Effect of boundary conditions on the non-linear forced vibration response of isotropic plates

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Abstract. The effect of boundary conditions on the linear and nonlinear steady state forced vibration response of rectangular isotropic plates have been investigated. For thin components subjected to dynamic forces the amplitude of vibration is comparable to its thickness which requires inclusion of nonlinear terms in the strain displacement relation. In the present work the non-linear governing equations of motion have been obtained using standard finite element procedure based on displacement field of FSDT. The governing equation of motion is solved using modified shooting method. The unstable portions of frequency response curves are obtained using continuation algorithm [1]. The non-linear responses are compared with linear response and very large difference between the two has been obtained. Unlike frequency domain methods such as Harmonic balance and IHB the present method does not require any assumption on participating modes. The time history, phase plane plots and FFT of the response have also been obtained to explain the physics of the peculiar nature of the nonlinear response.

1. Introduction

Plates are structures which have wide applications among engineering constructions. The knowledge of the dynamical behavior of the plates is important for their efficient design and maintenance. The dynamical response of the plate can change significantly due to the nonlinear terms at the equation of motion which become essential in the presence of large displacements.

Studies on the periodic responses of nonlinear dynamical systems are carried out mostly employing frequency domain or direct time domain integration methods. Both these methods are not sufficient to trace the unstable regions adequately. The harmonic balance or incremental harmonic balance methods lead to erroneous results if the correct number of harmonics are not included Sundarajan and Noah [1]. The direct time integration methods are computationally more involved for obtaining the steady state solutions of the large size weakly damped nonlinear systems and are unable to capture the unstable regions of the steady state response, Patel et al. [2], Ibrahim et al. [3]. Unlike the frequency domain methods, the number of equations does not depend on the number of harmonics. Also, the steady state solutions can be reached in significantly less number of iterations as compared to the direct time integration methods. Another feature of the shooting method is that it yields the monodromy matrix, which can be used to predict the stability of solution Nayfeh & Balachandran [4]. Modified shooting technique is proposed for predicting the periodic responses of non-linear systems directly from solution of second order equations by Ibrahim et al. [5].

From the literature it is evident that the effect of boundary conditions on the nonlinear steady state forced vibration response has not been investigated adequately. The current work presents numerical
methods for investigating the dynamical behavior of plates. The equation of motion of the plate is derived based on first order shear deformation plate theory and geometrical nonlinear terms are included. It is discretized by the finite element method and periodic responses are obtained by using shooting technique along with continuation scheme. The non-linear steady state forced vibration characteristics of isotropic plate subjected to transverse uniformly distributed harmonic force is investigated. Detailed parametric analysis has been done to explore the nonlinear dynamics of isotropic plates through frequency response curves, response history and FFT of the response.

2. Governing Equations and Solution Procedure
The geometry of an isotropic plate subjected to transverse harmonic force with frequency in the vicinity of the first fundamental mode has been represented in Figure 1.

\[
\begin{align*}
\{u\}^e &= \{u_0 \ v_0 \ w_0 \ \phi_x \ \phi_y \}^T \\
\{\varepsilon\} &= \begin{bmatrix} \varepsilon_p^L \\ \varepsilon_b \\ \varepsilon_s \end{bmatrix} + \begin{bmatrix} z\varepsilon_p^L \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_p^{NL} \end{bmatrix}
\end{align*}
\]

Where, \((u_0, v_0, w_0, \phi_x, \phi_y)\) are unknown functions to be determined. \((u_0, v_0, w_0)\) denote the displacements of a point on the plane \(z = 0\).

Using von Karman’s assumption for small strains and moderately large rotation, strain field in terms of mid-plane deformation can be written as:

\[
\{\varepsilon\} = \begin{bmatrix} \varepsilon_p^L \\ \varepsilon_b \\ \varepsilon_s \end{bmatrix} + \begin{bmatrix} z\varepsilon_p^L \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_p^{NL} \end{bmatrix}
\]
\[
\begin{align*}
\{ \varepsilon^L \} &= \begin{pmatrix}
\varepsilon^L_{xx} \\
\varepsilon^L_{yy} \\
\varepsilon^L_{xy} \\
\varepsilon^L_{yx}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial u_0}{\partial x} + z \frac{\partial \phi_x}{\partial x} + \frac{\partial w_0}{\partial x} + \phi_x \\
\frac{\partial u_0}{\partial y} + z \frac{\partial \phi_y}{\partial y} + \frac{\partial w_0}{\partial y} + \phi_y \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + \frac{\partial w_0}{\partial x} + \phi_x \\
\frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_x}{\partial x} + \frac{\partial w_0}{\partial y} + \phi_y
\end{pmatrix}, \\
\{ \varepsilon^{NL} \} &= \begin{pmatrix}
\varepsilon^{NL}_{xx} \\
\varepsilon^{NL}_{yy} \\
\varepsilon^{NL}_{xy} \\
\varepsilon^{NL}_{yx}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \\
\frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + \frac{\partial w_0}{\partial x} + \phi_x \\
\frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_x}{\partial x} + \frac{\partial w_0}{\partial y} + \phi_y
\end{pmatrix}
\end{align*}
\]

\[
\{ \varepsilon_3 \} = \begin{pmatrix}
\theta_x + \frac{\partial w_0}{\partial x} \\
\theta_y + \frac{\partial w_0}{\partial y}
\end{pmatrix}, \quad \{ \varepsilon_b \} = \begin{pmatrix}
\frac{\partial \theta_x}{\partial x} \\
\frac{\partial \theta_y}{\partial y} \\
\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}
\end{pmatrix}
\]

Using standard finite element assembly procedure and considering dissipative forces, the governing equation of motion can be written as [6]:

\[
[M] \{ \ddot{U} \} + [C] \{ \dot{U} \} + [K+(1/2)K_1(U)+(1/3)K_2(U)] \{ U \} = \{ F \} \tag{5}
\]

Where \([K]\) is the linear stiffness matrix, \([K_1]\) and \([K_2]\) are non-linear stiffness matrices linearly and quadratically dependent on the field variables.

Using Rayleigh proportional damping model, the damping matrix \([C]\) is taken as:

\[
[C] = \tilde{\alpha} [M] + \beta [K] \tag{6}
\]

where, \(\beta = \frac{\xi}{\omega_n}; \tilde{\alpha} = \xi \omega_n \) \((\xi: \text{modal damping factor}; \omega_n: \text{natural frequency});\). 

The governing equations of motion are solved using modified shooting method [6]. For unstable portion of the steady state response curves continuation scheme based on arc-length is employed. For shooting method the forcing frequency is incremented starting far away from resonance and the corresponding steady state is obtained. When the shooting method fails to give converged solution near sharp turning points the solution is continued by specifying arc length and finding the corresponding converged frequency. The details of the procedure is available in authors work [6-7] and are not elaborated here for the sake of brevity.

3. Results and Discussion

The linear as well as geometrically non-linear steady state periodic response of isotropic plates has been obtained. The effect of boundary conditions on linear and nonlinear steady state forced vibration response is analysed. A uniformly distributed transverse harmonic force \(F=F_0\cos \omega t\), with frequency in the vicinity of first fundamental mode is used for the study.

The material properties considered in the analysis are taken as: \(E = 71.7 \times 10^9 \text{ Pa}, \) density “\( \rho \)” = 2740 kg/m\(^3\), Poisson’s ratio, \(\nu=0.33\).

Different boundary conditions are:

All Clamped with immovable edge (CCCC) are:
\(v = u = w = \varphi_1 = \varphi_2 = 0; \quad \text{at } x = 0, \ L \text{ and } y = 0, \ b\)

Simply Supported-Clamped-Simply Supported-Clamped (SCSC) are:
\(v = w = \varphi_2 = 0; \quad \text{at } x = 0, \ L\)
\( v = u = w = \varphi_1 = \varphi_2 = 0; \) at \( y = 0, b \)

Clamped-Simple Supported-Clamped-Simply Supported (CSCS) are:

\( v = u = w = \varphi_1 = \varphi_2 = 0; \) at \( x = 0, L \)

\( u = w = \varphi_1 = 0; \) at \( y = 0, b \)

Before proceeding with the parametric studies the influence of discretization scheme on the free vibration characteristics has been carried out and based on the convergence study a mesh size of 5x5 elements for quarter plate with symmetric boundary condition has been used in the analysis.

The linear and nonlinear frequency response curve of rectangular plates are shown in Figure 2. It can be seen from the comparison of linear and nonlinear frequency response that the amplitude in linear case is considerably large compared to nonlinear case with linear peak amplitude for CSCS plate being more than 12 times the nonlinear peak amplitude. The hardening nonlinearity and the peak amplitudes of CCCC and SCSC plates are comparable whereas the peak amplitude is significantly greater for CSCS plates for both linear and nonlinear cases. The GNL frequency response in CCCC and SCSC cases reveals secondary peak in the forcing frequency range 1.5 to 2.1 due to modal interactions between first and higher modes.

![Figure 2](image-url)

(a) Geometrically Non-linear frequency response  
(b) Geometrically linear frequency response

Figure 2 Steady state forced vibration frequency response curves (\( L/b=2; b/h=100, b=0.5\ m, F_0=5\ kPa, \xi=0.010 \)).

The steady state response history (Figure 3) reveal equal half cycle time in tension and compression for all the boundary conditions for linear analysis as expected. However the cycle time is different in tension and compression with nonlinearity.

The phase plane plots (Figure 4) for nonlinear forced vibration corresponding to peak amplitude in the frequency response reveal asymmetry indicating presence of higher harmonics whereas for linear cases the phase plane is symmetrical indicating only fundamental harmonic contributions.

The FFT of the steady state forced vibration response history (Figure 5) reveal dominant odd order harmonic participation apart from fundamental harmonics for nonlinear cases.
Figure 3 Steady state response history corresponding to peak amplitude in the freq. response curves

Figure 4 Phase plane plots for nonlinear cases at peak amplitude.

Figure 5 FFT of the steady state response corresponding to peak amplitude in nonlinear frequency response curve.
4. Conclusion

The effect of boundary conditions on linear as well as geometrically non-linear forced vibration response of plates has been analyzed. The methodology adopted is computationally efficient and do not require a priori assumptions on modal participation. It is evident from the analysis that the peak amplitude is greater for CSCS followed by CCCC and SCSC cases. The peak amplitude is significantly greater for CSCS in linear analysis (for aspect ratio \(L/b=2\)). The GNL frequency response in CCCC and SCSC cases reveals secondary peak in the forcing frequency range 1.5 to 2.1 due to modal interactions between first and higher modes. The hardening nonlinearity is found to be greater for CCCC and SCSC plates compared to plates with CSCS boundary conditions. FFT of the nonlinear response reveal dominant odd order harmonic contributions.

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