On some topological indices of line graphs

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Abstract
Topological indices are real values assigned to graphs. Different topological indices have been defined so far. This paper discusses Arithmetico-geometrico index and Harmonic mean index of line graph of certain graphs including subdivision graphs.

Keywords
Arithmetico-geometrico index, Harmonic mean index, line graph, subdivision graph.

AMS Subject Classification
05C07, 05C76, 92E10.

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1. Introduction

Tracing its roots to Gutman and Trinajstic in 1972 [1], various topological indices are defined till date. The different topological indices are Wiener index [4], Arithmetic-geometrico index [3] and Harmonic mean index [2], to name a few. This paper intends to discuss the Arithmetico-geometrico index and Harmonic mean index of line graph of certain graphs including subdivision graphs.

Here we consider only simple connected graphs which are connected graphs without loops or multiple edges. For a graph $G$, $V(G)$ and $E(G)$ denote the set of all vertices and edges respectively. For a graph $G$ the degree of a vertex $v$ is the number of edges incident to $v$ and is denoted by $\delta(v)$. Here AG Index and HM index of line graph of some standard graphs and line graphs of some subdivision graphs are discussed. Some basic definitions used in the paper.

Definition 1.1. Arithmetico-geometrico topological index for a non-empty graph $G$ is denoted by $AG(G)$ and is defined as

$$AG(G) = \sum_{rs \in E(G)} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}},$$

where $\delta(r)$ and $\delta(s)$ denote the degrees of vertices of the edge $rs$ in $G$.

Definition 1.2. Harmonic mean index of a graph $G$ is denoted and defined as

$$HM(G) = \sum_{rs \in E(G)} \frac{2\delta(r)\delta(s)}{\delta(r) + \delta(s)}$$

where $\delta(r)$ and $\delta(s)$ denote the degrees of vertices of the edge $rs$ in $G$.

2. Main Results

Theorem 2.1. Let $\tau$ be the line graph of the subdivision graph of the wheel graph $W_{p+1}$. Then its AG index is

$$AG(\tau) = \frac{3 + p}{2} \sqrt{\frac{p}{3} + \frac{p^2 + 7p}{2}}$$

and its HM index is

$$HM(\tau) = \frac{1}{2(3+p)}[p^4 + 2p^3 + 33p^2 + 72p]$$

Proof. Consider wheel graph $W_{p+1}$ obtained by placing $K_1$ in the centre of $C_p$ and joining every edge of the cycle with $K_1$. Then $W_{p+1}$ will have $p+1$ vertices and $2p$ edges. Its subdivision graph can be obtained as inserting a vertex into each of the $2p$ edges, so that its has $3p+1$ vertices and $4p$ edges. The line graph of this subdivision graph contain as many vertices as the edges in the subdivision graph. Hence there are $4p$ vertices and we can easily see that there are $\frac{p^2 + 3p}{2}$ edges.

Let us denote the line graph by $\tau$. The edge set of $\tau$ can be partitioned into three compartments taking the degrees of their vertices into consideration. They are as follows:
1. $E_1 = \{rs \in E(\tau)|\delta(r) = 3 = \delta(s)\}$

2. $E_2 = \{rs \in E(\tau)|\delta(r) = 3, \delta(s) = p\}$

3. $E_3 = \{rs \in E(\tau)|\delta(r) = p = \delta(s)\}$

Note that $\delta(r), \delta(s)$ represent the degrees of $r$ and $s$ respectively. There are $4p$ edges in $E_1$, $p$ edges in $E_2$, and $\frac{p^2 - p}{2}$ edges in $E_3$.

Now, 

\[
AG(\tau) = \sum_{rs \in E(\tau)} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}}
\]

\[
= \sum_{rs \in E_1} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}} + \sum_{rs \in E_2} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}} + \sum_{rs \in E_3} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}}
\]

\[
= \sum_{rs \in E_1} \frac{3 + 3}{2\sqrt{3.3}} + \sum_{rs \in E_2} \frac{3 + p}{2\sqrt{3.3}} + \sum_{rs \in E_3} \frac{p + p}{2\sqrt{p \cdot p}}
\]

\[
= \sum_{rs \in E_1} 1 + \sum_{rs \in E_2} \frac{3 + p}{2\sqrt{3.3}} + \sum_{rs \in E_3} 1
\]

\[
= 4p + p \frac{3 + p}{2\sqrt{3.3}} + \frac{p^2 - p}{2}
\]

\[
= \frac{p^2 + 7p}{2} + \frac{3 + p}{2} \sqrt{\frac{p}{3}}
\]

\[
HM(\tau) = \sum \frac{2\delta(r)\delta(s)}{\delta(r) + \delta(s)}
\]

\[
= \sum_{rs \in E_1} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}} + \sum_{rs \in E_2} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}} + \sum_{rs \in E_3} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}}
\]

\[
= \sum_{rs \in E_1} \frac{2.3.3}{3 + 3} + \sum_{rs \in E_2} \frac{2.3.p}{3 + 3} + \sum_{rs \in E_3} \frac{2.p.p}{3 + p + p}
\]

\[
= 4p.3 + p.6p \frac{p^2 - p}{3 + p} + \frac{2p^2}{2p}
\]

\[
= 12p + \frac{6p^2}{3 + p} + \frac{(p^2 - p)p}{2}
\]

\[
= \frac{1}{2(3 + p)}[24p(3 + p) + 12p^2 + (3 + p)(p^3 - p^2)]
\]

\[
= \frac{1}{2(3 + p)}[p^4 + 2p^3 + 33p^2 + 72p]
\]

Hence the theorem.

**Theorem 2.2.** The AG index of line graph $\tau_1$ can be obtained as $AG(\tau_1) = 2(p + k - 3) + \frac{3 + 5\sqrt{3}}{2\sqrt{2}}$, where the parent tadpole graph has $C_p$ attached with a path of length $k$.

**Proof.** The $2(p + k) + 1$ edges of $\tau_1$ can be compartmentalised to four as follows: $E_1$ with one edge defined as $\{rs \in E(\tau_1)|\delta(r) = 1, \delta(s) = 2\}$

$E_2$ with $2(p + k) - 6$ edges defined as $\{rs \in E(\tau_1)|\delta(r) = 3, \delta(s) = p\}$

$E_3$ with $3$ edges defined as $\{rs \in E(\tau_1)|\delta(r) = 2, \delta(s) = 3\}$

and $E_4$ with $3$ edges defined as $\{rs \in E(\tau_1)|\delta(r) = \delta(s) = 3\}$

\[
\square
\]
Then

\[ AG(\tau_1) = \sum_{rs \in E(\tau_1)} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}} \]

\[ = \left\{ \sum_{rs \in E_1} + \sum_{rs \in E_2} + \sum_{rs \in E_3} + \sum_{rs \in E_4} \right\} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}} \]

\[ = \sum_{rs \in E_1} \frac{1+2}{2\sqrt{1.2}} + \sum_{rs \in E_2} \frac{2+2}{2\sqrt{2.2}} + \sum_{rs \in E_3} \frac{2+3}{2\sqrt{2.3}} \]

\[ + \sum_{rs \in E_4} \frac{3+3}{2\sqrt{3.3}} \]

\[ = \left( \frac{3}{2\sqrt{2}} \times 1 \right) + \left( 1 \times (2(p+k) - 6) \right) + \left( \frac{5}{2\sqrt{6}} \times 3 \right) \]

\[ + (3 \times 3) \]

\[ = \frac{3}{2\sqrt{2}} + (2(p+k) - 6) + \frac{5}{2\sqrt{2}} \]

\[ = (2(p+k) - 6) + \frac{3+5\sqrt{3}}{2\sqrt{2}} \]

\[ HM(\tau_1) = \sum \frac{2\delta(r)\delta(s)}{\delta(r) + \delta(s)} \]

\[ = \left( \sum_{rs \in E_1} + \sum_{rs \in E_2} + \sum_{rs \in E_3} + \sum_{rs \in E_4} \right) \frac{2\delta(r)\delta(s)}{\delta(r) + \delta(s)} \]

\[ = \frac{4}{3} \times (2(p+2k-6) + \frac{36}{5} + 9) \]

\[ = 4(p+k) - \frac{7}{15} \]

\[ 3. \textbf{Topological indices of line graphs of some standard graphs} \]

Line graph of cycles are cycles them self that is \( L(C_n) = C_n \) and \( AG(C_n) = n \).

Line graphs of paths are paths \( L(P_{n+1}) = P_n \) and line graphs of stars \( K_{1,n} \) is \( K_n \).

**Theorem 3.1.** Let line graphs of paths be denoted by \( L(P_{n+1}) = \mathcal{P}_n \), then for \( \mathcal{P}_n \) the following results can be obtained

\[ AG(\mathcal{P}_n) = n - 2 + \frac{3}{\sqrt{2}} \]

**Proof.** \( \mathcal{P}_n \) has two sets of edges, one set \( E_1 \) having the two end edges whose vertices are of degrees 1 and 2 and the second set \( E_2 \) with \( n-2 \) edges both of its vertices with degree 2. Hence

\[ AG(\mathcal{P}_n) = \sum_{rs \in E(\mathcal{P}_n)} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}} \]

\[ = \sum_{rs \in E_1} \frac{1+2}{2\sqrt{1.2}} + \sum_{rs \in E_2} \frac{2+2}{2\sqrt{2.2}} \]

\[ = n - 2 + \frac{3}{\sqrt{2}} \]

\[ \square \]

**Theorem 3.2.**

\[ AG(K_n) = n \]

where \( K_n \) is a complete graph on \( n \) vertices.

**Proof.** For \( K_n \) there are \( \frac{(n-1)(n-2)}{2} \) each incident with vertices of degree \( n-1 \). Hence

\[ AG(K_n) = \sum_{rs \in E(K_n)} \frac{\delta(r) + \delta(s)}{2\sqrt{\delta(r)\delta(s)}} \]

\[ = n \cdot \frac{(n-1) + (n-1)}{2\sqrt{(n-1)(n-1)}} = n \]

\[ \square \]

**References**

[1] Ivan Gutman and Nenad Trinajstic, Graph theory and molecular orbits. Total \( \pi \) electron energy of alternate hydrocarbons, *Chemical Physics Letters*, 17 No.4 (1972) 555-558.

[2] Suresh Singh G., Koshy N.J., Harmonic mean topological indices of graphs, *International Journal of Research in Engineering, Science and Management*, Volume 3, Issue-2 (2020)

[3] V.S. Shigehalli, R. Kanabur, Computation of new degree based topological indices of graphs, *Journal of Mathematics*, (2016)

[4] Wiener H., Structural determination of paraffin boiling points, *Journal of American Chem. Soc.*, 69 (1947) 17-20.