Square-Central and Artin-Schreier Elements in Division Algebras

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Abstract

We study the behavior of square-central elements and Artin-Schreier elements in division algebras of degree a power of two. We provide chain lemmas for such elements in division algebras of exponent 2 over 2-special fields of cohomological 2-dimension 2, and deduce a common slot lemma for tensor products of quaternion algebras over such fields. We also extend to characteristic 2 a theorem proven by Merkurjev for characteristic not 2 on the decomposition of any central simple algebra of exponent 2 and degree a power of 2 over a field of cohomological 2-dimension 2 as a tensor product of quaternion algebras.

Keywords: Quaternion algebras, Quadratic forms, Common slot lemma

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1. Introduction

A quaternion algebra over a field \(F\) is an algebra of the form

\[
[\alpha,\beta] = F[x, y : x^2 + x = \alpha, y^2 = \beta, yxy^{-1} = x + 1]
\]

for some \(\alpha \in F\) and \(\beta \in F^\times\) if \(F\) is of characteristic 2, and

\[
(\alpha,\beta) = F[x, y : x^2 = \alpha, y^2 = \beta, yxy^{-1} = -x]
\]

for some \(\alpha,\beta \in F^\times\) otherwise.

The common slot lemma for quaternion algebra states that for every two isomorphic quaternion algebras \((\alpha,\beta)\) and \((\alpha',\beta')\) there exists \(\beta'' \in F^\times\) such that

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$$(\alpha, \beta) \cong (\alpha, \beta''') \cong (\alpha', \beta'') \cong (\alpha', \beta')$$ (see [Jac96, Lemma 5.6.45 & Corollary 5.6.48]).

A noncentral element $v$ in a quaternion $F$-algebra is square-central if $v^2 \in F^\times$. In characteristic 2, a noncentral element $v$ is Artin-Schreier if $v^2 + v \in F$. In characteristic 2, for every two Artin-Schreier elements $v$ and $w$ in a quaternion algebra there exists a square-central element $t$ such that $vt + tv = tw + wt = t$. Otherwise, for every two square-central elements $v$ and $w$ in a quaternion algebra there exists another square-central element $t$ such that $vt = -tv$ and $tw = -wt$. The common slot lemma is an immediate result of this fact, which we refer to as the chain lemma for square-central or Artin-Schreier elements.

We extend the notions of square-central and Artin-Schreier elements and their chain lemmas to any algebra of degree a power of 2. The notion of common slot lemma extends naturally to any tensor product of quaternion algebras. In characteristic not 2, a common slot lemma and a chain lemma for square-central elements in tensor products of two quaternion algebras were provided in [Siv12] and [CV13], respectively. Equivalent results in characteristic 2 were provided in [Cha13]. In [Cha15] a common slot lemma was provided for tensor products of quaternion algebras over fields of cohomological 2-dimension 2. However, a chain lemma for the square-central elements was not provided, and the length of the chain was not computed.

In this paper we prove several facts about square-central and Artin-Schreier elements, and in particular we provide a chain lemma for them in division algebras of exponent 2 over 2-special fields (i.e. fields with no nontrivial odd degree extensions) of cohomological 2-dimension $\leq 2$. We then deduce the common slot lemma for tensor products of quaternion algebras over such fields and bound the length of the chains from above by 3, which is equal to the maximal length of such chains for biquaternion algebras over arbitrary fields. We make use of [Kah90, Theorem 3] on the decomposition of any central simple algebra of exponent 2 and degree a power of 2 over a field of cohomological 2-dimension 2 as a tensor product of quaternion algebras, which was proven in that paper for characteristic not 2 and we prove it here also for characteristic 2.

2. Preliminaries

Quadratic forms play a major role in the study of central simple algebras of exponent 2. By $I_q^F$ we denote the group of Witt equivalence classes of even dimensional nonsingular quadratic forms over the field $F$. Every such form is isometric to $[a_1, b_1] \perp \cdots \perp [a_m, b_m]$ for some integer $m$ and $a_1, b_1, \ldots, a_m, b_m \in F$ if the characteristic is 2, and $\langle a_1, \ldots, a_{2m} \rangle$ for some $a_1, \ldots, a_{2m} \in F^\times$ otherwise. The determinant (known also as the Arf invariant in characteristic 2) is denoted
by $\delta$ and defined as follows: In characteristic 2, $\delta$ maps $I_q F$ to the additive group $F/(a^2 + a : a \in F)$ by

$$\delta([a_1, b_1] \perp \cdots \perp [a_m, b_m]) = a_1 b_1 + \cdots + a_m b_m.$$  

Otherwise, it maps $I_q F$ to the multiplicative group $F^\times/(F^\times)^2$ by

$$\delta((a_1, \ldots, a_{2m})) = (-1)^m a_1 \cdots a_{2m}.$$  

(see [EKM08, Section 13]).

The subgroup of $I_q F$ of forms with trivial discriminant is denoted by $I^2_q F$. We follow the traditional abuse of notation of writing $f \in I^2_q F$ when we actually mean that the Witt equivalence class of $f$ belongs to $I^2_q F$. The Clifford invariant is denoted by $C$ and defined as follows: It maps $I^2_q F$ to $Br_2(F)$ by

$$C(f) = F[x_1, \ldots, x_{2m} : (u_1 x_1 + \cdots + u_{2m} x_{2m})^2 = f(u_1, \ldots, u_{2m}) u_1, \ldots, u_{2m} \in F]$$  

(see [EKM08, Section 14]). For each form $f$, $C(f)$ is called its Clifford algebra. Since $f \in I^2_q F$, $C(f)$ is always $M_2(E(f))$ for some central simple $F$-algebra $E(f)$ of degree $2^m - 1$.

The subgroup of $I^2_q F$ of forms with trivial Clifford algebra is denoted by $I^3_q F$. The cohomological 2-dimension of $F$ is by definition $\leq 2$ if $H^3(L, \mathbb{Z}/2\mathbb{Z}) = 0$ for every finite field extension $L/F$ (see [EKM08, Section 101]). The latter holds if and only if $I^2_q L = 0$ (see [EKM08, Fact 16.2]). If the cohomological 2-dimension of $F$ is $\leq 2$ then $I^2_q F \cong Br_2(F)$. In [Mer91, Theorem 4] examples of fields of cohomological 2-dimension 2 were constructed. The 2-special closure of such a field is also of cohomological dimension 2 (see [EKM08, Example 101.17]).

A field extension $K/F$ is excellent if for every form $f$ over $F$, there exists a quadratic form $f'$ over $F$ such that $f'$ is isometric to the anisotropic part of $f_K$. In particular every quadratic field extension is excellent (see [EKM08, Example 29.2] and [MM95, Lemma 1]). A field is said to be 2-special if it has no nontrivial field extensions of odd degree.

The following lemma will be used later on:

**Lemma 2.1.** For any field extension $K/F$, if $f \in I_q F$ and $f_K \in I^2_q K$ then there exists $f' \in I^2_q F$ such that $f'_K \cong f_K$.

**Proof.** Let $\delta$ be a representative of the discriminant of $f$.

Assume the characteristic of $F$ is 2. On one hand $\delta \in F$. On the other hand, $\delta = u^2 + u$ for some $u \in K$ because $f_K \in I^2_q K$. Now, $f = [a_1, b_1] \perp \cdots \perp [a_n, b_n]$
for some \(a_1, b_1, \ldots, a_n, b_n \in F\). Without loss of generality we can assume \(a_1 \neq 0\). Set \(f' = [a_1, b_1 + a_1^{-1} \delta] \perp \cdots \perp [a_n, b_n]\). Because of the discriminant, \(f' \in \mathcal{I}_q^2 F\).

We shall now prove that \(f'_K \simeq f_K\). Since all the summands are the same except the first one, we shall prove that under the scalar extension, the first summands in both forms are isometric.

\[
[a_1, b_1 + a_1^{-1} \delta]_K = a_1 x^2 + xy + (b_1 + a_1^{-1} \delta)y^2 = a_1(x^2 + xy + (a_1 b_1 + \delta)y^2) \simeq a_1((x + uy)^2 + (x + uy)y + (a_1 b_1 + \delta)y^2) = a_1(x^2 + xy + a_1 b_1 y^2) \simeq [a_1, b_1]_K
\]

(The first isometry is obtained by replacing \(y\) with \(a_1 y\).)

Assume the characteristic of \(F\) is not 2. On one hand \(\delta \in F^\times\). On the other hand, \(\delta = u^2\) for some \(u \in K^\times\) because \(f_K \in \mathcal{I}_q^2 K\). Now, \(f \simeq \langle a_1, \ldots, a_n \rangle\) for some \(a_1, \ldots, a_n \in F^\times\). Set \(f' = \langle \delta^{-1} a_1, a_2, \ldots, a_n \rangle\). Because of the discriminant, \(f' \in \mathcal{I}_q^2 F\). It is obvious that \(f'_K \simeq f_K\). \(\square\)

3. Square-central and Artin-Schreier elements

Let \(A\) be a division algebra of degree \(2^n\) over a field \(F\) for some \(n \geq 2\). In this section we prove a couple of facts about square-central and Artin-Schreier elements in \(A\) without restricting the cohomological dimension of \(F\). If \(F\) is of characteristic not 2 and \(x\) is square-central then any other element \(t\) in the algebra decomposes into \(t_0 + t_1\) such that \(t_0 = \frac{1}{2}(t + xt^{-1}x)\) commutes with \(x\) and \(t_1 = \frac{1}{2}(t - xt^{-1}x)\) anti-commutes with \(x\). If \(F\) is of characteristic 2 and \(x\) is Artin-Schreier then any other element \(t\) in the algebra decomposes into \(t_0 + t_1\) such that \(t_0 = xt + tx + t\) commutes with \(x\) and \(t_1 = xt + tx\) satisfies \(xt + t_1 = t_1\).

**Lemma 3.1.** Assume \(A\) is of degree 4 and exponent 2. Let \(x\) and \(x'\) be two commuting elements in \(A\). If the characteristic of \(F\) is 2 then assume \(x\) is either Artin-Schreier or square-central and \(x'\) is Artin-Schreier. Otherwise assume that \(x\) and \(x'\) are square-central. Then \(A\) decomposes as \(Q_1 \otimes Q_2\) such that \(x \in Q_1\) and \(x' \in Q_2\).

**Proof:** Assume \(F\) is of characteristic 2. Let \(\tau\) be an involution on \(A\) such that \(\tau(x') = x' + 1\) and either \(\tau(x) = x + 1\) (if \(x\) is Artin-Schreier) or \(\tau(x) = x\) (if \(x\) is square-central). Then \(\tau\) restricts to an involution of the second kind on the centralizer \(C(x')\), and it commutes with the canonical involution \(\gamma\) on this quaternion algebra. Therefore, \(\gamma \circ \tau\) is an automorphism of order 2 on \(C(x')\), and the algebra of fixed points is a quaternion algebra \(Q_1\) over \(F\) containing \(x\) and centralizing \(x'\). (See [KMRT98, Proposition 2.22].)

The proof in characteristic not 2 can be written in a similar way, or concluded from [CV13, Proposition 3.9]. \(\square\)
Proposition 3.2. Assume $F$ is 2-special. If $F$ is of characteristic 2 then for any two Artin-Schreier elements in $A$ there exists another element, either Artin-Schreier or square-central, commuting with them both. Otherwise, for any two square-central elements in $A$ there exists another square-central element commuting with them both.

Proof. Assume $F$ is of characteristic 2. Let $x, t$ be two Artin-Schreier elements in $A$. Now, $t = t_0 + t_1$ where $t_0$ commutes with $x$ and $t_1$ satisfies $xt_1 + t_1x = t_1$. The element $t^2 + t = t_0^2 + t_1t_0 + t_0t_1 + t_1^2 + t_1 + t_1$ is central, and therefore it commutes with $x$. However, $T = t_1t_0 + t_0t_1 + t_1$ satisfies $xT + Tx = T$, which means that $t_0t_1 + t_1t_0 = t_1$. The algebra $F[x,t]$ is therefore either a quaternion algebra over some field extension of $F$ or a field extension of $F$. In the second case the statement is trivial. In the first case, if the center of $F[x,t]$ is $F$ then $A = F[x,t] \otimes A_0$. Now $A_0$ is of degree $2^{n-1}$ and therefore it contains a field extension of degree $2^{n-1}$ of the center, and since $F$ is 2-special it contains a quadratic field extension of $F$ (see [EKM08, Proposition 101.15]), which is generated by either a square-central or Artin-Schreier element. This element commutes with $x$ and $t$. If the center of $F[x,t]$ is a nontrivial field extension of $F$ then it must be of degree a power of 2. Since $F$ is 2-special, the center of $F[x,t]$ must contain a quadratic field extension of $F$ which is generated by either a square-central or Artin-Schreier element. This element commutes with $x$ and $t$.

Assume $F$ is of characteristic different form 2. Let $x, t$ be two square-central elements in $A$. Now, $t = t_0 + t_1$ where $t_0$ commutes with $x$ and $t_1$ anti-commutes with $x$. The element $t^2 = t_0^2 + t_1t_0 + t_0t_1 + t_1^2$ is central, and therefore it commutes with $x$. However, $t_1t_0 + t_0t_1$ anti-commutes with $x$, which means that $t_1t_0 = -t_0t_1$. The algebra $F[x,t]$ is therefore either a quaternion algebra over some field extension of $F$ or a field extension of $F$. In the second case the statement is trivial. In the first case, if the center of $F[x,t]$ is $F$ then $A = F[x,t] \otimes A_0$. Now $A_0$ is of degree $2^{n-1}$ and therefore it contains a field extension of degree $2^{n-1}$ of the center, and since $F$ is 2-special it contains a quadratic field extension of $F$, which is generated by a square-central element. This element commutes with $x$ and $t$. If the center of $F[x,t]$ is a nontrivial field extension of $F$ then it must be of degree a power of 2. Since $F$ is 2-special, the center of $F[x,t]$ must contain a quadratic field extension of $F$ which is generated by a square-central element. This element commutes with $x$ and $t$. □

4. Over fields of cohomological dimension 2

Let $F$ be a field of cohomological 2-dimension $\leq 2$, and $A$ be a division algebra over $F$ of exponent 2 and degree $2^n$ for some $n \geq 2$. The following fact appeared in
Theorem 4.1. For any form $f \in I_q^2 F$, $E(f)$ is a division algebra if and only if $f$ is anisotropic. As a result, every algebra of exponent 2 and power $2^m$ is isomorphic to $E(f)$ for some $f \in I_q^2 F$, and therefore decomposes as the tensor product of quaternion algebras.

Proof. Assume the characteristic of $F$ is 2. First we show that every form $f \in I_q^2 F$ is universal (i.e. represents every element of $F$): Let $b \in F^\times$ and let $D(f)$ denote the set of elements represented by $f$. This set is known to be a group when $f$ is anisotropic. The form $\langle 1, b \rangle \otimes f = f \perp bf$ is in $I_q^3 F$. Therefore it has a nontrivial zero, which means that either $f$ has a zero or $f_1 + bf_2 = 0$ for some $f_1, f_2 \in D(f)$. If $f$ has a zero then it contains a hyperbolic subform, and so it is universal. Otherwise, since $f$ is round and $D(f)$ is a group, $b \in D(f)$.

Now we show that if $f \in I_q^2 F$ is an anisotropic form of dimension at least 6 and $f_K$ is isotropic for some $K = F[x : x^2 + x = a]$ then $f_K = \mathbb{H} \perp f_0$ where $\mathbb{H}$ is a hyperbolic plane and $f_0$ is anisotropic: Since $f_K$ is isotropic, $f = (f' \otimes [1, a]) \perp f_0$ for some bilinear form $f'$ and anisotropic form $f_0$. The dimension of $f'$ is equal to the Witt index of $f_K$. If the dimension of $f'$ is greater than 1 then it contains a subform of dimension 2, $\phi$, and then $\phi \otimes [1, a]$ is a proper subform of $f$. This proper subform is in $I_q^2 F$, which means that it is universal, and so $f$ is isotropic, contradiction.

Let $f$ be a form in $I_q^2 F$. Clearly if $f$ is isotropic then $E(f)$ is not a division algebra. Therefore assume that $f$ is anisotropic. We want to show that $E(f)$ is a division algebra. According to Springer’s classical theorem (see [EKM08, Corollary 18.5]) $f$ remains anisotropic under odd degree extensions. Therefore $f$ remains anisotropic under scalar extension to the 2-special closure of $F$, and it is enough to prove that $E(F)$ is a division algebra under such a scalar extension. Assume $F$ is 2-special then.

If $E(f)$ is split then because $I_q^2 F \cong Br_2(F)$, $f$ is hyperbolic. Assume then that $E(f)$ is nonsplit. Hence the dimension of $f$ is at least 4. If $f$ is of dimension 4 then it is known that $f$ is the norm form of the algebra $E(f)$, which is a quaternion algebra, and it is well known that the quaternion algebra is a division algebra if and only if its norm form is anisotropic. Therefore assume the dimension of $f$ is at least 6.

Let $m$ be half the dimension of the maximal nondegenerate anisotropic subform of $f$. For our convenience, we call $m$ the anisotropic length of $f$. We assume that for each $t < m$, if the anisotropic length of a form $\phi$ is $t$ then the index of $E(\phi)$ is $2^{t-1}$. Since $E(f)$ is nonsplit, it contains a nontrivial separable field extension of $F$. In particular, since $F$ is 2-special, $E(f)$ contains a quadratic separable field
extension $K$ of $F$ (see [EKM08, Proposition 101.15]). Since the dimension of $f$ is at least 6, the anisotropic length of $f_K$ is exactly $m - 1$. Therefore, according to the assumption, $E(f_K)$ is of index $2^{m-2}$. On the other hand, the index of $E(f)$ is at least twice the index of $E(q_K) \otimes K = E(q_K)$, which means that it must be $2^{m-1}$. Consequently, $E(f)$ is a division algebra.

The following theorem extends [Bar14, Theorem 3.3] which presents a similar decomposition on the symbol-level for division algebras of exponent 2 over fields cohomological 2-dimension 2 and characteristic not 2:

**Theorem 4.2.** Let $x$ and $x'$ be two commuting elements in $A$. If $F$ is of characteristic 2 then assume $x$ is either Artin-Schreier or square-central and $x'$ is Artin-Schreier. Otherwise assume that they are both square-central elements. Then $A = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_n$ such that $x \in Q_1$ and $x_2 \in Q_2$.

**Proof.** Let $B$ be the centralizer of $F[x,x']$ in $A$. It is enough to prove that $B = F[x,x'] \otimes B'$ for some central simple algebra $B'$ over $F$, because then $A = A_0 \otimes B'$ where $A_0$ is a degree 4 central simple algebra of exponent $\leq 2$ containing $x$ and $x'$ and therefore it decomposes as $Q_1 \otimes Q_2$ such that $x \in Q_1$ and $x' \in Q_2$ according to Lemma 2.1 and $B'$ decomposes as $Q_3 \otimes \cdots \otimes Q_n$ according to Theorem 4.1.

Let $f$ be the unique quadratic form in $I_q^2 F$ with $E(f) = A$. In particular, $f$ is anisotropic. However, $f_{[x]}$ is isotropic, because $A \otimes F[x]$ is not a division algebra. Since quadratic extensions are excellent, there exists $g \in I_q F$ such that $g_{[x]}$ is isometric to the anisotropic part of $f_{[x]}$, and according to Lemma 2.1 we can assume $g \in I_q^2 F$.

Now, $E(g_{[x]}) = C_A(F[x])$ and $g_{[x]}$ is anisotropic. However $g_{[x,x']}$ is isotropic. Assume $F$ is of characteristic 2. Then $g_{[x]} = \phi \perp d[1, x^2 + x']$ for some $d \in F[x]$ and $\phi \in I_q F[x]$ such that $\phi_{[x,x']}$ is the anisotropic part of $g_{[x,x']}$. Because of the cohomological dimension, $g_{[x]} \perp d^{-1} g_{[x]}$ is hyperbolic, which means that $g_{[x]} \simeq d^{-1} g_{[x]}$, and hence we can assume $d = 1$. Therefore $\phi$ is isometric to the anisotropic part of $g_{[x,x']}$ $\perp [1, x^2 + x']$. Since $g \perp [1, x^2 + x']$ is in $I_q F$ and $F[x]/F$ is excellent, there exists $\phi'$ in $I_q F$ such that $\phi'_{[x]}$ is isometric to $\phi$. Now, $\phi'_{[x,x']}$ is in $I_q^2 F[x,x']$, and therefore according to Lemma 2.1 there exists $\tau \in I_q^2 F$ such that $\tau_{[x,x']}$ is isometric to $\phi'_{[x,x']}$.

Consequently, $B$ is a restriction of $E(\tau)$ to $F[x,x']$.

Assume $F$ is characteristic different from 2. Then $g_{[x]} = \phi \perp d[1,- x^2]$ for some $d \in F[x]$ and $\phi \in I_q F[x]$ such that $\phi_{[x,x']}$ is the anisotropic part of $g_{[x,x']}$. Because of the cohomological dimension, $g_{[x]} \perp - d^{-1} g_{[x]}$ is hyperbolic, which means that $g_{[x]} \simeq d^{-1} g_{[x]}$, and hence we can assume $d = 1$. Therefore $\phi$ is isometric to the anisotropic part of $g_{[x]} \perp (-1, x^2)$. Since $g \perp (-1, x^2)$ is in $I_q F$ and $F[x]/F$ is excellent, there exists $\phi'$ in $I_q F$ such that $\phi'_{[x]}$ is isometric to $\phi$. Now, $\phi'_{[x,x']}$ is in $I_q^2 F[x,x']$, and therefore according to Lemma 2.1 there exists $\tau$
in $I_q^2F$ such that $\tau_{F[x,x']} \text{ is isometric to } \phi'_{F[x,x']}$. Consequently, $B$ is a restriction of $E(\tau)$ to $F[x,x']$. □

**Corollary 4.3.** If $F$ is of characteristic 2 then for every two commuting Artin-Schreier elements $x, x' \in A$ there exists a square-central element $z$ such that $xz + zx = x'z + zx' = z$. Otherwise, for every two commuting square-central elements $x, x' \in A$ there exists a square-central element $z$ anti-commuting with them both.

**Proof.** Since $A = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_n$ where $x \in Q_1$ and $x' \in Q_2$, there exist square-central elements $y \in Q_1$ and $y' \in Q_2$ such that $y^2, y'^2 \in F$ and $xy + yx = y$ and $x'y' + y'x' = y'$ if the characteristic is 2, and $yx = -xy$ and $y'x' = -x'y'$ otherwise. Take $z = yy'$. □

**Corollary 4.4.** If $F$ is of characteristic 2, $x \in A$ is square-central, $x' \in A$ is Artin-Schreier and $xx' = x'x$, then there exists a square-central element $z$ and an Artin-Schreier element $w$ such that $wx + xw = x$ and $wz + zw = z = x'z + zx' = xz + zx$. From here on we can apply Corollary 4.3 □

From now on assume that $F$ is also 2-special.

**Theorem 4.5.** If $F$ is of characteristic 2 then for every two Artin-Schreier elements $x$ and $x'$ in $A$ there exists either a chain $x, y_1, x_1, y_2, x'$ such that $x_1$ is Artin-Schreier, $y_1$ and $y_2$ are square-central and $xy_1 + y_1x = x_1y_1 + y_1x_1 = y_1$ and $xy_2 + y_2x' = x_1y_2 + y_2x_1 = y_2$, or a chain $x, y_1, x_1, y_2, x_2, x_3, x'$ with similar properties. Otherwise, characteristic not 2 then for every two square-central elements $x$ and $x'$ in $A$ there exists a chain of square-central elements $x = x_0, x_1, x_2, x_3, x_4 = x'$ such that $x_ix_{i+1} = -x_{i+1}x_i$.

**Proof.** Assume the characteristic is not 2. According to Lemma 3.2 there exists a square-central element $x_2$ which commutes with them both. According to Corollary 4.3 there exists a square-central element $x_1$ commuting with $x$ and $x_2$, and a square-central element $x_3$ commuting with $x_2$ and $x'$. The proof in characteristic 2 is similar, making use of Corollaries 4.3 and 4.4 □

**Theorem 4.6.** For every two isomorphic tensor products of quaternion algebras over $F$, $\otimes_{i=1}^n Q_i$ and $\otimes_{i=1}^n Q'_i$, there exists a chain $\otimes_{i=1}^n Q_i, \otimes_{i=1}^n Q_i', \otimes_{i=1}^n Q_i'', \otimes_{i=1}^n Q_i'''$, such that every two adjacent tensor products share a common slot.
Proof. If $\otimes_{i=1}^{n} Q_i$ is not a division algebra then it is isomorphic to $M_2(F) \otimes Q''_2 \otimes \cdots \otimes Q''_n$ for some quaternion algebras $Q''_i$. If the characteristic is 2, write $Q_1 = (a, b)$ and $Q'_1 = (a', b')$ for some $a, a' \in F$ and $b, b' \in F^{\times}$. Then $M_2(F)$ is isomorphic to both $[a, 1)$ and $[a', 1)$. Otherwise, write $Q_1 = (a, b)$ and $Q'_1 = (a', b')$ for some $a, a', b, b' \in F^{\times}$. Then $M_2(F)$ is isomorphic to both $(a, 1)$ and $(a', 1)$. The required chain is obtained as a result.

Assume $\otimes_{i=1}^{n} Q_i$ is a division algebra. Write $Q_1 = F[x, y : x^2 + x = a, y^2 = b, xy + yx = y]$ and $Q_2 = F[x', y' : x'^2 + x' = a', y'^2 = b', x'y' + y'x' = y']$ if the characteristic is 2, and $Q_1 = F[x, y : x^2 = a, y^2 = b, xy = -yx]$ and $Q'_1 = F[x', y' : x'^2 = a, y'^2 = b, x'y' = -y'x']$ otherwise. If the characteristic is 2 then there exists an element $z$, either square-central or Artin-Schreier, commuting with $x$ and $x'$. Therefore, according to Theorem 4.2, $A$ is isomorphic to $\otimes_{i=1}^{n} Q''_i$ where $x \in Q''_1$ and $z \in Q''_2$, and $A$ is also isomorphic to $\otimes_{i=1}^{n} Q''''_i$ where $x' \in Q''''_1$ and $z \in Q''''_2$. The required chain is obtained as a result.

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