Dynamical evolution of close-in super-earths tidally interacting with its host star near spin-orbit resonances

The case of Kepler-21 system

S. H. Luna\textsuperscript{1,2}, M. D. Melita\textsuperscript{2,4}, and H. D. Navone\textsuperscript{1,3}

\textsuperscript{1} Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, Av. Pellegrini 250, Rosario, Argentina.
\textsuperscript{2} Instituto de Astronomía y Física del Espacio (IAFE), CONICET-Universidad de Buenos Aires, Buenos Aires, Argentina.
\textsuperscript{3} Instituto de Física de Rosario (IFIR). CONICET-Universidad Nacional de Rosario. Rosario, Argentina.
\textsuperscript{4} Facultad de Ciencias Astronómicas y Geofísicas, Universidad Nacional de La Plata. Paseo del Bosque s/n. La Plata. Argentina.

ABSTRACT

Context. To date, many presumably rocky planets with masses between 1 – 10 M\textsubscript{\textit{J}} have been discovered in orbits very close to their host star and therefore tidal interaction is expected to be very strong in such cases.

Aims. Our goal is to study the orbital and rotational evolution of the binary system composed by a rocky planet and its host star considering tidal dissipation within both bodies, in order to characterise the dynamical evolution, particularly near spin-orbit resonances, by computing time rates and typical timescales of the main features of the tidal evolution, i.e. synchronization, circularization and planarization. We would also evaluate if tidal effects are detectable in reasonable timescales.

Methods. We make use the equations of motion that give, on one hand, the time evolution of the semimajor axis (a), the eccentricity (e) and both inclinations, that of the star’s orbit as seen from the planet (i\textsubscript{p}) and that of the planet as seen from the star (i\textsubscript{s}), and, on the other hand, the differential equations that drive the rotational evolution along the maximal inertia axis. We produced a set of regular equations that can be numerically stable for vanishing values of the eccentricity and inclinations. We focus our investigations around spin-orbit resonances, because the angular accelerations caused by the tidal and the triaxial torques combined can produce permanent captures. We select the Kepler-21b planet as an interesting example because of its proximity to the host star and, as known mass-radius relationships suggest that it is a rocky planet, we have the possibility to use reliable rheological models for terrestrial planets. Another characteristic of this system is that the stellar and orbital parameters relevant to our study are well determined.

Results. We found that, for the parameters and the initial conditions explored, the orbital evolution is practically independent of the rotational initial conditions but the dependency on the rheological parameters is noticeable. As the rotational evolution is much faster than the orbital one, we also noticed that, qualitatively, both the rotational and orbital evolution can be described completely in terms of the secular tidal evolution. Therefore, we could obtain reliable analytical estimations of characteristic timescales at least for a, e and i\textsubscript{p}. We found that since the timescale of orbital decay is longer than the others, the orbit will be circularized and planarized before the planet arrive to the Roche limit. We also found that tidal effects on the planet’s orbital period and on stellar rotation are negligible.

Key words. Celestial mechanics – Planet-Star interactions – Tidal interaction – Planets and satellites: dynamical evolution and stability – Planets and satellites: terrestrial planets – Planets and satellites: individual: Kepler-21b

1. Introduction

Recent developments and improvements in the field of exoplanet detection methods have allowed the discovery of many earth-sized planets, the so-called super-earths which have masses between 1 and 10 M\textsubscript{\textit{J}}. This has raised the interest in different but related topics, such as the dynamical and thermal evolution \cite{Renaud2018}, the presence of atmospheres \cite{Seager2010}, the internal structure and composition \cite{Valencia2006,Valencia2007} and the possibility that this kind of celestial bodies offer for harbouring life.

In this work we are interested in the first one of the aforementioned problems, i.e., the dynamical evolution of a binary system formed by a super-earth and its host star, modelled as bodies capable of experiencing internal stresses and energy dissipation. Our aim is to characterize the orbital and rotational evolution of such a system due to the tides that each celestial body raises on the other one, in particular regarding synchronization, orbital decay or expansion, circularization and planarization \cite{Veras2019}, the first one refers to the synchronization between the orbital and the rotational motion, i.e. when \( \dot{\theta} = n \) where \( \dot{\theta} \) is the spin rate of the considered body and \( n \) is the mean motion or the mean orbital frequency, which is defined by the Kepler’s third law:

\[ G (m_s + m_p) = n^2 a^3 \]  \hspace{1cm} (1)

* Contact: shluna@iafe.uba.ar
where \( G \) is the Universal Gravitation constant. Moreover, as we shall see later, there is the possibility that a rotating body may get trapped in higher-than-synchronous rotation states (Efroimsky 2012), and that the eccentricity plays a key role in the possibility of capture in spin-orbit resonances as was shown by Goldreich & Peale (1966); Efroimsky (2012); Makarov (2012); Makarov et al. (2012).

Resonant capture in spin-orbit resonances is a phenomenon that occurs due to the perturbation caused by the triaxiality of at least one of the bodies, characterized by its principal moments of inertia. Naturally, we will consider superearths as triaxial bodies because they probably have a rocky composition, in contraposition to the stars, which are expected to be fluid.

This work is organized as follows: In Sect. 2 we present the equation of motion, that have to be solved numerically in order to obtain the time evolution of the system. We also give explicitly the tidal and triaxial potentials and the rheological models for each component. We also derive regular form of the equations of motion. In Sect. 3 we discuss the choice of the values of the physical and rheological parameters involved and the initial conditions for the numerical integration. We define the characteristic timescales of the dynamical evolution and we evaluate the possibility of detecting tidal effects. We show how the dynamics of a system interacting only through tides can be characterized analytically. The results are presented in Sect. 4. Finally, in Sect. 5 we offer a final discussion and draw some conclusions.

2. Methods

In order to describe the dynamical evolution – including the orbital and rotational motions – of a binary system constituted by a rocky planet and its host star we will make use of the equations of motion derived by Boué & Efroimsky (2019), which are valid for the case when the tidal dissipation within both bodies is considered, as were implemented by Veras et al. (2019) for the study of the orbital and rotational evolution of systems formed by a rocky planet and a white dwarf but including the disturbing potential due to the permanent-triaxial shape of the planet.

We consider a system of two bodies, the planet and its host star. Each of them is thought as a nearly spherical and homogeneous mass distribution, the former of total mass \( m_p \) and radius \( R_p \), and the latter of total mass \( m_s \) and radius \( R_s \). The mutual orbit is parametrised by the six classical orbital elements: the semimajor axis \( a \), the eccentricity \( e \), the mean anomaly \( M \), the argument of pericenter \( \omega \), the inclination \( i \) and the longitude of the ascending node \( \Omega \). But, as we will consider the tidal dissipation within the two bodies, then the forces exerted on one of them by the other also has to be considered. This implies the use of two references systems corresponding to each body. In other words, when the force of the planet on the star is to be evaluated one has to consider the orbit of the former as seen from the latter, and vice versa when evaluating the force of the star on the planet. As both orbits are defined with respect to the corresponding equatorial planes of each body, then the aforementioned reference systems are related by a rotation from one system to an inertial reference system ad then to the other system attached to the other body. However, this rotations only affect the angles that give the orientation of the orbits in the space, i.e. \( \omega, i \) and \( \Omega \), while \( a, e \) and \( M \) are the same in both systems (Boué & Efroimsky 2019).

We assume that each body rotates about its maximal inertia axis, being \( C_s \) and \( C_p \) the corresponding moments of inertia of the star and the planet, respectively, being the axis perpendicular to the corresponding equatorial plane of each one. In this situation, the rotational state of the star and the planet is completely determined by their rotation angles \( \theta_s \) and \( \theta_p \) – which are reckoned from a fixed direction in an inertial reference system – and their spin rates \( \dot{\theta}_s \) and \( \dot{\theta}_p \), given the fact that we will neglect the time evolution of the precession and nutation angles. Under this considerations, the Euler equations reduces to only the one describing the rotation of each body about its spin axis.

2.1. Orbital evolution

The orbital dynamic is described by a set of Lagrange-type planetary equations which give the time evolution of the semimajor axis, the eccentricity and the mean anomaly, on one hand, and the inclination, the argument of the pericenter and the longitude of the ascending node of each body’s orbit as seen from the other. But, as we will neglect both the apsidal and precession rates because they have time scales much longer than that of the other elements and even much grater than the rotational evolution one, we will consider only the time evolution of \( a, e, i_s \) and \( i_p \), the corresponding equations are:

\[
\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial M}
\]

\[
\frac{de}{dt} = \frac{1 - e^2}{na^2 e} \frac{\partial R}{\partial M} - \frac{\sqrt{1 - e^2}}{na^2 e} \left( \frac{\partial R}{\partial \omega_s} + \frac{\partial R}{\partial \omega_p} \right)
\]

\[
\frac{di_s}{dt} = -\frac{\beta}{C_p \theta_p \sin i_s} \left( \frac{\partial R}{\partial \omega_s} - \cos i_s \frac{\partial R}{\partial \Omega_s} \right) - \frac{1}{na^2 \sqrt{1 - e^2} \sin i_s} \left( \frac{\partial R}{\partial \Omega_s} - \cos i_s \frac{\partial R}{\partial \omega_p} \right)
\]

\[
\frac{di_p}{dt} = \frac{\beta}{C_p \theta_p \sin i_p} \left( \frac{\partial R}{\partial \omega_p} - \cos i_p \frac{\partial R}{\partial \Omega_p} \right) - \frac{\cos(\omega_s - \omega_p)}{na^2 \sqrt{1 - e^2} \sin i_s} \frac{\partial R}{\partial \omega_s} - \frac{\cos(\omega_s - \omega_p)}{na^2 \sqrt{1 - e^2} \sin i_p} \frac{\partial R}{\partial \omega_p}
\]
\[
\frac{\text{d}p}{\text{d}t} = \frac{\beta}{C_s \theta_s \sin i_p} \left( \frac{\partial R}{\partial \omega_p} - \cos i \frac{\partial R}{\partial \zeta_p} \right) - \frac{1}{n a^2 \sqrt{1 - e^2 \sin i_p}} \left( \frac{\partial R}{\partial \Omega_p} - \cos i \frac{\partial R}{\partial \omega_p} \right) - \frac{\sin(\omega_p - \omega_s)}{n a^2 \sqrt{1 - e^2 \sin i_p}} \left( \frac{\partial R}{\partial \Omega_p} - \cos i \frac{\partial R}{\partial \omega_p} \right).
\]

where \( \beta = \frac{m_s m_p}{m_s + m_p} \) is the reduced mass, \( R \) is known as the disturbing function; while \( \omega_s, i_s \) and \( \Omega_s \) are the argument of pericenter, the inclination and the longitude of the ascending node of the star’s orbit as seen from the planet, and \( \omega_p, i_p \) and \( \Omega_p \) are the corresponding elements of the planet’s orbit as seen from the star.

As we have neglected the apsidal and nodal precessions rates, they can assume arbitrary values which for simplicity we will set them to zero. In consequence, the equations of orbital evolution becomes expressed as:

\[
\frac{\text{d}a}{\text{d}t} = \frac{2}{n a} \frac{\partial R}{\partial M},
\]

\[
\frac{\text{d}e}{\text{d}t} = \frac{1 - e^2}{n a^2} \frac{\partial R}{\partial M},
\]

\[
\frac{\text{d}i_s}{\text{d}t} = C_s \theta_s \sin i \left( \frac{\partial R}{\partial \omega_s} - \cos i \frac{\partial R}{\partial \Omega_s} \right) - \frac{1}{n a^2 \sqrt{1 - e^2 \sin i_s}} \left( \frac{\partial R}{\partial \Omega_s} - \cos i \frac{\partial R}{\partial \omega_s} \right),
\]

\[
\frac{\text{d}p}{\text{d}t} = \frac{\beta}{C_s \theta_s \sin i_p} \left( \frac{\partial R}{\partial \omega_p} - \cos i \frac{\partial R}{\partial \zeta_p} \right) - \frac{1}{n a^2 \sqrt{1 - e^2 \sin i_p}} \left( \frac{\partial R}{\partial \Omega_p} - \cos i \frac{\partial R}{\partial \omega_p} \right) - \frac{\sin(\omega_p - \omega_s)}{n a^2 \sqrt{1 - e^2 \sin i_p}} \left( \frac{\partial R}{\partial \Omega_p} - \cos i \frac{\partial R}{\partial \omega_p} \right)
\]

In what follows, we introduce some modifications to Eqs. (6), (7) and (8) in order to avoid numerical instabilities when the eccentricity and/or the inclination becomes too small. If we define the variables \( \xi = 1 - e^2 \) and \( x = \cos i \) then we have:

\[
\frac{\text{d}\xi}{\text{d}t} = -2\epsilon \frac{\text{d}e}{\text{d}t},
\]

\[
\frac{\text{d}x}{\text{d}t} = -\sin i \frac{\text{d}i_s}{\text{d}t}.
\]

If we now replace the expressions for \( \text{d}e/\text{d}t \) and \( \text{d}i/\text{d}t \) given by Ecs. (7), (8) and (9) in Ecs. (10) and (11), then the equations for the rotational evolution take the form:

\[
\frac{\text{d}a}{\text{d}t} = \frac{2}{n a} \frac{\partial R}{\partial M},
\]

\[
\frac{\text{d}e}{\text{d}t} = \frac{2}{n a^2} \frac{\partial R}{\partial M} - \sqrt{e^2} \frac{\partial R}{\partial M},
\]

\[
\frac{\text{d}x_s}{\text{d}t} = \frac{1}{n a^2 \sqrt{e^2}} \left( \frac{\partial R}{\partial \Omega_s} - x_s \frac{\partial R}{\partial \omega_s} \right) + \frac{1}{1 - x_s^2} \left( \frac{\partial R}{\partial \Omega_p} - x_p \frac{\partial R}{\partial \omega_p} \right) - \frac{\beta}{C_s \theta_p} \left( x_p \frac{\partial R}{\partial \Omega_p} - \frac{\partial R}{\partial \omega_p} \right),
\]

\[
\frac{\text{d}x_p}{\text{d}t} = \frac{1}{n a^2 \sqrt{e^2}} \left( \frac{\partial R}{\partial \Omega_p} - x_p \frac{\partial R}{\partial \omega_p} \right) + \frac{1}{1 - x_p^2} \left( \frac{\partial R}{\partial \Omega_s} - x_s \frac{\partial R}{\partial \omega_s} \right) + \frac{\beta}{C_s \theta_s} \left( x_s \frac{\partial R}{\partial \Omega_s} - \frac{\partial R}{\partial \omega_s} \right).
\]

It is worth mentioning that from the corresponding definitions it is easy to see that \( \xi \to 1 \) when \( e \to 0 \) and \( x_s, x_p \to 1 \) when \( i_s, i_p \to 0 \).

### 2.2. Rotational evolution

Under the assumptions given above, and as we have already pointed out, the time evolution of the rotation angle or sidereal time is given by the Euler’s equation corresponding to the axis perpendicular to the equatorial plane which, in general, is given by Boué & Erofimsky (2019):

\[
\frac{\text{d}^2 \theta}{\text{d}t^2} = \frac{\beta}{C} \frac{\partial R}{\partial \Omega_p}.
\]

It should be remembered that this equation and Eqs. (4) – (5) are obtained under the so called *gyroscopic approximation*. This means that if \( \psi, \varepsilon \) and \( \theta \) are the Eulerian angles specifying the orientation of the body at each instant of time – called precession, nutation and proper rotation or spin angles, respectively –, then we assume that \( \dot{\theta} \gg |\dot{\psi}|, |\dot{\varepsilon}| \). Furthermore, we directly assume for simplicity that \( \dot{\psi} = \dot{\varepsilon} = 0 \). We neglect the time variation of the precession and nutation angles. In consequence, the values of \( \dot{\psi} \) and \( \dot{\varepsilon} \) are arbitrary constants which, also for simplicity, we set them equal to zero. This correspond to the assumption that the rotation axis is perpendicular to the inertial reference plane or, in other words, we are assuming that the latter coincides with or is parallel to the equatorial plane of the considered body.

Another important and necessary assumption is that we also neglect the time variation of the moments of inertia of the considered bodies. This is justified because even though the tidal theory consider the deformation of celestial bodies – as we will see later – this deformations (considered linear) are always very small, i.e. \( \varepsilon < 10^{-6} \), where \( \varepsilon \) is a typical value of the strain (Erofimsky, 2018).
2.3. Disturbing potentials

The disturbing function $R$ present in the equations of motion is given by (Boué & Efroimsky 2019):

$$ R = -\frac{1}{\beta} \left[ m_1 \left[ U_\rho (r_1, r_1^*) + V_\rho (r_1^*) \right] + m_\rho U_\rho \left( -r_p, -r_p^* \right) \right] $$  \hspace{1cm} (17)

where $U_\rho (r_1, r_1^*)$ is the tidal disturbing potential generated by the (deformed) planet and $U_\rho \left( -r_p, -r_p^* \right)$ is the corresponding tidal potential generated by the star. As we are interested in the dynamical evolution near the spin-orbit resonances, we add the permanent-triaxiality-caused potential of the planet $V_\rho (r_1)$. The first one is given by (Efroimsky 2012):

$$ U_\rho (r_1, r_1^*) = -\frac{Gm_\rho}{a^*} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{h=0}^{\infty} \sum_{s=-\infty}^{\infty} U_\rho^{lm} rhs $$ \hspace{1cm} (18)

where,

$$ U_\rho^{lmphs} = \left( \frac{R_\rho}{a^*} \right)^{l+1} \left( \frac{R_\rho}{a^*} \right)^{(l-m)!} \left( 2 - \delta_{0m} \right) F_{lmphs}(\epsilon^*) G_{lmphs}(\epsilon) \left[ \cos \left( v_{lmphs} - m \theta_p^* \right) - \left( v_{lmphs} - m \theta_p \right) \right] K_{l \chi_{lmphs}}^p $$

$$ + \sin \left[ \left( v_{lmphs} - m \theta_p^* \right) - \left( v_{lmphs} - m \theta_p \right) \right] K_{l \chi_{lmphs}}^{p*} $$  \hspace{1cm} (19)

The tidal potential generated by the star is given by an expression analogous to Eq. (18) and (19):

$$ U_s \left( -r_p, -r_p^* \right) = -\frac{Gm_\rho}{a^*} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{h=0}^{\infty} \sum_{s=-\infty}^{\infty} U_s^{lmphs} $$ \hspace{1cm} (20)

with,

$$ U_s^{lmphs} = \left( \frac{R_s}{a^*} \right)^{l+1} \left( \frac{R_s}{a^*} \right)^{(l-m)!} \left( 2 - \delta_{0m} \right) F_{lmphs}(\epsilon^*) G_{lmphs}(\epsilon) \left[ \cos \left( v_{lmphs} - m \theta_p^* \right) - \left( v_{lmphs} - m \theta_p \right) \right] K_{l \chi_{lmphs}}^p $$

$$ + \sin \left[ \left( v_{lmphs} - m \theta_p^* \right) - \left( v_{lmphs} - m \theta_p \right) \right] K_{l \chi_{lmphs}}^{p*} $$  \hspace{1cm} (21)

where $\delta_{0m}$ is the Kronecker’s delta, $F_{lmphs}(\epsilon)$ and $G_{lmphs}(\epsilon)$ are the inclination functions (Goode & Wagner 2008) and the eccentricity functions (Giacaglia 1976), respectively $(\epsilon = p, s)$. The $v_{lmphs}^p$’s $(\text{with } k = h, p \text{ and } j = q, s)$ are linear combinations of $\omega, \Omega$ and $M$, expressed via:

$$ v_{lmphs}^p = (l - 2k) \omega_p + (l - 2k + j) M + \Omega_q $$

\hspace{1cm} (22)

and an analog expressions for $v_{lmphs}^{p*}$ and $v_{lmphs}^{p*}$.

Within the framework of the Darwin–Kaula formalism of bodily tides, one considers the presence of three bodies in order to take the tidal dissipation into account: the primary – whose shape and deformation is considered – and “two secondaries”, the tide-rising and the tidally-disturbed ones, considered to be point masses. In the case of the planet being the primary, the tide-rising secondary is located at $r_1^*$ while the tide disturbed one is at $r_1$. Conversely, when the star is considered to be the primary, then the tide-rising secondary is at $-r_p^*$ while the tide-disturbed one is at $-r_p$. It is worth to indicate that as we are considering a two-body system, in both cases the two secondaries coincides with each other, i.e. in the first case both secondaries coincide with the star and both coincide with the planet in the second case. By virtue of the Kaula (1961) transformation, the set of coordinates of each secondary is expressed through the corresponding set of orbital elements that parametrizes their orbits (Efroimsky & Williams 2009).

In the above expressions – Eqs. (18) and (20) – $K_{l, \chi_{lmphs}}^{p*}$ and $K_{l, \chi_{lmphs}}^p$ are called quality functions (Makarov 2012) and are given by:

$$ K_{l, \chi}^{p*} = \frac{3}{2l - 1} \left( \mathfrak{R} \left[ \hat{J}(\chi) \right] + A_l J \left( \mathfrak{R} \left[ \hat{J}(\chi) \right] \right) + \mathfrak{I} \left[ \hat{J}(\chi) \right] \right)^2 $$  \hspace{1cm} (23)

$$ K_{l, \chi}^p = \frac{3}{2l - 1} \left( \mathfrak{I} \left[ \hat{J}(\chi) \right] + A_l J \left( \mathfrak{R} \left[ \hat{J}(\chi) \right] \right) + \mathfrak{I} \left[ \hat{J}(\chi) \right] \right)^2 $$  \hspace{1cm} (24)

where $J$ is the compliance – inverse of the shear rigidity $\mu$ –, $\mathfrak{R}(\chi)$ and $\mathfrak{I}(\chi)$ denotes the real and imaginary parts of the complex number $\chi$. We have omitted the indices for the sake of simplicity. $\chi_{lmphs}^p$ are the tidal frequencies, i.e. are the physical frequencies at which the deformations are excited. They are defined as the absolute value of the tidal modes $\omega_{lmphs}^p$ (Efroimsky 2012, Efroimsky & Makarov 2013), mathematically:

$$ \chi_{lmphs}^p = |\omega_{lmphs}^p| = |(l - 2k) \omega_p^* + (l - 2k + j) M^* + m \left( \Omega_q^* - \theta_p^* \right)| $$  \hspace{1cm} (25)

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where $\theta_s^+$ is the spin rate of the considered primary. Very commonly, one can neglect both the apsidal and the nodal precessions, in which case the tidal modes can be expressed as:

$$\omega_{hmkl}^{\theta_s^+} \approx (l - 2k + j)n^+ - m\theta_v^+$$  \hspace{1cm} (26)

where $n_s^+$ is the mean orbital frequency of the tide-rising secondary. Consequently, the $\omega_{hmkl}^{\theta_s^+}$'s can be either positive or negative depending on whether the primary’s spin rate is slower or faster than the mean rate at which the tide-rising secondary describes its orbit, respectively, while the tidal frequencies are always positive. $A_l$ is given by:

$$A_l = \frac{2\dot{\theta}_v^+ + 4l + 3}{T_g \rho_v R_v}$$  \hspace{1cm} (27)

where $\rho_v$ is the mean density, $R_v$ is the mean radius of the primary and $g_v = Gm_v/R_v^2$ is the surface gravity. $\bar{J}(\chi)$ is the complex compliance and is defined by:

$$\bar{J}(\chi) = \int_0^\infty J(t - t') \exp[-i\chi(t - t')] \dq{t'}$$  \hspace{1cm} (28)

where the over-dot means differentiation with respect to $t'$. The functional form of the kernel $J(t - t')$ depends on the rheology of the material that composes the primary or, in other words, on the rheological model that describe the creep response of the body as a whole. Realistic materials are thought to have a mixture of elastic, viscous (or viscoelastic) and hereditary behaviours [Renaud & Henning 2018], i.e. the deformation is directly proportional to the stress and simultaneously depends on the time derivative of the latter and on its past values, respectively. Thus, $J(t - t')$ is given generally by:

$$J(t - t') = J(0)\Theta(t - t') + \text{viscous and hereditary terms}$$  \hspace{1cm} (29)

being $J(0)$ the value of the unrelaxed compliance – corresponding to the elastic response and inverse of the elastic rigidity $\mu(0)$ – and $\Theta(t - t')$ the Heaviside step function [Efroimsky 2012].

It is worth to indicate that the complex compliance enters in the constitutive equation that relates the stress and strain tensors. In a linear regime and under the assumption that the material of which the primary is made of is homogeneous incompressible and isotropic, the relationship between the components of the tensors, written in the frequency domain, is:

$$2\bar{u}_{\gamma\gamma}(\chi) = \bar{J}(\chi)\varepsilon_{\gamma\gamma}(\chi)$$  \hspace{1cm} (30)

where $\bar{u}_{\gamma\gamma}$ and $\bar{\varepsilon}_{\gamma\gamma}$ are the complex counterparts of the components of the strain and stress tensors, respectively. We refer the reader to the work by Efroimsky (2012) for a comprehensive explanation of this subject and detailed derivations of the equations given above.

The permanent-triaxiality-caused potential is a special case of the more general expression of the potential generated by the so-called Coefficients of the Gravitational Field (CGF), $C_{lm}$ and $S_{lm}$, which is given by (Frouard & Efroimsky 2017):

$$V_g(r_e) = -\frac{Gm_p}{a} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} V_{lmpq}^p r_e^{l-p}$$  \hspace{1cm} (31)

where,

$$V_{lmpq}^p = \frac{R_v^l}{a} F_{lmp}(i_c) G_{lmp} (e) \left[ C_{lm} \cos \left( \delta_{lm}^p - m\theta_p + \phi_{lm} \right) + S_{lm} \sin \left( \delta_{lm}^p - m\theta_p + \phi_{lm} \right) \right]$$  \hspace{1cm} (32)

where $\phi_{lm} = \left[ (-1)^{l-m} - 1 \right] \frac{\pi}{2}$. We only consider the triaxiality of the planet for two main reasons: the first one is that the tidal interaction is expected to affect more strongly its dynamical evolution than that of the star, in particular the rotational evolution where there exist probabilities that the planet gets trapped in a spin-orbit resonance. The second one is that we can reasonably neglect the triaxiality of a fluid body like a star.

### 2.4. Detailed expression of the equations of motion

The equations describing the orbital and rotational dynamics given in Eqs. (12)-(14) and (16) are quite general and therefore we consider necessary at this point to rewrite them in the form we have used them for our numerical simulations.

It can be seen in Eqs. (12)-(14) and (16) mentioned in the last paragraph, the equations of motion depends on the gradient of the disturbing function $R$, among other factors and so, the calculation of the derivatives of $R$ deserves some discussion.

As was pointed out in the preceding subsection, $r_e$ correspond to the position of the tide-disturbed secondary while the tide-rising one is located at $r_s^*$ when the planet is considered to be the primary body. Accordingly, $-r_p$ and $-r_s^*$ are the radius-vectors of the tide-disturbed and tide-rising secondaries, respectively, when the roles of the planet and the star are reversed with respect to the former case. It should be stressed that we always consider the motion of the tide-disturbed secondary and thus the general rule to be followed in order to find the aforementioned derivatives is the following: first, we have to differentiate the disturbing function with
respect to the orbital elements of the secondary and then identify both secondaries with each other, i.e. to set \( r_s = r_s' \) and \( r_p = r_p' \) in order to evaluate properly the force per unit mass each body exerts on the other (the planet and the star).

Let \( \psi_s \), be an equivalent notation for the angular elements \( \omega_s, M \) and \( \Omega_s \). Then, the derivatives of the disturbing function, given by Eq. (17), can be expressed in a general form as:

\[
\frac{\partial R}{\partial \psi_s} = -\frac{1}{\beta} \left[ m_0 \frac{\partial}{\partial \psi_s} \left( U_p + V_p \right) \frac{\partial \psi_s}{\partial \psi_s} + m_p \frac{\partial U_p}{\partial \psi_p} \frac{\partial \psi_p}{\partial \psi_p} \right]
\]

(33)

where the explicit dependence on the radius-vectors were omitted for simplicity. We should stress the fact that when the case \( \psi_s = M \) is considered, we have to keep in mind that both \( \psi_s \) and \( \psi_p \) are simultaneously equal to \( M \) and, in consequence, both terms inside the square brackets must be kept, i.e.:

\[
\frac{\partial R}{\partial M} = -\frac{1}{\beta} \left[ m_0 \left( \frac{\partial U_p}{\partial M} + \frac{\partial V_p}{\partial M} \right) + m_p \frac{\partial U_p}{\partial M} \right].
\]

(34)

As can be seen in Eqs. (18), (20) and (31) \( \omega_s, M \) and \( \Omega_s \) enter as the arguments of the cosine and sine functions through the \( v^p_{\text{link}} \). In consequence, the derivatives of the tidal and triaxiality-caused potentials can be expressed as:

\[
\frac{\partial U_p}{\partial \psi_s} = \frac{Gm_0}{a^3} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{h=0}^{\infty} \sum_{s=-\infty}^{\infty} \frac{\partial U^p_{\text{lmhs}}}{\partial \psi_s} \frac{\partial v^p_{\text{link}}}{\partial \psi_s},
\]

\[
\frac{\partial V_p}{\partial \psi_s} = \frac{Gm_p}{a} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{h=0}^{\infty} \sum_{s=-\infty}^{\infty} \frac{\partial V_p}{\partial \psi_s} \frac{\partial v^p_{\text{link}}}{\partial \psi_s},
\]

\[
\frac{\partial U_p}{\partial \psi_p} = \frac{Gm_0}{a^3} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{h=0}^{\infty} \sum_{s=-\infty}^{\infty} \frac{\partial U^p_{\text{lmhs}}}{\partial \psi_p} \frac{\partial v^p_{\text{link}}}{\partial \psi_p}.
\]

(35)

(36)

(37)

The corresponding derivatives of \( v^s_{\text{link}}, v^s_{\text{lmhs}} \) and \( v^p_{\text{link}} \) can be expressed as:

\[
\frac{\partial v^s_{\text{lmhs}}}{\partial \psi_s} = (l - 2h) \frac{\partial \omega_s}{\partial \psi_s} + (l - 2h + s) \frac{\partial M}{\partial \psi_s} + m \frac{\partial \Omega_s}{\partial \psi_s},
\]

(38)

\[
\frac{\partial v^s_{\text{lmhs}}}{\partial \psi_p} = (l - 2p) \frac{\partial \omega_p}{\partial \psi_p} + (l - 2p + q) \frac{\partial M}{\partial \psi_p} + m \frac{\partial \Omega_p}{\partial \psi_p},
\]

(39)

\[
\frac{\partial v^p_{\text{link}}}{\partial \psi_p} = (l - 2h) \frac{\partial \omega_p}{\partial \psi_p} + (l - 2h + s) \frac{\partial M}{\partial \psi_p} + m \frac{\partial \Omega_p}{\partial \psi_p}.
\]

(40)

After the derivatives have been calculated, we have to identify the tide-rising and the tide-disturbed secondaries in both the tidal potential of the planet and of the star. As the arguments of the sine and cosine functions remains the same, once the identification have been performed, we obtain:

\[
\left\{ v^s_{\text{lmhs}} - m \delta^s_{l} \right\} - \left\{ v^s_{\text{lmhs}} - m \delta^s_{l} \right\} = 2(h - p) \omega_s + (2h - 2p + q - s) M,
\]

(41)

\[
\left\{ v^p_{\text{lmhs}} - m \delta^p_{\ell} \right\} - \left\{ v^p_{\text{lmhs}} - m \delta^p_{\ell} \right\} = 2(h - p) \omega_p + (2h - 2p + q - s) M.
\]

(42)

The derivatives of the tidal potentials can be split into two types of terms, namely the secular and the oscillating ones. The former comprises the terms independent of \( M \), i.e. those for which \( h = p \) and \( q = s \), the rest being part of the latter type of terms. As can be easily seen, for the case of the secular terms the arguments of the sine and cosine given by Eqs. (41) and (42) are identically null and, in consequence, the derivatives of the disturbing function become independent of the quality function \( K_R(l, \chi) \).

As we are interested in the dynamical evolution due to those terms of the tidal potentials' gradient, we will take a closer view of them. Taking into account what was pointed out in the last paragraph, the secular terms of the expressions given by Eqs. (35) and (37) can be written as:

\[
\frac{\partial U_p}{\partial \psi_s}\text{sec} = -\frac{Gm_0}{a} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{h=0}^{\infty} \sum_{s=-\infty}^{\infty} \left( \frac{\partial U^p_{\text{lmhs}}}{\partial \psi_s} \frac{\partial v^p_{\text{link}}}{\partial \psi_s} \right)\text{sec},
\]

(43)

\[
\frac{\partial U_s}{\partial \psi_p}\text{sec} = -\frac{Gm_p}{a} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{h=0}^{\infty} \sum_{s=-\infty}^{\infty} \left( \frac{\partial U^p_{\text{lmhs}}}{\partial \psi_p} \frac{\partial v^p_{\text{link}}}{\partial \psi_p} \right)\text{sec},
\]

(44)

where,

\[
\frac{\partial U^p_{\text{link}}}{\partial v^p_{\text{link}}}\text{sec} = -\frac{R_0}{a} \frac{2^{l+1} (l - m)^2}{(l + m)!} (2 - \delta_{lm}) F^2_{\text{fkg}}(s) G^2_{\text{fkg}}(e) K_p^p \left( l, v^p_{\text{link}} \right)
\]

(45)

\[
\frac{\partial v^p_{\text{link}}}{\partial v^p_{\text{link}}}\text{sec}
\]

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and
\[
\left( \frac{\partial U^\gamma_{\text{imp}}}{\partial \nu_{\text{imp}}^p} \right)_{\sec} = -\frac{R_p}{a} \sum_{l=2}^{\infty} \frac{l^2+1}{2} \frac{(l-m)!}{(l+m)!} (2 - \delta_{lm}) F^3_{\text{imp}}(i_p) G^4_{\text{imp}}(e) K^2_l(\lambda_{\text{imp}}^p). \tag{46}
\]

For completeness we have to add:
\[
\frac{\partial V^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} = \left( \frac{R_p}{a} \right)^{2} F^2_{\text{imp}}(i_p) G^1_{\text{imp}}(e) \left[ -C_{\text{imp}} \sin(\nu_{\text{imp}}^p - m \theta_p + \phi_{\text{imp}}) + S_{\text{imp}} \cos(\nu_{\text{imp}}^p - m \theta_p + \phi_{\text{imp}}) \right]. \tag{47}
\]

Then, if we insert in Eq. \((33)\) the corresponding expression for the derivatives of the tidal potential of both bodies and the one generated by the CGF of the planet, given by Eqs. \((36), (43)\) and \((44)\), we arrive to:
\[
\frac{\partial R}{\partial \psi_e} = \frac{G}{\beta a} \left[ m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} \right)_{\sec} \frac{\partial V^\psi_{\text{imp}}}{\partial \phi_s} \frac{\partial V^\psi_{\text{imp}}}{\partial \phi_p} + m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \frac{\partial V^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} \frac{\partial V^\psi_{\text{imp}}}{\partial \phi_s} \frac{\partial V^\psi_{\text{imp}}}{\partial \phi_p} \right]. \tag{48}
\]

Finally, taking into account Eqs. \((38), (39)\) and \((40)\), the expressions of the derivatives of the disturbing function are:
\[
\frac{\partial R}{\partial M} = \frac{G m_p}{\beta a} \left[ m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} \right)_{\sec} (l-2p+q) + m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \frac{\partial V^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} (l-2p+q) \right]. \tag{49}
\]
\[
\frac{\partial R}{\partial \omega_e} = \frac{G m_p}{\beta a} \left[ m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} \right)_{\sec} (l-2p) \right]. \tag{50}
\]
\[
\frac{\partial R}{\partial \Omega_e} = \frac{G m_p}{\beta a} \left[ m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} \right)_{\sec} (l-2p) \right]. \tag{51}
\]
\[
\frac{\partial R}{\partial \psi_e} = \frac{G m_p}{\beta a} \left[ m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} \right)_{\sec} (l-2p) \right]. \tag{52}
\]

and
\[
\frac{\partial R}{\partial \Omega_p} = \frac{G m_p^2}{\beta a} \left[ m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} \right)_{\sec} m \right]. \tag{53}
\]

Eqs. \(12 - 15\) can now be rewritten as:
\[
\frac{da}{dt} = 2 n a \left[ m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} \right)_{\sec} (l-2p+q) + m_p \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \frac{\partial V^\psi_{\text{imp}}}{\partial \nu_{\text{imp}}^p} (l-2p+q) \right]. \tag{54}
\]
\[
\frac{d\vec{C}}{dt} = 2n \sqrt{\varepsilon} \left( \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \vartheta_{\text{impq}}} \right)_{\text{sec}} (l-2p) + \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial V_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p) \right)
\]
\[
+ \frac{m_p}{m_s} \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p) + \sqrt{\varepsilon} \left( \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p+q) + \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial V_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p+q) \right) \right)
\]
\[
, (55)
\]
\[
\frac{dx_s}{dt} = \frac{n}{\sqrt{\varepsilon} m_s} \left( \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \vartheta_{\text{impq}}} \right)_{\text{sec}} m - x_s \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p) \right)
\]
\[
+ \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p+q) + \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial V_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p+q) \right) \right)
\]
\[
, (56)
\]
\[
\frac{dx_p}{dt} = \frac{n}{\sqrt{\varepsilon} m_s} \left( \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \vartheta_{\text{impq}}} \right)_{\text{sec}} m - x_p \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p) \right)
\]
\[
+ \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p+q) + \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial V_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} (l-2p+q) \right) \right)
\]
\[
, (57)
\]
These are exactly the expressions we have implemented in our codes. It can be noted that Eqs. (54), (55), (56) and (57) can be rewritten as:
\[
\frac{da}{dt} = \left( \frac{da}{dt} \right)_{\text{tide}} + \left( \frac{da}{dt} \right)_{\text{tri}},
\]
\[
(58)
\]
\[
\frac{d\vec{C}}{dt} = \left( \frac{d\vec{C}}{dt} \right)_{\text{tide}} + \left( \frac{d\vec{C}}{dt} \right)_{\text{tri}},
\]
\[
(59)
\]
\[
\frac{dx_s}{dt} = \left( \frac{dx_s}{dt} \right)_{\text{tide}} + \left( \frac{dx_s}{dt} \right)_{\text{tri}},
\]
\[
(60)
\]
\[
\frac{dx_p}{dt} = \left( \frac{dx_p}{dt} \right)_{\text{tide}} + \left( \frac{dx_p}{dt} \right)_{\text{tri}},
\]
\[
(61)
\]
where the first term on the right hand side of Eqs. (58), (59), (60) and (61) are due to both the tidal potentials of the planet and the star and the second one are due to the permanent-triaxiality-caused potential of the planet. The double brackets means averaging over both the mean anomaly and the arguments of the pericenters (\(\omega_s\) and \(\omega_p\)), i.e. “the secular part” of each equation. Detailed expressions of the former terms can be found in the work by [Veras et al. (2019)] but it has to be stressed that they are completely equivalent to the ones given in our work.

In order to close the system, we have to add the equations for describing the rotational evolution of both the star and the planet, which are obtained from Eq. (16) where one has to replace the corresponding parameters of each body, the expression of the disturbing function and its corresponding derivatives given by Eq. (17), (52) and (53). The result is:
\[
\frac{d^2\theta_s}{dt^2} = -\frac{G m_p}{a C_s} \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \vartheta_{\text{impq}}} \right)_{\text{sec}} m
\]
\[
(62)
\]
and
\[
\frac{d^2\theta_p}{dt^2} = -\frac{G m_p}{C_p a} \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} m + \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial V_{\text{impq}}}{\partial \varphi_{\text{impq}}} \right)_{\text{sec}} m
\]
\[
(63)
\]
Before closing this subsection, let us rewrite the last equation as:
\[
\frac{d^2\theta_p}{dt^2} = \frac{1}{C_p} \left[ -\frac{G m_p}{a} \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial U_{\text{impq}}}{\partial \Omega_{\text{impq}}} \right)_{\text{sec}} m - \frac{G m_p}{a} \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\partial V_{\text{impq}}}{\partial \Omega_{\text{impq}}} \right)_{\text{sec}} m \right]
\]
\[
(64)
\]
By replacing Eq. (45) in the first term inside the square brackets on the right hand side of Eq. (64) we obtain:

\[
\langle T^\text{tide}_z \rangle = \frac{G m_s^2}{a} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{pq=0}^{\infty} \left( \frac{R_p}{a} \right)^{2l+1} \frac{(l-m)!}{(l+m)!} m F_{\text{imp}}^2(i_s) G_{lpq}^2(e) K_1^p \left( l, x_{lpq}^s \right),
\]

which is nothing but the secular part of the polar component of the tidal torque.

Now, let us consider the second term inside the square brackets on the right hand side of Eq. (64). If we replace in it the corresponding expression for the derivative of \( V_p(r) \) with respect to \( \Omega \), we arrive to:

\[
\langle T^\text{CGF}_z \rangle = -\frac{G m_s p}{a} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{pq=0}^{\infty} m \left( \frac{R_p}{a} \right)^{2l} F_{\text{imp}}(i_s) G_{lpq}(e) \left[ -C_{lm} \sin \left( \theta_{lpq} - m \theta_p + \phi_{lm} \right) + S_{lm} \cos \left( \theta_{lpq} - m \theta_p + \phi_{lm} \right) \right],
\]

which we can identify as the polar component of the torque associated with the permanent-figure of the planet.

Now, let us consider the \( l = 2 \) terms of Eq (65). It can be demonstrated that the CGF's, \( C_{2m} \) and \( S_{2m} \) with \( m = 0, 1, 2 \), are related to the components of the inertia tensor (Forge & Müller 2012). Let \( A, B \) and \( C \) be the principal moments of inertia corresponding to the maximal, intermediate and minimal principal axis, the latter being the one perpendicular to the equator’s plane and is \( A < B < C \). Then, we have:

\[
C_{20} = \frac{1}{2} m_p R_p^2 [A + B - 2C] = -J_2,
\]

where \( J_2 \) is known as the dynamic form factor, and

\[
C_{22} = \frac{B - A}{4 m_p R_p^2}.
\]

The others coefficients are proportional to the products of inertia, i.e. the non-diagonal components of the inertia tensor which we will neglect. For very low inclinations, as in the cases we will consider, only the inclination function for which, in general, is \( p = (l-m)/2 \) are non-zero (Gooding & Wagner 2008). Therefore, for \( l = 2 \) we will have that in the limit of \( i_s \to 0 \) only the non-vanishing inclination functions are \( F_{200}(i_s) \) and \( F_{220}(i_s) \) will not vanish (see Kaula 2000, Table 1). As can be seen, they coincide with the terms associated to the factors \( C_{20} \) and \( C_{22} \). Of particular interest for us are the latter term. Under the considerations given above, we can rewrite Eq. (66) as:

\[
\langle T^\text{tri}_{2l+2}(r) \rangle = \frac{3 G m_s}{2 a^3} (B - A) \sum_{q=-\infty}^{\infty} G_{20q}(e) \sin \left( \frac{\theta_{20q}}{2} - 2 \theta_p \right),
\]

which is the expression of the commonly known triaxiality-caused torque. Therefore, the equation for the rotational evolution of the planet – Eq. (64) – can now be rewritten in terms of the tidal and the latter torques, namely:

\[
\frac{d^2 \theta_p}{d t^2} = \frac{\langle T^\text{tide}_z \rangle + \langle T^\text{tri}_{2l+2}(r) \rangle}{C_p},
\]

which is completely analogous to Eqs. (58) – (61). The reason behind the above discussion is that Eq. (68) not only gives the equation governing the rotational dynamics of the planet, but also gives the clues for understanding the captures in spin-orbit resonances.

If we considered that the planet is an oblate body (i.e. \( A = B \)), then the triaxial torque would vanish and its long-term spin dynamics would be driven only by the secular terms of the tidal torque. In consequence, Eq. (68) could be rewritten as:

\[
\left\langle \frac{d^2 \theta_p}{d t^2} \right\rangle = 2 \frac{G m_s^2}{C_p a} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{pq=0}^{\infty} \left( \frac{R_p}{a} \right)^{2l+1} \frac{(l-m)!}{(l+m)!} m F_{\text{imp}}^2(i_s) G_{lpq}^2(e) K_1^p \left( l, x_{lpq}^s \right).
\]

### 3. Preliminary discussion

In this section we briefly explain the model we adopted and present the exoplanetary system studied in this work. Next, we discuss on the rheology of each component and on the values of physical and rheological parameters we assumed for the system we study and on the initial conditions for the numerical simulation we have performed, specially for those concerning the rotational evolution of the planet.
Table 1: Physical and orbital parameters of Kepler-21 system studied in this work

| System name   | $m_p$ [$M_\oplus$] | $R_p$ [$R_\oplus$] | $a$ [AU] | $e$ | $m_*$ [$M_\odot$] | $R_*$ [$R_\odot$] | $P_{rot}$ [days] | Reference            |
|---------------|---------------------|---------------------|----------|-----|-------------------|-------------------|-----------------|---------------------|
| Kepler 21     | 5.10                | 1.639               | 0.0427172| 0.02| 1.408             | 1.902             | 12.6            | López-Morales et al. (2016) |

3.1. Kepler-21 system

Kepler-21 star, also known as HD 179070, is a F6IV dwarf located at a distance of 352 light-years in the Lyra constellation. In Table 1 we show the orbital, planetary and stellar parameters which are relevant to our work. From the mass and radius of the planet it can be inferred that it has rocky composition, being its density approximately $6.4 \pm 2.1 \, g \, cm^{-3}$ (López-Morales et al. 2016). In Fig. 1 we show the mass-radius relationships according to several ice-mass fraction values (which is a measure of the water content of the planet) and considering a planetary composition similar to that of the Earth (Valencia et al. 2007a). It is also shown the corresponding position of some super-earths on the graph. The Earth would be placed on the origin of the graph. As can be noted, it is expected that Kepler-21b has an earth-like composition with an about 10% of water.

We have selected this system because of two main reasons, the first one is that due to the proximity of the components suggest that tidal effects are strong at least on the planet. The other one is that all the relevant physical and orbital parameters are well determined.

3.2. Model adopted

Let us recall at this point the principal assumptions we have posed before in order to build up the model we have used to carry out our study. First of all, we ignore the apsidal and nodal precessions of both orbits, that of the planet as seen from the star and that of the latter as seen from the former. This gives us the freedom to choose arbitrary values of the arguments of pericenters $\omega_s$ and $\omega_p$ and those of the longitudes of the ascending nodes $\Omega_s$ and $\Omega_p$. For simplicity, we set all of them equal to zero. Thus, we consider only the time evolution of $a$, $e$, $i_s$ and $i_p$.

Regarding the disturbing function, in our model we consider the secular terms of the tidal potential due to both the star and the planet. As we will consider low inclinations, we have pointed out before that only those terms associated with the $J_2$ and $C_{22}$ coefficient will matter. However, we will neglect the effect of the former and consider only the latter because of its noticeable influence in the vicinity of spin-orbit resonances.
In consequence, the quality function of the star $K$ both components together with those of the semimajor axis and the eccentricity of the system we have selected to apply the model conformed by a rocky planet and its host star. The reason behind we limit our study to Earth-like planets is twofold: On one hand, Due to the fact that the tidal theory we have adopted in this work is applicable to binary systems, we will then consider those worlds and, consequently, there exist a growing interest in researching if these celestial bodies offer the conditions that makes the life possible. On the other hand, if we can infer that the compositions of the studied planets are similar to that of the Earth, then we can assume that their rheology is also similar and then propose reliable models to describe its internal dissipation.

In Table 1 we show the values of the physical parameters involved in the simulations performed, i.e. the mass and the radius of both components together with those of the semimajor axis and the eccentricity of the system we have selected to apply the model described in Sect. 2. In what follows, we will first discuss on the rheology of the host stars and then on the rheology of the planets.

### 3.3. Rheology of the components

Due to the fact that the tidal theory we have adopted in this work is applicable to binary systems, we will then consider those conformed by a rocky planet and its host star. The reason behind we limit our study to Earth-like planets is twofold: On one hand, most recent developments in the field of planetary detection methods and in their accuracy have allowed the discovery of such worlds and, consequently, there exist a growing interest in researching if these celestial bodies offer the conditions that makes the life possible. On the other hand, if we can infer that the compositions of the studied planets are similar to that of the Earth, then we can assume that their rheology is also similar and then propose reliable models to describe its internal dissipation.

In Table 1 we show the values of the physical parameters involved in the simulations performed, i.e. the mass and the radius of both components together with those of the semimajor axis and the eccentricity of the system we have selected to apply the model described in Sect. 2. In what follows, we will first discuss on the rheology of the host stars and then on the rheology of the planets.

#### 3.3.1. Rheology of Sun-like stars

As was discussed by Veras et al. (2019) stars can be considered to be viscous bodies, despite the fact that they can present some magnetic rigidity. Therefore, the dominant rheological parameter is the shear viscosity $\eta_s$. From the work by Efroimsky (2012) we know that the complex compliance corresponding to viscous deformation is given by:

$$J(\chi) = -\frac{i}{\eta_s \chi}.$$  \hspace{1cm} (70)

Then, we have that its real and imaginary parts evidently are:

$$\Re \left[ J(\chi) \right] = 0 \quad \text{and} \quad \Im \left[ J(\chi) \right] = -\frac{1}{\eta_s \chi}.$$  \hspace{1cm} (71)

In consequence, the quality function of the star $K^*_{\text{I}}(l, \chi_{l,\text{mpq}})$, given in general form by Eq. (24), becomes Efroimsky(2015):

$$K^*_{\text{I}}(l, \chi_{l,\text{mpq}}) = \frac{3}{2} \left( \frac{1}{l-1} \right) \left( \frac{A_l \eta_s \chi_{l,\text{mpq}}}{1 + (A_l \eta_s \chi_{l,\text{mpq}})^2} \right)^2.$$  \hspace{1cm} (72)

#### 3.3.2. Rheology of Earth-like planets

The rheological response of rocky and icy celestial bodies like planets and satellites – including the possibility that they can have a subsurface ocean – is best modelled by the Andrade (1910) model, or even better by the Sundberg & Cooper (2010) model, as was pointed by Renaud & Henning (2018). Due to the complexity of the latter model and the fact that in general the rheological parameters are poorly constrained even for the Earth, we choose the former rheological model to describe the dissipative within the planet. The complex compliance for this rheology is Efroimsky(2012, 2015):

$$J(\chi) = J = -\frac{i}{\eta_s \chi} + J (i \chi \tau_\lambda)^{-\alpha} \Gamma (1 + \alpha) \quad \text{and} \quad J = \left[ 1 - \frac{i}{\chi \tau_M} + (i \chi \tau_\lambda)^{-\alpha} \Gamma (1 + \alpha) \right].$$  \hspace{1cm} (73)

where $\alpha$ is known as the Andrade parameter, $\Gamma$ is the gamma function, $\tau_M$ and $\tau_\lambda$ are the Maxwell and Andrade characteristic time scales, respectively. It worth to indicate that $\tau_M$ is defined as the ratio of the shear viscosity over the mean rigidity $\mu$, i.e.:

$$\tau_M = \frac{\eta_s}{\mu}.$$  \hspace{1cm} (74)

It should be noted that the first two terms in the first line of Eq. (73) correspond to a viscoelastic response which defines de Maxwell rheology, while the last term correspond to an hereditary reaction caused by dislocation process within the material under stress (Efroimsky, 2012), as was prescribed in Eq. (29). Andrade’s rheology can be applied to a wide range of materials, from rocky planets or satellites partially molten up to icy bodies, even if these bodies have a subsurface ocean. The Andrade’s parameter ranges between [0.15, 0.2] for the former kind of celestial bodies and between [0.2, 0.4] for the latter.

The real and imaginary parts of the complex compliance given by Eq. (73) are:

$$\Re \left[ J(\chi) \right] = \frac{J}{\chi \tau_M} \left[ \chi \Gamma_M + \chi^{1-\alpha} \tau_M \Gamma_\lambda \cos \left( \frac{\pi \alpha}{2} \right) \Gamma (1 + \alpha) \right],$$  \hspace{1cm} (75)

$$\Im \left[ J(\chi) \right] = \frac{J}{\chi \tau_M} \Re (\chi).$$  \hspace{1cm} (75)
and
\[
\mathcal{S} \left[ \mathcal{F}(\chi) \right] = -\frac{J}{\chi^2 \tau_M} \left[ 1 + \chi^{1-\alpha} \tau_M \tau_\alpha \sin \left( \frac{\tau_M}{2} \right) \Gamma (1 + \alpha) \right] \\
= \frac{J}{\chi^2 \tau_M} I(\chi).
\]

Then, again by virtue of Eq. (24), the quality function of the planet \( K_p^f \left( l, \chi_{\text{mpq}}^s \right) \) is given by:
\[
K_p^f \left( l, \chi_{\text{mpq}}^s \right) = -\frac{3}{2} \frac{1}{\tau_1 - 1} \frac{A_l \chi_{\text{mpq}} \tau_M I \left( \chi_{\text{mpq}}^s \right)}{\mathcal{R} \left( \chi_{\text{mpq}}^s \right) + A_l \chi_{\text{mpq}} \tau_M I^2 \left( \chi_{\text{mpq}}^s \right)}.
\]

3.4. Initial conditions and physical parameters

Apart from the values of the parameters given in Table I, we have on hand to set other some physical and rheological parameters and on the other to choose the values of the initial conditions of the orbital and rotational parameters in order to solve the differential equations of motion. In the following we show the values of the parameters we have set for each system.

With respect to the rheological parameters of the host star and the planet, we have only to set the value of the shear viscosity \( \eta_s \) of the former and the shear rigidity \( \mu \), the Maxwell time \( \tau_M \) and the Andrade’s parameter \( \alpha \) of the latter. Due to the lack of a better model, we will set \( \tau_\alpha = \tau_M \). In the first case, we take the value of the shear viscosity of the host star to be that estimated by Veras et al. (2019) for the Sun and Sun-like stars, which is \( \eta_s \approx 10^{12} \text{ Pa.s} \). In the second one, from Fig. 1 we can expect that the considered planet has a composition similar to that of the Earth and therefore we can take the corresponding values of its rheological parameters. For the Earth \( \mu = 0.8 \times 10^{10} \text{ Pa.s} \), which is the most reliable value among the rheological parameters involved. Having in mind Eq. (74) and the fact that the viscosity depends on the temperature, such that the former decreases when the latter increases, we then have that the Maxwell time is expected to decrease as the planet’s internal temperature increases. At the same time, planets internally differentiated such as the Earth, with partially molten cores, are expected to have the corresponding values of the Andrade’s parameter, i.e. \( \alpha \in [0.15, 0.2] \). Thus, due to this uncertainty we have performed numerical simulations with different values of the less constrained parameters as follows: \( \tau_M = 10, 50, 100 \text{ years and } \alpha = 0.15, 0.2, 0.3 \).

Concerning the rest of the physical parameters, the values of the mean densities and surface gravity of both the planet and the host star were computed using the data given in Table I. The maximum moments of inertia \( C_s \) and \( C_p \) of the star and the planet, respectively, are computed as:
\[
C_i = \frac{\xi_e}{m_p} R_s^2, 
\]

where \( \xi_e \) is called the moment of inertia factor. For a homogeneous solid sphere, its value is 2/5. But if the internal structure of rocky planets is considered, its value is a little lower, for example Zeng & Jacobsen (2017) showed that for both the Earth and Earth-like rocky planets, its value is approximately equal to 1/3. In consequence, we can take \( \xi_e = 2/5 \) and \( \xi_p = 1/3 \). The last physical parameter to be set is the degree of triaxiality, i.e. we need a value for \( (B - A)/C \). Such parameter can be estimated following the work by Rodríguez et al. (2012), namely:
\[
\frac{B - A}{C} \approx \frac{15}{4} \frac{m_s (R_s/a)^3}{m_p}. 
\]

Lastly, regarding the orbital parameters we have set the initial values of \( a \) and \( e \) as those given by Table I, correspondingly, as we are interested in the study of the fate of the considered system and in estimating characteristic timescales of the corresponding evolution. In this sense, we will set \( \xi_a = 5^\circ \) and \( \xi_e = 5^\circ \) in order to estimate planarization and obliquity damping time scales. Also from Table I we have computed the initial spin rate of the star. As the rotational state of the planet is completely unknown, we have explored several values in our numerical experiments. However, as we will see, it is very likely that the planet is already captured in synchronous rotation.

3.5. Semianalytical characterization of tidal evolution

In this work, we have studied the Kepler-21 exoplanetary system which are known to be formed by a planet – presumably a rocky one – and its host star. Its orbital and some physical parameters are shown in Table I.

Before performing any numerical integration, we can explore some features of the tidal evolution of the considered system. In Fig. 4 we show the graph of the angular acceleration of the planet due to the secular terms of the polar component of the tidal torque, given by Eq. (69), vs. the normalized angular velocity \( (\theta_p/n) \), where we have taken \( \tau_M = 50 \text{ yr} \) and \( \alpha = 0.2 \). It can be seen that to the right of the co-rotation regime \( (\theta_p/n = 1) \) or the 1:1 spin-orbit resonance, the acceleration is negative which means that the tidal torque tends to break the spin motion; while to the left of the same spin-orbit resonance, the angular acceleration is positive. We can also note that the probabilities of capture in higher spin-orbit resonances are very low for the current value of the eccentricity. This can be explained by the small value of the eccentricity of Kepler-21’s orbit. For very low inclinations, as the cases we will consider, and taking into account Eq. (69) along with \( l = 2, m = 2 \) and \( p = 0 \), \( \left\langle \theta_p \right\rangle \) can be expressed as:
\[
\left\langle \theta_p \right\rangle = \frac{3 G m_p^2 R_p^5}{2 C_p a^6} \sum_{q=2, q_{\text{max}}}^{q_{\text{max}}} G_{\text{pq}}^2 (a) K_p^f \left( 2, \chi_{220_p}^s \right)
\]

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easy to understand why the angular acceleration of the planet due to resonance compared with its variation around the others resonances. Eccentricity plays a key role in the possibility of capture in higher-than-synchronous resonances.

Let us suppose that initially \( \dot{\theta}_p = n \) and both the semimajor axis and the eccentricity decrease in time. Then, by virtue of the preceding discussion on the rotational evolution of the planet, we have that the latter spins-down passing through several spin-orbit resonances before arriving to the synchronization. As the spin rate evolves, by virtue of Fig. 4 we observe that from the initial state up to the 3:2 spin-orbit resonance the time derivatives of both the semimajor axis and the eccentricity are positive and thus both \( a \) and \( e \) grow in time. Then, from the 3:2 resonance up to the complete synchronization, \( e \) decreases while \( a \) continues increasing. Finally, when the planet arrives to the 1:1 spin-orbit resonance, we found by direct evaluation that

\[
\left\langle \frac{da}{dt} \right\rangle_{\text{tide}} \approx -3.21 \times 10^{-10} \text{ AU yr}^{-1}, \quad \text{and} \quad \left\langle \frac{de}{dt} \right\rangle_{\text{tide}} \approx -2.31 \times 10^{-9} \text{ yr}^{-1}.
\]

In other words, when \( \theta_p = n \) both the semimajor axis and the eccentricity decrease in time.

Analogous analysis can be done for the time evolution of the inclinations. Observing Fig. 5 we realize that \( \left\langle \frac{di}{dt} \right\rangle_{\text{tide}} \) is positive from \( \theta/n = 4 \) up to \( \theta/n = 2 \) and then becomes negative from the latter up to the 1:1 spin-orbit resonance. Instead, \( \left\langle \frac{d\dot{\theta}_p}{dt} \right\rangle_{\text{tide}} \) is negative along the entire interval except for very narrow intervals around the 2:1 and 1:1 resonances. Again, direct evaluation gives:

\[
\left\langle \frac{di}{dt} \right\rangle_{\text{tide}} \approx -0.081^\circ \text{ yr}^{-1}, \quad \text{and} \quad \left\langle \frac{d\dot{\theta}_p}{dt} \right\rangle_{\text{tide}} \approx -1.363 \times 10^{-7} \text{ yr}^{-1}.
\]
Of course, it worth to say that the preliminary results presented above depend on the values of the parameters, which were given before and were chosen in order to obtain a first estimation of the solution of the problem addressed in this work.

We close this subsection noting that as it is observed in Figs. 2, 4 and 5 all graphs has almost the same shape near each spin-orbit resonance where they have very sharp variations but differing by its amplitude and sign change from one side of the resonances to the other. This is due principally by the frequency dependence of the quality functions, as was pointed out by e.g. Makarov et al. (2012).

3.6. Characteristic time scales of dynamical evolution

We define the characteristic time scale of the evolution of an orbital element as that required for a certain change in that element to take place. Mathematically expressed, the characteristic time scale of the semimajor axis $\tau_a$, the eccentricity $\tau_e$, and the inclinations $\tau_i_s$ and $\tau_i_p$ are defined as:

$$\tau_a = (r_{\text{Roche}} - a) \left\langle \frac{da}{dt} \right\rangle^{-1},$$  \hspace{1cm} (81)

$$\tau_e = -e \left\langle \frac{de}{dt} \right\rangle^{-1},$$  \hspace{1cm} (82)

$$\tau_i_s = -i_s \left\langle \frac{di_s}{dt} \right\rangle^{-1},$$  \hspace{1cm} (83)

and

$$\tau_i_p = -i_p \left\langle \frac{di_p}{dt} \right\rangle^{-1},$$  \hspace{1cm} (84)

where $r_{\text{Roche}}$ is the Roche limit and the values of $a$ and $e$ are the current ones, while the values of $i_s$ and $i_p$ are those we have set for the numerical simulation. It can be noted that $\tau_a$ correspond to the time interval that elapses from the current value of $a$ up to the planet reaches the Roche radius, beyond which it will be completely destroyed by tidal forces. The latter is defined as (Murray & Dermott 1999):

$$r_{\text{Roche}} = R_p \left( \frac{m_p}{m_p} \right)^{\frac{1}{3}}.$$  \hspace{1cm} (85)

Of course, this definition does not take into account the rigidity of the material that compose the body but only considers self-gravity. Even so, it can be considered as an upper limit. For the Kepler-21 system, $r_{\text{Roche}} \approx 4.54 \times 10^{-3}$ AU. Replacing in Eq. (81) the corresponding values given in Table 1 and that of $\left\langle \left\langle \frac{da}{dt} \right\rangle \right\rangle$ tide, yields $\tau_a \approx 115.6 \times 10^6$ years.
Fig. 4. Evaluation of the time derivatives of the semimajor axis and the eccentricity from Eqs. (54) and (55) vs. its normalized spin rate, where we have zoomed in two neighborhoods around the 1:1 and 3:2 spin-orbit resonances.

Fig. 5. Evaluation of the time derivatives of the inclination of the planet’s and the star’s orbits reckoned from the corresponding equatorial plane of the star and of the planet, respectively, given by Eqs. (57) and (56) vs. the normalized spin rate of the latter, where we have zoomed in two neighborhoods around the 1:1 and 2:1 spin-orbit resonances.
The rest of the characteristic times, $\tau_c$, $\tau_i$ and $\tau_s$, correspond to complete circularization and planarization of the orbit with respect to both equators, respectively. Again, for Kepler-21 system we found: $\tau_c = 8.68 \times 10^6$, $\tau_i = 61.95$ and $\tau_s = 36.7 \times 10^6$ year. As the orbital decay timescale is larger than the others, when $a \approx r_{\text{Roche}}$ the orbit will be completely circularized and planarized. It worth to indicate that this preliminary results – the ones given in this and the latter Subsections – has to be obtained also by numerical integrations (see Sect. 4).

We can also compute time scales of the rotational evolution. Analogously to the definitions given before, we define the characteristic spin-down timescale of the planet by:

$$\tau_{\theta_p} = \frac{n - \dot{\theta}_p}{\dot{\theta}_p}. \quad (86)$$

In other words, this is the characteristic time for the planet to arrive to the synchronous rotation. Let us suppose again that $\dot{\theta}_p/n = 4$. Then, by direct evaluation of Eq. (63), we obtain $\dot{\theta}_p = 9.904 \text{ yr}^{-2}$ and consequently $\tau_{\theta_p} = 256$ years, which is much shorter than the corresponding timescales of orbital evolution. This result justify the analysis performed in Subsect. 3.5 where we evaluated the time derivatives of the orbital elements as a function only of the normalized spin rate of the planet.

The characteristic timescale of the stellar spin evolution can be computed in a similar way. Direct evaluation of Eq. (62) gives $\dot{\theta}_s \approx 1.9 \times 10^{-9} \text{ yr}^{-2}$. If we define the corresponding characteristic timescale as:

$$\tau_{\theta_s} = \frac{n - \dot{\theta}_s}{\dot{\theta}_s}. \quad (87)$$

and we take into account that the current spin rate of the star, computed from its rotational period, is $\dot{\theta}_s \approx 182.14$ yr$^{-1}$ then, by virtue of Eq. (57), we have that $\tau_{\theta_s} \approx 346 \text{ Gyr}$, which is nearly 25 times the age of the Universe. This indicates that tidal interaction has no appreciable effects on the stellar rotation.

### 3.7. Can the tidal effects be detected?

It may result interesting to have the possibility of contrasting the results obtained from numerical simulation with those from observations. Within the limits of this work, we can answer the question posed in the title of this subsection only partially. In recent years, great advances have been made in observational techniques that allow measurements of very good precision of physical and orbital parameters of planets orbiting stars other than the Sun. One of these quantities is the orbital period $P_{\text{orb}}$ of planets, which can be measured by observing several successive transits and is defined by:

$$P_{\text{orb}} = \frac{2\pi}{n}. \quad (88)$$

If we assume that the orbital parameters of a two body system evolve in time only due to the tides each body rises on the other – as was supposed in the last subsection –, then the orbital period will also evolve as a consequence of the time variation of the semimajor axis. This can be expressed mathematically as:

$$\frac{dP_{\text{orb}}}{dt} = -\frac{2\pi}{n^2} \frac{dn}{dt}. \quad (89)$$

From Eq. (1) we have:

$$\frac{dn}{dt} = -\frac{3}{2} \frac{na}{a^3} \frac{da}{dt}.$$

Then, Eq. (89) can be rewritten as:

$$\frac{dP_{\text{orb}}}{dt} = \frac{3\pi}{n} \frac{da}{dt}. \quad (90)$$

If we take the values of the parameters for Kepler 21 system given in Table I compute $n$ with Eq. (1) and also take the value of $\langle \frac{da}{dt} \rangle_{\text{tide}}$ given above for the time derivative of the semimajor axis and then replace them in Eq. (90) we get as result:

$$\langle \frac{dP_{\text{orb}}}{dt} \rangle_{\text{tide}} \approx -0.003879 \frac{s}{\text{yr}},$$

which is equivalent to a decrement of nearly 3.88 ms per year. In consequence, we should need more than 1000 years to measure a difference of about 4 seconds. The measured orbital period of planet Kepler-21b is 2.78578 days with an uncertainty of 0.00003 days which is equivalent to approximately 2.6 s. This result clearly shows the impossibility to detect tidal effects, at least by measuring the orbital period of the planet.
4. Numerical experiments and results

We have performed several numerical integrations in order to characterize the dynamical evolution of the system Kepler-21. First, we were interested in validating our code by verifying if the solutions of the equations of motion behave as expected in the light of the preceding discussion on the orbital and rotational evolution of the system Kepler-21, due only to the secular terms of the tidal potentials, offered in Subsect. 3.5. To that end, we have supposed that the planet was an oblate body, i.e. we have set \((B-A)/C = 0\). The rheological parameters were set as \(\tau_M = 50\) yrs and \(\alpha = 0.2\). We have also set the following initial conditions: \(\theta_i(0) = 0, \theta_s(0) = 0\), \(\theta_i(0)/n = 4.1\), \(a(0) = 0.0427172\) AU, \(e(0) = 0.02\), \(i_s(0) = 5^\circ\), \(i_t(0) = 5^\circ\). We then integrated numerically Eqs. (53), (55), (56), (57), (62) and (63) from \(t = 0\) to \(t = 1000\) yrs with an initial time step equal to the thousandth part of the current value of the orbital period \((dt = 10^{-3} P_{\text{orb}})\). In our numerical simulations we have made use of the subroutine \texttt{bsstep}\footnote{This subroutine is part of \texttt{gsl}, \url{https://www.gnu.org/software/gsl/}.} which implements the Bulirsch-Stoer integration scheme. The results, corresponding to the solid red line, are shown in Fig. 6, where the top left panel – labeled with (a) – shows the time evolution of the normalized spin rate of the planet. As can be observed, the latter slows down its rotation speed monotonically. From the initial conditions given before, it can be verified that the planet arrives to the 2:1, 3:2 and 1:1 spin-orbit resonances at \(t \approx 165\), \(t \approx 200\) and \(t \approx 231\) yrs., respectively, in good agreement with the estimated planet’s spin-down timescale discussed in Subsect. 3.6.

As another numerical check of our code, in panel 6b we have included a graph of the angular acceleration of the planet due to the secular terms of the tidal torque given by direct evaluation of Eq. (69) – dotted green line – and the corresponding ones obtained from the numerical integration (red dots). The apparent lack of point is because not all the values resulting from the integration were written into the output file. We observe good agreement between both sets of values. This comparison was helpful when we were searching an optimal initial integration step size.

Concerning the orbital evolution, panels 6c and 6d of Fig. 6 show the evolution of the semimajor axis and the eccentricity relative to its initial values, respectively. We can observe that both \(a\) and \(e\) evolve as expected from the corresponding discussion in Subsect. 3.5 and that, after the planet is trapped in 1:1 spin-orbit resonance, both elements evolve linearly. In what respects to time evolution of \(i_s\) and \(i_t\), we can see in panels 6e and 6f that they also evolve as expected from the aforementioned discussion. On one hand, the latter one decreases along the whole time interval of the integration, with some changes in its time derivative when the planet crosses \(a\) or gets captured in the 1:1 spin-orbit resonance. On the other hand, the logarithm of former one increases from its initial value up to the 2:1 spin-orbit resonance and then starts decreasing nearly linearly up to the 1:1 resonance, from which continues decreasing linearly but with a grater slope until \(t \approx 800\) years.

We then have performed a linear fit of the graphs of \(a\) and \(e\) vs. \(t\) within the interval \(t \in [232, 1000]\) years as another quantitative test of our code. We have obtained:

\[
\begin{align*}
\left\langle \frac{da}{dt} \right\rangle_{\text{fitted}}^{\text{tide}} & \approx -3.17 \times 10^{-10} \text{AU yr}^{-1}, \\
\left\langle \frac{de}{dt} \right\rangle_{\text{fitted}}^{\text{tide}} & \approx -2.31 \times 10^{-9} \text{yr}^{-1}
\end{align*}
\]

As can be seen, these results are consistent with those given in Subsect. 3.5. We have also performed a linear fit of \(\log_i\) evolution within the time interval \(t \approx 400\) yr on. The linear fit of the data over the corresponding time interval gives:

\[
\left\langle \frac{d\log_i}{dt} \right\rangle_{\text{fitted}}^{\text{tide}} \approx -5.88 \times 10^{-8} \circ \text{yr}^{-1},
\]

which is significantly different from the value obtained in Subsect. 3.5. However, we have to take into account that \(i_s\) has changed noticeably from its initial value becoming practically zero. If we re-evaluate the time derivative of \(i_p\) considering \(i_s = 1 \times 10^{-6} \circ\) we obtain:

\[
\left\langle \frac{d\log_i}{dt} \right\rangle_{\text{fitted}}^{\text{tide}} \approx -6.34 \times 10^{-8} \circ \text{yr}^{-1}.
\]

We then realize that the discrepancies are lesser than before. Even so, if we compute the corresponding timescale we found that \(\tau_{i_p} \approx 78 \times 10^6\) yr with the value of \(\frac{d\log_i}{dt} \rangle_{\text{fitted}}^{\text{tide}}\) and \(\tau_{i_s} \approx 85 \times 10^6\) yr with \(\left\langle \frac{d\log_i}{dt} \right\rangle_{\text{fitted}}^{\text{tide}}\), which are, at least, of the same order of magnitude.

We have also performed a linear fit of \(\log_i\) within the time interval where the graph behaves linearly, i.e. \(t \in [232, 800]\), in order to estimate the time derivative of \(i_s\). Thus, in the interval we have that \(\log_i = b t + h\). Then, the relationship between the fitting parameter \(b\) and \(\left\langle \frac{d\log_i}{dt} \right\rangle_{\text{fitted}}^{\text{tide}}\) is given by:

\[
\frac{d}{dt} \log_i = \frac{1}{\tau_i} \left\langle \frac{d\log_i}{dt} \right\rangle_{\text{fitted}}^{\text{tide}} = b.
\]

From the linear fit we get \(b \approx -0.0162\) yr\(^{-1}\) and this implies that

\[
\left\langle \frac{d\log_i}{dt} \right\rangle_{\text{fitted}}^{\text{tide}} \approx -0.081^\circ \text{yr}^{-1}
\]

which is in good agreement with the value computed in Subsect. 3.5. From these results we can also see that we will obtain approximately the same values of the orbital time scales for \(a\), \(e\) and \(i_s\).
Fig. 6: Dynamical evolution of the Kepler-21 system assuming that the planet is an oblate body (red line) and assuming $\tau_M = 50$ yr and $\alpha = 0.2$. Panel (a) shows the time evolution of the normalized spin rate of the planet, while panel (b) shows its angular acceleration due to the secular terms of the tidal torque vs. its normalized spin rate obtained from numerical integration (red points) and from direct evaluation of Eq. (69) (dotted green line). Panels (c) and (d) show the time evolution of the semimajor axis and the eccentricity, respectively, with respect to their initial values. Panel (e) show the time evolution of the logarithm of the inclination of the star’s orbit as seen from the planet, while panel (f) shows the corresponding evolution of the inclination of the latter’s orbit as seen form the former with respect to its initial value. We have rescaled the values of $\langle \dot{\theta}_p \rangle$, $a - a_0$, $e - e_0$ and $i_p - 5^\circ$ for clarity.
Table 2: Values of the triaxiality degree \((B - A)/C\) and rheological parameters, the Maxwell time \(\tau_M\) and the Andrade parameter \(\alpha\), used in the first set of integrations along with the estimated rates of change of the orbital elements, obtained by direct evaluation of the corresponding equations of motion due only to tidal interaction, and their corresponding characteristic timescales. The first column identifies each subset of parameters (characterized by the values of \((B - A)/C, \tau_M\) and \(\alpha\)) with the corresponding graph of the dynamical evolution shown in Fig. 7.

| Subset No. | \((B - A)/C\) | \(\tau_M\) [yr] | \(\alpha\) | \(\langle \frac{d\theta}{dt} \rangle_{\text{tide}}\) \([\times 10^{-10} \text{AU yr}^{-1}]\) | \(\langle \frac{d\alpha}{dt} \rangle_{\text{tide}}\) \([\times 10^{-9} \text{yr}^{-1}]\) | \(\langle \frac{dC}{dt} \rangle_{\text{tide}}\) \([\times 10^{-2} \text{yr}^{-1}]\) |
|-----------|---------------|---------------|--------|----------------------------------|----------------------------------|----------------------------------|
| 1         | \(5 \times 10^{-5}\) | 50            | 0.2    | \(-3.32103\)                    | \(114.930\)                       | \(-2.30499\)                     | \(-8.07088\)                     |
| 2         | 10            | 50            | 0.2    | \(-3.38069\)                    | \(112.901\)                       | \(-2.94879\)                     | \(-10.6585\)                     |
| 3         | 0.15          | 50            | 0.2    | \(-3.35710\)                    | \(113.695\)                       | \(-2.69470\)                     | \(-9.63583\)                     |
| 4         | \(6.99 \times 10^{-4}\) | 50            | 0.2    | \(-3.32103\)                    | \(114.930\)                       | \(-2.30499\)                     | \(-8.07088\)                     |
| 5         | 0.3           | 50            | 0.2    | \(-3.23521\)                    | \(117.978\)                       | \(-1.37866\)                     | \(-4.34871\)                     |
| 6         | 100           | 50            | 0.2    | \(-3.29946\)                    | \(115.681\)                       | \(-2.07231\)                     | \(-7.13567\)                     |
| 7         | \(1 \times 10^{-3}\) | 50            | 0.2    | \(-3.32103\)                    | \(114.930\)                       | \(-2.30499\)                     | \(-8.07088\)                     |

Next we have estimated the value of the triaxiality degree using Eq. (79). From the values given in Table 1 we have obtained: \((B - A)/C \approx 6.99 \times 10^{-4}\) and we used this value in the numerical integrations including the permanent-triaxiality-caused disturbing potential. We have divided the integrations into three different sets, each of them characterized by the initial conditions. The first one consisted of seven integrations using the same initial conditions as those used in the latter (in which we considered only the tides each body rises on the other) and over the same time interval, but with different values of \((B - A)/C\), the computed one, \((B - A)/C = 5 \times 10^{-5}\) – the estimated value for the Earth – and \((B - A)/C = 1 \times 10^{-3}\) and also varying the values of the rheological parameters, i.e. those of the Maxwell time \((\tau_M = 10, 50, 100 \text{ yrs.})\) and of the Andrade parameter \((\alpha = 0.15, 0.2, 0.3)\), as shown in Table 2. In the second set we performed five numerical integrations by varying the initial value of the planet’s spin rate, namely: \(\theta_p(0)/n = 4.1, 2.1, 1.6, 1.1, \text{and 1.0}\) and using \((B - A)/C = 6.99 \times 10^{-4}\), \(\tau_M = 50 \text{ yrs.}\) and \(\alpha = 0.2\). In the third one we performed ten numerical integration by varying the initial value of the planet’s angle of rotation in a similar way to that used for computing capture probabilities in spin-orbit resonances with the brute-force method (see Makarov et al. 2012). \(\theta_p(0) = 180^\circ (j/10)\) where \(j = 0, 1, \ldots, 9\), while its initial spin rate was set as \(\theta_p(0)/n = 4.1, \text{and with the same values of the other parameters and initial conditions.}\)

The results of the first set of integrations are shown in Fig. 7. At first sight, it can be observed that the variation of the triaxiality degree and of the rheological parameters has consequences more appreciable on the time evolution of the eccentricity and of the inclination of the star’s orbit as seen from the planet than on the other elements and the spin rate of the planet. We can see also that the time evolution of the orbital elements is nearly linear after the planet is captured in the 1:1 spin-orbit resonance. For that reason, we have performed a linear fit on the data obtained from its corresponding numerical integration over the time span in which each element evolve linearly, of course the exception is \(i_t\) whose logarithm evolves linearly along a time interval somewhat shorter than that of the other elements. The values of the slopes obtained correspond to each element’s time derivative – \(\langle \frac{d\theta}{dt} \rangle_{\text{tide}}, \langle \frac{d\alpha}{dt} \rangle_{\text{tide}}\), and \(\langle \frac{dC}{dt} \rangle_{\text{tide}}\) –, and are shown in Table 3 along with the corresponding characteristic timescales – \(\tau_{\theta}, \tau_{\alpha}\), and \(\tau_C\). The errors associated to the linear fit affects the last figure of each corresponding value. These results can be compared with those obtained from direct evaluation of the equations of motion considering only the tidal effects – \(\langle \frac{d\theta}{dt} \rangle_{\text{tide}}, \langle \frac{d\alpha}{dt} \rangle_{\text{tide}}, \text{and} \langle \frac{dC}{dt} \rangle_{\text{tide}}\) and the corresponding characteristic timescales given in Table 2. Given the strong dependence of \(\langle \frac{d\theta}{dt} \rangle_{\text{tide}}\) on \(i_t\), as was shown above, it will not be possible to obtain reliable values of the corresponding characteristic timescale as we avoid both computing its value from the equation of motion considering only the tidal interaction and fit the data obtained from numerical integrations. In panel 7b we show the graph of the resulting acceleration of the planet due to the secular terms of the tidal torque obtained from the numerical integrations with that obtained by direct evaluation of Eq. (69) for comparison.

Fig. 8 shows the results of the second and third sets of integrations. As can be seen, the orbital and rotational evolution is similar to that of Fig. 6. In panel 8a it can be noted that the planet result completely captured in the 1:1 spin-orbit resonance at different instants of time. Concerning the orbital evolution (panels 8b, 8c, and 8d), only the instants of time at which the planet pass through or is trapped in a spin-orbit resonance is affected. It can be observed in panel 8d which correspond to the results obtained from the third set of integrations, that over the time interval where the eccentricity evolves linearly, if we perform a linear fit of the data, we find that the slopes are all equal while only the point at which each line intercept the vertical axis is affected. Also as result of third set of integrations, we found that the normalized spin rate of the planet, \(a, e, \log (i_t), \text{and} i_t\) evolve practically in the same way as shown in Fig. 6 the differences being negligible. For that reason we omitted showing the corresponding graphs.
Fig. 7: Dynamical evolution of the Kepler-21 system as result of the first set of numerical integrations, using the parameters listed in Table 2. In panel (a) we show the time evolution of the normalized spin rate of the planet. Panel (b) shows the angular acceleration of the planet due to the secular terms of the tidal torque obtained from numerical integration and from direct evaluation of Eq. (69) versus its normalized spin rate. Panels (c) and (d) show the time evolution of the semimajor axis and the eccentricity with respect to their initial values. Panel (e) show the time evolution of the logarithm of the inclination of the star’s orbit as seen from the planet, while panel (f) shows the corresponding evolution of the inclination of the latter’s orbit as seen form the former with respect to its initial value. We have rescaled the values of $a - a(0)$, $e - e(0)$ and $i_p - 5^\circ$ for clarity. It is also shown the graph that corresponds to dynamical evolution assuming that the planet is oblate for comparison.
Fig. 8: Dynamical evolution of the Kepler-21 system as result of the second and third set of numerical integrations. In panels (a) we show the time evolution of the normalized spin rate of the planet. Panels (b) and (c) show the time evolution of the semimajor axis and the eccentricity with respect to their initial values. Panel (e) show the time evolution of the logarithm of the inclination of the star’s orbit as seen from the planet, while panel (f) shows the corresponding evolution of the inclination of the latter’s orbit as seen form the former with respect to its initial value. All of them correspond to the second set of integrations, while panel (d) correspond to the third one. We have rescaled the values of $\langle \dot{\theta}_p \rangle$, $a - a(0)$, $e - e(0)$ and $i_p - 5^\circ$ for clarity. It is also shown the graph that corresponds to dynamical evolution assuming that the planet is oblate for comparison.
Table 3: Values of the slopes obtained from linear fit of the time evolution of the orbital elements performed over the time span along which the planet is captured in the 1:1 spin-orbit resonance, corresponding to each subset of parameter’s values given by the second, third and fourth columns of Table 2. For each value of the time derivative of each orbital element its corresponding characteristic timescale is also shown.

| Subset No. | $\dot{a}/a$ [×10^{-10} AU yr^{-1}] | $\dot{i}/i$ [×10^{-9} yr^{-1}] | $\dot{e}/e$ [×10^{-2} yr^{-1}] |
|------------|----------------------------------|-------------------------------|-------------------------------|
|            | $\tau_{a}^{fit}$ [×10^6 yr]     | $\tau_{i}^{fit}$ [×10^6 yr]   | $\tau_{e}^{fit}$ [yr]         |
| 1          | $-3.17014$                      | $-2.31608$                    | $-8.0762$                     |
|            | $120.4$                         | $8.64$                        | $61.91$                       |
| 2          | $-3.18077$                      | $-2.96671$                    | $-10.6633$                    |
|            | $120.0$                         | $6.74$                        | $46.89$                       |
| 3          | $-3.17624$                      | $-2.70373$                    | $-9.6431$                     |
|            | $120.2$                         | $7.40$                        | $51.85$                       |
| 4          | $-3.16993$                      | $-2.31661$                    | $-8.0715$                     |
|            | $120.4$                         | $8.63$                        | $61.95$                       |
| 5          | $-3.15433$                      | $-1.38762$                    | $-4.3544$                     |
|            | $121.0$                         | $14.42$                       | $114.83$                      |
| 6          | $-3.16593$                      | $-2.08284$                    | $-7.1329$                     |
|            | $120.6$                         | $9.61$                        | $70.09$                       |
| 7          | $-3.18699$                      | $-2.31135$                    | $-8.0737$                     |
|            | $119.7$                         | $8.65$                        | $61.93$                       |

5. Final discussion and Conclusions

In this work we have studied the dynamical evolution of the Kepler-21 system considering the perturbations due to the secular terms of the tidal potentials of both the star and the planet and the short period terms of the permanent-triaxiality-caused potential of the latter. We were able to compute characteristic time scales of orbital evolution and have evaluated the possibility that the tidal effects can be detected by measuring the orbital period of Kepler-21b.

There are several topics that deserve particular attention. Let us consider first the results shown in Fig. 7 and in Tables 2 and 3. The time span for the planet to reach complete synchronization depends more strongly on the Maxwell time than on the Andrade parameter. This can also be verified by observing Fig. 7b. Then, after the planet is captured in synchronous rotation, the orbital elements evolve linearly in time but with different rates. The time derivatives of the semimajor axis and the inclination of the planet’s orbit as seen from the star are less affected by the changes mentioned before than those of the eccentricity and of the inclination of the star’s orbit as seen from the planet. We can observe that, as general tendency, the larger the Maxwell time and the larger the Andrade parameter the smaller the value of the time derivatives. Varying the triaxiality degree has no influence on the dynamical evolution.

In concordance with what we have pointed out in the last paragraph concerning the rotational evolution of Kepler-21b, we observe that the angular acceleration of the studied planet due to the secular terms of the tidal force depends more strongly on the Maxwell time than on the Andrade parameter, as can be observed in Fig. 7b. If the rigidity were held constant, then a shorter Maxwell time would correspond to a less viscous planet. We can notice in the same figure that for $\tau_M = 10$ yr the tidal angular acceleration becomes nearly proportional to the frequency, which reminds the constant phase-lag model (Singer 1968).

The estimation of the time derivatives of the orbital elements and the corresponding characteristic timescales produced considering only the tidal interaction turned out to be very similar to the ones obtained from numerical integrations where we considered both the tidal and the permanent-triaxiality perturbations. Thus, in principle, one could obtain nearly all the information concerning the characterization of orbital evolution of the system simply by evaluating the element’s time derivatives setting $\dot{\theta}_p = n$ as we have done in Sect. 4 at least for $a$, $e$ and $i_o$.

The initial spin rate does not have a major impact on the dynamics as it only affects the instant at which the planet arrives to the 1:1 spin-orbit resonance as Fig. 5a shows. Concerning the variation of the initial angle of rotation only affects the vertical level of the linear evolution of the eccentricity while the planet is captured in the 1:1 resonance leaving the slopes unchanged, as can be observed in Fig. 5d. Therefore, the dynamical evolution of the system turns out to be practically independent of the rotational initial conditions.

Finally, we found that tidal effects on the stellar rotation and on the time variation of the planet’s orbital period are negligible.

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Appendix A: “Modified” inclination functions and eccentricity functions

The inclination functions are defined by (Gooding & Wagner 2008):

\[ F_{\text{incl}}(i) = \frac{(l + m)!}{2^{l}(l + k)!} \sum_{j} (-1)^{j} \binom{l + k}{j} \binom{l - k}{l - m - j} \cos^{2l-m-k-2j} \left( \frac{i}{2} \right) \sin^{m+k+2j} \left( \frac{i}{2} \right) \]

(A.1)

where the limits of summation are max(0, k - m) and min(l - m, l + k). As a matter of convenience, this expression can be rewritten as a function of the variable \( x = \cos \theta \) defined above instead of the inclination solely. The result is:

\[ F_{\text{incl}}(x) = \frac{(l + m)!}{2^{l}(l + k)!} \left( \frac{1 + x}{2} \right)^{\frac{n}{2}} \left( \frac{1 - x}{2} \right)^{\frac{m}{2}} P_{l-m}^{m-k,m+k}(x) \]

(A.2)

where \( P_{n}^{\alpha,\beta}(x) \) are the Jacobi’s polynomial which are defined as (Szegö 1939):

\[ P_{n}^{\alpha,\beta}(x) = \sum_{m=0}^{n} \left( \frac{n + \alpha}{n - m} \right) \left( \frac{n + \beta}{n + m} \right) \left( \frac{x - 1}{2} \right)^{m} \left( \frac{x + 1}{2} \right)^{n-m} \]

(A.3)

The Eq. (A.2) is the one we have used in our numerical simulations and we call the \( F_{\text{incl}}(x) \) modified inclination function just to differentiate them from the traditional inclinations functions \( F_{\text{incl}}(i) \) defined by Kaula (1961). This functions are related by

\[ F_{\text{incl}}(i) = (-1)^{\frac{1}{2}(l-m+1)} F_{\text{incl}}(x) \]

where \( E(x) \) is the entire function (Gooding & Wagner 2010).

In what respects to the eccentricity functions, they are identical to the well known Hansen’s coefficients, \( X_{k}^{n,m}(e) \), in which general are defined as:

\[ \frac{r^{e}}{a} \exp(i m \nu) = \sum_{k} X_{k}^{n,m}(e) \exp(i k M) \]

(A.4)

where \( n, m \) and \( k \) are integers, \( r \) is the distance, \( \nu \) is the true anomaly and \( i = \sqrt{-1} \). For low to moderate eccentricities they can be computed as a power series of the eccentricity (Hughes 1981):

\[ X_{k}^{n,m}(e) = e^{m-d} \sum_{j=0}^{\infty} N_{r,s}^{m} e^{2j} \]

(A.5)

where \( r = j + \max(0, k-m), s = j + \max(0, m-k) \) and \( N_{r,s}^{m} \) are known as the Newcomb operators which can be computed recursively by (Proulx & McClain 1988):

\[ 4(r + s)N_{r,s}^{m+n} = 2(2m - n) N_{r-1,s}^{m,n+1} - 2(2m + n) N_{r,s-1}^{m,n+1} + (m - n) N_{r-2,s}^{m,n+2} - (m + n) N_{r,s-2}^{m,n+2} + 2(2r + 2s - 4 - n) N_{r-1,s-1}^{m+1,n+1} \]

(A.6)

If \( s > r \) then \( N_{r,s}^{m} = 0 \). The recursion is initialized by considering that \( N_{0,0}^{m,n} = 0 \) if \( r < 0 \) or \( s < 0 \) and by \( N_{0,0}^{0,m} = 1 \).

The relationship between Kaula’s eccentricity functions, \( G_{ipq}(e) \), and the Hansen coefficients is given by:

\[ G_{ipq}(e) = X_{(i+1)j-l-2p}(e) \]

(A.7)

Using Eq. (A.7) and Eq. (A.5), the \( G_{ipq}(e) \) can be computed by means of:

\[ G_{ipq}(e) = \sum_{j=0}^{\infty} N_{r,s}^{(i+1)l-l-2p} e^{2j+n|q|} \]

(A.8)

where \( r \) and \( s \) are given now by: \( r = j + \max(0, q) \) and \( s = j + \max(0, -q) \), respectively.

Of course, both the series in Eq. (A.4) and the one in Eq. (A.8) must be truncated to some order in the eccentricity to make the practical computation possible. This implies that for each pair of values of \( l \) and \( p \), which fixes the values of \( n = -(l + 1) \) and \( m = l - 2p + 1 \), it is necessary to determine a maximum value of \( k \) or, equivalently a maximum value of \( q \), because \( k = l - 2p + q \), say \( q_{\text{max}} \). Eq. (A.4) tell us that the function on the left hand side is expanded in Fourier series whose coefficients are the \( X_{k}^{n,m}(e) \)’s. So, one has to wonder how many terms of the Fourier series one has to keep in order to obtain a reliable representation of the function \( (r/a)^{e} \exp(i m \nu) \). To answer that question, we have implemented Eqs. (A.5) and (A.6) in a Fortran code in order to be able to evaluate both the right and the left hand sides of Eq. (A.4) and compute the root mean square (RMS) over the interval \([-\pi, \pi]\). The value of the RMS is used as an error control, i.e. for each set of values of \( l \leq l_{\text{max}}, p \) (with \( |p| \leq l \), \( e \), an initial trial value of \( q_{\text{max}} \) and a given tolerance – which we have set equal to \( 10^{-6} \) – the program we have written computes the Hansen coefficients corresponding to each value of \( q \), with \( |q| \leq q_{\text{max}} \) and verifies if the value of the RMS is less than the given tolerance. If the condition is not satisfied, the the code automatically increases the value of \( q_{\text{max}} \) by one and repeats the computation and performs the verification once again until the condition is satisfied. We have found that at least up to \( e = 0.2 \) it is safe to take \( q_{\text{max}} = 12 \), which is the value we have adopted.