Sharp High-dimensional Central Limit Theorems for Log-concave Distributions

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Abstract: Let $X_1, \ldots, X_n$ be i.i.d. log-concave random vectors in $\mathbb{R}^d$ with mean 0 and covariance matrix $\Sigma$. We study the problem of quantifying the normal approximation error for $W = n^{-1/2} \sum_{i=1}^{n} X_i$ with explicit dependence on the dimension $d$. Specifically, without any restriction on $\Sigma$, we show that the approximation error over rectangles in $\mathbb{R}^d$ is bounded by $C(\log^{13}(dn)/n)^{1/2}$ for some universal constant $C$. Moreover, if the Kannan–Lovász–Simonovits (KLS) spectral gap conjecture is true, this bound can be improved to $C(\log^3(dn)/n)^{1/2}$. This improved bound is optimal in terms of both $n$ and $d$ in the regime $\log n = O(\log d)$. We also give $p$-Wasserstein bounds with all $p \geq 2$ and a Cramér type moderate deviation result for this normal approximation error, and they are all optimal under the KLS conjecture. To prove these bounds, we develop a new Gaussian coupling inequality that gives almost dimension-free bounds for projected versions of $p$-Wasserstein distance for every $p \geq 2$. We prove this coupling inequality by combining Stein’s method and Eldan’s stochastic localization procedure.

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1 Introduction

Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $\mathbb{R}^d$ with mean 0 and covariance matrix $\Sigma = (\Sigma_{jk})_{1 \leq j, k \leq d}$. Set $W = n^{-1/2} \sum_{i=1}^{n} X_i$. The classical central limit theorem (CLT) states that $W$ converges in law to $N(0, \Sigma)$ as $n \to \infty$. This paper aims to quantify the convergence rate of this normal approximation with explicit dependence on the dimension $d$. It is known that this dependence is crucially determined by how to measure the distance between the law of $W$ and $N(0, \Sigma)$. In this paper, we primarily focus on the uniform distance over rectangles in $\mathbb{R}^d$. That is,

$$\rho(W, Z) = \sup_{A \in \mathcal{R}} |P(W \in A) - P(Z \in A)|,$$

where $Z \sim N(0, \Sigma)$ and $\mathcal{R} := \{ \prod_{j=1}^{d} [a_j, b_j] : -\infty < a_j < b_j < \infty \}$ is the set of rectangles in $\mathbb{R}^d$. The recent seminal work of Chernozhukov, Chetverikov and Kato [10, 13] has shown that, under mild regularity assumptions, one can get a non-trivial bound for $\rho(W, Z)$ even
when the dimension $d$ is much larger than the sample size $n$. When we allow $\Sigma$ to be degenerate, the currently best known general bound for $\rho(W, Z)$ is as follows: Suppose $\sigma^2 = \min_{1 \leq j \leq d} \Sigma_{jj} > 0$. Suppose also that there exists a constant $B > 0$ such that $\mathbb{E} \exp(|X_{1j}|/B) \leq 2$ and $\mathbb{E} X_{1j}^4 \leq B^2$ for all $j = 1, \ldots, d$, where $X_{1j}$ is the $j$-th component of $X_1$. Then, according to Theorem 2.1 in Chernozhukov et al. [15], we have

$$\rho(W, Z) \leq c \left( \frac{B^2 \log^5(dn)}{n} \right)^{1/4},$$

where $c$ is a constant depending only on $\sigma^2$. The bound (1.1) gives a meaningful estimate for $\rho(W, Z)$ even when $d$ is exponentially larger than $n$, but the dependence on $n$ does not match the classical Berry–Esseen rate $1/\sqrt{n}$. Recently, by exploiting the regularity of $Z$, several authors have succeeded in getting bounds with $1/\sqrt{n}$ rates up to $\log n$ factors when $\Sigma$ is non-degenerate; see Fang and Koike [25], Lopes [37], Kuchibhotla and Rinaldo [35], Chernozhukov, Chetverikov and Koike [16]. In particular, by Corollary 1.1 in Fang and Koike [25], if $X_1$ is log-concave (cf. Definition 1.1), then

$$\rho(W, Z) \leq \frac{C}{\sigma^2} \sqrt{\frac{\log^3 d}{n} \log n},$$

where $C$ is a positive universal constant and $\sigma^2$ is the smallest eigenvalue of the correlation matrix of $W$. The bound (1.2) is rate-optimal up to the $\log n$ factor because $\sqrt{\log^3 d/n} \rho(W, Z)$ does not vanish as $n \to \infty$ under appropriate growth conditions on $n$ and $d$ when the coordinates of $X_1$ are i.i.d. and follow a standardized exponential distribution; see Proposition 1.1 in Fang and Koike [25], Corollary 2.1 in Chernozhukov, Chetverikov and Koike [16] gives a similar bound to (1.2) without log-concavity when $X_{1j}$ are uniformly bounded.

In this paper, we show that a bound of the form $C \sqrt{\log^n (dn)/n}$ for some constants $C, a > 0$ is achievable even when $\Sigma$ is degenerate, provided that $X_1$ is log-concave. Remarkably, $C$ and $a$ can be taken universally and thus independently of $\Sigma$. In addition, if the Kannan–Lovász–Simonovits (KLS) conjecture is true, our bound is optimal in both $n$ and $d$ in the regime $\log n = O(\log d)$. To state the result formally, we introduce some definitions and notations.

**Definition 1.1** (Log-concavity). A probability measure $\mu$ on $\mathbb{R}^d$ is log-concave if

$$\mu(\theta A + (1 - \theta) B) \geq \mu(A)^\theta \mu(B)^{1-\theta}$$

for any non-empty compact sets $A$ and $B$ of $\mathbb{R}^d$ and any $\theta \in (0, 1)$. We say that a random vector $X$ in $\mathbb{R}^d$ is log-concave if its law $\mathcal{L}(X)$ is log-concave.

**Definition 1.2** (Poincaré constant). A probability measure $\mu$ on $\mathbb{R}^d$ is said to satisfy a Poincaré inequality if there exists a constant $\varpi \geq 0$ such that

$$\text{Var}_\mu(h) := \int h^2 d\mu - \left( \int hd\mu \right)^2 \leq \varpi \int |\nabla h|^2 d\mu$$

(1.3)
for every locally Lipschitz function $h : \mathbb{R}^d \to \mathbb{R}$ with $h \in L^2(\mu)$. Here,

$$|\nabla h(x)| := \limsup_{y \to x} \frac{|h(y) - h(x)|}{|y - x|}, \quad x \in \mathbb{R}^d.$$ 

The smallest constant $\varpi$ satisfying (1.3) is called the Poincaré constant of $\mu$ and denoted by $\varpi(\mu)$. By convention, we set $\varpi(\mu) := \infty$ if $\mu$ does not satisfy any Poincaré inequality. For a random vector $X$ in $\mathbb{R}^d$, we write $\varpi(X) = \varpi(\mathcal{L}(X))$.

We denote by $\text{LC}_d$ the set of isotropic (i.e., with zero mean and identity covariance) log-concave probability measures on $\mathbb{R}^d$. Define

$$\varpi_d := \sup_{\mu \in \text{LC}_d} \varpi(\mu).$$

The KLS conjecture suggests that $\varpi_d$ would be bounded by a universal constant. The currently best known bound is the following one due to Klartag and Lehec [32] (cf. Theorem 1.1 and Eq.(7) ibidem):

$$\varpi_d \leq C(1 + \log^{10} d).$$

(1.4)

Here and below, we use $C$ to denote positive universal constants, which may differ in different expressions. We refer to Alonso-Gutiérrez and Bastero [1] for more background of the KLS conjecture.

With these notations, our first main result is stated as follows:

**Theorem 1.1.** Let $n \geq 2$ be an integer. Let $X_1, \ldots, X_n$ be i.i.d. log-concave random vectors in $\mathbb{R}^d$ with mean 0 and covariance matrix $\Sigma$, and set $W = n^{-1/2} \sum_{i=1}^n X_i$. Let $Z \sim \mathcal{N}(0, \Sigma)$. Suppose that $\Sigma$ has rank $r \geq 1$. Then

$$\sup_{A \subset \mathbb{R}} |P(W \in A) - P(Z \in A)| \leq C\sqrt{\frac{\log^2(dn) \log(2d)}{n}} \cdot$$

(1.5)

The most remarkable feature of the bound (1.5) is that the right hand side is bounded by a quantity independent of $\Sigma$ because $\varpi_r \leq \varpi_d$. Note that the log-concavity itself does not impose any restriction on $\Sigma$ because it is invariant under affine transformation. In particular, the bound (1.5) holds even when $\Sigma$ is degenerate. Combining (1.5) with the estimate (1.4), we obtain a bound for $\rho(W, Z)$ with the rate $O(\sqrt{\log^{13}(dn)/n})$. In terms of the dependence on $n$, this improves the bound derived from (1.1). Moreover, if the KLS conjecture is true, Theorem 1.1 gives a bound for $\rho(W, Z)$ of the form $C\sqrt{\log^3(dn)/n}$. As shown by Proposition 1.1 in Fang and Koike [25], this bound is rate-optimal in both $n$ and $d$ when $\log n = O(\log d)$.

To prove Theorem 1.1, we construct a coupling of $W$ and $Z$ such that $\|u \cdot (W - Z)\|_p$ enjoys an almost dimension-free bound for any $u \in \mathbb{R}^d$ and $p \geq 1$, where $\cdot$ is the Euclidean inner product and $\|\cdot\|_p$ is the $L^p$-norm with respect to the underlying probability measure. See Section 2 for the precise result. Such a bound can be used to control the tail probability of $\max_{1 \leq j \leq d} |W_j - Z_j|/\sigma_j$ with $\sigma_j := \sqrt{\Sigma_{jj}}$. As illustrated by Lemma 2.1 in Chernozhukov, Chetverikov and Kato [12], we can derive a bound for the Kolmogorov distance between
max_{1 \leq j \leq d}(W_j - x_j)/\sigma_j and max_{1 \leq j \leq d}(Z_j - x_j)/\sigma_j for any x \in \mathbb{R}^d from such a control along with an anti-concentration inequality for max_{1 \leq j \leq d}(Z_j - x_j)/\sigma_j, and this leads to a bound for \rho(W, Z). Hence, our main technical contribution is derivation of the afore-mentioned coupling inequality for W and Z. This new coupling inequality is shown by combining Stein’s method and Eldan’s stochastic localization procedure as detailed in Section 3.

Our coupling inequality naturally leads to a bound for the $p$-Wasserstein distance between $W$ and $Z$ for any $p \geq 1$, which will be of independent interest. Let us recall the definition of the $p$-Wasserstein distance:

**Definition 1.3 ($p$-Wasserstein distance).** Let $\mu$ and $\nu$ be two probability measures on $\mathbb{R}^d$. For $p \geq 1$, the $p$-Wasserstein distance between $\mu$ and $\nu$ is defined as

$$W_p(\mu, \nu) = \left( \inf_\pi \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy) \right)^{1/p},$$

where $| \cdot |$ denotes the Euclidean norm and $\pi$ is a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mu$ and $\nu$. For two random vectors $X$ and $Y$ in $\mathbb{R}^d$, we write $W_p(X, Y) = W_p(\mathcal{L}(X), \mathcal{L}(Y))$.

**Theorem 1.2.** Under the same assumptions as Theorem 1.1, we have

$$W_p(W, Z) \leq C \sqrt{\text{tr}(\Sigma)} \left( \frac{p}{\sqrt{n}} \log(2r)^{p/2} \log(2r) + \frac{1}{\sqrt{\log(2r)}} \right)$$

for any $p \geq 1$. Moreover, if $\log^2(2r) \leq cn$ for some positive constant $c$, there exists a constant $C'$ depending only on $c$ such that

$$W_p(W, Z) \leq C' \frac{p^2 \sqrt{\text{tr}(\Sigma) \log(2r)}}{\sqrt{n}}$$

for any $p \geq 1$.

In view of (1.4), the condition $\log^2(2r) \leq cn$ will be a rather mild restriction. When $X_1$ has independent coordinates and $X_{1j}$ has a non-zero skewness $\gamma_j$ for all $j = 1, \ldots, d$, $W_p(W, Z)$ is lower bounded by $c \sqrt{\text{tr}(\Sigma)/n}$ with some positive constant $c$ depending only on $\min_{1 \leq j \leq d} |\gamma_j|$ in view of Theorem 1.1 in Rio [44]. Hence the bound (1.7) has optimal dependence on $n, d$ and $\Sigma$ if the KLS conjecture is true. Also, when $\Sigma = I_d$, (1.4) and (1.7) give an upper bound for $W_p(W, Z)$ of the form $Cp^2 \sqrt{d \log(2d)/n}$, which improves the currently best known bound $Cp^d/\sqrt{n}$ given by Theorem 3.3 in Fathi [27]. We remark that Theorem 4.1 in Courtade, Fathi and Pananjady [17] implies that the bound (1.7) for $p = 2$ and $\Sigma = I_d$ holds without the condition $\log^2(2r) \leq cn$. Indeed, by a simplified proof of Theorem 1.2 for the case $p = 2$ using Lemma 3.2, we can show $W_2(W, Z) \leq C' \sqrt{\text{tr}(\Sigma) \log(2r)}$, without the condition $\log^2(2r) \leq cn$. This bound is completely dimension-free if the KLS conjecture is correct.

Yet another application of our coupling inequality gives the following Cramér type moderate deviation result for $\max_{1 \leq j \leq d} W_j$:
Theorem 1.3. Under the same assumptions as Theorem 1.1, suppose additionally that \( \sigma_j > 0 \) for all \( j = 1, \ldots, d \). Set

\[
\overline{\sigma} = \max_{1 \leq j \leq d} \sigma_j, \quad \underline{\sigma} = \min_{1 \leq j \leq d} \sigma_j. \tag{1.8}
\]

Then, there exist universal constants \( c \in (0, 1) \) and \( C > 0 \) such that, for

\[
\frac{\overline{\sigma}^2 \omega_r \log^3(3d)}{\underline{\sigma}^2 n} \leq c, \quad 0 \leq x \leq \frac{\underline{\sigma}^2 n}{\overline{\sigma}^2 \omega_r},
\]

we have

\[
\frac{P(\max_{1 \leq j \leq d} W_j > x)}{P(\max_{1 \leq j \leq d} Z_j > x)} - 1 \leq C \left( 1 + \frac{x}{\underline{\sigma}} \right) \left( \log(\text{dn}) + \frac{x^2}{\overline{\sigma}^2} \right) \frac{\overline{\sigma} \sqrt{\omega_r}}{n}. \tag{1.9}
\]

Note that, applying the result to \((W^T, -W^T)^\top\), we can replace \( \max_{1 \leq j \leq d} W_j \) and \( \max_{1 \leq j \leq d} Z_j \) in (1.9) with \( \max_{1 \leq j \leq d} |W_j| \) and \( \max_{1 \leq j \leq d} |Z_j| \), respectively. Corollary 5.1 in Kuchibhotla, Mukherjee and Banerjee [34] gives a Cramér type moderate deviation result for \( \max_{1 \leq j \leq d} |W_j| \) with the bound of the form \( K \{(1 + x)^6 \log^{16}(3d)/n\}^{1/6} \) when coordinates of \( X_1 \) are sub-exponential, where \( K \) is a positive constant depending only on \( \overline{\sigma}, \underline{\sigma} \) and sub-exponential norms of \( X_{1j} \). In the meantime, denoting by \( K' \) a positive constant depending only on \( \overline{\sigma} \) and \( \underline{\sigma} \), we can bound the right hand side of (1.9) as

\[
K' \left( 1 + x \right) \left( \log(\text{dn}) + x^2 \right) \frac{\sqrt{\log^3(3d)}}{n} \leq \frac{\log^{10}(3d)}{n},
\]

\[
\leq K' \left( 2 + 2 \log^{3/2}(\text{dn}) + 3x^3 \right) \frac{\log^{10}(3d)}{n},
\]

\[
\leq \sqrt{3} K' \left( 4 + 4 \log^3(\text{dn}) + 9x^6 \right) \log^{10}(3d),
\]

where the first inequality follows by the elementary inequality \( x \log(\text{dn}) \leq x^3/3 + \log^{3/2}(dn)/(3/2) \leq x^3 + \log^{3/2}(dn) \), the second by \( \log(\text{dn}) \leq 1 + \log^{3/2}(\text{dn}) \) and \( x^2 \leq 1 + x^3 \), and the last by \( (a + b + c)^2 \leq 3(a^2 + b^2 + c^2) \) for any \( a, b, c \in \mathbb{R} \). Consequently, our bound improves Kuchibhotla, Mukherjee and Banerjee [34]’s one when \( X_1 \) are log-concave. Moreover, inspection of the proof of Proposition 1.1 in Fang and Koike [25] leads to the following result, showing that the bound (1.9) is sharp if the KLS conjecture is true.

Proposition 1.1. Let \( X = (X_{ij})_{i,j=1}^\infty \) be an array of i.i.d. random variables such that \( \mathbb{E} \exp(c|X_{ij}|) < \infty \) for some \( c > 0 \), \( \mathbb{E}X_{ij} = 0 \), \( \mathbb{E}X_{ij}^2 = 1 \) and \( \gamma := \mathbb{E}X_{ij}^3 \neq 0 \). Let \( W = n^{-1/2} \sum_{i=1}^n X_i \) with \( X_i := (X_{i1}, \ldots, X_{id})^\top \). Suppose that \( d \) depends on \( n \) so that \( (\log^3 d)/n \to 0 \) and \( d(\log^3 d)/n \to \infty \) as \( n \to \infty \). Also, let \( Z \sim N(0, I_d) \). Then there exists a sequence \( (x_n) \) of positive numbers such that \( x_n = o(n^{1/6}) \) as \( n \to \infty \) and

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \frac{\mathbb{P}(\max_{1 \leq j \leq d} W_j > x_n)}{\mathbb{P}(\max_{1 \leq j \leq d} Z_j > x_n)} - 1 > 0.
\]
Remark 1.1. Cramér’s original result in the univariate case gives a higher order asymptotic expansion of $P(W > x)$ for moderately large $x$ (see e.g. Petrov [41, Chapter VIII, Theorem 1]), while Theorem 1.3 concerns only the first order asymptotic expansion of a possible moderate deviation result. We refer to a result like Theorem 1.3 as a “Cramér type moderate deviation result” following the custom in Stein’s method literature (see e.g. Chapter 11 of Chen, Goldstein and Shao [9]).

Finally, for uniformly log-concave random vectors, we can remove dependence on the constant $\tau_r$. Following Saumard and Wellner [45], we define the uniform log-concavity as follows:

**Definition 1.4** (Uniform log-concavity). Let $\epsilon > 0$. A probability density function $q : \mathbb{R}^d \to [0, \infty)$ is said to be $\epsilon$-uniformly log-concave if there is a log-concave function $g : \mathbb{R}^d \to [0, \infty)$ such that $q(x) = g(x)e^{-\epsilon|x|^2/2}$ for every $x \in \mathbb{R}^d$.

A probability measure $\mu$ on $\mathbb{R}^d$ is said to be $\epsilon$-uniformly log-concave if it has an $\epsilon$-uniformly log-concave density. A random vector $X$ in $\mathbb{R}^d$ is said to be $\epsilon$-uniformly log-concave if its law is $\epsilon$-uniformly log-concave.

Our definition of $\epsilon$-uniform log-concavity is equivalent to strong log-concavity with variance parameter $\epsilon^{-1}$ in Saumard and Wellner [45, Definition 2.9]. Thus, if $q : \mathbb{R}^d \to [0, \infty)$ is a probability density function of the form $e^{-V}$ with $V : \mathbb{R}^d \to \mathbb{R}$ a $C^2$ function, then $q$ is $\epsilon$-uniformly log-concave if and only if $\text{Hess} V - \epsilon I_d$ is positive semidefinite; see Proposition 2.24 in Saumard and Wellner [45].

**Theorem 1.4.** Let $n \geq 2$ be an integer and $\epsilon \in (0, 1)$ a constant. Let $X_1, \ldots, X_n$ be i.i.d. isotropic $\epsilon$-uniformly log-concave random vectors in $\mathbb{R}^d$, and set $W = n^{-1/2} \sum_{i=1}^n X_i$. Let $Z \sim N(0, I_d)$. Then, there exist universal constants $c$ and $C$ such that

$$
\sup_{A \in \mathcal{R}} |P(W \in A) - P(Z \in A)| \leq C \sqrt{\frac{\log^2(2n) \log(2d)}{\epsilon n}}
$$

(1.10)

and

$$
W_p(W, Z) \leq C \sqrt{d} \left( \frac{p}{\sqrt{\epsilon n}} + \frac{p^{3/2}}{\epsilon n} \right)
$$

(1.11)

for any $p \geq 1$. Moreover, for

$$
\frac{\log^3(3d)}{\epsilon n} \leq c, \quad 0 \leq x \leq (\epsilon n)^{1/6},
$$

(1.12)

we have

$$
\left| \frac{P(\max_{1 \leq j \leq d} W_j > x)}{P(\max_{1 \leq j \leq d} Z_j > x)} - 1 \right| \leq C \frac{(1 + x)(\log(2n) + x^2)}{\sqrt{\epsilon n}}.
$$

(1.13)

For fixed $\epsilon$, the bounds (1.10), (1.11) and (1.13) are generally rate optimal by the same reasoning as above. In fact, when the coordinates of $X_1$ are i.i.d. and follows the scaled Weibull distribution with scale parameter 1 and shape parameter $\beta \geq 2$, then $X_1$ is $(\beta - 1)\{\Gamma(1 + 2/\beta) - \Gamma(1 + 1/\beta)^2\}$-uniformly log-concave and its coordinates have non-zero
skewness. We remark that the bound (1.2) is applicable in the setting of Theorem 1.4, but it leads to an extra log \( n \) factor compared to (1.10) in the high-dimensional regime \( \log n = O(\log d) \). Also, regarding the \( p \)-Wasserstein bound (1.11), the dependence on \( \varepsilon \) is improved compared to Theorem 3.4 in Fathi [27] when \( \varepsilon n \geq 1 \).

The remainder of the paper is organized as follows. In Section 2, we formulate our new coupling inequalities and prove the main results stated in the introduction. We prove the coupling inequalities in Section 3. Section 4 gives the proof of an auxiliary result to establish the Cramér type moderate deviation result.

**Notations.** For a random vector \( \xi \) in \( \mathbb{R}^d \) and \( p > 0 \), we write \( \|\xi\|_p = (\mathbb{E}|\xi|^p)^{1/p} \). For a matrix \( A \), \( \|A\|_{op} \) and \( \|A\|_{H.S.} \) denote the operator norm and the Hilbert-Schmidt norm of \( A \), respectively. For two \( d \times d \) matrices \( A \) and \( B \), we write \( A \preceq B \) or \( B \succeq A \) if \( B - A \) is positive semidefinite. We write \( \langle A,B \rangle_{H.S.} = \text{tr}(A^\top B) \) for their Hilbert-Schmidt inner product.

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**2 Projected Wasserstein bounds**

The proofs of the main results rely on the following “projected” Wasserstein bounds.

**Theorem 2.1.** Let \( \mu \) be a centered log-concave probability measure on \( \mathbb{R}^d \). Suppose that the covariance matrix \( \Sigma \) of \( \mu \) has rank \( r \geq 1 \). Then, for any integer \( n \geq 1 \), we can construct random vectors \( W \) and \( Z \) in \( \mathbb{R}^d \) such that \( W = n^{-1/2} \sum_{i=1}^n X_i \) with \( X_i \overset{i.i.d.}{\sim} \mu \), \( Z \sim N(0, \Sigma) \) and

\[
\|u \cdot (W - Z)\|_p \leq C|u| \left( \sqrt{\frac{\omega_r}{n}} + \frac{p^{3/2}}{\sqrt{n}} \frac{\log(2r)}{n} \right) (2.1)
\]

for all \( u \in \mathbb{R}^d \) and \( p \geq 1 \).

**Theorem 2.2.** Let \( \mu \) be an isotropic probability measure on \( \mathbb{R}^d \). Suppose that \( \mu \) is \( \varepsilon \)-uniformly log-concave for some \( \varepsilon > 0 \). Then, for any integer \( n \geq 1 \), we can construct random vectors \( W \) and \( Z \) in \( \mathbb{R}^d \) such that \( W = n^{-1/2} \sum_{i=1}^n X_i \) with \( X_i \overset{i.i.d.}{\sim} \mu \), \( Z \sim N(0, I_d) \) and

\[
\|u \cdot (W - Z)\|_p \leq C|u| \left( \frac{p}{\sqrt{\varepsilon n}} + \frac{p^{3/2}}{\varepsilon n} \right) (2.2)
\]

for all \( u \in \mathbb{R}^d \) and \( p \geq 1 \).

We prove these theorems in the next section.

**Remark 2.1.** It would be worth mentioning that Theorem 2.1 follows once we prove the corresponding bound for \( UW \) with a \( d \times d \) orthogonal matrix \( U \). This feature allows us to reduce the proof of Theorem 2.1 to the case \( \Sigma = I_d \). By contrast, such reduction is generally impossible if we directly bound the left hand side of (1.5) as in Fang and Koike [25] because the class of rectangles are not rotationally invariant.
In the remainder of this section, we prove the main results stated in the introduction using these coupling inequalities. Below we will frequently use the inequality \( \varpi_r \geq \varpi_r(N(0, I_r)) = 1 \) without reference.

**Proof of Theorem 1.1.** For two vectors \( x, y \in \mathbb{R}^d \), we write \( x \leq y \) if \( x_j \leq y_j \) for all \( j = 1, \ldots, d \). Then we have

\[
\sup_{A \in \mathcal{R}} |P(W \in A) - P(Z \in A)| = \sup_{x \in \mathbb{R}^{2d}} |P((W^T, -W^T)^T \leq x) - P((Z^T, -Z^T)^T \leq x)|.
\]

Also, the covariance matrix of \((W^T, -W^T)^T\) has rank \( r \). Moreover, \((X_i^T, -X_i^T)^T\) are log-concave by Proposition 3.1 in Saumard and Wellner [45]. Consequently, it suffices to prove

\[
\sup_{x \in \mathbb{R}^d} |P(W \leq x) - P(Z \leq x)| \leq C \sqrt{\frac{\log^2(\sigma n) \log(2d)}{n}}
\]
when \( d \geq 2 \). Also, since the left hand side is bounded by 1, we may assume

\[
\varpi_r \frac{\log^2(\sigma n) \log(2d)}{n} \leq 1
\]
without loss of generality.

Next, if \( \mathbb{E}W_j^2 = 0 \) for some \( j \), then \( W_j = Z_j = 0 \) a.s. Hence, with \( \mathcal{J} = \{j \in \{1, \ldots, d\} : \mathbb{E}W_j^2 \neq 0\} \), we have

\[
\sup_{x \in \mathbb{R}^d} |P(W \leq x) - P(Z \leq x)| = \sup_{x \in \mathbb{R}^{d'}} |P((W_j)_{j \in \mathcal{J}} \leq x) - P((Z_j)_{j \in \mathcal{J}} \leq x)|,
\]
where \( d' \) is the number of elements in \( \mathcal{J} \). Also, the covariance matrix of \((X_{ij})_{j \in \mathcal{J}}\) has rank \( r \). Moreover, \((X_{ij})_{j \in \mathcal{J}}\) are log-concave by Proposition 3.1 in Saumard and Wellner [45]. Consequently, without loss of generality, we may assume \( \mathcal{J} = \{1, \ldots, d\} \), i.e. \( \sigma_j = \sqrt{\mathbb{E}W_j^2} > 0 \) for all \( j = 1, \ldots, d \).

Again without loss of generality, we may assume that \( W \) and \( Z \) are the same as in Theorem 2.1. Fix \( x \in \mathbb{R}^d \) arbitrarily and set

\[
W^\vee := \max_{1 \leq j \leq d} \frac{W_j - x_j}{\sigma_j}, \quad Z^\vee := \max_{1 \leq j \leq d} \frac{Z_j - x_j}{\sigma_j}.
\]

Then we have

\[
P(W \leq x) - P(Z \leq x) = P(W^\vee \leq 0) - P(Z^\vee \leq 0).
\]
Let \( e_1, \ldots, e_d \) be the standard basis of \( \mathbb{R}^d \). For every \( j = 1, \ldots, d \), we apply the bound (2.1) with \( u = e_j/\sigma_j \) and then obtain

\[
\left\| \frac{W_j - Z_j}{\sigma_j} \right\|_p \leq C \frac{|\Sigma^{1/2} e_j|}{\sigma_j} \left( \frac{p^{3/2}}{\sqrt{n} \varpi_r} + \frac{p^{3/2}}{\sqrt{\varpi_r \log(2r)} n} \right)
\]
for any \( p \geq 1 \). Observe that

\[
\frac{|\Sigma^{1/2} e_j|^2}{\sigma_j^2} = \frac{e_j^T \Sigma e_j}{\sigma_j^2} = 1.
\]
Further, let \( p = \log(nd) \geq 1 \). Then
\[
\varpi r \log(2r) \frac{p^{3/2}}{n} \leq \varpi r \log(2d) \frac{p^{3/2}}{n} = \sqrt{\varpi r \frac{\log^2(dn)}{n}} \sqrt{\varpi r \frac{\log(dn) \log(2d)}{n}} \leq \sqrt{\varpi r \frac{\log^2(dn)}{n}},
\]
where we used (2.4) for the last inequality. In addition, note that
\[
\frac{\log^3(dn)}{n} \leq 4 \frac{\log^3 d + \log^3 n}{n} \leq 4(1 + (3/e)^3) < 36,
\]
where we used (2.4) and the elementary inequality \( \log n \leq (3/e)n^{1/3} \) in the second inequality. Hence we have
\[
\frac{\sqrt{\log 2}}{\varpi r \log(2r)} \frac{p^{5/2}}{n} \leq \frac{p^{5/2}}{n} = \sqrt{\frac{\log^2(dn)}{n}} \sqrt{\frac{\log^3(dn)}{n}} \leq 6 \sqrt{\varpi r \frac{\log^2(dn)}{n}}.
\]

Therefore, there exists a positive universal constant \( C_0 > 0 \) such that
\[
\max_{1 \leq j \leq d} \left\| \frac{W_j - Z_j}{\sigma_j} \right\|_p \leq C_0 \sqrt{\varpi r \frac{\log^2(dn)}{n}}.
\]

Also, for any \( \eta > 0 \),
\[
P(\|W^\vee - Z^\vee\| > \eta) \leq \eta^{-p} \mathbb{E} \max_{1 \leq j \leq d} \left| \frac{W_j - Z_j}{\sigma_j} \right|^p \leq \eta^{-p} \sum_{j=1}^d \mathbb{E} \left| \frac{W_j - Z_j}{\sigma_j} \right|^p \leq d \eta^{-p} \max_{1 \leq j \leq d} \mathbb{E} \left| \frac{W_j - Z_j}{\sigma_j} \right|^p.
\]

Therefore, taking \( \eta = eC_0 \sqrt{\varpi r \log^2(dn)/n} \), we obtain
\[
P(\|W^\vee - Z^\vee\| > \eta) \leq de^{-p} = \frac{1}{n}.
\]

Thus, by Lemma 2.1 in Chernozhukov, Chetverikov and Kato [12],
\[
|P(W \leq x) - P(Z \leq x)| \leq \sup_{t \in \mathbb{R}} P(\|Z^\vee - t\| \leq \eta) + \frac{1}{n}.
\]

Observe that
\[
P(\|Z^\vee - t\| \leq \eta) = P(t - \eta \leq Z^\vee \leq t + \eta) = P(Z^\vee \leq (t - \eta) + 2\eta) - P(Z^\vee < t - \eta).
\]

Thus, by Nazarov’s inequality (cf. Chernozhukov, Chetverikov and Kato [14]),
\[
\sup_{t \in \mathbb{R}} P(\|Z^\vee - t\| \leq \eta) \leq 2\eta (\sqrt{2 \log d} + 2) \leq 8 \sqrt{\log(2d)} \eta \leq C \sqrt{\varpi r \frac{\log^2(dn) \log(2d)}{n}}.
\]

All together, we obtain (2.3). \( \Box \)
Proof of Theorem 1.2. Without loss of generality, we may assume that \( W \) and \( Z \) are the same as in Theorem 2.1. Also, thanks to Jensen’s inequality, it suffices to consider the case \( p \geq 2 \). Let \( e_1, \ldots, e_d \) be the standard basis of \( \mathbb{R}^d \). For every \( j = 1, \ldots, d \), we apply the bound (2.1) with \( u = e_j \) and then obtain

\[
\|W_j - Z_j\|_p \leq C \sigma_j \left( \sqrt{\frac{p}{n}} + \frac{1}{\sqrt{\sigma_j \log(2r)}} \right),
\]

(2.5)

where we used the identity \(|\Sigma^{1/2} e_j|^2 = e_j^\top \Sigma e_j = \sigma_j^2\). Since

\[
W_p(W, Z) \leq \|W - Z\|_p \leq \left( \sum_{j=1}^d \|W_j - Z_j\|_p^2 \right)^{1/2},
\]

we obtain (1.6).

To prove (1.7), we may assume \( p \leq \sqrt{n} \) without loss of generality. In fact, since \( W \) is log-concave by Proposition 3.5 in Saumard and Wellner [45], we have by the reverse Hölder inequality (see e.g. Proposition A.5 in Alonso-Gutiérrez and Bastero [1])

\[
\|W\|_p \leq C p \mathbb{E}|W| \leq C p \sqrt{\mathbb{E}|W|^2} = C p \sqrt{\text{tr}(\Sigma)}.
\]

Also, since \( Z \) is Gaussian, we have \( \|Z\|_p \leq C \sqrt{p \text{tr}(\Sigma)} \) (cf. Lemma 6.3 in Fang and Koike [26]). Hence \( \|W - Z\|_p \leq C p \sqrt{\text{tr}(\Sigma)} \). Therefore, if \( p > \sqrt{n} \), the right hand side of (1.7) dominates \( C' p \sqrt{\text{tr}(\Sigma)} \), so (1.7) trivially holds with appropriate choice of \( C' \). Under the assumptions \( p \leq \sqrt{n} \) and \( \sigma_j \log^2(2r) \leq c \sqrt{n} \), (1.7) immediately follows from (1.6).

For the proof of Theorem 1.3, we use the following general result to derive a Cramér type moderate deviation from projected \( p \)-Wasserstein bounds:

**Proposition 2.1.** Let \( W \) be a random vector in \( \mathbb{R}^d \) and \( Z \) a Gaussian vector in \( \mathbb{R}^d \) with mean 0 and covariance matrix \( \Sigma \) such that \( \sigma_j > 0 \) for all \( j = 1, \ldots, d \). Suppose that

\[
\max_{1 \leq j \leq d} \|W_j - Z_j\|_p \leq A p^\alpha \Delta \quad \text{for all } 1 \leq p \leq p_0
\]

(2.6)

and

\[
\log d + |\log(\Delta/\sigma)| \leq p_0/2
\]

(2.7)

with some constants \( \alpha \geq 0 \), \( A > 0 \), \( p_0 \geq 1 \) and \( \Delta > 0 \). Define \( \bar{\sigma} \) and \( \underline{\sigma} \) as in (1.8). Assume also \( \Delta |\log d|^{\alpha+1/2} \leq B \bar{\sigma} \) for some constant \( B > 0 \). Then there exists a positive constant \( C \) depending only on \( \alpha \), \( A \) and \( B \) such that

\[
\left| \frac{P(\max_{1 \leq j \leq d} W_j > x)}{P(\max_{1 \leq j \leq d} Z_j > x)} - 1 \right| \leq C \left( 1 + \frac{x}{\underline{\sigma}} \right)^{1 + \log d} \left( 1 + \log \left( \frac{\Delta}{\bar{\sigma}} \right) \right) \left( 1 + \frac{x^2}{\bar{\sigma}^2} \right)^{\alpha} \frac{\Delta}{\bar{\sigma}}
\]

(2.8)

for all \( 0 \leq x \leq \min\{\bar{\sigma}(\Delta/\bar{\sigma})^{-1/(2\alpha+1)}, \underline{\sigma} \sqrt{p_0/2}\} \).
The proof of this proposition is given in Section 4. This result can be seen as a multi-dimensional extension of Theorem 2.1 in Fang and Koike [26] in terms of maxima, and it will be of independent interest. See Theorem 4.2 in Fang and Koike [26] for another multi-dimensional extension in terms of Euclidean norms.

**Remark 2.2.** In practice, the parameters $A, p_0$ and $\Delta$ in Proposition 2.1 will be determined in the following way. First, to deduce a meaningful bound from Proposition 2.1, we need to set $\Delta$ to a small value. Then, to make (2.7) hold, we need to take $p_0$ sufficiently large. However, as $p_0$ increases, we need to take $A\Delta$ large enough to make (2.6) hold. The adjustment by $A$ in (2.6) is useful to accomplish the last purpose.

**Proof of Theorem 1.3.** As in the proof of the previous results, we may assume that $W$ and $Z$ are the same as in Theorem 2.1. Then, by (2.5),

$$\max_{1 \leq j \leq d} \|W_j - Z_j\|_p \leq C\sigma \left( \sqrt{\frac{p_2}{\sqrt{n}}} + \frac{1}{\sqrt{n}} \log(2r) \frac{p^{3/2}}{n} + \frac{1}{\sqrt{n}} \log(2r) \frac{p^{5/2}}{n} \right)$$

for any $p \geq 1$. Hence, with $\alpha = 1$, $p_0 = 2 \min \{ n/(\sigma r \log(2r)), n^{1/3} \}$ and $\Delta = \sigma \sqrt{\sigma r/n}$, we have (2.6) for some universal constant $A$. Now assume

$$p_0 = 2 \min \left\{ \left( \frac{n}{\sigma r} \right)^{1/3}, \left( \frac{n}{\sigma r \log(2r)} \right)^{2/3}, n^{1/3} \right\} \geq 2(n/\sigma r)^{1/3}.$$  \hspace{1cm} (2.9)

Since $\Delta/\sigma = \sqrt{\frac{\sigma^2 \sigma r \log(3d)}{\sigma^2 n}} \leq 1$, we have

$$|\log(\Delta/\sigma)| \leq 3(\sigma/\Delta)^{1/3} \leq 3 \left( \frac{n}{\sigma r} \right)^{1/3} \leq 3 \left( \frac{n}{\sigma r} \right)^{1/3} \left( \frac{\sigma^2 \sigma r \log(3d)}{\sigma^2 n} \right)^{1/6}.$$

Therefore, if

$$\left( \frac{\sigma^2 \sigma r \log(3d)}{\sigma^2 n} \right)^{1/6} \leq \frac{1}{6},$$

then

$$\log d = n^{1/3} \left( \frac{\log d}{n} \right)^{1/3} \leq \left( \frac{\sigma^2 \sigma r \log(3d)}{\sigma^2 n} \right)^{1/3} \leq \left( \frac{\sigma^2 \sigma r \log(3d)}{\sigma^2 n} \right)^{1/6} \leq \frac{1}{36} \leq \frac{1}{2} \left( \frac{n}{\sigma r} \right)^{1/3}.$$

Further,

$$|\log(\Delta/\sigma)| \leq 3 \left( \frac{n}{\sigma r} \right)^{1/3} \frac{1}{6} = \frac{1}{2} \left( \frac{n}{\sigma r} \right)^{1/3}.$$

Hence we obtain

$$\log d + |\log(\Delta/\sigma)| \leq \left( \frac{n}{\sigma r} \right)^{1/3} \leq \frac{p_0}{2} \leq \frac{n}{2},$$

where the last inequality follows by (2.9). In this case we also have

$$\Delta \log d \leq \sqrt{\frac{\sigma^2 \sigma r \log(3d)}{n}} \leq \sigma.$$
\[ \log d + |\log \left( \frac{\Delta}{\sigma} \right)| = \log \left( \frac{d \sigma \sqrt{n}}{\sigma \sqrt{\varpi}} \right) \leq \log(n). \]

In addition,
\[ \sigma \left( \frac{\sigma^2 n}{\sigma^2 \varpi} \right)^{1/6} = \sigma (\Delta/\sigma)^{-1/3} = (\sigma^4/\sigma)^{1/3} (n/\varpi)^{1/6} \leq \sigma (n/\varpi)^{1/6} \leq \sigma \sqrt{p_0/2}, \]
where the last inequality follows by (2.9). Combining these estimates and Proposition 2.1 with \( B = 1 \) gives the desired result.

**Proof of Theorem 1.4.** Most parts of the proof are almost the same as the corresponding parts of Theorems 1.1–1.3, so we only describe necessary changes. Without loss of generality, we may assume that \( W \) and \( Z \) are the same as those in Theorem 2.2. We also assume \( d > 1 \) because the results for \( d = 1 \) follow from standard ones.

First, to prove (1.10), we may assume \( \log^2 (dn)/(n \varepsilon) \leq 1 \) without loss of generality. Then, with \( p = \log (dn) \), we have \( \max_j \|W_j - Z_j\|_p \leq C_0 p/\sqrt{n \varepsilon} \) for some universal constant \( C_0 > 0 \). Now, fix \( A = \prod_{j=1}^d [a_j, b_j] \in \mathcal{R} \) and set
\[ W^\vee := \max_{1 \leq j \leq d} \{(a_j - W_j) \vee (W_j - b_j)\}, \quad Z^\vee := \max_{1 \leq j \leq d} \{(a_j - Z_j) \vee (Z_j - b_j)\}. \]

Then we have
\[ |P(W \in A) - P(Z \in A)| = |P(W^\vee \leq 0) - P(Z^\vee \leq 0)| \leq \sup_{t \in \mathbb{R}} P(|Z^\vee - t| \leq \eta) + P(|W^\vee - Z^\vee| > \eta) \]
for any \( \eta > 0 \) by Lemma 2.1 in Chernozhukov, Chetverikov and Kato [12]. Since \( P(|W^\vee - Z^\vee| > \eta) \leq d n^{-p} \max_j E \|W_j - Z_j\|^p \), we can deduce (1.10) similarly to the proof of Theorem 2.1 by choosing \( \eta = e C_0 \log (dn)/\sqrt{n \varepsilon} \).

Next, the proof of (1.11) is a straightforward modification of that of Theorem 1.2.

Finally, to prove (1.13), we may assume \( \varepsilon n \geq 1 \) because we take \( c = 1/2 \) for (1.12). Observe that \( \sigma = \bar{\sigma} = 1 \) in the present setting. Then, we can verify that (2.6) and (2.7) hold with \( \alpha = 1, p_0 = 2 \varepsilon n, \Delta = 1/\sqrt{\varepsilon n} \) and some universal constant \( A > 0 \). Also, the condition \( \Delta (\log d)^{3/2} \leq B \) holds with \( B = \sqrt{c} \) by (1.12). Thus, (1.13) follows from Proposition 2.1.

**3 Proof of Theorems 2.1 and 2.2**

Given a probability density function \( q \) on \( \mathbb{R}^d \), we write \( \text{Cov}(q) = \text{Cov}(X) \), where \( X \) is a random vector in \( \mathbb{R}^d \) with density \( q \).

**3.1 Score bound**

The proof of Theorem 2.2 uses an analog of the so-called Stein kernel method to get a \( p \)-Wasserstein bound. This method relies on a score-based bound for the Wasserstein distance due to Otto and Villani [40] (see e.g. Eq.(3.8) of Ledoux, Nourdin and Peccati.
[36]), so we first develop a projected Wasserstein version of this bound. For later use, we develop such a bound for Markov kernels. Since Otto and Villani [40]’s proof is not constructive, it causes a measurability issue when applied to Markov kernels. To avoid this difficulty, it constructs an explicit coupling using the so-called Föllmer process. We refer to Eldan and Mikulincer [22] for background of the Föllmer process.

We fix a standard Gaussian vector $G$ in $\mathbb{R}^d$ independent of everything else. For a random vector $W$ in $\mathbb{R}^d$ and $t \in [0, 1)$, we set $W[t] := \sqrt{t}W + \sqrt{1-t}G$. It is straightforward to check that the law of $W[t]$ has a smooth density $f_{W[t]}$ with respect to $N(0, I_d)$. Moreover, $f_{W[t]}$ is strictly positive by Lemma 3.1 of Johnson and Suhov [30]. Therefore, we can define the score of $W[t]$ with respect to $N(0, I_d)$ by $\rho_{W[t]}(w) = \nabla \log f_{W[t]}(w)$, $w \in \mathbb{R}^d$.

**Proposition 3.1.** Let $P$ be a Markov kernel from a measurable space $(\mathcal{X}, \mathcal{A})$ to $\mathbb{R}^d$. Suppose that $P(x, \cdot)$ has a smooth density $f^x$ with respect to $N(0, I_d)$ for all $x \in \mathcal{X}$. Then, there exists a Markov kernel $Q$ from $(\mathcal{X}, \mathcal{A})$ to $\mathbb{R}^d \times \mathbb{R}^d$ satisfying the following conditions for any $x \in \mathcal{X}$:

1. For any Borel set $A$ in $\mathbb{R}^d$, $Q(x, A \times \mathbb{R}^d) = P(x, A)$ and $Q(x, \mathbb{R}^d \times A) = N(0, I_d)(A)$.
2. If $W$ and $Z$ are random vectors in $\mathbb{R}^d$ such that $(W, Z) \sim Q(x, \cdot)$, then

$$\|u \cdot (W - Z)\|_p \leq \int_0^1 \frac{1}{\sqrt{t}} \|u \cdot \rho_{W[t]}(W[t])\|_p dt$$

for any $p \geq 1$ and $u \in \mathbb{R}^d$.

**Proof.** Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in [0,1]}, Q)$ on which a $d$-dimensional standard $\mathbf{F}$-Brownian motion $Y = (Y_t)_{t \in [0,1]}$ is defined. For every $x \in \mathcal{X}$, define a process $M^x = (M^x_t)_{t \in [0,1]}$ as $M^x_t = P_{1-t}f^x(Y_t)$, where $P_{1-t}f^x(y) = \mathbb{E}f^x(y + \sqrt{1-t}G)$. Then, define a measure $P^x$ on $(\Omega, \mathcal{F})$ as $P^x(F) = \mathbb{E}_Q[M^x_11_F]$, i.e. $dP^x/dQ = M^x_1$ (cf. Eq.(19) in Eldan and Lee [20]). Also, define a process $B^x = (B^x_t)_{t \in [0,1]}$ as

$$B^x_t = Y_t - \int_0^t \nabla \log(P_{1-s}f^x)(Y_s)ds.$$

By Theorem 2.1 in Eldan and Lee [20], $P^x$ is a probability measure on $(\Omega, \mathcal{F})$, and $B^x$ is well-defined and a $d$-dimensional standard $\mathbf{F}$-Brownian motion under $P^x$. Moreover, the law of $Y_t$ has density $f^x$ with respect to $N(0, I_d)$ under $P^x$. We then define $Q(x, A) = P^x((Y_1, B^x_1) \in A) = \mathbb{E}_Q[1_A(Y_1, B^x_1)M^x_1]$ for any measurable set $A \subset \mathbb{R}^d \times \mathbb{R}^d$. It is evident from Fubini’s theorem that $Q$ is a Markov kernel from $(\mathcal{X}, \mathcal{A})$ to $\mathbb{R}^d \times \mathbb{R}^d$. Also, $Q$ satisfies condition (i) by construction. So it remains to check condition (ii) is satisfied.

Let $W$ and $Z$ be as in condition (ii). We have

$$\|u \cdot (W - Z)\|_p = \|u \cdot (Y_1 - B^x_1)\|_{L^p(P^x)} \leq \int_0^1 \|u \cdot \nabla \log(P_{1-t}f^x)(Y_t)\|_{L^p(P^x)} dt,$$

where the last inequality follows from (3.2) and the integral Minkowski inequality (see e.g. Proposition C.4 in Janson [29]). For every $t \in [0, 1]$, the law of $Y_t$ under $P^x$ is the same
as the law of $\sqrt{t}W[t] = tW + \sqrt{t(1-t)}G$. This is pointed out by Eldan and Mikulincer [22, Eq.(7)], but we can directly prove it as follows. For any bounded measurable function $h : \mathbb{R}^d \to \mathbb{R}$, we have

$$\mathbb{E}_{P^t}[h(Y_t)] = \mathbb{E}_Q[h(Y_t)M_1^t] = \mathbb{E}_Q[h(Y_t)f^x(Y_1)].$$

Let $G' \sim N(0, I_d)$ be independent of $G$. Recall that $Y$ is a standard Brownian motion in $\mathbb{R}^d$ under $Q$. Then, one can easily check that $(Y_t, Y_1)$ has the same law as $(tG' + \sqrt{t(1-t)}G, G')$ under $Q$. Hence

$$\mathbb{E}_{P^t}[h(Y_t)] = \mathbb{E}[h(tG' + \sqrt{t(1-t)}G)f^x(G')] = \mathbb{E}[h(tW + \sqrt{t(1-t)}G)],$$

where the last equality holds because $f^x$ is the density of $P(x, \cdot)$ with respect to $N(0, I_d)$. This proves the desired result. Therefore,

$$\|u \cdot (W - Z)\|_p \leq \int_0^1 \|u \cdot \nabla \log(P_{1-t}f^x)(\sqrt{t}W[t])\|_p dt.$$

Now, for any bounded measurable function $h : \mathbb{R}^d \to \mathbb{R}$, we have by definition

$$\mathbb{E}h(W[t]) = \mathbb{E}[h(\sqrt{t}G' + \sqrt{1-t}G)f^x(G')].$$

Since $(\sqrt{t}G' + \sqrt{1-t}G, G')$ has the same law as $(G', \sqrt{t}G' + \sqrt{1-t}G)$, we obtain

$$\mathbb{E}h(W[t]) = \mathbb{E}[h(G')f^x(\sqrt{t}G' + \sqrt{1-t}G)] = \mathbb{E}[h(G')P_{1-t}f^x(\sqrt{t}G')] = \mathbb{E}[h(G')P_{1-t}f^x'((\sqrt{t})w)].$$

This implies that $w \mapsto P_{1-t}f^x(\sqrt{t}w)$ is the density of the law of $W[t]$ with respect to $N(0, I_d)$. Consequently, $\rho_{W[t]}(w) = \sqrt{t} \nabla \log(P_{1-t}f^x)(\sqrt{t}w)$. Hence

$$\|u \cdot \nabla \log(P_{1-t}f^x)(\sqrt{t}W[t])\|_p = \|u \cdot \rho_{W[t]}(W[t])\|_p / \sqrt{t}.$$

All together, we obtain (3.1).

\[\square\]

### 3.2 Stein kernel bound

To bound the right hand side of (3.1), we use the notion of Stein kernel. Throughout this subsection, $\mu$ denotes a centered probability measure on $\mathbb{R}^d$.

**Definition 3.1** (Stein kernel). A $d \times d$ matrix valued measurable function $\tau$ on $\mathbb{R}^d$ is called a Stein kernel for $\mu$ if all entries of $\tau$ belong to $L^1(\mu)$ and

$$\int x \cdot h(x)\mu(dx) = \int \langle \tau(x), \nabla h(x) \rangle_{H.S.\mu}(dx) \tag{3.3}$$

for every compactly supported smooth function $h : \mathbb{R}^d \to \mathbb{R}^d$.

We say that $\tau$ is a Stein kernel for a centered random vector $W$ in $\mathbb{R}^d$ if it is a Stein kernel for the law of $W$. In this case, (3.3) reads as

$$\mathbb{E}[W \cdot h(W)] = \mathbb{E}[\langle \tau(W), \nabla h(W) \rangle_{H.S.}]$$. 

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Remark 3.1. The definition of Stein kernels here is the same as in Courtade, Fathi and Pananjady [17]. The same definition is also adopted in Nourdin, Peccati and Swan [39], Fathi [27] and Mikulincer and Shenfeld [38]. As discussed in the introduction of Courtade, Fathi and Pananjady [17], some articles use a slightly weaker version of (3.3) to define Stein kernels; see e.g. Ledoux, Nourdin and Peccati [36] and Fang and Koike [25]. We need the present stronger version to apply Lemma 2.9 in Nourdin, Peccati and Swan [39].

We obtain the following bound by a direct extension of the arguments in the proof of Ledoux, Nourdin and Peccati [36, Proposition 3.4]:

Lemma 3.1. Let \( W \) be a centered random vector in \( \mathbb{R}^d \). Suppose that \( W \) has a Stein kernel \( \tau \). Then

\[
\int_0^1 \frac{1}{\sqrt{t}} \| u \cdot \rho_{W[t]}(W[t]) \|_p dt \leq C \sqrt{p} \| (\tau(W)^\top - I_d)u \|_p
\]

(3.4)

for all \( u \in \mathbb{R}^d \) and \( p \geq 1 \).

Proof. By Lemma 2.9 in Nourdin, Peccati and Swan [39],

\[
\rho_{W[t]}(W[t]) = \frac{t}{\sqrt{1-t}} \mathbb{E}[(\tau(W) - I_d)G|W[t]] \text{ a.s.}
\]

for all \( t \in (0,1) \). Hence, by Jensen’s inequality,

\[
\int_0^1 \frac{1}{\sqrt{t}} \| u \cdot \rho_{W[t]}(W[t]) \|_p dt \leq \int_0^1 \frac{t}{\sqrt{1-t}} \| u \cdot (\tau(W) - I_d)G \|_p dt.
\]

Conditional on \( W \), \( u \cdot (\tau(W) - I_d)G \) follows the normal distribution with mean 0 and variance \( u^\top (\tau(W) - I_d)^\top (\tau(W) - I_d)u = |(\tau(W)^\top - I_d)u|^2 \). Moreover, when \( \zeta \) is a normal variable with mean 0 and variance \( \sigma^2 \), then \( \| \zeta \|_p = 2\sigma \Gamma((p+1)/2)/\sqrt{\pi} \) \( \leq C \sqrt{p} \sigma \), where the upper bound follows by Stirling’s formula. Consequently, we obtain

\[
\int_0^1 \frac{1}{\sqrt{t}} \| u \cdot \rho_{W[t]}(W[t]) \|_p dt \leq C \sqrt{p} \| (\tau(W)^\top - I_d)u \|_p.
\]

This completes the proof. \( \Box \)

To get the desired bounds from Lemma 3.1, we need to construct a Stein kernel \( \tau \) for \( \mu \) such that \( \int |\tau(x)^\top u|^p \mu(dx) \) enjoys a dimension-free bound (up to the constant \( \varpi_d \)) for any \( u \in \mathbb{R}^d \) and \( p \geq 1 \) when \( \mu \) is log-concave. If we additionally assume that \( \mu \) is uniformly log-concave, such a construction is given by Fathi [27]; see Corollary 2.4 ibidem. However, it is unclear whether this construction gives an appropriate bound in the log-concave case: Only an entry-wise bound is available in this case. Recently, Mikulincer and Shenfeld [38] developed another construction that would admit a bound for \( \int \| \tau(x) \|_{op}^p \mu(dx) \); see Theorem 1.5 and the proof of Theorem 5.7 ibidem. Their original bound is not dimension-free, but it implicitly depends on \( \varpi_d \), so it is improvable if the KLS conjecture is true.
However, inspection of their proof suggests that the bound would still contain a poly-log factor on the dimension even if we assume the KLS conjecture.

Here, our first key observation is that a Stein kernel constructed in Courtade, Fathi and Pananjady [17] enjoys a dimension-free bound for ∫ |τ(x)| u|^2| |μ(dx), provided that the KLS conjecture is true. We first recall the construction of Courtade, Fathi and Pananjady [17]. For every integer k ≥ 1, we define W^{1,2}_k(μ) as the closure of the set of all compactly supported smooth functions h : ℝ^d → ℝ^k in L^2(μ), with respect to the norm ∫ (∫ h|^2 + ∫ |∇h|^2 |_(H.S.) dμ. We regard W^{1,2}_k(μ) as a Hilbert space equipped with this norm. We also write W^{1,2}_k(μ) for the set of functions h ∈ W^{1,2}_k(μ) with ∫ h dμ = 0. If τ is a Stein kernel for μ and ∫ (x|^2 + ∫ |τ(x)|^2 |_(H.S.) dμ < ∞, it is evident by definition that (3.3) holds for any h ∈ W^{1,2}_d(μ).

Theorem 3.1 (Courtade, Fathi and Pananjady [17], Theorem 2.4). If ω(μ) < ∞, there exists a unique function ψ_μ ∈ W^{1,2}_d,0(μ) such that τ_μ := ∇ψ_μ is a Stein kernel for μ.

Lemma 3.2. Under the assumptions of Theorem 3.1, if X ∼ μ, then

E[τ_μ(X)^T u]^2 ≤ ω(μ)E[X · u]^2

for all u ∈ ℝ^d.

Proof. Observe that ω(μ) < ∞ implies E|X|^2 < ∞. Hence, we can apply (3.3) with τ = τ_μ and h(x) = uu^T ψ_μ(x), which yields

E[X · uu^T ψ_μ(X)] = E[(τ_μ(X)), uu^T τ_μ(X)]_{H.S.}

= E[tr(τ_μ(X)^T uu^T τ_μ(X))] = E[τ_μ(X)^T u]^2.

Hence, by the Cauchy–Schwarz inequality,

E[τ_μ(X)^T u]^2 ≤ E[X · u]^2E|^u^T ψ_μ(X)|^2.

By a standard approximation argument, one can easily verify that (1.3) with ω = ω(μ) holds for any h ∈ W^{1,2}_1(μ). Applying this inequality with h(x) = u^T ψ_μ(x), we obtain

E|u^T ψ_μ(X)|^2 ≤ ω(μ)E|u^T τ_μ(X)|^2 = ω(μ)E|τ_μ(X)^T u|^2.

Consequently,

E|τ_μ(X)^T u|^2 ≤ ω(μ)E|X · u|^2E|^τ_μ(X)^T u|^2.

This yields the desired result. □

A drawback of Courtade, Fathi and Pananjady [17]’s construction is that it is based on the Lax–Milgram theorem and thus implicit. So it is generally difficult to control moments higher than two. Fortunately, this is not the case when μ is uniformly log-concave: We can find an explicit representation of the function ψ_μ in Theorem 3.1, which allows us to get a dimension-free bound for ||τ_μ||_{op}:

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Lemma 3.3. Let $\mu$ be a centered probability measure on $\mathbb{R}^d$. Suppose that $\mu$ has a smooth, positive and $\varepsilon$-uniformly log-concave density for some $\varepsilon > 0$. Then the function $\tau_\mu$ in Theorem 3.1 satisfies $\|\tau_\mu\|_{op} \leq \varepsilon^{-1} \mu$-a.s.

Proof. By assumption, the density of $\mu$ is of the form $e^{-V}$ with $V : \mathbb{R}^d \to \mathbb{R}$ a $C^\infty$ function such that $\text{Hess} V \succeq \varepsilon I_d$. For every $x \in \mathbb{R}^d$, consider the following stochastic differential equation (SDE):

$$X^x_0 = x, \quad dX^x_t = -\nabla V(X^x_t) dt + \sqrt{2} dB_t, \quad t \geq 0. \quad (3.5)$$

This SDE has a unique strong solution. To see this, note that $-x \cdot \nabla V(x) \leq V(0) - V(x)$ for all $x \in \mathbb{R}^d$ because $V$ is convex. Hence, by Theorem 5.1 in Saumard and Wellner [45], there exists a constant $c > 0$ such that $-x \cdot \nabla V(x) \leq c$ for all $x \in \mathbb{R}^d$. Also, note that $\nabla V$ is locally Lipschitz. Therefore, the SDE (3.5) has a unique strong solution by Theorems 3.7 and 3.11 in Ethier and Kurtz [24, Chapter 5].

Let $X^x = (X^x_t)_{t \geq 0}$ be the solution to (3.5). Below we show that the function $x \mapsto \psi_t(x) := \int_0^t \mathbb{E} X^x_s ds$ converges to some function $\psi$ in the space $W^{1,2}_d(\mu)$ as $t \to \infty$, and $\psi = \psi_\mu$ $\mu$-a.s. Indeed, this fact follows from the general result of Arras and Houdré [2, Theorem 5.10] based on Dirichlet form theory (see also Arras and Houdré [3, Proposition 3.1]), but we give a proof without referring to Dirichlet forms for readers’ convenience.

Let $(T_t)_{t \geq 0}$ be the transition semigroup on $L^2(\mu)$ associated with the SDE (3.5). It is well-known that its generator is given by

$$L = -\nabla V \cdot \nabla + \nabla \cdot \nabla,$$

where $\nabla \cdot \nabla$ denotes the Laplacian. Using integration by parts, we have for any compactly supported smooth functions $g : \mathbb{R}^d \to \mathbb{R}$ and $h : \mathbb{R}^d \to \mathbb{R}$

$$\int_{\mathbb{R}^d} gLh d\mu = - \int_{\mathbb{R}^d} \nabla g \cdot \nabla h d\mu. \quad (3.6)$$

From this identity we obtain $\text{Dom}(L) \subset W^{1,2}_d(\mu)$. Also, by definition, (3.6) also holds for any $g \in W^{1,2}_d(\mu)$ and $h \in \text{Dom}(L)$. Further, by Proposition 9.2 in Ethier and Kurtz [24, Chapter 4], (3.6) implies that $\mu$ is a stationary distribution for $(T_t)_{t \geq 0}$.

Next we show that

$$\int |T_t g|^2 d\mu \leq e^{-2t/\varepsilon(\mu)} \int g^2 d\mu \quad (3.7)$$

for any $t \geq 0$ and $g \in L^2(\mu)$ with $\int g d\mu = 0$. Since $\text{Dom}(L)$ is dense in $L^2(\mu)$, it suffices to prove (3.7) when $g \in \text{Dom}(L)$. Then, by Proposition 1.5(b) in Ethier and Kurtz [24, Chapter 1],

$$\frac{d}{dt} T_t g = LT_t g. \quad (3.8)$$

Therefore,

$$\frac{d}{dt} \int |T_t g|^2 d\mu = 2 \int T_t g \cdot LT_t g d\mu = -2 \int |\nabla T_t g|^2 d\mu,$$
where the first identity follows from (3.8) and the second from (3.6). Since \( \int T_1 g d\mu = \int g d\mu = 0 \), we have
\[
\int |T_1 g|^2 d\mu \leq \varpi(\mu) \int |\nabla T_1 g|^2 d\mu.
\]

Consequently,
\[
\frac{d}{dt} \int |T_1 g|^2 d\mu \leq -\frac{2}{\varpi(\mu)} \int |T_1 g|^2 d\mu.
\]

Thus, we obtain (3.7) by Gronwall’s inequality.

Applying (3.7) with \( g(x) = x_j \) for every \( j = 1, \ldots, d \), we obtain
\[
\int |\mathbb{E}X_t^x|^2 \mu(dx) \leq e^{-2t/\varpi(\mu)} \int |x|^2 \mu(dx). \tag{3.9}
\]

Hence, for \( 0 < s < t \),
\[
\int |\psi_t - \psi_s|^2 d\mu \leq \int (t - s) \left( \int_s^t |\mathbb{E}X_u^x|^2 du \right) \mu(dx)
\leq \varpi(\mu)(t - s)(e^{-2s/\varpi(\mu)} - e^{-2t/\varpi(\mu)}) \int |x|^2 \mu(dx).
\]

Thus \( \int |\psi_t - \psi_s|^2 d\mu \to 0 \) as \( s, t \to \infty \). Meanwhile, by Theorem 39 in Protter [43, Chapter V], we can take a version of \( X^x \) such that the map \( x \mapsto X^x_t \) is differentiable for any \( t \geq 0 \) and its derivative \( \nabla_x X^x_t = (\partial_{x_1} X^x_t, \ldots, \partial_{x_d} X^x_t) \) satisfies
\[
\nabla_x X^x_t = I_d - \int_0^t \text{Hess } V(X^x_s) \nabla_x X^x_s ds.
\]

From this equation we can prove
\[
\| \nabla_x X^x_t \|_{op} \leq e^{-\varepsilon t}. \tag{3.10}
\]

To see this, fix \( u \in \mathbb{R}^d \) and set \( D_t = \nabla_x X^x_t u \). Then we have
\[
\frac{d}{dt} |D_t|^2 = 2 D_t \cdot \frac{d}{dt} D_t = -2D_t \text{Hess } V(X^x_t) D_t \leq -2\varepsilon |D_t|^2,
\]
where the last inequality follows from \( \text{Hess } V \succeq \varepsilon I_d \). Hence we obtain \( |D_t|^2 \leq |D_0|^2 e^{-2\varepsilon t} = |u|^2 e^{-2\varepsilon t} \) by Gronwall’s inequality. This implies (3.10). By (3.10), \( \psi_t \) is differentiable and \( \nabla \psi_t = \int_0^t \mathbb{E} \nabla_x X^x_s ds \). Hence, for \( 0 < s < t \),
\[
\int \| \nabla \psi_t - \nabla \psi_s \|_{H.S.}^2 d\mu \leq \int \left( \int_s^t \| \mathbb{E} \nabla_x X^x_u \|_{H.S.} d\mu \right)^2 \mu(dx)
\leq d \left( \int_s^t e^{-\varepsilon u} du \right)^2 \frac{d(e^{-\varepsilon s} - e^{-\varepsilon t})^2}{\varepsilon^2}.
\]

So \( \int \| \nabla \psi_t - \nabla \psi_s \|_{H.S.}^2 d\mu \to 0 \) as \( s, t \to \infty \). Consequently, \( \psi_t \) converges to some function \( \psi \) in the space \( W_d^{1,2}(\mu) \) as \( t \to \infty \).
To prove $\psi = \psi_\mu \mu$-a.s., we need to check $\int \psi d\mu = 0$ and $\nabla \psi$ is a Stein kernel for $\mu$. The former is immediate because $\int \mathbb{E}X_t^\mu dx = \int x\mu(dx) = 0$, where the first identity holds because $\mu$ is a stationary distribution for $(T_t)_{t\geq 0}$. Meanwhile, by Proposition 1.5(a) in Ethier and Kurtz [24, Chapter 1],
\[ \mathbb{E}X_t^\mu - x = L\psi_t(x) \mu\text{-a.s.,} \]
where the operator $L$ is applied to $\psi_t$ coordinate-wise. Thus, by (3.6), for any $g \in W^{1,2}_d(\mu)$,
\[ \int g(x) \cdot (\mathbb{E}X_t^\mu - x)\mu(dx) = -\int \nabla \psi_t \cdot \nabla g\mu. \]
By (3.9), $\int \|\mathbb{E}X_t^\mu\|^2\mu(dx) \to 0$ as $t \to \infty$. Therefore, letting $t \to \infty$ in the above identity, we obtain
\[ \int g(x) \cdot x\mu(dx) = \int \nabla \psi \cdot \nabla g\mu. \]
So $\nabla \psi$ is a Stein kernel for $\mu$. All together, $\psi = \psi_\mu \mu$-a.s.

Finally, since $\|\nabla \psi_t\|_{op} \leq \int_t^0 e^{-\varepsilon s}ds$ for all $t \geq 0$, we have $\|\nabla \psi\|_{op} \leq \varepsilon^{-1}$ $\mu$-a.s. This completes the proof. \hfill \Box

**Remark 3.2.** (a) The operator norm bound as in Lemma 3.3 holds for the other constructions mentioned above; see Fathi [27, Corollary 2.4] and Mikulincer and Shenfeld [38, Theorem 3.1(1-a)] What is important for the construction by Courtade, Fathi and Pananjady [17] is that it also satisfies the estimate in Lemma 3.2, i.e. a (nicely) dimension-free bound for $\mathbb{E}r_\mu(X)^\top u^2$ in the general log-concave case. This type of estimate plays a key role in the proof of Proposition 3.2 below because we need a precise estimate for the variance of $\tau_\mu(X)^\top u$ to derive an appropriate bound from an application of Rosenthal’s inequality. As discussed above, such an estimate is currently unavailable for other constructions.

(b) Fathi [27]'s construction is known to ensure the positive definiteness of the Stein kernel. This property is important in some applications; see Fathi and Mikulincer [28]. It is unclear whether the construction by Courtade, Fathi and Pananjady [17] always has this property. The proof of Lemma 3.3 implies that this happens when $f_0^t \mathbb{E}[\nabla X_s^\mu]ds$ is positive definite for sufficiently large $t$, but verification of this condition is not straightforward.

(c) As we mentioned in the proof of Lemma 3.3, the explicit representation of $\tau_\mu$ is already given in Arras and Houdré [2]; see Remark 5.11(ii) ibidem. Indeed, the bound $\|\tau_\mu\|_{op} \leq \varepsilon^{-1}$ also follows from this representation and gradient bounds for diffusion semigroups, e.g. Proposition 3.2.5 of Bakry, Gentil and Ledoux [4].

Combining these observations give the following upper bound for (3.4):

**Proposition 3.2.** Let $X_1, \ldots, X_n$ be centered independent random vectors in $\mathbb{R}^d$. Suppose that $X_i$ has a smooth, positive and $\varepsilon$-uniformly log-concave density for all $i = 1, \ldots, n$ and some $\varepsilon > 0$. Then, for any $d \times d$ matrix $A$, $W := n^{-1/2} \sum_{i=1}^n AX_i$ has a Stein kernel $\tau$ satisfying
\[
\|\tau(W)^\top - I_d\|_p \leq C\|A\|_{op} \frac{\sqrt{p \sum_{i=1}^n X_i u^\top u + \frac{p}{\varepsilon} A^\top u}}{n} \left( \left\| \frac{1}{n} \sum_{i=1}^n \Sigma_i - I_d \right\| u \right).
\]

(3.11)
for all \( u \in \mathbb{R}^d \) and \( p \geq 1 \), where \( \Sigma_i = A \text{Cov}(X_i)A^\top \).

**Proof.** For every \( i = 1, \ldots, n \), let \( \tau_i \) be the Stein kernel for \( X_i \) given by Theorem 3.1. By Lemma 3.3, \( \| \tau_i(X_i) \|_{op} \leq \varepsilon^{-1} \) a.s. Let

\[
\tau(w) := \frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^{n} A\tau_i(X_i)A^\top | W = w \right].
\]

It is straightforward to check that \( \tau \) is a Stein kernel for \( W \). We show that this \( \tau \) satisfies (3.11). Thanks to Jensen’s inequality, it suffices to prove (3.11) when \( p \geq 2 \). Since \( \mathbb{E}\tau_i(X_i) = \text{Cov}(X_i) \) for every \( i \) by the definition of Stein kernel, we have

\[
\| (\tau(W)^\top - I_d)u \|_p \leq \| (\tau(W)^\top - \mathbb{E}\tau(W)^\top)u \|_p + \left( \frac{1}{n} \sum_{i=1}^{n} \Sigma_i - I_d \right) u.
\]

By Jensen’s inequality,

\[
\| (\tau(W)^\top - \mathbb{E}\tau(W)^\top)u \|_p \leq \frac{1}{n} \left\| \sum_{i=1}^{n} A\{\tau_i(X_i)^\top - \mathbb{E}\tau_i(X_i)^\top\}A^\top u \right\|_p \leq \frac{\|A\|_{op} \| \sum_{i=1}^{n} \{\tau_i(X_i)^\top - \mathbb{E}\tau_i(X_i)^\top\} A^\top u \|_p}{n}.
\]

Applying Rosenthal’s inequality for random vectors in \( \mathbb{R}^d \) (cf. Eq.(4.2) in Pinelis [42]), we obtain

\[
\| (\tau(W)^\top - \mathbb{E}\tau(W)^\top)u \|_p \leq C \|A\|_{op} \left( \frac{p \sum_{i=1}^{n} \mathbb{E}\{\tau_i(X_i)^\top - \mathbb{E}\tau_i(X_i)^\top\}A^\top u |^2 + p \max_{1 \leq i \leq n} \| \{\tau_i(X_i)^\top - \mathbb{E}\tau_i(X_i)^\top\} A^\top u \|_p}{n} \right),
\]

where we used Lemma 3.2, \( \|\tau_i(X_i)\|_{op} \leq \varepsilon^{-1} \) a.s. and Jensen’s inequality to get the last line. Since \( \mathbb{E}|A^\top u \cdot X_i|^2 = (A^\top u)^\top \text{Cov}(X_i)A^\top u = u^\top \Sigma_i u \), we complete the proof. \( \square \)

The results obtained so far will be used in the following form:

**Proposition 3.3.** Let \( A \) be a \( d \times d \) matrix and \( (f_y)_{y \in \mathbb{R}^d} \) be a family of smooth, positive and \( \varepsilon \)-uniformly log-concave densities on \( \mathbb{R}^d \) with some constant \( \varepsilon > 0 \). Suppose that the map \( \mathbb{R}^d \times \mathbb{R}^d \ni (y, x) \mapsto f_y(x) \in [0, \infty) \) is measurable. For every \( y \in \mathbb{R}^d \), let \( \nu_y \) be the law of \( \xi - \mathbb{E}\xi \) with \( \xi \) a random vector in \( \mathbb{R}^d \) having density \( f_y \). Then, there exists a Markov kernel \( Q \) from \( (\mathbb{R}^d)^n \) to \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying the following conditions for any \( y = (y^{(i)})_{i=1}^{n} \in (\mathbb{R}^d)^n \):

(i) \( Q(y, \cdot \times \mathbb{R}^d) \) equals the law of \( A\{\sum_{i=1}^{n} \xi_i \} \), where \( \xi_1, \ldots, \xi_n \) are independent random vectors in \( \mathbb{R}^d \) such that \( L(\xi_i) = \nu_{y^{(i)}} \) for all \( i = 1, \ldots, n \).
(ii) $Q(y, \mathbb{R}^d \times \cdot)$ is the $d$-dimensional standard normal distribution.

(iii) If $W'$ and $Z'$ are random vectors in $\mathbb{R}^d$ such that $(W', Z') \sim Q(y, \cdot)$, then

$$
\|u \cdot (W' - Z')\|_p \leq \frac{C\|A\|_{op}}{n} \left( \sqrt{\frac{1}{n}} \sum_{i=1}^{n} \| \nu_{y^{(i)}} \| u^\top A \text{Cov}(f_{y^{(i)}}) A^\top u + \frac{p^{3/2}}{\varepsilon} |A^\top u| \right) + C\sqrt{p} \left( \frac{1}{n} \sum_{i=1}^{n} A \text{Cov}(f_{y^{(i)}}) A^\top - I_d \right) u
$$

for any $p \geq 1$ and $u \in \mathbb{R}^d$.

Proof. For every $y = (y^{(i)})_{i=1}^{n} \in (\mathbb{R}^d)^n$ and every Borel set $E \subset \mathbb{R}^d$, define $P(y, E) = P(An^{-1/2} \sum_{i=1}^{n} \xi_i \in E)$, where $\xi_1, \ldots, \xi_n$ are the same as in condition (i). Then, $P$ defines a Markov kernel from $(\mathbb{R}^d)^n$ to $\mathbb{R}^d$. Applying Proposition 3.1 to $P$, we can construct a Markov kernel $Q$ from $(\mathbb{R}^d)^n$ to $\mathbb{R}^d \times \mathbb{R}^d$ satisfying conditions (i) and (ii) for all $y \in (\mathbb{R}^d)^n$ and

$$
\|u \cdot (W' - Z')\|_p \leq \int_0^1 \frac{1}{\sqrt{t}} \|u \cdot \rho_{W' \mid t}(W' \mid t)\|_{p} dt
$$

with $(W', Z') \sim Q(y, \cdot)$ for any $p \geq 1$ and $u \in \mathbb{R}^d$. By Proposition 3.2, $W'$ has a Stein kernel $\tau$ satisfying

$$
\|\tau(W)^\top - I_d\| u\|_p \leq \frac{C\|A\|_{op}}{n} \left( \sqrt{\frac{1}{n}} \sum_{i=1}^{n} \| \nu_{y^{(i)}} \| u^\top A \text{Cov}(f_{y^{(i)}}) A^\top u + \frac{p^{3/2}}{\varepsilon} |A^\top u| \right) + \left( \frac{1}{n} \sum_{i=1}^{n} A \text{Cov}(f_{y^{(i)}}) A^\top - I_d \right) u.
$$

Combining these bounds with Lemma 3.1 shows that condition (iii) is satisfied. \qed

### 3.3 Proof of Theorem 2.2

First we prove the claim when $\mu$ has a smooth, positive density $f$. In this case, by Proposition 3.3 with $A = I_d$ and $f_y \equiv f$, we can construct random vectors $W$ and $Z$ in $\mathbb{R}^d$ such that $W = \frac{d}{n^{1/2}} \sum_{i=1}^{n} X_i$ with $X_i \overset{i.i.d.}{\sim} \mu$, $Z \sim \mathcal{N}(0, I_d)$ and

$$
\|u \cdot (W - Z)\|_p \leq \frac{C|u|\sqrt{p}}{n} \left( \sqrt{\frac{p}{n\overline{\nu}(\mu)}} + \frac{p}{\varepsilon} \right)
$$

for all $u \in \mathbb{R}^d$ and $p \geq 1$. By the Brascamp–Lieb inequality (see e.g. Proposition 10.1 in Saumand and Wellner [45]), we have $\overline{\nu}(\mu) \leq \varepsilon^{-1}$. Hence (2.2) holds.

For the general case, take a constant $a \in (0, 1)$ arbitrarily, and let $\mu^{a}$ be the law of the random vector $\sqrt{1 - a}X + \sqrt{a}G$, where $X \sim \mu$ and $G \sim \mathcal{N}(0, I_d)$ are independent. Clearly, $\mu^{a}$ is isotropic and has a smooth, positive density. Also, by Theorem 3.7 in Saumand and Wellner [45], $\mu^{a}$ is $\varepsilon/(1 - (1 - \varepsilon)a)$-uniformly log-concave. Hence we can construct random
vectors \( W^a \) and \( Z^a \) in \( \mathbb{R}^d \) such that \( W^a \overset{d}{=} n^{-1/2} \sum_{i=1}^{n} X_i \) with \( X_i \overset{i.i.d.}{\sim} \mu^a \), \( Z^a \sim N(0, I_d) \) and
\[
\|u \cdot (W^a - Z^a)\|_p \leq C|u| \left( \frac{p}{\sqrt{\varepsilon n}} + \frac{p^{3/2}}{\varepsilon n} \right)
\]
for all \( u \in \mathbb{R}^d \) and \( p \geq 1 \). In particular, the family \( \{(W^a, Z^a) : a \in (0, 1)\} \) is tight, so by Prohorov’s theorem there exists a sequence \((a_k)_{k=1}^{\infty}\) of numbers in \((0, 1)\) such that \( a_k \to 0 \) and \((W^{a_k}, Z^{a_k})\) converges in law to some pair \((W, Z)\) of random vectors in \( \mathbb{R}^d \) as \( k \to \infty \).

It is clear that \( W \overset{d}{=} n^{-1/2} \sum_{i=1}^{n} X_i \) with \( X_i \overset{i.i.d.}{\sim} \mu \) and \( Z \sim N(0, I_d) \). Also, by Theorem 3.4 in Billingsley [7],
\[
\|u \cdot (W - Z)\|_p \leq \liminf_{k \to \infty} \|u \cdot (W^{a_k} - Z^{a_k})\|_p \leq C|u| \left( \frac{p}{\sqrt{\varepsilon n}} + \frac{p^{3/2}}{\varepsilon n} \right)
\]
for all \( u \in \mathbb{R}^d \) and \( p \geq 1 \). So \( W \) and \( Z \) are desired ones. \( \square \)

### 3.4 Stochastic localization

A naive idea to use Proposition 3.2 for the proof of Theorem 2.1 is to approximate a log-concave distribution by a uniformly log-concave one. To be precise, given a log-concave random vector \( X \) in \( \mathbb{R}^d \) and a positive constant \( \varepsilon \), we wish to construct an \( \varepsilon \)-uniformly log-concave random vector \( X^\varepsilon \) such that \( \|u \cdot (X - X^\varepsilon)\|_p = O(\varepsilon) \) as \( \varepsilon \downarrow 0 \) for all \( u \in \mathbb{R}^d \) and \( p \geq 2 \). If this is possible, it is not difficult to see that we can obtain a bound of order \( 1/\sqrt{n} \) for (3.4) via Proposition 3.2. However, this approach seems hopeless because the currently best known bound for \( W_p(X, X^\varepsilon) \) is presumably the one given by Proposition 1 in Dalalyan, Karagulyan and Riou-Durand [18] and it is of order \( \varepsilon^{1/p} \). Instead, we take an alternative idea of using Eldan’s stochastic localization that enables us to express a log-concave distribution as a mixture of uniformly log-concave distributions and some “nice” distribution. The latter can be handled by the martingale embedding method developed in Eldan, Mikulincer and Zhai [23]. Below we detail this strategy.

Throughout this subsection, we assume \( d \geq 2 \). Let \( \mu \) be an isotropic log-concave probability measure on \( \mathbb{R}^d \). Suppose that \( \mu \) has a smooth, positive density \( f \) with respect to \( N(0, I_d) \). Consider the following SDE:
\[
Y_0 = 0, \quad dY_t = \nabla \log(P_{t-I}f)(Y_t)dt + dB_t, \quad t \in [0, 1], \tag{3.12}
\]
where \( B_t \) is a \( d \)-dimensional Brownian motion. By Theorem 2.1 in Eldan and Lee [20], this SDE has a weak solution \( Y = (Y_t)_{t \in [0, 1]} \) such that \( Y_1 \sim \mu \). Moreover, we can show that the solution to (3.12) is unique in law; see Theorem A.1 in Appendix A. \( Y \) is known as the Föllmer process in the literature.

For \( t \in (0, 1) \) and \( y \in \mathbb{R}^d \), define a probability density function \( f_{t,y} : \mathbb{R}^d \to [0, \infty) \) as
\[
f_{t,y}(x) = \frac{f(x)\phi_{tI_d}(x-y)}{P_{tI}f(y)} = \frac{f(x)e^{-|x-y|^2/(2t)}}{\int_{\mathbb{R}^d} f(z)e^{-|z-y|^2/(2t)}dz}, \quad x \in \mathbb{R}^d,
\]
where \( \phi_{tI_d} \) is the (Lebesgue) density of \( N(0, tI_d) \). For every \( t \in (0, 1) \), the conditional law of \( Y_1 \) given \( F_t := \sigma(Y_s : 0 \leq s \leq t) \) has density \( f_{1-t,Y_1} \). In fact, by Eq.(21) in Eldan and
Lee [20], there exists a probability measure $Q$ such that $Y$ is a $d$-dimensional standard Brownian motion under $Q$ and

$$P(Y_1 \in A|F_t) = \mathbb{E}_Q \left[ 1_A(Y_1) \frac{f(Y_1)}{P_{1-t}f(Y_1)} |F_t \right]$$

for every Borel set $A \subset \mathbb{R}^d$. Under $Q$, the conditional law of $Y_1$ given $F_t$ is $N(Y_t, (1-t)I_d)$. Consequently,

$$P(Y_1 \in A|F_t) = \int_{\mathbb{R}^d} 1_A(x) \frac{f(x)}{P_{1-t}f(Y_t)} \phi(1-t)I_d(x-Y_t)dx = \int_{A} f_{1-t,Y_t}(x)dx.$$

So the desired result follows. In particular, we have $\mathbb{E}[Y_1|F_t] = m(t, Y_t)$, where

$$m(t, y) := \int_{\mathbb{R}^d} x f_{1-t,y}(x)dx.$$

Since $\mu$ is log-concave, $\text{Hess}(\log f) \preceq I_d$, hence $\text{Hess}(\log f_{1-t,y}) \preceq -\frac{t}{1-t}I_d$ for all $t \in (0,1)$ and $y \in \mathbb{R}^d$. This means that $f_{1-t,y}$ is $t/(1-t)$-uniformly log-concave. Thus, conditional on $F_t$, $Y_1 - m(t, Y_t)$ is centered and $t/(1-t)$-uniformly log-concave, so the sum of its independent copies would be coupled with a suitable Gaussian vector by Proposition 3.3. Therefore, if we can couple the sum of independent copies of $m_t := m(t, Y_t)$ with a suitable Gaussian vector for a moderately small $t$, the proof of Theorem 2.1 will be complete. In this section, we accomplish this by a version of the martingale embedding method of Eldan, Mikulincer and Zhai [23].

Let $(Y^{(1)}, B^{(1)}), \ldots, (Y^{(n)}, B^{(n)})$ be independent copies of $(Y, B)$. By definition, $Y^{(i)}$ satisfies the following SDE for every $i = 1, \ldots, n$:

$$Y^{(i)}_0 = 0, \quad dY^{(i)}_t = \nabla \log(P_{1-t}f)(Y^{(i)}_t)dt + dB^{(i)}_t, \quad t \in [0,1].$$

Define $m^{(i)}_t := m(t, Y^{(i)}_t)$.

**Proposition 3.4.** There exists a universal constant $c_0 \geq 3$ such that, for any $0 \leq \varepsilon \leq (c_0 \varpi_d \log(2d))^{-1}$, we can construct a random vector $Z_{\varepsilon} \sim N(0, \text{Cov}(m_{\varepsilon}))$ on the same probability space where $Y^{(1)}, \ldots, Y^{(n)}$ are defined and such that

$$\left\| u \cdot \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n m^{(i)}_{\varepsilon} - Z_{\varepsilon} \right) \right\|_p \leq C\|u\| \varepsilon \left( \frac{p}{\sqrt{n}} + \frac{p^{5/2}}{n} \right)$$

for all $u \in \mathbb{R}^d$ and $p \geq 1$.

For the proof of Proposition 3.4, we use the following stochastic integral representation of $m_t$:

$$m_t = \int_0^t \Gamma_s dB_s, \quad \text{where} \quad \Gamma_t = \frac{\text{Cov}(f_{1-t,Y_t})}{1-t}.$$  

This representation is found in Eldan, Mikulincer and Zhai [23, Section 4], but we can directly verify it using Ito’s formula; see Lemma 13 in Eldan, Lehec and Shenfeld [21] for details.

We collect some properties of $\Gamma_t$ necessary for our proof. The first one is a lower bound of $\mathbb{E}\Gamma_t$:  

```
Lemma 3.4 (Eldan and Mikulincer [22], Corollary 3). If \( t \leq \frac{1}{2\omega(\mu) + 1} \), then \( \mathbb{E} \Gamma_t \succeq I_d/3 \).

To get moment bounds for \( \|\Gamma_t\|_{op} \), we employ a recent result of Klartag and Lehec [32]. For this purpose, we relate our notation to theirs. For \( t > 0 \) and \( \theta \in \mathbb{R}^d \), we define a probability density function \( p_{t,\theta} : \mathbb{R}^d \to [0, \infty) \) as

\[
p_{t,\theta}(x) = f_{1/(1+t),\theta/(1+t)}(x) = \frac{f(x)e^{-|x|^2/2}e^{\theta \cdot x - t|x|^2/2}}{\int_{\mathbb{R}^d} f(z)e^{-|z|^2/2}e^{\theta \cdot z - t|z|^2/2} dz}, \quad x \in \mathbb{R}^d.
\]

Note that this is the same notation as in Klartag and Lehec [32] because \( f \) is the density of \( \mu \) with respect to \( N(0, I_d) \). Then, as in Klartag and Lehec [32], set

\[
a(t, \theta) = \int_{\mathbb{R}^d} xp_{t,\theta}(x)dx, \quad A(t, \theta) = \text{Cov}(p_{t,\theta}).
\]

Also, define

\[
\tilde{B}_t = \int_0^{t/(1+t)} \frac{1}{1 - s} dB_s, \quad \theta_t = (1 + t)Y_{t/(1+t)}, \quad t \geq 0.
\]

Lemma 3.5. \((\tilde{B}_t)_{t \geq 0}\) is a standard Brownian motion in \( \mathbb{R}^d \). Moreover, \((\theta_t)_{t \geq 0}\) satisfies the following SDE:

\[
d\theta_t = a(t, \theta_t)dt + d\tilde{B}_t, \quad \theta_0 = 0. \tag{3.13}
\]

In addition, the solution to (3.13) is unique in law.

Proof. This fact is pointed out in Klartag and Putterman [33, Section 4.2]. We give a formal proof for the sake of completeness. First, we can easily check that the quadratic covariation matrix process of \((\tilde{B}_t)_{t \geq 0}\) is given by \([\tilde{B}, \tilde{B}]_t = tI_d\); hence, \((\tilde{B}_t)_{t \geq 0}\) is a standard Brownian motion in \( \mathbb{R}^d \) by Lévy’s characterization (see e.g. Theorem 40 in Protter [43, Chapter II]). Next, by a direct calculation, we have for any \( t \in (0, 1) \) and \( y \in \mathbb{R}^d \)

\[
\nabla \log(P_{1-t}f)(y) = \frac{m(t, y) - y}{1 - t}.
\]

Hence

\[
Y_t - B_t = \int_0^t \frac{m(s, Y_s) - Y_s}{1 - s} ds.
\]

Also, integration by parts gives

\[
\frac{Y_t}{1 - t} = \int_0^t \frac{1}{1 - s} dY_s + \int_0^t \frac{Y_s}{(1 - s)^2} ds.
\]

Consequently, for any \( t \geq 0 \),

\[
\tilde{B}_t = \int_0^{t/(1+t)} \frac{1}{1 - s} dB_s - \int_0^{t/(1+t)} \frac{m(s, Y_s) - Y_s}{(1 - s)^2} ds
\]

\[
= \theta_t - \int_0^{t/(1+t)} \frac{m(s, Y_s)}{(1 - s)^2} ds = \theta_t - \int_0^t m(u/(1 + u), Y_{u/(1+u)}) du.
\]
Since \(a(u, \theta) = m(u/(1+u), \theta/(1+u))\) for \(u > 0\) and \(\theta \in \mathbb{R}^d\) by definition, \((\theta_t)_{t \geq 0}\) satisfies (3.13).

Conversely, if \((\theta_t)_{t \geq 0}\) is a solution to (3.13) with a standard Brownian motion \((\tilde{B}_t)_{t \geq 0}\) in \(\mathbb{R}^d\), in a similar manner to the above, we can verify that
\[
B_t = \int_0^{t/(1-t)} \frac{1}{1+s} d\tilde{B}_s, \quad t \in [0, 1),
\]
is a standard Brownian motion in \(\mathbb{R}^d\) and \(Y_t = (1-t)\theta_t/(1-t)\) satisfies (3.12) for \(t \in [0, 1)\). Hence uniqueness in law for (3.13) follows from that for (3.12).

Thanks to Lemma 3.5, the process \((\theta_t)_{t \geq 0}\) has the same law as the one defined in Klartag and Lehec [32]; see Eq. (18) ibidem.

Let
\[
\kappa_d := \sup_{\mu \in \mathcal{LC}_d} \left\| \int_{\mathbb{R}^d} x_1 x_2^\top \mu(dx) \right\|_{H.S.},
\]
By Fact 6.1 in Eldan [19], there exists a positive universal constant \(C_0 > 0\) such that
\[
\kappa_d^2 \leq C_0 \omega_d.
\]

**Lemma 3.6.** There exist positive universal constants \(C\) and \(c\) such that
\[
\|\|\Gamma_s\|\|_p \leq Cp
\]
for any \(0 < s \leq (1/2) \wedge (c\kappa_d^2 \cdot \log d)^{-1}\).

**Proof.** Let \(A_t = A(t, \theta_t)\). By the proof of Klartag and Lehec [32, Corollary 5.4], there exist positive universal constants \(C\) and \(c\) such that
\[
E\|A_t\|_p^p \leq 2^p + Cp^p!
\]
for any \(0 < t \leq (c\kappa_d^2 \cdot \log d)^{-1}\). Next, recall that \(A_t = \text{Cov}(f_1/(1+t), Y_t/(1+t))\). Hence, for any \(s \in (0, 1)\), \(A_{s/(1-s)} = \text{Cov}(f_1-s, Y_s)\). Therefore, for \(0 < s \leq (1/2) \wedge (2c\kappa_d \cdot \log d)^{-1}\),
\[
E\|\Gamma_s\|_p^p = \frac{E\|A_{s/(1-s)}\|_p^p}{(1-s)^p} \leq 4^p + (2C)^p p^p! \leq 4^p + (2C)^p p!.
\]
So we obtain the desired result. \(\square\)

We will also need the notion of matrix geometric mean. For two positive definite matrices \(A\) and \(B\), their geometric mean is defined as
\[
A \# B := \sqrt{A} \sqrt{A^{-1/2}BA^{-1/2}} \sqrt{A}.
\]
\(A \# B\) is evidently positive definite. Also, we have \(A \# B = B \# A\) by Theorem 4.1.3 in Bhatia [6]. The matrix geometric mean is useful because of the following lemma.

**Lemma 3.7.** Let \(A\) and \(B\) be two \(d \times d\) positive definite matrices. Then
\[
A + B - 2A \# B \leq (A - B)A^{-1}(A - B).
\]
Proof. Using the definition of $A \# B$, we obtain
\[
A + B - 2A \# B = \sqrt{A} \left( I_d + A^{-1/2}BA^{-1/2} - 2\sqrt{A^{-1/2}BA^{-1/2}} \right) \sqrt{A}
\]
\[
= \sqrt{A} \left( I_d - A^{-1/2}BA^{-1/2} \right)^2 \sqrt{A}
\]
\[
\leq \sqrt{A} \left( I_d - A^{-1/2}BA^{-1/2} \right)^2 \sqrt{A}
\]
\[
= (\sqrt{A} - BA^{-1/2})(\sqrt{A} - A^{-1/2}B) = (A - B)A^{-1}(A - B).
\]

\[\square\]

Proof of Proposition 3.4. Recall that we assume $d \geq 2$, so $\log(2d) > 1$. Also, note that $\varpi_d \geq 1$ because $\varpi(N(0, I_d)) = 1$. Let $C_0$ and $c$ be the universal constants in (3.14) and Lemma 3.6, respectively. Then, we take $c_0 := \max\{cC_0, 3\}$. By construction, we have $c_0 \varpi_d \log(2d) \geq \max\{c\kappa^2 d \log d, 2\varpi_d + 1\}$.

For every $i = 1, \ldots, n$, we can write
\[
m_t^{(i)} = \int_0^t \Gamma_s^{(i)} dB_s^{(i)}, \quad \text{where} \quad \Gamma_t^{(i)} = \frac{\text{Cov}(f_1 - tX_i)}{1 - t}.
\]

Consider a continuous local martingale $M = (M_t)_{t \in [0, 1]}$ in $\mathbb{R}^d$ defined as
\[
M_t = \frac{1}{\sqrt{n}} \sum_{i=1}^n m_t^{(i)} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \Gamma_s^{(i)} dB_s^{(i)}, \quad t \in [0, 1].
\]

The quadratic covariation matrix process of $M$ is given by
\[
[M, M]_t = \frac{1}{n} \sum_{i=1}^n \int_0^t (\Gamma_s^{(i)})^2 ds = \int_0^t \bar{\Gamma}_s^2 ds,
\]
where
\[
\bar{\Gamma}_s := \sqrt{\frac{1}{n} \sum_{i=1}^n (\Gamma_s^{(i)})^2}.
\]

Since $\bar{\Gamma}_s$ is invertible and satisfies $\bar{\Gamma}_s^{-1} \bar{\Gamma}_s^2 \bar{\Gamma}_s^{-1} = I_d$, we can define a process $\tilde{B} = (\tilde{B}_t)_{t \in [0, 1]}$ as
\[
\tilde{B}_t = \int_0^t \bar{\Gamma}_s^{-1} dM_s, \quad t \in [0, 1].
\]

We evidently have $M_t = \int_0^t \Gamma_s dB_s$. Moreover, since $[\tilde{B}, \tilde{B}]_t = tI_d$ for all $t \in [0, 1]$, $\tilde{B}$ is a standard Brownian motion in $\mathbb{R}^d$ by Lévy’s characterization.

Next, since $0 \leq \mathbb{E}[(\Gamma_t - \mathbb{E}\Gamma_t)^2] = \mathbb{E}\Gamma_t^2 - (\mathbb{E}\Gamma_t)^2$, we have by Lemma 3.4
\[
\mathbb{E}\Gamma_t^2 \geq \frac{1}{9} I_d \quad \text{for any} \quad 0 \leq t \leq \varepsilon. \quad (3.15)
\]
In particular, $\mathbb{E}\Gamma_t^2$ is invertible for $0 \leq t \leq \varepsilon$. Define

$$U_t = \sqrt{\mathbb{E}[\Gamma_t^2]^{-1/2}\Gamma_t^2\mathbb{E}[\Gamma_t^2]^{-1/2}} \sqrt{\mathbb{E}[\Gamma_t^2]}^{-1}.$$ 

We can easily check that $U_t^T U_t = U_t U_t^T = I_d$ and $\mathbb{E}[\Gamma_t^2]#\Gamma_t^2 = \sqrt{\mathbb{E}[\Gamma_t^2]}U_t \Gamma_t$. In addition, we define a process $\tilde{B} = (\tilde{B}_t)_{t \in [0, \varepsilon]}$ as

$$\tilde{B}_t = \int_0^t U_s d\tilde{B}_s, \quad t \in [0, \varepsilon].$$

We have for all $t \in [0, \varepsilon]$

$$[\tilde{B}, \tilde{B}]_t = \int_0^t U_s U_s^T ds = t I_d.$$ 

Therefore, by Lévy’s characterization again, $\tilde{B}$ is a standard Brownian motion in $\mathbb{R}^d$. Hence

$$Z_\varepsilon := \int_0^{t_1} \sqrt{\mathbb{E}[\Gamma_t^2]}d\tilde{B}_t = \int_0^{t_1} \sqrt{\mathbb{E}[\Gamma_t^2]}U_t d\tilde{B}_t$$

defines a centered Gaussian vector in $\mathbb{R}^d$ such that

$$\text{Cov}(Z_\varepsilon) = \int_0^{t_1} \mathbb{E}[\Gamma_t^2]dt = \text{Cov}(m_\varepsilon).$$

We are going to bound $\|u \cdot (M_\varepsilon - Z_\varepsilon)\|_p$. Thanks to Jensen’s inequality, it suffices to consider the case $p \geq 2$. Since

$$u \cdot (M_\varepsilon - Z_\varepsilon) = \int_0^{t_1} u^T (\tilde{\Gamma}_t - \sqrt{\mathbb{E}[\Gamma_t^2]}U_t) d\tilde{B}_t,$$

we obtain by Proposition 4.2 in Barlow and Yor [5]

$$\|u \cdot (M_\varepsilon - Z_\varepsilon)\|_p \leq C \sqrt{p} \left\| \int_0^{t_1} u^T (\tilde{\Gamma}_t - \sqrt{\mathbb{E}[\Gamma_t^2]}U_t)(\tilde{\Gamma}_t - U_t^T \sqrt{\mathbb{E}[\Gamma_t^2]}u)dt \right\|_p$$

$$= C \sqrt{p} \left\| \int_0^{t_1} u^T (\tilde{\Gamma}_t^2 + \mathbb{E}[\Gamma_t^2] - \sqrt{\mathbb{E}[\Gamma_t^2]}U_t \tilde{\Gamma}_t - \Gamma_t U_t^T \sqrt{\mathbb{E}[\Gamma_t^2]}u)dt \right\|^{1/2}. $$

By construction, $\sqrt{\mathbb{E}[\Gamma_t^2]}U_t \tilde{\Gamma}_t = \mathbb{E}[\Gamma_t^2]#\tilde{\Gamma}_t^2$, which is a (random) symmetric matrix. Hence

$$\|u \cdot (M_\varepsilon - Z_\varepsilon)\|_p \leq C \sqrt{p} \left\| \int_0^{t_1} u^T (\tilde{\Gamma}_t^2 + \mathbb{E}[\Gamma_t^2] - 2\mathbb{E}[\Gamma_t^2]#\tilde{\Gamma}_t^2)dt \right\|^{1/2}. $$

(3.16)

Therefore, we obtain by Lemma 3.7

$$\|u \cdot (M_\varepsilon - Z_\varepsilon)\|_p \leq C \sqrt{p} \left\| \int_0^{t_1} u^T (\tilde{\Gamma}_t^2 - \mathbb{E}[\Gamma_t^2])\mathbb{E}[\Gamma_t^2]^{-1}(\tilde{\Gamma}_t^2 - \mathbb{E}[\Gamma_t^2])u)dt \right\|^{1/2}. $$
Then, by (3.15),
\[ \|u \cdot (M_\varepsilon - Z_\varepsilon)\|_p \leq C \sqrt{p} \left( \int_0^\varepsilon |(\tilde{\Gamma}_t^2 - \mathbb{E}[\Gamma_t^2])u|^2 dt \right)^{1/2}. \]

By the integral Minkowski inequality, we obtain
\[ \|u \cdot (M_\varepsilon - Z_\varepsilon)\|_p \leq C \sqrt{p} \int_0^\varepsilon \|((\tilde{\Gamma}_t^2 - \mathbb{E}[\Gamma_t^2])u)^2 dt. \]

To evaluate the integrand, observe that
\[ (\tilde{\Gamma}_t^2 - \mathbb{E}[\Gamma_t^2])u = \frac{1}{n} \sum_{i=1}^n (\Gamma_t^{(i)})^2 u - \mathbb{E}[(\Gamma_t^{(i)})^2 u]. \]

Hence, noting that sub-exponential tails are equivalent to linear growth of \(L^r\)-norms (cf. Vershynin [46, Proposition 2.7.1]), we have by Lemma 2.1 in Fang and Koike [26] and Lemma 3.6
\[ \|u \cdot (M_\varepsilon - Z_\varepsilon)\|_p \leq C |u| \left( \frac{\sqrt{p}}{n} + \frac{p^{5/2}}{n} \right). \]

Consequently,
\[ \|u \cdot (M_\varepsilon - Z_\varepsilon)\|_p \leq C |u| \varepsilon \left( \frac{\sqrt{p}}{n} + \frac{p^{5/2}}{n} \right). \]

This completes the proof.

### 3.5 Proof of Theorem 2.1

We will use the following simple fact about the Poincaré constant.

**Lemma 3.8.** Let \(X\) be a random vector in \(\mathbb{R}^d\). Then, for any \(k \times d\) matrix \(A\), \(\varpi(AX) \leq \|A\|_{op}^2 \varpi(X)\).

**Proof.** Fix a locally Lipschitz function \(h : \mathbb{R}^k \to \mathbb{R}\) arbitrarily. Define a function \(\tilde{h} : \mathbb{R}^d \to \mathbb{R}\) as \(\tilde{h}(x) = h(Ax)\) for \(x \in \mathbb{R}^d\). It is straightforward to check that \(\tilde{h}\) is locally Lipschitz and satisfies \(\|\nabla \tilde{h}(x)\| \leq \|A\|_{op} \|\nabla h(Ax)\|\) for all \(x \in \mathbb{R}^d\). Hence, we have \(\text{Var}[h(AX)] = \text{Var}[\tilde{h}(X)] \leq \varpi(X) \|A\|_{op}^2 \mathbb{E}[\nabla h(AX)^2]\) by the definition of \(\varpi(X)\). This implies the desired result.

**Proof of Theorem 2.1.** We divide the proof into two steps.

**Step 1.** First we prove the result when \(\Sigma = I_d\). Since \(\mu \times \mu\) is log-concave by Proposition 3.2 in Saumard and Wellner [45], the result for \(d = 1\) follows from that for \(d = 2\). Hence we may assume \(d \geq 2\).

By a similar argument as in the proof of Theorem 2.2, we may assume that \(\mu\) has a positive and smooth density. Let \(c_0\) be the universal constant in Proposition 3.4, and set
\( \varepsilon := (c_0 \omega d \log(2d))^{-1} \) and \( \Sigma_{\varepsilon} := \mathbb{E} \text{Cov}(f_{1-\varepsilon,Y_\varepsilon}) \). Note that \( \Sigma_{\varepsilon} \geq (1-\varepsilon) \mathbb{E} \Gamma_{\varepsilon} \geq \frac{2}{d} I_d \) by Lemma 3.4. In particular, \( \Sigma_{\varepsilon} \) is invertible. Also, for every \( y \in \mathbb{R}^d \), let \( \nu_{\varepsilon,y} \) be the law of \( \xi - \mathbb{E} \xi \) with \( \xi \) a random vector in \( \mathbb{R}^d \) having density \( f_{1-\varepsilon,y} \). Then, by Proposition 3.3 with \( A = \Sigma_{\varepsilon}^{-1/2} \) and \( f_y = f_{1-\varepsilon,y} \), there exists a Markov kernel \( \mathcal{Q} \) from \( (\mathbb{R}^d)^n \) to \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying the following conditions for any \( y = (y^{(i)})_{i=1}^n \in (\mathbb{R}^d)^n \):

(i) \( \mathcal{Q}(y, \cdot \times \mathbb{R}^d) \) equals the law of \( \Sigma_{\varepsilon}^{-1/2} n^{-1/2} \sum_{i=1}^n \xi_i \), where \( \xi_1, \ldots, \xi_n \) are independent random vectors in \( \mathbb{R}^d \) such that \( \mathcal{L}(\xi_i) = \nu_{\varepsilon,y^{(i)}} \) for all \( i = 1, \ldots, n \).

(ii) \( \mathcal{Q}(y, \mathbb{R}^d \times \cdot) \) is the \( d \)-dimensional standard normal distribution.

(iii) If \( W' \) and \( Z' \) are random vectors in \( \mathbb{R}^d \) such that \( (W', Z') \sim \mathcal{Q}(y, \cdot) \), then

\[
\|u \cdot (W' - Z')\|_p \leq C \frac{\|\Sigma_{\varepsilon}^{-1/2}\|_{op}}{n} \left( \frac{1}{n} \sum_{i=1}^n \mathcal{W}(\nu_{\varepsilon,y^{(i)}}) u^\top \Sigma_{\varepsilon}^{-1/2} \text{Cov}(f_{1-\varepsilon,y^{(i)}}) \Sigma_{\varepsilon}^{-1/2} u + \frac{p^{3/2}}{\varepsilon} \|\Sigma_{\varepsilon}^{-1/2} u\| \right) + \sqrt{p} \left( \frac{1}{n} \sum_{i=1}^n \Sigma_{\varepsilon}^{-1/2} \text{Cov}(f_{1-\varepsilon,y^{(i)}}) \Sigma_{\varepsilon}^{-1/2} - I_d \right) u \]

for any \( p \geq 1 \) and \( u \in \mathbb{R}^d \).

Note that \( \mathcal{W}(\nu_{\varepsilon,y}) \leq \mathcal{W}(\text{Cov}(f_{1-\varepsilon,y})) \|_{op} \) for any \( y \in \mathbb{R}^d \) by Lemma 3.8. We will use the bound in property (iii) after applying this inequality. Next, let \( Z_{\varepsilon} \) be as in Proposition 3.4. Define a probability distribution \( \Pi \) on \( \mathbb{R}^d \times \mathbb{R}^d \) as

\[
\Pi(A) = \mathbb{E} \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_A \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n m_{\varepsilon}^{(i)} + \sqrt{\Sigma_{\varepsilon} w, Z_{\varepsilon} + \sqrt{\Sigma_{\varepsilon} z}} \right) \mathcal{Q}((Y_{\varepsilon}^{(1)}, \ldots, Y_{\varepsilon}^{(n)}), dw dz) \right].
\]

Take random vectors \( W \) and \( Z \) in \( \mathbb{R}^d \) such that \( (W, Z) \sim \Pi \). Below we show that these \( W \) and \( Z \) are desired ones.

Recall that \( Y^{(1)}, \ldots, Y^{(n)} \) are independent copies of \( Y \), where \( Y \) is defined as (3.12). Moreover, recall that \( Y_1 \sim \mu, m_{\varepsilon} = \mathbb{E}[Y_1 | \mathcal{F}_\varepsilon] \) and the conditional law of \( Y_1 \) given \( \mathcal{F}_\varepsilon \) has density \( f_{1-\varepsilon,Y_\varepsilon} \). Hence \( W = n^{-1/2} \sum_{i=1}^n X_i \) with \( X_i \sim \mu \) by construction. Also, it is straightforward to see that \( Z \sim N(0, \text{Cov}(Z_{\varepsilon}) + \Sigma_{\varepsilon}) \). Then, since \( \mathbb{E}(Y_1 - m_{\varepsilon}) m_{\varepsilon}^\top = \mathbb{E}[\mathbb{E}[Y_1 - m_{\varepsilon} | \mathcal{F}_\varepsilon] m_{\varepsilon}^\top] = 0 \), we have

\[
I_d = \mathbb{E} Y_1 Y_1^\top = \mathbb{E}(Y_1 - m_{\varepsilon}) (Y_1 - m_{\varepsilon})^\top + \mathbb{E} m_{\varepsilon} m_{\varepsilon}^\top.
\]

By construction, we have \( \mathbb{E} m_{\varepsilon} m_{\varepsilon}^\top = \text{Cov}(m_{\varepsilon}) = \text{Cov}(Z_{\varepsilon}) \). Also,

\[
\mathbb{E}(Y_1 - m_{\varepsilon})(Y_1 - m_{\varepsilon})^\top = \mathbb{E}[\text{Cov}(Y_1 | \mathcal{F}_\varepsilon)] = \mathbb{E} \text{Cov}(f_{1-\varepsilon,Y_{\varepsilon}}) = \Sigma_{\varepsilon}.
\]

Therefore, \( I_d = \Sigma_{\varepsilon} + \text{Cov}(Z_{\varepsilon}) \), so \( Z \sim N(0, I_d) \). Finally, for any \( p \geq 1 \) and \( u \in \mathbb{R}^d \),

\[
\|u \cdot (W - Z)\|_p
\]
Also, by Lemma 2.1 in Fang and Koike \[94x508\] is \(\varepsilon/\sqrt{n}\), where the last line follows by Proposition 3.6 and property (iii). Now, recall that \(f_{1,\varepsilon,y}\) is \(\varepsilon/(1 - \varepsilon)\)-uniformly log-concave. Hence, by the Brascamp–Lieb inequality,

\[
\|\text{Cov}(f_{1,\varepsilon,y})\|_{op} \leq \frac{1}{\varepsilon}.
\]

Then, by Theorem 15.10 in Boucheron, Lugosi and Massart [8] and Lemma 3.6,

\[
\left\| \left( \sum_{i=1}^{n} \text{Cov}(f_{1,\varepsilon,y^{(i)})} - \Sigma_{\varepsilon} \right) u \right\|_p \leq C |u| \sqrt{n + \frac{p}{\varepsilon}}.
\]

Also, by Lemma 2.1 in Fang and Koike [26] and Lemma 3.6,

\[
\left\| \left( \frac{1}{n} \sum_{i=1}^{n} \text{Cov}(f_{1,\varepsilon,y^{(i)})} - \Sigma_{\varepsilon} \right) u \right\|_p \leq \frac{C |u|}{n} \left( \sqrt{p} n + p \right).
\]

Moreover, \(\|\Sigma_{\varepsilon}^{-1/2}\|_{op} \leq C\) since \(\Sigma_{\varepsilon} \geq \frac{2}{\varepsilon} I_d\). Consequently,

\[
\|u \cdot (W - Z)\|_p \leq \frac{C |u|}{\sqrt{\omega_d \log(2d)}} \frac{p^{5/2}}{n} + C |u| \left( \frac{\sqrt{\omega_d}}{\sqrt{n}} \frac{p}{\sqrt{n}} + \omega_d \log(2d) \frac{p^{3/2}}{n} \right).
\]

This gives (2.1).

**Step 2.** Next we consider the general case. Let \(\Sigma = U^T \Lambda U\) be a a spectral decomposition of \(\Sigma\), where \(U\) is a \(d \times d\) orthogonal matrix and \(\Lambda\) is a \(d \times d\) diagonal matrix. Without loss of generality, we may assume that the first \(r\) diagonal entries \(\lambda_1, \ldots, \lambda_r\) of \(\Lambda\) are positive and others are zero.

Let \(X \sim \mu\). Since \(\text{Cov}(UX) = \Lambda, (UX)_{j} = 0\) a.s. for \(j = r + 1, \ldots, d\). Let \(\nu\) be the law of the random vector \(((UX)_1/\sqrt{\lambda_1}, \ldots, (UX)_r/\sqrt{\lambda_r})^T\) in \(\mathbb{R}^r\). By construction, \(\nu\) is isotropic. Also, \(\nu\) is log-concave by Proposition 3.1 in Saumard and Wellner [45]. Thus, we can construct random vectors \(W_0\) and \(Z_0\) in \(\mathbb{R}^r\) such that \(W_0 \overset{d}{\sim} \sum_{i=1}^{r} Y_i + \nu, Z_0 \overset{d}{\sim} N(0, I_r)\) and

\[
\|v \cdot (W_0 - Z_0)\|_p \leq C |v| \left( \sqrt{\omega_r} \frac{p}{\sqrt{n}} + \omega_r \log(2r) \frac{p^{3/2}}{n} + \frac{1}{\sqrt{\omega_r \log(2r)}} \frac{p^{5/2}}{n} \right)\]
for all $v \in \mathbb{R}^r$ and $p \geq 1$. Now, define an $r \times d$ matrix as $S = (\text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}) O_{r,d-r})$, where $O_{r,d-r}$ denotes the $r \times (d - r)$ zero matrix. Then, set $\tilde{Y}_i = S^t Y_i$ for $i = 1, \ldots, n$. By construction, $\tilde{Y}_i$ has the same law as $UX$. Hence, $X_i := U^t \tilde{Y}_i \sim \mu$. Let $W = n^{-1/2} \sum_{i=1}^n X_i$. Also, set $Z = U^t S^t Z_0$. We have $Z \sim N(0, \Sigma)$ by construction. Finally, for any $u \in \mathbb{R}^d$ and $p \geq 1$,

$$
\|u \cdot (W - Z)\|_p = \|SUu \cdot (W_0 - Z_0)\|_p \\
\leq C|SUu| \left( \frac{p}{\sqrt{n}} + \frac{\log(2r)}{n} \right) \left( \frac{p^{5/2}}{\sqrt{n}} + \frac{1}{\sqrt{\log(2r)}} \right).
$$

Since $|SUu|^2 = u^t U^t S^t SUu = u^t U^t \Lambda U u = u^t \Sigma u = |\sqrt{\Sigma}u|^2$, we complete the proof.

\[\square\]

4 Proof of Proposition 2.1

We write $\phi$ and $\Phi$ for the density and distribution function of $N(0,1)$, respectively. The proof relies on the following lemma.

**Lemma 4.1.** Let $Z$ be as in Proposition 2.1. Then

$$
P \left( x - \varepsilon < \max_{1 \leq j \leq d} Z_j \leq x \right) \leq \frac{\varepsilon}{\sigma} (1 + x/\sigma) \exp(\varepsilon x/\sigma^2) P \left( \max_{1 \leq j \leq d} Z_j > x \right)
$$

for any $x \geq 0$ and $\varepsilon > 0$.

**Proof.** The proof is a modification of that of Theorem 3 in Chernozhukov, Chetverikov and Kato [11]. Without loss of generality, we may assume that the correlation coefficient between $Z_j$ and $Z_k$ is less than 1 whenever $j \neq k$. Set $\check{Z}_j := (Z_j - x)/\sigma_j + x/\sigma$ for $j = 1, \ldots, d$. Then

$$
P \left( x - \varepsilon < \max_{1 \leq j \leq d} Z_j \leq x \right) = P \left( \left\{ \bigcup_{j=1}^d \{ Z_j > x - \varepsilon \} \right\} \cap \left\{ \bigcap_{j=1}^d \{ Z_j \leq x \} \right\} \right)
$$

$$
= P \left( \left\{ \bigcup_{j=1}^d \{ \check{Z}_j > x/\sigma - \varepsilon/\sigma_j \} \right\} \cap \left\{ \bigcap_{j=1}^d \{ \check{Z}_j \leq x/\sigma \} \right\} \right)
$$

$$
\leq P \left( \frac{x - \varepsilon}{\sigma} < \max_{1 \leq j \leq d} \check{Z}_j \leq \frac{x}{\sigma} \right).
$$

Since $\check{Z}_j \sim N(\mathbb{E} \check{Z}_j, 1)$ and $\mathbb{E} \check{Z}_j = -x/\sigma_j + x/\sigma \geq 0$ for all $j$, by Lemma 5 in Chernozhukov, Chetverikov and Kato [11], $\max_{1 \leq j \leq d} \check{Z}_j$ has density of the form $f(z) = \phi(z)G(z)$, where $G$ is non-decreasing by Lemma 6 in Chernozhukov, Chetverikov and Kato [11]. Then, for any $z \in \mathbb{R}$,

$$
\int_{z}^{\infty} \phi(u)du G(z) \leq \int_{z}^{\infty} \phi(u)G(u)du = P \left( \max_{1 \leq j \leq d} \check{Z}_j > z \right).
$$

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Hence
\[
\sup_{z \in [(x-\varepsilon)/\sigma, x/\sigma]} f(z) \leq \phi((x-\varepsilon)/\sigma)G(x/\sigma) \leq \frac{\phi((x-\varepsilon)/\sigma)}{1 - \Phi(x/\sigma)}P^{\max_{1 \leq j \leq d} \tilde{Z}_j > x/\sigma}.
\]

By Birnbaum’s inequality, we have for all \( z \geq 0 \)
\[
\frac{\phi(z)}{1 - \Phi(z)} \leq \frac{2}{\sqrt{4 + z^2}} = \frac{\sqrt{4 + z^2} + z}{2} \leq 1 + z.
\]

Hence
\[
\frac{\phi((x-\varepsilon)/\sigma)}{1 - \Phi(x/\sigma)} \leq (1 + x/\sigma)\phi((x-\varepsilon)/\sigma) \leq (1 + x/\sigma)\exp(\varepsilon x/\sigma^2).
\]

Also,
\[
P^{\max_{1 \leq j \leq d} \tilde{Z}_j > x/\sigma} = P^{\bigcup_{j=1}^{d} \{ \tilde{Z}_j > x/\sigma \}} = P^{\max_{1 \leq j \leq d} Z_j > x}.
\]

Hence we obtain the desired result. \( \square \)

**Proof of Proposition 2.1.** In this proof, we use \( C \) to denote positive constants, which depend only on \( \alpha, A \) and \( B \) and may be different in different expressions. First we prove the claim when \( \Delta/\sigma < 1/e \). Set
\[
p = \log d + \log(\sigma/\Delta) + \frac{x^2}{\sigma^2}, \quad \varepsilon = Ap^\alpha \Delta e.
\]

Because \( \log d + |\log(\Delta/\sigma)| \leq p_0/2 \) and \( x \leq \sigma/\sqrt{p_0/2} \), we have \( p \leq p_0 \).

We have
\[
P^{\max_j W_j > x} \leq P^{\max_j Z_j > x - \varepsilon} + P^{\max_j W_j - \max_j Z_j \geq \varepsilon} = P^{\max_j Z_j > x} + P^{x - \varepsilon < \max_j Z_j \leq x} + P^{\max_j W_j - \max_j Z_j \geq \varepsilon}.
\]

By Markov’s inequality and assumption,
\[
P^{\max_j W_j - \max_j Z_j \geq \varepsilon} \leq e^{-p} \E^{\max_j W_j - \max_j Z_j \geq \varepsilon} \\
\leq e^{-p} \E^{\max_j W_j - Z_j \geq \varepsilon} \\
\leq e^{-p} d(\Delta p^\alpha \Delta)^p = de^{-p} = \frac{\Delta}{\sigma} e^{-\frac{x^2}{\sigma^2}}.
\]

Also, by Lemma 4.1,
\[
P^{x - \varepsilon < \max_j Z_j \leq x} \leq \frac{\varepsilon}{\sigma}(1 + x/\sigma)\exp(\varepsilon x/\sigma^2) P^{\max_j Z_j > x}.
\]
Hence

\[ P\left(\max_j W_j > x\right) \leq P\left(\max_j Z_j > x\right) + \frac{\varepsilon}{\sigma} (1 + x/\sigma) \exp(\varepsilon x/\sigma^2) P\left(\max_j Z_j > x\right) + \frac{\Delta}{\sigma} e^{-\frac{x^2}{2\sigma^2}}. \]

Similarly, we deduce

\[ P\left(\max_j Z_j > x\right) \]

\[ = P\left(\max_j Z_j > x + \varepsilon\right) + P\left(x < \max_j Z_j \leq x + \varepsilon\right) \]

\[ \leq P\left(\max_j W_j > x\right) + P\left(\max_j W_j - \max_j Z_j > \varepsilon\right) + P\left(x < \max_j Z_j \leq x + \varepsilon\right) \]

\[ \leq P\left(\max_j W_j > x\right) + \frac{\varepsilon}{\sigma} (1 + (x + \varepsilon)/\sigma) \exp(\varepsilon (x + \varepsilon)/\sigma^2) P\left(\max_j Z_j > x + \varepsilon\right) + \frac{\Delta}{\sigma} e^{-\frac{x^2}{2\sigma^2}}. \]

Consequently, we obtain

\[ \left| P\left(\max_j W_j > x\right) - P\left(\max_j Z_j > x\right) \right| \leq \frac{\varepsilon}{\sigma} (1 + (x + \varepsilon)/\sigma) \exp(\varepsilon (x + \varepsilon)/\sigma^2) P\left(\max_j Z_j > x\right) + \frac{\Delta}{\sigma} e^{-\frac{x^2}{2\sigma^2}}. \]

Since \( x \leq \sigma(\sigma/\Delta)^{1/(2\alpha+1)} \) and \( \log d \leq C(\sigma/\Delta)^{2/(2\alpha+1)} \), we have

\[ \varepsilon \leq C \Delta \left(\log d\right)^{\alpha} + \left\{ \log(\sigma/\Delta) \right\}^{\alpha} + \frac{x^{2\alpha}}{\sigma^{2\alpha}} \]

\[ \leq C \Delta \left(\log(\sigma/\Delta)\right)^{\alpha} + \left(\sigma/\Delta\right)^{2\alpha/(2\alpha+1)} \leq C \sigma(\Delta/\sigma)^{1/(2\alpha+1)}. \]  

(4.2)

Hence \( \exp(\varepsilon (x + \varepsilon)/\sigma^2) \leq C \) and \( \varepsilon/\sigma \leq C \). Also, letting \( J \) be an element of \( \{1, \ldots, d\} \) satisfying \( \sigma^2 = \mathbb{E} Z_j^2 \), we have

\[ P\left(\max_j Z_j > x\right) \geq P\left(Z_J > x\right) = 1 - \Phi(x/\sigma) \geq \frac{\phi(x/\sigma)}{1 + x/\sigma}. \]  

(4.3)

Hence

\[ \left| P\left(\max_j W_j > x\right) - P\left(\max_j Z_j > x\right) \right| \]

\[ \leq C \frac{\varepsilon}{\sigma} (1 + x/\sigma) P\left(\max_j Z_j > x\right) + \sqrt{2\pi} e^{-\frac{x^2}{2\sigma^2}} \left(1 + x/\sigma\right) \frac{\Delta}{\sigma} P\left(\max_j Z_j > x\right) \]

\[ \leq C \frac{\varepsilon}{\sigma} \left\{ \varepsilon (1 + x/\sigma) + \Delta \right\} P\left(\max_j Z_j > x\right). \]

This completes the proof of (2.8).

It remains to prove (2.8) when \( \Delta/\sigma > 1/e \). In this case, we have \( x/\sigma \leq C \). Thus, by (4.3),

\[ \frac{1}{P(\max_j Z_j > x)} \leq C. \]

So (2.8) holds because \( \Delta/\sigma > 1/e \).

\[ \square \]
A Uniqueness in law for (3.12)

**Theorem A.1.** Uniqueness in law holds for the SDE (3.12) when \( f \) is the smooth, positive density with respect to \( N(0, I_d) \) of a log-concave probability measure \( \mu \).

**Proof.** Let \( Y^{(i)} = (Y^{(i)}_t)_{t \in [0,1]}, i = 1,2, \) be weak solutions to (3.12). Since both \( Y^{(1)} \) and \( Y^{(2)} \) are continuous, it suffices to show that \( (Y^{(1)}_t)_{t \in [0,t_1]} \) and \( (Y^{(2)}_t)_{t \in [0,t_1]} \) have the same law for any fixed \( 1/2 < t_1 < 1 \). By Proposition 3.10 in Karatzas and Shreve [31, Chapter 5], this follows once we show that

\[
\int_0^{t_1} |\nabla \log P_{1-t} f(Y^{(i)}_t)|^2 dt < \infty \quad \text{a.s. for } i = 1,2.
\]

Set \( \rho = f \phi_{I_d} \) so that \( \rho \) is the Lebesgue density of \( \mu \). By Theorem 5.1 in Saumard and Wellner [45], there exist constants \( a, b > 0 \) such that \( \rho(x) \leq e^{-a|x|+b} \) for all \( x \in \mathbb{R}^d \). Also, observe that

\[
\nabla \log P_{1-t} f(y) = \frac{\nabla P_{1-t} f(y)}{P_{1-t} f(y)} = \frac{\int_{\mathbb{R}^d} \frac{2x-y}{2(1-t)} e^{-|2x-y-t|^2/(2(1-t))} dx}{\int_{\mathbb{R}^d} e^{-|2x-y-t|^2/(2(1-t))} dx} \quad \text{for } t \in (0,1), y \in \mathbb{R}^d.
\]

By this expression, for any \( \tau \in (0,t_1) \) and \( K > 0 \), we have

\[
\sup_{t \in [\tau,t_1], |y| \leq K} |\nabla \log P_{1-t} f(y)| < \infty. \tag{A.1}
\]

Moreover, if \( \tau \leq 1/2, \)

\[
\sup_{t \in [0,\tau], |y| \leq a/4} |\nabla \log P_{1-t} f(y)| \leq \frac{\int_{\mathbb{R}^d} |x|^{a/4} e^{-\frac{a}{2} |x|+b} dx}{\int_{\mathbb{R}^d} e^{-|2x|^2/(2(1-t))} dx} < \infty. \tag{A.2}
\]

Now, since \( Y^{(i)}_t \rightarrow Y^{(i)}_0 = 0 \) a.s. as \( t \downarrow 0 \), there is a random variable \( t_0 \in (0,1/2) \) such that \( \sup_{0 \leq t \leq t_0} |Y^{(i)}_t| \leq a/4. \) Then, noting \( \sup_{t \in [0,1]} |Y^{(i)}_t| < \infty \), we have \( \sup_{t \in [t_0,t_1]} |\nabla \log P_{1-t} f(Y^{(i)}_t)| < \infty \) a.s. by (A.1) and \( \sup_{t \in [0,t_0]} |\nabla \log P_{1-t} f(Y^{(i)}_t)| < \infty \) a.s. by (A.2). Consequently, we obtain

\[
\int_0^{t_1} |\nabla \log P_{1-t} f(Y^{(i)}_t)|^2 dt \leq \sup_{t \in [0,t_1]} |\nabla \log P_{1-t} f(Y^{(i)}_t)|^2 < \infty \quad \text{a.s.}
\]

This completes the proof. \( \square \)

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References

[1] D. Alonso-Gutiérrez and J. Bastero (2015). Approaching the Kannan-Lovász-Simonovits and variance conjectures. Springer.

[2] B. Arras and C. Houdré (2019). On Stein’s method for multivariate self-decomposable laws. *Electron. J. Probab.* **24**, 1–63.

[3] B. Arras and C. Houdré (2022). Covariance representations, $L^p$-Poincaré inequalities, Stein’s kernels and high dimensional CLTs. Preprint. Available at 
https://arxiv.org/abs/2204.01088.

[4] D. Bakry, I. Gentil and M. Ledoux (2014). *Analysis and geometry of Markov diffusion operators*. Springer.

[5] M.T. Barlow and M. Yor (1982). Semimartingale inequalities via the Garsia-Rodemich-Rumsey lemma, and applications to local times. *J. Functional Analysis* **49**, 198–229.

[6] R. Bhatia (2007). *Positive definite matrices*. Princeton University Press.

[7] P. Billingsley (1999). *Convergence of probability measures* (2nd edition). Wiley.

[8] S. Boucheron, G. Lugosi, and P. Massart (2013). *Concentration inequalities: A nonasymptotic theory of independence*. Clarendon Press, Oxford.

[9] L. H. Y. Chen, L. Goldstein, and Q.-M. Shao (2011). *Normal approximation by Stein’s method*. Springer.

[10] V. Chernozhukov, D. Chetverikov and K. Kato (2013). Gaussian approximations and multiplier bootstrap for maxima of sums of high-dimensional random vectors. *Ann. Statist.* **41**, 2786–2819.

[11] V. Chernozhukov, D. Chetverikov and K. Kato (2015). Comparison and anti-concentration bounds for maxima of Gaussian random vectors. *Probab. Theory Relat. Fields* **162**, 47–70.

[12] V. Chernozhukov, D. Chetverikov and K. Kato (2016). Empirical and multiplier bootstraps for suprema of empirical processes of increasing complexity, and related Gaussian couplings. *Stochastic Process. Appl.* **126**, 3632–3651.

[13] V. Chernozhukov, D. Chetverikov and K. Kato (2017a). Central limit theorems and bootstrap in high dimensions. *Ann. Probab.* **45**, 2309–2352.

[14] V. Chernozhukov, D. Chetverikov and K. Kato (2017b). Detailed proof of Nazarov’s inequality. Preprint. Available at 
https://arxiv.org/abs/1711.10696

[15] V. Chernozhukov, D. Chetverikov, K. Kato and Y. Koike (2022). Improved central limit theorem and bootstrap approximation in high dimensions. *Ann. Statist.* **50**, 2562–2586.
[16] V. Chernozhukov, D. Chetverikov and Y. Koike (2023). Nearly optimal central limit theorem and bootstrap approximations in high dimensions. *Ann. Appl. Probab.* **33**, 2374–2425.

[17] T. A. Courtade, M. Fathi and A. Pananjady (2019). Existence of Stein kernels under a spectral gap, and discrepancy bounds. *Ann. Inst. Henri Poincaré Probab. Stat.* **55**, 777–790.

[18] A. S. Dalalyan, A. Karagulyan and L. Riou-Durand (2022). Bounding the error of discretized Langevin algorithms for non-strongly log-concave targets. *J. Mach. Learn. Res.* **23**, 1–38.

[19] R. Eldan (2013). Thin shell implies spectral gap up to polylog via a stochastic localization scheme. *Geom. Funct. Anal.* **23**, 532–569.

[20] R. Eldan and J. R. Lee (2018). Regularization under diffusion and anticoncentration of the information content. *Duke Math. J.* **167**, 969–993.

[21] R. Eldan, J. Lehec and Y. Shenfeld (2020). Stability of the logarithmic Sobolev inequality via the Föllmer process. *Ann. Inst. Henri Poincaré Probab. Stat.* **56**, 2253–2269.

[22] R. Eldan and D. Mikulincer (2020). Stability of the Shannon–Stam inequality via the Föllmer process. *Probab. Theory Related Fields* **177**, 891–922.

[23] R. Eldan, D. Mikulincer and A. Zhai (2020). The CLT in high dimensions: quantitative bounds via martingale embedding. *Ann. Probab.* **48**, 2494–2524.

[24] S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley, 1986.

[25] X. Fang and Y. Koike (2021). High-dimensional central limit theorems by Stein’s method. *Ann. Appl. Probab.* **31**, 1660–1686.

[26] X. Fang and Y. Koike (2022). From $p$-Wasserstein bounds to moderate deviations. Preprint. Available at https://arxiv.org/abs/2205.13307

[27] M. Fathi (2019). Stein kernels and moment maps. *Ann. Probab.* **47**, 2172–2185.

[28] M. Fathi and D. Mikulincer (2022). Stability estimates for invariant measures of diffusion processes, with applications to stability of moment measures and Stein kernels. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **23**, 1417–1445.

[29] S. Janson (1997). *Gaussian Hilbert spaces*. Cambridge University Press.

[30] O. Johnson and Y. Suhov (2001). Entropy and random vectors. *J. Stat. Phys.* **104**, 145–165.

[31] I. Karatzas and S. E. Shreve (1998). *Brownian motion and stochastic calculus* (2nd edition). Springer.
[32] B. Klartag and J. Lehec (2022). Bourgain’s slicing problem and KLS isoperimetry up to polylog. *Geom. Funct. Anal.* **32**, 1134–1159.

[33] B. Klartag and E. Putterman (2021). Spectral monotonicity under Gaussian convolution. To appear in *Ann. Fac. Sci. Toulouse Math.* Preprint available at https://arxiv.org/abs/2107.09496

[34] A. K. Kuchibhotla, S. Mukherjee and D. Banerjee (2021). High-dimensional CLT: Improvements, non-uniform extensions and large deviations. *Bernoulli* **27**, 192–217.

[35] A. K. Kuchibhotla and A. Rinaldo (2020). High-dimensional CLT for sums of non-degenerate random vectors: $n^{-1/2}$-rate. *Preprint*. Available at https://arxiv.org/abs/2009.13673

[36] M. Ledoux, I. Nourdin and G. Peccati (2015). Stein’s method, logarithmic Sobolev and transport inequalities. *Geom. Funct. Anal.* **25**, 256–306.

[37] M. E. Lopes (2022). Central limit theorem and bootstrap approximation in high dimensions: Near $1/\sqrt{n}$ rates via implicit smoothing. *Ann. Statist.* **50**, 2492–2513.

[38] D. Mikulincer and Y. Shenfeld (2021). The Brownian transport map. *Preprint*. Available at https://arxiv.org/abs/2111.11521

[39] I. Nourdin, G. Peccati and Y. Swan (2014). Entropy and the fourth moment phenomenon. *J. Funct. Anal.* **266**, 3170–3207.

[40] F. Otto and C. Villani (2000). Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* **173**, 361–400.

[41] V. V. Petrov (1975). *Sums of independent random variables*. Springer.

[42] I. Pinelis (1994). Optimum bounds for the distributions of martingales in Banach spaces. *Ann. Probab.* **22**, 1679–1706.

[43] P. E. Protter (2005). *Stochastic integration and differential equations* (2nd edition). Springer.

[44] E. Rio (2011). Asymptotic constants for minimal distance in the central limit theorem. *Electron. Commun. Probab.* **16**, 96–103.

[45] A. Saumard and J. A. Wellner (2014). Log-concavity and strong log-concavity: A review. *Stat. Surv.* **8**, 45–114.

[46] R. Vershynin (2018). *High-dimensional probability*. Cambridge University Press.