Affine Toda solitons and automorphisms of Dynkin diagrams

Niall J. MacKay
Research Institute for Mathematical Sciences,
University of Kyoto,
Kyoto 606,
Japan.

William A. McGhee
Department of Mathematical Sciences,
University of Durham,
Durham DH1 3LE,
England.

Abstract

Using Hirota’s method, solitons are constructed for affine Toda field theories based on the simply-laced affine algebras. By considering automorphisms of the simply-laced Dynkin diagrams, solutions to the remaining algebras, twisted as well as untwisted, are deduced.
1 Introduction

Recent work has shown that soliton solutions can be constructed for affine Toda field theories based on the $a_n^{(1)}$ and $d_4^{(1)}$ algebras \cite{1} as well as the $c_n^{(1)}$ algebra \cite{2} when the coupling constant is purely complex. In the case of the $a_n^{(1)}$ theory N-soliton solutions have been constructed, whereas for $d_4^{(1)}$ and $c_n^{(1)}$ only static single solitons.

The purpose of this paper is to construct static single solitons for all of the remaining algebras, both twisted and untwisted. This is achieved by considering a generalisation of the field ansatz used in \cite{1}, although as in \cite{1} a special decoupling of the equations of motion is considered. It is found that once the soliton solutions for $d_n^{(1)}, e_6^{(1)}, e_7^{(1)}$ and $e_8^{(1)}$ are constructed, solutions for theories based on the other algebras follow by folding the simply-laced Dynkin diagrams. The solutions for $e_6^{(1)}$ and $e_7^{(1)}$ have been obtained independently by Hall \cite{3}.

For the simply-laced algebras the number of static solitons is found to be equal to the rank of the corresponding algebra. However by the method that will be employed here, for the non-simply-laced algebras a lesser number is found (as in \cite{2} for $c_n^{(1)}$). Also, for all the theories the mass ratios of the solitons\footnote{For $g_2^{(1)}$ and $c_n^{(1)}$ only one soliton is found and so mass ratios cannot be considered.} can be calculated and are found to coincide with the mass ratios of the fundamental particles in the real-coupling affine Toda theory \textit{(i.e.} those obtained by expanding the potential term of the Lagrangian density about its minimum \cite{4} \cite{9}).

The paper concludes with a discussion of some aspects of topological charge.

2 The equations of motion

The Lagrangian density of affine Toda field theory can be written in the form

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{\beta^2} \sum_{j=0}^{n} n_j (e^{\beta \alpha_j \cdot \phi} - 1).$$

The field $\phi(x,t)$ is an $n$-dimensional vector, $n$ being the rank of the finite Lie algebra $g$. The $\alpha_j$'s, for $j = 1, \ldots, n$ are the simple roots of $g$; $\alpha_0$ is chosen such that the inner
products among the elements of the set \( \{ \alpha_0, \alpha_j \} \) are described by one of the extended Dynkin diagrams. It is expressible in terms of the other roots by the equation

\[
\alpha_0 = - \sum_{j=1}^{n} n_j \alpha_j
\]

where the \( n_j \)'s are positive integers, and \( n_0 = 1 \). Both \( \beta \) and \( m \) are constants, \( \beta \) being the coupling constant.

The inclusion of \( \alpha_0 \) distinguishes affine Toda field theory from Toda field theory. Toda field theory is conformal and integrable, its integrability implying the existence of a Lax pair, infinitely many conserved quantities and exact solubility \cite{[4][5][6]} (for further references see \cite{[7]}). The extended root is chosen in such a way as to preserve the integrability of Toda field theory (though not the conformal property), with the enlarged set of roots \( \{ \alpha_0, \alpha_j \} \) forming an admissible root system \cite{[4]}.

Setting the coupling constant \( \beta \) to be purely complex, \( i.e. \) \( \beta = i\gamma \), the equations of motion are

\[
\partial^2 \phi - \frac{i m^2}{\gamma} \sum_{j=0}^{n} n_j \alpha_j e^{i\gamma \alpha_j \phi} = 0. \tag{2.1}
\]

Extending the idea of \cite{[1]}, consider the following substitution for the field \( \phi(x, t) \)

\[
\phi = - \frac{1}{i\gamma} \sum_{i=0}^{n} \eta_i \alpha_i \ln \tau_i
\]

which reduces (2.1) to the form

\[
\sum_{j=0}^{n} \alpha_j Q_j = 0
\]

where

\[
Q_j = \left( \frac{\eta_j}{\tau_j^2} (D^2_t - D^2_x) \tau_j - 2m^2 n_j \left( \prod_{k=0}^{n} \tau_k^{-\eta_k \alpha_k \cdot \alpha_j} - 1 \right) \right).
\]

\( D_x \) and \( D_t \) are Hirota derivatives, defined by

\footnote{For the simply-laced algebras the choice of \( \eta_i \) coincides with that of \cite{[1]}, namely \( \eta_i = 1 \). However, for the remaining algebras it is other choices of \( \eta_i \) which yield soliton solutions.}
\[ D_x^m D_t^n f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t) g(x', t') \bigg|_{x=x', t=t'}. \]

It will be assumed (cf. [4]) that \( Q_j = 0 \ \forall j \), although this is not the most general decoupling. (The existence of \( n + 1 \) \( \tau \)-functions (compared to the \( n \)-component field \( \phi \)) is due to the relationship between affine and conformal affine Toda theories [4].) Therefore,

\[ \eta_j (D_t^2 - D_x^2) \tau_j \cdot \tau_j - 2m^2 n_j \left( \prod_{k=0}^{n} \tau_k^{-\eta_k \alpha_k \cdot \alpha_j} - 1 \right) \tau_j^2 = 0. \quad (2.2) \]

In the spirit of Hirota’s method for finding soliton solutions [8], suppose

\[ \tau_j = 1 + \delta_j^{(1)} e^{\Phi \epsilon} + \delta_j^{(2)} e^{2\Phi \epsilon^2} + ... + \delta_j^{(p_j)} e^{p_j \Phi \epsilon^{p_j}} \]

where \( \Phi = \sigma(x - vt - \xi) \) and \( \delta_j^{(k)} \) \( 1 \leq k \leq p_j \), \( \sigma, v \) and \( \xi \) are arbitrary complex constants. The constant \( p_j \) is a positive integer and \( \epsilon \) an infinitesimal parameter. The method employed is to solve (2.2) at successive orders in \( \epsilon \), and then absorb \( \epsilon \) into the exponential.

At first order in \( \epsilon \), it is easily shown that

\[ \sum_{j=0}^{n} K_{ij} \delta_j^{(1)} = \frac{\sigma^2 (1 - v^2)}{m^2} \delta_i^{(1)} \]

where

\[ K_{ij} = \frac{\eta_j}{\eta_i} n_i \alpha_i \cdot \alpha_j. \]

Defining the matrices,

1. \( \eta = \text{diag}(\eta_0, \eta_1, ..., \eta_n) \)
2. \( N = \text{diag}(n_0, n_1, ..., n_n) \)
3. \( (C)_{ij} = \alpha_i \cdot \alpha_j \)

then \( \delta^{(1)} = (\delta_0^{(1)}, \delta_1^{(1)}, ..., \delta_n^{(1)})^T \) is an eigenvector of the matrix \( K \) where

\[ \eta K \eta^{-1} = NC \]

with eigenvalue \( \lambda \) where
\[ \sigma^2(1 - v^2) = m^2 \lambda. \]

As \( K \) and \( NC \) are similar, they share the same eigenvalues. Indeed for the a,d and e theories it has been shown \cite{10} that the squared masses of the fundamental Toda particles are eigenvalues of \( NC \). For the non-simply-laced theories, the eigenvalues of \( NC \) are also eigenvalues of a simply-laced theory and so are related to the squared masses of the non-simply-laced theory. As will be seen in section 5 this leads to the ratios of static energies of the solitons being equal to the ratios of the unrenormalized masses of the fundamental particles described by the Lagrangian fields.

It is straightforward to show that for \( \tau_j \) to be bounded as \( x \to \pm \infty \),

\[ n_0 \eta_j p_j = n_j \eta_0 p_0. \]

In all cases, \( \eta_j \) is chosen to be

\[ \eta_j = \frac{2}{\alpha_j \cdot \alpha_j}, \]

since this choice of \( \eta_j \) causes each \( \tau_j \) to be raised to a non-negative integer power in the equations of motion (2.2). So, for the simply-laced cases \( \eta_j = 1 \) and for single soliton solutions \( p_j = n_j \).

Finally, it is unnecessary to consider the solution corresponding to \( \lambda = 0 \), as it is always \( \phi = 0 \).

3 Affine Toda solitons for simply-laced algebras

The length of the longest roots will be taken to be \( \sqrt{2} \) for all cases. It is necessary fix the root lengths in this way, otherwise the parameters \( m \) and \( \beta \) in the equations of motion have to be rescaled. Also under this convention, the soliton masses are found to satisfy one universal formula.
3.1 The $a_n^{(1)}$ theory

The Dynkin diagram for $a_n^{(1)}$ is shown in Figure 3.1a.

The eigenvalues of the matrix $NC$ are

$$\lambda_a = 4 \sin^2 \left( \frac{\pi a}{n+1} \right).$$

![Figure 3.1a: Affine Dynkin diagram for $a_n^{(1)}$.](image)

With $\eta_j = 1 \ \forall j$, the equations of motion are

$$(D_t^2 - D_x^2) \tau_j \cdot \tau_j = 2m^2 (\tau_{j-1}\tau_{j+1} - \tau_j^2)$$
i.e. those of [1]. For the single soliton solutions $p_0 = 1$, giving

$$\tau_j = 1 + \omega^j e^\Phi$$

where $\omega$ is an $(n+1)^{th}$ root of unity. There are $n$ non-trivial solutions [1] (equal to the number of fundamental particles) with $\omega_a = \exp 2\pi ia/(n+1)$ where $1 \leq a \leq n$. These $n$ solutions to $a_n^{(1)}$ can be written in the form

$$\phi_{(a)} = -\frac{1}{i\gamma} \sum_{k=1}^r \alpha_j \ln \left( \frac{1 + w_a^j e^\Phi}{1 + e^\Phi} \right).$$

It was shown in [1] that $\phi_{(a)} \ (1 \leq a \leq n)$ can be associated with the $a$-th fundamental representation of $a_n^{(1)}$, and that different values of $Im \xi$ give rise to different topological charges. The topological charges are found to be weights of the particular representation. Therefore, strictly speaking the results presented here correspond to representatives from each class of solution, as the value of $\xi$ and so the topological charge, is not specified.
3.2 The $d_{n}^{(1)}$ theory

The equations of motion for $d_{4}^{(1)}$, whose Dynkin diagram is shown in Figure 3.2a, are slightly different to those for $d_{n}^{(1)}_{n \geq 5}$ and so will be considered separately.

![Affine Dynkin diagram for $d_{4}^{(1)}$.](image)

Figure 3.2a: Affine Dynkin diagram for $d_{4}^{(1)}$.

The eigenvalues of the matrix $NC$ are $\lambda = 2, 2, 2$ and 6. With $\eta_{j} = 1 \; \forall j$, the single soliton has $p_{j} = n_{j} \; \forall j$ and satisfies

$$
(D_{t}^{2} - D_{x}^{2})(\tau_{j} \cdot \tau_{j}) = 2m^{2}(\tau_{2} - \tau_{j}^{2}) \quad (j \neq 2)
$$

$$
(D_{t}^{2} - D_{x}^{2})(\tau_{2} \cdot \tau_{2}) = 4m^{2}(\tau_{0}\tau_{1}\tau_{3}\tau_{4} - \tau_{2}^{2}).
$$

If $\lambda=2$, three solutions are obtained [1]:

$$
\tau_{0} = \tau_{3} = 1 + e^{\Phi}
$$

$$
\tau_{2} = 1 + e^{2\Phi}
$$

$$
\tau_{1} = \tau_{4} = 1 - e^{\Phi}
$$

and cycles of the indices $(1, 3, 4)$.

If $\lambda=6$, one solution is obtained:

$$
\tau_{0} = \tau_{1} = \tau_{3} = \tau_{4} = 1 + e^{\Phi}
$$

$$
\tau_{2} = 1 - 4e^{\Phi} + e^{2\Phi}.
$$

The Dynkin diagram for $d_{n}^{(1)} (n \geq 5)$ is shown in Figure 3.2b.
In this case the eigenvalues of the matrix $NC$ are

$$\lambda_a = 8 \sin^2 \vartheta_a \quad \text{where} \quad \vartheta_a = \frac{a\pi}{2(n-1)} \quad (1 \leq a \leq n-2)$$

and $\lambda_{n-1} = \lambda_n = 2$.

With $\eta_j = 1 \forall j$, the single soliton has $p_j = n_j \forall j$ and satisfies the following equations

$$\begin{align*}
(D_t^2 - D_x^2)\tau_0 \cdot \tau_0 &= 2m^2(\tau_2 - \tau_0^2) \\
(D_t^2 - D_x^2)\tau_1 \cdot \tau_1 &= 2m^2(\tau_2 - \tau_1^2) \\
(D_t^2 - D_x^2)\tau_2 \cdot \tau_2 &= 4m^2(\tau_0\tau_1\tau_3 - \tau_2^2) \\
(D_t^2 - D_x^2)\tau_j \cdot \tau_j &= 4m^2(\tau_{j-1}\tau_{j+1} - \tau_j^2) \quad (3 \leq j \leq n-3) \\
(D_t^2 - D_x^2)\tau_{n-2} \cdot \tau_{n-2} &= 4m^2(\tau_{n-2}\tau_{n-3} - \tau_{n-2}^2) \\
(D_t^2 - D_x^2)\tau_{n-1} \cdot \tau_{n-1} &= 2m^2(\tau_{n-2} - \tau_{n-1}^2) \\
(D_t^2 - D_x^2)\tau_n \cdot \tau_n &= 2m^2(\tau_{n-2} - \tau_n^2).
\end{align*}$$

For $\lambda = 2$ it is found that

$$\delta_0^{(1)} = -\delta_1^{(1)} = 1, \quad \delta_j^{(1)} = 0, \quad \delta_j^{(2)} = (-1)^j \quad (2 \leq j \leq n-2)$$

and

$$\delta_{n-1}^{(1)} = -\delta_n^{(1)} = \pm 1 \quad \text{for even } n, \quad \delta_{n-1}^{(1)} = -\delta_n^{(1)} = \pm i \quad \text{for odd } n.$$

For $\lambda = \lambda_a$ ($1 \leq a \leq n-2$), whether $n$ is even or odd,

$$\delta_0^{(1)} = \delta_1^{(1)} = 1, \quad \delta_n^{(1)} = \delta_{n-1}^{(1)} = (-1)^a$$

and

$$\delta_j^{(1)} = \frac{2\cos((2j-1)\vartheta_a)}{\cos \vartheta_a}, \quad \delta_j^{(2)} = 1 \quad (2 \leq j \leq n-2).$$
3.3 The $\mathfrak{e}_6^{(1)}$ theory

The Dynkin diagram for $\mathfrak{e}_6^{(1)}$ is shown in Figure 3.3a.

![Affine Dynkin diagram for $\mathfrak{e}_6^{(1)}$.](image)

The eigenvalues of the matrix $NC$ are given by

$$\lambda_1, \lambda_6 = 3 - \sqrt{3}, \quad \lambda_2 = 2(3 - \sqrt{3})$$

and

$$\lambda_3, \lambda_5 = 3 + \sqrt{3}, \quad \lambda_4 = 2(3 + \sqrt{3}).$$

As in the other simply-laced cases $\eta_j = 1$ and $p_j = n_j$ $\forall j$ giving the equations of motion

$$(D_i^2 - D_x^2)\tau_a \cdot \tau_a = 2m^2(\tau_b - \tau_a^2)$$

$$(D_i^2 - D_x^2)\tau_b \cdot \tau_b = 4m^2(\tau_a \tau_4 - \tau_b^2)$$

$$(D_i^2 - D_x^2)\tau_4 \cdot \tau_4 = 6m^2(\tau_2 \tau_3 \tau_5 - \tau_4^2)$$

where $(a,b)=(0,2),(1,3)$ and $(6,5)$. A summary of the $\delta$-values for the six single soliton solutions is given in Table 3.3.

With reference to Table 3.3, the vector $\delta^{(1)} = (\delta_0^{(1)}, \delta_1^{(1)}, ..., \delta_6^{(1)})^T$ is an eigenvector of the matrix $K$, which is conjugate to $NC$. The terms $\delta_a^{(b)}$ $(b \geq 2)$ are coefficients of $e^{b\Phi}$ in $\tau_a$. As usual, the $\delta$-values corresponding to $\lambda = 0$ have not been included as they lead to a trivial solution.
Table 3.3: δ-values for $\varepsilon_{6}^{(1)}$

| $\lambda$ | $3 - \sqrt{3}$ | $3 - \sqrt{3}$ | $2(3 + \sqrt{3})$ | $2(3 - \sqrt{3})$ | $3 + \sqrt{3}$ | $3 + \sqrt{3}$ |
|------------|----------------|----------------|-------------------|-------------------|----------------|----------------|
| $\delta_{0}^{(1)}$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\delta_{1}^{(1)}$ | $\omega$ | $\omega^2$ | 1 | 1 | $\omega$ | $\omega^2$ |
| $\delta_{2}^{(1)}$ | $-(\lambda - 2)$ | $-(\lambda - 2)$ | $-(\lambda - 2)$ | $-(\lambda - 2)$ | $-(\lambda - 2)$ | $-(\lambda - 2)$ |
| $\delta_{3}^{(1)}$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\delta_{4}^{(1)}$ | $-\omega(\lambda - 2)$ | $-\omega^2(\lambda - 2)$ | $-(\lambda - 2)$ | $-(\lambda - 2)$ | $-\omega(\lambda - 2)$ | $-\omega^2(\lambda - 2)$ |
| $\delta_{5}^{(1)}$ | $\omega^2$ | $\omega$ | 1 | 1 | $\omega^2$ | $\omega$ |
| $\delta_{6}^{(1)}$ | 0 | 0 | $3(\lambda - 3)$ | $3(\lambda - 3)$ | 0 | 0 |
| $\delta_{7}^{(1)}$ | 0 | 0 | $3(\lambda - 3)$ | $3(\lambda - 3)$ | 0 | 0 |
| $\delta_{0}^{(2)}$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\delta_{1}^{(2)}$ | $-\omega^2(\lambda - 2)$ | $-\omega(\lambda - 2)$ | $-(\lambda - 2)$ | $-(\lambda - 2)$ | $-\omega^2(\lambda - 2)$ | $-\omega(\lambda - 2)$ |
| $\delta_{2}^{(2)}$ | $\omega$ | $\omega^2$ | 1 | 1 | $\omega$ | $\omega^2$ |
| $\delta_{3}^{(2)}$ | $\omega^2$ | $\omega$ | 1 | 1 | $\omega^2$ | $\omega$ |

3.4 The $\varepsilon_{7}^{(1)}$ theory

\[ \begin{array}{c}
\circ \alpha_2 \\
\circ \alpha_0 \quad \circ \alpha_1 \quad \circ \alpha_3 \quad \circ \alpha_4 \quad \circ \alpha_5 \quad \circ \alpha_6 \quad \circ \alpha_7
\end{array} \]

Figure 3.4a: Affine Dynkin diagram for $\varepsilon_{7}^{(1)}$.

For the $\varepsilon_{7}^{(1)}$ theory, whose Dynkin diagram is shown in Figure 3.4a, the non-zero eigenvalues of the matrix $N\alpha'$ are

\[
\lambda_1 = 8\sqrt{3} \sin \left( \frac{\pi}{18} \right) \sin \left( \frac{2\pi}{9} \right) \\
\lambda_2 = 8 \sin^2 \left( \frac{2\pi}{9} \right), \quad \lambda_3 = 8 \sin^2 \left( \frac{\pi}{3} \right) \\
\lambda_4 = 8\sqrt{3} \sin \left( \frac{7\pi}{18} \right) \sin \left( \frac{4\pi}{9} \right), \quad \lambda_5 = 8 \sin^2 \left( \frac{4\pi}{9} \right) \\
\lambda_6 = 8\sqrt{3} \sin \left( \frac{5\pi}{18} \right) \sin \left( \frac{\pi}{9} \right), \quad \lambda_7 = 8 \sin^2 \left( \frac{\pi}{9} \right).
\]

A summary of the δ-values for the seven soliton solutions for $\varepsilon_{7}^{(1)}$ is given in Table 3.4.
Table 3.4: $\delta$-values for $e_7^{(1)}$

| $\lambda$ | $\lambda_3$ | $\lambda_2, \lambda_5, \lambda_7$ | $\lambda_1, \lambda_4, \lambda_6$ |
|-----------|-------------|-------------------------------|-------------------------------|
| $\delta_0^{(1)}$ | 1           | 1                             | 1                             |
| $\delta_1^{(1)}$ | $-4$       | $-(\lambda - 2)$              | $-(\lambda - 2)$              |
| $\delta_2^{(2)}$ | 1           | 1                             | 1                             |
| $\delta_3^{(1)}$ | $-4$       | 0                             | $2(\lambda - 2)$              |
| $\delta_4^{(2)}$ | 1           | $-1$                          | 1                             |
| $\delta_5^{(1)}$ | 3           | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ |
| $\delta_6^{(2)}$ | 3           | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ |
| $\delta_7^{(3)}$ | 1           | 1                             | 1                             |
| $\delta_8^{(1)}$ | 4           | 0                             | $-(\lambda^2 - 6\lambda + 8)$  |
| $\delta_9^{(2)}$ | 6           | $2(\lambda - 1)$              | $2(2\lambda^2 - 9\lambda + 9)$ |
| $\delta_{10}^{(3)}$ | 4          | 0                             | $-(\lambda^2 - 6\lambda + 8)$  |
| $\delta_{11}^{(4)}$ | 1           | 1                             | 1                             |
| $\delta_{12}^{(1)}$ | 3           | $-\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ |
| $\delta_{13}^{(2)}$ | 3           | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ |
| $\delta_{14}^{(3)}$ | 1           | $-1$                          | 1                             |
| $\delta_{15}^{(1)}$ | $-4$       | $(\lambda - 2)$              | $-(\lambda - 2)$              |
| $\delta_{16}^{(2)}$ | 1           | 1                             | 1                             |
| $\delta_{17}^{(1)}$ | 1           | $-1$                          | 1                             |

In general, when solving the equations of motion, the $\delta$’s are found to be polynomials in the eigenvalues. However, for $e_7^{(1)}$ the calculation can be simplified by using the characteristic polynomial of $NC$, which for $\lambda = \lambda_2, \lambda_5, \lambda_7$, gives

$$\lambda^3 - 12\lambda^2 + 36\lambda - 24 = 0$$

and for $\lambda = \lambda_1, \lambda_4, \lambda_6$, gives

$$\lambda^3 - 18\lambda^2 + 72\lambda - 72 = 0.$$ 

Therefore, the $\delta$’s can be written as quadratic or linear polynomials in $\lambda$. 

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3.5 The $e_8^{(1)}$ theory

The Dynkin diagram for $e_8^{(1)}$ is shown in Figure 3.5a.

![Dynkin Diagram](image)

Figure 3.5a: Affine Dynkin diagram for $e_8^{(1)}$.

The eigenvalues of the matrix $NC$ are

$$
\begin{align*}
\lambda_1 &= 32\sqrt{3} \sin\left(\frac{\pi}{30}\right) \sin\left(\frac{\pi}{5}\right) \cos^2\left(\frac{\pi}{5}\right) \\
\lambda_2 &= 32\sqrt{3} \sin\left(\frac{\pi}{30}\right) \sin\left(\frac{\pi}{5}\right) \cos^2\left(\frac{\pi}{5}\right) \cos^2\left(\frac{7\pi}{30}\right) \\
\lambda_3 &= 8\sqrt{3} \sin\left(\frac{7\pi}{30}\right) \sin\left(\frac{2\pi}{5}\right) \\
\lambda_4 &= 512\sqrt{3} \sin\left(\frac{\pi}{30}\right) \sin\left(\frac{\pi}{5}\right) \cos^2\left(\frac{2\pi}{15}\right) \cos^4\left(\frac{\pi}{5}\right) \\
\lambda_5 &= 8\sqrt{3} \sin\left(\frac{13\pi}{30}\right) \sin\left(\frac{2\pi}{5}\right) \\
\lambda_6 &= 8\sqrt{3} \sin\left(\frac{11\pi}{30}\right) \sin\left(\frac{\pi}{5}\right) \\
\lambda_7 &= 32\sqrt{3} \sin\left(\frac{\pi}{30}\right) \sin\left(\frac{\pi}{5}\right) \cos^2\left(\frac{\pi}{30}\right) \\
\lambda_8 &= 8\sqrt{3} \sin\left(\frac{\pi}{30}\right) \sin\left(\frac{\pi}{5}\right).
\end{align*}
$$

A summary of the $\delta$-values for the eight soliton solutions for $e_8^{(1)}$ is given in Table 3.5. As in the previous case the characteristic polynomial of $NC$ can be used to simplify the expressions for the $\delta$-values. For $\lambda = \lambda_1, \lambda_2, \lambda_4, \lambda_7$,

$$\lambda^4 - 30\lambda^3 + 240\lambda^2 - 720\lambda + 720 = 0,$$

and for $\lambda = \lambda_3, \lambda_5, \lambda_6, \lambda_8$,

$$\lambda^4 - 30\lambda^3 + 300\lambda^2 - 1080\lambda + 720 = 0.$$

This factorisation of the characteristic polynomial was noted in [9].
Table 3.5: $\delta$-values for $\epsilon_8^{(1)}$

| $\lambda$ | $\lambda_1, \lambda_2, \lambda_4, \lambda_7$ | $\lambda_3, \lambda_5, \lambda_6, \lambda_8$ |
|-----------|-----------------|-----------------|
| $\delta_0^{(1)}$ | 1 | 1 |
| $\delta_1^{(1)}$ | $-\frac{1}{6}(\lambda^3 - 24\lambda^2 + 132\lambda - 192)$ | $\frac{1}{3}(\lambda^3 - 21\lambda^2 + 114\lambda - 84)$ |
| $\delta_1^{(2)}$ | 1 | 1 |
| $\delta_2^{(1)}$ | $\frac{1}{4}(\lambda^3 - 18\lambda^2 + 84\lambda - 108)$ | $\frac{1}{4}(\lambda^3 - 24\lambda^2 + 144\lambda - 108)$ |
| $\delta_2^{(2)}$ | $\frac{1}{4}(\lambda^3 - 18\lambda^2 + 84\lambda - 108)$ | $\frac{1}{4}(\lambda^3 - 24\lambda^2 + 144\lambda - 108)$ |
| $\delta_3^{(1)}$ | $\frac{1}{6}(\lambda^3 - 6\lambda^2 + 24)$ | $-\frac{1}{6}(5\lambda^3 - 102\lambda^2 + 540\lambda - 384)$ |
| $\delta_3^{(2)}$ | $\frac{2}{3}(5\lambda^3 - 60\lambda^2 + 225\lambda - 261)$ | $-\frac{2}{3}(5\lambda^3 - 24\lambda^2 + 135\lambda - 99)$ |
| $\delta_3^{(3)}$ | $\frac{1}{6}(\lambda^3 - 6\lambda^2 + 24)$ | $-\frac{1}{6}(5\lambda^3 - 102\lambda^2 + 540\lambda - 384)$ |
| $\delta_4^{(1)}$ | 1 | 1 |
| $\delta_4^{(2)}$ | $-\frac{1}{2}(\lambda - 2)(\lambda^2 - 6\lambda + 6)$ | $\lambda^2 - 9\lambda + 6$ |
| $\delta_4^{(3)}$ | $64\lambda^3 - 668\lambda^2 + 2214\lambda - 2325$ | $3\lambda^3 - 50\lambda^2 + 234\lambda - 165$ |
| $\delta_4^{(4)}$ | $-(303\lambda^3 - 3186\lambda^2 + 10614\lambda - 11180)$ | $-2(3\lambda^3 - 54\lambda^2 + 267\lambda - 190)$ |
| $\delta_4^{(5)}$ | $64\lambda^3 - 668\lambda^2 + 2214\lambda - 2325$ | $3\lambda^3 - 50\lambda^2 + 234\lambda - 165$ |
| $\delta_4^{(6)}$ | $-\frac{1}{2}(\lambda - 2)(\lambda^2 - 6\lambda + 6)$ | $\lambda^2 - 9\lambda + 6$ |
| $\delta_5^{(1)}$ | 1 | 1 |
| $\delta_5^{(2)}$ | $\frac{5}{12}(\lambda^3 - 12\lambda^2 + 48\lambda - 60)$ | $\frac{5}{12}(\lambda^3 - 18\lambda^2 + 84\lambda - 60)$ |
| $\delta_5^{(3)}$ | $\frac{5}{4}(11\lambda^3 - 116\lambda^2 + 384\lambda - 400)$ | $\frac{5}{4}(\lambda - 8)(3\lambda^2 - 26\lambda + 20)$ |
| $\delta_5^{(4)}$ | $\frac{5}{4}(11\lambda^3 - 116\lambda^2 + 384\lambda - 400)$ | $\frac{5}{4}(\lambda - 8)(3\lambda^2 - 26\lambda + 20)$ |
| $\delta_5^{(5)}$ | $\frac{5}{12}(\lambda^3 - 12\lambda^2 + 48\lambda - 60)$ | $\frac{5}{12}(\lambda^3 - 18\lambda^2 + 84\lambda - 60)$ |
| $\delta_6^{(1)}$ | 1 | 1 |
| $\delta_6^{(2)}$ | $-\frac{1}{3}(\lambda^3 - 12\lambda^2 + 36\lambda - 24)$ | $-\frac{1}{3}(\lambda^3 - 12\lambda^2 + 36\lambda - 24)$ |
| $\delta_6^{(3)}$ | $\frac{1}{6}(7\lambda^3 - 78\lambda^2 + 288\lambda - 324)$ | $\frac{1}{6}(7\lambda^3 - 108\lambda^2 + 468\lambda - 324)$ |
| $\delta_6^{(4)}$ | $-\frac{1}{3}(\lambda^3 - 12\lambda^2 + 36\lambda - 24)$ | $-\frac{1}{3}(\lambda^3 - 12\lambda^2 + 36\lambda - 24)$ |
| $\delta_7^{(1)}$ | 1 | 1 |
| $\delta_7^{(2)}$ | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ |
| $\delta_7^{(3)}$ | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ | $\frac{1}{2}(\lambda^2 - 6\lambda + 6)$ |
| $\delta_8^{(1)}$ | 1 | 1 |
| $\delta_8^{(2)}$ | $-(\lambda - 2)$ | $-(\lambda - 2)$ |
4 Folding and the non-simply-laced algebras

With the construction of the soliton solutions in the previous section, enough information has been gathered to deduce solutions to the non-simply laced algebras.

4.1 From $a_n^{(1)}$ to $c_n^{(1)}$.

The Dynkin diagram for $c_n^{(1)}$ is shown in Figure 4.1a.

![Affine Dynkin diagram for $c_n^{(1)}$](image)

Figure 4.1a: Affine Dynkin diagram for $c_n^{(1)}$.

The set of roots $\{\alpha'_i\}$ is expressible in terms of the roots $\{\alpha_i\}$ of $a_{2n-1}^{(1)}$ via

$$\alpha'_0 = \alpha_0, \quad \alpha'_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i}) \quad (1 \leq i \leq n-1), \quad \alpha'_n = \alpha_n. \quad (4.1.1)$$

This is the origin of the idea of ‘folding’, discussed in [11]: the diagram for $a_{2n-1}^{(1)}$ has been ‘folded’ using its symmetry under the reflection $0 \to 0, i \to 2n - i$ of the nodes. Generally, suppose the Dynkin diagram of a simply-laced algebra has some symmetry. The equations of motion then also have this symmetry, so that solitons with this symmetry as an initial condition preserve it as they evolve. Thus a solution for $a_{2n-1}^{(1)}$ can be written in terms of $\{\alpha'_i\}$ if $\tau_i = \tau_{2n-i}$. As will be seen these are solutions for $c_n^{(1)}$.

For $c_n^{(1)}$ the $\eta'_j$'s (all quantities relating to $c_n^{(1)}$ will be denoted by a prime) are given by

$$\eta'_0 = \eta'_n = 1 \quad \text{and} \quad \eta'_i = 2 \quad (i \neq 0, n)$$

so that for the single soliton solution $p'_j = 1 \forall j$. The equations of motion are then

$$(D^2_t - D^2_x)\tau'_0 \cdot \tau'_0 = 2m^2(\tau'^2_1 - \tau'^2_0)$$

$$(D^2_t - D^2_x)\tau'_j \cdot \tau'_j = 2m^2(\tau'_{j-1}\tau'_{j+1} - \tau'^2_j) \quad (1 \leq j \leq n-1)$$

$$(D^2_t - D^2_x)\tau'_n \cdot \tau'_n = 2m^2(\tau'^2_{n-1} - \tau'^2_n).$$

This set of equations is that for $a_{2n-1}^{(1)}$ with

$$\tau'_0 = \tau_0, \quad \tau'_n = \tau_n, \quad \text{and} \quad \tau'_j = \tau_j = \tau_{2n-j} \quad (1 \leq j \leq n-1) \quad (4.1.2)$$
and so the solutions to $c_n^{(1)}$ are those for $a_{2n-1}^{(1)}$ with the conditions (4.1.2) imposed. This leads to the requirement that

$$\omega_a^j = \omega_a^{2n-j} \quad \text{where} \quad 1 \leq j \leq n-1.$$ 

The only $a$ satisfying this equation is $a = n$, giving $\omega_a = -1$, i.e. the only non-trivial soliton of $a_{2n-1}^{(1)}$ surviving the folding procedure is that corresponding to the $n$-th spot on the Dynkin diagram (the trivial solution corresponding to the zeroth spot also survives). Therefore,

$$\tau'_j = 1 + (-1)^j e^\Phi$$

giving the soliton solution to $c_n^{(1)}$ as

$$\phi' = -\frac{1}{i\gamma} \left( 2 \sum_{j=1}^{n-1} \alpha_j \ln \left( \frac{1 + (-1)^j e^\Phi}{1 + e^\Phi} \right) + \alpha_n \ln \left( \frac{1 + (-1)^n e^\Phi}{1 + e^\Phi} \right) \right).$$

In fact, with the identification of roots in (4.1.1), $\phi' = \phi(n)$ where $\phi(n)$ is a soliton solution for $a_{2n-1}^{(1)}$. This turns out to be a common feature of solitons to the non-simply-laced theories - they are equal to a soliton of the corresponding simply-laced algebra.

4.2 From $d_n^{(1)}$ to $b_n^{(1)}$, $a_{2n-1}^{(2)}$, $a_{n+1}^{(2)}$, $a_{2n}^{(2)}$, and $g_2^{(1)}$

Turning first to the $b_n^{(1)}$ theory, which has Dynkin diagram shown in Figure 4.2a, the set of roots $\{\alpha'_i\}$ are expressible in terms of the roots $\{\alpha_i\}$ of $d_{n+1}^{(1)}$ via

$$\alpha'_i = \alpha_i \quad (0 \leq i \leq n-1), \quad \alpha'_n = \frac{1}{2}(\alpha_n + \alpha_{n+1}).$$

![Figure 4.2a: Affine Dynkin diagram for $b_n^{(1)}$.](image-url)
With \( \tau_i' = \tau_i, \tau_n' = \tau_n = \tau_{n+1} \), the equations of motion for \( d_{n+1}^{(1)} \) reduce to those for \( b_n^{(1)} \). The number of solutions is found to be \( n - 1 \) with eigenvalues of \( NC' \) equal to

\[
\lambda_a = 8 \sin^2 \left( \frac{a\pi}{2n} \right) \quad (1 \leq a \leq n - 1).
\]

In this case all the solutions of \( d_{n+1}^{(1)} \) survive except those corresponding to the Dynkin spots \( n \) and \( n + 1 \).

Solutions to theories based on twisted algebras such as \( a_{2n-1}^{(2)} \), shown in Figure 4.2b, need to be handled slightly differently. The roots of \( a_{2n-1}^{(2)} \) are obtainable from those of \( d_{2n}^{(1)} \). However, if we apply the previous procedure and identify \( \tau' \)'s in the equations of motion for \( d_{2n}^{(1)} \), they are found to be slightly different from those of \( a_{2n-1}^{(2)} \), in that the coefficient of \( m^2 \) differs. This is because the twisted algebras are obtained from symmetries of the simply-laced diagrams which involve the extended root, which is thus rescaled by folding.

\[
\begin{align*}
\alpha_0' &= 1/2(\alpha_0 + \alpha_{2n-1}), \\
\alpha_1' &= 1/2(\alpha_1 + \alpha_{2n}), \\
\alpha_n' &= \alpha_n \\
\alpha_i' &= 1/2(\alpha_i + \alpha_{2n-i}) \quad (2 \leq i \leq n - 1).
\end{align*}
\]

It is necessary, therefore, to consider the equations of motion of \( d_{2n}^{(1)} \) with the following identification of \( \tau \)-functions:

\[
\tau_0' = \tau_0 = \tau_{2n-1}, \quad \tau_1' = \tau_1 = \tau_{2n}, \quad \tau_n' = \tau_n \quad \text{and} \quad \tau_i' = \tau_i = \tau_{2n-i} \quad (2 \leq i \leq n - 1)
\]

As a result, solutions of \( a_{2n-1}^{(2)} \) are those of \( d_{2n}^{(1)} \) with eigenvalue \( \lambda^{(sl)} \), satisfying (4.2.1) and

\[
\sigma^2(1 - v^2) = \frac{1}{2}m^2\lambda^{(sl)} = m^2\lambda^{(tw)}.
\]
With this root convention \( \lambda^{(sl)} = 2\lambda^{(tw)} \), \( \lambda^{(tw)} \) being an eigenvalue of the extended Cartan matrix for \( a^{(2)}_{2n-1} \).

As a result, the case \( a^{(2)}_{2n-1} \) has \( n \) solutions corresponding to

\[
\lambda^{(tw)}_a = 4 \sin^2 \left( \frac{a\pi}{2n-1} \right) \quad (1 \leq a \leq n-1) \quad \text{and} \quad \lambda^{(tw)}_n = 1.
\]

The solitons of \( d^{(1)}_{2n} \) lost through folding are those corresponding to the \( k \)-th spot \((1 \leq k \leq 2n - 1, k \text{ odd})\) and one of the solitons corresponding to the \((2n-1)\)-th and \(2n\)-th spots.

This procedure generalises to the other twisted algebras.

Solitons for the \( d^{(2)}_{n+1} \) and \( a^{(2)}_{2n} \) theories are obtained from the \( d^{(1)}_{n+2} \) and \( d^{(1)}_{2n+2} \) theories respectively, whereas \( g^{(1)}_2 \) is obtained from \( d^{(1)}_4 \). The number of solitons in each case is \( n, n, \) and 1, respectively.

### 4.3 The remaining theories: \( f^{(1)}_4 \), \( d^{(3)}_4 \) and \( e^{(2)}_6 \)

In a similar manner to the previous two subsections, solitons can be obtained for \( f^{(1)}_4 \) and \( d^{(3)}_4 \) from \( e^{(1)}_6 \), and \( e^{(2)}_6 \) from \( e^{(1)}_7 \). The number of solitons in each case is two, two and three, respectively.

### 5 Soliton masses and topological charge

#### 5.1 Soliton mass

In [1] it was shown that the masses of the \( a^{(1)}_n \) solitons are given by

\[
M_a = \frac{2mh}{\beta^2} \sqrt{\lambda_a}, \quad (4.1)
\]

\( M_a \) being the mass of the soliton corresponding to eigenvalue \( \lambda_a \), and \( h \) the Coxeter number defined by

\[
h = \sum_{j=0}^{n} n_j.
\]

Since the masses of the fundamental Toda particles equal \( \sqrt{\lambda} \), the ratios of the soliton masses are equal to the ratios of the fundamental particles.
By considering the soliton momentum,

\[ M\tilde{\gamma}(v)v = -\int_{-\infty}^{\infty} dx \dot{\phi} \cdot \phi' \]

where \( \tilde{\gamma}(v) = (1 - v^2)^{-\frac{1}{2}} \) it is straightforward to confirm (case-by-case) that (4.1) holds for the solitons of the remaining simply-laced algebras.

Consider now the solitons belonging to the other algebras. For the untwisted algebras, as each soliton is also a solution of one of the simply-laced cases, equation (4.1) holds though with the Coxeter number equal to that of the simply-laced algebra (it is easily shown that the Coxeter number of an untwisted non-simply-laced algebra is equal to the Coxeter number of the algebra from which it is folded). Hence, (4.1) holds with the mass ratios being those of the fundamental particles. By relating a solution of a twisted algebra to a solution of the corresponding simply-laced algebra, the masses of the twisted solitons are readily seen to satisfy (4.1) also.

5.2 Topological charges

The topological charge of a soliton is defined as,

\[ t = \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \phi = \frac{\gamma}{2\pi} (\lim_{x \to \infty} - \lim_{x \to -\infty}) \phi(x,t). \]

Previous work [1] has shown that for \( a^{(1)}_n \) the topological charge of the soliton \( \phi_{(a)} \) is found to be a weight of the \( a \)-th fundamental representation. Different choices of \( Im \xi \) give rise to different topological charges. For the representations associated with the roots \( \alpha_1 \) and \( \alpha_n \) all weights occur as topological charges whereas for the other representations a lesser number are found. We shall use Dynkin labelling for representations, based on the diagrams in figures 3.1a and 3.2a.

As an example consider the case \( a^{(1)}_3 \). There are found to be three non-trivial solutions

\[ \phi_{(1)} = -\frac{1}{i\gamma} \sum_{k=1}^{3} \alpha_j \ln \left( \frac{1 + i\bar{\gamma} e^{\Phi}}{1 + e^{\Phi}} \right) \]

\[ \phi_{(2)} = -\frac{1}{i\gamma} \sum_{k=1}^{3} \alpha_j \ln \left( \frac{1 + (-1)i\bar{\gamma} e^{\Phi}}{1 + e^{\Phi}} \right) \]
\[ \phi_{(3)} = -\frac{1}{i\gamma} \sum_{k=1}^{3} \alpha_j \ln \left( \frac{1 + (-i)^j e^{\Phi}}{1 + e^{\Phi}} \right). \]

The solitons \( \phi_{(1)} \) and \( \phi_{(3)} \) have topological charges filling the first and third fundamental representations respectively, whereas the topological charges of \( \phi_{(2)} \) occur as only some of the weights of the second fundamental representation. It may be thought that all the single static solitons have not yet been found and that other solutions should exist having topological charges filling the rest of the second fundamental representation. This does not appear to be true, since it is possible to construct static double solitons which have some of the other weights as topological charges. The static double soliton for \( a_3^{(1)} \) made up of \( \phi_{(1)} \) and \( \phi_{(3)} \) has been studied and is found to have topological charges filling the adjoint representation \((1,0,1)\). Some consideration has also been given to the \( \phi_{(1)}-\phi_{(2)} \) static double soliton which has topological charges, not previously found, occurring as weights of the second fundamental representation. However, a study of this case is not yet complete.

Similar consideration has been given to \( d_4^{(1)} \). The solutions of section 3.2 corresponding to \( \lambda = 2 \) are found to have topological charges lying in the first, third and fourth fundamental representations. The solution corresponding to \( \lambda = 6 \) has topological charge lying in the second fundamental representation.

The static double soliton made up of the pair of solutions associated with the first and third fundamental representations are found to have topological charges in the representation \((1,0,1,0)\), as well as filling up the remainder of the fourth fundamental representation. Similar results hold for the static double solitons composed of the other two pairs of solitons having \( \lambda = 2 \).

It is clear that this aspect of the solitons requires a great deal more study.
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