Multi-robot Symmetric Rendezvous Search on the Line

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Abstract. We study the Symmetric Rendezvous Search Problem for a multi-robot system. There are \( n > 2 \) robots arbitrarily located on a line. Their goal is to meet somewhere on the line as quickly as possible. The robots do not know the initial location of any of the other robots or their own positions on the line. The symmetric version of the problem requires the robots to execute the same search strategy to achieve rendezvous. Therefore, we solve the problem in an online fashion with a randomized strategy. In this paper, we present a symmetric rendezvous algorithm which achieves a constant competitive ratio for the total distance traveled by the robots. We validate our theoretical results through simulations.

Keywords: Multi-robot systems · Rendezvous search · Symmetric rendezvous · Online planning

1 Introduction

There are various examples of the rendezvous search problem in real life: two friends who are separated in a shopping mall and want to meet again; a group of parachutists who want to meet after landing in a large field; a rescue helicopter looking for a lost hiker in the desert \[43\]. The common challenges in each example are that the searchers are unaware of the location of the others and no common meeting point has been decided a priori. As such, the searchers need to execute an online rendezvous search to meet at a common location as quickly as possible.

In this paper, we study the robotic version of rendezvous search problem: a team of robots whose locations are unknown to each other should meet somewhere in the environment in the least time possible. The robots have limited sensing capabilities and are operating in a large environment. The robots can only communicate when they are in close proximity of each other; i.e., the communication is only possible when the robots meet each other. There are many reasons why the robots may need to rendezvous. For example, in a scenario where the robots need share information with the others urgently, a rendezvous algorithm would lead to better coordinated planning \[36\].

There are two primary versions of the rendezvous search problem, depending on whether or not the robots can meet in advance of the search to agree on the
strategies that they will execute. In asymmetric rendezvous search, the robots can meet in advance and choose distinct roles for each robot. For example, one robot can wait while the other carries out an exhaustive search. This is different from symmetric rendezvous search, where the robots execute the same strategy, since they do not have a chance to agree on their roles. In this version, it is not necessary to implement a different strategy on each robot; thus, this makes it more appealing for programming large multi-robot systems.

In this paper, we study the symmetric version of the rendezvous search problem with $n > 2$ robots that are arbitrarily located on a line. For example, robots may be deployed in a linear environment such as a road, a corridor, a river, or a tree row. We consider a scenario where the robots are unaware of their own position along the line or the positions of any of the other agents. In fact, we consider the even more challenging scenario where even the initial distance between any pair of robots is unknown. We also do not assume that the robots know the directions leading to the other robots (i.e., a robot does not know whether the other robots are to its left, or right, or both). Each robot can only keep track of their own positions relative to their own starting positions. In the absence of any prior knowledge and global information, we propose an online strategy to be followed by all the robots. The strategy involves making random choices in the directions to move to break the symmetry of the search.

We analyze its performance using the notion of competitive ratio. The competitive ratio of an online algorithm is the worst-case ratio of the cost of the solution (i.e., distance traveled by the robots) found by the online algorithm to the cost of an optimal offline solution. For omniscient robots, the optimal offline algorithm would be to move toward the midpoint of the line segment with endpoints at the positions of the leftmost and rightmost robot. If an online algorithm has a constant competitive ratio, then it means it performs competitively with respect to omniscient robots even in the absence of prior knowledge.

The rendezvous search problem has been extensively studied in literature for linear search environments with the focus mostly on two-players rendezvous [3, 4, 5, 11, 12, 26, 29, 38, 42]. The studies on multi-player rendezvous [26, 33, 34] assume that the initial distances between the robots are known and the robots are placed equidistant apart. For the symmetric version, it is also assumed that when two players meet, they may exchange any information known to them at the time. In contrast, we do not make any of these assumptions in our study. Our main contribution in this paper is to study the symmetric version of the multi-robot rendezvous problem with unknown initial distance between any pair of robots. We present a randomized symmetric rendezvous algorithm which yields a constant competitive ratio.

2 Related Work

The rendezvous search problem generalizes the linear search problem in which a single searcher tries to find a stationary object located at an unknown initial distance. This problem is also known as the lost cow problem and was originally
The solutions provided for the multi-player linear rendezvous search problem in the literature assume that the distance between each pair of adjacent robots is known; in contrast, we consider it to be unknown. Lim et al. [34] studied the rendezvous of $m \leq n$ blind, unit speed players. The authors showed that $R_{2,2}^{m}$ is $47/48$ and $R_{n,m}^{s}$ is asymptotic to $n/2$, where $R_{n,m}^{a}$ and $R_{n,m}^{s}$ denote the least expected rendezvous time of $m$ players for the asymmetric and symmetric strategy, respectively. Prior to this study, Lim and Alpern focused on the asymmetric version of the same problem and minimizing the maximum time to rendezvous rather than the expected time [33]. The asymmetric value of the $n$-player minimax rendezvous time $M_n$ has an upper bound $n/2 + (n/\log n) + o(n/\log n)$. Gal [28] presented a simpler strategy for the problem in [33] and showed that the worst case meeting time has an asymptotic behavior of $n/2 + O(\log n)$. The rendezvous of only three agents is considered in [8, 11]. Baston [11] proposed a strategy that has a maximum rendezvous time of 3.5. Alpern and Lim [6] showed that Baston’s strategy is the only strategy to achieve this result.

The deterministic rendezvous of two asynchronous agents on the line was studied in [13, 35, 42]. To break the symmetry between the agents, [35] considered that each agent has a distinct identifier, called label. A label is a nonempty binary string which is known to the agent. The cost of the proposed algorithm is
\(O(D|L_{\text{min}}|^2)\) when \(D\) is known and \(O((D+|L_{\text{max}}|)^3)\) when \(D\) is unknown, where \(D\) is the initial distance between the agents, and \(|L_{\text{min}}|\) and \(|L_{\text{max}}|\) denote the lengths of the shorter and longer label of the agents, respectively. This bound was later improved by Stachowiak [42].

The rendezvous search problem was also studied in other types of environments such as a circle [23, 27], a graph [24, 22, 23, 33, 37, 44], and a plane [2, 8, 10, 14, 18, 39].

A generalization of the rendezvous search problem is the Gathering problem [1, 16, 17, 24, 25, 25, 30, 32, 41]. Gathering requires two or more robots in an environment to co-locate at one point in finite time. The robots operate in Wait-Look-Compute-Move cycles in which they decide on their moves viewing their surroundings and analyzing the configuration of robot locations. In contrast, rendezvous search assumes extremely limited sensing capabilities, for example only collision detection, hence the solution methods do not involve computing the positions of the other robots in the environment.

### 3 Problem Formulation

We consider \(n > 2\) identical robots that are placed arbitrarily on a line. The initial distance between every pair of robots is unknown. Moreover, the robots are unaware of their own initial positions and the initial positions of the others. We assume that each robot knows the number \(n\). Rendezvous can occur anywhere on the line, since no rendezvous location is fixed in advance of the search. We study the synchronous case in which we assume that the robots have synchronized clocks, thus, they start executing the algorithm at the same time.

Let \(x_j\) denote the initial position of robot \(R_j\) in order from left to right, where \(j \in 1, \ldots, n\). Let \(d\) be initial distance between the leftmost robot \(R_1\) and the rightmost robot \(R_n\). Without loss of generality, \(R_1\) is located at \(x = 0\) and \(R_n\) is located at \(x = d\). Based on the initial configuration of the robots, \(R_1\) and \(R_n\) are called the boundary robots and the robots that are located between \(R_1\) and \(R_n\) are called the internal robots. It is easy to see that each boundary robot has one neighbor while each internal robot has two neighbors.

We focus on minimizing the distance competitive ratio of our algorithm. Due to using symmetric strategies, the expected distance traveled is identical for all robots. Here the expectation is taken over the randomization in the strategy and over the number of robots. Therefore, we only analyze it for one robot.

### 4 Multi-robot Symmetric Rendezvous Algorithm

In this section, we present the multi-robot symmetric rendezvous algorithm \(\mathcal{MSR}\). Since the search is on the line, we maintain the invariant that the robots never cross paths. This implies that once the boundary robots (\(R_1\) and \(R_n\)) meet, then the algorithm terminates, since all robots would meet. Therefore, the main idea in \(\mathcal{MSR}\) is to find out if any robot pair can act as boundary robots and meet the rest of the robots to achieve rendezvous.
We assign a non-negative sequence \( f_{-1}, f_0, f_1, f_2, \ldots \) to each robot, where \( f_{-1} = 0 \) and \( f_i = r^i \), for \( i \geq 0 \). Robots use the same expansion radius \( r > 1 \) which is fixed throughout the algorithm. We derive the optimized value of \( r \) in the proof of Theorem 1. The algorithm proceeds in rounds indexed by integers \( i \geq 0 \). Each round consists of two consecutive phases: phase-1 and phase-2. The first direction the robot performs the search in round \( i \) refers to phase-1. We denote this direction by \( \text{dir}_1 \). In phase-2 of round \( i \), the robot searches the opposite direction of \( \text{dir}_1 \), which is denoted by \( \overline{\text{dir}}_1 \). Let the initial distance between \( R_1 \) and \( R_n \) be \( d = r^{k+\delta} \), where \( \delta \in (0,1] \) and \( k \in \mathbb{Z}^+ \). Note that \( \delta \) and \( k \) are unknown to the algorithm.

Our online algorithm is a hybrid of two search modes: randomized search and deterministic search. The robot executing MSR starts by performing a randomized search to break the symmetry but switches to a deterministic search once it meets a robot. A robot is called single if it has not met any other robot yet. Thus, at the beginning of the algorithm, all robots are single. As long as the robot is single, it executes the randomized search mode in which it flips a coin at the beginning of each round to decide its itinerary. If it tosses heads, then it moves right in phase-1 and left in phase-2; otherwise it moves left in phase-1 and right in phase-2. Rendezvous can never occur if the coin flips of all robots in each round after the start come up the same. This also implies that the robots will always be executing the randomized mode of the algorithm. The randomized mode adopts the symmetric rendezvous algorithm \( SR \) presented in our previous work for two robots [38].

When the single robot meets another single robot, it switches to the deterministic search mode. In this mode, the robot does not randomize its direction at the beginning of a round. Instead, it always starts the round in its deterministic direction \( \text{dir}_d \). If the single robot meets another single robot in phase-1 of round \( i \), then \( \text{dir}_d = \overline{\text{dir}}_1 \). However, if the single robot meets another single robot in phase-2 of round \( i \), then \( \text{dir}_d = \text{dir}_1 \). Once the robot is in the deterministic search mode, its search becomes similar to the one in the LostCow algorithm, with the difference that in our algorithm, the robot uses the first direction it moves in a round (phase-1) to search for any undiscovered mobile robots instead of a stationary object while it uses the second direction it moves (phase-2) to see whether the rest of the robots are gathered.

We now describe the motion pattern of robot \( R_j \). In round \( i \), \( R_j \) starts at one of \( x = x_j \pm f_{2i-1} \). If the robot is to first search the right direction in round \( i \), then it moves right to \( x = x_j + f_{2i} \) in phase-1 and left to \( x = x_j - f_{2i+1} \) in phase-2. If the robot is to first search the left direction in round \( i \), then it moves left to \( x = x_j - f_{2i} \) in phase-1 and right to \( x = x_j + f_{2i+1} \) in phase-2. Assuming that \( R_j \) does not meet any robot during round \( i \), its possible ending position is \( x = x_j \pm f_{2i+1} \). This is also the initial position for round \( i + 1 \). The possible itineraries of \( R_j \) based on its initial position in round \( i \) are shown in Fig. 1.

In our algorithm, the robot can also be in one of the following additional states beyond the initial single state: bound and internal. We determine these states based on the fact that an internal robot can meet another robot in both directions.
Deniz Ozsoyeller and Pratap Tokekar

Possible Itineraries in Round $i$ (Alg. MSR)

$$\text{dir}_i(i) = \text{right} \quad \text{dir}_{i-1}(i) = \text{left}$$

$$\text{dir}_i(i) = \text{right} \quad \text{dir}_{i-1}(i) = \text{right}$$

$$\text{dir}_i(i) = \text{left} \quad \text{dir}_{i-1}(i) = \text{right}$$

$$\text{dir}_i(i) = \text{left} \quad \text{dir}_{i-1}(i) = \text{left}$$

\[ x_j - f_{i+1} \quad x_j - f_i \quad x_j \quad x_j + f_i \quad x_j + f_{i+1} \]

Fig. 1: The possible itineraries of robot $R_j$ executing algorithm MSR. The arrows show the directions that the robot is moving in round $i$. The small square and circle icons show the beginning and end of a motion, respectively.

since it has two neighbors while a boundary robot can meet another robot in only one direction since it has only one neighbor. However, in MSR, although the robot in internal state is the actual internal robot in the initial configuration, the robot in boundary state only acts as one but does not have to be the actual boundary robot. Thus, we refer boundary as temporary boundary.

Next, we introduce the behaviours of the robot that meets another robot in a round. When single robots $R_j$ and $R_j'$ meet in round $i$, they both change their states to boundary. Recall that this is the last round before both robots switch to the deterministic search mode. Starting from round $i + 1$, the boundary robot $R_j$ no longer flips a coin but always starts a round in direction $d$. The robot in this state continues the search until it meets another boundary robot in direction $d$. When this happens, it changes its state to internal and waits until it sticks to a boundary robot. A sequence of consecutive internal robots is always surrounded by boundary robots which carry out the search in both directions. This also implies that both neighbors of an internal robot can be internal, or one of its neighbors can be internal while the other is boundary. Thus, a single robot can never encounter an internal robot. It is easy to see that the robots $R_1$ and $R_n$ can never be internal since they each have only one neighbor, hence can meet the other only in one direction. A boundary robot collects any single robot moving towards it. In other words, the single robot always sticks to and moves along with the boundary robot that it meets. Therefore, the neighbors of a robot at the initial configuration can change when the robot changes its state to boundary. For example, consider the consecutive robots $R_{j-1}$, $R_j$, and $R_{j+1}$ at the beginning. Although $R_{j+1}$ is not initially adjacent to $R_{j-1}$, if $R_j$ sticks to $R_{j+1}$, then $R_{j+1}$ becomes adjacent to
$R_{j-1}$. The pseudocode of $\mathcal{MSR}$ is presented in Algorithm 1 and Algorithm 2, and its finite state representation is shown in Fig. 2.

![Finite state machine representation of Algorithm $\mathcal{MSR}$](image)

Let $D_{i,1}(D_{i,2})$ denote the distance traveled in phase-1 (phase-2) of round $i$. If the direction in phase-1 of this round differs from the previous round, then $D_{i,1} = f_{2i} - f_{2i-1}$. Otherwise, $D_{i,1} = f_{2i} + f_{2i-1}$. Regardless of the direction in phase-1, $D_{i,2} = f_{2i} + f_{2i+1}$. Thus, the distance traveled (the length of an itinerary) in round $i$ is either $D_i = f_{2i+1} + 2f_{2i} - f_{2i-1}$ or $D_i = f_{2i+1} + 2f_{2i} + f_{2i-1}$. The maximum total time required for round $i$ is $T_i = T_{i,1} + T_{i,2}$, where $T_{i,1} = f_{2i} + f_{2i-1}$ and $T_{i,2} = f_{2i} + f_{2i+1}$.

In the synchronous case, robots start each phase of a round at the same time. This is managed by introducing waiting times in $\mathcal{MSR}$ algorithm. The waiting time in a phase is determined by the following cases: (1) The robot does not meet another robot. (2) The robot meets another robot but continues moving without waiting at the met location, which occurs when a boundary robot meets a single or an internal robot. (3) The robot meets another robot and waits at the met location till the end of the phase, which occurs when two boundary or two single robots meet. If case-1 or case-2 occurs, then $W_{i,1} = T_{i,1} - D_{i,1}$ and $W_{i,2} = T_{i,2} - D_{i,2}$. Let $D_{i,1,m}(D_{i,2,m})$ denote the distance traveled by the robot until it meets another robot in phase-1 (phase-2) of round $i$. Since any robot pair meets before they reach their ending positions in the phase, we have $D_{i,1,m}(D_{i,2,m}) < D_{i,1}(D_{i,2})$.

If case-3 occurs, then $W_{i,1} = T_{i,1} - D_{i,1,m}$ and $W_{i,2} = T_{i,2} - D_{i,2,m}$.

An example execution of Algorithm $\mathcal{MSR}$ when $d = 50$, $r = 1.28$, and $n = 10$ is presented in Fig. 3. Each robot has unit speed. Fig. 3 (Top) shows the positions of the robots until rendezvous. The small circles on the plot depict the meeting time of the robots. The rendezvous occurs after 6 rounds. Fig. 3 (Bottom) shows the robot meetings and the changes in their states during a round. Initially, all robots are in single (S) state. In phase-1 of round 2, $R_4$ and $R_5$ meet and change their states to boundary (B). This is followed, in the next phase, by the meetings of two pairs of single robots: $R_0$ and $R_1$, and $R_6$ and $R_8$, which also change their states to boundary. In round 3, single robots $R_4$, $R_6$, $R_8$ meet and stick to boundary robots $R_4$, $R_5$, and $R_7$, respectively. The ID of the robot that sticks to another robot is shown below this robot’s ID. In round 4, single $R_2$ meets boundary $R_1$ and sticks to it. Next, we see that the boundary robots $R_1$ and $R_4$ meet in their deterministic direction and become internal robots. The boundary robots $R_5$ and $R_6$ also meet in their deterministic direction and become internal robots. In phase-2 of round 5, $R_0$ meets and collects the waiting internal robots.
Algorithm 1: MSR, Multi-robot symmetric rendezvous search algorithm.

1 global
   // $S$: SINGLE; $B$: BOUNDARY,
   // $I$: INTERNAL; $N$: NOROBOT
2 state = $S$
3 meets = $N$
4 stick = 0
5 firstmeeting = 0
6 direction = 1
7 directionsmet[2] = {0}
8 RIGHT = 1, LEFT = -1
9 HEAD = 0, TAIL = 1
10 end global

   // $i$ denotes the current round.
11 $i = 0$
12 phase = 1
13 continues = true
14 while Rendezvous() $\neq$ true do
15   | if state == $S$ then
16      |   cointoss = Random(HEAD, TAIL)
17      |   direction = RIGHT
18      |   if cointoss == TAIL then
19      |      direction = LEFT
20      |   end
21   end
22
23   if stick $\neq$ 1 then
24      | if state $\neq$ $I$ then
25      |      phase = 1
26      |      continues = MoveTo(direction * f(2i), direction, phase, $i$)
27      |      if continues $\neq$ true then
28      |      |   phase = 2
29      |      |   continues = MoveTo(-1 * direction * f(2i + 1), -1 * direction, 
30      |      |      phase, $i$)
31      |      |   if continues $\neq$ true then
32      |      |      |   $i = i + 1$
33      |      |   end
34   end
35   | if state == $I$ then
36   |   WaitInternal() 
37   end
38
39   if stick == 1 then
40      | StickToRobot() 
41   end
42 end
Algorithm 2: MoveTo(destination, currdirection, phase, round)

1  dir = 0
2  prevpos = GetCurrentPos()
3
4  if currdirection == LEFT then
5    dir = 1
6  end
7  if firstmeeting == 1 then
8    direction = -1 * currdirection
9    firstmeeting = 0
10 end
11 while Reach(destination) ≠ true do
12   meets = CheckState()
13   if (state == S or state == B) and meets == N then
14     Move(currdirection)
15   else if state == S and meets == B then
16     directionsmet[dir] = 1
17     stick = 1
18     return false
19   else if state == S and meets == S then
20     state = B
21     directionsmet[dir] = 1
22     firstmeeting = 1
23     break
24   else if state == B and (meets == I or meets == S) then
25     StickToAndMove(currdirection, destination)
26   else if state == B and meets == B then
27     if directionsmet[dir] ≠ 1 then
28       directionsmet[dir] = 1
29       state = I
30       return false
31     else if directionsmet[dir] == 1 and Total() != n then
32       break
33     else if directionsmet[dir] == 1 and Total() == n then
34       rendezvous = 1
35     end
36   end
37 end
38 currpos = GetCurrentPos()
39 Dround,phase = DistanceTrav(prevpos, currpos)
40 Wrround,phase = Tround,phase - Drround,phase.
41 Wait(Wround,phase)
42 return true
$R_1$ and $R_4$, while $R_9$ meets and collects the waiting internal robots $R_8$ and $R_5$. Finally, the rendezvous occurs when robots $R_0$ and $R_9$ meet.

Sample Execution of Algorithm MSR when $d = 50$, $r = 1.28$, and $n = 10$

| Round | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| 0     | S | S | S | S | S | S | S | S | S | S |
| 1     | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2(p1) | B | B | B | B | S | S | S | S | S | S |
| 2(p2) | S | S | S | S | B | B | B | B | S | S |
| 3(p1) | S | S | S | S | B | B | B | S | S | S |
| 3(p2) | S | S | S | S | B | B | B | S | S | S |
| 4(p1) | S | S | S | S | S | S | B | B | B | B |
| 4(p2) | S | S | S | S | S | S | B | B | B | B |
| 5(p1) | S | S | S | S | S | S | S | S | S | S |
| 5(p2) | S | S | S | S | S | S | S | S | S | S |
| 5(p2) | S | S | S | S | S | S | S | S | S | S |

Fig. 3: Example Execution of Algorithm MSR.

5 Analysis of Algorithm MSR

In this section, we find an upper bound on the expected distance traveled by the robot and compare it with the optimal offline solution. Note that since our strategy is symmetric, the expected distance traveled is identical for all robots. In the offline setting, the robots know their positions on the line and the initial distance between the boundary robots. Thus, the solution would be for them to move toward $x = d/2$. Our main result is presented in Theorem[1]

Let $\alpha$ be the first round that any robot can travel far enough to reach the meeting location of any pair of robot. We divide the execution of the analysis into
two stages. *Stage-1* consists of rounds $0 \leq i \leq \alpha - 1$. We ignore the possibility of rendezvous occurring in this stage and bound the probability of rendezvous not occurring. *Stage-2* consists of rounds $i \geq \alpha$ in which the rendezvous can occur.

In Lemma 2 we show that if any pair of robots meet in Stage-2, then all robots meet. Rendezvous cannot occur in this stage as long as the coin flips of all robots come up the same, when all robots are still in the randomized search mode of MSR. Thus, we compute the expected distance traveled using an infinite sum.

Let $S_i$ denote the event that at least one adjacent single robot pair $p$ meets in round $i$. Here $p$ can be any adjacent robot pair. $S_i$ will be our proxy event for rendezvous. Let $M_i^j$ be the event that $R_j$ initially moves right and $M_i^{j+1}$ be the event that $R_j$ initially moves left in round $i$. The adjacent robots $R_j$ and $R_{j+1}$ can meet if event $E_1 = M_i^j \land M_i^{j+1}$ or $E_2 = M_i^j \land M_i^{j+1}$ occurs. The probability that a single robot meets an adjacent single or boundary robot in round $i$ is $P[E_1 \lor E_2] = 1/2$. On the other hand, $R_j$ and $R_{j+1}$ cannot meet when they initially move in the same direction; that is if event $E_3 = M_i^j \land M_i^{j+1}$ or $E_4 = \overline{M_i^j} \land \overline{M_i^{j+1}}$ occurs. Thus, we have

$$P[S_i] = \begin{cases} 0 & \text{for } i \leq \alpha - 1, \\ \frac{2^i}{2^{i+2}} & \text{for } i \geq \alpha. \end{cases}$$

Next, we calculate the value of round $\alpha$ in terms of $k$ and $r$. To upper bound the expected distance traveled, we consider the furthest possible meeting location $x_m$ for $p$. Assuming that one of the robots in $p$ can be one of the actual boundary robots, either $x_m \in (d, d + f_{2i-1})$ or $x_m \in (-f_{2i-1}, 0)$. Let this boundary robot be $R_n$. Then, robot $R_j$ is able to reach $x_m$ if $x_j + f_{2i+1} \geq d + f_{2i-1}$. In the worst case, $R_j = R_1$, so $x_j = x_1 = 0$. Therefore, we have

$$f_{2i+1} \geq d + f_{2i-1}, \quad (1)$$

which holds for $\alpha \geq k/2 + 1 - \log_{\alpha^2} r^2 - 1$. Henceforth, $\alpha = k/2 + \iota^*$, where $\iota^* = 1 + \lceil \log_{\alpha^2} r^2 - 1 \rceil$.

Let $A_i$ denote the event that the algorithm is still active in round $i$. We assume that Algorithm MSR is always active when $i < \alpha$, thus $P[A_{i < \alpha}] = 1$. In Lemma 2 we show that rendezvous cannot occur in round $i \geq \alpha$ only if all robots’ coin flips come up the same. Therefore, the probability that the algorithm is still active in round $i \geq \alpha$ is $P[A_{i \geq \alpha}] = P[S_0 \land \cdots \land S_{i-1}]$. Thus, we have

$$P[A_i] = \begin{cases} 1 & 0 \leq i \leq \alpha - 1, \\ \left(\frac{2^i}{2^{i+2}}\right)^{i-\alpha+1} & \alpha \leq i. \end{cases} \quad (2)$$

We continue by analyzing the distance traveled in Stage-1 and Stage-2. We start with the computation of the expected distance traveled during Stage-1 which encompasses the rounds $i < \alpha$.

**Lemma 1.** The expected distance traveled during Stage-1 satisfies

$$\sum_{i=0}^{\alpha-1} E[D_i | A_i]P[A_i] < r^{k+2\iota^*} \left(\frac{r + 2 + r^{-1}}{r^2 - 1}\right). \quad (3)$$
Proof. Since we consider that rendezvous cannot occur in this stage, \( P[A_0] = 1 \). The distance traveled by a robot in round \( i \) is \( D_i \leq f_{2i+1} + 2f_{2i} + f_{2i-1} \). Therefore,

\[
\sum_{i=0}^{\alpha-1} \mathbb{E}[D_i \mid A_i] P[A_i] < \sum_{i=0}^{\alpha-1} (f_{2i+1} + 2f_{2i} + f_{2i-1}) \cdot 1 < r^{k+2i} \left( \frac{r + 2 + r^{-1}}{r^2 - 1} \right)
\]

We now compute the expected distance traveled for all rounds \( i \geq \alpha \). Unlike Stage-1, rendezvous occurs in this stage with nonzero probability. In the next lemma, we show that the meeting of any pair of single robots always results in rendezvous in the next round.

![Diagram](image)

Fig. 4: Two possible configurations of the robots at the end of phase-1 of round \( i + 1 \), given that event \( S_i \) occurs in round \( i \), that is at least one adjacent single robot pair \( p \) meets in round \( i \). B refers to the bound robot and S refers to the single robot.

Lemma 2. Given that event \( S_i \) occurs in round \( i \geq \alpha \), then the robots always achieve rendezvous in round \( i + 1 \).

Proof. Consider that all robots are single at the beginning of round \( i \). [1] ensures that any adjacent single robot pair can meet if event \( E_1 \lor E_2 \) occurs between them in round \( i \geq \alpha \) and thus can become a bound robot pair. The actual boundary robots can never be internal. Moreover, each sequence of consecutive internal robots is always located between two bound robots which move in opposite deterministic directions. Therefore, it is guaranteed that once there is a bound robot pair in round \( i \), then there will always be at least one active bound robot pair until rendezvous.

First, we show that the rendezvous may not occur in round \( i \) if this is the first round that there is a bound robot pair \( p \). In \( MSR \), if pair \( p \) is formed in phase-1, then both robots in \( p \) wait till the end of this phase, and move away from each other in phase-2. Therefore, rendezvous cannot occur in this round.

Following an unsuccessful round \( i \), the robots will start round \( i + 1 \). Now, consider that we have at least one bound robot pair \( p \) at the beginning of round
At the end of phase-1 of round $i + 1$, the robots can have two possible configurations. In the first configuration, which is shown in Fig. 4(a), the rest of the robots are between the leftmost boundary robot $R_j$ and the rightmost boundary robot $R_j'$. This can occur in two ways. The first way is when $R_j$ and $R_j'$ are the actual boundary robots. The second way is when $R_j$ and $R_j'$ are not the actual boundary robots but have already met any single robot in their deterministic direction. That is, any robot that started to the left of $R_j$ and right of $R_j'$ would have already met $R_j$ and $R_j'$. Recall that a single robot sticks to the boundary robot that it meets. In phase-2 of the first configuration, $R_j$ and $R_j'$ move towards each other while collecting any robot between them, which would be already in internal state, thus the rendezvous occurs.

In the second configuration shown in Fig. 4(b), there can only be single robots in the deterministic directions of $R_j$ and $R_j'$. In this case, the single robots on the left side of $R_j$ move in the same direction as $R_j$, while the single robots on the right side of $R_j'$ move in the same direction as $R_j'$. Thus, the rendezvous cannot occur in phase-1. However, in phase-2, $R_j$ and $R_j'$ move towards each other while collecting any robots between them and start waiting before starting the next round. Meanwhile, the single robots move toward the waiting boundary robots $R_j$ and $R_j'$ and meet them, thus the rendezvous occurs. (1) also ensures that while the boundary robots are waiting for the end of phase-2 at their meeting location, a single robot moving towards this location would be able to meet them.

**Lemma 3.** The expected distance traveled during Stage-2 satisfies

$$\sum_{i=\infty}^{\infty} \left[ (\mathbb{E}[D_i | S_i] \mathbb{P}[S_i]) + (\mathbb{E}[D_i | S_i] \mathbb{P}[S_i]) \right] \mathbb{P}[A_i]$$

$$< r^{k+2i} \left( \frac{0.75r^3 + 1.5r^2 + 1.75r + r^{-1} + 2}{4 - r^2} \right)^i.$$  \hspace{1cm} (4)

**Proof.** If $S_i$ does not occur in round $i$, then the expected distance traveled in round $i$ is

$$\mathbb{E}[D_i | S_i] \leq f_{2i+1} + 2f_{2i} + f_{2i-1}.$$ \hspace{1cm} (5)

By Lemma 2, event $S_i$ in round $i$ results in rendezvous in round $i + 1$. Therefore, the distance traveled when $S_i$ occurs is equal to the distance traveled in rounds $i$ and $i + 1$. Hence, we have

$$\mathbb{E}[D_i | S_i] < (f_{2i+1} + 2f_{2i} + f_{2i-1}) + (f_{2(i+1)+1} + 2f_{2(i+1)} + f_{2(i+1)-1})$$

$$< r^{2i} \left( r^3 + 2r^2 + 2r + r^{-1} + 2 \right).$$ \hspace{1cm} (6)

The expected distance traveled during Stage-2 is the sum of the expected distance traveled in unsuccessful rounds and the expected distance traveled in the (final) successful round, which is given by

$$\sum_{i=\infty}^{\infty} \left[ (\mathbb{E}[D_i | S_i] \mathbb{P}[S_i]) + (\mathbb{E}[D_i | S_i] \mathbb{P}[S_i]) \right] \mathbb{P}[A_i].$$ \hspace{1cm} (7)
First, we compute the expected distance traveled in successful rounds during Stage-2 using (2) and (6), which is bounded by

\[ \sum_{i=\alpha}^{\infty} (E[D_i \mid S_i]P[S_i]) P[A_i] < 0.75r^{k+2r^*} \left( \frac{r^3 + 2r^2 + 2r + r^{-1} + 2}{4 - r^2} \right) \text{ for } n > 2. \]  

Next, we compute the expected total distance traveled during unsuccessful Stage-2 rounds using (2) and (5), which is bounded by

\[ \sum_{i=\alpha}^{\infty} (E[D_i \mid S_i]P[S_i]) P[A_i] < 0.25r^{k+2r^*} \left( \frac{r + 2 + r^{-1}}{4 - r^2} \right) \text{ for } n > 2. \]  

Therefore, we conclude that (7) is bounded by the sum of (8) and (9).

**Theorem 1.** For the choice of \( r = 1.28 \), MSR algorithm has the competitive ratio of 54.732 and the competitive ratio of 34.154 as \( n \to \infty \).

**Proof.** The expected distance traveled is obtained by adding the expressions in equations (3) and (4). Since the initial distance between the leftmost and rightmost robot on the line is \( d = r^{k+\delta} \), we first replace each occurrence of \( r^{k} \) with \( dr^{-\delta} \) and \( n \) with 3. Then, we divide it by \( d/2 \), which is the cost of the optimal offline solution. This expression is maximized at \( \delta = 0 \), and the coefficient of \( d \) in it is minimized when \( r = 1.28 \). As a result, we guarantee the competitive ratio of 54.732. As \( n \to \infty \), the distance traveled in Stage-2 approaches to 0; thus the competitive ratio of our algorithm approaches to 34.154.

### 6 Simulations

In this section, we study the performance of our algorithm in simulations and verify the bound obtained in Theorem 1. We implemented a multithreaded system using Java, where each robot is represented as a separate thread that executes the Algorithm MSR. Varying the parameters \( d, n, \) and \( r \), we report the results of the following: average distance competitive ratio, average number of rounds, average total time, average time competitive ratio, and average distance traveled. Each result is averaged over 100 trials, where each trial uses the maximum of the respective data among \( n \) robots.

Fig. 5(Left) and Fig. 5(Right) show the average distance competitive ratio as a function of \( n \) an \( d \) and as a function of \( n \) and \( r \), respectively. This ratio is obtained by dividing the maximum distance traveled by \( d/2 \), where \( d \) is the respective initial distance value tested in the simulations. In the left plot, \( n \) is varied between 4 and 64 while \( d \) is varied between 50 and 125. The robots are
distributed uniformly random on the line. We observe that the average distance competitive ratio is higher when \( n \) is small and stays constant as \( n \) and \( d \) increases. We further see that this ratio is smaller than our theoretically proved upper bound. Recall that the optimal \( r \) value obtained in Theorem 1 is 1.28. In the right plot, this value is shown with double stars. With respect to the change in \( n \) and when \( d \) is fixed, we can see that MSR performs the best for \( r = 1.28 \).

Next, in Fig. 6 we investigate the average number of rounds to achieve rendezvous with respect to the change in \( n \) and \( d \) (Left plot), also with respect to the change in \( n \) and \( r \) (Right plot). The average of total number of rounds stays constant as \( n \) increases and decreases as \( r \) increases. Since the number of rounds is proportional to the distance the traveled by the robots, in the left plot, we see that the average number of rounds increases as \( d \) increases, but stays constant with respect to the change in \( n \).

The total time required for rendezvous is equal to the sum of the total distance traveled and total waiting time. Figure 7 shows the results of the simulations for the average total time with respect to the change in \( n \). Left plot reports the results for various \( d \) when \( r = 1.28 \), whereas the right plot reports the results for various \( r \) values when \( d = 75 \). As expected, the total time increases as \( d \) increases and stays constant as \( n \) increases. In the right plot, we observe that \( r = 1.28 \) outperforms the other \( r \) values as \( n \) increases. Figure (Left) shows the time competitive ratio with respect to the change in \( n \) and \( d \). Comparing the distance and time ratios in Figures 6 and 7 we see that the distance competitive ratio is less than time competitive ratio.

Fig. 8 provides the average distance traveled for various \( n \) values with respect to the change in \( d \). As in the average total time results, we see that the average distance traveled increases as \( d \) increases and stays constant as \( n \) increases for \( r = 1.28 \). In the right plot, we compare the effect of equidistant and uniformly random initial placements of the robots on the performance of our algorithm. For various \( n \) values and \( d = 75 \), the average distance traveled is smaller when the robots are randomly located on the line compared to the other case. The performances of MSR in these two cases approach each other as \( n \) increases.

7 Conclusion

This paper addresses the symmetric rendezvous search problem with multiple robots on the line. In our problem formulation, the initial distance between any
A pair of robots is unknown to the robots. Moreover, the robots do not know their own positions and the positions of each other. For this problem, we introduced an online algorithm whose competitive ratio is 57.732 which in the asymptotic case becomes 34.154.

The algorithm presented here can be extended to the asynchronous case. In the asynchronous case, the robots do not need to start at the same time and use waiting time within the rounds. Our immediate future work is to present this extension and provide its theoretical performance guarantees and simulations.
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