CONTACT SINGULARITIES

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Abstract. A contact singularity is a normal singularity \((V, 0)\) together with a holomorphic contact form \(\eta\) on \(V \setminus \text{Sing} V\) in a neighbourhood of 0, i.e. \(\eta \wedge (d\eta)^r\) has no zero, where \(\dim V = 2r + 1\). The main result of this paper is that there are no isolated contact singularities.

1. Introduction

Let \((V, 0)\) be a normal complex singularity of dimension \(n := 2r + 1\). In analogy with the notion of symplectic singularities, which were studied by Beauville [Bea1, Bea2], we call \((V, 0)\) a contact singularity if there is a holomorphic 1-form on \(V \setminus \text{Sing} V\) in a neighbourhood of 0 such that \(\eta \wedge (d\eta)^r\) has no zero on \(V \setminus \text{Sing} V\).

The main result of this paper is that there are no isolated contact singularities, see 3.5.

The key tool in proving this is a Hodge theoretic result for rational singularities which may be of independent interest. Let \(\pi : X \to V\) be a resolution of an isolated rational singularity \((V, 0)\) such that the exceptional set \(E := \pi^{-1}(0)\) is an SNC (=simple normal crossing) divisor. It was shown by Steenbrink and van Straaten [St] that then every holomorphic \(p\)-form, say, \(\eta\) on \(V \setminus \{0\}\) extends to a holomorphic form on the resolution \(X\). In 2.1 we will show more strongly that for \(p \geq 1\) even \(\pi^*(\eta)|E = 0\) so that \(\pi^*(\eta)\) belongs to \(\Omega^p_X(E)(-E)\), where \(\Omega^p_X(E)\) denotes the sheaf of meromorphic \(p\)-forms with at most logarithmic poles along \(E\) which are holomorphic on \(X \setminus E\). This result should be viewed as a generalization of the fact that on a Fano manifold there are no non-zero holomorphic \(p\)-forms, \(p \geq 1\), see Sect. 2.

We add a few remarks about the contents of the various sections. Sect. 2 contains the aforementioned result about the extendability of differential forms on rational singularities. In Section 3 we study contact singularities. With the same arguments as in the case of symplectic singularities [Bea1, Bea2] they are Gorenstein and rational if \(\text{codim Sing} V \geq 3\), see 3.1. We treat the case of complete intersections more closely and show then the main result described above.

Sect. 4 contains a technical result on holomorphic forms for a special class of isolated singularities which is used in Sect. 5 to compute the number of moduli of an isolated symplectic singularity that is resolved after one blowing up. It turns out that in this case the number of moduli is a topological invariant given by \(\beta_2 - 1\), where \(\beta_2\) is the second Betti number of the exceptional set of the blowing up.

2. Differential forms on rational singularities

To motivate the main result of this section, let us first consider the situation that \((V, 0) \subseteq (\mathbb{C}^N, 0)\) is an isolated singularity of dimension \(n \geq 2\) that is given

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by homogeneous equations. Blowing up the point 0 ∈ V gives a resolution of singularities π : X → V with exceptional set, say, E, which is a projective manifold of dimension n − 1. If (V, 0) is rational and Gorenstein then by a result of Kempf [KKMS] Prop. on p. 50 a nowhere vanishing holomorphic n-form ω on V \ {0} extends to a holomorphic n-form on X which shows that ω_X ≅ O_X(kE) for some k ≥ 0. By adjunction
\[ ω_E ≅ ω_X(E) ≅ O_X((k + 1)E) \]
and so the sheaf ω_E^{-1} is ample. Hence E is a Fano manifold. In particular E admits no global holomorphic p-forms, p ≥ 1. If η is a holomorphic p-form on V \ {0} then by the results of [SSU] this form extends to a holomorphic p-form π^*(η) on X. As E is Fano, this form vanishes when restricted to E and so
\[ (π)(η) ∈ Γ(X, Ω_X^p(E)(−E)). \]
Our aim is to generalize this observation to non-homogeneous singularities. We will use the following notations.

Let (V, 0) be an arbitrary isolated singularity and consider as before a resolution of singularities π : X → V such that the exceptional set E := π^{-1}(0) is an SNC divisor.

The main result of this section is the following theorem.

**Theorem 2.1.** Assume that (V, 0) is a rational isolated singularity of dimension n ≥ 2 and let η be a holomorphic p-form on U := V \ {0}, where p ≥ 0. Then π^*(η) extends to a holomorphic p-form in Ω_X^p(E)(−E).

Thus, if ˜π : Y → V is any resolution of singularities and F := ˜π^{-1}(0) then ˜π^*(η) extends to a holomorphic form on Y such the restriction ˜π^*(η)|F vanishes as a section of Ω_F^{p+1}/Torsion. This follows easily by dominating Y by a resolution π : X → V for which E = π^{-1}(0) is a simple normal crossing divisor.

Before proving the main theorem we remind the reader of some basic results concerning mixed Hodge structures of isolated singularities, for a general reference of the theory of mixed Hodge structures see [El].

Let V be a contractible Stein space and assume that 0 ∈ V is the only singular point of V. Let π : X → V be as before a resolution of singularities such that E := π^{-1}(0) is an SNC divisor. By [Ste1, 1.9] (cf. also [El Sect. 5]) the local cohomology group H^p(0)(V, C) carries a mixed Hodge structure. As V is contractible this local cohomology group is isomorphic to H^{p−1}(U, C) for p ≠ 1 whereas H^1(0)(V, C) ≅ H^0(U, C)/C. Thus H^p(U, C) also carries a mixed Hodge structure. We have the following facts.

**Theorem 2.2.** (1) (Ste1 Sect. 1) The spaces H^p(E, C) and H^p(U, C) carry natural mixed Hodge structures, and the Hodge filtrations are given by
\[ F^p H^p(E, C) = H^0(E, Ω_X^p/E)(−E) \]
and
\[ F^p H^p(U, C) = H^0(E, Ω_X^p/E)(−E) \]
Moreover, the natural map
\[ H^p(E, C) ≅ H^p(X, C) → H^p(U, C) \]
is a morphism of mixed Hodge structures.
(2) ([SSS, 1.3 and 1.4]) If $\eta$ is a holomorphic $p$-form on $U$ and if $p \leq n - 2$ then $\pi^*(\eta)$ extends to a holomorphic $p$-form on $X$. If moreover $(V,0)$ has a rational singularity then the same is true for $p = n - 1$ and $p = n$.

Proof. (1) was stated in this form in [SSS, proof of 1.3]. We add a few remarks how to deduce it from [Ste1]. (a) and (c) are immediate consequences of [Ste1, 1.5 and 1.12], whereas (b) is an easy consequence of the description of the Hodge filtration given in [Ste1, 1.9]. Note that the complex $A\{0\}_c(V)$ constructed in [loc.cit., 1.8] does not compute the local cohomology $H^n(X,\mathcal{O}_X)$ (as was claimed there) but $H^{*-1}(U,\mathcal{O}_X)$ (the error comes from the isomorphism in the second paragraph of 1.8 which is only valid for $k \neq 1$).

Thus, if $\eta$ is as in 2.1, then the form $\pi^*(\eta)$ extends to a holomorphic form on $X$ by 2.2 (2). Hence 2.1 is a consequence of the following lemma.

Lemma 2.3. Under the assumptions of 2.1 we have that

$$H^0(X,\Omega^p_X) \cong H^0(X,\Omega^p_X(E)(-E)).$$

Proof. Consider the $W$-filtration associated to the mixed Hodge structure on the cohomology $H^*(E,\mathbb{C})$. Using the standard description of the $W$-filtration (see e.g. [Elz, 3.5]) it follows that

$$\text{Gr}_p^W H^p(E,\mathbb{C}) = \text{Im} \left( H^p(E,\mathbb{C}) \longrightarrow \bigoplus_i H^p(E_i,\mathbb{C}) \right),$$

where the sum is taken over all irreducible components $E_i$ of $E$. This module carries a Hodge structure of weight $p$. In a first step we will show that

$$F^p \text{Gr}_p^W H^p(E,\mathbb{C}) = 0,$$

where $F^p \text{Gr}_p^W H^p(E,\mathbb{C})$ is the $p$-th piece of the Hodge filtration. By Hodge symmetry this piece is isomorphic to

$$M := \frac{F^0 \text{Gr}_p^W H^p(E,\mathbb{C})}{F^1 \text{Gr}_p^W H^p(E,\mathbb{C})}.$$

Using the fact that a morphism of Hodge structures is strict (cf. [Elz, 1.3.7 (iii)]) the latter module is equal to the image of $\text{Gr}_p^W H^p(E,\mathbb{C})$ under the map $\beta$ in the diagram

$$\begin{array}{ccc}
H^p(X,\mathbb{C}) & \longrightarrow & H^p(E,\mathbb{C}) \\
\downarrow \text{can} & & \downarrow \text{can} \\
H^p(X,\mathcal{O}_X) & \longrightarrow & H^p(E,\mathcal{O}_E) \longrightarrow \bigoplus H^p(E_i,\mathcal{O}_{E_i}).
\end{array}$$

Here the vertical arrows are the canonical maps induced by the inclusion of the sheaves of locally constant functions into holomorphic functions. As $(V,0)$ has rational singularities the group $H^p(X,\mathcal{O}_X)$ vanishes. Thus the map $\alpha$ in the diagram is the zero map and so the image of $\text{Gr}_p^W H^p(E,\mathbb{C})$ under $\beta$ vanishes. Accordingly $M$ and then $F^p \text{Gr}_p^W H^p(E,\mathbb{C})$ also vanish, proving (1).

If $F^q := F^q H^p(E,\mathbb{C})$ is the Hodge filtration and $W_i := W_i H^p(E,\mathbb{C})$ is the weight filtration on $H^p(E,\mathbb{C})$ then

$$F^p \cap W_{p-1} = 0.$$
In fact, the Hodge filtration on each quotient $W_i/W_{i-1}$ is induced by the filtration $F^i \cap W_i$. As $W_i/W_{i-1}$ carries a pure Hodge structure of weight $i$ it follows that $F^p \cap W_i \subset F^p \cap W_{i-1}$ for all $i < p$ and so $F^p \cap W_{p-1} \subset F^p \cap W_{p-2} \subset \cdots \subset F^p \cap W_0 = 0$ vanishes.

From (1) and (2) it follows that

$$F^p H^p(E, \mathbb{C}) \cong F^p \text{Gr}^W_p H^p(E, \mathbb{C}) = 0.$$  

By \ref{subsec:2.2} (1) $F^p H^p(E, \mathbb{C}) \cong H^0(E, \Omega^p_X/\Omega^p_X(E)(-E))$ and so the latter module vanishes as well, proving our result. \qed

**Remark 2.4.** We note that \ref{subsec:2.3} is no longer true for non-isolated singularities as is already seen from the case of a product $V \times \mathbb{C}$.

Theorem \ref{subsec:2.3} has the following interesting consequence which will be used later.

**Proposition 2.5.** If $(V, 0)$ is a rational isolated singularity of dimension $n \geq 2$ then any closed holomorphic $p$-form $\eta$ on $V \setminus \{0\}$ with $1 \leq p \leq 2$ is exact, i.e. after shrinking $V$ as a neighbourhood of 0 we can find a $(p-1)$-form $\xi$ on $V \setminus \{0\}$ with $d(\xi) = \eta$.

In order to prove this result we need a few preparations. For an arbitrary complex space $X$ let $H^p_{DR}(X)$ denote the $p$-th “naïve” de Rham cohomology group of $X$

$$H^p_{DR}(X) := H^p(\Gamma(X, \Omega^*_X(\mathbb{C}))).$$

If $X$ is smooth then $H^p(X, \mathbb{C})$ is the $p$-th hypercohomology of the de Rham complex $\Omega^*_X/\mathbb{C}$ and so there is always a natural map

$$\alpha_p : H^p_{DR}(X) \to H^p(X, \Omega^*_X(\mathbb{C})) \cong H^p(X, \mathbb{C}).$$

Using the the spectral sequence $E_1^{pq} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}(X, \mathbb{C})$ it follows that the map $\alpha_0$ is always bijective and that $\alpha_1$ is injective; note that the spectral sequence induces an exact sequence of small order terms

$$0 \to H^1_{DR}(X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to H^2_{DR}(X).$$

We need the following simple observation.

**Lemma 2.6.** Assume that $X$ is smooth. If $H^1(X, \mathcal{O}_X) = 0$ then the natural map $H^1_{DR}(X) \to H^1(X, \mathbb{C})$ is bijective and $H^2_{DR}(X) \to H^2(X, \mathbb{C})$ is injective.

**Proof.** The first part follows immediately from the sequence of small order terms above. The second part is easily seen again from the spectral sequence; we leave the simple details to the reader. \qed

**Proof of \ref{subsec:2.3}.** We may assume that $V$ is a contractible Stein space with 0 as its only singular point. As before let $\pi : X \to V$ be a resolution of singularities with exceptional set $E := \pi^{-1}(0)$. Suppose that $\eta$ is a $p$-form on $U := V \setminus \{0\}$, where $p = 1$ or $p = 2$. By \ref{subsec:2.1} the form $\pi^*(\eta)$ extends to a holomorphic differential form on $X$ that lies in $\Gamma(X, \Omega^p_X(E)(-E))$. As $\eta$ is closed we can consider its associated cohomology class $[\pi^*(\eta)] \in H^p_{DR}(X)$. Its image in $H^p(X, \mathbb{C}) \cong H^p(E, \mathbb{C})$ is already contained in $F^p(H^p(E, \mathbb{C})) \subset H^p(E, \mathbb{C})$. Under the isomorphism

$$F^p(H^p(E, \mathbb{C})) \cong H^0(E, \Omega^p_X(E)/\Omega^p_X(E)(-E))$$

(see \ref{subsec:2.2} (1)) it is represented by the residue class of $\pi^*(\eta)$ in $\Omega^p_X(E)/\Omega^p_X(E)(-E)$ and so it is zero. Using \ref{subsec:2.6} it follows that the cohomology class of $[\pi^*(\eta)]$ in $H^p_{DR}(X, \mathbb{C})$ vanishes. Thus we can find a form $\xi \in \Gamma(X, \Omega^{p-1}_X)$ with $d(\xi) = \eta$, as required. \qed
3. Contact singularities

Recall that a contact form on a complex manifold \( X \) of dimension \( 2r + 1 \) is a holomorphic 1-form \( \eta \) on \( X \) such that

\[
\eta \wedge (d\eta)^r
\]

has no zero. By Darboux’s lemma, in suitable coordinates \( (z, x_1, \ldots, x_{2r}) \) such a form can be written as

\[
\eta = dz + \sum_{\rho=1}^{r} x_{2\rho-1} dx_{2\rho}.
\]

By a contact singularity we mean a normal singularity \( (V, 0) \) together with a 1-form \( \eta \in (\Omega^1 V)^{\wedge^r} \) which is a contact form on the regular part of \( V \). Note that the dimension of \( V \) is necessarily odd so that we can write

\[
\dim V = 2r + 1 \geq 3.
\]

In analogy with the case of symplectic singularities [Bea2] we have the following facts.

**Lemma 3.1.** If \( (V, 0) \) is a contact singularity then the following hold.

(a) \( (V, 0) \) is quasi-Gorenstein, i.e. \( \omega_{V, 0} \cong \mathcal{O}_{V, 0} \).

(b) If \( \text{codim} \text{Sing} V \geq 3 \) then \( (V, 0) \) is rational.

In particular it follows that a contact singularity with \( \text{codim} \text{Sing} V \geq 3 \) is always Gorenstein.

**Proof.** As \( \omega_V \) is generated on \( V_{\text{reg}} \) by the form \( \eta \wedge (d(\eta))^r \), the singularity is quasi-Gorenstein. If \( \text{codim} \text{Sing} V \geq 3 \) and \( \pi: X \to V \) is a resolution of singularities, then by [Fle2] the form \( \pi^*(\eta) \) extends to a holomorphic 1-form on \( X \) and so \( \pi^*(\eta \wedge (d(\eta))^r) \) extends to a holomorphic form in \( \omega_X \). Using [Fle1, 1.3] it follows that \( (V, 0) \) is rational.

Using 2.5 we can show that the product of a symplectic singularity with an affine line has the structure of a contact singularity. Recall that a normal singularity \( (V, 0) \) of dimension \( 2r \geq 2 \) is said to be symplectic (see [Bea2, 1.1]) if there is a closed 2-form \( \eta \) on \( U := \text{Reg} V \) such that \( (d(\eta))^{2r} \) has no zero on \( U \) near 0 and such that for a resolution of singularities \( \pi: X \to V \) the form \( \pi^*(\eta) \) extends to a holomorphic 2-form on \( X \). Note that by loc.cit. such a singularity is always rational.

**Proposition 3.2.** Let \( (V, 0) \) be an isolated symplectic singularity of dimension \( 2r \geq 2 \). Then \( (V \times \mathbb{C}, 0) \) is a contact singularity.

**Proof.** Let \( \eta \) be a symplectic form on \( V \setminus \{0\} \) so that \( \eta \) is a closed non-degenerate holomorphic 2-form. By 2.3 we can find a holomorphic 1 form \( \xi \) on \( V \setminus \{0\} \) with \( d(\xi) = \eta \). With \( z \) the local coordinate function on \( \mathbb{C} \), the form

\[
\omega := dz + \xi
\]

on \( (V \setminus \{0\}) \times \mathbb{C} \) then satisfies \( \omega \wedge (d\omega)^r = dz \wedge (\xi)^r \) and so it is non-degenerate everywhere. Thus \( \omega \) is a contact form on the singularity \( (V \times \mathbb{C}, 0) \), as required.

**Proposition 3.3.** A complete intersection \( (V, 0) \) with \( \text{codim} \text{Sing} V \geq 3 \) can never carry a contact structure.
Proof. In the case of a complete intersection singularity there is an exact sequence
\[ 0 \to \mathcal{O}_V^a \to \mathcal{O}_V^b \to \Omega_1^V \to 0, \]
where \( b \) is the embedding dimension of \((V,0)\) and \( a \) is the number of defining equations. As \( \mathcal{O}_{V,0} \) is Cohen-Macaulay and \( \operatorname{codim} \operatorname{Sing} X \geq 3 \) it follows easily from the Auslander-Buchsbaum theorem (see e.g. [Eis]) that \( \Omega_1^{V,0} \) is a reflexive module. Hence \( \omega \) is a section in \( \Omega_1^{V,0} \) and so \( \omega \wedge (d(\omega))^r \) is a section in
\[ \operatorname{Im}(\Omega_1^{V,0} \to \omega_{V,0}). \]
In particular the cokernel of \( \Omega_1^{V,0} \to \omega_{V,0} \) is 0 and so we can find \( f_1, \ldots, f_{2r+1} \) in the maximal ideal of \( \mathcal{O}_{V,0} \) such that the form \( df_1 \wedge \ldots \wedge df_{2r+1} \) generates \( \omega_{V,0} \). Choosing these elements generically we can achieve that the map \( f := (f_1, \ldots, f_{2r+1}) : (V,0) \to (\mathbb{C}^{2r+1},0) \) is finite. By construction this map is unramified in codimension 1. Using the theorem on purity of the branch locus (see, e.g. [FOV, 3.2.14]) it follows that \( f \) is unramified everywhere and so \( f \) is an isomorphism.

Remark 3.4. The same argument shows that \( \Omega_1^{V,0} \) cannot be reflexive if \((V,0)\) is not smooth and carries a contact structure.

An interesting application –and actually motivation– of our main theorem 2.1 is the following result.

Theorem 3.5. A non-smooth isolated singularity can never carry a contact structure.

Proof. Assume that \((V,0)\) is an isolated singularity that carries a contact structure which is given by the 1-form \( \eta \) in \((\Omega_1^{V})^\vee \). By definition, \((V,0)\) is then normal of dimension \( \geq 3 \). The form \( \delta := (d\eta)^r \) on \( U := V \setminus \{0\} \) can be considered as a vector field on \( U \) via the canonical isomorphism \( \Omega_2^U \cong \Theta_U \otimes \omega_U \cong \Theta_U \); note that \( \omega_U \cong \mathcal{O}_U \) by [3.3]. Such a vector field gives rise to a derivation \( \delta : A \to A \), where \( A := \mathcal{O}_{V,0} \). As the singularity is isolated \( \delta \) maps the maximal ideal of \( A \) into itself. Using equivariant resolution of singularities (see e.g., [BM]) we can find a resolution \( \pi : X \to V \) such that
(a) \( E := \pi^{-1}(0) \) is a simple normal crossing divisor in \( X \) and
(b) \( \delta \) extends to a vector field on \( X \) that is tangent to \( E \).
Using [3.4] the singularity is Gorenstein and rational. By [2.1] the form \( \pi^*(\eta) \) extends to a holomorphic form in \( \Omega_X^1 \langle E \rangle(-E) \). As by construction \( \delta \) can be considered as a section in \( \Theta_X(E) \) it follows that the contraction
\[ \langle \delta, \pi^*(\eta) \rangle \]
is a section in \( \mathcal{O}_X(-E) \). Hence on \( V \) the contraction \( \langle \delta, \eta \rangle \) must lie in the maximal ideal of \( A \) and so its zero set has codimension 1 in \( V \). Under the isomorphism \( \mathcal{O}_V \cong \omega_V \) the element \( \langle \delta, \eta \rangle \) corresponds to \( \eta \wedge (d\eta)^r \). Thus \( \eta \wedge (d\eta)^r \) has a zero set of codimension 1 as well, which gives a contradiction.

We do not know whether there are contact singularities with a singularity set of even dimension. We even do not know any example of a contact singularity which is not a product \( \mathbb{C} \times V \), where \( V \) is a symplectic singularity.
4. Extensions of vector fields

4.1. In this section we consider a non-smooth normal isolated singularity \((V, 0)\). Let \(\pi : X \to V\) be the blowing up of \(0 \in V\). We will always assume that \(X\) and the exceptional set \(E := \pi^{-1}(0)\) are smooth.

For the use in the next section we need the following result.

**Proposition 4.2.** The natural inclusion

\[
H^0(X, (\Lambda^p \Theta_X(E))^((p-1)E)) \hookrightarrow H^0(X, (\Lambda^p \Theta_X(E))(kE))
\]

is an isomorphism for all \(k \geq p - 1\).

For the proof we need the following simple observation.

**Lemma 4.3.** With \((V, 0)\) as in 4.1 and \(A := \mathcal{O}_{V,0}\) the following hold.

1. Every homomorphism \(f : \Omega^p_A \to A\) factors through the maximal ideal \(m_A\) of \(A\).
2. The image of the natural map \(\pi^*(\Omega^p_V) \to \Omega^p_X\) is just \(\Omega^p_X(E)(-pE)\). In particular

\[
\pi^*(\Omega^p_V)/\text{Torsion} \cong \Omega^p_X(E)(-pE).
\]

**Proof.** First consider the case \(p = 1\) in (1). A homomorphism \(f : \Omega^1_A \to A\) corresponds to a derivation \(\delta : A \to A\), and it is well known that such a derivation maps \(A\) into \(m_A\). To deduce (1) for \(p > 1\) assume that there is a surjective map \(f : \Omega^p_A \to A\).

For a sufficiently general form \(\eta \in \Omega^{p-1}_A\) the composed map \(\Omega^1_A \xrightarrow{\eta} \Omega^p_A \to A\) is as well surjective which contradicts the first part of the proof.

To deduce (2) it is sufficient to treat the case \(p = 1\); the general case then follows by taking exterior powers. In the case that \(V = \mathbb{C}^n\) this is easily verified by an explicit computation which we leave to the reader. In the general case let \((V, 0) \hookrightarrow (\mathbb{C}^n, 0)\) be an embedding. The blowing up \(X\) of \(V\) in \(0\) then embeds into the blowing up, say, \(Y\) of \(\mathbb{C}^n\) in \(0\), and \(E\) is the intersection of the exceptional set, say \(F\), of \(Y\) with \(X\). Hence the assertion follows from the commutative diagram with surjective vertical arrows

\[
\begin{array}{ccc}
\pi^*(\Omega^1_V) & \to & \Omega^1_Y(F)(-F) \\
\downarrow & & \downarrow \\
\pi^*(\Omega^1_V) & \to & \Omega^1_X(E)(-E).
\end{array}
\]

**Proof of (4.2).** A section \(\delta\) in \(H^0(X, (\Lambda^p(\Theta_X(E))(kE)))\) gives rise to a map

\[
\delta : \Omega^p_X(E)(-pE) \to \mathcal{O}_X((k - p)E).
\]

Composing \(\pi_*\delta\) with the natural map \(\Omega^p_V \to \pi_*(\Omega^p_X(E)(-pE))\) we get a map \(\Omega^p_V \to \mathcal{O}_V\), which by 4.3 (1) factors through the maximal ideal \(m_A\) at \(0\). As by 4.3 (2) \(\pi^*(\Omega^p_V) \to \Omega^p_X(E)(-pE)\) is surjective the image of \(\delta\) is contained in the ideal sheaf \(\mathcal{O}_X(-E) = m_A\mathcal{O}_V\) of \(E\) in \(\mathcal{O}_X\). Hence \(\delta\) is in \(H^0(X, (\Lambda^p\Theta_X(E))((p - 1)E))\), as required.

\(\square\)
5. Deformations of symplectic singularities

In this section let \((V, 0)\) be an isolated symplectic singularity with \(\dim V \geq 4\). We will always assume that the tangent cone \(C_0 V\) has an isolated singularity. The blowing up \(\pi : X \to V\) of \(V\) in 0 then is smooth with exceptional set \(\pi^{-1}(0)\) which is isomorphic to the projective tangent cone \(E := \mathbb{P}(C_0 V)\). Such singularities were completely classified by [Bea2]. We wish to supplement his result with the following observation.

**Proposition 5.1.** The dimension of the space of infinitesimal deformations \(T^{1}_{V,0}\) is given by \(b_2(E) - 1\).

Using the classification in [Bea2] it follows that \((V, 0)\) is rigid except when \(V\) is the set of \(n \times n\)-matrices of rank \(\leq 1\) and of trace 0.

The proof of this result will follow from 5.3 below.

Recall that a symplectic singularity of dimension \(\geq 4\) is always rational and Gorenstein [Bea2]. We need the following important facts shown essentially in loc.cit.

**Proposition 5.2.** Let \((V, 0)\) be an isolated non-smooth symplectic singularity of dimension \(2r\) with symplectic form \(\eta \in (\Omega^{2}_{V,0})^{\vee \vee}\). If the blowing up \(\pi : X \to V\) of \(0 \in V\) is smooth then the following hold.

1. \(\omega_X \cong \mathcal{O}_X((r-1)E)\) and \(\omega_E \cong \mathcal{O}_E(rE)\).
2. Multiplying with \(\eta^{r-1}\) gives an isomorphism of vector bundles
   \[
   (*) \quad \eta^{r-1} : \Omega^{1}_{X}(E)(-E) \to \Theta_{X}(E)
   \]
3. \(E\) is a rational homogeneous manifold.
4. \(\xi := \text{Res } \eta \in H^{0}(E, \Omega^{1}_{E}(-E))\) defines a contact structure on \(E\).
5. \((V, 0)\) is homogeneous, i.e. there is a locally closed embedding of \(V \hookrightarrow \mathbb{C}^{N}\) such that the ideal of \(V\) is given by homogeneous polynomials.

**Proof.** For the convenience of the reader we include simplified arguments for some of these facts. The second part of (1) follows from the adjunction formula. As the form \(\eta^{r}\) generates \(\omega_X\) outside of \(E\), we have \(\omega_X \cong \mathcal{O}_X(kE)\) for some \(k \geq 0\). By 2.4 the form \(\eta\) extends to a section in \(\Omega^{2}_{X}(E)(-E)\) whence \(\eta^{r}\) yields a section in \(\Omega^{2}_{X}(E)(-rE) \cong \omega_X(-r(1)E)\). This shows that \(k \geq r - 1\).

To show the converse inequality let us first derive (2). As by 4.3 \(\Omega^{1}_{X}(E)(-E)\) is globally generated the map \((*)\) factors through \(\Theta(E)\) by 4.2 (applied to the case \(p = 1\)). Hence we obtain a map as in \((*)\) which is an isomorphism outside \(E\). Taking the determinant of this map gives an inclusion

\[
\mathcal{O}_X((k+1)E - 2rE) \cong \det \Omega^{1}_{X}(E)(-E) \hookrightarrow \Theta_{X}(E) \cong \mathcal{O}_X(-(k+1)E)
\]

and so \(k + 1 - 2r \leq -(k+1)\), i.e. \(k + 1 \leq r\). Together with the inequality \(k \geq r - 1\) this proves (1). Moreover (2) follows as the determinant of \((*)\) is an isomorphism.

To deduce (3) note that by (2) and \(\Theta_{E}\) being a quotient of \(\Theta_{X}(E)\) is globally generated and so \(E\) is a rational homogeneous manifold. (4) is a consequence of the equation \(\text{Res } \eta^{r} = d(\xi) \wedge \xi^{r-1}\) in \(\Omega^{2r-1}_{E}(-rE)\) which is easily verified by a local computation using (1). For the proof of (5) we refer the reader to [Bea2].

**Proposition 5.3.** Let \((V, 0)\) be an isolated symplectic singularity of dimension \(n = 2r \geq 4\) and let \(\pi : X \to V\) be a resolution of singularities such that \(E := \pi^{-1}(0)\) is
a simple normal crossing divisor. Then there is an isomorphism
\[ T^1_{V,0} \cong H^1(X, \Omega^1_X(E)) \].

Proof. We may assume that \( V \) is a Stein space and that \( U := V \setminus \{0\} \) is non-singular. We will show that
(a) \( T^1_{V,0} \cong H^1(U, \Theta_U) \), and
(b) \( H^1(U, \Theta_U) \cong H^1(X, \Omega^1_X(E)) \).

To prove (a) we may assume that there is a closed embedding \( V \subseteq W \), where \( W \) is a Stein neighbourhood of \( 0 \) in \( \mathbb{C}^N \). The sheaf \( \Omega^1_W|V \cong \mathcal{O}_V^N \) is free and so
\[ \text{Ext}^1(V, \Omega^1_W|V, \mathcal{O}_V) = 0. \]

As \((V, 0)\) is symplectic, \( \mathcal{O}_{V,0} \) is rational and Gorenstein (see [Bea2, 1.3]). In particular, \( \text{depth}(\mathcal{O}_{V,0}) = \dim V \geq 3 \) and so by [Sche]
\[ H^1(U, \Theta_W|V) \cong H^2_{(0)}(V, \mathcal{O}_V^N) = 0. \]

There is a natural exact sequence
\[ 0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_W|V \to \Omega^1_V \to 0. \]

Taking the associated cohomology sequences with respect to the functors
\[ \text{Hom}_V(-, \mathcal{O}_V) \quad \text{and} \quad \text{Hom}_U(-, \mathcal{O}_U) \]
and using (1) and (2) we get a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}(\Omega^1_W|V, \mathcal{O}_V) & \rightarrow & \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_V) \\
\cong & & \cong \\
H^0(U, \Theta_W|V) & \rightarrow & H^0(U, (\mathcal{I}/\mathcal{I}^2)^\vee) \\
\beta & & \\
H^1(U, \Theta_U) & \rightarrow & H^1(U, \Theta_U) \\
\end{array}
\]

where we have used the fact that for locally free modules \( \mathcal{M} \) on \( U \) there are isomorphisms \( H^i(U, \mathcal{M}^\vee) \cong \text{Ext}^i_U(\mathcal{M}, \mathcal{O}_U) \) for all \( i \). Thus the map \( \beta \) in the diagram above is an isomorphism. As \( T^1_V \) is isomorphic to \( \text{Ext}^1_U(\Omega^1_V, \mathcal{O}_V) \), (a) follows.

To deduce (b) note first that the symplectic structure provides an isomorphism \( \Theta_U \cong \Omega^1_U \) and so
\[ H^1(U, \Theta_U) \cong H^1(U, \Omega^1_U) \]

Moreover, by the vanishing theorem for isolated singularities of Steenbrink [Ste2]
\[ H^q(X, \Omega^p_X(E)(-E)) = 0 \quad \text{for} \quad p + q > n. \]

By duality this is equivalent to
\[ H^q_E(X, \Omega^p_X(E)) = 0 \quad \text{for} \quad p + q < n, \]
where \( H^q_E(\ldots) \) denotes cohomology with support in \( E \). As \( U \cong X \setminus E \) there is an exact sequence
\[ H^1_E(X, \Omega^1_X(E)) \rightarrow H^1(X, \Omega^1_X(E)) \rightarrow H^1(U, \Omega^1_U) \rightarrow H^1_E(X, \Omega^1_X(E)), \]
in which the outer terms vanish because of (2)’. Together with (1) this proves (b).

\[ \square \]

Remark 5.4. The proof shows more generally that for any isolated singularity of dimension \( \geq 4 \) there is always an isomorphism \( H^1(U, \Omega^1_U) \cong H^1(X, \Omega^1_X(E)) \).

Now the remaining argument for the proof of [L] is provided by the following lemma.
Lemma 5.5. Let $(V,0)$ be as in \cite{Akh5} and let $\pi : X \to V$ be the blowing up of $V$ in $0$. Then $\dim \Omega^1_X(E) = b_2(E) - 1$.

Proof. As the singularity is homogeneous, $O_E(1) \cong O_X(-E)|E$ and $X$ is just the projective analytic space, say $\text{Proj}(S(O_E(1)))$, associated to the symmetric algebra $S(O_E(1)) = \bigoplus_{q \geq 0} O_E(q)$. Moreover an easy local computation shows that $\Omega^1_X(E)$ is the coherent $O_X$-module associated to the graded $S(O_E(1))$-module $\bigoplus_{q \geq 0} \Omega^1_X(-qE)|E$. This implies that

$$H^p(X, \Omega^1_X(E)) \cong \bigoplus_{q \geq 0} H^p(E, \Omega^1_X(-qE)|E).$$

It suffices to show that

(a) $H^p(E, \Omega^1_X(-qE)|E) = 0$ for $p > 0$ and $q > 0$.

(b) $\dim \Omega^1_X(E) = b_2(E) - 1$.

To deduce (a) note first that by \cite{Akh5} below the groups $H^p(E, \Theta_E(-qE))$ vanish if $p > 0$ and $q \geq 0$. Using the exact sequence

$$0 \to O_E(-qE) \to \Theta_X(E)(-qE)|E \to \Theta_E(-qE) \to 0$$

it follows that $H^p(E, \Theta_X(E)(-qE)|E)$ also vanishes if $p > 0$ and $q \geq 0$. By \cite{Akh5} (2) this gives

$$H^p(E, \Omega^1_X(E)(-q^1+1)|E) \cong H^p(E, \Theta_X(E)(-qE)|E) = 0$$

for $p > 0$ and $q \geq 0$, proving (a). To deduce (b), let us consider the exact sequence

$$0 \to \Omega^1_E \to \Omega^1_X(E)|E \to O_E \to 0$$

and its cohomology sequence

$$\mathbb{C} \cong H^0(E, O_E) \xrightarrow{\partial} H^1(E, \Omega^1_E) \to H^1(E, \Omega^1_X(E)|E) \to H^1(E, O_E) = 0.$$
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