EXISTENCE AND \(L^\infty\)-ESTIMATES FOR ELLIPTIC EQUATIONS INVOLVING CONVOLUTION

GRETA MARINO AND DUMITRU MOTREANU

ABSTRACT. In this paper, with a fixed \(p \in (1, +\infty)\) and a bounded domain \(\Omega \subset \mathbb{R}^N\), \(N \geq 2\), whose boundary \(\partial \Omega\) fulfills the Lipschitz regularity, we study the following boundary value problem
\[
- \text{div}\, A(x, u, \nabla u) + a|u|^{p-2}u = B(x, \rho \ast E(u), \nabla(\rho \ast E(u))) \quad \text{in } \Omega,
\]
\[
A(x, u, \nabla u) \cdot \nu = C(x, u) \quad \text{on } \partial \Omega,
\]
where \(A: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N\), \(B: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\), \(C: \partial \Omega \times \mathbb{R} \to \mathbb{R}\) are Carathéodory functions, \(a > 0\) is a constant, \(E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)\) is an extension operator related to \(\Omega\), and \(\rho\) is an integrable function on \(\mathbb{R}^N\). This is a novel problem that involves the nonlocal operator assigning to \(u\) the convolution \(\rho \ast E(u)\) of \(\rho\) with \(E(u)\). Under verifiable conditions, we prove the existence of a (weak) solution to problem (P) by using the surjectivity theorem for pseudomonotone operators. Moreover, through a modified version of Moser iteration up to the boundary initiated in [5, 6] we show that (any) weak solution to (P) is bounded.

1. INTRODUCTION

Let \(\Omega \subset \mathbb{R}^N\), \(N \geq 2\), be a bounded domain with a Lipschitz continuous boundary \(\partial \Omega\) and let \(p \in (1, +\infty)\) be a real number. It is well-known that there exists an extension operator \(E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)\) meaning that \(E\) is a linear map satisfying
\[
E(u)|_{\Omega} = u, \quad \forall u \in W^{1,p}(\Omega)
\]
and for which there exists a constant \(C = C(\Omega) > 0\) depending only on \(\Omega\) such that
\[
\|E(u)\|_{L^p(\mathbb{R}^N)} \leq C(\Omega)\|u\|_{L^p(\Omega)},
\]
\[
\|E(u)\|_{W^{1,p}(\mathbb{R}^N)} \leq C(\Omega)\|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega)
\]
(see [1, 2]). In the terminology of [1] such a map \(E\) is called a \((1, p)\)-extension operator for \(\Omega\). Generally, the extension operators are constructed by using reflection maps and partitions of unity. For the rest of the paper, we fix an extension operator \(E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)\).

We state the following boundary value problem
\[
- \text{div}\, A(x, u, \nabla u) + a|u|^{p-2}u = B(x, \rho \ast E(u), \nabla(\rho \ast E(u))) \quad \text{in } \Omega,
\]
\[
A(x, u, \nabla u) \cdot \nu = C(x, u) \quad \text{on } \partial \Omega,
\]
where \(a > 0\) is a constant, \(\nu(x)\) denotes the outer unit normal of \(\Omega\) at \(x \in \partial \Omega\), \(\rho \ast E(u)\) stands for the convolution product of some integrable function \(\rho\) on \(\mathbb{R}^N\) with \(E(u)\), and \(A, B, C\) are Carathéodory functions satisfying suitable \(p\)-structure growth conditions. Due to the presence of convolution, problem (1.1) is nonlocal. Furthermore, in the statement of problem (1.1) we have full dependence on the solution \(u\) and on its gradient \(\nabla u\), which makes the problem highly non-variational, so the variational methods are not applicable. The boundary condition in (1.1) is nonhomogeneous and includes the Robin boundary condition.

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The starting point of this work has been the elliptic problem in [7] with homogeneous Dirichlet boundary condition
\[-\Delta_p u - \mu \Delta_q u = f(x, \rho \ast u, \nabla (\rho \ast u)) \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega,\] (1.2)
involving the \(p\)-Laplacian \(\Delta_p\) and the \(q\)-Laplacian \(\Delta_q\) with \(1 < q < p < +\infty\), where for the first time the boundary value problem with convolution for solution and its gradient was considered. Any solution \(u \in W^{1,p}_0(\Omega)\) of (1.2) can be identified with \(E(u) \in W^{1,p}(\mathbb{R}^N)\) obtained by extension with zero outside \(\Omega\). In this case both \(\rho\) and \(u\) are integrable functions on \(\mathbb{R}^N\) and the convolution \(\rho \ast u\) in (1.2) makes sense. This is no longer possible for (1.1) because we have \(u \in W^{1,p}(\Omega)\) and the extension by zero outside \(\Omega\) generally does not produce an element of \(W^{1,p}(\mathbb{R}^N)\). Here is the essential point where the extension operator \(E\) is necessary in (1.1).

Finally, among papers involving quasilinear elliptic equations with convection term we can refer to [4].

The aim of this paper is two-fold: to establish an existence result for (1.1) and to provide a priori estimates for the solutions to (1.1) up to the boundary showing their uniform boundedness. The proof of existence of solutions to (1.1) relies on the theory of pseudomonotone operators and properties of convolution and extension operator. In order to prove a priori estimates for problem (1.1) and show the boundedness of its solutions we develop a modified version of Moser iteration originating in [5, 6].

First of all we recall that the critical exponents corresponding to \(p\) in \(\Omega\) and on \(\partial \Omega\) are denoted by \(p^*\) and \(p_\ast\), respectively (see Section 2).

For the existence result, our assumptions are as follows.

(A) The maps \(A: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N\), \(B: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\), and \(C: \partial \Omega \times \mathbb{R} \to \mathbb{R}\) are Carathéodory functions (i.e., they are measurable in the first variable and continuous in the others) satisfying the following conditions:

(A1) \(|A(x, s, \xi)| \leq a_1|\xi|^{p-1} + a_2|s|^{p-1} + a_3\) for a.e. \(x \in \Omega\),

(A2) \(A(x, s, \xi - \xi') \cdot (\xi - \xi') > 0\) for a.e. \(x \in \Omega\),

(A3) \(|B(x, s, \xi)| \leq a_4|\xi|^p - a_5\) for a.e. \(x \in \Omega\),

(A4) \(|B(x, s, \xi)| \leq f(x) + b_1|s|^\alpha_1 + b_2|\xi|^\alpha_2\) for a.e. \(x \in \Omega\),

(A5) \(|C(x, s)| \leq c_1|s|^\alpha_3 + c_2\) for a.e. \(x \in \partial \Omega\),

for all \(s \in \mathbb{R}\) and \(\xi, \xi' \in \mathbb{R}^N\), \(\xi \neq \xi'\), with positive constants \(a_i, b_j, c_k (i \in \{1, \ldots, 5\}, j, k \in \{1, 2\})\), with

\[\alpha_1, \alpha_2, \alpha_3 \in [0, p-1)\] (1.3)

and a nonnegative function \(f \in L^{r'}(\Omega)\) with \(r \in [1, p^*)\).

Assumptions (A1)-(A2) are the Leray-Lions conditions, while (A3) is a coercivity condition. In problem (1.2) we have \(A(x, s, \xi) = |\xi|^{p-2}\xi + \mu|\xi|^{q-2}\xi\), with \(1 < q < p < +\infty\) and \(\mu \geq 0\), which fulfills these assumptions. The maps \(B\) and \(C\) are only subject to the growth conditions (A4)-(A5).

By a (weak) solution to problem (1.1) we mean any function \(u \in W^{1,p}(\Omega)\) verifying

\[\int_\Omega A(x, u, \nabla u) \cdot \nabla \varphi dx + a \int_\Omega |u|^{p-2}u \varphi dx = \int_\Omega B(x, \rho \ast E(u), \nabla (\rho \ast E(u))) \varphi dx + \int_{\partial \Omega} C(x, u) \varphi d\sigma\] (1.4)

for all \(\varphi \in W^{1,p}(\Omega)\). Under assumptions (A), all the integrals in (1.4) are finite for \(u, \varphi \in W^{1,p}(\Omega)\), thus the definition of weak solution is meaningful. In the same spirit, \(u \in W^{1,p}_0(\Omega)\) is a (weak) solution to (1.2) if

\[\int_\Omega (|\nabla u|^{p-2} + \mu|\nabla u|^{q-2}) \nabla u \cdot \nabla \varphi dx = \int_\Omega B(x, \rho \ast u, \nabla (\rho \ast u)) \varphi dx\]

holds for every \(\varphi \in W^{1,p}_0(\Omega)\).
Theorem 1.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary $\partial \Omega$ endowed with the extension operator $E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ and let $\rho \in L^1(\mathbb{R}^N)$. If hypotheses (A) are satisfied, then there exists a (weak) solution to problem (1.1).

The proof of Theorem 1.1 is the object of Section 3.

Now we turn to the uniform boundedness of solutions to problem (1.1). We formulate the following hypotheses.

(H) The maps $\mathcal{A}: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B}: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\mathcal{C}: \partial \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the conditions

(H1) $|\mathcal{A}(x,s,\xi)| \leq a_1|\xi|^{p-1} + a_2|s|^{p^*-1} + a_3$ for a.e. $x \in \Omega$,

(H2) $\mathcal{A}(x,s,\xi) \cdot \xi \geq a_4|\xi|^p - a_5|s|^{p^*} - a_6$ for a.e. $x \in \Omega$,

(H3) $|\mathcal{B}(x,s,\xi)| \leq f(x) + b_1|s|^\alpha_1 + b_2|\xi|^{\alpha_2}$ for a.e. $x \in \Omega$,

(H4) $|\mathcal{C}(x,s)| \leq c_1|s|^{p^*-1} + c_2$ for a.e. $x \in \partial \Omega$,

for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, with nonnegative constants $a_i, b_j, c_k (i \in \{1, \ldots, 6\}, j, k \in \{1, 2\})$ and $\alpha_1, \alpha_2$ such that

$$0 \leq \alpha_1 < p^* - p, \quad 0 \leq \alpha_2 < \min \left\{ p - 1, \frac{p}{p^*}(p^* - p) \right\},$$

and a nonnegative function $f \in L^r(\Omega)$, with $r \in [1, p^*/p)$.

Theorem 1.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary $\partial \Omega$ endowed with the extension operator $E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ and let $\rho \in L^1(\mathbb{R}^N)$. Assume that hypotheses (H) are satisfied. Then, every (weak) solution $u \in W^{1,p}(\Omega)$ to problem (1.1) belongs to $L^\infty(\Omega)$ with the trace $\gamma u \in L^\infty(\partial \Omega)$.

The proof of Theorem 1.2 is given in Section 4.

Combining Theorems 1.1 and 1.2 we obtain the following existence result of bounded solutions to problem (1.1).

Corollary 1.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary $\partial \Omega$ endowed with the extension operator $E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ and let $\rho \in L^1(\mathbb{R}^N)$. Assume that hypotheses (A1)-(A3), (A4) with $\alpha_2$ as in (1.5), and (A5) are satisfied. Then, there exists a (weak) solution $u \in W^{1,p}(\Omega)$ to problem (1.1) which belongs to $L^\infty(\Omega)$ and whose trace $\gamma u$ is an element of $L^\infty(\partial \Omega)$.

Corollary 1.3 is a direct consequence of Theorems 1.1 and 1.2 noticing that Theorems 1.1 and 1.2 can be simultaneously applied.

We illustrate the applicability of our results by an example using the extension operator constructed in [2, page 275].

Example 1.4. Consider in $\mathbb{R}^2$ the rectangular domains $\Omega = (0, 1) \times (0, 1)$, $\Omega_1 = (0, 1) \times (-1, 1)$, $\Omega_2 = (-1, 1) \times (-1, 1)$, $\Omega_3 = (-1, 1) \times (-1, 3)$, $\Omega = (-1, 3) \times (-1, 3)$. We introduce the maps $R_1: W^{1,p}(\Omega) \to W^{1,p}(\Omega_1)$, $R_2: W^{1,p}(\Omega_1) \to W^{1,p}(\Omega_2)$, $R_3: W^{1,p}(\Omega_2) \to W^{1,p}(\Omega_3)$, and $R_4: W^{1,p}(\Omega_3) \to W^{1,p}(\Omega)$, respectively, by

$$(R_1u)(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_2 > 0, \\ u(x_1, -x_2) & \text{if } x_2 < 0 \end{cases}$$

for all $u \in W^{1,p}(\Omega)$ and $(x_1, x_2) \in \Omega$,

$$(R_2u)(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_1 > 0, \\ u(-x_1, x_2) & \text{if } x_1 < 0 \end{cases}$$
for all \( u \in W^{1,p}(\Omega_1) \) and \((x_1, x_2) \in \Omega_1\),
\[
(R_3 u)(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_2 < 1, \\ u(x_1, 2 - x_2) & \text{if } x_2 > 1 \end{cases}
\]
for all \( u \in W^{1,p}(\Omega_2) \) and \((x_1, x_2) \in \Omega_2\), and
\[
(R_4 u)(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_1 < 1, \\ u(2 - x_1, x_2) & \text{if } x_1 > 1 \end{cases}
\]
for all \( u \in W^{1,p}(\Omega_3) \) and \((x_1, x_2) \in \Omega_3\).

For a fixed \( \psi \in C^1(\hat{\Omega}) \) with \( \psi = 1 \) on \( \Omega \) and \( \text{supp } \psi \subset \hat{\Omega} \), the linear map \( E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2) \) which carries each \( u \in W^{1,p}(\Omega) \) to the function \( E u \in W^{1,p}(\mathbb{R}^2) \) obtained by extending \( \psi(R_4 \circ R_3 \circ R_2 \circ R_1 u) \) with zero outside \( \hat{\Omega} \) is an extension operator. Accordingly, given a constant \( a > 0 \), a function \( \rho \in L^1(\mathbb{R}^2) \) and a Carathéodory function \( B: \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) satisfying (H) and (1.5) we state the Neumann problem
\[
-\Delta_{\rho} u + a |u|^{p-2} u = B(x, \rho * E(u), \nabla (\rho * E(u))) \quad \text{in } \Omega,
\]
\[
|\nabla u|^{p-2} \nabla u \cdot \nu = 0 \quad \text{on } \partial \Omega.
\]
A frequent form of \( B \) is \( B(x, s, \xi) = g(s) + h(\xi) \). Our results apply to the stated problem.

The rest of the paper is organized as follows. Section 2 contains preliminaries to be used in the sequel. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2.

2. Preliminaries

The Euclidean norm of \( \mathbb{R}^N \) is denoted by \(| \cdot |\), while the notation \( \cdot \) stands for the standard inner product on \( \mathbb{R}^N \). By \(| \cdot |\) we also denote the Lebesgue measure on \( \mathbb{R}^N \). In the rest of the paper, for every \( r \in (1, +\infty) \) we denote by \( r' \) its Hölder conjugate, that is \( r' = \frac{r}{r-1} \).

For any \( r \in [1, \infty) \) and a domain \( \Omega \subset \mathbb{R}^N \), we denote by \( L^r(\Omega) \) and \( W^{1,r}(\Omega) \) the usual Lebesgue and Sobolev spaces equipped with the norms
\[
\|u\|_{L^r(\Omega)} = \left( \int_{\Omega} |u|^r \, dx \right)^{\frac{1}{r}},
\]
\[
\|u\|_{W^{1,r}(\Omega)} = \left( \int_{\Omega} |\nabla u|^r \, dx \right)^{\frac{1}{r}} + \left( \int_{\Omega} |u|^r \, dx \right)^{\frac{1}{r}}.
\]
(2.1)

Recall that the norm of \( L^\infty(\Omega) \) is
\[
\|u\|_{L^\infty(\Omega)} = \text{ess sup}_{\Omega} |u|.
\]

For any \( u \in W^{1,r}(\Omega) \), set \( u^\pm := \max\{\pm u, 0\} \), which yields
\[
u^\pm \in W^{1,r}(\Omega), \quad |\nu| = u^+ + u^-, \quad u = u^+ - u^-.
\]
(2.2)

By the Sobolev embedding theorem there exists a linear continuous embedding \( i: W^{1,r}(\Omega) \to L^{r^*}(\Omega) \), where the corresponding critical exponent \( r^* \) in the domain is given by
\[
r^* = \begin{cases} \frac{N r}{N - r} & \text{if } r < N, \\ +\infty & \text{if } r \geq N. \end{cases}
\]
The boundary \( \partial \Omega \) is endowed with the \((N-1)\)-dimensional Hausdorff (surface) measure. The measure of \( \partial \Omega \) is denoted by \(|\partial \Omega|\). The Lebesgue spaces \( L^s(\partial \Omega) \), with \( 1 \leq s \leq +\infty \), have the norms
\[
\|u\|_{L^s(\partial \Omega)} = \left( \int_{\partial \Omega} |u|^s \, d\sigma \right)^{\frac{1}{s}} \quad (1 \leq s < +\infty), \quad \|u\|_{L^\infty(\partial \Omega)} = \text{ess sup}_{\partial \Omega} |u|.
\]
There exists a unique linear continuous map \( \gamma : W^{1,r}(\Omega) \to L^r(\partial\Omega) \), called the trace map, characterized by \( \gamma(u) = u|_{\partial\Omega} \) whenever \( u \in W^{1,r}(\Omega) \cap C(\overline{\Omega}) \), where \( r_* \) is the corresponding critical exponent on the boundary defined as

\[
    r_* = \left\{ \begin{array}{ll}
        \frac{N-1}{N-r} & \text{if } r < N, \\
        +\infty & \text{if } r \geq N.
    \end{array} \right.
\]

As usual, the subspace of \( W^{1,r}(\Omega) \) consisting of zero trace elements is denoted \( W^{1,r}_0(\Omega) \). For the sake of notational simplicity, we drop the use of the symbol \( \gamma \) writing simply \( u \) in place of \( \gamma u \). We refer to [1] for the theory of Sobolev spaces.

The following propositions are useful in the proof of our boundedness result.

**Proposition 2.1.** ([5, Proposition 2.2]) Let \( u \in L^p(\Omega) \), \( 1 < p < +\infty \), be nonnegative. If it holds

\[
    \|u\|_{L^{\infty}(\Omega)} \leq C
\]

for a constant \( C > 0 \) and a sequence \( (\alpha_n) \subset \mathbb{R}_+ \) such that \( \alpha_n \to +\infty \) as \( n \to \infty \), then \( u \in L^\infty(\Omega) \).

**Proposition 2.2.** ([5, Proposition 2.4]) Let \( 1 < p < +\infty \) and let \( u \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \). Then, \( u \in L^\infty(\partial\Omega) \).

Recall that for \( \rho \in L^1(\mathbb{R}^N) \) and \( u \in W^{1,p}(\mathbb{R}^N) \), with \( 1 < p < +\infty \), the convolution \( \rho \ast u \) is defined by

\[
    (\rho \ast u)(x) := \int_{\mathbb{R}^N} \rho(x - y)u(y)dy \quad \text{for a.e. } x \in \mathbb{R}^N.
\]

The weak partial derivatives of the convolution \( \rho \ast u \) are expressed by

\[
    \frac{\partial}{\partial x_i}(\rho \ast u) = \rho \ast \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \ldots, N.
\]

Thanks to Tonelli’s and Fubini’s theorems as well as Hölder’s inequality, there hold

\[
    \|\rho \ast u\|_{L^r(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \|u\|_{L^r(\mathbb{R}^N)}
\]

for every \( r \in [1, p^*] \) and

\[
    \left\| \rho \ast \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)} \quad \text{for } i = 1, \ldots, N \tag{2.3}
\]

(see [2, Theorem 4.15]). Taking into account the fact that the function \( t \mapsto t^{1/2} \) is sublinear as well as the function \( t \mapsto t^p \) is convex on \((0, +\infty)\) and (2.3), it follows that

\[
    \|\nabla(\rho \ast u)\|_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla(\rho \ast u)|^p dx = \int_{\mathbb{R}^N} \left( \sum_{i=1}^N \left( \frac{\partial u}{\partial x_i} \right)^2 \right)^{p/2} dx
\]

\[
    \leq \int_{\mathbb{R}^N} \left( \sum_{i=1}^N \left( \frac{\partial u}{\partial x_i} \right) \right)^p dx \leq N^{p-1} \int_{\mathbb{R}^N} \left( \sum_{i=1}^N \left( \frac{\partial u}{\partial x_i} \right) \right)^p dx \leq N^p \|\rho\|_{L^1(\mathbb{R}^N)} \|\nabla u\|_{L^p(\mathbb{R}^N)}.
\]

Finally, we recall the main theorem on the pseudomonotone operators that will be used to prove our existence result. Let \( X \) be a reflexive Banach space endowed with the norm \( \| \cdot \| \). The norm convergence is denoted by \( \to \) and the weak convergence by \( \rightharpoonup \). We denote by \( X^* \) the topological dual of \( X \) and by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( X \) and \( X^* \). A map \( A : X \to X^* \) is called bounded if it maps bounded sets to bounded sets. It is said to be coercive if there holds

\[
    \lim_{\|u\| \to +\infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty.
\]

Finally, \( A \) is called pseudomonotone if \( u_n \rightharpoonup u \) in \( X \) and

\[
    \limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle \leq 0
\]
We consider the terms in (3.1) separately. First note that Hölder's inequality gives

\[ \langle Au, u - w \rangle \leq \liminf_{n \to +\infty} \langle Au_n, u_n - w \rangle, \quad \forall w \in X. \]

The surjectivity theorem for pseudomonotone operators reads as follows (see, e.g., [3]).

**Theorem 2.3.** Let \( X \) be a reflexive Banach space, let \( A: X \to X^* \) be a pseudomonotone, bounded and coercive operator, and let \( g \in X^* \). Then, there exists at least a solution \( u \in X \) to the equation \( Au = g \).

3. Proof of Theorem 1.1

Throughout the proof of the theorem, we will denote by \( C_i, i \in \mathbb{N} \), constants which depend on the given data.

With a fixed \( \rho \in L^1(\mathbb{R}^N) \) and an extension operator \( E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N) \), we introduce the nonlinear operator \( T: W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^* \) by

\[
(Tu, \varphi) = \int_{\Omega} A(x, u, \nabla u) \cdot \nabla \varphi dx + a \int_{\Omega} |u|^{p-2} u \varphi dx
- \int_{\Omega} B(x, \rho * E(u), \nabla (\rho * E(u))) \varphi dx - \int_{\partial \Omega} C(x, u) \varphi d\sigma
\]

for all \( u, \varphi \in W^{1,p}(\Omega) \). Assumption (A) guarantees that \( T \) is well defined.

Let us show that \( T \) is also bounded. Indeed, fix \( \varphi \in W^{1,p}(\Omega) \) such that \( \|\varphi\|_{W^{1,p}(\Omega)} \leq 1 \). Then,

\[
|\langle Tu, \varphi \rangle| \leq \int_{\Omega} |A(x, u, \nabla u)\|\nabla \varphi\| dx + a \int_{\Omega} |u|^{p-1}\|\varphi\| dx
+ \int_{\Omega} |B(x, \rho * E(u), \nabla (\rho * E(u)))\|\varphi\| dx + \int_{\partial \Omega} |C(x, u)\|\varphi\| d\sigma.
\]

We estimate the terms of the inequality above separately. First of all, observe that

\[
\int_{\Omega} |A(x, u, \nabla u)\|\nabla \varphi\| dx \leq \int_{\Omega} (a_1|\nabla u|^{p-1} + a_2|u|^{p-1} + a_3)|\nabla \varphi| dx
\leq a_1\|\nabla u\|_{L^p(\Omega)}^{p-1}\|\nabla \varphi\|_{L^p(\Omega)} + a_2\|u\|_{L^p(\Omega)}^{p-1}\|\nabla \varphi\|_{L^p(\Omega)}
+ a_3\|\varphi\|_{L^p(\Omega)}^{p-1}\|\nabla \varphi\|_{L^p(\Omega)}
\leq a_1\|\nabla u\|_{L^p(\Omega)}^{p-1} + a_2\|u\|_{L^p(\Omega)}^{p-1} + C_1,
\]

as well as

\[
a \int_{\Omega} |u|^{p-1}\|\varphi\| dx \leq a\|u\|_{L^p(\Omega)}^{p-1}\|\varphi\|_{L^p(\Omega)} \leq a\|u\|_{L^p(\Omega)}^{p-1}.
\]

Thanks to (A4) we also have

\[
\int_{\Omega} |B(x, \rho * E(u), \nabla (\rho * E(u)))\|\varphi\| dx \leq \int_{\Omega} (f(x) + b_1|\rho * E(u)|^{\alpha_1} + b_2|\nabla (\rho * E(u))|^{\alpha_2}) \|\varphi\| dx.
\]

We consider the terms in (3.5) separately. First note that Hölder's inequality gives

\[
\int_{\Omega} f(x)|\varphi| dx \leq \|f\|_{L^{r'}(\Omega)}\|\varphi\|_{L^r(\Omega)}
\leq \|f\|_{L^{r'}(\Omega)}\|\varphi\|_{L^r(\Omega)}\|\Omega\|^{\frac{r'}{r}}
\leq C_2.
\]
Moreover, exploiting the properties of $E$ and of the convolution and the Sobolev embedding we have

$$b_1 \int_\Omega |\rho * E(u)|^{\alpha_1} |\varphi| dx \leq \||\rho * E(u)||_{L^{p_1}(\Omega)}^{\alpha_1}
\leq C_3 \||\rho||_{L^1(\Omega)} \||E(u)||_{L^{p_1}(\Omega)}^{\alpha_1}
\leq C_4 \||\rho||_{L^1(\Omega)} \||u||_{L^{p_1}(\Omega)}^{\alpha_1}
\leq C_5 \||u||_{W^{1,p}(\Omega)}^{\alpha_1},$$

as well as

$$b_2 \int_\Omega |\nabla (\rho * E(u))|^{\alpha_2} |\varphi| dx \leq b_2 \||\nabla (\rho * E(u)||_{L^{p_2}(\Omega)}^{\alpha_2}
\leq C_6 \||\rho||_{L^2(\Omega)} \||\nabla E(u)||_{L^{p_2}(\Omega)}^{\alpha_2}
\leq C_7 \||\nabla E||_{L^{p_2}(\Omega)}^{\alpha_2} \leq C_8 \||u||_{W^{1,p}(\Omega)}^{\alpha_2}.$$ 

Finally, hypothesis (A5) gives the following estimate for the boundary term in (3.2)

$$\int_{\partial \Omega} |C(x,u)||\varphi| d\sigma \leq \int_{\partial \Omega} (c_1 |u|^{\alpha_3} + c_2) |\varphi| d\sigma
\leq c_1 \||u||_{L^{p_3}(\partial \Omega)}^{\alpha_3} \||\varphi||_{L^{p_3}(\partial \Omega)}^{\alpha_3}
\leq c_1 \||u||_{W^{1,p}(\Omega)}^{\alpha_3} + C_8.$$ 

Taking into account (3.3)-(3.9) and applying once again the Sobolev embedding, from (3.2) we derive

$$\|T u, \varphi\| \leq C_9 (\|u||_{W^{1,p}(\Omega)}^{\beta} + 1),$$

for all $\|\varphi\|_{W^{1,p}(\Omega)} \leq 1$, with $\beta := \max\{p - 1, \alpha_1, \alpha_2, \alpha_3\}$. This in turn implies

$$\|Tu\|_{W^{1,p}(\Omega)}^{\beta} \leq C_9 (\|u||_{W^{1,p}(\Omega)}^{\beta} + 1),$$

which shows that $T$ is bounded.

Now we prove that $T$ is pseudomonotone. Toward this, let $(u_n)_{n \in \mathbb{N}} \subset W^{1,p}(\Omega)$ be a sequence satisfying $u_n \rightarrow u$ for some $u \in W^{1,p}(\Omega)$ and

$$\limsup_{n \rightarrow +\infty} \langle Tu_n, u_n - u \rangle \leq 0.$$ 

By Hölder’s inequality and Rellich-Kondrachov compact embedding theorem it follows that, passing to a subsequence if necessary,

$$\int_\Omega |u_n|^{p-2} u_n (u_n - u) dx \leq \int_\Omega |u_n|^{p-1} u_n - u dx
\leq \||u_n||_{L^p(\Omega)}^{p-1} \|u_n - u||_{L^p(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$ 

With a similar argument already exploited in (3.6)-(3.8) we have

$$\int_\Omega |B(x, \rho * E(u_n), \nabla (\rho * E(u_n))) (u_n - u) dx|
\leq \int_\Omega |B(x, \rho * E(u_n), \nabla (\rho * E(u_n)))||u_n - u dx
\leq C_{10} \||u_n - u||_{L^r(\Omega)} + C_{11} \||u_n||_{W^{1,p}(\Omega)}^{\alpha_1} \||u_n - u||_{L^{p^* - \alpha_1}(\Omega)}
+ C_{12} \||u_n||_{W^{1,p}(\Omega)}^{\alpha_2} \||u_n - u||_{L^{p^*-\alpha_2}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$. Since

$$r, \frac{p^*}{p^* - \alpha_1}, \frac{p}{p - \alpha_2} < p^*,$$
we can apply the Rellich-Kondrachov compact embedding theorem to the previous estimate, which gives
\[
\lim_{n \to +\infty} \int_{\Omega} B(x, \rho \ast E(u_n), \nabla(\rho \ast E(u_n)))(u_n - u)\,dx = 0. \tag{3.12}
\]
Finally, assumption (A), Hölder’s inequality and the compactness of the trace mappings due to the inequalities
\[
p, \frac{p_* - \alpha_3}{p_*} < p_*,
\]
give
\[
\left| \int_{\partial \Omega} C(x, u_n)(u_n - u)\,ds \right| \leq \int_{\partial \Omega} (c_1|u_n|^{\alpha_3} + c_2)|u_n - u|\,ds \leq c_1 \|u_n\|^{\alpha_3}_{L^{p_1}(\partial \Omega)} \|u_n - u\|_{L^{\frac{p_2}{p - p_1}}(\partial \Omega)} + c_2 |\partial \Omega|^\frac{p - p_1}{p}\|u_n - u\|_{L^p(\partial \Omega)} \to 0 \quad \text{as } n \to \infty. \tag{3.13}
\]
If we gather (3.11), (3.12) and (3.13), in view of (3.1) then inequality (3.10) becomes
\[
\limsup_{n \to +\infty} \int_{\Omega} A(x, u_n, \nabla u_n)(u_n - u)\,dx \leq 0. \tag{3.14}
\]
Thanks to assumptions (A1)-(A3) it is allowed to invoke [3, Theorem 2.109]. Then (3.14) and the weak
convergence \(u_n \rightharpoonup u\) in \(W^{1,p}(\Omega)\) ensure the strong convergence \(u_n \to u\) in \(W^{1,p}(\Omega)\). Once the strong
convergence is achieved, it is straightforward to deduce from the continuity of the involved Nemytskii
maps that the nonlinear operator \(T\) is pseudomonotone.

The next step is to show that \(T\) is coercive. To this end, first observe that
\[
\langle Tv, v \rangle = \int_{\Omega} A(x, v, \nabla v) \cdot \nabla v\,dx + a \int_{\Omega} |v|^p\,dx + \int_{\partial \Omega} B(x, \rho \ast E(v), \nabla(\rho \ast E(v)))v\,ds + \int_{\partial \Omega} C(x, v)v\,ds. \tag{3.15}
\]
We estimate the terms of the inequality above separately. First of all thanks to assumption (A3) we have
\[
\int_{\Omega} A(x, v, \nabla v) \cdot \nabla v\,dx \geq \int_{\Omega} (a_4|\nabla v|^p - a_5)|v|\,dx = a_4\|\nabla v\|^{p}_{L^{p}(\Omega)} - a_5|\Omega|.
\]
Moreover, reasoning as in (3.6)-(3.8) we have
\[
\int_{\Omega} B(x, \rho \ast E(v), \nabla(\rho \ast E(v)))v \geq - \int_{\Omega} \left[ f(x) + b_1\rho \ast E(v) \right]^{\alpha_1} + b_2|\nabla(\rho \ast E(v))|^{\alpha_2}|v|\,dx \geq - C_{13} \|v\|_{L^{p}(\Omega)} - C_{14} \|v\|^{\alpha_1}_{W^{1,p}(\Omega)} \|v\|^{\frac{p-1}{p- \alpha_1}}_{L^{\frac{p}{p- \alpha_1}}(\Omega)} - C_{15} \|v\|^{\alpha_2}_{W^{1,p}(\Omega)} \|v\|^{\frac{p-1}{p- \alpha_2}}_{L^{\frac{p}{p- \alpha_2}}(\Omega)}
\]
as well as
\[
\int_{\partial \Omega} C(x, v)v\,ds \geq - \int_{\partial \Omega} (c_1|v|^{\alpha_3} + c_2)|v|\,ds \geq - c_1 \|v\|^{\alpha_3+1}_{L^{\alpha_3+1}(\partial \Omega)} - C_{16} \|v\|_{L^p(\partial \Omega)}.
\]
From (3.15) we easily derive
\[
\langle Tv, v \rangle \geq a_4\|\nabla v\|^{p}_{L^{p}(\Omega)} + a\|v\|^{p}_{L^{p}(\Omega)} - C_{17} \left[ \|v\|^{\alpha_1+1}_{W^{1,p}(\Omega)} + \|v\|^{\alpha_2+1}_{W^{1,p}(\Omega)} + \|v\|^{\alpha_3+1}_{W^{1,p}(\Omega)} + \|v\|_{W^{1,p}(\Omega)} + 1 \right].
\]
for every \( v \in W^{1,p}(\Omega) \). Then by virtue of hypothesis (1.3) we have
\[
\lim_{\|v\|_{W^{1,p}(\Omega)} \to +\infty} \frac{\langle Tv, v \rangle}{\|v\|_{W^{1,p}(\Omega)}^p} = +\infty,
\]
thus the coercivity of \( T \) ensues. We have already shown that the nonlinear operator \( T \) is bounded, pseudomonotone and coercive. Consequently, all the requirements of Theorem 2.3 are fulfilled. Therefore, there exists \( u \in W^{1,p}(\Omega) \) verifying \( T(u) = 0 \). Taking into account (3.1) it follows that \( u \) is a weak solution to problem (1.1), which completes the proof.

4. Proof of Theorem 1.2

Let \( u \in W^{1,p}(\Omega) \) be a weak solution to (1.1) for which we can admit that \( u \neq 0 \). First, we show that \( u \in L^r(\Omega) \) for every \( r \in [1, +\infty) \). According to (2.2) and to the fact that, in the nonlocal terms, the operator \( E \) and the convolution with \( \rho \) are linear maps, we can suppose that \( u \geq 0 \), otherwise we work with \( u^+ \) and \( u^- \). Moreover, throughout the proof we will denote by \( M_i, i \in \mathbb{N} \), constants which depend on the given data and possibly on the solution itself, and we will specify the dependance when it will be relevant.

Let \( h > 0 \) and set \( u_h(x) := \min\{u(x), h\} \) for \( x \in \Omega \). For every number \( \kappa > 0 \), choose \( \varphi = u_h u_h^{\kappa-1} \) as test function in (1.4). We note that
\[
\nabla \varphi = u_h^{\kappa p} \nabla u + \kappa p u_h^{\kappa p-1} \nabla u_h.
\]
Inserting such a \( \varphi \) in (1.4) gives
\[
\int_{\Omega} (A(x, u, \nabla u) \cdot \nabla u) u_h^{\kappa p} dx + \kappa p \int_{\Omega} (A(x, u, \nabla u) \cdot \nabla u_h) u_h^{\kappa p-1} u dx + a \int_{\Omega} u^p u_h^{\kappa p} dx
\]
\[
= \int_{\Omega} B(x, \rho * E(u), \nabla (\rho * E(u))) u_h^{\kappa p} dx + \int_{\partial \Omega} C(x, u) u_h^{\kappa p} ds.
\]
Applying condition (H2) yields
\[
\int_{\Omega} (A(x, u, \nabla u) \cdot \nabla u_h) u_h^{\kappa p} dx
\]
\[
\geq \int_{\Omega} \left[ a_4 |\nabla u|^p - a_5 u^{p^*} - a_6 \right] u_h^{\kappa p} dx
\]
\[
\geq a_4 \int_{\Omega} |\nabla u|^p u_h^{\kappa p} dx - (a_5 + a_6) \int_{\Omega} u^{p^*} u_h^{\kappa p} dx - a_6 |\Omega|
\]
and
\[
\int_{\Omega} (A(x, u, \nabla u) \cdot \nabla u_h) u_h^{\kappa p-1} u dx
\]
\[
= \int_{\{x \in \Omega : u(x) \leq h\}} (A(x, u, \nabla u) \cdot \nabla u_h) u_h^{\kappa p} dx
\]
\[
\geq \int_{\{x \in \Omega : u(x) \leq h\}} \left[ a_4 |\nabla u|^p - a_5 u^{p^*} - a_6 \right] u_h^{\kappa p} dx
\]
\[
\geq a_4 \int_{\{x \in \Omega : u(x) \leq h\}} |\nabla u|^p u_h^{\kappa p} dx - (a_5 + a_6) \int_{\Omega} u^{p^*} u_h^{\kappa p} dx - \kappa p a_6 |\Omega|.
\]
Note that in the last passage of both (4.2) and (4.3) we use the following fact
\[
u_h^{\kappa p} \leq u^{p^*} u_h^{\kappa p} + 1.
\]
Indeed, if \( u > 1 \), then \( u^{p^*} > 1 \), which implies that
\[
u_h^{\kappa p} \leq u^{p^*} u_h^{\kappa p} < u^{p^*} u_h^{\kappa p} + 1.
\]
If \( u \leq 1 \), then we refer to the definition of \( u_h := \min\{u(x), h\} \), and again distinguish among two cases.
If \( h > 1 \), then \( u_h(x) = u(x) \leq 1 \), and it follows that

\[
u_h^{\alpha} \leq 1 < 1 + w^r u_h^{\alpha},
\]

because \( w^r u_h^{\alpha} > 0 \). If \( h \leq 1 \), then \( u_h(x) = h \leq 1 \), and we have again

\[
u_h^{\alpha} \leq 1 < 1 + w^r u_h^{\alpha}.
\]

By means of condition (H3) we have

\[
\int_\Omega \mathcal{B}(x, \rho * E(u), \nabla(\rho * E(u))) u_h^{\alpha} dx 
\leq \int_\Omega (f(x) + b_1 |\rho * E(u)|^{\alpha_1} + b_2 |\nabla(\rho * E(u))|^{\alpha_2}) u_h^{\alpha} dx.
\] (4.4)

We estimate the terms on the right-hand side of (4.4) separately. First, through H"older's inequality we have

\[
\int_\Omega f(x) u_h^{\alpha} dx \leq \| f \|_r \left( \int_\Omega (u_h^{\alpha})^r dx \right)^{1/r} \leq M_1(1 + \| u_h^\alpha \|_{L^r(\Omega)}).
\] (4.5)

Moreover, we set \( r_1 := \frac{p^r}{p^r - \alpha_1} \) and \( r_2 := \frac{p^r}{p^r - \alpha_2} \). Making use of Hölder’s inequality, with an argument similar as in (3.7)-(3.8), we find that

\[
\int_\Omega |\rho * E(u)|^{\alpha_1} u_h^{\alpha} dx \leq \| \rho * E(u) \|_{L^r(\Omega)}^{\alpha_1} \| u_h^{\alpha} \|_{L^{r_1}(\Omega)}^{\alpha_1} 
\leq M_2 \| \rho * E(u) \|_{L^r(\Omega)}^{\alpha_1} \| u_h^{\alpha} \|_{L^{r_1}(\Omega)}^{\alpha_1} 
\leq M_3 \| \rho \|_{L^r(\Omega)}^{\alpha_1} \| u \|_{L^r(\Omega)}^{\alpha_1} \| u_h^{\alpha} \|_{L^{r_1}(\Omega)}^{\alpha_1} 
\leq M_4 \left(1 + \| u_h^{\alpha} \|_{L^{r_1}(\Omega)}^{\alpha_1} \right),
\] (4.6)

and

\[
\int_\Omega |\nabla(\rho * E(u))|^{\alpha_2} u_h^{\alpha} dx \leq M_5 \| \nabla(\rho * E(u)) \|_{L^r(\Omega)}^{\alpha_2} \| u_h^{\alpha} \|_{L^{r_2}(\Omega)}^{\alpha_2} 
\leq M_6 \| \rho \|_{L^r(\Omega)}^{\alpha_2} \| \nabla u \|_{L^r(\Omega)}^{\alpha_2} \| u_h^{\alpha} \|_{L^{r_2}(\Omega)}^{\alpha_2} 
\leq M_7 \left(1 + \| u_h^{\alpha} \|_{L^{r_2}(\Omega)}^{\alpha_2} \right),
\] (4.7)

where the constants \( M_4 \) and \( M_7 \) depend on the solution \( u \), precisely

\[
M_4 = M_4(\| u \|_{W^{1,p}(\Omega)}) \quad \text{and} \quad M_7 = M_7(\| \nabla u \|_{L^p(\Omega)}).
\] (4.8)

Via hypothesis (H4) we estimate

\[
\int_{\partial \Omega} C(x, u) u_h^{\alpha} d\sigma \leq \int_{\partial \Omega} (c_1 w^r u_h^{\alpha} + c_2) u_h^{\alpha} d\sigma 
\leq (c_1 + c_2) \int_{\partial \Omega} w^r u_h^{\alpha} d\sigma + c_2 |\partial \Omega|.
\] (4.9)

From (1.5) and the hypothesis on \( r \), we see that

\[
\tilde{r} := \max \{r, r_1, r_2\} < \frac{p^r}{p}.
\] (4.10)

Combining (4.1)-(4.7), (4.9), (4.10) results in

\[
a_4 \left( \int_\Omega |\nabla u|^p u_h^{\alpha} dx + \kappa p \int_{\{x \in \Omega : \ u(x) \leq h\}} |\nabla u|^p u_h^{\alpha} dx \right) 
\leq (\kappa p + 1)(a_5 + a_6) \int_\Omega w^r u_h^{\alpha} dx + (c_1 + c_2) \int_{\partial \Omega} u_h^{\alpha} d\sigma 
+ M_8 \| u_h^\alpha \|_{L^{p^r}(\Omega)} + M_9 (\kappa + 1),
\] (4.11)
with positive constants $M_8$ and $M_9$ independent on $\kappa$.

Notice that

$$
\int_{\Omega} |\nabla u|^p u_h^\kappa dx + \kappa \int_{\{x \in \Omega: u(x) \leq h\}} |\nabla u|^p u_h^\kappa dx
= \int_{\{x \in \Omega: u(x) > h\}} |\nabla u|^p u_h^\kappa dx + (\kappa + 1) \int_{\{x \in \Omega: u(x) \leq h\}} |\nabla u|^p u_h^\kappa dx
\geq \frac{\kappa p + 1}{(\kappa + 1)^p} \int_{\{x \in \Omega: u(x) > h\}} |\nabla (u_h^\kappa)|^p dx,
$$

thanks to Bernoulli’s inequality $(\kappa + 1)^p \geq \kappa p + 1$ and to the fact that $(\kappa + 1)^p > 1$. Therefore, (4.11) and (2.1) entail

$$
\frac{\kappa p + 1}{(\kappa + 1)^p} \|u_h^\kappa\|^p_{W^{1,p}(\Omega)} \leq \frac{\kappa p + 1}{(\kappa + 1)^p} \|u_h^\kappa\|^p_{L^p(\Omega)} + M_{10}(\kappa p + 1) \int_{\Omega} u^{p^*} u_h^\kappa dx
+ M_{11} \int_{\partial \Omega} u^{p^*} u_h^\kappa d\sigma + M_8 \|u_h^\kappa\|^p_{L^p(\Omega)} + M_9(\kappa + 1)
\leq M_{10}(\kappa p + 1) \int_{\Omega} u^{p^*} u_h^\kappa dx + M_{11} \int_{\partial \Omega} u^{p^*} u_h^\kappa d\sigma
+ M_{12} \left( \frac{\kappa p + 1}{(\kappa + 1)^p} + 1 \right) \|u_h^\kappa\|^p_{L^p(\Omega)} + M_9(\kappa + 1).
$$

(4.12)

We now aim to estimate the critical integrals on the right-hand side of (4.12). To this end, we set $A := u^{p^* - p}$ and $B := u^{p^* - p}$, and take $\Lambda, \Gamma > 0$. Then Hölder’s inequality and the Sobolev embedding give

$$
\int_{\Omega} u^{p^*} u_h^\kappa dx
= \int_{\{x \in \Omega: A(x) \leq \Lambda\}} A(u_h^\kappa)^p dx + \int_{\{x \in \Omega: A(x) > \Lambda\}} A(u_h^\kappa)^p dx
\leq \Lambda \int_{\{x \in \Omega: A(x) \leq \Lambda\}} (u_h^\kappa)^p dx
+ \left( \int_{\{x \in \Omega: A(x) > \Lambda\}} A^{\frac{p^* - p}{\kappa - p}} dx \right)^{\frac{p^* - p}{\kappa - p}} \left( \int_{\Omega} (u_h^\kappa)^p dx \right)^{\frac{p}{\kappa - p}}
\leq \Lambda \|u_h^\kappa\|^p_{L^p(\Omega)} + \left( \int_{\{x \in \Omega: A(x) > \Lambda\}} A^{\frac{p^* - p}{\kappa - p}} dx \right)^{\frac{p^* - p}{\kappa - p}} C_{\Omega}^p \|u_h^\kappa\|^p_{W^{1,p}(\Omega)}
$$

(4.13)
as well as
\[
\int_{\partial \Omega} u^p \cdot u_h^{sp} \, d\sigma \\
= \int_{\{x \in \partial \Omega : B(x) \leq \Gamma\}} B(u u_h^p) \, d\sigma + \int_{\{x \in \partial \Omega : B(x) > \Gamma\}} B(u u_h^p) \, d\sigma
\]
\[
\leq \Gamma \int_{\{x \in \partial \Omega : B(x) \leq \Gamma\}} (u u_h^p) \, d\sigma
\]
\[
+ \left( \int_{\{x \in \partial \Omega : B(x) > \Gamma\}} B \frac{1}{M} \, d\sigma \right)^{\frac{p-1}{p}} \left( \int_{\partial \Omega} (u u_h^p) \, d\sigma \right)^{\frac{1}{p}}
\]
\[
\leq \Gamma \|u u_h^p\|_{L^p(\partial \Omega)}^p + \left( \int_{\{x \in \partial \Omega : B(x) > \Gamma\}} B \frac{1}{M} \, d\sigma \right)^{\frac{p-1}{p}} c_{\partial \Omega}^p \|u u_h^p\|_{W^{1,p}(\Omega)}^p,
\]
with the embedding constants $C_{\Omega}$ and $c_{\partial \Omega}$. Moreover, if we set
\[
f_1(\Lambda) := \left( \int_{\{x \in \Omega : A(x) > \Lambda\}} A \frac{1}{M} \, dx \right)^{\frac{p-1}{p}}
\]
as well as
\[
f_2(\Gamma) := \left( \int_{\{x \in \partial \Omega : B(x) > \Gamma\}} B \frac{1}{M} \, d\sigma \right)^{\frac{p-1}{p}},
\]
we see that
\[
f_1(\Lambda) \to 0 \quad \text{as} \quad \Lambda \to 0 \quad \text{as well as} \quad f_2(\Gamma) \to 0 \quad \text{as} \quad \Gamma \to 0.
\]
From (4.12), taking into account (4.13)-(4.15) and applying Hölder’s inequality we have
\[
\frac{\kappa p + 1}{(\kappa + 1)^p} \|u u_h^p\|_{W^{1,p}(\Omega)}^p
\]
\[
\leq M_{13} \left( \kappa p + 1 \Lambda + 1 + \frac{\kappa p + 1}{(\kappa + 1)^p} \right) \|u u_h^p\|_{L^p(\Omega)}^p \tag{4.17}
\]
\[
+ M_{10}(\kappa p + 1) f_1(\Lambda) C_{\Omega}^p \|u u_h^p\|_{W^{1,p}(\Omega)}^p + M_{11} \Gamma \|u u_h^p\|_{L^p(\partial \Omega)}^p
\]
\[
+ M_{12} f_2(\Gamma) c_{\partial \Omega}^p \|u u_h^p\|_{W^{1,p}(\Omega)}^p + M_3(\kappa + 1).
\]
Taking into account (4.16) we can choose $\Lambda = \Lambda(\kappa, u), \Gamma = \Gamma(\kappa, u) > 0$ large enough in order to have
\[
M_{10}(\kappa p + 1) f_1(\Lambda) C_{\Omega}^p = \frac{\kappa p + 1}{4(\kappa + 1)^p} \quad \text{as well as} \quad M_{11} f_2(\Gamma) c_{\partial \Omega}^p = \frac{\kappa p + 1}{4(\kappa + 1)^p}.
\]
Then from (4.17) we have
\[
\frac{\kappa p + 1}{4(\kappa + 1)^p} \|u u_h^p\|_{W^{1,p}(\Omega)}^p
\]
\[
\leq M_{13} \left( \kappa p + 1 \Lambda(\kappa, u) + 1 + \frac{\kappa p + 1}{(\kappa + 1)^p} \right) \|u u_h^p\|_{L^p(\Omega)}^p \tag{4.18}
\]
\[
+ M_{11} \Gamma(\kappa, u) \|u u_h^p\|_{L^p(\partial \Omega)}^p + M_3(\kappa + 1),
\]
where both $\Lambda(\kappa, u), \Gamma(\kappa, u)$ depend on $\kappa$ and on the solution itself.

From this point we proceed as in [5, Theorem 3.1, Case I.1] with $\|u u_h^p\|_{L^p(\Omega)}$ replaced by $\|u u_h^p\|_{L^p(\Omega)}$, which gives us
\[
\|u\|_{L^{(\kappa + 1)p}(\Omega)} \leq M_{14}(\kappa, u)
\]
for any $\kappa > 0$, where $M_{14}(\kappa, u)$ is a positive constant which depends on $\kappa$ and on the solution $u$. Consequently, the claim that $u \in L^r(\Omega)$ for every $r \in [1, \infty)$ follows.
Once the $L^r(\Omega)$-bound is reached, the proof of the $L^r(\partial\Omega)$-boundedness is straightforward (see [5, Case I.2]).

We are now in a position to establish the $L^\infty$-boundedness of $u$. Taking advantage of (4.10), we fix $q_1 \in (p^\ast, p^*)$ and $q_2 \in (p, p_\ast)$. By Hölder’s inequality and the obtained $L^r$-bounds in $\Omega$ and on $\partial\Omega$, we can express (4.12) in the form
\[
\frac{\kappa p + 1}{(\kappa + 1)p} \|\varphi u_k\|_{W^{1,p}(\Omega)}^p \leq M_{15} \left( \frac{\kappa p + 1}{(\kappa + 1)p} \|\varphi u_k\|_{L^{q_2}(\Omega)}^p + \|\varphi u_k\|_{\nabla L^1(\Omega)}^\kappa + 1 \right) + M_{16} \|\varphi u_k\|_{L^{q_1}(\partial\Omega)}^p + M_{17}(\kappa + 1).
\]
Then, proceeding as in [5, Case II.1], arranging the constants and applying Hölder’s inequality, the Sobolev embedding and Fatou’s lemma we achieve
\[
\|u\|_{L^{\infty(\Omega)}} \leq M_{18},
\]
where $M_{18}$ is independent on $\kappa$ and $(\kappa_n + 1)p^* \to \infty$ as $n \to \infty$.

Therefore, we can invoke Proposition 2.1, whence $u \in L^\infty(\Omega)$. Finally, by Proposition 2.2, it follows that $\gamma u \in L^\infty(\partial\Omega)$. The proof is thus complete.

**Remark 4.1.** Hypothesis (H1) is not needed in the proof of Theorem 1.2, but it is necessary in order to have a well-defined weak solution as formulated in (1.4).

**Remark 4.2.** The bounds obtained in Theorems 1.2 depend on the data in assumption (H) and on the solution itself. The proof shows that the following estimate is valid
\[
\|u\|_{L^r(\Omega)} \leq M(\|u\|_{L^{r^*}(\Omega)}), \quad \forall r \geq 1,
\]
with a constant $M(\|u\|_{L^{r^*}(\Omega)})$ depending on $\|u\|_{L^p}$. The key step for proving estimate (4.18) is (4.8).

**Remark 4.3.** Once (4.18) is reached, an alternative reasoning to get the uniform boundedness of $u$ can be carried out as follows. Let $0 < t < \|u\|_{L^\infty(\Omega)}$, where a priori one can have $\|u\|_{L^\infty(\Omega)} = +\infty$. Setting
\[
\Omega_t = \{x \in \Omega : |u(x)| > t\},
\]
it is clear that
\[
\|u\|_{L^r(\Omega)} \geq \left( \int_{\Omega_t} |u|^r \, dx \right)^{\frac{1}{r}} \geq t |\Omega_t|^{\frac{1}{r}}, \quad \forall r \geq 1,
\]
so
\[
\liminf_{r \to \infty} \|u\|_{L^r(\Omega)} \geq t.
\]
Since $t \in (0, \|u\|_{\infty})$ is arbitrary, we deduce that
\[
\liminf_{r \to \infty} \|u\|_{L^r(\Omega)} \geq \|u\|_{L^\infty(\Omega)}.
\]
In view of estimate (4.18), the conclusion that $u \in L^\infty(\Omega)$ is achieved.

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(G. Marino) Technische Universität Chemnitz, Fakultät für Mathematik, Reichenhainer Straße 41, 09126 Chemnitz, Germany

E-mail address: greta.marino@mathematik.tu-chemnitz.de

(D. Motreanu) Département de Mathématiques, Université de Perpignan, 66860 Perpignan, France

E-mail address: motreanu@univ-perp.fr