DENSE SETS OF JOINT DISTRIBUTIONS APPEARING IN FILTRATION ENLARGEMENTS, STOCHASTIC CONTROL, AND CAUSAL OPTIMAL TRANSPORT

DANIEL LACKER

Abstract. It is well known that any pair of random variables \((X, Y)\) with values in Polish spaces, provided that \(Y\) is nonatomic, can be approximated in joint law by random variables \((X', Y)\) where \(X'\) is \(Y\)-measurable. This article surveys and extends some recent dynamic analogues of this result. For example, if \(X\) and \(Y\) are stochastic processes in discrete or continuous time, then, under a nonatomic assumption as well as a necessary and sufficient compatibility condition, one can approximate \((X, Y)\) in law in path space by processes of the form \((X', Y)\), where \(X'\) is adapted to the filtration generated by \(Y\). A similar approximation is valid for randomized stopping times. Natural applications include relaxations of (mean field) stochastic control and causal optimal transport problems as well as new characterizations of the immersion property for progressively enlarged filtrations.

1. Introduction

If \(X\) and \(Y\) are random variables with values in Polish spaces \(\mathcal{X}\) and \(\mathcal{Y}\), and if \(Y\) is nonatomic, then there is a sequence \((X_n)\) of \(Y\)-measurable random variables such that \((X_n, Y) \Rightarrow (X, Y)\), where \(\Rightarrow\) denotes convergence in law. In other words, the set of joint distributions concentrated on the graph of a function is dense in the set of all joint distributions with second marginal equal to the law of \(Y\). Moreover, the latter set is convex, and its set of extreme points is precisely the former. In fact, one may even demand that \(X_n\) has the same law as \(X\) for each \(n\). These (known) facts are reviewed and proven in Section 2.

The goal of this paper is to survey and elaborate on some recent results most naturally described as dynamic or non-anticipative analogues of those mentioned in the previous paragraph. Let us present immediately two such theorems, in discrete and continuous time:

**Theorem 1.1** (Discrete time). Consider two stochastic processes \(Y = (Y_1, \ldots, Y_N)\) and \(X = (X_1, \ldots, X_N)\) with values in Polish spaces \(\mathcal{Y}\) and \(\mathcal{X}\), respectively, where \(\mathcal{X}\) is homeomorphic to a convex subset of a locally convex space. Let \(\mathbb{F}^Y = (\mathcal{F}^Y_n)_{n=1}^N\) and \(\mathbb{F}^X = (\mathcal{F}^X_n)_{n=1}^N\) denote the filtrations generated by these processes. Suppose that the law of \(Y_1\) is nonatomic. Then the following are equivalent:

(i) \(X\) is compatible with \(Y\) in the sense that \(\mathcal{F}^X_n\) is conditionally independent of \(\mathcal{F}^Y_N\) given \(\mathcal{F}^Y_n\), for each \(n = 1, \ldots, N\).

(ii) There exists a sequence \(X^k = (X^k_1, \ldots, X^k_N)\) of \(\mathbb{F}^Y\)-adapted processes such that \((Y, X^k) \Rightarrow (Y, X)\) in \(\mathcal{Y}^N \times \mathcal{X}^N\).

**Theorem 1.2** (Continuous time). Consider two stochastic processes \(Y = (Y_t)_{t \geq 0}\) and \(X = (X_t)_{t \geq 0}\) with values in Polish spaces \(\mathcal{Y}\) and \(\mathcal{X}\), respectively, where \(\mathcal{X}\) is homeomorphic to a convex subset of a locally convex space. Assume \(X\) is continuous and \(Y\) is càdlàg. Let \(\mathbb{F}^Y = (\mathcal{F}^Y_t)_{t \geq 0}\) and \(\mathbb{F}^X = (\mathcal{F}^X_t)_{t \geq 0}\) denote the (unaugmented) filtrations generated by these processes. Assume either one of the following holds:

(a) \(X_0\) and \(Y_0\) are a.s. constant, and the law of \(Y_t\) is nonatomic for every \(t > 0\).

(b) The law of \(Y_0\) is nonatomic.
Then the following are equivalent:

(i) \( X \) is compatible with \( Y \) in the sense that \( F_t^X \) is conditionally independent of \( F_{\infty}^Y \) given \( F_t^Y \), for each \( t \geq 0 \).

(ii) There exists a sequence \( X^n \) of continuous \( F^Y \)-adapted processes such that \( (X^n, Y) \Rightarrow (X, Y) \) in \( C([0, \infty); \mathcal{X}) \times D([0, \infty); \mathcal{Y}) \), where \( C \) and \( D \) denote the continuous and Skorokhod path spaces, respectively.

The proofs of these claims are given in Section 3, in slightly more general forms, and with two alternatives to Theorem 1.2, when \( X \) has càdlàg or merely measurable trajectories. These statements can be framed in terms of the joint distribution of \( X, Y \), once we first show that the compatibility constraint (i) is closed in a suitable sense. In the context of Theorem 1.1, it follows that the (weak) closure of the set \( \Pi_0^c(\mu) \) of joint laws of \( (X, Y) \), where \( X \) ranges over \( F^Y \)-adapted processes and \( Y \sim \mu \), is exactly the set \( \Pi^c(\mu) \) of joint laws of \( (X, Y) \) with \( Y \sim \mu \) and satisfying the compatibility condition (i). As similar formulation is valid for Theorem 1.2 as long as one is careful about the two cases (a) and (b).

The set of joint laws \( \Pi^c(\mu) \) described above is always convex. In discrete time, \( \Pi_0^c(\mu) \) is precisely the set of extreme points of \( \Pi^c(\mu) \). But this fails in general in continuous time, and we give a counterexample built on a stochastic differential equation (SDE) which admits a weak solution but no strong solution. See Section 5 for details.

Analogously, we show in Section 4 that the set of stopping randomized stopping times are the closure of the set of true stopping times, in a joint-distriutional sense:

**Theorem 1.3.** Consider a càdlàg stochastic process \( Y = (Y_t)_{t \geq 0} \) with values in a Polish space \( \mathcal{Y} \), and let \( \tau \) be a random time. Assume \( Y_t \) is nonatomic for every \( t > 0 \). Let \( \mathbb{F} \) denote the augmented (complete and right-continuous) filtration generated by \( Y \). Then the following are equivalent:

(i) \( \tau \) is an \( \mathbb{F} \)-randomized stopping time in the sense that \( \mathbb{P}(\tau \leq t|\mathcal{F}_\infty) = \mathbb{P}(\tau \leq t|\mathcal{F}_t) \) a.s., for every \( t \geq 0 \).

(ii) There exists a sequence of \( \mathbb{F} \)-stopping times \( \tau_n \) such that \( (Y, \tau_n) \Rightarrow (Y, \tau) \) in \( D([0, \infty); \mathcal{Y}) \times [0, \infty] \).

It should be stressed that few if any of these theorems are completely new, appearing in special cases but with the same key proof ideas in [17, Lemma 3.11] and [18, Section 6], written up also in the recent book [16, Sections II.1.1.1, II.7.2.5]. This paper is intended more to survey and extend to the natural level of generality. The motivation for consolidating these kinds of results in a single, concise reference stems from their relevance in increasingly diverse areas of application, some of which we discuss in the next few paragraphs.

1.1. **Optimal control and stopping.** In the theory of stochastic optimal control, one often seeks to choose an adapted process in order to minimize some cost criterion, expressed as a continuous functional of the joint law of the control and noise processes. Typically, \( Y \) is an underlying noise process, and the control process \( X \) must be adapted to \( Y \). Framing the problem as optimization over the family \( \Pi_0^c(\mu) \) of admissible joint distributions, one is naturally inclined to take the closure of \( \Pi_0^c(\mu) \) to obtain a convenient topological setting, and this closure is precisely what we identified as \( \Pi^c(\mu) \) above. This relaxation is now standard in stochastic control, both in discrete time [12] and continuous time [33, 28, 37], though it is usually justified by extreme point arguments rather than density. In mean field contexts [15, 39, 44], however, the objective function depends nonlinearly on the law of the controlled process; extreme point arguments then break down, but one can still justify the relaxation by a density argument. See Section 6 for details.
A similar strategy applies in optimal stopping, leading to the notion of *randomized stopping time* (see [8, 22, 27]), where $\Pi_0^c(\mu)$ corresponds to true stopping times. This relaxation played an important role in recent analysis of American option pricing [9] and optimal Skorokhod embedding problems [10].

1.2. **Stochastic games.** In static (one-shot) games, it is well known that Nash equilibria typically exist among *mixed strategies*, in which agents independently randomize their actions, but not necessarily among *pure strategies*, in which agents choose (deterministic) actions. When stochastic factors are present, the natural analogue of a pure strategy is a measurable function from the stochastic factor to the action space, and a good notion of mixed strategy should again convexify and/or compactify the set of pure strategies. This leads, for instance, to the notion of *distributional strategy* in [42].

Similar ideas are useful in the analysis of dynamic stochastic games, where the natural analogue of a pure strategy is an adapted process. Indeed, special cases of the theorems announced above appeared first in the author’s work on mean field games [17, 18]. Therein, the topology of $\Pi^c(\mu)$ is well-suited to compactness arguments, whereas optimality criteria are more easily checked using the dense subset $\Pi_0^c(\mu)$. These ideas again proved fruitful in studying the limits of $n$-player games and control problems in [38, 39].

1.3. **Filtration enlargement.** The notion of compatibility mentioned in the theorems above has appeared in a variety of contexts before, particularly in the context of enlargement of filtration, though we roughly adopt the terminology of Kurtz [36]. The statement that $X$ is compatible with $Y$ is equivalent to the statement that every $\mathbb{F}^Y$-martingale is an $\mathbb{F}^{X,Y}$-martingale. We review this and other characterizations in Section 3.6. In the literature on enlargements of filtrations, this property goes by the name *H-hypothesis* or *immersion*. See [14] for its appearance in filtering, [23] for applications in credit risk, or the recent book [2]. Theorems 1.2 and 1.3 add to the pantheon of characterizations of the H-hypothesis.

1.4. **Optimal transport.** Optimal transport (see [50, 51, 47] for overviews) is a final domain of application for the kinds of results surveyed in this paper. Classically, letting $\Pi(\mu, \nu)$ denote the set of joint laws on $\mathcal{X} \times \mathcal{Y}$ with first marginals $\mu$ and second marginal $\nu$, the Kantorovich formulation of optimal transport is to optimize a functional of the form

$$J(\gamma) = \int_{\mathcal{X} \times \mathcal{Y}} c \, d\gamma$$

(1.1)

over $\gamma \in \Pi(\mu, \nu)$. This can be seen as a relaxation of the original Monge formulation, in which we optimize over the subset $\Pi_0(\mu, \nu)$ of couplings of the form $\mu(dx)\delta_{\phi(x)}(dy)$. When $\mu$ is nonatomic, $\Pi_0(\mu, \nu)$ is dense in $\Pi(\mu, \nu)$, showing that the Kantorovich formulation is a genuine relaxation of the Monge formulation, at least if $J$ is continuous on $\Pi(\mu, \nu)$. See Sections 2.2 and 2.3 for details. Notably, this cannot be proven by an extreme point argument, because in general $\Pi_0(\mu, \nu)$ is a proper subset of the set of extreme points of $\Pi(\mu, \nu)$. It is notoriously difficult to characterize the extreme points of $\Pi(\mu, \nu)$; to enter this rabbit hole, see [11, 48].

Recent developments on *causal optimal transport* are more in line with the focus of this paper on the dynamic setting. In the setting of the previous paragraph, the spaces $\mathcal{X}$ and $\mathcal{Y}$ are equipped with filtrations, and the coupling set $\Pi(\mu, \nu)$ is replaced by a subset $\Pi^c(\mu, \nu)$ consisting only of those joint laws satisfying a compatibility condition like those discussed above. Causal optimal transport, introduced by Lassalle in [40] and studied further in [1, 6, 7], pertains to the optimization of functionals as in (1.1) over $\Pi^c(\mu, \nu)$. In other words, this is an optimal transport problem with the additional constraint of compatibility. With the second marginal $\nu$ fixed, it remains an open question when (if ever) $\Pi_0^c(\mu, \nu)$ is dense in $\Pi^c(\mu, \nu)$, and this undermines the
utility of the Kantorovich formulation as a relaxation of the Monge formulation. See Section 3.5 for further discussion, including a remarkable partial result in this direction.

1.5. Organization of the paper. The paper is organized as follows. Section 2 is something of a warm-up, collecting a number of useful facts about the set of joint distributions on $\mathcal{X} \times \mathcal{Y}$ with either one or both of the marginals fixed. Most of the results here are folklore, but complete proofs are provided. Section 3 turns to the dynamic setting in which $\mathcal{X}$ and $\mathcal{Y}$ are replaced with path spaces; this is where proofs of (more general forms of) Theorems 1.1 and 1.2 are provided. Section 4 discusses randomized stopping times and proves Theorem 1.3. As mentioned briefly above, sets of joint distributions of the form of $\Pi^c(\mu)$ above are convex, and Section 5 discusses characterizations of their extreme points, as well as an interesting counterexample in continuous time. Lastly, Sections 6 highlights an application in stochastic optimal control.

Notation. Our notation throughout is as follows. We write $\mathbb{R}_+ = [0, \infty)$. Given a measurable space $(\Omega, \mathcal{F})$, we write $\mathcal{P}(\Omega, \mathcal{F})$ for the set of probability measures, and we abbreviate this to $\mathcal{P}(\Omega)$ when the $\sigma$-field is understood. Every topological space $\mathcal{X}$ is equipped with its Borel $\sigma$-field, and accordingly we write $\mathcal{P}(\mathcal{X})$ for the set of Borel probability measures on $\mathcal{X}$. Unless otherwise stated, we equip $\mathcal{P}(\mathcal{X})$ with the usual topology of weak convergence, induced by bounded continuous test functions. The set of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ is denoted $C(\mathcal{X}; \mathcal{Y})$. If the range space is $\mathbb{R}$, we write simply $C(\mathcal{X})$ instead of $C(\mathcal{X}; \mathbb{R})$.

2. The static case

This section serves as a warm-up, reviewing some known facts about weak convergence of joint distributions with either one or two fixed marginals for which concise proofs can be difficult to locate in the literature.

2.1. One fixed marginal. First we recall an important lemma on weak convergence. It is a consequence, for instance, of [29, Corollary 2.9], but we include a direct proof. In this section, we fix two Polish spaces $\mathcal{X}$ and $\mathcal{Y}$, as well as $\mu \in \mathcal{P}(\mathcal{X})$. Let

$$\Pi(\mu) = \{P \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : P(\cdot \times \mathcal{Y}) = \mu\}$$

(2.1)
denote the set of joint laws with first marginal $\mu$.

Lemma 2.1. Suppose $P, P_n \in \Pi(\mu)$, with $P_n \to P$ weakly. Let $\varphi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be bounded and satisfy the following:

1. $\varphi(\cdot, y)$ is measurable for each $y \in \mathcal{Y}$.
2. $\varphi(x, \cdot)$ is continuous for $\mu$-a.e. $x \in \mathcal{X}$.

Then $\int \varphi dP_n \to \int \varphi dP$.

Proof. Let $c > 0$ be such that $|\phi(x, y)| \leq c$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Fix $\epsilon > 0$, and use Prokhorov's theorem to find a compact set $K \subset \mathcal{Y}$ such that $P_n(\mathcal{X} \times K) \geq 1 - \epsilon$ for all $n$. Let $\mathcal{X}_0 \subset \mathcal{X}$ be a Borel set with $\mu(\mathcal{X}_0) = 1$ such that $\phi(x, \cdot)$ is continuous for every $x \in \mathcal{X}_0$. Consider the map $\Phi : \mathcal{X}_0 \to C(K)$ defined by $\Phi(x)(y) = \phi(x, y)$. It is easy to prove (and follows immediately from [4, Theorem 4.55]) that $\Phi$ is Borel measurable.

We next show that for each $\delta > 0$ there exists a continuous function $\phi_\delta : \mathcal{X} \times K \to \mathbb{R}$ such that $|\phi_\delta| \leq c$ pointwise and

$$\mu\{x \in \mathcal{X} : \phi_\delta(x, y) \equiv \phi(x, y), \forall y \in \mathcal{Y}\} \geq 1 - \delta.$$

First extend the domain of $\Phi$ to all of $\mathcal{X}$ by choosing arbitrarily some $f_0 \in C(K)$ and setting $\Phi(x) = f_0$ for $x \notin \mathcal{X}_0$, noting that $\Phi$ remains measurable. Next, apply Lusin's theorem [13, Theorem 7.1.13] to find, for each $\delta > 0$, a continuous function $\Phi_\delta : \mathcal{X} \to C(K)$ such that
\(\mu\{x \in \mathcal{X} : \Phi_\delta(x) = \Phi(x)\} \geq 1 - \delta\). Finally, define \(\phi_\delta(x, y) = (\Phi_\delta(x)(y) \wedge c) \vee (-c)\), where \(c\) was the bound on \(|\phi|\).

With these preparations out of the way, the proof proceeds first with the bound
\[
\left| \int_{\mathcal{X} \times \mathcal{Y}} \phi \, dP_n - \int_{\mathcal{X} \times \mathcal{Y}} \phi \, dP \right| \leq \left| \int_{\mathcal{X} \times K} \phi \, dP_n - \int_{\mathcal{X} \times K} \phi \, dP \right| + 2c \\
\leq \left| \int_{\mathcal{X} \times K} \phi_\delta \, dP_n - \int_{\mathcal{X} \times K} \phi_\delta \, dP \right| + (2\epsilon + 4\delta)c.
\]

Because \(\phi_\delta\) is continuous on the closed set \(\mathcal{X} \times K\), it admits a continuous extension \(\tilde{\phi}_\delta\) to all of \(\mathcal{X} \times \mathcal{Y}\) with \(|\tilde{\phi}_\delta| \leq c\) pointwise, by the Tietze extension theorem. Thus
\[
\left| \int_{\mathcal{X} \times \mathcal{Y}} \phi \, dP_n - \int_{\mathcal{X} \times \mathcal{Y}} \phi \, dP \right| \leq \left| \int_{\mathcal{X} \times \mathcal{Y}} \tilde{\phi}_\delta \, dP_n - \int_{\mathcal{X} \times \mathcal{Y}} \tilde{\phi}_\delta \, dP \right| + 4c(\epsilon + \delta).
\]

Finally, continuity of \(\tilde{\phi}_\delta\) and weak convergence of \(P_n\) to \(P\) imply
\[
\limsup_{n \to \infty} \left| \int_{\mathcal{X} \times \mathcal{Y}} \phi \, dP_n - \int_{\mathcal{X} \times \mathcal{Y}} \phi \, dP \right| \leq 4c(\epsilon + \delta).
\]

As \(\epsilon\) and \(\delta\) were arbitrary, the proof is complete. \(\Box\)

The following proposition, mentioned in the introduction, is folklore. We will prove a generalization in Proposition 2.7 based on the argument of [19, Theorem 2.2.3], though the writing of [19] is at a rather difficult level of generality. See also [5, Theorem 9.3].

**Proposition 2.2.** The following hold:

(i) If \(\mu\) is nonatomic, then the following set is dense in \(\Pi(\mu)\):
\[
\Pi_0(\mu) := \left\{ \mu(dx)\delta_\varphi(dy) \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : \varphi : \mathcal{X} \to \mathcal{Y} \text{ is measurable} \right\}.
\]

(ii) If \(\mu\) is nonatomic and \(\mathcal{Y}\) is homeomorphic to a convex subset of a locally convex space, then the following set is dense in \(\Pi(\mu)\):
\[
\left\{ \mu(dx)\delta_\varphi(dy) \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : \varphi : \mathcal{X} \to \mathcal{Y} \text{ is continuous} \right\}.
\]

**Proof.** The first claim will follow from Proposition 2.7. To prove the second claim, we must only show that any measurable function \(\varphi : \mathcal{X} \to \mathcal{Y}\) can be obtained as the \(\mu\)-a.s. limit of continuous functions. Assume without loss of generality that \(\mathcal{Y}\) is in fact a subset of a locally convex space \(\hat{\mathcal{Y}}\), endowed with the induced topology. By a form of Lusin’s theorem [13, Theorem 7.1.13], for each \(\epsilon > 0\) we may find a compact \(K_\epsilon \subset \mathcal{X}\) such that \(\mu(K_\epsilon^c) \leq \epsilon\) and the restriction \(\varphi|_{K_\epsilon} : K_\epsilon \to \mathcal{Y}\) is continuous. Using a generalization of the Tietze extension theorem due to Dugundji [21, Theorem 4.1], we may find a continuous function \(\hat{\varphi}_\epsilon : \mathcal{X} \to \hat{\mathcal{Y}}\) such that \(\hat{\varphi}_\epsilon = \varphi\) on \(K_\epsilon\) and such that the range \(\hat{\varphi}_\epsilon(\mathcal{X})\) is contained in the convex hull of \(\varphi|_{K_\epsilon}(\mathcal{X})\), which is itself contained in the convex set \(\mathcal{Y}\). We thus view \(\hat{\varphi}_\epsilon\) as a continuous function from \(\mathcal{X}\) to \(\mathcal{Y}\). Since \(\mu(\hat{\varphi}_\epsilon \neq \varphi) \leq \mu(K_\epsilon^c) \leq \epsilon\), we may find a subsequence of \(\hat{\varphi}_\epsilon\) which converges \(\mu\)-a.s. to \(\varphi\). \(\Box\)

**Remark 2.3.** Note that part (ii) of Proposition 2.2 fails in general when \(\mathcal{Y}\) fails to be (homeomorphic to) a convex set, which is most easily seen when \(\mathcal{Y}\) is a discrete space.

**Remark 2.4.** The two previous results can be stated in terms of *stable convergence*, a mode of convergence of probability measures on product spaces studied in detail in [29]. See also [13, Section 8.10(xi)] for more references. Even when \(\mathcal{X}\) is merely a measurable space, one may define the *stable topology* on \(\mathcal{P}(\mathcal{X} \times \mathcal{Y})\) as the coarsest topology such that \(P \mapsto \int \varphi \, dP\) is continuous for every bounded jointly measurable function \(\varphi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}\) such that \(\varphi(x, \cdot)\) is continuous for every fixed \(x \in \mathcal{X}\). An equivalent statement of Lemma 2.1 is that, when \(\mathcal{X}\) is a Polish space, the topologies of weak convergence and stable convergence agree on \(\Pi(\mu)\).
2.2. Two fixed marginals. Throughout this section, again assume $\mathcal{X}$ and $\mathcal{Y}$ are Polish spaces. We are now given marginals on both spaces, $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$, and we define the set of couplings:

$$\Pi(\mu, \nu) = \{ P \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : P(\cdot \times Y) = \mu, \ P(X \times \cdot) = \nu \}. \quad (2.3)$$

The first natural question is if a two-marginal analogue of Lemma 2.1 can hold. That is, if $P, P_n \in \Pi(\mu, \nu)$ with $P_n \to P$, and if $\phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is bounded and measurable, then we might expect $\int \phi dP_n$ to converge to $\int \phi dP$. It turns out that this does hold for $\phi$ of the form $\phi(x, y) = f(x)g(y)$ for bounded measurable functions $f$ and $g$, as is shown in Lemma 2.5 below. For general $\phi$, however, the subsequent example 2.6 illustrates what can go wrong. This lemma will not be used in the sequel, but it seems relevant to include here.

**Lemma 2.5.** Suppose $P_n \to P$, with $P_n, P \in \Pi(\mu, \nu)$. Then, if $f : \mathcal{X} \to \mathbb{R}$ and $g : \mathcal{Y} \to \mathbb{R}$ are bounded and measurable, we have $\int fg dP_n \to \int fg dP$.

**Proof.** By approximating $f$ uniformly by simple functions, we may assume that $f$ is itself simple. That is,

$$f(x) = \sum_{i=1}^{m} a_i 1_{A_i}(x),$$

were $m \geq 1$, $a_i \in \mathbb{R}$, and $(A_1, \ldots, A_m)$ is a Borel partition of $\mathcal{X}$. Fix $\epsilon > 0$. Let $\|\psi\|_\infty = \sup_{y \in \mathcal{Y}} |\psi(y)|$ for any $\psi : \mathcal{Y} \to \mathbb{R}$. By Lusin’s theorem, there is a continuous function $\phi : \mathcal{Y} \to \mathbb{R}$ such that $\|\phi\|_\infty \leq \|g\|_\infty$ and $\nu(\phi \neq g) \leq \epsilon$. Then

$$\left| \int_{A_i \times \mathcal{Y}} g(y)P_n(dx, dy) - \int_{A_i \times \mathcal{Y}} g(y)P(dx, dy) \right| \leq \left| \int_{A_i \times \mathcal{Y}} (g(y) - \phi(y))P_n(dx, dy) - \int_{A_i \times \mathcal{Y}} (g(y) - \phi(y))P(dx, dy) \right| + \left| \int_{A_i \times \mathcal{Y}} \phi(y)P_n(dx, dy) - \int_{A_i \times \mathcal{Y}} \phi(y)P(dx, dy) \right|.$$

The last term tends to zero thanks to Lemma 2.1. The other term is bounded by $\|g - \phi\|_\infty (P_n(A_i \times \{\phi \neq g\}) + P(A_i \times \{\phi \neq g\})) \leq 4\|g\|_\infty \epsilon$, because the second marginal of each $P_n$ (and of $P$) is $\nu$. This completes the proof. \(\square\)

**Example 2.6.** Suppose $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$ and $\mu = \nu = N(0, 1)$, where $N(0, 1)$ denotes the standard Gaussian law. Fix some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a two-dimensional standard Gaussian $X$, and define $X_n$ as the rotation of $X$ by $1/n$ radians. Both $P := \mathbb{P} \circ (X, X)^{-1}$ and $P_n := \mathbb{P} \circ (X, X_n)^{-1}$ belong to $\Pi(\mu, \nu)$, and $P_n \to P$. But if $\phi(x, y) = 1_{\{x=y\}}$, then $\int \phi dP_n = 0$ for all $n$ while $\int \phi dP = 1$.

Let $\Pi_0(\mu, \nu)$ denote the set of Monge couplings, defined as

$$\Pi_0(\mu, \nu) = \{ \mu(dx)\delta_{\varphi(x)}(dy) \in \Pi(\mu, \nu) : \varphi : \mathcal{X} \to \mathcal{Y} \text{ is measurable} \}. \quad (2.4)$$

That is, a coupling of $(\mu, \nu)$ belongs to $\Pi_0(\mu, \nu)$ if and only if it is concentrated on the graph of a measurable function. The following Proposition 2.7 shows that if $\mu$ is nonatomic then $\Pi_0(\mu, \nu)$ is dense in $\Pi(\mu, \nu)$, which gives Proposition 2.2 as a corollary. This is implicit in [45]:

**Proposition 2.7.** If $\mu$ is nonatomic, then $\Pi_0(\mu, \nu)$ is dense in $\Pi(\mu, \nu)$. 
Proof. Fix \( P \in \Pi(\mu, \nu) \). Let \( \pi_n = \{A_{n,i} : i = 1, \ldots, n\} \) be a sequence of partitions of \( \mathcal{X} \) such that \( \pi_n \subset \pi_{n+1} \) and \( \bigcup_n \pi_n \) generates the Borel \( \sigma \)-field. Let \( \eta_{|A} = \eta(A \cap \cdot) \) denote the trace of a measure \( \eta \) on a set \( A \).

Define the finite measure \( \tilde{\nu}^n_i(\cdot) = P(A_{n,i} \times \cdot) \), which has total mass \( \tilde{\nu}^n_i(\mathcal{Y}) = P(A_{n,i} \times \mathcal{Y}) = \mu(A_{n,i}) \). Because \( \mu \) is nonatomic, there exists by Borel isomorphism a measurable function \( \phi_n : A_n \rightarrow \mathcal{Y} \) such that \( \mu(A_{n,i}) \circ \phi_n^{-1} = \tilde{\nu}_i \). Now define \( \phi_n : \mathcal{X} \rightarrow \mathcal{Y} \) by setting \( \phi_n = \phi_{n,i} \) on \( A_{n,i} \) for each \( i \). We claim that \( \mu \circ \phi_n^{-1} = \nu \). Indeed, for a Borel set \( B \subset \mathcal{Y} \),

\[
\mu \circ \phi_n^{-1}(B) = \sum_{i=1}^n \mu(A_{n,i} \cap \phi_n^{-1}(B)) = \sum_{i=1}^n \mu(A_{n,i}) = \sum_{i=1}^n \tilde{\nu}_i(B) = \sum_{i=1}^n P(A_{n,i} \times B) = P(\mathcal{X} \times B) = \nu(B).
\]

Now define \( P_n(dx, dy) = \mu(dx)\delta_{\phi_n(x)}(dy) \). By construction, we have \( P_n \in \Pi(\mu, \nu) \) and

\[
\int_{A_{n,i} \times Y} f(y) P_n(dx, dy) = \int_{A_{n,i} \times Y} f(y) \tilde{\nu}_i^n(dy) = \int_{A_{n,i} \times Y} f(y) P(dx, dy)
\]

for every \( i = 1, \ldots, n \) and every bounded measurable \( f : \mathcal{Y} \rightarrow \mathbb{R} \). Hence, \( P_n = P \circ (\pi_n) \circ B(\mathcal{Y}) \), where \( B(\mathcal{Y}) \) denotes the Borel \( \sigma \)-field of \( \mathcal{Y} \). Because \( \pi_n \subset \pi_{n+1} \) for each \( n \) and \( \sigma(\bigcup_n \pi_n) = B(\mathcal{X}) \), it is straightforward to conclude that \( P_n \rightarrow P \). \( \square \)

Remark 2.8. In Proposition [2.7] when both marginals are nonatomic, the result can be refined so that the approximations are bimeasurable. Precisely, suppose we are given a pair of nonatomic random variables \( X \) and \( Y \) with values in Polish spaces \( \mathcal{X} \) and \( \mathcal{Y} \). Then there exists a sequence \((X_n, Y_n)\) of \( \mathcal{X} \times \mathcal{Y} \)-valued random variables such that \( X_n \) is \( \mathcal{X}_n \)-measurable, \( Y_n \) is \( \mathcal{Y}_n \)-measurable, \( X_n \overset{d}{=} X \), \( Y_n \overset{d}{=} Y \), and \((X_n, Y_n) \Rightarrow (X, Y)\). See [26] Proposition A.3 or [24] pp. 296 for proofs in the case where \( X \) and \( Y \) are uniform in \([0,1]\), which adapts easily to more general settings.

Remark 2.9. Results closely related to Proposition [2.7] are well known in the theory of copulas, mainly for \( \mathcal{X} = \mathcal{Y} = [0,1] \) and when \( \mu = \nu \) are both the uniform (Lebesgue) measure. In this case, a dense subset of \( \Pi(\mu, \nu) \) is given by the set of Monge couplings induced by piecewise continuous bijections, known as shuffles of min [41].

Remark 2.10. Combining Proposition [2.7] with a remarkable result of Oxtoby [43] Theorem 1] yields the following: Suppose \( \mathcal{X} \) is a connected \( n \)-dimensional manifold for some \( n \geq 2 \), and suppose \( \mu \in \mathcal{P}(\mathcal{X}) \) is nonatomic and charges every nonempty open set. Then a dense subset of \( \Pi(\mu, \mu) \) is given by the set of measures of the form \( \mu(dx)\delta_{\varphi(x)}(dy) \), where \( \varphi : \mathcal{X} \rightarrow \mathcal{X} \) is a homeomorphism and \( \mu \circ \varphi^{-1} = \mu \).

2.3. Optimal transport. We state for posterity two natural applications of the results of the previous section, both contained in the following Proposition, both of which are but slight appendages to known results. The first deals with attainment of the infimum in optimal transport problems, while the second shows that the Kantorovich transport problem is often a genuine relaxation of the Monge problem, in the sense that the infima agree. A different (and admittedly more useful) version of the latter result was shown in [45] Theorem B], generalizing [5] Theorem 2.1], and the boundedness assumption we impose can certainly be relaxed.

Proposition 2.11. Suppose \( \mathcal{X} \) and \( \mathcal{Y} \) are Polish spaces and \( \mu \in \mathcal{P}(\mathcal{X}) \), \( \nu \in \mathcal{P}(\mathcal{Y}) \). Suppose \( \Psi : \mathcal{X} \times \mathcal{Y} \rightarrow (-\infty, \infty] \) is of the form

\[
\Psi(x, y) = \psi(x, y) + g(x)h(y),
\]

where
for some bounded measurable functions \(g\) and \(h\) on \(\mathcal{X}\) and \(\mathcal{Y}\) and some jointly measurable function \(\psi: \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty]\) which is bounded from below and lower semicontinuous in its second variable; that is, \(\psi(x, \cdot)\) is lower semicontinuous for each \(x\). Then the infimum \(\inf_{P \in \Pi(\mu, \nu)} \int \Psi \, dP\) is attained. If, in addition, \(\psi\) uniformly bounded and continuous in its second variable, then

\[
\inf_{P \in \Pi(\mu, \nu)} \int \Psi \, dP = \inf_{P \in \Pi(\mu, \nu)} \int \Psi \, dP.
\]  

(2.5)

Proof. The set \(\Pi(\mu, \nu)\) is weakly compact, and it follows from (a simple extension of) Lemma 2.4 and Lemma 2.5 that \(P \mapsto \int \Psi \, dP\) is lower semicontinuous on \(\Pi(\mu, \nu)\). This proves the first claim. For the second, note that the map \(\Pi(\mu, \nu) \ni P \mapsto \int \Psi \, dP\) is continuous by Lemmas 2.4 and 2.5 under the additional assumption. By Proposition 2.7 \(\Pi_0(\mu, \nu)\) is dense in \(\Pi(\mu, \nu)\). \(\square\)

3. The dynamic case

We now extend the results of the previous section to the dynamic setting, first in discrete time, and then in continuous time. The first appearance of a result of this nature seems to be \([17]\) Lemma 3.11, which is in continuous time but implicitly contains the discrete time result proven in more generality in \([18]\) Proposition 6.2 for discrete time.

3.1. Discrete time. In the following, for a discrete-time stochastic process \((X_1, \ldots, X_N)\) we write \(\mathbb{F}^X = (\mathcal{F}^X_n)_{n=1,\ldots,N}\) for the corresponding natural filtration, namely \(\mathcal{F}_n^X = \sigma(X_1, \ldots, X_n)\). Recall that another process \((Y_1, \ldots, Y_n)\) is said to be adapted to \(\mathbb{F}^X\), or simply \(X\)-adapted, if \(Y_n\) is \(\mathcal{F}_n^X\)-measurable for every \(n\). We now prove an extension of Theorem 1.1.

**Theorem 3.1.** Let \(Z\) be a random variable with values in a Polish space \(Z\). Consider two stochastic processes \(Y = (Y_1, \ldots, Y_N)\) and \(X = (X_1, \ldots, X_N)\) with values in Polish spaces \(\mathcal{Y}\) and \(\mathcal{X}\), respectively, where \(\mathcal{X}\) is homeomorphic to a convex subset of a locally convex space. Suppose that the law of \(Y_1\) is nonatomic and that \(Y_n\) is \(Z\)-measurable for every \(n\). Then the following are equivalent:

(i) For each \(n\), \(\mathcal{F}_n^X\) is conditionally independent of \(Z\) given \(\mathcal{F}_n^Y\).

(ii) Then there exists a sequence \(X^{(k)} = (X^{(k)}_1, \ldots, X^{(k)}_N)\) of \((\mathcal{F}_n^Y)_{n=1,\ldots,N}\)-adapted processes such that \((Z, X^{(k)}) \Rightarrow (Z, X)\) in \(Z \times \mathcal{X}^N\).

(iii) Then there exists a sequence \(X^{(k)} = (X^{(k)}_1, \ldots, X^{(k)}_N)\) of \((\mathcal{F}_n^Y)_{n=1,\ldots,N}\)-adapted processes such that \((Z, X^{(k)}) \Rightarrow (Z, X)\) in \(Z \times \mathcal{X}^N\), and \(X^{(k)}\) is of the form \(X^{(k)} = f^{(k)}_n(Y_1, \ldots, Y_n)\) for some continuous functions \(f^{(k)}_n : \mathcal{Y}^n \rightarrow \mathcal{X}\).

The role of the random variable \(Z\) in Theorem 3.1 is to incorporate additional information that the process \(X\) must respect. This can be useful, for example, for stochastic control problems with partial information, illustrated in Section 6. The weak convergence \((Z, X^{(k)}) \Rightarrow (Z, X)\) in \(Z \times \mathcal{X}^N\) implies the weak convergence \((Y, X^{(k)}) \Rightarrow (Y, X)\) in \(\mathcal{Y}^N \times \mathcal{X}^N\), which follows quickly from Lemma 2.4 and the assumption that \(Y\) is \(Z\)-measurable. Notice also that Theorem 3.1 from the introduction follows immediately from Theorem 3.1 by choosing \(Z = \mathcal{Y}^N\) and \(Z = Y\).

**Proof of Theorem 3.1.** We begin by proving that (i) implies (ii), as this is the more challenging step. The proof is an inductive application of Proposition 2.2(ii). First, use Proposition 2.2 to find a sequence of continuous functions \(h^{(1)}_j : \mathcal{Y} \rightarrow \mathcal{X}\) such that \((Y_1, h^{(1)}_j(Y_1)) \Rightarrow (Y_1, X_1)\) as \(j \rightarrow \infty\). Let us show that in fact \((Z, h^{(1)}_j(Y_1)) \Rightarrow (Z, X_1)\). Let \(\varphi : Z \rightarrow \mathbb{R}\) be bounded and measurable, and let \(\psi : \mathcal{X} \rightarrow \mathbb{R}\) be continuous. Use Lemma 2.4 as well as the conditional independence of \(Z\)
and $X_1$ given $Y_1$ to get
\[
\lim_{j \to \infty} \mathbb{E}[\varphi(Z)\psi(h^i_1(Y_1))] = \lim_{j \to \infty} \mathbb{E} \left[ \mathbb{E}[\varphi(Z)|Y_1] \psi(h^i_1(Y_1)) \right] \\
= \mathbb{E}[\mathbb{E}[\varphi(Z)|Y_1] \psi(X_1)] \\
= \mathbb{E}[\varphi(Z)|Y_1, X_1] \psi(X_1)] \\
= \mathbb{E}[\varphi(Z)\psi(X_1)]
\]

This is enough to show that $(Z, h^i_1(Y_1)) \Rightarrow (Z, X_1)$ (see e.g. [25 Proposition 3.4.6(b)]).

We proceed inductively as follows. Abbreviate $Y^n := (Y_1, \ldots, Y_n)$ and $X^n := (X_1, \ldots, X_n)$ for each $n = 1, \ldots, N$, noting $Y^N = Y$. Suppose we are given $1 \leq n < N$ and continuous functions $g^i_k : \mathcal{Y}^k \to \mathcal{X}$, for $k \in \{1, \ldots, n\}$ and $j \geq 1$, satisfying
\[
\lim_{j \to \infty} (Z, g^i_1(Y^1), \ldots, g^i_n(Y^n)) = (Z, X_1, \ldots, X_n),
\]
where the convergence here and throughout the proof is in law. We will show that there exist continuous functions $h^i_k : \mathcal{Y}^k \to \mathcal{X}$ for each $k \in \{1, \ldots, n+1\}$ and $i \geq 1$ such that
\[
\lim_{i \to \infty} (Z, h^i_1(Y^1), \ldots, h^i_{n+1}(Y^{n+1})) = (Z, X_1, \ldots, X_{n+1}).
\]

By Proposition 2.2 there exists a sequence of continuous functions $\hat{g}^i : (\mathcal{Y}^{n+1} \times \mathcal{X}^n) \to \mathcal{X}$ such that
\[
\lim_{j \to \infty} (Y^{n+1}, X_1, \ldots, X_n, \hat{g}^i(Y^{n+1}, X^n)) = (Y^{n+1}, X_1, \ldots, X_n, X_{n+1}).
\]

Note that $Z$ and $(Y^n, X^n)$ are conditionally independent given $Y_1$. Using the same argument as above, it follows that in fact
\[
\lim_{j \to \infty} (Z, X_1, \ldots, X_n, \hat{g}^i(Y^{n+1}, X^n)) = (Z, X_1, \ldots, X_n, X_{n+1}).
\]

By continuity of $\hat{g}^i$, the limit (3.1) implies (using Lemma 2.1 and the fact that $Y^{n+1}$ is $Z$-measurable) that, for each $j$,
\[
\lim_{i \to \infty} (Z, g^i_1(Y^1), \ldots, g^i_n(Y^n), \hat{g}^i(Y^{n+1}, g^i_1(Y^1), \ldots, g^i_n(Y^n))) \\
= (Z, X_1, \ldots, X_n, \hat{g}^i(Y^{n+1}, X_1, \ldots, X_n)).
\]

Combining the two limits (3.3) and (3.4), we may find a subsequence $j_i$ such that
\[
\lim_{i \to \infty} (Z, g^i_1(Y^1), \ldots, g^i_n(Y^n), \hat{g}^i(Y^{n+1}, g^i_1(Y^1), \ldots, g^i_n(Y^n))) \\
= (Z, X_1, \ldots, X_n, X_{n+1}).
\]

Define $h^i_k := h^i_{k+1}$ for $k = 1, \ldots, n$ and $h^i_{n+1}(Y^{n+1}) := \hat{g}^i(Y^{n+1}, g^i_1(Y^1), \ldots, g^i_n(Y^n))$ to establish (3.2). This completes the proof that (i) implies (iii).

Clearly (iii) implies (ii). To prove (ii) implies (i), note first that (i) holds if and only if for every $n$ and every bounded measurable functions $f_n$, $h_n$, and $g$ on $\mathcal{X}^n$, $\mathcal{Y}^n$, and $Z$, respectively, we have
\[
\mathbb{E}[f_n(X^n)h_n(Y^n)(g(Z) - g_n(Y^n))] = 0,
\]
where $g_n(Y^n) := \mathbb{E}[g(Z)|\mathcal{F}^Y_n]$.

In fact, by a routine approximation, it is enough that this holds for continuous $f_n$. Hence, if a sequence of processes $X^{(k)} = (X^{(k)}_1, \ldots, X^{(k)}_N)$ satisfies (3.3) as well as $(X^{(k)}, Z) \Rightarrow (X, Z)$ for some process $X$, then using Lemma 2.1 we conclude that $X$ must satisfy (3.3) as well. To
conclude that (ii) implies (i), simply notice that any $Y$-adapted process $X$ trivially verifies (3.5) because $Y$ is assumed to be $Z$-measurable.

**Remark 3.2.** In Theorem 3.1 it is natural to wonder if (i) and (ii) remain equivalent without the assumption that $\mathcal{X}$ is convex. This would complete the analogy with the static case, Proposition 2.2. But it is unclear if this is possible, as our method of proof relies on the continuity of $g^j$ in passing to the limit in (3.4).

It is worth clarifying the natural topological formulation of Theorem 3.1 in the setting where $Z = \mathcal{Y}^N$ and $Z = Y$. On the space $\mathcal{Y}^N \times \mathcal{X}^N$, let $Y = (Y_1, \ldots, Y_N)$ and $X = (X_1, \ldots, X_N)$ denote the canonical processes, i.e., the projections onto $\mathcal{Y}^N$ and $\mathcal{X}^N$, respectively. Let $\mathbb{F}^Y = (\mathcal{F}_t^Y)_{t \geq 0}$ and $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$ denote the time-$n$ marginal law. Let $\Pi^c(\mu)$ denote the set of compatible joint laws, i.e., the set of probability measures on $\mathcal{Y}^N \times \mathcal{X}^N$ with first marginal $\mu$ and under which $\mathcal{F}^X_n$ is conditionally independent of $\mathcal{F}^Y_n$ given $\mathcal{F}^Y_n$, for each $n = 1, \ldots, N$. Let $\Pi^c_0(\mu)$ denote the subset of adapted joint laws, i.e., the set of probability measures on $\mathcal{Y}^N \times \mathcal{X}^N$ of the form

$$\mu(dy)\delta_{\hat{x}}(y)(dx),$$

where $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_N) : \mathcal{Y}^N \to \mathcal{X}^N$ is adapted in the sense that $\hat{x}^{-1}(C) \in \mathcal{F}^Y_n$ for each set $C \in \mathcal{F}^X_n$ and each $n \in \{1, \ldots, N\}$.

Notice that $P \in \mathcal{P}(\mathcal{Y}^N \times \mathcal{X}^N)$ belongs to $\Pi^c(\mu)$ if and only if $P \circ Y^{-1} = \mu$ and the following holds: For each $n$ and each bounded continuous functions $f$, $h$, and $g$ on $\mathcal{X}^n$, $\mathcal{Y}^n$, and $\mathcal{Y}^N$, respectively, we have

$$\mathbb{E}^P \left[ f(X_1, \ldots, X_n) h(Y_1, \ldots, Y_n) (g(Y_1, \ldots, Y_N) - \tilde{g}(Y_1, \ldots, Y_n)) \right] = 0,$$

where $\tilde{g}(Y_1, \ldots, Y_n) = \mathbb{E}^\mu[g(Y_1, \ldots, Y_N) | \mathcal{F}^Y_n]$.

It follows then from Lemma 2.1 that $\Pi^c(\mu)$ closed. Deduce from Theorem 1.1 that if $\mu_0$ is nonatomic then $\Pi^c(\mu)$ is precisely the closure of $\Pi^c_0(\mu)$.

### 3.2. Continuous time, continuous paths.

We now extend the results of the previous section to continuous time. There several natural choices of path space for our processes, and we will work mainly with continuous and càdlàg paths. Let $C([\mathbb{R}_+; \mathcal{X})$ denote the space of continuous functions from $\mathbb{R}_+$ to $\mathcal{X}$, endowed with the topology of uniform convergence on compacts. Let $D([\mathbb{R}_+; \mathcal{X})$ denote the Skorohod space of càdlàg functions1 from $\mathbb{R}_+$ to (a metric space) $\mathcal{X}$, endowed with the usual Skorohod $J_1$ topology. Define $C([0, T]; \mathcal{X})$ and $D([0, T]; \mathcal{X})$ analogously on finite time intervals.

For a continuous-time stochastic process $X = (X_t)_{t \geq 0}$, let $\mathbb{F}^X = (\mathcal{F}^X_t)_{t \geq 0}$ denote the filtration generated by $X$, i.e., $\mathcal{F}^X_t = \sigma(X_s : s \leq t)$. Recall that a process $Y = (Y_t)_{t \geq 0}$ is $\mathbb{F}^X$-adapted if $Y_t$ is $\mathcal{F}^X_t$-measurable for each $t \geq 0$. For any filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, define as usual

$$\mathcal{F}_\infty = \sigma\left( \bigcup_{t \geq 0} \mathcal{F}_t \right).$$

Filtrations are not augmented in any way unless explicitly stated.

**Assumption (A).** We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting the following:

1. A random variable $Z$ taking values in a Polish space $\mathcal{Z}$.
2. A filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the following:
   a. $\mathcal{F}_t \subset \sigma(Z)$, for every $t \geq 0$.

---

1As usual, a function $f$ defined on an interval in $\mathbb{R}_+$ is called càdlàg if it is right-continuous and the limit $\lim_{s \uparrow t} f(s)$ exists for every $t > 0$. 
(b) \((\Omega, \mathcal{F}_t)\) is a standard Borel space for each \(t \geq 0\).

(c) \((\Omega, \mathcal{F}_t, \mathbb{P})\) is nonatomic for each \(t > 0\).

(3) A stochastic process \(X = (X_t)_{t \geq 0}\) taking values in a Polish space \(\mathcal{X}\), which is homeomorphic to a convex subset of a locally convex space.

An additional bit of structure will allow us to add another useful equivalence in the following results and streamline the proof:

**Assumption (B).** We are given for each \(t \geq 0\) a random variable \(U_t\) with values in a Polish space \(\mathcal{U}_t\) such that \(\mathcal{F}_t = \sigma(U_t)\).

It is worth noting that, with Assumption (A) in force, Assumption (B) always holds with \(\mathcal{U}_t = [0, 1]\) for every \(t\), thanks to Borel isomorphism and Assumption (A.3). In other words, Assumption (B) imposes no additional structural constraints. Note that we make requirements of measurability in \(t\) of the process \((U_t)\).

**Example 3.3.** Typically, we are given some càdlàg process \(Y = (Y_t)_{t \geq 0}\) with values in some Polish space \(\mathcal{Y}\) such that \(\mathcal{F}_t = \mathcal{F}_{t, Y}^t = \sigma(Y_s : s \leq t)\) is the natural filtration. In Assumption (B) we may then set \(\mathcal{U}_t = D([0, t]; \mathcal{Y})\) and \(U_t = Y|_{[0, t]}\).

One last technical assumption is needed to avoid difficulties at time zero:

**Assumption (C).** Either \(X_0\) is a.s. \(\mathcal{F}_0\)-measurable, or \((\Omega, \mathcal{F}_0, \mathbb{P})\) is nonatomic.

**Theorem 3.4 (Continuous paths).** Under Assumptions (A), (B), and (C), the following are equivalent:

(i) For each \(t \geq 0\), \(\mathcal{F}_t^X\) is conditionally independent of \(Z\) given \(\mathcal{F}_t\).

(ii) There exists a dense set \(T \subset \mathbb{R}_+\) such that, for each \(t \in T\), \(\mathcal{F}_t^X\) is conditionally independent of \(Z\) given \(\mathcal{F}_t\).

(iii) There exists a sequence \(X^n\) of continuous \(\mathbb{F}\)-adapted processes such that \((X^n, Z) \Rightarrow (X, Z)\) in \(C(\mathbb{R}_+; \mathcal{X}) \times \mathbb{Z}\).

(iv) Statement (iii) holds, and, for each \(n\), \(X^n\) linear between some time points \(0 = t_0 < t_1 < \cdots < t_n\), and constant after time \(t_n\), with \(X^n_k = f_k(U_{t_{k-1}})\) for some continuous function \(f_k : \mathcal{U}_{t_{k-1}} \rightarrow \mathcal{X}\), for each \(k = 1, \ldots, n\).

**Proof.** Clearly (i) implies (ii) and (iv) implies (iii). We next turn to the proof that (ii) implies (iv).

First, using the fact that \(\mathcal{X}\) is a convex subset of a vector space, it is straightforward to construct a sequence of continuous \(\mathbb{F}_t^X\)-adapted processes, \(X^n\), such that \(X^n \Rightarrow X\) a.s. in \(C(\mathbb{R}_+; \mathcal{X})\) and also each \(X^n\) is piecewise linear with a deterministic and finite set of points of non-differentiability. Thus, it suffices to prove the claim under the assumption that \(X\) is of the form

\[
X_t = \frac{t - t_k}{t_{k+1} - t_k} H_k + \frac{t_{k+1} - t}{t_{k+1} - t_k} H_{(k-1)+}, \quad \text{for } t \in [t_k, t_{k+1}], \; k = 0, \ldots, m - 1,
\]

where \(m \in \mathbb{N}\), the times \(0 = t_0 < t_1 < \cdots < t_n\) are deterministic, and \(H_0, \ldots, H_{m-1}\) are some \(\mathcal{X}\)-valued random variables. Assumption (ii) implies that \(H_k\) is conditionally independent of \(Z\) given \(\mathcal{F}_{t_k} = \sigma(U_{t_k})\) for each \(k = 0, \ldots, m - 1\). We will apply Theorem 3.3 with \(Y_k = U_{t_k}\) for each \(k\). Apply Theorem 3.3 to find a sequence \(H^n = (H^n_k)_{k=1}^{m-1}\) of \((\mathcal{F}_{t_k})_{k=1}^{m-1}\)-adapted processes such that

\[
(Z, H^n_1, \ldots, H^n_{m-1}) \Rightarrow (Z, H_1, \ldots, H_{m-1}),
\]

in \(Z \times \mathcal{X}^{m-1}\). In Assumption (C) if \(X_0 = H_0\) is \(\mathcal{U}_0\)-measurable, then setting \(H^n_0 := H_0\) then we may use Lemma 2.1 to get

\[
(Z, H^n_0, H^n_1, \ldots, H^n_{m-1}) \Rightarrow (Z, H_0, H_1, \ldots, H_{m-1}),
\]

(3.6)
in $Z \times \mathcal{X}^m$. On the other hand, in Assumption (C), if $\mathcal{F}_0 = \sigma(U_0)$ is nonatomic, then an application of Theorem 3.1 directly gives us a sequence $H^n = (H^n_k)_{k=0}^{m-1}$ of $(\mathcal{F}_t)_{k=0}^{m-1}$-adapted processes such that (3.6) holds. In either case, we may now define

$$X^n_t = \frac{t-t_k}{t_{k+1}-t_k} H^n_{t} + \frac{t_{k+1}-t}{t_{k+1}-t_k} H^n_{t_{k+1}},$$

for $t \in [t_k, t_{k+1})$, $k = 0, \ldots, m-1$, and conclude that $(X^n, Z) \Rightarrow (X, Z)$ in $C(\mathbb{R}_+; \mathcal{X}) \times Z$. Note that in the construction of $H^n = (H^n_k)_{k=0}^{m-1}$, Theorem 3.1 allows us to take $H^n_k = f^n_k(U_k)$ for some continuous functions $f^n_k$ on $U_k$. Hence, we have proven that (ii) implies (iv).

With the proof that (ii) implies (iv) now complete, we complete the chain of implications by showing that (iii) implies (i), arguing as in the proof of Theorem 3.1. Note that (i) holds if and only if for every $t$ and every bounded measurable functions $f$, $h$, and $g$ on $C([0, t]; \mathcal{X})$, $D([0, t]; \mathcal{Y})$, and $Z$, respectively, we have

$$\mathbb{E}[f(X|\sigma([0, t])) h(U_t)(g(Z) - \tilde{g}(U_t))] = 0, \quad (3.7)$$

where $\tilde{g}(U_t) = \mathbb{E}[g(Z)|\mathcal{F}_t]$. By a routine approximation, it is enough that this holds only for continuous $f$. Hence, if a sequence of continuous processes $X^n$ satisfies (3.7), as well as $(X^n, Z) \Rightarrow (X, Z)$ for some process $X$, then using Lemma 2.1 we conclude that $X$ must satisfy (3.7) as well. To conclude that (iii) implies (i), simply notice that any $\mathbb{F}$-adapted process $X$ trivially satisfies (3.7). \hfill \Box

**Remark 3.5.** Recalling the setting of Example 3.3, we deduce Theorem 1.2 from Theorem 3.4 by taking $Z = D(\mathbb{R}_+; \mathcal{Y})$ and $Z = Y$.

As in the previous section, we close this subsection with a topological formulation of Theorem 3.1 in the case described in Example 3.3 with $Z = D(\mathbb{R}_+; \mathcal{Y})$ and $Z = Y$. On the space $D(\mathbb{R}_+; \mathcal{Y}) \times C(\mathbb{R}_+; \mathcal{X})$, let $Y = (Y_t)_{t \geq 0}$ and $X = (X_t)_{t \geq 0}$ denote the canonical processes, and let $\mathbb{F}^Y$ and $\mathbb{F}^X$ denote their natural filtrations. Fix $\mu \in \mathcal{P}(D(\mathbb{R}_+; \mathcal{Y}))$, and for each $t \geq 0$ let $\mu_t \in \mathcal{P}(\mathcal{Y})$ denote the time-$t$ marginal law. Let $\Pi^c(\mu)$ denote the set of compatible joint laws, i.e., the set of probability measures $P$ on $D(\mathbb{R}_+; \mathcal{Y}) \times C(\mathbb{R}_+; \mathcal{X})$ with first marginal $\mu$ and under which $\mathcal{F}^Y_t$ is conditionally independent of $\mathcal{F}^X_t$ given $F^Y_t$, for every $t \geq 0$. Let $\Pi^c_0(\mu)$ denote the subset of adapted joint laws, i.e., the set of probability measures of the form

$$\mu(dy)\delta_{\hat{x}(y)}(dx),$$

where $\hat{x} : D(\mathbb{R}_+; \mathcal{Y}) \to C(\mathbb{R}_+; \mathcal{X})$ is adapted in the sense that $\hat{x}^{-1}(C) \in \mathcal{F}^Y_t$ for every $t \geq 0$ and every set $C \in \mathcal{F}^X_t$.

Notice that $P \in \mathcal{P}(D(\mathbb{R}_+; \mathcal{Y}) \times C(\mathbb{R}_+; \mathcal{X}))$ belongs to $\Pi^c(\mu)$ if and only if $P \circ Y^{-1} = \mu$ and the following holds: For each $t \geq 0$, each bounded continuous function $f : C([0, t]; \mathcal{X}) \to \mathbb{R}$, and each bounded measurable functions $h$ and $g$ on $D([0, t]; \mathcal{Y})$ and $D(\mathbb{R}_+; \mathcal{Y})$, respectively, we have

$$\mathbb{E}^P \left[ f(X_{\wedge t}) h((Y_{\wedge t})(g(Y) - \tilde{g}(Y_{\wedge t})) \right] = 0,$$

where $\tilde{g}(Y_{\wedge t}) = \mathbb{E}^\mu[g(Y)|\mathcal{F}^Y_t]$. It follows from Lemma 2.1 that $\Pi^c(\mu)$ closed. For each $x \in \mathcal{X}$, define

$$\mathcal{P}_x = \{ P \in \mathcal{P}(D(\mathbb{R}_+; \mathcal{Y}) \times C(\mathbb{R}_+; \mathcal{X})) : P(X_0 = x) = 1 \}.$$

We then deduce the following facts from Theorem 1.1

(a) If $\mu_0$ is nonatomic, then the closure of $\Pi^c_0(\mu)$ is $\Pi^c(\mu)$.

(b) Suppose $\mu_t$ is nonatomic for each $t > 0$, and $\mu \circ Y_0^{-1} = \delta_{y_0}$ for some $y_0 \in \mathcal{Y}$. Then, for each $x \in \mathcal{X}$, the closure of $\mathcal{P}_x \cap \Pi^c_0(\mu)$ is precisely $\mathcal{P}_x \cap \Pi^c(\mu)$. 


3.3. Continuous time, càdlàg paths. An analogue of Theorem 3.4 holds when the processes $X$ is merely càdlàg, but there is a subtle breakdown in the implication $(iii) \Rightarrow (i)$. To clarify the correct analogue in the càdlàg case requires a more careful choice of filtrations. For any filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, let $\mathbb{F}_+ = (\mathcal{F}_{t+})_{t \geq 0}$ denote the right-continuous version, defined by $\mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s$. A filtration $\overline{\mathbb{F}} = (\mathcal{F}_t)_{t \geq 0}$ is complete if $\mathcal{F}_0$ contains all of the null sets of $\mathcal{F}_\infty$. A generic filtration $\mathbb{F}$ can of course be rendered complete by appending to $\mathcal{F}_0$ all of the $\mathcal{F}_\infty$-null sets, and the resulting filtration, denoted $\overline{\mathbb{F}}$, is called the completion of $\mathbb{F}$. It is well known that $\mathbb{F}_+ = (\overline{\mathbb{F}}_+)$; that is, the operations of “completion” and “making right-continuous” commute.

**Theorem 3.6** (Càdlàg paths). Suppose Assumptions (A), (B), and (C) hold. Consider the following statements:

(i) For each $t \geq 0$, $\mathcal{F}_t^X$ is conditionally independent of $Z$ given $\mathcal{F}_t$.

(ii) There exists a dense set $\mathbb{T} \subset \mathbb{R}_+$ such that, for each $t \in \mathbb{T}$, $\mathcal{F}_t^X$ is conditionally independent of $Z$ given $\mathcal{F}_t$.

(iii) There exists a sequence $X^n$ of càdlàg $\overline{\mathbb{F}}$-adapted processes such that $(X^n, Z) \Rightarrow (X, Z)$ in $D(\mathbb{R}_+; \mathcal{X}) \times \mathcal{Z}$.

(iv) Statement (iii) holds, and, for each $n$, $X^n$ piecewise constant between some time points $0 = t_0 < t_1 < \cdots < t_n$, and constant after time $t_n$, with $X^n_{t+} = f_k(U_{t+})$ for some continuous function $f_k : U_{t+} \to \mathcal{X}$, for each $k = 1, \ldots, n$.

(v) For every $t \geq 0$, $\mathcal{F}_{t+}^X$ is conditionally independent of $Z$ given $\mathcal{F}_{t+}$.

Then (i) $\Rightarrow$ (ii) $\iff$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). If the completion of the filtration $\mathbb{F}$ is right-continuous, then (i-v) are equivalent. In particular, this holds if $\mathbb{F}$ is generated by a Feller process.

The proof makes use of a simple lemma which will be useful again in Section 3.4.

**Lemma 3.7.** Suppose $\mathbb{F}$ and $\mathcal{G}$ are two filtrations, and suppose $\mathcal{H}$ is another $\sigma$-field. Suppose $\mathcal{F}_t$ is conditionally independent of $\mathcal{H}$ given $\mathcal{G}_t$, for every $t \in \mathbb{T}$, where $\mathbb{T} \subset \mathbb{R}_+$ is dense. Then $\mathcal{F}_t^+$ is conditionally independent of $\mathcal{H}$ given $\mathcal{G}_{t+}$, for every $t \in \mathbb{R}_+$.

**Proof.** Fix $t \geq 0$, and find $t_n \in \mathbb{T}$ such that $t_n > t$ and $t_n \downarrow t$. Let $A \in \mathcal{H}$ and $B \in \mathcal{F}_{t+}$. Then $B \in \mathcal{F}_{t_n}$ for all $n$, so

$$\mathbb{P}(B \mid \mathcal{G}_{t_n}) \mathbb{P}(A \mid \mathcal{G}_{t_n}) = \mathbb{P}(B \cap A \mid \mathcal{G}_{t_n}).$$

By backward martingale convergence, letting $t_n \downarrow t$ yields

$$\mathbb{P}(B \mid \mathcal{G}_{t+}) \mathbb{P}(A \mid \mathcal{G}_{t+}) = \mathbb{P}(B \cap A \mid \mathcal{G}_{t+}).$$

$\square$

**Proof of Theorem 3.6.** Clearly (i) implies (ii) and (iv) implies (iii). It follows from Lemma 3.7 that (ii) implies (v).

We begin with the proof that (ii) implies (iv), by first approximating $X$ by piecewise constant processes. Indeed, by an approximation, we may assume $X$ is of the form

$$X_t = \sum_{k=0}^{m} H_k 1_{[t_k, t_{k+1})}(t),$$

where $0 = t_0 < t_1 < \cdots < t_m = T$ are deterministic, with $t_{m+1} := \infty$. As in the proof of step 3 of Theorem 3.4, we can apply Theorem 3.1 to find a sequence $H^n = (H^n_k)_{k=0}^m$ of $(\mathcal{F}_{t+}^X)_{t \geq 0}$-adapted processes such that

$$(Z, H^n_0, \ldots, H^n_m) \Rightarrow (Z, H_0, \ldots, H_m),$$
in \(Z \times \mathcal{X}^{m+1}\), with each \(H^n_k\) of the form \(H^n_k = f^n_k(U_{tk})\) for some continuous function \(f^n_k : U_{tk} \rightarrow \mathcal{X}\). Define
\[
X^n_t = \sum_{k=0}^{m} H^n_k 1_{[t_k,t_{k+1})}(t),
\]
and conclude that \((X^n, Z) \Rightarrow (X, Z)\) in \(D(\mathbb{R}_+; \mathcal{X}) \times \mathcal{Z}\).

We next prove that (iii) implies (ii). For any \(\mathbb{F}\)-adapted process \(X\), it holds that
\[
\mathbb{E}[f(X_{\lambda t})h(Y_{\lambda t})(g(Z) - \tilde{g}(Y_{\lambda t}))] = 0, \quad (3.8)
\]
where \(\tilde{g}(Y_{\lambda t}) = \mathbb{E}[g(Z) | \mathcal{F}_t]\). For every \(t\) and every bounded functions \(f, h,\) and \(g\) on \(D(\mathbb{R}_+; \mathcal{X}), D(\mathbb{R}_+; \mathcal{Y}),\) and \(\mathcal{Z}\), respectively. Suppose we are given a sequence of continuous processes \(X^n\) such that \((X^n, Z) \Rightarrow (X, Z)\) and also (3.8) holds (for every \(t\) and every \(f, h, g\)). At this point, we are faced with the annoying fact that the restriction map
\[D(\mathbb{R}_+; \mathcal{X}) \ni x \mapsto x|_{[0,t]} \in D([0,t]; \mathcal{X})\]
is continuous at a point \(x \in D(\mathbb{R}_+; \mathcal{X})\) if and only if \(x\) is continuous at \(t\). To get around this, note that \(\mathbb{T} = \{ t \geq 0 : \mathbb{P}(X_t = X_{t-}) = 1 \}\) is dense in \(\mathbb{R}_+\) (see [25, Lemma 3.7.7]). Hence, for \(t \in \mathbb{T}\), and for \(f, h,\) and \(g\) as above, we use Lemma 2.1 to conclude
\[
0 = \lim_{n \rightarrow \infty} \mathbb{E}[f(X^n_{\lambda t})h(U_t)(g(Z) - \tilde{g}(U_t))] = \mathbb{E}[f(X_{\lambda t})h(U_t)(g(Z) - \tilde{g}(U_t))].
\]
This proves (ii).

Finally, we prove that (v) implies (i) under the additional assumption that the completion of \(\mathbb{F}\) is right-continuous. Since \(\mathcal{F}^X_t \subset \mathcal{F}^X_{t+}\), (v) implies that \(\mathcal{F}^X_t\) is conditionally independent of \(Z\) given \(\mathcal{F}_{t+}\), for every \(t \geq 0\). But every set of \(\mathcal{F}_{t+}\) differs from a set in \(\mathcal{F}_t\) by an \(\mathcal{F}_\infty\)-null set, and we deduce easily that \(\mathcal{F}^X_t\) is conditionally independent of \(Z\) given \(\mathcal{F}_t\), for every \(t \geq 0\). Finally, it is well known that the completed natural filtration of a Feller process is automatically right-continuous; see [46, Theorem I.47].

\[\square\]

**Remark 3.8.** Recalling the setting of example 3.3 by taking \(Z = D(\mathbb{R}_+; \mathcal{Y})\) and \(Y = \mathcal{Y}\) we can deduce from Theorem 3.6 an analogue of Theorem 1.2 in which \(X\) is càdlàg.

A topological formulation of Theorem 3.6 (and Theorem 3.10 in the following section) is possible, along the lines of the discussions at the end of Section 3.2. We do not bother to spell this out here, as it differs only in notation from the aforementioned discussion.

### 3.4. Continuous time, measurable paths

We state one more alternative of Theorems 3.4 and 3.6 in the case that \(X\) is not continuous or even càdlàg but merely measurable. For a Polish space \(\mathcal{X}\), let \(M(\mathbb{R}_+; \mathcal{X})\) denote the set of equivalence classes of a.e. equal measurable functions from \(\mathbb{R}_+\) to \(\mathcal{X}\). Endow \(M(\mathbb{R}_+; \mathcal{X})\) with the topology of convergence in measure, with \(\mathbb{R}_+\) equipped with the measure \(e^{-x}dx\) (any finite measure equivalent to Lebesgue measure would do). Note that \(M(\mathbb{R}_+; \mathcal{X})\) is a Polish space, and \(x^n \rightarrow x\) in \(M(\mathbb{R}_+; \mathcal{X})\) if and only if \(x^n|_{[0,t]} \rightarrow x|_{[0,t]}\) in Lebesgue measure for each \(t > 0\).

A measurable \(\mathcal{X}\)-valued process can always be naturally identified with an \(M(\mathbb{R}_+; \mathcal{X})\)-valued random variable, but we must be careful to define its natural filtration in a way that reflects the topological setting. For each \(t > 0\), we may view the restriction \(X|_{[0,t]}\) as a \(M([0,t]; \mathcal{X})\)-valued random variable, and we let \(\mathcal{F}^X_t = \sigma(X|_{[0,t]}) = \sigma(\{X|_{[0,t]} \in A\} : A \subset M([0,t]; \mathcal{X}) \text{ Borel}).\)
Equivalently, we may write
\[ *\mathcal{F}_t^X = \sigma \left( \int_0^t h(s, X_s) ds : \text{for bounded continuous } h : [0, T] \times \mathcal{X} \to \mathbb{R} \right). \]

A priori, this could be smaller than the \( \sigma \)-field \( \mathcal{F}_t^X = \sigma(X_s : s \leq t) \), because passing to the equivalence class has lost us some “pointwise” information. This filtration \( *\mathcal{F}_t^X = (\mathcal{F}_t^X)_{t \geq 0} \) seems to be the appropriate one to use, as evidenced by Theorem 3.10 below.

**Remark 3.9.** The two natural filtrations \( *\mathcal{F}^X \) and \( \mathcal{F}^X \) never disagree by too much, in the sense that one can always find a modification of \( X \) for which they agree. More precisely, let \( \mathcal{G} = (\mathcal{G}_t)_{t \geq 0} \) denote the natural filtration on \( M(\mathbb{R}_+; \mathcal{X}) \) defined by letting \( \mathcal{G}_t \) denote the \( \sigma \)-field generated by the restriction map \( M(\mathbb{R}_+; \mathcal{X}) \ni x \mapsto x|_{[0, t]} \in M([0, t]; \mathcal{X}) \). Then there exists a measurable map \( \hat{x} : \mathbb{R}_+ \times M(\mathbb{R}_+; \mathcal{X}) \to \mathcal{X} \) which is predictable with respect to \( \mathcal{G} \) and which satisfies \( \hat{x}(t, x) = x_t \) for a.e. \( t \), for each \( x \).

**Theorem 3.10 (Measurable paths).** Suppose Assumptions \([A]\) and \([B]\) hold. Then the following are equivalent:

(i) For every \( t \geq 0 \), \( *\mathcal{F}_t^X \) is conditionally independent of \( Z \) given \( \mathcal{F}_t \).

(ii) There exists a dense set \( T \subset \mathbb{R}_+ \) such that, for each \( t \in T \), \( *\mathcal{F}_t^X \) is conditionally independent of \( Z \) given \( \mathcal{F}_t \).

(iii) There exists a sequence \( X^n \) of measurable \( \mathcal{F} \)-adapted processes such that \( (X^n, Z) \Rightarrow (X, Z) \) in \( M(\mathbb{R}_+; \mathcal{X}) \times \mathcal{Z} \).

(iv) Statement (iii) holds, and, for each \( n \), \( X^n \) piecewise constant between some time points \( 0 = t_0 < t_1 < \cdots < t_n \), and constant after time \( t_n \), with \( X^n_{t_k} = f_k(U_{k-1}) \) for some continuous function \( f_k : U_{k-1} \to \mathcal{X} \), for each \( k = 1, \ldots, n \).

**Proof.** Clearly (i) implies (ii) and (iv) implies (iii). We begin by proving that (ii) implies (iv). By an approximation, we can reduce to the case where \( X \) is piecewise constant. Our generic piecewise constant process \( X \) can now be assumed to be of the form

\[ X_t = H_01_{[0]}(t) + \sum_{k=0}^{m-1} H_k1_{(t_k, t_{k+1})}(t), \]

where \( 0 = t_0 < t_1 < \cdots < t_m \) are deterministic, with \( t_i \in \mathbb{T} \) for each \( i \). Note that we choose \( X \) to be left-continuous so that it is predictable. Now, because the topology of \( M(\mathbb{R}_+; \mathcal{X}) \) is not sensitive to pointwise changes, we may take \( H_0 \) to be deterministic; that is, we may assume \( X_t \) is constant and deterministic on a neighborhood of the origin. For this reason, we do not need Assumption \([C]\), and we apply Theorem 3.1 to find a sequence \( H^n = (H^n_k)_{k=1}^{m-1} \) of \( (\mathcal{F}_{t_k})_{k=1}^{m-1} \)-adapted processes such that

\[ (Z, H^n_1, \ldots, H^n_{m-1}) \Rightarrow (Z, H_1, \ldots, H_{m-1}), \]

in \( \mathcal{Z} \times \mathcal{X}^{m-1} \). Set \( H^n_0 = H_0 \) and define

\[ X^n_t = H^n_01_{[0]}(t) + \sum_{k=0}^{m-1} H^n_k1_{(t_k, t_{k+1})}(t). \]

Conclude that \( (X^n, Z) \Rightarrow (X, Z) \) in \( M(\mathbb{R}_+; \mathcal{X}) \times \mathcal{Z} \).

The proof that (iii) implies (i) is exactly the same as the last paragraph of the proof of Theorem 3.1. Note that the restriction map \( x \mapsto x|_{[0, t]} \) is continuous from \( M(\mathbb{R}_+; \mathcal{X}) \) to \( M([0, t]; \mathcal{X}) \), so we run into none problems that appeared in the analogous step of Theorem 3.6. \( \square \)
3.5. **On the case of two fixed marginals.** In light of Proposition 2.7 it is natural to wonder if there are analogues of the results of this section in which the marginal law of the process $X$ is fixed. More precisely, in Theorem 3.11 does the equivalence between (i) and (ii) still hold if we require in (ii) that $X^{(k)} = X$ for each $k$? Such a result would be useful in the theory of causal optimal transport, discussed in the introduction (and see [10, 6, 1]). Indeed, with such a result we could derive an analogue of (2.5) in the causal context. But this seems out of reach of our method of proof, which relies on the continuous approximations of Proposition 2.2(ii).

There is, however, one remarkable positive result in this direction, due to Emery [21]. Proposition 2 therein shows the following: If $X$ and $Y$ are $F$-Brownian motions of the same dimension on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, then there exists a sequence $(X^n)$ of Brownian motions such that $\mathbb{P}^{X^{n}} = \mathbb{F}^Y$ for each $n$ and $(X^n, Y) \Rightarrow (X, Y)$. In other words, any joint Brownian motions can be approximated weakly in law by mutually adapted Brownian motions. The proof in [21] does not seem to generalize, as it relies heavily on the characterization of a centered multivariate Gaussian distribution by its covariance matrix.

3.6. **On the notion of compatibility.** This section has made frequent mention of a compatibility property, which we discuss here in more generality. There is an ever-growing list of interesting equivalent conditions, some of which are summarized in the following:

**Theorem 3.11** (Theorem 3 of [14], Theorem 2 of [3]). *On some probability space, consider two filtrations $\mathcal{G}$ and $\mathcal{F}$ with $\mathcal{F} \subset \mathcal{G}$. The following are equivalent:*

(i) $\mathcal{G}_t$ is conditionally independent of $\mathcal{F}_\infty$ given $\mathcal{F}_t$, for every $t$.

(ii) Every bounded $\mathcal{F}$-martingale is a $\mathcal{G}$-martingale.

(iii) Every $\mathcal{G}$-stopping time $\tau$ is a $\mathcal{F}$-pseudo stopping time, meaning $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$ for every uniformly integrable $\mathcal{F}$-martingale.

(iv) For every $t$ and every integrable $\mathcal{F}_\infty$-measurable $X$, $\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{G}_t]$ a.s.

(v) For every $t$ and every integrable $\mathcal{G}_t$-measurable $X$, $\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{F}_\infty]$ a.s.

The last two statements are easily seen to be restatements of (i). The condition (ii) is known as the H-hypothesis in the filtering literature [14], or often as the immersion property in the literature on progressive enlargements of filtration [31], whereas (iii) is a recent result of [2].

There is a natural additional characterization available in the case where $\mathcal{F} = \mathcal{F}^Y$ is the (unaugmented) filtration generated by a càdlàg process $Y$ with independent increments and taking values in, say, a separable Fréchet space. In this case, the conditions (i-v) are equivalent to the following:

(vi) The increments of $Y$ are independent with respect to the larger filtration $\mathcal{G}$. That is, $(Y_t - Y_s)_{t \geq s}$ is independent of $\mathcal{G}_s$ for every $s \geq 0$.

This is most easily checked to be equivalent to property (v), by using the fact that $\mathcal{F}_\infty$ decomposes as $\mathcal{F}_\infty = \mathcal{F}_t \vee \sigma(Y_s - Y_t : s \geq t)$ for each $t \geq 0$.

In a sense, the notion of compatibility appeared in the work of Yamada and Watanabe [52], which illustrates how it arises naturally with weak solutions of stochastic differential equations. In this context, one simply needs the driving Brownian motion to remain Brownian relative to a larger filtration which includes the solution process. Similarly, the work of Jacod and Mémin [30] on semimartingale-driven stochastic differential equations crucially uses compatibility, which they referred to as very good extension. See Kurtz [30] for a recent elaboration of this theme. In the study of existence of (stochastic) optimal controls, El Karoui et al. [33] make use of this notion under the name of natural extension.
4. Randomized stopping times

We next turn to analogues of Theorems 3.4 and 3.6 in which the process \( X \) is replaced with a stopping time. Given a random time \( \tau \) (i.e., a \([0, \infty)\]-valued random variable), its natural filtration \( \mathbb{F}^\tau = (\mathcal{F}_t)_{t \geq 0} \) is defined by

\[
\mathcal{F}_t^\tau = \sigma(\tau \land s) = \sigma(\{t \leq s\} : s \leq t).
\]

Note that \( \mathbb{F}^\tau \) is right-continuous, and also that \( \mathcal{F}_\infty^\tau = \sigma(\tau) \). Given a filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \), we say that a random time \( \tau \) is a \( \mathbb{F} \)-stopping time if, as usual, \( \{\tau \leq t\} \in \mathcal{F}_t \) for every \( t \). Given also a random variable \( Z \) with \( \mathcal{F}_t \subset \sigma(Z) \) for all \( t \geq 0 \), we say that a random time \( \tau \) is an \((\mathbb{F}, Z)\)-randomized stopping time if \( \mathcal{F}_t^\tau \) is conditionally independent of \( Z \) given \( \mathcal{F}_t \), for every \( t \geq 0 \). The following are easily seen to be equivalent:

1. \( \tau \) is an \((\mathbb{F}, Z)\)-randomized stopping time.
2. \( \mathbb{P}(\tau \leq t | Z) = \mathbb{P}(\tau \leq t | \mathcal{F}_t) \) a.s., for every \( t \geq 0 \).

As an important special case, suppose that the random variable \( Z \) satisfies \( \sigma(Z) = \mathcal{F}_\infty \). The we shorten “\((\mathbb{F}, Z)\)-randomized stopping time” to “\( \mathbb{F} \)-randomized stopping time,” meaning one of the following equivalent conditions holds:

1. \( \mathcal{F}_t^\tau \) is conditionally independent of \( \mathcal{F}_\infty \) given \( \mathcal{F}_t \), for every \( t \geq 0 \).
2. \( \mathbb{P}(\tau \leq t | \mathcal{F}_\infty) = \mathbb{P}(\tau \leq t | \mathcal{F}_t) \) a.s., for every \( t \geq 0 \).
3. Every \( \mathbb{F} \)-martingale is an \( \mathbb{F} \lor \mathbb{F}^\tau \)-martingale.

The following extends [18, Theorem 6.4]:

**Theorem 4.1.** Under Assumptions \( [A](1-2) \) and \( [B] \) consider the following statements:

(i) \( \tau \) is an \((\mathbb{F}, Z)\)-randomized stopping time.

(ii) There exists a sequence of \( \mathbb{F} \)-stopping times \( \tau_n \) such that \((Z, \tau_n) \Rightarrow (Z, \tau) \) in \( Z \times [0, \infty] \).

(iii) There exists a sequence of \( \mathbb{F}_+ \)-stopping times \( \tau_n \) such that \((Z, \tau_n) \Rightarrow (Z, \tau) \) in \( Z \times [0, \infty] \).

(iv) \( \tau \) is an \((\mathbb{F}_+, Z)\)-randomized stopping time.

Then the implications (i) \( \Rightarrow \) (ii) \( \Leftrightarrow \) (iii) \( \Leftrightarrow \) (iv) hold. If the completion of \( \mathbb{F} \) is right-continuous, then (i-iv) are equivalent.

**Remark 4.2.** Recalling the setting of Example 3.3 we deduce Theorem 4.1 from Theorem 4.1 by taking \( Z = D(\mathbb{R}_+; \mathcal{Y}) \) and \( Z = Y \).

The proof is somewhat involved, so we prepare with a series of lemmas.

**Lemma 4.3.** For every bounded \( \mathcal{F}_t^\tau \)-measurable random variable \( H \), there exists a sequence of uniformly bounded functions \( g_n : [0, \infty] \rightarrow \mathbb{R} \) such that \( g_n(\tau) \rightarrow H \) in \( L^1 \), \( g_n(\tau) \) is \( \mathcal{F}_t^\tau \)-measurable, and \( g_n \) is continuous at every point but \( t \).

**Proof.** First notice that being \( \mathcal{F}_t^\tau \)-measurable, \( H \) is necessarily of the form:

\[
H = h(\tau)1_{[0,t]}(\tau) + c1_{(t,\infty]}(\tau)
\]

for some bounded measurable function \( h : [0, t] \rightarrow \mathbb{R} \) and some \( c \in \mathbb{R} \). Now simply approximate \( h \) in \( L^1 \) by continuous functions \( h_n : [0, t] \rightarrow \mathbb{R} \), and define

\[
g_n(s) = h_n(s)1_{[0,t]}(s) + c1_{(s,\infty]}(s).
\]

\[\square\]

**Lemma 4.4.** Define a map \( \Phi : D(\mathbb{R}_+; [0, 1]) \rightarrow [0, \infty] \) by

\[
\Phi(h) = \inf\{t \geq 0 : h(t) \geq 1/2}\}

Then \( \Phi \) is continuous at every point \( h \) of the form \( h(\cdot) = 1_{[s, \infty]}(\cdot) \), where \( s \geq 0 \).
Proof. First assume \( h(t) = 1_{[s,\infty)}(t) \) for some fixed \( s \geq 0 \), and let \( h^n \in D(\mathbb{R}_+; [0,1]) \) with \( h^n \to h \). Note that \( \Phi(h) = s \). It is straightforward to check that \( \{ \Phi(h^n) : n \geq 1 \} \) is bounded. Suppose, along a subsequence, that \( \Phi(h^n) \to t \), for some \( t \geq 0 \). According to \( [25] \) Proposition 3.6.5 the set \( \{ h^n(\Phi(h^n)) : n \geq 1 \} \) is precompact, and the set of limit points is contained in \( \{ h(t^-), h(t) \} \). As \( h(t) \geq h(t^-) \) and \( h^n(\Phi(h^n)) \geq 1/2 \), we must have \( h(t) = 1 \), or equivalently \( t \geq s \). Similarly, for each \( \epsilon > 0 \), the set \( \{ h^n(\Phi(h^n) - \epsilon) : n \geq 1 \} \) is precompact, and the set of limit points is contained in \( \{ h((t - \epsilon)^-) - h(t - \epsilon) \} \). But \( h^n(\Phi(h^n) - \epsilon) < 1/2 \), and we conclude that \( h((t - \epsilon)^-) = 0 \) or equivalently \( s \geq t - \epsilon \). Thus \( \Phi(h^n) \to s = \Phi(h) \). \( \square \)

As a final preparation, we show how any \((\mathbb{F}_+, \mathbb{Z})\)-randomized stopping time can be approximated in a suitable sense by \((\mathbb{F}, \mathbb{Z})\)-randomized stopping times. First, we recall the well known non-randomized analogue:

**Lemma 4.5** (Corollary IV.58 of [20]). Let \( \mathbb{F} \) be any filtration, and let \( \tau \) be an \( \mathbb{F}_+ \)-stopping time. Then there exists a sequence of \( \mathbb{F} \)-stopping times such that \( \tau_n \downarrow \tau \) a.s.

**Proof.** Set \( \tau_n = \tau + 1/n \). \( \square \)

**Lemma 4.6.** Let \( \tau \) be an \((\mathbb{F}_+, \mathbb{Z})\)-randomized stopping time. There exists a sequence of \((\mathbb{F}, \mathbb{Z})\)-randomized stopping times \( \tau_n \) such that \( (\mathbb{Z}, \tau_n) \Rightarrow (\mathbb{Z}, \tau) \).

**Proof.** Define \( \tau_n = \tau + 1/n \). To see that \( \tau_n \) is an \((\mathbb{F}, \mathbb{Z})\)-randomized stopping time, simply note that

\[
\mathbb{P}(\tau_n \leq t \mid \mathbb{Z}) = \mathbb{P}(\tau \leq t - 1/n \mid \mathbb{F}_{(t-1/n)^+})
\]

\[
= \mathbb{P}(\tau_n \leq t \mid \mathbb{F}_{(t-1/n)^+}).
\]

Because \( \mathbb{F}_{(t-1/n)^+} \subset \mathbb{F}_t \subset \sigma(\mathbb{Z}) \), this implies \( \mathbb{P}(\tau_n \leq t \mid \mathbb{Z}) = \mathbb{P}(\tau_n \leq t \mid \mathbb{F}_t) \). \( \square \)

**Proof of Theorem 4.1.** Clearly (ii) implies (iii), because every \( \mathbb{F} \)-stopping time is an \( \mathbb{F}_+ \)-stopping time. Moreover, every \( \mathbb{F}_+ \) stopping time can be written as the decreasing (almost sure) limit of a sequence of \( \mathbb{F} \)-stopping times (see Lemma 4.5). Hence, (iii) implies (ii). To prove (iv) implies (iii), we can use Lemma 4.6 to approximate our \((\mathbb{F}_+, \mathbb{Z})\)-randomized stopping time by \((\mathbb{F}, \mathbb{Z})\)-randomized stopping times, and we can then use the fact that (i) implies (iii), which we prove next.

To show that (i) implies (iii), fix an \((\mathbb{F}, \mathbb{Z})\)-randomized stopping time \( \tau \). Repeating the construction of Lemma 4.6, we may approximate \( \tau \) a.s. by \((\mathbb{F}, \mathbb{Z})\)-randomized stopping times which almost surely do not take the value zero. Approximating then by \( \tau \wedge n \), we may further assume that \( \tau < \infty \) a.s. Hence, without loss of generality, we assume \( \mathbb{P}(\tau = 0) = \mathbb{P}(\tau = \infty) = 0 \). Define \( H_t = 1_{\{\tau \leq t\}} \), and note that \( H \) is a càdlàg \([0,1]\]-valued process with \( \mathbb{P}(H_0 = 0) = 1 \). Moreover, \( \mathbb{F}^H_t = \mathbb{F}^\tau_t \) is conditionally independent of \( \mathbb{F}^\infty \) given \( \mathbb{F}_t \), for every \( t \geq 0 \).

We may now apply Theorem 3.6 to find a sequence of càdlàg \( \mathbb{F} \)-adapted \([0,1]\]-valued processes \( H^n \) such that \( (\mathbb{Z}, H^n) \Rightarrow (\mathbb{Z}, H) \) in \( \mathbb{Z} \times D(\mathbb{R}_+; [0,1]) \). Let \( \tau_n = \Phi(H^n) \), where \( \Phi \) was defined in Lemma 4.4. Then \( \tau_n \) is an \( \mathbb{F} \)-stopping time because \( H^n \) is right-continuous and \( \mathbb{F} \)-adapted. By Lemma 4.4, \( H \) almost surely belongs to the set of continuity points of \( \Phi \), and we conclude using Lemma 2.1 that

\[
(\mathbb{Z}, \tau_n) = (\mathbb{Z}, \Phi(H^n)) \Rightarrow (\mathbb{Z}, \Phi(H)) = (\mathbb{Z}, \tau), \text{ in } \mathbb{Z} \times [0, \infty].
\]

We next show that (iii) implies (iv). Let \( \mathbb{T} = \{ t \geq 0 : \mathbb{P}(\tau = t) = 0 \} \), and note that \( \mathbb{T} \) is dense in \( \mathbb{R}_+ \). Fix \( t \in \mathbb{T} \), and let \( f_t = f_t(\tau) \) be a bounded \( \mathbb{F}^\tau_t \)-measurable random variable which is continuous at every point but \( t \). Let \( h_t(\mathbb{Z}) \) be bounded and \( \mathbb{F}^\tau_t \)-measurable, and let \( g(\mathbb{Z}) \) be bounded and \( \mathbb{Z} \)-measurable. Because \( \tau_n \) is an \( \mathbb{F}_+ \) stopping time, for each \( n \), we have

\[
0 = \mathbb{E} \left[ f_t(\tau_n) h_t(\mathbb{Z}) (g(\mathbb{Z}) - \mathbb{E}[g(\mathbb{Z}) \mid \mathbb{F}^\tau_t]) \right].
\]
Use Lemma [2.1] to pass to the limit to get

\[ 0 = \mathbb{E} \left[ f_t(\tau) h_t(Z) (g(Z) - \mathbb{E}[g(Z)|\mathcal{F}_{t+\epsilon}]) \right]. \]

This holds for every bounded \( Z \)-measurable \( g \), for every bounded \( \mathcal{F}_{t+\epsilon} \)-measurable \( h_t \), and every bounded \( \mathcal{F}_t \)-measurable \( f_t \) which is continuous at every point but \( t \). In fact, this is valid for every bounded measurable \( \mathcal{F}_t \)-measurable \( f_t \), thanks to Lemma 4.3. We conclude that, for \( t \in \mathbb{T}, \mathcal{F}_t \) is conditionally independent of \( Z \) given \( \mathcal{F}_{t+\epsilon} \). Use Lemma 3.7 to complete the proof of (iv). Finally, under the additional assumption that the completion of \( \mathcal{F} \) is right-continuous, it is clear that (iv) and (i) are equivalent. \( \square \)

5. Extreme points

In the previous section we saw in what sense adapted processes (resp. stopping times) are dense in the set of compatible processes (resp. randomized stopping times), in a certain joint-distributional sense. In this section we study the convex structure of the associated sets of joint distributions. In fact, these same dense subsets are often precisely the extreme points.

5.1. Transfer principles. We first recall a crucial lemma, often known as the transfer principle. We will see in Sections 5.2 and 5.3 how a transfer principle, when available, immediately provides the extreme points of sets of joint distributions with one fixed marginal.

Lemma 5.1 (Theorem 6.10 of [32]). Suppose \( X \) and \( Y \) are random elements of measurable spaces \( \mathcal{X} \) and \( \mathcal{Y} \), respectively, where \( \mathcal{X} \) is a Polish space. Then there exist a measurable function \( f : \mathcal{Y} \times [0,1] \to \mathcal{X} \) and (perhaps on an extension of the probability space) a uniformly distributed random variable \( U \), independent of \((X,Y)\), such that \((X,Y) \overset{d}= (f(Y,U),Y)\).

A dynamic (discrete-time) version of Lemma 5.1 follows by induction:

Lemma 5.2. Suppose \( X = (X_n)_{n=1}^N \) and \( Y = (Y_n)_{n=1}^N \) are stochastic processes with values in measurable spaces \( \mathcal{X} \) and \( \mathcal{Y} \), respectively, where \( \mathcal{X} \) is a Polish space. If \( X_n \) is conditionally independent of \( \mathcal{F}_n^X \) given \( \mathcal{F}_n^Y \), for every \( n \), then there exist measurable functions \( f_n : \mathcal{Y}^n \times [0,1] \to \mathcal{X} \) and independent uniform random variables \( (U_1,\ldots,U_N) \) such that, if \( Z = (Z_1,\ldots,Z_N) \) where

\[ Z_n = f_n(Y_1,\ldots,Y_n,U_1,\ldots,U_n), \]

then \((Z,Y) \overset{d}= (X,Y)\).

Proof. Abbreviate \( X^n = (X_1,\ldots,X_n) \) for \( n \in \mathbb{N} \) and similarly for other processes. By Borel isomorphism, it suffices to show that there exist measurable functions \( f_n : \mathcal{Y}^n \times [0,1] \to \mathcal{X} \) and independent uniform random variables \( (U_1,\ldots,U_N) \) such that, if \( Z = (Z_1,\ldots,Z_N) \) where

\[ Z_n = f_n(Y_1,\ldots,Y_n,U_1,\ldots,U_n), \]

then \((Y,Z)\) has the same law as \((Y,X)\), where \( U_k \) is independent of \((X^k,Y^N)\) for each \( k = 1,\ldots,n \). We prove this by induction, with the \( N = 1 \) case is covered by Lemma 5.1. Suppose the claim is proven for some \( N \). Thanks to Lemma 5.1 we may find an independent uniform \( U_{N+1} \) and a measurable function \( g : \mathcal{X}^N \times \mathcal{Y}^{N+1} \times [0,1] \to \mathcal{X} \) such that

\[ (X^N,Y^{N+1},g(X^N,Y^{N+1},U_{N+1})) \overset{d}= (X^N,Y^{N+1},X_{N+1}). \]

We may take \( U_1,\ldots,U_{N+1} \) to be independent of each other and of \( Y^{N+1} \), with also \( U_k \) independent of \( X^k \) for each \( k = 1,\ldots,N+1 \). Because \( X^N \) is conditionally independent of \( Y^{N+1} \) given \( Y^N \), we have \((X^N,Y^{N+1}) \overset{d}= (Z^N,Y^{N+1})\). Hence,

\[ (X^N,Y^{N+1},g(X^N,Y^{N+1},U_{N+1})) \overset{d}= (Z^N,Y^{N+1},g(Z^N,Y^{N+1},U_{N+1})). \]
Finally, define
\[ f_{N+1}(y_1, \ldots, y_{N+1}, u_1, \ldots, u_{N+1}) = g\left(f_1(y_1, u_1), \ldots, f_N(y_1, \ldots, y_N, u_1, \ldots, u_N), y_1, \ldots, y_{N+1}, u_{N+1}\right). \]

In continuous time, there seems to be no analogous transfer principle in general, as the following example illustrates.

**Example 5.3.** Let \( W = (W_t)_{t \geq 0} \) and \( X = (X_t)_{t \geq 0} \) be continuous processes, defined on some common filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), such that \( X \) and \( W \) are adapted to \( \mathbb{F} \), \( W \) is an \( \mathbb{F} \)-Wiener process, and \((W, X)\) satisfy the stochastic differential equation
\[ X_t = \int_0^t b(s, X)ds + W_t, \] (5.1)
where \( b: \mathbb{R}^+ \times C(\mathbb{R}^+) \to \mathbb{R} \) is the function of Tsirelson’s example \([49]\). We know then that \( \mathcal{F}_t^X \) is strictly larger than \( \mathcal{F}_t^W \) for some \( t \geq 0 \). Because \( W \) is a Wiener process with respect to \( \mathbb{F} \), it follows that \( \mathcal{F}_t^X \) is conditionally independent of \( \mathcal{F}_\infty^W \) given \( \mathcal{F}_t^W \), for every \( t \geq 0 \).

Now suppose also that \( X = F(W, U) \) a.s., where \( U \) is uniformly distributed on \([0, 1]\) and independent of \( W \), and where \( F: C(\mathbb{R}^+) \times [0, 1] \to C(\mathbb{R}^+) \) is an adapted function. By “adapted function” we mean that for every \((w, w', u, t) \in C(\mathbb{R}^+) \times C(\mathbb{R}^+) \times [0, 1] \times \mathbb{R}_+\) satisfying \( w_s = w'_s \) for all \( s \leq t \) we have \( F(w, u)(t) = F(w', u)(t) \). The independence of \( W \) and \( U \) ensures that \( W \) is a Brownian motion (in its own filtration) under the regular conditional measure \( P_u := \mathbb{P}(\cdot \mid U = u) \), for Lebesgue-a.e. \( u \in [0, 1] \). Now, under \( P_u \), it holds that \( X \) is adapted to the completion of \( \mathbb{F}^W \), that \( W \) is Brownian motion under \( \mathbb{F}^W \), and that the SDE (5.1) holds a.s. But this provides a strong solution of the SDE (5.1), which is a contradiction.

### 5.2. Extreme points in the static case

Here we review how the transfer principle leads to a well known description of the extreme points of the convex set \( \Pi(\mu) \) of probability measures on \( \mathcal{X} \times \mathcal{Y} \) with fixed first marginal \( \mu \). This is a good point to recall the definitions of \( \Pi(\mu) \) and \( \Pi_0(\mu) \) from (2.1) and (2.2), as well as \( \Pi(\mu, \nu) \) and \( \Pi_0(\mu, \nu) \) defined in (2.3) and (2.4).

**Proposition 5.4.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Polish spaces, and let \( \mu \in \mathcal{P}(\mathcal{X}) \). The set \( \Pi(\mu) \) is convex. Moreover, every \( P \in \Pi(\mu) \) can be written as \( P(\cdot) = \int_{\Pi_0(\mu)} m(\cdot)M(dm) \), for some probability measure \( M \) on \( \Pi_0(\mu) \).

**Proof.** Convexity is obvious. Fix \( P \in \Pi(\mu) \). By Lemma 5.1 there exists a measurable function \( f: \mathcal{X} \times [0, 1] \to \mathcal{Y} \) such that \((\mu \times \text{Leb}) \circ (x, u) \mapsto (x, f(x, u))]^{-1} = P \). Define \( P_u \in \Pi_0(\mu) \) by
\[ P_u(dx, dy) = \mu(dy)\delta_{f(x,u)}(dy). \]
Then, for any bounded measurable function \( \varphi \) on \( \mathcal{X} \times \mathcal{Y} \), we have
\[
\int_{\mathcal{X} \times \mathcal{Y}} \varphi(x,y)P(dx,dy) = \int_0^1 \int_{\mathcal{X}} \varphi(x,f(x,u))\mu(dx)du \\
= \int_0^1 \int_{\mathcal{X} \times \mathcal{Y}} \varphi(x,y)P_u(dx,dy)du.
\]

**Remark 5.5.** Recalling from Proposition 2.2 that \( \Pi_0(\mu) \) is dense in \( \Pi(\mu) \) when \( \mu \) is nonatomic, we have encountered a closed convex set \( \Pi(\mu) \) which is the closure of its extreme points. This is not as peculiar as it may at first seem, and an intriguing result of Klee \([35]\) shows that this situation is generic in a topological sense in infinite dimensional Banach spaces.
5.3. Extreme points in discrete time. We now extend Proposition \[5.4\] which requires a bit of notation but follows essentially the same proof, taking the dynamic form of the transfer principle Lemma \[5.2\] for granted.

Fix \( N \) throughout this section. Let \( \mathcal{Y} \) be a measurable space and \( \mathcal{X} \) a Polish space. Let \( Y = (Y_1, \ldots, Y_N) \) denote the canonical process (identity map) on \( \mathcal{Y}^N \), and let \( \mathbb{P}^Y = (\mathcal{F}_n^Y)_{n=1}^\infty \) denote its natural filtration. Similarly, let \( X = (X_1, \ldots, X_N) \) denote the canonical process (identity map) on \( \mathcal{X}^N \), and let \( \mathbb{P}^X = (\mathcal{F}_n^X)_{n=1}^\infty \) denote its natural filtration. Both canonical processes extend in the obvious way to \( \mathcal{Y}^N \times \mathcal{X}^N \). Similarly, both filtrations extend in the natural way to \( \mathcal{Y}^N \times \mathcal{X}^N \), e.g., by identifying \( \mathcal{F}_n^Y \) with \( \mathcal{F}_n^X \otimes \{0, \mathcal{X}^N\} \).

Fix a joint distribution \( \mu \) on \( \mathcal{Y}^N \). Let \( \Pi^c(\mu) \) denote the set of compatible joint laws, i.e., the set of probability measures \( P \) on \( \mathcal{Y}^N \times \mathcal{X}^N \) with first marginal \( \mu \) and under which \( \mathcal{F}_n^X \) is conditionally independent of \( \mathcal{F}_n^Y \), given \( \mathcal{F}_n^Y \), for every \( n = 1, \ldots, N \). Let \( \Pi^c_0(\mu) \) denote the subset of adapted joint laws, i.e., the set of probability measures on \( \mathcal{Y}^N \times \mathcal{X}^N \) of the form

\[
\mu(dy)\delta_{\hat{x}(y)}(dx),
\]

where \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_N) : \mathcal{Y}^N \to \mathcal{X}^N \) is adapted in the sense that \( \hat{x}_n(y_1, \ldots, y_N) \) depends only on \( y_1, \ldots, y_n \), for each \( n \), or equivalently \( \hat{x}^{-1}(C) \in \mathcal{F}_n^Y \) for every set \( C \in \mathcal{F}_n^X \). Note that Theorem \[1.1\] says that \( \Pi^c_0(\mu) \) is dense in \( \Pi^c(\mu) \) when the law of \( Y_1 \) is nonatomic and \( \mathcal{X} \) is convex.

The following theorem describes the convex structure of \( \Pi^c(\mu) \). Taking some Choquet theory for granted, this is equivalent to the recent \[7, Theorem 6.1\].

**Theorem 5.6.** The set \( \Pi^c(\mu) \) is convex. Moreover, every \( P \in \Pi^c(\mu) \) can be written as \( P(\cdot) = \int_{\Pi^c_0(\mu)} m(\cdot) M(d\mu) \), for some probability measure \( M \) on \( \Pi^c_0(\mu) \).

**Proof.** Convexity is straightforward after noticing that membership in \( \Pi^c(\mu) \) is characterized by a family of linear constraints; see \[3.5\]. Fix \( P \in \Pi^c(\mu) \). By Lemma \[5.2\], there exist measurable functions \( f_n : \mathcal{Y}^n \times [0, 1] \to \mathcal{X} \), for \( n = 1, \ldots, N \), such that, if \( f : \mathcal{Y}^N \times [0, 1] \to \mathcal{X}^N \) is defined by

\[
f(y_1, \ldots, y_N, u) = (f_1(y_1, u), \ldots, f_N(y_1, \ldots, y_N, u)),
\]

then \( (\mu \times \text{Leb}) \circ [(x, u) \mapsto (x, f(x, u))]^{-1} = P \). Define \( P_u \) by

\[
P_u(dy, dx) = \mu(dy)\delta_{f(x, u)}(dx),
\]

and note that the structure of \( f \) ensures that \( P_u \in \Pi^c_0(\mu) \). Finish as in the proof of Proposition \[5.4\]. \( \square \)

5.4. Extreme points in continuous time. Using Example \[5.3\] and adapting the proof of Theorem \[5.6\] we can show that the natural analogue of Theorem \[5.6\] fails in continuous time. Let us make this precise: Let \( \mathcal{X} = C(R_+) \). Let \( \mathbb{F}^i = (\mathcal{F}_t^i)_{t \geq 0} \) denote the two filtrations generated by the canonical processes \( (X^1, X^2) \) on \( \mathcal{X}^2 \). Let \( \Pi^c(\mathcal{W}) \) denote the set of joint laws on \( \mathcal{X}^2 \) compatible with Wiener measure. Precisely, let \( \mathcal{W} \) denote Wiener measure on \( \mathcal{X} \), and let \( \Pi^c(\mathcal{W}) \) denote the set of \( Q \in \mathcal{P}(\mathcal{X}^2) \) with first marginal \( \mathcal{W} \) such that \( \mathcal{F}_t^2 \) is conditionally independent of \( \mathcal{F}_t^1 \) given \( \mathcal{F}_t^1 \), for every \( t \geq 0 \). Let \( \Pi^c_0(\mathcal{W}) \) denote the subset of adapted joint laws, i.e., the set of probability measures on \( \mathcal{X}^2 \) of the form

\[
\mathcal{W}(d\omega)\delta_{F(\omega)}(dx),
\]

where \( F : \mathcal{X} \to \mathcal{X} \) is a measurable and adapted function in the sense that \( F(\omega)(t) = F(\omega')(t) \) whenever \( \omega(s) = \omega'(s) \) for all \( s \leq t \).

Next, recall the probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) and the processes \( W \) and \( X \) defined in Example \[5.3\] Let \( P = \mathbb{P} \circ (W, X)^{-1} \), and note that \( P \in \Pi^c(\mathcal{W}) \). We claim that there is no measurable
map \([0,1] \ni u \mapsto P_u \in \Pi_0^c(W)\) such that \(P(\cdot) = \int_0^1 P_u(\cdot) du\). Indeed, suppose there were such a map \(u \mapsto P_u\). By the disintegration theorem, we may write
\[
du P_u(d\omega, dx) = du \mathcal{W}(d\omega) \delta_{F(\omega, \cdot)}(dx),
\]
for some measurable function \(F : [0,1] \times X \to X\) with the property that \(F(u, \cdot)\) must be an adapted function for each \(u\). This would imply that we have an adapted map \(F\) for which \(\mathbb{P}(X = F(U, W)) = 1\). As explained in Example 5.3, this is impossible.

### 5.5. Extreme points of randomized stopping times

Although the previous section showed that the extreme point story breaks down in continuous time, this section shows that there is no such difficulty when working with stopping times as in Section \(\text{H}\). The description of extreme points described in Theorem 5.5 is not new (see, e.g., \([27, 22]\)), but we include it for the sake of completeness.

Let \((\Omega, \mathcal{F}, \mu)\) be an arbitrary probability space. Let \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) be a filtration on \(\Omega\), and with some abuse of notation let \(\mathbb{F}\) denote the same filtration extended to \(\overline{\Omega} := \Omega \times [0, \infty)\) by identifying \(\mathcal{F}_t\) with \(\mathcal{F}_t \otimes \{0, [0, \infty]\}\). Similarly, let \(\tau\) denote the identity map on \([0, \infty]\), and extend \(\tau\) to \(\overline{\Omega}\) by setting \(\tau(\omega, t) = t\). Let \(\mathbb{F}^\tau = (\mathcal{F}_t^\tau)_{t \geq 0}\) denote the filtration on \([0, \infty]\) defined by
\[
\mathcal{F}_t^\tau = \sigma(\{\tau \leq s\} : s \leq t),
\]
and again abuse notation by considering \(\mathbb{F}^\tau\) as a filtration on \(\overline{\Omega}\).

Let \(\Pi^c(\mu)\) denote the set of \(\mathbb{F}\)-randomized stopping time laws, defined more precisely as the set of \(P \in \mathcal{P}(\overline{\Omega}) = \mathcal{P}(\Omega \times [0, \infty])\) with first marginal \(\mu\) and with \(\mathcal{F}_t^\tau\) conditionally independent of \(\mathcal{F}_\infty\) given \(\mathcal{F}_t\), for every \(t \geq 0\). The latter constraint is equivalent to requiring \(P(\tau \leq t|\mathcal{F}_\infty) = P(\tau \leq t|\mathcal{F}_t)\) a.s., for every \(t\). Let \(\Pi_0^c(\mu)\) denote the subset of \(\mathbb{F}\)-stopping time laws, defined as
\[
\Pi_0^c(\mu) = \{\mu(d\omega)\delta_{\tau(\omega)}(dt) : \tau : \Omega \to [0, \infty)\text{ is an }\mathbb{F}\text{-stopping time}\}.
\]

Let \(\Pi_0^c(\mu)\) be a filtration on \(\overline{\Omega}\), and define \(\Pi_0^c(\mu) \ni P \mapsto P(C)\), for sets \(C \in \mathcal{F}_\infty \vee \mathcal{F}_\infty^\tau\).

**Theorem 5.7.** The set \(\Pi^c(\mu)\) is convex. Moreover, every \(P \in \Pi^c(\mu)\) can be written as \(P(\cdot) = \int_{\Pi_0^c(\mu)} m(\cdot) M(d\nu)\) for some probability measure \(M\) on \(\Pi_0^c(\mu)\).

**Proof.** To prove convexity is straightforward, simply note that \(P \in \Pi^c(\mu)\) if and only if \(P(\cdot \times [0, \infty]) = \mu(\cdot)\) and
\[
\int_{\Omega \times [0, \infty]} (f(\omega) - f_t(\omega)) 1_{[0,t]}(s) P(d\omega, ds) = 0,
\]
for every \(t \geq 0\) and every bounded \(\mathcal{F}_\infty\)-measurable function \(f\) on \(\Omega\), with \(f_t := \mathbb{E}^\mu[f|\mathcal{F}_t]\).

Let \(P \in \Pi^c(\mu)\), and disintegrate \(P(d\omega, dt) = \mu(d\omega)P(\omega, dt)\). Define \(A : \Omega \times [0,1] \to [0, \infty]\) by
\[
A(\omega, u) = \inf\{t \geq 0 : P(\omega, [0,t]) \geq u\}.
\]
Then \(P(\omega, \cdot) = \text{Leb} \circ A(\omega, \cdot)^{-1}\), and we conclude that for \(t \geq 0\)
\[
P(\omega, [0,t]) = \int_0^1 1_{A(\omega, u) \leq t} du = \int_0^1 \hat{P}_u(\omega, [0,t]) du,
\]
where we define \(P_u \in \Pi_0^c(\mu)\) for \(u \in [0,1]\) by setting \(\hat{P}_u(\omega, dt) = \delta_{A(\omega, u)}(dt)\) and \(P_u(d\omega, dt) = \mu(d\omega)\hat{P}_u(\omega, dt)\). This shows that \(P(\cdot) = \int_0^1 P_u(\cdot) du\). \(\square\)
6. Stochastic optimal control

In this section we illustrate how the notion of compatibility arises naturally in stochastic optimal control. Consider a standard Borel probability space \((\Omega, \mathcal{F}, P)\) supporting a càdlàg process \(Z\), with values in \(\mathbb{R}^d\), which is a semimartingale in its own filtration. Let \(T > 0\) denote the time horizon, and let \(A\) be a Polish space, called the action space. We are given a function

\[ g : [0, T] \times \Omega \times \mathbb{R}^d \times A \to \mathbb{R}^{d \times m}. \]

Assume the following:

1. \((g_1)\) \(g = g(t, \omega, x, a)\) is jointly continuous in \((x, a)\) and \(\mathbb{F}^Z\)-predictable in \((t, \omega)\).
2. \((g_2)\) \(g\) is Lipschitz in \(x\), uniformly in \((t, \omega, a)\).
3. \((g_3)\) There exists \(c > 0\) such that \(|g(t, \omega, x, a)| \leq c(1 + |x|)\), for all \((t, \omega, x, a)\).

For a filtration \(G\) on \(\Omega\), let \(\mathcal{A}(G)\) denote the set of \(G\)-predictable \(A\)-valued processes.

Let \(Y = (Y_t)_{t \geq 0}\) be another \(\mathbb{F}^Z\)-adapted càdlàg process, the role of which is to determine what information is available to the agent. Assume \(Y_t\) is nonatomic for every \(t > 0\). Now fix a filtration \(G\) satisfying \(\mathbb{F}^Z \subset G\) and \(G_t \subset \mathcal{F}\), and also \(G_t\) is conditionally independent of \(\mathcal{F}^Z\) given \(\mathcal{F}^Y_t\), for every \(t \geq 0\). Thanks to the implication \((i) \implies (ii)\) of Theorem \(3.11\), \(Z\) is a \(G\)-semimartingale. Hence, for every \(\alpha \in \mathcal{A}(G)\), there exists a unique (see [30, Theorem 4.5]) process \(X^\alpha = (X^\alpha_t)_{t \in [0, T]}\) satisfying

\[ dX^\alpha_t = g(t, X^\alpha_{t-}, \alpha_t) dZ_t, \quad X^\alpha_0 = x_0, \]

where \(x_0 \in \mathbb{R}^d\) is a fixed initial state.

Consider the optimization problem,

\[ \sup_{\alpha \in \mathcal{A}(\mathbb{F}^Y)} F(\mathcal{L}(X^\alpha, \alpha, Z)), \]

where \(\mathcal{L}(X^\alpha, \alpha, Z) = P \circ (X^\alpha, \alpha, Z)^{-1}\) and \(F : \mathcal{P}(D(\mathbb{R}^d_+; \mathbb{R}^d) \times M(\mathbb{R}^d_+, A) \times D(\mathbb{R}^d_+, \mathbb{R}^m)) \to \mathbb{R}\). The following theorem shows how to relax the \(\mathbb{F}^Y\)-adaptedness requirement:

**Theorem 6.1.** Suppose the restriction of \(F\) to the set \(\{\mathcal{L}(X^\alpha, \alpha, Z) : \alpha \in \mathcal{A}(G)\}\) is continuous. Then

\[ \sup_{\alpha \in \mathcal{A}(\mathbb{F}^Y)} F(\mathcal{L}(X^\alpha, \alpha, Z)) = \sup_{\alpha \in \mathcal{A}(G)} F(\mathcal{L}(X^\alpha, \alpha, Z)). \]  \(\quad(6.1)\)

**Proof.** The inequality \((\leq)\) is clear, as \(\mathcal{A}(\mathbb{F}^Y) \subset \mathcal{A}(G)\). To prove the reverse, it suffices to approximate an arbitrary \(\alpha \in \mathcal{A}(G)\) in a suitable sense. To apply Theorem \(3.10\) note that \(\mathcal{F}^\alpha_t\) is conditionally independent of \(Z\) given \(\mathcal{F}^\alpha_t\), for every \(t \geq 0\). Hence, there exists a sequence \(\alpha^n \in \mathcal{A}(\mathbb{F}^Y)\) such that \((\alpha^n, Z) \Rightarrow (\alpha, Z)\) in \(M(\mathbb{R}^d_+, A) \times D(\mathbb{R}^d_+, \mathbb{R}^d)\). Finally, it should not be surprising that we can conclude that \((X^{\alpha^n}, \alpha^n, Z) \Rightarrow (X^\alpha, \alpha, Z)\), e.g., by using the results of Jacod and Mémin [30], namely Theorems 2.25, 3.24, and 4.5 therein.

As a special case, if \(Y = Z\), then for \(G\) we may choose any filtration compatible to \(\mathbb{F}^Z\) in the sense of any of the equivalent conditions of Theorem \(3.11\). Note also that we could easily generalize Theorem \(6.1\) by replacing the Lipschitz assumption with some kind of weak uniqueness, and by weakening the continuity assumption on \(F\) to some order of Wasserstein-continuity as long as there are suitable moment estimates available for the processes \(X^\alpha\).

Traditionally, \(F\) is of the form \(F(P) = \int G dP\), where \(G\) is the sum of an integrated running reward and a terminal reward. When \(F\) is linear in this sense, an alternative proof of Theorem \(6.1\) would proceed by approximating a general control \(\alpha \in \mathcal{A}(G)\) by a piecewise constant one and then using the extreme point representation of Theorem \(5.6\) (cf. [34, Section 4]). For nonlinear functions \(F\), however, the extreme point argument fails, and one must resort to Theorem \(3.10\).
Optimal control problems with nonlinear $F$ have become increasingly relevant in recent years in the context of controlled McKean-Vlasov systems (see [15, 39, 44] and the references therein).

A key use of a result like Theorem 6.1 is in finding compactness to facilitate existence proofs. Theorem 3.11 shows that we can conduct the optimization over all compatible joint laws of $(\alpha, Z)$, and we have seen that the compatibility condition defines a closed set of probability measures. A final important step is to choose a better path space for the controls, as opposed to $M(\mathbb{R}_+; A)$, in which compact sets are scarce. This is typically done by working with relaxed controls, also known as Young measures, but we do not go through this here (see [33, 28, 37]).

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