I. INTRODUCTION

The interactions of fermions with solitons has led to the discovery of many novel and important field theoretic phenomena such as fractional charges [3,4] and quantum anomalies [2,4]. Fermionic zero modes on strings may play an important role in cosmology since they can be responsible for string superconductivity and vortons [6]. Although the standard electroweak theory does not contain any topological defects, “embedded defects” such as Z-strings are possible [11,12]. Fermionic (quark and lepton) zero modes on electroweak Z-strings have been studied in the literature [3,5] and can provide linked configurations of strings with anomalous values of baryonic charge and other quantum numbers [4,11]. Quark and lepton zero modes also have an important effect on the stability of the Z-string [11,12].

The work on fermionic zero modes on Z-strings has so far considered massless neutrinos. The conclusion of these studies has been [3,4,11,12] that there are no normalizable zero energy solutions for massless particles in the vortex background. Some attention had been given to massive neutrinos before, but in the framework of strings in Grand Unified models [20,21]. A reconsideration of both massive and massless neutrino zero modes on electroweak strings is needed. This is the aim of the present paper.

In Section 2, we present the electroweak standard model Lagrangian for leptons and find the neutrino equations of motion in a Z-string background. We find the solutions of these equations of motion, and show that a zero mode does exist as part of the continuum. In Section 3 we present the standard extension of the electroweak Lagrangian with a massive neutrino and derive the neutrino equations of motion. We solve the equations of motion in two asymptotic regimes (small and large r) and discuss the number of well-behaved solutions. We finally solve the equations numerically to obtain the zero mode solution.

There are several index theorems in the literature which specify the number of zero modes in a given Lagrangian. The index theorem in [22] does not apply to a massive neutrino case because it assumes that the source of all fermionic masses is the interaction with a single Higgs doublet (which is not true for the right-handed neutrino Majorana mass). The generalized index theorem in [23] and [24] also does not apply because it uses an ansatz for the spinor components not general enough to cover our case. The generalized index theorem in [24] applies to the massive neutrino with a Dirac mass and either a left- or a right-handed Majorana mass but not both.

II. THE STANDARD MODEL NEUTRINO

We consider first the Glashow-Weinberg-Salam electroweak standard model, and focus on the interactions of a single lepton family. The fermion content is a left-handed SU(2) lepton doublet, \( \Psi \equiv (\nu_L, e_L)^T \) and a right-handed electron singlet \( e_R^- \). Electrons acquire their (Dirac) mass from Yukawa coupling to the SU(2) doublet Higgs field \( \Phi \), while neutrinos are massless. (There is no right-handed neutrino to participate in a Dirac mass, and there is assumed to be no left-handed Majorana mass, since there is no fundamental Higgs triplet in the model.) The Lagrangian of the theory is:

\[
\mathcal{L}_{SM} = \frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + (D_{\mu} \Phi)^\dagger (D^\mu \Phi) - \lambda (\Phi^\dagger \Phi - \eta^2)^2 - i \bar{\Psi} \gamma^\mu D_{\mu} \Psi - i \bar{e}_R \gamma^\mu D_{\mu} e_R + \text{h.c.}
\]

Here

\[
D_{\mu} \Psi \equiv \left( \partial_\mu - i \frac{g}{2} \gamma^\mu W^a_{\mu} + i \frac{g'}{2} B_{\mu} \right) \Psi,
\]

\[
D_{\mu} e_R \equiv \left( \partial_\mu - i \frac{g}{2} \gamma^\mu W^a_{\mu} + i \frac{g'}{2} B_{\mu} \right) e_R
\]

and analogous expressions for the neutrino field. The neutrino mass matrix is given by

\[
M_{\nu} = \frac{g}{\sqrt{2}} \Phi \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}
\]

where the columns are left-handed neutrinos, electrons, and right-handed neutrinos.

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\[
D_\mu e_R \equiv (\partial_\mu + ig'_\mu) e_R \\
D_\mu \Phi \equiv \left(\partial_\mu - i\frac{g}{2}\tau^a W^a_\mu - i\frac{g'}{2} B_\mu\right) \Phi ,
\]
with
\[
W^a_\mu \equiv \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu , \\
F^a_\mu \equiv \partial_\mu B_\nu - \partial_\nu B_\mu
\]
where \(\tau^a\) are the Pauli matrices. As is conventional, we define
\[
Z_\mu \equiv \cos \theta_w W^3_\mu - \sin \theta_w B_\mu \\
A_\mu \equiv \sin \theta_w W^3_\mu + \cos \theta_w B_\mu
\]
where \(\theta_w\) is the weak mixing angle. The fermionic equations of motion for Lagrangian (1) are
\[
i\gamma^\mu D_\mu \Psi = h^\mu \Phi e_R \\
i\gamma^\mu D_\mu e_R = h^\mu \Phi \Psi_L .
\]
In the Z-string ansatz, \(A^\mu = W^1_\mu = W^2_\mu = 0\), \(\Phi = (0, \phi)^T\). In cylindrical coordinates \((r, \theta, z)\) the gauge and Higgs fields take the form
\[
q Z^\mu = \left(0, -\frac{v(r)}{r} \hat{e}_\theta\right) , \quad \phi = \eta f(r)e^{i\theta} \quad (5)
\]
where \(q \equiv \alpha/2 = \sqrt{g^2 + g'^2}/2\), and \(v(r)\) and \(f(r)\) are the Z-string profile functions.

With this ansatz, the neutrino equation of motion is:
\[
i\gamma^\mu D_\mu \nu_L = 0 \quad (6)
\]
A representation of the Dirac matrices in cylindrical coordinates is
\[
\gamma^r = \begin{pmatrix} 0 & e^{-i\theta} & 0 & 0 \\
-e^{i\theta} & 0 & 0 & 0 \\
0 & 0 & 0 & -e^{-i\theta} \\
0 & 0 & e^{i\theta} & 0 \end{pmatrix} , \quad (7)
\]
\[
\gamma^\theta = \begin{pmatrix} 0 & -ie^{-i\theta} & 0 & 0 \\
-ie^{i\theta} & 0 & 0 & 0 \\
0 & 0 & 0 & ie^{-i\theta} \\
0 & 0 & ie^{i\theta} & 0 \end{pmatrix} , \quad (8)
\]
\[
\gamma^0 = \begin{pmatrix} \tau^3 & 0 \\
0 & -\tau^3 \end{pmatrix} , \quad \gamma^z = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix} , \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix}
\]
In this representation, a left-handed Dirac fermion has the form: \(\nu_L^\alpha = (\alpha, \beta, -\alpha, -\beta)\).

A general solution to equation (6) can be written as a superposition of all modes with definite winding number,
\[
\alpha = \sum_{m=-\infty}^{\infty} \alpha_m(r) e^{ikz} e^{-i\omega t + im\theta} \quad (9)
\]
\[
\beta = \sum_{m=-\infty}^{\infty} i\beta_m(r) e^{ikz} e^{-i\omega t + im\theta}
\]
\(\alpha_m\) and \(\beta_m\) are, in general, complex functions of \(r\). We are interested in zero energy solutions to equation (6) such that all spinor components fall off outside the string core (large \(r\)) fast enough to be normalizable and are well-behaved at the origin (small \(r\)).

After setting \(k_z = 0\), we get a set of two recursive equations for the coefficients of \(\alpha\) and \(\beta\):
\[
\omega \alpha_{m-1} + \beta'_m + \frac{m + v}{r} \beta_m = 0 \quad (10)
\]
\[-\omega \beta_m + \alpha'_m - \frac{m - 1 + v}{r} \alpha_{m-1} = 0 \]
Let us consider the case \(m = 0\). For large \(r\), \(v \sim 1, f \sim 1\) and the closed system of equations is:
\[
\omega \alpha_1 + \beta'_0 + \frac{1}{r} \beta_0 = 0 \quad (11)
\]
\[-\omega \beta_0 + \alpha'_1 = 0 \]
Eliminating \(\alpha_1\) from the system we get Bessel’s equation for \(\beta\):
\[
\beta''_0 + \frac{1}{r} \beta'_0 + \beta_0 (\omega^2 - \frac{1}{r^2}) = 0 \quad (12)
\]
the general solution of which is given by
\[
\beta_0(\omega r) = A \sqrt{\omega} J_1(\omega r) + B \sqrt{\omega} Y_1(\omega r) \quad (13)
\]
Here \(J_1\) and \(Y_1\) are Bessel’s functions of the first and second kind, while \(A\) and \(B\) are constants. We see therefore that the \(k_z = 0\) solutions comprise a continuum labeled by \(\omega\). Using the normalization relation for Bessel’s functions:
\[
\int_0^\infty J_1(\omega r) J_1(\omega' r) r dr = \frac{1}{\omega} \delta(\omega' - \omega) .
\]
and a similar relation which holds for \(Y_1\), we see that the solution is properly \(\delta\)-function normalized. For \(\omega \neq 0\) and \(r\) large, \((13)\) is just a cylindrical wave \(\beta_0 \sim e^{\pm i\omega r/\sqrt{r}}\).
To see what happens with the zero mode, i.e. \(\omega = 0\), solution, we take \(r\) fixed but large enough for eq. (11) to be valid, say \(r \gg \eta^{-1}\). Thus, the limit \(\omega \rightarrow 0\) implies that \(\omega r \rightarrow 0\). Using the asymptotic expansion for Bessel’s function of small argument we get the neutrino wave function:
\[
\beta_0 = \frac{D}{\sqrt{\omega r}} \quad (15)
\]
where \(D\) is a constant independent of \(\omega\) and \(r\).

The neutrino equations for small \(r\) and \(m = 0\) are:
\[
\omega \alpha_{-1} + \beta'_0 + v_0 r \beta_0 = 0 \quad (16)
\]
\[-\omega \beta_0 + \alpha'_{-1} + \frac{1}{r} \alpha_{-1} = 0 \]
where we used \(v \sim v_0 r^2\) and \(f \sim f_0 r\) near the origin. Eliminating \(\alpha_{-1}\) from the system we get equation for \(\beta\):
\[ \beta_0' + \frac{1}{r} \beta_0 + \beta_0(\omega^2 + 2\nu_0) = 0 \]  

where \( \beta_0 \) is the only nontrivial solution of the system (10) in the zero mode limit, i.e. \( \omega r \to 0 \). The solutions for all other \( \alpha_m \) and \( \beta_m \) are not valid in the zero mode limit because of singular behavior at either small or large \( r \) which makes solutions not even Dirac delta function normalizable.

If we look at the solution for \( \beta_0 \) as an isolated zero mode, it is obvious not normalizable because the normalization integral diverges logarithmically \[14\]. But, we have seen that this state is actually part of a continuum spectrum of the theory and is Dirac delta function normalizable. Therefore it is a valid zero mode solution.

### III. BEYOND THE STANDARD MODEL

Recent evidence from the Super Kamiokande experiment \[25\] strongly suggests that neutrinos are very light, but not massless. A simple extension to the standard model which incorporates light neutrino masses is to add to each standard model family an \( SU(2)_L \times U(1)_Y \) singlet right-handed neutrino, and to incorporate both a Dirac mass for the neutrino and a Majorana mass for the right-handed neutrino. The origin of \( M_R \) is a vacuum expectation value of some \( SU(2) \) singlet Higgs field, acquired at some energy scale much larger than that of the electroweak scale. \( M_R \) can therefore be taken to be spatially homogeneous. The modified Lagrangian is

\[
\mathcal{L} = \mathcal{L}_{SM} + h \left[ \bar{\Psi}i\tau_2\Phi^*\nu_R - \frac{1}{\Sigma} \bar{\Phi}^T i\tau_1 \Phi \right] + \frac{1}{2} \bar{\nu}_R^T M_R \nu_R + \frac{1}{2} \bar{\nu}_L M_R \nu_L
\]

where \( \nu^c \equiv C\nu^T \) and \( \nu^T_R = (\gamma, \delta, \gamma, \delta) \). The charge conjugation matrix \( C \) in the representation we are working in is:

\[
C 
= i\gamma^2 \gamma^0 
= \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

The resulting neutrino equations of motion are:

\[
\begin{align*}
\bar{\Psi} \gamma^\mu D_\mu \nu_L &= h\phi^* \nu_R \\
\bar{\Psi} \gamma^\mu \partial_\mu \nu_R &= h\phi \nu_L + M_R(\nu_R)^c
\end{align*}
\]

The presence of the Majorana mass term, \( M_R \), prevents us from removing the angular dependence from the Dirac equations for the neutrino fields using the standard procedure \[13\]. Instead, we must consider a superposition of all modes with definite winding number. Extending the ansatz \[1\], we write

\[
\begin{align*}
\alpha &= \sum_{m=-\infty}^{\infty} \alpha_m(r)e^{ik_z z-\omega t+i\theta} \\
\beta &= \sum_{m=-\infty}^{\infty} \beta_m(r)e^{ik_z z-\omega t+i\theta} \\
\gamma &= \sum_{m=-\infty}^{\infty} \gamma_m(r)e^{ik_z z-\omega t+i\theta} \\
\delta &= \sum_{m=-\infty}^{\infty} \delta_m(r)e^{ik_z z-\omega t+i\theta}
\end{align*}
\]

After setting \( \omega = k_z = 0 \), we get a set of four recursive equations for the coefficients \( \alpha, \beta, \gamma, \delta \)

\[
\begin{align*}
\beta_m' + \frac{(m + v)}{r} \beta_m &= - h\eta \gamma - f \gamma_m \\
\gamma_m' - \frac{m}{r} \gamma_m - i M_R \gamma_{\ast}^{1-m} &= - h\eta \beta_m \\
\alpha_m' - \frac{(m + v)}{r} \alpha_m &= - h\eta \delta_{m+2} \\
\delta_m' + \frac{m}{r} \delta_m - i M_R \delta_{1-m} &= - h\eta \alpha_{m-2}
\end{align*}
\]

In the \( M_R = 0 \) limit, the equations are not recursively connected.

System (23) decomposes into two subsystems - (\( \beta, \gamma \)) and (\( \alpha, \delta \)). Let us first analyze the (\( \beta, \gamma \)) equations. From (23) we see that a self consistent set of equations consists of four equations for \( \beta_m, \beta_{-1-m}, \gamma_m \) and \( \gamma_{-1-m} \). The case \( m = 0 \) is the most interesting one because we already know that in the \( M_R = 0 \) case (e.g. electrons), zero modes have \( m = 0 \) \[14\]. Setting \( m = 0 \) in the first two equations of the system (23) we get:

\[
\begin{align*}
\beta_0' + \frac{v}{r} \beta_0 &= - h\eta \gamma_0 \\
\beta_{-1}' + \frac{(v - 1)}{r} \beta_{-1} &= - h\eta \gamma_{-1}
\end{align*}
\]

An analytic solution to the system (24) could not be found, but we can learn something about the structure of solutions by looking at their asymptotic behavior. Neglecting fermion back reaction to the string background, \( v \) and \( f \) are just the Nielsen-Olesen profiles and at large \( r \) behave as \( f(r \to \infty) \to 1, v(r \to \infty) \to 1 \). Assuming the asymptotic behavior \( \beta_0 \sim ae^{i\theta}, \gamma_0 \sim be^{i\theta}, \beta_{-1} \sim ce^{i\theta}, \gamma_{-1} \sim de^{i\theta} \) \((a, b, c, d \text{ are arbitrary complex numbers while } s, t, u \text{ and } l \text{ are complex numbers, with a nonpositive real part})\) we get following conditions:

\[
s = t = u^* = l^* \quad \text{and} \quad |M_R|^2 s^2 = (|h\eta|^2 - s^2)^2
\]

The solutions to this fourth order polynomial equation for \( s \) are given by
\[ s_{\pm}^2 = \frac{1}{2} \left[ |M_R|^2 + 2 |\hbar\eta|^2 \pm |M_R| \sqrt{|M_R|^2 + 4 |\hbar\eta|^2} \right] \] (26)

All four of these solutions for \( s \) are real. Two of these are positive, giving rise to exponentially growing modes and two are negative, giving rise to exponentially decaying modes. Only the latter are physically acceptable. In the physical limit of \( M_R \gg |\hbar\eta| \) we get \( s_+ = M_R \) and \( s_- = (\hbar\eta)^2/(2M_R) \), which is the standard “see-saw” result — outside the string core neutrino zero modes decay such that one decay channel is driven by the heavy neutrino (vacuum) mass eigenstate and another one is driven by the light one. In the limit of \( M_R/(\hbar\eta) \to 0 \) we have the standard result \( s = \pm |\hbar\eta| \).

If we want to match the exponentially decaying large \( r \) solutions in a given pair of equations \((\beta, \gamma)\) to the solutions at the origin, we must know how many solutions for small \( r \) are well-behaved for the set of \((\beta_0, \gamma_0, \beta_{-1}, \gamma_{-1})\). By well-behaved we mean nonsingular and single-valued. We require not only the functions, but also their derivatives to be single-valued in order to have a well-defined solution. Therefore, for a solution of the form \( r^p e^{iq \phi} \) the following must be true: (i) \( p \geq |q| \), (ii) both \( p \) and \( q \) are even or both are odd. Neglecting fermion back reaction on the string background, the asymptotic behavior of the profile functions is \( v \sim v_0 r^2 + v_3 r^4 + \ldots \) and \( f \sim f_0 r + f_2 r^3 + \ldots \) for small \( r \). Writing \( \beta_0 \sim ar^s \), \( \gamma_0 \sim br^s \), \( \beta_{-1} \sim cr^u \), \( \gamma_{-1} \sim dr^l \) \((a, b, c \text{ and } d \text{ are arbitrary complex numbers and } s, t, u \text{ and } l \text{ are nonnegative real numbers, } s \text{ and } t \text{ even, } u \text{ and } l \text{ odd})\) for small \( r \) we find that the leading orders are \( s = 0, t = 0, u = 1 \text{ and } l = 1 \).

System (24) is invariant under the parity transformation \((r \to -r)\), so all corrections to the solutions which are parity violating are zero. We can write the general well-behaved solution in the form:

\[
\begin{pmatrix}
\beta_0 \\
\gamma_0 \\
\beta_{-1} \\
\gamma_{-1}
\end{pmatrix}
= \begin{pmatrix}
 a_0 r^0 + a_2 r^2 + a_4 r^4 + \ldots \\
b_0 r^0 + b_2 r^2 + b_4 r^4 + \ldots \\
c_0 r^0 + c_2 r^2 + c_4 r^4 + \ldots \\
d_0 r^0 + d_2 r^2 + d_4 r^4 + \ldots
\end{pmatrix}
\] (27)

The structure of the system (24) is such that only three coefficients are independent (say \( a_0, b_0 \text{ and } d_0 \) are independent, while \( c_0 = iM_R b_0/2 \) and all other coefficient functions are functions of the first three) which says that there are three linearly independent well-behaved solutions. For the system of four linear first order differential equations we should have four solutions in total, so we conclude that one solution is not well-behaved, i.e. contains one or more terms such as \( r^p, p < 0 \) and/or \( r^p (\log(r))^q \).

With three well-behaved solutions at the origin and two well-behaved solutions at infinity there must be at least one solution which is well-behaved everywhere. Each of the two well-behaved solutions at infinity matches on to a unique linear combination of all four solutions at the origin where only one of them is bad, so there is always one linear combination of the two good solutions at infinity which does not have any contribution from the single bad solution at the origin. However, it could happen that there are two solutions which are well-behaved everywhere if each of the two good solutions at infinity matches exactly to a linear combination of the three good solutions at origin. (We call this possibility a “conspiracy”.) One must also be concerned that the regular solution for one spinor component, say \( \beta_0 \), might correspond to a solution of other spinor components which are not well-behaved. However, it is easy to show that if \( \beta_0 \to 0 \) as \( r \to \infty \) then \( \gamma_0, \beta_{-1}, \gamma_{-1} \to 0 \). Thus, barring conspiracies, we expect to find only one regular solution of the \((\beta_0, \gamma_0, \beta_{-1}, \gamma_{-1})\) equations.

Numerically solving system (24), we find a well-behaved solution (Figure 1). We took \( f(r) = \tanh(r) \) and \( v(r) = \tanh^2(r) \) for the string profile functions which gives the correct asymptotical behavior \( f \sim r, v \sim r^2 \) as \( r \to 0, f \sim 1, v \sim 1 \) as \( r \to \infty \) and the correct parity. We also set \( iM_R/(\hbar\eta) = 1.1 \). Although no other well-behaved numerical solutions were found, a “conspiracy” cannot be entirely dismissed because of possible numerical errors. However, more than one neutrino zero mode would lead to a mismatch between the number of neutrino and electron zero modes.

FIG. 1. Numerical solution of the system in eq. (24) showing \( \beta_0 \) (solid), \( \gamma_0 \) (short dash), \( \beta_{-1} \) (dot dash) and \( \gamma_{-1} \) (long dash). The solution is for \( iM_R/(\hbar\eta) = 1.1 \), \( f(r) = \tanh(r) \) and \( v(r) = \tanh^2(r) \).

Next we consider non-zero values of the parameter \( m \) in eq. (24). For the case \( m = 1 \) analysis of large \( r \) gives results similar to the \( m = 0 \) case — two exponentially growing and two exponentially decaying solutions, the latter ones with the expected large \( M_R \) “see-saw” behavior.

*An analysis in the case of electrons, similar to the one done for neutrinos, shows that there is only one well-behaved solution without the possibility for a conspiracy.
ior and the standard $M_R \to 0$ limit. For small $r$, the general well behaved solution is of the form:

$$\begin{pmatrix} \beta_1 \\ \gamma_1 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} a_0 r^3 + a_2 r^5 + a_4 r^7 + \ldots \\ b_0 r^1 + b_2 r^3 + b_4 r^5 + \ldots \\ c_0 r^2 + c_2 r^4 + c_4 r^6 + \ldots \\ d_0 r^2 + d_2 r^4 + d_4 r^6 + \ldots \end{pmatrix}$$  \tag{28}

where only two coefficients are independent (say $b_0$ and $c_0$, while $a_0 = -f_0 h\eta b_0/4$, $d_0 = i M_R b_0/4$, $c_2 = -(c_0 v_0 + h\eta f_0 i M_R b_0/4)/2$). With only two good asymptotic solutions, both at infinity and at the origin, we cannot say anything definite about existence of the zero modes in this sector. Depending on the specific matching coefficients between small and large $r$ solutions we could have none, one or two zero modes. However, numerical analysis shows that there are no nontrivial well-behaved solutions, i.e. $\beta_1 = \gamma_1 = \beta_2 = \gamma_2 = 0$ everywhere. Numerical experiments also indicates that a similar situation occurs for $m \geq 2$.

As a simple consequence of the fact that zero modes in $3 + 1$ dimension satisfy the relationship $\omega = \pm k$, they must be eigenstates of the $\gamma^0 \gamma^z$ operator. We found one neutrino zero mode with $\beta \neq 0$, $\gamma \neq 0$ and therefore, analysis of the $(\alpha, \delta)$ pair of the equations of the system (24) is not necessary. In order to be eigenstate of $\gamma^0 \gamma^z$ with the definite eigenvalue (in this case $-1$) the neutrino zero mode must have $\alpha = \delta = 0$.

IV. CONCLUSION

We have found that in the standard model a massless neutrino has one well-behaved zero mode on a Z-string which, being part of a continuum, is Dirac delta function normalizable.

We considered the minimal extension of the standard model Lagrangian which includes a phenomenologically valid massive neutrino, with both Dirac and Majorana mass terms. The homogeneous, non-winding Majorana mass term $M_R$ makes this case considerably different from the electron one. To accommodate the angular dependence of the equations of motion we must use an ansatz in which each spinor component is a superposition of at least two modes with definite winding number. As a consequence, we get a set of coupled equations. We solved the equations analytically in two asymptotic regimes, small and large $r$, and argued that in each sector determined by a definite winding number $m$ it is possible, in principle, to have one or more zero modes. Numerical analysis indicates that there is one well-behaved zero mode in the $m = 0$ sector and none in higher $m$ sectors.

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