Reduction of Feynman integrals in the parametric representation III: integrals with cuts

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Abstract Phase space cuts are implemented by inserting Heaviside theta functions in the integrands of momentum-space Feynman integrals. By directly parametrizing theta functions and constructing integration-by-parts (IBP) identities in the parametric representation, we provide a systematic method to reduce integrals with cuts. Since the IBP method is available, it becomes possible to evaluate integrals with cuts by constructing and solving differential equations.

1 Introduction

Feynman integrals with cuts are frequently encountered in perturbative calculations in high energy physics, especially while calculating various jet observables and event-shape distributions. Generally, cuts are implemented by inserting Heaviside theta functions in the integrands in the momentum space. The presence of theta functions largely complicates the calculations of Feynman integrals.

The most widely used technique to reduce Feynman integrals is the integration-by-parts (IBP) method [1,2]. However, it is not clear how to directly apply the regular IBP method to integrals with cuts. In a recent paper [3], theta functions were written as integrals of delta functions. The resulting integrals were reduced by combining the reverse unitarity [4] and the IBP method. By using this method, one has to introduce an extra scale for each theta function. Consequently, the reduction becomes much more complicated for integrals with several cuts. Thus the application of this method to more complicated integrals is far from trivial.

On the other hand, it was suggested that IBP identities can directly be derived in the parametric representation [5,6]. It can be shown that each momentum-space IBP identity [7] corresponds to a shift relation in the parametric representation [8]. Since a theta function has an integral representation quite similar to the Schwinger parametrization of a propagator, it is possible to directly parametrize theta functions and construct IBP identities in the parametric representation. In this paper, we show that the methods developed in Refs. [9,10] (referred to as paper I and paper II, respectively, hereafter) to parametrize and reduce tensor integrals can be applied to integrals with theta functions with slight modifications.

This paper is organized as follows. In Sect. 2, we show how to use the method developed in paper I and paper II to parametrize integrals with cuts and to construct IBP identities for them. Some detailed examples are provided in Sect. 3.

2 Parametrization and IBP identities

It is well known that a propagator can be parametrized by

\[
\frac{1}{D_{i}^{\lambda_{i}+1}} = \frac{e^{-\frac{\lambda_{i}+1}{2}i\pi}}{\Gamma(\lambda_{i}+1)} \times \int_{0}^{\infty} dx_{i} e^{i\lambda_{i}D_{i}} x_{i}^{\lambda_{i}}, \quad \text{Im}[D_{i}] > 0.
\]

(2.1)

Heaviside theta functions have a similar integral representation,

\[
\theta(D_{i}) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} dx_{i} \frac{e^{ixD_{i}}}{x_{i} + i0^{+}}.
\]

For future convenience, we define the function

\[
w_{\lambda}(u) \equiv e^{-\frac{\lambda+1}{2}i\pi} \int_{-\infty}^{\infty} dx \frac{1}{x^{\lambda+1}} e^{ixu}.
\]

(2.2)

It is easy to see that

\[
w_{0}(u) = 2\pi \theta(u),
\]
\[
w_{-1}(u) = 2\pi \delta(u),
\]
\[
w_{-2}(u) = 2\pi \delta'(u).
\]

With this representation, the standard procedure to parametrize Feynman integrals can easily be generalized to integrals with

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theta functions. Following the convention used in paper I and paper II, we have

\[ M \equiv e^{-\frac{L}{2}} \left[ \prod_{i=1}^{d} I_{1} dI_{2} \cdots dI_{L} w_{\lambda_{1}}(D_{1}) w_{\lambda_{2}}(D_{2}) \cdots w_{\lambda_{m}}(D_{m}) \right] \]

\[ \times \left[ \prod_{i=1}^{d} \frac{d^{\lambda_{i}+1}}{d^{\lambda_{i}+1}} \frac{1}{D_{m+1}^{\lambda_{m+1}+1}} \cdots \frac{1}{D_{n}^{\lambda_{n}+1}} \right] \]

\[ = s_{\lambda} \left[ I_{n+1} - \frac{1}{2} \sum_{i=1}^{d} (\lambda_{i} + \frac{1}{2}) \right] I(\lambda_{0}, \lambda_{1}, \ldots, \lambda_{n}). \]

(2.3)

where \( s_{\lambda} \) is the determinant of the \( d \)-dimensional metric, and \( \lambda_{n+1} \equiv -(L+1)\lambda_{0} - 1 + \sum_{i=0}^{n} \lambda_{i} - \sum_{i=m+1}^{n} (\lambda_{i} + 1) \), with \( \lambda_{0} \equiv -\frac{L}{2} \). We have the parametric integral

\[ I(\lambda_{0}, \ldots, \lambda_{n}) \equiv \int d\Pi^{(n+1)} \mathcal{F}(-n-1) \]

\[ = \frac{\Gamma(\lambda_{0})}{\prod_{i=1}^{n+1} \Gamma(\lambda_{i} + 1)} \times \int d\Pi^{(n+1)} x_{i}^{\lambda_{i}} \]

(2.4)

Here the measure is \( d\Pi^{(n)} \equiv \prod_{i=1}^{n+1} dx_{i} \delta(1 - \sum_{j=1}^{l} x_{j}) \), where the sum in the delta function runs over any nontrivial subset of \( \{x_{1}, x_{2}, \ldots, x_{n+1}\} \). The polynomial \( \mathcal{F}(x) \equiv F(x) + U(x) x_{n+1} \). \( U \) and \( F \) are symmetric polynomials, defined by \( U(x) \equiv \det A \), and \( F(x) \equiv U(x) (\sum_{i=1}^{L} A^{-1} i_{i} B_{i} - B_{j} - C) \). Polynomials \( A, B, \) and \( C \) are defined through \( \sum_{i=1}^{n} x_{i} D_{i} \equiv \sum_{i=1}^{L} A_{i} i_{i} l_{i} + 2 \sum_{i=1}^{L} B_{i} l_{i} + C \).

It should be noticed that in the definition of the parametric integral in Eq. (2.4), for a “propagator” \( w_{\lambda_{i}}(D_{i}) \), there is no corresponding gamma function in the prefactor. And the corresponding index \( \lambda_{i} \) can be both positive and negative.

Similar to the parametric IBP identities derived in paper I, we have

\[ 0 = \int d\Pi^{(n+1)} \frac{\partial}{\partial x_{i}} \mathcal{F}(-n-1), \quad i = 1, 2, \ldots, m. \]

(2.5a)

\[ 0 = \int d\Pi^{(n+1)} \frac{\partial}{\partial x_{i}} \mathcal{F}(-n-1) + \delta_{i,0} \int d\Pi^{(n)} \mathcal{F}(-n) \bigg|_{x_{i}=0} , \]

(2.5b)

\[ i = m + 1, m + 2, \ldots, n + 1. \]

We define the index-shifting operators \( R_{i}, D_{i}, \) and \( A_{i} \), with \( i = 0, 1, \ldots, n \), such that

\[ R_{i} I(\lambda_{0}, \ldots, \lambda_{i}, \ldots, \lambda_{n}) = (\lambda_{i} + 1) I(\lambda_{0}, \ldots, \lambda_{i} + 1, \ldots, \lambda_{n}), \]

\[ D_{i} I(\lambda_{0}, \ldots, \lambda_{i}, \ldots, \lambda_{n}) = I(\lambda_{0}, \ldots, \lambda_{i} - 1, \ldots, \lambda_{n}), \]

\[ A_{i} I(\lambda_{0}, \ldots, \lambda_{i}, \ldots, \lambda_{n}) = \lambda_{i} I(\lambda_{0}, \ldots, \lambda_{i}, \ldots, \lambda_{n}). \]

(2.6)

It is understood that

\[ I(\lambda_{0}, \ldots, \lambda_{i-1}, -1, \ldots, \lambda_{n}) \equiv \int d\Pi^{(n)} \mathcal{F}(-n) \bigg|_{x_{i}=0} . \]

3 Examples

We first consider the following simple but interesting example:

\[ I_{1} \left( \frac{d}{2}, \lambda_{1}, \lambda_{2} \right) \equiv \frac{i}{\pi d/2} \]

\[ \times \int d^{d}r w_{\lambda_{1}}(a^{2} - r^{2}) w_{\lambda_{2}}(a^{2} - (r - 2b)^{2}) \]

By using the method I described in paper II (cf. Eq. (3.17) therein), we get the following IBP identities:

\[ A_{1} - A_{2} - 4b^{2}D_{1} + 4b^{2}D_{2} + 2D_{1}D_{2} - D_{1}R_{2} \approx 0, \]

\[ 2A_{0} - 2A_{1} - A_{2} + 2a^{2}D_{1} + 2a^{2}D_{2} - 4b^{2}D_{2} - 2D_{2}R_{1} \approx 0. \]
Specifically, we consider the reduction of the integral \( I_1(-d/2, 0, 0) \). By solving IBP identities, we get

\[
I_{1a} = -\frac{i}{4\pi} d^{d-2} I_1(-d/2, 0, 0)
\]

\[
= \int d^d r \, \theta(a^2 - r^2) \, \theta(a^2 - (r - 2b)^2) 
\]

\[
= 4a^2 \int d^d r \, \delta(a^2 - r^2) \, \theta(a^2 - (r - 2b)^2) 
\]

\[
= \frac{16b^2(a^2 - b^2)}{d(d-1)} 
\]

\[
\times \int d^d r \, \delta(a^2 - r^2) \, \delta(a^2 - (r - 2b)^2) 
\]

\[
\equiv \frac{4a^2}{d} I_{1b} - \frac{16b^2(a^2 - b^2)}{d(d-1)} I_{1c}. 
\]

This result has an interesting geometric interpretation. It is easy to see that the integral \( I_{1a} \) is nothing but the volume of the intersection of two \( d \)-dimensional balls with a radius \( a \) separated by a distance of \( 2b \), as is shown in Fig. 1a. 2\( ab \) is the volume of this \( d \)-dimensional cone. Similarly, \( 8b\sqrt{a^2 - b^2} I_{1c} \) is the volume of the \( d \)-dimensional cone (with a flat bottom) shown in Fig. 1c. Hence Eq. (3.1) just tells us how to calculate the volume of the intersection of two balls.

By solving IBP identities, we get the following differentation equations:

\[
\frac{\partial}{\partial b^2} \left( I_{1b} \right) = \left( \frac{0}{0 - \frac{2b}{2b^2 - 2}} \right) \left( I_{1b} \right).
\]

It is easy to check that the solutions of these equations do agree with the result obtained by a direct calculation.

As a less trivial example, we consider the reduction of the integral

\[
I_2 = \frac{(2\pi)^6}{\pi^{d-4}} \times \int d^d l_1 d^d l_2 
\]

\[
\times \frac{\delta(l_{12}^+ \delta(l_{12}^- - a) \delta(l_{12}^- - b) \theta(l_1^- - l_1^-) \theta(l_2^+ - l_2^-))}{l_1^+ l_1^- (l_1^+ + l_2^-)}.
\]

Here the lightcone coordinates are used. That is, \( l_i^+ \equiv l_i \cdot n \), and \( l_i^- \equiv l_i \cdot \bar{n} \), with \( n^2 = \bar{n}^2 = 0 \), and \( n \cdot \bar{n} = 2 \). This integral is relevant for the calculation of the two-loop hemisphere soft functions [16]. This integral can be reduced to

\[
I_2 = -\frac{2}{(d-4)ab} \frac{(2\pi)^6}{\pi^{d}} 
\]

\[
\times \int d^d l_1 d^d l_2 
\]

\[
\times \frac{\delta(l_{12}^+ \delta(l_{12}^- - a) \delta(l_{12}^- - b) \theta(l_1^- - l_1^-) \theta(l_2^+ - l_2^-))}{l_1^+ l_1^- (l_1^+ + l_2^-)}
\]

\[
- \frac{1}{ab} \frac{(2\pi)^6}{\pi^{d}} \times \int d^d l_1 d^d l_2 
\]

\[
\times \frac{\delta(l_{12}^+ \delta(l_{12}^- - a) \delta(l_{12}^- - b) \theta(l_1^- - l_1^-) \theta(l_2^+ - l_2^-))}{l_1^+ l_1^- (l_1^+ - l_2^-)}.
\]

The detailed calculation is carried out by using a home-made Mathematica code. We have verified this result by explicit calculations of these integrals.
To validate our method, we have also applied this method to some practical calculations. For example, we reproduce the decay rate for the four-lepton decay $\gamma^* \rightarrow l\bar{l}l\bar{l}$, which can be obtained from the decay rate of the four-quark decay $\gamma^* \rightarrow q\bar{q}q\bar{q}$ [17,18] by stripping off some color factors. The detailed calculation is carried out as follows. We first generate IBP identities by using the method described in this paper. Then we solve these identities by using the package Kira [19].

4 Summary

By directly parametrizing Heaviside theta functions and constructing IBP identities in the parametric representation, we provide a systematic method to reduce integrals with cuts. We show that the methods developed in paper I and paper II to parametrize and to reduce regular Feynman integrals can be applied to integrals with cuts by slightly modifying the definitions of the index-shifting operators. Differential equations can also be constructed. Thus, in principle, the standard differential equation method can be used to evaluate integrals with cuts.

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