CONDENSATION AND METASTABLE BEHAVIOR OF NON-REVERSIBLE INCLUSION PROCESSES

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ABSTRACT. In this article, we perform quantitative analyses of the metastable behavior of an interacting particle system known as the inclusion process. For inclusion processes, it is widely believed that the system nucleates the condensation of particles because of the attractive nature of the interaction mechanism. The metastable behavior of the inclusion processes corresponds to the movement of the condensate on a suitable time scale, and the computation of the corresponding time scale and the characterization of the scaling limit of the condensate motion are the main problems in the study of metastability of inclusion processes. Previously, these problems were solved for reversible inclusion processes in [Bianchi, Dommers, and Giardinà, Electronic Journal of Probability, 22: 1-34, 2017], and the main contribution of the present study is to extend this analysis to a wide class of non-reversible inclusion processes. Non-reversibility is a major obstacle to analyzing such models, mainly because there is no closed-form expression of the invariant measure for the general case, and our main achievement is to overcome this difficulty. In particular, our results demonstrate that the time scale and limiting process of non-reversible inclusion processes are quantitatively and qualitatively different from those of reversible ones, respectively. We emphasize that, to the best of our knowledge, these results are the first rigorous quantitative results in the study of metastability when the invariant measure is not explicitly known. In addition, we consider the thermodynamic limit of metastable behavior of inclusion processes on large torus as in [Armendáriz, Grosskinsky, and Loulakis, Probability Theory and Related Fields, 169: 105-175, 2017]. For this model, we observe three different time scales according to the level of asymmetry of the model.

Key words and phrases. Metastability, condensation, interacting particle systems, inclusion process, non-reversible Markov chain.
1. Introduction

Metastability is a ubiquitous phenomenon that occurs in various stochastic systems, such as the small random perturbation of dynamical systems [8,13,22,24,25,27], low-temperature ferromagnetic systems [6, 9, 23, 26], and interacting particle systems consisting of sticky particles [1–3, 7, 10, 11, 15–18, 20, 29]. The present study focuses on quantitative analyses of the metastable behavior of an interacting particle system known as the inclusion process, whose precise mathematical formulation is given in Section 2.1.

1.1. Condensation of inclusion processes. The systems of particles interacting under the attractive interaction mechanism exhibit a phenomenon known as condensation, i.e., a macroscopically significant portion of the particles is concentrated at a site (cf. Definition 2.3). Over the last decade, comprehensive studies have been conducted to understand this phenomenon, especially for two representative stochastic particle systems: zero-range processes [1,3,15,18–20,29] and inclusion processes [2,7,10,11,16,17]. These studies have mainly focused on the following objectives:

- Establishing the existence of condensation by demonstrating that a large portion of the particles is located at only one site with dominating probability under the invariant measure of the dynamics.
Analyzing the metastable behavior of the condensate: once the appearance of the condensate has been successfully established, the next objective is to investigate the dynamical movements of the condensate. The successive movements of the condensate can be regarded as metastable transitions studied in the context of metastability (cf. [4, 5, 8]).

In this study, we attempt to achieve these objectives for inclusion processes, especially non-reversible ones, for which the invariant measure cannot be written in a closed-form.

Condensation of inclusion processes. The inclusion process is an interacting particle system that is expected to exhibit condensation, and it has recently attracted considerable interest in the study of metastability. The study of condensation of inclusion processes originated from the work [16] of Grosskinsky, Redig, and Vafayi, who demonstrated this phenomenon under either of the following conditions: reversible and doubly stochastic. Here, we emphasize that the invariant measures of inclusion processes can be written in an explicit form (cf. (2.8)) under either of these conditions, and the proof of the existence of condensation is based solely on this expression. The first contribution of the present study is to prove the condensation of a wide class of inclusion processes without such an explicit expression of the invariant measure. Moreover, we obtain sharp asymptotics for the mass of each metastable valley.

The metastable behavior of inclusion processes was firstly analyzed in [7, 17] for the reversible case. Meanwhile, owing to the lack of a closed-form expression for the invariant measure, the metastable behavior of non-reversible inclusion processes has not been analyzed rigorously thus far; the only existing study is [10], in which an asymmetric (i.e., non-reversible) model on a torus was addressed by computational methods. The second contribution of this study is to derive rigorous results on the metastable behavior of non-reversible inclusion processes by developing a sequence of novel computations for inclusion processes.

Further, we consider the thermodynamic limit of the condensate as in [1], for which the underlying lattice structure grows together with the number of particles. Thus, a suitable time-space rescaling of the condensate motion is expected to converge to a certain continuous process. The third contribution of this study is to demonstrate the existence of three different time scales for the above-mentioned thermodynamic limit according to the level of non-reversibility of the process and to characterize the limiting process in a precise manner. We
remark that, to the best of our knowledge, such an interesting phenomenon has never been observed in any other model.

**Main difficulty: non-reversibility.** The main challenge in the problems that we are going to consider in this study originates from the non-reversibility of processes. Quantitative analysis of the metastable behavior of non-reversible processes is a long-standing open question in the research of metastability because of the following two main difficulties associated with such processes:

1. Absence of the variational principle known as the Dirichlet–Thomson principle, which enables us to estimate the potential-theoretic quantities such as the capacity between metastable sets.

2. Absence of the explicit form of the invariant measure.

The first difficulty was recently resolved in [14, 30], in which the Dirichlet–Thomson principle for non-reversible Markov chains was established, and a manual for applying these generalized tools in the context of metastability was also developed in [24]. On the basis of these studies, numerous results of the analysis of metastability of non-reversible processes were presented in [20, 22–25, 29]. We remark that the models considered in these studies have a closed-form expression for the invariant measure; hence, the second difficulty mentioned above is not applicable.

In contrast to the first difficulty mentioned above, the second one remains a major obstacle. This is not merely a technical issue, as all existing tools for the analysis of metastable behavior use highly precise knowledge about the invariant measure in a neighborhood of the transition path between metastable sets. Therefore, general models without the closed-form expression of the invariant measure have not been addressed thus far. For instance, the Eyring–Kramers-type result for non-reversible diffusions considered by Freidlin and Wentzell [13] remains unresolved because of such a difficulty. **We emphasize that the present study provides the first metastability result that overcomes this difficulty.**

1.2. **Zero-range processes.** The most investigated particle system in the context of the condensation phenomenon is the sub-critical zero-range processes. We refer to [29] and the references therein for a comprehensive account of the long history of the investigation of condensation in zero-range processes. Here, we briefly review a part of this history to understand the state of the art of studies on the metastable behavior of interacting particle systems and to compare zero-range processes with inclusion processes.
Condensation in the zero-range process was firstly observed in [19]; since then, it has taken nearly a decade to complete to answer most of the relevant questions with sufficient generality. Among the various studies, we review those on the analysis of metastable behavior. First, Beltrán and Landim [3] analyzed the metastable behavior of reversible zero-range processes. As non-reversible zero-range processes have the same (closed-form) invariant measure as reversible ones, the analysis of metastability was extended to the non-reversible case in [20] and [29] on the basis of recent technologies for the analysis of the metastability of non-reversible processes. From these successful studies on non-reversible zero-range processes, one can infer that the study of non-reversible inclusion processes can be reduced to the study of the invariant measure. However, the main problem is that no existing tool can be applied without the closed-form of the invariant measure. In this study, we will overcome this problem by introducing a new way of analyzing inclusion processes.

Comparison between zero-range process and inclusion process. We conclude the introduction with a brief explanation of the intrinsic difference between the metastable behavior of zero-range processes and that of inclusion processes. Figure 1 shows a visualization of this difference. First, we explain the mechanism for the transition of the condensate for the zero-range process. Initially, a few particles are detached from the condensation of the zero-range process. These particles wander momentarily and finally form a small new condensate at a site that might be far away from the original condensate. Then, the movement of the full condensation is completed by sending particles from the original condensate to this new one as in Figure 1-(left). Because
of this mechanism, the condensate for the zero-range process has long-range movements. Meanwhile, for the simple inclusion process, the condensate moves only to its neighboring site all at once; hence, we have to move all the particles together to move the condensation to a distant location, as shown in Figure 1-(right). Our results will formulate this difference in a concrete form.

2. Condensation of Inclusion Processes

In this section, we introduce the inclusion process and explain the condensation phenomenon in a more concrete form. More precisely, we formulate the inclusion process in Section 2.1 and then review the known condensation results in Section 2.2. Here, we remark that our new results will be presented in Section 3.

2.1. Inclusion processes. The inclusion process is a particle system consisting of interacting random walks on a finite state set $S$. Thus, we should start by introducing the underlying random walk on $S$ constituting the inclusion process.

**Definition 2.1 (Underlying random walk).** The underlying random walk $(X(t))_{t \in [0, \infty)}$ is a continuous-time, irreducible Markov chain on $S$ with jump rate $r(\cdot, \cdot) : S \times S \rightarrow [0, \infty)$. Let $m(\cdot)$ denote the invariant measure of the Markov chain $X(\cdot)$. For the simplicity of the discussion, we set $r(x, x) = 0$ for all $x \in S$.

The inclusion process is defined as a continuous-time Markov chain on the set $^1\mathcal{H}_N \subseteq \mathbb{N}^S$ defined by

$$\mathcal{H}_N = \left\{ \eta = (\eta_x)_{x \in S} \in \mathbb{N}^S : \sum_{x \in S} \eta_x = N \right\}.$$  

Here, $\eta_x$ can be regarded as the number of particles at the site $x \in S$; hence, $\eta$ represents the particle configuration on $S$. For $\eta \in \mathcal{H}_N$ and $x, y \in S$, let $\sigma^{x,y}\eta \in \mathcal{H}_N$ denote the configuration obtained by sending a particle, if possible, from $x$ to $y$ in $\eta$. In other words, for $\eta$ with $\eta_x \geq 1$, define

$$(\sigma^{x,y}\eta)_z = \begin{cases} \eta_x - 1 & \text{if } z = x, \\ \eta_y + 1 & \text{if } z = y, \\ \eta_z & \text{otherwise}, \end{cases}$$

and we set $\sigma^{x,y}\eta = \eta$ if $\eta_x = 0$. Now, we are ready to define the inclusion process.

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$^1$In this article, $\mathbb{N}$ includes 0, i.e., $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$.

$^2$Writing $u, v \in T$ or $\{u, v\} \subseteq T$ implicitly implies that $u$ and $v$ are different.
Definition 2.2 (Inclusion process). Let \( \{d_N\}_{N \geq 1} \) be a sequence of positive real numbers converging to 0. The inclusion process \( \{\eta_N(t)\}_{t \geq 0} \) is a continuous-time Markov chain on \( \mathcal{H}_N \) associated with the generator \( \mathcal{L}_N \) given by

\[
(\mathcal{L}_N F)(\eta) = \sum_{x, y \in S} \eta_x (d_N + \eta_y) r(x, y) \{ F(\sigma^{x,y} \eta) - F(\eta) \}
\]

for all \( \eta \in \mathcal{H}_N \) and \( F : \mathcal{H}_N \to \mathbb{R} \).

Now, we briefly explain the dynamics of the inclusion process. According to the generator (2.1), for a configuration \( \eta \), a particle moves from site \( x \) to site \( y \) at the rate

\[
\eta_x (d_N + \eta_y) r(x, y) = \eta_x \eta_y r(x, y) + d_N \eta_x r(x, y).
\]

Thus, we can divide the dynamics into two components. The first one corresponding to the term \( \eta_x \eta_y r(x, y) \) denotes the attractive interaction of the system, as this term increases with \( \eta_y \), which means that particles are more likely to move to more occupied sites. Meanwhile, the dynamics corresponding to the term \( d_N \eta_x r(x, y) \) denotes the diffusive behavior of the particles. However, if the parameter \( d_N \) is sufficiently small, this diffusive dynamics is dominated by the attractive interaction; consequently, we can expect condensation of the particles at one site. However, it is the second type of dynamics that gives rise to the transition of the condensate; when this diffusive effect accumulates for a sufficiently long time, we may observe the movement of the condensate to another one.

We conclude this subsection by introducing several notations regarding the inclusion processes as follows:

- Let \( r_N(\cdot, \cdot) \) denote the transition rate kernel associated with the inclusion process \( \eta_N(\cdot) \), i.e.,

\[
r_N(\eta, \eta') = \begin{cases} \eta_x (d_N + \eta_y) r(x, y) & \text{if } \eta' = \sigma^{x,y} \eta, \\ 0 & \text{otherwise.} \end{cases}
\]

- Let \( \lambda_N(\cdot) \) denote the corresponding holding rate:

\[
\lambda_N(\eta) = \sum_{\eta' \in \mathcal{H}_N} r_N(\eta, \eta'),
\]

- Let \( p_N(\cdot, \cdot) \) denote the jump probability kernel:

\[
p_N(\eta, \eta') = \frac{r_N(\eta, \eta')}{\lambda_N(\eta)}.
\]
Let $\mathbb{P}_\eta = \mathbb{P}_\eta^N$ and $\mathbb{E}_\eta = \mathbb{E}_\eta^N$ be the law and the expectation with respect to the process $\eta_N(\cdot)$ starting from $\eta$, respectively.

We can readily verify that the inclusion process defined above is an irreducible Markov chain on $H_N$ and thus has a unique invariant measure on $H_N$. Let $\mu_N$ denote this unique invariant measure.

2.2. Condensation of inclusion processes. In this subsection, we summarize all the known results regarding the condensation of inclusion processes.

Condensation on metastable sets. To describe this condensation phenomenon in a more concrete form, we introduce the metastable set. In the context of inclusion processes, this metastable set is very simple, i.e., it is just a singleton set. For $x \in S$, let $\xi^x_N \in H_N$ denote the configuration with all the particles located at $x$, i.e.,

$$(\xi^x_N)_y = \begin{cases} N & \text{if } y = x, \\ 0 & \text{otherwise}. \end{cases}$$

For each $x \in S$, define the set $E^x_N$ by

$$E^x_N = \{ \xi^x_N \} = \{ \eta \in H_N : \eta_x = N \}.$$ 

This set is metastable in the sense that not only the rate of escaping from this set is extremely low but also the likelihood of returning to this set immediately after escape is extremely high.

For a subset $R$ of $S$, define

$$E_N(R) = \bigcup_{x \in R} E^x_N \quad \text{and} \quad E_N = E_N(S).$$

With these terminologies, we are now ready to formulate the condensation in a concrete form.

**Definition 2.3 (Condensation).** The inclusion process is said to exhibit condensation if

$$\lim_{N \to \infty} \mu_N(E_N) = 1;$$

and to exhibit condensation on $R \subseteq S$ if

$$\lim_{N \to \infty} \mu_N(E_N(R)) = 1.$$
If the condensation occurs, we define the *maximal condensing set* as
\[
S_* = \left\{ x \in S : \limsup_{N \to \infty} \mu_N(E^x_N) > 0 \right\} \neq \emptyset.
\] (2.5)
Hence, \( S_* \) denotes the smallest set on which the condensation occurs. Finally, we write the remainder set as
\[
\Delta_N = \mathcal{H}_N \setminus E_N(S_*).
\]

**Formula for invariant measure: two special conditions.** Now, we introduce two conditions for the underlying random walk defined in Definition 2.1 that enable us to write the invariant measure in an explicit form.

(Rev) The underlying random walk \( X(\cdot) \) is *reversible* with respect to its invariant measure, i.e.,
\[
m(x)r(x, y) = m(y)r(y, x) \quad \text{for all } x, y \in S,
\]
such that the inclusion process is also reversible with respect to its invariant measure \( \mu_N(\cdot) \).

(UI) The invariant measure \( m(\cdot) \) for the underlying random walk \( X(\cdot) \) is the *uniform measure on \( S \).*

To explain the invariant measure for these cases, we define several notations. On the basis of the invariant measure \( m(\cdot) \) for the underlying random walk, we introduce the following notations:
\[
M_* = \max\{m(x) : x \in S\} \quad \text{and} \quad S_{\text{max}} = \{x \in S : m(x) = M_*\}.
\] (2.7)
Finally, we introduce an auxiliary function \( w_N : \mathbb{N} \to (0, \infty) \) as
\[
w_N(n) = \frac{\Gamma(n + d_N)}{n!\Gamma(d_N)}, \quad n \in \mathbb{N},
\]
where \( \Gamma \) denotes the usual gamma function. Then, we deduce the following formula under (Rev) or (UI).

**Proposition 2.4.** Under the condition (Rev) or (UI), the invariant measure \( \mu_N(\cdot) \) can be written as
\[
\mu_N(\eta) = \frac{1}{Z_N} \prod_{x \in S} \left( \frac{m(x)}{M_*} \right)^{\eta_x} w_N(\eta_x) \quad \text{for all } \eta \in \mathcal{H}_N
\] (2.8)
where the partition function $Z_N$ is given by

$$Z_N = \sum_{\eta \in \mathcal{H}_N} \prod_{x \in S} \left( \frac{m(x)}{M_*} \right)^{\eta_x} w_N(\eta_x) .$$

We remark that $m(x)/M_* = 1$ for all $x \in S$ under the condition (UI). The proof for the case (Rev) is straightforward, as the following detailed balance condition holds:

$$\mu_N(\eta) r_N(\eta, \eta') = \mu_N(\eta') r_N(\eta', \eta) .$$

This implies that the inclusion process is also reversible with respect to $\mu_N(\cdot)$. For the case (UI), the proof is presented in [16, Theorem 2.1(a)]; nevertheless, we provide a short proof in Section 5 for the completeness of the study. Based on the explicit formula (2.8), the following result is established in [7, Proposition 2.1].

**Proposition 2.5.** Suppose that $\mu_N(\cdot)$ admits the formula (2.8) and that $\lim_{N \to \infty} d_N \log N = 0$. Then, it holds that

$$\lim_{N \to \infty} \mu_N(\mathcal{E}_N^x) = \frac{1}{|S_{\max}|} \text{ for all } x \in S_{\max} .$$

In other words, the inclusion process exhibits the condensation on $S_{\max}$; moreover, $S_* = S_{\max}$. In particular, for the case (UI), we have $S_* = S$.

Here, we emphasize that the proof of this proposition is based entirely on the formula (2.8). Without this expression, proving the condensation phenomenon becomes a completely non-trivial task; we confront this difficulty in this study.

## 3. Main Results

In this section, we explain the main results obtained in this article. Our primary concern is the metastable behavior of the condensate of the inclusion process. Rigorous analysis of this metastable behavior was previously restricted to the inclusion process satisfying (Rev). We will extend these results to

1. inclusion processes satisfying (UI) (cf. Section 3.2),
2. inclusion processes for which jump rate $r(\cdot, \cdot)$ is uniformly positive (cf. Section 3.3),
3. inclusion processes in the thermodynamic limit regime for which the underlying graph ($d$-dimensional discrete torus) grows together with the number of particles (cf. Section 3.4).
For these cases, the inclusion process can be non-reversible. In particular, for case (2), even the invariant measure cannot be written in an explicit form; hence, the existence of the condensation is unknown. We shall establish this existence of the condensation in Theorem 3.15.

3.1. Description of metastable behavior. Before explaining our main results, we briefly review the canonical methodology developed in [4, 5] for the description of the metastable behavior of the stochastic systems as a convergence of the so-called trace process. We explain this methodology in the context of inclusion processes for the convenience of the readers. The successive movements of condensate in the inclusion process can be regarded as a transition among the metastable sets $\mathcal{E}_x^S = \{\xi_x^S\}, x \in S_\star$, as the condensation at a site $x \in S \setminus S_\star$ will be mollified on a shorter time scale than the condensation at a site $x \in S_\star$. Hence, by identifying the state $\xi_x^S$ with $x$ and ignoring short excursions on $\Delta_N = \mathcal{H}_N \setminus \mathcal{E}_N(S_\star)$, the resulting dynamics converges to a Markov chain on $S_\star$ after suitable time-rescaling. In the context of metastability theory, this procedure is a canonical way of describing the metastable behavior, and it has been systematically established in [4, 5] on the basis of the martingale approach. In this approach, the procedure explained above is understood as the convergence of the so-called trace process. Hence, we now introduce the trace process in the context of the inclusion process.

Definition 3.1 (Trace process of the inclusion process). Fix a non-empty set $G \subseteq \mathcal{H}_N$ and define a (random) non-decreasing function by

$$T^G(t) = \int_0^t 1 \{\eta_N(s) \in G\} \, ds .$$

Let $S^G(t)$ be its generalized inverse:

$$S^G(t) = \sup \{s \geq 0 : T^G(s) \leq t\} .$$

Then, the trace process $\eta_N^G(\cdot)$ on $G$ is defined by

$$\eta_N^G(t) = \eta_N(S^G(t)) . \quad (3.1)$$

The trace process $\eta_N^G(t)$ on $G$ is obtained from $\eta_N(t)$ by turning off the clock when $\eta_N(\cdot)$ does not belong to $G$, since $S^G(\cdot)$ freezes the clock when the process $\eta_N(\cdot)$ escapes from $G$ and turns it back when $\eta_N(\cdot)$ returns to $G$. Therefore, the process $\eta_N^G(\cdot)$ becomes a random process on $G$ whose trajectory is obtained from that of $\eta(\cdot)$ by removing its
excursions on $G^c$. Then, it is well known that $\eta_N^G(\cdot)$ is a Markov chain on $G$ (cf. [4, Proposition 6.1]).

**Description of movements of condensate.** On the basis of the trace process constructed above, we are now ready to rigorously formulate the metastable behavior of the inclusion processes. Denote simply by

$$\eta_N^\ast(\cdot) = \eta_N^{E_N(S)}(\cdot)$$

the trace process on the metastable set $E_N(S) = \{\xi_N^x : x \in S\}$. For the sake of simplicity, define an identification function $\Psi : E_N(S) \rightarrow S$ as

$$\Psi(\xi_N^x) = x \text{ for } x \in S.$$

Using this function, we define a process $\{Y_N(t)\}_{t \geq 0}$ on $S$ by

$$Y_N(t) = \Psi(\eta_N^\ast(t)).$$

Thus, the process $Y_N(\cdot)$ is obtained by taking the label of the metastable set at which the process $\eta_N^\ast(\cdot)$ is staying. Since $\eta_N^\ast(\cdot)$ is a Markov chain, the process $Y_N(\cdot)$ is a Markov chain on $S$ as well. Now, the long-time movement of the condensate can be characterized by proving the convergence of the process $Y_N(\cdot)$ with a proper acceleration factor $\theta_N$ to a certain limiting Markov chain on $S$. Let $\{Y(t)\}_{t \geq 0}$ denote a continuous-time Markov chain on $S$, which is the candidate for the limiting Markov chain.

**Definition 3.2 (Description of metastable behavior).** Suppose that the inclusion process exhibits condensation in the sense of Definition 2.3. Then, the dynamical movement of the condensate of an inclusion process is said to be **described by a Markov chain** $\{Y(t)\}_{t \geq 0}$ on $S$ with scale $\theta_N$ if the law of the process $Y_N(\theta_N \cdot)$ starting from $\xi_N^x$ converges to that of $Y(\cdot)$ starting from $x$ for all $x \in S$, and if

$$\lim_{N \rightarrow \infty} \sup_{\eta \in E_N(S)} \mathbb{E}_\eta \left[ \int_0^T 1 \left\{ \eta_N(\theta_N s) \notin E_N(S) \right\} \, ds \right] = 0 \text{ for all } T > 0. \quad (3.3)$$

**Remark.** Note that the condition (3.3) implies that the inclusion process does not spend too much time outside the metastable sets and hence guarantees that there exist only fast transitions between the metastable sets. In general models, proving (3.3) is not a trivial issue; however, in the inclusion process case, it directly follows from the definition of condensation (Definition 2.3), as one can see from Proposition 4.1.

The main objective of this study is to prove the requirements of Definition 3.2 for a wide class of non-reversible inclusion processes. We also remark that this has been
achieved for reversible inclusion processes in [7]. Now, we review this result along with some conjectures regarding the non-reversible case.

**movements of condensate: reversible and non-reversible cases.** Now, we explain the known result and the conjectures for the limiting chain $Y(\cdot)$ and the factor $\theta_N$ appearing in Definition 3.2.

First, we define a Markov chain $Y^{rv}(t)$ on $S_*$ (cf. Proposition 2.5) with rate
\[
a^{rv}(x, y) = r(x, y) \text{ for all } x, y \in S_* .
\] (3.4)

Note that $r(\cdot, \cdot)$ is the jump rate of the underlying random walk $X(\cdot)$; thus, $Y^{rv}(\cdot)$ can be regarded as the restricted Markov chain of $X(\cdot)$ on $S_*$. We also remark that $Y^{rv}(\cdot)$ is not necessarily an irreducible chain. Further, we define
\[
\theta^{rv}_N = \frac{1}{d_N} .
\]

Then, in the terminology of Definition 3.2, the following result has been established in [7, Theorem 2.3].

**Theorem 3.3.** Suppose that the underlying random walk is reversible with respect to its invariant measure $\mu_N$ and that $\lim d_N \log N = 0$. Then, the movement of the condensate is described by a Markov chain $Y^{rv}(\cdot)$ on $S_* = S_{\text{max}}$ (cf. (2.7)) with scale $\theta^{rv}_N$.

For the non-reversible case, we expect a completely different result compared to the reversible case. Suppose that we have characterized the set $S_*$. Define $Y^{nrv}(\cdot)$ as a Markov chain on $S_*$ with rate
\[
a^{nrv}(x, y) = [r(x, y) - r(y, x)] 1 \{r(x, y) > r(y, x)\} \text{ for } x, y \in S_* ;
\] (3.5)

and define the time scale as
\[
\theta^{nrv}_N = \frac{1}{Nd_N} .
\] (3.6)

**Conjecture 3.4.** Suppose that $\lim d_N \log N = 0$. Then, the movement of the condensate is described by the Markov chain $Y^{nrv}(\cdot)$ with scale $\theta^{nrv}_N$.

The proof of Theorem 3.3 for the reversible case obtained in [7] is based on the potential theory of reversible Markov chains. Hence, it is tempting to adopt the recently developed potential theory of non-reversible Markov chains [14, 30] to investigate the non-reversible case. Indeed, we are able to do so in the case if the invariant measure $\mu_N(\cdot)$ admits the formula (2.8). However, instead of following this traditional approach,
we try to directly estimate the so-called mean-jump rate by exploiting several model-dependent features of the inclusion process. This is mainly because we wish to tackle the general case without the formula (2.8) on \( \mu_N \). Indeed, one of the main difficulties in the study of the non-reversible case is the lack of such an explicit formula for \( \mu_N \); in this case, it is even unclear what \( S_* \) is. Specifying \( S_* \) itself seems to be an extremely difficult problem.

Remark 3.5. In general, it is anticipated that the metastable transition of non-reversible dynamics occurs faster than that of its reversible counterpart. For instance, such a phenomenon has been verified for the stochastic discrete gradient descent [24], small random perturbation of dynamical systems [22], and zero-range processes [3, 20, 29]. These results show that the non-reversible dynamics is faster than the reversible one by a constant (i.e., \( O(1) \)) factor, while Conjecture 3.4 indicates that the non-reversible dynamics of the inclusion process is expected to be \( O(N) \) times faster than the reversible one.

Finally, suppose that the relation \( r(x, y) = r(y, x) \) holds for all \( x, y \in S_* \). In this case, we have \( a_{\text{nr}}^N(x, y) = 0 \) for all \( x, y \in S_* \); hence, Conjecture 3.4 implies that the scale \( \theta_{\text{nr}}^N = \frac{1}{Nd_N} \) is too short to observe the transitions. We expect that the correct scale for this case is \( \theta_{\text{rv}}^N = \frac{1}{d_N} \).

**Conjecture 3.6.** Suppose that \( r(x, y) = r(y, x) \) for all \( x, y \in S_* \) and that \( \lim d_N \log N = 0 \). Then, the movement of the condensate is described by the Markov chain \( Y_{\text{rv}}(\cdot) \) on \( S_* \) with scale \( \theta_{\text{rv}}^N \).

Here, we emphasize that Theorem 3.3 is a special case of this conjecture. To see this, observe that \( S_* = S_{\text{max}} \) for the reversible case; thus, we have

\[
    r(x, y) = \frac{m(y)}{m(x)} r(y, x) = \frac{M_\ast}{M_\ast} r(y, x) = r(y, x) \quad \text{for all } x, y \in S_{\ast}.
\]

This implies that, if the previous conjecture is true, the scale \( \theta_{\text{rv}}^N \) and the limiting Markov chain \( Y_{\text{rv}}(\cdot) \) appear in the reversible case because \( r(\cdot, \cdot) \) is symmetric on \( S_* = S_{\text{max}} \), and the reversibility is not a fundamental reason.

In this study, we verify the validity of Conjectures 3.4 and 3.6 for wide-class of non-reversible inclusion processes.

**Comments on the convergence of finite-dimensional marginal distributions.** Before proceeding to the main results of this article, we remark on the mode of convergence regarding Definition 3.2. Although the convergence of the trace process is natural in the
study of metastability, an alternative description has been presented [21], which does not need to recall the trace process in the description and is hence more intuitive to understand. To see this, define a map \( \hat{\Psi} : \mathcal{H}_N \rightarrow S_* \cup \{o\} \) as
\[
\hat{\Psi}(\eta) = \begin{cases} 
  x & \text{if } \eta = \xi_N^x \text{ with } x \in S_* , \\
  o & \text{otherwise ,}
\end{cases}
\]
and define a process \( \{\hat{Y}_N(t)\}_{t \geq 0} \) on \( S_* \cup \{o\} \) by
\[
\hat{Y}_N(t) = \hat{\Psi}(\eta_N(t)) .
\]

**Definition 3.7.** The dynamical movement of the condensate of an inclusion process is said to be described by a Markov chain \( \{Y(t)\}_{t \geq 0} \) on \( S_* \) with scale \( \theta_N \) in the finite-dimensional marginal sense if, for all \( k \in \mathbb{N} \), we have
\[
\lim_{N \to \infty} \mathbb{P}_{\xi_N^x} \left[ \hat{Y}_N(\theta_N t_1) \in A_1, \ldots, \hat{Y}_N(\theta_N t_k) \in A_k \right] = \mathbb{P}_x \left[ Y(t_1) \in A_1, \ldots, Y(t_k) \in A_k \right]
\]
for all \( 0 \leq t_1 < \cdots < t_k \) and \( A_1, \ldots, A_k \subseteq S_* \), where \( \mathbb{P}_x \) denotes the law of the process \( Y(\cdot) \) starting from \( x \).

To establish this convergence of marginal distributions from that of the trace process defined in Definition 3.2, it is known from [21, Proposition 2.1] that the verification of the following technical condition is sufficient:
\[
\lim_{N \to \infty} \limsup_{\delta \to 0} \sup_{2\delta \leq s \leq 3\delta} \sup_{\eta \in \mathcal{E}_N(S_*)} \mathbb{P}_{\eta} [\eta_N(\theta_N s) \notin \mathcal{E}_N(S_*)] = 0 . \tag{3.7}
\]
For the inclusion process, this condition is straightforward to check (cf. Proposition 4.1); thus, the convergence of the trace process immediately implies the convergence of the finite-dimensional distributions.

3.2. Main result 1: inclusion processes under condition (UI). In this subsection, we explain our result of the analysis of the metastable behavior of the inclusion process under the condition (UI). For this case, as mentioned in Proposition 2.4, the invariant measure admits the expression (2.8); therefore condensation occurs owing to Proposition 2.5. Moreover, as the invariant measure for the underlying random walk is uniform, we have \( S_* = S_{\max} = S \), i.e., condensation occurs on the entire state set \( S \).

The metastable behavior of the inclusion process for this case was known only when \( r(\cdot, \cdot) \) is completely symmetric (as in case (1) of the theorem below). The following theorem extends this result for the general case under (UI).
Theorem 3.8. Suppose that the underlying random walk satisfies the condition (UI) and that $\lim_{N \to \infty} d_N \log N = 0$.

1. Suppose that $r(x, y) = r(y, x)$ for all $x, y \in S$. Then, Conjecture 3.6 holds.
2. Suppose that $r(x, y) \neq r(y, x)$ for some $x, y \in S$. Then, Conjecture 3.4 holds.

We remark that, for case (1), the underlying random walk is reversible; hence, this result is a consequence of [7] (i.e., of Theorem 3.3 of the current paper). Our new result focuses on case (2), which is essentially the first rigorous analysis of the metastable behavior of non-reversible inclusion processes. The proof of this result relies on careful analysis of the mean-jump rates established in Section 4. We explain the proof in Section 5.

Inclusion processes on torus. An interesting example satisfying condition (UI) is the simple random walk on the discrete torus. Suppose that the underlying random walk is a simple random walk on the torus $T_L = \mathbb{Z}/(L\mathbb{Z})$ with jump rate

$$r(x, y) = \begin{cases} 
p & \text{if } y = x + 1 \pmod{L}, \\
1-p & \text{if } y = x - 1 \pmod{L}, \\
0 & \text{otherwise}. \end{cases}$$

As the uniform measure on $T_L$ is the invariant measure for this random walk, the condition (UI) is valid. We can prove that the dynamical transition of the condensate can be described as follows. For the simplicity we may assume that $p \geq 1/2$ since the case $p \leq 1/2$ can be treated in the same manner.

Corollary 3.9. Suppose that $\lim_{N \to \infty} d_N \log N = 0$. Then, the dynamical movement of the condensate for the inclusion process on $T_L$ defined above is described by the following limiting Markov chain and the time scale:

1. for $p = 1/2$, a Markov chain $\{Y^{\text{sym}}(t)\}_{t \geq 0}$ with jump rate

$$a^{\text{sym}}(x, y) = \begin{cases} 
1/2 & \text{if } |y - x| = 1, \\
0 & \text{otherwise}, \end{cases}$$

and scale $\theta_N^{\text{sym}} = 1/d_N$.

2. for $p > 1/2$, a Markov chain $\{Y^{\text{asym}}(t)\}_{t \geq 0}$ with jump rate

$$a^{\text{asym}}(x, y) = \begin{cases} 
2p - 1 & \text{if } y = x + 1, \\
0 & \text{otherwise}, \end{cases}$$
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and scale $\theta_N^{NVR} = 1/(Nd_N)$.

We note that the transition scale for the asymmetric case is $1/(Nd_N)$, and it is $O(N)$ times faster than that of the symmetric case, i.e., $1/d_N$. This observation verifies the statement in Remark 3.5. Furthermore, it is interesting that the limiting dynamics for the partially asymmetric case (i.e., $p \in (1/2, 1)$) is totally asymmetric.

3.3. Main result 2: inclusion processes with uniformly positive rates. As mentioned earlier in the introduction, the condensation of inclusion processes without condition (Rev) or (UI) is unknown. For instance, whether condensation occurs on $S$, i.e., $\lim_{N \to \infty} \mu_N(E_N) = 1$, is an open question. This is mainly because of the lack of the explicit formula of $\mu_N$. Under suitable assumptions, we now describe both static and dynamical analyses of condensation in such general cases.

Metastable behavior for general non-reversible inclusion processes. We assume first that the occurrence of the condensation has been verified, and then focus on the analysis of the metastable behavior. We will return to the condensation issue later in this subsection.

To prove Conjecture 3.4, we should first characterize $S_*$. To this end, let us consider an auxiliary Markov chain $(Z(t))_{t \geq 0}$ on $S$ with jump rate

$$b(x, y) = [r(x, y) - r(y, x)] 1\{r(x, y) > r(y, x)\} \text{ for all } x, y \in S,$$

(3.8)

which is an extension of $a_N^{NVR}(\cdot, \cdot)$ defined in (3.5) to the set $S$. Let $S_0$ denote the set of recurrent states (including absorbing states; refer to Figure 2) of the Markov chain $Z(t)$. We say that $S_0$ has only one irreducible component if the Markov chain $Z(t)$ restricted to $S_0$ is irreducible, i.e., for any $x, y \in S_0$, there exists some $k \geq 1$ such that

$$\sum_{z_1, \ldots, z_{k-1} \in S_0} b(x, z_1)b(z_1, z_2) \cdots b(z_{k-1}, y) > 0.$$

This assumption is equivalent to the uniqueness of the invariant measure for $Z(t)$, and for such a case $S_0$ is the support of the invariant measure. Then, the following result describes the metastable behavior of the inclusion process when $S_0$ has only one irreducible component.

Theorem 3.10. Suppose that condensation occurs and that $S_0$ defined above has only one irreducible component. Then, $S_* = S_0$ and Conjecture 3.4 holds.

Now, we turn to Conjecture 3.6. To this end, we assume that $r(x, y) = r(y, x)$ for all $x, y \in S_0$. Then, consider another auxiliary Markov chain $(Z_2(t))_{t \geq 0}$ on $S_0$ whose
The set $S_0$ is given by $S_0 = A \cup B$. In this case, $Z_1(\cdot)$ restricted to $S_0$ has two irreducible components $A$ and $B$; thus, it does not satisfy the condition of Theorem 3.10. The set $A$ is semi-attracting since $r(u, v) = r(v, u)$. (Middle) The set $S_0$ satisfies the condition of Theorem 3.10, since $S_0$ has only one irreducible component with respect to $Z_1(\cdot)$. (Right) The set $S_0$ is attracting; hence $S_0$ satisfies all the conditions of Theorem 3.12.

rate between $x \in S_0$ and $y \in S_0$ is just $r(x, y)$. We need to introduce additional simple concepts to state our result.

**Notation 3.11.** The set $A \subseteq S$ is called attracting if it holds that $r(x, y) < r(y, x)$ for all $x \in A$ and $y \in A^c$ with $r(x, y) + r(y, x) > 0$. Moreover, $A$ is called semi-attracting if it holds that $r(x, y) \leq r(y, x)$ for all $x \in A$ and $y \in A^c$ with $r(x, y) + r(y, x) > 0$.

We refer to Figure 2 for the illustration.

Note that attracting sets are semi-attracting as well. For the symmetric case, we obtain the following result.

**Theorem 3.12.** Suppose that condensation occurs and that the Markov chain $(Z_2(t))_{t \geq 0}$ on $S_0$ defined above is irreducible. Further, assume that $S_0$ is attracting. Then, $S_* = S_0$ and Conjecture 3.6 holds.
Remark. Since $S_* = S_0$ in this case, the Markov chain $Z_2(\cdot)$ is indeed $Y_{\text{TV}}(\cdot)$. The condition that $S_0$ is attracting is required to guarantee that $S_0$ is the set of states at which the transition occurs.

As a consequence of Theorems 3.10 and 3.12, we can provide the following non-trivial asymptotic limit of $\mu_N(\xi^*_x)$ for $x \in S_* = S_0$.

**Theorem 3.13.** Under the conditions of Theorem 3.10 (resp. Theorem 3.12), it holds that

$$
\lim_{N \to \infty} \mu_N(\xi^*_N) = \nu(x) \text{ for all } x \in S_*
$$

where $\nu(\cdot)$ is the unique invariant measure of the irreducible Markov chain $Y_{\text{TV}}(\cdot)$ (resp. $Y_{\text{rv}}(\cdot)$).

**Remark 3.14.** Several remarks regarding the irreducibility of $Z_1(\cdot)$ and $Z_2(\cdot)$ on $S_0$ are stated below.

1. When there exist multiple irreducible components of $Z_1(\cdot)$ on $S_0$, a certain linear combination of the invariant measure on each component is expected to equal the limit of $\mu_N$ on $E_N$. However, at this moment, it is unclear as to which linear combination is the correct one. Moreover, characterizing $S_*$ is not possible at this moment. The sites in $S \setminus S_0$ will be discarded in the long-time limit; however, it is unclear as to which sites of $S_0$ will survive, partially or completely, in the accelerated process. We shall not pursue this problem in the present study, and is left as a topic for future research.

2. The reversible case in which there exist multiple irreducible components of $Z_2(\cdot)$ on $S_0$ has been investigated in [7] for a specific form of the underlying graph. In these longer scaling limits, each irreducible component is expected to act as a single element in the limiting dynamics, and the long-time movement will occur among these component-wise elements. If the graph distance between these components is exactly 2, then the transition occurs in the second scale $N/d_N^2$. If the distance is greater than 2, then the transition occurs in the third scale $N^2/d_N^3$. However, such generality has not been analyzed even for the reversible inclusion process on general graphs.

**Condensation.** Previously, we analyzed the metastable behavior of inclusion processes by assuming that the condensation occurs. However, without the closed-form expression for the invariant measure, the verification of the condensation is not a simple task. Here, we prove the existence of condensation under the following assumption:
The jump rate of the underlying random walk is uniformly positive in the sense that
\[ r(x, y) > 0 \text{ for all } x, y \in S. \] (3.9)

With this assumption, we can establish the existence of condensation for inclusion processes. We emphasize that this is the first verification of the condensation for the inclusion process without explicit formula (2.8) for \( \mu_N \).

**Theorem 3.15.** Suppose that the assumption (UP) holds and
\[ \lim_{N \to \infty} d_N N^{\vert S \vert + 2} (\log N)^{\vert S \vert - 3} = 0. \] (3.10)
Then, the condensation occurs for the inclusion process, i.e.,
\[ \lim_{N \to \infty} \mu_N(\mathcal{E}_N) = 1. \] (3.11)

The following corollary is now immediate.

**Corollary 3.16.** Theorems 3.10, 3.12, and 3.13 hold under the conditions (UP) and (3.10).

The proof of Theorem 3.15 is given in Section 7 and relies on the results on mean-jump rates established in Section 4 along with a weak result on the nucleation of condensation stated below in Theorem 3.17.

In general, the nucleation regime explains the typical behavior of particles, starting from an arbitrary distribution among sites to condensation at a sole site. The only rigorous result regarding the nucleation was obtained in [17], where it was proved that the nucleation procedure of the inclusion process satisfying both (Rev) and (UI) can be explained by a Wright–Fisher-type slow-fast diffusion. We refer to [17] for further information on nucleation; although our nucleation result explained hereafter is much weaker, it is the first quantitative result in the study of nucleation of non-reversible inclusion processes. For \( \mathcal{A} \subseteq \mathcal{H}_N \), let \( \tau_{\mathcal{A}} = \tau_{\mathcal{A}}^N \) denote the hitting time of the set \( \mathcal{A} \) with respect to the inclusion process \( \eta_N(\cdot) \), and let \( \delta > 0 \) be an arbitrary fixed number. Define
\[ \mathcal{U}_N = \{ \eta \in \mathcal{H}_N : \eta_x \leq \delta \log N \text{ for some } x \in S \}. \]
Then, the nucleation result can be formulated as follow.
Theorem 3.17. Suppose that the assumption (UP) holds and $\lim_{N \to \infty} d_N \frac{N^2}{(\log N)^2} = 0$. Then, there exists a constant $C = C(\delta) > 0$ such that

$$\sup_{\eta \in \mathcal{H}_N} \mathbb{E}_\eta [\tau_{\mathcal{U}_N}] \leq CN .$$

Suppose that the inclusion process starts from a configuration containing $\Omega(N)^3$ particles at all sites. Then, the first stage of the nucleation of condensation is to empty a site, which can be deduced by studying the typical path to the set $\{\eta \in \mathcal{H}_N : \eta_x = 0 \text{ for some } x \in S\}$ and examining the mean of the hitting time. The theorem above provides a weak form of such a result, and its proof will be given in Section 7.4. It is strongly expected that the actual scale of the nucleation of particles is $O(\log N)$, which serves as an important topic of future research.

3.4. Main result 3: inclusion processes in the thermodynamic limit regime.

In the previous models, we fixed the state space $S$. In this subsection, we consider a slightly different model for which the space given by the multi-dimensional discrete torus grows together with the number of particles. Then, a suitable time-space rescaling of the movements of the condensate converges to a continuous process on a multi-dimensional torus; this type of result is referred to as the thermodynamic limit of condensation (cf. [1]).

The thermodynamic limit of condensation has been thoroughly studied for zero-range processes in [1, 28]. In [1], the thermodynamic limit of condensation of the symmetric zero-range process on the torus has been investigated by the martingale approach, and in [28], it has been generalized to the asymmetric zero-range process on the multi-dimensional torus via a new approach based on the solution of a Poisson equation. For the simple inclusion process, the thermodynamic limit of the inclusion process whose underlying random walk is either a symmetric or totally asymmetric random walk on the one-dimensional torus has been investigated in [10]. The authors used exquisitely constructed heuristic simulations to derive various time scales related to the nucleation regime of the process, which is divided into four parts: nucleation, coarsening, saturation, and stationary. Readers may refer to [10] for further details.

Our contribution to the study of condensation in the thermodynamic limit regime is to establish the scaling limit of the movement of condensation, and we find three different time scales according to the level of asymmetry. We explain these results in the remainder of this subsection.

\footnote{A number asymptotically lying between $c_1 N$ and $c_2 N$.}
**Model.** We start by introducing our model, which is distinguished from previous models by the characteristic that the underlying state space is growing. Recall that $\mathbb{T}_L = \mathbb{Z}/L\mathbb{Z}$ denotes a discrete torus of length $L$. Now, we consider the inclusion process consisting of $N$ interacting particles that move according to a random walk on the multi-dimensional torus $\mathbb{T}_L^d$ where $L$ and $N$ grow together such that

$$L \to \infty, \quad N = N_L \to \infty, \quad \text{and} \quad \frac{N}{L^d} \to \rho \quad \text{for some} \quad \rho > 0. \quad (3.12)$$

Henceforth, we assume that $\rho > 0$ is fixed and regard $N$ as a variable that is dependent on $L$; hence, the only control variable is $L$. With this convention, the condition (3.12) implies that the total density is maintained to be close to $\rho$ as $L \to \infty$.

To get a scaling limit, we will assume that the underlying system is a translation-invariant random walk on $\mathbb{T}_L^d$, i.e., the jump rate of the underlying random walk on $\mathbb{T}_L^d$ is given by

$$r(x, y) = h(y - x) \quad (3.13)$$

for some non-negative function $h : \mathbb{Z}^d \to [0, \infty)$ with compact support, i.e., there exists $M > 0$ such that $h(x) = 0$ if $|x| > M$. We assume that this random walk is irreducible, i.e., the support of $h$ spans $\mathbb{Z}^d$.

**Remark 3.18.** Now, we state several remarks on this model:

(1) It should be emphasized that the simple nearest-neighbor random walk on $\mathbb{T}_L^d$ is an example of the translation-invariant random walk.

(2) By the translation invariance, it can immediately be verified that the random walk satisfies the condition (UI), i.e., the invariant measure $m$ of the underlying random walk is the uniform measure on $\mathbb{T}_L^d$. Moreover, this random walk is reversible with respect to this invariant measure only when the function $h$ is symmetric, i.e., $h(x) = h(-x)$ for each $x \in \mathbb{Z}^d$.

(3) Throughout the remainder of this subsection, we shall implicitly assume $L > 2M$ so that the state space $\mathbb{T}_L^d$ is much larger than the support of $h$.

The inclusion process $\{\eta_L(t)\}_{t \geq 0}$ on $\mathbb{T}_L^d$ consisting of $N$ particles where $N$ and $L$ satisfy (3.12) is defined as a continuous-time Markov chain on the configuration space given by

$$\mathcal{H}_L = \left\{ \eta \in \mathbb{N}^{\mathbb{T}_L^d} : \sum_{x \in \mathbb{T}_L^d} \eta_x = N \right\}.$$
If the inclusion process consists of the translation-invariant underlying random walks described above, then the generator corresponding to the inclusion process is defined, for $f : \mathcal{H}_L \to \mathbb{R}$, by

$$(\mathcal{L}_L f)(\eta) = \sum_{x, y \in \mathbb{T}_L^d} \eta_x (d_L + \eta_y) r(x, y) \{f(\sigma^{x, y} \eta) - f(\eta)\} ; \quad \eta \in \mathcal{H}_L ,$$

where $\{d_L\}_{L=1}^\infty$ is a sequence of positive real numbers converging to 0. Let $\mathbb{P}_\eta^L$ and $\mathbb{E}_\eta^L$ denote the law and expectation with respect to the process $\eta_L(\cdot)$ starting at $\eta$, respectively.

**Condensation.** We are primarily interested in the limiting behavior of the condensate of the model explained above as $L$ tends to infinity. As before, define the metastable set corresponding to the condensation of the inclusion process as

$$\mathcal{E}_L^x = \{\xi_L^x\} \text{ for each } x \in \mathbb{T}_L^d ,$$

where $\xi_L^x$ denotes the configuration containing all the particles at site $x \in \mathbb{T}_L^d$. Write

$$\mathcal{E}_L = \bigcup_{x \in \mathbb{T}_L^d} \mathcal{E}_L^x . \quad (3.14)$$

Let $\mu_L(\cdot)$ denote the invariant measure for this model. As this model satisfies the condition (UI) as mentioned in (2) of Remark 3.18, we can use Proposition 2.4 to write the invariant measure as

$$\mu_L(\eta) = \frac{1}{Z_L} \prod_{x \in \mathbb{T}_L^d} w_L(\eta_x) ; \quad \eta \in \mathcal{H}_L , \quad (3.15)$$

where

$$w_L(n) = \frac{\Gamma(n + d_L)}{n! \Gamma(d_L)} , \quad n \in \mathbb{N} \text{ and } Z_L = \sum_{\eta \in \mathcal{H}_L} \prod_{x \in \mathbb{T}_L^d} w_L(\eta_x) .$$

Owing to this expression, we can prove the occurrence of condensation provided that $d_L$ converges to 0 sufficiently fast.

**Theorem 3.19.** Suppose that $\lim_{L \to \infty} d_L L^d \log L = 0$. Then, we have

$$\lim_{L \to \infty} \mu_L(\mathcal{E}_L) = 1 .$$

Consequently, by the symmetry of the invariant measure (3.15), we have

$$\mu_L(\mathcal{E}_L^x) = (1 + o_L(1)) \frac{1}{L^d} \text{ for all } x \in \mathbb{T}_L^d .$$
We remark that this result has been recently proved in [11, Proposition 2] using the technique of size-biased sampling. However, we propose an alternative proof of this theorem in Section 8.1 for the completeness of the article.

Description of metastable behavior. Now, we turn to the dynamics of the condensate. In this model, we rescale the state space so that we can identify $x \in \mathbb{T}^d_L$ as a point $L^{-1}x \in \mathbb{T}^d$. By rescaling the time appropriately, we expect the dynamics of the condensate to converge to a process on $\mathbb{T}^d$ as $L \to \infty$. Our result presented below verifies that three different time scales appear according to the level of asymmetry of the underlying random walk. To rigorously formulate this result, we start by defining a map $\Theta_L : \mathcal{E}_L \to \mathbb{T}^d$ by

$$\Theta_L(x_L) = \frac{x}{L}, \quad x \in \mathbb{T}^d_L.$$ 

Define a process \( \{Y_L(t)\}_{t \geq 0} \) on $\mathbb{T}^d$ by

$$Y_L(t) = \Theta_L(\eta_{E_L}^L(t)),$$

where $\eta_{E_L}^L(\cdot)$ is the trace process of $\eta_L(\cdot)$ on the set $\mathcal{E}_L$. The following is a variant of Definition 3.2.

**Definition 3.20.** The movement of the condensate of the inclusion process on $\mathbb{T}^d_L$ defined above is said to be described by a process $\{Y(t)\}_{t \geq 0}$ on $\mathbb{T}^d$ with scale $\theta_L$ if the following conditions hold simultaneously.

1. For each sequence $\{x_L\}_{L=1}^{\infty}$ such that $x_L \in \mathbb{T}^d_L$ for all $L \geq 1$ and $\lim_{L \to \infty} (x_L/L) = u$, the law of the rescaled trace process $Y_L(\theta_L \cdot)$ starting from $\xi^L_E$ converges to that of the process $Y(\cdot) + u$ on $\mathbb{T}^d$.

2. The excursions outside $\mathcal{E}_L$ are negligible at the scale $\theta_L$ in the sense that

$$\lim_{L \to \infty} \sup_{\eta \in \mathcal{E}_L} \mathbb{E}_\eta \left[ \int_0^T 1 \{ \eta_L(\theta_L s) \notin \mathcal{E}_L \} \, ds \right] = 0 \text{ for all } T > 0. \quad \text{(3.16)}$$

**Main results for thermodynamic limit of metastable behavior.** Let $v$ denote the mean displacement (hence, the velocity) of the underlying random walk:

$$v = \sum_{y \in \mathbb{Z}^d} h(y) y.$$

We decompose the model into three cases as follows:

1. If $v \neq 0$, the model is referred to as **totally asymmetric**.

2. If $v = 0$ and $h$ is not symmetric, then the model is referred to as **mean-zero asymmetric**.
(3) If $v = 0$ and $h$ is symmetric, then the model is referred to as symmetric.

Then, the relevant time scales for these three cases are different, as we will see below. The following is the first main result.

**Theorem 3.21 (The first time scale for the totally asymmetric case).** Suppose that $v \neq 0$ and assume that $\lim d_L L^{d+1} \log L = 0$. Then, the movement of the condensate of the inclusion process on $\mathbb{T}_L^d$ is described by the deterministic motion $V(t) = \rho vt$ with scale $\theta_L = 1/(d_L L^{d-1})$.

Note that the limiting dynamics $V(t)$ obtained in the last theorem is non-degenerate only when the dynamics is totally asymmetric, i.e., $v \neq 0$. Hence, if $v = 0$, we have to wait for more time to observe the transitions of the condensation. Now, we formulate this result in a rigorous form. For each $y \in \mathbb{R}^d$, let $y \otimes y$ denote the outer product, i.e., $y \otimes y = yy^\dagger$.\(^4\) Hence, $y \otimes y$ is a $d \times d$ matrix. Consider a non-negative symmetric matrix $S_1$ given by

$$S_1 = \rho \sum_{y \in \mathbb{Z}^d: h(y) > h(-y)} (h(y) - h(-y)) y \otimes y$$

and let $\Sigma_1$ denote its square root.\(^5\)

**Theorem 3.22 (The second time scale for the mean-zero asymmetric case).** Suppose that $v = 0$ and assume that $\lim d_L L^{d+2} \log L = 0$. Then, the movement of the condensate of the inclusion process on $\mathbb{T}_L^d$ is described by the Brownian motion with diffusion matrix $\Sigma_1$ and scale $\theta_L = 1/(d_L L^{d-2})$.

This theorem explains the diffusive behavior of condensation when the underlying random walk is mean-zero such that the local drift at the time scale $1/(d_L L^{d-1})$ is canceled out. However, note that the matrix $S_1$, and hence $\Sigma_1$ is a zero matrix when the underlying random walk is symmetric. This indicates that we still have to wait for more time to observe the macroscopic movements of the condensate for the symmetric case. Indeed, we should wait for much longer to observe these movements. To formulate this, define a positive definite matrix $S_2$ by

$$S_2 = \sum_{y \in \mathbb{Z}^d} h(y) y \otimes y ,$$

\(^4\)Given a matrix $A$, let $A^\dagger$ denote the transpose of $A$.

\(^5\)Let $U^\dagger \Lambda U$ denote the diagonalization of the symmetric matrix $S_1$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d)$. Define $\Lambda^{1/2} = \text{diag}(\lambda_1^{1/2}, \ldots, \lambda_d^{1/2})$ which is well defined since $S_1$ is non-negative definite. Then, $\Sigma_1$ is defined by $U^\dagger \Lambda^{1/2} U$. Note that $\Sigma_1 \Sigma_1 = S_1$. 
and let $\Sigma_2$ denote its square root.

**Theorem 3.23 (The third time scale for the symmetric case).** Suppose that $h(x) = h(-x)$ for all $x \in \mathbb{Z}^d$ and assume that $\lim d_LL^{2d+2}\log L = 0$. Then, the movement of the condensate of the inclusion process on $\mathbb{T}_L^d$ is described by the Brownian motion with diffusion matrix $\Sigma_2$ and scale $\theta_L = L^2/d_L$.

The proofs of Theorems 3.21, 3.22, and 3.23 are given in Section 8. We conclude this section with several remarks on these theorems regarding the metastable behavior of the inclusion process in thermodynamic limit regime.

**Remark 3.24.**

1. It should be noted that the limiting particle density $\rho$ affects the limiting dynamics of the asymmetric cases. This is mainly because the higher density facilitates the first escape of one particle from a condensate. Subsequently, the movement of the remaining particles occurs instantaneously because of the asymmetry of the system. However, for the symmetric case, this acceleration of the first jump by the higher $\rho$ is canceled out by the fact that we have to move more particles to the adjacent site for the higher $\rho$. These two effects are exactly matched for the symmetric case; consequently, the limiting dynamics becomes independent of $\rho$.

2. Our condition on $d_L$ for this theorem is sub-optimal for technical reasons. We believe that all the results must hold under the condition $\lim d_LL^d\log L = 0$ as in Theorem 3.19.

3. Condensation of the zero-range process in the thermodynamic regime exhibits phase transition in terms of $\rho$ (e.g., see [1]). More precisely, there exists $\rho_c > 0$ such that condensation occurs if and only if $\rho > \rho_c$. However, in the inclusion process, we do not observe such a phenomenon. We refer to [11, Proposition 1] for further details.

4. **Movements of Condensate: General Results**

In this section, we present general results regarding the metastable behavior of the inclusion process on the basis of the martingale approach of Beltrán and Landim developed in [4,5]. The primary contribution of this approach is to reduce the analysis of metastable behavior to an investigation of the scaling limit of the so-called mean-jump rate of the trace process on metastable sets. In the reversible case, the mean-jump
rate can be estimated on the basis of the Dirichlet–Thomson principle of the potential theory. For instance, it leads us to rigorous results for the metastable behavior of the reversible zero-range process [1,3] and the reversible inclusion process [7]. Based on recent developments [14,30] of the non-reversible version of the Dirichlet–Thomson principle along with the martingale approach developed in [5,24] for the non-reversible case, the metastable behavior has also been analyzed for the totally asymmetric zero-range process on the discrete torus in [20] and for general non-reversible zero-range processes in [29]. We emphasize that the explicit form of the invariant measure played a crucial role in these studies.

Although we do not have such a formula, we will provide estimate of the mean-jump rate in this section (cf. Proposition 4.3). To overcome the lack of knowledge about the invariant measure, we exploit the fact that the metastable set of inclusion process is a singleton and the mean-jump rate is thus reduced to a jump rate between these singletons.

The results obtained in this section directly imply Theorem 3.8 regarding the metastability of the inclusion processes under (UI), as we are aware of the appearance of condensation for this case. We explain this in Section 5. However, for the general case, considerable effort is required to prove the existence of condensation to apply the results obtained in this section. This will be done under the condition (UP) in Sections 7 and 6. We also discuss the thermodynamic limit in Section 8 on the basis of the results obtained in this section.

4.1. Applications of the martingale approach. We start by explaining the application of the martingale approach for the inclusion setting.

Preliminary: negligibility of excursions on $\Delta_N$.

As a preliminary step, we first verify the two conditions given by (3.3) and (3.7) for the inclusion process under static condensation.

**Proposition 4.1.** Suppose that the inclusion process exhibits condensation and let $S_*$ be the maximal condensing set defined in (2.5). Then, for any sequence $(\alpha_N)_{N=1}^\infty$ of positive real numbers, we have

$$\lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N(S_*)} \mathbb{E}_\eta \left[ \int_0^T 1 \{ \eta_N(\alpha_N s) \in \Delta_N \} \, ds \right] = 0 \text{ for all } T > 0 ,$$  

$$\lim_{\delta \to 0} \lim_{N \to \infty} \sup_{2\delta \leq s \leq 3\delta} \sup_{\eta \in \mathcal{E}_N(S_*)} \mathbb{P}_\eta [ \eta_N(\alpha_N s) \in \Delta_N ] = 0 ,$$
where $\Delta_N = \mathcal{E}_N(S_*)^c$. In other words, the two conditions given by (3.3) and (3.7) hold.

Proof. For $x \in S_*$, we have
$$
P_{\xi_N} \left[ \eta_N(\alpha_N s) \in \Delta_N \right] \leq \frac{1}{\mu_N(\mathcal{E}_N^c)} P_{\mu_N} \left[ \eta_N(\alpha_N s) \in \Delta_N \right] = \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^c)} = o(1),$$
where the last identity follows from $\mu_N(\Delta_N) = o(1)$ and (2.5). Now, (4.1) directly follows from the Fubini theorem, as does (4.2).

Application of the martingale approach to inclusion processes. For $A \subseteq S$, we consider the trace process $\eta_{E_N}^{c(A)}(\cdot)$ defined in (3.1), which is a Markov chain on $\mathcal{E}_N(A)$. Denote the jump rate of this Markov chain by $r_N^A(\cdot, \cdot) : \mathcal{E}_N(A) \times \mathcal{E}_N(A) \to [0, \infty)$. Such a jump rate is called the mean-jump rate in the context of metastability theory. With this notation, the following is a consequence of the martingale approach [4,5].

**Proposition 4.2.** Suppose that the inclusion process exhibits condensation and let $S_*$ be the maximal condensing set. In addition, we suppose that
$$
\lim_{N \to \infty} \theta_N r_N^{S_*}(\xi_N^x, \xi_N^y) = a(x, y) \text{ for all } x, y \in S_*
$$
for some $a : S_* \times S_* \to [0, \infty)$. Then, the movement of the condensate is described by a Markov chain on $S_*$ with rate $a(\cdot, \cdot)$ and scale $\theta_N$. Moreover, the same description holds in the sense of the finite-dimensional marginal explained in Definition 3.7.

Proof. We refer to [4, Theorem 2.5] and [5, Theorem 2.1] for the first part of the proposition. The requirements of these theorems hold because of Definition 2.3 and (4.3). The second part of the proposition follows from (4.2) of Proposition 4.1 and [21, Proposition 2.1].

**Estimation of the mean-jump rate.** In view of Proposition 4.2, the analysis of the metastable behavior of the inclusion process is reduced to find a suitable scaling limit of the form (4.3) for the mean-jump rates. Such a scaling limit stated as Proposition 4.3 below is the main result of this section. Write
$$
\ell_N = d_N \log N + q^N,
$$
where $q \in (0, 1)$ is a fixed constant that will be specified later in (4.10).

**Proposition 4.3.** Suppose that $\lim_{N \to \infty} d_N \log N = 0$. Fix a non-empty set $A \subseteq S$ and define
$$
r_N^A(x, y) = \frac{1}{d_N N} r_N^A(\xi_N^x, \xi_N^y) \text{ for } x, y \in A.
$$
(1) If $A$ is a semi-attracting set, we have
\[
r^{A}_N(x, y) = \begin{cases} 
(1 + O\left(\frac{1}{N} + \ell_N\right)) \left(r(x, y) - r(y, x)\right) & \text{if } r(x, y) > r(y, x) , \\
O\left(\frac{1}{N} + \ell_N\right) & \text{if } r(x, y) < r(y, x) , \\
\frac{1}{N}r(x, y) + O\left(\frac{1}{N} + \ell_N\right) & \text{if } r(x, y) = r(y, x) .
\end{cases}
\]
(4.4)

(2) If $A$ is an attracting set, we have
\[
r^{A}_N(x, y) = \begin{cases} 
(1 + O\left(\ell_N\right)) \left(r(x, y) - r(y, x)\right) & \text{if } r(x, y) > r(y, x) , \\
O\left(\ell_N\right) & \text{if } r(x, y) < r(y, x) , \\
\frac{1}{N}r(x, y) + O\left(\ell_N\right) & \text{if } r(x, y) = r(y, x) .
\end{cases}
\]
(4.5)

It should be noted that the only difference between (4.4) and (4.5) is the appearance of the additional $O(1/N)$-order error term. Note that the error term $O(1/N)$ can be ignored when we consider the time scale $\theta_{rv}^N = 1/(Nd_N)$. Hence, in view of the following lemma, part (1) of the previous theorem provides sufficient control regarding the proof of Theorem 3.10.

**Lemma 4.4.** The set $S_0$ defined right after (3.8) is a semi-attracting set.

**Proof.** By contrast, suppose that some $x \in S_0$ and $y \in S^c_0$ satisfy $r(x, y) > r(y, x)$. Pick an invariant measure $\pi$ of $Z_1(\cdot)$ such that $\pi(x) > 0$ and $\pi(y) = 0$. Then, we have
\[
0 = \sum_{z \in S} \pi(y)b(y, z) = \sum_{z \in S} \pi(z)b(z, y) \geq \pi(x)b(x, y) = \pi(x)(r(x, y) - r(y, x)) > 0 ,
\]
which is a contradiction. ■

Meanwhile, we cannot afford this error when we consider the time scale $\theta_{rv}^N = 1/d_N$. Hence, we need to assume the attractiveness of $A$ in Theorem 3.12 to eliminate this error in (4.5).

The remainder of this section is devoted to proving Proposition 4.3. We establish several preliminary estimates in Section 4.2, and the proof of Proposition 4.3 is then given in Section 4.3.

4.2. **Hitting times on the tubes.** A set playing a significant role in the estimate of $r^{A}_N(\xi^x_N, \xi^y_N)$ for $x, y \in A$ is the **tube** $A^{x,y}_N$ between $\xi^x_N$ and $\xi^y_N$ defined hereafter, as the transition from $\xi^x_N$ to $\xi^y_N$ takes place only along this tube with dominating probability.
Figure 3. Visualization of the objects introduced in Notation 4.5 when $S = \{x, y, z\}$ and $r(y, z) = r(z, y) = 0$.

**Notation 4.5 (Tube between metastable sets).** Here, we gather all the relevant notation related to the tube that will be frequently used in the remainder of this study. We refer to Figure 3 for the illustration of the notation introduced here.

- For $x, y \in S$, the tube $\mathcal{A}_{x, y}^N$ between $\xi_N^x$ and $\xi_N^y$ is defined by
  \[
  \mathcal{A}_{x, y}^N = \{\eta \in \mathcal{H}_N : \eta_x + \eta_y = N\}. 
  \]
  Note that this tube contains $\xi_N^x$ and $\xi_N^y$. Let $\hat{\mathcal{A}}_{x, y}^N$ denote the set obtained from $\mathcal{A}_{x, y}^N$ by removing these two extremal configurations:
  \[
  \hat{\mathcal{A}}_{x, y}^N = \{\eta \in \mathcal{A}_{x, y}^N : \eta_x, \eta_y \geq 1\} = \mathcal{A}_{x, y}^N \setminus \{\xi_N^x, \xi_N^y\}. 
  \]

- We denote the set $\mathcal{A}_{N, y}^x$ by $\{\zeta_{0, y}^x, \zeta_{1, y}^x, \ldots, \zeta_{N, y}^x\}$, where $\zeta_{i, y}^x$ denotes the configuration
  \[
  (\zeta_{i, y}^x)_z = \begin{cases} 
  N - i & \text{for } z = x, \\
  i & \text{for } z = y, \quad 0 \leq i \leq N, \\
  0 & \text{otherwise},
  \end{cases} \quad (4.6)
  \]
  Note that $\zeta_0^x = \xi_N^x$, $\zeta_N^x = \xi_N^y$, and $\hat{\mathcal{A}}_{N, y}^x = \{\zeta_{1, y}^x, \ldots, \zeta_{N, y}^x\}$.

\footnote{Indeed, it should be denoted by $\{\zeta_{0, N}^x, \zeta_{1, N}^x, \ldots, \zeta_{N, N}^x\}$; however, we have ignored the dependency on $N$ in the notation.}
• If \( x, y \in S \) satisfy \( r(x, y) + r(y, x) > 0 \), then we write \( x \sim y \). With this notation, we write

\[
A_N = \bigcup_{x, y \in S : x \sim y} A_{x,y}^N \quad \text{and} \quad \hat{A}_N = \bigcup_{x, y \in S : x \sim y} \hat{A}_{x,y}^N .
\] (4.7)

Note that \( A_N = \hat{A}_N \cup E_N \). The remainder set is denoted by \( R_N \):

\[
R_N = \mathcal{H}_N \setminus A_N .
\]

Finally, we define several constants for convenience:

\[
R_1 = \min \{ r(x, y) : x, y \in S \text{ such that } r(x, y) > 0 \} > 0 ,
\]

\[
R_2 = \max \{ r(x, y) : x, y \in S \} ,
\]

\[
\Lambda = \max \{ \lambda(x) : x \in S \} ,
\]

where \( \lambda(x) = \sum_{y \in S} r(x, y) \) denotes the holding rate of the underlying random walk. For \( x, y \in S \) satisfying \( x \sim y \), we write

\[
q_{x,y} = \min \{ r(x, y), r(y, x) \} \max \{ r(x, y), r(y, x) \} \in [0, 1] .
\] (4.9)

Then, we define

\[
q = \max \{ q_{x,y} : x, y \in S, x \sim y \text{ and } r(x, y) \neq r(y, x) \} < 1 .
\] (4.10)

For \( C \subseteq \mathcal{H}_N \), let \( \tau_C \) denote the hitting time of the set \( C \). If the set \( C = \{ \eta \} \) is a singleton, we write \( \tau_{\{\eta\}} \) simply as \( \tau_\eta \). In the remainder of this section, we fix \( A \subseteq S \) and \( x, y \in S \). Then, we define an event \( E_0 = E_0^{y,A} \) by

\[
E_0 = \{ \tau_{E_N^y} = \tau_{E_N^x(A)} \} .
\]

Now, we provide a sequence of lemmas regarding the probability of the event \( E_0 \). We remark that these lemmas are also valid for a wide class of events that depend only on the hitting times of subsets of \( (\hat{A}_{x,y}^N)^c \) such as \( \{ \tau_{E_N^x} < \tau_{E_N^y} \} \).

The first lemma asserts that, provided \( d_N \) is sufficiently small, the inclusion process on \( \hat{A}_{x,y}^N \) behaves as a nearest-neighbor random walk whose jump rate from \( \zeta_{x,y}^i \) to \( \zeta_{x,y}^{i+1} \) is \( r(x, y) \) and from \( \zeta_{x,y}^{i+1} \) to \( \zeta_{x,y}^i \) is \( r(y, x) \), especially when we are only concerned with the event \( E_0 \).

**Lemma 4.6.** Suppose that \( x, y \in S \) satisfy \( x \sim y \). Then, there exists \( C > 0 \) such that

\[
\left| \mathbb{P}_{\zeta_{x,y}^i} [E_0] - \frac{r(x, y)}{r(x, y) + r(y, x)} \mathbb{P}_{\zeta_{x,y}^{i+1}} [E_0] - \frac{r(y, x)}{r(x, y) + r(y, x)} \mathbb{P}_{\zeta_{x,y}^{i-1}} [E_0] \right| \leq C \frac{d_N}{i(N - i)} \]
By the Markov property, we have
\[
\text{where the last line follows from the definition (4.8). Similarly, by (4.12) and (4.13),}
\]
\[
\text{Thus, the holding rate at } \zeta_{\cdot}^{x,y} \text{ is given by}
\]
\[
\text{Hence, by (4.11) and (4.13),}
\]
\[
\text{where the last line follows from the definition (4.8). Similarly, by (4.12) and (4.13),}
\]
\[
\text{The last two bounds imply that}
\]
\[
\text{By the Markov property, we have}
\]
\[
\text{For } a, b \in \mathbb{Z}, \text{ the interval } \llbracket a, b \rrbracket \text{ denotes } [a, b] \cap \mathbb{Z}.
Finally, inserting the estimates (4.14), (4.15), and (4.16) into the last identity completes the proof.

On the basis of the previous estimate, we can estimate the probabilities \( \mathbb{P}_{\xi^+_i} [E_0] \) and \( \mathbb{P}_{\xi^+_{N-1}} [E_0] \) in terms of \( \mathbb{P}_{\xi^+_N} [E_0] \) and \( \mathbb{P}_{\xi^+_N} [E_0] \). We divide this estimate into three cases according to the relation between \( r(x, y) \) and \( r(y, x) \) as follows.

**Lemma 4.7.** Suppose that \( x, y \in S \) satisfy \( r(x, y) > r(y, x) > 0 \). Then, it holds that

\[
\left| \mathbb{P}_{\xi^+_i} [E_0] - \frac{q_{x,y} - q_{x,y}^N}{1 - q_{x,y}^N} \mathbb{P}_{\xi^+_N} [E_0] - \frac{1 - q_{x,y}}{1 - q_{x,y}^N} \mathbb{P}_{\xi^+_N} [E_0] \right| = O(d_N \log N) \quad \text{and}
\]

\[
\left| \mathbb{P}_{\xi^+_{N-1}} [E_0] - \frac{q_{x,y}^{N-1} - q_{x,y}^N}{1 - q_{x,y}^N} \mathbb{P}_{\xi^+_N} [E_0] - \frac{1 - q_{x,y}^{N-1}}{1 - q_{x,y}^N} \mathbb{P}_{\xi^+_N} [E_0] \right| = O(d_N) .
\]

**Proof.** Following (4.9) and Lemma 4.6, it holds for \( i \in [1, N-1] \) that

\[
\left| \mathbb{P}_{\xi^+_i} [E_0] - \frac{1}{1 + q_{x,y}} \mathbb{P}_{\xi^+_{i+1}} [E_0] - \frac{q_{x,y}}{1 + q_{x,y}} \mathbb{P}_{\xi^+_{i-1}} [E_0] \right| \leq C \frac{d_N N}{i(N - i)} .
\]

Write

\[
b_i = \mathbb{P}_{\xi^+_i} [E_0] - \mathbb{P}_{\xi^+_N} [E_0] \quad \text{for} \quad i \in [1, N] ;
\]

so that we can rewrite (4.17) as

\[
\left| b_{i+1} - q_{x,y} b_i \right| \leq C \frac{d_N N}{i(N - i)} (1 + q_{x,y}) ,
\]

and therefore, for \( i \in [1, N] \),

\[
\left| b_i - q_{i-1} b_1 \right| \leq C d_N N (1 + q_{x,y}) \sum_{j=1}^{i-1} \frac{q_{i-1-j}}{j(N - j)} .
\]

Since \( \mathbb{P}_{\xi^+_N} [E_0] - \mathbb{P}_{\xi^+_N} [E_0] = b_1 + \cdots + b_N \), the previous bound implies that

\[
\left| \mathbb{P}_{\xi^+_N} [E_0] - \mathbb{P}_{\xi^+_N} [E_0] - \sum_{i=1}^{N} q_{i-1} b_1 \right| \leq C d_N N (1 + q_{x,y}) \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{q_{i-1-j}}{j(N - j)}
\]

\[
\quad = C d_N N (1 + q_{x,y}) \sum_{j=1}^{N-1} \frac{1}{j(N - j)} \sum_{i=j+1}^{N} q_{i-1-j}
\]

\[
\quad \leq C d_N \sum_{j=1}^{N-1} \left[ \frac{1}{j} + \frac{1}{N - j} \right] \frac{1 + q_{x,y}}{1 - q_{x,y}} \leq C d_N \log N .
\]
From this computation, we can deduce that
\[
\left| b_1 - \frac{1 - q_{x,y}}{1 - q_{x,y}} (\mathbb{P}_{\xi_N^y} [E_0] - \mathbb{P}_{\xi_N^x} [E_0]) \right| \leq Cd_N \log N.
\]

By inserting \( b_1 = \mathbb{P}_{\xi_N^x} [E_0] - \mathbb{P}_{\xi_N^y} [E_0] \), we obtain the first estimate of the lemma. The second one can be proved similarly.

Now, we consider the second case in which the jump from \( y \) to \( x \) is excluded.

**Lemma 4.8.** Suppose that \( x, y \in S \) satisfy \( r(x, y) > r(y, x) = 0 \). Then, it holds that
\[
\left| \mathbb{P}_{\xi_N^x} [E_0] - \mathbb{P}_{\xi_N^y} [E_0] \right| = O(d_N \log N) \quad \text{and} \quad \left| \mathbb{P}_{\xi_N^x} [E_0] - \mathbb{P}_{\xi_N^y} [E_0] \right| = O(d_N).
\]

**Proof.** By Lemma 4.6, it holds that
\[
\left| \mathbb{P}_{\xi_N^{x,y}} [E_0] - \mathbb{P}_{\xi_N^{y,x}} [E_0] \right| \leq C \frac{d_N N}{i(N - i)} \quad \text{for all } i \in [1, N - 1].
\]

By inserting \( i = N - 1 \), we immediately obtain the second estimate. For the first estimate, it suffices to apply the triangle inequality such that
\[
\left| \mathbb{P}_{\xi_N^{x,y}} [E_0] - \mathbb{P}_{\xi_N^{y,x}} [E_0] \right| \leq \sum_{i=1}^{N-1} C \frac{d_N N}{i(N - i)} = O(d_N \log N).
\]

This completes the proof of the first estimate.

Now, we consider the last case, i.e., the symmetric case.

**Lemma 4.9.** Suppose that \( x, y \in S \) satisfy \( r(x, y) = r(y, x) > 0 \). Then, it holds that
\[
\left| \mathbb{P}_{\xi_N^{x,y}} [E_0] - \mathbb{P}_{\xi_N^{y,x}} [E_0] \right| = O(d_N \log N).
\]

**Proof.** For \( i \in [1, N - 1] \), write
\[
c_i = \mathbb{P}_{\xi_N^{x,y}} [E_0] - \mathbb{P}_{\xi_N^{y,x}} [E_0] - \frac{1}{N} (\mathbb{P}_{\xi_N^{x,y}} [E_0] - \mathbb{P}_{\xi_N^{y,x}} [E_0]).
\]

Then, we can observe that
\[
c_1 + \cdots + c_N = 0
\]
and that the left-hand side of (4.18) is \( |c_1| \). Thus, it suffices to show that \( |c_1| = O(d_N \log N) \).

By Lemma 4.6, it holds that
\[
\left| \mathbb{P}_{\xi_N^{x,y}} [E_0] - \frac{1}{2} \mathbb{P}_{\xi_N^{x,y}} [E_0] - \frac{1}{2} \mathbb{P}_{\xi_N^{y,x}} [E_0] \right| \leq C \frac{d_N N}{i(N - i)} \quad \text{for all } i \in [1, N - 1].
\]
By (4.19), this inequality can be written as

\[ |c_i - c_{i+1}| \leq C \frac{d_N N}{i(N - i)}. \]

Therefore, by the triangle inequality, we obtain

\[ |c_1 - c_i| \leq \sum_{j=1}^{i-1} |c_j - c_{j+1}| \leq C d_N \sum_{j=1}^{i-1} \frac{N}{j(N - j)} \leq C d_N \log N. \]

Hence, by (4.20),

\[ |Nc_1| = |Nc_1 - (c_1 + \cdots + c_N)| \leq \sum_{i=2}^{N} |c_1 - c_i| \leq C d_N N \log N. \]

This completes the proof of \( |c_1| = O(d_N \log N) \).

4.3. Proof of Proposition 4.3.

Proof of Proposition 4.3. Fix \( A \subseteq S \) and fix \( x, y \in A \). By [4, Corollary 6.2], we can write the jump rate \( r^A_N(\xi_N^x, \xi_N^y) \) as

\[
\begin{align*}
    r^A_N(\xi_N^x, \xi_N^y) &= r_N(\xi_N^x, \xi_N^y) + \sum_{\eta \in H_N \setminus E_N(A)} r_N(\xi_N^x, \eta) \mathbb{P}_\eta \left[ \tau_{E_N^y} = \tau_{E_N(A)} \right] \\
    &= \sum_{z: z \neq x} N d_N r(x, z) \mathbb{P}_{\xi_N^x \rightarrow \xi_N^z} [E_0].
\end{align*}
\]

Hence, it suffices to estimate \( \mathbb{P}_{\xi_N^x \rightarrow \xi_N^z} [E_0] \) for \( z \neq x \) with \( r(x, z) > 0 \) to estimate \( r^A_N(\xi_N^x, \xi_N^y) \).

Suppose first that \( z \neq y \). Then, we divide the estimate of \( \mathbb{P}_{\xi_N^x \rightarrow \xi_N^z} [E_0] \) into two cases:

(Case 1: \( z \in A \)) Since \( \mathbb{P}_{\xi_N^x} [E_0] = 0 \), we deduce from Lemmas 4.7, 4.8, and 4.9 that

\[ \mathbb{P}_{\xi_N^x \rightarrow \xi_N^z} [E_0] = O(d_N \log N). \]

(Case 2: \( z \notin A \)) We divide this case into two as following:

- If \( A \) is attracting, we have \( r(x, z) < r(z, x) \). Thus by Lemma 4.7 we obtain

\[ \mathbb{P}_{\xi_N^x \rightarrow \xi_N^z} [E_0] = O(q^N) \mathbb{P}_{\xi_N^y} [E_0] + O(d_N \log N) = O(\ell_N). \]
• If $A$ is semi-attracting, we only have $r(x, z) \leq r(z, x)$. Thus by Lemmas 4.7 and 4.9 we obtain

$$\mathbb{P}_{\xi^x_1}[E_0] = O\left(\frac{1}{N} + q^{N}\right) \mathbb{P}_{\xi^y_1}[E_0] + O(d_N \log N) = O\left(\frac{1}{N} + \ell_N\right). \quad (4.24)$$

Now it remains to estimate $\mathbb{P}_{\xi^{x, y}_1}[E_0]$ when $r(x, y) \neq 0$ to estimate (4.21). To this end, we consider four cases separately:

1. $r(x, y) > r(y, x) > 0$: By Lemma 4.7 and the fact that

$$\mathbb{P}_{\xi^x_N}[E_0] = 0 \text{ and } \mathbb{P}_{\xi^y_N}[E_0] = 1,$$

we have that

$$\left|\mathbb{P}_{\xi^{x, y}_1}[E_0] - \frac{q^{N,y}_x q^{N,y}_x}{1 - q^{N,y}_x} \cdot 0 - \frac{1 - q^{N,y}_x}{1 - q^{N,y}_x} \cdot 1\right| = O(d_N \log N).$$

Thus, we have that

$$\mathbb{P}_{\xi^{x, y}_1}[E_0] = 1 - \frac{q^{N,y}_x}{1 - q^{N,y}_x} + O(d_N \log N) = (1 + O(\ell_N))(1 - q^{N}_x y). \quad (4.26)$$

2. $r(y, x) > r(x, y) > 0$: By Lemma 4.7 and (4.25),

$$\left|\mathbb{P}_{\xi^{x, y}_1}[E_0] - \frac{1 - q^{N,y}_x}{1 - q^{N,y}_x} \cdot 0 - \frac{1 - q^{N,y}_x}{1 - q^{N,y}_x} \cdot 1\right| = O(d_N \log N).$$

Therefore, we obtain that

$$\mathbb{P}_{\xi^{x, y}_1}[E_0] = \frac{q^{N,y}_x - q^{N,y}_x}{1 - q^{N,y}_x} + O(d_N \log N) = O(\ell_N). \quad (4.27)$$

3. $r(x, y) > r(y, x) = 0$: By Lemma 4.8 and (4.25),

$$\mathbb{P}_{\xi^{x, y}_1}[E_0] = 1 + O(d_N \log N). \quad (4.28)$$

4. $r(x, y) = r(y, x) > 0$: By Lemma 4.9 and (4.25),

$$\left|\mathbb{P}_{\xi^{x, y}_1}[E_0] - \frac{N - 1}{N} \cdot 0 - \frac{1}{N} \cdot 1\right| = O(d_N \log N).$$

Hence, we can conclude that

$$\mathbb{P}_{\xi^{x, y}_1}[E_0] = \frac{1}{N} + O(d_N \log N). \quad (4.29)$$

Finally, we can combine (4.22)-(4.29) along with the identity (4.21) to complete the proof of the proposition. ■
5. **Metastable Behavior of Inclusion Processes under Condition (UI)**

In this section, we investigate the metastable behavior of the inclusion process under the condition (UI). We first show that the invariant measure $\mu_N(\cdot)$ admits the expression (2.8).

**Proof of Proposition 2.4 for case (UI).** It suffices to prove that, for $\eta \in H_N$,

$$
\sum_{x, y \in S: \eta_y \geq 1} \mu_N(\sigma^{y, x}\eta)(\sigma^{y, x}\eta)_x(d_N + (\sigma^{y, x}\eta)_y)r(x, y) = \mu_N(\eta) \sum_{x, y \in S} \eta_y(d_N + \eta_x)r(y, x) .
$$

(5.1)

Calculating the left-hand side of (5.1), it holds that

$$
\sum_{x, y \in S: \eta_y \geq 1} \mu_N(\sigma^{y, x}\eta)(\sigma^{y, x}\eta)_x(d_N + (\sigma^{y, x}\eta)_y)r(x, y) = \sum_{y \in S: \eta_y \geq 1} \sum_{x \in S} \mu_N(\sigma^{y, x}\eta)(\eta_x + 1)(d_N + \eta_y - 1)r(x, y)
$$

$$
= \mu_N(\eta) \sum_{y : \eta_y \geq 1} \sum_{x \in S} \eta_y(d_N + \eta_x)r(x, y) = \mu_N(\eta) \sum_{x, y \in S} (\eta_x\eta_y + d_N\eta_y)r(x, y) .
$$

Comparing to the right-hand side of (5.1), it suffices to show that

$$
\sum_{x, y \in S} \eta_y r(x, y) = \sum_{x, y \in S} \eta_y r(y, x) .
$$

This identity holds since

$$
\sum_{x, y \in S} \eta_y r(x, y) = \sum_{y \in S} \eta_y \sum_{x \in S} r(x, y) (\text{UI}) = \sum_{y \in S} \eta_y \sum_{x \in S} r(y, x) = \sum_{x, y \in S} \eta_y r(y, x) .
$$

Now, we can prove Theorem 3.8 by gathering the results obtained so far.

**Proof of Theorem 3.8.** As we have mentioned before, part (1) follows from the investigation of the reversible case. Hence, we shall only concentrate on part (2). By Propositions 2.4 and 2.5, we know that condensation occurs on the entire set $S$, i.e., $S = S_*$. Then, the condition (4.3) of Proposition 4.2 follows from Proposition 4.3 with $A = S$, with $\theta_N = \frac{1}{N^{d_N}}$ and

$$
a(x, y) = [r(x, y) - r(y, x)] 1 \{r(x, y) > r(y, x)\} \text{ for all } x, y \in S .
$$

These scale and limiting chain correspond to (3.5) and (3.6) of Conjecture 3.4, and the proof is completed.
6. Metastable Behavior of Inclusion Processes with Condensation

In this section, we are concerning on the metastable behavior of the condensate of non-reversible inclusion processes under the condition that the condensation occurs, namely Theorems 3.10, 3.12 and 3.13. By assuming several irreducibility conditions on the limiting Markov chain, we derive the followings in this section based on the results obtained in Section 4:

- the characterization of the maximal condensing set \( S^* \subseteq S \),
- the asymptotic limit of \( \mu_N(\xi^x) \) for \( x \in S^* \) as \( N \to \infty \),
- the limiting Markov chain on \( S^* \) describing the movement of condensate.

We prove these main results in Section 6.2 based on a lemma introduced in 6.1.

6.1. A preliminary lemma. In this short subsection, we introduce an elementary lemma. We believe that this result is not new, but we include the full proof since we were not able to find an exact reference that states the exact result that we need.

**Lemma 6.1.** Let \( (Z_N(\cdot))_{N=1}^\infty \) be a sequence of continuous-time Markov chains on a finite set \( \mathcal{G} \). Denote the jump rate and the invariant measure of \( Z_N(\cdot) \) by \( a_N(\cdot, \cdot) \) and \( \pi_N(\cdot) \), respectively. Suppose in addition that

\[
\lim_{N \to \infty} a_N(x, y) = a(x, y) \quad \text{for all } x, y \in \mathcal{G} .
\]

Then each limit point of \( \{\pi_N\} \) becomes an invariant measure for the Markov chain \( Z(\cdot) \) with jump rate \( a(\cdot, \cdot) \). Moreover, if \( Z(\cdot) \) admits the unique invariant measure \( \pi \), then we have that

\[
\lim_{N \to \infty} \pi_N(x) = \pi(x) \quad \text{for all } x, y \in \mathcal{G} .
\]

**Remark.** In the second statement above, note that we did not assume the irreducibility of \( Z(\cdot) \). However, the uniqueness of the invariant measure for \( Z(\cdot) \) is a crucial condition for this statement.

**Proof.** Suppose that a subsequence \( (\pi_{N_k})_{k=1}^\infty \) converges to \( \pi_0 \). Note that \( \pi_0 \) must be a probability measure on \( \mathcal{G} \) as well. Since \( \pi_{N_k} \) is an invariant measure for the chain \( Z_{N_k} \), we have

\[
\sum_{y \in \mathcal{G}} \pi_{N_k}(x) a_{N_k}(x, y) = \sum_{y \in \mathcal{G}} \pi_{N_k}(y) a_{N_k}(y, x) \quad \text{for all } x, y \in \mathcal{G} .
\]

By letting \( k \to \infty \) at the last identity, we obtain that

\[
\sum_{y \in \mathcal{G}} \pi_0(x) a(x, y) = \sum_{y \in \mathcal{G}} \pi_0(y) a(y, x) \quad \text{for all } x, y \in \mathcal{G} .
\]
Therefore, \( \pi_0 \) is an invariant measure of \( Z(\cdot) \). This concludes the first statement.

Next we consider the second statement. Since \( \{\pi_N : N \in \mathbb{N}\} \) is a bounded subset of \( \mathbb{R}^S \), we know that this set is precompact. Moreover, we have shown above that every convergent subsequence converges to an invariant measure of \( Z(\cdot) \), which should be \( \pi \) by the uniqueness assumption for this case. This completes the proof.

\[ \square \]

6.2. **Proof of main results.** Now, we are ready to prove Theorems 3.10, 3.12, and 3.13. We consider the asymmetric case and the symmetric case separately. Recall two Markov chains \((Z_1(t))_{t \geq 0}\) and \((Z_2(t))_{t \geq 0}\) and the set \( S_0 \subseteq S \) from Section 3.3. We start with the asymmetric case.

**Proof of Theorem 3.10 and the asymmetric case of Theorem 3.13.** We first prove Theorem 3.10 by using Proposition 4.2. It suffices to verify (4.3) and the fact that \( S^\circ = S_0 \).

We recall the invariant measure \( \nu \) of \( Y_{nrv}(\cdot) \) on \( S_0 \) (cf. Theorem 3.13), and the rate \( b(\cdot, \cdot) : S \times S \to [0, \infty) \) defined in (3.8). Recalling the remark after Notation 3.11, the set \( S_0 \) is semi-attracting. Thus, by Proposition 4.3, we know that

\[
\lim_{N \to \infty} \theta_{nrv}^N r_N^S(\xi^x_N, \xi^y_N) = b(x, y) \quad \text{for all } x, y \in S ,
\]

where \( \theta_{nrv}^N = \frac{1}{Nd_N} \). We assumed that the Markov chain \( Z_1(\cdot) \) with jump kernel \( b \) has the only irreducible component \( S_0 \), and this guarantees the uniqueness of the invariant measure of \( Z_1(\cdot) \), which will be denoted by \( \pi \). Since the invariant measure of the trace process \( \eta_{E_N}^S(\theta_{nrv}^N \cdot) \) is the conditioned measure \( \mu_N(\cdot | E_N) = \mu_N(\cdot) / \mu_N(E_N) \) on \( E_N \), we can deduce from Lemma 6.1 that

\[
\lim_{N \to \infty} \frac{\mu_N(\xi^x_N)}{\mu_N(E_N)} = \pi(x) \quad \text{for all } x \in S .
\]

Since condensation occurs, i.e., \( \lim_{N \to \infty} \mu_N(E_N) = 1 \), we obtain that

\[
\lim_{N \to \infty} \mu_N(\xi^x_N) = \pi(x) . \tag{6.3}
\]

Since \( S_0 \) is the unique irreducible component of the chain \( Z_1(\cdot) \), we know that \( \pi(x) = 0 \) for \( x \in S \setminus S_0 \), and that \( \pi(x) > 0 \) for \( x \in S_0 \). From this and (6.3), we can conclude that \( S_\ast = S_0 \). Next, using Proposition 4.3 with \( A = S_0 \), we obtain

\[
\lim_{N \to \infty} \theta_{nrv}^N r_N^{S_0}(\xi^x_N, \xi^y_N) = b(x, y) \quad \text{for all } x, y \in S_0 ,
\]

Hence, the jump rate of the speeded-up trace process \( \eta_{E_N(S_0)}^S(\theta_{nrv}^N \cdot) \) converges to \( b(\cdot, \cdot) \), by identifying \( \xi^x_N \) with \( x \), which gives (4.3). This concludes Theorem 3.10.
Finally, note that \( \pi \) conditioned on the irreducible component \( S_0 \) is the invariant measure of the Markov chain \( Z_1(\cdot) \) conditioned on \( S_0 \), which is indeed the Markov chain \( Y^{\text{irr}}(\cdot) \) defined in the paragraph preceding (3.5). Thus, we can conclude that \( \pi(x) = \nu(x) \) for \( x \in S_0 \) as well. This and (6.3) finish the proof of the asymmetric case of Theorem 3.13. \( \blacksquare \)

Now, we consider the symmetric case, for which the time scale is now \( 1/d_N \) instead of \( 1/(Nd_N) \).

**Proof for Theorem 3.12 and the symmetric case of Theorem 3.13.** Again we use Proposition 4.2; hence, we shall demonstrate (4.3) and the fact that \( S_* = S_0 \).

We first prove that condensation occurs on \( S_0 \). By Proposition 4.3, we know that

\[
\lim_{N \to \infty} \frac{1}{Nd_N} r_{N}^S(\xi_N^x, \xi_N^y) = b(x, y) \quad \text{for all } x, y \in S.
\]

Here, the Markov chain \( Z_1(\cdot) \) does not necessarily admit a unique invariant measure. Nevertheless, all the invariant measures of \( Z_1(\cdot) \) do share the characteristic that they should vanish on \( S \setminus S_0 \), which is clear from the definition of \( S_0 \). Hence, it follows from the first statement of Lemma 6.1 that

\[
\lim_{N \to \infty} \frac{\mu_N(\xi_N^x)}{\mu_N(\mathcal{E}_N)} = 0 \quad \text{for all } x \in S \setminus S_0.
\]

It follows from above and the assumption of condensation on \( S \) that \( \lim_{N \to \infty} \mu_N(\mathcal{E}_N(S_0)) = 1 \), so that condensation occurs on \( S_0 \).

Next, using part (1) of Proposition 4.3 with \( A = S_0 \), which is possible since \( S_0 \) is assumed to be attracting, we obtain that

\[
\lim_{N \to \infty} \theta_N^{xy} r_{N}^{S_0}(\xi_N^x, \xi_N^y) = r(x, y) \quad \text{for all } x, y \in S_0,
\]

which establishes (4.3).

Since the Markov chain \( Z_2(\cdot) \) on \( S_0 \) with jump kernel \( b \) is irreducible by the condition of the theorem, it admits the unique invariant measure \( \nu \) on \( S_0 \). Hence, Lemma 6.1 implies that

\[
\lim_{N \to \infty} \frac{\mu_N(\xi_N^x)}{\mu_N(\mathcal{E}_N(S_0))} = \nu(x) \quad \text{for all } x \in S_0.
\]

Since condensation occurs on \( S_0 \), this implies that \( \lim_{N \to \infty} \mu_N(\xi_N^x) = \nu(x) \); thus, \( S_* = S_0 \). Therefore, Theorem 3.13 is proved for the symmetric case. Finally, Theorem 3.12 is concluded via Proposition 4.2. \( \blacksquare \)
7. Condensation under Condition (UP)

In this section, we establish condensation of the inclusion process under the condition (UP), i.e., prove Theorem 3.15. With this result on the occurrence of condensation, the analysis of the metastable behavior, as well as the characterization of $S_*$ and asymptotic mass of the invariant measure, follows immediately from the results obtained in Section 6. We mention that we do not have an explicit formula of the invariant measure $\mu_N$ for this case as well, and hence all the proof should follow the ways that have never been explored before.

We assume the condition (UP) throughout this section, i.e., $r(x, y) > 0$ for all $x, y \in S$. We start by summarizing several sets that are repeatedly used in the proof of the main result of this section. We refer to Figure 4 for the illustration of these sets.

**Notation 7.1.** Let $R$ be a non-empty subset of $S$.

- Define the $R$-tube as
  \[ A_N^R = \{ \eta \in \mathcal{H}_N : \eta_x = 0 \text{ for all } x \in S \setminus R \} . \]
  For example, $A_N^S = \mathcal{H}_N$, $A_N^{\{x\}} = \mathcal{E}_N^x$, and $A_N^{\{x,y\}} = A_N^{x,y}$ for all $x, y \in S$. In view of the last example, we can regard $A_N^R$ as a natural extension of the tube $A_N^{x,y}$ introduced in Notation 4.5.

- We decompose each $R$-tube $A_N^R$ into its boundary $\partial A_N^R$ and the core $R_N^R$ where
  \[ \partial A_N^R = \{ \eta \in A_N^R : \eta_x = 0 \text{ for some } x \in R \} \text{ and } \]
  \[ R_N^R = \{ \eta \in A_N^R : \eta_x > 0 \text{ for all } x \in R \} . \]
  For example, we have $\partial A_N^{\{x,y\}} = \mathcal{E}_N^x \cup \mathcal{E}_N^y$ and $R_N^{\{x,y\}} = \mathcal{A}_N^{x,y}$.

- We further decompose the core $R_N^R$ into the inner core $I_N^R$ and the outer core $O_N^R$ where
  \[ I_N^R = \{ \eta \in R_N^R : \eta_x > \epsilon \log N \text{ for all } x \in R \} , \]
  \[ O_N^R = \{ \eta \in R_N^R : \eta_x \leq \epsilon \log N \text{ for some } x \in R \} , \]
  where $\epsilon$ is a small enough number that will be specified later (cf. (7.19)). We stress that $\epsilon$ does not depend on $N$. For the convenience of notation, we assume in this and the next subsections that $\epsilon \log N$ is an integer. (For general case, it suffices to replace all $\epsilon \log N$ below with $\lfloor \epsilon \log N \rfloor$.) For instance, a configuration $\eta$ belonging to $I_N^R$ does not have particles at $S \setminus R$ while have more than $\epsilon \log N$ particles at each site of $R$. Summing up, we decompose each $R$-tube $A_N^R$ into
Figure 4. Visualization of the notation introduced in Notation 7.1 when $S = \{x, y, z, w\}$ and $R = \{x, y, z\}$.

the following disjoint union:

$$A_N^R = \partial A_N^R \cup O_N^R \cup I_N^R.$$  

(7.1)

- Write $|S| = \kappa$. For $1 \leq k \leq \kappa$, we define

$$B_N^k = \bigcup_{R \subseteq S, |R| = k} A_N^R.$$  

Namely, $B_N^k$ is a collection of configurations that have at most $k$ sites with at least one particle. For instance, $B_N^1 = \mathcal{E}_N$, $B_N^2 = A_N$ (by the assumption (UP)), and $B_N^\kappa = \mathcal{H}_N$.

In this section, we are mainly focusing on the following proposition.

Proposition 7.2. Suppose that (3.10) holds. Then, for all $\ell \in [2, \kappa]$, we have that

$$\lim_{N \to \infty} \frac{\mu_N(B_N^\ell)}{\mu_N(B_N^{\ell-1})} = 1.$$  

(7.2)

Proof. We explain the proof based on the results that will be proved in the remaining part of this section. We prove this proposition by means of the backward induction on $\ell$ from $\kappa$ to 2. We note that the initial case $\ell = \kappa$ is proved by Propositions 7.3 and 7.7 (cf. discussion between (7.4) and (7.5)). Then, by the induction step proved in Proposition 7.15, the assertion of the proposition holds for all $\ell \in [2, \kappa]$. ■

With this proposition, Theorem 3.15 is immediate.
Proof of Theorem 3.15. Since $B_1^N = E_N$ and $B_\kappa^N = H_N$, it suffices to check that
\[
\lim_{N \to \infty} \frac{\mu_N(B_\kappa^N)}{\mu_N(B_1^N)} = 1. \tag{7.3}
\]
This is immediate from (7.2) and we are done. \hfill \blacksquare

Now, we explain our plan to prove the detailed ingredients appeared in the proof of Proposition 7.2. The initial step $\ell = \kappa$ for the backward induction is proved in Sections 7.1 and 7.2, and the induction step is established in Section 7.3. For the proof of these steps, an auxiliary Markov chain introduced in Definition 7.9 of Section 7.2 is crucially used. As a by-product of our investigation of the hitting time of this Markov chain carried out in Lemma 7.11, we prove the nucleation result presented as Theorem 3.17 in Section 7.4 as well.

7.1. Initial step (1): negligibility of the outer core. Now, we prove the case $\ell = \kappa$ for Proposition 7.2. Since $B_\kappa^N = A_S^N (= H_N)$ and $B_{\kappa-1}^N = \partial A_S^N$ by the definition of the boundary, it suffices to prove
\[
\lim_{N \to \infty} \frac{\mu_N(A_S^N)}{\mu_N(\partial A_S^N)} = 1. \tag{7.4}
\]
Since $\mu_N(A_S^N) = \mu_N(\partial A_S^N) + \mu_N(O_S^N) + \mu_N(I_S^N)$ by (7.1), it suffices to prove that
\[
\lim_{N \to \infty} \frac{\mu_N(O_S^N)}{\mu_N(\partial A_S^N)} = 0 \quad \text{and} \quad \lim_{N \to \infty} \frac{\mu_N(I_S^N)}{\mu_N(\partial A_S^N)} = 0. \tag{7.5}
\]
The proof of the latter one is considered in the next subsection, and we focus only on the former one in the current subsection. Thus, the main object now is to prove the following proposition.

Proposition 7.3. Suppose that $\lim_{N \to \infty} d_N N = 0$. Then, for sufficiently small $\epsilon > 0$, we have that
\[
\lim_{N \to \infty} \frac{\mu_N(O_S^N)}{\mu_N(\partial A_S^N)} = 0.
\]

To prove this, we decompose the outer core $O_S^N$ into more refined objects, and estimate each of them carefully.

Decomposition of outer core. For $x \in R \subseteq S$ and $k \in [1, N]$, we define
\[
C_R^N(x, k) = \{ \eta \in A_R^N : \eta_x = k \}.
\]
For instance, we have
\[ C^R_N(x, N) = E^x_N \] and \[ C^R_N(x, 0) = A^R_N\{x\} \, . \] (7.6)

Then, it holds that
\[ O^R_N \subseteq \bigcup_{x \in R} \bigcup_{k=1}^{\varepsilon \log N} C^R_N(x, k) \, ; \] (7.7)
thus,
\[ \mu_N(O^S_N) \leq \sum_{x \in S} \sum_{k=1}^{\varepsilon \log N} \mu_N(C^S_N(x, k)) \, . \] (7.8)

Hence, it suffices to estimate \( \mu_N(C^S_N(x, k)) \) for \( k \in [1, \varepsilon \log N] \) and \( x \in S \).

Estimation of \( \mu_N(C^S_N(x, k)) \). For \( k \in [0, N - 1] \) and \( x \in R \subseteq S \), we define
\[ F^R_N(x; k \to k + 1) = \sum_{\eta \in C^R_N(x, k), \zeta \in C^R_N(x, k + 1)} \mu_N(\eta) r_N(\eta, \zeta) \, , \]
\[ F^R_N(x; k + 1 \to k) = \sum_{\eta \in C^R_N(x, k + 1), \zeta \in C^R_N(x, k)} \mu_N(\eta) r_N(\eta, \zeta) \, . \]

Lemma 7.4. For all \( k \in [0, N - 1] \) and \( x \in S \), it holds that
\[ F^S_N(x; k \to k + 1) = F^S_N(x; k + 1 \to k) \, . \]

Proof. Since \( \mu_N \) is the invariant measure for the inclusion process, we have that,
\[ \sum_{x, y \in S} \mu_N(\eta) r_N(\eta, \sigma^{x,y}\eta) = \sum_{x, y \in S} \mu_N(\sigma^{x,y}\eta) r_N(\sigma^{x,y}\eta, \eta) \] for all \( \eta \in H_N \) . (7.9)

Here, we use the convention that \( r_N(\eta, \eta) = 0 \) for \( \eta \in H_N \). By summing (7.9) over \( \eta \in C^S_N(x, k) \),
\[ \sum_{\eta \in C^S_N(x, k), y, z \in S \setminus \{x\}} \mu_N(\eta) r_N(\eta, \sigma^{y,z}\eta) + \sum_{\eta \in C^S_N(x, k), y \in S \setminus \{x\}} \mu_N(\eta) r_N(\eta, \sigma^{y,x}\eta) \]
\[ + \sum_{\eta \in C^S_N(x, k), y \in S \setminus \{x\}} \mu_N(\eta) r_N(\eta, \sigma^{x,y}\eta) \]
\[ = \sum_{\eta \in C^S_N(x, k), y, z \in S \setminus \{x\}} \mu_N(\sigma^{y,z}\eta) r_N(\sigma^{y,z}\eta, \eta) + \sum_{\eta \in C^S_N(x, k), y \in S \setminus \{x\}} \mu_N(\sigma^{y,x}\eta) r_N(\sigma^{y,x}\eta, \eta) \]
\[ + \sum_{\eta \in C^S_N(x, k), y \in S \setminus \{x\}} \mu_N(\sigma^{x,y}\eta) r_N(\sigma^{x,y}\eta, \eta) \, . \] (7.10)
Note that the first summations in the respective sides of (7.10) are canceled out with each other. Therefore, (7.10) can be simply rewritten as
\[
F^S_N(x; k \to k+1) + F^S_N(x; k \to k-1) = F^S_N(x; k \to k+1) + F^S_N(x; k-1 \to k),
\]
where \(F^S_N(x; -1 \to 0)\) and \(F^S_N(x; 0 \to -1)\) are defined to be 0. Inserting \(k = 0\) to (7.11) implies
\[
F^S_N(x; 0 \to 1) = F^S_N(x; 1 \to 0).
\] (7.12)
Therefore, (7.11) and (7.12) along with induction on \(k\) finish the proof. ■

**Lemma 7.5.** For \(k \in [0, N - 1]\) and \(x \in S\), we have that
\[
\mu_N(C^S_N(x, k+1)) \leq \frac{R_2(k + d_N)(N - k)}{R_1(k + 1)(N - k - 1 + d_N)} \mu_N(C^S_N(x, k)),
\]
where the constants \(R_1\) and \(R_2\) are introduced in (4.8).

**Proof.** Looking at \(F^S_N(x; k \to k+1)\) more carefully, we get the following bound:
\[
F^S_N(x; k \to k+1) = \sum_{\eta \in C^S_N(x, k), \zeta \in C^S_N(x, k+1)} \mu_N(\eta) r_N(\eta, \zeta)
\]
\[
= \sum_{\eta \in C^S_N(x, k)} \sum_{y \in S \setminus \{x\}} \mu_N(\eta) r_N(\eta, \sigma^y, x)
\]
\[
= \sum_{\eta \in C^S_N(x, k)} \left[ \mu_N(\eta) \sum_{y \in S \setminus \{x\}} \eta_y(d_N + \eta_x)r(y, x) \right]
\]
\[
= (k + d_N) \sum_{\eta \in C^S_N(x, k)} \mu_N(\eta) \sum_{y \in S \setminus \{x\}} r(y, x)\eta_y
\]
\[
\leq R_2(k + d_N)(N - k) \mu_N(C^S_N(x, k)).
\] (7.13)

Similarly, we can get
\[
F^S_N(x; k+1 \to k) = \sum_{\eta \in C^S_N(x, k+1)} \sum_{y \in S \setminus \{x\}} \mu_N(\eta) r_N(\eta, \sigma^x, y\eta)
\]
\[
= \sum_{\eta \in C^S_N(x, k+1)} \left[ \mu_N(\eta) \sum_{y \in S \setminus \{x\}} \eta_x(d_N + \eta_y)r(x, y) \right]
\]
\[
\geq R_1(k + 1)(N - k - 1 + d_N) \mu_N(C^S_N(x, k+1)).
\] (7.14)
Combining (7.13), (7.14) with Lemma 7.4, we can complete the proof of the lemma. ■
In the proof above, it is crucial to have \( r(x, y) > 0 \) for all \( x, y \in S \) to deduce (7.14). Hence, the condition (UP) is critically used.

**Lemma 7.6.** For sufficiently small \( \epsilon > 0 \), we have that

\[
\sum_{k=1}^{\epsilon \log N} \mu_N(C^S_N(x, k)) \leq O(N d_N) \mu_N(A^S_N \{x\}) .
\]

*Proof.* Inserting \( k = 0 \) to Lemma 7.5 yields that, for some constant \( C_1 > 0 \),

\[
\mu_N(C^S_N(x, 1)) \leq C_1 d_N \mu_N(C^S_N(x, 0)) ,
\]

while inserting \( k \in [1, N - 2] \) provides us that for some constant \( C_2 > 0 \),

\[
\mu_N(C^S_N(x, k + 1)) \leq C_2 N \mu_N(C^S_N(x, k)) .
\]

Let \( C_0 = \max\{C_1, C_2\} \). Then, (7.15) and (7.16) imply that

\[
\mu_N(C^S_N(x, k)) \leq C_0^k d_N \mu_N(C^S_N(x, 0)) \quad \text{for} \quad k \in [1, N - 1] .
\]

Summing this up for \( k = 1, 2, \ldots, \epsilon \log N \), we get

\[
\sum_{k=1}^{\epsilon \log N} \mu_N(C^S_N(x, k)) \leq \frac{C_0^{(\epsilon \log N)+1} - C_0}{C_0 - 1} d_N \mu_N(C^S_N(x, 0)) .
\]

Take \( \epsilon \) small enough so that

\[
\frac{C_0^{(\epsilon \log N)+1} - C_0}{C_0 - 1} = O(N) .
\]

The proof is completed since \( C^S_N(x, 0) = A^S_N \{x\} \) by (7.6). \( \blacksquare \)

Now, we are ready to prove the main goal of this subsection.

**Proof of Proposition 7.3.** By (7.7) and the previous lemma, we get

\[
\mu_N(O^S_N) \leq \sum_{x \in S} \sum_{k=1}^{\epsilon \log N} \mu_N(C^S_N(x, k)) \leq C d_N N \sum_{x \in S} \mu_N(A^S_N \{x\}) .
\]

The proof is completed since

\[
\sum_{x \in S} \mu_N(A^S_N \{x\}) = \mu_N(\partial A^S_N) ,
\]

and since \( \lim_{N \to \infty} d_N N = 0 \) by the assumption of the proposition. \( \blacksquare \)
7.2. Initial step (2): negligibility of the inner core. In this subsection, we prove the negligibility of the inner core $I_S^N$ via the following proposition.

**Proposition 7.7.** Suppose that $\lim_{N \to \infty} d_N N^{\kappa + 2} (\log N)^{\kappa - 3} = 0$. Then, we have that

$$\lim_{N \to \infty} \frac{\mu_N(I_S^N)}{\mu_N(\partial A_S^N)} = 0.$$ 

The proof of this part is more demanding than that of the outer core, and we have to introduce a sequence of new concepts.

Define the closure and the (outer) boundary of $I_R^N$ for $R \subseteq S$ as

$$I_R^N = \{ \eta \in A_R^N : r_N(\zeta, \eta) > 0 \text{ for some } \zeta \in I_R^N \},$$

$$\partial I_R^N = I_R^N \setminus I_R^N.$$

Thus, $I_S^N$ consists of configurations $\eta$ such that $\eta_x \geq \epsilon \log N$ for all $x$ and there exists at most one $x \in S$ with $\eta_x = \epsilon \log N$, while $\partial I_S^N$ consists of configurations $\eta$ such that $\eta_x \geq \epsilon \log N$ for all $x$ and there exist exactly one $x \in S$ with $\eta_x = \epsilon \log N$. Therefore, we have the following decomposition for $\partial I_S^N$

$$\partial I_S^N \subseteq \bigcup_{x \in S} C_N^S(x, \epsilon \log N).$$

Therefore, by (7.17) and (7.19), we have that

$$\mu_N(\partial I_S^N) \leq \sum_{x \in S} \mu_N(C_N^S(x, \epsilon \log N)) \leq C N d_N \sum_{x \in S} \mu_N(C_N^S(x, 0)) = O(N d_N) \mu_N(\partial A_N^S). \tag{7.20}$$

Hence, Proposition 7.7 is the consequence of the following proposition.

**Proposition 7.8.** Suppose that $\lim_{N \to \infty} d_N N^{\kappa + 2} (\log N)^{\kappa - 3} = 0$. Then, we have that

$$\mu_N(I_S^N) = O(N^{\kappa - 2} (\log N)^{\kappa}) \mu_N(\partial I_S^N). \tag{7.21}$$

**Proof of Proposition 7.7.** By (7.20) and (7.21), we get that

$$\mu_N(I_S^N) = O(N^{\kappa - 2} (\log N)^{\kappa}) \mu_N(\partial I_S^N) = O(N^{\kappa - 1} (\log N)^{\kappa} d_N) \mu_N(\partial A_N^S).$$

By the condition $d_N N^{\kappa + 2} (\log N)^{\kappa - 3} = o_N(1)$, we are done. \hfill \blacksquare

The remaining part of this subsection is devoted to prove Proposition 7.8.

**Auxiliary Markov chain $\hat{\eta}_N^R(\cdot)$ and its hitting time estimate.** The crucial ingredient in the proof of Proposition 7.8 is an auxiliary discrete time Markov chain on $\hat{T}_N = $
\(I^S \cup \partial I^S\) and the estimate of the hitting time of the set \(\partial I^S\) when the chain starts from \(I^S\). To use these results at the induction step in Section 7.3, we will work on \(I^R\) for \(R \subseteq S\).

**Definition 7.9.** For \(R \subseteq S\), let \((\tilde{\eta}^R_N(t))_{t \in \mathbb{N}}\) denote the discrete-time Markov chain on \(I^R\) whose transition probability \(\tilde{p}^R_N\) is given by

\[
\tilde{p}^R_N(\eta, \sigma \cdot \eta) = \frac{\eta_y(d_N + \eta_x) r(y, x)}{\sum_{a, b \in R} \eta_a (d_N + \eta_b) r(a, b)} \text{ for } \eta, \sigma \cdot \eta \in \mathcal{I}^R, \tag{7.22}
\]

and set

\[
\tilde{p}^R_N(\eta, \eta) = 1 - \sum_{\zeta \neq \eta \notin \mathcal{I}^R} \tilde{p}^R_N(\eta, \zeta) \text{ for } \eta \in \partial \mathcal{I}^R. \tag{7.23}
\]

In other words, \(\tilde{\eta}^R_N(\cdot, \cdot)\) is attained from the discrete version of the inclusion process by changing the jump rate \(\eta_x(d_N + \eta_y) r(x, y)\) to \(\eta_y(d_N + \eta_x) r(y, x)\) and then restrict to \(I^R\). This chain is well-defined since \(\eta_x, \eta_y > 0\) for \(\eta \in \mathcal{I}^R\).

Let \(\tilde{L}^R_N\) denote the corresponding generator and by \(\tilde{E}^R_N\) the expectation with respect to the chain \(\tilde{\eta}^R_N(\cdot, \cdot)\) starting from \(\eta \in \mathcal{I}^R\). Finally, let \(\sigma_R := \tau_{\partial \mathcal{I}^R}\) be the hitting time the set \(\partial \mathcal{I}^R\) by the chain \(\tilde{\eta}^R_N(\cdot, \cdot)\). **Then, the primary purpose is to estimate** \(\tilde{E}^R_N[\sigma_R]\) **for** \(\eta \in \mathcal{I}^R\). The crucial step for this estimate is the following construction of a test function, which based on the so-called Gordan’s lemma.

**Lemma 7.10.** Suppose that \(R \subseteq S\) and \(\lim_{N \to \infty} d_N \frac{N^3}{(\log N)^2} = 0\). Then, there exist a constant \(C = C(\epsilon) > 0\) and a test function \(f_0 = f^R_0: \mathcal{I}^R_N \to \mathbb{R}\) such that

\[
\max_{\mathcal{I}^R_N} f_0 - \min_{\mathcal{I}^R_N} f_0 \leq C \log N, \text{ and } \tag{7.24}
\]

\[
(\tilde{L}^R_N f_0)(\eta) \geq \frac{\log N}{CN^3} \text{ for all } \eta \in \mathcal{I}^R_N. \tag{7.25}
\]

**Proof.** Fix \(R \subseteq S\) and consider a \(|R| \times |R|\) skew-symmetric matrix \(Q\) defined by

\[
Q_{x,y} = r(x, y) - r(y, x), \quad x \in R, y \in R.
\]

By Gordan’s lemma stated in Lemma 9.2 at the appendix, we have that

\[
\exists \alpha = (\alpha_x)_{x \in R} \in \mathbb{R}^{|R|} \text{ such that } (Q \alpha)_1, \ldots, (Q \alpha)_{|R|} < 0, \tag{7.26}
\]

or

\[
\exists \beta(\neq 0) = (\beta_x)_{x \in R} \in \mathbb{R}^{|R|} \text{ so that } \beta_1, \ldots, \beta_{|R|} \leq 0 \text{ and } Q \beta = 0. \tag{7.27}
\]
We consider these two cases separately.

**Case 1: (7.25)** In this case, define \( f_0 : \mathcal{I}_N^R \to \mathbb{R} \) as
\[
f_0(\eta) = \sum_{x \in R} \alpha_x \left(1 + \frac{1}{2} + \cdots + \frac{1}{\eta_x}\right).
\]

Then, for each \( \eta \in \mathcal{I}_N^R \),
\[
|f_0(\eta)| \leq C \sum_{x \in R} |\alpha_x| \log \eta_x \leq C' \log N;
\]
hence, the condition \(7.23\) follows immediately. To check the condition \(7.24\), we define
\[
w(\eta) = \sum_{a, b \in R} \eta_a (d_N + \eta_b) r(a, b),
\]
so that
\[
(\hat{L}^R_N f_0)(\eta) = \frac{1}{w(\eta)} \sum_{x, y \in R} \eta_y (d_N + \eta_x) r(y, x) \left(\frac{\alpha_y}{\eta_y + 1} - \frac{\alpha_x}{\eta_x}\right)
\]
\[
= \frac{1}{w(\eta)} \left\{ \sum_{x, y \in R} r(y, x) \eta_x \eta_y \left(\frac{\alpha_y}{\eta_y + 1} - \frac{\alpha_x}{\eta_x}\right) + O(d_N \frac{N}{\log N}) \right\}
\]
\[
= \frac{1}{w(\eta)} \left\{ \sum_{x, y \in R} r(y, x) \eta_x \eta_y \left(\frac{\alpha_y}{\eta_y} - \frac{\alpha_x}{\eta_x}\right) + O\left(\frac{N}{\log N}\right) + O(d_N \frac{N}{\log N}) \right\}.
\]

The seemingly not so serious last identity is indeed the main reason that we introduced the inner core \( \mathcal{I}_N^R \). The error coming from this identity is not able to control if \( \eta_y \) is close to 0. In this case the bound \( \eta_y \geq \epsilon \log N \) provides us the small error term of \( O(N/\log N) \).

Now the last summation can be computed as
\[
\sum_{x, y \in R} r(y, x)(\alpha_y \eta_x - \alpha_x \eta_y) = \sum_{x \in R} \left[ \eta_x \sum_{y \in R} \alpha_y \{r(y, x) - r(x, y)\}\right]
\]
\[
= \sum_{x \in R} \left[ \eta_x \sum_{y \in R} -Q_{x, y} \alpha_y\right]
\]
\[
= \sum_{x \in R} \eta_x (-Q\alpha)_x \geq \frac{N}{C}.
\]
where the last inequality is due to (7.25). Since \( w(\eta) = O(N^2) \), applying the last inequality to (7.27) verifies the condition (7.24).

(Case 2: (7.26)) Define \( f_0 : \mathbb{T}_N^R \to \mathbb{R} \) by

\[
f_0(\eta) = \sum_{x \in R} \beta_x \left( \frac{1}{2} + \cdots + \frac{1}{\eta_x} \right).
\]

Then, the condition (7.23) follows similarly as (Case 1). By a similar calculation, for \( \eta \in \mathbb{T}_N^R \),

\[
(\hat{L}_N^R f_0)(\eta) \geq \frac{1}{w(\eta)} \left[ \sum_{x \in R} \eta_x \cdot \beta_x \left( \frac{1}{2} + \cdots + \frac{1}{\eta_x} \right) \right] \geq \frac{1}{C} N^{-3} \log N,
\]

where the last inequality follows from the fact that \( w(\eta) = O(N^2) \) and the condition on \( d_N \) given at the statement of Lemma.

We remark that, at the first inequality of (7.28), the condition (UP) is strongly used again. Now, we estimate the expectation of the hitting time \( \sigma_R \) of the outer boundary \( \partial I_N^R \).

**Lemma 7.11.** Suppose that \( R \subseteq S \) and that \( \lim_{N \to \infty} d_N \frac{N^2}{(\log N)^2} = 0 \). Then, there exists \( C = C(\epsilon) > 0 \) such that

\[
\sup_{\eta \in \mathbb{T}_N^R} \mathbb{E}_\eta^R [\sigma_R] \leq C N^3.
\]

**Proof.** For \( f : \mathbb{T}_N^R \to \mathbb{R} \), we know that

\[
\mathcal{M}(n) = f(\hat{\eta}_N^R(0)) - f(\hat{\eta}_N^R(n)) = \sum_{k=0}^{n-1} (\hat{L}_N^R f)(\hat{\eta}_N^R(k)) ; n \in \mathbb{N}
\]
is a discrete-time martingale with initial value 0. Therefore, by the optional stopping theorem, we have for all $\eta \in I^R_N$ and $n \geq 0$ that

$$
\hat{E}_\eta^R \left[ f(\hat{\eta}_N^R(\sigma_R \wedge n)) \right] = f(\eta) + \hat{E}_\eta^R \left[ \sum_{k=0}^{(\sigma_R \wedge n)-1} (\hat{L}_N^R f)(\hat{\eta}_N^R(k)) \right].
$$

(7.29)

Now, we insert $f = f_0$ where $f_0$ is the test function obtained in Lemma 7.10. Using the bounds in (7.23) and (7.24), it holds for all $n \geq 0$ that

$$
C \log N \geq \hat{E}_\eta^R \left[ f_0(\hat{\eta}_N^R(\sigma_R \wedge n)) \right] - f_0(\eta)
$$

$$
= \hat{E}_\eta^R \left[ \sum_{k=0}^{(\sigma_R \wedge n)-1} (\hat{L}_N^R f_0)(\hat{\eta}_N^R(k)) \right] \geq \frac{\log N}{CN^3} \hat{E}_\eta^R [\sigma_R \wedge n].
$$

Thus, the proof is completed by letting $n \to \infty$. \hfill \blacksquare

**Remark 7.12.** A careful reading of the proofs shows that Lemmas 7.10 and 7.11 holds for any $\epsilon > 0$.

**Lemma 7.13.** Fix a set $R \subseteq S$ and a constant $\delta \geq 0$. Suppose that $\lim_{N \to \infty} d_N \frac{N^2}{(\log N)^2} = 0$, and that a function $f : I^R_N \to \mathbb{R}$ satisfies

$$
f(\eta) \leq \sum_{\zeta \in I^R_N} \hat{p}_N^R(\eta, \zeta) f(\zeta) + \delta \quad \text{for all } \eta \in I^R_N.
$$

(7.30)

Then, for each $\eta \in I^R_N$, we have that

$$
f(\eta) \leq \max_{\partial I^R_N} f + CN^3 \delta.
$$

**Proof.** Define $g : I^R_N \to \mathbb{R}$ by

$$
g(\eta) = \hat{E}_\eta^R \left[ f(\hat{\eta}_N^R(\sigma_R)) + \delta \sigma_R \right].
$$

For $\eta \in I^R_N$, the Markov property gives us that

$$
g(\eta) = \sum_{\zeta \in I^R_N} \hat{p}_N^R(\eta, \zeta) \hat{E}_\zeta^\eta \left[ f(\hat{\eta}_N^R(\sigma_R)) + \delta(\sigma_R + 1) \right]
$$

$$
= \sum_{\zeta \in I^R_N} \hat{p}_N^R(\eta, \zeta) g(\zeta) + \delta.
$$

(7.31)
Let \( h = f - g \). Then, by (7.30) and (7.31), we have that
\[
h(\eta) \leq \sum_{\zeta \in I^R_N} \hat{p}^R_N(\eta, \zeta) h(\zeta) \quad \text{for all } \eta \in I^R_N.
\]
On the other hand, we have \( h \equiv 0 \) on \( \partial I^R_N \) since \( \sigma_R = 0 \) on \( \partial I^R_N \). Therefore, since \( \hat{p}^R_N(\cdot) \) is irreducible, the maximum principle implies that \( h \leq 0 \), i.e., \( f \leq g \) on \( I^R_N \). Since \( g(\eta) \leq \max_{\partial I^R_N} f + \delta \bar{R}_R[\sigma_R] \) by the definition of \( g \), the proof is completed by Lemma 7.11. \( \blacksquare \)

Now, we define \( m : \mathcal{H}_N \to \mathbb{R} \) by
\[
m(\eta) = \mu_N(\eta) \prod_{x \in S} \eta_x.
\]
Then we can obtain the following estimate on \( m \) based on the maximum principle given in Lemma 7.13.

**Lemma 7.14.** There exists \( C = C(\epsilon) \) such that for each \( \eta \in I^S_N \),
\[
m(\eta) \leq \max_{\partial I^S_N} m + C \frac{d_N N^3}{(\log N)^2} \max_{I^S_N} m.
\]

**Proof.** We can deduce from (7.9) that, for each \( \eta \in I^S_N \),
\[
\sum_{x, y \in S} \mu_N(\eta) \eta_x(d_N + \eta_y) r(x, y) = \sum_{x, y \in S} \mu_N(\sigma^y x \eta)(\eta_x + 1)(d_N + \eta_y - 1) r(x, y).
\]
Inserting \( \mu_N(\eta) = m(\eta) \left( \prod_{x \in S} \eta_x \right)^{-1} \) and rearranging it yield that
\[
m(\eta) = \sum_{x, y \in S} \frac{\eta_x(d_N + \eta_y) r(x, y)}{\sum_{a, b \in S} \eta_a(d_N + \eta_b) r(a, b)} m(\sigma^y x \eta) \cdot
\]
By recalling the definition of \( \hat{p}^S_N \) (cf. (7.22)), we can rewrite the last identity as
\[
m(\eta) = \sum_{x, y \in S} \left[ 1 + \frac{d_N}{(d_N + \eta_y)(\eta_y - 1)} \right] \hat{p}^S_N(\eta, \sigma^y x \eta) m(\sigma^y x \eta) \cdot
\]
For \( \eta \in I^S_N \), we have
\[
\sum_{x, y \in S} \frac{d_N}{(d_N + \eta_y)(\eta_y - 1)} \hat{p}^S_N(\eta, \sigma^y x \eta) m(\sigma^y x \eta) \leq C d_N \frac{\max_{I^S_N} m}{(\log N)^2} \cdot
\]
since \( \eta_x \geq \epsilon \log N \) for all \( x \in S \). By (7.33) and (7.34), \( m \) satisfies
\[
m(\eta) \leq \sum_{\zeta \in \mathcal{T}_N^S} \hat{p}_N^S(\eta, \zeta) m(\zeta) + \frac{Cd_N}{(\log N)^2} \max_{\mathcal{T}_N^S} m. \tag{7.35}
\]
Hence, the proof is complete by Lemma 7.13 with \( R = S \) and \( f = m \). ■

Now, we are ready to prove Proposition 7.8 by combining results obtained in Lemmas 7.10-7.14.

Proof of Proposition 7.8. By Lemma 7.14,
\[
\mu_N(\mathcal{I}_N^S) = \sum_{\eta \in \mathcal{T}_N^S} \mu_N(\eta) = \sum_{\eta \in \mathcal{T}_N^S} \frac{m(\eta)}{\prod_{x \in S} \eta_x}
\leq \sum_{\eta \in \mathcal{T}_N^S} \frac{1}{\prod_{x \in S} \eta_x} \left\{ \max_{\partial \mathcal{T}_N^S} m + \frac{d_N}{(\log N)^2} CN^3 \max_{\mathcal{T}_N^S} m \right\} \tag{7.36}
\leq CN^{-1}(\log N)^{\kappa-1} \left\{ \max_{\partial \mathcal{T}_N^S} m + \frac{d_N}{(\log N)^2} CN^3 \max_{\mathcal{T}_N^S} m \right\},
\]
where the last line follows from Lemma 9.1 (note that \( \kappa = |S| \)). Recall the definition of \( m \) from (7.32) and note that
\[
\prod_{x \in S} \eta_x = \begin{cases} O(N^{\kappa-1} \log N) & \text{for } \eta \in \partial \mathcal{T}_N^S \text{ and}, \\ O(N^\kappa) & \text{for } \eta \in \mathcal{T}_N^S. \end{cases}
\]
Based on this, we can further deduce from (7.36) that
\[
\mu_N(\mathcal{I}_N^S) \leq CN^{\kappa-2}(\log N)^{\kappa} \max_{\partial \mathcal{T}_N^S} \mu_N + Cd_N N^{\kappa+2}(\log N)^{\kappa-3} \max_{\mathcal{T}_N^S} \mu_N
\leq CN^{\kappa-2}(\log N)^{\kappa} \mu_N(\partial \mathcal{T}_N^S) + Cd_N N^{\kappa+2}(\log N)^{\kappa-3} \mu_N(\mathcal{T}_N^S).
\]
By the condition on \( d_N \) given at the statement of the proposition, we complete the proof. ■

7.3. Induction step. Next, we consider the induction step. We shall prove the following two statements together by the backward induction: there exists \( C > 0 \) such that, for all \( i \in [2, \kappa] \),
\[
\lim_{N \to \infty} \frac{\mu_N(A_N^i)}{\mu_N(A_N^{i-1})} = 1, \tag{7.37}
\]
and for all $R \subseteq S$ with $|R| = i$, and $\forall z \in R$,
\[
\mu_N(C_N^R(z, 1)) \leq Cd_N \mu_N(A_N^{i-1}) .
\] (7.38)
Note that the initial case $i = \kappa$ for (7.37) is proven in Propositions 7.3, 7.7, and for (7.38) is proven in Lemma 7.5.

Now, we will assume the following condition throughout this subsection:
\[
\lim_{N \to \infty} d_N N^{\kappa+2}(\log N)^{\kappa-3} = 0 .
\] (7.39)

**Proposition 7.15.** Suppose that the induction hypotheses (7.37) and (7.38) hold for $i = \ell + 1$. Then, (7.37) and (7.38) hold for $i = \ell$ as well.

The overall outline of the proof is similar to the initial step, but several additional technical difficulties arise in the course of the proof. As before, we investigate the outer core and inner core separately in Lemmas 7.16 and 7.19, respectively.

**Estimation of the outer core.** For the outer core $\mathcal{O}_N^R$ with $R \subseteq S$, we will prove the following bound.

**Lemma 7.16.** For all $R \subseteq S$, it holds that
\[
\mu_N(\mathcal{O}_N^R) = o_N(1) \left[ \mu_N(\partial \mathcal{A}_N^R) + \mu_N(\mathcal{A}_N^R) \right] .
\]

We first prove two preliminary lemmas before proving this lemma. Recall the notions introduced after Proposition 7.7.

**Lemma 7.17.** For all $R \subseteq S$, $x \in R$, and $j \in [0, N-1]$, it holds that
\[
F_N^R(x; j+1 \to j) - F_N^R(x; j \to j+1) \leq Cd_N N \mu_N(A_N^R) .
\] (7.40)

**Proof.** By summing (7.9) over $\eta \in C_N^R(x, k)$,
\[
\sum_{\eta \in C_N^R(x, k)} \sum_{y \in (R \setminus \{x\})} \mu_N(\eta) r_N(\eta, \gamma^{y,x} \eta) + \sum_{\eta \in C_N^R(x, k)} \sum_{y \in (R \setminus \{x\})} \mu_N(\eta) r_N(\eta, \gamma^{y,x} \eta)
\]
\[
+ \sum_{\eta \in C_N^R(x, k)} \sum_{y \in (R \setminus \{x\})} \mu_N(\eta) r_N(\eta, \gamma^{x,y} \eta) + \sum_{\eta \in C_N^R(x, k)} \sum_{y \in (R \setminus \{x\})} \mu_N(\eta) r_N(\eta, \gamma^{x,y} \eta)
\]
\[
= \sum_{\eta \in C_N^R(x, k)} \sum_{y \in (R \setminus \{x\})} \mu_N(\sigma^{y,x} \eta) r_N(\sigma^{y,x} \eta, \eta) + \sum_{\eta \in C_N^R(x, k)} \sum_{y \in (R \setminus \{x\})} \mu_N(\sigma^{y,x} \eta) r_N(\sigma^{y,x} \eta, \eta)
\]
\[
+ \sum_{\eta \in C_N^R(x, k)} \sum_{y \in (R \setminus \{x\})} \mu_N(\sigma^{x,y} \eta) r_N(\sigma^{x,y} \eta, \eta) + \sum_{\eta \in C_N^R(x, k)} \sum_{y \in (R \setminus \{x\})} \mu_N(\sigma^{x,y} \eta) r_N(\sigma^{x,y} \eta, \eta) .
\] (7.41)
Compared to the corresponding computations in Lemma 7.4, the last terms in both sides of (7.41) should be handled in addition. The term in the left-hand side is bounded above by
\[ \sum_{\eta \in C} \mu_N(\eta) \mathcal{F}(\eta, x, k) = O(d_N N) \sum_{\eta \in C_N(x, k)} \mu_N(\eta) = O(d_N N) \mu_N(C_N(x, k)). \]

The term in the right-hand side of (7.41) is bounded below by 0. Hence, we can obtain from (7.41) that
\[ [F_N(x; k + 1 \rightarrow k) - F_N(x; k \rightarrow k + 1)] - [F_N(x; k \rightarrow k - 1) - F_N(x; k - 1 \rightarrow k)] \leq C d_N N \mu_N(C_N(x, k)). \]

By summing the bound over \( k = 0, 1, \ldots, j \), we obtain (7.40).

**Lemma 7.18.** There exists \( C > 0 \) such that for all \( R \subseteq S \), \( x \in R \), and \( k \in [0, N - 1] \), we have
\[ \mu_N(C_N(x, k + 1)) \leq C \frac{(k + d_N)(N - k)}{(k + 1)(N - k - 1 + d_N)} \mu_N(C_N(x, k)) + C d_N N \mu_N(A_N^R). \]

**Proof.** The proof is identical to Lemma 7.5 if we replace the role of Lemma 7.4 with that of Lemma 7.17.

Now, we prove Lemma 7.16 based on Lemmas 7.17 and 7.18.

**Proof of Lemma 7.16.** Fix \( x \in R \). Inserting \( k = 0 \) in Lemma 7.18 implies that there exists a constant \( C_1 > 1 \) such that
\[ \mu_N(C_N(x, 1)) \leq C_1 d_N \mu_N(C_N^R(x, 0)) + C_1 d_N \mu_N(A_N^R). \]

On the other hand, inserting \( k \in [1, N - 1] \) to Lemma 7.18 implies that there exists a constant \( C_2 > 1 \) such that
\[ \mu_N(C_N(x, k + 1)) \leq C_2 d_N \mu_N(C_N(x, k)) + C_2 d_N \mu_N(A_N^R). \]

Let \( C_0 = \max\{C_1, C_2\} \). Then, by (7.42) and (7.43), we obtain that
\[ \mu_N(C_N(x, k)) \leq C_0^k d_N \mu_N(C_N^R(x, 0)) + \frac{C_0^{k+1} - C_0}{C_0 - 1} d_N \mu_N(A_N^R) \quad ; \quad k \in [1, N - 1]. \]
Summing this up for $k \in [1, \epsilon \log N]$, it holds that

$$\sum_{k=1}^{\epsilon \log N} \mu_N(C_N^R(x, k)) \leq C \times C_0^{\epsilon \log N} d_N \left[ \mu_N(C_N^R(x, 0)) + \mu_N(A_N^R) \right].$$

Take $\epsilon$ small enough so that

$$C \epsilon \log N \ll N. \quad (7.45)$$

Therefore, by (7.7),

$$\mu_N(O_N^R) \leq \sum_{x \in R} \mu_N(C_N^R(x, k)) \leq Cd_N N \sum_{x \in R} \left\{ \mu_N(A_N^R(x)) + \mu_N(A_N^R) \right\}
= O(d_N N) \left[ \mu_N(\partial A_N^R) + \mu_N(A_N^R) \right].$$

This finishes the proof. ■

**Estimation of the inner core.** Next, we control the inner core $I_N^R$. The proof of the following lemma also relies on Lemma 7.11 regarding the estimate of the hitting time.

**Lemma 7.19.** Suppose that (7.38) holds for $i = \ell + 1$. Then, for all $R \subseteq S$ with $|R| = \ell$, we have that

$$\mu_N(I_N^R) = o_N(1) \left[ \mu_N(\partial A_N^R) + \mu_N(A_N^\ell) \right].$$

**Proof.** Fix $R \subseteq S$ and define $m^R : \mathcal{H}_N \to \mathbb{R}$ by

$$m^R(\eta) = \mu_N(\eta) \prod_{x \in R} \eta_x. \quad (7.46)$$

Similarly to Lemma 7.14, for $\eta \in I_N^R$, we get

$$\mu_N(\eta) = \sum_{y \in R, x \in S \setminus \{y\}} (\eta_x + 1)(d_N + \eta_y - 1)r(x, y) \sum_{a \in R, b \in S \setminus \{a\}} \eta_a(d_N + \eta_b) r(a, b) \mu_N(\sigma_{y,x}^x \eta).$$

By discarding the transitions escaping $R$ in the denominator of the right-hand side, we get

$$\mu_N(\eta) \leq \sum_{x, y \in R} (\eta_x + 1)(d_N + \eta_y - 1)r(x, y) \sum_{a, b \in R} \eta_a(d_N + \eta_b) r(a, b) \mu_N(\sigma_{y,x}^x \eta)
+ \sum_{y \in R, x \in S \setminus R} (d_N + \eta_y - 1)r(x, y) \sum_{a, b \in R} \eta_a(d_N + \eta_b) r(a, b) \mu_N(\sigma_{y,x}^x \eta). \quad (7.47)$$
By the assumption that (7.38) holds for \( i = \ell + 1 \), the last term is bounded by
\[
\sum_{x \in S \setminus R} \frac{C}{\log N} \mu_N(C^R_N(x)) \leq \frac{Cd_N}{\log N} \mu_N(A^t_N).
\]
Inserting this to (7.47) and using the formula (7.46) of \( \mathbf{m}^R \), we can deduce that
\[
\mathbf{m}^R(\eta) \leq \sum_{x, y \in R} \frac{\eta_x(d_N + \eta_y) r(x, y)}{\sum_{a, b \in R} \eta_a (d_N + \eta_b) r(a, b)} \mathbf{m}^R(\sigma^{y, x} \eta) + \frac{2d_N}{(\log N)^2} \max_{\mathbf{T}^R_N} \mathbf{m}^R + \frac{Cd_N N^\ell}{\log N} \mu_N(A^t_N). \tag{7.48}
\]

Now, as in the proof of Lemma 7.14 (cf. (7.33), (7.34), and (7.35)), we can obtain from the previous inequality that
\[
\mathbf{m}^R(\eta) \leq \sum_{x, y \in R} \mathbf{p}^R_N(\eta, \sigma^{y, x} \eta) \mathbf{m}^R(\sigma^{y, x} \eta) + \frac{2d_N}{(\log N)^2} \max_{\mathbf{T}^R_N} \mathbf{m}^R + \frac{Cd_N N^\ell}{\log N} \mu_N(A^t_N).
\]

Therefore, Lemma 7.13 with \( f = \mathbf{m}^R \) and Lemma 7.11 give that,
\[
\mathbf{m}^R(\eta) \leq \max_{\partial \mathbf{T}^R_N} \mathbf{m}^R + \max_{\mathbf{T}^R_N} \mathbf{m}^R \frac{Cd_N N^3}{(\log N)^2} + \frac{Cd_N N^{\ell+3}}{\log N} \mu_N(A^t_N). \tag{7.49}
\]
for all \( \eta \in \mathbf{T}^R_N \). Now recalling the definition (7.46) and applying Lemma 9.1,
\[
\mu_N(\mathbf{T}^R_N) \leq \sum_{\eta \in \mathbf{T}^R_N} \prod_{x \in R} \eta_x \left\{ \max_{\partial \mathbf{T}^R_N} \mathbf{m}^R + \max_{\mathbf{T}^R_N} \mathbf{m}^R \frac{Cd_N N^3}{(\log N)^2} + \frac{Cd_N N^{\ell+3}}{\log N} \mu_N(A^t_N) \right\}
\]
\[
\leq C N^{-\ell} \left\{ \left( \frac{(\log N)^{\ell-1}}{N} \right) \max_{\partial \mathbf{T}^R_N} \mu_N + d_N \frac{(\log N)^{\ell+3}}{(\log N)^2} \max_{\mathbf{T}^R_N} \mu_N + d_N \frac{(\log N)^{\ell+3}}{\log N} \mu_N(A^t_N) \right\}
\]
\[
= C N^{-\ell} \mu_N(\partial \mathbf{T}^R_N) + d_N N^{\ell+2} (\log N)^{3} \mu_N(\mathbf{T}^R_N)
\]
\[
+ C d_N N^{\ell+2} (\log N)^{\ell-2} \mu_N(A^t_N).
\]

By (7.39), we can finally deduce that
\[
\mu_N(\mathbf{T}^R_N) = O(N^{-\ell} (\log N)^{\ell}) \mu_N(\partial \mathbf{T}^R_N) + O(d_N N^{\ell+2} (\log N)^{\ell-2}) \mu_N(\mathbf{A}^t_N). \tag{7.50}
\]
Recall the notation defined after Proposition 7.3 to see that
\[
\partial \mathbf{T}^R_N \subseteq \bigcup_{x \in R} C^R_N(x, \epsilon \log N).
\]
Therefore by (7.44),
\[
\mu_N(\partial \mathbf{T}^R_N) \leq \sum_{x \in R} \mu_N(C^R_N(x, \epsilon \log N)) = O(d_N) \left[ \mu_N(\partial \mathbf{A}^R_N) + \mu_N(\mathbf{A}^R_N) \right]. \tag{7.51}
\]
(7.50) and (7.51) give
\[
\mu_N(\mathcal{I}_N^R) = O(d_N^\ell (\log N)^\ell)\mu_N(\partial \mathcal{A}_N^R) + O(d_N^{\ell+2}(\log N)^{\ell-2})\mu_N(\mathcal{A}_N^\ell).
\]
By (7.39), we finish the proof. ■

Proof of Proposition 7.15. Take \(R \subseteq S\) with \(|R| = \ell\). Since \(\mathcal{R}_N^R\) is decomposed into \(\mathcal{O}_N^R\) and \(\mathcal{I}_N^R\), and since \(\partial \mathcal{A}_N^R \subseteq \mathcal{A}_N^{\ell-1}\), we can derive from Propositions 7.16 and 7.19 that
\[
\mu_N(\mathcal{R}_N^R) = o_N(1) \left[ \mu_N(\partial \mathcal{A}_N^R) + \mu_N(\mathcal{A}_N^\ell) \right] \leq o_N(1) \left[ \mu_N(\mathcal{A}_N^{\ell-1}) + \mu_N(\mathcal{A}_N^\ell) \right].
\]
Summing the last bound over all \(R \subseteq S\) with \(|R| = \ell\) yields that
\[
\mu_N(\mathcal{A}_N^\ell) = o_N(1) \left[ \mu_N(\mathcal{A}_N^{\ell-1}) + \mu_N(\mathcal{A}_N^\ell) \right].
\]
We can deduce (7.37) with \(i = \ell\) from here. On the other hand, we can verify (7.38) with \(i = \ell\) from (7.37) and (7.44). To be more specific, for \(x \in R\), inserting \(k = 1\) in (7.44) gives us
\[
\mu_N(\mathcal{C}_N^R(x, 1)) \leq C d_N \mu_N(\mathcal{C}_N^R(x, 0)) + C d_N \mu_N(\mathcal{A}_N^R).
\]
Since \(\mathcal{C}_N^R(x, 0) \subseteq \mathcal{A}_N^{\ell-1}\) and \(\mu_N(\mathcal{A}_N^R) \leq \mu_N(\mathcal{A}_N^\ell) = (1 + o_N(1))\mu_N(\mathcal{A}_N^{\ell-1})\) by (7.37), we conclude that \(\mu_N(\mathcal{C}_N^R(x, 1)) \leq C d_N \mu_N(\mathcal{A}_N^{\ell-1})\) and thus conclude the proof of Proposition 7.15. ■

Remark 7.20. We remark that the final step in (7.2), i.e., \(\ell = 2\), can be proved in a completely independent way without assumption (UP), and with a much weaker assumption on \(d_N\). To be more specific, we can prove the following result:

**Theorem.** Suppose that \(\lim_{N \to \infty} d_N N \log N = 0\). Then, we have
\[
\lim_{N \to \infty} \frac{\mu_N(\mathcal{E}_N)}{\mu_N(\mathcal{A}_N)} = 1.
\]

Note that under condition (UP), this is exactly the case \(\ell = 2\) in (7.2). We omit the proof of this statement, and only remark that it can be proved by tracing the original process on \(\mathcal{A}_N\) and calculating the transition rates of the trace process, as done in Section 4.3.

7.4. **Proof of Theorem 3.17.** Now, we explain the proof of Theorem 3.17 whose main idea of proof is nearly identical to that of Lemma 7.11. Slight difference is that here we are dealing with the original continuous-time chain \(\eta_N(\cdot)\), instead of the reversed discrete-time chain \(\hat{\eta}_N^R(\cdot)\).
**Proof of Theorem 3.17.** We recall the definition of $U_N$ from the display before Theorem 3.17. Let us identify $\epsilon$ in the definition of $I_S^N$ with $\delta$ in the definition of $U_N$. Then, in the terminology introduced in this section, we have $U_N = (I_S^N)^c$ and thus $\tau_{U_N} = \tau_{\partial I_S^N}$ provided that the chain starts in $U_N^c$. Thus a deduction similar to that in Lemma 7.10 guarantees the existence of test function $g_0 : \mathcal{I}_N^S \to \mathbb{R}$ such that
\[
\max_{\mathcal{I}_N^S} g_0 - \min_{\mathcal{I}_N^S} g_0 \leq C \log N \quad \text{and} \quad (\mathcal{L}_N g_0)(\eta) \geq \frac{\log N}{CN} \quad \text{for all} \ \eta \in \mathcal{I}_N^S.
\]
Here, the denominator of the lower bound of $(\mathcal{L}_N g_0)(\eta)$ is $CN$ instead of $CN^3$, since there is no $w(\eta)$ term as in Lemma 7.10 in the calculation of the continuous-time generator $\mathcal{L}_N$. Let us consider an arbitrary extension of $g_0$ to a function on $\mathcal{H}_N$ and then consider the continuous-time martingale
\[
M_{g_0}(t) := g_0(\eta_N(t)) - g_0(\eta_N(0)) - \int_0^t (\mathcal{L}_N g_0)(\eta_N(s)) ds \quad ; \quad t \geq 0 .
\]
Then, proceeding as in Lemma 7.11, we can conclude that $\mathbb{E}_\eta[\tau_{U_N}] \leq CN$. ■

8. Inclusion Processes in Thermodynamic Limit Regime

In this section, we consider the inclusion process in the thermodynamic limit regime and prove the condensation (Theorem 3.19) and the metastable behavior (Theorems 3.21-3.23).

**Organization of the section.** In Section 8.1, we prove the existence of the condensation (Theorem 3.19), which is indeed not very far from that of the fixed $L$ case under (UI). On the other hand, the metastable behavior is more delicate than the fixed $L$ case, mainly because the limiting dynamic is now a continuous process on $\mathbb{T}^d$, while the trace process is a jump process on $\mathbb{T}^d_L$. The proof of this convergence is based on two ingredients: the convergence of the generator (Proposition 8.1) and the tightness (Proposition 8.3). These ingredients are obtained in Sections 8.2 and 8.3, respectively. Finally, we prove Theorems 3.21-3.23 in Section 8.4.

8.1. Condensation. We first establish condensation by proving Theorem 3.19. This should be distinguished from the former cases by the fact that the graph grows along with the number of particles. Although the proof is given in [11, Proposition 2], we present a proof here for the completeness of the article.
Proof of Theorem 3.19. Recall $\mathcal{E}_L$ from (3.14). Then, it suffices to show that

$$\lim_{L \to \infty} \frac{\mu_L(\mathcal{H}_L \setminus \mathcal{E}_L)}{\mu_L(\mathcal{E}_L)} = 0. \quad (8.1)$$

Since the inclusion process that we consider here satisfies the condition (UI), thanks to Proposition 2.4, the invariant measure of the process denoted by $\mu_L$ can be expressed explicitly by

$$\mu_L(\eta) = \frac{1}{Z_L} \prod_{x \in \mathbb{T}_{dL}^1} w_L(\eta_x), \quad \eta \in \mathcal{H}_L, \quad (8.2)$$

where

$$w_L(n) = \frac{\Gamma(n + d_L)}{n! \Gamma(d_L)}, \quad n \in \mathbb{N} \quad \text{and} \quad Z_L = \sum_{\eta \in \mathcal{H}_L} \prod_{x \in \mathbb{T}_{dL}^1} w_L(\eta_x).$$

We recall the following elementary inequality from [7, Lemma 3.1]:

$$\frac{d_L}{d_L + k \Gamma(d_L + 1)} \leq w_L(k) \leq \frac{d_L}{k} e^{d_L \log L} \quad \text{for all} \quad k \in [1, L]. \quad (8.3)$$

Since we assumed that $\lim_{L \to \infty} d_L L^d \log L = 0$, the previous inequality implies that

$$w_L(k) = (1 + o_L(1)) \frac{d_L}{k} \quad \text{uniformly for} \quad k \in [1, L]. \quad (8.3)$$

Decompose

$$\mathcal{H}_L \setminus \mathcal{E}_L = \bigcup_{i=2}^{L} \Delta_i \quad (8.4)$$

where, for each $i \in [2, L],

$$\Delta_i = \{ \eta \in \mathcal{H}_L : \text{exactly} \ i \ \text{coordinates of} \ \eta = (\eta_x)_{x \in \mathbb{T}_{dL}} \ \text{are non-zero} \}. \quad (8.4)$$

By (8.3) and the definition of $S_{n,k}$ in Lemma 9.1, for large enough $L$,

$$\mu_L(\Delta_i) \leq \frac{1}{Z_L} (2d_L)^i S_{N,i} d^d \binom{L}{i},$$

where the last term appears since there are $\binom{L}{i}$ ways to select $i$ coordinates that are non-zero. By Lemma 9.1, it holds for all large enough $L$ that

$$\mu_L(\Delta_i) \leq \frac{1}{Z_L} \frac{1}{3N \log(N + 1)} (6d_L \log(N + 1))^i \frac{L^d}{i} \leq \frac{1}{Z_L} \frac{1}{N \log N} (7d_L \log L^d)^i \frac{L^d}{i}. \quad (8.5)$$
For convenience, write $u_L = 7dL \log L$. Then, by combining (8.4) and (8.5), we obtain for all large enough $L$ that

$$
\mu_L(\mathcal{H}_L \setminus \mathcal{E}_L) = \sum_{i=2}^{L} \mu_L(\Delta_i) \leq \frac{1}{Z_L} \frac{1}{N \log N} \{(1 + u_L)^{L^d} - 1 - L^d u_L\}
\leq \frac{1}{Z_L} \frac{1}{N \log N} \{e^{L^d u_L} - 1 - L^d u_L\}
\leq \frac{1}{Z_L} \frac{1}{N \log N} (L^d u_L)^2,
$$

where the last inequality follows because $\lim_{L \to \infty} L^d u_L = 0$. Thus,

$$
\mu_L(\mathcal{H}_L \setminus \mathcal{E}_L) \leq \frac{C}{Z_L} d^2 L^d \log L. \quad (8.6)
$$

On the other hand, by the explicit formula (8.2) and the asymptotic (3.12), we have that

$$
\mu_L(\mathcal{E}_L) = L^d \times \frac{1}{Z_L} w_L(N) w_L(0)^{L^d-1} = (1 + o_L(1)) L^d \times \frac{1}{Z_L} \frac{d_L}{N} = (1 + o_L(1)) \frac{1}{\rho Z_L} d_L. \quad (8.7)
$$

Now, (8.1) is straightforward from (8.6) and (8.7).

8.2. **Convergence of the generator.** Now, we consider the metastable behavior associated with the condensation proved above. The generator $\mathcal{L}^{\tau_d}$ associated with the limiting object presented in Theorems 3.21-3.23 can be written as, for all sufficiently smooth $f : \mathbb{T}^d \to \mathbb{R}$,

$$(\mathcal{L}^{\tau_d} f)(x)
= \begin{cases}
\rho \left( \sum_{y \in \mathbb{Z}^d} h(y) y \right) \cdot \nabla f(x) & \text{for totally asym. case}, \\
\frac{\rho}{2} \sum_{y \in \mathbb{Z}^d, h(y) > h(-y)} (h(y) - h(-y)) y^\dagger [\nabla^2 f(x)] y & \text{for mean-zero asym. case}, \\
\frac{1}{2} \sum_{y \in \mathbb{Z}^d} h(y) y^\dagger [\nabla^2 f(x)] y & \text{for symmetric case},
\end{cases} \quad (8.8)
$$

where $(\nabla^2 f)(x)$ denotes the Hessian of $f$ at $x$. The main objective of this subsection is to prove the convergence of the generator of the trace process to the generator $\mathcal{L}^{\tau_d}$ in an appropriate sense as $L \to \infty$ (cf. Proposition 8.2). The proof of this result again relies on the asymptotics of the mean-jump rate.

Asymptotics of mean-jump rate. We start by introducing several notations related to the mean-jump rate. Recall that $\eta_{L}^{\mathcal{E}_L}(\cdot)$ denotes the trace process of $\eta_L(\cdot)$ on the set
\( \mathcal{E}_L \). We let \( r^\mathcal{E}_L(\cdot, \cdot), \mathcal{L}^\mathcal{E}_L, \) and \( \mu^\mathcal{E}_L \) denote the jump rate, the infinitesimal generator and the invariant measure of the trace process \( \eta^\mathcal{E}_L(\cdot) \), respectively. For \( x, y \in \mathbb{T}_L^d \), we write

\[
\mathbf{b}_L(x, y) = r^\mathcal{E}_L(\xi^x_L, \xi^y_L).
\]  

(8.9)

With these notation, we summarize the asymptotic relations for \( \mathbf{b}_L(\cdot, \cdot) \) which are immediate from Proposition 4.3.

**Proposition 8.1.** The followings hold for the inclusion process on \( \mathbb{T}_L^d \) with \( N \simeq \rho L^d \) particles:

1. for (either totally or mean-zero) asymmetric case, we have that

\[
\mathbf{b}_L(x, x + y) = \begin{cases} 
(1 + O(d_L \log L + q^N)) d_L N(h(y) - h(-y)) & \text{if } h(y) > h(-y), \\
O(d_L \log L + q^N) d_L N & \text{otherwise.}
\end{cases}
\]  

(8.10)

2. for symmetric case, we have that

\[
\mathbf{b}_L(x, x + y) = \begin{cases} 
(h(y) + O(d_L L^d \log L + L^d q^N)) d_L & \text{if } h(y) = h(-y) > 0, \\
O(d_L L^d \log L + L^d q^N) d_L & \text{otherwise.}
\end{cases}
\]  

(8.11)

Convergence of generator of speeded-up trace process. Now, we are ready to proceed to the main result regarding the convergence of the generator. We are primarily interested in the convergence of the speeded-up (Markov) process defined by

\[
W_L(t) = Y_L(\theta_L t),
\]  

(8.12)

where

\[
\theta_L = \begin{cases} 
1/(d_L L^{d-1}) & \text{for totally asymmetric case,} \\
1/(d_L L^{d-2}) & \text{for mean-zero asymmetric case,} \\
L^2/d_L & \text{for symmetric case.}
\end{cases}
\]

Let \( \mathcal{L}^W_L \) denote the infinitesimal generator associated with the continuous-time Markov chain \( W_L(\cdot) \). Then, we can write this generator as, for all \( F: \mathbb{T}^d \rightarrow \mathbb{R} \),

\[
(\mathcal{L}^W_L: F)\left(\frac{x}{L}\right) = \theta_L \mathcal{L}^\mathcal{E}_L (F \circ \Theta_L)(\xi^x_L)
\]

\[
= \theta_L \sum_{y \in \mathbb{T}_L^d} \mathbf{b}_L(x, x + y) \left\{ F\left(\frac{x + y}{L}\right) - F\left(\frac{x}{L}\right) \right\}.
\]  

(8.13)

The following is the main result of the current subsection.
Proposition 8.2. Under the conditions of Theorems 3.21-3.23, it holds for all $f \in C^3(\mathbb{T}^d)$ that

$$\lim_{L \to \infty} \sup_{x \in \mathbb{T}_L^d} \left| (\mathcal{L}^{W_L} f) \left( \frac{x}{L} \right) - (\mathcal{L}^{T_d} f) \left( \frac{x}{L} \right) \right| = 0.$$  

Proof. We fix $f \in C^3(\mathbb{T}^d)$ and consider three cases separately.

(Case 1: Totally asymmetric case) For this case, $\theta_L = 1/(d_L L^{d-1})$. Hence, by (8.13) and by part (1) of Proposition 8.1, we can deduce that

$$\left( \mathcal{L}^{W_L} f \right)(\xi_L) - \left( \mathcal{L}^{T_d} f \right)\left( \frac{x}{L} \right) = \frac{1}{d_L L^{d-1}} \sum_{y \in \mathbb{T}_L^d} b_L(x, x + y) \left\{ f \left( \frac{x + y}{L} \right) - f \left( \frac{x}{L} \right) \right\} - \rho \sum_{y \in \mathbb{Z}^d} h(y) y \cdot \nabla f \left( \frac{x}{L} \right)$$

$$= \sum_{y \in \mathbb{Z}^d : h(y) > h(-y)} \frac{N}{L^d} \left( h(y) - h(-y) \right) y \cdot \nabla f \left( \frac{x}{L} \right) + o_L(1) - \rho \sum_{y \in \mathbb{T}_L^d} h(y) y \cdot \nabla f \left( \frac{x}{L} \right).$$

The second equality holds by the first-order Taylor expansion and $\lim_{L \to \infty} d_L L^{d+1} \log L = 0$. Since $N/L^d \to \rho$, the last line converges to 0 as $L \to \infty$ and we are done.

(Case 2: Mean-zero asymmetric case) For this case, $\theta_L = 1/(d_L L^{d-2})$; thus, by (8.13) and part (1) of Proposition 8.1, we obtain that

$$\left( \mathcal{L}^{W_L} f \right)(\xi_L) - \left( \mathcal{L}^{T_d} f \right)\left( \frac{x}{L} \right) = \frac{1}{d_L L^{d-2}} \sum_{y \in \mathbb{T}_L^d} b_L(x, x + y) \left[ f \left( \frac{x + y}{L} \right) - f \left( \frac{x}{L} \right) \right] - \left( \mathcal{L}^{T_d} f \right)\left( \frac{x}{L} \right)$$

$$= \frac{1}{d_L L^{d-2}} \sum_{y \in \mathbb{Z}^d : h(y) > h(-y)} d_L N(\theta_L - h(-y)) \left[ f \left( \frac{x + y}{L} \right) - f \left( \frac{x}{L} \right) \right] + o_L(1) - \left( \mathcal{L}^{T_d} f \right)\left( \frac{x}{L} \right).$$

In this case, unlike in (Case 1), the first-order terms at the Taylor expansion cancel out each other. Thus, we apply the second-order Taylor expansion to get

$$\frac{N}{2L^d} \sum_{y \in \mathbb{Z}^d : h(y) > h(-y)} (h(y) - h(-y)) y^T \nabla^2 f \left( \frac{x}{L} \right) y - \left( \mathcal{L}^{T_d} f \right)\left( \frac{x}{L} \right) + o_L(1).$$

This concludes the proof for this case since $N/L^d \to \rho$. 
(Case 3: Symmetric case) For this case, \( \theta_L = L^2/d_L \). Thus by (8.13) and by part (2) of Proposition 8.1, we obtain

\[
(L^{W_L} f)(\xi^L) - (L^{T^d} f)\left(\frac{x}{L}\right) = \frac{L^2}{d_L} \sum_{y \in \mathbb{T}_L^d} b_L(x, x + y) \left[f\left(\frac{x+y}{L}\right) - f\left(\frac{x}{L}\right)\right] - (L^{T^d} f)\left(\frac{x}{L}\right),
\]

\[
= \frac{L^2}{2d_L} \sum_{y \in \mathbb{Z}^d} d_L h(y) \left[f\left(\frac{x+y}{L}\right) + f\left(\frac{x-y}{L}\right) - 2f\left(\frac{x}{L}\right)\right] - (L^{T^d} f)\left(\frac{x}{L}\right) + o_L(1).
\]

Note that the last error term is \( o_L(1) \), since \( \lim_{L \to \infty} d_L L^2 + 2 \log L = 0 \). Hence, we apply the second-order Taylor expansion to deduce that the last expression is equal to

\[
\frac{1}{2} \sum_{y \in \mathbb{Z}^d} h(y) y^\dagger \nabla f^2\left(\frac{x}{L}\right) y - (L^{T^d} f)\left(\frac{x}{L}\right) + o_L(1).
\]

This finishes the proof the definition of \( L^{T^d} \).

8.3. Tightness. The last ingredient for the proof of the convergence stated in part (1) of Definition 3.20 is the tightness of the process \( W_L(t) = Y_L(\theta_L t) \). Let \( Q^L_{\xi_L, \eta} \), \( \eta \in E_L \) denote the law of the process \( W_L(\cdot) \) on the path space \( D([0, \infty), \mathbb{T}^d) \) when the inclusion process starts from \( \eta \), i.e., associated with the law \( \mathbb{P}^L_{\eta} \).

Proposition 8.3. Let \( (x_L)_{L=1}^\infty \) be a sequence such that \( x_L \in \mathbb{T}^d_L \) for all \( L \geq 1 \). Then, under the conditions of Theorems 3.21-3.23, the sequence \( \{Q^L_{\xi_L, \eta}\}_{L \geq 1} \) of path measures is tight in \( D([0, \infty), \mathbb{T}^d) \).

The natural way of proving this proposition is to use the Aldous criterion. Of course, we found a proof of the tightness based on this criterion, but controlling errors coming from the non-regularity of distance function \( d(x, 0) = |x| \) around 0 requires complicated computations based on the large-deviation principle and the local central limit theorem for the random walk on the discrete torus. Instead, we realized that the criterion presented as Proposition 9.3 is more adequate to apply, in that it only considers smooth functions \( F \), which guarantees sufficiently small error terms via Taylor expansion.

Proof of Proposition 8.3. The condition (1) of Proposition 9.3 is straightforward, since \( \mathbb{T}^d \) is compact. Now let us check the condition (2). To this end, fix \( f \in C^\infty_c(\mathbb{T}^d) \) and
\( \delta > 0 \). Then, by the martingale problem associated with the Markov chain \( W_L(\cdot) \), we know that the process given by

\[
M^L_f(t) = f(W_L(t)) - f(W_L(0)) - \int_0^t (L^W_L f)(W_L(s)) ds \quad (8.14)
\]

is a \( \mathbb{Q}^L \)-martingale. Let \( (\mathcal{F}^L_t)_{t \geq 0} \) denote the canonical filtration associated with the process \( W_L(\cdot) \) and by \( E^L_\eta \) the expectation associated with \( \mathbb{Q}^L_\eta \). Then, the previous observation implies that, for all \( t \geq 0 \) and \( 0 \leq u \leq \delta \), we have that

\[
E^L_{\xi^L_{t \wedge \delta}} \left[ f(W_L(t + u)) - f(W_L(t)) \right] = E^L_{\xi^L_{t \wedge \delta}} \left[ \int_t^{t+u} (L^W_L f)(W_L(s)) ds \right].
\]

Hence, in view of Proposition 9.3, it suffices to check

\[
\lim_{\delta \to 0} \limsup_{L \to \infty} E^L_{\xi^L_{t \wedge \delta}} \sup_{0 \leq u \leq \delta} \left| \int_t^{t+u} (L^W_L f)(W_L(s)) ds \right| = 0. \quad (8.15)
\]

By Proposition 8.2, it suffices to prove that

\[
\lim_{\delta \to 0} \limsup_{L \to \infty} E^L_{\xi^L_{t \wedge \delta}} \sup_{0 \leq u \leq \delta} \left| \int_t^{t+u} (L^T_L f)(W_L(s)) ds \right| = 0.
\]

This is obvious since \( L^T_L f \) is a bounded function on \( T^d \). \( \blacksquare \)

8.4. Proof of the main results.

**Proof of Theorems 3.21-3.23.** Fix a sequence \( (x_L)_{L=1}^{\infty} \) that satisfies \( x_L \in T_L^d \) for all \( L \geq 1 \) and \( \lim_{L \to \infty} (x_L/L) = u \), as in part (1) of Definition 3.20. For simplicity, we write \( \mathbb{Q}^L = \mathbb{Q}^L_{\xi^L_{t \wedge \delta}} \) and \( E^L = E^L_{\xi^L_{t \wedge \delta}} \).

Let us first identify the limit points of the sequence \( \{\mathbb{Q}^L\}_{L \geq 1} \). Let \( \mathbb{Q} \) denote an arbitrary limit point of \( \{\mathbb{Q}^L\}_{L \geq 1} \). Fix \( f \in C^3(T^d) \) and consider

\[
M_f(t) = f(\omega(t)) - f(\omega(0)) - \int_0^t (L^T_L f)(\omega(s)) ds; \quad t \geq 0,
\]

where \( \omega(t) \) is the canonical coordinate process on \( D([0, \infty), T^d) \). Then, we claim that \( (M_f(t))_{t \geq 0} \) is a \( \mathbb{Q} \)-martingale, i.e.,

\[
E^\mathbb{Q} [g((\omega(u) : 0 \leq u \leq s)) (M_f(t) - M_f(s))] = 0 \quad (8.16)
\]

for all \( 0 \leq s \leq t \) and for all bounded, continuous function \( g \) on \( D([0, s], T^d) \). To prove (8.16), we recall the \( \mathbb{Q}^L \)-martingale \( M^L_f(t) \) defined in (8.14) so that we have

\[
E^L [g((\omega(u) : 0 \leq u \leq s)) (M^L_f(t) - M^L_f(s))] = 0. \quad (8.17)
\]
By Proposition 8.2, we have
\[
\lim_{L \to \infty} \left| M_L^f(t) - f(W_L(t)) - f(W_L(0)) - \int_0^t (\mathcal{L}^{\mathbb{T}^d} f)(W_L(s))\,ds \right| = 0 \tag{8.18}
\]
for all \( t \geq 0 \), and hence by (8.17) and (8.18), we obtain that
\[
\lim_{L \to \infty} E^L_{\eta} [g(\omega(u) : 0 \leq u \leq s)) (M_f(t) - M_f(s))] = 0 . \tag{8.19}
\]
Therefore, the proof of (8.16) is completed if we can establish the following limit:
\[
E^L_{\eta} [g((\omega(u) : 0 \leq u \leq s)) (M_f(t) - M_f(s))] \to E^Q_{\eta} [g((\omega(u) : 0 \leq u \leq s)) (M_f(t) - M_f(s))] \text{ as } L \to \infty . \tag{8.20}
\]
This is not trivial since the map \( H : \omega \mapsto g((\omega(u) : 0 \leq u \leq s)) (M_f(t) - M_f(s)) \) is not continuous on \( D([0, \infty), \mathbb{T}^d) \). However, in [1, Proposition 3.2], this limiting procedure has been robustly confirmed and can be applied to our situation as well. Thus, the claim is proved. It completes the identification of limit points since the solution of the martingale problem is unique and since \( C^3(\mathbb{T}^d) \) consists the core of the generator \( \mathcal{L}^{\mathbb{T}^d} \) given in (8.8) because \( \mathbb{T}^d \) is compact. Finally, along with the tightness established in Proposition 8.3, we can conclude the convergence of the process \( W_L(\cdot) \) to \( Y(\cdot) + u \) where \( Y(\cdot) \) is the process generated by \( \mathcal{L}^{\mathbb{T}^d} \) and starting at 0. This finally completes the verification of part (1) of Definition 3.20.

Now, we turn to part (2) of Definition 3.20, i.e., we prove
\[
\lim_{L \to \infty} \sup_{\eta \in \mathcal{E}_L} \mathbb{E}^L_{\eta} \left[ \int_0^t \mathbb{1}_{\mathcal{H}_L \setminus \mathcal{E}_L}(\eta_L(\theta_Ls))\,ds \right] = 0 \text{ for all } t > 0 . \tag{8.21}
\]
To this end, let us first fix \( x \in \mathbb{T}^d_L \) and \( t > 0 \). Then, by the translation invariance of the model, we have
\[
\mathbb{E}^L_{\xi_t^L} \left[ \int_0^t \mathbb{1}_{\mathcal{H}_L \setminus \mathcal{E}_L}(\eta_L(\theta_Ls))\,ds \right] = \mathbb{E}^L_{\mu_L} \left[ \int_0^t \mathbb{1}_{\mathcal{H}_L \setminus \mathcal{E}_L}(\eta_L(\theta_Ls))\,ds \right]
\]
since the invariant measure \( \mu_L^\xi(\cdot) \) of the trace process is a uniform measure on \( \mathcal{E}_L = \{\xi_t^L : x \in \mathbb{T}^d_L\} \). Now, we can deduce from Fubini theorem that
\[
\mathbb{E}^L_{\mu_L} \left[ \int_0^t \mathbb{1}_{\mathcal{H}_L \setminus \mathcal{E}_L}(\eta_L(\theta_Ls))\,ds \right] \leq \frac{1}{\mu_L(\mathcal{E}_L)} \mathbb{E}^L_{\mu_L} \left[ \int_0^t \mathbb{1}_{\mathcal{H}_L \setminus \mathcal{E}_L}(\eta_L(\theta_Ls))\,ds \right]
\]
\[
= \frac{1}{\mu_L(\mathcal{E}_L)} \int_0^t \mu_L(\mathcal{H}_L \setminus \mathcal{E}_L) .
\]
Thus, (8.21) follows from static condensation established in Theorem 3.19. \( \blacksquare \)
9. Appendix

In the appendix, we collect several known results for the completeness of the article.

9.1. A lemma on the sum of reciprocals. The following elementary lemma is repeatedly used throughout the article.

**Lemma 9.1.** For integers \( n \geq k \geq 1 \), define

\[
A_{n,k} = \left\{ (a_1, \ldots, a_k) \in \mathbb{N}^k : a_1, \ldots, a_k \geq 1 \quad \text{and} \quad \sum_{i=1}^{k} a_i = n \right\},
\]

and define

\[
S_{n,k} = \sum_{(a_1, \ldots, a_k) \in A_{n,k}} \prod_{i=1}^{k} \frac{1}{a_i}.
\]

Then, it holds that

\[
S_{n,k} \leq \frac{(3 \log(n + 1))^{k-1}}{n} \quad \text{for all } n \geq k \geq 1.
\]  

(9.1)

**Proof.** We proceed by the mathematical induction on \( k \). Note that the inequality (9.1) is trivial for the initial case \( k = 1 \). Now, we fix \( k \geq 2 \) and assume that (9.1) holds for \( S_{n,\ell} \) with \( \ell = k - 1 \) and \( n \geq \ell \). Then, look at the inequality for \( S_{n,k} \) for some fixed \( n \).

Since \( a_k \) can take values from 1 to \( n - (k - 1) \), we can write

\[
S_{n,k} = \sum_{m=1}^{n-(k-1)} \sum_{(a_1, \ldots, a_{k-1}) \in A_{n-m,k-1}} \frac{1}{m} \prod_{i=1}^{k} \frac{1}{a_i} = \sum_{m=1}^{n-(k-1)} \frac{1}{m} S_{n-m,k-1}.
\]

Thus, by the induction hypothesis, we get that

\[
S_{n,k} \leq \sum_{m=1}^{n-(k-1)} \frac{1}{m} \left( \frac{3 \log(n - m + 1)}{n - m} \right)^{k-2} \leq \left( \frac{3 \log(n + 1)}{n} \right)^{k-2} \sum_{m=1}^{n-(k-1)} \frac{1}{m(n-m)}. \quad (9.2)
\]

The proof of the inequality (9.1) is completed since the last summation can be estimated by

\[
\sum_{m=1}^{n-(k-1)} \frac{1}{m(n-m)} = \frac{1}{n} \sum_{m=1}^{n-(k-1)} \left( \frac{1}{m} + \frac{1}{n-m} \right) \leq \frac{3}{n} \log(n + 1).
\]  

(9.3)

Inserting (9.3) to (9.2) finishes the proof of the induction step, and thus concludes the proof.
9.2. **Gordan’s lemma.** The following elementary lemma is used in the proof of Lemma 7.10. This lemma has many equivalent statements, which include the one known as Farkas’ lemma.

**Lemma 9.2 (Gordan’s lemma).** Let $A$ be an $m \times n$ matrix for integers $m, n \geq 1$. Then, exactly one of the following statements holds.

- There exists a vector $\alpha \in \mathbb{R}^m$ such that all the components of $A^\intercal \alpha$ are positive.
- There exists a vector $0 \neq \beta \in \mathbb{R}^n$ such that all the components of $\beta$ are non-positive and such that $A\beta = 0$.

**Proof.** We refer to e.g., [12, Section 3].

9.3. **A criterion for the tightness.** We introduce a criterion for the tightness of the random process which is used in the proof of tightness of the speeded-up trace process in the thermodynamic limit case in Section 8. This criterion is thoroughly explained in [31], and is also used in [17] to prove the metastable behavior of symmetric inclusion processes.

**Proposition 9.3.** For each $N \geq 1$, let $X^N$ be a continuous-time Markov chain on $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$, and let $\mathcal{F}^N_t, t \geq 0$ be its natural filtration. Fix $\{x_N\}_{N \geq 1} \subseteq \Omega$ and let $P_{x_N}$ and $E_{x_N}$ denote the law and expectation of $X^N$ starting at $x_N$, respectively. Then, the collection of laws $\{P_{x_N}\}_{N \geq 1}$ is tight in the path space $D([0, \infty); \Omega)$ provided that both of the following conditions hold.

1. The sequence $\{X^N\}_{N \geq 1}$ is stochastically bounded in $D([0, \infty); \Omega)$.
2. For all $F \in C^\infty_c(\Omega)$, there exists a family of non-negative random variables $Z_N(\delta, F), \delta > 0$, such that, for all $t \geq 0$ and $0 \leq u \leq \delta$,

$$\left| E_{x_N} \left[ F(X^N_{t+u}) - F(X^N_t) \bigg| \mathcal{F}^N_t \right] \right| \leq E_{x_N} \left[ Z_N(\delta, F) \bigg| \mathcal{F}^N_t \right] \quad P_{x_N} \text{-a.s.}, \quad (9.4)$$

and

$$\lim_{\delta \to 0^+} \limsup_{N \to \infty} E_{x_N} Z_N(\delta, F) = 0. \quad (9.5)$$

**Proof.** See [31, Lemma 3.11] for the proof for the Euclidean case, i.e., $\Omega = \mathbb{R}^d$. The proof for the case $\Omega = \mathbb{T}^d$ is obviously the same with that of the Euclidean space.

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