TORSION SUBGROUPS OF ELLIPTIC CURVES OVER QUADRATIC CYCLOTOMIC FIELDS IN ELEMENTARY ABELIAN 2-EXTENSIONS

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Abstract. Let \( K \) denote the quadratic field \( \mathbb{Q}(\sqrt{d}) \) where \( d = -1 \) or \(-3\) and let \( E \) be an elliptic curve defined over \( K \). In this paper, we analyze the torsion subgroups of \( E \) in the maximal elementary abelian 2-extension of \( K \).

1. Introduction

Finding the set of rational points on a curve is one of the fundamental problems in number theory. Given a number field \( K \) and an algebraic curve \( C/K \), the set, \( C(K) \), of points on \( C \) which are defined over \( K \), has the following properties depending on the genus of the curve.

1. If \( C \) has genus 0, then \( C(\mathbb{Q}) \) is either empty or infinite.
2. If \( C \) has genus greater than 1, then \( C(\mathbb{Q}) \) is either empty or finite.

(Faltings's theorem)

Assume \( C \) has genus 1. If \( C(K) \) is not empty, then it forms a finitely generated abelian group, proven by Luis Mordell and André Weil. Thus, given an elliptic curve \( E/K \), the group \( E(K) \) has the structure

\[ E(K)_{\text{tors}} \oplus \mathbb{Z}^r. \]

Here \( E(K)_{\text{tors}} \) is the finite part of this group \( E(K) \), called the torsion subgroup and the integer \( r \) is called the rank of \( E \) over the number field \( K \).

In this paper, we will study the elliptic curves over the quadratic cyclotomic fields and how their torsion subgroups grow in the compositum of all quadratic extensions of the base field.

First, we summarize the results obtained so far on \( E(K)_{\text{tors}} \) for a number field \( K \). Mazur [13] showed that the only groups that can be realized as the torsion subgroups of elliptic curves defined over \( Q \) are the following:

\[ \mathbb{Z}/m\mathbb{Z} \text{ for } 1 \leq m \leq 12, m \neq 11, \text{ or } \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2m\mathbb{Z} \text{ for } 1 \leq m \leq 4. \]

Similarly, the list for the torsion groups of elliptic curves defined over a quadratic field has been given by S. Kamienny [6], M.A. Kenku, and F. Momose [8].

\[ \mathbb{Z}/m\mathbb{Z} \text{ for } 1 \leq m \leq 18, m \neq 17, \text{ or } \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2m\mathbb{Z} \text{ for } 1 \leq m \leq 6, \]
\[ \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \text{ for } m = 1, 2, \text{ and } \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \]

If one fixes the quadratic field \( K \), it is very likely that one will have a smaller list. In fact, the groups \( \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) and \( \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \) are only realized when \( K = \mathbb{Q}(\sqrt{-3}) \) where as the group \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \) is only realized over the field \( \mathbb{Q}(i) \) since they contain the roots of unity for 3 and 4 respectively (See Weil pairing). On the other hand, Filip Najman [15] has proved that

\begin{enumerate}
  \item If \( K = \mathbb{Q}(i) \), then \( E(K)_{\text{tors}} \) is either one of the groups from Mazur’s theorem or \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \).
  \item If \( K = \mathbb{Q}(\sqrt{-3}) \), then \( E(K)_{\text{tors}} \) is either one of the groups from Mazur’s theorem or \( \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) or \( \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \).
\end{enumerate}

One may also ask how the torsion subgroups of elliptic curves over a given number field \( K \) grow over the compositum of all the quadratic extensions of \( K \).

Let \( F \) be the maximal elementary abelian two extension of \( K \), i.e.,

\[ F := K[\sqrt{d} : d \in \mathcal{O}_K] \]

where \( \mathcal{O}_K \) denotes the ring of integers of \( K \). The problem of finding \( E(F)_{\text{tors}} \) where \( K = \mathbb{Q} \) has been studied by Michael Laska, Martin Lorenz [11], and Yasutsugu Fujita [5, 4]. Laska and Lorenz described a list of 31 possible groups and Fujita proved that the list of 20 different groups is complete.

Our main theorem generalizes the results of Laska, Lorenz and Fujita to the case where \( K \) is a quadratic cyclotomic field. We find that (See Theorem 11.1)

\begin{enumerate}
  \item If \( K = \mathbb{Q}(i) \), then \( E(F)_{\text{tors}} \) is isomorphic to one of the following groups:
    \[ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z} \quad (N = 2, 3, 4, 5, 6, 8) \]
    \[ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4N\mathbb{Z} \quad (N = 2, 3, 4) \]
    \[ \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z} \quad (N = 2, 3, 4, 6, 8) \]
    or \( \{1\}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/15\mathbb{Z} \).
  \item If \( K = \mathbb{Q}(\sqrt{-3}) \), then \( E(F) \) is either isomorphic to one of the groups listed above or
    \[ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z}. \]
\end{enumerate}

We first study the points on various modular curves and use these results in §5 and in §7 to prove Theorem 7.6 which gives us a list of possible torsion subgroups. The main result of §5 is Proposition 5.5 where we give the possible odd order subgroups of \( E(F) \). §6 is concerned with finding non-trivial solutions to Fermat’s quartic equation which is crucial to rule out the subgroup \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z} \). All of these results together leads us to Theorem 7.6.

In §8 we analyze the growth of each non-cyclic torsion subgroup over the base field which helps us eliminate some of the groups given in Theorem 7.6. Analyzing the case where \( E(K) \) is cyclic, we obtain more restrictions on the torsion subgroups in §10 and we prove our main result Theorem 11.1.
2. Background

If $E$ is an elliptic curve given by the model
\[ y^2 = x^3 + ax + b, \]
then the quadratic twist of $E$ by $d \in K$ is
\[ E^{(d)} : y^2 = x^3 + ad^2 x + bd^3 \]
and it is isomorphic to $dy^2 = x^3 + ax + b$. Note that if $d$ is a square in $K$, then $E^{(d)}$ and $E$ are isomorphic over $K$.

To compute the rank of elliptic curves over quadratic fields, we make use of the following Lemma.

**Lemma 2.1** ([11, Corollary 1.3]). Let $d$ be a square-free integer. Then for an elliptic curve $E/Q$, the following holds:
\[ \text{rank}(E(Q(\sqrt{d}))) = \text{rank}(E(Q)) + \text{rank}(E^{(d)}(Q)). \]

Throughout the paper, we compute the rank of $E(Q)$ and $E^{(d)}(Q)$ on Magma [1] and use Lemma 2.1 to compute the rank of $E(Q(\sqrt{d}))$. The torsion subgroup of a given elliptic curve is also computed using Magma.

Let $K$ be a number field. Given an elliptic curve $E/K$, we call a subgroup $C$ of $E(C)$ as $K$-rational if there exists an elliptic curve $E'/K$ and an isogeny $\phi : E \to E'$ defined over $K$ such that $C = \ker \phi$. Equivalently, $C$ is $K$-rational if it is invariant under the action of $\text{Gal}(\bar{K}/K)$.

For our later purposes, we will state the following result of Newman, which tells us about the existence of $K$-rational subgroups of certain degrees in quadratic extensions of $K$.

**Theorem 2.2** ([17, Theorem 8]). Let $E/K$ be an elliptic curve. Then $E(C)$ has no $K$-rational cyclic subgroups of order $24$, $35$ or $45$ defined over a quadratic extension of $K$. Moreover, if $E$ is defined over $Q(\sqrt{-3})$, $E$ does not have a $K$-rational cyclic subgroup of order $20$, $21$ or $63$ defined over a quadratic extension of $Q(\sqrt{-3})$.

**Remark 2.3.** The proof of Theorem 8 in [17] shows that the modular curve $X_0(n)(K)$ has no non-cuspidal points for $n = 24$, $35$ or $45$. Moreover, if $K = Q(\sqrt{-3})$, then $X_0(20)(K)$ also does not have any non-cuspidal points.

We will use the following result to determine the odd torsion subgroup in an elementary abelian 2-extension of a field. Remember that the multiplication by $n$ map on an elliptic curve is denoted by $[n]$.

**Lemma 2.4** ([11, Corollary 1.3]). Let $E$ be an elliptic curve over the field $k$ and let $L$ be an elementary abelian 2-extension of $k$ of degree $2^m$, i.e.,
the Galois group of $L/k$ is an elementary abelian 2-group of rank $m$. If $E(k)_{2'} = \{ P \in E(k) : [n]P = 0 \text{ for some odd } n \}$, then
\[ E(L)_{2'} \cong E^{(d_1)}(k)_{2'} \oplus \ldots \oplus E^{(d_m)}(k)_{2'}, \]
for suitable $d_i : i = 1, \ldots, m$ in $\mathcal{O}_k$. Furthermore, the image of each summand $E^{(d)}(k)_{2'}$ is a $k$-rational subgroup of $E(L)$.

In §7, we will use the correspondence between the lattices in $\mathbb{C}$ and the elliptic curves over $\mathbb{C}$ to describe some explicit isogenies.

Now, let $L_1$ and $L_2$ be lattices inside $\mathbb{C}$ such that $\alpha L_1 \subset L_2$ for some $\alpha$ in $\mathbb{C}$. Then the map $\mathbb{C}/L_1 \to \mathbb{C}/L_2$ induced by
\[ z \mapsto \alpha z \]
defines an isogeny in $\text{Hom}(E_1, E_2)$ where $E_1, E_2$ are the elliptic curves corresponding to the lattices $L_1, L_2$, as in [22, VI, Proposition 3.6b]. We will denote this isogeny by $[\alpha]_{E_1, E_2}$. When $E_1 = E_2$, we will use $[\alpha]_{E_1}$ and we drop the domain and the target when it is clear from the context.

### 3. Elementary Abelian 2-Extensions

For the rest of the paper, let $K$ be the quadratic field $\mathbb{Q}(\sqrt{d})$ for $d = -1, -3$ and let $F$ be the field
\[ K(\sqrt{d} : d \in \mathcal{O}_K). \]
The field $F$ is called the maximal elementary abelian 2-extension of $K$ since its Galois group is an elementary abelian 2-group and it is maximal with respect to this property. Let $E/K$ be an elliptic curve given by $y^2 = f(x)$. We make the following quick observations.

1. If $f$ is irreducible in $K$, then it remains irreducible over the field $F$. Otherwise, $f$ has a root $\alpha$ in $F$ and the degree of $K(\alpha)$ over $K$ is divisible by 3 but it is not possible since $K(\alpha)$ is contained in $F$.

2. If $E(K)$ does not have a point of order 2, then $E(F)$ cannot have a point of order 2 either. This simply follows from the fact that the points of order 2 on the elliptic curve $E$ are given by the zeros of $f$, i.e.,
\[ \{(\alpha, 0) : f(\alpha) = 0\}. \]

Therefore, if $E(K)$ does not have a point of order 2, then $f$ is irreducible over $K$ and the claim follows from the first observation.

3. If the $j$-invariant of $E$ is not 0 or 1728, then for any elliptic curve $E'/K$ isomorphic to $E$, we have $E(F) \cong E'(F)$ since $E$ and $E'$ are isomorphic over a quadratic extension of $K$ (hence also over $F$).

4. Let $L$ be a finite elementary abelian extension of $K$. Then for a prime $\mathfrak{P} \in \mathcal{O}_L$ above $\mathfrak{p}$ in $\mathcal{O}_K$, the residue field of $\mathcal{O}_L$ is at most a quadratic extension of the residue field of $\mathcal{O}_K/\mathfrak{p}$ since the Galois group of the finite field $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ is cyclic for every $n$ and $q$. 
4. Rational Points on the Modular Curve $Y_0(n)$

**Proposition 4.1.** Let $E/K$ be an elliptic curve. Then $E(\mathbb{C})$ has no $K$-rational cyclic subgroups of order 32, 36 or 64.

*Proof.* An affine model for the modular curve $X_0(32)$ is given in [25, p 503] as

$$y^2 = x^3 + 4x.$$

Ogg’s theorem [19] tells us that $X_0(32)$ has the following cusps.

| $d$ | $\phi(d, 32/d)$ |
|-----|-----------------|
| 1   | 1               |
| 2   | 2               |
| 4   | 4               |
| 8   | 8               |
| 16  | 16              |
| 32  | 32              |

We see that $X_0(32)$ has 8 cusps; four of them defined in $\mathbb{Q}$ and the other four defined in $\mathbb{Q}(i)$. We compute on Magma that $X_0(32)$ has only 8 points over $\mathbb{Q}(i)$ and has 4 points over $\mathbb{Q}(\sqrt{-3})$, hence they are all cuspidal.

This shows that there are no elliptic curves $E/K$ with a $K$-rational cyclic subgroup of order 32.

Similarly, an affine model for the modular curve $X_0(36)$ is also given in [25, p 503] as

$$y^2 = x^3 + 1$$

We again apply Ogg’s method to compute the cusps of the modular curve $X_0(36)$.

| $d$ | $\phi(d, 36/d)$ |
|-----|-----------------|
| 1   | 1               |
| 2   | 2               |
| 3   | 3               |
| 4   | 4               |
| 6   | 6               |
| 9   | 9               |
| 12  | 12              |
| 18  | 18              |
| 36  | 36              |

The table shows that $X_0(36)$ has 12 cusps; six of them defined over $\mathbb{Q}$ and the remaining six are defined over $\mathbb{Q}(\sqrt{-3})$. We find that it has 6 points over $\mathbb{Q}(i)$ and 12 points over $\mathbb{Q}(\sqrt{-3})$.

This shows that there are no non-cuspidal $K$-points on $X_0(36)$ and so there are no elliptic curves $E/K$ with a $K$-rational cyclic subgroup of order 36.

Finally, $E(\mathbb{C})$ has no $K$-rational cyclic subgroup of order 64 since otherwise it would induce a $K$-rational cyclic subgroup of order 32. $\square$

Now, we will study the $K$-rational points on $X_0(20)$ and $X_0(27)$ to prove Proposition 5.5 and also to prove the results of §7 later.

### 4.1. The modular curve $X_0(20)$

Let $K = \mathbb{Q}(i)$. An equation for the modular curve $X_0(20)$ is given in [25] as

$$y^2 = (x + 1)(x^2 + 4).$$

It is known that there are no cyclic 20-isogenies defined over $\mathbb{Q}$, see Theorem 2.1 in [11], hence $X_0(20)(\mathbb{Q})$ has only cusps. Ogg’s method tells us that there are only 6 of them. Then we compute on Magma that $X_0(20)(K)$ has 12 points and they are listed as

$\{O, (-1, 0), (0, \pm 2), (4, \pm 10), ((\pm 2i, 0), (2i - 2, \pm (2i + 4)), (-2i - 2, \pm (2i - 4))\}$.
This shows that there are 6 non-cuspidal points on \( X_0(20)(K) \). We will study these points in more detail in Proposition 7.2.

4.2. The modular curve \( X_0(27) \). A model for the modular curve \( X_0(27) \) is given in [25] as

\[ y^2 + y = x^3 - 7. \]

Again by Ogg’s method, we find that \( X_0(27) \) has 6 cusps; four of them defined over \( \mathbb{Q}(\sqrt{-3}) \) and the other two defined over \( \mathbb{Q} \).

Now let \( K = \mathbb{Q}(\sqrt{-3}) \). We compute that \( E(K) \) has 9 points and hence there are 3 non-cuspidal points on \( X_0(27) \) defined over \( K \).

Similarly if \( K = \mathbb{Q}(i) \), the group \( E(K) \) is also finite and it has 3 points which shows that there is only one non-cuspidal point on \( X_0(27) \) defined over \( \mathbb{Q}(i) \), in fact defined over \( \mathbb{Q} \).

Let \( E_1 \) be the elliptic curve associated with the lattice \([1, \frac{1+\sqrt{-27}}{2}]\). Then

\[ \sqrt{-27} : E_1 \to E_1 \]

and

\[ \frac{9 + \sqrt{-27}}{2} : E_1 \to E_1 \]

define endomorphisms of \( E_1 \) and they are cyclic of degree 27. Moreover \( E_1 \) has complex multiplication by the order \( \mathbb{Z}[\frac{1+\sqrt{-27}}{2}] \) and it is given in [3, p.261] that

\[ j(E_1) = -2^{15}5^33. \]

Therefore the endomorphisms listed above are defined over \( \mathbb{Q}(\sqrt{-3}) \) by [21, Theorem 2.2]. We find a model for \( E_1 \) in the database [2]; the elliptic curves over \( \mathbb{Q} \) with complex multiplication, as

\[ y^2 + y = x^3 - 270x - 1708. \]

Hence any elliptic curve defined over \( K \) with a \( K \)-rational cyclic subgroup of order 27 is a quadratic twist of \( E_1 \).

5. Odd Torsion

Using Lemma 2.4, we see that the odd primes dividing the order of a point in \( E(F) \) can only be 3, 5 or 7. We will prove in Proposition 5.1 and Proposition 5.4 that \( E(F) \) does not have a point of order 21 for \( K = \mathbb{Q}(\sqrt{d}) \) for \( d = -1, -3 \).

Proposition 5.1. Let \( E \) be an elliptic curve defined over \( K = \mathbb{Q}(\sqrt{-3}) \). Then \( E(F) \) has no point of order 21.

Proof. Assume \( E(F) \) has a subgroup of order 21. Then by Lemma 2.4 replacing \( E \) by a twist if necessary, we may assume that \( E(K) \) has a point of order 3 and \( E^{(d)}(K) \) has a point of order 7 for some \( d \) in \( \mathcal{O}_K \), hence \( E \) has a subgroup of order 21 over a quadratic extension of \( K \) and it is \( K \)-rational by Lemma 2.4 Theorem 2.2 shows that it is not possible. \( \square \)
Remark 5.2. The modular curve $X_0(21)$ is an elliptic curve with Mordell-Weil rank 1 over $\mathbb{Q}(i)$. Hence $X_0(21)$ cannot be immediately used to determine whether an elliptic curve $E$ can have a subgroup of order 21 defined over the field $F$.

We will need the following result in the proof of Proposition 5.4.

**Theorem 5.3** ([17, Theorem 7]). Let $K$ be a quadratic field and let $E/K$ be an elliptic curve. If $j(E) = 0$ and $p > 3$ is a prime, then $E(K)_{\text{tor}}$ has no element of order $p$. If $j(E) = 1728$ and $p > 2$ is a prime, then $E(K)_{\text{tor}}$ has no element of order $p$.

**Proof.** The proof uses the techniques from [12]. □

**Proposition 5.4.** Let $E$ be an elliptic curve defined over $\mathbb{Q}(i)$. Then $E(F)$ does not have a subgroup of order 21.

**Proof.** Let $E$ be an elliptic curve defined over $K = \mathbb{Q}(i)$ and suppose that $E(F)$ has a subgroup of order 21. We may assume that $E(K)$ has a point of order 7 (by replacing with a twist if necessary) by Lemma 2.4. It can be found in [10, Table 3, p 217] that an elliptic curve with a point of order 7 is isomorphic to

$$E_t : y^2 + (1 - c)xy - by = x^3 - bx^2$$

where $b = t^3 - t^2$ and $c = t^2 - t$ for some $t \neq 0, 1$ in $K$. Therefore $E$ is isomorphic to $E_t$ for some $t \in K$. Moreover, either the $j$ invariant of $E$ is 0 or 1728, or the isomorphism is defined over a quadratic extension of $K$. By [17, Theorem 7], we know that an elliptic curve with $j$ invariant 0 or 1728 cannot have a $K$-point of order 7, hence $E$ is a quadratic twist of $E_t$ and if $E(F)$ has a point of order 21, then so has $E_t$; hence we may assume $E$ is $E_t$.

We compute the third division polynomial of the elliptic curve $E_t$ as the following.

$$\psi(x, t) = x^4 + \left(\frac{1}{3} t^4 - 2t^3 + t^2 + \frac{2}{3} t + \frac{1}{3}\right) x^3 + \left(t^5 - 2t^4 + t^2\right) x^2 + \left(t^6 - 2t^5 + t^4\right) x + \left(-\frac{1}{3} t^9 + t^8 - t^7 + \frac{1}{3} t^6\right).$$

Now $E$ has a point $P$ of order 3 defined in a quadratic extension of $K$ and the subgroup generated by $P$ is $K$-rational by Lemma 2.4. We claim that $x(P)$, the $x$-coordinate of the point $P$ must be in $K$ which forces the equation $\psi(x, t) = 0$ to have a root in $K$. The only points in $\langle P \rangle$ are $P$, $-P$ and the point at infinity. Since $x(P) = x(-P)$, it follows that $\sigma(x(P)) = x(\sigma(P)) = x(P)$ and hence $x(P)$ is invariant under the action of the Galois group. Now, the pair $(E, P)$ corresponds to a point $(x_0, s)$ on the curve $C$ given by the equation

$$C : \psi(x, t) = 0$$
where $E = E_s$ and $x_0$ is the $x$-coordinate of the point $P$ of order 3. Therefore it is enough to find $C(K)$, the set of $K$-points on $C$. The curve $C$ is birational (over $\mathbb{Q}$) to the hyperelliptic curve

$$\bar{C} : y^2 = f(u)$$

where

$$f(u) = u^8 - 6u^6 + 4u^5 + 11u^4 - 24u^3 + 22u^2 - 8u + 1$$

and $C$ and $\bar{C}$ are isomorphic over $K$ outside the set of singularities which is $\{(0, 0), (0, 1)\}$. Note that we require $t$ to be different than 0 or 1 in $[1]$, hence it is enough to find $\bar{C}(K)$. The polynomial $f(u)$ factors as

$$f(u) = (u^2 - u + 1)(u^6 + u^5 - 6u^4 - 3u^3 + 14u^2 - 7u + 1)$$

Let $g$ and $h$ denote the factors of $f$:

$$g(u) = u^2 - u + 1$$

$$h(u) = u^6 + u^5 - 6u^4 - 3u^3 + 14u^2 - 7u + 1.$$  

Using the Descent Theorem ([23, Theorem 11]; one can also look at Example 9 and 10 in [23]), it is enough to find the points on the unramified coverings $\tilde{C}_d$ of $\bar{C}$, which are given as the intersection of two equations in $K^3$:

$$w^2 = dg(u) = d(u^2 - u + 1)$$

and

$$z^2 = dh(u) = d(u^6 + u^5 - 6u^4 - 3u^3 + 14u^2 - 7u + 1)$$

where $d$ is a square-free number in $O_K$ dividing the resultant of $g(u)$ and $h(u)$, which is 112. Therefore $d$ belongs to the set

$$\{1, i, (1 + i), 7, i(1 + i), 7(1 + i), 7i, 7i(1 + i)\}$$

If we exclude the cases $d = 1$ and $d = 7i$, reduction of $d$ takes values $\{2, 3, 2, 1, 1, 2\}$ and $\{3, 4, 2, 2, 3, 4\}$ with respect to the ideal $(2 - i)$ and $(2 + i)$ respectively. We will reduce the curve

$$z^2 = d(u^6 + u^5 - 6u^4 - 3u^3 + 14u^2 - 7u + 1)$$

at $(2 - i)$ for the values of $d = i, (1 + i), 7, 7i(1 + i)$ and similarly reduce it at $(2 + i)$ for the values $d = i(1 + i), 7(1 + i)$. In each case described above, $z^2 = dh(u)$ reduces to either

$$z^2 = 2(u^6 + u^5 - 6u^4 - 3u^3 + 14u^2 - 7u + 1).$$

or

$$z^2 = 3(u^6 + u^5 - 6u^4 - 3u^3 + 14u^2 - 7u + 1).$$

A quick computation on Magma shows that neither of these equations has a solution over $\mathbb{F}_5$, hence there are no $K$-points on $\tilde{C}_d$ for $d \neq 1, 7i$.

Let $d = 7i$. Magma computes 0 as an upper bound for the Mordell-Weil rank of the Jacobian of the curve

$$z^2 = 7i \left( u^6 + u^5 - 6u^4 - 3u^3 + 14u^2 - 7u + 1 \right),$$
hence the rank of the Jacobian of $z^2 = 7i h(u)$ is zero. Moreover, we compute on Magma that the Jacobian of the hyperelliptic curve $z^2 = (7i) h(u)$ has 79 and 171 points respectively over the finite fields $\mathbb{F}_5$ and $\mathbb{F}_{13}$ (reduced at $(2 - i)$ and $(2 - 3i)$) which proves that the torsion subgroup of the Jacobian of $z^2 = (7i) h(u)$ over $K$ is trivial. Therefore $\tilde{C}_d(K) = \emptyset$ for $d = 7i$. Hence, if there is a point on the curve $\tilde{C}(K)$, it must arise from the covering $\tilde{C}_1(K)$.

Now we may assume that $d = 1$. We would like to find the $K$-points on the curve

$$C_2 : z^2 = h(u) = u^6 + u^5 - 6u^4 - 3u^3 + 14u^2 - 7u + 1.$$  

Magma computes that the rank of $J(C_2)(\mathbb{Q})$ is 2 and also that 2 is an an upper bound for $J(C_2)(K)$, therefore the rank of $J(C_2)$ over $K$ is equal to its rank over $\mathbb{Q}$.

Similar to the case $d = 7i$, the reduction of $C_2$ at the good primes $(2 - i)$ and $(2 - 3i)$ has 79 and 171 points over $\mathbb{F}_5$ and $\mathbb{F}_{13}$ respectively, hence $J(C_2)(K)_{\text{tors}}$ is also trivial.

Let $J$ denote $J(C_2)$. We claim that $J(\mathbb{Q}) = J(K)$. Since $J(\mathbb{Q})$ has rank 2, we can find $x, y \in J(\mathbb{Q})$ such that $J(\mathbb{Q})$ is generated by $x$ and $y$ as an abelian group. Let $\sigma$ be the generator of the $\text{Gal}(\bar{K}/\mathbb{Q})$ and assume that $\sigma(x) = ax + by$ for some $a, b \in \mathbb{Z}$.

Let $x' = nx + my$ and $y' = rx + sy$ be the generators of $J(\mathbb{Q})$. Then $sx' - my'$ is not zero since $x', y'$ are the generators of a free abelian group. So a multiple of $x$ (namely $(sn - mr)x$) is in $J(\mathbb{Q})$ and it is fixed by $\sigma$. Then we obtain $lax + lby = lx$ (we use $l$ for $sn - mr$ above to simplify the notation). Since $J(\mathbb{Q})$ has trivial torsion, it implies that $(a - 1)x + by = 0$. We conclude that $a = 1$ and $b = 0$, i.e., $x$ in $J(\mathbb{Q})$. A similar argument shows that $y$ is also in $J(\mathbb{Q})$, consequently $J(K) = J(\mathbb{Q})$.

We claim that $C_2(\mathbb{Q}) = C_2(K)$. Let $P$ be a point in $C_2(K)$. If $P_0$ denotes the point $[0 : 1 : 1]$ on $C_2$, then $[P - P_0]$ represents a point in $J(K)$ which equals to $J(\mathbb{Q})$. If $P'$ denotes the Galois conjugate of $P$, then $[P' - P_0]$ must be equal to $[P - P_0]$, since a point in $J(\mathbb{Q})$ is invariant under the action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Hence $P = P'$ and $P$ is in $C(\mathbb{Q})$ which proves that $C_2(\mathbb{Q}) = C_2(K)$.

Now we will show that $\tilde{C}(\mathbb{Q}) = \tilde{C}(\mathbb{Q}(i))$. Remember that the curve $\tilde{C}$ is given by

$$y^2 = u^6 - 6u^6 + 4u^5 + 11u^4 - 24u^3 + 22u^2 - 8u + 1$$

and we showed by the Descent theorem that a point $(u, y)$ in $\tilde{C}(K)$ corresponds to a point on the intersection of

$$C_1 : w^2 = u^2 - u + 1$$

and

$$C_2 : z^2 = u^6 + u^5 - 6u^4 - 3u^3 + 14u^2 - 7u + 1$$
such that \( y = wz \) where \((u, w) \in C_1(K)\) and \((u, z) \in C_2(K)\). The first equation
\[
w^2 = u^2 - u + 1 = (u - 1/2)^2 + 3/4
\]
implies that if \((u, w)\) is a point on \(C_1(K)\) with \(u\) in \(Q\), then \(w\) is also in \(Q\). Hence, we showed that if \((u, y)\) is a point on \(\tilde{C}(K)\), then \(u, z\) are both in \(Q\) since \(C_2(K) = C_2(Q)\) and by the above argument, \(w\) is also in \(Q\). Therefore \(y = wz\) is also in \(Q\).

To summarize, we proved our claim that \(\tilde{C}(K) = \tilde{C}(Q)\). This implies that a pair \((E, P)\) on \(C(K)\) corresponds to a point \((u, y)\) on \(\tilde{C}(Q)\) and therefore to a point in \(C(Q)\). However, if \(E\) is defined over \(Q\) and \(x(P) \in Q\), then \(P\) is in \(E(Q(\sqrt{d}))\) for some \(d \in Q\). We know by [11] that \(E\) does not have a subgroup of order 21 defined over a quadratic extension of \(Q\). Therefore there is no elliptic curve \(E\) defined over \(K = Q(i)\) such that \(\mathbb{Z}/21\mathbb{Z} \subset E(F)\). □

**Proposition 5.5.** Let \(K\) be a quadratic cyclotomic field and let \(E\) be an elliptic curve defined over \(K\). Then \(E(F)_{2'}\) is isomorphic to one of the following groups:
\[
\mathbb{Z}/N\mathbb{Z}\text{ for } N \in \{1, 3, 5, 7, 9, 15\}\text{ or } \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.
\]

*Proof.* By Lemma 2.4 and [15] Theorem 2, we see that the odd numbers dividing the order of \(E(F)_{\text{tors}}\) are products of 3, 5, 7 and 9. Since \(F\) does not contain a primitive \(n\)-th root of unity for \(n = 5, 7\) or 9, \(\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}\) or \(\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}\) can not be isomorphic to a subgroup of \(E(F)\) by the Weil pairing.

If \(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}\) or \(\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}\) is a subgroup of \(E(F)\), then it is Galois invariant by Lemma 2.4, hence by Proposition 5.4 and Proposition 5.1, this is not possible. (Note that if \(E\) has a Galois invariant subgroup of order 63, then it also has a Galois invariant subgroup of order 21 which is not possible for \(\mathbb{Q}(i)\) by Proposition 5.4 and for \(\mathbb{Q}(\sqrt{-3})\) by Proposition 5.1.) Similarly, \(\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}\) and \(\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}\) are also not possible by Proposition 2.2 (See also Remark 2.3 following Proposition 2.2).

If \(E(F)\) contains a subgroup isomorphic to \(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}\), then by Lemma 2.4, we may assume that \(E(K)\) has a point \(P\) of order 9 and has an additional \(K\)-rational subgroup \(C\) of order 3 arising from a twist, in other words, the subgroup \(C\) is defined in a quadratic extension \(K(\sqrt{d})\) of \(K\). We will show that \(E[3]\) is a subset of \(E(K)\) and obtain a contradiction since \(E(K)\) can not have a subgroup of order 27 by [15] Theorem 2.

Let \(\sigma\) be the generator of the Galois group of \(K(\sqrt{d})\) over \(K\). Then the image of \(\sigma\) under the map (Galois action on the set of 3-torsion points)
\[
\text{Gal}(K(\sqrt{d})/K) \to \text{GL}_2(\mathbb{F}_3)
\]
is \[
\begin{bmatrix}
1 & \alpha \\
0 & \beta
\end{bmatrix}
\]
for some \(\alpha, \beta\) in \(\mathbb{F}_3\). Note here that \(\beta\) is 1 modulo 3 since \(K\) contains a third root of unity \(\zeta_3\), \(\beta\) is the determinant of the matrix of \(\sigma\).
and $\sigma(\zeta_3) = \zeta_3^3$. The fact that $\sigma$ has order 2 tells us that $\alpha = 0$ and $E[3]$ is a subset of $E(K)$ as we claimed.

Hence $E(F)$ can not have a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$.

Finally, $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ can not be isomorphic to a subgroup of $E(F)$ either. Otherwise, $E(F)$ has a $K$-rational subgroup of order 15 and by [16, Lemma 7], $E$ is isogenous to an elliptic curve with a cyclic $K$-rational subgroup of order 45 contradicting Proposition 2.2. See [11, Proposition 2.2] for the case $K = \mathbb{Q}$.

\[ \square \]

6. Solutions to the Equation $x^4 + y^4 = 1$.

In this section, we will find points on the affine curve $x^4 + y^4 = 1$ which is an affine model for $Y_0(64)$ [7, Proposition 2]. The results of Theorem 6.2 will be used in Proposition 7.3.

We will call a solution $(x,y)$ (resp. $(x,y,z)$) of a Diophantine equation trivial if $xy = 0$ (resp. $xyz = 0$) and non-trivial otherwise.

Lemma 6.1. Let $K = \mathbb{Q}(\sqrt{d})$ for $d = -1$ or $-3$.

1. The only solutions of $x^4 + y^2 = 1$ defined over $K$ are trivial.
2. The only solutions of $x^4 - y^4 = z^2$ defined over $K$ are trivial.
3. The only solutions of $x^4 + y^4 = z^2$ defined over $\mathbb{Q}(i)$ are trivial.

Proof.

1. We can find a rational map between $C : x^4 + y^2 = 1$ and $E : y^2 = x^3 + 4x$ such that

$$ f : (x,y) \mapsto \left( \frac{2x^2}{1-y}, \frac{4x}{1-y} \right). $$

Since the $K$-valued points on $C$ maps to $K$-valued points on $E$, it is enough to look at the inverse images of $K$-points on $E$ and the points where the map $f$ is not defined. We previously computed in the proof of Proposition 4.1 that

$$ E(\mathbb{Q}(\sqrt{-3})) = \{(0,0), (2,\pm 4)\} $$

and

$$ E(\mathbb{Q}(i)) = \{(0,0), (2,\pm 4), (\pm 2i, 0), (-2, \pm 4i)\} $$

We easily compute that the inverse image of $E(\mathbb{Q}(i))$ under $f$ is the set

$$ \{(0,-1), (\pm i, 0), (\pm 1, 0)\}. $$

The points where $f$ is not defined are the points where $y = 1$ and so we obtain the point $(0,1)$. All of the solutions we found are trivial.

2. Let $(a, b, c)$ be a solution to $x^4 - y^4 = z^2$ over $K$. Assume $a^2 \neq c$, then $\left( \frac{2b^2}{a^2-c}, \frac{4ab}{a^2-c} \right)$ is a point on $E : y^2 = x^3 + 4x$. As we did earlier, we simply compute the points $(a, b, c)$ for each point in $E(K)$ (which is described in the first part) and the points $(a, b, c)$ with $a^2 = c$ to show that $abc = 0$. 


(3) Let
\[ f : (x^4 + y^4 = z^2) \rightarrow (y^2 = x^3 - 4x) \]
\[ (x, y, z) \mapsto \left(\frac{-2x^2}{y^2 - z}, \frac{4xy}{y^2 - z}\right) \]
A quick computation on Magma shows that the affine curve \( y^2 = x^3 - 4x \) has only 3 points defined over \( \mathbb{Q}(i) \) and they are \((0, 0)\) and \((\pm 2, 0)\). Hence if \((a, b, c)\) is a solution to \( x^4 + y^4 = z^2 \) over \( \mathbb{Q}(i) \), then either \( a = 0 \) or \( b = 0 \). Notice also that if \( f \) is not defined at \((a, b, c)\), then \( a = 0 \). Hence the only solutions defined over \( \mathbb{Q}(i) \) are trivial.

\[ \square \]

**Theorem 6.2.** Let \( K = \mathbb{Q}(i) \). Assume that \( x^4 + y^4 = 1 \) has a solution in a quadratic extension \( L \) of \( K \). Then \( L = K(\sqrt{-7}) \) and the only solutions are
\[ (\epsilon_1, 0), (\pm 1, 0), (0, \epsilon_1), (0, \pm 1) \]
where
\[ \epsilon_{1,2} = \pm i \text{ or } \pm 1, \quad \epsilon_3 = \pm 1, \quad \text{and} \quad i = \sqrt{-1}. \]

**Proof.** Mordell (Chapter 14, Theorem 4 of [14]) proves that if \( x^4 + y^4 = 1 \) has a nontrivial solution in a quadratic extension \( L \) of \( \mathbb{Q} \), then \( L = \mathbb{Q}(\sqrt{-7}) \). We will use his technique to show that this result still holds if one replaces \( \mathbb{Q} \) with \( \mathbb{Q}(i) \). Let \( L \) be a quadratic extension of \( K \) and let \((a, b)\) be a solution in \( L \). Then we can find \( t \in L \) such that
\[ a^2 = \frac{1 - t^2}{1 + t^2}, \quad b^2 = \frac{2t}{1 + t^2}. \]

We will analyze the equation in two cases:

1. \( t \) is in \( K \)
2. \( t \) is not in \( K \).

Assume \( t \) is in \( K \). If \( a \) (resp. \( b \)) is in \( K \), then \((a, b^2)\) (resp. \((a^2, b)\)) gives a solution to the equation \( x^4 + y^2 = 1 \) (resp. \( x^2 + y^4 = 1 \)) and Lemma 6.1 tells us that \((a, b)\) is trivial. If neither \( a \) nor \( b \) is in \( K \), then \( a = a_1 \sqrt{w} \) and \( b = b_1 \sqrt{w} \) for some \( a_1, b_1, \text{ and } w \) in \( K \) since \( a^2, b^2 \in K \), hence \((a_1, b_1, 1/w)\) is a solution to \( x^4 + y^4 = z^2 \). Again by Lemma 6.1, this is not possible.

Assume \( t \notin K \). Since \( t \) is in \( L \), \( K(t) \) is contained in \( L \) and thus \( L = K(t) \).

Let \( F(z) \) be the minimal polynomial of \( t \) over \( K \) and let us define \( X, Y \) as follows:
\[ X = (1 + t^2)ab, \quad Y = (1 + t^2)b \]
Then, it is easy to see that \( X^2 = 2t(1 - t^2) \) and \( Y^2 = 2t(1 + t^2) \). Since \( X, Y \) are in \( L \), then there are \( c, d, e, f \in K \) such that
\[ X = c + dt, \quad Y = e + ft \]
Let us define the polynomials $g(z)$ and $h(z)$ as follows:

$$g(z) = (c + dz)^2 - 2z(1 - z^2)$$

$$h(z) = (e + fz)^2 - 2z(1 + z^2).$$

Then $g(t) = h(t) = 0$ because $X^2 = 2t(1 - t^2)$ and $Y^2 = 2t(1 + t^2)$. Since $F(z)$ is the minimal polynomial of $t$ over $K$, $F(z)$ must divide both $g(z)$ and $h(z)$. It follows that $g$ and $h$ both have exactly one root over $K$ (not necessarily the same root) since $g$ and $h$ are cubic polynomials. Let $u$ and $v$ denote the roots of $g$ and $h$ respectively; that is,

$$g(z) = 2(z - u)F(z) \text{ and } h(z) = -2(z - v)F(z).$$

Then $(-2u, 2(c + du))$ is a point on the affine curve $E_1 : y^2 = x^3 - 4x$. We previously computed the points on $E_1(K)$ in the proof of Proposition 6.1. There are three of them: $(0, 0), (2, 0), (-2, 0)$, hence $u$ is either $0, 1$ or $-1$.

Similarly, $(2v, 2(e + f v))$ is a point on the affine curve $E_2 : y^2 = x^3 + 4x$. Thus using our previous computations of $E_2(K)$, we see that the only possible solutions for $(2v, 2(e + f v))$ are

$$\{(0, 0), (2i, 0), (-2i, 0), (2, 4), (2, -4), (-2, 4i), (-2, -4i)\}.$$

This shows that $v$ is either $0, \pm 1$ or $\pm i$. We will first compute $F(z)$ for each value of $u$ using the equality $g(z) = 2(z - u)F(z) = (c + dz)^2 - 2z(1 - z^2)$.

1. If $u = 0$, then $g(0) = c^2 = 0$ and hence $c = 0$. We obtain that

$$g(z) = 2z(z^2 + \frac{d^2}{2}z - 1)$$

and

$$F(z) = z^2 + \frac{d^2}{2}z - 1.$$  

We compute $h(z) = -2(z - v)(z^2 + \frac{d^2}{2}z - 1)$. Notice that the constant term of $h(z)$ is $-2v$. On the other hand, $h(z) = (e + fz)^2 - 2z(1 + z^2)$ and hence the constant term is $e^2$. Therefore $e^2 = -2v$ and this is satisfied only if $v = 0$ or $\pm i$.

(a) If $v = 0$, then

$$F(z) = z^2 - \frac{f^2}{2}z + 1$$  

which contradicts with $F(z) = z^2 + \frac{d^2}{2}z - 1$ since the constant terms are not equal.

(b) If $v = i$, then using the equality $h(i) = 0$, we find that

$$F(z) = z^2 + z(-\frac{f^2}{2} + i) + i \frac{f^2}{2} = z^2 + \frac{d^2}{2}z - 1$$

and hence $-\frac{f^2}{2} + i = \frac{d^2}{2}$ and $i \frac{f^2}{2} = -1$. It follows that $f^2 = 2i$ and $d = 0$ and we arrive at a contradiction since $F$ has to be irreducible over $K$.  

(c) If \( v = -i \), then we find that
\[
F(z) = z^2 + z\left(-\frac{f^2}{2} - i\right) - i\frac{f^2}{2} = z^2 + \frac{d^2}{2}z - 1
\]
which forces \( d \) to be 0 and we arrive at a contradiction.

(2) If \( u = 1 \), then \( g(1) = (c + d)^2 = 0 \) and hence \( c = -d \). We see that
\[
g(z) = d^2(z - 1) + 2z(z^2 - 1)
\]
and we find that
\[
F(z) = z^2 + z\left(\frac{d^2}{2} + 1\right) - \frac{d^2}{2}
\]
which tells us that the constant term of \( h(z) \) is \( e^2 = -d^2v \). This equation has solutions in \( K \) only if \( v = 0 \) or \( v = \pm 1 \).

(a) If \( v = 0 \), we computed \( F(z) \) in (2). We obtain
\[
F(z) = z^2 + z\left(\frac{d^2}{2} + 1\right) - \frac{d^2}{2}.
\]
The equality of the constant terms gives us \( -d^2 = 2 \) and we obtain a contradiction.

(b) If \( v = 1 \), then \((e + f)^2 = 2(1 + 1) = 4 \). Now the long division of \( h(z) \) by \((z - 1)\) gives us that
\[
F(z) = z^2 - \left(\frac{f^2}{2} - 1\right)z + \frac{e^2}{2}.
\]
On the other hand, \( F(z) = z^2 + z\left(\frac{d^2}{2} + 1\right) - \frac{d^2}{2} \), thus \( e^2 = -d^2 = f^2 \) and since \((e + f)^2 = 4 \), we get \( e^2 = f^2 = 1 \) and
\[
F(z) = z^2 + z\left(\frac{d^2}{2} + 1\right) - \frac{d^2}{2}.
\]

(c) Similarly if \( v = -1 \), we obtain \((e - f)^2 = -4 \). Similar to the previous part, the long division of \( h(z) \) by \((z + 1)\) produces
\[
F(z) = z^2 - \left(\frac{f^2}{2} + 1\right)z - \frac{e^2}{2}.
\]
On the other hand, \( F(z) = -2(z + 1)(z^2 + z\left(\frac{d^2}{2} + 1\right) - \frac{d^2}{2}) \).
\[
e^2 = d^2 \text{ and } d^2 + f^2 = -4
\]
which implies that \( e = 0 \) or \( f = 0 \). If \( e = 0 \), then \( d = 0 \) and we get a contradiction. If \( f = 0 \), then \( e^2 = d^2 = -4 \) and we compute
\[
F(z) = z^2 - z + 2.
\]

(3) If \( u = -1 \), then \( g(-1) = (c - d)^2 = 0 \) and hence \( c = d \). We see that
\[
F(z) = z^2 + z\left(\frac{d^2}{2} - 1\right) + \frac{d^2}{2}.
\]
Constant coefficient of \( h(z) \) equals to \( 2v\left(\frac{d^2}{2}\right) = e^2 \), thus \( v \) has to be a square in \( K \). Therefore, \( v \) can be 0 or \( \pm 1 \) and we computed \( F(z) \) for each of these values in (2), (3), and (5).
(a) If \( v = 0 \), then
\[
F(z) = z^2 + z \left( \frac{d^2}{2} - 1 \right) + \frac{d^2}{2} = F(z) = z^2 - \frac{f^2}{2} z + 1.
\]
we find \( d^2 = 2 \) which is not possible since \( d \) is in \( K \).
(b) If \( v = 1 \), then
\[
F(z) = z^2 + z \left( \frac{d^2}{2} - 1 \right) + \frac{d^2}{2} = z^2 - \left( \frac{f^2}{2} - 1 \right) z + \frac{e^2}{2}
\]
which implies that \( d^2 = e^2 \) and \( d^2 + f^2 = 4 \). We see that \( f = 0 \)
and \( d^2 = 4 \). In this case, we compute
\[
F(z) = z^2 + z + 2.
\] (7)
(c) If \( v = -1 \), then we have
\[
F(z) = z^2 + z \left( \frac{d^2}{2} - 1 \right) + \frac{d^2}{2} = z^2 - \left( \frac{f^2}{2} + 1 \right) z - \frac{e^2}{2}.
\]
In this case, we compute that \( d^2 = 1 \) and hence
\[
F(z) = z^2 - z + 1 + \frac{1}{2}.
\] (8)
To summarize, we showed that \( F(z) \) is one of the following polynomials:
\[
z^2 + \frac{z}{2} + \frac{1}{2}, \quad z^2 - z + 2, \quad z^2 + z + 2, \quad z^2 - \frac{z}{2} + \frac{1}{2}.
\]
One can easily check that the splitting field \( L/K \) of each polynomial listed
above is \( K(\sqrt{-7}) \). Now we will find all non-trivial solutions to the equation
\( x^4 + y^4 = 1 \) over the field \( L = K(\sqrt{-7}) \).
Remember that we started with a solution \((a, b)\) in \( L \) and constructed \( X \)
and \( Y \). It is easy to see that \( a = X/Y = \frac{c + dt}{e + ft} \) and \( b = Y / (1 + t^2) = \frac{e + ft}{1 + t^2} \).
In the following, we use the notation \( w = \sqrt{-7} \) for simplicity. Also \( \delta_j \)
for \( j = 1, 2, 3 \) denote the integers such that \( \delta_j^2 = 1 \).
(1) We found \( F(z) = z^2 + z + 2 \) in (7) with conditions that \( f = 0 \),
\( d^2 = e^2 = 4 \) and \( c = d \). The roots of \( F(z) \) are \( t = (-1 + \delta_3 w)/2 \)
and we compute
\[
a = \delta_1(1 + \delta_3 w)/2.
\]
Then we find \( b \) as \( \delta_2(-1 + \delta_3 w)/2 \).
(2) Similarly, we obtained \( F(z) = z^2 - z + 2 \) in (6) with the conditions
\( c = -d, e^2 = d^2 = -4 \) and \( f = 0 \). In this case, we find \( t = (1 + \delta_3 w)/2 \)
and
\[
a = \delta_1(-1 + \delta_3 w)/2.
\]
Similar to the first case, \( b = i\delta_2(1 + \delta_3 w)/2 \).
(3) The polynomial \( F(z) = z^2 + \frac{z}{2} + \frac{1}{2} \) we found in (4) produces \( t =
(-1 \pm w)/4 \) and we compute
\[
a = \delta_1 i(-1 + \delta_3 w)/2, \quad b = \delta_2(1 + \delta_3 w)/2.
\]
(4) Similarly, the polynomial $F(z) = z^2 - \frac{5}{2} + \frac{1}{2}$ in $[8]$ produces $t = (1 \pm w)/4$ and

$$a = \delta_1 i(1 + \delta_3 w)/2, \quad b = \delta_2 i(-1 + \delta_3 w)/2.$$  

Hence the result follows.  

\[ \square \]

**Remark 6.3.** Unfortunately, the method of the proof of Theorem 6.2 does not apply to the solutions over the field $\mathbb{Q}(\sqrt{-3})$ since there are non-trivial solutions to the equation $u^4 + v^4 = z^2$ over $\mathbb{Q}(\sqrt{-3})$.

7. Restrictions on the Torsion Subgroups

In the proof of Proposition 7.2, we will explicitly describe three cyclic 20 isogenies. To show that their field of definition is $\mathbb{Q}(i)$, we will first need the following result.

**Proposition 7.1.** Let $E_1$ and $E_2$ be the elliptic curves associated to the lattices $[1, i]$ and $[1, 2i]$. Then any isogeny in $\text{Hom}(E_1, E_2)$ is defined over the field $\mathbb{Q}(i)$.

**Proof.** Let $\lambda : E_1 \rightarrow E_2$ be an isogeny, then $\lambda = [\alpha]_{E_1,E_2}$ for some $\alpha \in \mathbb{C}$ with $\alpha$ and $\alpha i$ both in $[1, 2i]$ since $\alpha[1, i]$ is contained in $[1, 2i]$. It follows that $\alpha = 2a + 2bi$ for some integers $a$ and $b$. Hence $\text{Hom}(E_1, E_2)$ is isomorphic to $2(\mathbb{Z}[i])$ as an additive group and it is enough to show that the isogenies $[2]_{E_1,E_2}$ and $[2i]_{E_1,E_2}$ are defined over $\mathbb{Q}(i)$.

We first note that both the lattices $[1, i]$ and $[1, 2i]$ have $j$-invariant in $\mathbb{Q}$, since their endomorphism rings $\mathbb{Z}[i]$ and $\mathbb{Z}[2i]$ have class number one. Hence we may assume that $E_1, E_2$ are defined over $\mathbb{Q}$.

We see that $[2]_{E_1,E_2}$ and $[2i]_{E_1,E_2}$ both define isogenies of degree 2. Since $E_1$ has $j$ invariant 1728, it has a model of the form

$$y^2 = x^3 + dx$$

for some $d$. The elliptic curve $E_1$ has at least one 2-isogeny defined over $\mathbb{Q}$ since $(0, 0)$ is a point on $E_1$. Note that $[6 + 2i]_{E_1,E_2}$ is a cyclic isogeny of degree 20 in $\text{Hom}(E_1, E_2)$. If $[2]_{E_1,E_2}$ and $[2i]_{E_1,E_2}$ are both defined over $\mathbb{Q}$, then $[6 + 2i]$ is also defined over $\mathbb{Q}$ but there is no elliptic curve over $\mathbb{Q}$ with a rational cyclic 20-isogeny. See [11] Theorem 2.1. Hence the isogenies $[2]_{E_1,E_2}$ and $[2i]_{E_1,E_2}$ cannot be both defined over $\mathbb{Q}$.

Assume $[2i]_{E_1,E_2}$ is not defined over $\mathbb{Q}$, then it is defined over $L$ (more precisely $\mathbb{Q}(\sqrt{-d})$). We will show that the field $L$ is $\mathbb{Q}(i)$. Composing the following isogenies of $E_1$ and $E_2$

$$E_1 \xrightarrow{[2i]} E_2 \xrightarrow{[i]} E_1,$$

we obtain the endomorphism $[-2]_{E_1}$ of $E_1$ which is defined over $\mathbb{Q}$. This implies that $[i]_{E_2,E_1}$ is defined over $L$. Then the endomorphism $E_1 \xrightarrow{[2]} E_2 \xrightarrow{[i]} E_1$
is defined over $L$. On the other hand, the endomorphism given above is
defined over $\mathbb{Q}(i)$ since the field of definition of $E_1 \xrightarrow{[i]} E_1$ is $\mathbb{Q}(i)$ and $E_1 \xrightarrow{[2]} E_1$ is defined over $\mathbb{Q}$. Thus $L$ must equal to $\mathbb{Q}(i)$. (Notice here that the endomorphism $[2]E_1$ is the multiplication by 2 map on $E_1$ and hence its field of definition is $\mathbb{Q}$.)

One can do a similar discussion for the case $[2i]E_1,E_2$ is defined over $\mathbb{Q}$.

□

Proposition 7.2. Let $E$ be an elliptic curve over $K = \mathbb{Q}(i)$ and let $F$ be the maximal elementary abelian 2-extension of $K = \mathbb{Q}(i)$. Then $E(F)$ cannot have a cyclic $K$-rational subgroup of order 20.

Proof. Let $E_1$ and $E_2$ be the elliptic curves with the endomorphism rings

$\mathbb{Z}[i]$ and $\mathbb{Z}[2i]$

respectively and let $\lambda_i$ be the following endomorphisms

$\lambda_1 = [4 + 2i] : E_2 \to E_2,$
$\lambda_2 = [6 + 2i] : E_1 \to E_2,$
$\lambda_3 = [6 - 2i] : E_1 \to E_2.$

for $i = 1, 2, 3$. Let $\tilde{\lambda}_i$ denote the dual of the isogeny $\lambda_i$ for each $i = 1, 2, 3$.

The elliptic curves $E_1$ and $E_2$ have $j$-invariant in $\mathbb{Q}$ since their endomorphism rings are orders of class number one in the field $\mathbb{Q}(i)$ and hence the isogeny $\lambda_1$ is defined over $\mathbb{Q}(i)$ by [21] Theorem 2.2 and $\lambda_2,\lambda_3$ are defined over $\mathbb{Q}(i)$ by Proposition 7.1. Consequently, the 6 non-cuspidal points (see §4.1) on $X_0(20)$ over $K$ represent the elliptic curves $E_1$ and $E_2$ up to isomorphism, that is, if an elliptic curve defined over $K$ has a cyclic $K$-rational 20-isogeny, it is isomorphic to $E_1$ or $E_2$.

Let $E/K$ be an elliptic curve such that $E(F)$ contains a cyclic $K$-rational subgroup $C$ of order 20. Then $E$ is isomorphic to $E_1$ or $E_2$.

Assume $E$ is isomorphic to $E_1$, then $E$ has $j$-invariant 1728 and a quadratic twist $E^{(d)}(K)$ of $E$ has a point of order 5 by Lemma 2.4, a contradiction to Theorem 5.3 since $E^{(d)}$ also has $j$-invariant 1728.

Assume now that $E$ is isomorphic to $E_2$. The $j$-invariant of $E_2$ is given in [3] as

$$j(E_2) = 11^3.$$ 

We find a model for the elliptic curve with this $j$-invariant in [2] as:

$$E' : y^2 = x^3 - 11x - 14.$$ 

The elliptic curve with the given equation has good reduction at 3 and it has 64 points over the finite field $F_{81}$. Since $E'(F)[5]$ has to inject into $E'(F_{81})$ and 5 does not divide 64, $E'(F)$ can not have a point of order 5.

Now, the elliptic curve $E$ is a quadratic twist of $E'$ and $E'(F) \simeq E(F)$, hence we arrive at a contradiction. □
We use the following result to show that there exists no elliptic curve $E$ defined over $\mathbb{Q}(i)$ such that $E(F)$ contains a $K$-rational subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z}$ (Proposition 7.5).

**Proposition 7.3.** Let $E$ be an elliptic curve defined over $\mathbb{Q}(i)$. Assume $E$ has a cyclic isogeny of degree 64 defined over a quadratic extension of $\mathbb{Q}(i)$, then the $j$-invariant of $E$ is integral.

**Proof.** Kenku [7, Proposition 2] shows the modular curve $X_0(64)$ has the affine equation

$$x^4 + y^4 = 1$$

and that it has 12 cusps; the points at infinity [20, p.1].

Theorem 6.2 proves that any point of $Y_0(64)$ defined over a quadratic extension of $\mathbb{Q}(i)$ is in $\mathbb{Q}(\sqrt{-7}, i)$ and there are 32 such points.

Let $E_i$ for $i = 1, 2$ be the elliptic curves with complex multiplication by the orders $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ and $\mathbb{Z}[\sqrt{-7}]$ respectively and let $E_3$ be the elliptic curve associated to the lattice $[1, 2\sqrt{-7}]$. Notice that $E_3$ also has complex multiplication by the order $\mathbb{Z}[2\sqrt{-7}]$. Now let $\alpha_i$ for $i = 1, 2, 3, 4$ as follows:

$$\alpha_1 = \frac{9 + 5\sqrt{-7}}{2}, \quad \alpha_2 = 1 + 3\sqrt{-7},$$

$$\alpha_3 = 6 + 2\sqrt{-7}, \quad \alpha_4 = 10 + 2\sqrt{-7}$$

Then $\alpha_i$ defines an endomorphism $\lambda_i = [\alpha_i]_{E_i}$ of $E_i$ for every $i = 1, 2, 3$ and $\alpha_4$ defines an isogeny $\lambda_4 = [\alpha_4]_{E_2, E_3}$ from $E_2$ to $E_3$. Let $\bar{\lambda}_i$ be the dual of $\lambda_i$ for each $i = 1, 2, 3, 4$ and let $C_i$ be the kernel of $\lambda_i$. Similarly let $\bar{C}_i$ be the kernel of $\bar{\lambda}_i$. The $j$-invariants of $E_1$ and $E_2$ are given in [3] as

$$j(E_1) = -3^35^3 \quad \text{and} \quad j(E_2) = 3^35^317^3.$$ 

The isogeny $\lambda_i$ is cyclic of degree 64 for each $i$ and [21, Theorem 2.2] tells us that $\lambda_1$ and $\lambda_2$ are defined over $\mathbb{Q}(\sqrt{-7})$. See also [7, Lemma 1].

The class number of the order $O = \mathbb{Z}[2\sqrt{-7}]$ can be computed by the formula in [3, Theorem 7.24]; the conductor $f$ of $O$ is 4 and the class number of $O$ is $h = 2$, hence $E_3$ can be defined over a quadratic extension of $\mathbb{Q}$. Let $L = \mathbb{Q}(\sqrt{-7})(j(E_3))$ be the ring class field of the order $O$. We know the following facts: (See also [3, Proposition 9.5, p.184].)

1. The minimal polynomial $f$ of $j(E_3)$ is in $\mathbb{Z}[x]$ and it is of degree $h(O) = 2$.
2. All primes of $\mathbb{Q}(\sqrt{-7})$ that ramify in $L$ must divide $4\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$. 

First property tells us that $L = \mathbb{Q}(\sqrt{-7}, \sqrt{d})$ for some square-free $d \in \mathbb{Q}$.

Consider the following diagram of field extensions.

\[
\begin{array}{ccc}
L = \mathbb{Q}(\sqrt{-7}, \sqrt{d}) & & \\
\downarrow & & \downarrow \\
\mathbb{Q}(\sqrt{-7}) & & \mathbb{Q}(\sqrt{d}) \\
\downarrow & & \downarrow \\
\mathbb{Q} & & \\
\end{array}
\]

If a prime in \(\mathbb{Q}\) ramifies in \(\mathbb{Q}(\sqrt{d})\), then it is either 2 (by the second property given above) or 7 (by the fact that the only prime ramifies in \(\mathbb{Q}(\sqrt{-7})\) is 7). Since \(L\) contains \(\sqrt{-7}\), we may assume that \(d\) and 7 are relatively prime. Hence \(d\) is \(\pm 2\) or \(-1\) and

\[
L = \mathbb{Q}(\sqrt{-7}, \sqrt{\pm 2}) \quad \text{or} \quad L = \mathbb{Q}(\sqrt{-7}, i).
\]

We will use [3, Theorem 9.2, p.180] to determine \(L\). Let \(p = 29\). Then \(p\) can be written as \(p = 1^2 + 28\). We have the following:

(1) \(-28\) is a quadratic residue modulo 29.

(2) The polynomial \(x^2 + 1\) has a solution modulo \(p\) whereas \(x^2 \pm \sqrt{-7}\) does not. We see that

\[
L = \mathbb{Q}(\sqrt{-7}, i).
\]

This shows that we may assume \(E_3\) is defined over the field \(\mathbb{Q}(i)\). Thus by Theorem [21 Theorem 2.2], \(\lambda_3\) is defined over \(\mathbb{Q}(\sqrt{-7}, i)\). Now we will show that \(\lambda_4\) is also defined over \(\mathbb{Q}(\sqrt{-7}, i)\).

Similar to Proposition [7.1] we can show that \(\text{Hom}(E_2, E_3)\) is isomorphic to \(2(\mathbb{Z}[\sqrt{-7}])\) and it is generated by the isogenies \([2]_{E_2, E_3}\) and \([2\sqrt{-7}]_{E_2, E_3}\). It follows that these isogenies cannot be both defined over \(\mathbb{Q}(i)\) since they generate \(\text{Hom}(E_2, E_3)\). Otherwise \([10 + 2\sqrt{-7}]_{E_2, E_3}\) would be a cyclic 64-isogeny over \(\mathbb{Q}(i)\) but there are no such isogenies defined over \(\mathbb{Q}(i)\) by Proposition [4.1]. Since \([2\sqrt{-7}]_{E_2}\) is an endomorphism of \(E_2\) defined over \(\mathbb{Q}(\sqrt{-7})\) (and not over \(\mathbb{Q}\)), similar to Proposition [7.1] we see that either \([2]_{E_2, E_3}\) or \([2\sqrt{-7}]_{E_2, E_3}\) is defined over \(\mathbb{Q}(i, \sqrt{-7})\).

Thus \((E_i, C_i)\) and \((E_i, \bar{C}_i)\) are rational over \(L\) and they represent 8 different points on \(Y_0(64)(L)\).

Let \((E_i, C_i)\) correspond to the point \((X(w_i), Y(w_i))\) on

\[
x^4 + y^4 = 1.
\]

Denote by \(W_{64}\) the modular transformation corresponding to the matrix

\[
\begin{bmatrix}
0 & -1 \\
64 & 0
\end{bmatrix}.
\]
Then $W_{64}$ acts as an involution on $X_0(64)$. Furthermore, we know by [7] Proof of Lemma 1] that $W_{64}$ sends $(E_i, C_i)$ to $(E_i, \bar{C}_i)$. Using the transformation formula for $\eta(z)$, Kenku shows that $W_{64}$ maps $(x, y)$ to $(y, x)$. Hence if $(E_i, C_i)$ correspond to the point $(X(w_i), Y(w_i))$ on $x^4 + y^4 = 1$, then $(E_i, \bar{C}_i)$ correspond to $(Y(w_i), X(w_i))$. Using the transformation formula for $\eta(z)$ again, we see that

$$\begin{bmatrix} 1 & 0 \\ 16 & 1 \end{bmatrix} : (x, y) \mapsto (-ix, y)$$

$$\begin{bmatrix} 1 & 0 \\ 32 & 1 \end{bmatrix} : (x, y) \mapsto (-x, y)$$

$$\begin{bmatrix} 1 & 0 \\ 48 & 1 \end{bmatrix} : (x, y) \mapsto (ix, y).$$

Since these matrices are in $SL_2(\mathbb{Z})$, their actions do not change the isomorphism class of the elliptic curve corresponding to the given point on $Y_0(64)$, hence the remaining 24 points on $Y_0(64)$ (over $K(\sqrt{-7})$) correspond to some pair $(E, C)$ where $E$ is isomorphic to $E_1, E_2$ or $E_3$.

This proves that an elliptic curve with a cyclic isogeny of degree 64 over a quadratic extension of $\mathbb{Q}(i)$ has complex multiplication and thus its $j$-invariant is integral.

\[ \square \]

Remark 7.4. Over $K = \mathbb{Q}(\sqrt{-3})$, we find a point $(2/\sqrt{5}, \sqrt{-3}/\sqrt{5})$ on the curve $x^4 + y^4 = 1$ and a computation on Magma shows that it produces an elliptic curve with a non-integral $j$-invariant.

Proposition 7.5. Let $K = \mathbb{Q}(\sqrt{d})$ for $d = -1$ or $-3$ and let $F$ be the maximal elementary abelian extension of $K$. Assume that $E$ is an elliptic curve defined over $K$. Then $E(F)$ does not contain a rational subgroup isomorphic to one of the following groups:

- $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$,
- $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.

Moreover, if $K = \mathbb{Q}(i)$, then $E(F)$ does not contain a $K$-rational subgroup isomorphic to the group $\mathbb{Z}/32\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. For each group except $\mathbb{Z}/32\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ given in the statement of the proposition, the proof follows the proof of [11] Proposition 2.4 by using Proposition 4.1, Proposition 7.2, Theorem 2.2, and Remark 2.3. Assume that $E(F)$ has a $K$-rational subgroup $V \simeq \mathbb{Z}/32\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We will slightly modify the argument given in the proof of Proposition 2.4 in [11]. Let $A$ denote the subgroup of $\text{Aut}(V)$ given by the action of $\text{Gal}(\bar{K}/K)$ on $V$. Writing $V$ as $L \oplus S$, with $L \simeq \mathbb{Z}/32\mathbb{Z}$ and $S \simeq \mathbb{Z}/2\mathbb{Z}$, we know by
Lemma 1.5 in [11] that there exist a subgroup $A_0$ of $A$ with index 2 which stabilizes $L$ and $S$. Now

$$\phi : E \to E' = E/S$$

gives us a pair $(E', L')$ where $L' = \phi^{-1}(L)$ is of order 64 and $\phi'$ is the dual isogeny of $\phi$. Thus, the pair $(E', L')$ corresponds to a point on $Y_0(64)$, defined over a quadratic extension of $K$. By Lemma 7.3, the $j$-invariant of $E'$ is integral and thus $E'$ has either good or additive reduction [22, Proposition 5.5, p.181]. By [22, Corollary 7.2, p.185], $E$ also has either good or additive reduction.

Let $L$ be the field generated by $V \subset E(F)$, let $\mathfrak{p}$ be a prime of $\mathcal{O}_L$ above $p = 2 - i$ and let $\tilde{E}$ denote the reduction of $E$ at prime $\mathfrak{p}$. If $E$ has good reduction at $\mathfrak{p}$, then $(E/L)_{(2)}$ injects into $\tilde{E}(\mathbb{F}_{25})$. By the Weil conjectures, we see that $|\tilde{E}(\mathbb{F}_{25})| \leq 36$ which is not possible since $E(L)$ must have at least 64 points. Assume $E$ has additive reduction at $\mathfrak{p}$. Let $\tilde{E}$ denote the reduction of $E$ and let $\tilde{E}_{ns}$ denote the set of non-singular points of $\tilde{E}$. Define

$$E_1(L) = \{ P \in E(L) : \tilde{P} = \tilde{O} \}$$

and

$$E_0(L) = \{ P \in E(L) : \tilde{P} \in \tilde{E}_{ns} \}$$

By [22, Proposition 3.1(a), p.176], $E_1(L)$ has no non-trivial points of order $2^k$ for any $k$. Then $E_0(L)_{(2)}$ injects into $\tilde{E}_{ns}(\mathbb{F}_{25})$. However, $\tilde{E}_{ns}(\mathbb{F}_{25})$ is isomorphic to the additive group $\mathbb{F}_{25}$ and hence $E_0(L)_{(2)}$ is trivial. We know by [24, Addendum to Theorem, §6] that

$$[E(L) : E_0(L)] \leq 4.$$ 

Therefore, $E(L)_{(2)}$ has at most 4 elements which contradicts our assumption that $E(L)$ has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z}$.

\[ \square \]

**Theorem 7.6.** Let $K$ be a quadratic cyclotomic field and let $F$ be the maximal elementary abelian extension of $K$. Assume that $E$ is an elliptic curve defined over $K$.

1. If $K = \mathbb{Q}(i)$, then $E(F)_{tors}$ is isomorphic to one of the following groups:
   
   $\mathbb{Z}/2^{b+r}\mathbb{Z} \oplus \mathbb{Z}/2^b\mathbb{Z}$
   
   $(b = 1, 2, 3$ and $r = 0, 1, 2, 3)$
   
   $\mathbb{Z}/2^{b+r}\mathbb{Z} \oplus \mathbb{Z}/2^b\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
   
   $(b = 1, 2, 3$ and $r = 0, 1)$
   
   $\mathbb{Z}/2^b\mathbb{Z} \oplus \mathbb{Z}/2^b\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
   
   $(b = 1, 2, 3)$
   
   $\mathbb{Z}/2^b\mathbb{Z} \oplus \mathbb{Z}/2^b\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$
   
   $(b = 1, 2, 3)$
   
   or $1, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/15\mathbb{Z}$, and $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

2. If $K = \mathbb{Q}(\sqrt{-3})$, then $E(F)$ is either isomorphic to one of the above groups or

   $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/64\mathbb{Z}$.
Proof. The proof follows from [11, Proof of Theorem 2.5] using Proposition 7.5, Proposition 4.1, Lemma 7.2, Proposition 5.5 and Theorem 2 in [15]. □

8. $E(K)_{\text{tors}}$ is Non-Cyclic

In this section, we study $E(F)_{\text{tors}}$ when the torsion subgroup of $E$ over $K$ is not cyclic. Theorem 2 in [15] shows that if $E(K)_{\text{tors}}$ is not cyclic, then it is isomorphic to one of the following groups.

\[ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z} \quad \text{for} \quad n = 1, 2, 3, 4 \quad \text{and (only if $K = \mathbb{Q}(i)$)} \quad \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \]

We study each of these groups separately. The Proposition 8.5, Proposition 8.6, Proposition 8.7, Proposition 8.8 and Proposition 8.9 are the main results of this section.

Let $E$ be an elliptic curve over a number field $k$ with $E[2] \subset E(k)$. We know that the points of order 2 on $E$ are given by the roots of the polynomial $f(x)$ where $E$ is defined by $y^2 = f(x)$. Therefore, we may assume that $E$ is of the form

\[ y^2 = (x - \alpha)(x - \beta)(x - \gamma) \]

with $\alpha, \gamma, \beta$ in $k$. We will use the following results very often in this section. Remember that $F$ denotes the maximal elementary abelian extension of $k$.

Lemma 8.1 ([9, Theorem 4.2]). Let $k$ be a field of characteristic not equal to 2 or 3 and let $E$ an elliptic curve over $k$ given by

\[ y^2 = (x - \alpha)(x - \beta)(x - \gamma) \]

with $\alpha, \beta, \gamma$ in $k$. For $P = (x, y)$ in $E(k)$, there exists a $k$-rational point $Q$ on $E$ such that $[2]Q = P$ if and only if $x - \alpha, x - \beta$ and $x - \gamma$ are all squares in $k$.

In this case if we fix the sign of the square roots of $x - \alpha, x - \beta, x - \gamma$, then the $x$-coordinate of $Q$ equals to either

\[ \sqrt{x - \alpha} \sqrt{x - \beta} \pm \sqrt{x - \alpha} \sqrt{x - \gamma} \pm \sqrt{x - \beta} \sqrt{x - \gamma} + x \]

or

\[ -\sqrt{x - \alpha} \sqrt{x - \beta} \pm \sqrt{x - \alpha} \sqrt{x - \gamma} \mp \sqrt{x - \beta} \sqrt{x - \gamma} + x. \]

See also the proof of Theorem 4.2 in [9].

Theorem 8.2 ([18, Theorem 9]). Let $K$ be a number field and let $E/K$ be an elliptic curve with full 2-torsion. Then $E$ has a model of the form $y^2 = x(x + \alpha)(x + \beta)$ where $\alpha, \beta \in \mathcal{O}_K$.

1. $E(K)$ has a point of order 4 if and only if $\alpha, \beta$ are both squares, $-\alpha, \beta - \alpha$ are both squares, or $-\beta, \alpha - \beta$ are both squares in $K$.
2. $E(K)$ has a point of order 8 if and only if there exist $d \in \mathcal{O}_K$, $d \neq 0$ and a Pythagorean triple $(u, v, w)$ such that

\[ \alpha = d^2 u^4, \quad \beta = d^2 v^4, \]

or we can replace $\alpha, \beta$ by $-\alpha, \beta - \alpha$ or $-\beta, \alpha - \beta$ as in the first case.
By Theorem 8.2, we may assume that an elliptic curve $E$ with full 2-torsion has the model $y^2 = x(x + a)(x + b)$ with $a, b \in \mathcal{O}_K$. We denote this curve by $E(a, b)$. Then $E(a, b)$ is isomorphic (over $K$) to $E(-a, b - a)$ and $E(-b, a - b)$ by the isomorphisms

$$(x, y) \mapsto (x + a, y) \text{ and } (x, y) \mapsto (x + b, y)$$

respectively. Assume $E$ has a point of order 4 in $E(F)$. Then using Theorem 8.2 together with the isomorphisms between $E(a, b)$, $E(-a, b - a)$, and $E(-b, a - b)$, we may assume that there is a point $Q$ such that $[2]Q = (0, 0)$ and that $a$ and $b$ are both squares. Notice that with a similar discussion, we may assume that if $E(K)$ has a point of order 8, then $M = u^4$ and $N = v^4$ for some $u, v \in \mathcal{O}_K$ such that $u^2 + v^2$ is a square in $K$ (replacing $E$ by a quadratic twist if necessary).

The following result is on the classification of twists of elliptic curves over $K$. We will use this result very often in this section.

**Theorem 8.3** ([13 Theorem 15]). Let $K = \mathbb{Q}(\sqrt{D})$ with $D = -1, -3, d \in K$ a non-square and let $E/K$ an elliptic curve with full 2-torsion. Then,

1. If $E(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, then $E^d(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
2. If $E(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, then $E^d(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
3. If $E(K)_{tor} \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, then $K = \mathbb{Q}(i)$ and $E^d(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
4. If $E(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, then $E^d(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ unless $K = \mathbb{Q}(\sqrt{-3})$ and $d = -1$ in which case $E^d(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
5. If $E(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $E^d(K)_{tor} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for almost all $d$.

For the rest of §8, $E$ will always denote an elliptic curve given by the equation

$E(a, b) : y^2 = x(x + a)(x + b), \ a, b \in K$.

Furthermore, we will assume that the greatest common divisor $(a, b)$ (defined up to a unit) is square-free. Otherwise, we may replace it by the quadratic twist $E(d^2)$ of $E$ where $d^2$ divides both $a$ and $b$. Also, $[n]$ denotes the multiplication by $n$ on the elliptic curve $E$.

We will need the following lemma for the proof of Proposition 8.5.

**Lemma 8.4.** Let $\alpha$ be in $K$. If $\sqrt{\alpha}$ is a square in $F$, then $\alpha$ or $-\alpha$ is a square in $K$. If $K = \mathbb{Q}(i)$, then $\alpha$ must be a square in $K$.

**Proof.** Let $w \in F$ be such that $w^2 = \sqrt{\alpha}$. Then $w$ is a root of the polynomial $f(x) = x^4 - \alpha$ defined over $K$. Since $f$ has one root in $F$ and $F$ is a Galois extension of $K$, $f$ splits in $F$. If $f$ is reducible over $K$, then it has to be product of two quadratic polynomials, otherwise 3 has to divide the order of $Gal(F/K)$. Let us write $f$ as a product of two polynomials.

$$x^4 - \alpha = (x^2 + ex + f)(x^2 + cx + d)$$
for $e, f, c, d \in K$. Then $ec + f + d = 0$, $ed + fc = 0$ and $c + e = 0$. Therefore, replacing $e$ by $-c$, we obtain $c(f - d) = 0$. So, either $c = 0$ or $f = d$. If $c = 0$, then $f = -d$ and so $fd = -\alpha$ implies that $\alpha = f^2$ in $K$. If $f = d$, then $\alpha = -fd$, so $-\alpha$ is a square in $K$.

Now assume that $f$ is irreducible over $K$. The Galois group of $f$ over $K$ is an elementary abelian 2-group since it is a quotient of Gal($F/K$). However, an elementary abelian 2-subgroup of $S_4$ is either of order 2 or the Klein four-group $V_2$. Hence it must be isomorphic to $V_2$ and in particular, it has order 4. If $f$ remains irreducible over $K(i)$, then we obtain an automorphism of order 4 in Gal($K(w)/K(i)$); namely $w \mapsto iw$ which contradicts the fact that Galois group is $V_2$. Therefore, if $K = \mathbb{Q}(i)$, it can not be irreducible and we may assume that $K = \mathbb{Q}(\sqrt{-3})$. Then $f$ has to be reducible over $K(i)$ and by our previous discussion, we see that $\alpha$ or $-\alpha$ is a square in $K(i)$. If $\alpha = \pm d^2$ with $d \in K(i)$, then $d = bi$ for some $b \in K$ and $\alpha = \pm b^2$.

Proposition 8.5. Let $E : y^2 = x(x + a)(x + b)$ be an elliptic curve defined over $K$. Assume that $E(K)_{tors} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.

1. If $K = \mathbb{Q}(i)$, then $E(F)_{tors}$ is either isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z}$.

2. If $K = \mathbb{Q}(\sqrt{-3})$, then $E(F)$ is isomorphic to one of the groups above or $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/64\mathbb{Z}$.

Proof. Since any number in $K$ is a square in $F$, $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z} \subset E(F)$ by Lemma 8.1. By Theorem 8.2, we may assume that $a = u^4$ and $b = v^4$ for some $u, v \in \mathcal{O}_K$ such that $u^4 + v^2 = w^2$ for some $w \in K$. We will show that $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \not\subset E(F)$. Let $Q_2 = (x, y)$ be a point of order 4 such that $[2]Q_2 = (-a, 0)$. By Lemma 8.1, we compute that $x$ equals to one of the followings:

$$\pm \sqrt{-u^4 + 0\sqrt{-u^4 + v^4} - u^4}.$$

If $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \subset E(F)$, then there is a point $Q_3$ in $E(F)$ such that $[2]Q_3 = Q_2$ and by Lemma 8.1, $x + u^4$ is a square in $F$, i.e.,

$$x + u^4 = \pm \sqrt{-u^4 \sqrt{(-u^4 + v^4)} = \pm u^2 \sqrt{u^4 - v^4}}$$

is a square in $F$. Since $u^2$ and $-u^2 = (iu)^2$ are both squares in $F$, $\sqrt{u^4 - v^4}$ is also a square in $F$. By Lemma 8.4, we see that $u^4 - v^4$ or $v^4 - u^4$ has to be a square in $K$. Therefore, there exist a $t \in K$ such that $(u, v, t)$ or $(v, u, t)$ satisfy the equation $x^4 - y^4 = z^2$ and by Lemma 6.1, $wrt = 0$. However, $u$ or $v$ can not be zero since $E$ is non-singular, hence $t$ must be zero which means $a$ equals to $b$; also contradicts to $E$ being nonsingular. Hence this shows that $E(F)$ does not contain a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Theorem 8.3 implies that $E(F)_{2'} = 0$ and the result follows from Theorem 7.6.

See [4, Proposition 4.1] for a similar result over $\mathbb{Q}$. \qed
Proposition 8.6. Assume \( E(K)_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \). Then \( E(F)_{\text{tors}} \) is isomorphic to \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \).

Proof. By Theorem 8.3 all (non-trivial) quadratic twists of \( E \) have torsion subgroup isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \), hence, the odd part of \( E(F)_{\text{tors}} \) must be isomorphic to \( \mathbb{Z}/3\mathbb{Z} \). By Lemma 8.1, we also have that \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \subset E(F) \) since \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \subset E(K) \). Hence \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \subset E(F) \).

To prove the statement, we need to show there is no point of order 8 in \( E(F) \). Let \( P_2 \) be in \( E(F) \) such that \( [2]P_2 = P_1 = (0,0) \). We compute the \( x \)-coordinate of \( P_2 \) by Lemma 8.1 as

\[
x(P_2) = \pm \sqrt{ab}.
\]

Then, \( P_2 \) is either \( (\sqrt{ab}, \pm \sqrt{ab}(\sqrt{a} + \sqrt{b})) \) or \( (-\sqrt{ab}, \pm \sqrt{ab}(-\sqrt{a} + \sqrt{b})) \).

Suppose that \( P_2 \) is in \( [2]E(F) \). Then again by Lemma 8.1 \( \sqrt{ab} \) has to be a square in \( F \) and by Lemma 8.4 \( ab \) or \( -ab \) is a square in \( K \). Suppose that \( ab \) is a square in \( K \). Then \( a = da'^2 \) and \( b = db'^2 \) where \( (a', b') \) is a unit. Then

\[
P_2 = (\epsilon_1 da'b', \epsilon_2 da'b' (\epsilon_1 a' \sqrt{d} + b' \sqrt{d} ))
\]

where \( \epsilon_1^2 = \epsilon_2^2 = 1 \) and the point

\[
(\epsilon_1 da'b', \epsilon_2 da'b' (\epsilon_1 a' + b'))
\]

defines a point of order 4 in \( E(d)(K) \). However by Theorem 8.3 we know that any quadratic twist of \( E \) has torsion subgroup isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) over \( K \), hence \( ab \) is not a square.

Now assume \( -ab \) is a square. Let \( a = da'^2 \) and \( b = -db'^2 \). If \( P_2 \) is in \( [2]E(F) \), then \( x(P_2) + a \) is also a square in \( F \).

\[
x(P_2) + a = \pm \sqrt{ab} + a = \pm (da'b') i + da'^2 = da'(a' \pm b' i)
\]

Hence \( a' + b'i = u^2 s \) for some \( u \) in \( K(i) \) and \( s \in K \). We also see that \( a' - b'i = \bar{u}^2 s \) where \( \bar{u} \) denotes the Galois conjugate of \( u \). Then

\[
(a - b)/d = a'^2 + b'^2 = (u\bar{u})^2 s^2
\]

is a square in \( K \). Consider the curve

\[
E' = E(a - b, -b) : y^2 = x(x + a - b)(x - b).
\]

Taking the quadratic twist of \( E' = E(a - b, -b) \) by \( d \), we obtain

\[
E'^{(d)} : y^2 = x(x + d(a - b))(x - db).
\]

Notice that \( d(a - b) = d^2(a'^2 + b'^2) \) and \( -db = d^2b'^2 \) are squares in \( K \). Hence \( E'^{(d)} \) has a point of order 4 by Lemma 8.1. However, this is not possible by Theorem 8.3 since \( E \) and \( E' \) are isomorphic over \( K \). Hence \( P_2 \) is not in \( [2]E(F) \).

Using the isomorphism between \( E(-a, b - a), E : y^2 = x(x + a)(x + b), \) and \( E(a, b) \) we described earlier, one can show that there is no point \( P \) of order 8 in \( E(F) \) which proves that \( E(F) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \).

See [4, Proposition 4.3] for a similar result over \( \mathbb{Q} \).
Proposition 8.7. Let $K = \mathbb{Q}(i)$. If $E(K)_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, then $E(F)_{\text{tors}}$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.

Proof. Suppose that $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \subset E(K)$. By Theorem 8.2 we may assume that $a = s^2$ and $b = t^2$ for some $s, t \in K$. Let

$$P_1 = (0, 0) \text{ and } Q_1 = (-s^2, 0)$$

as before and

$$x(P_2) = \pm st \text{ and } x(Q_2) = \pm s\sqrt{s^2 - t^2} - s^2$$

such that $[2]P_2 = (0, 0)$ and $[2]Q_2 = Q_1$.

Since $Q_2$ has order 4, it must be in $E(K)$ which forces $s^2 - t^2$ to be a square in $K$. Let $r$ be in $K$ such that

$$s^2 - t^2 = r^2.$$ 

Hence, we compute that $Q_2$ equals to

$$(sr - s^2, \pm isr(r - s)) \text{ or } (-sr - s^2, \pm isr(r + s)).$$

Let $P_2$ denote the point with $x(P_2) = st$ and let $Q_2$ denote the point with $x(Q_2) = (sr - s^2, \pm isr(r - s))$. By Lemma 9, we know that $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \subset E(F)$. We want to show that $\mathbb{Z}/16\mathbb{Z} \not\subset E(F)$. Using Lemma 8.1 we find a point $P_3$ such that $[2]P_3 = P_2$ and

$$x(P_3) = 1/2\sqrt{st} \sqrt{3 + \sqrt{5 + t + s^2}}.$$ 

Assume $P_3$ is in $[2]E(F)$. Then $st$ is a square in $K$ by Lemma 8.1 and Lemma 8.4. Since $s$ and $t$ are relatively prime, either $s$ and $t$ are both squares or they are both $i$ times a square. Now, let $u, v$ be in $K$ such that $s = iu^2, t = iv^2$. Then the equation $s^2 - t^2 = r^2$ gives us $-u^4 + v^4 = r^2$ which has no non-trivial solutions over $K$ by Lemma 2. Similarly, if $s, t$ are both squares, then $u^4 - v^4 = r^2$ which proves that $P_3 \notin [2]E(F)$.

Similarly using Lemma 8.1 we can find another point $Q_3$ where $[2]Q_3 = Q_2$ such that

$$x(Q_3) + s^2 = \sqrt{(sr - s^2)sr} + \sqrt{(sr - s^2)(sr - r^2)} + \sqrt{(sr - r^2)sr} + sr$$

$$= s\sqrt{r^2 - s} + \sqrt{sr} \sqrt{-(r - s)^2} + r\sqrt{s - r} + sr$$

$$= -(-s)\sqrt{r^2 - s} + \sqrt{r^2} \sqrt{-(r - s)^2} + r\sqrt{-s} \sqrt{r - s} = -(-s)r$$

$$= 1/2\sqrt{sr} \sqrt{r^2 - s - \sqrt{-s}}^2.$$ 

If $Q_3$ is in $[2]E(F)$, then $x(Q_3) + s^2$ is a square in $F$. By lemma 8.4 $sr$ is a square in $K$. Hence as we discussed earlier, $s$ and $r$ are either both squares or both $i$ times a square. In both cases, we obtain a non-trivial solution for the equation $x^4 - y^4 = z^2$ which is not possible. Therefore, $Q_3$ is not in $[2]E(F)$.

We will show next that there are no points of order 16 in $E(F)$. Let $R_3$ be in $E(F)$ such that $[2]R_3 = P_2 + Q_2 = R_2$, where $[2]R_2 = (-t^2, 0)$. We find that

$$x(R_3) + t^2 = (1/2)\sqrt{t^2}(\sqrt{r} + \sqrt{r - s} \sqrt{r + t^2}$$.
Hence if $R_3$ is in $[2]E(F)$, then $tri$ has to be a square. This leads to a contradiction to Lemma 6.1 as earlier. Hence $R_3$ is not in $[2]E(F)$ either.

Notice that $R_3 = [a]P_3 + [b]Q_3$ for some odd integers $a, b$. We will assume for simplicity that $a = b = 1$. (The following discussion can be modified for general $a, b$.)

Assume that there is a point $P$ of order 16 in $E(F)$. Then $2P = [k]P_3 + [l]R_3$ for some $k, l \in \mathbb{Z}$. Define

$$Q = \begin{cases} 
[(k - 1)/2]P_3 + [(l)/2]R_3 & \text{if } k \text{ is odd, } l \text{ is even} \\
[k/2]P_3 + [(l - 1)/2]R_3 & \text{if } l \text{ is odd, } k \text{ is even} \\
[(k + 1)/2]P_3 + [(l - 1)/2]R_3 & \text{if } k, l \text{ are both odd}
\end{cases}$$

Then, $[2](P - Q)$ is either $P_3, R_3$ or $Q_3$ which is not possible as we showed earlier. Hence, $\mathbb{Z}/16\mathbb{Z} \not\subset E(F)$ and $E(F) \simeq \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.

\[\square\]

**Proposition 8.8.** Assume that $E(K)_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

1. Let $K = \mathbb{Q}(i)$. Then $E(F)$ does not contain a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.
2. Let $K = \mathbb{Q}(\sqrt{-3})$. Then $E(F)$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ only if $E^{-1}(K)$ has a point of order 4 in which case $E(F) \simeq \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.

**Proof.** By Lemma 8.1, we know that $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \subset E(F)$. We first determine when $E(F)$ has a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. By Theorem 8.2, we may assume that $a = s^2$ and $b = t^2$ for some $s, t \in O_K$ relatively prime. Suppose that $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \subset E(F)$.

Let $Q_1 = (-s^2, 0)$ and let $Q_2$ be in $E(F)$ so that $[2]Q_2 = Q_1$. We compute

$$x(Q_2) = -s^2 + s\sqrt{s^2 - t^2}.$$

Then there is a point $Q_3 \in E(F)$ such that $[2]Q_3 = Q_2$. Hence by Lemma 8.1

$$x(Q_2) + s^2 = s\sqrt{s^2 - t^2}$$

is a square in $F$ which implies that $\sqrt{s^2 - t^2}$ is a square in $F$. Hence either $s^2 - t^2$ or $t^2 - s^2$ is a square in $K$ by Lemma 8.4.

Let $K = \mathbb{Q}(i)$ and let $s^2 - t^2 = r^2$ for some $r \in K$. Then we compute that

$$Q_2 = (sr - s^2, isr(r-s)).$$

The points $P_2$ and $Q_2$ generate $E[4]$ and they are both in $E(K)$. This contradicts the fact that $E(K) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Note that there is no need to consider the case $t^2 - s^2$ is a square separately since $-1$ is a square in $K$.

Now we may assume that $K = \mathbb{Q}(\sqrt{-3})$. If $s^2 - t^2 = r^2$, then $Q_2 = (-s^2 + sr, isr(r-s))$ in $E(K(i))$ induces a point of order 4 on the quadratic twist $E'^{-1}$ of $E$, namely $(-s^2 + sr, sr(r-s))$. Similarly, if $s^2 - t^2 = -r^2$, then $R_2 = (-t^2 + rt, irt(r-t))$ gives rise to a $K$-point of order 4 on $E'^{-1}$.

\[\square\]
Hence we conclude that $E(F)$ does not contain the full 8-torsion if $E^{(-1)}(K)$ does not have a point of order 4.

Assume $E^{(-1)}(K)$ has a point of order 4. Then we can show $E(F)$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ by computing points of order 8 as we did in the proof of Proposition 8.7.

Next we will show that there is no point of order 16 in $E(F)$. We know that there is a point $P_3$ in $E(F)$ such that $[2]P_3 = P_2$. We will show that $P_3 \notin [2]E(F)$. We compute

$$x(P_3) = (1/2)\sqrt{st}(\sqrt{s} + \sqrt{t} + \sqrt{s+t})^2.$$ 

Assume $P_3$ is in $[2]E(F)$. Then $\sqrt{st}$ is a square in $F$ and so $st$ or $-st$ is a square in $K$. Since $s$ and $t$ are relatively prime, $s = \pm du^2$ and $t = dv^2$ for some unit $d$ in $O_K$. The only square-free units in $O_K$ are \{\pm1\}, hence $d = \pm1$. In each case we obtain a non-trivial solution (over $K$) to the equation $s^2 - t^2 = z^2$ since $s^2 - t^2$ or $t^2 - s^2$ is a square. This is not possible by Lemma 2. This shows that $P_3$ is not in $[2]E(F)$.

Let $s^2 - t^2 = r^2$. Then using the computations of points $Q_3$ and $R_3$ in the proof of Proposition 8.7, we see that $Q_3$ and $R_3$ can not be in $[2]E(F)$. This can be shown with a similar argument we used to show $P_3$ is not in $[2]E(F)$. With a similar discussion to the proof of Proposition 8.7 we construct a point $Q$ in $E(F)$ and show that $E(F)$ does not contain a point of order 16.

If $s^2 - t^2 = -r^2$, for a similar argument, use $x(Q_3)$ and $x(R_3)$ which we computed in Proposition 8.7.

See [3] Proposition 4.6 for a similar result over $\mathbb{Q}$.

\begin{theorem}
Let $K$ be the quadratic field $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$. Assume $E(K)_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then $E(F)_{\text{tors}}$ is isomorphic to one of the groups listed in Proposition 8.5, Proposition 8.6, Proposition 8.8, Proposition 8.7, or the group $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

\end{theorem}

\begin{proof}
A quadratic twist of $E$ can have torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Note that since $E$ and $E^{(d)}$ are isomorphic over a quadratic extension of $K$,

$$E(F)_{\text{tors}} \simeq E^{(d)}(F)_{\text{tors}}.$$ 

Hence if $E^{(d)}(K)_{\text{tors}} \notin \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for some $d \in K$, then $E(F)_{\text{tors}}$ will be one of the groups listed in Proposition 8.5, Proposition 8.6, Proposition 8.8, Proposition 8.7. Therefore we may assume that $E^{(d)}(K) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for all $d \in K$. The rest of the proof is same as the proof of Proposition 8.6.

See [3] Proposition 4.5 for a similar result over $\mathbb{Q}$.

\end{proof}

9. $E(K)_{\text{tors}}$ is Cyclic

Let $E : y^2 = f(x)$ be an elliptic curve with $E(K)_{\text{tors}} \simeq \mathbb{Z}/N\mathbb{Z}$. If $N$ is odd, then there is no point of order 2 in $E(K)$. Since the 2-torsion points
on $E$ are $(\alpha_i, 0)$ where $\alpha_i$ are the roots of $f$, $E(K)_{\text{tors}}$ being odd implies $f$ is irreducible over $K$. Therefore, $f$ is irreducible over $F$. Then $E(F)_{\text{tors}}$ is also odd and we analyzed this case in [5]. Hence, we assume that $N$ is even.

We will need the following lemma to show that $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ is not contained in $E(F)$ when $E(K)$ is cyclic.

**Lemma 9.1.** Let $K$ be the field $\mathbb{Q}(\sqrt{-3})$ and let $F$ be the maximal elementary abelian 2-extension of $K$. Suppose $a \in K$. Then $\sqrt{ai}$ can not be a square in $F$.

**Proof.** Suppose that $\sqrt{ai}$ is a square in $F$. Then the proof of Lemma 8.4 shows that $ai$ is a square in $K(i)$. Then

$$ai = b^2(1 + i)^2 \text{ or } ai = b^2(1 - i)^2 \text{ with } b \in K.$$ 

Then $a = \pm 2b^2$. Hence $\sqrt{ai}$ is equal to $b(1 \pm i)$ and we obtain that $(1 \pm i)$ is a square in $F$. Now let $\beta = \sqrt{1 + i}$. Hence $(\beta^2 - 1)^2 + 1 = 0$ and we see that $\beta$ is a root of the polynomial

$$f(x) = x^4 - 2x^2 + 2.$$ 

We observe that the degree of the splitting field of $f$ is 8 and that $F$ has to contain the splitting field of $f$ since it contains $\beta$, hence the Galois group of $f$ over $K$ has to be an elementary abelian 2-group. Notice that the Galois group of $f$ is a subgroup of $S_4$. Since $S_4$ does not have any elementary abelian 2-subgroup of order 8, we get a contradiction. Hence neither $\sqrt{1 + i}$ nor $\sqrt{1 - i}$ is a square in $F$ and this proves that $\sqrt{ai}$ can not be a square in $F$ for any $a \in K$. 

\[\Box\]

**Proposition 9.2.** Let $E$ be an elliptic curve over $K$ and suppose that $E(K)_{\text{tors}} \cong \mathbb{Z}/2N\mathbb{Z}$ for some integer $N$.

1. If $K = \mathbb{Q}(i)$, then $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \not\subset E(F)$.
2. If $K = \mathbb{Q}(\sqrt{-3})$, then $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \not\subset E(F)$.

**Proof.** Suppose $E$ is given by the equation $y^2 = f(x)$. Then $f$ has exactly one root in $K$. Without loss of generality, we may assume that $E$ is given by

$$y^2 = f(x) = x(x - \alpha)(x - \bar{\alpha})$$ 

where $\bar{\alpha}$ denoted the complex conjugate of $\alpha$. We may write $\alpha$ and $\bar{\alpha}$ as $a + b\sqrt{c}$ and $a - b\sqrt{c}$ for some square-free $c$ since they are defined over a quadratic field. Then

$$\alpha - \bar{\alpha} = 2b\sqrt{c}.$$ 

If the point $Q_1 = (\alpha, 0)$ is in $[2]E(F)$, then by Lemma 8.1 $\alpha - \bar{\alpha}$ is a square in $F$ and hence either $c$ or $-c$ is a square by Lemma 8.4. If $K = \mathbb{Q}(i)$, then $Q_1 \not\in [2]E(F)$ which proves that $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \not\subset E(F)$ when $K = \mathbb{Q}(i)$.

Let $K = \mathbb{Q}(\sqrt{-3})$. If $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \subset E(F)$, then $c = -1$ and we may assume that $\alpha = a + bi$. Let $P_3$ be a point such that $[2]P_3 = (0, 0)$. Suppose that $E[8] \subset E(F)$. Then $P_2$ is in $[2]E(F)$ and Theorem 8.2 implies that
\[ x(P_2) = \sqrt{\alpha \bar{\alpha}}, \quad x(P_2) = \alpha \quad \text{and} \quad x(P_2) = \bar{\alpha} \] are all squares in \( F \). Hence by Lemma 8.4

\[ \alpha \bar{\alpha} = a^2 + b^2 = d^2 \quad \text{or} \quad \alpha \bar{\alpha} = a^2 + b^2 = -d^2 \]

for some \( d \) in \( K \). We obtain either

\[ (x(P_2) - \alpha)(x(P_2) - \bar{\alpha}) = 2d(d - a) = e^2 \tag{9} \]

or

\[ (x(P_2) - \alpha)(x(P_2) - \bar{\alpha}) = 2d(d + ai) = f^2 \tag{10} \]

for some \( e, f \in K \). We can parametrize \( a, b, d \) as

\[ a = k(m^2 - n^2), \quad b = 2kmn, \quad \text{and} \quad d = k(m^2 + n^2) \]

for some \( k, m, n \in \mathcal{O}_K \). We set \( d = ki(m^2 + n^2) \) if \( a^2 + b^2 = -d^2 \).

Equation 9 and 10 gives us either \( m^2 + n^2 \) or \( 2(m^2 + n^2) \) is a square in \( K \). Suppose \( 2(m^2 + n^2) \) is a square in \( K \), then \( m^2 + n^2 \) is divisible by 2 and so is \( m^2 - n^2 \). This means that \( a, b, d \) are all divisible by 2. (Notice that 2 remains as a prime in \( \mathcal{O}_K \)). In this case, we can replace \( E \) by the quadratic twist \( E^{(2)} \) of \( E \) since \( E(F) \simeq E^{(2)}(F) \) and they both have cyclic torsion subgroup over \( K \).

Therefore it is enough to consider the case where \( a, b \) are not both divisible by 2. We will assume that \( m^2 + n^2 \) is a square in \( K \), then we compute \( x(Q_2) \) where \( Q_1 = (\alpha, 0) = [2]Q_2 \). We know that

\[ x(Q_2) - \alpha = \sqrt{\alpha (\alpha - \bar{\alpha})} = 2(m - ni) \sqrt{mn} \]

is a square in \( F \), hence \( \sqrt{mn} \) has to be a square in \( F \) but this is not possible by Lemma 9.1. Note that switching the parametrization of \( a \) and \( b \) does not change the result. \( \square \)

**Proposition 9.3** ([4, Lemma 13]). Let \( E(K) \) be cyclic. Then \( E(F) \) contains a point of order 4 if and only if there exist a \( d \in \mathcal{O}_K \) such that \( E^{(d)}(K) \) has a point of order 4.

**Proof.** The proof follows the proof of the statement when \( K = \mathbb{Q} \) and it is given in [4, Lemma 13]. \( \square \)

10. More Restrictions on the Torsion Subgroups

**Proposition 10.1.** Let \( E \) be an elliptic curve over \( K \). Then \( E(F) \) cannot have a subgroup isomorphic to \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \).

**Proof.** Suppose that \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \subset E(F) \). Assume that \( E(K) \) is cyclic. Then by Proposition 9.3, \( E^{(d)}(K) \) has a point of order 4 for some \( d \in K \). Since \( E \) and \( E^{(d)} \) are quadratic twists, \( E(F) \simeq E^{(d)}(F) \) and hence \( E^{(d)}(F) \) has a Galois invariant subgroup of order 20 which is not possible by Proposition 4.1. Similarly, the result holds in the case where \( E(K) \) contains the full 2-torsion by our results in §8. \( \square \)
Proposition 10.2. Let $E$ be an elliptic curve over $K$. Then $E(F)$ cannot have a subgroup isomorphic to $\mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$.

Proof. If $E(K)$ contains $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $E(F)$ cannot have a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ by the results of [8]. Hence we may assume that $E(K)$ is cyclic.

If $K = \mathbb{Q}(i)$, then $E[4]$ is not contained in $E(F)$ by Lemma 9.2.

Let $K = \mathbb{Q}(\sqrt{-3})$. We may assume that $E$ has a point $P$ of order 4 by Lemma 9.3 (replacing $E$ by a twist if necessary) and a $K$-rational subgroup $C$ of order 3 by Lemma 2.4. Let $\phi : E \to E' := E/C$, then $E'$ has a cyclic isogeny of order 9 defined over $K$ by [16] Lemma 7 since $E$ has an additional $K$-rational 3-cycle. The image of $P$, $\phi(P)$ is in $E'(K)$ and it is of order 4 since the order of $\ker(\phi)$ is relatively prime to 4. Then $E'$ has a $K$-rational subgroup of order 36 which is not possible by Proposition 4.1. Hence, there is no such curve over the fields $K$. □

Proposition 10.3. Let $E$ be an elliptic curve over $K$. Then $E(F)$ cannot have a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z}$.

Proof. Suppose that $E$ is an elliptic curve defined over $K$ with $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z} \subset E(F)$. Then Proposition 8.6, Proposition 8.9, and Lemma 9.3 implies that we may assume that $E(K)$ has a point of order 4. Let $P_2$ denote such a point in $E(K)$.

We pick generators $x, y$ for the 2-adic Tate module $T_2(E)$ of $E$ such that $x \equiv P_2 \pmod{4}$. Notice that $E[4] \subset E(F)$. Then the 2-adic representation of the group $\text{Gal}(\overline{F}/F)$ is given as follows:

$$\text{Gal}(\overline{F}/F) \to \text{Aut}(T_2(E))$$

$$\rho_2 : \sigma \mapsto \begin{pmatrix} 1 + 4a_\sigma & 4c_\sigma \\ 4b_\sigma & 1 + 4d_\sigma \end{pmatrix}$$

(11)

for $a_\sigma, b_\sigma, c_\sigma$ and $d_\sigma$ in $\mathbb{Z}_2$. Note that $F$ contains a primitive 8th root of unity. Hence

$$\det(\rho_2(\sigma)) \equiv 1 \pmod{8}$$

since $\zeta_8^{\det(\rho_2(\sigma))} = \sigma(\zeta_8) = \zeta_8$. Computing the determinant, we obtain that $a_\sigma + d_\sigma \equiv 0 \pmod{2}$. Let $E' = E/(P_1)$ where $P_1 = [2]P_2$ and $\phi$ be the morphism $E \to E'$. We will choose the generators of $T_2(E')$ as $x'$ and $y'$ where $2x' = \phi(x)$ and $y' = \phi(y)$. Hence, we find the 2-adic representation of $\text{Gal}(\overline{F}/F)$ on $T_2(E')$ as:

$$\rho'_2 : \sigma \mapsto \begin{pmatrix} 1 + 4a_\sigma & 8c_\sigma \\ 2b_\sigma & 1 + 4d_\sigma \end{pmatrix}$$

(12)

Since $E(K)$ has a point of order 4, $E'(K)$ contains full 2-torsion. (see [22] Example 4.5). Hence by Lemma 8.1, the full 4-torsion $E'[4]$ is contained in $E'(F)$. Hence the representation in (12) tells us that $b_\sigma$ is divisible by 2 for every $\sigma \in \text{Gal}(\overline{F}/F)$. Let $b_\sigma = 2b'_\sigma$. 


Also notice that \( E'(F) \) must have a point of order 16 since \( E(F) \) has a point of order 32. Let \( kx' + ly' \pmod{16} \) be such a point for some \( k, l \in \mathbb{Z} \) and at least one of \( k, l \) is not divisible by 2. Since this point is in \( E(F) \), it is fixed under the action of \( \text{Gal}(\overline{F}/F) \). Then we obtain from the representation in (12) that

\[
(1 + 4a_\sigma)k + 8c_\sigma l \equiv k \pmod{16} \quad (13)
\]

\[
(4b'_\sigma)k + (1 + 4d_\sigma)l \equiv l \pmod{16} \quad (14)
\]

Assume \( a_\sigma \) is a unit in \( \mathbb{Z}_2 \) for some \( \sigma \), then so is \( d_\sigma \) since \( a_\sigma + d_\sigma \equiv 0 \pmod{2} \). An easy computation shows that \( k \) and \( l \) are both congruent to 0 modulo 2 which is a contradiction.

Hence \( a_\sigma \equiv 0 \pmod{2} \) for all \( \sigma \in \text{Gal}(\overline{F}/F) \) and so is \( d_\sigma \). Proposition 9.2 together with the results of \( \S \cdot 8 \) imply that either \( E(K) \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \) or \( E(F) \) does not contain \( E[8] \). If \( E(K) \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \), then we showed in Proposition 8.7 that it can not have a point of order 16.

Hence, we may assume that \( E(F) \) does not contain \( E[8] \) and it implies that \( b'_\sigma \) is not divisible by 2 for some \( \sigma_1 \). See the representation in (12).

Once again using (13) and (14), we compute that \( c_\sigma \equiv 0 \pmod{2} \) for all \( \sigma \). Then the representation of \( T_2(E) \) in (11) implies that \( E[8] \) is contained in \( E(F) \) and we get a contradiction. See [4] for the case \( K = \mathbb{Q} \).

Corollary 10.4. Let \( E : y^2 = x(x + a)(x + b) \) be an elliptic curve defined over \( K \). Assume that \( E(K)_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \). Then

\( E(F)_{\text{tors}} \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z} \).

Proof. The statement follows from the Proposition 8.5 and Proposition 10.3.

Proposition 10.5. Let \( E/K \) be an elliptic curve. Then \( E(F) \) cannot be isomorphic to \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \).

Proof. We will proceed the same way as in Proposition 10.3. The representation of \( \text{Gal}(\overline{F}/F) \) on \( T_2(E) \) and \( T_2(E') \) is same as given in the proof of Proposition 10.3. Since \( E'(F) \) contains full 2 torsion and also a point of order 3, \( E'(F) \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \) by Proposition 8.6. Hence \( b_\sigma \equiv 0 \pmod{2} \) for all \( \sigma \). If \( a_\sigma \equiv 0 \pmod{2} \) for all \( \sigma \), then so is \( d_\sigma \) and \( y' \pmod{8} \) is stabilized under the action of \( \text{Gal}(\overline{F}/F) \). However, \( E'(F) \) does not have a point of order 8. Hence \( a_\sigma \) is not divisible by 2 for some \( \sigma_1 \). Then similar to Proposition 10.3, we obtain congruences as in (13) and (14) (replacing modulo 16 by 8), we get a contradiction.

11. Main Result

Theorem 11.1. Let \( K \) be a quadratic cyclotomic field, let \( E \) be an elliptic curve over \( K \), and let \( F \) be the maximal elementary abelian 2-extensions of \( K \).
If \( K = \mathbb{Q}(i) \), then \( E(F)_{\text{tors}} \) is isomorphic to one of the following groups:

\[
\begin{align*}
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z} & \quad (N = 2, 3, 4, 5, 6, 8) \\
\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4N\mathbb{Z} & \quad (N = 2, 3, 4) \\
\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z} & \quad (N = 2, 3, 4, 6, 8)
\end{align*}
\]

or \{1\}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/15\mathbb{Z}.

(2) If \( K = \mathbb{Q}(\sqrt{-3}) \), then \( E(F) \) is either isomorphic to one of the groups listed above or

\[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z}.\]

Proof. We begin with the list given in Theorem 7.6. Suppose that \( E(F)_{(2)} \neq 1 \). If \( E(F)^{\prime} = 1 \), then either \( \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \) is not contained in \( E(F) \) or \( E(F) \simeq \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \) by our results in [8] and Proposition 9.2. Then with the notation of Theorem 7.6, if \( b = 3 \), then \( r = 0 \). By Proposition 10.3, if \( b = 2 \), then \( r \leq 2 \). We obtain the groups:

\[
\begin{align*}
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z} & \quad \text{for } N = 1, 2, 4, 8, \\
\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4N\mathbb{Z} & \quad \text{for } N = 1, 2, 4, \\
\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}
\end{align*}
\]

and \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z} \) if \( K = \mathbb{Q}(\sqrt{-3}) \).

Suppose \( E(F)^{\prime} \simeq \mathbb{Z}/3\mathbb{Z} \). If \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) is contained in \( E(K) \), then \( E(K) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \) and we showed that \( E(F) \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \). Otherwise, we know that \( E(F) \) cannot contain \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \) if \( K = \mathbb{Q}(i) \). Similarly \( E(F) \) cannot contain \( \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \) if \( K = \mathbb{Q}(\sqrt{-3}) \). Along with Proposition 10.5, we are left with three possible groups:

\[
\begin{align*}
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} & \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}
\end{align*}
\]

The case \( E(F)^{\prime} \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) follows from Proposition 10.2, Proposition 9.2 and Theorem 7.6. The only possible group is

\[\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}.\]

Similarly when \( E(F)^{\prime} \simeq \mathbb{Z}/5\mathbb{Z} \), it follows from Proposition 9.2, Lemma 10.1 and Theorem 7.6 that the only option is

\[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}.
\]

The case where \( E(F)_{(2)} = 1 \) was studied in the first section and we found the groups

\[
\{1\}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/15\mathbb{Z} \text{and } \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.
\]

\[\square\]

Remark 11.2. Every group we listed in Theorem 11.1 except \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/32\mathbb{Z} \) also appears as the torsion subgroup of some elliptic curve defined over \( \mathbb{Q} \) in its maximal elementary abelian 2 extensions. We were able to prove
neither the nonexistence of an elliptic curve defined over $\mathbb{Q}(\sqrt{-3})$ with such a subgroup in $E(F)$ nor give an example of such a curve.

Acknowledgements. The author wishes to thank Sheldon Kamienny for suggesting this problem and for his kind support. Samir Siksek provided valuable insight for a part of the proof of Theorem 5.4. This work also greatly benefited from conversations with Burton Newman and Jennifer Balakrishnan. We also thank the anonymous referee for the remarks on the previous draft.

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