FLUCTUATIONS OF CYCLES IN A FINITE CW COMPLEX

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ABSTRACT

We use algebraic topology to study the stochastic motion of cellular cycles in a finite CW complex. Inspired by statistical mechanics, we introduce a homological observable called the average current. The latter measures the average flux of the probability in the process. In the low temperature, adiabatic limit, we prove that the average current fractionally quantizes, in which the denominators are combinatorial invariants of the CW complex.

1. Introduction

The interplay between dynamical systems and other branches of mathematics is more than a century old. One of the early prototype results in differential topology, the Poincaré–Hopf theorem, equates the Euler characteristic of a compact smooth manifold with the enumeration of zeros of a generic vector field. In the 1930s, Marston Morse generalized this result, in what came to be known as the Morse inequalities, using gradient dynamics. A more recent

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interaction stems from Hamiltonian dynamics. The latter has inspired insights in the study of symplectic manifolds, enumerative geometry, string theory and algebraic topology.

The goal of this paper is to erect yet another scaffold, one that will link the fields of stochastic dynamics, enumerative combinatorics and algebraic topology. Our investigation concerns the random motion of cellular \((k - 1)\)-cycles in a finite CW connected CW complex \(X\) of dimension \(d\) with \(k \leq d\). We associate to this random motion a \(k\)-dimensional real homology class, the average current, which is a kind of measurement of the flow of probability. The average current is subsequently used to relate the process to the algebraic topology of \(X\). From now on, we assume \(k = d\). This represents no loss in generality as we can replace \(X\) by its \(k\)-skeleton, if necessary.

1.1. Markov chains. A continuous time Markov chain consists of:

- a locally finite directed graph 
  \[ \Gamma = (\Gamma_0, \Gamma_1), \]
  called a state diagram;
- an assignment of a continuous function 
  \[ k_\alpha : \mathbb{R} \to [0, \infty), \]
  to each edge \(\alpha\).

The function \(k_\alpha\) is called the transition rate of the edge \(\alpha\). It is to be interpreted as the instantaneous rate of change of probability in jumping from the source state of \(\alpha\) to its target state (cf. [Re, Ch. 5]).

Remark 1.1: In the mathematical literature on Markov chains, the transition rates are often assumed to be constant. We will not make this assumption.

Let \(s = (s_0, s_1) : \Gamma_1 \to \Gamma_0 \times \Gamma_0\) be the function which assigns to \(\alpha\) the ordered pair consisting of its source and target. Then the rates define a time dependent square matrix \(H = H(t)\), as follows. For \(i \neq j\), set 
\[
   h_{ij} = \sum_{s(\alpha) = (i, j)} k_\alpha.
\]
Then the matrix entries of \(H\) are given by
\[
   H_{ij} = \begin{cases} 
   h_{ij}, & i \neq j; \\
   -\sum_{\ell \neq j} h_{\ell j}, & i = j,
   \end{cases}
\]
where the indices range over \( i, j \in \Gamma_0 \). The matrix \( H \) is called the \textbf{master operator}; by design, the sum of its entries in each column is zero. Associated with \( H \) is a linear, first order ordinary differential equation

\[
p'(t) = Hp(t),
\]

in which \( p(t) \) is a one-parameter family of (probability) distributions on the set of vertices \( \Gamma_0 \). Equation (1) is called the \textbf{Kolmogorov equation} or the \textbf{master equation} \cite{vK, ch. 4}, \cite{Re, §5.4}. If \( p(t) \) solves the Kolmogorov equation and \( p(0) \) is an initial probability distribution on \( \Gamma_0 \), then, using the definition of \( H \), it is straightforward to check that \( p(t) \) is a probability distribution for every \( t > 0 \). Consequently, one may regard the solutions to (1) as describing a flow on the space of probability distributions on \( \Gamma_0 \).

1.2. \textbf{Markov CW chains.} Let \( X \) be a connected finite CW complex as above. With respect to the choice of auxiliary data, we shall associate to \( X \) a continuous time Markov chain called a \textbf{Markov CW chain}. Part of the auxiliary structure includes a fixed choice of integer \((d-1)\)-cycle \( z_0 \in Z_{d-1}(X; \mathbb{Z}) \).

We initially postulate that the allowed states of a Markov CW chain are the cellular \((d-1)\)-cycles in \( X \) over the integers that are homologous to \( z_0 \). In other words, the set of allowed states is the coset

\[
z_0 + B_{d-1}(X; \mathbb{Z}),
\]

where \( B_{d-1}(X; \mathbb{Z}) \) is the abelian group of cellular \((d-1)\)-boundaries.

A transition from one state to another is provided by an \textbf{elementary homology}. That is, a transition from a state \( z \) to a state \( z' \) requires a choice of \( d \)-cell \( \alpha \) and a choice of \((d-1)\)-cell \( f \) such that

\[
\langle z, f \rangle \neq 0 \quad \text{and} \quad \langle \partial \alpha, f \rangle \neq 0,
\]

where \( \langle -, - \rangle \in \mathbb{Z} \) denotes the incidence number. Furthermore, the target state \( z' \) is obtained from \( z \) by adding \( \partial \alpha \), the boundary of \( \alpha \), with the proper coefficient \( u \in \mathbb{Z} \):

\[
z' = z + u\partial \alpha
\]

(the coefficient \( u \) is too technical to describe here; see Definition 4.2 for the precise statement).
Remark 1.2: If \( d = 1 \), then the state diagram \( \Gamma \) of a Markov CW chain is the double of \( X \). The latter is the directed graph having the same vertices as \( X \), with a pair of directed opposing edges for each 1-cell of \( X \). In particular, when \( d = 1 \), the state diagram is finite, and the coefficient \( u \) appearing in (2) is always \( \pm 1 \).

However, if \( d > 1 \), there will generally be an infinite number of states, but only finitely many are directly accessible from any given state by a single transition. Consequently, for \( d > 1 \) the graph \( \Gamma \) is locally finite (but possibly globally infinite). Furthermore, the states which are inaccessible from \( z_0 \) by a sequence of transitions are considered as being decoupled from the process; for this reason we will omit them.

The last step in the description of the Markov CW chain involves the choice of rates. The method we use is inspired by considerations in statistical mechanics. First we fix a real number \( \beta > 0 \) which we think of as inverse temperature. Then we define the space of parameters to be the real vector space \( M_X \) whose vectors are pairs

\[ (E_\bullet, W_\bullet) \]

in which \( E_\bullet : X_{d-1} \to \mathbb{R} \) and \( W_\bullet : X_d \to \mathbb{R} \) are functions. Hence, a vector of \( M_X \) consists of a choice of a real number label \( E_f \) for every \((d-1)\)-cell \( f \) and a real number label \( W_\alpha \) for each \( d \)-cell \( \alpha \). A vector \( (E_\bullet, W_\bullet) \in M_X \) is sometimes referred to as a system of weights for \( X \).

A driving protocol is a smooth map

\[ \lambda : \mathbb{R} \to M_X, \]

i.e., a 1-parameter family of weights \( \lambda(t) = (E_\bullet(t), W_\bullet(t)) \). Then for each edge of the form \((f, \alpha)\) appearing in \( \Gamma \), we define the transition rate at time \( t \) to be the real number

\[ e^{\beta(E_f(t)-W_\alpha(t))}. \]

Summarizing, the Markov CW chain, whose construction is outlined above, depends on a choice of 4-tuple

\[ (X, z_0, \beta, \lambda) \]

in which

- \( X \) is a finite connected CW complex;
- \( z_0 \) is cellular \((d-1)\)-cycle;
- \( \beta > 0 \) is a real number;
- \( \lambda \) is a driving protocol.
1.3. **Periodic driving.** This paper will focus on periodic driving, where we will see how to associate an algebraic topological invariant that may be non-trivial (cf. [CKS] in the $d = 1$ case). For a real number $\tau_D > 0$, we say that a driving protocol $\lambda$ is $\tau_D$-periodic if $\lambda(t) = \lambda(t + \tau_D)$ for every $t \in \mathbb{R}$. In this case, $\lambda$ amounts to a choice of pair

$$(\tau_D, \gamma),$$

where $\gamma : [0, 1] \to M_X$ is the smooth loop defined by

$$\gamma(t) := \lambda(\tau_D t).$$

Given a $\tau_D$-periodic driving protocol $\lambda = (\tau_D, \gamma)$, the master equation may be rewritten in the form

$$p'(t) = \tau_D \mathcal{H} p(t), \quad p(0) = z_0,$$

where $\mathcal{H}$ is the master operator which is now defined using $\gamma$ in place of $\lambda$ (cf. [CKS]). Changing the value of $\tau_D$ is called **driving time rescaling**.

The initial value problem (3) has a unique formal solution $\varrho = \varrho(t)$ which is a one-parameter family of probability distributions on the set of vertices $\Gamma_0$. Taking the weighted sum defined by $\varrho$, we obtain the expectation

$$E[\varrho] = \sum_{z \in \Gamma_0} \varrho_z z \quad (or \quad first \quad moment),$$

where $\rho_z$ denotes the component of $\rho$ at the vertex $z$. We will see that the series (4) converges to a well-defined family of real $(d - 1)$-cycles, i.e., for every $t$, we have $E[\varrho](t) \in Z_{d-1}(X; \mathbb{R})$, where $Z_{d-1}(X; \mathbb{R})$ is the vector space of real cellular $(d - 1)$-cycles on $X$. Applying the time-dependent biased coboundary operator

$$\partial^*_{E,W} = e^{-\beta W} \partial^* e^{\beta E} : C_{d-1}(X; \mathbb{R}) \to C_d(X; \mathbb{R})$$

(cf. (10) below; here $C_*(X; \mathbb{R})$ denotes the real cellular chain complex of $X$) to $E[\varrho]$ and integrating, we obtain a real cellular $d$-chain

$$Q := \int_0^1 \partial^*_{E,W} E[\varrho] \, dt,$$

which can be viewed as the **average current** of the process; it depends on the triple $(\beta, \tau_D, \gamma)$. The explanation for the terminology is that $\partial^*_{E,W} E[\varrho]$ measures the flux of the expected value, and the displayed integral is just the average value over time of this flux.
When the driving time \( \tau_D \) is sufficiently large, the expected value \( E[\varrho(t)] \) will be 1-periodic (cf. Theorem B). Then \( Q \) is a real \( d \)-cycle for large \( \tau_D \) (by Theorem A below and the fundamental theorem of calculus). Since \( X \) has dimension \( d \), there are no \( d \)-boundaries, so the group of real \( d \)-cycles coincides with the homology group \( H_d(X; \mathbb{R}) \). Consequently, the average current will be a real homology class:

\[
Q \in H_d(X; \mathbb{R}),
\]

which we view as a characteristic class for the Markov CW chain. Summarizing, we have associated a \( d \)-dimensional homology class to a Markov CW chain which depends on the data \((\gamma, \tau_D, \beta)\). The current work will investigate the properties of this homology class after taking limits in the driving time \( \tau_D \) and the inverse temperature \( \beta \).

1.4. STATEMENT OF RESULTS. Suitably understood, the master operator is a biased (i.e., weighted) Laplacian. However, there are analytical difficulties in working with the master equation directly, as the space of distributions on the set of states is typically infinite dimensional. Fortunately, the expectation of the formal solution of the master equation also satisfies a Kolmogorov-type equation whose dynamical operator acts on the finite-dimensional vector space of cellular \((d - 1)\)-chains \( C_{d-1}(X; \mathbb{R}) \), as we will now explain.

Given a triple \((\gamma, \tau_D, \beta)\) as above, the biased Laplacian

\[
H : C_{d-1}(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R})
\]

is the operator given by

\[
-\partial \partial_{E,W}^* = -\partial e^{-\beta W} \partial^* e^{\beta E},
\]

where \( \partial \) is the boundary operator and \( \partial^* \) is its formal adjoint (i.e., the coboundary operator). Note that \( H \) is defined in terms of \( \gamma \) and \( \beta \); in particular \( H \) is time-dependent. Furthermore, \( H \) is negative semi-definite. The dynamical equation is given by

\[
\dot{q} = \tau_D H q.
\]

THEOREM A (Expectation Dynamics): Let \( \varrho(t) \) be the formal solution of the master equation (3) with initial value \( \varrho(0) = z_0 \). Then its expectation

\[
\rho(t) := E[\varrho(t)]
\]

is the unique solution to the dynamical equation (7) with respect to the initial condition \( \rho(0) = z_0 \).
Theorem A is the cornerstone of our investigation: it relates the evolution of a state of the process to the evolution of its first moment. The latter is more directly related to the topology of the CW complex $X$.

The next step of the program is to analyze $\rho$ under two limits on the process. The first of these is the \textbf{adiabatic limit}, in which $\tau_D \to \infty$. The term “adiabatic” appreciates the sufficiently slow variation of the parameters.

\textbf{Theorem B (Adiabatic Theorem):} There exists a positive real number

$$\tau_0 = \tau_0(\beta, \gamma)$$

such that for all $\tau_D > \tau_0$, a 1-periodic solution $\rho_{\tau_D}$ of the dynamical equation (7) exists and is unique. Furthermore,

$$\lim_{\tau_D \to \infty} \rho_{\tau_D} = \rho^B,$$

where $\rho^B = \rho^B(\gamma, \beta)$ is the Boltzmann distribution at $[z_0] \in H_{d-1}(X; \mathbb{R})$ (cf. Definition 3.13).

In particular, Theorem B shows that the long time behavior of the process is no longer dynamical in nature. Furthermore, the limiting cycle is given by the Boltzmann distribution [CCK2], which can be interpreted as the unique “harmonic” cycle on this class (see Theorem 3.12). This cycle is a weighted average over the cycles representing the homology class, so Theorem B is a kind of ergodic theorem for the expectation of our Markov process.

\textbf{Remark 1.3:} We reiterate that Theorem B refers to the first moment of the process, whereas the actual process typically has no limiting distribution.

Recall that the average current $Q$ is defined in terms of the parameters $(\tau_D, \gamma, \beta)$. In what follows set $Q = Q_{\tau_D}$ to emphasize its dependence on $\tau_D$. Set

$$Q^B := \lim_{\tau_D \to \infty} Q_{\tau_D}.$$ 

Theorem B implies that $Q^B$ is well-defined and depends only on Boltzmann distribution $\rho^B$.

The second limit we are interested in is the \textbf{low temperature limit}, under which $\beta \to \infty$. The main result of this paper is stated in the low temperature, adiabatic limit. In what follows we write $Q^B_\beta$ to indicate the dependence of $Q^B$ on $\beta$.
Theorem C (Quantization): Assume $X$ is a connected finite CW complex of dimension $d$. For a “good” periodic driving protocol $(\tau_D, \gamma)$, the low temperature, adiabatic limit of the average current is well-defined and fractionally quantizes, i.e., there is a positive integer $\delta$ such that
\[
\lim_{\beta \to \infty} Q^B_\beta \in H_d(X; \mathbb{Z}[\frac{1}{\beta}]) \subset H_d(X; \mathbb{R}).
\]
Moreover, the $\delta$ is a combinatorial invariant of $X$ (cf. Theorem 7.7).

We consider Theorem C to be the main result of this manuscript.

Remarks 1.4: (1). The term “good” that appears in Theorem C is a generic property: it refers to those driving protocols whose image lies in a suitable open and dense topological subspace of $\mathcal{M}_X$. More precisely, the subspace we take, denoted by $\tilde{\mathcal{M}}_X$, is the subset of pairs $(E, W)$ in which either $E: X_{d-1} \to \mathbb{R}$ or $W: X_d \to \mathbb{R}$ is one-to-one.

(2). If $d = 1$, then $\delta = 1$ and Theorem C recovers a version of the integral quantization result of [CKS, thm. A]. In higher dimensions, the appearance of torsion phenomena in the integral homology of $X$ is partly responsible for the inversion of the number $\delta$. More precisely, the $\delta$ is a product of the $d$-spanning tree numbers and the $(d - 1)$-spanning co-tree numbers associated with the CW complex $X$, as in [CCK1],[CCK2]. For example, if $X$ is the real projective plane $\mathbb{R}P^2$ with standard cell structure (i.e., having a single cell in each dimension $\leq 2$), then $\delta = 4$.

1.5. An example. For $d \geq 2$, let $c: S^{d-1} \to S^{d-1} \vee S^{d-1}$ be the $(d-2)$-fold suspension of the map $S^1 \to S^1 \vee S^1$ which is given by the loop multiplication $xy^{-1}$, where $x, y$ denote the two inclusions of $S^1$ into $S^1 \vee S^1$.

Let
\[
X = (S^{d-1} \vee S^{d-1}) \cup (D^d \amalg D^d)
\]
be the CW complex of dimension $d$ given by attaching two $d$-cells to $S^{d-1} \vee S^{d-1}$, each one using the map $c$. Then
\[
H_d(X; \mathbb{Z}) \cong H_{d-1}(X; \mathbb{Z}) \cong \mathbb{Z}.
\]
Denote the $(d - 1)$-cells of $X$ by $f_1, f_2$ and the $d$-cells by $e_1, e_2$. We take $x \in H_{d-1}(X; \mathbb{Z})$ to be the generator defined by $f_1$. 

Let $W_1: [0, 1] \to [-1, 1]$ be any smooth function which vanishes for $t \in \{0, 1/2, 1\}$ and which satisfies $W_1(t) < 0$ for $t \in (0, 1/2)$ and $W_1(t) > 0$ for $t \in (1/2, 1)$. We take $W_1$ to be a one-parameter family of weights for the $d$-cell $e_1$. Set

$$W_2(t) := -W_1(t),$$

providing a family of weights for the $d$-cell $e_2$. Let $E_1: [0, 1] \to [-1, 1]$ be any smooth function such that $E_1(1/2) > 1$ and $E_1(0) = E_1(1) = -1$. Set $E_2(t) = -E_1(t)$. Then

$$\gamma(t) := (E_\bullet(t), W_\bullet(t))$$

defines a good 1-periodic continuous driving protocol on $X$. If we additionally assume $E_i'(0) = E_i'(1)$, $W_i'(0) = W_i'(1)$ for $i = 1, 2$, then $\gamma$ will be smooth.

**Theorem D:** With respect to the above choices, the average current in the low temperature, adiabatic limit coincides with the generator of $H_d(X; \mathbb{Z})$ given by the cycle $e_1 - e_2$.

**Remarks 1.5:** (1). Integer coefficients occur in Theorem D, since in this example $\delta = 1$, where $\delta$ is as in Theorem C.

(2). In the above example, the space of parameters $\mathcal{M}_X$ is a real vector space of dimension four. The topological subspace of “bad” parameters has codimension two. The low temperature adiabatic limit of the average current can be interpreted as the linking number of the good driving protocol $\gamma: S^1 \to \mathcal{M}_X$ with the subspace of bad parameters.

1.6. **Related Work.** This paper is an extension of the program introduced in [CKS] to higher dimensions. A topological study of continuous time random walks on graphs was performed there, and an explicit result regarding the long time behavior of trajectories was obtained. The results of that paper were proved using Kirchhoff’s theorems on the flow of current in an electrical circuit. While the motivations of the papers are similar, the generalization to higher dimensions introduces formidable technicalities.

Other authors have considered generalizations of random walks to higher dimensional simplicial complexes. Parzanchevski and Rosenthal [PR] define the $(p$-lazy) $k$-walk to be a Markov particle process on the set of $k$-simplices of a simplicial complex. In their setup, a $k$-simplex transitions to another $k$-simplex through a co-face, and they relate this to the ‘up-down’ component of the $k$-Laplacian. Mukherjee and Steenbergen [MS] consider a stochastic process...
where a \( k \)-simplex transitions to another \( k \)-simplex via a face in the simplicial complex. The latter is related to the ‘down-up’ component of the Laplacian, and the two processes are dual to one another. Rosenthal [Ro] also defined a simplicial branching random walk (SBRW), which modifies the \( k \)-walk so that a \( k \)-simplex transitions to all neighbors of an adjacent \((k + 1)\)-cell instead of a single neighbor. Our notion of Markov CW chain is closest to the SBRW of [Ro], but is still distinct.

The continuous time Markov chain considered in this paper is both time-inhomogeneous and not uniform (due to non-trivial values of \( E \) and \( W \)). Even if we restrict to the embedded discrete time process, take trivial weights, and use a simplicial complex in which we collapse out the \((k - 2)\)-skeleton (so every \((k - 1)\)-cell is a cycle), our process is still distinct from the \( k \)-walk and the SBRW. In the \( k \)-walk of [PR] and [MS], a simplex transitions to a single neighbor, instead of all adjacent neighbors as in the Markov CW chain. In the language of [Ro], the SBRW treats each of the \( z_b = \langle z, b \rangle \) ‘particles’ on a \( k \)-cell \( b \) independently. This is in contrast to the Markov CW chain, where all \( z_b \) ‘particles’ move together (see Figure 1).

Our definition of a Markov CW chain is derived from the notion of a Langevin process in statistical mechanics [vK, Ch. 9]. We imagine a smooth, compact, Riemannian manifold \((M, g)\) together with two additional pieces of data. The first is a Morse function \( f : M \to \mathbb{R} \) that satisfies the Morse–Smale transversality condition. Hence, the Morse–Smale chain complex is defined. The second is a stochastic vector field on \( M \) that possesses Gaussian and Markovian statistics. From a statistical mechanics perspective, the stochastic vector field arises from coupling our dynamical system to a ‘bath,’ i.e., another dynamical system with an enormous number of degrees of freedom.

These two ingredients can be used to define a stochastic flow on \( M \) (see, e.g., [K]). An initial embedded \( k \)-dimensional submanifold will then evolve according to the appropriate stochastic differential equation, also known as a Langevin equation. The setting of this manuscript is concerned with the low-temperature limit of these continuous processes under which the stochastic motion becomes more deterministic and is restricted to the associated Morse CW decomposition of \( M \) given by the unstable manifolds of \( f \) (cf. [Q]). In this way, the CW complexes considered here originate from the Morse–Smale CW decompositions of smooth manifolds (this is the basis for Hypothesis 2.1). It is worth noting that the smooth setting is what distinguishes our process from
those already appearing in the literature. The embedded submanifold is a $k$-cycle in bordism homology, forcing the state space to consist of $k$-cycles instead of $k$-cells. If the stochastic diffeomorphism pushes a portion of the $k$-cycle $z$ over a $(k+1)$-cell and onto all adjacent $k$-cells, it must do so uniformly. That is, all $z_b = \langle z, b \rangle$ ‘particles’ on a $k$-cell $b$ move together, not independently.

Figure 1. The motivation for an elementary transition of the Markov CW chain. The initial cycle has incidence $j$ with one face of a 2-cell. The stochastic vector field pushes the cycle off the face and across the entire 2-cell. Only the first and last pictures take place in the process on the CW complex; the intermediate figure is the smooth manifold picture which motivates our definition.

OUTLINE. Section 2 is about language. In section 3 we review some material on spanning trees and spanning co-trees that appears in our earlier papers [CCK1], [CCK2]. In section 4 we define the Markov CW chain, derive its basic properties and then give a proof of Theorem A. The proof of the Adiabatic Theorem appears in section 5. In section 6, we determine the low-temperature limit of the time-dependent Boltzmann distribution. Section 7 contains the proof of the Quantization Theorem. Section 8 validates the example (Theorem D).

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2. Preliminaries

2.1. Notation. Many of the functions appearing in this paper depend on several variables. To avoid clutter we typically avoid displaying the function arguments. However, when a particular variable is to be emphasized we display it. For example, if \( f(t, x, y, z) \) is a function of four variables, we typically write it as \( f \). When we wish to emphasize some of the variables, say \( t, z \), we write \( f = f_{t, z} \) or \( f = f(t, z) \). If \( f \) is differentiable in the \( t \)-variable, where \( t \) is viewed as time, we often write \( \dot{f} \) for its time derivative (i.e., \( \frac{\partial f}{\partial t} \)).

2.2. CW complexes. Let \( X \) be a finite CW complex of fixed dimension \( d \geq 1 \). We denote the \( k \)-skeleton of \( X \) by \( X^{(k)} \), and the set of \( k \)-cells by \( X_k \). We are primarily interested in the top dimensions of \( X \) (\( d \) and \( d - 1 \)), but our results hold for any intermediate dimension \( k \) by truncation to \( X^{(k)} \).

Recall that the CW structure of \( X \) is specified inductively by attaching cells of increasing dimension. The \( k \)-skeleton is formed from the \( (k - 1) \)-skeleton by means of attaching maps

\[
S^{k-1}_\alpha \xrightarrow{\varphi_\alpha} X^{(k-1)}
\]

where \( \alpha \) indexes the set of \( k \)-cells to be attached. Then

\[
X^{(k)} = X^{(k-1)} \cup \bigsqcup \alpha D^k_\alpha,
\]

where the disjoint union is amalgamated along the attaching maps.

2.3. The cellular chain complex. For a commutative ring \( A \), let

\[
C_k := C_k(X; A)
\]

denote the free \( A \)-module with basis \( X_k \). In this section, \( A = \mathbb{R} \) is the field of real numbers. For now, equip \( C_k \) with the standard inner product \( \langle -, - \rangle \) by declaring \( X_k \) to be an orthonormal basis. Recall that \( C_* \) forms a chain complex of abelian groups (vector spaces when \( A = \mathbb{R} \)), in which the effect of the boundary operator \( \partial : C_k \to C_{k-1} \) on a \( k \)-cell \( \alpha \) is

\[
\partial \alpha = \sum_{j \in X_{k-1}} b_{\alpha, j} j,
\]

where \( b_{\alpha, j} := \langle \partial \alpha, j \rangle \) is the incidence number of \( \alpha \) and \( j \); this is a finite sum. The incidence number can be explicitly described by means of the attaching
maps: $b_{\alpha,j}$ is the degree of the composite

$$S^{k-1}_\alpha \xrightarrow{\varphi_{\alpha}} X^{(k-1)} \rightarrow X^{(k-1)}/X^{(k-2)} \cong \bigvee_i S^{k-1}_i \rightarrow S^{k-1}_j,$$

where the last map is given by projection onto the wedge summand corresponding to cell $j$.

We will assume $X$ comes equipped with the following auxiliary structure:

**Hypothesis 2.1:** Let $k$ be an integer satisfying $1 \leq k \leq d$. For every $k$-cell $\alpha$ and $(k-1)$-cell $j$, we fix a choice of finite set $X(\alpha,j)$ such that the cellular boundary map $\partial: C_k \rightarrow C_{k-1}$ admits an additional decomposition:

$$b_{\alpha,j} = \sum_{\varepsilon_{\alpha,j} \in X(\alpha,j)} (-1)^{\chi(\varepsilon_{\alpha,j})},$$

where $\chi \in \{0, 1\}$.

**Remarks 2.2:** (1). Clearly, one can always make choices so that the hypothesis is satisfied. However, in the main cases of interest the decomposition comes for free.

In fact, the hypothesis is inspired by properties of the boundary map in the Morse–Smale complex of a Morse function $f: M \rightarrow \mathbb{R}$ on a compact Riemannian manifold $M$ satisfying the Morse–Smale transversality condition. In the Morse–Smale case, we take $X(\alpha,j)$ to be the (finite) set of flow lines between the corresponding critical points of index $k$ and $k-1$, $b_{\alpha,j}$ is a signed sum over the flow lines, with $(-1)^{\chi(\varepsilon)}$ the sign of flow line $\varepsilon$.

(2). There are other cases of interest in which the hypothesis holds without additional choices: if $X$ is a regular CW complex, connected polyhedron, or simplicial complex, then the hypothesis holds with $|X(\alpha,j)| = 1$ for every $\alpha$ and $j$, and therefore, $b_{\alpha,j} = \pm 1$.

The **coboundary operator** $\partial^*: C_{k-1} \rightarrow C_k$ is the formal adjoint to the boundary operator in the standard inner products. Explicitly,

$$\partial^* j = \sum_{\alpha \in X_d} b^*_{j,\alpha} \alpha,$$

for any $(k-1)$-cell $j$, where $b^*_{j,\alpha} := b_{\alpha,j}$.

We write $H_*(X; A)$ for the **cellular homology** of $X$ with coefficients in $A$. That is,

$$H_k(X; A) := Z_k/B_k,$$
where $Z_k$, the group of $k$-cycles, is the kernel of the homomorphism $\partial: C_k \rightarrow C_{k-1}$, and $B_k$, the group of $k$-boundaries, is the image of $\partial: C_{k+1} \rightarrow C_k$. The quotient is well-defined since $\partial \circ \partial = 0$.

2.4. The biased coboundary operator. Fix a system of weights $(E_\bullet, W_\bullet)$ as well as a real number $\beta > 0$. One may then associate (diagonal) operators

$$e^{\beta E}: C_{d-1}(X; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R})$$

and

$$e^{\beta W}: C_d(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R})$$

by

$$j \mapsto e^{\beta E} j \quad \text{and} \quad \alpha \mapsto e^{\beta W} \alpha,$$

for $j \in X_{d-1}$ and $\alpha \in X_d$. We use these to equip $C_d(X; \mathbb{R})$ and $C_{d-1}(X; \mathbb{R})$ with modified inner products: for $i, j \in X_{d-1}$ and $\alpha, \gamma \in X_d$, set

$$\langle i, j \rangle_E := e^{\beta E} \delta_{ij} \quad \text{and} \quad \langle \alpha, \gamma \rangle_W := e^{\beta W} \delta_{\alpha\gamma},$$

where in this case $\delta$ denotes the Kronecker delta. The modified inner products are then given by extending these formulas bilinearly.

If we define the formal adjoint of $\partial$ using the modified inner products, we obtain the biased coboundary operator; explicitly,

$$\partial_{E,W}^* := e^{-\beta W} \partial^* e^{\beta E},$$

where $\partial^*$, the standard coboundary operator, is the formal adjoint with respect to the standard inner products.

3. Combinatorial structures

We briefly recall the properties of spanning trees and spanning co-trees in this section. We do not present any new results in this section and we refer the reader to [CCK1] and [CCK2] for a more complete treatment. For a finite complex $Y$, let $\beta_k(Y)$ be the $k$-th Betti number, i.e., the rank of $H_k(Y; \mathbb{Q})$.

3.1. Spanning trees.

Definition 3.1: Assume $\dim X = d \geq 1$ and let $1 \leq k \leq d$. A $k$-spanning tree for $X$ is a subcomplex $i: T \subset X$ such that

- $H_k(T; \mathbb{Z}) \cong 0$, and
- $\beta_{k-1}(T) = \beta_{k-1}(X)$, and
- $X^{(k-1)} \subset T \subset X^{(k)}$.

When $k = d$, we simplify the terminology to spanning tree.
Remark 3.2: If \( d = k = 1 \) then the above coincides with the usual notion of spanning tree in a connected graph.

Definition 3.3: A \( k \)-cell \( b \in X_k \) is said to be **essential** if there exists a \( k \)-cycle \( z \in Z_k(X; \mathbb{R}) \) such that \( \langle z, b \rangle \neq 0 \).

Removing an essential \( k \)-cell from \( X^{(k)} \) results in a complex in which \( \beta_k \) decreases by one and \( \beta_{k-1} \) is fixed [CCK1, Lemma 2.2]. Every \( k \)-spanning tree can therefore be constructed by iteratively removing essential \( k \)-cells from \( X^{(k)} \).

Definition 3.4: Let \( k = d \). For a spanning tree \( T \), define a linear transformation

\[
\varsigma_T: B_{d-1}(X; \mathbb{Q}) \to C_d(T; \mathbb{Q})
\]

as follows: \( \varsigma_T(b) \) is the unique \( d \)-chain in \( T \) so that \( \partial \varsigma_T(b) = b \).

The \( d \)-chain \( \varsigma_T(b) \) exists since

\[
B_{d-1}(T; \mathbb{Q}) = B_{d-1}(X; \mathbb{Q})
\]

for every spanning tree \( T \). The chain is unique since the difference of any two distinct \( d \)-chains with boundary \( b \) would give rise to a non-trivial \( d \)-cycle in \( T \), for which there are none.

Definition 3.5: For a given system of weights \((E_\bullet, W_\bullet)\) on \( X \), the **weight** of a spanning tree \( T \) is the positive real number

\[
w_T := \theta_T^2 \prod_{\alpha \in T_d} e^{-\beta W_\alpha},
\]

where \( \theta_T \) denotes the order of the torsion subgroup of \( H_{d-1}(T; \mathbb{Z}) \).

Theorem 3.6 (cf. [CCK1, thm. A]): With respect to the modified inner product \( \langle -,- \rangle_W \), an orthogonal splitting to the boundary operator

\[
\partial: C_d(X; \mathbb{R}) \to B_{d-1}(X; \mathbb{R})
\]

is given by

\[
\mathcal{A} := \frac{1}{\Delta} \sum_T w_T \varsigma_T,
\]

where the sum is over all spanning trees, and \( \Delta = \sum_T w_T \).
Remark 3.7: The map $\mathcal{A}$ is an orthogonal splitting of $\partial$ in the short exact sequence

$$0 \to Z_d(X; \mathbb{R}) \to C_d(X; \mathbb{R}) \xrightarrow{i} B_{d-1}(X; \mathbb{R}) \to 0,$$

with respect to the modified inner product $\langle -, - \rangle_W$. It follows that

$$I - \mathcal{A}\partial : C_d(X; \mathbb{R}) \to Z_d(X; \mathbb{R})$$

gives the orthogonal projection of $i$. The latter operator was constructed explicitly and studied in [CCK1].

3.2. Spanning co-trees.

Definition 3.8: Assume $\dim X = d \geq 1$. Fix an integer $k$ with $0 \leq k \leq d$. A $k$-spanning co-tree for $X$ is a subcomplex $j : L \subset X$ such that

- $j_* : H_k(L; \mathbb{Q}) \to H_k(X; \mathbb{Q})$ is an isomorphism,
- $\beta_{k-1}(L) = \beta_{k-1}(X)$, and
- $X^{(k-1)} \subset L \subset X^{(k)}$.

When $k = d - 1$ we shorten the terminology to spanning co-tree.

Remark 3.9: Similar to $k$-spanning trees, $k$-spanning co-trees are shown to exist by removing certain $k$-cells from $X^{(k)}$.

Note that a 0-spanning co-tree is just a 0-cell of $X$. There is only one $d$-spanning co-tree given by $X$.

We now restrict to the case $k = d - 1$. Since a spanning co-tree $L$ has no $d$-cells, the relative homology group $H_{d-1}(X, L; \mathbb{Q})$ is trivial. It follows that $H_{d-1}(X, L; \mathbb{Z})$ is finite; let $a_L$ denote its order. Note that the composite

$$\phi_L : Z_{d-1}(L; \mathbb{Z}) \xrightarrow{\cong} H_{d-1}(L; \mathbb{Z}) \to H_{d-1}(X; \mathbb{Z})$$

is a rational isomorphism since $L$ has no $d$-cells.

Definition 3.10: With $L$ as above, let $\psi_L : H_{d-1}(X; \mathbb{Q}) \to Z_{d-1}(X; \mathbb{Q})$ denote the composite

$$H_{d-1}(X; \mathbb{Q}) \xrightarrow{(\phi_L \otimes \mathbb{Q})^{-1}} Z_{d-1}(L; \mathbb{Q}) \xrightarrow{j_*} Z_{d-1}(X; \mathbb{Q}).$$
Definition 3.11: For a given system of weights $(E, W)$ on $X$, the weight of a spanning co-tree $L$ is the positive real number

$$b_L = a_L^2 \prod_{b \in L_{d-1}} e^{-\beta E_b}.$$ 

Theorem 3.12 ([CCK2, thm. A]): With respect to the modified inner product $\langle -, - \rangle_E$, the orthogonal splitting to the quotient homomorphism

$$Z_{d-1}(X; \mathbb{R}) \to H_{d-1}(X; \mathbb{R})$$

is given by

$$\rho^B(E) = \rho^B = \frac{1}{\nabla} \sum_L b_L \psi_L,$$ 

where the sum is over all spanning co-trees $L$, and $\nabla = \sum_L b_L$.

Definition 3.13 (cf. [CCK2, defn. 1.12]): Let $x \in H_{d-1}(X; \mathbb{Z})$ be an integer homology class. The Boltzmann distribution at $x$ is the real $(d-1)$-cycle

$$\rho^B(x) := \frac{1}{\nabla} \sum_L b_L \psi_L(\bar{x}) \in Z_{d-1}(X; \mathbb{R}),$$

where $\bar{x} \in H_{d-1}(X; \mathbb{Q})$ is the image of $x$ under the homomorphism

$$H_{d-1}(X; \mathbb{Z}) \to H_{d-1}(X; \mathbb{Q}).$$

Remark 3.14: For a spanning tree $T$, let $A \subset \mathbb{Q}$ be a ring in which $\theta_T$ is a unit. An elementary diagram chase involving the long exact sequence in homology of the pair $(X, T)$ implies that the linear transformation $\varsigma_T$ uniquely lifts to a homomorphism

$$B_{d-1}(X; A) \to C_d(T; A).$$

Similarly, for any spanning co-tree $L$, if the $a_L$ is a unit in $A$, then $\psi_L$ uniquely lifts to a homomorphism

$$H_{d-1}(X; A) \to Z_{d-1}(X; A).$$

Remark 3.15: The Boltzmann distribution is the unique ‘harmonic form’ on $X$ as specified by combinatorial Hodge theory (see [CCK2]). Remark 3.14 specifies the minimal coefficients under which the harmonic form of a homology class will exist.
Example 3.16: Let $X$ denote the torus with CW structure given by four 0-cells, eight 1-cells, and four 2-cells, shown in Figure 2. We make the usual identifications of opposite sides in this picture, although this is not shown explicitly. Instead, the displayed arrows label a chosen orientation.

This complex has four 2-spanning trees, given by removing any single 2-cell. There are thirty-two 1-spanning trees, obtained by subtracting the 24 loops of $X^{(1)}$ from the 56 possible choices of 3 edges.

On the other hand, there are thirty-two 1-spanning co-trees, and four 0-spanning co-trees (cf. Figure 3). These statements can be obtained by careful enumeration or by using Theorem [CCK2, Corollary D].

4. The process

In this section, we construct a Markov CW chain given a system of weights on $X$. We continue to assume $X$ is a connected finite CW complex of dimension $d \geq 1$. Our recipe makes use of Hypothesis 2.1 in the case $k = d$.

4.1. The cycle-incidence graph.

Definition 4.1: For an integer $(d - 1)$-cycle $z_0 \in Z_{d-1}(X; \mathbb{Z})$, let

$$Z_{d-1}^z(X; \mathbb{Z}) = z_0 + B_{d-1}(X; \mathbb{Z})$$

denote the coset consisting of the integral $(d - 1)$-cycles that are homologous to $z_0$. 
(a) Three distinct 1-spanning trees of the torus, out of the 32 total.

Figure 3

Definition 4.2: Consider the directed graph $G$ defined as follows. The vertices of $G$ are given by integer $(d-1)$-cycles $z$ homologous to $z_0$, i.e.,

$$z \in Z_{d-1}^z(X; \mathbb{Z}).$$

A directed edge of $G$ with source $z$ is specified by a 4-tuple

$$e := (\alpha, f, \varepsilon_{\alpha, f}, z)$$

with $\alpha \in X_d$, $f \in X_{d-1}$, and $\varepsilon_{\alpha, f} \in X(\alpha, f)$, satisfying the following:

- $\langle z, f \rangle \neq 0$,
- $\langle \partial \alpha, f \rangle = \sum_{\varepsilon \in X(\alpha, f)} (-1)^{\chi(\varepsilon)} \neq 0$, and
- $z' = z - (-1)^{\chi(\varepsilon_{\alpha, f})} \langle z, f \rangle \partial \alpha$.

In the above, the target of the edge $e$ is defined to be $z'$. To indicate this, we sometimes write

$$z = s(e), \quad z' = t(e).$$

The cycle-incidence graph

$$\Gamma := \Gamma_{X, z_0}$$

is the directed subgraph of $G$ given by the directed path component of $z_0$. That is, a vertex $z$ lies in $\Gamma$ if there exists a finite sequence of directed edges $z_0 \to z_1 \to \cdots \to z_k \to z$, i.e., there is a finite directed path from $x$ to $z$. An edge belongs to $\Gamma$ if and only if it occurs in such a path.
Figure 4. Two elementary transitions on the torus of Example 3.16. According to the orientations of Figure 2, the initial cycle first jumps across the 2-cell $A$ and then the 2-cell $B$, resulting in the displayed cycles.

The cycle-incidence graph is the state diagram of the Markov CW chain described in the introduction, in which the cycle $z_0$ represents an initial condition. For a particular choice of $(d-1)$-cell $f$ incident to $z$ and $d$-cell $\alpha$ incident to $f$, the cycle $z$ can ‘hop’ across the $d$-cell $\alpha$, to form a new cycle

$$z' := z - (-1)^{\chi(\epsilon, f)} \langle z, f \rangle \partial \alpha.$$ 

This type of jump is known as an elementary transition. Informally, an elementary transition consists of the cycle $z$ completely ‘jumping off’ of the cell $f$ across $\alpha$ to form the new cycle $z'$ (cf. Figures 1 and 4).

Remark 4.3: Typically, the newly formed cycle $z'$ will still have non-zero incidence with the $(d-1)$-cell $f$ that is used in defining the elementary transition. There is one notable exception to this: when $|X(\alpha, f)| = 1$ we have $b_{\alpha, f} = \pm 1$. Consequently,

$$\langle z', f \rangle = \langle z, f \rangle - (-1)^{\chi(\epsilon, f)} \langle z, f \rangle \sum_k b_{\alpha, k} \langle k, f \rangle$$

$$= \langle z, f \rangle - (-1)^2 \chi(\epsilon, f) \langle z, f \rangle = 0.$$ 

Therefore, in this case $z'$ will have trivial incidence with $f$.

When $\dim X = d = 1$, it is not hard to identify the directed graph $\Gamma$ provided that the initial state is a vertex. Define the double $DX$ of $X$ to be the directed graph with the same set of vertices, where a directed edge is specified by a pair

$$(i, \alpha) \in X_0 \times X_1.$$
such that \( i \) is an endpoint \( \alpha \). We also assume that \( \alpha \) has distinct endpoints. We take the initial state \( z_0 \) to be any vertex of \( X \). We also remind the reader that \( X \) is assumed to be finite and connected.

**Lemma 4.4:** With respect to the above assumptions, \( \Gamma = DX \).

**Remark 4.5:** In this case, we are implicitly taking the finite sets \( X(\alpha, f) \) to be singletons since \( b_{\alpha, f} = \pm 1 \) in the case of graphs.

**Proof of Lemma 4.4.** Let \( \alpha \) be an edge of \( X \) and write \( \partial \alpha = j - i \) for the value of the boundary operator at \( \alpha \), where \( i \) and \( j \) are distinct vertices given by the endpoints of \( \alpha \). Then the directed edge \((i, \alpha)\) determines an elementary transition from \( i \) to \( j \) given by the equation

\[
j = i + b_{\alpha, i} \partial \alpha
\]

where in this case \( b_{\alpha, i} = -1 \). Similarly \((j, \alpha)\) provides an elementary transition from \( j \) to \( i \) given by

\[
i = j + b_{\alpha, j} \partial \alpha
\]

where \( b_{\alpha, j} = +1 \). It is straightforward to check that every elementary transition with source/target \( i \) is given by the above. Since the initial state is a vertex, the above also shows every other state arising from a sequence of elementary transitions is also a vertex. Furthermore, as \( X \) is connected, every vertex can be reached by such a sequence. It follows that \( \Gamma = DX \). \( \blacksquare \)

### 4.2. The Rates

Let \((\tau_D, \gamma)\) be a driving protocol. Then

\[
\gamma(t) := (E_\bullet(t), W_\bullet(t))
\]

where \(E_\bullet : X_{d-1} \to \mathbb{R}\) and \(W_\bullet : X_d \to \mathbb{R}\) are one-parameter families of weights. Let \(\beta > 0\) be a real number representing inverse temperature.

Let \(e = (\alpha, f, \epsilon_{\alpha, f}, z)\) be a directed edge of \(\Gamma\). The number

\[
k_{\alpha, f}(t) := e^{\beta(E_f(t) - W_\alpha(t))}
\]

will be taken as the transition rate along \(e\) at time \(t\). In what follows, we sometimes denote the pair \((\alpha, f)\) by \((\alpha_e, f_e)\). Let the collection of such rates be denoted by \(k_\bullet\). Then, the pair

\[
(\Gamma, k_\bullet)
\]

completes the description of the Markov CW chain.
4.3. The Master Equation. The rates give rise to a time-dependent evolution operator $\mathcal{H}$ operating on the vector space $C_0(\Gamma; \mathbb{R})$ of 0-chains, where for $z \in \Gamma_0$ we have

\begin{equation}
\mathcal{H}(z) := \sum_{e \in \Gamma_1, s(e) = z} k_{\alpha_e, f_e} \cdot (t(e) - z).
\end{equation}

Note that the sum is finite since the vertices of $\Gamma$ have finite valence.

Consider the obvious embedding

$C_0(\Gamma; \mathbb{R}) \subset C^0(\Gamma; \mathbb{R})$

from 0-chains to 0-cochains, i.e., functions $\Gamma_0 \to \mathbb{R}$ which we regard as “distributions.” Extend $\mathcal{H}$ to act on $C^0(\Gamma; \mathbb{R})$ as follows: given a distribution $p: \Gamma_0 \to \mathbb{R}$ define

\begin{equation}
\mathcal{H}(p)(z) = \sum_{w \in \Gamma_0} \mathcal{H}_{z,w} p(w),
\end{equation}

where $\mathcal{H}_{z,w}$ denotes the $(z, w)$-matrix entry of $\mathcal{H}$. Again, the sum is finite since the number of non-trivial entries in every row and column is finite.

The evolution of the process is described by the master equation

\begin{equation}
\dot{p} = \tau_D \mathcal{H} p, \quad p(0) = p_0,
\end{equation}

where $p(t)$ is a one-parameter family of 0-cochains. In what follows, we choose the initial distribution $p_0$ to be

$p_0(z) = \begin{cases} 1, & z = z_0, \\ 0, & \text{otherwise}, \end{cases}$

where $z_0 \in \Gamma_0$ is a fixed vertex.

Example 4.6: Assume $\dim X = d = 1$. Then by Lemma 4.4, $\Gamma = DX$. Hence

$C^0(\Gamma; \mathbb{R}) \cong C_0(X; \mathbb{R})$

canonically. We choose the initial state $z_0$ to be any vertex of $X$. In this case $\mathcal{H}$ is identified with the biased Laplacian $-\partial \partial_{E,W}$ acting on $C_0(X; \mathbb{R})$ and the process coincides with the one of [CKS].
4.4. THE TRAJECTORY SPACE. Let Γ and \( k \) be as above and let \( t > 0 \) be given. A trajectory of length \( n \) consists of a directed path

\[
z_0 \xrightarrow{e_1} z_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} z_{n-1} \xrightarrow{e_n} z_n
\]
together with jump times \( 0 := t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t_{n+1} =: t \), where \( e_k = (\alpha_k, f_k, \varepsilon_k, z_k) \) is a directed edge of Γ from \( z_k \) to \( z_{k+1} \). We use the notation

\((z_*, e_*, t_*)\)
to refer to this trajectory.

Define the escape rate at a vertex \( z \in \Gamma_0 \) over the interval \([t, t']\) by the expression

\[
u_z(t, t') = \exp \left( - \sum_{e \in X(z)} \int_t^{t'} \tau_D k_{\alpha_e, f_e}(s) \, ds \right),
\]
where \( X(z) \subset X_1 \) is the set of directed edges having terminal vertex \( z \), and \( k_{\alpha_e, f_e} \) is the transition rate across the directed edge \( e \) (cf. Eq (14)).

**Definition 4.7:** With respect to the initial distribution \( p_0 \) defined above, the probability density of the trajectory \((z_*, e_*, t_*)\) is

\[
f[z_*, e_*, t_*] := \prod_{m=1}^{n+1} u_{z_m}(t_{m-1}, t_m) \prod_{m=1}^{n} \tau_D k_{\alpha_{e_m}, f_{e_m}}(t_m).
\]

Finally, given that the process is at state \( z_0 \) at time 0, the probability that the process is in state \( z \) at time \( t \) is

\[
P[z; t] := \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \sum_{(z_*, e_*, t_*)} f[z_*, e_*, t_*] \sum_{z_n = z}
\]
where the summation on the far right runs over the set of trajectories of length \( n \) that begin in \( z_1 \) and terminate in \( z \).

**Proposition 4.8:** For every \( t \geq 0 \), the function \( z \mapsto P[z; t] \) is a probability distribution.

**Proof.** We only provide a sketch of the proof here, and refer the reader to [CCK4] for the details. We first show that series (19) converges. Note that the series consists of positive terms. As there are finitely many rates, the graph Γ
has a global bound on its degree, the numbers \( t_j \) are bounded, it follows that
the \( n^{th} \) term of (19) is bounded by
\[
x^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_1 \cdots dt_n = \frac{(xt)^n}{n!},
\]
for a judicious choice of \( x > 0 \) (which depends on \( \beta \) and \( \tau_D \)). The series \( \sum \frac{(xt)^n}{n!} \)
converges to \( e^{xt} \). Hence, by the comparison test, the series (19) converges.

To conclude the proof, we need to explain why the series
\[
\sum_z P[z; t]
\]
converges to 1.

We first show that the expression (19) is a formal solution to the master equation. Set \( A := \tau_D \mathcal{H} \). With this notation the master equation becomes \( \dot{p} = Ap \)
with \( p(0) = p_0 \).

Rewrite \( A \) as \( A^0 + A^1 \), where \( A^0 \) is the diagonal matrix and \( A^1 \) equals 0 along
the diagonal. Now set \( A_\epsilon = A^0 + \epsilon A^1 \). We then consider the equation
\[
\dot{p} = A_\epsilon p, \quad p(0) = p_0,
\]
where we assume the solution has the form
\[
p^0 + \epsilon p^1 + \epsilon^2 p^2 + \cdots,
\]
where \( p^0 \) solves the equation \( \dot{q} = A_0 q \). If we substitute the expression (21)
into the equation (20), expand both sides, equate the coefficients of \( \epsilon^j \) for
\( j = 0, 1, \ldots \), and set \( \epsilon = 1 \), we tediously but straightforwardly arrive at the
expression (19). Hence, (19) is the formal solution to the master equation (17).

To complete the proof it is enough to show that the formal solution \( q(t) \) to
the master equation is a probability distribution for each \( t \). Let \( u \) be the row
vector whose value at every vertex of \( \Gamma \) is 1. Multiplying both sides of (17) by \( u \)
on the left, we obtain
\[
u \cdot \dot{p} = u \cdot (Ap) = (uA) \cdot p = 0,
\]
since the sum of the entries in any column of \( A \) vanishes. As \( u \cdot \dot{p} = 0 \), we
infer that the formal solution \( p(t) \) is such that \( u \cdot p(t) = c \) for some constant \( c \).
Since \( u \cdot p(0) = 1 \), it follows that \( c = 1 \). Consequently, the formal solution
\( q(t) := P[z, t] \) is a probability distribution for all \( t \).

The proof of Proposition 4.8 also established the following result.
Corollary 4.9: The function $P[z, t]$ is the solution to the master equation (17).

Remark 4.10 (The associated Markov process): Denote the space of trajectories by $\mathcal{P}_\Gamma$. Then the probability density function $f$ gives rise to a probability on distribution on $\mathcal{P}_\Gamma$. Moreover, for each $t \geq 0$, one has a random variable $X_t: \mathcal{P}_\Gamma \to \Gamma_0$, which assigns to a trajectory $(z_\bullet, e_\bullet, t_\bullet)$ the vertex $z_j$ satisfying the inequality $t_j \leq t < t_{j+1}$ for $0 \leq j \leq n$ (here we employ the convention $t_{n+1} = \infty$). Then $X_t$ is a stochastic process satisfying the Markov condition.

4.5. Expectation. The expectation of a 0-cochain $p: \Gamma_0 \to \mathbb{R}$ is the formal sum

$$\mathbb{E}[p] := \sum_{z \in \Gamma_0} p(z)z.$$

We will give criteria for deciding when such an expression exists as an element of $Z_{d-1}(X; \mathbb{R})$.

Let $\Sigma = 2^{\Gamma_0}$ be the $\sigma$-algebra of all subsets of $\Gamma_0$. For fixed $b \in X_{d-1}$, the function

$$\mu_b(A) := \sum_{z \in A} \langle z, b \rangle, \quad A \in \Sigma$$

is a signed measure on $(\Gamma_0, \Sigma)$.

For a function $p: X_{d-1} \to \mathbb{R}$, i.e., a 0-cochain, we consider the series

$$\sum_{z \in \Gamma_0} p(z)\langle z, b \rangle := \int p \, d\mu_b$$

(22)

$$:= \int p \, d\mu_b^+ - \int p \, d\mu_b^-$$

(i.e., the Lebesgue integral over a discrete measure space), where $\mu_b = \mu_b^+ - \mu_b^-$ is the Hahn–Jordan measure decomposition [H] of $\mu_b$, in which $\mu_b^\pm$ are the unsigned measures

$$\mu_b^+(A) := \sup\{\mu(B) \mid B \subset A\},$$

$$\mu_b^-(A) := \sup\{-\mu(B) \mid B \subset A\}.$$

Definition 4.11: A 0-cochain $p \in C^0(\Gamma; \mathbb{R})$ is good if the integrals

$$\int p \, d\mu_b^\pm$$

are finite for all $b \in X_{d-1}$. In particular, the series (22) converges for good 0-cochains $p$ and for every $b \in X_{d-1}$. 
Lemma 4.12: If \( p \) is good, then its expectation \( \mathbb{E}[p] \) defines an element of \( Z_{d-1}(X; \mathbb{R}) \).

Proof. By the identity \( z = \sum_b \langle z, b \rangle b \), we infer

\[
\mathbb{E}[p] = \sum_z p(z)z = \sum_b \left( \sum_z p(z) \langle z, b \rangle \right) b,
\]

where the outer summation is finite. By hypothesis, the inner summation converges. It follows that \( \mathbb{E}[p] \) defines an element of \( C_{d-1}(X; \mathbb{R}) \). But clearly, this element is a cycle.

Example 4.13: Assume \( \dim X = 1 \) and choose \( z_0 \) to be any vertex of \( X \). Then \( \Gamma = DX \) by Lemma 4.4 and every \( p \in C^0(\Gamma; \mathbb{R}) \cong C_0(X; \mathbb{R}) \) is good. With respect to this identification, the expectation \( \mathbb{E}: C^0(\Gamma; \mathbb{R}) \to C_0(X; \mathbb{R}) \) is the identity homomorphism.

We now return to \( \varrho(t) := P[z, t] \), the formal solution of the master equation (17). Set

\[
\rho(t) := \mathbb{E}[\varrho(t)],
\]

so that \( \rho(t) \) is the expected value of \( P[z, t] \) with respect to \( z \).

Remark 4.14: With the identity \( z = \sum_b \langle z, b \rangle b \), we put the following norm on cycles:

\[
||z|| = \sum_{b \in X_{d-1}} |\langle z, b \rangle|.
\]

Lemma 4.15: The 1-parameter of family of 0-cochains \( \varrho(t) \) is good. In particular, the expected value \( \rho(t) = \mathbb{E}[\varrho(t)] \) defines a 1-parameter family of elements of \( Z_{d-1}(X; \mathbb{R}) \).

Proof. Note the inequality \( |\langle z, b \rangle| \leq ||z|| \) holds for every \( b \in X_{d-1} \). Set \( \rho_t := \rho(t) \), and recall that we have fixed a vertex \( z_0 \in \Gamma_0 \) in defining \( \varrho_t \) via (17). Then for each \( t \), it will be enough to prove that the series

\[
\sum_{z \in \Gamma_0} \varrho_t(z) ||z||
\]
converges. We filter \( z \in \Gamma_0 \) by the number of edges in a minimal path from \( z \) to \( z_0 \); call this number \( u(z) \). The previous display can then be rewritten as

\[
\sum_{n=0}^{\infty} \sum_{u(z)=n} \rho_t(z) ||z||.
\]

The graph \( \Gamma \) possesses the following global finiteness property: there is a number \( c > 0 \) such that the valence of any vertex of \( \Gamma \) is at most \( c \). In particular, the number of directed paths of length \( n \) which start at a given vertex is at most \( c^n \). Using this observation, the proof of Proposition 4.8, and Taylor’s remainder theorem, there is a \( w > 0 \) (which depends on \( t, c, \beta \) and \( \tau_D \)) such that

\[
\sum_{u(z)=n} \rho_t(z) ||z|| \leq \frac{e^{w}w^n}{n!}.
\]

Consequently,

\[
\sum_{n=0}^{\infty} \sum_{u(z)=n} \rho_t(z) ||z|| \leq e^w \sum_{n=0}^{\infty} \frac{w^n}{n!} = e^{2w}.
\]

4.6. THE DYNAMICAL EQUATION.

**Definition 4.16:** For a periodic driving protocol \((\tau_D, \gamma)\) with \( \gamma(t) = (E(t), W(t)) \), the **dynamical operator**

\[
H(t) : C_{d-1}(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R})
\]

is defined by

\[
H = -\partial e^{-\beta W} \partial^* e^{\beta E}.
\]

**Remark 4.17:** If \( f \in X_{d-1} \) then

\[
H(f) = \sum_{\alpha \in X_d} k_{\alpha, f} \langle f, \partial \alpha \rangle \partial \alpha
\]

\[
= \sum_{\alpha \in X_d} (-1)^{\chi(\epsilon_{\alpha, f})} k_{\alpha, f} \partial \alpha,
\]

where \( k_{\alpha, f} := e^{\beta(E_f-W_{\alpha})} \).

**Definition 4.18:** The **dynamical equation** is

\[
\dot{p} = \tau_D H p.
\]
In the above, it is implicitly assumed that the initial value \( p(0) \in C_{d-1}(X; \mathbb{R}) \) of a solution is a cycle representing a fixed homology class.

**Lemma 4.19:** If \( p \in C^0(\Gamma; \mathbb{R}) \) is good, then \( \mathbb{E}[\mathcal{H}(p)] \) defines an element of \( Z_{d-1}(X; \mathbb{R}) \). Furthermore, the following identity holds formally:

\[
\mathbb{E}[\mathcal{H}(p)] = H(\mathbb{E}[p]).
\]

**Proof.** If \( p \) is good, then \( \mathbb{E}[p] \) converges to an element of \( Z_{d-1}(\Gamma; \mathbb{R}) \) (cf. Lemma 4.12). The linear transformation \( H \) is continuous since it acts on a finite dimensional vector space. It follows that \( H(\mathbb{E}[p]) \) also converges. By a straightforward calculation using (16) and (23), both \( \mathbb{E}[\mathcal{H}(p)] \) and \( H(\mathbb{E}[p]) \) are given by the expression

\[
\sum_{z \in \Gamma_0} \sum_{e \in \Gamma_1} (-1)^{\chi(\epsilon_{\alpha_e}, f_e)} k_{\alpha_e, f_e} p(z, f_e, z) \partial \alpha_e.
\]

Hence, \( \mathbb{E}[\mathcal{H}(p)] = H(\mathbb{E}[p]) \) and \( \mathbb{E}[\mathcal{H}(p)] \) is convergent. In particular, \( \mathbb{E}[\mathcal{H}(p)] \) defines an element of \( Z_{d-1}(X; \mathbb{R}) \).

From this last result we readily deduce Theorem A:

**Corollary 4.20:** The family of cycles \( \rho(t) \in Z_{d-1}(X; \mathbb{R}) \) is the unique solution to the dynamical equation (24) having initial value \( z_0 \).

**Proof.** Set \( P = P[z, t] \). By Lemma 4.15, \( \rho(t) = \mathbb{E}[P[z, t]] \) converges and thus differentiation commutes with expectation. Application of Lemma 4.19 and Corollary 4.9 yields

\[
\dot{\rho} = \mathbb{E}[\dot{P}] = \mathbb{E}[\tau_D \mathcal{H}P] = \tau_D \mathbb{E}[\mathcal{H}P] = \tau_D H \mathbb{E}[P] = \tau_D H \rho.
\]

**5. The adiabatic theorem**

In this section, we state and prove the Adiabatic Theorem (Theorem B) for the Markov CW chain on \( X \). The adiabatic theorem states that for slow enough driving, a periodic solution to the dynamical equation exists and is unique. Our proof is similar to that of [CKS], but modified appropriately to the higher dimensional setting.
5.1. Formal solution. The dynamical equation is a first order linear system of differential equations, and so specifying an initial condition guarantees the existence of a unique solution \([A]\). We introduce the time-ordered exponential \(U(t, t_0)\) for \(0 \leq t_0 \leq t \leq 1\), which uniquely solves the initial value problem

\[
\frac{d}{dt} U(t, t_0) = \tau_D H(t) U(t, t_0) \quad U(t_0, t_0) = I.
\]

Explicitly,

\[
U(t, t_0) = \lim_{N \to \infty} e^{\varepsilon \tau_D H(t_N)} e^{\varepsilon \tau_D H(t_{N-1})} \cdots e^{\varepsilon \tau_D H(t_0)},
\]

where \(\varepsilon = t/N\) and \(t_j = j\varepsilon\). The expression

\[
\rho(t) = U(t, 0) \rho(0) = (\lim_{N \to \infty} e^{\varepsilon \tau_D H(t_N)} e^{\varepsilon \tau_D H(t_{N-1})} \cdots e^{\varepsilon \tau_D H(t_0)}) \rho(0)
\]

gives the formal solution to the dynamical equation (24) for \(\rho(0) = z_0\). The time-ordered exponential is often denoted

\[
\hat{T} \exp \left( \tau_D \int_{t_0}^{t} H(\tau) d\tau \right) := U(t, t_0),
\]

in analogy with the solution to a one-dimensional differential equation.

**Definition 5.1:** For an operator \(A: V \to V\) on a finite-dimensional real inner product space \(V\), let

\[
|A| := \sup_{v \neq 0} \frac{|Av|}{|v|} = \sup_{|v|=1} |Av|
\]

be the standard operator norm. If \(A\) is self-adjoint, then \(|A| = \lambda\), where \(\lambda\) is the maximum of the absolute value of the eigenvalues of \(A\).

In what follows, we think of \(U(t, t_0)\) as acting on \(B_{d-1}(X; \mathbb{R})\), where the latter is equipped with the norm arising from the restriction of the modified inner product \(\langle - , - \rangle_{E(t)}\).

**Lemma 5.2:** Let \((\tau_D, \gamma)\) be a driving protocol. There exists a positive constant \(\lambda\) so that for all \(t > t_0 \in [0, 1]\),

\[
|U(t, t_0)| < e^{-\lambda \tau_D (t-t_0)}.
\]

**Proof.** For \(t \in [0, 1]\), let \(A(t) = \tau_D H(t)\) acting on \(B_{d-1}(X; \mathbb{R})\). Then \(A(t)\) is negative definite and self-adjoint with respect to the restriction of the inner product \(\langle - , - \rangle_{E(t)}\) to \(B_{d-1}(X; \mathbb{R})\). By compactness there is a \(\lambda > 0\) such that \(-\lambda\) is greater than or equal to all eigenvalues of \(A(t)\) for every \(t \in [0, 1]\).
Let $C$ be the constant operator given by $Cv = -\tau_D \lambda v$ and let $U_C(t, t_0)$ be the evolution operator for $C$. Then

$$|U_C(t, t_0)| = e^{-\lambda \tau_D (t-t_0)}.$$ 

But clearly, $|U(t, t_0)| \leq |U_C(t, t_0)|$. ■

**Proof of Theorem B.** Write $\rho^B(t) = \rho^B(\gamma(t), \beta)$ for the time-dependent Boltzmann distribution. Then $\rho^B$ is 1-periodic, since $\gamma$ is. Let $\rho(t)$ denote a solution to the dynamical equation Eq. (24) with initial value $\rho^B(0)$. Then

$$\rho(t) = \rho^B(t) + \xi(t),$$

where $\xi: [0, 1] \to B_d-1(X; \mathbb{R})$ is a path. Hence, $\rho(t)$ is 1-periodic precisely when $\xi(t)$ is 1-periodic. Observe that $\xi(t)$ depends on $\tau_D$ whereas $\rho^B$ does not. However, the values of $\rho(0)$ and $\xi(0)$ are independent of $\tau_D$.

Apply the dynamical operator to this solution. Then the dynamical equation becomes

$$\dot{\xi} = \tau_D H \xi - \dot{\rho}^B. \tag{25}$$

The solution to equation (25) is then

$$\xi(t) = U(t, 0)\xi(0) - \int_0^t U(t, t')\dot{\rho}^B dt'. \tag{26}$$

Evaluating at $t = 1$, the requirement for $\rho$ to be 1-periodic is equivalent to demanding that the equation

$$(I - U(1, 0))\xi(0) = -\int_0^1 U(1, t')\dot{\rho}^B dt'$$

is satisfied. As $\tau_D$ is made large, the non-zero eigenvalues of $\tau_D H(t)$ tend to $-\infty$. Hence, by compactness, it follows that there is a $\tau_0 > 0$ such that the operator $I - U(1, 0)$ is invertible for $\tau_D \geq \tau_0$. Then

$$\xi(0) = -(I - U(1, 0))^{-1} \int_0^1 U(1, t')\dot{\rho}^B(t') dt'. \tag{27}$$

In particular, the periodic solution $\rho(t)$ exists and is unique for $\tau_D \geq \tau_0$. 
As for the adiabatic limit, it suffices to show that $|\xi(t)| \to 0$ as $\tau_D \to \infty$. From Eq. (26), we have

$$
|\xi(t)| \leq |U(t,0)||\xi(0)| + \int_0^t |U(t,t')||\rho^B(t')|dt'
\leq e^{-\lambda\tau_D t}|\xi(0)| + \int_0^t e^{-\lambda\tau_D (t-t')}|\rho^B(t')|dt'
\leq e^{-\lambda\tau_D t}|\xi(0)| + c \int_0^t e^{-\lambda\tau_D (t-t')}dt',
$$

where $c \geq |\rho^B(t')|$ is any upper bound for all $t' \in [0, 1]$. Consequently, $|\xi(t)| \to 0$ when $\tau_D \to \infty$. \hfill \blacksquare

6. The low temperature limit

For fixed $(E,W,\beta)$, the Boltzmann distribution can be regarded as a homomorphism of vector spaces

$$
\rho^B: H_{d-1}(X;\mathbb{R}) \to Z_{d-1}(X;\mathbb{R}),
\quad x \mapsto \rho^B(x).
$$

Recall that $\rho^B$ is dependent on the parameters $(E,W,\beta)$.

If $E$ is one-to-one, then the functional

$$
L \mapsto \sum_{i \in L_{d-1}} E_i
$$

has a unique minimum for some spanning co-tree $L^\mu$. In this case, we say $L^\mu$ is the minimal spanning co-tree for $E$.

**Lemma 6.1:** Suppose $E$ is one-to-one. Then the low temperature limit of $\rho^B$ is supported on the minimal spanning co-tree $L^\mu$, i.e.,

$$
\lim_{\beta \to \infty} \rho^B_\beta = \psi_{L^\mu},
$$

and the convergence is uniform.

**Proof.** This follows from [CCK2, cor. B], but we now include some details. Since the domain of $\rho^B$ is compact, uniform convergence follows from pointwise convergence. We proceed by studying the components of $\rho^B$ individually.
Let $L$ be a spanning co-tree. Multiply the numerator and denominator of the component

$$\rho_B := b_L \nabla \psi_L$$

by the expression $\exp(-\beta \sum_{a \in L_{d-1}^\mu} E_a)$ to obtain

$$\rho_B^L = \frac{a_L^2 \exp\{-\beta(\sum_{b \in L_{d-1}^\mu} E_b - \sum_{a \in L_{d-1}^\mu} E_a)\} \psi_L}{\sum_K a_K^2 \exp\{-\beta(\sum_{e \in K_{d-1}^\mu} E_e - \sum_{a \in L_{d-1}^\mu} E_a)\}},$$

(28)

where the left-most sum in the denominator is taken over all spanning co-trees. Since $L^\mu$ is minimal, the numerator tends to zero for all $L \neq L^\mu$. When $L = L^\mu$, the difference of sums vanishes and the numerator tends to $a_L^2 \psi_L^\mu$. The same argument is true for the sum in the denominator, in which case we have

$$\lim_{\beta \to \infty} \rho_B^L = \frac{a_L^2 \psi_L^\mu}{a_L^2} = \psi_L^\mu. \quad \blacksquare$$

Similarly, if $W$ is one-to-one, then just as for spanning co-trees, the functional on the set of spanning trees given by

$$T \mapsto \sum_{\alpha \in T} W_\alpha$$

has a unique minimum $T^\mu$, henceforth called the minimal spanning tree. Recall from Remark 3.7 that the operator $A = \frac{1}{\Delta} \sum w_{T\leq T}$ is the orthogonal section of the boundary operator $\partial: C_d(X; \mathbb{R}) \to B_{d-1}(X; \mathbb{R})$ in the modified inner product $\langle -, - \rangle_W$. Then an argument analogous to Lemma 6.1, which we omit, yields the following result.

**Lemma 6.2:** Assume $W$ is one-to-one. Then the low temperature limit of the operator $A$ is supported on the minimal spanning tree $T^\mu$, i.e.,

$$\lim_{\beta \to \infty} A = \varsigma_T^\mu,$$

and the convergence is uniform.

We now turn to the time-dependent case. Assume $(\tau_D, \gamma)$ is a driving protocol where $\gamma(t) = (E(t), W(t))$. 
**Proposition 6.3:** Let $L$ be a spanning co-tree and let $E$ be one-to-one for all $t$. The $L$-component of the time derivative of the Boltzmann distribution tends to 0 uniformly in the low temperature limit.

**Proof.** A tedious but straightforward computation of the time derivative of Eq. (13) gives

\[
\dot{\rho}_L = \frac{\beta a_L^2 \exp(-\beta \sum_{b \in L} E_b) \sum_K [a_K^2 \exp(-\beta \sum_{a \in K} E_a)](\sum_{a \in K} \dot{E}_a - \sum_{b \in L} \dot{E}_b)]}{[\sum_K a_K^2 \exp(-\beta \sum_{a \in K} E_a)]^2} \psi_L.
\]

(29)

For convergence in the low temperature limit, we only need to verify the statement point-wise since $[0, 1]$ is compact, and it suffices check the statement for each component $\dot{\rho}_L$. First, multiply the numerator and denominator of Eq. (29) by $\exp\{-2 \sum_{b \in L} E_b\}$ to get

\[
\dot{\rho}_L = \frac{\beta a_L^2 \sum_K a_K^2 \exp\{-\beta (\sum_{a \in K} E_a - \sum_{b \in L} E_b)\}(\sum_{a \in K} \dot{E}_a - \sum_{b \in L} \dot{E}_b)]}{[\sum_K a_K^2 \exp\{-\beta (\sum_{a \in K} E_a - \sum_{b \in L} E_b)\}]^2}.
\]

(30)

There are two cases to consider: either $L$ is the minimal spanning co-tree or it is not.

If $L$ is the minimal spanning co-tree, so that

\[
\sum_{b \in L} E_b < \sum_{a \in K} E_a
\]

for every other spanning co-tree $K$, then the denominator of Eq. (30) is given by

\[
\left[ a_L^4 + \sum_{K \neq L} a_K^2 \exp\left(-\beta \left( \sum_{a \in K} E_a - \sum_{b \in L} E_b \right)\right) \right]^2,
\]

which tends to $a_L^4 < \infty$ as $\beta \to \infty$. As for the numerator of Eq. (30), when $L = K$, we have $\sum_{a \in K} \dot{E}_a = \sum_{b \in L} \dot{E}_b$ and the numerator is exactly zero. If $L \neq K$, then the exponential factor is negative and tends to zero as $\beta \to \infty$.

If $L$ is not the minimal spanning co-tree, then some other spanning co-tree will be minimal. Therefore, at least one of the exponents $-\beta (\sum E_a - \sum E_b)$ will be positive. Since the denominator is squared, Eq. (29) is dominated by $A\beta/e^{B\beta}$ for some constants $A$ and $B$ with $B > 0$ for large $\beta$. It is easy to see this expression tends to zero as $\beta \to \infty$. ■
7. Current generation

As above we fix a cycle $z_0 \in Z_{d-1}(X; \mathbb{Z})$. For a periodic driving protocol $(\tau_D, \gamma)$, assume $\tau_D$ large enough so a unique 1-periodic solution $\rho(t)$ to Eq. (24) exists (cf. Theorem B). Recall the biased coboundary operator

$$\partial^*_{E,W} = e^{-\beta W} \partial^* e^{\beta E}.$$ 

**Definition 7.1:** For a periodic driving protocol $(\tau_D, \gamma)$ and $\beta > 0$, the **current density** at $t \in [0, 1]$ is defined as

$$J(t) := \tau_D \partial^*_{E,W} \rho(t) \in C_d(X; \mathbb{R}),$$

where $\rho(t)$ is the unique periodic solution to the dynamical equation (24). The average current is

$$Q = \int_0^1 J(t) dt.$$ 

Note that $J$ satisfies the continuity equation $\partial J = -\dot{\rho}$. When $\tau_D$ is sufficiently large, $Q$ defines a real $d$-dimensional homology class. To see this, apply $\partial$ to Eq. (32) to find

$$\partial Q = \tau_D \int_0^1 \partial \partial^*_{E,W} \rho(t) dt$$

$$= -\tau_D \int_0^1 \dot{\rho} dt$$

$$= \tau_D (\rho(0) - \rho(1))$$

$$= 0,$$

since $\rho$ is 1-periodic. Consequently, for $\tau_D$ sufficiently large, $Q$ is a $d$-cycle.

**Lemma 7.2:** The current density $J$ coincides with the expression

$$A(\dot{\rho}),$$

where $A$ is the operator of Eq. (12) and $\rho$ is the periodic solution of the dynamical equation.

**Proof.** Consider the set of all $w(t) \in C_d(X; \mathbb{R})$, with $t \in [0, 1]$ satisfying

- $\partial w = -\dot{\rho}$, and
- $\langle w(t), z \rangle_{W(t)} = 0$ for all $z \in Z_d(X; \mathbb{R})$, and $t \in [0, 1]$. 
Then any \( w \in C_d(X; \mathbb{R}) \) satisfying the above two conditions is necessarily unique. From the definition of \( J \), the first condition is verified by Eq. (24), and the second condition follows from the definition of the modified inner product.

It therefore suffices to show that the above two conditions are satisfied by the expression \( A(\dot{\rho}) \). The first condition follows from the fact that \( A \) is a section of \( \partial \), whereas the second follows from the fact that \( A \) gives an orthogonal splitting.

**Corollary 7.3:** If \( \gamma \) is constant, then \( Q = 0 \).

**Proof.** The weights appearing in \( A \) are time-independent since \( \gamma \) is constant. By Lemma 7.2,

\[
Q = \int_0^1 A(\dot{\rho}) \, dt = A\left( \int_0^1 \dot{\rho} \, dt \right) = A(0) = 0,
\]

since \( \rho \) is 1-periodic. 

7.1. **Quantization.** Current quantization occurs when the parameters are restricted to the generic subspace of **good** parameters (compare [CKS]). This space admits a decomposition

\[
\mathcal{M}_X = U \cup V,
\]

where \( U \) denotes the subspace of parameters where \( E \) is one-to-one, and \( V \) denotes the subspace where \( W \) is one-to-one. Both \( U \) and \( V \) are open subspaces.

**Definition 7.4:** Let \( L\mathcal{M}_X \) denote space of smooth unbased loops \( \gamma : [0, 1] \to \mathcal{M}_X \) in the Whitney \( C^\infty \) topology. Such a \( \gamma \) is called a **loop of good parameters** and the pair \((\tau_D, \gamma)\) is called a **good driving protocol**.

For a closed subinterval \( I \subset [0, 1] \), the contribution along \( I \) to the average current is given by the expression

\[
Q|_I = \int_{t \in I} J(t) \, dt.
\]

We now choose a subdivision of \([0, 1]\) such that the image of each segment under \( \gamma \) lies in either \( U \) or in \( V \). More precisely, we choose

\[
0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1
\]
a subdivision and set $I_j := [t_j, t_{j+1}]$. By taking the subdivision sufficiently fine and amalgamating contiguous segments if necessary, we may assume that

(i) $\gamma(I_j) \subset U$, or
(ii) $\gamma(I_j) \subset V$ and $\gamma(\partial I_j) \subset U$,

for every $j$. The segments satisfying (i) are said to be of type $U$ and those satisfying (ii) are of type $V$. Then trivially

\begin{equation}
Q = \sum_{k=0}^{n-1} \int_{I_k} J(t) \, dt.
\end{equation}

Theorem B implies $\lim_{\tau_D \to \infty} J_{\tau_D} = A(\dot{\rho}^B)$. Consequently,

\begin{equation}
Q^B := \lim_{\tau_D \to \infty} Q(\tau_D, \beta) = \int_{0}^{1} A(\dot{\rho}^B) \, dt.
\end{equation}

**Lemma 7.5:** Suppose that $I$ is of type $U$. In the low temperature limit, the contribution to $Q^B$ along $I$ is trivial.

**Proof.** By Lemma 7.2 and (34) the average current along $I$ in the adiabatic limit is given by

$$\int_{I} A(\dot{\rho}^B) \, dt.$$ 

Since $E$ is one-to-one on segments of type $U$, Proposition 6.3 implies that $\dot{\rho}^B \to 0$ uniformly in the low temperature limit. Consequently, $A(\dot{\rho})$ also tends to zero. 

**Lemma 7.6:** Suppose that $I = [u, v]$ is of type $V$. In the low temperature limit, the contribution to $Q^B$ along $I$ lies in $C_d(X; \mathbb{Z}[\frac{1}{\delta_I}])$,

where

$$\delta_I := \theta_{T^\mu} a_{L^\mu(u)} a_{L^\mu(v)},$$

in which

- $T^\mu$ is the unique minimal spanning tree on $I$,
- $L^\mu(t)$ is the unique minimal spanning co-tree at $\gamma(t)$ for $t = u, v$, and
- the integers $\theta_T$ and $a_L$ are defined in §3.
Proof. By Lemma 7.2, the average current along $I$ in the adiabatic limit is given by the expression

$$\int_I A(\dot{\rho}^B) \, dt.$$ 

Since $I$ is of type $V$, Lemma 6.2 implies that $A \to \varsigma_T^\mu$ uniformly on $I$ as $\beta \to \infty$.

Therefore, the contribution to the low temperature limit of the Boltzmann current along $I$ is given by

$$\lim_{\beta \to \infty} Q^B|_I = \lim_{\beta \to \infty} \int_u^v A(\dot{\rho}^B) \, dt = \varsigma_T^\mu \left( \lim_{\beta \to \infty} \int_u^v \dot{\rho}^B \, dt \right) = \varsigma_T^\mu (\psi_{L^\mu(v)} - \psi_{L^\mu(u)})[z_0] \quad \text{by Lemma 6.1.}$$

Note that the difference $\psi_{L^\mu(v)} - \psi_{L^\mu(u)}$ takes image in $B_{d-1}(X; Z[\frac{1}{\delta_I}])$ since its projection to $H_{d-1}(X; Z[\frac{1}{\delta_I}])$ is trivial. Hence, the displayed composition makes sense.

By Remark 3.14, we have a well-defined homomorphism

$$\varsigma_T^\mu : B_{d-1}(X; Z[\frac{1}{\delta_I}]) \to C_d(X; Z[\frac{1}{\delta_I}]).$$

Similarly, the same remark shows that the difference

$$\psi_{L^\mu(v)} - \psi_{L^\mu(u)} : H_{d-1}(X; Z[\frac{1}{\delta_I}]) \to B_{d-1}(X; Z[\frac{1}{\delta_I}])$$

is well-defined. Consequently, $\varsigma_T^\mu (\psi_{L^\mu(v)} - \psi_{L^\mu(u)})$ is defined as a homomorphism

$$H_{d-1}(X; Z[\frac{1}{\delta_I}]) \to C_d(X; Z[\frac{1}{\delta_I}]).$$

Applying this homomorphism to $[z_0]$ gives the conclusion. \hfill \Box

The following is now a straightforward consequence of the previous two lemmas together with Remark 3.14.

**Theorem 7.7 (Quantization):** Let $X$ be a connected finite CW complex $X$ of dimension $d$. Let $(\tau_D, \gamma)$ be a good driving protocol.

Then the low temperature, adiabatic limit of the average current $Q$ of $(\tau_D, \gamma)$ is well-defined and lies in the fractional lattice

$$H_d(X; Z[\frac{1}{\delta}]) \subset H_d(X; \mathbb{R}),$$
in which
\[
\delta := \prod_{L,T} a_L \theta_T,
\]
where \(L\) ranges over all spanning co-trees in dimension \(d - 1\) and \(T\) ranges over all spanning trees in dimension \(d\).

Remark 7.8: The factors appearing in \(\delta\) have a combinatorial significance. For example, in [CCK1] we derived a new identity for Milnor’s variant of Reidemeister torsion [M] that involves the factors \(\theta_T\) as \(T\) ranges over the \(d\)-spanning trees in every dimension \(1 \leq d \leq \text{dim} X\). As will be explained elsewhere, the improved higher matrix-tree theorem of [CCK2, cor. D] may be used to further simplify our identity by making use of the factors \(a_L\).

Remark 7.9: The space of good parameters can be extended to a space of robust parameters and one still obtains a quantization of the average current, as was done for graphs in [CKS]. This is explained in work [CCK3].

8. The example

Let \(X\) and \(\gamma = (E_\bullet, W_\bullet)\) be as in Theorem D, where \(\gamma\) is 1-periodic. Figure 5 indicates the ordering of the weights over the unit interval:

| \(E_1 < E_2\) | \(W_1 < W_2\) | \(E_1 > E_2\) | \(W_2 < W_1\) | \(E_1 < E_2\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0               | \(\frac{1}{2}\)  | 1               |

Figure 5. The ordering of the weights over the unit interval.

Let \(L_i \subset X\) be the \((d - 1)\)-sphere determined by the \((d - 1)\)-cell \(f_i\). Then \(L_1, L_2\) are the spanning co-trees of \(X\). Let \(T_i \subset X\) be the spanning tree of \(X\) given by attaching the \(d\)-cell \(e_i\) to the \((d - 1)\)-skeleton.

The ordering \(E_1 < E_2\) associates the spanning co-tree \(L_1\) at \(t = 0, 1\). The ordering \(E_2 > E_1\) associates the spanning co-tree \(L_2\) at \(t = 1/2\). The ordering \(W_1 < W_2\) associates the spanning tree \(T_1\) on \((0, 1/2)\) and the ordering \(W_2 < W_1\) associates the spanning tree \(T_2\) on \((1/2, 1)\). Figure 6 gives the corresponding schematic with spanning tree/co-tree labels replacing the inequalities of weights.
For the rest of the argument, we assume homology is taken with integer coefficients. The generator of $H_{d-1}(X)$ is given by $f_i$ lying in $H_{d-1}(L_i) = Z_{d-1}(L_i)$, and by the definition of spanning co-tree these are uniquely defined. By the definition of spanning tree, there is a unique $d$-chain $e_1 \in H_d(T_1) \subset C_d(T_1)$ which bounds the difference $f_2 - f_1 \in C_{d-1}(X)$ along $[0, 1/2]$. Similarly, $-e_2 \in H_d(T_2)$ uniquely bounds the difference $f_1 - f_2$ along $[1/2, 1]$. Then using equation (33) and following the proof of Theorem C, the average current in the low temperature, adiabatic limit is given by the sum of the two bounding $d$-chains, i.e., $e_1 + (-e_2) = e_1 - e_2$.  

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