LIMITS OF SPECIAL WEIERSTRASS POINTS

C. Cumino\textsuperscript{a}, E. Esteves\textsuperscript{b} and L. Gatto\textsuperscript{a}\textsuperscript{1}

\textsuperscript{a} Dipartimento di Matematica, Politecnico di Torino, C.so Duca degli Abruzzi 24, 10129 Torino, Italy

\textsuperscript{b} Instituto Nacional de Matemática Pura e Aplicada, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil

1. INTRODUCTION

1.1. (Our main result) Let $X \cup_P Y$ be the union of two general connected, smooth, nonrational curves $X$ and $Y$ intersecting transversally at a point $P$. Assume that $P$ is a general point of $X$ or of $Y$. Our main result is Theorem 3.4, which, in a simplified way, says:

Let $Q \in X$. Then $Q$ is the limit of special Weierstrass points on a family of smooth curves degenerating to $C$ if and only if $Q \neq P$ and either of the following conditions hold: $Q$ is a special ramification point of the linear system $|K_X + (g_Y + 1)P|$, or $Q$ is a ramification point of the linear system $|K_X + (g_Y + 1 + j)P|$ for $j = \pm 1$ and $P$ is a Weierstrass point of $Y$.

Above, $g_Y$ stands for the genus of $Y$ and $K_X$ for a canonical divisor of $X$.

As an application, we use Theorem 3.4 to recover in a unified way computations made by Diaz and Cukierman of certain divisor classes in the moduli spaces of stable curves; see Subsections 1.2 and 1.3 and Sections 5 and 6.

1.2. (Motivation) In order to understand how the above result fits in the literature on the subject, we must recall that in the last two decades several papers on limits of Weierstrass points and linear series on stable curves appeared. The investigations about these topics were initially aimed to prove existence theorems (about, e.g., distinguished linear series on smooth curves) or to do enumerative geometry, in the sense of [15], on the moduli space of genus-$g$ stable curves $\overline{M}_g$. For instance, in the beginning of the eighties, Harris and Mumford [12] proved that the moduli space $\overline{M}_g$ is of

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general type for \( g \) odd and \( g \geq 23 \), doing computations on \( \text{Pic}(\overline{\mathcal{M}}_g) \), the Picard group of the moduli functor.

The same techniques were successfully used by Diaz [6] to compute the class \( \overline{E}_{g,-1} \) (there named \( D_{g-1} \)) of the closure of the locus of curves having an exceptional (here called special) Weierstrass point of type \( g-1 \); see Subsection 5.1 for a precise definition. A Weierstrass point \( Q \) on a smooth curve of genus \( g \) is said to be of type \( g-1 \) if \( \dim |(g-1)Q| \geq 1 \), and of type \( g+1 \) if \( \dim |(g+1)Q| \geq 2 \). Diaz computed the class \( \overline{E}_{g,-1} \) by intersecting it with certain test curves entirely contained in the boundary of \( M_g \) in \( \overline{M}_g \). This way he got relations among the coefficients of the expression of \( \overline{E}_{g,-1} \) in terms of generators of \( \text{Pic}(\overline{\mathcal{M}}_g) \). These test curves were induced by one-parameter families \( F_i \to X_i \) of curves given as follows: start with a general smooth curve \( X_i \) of genus \( g-i \), for each \( i \in \{1, \ldots, g-1\} \), and a general smooth pointed curve \( (Y_i, B_i) \) of genus \( i \); then the fiber \( (F_i)_P \) over \( P \in X_i \) is \( X_i \cup P Y_i \), the point \( B_i \in Y_i \) being identified with \( P \in X_i \). This can be seen as a curve in \( \overline{M}_g \) via a nonconstant map \( \gamma_i : X_i \to \overline{M}_g \).

The crux of Diaz’s method was to evaluate \( \int_X \gamma_i^* \overline{E}_{g,-1} \), which amounts to knowing, with multiplicities, for how many pairs \( (P, R) \) with \( P \in X_i \) and \( R \in X_i \cup P Y_i \) there is a family of smooth curves degenerating to \( X_i \cup P Y_i \) with Weierstrass points of type \( g-1 \) converging to \( R \). This was done in [6] by using the theory of admissible coverings introduced and developed in [12]. So half of our Theorem 3.4 is in [6].

After Diaz’s work, it was natural to ask what the limits of special Weierstrass points of type \( g+1 \) are, the other half of Theorem 3.4. In fact, soon afterwards, Cukierman [3] computed the class \( \overline{E}_{g,1} \) of the closure of the locus of curves having a Weierstrass point of type \( g+1 \); see Subsection 5.1. However, his method was not based on test curves, but on a Hurwitz formula with singularities. (He used Diaz’s result as well.) Also, the theory of admissible coverings could not be effectively used, as the condition defining \( \overline{E}_{g,1} \) is not about the existence of a pencil, but of a net. Of course, once we have an expression for \( \overline{E}_{g,1} \) in terms of the generators of \( \text{Pic}(\overline{\mathcal{M}}_g) \), we can evaluate it along the \( \gamma_i \). But we cannot infer what the limits of Weierstrass points of type \( g+1 \) on \( X_i \cup P Y_i \) are just from their number.

Our Theorem 3.4 fills this gap. To show the “only if” part of it is not hard. To show the “if” part we use limit linear series on two-parameter families of curves, instead of admissible coverings.

1.3. (Application) Our Theorem 3.4 can be used to compute the classes \( \overline{E}_{g,-1} \) and \( \overline{E}_{g,1} \) in a unified and conceptually simpler way. Also, we do not need to worry about multiplicities, an usual nuisance of the method of test curves.

In brief, here is how. First of all, we consider another divisor class on \( \overline{M}_g \), the class \( \overline{SW}_g \) of the closure of the locus of curves having a special Weierstrass point, either of type \( g-1 \) or of type \( g+1 \); see Subsection 6.1 for
a more precise definition. It turns out that $\overline{SW}_g$ is much easier to compute. An expression for it, in terms of the generators of $\text{Pic}(\mathcal{M}_g)$, appeared already in [10], though there are multiplicity issues due to the method of test curves.

Here we compute $\overline{SW}_g$ in Theorem 5.3 directly, by intersecting $\overline{SW}_g$ with a general curve in $\mathcal{M}_g$. No multiplicity issues arise. Of vital importance in this computation is Theorem 4.2, which essentially describes the limits of Weierstrass points on a general irreducible uninodal curve. This description is much finer than the one found in [9], Thm. A2.1, p. 60, for instance. For the proof of Theorem 4.2 we use the theory of limit linear series for curves that are not of compact type, developed in [8].

Then we show that $\overline{SW}_g = E_{g,-1} + E_{g,1}$. This follows from our Proposition 6.1. This is something to be expected, from a purely set-theoretic point of view, but nevertheless, because of multiplicity issues, is not immediate and had to be proved.

Now we use the test curves given by the $\gamma_i$. Having the expression for $\overline{SW}_g$ allows us to compute $\int_X \gamma_i^* \overline{SW}_g$, which gives us the sum

$$\int_X \gamma_i^* E_{g,-1} + \int_X \gamma_i^* E_{g,1}$$

for each $i = 1, \ldots, g - 1$. For each $j = -1, 1$, let $e_{i,j}$ denote the number of pairs $(P, R)$ with $P \in X_i$ and $R \in X_i \cup P Y_i$ such that there is a family of smooth curves degenerating to $X_i \cup P Y_i$ with Weierstrass points of type $g + j$ converging to $R$. Theorem 3.4 tells us what these pairs are. Their number is computed in [4], Thm. 5.6. Then

$$(1) \quad \int_X \gamma_i^* E_{g,j} \geq e_{i,j}.$$ 

In principle, the inequality may be strict because of multiplicity issues. However, a simple computation yields

$$\int_X \gamma_i^* \overline{SW}_g = e_{i,-1} + e_{i,1}.$$ 

Thus equality holds in (1). From this equality, for each $j = -1, 1$ and each $i = 1, \ldots, g - 1$, the classes $E_{g,-1}$ and $E_{g,1}$ are computed in the usual way, like in [6]; see Subsection 6.2 for more details.

1.4. (Layout) In Section 2 we present a few preliminaries on ramification schemes, deformations of curves and limit linear series. In Section 3 we describe limits of special Weierstrass points on reducible uninodal curves, and in Section 4 we describe limits of Weierstrass points on irreducible uninodal curves. In Section 5 we compute $\overline{SW}_g$ and in Section 6 we compute $E_{g,-1}$ and $E_{g,1}$.

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2. Preliminaries

2.1. (Ramification points) A (nodal) curve is a connected, reduced, projective scheme of dimension 1 over \( \mathbb{C} \) whose only singularities are nodes, i.e. ordinary double points. The canonical sheaf, or dualizing sheaf of a curve \( C \) will be denoted \( \omega_C \). By the hypothesis on the singularities of \( C \), the sheaf \( \omega_C \) is a line bundle. The (arithmetic) genus of \( C \), i.e. \( h^0(C, \omega_C) \), will be denoted \( g_C \).

Let \( C \) be a smooth curve, and \( V \) a linear system of dimension \( r \) of sections of a line bundle \( L \) on \( C \). For each \( P \in C \) and each nonnegative integer \( a \), let \( V(-aP) \subseteq V \) be the linear subsystem of sections of \( V \) vanishing at \( P \) with multiplicity at least \( a \). We call \( P \) a ramification point of \( V \) if \( \dim V(-rP) \geq 1 \); otherwise we call \( P \) an ordinary point of \( V \). A ramification point \( P \) of \( V \) is said to be special of type \( r-1 \) if \( \dim V(-(r-1)P) \geq 2 \), and special of type \( r+1 \) if \( \dim V(-(r+1)P) \geq 1 \). A special ramification point of \( V \) is a special ramification point of type \( r-1 \) or \( r+1 \).

The orders of vanishing at \( P \) of the sections of \( L \) in \( V \) can be ordered increasingly. We call this increasing sequence the order sequence of \( V \) at \( P \). The point \( P \) is ordinary (resp. a Weierstrass point) if \( P \) is an ordinary point (resp. a ramification point) of the canonical system, i.e. the complete system of sections of \( \omega_C \).

2.2. (Ramification schemes) Let \( p: X \to S \) be a smooth, projective map of schemes whose fibers are curves. For each integer \( i \geq 0 \), and each invertible sheaf \( L \) on \( X \), let \( J^i_p(L) \) denote the relative sheaf of jets, or principal parts, of order \( i \) of \( L \). The sheaf \( J^i_p(L) \) is locally free of rank \( i+1 \). Also, there is a natural evaluation map, \( e_i: p^*p_*L \to J^i_p(L) \), which locally, after trivializations are taken, is represented by a Wronskian matrix of functions and their derivatives up to order \( i \).

There is a natural identification \( J^0_p(L) = L \). Furthermore, for each integer \( i > 0 \) there is a natural exact sequence of the form:

\[
0 \to \omega_p^{\otimes i} \otimes L \to J^i_p(L) \to J^{i-1}_p(L) \to 0,
\]

where \( \omega_p \) is the relative dualizing sheaf of \( p \). The truncation maps \( r_i \) are compatible with the evaluation maps, that is, \( e_{i-1} = r_i \circ e_i \) for each \( i > 0 \).

Sheaves satisfying the same properties as the \( J^i_p(L) \) above can be constructed if \( p \) is only a flat, projective map whose fibers are (nodal) curves. In addition, they coincide on the smooth locus of \( p \) with the corresponding sheaves of jets. These sheaves appeared in \([9\), \([7\), \([13\) and \([14\). We will use the same notation, \( J^i_p(L) \), for these sheaves.
So, more generally, let \( p : \mathcal{X} \to S \) be a flat, projective map whose fibers are curves of genus \( g \). Let \( \mathcal{L} \) be an invertible sheaf on \( \mathcal{X} \), and \( \nu : \mathcal{V} \to p_*\mathcal{L} \) any map from a locally free sheaf \( \mathcal{V} \) of constant positive rank, say \( r + 1 \) for a certain integer \( r \geq 0 \). For each integer \( i \geq 0 \), consider the natural evaluation map,

\[
u_i : p^*\mathcal{V} \to p^*p_*\mathcal{L} \to J_i^1(\mathcal{L}).\]

We call the degeneracy scheme of \( \nu_{r+j} \), for \( j = -1, 1 \), the special ramification scheme of type \( r + 1 + j \) of \( (\mathcal{V}, \mathcal{L}) \), and denote it by \( \text{VE}_j(\mathcal{V}, \mathcal{L}) \). We call the degeneracy scheme of \( \nu_r \) the ramification scheme of \( (\mathcal{V}, \mathcal{L}) \), and denote it by \( W(\mathcal{V}, \mathcal{L}) \).

If \( S := \text{Spec}(\mathbb{C}) \), if \( \mathcal{X} \) is smooth, and if \( \nu \) is injective, then the support of the scheme \( W(\mathcal{V}, \mathcal{L}) \) is the set of ramification points of the linear system \( H^0(\mathcal{S}, \mathcal{V}) \subseteq H^0(\mathcal{X}, \mathcal{L}) \) of sections of \( \mathcal{L} \). Also, the support of \( \text{VE}_j(\mathcal{V}, \mathcal{L}) \) is the set of special ramification points of type \( r + 1 + j \) of the same linear system, for \( j = -1, 1 \).

The map \( \nu_r \) is a map of locally free sheaves of the same rank \( r + 1 \). Taking determinants, \( \nu_r \) induces a Wronskian section \( w_p \) of the invertible sheaf

\[
W := \bigwedge^{r+1} j^*_p(\mathcal{L}) \otimes \left( \bigwedge^{r+1} p^*\mathcal{V} \right)^\vee.
\]

Using the truncation sequences \( \text{[2]} \), we get

\[
W \cong \omega_p^{\otimes \left( \frac{r+1}{2} \right)} \otimes \mathcal{L}^\otimes r+1 \otimes \left( \bigwedge^{r+1} p^*\mathcal{V} \right)^\vee.
\]

Locally, after trivializations are taken, \( w_p \) corresponds to a Wronskian determinant of a sequence of \( r + 1 \) functions. Its zero scheme is the ramification scheme of \( (\mathcal{V}, \mathcal{L}) \).

The formation of the ramification scheme is functorial in the following sense: Suppose there are an invertible sheaf \( \mathcal{L}' \) on \( \mathcal{X} \), a locally free sheaf \( \mathcal{V}' \) of rank \( r + 1 \) on \( \mathcal{S} \), a map \( \psi : \mathcal{V}' \to \mathcal{L}' \), and a commutative diagram of maps of the form:

\[
\begin{array}{ccc}
\mathcal{V}' & \xrightarrow{\nu'} & p_*\mathcal{L}' \\
\downarrow \null & & \downarrow \null \\
\mathcal{V} & \xrightarrow{\nu} & p_*\mathcal{L}.
\end{array}
\]

Let \( V \subseteq S \) be the ramification scheme of \( \mu \), and \( Y \subseteq \mathcal{X} \) that of \( \psi \). (So \( \text{Im}(\psi) = \mathcal{I}_{Y \mid \mathcal{X}} \otimes \mathcal{L} \).) Then

\[
\mathcal{I}_{W \cap [X]} \mathcal{I}_{Y \cap [X]}^{r+1} = \mathcal{I}_{V \cap [X]} \mathcal{I}_{W \cap [X]},
\]

where \( W := W(\mathcal{V}, \mathcal{L}) \) and \( W' := W(\mathcal{V}', \mathcal{L}') \).

By derivation, the section \( w_p \) induces a global section \( w'_p \) of the rank-2 locally free sheaf \( J^1_0(W) \). We will call the zero scheme of this section the special ramification scheme of \( (\mathcal{V}, \mathcal{L}) \), and denote it by \( \text{VSW}(\mathcal{V}, \mathcal{L}) \). Notice
that the irreducible components of the ramification scheme have codimension at most 1 in $\mathcal{X}$, while those of the special ramification schemes have codimension at most 2. Also, a local analysis of the matrices representing the maps $u_i$ shows that, set-theoretically,

$$VSW(\mathcal{V}, \mathcal{L}) = VE_{-1}(\mathcal{V}, \mathcal{L}) \bigcup VE_1(\mathcal{V}, \mathcal{L}).$$

Let $T$ be an $S$-scheme, and let $p_T: \mathcal{X}_T \to T$ denote the induced family by base extension. For each coherent sheaf $\mathcal{F}$ on $S$ (resp. $\mathcal{X}$), let $\mathcal{F}_T$ denote its pullback to $T$ (resp. $\mathcal{X}_T$). Then $\nu$ induces a map

$$\nu_T: \mathcal{V}_T \to (p_*)\mathcal{L}_T \to p_T^*\mathcal{L}_T,$$

and the (special) ramification scheme(s) of $(\mathcal{V}, \mathcal{L})$ pull back to the (special) ramification scheme(s) of $(\mathcal{V}_T, \mathcal{L}_T)$. Furthermore, if $Y \subseteq \mathcal{X}_T$ is a $T$-flat closed subscheme whose fibers over $T$ are subcurves of the fibers of $p$, then the (special) ramification scheme(s) of $(\mathcal{V}, \mathcal{L})$ coincide on $Y = (\mathcal{V} \cap \mathcal{X}_T - Y)$ with the corresponding (special) ramification scheme(s) of $(\mathcal{V}_T, \mathcal{L}_T|_Y)$.

In case $\mathcal{L}$ is the relative dualizing sheaf of $p$, and $\mathcal{V} = p_*\mathcal{L}$, the ramification schemes and special ramification schemes are called Weierstrass schemes and special Weierstrass schemes. In addition, we set

$$W(p) := W(\mathcal{V}, \mathcal{L}), \quad VSW(p) := VSW(\mathcal{V}, \mathcal{L}),$$

and $VE_j(p) := VE_j(\mathcal{V}, \mathcal{L})$ for $j = -1, 1$.

2.3. (Smoothings) Let $C$ be a curve. A smoothing of $C$ consists of two data: a flat, projective map $p: \mathcal{X} \to S$ to $S := \text{Spec}(\mathbb{C}[t])$ with smooth generic fiber, and an isomorphism between the special fiber and $C$. The smoothing is called regular if the total space $\mathcal{X}$ is a regular scheme.

Let $p: \mathcal{X} \to S$ be a smoothing of $C$, and identify the special fiber with $C$ with the given isomorphism. Since the general fiber is smooth, for each node $P$ of $C$, there are a nonnegative integer $k$ and a $\mathbb{C}[t]$-algebra isomorphism

$$\hat{\mathcal{O}}_{\mathcal{X}, P} \cong \frac{\mathbb{C}[t, x, y]}{(xy - t^{k+1})}.\tag{3}$$

We call $k$ the singularity type of $P$ in $\mathcal{X}$, and set $k(P) := k$. Notice that $k(P) = 0$ if and only if $\mathcal{X}$ is regular at $P$.

If $E \subseteq C$ is an irreducible component, then $E$ is not necessarily a Cartier divisor of $\mathcal{X}$. However, let $m_E$ be the least common multiple of the $k(P) + 1$ for all $P \in E \cap C - E$. Then there is a natural effective Cartier divisor on $\mathcal{X}$ whose associated 1-cycle is $m_E[E]$; call this divisor $E_P$. At a node $P \in E \cap C - E$, with $\hat{\mathcal{O}}_{\mathcal{X}, P}$ of the form (3), it is given by $x^n = 0$, for $n := m_E/(k + 1)$, if $x = 0$ is the equation defining the subcurve $E \subseteq C$. Notice that $E$ itself is a Cartier divisor of $\mathcal{X}$ if and only if $m_E = 1$.

For each integer $d > 0$, let $S \to S$ be the map defined by taking $t$ to $t^d$, and let $p_d: \mathcal{X}_d \to S$ be the smoothing induced by base change. The special
fiber of \( p \) is equal to that of \( p \). But, for a node \( P \in C \), if \( k \) is the singularity type of \( P \) in \( X \), then \((k+1)d - 1\) is the singularity type of \( P \) in \( X_d \).

Suppose \( C \) is semistable. There are a smoothing \( \overline{\mathcal{X}}: X \to S \) and an \( S \)-map \( b: \mathcal{X} \to \overline{\mathcal{X}} \) that blows down (collapses) all rational smooth components \( E \) of \( C \) which intersect \( C - E \). In fact, just let

\[
\overline{\mathcal{X}} := \text{Proj} \left( \bigoplus_{i \geq 0} H^0(X, \omega_p^{\otimes i}) \right),
\]

where \( \omega_p \) is the relative dualizing sheaf of \( p \). However the singularity types of \( \overline{\mathcal{X}} \) are bigger than those of \( \mathcal{X} \): if \( \overline{C} := b(C) \), and \( P \in \overline{C} \) is a node obtained by blowing down a chain of \( r \) smooth rational curves, and \( k_0, k_1, \ldots, k_r \) are the singularity types in \( \mathcal{X} \) of the nodes of \( C \) on that chain, then the singularity type of \( P \) in \( \overline{\mathcal{X}} \) is \( k_0 + k_1 + \cdots + k_r + r \).

In certain circumstances, it might be interesting to avoid blowing down some of the rational components of \( C \) in a construction as above. This is possible after base change. With a base change we may produce sections \( \Sigma_i \subset \mathcal{X} \) of \( p \) through its smooth locus intersecting the components we do not want to blow down. Then just do the above construction with \( \omega_p \) replaced by \( \omega_p(\Sigma_i) \).

So, given a node \( P \) of \( C \), and positive integers \( m_0, \ldots, m_n \), it is possible, with base changes, blowups and blowdowns, to find an integer \( d > 0 \), a smoothing \( \tilde{p}: \tilde{\mathcal{X}} \to S \) and an \( S \)-map \( b: \tilde{\mathcal{X}} \to \mathcal{X}_d \) such that \( \tilde{p} = b \circ p_d \) and:

1. \( b \) is an isomorphism off \( P \).
2. \( b^{-1}(P) \) is a chain of \( n \) smooth rational components of the special fiber \( \tilde{C} \) of \( \tilde{p} \).
3. The singularity types in \( \tilde{\mathcal{X}} \) of the nodes \( \tilde{P}_0, \ldots, \tilde{P}_n \) of \( \tilde{C} \) on \( b^{-1}(P) \), ordered in sequential order, are \( \ell m_0 - 1, \ldots, \ell m_n - 1 \) for a certain integer \( \ell > 0 \). (In fact,

\[
\ell(m_0 + \cdots + m_n) = (k+1)d,
\]

where \( k \) is the singularity type of \( P \) in \( \mathcal{X} \)).

2.4. (Limit linear series) Let \( C \) be a curve, and \( p: \mathcal{X} \to S \) a smoothing of \( C \). Identify \( C \) with the closed fiber of \( p \), and denote by \( X_0 \) the general fiber.

Let \( \mathcal{L} \) be an invertible sheaf on \( \mathcal{X} \). Since \( p \) is flat, \( H^0(\mathcal{X}, \mathcal{L}) \) is a torsion-free \( \mathbb{C}[\!(t)\!] \)-module, whence free. Let \( V \subseteq H^0(\mathcal{X}, \mathcal{L}) \) be a \( \mathbb{C}[\!(t)\!] \)-submodule. Then also \( V \) is free, say of rank \( r + 1 \), for a certain integer \( r \geq 0 \). Assume \( V \) is saturated, i.e. \( (V : (t)) = V \). Letting \( V_* \) be the subspace of \( H^0(X_0, \mathcal{L}|_{X_0}) \) generated by \( V \), we have that \( V \) is saturated if and only if \( V = V_* \cap H^0(\mathcal{X}, \mathcal{L}) \).

In our applications we will actually have \( V = H^0(\mathcal{X}, \mathcal{L}) \), so saturated.

Let \( \bar{R} \subset \mathcal{X} \) be the ramification scheme of \( (V \otimes \mathcal{O}_S, \mathcal{L}) \), as defined in Subsection 2.2. Since \( X_0 \) is smooth, \( \bar{R} \) is indeed a divisor. However, \( \bar{R} \) may not intersect \( C \) properly, as \( R \) may contain in its support a component of \( C \). Nevertheless, let \( \overline{\bar{R}} := \bar{R} \cap \mathcal{X} \). Then \( \overline{\bar{R}} \) intersects \( C \) properly. The intersection, \( \partial \overline{\bar{R}} := \overline{\bar{R}} \cap C \) is called the limit ramification scheme.
In [8] it is shown how to compute the 0-cycle $[\partial R]$ associated to $\partial R$ when $p$ is regular. We review this below.

Let $C_1, \ldots, C_n$ be the irreducible components of $C$. Since $C$ is connected, for each $i = 1, \ldots, n$ there is an invertible sheaf $\mathcal{L}_i$ on $X$ of the form
\[
\mathcal{L}_i = \mathcal{L} \otimes \mathcal{O}_X(\sum_{m} a_{i,m} C^p_m), \quad a_{i,m} \in \mathbb{Z},
\]
such that the restriction map
\[
H^0(X, \mathcal{L}_i) \to H^0(C_i, \mathcal{L}_i|_{C_i})
\]
has kernel $tH^0(X, \mathcal{L}_i)$. (The divisors $C^p_m$ are as explained in Subsection 2.3.)

There is a natural identification $\mathcal{L}_i|_{X_i} = \mathcal{L}|_{X_i}$. Using it, set
\[
V_i := H^0(X, \mathcal{L}_i) \cap V_* \subseteq H^0(X, \mathcal{L}|_{X_i}).
\]
Then also $V_i$ is saturated and free of rank $r + 1$. (In fact, $V_* = V_*$.)

Let $V_i \subseteq H^0(C_i, \mathcal{L}_i|_{C_i})$ be the image of $V_i$ under [4]. Since $V_i$ is saturated, and [4] has kernel $tH^0(X, \mathcal{L}_i)$, the dimension of $V_i$ is $r + 1$. We call $(V_i, \mathcal{L}_i|_{C_i})$ a limit linear system on $C_i$.

Let $R_i \subseteq C_i$ be the ramification scheme of $(\overline{V}, \mathcal{L}_i|_{C_i})$, as defined in Subsection 2.2. Put $R'_i := R_i - R_i \cap \overline{C} - C_i$. Then
\[
[\partial R] = [R'_1] + \cdots + [R'_n].
\]
Furthermore, if $p$ is regular, then
\[
[\partial R] = \sum_{i=1}^{n} [R_i] + \sum_{i<j} \sum_{P \in C_i \cap C_j} (r + 1)(r - \ell_{i,j})[P],
\]
where $\ell_{i,j} := a_{i,j} + a_{j,i} - a_{i,i} - a_{j,j}$ for each distinct $i, j = 1, \ldots, n$.

When $\mathcal{L} = \omega$ and $V = H^0(X, \mathcal{L}_i)$, the limit ramification scheme is called the limit Weierstrass scheme, and denoted $\partial W_p$; also, a limit linear system is called a limit canonical system.

Let $P$ be a nonsingular point of $C$, and $\Gamma \subset X$ a section of $p$ intersecting $C$ at $P$. Say, $P \in C_i$. Let $P_*$ be the rational point of $X_i$ cut out by $\Gamma$. Then the behaviour of $(V_i, \mathcal{L}_i|_{X_i})$ at $P_*$ is partially captured by that of $(\overline{V}, \mathcal{L}_i|_{C_i})$ at $P$. For instance, we have semicontinuity:
\[
\dim \overline{V}_i(-aP) \geq \dim C((t)) V_*(-aP) \quad \text{for each } a = 0, 1, \ldots.
\]

In fact, let $m := \dim C((t)) V_*(-aP)$. Since $V_* = V_*$, we may choose a $\mathbb{C}[t]$-basis $\sigma_1, \ldots, \sigma_m$ of $V_i \cap V_*(-aP_*)$. The images $\overline{\sigma}_i$ in $\overline{V}_i$ vanish at $P$ with multiplicity at least $a$ as well. If there is a nonzero $m$-tuple $(c_1, \ldots, c_m) \in \mathbb{C}^m$ such that $c_1 \sigma_1 + \cdots + c_m \sigma_m = 0$, then
\[
c_1 \sigma_1 + \cdots + c_m \sigma_m = t \sigma
\]
for some $\sigma \in V_i$, because [4] has kernel $tH^0(X, \mathcal{L}_i)$ and $V_i$ is saturated. Because of (6), also $\sigma \in V_i \cap V_*(-aP_*)$, and hence $\sigma = b_1 \sigma_1 + \cdots + b_m \sigma_m$ for certain $b_k \in \mathbb{C}[[t]]$. Plugging this expression in (6), we get a nontrivial relation for the sections $\sigma_i$, an absurd.
In particular, if $P_*$ is a special ramification point of type $r + j$ of $V_*$, for $j = -1$ or $j = 1$, then so is $P$ with respect to $V_\alpha$. When $\mathcal{L} = \omega$ and $V = H^0(X, \omega)$ we say that $P$ is the limit of a special Weierstrass point of type $g + j$ along $p$.

3. General reducible curves

**Proposition 3.1.** Let $X$ and $Y$ be two smooth nonrational curves. Let $A \in X$ and $B \in Y$, and let $C$ be the uninodal curve union of $X$ and $Y$ with $A$ identified with $B$. Let $p: X \to S$ be a smoothing of $C$. Then the following statements hold:

(i) If $B$ is at most a simple Weierstrass point of $Y$, then there is a vector subspace $V \subseteq H^0(X, \omega_X((g_Y + 2)A))$ of codimension 1 containing $H^0(X, \omega_X((g_Y + 2)A))$ such that $(V, \omega_X((g_Y + 2)A))$ is a limit canonical system on $X$. Furthermore, if $B$ is an ordinary point of $Y$, then $A$ is a base point of this system, i.e. $V = H^0(X, \omega_X((g_Y + 1)A))$.

(ii) If $A$ and $B$ are at most simple Weierstrass points of $X$ and $Y$, with at least one of them ordinary, then the limit Weierstrass scheme contains the node of $C$ with multiplicity at most 1.

**Proof.** Assume $B$ is at most a simple Weierstrass point of $Y$. Let $\ell$ be an integer, and set $\mathcal{L} := \omega_p((\ell + 1)A)$, where $\omega_p$ is the relative dualizing sheaf of $p$. Then

\begin{align*}
(7) & \quad \mathcal{L}|_X \cong \omega_X((\ell + 1)A) \quad \text{and} \quad \mathcal{L}|_Y \cong \omega_Y((1 - \ell)B).
\end{align*}

In addition, the following natural sequences are exact:

\begin{align*}
(8) & \quad 0 \to \mathcal{L}|_X(-A) \to \mathcal{L}|_C \to \mathcal{L}|_Y \to 0, \\
(9) & \quad 0 \to \mathcal{L}|_Y(-B) \to \mathcal{L}|_C \to \mathcal{L}|_X \to 0.
\end{align*}

By Riemann–Roch, $h^0(\mathcal{L}|_X) \geq g$ if and only if $\ell \geq g_Y$. On the other hand, by the hypothesis on $B$, we have $h^0(Y, \mathcal{L}|_Y) = \max(0, g_Y + 1 - \ell)$ for $\ell \neq g_Y + 1$, whereas for $\ell = g_Y + 1$ either $h^0(Y, \mathcal{L}|_Y) = 0$ if $B$ is an ordinary point of $Y$, or else $h^0(Y, \mathcal{L}|_Y) = 1$.

Set $\mathcal{M} := \omega_p((g_Y + 1)A)$ and $\mathcal{N} := \omega_p((g_Y + 1)A)$. From (7) for $\mathcal{L} := \mathcal{M}$, and Riemann–Roch, since $g_Y > 0$ we have $H^1(X, \mathcal{M}|_X(-A)) = 0$. So, the exactness of (8) for $\mathcal{L} := \mathcal{M}$ implies

\begin{align*}
& h^0(C, \mathcal{M}|_C) = h^0(X, \mathcal{M}|_X(-A)) + h^0(Y, \mathcal{M}|_Y) = (g - 1) + 1 = g.
\end{align*}

Thus the restriction $H^0(X, \mathcal{M}) \to H^0(C, \mathcal{M}|_C)$ is surjective.

Consider now the restriction map

\begin{align*}
& \alpha: H^0(C, \mathcal{M}|_C) \to H^0(X, \omega_X((g_Y + 1)A)).
\end{align*}

It follows from the exactness of (8) that $\alpha$ contains $H^0(X, \omega_X((g_Y A))$ in its image, and from the exactness of (9) that the kernel of $\alpha$ is isomorphic to $H^0(Y, \omega_Y(-g_Y B))$. Thus $\alpha$ is injective, hence bijective, if and only if $B$ is
an ordinary point of $Y$. In this case, the complete linear system of sections of $\omega_X((g_Y + 1)A)$ is a limit canonical system on $X$. On the other hand, if $B$ is a (simple) Weierstrass point of $Y$, the image of $\alpha$ is the subspace $H^0(X, \omega_X((g_Y + 1)A))$.

Applying (7) for $L := N$, as $B$ is at most a simple Weierstrass point of $Y$, we get $H^0(Y, N|_Y(-B)) = 0$. So, the natural map
\[ \beta: H^0(C, N|_C) \to H^0(X, \omega_X((g_Y + 2)A)) \]
is injective. The maps $\alpha$ and $\beta$ fit in a commutative diagram of the form
\[ \begin{array}{ccc}
H^0(X, \mathcal{M}) & \longrightarrow & H^0(C, \mathcal{M}|_C) \\
\downarrow & & \downarrow \\
H^0(X, \mathcal{N}) & \longrightarrow & H^0(C, \mathcal{N}|_C) \\
& & \downarrow \\
& & H^0(X, \omega_X((g_Y + 2)A)),
\end{array} \]
where the horizontal maps are induced by restriction, and the vertical maps are induced from the inclusion $\mathcal{M} \to \mathcal{N}$. Since $\beta$ is injective, the image $V$ of the bottom composition has codimension 1 in $H^0(X, \omega_X((g_Y + 2)A))$, and $(V, \omega_X((g_Y + 2)A))$ is a limit canonical system on $X$. From the diagram, $V$ contains the image of the top composition, which is $H^0(X, \omega_X((g_Y + 1)A))$ if $B$ is an ordinary point of $Y$, and is $H^0(X, \omega_X(g_Y A))$ otherwise. In the former case, by dimension considerations, $V = H^0(X, \omega_X((g_Y + 1)A))$. This finishes the proof of (9).

Let us prove (10). Without loss of generality, we may assume that $A$ is an ordinary point of $X$. Then, from Plücker formula, the ramification divisor of the complete linear system of sections of $\omega_X((g_Y + 1)A)$ has degree
\[ (2g_Y + g_Y - 1)g + (g_Y - 1)g(g - 1) - gX \]
on $X - A$. On the other hand, since $B$ is at most a simple Weierstrass point of $Y$, also by Plücker formula, the ramification divisor of the complete linear system of sections of $\omega_Y((g_X + 1)B)$ has degree
\[ (2g_Y + g_X - 1)g + (g_Y - 1)g(g - 1) - w_B \]
on $Y - B$, where $w_B = g_Y$ if $B$ is an ordinary point, or else $w_B = g_Y + 1$.

Suppose first that $B$ is an ordinary point of $Y$. Then, by the already proved (9), the limit Weierstrass scheme $\partial W_p$ has degree away from the node equal to the sum of (10) and (11), with $w_B = g_Y$. But this sum is $g^3 - g$. So the node of $C$ is not contained in $\partial W_p$.

Finally, suppose that $B$ is a (simple) Weierstrass point of $Y$. Then, by (10), there is a vector subspace $V \subset H^0(X, \omega_X((g_Y + 2)A))$ of dimension $g$ containing $H^0(X, \omega_X(g_Y A))$ such that $(V, \omega_X((g_Y + 2)A))$ is a limit canonical system. Since $A$ is an ordinary point of $X$, Plücker formula yields that the ramification divisor of $(V, \omega_X((g_Y + 2)A))$ has degree
\[ (2g_X + g_Y)g + (g_X - 1)g(g - 1) - w_A \]
on $X - A$, where $w_A = 2g_X + g_Y - 1$ if $V \neq H^0(X, \omega_X((g_Y + 1)A))$, and $w_A = 2g_X + g_Y$ otherwise. At any rate, using $w_B = g_Y + 1$, the sum of (11) and (12) is at least $g^3 - g - 1$, with equality only if $w_A = 2g_Y + g_X$. So $\partial W'_\rho$ can only contain the node of $C$ with multiplicity 1.

Lemma 3.2. Let $X$ be a general curve, $P \in X$ a general point, and $n$ a positive integer. Let $Q \in X - P$. Then the following statements are equivalent, for each $j = -1, 1$:

(i) The point $Q$ is a ramification point of the complete system of sections of $\omega_X((n + 1 + j)P)$.

(ii) There is a unique subspace $V \subseteq H^0(X, \omega_X((n + 2)P))$ of codimension 1 containing $H^0(X, \omega_X(nP))$ but different from $H^0(X, \omega_X((n + 1)P))$ such that $Q$ is a special ramification point of $(V, \omega_X((n + 2)P))$ of type $g_X + n + j$.

Proof. Set $g := g_X + n$. Also, set

(13) $V' := H^0(\omega_X(nP)) + H^0(\omega_X((n + 2)P - gQ)) \subseteq H^0(\omega_X((n + 2)P))$.

Since $(X, P)$ is general, by Prop. 3.1 of [4], all the ramification points but $P$ of the complete linear system of sections of $\omega_X(nP)$ or $\omega_X((n + 2)P)$ are simple. Then

(14) $h^0(X, \omega_X((n + 2)P - gQ)) = 1$ and $h^0(X, \omega_X(nP - gQ)) = 0$.

Thus the sum in (13) is direct, and $V'$ has dimension $g$. In addition,

(15) $V'(-iQ) = H^0(X, \omega_X(nP - iQ)) \oplus H^0(X, \omega_X((n + 2)P - gQ))$

for each $i = 0, 1, \ldots, g$, and thus

(16) $\dim V'(-iQ) = h^0(X, \omega_X(nP - iQ)) + 1$ for each $i = 0, 1, \ldots, g$.

Suppose first that (15) holds. Then either

(17) $h^0(X, \omega_X(nP - (g - 1)Q)) \geq 1$,

in which case (16) implies that $\dim V'(-(g - 1)Q) \geq 2$, and hence $Q$ is a special ramification point of type $g - 1$ of $(V', \omega_X((n + 2)P))$; or

(18) $h^0(X, \omega_X((n + 2)P - (g + 1)Q)) \geq 1$,

in which case (16) implies that $V'(-(g + 1)Q) \neq 0$, and hence $Q$ is a special ramification point of type $g + 1$ of $(V', \omega_X((n + 2)P))$. Notice that $V'$ cannot be $H^0(X, \omega_X((n + 1)P))$ because, since $(X, P)$ is general, the complete linear system of sections of $\omega_X((n + 1)P)$ has no special ramification points but $P$, by Prop. 3.1 of [4].

For the uniqueness, just observe that, if $Q$ is a ramification point of $(V', \omega_X((n + 2)P))$, for a subspace $V$ as described in (15), then (17) implies that $V \supseteq H^0(X, \omega_X((n + 2)P - gQ))$, and hence $V \supseteq V'$. Since both $V$ and $V'$ have dimension $g$, we have $V = V'$. 

□
Finally, suppose (17) holds. As we saw above, necessarily $V = V'$. So, $Q$ is a special ramification point of $(V, \omega_X((n + 2)P))$ of type $g + j$ if and only if $\dim V'(- (g + 1)P) \geq 1$. Using (15) with $i = g - 1$, we see that the former inequality occurs if and only if (17) holds, i.e., if and only if $Q$ is a ramification point of the complete linear system of sections of $\omega_X(nP)$. On the other hand, since $H^0(X, \omega_X(nP - gQ)) = 0$, the latter inequality occurs if and only if (18) holds, i.e., if and only if $Q$ is a ramification point of the complete linear system of sections of $\omega_X((n + 2)P)$.

Proposition 3.3. Let $Y$ be a general smooth curve, $\Delta \subset Y \times Y$ the diagonal, and $p_1$ and $p_2$ the projection maps from $Y \times Y$ onto the indicated factors. Set $E := (p_2^*\omega_Y)(- (g_Y - 1)\Delta)$ and $\mathcal{E} := p_{1*}L$. Then $\mathcal{E}$ is invertible, and the degeneracy scheme of the evaluation map $p_1^*\mathcal{E} \to \mathcal{L}$ intersects $\Delta$ transversally along the Weierstrass points of $Y$.

Proof. By [4], Cor. 3.3, the Weierstrass points of the general curve are simple. Thus $h^0(Y, \omega_Y(-(g_Y - 1)P)) = 1$ for each $P \in Y$, and hence $\mathcal{E}$ is invertible. Let $Z$ denote the degeneracy scheme of the evaluation map $p_1^*\mathcal{E} \to \mathcal{L}$. For each $P \in Y$, the intersection $Z \cap p_1^{-1}(P)$ is the ramification scheme of the complete linear system of sections of $\omega_Y(-(g_Y - 1)P)$. Thus $Z \cap p_1^{-1}(P)$ is finite and contains $(P, P)$ if and only if $P$ is a Weierstrass point of $Y$. We need only show now that $Z$ intersects $\Delta$ transversally, which will follow from showing that the intersection number $Z \cdot \Delta$ is $g_Y^3 - g_Y$.

Let $\delta : Y \to Y \times Y$ be the diagonal map. We have $\delta^* \mathcal{O}_{Y \times Y}(- \Delta) = \omega_Y$. Thus

$$Z \cdot \Delta = \deg Z = \deg(\mathcal{O}_{Y \times Y}(- \Delta) - c_1(\mathcal{E})).$$

Now, since $Y$ has at most simple Weierstrass points, for each $i = 0, \ldots, g_Y - 2$ the natural exact sequence

$$0 \to p_{1*}p_2^*\omega_Y(-(i + 1)\Delta) \to p_{1*}p_2^*\omega_Y(-i\Delta) \to \omega_Y \otimes \delta^* \mathcal{O}_{Y \times Y}(-i\Delta) \to 0$$

is exact. As $c_1(p_{1*}p_2^*\omega_Y) = 0$ and $\delta^* \mathcal{O}_{Y \times Y}(- \Delta) = \omega_Y$, we get

$$c_1(\mathcal{E}) = -(c_1(\omega_Y) + c_1(\omega_Y^{\otimes 2}) + \cdots + c_1(\omega_Y^{\otimes g_Y - 1})).$$

Thus

$$Z \cdot \Delta = \sum_{i=1}^{g_Y} i \deg(c_1(\omega_Y)) = \binom{g_Y + 1}{2}(2g_Y - 2) = g_Y^3 - g_Y.$$

Theorem 3.4. Let $X$ and $Y$ be two general smooth nonrational curves. Let $A \in X$ and $B \in Y$, and let $C$ be the uninodal curve union of $X$ and $Y$ with $A$ identified with $B$. Set $g := g_C$. Suppose that either $A$ is a general point of $X$ or $B$ is a general point of $Y$. Let $Q \in C$ lying on $X$. Then, for each $j = -1, 1$, the point $Q$ is the limit of a special Weierstrass point of type $g + j$. 

□
along a smoothing of $C$ if and only if $Q$ is not the node of $C$, and either of the following two situations occur:

(i) $Q$ is a special ramification point of type $g+j$ of the complete linear system of sections of $\omega_X((gy + 1)A)$

(ii) $Q$ is a ramification point of the complete linear system of sections of $\omega_X((gy + 1 + j)A)$ and $B$ is a Weierstrass point of $Y$.

Proof. We prove the “only if” part of the statement first. Let $p: X \to S$ be a smoothing of $C$, as indicated in Figure 1 below, such that $Q$ is the limit of a Weierstrass point of type $g+j$ along $p$. In particular, $Q$ appears with multiplicity at least 2 in the limit Weierstrass scheme $\partial W_p$.

![Figure 1. The smoothing.](image)

Since $X$ and $Y$ are general, their Weierstrass points are simple. Also, since either $A$ or $B$ is general, either $A$ or $B$ is ordinary. Thus, it follows from Proposition 3.1 item (ii), that $Q$ is not the node of $C$.

Suppose first that $B$ is an ordinary point of $Y$. Then, by Proposition 3.1 item (i), the system of sections of $\omega_X((gy + 2)A)$ vanishing at $A$ is a limit canonical system, and hence (i) holds.

On the other hand, suppose that $B$ is a Weierstrass point of $Y$. By Proposition 3.1 item (i), there is a vector subspace $V \subset H^0(X, \omega_X((gy + 2)A))$ of codimension 1 containing $H^0(X, \omega_X(gyA))$ such that $(V, \omega_X((gy + 2)A))$ is a limit canonical system, and hence admits $Q$ as a special ramification point of type $g+j$. Now, (ii) follows from Lemma 3.2.

For the “if” part of the proof, we will construct smoothings as convenient slices of certain 2-parameter families.

Suppose $Q$ is not the node of $C$. Suppose first that (i) holds. Then $g_X \geq 2$. Also, it follows from Prop. 3.1 in [4] that $A$ is not a general point of $X$. So, by hypothesis, $B$ is a general point of $Y$, whence an ordinary point.

We will first deform $C$ by letting $A$ vary to a general point. More precisely, let $\Delta \subseteq X \times X$ be the diagonal, and consider the union $U$ of $X \times X$ with $Y \times X$ with $\Delta$ naturally identified with $\{B\} \times X$. Let $q: U \to X$ be the projection onto the second factor. Since $X$ is nonsingular, we may identify the complete local ring of $X$ at $A$ with $\mathbb{C}[[t]]$, and let $\tilde{q}: \tilde{U} \to S$ be the family induced over $S := \text{Spec}(\mathbb{C}[[t]])$ by base change.
Let $V := \mathbb{C}[[t_1, t_2, \ldots, t_N]]$ be the base of the universal deformation space of $C$, where $t_1 = 0$ corresponds to equisingular deformations. The map $\tilde{q}$ corresponds to a local homomorphism $h: V \to C[[t]]$ such that $h(t_1) = 0$. Since $g_X \geq 2$, the map $\tilde{q}$ is not, even infinitesimally, a constant family. So there is $j \geq 2$ such that $h(t_j)$ generates $tC[[t]]$. We may assume that $j = 2$ and, after a harmless reparameterization, that $h(t_2) = t$. Letting $p_i(t) := h(t_i)$ for each $i \geq 3$, we have $h(t_i - p_i(t_2)) = 0$ for each $i \geq 3$.

Consider the two-parameter subfamily of the universal deformation of $C$ given precisely by the equations $t_i - p_i(t_2) = 0$ for $i = 3, \ldots, N$. Identify the base of this family with $S_2 := \text{Spec}(C[[t_1, t_2]])$, and let $u: T \to S_2$ denote the map giving the family, which is depicted in Figure 2 below.

![Figure 2. The first family.](image)

Notice that $T$ is a regular threefold. Let $E \subset S_2$ be the Cartier divisor given by $t_1 = 0$. The slice $u_E: u^{-1}(E) \to E$ is precisely $\tilde{q}$, under the identification $t_2 = t$. Hence, the pullback $\pi^* E$ is the sum of two effective Cartier divisors, $X_E$ and $Y_E$, the first isomorphic to $X \times E$, the second to $Y \times E$, whose intersection on $Y \times E$ is $B \times E$, and on $X \times E$ is the graph $\Sigma$ of a nonconstant map $E \to X$, sending the special point $o \in E$ to $A$, and the general point $e \in E$ to the general point of $X$.

Let $M := \omega_u(g_Y Y_E)$, where $\omega_u$ is the relative dualizing sheaf of $u$. Then
\begin{equation}
M|_{X_E} \cong \omega_{X_E/E}((g_Y + 1)\Sigma).
\end{equation}

A fiberwise check, as done in the proof of Proposition 3.1, shows that $u_* M$ is locally free of rank $g$, with formation commuting with base change. In addition, since $B$ is an ordinary point of $Y$, the natural map
\[\gamma: (u_* M)|_E \to u_{E*}(M|_{X_E})\]

is an isomorphism.

Form the special ramification scheme $Z \subseteq T$ of type $g+j$ of $(u_* M, M)$, as explained in Subsection 2.2. Since $\gamma$ is an isomorphism, $Z$ agrees on $X_E - \Sigma$
with the special ramification scheme of type $g+j$ of $(u_E^*(\mathcal{M}|_{X_E}), \mathcal{M}|_{X_E})$. Because of (19), the fact that $\Sigma \cap u^{-1}(o) = \{A\}$, and the hypothesis on $Q$, we have that $Q$ is an isolated point of $Z \cap u^{-1}(o)$. Furthermore, since the general point of $\Sigma$ is the general point of $X \times \{e\}$, Prop. 3.1 in [4] yields $Z \cap u^{-1}(e) \subseteq Y \times \{e\}$.

Since $Z$ is defined locally by two regular functions, there is an irreducible curve $N \subseteq Z$ containing $Q$. Since $Q$ is an isolated point of $Z \cap u^{-1}(o)$, and $Z \cap u^{-1}(e) \subseteq Y \times \{e\}$, the general point of $N$ must lie on a smooth fiber of $u$, and hence be a special Weierstrass point of type $g+j$ of that fiber. So $Q$ is the limit of a special Weierstrass point of type $g+j$.

Suppose now that (ii) holds. In particular, $B$ is a Weierstrass point of $Y$, and hence $g_Y \geq 2$. Letting $B$ vary, we may construct a two-parameter family similar to the one constructed in the first case. Thus we get a family of curves $u: T \to S_2$ over $S_2 = \text{Spec}(\mathbb{C}[[t_1, t_2]])$ such that $T$ is a regular threefold, and the pullback $u^*E$ of the Cartier divisor $E \subseteq S_2$ given by $t_1 = 0$ is the sum of two effective Cartier divisors, $X_E$ and $Y_E$, the first isomorphic to $X \times E$, the second to $Y \times E$, whose intersection on $X \times E$ is $A \times E$, and on $Y \times E$ is the graph $\Sigma$ of a nonconstant map $E \to Y$, sending the special point $o \in E$ to $B$, and the general point $e \in E$ to the general point of $Y$.

Let $S_2 \to S_2$ be the blowup map of $S_2$ at $o$, and denote by $F \subseteq S_2$ the exceptional divisor. Abusing notation, we denote the strict transform of $E$ by $\overline{E}$ as well, and let $o$ denote the point of intersection of $E$ and $F$. The fibered product $T \times_{S_2} \overline{S}_2$ is singular only at the node of the fiber over $o$ of the second projection $T \times_{S_2} \overline{S}_2 \to \overline{S}_2$.

Let $\overline{T}$ be the blowup of $T \times_{S_2} \overline{S}_2$ along the subscheme $Y_E \times_{S_2} \overline{S}_2 \subseteq T \times_{S_2} \overline{S}_2$. A local analysis shows that $\overline{T}$ is smooth. Denote by $\overline{X}_E$ and $\overline{Y}_E$ the strict transforms in $\overline{T}$ of $X_E \times_{S_2} E$ and $Y_E \times_{S_2} E$. Let also $\overline{X}_F$ and $\overline{Y}_F$ denote the strict transforms of $X \times F$ and $Y \times F$. Let $\overline{u}: \overline{T} \to \overline{S}_2$ be the induced map. The fiber $\overline{T}_o := \overline{u}^{-1}(o)$ consists of three components: two of them disjoint and naturally identified with $X$ and $Y$, and the remaining, say $R$, isomorphic to a line and meeting $X$ and $Y$ transversally at $A$ and $B$. A local analysis shows that $\overline{X}_E \cap \overline{T}_o = X$ and $\overline{Y}_E \cap \overline{T}_o = Y \cup R$, while $\overline{X}_F \cap \overline{T}_o = X \cup R$ and $\overline{Y}_F \cap \overline{T}_o = Y$. Figure 3 below describes the family given by $\overline{u}$.

For each $z \in E \cup F$, let $X_z$ and $Y_z$ denote the components of $\overline{u}^{-1}(z)$ that are base extensions of $X$ and $Y$.

Let $\omega_\overline{u}$ be the relative dualizing sheaf of $\overline{u}$. Let

$$\mathcal{M} := \omega_{\overline{u}}(g_Y(\overline{Y}_E + \overline{Y}_F)), \quad \mathcal{N} := \mathcal{M}(\overline{Y}_E), \quad \mathcal{P} := \mathcal{N}(\overline{Y}_E).$$

Clearly, $\mathcal{M} \subset \mathcal{N} \subset \mathcal{P}$. The restriction $\mathcal{P}|_{\overline{X}_F}$ is the pullback of $\omega_X((g_Y+2)A)$ under the composition $\overline{X}_F \to X \times F \to X$. Thus

$$\overline{u}_*(\mathcal{P}|_{\overline{X}_F}) = H^0(X, \omega_X((g_Y+2)A) \otimes \mathcal{O}_F).$$
Figure 3. The second family.

and in particular $\tilde{u}_*(\mathcal{P}|_{\tilde{X}_F})$ is a locally free $\mathcal{O}_F$-module of rank $g + 1$.

We claim that the natural composition

$$\delta: (\tilde{u}_*\mathcal{N}|_F \rightarrow (\tilde{u}_*\mathcal{P})|_F \rightarrow \tilde{u}_*(\mathcal{P}|_{\tilde{X}_F})$$

is injective with invertible cokernel, and that, as $f$ ranges in $F - o$, the image $V_f$ of $\delta(f)$ ranges through all subspaces of $H^0(X, \omega_X((g_Y + 2)A))$ of dimension $g$ containing $H^0(X, \omega_X(g_Y A))$ but $H^0(X, \omega_X((g_Y + 1)A))$. In particular, $(\tilde{u}_*\mathcal{N})|_F$ is locally free of rank $g$.

Once the claim is established, we proceed as in the first case. Indeed, a fiberwise analysis shows that $\tilde{u}_*\mathcal{N}$ is locally free of rank $g$ on $\tilde{S}_2 - F$. Thus, from the claim, $\tilde{u}_*\mathcal{N}$ is locally free of rank $g$ everywhere. Form the special ramification scheme $Z$ of type $g + j$ of $(\tilde{u}_*\mathcal{N}, \mathcal{N})$. For each $f \in F - o$, since $\delta(f)$ is injective, $Z$ agrees on $X_f - A$ with the special ramification scheme of type $g + j$ of $(V_f, \omega_X((g_Y + 2)A))$.

Now, by Lemma 3.2 there is a subspace $V \subseteq H^0(X, \omega_X((g_Y + 2)A))$ of codimension 1 with $V \supseteq H^0(X, \omega_X(g_Y A))$ but $V \neq H^0(X, \omega_X((g_Y + 1)A))$ such that $Q$ is a special ramification point of $(V, \omega_X((g_Y + 2)A))$ of type $g + j$. From the claim there is $f \in F - o$ such that $V = V_f$. So, viewing $Q$ as a point of $X_f$, we have $Q \in Z$.

Since all irreducible components of $Z$ have codimension at most 2 in $\tilde{T}$, there is an irreducible curve $N \subseteq Z$ passing through $Q \in X_f$. Now, only finitely many points of $X$ can be special ramification points of type $g + j$ of a linear system like $V$, namely, by Lemma 3.2 the ramification points of the complete linear system of sections $\omega_X((g_Y + 1 + j)A)$. But, again by Lemma 3.2 each of these points is a special ramification point of a unique
Thus, for all but finitely many $f \in F - o$, the image $V_f$ has no special ramification points of type $g + j$. Hence $N$ intersects only finitely many fibers of $\tilde{u}$ over $F$. So the general point of $N$ must be on a smooth fiber of $\tilde{u}$, and hence be a special Weierstrass point of type $g + j$ of that fiber. So $Q$ is the limit of a special Weierstrass point of type $g + j$.

Now, let us establish the claim. First, a fiberwise analysis shows that $\tilde{u}_*M$ is locally free of rank $g$, and that $R^1\tilde{u}_*M$ is invertible, both with formation commuting with base change. Consider the long exact sequence in higher direct images:

$$0 \to \tilde{u}_*M \to \tilde{u}_*N \to \tilde{u}_*(N|_{\tilde{Y}_F}) \to R^1\tilde{u}_*M \to R^1\tilde{u}_*N \to R^1\tilde{u}_*(N|_{\tilde{Y}_F}) \to 0.$$  

Since $R^1\tilde{u}_*M$ is invertible, and $\tilde{u}_*(N|_{\tilde{Y}_F})$ is supported on $F$, the middle map above is zero, breaking up the long sequence in two short exact sequences,

$$0 \to \tilde{u}_*M \to \tilde{u}_*N \to \tilde{u}_*(N|_{\tilde{Y}_E}) \to 0,$$

$$0 \to R^1\tilde{u}_*M \to R^1\tilde{u}_*N \to R^1\tilde{u}_*(N|_{\tilde{Y}_E}) \to 0.$$  

The exactness of the first sequence shows the surjectivity of the natural map $(\tilde{u}_*N)|_F \to \tilde{u}_*(N|_{\tilde{Y}_E})$. Now, a fiberwise analysis, using that $B$ is a simple Weierstrass point of $Y$, shows that $R^1\tilde{u}_*(N|_{\tilde{Y}_E})$ is a locally free $O_F$-module of rank 2. So, since $R^1\tilde{u}_*M$ is also locally free, the exactness of the second sequence above implies that, for each Cartier divisor $G \subset \tilde{S}_2$ intersecting $F$ properly, the natural multiplication-by-$G$ map $(R^1\tilde{u}_*N)(-G) \to R^1\tilde{u}_*N$ is injective, and hence the natural map

$$\delta_G: \tilde{u}_*N|_G \longrightarrow \tilde{u}_*(N|_{\tilde{u}^{-1}(G)})$$

is an isomorphism. This isomorphism allows us to work with slices of the family $\tilde{u}$ that intersect $F$ properly.

In particular, for each $f \in F - \{o\}$, let $G \subset \tilde{S}_2$ be a smooth curve passing through $f$, and whose general point lies on $\tilde{S}_2 - (E \cup F)$. So we have a smoothing $\tilde{u}_G: \tilde{u}^{-1}(G) \to G$ of the fiber $G$, and we can also choose $G$ such that $\tilde{u}_G$ is regular. Then, as we saw in the proof of Proposition 3.1, the natural map

$$\delta_{G,f}: (\tilde{u}_G)_*(N|_{\tilde{u}^{-1}(G)})(f) \to H^0(X_f,N|_{X_f})$$

is injective and, under the isomorphism $N|_{X_f} \cong \omega_{X_f}((g_Y + 2)A)$, its image is a $g$-dimensional subspace of $H^0(X,\omega_X((g_Y + 2)A))$ that contains $H^0(X,\omega_X(g_Y A))$ but is different from $H^0(X,\omega_X((g_Y + 1)A))$. Now, since $\delta(f) = \delta_{G,f} \circ \delta_G(f)$, and $\delta_G$ is an isomorphism, $\delta(f)$ has the same properties.

To understand what happens at $o$, consider the slice of $\tilde{u}$ over $E$. We claim the natural map

$$\eta: \tilde{u}_*(N|_{\tilde{u}^{-1}(E)}) \longrightarrow \tilde{u}_*(N|_{\tilde{X}_E})$$

is an isomorphism. Indeed, let $\Sigma_E := \tilde{X}_E \cap \tilde{Y}_E$. Since $\delta_E$ is an isomorphism, applying the long exact sequence in higher direct images to the exact
sequence
\[ 0 \to \mathcal{N}|_{\tilde{X}_E}(-\Sigma_E) \to \mathcal{N}|_{\tilde{u}^{-1}(E)} \to \mathcal{N}|_{\tilde{Y}_E} \to 0 \]
we get that the natural map \((\tilde{u}_*\mathcal{N})|_E \to \tilde{u}_*(\mathcal{N}|_{\tilde{Y}_E})\) is surjective, and that
the image of \(\eta\) contains \(\tilde{u}_*(\mathcal{N}|_{\tilde{X}_E}(-\Sigma_E))\). Thus, to show our last claim we
need only show that the natural map
\[ \epsilon: \tilde{u}_*(\mathcal{N}|_{\tilde{Y}_E}) \to \tilde{u}_*(\mathcal{N}|_{\Sigma_E}) \]
is an isomorphism.

Since the map \(\Sigma_E \to E\) is an isomorphism, \(\tilde{u}_*(\mathcal{N}|_{\Sigma_E})\) is locally free of
rank 1. Also \(\tilde{u}_*(\mathcal{N}|_{\tilde{Y}_E})\) is locally free of rank 1, because it is so over the
generic point \(e \in E\). Since the point in the intersection \(\Sigma_E \cap \tilde{u}^{-1}(o)\) is not
a Weierstrass point of \(Y\), the map \(\epsilon(e)\) is an isomorphism.

To show that also \(\epsilon(o)\) is an isomorphism, it amounts to show that the
point in \(\Sigma_o := \Sigma_E \cap \tilde{u}^{-1}(o)\) of intersection of \(X_o\) and \(R\) is not a limit
ramification point of \((\tilde{u}_*\mathcal{N})|_{\tilde{Y}_E}, \mathcal{N}|_{\tilde{Y}_E}\). This is indeed the case, since \(\tilde{Y}_E\)
the blowup of \(Y \times E\) at \((B, o)\), and \(\Sigma_E\) is the strict transform of the graph
of the map \(E \to Y\) obtained by considering the identity map of \(Y\) locally
analytically at \(B\). So, the transversality stated in Proposition 4.3 shows that
\(\Sigma_o\) is not a limit ramification point.

Finally, since \(\eta\) and \(\delta_E\) are isomorphisms, it follows that \(\delta(o)\), which is
the composition of the isomorphism \(\eta(o) \circ \delta_E(o)\) with the inclusion
\[ H^0(X, \omega_X((g_Y + 1)A)) \to H^0(X, \omega_X(g_Y + 2)A), \]
is injective of rank \(g\). So, \(\delta(f)\) is injective of rank \(g\) for every \(f \in F\),
and hence \(\delta\) is injective with invertible cokernel. Moreover, as the image
of \(\delta(o)\) is different from that of \(\delta(f)\) for \(f \in F - o\), then, as \(f\) varies in
\(F - o\), the image \(V_f\) of \(\delta(f)\) varies through all the codimension-1 subspaces
of \(H^0(X, \omega_X((g_Y + 2)A))\) containing \(H^0(X, \omega_X(g_Y A))\), with the exception
of \(H^0(X, \omega_X((g_Y + 1)A))\). The proof of the claim is complete. \(\square\)

4. General irreducible singular curves

**Proposition 4.1.** Let \(a\) and \(b\) be positive integers. Let \(X\) be a general smooth
curve of genus \(g \geq 0\), and \(P\) and \(Q\) general points on \(X\). Then the complete
linear system of sections of \(\omega_X(aP + bQ)\) has only simple ramification points,
and \(P\) and \(Q\) are not among them.

**Proof.** If \(g = 0\), all complete linear systems on \(X\) have no ramification points.
If \(g = 1\), the curve \(X\) can be any curve of genus 1, as long as \(P - Q\) is neither
\(a\)-torsion nor \(b\)-torsion in the Jacobian variety.

Assume \(g > 1\). Let \(i < g\) be any positive integer, and put \(j := g - i\).
Let \(Y\) and \(Z\) be two general smooth curves, \(Y\) of genus \(i\), and \(Z\) of genus
\(j\), and let \(A\) and \(M\) be general points of \(Y\), and \(B\) and \(N\) general points
of \(Z\). By induction on the genus, we may assume the statement of the
proposition holds for \((Y, A, M)\) and \(\omega_Y (aA + (b + j)M)\), and for \((Z, B, N)\) and \(\omega_Z (bB + (a + i)N)\).

Let \(X_0\) be the nodal curve of genus \(g\) given as the union of \(Y\) and \(Z\), with \(M\) identified with \(N\). Since \(X_0\) is nodal, and \(A\) and \(B\) are nonsingular points of \(X_0\), there are a regular smoothing \(p: X \to S\) of \(X_0\), and sections \(\Gamma, \Delta \subset X\) such that, identifying the closed fiber of \(p\) with \(X_0\), we have \(\Gamma \cap X_0 = \{A\}\) and \(\Delta \cap X_0 = \{B\}\).

Let \(\mathcal{X}_*\) denote the general fiber of \(p\). Let \(P\) and \(Q\) be the points of intersection of \(\Gamma\) and \(\Delta\) with \(\mathcal{X}_*\). The 2-pointed curve \((\mathcal{X}_*, P, Q)\) is defined over a finitely generated field extension of \(\mathbb{Q}\), and hence can be viewed as a 2-pointed complex curve. We claim the statement of the proposition holds for this two-pointed curve.

To prove our claim, let \(\omega_p\) be the relative dualizing sheaf of \(p\). Let \(W_* \subset \mathcal{X}_*\) be the ramification divisor of the complete linear system of sections of \(\omega_p (a\Gamma + b\Delta)|_{\mathcal{X}_*}\). We need only show that \(W_*\) is reduced, and does not contain \(P\) or \(Q\) in its support. For this, it is enough to show that the limit Weierstrass scheme \(\partial W\) is reduced and does not contain \(A\) or \(B\) in its support.

Since \(\mathcal{X}\) is regular, \(Y\) and \(Z\) are Cartier divisors. Set
\[
\mathcal{L}_1 := \omega_p (a\Gamma + b\Delta + (b + j - 1)Z), \quad \mathcal{L}_2 := \omega_p (a\Gamma + b\Delta + (a + i - 1)Y).
\]

Then
\[
\mathcal{L}_1|_Y = \omega_Y (aA + (b + j)M), \quad \mathcal{L}_1|_Z = \omega_Z (bB + (2 - b - j)N),
\]
\[
\mathcal{L}_2|_Z = \omega_Z (bB + (a + i)N), \quad \mathcal{L}_2|_Y = \omega_Y (aA + (2 - a - i)M).
\]

Due to the generality of \(M\) and \(N\), we have
\[
h^0 (Y, \mathcal{L}_2|_Y (-M)) = h^0 (Z, \mathcal{L}_1|_Z (-N)) = 0,
\]
and hence the natural maps
\[
\frac{H^0 (\mathcal{X}, \mathcal{L}_1)}{tH^0 (\mathcal{X}, \mathcal{L}_1)} \longrightarrow H^0 (Y, \mathcal{L}_1|_Y) \quad \text{and} \quad \frac{H^0 (\mathcal{X}, \mathcal{L}_2)}{tH^0 (\mathcal{X}, \mathcal{L}_2)} \longrightarrow H^0 (Y, \mathcal{L}_2|_Z)
\]
are injective. They are actually isomorphisms, since \(H^0 (\mathcal{X}, \mathcal{L}_i)\) is free of rank \(g + a + b - 1\), and
\[
h^0 (Y, \mathcal{L}_1|_Y) = h^0 (Z, \mathcal{L}_2|_Z) = g + a + b - 1,
\]
by the Riemann–Roch theorem.

Since
\[
\mathcal{L}_2 \cong \mathcal{L}_1 ((g + a + b - 2)Y),
\]
it follows that \([\partial W] = [R_1] + [R_2]\), where \(R_1\), resp. \(R_2\), is the ramification divisor of the complete linear system of sections of \(\omega_Y (aA + (b + j)M)\), resp. \(\omega_Z (bB + (a + i)N)\); see Subsection 2.4. By induction, \(R_1\) and \(R_2\) are reduced and, viewed as subschemes of \(X_0\), disjoint. So \(\partial W\) is reduced. In addition,
A and B are not in the supports of \( R_1 \) and \( R_2 \), and thus are not in support of \( \partial W \) either.

**Theorem 4.2.** Let \( C \) be a nodal curve of genus \( g \geq 2 \) that is union of a smooth curve \( X \) of genus \( g - 1 \) and a chain \( E = (E_1, \ldots, E_{g-1}) \) of \( g - 1 \) rational curves. Suppose \( X \) meets \( E \) only in \( E_1 \) and \( E_{g-1} \), at points \( A \) and \( B \), respectively. Suppose \( X \) is general, and \( A \) and \( B \) are general points of \( X \). Let \( p : X \to S \) be a regular smoothing of \( C \) and \( W(p) \) its Weierstrass scheme. Then \( W(p) \) is a Cartier divisor, and the difference

\[
W(p) - \sum_{i=1}^{g-1} \frac{i(g-i)g}{2} E_i
\]

is effective and intersects each fiber of \( p \) transversally. In particular, the limit Weierstrass scheme of \( p \) is reduced and contains no node of \( C \) in its support.

**Proof.** Figure 4 below describes the curve \( C \).

![Figure 4. The nodal curve.](image)

We will prove the second statement first. Let \( \omega_p \) be the relative dualizing sheaf of \( p \). By adjunction, \( \omega_p \) restricts to the trivial sheaf on each \( E_i \) and to \( \omega_X(A+B) \) on \( X \). So, each global section of \( \omega_p \) that vanishes on \( X \) vanishes on the whole fiber \( C \), and hence the restriction map

\[
H^0(X, \omega_p) \to H^0(X, \omega_X(A+B))
\]

has image of dimension \( g \). Since \( h^0(X, \omega_X(A+B)) = g \) by the Riemann–Roch theorem, the restriction map is surjective. So the complete linear system of sections of \( \omega_X(A+B) \) is a limit canonical system on \( X \).

By Proposition 4.1, the ramification points of \( H^0(X, \omega_X(A+B)) \) are simple. So, the limit Weierstrass scheme \( \partial W \) of \( f \) is reduced on \( X - \{A,B\} \). Also, since neither \( A \) nor \( B \) is a ramification point of \( H^0(X, \omega_X(A+B)) \), again by Proposition 4.1, Plücker formula yields

\[
(20) \quad \deg(\partial W \cap X - \{A,B\}) = g^3 - g^2.
\]
For each $i = 1, \ldots, g - 1$ set
\begin{equation}
L_i := \omega_p \left(-i(g - i)E_i - \sum_{m=1}^{i-1} m(g - i)E_m - \sum_{m=i+1}^{g-1} (g - m)iE_m\right).
\end{equation}

Notice that $L_i$ has degree $g$ on $E_i$, zero on each $E_m$ for $m \neq i$, and $L_i|_X \cong \omega_X \left(- (g - i - 1)A - (i - 1)B\right)$.

Since $A$ and $B$ are general points of $X$, we have $h^0(X, L_i|_X) = 1$ and
\begin{equation}
h^0(X, L_i|_X(-A)) = h^0(X, L_i|_X(-B)) = h^0(X, L_i|_X(-A - B)) = 0.
\end{equation}

Let $V_i$ be the image of

\[
H^0(X, L_i) \rightarrow H^0(E_i, L_i|_{E_i})
\]

Since $h^0(X, L_i|_X(-A - B)) = 0$, and $\deg L_i|_{E_m} = 0$ for $m \neq i$, each global section of $L_i$ that vanishes on $E_i$ vanishes on the whole $C$. Thus $(V_i, L_i|_{E_i})$ is a limit canonical system on $E_i$.

Let $P$ and $Q$ be the nodes of $C$ in $E_i$. A section in $V_i$ that vanishes at $P$ (or $Q$) is the restriction of a global section of $L_i$ that vanishes at $P$ (or $Q$). Since $\deg L_i|_{E_m} = 0$ for $m \neq i$, it follows from (22) that this global section vanishes on all components of $C$ but possibly $E_i$. In particular, its restriction in $V_i$ vanishes at $P$ and $Q$. In other words,
\begin{equation}
V_i(-P) = V_i(-Q) = V_i(-P - Q) = H^0(E_i, L_i|_{E_i}(-P - Q)),
\end{equation}

where the last equality follows from dimension considerations.

By (23), the system $V_i$ contains a complete subsystem of codimension 1, namely $H^0(E_i, L_i|_{E_i}(-P - Q))$. Since complete systems on the projective line have no ramification points, the order sequence of $V_i$ at each point of $E_i$ starts with $0, \ldots, g - 2$. The last order can only be $g - 1$ or $g$, since $L_i|_{E_i}$ has degree $g$. Thus all ramification points of $V_i$ are simple. In addition, since $P$ and $Q$ are not ramification points of $H^0(E_i, L_i|_{E_i}(-P - Q))$, it follows from (23) that they are not ramification points of $(V_i, L_i|_{E_i})$ either.

Thus $\partial W$ is reduced on $E_i - \{P, Q\}$ and, since neither $P$ nor $Q$ is a ramification point of $(V_i, L_i|_{E_i})$, Plücker formula yields
\begin{equation}
\deg(\partial W \cap E_i - \{P, Q\}) = g.
\end{equation}

Finally, since there are $g - 1$ rational components in $C$, we get
\[
g^3 - g = \deg(\partial W \cap C) \geq g^3 - g^2 + (g - 1)g = g^3 - g,
\]
where the inequality follows from combining (20) and (24) for $i = 1, \ldots, g - 1$. The inequality is thus an equality, showing that we accounted for all points in the support of $\partial W$, and hence that all of them appear with multiplicity 1. The second statement is proved.

To prove the first statement, we will consider a filtration $L_{i,j}$ of subsheaves of $\omega_p$ containing $L_i$, defined below.
For each \( i = 1, \ldots, g - 1 \) and each \( j = 0, 1, \ldots, i(g - i) - 1 \), let \( k, k', \ell, \ell' \) be integers such that
\[
(25) \quad j = ki + \ell = k'(g - i) + \ell', \quad 0 \leq k, \ell' \leq g - i - 1, \quad 0 \leq k', \ell \leq i - 1,
\]
and put
\[
(26) \quad c_{i,j,m} := km + \max(0, \ell - i + m + 1), \quad m = 1, \ldots, i,
\]
\[
(27) \quad c'_{i,j,m} := k'(g - m) + \max(0, \ell' + i - m + 1), \quad m = i, \ldots, g - 1.
\]
Notice that \( c_{i,j,i} = c'_{i,j,i} = j + 1 \). Finally, set
\[
D_{i,j} := \sum_{m=1}^{i} c_{i,j,m}E_m + \sum_{m=i+1}^{g-1} c'_{i,j,m}E_m,
\]
and put \( L_{i,j} := \omega_f(-D_{i,j}) \). Notice that
\[
\mathcal{L}_i = \mathcal{L}_{i,i(g-i)-1}.
\]
For each \( i = 1, \ldots, g - 1 \) set \( D_{i,-1} := 0 \) and \( \mathcal{L}_{i,-1} := \omega_f \). And for each \( j = 0, 1, \ldots, i(g - i) - 1 \) set \( F_{i,j} := D_{i,j} - D_{i,j-1} \). Then \( \mathcal{L}_{i,j} = \mathcal{L}_{i,j-1}(-F_{i,j}) \).
It follows from (25), (26) and (27) that
\[
F_{i,j} = E_{i-\ell} + E_{i-\ell+1} + \cdots + E_i + E_{i+1} + \cdots + E_{i+\ell'}.
\]
Using this, it can also be shown, by induction on \( j \), that
\[
\mathcal{L}_{i,j}|_{X} = \begin{cases} 
\omega_X((1 - k)A + (1 - k')B) & \text{if } \ell \neq i - 1 \text{ and } \ell' \neq g - i - 1, \\
\omega_X(-kA + (1 - k')B) & \text{if } \ell = i - 1 \text{ and } \ell' \neq g - i - 1, \\
\omega_X((1 - k)A - k'B) & \text{if } \ell \neq i - 1 \text{ and } \ell' = g - i - 1, \\
\omega_X(-kA - k'B) & \text{if } \ell = i - 1 \text{ and } \ell' = g - i - 1.
\end{cases}
\]
and
\[
\deg \mathcal{L}_{i,j}|_{E_m} = \begin{cases} 
k + k' + 2 & \text{if } m = i, \\
-1 & \text{if } m = i - \ell - 1 \text{ or } m = i + \ell' + 1, \\
0 & \text{otherwise}.
\end{cases}
\]
It follows that
\[
(28) \quad h^0(F_{i,j}, \mathcal{L}_{i,j-1}|_{F_{i,j}}) = k + k' + 1,
\]
and, setting \( \hat{F}_{i,j} := \overline{C - F_{i,j}} \),
\[
h^0(\hat{F}_{i,j}, \mathcal{L}_{i,j}|_{\hat{F}_{i,j}}) = h^0(X, \omega_X(-kA - k'B)).
\]
Since \( A \) and \( B \) are general points of \( X \), and \( k + k' \leq g - 2 \), it follows that
\[
(29) \quad h^0(\hat{F}_{i,j}, \mathcal{L}_{i,j}|_{\hat{F}_{i,j}}) = g - 1 - k - k' .
\]
Consider now the natural short exact sequence,
\[
(30) \quad 0 \to \mathcal{L}_{i,j} \to \mathcal{L}_{i,j-1} \to \mathcal{L}_{i,j-1}|_{F_{i,j}} \to 0.
\]
Restricting it to $C$, we obtain the short exact sequence

$$0 \to L_{i,j}\big|_{\bar{F}_{i,j}} \to L_{i,j-1}|_C \to L_{i,j-1}|_{F_{i,j}} \to 0.$$  

So, putting together (28) and (29), we get

$$h^0(C, L_{i,j-1}|_C) \leq h^0(F_{i,j}, L_{i,j-1}|_{F_{i,j}}) + h^0(\bar{F}_{i,j}, L_{i,j}|_{\bar{F}_{i,j}}) = g.$$  

By semicontinuity, equality holds above. Also, each of the restriction maps

in the composition below is surjective:

$$(31) \quad H^0(X, L_{i,j}) \longrightarrow H^0(C, L_{i,j-1}|_C) \longrightarrow H^0(F_{i,j}, L_{i,j-1}|_{F_{i,j}}).$$  

As a consequence, not only is the sequence derived from (30),

$$(32) \quad 0 \to H^0(X, L_{i,j}) \to H^0(X, L_{i,j-1}) \to H^0(F_{i,j}, L_{i,j-1}|_{F_{i,j}}) \to 0,$$  

left-exact, but also right-exact.

Now, the inclusion map

$$H^0(X, L_{i,j}) \longrightarrow H^0(X, L_{i,j-1})$$  

is a map of free $\mathbb{C}[|t|]$-modules of the same rank $g$. Because of the exactness in (32), and because of (28), the determinant of this map is an element of $\mathbb{C}[|t|]$ of order $k + k' + 1$. Summing $k + k' + 1$ for $j = 0, 1, \ldots, i(g - i) - 1$, we get that the determinant of the inclusion

$$H^0(X, L_{i}) \longrightarrow H^0(X, \omega_{p})$$  

has order $i(g - i)g/2$. Thus, using (21), and using the functoriality of the ramification scheme (see Subsection 2.2), we get

$$\text{mult}_{E_i}(W(p)) + \frac{i(g - i)g}{2} = gi(g - i) + \text{mult}_{E_i}(W(p, L_i, L_i)).$$  

However, since the kernel of the restriction $H^0(X, L_{i}) \to H^0(E_{i}, L_{i}|_{E_{i}})$ is $tH^0(X, L_{i})$, we have that $\text{mult}_{E_i}(W(p, L_i, L_i)) = 0$. Thus

$$\text{mult}_{E_i}(W(p)) = gi(g - i) - \frac{i(g - i)g}{2} = \frac{i(g - i)g}{2},$$  

as stated. The transversality in the statement is equivalent to the fact that the limit Weierstrass divisor is reduced. \hfill \Box

**Remark 4.3.** Keep the setup of Theorem 4.3, but do not assume that $p$ is regular. If the singularity types of the nodes of $C$ in $X$ are equal, the same conclusions holds, with the only difference that we replace the $E_{i}$ by the $E'_{p}$ defined in Subsection 2.3. In fact, making these substitutions, the proof given above works word by word.

**Theorem 4.4.** Let $g$ be a positive integer. Let $X$ be a general smooth curve of genus $g - 1$, and $A$ and $B$ general points of $X$. Let $C$ be the nodal curve of genus $g$ obtained from $X$ by identifying $A$ and $B$. Then no point of $C$ is a limit of special Weierstrass points on a family of smooth curves degenerating to $C$. 
Proof. The statement is true if \( g = 1 \), because an elliptic curve has no Weierstrass points. Suppose \( g > 1 \) now. Let \( p: X \to S \) be a smoothing of \( C \). We claim that the geometric general fiber has no special Weierstrass point.

So, as seen in Subsection 2.3, we may replace \( p \) by a smoothing \( \tilde{p}: \tilde{X} \to S \), whose special fiber is the curve \( C \) described in Theorem 4.2, whose nodes have equal singularity types in \( \tilde{X} \). Then, by Theorem 4.2 and Remark 4.3 thereafter, the limit Weierstrass divisor of \( \tilde{p} \) is reduced.

\[ \bigcirc \]

5. The special ramification locus

5.1. (Special ramification loci) Let \( \overline{M}_g \) be the coarse moduli space of stable curves of genus \( g \geq 4 \), and \( \overline{M}_{g,1} \) that of pointed curves. Let \( \pi: \overline{M}_{g,1} \to \overline{M}_g \) be the forgetful map.

Let \( M_g \subset \overline{M}_g \) denote the open locus of smooth curves and \( M^0_g \subset M_g \) that of smooth curves without nontrivial automorphisms. Set \( M_{g,1} := \pi^{-1}(M_g) \) and \( M^0_{g,1} := \pi^{-1}(M^0_g) \), and let \( \pi: M_{g,1} \to M_g \) and \( \pi^0: M^0_{g,1} \to M^0_g \) be the restrictions of \( \pi \). Recall that \( M^0_g \) is a fine moduli space, with universal family \( \pi^0 \). Also, since \( g \geq 4 \), the boundary of \( M^0_g \) in \( M_g \) has codimension at least 2; see [11], p. 53.

Let \( VSW_g \) denote the closure of \( VSW(\pi^0) \) in \( \overline{M}_{g,1} \); see Subsection 2.2.

Also, let \( \overline{E}_{g,j} \) denote the closure of \( E_{g,j}(\pi^0) \) in \( \overline{M}_{g,1} \) for each \( j \).

Notice that \( VSW_g \) and the schemes \( \overline{E}_{g,j} \) have pure codimension 2 in \( \overline{M}_{g,1} \). Indeed, their intersections with \( M^0_{g,1} \) are determinantal of codimension at most 2. Since \( VSW(\pi^0) \subseteq VSW(\pi^0) \) set-theoretically for each \( j \), it is enough to show that an irreducible component \( U \) of \( VSW(\pi^0) \) cannot have codimension smaller than 2. And indeed, since the restriction \( \pi^0: VSW(\pi^0) \to M^0_g \) is finite, if \( \text{codim}(U, M^0_{g,1}) \leq 1 \), then \( \pi^0(U) = M^0_g \). However, this is not possible because a general curve has no special Weierstrass points; see [4], Prop. 3.1.

We observe that \( VSW_g \cap M_{g,1} \) parameterizes the smooth pointed curves \( (C, P) \) such that \( P \) is a special Weierstrass point of \( C \). Indeed, there is a smooth, projective map \( p: X \to S \) whose fibers are curves of genus \( g \), and such that the horizontal maps in the naturally induced commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi_1} & M_{g,1} \\
p \downarrow & & \downarrow \pi \\
S & \xrightarrow{\phi} & M_g
\end{array}
\]

are finite and surjective; see [11], Lemma 3.89, p. 142. The image of \( VSW(p) \) under \( \phi_1 \) parameterizes the smooth pointed curves \( (C, P) \) such that \( P \) is a
special Weierstrass point of \( C \). We need to show that

\[
\phi_1(VSW(p)) = VSW_g \cap M_{g,1}.
\]

Let \( S^0 := \phi^{-1}(M_0^0) \). Since \( \phi \) is finite and surjective, the boundary of \( S^0 \) in \( S \) has codimension 2. Reasoning as in the last paragraph, we see that the irreducible components of \( VSW(p) \) meet \( X_0 : = \pi^{-1}(M_0^0) \). Thus \( VSW(p) \) is set-theoretically the closure of \( VSW(p^0) \) in \( X \), where \( p^0 := p|_{X^0} : X^0 \to S^0 \). Then

\[
\phi_1(VSW(p)) = \phi_1(VSW(p^0)) \cap M_{g,1} = VSW(\pi^0) \cap M_{g,1} = VSW_g \cap M_{g,1}.
\]

An analogous reasoning shows that \( VE_{g,j} \cap M_{g,1} \) parameterizes the smooth pointed curves \((C, P)\) such that \( P \) is a special Weierstrass point of \( C \) of type \( g + j \), for \( j = -1, 1 \).

Set \( E_{g,j} := \pi_*[VE_{g,j}] \) for \( j = -1, 1 \) and \( SW_g = \pi_*[VSW_g] \). Since \( \pi_*|_{VSW_g} \) is generically finite, \( SW_g \) and the \( E_{g,j} \) are cycles of pure codimension 1 of \( \overline{M}_g \).

5.2. (The Picard group) Let \( \overline{M}_g \) be the coarse moduli space of stable curves of genus \( g \geq 3 \). Since \( \overline{M}_g \) has only finite quotient singularities, the group of codimension-1 cycle classes of \( \overline{M}_g \) with rational coefficients is isomorphic to the Picard group with rational coefficients, \( \text{Pic}(\overline{M}_g) \otimes \mathbb{Q} \). This group is freely generated by the tautological class \( \lambda \) and the boundary classes \( \delta_0, \delta_1, \ldots, \delta_{\lfloor g/2 \rfloor} \); see \([1] \), Thm. 1, p. 154 and \([11] \), Prop. 3.88, p. 141. 

**Theorem 5.3.** Let \( g \geq 4 \). The following formula holds in \( \text{Pic}(\overline{M}_g) \otimes \mathbb{Q} \):

\[
SW_g = (3g^4 + 4g^3 + 9g^2 + 6g + 2)\lambda - \frac{g(g + 1)(2g^2 + g + 3)}{6} \delta_0 \sum_{i=1}^{\lfloor g/2 \rfloor} i(g - i) \delta_i.
\]

**Proof.** The above formula was shown in \([10] \), Thm. 5.1, p. 44, using the method of test curves. Here we will show how to obtain it directly.

Let \( p : X \to S \) be a flat, projective map over a smooth, projective curve \( S \) whose fiber \( X_s \) over each \( s \in S \) is a stable curve of genus \( g \). Assume \( p \) has finitely many singular fibers, all of them unimodal, and that each of its nonsingular fibers has no nontrivial automorphisms. Also, assume the general fiber of \( p \) is a general curve of genus \( g \), and in particular has no special Weierstrass points, and each of the singular fibers of \( p \) is general, among singular fibers of like nature.

For each \( s \in S \) such that \( X_s \) is singular, let \( P_s \) denote the unique node of \( X_s \), and let \( k_s \) be the singularity type of \( P_s \) in \( X \). Let \( S_0 \) be the set
of \( s \in S \) such that \( X_s \) is singular and irreducible. In addition, for each \( i = 1, \ldots, \lfloor g/2 \rfloor \), let
\[
S_i := \{ s \in S \mid X_s \text{ contains a component of genus } i \}.
\]

Up to replacing \( S \) by a finite covering we may assume that \( k_s + 1 \) is divisible by \( g \) for each \( s \in S_0 \).

Let \( \lambda' := c_1(p_*\omega_p) \), where \( \omega_p \) is the relative dualizing sheaf of \( p \), and set
\[
\delta_i' := \sum_{s \in S_i} (k_s + 1)[s]
\]
for \( i = 0, 1, \ldots, \lfloor g/2 \rfloor \).

Let \( p^0 \) the restriction of \( p \) over its smooth locus. Since the general fiber of \( p \) has no special Weierstrass points, \( VSW(p^0) \) is finite. Consider the zero cycle \( SW(p^0) := p_*[VSW(p^0)] \). Viewing \( SW(p^0) \) as a divisor class in \( \text{Pic}(S) \otimes \mathbb{Q} \), to show the statement of the theorem we need only show that in \( \text{Pic}(S) \otimes \mathbb{Q} \) the class of \( SW(p^0) \) satisfies an equation similar to that of \( \overline{\text{W}}_{\nu} \), but with \( \delta \) replaced by \( \lambda' \) and the \( \delta_i \).

By considering blowups and blowdowns, we may find a map of schemes \( \beta: \tilde{X} \to X \) such that

1. \( \beta \) is an isomorphism away from the points \( P_s \) for \( s \in S_0 \);
2. for each \( s \in S_0 \), the fiber \( \tilde{X}_s := (p \circ \beta)^{-1}(s) \) is the nodal curve that is the union of the normalization of \( X_s \) and a chain of \( g - 1 \) rational curves connecting the branches over \( P_s \), and \( \beta: \tilde{X}_s \to X_s \) is the map collapsing the chain to \( P_s \);
3. the singularity type in \( X \) of each of the nodes of \( \tilde{X}_s \) is \((k_s + 1)/g - 1\) for each \( s \in S_0 \).

Let \( \tilde{p} := p \circ \beta \). For each \( s \in S_0 \), let \( \tilde{k}_s := (k_s + 1)/g - 1 \), let \( \tilde{X}_s := \tilde{X}_s \cap \tilde{X}_s \) be the normalization of \( X_s \), and let \( E_s = (E_{s,1}, \ldots, E_{s,g-1}) \) be the chain of rational components of \( \tilde{X}_s \). Also, for each \( i \geq 1 \) and each \( s \in S_i \), let \( Y_s \) denote the component of the fiber \( \tilde{X}_s \) of genus \( i \) and \( Z_s \) that of genus \( g - i \). Notice that \( Y_s \) and \( Z_s \) are Cartier divisors of \( X \) such that
\[
Y_s \cdot Z_s = Z_s \cdot Y_s = 1.
\]
Likewise, \( (\tilde{X}_s)^{\tilde{p}} \) and the \( E_{s,i} \) are Cartier divisors of \( \tilde{X} \) such that
\[
(\tilde{X}_s)^{\tilde{p}} + \sum_{i=1}^{g-1} E_{s,i}^{\tilde{p}} = (\tilde{k}_s + 1)\tilde{p}^* (s),
\]
and
\[
(\tilde{X}_s)^{\tilde{p}} \cdot E_{s,i} = E_{s,i}^{\tilde{p}} \cdot \tilde{X}_s = \begin{cases} 1 & \text{if } i = 1 \text{ or } i = g - 1, \\ 0 & \text{otherwise} \end{cases}
\]
for each \( i = 1, \ldots, g - 1 \), while for each distinct \( i, j = 1, \ldots, g - 1 \),
\[
E_{s,i}^p \cdot E_{s,j} = \begin{cases} 1 & \text{if } |j - i| = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( \tilde{p} := p \circ \beta \) and consider the Weierstrass divisor \( \tilde{W}p \). Let
\[
W := \tilde{W}p - \sum_{i=1}^{[g/2]} \sum_{s \in S_i} \left( \binom{g - i + 1}{2} Y_{s}^{\tilde{p}} + \binom{i + 1}{2} Z_{s}^{\tilde{p}} \right)
- \sum_{s \in S_0} \sum_{i=1}^{g-1} \frac{i(g-i)}{2} E_{s,i}^p.
\]

We claim that \( W \) is effective and intersects transversally each singular fiber of \( \tilde{p} \).

Indeed, the claim can be checked infinitesimally around each \( s \in S \) for which \( \tilde{X}_s \) is singular. So, it is possible to treat the fibers over \( S_0 \) and over the \( S_i \) for \( i > 0 \) independently.

First, Cukierman showed in [3], Prop. 2.0.8, p. 325, that \( W \) is effective on a neighborhood of the fiber over any \( s \in S_i \), for \( i > 0 \), and that \( W \) intersects properly this fiber. (In fact, Cukierman assumed \( \tilde{p} \) regular, but his proof goes through in our more general situation.) The intersection of \( W \) with the fiber is the limit Weierstrass divisor, which can be computed using Formula (5), as in the proof of Theorem 4.2, and shown to be reduced. So \( W \) intersects transversally each fiber \( \tilde{X}_s \) for each \( s \in S_i \) and each \( i = 1, \ldots, [g/2] \).

Finally, our Theorem 4.2 coupled with Remark 4.3 shows that \( W \) is effective on a neighborhood of the fiber over any \( s \in S_0 \), and intersects that fiber transversally. This finishes the proof of the claim.

Since \( W \) intersects transversally each singular fiber of \( \tilde{p} \), its branch locus over \( S \) is simply \( \text{VSW}(p^0) \). Since \( \text{VSW}(p^0) \) has codimension 2 in \( \tilde{X} \), we have
\[
[\text{VSW}(p^0)] = c_2(J^2_1(\mathcal{O}_{\tilde{X}}(W))) = c_1(W)(c_1(W) + c_1(\omega_{\tilde{p}})).
\]

In addition, \( c_1(W) \) can be computed from the definition of \( W \), since, by Plücker formula,
\[
c_1(W_{\tilde{p}}) = \left( \frac{g + 1}{2} \right) c_1(\omega_{\tilde{p}}) - \tilde{p}^* \lambda'.
\]

It is now a simple but tedious task, using the intersection theory of \( \tilde{X} \), and the formula (see [11], Formula 3.110, p. 158)
\[
\tilde{p}_*(c_1(\omega_{\tilde{p}})^2) = 12\lambda' - \delta'_0 - \delta'_1 - \cdots - \delta'_{[g/2]},
\]
to compute \( \text{SW}(p^0) = p_*[\text{VSW}(p^0)] \) and obtain the stated formula. \( \square \)
6. The special ramification loci of type $g + j$

**Proposition 6.1.** Let $g \geq 4$. The following statements hold:

1. $\overline{V_E}_{g,j} \subseteq VSW_g$ for $j = -1, 1$.
2. Set-theoretically, $\overline{V_E}_{g,-1} \cup \overline{V_E}_{g,1} = VSW_g$.
3. In the cycle group of codimension-2 cycles of $\overline{M}_{g,1}$,
   \[ [VSW_g] = [V_{E_{g,-1}}] + [V_{E_{g,1}}]. \]

**Proof.** The statements are local, and can be checked on a neighborhood of a point of $M_{g,1}^0$, parameterizing a pointed stable curve $(C,P)$.

Locally, the scheme $V_{E_{1}}(\pi_0)$ is given by all maximal minors of a (Wronskian) matrix of regular functions of the form
\[ M = \begin{bmatrix} A \\ c \\ d \end{bmatrix}, \]
where $A$ is a matrix with $g$ columns and $g - 1$ rows, and $c$ and $d$ are row vectors of size $g$. Furthermore, $V_{E_{-1}}(\pi_0)$ is given by all maximal minors of the matrix $A$, and $VSW(\pi_0)$ is given by the determinants of the square submatrices
\[ M_1 := \begin{bmatrix} A \\ c \end{bmatrix} \quad \text{and} \quad M_2 := \begin{bmatrix} A \\ d \end{bmatrix}. \]

Since the determinants of $M_1$ and $M_2$ are also maximal minors of $M$, it is clear that $VSW(\pi_0) \supseteq V_{E_{1}}(\pi_0)$. On the other hand, expanding these two determinants by the last rows, we see that they belong to the ideal of maximal minors of $A$. Thus $VSW(\pi_0) \supseteq V_{E_{-1}}(\pi_0)$ as well. Statement 1 is proved.

As for Statement 2, just observe that if $A$ has rank $g - 1$ at $(C,P)$, and thus $(C,P)$ is not in $V_{E_{-1}}(\pi_0)$, then the vanishing of the determinants giving $VSW(\pi_0)$ at $(C,P)$ says that the two last rows of $M$ depend linearly on the rows of $A$, and hence $M$ has rank $g - 1$, yielding that $(C,P)$ is in $V_{E_{1}}(\pi_0)$.

Finally, for the last statement we may resort to Lemma 5.3 in [4]. To apply this lemma, and immediately get the last statement, we need only check that $M_1$ has rank at least $g - 1$ at $(C,P)$, or equivalently, that $h^0(C, \omega_C(-gP)) \leq 1$. However, it follows from [5], Thm. 4.13, p. 918, and Claim 3 on p. 920, that the subset of $M_{g,1}$ parameterizing pointed curves $(C,P)$ such that $h^0(C, \omega_C(-gP)) \geq 2$ has codimension at least 3. Since the equality we want to prove involves codimension-2 cycles, we may indeed assume that $h^0(C, \omega_C(-gP)) \leq 1$. \qed

**6.2. (Computing the classes $E_{g,j}$)** We may write
\[ E_{g,j} = a_j \lambda - b_{j,0} \delta_0 - b_{j,1} \delta_1 - \cdots - b_{j,\lfloor g/2 \rfloor} \delta_{\lfloor g/2 \rfloor} \in \text{Pic}(\overline{M}_g) \otimes \mathbb{Q} \]
for $j = -1, 1$, where the $a_j$ and the $b_{j,\ell}$ are rational numbers to be computed.
The coefficients $a_{-1}$ and $a_1$ were determined by using Porteous formula on a general family of smooth curves in [6], Thm. 4.33, p. 21 and Thm. A1.4, p. 59:

\[(33) \quad a_{-1} = \frac{g^2(g-1)(3g-1)}{2} \quad \text{and} \quad a_1 = \frac{(g+1)(g+2)(3g^2 + 3g + 2)}{2}.
\]

To compute the remaining numbers, we use test families. Our first family, $p_0: X_0 \to S_0$, is constructed by taking a general pencil of plane cubics passing through a fixed point, and adding to each member of the pencil a general smooth curve of genus $g-1$ meeting the cubic transversally at the fixed point on the cubic and at a fixed general point on the curve of genus $g-1$; see [11], Ex. 3.140, p. 173. It follows from Theorem 3.4 and [4], Prop. 3.1, that for a nonsingular member of the pencil, the resulting stable curve does not contain any limit of special Weierstrass points. Thus $\int_{S_0} SW_g \geq 0$, with strict inequality if and only if there is a fiber of $p_0$ represented by a point in the support of $SW_g$. However, a quick computation, using the formula for $SW_g$ in Theorem 5.3, yields

\[(34) \quad \int_{S_0} SW_g = 0.
\]

So, in particular,

\[\int_{S_0} E_{g,j} = 0 \quad \text{for } j = -1, 1,
\]

yielding the relations

\[(35) \quad a_j - 12b_{j,0} + b_{j,1} = 0 \quad \text{for } j = -1, 1.
\]

(These relations were obtained directly by Diaz [6], Lemma 7.2, p. 40, for $j = -1$, and by Gatto [10], p. 67, for $j = 1$, and from them Gatto concluded [24]. Here we proceed in the opposite way.) It is thus enough to compute the $b_{j,i}$ for $j = -1, 1$ and $i \geq 1$.

For each $i = 1, \ldots, \lfloor g/2 \rfloor$, let $X$ be a general smooth curve of genus $g-i$, let $Y$ be a general smooth curve of genus $i$, and $B \in Y$ a general point. Identifying the diagonal $\Delta \subset X \times X$ with $\{B\} \times X \subset Y \times X$ in the natural way, we get a flat, projective map $p_i: F_i \to X$ whose fiber over each $P \in X$ is the uninodal stable curve union of $X$ and $Y$ with $P$ and $B$ identified; denote by $X \cup_P Y$ this fiber. Let $\gamma_i: X \to \overline{M}_g$ be the induced map.

The crux of the method is to compute the degree of the pullback $\gamma_i^*E_{g,j}$ for $j = -1, 1$. First, we claim that the number of points $Q \in X \cup_P Y$ for all $P \in X$ which are limits of special Weierstrass points is a lower bound for this degree.

Indeed, since $E_{g,j} = \pi_*[VE_{g,j}]$, the support of the cycle $E_{g,j}$ parameterizes the curves which are limits of smooth curves having a special Weierstrass point. So, by Theorem 5.4, a curve $X \cup_P Y$ is parameterized in this support only if either the complete linear system of sections of $\omega_X((i+1)P)$ or that of $\omega_Y((g-i+1)B)$ has special ramification points, other than $P$ or $B$, or $P$
is a Weierstrass point of $X$. However, by [4], Prop. 3.1, since $B$ is general, the linear system $H^0(Y, \omega_Y((g+i+1)B))$ has no special ramification points other than $B$, and the same is true for $H^0(X, \omega_X((i+1)P))$ for a general $P$. Thus $\gamma_i(X)$ intersects the support of $E_{g,j}$ in finitely many points.

To compute $\gamma_i^*E_{g,j}$ we do as follows: since $\overline{M}_g$ has finite quotient singularities, there is an integer $n > 0$ such that $nE_{g,j} = [D]$ for some effective Cartier divisor $D$; then $\gamma_i^*E_{g,j} = (1/n)[\gamma_i^{-1}D]$. Since $\gamma_i^{-1}D$ is finite, to compute it we need only work infinitesimally around each $P \in X$. So, let $\tilde{p}_i: \tilde{F}_i \to \tilde{X}$ denote the base change of $p_i$ to $\tilde{X} := \text{Spec}(\mathcal{O}_{X,P})$, and $\tilde{\gamma}_i: \tilde{X} \to \overline{M}_g$ the corresponding map. Let $u: U \to T$ be the universal deformation of $X \cup_P Y$. Here, $T$ is the spectrum of a ring of power series, whence regular. Let $\beta: T \to \overline{M}_g$ be the induced map. Because of the universal property of $u$, there is a map $\alpha: \tilde{X} \to T$ such that $\tilde{\gamma}_i = \beta \circ \alpha$. We will first describe $\beta^{-1}(D)$, or the cycle $[\beta^{-1}(D)]$, which amounts to the same because $T$ is regular.

Let $T^0 \subset T$ be the open subscheme parameterizing the smooth fibers of $u$ without nontrivial automorphisms. Set $U^0 := u^{-1}(T^0)$ and denote by $u^0: U^0 \to T^0$ the induced map. The map $\beta$ restricts to a map $\beta^0: T^0 \to M^0_g$. Since the boundary of $M^0_g$ in $\overline{M}_g$ has codimension at least 2, to compute $[\beta^{-1}(D)]$ we need only describe $(\beta^0)^{-1}(D \cap M^0_g)$. But $E_{g,j} \cap M^0_g$ is equal to $\pi^0_0[VE_j(x^0)]$, and is a Cartier divisor of $M^0_g$, because $M^0_g$ is smooth. Moreover, since both $T$ and $M^0_g$ are regular, and since the diagram

$$
\begin{array}{ccc}
U^0 & \longrightarrow & M^0_{g,1} \\
r^0 \downarrow & & \downarrow \pi^0 \\
T^0 & \longrightarrow & M^0_g
\end{array}
$$

is Cartesian, and the formation of $VE_j(\cdot)$ commutes with base change, we have

$$(\beta^0)^*(E_{g,j} \cap M^0_g) = u^0_0[VE_j(u_0)].$$

Thus

$$[(\beta^0)^{-1}(D \cap M^0_g)] = nu^0_0[VE_j(u^0)],$$

and so $[\beta^{-1}(D)] = nu_0[u_0[VE_j(u^0)]]$, where $[VE_j(u^0)]$ denotes the closure of $VE_j(u^0)$ in $U$. It follows that

$$\tilde{\gamma}_i^*E_{g,j} = (1/n)[\tilde{\gamma}_i^{-1}(D)] = (1/n)[\alpha^{-1}\beta^{-1}(D)] = \alpha^*u_0[VE_j(u^0)].$$

Finally, since $u$ is universal, we have a Cartesian diagram,

$$
\begin{array}{ccc}
\tilde{F}_i & \longrightarrow & \tilde{U} \\
\tilde{p}_i \downarrow & & \downarrow u \\
\tilde{X} & \longrightarrow & \tilde{T},
\end{array}
$$
that shows that
\[ (36) \quad \tilde{\gamma}_i^* \mathcal{E}_{g,j} = \tilde{\mu}_i \alpha_1^* [\mathcal{V} E_j(u^0)]. \]

Thus, it follows from (36) that the multiplicity of \( \tilde{\gamma}_i^* \mathcal{E}_{g,j} \) (at the closed point of \( \tilde{X} \)) is at least the number of points \( Q \in X \cup P Y \) whose image in \( U \) under \( \alpha_1 \) lies in \( \mathcal{V} E_j(u^0) \). Now, since the singularities of \( \mathcal{M}_g \) are quotient, \( \hat{O}_{T,0} \) is a finite \( \hat{O}_{\mathcal{M}_g} \)-module, where \( C := X \cup P Y \), and 0 is the closed point of \( T \). Thus \( T^0 \) has codimension 2 in \( T \) as well. Reasoning as in Subsection 5.1, we can show that, as a set, \( \mathcal{V} E_j(u^0) \) is also the closure of the analogous subscheme defined for the subfamily of \( u \) consisting of all smooth fibers. So the multiplicity of \( \tilde{\gamma}_i^* \mathcal{E}_{g,j} \) is at least the number of points \( Q \in X \cup P Y \) that are limits of special Weierstrass points. Our claim is thereby proved.

For each \( j = -1, 1 \), let \( d_{j,i} \) be the number of points \( (P, Q) \in X \times X \) such that \( Q \neq P \) and \( Q \) is a special ramification point of type \( g+j \) of the complete linear system of sections of \( \omega_X((i+1)P) \). Also, let \( d'_{j,i} \) be the number of ramification points different from \( B \) of the complete linear system of sections of \( \omega_Y((g - i + 1 + j)B) \), and \( d'' \) be the number of Weierstrass points of \( X \). For each \( j = -1, 1 \), set
\[ e_{j,i} = d_{j,i} + d'' d'_{j,i}. \]

By Theorem 3.4, the number of points \( Q \in X \cup P Y \), for all \( P \in X \), that are limits of special Weierstrass points is \( e_{j,i} \). Thus
\[ (37) \quad \int_X \gamma_i^* \mathcal{E}_{g,j} \geq e_{j,i}. \]

Now, since \( Y \) and \( B \) are general, by [4, Prop. 3.1], the complete linear system of sections of \( \omega_Y((g - i + 1 + j)B) \) has no special ramification points, other than \( B \) with weight \( i \). Also, since \( X \) is general, \( X \) does not have any special Weierstrass points; see [4, Cor. 3.3]. Thus, by Plücker formula,
\[ d'_{j,i} = (g+j)(g+i+j-1) + (i-1)(g+j)(g+j-1) - i = (g+j)^2 - i, \]
and
\[ d'' = (g - i - 1)(g - i + 1). \]

In addition, by [4, Thm. 5.6],
\[ d_{j,i} = (g - i)(g - i - 1) \left( (g+j)^2(i+1)^2 - (g - i + j)^2 \right). \]

Thus
\[ e_{j,i} = i(g - i)(g - i - 1) \left( (g+j)^2(g+3) + 2(g+j) - (g + 1) \right). \]

Now, using Theorem 5.3 and that \( \gamma_i^* \delta_j = 0 \) for every \( j \neq i \), while
\[ \gamma_i^* \lambda = 0 \quad \text{and} \quad \int_X \gamma_i^* \delta_i = 2(1 - g + i), \]
a simple computation yields:

\[(38) \int_X \gamma_i^* SW_g = e_{-1,i} + e_{1,i}.\]

Using Proposition [6.1] and using \((37)\) for \(j = -1, 1\) and \((38)\), we get

\[e_{-1,i} + e_{1,i} = \int_X \gamma_i^* SW_g = \int_X \gamma_i^* \overline{E}_{g,-1} + \int_X \gamma_i^* \overline{E}_{g,1} \geq e_{-1,i} + e_{1,i}.\]

Thus \(\int_X \gamma_i^* \overline{E}_{g,j} = e_{j,i}\) for \(j = -1, 1\), and hence we may get \(b_{j,i}\):

\[(39) \quad b_{j,i} = i(g - i)\left(\left(g + j\right)^2\left(g + 3\right) + 2(g + j) - (g + 1)\right).\]

Finally, using \((38), (39)\) and the relations \((35)\), formulas for \(b_{-1,0}\) and \(b_{1,0}\) can be obtained.

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