D3-branes on partial resolutions of abelian quotient singularities of Calabi-Yau threefolds

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ABSTRACT

We investigate field theories on the worldvolume of a D3-brane transverse to partial resolutions of a $\mathbb{Z}_3 \times \mathbb{Z}_3$ Calabi-Yau threefold quotient singularity. We deduce the field content and lagrangian of such theories and present a systematic method for mapping the moment map levels characterizing the partial resolutions of the singularity to the Fayet-Iliopoulos parameters of the D-brane worldvolume theory. As opposed to the simpler cases studied before, we find a complex web of partial resolutions and associated field-theoretic Fayet-Iliopoulos deformations. The analysis is performed by toric methods, leading to a structure which can be efficiently described in the language of convex geometry. For the worldvolume theory, the analysis of the moduli space has an elegant description in terms of quivers. As a by-product, we present a systematic way of extracting the birational geometry of the classical moduli spaces, thus generalizing previous work on resolution of singularities by D-branes.

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Introduction

In this paper we perform a systematic study of the worldvolume field theory of one D3-brane transverse to partial resolutions of a Calabi-Yau quotient singularity, by generalizing and improving on the approach pioneered in [1]. Particular examples of such systems have been studied before in [1, 2], where it was shown that they exhibit interesting phenomena such as topology change and projection of non-geometric phases. The main purpose of the present paper is to improve on the previous analysis of such systems, and to give a systematic and computationally efficient way to approach more complicated singularities. As a simple illustration of our methods, we re-analyze the case of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ Gorenstein singularities (a first analysis of which was presented in [2]), then we proceed to the analysis of the new and considerably more complicated case of a D3-brane transverse to a $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ singularity.

The motivation of the present work is to prepare the ground for a detailed analysis of the conformal field theory on a large number of D3-branes transverse to partial resolutions of a $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ Gorenstein singularity [4]. Such an analysis is motivated by investigations of the AdS/CFT conjecture [3] for spaces of the form $\text{AdS}_5 \times X_5$ with $X_5$ an Einstein-Sasaki five-manifold describing the angular part of the tangent cone to a partial resolution of such a singularity. As explained in [5, 6], the partial resolutions of interest for the AdS/CFT conjecture are those for which the tangent cone at the singular point can be written as a complex cone over a del Pezzo surface, so that $X_5$ is a circle bundle over the del Pezzo. Since one knows how to obtain the associated field theory only in the case when the partial resolution is toric, one is currently restricted to toric del Pezzo’s, which amounts to considering only the surfaces $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $dP_1, dP_2, dP_3$ (the blow-ups of $\mathbb{P}^2$ at one, two and three points respectively). Of these, only $dP_3$ and $F_0$ are known to admit a circle bundle $X_5$ over themselves which carries a regular Einstein-Sasaki structure, a result which follows from the work of Tian and Yau on the positive case of the Calabi conjecture [7, 8, 9]. Their investigations showed that all del Pezzo surfaces $dP_k$ with $k \geq 3$ admit a Kähler-Einstein metric of positive curvature. This allows one to prove the existence of regular Einstein-Sasaki five-manifolds $X_5$ which form the total space of a circle bundle over $sP_3$. The case of $F_0$ follows from more elementary results.

From the work of [1], it is also known that $dP_1$ and $dP_2$ do not admit such metrics, which raises the difficult mathematical question of whether a (possibly singular) associated ‘five-manifold’, carrying a (possibly non-regular) Einstein-Sasaki structure exists, and the physical question of what is the status of the AdS/CFT conjecture in such a situation. It is known (see, for example [10, 11]) that the complex cone over each toric del Pezzo surface can be realized as the tangent cone to some partial resolution of $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$. However, it is in principle possible that such partial resolutions are simply inaccessible from the point of view of the worldvolume field theory of a D3-brane transverse to our space, namely that they cannot be realized physically by turning on

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1We exclude the cases $X_5 = S^5$ and $X_5 = S^5/\mathbb{Z}_3$, which can be studied by direct methods.
Fayet-Iliopoulos parameters in our theory. As discussed in [1, 2], the partial resolutions of the singularity are realized as moduli spaces of the D-brane theory, in the presence of certain Fayet-Iliopoulos terms. Namely, the space of all possible Fayet-Iliopoulos parameters admits a partition into cones, and the complex structure of the classical moduli space of the worldvolume theory does not change as long as the Fayet-Iliopoulos parameters remain inside of a given cone. However, there is no apriori reason to expect that all partial resolutions can be obtained as moduli spaces in this manner. In other words, it is in principle possible that some partial resolutions are never realized in the field theory in the way outlined above, no matter how one chooses the Fayet-Iliopoulos terms. One purpose of the present paper is to test this possibility directly, by investigating the full list of field theoretic realizations of the partial resolutions of interest. As we will discover, these partial resolutions are in fact realized for some choices of Fayet-Iliopoulos terms. This shows that the solution of the puzzle is necessarily more subtle. The complexity of the analysis is quite pronounced, in marked contrast with cases considered before. Therefore, it turns out to be necessary to refine the techniques available for studying the problem, and to follow a systematic approach towards its resolution.

The structure of this paper is as follows. In Section 1, we present an overview of our results for the case of $\mathbb{Z}_3 \times \mathbb{Z}_3$, in nontechnical language. In Section 2, we give a systematic presentation of our method in a very general context. Since this approach leads to cumbersome notation, we illustrate the abstract methods by an application to the example of a D3-brane transverse to a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient singularity and its partial resolutions. In Section 3, we give a systematic presentation of the case $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$. Section 4 presents our conclusions. The appendix lists certain data relevant to the model $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$. Throughout the paper, we assume some familiarity with toric geometry as well as with some basic concepts of algebraic and symplectic geometry.

1 Overview of the case $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$

1.1 The geometric realization of the complex cones over $F_0, dP_1, dP_2$ and $dP_3$

In [3] it was shown how a certain partial resolution of the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity (which leads to the conifold singularity) can be realized in the moduli space of D-branes. This approach was used in [4] to deduce the worldvolume theory of $N$ parallel D3 branes transverse to a conifold singularity, as well as the corresponding worldvolume theories at certain other conical singularities which can be obtained as partial resolutions of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. In this manner, a systematic procedure was presented for deducing the CFT side of the AdS/CFT correspondence in the case of nonspherical horizons. In particular, the results of [5] (originally obtained by making use of arguments entirely dependent on the high symmetry of the conifold), were reobtained in a systematic
manner and generalized. In this section, we will apply methods similar to [6] to the conical singularities discussed in the introduction.

As it turns out, each of these singularities can be realized as a partial resolution of $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$. To explain this, consider a set of generators $g_1 = (\hat{1}, \hat{0}), g_2 = (\hat{0}, \hat{1})$ of the group $\mathbb{Z}_3 \times \mathbb{Z}_3$. We choose the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on $\mathbb{C}^3$ to be given by:

$$\begin{align*}
R(g_1) : (X, Y, Z) &\mapsto (\omega X, \omega^{-1} Y, Z) \\
R(g_2) : (X, Y, Z) &\mapsto (\omega X, Y, \omega^{-1} Z),
\end{align*}$$

(1)

where $\omega = e^{\frac{2\pi i}{3}}$. As is the case with any abelian quotient singularity, this is an affine toric variety. Thus, it can be described by a cone $C$ in $\mathbb{R}^3$ which cuts the plane $x + y + z = 1$ along a convex polygon. In our case, this polygon is a triangle with vertices $v_1, v_2, v_3$ which contains 7 other integral points $\{w_1, \ldots, w_7\}$, only one of which lies in its interior. We label these vectors as follows (see Figure1(a)):

$$\begin{align*}
v_1 &= (0, 3) & v_2 &= (0, 0) & v_3 &= (3, 0) & w_1 &= (0, 2) & w_2 &= (0, 1) \\
w_3 &= (1, 0) & w_4 &= (2, 0) & w_5 &= (1, 2) & w_6 &= (2, 1) & w_7 &= (1, 1).
\end{align*}$$

(2)
To find the associated symplectic quotient description of this toric singularity, con-
Consider the basis of linear relations between these vectors given by

\[ v_3 - 3 w_3 + 2 v_2 = 0 \]
\[ w_6 - 2 w_3 - w_2 + 2 v_2 = 0 \]
\[ w_4 - 2 w_3 + v_2 = 0 \]
\[ w_5 - w_3 - 2 w_2 + 2 v_2 = 0 \]
\[ w_7 - w_3 - w_2 + v_2 = 0 \]
\[ v_1 - 3 w_2 + 2 v_2 = 0 \]
\[ w_1 - 2 w_2 + v_2 = 0 \]

(3)

Introduce homogeneous coordinates \(x_i, y_j\) on \(\mathbb{C}^{10}\) corresponding to \(v_i, w_j\). Then the \(U(1)^7\)-action is determined by the matrix of linear relations:

\[
\begin{bmatrix}
  x_3 & y_6 & y_4 & y_5 & y_7 & y_3 & x_1 & y_2 & x_2 \\
  1 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 2 \\
  0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 2 \\
  0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -2 & 2 \\
  0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -3 & 2 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1
\end{bmatrix}
\]

(4)

while the associated moment map equations, for a central level \(\zeta = (\zeta_1...\zeta_7)\) are:

\[ |x_3|^2 - 3|y_3|^2 + 2|x_2|^2 = \zeta_1 \]
\[ |y_6|^2 - 2|y_3|^2 - |y_2|^2 + 2|x_2|^2 = \zeta_2 \]
\[ |y_4|^2 - 2|y_3|^2 + |x_2|^2 = \zeta_3 \]
\[ |y_5|^2 - |y_3|^2 - 2|y_2|^2 + 2|x_2|^2 = \zeta_4 \]
\[ |y_7|^2 - |y_3|^2 - |y_2|^2 + |x_2|^2 = \zeta_5 \]
\[ |x_1|^2 - 3|y_2|^2 + 2|x_2|^2 = \zeta_6 \]
\[ |x_1|^2 - 2|y_2|^2 + |x_2|^2 = \zeta_7 \]

(5)

If all \(\zeta_i = 0\), then we obtain the unresolved \(\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3\) orbifold singularity.

We consider only linear relations such that the sum of all charge vectors equals zero. Equivalently, we consider linear relations between the lifts of the vectors in Figure 1(a) to the plane \(z = 1\) in \(\mathbb{R}^3\).
Conversely, for generic values of $\zeta_i$, the singularity is completely resolved and the symplectic quotient leads to a smooth manifold. However, if the $\zeta_i$ lie in particular cones of some codimension, the symplectic quotient is singular. The singularity is determined by the cone, and is the same for all points in its interior.

As an extreme example, consider the one-dimensional cone in which all $\zeta_i = 0$ except for $\zeta_5 > 0$. In this cone, we see from (5) that there is a vicinity of the origin of the space of homogeneous coordinates on which $y_7$ cannot vanish (this coordinate corresponds to the center of the triangle in figure 1(a)). We can thus perform explicitly one of the $U(1)$ quotients by fixing its value to be real and positive. Since its value is then determined by the remaining homogeneous variables, we have effectively eliminated one coordinate and one quotient from the problem. Note that the higher the dimension of the cone in which the $\zeta_i$ lie, the larger is the number of coordinates we can eliminate, and the milder the singularity we obtain. The extreme case of the $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ singularity corresponds to $\zeta = 0$ (a zero-dimensional cone), while a complete resolution corresponds to a ‘generic’ vector $\zeta$, i.e. one which belongs to a seven-dimensional cone.

The procedure described above gives a symplectic description of the various resolutions.\(^3\) It is not hard to see that, with appropriate choices of levels $\zeta_i$, one can realize the singularity corresponding to any subpolygon of the triangle shown in Figure 1(a).

For the geometrically-inclined reader, we mention that the discussion above is a concrete realization of very general results of Symplectic Geometry and Geometric Invariant Theory. Such results assure us that the space of moment map levels has a canonical partition into conical chambers, which together form a fan $\Psi$ in $\mathbb{R}^7(\zeta)$. As $\zeta$ varies inside of a given chamber, the algebraic structure of our toric variety does not change (even though the Kähler metric induced by the symplectic reduction changes). When $\zeta$ crosses a wall separating two chambers, the toric variety undergoes a birational transformation known as a toric flop; the variety is singular precisely when $\zeta$ lies on a wall (this is the same mathematical structure which underlies topology change in the moduli space of conformal field theories [13]). The procedure we just described gives one practical way of identifying these walls and chambers.

The complex cones over the del Pezzo surfaces of interest are well-known to be affine toric varieties themselves [10], and are described by the polygons listed in Figure 1. The particular presentations shown there allow us to realize these cones as tangent cones to partial resolutions of the $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ singularity. This amounts to viewing the polygons as inscribed in the triangle of Figure 1(a) and turning on appropriate levels $\zeta$ in order to eliminate precisely the integral points which lie on the triangle.

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\(^3\)To be rigorous, this procedure yields not the partial resolutions but their tangent cones. In fact, the partial resolutions are only quasiprojective varieties, while the tangent cones are affine. As discussed for example in [12], the partial resolutions are associated to triangulations of the first set of points in Figure 1, and therefore their fans are not of affine type, i.e. do not coincide with the fan of all faces of a single 3-dimensional cone. However, the procedure above suffices to correctly identify the chambers in $\mathbb{R}^7(\zeta)$ parameterizing the resolutions. Note that the singularities we wish to realize (the complex cones over the toric del Pezzo surfaces) are affine toric varieties, and therefore they will only correspond to the tangent cones to the partial resolutions we will identify below.
but not on the del Pezzo polytope. The subspaces of \( \mathbb{R}^7(\zeta) \) leading to the desired partial resolutions, as well as the corresponding charge matrices describing these as toric varieties are listed in Table 1.

| Cone over | Relations | Charge Matrix |
|-----------|-----------|---------------|
| \( F_0 \) | \( \zeta_2 - 2\zeta_5, \zeta_4 - 2\zeta_5 = 0 \) | \( \begin{pmatrix} y_6 & y_5 & y_7 & y_3 & y_2 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \end{pmatrix} \) |
| \( dP_1 \) | \( \zeta_2 - 3\zeta_5 + \zeta_7, \zeta_3 - 2\zeta_5 + \zeta_7 = 0 \) | \( \begin{pmatrix} y_6 & y_5 & y_7 & y_3 & y_1 \\ 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & -2 & 0 & 1 \end{pmatrix} \) |
| \( dP_2 \) | \( \zeta_2, \zeta_4, \zeta_5 = 0 \) | \( \begin{pmatrix} y_6 & y_5 & y_7 & y_3 & y_2 & x_2 \\ 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} \) |
| \( dP_3 \) | \( \zeta_2 - 2\zeta_7, \zeta_3 - \zeta_7, \zeta_4 - 2\zeta_7 = 0 \) | \( \begin{pmatrix} y_6 & y_5 & y_7 & y_3 & y_1 & y_2 \\ 1 & 0 & 0 & 0 & -2 & -2 & 3 \\ 0 & 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} \) |

Table 1. Charge matrices for partial resolutions of the \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) orbifold.

Being affine varieties, the complex cones over the del Pezzo surfaces of interest can be presented as the affine spectrum \( \text{Spec}(R) \) of their coordinate ring \( R \). This ring admits a presentation \( R = \mathbb{C}[z]/I \) with \( \mathbb{C}[z] \) a polynomial ring and \( I \) an ideal of relations. In the context of toric geometry, the generators \( z_i \) are the invariant coordinates under the complex torus action in the holomorphic quotient description, while \( I \) is the ideal of monomial relations constraining them. It is well-known that the invariants \( z \) and monomial relations can be determined as follows. If \( C \) is the cone over the polygon \( P \) describing our variety (where the polygon is embedded in the affine plane \( x + y + z = 1 \) of \( \mathbb{R}^3 \)), then one considers the dual cone \( C^v \), which is the cone over the dual (polar) polygon \( P^v \). While the cone \( C \) and the polygon \( P \) are appropriate for the holomorphic quotient description of the toric variety, their duals \( C^v, P^v \) are appropriate for the description in terms of invariants \( z \) and monomial relations \( I \). More precisely, there will be one invariant \( z_i \) associated to each integral point of the dual polygon \( P^v \). Moreover, any integral linear relation between the primitive integral vectors lying in the cone \( C^v \) corresponds to a monomial relation among the variables.
z. Explicitly, a linear relation of the form $\sum_i a_i u_i$ between the vectors $u_i$ associated to the invariant coordinates $z_i$ yields the monomial relation $\Pi_i z_i^{a_i} = 1$. Hence the entire information characterizing the ring $R$ is contained in the dual polygons, which are listed in Figure 2.

![Figure 2(a). $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold](image)

Figure 2(b). Cone over $F_0$.

![Figure 2(c). Cone over $dP_1$](image)

Figure 2(d). Cone over $dP_2$.

![Figure 2(e). Cone over $dP_3$](image)

Figure 2(e). Cone over $dP_3$.

Figure 2. Geometric picture of the generators of $R$ and the relations between them.

### 1.2 Partial resolutions of the moduli space

The moduli space for one D-brane in the presence of Fayet-Iliopoulos terms can be analyzed by the methods pioneered in [1]. In the case of $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$, the main points of the analysis are as follows. The classical moduli space in the presence of Fayet-Iliopoulos terms is obtained by imposing the D- and F-flatness constraints and further
dividing by the gauge group of the worldvolume theory, which in this case turns out to be the torus $U(1)^8$. If we denote the solution set of the F-flatness constraints by $Z$, then the symplectic quotient which gives the moduli space (where the D-flatness constraints play the role of moment map equations) is isomorphic, as a complex variety, with the holomorphic quotient of $Z$ by the complexification $(\mathbb{C}^*)^8$ of the gauge group. For one D-brane on an abelian quotient singularity, the set $Z$ (which in the case $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ is complex eleven-dimensional, see Section 3) has the structure of an algebraic variety defined by a collection of monomial relations. Therefore, $Z$ is an affine toric variety, and by virtue of \cite{15} it admits an alternate presentation as a holomorphic quotient:

$$Z = \mathbb{C}^{42}/(\mathbb{C}^*)^{31}.$$ (6)

The procedure for precisely identifying the quotient above (the mysterious numbers 42,31 and the precise $(\mathbb{C}^*)^{31}$ action) is slightly technical and is explained in detail in section 2. Then the moduli space can be obtained in the form:

$$\mathcal{M}_\xi = (Z - S_\xi)/(\mathbb{C}^*)^8 = (\mathbb{C}^{42} - Z_\xi)/(\mathbb{C}^*)^{39},$$ (7)

where $S_\xi, Z_\xi$ are certain ‘exceptional sets’ which have complex codimension at least two in the respective spaces. Note that $Z_\xi$ depends on the choice of Fayet-Iliopoulos parameters $\xi$, which explains the dependence of the complex geometry of the moduli space on these parameters. In symplectic quotient language, the symplectic reduction of $Z$ by $U(1)^8$ is performed at a level characterized by the Fayet-Iliopoulos parameters $\xi \in \mathbb{R}^8$. When re-writing $Z$ itself as a symplectic quotient (with zero moment map level $0 \in \mathbb{R}^{31}$) we are rewriting $\mathcal{M}_\xi$ as a double symplectic quotient. To arrive at a presentation equivalent to the last entry in (6), we then rewrite this double quotient as a single symplectic quotient, with certain ‘overall’ moment map levels $\eta(\xi) \in \mathbb{R}^{39}$. The crucial point of this analysis is that the levels $\eta$ are not arbitrary in $\mathbb{R}^{39}$, but are determined by Fayet-Iliopoulos terms $\xi \in \mathbb{R}^8$. Therefore, as $\xi$ covers the whole of $\mathbb{R}^8$, the parameters $\eta(\xi)$ will cover only a subspace of $\mathbb{R}^{39}$. As shown in Sections 2 and 3, the precise relation between $\eta$ and $\xi$ is given by an injective linear map:

$$\mathbb{R}^8(\xi) \xrightarrow{w} \mathbb{R}^{39}(\eta),$$ (8)

such that $\eta(\xi) = w\xi$. Therefore, $\eta(\xi)$ will only vary inside of an 8-dimensional subspace $W$ of $\mathbb{R}^{39}$. This non-genericity of the level $\eta$ is responsible for the fact that the symplectic quotient description $\mathcal{M}_\xi = \{x \in \mathbb{C}^{42} | \mu(x) = \eta(\xi)\}/U(1)^{39}$ is not minimal, in the sense that, depending on the sign of the various components of $\eta(\xi)$, it will in general be possible to eliminate many of the homogeneous variables of $\mathbb{C}^{42}$ essentially by the same procedure as that discussed in subsection 1.1. In the holomorphic quotient language, this is reflected by the fact that the toric generators one obtains for the description $(\mathbb{C}^{42} - Z_\xi)/(\mathbb{C}^*)^{39}$ appear with multiplicities, so the toric description is non-minimal as well. In order to obtain a minimal description, one must eliminate all multiplicities in the list of toric generators or, equivalently, eliminate the redundant
homogeneous coordinates in the manner outlined above. In our case, this reduces the full quotient to the form \((\mathbb{C}^{10} - F_\xi)/((\mathbb{C}^*)^7)\), which is precisely what we would expect from the geometric description of the partial resolutions. This reduced quotient has a symplectic quotient description as well, and we let \(\zeta(\xi) \in \mathbb{R}^7\) denote the associated moment map levels.

As in the previous subsection, which homogeneous coordinates can be eliminated depends on the sign of the various components of \(\eta(\xi)\). A systematic analysis shows that this procedure defines a certain fan \(\Sigma\) in \(\mathbb{R}^{39}(\eta)\). Since \(w\) is an injective linear function, it follows that there exists a similar division of \(\mathbb{R}^8(\xi)\) into cones \(\Xi_I\) obtained by taking the inverse image of the cones \(\Sigma\) through the map \(w\) (geometrically, this amounts to intersecting the system of cones \(\Sigma\) of \(\mathbb{R}^{39}\) with the subspace \(W\)). Which homogeneous coordinates of the full quotient can be eliminated, and hence the minimal description of \(\mathcal{M}_\xi\) as a toric variety will thus depend on where \(\xi\) lies in \(\mathbb{R}^8\) with respect to the cones \(\Xi\). Note that not all of the cones \(\Xi_I\) will be 8-dimensional. In fact, it is not hard to see that those cones \(\Xi_I\) which have dimension less than 8 will form faces of the 8-dimensional ones.

As we discuss in Sections 2 and 3, the reduction of the quotient \((\mathbb{C}^{12} - Z_\xi)/(\mathbb{C}^*)^{39}\) to the quotient \((\mathbb{C}^{10} - F_\xi)/(\mathbb{C}^*)^7\) induces a piecewise-linear map

\[
\mathbb{R}^{39}(\eta) \xrightarrow{\pi} \mathbb{R}^7(\zeta),
\]

such that \(\zeta = \pi(\eta)\). Therefore, the dependence of \(\zeta\) on \(\xi\) is given by the composite map:

\[
\phi := \pi \circ w : \mathbb{R}^8(\xi) \rightarrow \mathbb{R}^7(\zeta),
\]

so that \(\zeta = \phi(\xi)\). Clearly \(\phi\) is a piecewise-linear map, which is linear on each of the cones \(\Xi_I\). Note that \(\phi\) need not have different linear expressions on each of the cones \(\Xi_I\).

The discussion above shows that understanding how to realize the various partial resolutions of \(\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3\) in terms of the worldvolume theory of the D3-brane amounts to computing the map \(\phi\). Indeed, this map connects the Fayet-Iliopoulos parameters \(\xi\) of the D-brane worldvolume theory to the toric levels \(\zeta\) discussed in section 2. Therefore, given a choice of partial resolution (described in the language of section 2 by a choice of \(\zeta\)), the map \(\phi\) allows us to determine if and how this geometric choice is physically realized in the D-brane theory.

Clearly \(\phi\) must take the fan \(\Xi\) into a refinement of the natural fan \(\Psi\) carried by \(\mathbb{R}^7(\zeta)\), which was discussed in the previous subsection. Hence the fan \(\Xi\) is a refinement of the preimage of \(\Psi\) through \(\phi\). The full information about the D-brane realization of the various partial resolutions is encoded in this preimage. By using our methods, one can in principle reconstruct the preimage from the smaller pieces provided by the cones \(\Xi_I\).

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4In technical terms, the fan associated to \(\phi\) as a piecewise-linear function is subordinate to the fan defined by the cones \(\Xi_I\), i.e. this later fan is a refinement of the fan of \(\phi\).
A systematic procedure for determining the map $\phi$ is developed in the next section. In the case of $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$, the complexity of the computation turns out to be markedly greater than in cases considered before [1, 2]. As we explain in Section 2, the complexity is characterized by an integer $c$, which depends on the combinatorial data of the problem, and for which we do not have an analytic expression. In a typical example considered in [1], such as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient singularity, one had $c = 9$, while our case we have $c = 42$.

The result of the analysis is that each of the complex cones over $F_0, dP_1, dP_2$ and $dP_3$ is indeed realized in the moduli space of the worldvolume theory. The details of how this conclusion can be reached are discussed in the next sections.

2 The classical moduli space for one D-brane transverse to an abelian Calabi-Yau quotient singularity

In this section, we spell out in detail our algorithm for the analysis of the moduli space of one D-brane on an abelian quotient singularity.

The algorithm presented here is essentially a systematic version of the approach taken in [1] and is carried out in the language of the quiver formalism [17] (see [18] for a discussion in the context of D-brane moduli spaces), which turns out to be a very effective way of implementing the projection conditions and analyzing the D- and F-flatness constraints. The main novel result is the construction of the map $\phi$ in subsection 2.8. As a simple illustration, we re-consider the realization of the conifold in the moduli space of a D-brane transverse to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ Calabi-Yau quotient singularity [2].

2.1 The quotient group and its representation on $\mathbb{C}^3$

Consider a finite abelian group $\Gamma$. Choosing a system of generators allows us to present the group as:

$$\Gamma = \mathbb{Z}_{t_1} \times \cdots \times \mathbb{Z}_{t_d},$$

where the integers $t_1, t_2 \geq 2$ are the torsion indices of $\Gamma$. The elements of the group are then written in the form:

$$u = (u_1 \ldots u_d),$$

with $u_a \in \mathbb{Z}_{t_a}$ ($a = 1 \ldots d$), while the group operation becomes:

$$u + v = ( (u_1 + v_1) \bmod t_1, \ldots, (u_d + v_d) \bmod t_d ).$$

$c$ is the number of facets of the cone of exponents associated to the F-flatness constraints, as discussed in subsection 2.7.
The irreducible representations of our group are given by its characters \( \chi \in \text{Hom}(\Gamma, U(1)) \), which are in one to one correspondence with the elements of \( \Gamma \). Indeed, every character is of the form:

\[
\chi_w(u) = \prod_{a=1}^{d} e^{2\pi i w_a} \quad (u \in \Gamma),
\]

(14)

where \( w = (w_1...w_d) \) is an element of \( \Gamma \) called the weight associated to \( \chi \). In fact, this correspondence gives an isomorphism between \( \Gamma \) and the multiplicative group of its characters:

\[
\chi_{w+w'} = \chi_w \chi_{w'}
\]

\[
\chi_{w}^{-1} = \chi_{-w}.
\]

(15)

We let \( \Gamma \) act on \( \mathbb{C}^3 \) by a special unitary representation \( \theta \). Since this representation decomposes into 3 irreducibles, we can always find coordinates \((x^1, x^2, x^3)\) on \( \mathbb{C}^3 \) such that \( \theta \) takes the form:

\[
(x^1, x^2, x^3) \xrightarrow{(u \in \Gamma)} \theta(u)(x) = (\chi_{w_1}(u)x^1, \chi_{w_2}(u)x^2, \chi_{w_3}(u)x^3).
\]

(16)

Special unitarity requires that \( w_1 + w_2 + w_3 = 0 \) (a relation which has to be understood as holding in the group \( \Gamma \)).

**Example:** The \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) quotient singularity

The group \( \Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \) has \( d = 2 \) torsion indices \( t_1 = 2, t_2 = 2 \). Consider its action on \( \mathbb{C}^3 \) given by the weights \( w_1 = (1, 1), w_2 = (1, 0), w_3 = (0, 1) \) (which do satisfy \( w_1 + w_2 + w_3 = (0, 0) \) in our group) and associated characters:

\[
\chi_1(u_1, u_2) = e^{i\pi(u_1+u_2)} = (-1)^{(u_1+u_2)}
\]

\[
\chi_2(u_1, u_2) = e^{i\pi u_1} = (-1)^{u_1}
\]

\[
\chi_3(u_1, u_2) = e^{i\pi u_2} = (-1)^{u_2},
\]

(17)

with \( u = (u_1, u_2), u_1, u_2 = 0, 1 \). The generators \((1, 0), (0, 1)\) of the group act as:

\[
(x_1, x_2, x_3) \xrightarrow{(1,0)} (-x_1, -x_2, x_3)
\]

\[
(x_1, x_2, x_3) \xrightarrow{(0,1)} (-x_1, x_2, -x_3)
\]

(18)

### 2.2 Solving the projection constraints

The low-energy theory on the worldvolume of a D-brane transverse to a quotient singularity is given by a projection of a supersymmetric gauge theory with 16 supercharges\[^{[16]}\]. The gauge group is determined by the singularity. The projection will be described in more detail below. It results in an Abelian gauge theory with a reduced supersymmetry and, when appropriate, a superpotential. The classical moduli space of supersymmetric vacua of this theory is found to be the tangent cone to the transverse space; this is
reasonable, since motion of the brane along this space is a supersymmetry-preserving
deformation, while the low-energy limit restricts us to small motions. The gauge theory
admits Fayet-Iliopoulos terms; these are expected to parameterize the deformations of
the singularity itself. How this works is the subject of this paper, and will be made
clear in what follows. We note here that this picture is a bit imprecise. As noted in
[14, 6] the Abelian gauge symmetry is in fact broken by twisted closed-string modes.
As described in detail in [6], this fact may safely be ignored in our discussion below.

Let \(|\Gamma|\) denote the number of elements of our group. The low-energy theory for
a D3-brane near this singularity is found as a projection of the theory of
\(|\Gamma|\) branes
moving on the covering space. This is an \(\mathcal{N} = 4\) theory with \(U(|\Gamma|)\) gauge symmetry,
containing (in \(\mathcal{N} = 1\) language) a vector multiplet and three chiral multiplets in the
adjoint representation of the gauge group. As is by now familiar, the three chiral
multiplets represent a nonabelian version of the positions of the \(|\Gamma|\) branes in the
three complex dimensional transverse space. The quotient theory is obtained by a
projection, restricting attention to fields invariant under the action of \(\Gamma\), which we lift
to the Chan-Paton indices via its regular representation.

We label the entries of the adjoint fields by

\[
X^i = (X^i_{v',v})_{v',v \in \Gamma} .
\]

The spacetime indices of the matrices \(X^i\) transform in the representation \([16]\) and the
chiral fields surviving the projection are thus those modes which satisfy

\[
R(u)X^i R(u)^{-1} = \chi_{-w_i}(u)X^i ,
\]

where \(R(u)\) is the regular representation of \(\Gamma\). Since \(\Gamma\) is abelian, \(R\) decomposes into
the sum of all irreducible representations, each taken with multiplicity one. Therefore,
we can always assume (after performing a unitary transformation \(X^i \rightarrow UX^i U^{-1}\))
that \(R(u)\) are given by the diagonal matrices:

\[
R(u)_{v',v} = \chi_v(u)\delta_{v',v} \quad (v', v \in \Gamma) .
\]

Then the projection conditions take the form:

\[
\chi_{v' - v + w_i}(u)X^i_{v',v} = X^i_{v',v} ,
\]

which require that all elements of \(X^i\) must be zero except for the following entries:

\[
x^i(v) := X^i_{v - w_i,v} .
\]

The set of surviving fields \(x^i(v)\) can be elegantly described in the language of
graph theory[17]. For this, consider a set of points (nodes) which are in one to one
correspondence with the elements of \(\Gamma\). For each node \(v \in \Gamma\), and for each \(i = 1..3\),
draw an edge from \(v\) to \(v - w_i\). Such an edge is associated with a surviving component
\(x^i(v)\) of the matrix \(X^i\) and will be called an edge of type \(i\). If \(w_i = 0\), then the edge

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connects the node \( v \) with itself, and is thus a loop, which cannot carry an orientation. If \( w_i \neq 0 \), the edge is given the orientation from \( v \) to \( v - w_i \) (such oriented edges will be called arrows). The graph \( Q \) thus obtained is called the McKay quiver of the quotient singularity \( \mathbb{C}^3/\Gamma \). We denote the set of its nodes by \( Q_0 \approx \Gamma \) and the set of its edges by \( Q_1 \). Since we have 3 types of edges leaving each vertex, the total number of edges in the graph is equal to \( 3|\Gamma| \). The McKay quiver can be thought of as the superposition of 3 graphs \( Q_i \) \((i = 1..3)\), where \( Q_i \) is obtained from \( Q \) by keeping only the edges of type \( i \). The edges of \( Q_i \) represent the surviving components of the matrix \( X_i \). In the absence of the projection conditions, \( Q_i \) would coincide with the full graph on the set of nodes \( \Gamma \), i.e. it would contain an edge connecting any two elements of \( \Gamma \) (in particular, it would contain a loop at each vertex). The projection conditions eliminate some of these edges, in a manner controlled by the weight \( w_i \). The surviving components of \( X_i \) can be read from the quiver as follows. If \( a \in Q_1 \) is an edge of \( Q \), then the associated field is given by:

\[ x(a) := x^{\text{type}(a)}(\text{tail}(a)) = X_{\text{head}(a),\text{tail}(a)}^{\text{type}(a)} \ , \]

where \( \text{tail}(a), \text{head}(a) \in \Gamma \) are the tail, respectively head of the edge \( a \) (in case \( a \) is a loop at a node \( v \), we define \( \text{tail}(a) = \text{head}(a) \) to be given by \( v \); otherwise, \( a \) is an arrow going from \( \text{tail}(a) \) to \( \text{head}(a) \)). Throughout this paper, we will represent arrows of type 1 by light grey lines, arrows of type 2 by dark grey lines and arrows of type 3 by black lines. In a color postscript rendering, these arrows appear respectively as green, blue and red.

**Example:** \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) singularity

In this case, we have \( |\Gamma| = 4 \) so the quiver will have 4 nodes.

The edges \( v \to v - w_i \) are:

1) type 1 (the surviving components of \( X^1 \)):

\[ (0,0) \to (1,1), \quad (0,1) \to (1,0), \quad (1,0) \to (0,1), \quad (1,1) \to (0,0) \ ; \]

2) type 2 (the surviving components of \( X^2 \)):

\[ (0,0) \to (1,0), \quad (0,1) \to (1,1), \quad (1,0) \to (0,0), \quad (1,1) \to (0,1) \ ; \]

3) type 3 (the surviving components of \( X^3 \)):

\[ (0,0) \to (0,1), \quad (0,1) \to (0,0), \quad (1,0) \to (1,1), \quad (1,1) \to (1,0) \ . \]

Indexing the group elements as follows:

\[ (0,0) \leftrightarrow 1 \]
\[ (1,0) \leftrightarrow 2 \]
\[ (0,1) \leftrightarrow 3 \]
\[ (1,1) \leftrightarrow 4 \]
we can rewrite the edges as:

(1) type 1:
\[ 1 \rightarrow 4, \ 3 \rightarrow 2, \ 2 \rightarrow 3, \ 4 \rightarrow 1 \]; \quad (28)

(2) type 2:
\[ 1 \rightarrow 2, \ 3 \rightarrow 4, \ 2 \rightarrow 1, \ 4 \rightarrow 3 \]; \quad (29)

(3) type 3:
\[ 1 \rightarrow 3, \ 3 \rightarrow 1, \ 2 \rightarrow 4, \ 4 \rightarrow 2 \]. \quad (30)

The quiver is drawn below:

![Quiver Diagram](image)

Figure 3. The quiver for the isolated $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient singularity.

The arrows of various types are drawn with different shades of gray: light gray for type 1 (fields $X_{uv}$), dark for type 2 (fields $Y_{uv}$), medium for type 3 (fields $Z_{uv}$). In color rendering, these correspond to 3 different colors: green for type 1, blue for type 2, red for type 3.

2.3 The action of the surviving gauge group

The gauge fields in the theory do not carry transverse space indices, so the projection conditions on these are:

\[ R(u)UR^{-1}(u) = U \]. \quad (31)
If we index the entries of $U$ by the elements of $\Gamma$:

$$U = (U_{\nu', \nu})_{\nu', \nu \in \Gamma},$$

then (31) becomes:

$$\chi_{\nu'-\nu}(u) U_{\nu', \nu} = U_{\nu', \nu},$$

which shows that the surviving entries of $U$ are given by:

$$U_{\nu} = U_{\nu, \nu} \in U(1).$$

Therefore, the projected gauge group is given by diagonal unitary matrices and is isomorphic with $\Pi_{v \in \Gamma} U(1) \approx U(1)^{\Gamma}$. Its action on the surviving fields $x^i(u)$ is:

$$x^i(u) \xrightarrow{\sum_{\nu \in \Gamma}} U_{u-w_i, u}^{-1} x^i(u) = e^{i(\phi_{u-w_i} - \phi_u)} x^i(u)$$

where we wrote $U_v = e^{i\phi_v}$ with $\phi_v \in \mathbb{R}$. Therefore, the charge of $x^i(u)$ with respect to the $v$-th $U(1)$ factor is:

$$q_v^i(u) = \delta_{v, u-w_i} - \delta_{v, u}.$$

Note that the diagonal subgroup $U(1)_{\text{diag}}$ given by $U_v = e^{i\phi_v}$ with $\phi_v = \phi$ independent of $v$ acts trivially on all surviving fields, so the effective gauge group is given by $G = U(1)^{\Gamma}/U(1)_{\text{diag}}$.

The action (35) can be translated into quiver language as follows. If $a \in Q_1$ is an edge of the quiver, then the field $x(a)$ given by (24) transforms as:

$$x(a) \xrightarrow{\sum_{\nu \in \Gamma}} e^{i(\text{head}(a)-\text{tail}(a))\phi} x(a).$$

One can imagine the various $U(1)$ factors of the surviving gauge group to be sitting at the nodes of the quiver, having the natural action (37) on the variables $x(a)$ associated to its edges. Note that each edge is charged with respect to the $U(1)$ factors associated to its terminal points (its tail and its head). The action of $U(1)^{\Gamma}$ on $x(a)$ is encoded by the $|\Gamma| \times 3|\Gamma|$ matrix of charges $d = (d_{v,a})_{v \in \Gamma, a \in Q_1}$ with entries:

$$d_{v,a} = \delta_{v, \text{head}(a)} - \delta_{v, \text{tail}(a)}.$$

(This is just another way of writing equation (36)). In graph-theoretic language, $d$ is the incidence matrix of the quiver, where the incidence index of an arrow $a$ on a node $v$ is $-1$ if $v = \text{tail}(a)$, $+1$ if $v = \text{head}(a)$ and zero otherwise, while the incidence index of a loop with any node (including the node where the loop sits) is defined to be zero.

2.4 The D-flatness constraints

The moment map $\mu : \mathbb{C}^{Q_1} \to \mathbb{R}^{\Gamma}$ for the action of $U(1)^{\Gamma}$ on the space $\mathbb{C}^{Q_1}$ of all surviving fields is given by:

$$\mu_v(x) = \sum_{i=1...3} \sum_{u \in \Gamma} q_v^i(u)|x_i^i(u)|^2$$

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which in quiver language becomes:

\[
\mu_v(x) = \sum_{a \in Q_1} d_{v,a} |x(a)|^2 = \sum_{a \in Q_1, \text{head}(a) = v} |x(a)|^2 - \sum_{a \in Q_1, \text{tail}(a) = v} |x(a)|^2 ,
\]

(40)

or, in matrix form:

\[
\mu = d p ,
\]

(41)

where \( p \) is the vector in \( \mathbb{R}^{Q_1} \) with components:

\[
p_a := |x(a)|^2 .
\]

(42)

The D-flatness constraints in the presence of Fayet-Iliopoulos terms \( (\xi_v) \) \((v \in \Gamma)\) read:

\[
\mu_v(x) = \xi_v \text{ for all } v \in \Gamma .
\]

(43)

Note that the moment map satisfies the condition:

\[
\sum_{v \in \Gamma} \mu_v(x) = 0 ,
\]

(44)

which is a consequence of the trivial action of \( U(1)_{\text{diag}} \). This is also reflected in the structure of the incidence matrix \( d \), by the fact that the sum of all of its rows is zero. Consistency requires that the Fayet-Iliopoulos terms also satisfy:

\[
\sum_{v \in \Gamma} \xi_v = 0 .
\]

(45)

Therefore, we can always express \( \xi_0 \) (where 0 is the identity element of \( \Gamma \)) as:

\[
\xi_0 = - \sum_{v \in \Gamma - \{0\}} \xi_v .
\]

(46)

It follows that the space of allowed Fayet-Iliopoulos parameters is in fact only \(|\Gamma| - 1\)-dimensional, and will be denoted by \( \mathbb{R}^{|\Gamma|-1}(\xi) \), where the vector \( \xi \) is given by \( \xi = (\xi_v)_{v \in \Gamma - \{0\}} \). Similarly, the first equation of the system (R) (the one corresponding to \( v = 0 \)) is a consequence of the other \( |\Gamma| - 1 \) equations. Eliminating it allows us to rewrite the moment map conditions in the form:

\[
\Delta p = \xi ,
\]

(47)

where \( \Delta \) is the \((|\Gamma| - 1) \times 3|\Gamma|\) matrix obtained by deleting the first row of \( d \).

**Example:** \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) singularity
With the above enumeration of the group elements, the incidence matrix is:

\[
d = \begin{bmatrix}
-1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\
\end{bmatrix},
\]  
(48)

while the matrix \( \Delta \) is given by (the first row of \( d \) corresponds to the neutral element of the group with our choice of enumeration):

\[
\Delta = \begin{bmatrix}
0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\
\end{bmatrix}.
\]  
(49)

### 2.5 The F-flatness constraints

The \( N = 4 \) theory includes a superpotential:

\[
W = \epsilon_{ijk} \text{Tr}(X^i X^j X^k). 
\]  
(50)

This can be expressed in terms of the surviving fields as

\[
W = -\epsilon_{ijk} \sum_{v \in \Gamma} x^k(v - w_i - w_j)x^j(v - w_i)x^i(v).
\]  
(51)

Supersymmetric vacua of the \( N = 4 \) theory satisfy the F-flatness conditions:

\[
[X^i, X^j] = 0 \quad (1 \leq i < j \leq 3).
\]  
(52)

Taking (23) into account these reduce to:

\[
x^j(v - w_i)x^i(v) - x^i(v - w_j)x^j(v) = 0, \text{ for all } 1 \leq i < j \leq 3 \text{ and } v \in \Gamma,
\]  
(53)

in agreement with the condition that (51) is stationary.

\textsuperscript{6}Note that the trace in (50) closes on the projected fields precisely due to the special unitarity condition \( w_1 + w_2 + w_3 = 0 \).
The constraints (53) can be viewed as monomial relations:
\[ x^j(v - w_i)x^i(v)[x^i(v - w_j)]^{-1}[x^j(v)]^{-1} = 1 \quad (54) \]

among the surviving fields. The solution of these equations defines a complex subvariety \( Z \) of the space \( \mathbb{C}Q_1 \) of all variables \( \{x^i(v)\} \), which we call the variety of commuting matrices. This algebraic variety can be described by the set of all its regular functions, which are obtained by restricting the regular functions defined on \( \mathbb{C}Q_1 \) (which are just the polynomials in the variables \( \{x^i(v)\} \)) to the subset \( Z \). The restriction of such a polynomial is achieved by simplifying each of its monomials with respect to the relations (54). If \( x = (x^i(v))_{i=1,3,v \in \Gamma} \) denotes the general point of \( \mathbb{C}Q_1 \), then a monomial of the polynomial ring \( \mathbb{C}[x] \) has the form \( \Pi_i,v[x^i(v)]^{s^i(v)} \), with \( s^i(v) \) some nonnegative integer exponents. For simplicity of notation, we will write this as \( x^s \), where \( s = (s^i(v))_{i=1,3,v \in \Gamma} \) is a vector in the lattice \( ZQ_1 \) whose components are nonnegative. The affine space \( \mathbb{C}Q_1 \) can be described by the collection of all such exponents, i.e. by the set of those points \( m \) of the lattice \( ZQ_1 \) which belong to the cone \( C \subset \mathbb{R}Q_1 \) defined by the inequalities \( m^i(v) \geq 0 \) for all \( i,v \) (\( C \) is the first ‘octant’ in \( \mathbb{R}Q_1 \)).

On the other hand, the relations (54) are equivalent with:
\[ x^m = 1 \quad \text{for all} \quad m \in R \quad , \quad (55) \]

where \( R \) is the sublattice of \( ZQ_1 \) (which we will call the lattice of relations) spanned by the vectors:
\[ r^{(ij)} := e^j(v - w_i) + e^i(v) - e^i(v - w_j) - e^j(v) \quad , \quad (56) \]
where \( (e^i(v))_{i=1,3,v \in \Gamma} \) is the canonical basis of \( ZQ_1 \). In general, the \( 3|\Gamma| \) vectors (56) are not linearly independent. In fact, it is not hard to see that they span a subspace of dimension \( \rho = 2(|\Gamma| - 1) \), so that the lattice of relations has rank \( \rho \).

Picking such an integral basis \( \{r_1..r_\rho\} \) of \( R \), we can form a \( \rho \) by \( 3|\Gamma| \) matrix \( \hat{R} \) whose rows are given by the components of \( r_1..r_\rho \). The conditions (54) show that two

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Figure 4. Pictorial description of the F-flatness constraints.
The arrows on opposite sides have the same types.
monomials $x^s, x^{s'}$ of the ambient space $\mathbb{C}^{Q_1}$ will restrict to the same function on $\mathcal{Z}$ if the (nonnegative) integral vectors $s, s' \in \mathbb{Z}^{Q_1}$ are such that $s - s'$ belongs to the sublattice $R$. Therefore, one can describe $\mathcal{Z}$ by considering the collection of those points of the quotient lattice $\mathbb{Z}^{Q_1}/R$ which lie in the image $\Sigma$ of the cone $C$ via the natural projection map $p : \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_1}/R$. The lattice $\mathbb{Z}^{Q_1}/R$ can be identified with the orthogonal complement $M$ of the lattice $R$ in $\mathbb{Z}^{Q_1}$. Clearly $M$ coincides with the set of integral vectors which lie in the kernel of the matrix $\hat{R}$. Therefore, a basis of the lattice $M$ can be obtained by computing a basis of the integral kernel of $\hat{R}$. If $v_1..v_{|\Gamma|+2}$ is such a basis, then one can form a $3|\Gamma|$ by $|\Gamma|+2$ matrix $K$ whose columns are given by the the components of $v_1..v_{|\Gamma|+2}$. Then it is not hard to see that the projection map $p : \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_1}/R$ can be identified with the transpose $K^t$ of the matrix $K$, viewed as a linear map from $\mathbb{Z}^{Q_1}$ to $M$. Therefore, once one has computed the matrix $K$, using its columns as a basis of $M$ identifies $M$ with $\mathbb{Z}^{|\Gamma|+2}$ and the projection $p(C)$ with the cone $\Sigma = K^t(C)$ generated by the columns of $K^t$, i.e. by the rows of $K$. In conclusion, the toric variety $\mathcal{Z}$ is described by the cone of exponents in $\mathbb{Z}^{|\Gamma|+2}$ which is generated by the rows of $K$.

**Example:** $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity

In this case, the number of independent monomial relations is $2(|\Gamma| - 1) = 6$, so the complex dimension of the variety of commuting matrixes is $\dim \mathcal{Z} = |\Gamma| + 2 = 6$. The matrix of an integral basis of the lattice of monomial relations is:

$$
\hat{R} = 
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & 2 & -1 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1
\end{bmatrix}
$$

(57)
and a basis of its integral kernel is given by the columns of:

\[
K = \begin{bmatrix}
1 & 1 & -1 & 1 & -2 & 1 \\
1 & 0 & 0 & 1 & -1 & 0 \\
1 & 1 & -1 & 0 & -1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

(58)

The lattice \(M\) is isomorphic with \(\mathbb{Z}^6\) and the 12 vectors given by the rows of \(K\) generate the cone of exponents \(\Sigma \subset \mathbb{R}^6\) of the toric variety \(Z\).

### 2.6 The action of the projected gauge group on the variety of commuting matrices

Let \(q := |\Gamma| - 1\) denote the rank of the effective gauge group. A monomial \(x^m\) of the ambient space has a \(U(1)^q\) charge given by the \(q\)-vector \(\Delta m\). Since the exponent vectors are identified modulo the lattice \(R\) when imposing the F-flatness constraints, we can describe the restriction of the \(U(1)^q\) action to the variety of commuting matrices by the descent of the map \(\Delta : \mathbb{Z}^{Q_1} \to \mathbb{Z}^q\) to a map \(V : M \to \mathbb{Z}^q\). The fact that \(d\) does indeed factor through the projection \(p : \mathbb{Z}^{Q_1} \to M\) follows from the fact that the relations (56) obviously lie in the kernel of \(\Delta\). Since we identify \(p\) with \(K^t\), it follows that \(V\) can be obtained as the unique matrix satisfying the condition:

\[
VK^t = \Delta
\]

(59)

Note that \(V\) has dimensions \(q \times (|\Gamma| + 2)\).

**Example:** \(\mathbb{Z}_2 \times \mathbb{Z}_2\) singularity
The integral matrix $V$ satisfying $VK^t = \Delta$ is given by:

$$V = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ \end{bmatrix}$$

(60)

2.7 The degenerate holomorphic quotient presentation of the moduli space

Once we have obtained the generators $K$ of the cone of exponents for $Z$ and the restriction $V$ of the $U(1)^q$ action to $Z$, a symplectic quotient presentation of the moduli space can be obtained by the methods of [1]. We will simply summarize the steps of that construction, referring the reader to [1] for details.

Since $Z$ is defined by monomial relations inside of the affine space $\mathbb{C}^{Q_1}$, it will be an affine toric variety (of complex dimension $|\Gamma| + 2$) whose toric generators are the generators of the cone $\Sigma^v$ dual to the cone of exponents $\Sigma$. If $c$ is the number of these vectors and $T$ is the $(|\Gamma| + 2) \times c$ matrix having them as its columns, then one can construct a $(c - |\Gamma| - 2) \times c$ matrix of charges $Q$ whose rows form an integral basis for the kernel of $T$. Note that we do not have an analytic expression for the number $c$ of toric generators, since this number depends in a complicated way on the combinatorial properties of the cone $\Sigma$.

At this stage, we have a toric variety $Z$, presented as a holomorphic quotient $\mathbb{C}^c/(\mathbb{C}^*)^{c-|\Gamma|-2}$, which is further divided by the $(\mathbb{C}^*)^q = (\mathbb{C}^*)^{|\Gamma|-1}$ action given by the charges encoded by the rows of $V$. This double quotient can be reduced to the form $\mathbb{C}^c/(\mathbb{C}^*)^{c-3}$ by choosing a lift of the $(\mathbb{C}^*)^q$ action to the space $\mathbb{C}^c$. Such a lift can be given by first choosing a left inverse $U$ of the matrix $T^t$ (i.e. $UT^t = id$, with $U$ a $(|\Gamma| + 2) \times c$ matrix) and then lifting the $(\mathbb{C}^*)^q$ action to $\mathbb{C}^c$ as the action specified by the charge matrix $VU$. It follows that the moduli space can be described by the holomorphic quotient $\mathbb{C}^c/(\mathbb{C}^*)^{c-3}$ associated to the $(c-3) \times c$ charge matrix $Q_{total}$ obtained by stacking $Q$ and $VU$. In what follows, we will always place the matrix $Q$ above the matrix $VU$ when constructing the matrix $Q_{total}$.

**Example:** $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity

---

\(^7c\) coincides with the number of facets of $\Sigma$. This is the integer mentioned in subsection 1.2, which essentially controls the computational complexity of the problem.
A choice for the matrices $T, U, Q, Q_{\text{total}}$ is:

$$T = \begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}$$

$$Q_{\text{total}} = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & -1 & 0 & -2 & 1 & 1 \\
0 & 1 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1
\end{bmatrix}$$

$$U = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}$$

$$Q = \begin{bmatrix}
1 & 0 & -1 & 1 & -1 & 0 & -2 & 1 & 1 \\
0 & 1 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1
\end{bmatrix}$$

In particular, in this example we have $c = 9$.

### 2.8 The reduction of the holomorphic quotient

To simplify notation in the sequel, we let $r = c - 3$ denote the number of rows of $Q_{\text{total}}$. Taking the transpose of the kernel of $Q_{\text{total}}$ gives a $3 \times c$ matrix $G_{\text{total}}$ such that $Q_{\text{total}}(G_{\text{total}})^t = 0_{r \times 3}$ (note that the columns of $G_{\text{total}}$, which play the role of toric generators, are naturally associated with the columns of $Q_{\text{total}}$). The rows of $Q_{\text{total}}$ form a basis of integral linear relations among these generators. In general, the holomorphic quotient description of the moduli space given by the charge matrix $Q_{\text{total}}$ is degenerate (not minimal) in the sense that the toric generators are not all distinct.
In order to make the following discussion clear, note that one can always reorder the columns of $G_{total}$ such that identical generators appear consecutively (for example, one can sort $G_{total}$ in decreasing lexicographic order on its columns, a convention which we will follow everywhere in this paper). While performing this rearrangement, one must also reorder the columns of $Q_{total}$ accordingly, since each generator is associated to such a column. Hence we let $G_t$ be the matrix obtained from $G_{total}$ by sorting its columns in decreasing lexicographic order, and $Q_t$ be the matrix obtained from $Q_{total}$ by performing the same permutation on its columns.

The columns of $G_t$ will form a number $b$ of blocks $\Gamma^{(1)} ... \Gamma^{(b)}$ (appearing in $G_t$ in this order), such that each block $\Gamma^{(k)}$ consists of $m_k$ copies of the same column $\gamma^{(k)}$, and such that the 3-vectors $\gamma^{(1)} ... \gamma^{(b)}$ are all distinct. Hence each block $\Gamma^{(k)}$ is a matrix of dimensions $3 \times m_k$. In general, some of these will consist of one column only (then $m_k = 1$), while other blocks will be multiple, i.e. will consist of $m_k \geq 2$ repetitions of $\gamma^{(k)}$. We let $n$ be the number of non-multiple blocks and $s$ the number of multiple blocks, so that $n + s = b$. Then the list of all toric generators consists of $m_1$ copies of $\gamma^{(1)}$, $m_2$ copies of $\gamma^{(2)} ... m_b$ copies of $\gamma^{(b)}$. It is convenient to index these as $\gamma_i^{(k)}$ ($i = 1..m_k, k = 1..b$) where $\gamma_1^{(k)} = ... = \gamma_{m_k}^{(k)} = \gamma^{(k)}$ are the $m_k$ copies of $\gamma^{(k)}$ appearing in the block $\Gamma^{(k)}$. We also index the homogeneous coordinates of the holomorphic quotient by $z_i^{(k)}$ ($k = 1..b, i = 1..m_k$) where $z_i^{(k)}$ corresponds to the column $\gamma_i^{(k)}$ of $Q_t$. If we also define $p_i^{(k)} = |z_i^{(k)}|^2 \geq 0$, we can write the moment map equations for the symplectic description of our quotient as:

$$Q_t p = \tilde{\xi},$$

(63)

where:

$$p = \begin{bmatrix} p_1^{(1)} \\ \vdots \\ p_{m_1}^{(1)} \\ \vdots \\ p_1^{(b)} \\ \vdots \\ p_{m_b}^{(b)} \end{bmatrix} = \begin{bmatrix} p^{(1)} \\ \vdots \\ p^{(b)} \end{bmatrix},$$

(64)

with $p^{(k)} = \begin{bmatrix} p_{1}^{(k)} \\ \vdots \\ p_{m_k}^{(k)} \end{bmatrix}$ and where $\tilde{\xi}$ is the $r$-vector whose first $r - q$ components are zero and whose last $q$ components are given by the vector of D-brane Fayet-Iliopoulos parameters $\xi$.

The rows of the matrix $Q_t$ form a basis for the lattice $S$ of integral linear relations among the toric generators. In fact, this is the only piece of information needed to reconstruct the holomorphic quotient, and any basis of the lattice $S$ contains the same data. In particular, one always has the freedom to perform invertible row operations...
on $Q_t$. Such a transformation (which corresponds to a change of integral basis of $S$) has the form:

$$Q_t \rightarrow WQ_t$$

(65)

with $W$ an $r \times r$ integral matrix which is invertible over the integers. The presence of multiplicities in the list of toric generators implies that one can always find a transformation $W$ which brings the matrix $Q_t$ to a form which is particularly suited for discussing the reduction procedure. This ‘canonical’ form can be found as follows. Since each multiple generator $\gamma^{(k)}$ appears $m_k \geq 2$ times, we can always find $m_k - 1$ elements of the lattice $S$ which correspond to the following $m_k - 1$ obvious relations:

$$\gamma_1^{(k)} = \gamma_2^{(k)}$$
$$\gamma_2^{(k)} = \gamma_3^{(k)}$$
$$\ldots$$
$$\gamma_{m_k-1}^{(k)} = \gamma_{m_k}^{(k)}$$

(66)

Certainly not all relations among $\gamma_i^{(k)}$ are of this form, but it is clear that all of the remaining relations must come from whatever linear relations exist among the distinct vectors $\gamma^{(1)} \ldots \gamma^{(b)}$ obtained by eliminating all multiplicities in our list. A basis for the latter type of relations is given by the rows of the reduced charge matrix $Q_{\text{reduced}}$, which is defined as follows. Define the matrix $G_{\text{reduced}}$ to be obtained from $G_{\text{total}}$ by removing all column multiplicities (that is, $G_{\text{reduced}}$ consists of the distinct columns $\gamma^{(1)} \ldots \gamma^{(b)}$ in this order (the inverse lexicographic order, in our conventions). Then $Q_{\text{reduced}}$ is any matrix whose rows give an integral basis for the kernel of $G_{\text{reduced}}$. Note that $Q_{\text{reduced}}$ has $b = n + s$ columns and let $p$ be the number of its rows. We clearly have $p = b - 3$. We conclude that one can always find a basis of linear relations among $\gamma_i^{(k)}$ which consists of all of the ‘equality’ relations (66) together with the relations given by the rows of $Q_{\text{reduced}}$.

Remember that $q = |\Gamma| - 1$ denotes the number of $U(1)$ factors of the projected gauge group on the worldvolume (which coincides with the number of rows of the matrix $V$). The arguments above show that $Q_t$ can always be brought to the form:

$$Q_{\text{can}} = \begin{pmatrix}
V^{(1)} & V^{(2)} & V^{(3)} & V^{(4)} & \ldots & V^{(b)} \\
c(m_1) & 0_{(m_1-1)\times m_2} & 0_{(m_1-1)\times m_3} & 0_{(m_1-1)\times m_4} & \ldots & 0_{(m_1-1)\times m_b} \\
0_{(m_2-1)\times m_1} & c(m_2) & 0_{(m_2-1)\times m_3} & 0_{(m_2-1)\times m_4} & \ldots & 0_{(m_2-1)\times m_b} \\
0_{(m_3-1)\times m_1} & 0_{(m_3-1)\times m_2} & c(m_3) & 0_{(m_3-1)\times m_4} & \ldots & 0_{(m_3-1)\times m_b} \\
0_{(m_4-1)\times m_1} & 0_{(m_4-1)\times m_2} & 0_{(m_4-1)\times m_3} & c(m_4) & \ldots & 0_{(m_4-1)\times m_b} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0_{(m_b-1)\times m_1} & 0_{(m_b-1)\times m_2} & 0_{(m_b-1)\times m_3} & 0_{(m_b-1)\times m_4} & \ldots & c(m_b)
\end{pmatrix}
$$

(67)

That is, the determinant of $W$ must be $\pm 1$.  

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where we defined the canonical \((m - 1) \times m\) block \(c(m)\) to be given by:

\[
c(m) := \begin{bmatrix}
1 & -1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & -1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & -1
\end{bmatrix}
\] (68)

if \(m \geq 1\) and to be a ‘zero by 1 block (i.e. a missing element in a column) if \(m = 1\) (similarly, \(0_{(m-1)\times m}\) is defined to be a missing element in a column if \(m = 1\)). The \(p \times m_k\) blocks \(V^{(k)}\) appearing in \(Q_{\text{can}}\) are defined by:

\[
V^{(k)} = \begin{bmatrix}
v^{(k)}_1 & 0 & 0 & \ldots & 0 \\
v^{(k)}_2 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v^{(k)}_p & 0 & 0 & \ldots & 0
\end{bmatrix}
\] (69)

(no zero columns are present in the case \(m_k = 1\)), where the \(p\)-vector \(v^{(k)} = \begin{bmatrix} v^{(k)}_1 & \ldots & v^{(k)}_p \end{bmatrix}\) is the \(k\)-th column of \(Q_{\text{reduced}}\). Note that the rows of \(Q_{\text{can}}\) are naturally divided into \(s + 1\) blocks, with the first block consisting of the first \(p\) rows and the next \(s\) blocks each consisting of \(m_k - 1\) rows for those \(m_k\) which are different from 1. In particular, we have \(r = p + \sum_{k=1,b} (m_k - 1)\) and \(c = \sum_{k=1,b} m_k\). If we let \(M = \sum_{k=1,b} m_k \geq 2\), then we can write the row and column dimensions of \(Q_{\text{can}}\) as \(r = p + M\) and \(c = b + M\), and since \(r = c - 3\) we deduce that \(p = b - 3\).

The transformation (65) performed in order to bring \(Q_t\) to the form \(Q_{\text{can}}\) brings the moment map equations (63) to the form:

\[
Q_{\text{can}} p = \eta ,
\] (70)

with the vector \(\eta \in \mathbb{R}^c\) given by:

\[
\eta = W_0 \xi ,
\] (71)

where \(W_0\) is the \(r \times q\) matrix obtained by keeping only the last \(q\) columns of \(W\).

Since the rows of \(Q_{\text{can}}\) are naturally divided into blocks, we divide the \(r = p + M\)-vector \(\eta\) accordingly into a \(p\)-vector \(\eta^{(0)}\) and \(s\) \((m_k - 1)\)-vectors \(\eta^{(k)}\) (for those \(k\) associated to multiple blocks, which we will call \(k_1..k_s\)) such that:

\[
\eta = \begin{bmatrix}
\eta^{(0)} \\
\eta^{(k_1)} \\
\vdots \\
\eta^{(k_s)}
\end{bmatrix} .
\] (72)
Then we can write:

\[
\eta^{(0)} = w^{(0)} \xi \\
\eta^{(k)} = w^{(k)} \xi \quad \text{for all } k \text{ associated to multiple blocks},
\]

where \( w^{(0)} \) is the \( p \times q \) matrix formed by the first \( p \) rows of \( W_0 \), and \( w^{(k)} \) (for those \( k \) corresponding to multiple blocks) are the \( (m_k - 1) \times q \) matrices given by the rows of \( W_0 \) associated to the other row blocks of \( Q_{can} \).

In order to reduce the holomorphic quotient, we must eliminate \( m_k - 1 \) homogeneous variables out of the \( m_k \) variables associated with each multiple block \( (m_k \geq 2) \). This is possible provided that the levels \( \eta \) of the moment map are such that the \( m_k - 1 \) variables to be eliminated in each multiple block are assured to be nonzero. Due to the structure of \( Q_{can} \), the various canonical blocks are ‘decoupled’ from each other, and we can discuss reduction within each canonical block separately. Indeed, the moment map equations (70) have the form:

\[
Q_{reduced} \begin{bmatrix} p_1^{(0)} \\ p_1^{(1)} \\ \vdots \\ p_1^{(b)} \end{bmatrix} = \eta^{(0)}
\]

\[
c(m_k) \begin{bmatrix} p_1^{(k)} \\ p_2^{(k)} \\ \vdots \\ p_{m_k}^{(k)} \end{bmatrix} = \eta^{(k)}
\]

(74) (75)

where \( k \) runs over all multiple blocks. Let us consider the equations involving the homogeneous variables \( z_1^{(k)} \ldots z_{m_k}^{(k)} \) associated with the multiple block \( k \). The equations \( c(m_k)p^{(k)} = \eta^{(k)} \) can be solved in terms of \( p_1^{(k)} \) and \( \eta^{(k)} \):

\[
p_1^{(k)} = p_1^{(k)}
\]

\[
p_2^{(k)} = p_1^{(k)} - \eta_1^{(k)}
\]

\[
p_3^{(k)} = p_1^{(k)} - \eta_1^{(k)} - \eta_2^{(k)}
\]

\[\ldots\]

\[
p_i^{(k)} = p_1^{(k)} - \eta_1^{(k)} - \ldots - \eta_{j-1}^{(k)}
\]

\[\ldots\]

\[
p_{m_k}^{(k)} = p_1^{(k)} - \eta_1^{(k)} - \ldots - \eta_{m_k-1}^{(k)}.
\]

The values of \( p_1^{(k)} \) are constrained by the conditions \( p_1^{(k)} \geq 0, p_2^{(k)} \geq 0, \ldots, p_{m_k}^{(k)} \geq 0 \), which are equivalent to:

\[
p_1^{(k)} \geq \max(0, \eta_1^{(k)}, \eta_1^{(k)} + \eta_2^{(k)}, \ldots, \eta_1^{(k)} + \ldots + \eta_{m_k-1}^{(k)}).
\]

(77)
If the maximum in the right hand side is attained precisely at \( \eta_1^{(k)} + \ldots + \eta_{i-1}^{(k)} \) (and at no other point), i.e. if the strict inequalities:

\[
\begin{align*}
\eta_1^{(k)} + \ldots + \eta_{i-1}^{(k)} &> 0, \\
\eta_1^{(k)} + \ldots + \eta_{i-1}^{(k)} &> \eta_1^{(k)}, \\
\eta_1^{(k)} + \ldots + \eta_{i-1}^{(k)} &> \eta_1^{(k)} + \eta_2^{(k)} \\
\ldots \\
\eta_1^{(k)} + \ldots + \eta_{i-1}^{(k)} &> \eta_1^{(k)} + \ldots + \eta_{i-2}^{(k)} \\
\eta_1^{(k)} + \ldots + \eta_{i-1}^{(k)} &> \eta_1^{(k)} + \ldots + \eta_{i-1}^{(k)}
\end{align*}
\]

(78) hold, then equation (77) assures that \( p_j^{(k)} = p_1^{(k)} - \sum_{l=1, j-l \neq i} \eta_l^{(k)} > 0 \) for all \( j \neq i \), so that we can eliminate all homogeneous variables associated to the canonical block except for that associated to its \( i \)-th column. (If the maximum is attained precisely at 0, then \( p_j^{(k)} > 0 \) for all \( j \neq 1 \) and we can eliminate all variables except for that associated to the first column. This case corresponds to \( i = 1 \)). The inequalities (78) (taken for all values of \( i \) in turn) divide the space \( \mathbb{R}^{m_k-1}(\eta^{(k)}) \) of values of \( \eta^{(k)} \) into \( m_k \) distinct chambers \( \sigma_i^{(k)} \) \( (i = 1 \ldots m_k) \), which are maximal-dimensional polyhedral cones in \( \mathbb{R}^{m_k-1}(\eta^{(k)}) \). If \( \eta^{(k)} \) belongs to the interior of \( \sigma_i^{(k)} \), then we can eliminate all of the homogeneous variables \( z_j^{(k)} \) except for the variable \( z_i^{(k)} \). In order to perform this reduction, we must first eliminate the variables \( z_j^{(k)} \) \( (j \neq i) \) from the first equations of (74). This can be done as follows. Since the the rows of \( c(m_k) \) are linearly independent, there exists a unique \( p \times (m_k - 1) \) matrix \( F_i(m_k) \) such that:

\[
F_i(m_k)c(m_k) = \begin{bmatrix}
-v_1^{(k)} & 0 & \ldots & 0 & v_1^{(k)} & 0 & \ldots & 0 \\
v_2^{(k)} & 0 & \ldots & 0 & v_2^{(k)} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-v_p^{(k)} & 0 & \ldots & 0 & v_p^{(k)} & 0 & \ldots & 0
\end{bmatrix}
\]

(79) (for \( i = 1 \), the matrix on the right hand side is defined to be the null \( p \times (m_k - 1) \) matrix). In fact, it is not hard to see that \( F_i(m_k) \) is given by:

\[
F_i(m_k) = \begin{bmatrix}
-v_1^{(k)} & -v_1^{(k)} & \ldots & -v_1^{(k)} & 0 & 0 & \ldots & 0 \\
-v_2^{(k)} & -v_2^{(k)} & \ldots & -v_2^{(k)} & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-v_p^{(k)} & -v_p^{(k)} & \ldots & -v_p^{(k)} & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

(80)

where the first \( i - 1 \) columns are copies of the vector \( v^{(k)} \) and the other columns are zero (for \( i = 1 \), \( F_1(m_k) \) is defined to be the null \( p \times (m_k - 1) \) matrix). Multiplying
\( c(m_k) \) to the left by \( F_i(m_k) \) and adding the result to \( V^{(k)} \) produces the matrix:

\[
\begin{bmatrix}
0 & 0 & \ldots & 0 & v_i^{(k)} & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & v_i^{(k)} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & v_i^{(k)} & 0 & \ldots & 0
\end{bmatrix}
\]

\( \text{(81)} \)

Hence performing this invertible row operation allows us to bring \( Q_{\text{can}} \) to a form in which all entries associated to the homogeneous variables \( z_j^{(k)} \) \( (j \neq i) \) are zero except for those appearing in the canonical block \( c(m_k) \). Once \( Q_{\text{can}} \) has been brought to this form, and once we know that \( \eta^{(k)} \) lies in the interior of the cone \( \sigma_i^{(k)} \) (so that \( z_j^{(k)} \neq 0 \) for all \( j \neq i \)), then we can eliminate these variables by using the \( (\mathbb{C}^*)^{m_k-1} \) subgroup associated with the rows of \( c(m_k) \) in order to set each of them equal to \( 0 \). Since these variables have been eliminated from the rest of the charge matrix, this will not have any effect on the remaining part of the holomorphic quotient. Note, however, that performing the row operations \( V^{(k)} \to V^{(k)} + F_i(m_k)c(m_k) \) (which is needed in order to eliminate our variables from the first \( p \) rows of \( Q_{\text{can}} \)) will induce a redefinition of \( \eta^{(0)} \) given by:

\[
\eta^{(0)} \to \eta^{(0)} + F_i(m_k)\eta^{(k)} .
\]

\( \text{(82)} \)

Applying the above discussion to each of the \( s \) multiple blocks leads to the following pattern of reduction. For each multiple block \( k \) \( (m_k \geq 2) \) we have a partition of the space \( \mathbb{R}^{m_k-1}(\eta^{(k)}) \) into \( m_k \) chambers \( \sigma_i^{(k)} \) \( (i = 1..m_k) \) which adjoin along common faces. If, for each \( k \) with \( m_k \geq 2 \), \( \eta^{(k)} \) lies in the interior of one of these chambers \( \sigma_i^{(k)} \), then we can reduce each of the multiple blocks \( k \) to its \( i_k \)-th column. This is achieved by performing the following invertible row operation on \( Q_{\text{can}} \):

\[
Q_{\text{can}} \to Q'_{\text{can}} = \begin{bmatrix}
F_{i_1}(m_1) & 0 & \ldots & 0 \\
0 & F_{i_2}(m_2) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & F_{i_s}(m_s)
\end{bmatrix} Q_{\text{can}} ,
\]

\( \text{(83)} \)

which replaces the first \( p \) components \( \eta^{(0)} \) of \( \eta \) with the \( p \)-vector given by:

\[
\zeta = \zeta_i = \eta^{(0)} + \sum_{k = 1..b \atop m_k \neq 1} F_{i_k}(m_k)\eta^{(k)} .
\]

\( \text{(84)} \)

Performing the reduction of \( z_j^{(k)} \) \( (j \neq i_k) \) eliminates all except for the first \( p \) rows of \( Q'_{\text{can}} \) and all of its columns except for the non-multiple columns and those containing the \( i_k \)-th column of each multiple block. The result is a toric variety based on the

\(^{9}\text{Here we are tacitly using the well-known equivalence between the symplectic and holomorphic quotient.}\)
Iliopoulos parameters \( \xi \) and \( \phi \).

The crucial equations (83) can be described with the help of a piecewise-linear function \( \pi : \mathbb{R}^r \to \mathbb{R}^p \) which we define as follows. For each set of \( s \) indices \( i_1 = 1..m_{k_1}..i_s = 1..m_{k_s} \) (remember that \( k_1..k_s \in \{1..b\} \) index the multiple blocks), we let \( \Sigma_{i_1..i_s} \) be the cone (or ‘wedge’) in \( \mathbb{R}^r \) defined by:

\[
\Sigma_{i_1..i_s} = \mathbb{R}^P \times \sigma^{(1)}_{i_1} \times \ldots \times \sigma^{(s)}_{i_s}.
\]

The collection of these cones (which has \( \Pi \ k = 1..b \ m_k = \Pi_{k=1..b} m_k \) elements) divides the space \( \mathbb{R}^r(\eta) \) into chambers which adjoin along common walls. If \( \eta \) belongs to the chamber \( \Sigma_{i_1..i_s} \), then the value of \( \pi \) at \( \eta \) is given by the linear expression:

\[
\pi(\eta) = \eta(0) + \sum_{k = 1..b} \sum_{m_k \neq 1} F_{i_k}(m_k)\eta^{(k)}
\]

which appears in the right hand side of (84). It is obvious that these expressions agree on the walls, so that \( \pi \) is a continuous piecewise linear function. In fact, one can give an analytic expression for \( \pi \), if one notices that:

\[
F_{i_k}(m_k)\eta^{(k)} = -\left( \sum_{j=1..i_k-1} \eta_{ij}^{(k)}\right) v^{(k)},
\]

where the sum is defined to be zero if \( i_k = 1 \). Since for \( \eta \in \Sigma_{i_1..i_s} \), we have \( \eta^{(k)} \in \sigma_i^{(k)} \), and since by the definition of the cones \( \sigma_i^{(k)} \) this implies that:

\[
\sum_{j=1..i_k-1} \eta_{ij}^{(k)} = \max(0, \eta_1^{(k)}, \eta_2^{(k)}, \ldots, \eta_1^{(k)} + \ldots + \eta_{m(k)-1}^{(k)}),
\]

it follows that for any \( \eta \in \mathbb{R}^r \), the value of \( \pi \) at \( \eta \) is given by:

\[
\pi(\eta) = \eta(0) - \sum_{k = 1..b} \max(0, \eta_1^{(k)}, \eta_2^{(k)}, \ldots, \eta_1^{(k)} + \ldots + \eta_{m(k)-1}^{(k)}) v^{(k)}.
\]

The important information for us is the map \( \phi : \mathbb{R}^Q \to \mathbb{R}^p \) from the D-brane Fayet-Iliopoulos parameters \( \xi \) to the effective moment map levels \( \zeta \) in the reduced toric presentation of the moduli space. This immediately follows from the above and from (87):

\[
\phi(\xi) = w^{(0)} - \sum_{k = 1..b} \max(0, \sum_{j=1..q} w_{1j}^{(k)} \xi_j, \sum_{j=1..q} \left( w_{1j}^{(k)} + w_{2j}^{(k)} \right) \xi_j, \ldots, \sum_{j=1..q} \left( w_{1j}^{(k)} + \ldots + w_{m(k)-1}^{(k)} \right) \xi_j) v^{(k)}.
\]
In conclusion, the relation between the D-brane Fayet-Iliopoulos parameters and the effective moment map levels is given by the piecewise-linear function $\phi$.

Since $\phi$ is clearly continuous, its linear regions form a subdivision of $\mathbb{R}^q(\xi)$ into chambers which are polyhedral cones. These chambers can be found from (78) as follows. For each $k$ associated to a multiple block, consider the vectors $e_1(m_k)\ldots e_m(m_k)$ in $\mathbb{R}^{m_k-1}(\eta^{(k)})$ given by:

$$e_j(m_k) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (91)$$

where the first $j - 1$ entries are equal to 1 (for $j = 1$, we define $e_1(m_k)$ to be the null $(m_k - 1)$-vector). Then the equations (78) (for a fixed $i \in \{1..m_k\}$) can be rewritten as follows:

$$\langle e_i(m_k) - e_j(m_k), \eta^{(k)} \rangle > 0 \text{ for all } j = 1..m_k, j \neq i. \quad (92)$$

Using $\eta^{(k)} = w^{(k)}\xi$ reduces these equations to:

$$\langle f_i^{(k)} - f_j^{(k)}, \xi \rangle > 0 \text{ for all } j = 1..m_k, j \neq i, \quad (93)$$

where the $q$-vectors $f_i^{(k)}$ are given by:

$$f_i^{(k)} = [w^{(k)}]^t e_i(m_k). \quad (94)$$

It is now clear that $\phi(\xi)$ will belong to $\Sigma_{i_1..i_s}$ if and only if $\xi$ belongs to the cone $\Xi_{i_1..i_s} = \Xi_{i_1}^{(k_1)} \cap \ldots \cap \Xi_{i_s}^{(k_s)}$, where $\Xi_{i}^{(k)}$ is the cone in $\mathbb{R}^q(\xi)$ defined by the $m_k - 1$ inequalities (92). In general, the set of all cones $\Xi_{i_1..i_s}$ will be a refinement of the true linear chamber structure of $\phi$, since the action of the matrices $[w^{(k)}]^t$ may ‘collapse’ or identify some of the cones $\Xi_{i}^{(k)}$.

### 2.9 Determining the preimage of an effective wall

For the purposes of the present paper, an important question is the following. Given a convex subset $H$ of the space of effective moment map levels $\mathbb{R}^p(\zeta)$, what is its preimage via the map $\phi$? In particular, is this preimage nonzero? That is, are the values of $\zeta$ associated to $H$ indeed realized by the D-brane theory?.

---

10More precisely, these regions form an integral polyhedral fan in $\mathbb{R}^q(\xi)$. 
Once the map $\phi$ has been identified, this question can be answered as follows. For simplicity of notation, consider only the case when $H$ is determined by a set of linear equations:

$$\langle a(t), \zeta \rangle = 0 \quad (t = 1..g).$$

(95)

Then $\phi(\xi)$ belongs to $H$ if and only if $\langle a(t), \phi(\xi) \rangle = 0$ for all $t$. In order to solve these conditions, one can simply look for the solution in each of the cones $\Xi_{i_1..i_s}$, where $\phi$ is given by a linear expression. Fixing such a cone, it is not hard to see that the set $\phi^{-1}(H) \cap \Xi_{i_1..i_s}$ consists of all values of $\xi$ which satisfy the inequalities (93) together with the equalities:

$$\langle c(t)_{i_1..i_s}, \xi \rangle = 0 \quad , \quad (t = 1..g)$$

(96)

where the components $\alpha = 1..q$ of the $q$-vector $c(t)_{i_1..i_s}$ are given by:

$$c(t)_{i_1..i_s}^\alpha = \sum_{\beta=1..p} w_{j_0}^{(0)} a(t)^{\beta} - \sum_{k=1..b} \sum_{j=1..i_k-1} \sum_{m_k \neq 1} w_{j_0}^{(k)} \langle v^{(k)}, a(t) \rangle.$$  

(97)

For each $i_1..i_k$, this system gives a subcone of the cone $\Xi_{i_1..i_s}$ (the cut of this cone with the subspace (96)). Running over all such cuts allows for a complete solution of the problem, although the resulting presentation of the solution need not be the most economical one.

**Example:** $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity

In this case, a basis for the kernel of the matrix $Q_{\text{total}}$ of (61) is given by the rows of:

$$G_{\text{total}} = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 1
\end{bmatrix}.$$  

(98)

Sorting the columns of $G_{\text{total}}$ in decreasing lexicographic order gives the matrix:

$$G_t = \begin{bmatrix}
2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2
\end{bmatrix},$$  

(99)

In general, $H$ is given by a set of linear equalities and inequalities; the exposition can be generalized immediately to this situation.
while doing the same permutation on the columns of $Q_{total}$ gives:

$$Q_t = \begin{bmatrix}
-2 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & 0 \\
-1 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & 0 \\
1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\
\end{bmatrix}. \quad (100)$$

The matrix $G_t$ has only 6 distinct columns, appearing with multiplicities:

$$m_1 = 1 \quad m_2 = 2 \quad m_3 = 2 \quad m_4 = 1 \quad m_5 = 2 \quad m_6 = 1 \quad (101)$$

Thus $G_t$ is formed of 4 blocks of columns given as follows:

$$\Gamma^{(1)} = \begin{bmatrix}
2 \\
-1 \\
0
\end{bmatrix} \quad \Gamma^{(2)} = \begin{bmatrix}
1 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix} \quad \Gamma^{(3)} = \begin{bmatrix}
1 & 1 \\
-1 & -1 \\
1 & 1
\end{bmatrix} \quad \Gamma^{(4)} = \begin{bmatrix}
0 \\
0 \\
1 & 1
\end{bmatrix} \quad \Gamma^{(5)} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 1
\end{bmatrix} \quad \Gamma^{(6)} = \begin{bmatrix}
0 \\
0 \\
-1 & 2
\end{bmatrix}. \quad (102)$$

The multiple blocks are $\Gamma^{(2)}$, $\Gamma^{(3)}$ and $\Gamma^{(5)}$. Keeping only one copy of each distinct column (without changing the decreasing lexicographic order) gives the matrix:

$$G_{red} = \begin{bmatrix}
2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}. \quad (103)$$

The columns of $G_{red}$ generate the cone over the following two-dimensional lattice polytope which lies in the hyperplane $z = 1$ of $\mathbb{R}^3$: 34
Figure 5. The polytope associated to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient singularity.

A basis of linear relations between the columns of $G_{\text{red}}$ is given by the rows of the reduced charge matrix:

\[
Q_{\text{red}} = \begin{bmatrix}
1 & 0 & -2 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & -2 & 1
\end{bmatrix},
\tag{104}
\]

whose number of rows is $p = 3$. The canonical form of $Q_{\text{total}}$ is:

\[
Q_{\text{can}} = \begin{bmatrix}
1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0
\end{bmatrix},
\tag{105}
\]

and it involves three copies of the canonical block:

\[
c(2) = \begin{bmatrix}
1 \\
-1
\end{bmatrix}.
\tag{106}
\]
The transition matrix $W$ satisfying $WQ_{\text{total}} = Q_{\text{can}}$ is given by:

$$W = \begin{bmatrix}
1 & -2 & 1 & -1 & -1 & 0 \\
1 & -1 & 1 & -1 & -1 & -1 \\
0 & 1 & 1 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & -1 \\
-1 & 2 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}$$

and has determinant $+1$. Its last $q = |\Gamma| - 1 = 3$ columns give the matrix:

$$W_0 = \begin{bmatrix}
-1 & -1 & 0 \\
-1 & -1 & -1 \\
-1 & 0 & -1 \\
0 & -1 & -1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}$$

which is formed of 4 row blocks:

$$w^{(0)} = \begin{bmatrix}
-1 & -1 & 0 \\
-1 & -1 & -1 \\
-1 & 0 & -1
\end{bmatrix}$$

(associated to the first $p = 3$ rows),

$$w^{(2)} = \begin{bmatrix}
0 & -1 & -1
\end{bmatrix}$$

(associated to the next $m_2 - 1 = 1$ rows),

$$w^{(3)} = \begin{bmatrix}
1 & 1 & 0
\end{bmatrix}$$

(associated to the next $m_3 - 1 = 1$ rows),

$$w^{(5)} = \begin{bmatrix}
1 & 0 & 1
\end{bmatrix}$$

(associated to the next $m_5 - 1 = 1$ rows). The piecewise-linear function $\phi$ is given by:

$$\phi(\xi) = \begin{bmatrix}
-\xi_1 - \xi_2 \\
-\xi_1 - \xi_2 - \xi_3 \\
-\xi_1 - \xi_3
\end{bmatrix} - \max[0, -\xi_2 - \xi_3] \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} - \max[0, \xi_1 + \xi_2] \begin{bmatrix}
-2 \\
-1 \\
0
\end{bmatrix} - \max[0, \xi_1 + \xi_3] \begin{bmatrix}
0 \\
-1 \\
-2
\end{bmatrix},$$

(113)
where we used the vectors:

\[
v^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v^{(3)} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \quad v^{(5)} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix},
\]

(114)
given by the second, third and fifths columns of \(Q_{\text{reduced}}\). This function has 8 linear pieces, which are the octants in \(\mathbb{R}^3(\xi)\) determined by the vectors (see Figure 6):

\[
g_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.
\]

(115)

![Figure 6. The linear regions of \(\phi\).](image)

The images of these vectors and their opposites under the map \(\phi\) are:

\[
f_1 = \phi(\pm g_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad f_2 = \phi(\pm g_2) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad f_3 = \phi(\pm g_3) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},
\]

(116)

so that all of the 8 linear chambers of \(\phi\) in \(\mathbb{R}^3(\xi)\) are mapped onto the cone in \(\mathbb{R}^3(\zeta)\) generated by \(f_1, f_2\) and \(f_3\) (these vectors are drawn in Figure 7).
Figure 7. The image of the linear regions of $\phi$.

The conifold singularity can be realized in the region $\zeta_2 = 0, \zeta_1 > 0, \zeta_3 > 0$ of $\mathbb{R}^3(\zeta)$, which defines a two-dimensional cone in the space of effective moment map levels (this particular realization corresponds to eliminating the points 1 and 4 in Figure 5). The preimage of this cone via the map $\phi$ is the union of the boundaries of the opposite cones $\Xi_1 = \langle h_1, h_2, h_3, h_4 \rangle_+$ and $\Xi_2 = -\Xi_1 = \langle -h_1, -h_2, -h_3, -h_4 \rangle_+$ in $\mathbb{R}^3(\xi)$, given by the generators:

$$
\begin{align*}
    h_1 &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \\
    h_2 &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \\
    h_3 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\
    h_4 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\end{align*}
$$

This region in $\mathbb{R}^3(\xi)$ is shown in Figure 8. The conifold transition is realized in the D-brane theory when we vary $\xi$ such as to cross this boundary.
3 The case $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$

In this section, we apply the algorithm discussed above to the example of interest in this paper – the worldvolume realization of partial resolutions of the $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient singularity.

The group $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ has $d = 2$ torsion indices $t_1 = 3$, $t_2 = 3$. We consider its action on $\mathbb{C}^3$ given by the weights $w_1 = (1,1)$, $w_2 = (2,0)$, $w_3 = (0,3)$ (which satisfy $w_1 + w_2 + w_3 = (0,0)$ in our group).

In this case, we have $|\Gamma| = 9$ so the quiver will have 9 nodes. The edges $v \to v - w_i$ can be obtained as above. Indexing the group elements as follows:

$$(0,0) \leftrightarrow 1$$

$$(0,1) \leftrightarrow 4$$

$$(0,2) \leftrightarrow 7$$

Figure 8. The realization of the conifold region in the space of D-brane Fayet-Iliopoulos parameters.
we obtain the quiver drawn below:

Figure 9. The quiver describing the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold theory.

With the above enumeration of the group elements, one obtains the $9 \times 27$ incidence matrix $d$ and the $8 \times 27$ matrix $\Delta$ given in the appendix (the first row of $d$ corresponds to the neutral element of the group with our choice of enumeration).

The number of independent monomial relations is $2(|\Gamma| - 1) = 16$, so the complex dimension of the variety of commuting matrixes is $\dim \mathcal{Z} = |\Gamma| + 2 = 11$. The matrix
of an integral basis of the lattice of monomial relations is:

\[
R = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[(118)\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[(119)\]

\[
K = \begin{bmatrix}
1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 & 2 & -1 & 1 & 0 & -1 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\
1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[(120)\]

The lattice \( M \) is isomorphic with \( \mathbb{Z}^{11} \) and the 16 vectors given by the rows of \( K \) generate the cone of exponents \( \Sigma \subset \mathbb{R}^{11} \) of the toric variety \( Z \).

The integral matrix \( V \) satisfying \( VK^T = \Delta \) is given by:

\[
V = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[(121)\]

A choice for the matrices \( T, U, Q, Q_{total} \) results in the manner explained above. The types of these matrices are:

\[
T : 11 \times 42 \quad U : 11 \times 42 \quad Q : 31 \times 42 \quad Q_{total} : 39 \times 42
\]
The matrix $G_t$ has only 6 distinct columns, appearing with multiplicities:

\[
\begin{align*}
m_1 &= 1 & m_2 &= 3 & m_3 &= 3 & m_4 &= 3 & m_5 &= 21 \\
m_6 &= 3 & m_7 &= 1 & m_8 &= 3 & m_9 &= 3 & m_{10} &= 1 .
\end{align*}
\]

Thus $G_t$ has 7 multiple blocks and 3 non-multiple columns. Keeping only one copy of each distinct column (without changing the decreasing lexicographic order) gives the matrix:

\[
G_{red} = \begin{bmatrix}
3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 1 & 0 & -1 & 2 & 1 & 0 & -1 \\
-1 & -1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 2
\end{bmatrix} .
\]

This allows us to present the moduli space as a 3-dimensional toric variety with 10 toric generators and matrix of charges:

\[
Q_{red} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 2 \\
0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -2 & 2 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1
\end{bmatrix} ,
\]

which has $p = 7$ rows. The canonical form of $Q_{total}$, as well as the transition matrix $W$ (which has determinant $-1$) are listed in the appendix. The the $39 \times 8$ matrix $W_0$ given by the last 8 columns of $W$ can now be used to determine the piecewise-linear function $\phi$, which we display in the appendix. Our procedure above gives 15309 cones $\Xi$ in $\mathbb{R}^8(\xi)$, which form a refinement of the fan associated to $\phi$. By intersecting them with the preimages of the regions $H$ of $\mathbb{R}^7(\zeta)$ listed in Table 1, we find:

(a) 936 maximal-dimensional cones in $\phi^{-1}(H_{F_0})$ leading to the complex cone over $F_0$

(b) 864 maximal-dimensional cones in $\phi^{-1}(H_{dP_1})$ leading to the complex cone over $dP_1$

(c) 1152 maximal-dimensional cones in $\phi^{-1}(H_{dP_2})$ leading to the complex cone over $dP_2$

(d) 1602 maximal-dimensional cones in $\phi^{-1}(H_{dP_3})$ leading to the complex cone over $dP_3$

We consider each case in turn:

(a) A maximal-dimensional cone in $\phi^{-1}(H_{F_0})$ is generated by the columns of the
matrix:
\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & -1 & -1 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 \\
\end{bmatrix}
\]

(125)

and maps to the cone in $\mathbb{R}^7(\zeta)$ generated by the columns of:
\[
\begin{bmatrix}
2 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 2 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 2 & 1 & 2 & 0 \\
0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

(126)

(b) One maximal-dimensional cone in this class is generated by the columns of the matrix:
\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 \\
\end{bmatrix}
\]

(127)

and maps to the cone in $\mathbb{R}^7(\zeta)$ generated by the columns of:
\[
\begin{bmatrix}
0 & 4 & 2 & 0 & 1 \\
0 & 3 & 2 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
1 & 2 & 2 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
2 & 1 & 2 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

(128)

(c) One maximal-dimensional cone in this class is generated by the columns of the
matrix:
\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 \\
-1 & -1 & 0 & 0
\end{bmatrix}
\]

and maps to the cone in $\mathbb{R}^7(\zeta)$ generated by the columns of:
\[
\begin{bmatrix}
2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

(d) One maximal-dimensional cone in this class is generated by the columns of the matrix:
\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 \\
-1 & -1 & 0
\end{bmatrix}
\]

and maps to the cone in $\mathbb{R}^7(\zeta)$ generated by the columns of:
\[
\begin{bmatrix}
2 & 1 & 0 \\
2 & 0 & 0 \\
1 & 0 & 0 \\
2 & 0 & 0 \\
1 & 0 & 0 \\
2 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

4 Conclusions

We considered the status of the AdS/CFT conjecture for nontrivial horizons built as $U(1)$ bundles over toric del Pezzo surfaces. By explicit computation, we discovered
that all such geometries (including those cases for which the existence of an Einstein-Sasaki structure is problematic) can in fact be realized in the moduli space of D3-branes transverse to a Calabi-Yau quotient singularity. By investigating the classical moduli spaces of the associated worldvolume theories, we discovered a highly intricate situation, whose complexity is markedly greater than in cases considered before. This required the development of a systematic approach to the problem, thus improving on the methods presented in [1].

In this paper, we confined ourselves to geometric aspects and to classical properties of the moduli space of the associated field theories. The quantum-mechanical aspects of these theories and especially of their conformal limits are currently under investigation [2].
Various matrices relevant for the case $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$:

$$d = \left[ \begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\Delta = \left[ \begin{array}{ccccccccc} 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$T = \left[ \begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
\[
\phi(\xi) = \begin{bmatrix}
\xi_2 - \xi_5 + \xi_6 - \xi_9 - 3 M_1 \\
-\xi_3 + \xi_5 - \xi_6 - \xi_7 - \xi_9 - 2 M_2 + 2 M_3 + M_4 \\
\xi_2 + \xi_3 + \xi_7 - M_5 + 2 M_6 \\
\xi_2 + \xi_5 + \xi_6 - \xi_9 - M_7 + M_8 + 2 M_9 \\
-\xi_7 - \xi_9 - M_{10} + M_{11} + M_{12} \\
\xi_4 + \xi_5 + \xi_6 - \xi_8 - \xi_9 - 3 M_{13} \\
\xi_4 + \xi_5 + \xi_6 - M_{14} + 2 M_{15}
\end{bmatrix}
\]

where

\[M_1 = \max(0, \xi_3 + \xi_6 + \xi_9, -\xi_2 - \xi_5 - \xi_9)\]
\[M_2 = \max(-\xi_2 - \xi_6 - \xi_7 - \xi_9 - \xi_8 - \xi_9, 0, -\xi_2 - \xi_6 - \xi_7)\]
\[M_3 = \max(0, \xi_3 + \xi_6 + \xi_9, -\xi_2 - \xi_5 - \xi_9)\]
\[M_4 = \max(0, \xi_7 + \xi_8 + \xi_9 - \xi_4 - \xi_5 - \xi_9)\]
\[M_5 = \max(0, \xi_3 + \xi_6 + \xi_9, \xi_2 + \xi_3 + \xi_6 + \xi_8 + \xi_9)\]
\[M_6 = \max(0, \xi_3 + \xi_6 + \xi_9, -\xi_2 - \xi_5 - \xi_9)\]
\[M_7 = \max(0, \xi_2 + \xi_6 + \xi_7, \xi_2 + \xi_3 + \xi_4 + \xi_6 + \xi_7 + \xi_9)\]
\[M_8 = \max(0, \xi_3 + \xi_6 + \xi_9, -\xi_2 - \xi_5 - \xi_9)\]
\[M_9 = \max(0, \xi_7 + \xi_8 + \xi_9, -\xi_2 - \xi_5 - \xi_9)\]
\[M_{10} = \max(\xi_3, 0, -\xi_2 - \xi_4 - \xi_5 - \xi_7 - \xi_9, -\xi_4 - \xi_5 - \xi_7 - \xi_9, -\xi_4 - \xi_5 - \xi_7 - \xi_9 - \xi_8, -\xi_2 - \xi_5 - \xi_7 - \xi_9, -\xi_5 - \xi_7 - \xi_9, -\xi_2 - \xi_5 - \xi_7 - \xi_9, -\xi_4 - \xi_5 - \xi_7 - \xi_9 + \xi_3 + \xi_6 + \xi_8 + \xi_9, -\xi_2 - \xi_4, \xi_5 - \xi_6 - \xi_7 - \xi_9)\]
\[M_{11} = \max(0, \xi_3 + \xi_6 + \xi_9, -\xi_2 - \xi_5 - \xi_9)\]
\[M_{12} = \max(0, \xi_7 + \xi_8 + \xi_9, -\xi_4 - \xi_5 - \xi_9)\]
\[M_{13} = \max(0, \xi_7 + \xi_8 + \xi_9, -\xi_4 - \xi_5 - \xi_9)\]
\[M_{14} = \max(0, \xi_7 + \xi_8 + \xi_9, \xi_4 + \xi_5 + \xi_6 + \xi_7 + \xi_8 + \xi_9)\]
\[M_{15} = \max(0, \xi_7 + \xi_8 + \xi_9, -\xi_4 - \xi_5 - \xi_9)\]
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