Vacuum polarization and classical self-action near higher-dimensional defects

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We analyze the gravity-induced effects associated with a massless scalar field in a higher-dimensional spacetime being the tensor product of \((d-n)\)-dimensional Minkowski space and \(n\)-dimensional spherically/cylindrically-symmetric space with a solid/planar angle deficit. These spacetimes are considered as simple models for a multidimensional global monopole (if \(n \geq 3\)) or cosmic string (if \(n = 2\)) with \((d - n - 1)\) flat extra dimensions. Thus, we refer to them as conical backgrounds. In terms of the angular deficit value, we derive the perturbative expression for the scalar Green’s function, valid for any \(d \geq 3\) and \(2 \leq n \leq d - 1\), and compute it to the leading order. With the use of this Green’s function we compute the renormalized vacuum expectation value of the field square \(\langle \phi^2(x) \rangle_{\text{ren}}\) and the renormalized vacuum averaged of the scalar-field’s energy-momentum tensor \(\langle T_{MN}(x) \rangle_{\text{ren}}\) for arbitrary \(d\) and \(n\) from the interval mentioned above and arbitrary coupling constant to the curvature \(\xi\).

In particular, we revisit the computation of the vacuum polarization effects for a non-minimally coupled massless scalar field in the spacetime of a straight cosmic string.

The same Green’s function enables to consider the old purely classical problem of the gravity-induced self-action of a classical pointlike scalar or electric charge, placed at rest at some fixed point of the space under consideration.

To deal with divergences, which appear in consideration of the both problems, we apply the dimensional-regularization technique, widely used in quantum field theory (QFT). The explicit dependence of the results upon the dimensionalities of both the bulk and conical submanifold, is discussed.

1. INTRODUCTION

Through the last decades the higher-dimensional generalizations of known four-dimensional solutions in General Relativity (GR) became the object of intense research in the context of widely developing higher-dimensional theories. It is enough to mention the possibility of the mini-black-hole creation in the high energy physics experiments [1]. Experimental confirmation of such a creation is considered as one of tests on the existence of extra dimensions, or it has to set new bounds on the parameters of the multidimensional theories predicting the existence of mini-black-holes. Though at present, there are no confirmations of the extra-dimension existence [2], the modern theories stimulated the research of the GR in \(d > 4\) spacetime dimensions. This implies not only the search of new solutions, but also the research of the higher-dimensional generalizations of the known four-dimensional solutions. The partial goal of such research is to clarify, which predictions by GR are proper for four dimensions only, and which ones are universal and extended to higher dimensions. At the other hand, it is expected that the research of higher-dimensional generalizations allows to shed light on some peculiarities of the standard four-dimensional theory and assists in the better understanding of the latter. This research assumes not only the study of geometric features of higher-dimensional solutions, but also the study of particularities of the classical/quantum matter dynamics on their background.

The standard problems of research within the field theory on the curved background, to which the physicists return through decades, are the effects of the induced by gravity vacuum polarization and the problem of self-action of the classical charged particle. These two problems, weakly related at the first glance, in fact have a number of
common features. The main of those is that the both problems are determined by the appropriate Green’s function being the solution of partial differential equation, which is sensitive to the global structure of the manifold. Thus, the both effects become essentially non-local. Furthermore, for the elimination of divergences arising in the both cases, one uses the same techniques.

The present work is devoted to the consideration of gravity-induced effects of the vacuum polarization of a massless scalar field and the self-action of a scalar or electric charge on the ultrastatic spacetime being the product of \((d - n)\)-dimensional Minkowski spacetime and \(n\)-dimensional spherically-symmetric space with an angular deficit.

We will be concentrated on the computation of the renormalized vacuum expectation values (VEV) for \(\langle \phi^2(x) \rangle_{\text{ren}}\) and \(\langle T_{MN}(x) \rangle_{\text{ren}}\), as well as of calculation of the renormalized self-energy \(U_{\text{ren}}(x)\) and self-force \(F_{\text{ren}}(x)\) of the static scalar or electric charge. For the regularization of formally diverging expressions we will use the dimensional-regularization technique.

The paper is organized as follows: Introduction is the first section. In the Second section, the Setup, we briefly present the background metric with angular deficit in arbitrary spacetime dimension and derive the initial expressions for the subsequent computation of classical self-force and vacuum averages. The perturbation theory we use, is described in the Section \(\text{III}\) where we also construct the approximated Green’s function. The Section \(\text{IV}\) is devoted to the computation of renormalized vacuum averaged \(\langle \phi^2(x) \rangle_{\text{ren}}\) in the dimensional-regularization scheme. The comparison with the analogous results known in the literature, is presented. The renormalized stress-energy tensor is computed in the Section \(\text{V}\). The classical self-energy and self-force of a pointlike scalar or electric charge in the spacetime-at-hand, are computed in the Section \(\text{VI}\). In the Section \(\text{VII}\) we discuss the special case of an infinitely thin cosmic string. We show that there is a some ambiguity in the previous calculations and propose an alternative approach to the problem. In the last Section \(\text{VIII}\) the Conclusion, we summarize the results and prospects. Useful integrals are given in the single Appendix.

We use the units \(G = c = \hbar = 1\) and metric with the signature \((-,-,+,...,+).\)

2. SETUP

In the model we consider quantized or classical massless scalar field \(\phi\), living in the static \(d\)-dimensional bulk with \(n\)-dimensional submanifold with solid or planar angular deficit. This \(n\)-dimensional subspace may be considered as created by the \(n\)-dimensional global monopole (for \(n \geq 3\)) or as a straight cosmic string (for \(n = 2\)).

First we overview the background geometry.

A. Background of the cosmic string and the global monopole, and their higher-dimensional analogues

The metric of a straight infinitely thin cosmic string with a mass per unit length \(\mu\), located along the \(z\)-axis in four spacetime dimensions, in cylindric coordinates reads

\[
ds^2 = -dt^2 + dz^2 + dρ^2 + β^2 ρ^2 dϕ^2 ,
\]

where \(β = 1 - 4G\mu\). (For the review of the formation, evolution and geometry of topological defects and some physical effects near them see \([3, 4]\) and Refs therein). The corresponding Riemann tensor vanishes everywhere except the symmetry axis \(ρ = 0\), where it has a \(δ\)-like singularity \([5]\). Straight string does not affect the local geometry of the spacetime, its effect on matter fields is purely topological, and the dimensionless parameter \(G\mu\) is the only parameter which measures the effect of conical structure on the dynamics of classical and quantized matter.

In some applications it is more appropriate to use coordinates \((t, x, y, z)\), which are conformally Cartesian on the plane transverse to the string. With the radial-coordinate transformation \(ρ \to r\) as

\[
ρ = \frac{r_0}{β} \left(\frac{r}{r_0}\right)^β , \quad x^1 = r \cos ϕ , \quad x^2 = r \sin ϕ ,
\]
where $r_0$ is an arbitrary scale with the length dimensionality, the line element (2.1) takes the form

$$ds^2 = -dt^2 + dz^2 + e^{-2(1-\beta)\ln(r/r_0)}\delta_{ab}\,dx^a dx^b,$$

where $r^2 = \delta_{ab} x^a x^b$, $a, b = 1, 2$.

The idea to use the conformal coordinates was put forward in the framework of a low-dimensional gravity. In this case it gives the possibility to find a self-consistent solution for the metric of a multi-center space, i.e. a static $(2 + 1)$-dimensional spacetime of $N$ point masses. Later it was shown that the line element of a multi-center spacetime can be generalized for the case of $N$ parallel cosmic strings. The same idea enables to obtain the explicit solutions of the problem of topological self-action in the multicenter and multistring spacetimes, and provides an appropriate framework for consideration of the vacuum polarization effect in the spacetime of multiple cosmic strings and in particular, the vacuum Casimir-like interaction of parallel strings.

One can consider the generalization of the metric (2.1) and (2.2) for a spherically symmetric case, when any plane containing the center of symmetry and dividing the space into two equal parts is a cone with the angular deficit $\delta \varphi = 2\pi(1 - \beta)$

$$ds^2 = -dt^2 + d\varrho^2 + \beta^2 \varrho^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

This metric describes an ultrastatic spherically symmetric spacetime with the solid angle deficit equal to $4\pi(1 - \beta^2)$.

Expression (2.3) approximates the metric of a global monopole. Strictly speaking, the metric of a global monopole contains a mass term, but this term is too small to be of importance on astrophysical scale.

As in the string case, there is a possibility to use conformally Cartesian coordinates on the section $t = \text{const}$ of the spacetime (2.3). After redefinition of the radial coordinate $\beta \varrho = r_0 (r/r_0)^{\beta}$ the metric of the spatial sector of the above line element takes the conformally Euclidean form. Thus, we can introduce a set of Cartesian coordinates $\{x^i\}, i = 1, 2, 3$ with usual relation with the spherical coordinates $r, \theta, \varphi$. In these coordinates metric (2.3) reduces to the form

$$ds^2 = -dt^2 + e^{-2(1-\beta)\ln(r/r_0)}\delta_{ik}dx^i dx^k,$$

where $r^2 = \delta_{ik} x^i x^k$, $i, k = 1, 2, 3$.

We see that both conical defects have no Newtonian potential and exert no gravitational force on the surrounding matter. For both defects their gravitational properties are determined by the deficit angle only. The main difference of a global monopole from the case of a cosmic string is that the monopole spacetime is not locally flat, and its gravitational field provides a tidal acceleration which is proportional to $r^{-2\beta}$.

Below we will consider multidimensional generalization of the spaces (2.2) and (2.4), with arbitrary number of conical and flat spatial dimensions. The corresponding metric reads:

$$ds^2 = g_{MN} \, dx^M dx^N = -dt^2 + dx^2_{d-1} + \ldots + dx^2_{n+1} + e^{-2(1-\beta)\ln r} \delta_{ik} dx^i dx^k,$$

with $r^2 = \delta_{ik} x^i x^k$ and $i, k = 1, \ldots, n$ while $M, N, \ldots = 0, 1, \ldots, d - 1$. Here $d \geq 3$ and $2 \leq n \leq d - 1$. Without loss of generality we put $r_0$ equal to unity.

The spacetime with metric (2.5) represents the tensor product of the $(d - n)$-dimensional Minkowski space and the $n$-dimensional centro-symmetric conformally flat space with a solid angle deficit equal to $\delta \Omega = 2(1 - \beta^2) \pi^{n/2}/\Gamma (n/2)$, if $n \geq 3$, or planar angular deficit equal to $\delta \varphi = 2\pi(1 - \beta)$, if $n = 2$.

The corresponding Ricci tensor and the scalar curvature are determined by the conical sector only:

$$R_{ik} = 2\pi(1 - \beta) \delta^2(\varrho) \delta_{ik}, \quad R = 4\pi(1 - \beta) r^{2(1-\beta)} \delta^2(\varrho), \quad n = 2;$$

$$R_{ik} = (1 - \beta^2)(n - 2) \frac{r^2 \delta_{ik} - x_i x_k}{r^4}, \quad R = (1 - \beta^2)(n - 1)(n - 2) \frac{1}{r^{2\beta}}, \quad n \geq 3. \quad (2.6)$$

For these spaces and corresponding Green’s functions we will use the notations $(d, n)$ and $G(x, x' | d, n)$. Notice, in these notations, the spacetime of a straight infinitely thin cosmic string and that one of a point global monopole in four spacetime dimensions have the type (4, 2) and (4, 3), respectively.
So, (2.5) represents the multidimensional generalization of the four-dimensional solutions obtained in [13, 14] and [15, 16], correspondingly.

For the first time metric of the form (2.5) with two-dimensional conical subspace ($n = 2$) was considered in the paper [17]. Later a number of solutions for a coupled system of the Einstein equation and the equations of motion for $n$ scalars was found and analyzed in [18]. It was shown, that the $n \geq 3$ solution with equal-to-zero cosmological constant has approximately the form (2.5) (in our coordinates). Thus, the metric (2.5) describes the conical defects which live in a $d-$dimensional bulk, having a flat $(d - n - 1)$-brane as a core. Some tiny QFT effects have been found on these backgrounds for some particular dimensionalities of the bulk dimension $d$ and the dimension of the conical subspace $n$. The vacuum polarization effects for a massless scalar and fermionic fields on the higher-dimensional monopole/string spacetime were investigated in [19, 20] and [12, 17, 21, 22]. In [23] the authors analyze the vacuum fluctuations of a quantum bosonic and fermionic currents induced by a magnetic flux running along the string. In this paper we continue the investigation of quantum and classical field-theoretical processes on the generalized background (2.5).

The geometry of the spacetime under consideration is simple enough and the metric does not contain any dimensional parameters. Nevertheless we cannot compute explicitly Green’s function $G(x, x′ | d, n)$ in a workable closed form. So, we restrict our consideration by the particular case of a small angular deficit; in what follows, we put $(1 - \beta) \ll 1$. It enables us to obtain perturbatively the universal expression for the Green’s function, which is valid for any $d$ and $n$ and for any value of the coupling constant $\xi$.

### B. Self-energy of a pointlike charge in a static spacetime: formalism

Let us consider a massless scalar field $\phi$ with a source $j$ in a static $d-$dimensional spacetime with the metric

$$ds^2 = g_{MN} dx^M dx^N = g_{00} dt^2 + g_{\mu\nu} dx^\mu dx^\nu, \quad g_{00} < 0.$$  \hspace{1cm} (2.7)

In this subsection the small Greek indices $\mu, \nu, \ldots$ run over all spatial coordinates $1, 2, \ldots, d - 1$.

The interaction of scalar field with the bulk curvature $R$ is introduced via coupling $\xi$, while interaction with charges is introduced by the charge density $j(x)$ in a standard way:

$$S_{\text{tot}} = -\frac{1}{2} \int d^dx \sqrt{-g} \left( \phi_{;M} \phi^{;M} + \xi R \phi^2 - 2\phi j \right) + S_j.$$  \hspace{1cm} (2.8)

$S_j$ is the action for a charged matter.

From (2.8) one derives the equation of motion for scalar field:

$$\partial_M \left( \sqrt{-g} g^{MN} \partial_N \phi \right) - \xi \sqrt{-g} \phi R = -\sqrt{-g} j$$ \hspace{1cm} (2.9)

In the static case, when $\partial_0 \phi = 0 = \partial_0 g_{MN}$, and pointlike charge $q$ placed at a fixed spatial point $x$ it reads

$$\partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) - \xi R \phi = -\sqrt{-g} j$$ \hspace{1cm} (2.10)

where

$$j(x') = q \frac{\delta^{d-1}(x - x')}{\sqrt{-g}}.$$ \hspace{1cm} (2.11)

The field energy in a static spacetime reads

$$U = -\int T^0_0 \sqrt{-g} d^{d-1}x,$$ \hspace{1cm} (2.12)

where $T^0_0$ stands for zero-zero component of the energy-momentum tensor, which for the scalar field is derived from the action (2.8) and given by

$$T^N_M = (1 - 2\xi) \phi_{;M} \phi_{;N} + \frac{4\xi - 1}{2} \phi_{;L} \phi_{;L} \delta^N_M - 2\xi \phi_{;M} \phi_{;N} + 2\xi \phi \Box \phi \delta^N_M + \frac{\xi}{2} \left( 2R^N_M - R \delta^N_M \right) \phi^2.$$ \hspace{1cm} (2.13)
Note that the interaction part of the action does not contribute to the field energy-momentum tensor. It is particularly obvious in the case under consideration since for a pointlike charge with the source (2.11) the Lagrangian density reads $\mathcal{L}_{\text{int}} = \sqrt{-g} \phi j = q \delta^{d-1}(x - x')$ and does not depend on the metric.

Making use the fact that the field and the metric are static we have

$$T_0^0 = -\frac{1}{2} \left[ \frac{4\xi}{g} \phi, \phi \right] + 2\xi \phi \left[ \sqrt{-g} g^{\mu \nu} \partial_\mu \phi \right] + \xi \left( R_0^0 - \frac{1}{2} R \right) \phi^2.$$  

Substituting $T_0^0$, the scalar-field energy is given by

$$U_{sc} = \frac{1}{2} \int d^{d-1}x \partial_\mu \left[ \sqrt{-g} g^{\mu \nu} \phi \partial_\nu \phi \right] - \frac{1}{2} \int d^{d-1}x \left[ \phi \partial_\mu \left( \sqrt{-g} g^{\mu \nu} \partial_\nu \phi \right) + \sqrt{-g} \xi (2R_0^0 - R) \phi^2 \right].$$

and integrating with help of the Gaus’ theorem, only the second integral survives. Simplifying and taking help of the field equation (2.10), (2.12) becomes

$$U_{sc} = \frac{1}{2} \int d^{d-1}x \sqrt{-g} \left[ \phi j - 2\xi R_0^0 \phi^2 \right].$$  \hspace{1cm} (2.14)

The corresponding form via Green’s function of the Eq. (2.10) reads:

$$U_{sc} = \frac{1}{2} \int d^{d-1}x d^{d-1}x' \sqrt{g(x)g(x')} j(x) G(x, x') j(x') - \xi \int d^{d-1}x \sqrt{-g} R_0^0 \phi^2,$$

where $G(x, x')$ satisfies

$$\partial_\mu \left( \sqrt{-g} g^{\mu \nu} \partial_\nu G(x, x') \right) - \xi R \sqrt{-g} G(x, x') = -\sqrt{-g} \delta^{d-1}(x, x').$$  \hspace{1cm} (2.15)

Thus for a point charge localized at the point $x$ of the spacetime from Eq.(2.11) we get

$$U_{sc}(x) = \frac{1}{2} q^2 G(x, x) - \xi \int d^{d-1}x \sqrt{-g} R_0^0 \phi^2.$$  \hspace{1cm} (2.16)

Note, that for a general case of a static spacetime one has $g = g_{00} \det(g_{\mu \nu})$, while $R$ in Eq.(2.15) stands for the scalar curvature of the whole $d$-dimensional space.

In addition, if the spacetime is ultrastatic (i.e. $g_{00} = -1$), then $g = - \det(g_{\mu \nu})$, $R_0^0 = 0$, and (2.16) takes the form

$$U_{sc}(x) = \frac{1}{2} q^2 G(x, x),$$

where $G$ is the solution of the equation

$$\partial_\mu \left( \sqrt{g} g^{\mu \nu} \partial_\nu G(x, x') \right) - \xi \mathcal{R} \sqrt{g} G(x, x') = -\sqrt{g} \delta^{d-1}(x, x')$$  \hspace{1cm} (2.17)

where $g = \det(g_{\mu \nu})$ and $\mathcal{R}$ is the corresponding scalar curvature. That is, $G$ is the Green’s function on the $(d - 1)$-dimensional space with the metric $g_{\mu \nu}$ with Euclidean signature and the curvature $\mathcal{R}.$

Now let us suppose that there exists at least one flat extra spatial dimension, say $x_{d-1}$. Then formally identifying $ix_{d-1} = t$, one notices that the equation (2.15) for the static Green’s function coincides with the full field equation (3.1) for the Euclidean Green’s function $G^E(x, x’ \mid d - 1, n)$ in the spacetime with $(d - 1)$ spacetime dimensions and $n$-dimensional conical subspace. Finally, with the use of well-known relation between Euclidean $G^E$ and Feynman $G^F$ Green’s functions, we obtain, that in this case

$$U_{sc} = \frac{1}{2} q^2 G^E(x, x \mid d - 1, n) = -\frac{i}{2} q^2 G^E(x, x \mid d - 1, n), \hspace{1cm} F = -\left( \frac{r}{r_0} \right)^{\beta} \frac{\delta U}{\delta r}. $$  \hspace{1cm} (2.18)

One can study the self-energy of a static electric charge along the same lines.
In this case the solution of the Maxwell equations is static, with the \(d\)-potential \(A_M = (A_0(x), 0, \ldots, 0)\) if the current equals \(J^M = (J(x), 0, \ldots, 0)\). The only nontrivial component of the Maxwell equations is
\[
\partial \mu \left( \sqrt{-g} g^{00} g^\mu \nu \partial_\nu A_0 \right) = -\sqrt{-g} J^M,
\]
and for the electrostatic self-energy one obtains (e.g. see [24])
\[
U_{el} = -\frac{1}{2} \int d^{d-1}x \sqrt{-g} A_0 J = \frac{1}{2} \int d^{d-1}x \sqrt{-g} \int d^{d-1}x' \sqrt{-g} J(x) G(x, x') J(x'),
\]
where Green’s function of the Eq. (2.19) is defined as the solution of
\[
\partial \mu \left( \sqrt{-g} g^{00} g^\mu \nu \partial_\nu G(x, x') \right) = \sqrt{-g} \delta^{d-1}(x, x').
\]
So, for the point charge, when the charge density \(J = e \delta^{d-1}(x, x')\), we obtain
\[
U_{el} = \frac{1}{2} e^2 G(x, x).
\]
In the particular case of an ultrastatic space Eq. (2.21) takes the form
\[
\partial \mu \left( \sqrt{-g} g^{00} g^\mu \nu \partial_\nu G(x, x') \right) = -\sqrt{-g} \delta^{d-1}(x, x').
\]
This equation coincides with Eq. (2.10) if \(\xi = 0\). Using this fact one finds that
\[
U_{el} = \frac{1}{2} e^2 G^E(x, x | d - 1, n)_{\xi=0}.
\]
Consequently, on the background under consideration the electrostatic self-energy can be obtained from the scalar one if we put \(\xi = 0\) and replace \(q^2\) by \(e^2\).

The spacetime of interest here (2.5), i.e. \(d\)-dimensional spacetime with \(n\)-dimensional subspace with a solid or planar angle deficit, satisfies the ultrastaticity condition, so we will use simple formulae (2.18, 2.23) for it.

3. GREEN’S FUNCTION: PERTURBATION THEORY

For our background metric (2.5) the exact Green’s function is unknown. Taking into account the fact that \((1 - \beta) \ll 1\) we make use of the perturbation-theory techniques. The Feynman propagator for the scalar field in curved background satisfies the equation
\[
L(x, \partial) G^F(x, x' | d, n) = -\delta^d(x - x'),
\]
where \(L(x, \partial)\) stands for the field-equation operator and determined by the background metric.

Following Schwinger [25], we rewrite eq. (3.1), in the operator form
\[
\mathcal{L} G = -1, \quad G = -\mathcal{L}^{-1}.
\]
If operator \(\mathcal{L}\) allows to be expressed as \(\mathcal{L} = \mathcal{L}_0 + \delta\mathcal{L}\), where \(\delta\mathcal{L}\) is considered as a small perturbation, then representing the solution of eq. (3.2) in the form \(G = G_0 + \delta G\), with \(G_0 = -\mathcal{L}_0^{-1}\) being the unperturbed Green’s function, one obtains
\[
G = \left[ -\mathcal{L}_0 (1 - G_0 \delta \mathcal{L}) \right]^{-1} = G_0 + G_0 \delta \mathcal{L} G_0 + G_0 \delta \mathcal{L} G_0 \delta \mathcal{L} G_0 + \ldots .
\]

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1 In what follows we define the Feynman propagator as \(G^F(x, x') = i \langle T [\phi(x) \phi(x')] \rangle_{\text{vac}}\).
In the case under consideration \( \mathcal{L}_0 \) is determined by the zeroth order in the small quantity \((1 - \beta)\), hence

\[
\mathcal{L}_0(x, \partial) = \partial^2, \quad \partial^2 \equiv \eta^{MN} \partial_M \partial_N.
\]

The perturbation operator

\[
\delta \mathcal{L}(x, \partial) = \partial_M \left( \sqrt{-g} g^{MN} \partial_N \right) - \partial^2 - \sqrt{-g} \xi R
\]

to the first order in \((1 - \beta)\) reads:

\[
\delta \mathcal{L}(x, \partial) = n \alpha(r) \left( \partial_0^2 - \sum_{N=n+1}^{d-1} \partial_N^2 \right) - (n - 2) \sum_{i=1}^{n} \left[ \alpha(r) \partial_i^2 + (\partial_i \alpha(r)) \partial_i \right] - \xi \gamma(r).
\]

In order to compactify our equations below, let us introduce the notation

\[
\beta' = 1 - \beta.
\]

With the use of this notation

\[
\alpha(r) = \beta' \ln r
\]

and

\[
\gamma(r) = \begin{cases} 
4\pi \beta' \delta^2(r), & n = 2; \\
2(n-1)(n-2)\beta'/r^2, & n \geq 3. 
\end{cases}
\]

with \( r = (x_1, x_2, \ldots, x_n) \).

In the problem-at-hand the function \( G_F^0(x, x') = \langle x | G_0 | x' \rangle = -\langle x | \partial^{-2} | x' \rangle \) in Fourier basis takes the form:\n
\[
G_F^0(x - x') = \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip(x-x')}}{p^2 - i\epsilon},
\]

where \( p^2 = p^2 - (p^0)^2 \) and \( px = px^0 \).

For the first-order correction to the Green’s function from (3.3) we get the following expression:

\[
G_F^1(x, x' | d, n) = \langle x | G_0 \delta \mathcal{L} G_0 | x' \rangle = \int \frac{d^dq}{(2\pi)^d} e^{iqx} \int \frac{d^dp}{(2\pi)^d} e^{ip(x-x')} \frac{\delta \mathcal{L}(q, ip)}{|p^2 - i\epsilon| \left[ (p + q)^2 - i\epsilon \right]},
\]

where \( \delta \mathcal{L}^{(1)}(q, ip) \) is defined as:

\[
\delta \mathcal{L}(q, ip) = \int d^dx e^{-iqx} \left[ \delta \mathcal{L}(x, \partial) |_{\partial_j \rightarrow ip_j} \right].
\]

Here one implies that the differential operator \( \delta \mathcal{L}(x, \partial) \) is prepared to the form where all differential operators stand before (at right-hand side from) the coordinate functions, and further one performs the substitution \( \partial_j \rightarrow ip_j \) and calculates the Fourier-transform, considering \( p_j \) as parameters.

In our problem the perturbation operator reads (3.5),

\[
\delta \mathcal{L}(q, ip) = \left[ np^2 - 2p^2 + (n-2)(qp) \right] \mathcal{F}[\alpha](q) - \xi \mathcal{F}[\gamma](q),
\]

\( \mathcal{F}[\varphi(x)](q) = \int d^dx \varphi(x) e^{-iqx} \).
where \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \) are \( n \)-dimensional conformal vectors with the Euclidean scalar product \((qp) \equiv \delta_{ik}q^kp^k\), while \( p^2 \equiv \eta_{MN}p^Mp^N\).

Making use of the explicit form of operator \( \delta \mathcal{L}(x, \partial) \), substitution of (3.9) into eq. (3.7) yields

\[
G^F(x, x' | d, n) = G_0^F(x - x') + \int \frac{dq}{(2\pi)^d} e^{iqx} \int \frac{dp}{(2\pi)^d} \frac{e^{ip(x-x')}}{[p^2 - i\varepsilon][(p + q)^2 - i\varepsilon]} \times \left([np^2 - 2p^2 + (n - 2)(qp)]\mathcal{F}[\alpha]|(q) - \xi\mathcal{F}[\gamma](q)\right).
\]

Taking into account that formulae for the background curvature (3.6) differ for cases \( n = 2 \) and \( n \geq 3 \), we consider here the generic case of a global monopole, while the case of a cosmic string is delegated to the Section 7 below.

In this case (3.10) takes the form

\[
G^F(x, x' | d, n) = G_0^F(x - x') - \frac{\Gamma(n/2)}{2\pi^{n/2}} \int d^nq \frac{e^{iqx}}{(q^2)^{n/2}} \int \frac{dp}{(2\pi)^d} \frac{e^{ip(x-x')}}{[(p + q)^2 - i\varepsilon]} \times \left([np^2 - 2p^2 + (n - 2)(qp)] + 2\xi(n - 1)q^2\right).
\]

where we use the following well-defined Fourier-transforms [26]:

\[
\mathcal{F}[\ln r](q) = -\frac{2^{d-1}}{\pi^{n/2-d}} \frac{\Gamma(n/2)}{(q^2)^{n/2}} \delta(q^0) \prod_{N=n+1}^{d-1} \delta(q^N),
\]

\[
\mathcal{F}[r^{-\lambda}](q) = \frac{2^{d-\lambda}}{\pi^{n/2-d}} \frac{\Gamma[(n - \lambda)/2]}{\Gamma[\lambda/2]} \frac{1}{(q^2)^{(n-\lambda)/2}} \prod_{N=n+1}^{d-1} \delta(q^N).
\]

In Eq. (3.11) and in all subsequent equations \( q \equiv (0, q, 0, 0 \ldots 0) \).

All the quantities we are interested in, are expressed via the Feynman propagator \( G^F(x, x' | d, n) \) and its derivatives, evaluated in coincident points. The corresponding expressions diverge, and for their evaluation we make use of the dimensional-regularization method (see, e.g. [27]).

The dimensional regularization consists in the replacement of the determining function \( G(x, x) \) by \( G_{\text{reg}}(x, x) \), corresponding formally to the Green’s function in \( D = (d - 2\varepsilon) \) dimensions. The subsequent renormalization includes the splitting of \( G_{\text{reg}}(x, x) \) onto two parts; the first one diverges as \( \varepsilon \to 0 \), while the other is finite. The renormalization finishes with the neglect of the divergent part \( G_{\text{div}}(x, x) \), with subsequent computation of the limit \( \varepsilon \to 0 \). But as it was remarked by Hawking [28], in the case of a curved space this procedure may be ambiguous, because in general there can be a variety of different ways of performing the analytic continuation from \( d \) to \( D \) dimensions. The simplest way is to take the product of the initial \( d \)-dimensional spacetime with a flat space with \( D - d \) dimensions with subsequent analytic continuation with respect to the extra dimensions.

Fortunately, the spacetime of interest here, has originally the structure demanded by this prescription. So, according to Hawking’s prescription, we will define \( G_{\text{ren}}^F(x, x | d, n) \) as a limit

\[
G_{\text{ren}}^F(x, x | d, n) = \lim_{\varepsilon \to 0} \left[G_{\text{reg}}^F(x, x | D, n) - G_{\text{div}}^F(x, x | D, n)\right].
\]

As it was shown by Hawking, results obtained by this prescription are in agreements with those ones obtained with help of the method of generalized \( \zeta \)-function.

### 4. Renormalized \( \langle \phi^2(x) \rangle \)

Now proceed to a perturbative expression for the regularized value of vacuum averaged \( \langle \phi^2(x) \rangle_{\text{ren}} \):

We define the Feynman propagator as \( G^F(x, x') = i\langle \mathcal{T} \{ \phi(x) \phi(x') \} \rangle_{\text{vac}} \). So,

\[
\langle \phi^2(x) \rangle_{\text{ren}} = -i G_{\text{ren}}^F(x, x | d, n) = G_{\text{ren}}^E(x, x | d, n).
\]
The first problem, arising here, is an expression arising in the zeroth order in $\beta'$. Indeed, for the contribution from the first term on the right hand side of (3.11) to the Green’s function taken in the limit of coincidence points, we have the formally divergent expression

$$G^F_0(x,x) = -\frac{1}{(2\pi)^d} \int \frac{d^dp}{p^2}.$$ 

However, all integrals of the form

$$\int d^dp \frac{P_{i_1}...P_{i_k}}{p^2},$$

which diverge in UV- or/and in IR-limits and correspond to the «tadpole»-type diagrams in QFT, are set to have zero value (no tadpole prescription) within the dimensional-regularization technique (see, e.g. [29]). According to this prescription we shall put all terms of the form (4.2) equal to zero.

Thus, in the case $d \geq 4$, $3 \leq n \leq d - 1$ and arbitrary value of the coupling constant $\xi$, for the first non-vanishing contribution to the coincidence-points Green’s function one obtains from eq. (3.11):

$$G^F(x,x | d,n) = \beta' \Gamma(n/2) \frac{2}{\pi^{n/2}} \int d^nq \frac{e^{i\phi_0}}{(q^2)^{n/2}} \int d^dp \frac{2p^2 - (n-2)(qp) - 2\xi(n-1)q^2}{(p^2 - i\epsilon)((p+q)^2 - i\epsilon)}.$$ 

The integral over $d^dp$ diverges. However, it has a standard form for the QFT. Within the framework of the dimensional regularization one performs the Wick rotation

$$p^0 \rightarrow ip^0_\xi, \quad d^dp \rightarrow id^dp_E$$

and replaces the integral over $d^dp_E$ by the expression that formally corresponds to integration over a $(D - 2\varepsilon)$-dimensional $p_E$-space:

$$\int \frac{d^dp_E}{(2\pi)^D} \rightarrow i \mu^{2\varepsilon} \int \frac{d^Dp_E}{(2\pi)^D}.$$ 

An arbitrary parameter $\mu$ with the dimension of reciprocal length is introduced to preserve the dimensionality of the regularized expression.

Computational technique for these integrals is well-developed (e.g., see [29]) and we obtain:

$$\int \frac{d^Dp_E}{(2\pi)^D} \frac{2p^2 - (n-2)(qp) - 2\xi(n-1)q^2}{p_E^2 (p+q)^2} = \left(1 - \frac{\xi}{\xi_D}\right) \frac{2(n-1) \Gamma^2(D/2)}{(4\pi)^{D/2} \Gamma(2-D/2)},$$

where we have denoted

$$\xi_D \equiv \frac{D-2}{4(D-1)}.$$ 

Notice, when $\varepsilon = 0$ and $\xi_D = \xi_d$ the field equation for a massless scalar field $\phi$ is invariant under conformal transformations of the metric.

For even $d$ the expression (4.3) has a simple pole at $\varepsilon = 0$, and under the removal of regularization the divergence in $G^F_{\text{reg}}(x,x | D,n)$ may arise due to this pole, or due to the $d^nq$-integration, or due to the both reasons simultaneously.

Let consider this question in more details.

Substituting (4.5) into (4.3) and making use of the integral (3.12) for the regularized vacuum mean $\langle \phi^2(x) \rangle$ we obtain (for all $3 \leq n \leq (d - 1)$) the following expression:

$$\langle \phi^2(x) \rangle_{\text{reg}} = -iG^F_{\text{reg}}(x,x | D,n) = \mu^{2\varepsilon} \beta' \frac{n-1}{4\pi^{D/2}} \frac{\Gamma(n/2) \Gamma^2(D/2)}{\Gamma(D)} \left(\xi_D - 1\right) \frac{\Gamma(-\frac{D-2}{2})}{\Gamma(-\frac{D-n-2}{2})} \frac{1}{\pi^{D-2}}.$$ 

---

3 For brief reference, we overview derivation of some of them in the Appendix A.
We see that the behavior of the regularized VEV \( \langle \phi^2(x) \rangle_{\text{reg}} \) in the limit \( \varepsilon \to 0 \) is determined by the factor
\[
\Gamma\left(-\frac{D-2}{2}\right)/\Gamma\left(-\frac{D-n-2}{2}\right)
\]
and, therefore, depends significantly upon the parity of the dimensionality both of the entire \( d \)-dimensional bulk and of its \( n \)-dimensional conical subspace.

Let consider all possible cases.

- **even \( d \), odd \( n \).** In this case \( (d-n-2)/2 \) is semi-integer, so Gamma-function in denominator (4.7) takes its finite and nonzero value. Whereas the Gamma-function \( \Gamma(1-D/2) \) in the numerator of eq. (4.13) has a simple pole in \( \varepsilon = 0 \), thus when the regularization removed, the separation of divergent part may be performed with help of the Laurent expansion
\[
\Gamma(-m+\varepsilon) = \frac{(-1)^m}{m!} \left( \frac{1}{\varepsilon} - \gamma + H_m + \mathcal{O}(\varepsilon) \right),
\]
where \( \gamma \) is the Euler’s constant, and \( H_m = \sum_{k=1}^{m} k^{-1} \) is the \( m \)-th harmonic number.
We obtain now
\[
\langle \phi^2(x) \rangle_{\text{div}} = -i G^F_{\text{div}}(x,x|d,n) = \frac{(-1)^{d/2}}{\varepsilon} \frac{\beta'(n-1) \Gamma(n/2) \Gamma^2(d/2)}{2\pi^{d/2}(d-n)} \frac{\Gamma(d)}{\Gamma\left(-\frac{d-n}{2}\right)} \left( \frac{\xi}{\xi_d} - 1 \right) \frac{1}{r^{d-2}}.
\]
Notice, in the case of a conformal coupling \( \langle \phi^2(x) \rangle_{\text{div}} \) vanishes.

Separation of the finite part of the regularized expression (4.8) is achieved by the following expansions:
\[
\frac{\xi}{\xi_D} - 1 = \left( \frac{\xi}{\xi_d} - 1 \right) + \varepsilon \frac{8\xi}{(d-2)^2} + \mathcal{O}(\varepsilon^2), \quad \frac{f(D)\mu^\varepsilon}{r^{d-2}} = \frac{f(d)}{r^{d-2}} \left[ 1 + 2\varepsilon \left( \ln \mu r - \frac{f'(d)}{f(d)} \right) + \mathcal{O}(\varepsilon^2) \right],
\]
where
\[
f(z) = \frac{\Gamma^3(z/2)}{\pi^{z/2} \Gamma(z) \Gamma\left(\frac{z+n-2}{2}\right)},
\]
that leads to the final result
\[
\langle \phi^2(x) \rangle_{\text{ren}} = (-1)^{(n-1)/2} \beta'(n-1) \Gamma(n/2) \Gamma(d/2) \Gamma(d/2) \left[ \left( \frac{\xi}{\xi_d} - 1 \right) \ln \tilde{\mu} r + \frac{1}{(d-1)(d-2)} \right] \frac{1}{r^{d-2}}.
\]
The constant \( \tilde{\mu} \) here is a renormalized value of the constant \( \mu \) introduced above:
\[
\tilde{\mu} = \mu \exp \left( \frac{f'(d)}{f(d)} + \frac{H_{d-2} - \gamma}{2} + \frac{1}{(d-1)(d-2)} \right).
\]
Notice, with the conformal coupling the logarithmic term and the uncertainty related with the arbitrary constant \( \tilde{\mu} \) in it, disappear from \( \langle \phi^2(x) \rangle_{\text{ren}} \).

Separately, we consider the case of higher-dimensional monopole, where \( n = (d-1) \). Then from eq. (4.9), making use of well-known formulae on Gamma-function
\[
\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin \pi x}, \quad \Gamma(2x) = \frac{2^{2x-1} \Gamma(x) \Gamma(x+\frac{1}{2})}{\sqrt{\pi}},
\]
we have:
\[
\langle \phi^2(x) \rangle_{\text{ren}} = (-1)^{d/2-1} \beta'(d-2) \Gamma(d/2) \frac{\Gamma(d/2)}{2^{d-1}\pi^{d/2}(d-1)} \left[ \left( \frac{\xi}{\xi_d} - 1 \right) \ln \tilde{\mu} r + \frac{1}{(d-1)(d-2)} \right] \frac{1}{r^{d-2}}.
\]
In particular, for the spacetime types \((4,3)\) and \((6,5)\) one obtains:
\[
\langle \phi^2(x) \rangle_{\text{ren}} = -\frac{\beta'}{12\pi^2r^2} \left( 6\xi - 1 \right) \ln \tilde{\mu} r + \frac{1}{6}, \quad d = 4; \quad (4.12)
\]
\[
\langle \phi^2(x) \rangle_{\text{ren}} = \frac{\beta'}{20\pi^3r^4} \left[ 5\xi - 1 \right] \ln \tilde{\mu} r + \frac{1}{20}, \quad d = 6,
\]
\[
(4.13)
\]
that coincides (with the accuracy required) with the results of \cite{30} and \cite{19}, respectively.\footnote{A note to be added: our result \cite{11} coincides with that one of \cite{30} numerically, since in the cited work it is the numerical computation that was done for several introduced (within their computational scheme) integrals (namely, \cite{24} eqns.\( (2.18, 2.19) \)). However, these integrals may be computed analytically; doing this, the result coincides with our (to the leading in \( \beta' \) order we are interested here).}

Finally, the renormalized VEV of \( \langle \phi^2(x) \rangle \) for 3-dim monopole in six dimensions reads

\[
\langle \phi^2(x) \rangle_{\text{ren}} = -\frac{\beta'}{120 \pi^3 r^4} \left[ (3 \xi - 1) \ln \mu r + \frac{1}{20} \right], \quad d = 6, n = 3. \tag{4.14}
\]

- **odd \( d \) and \( n \).** Here as \( \varepsilon \to 0 \) the Gamma-function \( \Gamma(1 - D/2) \) in numerator is finite, while \( \Gamma(1 - (D - n)/2) \) in the denominator of \( \text{(4.10)} \) has a simple pole. Therefore, in the lowest in \( \beta' \) order \( \langle \phi^2(x) \rangle_{\text{ren}} \) vanishes.\footnote{Unfortunately, we may say nothing about the result in the second order: whether it also vanishes, or has the finite value. Probably, the non-perturbative approach in some particular case \((d, n)\) of this type may shed light on this problem. Investigation of these effects lies beyond the mainline of our work here and hopefully will be considered later.}

- **odd \( d \), even \( n \).** In this case the both Gamma-functions, \( \Gamma(1 - D/2) \) and \( \Gamma(1 - (D - n)/2) \) in \( \text{(4.7)} \), are finite, hence \( \langle \phi^2(x) \rangle_{\text{div}} = 0 \) and after some algebra \( \text{(4.10)} \) we arrive at

\[
\langle \phi^2(x) \rangle_{\text{ren}} = (-1)^{n/2} \frac{\beta'}{4 \pi^{d/2}} \frac{(n - 1) \Gamma(n/2) \Gamma^2(d/2)}{\Gamma(d)} \frac{d - n}{2} \frac{(\xi - 1)}{(\xi_d - 1)} \frac{1}{r^{d-2}}. \tag{4.15}
\]

Hence for the \( d \)-dimensional monopole, \((d, d - 1)\)-spacetime, we have

\[
\langle \phi^2(x) \rangle_{\text{ren}} = (-1)^{n/2} \frac{(d - 2) \Gamma(d/2)}{2 d \pi^{d/2 - 1} (d - 1) r^{d-2}} \frac{(\xi - 1)}{(\xi_d - 1)}. \tag{4.16}
\]

In particular, for the five-dimensional monopole \((d = 5, n = 4)\) VEV \( \langle \phi^2(x) \rangle_{\text{ren}} \) takes the form

\[
\langle \phi^2(x) \rangle_{\text{ren}} = \frac{\beta'}{2 \pi^{d/2}} \frac{3(16 \xi - 3)}{2^{d/2} \pi^3}, \tag{4.17}
\]

which coincides with the results of the papers \cite{19, 20}.

- **even \( d \) and \( n \).** Here the simple pole of \( \Gamma(1 - D/2) \) in numerator \( \text{(4.7)} \) is compensated by that one of the Gamma-function \( \Gamma(-(D - n)/2) \) in denominator. The result of \( \text{(4.7)} \) at \( \varepsilon = 0 \), thereby, equals the ratio of the corresponding residuals. Moreover, as in the previous case, the divergent part \( \langle \phi^2(x) \rangle_{\text{div}} \) vanishes, and VEV equals

\[
\langle \phi^2(x) \rangle_{\text{ren}} = (-1)^{n/2} \frac{(n - 1) \Gamma(d/2) \Gamma^2(n/2)}{\Gamma(d)} \frac{d - n}{2} \frac{(\xi - 1)}{(\xi_d - 1)} \frac{1}{r^{d-2}}, \tag{4.18}
\]

and thus \( \langle \phi^2(x) \rangle_{\text{ren}} \) vanishes (in lowest in \( \beta' \) order) for the case of the conformal scalar field.

A direct comparison of \( \text{(4.15)} \) with the formula \( \text{(4.10)} \) shows, that in the interested accuracy the two cases with even conical subdimensionality \( n \) can be combined into the unified one, despite the intermediate formulae were based on the drastically different behavior of the Gamma-function. However, for odd \( n \) the result depends significantly upon the parity of the bulk’s dimensionality.

Summarizing, in this section we have computed the renormalized vacuum averaged \( \langle \phi^2 \rangle_{\text{ren}} \) for a massless scalar field on the generalized background \( \text{(2.6)} \). We have made the computation up to the first order in \( \beta' \) but for arbitrary values of the coupling constant \( \xi \) and for any dimension of the space \( d \geq 4 \) and any dimension of its conical subspace in the interval \( 3 \leq n \leq d - 1 \). For doing so we have used perturbation technique combined with the method of dimensional regularization. For the case with even \( d \) and odd \( n \) (in particular, for the four dimensional global monopole) it is the logarithmic factor \( \ln \mu r \) that has the crucial significance for the field with nonconformal factor, since all finite non-logarithmic terms may be absorbed by the finite renormalization of \( \mu \).

The methods presented in this section, may be used to compute the renormalized mean value of the energy-momentum tensor in a similar way.
5. RENORMALIZED ENERGY-MOMENTUM TENSOR

The total energy-momentum tensor derived from the action (2.8), is given by (2.13). In terms of the Green’s function, the regularized VEV of the energy-momentum tensor is given by

$$(T_{MN}(x))_{\text{reg}} = -i \lim_{x' \to x} D_{MN} G_{\text{reg}}^{\phi}(x, x'),$$  \hfill (5.1)

where $D_{MN}$ stands for the appropriate differential operator ($\nabla^M$ and $\nabla^{M'}$ denote the covariant derivative over $x^M$ and $x^{M'}$, respectively):

$$D_{MN} = (1 - 2\xi)\nabla_M \nabla_{N'} + \frac{1}{2} (4\xi - 1) \nabla_L \nabla^{L'} g_{MN} + \xi \left[ R_{MN} - \frac{1}{2} R g_{MN} + 2 \nabla_L \nabla^L g_{MN} - 2 \nabla_M \nabla_N \right].$$

Taking into account the special significance of the minimally coupled field, and in order to dilute routine computations, it is natural to compute the renormalized vacuum momentum density separately for different powers of $\xi$. We start to separate $\xi$—terms already from definition: thereby we can split energy-momentum tensor as

$$T_{MN} = T^{(0)}_{MN} + \xi T^{(\xi)}_{MN},$$

with

$$T^{(0)}_{MN} = \phi_{,M} \phi_{,N} - \frac{1}{2} g_{MN} \phi_{,L} \phi^{,L}$$

$$T^{(\xi)}_{MN} = -2\phi_{,M} \phi_{,N} + 2\phi_{,L} \phi^{,L} g_{MN} - 2\phi_{,MN} \phi - 2 \phi \Box \phi g_{MN} + \frac{1}{2} \left( 2R_{MN} - R g_{MN} \right) \phi^2. \hfill (5.2)$$

Each term here contains a quadratic form on $\phi$ and, therefore, can be derived from the Feynman propagator. Hence we may apply our point-splitting procedure for the derivatives combined with the perturbation-theory scheme, to reveal the linear on $\beta'$ contributions.

A note to be mentioned: $T^{(\xi)}_{MN}$ contains the second covariant derivatives; computing them, one needs in the corresponding Christoffel symbols. In the coordinates specified, all non-vanishing Christoffel symbols are of order $O(\beta')$. Given that the zeroth (in $\beta'$) order of the Green’s function vanishes in our scheme (as no tadpole prescription), the retaining of the Christoffel-part contribution yields the order $O(\beta^2)$, i.e. exceeds the necessary accuracy. Hence we can neglect these terms and consider derivatives as «flat».

Repeating the steps to construct the Green’s function, the 1st-order operator correction $\delta \mathcal{L}(x, \partial)$ also can be split as $\delta \mathcal{L}(x, \partial) = \delta \mathcal{L}^{(0)}(x, \partial) + \xi \delta \mathcal{L}^{(\xi)}(x, \partial)$ with $^6$

$$\delta \mathcal{L}^{(0)}(x, \partial) = -n\alpha(r) \partial_\sigma \partial^\sigma - (n - 2) \left[ \alpha(r) \partial_\sigma \partial^\sigma + \left( \partial_\sigma \alpha(r) \right) \partial^\sigma \right],$$

$$\delta \mathcal{L}^{(\xi)}(x, \partial) = -R(r). \hfill (5.3)$$

In what follows, the energy-momentum VEV in the first non-vanishing order reads schematically:

$$T_{MN} = 0 T_{MN} + 1 T_{MN} \xi + 2 T_{MN} \xi^2, \hfill (5.4)$$

where

$$0 T_{MN} = T^{(0)}_{MN} [\delta \mathcal{L}^{(0)}]$$

$$1 T_{MN} = T^{(0)}_{MN} [\delta \mathcal{L}^{(\xi)}] + T^{(\xi)}_{MN} [\delta \mathcal{L}^{(0)}]$$

$$2 T_{MN} = T^{(\xi)}_{MN} [\delta \mathcal{L}^{(\xi)}]. \hfill (5.5)$$

---

$^6$ Within this section the index $\sigma$ runs over all «flat» indices: $\sigma = 0, n + 1, ..., d - 1$. 
Performing the Fourier-transforms in (5.3), the $\xi$-separation in $\delta L(q, ip)$ reads effectively
\[
\delta L^{(0)}(q, ip) = -2^{n-1}\pi^{n/2}(2\pi)^{D-n}\Gamma(n/2)\delta^{D-n}(q^r) - \frac{2p^2 + (n-2)qp}{|q|^n},
\]
and
\[
\delta L^{(\ell)}(q, ip) = -2^{n}(n-1)\pi^{n/2}(2\pi)^{D-n}\Gamma(n/2)\delta^{D-n}(q^r)|q|^{-(n-2)},
\]
so the latter actually does not depend upon $p^M$.

**A. Computation of $\langle T_{MN}(x)\rangle_{\text{ren}}$ with minimal coupling**

Starting from (5.7) and proceeding along the same lines as for $\langle \phi^2 \rangle$, we obtain:
\[
\langle \delta T^{(0)}_{MN}(x) \rangle_{\text{reg}} = \beta' \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} e^{iqx} \frac{\delta L^{(0)}(q, ip)}{|p^2 - i\epsilon| |(p + q)^2 - i\epsilon|} \left( p_{MPN} + q_{MPN} - \frac{1}{2} \eta_{MN} q p \right).
\]

After the integration with help of integrals of Appendix A, $\langle \delta T_{MN}(x) \rangle_{\text{reg}}$ reads
\[
\langle \delta T^{(0)}_{MN}(x) \rangle_{\text{reg}} = -\beta' \frac{\Gamma(n/2) \Gamma^{2}(D/2) \Gamma^{3}(D/2) \Gamma \left( -\frac{D-2}{2} \right)}{2^D \pi^{3/2} (D-n)! (D+1)!} \frac{1}{\eta_{MN}} \times [ -A_D q^\alpha \eta_{MN} + (n-1)(D^2 - 2D - 2) q^2 \eta_{MN} + 2q^2 \eta_{MN}],
\]
with $A_D \equiv D(D-n) + (n-2)(D-2)(D+1)$. Hereafter the «tilded» quantity with indices means that it equals the corresponding tensor with no tilde for conical-subspace index, and vanishes in the opposite case.

Integrating the remaining Fouriers, one arrives at
\[
\langle \delta T^{(0)}_{MN}(x) \rangle_{\text{reg}} = \beta' \frac{\Gamma(n/2) \Gamma^{2}(D/2) \Gamma^{3}(D/2) \Gamma \left( -\frac{D-2}{2} \right)}{2^D \pi^{3/2} (D-n)! (D+1)!} \frac{1}{\eta_{MN}} \times [ -A_D q^\alpha \eta_{MN} + (n-1)(D^2 - 2D - 2) \eta_{MN} - 2\eta_{MN}],
\]

**B. Computation of $\xi$-terms**

Starting with the effective Fourier transforms (5.7) and (5.8), for the $^{1}T_{MN}$-contributions we have explicitly:
\[
T_{MN}^{(0)}[\delta L^{(\ell)}] = \beta' \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} e^{iqx} \delta L^{(\ell)}(q) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{|p^2 - i\epsilon| |(p + q)^2 - i\epsilon|} \left( p_{MPN} + q_{MPN} - \frac{1}{2} \eta_{MN} q p \right)
\]
\[
T_{MN}^{(0)}[\delta L^{(0)}] = 2\beta' \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} e^{iqx} \frac{\delta L^{(0)}(q, ip)}{|p^2 - i\epsilon| |(p + q)^2 - i\epsilon|} (-q_{MPN} + \eta_{MN} q p).
\]

Substituting (5.8) into (5.12) and integrating over $p$ and $q^r$, we obtain:
\[
T_{MN}^{(0)}[\delta L^{(\ell)}] = -\beta'(n-1)(D-2) \Gamma(n/2) \Gamma^{2}(D/2) \Gamma \left( -\frac{D-2}{2} \right) \int d^3 q \frac{e^{iqx}}{|q|^2 + n - D} \left( \tilde{q}_M \tilde{q}_N - |q|^2 \eta_{MN} \right).
\]
Substituting (5.7) into (5.13) and integrating over \( p \) and \( q^\alpha \), one concludes
\[
T^{(\xi)}_{MN} \left[ \delta \mathcal{L}^{(0)} \right] = T^{(0)}_{MN} \left[ \delta \mathcal{L}^{(\xi)} \right].
\] (5.15)

Thus combining (5.14) with (5.15) and integrating, for the regularized \( ^1 T_{MN} \) we arrive at
\[
\langle ^1 T_{MN} \rangle_{reg} = \beta \frac{(n-1)(D-2)\Gamma(n/2)\Gamma^3(D/2) \Gamma \left( -\frac{D-2}{2} \right)}{\pi^{D/2} \Gamma(D) \Gamma \left( -\frac{D-n}{2} \right)} \left[ \frac{1}{D-n} \left( \tilde{\eta}_{MN} - D \frac{\tilde{x}_M \tilde{x}_N}{r^2} \right) + \eta_{MN} \right] \frac{1}{r^D}. \tag{5.16}
\]

**Computation of \( \xi^2 \)-term.** The term under interest here, is given by
\[
\langle T^{(\xi)}_{MN} \left[ \delta \mathcal{L}^{(\xi)} \right] \rangle = 2\beta \int \frac{d^Dq}{(2\pi)^D} \int \frac{Dp}{(2\pi)^D} \mathcal{L}^{(\xi)}(q,ip) \left[ \frac{1}{p^2 - i\varepsilon} \right] \left[ (qMq_N + qMp_N + \eta_{MN} qp) \right]. \tag{5.17}
\]
Integrating and substituting it with (5.8) into (5.6), we obtain:
\[
\langle T^{(\xi)}_{MN} \left[ \delta \mathcal{L}^{(\xi)} \right] \rangle_{reg} = -\beta \frac{(n-1)(D-1)\Gamma(n/2)\Gamma^2(D/2) \Gamma \left( -\frac{D-2}{2} \right)}{2\pi^{D-1} \Gamma(D)} \int d^nq \left[ \frac{1}{\left( q^2 + n - D \right)} \left( \eta_{MN} - \tilde{q} \tilde{M} \tilde{n} \right) \right]. \tag{5.18}
\]
Comparing it with (5.14) and taking into account (5.15), one concludes:
\[
\langle ^2 T_{MN} \rangle_{reg} = -\frac{1}{2\xi_D} \langle ^1 T_{MN} \rangle_{reg}, \tag{5.19}
\]
so their ratio does not depend on the conical subdimensional \( n \).

Integrating the last Fourier integral, we arrive at
\[
\langle ^2 T_{MN} \rangle_{reg} = -\beta \frac{2(n-1)(D-1)\Gamma(n/2)\Gamma^3(D/2) \Gamma \left( -\frac{D-2}{2} \right)}{\pi^{D/2} \Gamma(D) \Gamma \left( -\frac{D-n}{2} \right)} \left[ \frac{1}{D-n} \left( \tilde{\eta}_{MN} - D \frac{\tilde{x}_M \tilde{x}_N}{r^2} \right) + \eta_{MN} \right] \frac{1}{r^D},
\]
therefore the combined regularized contribution of the \( \xi \)-terms equals
\[
\langle T_{MN} - ^0 T_{MN} \rangle_{reg} = \beta \frac{(D-2)\Gamma(n/2)\Gamma^3(D/2) \Gamma \left( -\frac{D-2}{2} \right)}{(n-1)^{D-1} \pi^{D/2} \Gamma(D) \Gamma \left( -\frac{D-n}{2} \right) r^D} \left[ \frac{1}{D-n} \left( \tilde{\eta}_{MN} - D \frac{\tilde{x}_M \tilde{x}_N}{r^2} \right) + \eta_{MN} \right] \frac{1}{D + 1} \xi \left( 1 - \frac{\xi}{2\xi_D} \right). \tag{5.20}
\]

**C. Summary**

Combining (5.11) and (5.20), we obtain for the regularized value of energy-momentum VEV:
\[
\langle T_{MN} \rangle_{reg} = \frac{C \mu^D \beta'}{r^D} \left[ \frac{(8(D-1)(n-1)}{D-n} (\xi - \xi_D)^2 + \frac{1}{D^2 - 1} \left( D \frac{\tilde{x}_M \tilde{x}_N}{r^2} - \tilde{\eta}_{MN} - (D-n)\eta_{MN} \right) + \eta_{MN} - \tilde{\eta}_{MN} \right] \frac{1}{D + 1} \tag{5.21}
\]
with
\[
C = \frac{\Gamma(n/2)\Gamma^3(D/2) \Gamma \left( -\frac{D-2}{2} \right)}{4\pi^{D/2} \Gamma(D) \Gamma \left( -\frac{D-n}{2} \right)}. \tag{5.22}
\]

We see that the classification on parity is based on the factor \( \Gamma \left( -\frac{D-2}{2} \right)/\Gamma \left( -\frac{D-n}{2} \right) \). Given that \( d - n \geq 1 \), the first pole of \( \Gamma \)-function in denominator happens at \( d = n + 2 \), we return exactly to the same dimensionality splitting as for \( \langle \phi^2 \rangle_{reg} \).

Hereafter it is more useful to consider the non-vanishing components of \( T_{MN} \) separately:

1. The regularized vacuum energy density \( \langle T_{00}(x) \rangle_{reg} \) (as well as flat-sector spatial diagonal components \( \langle T_{\alpha \beta}(x) \rangle \)):
\[
\langle T_{00}(x) \rangle_{reg} = \frac{\mu^D}{r^D} \left[ \frac{C(n-1)\beta'}{r^D} \left[ 8(D-1)(\xi - \xi_D)^2 + \frac{1}{D^2 - 1} \right] = -\langle T_{\alpha \beta}(x) \rangle_{reg}. \tag{5.22}
\]
2. The conical-subspace components \( \langle T_{ik}(x) \rangle_{\text{reg}} \)

\[
(T_{ik}(x))_{\text{reg}} = \frac{C_{\mu}^2}{r^{D+2}} \left[ \frac{8(D-1)(n-1)}{D-n} (\xi - \xi_D)^2 + \frac{1}{D^2-1} \right] \left( D x_i x_k - (D-n+1) r^2 \delta_{ik} \right). \tag{5.23}
\]

With respect to the parity of \( D \) and \( n \) one distinguishes the following cases:

- **\( d \) even, \( n \) odd.** The regularization removal \( 5.22 \) is achieved in analogy with \( \langle \delta^2 \rangle_{\text{reg}} \): the pole of the Gamma-function \( \Gamma \left( -\frac{D-2}{2} \right) \) in the numerator gives rise to the corresponding divergent part (as \( \epsilon \to 0 \))

\[
\langle T_{MN}(x) \rangle_{\text{div}} = \frac{(-1)^{d/2+1} \Gamma(n/2) \Gamma^2(d/2)}{4 \pi^{d/2} \Gamma(d) \Gamma \left( -\frac{d-2}{2} \right)} \frac{\beta'}{r^d} \Theta_{MN} \tag{5.24}
\]

\[
\Theta_{MN} = \left[ \frac{8(D-1)(n-1)}{D-n} (\xi - \xi_D)^2 + \frac{1}{d^2-1} \right] \left( \frac{\tilde{x}_M \tilde{x}_N}{r^2} - \tilde{\eta}_{MN} - (d-n) \eta_{MN} \right) + \frac{1}{d+1} (\eta_{MN} - \tilde{\eta}_{MN}),
\]

and to finite logarithmic and non-logarithmic terms.

In order to reveal the finite part, we have to point out the following observation: as we seen in the \( \text{Section 4} \) the divergent part corresponding to the pole of a Gamma-function, is accompanied with the logarithmic term in the finite part, and there is some arbitrariness in the non-logarithmic term, related with the finite renormalization of logarithmic scale factor. Here we renormalize the tensor quantity, but the Gamma-function \( \Gamma \left( -\frac{D-2}{2} \right) \), which gives a pole, sits in the common factor \( C \) in \( 5.21 \), while the tensor part is regular. Also taking into account that the finite logarithmic shift due to expansion of \( C \) is also common for the whole tensor, we expand \( C \) in \( \epsilon \) independent of the tensor structure, thus we have the **unified** logarithmic scale factor \( \tilde{\mu} \) for all components of \( T_{MN} \), while the tensor part in \( 5.21 \) has to be expanded additionally.

Thus for the renormalized tensor we write generically

\[
\langle T_{MN}(x) \rangle_{\text{ren}} = \frac{(-1)^{d/2-1} \Gamma(n/2) \Gamma^2(d/2)}{4 \pi^{d/2} \Gamma(d) \Gamma \left( -\frac{d-2}{2} \right)} \frac{\beta'}{r^d} \left[ 2 \Theta_{MN} \ln \tilde{\mu} r + A_{MN} \right]. \tag{5.25}
\]

It also allows the logarithmic-scale finite shift, but within the scalar transformation. In other words, for the scale change \( \mu \to \mu' \) there is an uniparametrical arbitrariness in \( A_{MN} \) in the generic form

\[
A'_{MN} = A_{MN} + 2 \Theta_{MN} \ln \frac{\mu}{\mu'}. \tag{5.26}
\]

Expanding

\[
\frac{8(n-1)(\xi - \xi_D)^2}{(D-1)^{-1}(D-n)} + \frac{1}{D^2-1} = \left[ \frac{8(n-1)(\xi - \xi_D)^2}{(D-1)^{-1}(D-n)} + \frac{1}{d^2-1} \right] + \left( \frac{4(n-1)(\xi - \xi_D)^2}{d-n} \right)^2 - \frac{1}{(d+1)^2} \epsilon + O(\epsilon^2)
\]

and fixing logarithmic scale as before (as \( \tilde{\mu} \), implying the absorption of all \( D \)-dependent coefficients in \( C \)) one obtains

\[
A_{MN} = \left( 4\xi - 1 \right)^2 - \frac{1}{(d+1)^2} \left( n - 1 \right) \eta_{MN} - \left( \frac{4(n-1)(\xi - \xi_D)^2}{d-n} \right)^2 + \frac{1}{(d+1)^2} \tilde{\eta}_{MN} + \\
+ \left[ \left( \frac{4(n-1)(\xi - \xi_D)^2}{d-n} \right)^2 - (n-1)(4\xi - 1)^2 + \frac{1}{(d+1)^2} \right] \frac{\tilde{x}_M \tilde{x}_N}{r^2}. \tag{5.27}
\]

For the renormalized vacuum energy density we obtain:

\[
\langle T_{00}(x) \rangle_{\text{ren}} = \frac{\beta' \left( -\frac{1}{d-1} \right)^{d/2} \Gamma(n/2) \Gamma^2(d/2)}{4 \pi^{d/2} \Gamma(d) \Gamma \left( -\frac{d-2}{2} \right)} \left[ \frac{2}{d^2-1} - 16(D-1)(\xi - \xi_D)^2 \right] \ln \tilde{\mu} r + \left( 4\xi - 1 \right)^2 - \frac{1}{(d+1)^2} \right] \frac{1}{r^d}.
\]

Not hard to conclude that for the values of a curvature-coupling

\[
\xi = \xi_D \pm \frac{1}{d-1} \sqrt{\frac{1}{8(d+1)}}
\]
the renormalized density $\langle T_{00}(x)\rangle_{\text{ren}}$ does not contain the logarithmic term and thereby does not depend upon the arbitrary constant $\bar{\mu}$, while the divergent part vanishes: $\langle T_{00}(x)\rangle_{\text{div}} = 0$.

The renormalized $\langle T_{ik}(x)\rangle$ reads:

$$
\langle T_{ik}(x)\rangle_{\text{ren}} = \beta' \frac{(-1)^{d/2-1}(n/2)\Gamma^2(d/2)}{2\pi^{d/2} \Gamma(d) \Gamma(-d+n/2)} \left[ \left( \frac{8(d-1)(n-1)}{d-n} \right) (\xi - \xi_d)^2 + \frac{1}{d^2} \right] \times \left( d x_i x_k - (d-n+1)r^2 \delta_{ik} \right) \ln \bar{\mu} r + \frac{1}{2} A_{ik} \right] \frac{1}{r^{d+2}}.
$$

(5.28)

In the case (4.3) the expression (5.25) reduces to

$$
\langle T_{MN}\rangle_{\text{ren}} = \frac{\beta'}{8\pi^2r^4} \left[ \left( \frac{8}{3} \xi - \frac{1}{6} \right)^2 + \frac{1}{90} \right] \left( \frac{4}{d^2} - \frac{8(d-1)(\xi - \xi_d)^2}{d-1} \right) \ln \bar{\mu} r + \frac{4\xi}{d-1} + \frac{d^3-1}{(d^2-1)^2} r^d.
$$

(5.29)

Furthermore, due to the (theoretical) arbitrariness of the constant $\bar{\mu}$, the non-logarithmic $\xi^2$–terms may be absorbed by the logarithm, introducing the new constant $\bar{\mu}'$:

$$
\langle T_{00}(x)\rangle_{\text{ren}} = \beta' \frac{(-1)^{d/2}(d/2)}{2\pi^{d/2} \Gamma(d) \Gamma(-d+1)} \left[ \left( \frac{1}{d^2} - \frac{8(d-1)(\xi - \xi_d)^2}{d-1} \right) \ln \bar{\mu}' r - \frac{4\xi}{d-1} + \frac{d^3-1}{(d^2-1)^2} r^d. \right.
$$

\text{In accord with (5.26), this finite shift } \bar{\mu} \to \bar{\mu}' = e^{-1/(d+1)}\bar{\mu} \text{ generates the corresponding shift } A_{ik} \to A'_{ik} \text{ of the spatial (in the conical sector) components.}

For higher-dimensional monopole ($n = d - 1$) equation (5.25) reduces to

$$
\langle T_{00}(x)\rangle_{\text{ren}} = \frac{\beta'}{4\pi^2r^4} \left[ \left( \frac{4}{d} - \frac{21}{100} \right) \ln \bar{\mu}' r + \frac{2}{9} \left( \xi - \frac{21}{100} \right) \right] \frac{1}{r^d},
$$

$$
\langle T_{ik}(x)\rangle_{\text{ren}} = \frac{\beta'}{4\pi^2r^4} \left[ \left( \frac{8}{3} \xi - \frac{1}{6} \right)^2 + \frac{1}{90} \right] \left( 2x_i x_k - r^2 \delta_{ik} \right) \ln \bar{\mu}' r + \frac{1}{24} A'_{ik} \right] \frac{1}{r^d}.
$$

(5.30)

Now we can compare our result (5.30) with the linear-in-$\beta'$ part of the corresponding expression in (30), applied to the spacetime-at-hand.

The logarithmic expression in (30) within our accuracy\footnote{It implies that we have neglected $O(R^2)$–terms.} generically is given by

$$
\langle T_{MN}(x)\rangle_{\text{log}} = \frac{1}{160\pi^2} \left[ \left( \frac{1}{3} - \frac{10}{3} \xi + 10\xi^2 \right) R_{MN} - \frac{1}{6} \square R_{MN} + \left( \frac{1}{4} + \frac{10}{3} \xi - 10\xi^2 \right) g_{MN} \square R \right] \ln \bar{\mu} r,
$$

(5.31)

while the non-logarithmic one is arbitrary. Substituting the Ricci tensor and Ricci-scalar (2.6), and making use of

$$
R_{;MN} = \frac{4(1 - \beta^2)}{r^4} \left( \frac{\bar{\mu}^2}{r^2} - \tilde{\eta}_{MN} \right), \quad \square R_{MN} = 4(1 - \beta^2) \frac{\bar{\mu}^2}{r^6},
$$

one concludes that our expression (5.29) has a discrepancy with (5.31) by factor of two, for all monomials $\eta_{MN}, \tilde{\eta}_{MN}$ and $\tilde{\xi}_M \tilde{\xi}_N$, respectively. Meanwhile, the corresponding expression for $\langle \phi^2 \rangle$ perfectly matches. Such a discrepancy implies necessity of re-derivation of the generic expression in the work (32) (actually referred by (30)). Following
their ideology, based on the de-Witt-Schwinger kernel, we could fix some inaccuracy of these works\textsuperscript{8}. Thus we think that if take into account the fixing coefficient, our result \cite{24,32} coincides with the generic one in the logarithmic term, whereas it contains information about the non-logarithmic term.

- **d and n odd.** Now $\Gamma \left(-\frac{d-2}{2}\right)$ in the numerator is regular, while $\Gamma \left(-\frac{d-n}{2}\right)$ in the denominator is infinite, hence the total renormalized $\langle T_{MN}(x) \rangle$ vanishes:

$$\langle T_{MN}(x) \rangle_{\text{ren}} = 0,$$

in accord with the corresponding value of $\langle \phi^2 \rangle$ \textsuperscript{9}.

- **d odd, n even.** Here both $\Gamma \left(-\frac{d-2}{2}\right)$ in the numerator and $\Gamma \left(-\frac{d-n}{2}\right)$ in the denominator are regular, with semi-integer arguments, so $\langle T_{MN}(x) \rangle_{\text{div}} = 0$ and we have simply

$$\langle T_{MN}(x) \rangle_{\text{ren}} = \frac{\Gamma(n/2)\Gamma^{2}(d/2) \Gamma \left(-\frac{d-n}{2}\right) \beta'}{4\pi^{d/2}\Gamma(d) \Gamma \left(-\frac{d-n}{2}\right)} \Theta_{MN}.$$

Transforming it with help of \textsuperscript{4,10}, one obtains

$$\langle T_{MN}(x) \rangle_{\text{ren}} = (-1)^{n/2-1} \frac{\Gamma(n/2)\Gamma^{2}(d/2) \Gamma \left(-\frac{d-n}{2}\right) \beta'}{4\pi^{d/2}\Gamma(d)} \Theta_{MN}.$$

In particular, for the $d-$dimensional monopole ($n = d - 1$) the renormalized energy-momentum tensor reads:

$$\langle T_{MN}(x) \rangle_{\text{ren}} = \frac{\pi^{1-d/2}\Gamma(d/2) \beta'}{\sqrt{4\pi^{2}}} \left[8(d-2)(\xi - \xi_0)^2 + \frac{d + 1}{(d^2 - 1)^2} \left(d \frac{\tilde{x}_M \tilde{x}_N}{r^2} - \eta_{MN} - \eta_{MN} \right) + \frac{\eta_{MN} - \eta_{MN}}{d^2 - 1}\right].$$

- **d and n even.** Here both $\Gamma \left(-\frac{d-2}{2}\right)$ in the numerator and $\Gamma \left(-\frac{d-n}{2}\right)$ in the denominator are singular, so their ratio is determined by the ratio of corresponding residuals \textsuperscript{4,8}.

Thus $\langle T_{MN}(x) \rangle_{\text{div}} = 0$, and

$$\langle T_{MN}(x) \rangle_{\text{ren}} = (-1)^{n/2-1} \frac{\Gamma(n/2)\Gamma^{2}(d/2) \Gamma \left(-\frac{d-n}{2}\right) \beta'}{4\pi^{d/2}\Gamma(d)} \Theta_{MN}.$$

Again, the formulae \textsuperscript{5,33} and \textsuperscript{5,34} are identical, and represent the unified expression for even $n$, like it was for $\langle \phi^2 \rangle$.

Summarizing, in this section we have computed the renormalized vacuum averaged $\langle T_{MN} \rangle_{\text{ren}}$ of the massless scalar field in the background of (global) monopole up to the first order in $\beta'$. Computing along the same ideology as in previous section, we obtain the same splitting with respect to the parity of a dimensionalities of the total spacetime and its deficit-angle submanifold. Here the most actual case with even $d$ and odd $n$ (in particular, for the $(4,3)$-type of a spacetime) demands the more accuracy working with logarithms, due to the tensorial structure of $\langle T_{MN} \rangle_{\text{reg}}$. The logarithmic mass-scale change generates the uniparametric equivalence class of the non-logarithmic symmetric tensors $A_{MN}^{10}$, representing the linear shell of monomials $\eta_{MN}$, $\tilde{\eta}_{MN}$ and $\tilde{x}_M \tilde{x}_N$. For definite value of $\xi$, the logarithmic term and corresponding logarithmic uncertainty can be removed from $\langle T_{00} \rangle_{\text{ren}}$. However, contrary to the case of $T_{00}$, no value of coupling $\xi$ kills the logarithmic term arising in $\langle T_{ik} \rangle_{\text{ren}}$ since both terms in the parenthesis of \textsuperscript{5,23} are positive. Finally, no value of $\xi$ eliminates the logarithmic arbitrariness both in $\langle \phi^2 \rangle_{\text{ren}}$ and in $\langle T_{MN} \rangle_{\text{ren}}$ simultaneously.

The other cases of $d$ and $n$ are similar to those ones of $\langle \phi^2 \rangle_{\text{ren}}$.

In the next section we show that the Green’s function obtained above, enables to consider the well-known purely classical problem of a gravity-induced self-action on a charge placed at fixed point of the space under consideration.

\textsuperscript{8} Actually, the pre-logarithmic expression \textsuperscript{5,31}, multiplied by 2, coincides with the pre-logarithmic coefficient in logarithmically-divergent part of the corresponding expression by Christensen \textsuperscript{24} for renormalized VEV for massive scalar field’s energy-momentum tensor.

\textsuperscript{9} See the footnote \textsuperscript{4} on page \textbf{17}.

\textsuperscript{10} Contrary to the result of \textsuperscript{31} where this matrix is symmetric but arbitrary.
6. STATIC SELF-ENERGY AND SELF-FORCE OF A POINTLIKE CHARGE

As it was concluded in (2.18) and (2.23), the self-energy of a scalar (q) or electric (e) point charge in an ultrastatic $d-$dimensional spacetime is determined by the coincidence-limit of the Euclidean Green’s function on the spacetime with the dimensionality $(d-1)$:

$$U_{sc}(x) = \frac{q^2}{2} G^{E}_{\text{reg}}(x,x|d-1,n), \quad U_{el}(x) = \frac{e^2}{2} G^{E}_{\text{ren}}(x,x|d-1,n) \bigg|_{\xi=0}. \quad (6.1)$$

The relation between self-energy and the self-force is given by (2.16). Taking into account that for the self-energy the first non-vanishing order is $O(\beta')$, one obtains to the lowest order simply

$$\mathbf{F}_{\text{ren}} = -\text{grad} U_{\text{ren}}.$$  

Moreover, simple relation between scalar and electrostatic self-energy (2.23) enables to restrict the consideration by the scalar one.

According to (4.6), the regularized scalar gravity-induced self-energy is given by

$$U_{\text{reg}} = q^2 \mu' \frac{n-1}{8 \pi^{(D-1)/2}} \Gamma(n/2) \Gamma^{3(\frac{D-1}{2})} \left( \frac{\xi}{\Gamma(D-1)} \right) \left( \frac{\xi}{\Gamma(\frac{3-D+n}{2})} \right) \frac{1}{r^{D-3}}. \quad (6.2)$$

Now the classification is determined basically by the factor

$$\Gamma\left( -\frac{D-3}{2} \right) / \Gamma\left( \frac{3-D+n}{2} \right). \quad (6.3)$$

With respect to the parity of $d$ and $n$ one distinguishes the following cases:

- $d$ even, $n$ odd. The Gamma-function $\Gamma(-\frac{D-3}{2})$ is regular, while $\Gamma(\frac{3-D+n}{2})$ tends to its pole (unless $D-n = 1$). Therefore the renormalized self-energy and the self-force vanish generically in this case:

$$U_{\text{ren}} = 0, \quad \mathbf{F}_{\text{ren}} = 0.$$  

For the exceptional case $d - n = 1$ both Gamma-functions are regular, hence

$$U_{\text{ren}} = -q^2 \beta' (-1)^{d/2} \frac{d-2}{8 \pi^{(d-3)/2}} \Gamma^{3(\frac{d-1}{2})} \left( \frac{\xi}{\Gamma(d-1)} \right) \left( \frac{\xi}{\Gamma(\frac{3-D+n}{2})} \right) \frac{1}{r^{D-3}}. \quad (6.4)$$

The corresponding self-force is given by

$$\mathbf{F}_{\text{ren}} = -q^2 \beta' (-1)^{d/2} \frac{(d-2)(d-3)}{8 \pi^{(d-3)/2}} \Gamma^{3(\frac{d-1}{2})} \left( \frac{\xi}{\Gamma(d-1)} \right) \left( \frac{\xi}{\Gamma(\frac{3-D+n}{2})} \right) \frac{r}{r^{D-1}}. \quad (6.5)$$

Thus, at $\xi = \xi_{d-1} = (d-3)/(d-2)$ the renormalized self-energy and self-force vanish.

In particular case of the (4,3)-spacetime one obtains

$$U_{\text{ren}} = -q^2 \beta' \frac{\pi (8 \xi - 1)}{2^6}, \quad \mathbf{F}_{\text{ren}} = -q^2 \beta' \frac{\pi (8 \xi - 1)}{2^6} \frac{r}{r^3}. \quad (6.6)$$

Thus, the pointlike charge feels the monopole as a point charge with the magnitude $2^{-4} \beta' (8 \xi - 1) \pi^2 q$ localized at the point $r = 0$. For values $\xi > \xi_3 = 1/8$ the self-force is attractive (in particular, for the conformal coupling, $\xi = \xi_4 = 1/6$), while for values $\xi < 1/8$ the self-force is repulsive.

In the case of electrostatic self-action (according to the eq. (2.23) one has to put $\xi = 0$ in (6.6) and replace $q^2$ by $e^2$) our result (6.6) coincides with the one of the paper [35].

- $d$ and $n$ odd. In this case the Gamma-function $\Gamma(-\frac{D-3}{2})$ is singular, while $\Gamma(\frac{3-D+n}{2})$ is regular. This leads to the non-zero diverging part, and the finite renormalized value of the self-energy takes the form

$$U_{\text{ren}} = (-1)^{(n+3)/2} q^2 \beta' \frac{n-1}{8 \pi^{(d+1)/2}} \Gamma^{3(\frac{d-1}{2})} \left( \frac{\xi}{\Gamma(d-1)} \right) \left( \frac{\xi}{\Gamma(\frac{3-D+n}{2})} \right) \ln \mu r + \frac{1}{(d-2)(d-3)} \frac{1}{r^{D-3}}. \quad (6.7)$$
with arbitrary $\tilde{\mu}$.

The corresponding self-force reads

$$F_{\text{ren}} = (-1)^{(n+3)/2} q^2 \beta' \frac{n-1}{8 \pi^{(d-1)/2}} \frac{\Gamma(n/2) \Gamma^2 \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d-n-1}{2} \right)}{\Gamma(d-1)} \left[ \frac{\xi}{\xi_{d-1}} - 1 \right] \left((d-3) \ln \tilde{\mu} r - 1 + \frac{1}{(d-2)}\right) \frac{r}{r^{d-3}}. \quad (6.8)$$

For $\xi = \xi_{d-1}$ the result becomes free of uncertainty.

- **$d$ odd, $n$ even.** Here the Gamma-function $\Gamma\left(\frac{d-D+n}{2}\right)$ is singular, while $\Gamma\left(\frac{3-D+n}{2}\right)$ is also singular, unless $d = n + 1$. Hence, in the generic case the divergent part of the self-energy vanishes, and $U_{\text{ren}}$ is determined by the ratio of corresponding residuals:

$$U_{\text{ren}} = (-1)^n q^2 \beta' \frac{(n-1)(d-3)}{8 \pi^{(d-1)/2}} \frac{\Gamma(n/2) \Gamma^2 \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d-n-1}{2} \right)}{\Gamma(d-1)} \left( \frac{\xi}{\xi_{d-1}} - 1 \right) \frac{1}{r^{d-3}}. \quad (6.9)$$

Corresponding self-force equals

$$F_{\text{ren}} = (-1)^n q^2 \beta' \frac{(n-1)(d-3)}{8 \pi^{(d-1)/2}} \frac{\Gamma(n/2) \Gamma^2 \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d-n-1}{2} \right)}{\Gamma(d-1)} \left( \frac{\xi}{\xi_{d-1}} - 1 \right) \frac{r}{r^{d-3}}. \quad (6.10)$$

In the exceptional case of the higher-dimensional monopole ($d = n + 1$) the denominator $\Gamma\left(\frac{3-D+n}{2}\right)$ is regular, hence we return to the logarithmic case: along the same lines as previously we obtain

$$U_{\text{ren}} = q^2 \beta' \frac{(n-1)(d-3)}{8 \pi^{(d-1)/2}} \frac{\Gamma(n/2) \Gamma^2 \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d-n-1}{2} \right)}{\Gamma(d-1)} \left((d-2) \left( \frac{\xi}{\xi_{d-1}} - 1 \right) \ln \tilde{\mu} r + \frac{1}{d-3}\right) \frac{1}{r^{d-3}}, \quad (6.11)$$

$$F_{\text{ren}} = q^2 \beta' \frac{(n-1)(d-3)}{8 \pi^{(d-1)/2}} \frac{\Gamma(n/2) \Gamma^2 \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d-n-1}{2} \right)}{\Gamma(d-1)} \left((d-2) \left( \frac{\xi}{\xi_{d-1}} - 1 \right) \left((d-3) \ln \tilde{\mu} r - 1\right) + 1\right) \frac{r}{r^{d-3}}. \quad (6.12)$$

- **$d$ and $n$ even.** Here both Gamma-functions in (6.3) are regular, hence the divergent part vanishes and after transformations with the help of (6.10) we have just

$$U_{\text{ren}} = (-1)^n q^2 \beta' \frac{n-1}{8 \pi^{(d-1)/2}} \frac{\Gamma(n/2) \Gamma^2 \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d-n-1}{2} \right)}{\Gamma(d-1)} \left( \frac{\xi}{\xi_{d-1}} - 1 \right) \frac{1}{r^{d-3}},$$

$$F_{\text{ren}} = (-1)^n q^2 \beta' \frac{(n-1)(d-3)}{8 \pi^{(d-1)/2}} \frac{\Gamma(n/2) \Gamma^2 \left( \frac{d-1}{2} \right) \Gamma \left( \frac{d-n-1}{2} \right)}{\Gamma(d-1)} \left( \frac{\xi}{\xi_{d-1}} - 1 \right) \frac{r}{r^{d-3}}. \quad (6.13)$$

To summarize: based on the formal relation of the Feynman propagator with Euclidean Green’s function in the coincidence-point limit, we have expressed the regularized self-action via regularized Green’s function of the previous dimensionality. As before, the consideration splits onto four characteristic cases of parities $d$ and $n$, though here one meets the significant exceptions of the monopole background with no flat spatial dimensions ($n = d-1$). In the most cases the self-action looks like the flat-space Coulomb interaction of a charge $q$ with a charge $\sim (\xi - \xi_{d-1}) q$ placed into the monopole position, and vanishes for the particular value $\xi = \xi_{d-1}$ of the curvature coupling. In the case of odd $d$ while $n$ is odd or equal to $d - 1$, there is an additional logarithmic multiplier, which depends on the arbitrary parameter $\tilde{\mu}$.

Finally, comparing (6.13) with (6.3) and (6.10), we notice that for even $n$ the cases with even and odd $d$ can be combined into the unified formula (except for the full-hyperspace monopole case), in accord with the previous computations of the renormalized $\langle \phi^2 \rangle$ and $\langle T_{MN} \rangle$.

**7. VACUUM POLARIZATION NEAR COSMIC STRING REVISITED**

Now consider the particular case of a two-dimensional ($n = 2$) conical subspace. If $d = 3(4)$ this space is the spacetime of a point mass (infinitely thin straight cosmic string).
This problem was considered in a series of papers. The primary goal of our consideration is to show that there is some ambiguity in previous calculations in the case of a non-minimally coupled massless scalar field.

Indeed, in calculations [12, 36, 40] the starting point is the expression (5.1) with operator $D_{MN}$, whose form is determined by the classical expression for the energy-momentum tensor and thus includes the $\xi$—dependent terms. Whereas as a Green’s function the authors used the Green’s function for a minimally coupled scalar field. Thus, it was supposed that one can extract a $\delta^2$—like potential from the wave equation, arguing it by the fact that the space is flat everywhere outside the point mass/cosmic string. This Green’s function does not depend on $\xi$ and in the limit $\beta \to 1$ tends to the flat Green’s function $G^F_0(x-x')$, which is the solution of the equation

$$\eta^{MN} \partial_M \partial_N G^F_0(x-x') = -\delta^d(x-x').$$

(7.1)

On the other hand, we can start from the explicit equation

$$\sqrt{-g} \left[ \Box - \xi R \right] G^F_\xi(x,x') = -\delta^d(x-x').$$

(7.2)

In the coordinates of usage here, the potential reads

$$\gamma(r) = \sqrt{-g} \xi R = 4\pi \beta' \xi \delta^2(r), \quad r = (x^1, x^2).$$

(7.3)

In the Eq. (7.2) there are two independent parameters, namely $\beta'$ and $\xi$. Suppose, that there exists a limit of the Green’s function $G^F_\xi$, when

$$\beta' \to 0, \quad \xi \to \infty, \quad \lambda \equiv 4\pi \xi \beta' = \text{const}.$$  

Let us denote it as $G^F_\lambda$. In this limit Eq. (7.2) takes the form

$$\left[ \eta^{MN} \partial_M \partial_N - \lambda \delta^2(x) \right] G^F_\lambda(x,x') = -\delta^d(x-x').$$

(7.4)

It is obvious that, if the limit does exist, $G^F_\lambda$ can not be equal to the flat-space Green’s function $G^F_0$.

The corresponding equation for the scalar field $\phi$ can be reduced to a stationary two-dimensional Schrödinger-like equation with a planar $\delta^2$—function potential. Equations of this kind have been widely discussed in the literature. It was shown that these interactions require regularization and infinite renormalization of the coupling constant and lead to non-trivial physical results. Alternatively, one can follow more satisfactory approach based on a self-adjoint extension of a noninteracting Hamiltonian, defined on a space with one extracted point (see [41, 42] and Refs therein).

We think, that the example above demonstrates the necessity to revise the vacuum polarization effects on manifolds with $\delta^2$—like singularities. This problem demands consideration in more detail. Here we restrict ourselves by the consideration of this problem in the framework of the perturbation approach.

Thus, we start from the expression (5.10) with the potential $\gamma$ defined by the Eq. (7.3). The Fourier transform of this potential has the form

$$\mathcal{F}[\gamma(r)] = 4\pi \beta'(2\pi)^{d-2} \delta(q^0) \prod_{N=3}^{d-1} \delta(q^N).$$

(7.5)

Substituting (7.3) into Eq. (5.10), we obtain that with our accuracy

$$G^F(x,x'|d,2) = G^F_0(x-x') + \frac{\beta'}{\pi} \int d^2q \frac{e^{iqx}}{q^2} \int \frac{d^dp}{(2\pi)^d} e^{ip(x-x')} \frac{p^2 - \xi q^2}{|p^2 - \xi| [(p+q)^2 - \xi]}.$$  

(7.6)

Starting from (7.3) and proceeding along the same line as in the previous sections, we obtain:

$$\langle \phi^2(x) \rangle_{\text{ren}} = -i G^F_{\text{ren}}(x,x|d,2) = -\frac{\beta'}{2\pi^{d/2}} \left( \frac{\xi}{\xi_d} - 1 \right) \frac{\Gamma^3(d/2)}{(d-2)\Gamma(d)} \frac{1}{r^{d-2}}.$$  

(7.7)
for the renormalized vacuum expectation value of the field square and

\[
\langle T_{00}\rangle_{\text{ren}} = \beta' \frac{\Gamma^3(d/2)}{4 \pi^{d/2} \Gamma(d)} \left( \frac{8(d-1)(\xi - \xi_d)^2}{d^2 - 1} - \frac{1}{d^2 - 1} \right) \frac{1}{r^d},
\]

\[
\langle T_{11}\rangle_{\text{ren}} = \beta' \frac{\Gamma^3(d/2)}{4 \pi^{d/2} \Gamma(d)} \left[ \frac{2(\xi - \xi_d)^2}{\xi_d} + \frac{1}{d^2 - 1} \right] \left( x_1^2 - (d-1)x_2^2 \right) \frac{1}{r^{d+2}},
\]

\[
\langle T_{22}\rangle_{\text{ren}} = \langle T_{11}\rangle_{\text{ren}}|_{x_1 = x_2},
\]

\[
\langle T_{12}\rangle_{\text{ren}} = \langle T_{21}\rangle_{\text{ren}} = \beta' \frac{d \Gamma^3(d/2)}{4 \pi^{d/2} \Gamma(d)} \left[ \frac{2(\xi - \xi_d)^2}{\xi_d} + \frac{1}{d^2 - 1} \right] x_1 x_2 \frac{1}{r^{d+2}},
\]

\[
\langle T_{\alpha\beta}\rangle_{\text{ren}} = -\delta_{\alpha\beta} \langle T_{00}\rangle_{\text{ren}}, \quad \alpha, \beta, ... = n + 1, ..., d - 1
\]

for the nonzero components of \( \langle T_{MN}\rangle_{\text{ren}} \).

The corresponding classical gravity-induced scalar self-energy and self-force are given by

\[
U_{\text{ren}} = -q^2 \beta' \frac{\Gamma^2(d-1)}{8 \pi^{(d-1)/2} \Gamma(d-1)} \left( \frac{\xi}{\xi_{d-1}} - 1 \right) \frac{1}{r^{d-3}},
\]

\[
F_{\text{ren}} = -\frac{q^2 \beta'}{4 \pi^{(d-1)/2} \Gamma(d-1)} \frac{\Gamma^3(d-1)}{\Gamma(d)} \left( \frac{\xi}{\xi_{d-1}} - 1 \right) \frac{r}{r^{d-1}}.
\]

Thus in any dimension the self-force is attractive for \( \xi > \xi_{d-1} \), repulsive vice versa, and equal to zero if \( \xi = \xi_{d-1} \).

Our results coincide with the ones of the papers [8, 10–12, 36–40] in the case of a minimally coupled scalar field in the three-/four-dimensional spacetime, but differ from those if \( \xi \neq 0 \). As it was mentioned above, this distinction is a consequence of the fact that Green’s function satisfies Eq. (7.2). The latter contains an additional a two-dimensional \( \delta^2(\mathbf{x}) \)-potential which was not taken into account in the cited papers.

However, our result for \( \langle T_{MN}\rangle_{\text{ren}} \) coincides with [36, 40] also for the particular value \( \xi = \xi_4 = 1/6 \). This occasional coincidence follows from the fact that the sum \( \langle T_{MN}^{(0)}[\delta \mathcal{L}(\xi)] + T_{MN}^{(1)}[\delta \mathcal{L}(\xi)] \rangle_{\text{reg}} \), representing the discrepancy, is proportional to \( \xi (\xi - \xi_D) \). If the divergent part vanishes, that is the case for the cosmic string, then the latter equals \( \xi (\xi - \xi_d) \) and thus vanishes for the conformal coupling also.

Notice, our results (7.1), (7.2) and (1.3) coincide with the \( n \to 2 \) limit of the results obtained in the Sections 4 and 5. First of all, as we mentioned, the monopole’s results for even \( n \) are combined (for odd and even \( d \)). Furthermore, we see that the expressions (1.15), (1.18), (5.3), (5.4), (6.13), (6.10) and (6.13) are regular at \( n = 2 \), though the initial expression for Ricci-scalar was singular in this limit (3.6), representing the only difference.

Next, we observe that this difference disappears after the first Fourier-transform. Indeed, the Fourier-transform of the string’s Ricci-scalar perfectly coincides with the (regular) \( n \to 2 \) limit of the corresponding Fourier-transform of the monopole’s Ricci-scalar, readily computed with help of (3.12). Since the rest of computation is the same for the cosmic string and global monopole, no wonder that we have obtained coincidence in the final formulae.

8. CONCLUSION

On curved backgrounds being multidimensional generalizations of the well-known four-dimensional cosmic string/global monopole, we have considered two, disconnected at the first glance, problems. Namely, the gravity-induced vacuum polarization of a massless scalar field and the classical self-action of a static scalar or electric charge. However, the technique to solve the problems under consideration turned out to be similar, since it refers to getting the compact workable expression for the Green’s function and its derivatives in the coincidence-point limit for all \( d \geq 3 \) and \( 2 \leq n \leq d - 1 \), representing our primary particular goal. For this purpose we use the methods of perturbation theory. Taking into account the actual smallness of the angle deficit, we have performed computations in first order with respect to the angular deficit value. Since, in principle, both the vacuum expectation values and the classical self-energy are divergent, for regularization and renormalization of these quantities we adapted the dimensional-regularization method.

The zeroth computational order is determined by the Minkowskian Green’s function and completely consists of the tadpole-like contributions (1.2). In quantum field theory, the appearance of divergences produced by tadpoles,
is explained by the fact that the perturbation theory is constructed with respect to a nonphysical vacuum, while their elimination is explained by the necessity of redefining the vacuum state. In the framework of self-action, it is of interest to understand why similar divergences appear in the classical theory too. Following the prescriptions of the quantum field theory, we assumed all expressions of the form (4.4) to be equal to zero. The motivation for this recipe is not associated in any way with the quantum theory. Actually, it relies on the absence of dimensional parameters in the corresponding expression and, as a consequence, on the impossibility to assign some reasonable finite value, except zero, to such integrals under regularization. Therefore, this rule is also equally applicable within the classical field theory.

The desired effects are computed in the first in \( \beta' \) order. Already starting from the Green’s function, for all of our computational tasks we meet the characteristic ratio of two Gamma-functions, which splits the consideration of all \((d, n)\)-types onto four characteristic cases, depending on parities of \(d\) and \(n\). The poles of Gamma-function may arise in numerator, in denominator, or in both. However, in the very end of computation one can combine all formulae with even \(n\) (for arbitrary \(d\)) into the unified case.

With help of the regularized Green’s function we have computed the renormalized vacuum averaged \(\langle \phi^2 \rangle_{\text{ren}}\) and \(\langle T_{MN} \rangle_{\text{ren}}\) for a massless scalar field coupled with the generalized conical background (2.5) via an arbitrary coupling \(\xi\). The expressions for vacuum averaged \(\langle \phi^2 \rangle_{\text{ren}}\), corresponding to all characteristic cases (with our accuracy), vanish at \(\xi = \xi_d\). In the case with even \(d\) and odd \(n\) (in particular, for the (4,3)-type of a spacetime) the VEVs of \(\langle \phi^2 \rangle_{\text{ren}}\) and \(\langle T_{MN} \rangle_{\text{ren}}\) contain logarithmic factor. We are in agreement with [20, 30] in the pre-logarithmic coefficient. Concerning the non-logarithmic term in \(\langle T_{MN} \rangle_{\text{ren}}\), we restrict its arbitrariness by the single arbitrary parameter, fixing the more wide freedom in [30].

For the self-action, in addition to the four basic characteristic cases of parities \(d\) and \(n\), there is a significant exception of the monopole background \((n = d - 1)\). In the most cases the self-action represents the Coulomb-like field with «charge» \((\xi - \xi_{d-1})\) and vanishes for the particular value \(\xi = \xi_{d-1}\) of the curvature coupling. Also it should be mentioned that \((\xi = 0)\) the gravity-induced self-energy and the self-force of the point-like static electric charge \(e\) can be obtained from our expressions by the formal identification \(q^2 \rightarrow e^2\), since the spacetime-at-hand is ultrastatic, and the defining expressions for spatial scalar and vectorial Green’s functions coincide.

We’d like to emphasize that within our scheme, the appearance of the mass-dimensionful term inside the logarithm is related neither with the arbitrary scale factor \(r_0\) coming from the cartesian coordinates (2.3), nor with any length/mass of the problem-at-hand since the latter is absent\(^{11}\). The logarithmic scale factor follows from the regularization and its value, in principle, is arbitrary.

Making use of the same approach, but applied to the delta-like interaction in the infinitely thin straight cosmic string, we have computed the effects under consideration. The results coincide with the known in literature [36, 37, 38, 40] only for minimal and conformal coupling, while for other values of \(\xi\) they do not coincide already in the first (in \(\beta'\)) order. We refer this discrepancy to the missing of the \(\xi\)-correction inside the Green’s function. In computation of \(\langle T_{MN} \rangle_{\text{ren}}\) to the first order, this difference is reflected in terms \(T_{MN}^{(0)}[\delta \mathcal{L}^{(\xi)}]\) and \(T_{MN}^{(4)}[\delta \mathcal{L}^{(\xi)}]\). If to ignore these two in our scheme and retain the two remaining in (5.5), one would obtain the old answer.

We have shown that up to first order, in our Fourier-transform language the results for the cosmic string spacetime can be obtained as the smooth limit of corresponding results for global monopole. From this framework, it represents the problem of independent interest, whether this coincidence takes place only in the linear-in-\(\beta'\) order, or being the first non-vanishing part of the nonperturbative limit.

Finally, the usage of the Perturbation Theory restricts the applicability by the requirement on smallness of the angular deficit. However, this approach is relatively simple (to the order under consideration) and allows to take an advantage of well-developed in QFT methods. In result, it allowed to obtain the final expressions valid for arbitrary \(2 \leq n \leq (d - 1)\) and \(d \geq 3\), which, in its turn, verified the particular cases also, what helped to justify/fixed

\(^{11}\) For the real cosmic string one has its real width, but the results for the cosmic string within our model do not concern logarithms.
the corresponding known results.

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Appendix A: Basic integrals

Here we give the derivation of basic integrals in the dimensional regularization scheme we use. Such a scheme is somewhat common in QFT but unusual in the classical theory, so it may be instructive to briefly derive useful integrals. The integrals are well-defined for Euclidean propagators (with imaginary time), and analytically generalized for the case of Minkowski metric. Here we rewrite the Fourier-transforms (3.12) in \( d \) dimensions:

\[
\mathcal{F}[|r|^{-\lambda}] (k) = 2^{d-\lambda} \pi^{d/2} \frac{\Gamma\left(\frac{d-\lambda}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)} \frac{1}{|k|^{d-\lambda}},
\]

implying the Euclidean scalar products inside.

The scalar single-propagator integral is defined as

\[
J^{(1)} = \int d^d p \frac{1}{(2\pi)^d} \frac{1}{p^2 - i\varepsilon}.
\]

Hereafter the right superscript labels the number of propagators. Passing to the spherical coordinates, one obtains the integral transformation with kernel \( p^{d-3} \) acting on test function. It is shown that the latter equals zero in distributional sense, as well as

\[
J^{(1)}_{i_1 \ldots i_k} \equiv \int d^d p \frac{1}{(2\pi)^d} \frac{p_{i_1} \ldots p_{i_k}}{p^2 - i\varepsilon} = 0.
\]

As it is well-known, this value is advocated as the absence of the parameter, upon which \( J^{(1)} \) could depend explicitly, since the only variable \( p \) in the integrand is integration one.

The scalar two-propagator integral is defined as

\[
J^{(2)} = \int d^d p \frac{1}{(2\pi)^d} \frac{1}{|p + q|^2 - i\varepsilon}.
\]

After the Wick rotation \( p_0 = ip_E; q_0 = iq_E \) we have the analogous Euclidean integral. Thus consider the two following integrals with Euclidean scalar product:

\[
J(q) \equiv \int d^d p \frac{1}{(2\pi)^d} \frac{1}{p^2 + q^2 - i\varepsilon}; \quad I \equiv \int d^d p d^d q \frac{e^{ix(p+q)}}{(2\pi)^{2d}} \frac{1}{p^2 q^2},
\]

with \( J^{(2)} = iJ(q) \), indeed. Being split on the product of identical integrals, \( I \) equals

\[
I = (I_0)^2, \quad I_0 \equiv \int \frac{d^d p}{(2\pi)^d} \frac{e^{ipx}}{p^2}.
\]

Making use of Fourier-transform (A.1), \( I \) is given by

\[
I = \frac{1}{16\pi^d} \frac{\Gamma^2\left(\frac{d-2}{2}\right)}{R^{2(d-2)}}, \quad R \equiv \sqrt{x^2}.
\]

Now change variable \( q \rightarrow p + q \) in \( I \) (A.5):

\[
I = \int d^d p d^d q \frac{e^{ixq}}{(2\pi)^{2d} p^2 (p + q)^2} = \mathcal{F}^{-1}[J(q)](x).
\]
Thus substituting (A.7) into (A.6) and taking the direct Fourier-transform with help of (A.1), \( J^{(2)}(q) \) equals

\[
J(q) = \frac{\Gamma^2 \left( \frac{d-2}{2} \right)}{16\pi^d} \int r^{-2(d-2)} \left( q \right) = \frac{\Gamma^2 \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-4}{2} \right)}{4\pi^{(d-2)/2} \Gamma(d-2)} |q|^{d-4}.
\]

(A.8)

Restoring the Minkowskian \( q^0 \), after the \( \Gamma \)-function transformations the \( J^{(2)} \) is given by

\[
J^{(2)} = -\frac{i2(d-1)}{(4\pi)^{d/2}} \frac{\Gamma(d/2) \Gamma \left( -\frac{d-2}{2} \right)}{\Gamma(d)} (q^2)^{d/2-2}.
\]

(A.9)

In the form (A.9) the arguments of all Gamma-functions do not intersect zero at \( n \geq 3 \). This result is identical to the ones given in a series of QFT textbooks and derived via Feynman parametrization, but remarkably, we have used just the single basic Fourier-integral (A.1).

• Vectorial integral is defined as

\[
J^{(2)}_M = \int \frac{d^4p}{(2\pi)^d} \frac{p_M}{[p^2 - i\varepsilon][(p + q)^2 - i\varepsilon]}.
\]

(A.10)

Obviousiy, the result has to be proportional to \( q_M \) as to the only available input vector in the problem-at-hand: \( J^{(2)}_M(q) = A_1(q)q_M \). Contracting this equality with \( q^M \) and representing \( p \cdot q = (1/2) [(p + q)^2 - p^2 - q^2] \), one uses (A.3) and (A.9) to determine the scalar \( A_1 \). Thus the result turns out to be

\[
J^{(2)}_M = -\frac{1}{2} J^{(2)} q_M = \frac{d-1}{4\pi} \frac{\Gamma^2(d/2) \Gamma \left( -\frac{d-2}{2} \right)}{\Gamma(d)} (q^2)^{d/2-2} q_M.
\]

(A.11)

• Tensorial integral is defined as

\[
J^{(2)}_{MN} = \int \frac{d^4p}{(2\pi)^d} \frac{p_{MPN}}{[p^2 - i\varepsilon][(p + q)^2 - i\varepsilon]}.
\]

(A.12)

Being the symmetric 2-rank tensor, the latter should be expressible via the flat metric \( \eta_{MN} \) and \( q_M q_N \)-monomial: \( J^{(2)}_{MN} = A_2 q^2 \eta_{MN} + A_3 q_M q_N \). Taking the trace and substituting (A.3), one obtains the relation \( nA_2 + A_3 = 0 \). Projecting (A.12) on \( q^N \) and realizing the same strategy as before, one gets the second constraint:

\[
(A_2 + A_3) q_M = \frac{1}{2} J^{(2)}_M = \frac{1}{4} q_M J^{(2)}.
\]

Resolving these two, the value of integral (A.12) is given by

\[
J^{(2)}_{MN} = -\frac{J^{(2)} q^2 \eta_{MN} - d q_M q_N}{4(d-1)}
\]

(A.13)

with trace \( \eta_{MN} J^{(2)}_{MN} = 0 \).

• Appealing to the same computational arguments, the three- and four-index integrals

\[
J^{(2)}_{MNK} = \int \frac{d^4p}{(2\pi)^d} \frac{p_{MPNKP}}{[p^2 - i\varepsilon][(p + q)^2 - i\varepsilon]}, \quad J^{(2)}_{MNKL} = \int \frac{d^4p}{(2\pi)^d} \frac{p_{MPNPPL}}{[p^2 - i\varepsilon][(p + q)^2 - i\varepsilon]}
\]

(A.14)

with help of symmetry and some combinatorics, are given by

\[
J^{(2)}_{MNK} = -\frac{J^{(2)}}{8(d-1)} \left[ (d+2) q_M q_N q_K - q^2 \left( q_M \eta_{NK} + q_N \eta_{MK} + q_K \eta_{MN} \right) \right]
\]

(A.15)

and

\[
J^{(2)}_{MNKL} = \frac{J^{(2)}}{16(d^2-1)} \left[ - (d+2) q^2 \left( q_M q_N q_K q_L + q_M q_N q_L q_K + q_M q_L q_K q_N + q_N q_M q_K q_L + q_N q_L q_K q_M + q_K q_L q_M q_N \right) 
\]

\noindent \[ + (d+4)(d+2) q_M q_N q_K q_L + (q^2)^2 \left( \eta_{MN} \eta_{KL} + \eta_{MK} \eta_{NL} + \eta_{ML} \eta_{NK} \right) \right],
\]

(A.16)
respectively.

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