COVARIANT DIFFERENTIAL AND INTEGRAL CALCULI
FOR LATTICE (l,q)-DEFORMED FIELDS

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Abstract
Using the Hecke $\hat{R}$-matrix, we give a definition of the lattice $(l,q)$-deformed $n$-component boson and Grassmann fields. Here $l$ is a deformation parameter for the commutation relations of "values" of these fields in two arbitrary lattice sites and $q$ is a deformation parameter for $n$-component $q$-boson or $q$-Grassmann variable. In framework of the Wess-Zumino approach to the non-commutative differential calculus the commutation relations between differentials and derivatives of these fields are determined. The $SL_q(n,C)$-invariant generalization of the Berezin integration for the lattice $n$-component $(l,q)$-Grassmann field is suggested. We show that the Gaussian functional integral for this field is expressed through the $(l,q)$-deformed counterpart of the Pfaffian.

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1. Introduction

During the two last decades the interplay between approaches to investigation of lattice field theories (as a method of regularization of quantum field theory) and critical phenomena in the lattice spin systems influenced considerably on understanding of the non-perturbative phenomena in the quantum field theory. On the other hand the possibility of the partition function representation of the lattice spin systems through partition function of the lattice field theories allows to describe the critical behaviour of these systems by means of a quantum field theory. So, for example, in the critical point partition function of two-dimensional Ising model can be represented as partition function of the lattice massless majorana fermion field theory [1-3]. In the continuum limit the masslessness of the majorana fermion is a consequence of conformal invariance of the critical fluctuations. As it was shown by Belavin, Polyakov and Zamolodchikov in [4], this invariance allows to assign to each universality class of the critical behaviour of exactly solvable lattice systems the corresponding two-dimensional quantum conformal field theory. This result is connected with the fact that, in critical points, such systems, besides the invariance with respect to the global conformal transformations (SL(2,C) group) possess larger symmetry, that is, the space of eigenvectors of the transfer matrix has an invariance with respect to the infinite-dimensional Virasoro algebra.

However, as soon as the correlation radius of critical fluctuations is finite nearby the critical point, the conformal invariance is broken. Nevertheless, as it was shown by Zamolodchikov in [5], an exactly solvable two-dimensional quantum field theory can still possess infinitely many integrals of motion, while being nearby the fixed point (this point corresponds to a quantum conformal field theory). The infinite number integrals of motions was found [6] near the critical point in the eight-vertex model in the point where this model corresponds to the two noninteracting Ising sub-lattices. Recently Jimbo and Miwa with collaborators showed in the series of papers [7,8] that the eigenvectors of the corner transfer-matrix for the six-vertex model form the irreducible representation space of $U_q(\hat{sl}_2)$ at $-1 < q < 0$. Moreover, in [9,10] was shown, that the eigenvectors space of the corner transfer-matrix for the RSOS model and, in particular, for the two-dimensional Ising model [10] can also possess this symmetry.
These results allow one to assume that the partition function of the two-dimensional exactly solvable lattice model can be expressed through the partition function of some $SL_q(2, C)$-invariant lattice $q$-deformed field theory, Fock space of which being constructed using the representation theory of $U_q(sl_2)$.

The attempt to realize this idea was suggested in [3] where the representation of the partition function of the two-dimensional Ising model in the form of the $SL_q(2, R)$-invariant functional integral over the lattice real $q$-fermion field ($q = -1$) was found. The following form of commutation relations for the lattice $q$-fermion field was proposed in [3]:

$$\xi^\alpha_i \xi^\beta_j = -q \hat{R}^{\alpha\beta}_{\gamma\rho} \hat{P}^{kl}_{ij} \xi^\gamma_k \xi^\rho_l$$

where $\hat{P}^{kl}_{ij} = \delta^k_i \delta^l_j + \delta^k_j \delta^l_i$ is the permutation matrix and the values of a latin indices goes over all values of variable $r$ which numerates lattice sites ($r = r_1, r_2, r_3, ..., r_N$). These commutation relations imply the independence from relative arrangement of sites and, therefore, essentially restrict the domain of values of $q$. In [3] was shown that (1.1) requires $q = \pm 1$.

In this paper we suggest a definition of lattice $(l, q)$-deformed boson and Grassmann fields for arbitrary $q$ and $l$ ($q^2 \neq -1$ in the last case). Here $l$ is a deformation parameter for the commutation relations of "values" of these fields in two arbitrary lattice sites and $q$ is a deformation parameter for $n$-component $q$-boson or $q$-Grassmann variable.

This paper is organized as follows. In section 2 the commutation relations for lattice $(l, q)$-deformed fields are defined. In section 3 we determine differential calculus for these fields. In section 4 we define the $SL_q(2, C)$-invariant extention of Berezin integration for the lattice $(l, q)$-Grassmann field. We show that Gaussian functional integral for the lattice $(l, q)$-Grassmann field is expressed in terms of $(l, q)$-deformed counterpart of the Pfaffian.

2. Lattice $(l, q)$-deformed fields

In this section we define the lattice $(l, q)$-deformed boson and Grassmann fields. Let us briefly recall the known facts about quantum hyperplanes $A_q^{n|0}$ and $A_q^{0|n}$ [12, 13], which for simplicity we will denote as quantum vector spaces $\mathcal{V}^{(n)}$ and $\Xi^{(n)}$ correspondingly. Define coordinate functions for $\mathcal{V}^{(n)}$ as $v^\alpha$ and correspondingly $\xi^\alpha$ for $\Xi^{(n)}$ ($\alpha = 1, 2, ..., n$) . These coordinate functions satisfy the commutation relations which can write by means of the linear map:

$$\hat{R} : \mathcal{V}^{(n)} \otimes \mathcal{V}^{(n)} \longrightarrow \mathcal{V}^{(n)} \otimes \mathcal{V}^{(n)} , \quad \hat{R}(v^\alpha \otimes v^\beta) \equiv \hat{R}^{\alpha\beta}_{\gamma\rho} v^\gamma \otimes v^\rho = q v^\alpha \otimes v^\beta , \quad (2.1)$$
\[ \hat{R} : \Xi^{(n)} \otimes \Xi^{(n)} \rightarrow \Xi^{(n)} \otimes \Xi^{(n)}, \quad \hat{R}(\xi^\alpha \otimes \xi^\beta) \equiv \hat{R}^{\alpha\beta}_{\gamma\rho} \xi^\gamma \otimes \xi^\rho = -\frac{1}{q} \xi^\alpha \otimes \xi^\beta, \]  

(2.2)

where \( \otimes \) denotes the tensor product of the quantum spaces, deformation parameter \( q \) in general case is a complex number and the linear map \( \hat{R} \) is determined by the symmetric \( \hat{R} \)-matrix of size \( n^2 \times n^2 \) for the quantum matrix group \( SL_q(n,C) \) (q-deformed \( SL(n,C) \)) [11-13]. The explicit form of \( \hat{R} \) is given in [13], where we see that

\[ \hat{R}^{\alpha\beta}_{\gamma\rho} = \hat{R}^{\gamma\rho}_{\alpha\beta}, \quad \hat{R}^{\alpha\beta}_{\gamma\rho} = \delta^\alpha_\rho \delta^\beta_\gamma (1 + (q - 1)\delta^\alpha_\beta) + \delta^\alpha_\gamma \delta^\beta_\rho (q - q^{-1})\Theta^{\alpha\beta}, \]  

(2.3a)

\[ \Theta^{\alpha\beta} = \begin{cases} 
1, & \text{if } \alpha < \beta \\
0, & \text{if } \alpha \geq \beta 
\end{cases}. \]  

(2.3b)

This matrix has eigenvalues \( q \) and \( -q^{-1} \) and satisfies the quantum Yang-Baxter equation [11-13]

\[ \hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}, \]  

(2.4a)

and the Hecke equation

\[ \hat{R}^2 - (q - q^{-1})\hat{R} - \hat{I} = (\hat{I} + q\hat{R})(\hat{I} - \frac{1}{q}\hat{R}) = 0, \]  

(2.4b)

where \( \hat{I} = I^{\alpha\beta}_{\gamma\rho} = \delta^\alpha_\gamma \delta^\beta_\rho \) is unit matrix. For the \( \hat{R} \)-matrix we can write the following representation

\[ \hat{R} = q \hat{S}_q - q^{-1} \hat{A}_q, \]  

where matrices

\[ \hat{A}_q = \frac{q^2}{1 + q^2} (\hat{I} - \frac{1}{q}\hat{R}), \]  

(2.5a)

\[ \hat{S}_q = \frac{1}{1 + q^2} (\hat{I} + q\hat{R}) \]  

(2.5b)

are orthonormal projections (relative to the eigenvalues \( -q^{-1} \) and \( q \) respectively)

\[ \hat{A}_q \cdot \hat{S}_q = \hat{S}_q \cdot \hat{A}_q = 0, \quad \hat{A}_q^2 = \hat{A}_q, \quad \hat{S}_q^2 = \hat{S}_q. \]  

These projections are the quantum analogs of the classical \( (q = 1) \) antisymmetrizer and symmetrizer for tensors with two indeces.

Using these matrices, we can write commutation relations (2.1), (2.2) in the form

\[ \hat{A}_q^{\alpha\beta} v^\gamma \otimes v^\rho = 0, \]  

(2.6)
\[ \hat{S}^{\alpha\beta}_{\gamma\rho} \xi^\gamma \otimes \xi^\rho = 0. \]  

(2.7a)

Let us write last relation in components \((q^2 \neq -1)\)

\[ \xi^\alpha \otimes \xi^\beta = -q \xi^\beta \otimes \xi^\alpha \quad \text{for} \alpha > \beta, \quad (\xi^\alpha)^2 = 0. \]  

(2.7b)

Since at \(q = 1\) it gives the commutation relations for the \(n\)-component classical Grassmann variable \(z^\alpha\)

\[ (z^\alpha)^2 = 0, \quad z^\alpha z^\beta + z^\beta z^\alpha = 0, \]

we will consider \(\xi\) as an \(n\)-component \(q\)-Grassmann variable.

Commutation relations (2.6) and (2.7) are invariant with respect to transformation of \(\xi\) and \(v\) by the quantum matrix \(\hat{A} \in SL_q(n, C)\) [13]

\[ \tilde{v}^\alpha = \hat{A}^\alpha_\beta \otimes v^\beta, \quad \tilde{\xi}^\alpha = \hat{A}^\alpha_\beta \otimes \xi^\beta, \]  

(2.8)

where matrix elements \(\hat{A}^\alpha_\beta\) commute with components \(\xi^\alpha\) and \(v^\alpha\) and belong to the associative algebra of functions on the quantum group \(SL_q(n)\), which we will denote as \(\mathcal{A}\) or \(Fun_q(SL(n))\) [13]. This algebra is Hopf algebra. It means that in this algebra the following maps are defined:

a) comultiplication \(\triangleright\)

\[ \mathcal{A} \xrightarrow{\triangleright} \mathcal{A} \otimes \mathcal{A} : \quad \triangleright (A^\alpha_\beta) = A^\alpha_\gamma \otimes A^\gamma_\beta, \]

where symbol \(\otimes\) denote the tensor product of quantum space,

b) counit \(\varepsilon\)

\[ \mathcal{A} \xrightarrow{\varepsilon} \mathbb{C} : \quad \varepsilon (A^\alpha_\beta) = \delta^\alpha_\beta, \]

where \(\mathbb{C}\) is a complex numbers,

c) antipod \(i\)

\[ \mathcal{A} \xrightarrow{i} \mathcal{A} : \quad i (A^{\alpha}_\beta) = (-q)^{\alpha-\beta} \tilde{A}^{\beta}_\alpha, \]

where \(\tilde{A}^{\beta}_\alpha\) is quantum minor for matrix element \(A^{\alpha}_\beta\). For the quantum matrix \(\hat{A} \in SL_q(n, C)\) we have

\[ \tilde{A}^{\sigma}_\beta = \sum_{\sigma \in S_{n-1}} (-q)^{l(\sigma)} A^{1}_{\sigma_1} ... A^{\alpha-1}_{\sigma_{\alpha-1}} A^{\alpha+1}_{\sigma_{\alpha+1}} ... A^{n}_{\sigma_n} \]

and \(i(\hat{A}) \cdot \hat{A} = \hat{A} \cdot i(\hat{A}) = \hat{1}\) [13], where “.” denotes the matrix multiplication.

Here the sum extends over the symmetric group \(S_{n-1}\), \(l(\sigma)\) is the length (the
number of inversion) of the substitution \( \sigma = (\sigma_1, ..., \sigma_{\alpha-1}, \sigma_{\alpha+1}, ..., \sigma_n) = \sigma(1, ..., \beta - 1, \beta + 1, ..., n) \)

d) multiplication map \( m \)

\[
A \otimes A \xrightarrow{m} A : \quad m(A^\alpha_\beta \otimes A^\gamma_\rho) = A^\alpha_\beta A^\gamma_\rho,
\]

e) matrix elements \( A^\alpha_\beta \) satisfy commutation relations

\[
\hat{R}^{\alpha\beta}_{\gamma\rho} A^\gamma_\mu A^\rho_\nu = A^\alpha_\gamma A^\beta_\rho \hat{R}^{\gamma\rho}_{\mu\nu}. \tag{2.9}
\]

These maps permit to define the following operations with the quantum matrix \( \hat{A} \in SL_q(n, C) \):

a) \( \triangle(\hat{A}) = \hat{A} \hat{\otimes} \hat{A} \) defines the rule of multiplication of quantum matrices and matrix elements of \( \hat{A} \hat{\otimes} \hat{A} \) have form \( A^\alpha_\gamma \otimes A^\gamma_\rho \) (\( \hat{\otimes} \) denote the tensor product of quantum spaces together with usual matrix multiplication),

b) \( \varepsilon(\hat{A}) = \hat{1} \) defines a unit matrix in \( SL_q(n, C) \), \( \hat{1} = \delta^\alpha_\beta \),

c) \( i(\hat{A}) = \hat{A}^{-1} \) defines the inverse matrix,

d) \( m(\hat{A} \otimes \hat{A}) = \hat{A} \hat{\otimes} \hat{A} \) defines usual tensor multiplication for quantum matrices and matrix elements of \( \hat{A} \hat{\otimes} \hat{A} \) have form \( A^\alpha_\beta A^\gamma_\rho \),

e) the quantum determinant of the quantum matrix \( \hat{A} \) is determined by relation

\[
(A^\alpha_\beta_1 A^\alpha_\beta_2 A^\alpha_\beta_3 \cdots A^\alpha_\beta_n) \otimes (\xi^{\beta_1} \otimes \xi^{\beta_2} \otimes \xi^{\beta_3} \otimes \cdots \otimes \xi^{\beta_n}) = det_q \hat{A} (\xi^{\alpha_1} \otimes \xi^{\alpha_2} \otimes \xi^{\alpha_3} \otimes \cdots \otimes \xi^{\alpha_n}) \tag{2.10}
\]

and satisfies property

\[
\triangle(det_q \hat{A}) = det_q(\hat{A} \hat{\otimes} \hat{A}) = det_q(\hat{A}) \otimes det_q(\hat{A}).
\]

For \( SL_q(n) \)

\[
det_q \hat{A} = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} A^{1}_{\sigma_1} ... A^{n}_{\sigma_n} = 1. \tag{2.11}
\]

Using (2.9), it is not hard to show that \( q \)-determinant commutes with matrix elements \( A^\alpha_\beta \) [13].

Now let us consider the definition of the \((l, q)\)-deformed lattice fields. At first define the lattice \( n \)-component \((l, q)\)-Grassmann field. For this consider \( d \)-dimensional hypercubic lattice. The lattice sites are numerated by radius vector \( r = (p_1, p_2, \ldots, p_d) \), where \( \{p_i\} \) are integer numbers and a full number of sites are \( N \). For convenience the lattice constant \( a \) is fixed to be equal to unit. Let \( \psi^\alpha_{r'} \in \Psi^{(n)}_{r'} \) be the \( n \)-component \( q \)-Grassmann variable \( \psi^\alpha \) on the site
\( r' \), which we will consider as a ”value” of the \( n \)-component \((l, q)\)-Grassmann field \( \psi_r^\alpha \) on site \( r' \). The definition of the lattice \((l, q)\)-Grassmann field requires information about the commutation relations of such ”values” in two arbitrary lattice sites \( r_i = (m_1, m_2, \ldots, m_d) \) and \( r_j = (k_1, k_2, \ldots, k_d) \). In this paper we will suppose the “quasi”-one-dimensional ordering for the lattice sites: \( r_j > r_i \), if \( k_1 - m_1 > 0 \) and the others \( k_i - m_i \) are arbitrary; if \( k_1 - m_1 = 0 \), then \( r_j > r_i \), if \( k_2 - m_2 > 0 \) and so on, and \( r_1 < r_2 < r_3 < \ldots < r_N \).

Let us write these commutation relations in the form which generalize of (2.7b) in simple way

\[
\begin{align*}
\psi_{r_i}^\alpha \psi_{r_j}^\beta &= -q \psi_{r_j}^\beta \psi_{r_i}^\alpha, \quad (\psi_{r_i}^\alpha)^2 = 0, \\
\psi_{r_j}^\alpha \psi_{r_j}^\beta &= -q \psi_{r_j}^\beta \psi_{r_j}^\alpha, \quad (\psi_{r_j}^\alpha)^2 = 0, \\
\psi_{r_j}^\alpha \psi_{r_i}^\alpha &= -l \psi_{r_i}^\alpha \psi_{r_j}^\alpha, \quad \alpha > \beta, \ r_j > r_i,
\end{align*}
\]  

(2.12)

where \((q^2 \neq -1)\) and deformation parameter \(l\) is a complex number. At \( l = q = 1 \) we obtain usual lattice Grassmann field.

It is not hard to note that commutation relations (2.12) are covariant with respect to transformation

\[
\tilde{\psi}_i^\alpha = A^\alpha_\beta \otimes \psi_i^\beta, \quad \hat{A} \in SL_q(n, C),
\]

where \( SL_q(n, C) \) is the quantum matrix group determining the internal properties of the field \( \psi_r^\alpha \) and

\[
\tilde{\psi}_i^\alpha = L_i^k \otimes \psi_k^\alpha, \quad \hat{L} \in SL_1(n, C),
\]

where the space quantum matrix group \( SL_1(n, C) \) determines the commutation properties of the lattice \( q \)-Grassmann field \( \psi_r^\alpha \) in different lattice sites.

Let us denote the quantum vector space which is generated by \( nN \)-component \( q \)-Grassmann vector \( \psi_r^\alpha \) as \( \Psi \) (\( N \) is the number of sites on lattice). Commutation relations (2.12) require that we must consider quantum vector space \( \Psi \) as the Hecke sum [14,15] relative to some matrix \( \hat{Q} \) which is defined by (2.12)

\[
\Psi = \Psi_{r_1}^{(n)} \otimes \hat{Q} \Psi_{r_2}^{(n)} \otimes \hat{Q} \cdots \otimes \hat{Q} \Psi_{r_{N-1}}^{(n)} \otimes \hat{Q} \Psi_{r_N}^{(n)}. \quad (2.13)
\]

For definition of Hecke sum (2.13) and its \( \hat{R} \)-matrix let us consider linear map \( \hat{Q}: \Psi_{r_i}^{(n)} \otimes \Psi_{r_j}^{(n)} \longrightarrow \Psi_{r_j}^{(n)} \otimes \Psi_{r_i}^{(n)} \)

\[
\begin{align*}
\hat{Q}(\psi_{r_i}^\alpha \otimes \psi_{r_j}^\beta) &\equiv \hat{Q}^\alpha_\gamma \psi_{r_i}^\gamma \otimes \psi_{r_j}^\rho = -\frac{1}{l} \psi_{r_j}^\alpha \otimes \psi_{r_i}^\beta, \\
\hat{Q}^{-1}(\psi_{r_j}^\alpha \otimes \psi_{r_i}^\beta) &\equiv (\hat{Q}^{-1})^\alpha_\gamma \psi_{r_j}^\gamma \otimes \psi_{r_i}^\rho = -l \psi_{r_i}^\alpha \otimes \psi_{r_j}^\beta.
\end{align*}
\]  

(2.14)
where \( r_i < r_j \ (i < j) \). In our case \( \hat{Q} \) is the matrix of size \( n^2 \times n^2 \)

\[
\hat{Q}_{\gamma\rho}^{\alpha\beta} = \delta_\gamma^\alpha \delta_\rho^\beta \delta^{\alpha\beta} + q^{-1} \delta_\rho^\alpha \delta_\gamma^\beta \Theta^{\alpha\beta} + q \delta_\rho^\alpha \delta_\gamma^\beta \Theta^{\beta\alpha},
\]  

(2.15)

and \( \Theta^{\alpha\beta} \) is defined in (2.3b). Note that \( \hat{Q}^{-1} = \hat{Q} \). Let us show explicit form of the \( Q \)-matrix for \( n = 2 \):

\[
\hat{Q} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & q & 0 \\
0 & q^{-1} & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

By analogy with (2.2), using the matrices \( \hat{R} \) and \( \hat{Q} \), we can define the \( \hat{R} \)-matrix as linear map \( \hat{R} : \Psi \otimes \Psi \rightarrow \Psi \otimes \Psi \):

\[
\hat{R}(\psi_i^\alpha \otimes \psi_j^\beta) = \hat{R}^{\alpha\beta}_{\gamma\rho \ i \ j} \psi_k^\gamma \otimes \psi_l^\rho = -\hat{I}^{\alpha\beta}_{\gamma\rho \ i \ j} \psi_k^\gamma \otimes \psi_l^\rho,
\]

(2.16)

where here and later on the values of a latin indeces goes over all values of the variable \( r \ (r = r_1, r_2, r_3, ..., r_N) \) and the \( \hat{R} \)-matrix has the block structure:

\[
\hat{R}^{\alpha\beta}_{\gamma\rho \ i \ j} = \hat{R}^{\alpha\beta}_{\gamma\rho \ i \ j} \delta_i^k \delta_j^l + (l - l^{-1}) I^{\alpha\beta}_{\gamma\rho} \delta_i^k \delta_j^l \Theta^{kl} + \hat{Q}^{\alpha\beta}_{\gamma\rho} \delta_i^k \delta_j^l \Theta^{kl}.
\]

(2.17)

\( \hat{I} \) is the diagonal matrix

\[
\hat{I}^{\alpha\beta}_{\gamma\rho \ i \ j} = \frac{1}{q} \delta_\gamma^\alpha \delta_\rho^\beta \delta_i^k \delta_j^l - \frac{1}{l} \delta_\gamma^\alpha \delta_\rho^\beta \delta_i^k \delta_j^l (\delta_{ij} - 1),
\]

(2.18)

\[
\hat{I}^{\alpha\beta}_{\gamma\rho \ i \ i} \psi_k^\gamma \otimes \psi_l^\rho = \frac{1}{q} \psi_i^\alpha \otimes \psi_i^\beta,
\]

\[
\hat{I}^{\alpha\beta}_{\gamma\rho \ i \ j} \psi_k^\gamma \otimes \psi_l^\rho = \frac{1}{l} \psi_i^\alpha \otimes \psi_j^\beta, \quad i \neq j.
\]

In (2.16) for product \( \Psi \otimes \Psi \) we imply the following ordering:

\[
\Psi \otimes \Psi = \Psi_{r_1}^{(n)} \otimes \Psi_{r_1}^{(n)} + \Psi_{r_1}^{(n)} \otimes \Psi_{r_2}^{(n)} + \Psi_{r_2}^{(n)} \otimes \Psi_{r_1}^{(n)} + \Psi_{r_2}^{(n)} \otimes \Psi_{r_2}^{(n)} + \Psi_{r_1}^{(n)} \otimes \Psi_r^{(n)} + \Psi_{r_2}^{(n)} \otimes \Psi_r^{(n)} + \cdots + \Psi_r^{(n)} \otimes \Psi_r^{(n)}.
\]

(2.19)
At such ordering the $\hat{R}$-matrix has block structure and, for example, for arbitrary two sites $r_i$ and $r_j$ ($r_i < r_j$) the block $\hat{R}_{(i,j)}$ of the $\hat{R}$-matrix acting on the sum $\Psi_{r_i}^{(n)} \otimes \Psi_{r_i}^{(n)} \oplus \Psi_{r_i}^{(n)} \otimes \Psi_{r_j}^{(n)} \oplus \Psi_{r_i}^{(n)} \oplus \Psi_{r_j}^{(n)} \otimes \Psi_{r_j}^{(n)}$ has the following form:

$$
\tilde{R}_{(i,j)} = \begin{pmatrix}
\hat{R} & 0 & 0 & 0 \\
0 & (l-l^{-1})\hat{I} & \hat{Q} & 0 \\
0 & \hat{Q} & 0 & 0 \\
0 & 0 & 0 & \hat{R}
\end{pmatrix}.
$$

(2.20)

It is not hard to check that the $\hat{R}$-matrix satisfies the quantum Yang-Baxter equation

$$
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23},
$$

(2.21)

and the Hecke equation

$$
\hat{R}^2 - (\hat{J} - \hat{I})\hat{R} - \hat{J} = (\hat{J} + \hat{J} \cdot \hat{R}) \cdot (\hat{J} - \hat{I} \cdot \hat{R}) = 0,
$$

(2.22)

where $\hat{J} = J_{\gamma \rho}^{\alpha \beta k l} = I_{\gamma \rho}^{\alpha \beta} I_{i j}^{k l}$ is unit matrix, $\hat{J}$ is the diagonal matrix

$$
\hat{J}^{\alpha \beta k l}_{\gamma \rho i j} = q \delta_{\gamma}^\alpha \delta_{\rho}^\beta \delta_{k}^i \delta_{l}^j \delta_{i j} - l \delta_{\gamma}^\alpha \delta_{\rho}^\beta \delta_{k}^i \delta_{l}^j (\delta_{i j} - 1),
$$

(2.23)

and $\hat{J} + \hat{J} \cdot \hat{R}$. By analogy with (2.5a) and (2.5b), from (2.22) we can define the quantum $(l, q)$-antisymmetrizer

$$
\hat{A}^{(l, q)} = (\hat{J} - \hat{I} \cdot \hat{R})
$$

(2.24)

and the quantum $(l, q)$-symmetrizer

$$
\hat{S}^{(l, q)} = (\hat{J} + \hat{J} \cdot \hat{R}),
$$

(2.25)

which are orthogonal projections

$$
\hat{A}^{(l, q)} \cdot \hat{S}^{(l, q)} = \hat{S}^{(l, q)} \cdot \hat{A}^{(l, q)} = 0,
$$

$$
(\hat{A}^{(l, q)})^2 = \hat{I} \cdot (\hat{I} + \hat{J}) \cdot \hat{A}^{(l, q)},
$$

$$
(\hat{S}^{(l, q)})^2 = \hat{J} \cdot (\hat{I} + \hat{J}) \cdot \hat{S}^{(l, q)}.
$$

(2.26)
The quantum symmetrizer \( \hat{S}^{(l,q)} \) permits to define the commutation relations for the lattice \( n \)-component \((l,q)\)-Grassmann field \( \psi^\alpha_r \) \((q^2 \neq -1)\) which include (2.12)

\[
\hat{S}^{(l,q)} \cdot (\psi \otimes \psi) = 0, \quad \text{or} \quad \psi^\alpha_i \otimes \psi^\beta_j + (\hat{J} \cdot \hat{R})^{\alpha \beta kl}_{\gamma \rho ij} \psi^\gamma_k \otimes \psi^\rho_l = 0, \tag{2.27}
\]

or in components

\[
\begin{align*}
\psi^\alpha_i \otimes \psi^\beta_j &= -q \psi^\beta_j \otimes \psi^\alpha_i, \\
\psi^\alpha_i \otimes \psi^\alpha_j &= -l \psi^\alpha_j \otimes \psi^\alpha_i, \\
\psi^\alpha_i \otimes \psi^\beta_j &= -l q \psi^\beta_j \otimes \psi^\alpha_i, \\
\psi^\beta_j \otimes \psi^\alpha_i &= -\frac{l}{q} \psi^\alpha_i \otimes \psi^\beta_j,
\end{align*}
\tag{2.28}
\]

where \( r_i < r_j \) and \( \alpha > \beta \).

The quantum antisymmetrizer \( \hat{A}^{(l,q)} \) determines the commutation relations for the lattice \( n \)-component \((l,q)\)-boson field \( \varphi = \{ \varphi^\alpha_r \}\)

\[
\hat{A}^{(l,q)} \cdot (\varphi \otimes \varphi) = 0 \quad \text{or} \quad \varphi^\alpha_i \otimes \varphi^\beta_j - (\hat{J} \cdot \hat{R})^{\alpha \beta kl}_{\gamma \rho ij} \varphi^\gamma_k \otimes \varphi^\rho_l = 0, \tag{2.29}
\]

or in components

\[
\begin{align*}
\varphi^\alpha_i \otimes \varphi^\beta_j &= \frac{1}{q} \varphi^\beta_j \otimes \varphi^\alpha_i, \\
\varphi^\alpha_i \otimes \varphi^\alpha_j &= \frac{1}{l} \varphi^\alpha_j \otimes \varphi^\alpha_i, \\
\varphi^\alpha_i \otimes \varphi^\beta_j &= \frac{q}{l} \varphi^\beta_j \otimes \varphi^\alpha_i, \\
\varphi^\beta_j \otimes \varphi^\alpha_i &= \frac{1}{l q} \varphi^\alpha_i \otimes \varphi^\beta_j.
\end{align*}
\tag{2.30}
\]

Here again \( r_i < r_j \) and \( \alpha > \beta \). In this case \( \hat{Q} \)-matrix (2.15) is acting by the following way:

\[
\hat{Q}(\varphi^\alpha_i \otimes \varphi^\beta_j) \equiv \hat{Q}^{\alpha \beta}_{\gamma \rho} \varphi^\gamma_i \otimes \varphi^\rho_j = l \varphi^\alpha_j \otimes \varphi^\beta_i, \\
\hat{Q}^{-1}(\varphi^\alpha_i \otimes \varphi^\beta_j) \equiv (\hat{Q}^{-1})^{\alpha \beta}_{\gamma \rho} \varphi^\gamma_j \otimes \varphi^\rho_i = \frac{1}{l} \varphi^\alpha_i \otimes \varphi^\beta_j.
\]

Note that \( \hat{R} \)-matrix (2.24) in which we substitute the transpose matrix \( \hat{Q}^t \) instead of \( \hat{Q} \)-matrix (2.15) also satisfies the quantum Yang-Baxter equation (2.21) and the Hecke equation (2.22). In this case commutation relations
(2.28) for the lattice $n$-component $(l,q)$-Grassmann field and (2.30) for the lattice $n$-component $(l,q)$-boson field take on the following form

$$
\psi^\alpha_{r_i} \otimes \psi^\beta_{r_i} = -q \psi^\beta_{r_i} \otimes \psi^\alpha_{r_i},
$$

$$
\psi^\alpha_{r_j} \otimes \psi^\alpha_{r_i} = -l \psi^\alpha_{r_i} \otimes \psi^\alpha_{r_j},
$$

$$
\psi^\alpha_{r_j} \otimes \psi^\beta_{r_i} = -\frac{l}{q} \psi^\beta_{r_i} \otimes \psi^\alpha_{r_j},
$$

$$
\psi^\beta_{r_j} \otimes \psi^\alpha_{r_i} = -l q \psi^\alpha_{r_i} \otimes \psi^\beta_{r_j},
$$

and

$$
\varphi^\alpha_{r_i} \otimes \varphi^\beta_{r_i} = \frac{1}{q} \varphi^\beta_{r_i} \otimes \varphi^\alpha_{r_i},
$$

$$
\varphi^\alpha_{r_j} \otimes \varphi^\alpha_{r_i} = \frac{1}{l} \varphi^\alpha_{r_i} \otimes \varphi^\alpha_{r_j},
$$

$$
\varphi^\alpha_{r_j} \otimes \varphi^\beta_{r_i} = \frac{1}{l q} \varphi^\beta_{r_i} \otimes \varphi^\alpha_{r_j},
$$

$$
\varphi^\beta_{r_j} \otimes \varphi^\alpha_{r_i} = \frac{q}{l} \varphi^\alpha_{r_i} \otimes \varphi^\beta_{r_j},
$$

where $r_i < r_j$ and $\alpha > \beta$.

Now let us consider a general linear transformation of the lattice field $\psi^\alpha_{r_i}$ by the matrix $\hat{F}$

$$
\Psi \xrightarrow{\delta} \hat{F} \otimes \Psi : \tilde{\psi} = \delta(\psi) = \hat{F} \otimes \psi, \quad \tilde{\psi}^\alpha_{r_i} = \delta(\psi^\alpha_{r_i}) = F^\alpha_{\beta i} \otimes \psi^\beta_{r_i},
$$

(2.31)

and suppose that the matrix elements $F^\alpha_{\beta i}$ belong to the associative algebra of functions $\mathcal{F}$ for which are defined the following operations:

a) comultiplication $\Delta$

$$
\mathcal{F} \xrightarrow{\Delta} \mathcal{F} \otimes \mathcal{F} : \quad \Delta(F^\alpha_{\beta i}) = F^\alpha_{\rho i} \otimes F^\rho_{\beta l},
$$

b) counit $\varepsilon$

$$
\mathcal{F} \xrightarrow{\varepsilon} \mathcal{C} : \quad \varepsilon(F^\rho_{\beta i}) = \delta^\alpha_{\beta} \delta^k_{i},
$$

where $\mathcal{C}$ is a complex numbers,

c) antipod $i$

$$
\mathcal{F} \xrightarrow{i} \mathcal{F} : \quad i(F^\alpha_{\beta i}) = (-q)^{(\alpha-\beta)+(k-i)} \tilde{F}^\beta_{\alpha k},
$$

where $\tilde{F}^\beta_{\alpha k}$ is the quantum minor for matrix element $F^\alpha_{\beta i}$.
d) multiplication map \( m \)
\[
\mathcal{F} \otimes \mathcal{F} \xrightarrow{m} \mathcal{F} : \quad m(F^\alpha_k \otimes F^\rho_l) = F^\alpha_k F^\rho_l.
\]
e) matrix elements \( F^\alpha_k \) satisfy the commutation relations
\[
\hat{R}^\alpha_{\beta k l} F^\gamma_{\mu m} F^\rho_{\nu n} = F^\alpha_{\gamma i} F^\beta_{\rho j} \hat{R}^\gamma_{\mu \nu k l}.
\]
By analogy with (2.10) one defines \((l,q)\)-determinant for matrix \( \hat{F} \):
\[
\det_{(l,q)}(\hat{F}) = \det_{(l,q)}(\hat{A}) = 1.
\]
Then it is not hard show that \( \mathcal{F} \) is algebra of functions \( \text{Fun}_{(l,q)}(GL(n) \times GL(N)) \) and commutation relations (2.28) and (2.29) are covariant with respect to transformation (2.31). This algebra contains the algebras \( \text{Fun}_q(GL(n)) \) and \( \text{Fun}_l(GL(N)) \) as subalgebras.

For further we restrict ourselves to such transformations (2.31) which do not depend from the coordinates of lattice sites
\[
F^\alpha_k = A^\alpha_\gamma \delta^k_i.
\]
Let us suppose that \( \det_q \hat{A} = 1 \). Then from commutation relations (2.32) and the expression (2.17) for \( \hat{R} \) it follows that \( A^\alpha_\gamma \in \text{Fun}_q(SL(n)) \). It means that \( \mathcal{A} \in \mathcal{F} \). Really, taking into account (2.34), from (2.33) and (2.10) it is not hard to show that
\[
\det_{(l,q)}(\hat{F}) = (\det_q \hat{A})^N = 1.
\]

3. Differential calculus for the lattice \((l,q)\)-deformed fields

The various approaches to the noncommutative differential geometry on the quantum spaces have been considered in [16-20] (for a review see [20]). In this section, following the Wess and Zumino approach [16,17], we consider the differential calculus for lattice \((l,q)\)-deformed fields defined in previous section. At first let us determine differential calculus for the lattice \((l,q)\)-Grassmann field \( \psi \) satisfying commutation relations (2.27).
Define a exterior differential $d$ on $\Psi$ satisfying the usual properties such as

$$d^2 = 0$$  \hspace{1cm} (3.1)

and the Leibniz rule

$$d(fg) = (df)g + (-1)^{|f|}f(dg),$$  \hspace{1cm} (3.2)

where $|f|$ denotes the parity of $f$ (for example, $|\psi| = -1$) and $f$ and $g$ are a functions of the $(l,q)$-Grassmann field $\psi$. Under these functions we understand their formal expansion in power series of $\psi$, which have the finite number of members because of the nilpotency condition (2.13) for $\psi^\alpha_r$. We will consider $\{\psi^\alpha_r\}$ as the basic elements for $\Psi^{(n)}$. In general case for derivation of the commutation relations between functions, differentials and derivatives it is necessary to consider their action on arbitrary function $f(\psi)$. However for $\Psi^{(n)}$, in consequence of the finite number of members of the power series and the form of commutation relations (2.28) we can restrict ourselves to study of their action on basic element $\{\psi^\alpha_r\}$.

The action of a exterior differential on this field is

$$d(\psi^\alpha_r \psi^\beta_{r'}) = (d\psi^\alpha_r) \psi^\beta_{r'} - \psi^\alpha_r (d\psi^\beta_{r'}).$$  \hspace{1cm} (3.3)

Define the derivative operator

$$\partial^k = \frac{\partial}{\partial \psi^k}, \quad \partial^r_l \psi^\beta = \delta^\beta_l \delta^r_k, \quad d = d\psi^\alpha \partial^k.$$  \hspace{1cm} (3.4)

Write the commutation relations between the lattice field $\psi$ and its differential $d\psi$ in the following general form

$$\psi \otimes d\psi = \hat{C} \cdot (d\psi \otimes \psi) = (\hat{J} + \hat{B}) \cdot (d\psi \otimes \psi),$$  \hspace{1cm} (3.5)

where $\hat{C}$ and $\hat{B}$ are some numerical matrices, which we will find from the consistency condition of (3.5) with (2.27).

Acting by exterior differential $d$ on (2.27)

$$\hat{S}^{(l,q)} \cdot (\psi \otimes d\psi - d\psi \otimes \psi) = 0,$$

and substituting in this equation (3.5) we obtain the following consistency condition

$$\hat{S}^{(l,q)} \cdot \hat{B} = 0.$$  \hspace{1cm} (3.6)
Using the orthogonality property of $\hat{A}^{(l,q)}$ and $\hat{S}^{(l,q)}$ (2.26) we suggest the following anzats for solution this condition

$$\hat{B} = a_1 \hat{A}^{(l,q)}.$$  

Requiring that commutation relation (3.5) at $\alpha \neq \beta$, $i = k$ coinsides with

$$\psi_\alpha^\gamma d\psi_\beta^\rho - \frac{1}{q} \hat{R}^{\alpha \beta}_{\gamma \rho} d\psi_\gamma^\gamma d\psi_\beta^\rho = 0,$$  

we can determinate the coefficient $a_1$: $a_1 = -1$. Then

$$\hat{C} = \hat{I} \cdot \hat{R}$$  

and we obtain

$$\psi_\alpha^\gamma d\psi_\beta^\rho - (\hat{I} \cdot \hat{R})^{\alpha \beta}_{\gamma \rho} d\psi_\gamma^\gamma d\psi_\beta^\rho = 0.$$  

Note that the derivatives defined by relation (3.4) do not satisfy the Leibniz rule. Indeed using (3.3) and (3.8), we obtain

$$d(\psi_\alpha^\gamma \psi_\beta^\rho) = (d\psi_\alpha^\gamma) \psi_\beta^\rho - \psi_\alpha^\gamma (d\psi_\beta^\rho) =$$

$$d\psi_\gamma^\gamma (\partial_\gamma^k \psi_\alpha^\gamma) \psi_\beta^\rho - \psi_\alpha^\gamma d\psi_\gamma^\gamma (\partial_\gamma^k \psi_\beta^\rho) =$$

$$d\psi_\gamma^\gamma [(\partial_\gamma^k \psi_\alpha^\gamma) \psi_\beta^\rho - (\hat{I} \cdot \hat{R})^{\alpha \delta}_{\gamma \rho} \psi_\delta^m (\partial_\gamma^l \psi_\beta^\rho)] =$$

$$d\psi_\gamma^\gamma \partial_\gamma^k (\psi_\alpha^\gamma \psi_\beta^\rho).$$

Hence it follows

$$\partial_\gamma^k (\psi_\alpha^\gamma \psi_\beta^\rho) = (\partial_\gamma^k \psi_\alpha^\gamma) \psi_\beta^\rho - (\hat{I} \cdot \hat{R})^{\alpha \delta}_{\gamma \rho} \psi_\delta^m (\partial_\gamma^l \psi_\beta^\rho).$$  

(3.9)

If consider $\partial_\alpha^i$ and $\psi_\beta^\rho$ as operators, acting on some function $f(\psi)$, then from (3.9) we obtain commutation relations

$$\partial_\gamma^k \psi_\alpha^\gamma + (\hat{I} \cdot \hat{R})^{\alpha \delta}_{\gamma \rho} \psi_\delta^m \partial_\beta^j = \delta_\gamma^\alpha \delta_\beta^k.$$  

(3.10)

Let us assume that the matrix $\hat{D}$ determines the commutation relations between derivatives $\partial_\gamma^k$ and differentials $d\psi_\beta^\rho$

$$\partial_\gamma^k d\psi_\alpha^\gamma - \hat{D}^{\alpha \beta}_{\gamma \rho} d\psi_\alpha^\gamma d\psi_\beta^\rho = 0.$$  

(3.11)
To find the matrix $\hat{D}$ let us consider (3.11) as the operator identity and act by them on $\psi^\delta_m$. Using (3.8) and (3.10), we obtain

\[
\partial_k^\delta d\psi^\alpha_j \psi^\delta_m - \hat{D}^\alpha\beta k i d\psi^\rho_i (\partial^l_\beta \psi^\delta_m) = \\
[(\hat{I} \cdot \hat{\mathcal{R}})^{-1}]^\alpha\delta i l (\partial^k_\gamma \psi^\mu_i) d\psi^\nu_l - \hat{D}^\alpha\beta k i d\psi^\rho_i (ibl \psi^\delta_m) = \\
[(\hat{I} \cdot \hat{\mathcal{R}})^{-1}]^\alpha\delta i l (\partial^k_\gamma \psi^\mu_i) d\psi^\nu_l - \hat{D}^\alpha\beta k i d\psi^\rho_i (\delta^\delta i l) = \\
[[[(\hat{I} \cdot \hat{\mathcal{R}})^{-1}]^\alpha\delta k i l - \hat{D}^\alpha\beta k i l] d\psi^\rho_i = 0.
\]

It implies $\hat{D} = (\hat{I} \cdot \hat{\mathcal{R}})^{-1}$ and we get

\[
d\psi^\gamma_i \partial^k_\alpha - (\hat{I} \cdot \hat{\mathcal{R}})^\alpha\delta k l \partial^j_\delta (d\psi^\rho_i) = 0. \tag{3.12}
\]

In order to determine the commutation relations between the derivatives we use the following anzats

\[
\hat{G}^\alpha\beta k l \partial_i^\alpha \partial_j^\beta = 0, \tag{3.13}
\]

where $\hat{G}$ is some numerical matrix. Multilying this equation from right by $\psi^\delta_m$ and commuting $\psi^\delta_m$ through to the left and requiring that terms linear in the derivatives are canceled, we find the consistency condition

\[
(\hat{J} - \hat{I} \cdot \hat{\mathcal{R}}) \cdot \hat{G} = 0 \quad \text{or} \quad \hat{A}^{(l,q)} \cdot \hat{G} = 0. \tag{3.14}
\]

Using orthogonality property (2.26) and again requiring that the commutation relation (3.13) at $\alpha \neq \beta$, $i = k$ has to coincide with

\[
\partial^i_\alpha \partial^i_\beta + q \hat{R}^{\gamma\rho}_{\alpha\beta} \partial^i_\gamma \partial^i_\rho = 0,
\]

we find:

\[
\hat{G} = \hat{S}^{(l,q)}.
\]

Hence we obtain commutation relations:

\[
\partial^r_i \partial^r_i = -q \partial^r_i \partial^r_i, \\
\partial^r_j \partial^r_i = -l \partial^r_i \partial^r_j, \\
\partial^r_j \partial^r_i = -l q \partial^r_i \partial^r_j, \\
\partial^r_j \partial^r_j = -\frac{l}{q} \partial^r_i \partial^r_i, \tag{3.15}
\]

where again $\alpha > \beta$ and $r_j > r_i$ or in covariant form

\[
\partial^k_\alpha \partial^l_\beta + (\hat{J} \cdot \hat{\mathcal{R}})^{\gamma\rho k l}_{\alpha\beta} \partial^i_\gamma \partial^j_\rho = 0. \tag{3.16}
\]
Now it is not hard to find commutation relations between differentials. For this aim, acting by operator $d$ on (3.8) and using the relation

$$d^2(\psi^\alpha_r \psi^\beta_r) = d(d\psi^\alpha_r \psi^\beta_r - \psi^\alpha_r d\psi^\beta_r) = d(d\psi^\alpha_r \psi^\beta_r) - d\psi^\alpha_r d\psi^\beta_r = 0,$$

we obtain

$$d\psi^\alpha_i d\psi^\beta_j - (\hat{I} \cdot \hat{R})_{\alpha\beta}^{k l} d\psi^\gamma_k d\psi^\rho_l = 0. \quad (3.17)$$

By analogy with the determination of the commutation relations (3.8), (3.10), (3.12), (3.16), (3.17) for the lattice $n$-component $(l,q)$-Grassmann field we can find analogous commutation relations for the lattice $n$-component $(l,q)$-boson field which satisfies commutation relations (2.29). Here we give the final result:

$$\varphi^\alpha_i d\varphi^\beta_j - (\hat{J} \cdot \hat{R})_{\alpha\beta}^{k l} d\varphi^\gamma_k \varphi^\rho_l = 0, \quad (3.18)$$

$$\partial^\gamma \varphi^\alpha_i - (\hat{J} \cdot \hat{R})_{\alpha\beta}^{k l} \varphi^\rho_l \partial^j \delta^\alpha_k = \delta^\alpha_i, \quad (3.19)$$

$$d\varphi^\gamma_i \partial^k \varphi^\beta_j - (\hat{J} \cdot \hat{R})_{\alpha\beta}^{k l} \varphi^\beta_l \partial^j (d\varphi^\rho_l) = 0, \quad (3.20)$$

$$\partial^k \partial^l \varphi^\alpha_i - (\hat{J} \cdot \hat{R})_{\alpha\beta}^{k l} \partial^i \partial^j (d\varphi^\rho_l) = 0, \quad (3.21)$$

$$d\varphi^\alpha_i d\varphi^\beta_j + (\hat{J} \cdot \hat{R})_{\alpha\beta}^{k l} d\varphi^\gamma_k d\varphi^\rho_l = 0, \quad (3.22)$$

where we use the notation

$$\partial^k = \frac{\partial}{\partial \varphi^\alpha_k}, \quad \partial^l \varphi^\beta_k = \delta^\beta \delta^l_k, \quad d = d\varphi^\alpha_k \partial^k. \quad (3.23)$$

4. The elements of integral calculus for the lattice $(l,q)$-Grassmann fields

In this section we determine the generalization of the Berezin integration for the lattice $(l,q)$-Grassmann field defined in previous sections.

At first let us define the $SL_q(n,C)$-invariant generalization of the Berezin integration on the quantum vector space $\Xi^{(n)}$. For this aim we will use the results of the paper [3] where representation of the partition function of the two-dimensional Ising model in form of the $q$-Grassmann functional integral ($q = -1$) was found. After calculation of this integral for the Ising model with nearest neighbour interaction the known solution of the model was obtained in [3].

Let us define the volume form which is invariant with respect to transformation from $SL_q(n,C)$. For that it is necessary to introduce the operation
of the exterior multiplication " $\wedge$ ". Make this, using the $q$-symmetrizer $\hat{S}_q \ (2.5b)$

$$\hat{S}_q^{\alpha\beta} \, d\xi^\gamma \wedge d\xi^\rho \equiv 0. \quad (4.1)$$

This definition permits to determine the commutation relations between the differentials $d\xi^\alpha$ relative to the exterior multiplication

$$d\xi^\alpha \wedge d\xi^\beta = -q \, \hat{R}^{\alpha\beta}_{\gamma\rho} \, d\xi^\gamma \wedge d\xi^\rho. \quad (4.2)$$

Using this relation, it is easy to show that the volume form $d\xi^n \wedge d\xi^{(n-1)} \wedge \ldots d\xi^2 \wedge d\xi^1$ is invariant with respect to the transformation $\hat{A} \in SL_q(n, C)$

$$d\tilde{\xi}^n \wedge d\tilde{\xi}^{(n-1)} \wedge \ldots d\tilde{\xi}^2 \wedge d\tilde{\xi}^1 = \det_q \hat{A} \, d\xi^n \wedge d\xi^{(n-1)} \wedge \ldots d\xi^2 \wedge d\xi^1 = d\xi^n \wedge d\xi^{(n-1)} \wedge \ldots d\xi^2 \wedge d\xi^1. \quad (4.3)$$

Taking into account this invariance and requiring that at $q = 1$ we must obtain the rules of Berezin integration [21] for usual Grassmann variable, one defines

$$\int d\xi^n \wedge d\xi^{(n-1)} \wedge \ldots d\xi^2 \wedge d\xi^1 = 0. \quad (4.5)$$

Note that the differentials in our volume form (4.3) are different from Berezin differentials ($q = 1$) since the latter is transformed by the reciprocal matrix. We are grateful to D.V. Volkov for this remark.

In consequence of commutation relations (2.2) for $\xi^\alpha$, it is easy to show, that product $\xi^n \xi^{(n-1)} \ldots \xi^2 \xi^1$ is also invariant with respect to transformations from $SL_q(n, C)$

$$\tilde{\xi}^1 \tilde{\xi}^2 \ldots \tilde{\xi}^{(n-1)} \tilde{\xi}^n = \xi^1 \xi^2 \ldots \xi^{(n-1)} \xi^n.$$

Using this invariance and (4.3), and again assuming the right limit at $q = 1$, one defines

$$\int d\xi^n \wedge d\xi^{(n-1)} \wedge \ldots d\xi^2 \wedge d\xi^1 \xi^1 \xi^2 \ldots \xi^{(n-1)} \xi^n = 1. \quad (4.6)$$

Analogous reasoning leads to the definition

$$\int d\xi^n \wedge d\xi^{(n-1)} \wedge \ldots d\xi^2 \wedge d\xi^1 \xi^1 \xi^2 \ldots \xi^{(n-1)} = 0. \quad (4.7)$$

The definitions (4.5)–(4.7) are able to write in the following covariant form

$$\int d\xi^n \wedge d\xi^{(n-1)} \wedge \ldots d\xi^2 \wedge d\xi^1 \xi^1 \xi^2 \ldots \xi^{(n-1)} \xi = \varepsilon^{\alpha_1 \alpha_2 \ldots \alpha_{(n-1)} \alpha_n}, \quad (4.8)$$
Here tensor $\varepsilon^{\alpha_1 \alpha_2 \ldots \alpha_{(n-1)} \alpha_n}$ is defined by the following rules:

$$\varepsilon^{1 \ 2 \ 3 \ \ldots\ (n-1) \ n} = 1, \quad (4.9)$$

and the remaining components are defined by the coefficients appearing in the l.h.s of the relation

$$\xi^{\alpha_1} \xi^{\alpha_2} \ldots \xi^{\alpha_{(n-1)}} \xi^{\alpha_n} = \varepsilon^{\alpha_1 \alpha_2 \ldots \alpha_{(n-1)} \alpha_n} \xi^1 \xi^2 \ldots \xi^{(n-1)} \xi^n \quad (4.10)$$

after reordering of its the l.h.s. to the r.h.s. by means of the commutation relations (2.2) for every specific set of the values of indices.

Note that in [23,24], using differential and integral calculi on the quantum plane which are invariant with respect to quantum inhomogeneous Euclidean group $E(2)_q$, the holomorphic representation of $q$-deformed path integral for the quantum mechanical evolution operator kernel of q-oscillator was constructed.

For the calculation of the Gaussian integral over the $n$-component $q$-Grassmann variable we will use the following definition of $q$-deformed Pfaffian (which is a generalization of definition of usual Pfaffian [25] for case of $q$-deformed commutation relations).

Consider the quadratic form

$$w = \sum_{\alpha<\beta} a^{\alpha\beta} \xi^\alpha \xi^\beta = \frac{1}{2} \sum_{\alpha,\beta} \hat{a}^{\alpha\beta} \xi^\alpha \xi^\beta, \quad (4.11)$$

where matrix elements $\hat{a}^{\alpha\beta}$ commute between themselves and $q$-antisymmetric matrix $\hat{a}$ has form

$$\hat{a} = \begin{bmatrix}
0 & \hat{a}^{12} & \hat{a}^{13} & \ldots & \hat{a}^{1n} \\
-q^{-1}\hat{a}^{12} & 0 & \hat{a}^{23} & \ldots & \hat{a}^{2n} \\
-q^{-1}\hat{a}^{13} & -q^{-1}\hat{a}^{23} & 0 & \ldots & \hat{a}^{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-q^{-1}\hat{a}^{1n} & -q^{-1}\hat{a}^{2n} & -q^{-1}\hat{a}^{3n} & \ldots & 0
\end{bmatrix}. \quad (4.11a)$$

Here the matrix elements arranged down the diagonal are determined by means of relations (2.2). Define $q$-Pfaffian of matrix $\hat{a}$ through

$$\frac{1}{(\frac{n}{2})!} w^{\frac{q}{2}} = Pf_q(\hat{a}) \xi^1 \xi^2 \ldots \xi^{(n-1)} \xi^n. \quad (4.12)$$
Hence, taking into account commutation relations (2.2), we get, for example, at \( n = 4 \)

\[
P f_q(\hat{a}) = \frac{1}{2}(1 + q^4)\hat{a}^{12} \hat{a}^{34} - \frac{1}{2}q(1 + q^2)\hat{a}^{13} \hat{a}^{24} + q^2\hat{a}^{14} \hat{a}^{23}.
\]

Using (4.12), it is not hard to show that

\[
\int d\xi^n \wedge d\xi^{(n-1)} \wedge ...d\xi^2 \wedge d\xi^1 \exp\left\{ \frac{1}{2} \sum_{\alpha, \beta} \hat{a}^{\alpha\beta} \xi^\alpha \xi^\beta \right\} = P f_q(\hat{a}).
\]

(4.13)

Now let us define the integral calculus for for the lattice \((l, q)\)-Grassmann field \( \psi = \{\psi^\alpha_r\} \) which satisfy commutation relations (2.28). Recall that the general linear transformation for this field is transformation by the quantum matrix \( \hat{F} \in Fun_{(l, q)}(GL(N) \times GL(n)) \). Following to (2.34), we restrict ourselves to transformations \( \hat{F} \) which do not depend from coordinates of lattice sites and belong to \( Fun_q(SL(n)) \) that it means \( det_q(\hat{F}) = 1 \) (2.35).

By analogy with (4.1), (4.2), let us define the commutation relations between the differentials \( d\psi^\alpha_r \) relative to the exterior multiplication

\[
(\hat{S}^{(l, q)})_{q \gamma \rho \ i \ j} d\psi^\gamma_k \wedge d\psi^\rho_l = \hat{J}^{\alpha \beta \ k \ l} \psi_i^\alpha \wedge \psi_j^\beta + (\hat{R} \cdot \hat{R})^{\alpha \beta \ k \ l} d\psi^\gamma_k \wedge d\psi^\rho_l \equiv 0.
\]

(4.14)

or in components

\[
\begin{align*}
d\psi_{r_i}^\alpha \wedge d\psi_{r_i}^\beta &= -q d\psi_{r_i}^\beta \wedge d\psi_{r_i}^\alpha, \\
d\psi_{r_j}^\alpha \wedge d\psi_{r_i}^\alpha &= -l d\psi_{r_j}^\alpha \wedge d\psi_{r_i}^\alpha, \\
d\psi_{r_j}^\alpha \wedge d\psi_{r_i}^\beta &= -l q d\psi_{r_i}^\beta \wedge d\psi_{r_j}^\alpha, \\
d\psi_{r_i}^\alpha \wedge d\psi_{r_j}^\beta &= -q l d\psi_{r_j}^\beta \wedge d\psi_{r_i}^\alpha,
\end{align*}
\]

(4.15)

where again \( \alpha > \beta \) and \( r_j > r_i \).

The exterior multiplication permits to define the \( SL_q(n, C) \)-invariant volume form \( D\psi \) with respect to the transformation by the quantum matrix \( \hat{F} \) (2.34):

\[
D\psi = d\psi^n_{r_N} \wedge d\psi^{(n-1)}_{r_N} \wedge ...d\psi^2_{r_N} \wedge d\psi^1_{r_N} \wedge ...
\]

\[
d\psi^n_{r_2} \wedge d\psi^{(n-1)}_{r_2} \wedge ...d\psi^2_{r_2} \wedge d\psi^1_{r_2} d\psi^n_{r_1} \wedge d\psi^{(n-1)}_{r_1} \wedge ...d\psi^2_{r_1} \wedge d\psi^1_{r_1}.
\]

(4.16)

Then we can define the \( SL_q(n, C) \)-invariant generalization of the Berezin integration for the lattice \((l, q)\)-deformed field \( \psi^\alpha_r \) by the following relation

\[
\int D\psi \psi^{\alpha_1}_{i_1} \psi^{\alpha_2}_{i_2} ... \psi^{\alpha_{n-1}}_{i_{n-1}} \psi^{\alpha_n}_{i_n} ... \psi^{\delta_1}_{i_N} \psi^{\delta_2}_{i_N} ... \psi^{\delta_{n-1}}_{i_N} \psi^{\delta_n}_{i_N} =
\]

\[
\varepsilon^{\alpha_1 \alpha_2 ... \alpha_{n-1} \alpha_n ... \delta_1 \delta_2 ... \delta_{n-1} \delta_n}.
\]

(4.17)
where tensor \( \hat{\varepsilon} \) is defined by rules:

\[
\varepsilon^{1 \ldots (n-1) n \ldots 1 \ldots (n-1) n}_{r_1 r_2 \ldots r_{N-1} r_N} = 1,
\]

and the remaining components are defined by the coefficients appearing in the l.h.s. of the relation

\[
\psi^{\alpha_1}_{i_1} \psi^{\alpha_2}_{i_1} \cdots \psi^{\alpha_{n-1}}_{i_1} \psi^{\alpha_n}_{i_1} \psi^{\delta_1}_{i_N} \psi^{\delta_2}_{i_N} \cdots \psi^{\delta_{n-1}}_{i_N} \psi^\delta_{i_N} = \\
\varepsilon^{\alpha_1 \alpha_2 \cdots \alpha_{n-1} \alpha_n \delta_1 \delta_2 \cdots \delta_{n-1} \delta_n}_{i_1 i_2 \cdots i_{N-1} i_N} \psi^{1}_{r_1} \psi^{2}_{r_1} \cdots \psi^{n-1}_{r_1} \psi^{n}_{r_1} \psi^{1}_{r_N} \psi^{2}_{r_N} \cdots \psi^{n-1}_{r_N} \psi^{n}_{r_N}
\]
after reordering of its the l.h.s. to the r.h.s. by means of the commutation relations (2.28) for every specific set of the values of indices.

For the calculation of the Gaussian integral over the lattice \((l, q)\)-Grassmann field we will use the following definition of \((l, q)\)-Pfaffian which is similar to (4.11).

Consider the quadratic form

\[
w = \sum_{\alpha<\beta i<k} b^{\alpha\beta}_{i k} \psi^\alpha_i \psi^\beta_k = \frac{1}{2} \sum_{\alpha,\beta,i,k} \hat{b}^{\alpha\beta}_{i k} \psi^\alpha_i \psi^\beta_k,
\]

where matrix elements \( \hat{b}^{\alpha\beta}_{i k} \) commute between themselves and \((l, q)\)-antisymmetric matrix \( \hat{b} \) is determined by means of the commutation relations (2.29), (2.30) and has the form which is similar to (4.11a). Define a \((l, q)\)-Pfaffian of matrix \( \hat{b} \) through

\[
\frac{1}{(nN^2)!} w^{\frac{nN}{2}} = P f_{(l,q)}(\hat{b}) \psi^{1}_{r_1} \psi^{2}_{r_1} \cdots \psi^{n-1}_{r_1} \psi^{n}_{r_1} \psi^{1}_{r_N} \psi^{2}_{r_N} \cdots \psi^{n-1}_{r_N} \psi^{n}_{r_N}
\]

Using (4.17) and (4.19), it is not hard to show that

\[
\int \mathcal{D}\psi \exp \left\{ \frac{1}{2} \sum_{\alpha,\beta,i,k} \hat{b}^{\alpha\beta}_{i k} \psi^\alpha_i \psi^\beta_k \right\} = P f_{(l,q)}(\hat{b}).
\]

In conclusion let us emphasize that for our generalization of the Berezin integration the invariance of volume form (4.16) with respect to transformation by matrix from \(SL_q(n, C)\) is essential. It is interest to construct the rules of the \(GL_q(n, C)\)-covariant Berezin integration. The results of research of this point we intend to publish in the following paper.
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