Research Article

On the Discrete Orlicz Electrostatic q-Capacitary Minkowski Problem

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We establish the existence of solutions to the Orlicz electrostatic q-capacitary Minkowski problem for polytopes. This contains a new result of the discrete $L_p$ electrostatic q-capacitary Minkowski problem for $p < 0$ and $1 < q < n$.

1. Introduction

The Orlicz Brunn-Minkowski theory was originated from the works of Ludwig [1], Ludwig and Reitzner [2], and Lutwak et al. [3, 4]. Hereafter, the new theory has quickly become an important branch of convex geometry (see, e.g., [5–10]). A special case of the theory is the $L_p$ Brunn-Minkowski theory, which is credited to Lutwak [11, 12] and attracted increasing interest in recent years (see, e.g., [13–20]).

It is well known that the $L_1$ Brunn-Minkowski theory is the classical Brunn-Minkowski theory. One of the cornerstones of the classical Brunn-Minkowski theory is the Minkowski problem. More than a century ago, Minkowski himself solved the Minkowski problem for discrete measures [21]. The complete solution for arbitrary measures was given by Aleksandrov [22] and Fenchel and Jessen [23]. The regularity was studied by, e.g., Lewy [24], Nirenberg [25], Pogorelov [26], Cheng and Yau [27], and Caffarelli et al. [28]. A generalization of the Minkowski problem is the $L_p$ Minkowski problem in the $L_p$ Brunn-Minkowski theory, which has been extensively studied (see, e.g., [29–49]). Naturally, the corresponding Minkowski problem in the Orlicz Brunn-Minkowski theory is called the Orlicz Minkowski problem which was first investigated by Haberl et al. [50] for even measures. Today, great progress has been made on it (see, e.g., [51–60]). The present paper is aimed at dealing with the Orlicz capacitary Minkowski problem.

The electrostatic $q$-capacitary measure $\mu_q(\Omega, \cdot)$ (see [61]) of a bounded open convex set $\Omega$ in $\mathbb{R}^n$ is the measure on the unit sphere $S^{n-1}$ defined for $\omega < S^{n-1}$ and $1 < q < n$ by

$$\mu_q(\Omega, \omega) = \int_{g^{-1}(\omega)} |\nabla U|^q d\mathcal{H}^{n-1}, \quad (1)$$

where $g^{-1}: S^{n-1} \rightarrow \partial \Omega$ (the boundary of $\Omega$) denotes the inverse Gauss map, $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure, and $U$ the $q$-equilibrium potential of $\Omega$.

A convex body $K$ is a compact convex set without empty interior in the $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$, and let $\mathcal{K}^n_o$ denote the set of convex bodies with the origin in their interiors. The support function (see [62, 63]) of $K \in \mathcal{K}^n$ is defined for $u \in S^{n-1}$ by

$$h_K(u) = h(K, u) = \max \{x \cdot u : x \in K\}, \quad (2)$$

where $x \cdot u$ denotes the standard inner product of $x$ and $u$. Note that $h(cK, u) = ch(K, u)$ for $c > 0$.

Let $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a given continuous function. For $1 < q < n$ and $K \in \mathcal{K}^n_o$, the Orlicz electrostatic $q$-capacitary measure $\mu_q^\varphi(\Omega, \cdot)$ is defined for $\omega < S^{n-1}$ and $1 < q < n$ by

$$\mu_q^\varphi(\Omega, \omega) = \int_{g^{-1}(\omega)} |\nabla U|^q d\mathcal{H}^{n-1} \varphi(U), \quad (3)$$

where $\varphi$ is a function on $[0, \infty)$.

The electrostatic $q$-capacitary Minkowski problem in the Orlicz Brunn-Minkowski theory is the following:

Given a bounded open convex set $\Omega \subset \mathbb{R}^n$, find a convex body $K \in \mathcal{K}^n$ such that

$$\mu_q^\varphi(\Omega, \cdot) = \mu_q^\varphi(K, \cdot).$$

This problem has been extensively studied by many authors (see, e.g., [51–59]). The present paper is aimed at dealing with the Orlicz capacitary Minkowski problem.
-capacitary measure, \( \mu_{p,q}(K,\cdot) \), of \( K \) is defined by

\[
d\mu_{p,q}(K,\cdot) = \varphi(h_K) d\mu_q(K,\cdot).
\]  

(3)

When \( \varphi(s) = s^p \) with \( p \in \mathbb{R} \), the Orlicz electrostatic \( q \)-capacitary measure becomes the following \( L_p \)-electrostatic \( q \)-capacitary measure introduced by Zou and Xiong [64]:

\[
d\mu_{p,q}(K,\cdot) = h_{K,p}^{-1} d\mu_q(K,\cdot).
\]  

(4)

The Minkowski problem characterizing the Orlicz electrostatic \( q \)-capacitary measure, proposed in [65], is the following.

1.1. The Orlicz Electrostatic \( q \)-Capacitary Minkowski Problem. Let \( 1 < q < n \). Given a continuous function \( \varphi : (0,\infty) \rightarrow (0, \infty) \) and a finite Borel measure \( \mu \) on \( S^{n-1} \), what are the necessary and sufficient conditions so that \( \mu = c \mu_{p,q}(K,\cdot) \) for some convex body \( K \) and constant \( c > 0 \)?

Let \( \varphi \) be a constant function. When \( q = 2 \), the Orlicz Minkowski-type problem is the classical electrostatic capacitary Minkowski problem. In the paper [66], Jerison established the existence of a solution to the electrostatic capacitary Minkowski problem. In a subsequent paper [67], he gave a new proof of the existence using a variational approach. The uniqueness was proved by Caffarelli et al. [68], and the regularity was given in [66]. When \( 1 < q < n \), the Orlicz Minkowski-type problem is the electrostatic \( q \)-capacitary Minkowski problem posed in [61]. The existence and regularity for \( 1 < q < 2 \) and the uniqueness for \( 1 < q < n \) of its solutions were proved in [61], and the existence for \( 2 < q < n \) was recently solved in [69].

Let \( \varphi(s) = s^p \) with \( p \in \mathbb{R} \). Then, the Orlicz Minkowski-type problem is the \( L_p \)-electrostatic \( q \)-capacitary Minkowski problem introduced by Zou and Xiong [64]. In [64], they completely solved the \( L_p \)-electrostatic \( q \)-capacitary Minkowski problem for the case \( p > 1 \) and \( 1 < q < n \). It is generally known that when \( p < 1 \), the \( L_p \)-Minkowski problem becomes much harder. Actually, the \( L_p \)-electrostatic \( q \)-capacitary Minkowski problem for the case \( p < 1 \) and \( 1 < q < n \) is also very difficult. Therefore, it is worth mentioning that an important breakthrough of the problem for the case \( 0 < p < 1 \) and \( 1 < q < 2 \) was made by Xiong et al. [70] for discrete measures.

The existence of the Orlicz electrostatic \( q \)-capacitary Minkowski problem was first investigated by Hong et al. [65]. As a consequence, in [65], they obtained a complete solution (including both existence and uniqueness) to the \( L_p \)-electrostatic \( q \)-capacitary Minkowski problem for the case \( p > 1 \) and \( 1 < q < n \), which was independently solved by Zou and Xiong [64].

We observe the statement above. At present, there is no result about the \( L_p \)-electrostatic \( q \)-capacitary Minkowski problem for the case \( p < 0 \) and \( 1 < q < n \). In this paper, we study the Orlicz electrostatic \( q \)-capacitary Minkowski problem including it.

A finite set \( E \) of \( S^{n-1} \) is said to be in general position if \( E \) is not contained in a closed hemisphere of \( S^{n-1} \) and any \( n \) elements of \( E \) are linearly independent.

A polytope in \( \mathbb{R}^n \) is the convex hull of a finite set of points in \( \mathbb{R}^n \) provided that it has positive \( n \)-dimensional volume. The convex hull of a subset of these points is called a facet of the polytope if it lies entirely on the boundary of the polytope and has positive \( (n-1) \)-dimensional volume.

Our main theorem is stated as follows.

**Theorem 1.** Suppose \( \varphi : (0,\infty) \rightarrow (0, \infty) \) is continuously differentiable and strictly increasing with \( \varphi(s) \rightarrow \infty \) as \( s \rightarrow \infty \) such that \( \varphi(t) = \int_{t}^{\infty} (1/\varphi(s)) ds \) exists for \( t > 0 \) and \( \lim_{t \rightarrow 0^+} \varphi(t) = \infty \). Let \( \mu = \sum_{i=1}^{N} a_i \delta_{u_i} \), where \( a_1, \ldots, a_N > 0 \), the unit vectors \( u_1, \ldots, u_N \in S^{n-1} \) are in general position, and \( \delta_{u_i} \) is the Dirac delta. Then, for \( 1 < q < n \), there exist a polytope \( P \) and constant \( c > 0 \) such that

\[
\mu = c \mu_{p,q}(P,\cdot).
\]

(5)

When \( \varphi(s) = s^p \) with \( p < 0 \), and \( \varphi(t) = -(1/p)t^p \), which satisfy the assumptions of Theorem 1, we obtain the following.

**Corollary 2.** Let \( p < 0 \) and \( 1 < q < n \). Suppose \( \mu \) is a discrete measure on \( S^{n-1} \), and its supports are in general position. If \( p + q \neq n \), then there exists a polytope \( P_0 \) such that \( \mu = \mu_{P_0}(P_0,\cdot) \); if \( p + q = n \), then there exist a polytope \( P \) and constant \( c > 0 \) such that \( \mu = c \mu_{p,q}(P,\cdot) \).

Obviously, this corollary makes up for the existing results for the \( L_p \)-electrostatic \( q \)-capacitary Minkowski problem, to some extent.

The rest of this paper is organized as follows. In Section 2, some of the necessary facts about convex bodies and capacity are presented. In Section 3, a maximizing problem related to the Orlicz electrostatic \( q \)-capacitary Minkowski problem is considered and its corresponding solution is given. In Section 4, we give the proofs of Theorem 1 and Corollary 2.

2. Preliminaries

2.1. Basics regarding Convex Bodies. For quick later reference, we list some basic facts about convex bodies. Good general references are the books of Gardner [62] and Schneider [63].

The boundary and interior of \( K \in \mathbb{R}^n \) will be denoted by \( \partial K \) and \( \text{int } K \), respectively. \( B = \{ x \in \mathbb{R}^n : \sqrt{x \cdot x} \leq 1 \} \) denotes the unit ball. The volume, the \( n \)-dimensional Lebesgue measure, of a convex body \( K \in \mathbb{R}^n \) is denoted by \( V(K) \), and the volume of \( B \) is denoted by \( \omega_n \). We will write \( C(S^{n-1}) \) for the set of continuous functions on \( S^{n-1} \) and \( C^*(S^{n-1}) \) for the set of positive functions in \( C(S^{n-1}) \).

For \( x \in \partial K \) with \( K \in K_n \), \( g_K(x) \) is the Gauss map of \( K \), which is the family of all unit exterior normal vectors at \( x \). In particular, \( g_K(x) \) consists of a unique vector for \( H^{n-1} \).
almost all $x \in \partial K$. The surface area measure of $K$ is a Borel measure on $S^{n-1}$ defined for a Borel set $\omega \subset S^{n-1}$ by

$$S(K, \omega) = \int_{x \in dS^1(\omega)} dH_n(x). \quad (6)$$

For $f \in C^1(S^{n-1})$, the Aleksandrov body associated with $f$, denoted by $[f]$, is the convex body defined by

$$[f] = \bigcap_{u \in S^{n-1}} \{ \xi \in \mathbb{R}^n : \xi \cdot u \leq f(u) \}. \quad (7)$$

It is easy to see that $h_{[f]} \leq f$ and $[h_K] = K$ for $K \in \mathbb{K}$. The Hausdorff distance of two convex bodies $K, L \in \mathbb{K}$ is defined by

$$\delta(K, L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|. \quad (8)$$

For a sequence of convex bodies $K_i \in \mathbb{K}$ and a convex body $K \in \mathbb{K}$, we have $\lim_{i \to \infty} K_i = K$ provided that

$$\delta(K_i, K) \to 0, \quad (9)$$

as $i \to \infty$.

For $K \in \mathbb{K}$ and $u \in S^{n-1}$, the support hyperplane $H(K, u)$ of $K$ at $u$ is defined by

$$H(K, u) = \{ x \in \mathbb{R}^n : x \cdot u = h(K, u) \}, \quad (10)$$

the half-space $H^+(K, u)$ at $u$ is defined by

$$H^+(K, u) = \{ x \in \mathbb{R}^n : x \cdot u \leq h(K, u) \}, \quad (11)$$

and the support set $F(K, u)$ at $u$ is defined by

$$F(K, u) = K \cap H(K, u). \quad (12)$$

Suppose that $P$ is the set of polytopes in $\mathbb{R}^n$ and the unit vectors $u_1, \cdots, u_N$ are in general position. Let $P(u_1, \cdots, u_N)$ be the subset of $P$. If $P \in P$ with

$$P = \bigcap_{k=1}^N H^+(P, u_k), \quad (13)$$

then $P \in P(u_1, \cdots, u_N)$. Obviously, if $P_i \in P(u_1, \cdots, u_N)$ and $P_i$ converges to a polytope $P$, then $P \in P(u_1, \cdots, u_N)$. Let $P_N(u_1, \cdots, u_N)$ be the subset of $P(u_1, \cdots, u_N)$ that any polytope in $P_N(u_1, \cdots, u_N)$ has exactly $N$ facets.

2.2. Electrostatic $q$-Capacity and $q$-Capacitary Measure. Here, we collect some notion and basic facts on electrostatic $q$-capacity and $q$-capacitary measure (see [61, 64, 70]).

Let $E$ be a compact set in $n$-dimensional Euclidean space $\mathbb{R}^n$. For $1 < q < n$, the electrostatic $q$-capacity, $C_q(E)$, of $E$ is defined (see [61]) by

$$C_q(E) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^q dx : u \in C_c^\infty(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on } E \right\}. \quad (14)$$

where $C_c^\infty(\mathbb{R}^n)$ is the set of smooth functions with compact supports. When $q = 2$, the electrostatic $q$-capacity becomes the classical electrostatic capacity $C_q(E)$.

For $K \in \mathbb{K}$ and $1 < q < n$, we need the following isocapacitary inequality which is due to Maźya [71]:

$$V(K)^{(n-q)/n} \leq \left( \frac{q-1}{n-q} \right)^{q-1} \frac{n\omega_{n-1}^n}{\omega_{n-1}} C_q(K). \quad (15)$$

The following lemma (see [64, 70]) gives some basic properties of the electrostatic $q$-capacity.

**Lemma 3.** Let $E$ and $F$ be two compact sets in $\mathbb{R}^n$ and $1 < q < n$.

(i) If $E \subset F$, then

$$C_q(E) \leq C_q(F). \quad (16)$$

(ii) For $\lambda > 0$,

$$C_q(\lambda E) = \lambda^{n-q} C_q(E). \quad (17)$$

(iii) For $x_0 \in \mathbb{R}^n$,

$$C_q(E + x_0) = C_q(E). \quad (18)$$

(iv) The functional $C_q(\cdot)$ is continuous on $\mathbb{K}$ with respect to the Hausdorff metric.

The following lemma is some basic properties of the electrostatic $q$-capacitary measure (compare [64, 70]).

**Lemma 4.** Let $K \in \mathbb{K}$ and $1 < q < n$.

(i) For $\lambda > 0$,

$$\mu_q(\lambda K, \cdot) = \lambda^{n-q-1} \mu_q(K, \cdot) \quad (19)$$

(ii) For $x_0 \in \mathbb{R}^n$,
\[ \mu_q(K + x_n \cdot ) = \mu_q(K, \cdot ) \]  
(20)

(iii) For \( k_j, K \in \mathbb{R}^n \), if \( k_j \rightarrow K \), then

\[ \mu_q(k_j, \cdot ) \rightarrow \mu_q(K, \cdot ) \]  
(21)

weakly as \( j \rightarrow +\infty \)

(iv) The measure \( \mu_q(k_j, \cdot ) \) is absolutely continuous with respect to the surface area measure \( S(K, \cdot ) \)

The following variational formula given in [61] of electrostatic \( q \)-capacity is critical.

**Lemma 5.** Let \( I \subset \mathbb{R} \) be an interval containing \( 0 \) in its interior, and let \( h_i(u) = h(t, u) : I \times S^{n-1} \rightarrow (0,\infty) \) be continuous such that the convergence in

\[ h'(0, u) = \lim_{t \rightarrow 0} \frac{h(t, u) - h(0, u)}{t} \]  
(22)

is uniform on \( S^{n-1} \). Then,

\[ \frac{dC_q([h_1])}{dt}_{|t=0} = (q - 1) \int_{S^{n-1}} h'(0, u) d\mu_q([h_0], u). \]  
(23)

### 3. An Associated Maximization Problem

In this section, we solve a maximization problem, and its solution is exactly the solution in Theorem 1.

Suppose \( \phi \) satisfies the assumptions of Theorem 1 and the unit vectors \( u_1, \ldots, u_N \) are in general position. For \( \alpha_1, \ldots, \alpha_N > 0 \) and \( P \in \mathcal{P}(u_1, \ldots, u_N) \), define the function, \( \Phi_P : \text{int } P \rightarrow \mathbb{R} \), by

\[ \Phi_P(\xi) = \sum_{k=1}^{N} \alpha_k h(P, u_k) - \xi \cdot u_k. \]  
(24)

Let \( 1 < q < n \). We consider the following maximization problem:

\[ \sup \left\{ \min_{\xi \in \text{int } P} \Phi_P(\xi) : C_q(Q) = 1, Q \in \mathcal{P}(u_1, \ldots, u_N) \right\}. \]  
(25)

The solution to problem (25) is given in Theorem 9. Its proof requires the following three lemmas which are similar to those in [58].

**Lemma 6.** Suppose \( \phi : (0,\infty) \rightarrow (0,\infty) \) is continuously differentiable and strictly increasing with \( \phi(s) \rightarrow \infty \) as \( s \rightarrow \infty \) such that \( \phi(t) = \int_t^\infty (1/\phi(s)) ds \) exists for \( t > 0 \) and \( \lim_{t \rightarrow 0} \phi(t) = \infty \). For \( \alpha_1, \ldots, \alpha_N > 0 \), if the unit vectors \( u_1, \ldots, u_N \in S^{n-1} \) are in general position, then there exists a unique \( \xi_\phi(P) \in \text{int } P \) such that

\[ \Phi_P(\xi_\phi(P)) = \min_{\xi \in \text{int } P} \Phi_P(\xi). \]  
(26)

**Proof.** Since \( \phi : (0,\infty) \rightarrow (0,\infty) \) is continuously differentiable and strictly increasing, we have for \( t > 0 \),

\[ \phi''(t) = \frac{\phi'(t)}{\phi^p(t)} > 0. \]  
(27)

Therefore, \( \phi \) is strictly convex on \((0,\infty)\).

Let \( 0 < \lambda < 1 \) and \( \xi_1, \xi_2 \in \text{int } P \). Then,

\[ \lambda \Phi_P(\xi_1) + (1 - \lambda) \Phi_P(\xi_2) = \lambda \sum_{k=1}^{N} \alpha_k h(P, u_k) - \xi_1 \cdot u_k \]

\[ + (1 - \lambda) \sum_{k=1}^{N} \alpha_k h(P, u_k) - \xi_2 \cdot u_k \]

\[ = \sum_{k=1}^{N} \alpha_k \lambda h(P, u_k) - \xi_1 \cdot u_k \]

\[ + (1 - \lambda) \alpha_k h(P, u_k) - \xi_2 \cdot u_k \]

\[ \geq \sum_{k=1}^{N} \alpha_k h(P, u_k) - (\lambda \xi_1 + (1 - \lambda) \xi_2) \cdot u_k \]

\[ = \Phi_P(\lambda \xi_1 + (1 - \lambda) \xi_2). \]  
(28)

Equality holds if and only if \( \xi_1 \cdot u_k = \xi_2 \cdot u_k \) for all \( k = 1, \ldots, N \). Since \( u_1, \ldots, u_N \) are in general position, \( \mathbb{R}^n = \text{lin} \{u_1, \ldots, u_N\} \) which is the smallest linear subspace of \( \mathbb{R}^n \) containing \{\( u_1, \ldots, u_N \)\}. Thus, \( \xi_1 = \xi_2 \). Namely, \( \Phi_P \) is strictly convex on \( \text{int } P \).

Since \( P \in \mathcal{P}(u_1, \ldots, u_N) \), it follows that for any \( x \in \partial P \), there exists a \( u_i \in \{u_1, \ldots, u_N\} \) such that

\[ h(P, u_i) = x \cdot u_i. \]  
(29)

Note that \( \phi \) is strictly decreasing on \((0,\infty)\) and \( \phi(t) = \infty \). Then, \( \Phi_P(\xi) \rightarrow \infty \) whenever \( \xi \in \text{int } P \) and \( \xi \rightarrow x \). This together with the strict convexity of \( \Phi_P \) means that there exists a unique interior point \( \xi_\phi(P) \) of \( P \) such that

\[ \Phi_P(\xi_\phi(P)) = \min_{\xi \in \text{int } P} \Phi_P(\xi). \]  
(30)

**Lemma 7.** Suppose \( \alpha_1, \ldots, \alpha_N > 0 \), the unit vectors \( u_1, \ldots, u_N \in S^{n-1} \) are in general position, and \( \phi : (0,\infty) \rightarrow (0,\infty) \) is continuously differentiable and strictly increasing with \( \phi(s) \rightarrow \infty \) as \( s \rightarrow \infty \) such that \( \phi(t) = \int_t^\infty (1/\phi(s)) ds \) exists for \( t > 0 \) and \( \lim_{t \rightarrow 0} \phi(t) = \infty \). If \( P \in \mathcal{P}(u_1, \ldots, u_N) \) converges to a polytope \( P \), then \( \lim_{t \rightarrow \infty} \xi_\phi(P) = \xi_\phi(P) \) and

\[ \lim_{t \rightarrow \infty} \Phi_P(\xi_\phi(P)) = \Phi_P(\xi_\phi(P)). \]  
(31)

**Proof.** Since \( P_j \rightarrow P \) and \( \xi_\phi(P_j) \in \text{int } P_j \), it follows that \( \xi_\phi(P_j) \) is bounded. Let \( \xi_\phi(P_j) \) be a subsequence of \( \xi_\phi(P_j) \) with \( \lim_{j \rightarrow \infty} \xi_\phi(P_j) = \xi_0 \). We first show that \( \xi_0 \in \text{int } P \) by contradiction.
Assume \( \xi_0 \in \partial P \). Then, \( \lim_{j \to \infty} \Phi_{P_j}(\xi_0(P_j)) = \infty \), which contradicts the fact that

\[
\lim_{j \to \infty} \Phi_{P_j}(\xi_0(P_j)) \leq \lim_{j \to \infty} \Phi_{P_j}(\xi_0(P)) = \Phi_{P}(\xi_0(P)) < \infty.
\]

(32)

We next show that \( \xi_0 = \xi_0(P) \). Let \( \xi_0 \neq \xi_0(P) \). Then,

\[
\lim_{j \to \infty} \Phi_{P_j}(\xi_0(P_j)) = \Phi_{P}(\xi_0) > \Phi_{P}(\xi_0(P)) = \lim_{j \to \infty} \Phi_{P_j}(\xi_0(P)).
\]

(33)

This contradicts the fact that

\[
\lim_{j \to \infty} \Phi_{P_j}(\xi_0(P_j)) \leq \lim_{j \to \infty} \Phi_{P_j}(\xi_0(P)).
\]

(34)

This means that \( \lim_{i \to \infty} \xi_0(P_i) = \xi_0(P) \) and

\[
\lim_{i \to \infty} \Phi_{P_i}(\xi_0(P_i)) = \Phi_{P}(\xi_0(P)).
\]

(35)

**Lemma 8.** Suppose \( \alpha_1, \ldots, \alpha_N > 0 \), the unit vectors \( u_1, \ldots, u_N \) \( \in \mathbb{S}^{N-1} \) are in general position, and \( \varphi : (0, \infty) \to (0, \infty) \) is continuously differentiable and strictly increasing with \( \varphi(s) \to \infty \) as \( s \to \infty \) such that \( \varphi(t) = \int_0^t (1/\varphi(s)) \, ds \) exists for \( t > 0 \) and \( \lim_{t \to 0} \varphi(t) = \infty \). Let \( P \in \mathcal{P}(u_1, \ldots, u_N) \) and \( \delta \geq 0 \) be small enough such that for \( k_0 \in \{1, \ldots, N\}, \)

\[
P_\delta = P \cap \{ x : x \cdot u_{k_0} \leq h(P, u_{k_0}) - \delta \} \in \mathcal{P}(u_1, \ldots, u_N).
\]

(36)

If the continuous function \( \lambda : (0, \infty) \to (0, \infty) \) is continuously differentiable on \( (0, \infty) \) and \( \lim_{\delta \to 0} \lambda'(\delta) \) exists, then \( \xi(\delta) = \xi_{\phi}(\lambda(\delta)P_\delta) \) has a right derivative, denoted by \( \xi'_0(0) \), at 0.

**Proof.** The proof is based on the ideas of Wu et al. [58]. Let \( \delta \geq 0 \) be small enough and

\[
\Phi(\delta) = \min_{\xi \in \text{int}(\lambda(\delta)P_\delta)} \frac{N}{k=1} \alpha_k \phi(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k)
\]

(37)

\[
= \sum_{k=1}^N \alpha_k \phi(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k).
\]

From this and the fact that \( \xi(\delta) \) is an interior point of \( \lambda(\delta)P_\delta \), it follows that for \( i = 1, \ldots, N, \)

\[
\sum_{k=1}^N \alpha_k \phi(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k)u_{k,i} = 0,
\]

(38)

where \( u_k = (u_{k,1}, \ldots, u_{k,n})^T \).

Let

\[
F_i(\delta, \xi_1, \ldots, \xi_n) = \sum_{k=1}^N \alpha_k \phi'(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k)u_{k,i}
\]

(39)

for \( i = 1, \ldots, n, \) where \( \xi = (\xi_1, \ldots, \xi_n) \). Then,

\[
\frac{\partial F_i}{\partial \xi_j} = -\sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k)u_{k,i}u_{k,j},
\]

(40)

\[
\frac{\partial F_i}{\partial \delta} = \sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k)u_{k,i}\lambda'(\delta)h(P_\delta, u_k)
\]

\[
- \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \delta \lambda(\delta) - \xi \cdot u_k)u_{k,i}\lambda(\delta).
\]

(41)

Let \( F = (F_1, \ldots, F_n) \). Then,

\[
\left( \frac{\partial F}{\partial x} \right)_{\xi=0} = -\sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k)u_{k,i}u_{k,j},
\]

(42)

where \( u_ku_k^T \) is an \( n \times n \) matrix.

Since \( u_1, \ldots, u_N \) are in general position, \( \mathbb{R}^n = \text{lin}(u_1, \ldots, u_N) \). Thus, for any \( x \in \mathbb{R}^n \) with \( x \neq 0 \), there exists a \( u_{k_o} \in \{ u_1, \ldots, u_N \} \) such that \( u_{k_o} \cdot x \neq 0 \). Note that \( \phi'' > 0 \). Then, we have

\[
x^T \left( -\sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k)u_{k,i}u_{k,j} \right) x
\]

\[
= -\sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k)(x \cdot u_k)^2
\]

\[
\leq -\alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k)(x \cdot u_k)^2 < 0.
\]

(43)

Therefore, \( \left( \frac{\partial F}{\partial \xi} \right)_{\xi=0} = 0 \) is negative definite. Thus,

\[
\det \left( \frac{\partial F}{\partial \xi} \right)_{\xi=0} < 0.
\]

(44)

From this, the fact that for \( i = 1, \ldots, n \), \( F_i(\delta, \xi_1, \ldots, \xi_n) = 0 \) follows by (38), the fact that \( \partial F_i/\partial \xi_j \) is continuous on \( \xi \) and \( \delta \) for all \( 1 \leq i, j \leq n \), and the implicit function theorem, it follows that \( \xi(\delta) = \xi_\phi(\lambda(\delta)P_\delta) \) is continuously differentiable on a neighbourhood of \( \delta \) small enough. Thus, \( \xi(\delta) \) is continuously differentiable for small enough \( \delta > 0 \), and
Theorem 9. Suppose \( \alpha, \cdots, \alpha_N > 0 \) and the unit vectors \( u_1, \cdots, u_N \in S^{n-1} \) are in general position. Let \( \varphi : (0,\infty) \to (0,\infty) \) be continuously differentiable and strictly increasing with \( \varphi'(s) \to \infty \) as \( s \to \infty \) such that \( \varphi(t) = \int_1^t 1/\varphi(s) \, ds \) exists for \( t > 0 \) and \( \lim_{t \to 0} \varphi(t) = \infty \). Then, there exists a polytope \( P \in P_N \{ u_1, \cdots, u_N \} \) such that \( \xi_\varphi(P) = 0, C_q(P) = 1, \) and \( \Phi_P(o) = \sup \{ \min_{\xi \in \mathbb{R}^n} \Phi_{\xi}(\xi): C_q(\xi) = 1, \xi \in P(u_1, \cdots, u_N) \} \).

Proof. For \( x \in \mathbb{R}^n \) and \( P \in P(u_1, \cdots, u_N) \), we first show

\[
\Phi_{\xi \varphi}(\xi_\varphi(P + x)) = \Phi_{\xi}(\xi_\varphi(P)).
\]

From Lemma 6 and definition (24), we have

\[
\Phi_{\xi \varphi}(\xi_\varphi(P + x)) = \min_{\xi \in \mathbb{R}^n} \Phi_{\xi \varphi}(\xi) = \min_{\xi \in \mathbb{R}^n} \sum_{k=1}^N \alpha_k \varphi(\xi(P + x) - \xi \cdot u_k) = \min_{\xi \in \mathbb{R}^n} \sum_{k=1}^N \alpha_k \varphi(\xi(P) - \xi \cdot u_k) = \Phi_{\xi}(\xi_\varphi(P)).
\]

Therefore, by (49) and (iii) of Lemma 2.1, we can choose a sequence \( P_i \in P(u_1, \cdots, u_N) \) with \( \xi_\varphi(P_i) = o \) and \( C_q(P_i) = 1 \) such that

\[
\lim_{i \to \infty} \Phi_{\xi_\varphi}(o) = \sup \{ \min_{\xi \in \mathbb{N}} \Phi_{\xi}(\xi): C_q(\xi) = 1, \xi \in P(u_1, \cdots, u_N) \}.
\]

We next prove that \( P_i \) is bounded. Assume that \( P_i \) is not bounded. Since the unit vectors \( u_1, \cdots, u_N \) are in general position, from the proof of ([45], Theorem 4.3), we see \( V(P) \) is not bounded. However, from (16), and noting that \( C_q(P) = 1 \), we have

\[
V(P) \leq \left( \frac{q-1}{n-q} \right)^{(n-1)(n-q)} (n\omega_n)^{-n(n-q)},
\]

which is a contradiction. Therefore, \( P_i \) is bounded.

By the Blaschke selection theorem, we can assume that a subsequence of \( P_i \) converges to a polytope \( P \in P_N \{ u_1, \cdots, u_N \} \). Thus, from (iv) of Lemma 3 and Lemma 7, it follows that \( C_q(P) = 1, \xi_\varphi(P) = o, \) and

\[
\Phi_{\xi}(o) = \sup \{ \min_{\xi \in \mathbb{R}^n} \Phi_{\xi}(\xi): C_q(\xi) = 1, \xi \in P(u_1, \cdots, u_N) \}.
\]

We now prove that \( P \in P_N \{ u_1, \cdots, u_N \} \), i.e., \( F(P, u_i) \) are facets for all \( i = 1, \cdots, N \). If not, there exists an \( i_0 \in \{ 1, \cdots, N \} \) such that \( F(P, u_{i_0}) \) is not a facet of \( P \). Choose \( \delta \geq 0 \) small enough so that the polytope

\[
P_\delta = P \cap \{ x: x \cdot u_{i_0} \geq \lambda(P, u_{i_0}) - \delta \} \in P(u_1, \cdots, u_N).
\]

Let \( \lambda(\delta) = C_q(P_\delta)^{(1/\delta)} \). Then, \( \lambda(\delta)P_\delta \in P(u_1, \cdots, u_N) \), \( C_q(\lambda(\delta)P_\delta) = 1 \) follows by (ii) of Lemma 3, and \( \lambda(\delta) \) is continuous in \([0, \infty)\). Since for any \( \delta \to 0 \), there is that \( \lambda(\delta) \to \lambda \), \( P_\delta \to P \), it follows from Lemma 7 that \( \xi_\varphi(\lambda(\delta)P_\delta) \to \xi_\varphi(P) = o \). This implies

\[
\lim_{\delta \to 0} \xi_\varphi(\lambda(\delta)P_\delta) = o.
\]

Let

\[
r_{i_0}(u) = \begin{cases} 1, & u = u_{i_0} \\ 0, & u \neq u_{i_0} \end{cases}
\]

for \( u \in S^{n-1} \). Then, from Lemma 5, we have for small enough \( \delta \geq 0, \)

\[
\frac{dC_q(P_\delta)}{d\delta} = \frac{dC_q([h(P)])}{d\delta} = \lim_{t \to 0} \frac{C_q([h(P) + tr_{i_0}]) - C_q([h(P)])}{t} = (q-1) \int_{S^{n-1}} r_{i_0}(u) d\mu_q(P_\delta, u) = (q-1)\mu_q(P_\delta, u_{i_0}).
\]
Thus, from (iii) and (iv) of Lemma 4, it follows that \( C_q(P_δ) \) is continuously differentiable for every \( δ > 0 \), and
\[
\lim_{δ \to 0^+} \frac{dC_q(P_δ)}{dδ} = 0. \tag{58}
\]
These imply that
\[
\lambda'(δ) = -\frac{1}{n-q} C_q(P_δ)^{(1/(n-q))-1} \frac{dC_q(P_δ)}{dδ} \tag{59}
\]
is continuous for every \( δ > 0 \), and
\[
\lim_{δ \to 0^+} \lambda'(δ) = 0. \tag{60}
\]
Therefore, \( λ(δ) = C_q(P_δ)^{(1/(n-q))} \) satisfies the conditions of Lemma 8. Noting that \( ξ(δ) = ξ(λ(δ)P_δ) \), we see \( ξ(0) \) exists.
Recall
\[
Φ(δ) = \sum_{k=1}^{N} α_κϕ(λ(δ)h(P_δ, u_k) - ξ(δ) ⋅ u_k). \tag{61}
\]
From this and (38), we have
\[
\sum_{k=1}^{N} α_κϕ'(h(P, u_k))u_k = 0. \tag{62}
\]
Thus, it follows from (60), (61), and (62) that
\[
\frac{dΦ(δ)}{dδ} \bigg|_{δ=0^+} = -α_κϕ'(h(P, u_k)) \quad - \sum_{k=1}^{N} α_κϕ'(h(P, u_k))\left(ξ'(0) ⋅ u_k\right)
\]
\[
= -α_κϕ'(h(P, u_k)) - ξ'(0) \cdot \sum_{k=1}^{N} α_κϕ'(h(P, u_k))u_k
\]
\[
= -α_κϕ'(h(P, u_k)) > 0. \tag{63}
\]
This means
\[
\lim_{δ \to 0^+} \frac{Φ(δ) - Φ(0)}{δ} > 0. \tag{64}
\]
Therefore, there exists a \( δ_0 > 0 \) small enough such that
\[
Φ(δ_0) > Φ(0). \tag{65}
\]
This together with (61) has
\[
Φ_{λ(δ_0)P_δ}(ξ(λ(δ_0)P_δ)) > Φ_P(ξ(P)) = Φ_P(0). \tag{66}
\]
Note that \( λ(δ_0) = C_q(P_δ)^{-1/(n-q)} \). Let \( P_0 = λ(δ_0)P_δ_0 - ξ(λ(δ_0)P_δ_0) \). Then, \( P_0 ∈ P(u_1, \ldots, u_N) \), \( C_q(P_0) = 1, ξ(P_0) = 0 \), and
\[
Φ_{P_0}(0) > Φ_P(0). \tag{67}
\]
This contradicts (53). Thus, \( P ∈ P_N(u_1, \ldots, u_N) \).

4. Solving the Orlicz Electrostatic \( q \)-Capacitary Minkowski Problem

**Proof of Theorem 1.** By Theorem 9, there exists a polytope \( P ∈ P_N(u_1, \ldots, u_N) \) with \( ξ(P) = 0 \) and \( C_q(P) = 1 \) such that
\[
Φ_P(0) = \sup \{ \min_{ξ ∈ int q} Φ(ξ): C_q(ξ) = 1, ξ ∈ P(u_1, \ldots, u_N) \}. \tag{68}
\]
For \( γ_1, \ldots, γ_N ∈ R \), choose \( |t| \) small enough so that the polytope \( P_t \), defined by
\[
P_t = \left\{ x: x ⋅ u_i ≤ h(P, u_i) + tγ_i \right\} \tag{69}
\]
has exactly \( N \) facets. Then, \( h(P_t, u_i) = h(P, u_i) + tγ_i \) for \( i = 1, \ldots, N \). Let
\[
β(t) = C_q(P_t)^{1/(n-q)}. \tag{70}
\]
Then, \( β(t)P_t ∈ P_N(u_1, \ldots, u_N) \) and \( C_q(β(t)P_t) = 1 \). By Lemma 5 and (iv) of Lemma 4, we obtain
\[
β'(0) = -\frac{1}{n-q} \frac{dC_q(P_t)}{dt} \bigg|_{t=0} = -\frac{q-1}{n-q} \sum_{j=1}^{N} γ_ju_j \tag{71}
\]
Define \( ξ(t) = ξ(β(t)P_t) \) and
\[
Φ(t) = \min_{ξ ∈ int q (β(t)P_t)} Φ(ξ) = \sum_{k=1}^{N} α_κϕ(β(t)h(P_t, u_k) - ξ(t) ⋅ u_k) \tag{72}
\]
Since \( ξ(t) \) is an interior point of \( β(t)P_t \), this has
\[
\sum_{k=1}^{N} α_κϕ'(β(t)h(P_t, u_k) - ξ(t) ⋅ u_k)u_k = 0, \tag{73}
\]
for \( i = 1, \ldots, n \), where \( u_k = (u_{k,1}, \ldots, u_{k,n})^T \). Note that \( ξ(0) \) is the origin. Then, setting \( t = 0 \) in (73), we have
\[
\sum_{k=1}^{N} α_κϕ'(h(P, u_k))u_k = 0, \tag{74}
\]
for \(i = 1, \ldots, n\). Hence, \n
\[
\sum_{k=1}^{N} \alpha_k \phi' \left( h(P, u_k) \right) u_k = 0.
\] (75)

Let \n
\[
F_j(t, \xi_1, \ldots, \xi_n) = \sum_{k=1}^{N} \alpha_k \phi' \left( \beta(t) h(P, u_k) \right) - (\xi, u_{k,1} + \cdots + \xi_{n, u_k}) u_k,
\] (76)

for \(i = 1, \ldots, n\). Then, \n
\[
\frac{\partial F}{\partial t} \bigg|_{(t, \xi_1, \ldots, \xi_n)} = \sum_{k=1}^{N} a_k \phi' \left( \beta(t) h(P, u_k) \right) \left( t \frac{\partial F}{\partial t} \bigg|_{(t, \xi_1, \ldots, \xi_n)} \right)
\]

\[
- (\xi, u_{k,1} + \cdots + \xi_{n, u_k}) \phi \left( h(P, u_k) \right) u_k
\]

\[
= \sum_{k=1}^{N} a_k \phi' \left( \beta(t) h(P, u_k) \right) - (\xi, u_{k,1} + \cdots + \xi_{n, u_k}) u_k, u_k.
\] (77)

Thus, \n
\[
\left( \frac{\partial F}{\partial \xi} \bigg|_{(t, \xi_1, \ldots, \xi_n)} \right)_{n \times n}
= - \sum_{k=1}^{N} \alpha_k \phi'' \left( h(P, u_k) \right) u_k u_k^T,
\] (78)

where \(u_k u_k^T\) is an \(n \times n\) matrix.

Since \(u_1, \ldots, u_N\) are in general position, \(\mathbb{R}^n = \text{lin} \{u_1, \ldots, u_n\}\). Thus, for any \(x \in \mathbb{R}^n\) with \(x \neq 0\), there exists a \(u_k \in \{u_1, \ldots, u_n\}\) such that \(u_k \cdot x \neq 0\). Note that \(\phi'' > 0\). Then, we have \n
\[
x^T \left( - \sum_{k=1}^{N} \alpha_k \phi'' \left( h(P, u_k) \right) u_k u_k^T \right) x
\]

\[
= - \sum_{k=1}^{N} \alpha_k \phi'' \left( h(P, u_k) \right) (x \cdot u_k)^2
\]

\[
\leq - \alpha_k \phi'' \left( h(P, u_k) \right) (x \cdot u_k)^2 < 0.
\] (79)

Hence, \(\partial F/\partial \xi\big|_{(t, \xi_1, \ldots, \xi_n)}\) is negative definite. This implies \(\det (\partial F/\partial \xi)_{(t, \xi_1, \ldots, \xi_n)} \neq 0\). By this, the facts that for all \(i = 1, \ldots, n, F_i(0, \ldots, 0) = 0\) follows by (74) and \(\partial F_i/\partial \xi_j\) is continuous on \(t\) and \(\xi\) for all \(1 \leq i, j \leq n\), and for the implicit function theorem, it follows that \n
\[
\xi'(0) = \left( \xi'_1(0), \ldots, \xi'_n(0) \right)
\] (80)

exists. Since \(\Phi(0)\) is a maximizer of \(\Phi(t)\), from (71), (72), and (75), we get \n
\[
0 = \phi'(0) \leq \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k)) \left( \beta'(0) h(P, u_k) + \gamma_k - \xi'(0) \cdot u_k \right)
\]

\[
= \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k)) \left[ - q^{-1} \frac{\sum_{j=1}^{N} \gamma_j \mu_j(P, u_j)}{n-q} h(P, u_k) + \gamma_k \right]
\]

\[
- \xi'(0) \cdot \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k)) u_k = \sum_{k=1}^{N} \alpha_k \phi'(h(P, u_k)) \gamma_k
\]

\[
= \sum_{k=1}^{N} \left[ \alpha_k \phi'(h(P, u_k)) - \frac{q}{n-q} \left( \sum_{j=1}^{N} \alpha_k \phi'(h(P, u_j)) h(P, u_j) \right) \mu_j(P, u_k) \right] \gamma_k.
\] (81)

Since \(\gamma_1, \ldots, \gamma_N\) are arbitrary, \n
\[
\alpha_k = - \frac{q}{n-q} \left( \sum_{j=1}^{N} \alpha_k \phi'(h(P, u_j)) h(P, u_j) \right) \frac{1}{\phi(h(P, u_k))} \mu_j(P, u_k),
\] (82)

for \(k = 1, \ldots, N\). Let \n
\[
c = \frac{q}{n-q} \left( \sum_{j=1}^{N} \alpha_k \phi'(h(P, u_j)) h(P, u_j) \right) \phi(h(P, u_k)).
\] (83)

Then, for \(k = 1, \ldots, N\), \n
\[
\alpha_k = c \phi(h(P, u_k)) \mu_j(P, u_k),
\] (84)

i.e., \n
\[
\mu = c \phi(\cdot).
\] (85)

This completes the proof.

**Proof of Corollary 2.** Let \(\phi(s) = s^{1-p}\) with \(p < 0\) in Theorem 1. Then, \(\phi(t) = -(1/p)t^p\) for \(t > 0\) and \(\lim \phi(t) = \infty\). Therefore, we see \(\phi\) and \(\phi\) satisfy the conditions of Theorem 1. Thus, from the theorem, (3), and (4), we obtain \n
\[
\mu = c h_{p-1}^{-1}(\cdot) \mu_q(P, \cdot) = c \mu_q(P, \cdot).
\] (86)

If \(p + q \neq n\), then from (i) of Lemma 4, we have \n
\[
\mu = \mu_q \left( e^{1/(n-p-q)} P, \cdot \right).
\] (87)

Let \(P_0 = e^{1/(n-p-q)} P\). Then, our desired result is given. If \(p + q = n\), then (86) is just the desired result.
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors’ Contributions

All authors contributed equally to this work. All authors have read and approved the final manuscript.

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References

[1] M. Ludwig, “General affine surface areas,” Advances in Mathematics, vol. 224, no. 6, pp. 2346–2360, 2010.
[2] M. Ludwig and M. Reitzner, “A classification of $SL(n)$ invariant valuations,” Annals of Mathematics, vol. 172, no. 2, pp. 1219–1267, 2010.
[3] E. Lutwak, D. Yang, and G. Zhang, “Orlicz centroid bodies,” Journal of Differential Geometry, vol. 84, no. 2, pp. 365–387, 2010.
[4] E. Lutwak, D. Yang, and G. Zhang, “Orlicz projection bodies,” Advances in Mathematics, vol. 223, no. 1, pp. 220–242, 2010.
[5] R. J. Gardner, D. Hug, and W. Weil, “The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities,” Journal of Differential Geometry, vol. 97, no. 3, pp. 427–476, 2014.
[6] R. J. Gardner, D. Hug, W. Weil, and D. Ye, “The dual Orlicz-Brunn-Minkowski theory,” Journal of Mathematical Analysis and Applications, vol. 430, no. 2, pp. 810–829, 2015.
[7] D. Xi, H. Jin, and G. Leng, “The Orlicz Brunn-Minkowski inequality,” Advances in Mathematics, vol. 260, pp. 350–374, 2014.
[8] B. Zhu, J. Zhou, and W. Xu, “Dual Orlicz-Brunn-Minkowski theory,” Advances in Mathematics, vol. 264, pp. 700–725, 2014.
[9] D. Zou and G. Xiong, “Orlicz-John ellipsoids,” Advances in Mathematics, vol. 265, pp. 132–168, 2014.
[10] D. Zou and G. Xiong, “Orlicz-Legendre ellipsoids,” Journal of Geometric Analysis, vol. 26, no. 3, pp. 2474–2502, 2016.
[11] E. Lutwak, “The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem,” Journal of Differential Geometry, vol. 38, no. 1, pp. 131–150, 1993.
[12] E. Lutwak, “The Brunn-Minkowski-Firey Theory II,” Advances in Mathematics, vol. 118, no. 2, pp. 244–294, 1996.
[13] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, “The log-Brunn-Minkowski inequality,” Advances in Mathematics, vol. 231, no. 3–4, pp. 1974–1997, 2012.
[14] S. Campi and P. Gronchi, “The $L_p$ Busemann-Petty centroid inequality,” Advances in Mathematics, vol. 167, pp. 128–142, 2002.
[15] S. Campi and P. Gronchi, “On the reverse $L_p$ Busemann-Petty centroid inequality,” Mathematika, vol. 49, pp. 1–11, 2002.
[16] C. Haberl and F. Schuster, “General $L_p$ affine isoperimetric inequalities,” Journal of Differential Geometry, vol. 83, pp. 1–26, 2009.
[17] C. Haberl and F. E. Schuster, “Asymmetric affine $L_p$ Sobolev inequalities,” Journal of Functional Analysis, vol. 257, no. 3, pp. 641–658, 2009.
[18] E. Lutwak, D. Yang, and G. Zhang, “$L_p$ Affine Isoperimetric Inequalities,” Journal of Differential Geometry, vol. 56, no. 1, pp. 111–132, 2000.
[19] E. Lutwak, D. Yang, and G. Zhang, “$L_p$ John Ellipsoids,” Proceedings of the London Mathematical Society, vol. 90, no. 2, pp. 497–520, 2005.
[20] E. Werner and D. Ye, “New $L_p$ affine isoperimetric inequalities,” Advances in Mathematics, vol. 218, no. 3, pp. 762–780, 2008.
[21] H. Minkowski, “Allgemeine Lehrsätze über die konvexe Polyeder,” Nachrichten von der Königlichen Gesellschaft der Wissenschaften, vol. 1897, pp. 198–219, 1897.
[22] A. D. Aleksandrov, “On the theory of mixed volumes. III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies,” Matematicheskii Sbornik, vol. 3, pp. 27–46, 1938.
[23] W. Fenchel and B. Jessen, “Mengenfunktionen und konvexe Körper,” Danske Videnskabernes Selskab Mathematisk-fysiske Meddelelser, vol. 16, pp. 1–31, 1938.
[24] H. Lewy, “On differential geometry in the large, I (Minkowski’s problem),” Transactions of the American Mathematical Society, vol. 43, no. 2, pp. 258–270, 1938.
[25] L. Nirenberg, “The Weyl and Minkowski problems in differential geometry in the large,” Communications on Pure and Applied Mathematics, vol. 6, no. 3, pp. 337–394, 1953.
[26] A. V. Pogorelov, The Minkowski Multidimensional Problem, V. H. Winston & Sons, Washington, D.C., 1978.
[27] S. Y. Cheng and S. T. Yau, “On the regularity of the solution of then-dimensional Minkowski problem,” Communications on Pure and Applied Mathematics, vol. 29, no. 5, pp. 495–516, 1976.
[28] L. Caffarelli, L. Nirenberg, and J. Spruck, “The Dirichlet problem for nonlinear second-order elliptic equations I. Monge-Ampère equation,” Communications on Pure and Applied Mathematics, vol. 37, no. 3, pp. 369–402, 1984.
[29] G. Bianchi, K. J. Böröczky, A. Colesanti, and D. Yang, “The $L_p$-Minkowski problem for $-n < p < 1$,” Advances in Mathematics, vol. 341, pp. 493–535, 2019.
[30] K. J. Böröczky and F. Fodor, “The $L_p$ dual Minkowski problem for $p > 1$ and $q > 0$,” Journal of Differential Equations, vol. 266, no. 12, pp. 7980–8033, 2019.
[31] K. J. Böröczky and H. T. Trinh, “The planar $L_p$-Minkowski problem for $0 < p < 1$,” Advances in Applied Mathematics, vol. 87, pp. 58–81, 2017.
[32] K. J. Böröczky, P. Hegedüs, and G. Zhu, “On the discrete logarithmic Minkowski problem,” International Mathematics Research Notices, vol. 2016, no. 6, pp. 1807–1838, 2016.
[33] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, “The logarithmic Minkowski problem,” Journal of the American Mathematical Society, vol. 26, no. 3, pp. 831–852, 2013.
[34] W. Chen, “$L_p$ Minkowski problem with not necessarily positive data,” Advances in Mathematics, vol. 201, no. 1, pp. 77–89, 2006.
[35] S. Chen, Q.-R. Li, and G. Zhu, “On the $L_p$ Monge-Ampère equation,” Journal of Differential Equations, vol. 263, pp. 4997–5011, 2017.

[36] K. S. Chou and X. J. Wang, “The $L_p$-Minkowski problem and the Minkowski problem in centroaffine geometry,” Advances in Mathematics, vol. 205, no. 1, pp. 33–83, 2006.

[37] P. Guan and C. Lin, "On equation det (u_{ij} + \delta_{ij}u) = t^d f on $S^n$," preprint.

[38] Y. Huang and Q. Lu, "On the regularity of the $L_p$ Minkowski problem,” Advances in Applied Mathematics, vol. 50, pp. 268–280, 2013.

[39] Y. Huang and Y. Zhao, "On the $L_p$ dual Minkowski problem,” Journal of Differential Geometry, vol. 101, no. 1, pp. 499–511, 2015.

[40] D. Hug, E. Lutwak, D. Yang, and G. Zhang, "On the $L_p$ Minkowski Problem for Polytopes," Discrete & Computational Geometry, vol. 33, no. 4, pp. 699–715, 2005.

[41] H. Jian, J. Lu, and X. J. Wang, "Nonuniqueness of solutions to the $L_p$-Minkowski problem,” Advances in Mathematics, vol. 281, pp. 845–856, 2015.

[42] J. Lu and X.-J. Wang, "Rotationally symmetric solution to the $L_p$ Minkowski problem,” Journal of Differential Equations, vol. 254, pp. 983–1005, 2013.

[43] E. Lutwak, D. Yang, and G. Zhang, "On the $L_p$-Minkowski problem,” Transactions of the American Mathematical Society, vol. 356, no. 11, pp. 4359–4370, 2004.

[44] E. Lutwak, D. Yang, and G. Zhang, "$L_p$ dual curvature measures,” Advances in Mathematics, vol. 262, pp. 909–931, 2014.

[45] G. Zhu, "The logarithmic Minkowski problem for polytopes,” Advances in Mathematics, vol. 262, pp. 932–960, 2014.

[46] G. Zhu, "The centro-affine Minkowski problem for polytopes,” Journal of Differential Geometry, vol. 101, no. 1, pp. 159–174, 2015.

[47] G. Zhu, "The $L_p$ Minkowski problem for polytopes with $0 < p < 1$,” Journal of Functional Analysis, vol. 269, no. 4, pp. 1070–1094, 2015.

[48] G. Zhu, "The $L_p$ Minkowski problem for polytopes with $p < 0$,” Indiana University Mathematics Journal, vol. 66, no. 4, pp. 1333–1350, 2017.

[49] G. Zhu, "Continuity of the solution to the $L_p$ Minkowski problem,” Proceedings of the American Mathematical Society, vol. 145, no. 1, pp. 379–386, 2017.

[50] C. Haberl, E. Lutwak, D. Yang, and G. Zhang, "The even Orlicz Minkowski problem,” Adv. Math., vol. 224, no. 6, pp. 2485–2510, 2010.

[51] G. Bianchi, K. J. Böröczky, and A. Colesanti, "The Orlicz version of the $L_p$ Minkowski problem on $S^{n-1}$ for $-n < p < 1$,” Advances in Applied Mathematics, vol. 111, article 101937, 2019.

[52] R. J. Gardner, D. Hug, S. Xing, and D. Ye, "General volumes in the Orlicz–Brunn–Minkowski theory and a related Minkowski problem II,” Calculus of Variations and Partial Differential Equations, vol. 59, no. 1, p. 15, 2020.

[53] R. J. Gardner, D. Hug, W. Weil, S. Xing, and D. Ye, "General volumes in the Orlicz–Brunn–Minkowski theory and a related Minkowski problem I,” Calculus of Variations and Partial Differential Equations, vol. 58, no. 1, p. 12, 2019.

[54] Q. Huang and B. He, "On the Orlicz Minkowski problem for polytopes,” Discrete & Computational Geometry, vol. 48, no. 2, pp. 281–297, 2012.

[55] H. Jian and J. Lu, "Existence of solutions to the Orlicz–Minkowski problem,” Advances in Mathematics, vol. 344, pp. 262–288, 2019.

[56] S. Yijing, "Existence and uniqueness of solutions to Orlicz Minkowski problems involving $0 < p < 1$,” Advances in Applied Mathematics, vol. 101, pp. 184–214, 2018.

[57] Y. Wu, D. Xi, and G. Leng, "On the discrete Orlicz Minkowski problem,” Journal of Mathematical Analysis and Applications, vol. 371, no. 3, pp. 1795–1814, 2019.

[58] Y. Wu, D. Xi, and G. Leng, "On the discrete Orlicz Minkowski problem II,” Geometriae Dedicata, vol. 205, no. 1, pp. 177–190, 2020.

[59] S. Xing and D. Ye, "On the general dual Orlicz Minkowski problem,” Indiana University Mathematics Journal, in press.

[60] B. Zhu, S. Xing, and D. Ye, "The dual Orlicz Minkowski problem,” Journal of Geometric Analysis, vol. 28, no. 4, pp. 3829–3855, 2018.

[61] A. Colesanti, K. Nyström, P. Salani, J. Xiao, D. Yang, and G. Zhang, "The Hadamard variational formula and the Minkowski problem for $p$-capacity,” Advances in Mathematics, vol. 285, pp. 1511–1588, 2015.

[62] R. J. Gardner, Geometric Tomography, Cambridge Univ. Press, New York, Second edition, 2006.

[63] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Cambridge Univ. Press, New York, Second edition, 2014.

[64] D. Zou and G. Xiong, "The $L_p$ Minkowski problem for the electrostatic $p$-capacity,” Journal of Differential Geometry, in press.

[65] H. Hong, D. Ye, and N. Zhang, "The $p$-capacitary Orlicz Hadamard variational formula and Orlicz Minkowski problems,” Calculus of Variations and Partial Differential Equations, vol. 57, pp. 1–31, 2018.

[66] D. Jerison, "A Minkowski problem for electrostatic capacity,” Acta Mathematica, vol. 176, no. 1, pp. 1–47, 1996.

[67] D. Jerison, "The direct method in the calculus of variations for convex bodies,” Advances in Mathematics, vol. 122, no. 2, pp. 262–279, 1996.

[68] L. A. Caffarelli, D. Jerison, and E. H. Lieb, "On the case of equality in the Brunn-Minkowski inequality for capacity,” Advances in Mathematics, vol. 117, no. 2, pp. 193–207, 1996.

[69] M. Akman, J. Gong, J. Hineman, J. Lewis, and A. Vogel, "The Brunn-Minkowski inequality and a Minkowski problem for nonlinear capacity,” preprint.

[70] G. Xiong, J. Xiong, and L. Xu, "The $L_p$ capacitary Minkowski problem for polytopes,” Journal of Functional Analysis, vol. 277, no. 9, pp. 3131–3155, 2019.

[71] V. Maz’ya, "Conductor and capacity inequalities for functions on topological spaces and their applications to Sobolev-type imbeddings,” Journal of Functional Analysis, vol. 224, no. 2, pp. 408–430, 2005.