Communication lower bounds for nested bilinear algorithms via rank expansion of Kronecker products

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Abstract. We develop lower bounds on communication in the memory hierarchy or between processors for nested bilinear algorithms, such as Strassen’s algorithm for matrix multiplication. We build on a previous framework that establishes communication lower bounds by use of the rank expansion, or the minimum rank of any fixed size subset of columns of a matrix, for each of the three matrices encoding a bilinear algorithm. This framework provides lower bounds for a class of dependency directed acyclic graphs (DAGs) corresponding to the execution of a given bilinear algorithm, in contrast to other approaches that yield bounds for specific DAGs. However, our lower bounds only apply to executions that do not compute the same DAG node multiple times. Two bilinear algorithms can be nested by taking Kronecker products between their encoding matrices. Our main result is a lower bound on the rank expansion of a matrix constructed by a Kronecker product derived from lower bounds on the rank expansion of the Kronecker product’s operands. We apply the rank expansion lower bounds to obtain novel communication lower bounds.
for nested Toom-Cook convolution, Strassen’s algorithm, and fast algorithms for contraction of partially symmetric tensors.

**Keywords** Communication lower bounds · Bilinear algorithm · Kronecker product · Rank expansion · Strassen’s algorithm · Convolution · Tensor contraction

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1 Introduction

In high-performance computing, communication cost (i.e., data movement across the memory hierarchy and/or between processors) has been shown to be more time-consuming and energy-draining than arithmetic costs [26]. And the gap is expected to continue to grow. The increased utilization of hierarchical memory and mesh topology (to address physical constraints in hardware, e.g., speed-of-light and voltage limitations) favors methods exploiting data locality, further motivating well-designed communication patterns [11, 12]. Therefore, it is imperative to design algorithms that minimize communication. Communication lower bounds provide a theoretical limit and guide the design of algorithms that minimize communication [5, 17, 22].

The pioneering work by Yao [37] introduced communication lower bounds for computing a boolean function between processors by viewing the computation as a decision tree. Later, Hong and Kung introduced the study of communication lower bounds for several algorithms in the memory hierarchy by modeling the computation as a dependency directed acyclic graph, or dependency DAG, and representing the data access patterns through a red-blue pebble game [22]. Since then, new techniques have been developed to derive more lower bounds. For nested loop programs with relatively simple access patterns, such as the classical matrix multiplication and LU factorization, volumetric inequalities such as the Loomis-Whitney inequality [29] or the more general Hölder-Brascamp-Lieb inequalities [13, 21] can be used to derive communication lower bounds [3, 5, 8, 16, 23]. However, fast algorithms such as Strassen’s subcubic matrix multiplication algorithm [36] have complicated data access patterns, so volumetric inequalities do not solely suffice in this setting.

Another approach is to directly study the dependency DAG. This approach applies graph-theoretic tools, such as the graph expansion, dominating sets, or graph partitioning by eigenvalues [6, 9, 10, 12, 15, 22, 24]. In general, this family of techniques aims to uncover bounds on the size of minimum cuts or vertex separators of the dependency DAG. To derive closed-form communication lower bounds, one needs additional constraints or a priori information on the dependency DAG. For example, the graph expansion argument [6] requires the DAG to have bounded degrees to derive sharp lower bounds, and the graph
partitioning argument [24] requires one to know the eigenvalues of the corresponding Laplacian matrix to derive analytic lower bounds. Furthermore, the graph-theoretic techniques must fix a dependency DAG. However, most algorithms admit algebraic reorganizations (i.e., computation of different partial sums) that change the dependency graph and may be more communication efficient in a particular setting.

By working with more abstract algorithm representations, a larger space of admissible dependency graphs can be considered simultaneously. Hypergraphs have been used to capture potential orderings of partial sums [7], while bilinear algorithms [31] provide a more powerful abstraction for problems that can be posed as bilinear maps on two input sets. Many important numerical problems fall under this category, including matrix multiplication, convolution, and symmetric tensor contractions, and all known fast algorithms for these problems can be expressed as bilinear algorithms.

A bilinear algorithm \((A, B, C)\) with \(A \in \mathbb{C}^{m_A \times R}, B \in \mathbb{C}^{m_B \times R},\) and \(C \in \mathbb{C}^{m_C \times R}\) computes \(f(x, y) = C[(A^T x) \odot (B^T y)],\) where \(\odot\) is the Hadamard product (elementwise or bilinear). The value \(R\) is called the rank of the bilinear algorithm. When a subset of columns from \(A, B,\) or \(C\) is a low-rank matrix, then the communication costs can be reduced for executing this portion of the computation. To see why, let \(P\) consist of a subset of \(k\) different columns from an identity matrix of dimension \(R\) so that a portion of the bilinear algorithm associated with \(k\) of the \(R\) bilinear products is \(CP^T (AP^T x) \odot (BP^T y).\)

We see that \(\text{rank}(AP), \text{rank}(BP),\) and \(\text{rank}(CP)\) bound the minimum number of linear combinations of inputs needed from \(x, y,\) and the amount of output information produced, respectively, in computing this portion of the bilinear algorithm. Lower bounds on the growth of this rank with \(k,\) i.e., the rank expansion, yield lower bounds on communication for any execution DAG of the bilinear algorithm. Such an execution DAG must only compute linear combinations of inputs or the bilinear forms, while the order of operations and intermediate values can be arbitrarily specified. See [35, Definition 4.1] for more details. In particular, we do not restrict the order in which products and sums are carried out nor prohibit intermediate values from being reused between different outputs, such as in the independent evaluation model studied by Hong and Kung [22]. The rank expansion of a matrix also characterizes its Kruskal rank [27], which is the smallest \(k\) for which any \(k\) columns of \(A\) are linearly independent, i.e., \(\text{rank}(AP) = k\) for any choice of \(P.\) We formally define bilinear algorithms and their motivation for developing fast bilinear algorithms in Section 2.

We focus on nested bilinear algorithms [31], which are bilinear algorithm constructed via Kronecker products of matrices encoding the two factor bilinear algorithms: \((A_1 \otimes A_2, B_1 \otimes B_2, C_1 \otimes C_2).\) This abstraction captures both recursive and higher-order methods for matrix multiplication, polynomial multiplication, convolution, tensor contractions, as well as other algorithms. We show in general the rank expansion for the matrices defining a nested bilinear algorithm is based on the rank expansion of its factors. We prove that for a certain class of rank expansion lower bounds \(\sigma_A\) and \(\sigma_B\) for \(A\) and \(B,\) respec-
tively, there exists a rank expansion lower bound $\sigma_C$ for $C = A \otimes B$ given by

$$\sigma_C(k) = \min_{k_A \in [1, n_A], k_B \in [1, n_B], k_A k_B = k} \sigma_A(k_A) \sigma_B(k_B),$$

where $n_X = \#\text{cols}(X)$. This result is a generalization of the identity, $\text{rank}(A \otimes B) = \text{rank}(A) \text{rank}(B)$.

A formal overview of our results is described in Section 3.

To prove our result, we start by introducing the grid framework in Section 4. This framework provides a visual interpretation for our matrix rank analysis. Using the grid framework, we show it suffices to consider the rank of a subset of columns of $A \otimes B$ that has a compact geometric structure on a 2D grid. With this structure, we prove two main theorems in Section 5. First, we show how to use the compact geometric structure to lower bound the rank of a matrix by solving a nontrivial discrete optimization problem. Second, to simplify the optimization problem, we apply a continuous relaxation. We can bound the solution to the continuous optimization problem by considering the optimal shape of the subgrid for general and more restricted $\sigma_A$ and $\sigma_B$. In Section 5, we derive our main result, $(\sigma_C(k) \leq \sigma_A(k_A) \sigma_B(k_B))$ by reducing general subgrids to rectangles. We also sharpen these lower bounds in Appendix A by instead considering an L-shaped geometry (we reserve these results for the appendix as we do not employ them for any of the applications considered).

Equipped with the general bounds derived in Section 5, we apply our framework to fast algorithms for matrix multiplication, convolution, and partially symmetric tensor contractions in Section 6. We obtain lower bounds on both sequential communication (communication in a two-level memory hierarchy) as well as parallel communication (communication between processors in a distributed-memory computer with a fully connected network). The latter bounds can be translated to the LogGP and BSP model [35]. Our lower bounds are all novel in that they consider a larger space of algorithms than previous works. We obtain the first communication lower bounds for nested symmetry preserving tensor contraction algorithms [34], lower bounds for multi-dimensional and recursive Toom-Cook (i.e., convolution) that match previously known bounds [10, 15], and lower bounds for Strassen’s algorithm, which are asymptotically lower than previous results [6, 9]. See Table 1 for a comparison between previously known lower bounds and the lower bounds derived in this paper.

### Table 1: Communication lower bounds for the Strassen’s fast matrix-matrix multiplication and nested Toom-k for 1D convolution.

| Algorithm       | Previous (S) | Previous (P) | This Paper (S) | This Paper (P) |
|-----------------|--------------|--------------|----------------|----------------|
| Strassen’s      | $\frac{n^{\log_2(7)}}{M^{\log_2(7)-1}}$ | $\frac{\sqrt{n}}{M^{\log_2(4)}}$ | $\frac{n^{\log_2(7)}}{M^{\log_2(5)-1}}$ | $\frac{n^{\log_3(7)}}{M^{\log_3(2)}}$ |
| Recursive       | $\frac{n^{\log_2(2k-1)}}{M^{\log_2(2k-1)-1}}$ | $\frac{n}{M^{\log_2(2k-1)-1}}$ | $\frac{n}{M^{\log_2(2k-1)-1}}$ | $\frac{n}{M^{\log_2(2k-1)-1}}$ |
| convolution     | $\frac{n^{\log_2(2k-1)}}{M^{\log_2(2k-1)-1}}$ | $\frac{n}{M^{\log_2(2k-1)-1}}$ | $\frac{n}{M^{\log_2(2k-1)-1}}$ | $\frac{n}{M^{\log_2(2k-1)-1}}$ |
2 Notation, Definitions, and Preliminaries

2.1 Notational Conventions

We will denote \( \mathbb{N} = \{1, 2, \ldots\} \) to be the natural numbers and \( \mathbb{R}_+ \) as the set of nonnegative reals. For any \( n \in \mathbb{N} \), we write \([n] = \{1, 2, \ldots, n\}\).

We denote the pseudoinverse of an increasing function \( f \) as

\[
\tilde{f}(x) = \sup \{ k : f(k) \leq x \}. \tag{1}
\]

The Kronecker product and Hadamard (entrywise) product are, respectively, \( \otimes \) and \( \odot \). Finally, for convenience we make the following definition.

Definition 1. Let \( \{e_1, \ldots, e_n\} \) be the standard basis vectors of \( \mathbb{R}^n \). For \( k \in [n] \), define

\[
P_n^{(k)} = \left\{ (e_{i_1} | \ldots | e_{i_k}) : 1 \leq i_1 < \ldots < i_k \leq n \right\}.
\]

In other words, \( P_n^{(k)} \) is the collection of operators \( P \) such that \( AP \) selects \( k \) columns of \( A \).

2.2 Bilinear Algorithms

The target of the communication lower bound framework studied in [34,35] is analysis of bilinear algorithms. We now formally define these.

Definition 2. A bilinear algorithm is defined by a matrix triplet,

\[
(A, B, C),
\]

(where \( A \in \mathbb{R}^{m_A \times R} \), \( B \in \mathbb{C}^{m_B \times R} \), and \( C \in \mathbb{C}^{m_C \times R} \)), which takes in two inputs \( x \in \mathbb{C}^{m_A} \) and \( y \in \mathbb{C}^{m_B} \) and computes an output \( z \in \mathbb{C}^{m_C} \) where

\[
z = f(x, y) = C[(A^T x) \odot (B^T y)].
\]

Here, \( R \) is referred to as the rank of the bilinear algorithm. We refer to the multiplication \( A^T x \) and \( B^T y \) as the encoding step and the multiplication with \( C \) as the decoding step. Similarly, matrices \( A \) and \( B \) may be referred to as the encoding matrix while \( C \) is the decoding matrix.

The power of this framework is the ability to explicitly express the use of recursion in a bilinear algorithm via Kronecker (tensor) products.

Definition 3. A nested bilinear algorithm is a bilinear algorithm whose matrix triplet is defined by Kronecker products, i.e.,

\[
\left( \bigotimes_{i=1}^{\tau} A_i, \bigotimes_{i=1}^{\tau} B_i, \bigotimes_{i=1}^{\tau} C_i \right).
\]
Nested bilinear algorithms are the main tool to develop fast bilinear algorithms. For example, instead of using the naïve eight elementwise products, Strassen’s algorithm for $2 \times 2$ matrix multiplication only requires seven \cite{36}. Recursively applying the bilinear algorithm via a nested bilinear algorithm yields the well-known Strassen’s algorithm with $O(n^{\log_2(7)})$ computational complexity. We refer to the survey \cite{31} for a detailed discussion.

Communication lower bounds for bilinear algorithms can be reasoned by exploiting the sparsity pattern of the encoding and decoding matrices. This approach has found success for the naïve $O(n^3)$ matrix multiplication by utilizing volumetric arguments, such as the Hölder-Brascamp-Lieb inequalities \cite{13,21}, since the sparsity structure of encoding and decoding matrices from the naïve algorithm is relatively simple, i.e., one nonzero per column of the matrix. In contrast, the respective matrices for Strassen’s algorithm have a dense structure, so volumetric inequalities do not solely suffice here.

### 2.3 Communication Cost Lower Bounds for Bilinear Algorithms

In this paper, we derive lower bounds on communication in two standard settings: sequential and parallel. In the sequential setting, we consider a fast but small memory of size $M$ (e.g., cache), where computation is performed, and slow but large memory (e.g., main memory). In the parallel setting, we consider $P$ processors that communicate over a network and bound the largest amount of data sent or received by any of the $P$ processors. In both cases, we quantify communication cost in the number of data elements, which may be elements of the input or output, as well as intermediate values (partial sums of inputs or of products of inputs). Finally, our computational model prohibits recomputation in the bilinear algorithm, meaning that if a bilinear product has already been computed and is not in fast memory or the current processor, the algorithm must communicate the value instead of recomputing it \cite{35}. This restriction stems from the assumption in proofs of the general lower bounds framework we build on (specifically, Lemma 5.2 in \cite{35}). Extension of these communication lower bounds to permit recomputation is of interest and has been explored for related problems \cite{9,10,15,22}. For integer multiplication and matrix multiplication, past works have shown that the best known lower bounds hold even when allowing recomputation \cite{9,10,15,30}.

Lower bounds for both settings are studied in the work of \cite{35}. They are derived by establishing the expansion bound, which we define next. Recall that $R$ is the associated rank of a bilinear algorithm and $\mathcal{P}_R^{(k)}$ is the collection of operators selecting $k$ columns from a matrix with $R$ columns (Definition 1).

**Definition 4.** The bilinear algorithm $(A, B, C)$ with rank $R$ has expansion $E^*: \mathbb{N}^3 \rightarrow \mathbb{N}$ if for any triplet $k_A, k_B, k_C \in [R]$,

$$
E^*(k_A, k_B, k_C) = \max_{P \in \mathcal{P}_R^{(k)}, \forall k} \{ \#\text{cols}(P) : \text{rank}(AP) \leq k_A, \text{rank}(BP) \leq k_B, \text{rank}(CP) \leq k_C \}.
$$
Likewise, the same bilinear algorithm has non-decreasing (in all variables) expansion bound $E : \mathbb{N}^3 \mapsto \mathbb{R}$ if $E$ upper bounds $E^*$, i.e.,

$$E^*(k_A, k_B, k_C) \leq E(k_A, k_B, k_C), \quad \forall k_A, k_B, k_C \in [R].$$

We relaxed the expansion bound from $[34,35]$ to be from $\mathbb{N}^3 \mapsto \mathbb{R}$ instead of $\mathbb{N}^3 \mapsto \mathbb{N}$, which can be done without loss of generality by rounding down non-integer values. The expansion function quantifies the largest subset of the bilinear algorithm we can complete when given a limited set of data, whereas the expansion bound is an easier-to-compute upper bound on the expansion function. These functions are closely related to the edge expansion of a graph $G$, a function previously used to derive communication lower bounds via the graph expansion framework $[6,10,12,22]$. The edge expansion helps to lower bound the number of edges leaving any sufficiently small subgraph of $G$ relative to the number of vertices in the subgraph. Similarly, the expansion bound helps to lower bound the rank of any subset of a bilinear algorithm relative to the size of the subset.

Equipped with this quantity, $[35]$ develops communication lower bounds in both the sequential and parallel setting. We start with the sequential setting.

**Proposition 5 (Theorem 5.3 [35])** Given a bilinear algorithm $(A, B, C)$ (where $A \in \mathbb{C}^{m_A \times R}$, $B \in \mathbb{C}^{m_B \times R}$, and $C \in \mathbb{R}^{m_C \times R}$), a corresponding expansion bound function $E$, and a sequential model with fast memory of size $M$, then any procedure for computing the bilinear algorithm must communicate at least

$$\max \left\{ \frac{2RM}{E_{\text{max}}(M)}, m_A + m_B + m_C \right\},$$

data elements between fast and slow memory, where

$$E_{\text{max}}(M) = \max_{r(A), r(B), r(C) \in \mathbb{N}} \left\{ E(r(A), r(B), r(C)) \right\}$$

and $E_{\text{max}}(M)$ is assumed to be increasing and convex.

The second term $m_A + m_B + m_C$ accounts for the reading of the input and writing of the output. Next, we state the parallel communication lower bound. To be consistent with $[35]$, we assume the algorithm is storage-balanced: at the beginning of the algorithm, the input is evenly distributed among $P \in \mathbb{N}$ processors. In the end, the output is evenly distributed among the processors.

**Proposition 6 (Theorem 5.4 [35])** Given a bilinear algorithm $(A, B, C)$ (where $A \in \mathbb{C}^{m_A \times R}$, $B \in \mathbb{C}^{m_B \times R}$, and $C \in \mathbb{R}^{m_C \times R}$), a corresponding expansion bound function $E$, and a parallel model with $P$ processors, then any storage-balanced procedure for computing the bilinear algorithm must communicate at least

$$r(A) + r(B) + r(C)$$
data elements between processors, where $r(A), r(B), r(C) \in \mathbb{N}$ satisfy

$$\frac{R}{P} \leq \mathcal{E}\left(\frac{r(A)}{P} + \frac{m_A}{P}, \frac{r(B)}{P} + \frac{m_B}{P}, \frac{r(C)}{P} + \frac{m_C}{P}\right).$$

Thus, expansion lower bound $\mathcal{E}$ yields lower bounds on sequential and parallel communication cost directly via these two propositions.

### 2.4 Rank Expansion Bounds

To characterize the expansion lower bound needed for our communication bounds ($\mathcal{E}$), we consider the rank of subsets of columns of each individual encoding/decoding matrix. Recall that $P \in \mathcal{P}_k(n)$ selects a subset of $k$ columns.

**Definition 7.** The rank expansion of $A \in \mathbb{C}^{m \times n}$, $\tilde{\sigma} : [n] \mapsto \mathbb{N}$, is defined as

$$\tilde{\sigma}(k) = \min_{P \in \mathcal{P}_k(n)} \{\text{rank}(AP)\}.$$

Let $\tilde{\sigma}_A$, $\tilde{\sigma}_B$, and $\tilde{\sigma}_C$ be rank expansion functions for matrices $A$, $B$, and $C$, respectively. Since, for any $P \in \mathcal{P}_k(n)$, $\tilde{\sigma}_A(\#\text{col}(P)) \leq \text{rank}(AP)$ (and similar for $B$, $C$), these rank expansion functions yield an expansion bound $\mathcal{E}$ (Definition 4) of the form,

$$\mathcal{E}(r(A), r(B), r(C)) := \min\{\tilde{\sigma}_A(r(A)), \tilde{\sigma}_B(r(B)), \tilde{\sigma}_C(r(C))\},$$

(2)

for any $r(A), r(B), r(C) \in \mathbb{N}$, where recall the dagger is the pseudoinverse defined in (1). While relatively easy to compute, the proposed expansion bound above may not be tight with the expansion function $\mathcal{E}$, as shown in the example below.

**Example 1** Consider the bilinear algorithm $(A, B, C)$ for standard matrix multiplication of two $n \times n$ matrices. Matrices $A$, $B$, and $C$ each contain $n$ copies of each column from an identity matrix of dimension $n^2$ (see [35, Lemma B.1] for an idea on the sparsity pattern). Then for any $r(A), r(B), r(C) \in [n^2]$, the Loomis-Whitney inequality [35, Lemma B.1] produces an expansion bound

$$\mathcal{E}_{\text{LW}}(r(A), r(B), r(C)) := \sqrt{r(A)r(B)r(C)},$$

while one can show (2) simplifies to

$$\mathcal{E}_{\text{RankExp}}(r(A), r(B), r(C)) := n \cdot \min\{r(A), r(B), r(C)\}.$$

When the fast memory size is not large, i.e., $\max\{r(A), r(B), r(C)\} \ll n^2$, then $\mathcal{E}_{\text{LW}}$ is tighter. The gap between the two expansion bounds arises because a subset of columns from one matrix, say $A$, may be low rank while the same subset for another matrix, say $B$, may be nearly full rank. The Loomis-Whitney inequality seems to account for this low-rank property while (2) does not.
However, the main advantage of the proposed expansion bound (2) is that it can derive bounds for any bilinear algorithm. In contrast, the Loomis-Whitney inequality requires simple access patterns [5], which excludes its application for fast algorithms such as Strassen’s algorithm [36].

As defined, the rank expansion is a discrete function, which makes it challenging to derive simple closed-form expressions for its pseudoinverse. Instead, we seek to lower bound \( \tilde{\sigma} \) by a continuous increasing function \( \sigma \).

**Definition 8.** Let \( A \in \mathbb{C}^{m \times n} \) with rank expansion \( \tilde{\sigma} \). A rank expansion lower bound \( \sigma \) for \( A \) is a continuous, nonnegative, and increasing function \( \sigma \) on \( \mathbb{R}^+ \) such that \( \sigma(k) \leq \tilde{\sigma}(k) \) for all \( k \in [n] \).

For a nested bilinear algorithm (Definition 3), we seek to obtain \( \sigma_A \) from the rank expansion \( \sigma_i \) of each term \( A_i \) in the Kronecker product defining \( A \) (and similar for \( B, C \)). Often \( \sigma_i \) is easy to obtain, e.g., for Strassen’s algorithm each \( A_i \) is the same 4-by-7 matrix, while in other cases each \( A_i \) may vary in size but may have a known/simple rank expansion.

3 Summary of General Lower Bound Results

As motivated above, we seek a rank expansion lower bound \( \sigma_C \) for \( C = A \otimes B \) given rank expansion lower bounds \( \sigma_A \) and \( \sigma_B \). We state our first main result below. Recall the pseudoinverse from (1), denoted with a dagger.

**Theorem 9** Suppose \( \sigma_A \) and \( \sigma_B \) are concave rank expansions lower bounds for \( A \in \mathbb{C}^{m_A \times n_A} \) and \( B \in \mathbb{C}^{m_B \times n_B} \), respectively, and \( \sigma_A(0) = \sigma_B(0) = 0 \). Let \( d_A = \sigma_A^1(1) \), \( d_B = \sigma_B^1(1) \). Then

\[
\sigma_C(k) = \min_{k_A \geq d_A, k_B \geq d_B} \sigma_A(k_A) \cdot \sigma_B(k_B)
\]

is a rank expansion lower bound for \( C = A \otimes B \).

The proof of this result is lengthy. We delay it until Section 5.3. Recursively applying the above theorem to \( C = \otimes_{i=1}^{p} A_i \), where \( p \geq 3 \), requires solving a non-trivial optimization problem. To derive simple rank expansion lower bounds for nested bilinear algorithms, we require the rank expansion lower bounds \( \sigma_i \) for each \( A_i \) to be log-log concave.

**Definition 10.** We say function \( f(x) \) is log-log concave (resp. convex) if \( \ln(f) \) is concave (resp. convex) in \( \ln(x) \).

The class of log-log concave functions contains many functions that can serve as tight rank expansion lower bounds, such as logarithms and polynomials with a leading term that has an exponent greater than or equal to 1. For further discussions, see Section 5.4. For log-log concave rank expansion functions, we obtain the following bound for nested bilinear algorithms.
Theorem 11 Let $\sigma_i$ be a rank expansion lower bound of $A_i \in \mathbb{R}^{m_i \times n_i}$ for $i = 1, 2, \ldots, p$ ($p \geq 2$), and let $C = \bigotimes_{i=1}^{p} A_i$. If $\sigma_i$ are concave and log-log concave and satisfy $\sigma_i(0) = 0$, then

$$\sigma_C(k) = \min_j \left\{ \sigma_j \left( \frac{k}{\prod_{i \neq j} d_i} \right) \right\},$$

where $d_i = \sigma_i^+(1)$, is a concave and log-log concave rank expansion lower bound of $C$. In particular, if $\sigma_i(k) = (k/k_i)^{q_i}$ with $k_i \geq 1$ and $q_i \in (0, 1]$, then

$$\sigma_C(k) = \left( \frac{k}{\prod_i k_i} \right)^{\min_j q_j}.$$  \hspace{1cm} (4)

If $\sigma_i(k) = a_i \ln(b_i k + 1)$, then

$$\sigma_C(k) = a \ln(b k + 1),$$

where $a = \min_i a_i$ and $b = \frac{e^{1/a} - 1}{\prod_{i=1}^{p} b_i^{-1} (e^{1/a} - 1)}$. The proof is given in Section 5.4. In Section 6, we illustrate the application of these results and derive new lower bounds for the communication cost of three bilinear algorithms (see Section 1 for a high-level description of our application contributions). Theorems 9 and 11 are easy to use and capable of producing nontrivial tight lower bounds. However, one potential problem is that they both require defining the lower bound $\sigma$ on the entire range $\mathbb{R}^+$, which requires extrapolating $\sigma$ beyond the intended domain $[0, n]$ (e.g., in Theorem 9, the optimal $k_A, k_B$ may have $k_A \geq n_A$ or $k_B \geq n_B$). In Appendix A, we derive lower bounds $\sigma_C$ that do not require extrapolating $\sigma_A$ and $\sigma_B$ to $\mathbb{R}^+$, which often results in tighter lower bounds on $\sigma_C$.

4 Rank Analysis of Kronecker Product via Grid Expansion

In this section, we introduce a grid representation to analyze the rank of a column-wise submatrix $CP$ in $C = A \otimes B$. The main idea is to represent columns of $CP$ as a subset in the 2-D grid representing columns of $A$ and $B$. We then manipulate the grid representation, compactifying the set of grid points, while keeping intact any low-rank structure. After the combinatorial arguments in this section, we derive rank bounds (including our main result, Theorem 9) by continuous analysis of the resulting compact geometric structure formed by the grid points.

4.1 Grid Framework

We first introduce our grid representation and then define the notions of a grid basis and a compact grid. A basis is a set of grid points that represents a linearly independent set of columns that span all points in a given grid. For compact grids, we show that there is a basis with a simple reducible structure.
4.1.1 Grid Representation

Let \( A \in \mathbb{C}^{m_A \times n_A} \) and \( B \in \mathbb{C}^{m_B \times n_B} \) be arbitrary matrices, and let \( C = A \otimes B \) be given by

\[
C = [a_1 \otimes B \; \cdots \; a_i \otimes B \; \cdots \; a_{n_A} \otimes B].
\]

The column \( c_k \) from \( C \) is defined as \( a_i \otimes b_j \) for some columns \( a_i \) and \( b_j \). Thus, we will refer to a column \( c_k \) by the tuple \((i, j)\).

Now recall that \( P_n^{(k)} \) is the set of matrices comprised of \( k \) different columns from an identity matrix of size \( n \). For any \( P \in P_n^{(k)} \), \( CP \) then contains \( k \) column vectors of \( C \). Thus, \( CP \) can be identified (up to a column reordering) by a set of \( k \) tuples. In particular, we can view this set of tuples as a set of grid points from an \( n_A \times n_B \) grid, as shown in Figure 1. Note that we use the Cartesian indexing system, not the array indexing system! We call this set of grid points the grid representation of \( CP \).

![Fig. 1: Let \( A \in \mathbb{C}^{m_A \times 3} \), \( B \in \mathbb{C}^{m_B \times 4} \), and \( C = A \otimes B \). The subset of columns \( \{a_1 \otimes b_1, a_1 \otimes b_2, a_2 \otimes b_2, a_3 \otimes b_4\} \) from \( C \) is represented by the black-filled columns on the figure on the left. This subset of columns is equivalent to the matrix product \( CP \) for some \( P \in P_{12}^{(4)} \). The grid representation of \( CP \) is the 3 \times 4 grid on the right. Each black-filled circle represents a selected column of \( C \), and its \((i, j)\) index is the column from \( A \) and \( B \) defining that column.](image)

For a grid \( G \subseteq [n_A] \times [n_B] \), we write \( G_{[i, \cdot]} = G \cap \{i\} \times [n_B] \) and \( G_{[\cdot, j]} = G \cap ([n_A] \times \{j\}) \) to be the set of grids points in column \( i \) and row \( j \) of \( G \), respectively. We denote the size of a grid \( G \) by \(|G|\). When it is not ambiguous, we use \((i, j)\) and \( a_i \otimes b_j \) interchangeably to denote a column of \( C \), and associate a grid \( G \) with a submatrix of \( C \), referring to the subset of column vectors represented by \( G \). When we say \( \text{span \{G\}} \), we mean the space spanned by column vectors represented by grid \( G \), and we call the dimension of this space by \( \text{rank}(G) \).

We endow the grid points with the colexicographic order, where \((i, j)\) precedes \((i', j')\) if \( i < i' \) or if \( i = i' \) and \( j < j' \). We then denote by \( [i, j] \) the set of points \((k, \ell) \leq (i, j)\) in this order, and we write \( ((i, j)) = [i, j] \setminus \{(i, j)\} \). We
Algorithm 1 Constructing a unique basis

1: function BasisSelection(Grid \( G \))
2: \( B \leftarrow \{ \} \)
3: for \((i, j) \in G\) using colexicographic traversal order do
4: \( \text{if } \ (i, j) \notin \text{span } \{B\} \text{ then} \)
5: \( B \leftarrow B \cup \{(i, j)\} \quad \triangleright \text{Add grid point if it is “new” to the span} \)
6: \( \text{return } B \)

We naturally extend this ordering to two grids by sorting their points and then comparing the first pair, second pair, and so on.

In order to analyze the rank of \( CP \), we introduce the notion of a basis, the maximal linearly independent set of columns in \( CP \), as defined below.

**Definition 12.** Let \( G \) be the grid representation of \( CP \). The basis of \( G \) is a subgrid \( B \subseteq G \), such that column vectors in \( B \) comprise a maximal linearly independent set of column vectors in \( G \), and \( B \) is minimal in the colexicographic order among all such subgrids of \( G \).

Algorithm 1 illustrates this definition and is one concrete way to find the basis. We traverse the grid in the colexicographic order and add point \((p, q)\) to the basis if it is not in the span of \((p, q)\).

In order to describe which columns in \( A \) and \( B \) are involved in \( CP \) and simplify the notations, we define the following projections.

**Definition 13.** The A-projection of the grid \( G \), denoted by \( P_A(G) \), is the set of indices \( I \subseteq [n_A] \) such that \( G_{[i, \cdot]} \neq \emptyset \). Likewise, the B-projection, \( P_B(G) \), is the set of indices \( J \subseteq [n_B] \) such that \( G_{[\cdot, j]} \neq \emptyset \).

The A-projection and B-projection can be viewed as the shadow of the grid \( G \) onto the x-axis (for columns of \( A \)) and the y-axis (for columns of \( B \)), respectively. We provide an example in Figure 2.

![Fig. 2: A grid representation for a CP where \( C = A \otimes B \). The A-projection is the set \( \{1, 2, 4, 5, 7\} \), or the set of indices on the x-axis highlighted in bold italics and with a gray ball. Similarly, the B-projection is the set \( \{1, 3, 4\} \).](image-url)
4.1.2 Compact Dense Grids

Now we introduce a class of structured grids that is the key to our analysis: compact dense grids (CDGs).

**Definition 14.** A grid $G$ is a dense grid if for every $i$, $G_{[i,\cdot]} = \{i\} \times [k]$ for some $k \geq 0$. A grid $G$ is a compact dense grid (CDG) if it is a dense grid and $|G_{[i,\cdot]}|$ is non-increasing in $i$.

We show the difference between a non-dense grid, a dense grid, and a CDG in Figure 3. We can transform any arbitrary grid into a dense grid by collapsing, i.e., letting the grid points fall vertically, as shown in the same figure. We make this concrete below.

**Definition 15.** Let $G$ be an arbitrary grid. Then a vertical collapse (VCollapse) of $G$ produces a dense grid $D$, where

$$D_{[i,\cdot]} = \begin{cases} \{i\} \times |G_{[i,\cdot]}| : G_{[i,\cdot]} \neq \emptyset \\ \emptyset : G_{[i,\cdot]} = \emptyset \end{cases}$$

Given any grid $G$, we can vertically collapse it to a dense grid $D_0$. Then with a reordering of the columns of matrix $A$ (the $x$-axis), we can produce a CDG $D$ from $D_0$. This process is illustrated in Figure 3. These two operations commute, and note that only the VCollapse step may change the rank of $G$ (or equivalently $CP$).

![Fig. 3: Example a non-dense grid (left), dense grid (center), and CDG (right). The left grid is non-dense since the $B$-projection of the second column equal is $\{2, 3, 4\}$. The action of a vertical collapse ensures the resulting grid (center) is dense.](image)

4.1.3 Reducible Structure in the Basis of a Compact Dense Grid

Here we establish the most important property of a CDG – its basis has a reducible structure. This is one of the key steps towards the main theorems. We start with two lemmas. As before, we denote the columns of $A$ and $B$ by $\{a_i\}$ and $\{b_j\}$. We identify column vectors of $C = A \otimes B$ in the 2-dimensional grid.
Lemma 16 Let \( u \in \mathbb{C}^n, v \in \mathbb{C}^m \). Let \( \{a_i\}_{i=1}^p \subseteq \mathbb{C}^n \), \( \{b_j\}_{j=1}^q \subseteq \mathbb{C}^m \) be sets of vectors. Then \( u \otimes v \in \text{span} \{a_i \otimes b_j\} \) if and only if \( u \in \text{span} \{a_i\} \) and \( v \in \text{span} \{b_j\} \).

Proof. The result follows from the basis for a tensor product of finite-dimensional vector spaces. See for instance [19].

Lemma 17 If \( a_p \notin \text{span} \{a_i\}_{i<p} \) and \( F \subseteq [(p,q)] \) is a set such that \( (p,q) \in \text{span} \{F_{[p,\cdot]}\} \), then \( (p,q) \in \text{span} \{F_{[\cdot,q]}\} \).

Proof. Let \( F_{[p,\cdot]} = \{(p,j_m)\}_{m} \). Let \( V' = \text{span} \{b_{jm}\} \) and \( V = \text{span} \{a_p \otimes b_{jm}\} \). Let \( P_X, P_Y \) be orthogonal projections. Let \( u = (I - P_Y)(a_p \otimes b_q) = a_p \otimes (I - P_Y) b_q \). If \( u \neq 0 \), then \( u \) is in the span of \( F \setminus \{(p,j_m)\}_{m} \), but this is impossible by Lemma 16, since \( a_p \notin \text{span} \{a_i\}_{i<p} \). Thus, \( u = 0 \) and \( (p,q) \in \text{span} \{F_{[p,\cdot]}\} \).

Now we are ready to prove the following key result on the structure of the basis of a CDG.

Proposition 18 Let \( D \) be a CDG and its basis be \( B_D \). Let the \( A \) and \( B \)-projection of \( B_D \) be \( X = P_A(B_D) \) and \( Y = P_B(B_D) \), respectively. Then
\[(i) \ (1, j) \in B_D \text{ if and only if } b_j \notin \text{span} \{b_k \}_{k \in Y \cap [j-1]} , \]
\[(ii) \ (i, 1) \in B_D \text{ if and only if } a_i \notin \text{span} \{a_k \}_{k \in X \cap [i-1]} , \]
\[(iii) \ B_D = (X \times Y) \cap D , \]
\[(iv) \ \text{For each } p \in X, (B_D)_{[p,\cdot]} \text{ spans } D_{[p,\cdot]} , \]
\[(v) \ \text{For each } q \in Y, (B_D)_{[\cdot,q]} \text{ spans } D_{[\cdot,q]} . \]

Proof. Part (i) is clear since the basis has minimal colexicographic order (see Algorithm 1).

For part (ii), the “if” direction follows from Lemma 16, which implies \((i, 1)\) is not spanned by preceding columns \(( (i, 1) \), and we conclude using the minimal colexicographic order property of the basis. For the “only if” part, note that \((i, 1) \in B_D \) implies \((i, 1) \notin \text{span} \{ (k, 1) \}_{k \in X \cap [i-1]} \). The conclusion then follows.

For part (iii) if \( D \) only involves a single column of the 2-D grid, then it is clear. Now suppose that CDG \( D \) involves at least \( p > 1 \) columns of the 2-D grid.

Case 1: If \( a_p \) is spanned by \( \{a_i\}_{i<p} \), then \( \forall q, (p,q) \) is spanned by preceding points \( \{(i,q)\}_{i<p} \subseteq ((p,q)) \subseteq D \), where the last inclusion is from the CDG structure of \( D \). Thus, \((p,q)\) is not added to \( B_D \) for any \( q \).

Case 2: Now suppose \( a_p \) is not spanned by \( \{a_i\}_{i<p} \). For a fixed \( q \), if \( b_q \notin \text{span} \{b_j\}_{j<q} \), then by Lemma 17, \((p,q)\) is not spanned by preceding columns \((p,q)\) and is thus added to the basis. If \( b_q \in \text{span} \{b_j\}_{j<q} \), then \((p,q)\) is spanned by \((p,j)\) \( j<q \subseteq ((p,q)) \subseteq D \). Thus \((p,q)\) \( \notin B_D \).

Case 1 means that if \( a_p \) is spanned by \( \{a_i\}_{i<p} \), then we skip this column in forming \( B_D \). Case 2 implies that if we do not skip the \( p \)th column, then we add
Fig. 4: An illustration of the structure of $B_D$.

Point $(p,q)$ to $B_D$ if and only if $b_q \notin \text{span}\{b_j\}_{j<q}$ if and only if $(1,q) \in B_D$. Thus, $B_D = (X \times Y) \cap D$.

Parts (iv) and (v) are then clear. Part (iv) follows from (i), (iii) and part (v) follows from (ii), (iii). This completes the proof.

Graphically, this proposition says that $B_D$ should look like the one in Figure 4, and for non-empty rows (columns) in $B_D$, the selected vectors span the vectors in that row (column) of $D$.

4.2 Grid Compactification and Expansion

Having shown in Proposition 18 that the basis for a CDG has a simple structure, our goal is now to bound the basis size (rank) of an arbitrary grid using that of a CDG constructed by collapsing that grid (via VCollapse, see Definition 15). To achieve this, we first define a notion of CDG expansion, which serves to upper bound the number of columns in $C$ spanned by a basis of a CDG shape. We then show that a subset of the basis of a collapsed CDG may be used to lower bound the rank of an arbitrary grid while having an expansion that upper bounds the size of the grid. That is, for any grid $G$, we find a CDG $S$ such that

$$|S| \leq \text{rank}(G) \quad \text{and} \quad |G| \leq |\text{GridExp}(S)|.$$

It then suffices to bound $|\text{GridExp}(S)|$ with $|S|$. Since the expansion we consider preserves a simple geometric structure, we achieve this latter bound by continuous analysis in Section 5.

4.2.1 Basis Expansion

We start by defining GridExp, or the grid expansion, on CDGs. For a CDG-shaped basis, this upper bounds the number of columns in $C$ the basis can
span. The definition consists of two steps: VExpansion and HExpansion. They commute with each other.

**Definition 19.** Let $\sigma_A$ and $\sigma_B$ be rank expansion lower bounds of $A$ and $B$.

For a CDG $S$, VExpansion($S$) is a CDG with the same number of columns of $S$. Each column is of size

$$|\text{VExpansion}(S)_{[i, \cdot]}| = \min \left\{ n_B, \lfloor \sigma_B^\dagger(|S_{[i, \cdot]}|) \rfloor \right\}.$$ 

HExpansion($S$) is a CDG with the same number of rows of $S$. Each row is of size

$$|\text{HExpansion}(S)_{[\cdot, j]}| = \min \left\{ n_A, \lfloor \sigma_A^\dagger(|S_{[\cdot, j]}|) \rfloor \right\}.$$ 

We define GridExp($S$) as the CDG

$$\text{GridExp}(S) = \text{HExpansion}(\text{VExpansion}(S)).$$

These expansions depend on the rank expansion of $A$ and $B$, $\sigma_A$ and $\sigma_B$ (Definition 7). When $\sigma_A$ and $\sigma_B$ are understood from the context, we often abbreviate this dependency. VExpansion grows $S$ vertically as much as possible such that VExpansion($S$) can have the same span as $S$. Similarly, HExpansion grows horizontally. See steps 3 and 4 in Figure 5 for an illustration.

![Fig. 5: Grid expansion of a pre-CDG (Definition 20) $S$ on a 4 × 4 grid where $\sigma_A^\dagger = \sigma_B^\dagger = f$, and $|f(1)| = 1$, $|f(2)| = 3$, $|f(3)| = 4$, and $|f(4)| = 5$. Through a column (Step 1, denoted by a 1 with a circle) and row (Step 2) reordering, we produce a CDG. We then apply a vertical expansion (Step 3), resulting in two additional points in dark blue. We finish with a horizontal expansion (Step 4), resulting in three more points in light blue.](image)

Note that GridExp($S$) only depends on the shape of $S$, and it is independent of the column vectors represented by $S$. Therefore, we can compute GridExp for a pre-compact dense grid as defined below.
**Definition 20.** A grid \( S \) is a pre-compact dense grid (pre-CDG) if \( S \) becomes a CDG after a column and row reordering of the grid. Denote this CDG by CDG(\( S \)). Then we extend the definition of GridExp to a pre-CDG \( S \) by

\[
\text{GridExp}(S) := \text{GridExp}(\text{CDG}(S)).
\]

This is well-defined since the shape of CDG(\( S \)) is unique. See Figure 5 for an example of converting a pre-CDG to a CDG.

### 4.2.2 Bounding General Grid Rank from a Compact Basis

Now we are ready to prove the main result of the discrete step.

**Lemma 21.** For any grid \( G \subseteq [n] \times [m] \), there exists a pre-CDG \( S \) such that

1. \( |S| \leq \text{rank}(G) \),
2. \( |G| \leq |\text{GridExp}(S)| \).

**Proof.** Since \( \text{rank}(G) \), \( |G| \), and the pre-CDG structure are invariant under column permutations on the grid, we may reorder the columns and assume \( D = \text{VCollapse}(G) \) is a CDG.

**Step 1. Construction of pre-CDG \( S \).** Let \( B_G \) and \( B_D \) be the bases of \( G \) and \( D \), respectively. Construct grid \( S \subseteq B_D \) in the following way. Let \( X = P_A(B_D) \). For each column index \( p \in X \), we remove grid points in \( B_{D[p, \cdot]} \) from the top until there are at most \( r_p = \text{rank}(G_{[p, \cdot]}) \) points left (we may not need to remove anything). Repeat this for each column, and let \( S \) be the resulting subgrid of \( B_D \). Clearly, \( S \) is a pre-CDG since \( B_D \) is a pre-CDG by Proposition 18.

**Step 2. Proof of (i).** Let \( B_G^k \) be the basis for \( G \cap ([k] \times [m]) \). With increasing number of columns \( k \), we have

\[
B_G^1 \subseteq B_G^2 \subseteq \cdots \subseteq B_G^n = B_G.
\]

Similarly, define \( S^k = S \cap ([k] \times [m]) \). We show that \( |S^k| \leq |B_G^k| \) for all \( k \) by an induction on \( k \). As defined in step 1, let \( r_p = \text{rank}(G_{[p, \cdot]}) \).

The base case \( |S^1| \leq r_1 = |B_G^1| \) is immediate. Now suppose \( |S^i| \leq |B_G^i| \) for all \( i \leq k \). We wish to show \( |S^{k+1}| \leq |B_G^{k+1}| \). Indeed, if \( a_{k+1} \in \text{span}\{a_i\}_{i \leq k} \), then by Proposition 18, \( |S^{k+1}| = |S^k| \), so \( |S^{k+1}| = |S^k| \leq |B_G^k| \leq |B_G^{k+1}| \). If \( a_{k+1} \notin \text{span}\{a_i\}_{i \leq k} \), we have \( |S^{k+1}| - |S^k| \leq r_{k+1} \). It is sufficient to show \( |B_G^{k+1}| - |B_G^k| \geq r_{k+1} \). To that end, when \( a_{k+1} \notin \text{span}\{a_i\}_{i \leq k} \), by Lemma 17, we must add \((k+1, q)\) to the basis (because of Algorithm 1) if \( b_q \notin \text{span}\{b_j : \exists (p, j) \in G \text{ s.t. } j < q\} \). Thus, we have to add at least a maximal linearly independent set of vectors \( G_{[k+1, \cdot]} \) to the basis (but may include more).

Hence, \( |B_G^{k+1}| - |B_G^k| \geq r_{k+1} \). Combining the two cases, we conclude \( |S^k| \leq |B_G^k| \) for all \( k \) in particular, \( |S| = |S^n| \leq |B_G^n| = |B_G| \) as desired.

**Step 3. Proof of (ii).** Let \( R_c \) and \( R_r \) be respectively the column and row reordering after which \( S \) becomes a CDG. That is, CDG(\( S \)) = \( R_c(R_r(S)) \). Let
for every $j$ this will imply for any $S$ such that for any CDG $S$ lower bound for $A$. 

Fix a column $i \in P_A(R_c(B_D))$. Denote $i'$ as the index of this column before applying $R_c$. Denote 

$$s = |CDG(S)_{[i,\cdot]}| = |S_{[i',\cdot]}|,$$

$$g = |R_c(G)_{[i,\cdot]}| = |G_{[i',\cdot]}|,$$

$$d = |R_c(D)_{[i,\cdot]}| = |D_{[i',\cdot]}|.$$ 

Since $S^+$ and $R_c(D)$ are both dense grids, it suffices to check $d \leq |S^+_{[i,\cdot]}|$. Clearly, $g = d$. If no point was removed from $B_D$ to form $S$ in column $i'$, then by part (iv) of Proposition 18 and the fact $\sigma_B$ is a lower bound on the rank expansion for $B$, $d \leq |S^+_{[i,\cdot]}|$. If grid points were removed from $B_D$ in column $i'$, then $s = r_{i'}$. Now $|S^+_{[i,\cdot]}| = |\sigma_B^+(s)| = |\sigma_B^+(r_{i'})| \geq g = d$. Thus, in both cases, $R_c(D)_{[i,\cdot]} \subseteq S^+_{[i,\cdot]}$.

The above shows for all $i \in P_A(R_c(B_D))$, The $i$th column $R_c(D)_{[i,\cdot]}$ is covered by $S^+_{[i,\cdot]}$. In particular, for each $j \in P_B(D)$, $R_c(B_D)_{[\cdot,j]}$ is covered by $S^+_{[\cdot,j]}$. By part (v) of Proposition 18 and the fact that $\sigma_A$ is a rank expansion lower bound for $A$, we have $|HExpansion(S^+)_{[\cdot,j]}| \geq |D_{[\cdot,j]}|$. Since this holds for every $j \in P_B(D)$, we conclude $|GridExpansion(S)| \geq |D| = |G|$ as desired. 

With this established, it remains to find a strictly increasing function $\phi$ such that for any CDG $S$, $|GridExpansion(S)| \leq \phi(|S|)$. Together with Lemma 21, this will imply for any $G$, there exists some pre-CDG $S$,

$$\phi(rank(G)) \geq \phi(|S|) = \phi(|CDG(S)|) \geq |GridExpansion(CDG(S))| \geq |G|.$$ 

Thus, $\sigma_C = \phi^{-1}$ is a valid rank expansion. We find such a function $\phi$ in the next section.

5 Rank Expansion Bounds via Continuous Analysis of Compact Grid Expansion

Finding a tight upper bound on $|GridExpansion(S)|$ (given $\sigma_A$ and $\sigma_B$) for all CDG $S$ with a given size can be a hard discrete optimization problem for arbitrary $\sigma_A$, $\sigma_B$, and CDG $S$. In order to find an easy-to-apply bound, we relax the integer-grid assumption and work with the so-called “stairs” in $\mathbb{R}^2$. This is made precise in Section 5.1 below. As required by Theorem 9 and Theorem 11, we assume

$$\sigma_A(0) = \sigma_B(0) = 0 \text{ and they are concave.} \tag{6}$$

Throughout this section, we assume in addition

$$\sigma_A \text{ and } \sigma_B \text{ are strictly increasing } C^1 \text{ functions.} \tag{7}$$
This additional assumption (7) will not make the resulting bound $\sigma_C$ worse through a density argument on these functions. Recall the pseudoinverse function denoted with a dagger, as defined in (1). Now we may simplify our notation and denote
\[ f(x) := \sigma_A^{-1}(x) = \sigma_A^+(x), \]
\[ g(x) := \sigma_B^{-1}(x) = \sigma_B^+(x). \]

These are strictly increasing convex functions with $f(0) = g(0) = 0$. We will assume these notations in the following context unless otherwise explained. To simplify notations, for arbitrary functions $h_1$ and $h_2$, we use the notation
\[ h_1(x) \propto h_2(x) \] (9) if there exists $c > 0$ such that $h_1(x) = c \cdot h_2(x)$.

5.1 The Stair Relaxation

We identify a CDG with a stair-like area in $\mathbb{R}^2$ — each grid point $(i, j)$ now represents a unit square $[i-1, i] \times [j-1, j]$ in $\mathbb{R}^2$. An illustration can be found in Figure 6. This stair structure consists of contiguous rectangles with decreasing heights and varying widths. We will refer to these rectangles as steps. The colloquial name of stairs represents the visual similarity to the stairs we see in real life in multi-story buildings. Now we define the continuous version of GridExp on stairs.

**Definition 22.** The expansion size of a CDG $S$ (with respect to $f$ and $g$) is defined as
\[ \langle S \rangle = \int_{S \subseteq \mathbb{R}^2} df(x)dg(y). \] (10)

We identify the grid $S$ with a stair in $\mathbb{R}^2$ in the integration.

The expansion size $\langle S \rangle$ now serves as an upper bound of $|\text{GridExp}(S)|$, as stated below.

**Lemma 23** For any CDG $S$, we have $|\text{GridExp}(S)| \leq \langle S \rangle$.

**Proof.** Let $S^+ = \text{VExpansion}(S)$ and $S^{++} = \text{HExpansion}(S^+)$. By definition of these expansions,
\[ |\text{VExpansion}(S)[i, \cdot]| = \min \left\{ n_B, \left\lfloor \frac{\sigma_B^+(|S[i, \cdot]|)}{\sigma_B^+(|S[i, \cdot]|)} \right\rfloor \right\} \leq g(|S[i, \cdot]|), \]
\[ |\text{HExpansion}(S^+)[\cdot, j]| = \min \left\{ n_A, \left\lfloor \frac{\sigma_A^+(|S^+[\cdot, j]|)}{\sigma_A^+(|S^+[\cdot, j]|)} \right\rfloor \right\} \leq f(|S^+[\cdot, j]|). \]

When these are indeed equalities, because of the stair structure in $S^{++}$ and the fact $f(0) = g(0) = 0$, we have
\[ |\text{GridExp}(S)| = |S^{++}| = \langle S \rangle. \]

Therefore, the conclusion follows.
We are left to bound \( \langle S \rangle \) given \( |S| = \int_S dxdy \). To that end, we further relax the corner points of \( S \) from being integer coordinates to general points in (the positive quadrant of) \( \mathbb{R}^2 \). The integral version of definitions of \( \langle S \rangle \) and \( |S| \) extend naturally to a non-integral stair. The decreasing stair structure is still preserved, i.e., \( S \) is the region below a decreasing step function in the first quadrant. In the next section, we prove that \( \langle S \rangle \) is maximized when \( S \) is a single rectangle when its size \( |S| \) is fixed.

5.2 Reduce Stairs to Rectangles

First, we introduce the merge operation that reduces the number of steps (rectangles) in the stair by 1.

**Definition 24.** Let \( S \) be a stair with at least 2 steps indexed by 1, 2, ..., (from left to right). Fix two consecutive steps \( k \) and \( k + 1 \). Denote \( u \) as the difference in their heights. There are two ways of changing the shape of \( S \) with \( |S| \) and its stair structure unchanged:

(i) increase \( u \): until step \( k \) has the same height of stair \( k - 1 \) (if \( k > 1 \)) or step \( k + 1 \) has the same height of stair \( k + 2 \) (or down to the ground if it is the last step).

(ii) decrease \( u \): until step \( k \) and \( k + 1 \) reaches the same height (\( u = 0 \)).

Denote the resulting stair from the first approach as \( M_1 \) and the second approach as \( M_2 \). The merge operation (on a given stair \( S \)) is defined as

\[
\text{Merge}(k, k + 1) = \begin{cases} 
M_1 & \text{if } \langle M_1 \rangle \geq \langle M_2 \rangle \\
M_2 & \text{if } \langle M_1 \rangle < \langle M_2 \rangle
\end{cases}
\]

An illustration of the merge step is given in Figure 6 below. The next step is to show that a merge always increases the expansion size.

**Lemma 25** Let \( S \) be a CDG on the grid with at least two steps. Identify \( S \) as a stair. Consider the merge of steps \( k \) and \( k + 1 \). The merge operation varies their height difference while keeping the area \( |S| \) unchanged. Denote \( E(u) \) as the (unique) stair when their height difference is \( u \). Then for any \([a, b] \subseteq [0, +\infty)\),

\[
\max_{u \in [a, b]} \langle E(u) \rangle = \max \{ \langle E(a) \rangle, \langle E(b) \rangle \}.
\]

Hence, for any \( k \) and \( u \) in the merge,

\[
\langle E(u) \rangle \leq \langle \text{Merge}(k, k + 1) \rangle = \max \{ \langle M_1 \rangle, \langle M_2 \rangle \}.
\]

Consequently, by carrying out a sequence of Merge, for any CDG \( S \) with size \( |S| = t \),

\[
\langle S \rangle \leq \max_{t_A, t_B \geq 1, \ t_A t_B = t} f(t_A)g(t_B).
\]

(11)
Fig. 6: Stair $S$ consists of four steps in decreasing height, and we want to “remove” a step without changing the area $|S|$. Consider the operation $\text{Merge}(2, 3)$. The original shape of $S$ (relevant to this merge) is highlighted in green. The pink stair represents $M_1$, produced by increasing $u$ to the maximum. The blue stair represents $M_2$, produced by decreasing $u$ to 0.

**Proof.** We start by introducing some notations. Consider the stair $E(u)$. As depicted in Figure 6, let $x_0 < x_1 < x_2$ be the horizontal splits between steps $(k-1, k)$ (or $x_0 = 0$ if $k = 1$), $(k, k+1)$, and $(k+1, k+2)$ (or ground if $k + 1$ is the last step). Let $y_0 < y_1 < y_2 < y_3$ be the height of steps $k-1$, $k$, $k+1$, $k+2$ (we set $y_0 = 0$ if $k+1$ is the last step, and $y_3 = +\infty$ if $k = 1$). Since $y_2 - y_1 = u$, using the equi-area relation, we have

$$
\begin{align*}
  y_1 &= \bar{y} - \frac{x_2 - x_0}{x_2 - x_1} u, \\
  y_2 &= \bar{y} + \frac{x_2 - x_0}{x_2 - x_1} u,
\end{align*}
$$

(12)

where $\bar{y}$ is the height of the new $k$th stair in $M_2$ when $u = 0$, which is independent of $u$.

The expansion size can be computed to be

$$
\langle E(u) \rangle = C + [f(x_1) - f(x_0)] [g(y_2(u)) - g(y_0)] + [f(x_2) - f(x_1)] [g(y_1(u)) - g(y_0)],
$$

where $C$ is independent of $u$, and other terms represent the area of the green region in Figure 6. Taking the derivative with respect to $u$ and using the relations in (12), we obtain

$$
\frac{d}{du} \langle E(u) \rangle = k_1 g'(y_2) - k_2 g'(y_1).
$$
Applying the relation (12) once more, we have
\[ \frac{d}{du} \langle E(u) \rangle = 0 \iff g' \left( \bar{y} - \frac{x_1 - x_0}{x_2 - x_0} u \right) = \frac{k_1}{k_2} \cdot g' \left( \bar{y} + \frac{x_2 - x_1}{x_2 - x_0} u \right). \]

By convexity of \( g \), the right-hand side is increasing in \( u \) whereas the left-hand side is decreasing in \( u \). Thus, there is at most one critical point for \( \langle E(u) \rangle \) on \( u \geq 0 \). Now we show that
\[ \frac{d}{du} \langle E(u) \rangle|_{u=0} \leq 0. \] (13)

Indeed, when \( u = 0 \), \( y_1 = y_2 = \bar{y} \), and by convexity of \( f \), \( k_1 \leq k_2 \) by the slopes of secant lines of \( f \). Therefore,
\[ \frac{d}{du} \langle E(u) \rangle|_{u=0} = (k_1 - k_2)g'(\bar{y}) \leq 0. \]

Thus, \( \langle E(u) \rangle \) cannot have a local maximum in any \((a, b) \subseteq \mathbb{R}_+\). Applying this to the case of \( \text{Merge}(k, k+1) \), the maximum of \( \langle E(u) \rangle \) is either \( \langle M_1 \rangle \) or \( \langle M_2 \rangle \).

We repeatedly apply \( \text{Merge}(2,3) \) until the grid has only two steps (if there are only 1 or 2 steps to begin with, we skip this step). Then we apply \( \text{Merge}(1,2) \), yielding a \( t_A \times t_B \) rectangle, with size \( t = t_A \times t_B \). This rectangle has a width and height of at least 1. This is because we start from a CDG \( S \) on the grid, so the first and last step of \( S \) has a height and width of at least 1. Hence, in the last merge, the two steps are of height and width of at least 1. Hence, the optimization problem (11) is a valid upper bound of \( \langle S \rangle \), which maximizes the grid expansion over all rectangular-shaped grids \( S \) with an area at least \( t \) and width and height at least 1. \[ \blacksquare \]

As motivated at the end of Section 4.2, Lemma 23 and Lemma 25 give us a suitable function \( \phi \) such that for any CDG \( S \), \( |\text{GridExp}(S)| \leq \phi(|S|) \),
\[ \phi(t) = \max_{t_A, t_B \geq 1, t_A t_B = t} f(t_A)g(t_B). \] (14)

To conclude the proof of Theorem 9, it remains to prove the proposed expansion lower bound
\[ \sigma_C(k) = \min_{k_A \geq d_A, k_B \geq d_B} \sigma_A(k_A) \cdot \sigma_B(k_B) \]
is indeed a lower bound of \( \phi^{-1} \). In fact, in the next section, we prove that it is optimal, in that \( \sigma_C = \phi^{-1} \).
5.3 Proof of Main Expansion Bound (Theorem 9)

We now provide the final step toward proving Theorem 9, showing that the inverse of the growth function $\phi$ gives us the rank expansion function, $\sigma_C = \phi^{-1}$.

Lemma 26 Suppose $\sigma_A$ and $\sigma_B$ are concave, strictly increasing, and smooth with $\sigma_A(0) = \sigma_B(0) = 0$. Let $f = \sigma_A^{-1}$, $g = \sigma_B^{-1}$, $d_A = f(1)$, $d_B = g(1)$. Define

$$\sigma_C(k) = \min_{k_A \geq d_A, k_B \geq d_B} \sigma_A(k_A) \cdot \sigma_B(k_B),$$

$$\phi(t) = \max_{t_A, t_B \geq 1} f(t_A)g(t_B).$$

Then $\sigma_C = \phi^{-1}$ on $[d_A d_B, +\infty)$.

Proof. To begin with, since $f$ and $g$ are strictly increasing, so is $\phi$, and thus $\phi$ is invertible. Also, $\sigma_C$ is strictly increasing on $[d_A d_B, +\infty)$ because both $\sigma_A$ and $\sigma_B$ are strictly increasing.

Step 1. $\sigma_C \leq \phi^{-1}$. Assume by contradiction there is some $k \geq d_A d_B$ where $\sigma_C(k) > \phi^{-1}(k)$. Then there exists a $k'$ such that

$$k' := \phi(\sigma_C(k)) = \phi(\phi^{-1}(k)) = k.$$

From the above inequality, this means that there exists $t_A, t_B \geq 1$ such that $t_A t_B = \sigma_C(k)$ and $\sigma_A^{-1}(t_A) \sigma_B^{-1}(t_B) \geq k'$. This implies

$$\sigma_C(k) < \sigma_C(k')$$

$$= \min_{k_A \geq d_A, k_B \geq d_B} \sigma_A(k_A) \cdot \sigma_B(k_B)$$

$$\leq \sigma_A(\sigma_A^{-1}(t_A)) \cdot \sigma_B(\sigma_B^{-1}(t_B))$$

$$= \sigma_C(k'),$$

which contradicts that $\sigma_C$ is strictly increasing.

Step 2. $\phi^{-1} \leq \sigma_C$. We show that if $\sigma_C(k) = t$ with $k \geq d_A d_B$, then $\phi(t) \geq k$. Indeed, when $k \geq d_A d_B$, $\sigma_C(k) = t$ implies

$$\sigma_C(k) = \min_{k_A \geq d_A, k_B \geq d_B} \sigma_A(k_A) \cdot \sigma_B(k_B) = t.$$

Note the equality constraint under the min. Let $t_A = \sigma_A(k_A) \geq 1$ and $t_B = \sigma_B(k_B) \geq 1$. Then we have found

$$t_A \geq 1, t_B \geq 1, t_A t_B = t,$$

such that $\sigma_A^{-1}(t_A) \sigma_B^{-1}(t_B) = k$.

By definition of $f$, $g$, and $\phi$, we have $\phi(t) \geq k$, and the proof is complete. ■

Finally, we put all the pieces together to give the full proof of Theorem 9. For convenience, we restate the theorem here.
Theorem 9 Suppose $\sigma_A$ and $\sigma_B$ are concave rank expansions lower bounds for $A \in \mathbb{C}^{m_A \times n_A}$ and $B \in \mathbb{C}^{m_B \times n_B}$, respectively, and $\sigma_A(0) = \sigma_B(0) = 0$. Let $d_A = \sigma_A^f(1)$, $d_B = \sigma_B^f(1)$. Then

$$\sigma_C(k) = \min_{k_A \geq d_A, k_B \geq d_B} \sigma_A(k_A) \cdot \sigma_B(k_B)$$

is a rank expansion lower bound for $C = A \otimes B$.

Proof. First, for $k \leq d_A d_B$, the bound is trivial since $\sigma_C(k) = \sigma_A(d_A) \cdot \sigma_B(d_B) = 1$, which is clearly a lower bound of the rank of any nonzero matrix.

Let us consider $k \geq d_A d_B$. Through a density argument, we may assume $\sigma_A$ and $\sigma_B$ are strictly increasing and smooth. For any grid $G$ of size $k$, by Lemma 21, we can find a pre-CDG $S$ such that

$$|S| \leq \text{rank}(G),$$

$$k = |G| \leq |\text{GridExp}(S)|.$$

By Lemma 23 and Lemma 25, we have

$$|\text{GridExp}(S)| = |\text{GridExp}(\text{CDG}(S))| \leq \langle \text{CDG}(S) \rangle,$$

$$\langle \text{CDG}(S) \rangle \leq \max_{t_A, t_B \geq 1} f(t_A) g(t_B) =: \phi(|S|),$$

where $f = \sigma_A^{-1}, g = \sigma_B^{-1}$. Finally, combining all the steps, since $\phi$ is increasing,

$$|G| \leq \phi(|S|) \leq \phi(\text{rank}(G)).$$

Therefore, Lemma 26 tells us

$$\text{rank}(G) \geq \phi^{-1}(|G|) = \sigma_C(|G|).$$

The proof is then complete.

5.4 Proof of Nested Expansion Bound (Theorem 11)

We now seek to extend the nested rank expansion lower bound (Theorem 9), to cases when $C = \otimes_{p=1}^{p} A_i$ contains $p > 2$ terms. Theorem 9 may not be easy to apply repeatedly, since the resulting bound $\sigma_C$ does not satisfy the assumptions applied to $\sigma_A$ and $\sigma_B$, as $\sigma_C(0) = 0$ and need not be concave. However, we can circumvent these issues when $\sigma_A$ and $\sigma_B$ are log-log concave (see Definition 10). For such rank expansion functions, we give a simple form for the rank expansion of a $p$-term Kronecker product in Theorem 11. Before going into its proof, we summarize some properties of log-log convex (concave) functions. For a more detailed reference on log-log convex functions, see [2],
Proposition 27: \( f(x) \) is log-log convex (resp. concave) if and only if \( \ln f(e^x) \) is convex (resp. concave) in \( x \). Thus, if \( \{f_i(x)\}_{i \in I} \) and \( \{g_i(x)\}_{i \in I} \) are, respectively, two collections of log-log convex and concave functions, then \( \sup_i f_i \) and \( \prod_i f_i \) are log-log convex functions, and \( \inf_i g_i \) and \( \prod_i g_i \) are log-log concave functions.

Proposition 28: If \( f : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) is invertible and log-log convex (resp. concave), then \( f^{-1} \) is log-log convex (resp. convex).

Proposition 29: Let \( f \) be a log-log convex (resp. concave) function on \([a, b] \). Then for any \( \varepsilon > 0 \), there exists a smooth log-log convex (resp. concave) function \( g \) on \([a, b]\) with \( \|f - g\|_{L^\infty([a, b])} < \varepsilon \).

This function class contains many useful functions for our work. We provide a couple of increasing, concave, and log-log concave functions below.

Proposition 30: For the following choices of \( f \), \( f(x) \) is concave and log-log concave, and for any \( t \in \mathbb{R}_+ \), \( f(t) - f(t - x) \) is log-log convex on \((0, t)\):
(i) \( f(x) = a \ln(bx + 1) \), \( a > 0 \), \( b > 0 \);
(ii) \( f(x) = ax^p \), \( a > 0 \), \( p \leq 1 \).

For a more comprehensive list of functions, see again [2]. Note that not all positive increasing log-log concave functions are concave, for example, \( f(x) = x \ln(1 + x) \) is convex but log-log concave by Proposition 27 and Proposition 30. The log-log convexity of \( f(t) - f(t - x) \) will be useful later in Appendix A.2 to simplify optimization problems.

Assuming that \( \sigma_A \) and \( \sigma_B \) are log-log concave, we can further reduce the minimization problem in Theorem 9.

Lemma 31: If \( f \) and \( g \) are positive increasing log-log concave functions on \( \mathbb{R}_+ \), then for any positive numbers \( a, b, c, d, k \) with \( b \) and \( d \) being possibly infinite,

\[
\min_{k_A \in [a, b], \ k_B \in [c, d], \ k_A k_B \geq k} f(k_A) \cdot g(k_B) = \min_{k_A \in [a, b], \ k_B \in [c, d], \ k_A k_B \geq k} f(k_A) \cdot g(k_B).
\]

Proof. By monotonicity, when \( k \leq ac \), the minimum is \( f(a)g(c) \), so we are done. When \( k \geq ac \), fixing \( k \), the optimization is equivalent to

\[
\min_{k_A \in [a, b], \ k_B \in [c, d], \ k_A k_B = k} f(k_A)g(k_B) = \min_{x \in [a, b], \ k/x \in [c, d]} f(x)g(k/x).
\]

By monotonicity, for a finite \( k \), we can always assume \( b \) and \( d \) are finite large numbers. Through a density argument using Proposition 29, we can further assume that \( f \) and \( g \) are smooth and strictly increasing on the intervals \([a, b]\) and \([c, d]\), respectively. Let \( h(x) = f(x)g(k/x) \), then

\[
h'(x) = \frac{1}{x} f(x)g(k/x) \left[ \frac{x f'(x)}{f(x)} - \frac{(k/x)g'(k/x)}{g(k/x)} \right].
\]
When \( f \) and \( g \) are log-log concave, by Proposition 27, \( F(x) = \ln f(e^x) \) and \( G(x) = \ln g(e^x) \) are concave. Note that
\[
\frac{xf'(x)}{f(x)} = F'(\ln x), \quad \frac{(k/x)g'(k/x)}{g(k/x)} = G'(\ln k - \ln x).
\]
Hence \( h'(x) \propto F'(\ln x) - G'(\ln k - \ln x) \) (\( \propto \) means positively proportional to, see (9)). Since \( F' \) and \( G' \) are decreasing, \( h(x) \) cannot have a local minima. Thus, the minimum of \( h \) is attained on the boundary. ■

With this simplification, we can prove Theorem 11. Again, for convenience, we restate it here.

**Theorem 11** Let \( \sigma_i \) be a rank expansion lower bound of \( A_i \in \mathbb{R}^{m_i \times n_i} \) for \( i = 1, 2, \ldots, p \) (\( p \geq 2 \)), and let \( C = \bigotimes_{i=1}^{p} A_i \). If \( \sigma_i \) are concave and log-log concave and satisfy \( \sigma_i(0) = 0 \), then
\[
\sigma_C(k) = \min_j \left\{ \sigma_j \left( \frac{k}{\prod_{i \neq j} d_i} \right) \right\},
\]
where \( d_i = \sigma_i^1(1) \), is a concave and log-log concave rank expansion lower bound of \( C \). In particular, if \( \sigma_i(k) = (k/k_i)^{q_i} \) with \( k_i \geq 1 \) and \( q_i \in (0, 1] \), then
\[
\sigma_C(k) = \left( \frac{k}{\prod_i k_i} \right)^{\min_j q_j}.
\]
If \( \sigma_i(k) = a_i \ln(b_i k + 1) \), then
\[
\sigma_C(k) = a \ln(bk + 1),
\]
where \( a = \min_i a_i \) and \( b = \frac{e^{1/a} - 1}{\prod_{i=1}^{n_i} b_i^{-1}(e^{1/a} - 1)} \).

**Proof.** Equation (3) will follow via a proof by induction on \( p \). In the base case \( p = 2 \), we have by Theorem 9 and Lemma 31,
\[
\sigma_C(k) = \min \{ \sigma_1(k/d_2), \sigma_2(k/d_1) \}
\]
is a valid concave and log-log concave rank expansion lower bound for \( C \). For \( p > 2 \), let \( B = \bigotimes_{i<p} A_i \). By the base case and the induction hypothesis,
\[
\sigma_C(k) = \min \{ \sigma_B(k/d_p), \sigma_p(k/d_B) \}
\]
\[
= \min \left\{ \min_{j<p} \left\{ \sigma_j \left( \frac{k/d_p}{\prod_{i \neq j, i<p} d_i} \right) \right\}, \sigma_p \left( \frac{k}{\prod_{i<p} d_i} \right) \right\}
\]
\[
= \min_{j \leq p} \left\{ \sigma_j \left( \frac{k}{\prod_{i \neq j, i \leq p} d_i} \right) \right\}.
\]
where we used the fact that $d_B = \sigma_B(1) = \prod_{i < j} d_i$, which is readily verified by an induction. Thus (3) is established, and it is clear that $\sigma_C$ is again concave and log-log concave.

If $\sigma_i(k) = (k/k_i)^{a_i}$, then $d_i = k_i$, and

$$\sigma_j \left( \frac{k}{\prod_{i \neq j} d_i} \right) = \left( \frac{k/d_j}{\prod_{i \neq j} d_i} \right)^{q_j} = \left( \prod_i k_i \right)^{q_j}.$$ 

The result follows after a trivial minimization.

For $\sigma_i(k) = a_i \ln(b_i k + 1)$, we proceed by induction on $p$. We assume that $a_1 \geq \ldots \geq a_{p-1} \geq a_p$, and again let $B = \bigotimes_{i < p} A_i$. Then

$$\sigma_C(k) = \min \left\{ \sigma_B(k/d_p), \sigma_p(k/d_B) \right\} = \min \left\{ a_{p-1} \ln \left( b_B k_{d_p} + 1 \right), a_p \ln \left( b_p k_{d_B} + 1 \right) \right\},$$

where $d_p = \sigma_p(1) = b_p^{-1} e^{1/a_p} - 1$ and $d_B = \sigma_B(1) = b_B^{-1} e^{1/a_{p-1}} - 1$, and so by the induction hypothesis, $b_p = \prod_{i < p} b_i^{-1} (e^{1/a_i} - 1)$.

When $k < d_B d_p$, $\sigma_C(k) \leq \sigma_C(k)$ is a valid rank expansion lower bound for any nonzero matrix. Consider $k \geq d_B d_p$. Let $k = \gamma d_B d_p$, $\gamma \geq 1$, we have

$$\sigma_C(k) = \min \left\{ a_{p-1} \ln \left( e^{1/a_{p-1}} - 1 \gamma + 1 \right), a_p \ln \left( e^{1/a_p} - 1 \gamma + 1 \right) \right\}. \quad (15)$$

We prove that the second term in the min above is smaller than the first term. With this established, one can directly verify that $\sigma_C$ coincides with the proposed function given by (5), and the proof is then complete.

To that end, consider $g(x) = \frac{h(x)}{x}$, where $h(x) = \ln \left( \gamma (e^x - 1) + 1 \right)$. The two terms in the minimization of (15) are, respectively, $g(1/a_{p-1})$ and $g(1/a_p)$. Now, it suffices to show that $g$ is decreasing on $\mathbb{R}_+$. Indeed, note that $h(0) = 0$, and one can check that $h$ is a concave function on $\mathbb{R}_+$ when $\gamma \geq 1$. Thus $g(x)$ is the secant slope of $h$ between $x_1 = 0$ and $x_2 = x$, which is decreasing in $x$.

6 Applications

We apply the rank expansion lower bounds from the previous section to derive communication lower bounds for several fast bilinear algorithms: Strassen’s fast matrix-matrix multiplication, fast convolution, and partially symmetric tensor contractions. We consider both sequential and parallel communication lower bounds, as we defined in Section 2.3. Throughout this section, we will use the set $\mathcal{P}^{(k)}_n$, which recall is the of matrices with $k$ different columns from an identity matrix of size $n$. 


6.1 Fast Matrix Multiplication

Communication lower bounds for classical matrix multiplication are usually derived by explicitly reasoning about potential partial sums and applying the Loomis-Whitney inequality [23, 29]. The expansion bound – which recall from Definition 4 bounds the largest portion of a bilinear one can complete given a subset of data – for classical matrix multiplication is the function (see, e.g., [35, Lemma B.1] or [5, Lemma 2.1]),

$$E(r(A), r(B), r(C)) = \sqrt{r(A)r(B)r(C)},$$

where \(r(A), r(B), \) and \(r(C)\) are the number of elements from the first input, second input, and output stored in cache or on a local machine, respectively. Strassen’s algorithm [36], the most practical known fast algorithm [4] among those that achieve subcubic complexity, corresponds to a more nontrivial bilinear algorithm than classical matrix multiplication.

We derive communication lower bounds for the bilinear algorithm given by Strassen’s approach, which considers any other computational DAG. These DAGs include ones that do not follow the recursive structure of Strassen’s algorithm; they need only compute the same scalar products at the base case level of recursion. In particular, the operands can be computed by any other additions or linear combinations. First, recall that the bilinear algorithm for Strassen’s algorithm is as follows.

**Definition 32 (Bilinear Algorithm for Strassen’s Matrix Multiplication).**

\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & -1
\end{bmatrix},
B = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 0 
\end{bmatrix},
C = \begin{bmatrix}
1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

In view of these two encoding and decoding matrices, we construct a rank expansion lower bound for the bilinear algorithm \((A, B, C)\).

**Lemma 33** For a nested bilinear algorithm \(\bigotimes_{i=1}^{\tau} A, \bigotimes_{i=1}^{\tau} B, \bigotimes_{i=1}^{\tau} C\), where \(\tau \in \mathbb{Z}_+\), and \((A, B, C)\) is the bilinear algorithm for Strassen’s base algorithm, then the function,

$$E(r(A), r(B), r(C)) = \min(r(A), r(B), r(C)) \log_2(3)$$

is an expansion bound function for the nested bilinear algorithm.

**Proof.** We first focus on the matrix \(A\) and will later briefly address matrices \(B\) and \(C\).

By direct calculations (e.g., numerical calculations), the smallest rank for a matrix formed by any subset of \(k \in [7]\) different columns from the matrix \(A\) is 1, 2, 3, 3, 4, 4 for \(k = 1, \ldots, 7\), respectively. Now, define the function

$$\sigma(k) = k \log_2(2).$$
One can directly check the following holds for any $k \in \{7\}$:

$$\sigma(k) \leq \min_{P \in \mathcal{P}^7\{k\}} \{\text{rank}(AP)\}.$$  

This shows $\sigma(k)$ is a rank expansion lower bound for $A$ (Definition 8). Indeed, $\sigma$ is also concave, log-log concave (see Proposition 30), and satisfies $\sigma(0) = 0$. Also, recalling the definition of the pseudoinverse from (1), one can verify $d_A \equiv \sigma^\dagger(1) = 1$. This permits us to apply Proposition 11 with $q \equiv q_i = \log_3(2) < 1$ to ensure the rank expansion lower bound for $\bigotimes_{i=1}^7 A$ for any $\tau \in \mathbb{N}$ and all $k \in \{7\}$ is

$$\sigma(k) = k^q.$$  

By using a nearly identical calculation, one can confirm $\sigma$ is a rank expansion lower bound for $\bigotimes_{i=1}^7 B$ and $\bigotimes_{i=1}^7 C$ for all $\tau \in \mathbb{N}$. We can then conclude for any $k \in \{7\}$ and all $P \in \mathcal{P}^7\{k\}$,

$$\sigma(k) \leq \min \{\text{rank}(\bigotimes_{i=1}^7 A|P), \text{rank}(\bigotimes_{i=1}^7 B|P), \text{rank}(\bigotimes_{i=1}^7 C|P)\}.$$  

Applying the monotone function $\sigma^{-1}(k) = k_0^\log_3$ to both sides and recalling the choice of the expansion bound function $E$, we get for all $P \in \mathcal{P}^7\{k\}$,

$$\#\text{cols}(P) \leq E\left(\text{rank}(\bigotimes_{i=1}^7 A|P), \text{rank}(\bigotimes_{i=1}^7 B|P), \text{rank}(\bigotimes_{i=1}^7 C|P)\right).$$  

Thus, $E$ is an expansion bound by Definition 4 (c.f. (2)).

Next, we apply Lemma 33 to derive a sequential communication lower bound.

**Corollary 34** Given square matrices of size $n \in \mathbb{N}$ (assumed to be a power of 2) and fast memory of size $M$, the sequential communication cost of Strassen’s fast matrix multiplication algorithm is at least

$$\max \left\{\frac{2n^{\log_2(7)}}{M^{\log_2(3)}} \cdot M, 3n^2\right\}.$$  

**Proof.** By Proposition 5 and noting the size of the inputs and output are $n^2$, the sequential communication lower bound of the nested bilinear algorithm is

$$\max \left\{\frac{2RM}{\mathcal{E}^{\max}(M)}, 3n^2\right\}, \quad (16)$$  

where the rank of the nested bilinear algorithm is $R = 7^{\log_2(n)} = n^{\log_2(7)}$. With the help of the expansion bound from Lemma 33, the function $\mathcal{E}^{\max}(M)$ (from Proposition 5) can be expressed as

$$\mathcal{E}^{\max}(M) = \max_{r(A), r(B), r(C) \in \mathbb{N}, \ r(A) + r(B) + r(C) = 3M} \mathcal{E}(r(A), r(B), r(C))$$  

$$= \max_{r(A), r(B), r(C) \in \mathbb{N}, \ r(A) + r(B) + r(C) = 3M} \min(r(A), r(B), r(C))^\log_2(3)$$  

$$= M^{\log_2(3)}.$$
This function is strictly increasing (over the nonnegative reals) and convex, hence, we can substitute it into (16).

In contrast, the existing sequential communication lower bound for the standard Strassen’s algorithm computational DAG [6] is

\[
\max \left\{ \frac{r^{\log_2(7)}}{M^{\log_4(7)}} \cdot M, 3n^2 \right\}.
\]

Remark 1 We suspect Corollary 34’s lower bound is not tight. The gap may arise because the rank expansion bound of a bilinear algorithm and communication complexity are not tight for certain bilinear algorithms \((A, B, C)\) where only a small subset of columns are low rank in \(A\) (or \(B\) or \(C\)) while the remaining are nearly full rank. For example, the subset of columns with indices \(\{1, 2, 4\}\) from \(A\) for Strassen’s bilinear algorithm (Definition 32) together have rank 2 while any 3 columns from the remaining 4 columns are full rank. The proof of Proposition 5 from [35] performs a worst-case analysis where if there exists a subset of columns that is low rank in, for example \(A\), then the low-rank structure is assumed to persist in any remaining subset of columns in \(A\).

Next, we consider lower bounds on the parallel communication cost.

**Corollary 35** Given square matrices of size \(n\) (assumed to be a power of 2) and \(P\) processors, the parallel communication cost of Strassen’s fast matrix multiplication algorithm is at least

\[
3 \cdot \left( \frac{n^{\log_3(7)}}{P^{\log_3(2)}} - \frac{n^2}{P} \right).
\]

**Proof.** By Proposition 6 and noting the size of the inputs and output are \(n^2\), the communication lower bound of the nested bilinear algorithm is no smaller than the sum of some natural numbers \(r(A), r(B), r(C) \in \mathbb{N}\), which satisfy,

\[
\frac{R}{P} \leq \mathcal{E}(r(A) + \frac{n^2}{P}, r(B) + \frac{n^2}{P}, r(C) + \frac{n^2}{P}).
\]

(17)

Lemma 33 provides an expansion bound \(\mathcal{E}\) for Strassen’s algorithm, yielding

\[
\frac{R}{P} \leq \min \left( r(A) + \frac{n^2}{P}, r(B) + \frac{n^2}{P}, r(C) + \frac{n^2}{P} \right)^{\log_3(3)}.
\]

Since the rank is \(R = n^{\log_2(7)}\), the communication cost is at least

\[
c(A) + c(B) + c(C) \geq 3 \cdot \left( \frac{n^{\log_3(7)}}{P^{\log_3(2)}} - \frac{n^2}{P} \right).
\]

\[
\]
6.2 Convolution

Given a set of distinct nodes \( \{x_i\}_{i=1}^m \), where \( x_i \in \mathbb{C} \), the corresponding Vandermonde matrix is \( V_m \in \mathbb{C}^{m \times n} \), where \( V_{m,i,j} = x_i^{j-1} \). To compute the discrete convolution between two vectors \( f, g \in \mathbb{C}^k \), we use the Toom-\( k \) bilinear algorithm, \( F = (V_{2k-1}^k)^T, (V_{2k-1}^k)^T, (V_{2k-1}^{2k-1})^{-1} \) [25]. The term Toom-\( k \) refers to a particular bilinear algorithm for computing the convolution between vectors of size \( k \), and it belongs to a broader class of convolution algorithms known as Toom-Cook. For example, the discrete Fourier transform (DFT) is a special case of Toom-Cook, hence our lower bounds apply to fast Fourier transform (FFT)-based approaches for convolution.

It should be noted convolution can be applied to both integer and polynomial multiplication (i.e., Toom-Cook). While they are similar, a key difference between the two is that the former includes a carry-over step. While our analysis does not consider this carry-over step, we mention that this step does not impact the communication lower bound of integer multiplication as in previous analysis [15].

By nesting Toom-Cook bilinear algorithms, one can utilize split-nesting schemes to derive algorithms for 1D convolution that are more stable and computationally efficient [1, 33], as well as to compute multidimensional convolution [28, 32]. This, however, results in bilinear algorithms whose matrices are no longer Vandermonde. In view of this, we apply Proposition 11 to deduce rank expansion lower bounds for convolution 

\[ k = \min_i k_i. \]  

Lemma 36 For a nested bilinear algorithm \( \bigotimes_{i=1}^r A_i, \bigotimes_{i=1}^r B_i, \bigotimes_{i=1}^r C_i \), where \( \tau \in \mathbb{Z}_+ \), \( (A_i, B_i, C_i) \) is the bilinear algorithm for Toom-\( k_i \), then the function,

\[ E_{\tau(A), \tau(B), \tau(C)} = \min(\tau(A), \tau(B)) \log_2(2^{k_i}-1), \]

is an expansion bound function for the bilinear algorithm.

Proof. Since \( C_i = V_{2k_i-1}^{2k_i-1} \) is invertible, the function

\[ \sigma_C(t) = t \leq \min_{P \in P_{2k_i-1}} \{ \text{rank}(C_i P) \} \]

is a rank expansion lower bound by Definition 8.

Furthermore, \( A_i = B_i = V_{2k_i-1}^{2k_i-1} \) are Vandermonde matrices with unique nodes \( \{x_j\}_{j=1}^{2k_i-1} \). Thus, \( (V_{2k_i-1}^{2k_i-1})^T P \) is the transpose of a Vandermonde matrix with a subset of nodes from \( \{x_i\} \), which must also consist of unique nodes. Hence, \( (V_{2k_i-1}^{2k_i-1})^T P \) is full rank. Using this observation and the fact \( \log_2(x) \) is increasing w.r.t. \( x \geq 1 \) for any fixed \( a \geq 1 \) as well as recalling \( k \) from (18), one can directly verify

\[ \sigma(t) = t \log_2(2^{k_i}-1) \leq \min_{P \in P_{2k_i-1}} \{ \text{rank}(A_i P) \}, \]
is a rank expansion lower bound for $A_i$ as well as for $B_i$.

Like in the proof for Lemma 33, one can verify $\sigma$ is concave, log-log concave, and satisfies $\sigma(0) = 0$, as does $\sigma_C$. Also, $d_A \equiv \sigma(1) = 1$. Thus, one can recursively apply Theorem 11 to confirm $\sigma$ is a rank expansion lower bound for $\bigotimes_{i=1}^{\tau} A_i$ and $\bigotimes_{i=1}^{\tau} B_i$, and $\sigma_C$ is for $\bigotimes_{i=1}^{\tau} C_i$. Thus, with $K = \prod_i k_i$, any $\ell \in [K]$, and all $P \in \mathcal{P}_K^{(\ell)}$,

$$\sigma(\ell) = \ell^{\log_2(\ell_1)} \leq \min \{ \text{rank}(\bigotimes_{i=1}^{\tau} A_i P), \text{rank}(\bigotimes_{i=1}^{\tau} B_i P) \}.$$  

Applying the monotone function $\sigma^{-1}(\ell) = \ell^{\log_2(2^{k_1} - 1)}$ to both sides and recalling the choice of the expansion bound function $E$, we get for all $P \in \mathcal{P}_K^{(\ell)}$,

$$\#cols(P) \leq E \left( \text{rank}(\bigotimes_{i=1}^{\tau} A_i P), \text{rank}(\bigotimes_{i=1}^{\tau} B_i P), \text{rank}(\bigotimes_{i=1}^{\tau} C_i P) \right).$$

Thus, $E$ is an expansion bound by Definition 4 (c.f. (2)).

We now state a communication lower bound when nesting the same Toom-$k$ bilinear algorithm. Note that these bounds hold for both multidimensional and recursive (1D) convolution. The latter is simply multidimensional convolution plus a recomposition step, where the recomposition is applied via a pre-multiplication by the linear operator $Q \in \{0, 1\}^{2n-1 \times (2^k-1)^\tau}$ (see Section 7.3 in [25]) to $C^\otimes \tau$. Because the rank of the $C^\otimes \tau$ matrix, the only matrix in the bilinear algorithm that is affected by the recomposition matrix $Q$, is absent in the expansion bound of Lemma 36, the expansion bound still holds for recursive (1D) convolution.

First, we focus on sequential communication, or communication between a fast and slow memory (e.g., cache and memory). In the special 1D case, where Toom-$n$ is recursively applied to an input vector of size $N = n^d$, the following result matches previously established lower bounds [10]. The proof is a direct result from Proposition 5 and Lemma 36 (and follows similarly to the proof for Corollary 34). Hence, we skip the proof.

**Corollary 37** Given two $d$-dimensional tensors where each mode length is $n$ and fast memory of size $M$, the sequential communication cost of discrete convolution using a nested Toom-$n$ bilinear algorithm is at least

$$\max \left\{ \frac{(2n-1)^d}{M^{\log_2(n-1)}}, M, 2^{n^d + (2n-1)^d} \right\}.$$

Next, we derive parallel communication (i.e., communication between parallel processors) lower bounds for multidimensional convolution, which asymptotically match previously established lower bounds [15]. Again, the result follows from Proposition 6 and Lemma 36, hence we skip the proof.

**Corollary 38** Given two $d$-dimensional tensors where each mode length is $n$ and $P$ processors, the parallel communication costs of discrete convolution using a nested Toom-$n$ bilinear algorithm is at least

$$2 \cdot \left( \frac{n^d}{P^{\log_2(n-1)}} - \frac{n^d}{P} \right).$$
6.3 Partially Symmetric Tensor Contractions

A tensor contraction, which generalizes matrix multiplication, are tensor products summed over a subset of modes (indices) from the tensors. Tensor contractions are also equivalent to a matrix multiplication between the unfolded tensors, where a subset of tensor modes are mapped to either row or column indices of a matrix [18, 35]. However, special fast bilinear algorithms become possible when the tensors have symmetry [34]. Symmetric tensors are invariant under all permutations of their indices, e.g., $t_{ijk} = t_{jik} = \ldots$. For example, a product of a symmetric matrix $A$ and a vector $b$ can be computed from $n(n+1)/2$ scalar products, mostly of the form $a_{ij}(b_i + b_j)$, yielding a bilinear algorithm with rank $R = n(n+1)/2$. The main application of such symmetry preserving algorithms is to lower the cost of contraction algorithms for partially symmetric tensors (tensors that are equivalent only under permutations of a subset of their indices). For example, given a partially symmetric tensor $T$ with symmetry $t_{ijab} = t_{jiab}$, the contraction $u_{iac} = \sum_{jb} t_{ijab}v_{jbc}$ can be performed with 2 times fewer operations to leading order than in the nonsymmetric case, by nesting a symmetry preserving algorithm for a symmetric vector product (corresponding to contraction indices $i$ and $j$) with an algorithm for matrix multiplication (corresponding to contraction indices $a, b, c$).

Tensors with partial symmetry are prevalent in quantum chemistry methods, a core application domain of higher-order tensor contractions [20]. By analysis of these bilinear algorithms’ rank expansions, communication lower bounds have been established showing that such symmetry preserving algorithms require asymptotically more communication for some contractions of symmetric tensors [35]. Our results allow us to derive the first communication lower bounds for nested symmetry preserving algorithms, yielding lower bounds on communication costs for symmetry preserving contraction algorithms on partially symmetric tensors.

The lemma below follows from the analysis in the proof of Lemma 6.3 in [35].

**Lemma 39** For the bilinear algorithm $(A, B, C)$ corresponding to symmetry preserving contraction of symmetric tensors of order $s + v$ and $v + t$ over $v$ indices, we can lower bound the rank expansion of each encoding matrix as follows: $\sigma_A(k) = k^{(s+v)/(s+t+v)} / s + t + \binom{v}{2}$, $\sigma_B(k) = k^{(v+t)/(s+t+v)} / s + t + \binom{v}{2}$, and $\sigma_C(k) = k^{(s+t)/(s+t+v)} / s + t + \binom{v}{2}$.

We derive lower bounds for the nesting of multiple symmetry preserving algorithms, as well as nesting of a symmetry preserving algorithm with a nonsymmetric contraction algorithm. In the former case, we consider nesting of two arbitrary symmetry preserving algorithms.

**Lemma 40** For the bilinear algorithm $(A \otimes U, B \otimes V, C \otimes W)$, where $(A, B, C)$ is a symmetry preserving contraction of symmetric tensors of order $s + v$ and $v + t$ over $v$ indices and all dimensions equal to $n_1$, while $(U, V, W)$ is a symmetry preserving contraction of symmetric tensors of order $s' + v'$ and $v' + t'$
over \( v' \) indices with all dimension equal to \( n'_2 \), we can lower bound the rank expansion of \( A \otimes U \) by

\[
\sigma_{A \otimes U}(k) \geq \min_{k_1, \ldots, k_s, k'_1, \ldots, k'_{s'}} \frac{1}{\binom{n_1 + s + t}{t'} \binom{n'_2 + s' + t'}{t'}} \cdot k_1^{s+s'} \cdot k'_1^{s'+s''}
\]

as well as similar bounds for \( B \otimes V \) and \( C \otimes W \).

**Proof.** The theorem follows by application of Theorem 11 on the rank expansion lower bounds given by Lemma 39. Note that in the first inequality, we restrict \( k_1 \) and \( k_2 \) to \([1, n_1 + s + t] \) and \([1, n'_2 + s' + t'] \), respectively, which weakens the bound up to a constant. \( \blacksquare \)

The rank expansion lower bound \( \sigma_{A \otimes U} \) in Lemma 40 generalizes to nestings of three or more symmetry preserving bilinear algorithms. This rank expansion lower bound implies parallel and sequential communication for nested bilinear algorithms follow immediately from those of the nested parts. These communication lower bounds ascertain that standard approaches for tiling nested loops can asymptotically minimize communication done in the execution of the bilinear algorithm.

We also consider nestings of a symmetry preserving algorithm with a standard (nonsymmetric) tensor contraction. Nonsymmetric tensor contractions are equivalent to matrix-matrix products with an appropriate choice of matrix dimensions. Like for symmetric tensor contractions, we assume tensor dimensions are equal size and classify contractions by the tuple \((s, t, v)\). When one of \( s, t, v \) is zero, the number of products needed to compute the contraction matches the size of the largest tensor \((n_1 + s + t, n'_2 + s' + t')\), and the corresponding bilinear encoding matrix is (some permutation of) the identity matrix, with a rank expansion of \( \tilde{\sigma}(k) = k \). When this is the case for nonsymmetric contractions, tight communication lower bounds thereof can be derived just by considering the rank expansion of a single matrix. For the symmetry preserving algorithm, for any choice of \( s, t, v \), a tight communication lower bound can be derived from the rank expansion of just one of the matrices. For further details on these lower bounds and optimal algorithms, see [35] (Section 7 discusses upper bounds). Consequently, we can use our general lower bounds on the rank expansion of a Kronecker product of matrices to derive communication lower bounds that we expect are tight by restricting the type of nonsymmetric contraction performed.

**Lemma 41** Consider the bilinear algorithm \((A \otimes U, B \otimes V, C \otimes W)\), where \((A, B, C)\) is a symmetry preserving contraction of symmetric tensors of order \( s + v \) and \( v + t \) over \( v \) indices and all dimensions equal to \( n_1 \), while \((U, V, W)\) is a contraction of nonsymmetric tensors of order \( s' + v' \) and \( v' + t' \) over \( v' \) dimensions.
indices with all dimension equal to $n'_2$. If $t' = 0$, we can lower bound the rank expansion of $A \otimes U$ by

$$\sigma_{A \otimes U}(k) = \sigma_A(k) = \frac{1}{(s+t+v)} k^{(s+v)/(s+t+v)}.$$ 

Similar bounds hold for $B \otimes V$ (if instead of $t' = 0$, we have $s' = 0$) and $C \otimes W$ (if instead of $t' = 0$, we have $v' = 0$).

The expansion bound in Lemma 41 implies that we can obtain a lower bound on parallel and sequential communication cost on nested algorithms composed of symmetry preserving algorithms and a nonsymmetric tensor contraction where one of $s', t', v'$ is zero. Given a rank expansion lower bound for one of the inputs or the output, e.g., $\sigma_{A \otimes U}$, we obtain a sequential communication lower bound of the form,

$$\Omega(n^{s+t+v+s'+t'+v'} M/\sigma_{A \otimes U}^{-1}(M)),$$

and a parallel communication lower bound of

$$\Omega(\sigma_{A \otimes U}(n^{s+t+v+s'+t'+v'}/p) - n^{s+v+s'+v'}/p).$$

For nested symmetry preserving algorithms, the greatest of the three communication lower bounds (based on $\sigma_{A \otimes U}$, $\sigma_{B \otimes V}$, or $\sigma_{C \otimes W}$) would be asymptotically attainable for sufficiently small $M, p$ with standard approaches for multidimensional loop tiling [14]. While the bound Lemma 41 should also be asymptotically attainable for many contractions, but not all, as preclusion of $t' > 0$ implies we do not provide bounds for communication associated of all inputs/outputs in a particular contraction. The new communication lower bounds imply that for some partially symmetric tensor contractions, the use of the symmetry preserving algorithm may require asymptotically more communication than if the symmetry was ignored. However, these contractions involve high order tensors. The example below is among the simplest possible cases.

**Example 2** Consider the contraction,

$$c_{im} = \sum_{j,k,l} a_{ijkl} b_{jklm},$$

where $a_{ijkl}$ is symmetric under any permutation of $(i,j,k,l)$ and $b_{jklm}$ is symmetric under any permutation of $(j,k,l)$. Here, a symmetry preserving algorithm with $s = 1, v = 3, t = 0$ may be nested with a nonsymmetric contraction with $s' = 0, v' = 0, t' = 1$. By Lemma 41, we obtain the following rank expansion lower bound,

$$\sigma_{B \otimes V}(k) = \frac{1}{(s+t+v)} k^{(t+v)/(s+t+v)} = (1/4) k^{3/4}.$$
Assume the dimension of each mode of the tensor (range of each index) is \( n_1 = n_2 = n \). Overall, this algorithm would then require 4 times fewer products \( (n^5/24 \text{ to leading order}) \) than if only considering symmetry in \((j,k,l)\) and performing classical matrix multiplication with dimensions \( n \times \binom{n+2}{3} \times n \) (requiring \( n^5/6 \) products to leading order). However, by our new lower bounds, it would require \( \Omega(n^5/M^{1/3}) \) sequential communication. On the other hand, for sufficiently small \( M \), the matrix-multiplication-based approach requires only \( O(n^5/M^{1/2}) \) sequential communication.

7 Conclusion

We develop a new framework to ascertain communication lower bounds for any bilinear algorithm via the rank expansion of a matrix, or the minimum rank of any submatrix of fixed size. Unlike previous works which assume a particular computational DAG, our lower bounds consider a larger space of permissible computational DAGs. Our new communication lower bounds for recursive convolution match previous bounds \([10,15]\), suggesting that any algebraic reorganization of convolution cannot reduce communication costs in this setting.

We note two limitations in our analysis that prevents us from obtaining tight lower bounds for standard nested matrix multiplication as well as some partially symmetric tensor contractions. First, as described in Example 1, we separately bound the ranks of the matrices from the nested bilinear algorithm to derive expansion bounds. However, this can derive expansion bounds which are much larger than the expansion function (which bounds the rank simultaneously), especially when the matrices have an extremely low-rank structure. Second, there seems to be a gap between converting rank expansion bounds to communication complexity. As discussed in Remark 1, the proof of the communication lower bound via rank expansions assumes if there is a low-rank structure in a matrix of a bilinear algorithm, that low-rank structure persists in any subset of columns, which is not true for Strassen’s algorithm. Therefore, to get a tight lower bound, we hypothesize one needs to separately handle low-rank and nearly full-rank subsets of columns in the matrices.

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A Appendix: Improved Lower Bounds by Limiting the Grid Expansion

Let $A \in \mathbb{C}^{m \times n_A}$, $B \in \mathbb{C}^{m \times n_B}$, and $C = A \otimes B$. As mentioned after the main theorems in Section 3, bounds $\sigma_C$ in those results require defining $\sigma_A$ and $\sigma_B$ beyond $n_A$ and $n_B$. This extrapolation can lead to a loose lower bound $\sigma_C$ even if $\sigma_A$ and $\sigma_B$ are tight. We illustrate this phenomenon in the next example.

Example 3 Consider the case $A = B \in \mathbb{R}^{4 \times 7}$ with rank and Kruskal rank (maximum $k$ such that any $k$ different columns are linearly independent [27]) being 4. For example,

$$A = B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 & 1 & 8 & 27 \end{bmatrix}.$$  

Denote the $i$th columns of $A$ and $B$ as $a_i$ and $b_j$, respectively. Now $\tilde{\sigma}_A(x) = \tilde{\sigma}_B(x) = \min \{ x, 4 \}$. Let $C = A \otimes B$, and we seek a rank expansion lower bound $\sigma_C(k)$ for $C$.

When $k = 13$, it is not hard to check that $\tilde{\sigma}_C(k) = 7$, which is attained by submatrix $CP = \{ a_i \otimes b_j : i = 1 \text{ or } j = 1 \}$. If we naturally take $\sigma_A(x) = \sigma_B(x) = \min \{ x, 4 \}$ (on $\mathbb{R}$), Theorem 9 gives $\sigma_C(13) = 4$. If we take instead $\sigma_A(x) = \sigma_B(x) = x^{\ln 4 / \ln 7}$, Theorem 9 gives $\sigma_C(13) \approx 6.2$, which is optimal after rounding up to integer. Although $x^{\ln 4 / \ln 7}$ is not as tight as $\min \{ x, 4 \}$ in the range $x \in [0, 7]$, its extrapolation on $x \geq 7$ is greater than that of $\min \{ x, 4 \}$.

Indeed, when $x \leq 7$, using $\min \{ x, 4 \}$ offers a tighter bound than $x^{\ln 4 / \ln 7}$. For example, when $k = 5$, with $\sigma_A(x) = \sigma_B(x) = \min \{ x, 4 \}$, $\sigma_C(5) = 4 = \tilde{\sigma}_C(5)$, but with $\sigma_A(x) = \sigma_B(x) = x^{\ln 4 / \ln 7}$, $\sigma_C(5) \approx 3.1$.

We see from above that a tighter rank expansion lower bounds on $A$ and $B$ may lead to a looser rank expansion lower bound $\sigma_C$. Thus, there is an opportunity to improve the established bound in Theorem 9. To achieve this improvement and avoid the behavior in the example above, we derive a lower bound $\sigma_C$ that evaluates $\sigma_A$ and $\sigma_B$ on their intended domains of $[0, n_A]$ and $[0, n_B]$, respectively.

A.1 The $L$-shaped bound

The derivation of the new rank expansion lower bound $\sigma_C$ is not much different from the main theorems. The only difference is in the continuous relaxation step (Section 5). Recall that in the discrete step (see Section 4.2.1 and definitions therein), we constructed the pre-CDG $S$ that is a subgrid of the basis of $D = \text{VCollapse}(G)$. Thus, we know

$$\text{CDG}(S) \subseteq [0, \sigma_A(n_A)] \times [0, \sigma_B(n_B)].$$
To simplify the notation, we will denote

\[ r_A := \sigma_A(n_A), \quad r_B := \sigma_B(n_B), \quad S := \text{CDG}(S). \]

In the continuous relaxation step, we carried out Merge(2, 3) (Definition 24) until \( S \) becomes a 2-step CDG, and then Merge(1, 2) to make it a rectangle. In fact, before the last step Merge(1, 2), the entire grid remains a subgrid of \([0, r_A] \times [0, r_B]\). To avoid leaving this domain (so that \( \text{GridExp}(S) \) remains in \([0, n_A] \times [0, n_B]\)), we stop Merge(1, 2) if either the first or second step hits the boundary of the domain, which leads to an L-shaped grid (a 2-step CDG). We refer to this stoppage of Merge(1, 2) as early stopping.

**Definition 42.** The L-shaped CDG \( S = L(x_1, y_1; x_2, y_2) \) is the 2-step CDG with horizontal edges at \( y_1, y_2 \) and vertical edges at \( x_1, x_2 \). By convention, we require \( 0 < x_1 < x_2, 0 < y_1 < y_2 \).

**Definition 43.** Fix the values \( r_A = \sigma_A(n_A) \) and \( r_B = \sigma_B(n_B) \). We denote the collection of L-shaped grids of size \( t \) that touch the boundaries as

\[ \mathcal{L}(t) = \{ L = L(x_1, y_1; r_A, r_B) : |L| = t \}. \]

Denote the collection of rectangle grids of size \( t \) within \([0, r_A] \times [0, r_B]\) as

\[ \mathcal{R}(t) = \{ R = [0, x] \times [0, y] : x \in [1, r_A], \quad y \in [1, r_B], \quad |R| = t \}. \]

The L-shaped bound we derive in this section extends the bounds in the main theorems. Thus, we will work under the same assumptions on \( \sigma_A \) and \( \sigma_B \). See equations (6) and (7).

It is not hard to check that the early-stopped merge also results in an upper bound of \( \langle S \rangle \) as stated in the lemma below, which serves as the analogue of Lemma 25.

**Lemma 44** Let \( S \) be a CDG in \([0, r_A] \times [0, r_B]\) of size \( |S| = t \). Then

\[ |\text{GridExp}(S)| \leq \max_{M \in \mathcal{L}(t) \cup \mathcal{R}(t)} \langle M \rangle. \tag{19} \]

**Proof.** The proof is similar to that of Lemma 25, where as long as the grid has at least three steps, we apply Merge(2, 3) repeatedly. This produces an L-shaped grid within \([0, r_A] \times [0, r_B]\). However, in the last merge operation, Merge(1, 2), we stop increasing \( u \), the height difference between steps 1 and 2, if step 1 reaches height \( r_B \) before step 2 reaches the ground. This may give an L-shaped grid \( L(x_1, y_1; x_2, r_B) \). By Lemma 25, either this L-shaped grid or a rectangle grid is an upper bound of \( \langle S \rangle \). If the rectangle one is an upper bound, then we are done since it is in \( \mathcal{R}(t) \).

Next, consider the case the upper bound is \( L(x_1, y_1; x_2, r_B) \). Similar to how we increased/decreased the height of the steps, we can horizontally merge the two horizontal layers of the 2-step stair, and we similarly disallow the width of the lower layer to go beyond \( r_A \). This may result in an L-shaped grid \( L(x_1', y_1, r_A, r_B) \). After these two merges, the resulting grid may either be a rectangle from \( \mathcal{R}(t) \), or the L-shaped grid above, which is from \( \mathcal{L} \). The proof is thus complete. \[ \blacksquare \]
With this new continuous relaxation step, we analogously derive the corresponding bound \( \sigma_C \), which includes the \( L \)-shaped bound, in the following theorem.

**Theorem 45** Suppose functions \( \sigma_A \) and \( \sigma_B \) are concave rank expansion lower bounds of \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{m \times n} \), respectively, with \( \sigma_A(0) = \sigma_B(0) = 0 \). Let \( r_A = \sigma_A(n_A) \), \( r_B = \sigma_B(n_B) \), \( d_A = \sigma_A^\dagger(1) \), and \( d_B = \sigma_B^\dagger(1) \). Define

\[
R_C(k) = \min_{k_A \in [d_A, n_A], k_B \in [d_B, n_B]} \sigma_A(k_A) \cdot \sigma_B(k_B),
\]

and

\[
L_C(k) = \min_{k_A \in [d_A, n_A], k_B \in [0, n_B], k_A + k_B = n_B} \sigma_A(k_A) r_B + \sigma_B(k_B) r_A - \sigma_A(k_A) \cdot \sigma_B(k_B). \tag{20}
\]

Then \( \sigma_C(k) = \min \{ L_C(k), R_C(k) \} \) is a rank expansion lower bound of \( C = A \otimes B \). When \( R_C(k) \leq \max \{ r_A, r_B \} \), \( R_C \) is a rank expansion lower bound for \( C \).

*Proof.* Again, using a density argument on the functions, we hereafter assume \( \sigma_A \) and \( \sigma_B \) are strictly increasing smooth functions so that they are invertible. We start by showing \( \sigma_C \) is continuous and increasing, as required by the definition of a rank expansion lower bound. Since \( R_C \) and \( L_C \) are continuous, so is \( \sigma_C \). Clearly, \( R_C \) is increasing. As for \( L_C \), we have the equivalent definition,

\[
L_C(k) = \min_{k_A \in [d_A, n_A], k_B \in [0, n_B], (n_A-k_A)(n_B-k_B) = n_A n_B - k} (r_A - \sigma_A(k_A)) (r_B - \sigma_B(k_B)).
\]

Thus, as \( k \) increases, one can increase \( k_A \) or \( k_B \) so that \( L_C \) increases. Hence, \( L_C \) is also an increasing function. Consequently, \( \sigma_C \) is an increasing function.

To prove \( \sigma_C \) is a lower bound of the rank, we express the right-hand side of (19) as a function \( \phi(t) \) as in the proof of Theorem 9. We compute the maximal expansion size of stairs in \( R(t) \) and \( L(t) \) below:

\[
\phi_R(t) = \max_{t_A \in [1, r_A], t_B \in [1, r_B]} \sigma_A^{-1}(t_A) \sigma_B^{-1}(t_B);
\]

\[
\phi_L(t) = \max_{x_1 \in [0, n_A], y_1 \in [0, n_B]} n_A \sigma_B^{-1}(y_1) + n_B \sigma_A^{-1}(x_1) - \sigma_A^{-1}(x_1) \sigma_B^{-1}(y_1). \tag{21}
\]

Then we have

\[
\phi(t) := \max \{ \phi_R(t), \phi_L(t) \} = \max_{M \in L(t) \cup R(t)} \langle M \rangle.
\]

When \( k \leq d_A d_B \), \( \sigma_C(k) \leq R_C(k) = 1 \), which is clearly a lower bound of the rank of a nonzero matrix. It remains to show the proposed function \( \sigma_C \) satisfies \( \sigma_C = \phi^{-1} \) on \([d_A d_B, +\infty)\).
Step 1. $R_C = \phi_R^{-1}$ on $(d_Ad_B, +\infty)$. This proof is the same as the proof for Lemma 26. All arguments carry through with the new constraints $t_A \leq r_A$, $t_B \leq r_B$, and $k_A \leq n_A$, $k_B \leq n_B$.

Step 2. $L_C = \phi_L^{-1}$. We will repeat the proof for Lemma 26 for this $L$-shaped case. First, we show $L_C \leq \phi_L^{-1}$. Assume by contradiction $\phi_L(\sigma_C(k)) > k$ for some $k$. Then there exist $x_1 \in [0, r_A]$, $y_1 \in [0, r_B]$, and $r_Bx_1 + r_Ay_1 - x_1y_1 = \sigma_C(k)$ such that

$$k' = n_A\sigma_B^{-1}(y_1) + n_B\sigma_A^{-1}(x_1) - \sigma_A^{-1}(x_1)\sigma_B^{-1}(y_1) > k.$$

Consequently, as $L_C$ is strictly increasing,

$$\sigma_C(k) \leq L_C(k) < L_C(k') = \min_{k_A \in [0,n_A], k_B \in [0,n_B], n_Ak_A + n_Bk_B = k'} r_B\sigma_A(k_A) + r_A\sigma_B(k_B) - \sigma_A(k_A) \cdot \sigma_B(k_B) \leq r_B\sigma_A(\sigma_A^{-1}(x_1)) + r_A\sigma_B(\sigma_B^{-1}(y_1)) - \sigma_A(\sigma_A^{-1}(x_1)) \cdot \sigma_B(\sigma_B^{-1}(y_1)) = r_Bx_1 + r_Ay_1 - x_1y_1 = \sigma_C(k).$$

This is absurd. Next, we show the other direction, $\phi_L^{-1} \leq L_C$, by showing that if $L_C(k) = t$, then $\phi_L(t) \geq k$. Indeed, if $L_C(k) = t$, then let $x_1 = \sigma_A(k_A) \in [0, r_A]$ and $y_1 = \sigma_B(k_B) \in [0, r_B]$. Then we have both

$$\sigma_A^{-1}(x_1)r_B + \sigma_A^{-1}(y_1)r_A - \sigma_A^{-1}(x_1)\sigma_B^{-1}(y_1) = k,$n_Ak_A + n_Bk_B = k'$

$$r_Bx_1 + r_Ay_1 - x_1y_1 = t.$$

Therefore,

$$\phi_L(t) \geq \sigma_A^{-1}(x_1)r_B + \sigma_A^{-1}(y_1)r_A - \sigma_A^{-1}(x_1)\sigma_B^{-1}(y_1) = k.$$

Hence, $L_C = \phi_L^{-1}$.

We now conclude $\sigma_C = \min \{R_C, L_C\}$ is the inverse of $\phi = \max \{\phi_R, \phi_L\}$. This is straightforward with steps 1 and 2, since all 4 functions here are positive increasing functions on $\mathbb{R}_+$. We have established that $\sigma_C$ is a valid rank expansion lower bound for $C$.

Finally, to see why when $R_C(k) \leq \max \{r_A, r_B\}$ implies that $R_C$ is a rank expansion lower bound, we consider the final merge operation $\text{Merge}(1, 2)$. The $L$-shaped bound will only come into play if we need to increase the height difference $u$, and step 1 reaches height $r_B$ before step 2 goes down to the ground. However, each of the 2 steps is of width at least 1, since the original stair $S$ corresponds to a CDG. This means step 1 will never reach $r_B$ before step 2 reaches the ground if $|S| \leq r_B$. Similarly, since $r_A$ and $r_B$ are interchangeable (by considering $C = B \otimes A$), we see that early stopping the merge and the $L$-shaped grid will never come into play if $|S| \leq \max \{r_A, r_B\}$.

Suppose $G$ is a grid and it produces a (pre-) CDG $S$ with size $|S| \leq \max \{r_A, r_B\}$. Then since we do not need to early stop the merge,$$\phi_R(\text{rank}(G)) \geq \phi_R(|S|) \geq |\text{GridExp}(S)| \geq |G|.$$
Hence, \( \text{rank}(G) \geq \delta_R^{-1}(k) = R_C(k) \), which establishes that \( R_C \) is a rank expansion lower bound in this case.

The new bound in Theorem 45 is more complicated than Theorem 9, which does not involve \( L \)-shaped grids. However, the new bound derived in this section is often tighter. Indeed, let \( \sigma^R_C \) and \( \sigma^{R+L}_C \) be, respectively, the rank expansion lower bounds derived from Theorem 9 and Theorem 45. Recall the \( \phi \) function we used when proving Theorem 9, shown below

\[
\phi^{\text{prev}}(t) = \max_{t_A, t_B \geq 1, t_A + t_B = t} \sigma^{-1}_A(t_A)\sigma^{-1}_B(t_B). \tag{22}
\]

Denote \( \phi^{\text{new}} = \max \{ \phi_R, \phi_L \} \) as the new \( \phi \) function used in the proof of Theorem 45 above. Since \( \sigma^R_C = (\phi^{\text{prev}})^{-1} \) and \( \sigma^{R+L}_C = (\phi^{\text{new}})^{-1} \), if \( \phi^{\text{prev}} \geq \phi^{\text{new}} \), the new bound \( \sigma^{L+R}_C \) is then tighter (greater) than \( \sigma^R_C \).

When \( \phi_L \leq \phi_R \), it is clear \( \phi^{\text{prev}} \geq \phi^{\text{new}} \), since \( \phi_R \) and \( \phi^{\text{prev}} \) are maximizing the same function and \( \phi_R \) has a smaller feasible region, so \( \phi^{\text{prev}} \geq \phi = \phi^{\text{new}} \). When \( \phi_L > \phi_R \), this is not clearly true. When the maximizer \( x_1 \) in \( \phi_L \) is at least 1, then we can prove \( \phi^{\text{prev}} \geq \phi_L \), so \( \sigma^{L+R}_C \geq \sigma^R_C \). We can require \( x_1 \geq 1 \) by early stopping the horizontal merge step in the proof of Lemma 44, but this will make the bound too complicated to state.

The bound in Theorem 45 does not require defining \( \sigma_A, \sigma_B \) beyond \( n_A \) and \( n_B \). Thus, it also resolves the undesirable phenomenon in Example 3. It provides a tighter rank expansion lower bound for \( C = A \otimes B \) when given tighter rank expansion lower bounds for \( A \) and \( B \). This is not always the case with Theorem 9 as shown in Example 3.

Suppose we have two rank expansion lower bounds \( \sigma_A, \hat{\sigma}_A \) for \( A \) and \( \sigma_B, \hat{\sigma}_B \) for \( B \). If \( \sigma_{A,B} \geq \sigma_{A,B} \) on \( [0, n_{A,B}] \), then the corresponding functions \( R_C \geq R_C \) on \( [0, n_{A,B}] \). If in addition \( \sigma_{A,B}(n_{A,B}) = \sigma_{A,B}(n_{A,B}) \), for example, both are equal to the true rank of \( A \) and \( B \), then also \( L_C \geq L_C \) on \( [0, n_{A,B}] \).

Despite these properties of the new bound in Theorem 45, Theorem 9 and Theorem 11 give much simpler bounds and can be easily applied recursively to derive a lower bound on the rank expansion for \( C = \bigotimes_{i=1}^p A_i \), \( p \geq 3 \).

The log-log convexity assumption simplifies Theorem 9. Similar assumptions can also simplify the bound in Theorem 45. Under such assumptions, we can show \( R_C \leq L_C \), which removes the need of \( L_C \) function. The obtained rank expansion lower bound is thus tighter than the one in Theorem 9 as discussed above. This is the main topic of the next section.

A.2 Simplifying the \( L \)-shaped bound

The main goal is to understand when \( R_C \leq L_C \), and thus \( \sigma_C = R_C \). We will show this is the case when \( \sigma_A \) and \( \sigma_B \) satisfy an appropriate log-log convexity condition (see Definition 10 for the definition and Section 5.4 for the basic properties). We will continue using the notations introduced in the previous section. Recall that \( r_A = \sigma_A(n_A) \) and \( r_B = \sigma_B(n_B) \).
Theorem 46 Let everything be defined as in Theorem 45. Suppose in addition functions \( r_A - \sigma_A(n_A - x) \) and \( r_B - \sigma_B(n_B - x) \) are log-log convex on \((0, n_A)\) and \((0, n_B)\), respectively. Then, \( R_C(k) \leq L_C(k) \), and consequently,

\[
\sigma_C(k) = \min_{k_A \in [d_A, n_A], k_B \in [d_B, n_B], k_Ak_B \geq k} \sigma_A(k_A) \cdot \sigma_B(k_B) \tag{23}
\]

is a rank expansion lower bound of \( C = A \otimes B \). This bound is tighter than the one in Theorem 9.

Proof. As defined in (21) during the proof of Theorem 45, the size of an \( L \)-shaped CDG after a grid expansion can be bounded by

\[
\phi_L(t) = \max_{x_1 \in [0, r_A], y_1 \in [0, r_B], r_B x_1 + r_A y_1 = t} n_A \sigma_B^{-1}(y_1) + n_B \sigma_A^{-1}(x_1) - \sigma_A^{-1}(x_1) \sigma_B^{-1}(y_1).
\]

Using a density argument, we assume that \( \sigma_A^{-1} \) and \( \sigma_B^{-1} \) are smooth and strictly increasing on \([0, r_A]\) and \([0, r_B]\). According to Theorem 45, when \( R_C(k) \leq \max \{r_A, r_B\} \), \( R_C \) is indeed a rank expansion lower bound of \( C \). Thus, hereafter we only consider the case \( R_C(k) = t \geq \max \{r_A, r_B\} \). To show \( R_C \leq L_C \), it suffices to show \( \phi_R(t) \geq \phi_L(t) \) when \( t \geq \max \{r_A, r_B\} \). Hence, we assume \( t \geq \max \{r_A, r_B\} \) in the proof below.

We show that when \( \sigma_A^{-1} \) and \( \sigma_B^{-1} \) satisfy the log-log convexity assumption, \( \phi_L(t) \) is maximized at either \( x_1 = 0 \) or \( y_1 = 0 \). To begin with, note that when \( x_1 = 0 \),

\[
\phi_L(t) = \sigma_A^{-1}(r_A) \sigma_B^{-1}(t/r_A),
\]

and when \( y_1 = 0 \),

\[
\phi_L(t) = \sigma_A^{-1}(t/r_B) \sigma_B^{-1}(r_B).
\]

Since \( t/r_A \geq 1 \) and \( t/r_B \geq 1 \), then the pairs \((r_A, t/r_A)\) and \((t/r_B, r_B)\) are feasible in the optimization, and so

\[
\phi_R(t) = \max_{t_A \in [1, r_A], t_B \in [1, r_B], t_A t_B = t} \sigma_A^{-1}(t_A) \sigma_B^{-1}(t_B).
\]

Thus, if \( \phi_L(t) \) is indeed maximized at \( x_1 = 0 \) or \( y_1 = 0 \), then we have \( \phi_L \leq \phi_R \), and \( R_C = \phi_R^{-1} \) is a valid rank expansion lower bound.

To that end, let us reuse the notation \( f = \sigma_A^{-1} \) and \( g = \sigma_B^{-1} \) for sake of simplicity. We introduce following functions on \((0, r_A)\) and \((0, r_B)\), respectively:

\[
\hat{f}(x) = \frac{f(r_A) - f(r_A - x)}{x f'(r_A - x)} \quad \text{and} \quad \hat{g}(x) = \frac{g(r_B) - g(r_B - x)}{x g'(r_B - x)}.
\]

Since \( r_A - \sigma_A(n_A - x) \) is log-log convex and increasing on \([0, n_A]\), its inverse, \( p(x) \equiv f(r_A) - f(r_A - x) \), is log-log concave and increasing on \([0, r_A]\). Therefore,

\[
\frac{d}{dx} \ln p(e^x) = \frac{e^x p'(e^x)}{p(e^x)}
\]
is positive and decreasing in $x$. Thus,

$$\hat{f}(x) = \frac{p(x)}{xp'(x)}$$

is increasing in $x$. Similarly, $\hat{g}(x)$ is also increasing.

Writing $x = x_1$, we can rewrite $\phi_L$ as

$$\phi_L(t) = \max_{x_1 \in [0,r_A], \ y_1 \in [0,r_B], \ (r_A-x_1)(r_B-y_1) = r_Ar_B-t} n_An_B - (n_A - f(x_1))(n_B - g(y_1))$$

$$= n_An_B - \min_{x \in [0,r_A - \frac{r_Ar_B - t}{r_B}]} (f(r_A) - f(x)) \biggl(g(r_B) - g \left( r_B - \frac{r_Ar_B - t}{r_A - x} \right) \biggr).$$

In the trivial cases $t = r_Ar_B$, one can directly verify $\phi_R = \phi_L$. Now assume $t < r_Ar_B$, and thus $x < r_A$. For simplicity, we write $r_Ar_B - t = c > 0$, $\frac{r_A}{r_B} = s_x \in [0, \frac{r_A}{r_B}]$. It suffices to show

$$\arg\min_{x \in (0, r_A - \frac{c}{r_B})} \left\{ h(x) : = (f(r_A) - f(x)) \left( g(r_B) - g \left( r_B - \frac{r_Ar_B - t}{r_A - x} \right) \right) \right\} \in \{0, r_A - \frac{c}{r_B}\}.$$

The function $h$ is $C^1$ on $[0, r_A - \frac{c}{r_B}]$, and

$$h'(x) = [f(r_A) - f(x)] \cdot \frac{ds_x}{dx} \cdot g'(r_B - s_x) - f'(x)[g(r_B) - g(r_B - s_x)]$$

$$= s_x \left[ \frac{f(r_A) - f(x)}{r_A - x} \cdot g'(r_B - s_x) - f'(x) \frac{g(r_B) - g(r_B - s_x)}{s_x} \right].$$

Since $f$ and $g$ are convex and strictly increasing, $f' > 0$, $g' > 0$ on interval $x \in (0, r_A - \frac{c}{r_B})$, so on this interval,

$$h'(x) = s_x f'(x)g'(r_B - s_x) \left( \hat{f}(r_A - x) - \hat{g} \left( \frac{c}{r_A - x} \right) \right)$$

$$\propto \hat{f}(r_A - x) - \hat{g} \left( \frac{c}{r_A - x} \right).$$

By monotonicity of $\hat{f}$ and $\hat{g}$, $h$ can only be increasing, increasing then decreasing, or decreasing on $(0, r_A - \frac{c}{r_B})$. In any case, $h$ can only attain minimum on the boundary $x \in \{0, r_A - c/r_B\}$. The proof is then complete.

In the example below, we show that when $r_A - \sigma_A(n_A - x)$ and $r_B - \sigma_A(n_B - x)$ are not log-log convex, i.e., when $f(r_A) - f(r_A - x)$ and $g(r_B) - g(r_B - x)$ are not log-log concave, it is possible to have $L_C \geq R_C$. Thus, Theorem 45 cannot always be simplified to Theorem 46.
Example 4 In general, given that $f = \sigma_A^{-1}$ is convex, strictly increasing, and $f(0) = 0$, we cannot ignore the L-shaped grids in maximization. A counterexample is given below.

Consider $r_A = 5$ and

$$
\sigma_A^{-1}(x) = f(x) = \begin{cases} 
  x & x \leq 3 \\
  \frac{5}{2} + \frac{1}{2}e^{2(x-3)} & 3 < x \leq 4 \\
  \frac{5}{2} + \frac{1}{2}e^2 + e^2(x-4) & 4 < x \leq 5 
\end{cases}.
$$

Then one can check that $f(r_A) - f(r_A - x)$ is not log-log concave and $\hat{f}$ is not monotonically increasing.

We demonstrate that in this case the L-shaped grids cannot be discarded. Let us take $g = f$. Consider grid $S = L(1,1;5,5)$ of size 9. We have $\langle S \rangle \approx 26.17$. With a rectangle of area 9 inside the $5 \times 5$ region, the maximum expansion size, rounding up to an integer, is $\lceil \phi_R(9) \rceil = \lceil f(5) \cdot f(9/5) \rceil = 25 < \lceil \phi_L(9) \rceil$. Thus in this case we have to return to Theorem 45 or Theorem 9.

To see how Theorem 46 improves the bound in Theorem 9, we give the following example. However, despite that the bound in Theorem 46 is tighter, it is often no longer concave, and it is in general not possible to apply the bound in Theorem 46 recursively as in Theorem 11.

Example 5 First consider $C = A \otimes B$, where $\sigma_A(k) = k^{1/2}$ and $\sigma_B(k) = k^{1/4}$. If we apply Theorem 9 or equivalently Theorem 11, we find a rank expansion lower bound,

$$
\sigma^\text{prev}_C(k) = k^{1/4}.
$$

Now if we turn to Theorem 46 (or Corollary 47 stated after this example), for $k > n_B$, we get a different rank expansion lower bound,

$$
\sigma^\text{new}_C(k) = \left( \frac{k}{n_B} \right)^{1/2} \cdot n_B^{1/4} = n_B^{-1/4} \cdot k^{1/2},
$$

and when $k \leq n_B$, $\sigma^\text{new}_C(k) = \sigma^\text{prev}_C(k)$. The bound is improved by a factor of $k^{1/4}$. Numerically, with $n_A = n_B = 100$, $k = 10n_A$, we have $\lceil \sigma^\text{prev}_C(k) \rceil = 6$, whereas $\sigma^\text{new}_C(k) = 10$. However, note that although $\sigma^\text{new}_C$ is continuous, it is no longer concave. Hence, it is not possible to apply the new bound recursively.

For another illustration, let us use logarithms for the rank expansion lower bound. Let $C = A \otimes A$ with $\sigma_A(k) = \ln(k+1)$. From Theorem 11 (or Theorem 9), we get

$$
\sigma^\text{prev}_C(k) = \ln \left( \frac{k}{e-1} + 1 \right).
$$

Using the new bound by Theorem 46 (or Corollary 47), when $k/n_A \geq e - 1$, we have

$$
\sigma^\text{new}_C(k) = \ln(n_A + 1) \cdot \ln(k/n_A + 1).
$$
Thus in this regime,

\[
\exp(\sigma_{C_{\text{prev}}}^\text{prev}(k)) = \frac{k}{e-1} + 1; \quad (24)
\]

\[
\exp(\sigma_{C_{\text{new}}}^\text{new}(k)) = \left(\frac{k}{n_A} + 1\right)^{\ln(n_A+1)}. \quad (25)
\]

If we choose \( k \) to be proportional to \( n_A \), (25) eventually surpasses (24) as \( n_A \to \infty \), and thus we obtain a tighter lower bound, \( \sigma_{C_{\text{new}}}^\text{new} \), on the rank expansion. Numerically, with \( n_A = 100 \), \( k = 10n_A \), we have (24) \( \approx 583 \), (25) \( \approx 63996 \). Thus, \( \lceil \sigma_{C_{\text{prev}}}^\text{prev}(k) \rceil = 7 \), whereas \( \lceil \sigma_{C_{\text{new}}}^\text{new}(k) \rceil = 12 \).

Finally, recall that we can further simplify the optimization problem in \( R_C \) using Lemma 31 when \( \sigma_A \) and \( \sigma_B \) are log-log concave. By combining Theorem 46 with Lemma 31, we have an even simpler expression of the rank expansion lower bound.

**Corollary 47** Let everything be defined as in Theorem 45, if \( \sigma_A(x) \), \( \sigma_B(x) \) are log-log concave, \( r_A - \sigma_A(n_A - x) \) and \( r_B - \sigma_B(n_B - x) \) are log-log convex, on \( (0, n_A) \) and \( (0, n_B) \) respectively, e.g., polynomials and logarithm functions listed in Proposition 30, then

\[
R_C(k) = \min_{k_A \in [d_A, n_A], k_B \in [d_B, n_B], k_A \in \{d_A, n_A\} \text{ or } k_B \in \{d_B, n_B\}} \sigma_A(k_A) \cdot \sigma_B(k_B)
\]

is a rank expansion lower bound for \( C \).