Closed Universe in Mirage Cosmology

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Abstract

We study the cosmological evolution of the closed universe on a spherical probe brane moving in the $\text{AdS}_m \times S^n$ background and the near-horizon background of the dilatonic D-branes. The Friedmann equations describing the evolution of the brane universe, and the effective energy density and pressure simulated on the probe brane due to its motion in the curved background spacetime are obtained and analyzed. We also comment on the relevance of the spherical probe brane to the giant graviton for the special value of the probe energy.

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1 Introduction

Recently, there has been active investigation on the possibility that our four-dimensional universe might be a three-brane embedded in higher-dimensional spacetime. Such idea is motivated by the recent proposal \cite{1, 2, 3, 4, 5, 6} on solving hierarchy problems in particle physics and opens up the possibility of probing the extra dimensions in the near future. According to the brane-world scenario, fields of the Standard Model, which are assumed to arise as fluctuations of branes in string theories, are confined to the three-brane, while gravity can freely propagate in the bulk spacetime. Even when the extra space is infinite, the special property of warped spacetime renders the gravity to be localized around the three-brane \cite{4, 5, 6}.

Lots of work on cosmological models based on the brane-world idea has been done, e.g. Refs. \cite{7, 8, 9, 10, 11, 12, 13}. Most of models assume that the three-brane, in which our four-dimensional universe is embedded, evolves with time due to the time evolution of energy density of the real matter confined within the three-brane. In this paper, we follow a different approach based on the idea that the cosmological evolution of our four-dimensional universe is due to the geodesic motion of the (probe) universe three-brane in the curved background of other branes in the bulk \cite{14, 15, 16, 17}. In this approach, the motion in ambient space induces cosmological evolution of the four-dimensional universe on the probe three-brane, simulating various kinds of matter responsible for expansion of brane universe \cite{17}. One can view such simulated matter or ‘mirage’ matter as the contribution of other branes, on which hidden gauge interactions are localized, and bulk background fields to the cosmological evolution of our four-dimensional universe.

The previous works \cite{17, 18, 19, 20, 21, 22} on mirage cosmology assume that the probe universe brane is planar, thereby the metric on the brane universe becomes that of the expanding flat universe (with zero spatial curvature). In this paper, we consider spherical probe branes wrapped around the sphere part of various background spacetimes in string theories. Since the probe universe brane is spherical, the metric on the brane universe becomes that of the expanding closed universe (with positive spatial curvature). Just as in the planar probe brane case, the mirage energy density consists of various terms simulating massless scalars, radiation and superluminal matter (outside of the range of the causality condition). Furthermore, the (total) mirage energy density and pressure simulated on the brane universe do not stay always positive during the course of cosmological evolution. One of the consequences of this fact is that even if the brane universe is described by the Robertson-Walker metric with positive spatial curvature it can expand indefinitely. Also, since the positivity conditions on the energy density and the pressure of the singularity theorems of cosmology are violated, one can hope that the initial singularity of cosmology may be cured within such framework.

The spherical probe branes are of interest also because of their relevance to the
giant gravitons, proposed to give a bulk interpretation of the stringy exclusion principle. When the spherical probe brane has the same energy as that of a massless particle with the same angular momentum and has the stable radius, one can view it as gravitons blown up into a sphere. Therefore, the mirage cosmology on a spherical probe brane with constant radius can be regarded as an example on a toy cosmological model on a (fuzzy) noncommutative sphere. We find that the Friedmann equation does not make sense when the giant graviton has different radius from the one determined by its angular momentum. This implies that the giant graviton is stable against its radius perturbation and it is not possible for a point-like graviton to make classical transition to the giant graviton through gradual increase in its radius.

The paper is organized as follows. In section 2, we discuss the relevant aspects of the standard closed expanding universe in arbitrary spacetime dimensions. Then, we study the closed universe mirage cosmology in the background of the AdS × S spacetime in section 3, and in the near-horizon background of the dilatonic D-branes in section 4.

2 Expanding Closed Universe

Generally, the induced metric on a spherical probe p-brane moving in the sphere part of the background spacetime can be put into the following standard form for the expanding closed universe in (p + 1)-dimensions:

\[ ds^2 = -d\eta^2 + g(\eta)d\Omega_p^2, \]

where the metric \( d\Omega_p^2 \) on a unit p-sphere is parameterized by the angular coordinates \( \theta_1, ..., \theta_p \) as follows:

\[ d\Omega_p^2 = d\theta_1^2 + \sin^2 \theta_1[d\theta_2^2 + \sin^2 \theta_2(d\theta_3^2 + \cdots + \sin^2 \theta_{p-1}d\theta_p^2)]. \]

One can further put the metric (1) on the brane universe into the following form for the Robertson-Walker metric with positive spatial curvature by redefining the coordinate as \( \chi = \sin \theta_1 \):

\[ ds^2 = -d\eta^2 + g(\eta) \left[ \frac{d\chi^2}{1 - \chi^2} + \chi^2 d\Omega_{p-1}^2 \right], \]

where the metric \( d\Omega_{p-1}^2 \) is parameterized by \( \theta_2, ..., \theta_p \). So, in this section, we discuss the relevant aspects of the standard expanding closed universe in (p + 1)-dimensions.

To derive the Friedmann equations for the expanding brane universe on a spherical probe p-brane, we define the scale factor \( a \) as \( a^2 = g \) with the Hubble parameter defined as \( H = \dot{a}/a \), where the overdot stands for the derivative with respect to the cosmic
We assume that the effective energy density $\rho_{\text{eff}}$ and pressure $\varphi_{\text{eff}}$ (as measured in the rest frame), simulated on the probe brane due to its motion in the source brane background, are perfect fluids, whose energy momentum tensor is generally given by

$$T_{\mu\nu} = (\varphi_{\text{eff}} + \rho_{\text{eff}})U_\mu U_\nu + \varphi_{\text{eff}} g_{\mu\nu},$$

where $U_\mu$ is the $(p+1)$-velocity of the fluid. From the Einstein’s equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor of the induced metric (3), one obtains the following Friedmann equations in the $(p+1)$-dimensional universe on the probe brane:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{16\pi}{p(p-1)} \varphi_{\text{eff}} - \frac{1}{a^2},$$

$$\frac{\ddot{a}}{a} = \left[1 + \frac{1}{2} a \frac{\partial}{\partial a}\right] \left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi}{p(p-1)} (p-2) \rho_{\text{eff}} + p\varphi_{\text{eff}},$$

where the explicit expression for $\left(\frac{\dot{a}}{a}\right)^2$ can be found by making use of the equations for the probe $p$-brane motion and the defining equation for the cosmic time $\eta$.

From the Friedmann equations (3), one obtains the following expressions for the effective matter density and pressure on the brane universe:

$$\rho_{\text{eff}} = \frac{p(p-1)}{16\pi} \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\right],$$

$$\varphi_{\text{eff}} = -\frac{p-1}{16\pi} \left[\left(p + a \frac{\partial}{\partial a}\right) \left(\frac{\dot{a}}{a}\right)^2 + \frac{p-2}{a^2}\right].$$

The scalar curvature of the brane universe is

$$\mathcal{R} = 2p \left[\frac{\ddot{a}}{a} + \frac{p-1}{2} \left(\frac{\dot{a}}{a}\right)^2 + \frac{p-1}{2} \frac{1}{a^2}\right] = p \left[\left(p + 1 + a \frac{\partial}{\partial a}\right) \left(\frac{\dot{a}}{a}\right)^2 + \frac{p-1}{a^2}\right] = 4(p-1)\pi (\rho_{\text{eff}} - p\varphi_{\text{eff}}).$$

In particular for the $p = 3$ case, the effective pressure and the scalar curvature have the following simple expressions in terms of the effective matter density:

$$\varphi_{\text{eff}} = -\rho_{\text{eff}} - \frac{1}{3} a \frac{\partial \rho_{\text{eff}}}{\partial a},$$

$$\mathcal{R} = 8\pi (4 + a \frac{\partial}{\partial a}) \rho_{\text{eff}}.$$

When the effective energy density and pressure satisfy the equation of state of the form

$$\varphi_{\text{eff}} = w \rho_{\text{eff}},$$

where $w$ is a time-independent constant, the conservation of energy equation $0 = \nabla_\mu T^\mu_0 = -\partial_0 \rho_{\text{eff}} - p(\rho_{\text{eff}} + \varphi_{\text{eff}}) \frac{\dot{a}}{a}$ can be integrated to give

$$\rho_{\text{eff}} \propto a^{-p(1+w)}.$$
So, the behavior of the effective energy density for particularly interesting cases of cosmological fluids are as follows. For the (collisionless, nonrelativistic) dust or matter dominated case, for which the pressure is negligible and therefore \( w = 0 \), \( \varrho_{\text{eff}} \propto a^{-p} \). For the radiation or relativistic particles, for which \( T^\mu{}_{\mu} = 0 \) and therefore \( w = 1/p \), \( \varrho_{\text{eff}} \propto a^{-(p+1)} \). For the vacuum case, for which \( \varrho_{\text{eff}} = -\varpi_{\text{eff}} \) or \( w = -1 \), \( \varrho_{\text{eff}} \) is independent of \( a \). The causality restricts \( w \) to be \( |w| \leq 1 \). So, \( \varrho_{\text{eff}} \) satisfying the causality condition can decrease at most as fast as \( \propto a^{-2p} \) as \( a \) increases. This extreme case is characteristic of a massless scalar, for which \( \varrho_{\text{eff}} = \varpi_{\text{eff}} \) or \( w = 1 \).

3 Brane Cosmology in the AdS \(_n \times S^m \) Background

In this section, we study the mirage cosmology on a spherical probe \( p \)-brane moving in the bulk background of the AdS \(_m \times S^n \) spacetime. The radii for AdS \(_m \) and \( S^n \) are respectively \( \tilde{L} \) and \( L \). This background spacetime can be obtained from string theory or M-theory for the \((p,n) = (3,5), (2,7), (5,4)\) cases with \( L = \frac{n-3}{2} \tilde{L} \) as the near-horizon geometries of the D3-, M2- and M5-branes. Other background spacetime in string theories corresponding to the near-horizon geometries of dilatonic D-branes will be considered in the next section. We consider the two separate cases of the probe brane wrapping (\( i \)) \( S^p \) in \( S^n \) with \( n = p + 2 \) and (\( ii \)) \( S^p \) in the AdS \(_m \) space with \( m = p + 2 \). For the former case, we parameterize the metric of the AdS \(_m \times S^{p+2} \) background spacetime as

\[
d s^2 = \frac{u^2}{L^2} \left[ -dt^2 + dx_1^2 + \cdots + dx_{m-2}^2 \right] + \tilde{L}^2 \frac{du^2}{u^2} + L^2 d\Omega^2_{p+2},
\]

where we choose to parameterize the metric \( d\Omega^2_{p+2} \) on a unit \( S^{p+2} \) in the following way:

\[
d\Omega^2_{p+2} = \frac{1}{1 - \rho^2} d\rho^2 + (1 - \rho^2) d\phi^2 + \rho^2 d\Omega^2_p,
\]

with the metric \( d\Omega^2_p \) on a unit \( S^p \) parameterized by the angular coordinates \( \theta_1, \ldots, \theta_p \) as in Eq. (2). For the cases in which such background spacetime can be obtained as the near-horizon geometry of maximally supersymmetric \( p \)-brane solution of the supergravity theories, applying the Hodge-dual transformation when necessary, one has the following \( (p + 1) \)-form potential whose magnetic field flux threads the \( S^{p+2} \) part of the bulk spacetime:

\[
A_{\phi\theta_1\ldots\theta_p} = L^{p+1} \rho^{p+1} \sin^{p-1} \theta_1 \cdots \sin \theta_{p-1} \equiv L^{p+1} \rho^{p+1} \epsilon_{\theta_1\ldots\theta_p},
\]

where \( \epsilon_{\theta_1\ldots\theta_p} \) is the volume form of a unit \( S^p \). For the latter case, we parameterize the metric of the AdS\(_{p+2} \times S^n \) spacetime as follows:

\[
d s^2 = - \left( 1 + \frac{r^2}{L^2} \right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega^2_p + L^2 d\Omega^2_n,
\]
where we parameterize the metric of the unit $S^n$ as $d\Omega_n^2 = h_{ij}(\varphi) d\varphi^i d\varphi^j$ ($i, j = 1, \ldots, n$) and the metric of the unit $S^p$ as in Eq. (4). The $(p+1)$-form potential whose electric field flux threading the AdS part of the spacetime is given by
\[
A_{\theta_1 \ldots \theta_p}^{p+1} = -\frac{r^{p+1}}{L} \sin^{p-1} \theta_1 \cdots \sin \theta_{p-1} \equiv -\frac{r^{p+1}}{L} \epsilon_{\theta_1 \cdots \theta_p}.
\]

### 3.1 Probe brane in $S^n$

We consider the spherical probe $p$-brane which wraps $S^p$ in the $S^{p+2}$ part of the background $\text{AdS}_m \times S^{p+2}$ spacetime. Since the $\text{AdS}_{p+2} \times S^n$ background generically does not have dilaton field, the action for the probe $p$-brane has the form
\[
S_p = -T_p \int d^{p+1} \xi \sqrt{-\det G} + T_p \int d^{p+1} \xi \hat{A}^{p+1}.
\]

In the static gauge, the spatial worldvolume coordinates $\xi^a$ ($a = 1, \ldots, p$) of the probe $p$-brane are identified with the coordinates of the unit $S^p$ in the $S^{p+2}$ part of the source background. We assume that the transverse target space coordinates of the probe $p$-brane depend on the time coordinate $\xi^0 = t$, only, i.e. the probe brane does not oscillate. Due to the translational symmetry of the source brane along the $x_i$-directions, the momentum of the probe along these directions is conserved, which always enables one go to the frame in which $\partial_i x_i = 0$. After Eqs. (11) and (13) are substituted and the coordinates of $S^p$ are integrated, the probe action takes the following form:
\[
S_p = -T_p V_p \int dt L^p \rho^p \left[ \frac{u^2}{L^2} - \frac{\tilde{L}^2}{u^2} (\partial_t u)^2 - \frac{L^2}{1 - \rho^2} (\partial_t \rho)^2 - L^2 (1 - \rho^2) (\partial_t \phi)^2 \right]
\]
\[
+ T_p V_p \int dt L^{p+1} \rho^{p+1} \partial_t \phi,
\]
where $V_p = 2\pi \frac{p!}{\Gamma\left(\frac{p+1}{2}\right)}$.

To obtain the equations describing the probe brane motion, we consider the following canonical momenta and Hamiltonian of the probe brane:
\[
P_u = \frac{\partial \mathcal{L}}{\partial (\partial_t u)} = \frac{T_p V_p L^p \tilde{L}^2 \rho^p \partial_t u}{u^2 \left( \frac{u^2}{L^2} - \frac{\tilde{L}^2}{u^2} (\partial_t u)^2 - \frac{L^2}{1 - \rho^2} (\partial_t \rho)^2 - L^2 (1 - \rho^2)(\partial_t \phi)^2 \right)},
\]
\[
P_\rho = \frac{\partial \mathcal{L}}{\partial (\partial_t \rho)} = \frac{T_p V_p L^{p+2} \rho^p (1 - \rho^2) \partial_t \rho}{(1 - \rho^2) \left( \frac{u^2}{L^2} - \frac{\tilde{L}^2}{u^2} (\partial_t u)^2 - \frac{L^2}{1 - \rho^2} (\partial_t \rho)^2 - L^2 (1 - \rho^2)(\partial_t \phi)^2 \right)},
\]
\[
P_\phi = \frac{\partial \mathcal{L}}{\partial (\partial_t \phi)} = \frac{T_p V_p L^{p+2} \rho^p (1 - \rho^2) \partial_t \phi}{\sqrt{\frac{u^2}{L^2} - \frac{\tilde{L}^2}{u^2} (\partial_t u)^2 - \frac{L^2}{1 - \rho^2} (\partial_t \rho)^2 - L^2 (1 - \rho^2)(\partial_t \phi)^2}} + T_p V_p L^{p+1} \rho^{p+1},
\]
\[
H = E = P_u \partial_t u + P_\rho \partial_t \rho + P_\phi \partial_t \phi - \mathcal{L}.
\]
We will find in the below that the cosmic scale factor is given by \( a = L\rho \), which implies that \( \partial_t \rho \) has to be nonzero in order for the brane universe to evolve with time. We also assume from now on that \( \partial_t u = 0 \), i.e. \( u = u_0 = \text{const.} \). Then, making use of the fact that the energy \( E \) and the angular momentum \( P_\phi \) of the probe brane is conserved, we obtain the following equations describing the motion of the probe brane:

\[
\begin{align*}
(\partial_t \rho)^2 &= \frac{u_0^2 (1 - \rho^2)}{L^2 L^2} \left[ 1 - \frac{(P_\phi - T_p V_p L^{p+1} \rho^{p+1})^2 + T_p V_p^2 L^{2(p+1)} \rho^{2p}(1 - \rho^2)}{E^2 L^2 L^2 u_0^{-2}(1 - \rho^2)} \right], \\
(\partial_t \phi)^2 &= \frac{(P_\phi - T_p V_p L^{p+1} \rho^{p+1})^2 u_0^4}{E^2 L^4 (1 - \rho^2)^2}.
\end{align*}
\]

The metric on the brane universe is given by the following induced metric on the probe \( p \)-brane:

\[
\begin{align*}
\mathrm{d}s^2 &= -\frac{u_0^2}{L^2} - \frac{L^2 (\partial_t \rho)^2}{1 - \rho^2} - L^2 (1 - \rho^2)(\partial_t \phi)^2 \mathrm{d}t^2 + L^2 \rho^2 \mathrm{d}\Omega_p^2 \\
&= -\frac{T_p V_p^2 L^{2p} u_0^4}{E^2 L^4} \rho^{2p} \mathrm{d}t^2 + L^2 \rho^2 \mathrm{d}\Omega_p^2,
\end{align*}
\]

where Eq. (19) is used to simplify the expression. In terms of the cosmic time \( \eta \) defined through

\[
\mathrm{d}\eta^2 = \frac{T_p V_p^2 L^{2p} u_0^4}{E^2 L^4} \rho^{2p} \mathrm{d}t^2,
\]

the induced metric takes the standard form (4) or (3) for the expanding closed universe with \( g = a^2 = L^2 \rho^2 \).

The Friedman equations describing the expanding universe on the probe \( p \)-brane can be obtained by substituting Eqs. (19) and (21) into the following:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{\rho^2} \left( \frac{\mathrm{d}t}{\mathrm{d}\eta} \right)^2 \left( \frac{\rho}{\mathrm{d}t} \right)^2, \quad \frac{\ddot{a}}{a} = \left[ 1 + \frac{1}{2} a \frac{\partial}{\partial a} \right] \left( \frac{\dot{a}}{a} \right)^2.
\]

The resulting expressions for Friedmann equations in terms of the cosmic scale factor are

\[
\begin{align*}
\left( \frac{\dot{a}}{a} \right)^2 &= \frac{1}{L^2} \left[ (E^2 - \bar{P}_\phi^2) \left( \frac{L}{a} \right)^{2(p+1)} - E^2 \left( \frac{L}{a} \right)^{2p} + 2\bar{P}_\phi \left( \frac{L}{a} \right)^{p+1} - \left( \frac{L}{a} \right)^{2} \right], \\
\frac{\ddot{a}}{a} &= -\frac{1}{L^2} \left[ p(E^2 - \bar{P}_\phi^2) \left( \frac{L}{a} \right)^{2(p+1)} - (p - 1)E^2 \left( \frac{L}{a} \right)^{2p} + (p - 1)\bar{P}_\phi \left( \frac{L}{a} \right)^{p+1} \right],
\end{align*}
\]
where we introduced \( N \equiv T_p V_p L^{p+1}, \bar{P}_\phi \equiv P_\phi/N \) and \( \bar{E} \equiv E \bar{L} L/(N \nu_0) \) to simplify the expressions.

The following effective energy density and pressure on the brane universe are obtained by substituting Eq. (23) into Eq. (6):

\[
\varrho_{\text{eff}} = \frac{p(p-1)}{16\pi L^2} \left[ (\bar{E}^2 - \bar{P}_\phi^2) \left( \frac{L}{a} \right)^{2(p+1)} - \bar{E}^2 \left( \frac{L}{a} \right)^{2p} + 2\bar{P}_\phi \left( \frac{L}{a} \right)^{p+1} \right],
\]

\[
\varphi_{\text{eff}} = \frac{p - 1}{16\pi L^2} \left[ (p+2)(\bar{E}^2 - \bar{P}_\phi^2) \left( \frac{L}{a} \right)^{2(p+1)} - p\bar{E}^2 \left( \frac{L}{a} \right)^{2p} + 2\bar{P}_\phi \left( \frac{L}{a} \right)^{p+1} \right]. \tag{25}
\]

The following scalar curvature of the brane universe is obtained by substituting Eq. (23) into Eq. (7):

\[
R = \frac{p}{L^2} \left[ (p+1)(\bar{P}_\phi^2 - \bar{E}^2) \left( \frac{L}{a} \right)^{2(p+1)} + (p-1)\bar{E}^2 \left( \frac{L}{a} \right)^{2p} \right]. \tag{26}
\]

As can be seen in Eq. (25), the effective energy density \( \varrho_{\text{eff}} \) simulated on the probe brane consists of (i) acausal term \( \sim a^{-2(p+1)} \), (ii) term \( \sim a^{-2p} \) simulating a massless scalar and (iii) term \( \sim a^{-(p+1)} \) simulating radiation or relativistic matter. At very early times of the evolution, when the probe brane is very close to the source brane (i.e. \( a \ll 1 \)), the acausal term dominates the effective energy density and the brane expansion, causing superluminal “shocks”. As the brane universe expands, the massless scalar term \( \sim a^{-2(p+1)} \) takes over, and then finally at later stage of the evolution the probe brane motion simulates radiation-dominated universe. Note, since the coordinate \( \rho \equiv a/L \) in the metric (12) is restricted to take value \( 0 \leq \rho \leq 1 \), the brane universe cannot expand indefinitely. As can be seen from Eq. (23), the cosmic scale factor \( a \) can never reach \( L \) (since the RHS of Eq. (23) becomes negative when \( a = L \)), but reaches some maximum value \( a_{\text{max}} < L \) and starts decreasing (since \( \dot{a} < 0 \) when \( \dot{a} = 0 \)), eventually the brane universe crunching to zero size. The scalar curvature (26) diverges when \( a = 0 \) (initial singularity). However, from the higher dimensional perspective (of the bulk background spacetime), such initial singularity corresponds to a perfectly regular point in \( S^{p+2} \), which is regular everywhere, thereby the initial singularity of the brane universe is resolved through higher-dimensional embedding.

It is argued in Ref. [23] that gravitons moving in the AdS\(_m \times S^n\) spacetime along the \( S^n \) part blow up into spherical \((n-2)\)-brane of increasing size as their angular momentum increases by showing that the spherical \((n-2)\)-brane with the stable radius (that does not change with time) for a give angular momentum has the same energy as that of graviton with the same angular momentum. So, when the energy \( E \) of the probe \( p \)-brane (under consideration in this paper) has the same energy as that of graviton with the same angular momentum \( P_\phi \), one can view the probe as the 'giant gravitons'. To determine the criterion for the probe brane to become the expanded gravitons, we
consider the Hamiltonian in Eq. (28) expressed in terms of conjugate momenta in the following way:

$$H = \frac{u}{L} \left[ \frac{P_\phi^2}{L^2} + \frac{P_u^2}{L^2 / u^2} + \frac{P_\rho^2}{L^2 / (1 - \rho^2)} + \frac{(\rho P_\phi - N \rho^p)^2}{L^2 (1 - \rho^2)} \right]^{\frac{1}{2}}. \quad (27)$$

Note, we are interested in the probe motion with $u = u_0 = \text{const}$ in this subsection, so $P_u = 0$ in the above Hamiltonian. Since we are looking for the probe motion with the stable radius $\rho$, $P_\rho = 0$. For such case, the energy takes the following minimum value

$$H = \frac{u_0 P_\phi}{L}, \quad (28)$$

when $\rho$ takes the equilibrium value determined by

$$P_\phi = N \rho^{p-1}. \quad (29)$$

The minimum energy (28) of the probe $p$-brane coincides with the energy of a massless particle, namely graviton, carrying the same angular momentum $P_\phi$ on $S^{p+2}$, thereby the probe brane behaves like a massless particle or a graviton for such case. Since the allowed range of the coordinate $\rho$ in the metric (12) is $0 \leq \rho \leq 1$, the maximum value of angular momentum for such particular probe motion is $P_\phi = \bar{N}$, as explained in Ref. [23] as the bulk origin of the stringy exclusion principle [24]. We notice that when the energy $E$ of the probe takes the minimum value (28) the acausal terms in Eqs. (23-26) vanish, since $|\bar{E}| = |\bar{P}_\phi|$ for such case. The first Friedmann equation (23) then takes the following form:

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{L^2} \left[ \bar{P}_\phi \left( \frac{L}{a} \right)^p - \frac{L}{a} \right]^2. \quad (30)$$

The RHS of this equation becomes negative unless $a = L \bar{P}_\phi^{1/(p-1)}$, which corresponds to the radius $\rho_0 = (P_\phi/N)^{1/(p-1)}$ of the giant graviton. From this, one can see that the giant graviton is stable against its radius perturbation. Eq. (30) also implies that classical transition from a point-like graviton to the giant graviton cannot occur through gradual increase in the size of a graviton. The transition is possible only through the quantum tunneling between two vacua, one corresponding to a point-like graviton and the other the giant graviton, described by instantons, as was proposed in Refs. [26, 27].

### 3.2 Probe brane in AdS

We consider the spherical probe $p$-brane which wraps $S^p$ in the AdS$_{p+2}$ part of the AdS$_{p+2} \times S^n$ background space. The action for the probe brane has the form (10).
After Eqs. (14) and (13) are substituted and the coordinates of $S^p$ are integrated, the probe action (14) in the static gauge takes the following form:

$$S_p = -T_p V_p \int dt \frac{r^p}{L} \sqrt{1 + \frac{r^2}{L^2} - \left(\frac{\partial r}{L} \right)^2 - L^2 h_{ij} \partial_t \varphi^i \partial_t \varphi^j + T_p V_p \int dt \frac{r^{p+1}}{L}},$$  \hspace{1cm} (31)

where the sign of the Wess-Zumino term has been reversed, corresponding to choosing the opposite brane charge.

To obtain the equations describing the probe brane motion, we consider the following conjugate momenta and Hamiltonian of the probe $p$-brane:

$$P_r = \frac{\partial \mathcal{L}}{\partial (\partial_r r)} = \frac{T_p V_p r^p \partial_r r}{(1 + \frac{r^2}{L^2}) \sqrt{1 + \frac{r^2}{L^2} - \left(\frac{\partial r}{L} \right)^2 - L^2 h_{ij} \partial_t \varphi^i \partial_t \varphi^j}},$$

$$P_i = \frac{\partial \mathcal{L}}{\partial (\partial_r \varphi^i)} = \frac{T_p V_p L^2 r^p h_{ij} \partial_t \varphi^j}{(1 + \frac{r^2}{L^2}) \sqrt{1 + \frac{r^2}{L^2} - \left(\frac{\partial r}{L} \right)^2 - L^2 h_{ij} \partial_t \varphi^i \partial_t \varphi^j}},$$

$$H = E = P_r \partial_t r + P_i \partial_t \varphi^i - \mathcal{L}$$

$$= T_p V_p \left[ \frac{r^p \left(1 + \frac{r^2}{L^2}\right)}{(1 + \frac{r^2}{L^2}) \sqrt{1 + \frac{r^2}{L^2} - \left(\frac{\partial r}{L} \right)^2 - L^2 h_{ij} \partial_t \varphi^i \partial_t \varphi^j}} - \frac{r^{p+1}}{L} \right].$$  \hspace{1cm} (32)

Making use of the fact that the energy $E$ and the total angular momentum $h_{ij} P_i P_j = \ell^2$ of the probe are conserved, from Eq. (32) one obtains the following equation describing the radial motion of the probe:

$$\left(\frac{\partial r}{L} \right)^2 = \left(1 + \frac{r^2}{L^2}\right)^2 \left[1 - \frac{1 + \frac{r^2}{L^2} - \frac{T_p^2 V_p^2 L^2 r^{2p+2}}{E}}{1 + \frac{r^2}{L^2}} \frac{T_p^2 V_p^4 L^2}{T_p^2 V_p^2 L^2} \ell^2 \right],$$  \hspace{1cm} (33)

along with

$$h_{ij} \partial_t \varphi^i \partial_t \varphi^j = \frac{\left(1 + \frac{r^2}{L^2}\right)^2 \ell^2}{T_p^2 V_p^2 L^4 \left(1 + \frac{r^2}{L^2} \frac{E}{T_p V_p}\right)^2}. \hspace{1cm} (34)$$

The metric of the brane universe is given by the following induced metric on the probe $p$-brane:

$$d\tilde{s}^2 = - \left[1 + \frac{r^2}{L^2} - \left(\frac{\partial r}{L} \right)^2 - L^2 h_{ij} \partial_t \varphi^i \partial_t \varphi^j \right] dt^2 + r^2 d\Omega_p^2$$

$$= - \left(\frac{r^{p+1}}{L} + \frac{E}{T_p V_p}\right)^2 dt^2 + r^2 d\Omega_p^2, \hspace{1cm} (35)$$
where we used Eqs. (33) and (34) to simplify the expression. In terms of the cosmic time $\eta$ defined through

$$d\eta^2 = \frac{(1 + \frac{r^2}{\tilde{L}^2})^2 \, r^{2p}}{\left(\frac{r^{p+1}}{L} + \frac{\tilde{E} T_p V_p}{r^p}\right)^2} dt^2,$$

the induced metric (35) takes the standard form (1) for the metric of the expanding closed universe with $g(\eta) = r^2(\eta)$ (therefore, the cosmic scale factor is $a = \sqrt{g} = r$). Here, just as in the previous section, $d\Omega^2_p$ is given by Eq. (2) and through the redefinition of the coordinate $\chi = \sin \psi$ the induced metric can be put into the standard form (3) for the Robertson-Walker metric.

The Friedman equations describing the expanding brane universe on the spherical probe $p$-brane can be obtained by substituting Eqs. (33) and (36) into the following:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{r^2} \left(\frac{dt}{d\eta}\right)^2 \left(\frac{dr}{dt}\right)^2,$$

$$\ddot{a} = \left[1 + \frac{1}{2} a \frac{\partial}{\partial a}\right] \left(\frac{\dot{a}}{a}\right)^2.$$ (37)

The resulting expressions are given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{L^2} \left[\left(\tilde{E}^2 - \tilde{\ell}^2\right) \left(\tilde{L}/a\right)^{2(p+1)} - \tilde{\ell}^2 \left(\tilde{L}/a\right)^{2p} + 2\tilde{E} \left(\tilde{L}/a\right)^{p+1} - \left(\tilde{L}/a\right)^2\right],$$ (38)

$$\ddot{a} = -\frac{1}{L^2} \left[p(\tilde{E}^2 - \tilde{\ell}^2) \left(\tilde{L}/a\right)^{2(p+1)} - (p-1)\tilde{\ell}^2 \left(\tilde{L}/a\right)^{2p} + (p-1)\tilde{E} \left(\tilde{L}/a\right)^{p+1}\right],$$ (39)

where we introduced $\tilde{N} \equiv T_p V_p L \tilde{L}^p$, $\tilde{\ell} \equiv \ell/\tilde{N}$ and $\tilde{E} \equiv E L / \tilde{N}$ to simplify the expressions. These Friedmann equations have the similar forms as those (23,24) associated with the probe brane moving in the $S^n$ part of the bulk background except that the massless scalar and the radiation terms are now respectively controlled by the angular momentum and the energy.

By substituting Eq. (38) into Eq. (3), one obtains the following effective energy density and pressure on the probe brane:

$$\rho_{\text{eff}} = \frac{p(p-1)}{16\pi L^2} \left[\left(\tilde{E}^2 - \tilde{\ell}^2\right) \left(\tilde{L}/a\right)^{2(p+1)} - \tilde{\ell}^2 \left(\tilde{L}/a\right)^{2p} + 2\tilde{E} \left(\tilde{L}/a\right)^{p+1}\right].$$

$$\varrho_{\text{eff}} = \frac{p-1}{16\pi L^2} \left[(p+2)(\tilde{E}^2 - \tilde{\ell}^2) \left(\tilde{L}/a\right)^{2(p+1)} - p\tilde{\ell}^2 \left(\tilde{L}/a\right)^{2p} + 2\tilde{E} \left(\tilde{L}/a\right)^{p+1}\right].$$ (40)

The scalar curvature of the brane universe is

$$\mathcal{R} = \frac{p}{L^2} \left[(p+1)(\tilde{E}^2 - \tilde{\ell}^2) \left(\tilde{L}/a\right)^{2(p+1)} + (p-1)\tilde{\ell}^2 \left(\tilde{L}/a\right)^{2p}\right].$$ (41)
The effective energy density in Eq. (40) consists of (i) acausal term $\sim a^{-2(p+1)}$, (ii) term $\sim a^{-2p}$ simulating a massless scalar and (iii) term $\sim a^{-(p+1)}$ simulating radiation or relativistic matter. Just as in the previous subsection, the brane universe expansion is dominated by the acausal term during the very early stage of cosmological evolution and as the brane universe expands the massless scalar term and then the radiation term takes over. Unlike the case in the previous subsection, the coordinate $r = a$ in the metric (14) can take arbitrarily large values. So, the brane universe can expand indefinitely even if it is described by the metric (1) of the expanding closed universe. This is because the effective pressure $\mathcal{P}_{\text{eff}}$ does not always stay positive in the course of cosmic evolution of the brane universe. When $a = 0$, the scalar curvature (41) diverges, corresponding to the initial singularity. From the higher-dimensional perspective, this initial singularity corresponds to a regular point in the AdS part of the AdS$_{p+2} \times S^n$ background spacetime.

It is argued [26, 27] that gravitons can expand also into the spherical part of the AdS portion of the AdS$\times S^n$ spacetime. To determine the condition for the spherical probe $p$-brane (moving in the AdS portion of the AdS$\times S^n$ spacetime) to become giant gravitons, we consider the following Hamiltonian for the probe brane in equilibrium radius $r$, i.e. Eq. (32) with $\partial_t r = 0$:

$$H = \frac{1}{T_p V_p} \left[ \frac{1}{L^2} \left( 1 + \frac{r^2}{L^2} \right) \left( \ell^2 + T_p^2 V_p^2 L^2 r^{2p} \right) - \frac{r^{p+1}}{L} \right]$$

$$= \frac{N}{L} \left[ \sqrt{1 + \frac{r^2}{L^2} \left( \frac{\tilde{\ell}^2 + r^{2p}}{L^{2p}} \right) - \frac{r^{p+1}}{L^{p+1}}} \right].$$

(42)

The Hamiltonian takes the following stable minimum value

$$H = \frac{\ell}{L},$$

when $r$ takes the equilibrium value determined by

$$\left( \frac{r}{L} \right)^{p-1} = \tilde{\ell}.$$  

(44)

The minimum energy (43) coincides with the energy of a massless particle with the angular momentum $\ell$ moving in the AdS space, thereby the spherical probe $p$-brane with the energy (43) can be identified as gravitons expanded into a $p$-sphere. On the other hand, since the range of radial coordinate $r$ in the metric (14) is not restricted, the probe angular momentum $\ell$ satisfying (44), i.e. $\ell = N (r / \tilde{L})^{p-1} = T_p V_p L \tilde{L} r^{p-1}$, can take any arbitrarily large values [26, 27]. As in the previous subsection, the minimum energy (43) corresponds to the critical energy for which the acausal terms in Eqs. (38-41) vanish, i.e. $|\tilde{E}| = |\tilde{\ell}|$ and the first Friedmann equation (38) takes the form (30) with $\tilde{P}_\phi$ and $L$ respectively replaced by $\tilde{\ell}$ and $\tilde{L}$.
4 Brane Cosmology in Near-Horizon Background of Dilatonic D-brane

In this section, we study the mirage cosmology on the spherical probe Dp-brane moving in the bulk background of the near-horizon region of the source D(6 − p)-brane, where the probe brane wraps $S^p$ in the transverse space of the source brane.

The solution for the source D(6 − p)-brane magnetically charged under the RR $(p + 1)$-form potential $A^{p+1}$ has the form

$$ds^2 = -g_{tt}dt^2 + \sum_{i=1}^{6-p} g_{ii}dx_i^2 + g_{rr}dr^2 + h(r)r^2d\Omega_{p+2}^2,$$

$$e^\Phi = \left(\frac{L_p}{r}\right)^{\frac{p+1}{4}} \frac{1}{A_{p+1}}, \quad A^{p+1} = L_p^{p+1} \rho^{p+1} \epsilon_{\theta_1...\theta_p},$$

$$g_{tt} = \left(\frac{r}{r_p}\right)^{\frac{p+1}{2}}, \quad g_{ii} = \left(\frac{r}{r_p}\right)^{\frac{p+1}{2}}, \quad g_{rr} = \left(\frac{L_p}{r}\right)^{\frac{p+1}{2}}, \quad h(r) = \left(\frac{L_p}{r}\right)^{\frac{p+1}{2}}, \quad (45)$$

where $d\Omega_{p+2}^2$ is parameterized as in Eq. (12). $L_p$ can be expressed in terms of the number $N$ of D$(6 - p)$-branes and the tension $T_p$ of the probe D$p$-brane as $L_p^{p+1} = N/(T_p V_p)$.

The action for the probe D$p$-brane in the absence of the NS $B$-field and the world-volume $U(1)$ gauge field is given by

$$S_p = -T_p \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det \hat{G}_{\alpha\beta}} + T_p \int d^{p+1}\xi \hat{A}^{p+1}, \quad (46)$$

where $\hat{G}_{\alpha\beta}$ and $\hat{A}^{p+1}$ are respectively the pullbacks of the metric and the RR $(p+1)$-form potential. In the static gauge, the spatial worldvolume coordinates $\xi^a (a = 1, ..., p)$ of the probe D$p$-brane are identified with the coordinates of the unit $S^p$ in the transverse space of the source D$(6 - p)$-brane. As in the previous section, we assume that the transverse target space coordinates of the probe $p$-brane depend on the time coordinate $\xi^0 = t$, only. We consider the frame in which $\partial_t x_i = 0$. After Eq. (13) is substituted and the coordinates of $S^p$ are integrated, the probe action (46) takes the following form:

$$S = -T_p V_p \int dt e^{-\Phi}[h(r)r^2\rho^2]^\frac{p-1}{2} \sqrt{g_{tt} - g_{rr}(\partial_t r)^2 - g_{\rho\rho}(\partial_t \rho)^2 - g_{\phi\phi}(\partial_t \phi)^2 + N} \int dt \rho^{p+1}\partial_t \phi. \quad (47)$$

The conjugate momenta and the Hamiltonian of the probe brane are given by

$$P_r = \frac{\partial\mathcal{L}}{\partial(\partial_t r)} = \frac{T_p^r}{\sqrt{g_{tt} - g_{rr}(\partial_t r)^2 - g_{\rho\rho}(\partial_t \rho)^2 - g_{\phi\phi}(\partial_t \phi)^2}} (h(r^2\rho^2)^\frac{p-1}{2} g_{rr}\partial_t r),$$

$$P_\rho = \frac{\partial\mathcal{L}}{\partial(\partial_t \rho)} = \frac{T_p^\rho}{\sqrt{g_{tt} - g_{rr}(\partial_t r)^2 - g_{\rho\rho}(\partial_t \rho)^2 - g_{\phi\phi}(\partial_t \phi)^2}} (h(r^2\rho^2)^\frac{p-1}{2} g_{\rho\rho}\partial_t \rho).$$
\[ P_\phi = \frac{\partial \mathcal{L}}{\partial (\partial_t \phi)} = \frac{T_p V_p e^{-\Phi}}{\sqrt{g_{tt} - g_{rr}(\partial_t r)^2 - g_{\rho\rho}(\partial_t \rho)^2 - g_{\phi\phi}(\partial_t \phi)^2}}(hr^2 \rho^2) g_{\phi\phi} \partial_t \phi + N \rho^{p+1}, \]

\[ H = E = P_\phi \partial_t r + P_p \partial_t \rho + P_\phi \partial_t \phi - \mathcal{L} \]

\[ = \frac{T_p V_p e^{-\Phi} (hr^2 \rho^2)^2 g_{tt}}{\sqrt{g_{tt} - g_{rr}(\partial_t r)^2 - g_{\rho\rho}(\partial_t \rho)^2 - g_{\phi\phi}(\partial_t \phi)^2}}. \]  

The metric of the brane universe is given by the following induced metric on the probe Dp-brane:

\[ ds^2 = -[g_{tt} - g_{rr}(\partial_t r)^2 - g_{\rho\rho}(\partial_t \rho)^2 - g_{\phi\phi}(\partial_t \phi)^2] dt^2 + h(r) r^2 \rho^2 d\Omega_p^2. \]  

By defining the cosmic time \( \eta \) through

\[ d\eta^2 = [g_{tt} - g_{rr}(\partial_t r)^2 - g_{\rho\rho}(\partial_t \rho)^2 - g_{\phi\phi}(\partial_t \phi)^2] dt^2, \]

one can put the induced metric (49) into the standard form (1) or (3) for the expanding closed universe with \( g(\eta) = h(r(\eta)) r^2(\eta) \rho^2(\eta) \). Since the scale factor is given by \( a = \sqrt{g} = h^{1/2}(r) r \rho \), the Friedmann equations are given by

\[ \left( \frac{\dot{a}}{a} \right)^2 = \left[ \frac{\dot{r}}{r} \left( \frac{1}{2} \frac{h'}{h} r + 1 \right) + \frac{\dot{\rho}}{\rho} \right]^2, \quad \frac{\ddot{a}}{a} = \left[ 1 + \frac{1}{2} a \frac{\partial}{\partial a} \left( \frac{\dot{a}}{a} \right) \right] \left( \frac{\dot{a}}{a} \right)^2, \]  

where the prime stands for derivative with respect to \( r \). In studying the Friedman equations, in the following we consider the two particular cases of the probe motion with either \( \dot{r} = 0 \) or \( \dot{\rho} = 0 \).

First, when \( \partial_t r = 0 \), i.e. \( r = r_0 = \text{const} \), from Eq. (48) one obtains the following equations of motion for the probe brane:

\[ (\partial_t r)^2 = \frac{g_{tt} - (P_\phi - N \rho^{p+1})^2 + T_p^2 V_p^2 e^{-2\Phi} (h_0 r_0^2 \rho^2)^p g_{\phi\phi}}{E^2 g_{\phi\phi}/g_{tt}}, \]

\[ (\partial_t \phi)^2 = \frac{(P_\phi - N \rho^{p+1})^2 g_{tt}^2}{E^2 g_{\phi\phi}^2}, \]  

where \( h_0 \equiv h(r_0) \). The metric (49) on the brane universe is simplified to

\[ ds^2 = -\frac{T_p^2 V_p^2}{E^2} (h_0 r_0^2 \rho^2)^p e^{-2\Phi} g_{tt}^2 dt^2 + h_0^2 r_0^2 \rho^2 d\Omega_p^2, \]  

and therefore the cosmic time \( \eta \) is defined through

\[ d\eta^2 = \frac{T_p^2 V_p^2}{E^2} (h_0 r_0^2 \rho^2)^p e^{-2\Phi} g_{tt}^2 dt^2. \]
Substituting Eqs. (52) and (54) into Eq. (51), one obtains the following form of the Friedman equation:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{L_p^2} \left[ (\tilde{E}^2 - \tilde{P}_\phi^2) \left( \frac{\tilde{L}_p}{a} \right)^{2(p+1)} + \tilde{E}^2 \left( \frac{\tilde{L}_p}{a} \right)^{2p} + 2\tilde{P}_\phi \left( \frac{\tilde{L}_p}{a} \right)^{p+1} - \left( \frac{\tilde{L}_p}{a} \right)^2 \right],
\]

(55)

where we introduced \((\tilde{L}_p/r_0)^4 \equiv (L_p/r_0)^{p+1}\), \(\tilde{E} \equiv E L_p^2 / (N r_0)\) and \(\tilde{P}_\phi \equiv P_\phi / N\) to simplify the expression. This is the same form as the Friedman equation (23) describing the expanding spherical \(p\)-brane moving in the \(S^{p+2}\) part of the \(\text{AdS}_m \times S^{n+2}\) spacetime. So, brane expansion in the near-horizon background of the dilatonic D-brane is qualitatively the same as that in the \(\text{AdS}_m \times S^n\) background. It is argued in Ref. [28] that giant graviton can also be realized within backgrounds other than the \(\text{AdS} \times S^n\) spacetime, provided some condition on the dilaton field and \(h(r)\) in the metric in Eq. (45) is satisfied, as we summarize in the following. The Hamiltonian in Eq. (48) can be expressed in terms of the conjugate momenta in the following form:

\[
H = \sqrt{g_{tt}} \left[ \frac{P_\phi^2}{h(r)r^2} + \frac{P_r^2}{g_{rr}} + \frac{P_\rho^2}{g_{\rho\rho}} + \left( \rho P_\phi - N \rho^p \right)^2 \right]^{\frac{1}{2}},
\]

(56)

provided the condition \(T_p V_p e^{-\Phi} (h(r)r^2)^{\frac{p+1}{2}} = N\) is satisfied. This condition is satisfied by the solution (45) for the dilatonic D-brane in the near-horizon region. For a given angular momentum \(P_\phi\) and with \(\partial_r r = 0\), the Hamiltonian takes the following minimum value

\[
H = \sqrt{g_{tt}} \frac{P_\phi}{r \sqrt{h}},
\]

(57)

provided \(\rho\) takes the equilibrium value (i.e. \(P_\rho = 0\)) determined by

\[
P_\phi = N \rho^{p-1}.
\]

(58)

This minimum energy coincides with the energy of a massless particle, i.e. graviton, with angular momentum \(P_\phi\) on \(S^{p+2}\). For our case, such minimum energy is given by

\[
H = E = \frac{1}{r_0} \left( \frac{r_0}{L_p} \right)^{\frac{p+1}{2}} P_\phi = \frac{r_0}{L_p^2} P_\phi,
\]

(59)

which coincides with the condition \(|\tilde{E}| = |\tilde{P}_\phi|\) for the acausal term in the Friedmann equation (55) to vanish, just as in the cases in the previous section.

Second, we consider the \(\partial_r \rho = 0\) (i.e. \(\rho = \rho_0 = \text{const}\)) case. This case corresponds to the spherical probe \(p\)-brane with constant radius \(\rho_0 = (P_\phi/N)^{1/(p-1)}\) (i.e. the giant graviton) moving along the \(r\)-direction of the source brane. The probe brane takes the following energy that coincides with the energy of a massless particle:

\[
H = \sqrt{g_{tt}} \left[ \frac{P_\phi^2}{h(r)r^2} + \frac{P_r^2}{g_{rr}} \right]^{\frac{1}{2}}.
\]

(60)
It is interesting to note that although the spherical brane has constant radius the brane universe evolves with time due to its motion along the \( r \)-direction in the source background (cf. the cosmic scale factor is given by \( a = \sqrt{h(r)} r \rho_0 \)). One can view the \( \partial_t \rho = 0 \) case as a toy cosmological model on a fuzzy noncommutative sphere. From Eq. \((48)\) with \( \partial_t \rho = 0 \), one obtains the following equations of motion for the probe brane:

\[
(\partial_r)^2 = \frac{g_{rr}}{g_{tt}} \left[ 1 - \frac{(P_\phi - N p_0^{p+1})^2 + T^2 p^2 (h r^2 \rho_0^2)^p e^{-2\Phi} g_{\phi\phi}}{E^2 g_{tt} / g_{tt}} \right],
\]

\[
(\partial_\phi)^2 = \frac{(P_\phi - N p_0^{p+1})^2 g_{tt}}{E^2 g_{\phi\phi}}. \tag{61}
\]

The brane universe metric \((49)\) is then simplified to

\[
d\tilde{s}^2 = \frac{T^2 p^2}{E^2} (h r^2 \rho_0^2)^p e^{-2\Phi} g_{tt} dt^2 + h^2 r^2 \rho_0^2 d\Omega_p^2,
\]

and therefore the cosmic time \( \eta \) is defined through

\[
d\eta^2 = \frac{T^2 p^2}{E^2} (h r^2 \rho_0^2)^p e^{-2\Phi} g_{tt} dt^2. \tag{63}
\]

So, the Friedman equations \((51)\) take the forms:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{(p - 3)^2}{16 L_p^2} \left[ \tilde{E}^2 \tilde{P}_\phi^{-2\frac{p-1}{p-3}} \left( \frac{a}{L_p} \right)^{2\frac{p+1}{p-3}} - \left( \frac{L_p}{a} \right)^2 \right], \tag{64}
\]

\[
\frac{\ddot{a}}{a} = \frac{(p - 1)(p - 3)}{8} \tilde{P}_\phi^{-2\frac{p-1}{p-3}} \tilde{E}^2 \left( \frac{a}{L_p} \right)^{2\frac{p+1}{p-3}}, \tag{65}
\]

where we introduced \( \tilde{E} \equiv E L_p / N \) and \( \tilde{P}_\phi \equiv P_\phi / N \) to simplify the expressions. The effective energy density and pressure on the brane universe is therefore given by

\[
\varrho_{\text{eff}} = \frac{p(p - 1)}{256 \pi L_p^2} \left[ (p - 3)^2 \tilde{E}^2 \tilde{P}_\phi^{-2\frac{p-1}{p-3}} \left( \frac{a}{L_p} \right)^{2\frac{p+1}{p-3}} - (p + 1)(p - 7) \left( \frac{L_p}{a} \right)^2 \right],
\]

\[
\varphi_{\text{eff}} = \frac{p - 1}{256 \pi L_p^2} \left[ (p - 3)(p^2 - p + 2) \tilde{E}^2 \tilde{P}_\phi^{-2\frac{p-1}{p-3}} \left( \frac{a}{L_p} \right)^{2\frac{p+1}{p-3}} - (p + 1)(p - 2)(p - 7) \left( \frac{L_p}{a} \right)^2 \right]. \tag{66}
\]

The scalar curvature \((67)\) of the brane universe takes the following form:

\[
R = \frac{p(p - 1)}{16 L_p^2} \left[ (p - 3) \tilde{E}^2 \tilde{P}_\phi^{-2\frac{p-1}{p-3}} \left( \frac{a}{L_p} \right)^{2\frac{p+1}{p-3}} - (p + 7) \left( \frac{L_p}{a} \right)^2 \right]. \tag{67}
\]
From Eq. (64), one can see that the brane universe cannot reach zero size (since the RHS becomes negative when \( a = 0 \)) when \( p > 3 \). So, the brane universe starts from some finite size where \( \dot{a} = 0 \) and expands indefinitely (since \( \ddot{a} \) is always positive). [As in section 3.2, despite having positive spatial curvature the brane universe expands indefinitely, because and \( \varphi_{eff} \) and \( \varphi_{eff} \) does not stay positive all the time during the expansion.] Also, what is extraordinary about this case is that the expansion of the brane universe accelerates with time, i.e., the brane universe inflates, rather than slowing down, since \( \dot{a} > 0 \) all the time. Since the brane universe cannot reach zero size, the spacetime curvature (67) remains finite at all time during the cosmological evolution. Such violation of the singularity theorems of cosmology is possible because the mirage energy density and pressure do not satisfy the positivity conditions of the theorems.

References

[1] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, “The hierarchy problem and new dimensions at a millimeter,” Phys. Lett. B429 (1998) 263, [hep-ph/9803315].

[2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, “New dimensions at a millimeter to a Fermi and superstrings at a TeV,” Phys. Lett. B436 (1998) 257, [hep-ph/9804398].

[3] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, “Phenomenology, astrophysics and cosmology of theories with submillimeter dimensions and TeV scale quantum gravity,” Phys. Rev. D59 (1999) 086004, [hep-ph/9807344].

[4] L. Randall and R. Sundrum, “A large mass hierarchy from a small extra dimension,” Phys. Rev. Lett. 83 (1999) 3370, [hep-ph/9905221].

[5] L. Randall and R. Sundrum, “An alternative to compactification,” Phys. Rev. Lett. 83 (1999) 4690, [hep-th/9906064].

[6] J. Lykken and L. Randall, “The shape of gravity,” JHEP 0006 (2000) 014, [hep-th/9908076].

[7] G. Dvali and S.H.H. Tye, “Brane inflation,” Phys. Lett. B450 (1999) 72, [hep-ph/9812483].

[8] P. Binetruy, C. Deffayet and D. Langlois, “Non-conventional cosmology from a braneworld,” Nucl. Phys. B565 (2000) 269, [hep-th/9905012].

[9] J.M. Cline, C. Grojean and G. Servant, “Cosmological expansion in the presence of extra dimensions,” Phys. Rev. Lett. 83 (1999) 4245, [hep-ph/9906528].

[10] C. Csaki, M. Graesser, C. Kolda and J. Terning, “Cosmology of one extra dimension with localized gravity,” Phys. Lett. B462 (1999) 34, [hep-ph/9906513].

[11] D.J. Chung and K. Freese, “Cosmological challenges in theories with extra dimensions and remarks on the horizon problem,” Phys. Rev. D61 (2000) 023511, [hep-ph/9906542].
[12] E.E. Flanagan, S.H. Tye and I. Wasserman, “A cosmology of the brane world,” Phys. Rev. D62 (2000) 024011, hep-ph/9909373.

[13] P. Kanti, I.I. Kogan, K.A. Olive and M. Pospelov, “Cosmological 3-brane solutions,” Phys. Lett. B468 (1999) 31, hep-ph/9909481.

[14] H.A. Chamblin and H.S. Reall, “Dynamic dilatonic domain walls,” Nucl. Phys. B562 (1999) 133, hep-th/9903225.

[15] A. Chamblin, M.J. Perry and H.S. Reall, “Non-BPS D8-branes and dynamic domain walls in massive IIA supergravities,” JHEP 9909 (1999) 014, hep-th/9908047.

[16] P. Kraus, “Dynamics of anti-de Sitter domain walls,” JHEP 9912 (1999) 011, hep-th/9910149.

[17] A. Kehagias and E. Kiritsis, “Mirage cosmology,” JHEP 9911 (1999) 022, hep-th/9910174.

[18] E. Papantonopoulos and I. Pappa, “Type 0 brane inflation from mirage cosmology,” hep-th/0001183.

[19] J.Y. Kim, “Dilaton-driven brane inflation in type IIB string theory,” Phys. Rev. D62 (2000) 065002, hep-th/0004155.

[20] J.Y. Kim, “Brane inflation in tachyonic and non-tachyonic type 0B string theories,” hep-th/0009175.

[21] E. Papantonopoulos and I. Pappa, “Cosmological evolution of a brane universe in a type 0 string background,” hep-th/0010014.

[22] D. Youm, “Brane inflation in the background of D-brane with NS $B$ field,” hep-th/0011024.

[23] J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the giant gravitons from anti-de Sitter space,” JHEP 0006 (2000) 008, hep-th/0003075.

[24] J. Maldacena and A. Strominger, “AdS$_3$ black holes and a stringy exclusion principle,” JHEP 9812 (1998) 005, hep-th/9804085.

[25] J. Madore, “The Fuzzy sphere,” Class. Quant. Grav. 9 (1992) 69.

[26] A. Hashimoto, S. Hirano and N. Itzhaki, “Large branes in AdS and their field theory dual,” JHEP 0008 (2000) 051, hep-th/0008014.

[27] M.T. Grisaru, R.C. Myers and O. Tafjord, “SUSY and Goliath,” JHEP 0008 (2000) 040, hep-th/0008013.

[28] S.R. Das, S.P. Trivedi and S. Vaidya, “Magnetic moments of branes and giant gravitons,” JHEP 0010 (2000) 037, hep-th/0008203.