Hermitian Matrix Model with Plaquette Interaction

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Abstract

We study a hermitian \((n+1)\)-matrix model with plaquette interaction, \(\sum_{i=1}^{n} M_{i}A_{i}M_{i}A_{i}\). By means of a conformal transformation we rewrite the model as an \(O(n)\) model on a random lattice with a non polynomial potential. This allows us to solve the model exactly. We investigate the critical properties of the plaquette model and find that for \(n \in ]-2,2]\) the model belongs to the same universality class as the \(O(n)\) model on a random lattice.

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1 Introduction

Despite the fact that great progress has been made in solving matrix models in recent years many interesting models remain unsolved. One important class of models for which an exact solution is still lacking is models with “plaquette type” interactions. Lattice gauge theories like the Weingarten model [1, 2] and the Kazakov-Migdal model [3] are typical examples of such models but recently also plaquette type models without gauge degrees of freedom have attracted attention, namely as generating functionals for Meander numbers [4]. In the present paper we will consider the following model

\[ Z = e^{N^2 F} = \int dM \prod_{i=1}^{n} dA_i \exp \left\{ -N \text{tr} \left( V(M) + \frac{1}{2} \sum_{i=1}^{n} A_i^2 - \frac{1}{2} \sum_{i=1}^{n} MA_iMA_i \right) \right\} \]  (1.1)

where all the matrices are hermitian and \( V(M) \) is an arbitrary polynomial potential. When \( n = 1 \) this model shows a large degree of similarity with the 2-dimensional reduced Weingarten model [1, 2] which is given by

\[ Z = \int dA \exp \left\{ -N \text{tr} \left[ \sum_{\mu=1}^{2} A_\mu^\dagger A_\mu - g \sum_{\mu,\nu=1}^{2} (A_\mu A_\nu A_\mu^\dagger A_\nu + h.c.) \right] \right\} \]  (1.2)

However, the two models are not equivalent. A model equivalent to (1.1) for \( n = 1 \) involving complex matrices is

\[ Z = \int dM_1 dM_2 dA^\dagger dA \exp \left\{ -N \text{tr} \left[ V(M_1) + \frac{1}{2} M_2^2 + A^\dagger A - M_1 A^\dagger M_2 A \right] \right\} \]

where \( M_1 \) and \( M_2 \) are hermitian and \( A \) is complex. Our model (1.1) also shows some similarity with matrix models generating Meander numbers [4]. Its interaction is of the type needed for such models. However, our model is too simple to provide a generating functional for Meander numbers. For that purpose one must be able to work also with an arbitrary number of \( M \)-matrices. Let us finish by mentioning that our solution of the model (1.1) gives the solution to a simple three-matrix problem, namely the following

\[ Z = \int dA dB dC \exp \left\{ -N \text{tr} \left( V(A) + \frac{1}{2} B^2 + \frac{1}{2} C^2 - gABC \right) \right\} \]  (1.3)

The partition function (1.3) can be brought on the form (1.1) (with \( n = 1 \)) by integrating out one of the three matrices. In reference [4] the model (1.3) with \( V(A) = \frac{1}{2} A^2 \) was studied numerically.

The paper is organized as follows. In section 2 we derive the saddle point equation of the model (1.1) and argue that it has the same structure as that of the \( O(n) \) model.
on a random lattice \[6\]. Then in section 3 we explicitly transform the model into an \(O(n)\) model with a somewhat unconventional potential. Exploiting the already known exact solution of the \(O(n)\) model on a random lattice \[7\], we hereafter in section 4 write down the solution of the present model. In section 5 we specialize to a quadratic potential and perform a detailed analysis of this case. In particular we investigate the critical properties of the model and find that for \(n \in [-2,2]\) the model (1.1) belongs to the same universality class as the ordinary \(O(n)\) model on a random lattice. Section 6 contains our conclusion and outlook. Finally in an appendix we comment on the Virasoro algebra structure carried by our model.

2 The saddle point equation

Let us carry out the gaussian integration over the \(A\)-matrices in (1.1). This gives

\[
Z = \int dM \exp \left\{ -N \operatorname{tr} V(M) \right\} \det (I \otimes I - M^T \otimes M)^{-n/2}. \tag{2.1}
\]

where \(M^T\) is the transpose of \(M\). Next, let us diagonalize the \(M\)-matrices and integrate out the angular degrees of freedom. This leaves us with the following integral over the eigenvalues, \(\{\lambda_i\}\), of the matrix \(M\)

\[
Z \propto \int \prod_i d\lambda_i \exp \left\{ -N \sum_j V(\lambda_j) \right\} \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{j,k} (1 - \lambda_j \lambda_k)^{-n/2}. \tag{2.2}
\]

In the limit \(N \to \infty\) the eigenvalue configuration is determined by the saddle point of the integral above \[6\]. The corresponding saddle point equation reads

\[
NV'(\lambda_i) = 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} + n \sum_j \frac{\lambda_j}{1 - \lambda_j \lambda_i} \tag{2.3}
\]

or

\[
V'(\lambda_i) + \frac{n}{\lambda_i} = 2 \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} + n \frac{1}{N \lambda_i^2} \sum_j \frac{1}{\lambda_i - \lambda_j}. \tag{2.4}
\]

Following \[9\] we now introduce an eigenvalue density \(\rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i)\) which in the limit \(N \to \infty\) becomes a continuous function. As is clear from equation (2.2) the model becomes singular if one of the eigenvalues approaches +1 or −1. We shall hence solve the model with the requirement that the support of the eigenvalue distribution does not include these points. To be precise we will assume that the eigenvalues are confined to one interval, \([\alpha, \beta]\), \(-1 < \alpha \leq \beta < 1\) and that the corresponding eigenvalue distribution is normalized to one. Using the hence obtained solution we will afterwards investigate what happens when, say, \(\alpha\) approaches 1. Again following \[9\] we introduce
the one-loop correlator $\Omega(p)$ by
\[
\Omega(p) = \int_{a}^{b} d\mu \frac{\rho(\mu)}{p - \mu}.
\] (2.5)

In terms of the one-loop correlator the saddle point equation (2.4) can be written as
\[
V'(p) + \frac{n}{p} = \Omega(p + i0) + \Omega(p - i0) + n \frac{1}{p^2} \Omega\left(\frac{1}{p}\right), \quad p \in [\alpha, \beta]
\] (2.6)
or with $\nabla'(p) = pV'(p) + n$ and $\nabla(p) = p\Omega(p)$
\[
\nabla(p + i0) + \nabla(p - i0) + n\frac{1}{p} = \nabla'(p), \quad p \in [\alpha, \beta].
\] (2.7)

This equation, in analogy with the saddle point equation of the $O(n)$ model on a random lattice, involves two cuts. First there is the cut of the function $\nabla(p)$. This cut is the physical cut, i.e. the support of the eigenvalue distribution corresponding to the matrix $M$. In addition to the physical cut another cut turns up in the saddle point equation, namely the cut of the function $\Omega(\frac{1}{p})$. The singular behaviour referred to above corresponds to the situation where the two cuts merge.

3 Transformation to $O(n)$ model

In order to fully exploit the similarity of our model with the $O(n)$ model on a random lattice we will explicitly bring it on the $O(n)$ model form. For that purpose, let us perform the following redefinitions of our matrix fields
\[
M \rightarrow \frac{1 - X}{1 + X}, \quad A_i \rightarrow \frac{1}{2} (1 + X)^{1/2} S_i (1 + X)^{1/2}
\] (3.1)
i.e.,
\[
dM \rightarrow [\det(1 + X)]^{-2N} dX, \quad dA_i \rightarrow [\det(1 + X)]^N dS_i.
\] (3.2)

Inserting these expressions in our original partition function we get
\[
Z \propto \int dX \prod_{i=1}^{n} dS_i \exp \left\{-N \text{tr} \left( V \left( \frac{1 - X}{1 + X} \right) + (2 - n) \log(1 + X) + X \sum_{i=1}^{n} S_i^2 \right) \right\}.
\] (3.3)

This model is nothing but the $O(n)$ model on a random lattice with the somewhat unconventional potential
\[
U(p) = V \left( \frac{1 - p}{1 + p} \right) + (2 - n) \log(1 + p) \equiv U_0(p) + (2 - n) \log(1 + p).
\] (3.4)

The saddle point equation for this model reads
\[
W(p + i0) + W(p - i0) + nW(-p) = U'(p),
\] (3.5)
where \( W(p) \) is the one-loop correlator of the \( X \)-field. The physical cut now extends from \( a = (1 + \alpha)/(1 - \alpha) \) to \( b = (1 + \beta)/(1 - \beta) \) and the unphysical cut is the mirror image with respect to zero of the physical cut. We note that the point \( p = 1 \) lies on the physical cut and the point \( p = -1 \) on the unphysical cut. As before we expect some kind of singularity to occur when the physical and the unphysical cut merge and we will have to assume that \( a > 0 \). Since the potential includes a logarithmic term we might also expect a Penner like singularity to appear, i.e. a singularity corresponding to the physical and the unphysical cut degenerating to respectively the point \( p = +1 \) and the point \( p = -1 \). 

4 The solution

In [7] an exact contour integral representation of the 1-loop correlator of the \( O(n) \) model on a random lattice was written down. The derivation of the exact formula was based on the assumption of the potential of the model being polynomial. However, it is easy to convince oneself that the formulas remain valid (when written in the appropriate way) as long as the potential or rather its derivative does not have any singularities which intervene with the physical cut of the one-loop correlator. For our model the only singular point of \( U'(p) \) is \( p = -1 \) (cf. to equation (3.4)) and, as argued earlier, this point always lies on the unphysical cut. Hence we can take over the solution of the \( O(n) \) model on a random lattice from [7]. Let us remind the reader of the structure of the solution. First, it is convenient to decompose the 1-loop correlator into a regular part, \( W_r(p) \), having no cut, and a singular part, \( W_s(p) \)

\[
W(p) = W_r(p) - W_s(p). \tag{4.1}
\]

It follows from equation (3.5) that \( W_r(p) \) is given by

\[
W_r(p) = \frac{2U''(p) - nU'(-p)}{4 - n^2} \tag{4.2}
\]

while \( W_s(p) \) is a solution to the homogeneous saddle point equation. As shown in reference [7] any solution of the homogeneous saddle point equation can be parametrized in terms of two auxiliary functions, \( G(p) \) and \( \tilde{G}(p) \). More precisely, any such solution, \( S(p) \), can be written as

\[
S(p) = \mathcal{A}(p)G(p) + \mathcal{B}(p)p\tilde{G}(p) \tag{4.3}
\]

where \( \mathcal{A}(p) \) and \( \mathcal{B}(p) \) are regular but not necessarily entire functions. The function \( G(p) \) is defined by the following three requirements
1. \( G(p) \) is a solution of the homogeneous saddle point equation corresponding to 
\( n = 2 \cos(\nu \pi) \), i.e.,
\[
G(p + i0) + G(p - i0) + nG(-p) = 0.
\]

2. \( G(p) \) is analytic in the complex plane except for the cut \([a, b]\) and behaves as 
\( (p - a)^{-1/2} \) and \( (p - b)^{-1/2} \) in the vicinity of \( a \) and \( b \).

3. \( G(p) \) has the following asymptotic behaviour
\[
G(p) \sim \frac{1}{2 \cos(\nu \pi/2)} \frac{1}{p}, \quad p \to \infty.
\]

These three requirements are enough to determine \( G(p) \) uniquely and a completely explicit expression for \( G(p) \) in terms of theta-functions can be written down [8]. We shall not need the detailed form of \( G(p) \) for the following but let us mention that, as is obvious from the definition, \( G(p) \) does not contain any explicit reference to the matrix model coupling constants. Furthermore the dependence of \( G(p) \) on \( n \) appears explicitly only via a parameter \( e \) given by
\[
e = a \text{ sn} \left( i(1 - \nu)K', k \right), \quad k = \frac{a}{b}.
\]

The function \( \tilde{G}(p) \) is defined in a way analogously to \( G(p) \). Only \( \nu \) is replaced by \( 1 - \nu \). Hence \( \tilde{G}(p) \) is a solution of the homogeneous saddle point equation with \( n \) replaced by \( -n \). Now, if \( \tilde{G}(p) \) is a solution of the homogeneous saddle point equation corresponding to \( -n \) then obviously \( p\tilde{G}(p) \) is a solution to the original saddle point equation. This explains the appearance of the combination \( p\tilde{G}(p) \) in relation (4.3). In a compact form the full one-loop correlator, \( W(p) \), can be written as (cf. to [7])
\[
W(p) = \frac{1}{4 - n^2} \left\{ ip \tilde{G}(p) \int_C \frac{d\omega}{2\pi i} \frac{U'(\omega)}{p^2 - \omega^2} \left[ (p^2 - e^2) i\tilde{G}(\omega) + \frac{\sqrt{e}}{e} \omega \tilde{G}(\omega) \right] \right. \\
\left. - G(p) \int_C \frac{d\omega}{2\pi i} \frac{U'(\omega)}{p^2 - \omega^2} \left[ (p^2 - \tilde{e}^2) \omega G(\omega) + p^2 \frac{\sqrt{\tilde{e}}}{\tilde{e}} i\tilde{G}(\omega) \right] \right\}
\]

where the contour \( C \) encircles the physical cut \([a, b]\) but not the points \( \omega = \pm p \) and where \( \sqrt{e} \) is defined by
\[
\sqrt{e} = \sqrt{(e^2 - a^2)(e^2 - b^2)} = -ab \text{ cn} \left( i(1 - \nu)K' \right) \text{ dn} \left( i(1 - \nu)K' \right).
\]
integral (4.5) in a specific case, the most convenient line of action is to deform the contour into several different contours encircling respectively the points \( \omega = \pm p \) and the various singularities of \( U'(p) \). The contribution from the poles \( \omega = \pm p \) then gives rise to the regular part of \( W(p) \) while the contribution from singularities of \( U'(p) \) gives the singular part of \( W(p) \). The expression (4.5) must be supplemented by a set of boundary equations which determine the endpoints of the physical cut, \( a \) and \( b \). These equations read

\[
\oint_C \frac{d\omega}{2\pi i} V'(\omega) \tilde{G}(\omega) = 0, \tag{4.7}
\]

\[
\oint_C \frac{d\omega}{2\pi i} \omega V'(\omega) \tilde{G}(\omega) = \frac{2 - n}{\sqrt{1 + \frac{n}{2}}}, \tag{4.8}
\]

and ensure the correct asymptotic behaviour of the one-loop correlator, namely \( W(p) \sim 1/p \) as \( p \to \infty \). In [7] it was shown that for the ordinary \( O(n) \) model on a random lattice the higher genera contributions to the correlators and the free energy simplify considerably if one expresses the \( p \)-dependence via a set of basis functions \( \{ G_a^{(k)}(p), G_b^{(k)}(p) \} \) and the dependence on the coupling constants via a set of moment variables \( \{ M_k, J_k \} \). Needless to say that a similar simplification can be obtained in the present case.

5 The quadratic potential

For simplicity, let us now restrict ourselves to the case where the potential \( V(M) \) in (1.1) is given by

\[
V(M) = \frac{1}{2T} M^2. \tag{5.1}
\]

The analysis of the general case can be done along the same lines. For \( U(p) \) in equation (3.4) we then obviously have

\[
U(p) = \frac{1}{2T} \left( \frac{1 - p}{1 + p} \right)^2 + (2 - n) \log(1 + p). \tag{5.2}
\]

5.1 The boundary equations

Inserting (5.2) into the boundary equation (4.7) we get

\[
(2 - n) \tilde{G}(-1) - \frac{2}{c} \left[ \frac{\partial^2}{\partial p^2} - \frac{\partial}{\partial p} \right] \tilde{G}(p) \bigg|_{p=-1} = 0. \tag{5.3}
\]

Here the first term comes from the pole at \( w = -1 \) in the logarithmic term and the second from the pole at \( \omega = -1 \) in \( U'_0(p) \). There is no contribution from infinity. Next inserting the expression for \( U'(p) \) into (4.8) we get

\[
(2 - n) G(-1) + \frac{6}{c} \frac{\partial}{\partial p} G(p) \bigg|_{p=-1} - \frac{2}{c} \left[ \frac{\partial^2}{\partial p^2} + 1 \right] G(p) \bigg|_{p=-1} = 0, \tag{5.4}
\]
where the first term comes from the pole at \( \omega = -1 \) in the logarithmic term and the two next from the pole at \( \omega = -1 \) in \( U_0(p) \). In this case we do have a contribution from infinity but it cancels with the constant on the right hand side of the original equation. To proceed we need to know \( \frac{\partial}{\partial p} G(p) \) and \( \frac{\partial}{\partial p^2} G(p) \). These can be found by exploiting the fact that any solution to the homogeneous saddle point equation corresponding to \( n = 2 \cos(\nu \pi) \) has a parametrization of the form (1.3) (and similarly for the saddle point equation corresponding to \( n = 2 \cos((1 - \nu)\pi) \)). The exact form of the parametrization is determined by the requirements on the analyticity properties and the asymptotic behaviour of the functions in question (cf. to [7]). The result reads

\[
\frac{\partial}{\partial p} G(p) = \frac{1}{(p^2 - a^2)(p^2 - b^2)} \left\{ p \left( e^2 - p^2 - \alpha \frac{\sqrt{e}}{e} \right) G(p) + i \left( \alpha p^2 + e^2 \alpha \right) \tilde{G}(p) \right\},
\]

(5.5)

\[
\frac{\partial^2}{\partial p^2} G(p) = \frac{1}{(p^2 - a^2)^2(p^2 - b^2)^2} \times \left\{ 2p^6 + \left( a^2 + b^2 - 5 \left( e^2 - \alpha \frac{\sqrt{e}}{e} \right) - \alpha \tilde{e} \right) p^4 \\
+ \left( (a^2 + b^2) \left( 2e^2 - \alpha \frac{\sqrt{e}}{e} - \alpha^2 \right) - 4a^2b^2 \right) p^2 + a^2b^2 \left( e^2 - \alpha \tilde{e} - \alpha \frac{\sqrt{e}}{e} \right) \right\} G(p)
\]

\[
+ \left[ 4\alpha p^4 + \left( 6e^2 \alpha - (a^2 + b^2) \right) p^2 - 3e^2 \alpha (a^2 + b^2) - 2a^2b^2 \right] (-i)p\tilde{G}(p) \right\} \quad (5.6)
\]

where

\[
\alpha = b \left( Z (i(1 - \nu)K', k) + i(1 - \nu)\frac{\pi}{2K} \right), \quad k = \frac{a}{b}.
\]

(5.7)

Exploiting the explicit expression for \( G(p) \) found in [8] one can derive the following useful relation between \( G(-1) \) and \( \tilde{G}(-1) \)

\[
\tilde{G}(-1) = -i\sqrt{K} \text{sn}(i\nu K', k)G(-1) = -i \left( \frac{1}{\sqrt{ab}} \right)^{-1} \tilde{e} G(-1).
\]

(5.8)

Now, let us for a moment go back to our original model (1.1). With a quadratic potential the model is invariant under the transformation \( M \to -M \) and the eigenvalues of the matrix \( M \) must hence live on an interval of the type \([-\alpha, \alpha]\). This means that for the support \([a, b]\) of the eigenvalue distribution of the matrix \( X \), defined in (3.1), we have \( b = \frac{1}{a} \). Exploiting (5.8) and setting \( b = 1/a \) and we get from (5.5) and (5.6)

\[
\frac{\partial}{\partial p} G(p) \bigg|_{p=-1} = -\frac{a^2}{(1 - a^2)^2} \left\{ \left( \alpha \frac{\sqrt{e}}{e} - e^2 + 1 \right) + (\alpha \tilde{e} - e\tilde{e}) \right\} G(-1),
\]

(5.9)

\[
\frac{\partial^2}{\partial p^2} G(p) \bigg|_{p=-1} = \frac{a^2}{(1 - a^2)^2} \left\{ 1 - \alpha \tilde{e} + 3e\tilde{e} + 2e^2 - \alpha^2 - \alpha \frac{\sqrt{e}}{e} \right\} G(-1).
\]

(5.10)
Finally inserting (5.9) and (5.10) into (5.3) and (5.4) both equations reduce to

\[(2 - n) - \frac{2a^2}{(1 - a^2)^2} \frac{1}{T} \left\{ 2 + 2\tilde{e}\alpha + \tilde{e}^2 - \tilde{\alpha}^2 \right\} = 0. \quad (5.11)\]

### 5.2 The string susceptibility

In this section we will determine the quantity \( \frac{d}{dT} T^3 \frac{dF}{dT} \) which we will make use of later when investigating the critical behaviour of the model. Here \( F \) stands for the genus zero contribution to the free energy of our model. The quantity \( \frac{d}{dT} T^3 \frac{dF}{dT} \) is related to the string susceptibility \( U(T) = \frac{d^2}{dT^2} (T^2 F) \) by

\[
\frac{d^2}{dT^2} \left( T^3 \frac{dF}{dT} \right) = T \frac{dU(T)}{dT}. \quad (5.12)
\]

By direct computation we find

\[
T^2 \frac{dF}{dT} = \frac{1}{2} \left\langle \frac{1}{N} \text{tr} \left( \frac{1 - X}{1 + X} \right)^2 \right\rangle = \frac{1}{2} \oint_{c_1} \frac{d\omega}{2\pi i} \left( \frac{1 - \omega}{1 + \omega} \right)^2 W(\omega). \quad (5.13)
\]

Multiplying by \( T \) and differentiating once more gives

\[
\frac{d}{dT} \frac{d}{dT} \frac{d}{dT} \left( T^3 \frac{dF}{dT} \right) = \frac{1}{2} \oint_{c_1} \frac{d\omega}{2\pi i} \left( \frac{1 - \omega}{1 + \omega} \right)^2 \frac{d}{dT} (TW(\omega)) \quad (5.14)
\]

Now, it follows from (3.3) that \( \frac{d}{dT} (TW(p)) \) fulfills the following equation

\[
\frac{d}{dT} (TW(p + i0)) + \frac{d}{dT} (TW(p - i0)) + n \frac{d}{dT} (TW(-p)) = \hat{V}^\prime(p) \quad (5.15)
\]

where

\[
\hat{V}^\prime(p) = (2 - n) \frac{1}{1 + p}. \quad (5.16)
\]

Furthermore we obviously have for the asymptotic behaviour

\[
\frac{d}{dT} (TW(p)) \sim \frac{1}{p}, \quad \text{as} \quad p \to \infty \quad (5.17)
\]

and as regards the analyticity structure, \( \frac{d}{dT} (TW(p)) \) must be analytic in the complex plane outside the support of the eigenvalue distribution and behave as

\[
\frac{d}{dT} (TW(p)) \sim (p - a)^{-1/2}, \quad (p - b)^{-1/2} \quad \text{for} \quad p \to a, b \quad (5.18)
\]

Let us introduce the following notation

\[
W_T(p) \equiv \frac{d}{dT} (TW(p)) \quad (5.19)
\]
and let us split $W_T(p)$ in a regular part, $W^r_T(p)$, and a singular part, $W^s_T(p)$, i.e.

$$W_T(p) = W^r_T(p) - W^s_T(p)$$

(5.20)

where $W^r_T(p)$ does not have any cut. Then we have from (5.15)

$$W^r_T(p) = \frac{2\hat{V}'(p) - n\hat{V}'(-p)}{4 - n^2} = \frac{1}{1 - p^2} \left\{ \frac{2 - n}{2 + n} - p \right\}.$$  \hspace{1cm} (5.21)

The singular part of $W_T(p)$ is a solution of the homogeneous saddle point equation and as any other such solution has a parametrization of the form (4.3). Since $W^r_T(p)$ has poles at $p = \pm 1$, $W^s_T(p)$ must likewise have poles here because the full function $W_T(p)$ should be analytic outside the support of the eigenvalue distribution. Therefore we can write

$$W^s_T(p) = \frac{1}{1 - p^2} \left\{ A(p^2)G(p) + pB(p^2)\tilde{G}(p) \right\}$$

(5.22)

where $A(p^2)$ and $B(p^2)$ are now entire functions. From the requirement (5.17) on the asymptotic behaviour and the expression (5.21) for $W^r_T(p)$ one can conclude that $A(p^2)$ and $B(p^2)$ must be constants. Hence we have

$$W_T(p) = \frac{1}{1 - p^2} \left\{ \frac{2 - n}{2 + n} - p + AG(p) + pB\tilde{G}(p) \right\}$$

(5.23)

and the constants $A$ and $B$ are determined by the requirement that the poles at $p = \pm 1$ should vanish, i.e.

$$\frac{2 - n}{2 + n} - 1 + AG(1) + B\tilde{G}(1) = 0,$$

(5.24)

and

$$\frac{2 - n}{2 + n} + 1 + AG(-1) - B\tilde{G}(-1) = 0.$$  \hspace{1cm} (5.25)

The solution reads

$$A = \frac{1}{2 + n} \left\{ \frac{2n\tilde{G}(-1) - 4\tilde{G}(1)}{G(1)\tilde{G}(-1) + G(-1)\tilde{G}(1)} \right\},$$

(5.26)

$$B = \frac{1}{2 + n} \left\{ \frac{2nG(-1) + 4G(1)}{G(1)\tilde{G}(-1) + G(-1)\tilde{G}(1)} \right\}.$$  \hspace{1cm} (5.27)

Going back to (5.14) we can write

$$\frac{d}{dT} T^3 \frac{dF}{dT} = \frac{1}{2} \int_{C_1} \frac{d\omega}{2\pi i} \left( \frac{1 - \omega}{1 + \omega} \right)^2 \frac{1}{1 - \omega^2} \left\{ AG(\omega) + \omega B\tilde{G}(\omega) \right\}$$

$$= \frac{1}{2} \int_{C_1} \frac{d\omega}{2\pi i} \frac{1 - \omega}{(1 + \omega)^3} \left\{ AG(\omega) + \omega B\tilde{G}(\omega) \right\}$$

$$= \frac{A}{2} \left\{ \frac{\partial^2}{\partial p^2} - \frac{\partial}{\partial p} \right\} G(p) \bigg|_{p=1} + \frac{B}{2} \left\{ 3\frac{\partial}{\partial p} - \frac{\partial^2}{\partial p^2} - 1 \right\} \tilde{G}(p) \bigg|_{p=1}$$

$$= \frac{1}{2 + n} \left( \frac{a^2}{(1 - a^2)^2} \right) \left\{ 2 + e^2 - \alpha^2 + 2\tilde{a}e \right\}.$$  \hspace{1cm} (5.28)
5.3 The critical behaviour

As argued earlier our model becomes singular as $a \to 0$ (cf. to sections 2 and 3). Below we will investigate the nature of the critical behaviour associated with this singularity.

In analogy with what was the case for the ordinary $O(n)$ model on a random lattice the present model only has a well defined scaling behaviour as $a \to 0$ if $n \in [-2, 2]$ and we will restrict ourselves to considering this range of $n$ values. One might also try to look for a critical point associated with $a \to 1$ (cf. to equation (5.11)), i.e. with the physical and the unphysical cut degenerating to the two points $+1$ and $-1$. Due to the analogy with the Penner potential [10] one might expect that having $a = 1$ (apart from at $c = 0$) is possible only for a particular value of $n$. (If the analogy were perfect it would be $n = 1$). However, we find that the equation $a = 1$ only has the trivial solution $c = 0$ regardless of the value of $n$.

5.3.1 The case $n \in [-2, 2]$ 

Let us consider the singular behaviour which occurs as $a \to 0$. First, let us fix $n$ and determine the critical value of $T$ as a function of $n$. By analysing the $k \to 0$ limit of the various elliptic functions which enter the equation (5.11) one concludes that in the limit $a \to 0$ the dominant term in the curly bracket is $\tilde{\alpha}^2$ and that

$$\tilde{\alpha} \sim i\nu \frac{1}{a}, \quad \text{as} \quad a \to 0.$$  

(5.29)

Hence the critical value, $T_*$ of $T$ is given by

$$(2 - n) - \frac{2\nu^2}{T_*} = 0$$  

(5.30)

or

$$T_* = \frac{\nu^2}{2\sin^2(\nu\pi/2)}.$$  

(5.31)

In particular we see that $T_*$ is always positive and greater than $2/\pi^2$. For $n = 1$ we get $T_* = \frac{2}{9}$. In reference [5] a numerical determination of this quantity gave $\frac{1}{T_*} = 4.504$.

The next to leading order term in the curly bracket in equation (5.11) comes from the term $2\tilde{c}\tilde{\alpha}$ which behaves as

$$2\tilde{c}\tilde{\alpha} \sim -\frac{4}{a^2} q^{(1-\nu)/2} \sim -\frac{4}{a^2} \left(\frac{a^2}{4}\right)^{1-\nu}, \quad a \to 0.$$  

(5.32)

From this we conclude that

$$T_* - T \sim a^{2-2\nu}.$$  

(5.33)

Now, let us take a look at the expression (5.28) for $\frac{d}{dt} T^3 \frac{dF}{dt}$. Here the leading order contribution comes from the term $\alpha^2$ and is of order $a^0$. The next to leading order
term comes from $2\hat{\alpha}e$ and is of order $a^{2\nu}$ (cf. to equation (5.32)). Bearing in mind the relation (5.12) we get using (5.33)

$$U(T) \sim (T_* - T)^{1/\nu}. \quad (5.34)$$

This means that

$$\gamma_{str} = -\frac{\nu}{1 - \nu}. \quad (5.35)$$

### 5.3.2 The cases $n = \pm 2$

For $n = \pm 2$ the relations (5.11) and (5.28) contain divergent terms. However, the limits $n \to \pm 2$ of these relations are well defined.

**The case $n = +2$:** Taking the limit $n \to 2$ in (5.11) one arrives at the following equation

$$\pi^2 - \frac{2}{(1 - a^2)^2 T} \left\{ \left( E' + a^2 K' \right)^2 - a^2 \left( 1 + a^2 \right)^2 K'^2 \right\} = 0. \quad (5.36)$$

This reproduces the result (5.31) that $T_* = 2/\pi^2$ for $n = 2$. In the limit $a \to 0$ the next to leading order contribution in the curly bracket comes from the term $(aK')^2$ which behaves as $(a \log a)^2$. This gives

$$T_* - T \sim (a \log a)^2 \quad (5.37)$$

Furthermore, in the limit $n \to 2$ the relation (5.28) reads

$$\frac{d}{dT} T^3 \frac{dT}{dF} = \frac{1}{4 (1 - a^2)^2} \left\{ (1 + a^2)^2 - 4 \frac{E'}{K'} \right\}. \quad (5.38)$$

Letting $a \to 0$ we get

$$\frac{d}{dT} T^3 \frac{dT}{dF} \sim \frac{1}{\log a}. \quad (5.39)$$

The results (5.37) and (5.39) coincide with those for the ordinary $O(2)$ model on a random lattice.

**The case $n = -2$:** For $n = -2$ the relation (5.11) reduces to

$$2 - \frac{1}{(1 - a^2)^2 T} \left\{ (1 + a^2)^2 - 4 \frac{E'}{K'} \right\} = 0 \quad (5.40)$$

which in accordance with (5.31) gives that $T_* = \frac{1}{2}$. Furthermore it follows that in the limit $a \to 0$

$$T_* - T \sim -\frac{1}{\log a}. \quad (5.41)$$
The relation (5.28) takes the following form when $n = -2$

$$\frac{d}{dT} T^3 \frac{dF}{dT} = \frac{1}{(1-a^2)^2} \frac{1}{\pi^2} \left\{ \left( E' + a^2 K' \right)^2 - a^2 (1 + a^2)^2 K'^2 \right\}. \quad (5.42)$$

In the limit $a \to 0$ we find

$$\frac{d}{dT} T^3 \frac{dF}{dT} \sim -(a \log a)^2. \quad (5.43)$$

We note that the results (5.41) and (5.43) do not coincide with those of the ordinary $O(-2)$ model on a random lattice which (for gaussian potential) does not have any singular points.

6 Conclusion and outlook

We have solved exactly a hermitian $(n+1)$-matrix model with plaquette interaction. For $n \in [-2, 2]$ the model was shown to belong to the same universality class as the $O(n)$ model on a random lattice. In particular this result confirms the speculation of reference [5] that the critical point of the model (1.3) describes the same physics as the critical point of the $O(1)$ model on a random lattice. Using equation (5.28) it is easy to see that the plaquette model has no singular points (with $T$ finite) for $n < -2$ and that for $n > 2$ the points given by $\bar{\nu} K' = 2mK$, where $\nu = i\bar{\nu}$, are singular. We expect that in analogy with the ordinary $O(n)$ model, the solution of the plaquette model breaks down at the first of these points, $\bar{\nu} K' = 2K$, and that the critical index $\gamma_{str}$ takes the value $+\frac{1}{2}$ at this singularity. Although our model is much simpler than general lattice gauge models and matrix models generating Meander numbers our results may be taken as an indication that elliptic functions might provide a convenient parametrization of such models.

Our solution of the plaquette model contains the solution of a certain three colour problem on a random lattice [5]. The classical three colour problem due to Baxter [11] consists in enumerating all possible ways of colouring with three different colours the links of a 2D regular three coordinated lattice in such a way that no two links which meet at the same vertex carry the same colour. The problem can also be understood as the problem of counting all possible foldings of the 2D regular triangulated lattice [12]. Obviously the partition function (1.3) (with $V(A) = \frac{1}{2}A^2$) generates random lattices with links of three different colours where no two links radiating from the same vertex have the same colour. Due to the matrix nature of the fields, however, in the present case the cyclic order of the three colours around a vertex will always be the same. In order to lift this constraint we would have to introduce two interaction vertices, $\text{tr} A B C$ and $\text{tr} A C B$. The quartic interaction term in the resulting two matrix model
would then look like \( c (\text{tr} ABAB + \text{tr} A^2B^2) \) [4]. Unfortunately an exact solution of a model with this type of interaction is still lacking. Let us mention in this connection that a somewhat similar interaction term, namely \( c (\text{tr} ABAB + 2\text{tr} A^2B^2) \) appears in a matrix model describing an Ising spin system living on the vertices of a randomly quadrangulated surface [13].

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**Appendix A  The Virasoro constraints**

Like the \( O(n) \) model on a random lattice our model can be understood as a deformation of the one-matrix (\( n=0 \)) model and obeys a set of Virasoro constraints obtainable from those of the \( n = 0 \) model by a canonical transformation [14]. Let us rewrite (2.1) as

\[
Z_n = \int dM \exp \left\{ \text{tr} \sum_{i=1}^{\infty} t_i M^i \right\} \exp \left\{ \frac{n}{2} \sum_{k=1}^{\infty} \frac{1}{k} \left( \text{tr} M^k \right)^2 \right\}.
\]

(A.1)

Introducing a differential operator, \( H \), by

\[
H = n \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\partial}{\partial t_k} \right)^2
\]

(A.2)

we have

\[
Z_n = e^H Z_0.
\]

(A.3)

The \( n = 0 \) model obeys the Virasoro constraints \( L_m Z_0 = 0, \ m \geq -1 \) where

\[
L_m = \sum_{k=0}^{m} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{m-k}} + \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{m+k}}
\]

(A.4)

and

\[
\frac{\partial}{\partial t_0} Z_0 = NZ_0.
\]

(A.5)

The general model obeys the Virasoro constraints \( \tilde{L}_m Z_n = 0, \ m \geq -1 \) with

\[
\tilde{L}_m = e^H L_m e^{-H} = L_m + n \sum_{k=0}^{\infty} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{m+k}}.
\]

(A.6)

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