PROJECTIVE REPRESENTATIONS OF FUNDAMENTAL GROUPS OF QUASIPROJECTIVE VARIETIES: A REALIZATION AND A LIFTING RESULT.

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Abstract. We discuss two results about projective representations of fundamental groups of quasiprojective varieties. The first is a realization result which, under a nonresonance assumption, allows to realize such representations as monodromy representations of flat projective logarithmic connections. The second is a lifting result: any representation as above, after restriction to a Zariski open set and finite pull-back, can be lifted to a linear representation.

1. INTRODUCTION

In this note, we study projective representations $\rho : \pi_1(X\setminus H) \to \text{PGL}_m(\mathbb{C})$, for $X$ a projective complex variety and $H$ an algebraic hypersurface in $X$.

If $X$ is smooth and $H$ is normal crossing, under some nonresonance assumption, we show that $\rho$ can be realized as the monodromy representation of a flat logarithmic projective connection; we refer to this as the realization result. This allows to extend to $X$ the analytic $\mathbb{P}^{m-1}$-bundle over $X \setminus H$ which underlies the suspension of $\rho$. Thanks to this, and an algebraization result of Serre [Ser58], we can derive a second result (lifting result): with no smoothness and normal crossing assumptions for $X$ and $H$, any $\rho : \pi_1(X\setminus H) \to \text{PGL}_m(\mathbb{C})$ is the projectivization of a linear representation, up to adding components to $H$ and pulling back by a generically finite morphism $Y \to X$. Contrary to the first, this second result is not new; it is a well known fact in étale cohomology that any class in $H^2(X\setminus H, \mathbb{Z}/m\mathbb{Z})$ can be made trivial after the two operations mentioned above. Yet, it seems of interest to show how it can be derived quickly from the realization result.

We plan to use the lifting result in a future paper about algebraic isomonodromic deformations.

The proof of the realization result is an adaptation of the work of Deligne [Del70] on the Riemann-Hilbert problem. For explicitness of basic ideas in this field, we will refer to [Bri04].

2. FLAT PROJECTIVE CONNECTIONS

2.1. Holomorphic connections.

Definition 2.1. Let $m > 0$. Let $X$ be a complex manifold. A $\mathbb{P}^{m-1}$-bundle on $X$ is a holomorphically locally trivial bundle on $X$, $\pi : P \to X$ with fiber the complex $(m - 1)$-dimensional projective space $\mathbb{P}^{m-1}$.

Definition 2.2. Let $X$ be a complex manifold. A holomorphic flat projective connection $\nabla$ on the $\mathbb{P}^{m-1}$-bundle $\pi : P \to X$ is a regular codimension $m - 1$
holomorphic foliation on $P$, transversal to any fiber of $\pi$. Let $\star \in X$. The monodromy representation of $\nabla$

$$\rho : \pi_1(X, \star) \to \text{Aut}(\pi^{-1}(\star))$$

is defined as follows. For any loop $\alpha(t)$ in $X$ with base point $\star$, for any $y \in \pi^{-1}(\star)$, there is a unique lifting path $\tilde{\alpha}_y(t)$ of $\alpha(t)$, with $\tilde{\alpha}_y(0) = y$ and contained in a leaf of $\nabla$; we set $\rho(\alpha)$ to be the automorphism of $\pi^{-1}(\star)$ which satisfies $\rho(\alpha)(y) = \tilde{\alpha}_y(1)$ for every $y \in \pi^{-1}(\star)$.

Strictly speaking, this map is an antirepresentation, but we maintain the usual shortcut of “monodromy representation”. Also, in effective computations, we are lead to use an isomorphism $\pi^{-1}(\star) \xrightarrow{\phi} \mathbb{P}^{m-1}$ and replace $\rho$ by $\tilde{\rho} : \pi_1(X, \star) \to \text{PGL}_m(\mathbb{C})$ given by $\tilde{\rho}(\alpha) = \phi \circ \rho(\alpha) \circ \phi^{-1}$. We also make the abuse of language of naming $\tilde{\rho}$ the monodromy representation of $\nabla$. As $\phi$ is arbitrary, $\tilde{\rho}$ is well defined only up to conjugation by an element of $\text{PGL}_m(\mathbb{C})$.

It is well known that any $\rho : \pi_1(X, \star) \to \text{PGL}_m(\mathbb{C})$ can be realized as the monodromy representation of a unique (up to bundle isomorphism) flat projective connection, see [CLNS5].

For any flat holomorphic linear connection $D : V \to \Omega^1_X \otimes V$ on a vector bundle $V$ over $X$, the foliation induced by horizontal sections (i.e. sections $s$ such that $Ds = 0$) on the total space $V$ descends to a flat projective connection $\nabla = \mathbb{P}(D)$ on $\mathbb{P}(V)$. We call $\mathbb{P}(D)$ the projectivization of $D$.

Any flat connection is locally the trivial one on the trivial bundle. To this respect, any flat projective connection is locally the projectivization of a flat linear connection. Also we have a form of uniqueness.

**Lemma 2.3.** Let $D_1, i = 1, 2$ be two flat holomorphic connections on the same vector bundle $V$, with equal traces

$$\text{tr}(D_1) = \text{tr}(D_2) : \text{det}(V) \to \Omega^1_X \otimes \text{det}(V).$$

Then $\mathbb{P}(D_1) = \mathbb{P}(D_2)$ if and only if $D_1 = D_2$.

**Proof.** Using local trivializations, it suffices to check the result for the trivial bundle $V = \mathcal{O}^m$.

Let $\omega = (\omega_{i,j})$ be a size $m$ square matrix with coefficients in $\Omega^1_X(X)$ and define $D(y) = dy - \omega \cdot y$ for any vector valued holomorphic function $y = (y_1, \ldots, y_m)^t \in \mathcal{O}^m$, we suppose $D$ is flat, that is $d\omega = \omega \wedge \omega$. We have a system of differential equations that define $\mathbb{P}(D)$ in the affine chart $y_m \neq 0$, setting $z_i = y_i/y_m, i = 1, \ldots, m - 1$ we find:

$$dz_i = \omega_{i,m} + z_i(\omega_{i,i} - \omega_{m,m}) + \sum_{k=1, k \neq i}^{m-1} \omega_{i,k}z_k - \sum_{k=1}^{m-1} \omega_{m,k}z_iz_k, \ i = 1, \ldots, m - 1.$$  

We see that the coefficients $\omega_{i,k}, k \neq i$ of $\omega$ are determined by this system; so do the differences $\Delta_i := \omega_{i,i} - \omega_{m,m}$. The family $(\Delta_i)$ and $\text{trace}(\omega)$ determine $m \cdot \omega_{m,m} = \text{trace}(\omega) - \sum_{i=1}^{m-1} \Delta_i$, and subsequently every $\omega_{i,i}$. $\square$

**Proposition 2.4.** Let $X$ be a complex manifold and $\nabla$ be a holomorphic flat projective connection on $\mathbb{P}(\mathcal{O}^m_X)$, then $\nabla = \mathbb{P}(D)$ for a unique holomorphic flat linear trace free connection $D : \mathcal{O}^m_X \to (\Omega^1_X)^m$. 

By trace free we mean $D(y) = dy - \omega \cdot y$ with $\text{trace}(\omega) = 0$.

**Proof.** We can cover $X$ with open sets $U_i$ such that the connection is trivializable on $U_i$; taking $U_i$ small enough, this means there exist holomorphic maps $G_i : U_i \to \text{SL}_m(\mathbb{C})$ such that, for $\phi_i = id \times \mathbb{P}G_i$, $\nabla|_{U_i} = \phi_i(\mathbb{P}(d))$, where $d$ is the trivial linear connection on $\mathcal{O}_{U_i}^m$. Also we can define flat linear connections by $D_i := \psi_i^*d$, where $\psi_i = id \times G_i$. The connections $D_i$ are trace free because so does $d$ and the matrices $G_i$ take values in $\text{SL}_m(\mathbb{C})$. For clarity, let us draw a commutative diagram.

$$
(U_i \times \mathbb{P}^{m-1}, \nabla) \xrightarrow{P} (U_i \times \mathbb{C}^m, D_i) \\
\downarrow \phi_i \downarrow \psi_i \\
(U_i \times \mathbb{P}^{m-1}, \mathbb{P}(d)) \xrightarrow{P} (U_i \times \mathbb{C}^m, d)
$$

Let $U_{i,j} := U_i \cap U_j$. If $U_{i,j} \neq \emptyset$, the connections $D_i|_{U_{i,j}}$ and $D_j|_{U_{i,j}}$ are both trace free connections on the trivial bundle with projectivization $\nabla|_{U_{i,j}}$, thus they are equal by lemma 2.3. This means the connection $D_i$ extends to a flat holomorphic connection $D$ on the trivial rank $m$ vector bundle over $X$ with $\mathbb{P}(D) = \nabla$. We have proved existence of the sought $D$, uniqueness follows from lemma 2.3. □

2.2. Logarithmic extensions.

**Definition 2.5.** Let $X$ be a complex manifold and $H$ an analytic hypersurface. Let $P \to X$ be a $\mathbb{P}^{m-1}$-bundle on $X$. A **logarithmic flat projective connection** on $P$, with poles in $H$, is a singular holomorphic codimension $m - 1$ foliation $\nabla$ on $P$ with the following properties.

1. The foliation $\nabla$ restricts to a holomorphic flat projective connection on $P|_{X \setminus H}$.
2. For every $x \in H$, there exists a neighborhood $U$ of $x$ and a flat logarithmic connection $D$ on the trivial rank $m$ vector bundle over $U$ with poles in $H$, such that there exists a bundle isomorphism $\phi : P|_U \to \mathbb{P}(\mathcal{O}_U^m)$ satisfying $\phi^*\mathbb{P}(D)|_{U \setminus H} = \nabla|_{P|_{U \setminus H}}$.

We define the **monodromy representation** of $\nabla$ to be the one of $\nabla|_{P|_{X \setminus H}}$.

Let us introduce a property $\mathcal{P}_m(M)$ for an element $M \in \text{PGL}_m(\mathbb{C})$.

$$\mathcal{P}_m(M) : \begin{cases}
\text{For any } \tilde{M} \in \text{GL}_m(\mathbb{C}) \text{ with } \mathbb{P}(\tilde{M}) = M, \\
\text{for any two eigenvalues } \lambda_1, \lambda_2 \text{ of } \tilde{M}, \\
\lambda_1^m = \lambda_2^m \Rightarrow \lambda_1 = \lambda_2.
\end{cases}$$

Of course it suffices to check this condition for only one lift $\tilde{M} \in \text{GL}_m(\mathbb{C})$.

For $X$ a complex manifold and $H$ a hypersurface in $X$, if $H_j$ is a component of $X$ we call $\alpha \in \pi_1(X \setminus H, \ast)$ a simple loop around $H_j$ if $\alpha$ is conjugate by a path to $(x, z)(t) = (e^{2\pi t}, z_0)$ for a coordinate patch $(x, z_1, \ldots, z_l)$ of $X$ centered at a point of $\{x = 0\} \subset H_i$. Our realization result is the following.

**Theorem 2.6.** Let $X$ be a complex manifold. Let $H$ be a normal crossing analytic hypersurface on $X$. Let $\rho : \pi_1(X \setminus H, \ast) \to \text{PGL}_m(\mathbb{C})$ be an antirepresentation. Suppose, for every simple loop $\alpha \in \pi_1(X \setminus H, \ast)$ around any component of $H$, we have $\mathcal{P}_m(\rho(\alpha))$. 

Lemma 2.7. Let $\omega$ be a matrix of meromorphic 1-forms on a neighborhood $U$ of $0$ in $\mathbb{C}^n$, with coordinates $x_1, \ldots, x_n$. Suppose $d\omega = \omega \wedge \omega$. Suppose the only pole of $\omega$ is $x_1 = 0$ and $\omega = A_1 \frac{dx_1}{x_1} + \tau$ for a holomorphic $\tau$, for $A_1$ a non resonant matrix, then there exists a neighborhood $V \subset U$ of $0$, such that the connection $D$ on $\mathcal{O}_V^n$ defined by $D(y) = dy - \omega \cdot y$ is isomorphic to $D_{A_1|V}$.

Proof. The result for only one variable is well known and allows to suppose $\tau|(x_2, \ldots, x_n) = 0$. Then, coincidence of monodromy yields the required isomorphism outside $x_1 = 0$. Finally, our nonresonance assumption allows to extend the isomorphism holomorphically at $x_1 = 0$ by lemma 2.7.

Lemma 2.9. Let $A_1 \in M_m(\mathbb{C})$. Suppose $mA_1$ is nonresonant. Let $U$ be a neighborhood of $0$ in $\mathbb{C}^n$ and suppose we have an automorphism $\phi$ of the holomorphic projective connection $\nabla = \mathbb{P}(D_{A_1|U\setminus\{x_1=0\}})$, then $\phi$ extends to an an automorphism of the trivial $\mathbb{P}^{m-1}$-bundle over $U$.

Proof. We can suppose $U$ is a polydisk. The automorphism $\phi$ is of the form $(x, z) \mapsto (x, G(x) \cdot z)$ for a holomorphic function $G : U \setminus \{x_1 = 0\} \to \text{PGL}_m(\mathbb{C})$.

If the covering $\pi : V \to U$ is defined by $(u_1, \ldots, u_n) \mapsto (x_1, \ldots, x_n) = (u_1^n, u_2, \ldots, u_n)$, there exists a holomorphic function $H : V \setminus \{u_1 = 0\} \to \text{SL}_m(\mathbb{C})$ satisfying $\mathbb{P}(H(y)) = G \circ \pi(y)$. This function induces an automorphism $(u, z) \mapsto (u, H(u) \cdot z)$ of the pull-back $\mathbb{P}(D_{mA_1|V\setminus\{u_1=0\}})$ of $\nabla$ by $\pi$.
Also, by lemma 2.3, \((u, y) \mapsto (u, H(u) \cdot y)\) is an automorphism of \(D_{mA_1}\).

By hypothesis \(mA_1\) is nonresonant, thus \(H\) and \(H^{-1}\) extend to holomorphic functions on \(V\), by lemma 2.4. Thus \(G = \pi \circ \mathbb{P}H\) also extends as desired. \(\square\)

**Proof of theorem 2.6.** Let \((H_i)_{i \in I}\) be the components of \(H\). Let \(\alpha_i \in \pi_1(X \setminus H, \ast)\) be a simple loop turning counterclockwise around \(H_i\). For any \(i\) choose a lift of \(\rho(\alpha_i)\) \(M_i \in SL_m(\mathbb{C})\) and \(A_i \in M_m(\mathbb{C})\), with real parts of its eigenvalues \(\mu\) satisfying \(0 \leq \text{Re}(\mu) < 1\), such that \(D_{A_i}\) has monodromy \(M_i\). Thanks to \(\mathcal{P}(\rho(\alpha_i))\), \(mA_i\) is automatically nonresonant.

Let \(p \in H\), and let \(H_{ij}, j = 1, \ldots, k\) be the components of \(H\) which contain \(p\). Because of normal crossings, there exist conjugates \(\gamma_j\) of \(\alpha_i, 1 \leq j \leq k\), which commute pairwise. Choose lifts \(N_j \in SL_m(\mathbb{C})\) of \(\rho(\gamma_j)\), conjugate to \(M_i\). The abelianity of \(\langle (\mathbb{P}N_j)_j \rangle \subset PGL_m(\mathbb{C})\) gives \(N_jN_j^{-1} = \lambda N_j\) with \(\lambda^m = 1\), but \(\mathcal{P}(\rho(\gamma_j))\) yields \(\lambda = 1\) and we have abelianity of \(\langle (N_j)_j \rangle \subset GL_m(\mathbb{C})\).

Because of the latter abelianity and [Bri04] lemmme 2], on a neighborhood \(U_p\) of \(p\), there exists a linear flat connection \(D_p\) with residues conjugate to \(A_i\) on \(H_{ij}\), given by a model \(D_{(B_i, \ldots, B_i)}\); set \(\nabla_p = \mathbb{P}(D_p)\). On \(U_0 = X \setminus H\) take \(\nabla_0\) a projective flat connection with monodromy \(\rho\). We denote \(P_p\), \(P_0\) the underlying bundles of \(\nabla_p, \nabla_0\) respectively. Also, set \(U_{p,0} := U_p \cap U_0 = U_p \setminus H\), \(U_{p,q} := U_p \cap U_q\).

By conjugacy of monodromies, we have isomorphisms \(\phi_{p,0} : P_{0|U_{p,0}} \rightarrow P_{p|U_{p,0}}\), such that \(\phi_{p,0}^* \nabla_{p|U_{p,0}} = \nabla_{0|U_{p,0}}\).

Define \(\phi_{0,p} = \phi_{p,0}^{-1}\) and denote \(H_0\) the set of singular point of \(H\). By theorem 2.8 and lemma 2.9 the composition \(\phi_{0,p} \circ \phi_{0,q}\) extends to an isomorphism \(P_{q|U_{p,q}}|H_0 \simeq P_{p|U_{p,q}}|H_0\), then it extends to \(\phi_{p,q} : P_{q|U_{p,q}} \rightarrow P_{p|U_{p,q}}\) because \(H_0\) has codimension > 1 in \(U_{p,q}\). By definition, the functions \(\phi_{i,j}\) satisfy the cocycle relations and define a \(\mathbb{P}^{m-1}\)-bundle \(P\). The local connections \(\nabla_j\) on \(P\) satisfy \(\phi_{i,j}^* \nabla_i|U_{i,j} = \nabla_j|U_{i,j}\) and give the sought flat projective logarithmic connection on \(P\).

\(\square\)

3. Lifting result

We will (re)prove the following.

**Theorem 3.1.** Let \(X\) be an irreducible projective complex variety and \(H\) an algebraic hypersurface in \(X\). Let \(\ast\) be a smooth point of \(X \setminus H\). For any representation \(\rho : \pi_1(X \setminus H, \ast) \rightarrow PSL_m(\mathbb{C})\), there exists a hypersurface \(H_{\rho}\) with \(H \subset H_{\rho}, \ast \notin H_{\rho}\) and a generically finite morphism \(f_{\rho} : (Y_{\rho}, \ast_{\rho}) \rightarrow (X, \ast)\) of projective varieties with basepoints, such that \(Y_{\rho}\) is smooth and the pull-back

\[ f_{\rho}^*\rho : \pi_1(Y_{\rho} \setminus f_{\rho}^{-1}(H_{\rho}), \ast_{\rho}) \rightarrow PSL_m(\mathbb{C}) \]

lifts to \(SL_m(\mathbb{C})\), that is \(f_{\rho}^*\rho = \mathbb{P}\hat{\rho}\), for a representation

\[ \hat{\rho} : \pi_1(Y_{\rho} \setminus f_{\rho}^{-1}(H_{\rho}), \ast_{\rho}) \rightarrow SL_m(\mathbb{C})\].

**Proof.** By resolution of singularities, after some birational morphism we can suppose \(X\) is smooth and \(H\) is normal crossing; we make this assumption in the sequel.
Let \((H_i)\) be the irreducible components of \(H\) and let \(\alpha_i \subset \pi_1(X \setminus H, \ast)\) be a simple loop around \(H_i\). Take a lift \(M_i \in \text{SL}_m(\mathbb{C})\) for \(\rho(\alpha_i)\). Consider the finite set \(S_i\) whose elements are finite order quotients \(\lambda/\mu\) of eigenvalues \(\lambda, \mu\) of \(M_i\). Set \(S := \cup_i S_i\) and let \(O\) be the set given by the orders of the elements of \(S\). Let \(\nu := \text{lcm}(\nu_i)\). Let \(r : X_1 \to X\) be a degree \(\nu\) ramified covering of \(X\) with ramification index at \(H_i\) equal to \(\nu\) and such that \(H_1 = r^{-1}(H)\) is a normal crossing divisor; existence of such an \(r\) is given, for example, by [Laz04, Prop. 4.1.12].

Then, we can apply our realization theorem 2.2.6 there exists a flat projective logarithmic connection \(\nabla\) with poles in \(H_1\) with monodromy \(r^*\rho\). Let \(P\) be the underlying analytic locally trivial \(\mathbb{P}^{m-1}\)-bundle of \(\nabla\) and take \(\ast_1 \in r^{-1}(\ast)\). By Serre, [Ser58, thm. 3 p. 34], \(P\) is the analytification of an algebraic locally trivial \(\mathbb{P}^{m-1}\)-bundle: \(\ast_1\) has a Zariski neighborhood \(U_1 \subset X_1\), \(U_1 = X_1 \setminus \overline{H}_1\) such that there exists a finite étale algebraic covering \(q : (U_2, \ast_2) \to (U_1, \ast_1)\) satisfying that \(q^*P\) is trivial. We can suppose \(H_1 \subset H_1\) and \(U_1\) is affine, which we do.

Then, we have an embedding \(U_2 \subset Y, \overline{U_2} = Y\), in a smooth projective \(Y\) such that the algebraic map \(q\) extends to a morphism \(q : Y \to X_1\). By triviality of \(q^*(P)|_{U_2}\) and proposition 2.4, \(\nabla_2 := q^*(\nabla|_{U_1})\) lifts to a trace free flat linear connection over \(U_2\). For this reason, the monodromy representation of \(\nabla_2\) lifts to \(\text{SL}_m(\mathbb{C})\). By construction this monodromy is \(q^*r^*\rho\). Hence, if we define, \(\ast_\rho := \ast_2, Y_\rho := Y, f_\rho := r \circ q\) and \(H_\rho\) to be the codimension 1 part of \(r \circ q(Y \setminus U_2)\), we have the situation announced in the theorem. \(\square\)

4. Acknowledgements

We thank Carlos Simpson for informations about the state of the art for theorem 3.1. We thank FIRB project “Geometria differenziale e teoria geometrica delle funzioni”, Marco Abate and Jasmin Raissy for their hospitality in Pisa.

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