxeter group with generating set \(S'\)) coincides with the partial order induced from the Bruhat order on \(W\).

With respect to the Bruhat order on \(W\), each coset \(W'w\) has a unique minimal element \(\tilde{w}\). For any \(w' \in W'\) and \(t \in T \cap W'\), we have \(\ell(tw') < \ell(w')\) if and only if \(\ell(tw\tilde{w}) < \ell(w'\tilde{w})\).
Proof. For \( t \in T \), \( w \in W \), the condition \( \ell(tw) < \ell(w) \) is equivalent to \( w^{-1}(\alpha) < 0 \), where \( \alpha \) is the positive root associated with \( t \). This implies the first part of the lemma. To prove the second part, choose \( \tilde{w} \) to be the element of minimal length in \( W'w \) (if there are several such, pick any). Take any reflection \( t \in T \cap W' \) and the corresponding positive root \( \alpha \). Then \( \ell(t\tilde{w}) > \ell(\tilde{w}) \) and therefore \( \tilde{w}^{-1}(\alpha) > 0 \). Thus \( \tilde{w}^{-1} \) maps every positive root that corresponds to a reflection in \( W' \) into a positive root (and every negative into a negative). Hence
\[
\ell(tu') < \ell(u') \iff (u')^{-1}(\alpha) < 0 \iff \tilde{w}^{-1}(u')^{-1}(\alpha) < 0 \iff \ell(tu'\tilde{w}) < \ell(u'\tilde{w}) ,
\]
as desired. \( \square \)

Remark 5.2. Let \( W' \) be a dihedral subgroup of \( W \). The second part of Lemma 5.1 implies that the action of the mixed Bruhat operators participating in the Yang-Baxter equation for \( W' \) restricted to each invariant subspace \( k[W'] \) is canonically isomorphic to their action on \( k[W'] \) via the linear isomorphism \( w' \mapsto w'\tilde{w} \), where \( \tilde{w} \) is the unique minimal element of \( W'w \). In turn, the action on \( k[W'] \) can be described quite explicitly using the first part of Lemma 5.1: the operators act as if \( W' \) was the whole group.

5.2. Example: solution for the symmetric group. Let \( W \) be the symmetric group \( S_n \). For the convenience of the reader (and also to motivate subsequent constructions), we will first treat this special case in complete detail, and later use it as a prototype for the general case.

Let \( W' \) be the 6-element dihedral reflection subgroup of \( W = S_n \) generated by the reflections \( a = (ij) \) and \( b = (jk), 1 \leq i < j < k \leq n \). A left coset of \( W' \) consists of the elements
\[
\begin{align*}
\tilde{w} &= \cdots i \cdots j \cdots k \cdots , \\
a\tilde{w} &= \cdots j \cdots i \cdots k \cdots , \\
b\tilde{w} &= \cdots i \cdots k \cdots j \cdots , \\
ab\tilde{w} &= \cdots j \cdots k \cdots i \cdots , \\
bab\tilde{w} &= \cdots k \cdots i \cdots j \cdots , \\
aba\tilde{w} &= \cdots k \cdots j \cdots i \cdots ,
\end{align*}
\]
where all entries besides \( i, j, \) and \( k \) are as in \( \tilde{w} \); here \( \tilde{w} \) is the minimal element of the coset. The partial order induced on \( W'\tilde{w} \) from the Bruhat order on \( S_n \) is canonically isomorphic to the Bruhat order on the symmetric group of permutations of three elements \( i, j, \) and \( k \). See Figure 1.

![Figure 1. The Bruhat order on the coset \( W'\tilde{w} \) in the symmetric group](image-url)
The restrictions of the operators $M_{ij}$, $M_{ik}$, and $M_{jk}$ to the invariant 6-dimensional subspace $k[W'w]$ spanned by the permutations (5.2) is readily computed using Definition 3.1 and Remark 5.2. For example, let us compute $M_{ik}(baw)$. We have: $(ik) = bab$, $(ik) \cdot baw = bw$, $\ell(baw) < \ell(baw)$, implying $M_{ik}(baw) = q_{ik} b w$.

Analogous considerations show that in the linear basis of $k[W'w]$ formed by the elements $\tilde{w}, a\tilde{w}, b\tilde{w}, ab\tilde{w}, ba\tilde{w}, aba\tilde{w}$ (in this order), the restrictions of the operators $M_{ij}$, $M_{ik}$, and $M_{jk}$ are given by the following matrices:

\[
M_{ij} = \begin{bmatrix}
0 & q_{ij} & 0 & 0 & 0 & 0 \\
p_{ij} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_{ij} & 0 & 0 & 0 \\
0 & 0 & p_{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_{ij} & 0 \\
0 & 0 & 0 & 0 & 0 & p_{ij}
\end{bmatrix},
M_{jk} = \begin{bmatrix}
0 & 0 & q_{jk} & 0 & 0 & 0 \\
p_{jk} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_{jk} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{jk} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_{jk} & 0 & 0
\end{bmatrix},
M_{ik} = \begin{bmatrix}
0 & 0 & 0 & 0 & q_{ik} & 0 \\
0 & 0 & 0 & q_{ik} & 0 & 0 \\
0 & 0 & 0 & 0 & q_{ik} & 0 \\
p_{ik} & 0 & 0 & 0 & 0 & 0 \\
0 & p_{ik} & 0 & 0 & 0 & 0 \\
p_{ik} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

We are now prepared to write the conditions under which the operators $R_\tau = 1 + \varepsilon M_\tau$ satisfy the type $A$ Yang-Baxter equation (2.4). The terms of degrees 0 and 1 in $\varepsilon$ are clearly the same on both sides of (2.4). Equating the quadratic terms gives the classical Yang-Baxter equation (5.3)

\[(M_{ij}, M_{jk}) = [M_{jk}, M_{ik}] + [M_{ik}, M_{ij}]
\]

(here $[A, B] = AB - BA$ stands for the commutator), while equating the cubic terms gives the quantum Yang-Baxter equation for the $M_\tau$:

\[(M_{ij}, M_{ik})M_{jk} = M_{jk}M_{ik}M_{ij} \quad \text{if } i < j < k.
\]

Substituting (5.2) into (5.3), we obtain, upon simplifications, the following system of equations:

\[
\begin{cases}
-q_{ij}q_{jk} + p_{jk}q_{ik} + q_{ik}q_{ij} = 0; \\
q_{ij}q_{jk} - q_{jk}q_{ik} - q_{ik}p_{ij} = 0; \\
p_{ij}q_{jk} - q_{jk}p_{ik} - q_{ik}p_{ij} = 0; \\
-q_{ij}p_{jk} + p_{jk}q_{ik} + p_{ik}q_{ij} = 0; \\
p_{ij}p_{jk} - q_{jk}p_{ik} - p_{ik}p_{ij} = 0; \\
-p_{ij}p_{jk} + p_{jk}p_{ik} + p_{ik}q_{ij} = 0.
\end{cases}
\]

Making the same substitution into (5.4), we obtain a single equation $q_{ij}p_{ik}q_{jk} = p_{ij}q_{ik}p_{jk}$, which actually follows from (5.5); indeed, multiply the first equation in (5.5) by $p_{ik}$, the last one—by $q_{ik}$, and subtract.

We thus arrived at the following result.
Proposition 5.3. The operators $R_{ij}$ given by (3.7) satisfy the type A quantum Yang-Baxter equations (2.3)–(2.4) if and only if the parameters $\{p_{ij}\}$ and $\{q_{ij}\}$ satisfy the equations (5.5).

It is possible to use equations (5.5) to provide a complete parametric description of all solutions of the Yang-Baxter equations of type $A$ that have the form (3.2). However, this description is quite cumbersome because of the many degenerate cases where lots of parameters $p_{ij}$ and $q_{ij}$ vanish. Instead, we will now explicitly describe the particularly simple family of solutions that is obtained in the “generic” case.

Suppose for a moment that $q_{ij} \neq 0$ for any $i$ and $j$. Adding the first two equations in (5.5) and dividing by $q_{ik}$, we obtain $p_{ij} - q_{ij} = p_{jk} - q_{jk}$. Similarly, the second and third equations lead to $p_{ij} - q_{ij} = p_{ik} - q_{ik}$. This observation prompts the following consideration.

Let us assume that the parameters $p_{ij}$ and $q_{ij}$ are related by

$$p_{ij} = q_{ij} + \kappa,$$

where $\kappa$ is a scalar constant that does not depend on $i$ and $j$. This assumption (motivated in the preceding paragraph) immediately leads to substantial simplifications: substituting (5.6) into (5.5) reduces this system of equations to a single equation

$$(5.7) \quad q_{ik}(q_{ij} + q_{jk} + \kappa) = q_{ij}q_{jk}$$

—or, if you like, to

$$(5.8) \quad p_{ik}(p_{ij} + p_{jk} - \kappa) = p_{ij}p_{jk}.$$

We conclude that whenever (5.6) and (5.7) are satisfied by the collections of parameters $p_{ij}$ and $q_{ij}$, the corresponding mixed Bruhat operators of type $A$ give rise to a solution of the Yang-Baxter equations.

Let us denote $q_i = q_{i,i+1}$ and $p_i = p_{i,i+1} = q_{i,i+1} + \kappa$ and then use (5.6)–(5.8) to compute all the $q_{ij}$ and $p_{ij}$ recursively. The prototypical example is $W = S_3$, in which case we have

$$q_{13} = \frac{q_1q_2}{q_1 + q_2 + \kappa} = \frac{\kappa q_1q_2}{p_1p_2 - q_1q_2},$$

$$p_{13} = \frac{p_1p_2}{p_1 + p_2 - \kappa} = \frac{\kappa p_1p_2}{p_1p_2 - q_1q_2}.$$

Continuing in the same fashion leads us to the formulas (3.8)–(3.9); once those formulas are written down, proving them by induction on $j - i$ is a matter of routine verification.

Remark 5.4. Observe that, for any choice of parameters $p_{ij}$ and $q_{ij}$, the operators $M_{ij}$ defined by (3.1) are, up to a scalar, involutions: $M_{ij}^2 = p_{ij}q_{ij}$. Now suppose that the $q_{ij}$ and $p_{ij}$ are given by (3.8)–(3.9). Then the $M_{ij}$ (hence the normalized elements $(p_{ij}q_{ij})^{-1/2}M_{ij}$) satisfy the Yang-Baxter equation $M_{ij}M_{ik}M_{jk} = M_{jk}M_{ik}M_{ij}$—but not the braid relation

$$M_{i,i+1}M_{i+1,i+2}M_{i,i+1} = M_{i+1,i+2}M_{i,i+1}M_{i,i+1}.$$

However, one can check that in the special case $q_1 = \cdots = q_{n-1}$ the latter condition is satisfied, and we therefore obtain a representation of the symmetric group $S_n$. 

5.3. **Proof of Theorem 3.3.** We will use our type $A$ solution as a model. Let $W'$ be a dihedral subgroup of $W$, with the set of canonical generators $S' = \{a, b\}$. Thus $(W', S')$ is a Coxeter system of type $A_2$, $B_2$, or $G_2$.

Let $T' = T \cap W'$ be the set of reflections $\tau \in W'$; these reflections correspond to the operators $R_{\tau}$ involved in the Yang-Baxter equation (2.2) associated with $W'$. For any left coset $W'w$, the subspace $k[W'w]$ is invariant under the action of all operators $R_{\tau}$, $\tau \in T'$. By Lemma 5.1, the coset $W'w$ is in canonical bijection with $W'$, giving rise to a canonically labelled basis in the subspace $k[W'w]$. Furthermore, the matrices of the operators $R_{\tau}$, $\tau \in T'$, restricted to $k[W'w]$, do not depend on the choice of a coset. We can explicitly write down these matrices of size $6 \times 6$, $8 \times 8$, or $12 \times 12$ in terms of the corresponding parameters $p_{\tau}$ and $q_{\tau}$, in complete analogy with (5.2).

If $W'$ is of type $A_2$, then we obtain the matrices whose only difference from (5.2) is in notation: we have to replace the subscripts $ij$, $jk$, and $ik$ by the reflections $a$, $b$ and $aba = bab$, respectively. This leads to a system of equations of the form (5.3). In view of (5.6) and (5.7), these equations will be satisfied if we impose the condition

\begin{equation}
(5.9) \quad p_{\tau} = q_{\tau} + \kappa_{\tau}, \quad \tau \in T,
\end{equation}

where $\kappa_{\tau}$ only depends on whether $\tau$ corresponds to a long or a short root, and require that

\begin{equation}
(5.10) \quad q_a q_b = q_{aba} (q_a + q_b + \kappa_a)
\end{equation}

whenever $a, b \in T$ are canonical generators for a dihedral subgroup of type $A_2$.

If $W'$ is of type $B_2$, then the set $T'$ consists of four reflections $a$, $b$, $aba$, and $bab$. Labelling the basis of the invariant 8-dimensional subspace by the elements of $W' = \{e, a, b, ba, ab, aba, bab, abab\}$ (in this order), we obtain the matrices

\begin{equation}
M_a = \begin{bmatrix}
0 & q_a & 0 & 0 & 0 & 0 & 0 & 0 \\
p_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_a & 0 & 0 & 0 \\
0 & 0 & p_a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q_a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q_a & 0
\end{bmatrix} \quad \text{and} \quad M_b = \begin{bmatrix}
0 & 0 & q_b & 0 & 0 & 0 & 0 & 0 \\
p_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q_b & 0 \\
0 & 0 & 0 & 0 & q_b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_b & 0
\end{bmatrix},
\end{equation}

and, in a similar way, the matrices $M_{aba}$ and $M_{bab}$. Substituting these matrices into the type $B_2$ Yang-Baxter equation

\begin{equation}
(1 + M_a)(1 + M_{aba})(1 + M_{bab})(1 + M_b) = (1 + M_b)(1 + M_{bab})(1 + M_{aba})(1 + M_a),
\end{equation}

we obtain a system of equations for the 8 parameters $p_{\tau}$ and $q_{\tau}$ corresponding to $\tau \in \{a, b, aba, bab\}$. If we make an assumption (5.4), this system of equations collapses into the single equation

\begin{equation}
(5.11) \quad q_a q_b = q_{aba} q_{bab} + q_{aba} q_{bab} + \kappa_a q_{aba} + \kappa_b q_{bab},
\end{equation}

which we want to be satisfied whenever $a, b \in T$ are canonical generators for a dihedral subgroup of type $B_2$.

For type $G_2$, we have 12 parameters $p_{\tau}$ and $q_{\tau}$. Assuming (5.9), we express everything in terms of the 6 parameters $q_{\tau}$ and the 2 parameters $\kappa_{\tau}$ (for the short
and long roots, respectively). Making a substitution into the Yang-Baxter equation of type $G_2$, we obtain the two equations

\begin{equation}
q_aq_b = q_aq_{aba} + q_{aba}q_{aba} + q_{aba}q_{bab} + q_{bab}q_b + q_{bab}q_b
+ \kappa_a q_{aba} + \kappa_a q_{bab} + \kappa_b q_b + \kappa_b q_{aba}
\end{equation}

and

\begin{align*}
-q_aq_{bab} + q_aq_{aba} - q_aq_{aba} + q_{aba}q_{bab} + q_{bab}q_{aba} \\
+ q_aq_{aba}q_{bab} - q_aq_{aba}q_{aba} - q_aq_{aba}q_{bab} - q_aq_{aba}q_{aba} \\
+ q_aq_{aba}q_{bab} + q_{aba}q_{aba}q_{bab} + q_{aba}q_{aba}q_{aba} \\
+ q_aq_{aba}q_{bab} + q_{aba}q_{bab}q_{aba} + q_{aba}q_{bab}q_{aba} \\
\end{align*}

\begin{equation}
+ \kappa_a q_{aba} - q_aq_{bab}q_a + q_{aba}q_{bab} + q_{aba}q_{bab}
+ q_{aba}q_{bab}q_{aba} + q_{aba}q_{bab}q_{aba}
+ q_{aba}q_{bab}q_{aba} + q_{aba}q_{bab}q_{aba}
+ q_{aba}q_{bab}q_{aba} + q_{aba}q_{bab}q_{aba}
\end{equation}

\begin{equation}
+ \kappa_a q_{aba}q_{bab} + \kappa_a q_{aba}q_{bab} + \kappa_b q_{aba}q_{bab} = 0.
\end{equation}

We are now fully prepared to complete the proof of Theorem 3.3, which amounts to checking the equations (5.10), (5.11), and (5.12)–(5.13) for every dihedral subgroup $W'$ of type $A_2$, $B_2$, or $G_2$, respectively. This is a straightforward verification. Let $a$ and $b$ be the canonical generators of $W'$, and let $\alpha$ and $\beta$ be the corresponding positive roots. For $W'$ of type $A_2$, we have $\kappa_a = \kappa_b$, and equation (5.10) becomes

\[
\frac{E_2(\alpha)}{E_1(\alpha) - E_2(\alpha)} \frac{E_2(\beta)}{E_1(\beta) - E_2(\beta)} = \frac{E_2(\alpha + \beta)}{E_1(\alpha + \beta) - E_2(\alpha + \beta)} \left( \frac{E_2(\alpha)}{E_1(\alpha) - E_2(\alpha)} + \frac{E_2(\beta)}{E_1(\beta) - E_2(\beta)} + 1 \right),
\]

which is immediately checked using that $E_1$ and $E_2$ are multiplicative (cf. 3.3). Let $W'$ be of type $B_2$. Note that equation (5.11) is invariant under interchanging $a$ and $b$. Therefore without loss of generality we may assume that $\alpha$ is short while $\beta$ is long. Then $aba$ and $bab$ correspond to positive roots $2\alpha + \beta$ (long) and $\alpha + \beta$ (short), respectively. Substituting (3.6) into (5.11) and factoring out $\kappa_a \kappa_b$, we obtain

\[
\frac{E_2(\alpha)}{E_1(\alpha) - E_2(\alpha)} \frac{E_2(\beta)}{E_1(\beta) - E_2(\beta)} = \frac{E_2(\alpha + \beta)}{E_1(\alpha + \beta) - E_2(\alpha + \beta)} \frac{E_2(2\alpha + \beta)}{E_1(2\alpha + \beta) - E_2(2\alpha + \beta)} + \frac{E_2(2\alpha + \beta)}{E_1(2\alpha + \beta) - E_2(2\alpha + \beta)} \frac{E_2(\alpha + \beta)}{E_1(\alpha + \beta) - E_2(\alpha + \beta)}
\]

\[
+ \frac{E_2(\alpha + \beta)}{E_1(\alpha + \beta) - E_2(\alpha + \beta)} \frac{E_2(\beta)}{E_1(\beta) - E_2(\beta)} + \frac{E_2(\alpha + \beta)}{E_1(\alpha + \beta) - E_2(\alpha + \beta)} \frac{E_2(\beta)}{E_1(\beta) - E_2(\beta)} + \frac{E_2(\alpha + \beta)}{E_1(\alpha + \beta) - E_2(\alpha + \beta)},
\]

which is easily checked using (3.3). The case $G_2$ is verified in a similar way (preferably with the help of a computer).
6. Tilted Bruhat Orders

We will now apply the results of Section 4 to the combinatorics of the Coxeter system \((W, S)\). Our main tool will be the following specialization of quantum Bruhat operators (4.4).

**Corollary 6.1.** Let

\[
Q_\tau(w) = \begin{cases} 
\tau w & \text{if } \ell(\tau w) = \ell(w) + 1; \\
\tau w & \text{if } \ell(\tau w) = \ell(w) - \ell(\tau) \text{ and } \ell(\tau) = 2ht(\alpha) - 1; \\
0 & \text{otherwise}.
\end{cases}
\]

Then the operators

\[
R_\tau = 1 + \varepsilon Q_\tau
\]

satisfy the Yang-Baxter equations.

**Proof.** In Corollary 4.1, set \(E(\alpha) = 1\). □

**Definition 6.2.** Motivated by (6.1), let us introduce the following digraph \(D(W)\). The vertices of \(D(W)\) are the elements of the group \(W\). For \(u \in W\) and \(\tau \in T\), we put a directed edge from \(u\) to \(v = \tau u\) if either \(\ell(v) = \ell(u) + 1\) or \(\ell(v) = \ell(u) - \ell(\tau)\) and \(\ell(\tau) = 2ht(\alpha) - 1\), where \(\alpha\) is the corresponding positive root. In other words, \((u, \tau u)\) is an edge in \(D(W)\) if multiplying \(u\) by \(\tau\) on the left either increases the length of \(u\) by as little as possible or decreases the length of \(u\) by as much as possible, given the height of \(\alpha\).

Once a reflection ordering \(\varphi\) for \(W\) is chosen, we label the edges of \(D(W)\) by assigning label \(\varphi(\tau)\) to an edge \((u, v)\) with \(v = \tau u\). We will write \(u \xrightarrow{m} v\) to denote that \((u, v)\) is an edge in \(D(W)\) labelled by \(m\).

**Example 6.3.** Consider a Weyl group of type \(B_2\). This is the first instance where the condition \(\ell(\tau) = 2ht(\alpha) - 1\) comes into play. Let \(a\) and \(b\) be the generators of \(W\) that correspond to the simple roots \(\alpha\) (short) and \(\beta\) (long), respectively. Then the reflections \(a, b,\) and \(bab\) satisfy this condition, while \(aba\) does not (see Figure 2).

\[
\begin{align*}
ht(\alpha + \beta) &= 2 \\
\ell(b) &= 1 \\
ht(\beta) &= 1 \\
\ell(aba) &= 3 \\
\ell(bab) &= 3 \\
ht(2\alpha + \beta) &= 3 \\
\ell(ba) &= 3 \\
ht(\alpha) &= 1 \\
\ell(a) &= 1
\end{align*}
\]

**Figure 2. Root system \(B_2\)**

We hence disallow down-directed edges that correspond to multiplying by \(aba\) (on the left). The resulting graph \(D(W)\), for the reflection ordering \(a < aba < bab < b\), is shown in Figure 3.

Notice that the construction of the digraph \(D(W)\) depends on the root system \(\Phi\), not just on the Weyl group \(W\). Thus, for example, digraphs of types \(B_n\) and \(C_n\) will differ from each other.
Definition 6.4. For $u, v \in W$, let $\ell(u, v)$ denote the length of the shortest path in $D(W)$ from $u$ to $v$. In particular, $\ell(e, v) = \ell(v)$ is the usual length function, where $e$ denotes the identity element in $W$; moreover, $\ell(u, v) = \ell(v) - \ell(u)$ whenever $u \leq v$ in the Bruhat order. The tilted Bruhat interval between $u$ and $v$ is the set

$$D(u, v) = \{ w \in W : \ell(u, w) + \ell(w, v) = \ell(u, v) \},$$

equipped with the following partial order: $w_1 \preceq_w w_2$ if and only if $\ell(u, w_1) + \ell(w_1, w_2) + \ell(w_2, v) = \ell(u, v)$.

Thus $D(u, v)$ is a graded poset whose Hasse diagram is the minimal subgraph of $D(W)$ containing all directed paths from $u$ to $v$ that have the smallest possible length. If $u \leq v$ in the Bruhat order, then $D(u, v) = [u, v] = \{ w : u \leq w \leq v \}$, explaining our choice of terminology. Note that the intervals $D(u, v)$ and $D(v, u)$ are by no means dual posets; for example, in Figure 3 the interval $D(e, wo)$ is the whole Bruhat order, while $D(wo, e)$ has only four vertices (see Figure 4).

Let us also define the tilted Bruhat order $D_\circ(W)$ as a graded partial order on $W$ and the following order relation: $w_1 \preceq_\circ w_2$ if and only if $\ell(u, w_1) + \ell(w_1, w_2) = \ell(u, w_2)$.

Thus $w_1 \preceq_\circ w_2$ if and only if there exists a shortest path from $u$ to $w_2$ that passes through $w_1$. Note that any interval in this poset (or in any $D(u, v)$) is again a tilted Bruhat interval between corresponding vertices. $D_e(W)$ is the usual Bruhat order.

Any choice of reflection ordering induces edge labelling of the Hasse diagrams of $D(u, v)$ and $D_\circ(W)$ inherited from $D(W)$. Figure 3 shows the tilted Bruhat interval $D(ab, a)$ for $W$ of type $B_2$, with the same conventions as in Figure 3. (It also happens to coincide with the tilted Bruhat order $D_{ab}(W)$.) Figure 5 shows an example of a tilted Bruhat order which is not pure (i.e., does not have a 1).

Our main combinatorial result is an extension of certain fundamental property of Bruhat orders to their “tilted analogues” introduced in Definition 6.4. Let us first review the known facts.
Recall \cite{21} that a finite graded poset with $\hat{0}$ and $\hat{1}$ (resp. with $\hat{0}$) is called Eulerian (resp. lower Eulerian) if its Möbius function \cite{20} is given by
\[
\mu(x, y) = (-1)^{\text{rank}(y) - \text{rank}(x)}
\]
for any $x \leq y$. A well known (but non-trivial—cf. \cite{2, 3, 18}) theorem of Verma \cite{23, 24} asserts that any interval in the Bruhat order of any Coxeter group is Eulerian. To our knowledge, no simple proof of this result is known, except for the special case $x = e$ (see Lascoux \cite{19, Lemma 1.13}). The story of Verma’s theorem is described in \cite{16, p. 176}. Remarkably, it can be strengthened as follows: any Bruhat interval is actually a face poset of a shellable regular CW sphere (see Björner \cite{1, Theorem 5.1} and Björner-Wachs \cite{2, Theorem 4.2}); hence it is also Cohen-Macaulay \cite{4}.

All the statements mentioned in the preceding paragraph are implied by the following “lexicographic shellability” result conjectured by Björner and proved by Dyer \cite{12, Proposition 4.3} for an arbitrary Coxeter group. (This requires a more general definition of a reflection ordering, not needed in this paper.)

**Theorem 6.5.** \cite{12} Let $u, v \in W$, $u \leq v$. Then, for any reflection ordering, there exists a unique label-increasing (and, by reversal of the ordering, also unique label-decreasing) maximal chain from $u$ to $v$ in the Bruhat order of $W$. The sequence of labels associated with this chain is lexicographically minimal (resp. lexicographically maximal) among all maximal chains from $u$ to $v$.

We generalize this result (in the case of a Weyl group) as follows.
Theorem 6.6. Fix a reflection ordering \( \varphi \) in a Weyl group \( W \).

1. For any pair of elements \( u, v \in W \), there is a unique path from \( u \) to \( v \) in the directed graph \( D(W) \) such that its sequence of labels is strictly increasing (resp. strictly decreasing).

2. The unique label-increasing (resp. label-decreasing) path from \( u \) to \( v \) has the smallest possible length \( \ell(u, v) \). Moreover it is lexicographically minimal (resp. lexicographically maximal) among all shortest paths from \( u \) to \( v \).

3. For any \( u \in W \),
\[
R_{\varphi^{-1}(1)} \cdots R_{\varphi^{-1}(N)}(u) = \sum_{v \in W} \epsilon^{\ell(u,v)} v,
\]
where the \( R_\tau \) are given by (6.1)–(6.2); as before, \( N = \ell(w_0) \).

Proof. We first note that part 1 of the theorem is equivalent to the special case \( \epsilon = 1 \) of (6.3). Indeed, comparing our definition of the digraph \( D(W) \) to (6.1), we see that \((u, v)\) is an edge in \( D(W) \) if and only if \( v = Q_\tau(u) \) for some \( \tau \in T \), in which case \((u, v)\) is labelled by \( \varphi(\tau) \). Thus the identity (6.3), with \( \epsilon = 1 \), asserts existence and uniqueness of the label-decreasing path.

Let us denote by \( T \) the specialization of the operator \( R_{\varphi^{-1}(1)} \cdots R_{\varphi^{-1}(N)} \) obtained by setting \( \epsilon = 1 \). By Proposition 2.4 and Corollary 6.1, the operator \( T \) does not depend on the choice of reflection ordering \( \varphi \).

We will identify an element \( w \in W \) with the linear operator \( u \mapsto w u \) in \( k[W] \). Let \( s \in S \). Then (6.1) gives \( Q_s = s \), implying
\[
(1 + Q_s)s = 1 + Q_s.
\]
Since there exists a reduced decomposition of \( w_0 \) that ends in \( s \), there also exists a reflection ordering \( \varphi \) such that \( \varphi^{-1}(N) = s \) (cf. (2.1)). Hence (6.4) implies that
\[
Ts = \left( \prod_{i=1}^{N-1} (1 + Q_{\varphi^{-1}(i)}) \right) (1 + Q_s) s = T.
\]
It follows that, more generally, \( Tw = T \) for all \( w \in W \). Analogously one shows that \( wT = T \) for all \( w \in W \). These equations can be interpreted as saying that the matrix of \( T \) in the basis \( W \) of \( k[W] \) is invariant under permutations of rows and columns. Hence there exists a constant \( c \) such that, for any \( u \in W \),
\[
T(u) = c \sum_{v \in W} v.
\]
On the other hand, it is clear from (6.3) that the coefficient of \( w_0 \) in \( T(e) \) is \( \leq 1 \), where \( e \in W \) is the identity element. Since \( c \) is obviously a positive integer, we conclude that \( c = 1 \), and part 1 is proved.

To prove the rest, we will need the following lemma that generalizes the corresponding result for the ordinary Bruhat order (see, e.g., [L, Lemma 4.1]).

Lemma 6.7. Assume that
\[
u, x, v \in W, \quad u \xrightarrow{k} x \xrightarrow{l} v, \quad k > l.
\]
Then there exists \( y \in W \) such that (cf. Figure 2)
\[
u \xrightarrow{m} y \xrightarrow{n} v, \quad l < n < m < k.
\]
Proof. Consider the dihedral group $W'$ generated by the reflections $\varphi^{-1}(k)$ and $\varphi^{-1}(l)$. Define the operators $Q_\tau$ and $R_\tau$ by (6.1)–(6.2), and write down the Yang-Baxter equation (2.2) for $W'$, so that the order of the terms in the left-hand side was compatible with the reflection order $\varphi$. Thus the sequence of reflections appearing in the left-hand side is label-increasing, while the one in the right-hand side is label-decreasing. Apply the left-hand side to $u$, and take the coefficient of $\varepsilon^2 v$. This will be the number of label-decreasing paths in $D(W')$ from $u$ to $v$ that have length 2 and stay within the coset $W'u$. We know one such path, namely $u \overset{k}{\rightarrow} x \overset{l}{\rightarrow} v$. By the Yang-Baxter equation, there should also be a label-increasing path of length 2 from $u$ to $v$ that stays within $W'u$; let us denote it by $u \overset{m}{\rightarrow} y \overset{n}{\rightarrow} v$. It remains to check that $k > m$ and $l < n$. These two statements are completely analogous to each other, so we will only show how to prove the first one. Suppose that, on the contrary, $k < m$. Then $l < k < m < n$, which in particular means that the four reflections labelled by $l, k, m, n$ are all distinct. If $W'$ is of type $A_2$, this already brings the desired contradiction, since in that case there are only three reflections in $W'$. If $W'$ is of type $B_2$, with canonical generators $a$ and $b$ (say, $\varphi(a) < \varphi(b)$), then there are four reflections in $W'$, and therefore $l, k, m, n$ correspond to $a, aba, bab, b$, respectively. But this would imply that $v = a \cdot aba \cdot u = b \cdot bab \cdot u$, a contradiction. The remaining case $W' = W = G_2$ is checked directly.

We can now complete the proof of Theorem 6.6 using an argument borrowed from [12]. Among all shortest paths in $D(W')$ from $u$ to $v$, let

$$u = w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_d = v$$

be the one whose label sequence is lexicographically minimal. To prove part 2 of the theorem, we need to show is that this path is label-increasing. Suppose otherwise, i.e., for some $i \in \{2, \ldots, d-1\}$, we have $w_{i-1} \overset{k}{\rightarrow} w_i \overset{l}{\rightarrow} w_{i+1}$ with $k > l$. (We cannot have $k = l$ since this would create a loop, and the path would not be shortest.) Then, by Lemma 6.7, there exists $y \in W$ such that $w_{i-1} \overset{m}{\rightarrow} y \overset{n}{\rightarrow} w_{i+1}$ and $m < k$. Thus replacing $w_i$ by $y$ in (6.7) produces a chain with lexicographically smaller sequence of labels—a contradiction.

Finally, part 3 follows from parts 1 and 2.

In the terminology of [3], Theorem 6.6 asserts that the tilted Bruhat order (hence any tilted Bruhat interval $D(u, v)$) is EL-shellable (hence CL-shellable), with the EL-shelling provided by any reflection ordering (and therefore by its reversal as well).

Theorem 6.6 implies the following generalization of Verma’s theorem and its refinements mentioned above.
Corollary 6.8. Each tilted Bruhat order $D_u(W)$ of a Weyl group $W$ is a lexicographically shellable lower Eulerian poset.

As a consequence, any tilted Bruhat interval is a face poset of a shellable regular CW sphere. In particular, it is Eulerian and Cohen-Macaulay.

Proof. By [1, Proposition 4.5], Theorem 6.6 implies that $D(u, v)$ is a face poset of a regular CW sphere. Such posets are well known to be both Eulerian and Cohen-Macaulay; see, e.g., Stanley [21, Section 1].

The Eulerian property can also be deduced directly from Theorem 6.6 as follows. By a simple counting argument (cf. [4, Corollary 2.3]), the values of the Möbius function can be computed from an EL-shelling by

$$
\mu(u, v) = (-1)^{\text{rank}(y) - \text{rank}(x)} \cdot \text{(number of label-decreasing chains from } u \text{ to } v).$$

In our case, there is exactly one such chain, and the Eulerian property follows.

Acknowledgments

We thank Anders Björner, Alain Lascoux, and Richard Stanley for useful comments and conversations. Part of our work was carried out while the first two authors were participating in the “Combinatorics” program at MSRI.

References

[1] A. Björner, Posets, regular CW complexes and Bruhat order, Europ. J. Combin. 5 (1984), 7–16.
[2] A. Björner and M. Wachs, Bruhat order of Coxeter groups and shellability, Adv. in Math. 43 (1982), 87–100.
[3] A. Björner and M. Wachs, On lexicographically shellable posets, Trans. Amer. Math. Soc. 277 (1983), 323–341.
[4] A. Björner, A. M. Garsia, and R. P. Stanley, An introduction to Cohen-Macaulay partially ordered sets, in Ordered Sets (I. Rival, ed.), Reidel, 1982, 583–615.
[5] J. Carrell, Quantum cohomology of $G/P$. Notes from three lectures of Dale Peterson, preprint, 5 pp., 1996.
[6] I. V. Cherednik, Factorizing particles on a half line, and root systems, Teoret. Mat. Fiz. 61 (1984), no. 1, 35–44.
[7] I. V. Cherednik, Quantum Knizhnik-Zamolodchikov equations and affine root systems, Commun. Math. Phys. 150 (1992), 109–136.
[8] V. V. Deodhar, Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function, Invent. Math. 39 (1977), 187-198.
[9] V. V. Deodhar, A note on subgroups generated by reflections in Coxeter groups Arch. Math. 53 (1989), 543–546.
[10] M. Dyer, Reflection subgroups of Coxeter systems, J. Algebra 135 (1990), 57–73.
[11] M. Dyer, On the “Bruhat graph” of a Coxeter system, Comp. Math. 78 (1991), 185-191.
[12] M. Dyer, Hecke algebras and shellings of Bruhat intervals, Comp. Math. 89 (1993), 91-115.
[13] S. Fomin, S. Gelfand, and A. Postnikov, Quantum Schubert polynomials, J. Amer. Math. Soc. 10 (1997), 565–596.
[14] S. Fomin and A. N. Kirillov, Quadratic algebras, Dunkl elements, and Schubert calculus, preprint CRM-2472, April 1997; to appear in Progress in Geometry.
[15] W. Fulton and R. Pandharipande, Notes on stable maps and quantum cohomology, preprint alg-geom/9608011.
[16] J. E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Univ. Press, 1990.
[17] Yang-Baxter equation in integrable systems (M. Jimbo, ed.), Advanced Series in Mathematical Physics 10, World Scientific, 1989.
[18] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), 165-184.
[19] A. Lascoux, Anneau de Grothendieck de la variété de drapeaux, The Grothendieck Festschrift, Vol. III, 1–34, Progr. Math. 88, Birkhäuser, 1990.
[20] G.-C. Rota, On the foundations of combinatorial theory I. Theory of Möbius functions, in: Gian-Carlo Rota on combinatorics, Birkhäuser, 1995, pp. 3-31.
[21] R. P. Stanley, A survey of Eulerian posets, in: Polytopes: abstract, convex and computational, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 440 (1994), 301-333.
[22] R. P. Stanley, Enumerative Combinatorics, vol. 1, 2nd edition, Cambridge Univ. Press, 1997.
[23] D.-N. Verma, Möbius inversion for the Bruhat order on a Weyl group, Ann. Sci. École Norm. Sup. 4 (1971), 393-398.
[24] D.-N. Verma, A strengthening of the exchange property of Coxeter groups, preprint, 1972.
[25] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Phys. Rev. Lett. 19 (1967), 1312-1314.

Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: brenti@mat.uniroma2.it

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
E-mail address: fomin@math.mit.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
E-mail address: apost@math.mit.edu