TROPICAL SCHEMES, TROPICAL CYCLES, AND VALUATED MATROIDS

DIANE MACLAGAN AND FELIPE RINCÓN

Abstract. We show that the weights on a tropical variety can be recovered from the tropical scheme structure proposed in [GG13], so there is a well-defined Hilbert-Chow morphism from a tropical scheme to the underlying tropical cycle. For a subscheme of projective space given by a homogeneous ideal $I$ we show that the Giansiracusa tropical scheme structure contains the same information as the set of valuated matroids of the vector spaces $I_d$ for $d \geq 0$. We also give a combinatorial criterion to determine whether a given relation is in the congruence defining the tropical scheme structure.

1. Introduction

The tropicalization of a subvariety $Y$ in the $n$-dimensional algebraic torus $T$ is a polyhedral complex $\text{trop}(Y)$ that is a “combinatorial shadow” of the original variety. Some invariants of $Y$, such as the dimension, are encoded in $\text{trop}(Y)$. The complex $\text{trop}(Y)$ comes equipped with positive integer weights on its top-dimensional cells, called multiplicities, that make it into a tropical cycle. This extra information encodes information about the intersection theory of compactifications of the original variety $Y$; see for example [KP11].

In [GG13] the authors propose a notion of tropical scheme structure for tropical varieties, which takes the form of a congruence on the semiring of tropical polynomials (see §2). When $Y \subset T$ is a subscheme defined by an ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ this congruence is denoted by $\text{Trop}(I)$. In [GG13] the tropical scheme structure is defined in the slightly more general context of $\mathbb{F}_1$-schemes.

In this paper we investigate the relation between these tropical schemes, ideals in the semiring of tropical polynomials, and the theory of valuated matroids introduced by Dress and Wenzel [DW92]. We also show that the tropical cycle of a scheme can be reconstructed from the corresponding congruence.

Our first result is the following.

Theorem 1.1. Let $K$ be field with a valuation $\text{val}: K \to \mathbb{R} := \mathbb{R} \cup \{\infty\}$, and let $Y$ be a subscheme of $T \cong (K^*)^n$ defined by an ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then any of the following three objects determines the others:

1) The congruence $\text{Trop}(I)$ on the semiring $S := \mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of tropical Laurent polynomials;

2) The ideal $\text{trop}(I)$ in $S$;

3) The set of valuated matroids of the vector spaces $I_d^h$, where $I^h \subset K[x_0, \ldots, x_n]$ is the homogenization of the ideal $I$, and $I_d^h$ is its degree $d$ part.
When the valuation on $K$ is trivial, this says that the tropical scheme structure $Trop(I)$ is equivalent to the information of the supports of all polynomials in $I$, and also of the (standard) matroids of the vector spaces $I^h$. Theorem 1.1 is mostly proved in Section 2, though we postpone the discussion of valuated matroids, including recalling their definition, to Section 4. The version proved there (Theorem 4.2) also holds for a subscheme $Z \subset \mathbb{P}^n$ given by a homogeneous ideal in $K[x_0, \ldots, x_n]$.

In Section 3 we show that the tropical cycle structure on a tropical variety $\text{trop}(Y)$ can be recovered from its tropical scheme structure, answering the question raised in [GG13, Remark 7.2.3].

**Theorem 1.2.** Let $Y \subset T$ be a subscheme defined by an ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The multiplicities of the maximal cells in the tropical variety $\text{trop}(Y)$ can be recovered from the congruence $Trop(I)$.

The classical Hilbert-Chow morphism takes a subscheme of $\mathbb{P}^n$ to the associated cycle in the Chow group of $\mathbb{P}^n$. Theorem 1.2 can thus be thought of as a tropical version of this morphism.

Finally, in Section 4 we investigate in more depth the structure of the congruence $Trop(I)$, and use ideas from valuated matroids and tropical linear spaces to characterize when a relation lives in $Trop(I)$. We also show that any tropical polynomial has a distinguished representative in its equivalence class in $Trop(I)$, and give a combinatorial procedure to compute it.

**Acknowledgements.** Both authors were partially supported by EPSRC grant EP/1008071/1. We thank Florian Block for helpful conversations about matroids, and Jeff Giansiracusa for discussion on [GG13] and comments on an earlier draft of this paper. The first author also thanks the Max Planck Institute for Mathematics for hospitality while some of this paper was written.

## 2. Tropical varieties and their scheme structure

In this section we recall the necessary background on tropical geometry and the definition of the tropical scheme structure proposed in [GG13]. We also develop some fundamental properties of these congruences, leading to part of the proof of Theorem 1.1.

Throughout this paper we denote by $\mathbb{R}$ the tropical semiring (or min-plus algebra)

$$\mathbb{R} := (\mathbb{R} \cup \{\infty\}, \min, +),$$

and by $\mathbb{B}$ its Boolean subsemiring consisting of $\{0, \infty\}$ with the induced operations. We denote by

$$S := \mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \quad \text{and} \quad \tilde{S} := \mathbb{R}[x_0, \ldots, x_n]$$

the semirings of tropical Laurent polynomials and tropical polynomials in the variables $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x} = (x_0, \ldots, x_n)$, respectively. Elements of $S$ or $\tilde{S}$ are (Laurent) polynomials with coefficients in $\mathbb{R}$ where all operations are to be interpreted tropically. Explicitly, if $F \in \tilde{S}$ then $F$ has the form $F(\mathbf{x}) = \min_{u \in \mathbb{N}^{n+1}}(a_u + \mathbf{x} \cdot \mathbf{u})$, where $a_u \in \mathbb{R}$ and all but finitely many of the $a_u$ equal $\infty$. Elements of $S$ have the
form $F(x) = \min_{u \in \mathbb{Z}^n}(a_u + x \cdot u)$, where again $a_u \in \mathbb{R}$ and all but finitely many $a_u$ equal $\infty$. Note that elements of $S$ and $\hat{S}$ are regarded as tropical polynomials, not functions. By this we mean that $F(x) = \min(2x, 0)$ and $G(x) = \min(2x, 1 + x, 0)$ are different as elements of $S$, even though $F(w) = G(w)$ for all $w \in \mathbb{R}$.

We adopt the notational convention that lower case letters denote elements of the conventional (Laurent) polynomial ring with coefficients in $K$ and upper case letters denote tropical (Laurent) polynomials with coefficients in $\overline{\mathbb{R}}$. Tropical polynomials are always written using standard arithmetic.

The support of a (Laurent) polynomial $f = \sum c_u x^u$ is the subset of $\mathbb{N}^{n+1}$ (respectively $\mathbb{Z}^n$) defined by $\text{supp}(f) := \{u : c_u \neq 0\}$. Similarly, for a tropical (Laurent) polynomial $F = \min(a_u + x \cdot u)$ we write $\text{supp}(F) := \{u : a_u \neq \infty\}$. We call $a_u$ the coefficient in $F$ of the monomial $u$.

Fix a field $K$ with a valuation $\text{val}: K \to \overline{\mathbb{R}}$. We write $R$ for the valuation ring $\{a \in K : \text{val}(a) \geq 0\}$, and $\mathbb{k}$ for the residue field $R/\{a \in K : \text{val}(a) > 0\}$.

A polynomial $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ gives rise to a tropical polynomial $\text{trop}(f) \in S$ as follows. If $f = \sum_{u \in \mathbb{Z}^n} c_u x^u$, then

$$\text{trop}(f) := \min(\text{val}(c_u) + x \cdot u).$$

The tropical hypersurface defined by $f$ is

$$\text{trop}(V(f)) := \{w \in \mathbb{R}^n : \text{the minimum in } \text{trop}(f)(w) \text{ is achieved at least twice}\}.$$

The tropicalization of a variety $Y \subset (K^*)^n$ defined by an ideal $I$ is

$$\text{trop}(Y) := \bigcap_{f \in I(Y)} \text{trop}(V(f)).$$

For more details on tropical varieties see [MS13].

Classically, a subscheme of the $n$-dimensional torus $T$ is defined by an ideal in the Laurent polynomial ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. There are two possible ways to tropicalize this. The first gives an ideal in the semiring $S$ of tropical Laurent polynomials.

**Definition 2.1.** Let $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be an ideal. The ideal $\text{trop}(I)$ in the semiring $S$ is generated by the tropical polynomials $\text{trop}(f)$ for $f \in I$:

$$\text{trop}(I) := \langle \text{trop}(f) : f \in I \rangle.$$

The definition of $\text{trop}(I)$ is the same for an ideal in $K[x_0, \ldots, x_n]$.

Note that if the value group $\Gamma := \text{im} \text{val} \subset K$ equals all of $\overline{\mathbb{R}}$ and the residue field $\mathbb{k}$ is infinite then every tropical polynomial in the ideal $\text{trop}(I)$ has the form $\text{trop}(f)$ for some $f \in I$. Indeed, in that case $\min(a + x \cdot u) + \text{trop}(f) = \text{trop}(cx^u f)$ for any $c \in K$ with $\text{val}(c) = a$, and $\min(\text{trop}(f), \text{trop}(g)) = \text{trop}(f + \alpha g)$ for a sufficiently general $\alpha \in K$ with $\text{val}(\alpha) = 0$.

A different approach to tropicalizing the scheme defined by $I$ is given in [GG13]. Here the ideal $I$ gives rise to a congruence on $S$. This is an equivalence relation on $S$ that is closed under tropical addition and tropical multiplication. In standard operations, this means that $F_1 \sim G_1$ and $F_2 \sim G_2$ imply that $\min(F_1, F_2) \sim \min(G_1, G_2)$ and $(F_1 + F_2) \sim (G_1 + G_2)$. If $\phi: S \to R$ is a semiring homomorphism, then
\{ F \sim G : \phi(F) = \phi(G) \} is a congruence, and all congruences on \( S \) arise in this fashion. This is a key reason to consider congruences instead of only ideals. For a subset \( \{ (F_\alpha, G_\alpha) : \alpha \in A \} \) of \( S \times S \) there is a smallest congruence on \( S \) containing \( F_\alpha \sim G_\alpha \) for all \( \alpha \in A \), which we denote by \( \langle F_\alpha \sim G_\alpha \rangle_{\alpha \in A} \). All these notions also make sense for the semiring \( \tilde{S} \).

The following definitions are taken from Definitions 5.1 and 6.1.4 of \cite{GG13}.

**Definition 2.2.** Let \( F \) be a tropical (Laurent) polynomial. For \( v \in \text{supp}(F) \) we write \( \hat{F}^v \) for the tropical polynomial obtained by removing the term involving \( v \) from \( F \). Explicitly, if \( F = \min(a_u + x \cdot u) \), then
\[
\hat{F}^v := \min_{u \neq v}(a_u + x \cdot u).
\]
The bend relations of \( F \) are:
\[
B(F) := \{ F \sim \hat{F}^v : v \in \text{supp}(F) \}.
\]
Given an ideal \( I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), the scheme-theoretic tropicalization of \( I \) is the congruence on \( S \)
\[
\mathcal{Trop}(I) := \langle B(\text{trop}(f)) : f \in I \rangle.
\]
We use the same definition for \( \mathcal{Trop}(I) \) if \( I \) is a homogeneous ideal in \( K[x_0, \ldots, x_n] \).

In [GG13] the authors show that the tropical variety of an ideal \( I \) can be recovered from the congruence \( \mathcal{Trop}(I) \) as
\[
\text{trop}(V(I)) = \text{Hom}(S/\mathcal{Trop}(I), \mathbb{R}),
\]
where the homomorphisms are semiring homomorphisms. Explicitly, this means that
\[
\text{trop}(V(I)) = \{ w \in \mathbb{R}^n : \text{trop}(f)(w) = \text{trop}(f \hat{v})(w) \text{ for all } f \in I, v \in \text{supp}(f) \},
\]  
\hspace{1cm} (2.1)

**Remark 2.3.** When \( I \) is a binomial ideal, an equivalent congruence appears in the work of Kahle and Miller \cite{KM11}. For a binomial ideal \( I \subset K[x_1, \ldots, x_n] \) they define a congruence on the monoid \( \mathbb{N}^n \) generated by the relations \( \{ u \sim v : \exists \lambda \in K^* \text{ such that } x^u - \lambda x^v \in I \} \). These relations, with the addition of \( u \sim \infty \) whenever \( x^u \in I \), generate the congruence \( \mathcal{Trop}(I) \) on \( \mathbb{B}[x_1, \ldots, x_n] \) when \( K \) has the trivial valuation.

In the rest of this section we develop some basic properties of these congruences, leading to a proof of part of Theorem 1.1. We will make repeated use of the following result on congruences on \( S \). By a monomial in \( S \) we mean a tropical polynomial whose support has size one.

**Lemma 2.4.** The congruence \( \langle F_\alpha \sim G_\alpha \rangle_{\alpha \in A} \) on \( S \) is equal to the transitive closure of the set \( U \) of relations of the form
\[
\min(M + F_\alpha, H) \sim \min(M + G_\alpha, H),
\]
where \( \alpha \in A, H \in S, \) and \( M \) is a monomial in \( S \).
Thus the congruence $\text{Trop}(I)$ is equal to the transitive closure of the set of relations of the form
\[
\min(a + F_\psi, H) \sim \min(a + F, H)
\]
and their reverse, where $F = \text{trop}(f)$ for some $f \in I$, $v \in \text{supp}(f)$, $a \in \mathbb{R}$, and $H \in S$.

Proof. By \cite[Lemma 2.4.5]{GG13} we know that $(F_\alpha \sim G_\alpha)_{\alpha \in A}$ is the transitive closure of the subsemiring of $S \times S$ generated by the elements $F_\alpha \sim G_\alpha$, $G_\alpha \sim F_\alpha$, and $1 \sim 1$. We first show that this is in fact the transitive closure $T$ of the $S$-subsemimodule $N$ (as opposed to the $S$-subsemiring) of $S \times S$ generated by these elements. Let $F \sim G$ and $F' \sim G'$ be elements of $T$. We will show that their tropical product is also in $T$, so $T$ is a subsemiring of $S \times S$, as desired. By definition, there exist chains $F = H_0 \sim H_1 \sim \cdots \sim H_l \sim H_{l+1} = G$ and $F' = H'_0 \sim H'_1 \sim \cdots \sim H'_{l'} \sim H'_{l'+1} = G'$ of relations in $N$. We may assume that $l = l'$. The fact that $F + F' \sim G + G'$ is in $T$ follows from the chain of relations in $N$
\[
F + F' \sim H_1 + F' \sim H_1 + H'_1 \sim H_2 + H'_1 \sim H_2 + H'_2 \sim \cdots \sim H_k + H'_k \sim G + H'_k \sim G + G'.
\]

We now prove that all relations in $N$ are in the transitive closure of the set $U$. Any relation in $N$ has the form
\[
\min\left(\min_{i=1}^s (Q_i + F_i), Q\right) \sim \min\left(\min_{i=1}^s (Q_i + G_i), Q\right),
\]
where all the $Q_i$ are in $S$, all the relations $F_i \sim G_i$ are in $(F_\alpha \sim G_\alpha)_{\alpha \in A}$, and $Q \in S$. By allowing some of the relations $F_i \sim G_i$ to be equal, we can assume that the $Q_i$ are monomials in $S$. For $l = 0, 1, \ldots, s$, let $H_l \in S$ be defined by
\[
H_l := \min\left(\min_{i=1}^l (Q_i + G_i), \min_{i=1}^s (Q_i + F_i), Q\right).
\]
Note that $H_0 \sim H_1 \sim \cdots \sim H_s$ is a chain of relations in $U$. The relation (2.2) is simply $H_0 \sim H_s$, so it is in the transitive closure of $U$.

The last claim of the lemma follows from the fact that $\text{Trop}(I)$ is generated by the relations $\text{trop}(f) \sim \text{trop}(f)_\psi$ for $f \in I$. If $f \in I$ then $x^u f \in I$, so we may replace the tropical monomial $M$ by a scalar $a$. \hfill \Box

Remark 2.5. If the value group $\Gamma = \text{im \ val}$ equals all of $\mathbb{R}$ then for all scalars $a \in \mathbb{R}$ we can find $\alpha \in K$ with $\text{val}(\alpha) = a$, so $a + \text{trop}(f) = \text{trop}(\alpha f)$. Therefore, in this case the congruence $\text{Trop}(I)$ can be described as the transitive closure of the set of relations of the form $\min(\text{trop}(f)_\psi, H) \sim \min(\text{trop}(f), H)$ and their reverse, where $f \in I$, $v \in \text{supp}(f)$, and $H \in S$.

The following proposition is the key technical result that is needed to prove Theorem 1.2 and parts of Theorem 1.1.

Proposition 2.6. Let $I$ be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $F \sim G$ be a relation in the congruence $\text{Trop}(I)$ on $S$, where $F = \min(\alpha_u + x \cdot u)$ and $G = \min(\beta_u + x \cdot u)$. Then there is a chain $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ of relations in $\text{Trop}(I)$ satisfying the following two properties.
(a) Each $F_i \sim F_{i+1}$ has the form $\min(m + \trop(g), H) \sim \min(m + \trop(g)\psi, H)$ or the reverse, for some $g \in I$, $H \in S$, and $m \in \mathbb{R}$.

(b) The coefficient $\gamma_{u,i}$ of $u$ in $F_i$ equals either $\alpha_u$ or $\beta_u$.

Proof. By Lemma 2.4 there is a chain $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ of relations in $\Trop(I)$ with the property that for each $i$ we have $F_i \sim F_{i+1}$ equal to $\min(m_i + \trop(g_i), H_i) \sim \min(m_i + \trop(g_i)\psi, H_i)$ or the reverse, for some polynomial $g_i \in I$, $v \in \supp(g_i)$, $H_i \in S$, and $m_i \in \mathbb{R}$. We now show that we can modify this chain to get a chain where the coefficients have the required form. We represent the given chain by a path of length $s+1$ with vertices labelled by the $F_i$ and an oriented edge labelled by $v$ from $\min(m_i + \trop(g_i), H_i)$ to $\min(m_i + \trop(g_i)\psi, H_i)$.

We claim that we can locally modify the path by switching the order of adjacent edges or amalgamating edges if the labels agree, in the following six ways:

1. $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$ can be replaced by $F_{i-1} \xleftarrow{v'} F_i \xrightarrow{v'} F_{i+1}$,
2. $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$ can be replaced by $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$,
3. $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$ can be replaced by $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$,
4. $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$ can be replaced by one of $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$, $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$, or $F_{i-1} = F_{i+1}$,
5. $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$ can be replaced by $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$, and
6. $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$ can be replaced by $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$.

By repeated use of the first of these operations we may assume that all left-pointing arrows in the path come before all right-pointing arrows. If $v$ appears as an arrow label on more than one arrow, by repeated use of the second and third operations we may assume that the left-pointing arrows labelled by $v$ are the last left-pointing arrows, and the right-pointing arrows labelled by $v$ are the first right-pointing arrows. By repeated use of the last three operations we can then replace these arrows by at most one arrow labelled by $v$. In this fashion we get a new chain $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ where each arrow label occurs exactly once. As the coefficient of $v$ in $F_i$ equals that in $F_{i+1}$ unless the arrow between $F_i$ and $F_{i+1}$ is labelled by $v$, this means that the coefficient of $v$ changes at most once in the path from $F$ to $G$, so for all $F_i$ the coefficient of $v$ equals the coefficient of $v$ in either $F$ or $G$.

It thus suffices to prove the possibility of the six arrow replacements. In each case we make use of the fact that the coefficients of $F_{i-1}, F_i$ and $F_{i+1}$ all agree for monomials $u \neq v, v'$.

Case $F_{i-1} \xleftarrow{v} F_i \xrightarrow{v} F_{i+1}$. By assumption $F_{i-1} = \min(m_{i-1} + \trop(g_{i-1}), H_{i-1})$, $F_i = \min(m_{i-1} + \trop(g_{i-1})\psi, H_{i-1}) = \min(m_{i-1} + \trop(g_{i-1})\psi, H_{i+1})$, and $F_{i+1} = \min(m_{i-1} + \trop(g_{i-1}), H_{i+1})$, for some $g_{i-1}, g_{i+1} \in I$, $v \in \supp(\trop(g_{i-1}))$, $v' \in \supp(\trop(g_{i+1}))$, $m_{i-1}, m_{i+1} \in \mathbb{R}$, and $H_{i-1}, H_{i+1} \in S$. Let $c_v$ be the coefficient of $x^v$ in $g_i$, and let $d_{\psi}$ be the coefficient of $x^\psi$ in $g_{i+1}$. Let $H'_{i-1} = \min(H_{i-1}, m_{i-1} + \val(d_v) + x \cdot v')$, and $H'_{i+1} = \min(H_{i+1}, m_{i-1} + \val(c_v) + x \cdot v')$. Set $F_i' = \min(m_{i-1} + \trop(g_{i-1}), H'_{i-1})$, and note that this equals $\min(m_{i-1} + \trop(g_{i-1}), H'_{i+1})$. In addition, $F_{i-1} = \min(m_{i-1} + \trop(g_{i+1})\psi, H_{i+1})$ and $F_{i+1} = \min(m_{i-1} + \trop(g_{i-1})\psi, H'_{i-1})$. We then have the relationship $F_{i-1} \xleftarrow{v'} F_i' \xrightarrow{v'} F_{i+1}$ as required.
• **Case** \( F_{i-1} \not\rightarrow F_i \not\rightarrow F_{i+1} \). By construction we have \( F_{i-1} = \min(m_{i-1} + \text{trop}(g_{i-1}), H_{i-1}) \), \( F_i = \min(m_i + \text{trop}(g_{i-1}), H_{i-1}) = \min(m_i + \text{trop}(g_i), H_i) \), and \( F_{i+1} = \min(m_{i+1} + \text{trop}(g_i), H_i) \) for some \( g_{i-1}, g_i \in I, H_{i-1}, H_i \in S \), and \( m_{i-1}, m_i, m_{i+1} \in \mathbb{R} \). Write \( g_{i-1} = \sum c_u x^u \), and \( g_i = \sum d_u x^u \).

Set \( H'_i = \min(H_i, m_{i-1} + \text{val}(c_i) + x \cdot v) \). We have \( F_{i-1} = \min(m_i + \text{trop}(g_i), H'_i) \).

Set \( F'_i = \min(m_i + \text{trop}(g_i), H'_i) \). We now have two further subcases. Let \( b_{\nu'} \) be the coefficient of \( v' \) in \( H_i \).

1. \( b_{\nu'} \leq m_{i-1} + \text{val}(c_{\nu'}) \). Set \( H'_{i-1} = \min((H_{i-1})_{\nu'}, b_{\nu'} + x \cdot v') \). In this case \( F'_i \) equals \( \min(m_{i-1} + \text{trop}(g_{i-1}), H'_{i-1}) \). We then have \( F_{i+1} = \min(m_{i+1} + \text{trop}(g_i), H'_i) \), and thus \( F_{i-1} \not\rightarrow F'_i \not\rightarrow F_{i+1} \) as required.

2. \( b_{\nu'} > m_{i-1} + \text{val}(c_{\nu'}) \). This implies in particular that \( c_{\nu'} \neq 0 \). Let \( h = g_{i-1} - (c_{\nu'}/d_{\nu'}) g_i = \sum (c_u - d_u(c_{\nu'}/d_{\nu'})) x^u \). By construction \( v' \not\in \text{supp}(h) \).

Set \( H'_{i-1} = F_{i+1} \).

We claim that \( F'_i = \min(m_{i-1} + \text{trop}(h), H'_{i+1}) \). The coefficient of \( v' \) in \( F_i \) is \( m_i + \text{val}(d_{\nu'}) \), since \( F_i \neq F_{i+1} \), so comparing the two different expressions for \( F_i \) we see that \( m_i + \text{val}(d_{\nu'}) \leq m_{i-1} + \text{val}(c_{\nu'}) \). Thus \( \text{val}(c_{\nu'}/d_{\nu'}) \geq m_i - m_{i-1} \). The coefficient of \( u \) in \( m_{i-1} + \text{trop}(h) \) is \( m_{i-1} + \text{val}(c_u - d_u(c_{\nu'}/d_{\nu'})) \), which is at least \( m_{i-1} + \text{val}(c_u), m_i + \text{val}(d_u) \). For \( u \not\in \nu', \nu' \) both terms in this minimum are at least the coefficient of \( u \) in \( F_{i-1} \), which equals that in \( F_{i+1} = H'_{i+1} \). For \( u = v, m_{i-1} + \text{val}(c_{\nu'}) < m_i + \text{val}(d_{\nu'}) \), since \( F_{i-1} \neq F_i \), so \( \text{val}(c_{\nu'}/d_{\nu'}) = \text{val}(c_{\nu'}) \). The coefficient of \( v \) in \( \min(m_{i-1} + \text{trop}(h), H'_{i+1}) \) is then equal to \( m_{i-1} + \text{val}(c_{\nu'}) \), which equals the coefficient in \( F'_i \). Finally, the coefficient of \( v' \) is \( b_{\nu'} \), which is also the coefficient of \( v' \) in \( F'_i \).

Since the coefficient of \( v \) is \( m_{i-1} + \text{val}(c_{\nu'}) < \infty \), we have \( v \in \text{supp}(\text{trop}(h)) \), and thus \( F_{i+1} = \min(m_{i-1} + \text{trop}(h), H'_{i+1}) \). This again gives the relation \( F_{i-1} \not\rightarrow F'_i \not\rightarrow F_{i+1} \).

• **Case** \( F_{i-1} \not\rightarrow F_i \not\rightarrow F_{i+1} \). This is identical to the previous case with the roles of \( F_{i-1} \) and \( F_{i+1} \) reversed.

• **Case** \( F_{i-1} \not\rightarrow F_i \not\rightarrow F_{i+1} \). By construction \( F_i = \min(m_i + \text{trop}(g_i), H_i) = \min(m_i + \text{trop}(g'_i), H'_i) \), \( F_{i-1} = \min(m_i + \text{trop}(g_i), H_i) \), and \( F_{i+1} = \min(m_i + \text{trop}(g_i), H_i) \). Let \( \gamma_{v,j} \) be the coefficient of \( v \) in \( F_j \), for \( j = i-1, i, i+1 \). We have \( \gamma_{v,j+1} > \gamma_{v,i} \). If \( \gamma_{v,i+1} = \gamma_{v,i} \), then \( F_{i-1} = F_{i+1} \), so we may replace \( F_{i-1} \not\rightarrow F_i \not\rightarrow F_{i+1} \) by just \( F_{i-1} = F_{i+1} \). If \( \gamma_{v,i-1} < \gamma_{v,i+1} \), then \( \gamma_{v,i+1} \). Then \( F_{i-1} = \min(\gamma_{v,i-1} - m_i + \text{trop}(g'_i), H_i) \), and \( F_{i+1} = \min(\gamma_{v,i+1} - m_i + \text{trop}(g'_i), H_i) \). So we can replace \( F_{i-1} \not\rightarrow F_i \not\rightarrow F_{i+1} \) by \( F_{i-1} \not\rightarrow F_{i+1} \). If \( \gamma_{v,i-1} > \gamma_{v,i+1} \), then with the same construction we can replace \( F_{i-1} \not\rightarrow F_i \not\rightarrow F_{i+1} \) by \( F_{i-1} \not\rightarrow F_{i+1} \).

• **Case** \( F_{i-1} \not\rightarrow F_i \not\rightarrow F_{i+1} \). By construction \( F_{i-1} = \min(m_{i-1} + \text{trop}(g_{i-1}), H_{i-1}) \), \( F_i = \min(m_i + \text{trop}(g_{i-1}), H_{i-1}) = \min(m_i + \text{trop}(g_i), H_i) \), and \( F_{i+1} = \min(m_i + \text{trop}(g_i), H_i) \). Set \( H_i = F_{i+1} \). Then \( F_{i-1} = \min(m_{i-1} + \text{trop}(g_{i-1}), H_i) \), and \( F_{i+1} = \min(m_{i-1} + \text{trop}(g_{i-1}), H_i) \). So we may replace \( F_{i-1} \not\rightarrow F_i \not\rightarrow F_{i+1} \) by \( F_{i-1} \not\rightarrow F_{i+1} \).
• **Case** $F_{i-1} \prec F_i \preceq F_{i+1}$. This is identical to the previous case, with the roles of $F_{i-1}$ and $F_{i+1}$ reversed. □

The first of the six cases of this proof is the only one that cannot be reversed. Indeed, in the congruence $\mathcal{Trop}( (x+y) )$ we have the relations $x \sim \min(x, y) \sim y$ but neither $x \sim \infty$ nor $y \sim \infty$.

A congruence $J$ on $\mathcal{S}$ or $S$ is *homogeneous* with respect to a grading by $\deg(x_i) = \delta_i \in \mathbb{Z}$ if $J$ is generated by relations of the form $F \sim G$ where $F$ and $G$ are both homogeneous of the same degree. For a tropical polynomial $F$ we write $F_d$ for its homogeneous component of degree $d$, where $d \in \mathbb{Z}$.

**Proposition 2.7.** Let $J$ be a homogeneous congruence on $\mathcal{S}$ or $S$. If $F \sim G \in J$, then $F_d \sim G_d \in J$ for all $d \in \mathbb{Z}$.

*Proof.* Let $\mathcal{J} = \{ F_\alpha \sim G_\alpha : \alpha \in A \}$ be a homogeneous generating set for $J$, and fix $F \sim G \in J$. By Lemma 2.4 there is a chain $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ of relations in $J$ where $F_i \sim F_{i+1}$ has the form $\min(M_i + H_i, P_i) \sim \min(M_i + H_{i+1}, P_i)$ with $H_i \sim H_{i+1} \in \mathcal{J}$, $M_i$ a monomial in $S$, and $P_i \in S$. The $d$th graded piece of $F_i \sim F_{i+1}$ either has the form $Q_i \sim Q_i$ or $\min(M_i + H_i, Q_i) \sim \min(M_i + H_{i+1}, Q_i)$ where $Q_i$ is homogeneous of degree $d$. Thus $(F_i)_d \sim (F_{i+1})_d \in J$, so $F_d \sim G_d \in J$. □

We will also need the notion of the homogenization of a congruence on $S$. This will play the same role as the homogenization of an ideal in the usual Laurent polynomial ring with coefficients in $K$. Geometrically, this is the tropical analogue of taking the projective closure of a subvariety of $(K^*)^n$.

Recall that the homogenization of a polynomial $f = \sum c_u x^u \in K[x_1, \ldots, x_n]$ is the polynomial $\tilde{f} := \sum c_u x^u x_0^{\deg(f) - |u|} \in K[x_0, \ldots, x_n]$, where $|u| = u_1 + \cdots + u_n$, and $\deg(f)$ is the maximum of $|u|$ for which $c_u \neq 0$. For an ideal $I \subset K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$, its homogenization is the ideal $I^h = \langle \tilde{f} : f \in I \cap K[x_1, \ldots, x_n] \rangle$.

**Definition 2.8.** Given $F = \min(a_u + x \cdot u) \in S$, we denote by $\deg(F)$ the maximum $\max(|u| : a_u \neq \infty)$. If $\text{supp}(F) \subset \mathbb{N}^n$, we write $\tilde{F}$ for the tropical polynomial in $\mathcal{S}$ given by

$$\tilde{F} := \min( a_u + x \cdot u + (\deg(F) - |u|)x_0 ) .$$

If $F \sim G$ is a relation, where $F, G$ are tropical polynomials (with $\text{supp}(F), \text{supp}(G) \subset \mathbb{N}^n$) satisfying $\deg(F) \geq \deg(G)$, its homogenization is

$$\tilde{F} \sim \tilde{G} := \tilde{F} \sim ( \tilde{G} + (\deg(F) - \deg(G))x_0 ) .$$

Let $J$ be a congruence on $S$. The homogenization $J^h$ of $J$ is the congruence

$$J^h := \langle \tilde{F} \sim \tilde{G} : F \sim G \in J \text{ and } \text{supp}(F), \text{supp}(G) \subset \mathbb{N}^n \rangle .$$

**Proposition 2.9.** Let $I$ be an ideal in $K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$, and let $I^h \subset K[x_0, \ldots, x_n]$ be its homogenization. Then we have the equality of congruences on $\mathcal{S}$

$$\mathcal{Trop}(I^h) = \mathcal{Trop}(I)^h .$$
Proof. Let $f$ be a homogeneous polynomial in $I^h$. Write $g = f|_{x_0=1}$. Note that $g \in I$, and $f = x_0^n \tilde{g}$ for some $a \geq 0$. For a monomial $x^u \in K[x_0, \ldots, x_n]$ write $u'$ for the projection of $u$ onto the last $n$ coordinates.

Choose $u \in \text{supp}(f)$, and consider the relation $\text{trop}(f) \sim \text{trop}(f)u \in \mathcal{T}rop(I^h)$. The homogenization of the relation $\text{trop}(g) \sim \text{trop}(g)u' \in \mathcal{T}rop(I)$ is

$$\widetilde{\text{trop}}(g) \sim (\deg(g) - \deg(gu'))x_0 + \text{trop}(g)u'.$$

Adding $ax_0 \sim ax_0$ to this relation gives the relation $\text{trop}(f) \sim \text{trop}(f)u$. Since these relations generate $\mathcal{T}rop(I^h)$, it follows that $\mathcal{T}rop(I^h) \subset \mathcal{T}rop(I)^h$.

For the converse, it suffices to consider a relation of the form $F \sim \sim G$ for $F \sim G \in \mathcal{T}rop(I)$ with both $F, G$ non-Laurent tropical polynomials, and show that it is a relation in $\mathcal{T}rop(I^h)$. By Proposition 2.6 we can find a chain $F = F_0 \sim F_1 \sim \cdots \sim F_s \sim F_{s+1} = G$ with $F_i \sim F_{i+1} \in \mathcal{T}rop(I)$ of the form $\min(a_i + \text{trop}(h_i), H_i) \sim \min(a_i + \text{trop}(h_i), H_i)$ for some $h_i \in I$, $a_i \in \mathbb{R}$, and $H_i \in S$, and for which the coefficient of $u$ in $F_i$ equals the coefficient of $u$ in either $F$ or $G$. This latter condition implies that if $u \notin \text{supp}(F) \cup \text{supp}(G)$ then $u \notin \text{supp}(F_i)$, so in particular each $F_i$ is a non-Laurent tropical polynomial and $\deg(F_i) \leq \max(\deg(F), \deg(G))$. The homogenization of the relation $\min(a_i + \text{trop}(h_i), H_i) \sim \min(a_i + \text{trop}(h_i), H_i)$ equals

$$\min(a_i + \text{trop}(h_i) + bx_0, H_i + dx_0) \sim \min(a_i + \text{trop}(h_i) + bx_0, H_i + dx_0),$$

where $v' \in \mathbb{N}^{n+1}$ has last $n$ coordinates equal to $v$, and the numbers $b, d$ satisfy $b = \max(\deg(H_i) - \deg(h_i), 0)$ and $d = \max(\deg(h_i) - \deg(H_i), 0)$. Since $h_i \in I^h$, we have $\text{trop}(h_i) \sim \text{trop}(h_i)v' \in \mathcal{T}rop(I^h)$, and so $F_i \sim \sim F_{i+1} \in \mathcal{T}rop(I^h)$.

Each relation $F_i \sim \sim F_{i+1}$ is homogeneous of degree at most $\max(\deg(F), \deg(G))$. The righthand side of $F_{i-1} \sim F_i$ and the lefthand side of $F_i \sim \sim F_{i+1}$ are either identical or differ by a multiple $bx_0$, with $b \in \mathbb{N}$ equal to the difference between their degrees. Thus we can add a multiple of $x_0$ to both sides of the lower degree relation to get two relations whose adjacent terms coincide. Doing this for the string $F_0 \sim F_1, \ldots, F_s \sim \sim F_{s+1}$ gives a chain of relations in $\mathcal{T}rop(I^h)$ of the same degree, whose first entry is $ax_0 + \tilde{F}$ and whose last entry is $bx_0 + \tilde{G}$ for some $a, b \in \mathbb{N}$ with at most one of $a$ and $b$ nonzero. Taking the transitive closure we get $ax_0 + \tilde{F} \sim bx_0 + \tilde{G} \in \mathcal{T}rop(I^h)$. This relation equals $F \sim \sim G$, which completes the proof. \hfill $\Box$

Note that the use of Proposition 2.6 was key in the proof of Proposition 2.9.

We are now in position to prove the equivalence (1) $\Leftrightarrow$ (2) of Theorem 1.1 from the introduction.

Proof of (1) $\Leftrightarrow$ (2) of Theorem 1.1. We first show that the ideal $\text{trop}(I)$ determines the congruence $\mathcal{T}rop(I)$. For a tropical polynomial $F \in \text{trop}(I)$ and $u \in \text{supp}(F)$ we can form the relation $F \sim F_u$. Any $F \in \text{trop}(I)$ has the form $\min_{1 \leq i \leq s}(a_i + \text{trop}(f_i))$ for some $f_1, \ldots, f_s \in I$ and $a_i \in \mathbb{R}$. The polynomial $F_u$ is then $\min(a_i + \text{trop}(f_i)u)$, where we set $\text{trop}(f_i)u = \text{trop}(f_i)$ if $u \notin \text{supp}(f_i)$. Thus $F \sim F_u$ equals the minimum of $(a_i + \text{trop}(f_i)) \sim (a_i + \text{trop}(f_i)u)$ for $1 \leq i \leq s$, so it lies in $\mathcal{T}rop(I)$. The congruence $\mathcal{T}rop(I)$ is generated by relations of the form $\text{trop}(f) \sim \text{trop}(f)u$ for $f \in I$. These
are of the form $F \sim F_\mathbf{u}$ for $F \in \text{trop}(I)$, so $\mathcal{T}rop(I)$ equals the congruence generated by $\{F \sim F_\mathbf{u} : F \in \text{trop}(I), \mathbf{u} \in \text{supp}(F)\}$.

Conversely, Proposition 2.9 implies that the congruence $\mathcal{T}rop(I)$ determines the congruence $\mathcal{T}rop(I^h)$ on $\mathcal{S}$, where $I^h \subset K[x_0, \ldots, x_n]$ is the homogenization of the ideal $I$. We may regard any homogeneous polynomial $f \in I^h_d$ as a linear form $l_f$ on the affine space $\mathbb{A}^{(n+d)}$ whose coordinates are indexed by the monomials of degree $d$ in $K[x_0, \ldots, x_n]$. Similarly, we may regard a homogeneous tropical polynomial $F$ of degree $d$ as a tropical linear form on $\mathbb{R}^{(n+d)}$. Let $L_d$ be the subspace of $\mathbb{A}^{(n+d)}$ on which the linear forms $l_f$ vanish for all $f \in I^h_d$, and let $\ell_d \subset \mathbb{R}^{(n+d)}$ be its tropicalization. We have

$$\ell_d = \{\mathbf{z} \in \mathbb{R}^{(n+d)} : \text{the minimum in } \text{trop}(l_f)(\mathbf{z}) \text{ is achieved twice for all } f \in I^h_d\}$$

$$= \{\mathbf{z} \in \mathbb{R}^{(n+d)} : \text{trop}(l_f)(\mathbf{z}) = \text{trop}(l_f)_{\mathbf{u}}(\mathbf{z}) \text{ for all } f \in I^h_d, \mathbf{u} \in \text{supp}(f)\}$$

$$= \{\mathbf{z} \in \mathbb{R}^{(n+d)} : l_{\text{trop}(f)}(\mathbf{z}) = l_{\text{trop}(f)_{\mathbf{u}}}(\mathbf{z}) \text{ for all } f \in I^h_d, \mathbf{u} \in \text{supp}(f)\},$$

so $\mathcal{T}rop(I^h)$ determines the collection of tropical linear spaces $\{\ell_d\}_{d \geq 0}$. Furthermore, the tropical linear space $\ell_d$ determines its dual tropical linear space $\ell_d^\perp$, which is the tropicalization of the linear subspace $L_d^\perp$ orthogonal to $L_d$. A vector lies in $L_d^\perp$ if and only if it is the coefficient vector of a polynomial $f \in I^h_d$. It follows that the tropical linear space $\ell_d^\perp$ is equal to the degree $d$ part $\text{trop}(I^h)_d$ of the ideal $\text{trop}(I^h)$. Thus $\mathcal{T}rop(I^h)$ determines $\text{trop}(I^h)$, and since $\text{trop}(I)$ is the ideal in $\mathcal{S}$ generated by $\{F|_{x_0=0} : F \in \text{trop}(I^h)\}$, it also determines $\text{trop}(I)$. $\square$

**Remark 2.10.** The tropical linear spaces $\ell_d$ also encode the valuated matroids of the vector spaces $I^h_d$, so we can see the third equivalence of Theorem 1.1 from the previous argument as well. This is elaborated on in Section 4.

### 3. Multiplicities

In this section we prove Theorem 1.2. The strategy is to define a Gröbner theory for congruences on the semiring of tropical polynomials, which lets us determine the multiplicities from the tropical scheme.

We first recall the definition of multiplicity for maximal cells of a tropical variety. For an irreducible $d$-dimensional subvariety $Y \subset (K^*)^n$ the tropical variety $\text{trop}(Y) \subset \mathbb{R}^n$ is the support of a pure $d$-dimensional $\Gamma$-rational polyhedral complex. This means that $\text{trop}(Y)$ is the union of a set $\Sigma$ of $d$-dimensional polyhedra of the form $\{\mathbf{w} \in \mathbb{R}^n : A\mathbf{w} \leq \mathbf{b}\}$ where $A \in \mathbb{Q}^{r \times n}$ and $\mathbf{b} \in \Gamma^r$ for some $r \in \mathbb{N}$, and these polyhedra intersect only along faces. See [MS13, Chapter 3] for more details.

Let $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ideal of $Y$. Fix a group homomorphism $\Gamma \to K^*$, which we write $w \mapsto t^w$, satisfying $\text{val}(t^w) = w$. This may require replacing $K$ by an extension field; see [MS13, Chapter 2]. For $a$ in the valuation ring $R$ we write $\overline{a}$ for its image in $K$. Fix $\mathbf{w}$ in the relative interior of a $d$-dimensional polyhedron $\sigma \in \Sigma$. We denote by $\text{in}_\mathbf{w}(I) \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ the initial ideal of $I$ with respect to $\mathbf{w}$, in the sense described in [MS13, §2.4]. This is the ideal $\text{in}_\mathbf{w}(I) := \langle \text{in}_\mathbf{w}(f) : f \in I \rangle,$
where for \( f = \sum c_u x^u \) the initial form \( \text{in}_w(f) \) equals \( \sum_{\text{val}(c_u) = \gamma} t^{-\text{val}(c_u)} c_u x^u \), with \( \gamma = \min(\text{val}(c_u) + w \cdot u) = \text{trop}(f)(w) \).

The multiplicity of \( w \) is the multiplicity of the initial ideal \( \text{in}_w(I) \):

\[
\text{mult}(w) := \sum_P \text{mult}(P, \text{in}_w(I)),
\]

where the sum is over the minimal associated primes of \( \text{in}_w(I) \), and \( \text{mult}(P, \text{in}_w(I)) \) is the multiplicity of the associated primary component. See [MS13, Chapter 4] for more details. If coordinates on the torus \((K^*)^n \) have been chosen so that \( \text{in}_w(I) \) has a generating set involving only the variables \( x_{d+1}, \ldots, x_n \), then

\[
\text{mult}(w) = \dim_k \left( \mathbb{k}[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}] / \text{in}_w(I) \cap \mathbb{k}[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}] \right)
\]

(see [MS13, Lemma 3.44]).

We now extend the definition of initial ideals to congruences on \( \tilde{S} \) and \( S \).

**Definition 3.1.** Let \( F = \min(a_u + x \cdot u) \in \tilde{S} \) and \( w \in \mathbb{R}^{n+1} \). The initial form of \( F \) with respect to \( w \) is the tropical polynomial in \( \mathbb{B}[x_0, \ldots, x_n] \)

\[
\text{in}_w(F) := \min_{a_u + w \cdot u = F(w)} (x \cdot u).
\]

For \( G = \min(b_u + x \cdot u) \in \tilde{S} \), let \( \gamma = \min(F(w), G(w)) \). The initial form of the relation \( F \sim G \) with respect to \( w \) is the relation

\[
\text{in}_w(F \sim G) := \min_{a_u + w \cdot u = \gamma} (x \cdot u) \sim \min_{b_u + w \cdot u = \gamma} (x \cdot u).
\]

Note that if \( F(w) = G(w) \) then this is \( \text{in}_w(F) \sim \text{in}_w(G) \), but if \( F(w) < G(w) \) then this is \( \text{in}_w(F) \sim \infty \).

For a congruence \( J \) on \( \tilde{S} \), the initial congruence of \( J \) with respect to \( w \) is the congruence on \( \mathbb{B}[x_0, \ldots, x_n] \)

\[
\text{in}_w(J) := \langle \text{in}_w(F \sim G) : F \sim G \in J \rangle.
\]

The initial form with respect to \( w \in \mathbb{R}^n \) of a relation between tropical Laurent polynomials and the initial congruence of a congruence on \( S \) are defined analogously.

**Example 3.2.** Consider \( S = \mathbb{R}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}] \), and let \( F = \min(0+x, 1+y, 2+z) \in S \). For \( u = (1, 0, 0) \) we have the relation \( F \sim F_u \), which is \( \min(0 + x, 1 + y, 2 + z) \sim \min(1 + y, 2 + z) \). If \( w = (2, 1, 3) \), the initial form \( \text{in}_w(F \sim F_u) \) of this relation is \( \min(x, y) \sim y \). For \( w = (1, 2, 2) \) the initial form is \( x \sim \infty \).

As in standard Gröbner theory, the initial congruence of a congruence generated by \( \{F_\alpha \sim G_\alpha\}_{\alpha \in A} \) for some set \( A \) is not necessarily generated by \( \{\text{in}_w(F_\alpha \sim G_\alpha)\}_{\alpha \in A} \). For example, for \( w = (0, 1, 2) \) and the congruence \( J \) on \( \mathbb{R}[x, y, z] \) generated by \( \{x \sim y, x \sim z\} \), we have \( y \sim z \in J \), so \( y \sim \infty \in \text{in}_w(J) \). However, the initial form of both \( x \sim y \) and \( x \sim z \) is \( x \sim \infty \), and \( y \sim \infty \not\in \langle x \sim \infty \rangle \).

Definition 3.1 is designed to commute with tropicalization of polynomials, as the following lemma shows.
Lemma 3.3. For \( f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) and \( w \in \mathbb{R}^n \) we have
\[
in_w(\text{trop}(f)) = \text{trop}(in_w(f)).
\]
The same holds for \( f \in K[x_0, \ldots, x_n] \) and \( w \in \mathbb{R}^{n+1} \).

Proof. Suppose \( f = \sum c_u x^u \) with \( c_u \in K \), so \( \text{trop}(f) = \min(\text{val}(c_u) + x \cdot u) \). Let \( \gamma = \text{trop}(f)(w) \). By definition, \( \text{in}_w(f) = \sum_{\text{val}(c_u) + w \cdot u = \gamma} \frac{t^{\text{val}(c_u)} c_u x^u}{\mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]} \). Thus \( \text{trop}(\text{in}_w(f)) = \min(\text{val}(c_u) + w \cdot u = \gamma(x \cdot u)) = \text{in}_w(\text{trop}(f)) \), as claimed.

The first key result of this section is the following, which says that taking congruences commutes with taking initial ideals.

Proposition 3.4. Let \( I \) be an ideal in \( K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \). Then for \( w \in \mathbb{R}^n \) we have
\[
in_w(Trop(I)) = Trop(in_w(I)).
\]

Proof. Fix \( w \in \mathbb{R}^n \). The congruence \( Trop(in_w(I)) \) is generated by relations of the form \( \text{trop}(g) \sim \text{trop}(g) \cdot v \) for \( g \in \text{in}_w(I) \) and \( v \in \text{supp}(g) \). We first note that we can write \( g = \sum \text{in}_w(f_i) \) for some \( f_i \in I \) with \( \text{supp}(\text{in}_w(f_i)) \cap \text{supp}(\text{in}_w(f_j)) = \emptyset \) if \( i \neq j \). Indeed, if \( g = \sum a_i x^u \cdot \text{in}_w(f_i) \) for \( a_i \in \mathbb{k} \) and \( f_i \in I \), then for \( c_i \in R \) with \( c_i = a_i \) we have \( g = \sum \text{in}_w(c_i x^u f_i) \), so we may assume that \( u_i = 0 \) and \( a_i = 1 \). If the minimum in both \( \text{trop}(f_i)(w) \) and \( \text{trop}(f_j)(w) \) is achieved at the term involving \( u \), where the coefficient of \( x^u \) in \( f_i \) is \( c \) and the coefficient in \( f_j \) is \( d \), then \( \gamma := \text{trop}(f_j)(w) - \text{trop}(f_i)(w) = \text{val}(d) - \text{val}(c) \in \Gamma \), and we can find \( \alpha \in K \) with \( \text{val}(\alpha) = \text{val}(d) - \text{val}(c) \) and \( \alpha t^{-\text{val}(\alpha)} = 1 \). Then we have \( h = f_j + \alpha f_i \in I \), and \( \text{in}_w(h) = \text{in}_w(f_i) + \text{in}_w(f_j) \). We may thus replace \( f_i, f_j \) by \( h \), and repeat this procedure until the supports of the \( \text{in}_w(f_i) \) are disjoint. Note that this implies that \( \text{trop}(g) \sim \text{min}(\text{trop}(\text{in}_w(f_i))) \).

Now, for \( v \in \text{supp}(\text{in}_w(f_i)) \) we can write \( H = \min_{i=2}^\infty(\text{trop}(\text{in}_w(f_i))) \), so \( \text{trop}(g) \sim \text{trop}(g) \cdot v \) is equal to \( \text{min}(\text{trop}(\text{in}_w(f_i)), H) \sim \text{min}(\text{trop}(\text{in}_w(f_i)) \cdot v, H) \). This shows that \( Trop(\text{in}_w(I)) \) is generated by relations of the form \( \text{trop}(\text{in}_w(f)) \sim \text{trop}(\text{in}_w(f)) \cdot v \). Since \( \text{min}(\text{trop}(f)(w), \text{trop}(f) \cdot v(w)) = \text{trop}(f)(w) \), we have that \( \text{in}_w(\text{trop}(f) \sim \text{trop}(f) \cdot v) = \text{in}_w(\text{trop}(f)) \sim \text{in}_w(\text{trop}(f)) \cdot v \), so by Lemma 3.3, \( \text{in}_w(\text{trop}(f) \sim \text{trop}(f) \cdot v) = \text{trop}(\text{in}_w(f)) \cdot v \). Note that the term \( \text{trop}(\text{in}_w(f)) \cdot v \) may equal \( \infty \). This proves the containment \( \text{in}_w(Trop(I)) \supseteq Trop(\text{in}_w(I)) \).

For the reverse inclusion, let \( (F' \sim G') = \text{in}_w(F \sim G) \) be a generator of the congruence \( \text{in}_w(Trop(I)) \), where \( F \sim G \in Trop(I) \). Fix a chain \( F = F_0 \sim F_1 \sim \cdots \sim F_s = F_{s+1} = G \) in \( Trop(I) \) with \( F_i = \min(m_i + \text{trop}(g_i), H_i) \) and \( F_{i+1} = \min(m_i + \text{trop}(g_i) \cdot v, H_i) \) (or the reverse), satisfying the conditions of Proposition 2.6. In particular, we have \( \gamma := \min(F(w), G(w)) \leq F_i(w) \) for all \( i \). For any \( F_i = \min(a_u + x \cdot u) \) in this chain, define \( F'_i := \min_{a_u + x \cdot u = \gamma} (x \cdot u) \). Note that \( F'_i \) might be equal to \( \infty \). We claim that the chain \( F' = F'_0 \sim F'_1 \sim \cdots \sim F'_s \sim F'_{s+1} = G' \) is a chain of relations in \( Trop(\text{in}_w(I)) \). It follows that \( F' \sim G' \in Trop(\text{in}_w(I)) \), completing the proof.

To prove the claim, consider first the case where \( F_i(w) = F_{i+1}(w) = \gamma \) for some \( i \). If \( m_i + \text{trop}(g_i)(w) > H_i(w) = \gamma \) then \( (F_i \sim F_{i+1}) = (\text{in}_w(H_i) \sim \text{in}_w(H_i)) \). If
$m_i + \text{trop}(g_i)(w) = H_i(w) = \gamma$ then $F'_i \sim F'_{i+1}$ equals either

$$\min(\text{in}_w(\text{trop}(g_i)), \text{in}_w(H_i)) \sim \min(\text{in}_w(\text{trop}(g_i)_u, \text{in}_w(H_i)),$$

where we note that $\text{in}_w(\text{trop}(g_i))_u$ may equal $\infty$. If $\gamma = m_i + \text{trop}(g_i)(w) < H_i(w)$ then $F'_i \sim F'_{i+1}$ is equal to $\text{in}_w(\text{trop}(g_i)) \sim \text{in}_w(\text{trop}(g_i))_u$. In all cases, Lemma 3.3 ensures that the relation $F'_i \sim F'_{i+1}$ is in $\mathcal{Trop}(\text{in}_w(I))$. Now, suppose that $F_i(w) < F_{i+1}(w)$. If $\gamma = F_i(w)$ then $\text{trop}(g_i)(w) < H_i(w)$ and $\text{in}_w(g_i)$ is a monomial. This means that $(F'_i \sim F'_{i+1}) = (\text{trop}(\text{in}_w(g_i)) \sim \infty) \in \mathcal{Trop}(\text{in}_w(I))$. Finally, if $\gamma < F_i(w)$ then $F'_i \sim F'_{i+1}$ is the relation $\infty \sim \infty$, which is in $\mathcal{Trop}(\text{in}_w(I))$. \qed

Note that the second condition in Proposition 2.6 was crucial in this proof.

We are now ready to prove Theorem 1.2. The proof requires understanding the effect of changes of coordinates on tropical varieties and congruences. The group $GL(n, \mathbb{Z})$ acts on $S$ by monomial change of coordinates. Explicitly, a matrix $A$ sends a tropical polynomial $f(x) = \min(a_u + x \cdot u)$ to $\min(a_u + x \cdot Au) = f(A^T x)$. We write $A \cdot f$ for this transformed polynomial. If $J$ is a congruence on $S$ then $A \cdot J$ is the congruence generated by $\{A \cdot f \sim A \cdot g : f \sim g \in J\}$. This action is the tropicalization of the action of $GL(n, \mathbb{Z})$ on $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ that sends any monomial $x^u$ to $x^{Au}$. Moreover, the action commutes with tropicalization: We have $\text{trop}(A \cdot f) = A \cdot \text{trop}(f)$. In particular, this implies that if $I$ is an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ then $\text{trop}(V(A \cdot I)) = A \cdot \text{trop}(V(I))$; see [MS13, Corollary 3.2.13].

**Proof of Theorem 1.2.** Let $w$ lie in the relative interior of a maximal cell $\sigma$ of the tropical variety $\text{trop}(V(I))$, and let $L = \text{span}(w - w' : w' \in \sigma)$. After a monomial change of coordinates we may assume that $L = \text{span}(e_1, \ldots, e_d)$. By [MS13, Lemma 3.3.7] we have $L = \text{trop}(V(\text{in}_w(I)))$, so Equation (2.1) implies that $L$ can be recovered from the congruence $\mathcal{Trop}(\text{in}_w(I)) = \text{in}_w(\mathcal{Trop}(I))$. This means that $L$ is determined by $\text{in}_w(\mathcal{Trop}(I))$, and thus by $\mathcal{Trop}(I)$.

By [MS13, Corollary 2.4.9] the initial ideal $\text{in}_w(I)$ is homogeneous with respect to the $\mathbb{Z}^d$-grading by $\deg(x_i) = e_i$ for $1 \leq i \leq d$ and $\deg(x_i) = 0$ otherwise, so it has a generating set $f_1, \ldots, f_r$ where $f_i \in k[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$. By [MS13, Lemma 3.4.6], the multiplicity of $\sigma$ equals the dimension $\dim_k(R'/(\text{in}_w(I) \cap R'))$, where $R' = k[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $\text{in}_w(I)^h \subset k[x_0, \ldots, x_n]$ be the homogenization of the ideal $\text{in}_w(I) \cap k[x_1, \ldots, x_n]$. Note that since $R'/(\text{in}_w(I) \cap R')$ is zero-dimensional, the Hilbert polynomial of $k[x_0, x_{d+1}, \ldots, x_n]/(\text{in}_w(I)^h)$ is equal to the constant polynomial $\dim_k(R'/(\text{in}_w(I) \cap R'))$, and thus equals $\text{mult}(w)$. By [GG13, Theorem 7.1.5] the Hilbert polynomial of a homogeneous ideal $J$ can be recovered from its tropicalization $\mathcal{Trop}(J) \subset \hat{S}$, so to show that $\text{mult}(w)$ can be recovered from $\mathcal{Trop}(I)$ it is enough to show that $\mathcal{Trop}(\text{in}_w(I)^h)$ can be recovered from $\mathcal{Trop}(I)$. By Proposition 2.9 we have $\mathcal{Trop}(\text{in}_w(I)^h) = \mathcal{Trop}(\text{in}_w(I))^h$, and by Proposition 3.4 we have $\mathcal{Trop}(\text{in}_w(I))^h = \text{in}_w(\mathcal{Trop}(I))^h$, so the result follows. \qed

We can thus recover the tropical cycle from the tropical scheme. This can be considered as a tropicalization of the Hilbert-Chow morphism that takes a scheme to the underlying cycle.
4. Tropical schemes and valuated matroids

In this section we investigate in more depth the structure of the equivalence classes of \( \text{Trop}(I) \). We restrict our attention to the case where \( I \) is a homogeneous ideal in the polynomial ring \( K[x_0, \ldots, x_n] \); an understanding in this case extends to ideals in \( K[x_1^\pm, \ldots, x_n^\pm] \) using Proposition 2.9. We prove that any homogeneous tropical polynomial \( F \in \mathcal{S} \) has a distinguished representative in its equivalence class, and we give a computationally tractable description of it. The combinatorial machinery that naturally keeps track of the information contained in the congruence \( \text{Trop}(I) \) is that of valuated matroids.

Valuated matroids are a generalization of the notion of matroids that were introduced by Dress and Wenzel in [DW92]. Our sign convention is, however, the opposite of theirs. For basics of standard matroids, see, for example, [Oxl92].

Let \( E \) be a finite set, and let \( r \in \mathbb{N} \). Denote by \( \binom{E}{r} \) the collection of subsets of \( E \) of size \( r \). A valuated matroid \( \mathcal{M} \) on the ground set \( E \) is a function \( p : \binom{E}{r} \to \mathbb{R} \) satisfying the following properties.

1. There exists \( B \in \binom{E}{r} \) such that \( p(B) \neq \infty \).
2. Tropical Plücker relations: For every \( B, B' \in \binom{E}{r} \) and every \( u \in B - B' \) there exists \( v \in B' - B \) with
   \[
   p(B) + p(B') \geq p(B - u \cup v) + p(B' - v \cup u).
   \]
The support
   \[
   \text{supp}(p) := \{ B \in \binom{E}{r} : p(B) \neq \infty \}
   \]
is the collection of bases of a rank \( r \) matroid on the ground set \( E \), called the underlying matroid \( \mathcal{M} \) of \( \mathcal{M} \). The function \( p \) is called the basis valuation function of \( \mathcal{M} \).

We denote by \( M_d \) the set of monomials of degree \( d \) in the variables \( x_0, \ldots, x_n \). Any homogeneous polynomial \( f \in K[x_0, \ldots, x_n] \) of degree \( d \) can be regarded as a linear form \( l_f \) on the \( K \)-vector-space \( V_d \) with basis \( M_d \). Let \( I_d \) be the degree \( d \) part of the ideal \( I \). Consider the linear subspace
   \[
   L_d := \{ y \in V_d : l_f(y) = 0 \text{ for all } f \in I_d \} \subset V_d.
   \]
Under the pairing \( \langle \cdot, \cdot \rangle : K[x_0, \ldots, x_n]_d \times V_d \to K \) defined by \( \langle f, y \rangle := l_f(y) \), the linear subspace \( L_d \) is orthogonal to \( I_d \). Let \( r_d = \dim(L_d) \). The linear subspace \( L_d \) determines a point in the Grassmannian \( \text{Gr}(r_d, V_d) \). The coordinates of this point in the Plücker embedding of \( \text{Gr}(r_d, V_d) \) into \( \mathbb{P}^N \), where \( N := \binom{M_d}{r_d} - 1 \), are called the Plücker coordinates of \( L_d \) (and dually of \( I_d \)). They are indexed by subsets of \( M_d \) of size \( r_d \).

**Definition 4.1.** The valuated matroid \( \mathcal{M}(I_d) \) of \( I_d \) is the function \( p_d : \binom{M_d}{r_d} \to \mathbb{R} \) given by setting \( p_d(B) \) to be the valuation of the Plücker coordinate of \( L_d \in \text{Gr}(r_d, \binom{M_d}{r_d}) \) indexed by \( B \). We denote by \( \overline{\mathcal{M}}(I_d) \) the underlying matroid of \( \mathcal{M}(I_d) \).

A valuated matroid that comes from taking the valuation of Plücker coordinates is called realizable. The function \( p_d \) is the tropical Plücker vector associated to the tropical linear space \( \text{trop}(L_d) \); it completely determines \( \text{trop}(L_d) \) [SS04, Theorem].
While we will only deal with realizable valuated matroids in what follows, all the statements and proofs in this section can be carried out using only matroid-theoretic techniques, and do not require realizability.

Usual matroids have several different “cryptomorphic” definitions, and the same is true for valuated matroids. In the underlying matroid $\mathcal{M}(I_d)$, a subset of monomials $A \subset M_d$ is dependent if and only if there exists a polynomial $h \in I_d$ with supp$(h) \subset A$. Thus $C \subset M_d$ is a circuit of $\mathcal{M}(I_d)$ if and only if $C = \text{supp}(h)$ for some $h \in I_d$ of minimal support. A tropical polynomial $H \in \text{trop}(I)_d$ is called a vector of the valuated matroid $\mathcal{M}(I_d)$. Such an $H$ has the form $\min(a_i + \text{trop}(f_i) : 1 \leq i \leq s)$ for some $f_i \in I_d$ and $a_i \in \mathbb{R}$. Vectors of minimal support are called valuated circuits of $\mathcal{M}(I_d)$. These all have the form $H = a + \text{trop}(h)$ for some $h \in I_d$ of minimal support and $a \in \mathbb{R}$. If $H$ and $G$ are valuated circuits of $\mathcal{M}(I_d)$ with the same support then there exists some $a \in \mathbb{R}$ such that $H = a + G$. The set of vectors and the set of valuated circuits of $\mathcal{M}(I_d)$ each separately determines $\mathcal{M}(I_d)$; see [MT01, Theorem 3.3].

With these definitions in place, we can now finish the proof of Theorem 1.1. We restate it in a slightly generalized form, allowing more general projective schemes.

**Theorem 4.2.** Let $Z \subset \mathbb{P}^n$ be a subscheme defined by a homogeneous ideal $I \subset K[x_0, \ldots, x_n]$. Then any of the following three objects determines the others:

1. The congruence $\text{Trop}(I)$ on $\tilde{S}$;
2. The ideal $\text{trop}(I)$ in $\tilde{S}$;
3. The set of valuated matroids $\{\mathcal{M}(I_d)\}_{d \geq 0}$, where $I_d$ is the degree $d$ part of $I$.

Thus if $Y \subset T \cong (K^*)^n$ is a subscheme given by an ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and $I^h \subset K[x_0, \ldots, x_n]$ is the ideal of the projective closure $\overline{Y} \subset \mathbb{P}^n$ of $Y$, then the ideal $\text{trop}(I) \subset S$ and the set of valuated matroids $\mathcal{M}(I_d^h)$ for $d \geq 0$ determine each other.

**Proof.** The proof that (1) determines (2) given at the end of Section 2 included the proof for general homogeneous ideals, as we never used that $I^h$ was a homogenization. The proof given there that (2) determines (1) is also valid for homogeneous ideals.

The elements of $\text{trop}(I)_d$ are the vectors of the valuated matroid $\mathcal{M}(I_d)$, so $\text{trop}(I)$ determines and is determined by the set of valuated matroids $\{\mathcal{M}(I_d)\}_{d \geq 0}$. This shows (2) $\iff$ (3).

When $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the ideal $\text{trop}(I^h)$ in $\tilde{S}$ is the homogenization of the ideal $\text{trop}(I)$ in $S$, and also $\text{trop}(I) = \text{trop}(I^h)|_{x_0=0}$. This shows that $\text{trop}(I)$ determines $\text{trop}(I^h)$ and conversely, so the last part follows from the first. \hfill $\Box$

We now investigate in more depth the structure of the equivalence classes of $\text{Trop}(I)$. In what follows, for any homogeneous tropical polynomial $F \in \tilde{S}_d$ and any $u \in M_d$, we denote by $F^u$ the coefficient of $F$ corresponding to the monomial $u$. For $F, G \in \tilde{S}_d$, we say that $F \leq G$ if the inequality holds coefficient-wise, so $F^u \leq G^u$ for all $u \in M_d$.

We will restrict our attention to the case where the subspace $I_d$ contains no monomial, so the matroid $\mathcal{M}(I_d)$ is a loopless matroid. When $I$ does contain a monomial $g = ax^u$, the congruence $\text{Trop}(I)$ contains the relation $\text{trop}(g) \sim \text{trop}(g)_u = \infty$. For
any tropical polynomial $P \in \tilde{S}_d$ and any $\lambda \in \mathbb{R}$, the relation $\min(P, \lambda + x \cdot u) \sim P$ is then in $\mathcal{Trop}(I)$. This implies that the equivalence class of a tropical polynomial $F \in \tilde{S}_d$ does not depend on the coefficient of the monomial $u$, so we would not lose information by ignoring this coefficient.

Let $F = \min_{u \in M_d}(F^u + x \cdot u)$ be a homogeneous tropical polynomial of degree $d$. For any circuit $C \subseteq M_d$ of $\mathcal{M}(I_d)$ and any $u \in C$, let $G_{C,u}$ be the valuated circuit of $\mathcal{M}(I_d)$ satisfying $\text{supp}(G_{C,u}) = C$ and $G^u_{C,u} = 0$. Furthermore, let

$$\lambda_{C,u} := \max_{v \in C-u} (F^v - G^v_{C,u}) \in \mathbb{R}. \quad (4.1)$$

The subtraction here is in ususal arithmetic, where we follow the convention that $\infty - a = \infty$ for $a \in \mathbb{R}$. Since $v \in C$ we have $G^v_{C,u} < \infty$. The assumption that $\mathcal{M}(I_d)$ is loopless ensures that this maximum is over a nonempty set, so $\lambda_{C,u} \in \mathbb{R}$. Equivalently, $\lambda_{C,u}$ satisfies

$$\lambda_{C,u} = \min (\lambda \in \mathbb{R} : \lambda + (G_{C,u})_u \geq F). \quad (4.2)$$

We define the tropical polynomial $\pi(F) \in \tilde{S}_d$ to be the tropical sum

$$\pi(F) := \min \left( F, \min_{u \in C \subseteq M_d} (\lambda_{C,u} + G_{C,u}) \right),$$

where the inner minimum is taken over all circuits $C$ of $\mathcal{M}(I_d)$ and all $u \in C$. The coefficient of $v$ in $\pi(F)$ is

$$\pi(F)^v = \min \left( F^v, \min_{v \in C \subseteq M_d} (\lambda_{C,v}) \right), \quad (4.3)$$

where the inner minimum is only over those circuits $C$ containing $v$.

**Example 4.3.** Consider the ideal $I = \langle x + y + tz, x + y + t^2w \rangle$ in $\mathbb{C}[[t]][x,y,z,w]$. The underlying matroid $\mathcal{M}(I_1)$ in degree one has ground set $M_1 = \{x, y, z, w\}$, and circuits $\{x, y, z\}$, $\{x, y, w\}$, and $\{z, w\}$. The valuated matroid $\mathcal{M}(I_1)$ has valuated circuits $\min(x, y, 1 + z)$, $\min(x, y, 2 + w)$, and $\min(z, 1 + w)$. Consider the tropical polynomial $F = \min(x, 1 + y) \in \mathbb{R}[x, y, z, w]$. The polynomial $\pi(F)$ is equal to

$$\pi(F) = \min(F, 1 + \min(x, y, 1 + z), 1 + \min(x, y, 2 + w), \infty + \min(z, 1 + w))$$

$$= \min(x, 1 + y, 2 + z, 3 + w).$$

Similarly, for the tropical polynomial $F' = 2 + w \in S$ we have

$$\pi(F') = \min(F', \infty + \min(x, y, 1 + z), \infty + \min(x, y, 2 + w), 1 + \min(z, 1 + w))$$

$$= \min(1 + z, 2 + w). \quad \diamondsuit$$

The following proposition shows that $\pi(F)$ is the coefficient-wise smallest tropical polynomial in the equivalence class of $F$ in $\mathcal{Trop}(I)$. It is thus a distinguished representative of the equivalence class.

**Proposition 4.4.** The map $\pi : \tilde{S}_d \rightarrow \tilde{S}_d$ satisfies the following properties:

(a) $\pi(F) \leq F$.
(b) $\pi(\pi(F)) = \pi(F)$.
(c) $F \sim \pi(F) \in \mathcal{Trop}(I)$. 

(d) \( F \sim F' \in \mathcal{T}rop(I) \iff \pi(F) = \pi(F') \).

In the proof of Proposition 4.4 we will make use of the following facts about valuated circuits:

1. If \( H \) is a vector of \( M(I_d) \) with \( u \in \text{supp}(H) \) then there is a valuated circuit \( G \) with \( G^u = H^u \) and \( G \geq H \).

2. If \( H \) is a vector and \( G \) is a valuated circuit of \( M(I_d) \) with \( H^u = G^u < \infty \) and \( H' > G' \), then there is a valuated circuit \( G' \) of \( M(I_d) \) with \( G' \geq \min(H,G) \), \( G'^u = G'^v \), and \( G'^u = \infty \).

Fact (1) follows from Theorems 3.4 and 3.5 of [MT01] and the definition given there of the function \( \phi_{\chi \rightarrow \chi'}(\chi) \). Fact (2) is a combination of Fact (1) and the valuated circuit elimination axiom [MT01, Theorem 3.1 (VCE)].

Proof of Proposition 4.4. Property (a) follows directly from the definition, since \( F \) is a tropical summand of \( \pi(F) \). Property (b) follows from properties (c) and (d), which we now prove. In order to show that Property (c) holds, fix an enumeration \( \{(u_1, C_1), \ldots, (u_s, C_s)\} \) of the set \( \{(u, C) : C \text{ is a circuit of } M(I_d) \text{ and } u \in C\} \). For \( 0 \leq i \leq s \), set

\[
H_i := \min \left( F, \min_{1 \leq j \leq i} (\lambda_{C_j, u_j} + G_{C_j, u_j}) \right),
\]

so that \( H_0 = F \) and \( H_s = \pi(F) \). By Equation (4.2), for any \( i \) we have \( H_{i-1} \leq F \leq \lambda_{C_i, u_i} + (G_{C_i, u_i})_{\hat{u}} \). Since \( \mathcal{T}rop(I) \) is a congruence, the relation

\[
H_{i-1} = \min (H_{i-1}, \lambda_{C_i, u_i} + (G_{C_i, u_i})_{\hat{u}}) \sim \min (H_{i-1}, \lambda_{C_i, u_i} + G_{C_i, u_i}) = H_i
\]

is in \( \mathcal{T}rop(I) \). The result follows from transitivity.

We now prove Property (d). If \( \pi(F) = \pi(F') \) then by Property (c) we have \( F \sim F' \sim F' \sim \pi(F') \sim \pi(F') \). In order to prove the converse statement, by Lemma 2.4 it is enough to show that \( \pi(\min(H, P)) = \pi(\min(H, P)) \) for any vector \( H \) of \( M(I_d) \), \( u \in \text{supp}(H) \), and \( P \in \tilde{S}_d \). Set

\[
F := \min(H, P) \quad \text{and} \quad F' := \min(H, P).
\]

Note that \( F \) and \( F' \) can only differ in the coefficient corresponding to the monomial \( u \). We will assume that \( F^u = H^u < P^u \), as otherwise \( F = F' \).

For any circuit \( C \) of \( M(I_d) \) and any \( u' \in C \), let \( \lambda_{C, u'} \in \mathbb{R} \) be as in Equation (4.1). Let \( \lambda_{C, u'}' \) be defined analogously for the tropical polynomial \( F' \). Since \( F \preceq F' \), we have \( \lambda_{C, u'} \leq \lambda_{C, u'}' \). It follows that \( \pi(F) \leq \pi(F') \). Since \( F^v = F'^v \) for \( v \neq u \), we see from Equation (4.1) that \( \lambda_{C, u} = \lambda_{C, u}' \). Thus Equation (4.3) implies that \( \pi(F)^u = \pi(F')^u \).

Suppose that \( \pi(F)^v < \pi(F')^v \) for some \( v \neq u \). By Equation (4.3), there must be a circuit \( C' \) of \( M(I_d) \) with \( v \in C' \) and both \( \lambda_{C, v} < \pi(F')^v \leq \lambda_{C, v}' \) and \( \lambda_{C, v} < F'^v = F'^v \leq H'^v \). The maximum in Equation (4.1) must then be achieved at the coefficient of \( u \), as this the only coefficient for which \( F \) and \( F' \) differ, so \( \lambda_{C, v} = F^u - G^u_{C, v} \). Note that this implies in particular that \( u \in C \). By Fact (2) applied to the vector \( H \) and the valuated circuit \( \lambda_{C, v} + G_{C, v} \), there is a valuated circuit \( G' \) of support \( C' \) with
4.4. Let \( B \) be a basis of \( F \) such that \( \pi(B) \neq \infty \) if and only if \( A \) is an independent set of \( M(I_d) \). Given any subset \( E \subset M_d \), the restriction of the function \( p_d \) to the maximal independent subsets of \( E \) gives rise to a valuated matroid on the set \( E \), called the restriction \( M(I_d)|E \) of \( M(I_d) \) to \( E \).

Now, suppose \( F \) is a homogeneous tropical polynomial of degree \( d \). Let \( E_F \) be the closure of \( \text{supp}(F) \) in the matroid \( M(I_d) \). If \( \text{supp}(F) = E_F \), it follows from Equation (4.3) and [Cor13, Section 4] that \( \pi(F) \) is the tropical projection of \( F \in \mathbb{R}^{E_F} \) onto the tropical linear space in \( \mathbb{R}^{E_F} \) corresponding to the valuated matroid \( \underline{M}(I_d)|E_F \), but taking tropical sum to be max instead of \( \min \). If \( \text{supp}(F) \subsetneq E_F \) then we have to be more careful: The tropical polynomial \( \pi(F) \) is the tropical projection of \( F \) after substituting the coefficients corresponding to monomials in \( E_F - \text{supp}(F) \) by large enough real numbers.

Tropical projections onto tropical linear spaces have been studied in [Ard04, Cor13, Rin13]. Using those results one can obtain a description of \( \pi(F) \) amenable to computational purposes, as we now describe. For any basis \( B \) of \( E_F \) (i.e., a maximal independent set contained in \( E_F \)), define

\[
w_F(B) := p_d(B) + \sum_{u \in B} F^u \in \mathbb{R}.
\]

Let \( B_F \) be a basis of \( E_F \) such that \( w_F(B_F) \) is minimal among all bases \( B_F \) of \( E_F \). Note that the value of \( w_F(B_F) \) is finite, so \( B_F \subset \text{supp}(F) \). For any \( u \in E_F - B_F \) there exists a unique circuit \( C(B_F, u) \) of \( \underline{M}(I_d) \) contained in \( B_F \cup u \), called the fundamental circuit of \( u \) over \( B_F \). It is equal to

\[
C(B_F, u) = \{ v \in B_F : B_F \cup u - v \text{ is independent in } \underline{M}(I_d) \} \cup u.
\]
With this notation in place, [Cor13, Section 4, Proposition 5] implies that the coefficients of $\pi(F)$ are given by

$$
\pi(F)^u = \begin{cases} 
F^u & \text{if } u \in B_F, \\
\max_{v \in C(B_F, u)} (F^v - p_d(B_F \cup u - v) + p_d(B_F)) & \text{if } u \in E_F - B_F, \\
\infty & \text{if } u \notin E_F.
\end{cases}
$$

The computation of the coefficients $\pi(F)^u$ using this description involves computing a maximum over only one circuit of $M(I_d)$. This makes it computationally much simpler than formula (4.3), assuming that we know the function $p_d$.

**References**

[Ard04] Federico Ardila, *Subdominant matroid ultrametrics*, Ann. Comb. 8 (2004), no. 4, 379–389.

[Cor13] Eduardo Corel, *Gérard-Levelt membranes*, J. Algebraic Combin. 37 (2013), no. 4, 757-776.

[DW92] Andreas Dress and Walter Wenzel, *Valuated matroids*, Adv. Math. 93 (1992), no. 2, 214–250.

[GG13] Jeffrey Giansiracusa and Noah Giansiracusa, *Equations of tropical varieties*. Preprint. arXiv:1308.0042.

[KM11] Thomas Kahle and Ezra Miller, *Decompositions of commutative monoid congruences and binomial ideals*, 2011. Preprint. arXiv:1107.4699.

[KP11] Eric Katz and Sam Payne, *Realization spaces for tropical fans*, Combinatorial aspects of commutative algebra and algebraic geometry, Abel Symp., vol. 6, Springer, Berlin, 2011, pp. 73–88.

[MS13] Diane Maclagan and Bernd Sturmfels, *Introduction to Tropical Geometry*. Available at http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.html.

[Mur97] Kazuo Murota, *Matroid valuation on independent sets*, J. Combin. Theory Ser. B 69 (1997), no. 1, 59–78.

[MT01] Kazuo Murota and Akihisa Tamura, *On circuit valuation of matroids*, Adv. in Appl. Math. 26 (2001), no. 3, 192–225.

[Oxl92] James G. Oxley, *Matroid theory*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1992.

[Rin13] Felipe Rincón, *Local tropical linear spaces*, Discrete Comput. Geom. 50 (2013), no. 3, 700–713.

[SS04] David Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. Geom. 4 (2004), no. 3, 389–411.

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom.

E-mail address: (D.Maclagan/E.F.Rincon)@warwick.ac.uk