Interlacing of positive real zeros of Bessel functions

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Abstract
We unify the three distinct inequality sequences (Abramowitz ans Stegun (1972) [1,9.5.2]) of positive real zeros of Bessel functions into a single one.

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1 Introduction and results

Consider the Bessel differential equation

$$z^2 \frac{d^2 C_\nu(z)}{dz^2} + z \frac{dC_\nu(z)}{dz} + (z^2 - \nu^2)C_\nu(z) = 0.$$  (1)

The general solution is $C_\nu(z) = J_\nu(z) \cos \alpha - Y_\nu(z) \sin \alpha$, a linear combination of $J_\nu(z)$ and $Y_\nu(z)$ being the Bessel functions of the first and second kind (defined e.g. in Ref. [1]).

When $\nu$ is real, the Bessel functions $J_\nu(x)$, $Y_\nu(x)$ each have an infinite number of positive real zeros, all of which are simple with the possible exception of $x = 0$ [1] (see also [3], Bessel-Lommel and Rolle’s theorems). For non-negative order $\nu$ the $s$th positive real zeros of these functions are denoted by $j_{\nu,s}$, $y_{\nu,s}$, $j'_{\nu,s}$, $y'_{\nu,s}$, respectively, except that $x = 0$ is counted as the first zero of $J'_0(x)$. Since $J'_0(x) = -J_1(x)$, it follows that $j'_{0,1} = 0$, $j'_{0,s} = j_{1,s-1}$, $(s = 2, 3, ...)$ [1].

The following results are widely known [1] [3] [4] about the zeros of the Bessel functions.

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Theorem 1.1. The positive real zeros of the functions $J_\nu(x)$, $J_{\nu+1}(x)$, $Y_\nu(x)$, $Y_{\nu+1}(x)$, $J'_\nu(x)$ and $Y'_\nu(x)$ interlace according to the three distinct inequalities:

\[ j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \ldots \]  
(2)

\[ y_{\nu,1} < y_{\nu+1,1} < y_{\nu,2} < y_{\nu+1,2} < y_{\nu,3} < \ldots \]  
(3)

\[ \nu < j'_\nu < y_{\nu,1} < j_{\nu,1} < j'_\nu < y_{\nu,2} < j_{\nu,2} < j'_\nu < j_{\nu,3} < \ldots \]  
(4)

Furthermore, the positive real zeros of $J'_\nu(x)$, $J'_{\nu+1}(x)$ and that of $Y'_\nu(x)$, $Y'_{\nu+1}(x)$ are interlaced:

\[ j'_{\nu,1} < j'_{\nu+1,1} < j'_{\nu,2} < j'_{\nu+1,2} < j'_{\nu,3} < \ldots \]  
(5)

\[ y'_{\nu,1} < y'_{\nu+1,1} < y'_{\nu,2} < y'_{\nu+1,2} < y'_{\nu,3} < \ldots \]  
(6)

Note that Eqs. (4)-(6) were established only recently in Ref. [4] where a particular inverse scattering problem was studied.

In addition to Theorem 1.1 the following auxiliary relations can be found.

Proposition 1.2. For non-negative orders, i.e. $\nu \geq 0$

\[ j_{\nu+1,s} < j'_{\nu,s+1}, \quad s = 1, 2, \ldots \]  
(7)

\[ y_{\nu+1,s} < y'_{\nu,s}, \quad s = 1, 2, \ldots \]  
(8)

Eq. (7) emerged previously when studying a particular inverse scattering problem [2] and also in Ref. [5] independently of the authors.

Now, with the aid of Proposition 1.2 it is possible to unify Eqs. (2), (3) and (4) into a single one. In addition we obtain a simple proof of Eqs. (5) and (6).

We shall formulate our main result in a slightly generalized way, including also an interesting breaking condition.

Theorem 1.3 (Interlacing of positive real zeros of the Bessel functions). The positive real zeros of the Bessel functions $J_\nu(x)$, $J'_\nu(x)$, $Y_\nu(x)$, $Y'_\nu(x)$, $J_{\nu+\varepsilon}(x)$, $Y_{\nu+\varepsilon}(x)$, $0 < \varepsilon \leq 1$, are interlaced according to the inequalities

\[ j'_{\nu,s} < y_{\nu,s} < y_{\nu+\varepsilon,s} < y'_{\nu,s} < j_{\nu,s} < j'_{\nu,s+1} < \ldots \quad s = 1, 2, \ldots, \nu \geq 0. \]  
(9)

For $\varepsilon > 1$ this interlacing property is destroyed.

Eqs. (5) and (6) can be generalized to

\[ j'_{\nu,1} < j'_{\nu+\varepsilon,1} < j'_{\nu,2} < j'_{\nu+\varepsilon,2} < j'_{\nu,3} < \ldots \]  
(10)

\[ y'_{\nu,1} < y'_{\nu+\varepsilon,1} < y'_{\nu,2} < y'_{\nu+\varepsilon,2} < y'_{\nu,3} < \ldots \]  
(11)

with $0 < \varepsilon \leq 1$ and $\nu \geq 0$. We note that the latter two inequalities cannot be integrated with our unified interlacing inequality (9). While both $j'_{\nu,s} < j'_{\nu+\varepsilon,s} < y_{\nu+\varepsilon,s}$ and $y'_{\nu,s} < y'_{\nu+\varepsilon,s} < j_{\nu+\varepsilon,s}$ hold, numerical counterexamples can easily be constructed for the non-existence of a uniform inequality between both $j'_{\nu+\varepsilon,s}$ and $y_{\nu,s}$ (for which $j'_{\nu,s} < y_{\nu,s} < y_{\nu+\varepsilon,s}$ applies), and $y'_{\nu+\varepsilon,s}$ and $j_{\nu,s}$ (for which $y'_{\nu,s} < j_{\nu,s} < j_{\nu+\varepsilon,s}$ applies) for all $s = 1, 2, \ldots$ and $\nu > 0$, $0 < \varepsilon \leq 1$. 

2
2 Proofs

We start by proving Eqs. (10) and (11) depending on Theorem 1.3. The first inequality is trivial due to the monotonicity of \( j'_{\nu,s} \) in \( \nu \) for \( \nu \geq 0 \) (see [3]).

For proving the second inequality take the sequence of Theorem 1.3 at some arbitrary \( s \) for \( \nu, \nu + 1 \) and for \( \nu + 1, \nu + 2 \) with \( \varepsilon = 1 \):

\[
\begin{align*}
\quad j_{\nu,s} & < y_{\nu,s} < y_{\nu+1,s} < j'_{\nu,s} < j_{\nu+1,s} < j'_{\nu,s+1} \\
\quad j'_{\nu+1,s} & < y_{\nu+1,s} < y_{\nu+2,s} < y'_{\nu+1,s} < j_{\nu+2,s} < j'_{\nu+1,s+1} \\
\end{align*}
\]

From the first one we have \( j_{\nu+1,s} < j'_{\nu,s+1} \) and from the second one we have \( j'_{\nu+1,s} < j'_{\nu,s+1} \) of Eq. (5) for the derivative function \( J'_\nu(x) \). For \( \varepsilon < 1 \) the relation follows from the the monotonicity of \( j'_{\nu,s} \) in \( \nu \).

For the positive zeros of the derivative function \( Y'_\nu(x) \) a similar reasoning can be presented. \( y_{\nu,s} < y'_{\nu+s,\varepsilon} \) is trivial due to the monotonicity in \( \nu \). Use Theorem 1.3 for \( \nu, \nu + 1 \) and for \( \nu + 1, \nu + 2 \) with \( \varepsilon = 1 \). . . < \( y_{\nu+1,s} < \) . . . < \( y'_{\nu,s+1} < \).

and combine them to get the relations \( y_{\nu+1,s} < y'_{\nu,s+1} \) contained in Eq. (6). Again the monotonicity of \( y_{\nu,s} \) in \( \nu \) implies the non-trivial inequalities \( y_{\nu+s,\varepsilon} < y'_{\nu+s,\varepsilon+1} \) for \( \varepsilon < 1 \).

Eq. (7) of Proposition 1.2 was already proven, independently of each other, in Refs. [2] and [5] therefore its proof is omitted here. The proof of Eq. (8) is elementary and based on the analysis of intervals on which both \( Y'_{\nu+1}(x) \) and \( Y'_\nu(x) \) take the same and the opposite sign. Of course, by definition, \( Y'_{\nu+1}(x) \) and \( Y'_\nu(x) \) each keeps sign in the intervals defined by their two consecutive zeros, respectively. That is \( Y'_{\nu+1}(x) \) keeps the sign in the interval

\[
y_{\nu+1,s} < x < y_{\nu+1,s+1}, \quad s = 1, 2, \ldots,
\]

and \( Y'_\nu(x) \) does it in

\[
y_{\nu,s} < x < y_{\nu,s+1}, \quad s = 1, 2, \ldots.
\]

However, since \( Y'_\nu(x \to 0) = -\infty \) for \( \nu \geq 0 \) and using Eq. (3) [which implies that \( y_{\nu,s} < y_{\nu+1,s} \) and \( y_{\nu,s+1} < y_{\nu+1,s+1} \)], one concludes by induction that both \( Y'_{\nu+1}(x) \) and \( Y'_\nu(x) \) keep the same sign in the common intervals

\[
y_{\nu+1,s} < x < y_{\nu,s+1}, \quad s = 1, 2, \ldots,
\]

whereas in

\[
y_{\nu,s} < x < y_{\nu+1,s}, \quad s = 1, 2, \ldots,
\]

the signs do differ. Now let us take the recurrence relation

\[
C'_\nu(x) = -C'_{\nu+1}(x) + \frac{\nu}{x} C_\nu(x)
\]
with \( C = Y \) at \( x = y'_{\nu,s} \). It yields

\[
Y_{\nu+1}(y'_{\nu,s}) = \frac{\nu}{y'_{\nu,s}} Y_{\nu}(y'_{\nu,s}), \tag{14}
\]

i.e. the signs of \( Y_{\nu+1}(x) \) and \( Y_{\nu}(x) \) coincide at \( x = y'_{\nu,s} \). But, because of (13) [which tells that \( y'_{\nu,s} \) lies within the interval \( y_{\nu,s} < x < y_{\nu,s+1} \)], the content of Eq. (14) means also that \( y'_{\nu,s} \) must be in the common intervals given above by (12). This completes the proof of Proposition 1.2 for \( y_{\nu+1,s} < y'_{\nu,s} \).

The proof of the first part of Theorem 1.3 is also elementary and follows from the application of the two relations of Proposition 1.2 [being previously unknown] in conjunction with the three distinct inequalities (2), (3) and (4) [being already known, i.e. from Ref. [1]]. The case \( 0 < \varepsilon < 1 \) is immediately implied by the well-known property of \( j_{\nu,s}'s \) and \( y_{\nu,s}'s \) that, for a fixed \( s \), they are strictly increasing functions of \( \nu \) if \( \nu \geq 0 \) [3].

The negative statement for \( \varepsilon > 1 \) can be deduced from \( y_{\nu+\varepsilon,s} > j_{\nu,s} \), that is from the violation of the prescribed relation between the third and fifth term in the inequality sequence of Theorem 1.3 [4]. In Ref. [2] it was proven that the Wronskian \( W_{\nu\mu}(x) \equiv J_{\nu}(x)Y_{\mu}'(x) - J_{\nu}'(x)Y_{\mu}(x) \neq 0 \) for \( x \in (0, \infty) \) if and only if \( 0 < |\nu - \mu| \leq 1 \) is maintained (\( \nu \neq \mu \)). One of the ideas in that proof is that the set of extremal points of \( W_{\nu\mu}(x) \) is \( \{j_{\nu,s}\}_{s=1}^{\infty} \cup \{y_{\mu,s}\}_{s=1}^{\infty} \) and it has been unveiled that the inequality sequences \( y_{\mu,s} < j_{\nu,s} < y_{\mu,s+1} < j_{\nu,s+1}, s = 1, 2, \ldots \) hold if and only if all the local extrema are of the same sign, i.e. \( W_{\nu\mu}(x) \neq 0, x \in \mathbb{R}^+ \). These relations are exactly the same that we are studying here with \( \mu = \nu + \varepsilon \). Since for \( \varepsilon > 1 \) \( |\nu - \mu| \leq 1 \) cannot hold and there is at least one root of \( W_{\nu\mu}(x) \) the inequalities must be violated for some \( s \). Now our proof is complete.

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