Dependent Stopping Times

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ABSTRACT

Stopping times are used in applications to model random arrivals. A standard assumption in many models is that they are conditionally independent, given an underlying filtration. This is a widely useful assumption, but there are circumstances where it seems to be unnecessarily strong. We use a modified Cox construction along with the bivariate exponential introduced by Marshall and Olkin (1967) to create a family of stopping times, which are not necessarily conditionally independent, allowing for a positive probability for them to be equal. We indicate applications to modeling COVID-19 contagion (and epidemics in general), civil engineering, and to credit risk.

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1 INTRODUCTION

Stopping times are ubiquitous in the modeling of randomly occurring phenomena, from bus arrivals, customers arriving at a restaurant, to the time a cancer metastasizes, to the time a person exhibits symptoms of COVID-19 or some other contagious disease. They are also used to model metal fatigue (for example in airplane fuselages), and more generally the arrival of catastrophes, such as the collapse of condominium towers in Surfside, Florida (near Miami Beach), or for the time of a train derailment on poorly maintained tracks, such as the recent one of an Amtrak train near Joplin, Montana. A particularly common use of them is in the theory of credit risk within the discipline of Mathematical Finance.

These models all have a common structure. Let \( \tau \) be a stopping time on a filtered complete probability space \( (\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0}) \), satisfying the “usual conditions.” \( \tau \) models the random time when the event in question occurs. One can try to get a handle on estimating when \( \tau \) occurs by studying its compensator. In particular, the process \( 1_{\{t \geq \tau\}} \) is an increasing, adapted process since \( \tau \) is assumed to be a stopping time. As such it is a submartingale, and by (for example) the Doob-Meyer Decomposition Theorem, we know there exists a unique, increasing, right continuous, predictably measurable process \( A(t) \), with \( A(0) = 0 \) and such that

\[
1_{\{t \geq \tau\}} - A(t) = \text{a martingale} \tag{1}
\]

A common (and by now relatively standard) way to construct a stopping time \( \tau \) which has a given compensator \( A \) is via what is known as a Cox construction. Given a right continuous, adapted, increasing process \( A(t) \) with \( A(0) = 0 \), one posits the existence of an independent exponential random variable \( Z \), with parameter 1, independent of the process \( A \), and then one constructs \( \tau \) as:

\[
\tau = \inf_{t \geq 0} \{ A(t) \geq Z \} \tag{2}
\]

It is simple to see that \( A \) is the compensator of of the stopping time \( \tau \) given in (2), relative to an appropriate filtration.

Often the process \( A \) is taken to have continuous sample paths, and even to be of the form

\[
A(t) = \int_0^t \alpha_s ds \tag{3}
\]

for some adapted process \( \alpha \). Sufficient conditions for \( A \) to be of the form (3) are known (see, e.g., Ethier and Kurtz [7], Guo and Zeng [10], Zeng [24], and for a general result, Janson et al. [12]).

Problems arise with this standard theory when one has more than one stopping time, if one is modeling more than one random event. Then, typically one assumes that the two (or more) stopping times \( \tau_1 \) and \( \tau_2 \), constructed using two independent exponential random variables \( Z_1 \) and \( Z_2 \), are conditionally independent, given the entire filtration \( (\mathcal{F}_t)_{t \geq 0} \). This creates the property that \( P(\tau_1 = \tau_2) = 0 \). For many, even most, situations such a model is perfectly appropriate.
In this paper we are concerned with models where one can have $P(\tau_1 = \tau_2) > 0$, and some of the ramifications of such a model. Such a model arises in applications when $\eta_1$ and $\eta_2$ are constructed as Cox processes with independent exponentials $Z_1$ and $Z_2$, but with an added complication: there is a third stopping time $\eta_3$, and $\tau_1 = \eta_1 \wedge \eta_3$, while $\tau_2 = \eta_2 \wedge \eta_3$. This situation arises naturally in Credit Risk for example, as we indicate in the applications, but also in other domains, such as civil engineering, and disease contagion. The resulting stopping times of interest, $\tau_1$ and $\tau_2$, are no longer conditionally independent, and in simple cases, the bivariate exponential distribution of Marshall and Olkin [19] comes into play. These models do not have densities in $\mathbb{R}_+ \times \mathbb{R}_+$, leading to a two dimensional cdf with a singular component. We explore the consequences of such a phenomenon in some detail, and we explain its utility for various kinds of applications, caused by the confluence of stopping times that arise naturally in the modeling of random events.

Cox and Lewis [5] studied the case of multiple event types occurring in a continuum, but differently from us, they do not generalize the model proposed by Marshall and Olkin [19]. Moreover, they mostly consider “regular” processes, i.e., where events cannot occur at the same time, which is an important contribution of our work. Diggle and Milne [6] also proposed a multivariate version of the Cox Process but their model does not allow for $P(\tau_1 = \tau_2) > 0$ either. Another example of a multivariate point process is given in Brown et al. [3].

There is existing work on modeling simultaneous defaults, but differently from us, under Merton’s structural risk model (see Li [17]). Kay Giesecke, in a seminal paper concerning Credit Risk, published in 2003, was the first (to our knowledge) to consider the Marshall and Olkin model of the bivariate exponential. In this paper we develop the ideas present in [8] and go far beyond them. Another closely related work is Lindskog and McNeil [18] who consider a Poisson shock model of arbitrary dimension with both fatal and not-necessarily-fatal shocks.

There are also other types of generalizations of Cox Processes (see Gueye and Jeanblanc [9]), where they generalize, not the number of stopping times, but the form of the process $A_t$. They assume that $A_t$ is not necessarily of the form given in equation (3).

The organization of this paper is as follows. Section 2 presents the survival function of two conditionally dependent (as opposed to conditionally independent) stopping times, an interpretation of it, a decomposition of it into its singular and absolutely continuous parts, and a series of results exploring the special properties such a modeling approach has. For example, not only do we treat the case where $P(\tau_1 = \tau_2) > 0$, but more generally we study when the two stopping times are ‘close’ to each other, in various metrics. Section 3 provides an extension to an arbitrary (but finite) number of such stopping times. Section 4 shows the applicability of our results by providing examples in epidemiology (such as the case of COVID-19 and its variants), to civil engineering (e.g., the recent condo collapse of Champlain Towers in Florida), and to credit catastrophe risk.
2 THE SURVIVAL FUNCTION

As a starting point, we will consider the case of having 2 stopping times, which we will generalize to $n$ stopping times in section 3. Let us fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ large enough to support a $R^d$-valued càdlàg stochastic process $X = \{X_t, t \geq 0\}$ and 3 independent exponential random variables with parameter 1 ($Z_i, i = 1, 2, 3$). Then, define:

$$\tau_1 := \min (\eta_1, \eta_3) \quad \text{and} \quad \tau_2 := \min (\eta_2, \eta_3)$$

where

$$\eta_i := \inf \{s : A_i(s) \geq Z_i\}, \quad A_i(s) := \int_0^s \alpha_i(r)dr, \quad Z_i \overset{i.i.d.}{\sim} \text{Exp}(1) \quad (5)$$

**Theorem 1** (Survival function). Suppose $\alpha_i : R^d \rightarrow [0, \infty)$ for $i = 1, 2, 3$, usually known as the intensity, are non-random positive continuous functions, which implies that $A_i(s)$ are continuous and strictly increasing for any $s \geq 0$. Assume further that $\lim_{s \rightarrow \infty} A(s) = \infty$.

Then,

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \exp [-A_1(s) - A_2(t) - A_3(s \vee t)]$$

**Proof.** By definition of $\tau_1$ and $\tau_2$, we have,

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \mathbb{P}(\min(\eta_1, \eta_3) > s, \min(\eta_2, \eta_3) > t)$$

$$= \mathbb{P}(\eta_1 > s) \mathbb{P}(\eta_2 > t) \mathbb{P}(\eta_3 > s, \eta_3 > t)$$

$$= \mathbb{P}(\eta_1 > s) \mathbb{P}(\eta_2 > t) \mathbb{P}(\eta_3 > \max(s, t))$$

$$= \exp [-A_1(s) - A_2(t) - A_3(s \vee t)]$$

$$= \begin{cases} \exp [-A_1(s) - A_2(t) - A_3(s)] & s > t \\ \exp [-A_1(s) - A_2(t) - A_3(t)] & s < t \end{cases}$$

From Theorem 1, it is clear that $\tau_1$ is not independent of $\tau_2$ and, as we will show in subsection (2.2), that their joint distribution has an absolutely continuous and a non-trivial singular part, which means that,

$$\mathbb{P}(\tau_1 = \tau_2) > 0 \quad (6)$$

Note that Theorem 1 is a generalization of the bivariate exponential (BVE) distribution introduced by Marshall and Olkin [19]. In the specific case where $\alpha_i(s) = \alpha_i$. i.e., the intensity is constant, we recover the Marshall and Olkin BVE with parameter $(\alpha_1, \alpha_2, \alpha_3)$. This is $(\tau_1, \tau_2) \sim \text{BVE}(\alpha_1, \alpha_2, \alpha_3)$.

Now, to make $A_i(s)$ random, yet increasing processes, we provide the following Corollary.

**Corollary 1.1.** Let $\alpha_i(u) := \alpha_i(X_u)$, where $\alpha_i(\cdot)$ is a positive continuous function and $X$ is an $R^d$-valued stochastic process adapted to the filtration $\mathbb{F}$ and independent of $Z_1, Z_2$ and $Z_3$. Then,

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \mathbb{E}(\exp [-A_1(s) - A_2(t) - A_3(s \vee t)])$$
Proof. By a similar calculation as in Theorem 1, we get

$$\mathbb{P}(\tau_1 > s, \tau_2 > t|(X_u)_{0\leq u \leq (s\wedge t)}) = \exp [-A_1(s) - A_2(t) - A_3(s \lor t)]$$

The result follows by taking an expectation.

By construction, if \(\alpha_i(s)\) are not random:

$$\mathbb{P}(\tau_i > s) = \mathbb{P}(\min(\eta_i, \eta_3) > s) = \mathbb{P}(\eta_i > s)\mathbb{P}(\eta_3 > s)$$

$$= \exp \left[ - \int_0^s (\alpha_i + \alpha_3)(r)dr \right]$$  \(7\)

In the case where \(\alpha_i(u) = \alpha_i(X_u)\), i.e. a random intensity, we have:

$$\mathbb{P}(\tau_i > s) = \mathbb{E}\left( \exp \left[ - \int_0^s (\alpha_i + \alpha_3)(r)dr \right] \right)$$  \(8\)

which coincides with the marginal distribution in the Cox construction, see Lando [15] [16].

This is, each of the stopping times is a Cox process. However, by construction, they are not independent of each other.

Based on this model for \(\tau_1\) and \(\tau_2\), we might state the informational setup in the following way, which is a slight generalization of the one introduced in Lando [15]:

$$\mathcal{F}_t = \sigma \{X_s, 0 \leq s \leq t\}$$

$$\mathcal{G}^1_t = \sigma \{1_{\{\tau_1 \leq s\}}, 0 \leq s \leq t\}$$

$$\mathcal{G}^2_t = \sigma \{1_{\{\tau_2 \leq s\}}, 0 \leq s \leq t\}$$

$$\mathcal{H}_t = \mathcal{F}_t \lor \mathcal{G}^1_t \lor \mathcal{G}^2_t$$  \(9\)

### 2.1 Interpretation of Joint Distribution

A way to interpret the distribution given in Theorem 1 is the following: Suppose we have a two-component system where each component’s life is represented by \(\tau_1\) and \(\tau_2\) respectively. Any component dies after receiving a shock. Shocks are governed, in the most general case, by 3 independent Cox processes \(\Lambda_1(t, \alpha_1(X))\), \(\Lambda_2(t, \alpha_2(X))\) and \(\Lambda_3(t, \alpha_3(X))\). In a lesser general case, \(\alpha_i(X_t) = \alpha_i(t)\), i.e., the intensity is varying with time, but in a non-random way. Events in the process \(\Lambda_1(t, \alpha_1(X))\) are shocks to component 1, events in the process \(\Lambda_2(t, \alpha_2(X))\) are shocks to component 2 and events in the process \(\Lambda_3(t, \alpha_3(X))\) are shocks to both components.

Hence, we get,

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \mathbb{E}\left[ \mathbb{P}(\tau_1 > s, \tau_2 > t|(X_u)_{0\leq u \leq (s\lor t)}) \right]$$
\[ E_1 P_1 (s, X_1) = E_2 P_2 (t, X_2) = 0 \]
\[ P_3 (s \lor t, X_3) = 0 \]
\[ E_3 \exp (-A_1(s) - A_2(t) - A_3(s \lor t)) \]  

Which coincides with Corollary 1.1. In the second line of the previous expression \( \tilde{P} (\cdot) \) means \( \mathbb{P} (X_u \leq u \leq \nu v) \)

### 2.2 Decomposition of the Joint Distribution

As found in Theorem 1 under the case of a non-random intensity \( \alpha_i(t) \), we have that the joint survival function is:

\[ F_{(\tau_1, \tau_2)}(s, t) = \exp [-A_1(s) - A_2(t) - A_3(s \lor t)] \]

**Theorem 2.** Under the conditions of Theorem 1, the absolutely continuous and singular parts of \( F_{(\tau_1, \tau_2)}(s, t) \) are given by:

\[ F_{(\tau_1, \tau_2)}(s, t) = \beta F_a(s, t) + (1 - \beta) F_s(s, t) \]  

where

\[ \beta = \int _0 ^\infty (\alpha_1(s) + \alpha_2(s)) \exp [-A_1(s) - A_2(s) - A_3(s)] \, ds \]

\[ F_a(s, t) = \frac{1}{\beta} \left[ e^{-A_1(s) - A_2(t) - A_3(s \lor t)} - \int _s ^\infty \alpha_3(x) \exp \left( -\sum _i ^3 A_i(x) \right) \, dx \right] \]

\[ F_s(s, t) = \frac{1}{1 - \beta} \left[ \int _s ^\infty \alpha_3(x) \exp [-A_1(x) - A_2(x) - A_3(x)] \, dx \right] \]

**Proof.** Let \( \Lambda_i \) be the waiting time to the 1st event in the process \( \Lambda_i(t, \alpha_i(t)) \) defined in section 2.1. Then, \( F_{(\tau_1, \tau_2)}(s, t) \) can be written as,

\[ F_{(\tau_1, \tau_2)}(s, t) = \mathbb{P} (\tau_1 > s, \tau_2 > t | B) \mathbb{P} (B) + \mathbb{P} (\tau_1 > s, \tau_2 > t | B^c) \mathbb{P} (B^c) \]

where \( B = \{ \Lambda_3 > \min(\Lambda_1, \Lambda_2) \} \) and \( B^c = \{ \Lambda_3 \leq \min(\Lambda_1, \Lambda_2) \} \)

Now,

\[ \mathbb{P} (B^c) = \mathbb{P} (\Lambda_3 \leq \Lambda_1, \Lambda_3 \leq \Lambda_2) \]

\[ = \mathbb{P} (\Lambda_3 \leq \Lambda_1 \leq \Lambda_2) + \mathbb{P} (\Lambda_3 \leq \Lambda_2 \leq \Lambda_1) \]

Recall \( \Lambda_i \) are independent with density equal to \( f_{\Lambda_i}(s) = \alpha_i(s) \exp (-A_i(s)) \)

\[ = \int _0 ^\infty \int _0 ^\infty \alpha_1(x) \alpha_2(y) \alpha_3(z) \exp [-A_1(x) - A_2(y) - A_3(z)] \, dy \, dx \, dz \]
\[ + \int_0^\infty \int_0^\infty \int_0^\infty \alpha_1(x)\alpha_2(y)\alpha_3(z) \exp[-A_1(x) - A_2(y) - A_3(z)] \, dxdydz \]

\[ = \int_0^\infty \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] \, ds \quad (13) \]

From this \( \mathbb{P}(B) = \int_0^\infty (\alpha_1(s) + \alpha_2(s)) \exp[-A_1(s) - A_2(s) - A_3(s)] \, ds \)

Since \( \mathbb{P}(\min(A_1, A_2, A_3) > t) = \prod_{i=1}^3 \mathbb{P}(A_i > t) = \exp[-\sum_{i=1}^3 A_i(t)] \), we get

\[ \mathbb{P}(\tau_1 > s, \tau_2 > t|B^c) = \mathbb{P}[A_1 > s, A_2 > t, A_3 > (s \vee t)|A_3 \leq \min(A_1, A_2)] \]

\[ = \mathbb{P}[A_3 > (s \vee t)|A_3 \leq \min(A_1, A_2)] \]

\[ = \mathbb{P}[(s \vee t) < A_3 \leq \min(A_1, A_2)] \]

\[ = \frac{\int_{s \vee t}^{\infty} \alpha_3(x) \exp[-A_1(x) - A_2(x) - A_3(x)] \, dx}{\int_0^\infty \alpha_3(x) \exp[-A_1(x) - A_2(x) - A_3(x)] \, dx} \quad (14) \]

With all these elements, \( \mathbb{P}(\tau_1 > s, \tau_2 > t|B) \) is obtained by subtraction. We can verify that

\( \mathbb{P}(\tau_1 > s, \tau_2 > t|B^c) \) is a singular distribution since its mixed second partial derivative is zero when \( s \neq t \). Conversely, \( \mathbb{P}(\tau_1 > s, \tau_2 > t|B) \) is absolutely continuous since its mixed second partial derivative is a density.

Remark. The value of \( \mathbb{P}(B^c) \) corresponds to \( \mathbb{P}(\tau_1 = \tau_2) \). This is,

\[ \mathbb{P}(\tau_1 = \tau_2) = \int_0^\infty \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] \, ds \quad (15) \]

Corollary 2.1. Let \( \alpha_i(u) := \alpha_i(X_u) \), where \( \alpha_i(\cdot) \) is a positive continuous function and \( X \) is an \( \mathbb{R}^d \)-valued stochastic process adapted to the filtration \( \mathcal{F} \) and independent of \( Z_1, Z_2 \) and \( Z_3 \). Then,

\[ \mathbb{P}(\tau_1 = \tau_2) = \mathbb{E} \left( \int_0^\infty \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] \, ds \right) \]

Proof. By a similar calculation to get \( \mathbb{P}(B^c) \) in the proof of Theorem 2 we get

\[ \mathbb{P}(\tau_1 = \tau_2|X_u \geq 0) = \int_0^\infty \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] \, ds \]

The result follows by taking an expectation.

\[ \square \]

2.3 Estimating the Probability of Equality in Two Stopping Times

Now, we are interested in finding estimates for \( \mathbb{P}(\tau_1 = \tau_2) \) (see equation (15) in section 2.2) under different assumptions for \( \alpha_i(\cdot) \) (the intensity) or \( A_i(s) \) (the compensator). As in Theorem 1 we assume \( \lim_{s \to \infty} A_i(s) = \infty \) for every \( i = 1, 2, 3 \). Note that, although \( \alpha_i(\cdot) \) are not random functions by themselves, whenever we say \( \alpha_i(\cdot) \) are random, we mean that they are functions of the underlying stochastic process \( X \), i.e., \( \alpha_i(s) = \alpha_i(X_s) \).
Proposition 2.1 (Constant intensity). If $\alpha_i(s) = \alpha_i \in \mathbb{R}^+$ for all $s \geq 0$, it follows that:

$$
P(\tau_1 = \tau_2) = \int_0^\infty \alpha_3 \exp \left[ -(\alpha_1 + \alpha_2 + \alpha_3) s \right] ds
$$

$$
= \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}
$$

Proposition 2.2 (Same intensity). If $\alpha_1(s) = \alpha_2(s) = \alpha_3(s) =: \alpha(s)$ for all $s \geq 0$, no matter whether $\alpha_i(\cdot)$ are random or not, we have that $A_1(s) = A_2(s) = A_3(s) =: A(s)$ for all $s \geq 0$. Moreover we get,

$$
P(\tau_1 = \tau_2) = \frac{1}{3}
$$

Proof.

$$
P(\tau_1 = \tau_2) = \int_0^\infty \alpha(s) \exp \left[ -3A(s) \right] ds
$$

Setting $u = A(s)$, then $du = \alpha(s)ds$

$$
= \int_0^\infty e^{-3u} du = \frac{1}{3}
$$

Even in the case when $\alpha_i(s)$ are random processes, we get the same result. This is because:

$$
P(\tau_1 = \tau_2 \mid (X_u)_{u \geq 0}) = \int_0^\infty \alpha(s) \exp \left[ -3A(s) \right] ds
$$

By using the change of variable technique and taking a further expectation

$$
P(\tau_1 = \tau_2) = \frac{1}{3}
$$

Proposition 2.3 (Proportional intensity). If $\alpha_i(s) = a_i \alpha_3(s)$ for $i = 1, 2$, for all $s > 0$, and $a_i$ positive constants, then:

$$
P(\tau_1 = \tau_2) = \frac{1}{a_1 + a_2 + 1}
$$

Proof. It follows by a similar proof as Proposition 2.2.

Proposition 2.4 (Bounded non-random intensity). If $\ell_i \beta(s) \leq \alpha_i(s) \leq u_i \beta(s)$ for all $s \geq 0$, for $\beta(\cdot)$ positive and integrable with $\int_0^\infty \beta(s) ds = \infty$, where $\ell_i, u_i \in \mathbb{R}^+$ and $0 < \ell_i < u_i < \infty$. Then,

$$
\frac{\ell_3}{u_1 + u_2 + u_3} \leq P(\tau_1 = \tau_2) \leq \frac{u_3}{\ell_1 + \ell_2 + \ell_3}
$$
Proof. $\ell_i \beta(s) \leq \alpha_i(s) \leq u_i \beta(s)$ implies

$$\ell_i \int_0^s \beta(u) du \leq A_i(s) \leq u_i \int_0^s \beta(u) du \text{ for all } s \geq 0$$

From equation (15) and the given bounds on $\alpha_i(s)$:

$$\mathbb{P}(\tau_1 = \tau_2) = \int_0^\infty \alpha_3(s) \exp \left[ -A_1(s) - A_2(s) - A_3(s) \right] ds$$

$$\leq \int_0^\infty u_3 \beta(s) \exp \left[ -(\ell_1 + \ell_2 + \ell_3) \int_0^s \beta(u) du \right] ds$$

$$= \frac{u_3}{\ell_1 + \ell_2 + \ell_3}$$

Similarly,

$$\mathbb{P}(\tau_1 = \tau_2) = \int_0^\infty \alpha_3(s) \exp \left[ -A_1(s) - A_2(s) - A_3(s) \right] ds$$

$$\geq \int_0^\infty \ell_3 \beta(s) \exp \left[ -(u_1 + u_2 + u_3) \int_0^s \beta(u) du \right] ds$$

$$= \frac{\ell_3}{u_1 + u_2 + u_3}$$

\qed

**Proposition 2.5** (Bounded random intensity). Assume $\ell_i \beta(X_s) \leq \alpha_i(X_s) \leq u_i \beta(X_s)$ a.s. for all $s \geq 0$, for $\beta(\cdot)$ positive and integrable with $\int_0^\infty \beta(X_s) ds = \infty$ a.s., where $\ell_i, u_i$ are positive real random variables such that $0 < \ell_i < u_i < \infty$. Then, we get

$$\mathbb{E} \left[ \frac{\ell_3}{u_1 + u_2 + u_3} \right] \leq \mathbb{P}(\tau_1 = \tau_2) \leq \mathbb{E} \left[ \frac{u_3}{\ell_1 + \ell_2 + \ell_3} \right]$$

Proof. This follows by a conditioning argument:

$$\mathbb{P}(\tau_1 = \tau_2 | (X_u)_{u \geq 0}) = \int_0^\infty \alpha_3(X_s) \exp \left[ -A_1(s) - A_2(s) - A_3(s) \right] ds$$

$$\leq \int_0^\infty u_3 \beta(X_s) \exp \left[ -(\ell_1 + \ell_2 + \ell_3) \int_0^s \beta(X_u) du \right] ds$$

$$= \frac{u_3}{\ell_1 + \ell_2 + \ell_3}$$

And then take an expectation. We obtained the other bound in a similar way. \qed

We could analyze different scenarios for the previous 2 propositions. For example, suppose $\ell_3$ is close to $u_3$ and $u_1 + u_2$ is close to 0 (i.e., $u_1 \approx 0$ and $u_2 \approx 0$). These conditions imply
that $\alpha_3(s)$ is almost a constant and that $\alpha_1(s)$ and $\alpha_2(s)$ are close to 0. Then $\frac{\ell_3}{u_1 + u_2 + \ell_3}$ and $\frac{u_3}{\ell_1 + \ell_2 + \ell_3}$ are approximately 1. Hence,

$$P(\tau_1 = \tau_2) \approx 1$$

Another way to ensure the previous approximation is by letting $\ell_3$ be close to $u_3$ and $\ell_3 \gg u_1 + u_2$ (where $\gg$ stands for “much greater than”).

**Proposition 2.6** (Bounded sum of non-random compensators). If

$$\ell \leq A_1(s) + A_2(s) < u$$

for all $s \geq 0$ where $\ell, u \in \mathbb{R}^+$ and $0 < \ell < u < \infty$, then $e^{-u} \leq \exp[-A_1(s) - A_2(s)] \leq e^{-\ell}$. Hence,

$$e^{-u} \leq P(\tau_1 = \tau_2) \leq e^{-\ell}$$

**Proof.** $\ell \leq A_1(s) + A_2(s) < u$ implies

$$e^{-u} \leq \exp[-A_1(s) - A_2(s)] \leq e^{-\ell} \text{ for all } s \geq 0$$

Then,

$$P(\tau_1 = \tau_2) = \int_0^\infty \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] \, ds$$

$$\leq \int_0^\infty \alpha_3(s) \exp[-\ell - A_3(s)] \, ds = e^{-\ell}$$

Similarly for the lower bound $e^{-u}$

**Proposition 2.7** (Bounded sum of random compensators). If $\ell \leq A_1(s) + A_2(s) < u$ for a.s. all $s \geq 0$ where $\ell, u$ are positive real random variables such that $0 < \ell < u < \infty$. Then, by a conditioning argument and by taking an expectation, we get:

$$E[e^{-u}] \leq P(\tau_1 = \tau_2) \leq E[e^{-\ell}]$$

**Proposition 2.8** (Bounded sum of non-random compensators by another compensator). If $\ell A_3(s) \leq A_1(s) + A_2(s) \leq u A_3(s)$ for all $s \geq 0$ where $\ell, u \in \mathbb{R}^+$ and $0 < \ell < u < \infty$. Then,

$$\frac{1}{u + 1} \leq P(\tau_1 = \tau_2) \leq \frac{1}{\ell + 1}$$

This is also true if $\ell \alpha_3(s) \leq \alpha_1(s) + \alpha_2(s) \leq u \alpha_3(s)$ for all $s \geq 0$

**Proof.**

$$P(\tau_1 = \tau_2) = \int_0^\infty \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] \, ds$$

$$\leq \int_0^\infty \alpha_3(s) \exp[-(\ell + 1) A_3(s)] \, ds = \frac{1}{\ell + 1}$$

Similarly for the lower bound $\frac{1}{u + 1}$
Proposition 2.9 (Bounded sum of random compensators by another compensator). If \( \ell A_3(s) \leq A_1(s) + A_2(s) \leq u A_3(s) \) for a.s. all \( s \geq 0 \) where \( \ell, u \) are positive real random variables such that \( 0 < \ell < u < \infty \). Then, by a conditioning argument and by taking an expectation, we get:
\[
E \left[ \frac{1}{u+1} \right] \leq \mathbb{P}(\tau_1 = \tau_2) \leq E \left[ \frac{1}{\ell+1} \right]
\]

Proposition 2.10 (Intensity bounded by non-random sum of intensities). If \( \ell [\alpha_1(s) + \alpha_2(s)] \leq \alpha_3(s) \leq u [\alpha_1(s) + \alpha_2(s)] \) for all \( s \geq 0 \) where \( \ell, u \in \mathbb{R}^+ \) and \( 0 < \ell < u < \infty \). Then,
\[
\frac{\ell}{u+1} \leq \mathbb{P}(\tau_1 = \tau_2) \leq \frac{u}{\ell+1}
\]
For this inequality to be meaningful we need \( u < \ell + 1 \)

Proof.
\[
\mathbb{P}(\tau_1 = \tau_2) = \int_0^\infty \alpha_3(s) \exp [-A_1(s) - A_2(s) - A_3(s)] ds \leq \int_0^\infty u [\alpha_1(s) + \alpha_2(s)] \exp [- (\ell + 1)(A_1(s) + A_2(s))] ds = \frac{u}{\ell+1}
\]
Similarly for the lower bound \( \frac{\ell}{u+1} \)

Proposition 2.11 (Intensity bounded by random sum of intensities). If \( \alpha_i(X_s) \) are random processes, we require \( \ell [\alpha_1(X_s) + \alpha_2(X_s)] \leq \alpha_3(X_s) \leq u [\alpha_1(X_s) + \alpha_2(X_s)] \) for a.s. all \( s \geq 0 \) where \( \ell, u \) are positive real random variables such that \( 0 < \ell < u < \infty \). Then, by a conditioning argument and by taking an expectation, we get:
\[
E \left[ \frac{\ell}{u+1} \right] \leq \mathbb{P}(\tau_1 = \tau_2) \leq E \left[ \frac{u}{\ell+1} \right]
\]

2.4 Conditional Probabilities

Through this section, let \( \tilde{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot | (X_u)_{0\leq u}) \).

Proposition 2.12.
\[
\mathbb{P}(\tau_1 = \tau_2, \tau_1 \leq t) = E \left[ \int_0^t \alpha_3(s) \exp [-A_1(s) - A_2(s) - A_3(s)] ds \right]
\]

Proof. By the definition of \( \tau_i \), we have
\[
\tilde{\mathbb{P}}(\tau_1 = \tau_2, \tau_1 > t) = \tilde{\mathbb{P}}(\min(\eta_1, \eta_3) = \min(\eta_2, \eta_3), \min(\eta_1, \eta_3) > t) = \tilde{\mathbb{P}}(\min(\eta_1, \eta_3) = \eta_3 = \min(\eta_2, \eta_3), \min(\eta_1, \eta_3) > t)
\]
Proposition 2.13.

Recall \( \eta_i \) under \( \tilde{P} \) are independent with density equal to \( \alpha_i(s) \exp(-A_i(s)) \)

\[
\begin{align*}
&= \int \int \int \alpha_1(x)\alpha_2(y)\alpha_3(z) \exp[-A_1(x) - A_2(y) - A_3(z)] dydxdz \\
&\quad + \int \int \int \alpha_1(x)\alpha_2(y)\alpha_3(z) \exp[-A_1(x) - A_2(y) - A_3(z)] dx dy dz
\end{align*}
\]

This implies, \[
\tilde{P}(\tau_1 = \tau_2, \tau_1 \leq t) = \int_0^t \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] ds
\]

Take an expectation to get the result.

Remark. When taking the limit \( t \to \infty \) in the previous proposition, we recover, as expected, the result of Corollary 2.11.

**Proposition 2.13.**

\[
P(\tau_1 = \tau_2 | \tau_1 \leq t) = \mathbb{E} \left[ \frac{\int_0^t \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] ds}{1 - \exp[-A_1(t) - A_3(t)]} \right]
\]

**Proof.** Recall that, \[
\tilde{P}(\tau_1 \leq t) = 1 - \exp[-A_1(t) - A_3(t)]
\]

Using proposition 2.12, we get that

\[
\tilde{P}(\tau_1 = \tau_2 | \tau_1 \leq t) = \frac{\tilde{P}(\tau_1 = \tau_2, \tau_1 \leq t)}{\tilde{P}(\tau_1 \leq t)}
\]

\[
= \frac{\int_0^t \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] ds}{\int_0^t (\alpha_1(s) + \alpha_3(s)) \exp[-A_1(s) - A_3(s)] ds}
\]

The result follows by taking an expectation.

**Proposition 2.14.**

\[
P(\tau_1 = \tau_2 | \tau_1 \leq t, \tau_2 \leq t) = \mathbb{E} \left[ \frac{\int_0^t \alpha_3(s) \exp[-A_1(s) - A_2(s) - A_3(s)] ds}{1 - e^{-A_3(t)} (e^{-A_2(t)} + e^{-A_1(t)} - e^{-A_1(t) - A_2(t)})} \right]
\]
Proof. Note that:

1. The events \( \{ \tau_1 = \tau_2, \tau_1 \leq t, \tau_2 \leq t \} \) and \( \{ \tau_1 = \tau_2, \tau_1 \leq t \} \) are equal

2. \( \{ \tau_1 \leq t, \tau_2 \leq t \}^c = \{ \tau_1 > 0, \tau_2 > t \} \cup \{ \tau_1 > t, \tau_2 > 0 \} \cup \{ \tau_1 > t, \tau_2 > t \} \)

Hence we have:

\[
\tilde{P}(\tau_1 = \tau_2 | \tau_1 \leq t, \tau_2 \leq t) = \frac{\tilde{P}(\tau_1 = \tau_2, \tau_1 \leq t)}{\tilde{P}(\tau_1 \leq t, \tau_2 \leq t)} = \frac{\int_0^t \alpha_3(s) \exp \left[ -A_1(s) - A_2(s) - A_3(s) \right] ds}{1 - e^{-A_3(t)} (e^{-A_2(t)} + e^{-A_1(t)} - e^{-A_1(t)-A_2(t)})}
\]

Take an expectation to get the result

\[ \square \]

2.5 Distance Between Stopping Times

Proposition 2.15. For \( s < t \),

\[
\mathbb{P}(s < \tau_1 \leq t, s \leq \tau_2 \leq t) = \mathbb{E} \left[ e^{-(A_1+A_2+A_3)(s)} + e^{-(A_1+A_2+A_3)(t)} - e^{-A_1(s)-A_2(t)-A_3(t)} - e^{-A_1(t)-A_2(s)-A_3(t)} \right]
\]

In particular, if \( s = r - \varepsilon \) and \( t = r + \varepsilon \), we get

\[
\mathbb{P}(r - \varepsilon < \tau_1 \leq r + \varepsilon, r - \varepsilon \leq \tau_2 \leq r + \varepsilon) = \mathbb{E} \left[ e^{-(A_1+A_2+A_3)(r-\varepsilon)} + e^{-(A_1+A_2+A_3)(r+\varepsilon)} - e^{-A_1(r-\varepsilon)-A_2(r+\varepsilon)-A_3(r+\varepsilon)} - e^{-A_1(r+\varepsilon)-A_2(r-\varepsilon)-A_3(r+\varepsilon)} \right]
\]

Proof. It follows by noticing that:

\[
\mathbb{P}(s < \tau_1 \leq t, s \leq \tau_2 \leq t) = \mathbb{P}(\tau_1 > s, \tau_2 > s) - \mathbb{P}(\tau_1 > s, \tau_2 > t) - \mathbb{P}(\tau_1 > t, \tau_2 > s) + \mathbb{P}(\tau_1 > t, \tau_2 > t)
\]

Then condition on \((X_t)_{t \geq 0}\), use Theorem [1] and take an expectation

\[ \square \]

The next proposition shows that the probability of the two stopping time happening over the next time interval \((t, t + \varepsilon)\) with \( \varepsilon \approx 0 \) is equal to the expectation of the common intensity \((\alpha_3)\) at time \( t \).

Proposition 2.16.

\[
\lim_{\varepsilon \to 0} \frac{\mathbb{P}(\tau_1 \in (t, t + \varepsilon], \tau_2 \in (t, t + \varepsilon) | \tau_1 > t, \tau_2 > t)}{\varepsilon} = \mathbb{E}(\alpha_3(X_t))
\]
Proof. Note that
\[
\hat{P}(\tau_1 \in (t, t + \varepsilon], \tau_2 \in (t, t + \varepsilon] | \tau_1 > t, \tau_2 > t) = \frac{\hat{P}(\tau_1 \in (t, t + \varepsilon], \tau_2 \in (t, t + \varepsilon])}{\hat{P}(\tau_1 > t, \tau_2 > t)}
\]
For the numerator, using Proposition 2.15 with \(t\) and \(t + \varepsilon\),
\[
\hat{P}(\tau_1 \in (t, t + \varepsilon], \tau_2 \in (t, t + \varepsilon]) = \exp \left[ - (A_1 + A_2 + A_3) (t) \right] - \exp \left[ - (A_1 + A_3) (t + \varepsilon) - A_2 (t) \right] + \exp \left[ - (A_1 + A_2 + A_3) (t + \varepsilon) \right]
\]
Dividing by \(\hat{P}(\tau_1 > t, \tau_2 > t) = \exp \left[ - A_1 (t) - A_2 (t) - A_3 (t) \right]\), we get:
\[
\hat{P}(\tau_1 \in (t, t + \varepsilon], \tau_2 \in (t, t + \varepsilon] | \tau_1 > t, \tau_2 > t) = \frac{\exp \left[ - \int_t^{t+\varepsilon} (\alpha_2 + \alpha_3) (X_u) du \right] - \exp \left[ - \int_t^{t+\varepsilon} (\alpha_1 + \alpha_3) (X_u) du \right]}{\exp \left[ - \int_t^{t+\varepsilon} (\alpha_1 + \alpha_2 + \alpha_3) (X_u) du \right]}
\]
Hence, by using L'Hôpital's rule, we get:
\[
\lim_{\varepsilon \to 0} \frac{\hat{P}(\tau_1 \in (t, t + \varepsilon], \tau_2 \in (t, t + \varepsilon] | \tau_1 > t, \tau_2 > t)}{\varepsilon} = \frac{(\alpha_2 + \alpha_3) (X_t) + (\alpha_1 + \alpha_3) (X_t) - (\alpha_1 + \alpha_2 + \alpha_3) (X_t)}{\varepsilon} = \alpha_3 (X_t).
\]
Moreover, given \(\hat{P}(\tau_1 \in (t, t + \varepsilon], \tau_2 \in (t, t + \varepsilon] | \tau_1 > t, \tau_2 > t)\) is bounded by 1 and a conditioning argument, we can conclude that
\[
\lim_{\varepsilon \to 0} \frac{\mathbb{P}(\tau_1 \in (t, t + \varepsilon], \tau_2 \in (t, t + \varepsilon] | \tau_1 > t, \tau_2 > t)}{\varepsilon} = \mathbb{E} (\alpha_3 (X_t)).
\]
This Proposition motivates the following result

**Proposition 2.17.** If \(s < t\)
\[
\lim_{\varepsilon \to 0} \frac{\mathbb{P}(\tau_1 \leq s + \varepsilon, \tau_2 \leq t + \varepsilon | \tau_1 > s, \tau_2 > t)}{\varepsilon^2} = \mathbb{E} \left[ \alpha_1 (X_s) (\alpha_2 (X_t) + \alpha_3 (X_t)) \right]
\]

Proof. As \(s < t\), we can always find a sufficiently small \(\varepsilon\) such that \(s + \varepsilon < t\). Hence, without loss of generality, we assume \(s < s + \varepsilon < t < t + \varepsilon\) and we proceed as in the proof of Proposition 2.16 but differently from there, we use L'Hôpital's rule twice to get the desired limit.

\[\square\]
The next couple of propositions provide a measure, in two different metrics, of how close the two stopping are from each other.

**Proposition 2.18.** (Distance in probability)

\[
\mathbb{P}(|\tau_1 - \tau_2| \leq \varepsilon) = 1 - \mathbb{E}\left[ \int_0^\infty \alpha_1(x)e^{-A_1(x)-(A_2+A_3)(x+\varepsilon)} \, dx \right. \\
+ \int_0^\infty \alpha_2(x)e^{-A_2(x)-(A_1+A_3)(x+\varepsilon)} \, dx \right]
\]

**Proof.** Let \( \tau_1 := \min(\tau_1, \tau_2) \) and \( \tau_2 := \max(\tau_1, \tau_2) \). Similarly, \( \eta_1 := \min(\eta_1, \eta_2, \eta_3) \), \( \eta_2 := \max(\eta_1, \eta_2, \eta_3) \), and \( \eta_3 \) be the second largest from \((\eta_1, \eta_2, \eta_3)\). Then,

\[
\mathbb{P}(\tau_2 > \tau_1 + \varepsilon) = \mathbb{P}(\eta_3 \geq \eta_2 > \eta_1 + \varepsilon, \eta_3 \geq \eta_2)
\]

\[= \int_0^\infty \int_0^{\infty} f_1(x_1) f_2(x_2) f_3(x_3) \, dx_1 \, dx_2 \, dx_3
\]

where \( f_j \) stands for the density of \( \eta_j \) under \( \mathbb{P} \), i.e., \( f_j(y) = \alpha_j(y)e^{-A_j(y)} := \alpha_j(X_y)e^{-A_j(y)} \)

\[= \int_0^\infty \int_0^{\infty} f_1(x_1) f_2(x_2)e^{-A_3(x_2)} \, dx_2 \, dx_1 + \int_0^\infty f_2(x_1) \int_0^{\infty} f_1(x_2)e^{-A_3(x_2)} \, dx_2 \, dx_1
\]

\[+ \int_0^\infty f_1(x_1) \int_0^{\infty} f_3(x_2)e^{-A_2(x_2)} \, dx_2 \, dx_1
\]

\[+ \int_0^\infty f_2(x_1) \int_0^{\infty} f_3(x_2)e^{-A_1(x_2)} \, dx_2 \, dx_1
\]

\[= \int_0^\infty \int_0^{\infty} \left[ \alpha_2(x_2) + \alpha_3(x_2) \right] e^{-A_2(x_2)-A_3(x_2)} \, dx_2 \, dx_1
\]
$$+ \int_0^{\infty} f_2(x_1) \int_{x_1+\varepsilon}^{\infty} [\alpha_1(x_2) + \alpha_3(x_2)] e^{-A_1(x_2) - A_3(x_2)} \, dx_2 \, dx_1$$

$$= \int_0^{\infty} \alpha_1(x_1) e^{-A_1(x_1) - (A_2 + A_3)(x_1+\varepsilon)} \, dx_1 + \int_0^{\infty} \alpha_2(x_1) e^{-A_2(x_1) - (A_1 + A_3)(x_1+\varepsilon)} \, dx_1$$

Use this expression, along with the following equality and then take an expectation to get the desired result

$$\tilde{\mathbb{P}}(|\tau_1 - \tau_2| \leq \varepsilon) = 1 - \tilde{\mathbb{P}}(\tau(2) - \tau(1) > \varepsilon)$$

**Proposition 2.19.** ($L^2$ distance)

If $\mathbb{E}(\tau_i^2) < \infty$ and $\lim_{x \to \infty} x^2 e^{-A_i(x)} = 0$ for $i = 1, 2$, we have

$$\mathbb{E}[(\tau_1 - \tau_2)^2] = 2 \left[ \int_0^{\infty} x e^{-A_1(x) - A_3(x)} \, dx + \int_0^{\infty} x e^{-A_2(x) - A_3(x)} \, dx ight.$$

$$- \int_0^{\infty} \int_0^{y} e^{-A_1(x) - A_2(y) - A_3(y)} \, dxdy - \int_0^{\infty} \int_y^{\infty} e^{-A_1(x) - A_2(y) - A_3(x)} \, dxdy \right]$$

**Proof.** Let $\tilde{\mathbb{E}}(\cdot) := \mathbb{E}(\cdot|(X_u)_{u \geq 0})$. $F_{\tau_i}$ and $\overline{F}_{\tau_i}$ stand for the cumulative and the survival distribution functions of $\tau_i$ given $(X_u)_{u \geq 0}$. Similarly, $F(x, y)$ and $\overline{F}(x, y)$ stand for the cumulative and the survival joint distribution functions of $\tau_1, \tau_2$ given $(X_u)_{u \geq 0}$.

We expand the square and handle each term separately. For $\tilde{\mathbb{E}}(\tau_i^2) < \infty$, we use integration by parts in the following way:

$$\tilde{\mathbb{E}}(\tau_i^2) = \int_0^{\infty} x^2 \, dF_{\tau_i}(x)$$

$$= 2 \int_0^{\infty} \overline{F}_{\tau_i}(x) \, dx$$

$$= 2 \int_0^{\infty} x e^{-A_i(x) - A_3(x)} \, dx$$

To find $\tilde{\mathbb{E}}(\tau_1 \tau_2)$ we exploit the result of Young on integration by parts in two or more dimensions. If $G(0, y) \equiv 0 \equiv G(y, 0)$ and $G$ is of bounded variation on finite intervals, then:

$$\int_0^{\infty} \int_0^{\infty} G(x, y) \, dF(x, y) = \int_0^{\infty} \int_0^{\infty} \overline{F}(x, y) \, dG(x, y)$$

This equality implies that, for $i, j > 0$:

$$\int_0^{\infty} \int_0^{\infty} x^i y^j \, dF(x, y) = \int_0^{\infty} \int_0^{\infty} i j x^{i-1} y^{j-1} \overline{F}(x, y) \, dxdy$$
Remark. In the case of independence of $\tau_1$ and $\tau_2$, hence,

$$\mathbb{E}(\tau_1 \tau_2) = \int_0^\infty \int_0^\infty x y dF_{(\tau_1, \tau_2)}(x, y)$$

$$= \int_0^\infty \int_0^y e^{-A_1(x) - A_2(y) - A_3(x,y)} dxdy$$

$$= \int_0^\infty \int_y^\infty e^{-A_1(x) - A_2(y) - A_3(y)} dxdy + \int_0^\infty \int_y^\infty e^{-A_1(x) - A_2(y) - A_3(x)} dxdy \quad (18)$$

The result follows by taking an expectation.

**Remark.** In the case of independence of $\tau_1$ and $\tau_2$, as $A_3(s) = 0$ for all $s \geq 0$, we get that:

$$\mathbb{E}[(\tau_1 - \tau_2)^2] = 2 \left[ \int_0^\infty xe^{-A_1(x)} dx + \int_0^\infty xe^{-A_2(x)} dx \right.$$  

$$- \int_0^\infty e^{-A_1(x)} dx \int_0^\infty e^{-A_2(x)} dx \right] \quad (19)$$

### 3 Generalization to K Stopping Times

Given the interpretation introduced in section 2.1, there is a natural way to extend our model to more than two stopping times. We explicitly motivate and present the case of 3 stopping times. Suppose we have a three-component system where each component’s life is represented by $\tau_1$, $\tau_2$, and $\tau_3$ respectively. Any component dies after receiving a shock, which are governed, in the most general case, by 6 independent Cox processes:

$$\Lambda_1(t, \alpha_1(X)), \Lambda_2(t, \alpha_2(X)), \Lambda_3(t, \alpha_3(X)), \Lambda_{(1,2)}(t, \alpha_{(1,2)}(X)), \Lambda_{(2,3)}(t, \alpha_{(2,3)}(X)), \Lambda_{(1,3)}(t, \alpha_{(1,3)}(X)) \text{ and } \Lambda_{(1,2,3)}(t, \alpha_{(1,2,3)}(X))$$

In a less general case, $\alpha_i(X_t) = \alpha_i(t)$, i.e., the intensity is varying with time, but in a non-random way. Events in the process $\Lambda_1(t, \alpha_1(X))$ are shocks to only component 1, events in the process $\Lambda_2(t, \alpha_2(X))$ are shocks to only component 2, events in the process $\Lambda_3(t, \alpha_3(X))$ are shocks to only component 3, events in the process $\Lambda_{(i,j)}(t, \alpha_{(i,j)}(X))$ are shocks to component $i$ and $j$, and events in the process $\Lambda_{(1,2,3)}(t, \alpha_{(1,2,3)}(X))$ are shocks to the 3 components.

In this way, considering $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot|(X_u)_{u \geq 0})$ and $A_j(s) = \int_0^s \alpha_j(X_u) du$, we have:

$$\mathbb{P}(\tau_1 > s_1, \tau_2 > s_2, \tau_3 > s_3) = \mathbb{E}\left(\tilde{\mathbb{P}}[\Lambda_1(s_1, \alpha_1(X)) = 0]\right)$$

$$\tilde{\mathbb{P}}[\Lambda_2(s_2, \alpha_2(X)) = 0] \tilde{\mathbb{P}}[\Lambda_3(s_3, \alpha_3(X)) = 0] \tilde{\mathbb{P}}[\Lambda_{(1,2)}(s_1 \cup s_2, \alpha_{(1,2)}(X)) = 0]$$

$$\tilde{\mathbb{P}}[\Lambda_{(1,3)}(s_1 \cup s_3, \alpha_{(1,3)}(X)) = 0] \tilde{\mathbb{P}}[\Lambda_{(2,3)}(s_2 \cup s_3, \alpha_{(2,3)}(X)) = 0]$$

$$\tilde{\mathbb{P}}[\Lambda_{(1,2,3)}(s_1 \cup s_2 \cup s_3, \alpha_{(1,2,3)}(X)) = 0]$$

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By using a similar technique we can generalize to any number $K$ of stopping times and in section 4.1, we will present an application of this natural extension of our model. However, as the number of stopping times $K$ increases, handling the expression presented in equation (20) becomes cumbersome. Hence, in the following, we propose a slightly different approach to generalize to more than two stopping times. We should note that this approach is fully studied in Jarrow et al. [13] and here we just limit ourselves to list the most important results obtained in there.

Suppose that we have a $K$-component system where each component’s life is represented by $\tau_1, \tau_2, \ldots, \tau_K$. Any component dies after receiving a shock, which are governed by $K + 1$ independent Cox processes:

$$\Lambda_1(t, \alpha_1(X)), \Lambda_2(t, \alpha_2(X)), \ldots, \Lambda_K(t, \alpha_K(X)), \Lambda(\bar{R})(t, \alpha(1,\ldots,K)(X))$$

Events in the process $\Lambda_i(t, \alpha_i(X))$ are shocks to only component $i$ and events in the process $\Lambda(\bar{R})(t, \alpha(1,\ldots,K)(X))$ are shocks to all $K$ components.

Considering as before: $\tilde{P}(\cdot) := \mathbb{P}(\cdot | (X_u)_{u \geq 0})$ and $A_j(s) = \int_0^s \alpha_j(X_u)du$, we have:

$$\mathbb{P}(\tau_1 > s_1, \tau_2 > s_2, \ldots, \tau_K > s_K) = \mathbb{E}\left(\prod_{i=1}^{K} \tilde{P}[\Lambda_i(s, \alpha_i(X)) = 0] \times \tilde{P}[\Lambda(1,2,\ldots,K)(\max(s_1,\ldots,s_K), \alpha(1,2,\ldots,K)(X)) = 0]\right)$$

$$= \mathbb{E}\left(\exp\left[-\sum_{i=1}^{K} A_i(s_i) - A(1,2,\ldots,K)(\max(s_1,\ldots,s_K))\right]\right) \quad (21)$$

An alternative way to obtain the joint survival function given in equation (21) is by the following argument. Suppose we have $K + 1$ independent exponential random variables each with parameter 1, i.e. $(Z_i, i = 0,\ldots,K)$, which help us to define $K + 1$ stopping times $(\eta_0, \eta_1, \ldots, \eta_K)$ in the following way:

$$\eta_i := \inf\{s : A_i(s) \geq Z_i\} \quad (22)$$

where $A_i(s) = \int_0^s \alpha_i(X_r)dr$ and $Z_i \sim \text{Exp}(1)$ for $i = 0, 1, 2, \ldots, K$.

Then, we define $K$ stopping times as

$$\tau_i = \min(\eta_0, \eta_i) \quad (23)$$

for $i = 1, \ldots, K$. Then, as stated in Jarrow et al. [13]:

$$= \mathbb{E}\left(\exp\left[-A_1(s_1) - A_2(s_2) - A_3(s_3) - A(1,2)(s_1 \lor s_2) - A(1,3)(s_1 \lor s_3) - A(2,3)(s_2 \lor s_3) - A(1,2,3)(s_1 \lor s_2 \lor s_3)\right]\right) \quad (20)$$
\[ P(\tau_1 > s_1, \tau_2 > s_2, \ldots, \tau_K > s_K) = \]
\[ E \left[ \exp \left( -\sum_{i=1}^{K} A_i(s_i) - A_0(\max(s_1, s_2, \ldots, s_K)) \right) \right] \]

(24)

which coincides with (21) by setting \( A_{(1,2,\ldots,K)}(\cdot) = A_0(\cdot) \). This joint survival function allows us to conclude that the probability of at least two stopping times being \( \varepsilon \)-close is:

\[ P(|\tau_i - \tau_j| < \varepsilon \text{ for some } (i, j) \in \{1, \ldots, K\} \times \{1, \ldots, K\}, i \neq j) = \]
\[ 1 - E \left[ \sum_{j \in P} \int_{0}^{\infty} \int_{x_1+\varepsilon}^{\infty} \int_{x_2+\varepsilon}^{\infty} \int_{x_{K-1}+\varepsilon}^{\infty} f_{j_1}(x_1) f_{j_2}(x_2) \ldots f_{j_{K-2}}(x_{K-2}) \right. \]
\[ \left. \int_{x_{K-2}+\varepsilon}^{\infty} f_{j_{K-1}}(x_{K-1}) \exp \left[ -A_{j_{K}}(x_{K-1} + \varepsilon) - A_0(x_{K-1} + \varepsilon) \right] dx_{K-1} dx_{K-2} \ldots dx_3 dx_2 dx_1 \right] \]

(25)

We can also estimate bounds for the previous probability:

\[ P(|\tau_i - \tau_j| < \varepsilon \text{ for some } (i, j)) \leq \]
\[ \binom{K}{2} - \sum_{i=1}^{K} \sum_{j \neq i} E \left[ \int_{0}^{\infty} \alpha_i(X_x) e^{-A_i(x) - A_j(x+\varepsilon) - A_0(x+\varepsilon)} dx \right] \]

(26)

\[ P(|\tau_i - \tau_j| < \varepsilon \text{ for some } (i, j)) \geq \]
\[ 1 - E \left[ e^{-A_0((K-1)\varepsilon)} \sum_{j \in P} \exp \left( -\sum_{i=1}^{K-1} A_{j_{i+1}}(i\varepsilon) \right) \right] \]

(27)

Moreover, we can also conclude that the previous probability (i.e., equation (25)) is increasing in \( K \). This is:

\[ P(|\tau_i - \tau_j| < \varepsilon \text{ for some } (i, j) \in \{1, \ldots, K\} \times \{1, \ldots, K\}, i \neq j) < \]
\[ P(|\tau_i - \tau_j| < \varepsilon \text{ for some } (i, j) \in \{1, \ldots, K+1\} \times \{1, \ldots, K+1\}, i \neq j) \] \hspace{1cm} (28)

What is more, this probability has a limit:

\[ \lim_{K \to \infty} P(|\tau_i - \tau_j| < \varepsilon \text{ for some } (i, j) \in \{1, \ldots, K\} \times \{1, \ldots, K\}, i \neq j) = 1. \]

(29)

Finally, letting \( \tau_K = \max\{\tau_1, \ldots, \tau_K\} \), we can get the probability of all stopping times occurring before any \( \varepsilon > 0 \)

\[ P(\tau_K \leq \varepsilon) = 1 + E \left[ e^{-A_0(\varepsilon)} \left( \prod_{i=1}^{K} (1 - e^{-A_i(\varepsilon)}) - 1 \right) \right] \]

(30)

For more results and for the proofs, see Jarrow et al. \[13\].
4 APPLICATIONS

4.1 Application to Epidemiology

Suppose we are interested in knowing what is the probability of \( n \) people getting infected with COVID-19 at the exact same time. The time to infection of each person can be modeled as a stopping time. Some current models (see Britton and Pardoux [2]) assume independence of these stopping times and thus the probability of them being equal is 0. However, using our model we can weaken the independence assumption and conclude that:

**Theorem 3.** If \((\tau_1, \tau_2, \ldots, \tau_n)\) follows the joint distribution described in Section 3, then:

\[
\mathbb{P}(\tau_1 = \tau_2 = \cdots = \tau_n) = \mathbb{E} \left[ \int_0^\infty \alpha_{(1,2,\ldots,n)}(u) e^{-\sum_{k=1}^n \sum_{i \in C_k} A_i(u)} \, du \right]
\]

where each \( \alpha_i(u) \) is the intensity of the processes described in Section 3, and \( A_i(u) = \int_0^u \alpha_i(s) \, ds \). The innermost sum in the exponent is taken over all the possible combinations \( \binom{n}{k} \). For example, if \( k = 3, j \) could be \((1,3,5), (2,3,n)\), etc; if \( k = n, j \) can only be \((1,2,\ldots,n)\).

**Remark.** To be more specific if \( n = 3 \), then we get,

\[
\mathbb{P}(\tau_1 = \tau_2 = \tau_3) = \mathbb{E} \left[ \int_0^\infty \alpha_{(1,2,3)}(u) \exp \left[ -A_1(u) - A_2(u) - A_3(u) - A_{(1,2)}(u) - A_{(1,3)}(u) - A_{(2,3)}(u) - A_{(1,2,3)}(u) \right] \, du \right]
\]

**Proof.** Using the notation from Section 3, events in the process \( \Lambda_{(1,2,\ldots,n)} \) are shocks to the \( n \) components of the system. Hence

\[
\{\tau_1 = \tau_2 = \cdots = \tau_{n-1} = \tau_n\} = \{\Lambda_{(1,2,\ldots,n)} \leq \min(\Lambda_1, \ldots, \Lambda_n, \Lambda_{(1,2)}, \ldots, \Lambda_{(n,n)}, \Lambda_{(1,2,3)}, \ldots, \Lambda_{(1,2,\ldots,n-1)})\}
\]

Then, we find the distribution of:

\[
M := \min(\Lambda_1, \ldots, \Lambda_n, \Lambda_{(1,2)}, \ldots, \Lambda_{(n,n)}, \Lambda_{(1,2,3)}, \ldots, \Lambda_{(1,2,\ldots,n-1)})
\]

under the measure \( \tilde{\mathbb{P}}(\cdot) \) which stands for \( \mathbb{P}(\cdot|(X_s)_{s \geq 0}) \)

\[
\tilde{\mathbb{P}}[M > t] = \mathbb{P}[\Lambda_1 > t, \ldots, \Lambda_n > t, \ldots, \Lambda_{(n,n)} > t, \Lambda_{(1,2,3)} > t, \ldots, \Lambda_{(1,2,\ldots,n-1)} > t] = \exp \left[ -A_1(t) - \cdots - A_n(t) - A_{(1,2)}(t) - \cdots - A_{(n,n)}(t) - A_{(1,2,3)}(t) - \cdots - A_{(1,2,\ldots,n-1)}(t) \right]
\]
Hence, $M$, given $(X_s)_{s \geq 0}$, has a continuous distribution with density equal to:

$$f_M(t) = \left[ \sum_{k=1}^{n-1} \sum_{j \in C_k} \alpha_j(t) \right] e^{-\sum_{k=1}^{n-1} \sum_{j \in C_k} \alpha_j(t) A_j(t)}$$

(34)

The innermost sums are taken over all the possible combinations $\binom{n}{k}$. Then,

$$
\mathbb{P}[\Lambda_{1,2,\ldots,n} < M] = \int_0^\infty \int_0^\infty \alpha_{(1,2,\ldots,n)}(x)e^{-\Lambda_{(1,2,\ldots,n)}(x)} f_M(y) dy \, dx \\
= \int_0^\infty \alpha_{(1,2,\ldots,n)}(u)e^{-\sum_{k=1}^n \sum_{j \in C_k} A_j(u)} du
$$

(35)

The result follows by taking an expectation.

**Corollary 3.1.** If $\alpha_i(X_t) = \alpha(X_t)$ for all $i = 1, 2, \ldots, n$; $\alpha_{(i,j)}(X_t) = \frac{1}{2} \alpha(X_t)$ for all size 2 combinations in $\binom{n}{2}$; $\alpha_{(i,j,k)}(X_t) = \frac{1}{3} \alpha(X_t)$ for all size 3 combinations in $\binom{n}{3}$; $\ldots$; $\alpha_{(1,2,\ldots,n)}(X_t) = \frac{1}{n} \alpha(X_t)$. Then, even if $\alpha(X_t)$ is a random process, we have:

$$
\mathbb{P}(\tau_1 = \tau_2 = \cdots = \tau_n) = \frac{1}{n} \left[ \frac{\binom{n}{1}}{1} + \frac{\binom{n}{2}}{2} + \cdots + \frac{\binom{n}{n}}{n} \right]
$$

(36)

**Corollary 3.2.** If $\alpha_i(X_t) = \alpha(X_t)$ for all $i = 1, 2, \ldots, n$; $\alpha_{(i,j)}(X_t) = 2 \alpha(X_t)$ for all size 2 combinations in $\binom{n}{2}$; $\alpha_{(i,j,k)}(X_t) = 3 \alpha(X_t)$ for all size 3 combinations in $\binom{n}{3}$; $\ldots$; $\alpha_{(1,2,\ldots,n)}(X_t) = n \alpha(X_t)$. Then, even if $\alpha(X_t)$ is a random process, we have:

$$
\mathbb{P}(\tau_1 = \tau_2 = \cdots = \tau_n) = n \left[ \frac{1}{\binom{n}{1} + 2 \binom{n}{2} + \cdots + n \binom{n}{n}} \right]
$$

$$
= \frac{1}{2n-1}
$$

(37)

**4.2 Application to Engineering**

A classic problem in Operations Research, basically studied in queuing theory, is that of a complicated machine. The machine fails if one of its key parts fails. Knowing this, designers create a certain redundancy by doubling key components, so that if one fails, there is a back-up ready to assume its duties. To save money, however, one can have one back-up for two components. Suppose $\eta_1$ is the (random) failure time of one component, and $\eta_2$ is the (again, random) failure time of the second component. If they fail at the same time, then the machine itself will fail, since the solitary back-up cannot replace both components simultaneously. One usually considers such a situation to be unlikely, even very unlikely. If it were to happen, however, we would be interested in $P(\eta_1 = \eta_2)$. In the conventional models, $P(\eta_1 = \eta_2)$ is zero, since they are each exponential, and conditionally independent. However if we consider a third time, $\eta_3$, and if this third stopping time is the (once again,
random) time of an external shock (such as the failure of the air conditioning unit, or a power failure with a surge when the power resumes, etc.), and if we let
\[ \tau_1 = \eta_1 \land \eta_3 \quad \text{and} \quad \tau_2 = \eta_2 \land \eta_3 \] (38)
then we are in the case where \( P(\tau_1 = \tau_2) = \alpha > 0 \), and in special cases we can calculate the probability \( \alpha \) with precision, and in other, more complicated situations, we can give upper and lower bounds, both, for \( \alpha \).

A different kind of example, exemplified by the recent, and quite dramatic example is that of the collapse of the Champlain Towers South, in Surfside, Florida (just north of Miami Beach). The twelve story towers fell at night (1:30AM), and killed 98 people who were in their apartments at the time, and presumably even in their beds. The towers had a pool deck above a covered garage. The towers holding up the pool deck were too thin, and not strong enough to withstand the stresses imposed on them over four decades. Water was seen pouring into the parking garage only minutes before the collapse.

In the main building that collapsed, structural columns were too narrow to accommodate enough rebar, meaning that contractors had to choose between cramming extra steel into a too-small column (which can create air pockets that accelerate corrosion) or inadequately attaching floor slabs to their supports. Our model would have several stopping times, each one representing the failure of a different component of the structure. Choosing two important ones, such as: (1) The corrosion of the rebar supports within the concrete, due to the salt air and massive strains due to violent weather, which plagues the Florida coast during hurricane season; (2) The use of a low quality grade of concrete, violating regulations of the local government in the construction of the towers, leading to concrete integrity decay due to 40 years of seaside weather.

One way to model this is to take a vector of two Cox constructions, using two independent exponentials \( Z_1 \) and \( Z_2 \) to construct our failure times \( \tau_1 \) and \( \tau_2 \). This then gives us that \( \tau_1 \) and \( \tau_2 \) are conditionally independent, given the underlying filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \). This leads to \( P(\tau_1 = \tau_2) = 0 \), even if they have the exact same compensators.

Other models have been proposed, such as the general framework presented in the book of Anna Aksamit and Monique Jeanblanc [1], where the random variables \( Z_1 \) and \( Z_2 \) are multivariate exponentials, but with a joint density describing how they relate to each other. However, even in this more general setting, we have \( P(\tau_1 = \tau_2) = 0 \).

Assuming it is the stopping times occurring simultaneously that causes the collapse, we want a model that allows us to have \( P(\tau_1 = \tau_2) > 0 \). Let’s assume we have three standard Cox Constructions, with independent exponentials \((Z_1; Z_2; Z_3)\), with different compensators \((\int_0^t \alpha_1^1 ds, \int_0^t \alpha_2^2 ds, \int_0^t \alpha_3^3 ds)\).

Call the three stopping times \( \eta_1, \eta_2, \eta_3 \). The time \( \eta_3 \) could be anything, such as a hurricane putting heavy stress on the building, or an earthquake, or some other external factor. (Current forensic analysis, which is ongoing as we write this, suggests that the collapse of the pool-deck created a seismic shock sufficient to precipitate the collapse of the south tower, and
weakened the structural integrity of the north tower to such an extent that it was condemned and then deliberately destroyed.) However in this case we can take \( \eta_3 \) to be the time of the flooding of the parking structure under the swimming pool and pool deck in general.

Simulations commissioned by the newspaper *The Washington Post* and done by a team led by Khalid M. Mosalam of the University of California, Berkeley, show how that might have happened and indicate that it is a plausible scenario (Swaine et al. [21]).

As in (38), we define
\[
\tau_1 = \eta_1 \land \eta_3 \quad \text{and} \quad \tau_2 = \eta_2 \land \eta_3, \tag{39}
\]
where we take the stopping time \( \eta_3 \) to be the time of the collapse of the pool deck. As the noted engineer H. Petroski has pointed out [20] it is often the case that multiple things happen at once in order to precipitate a disaster such as the fall of the Champlain Tower South. Indeed, the forensic engineer R. Leon of Virginia Tech is quoted in *American Society of Civil Engineers, 2021* [22]. The quote of Professor Leon is as follows: “I think it is way too early to tell,” said Roberto Leon, P.E., F.SEI, Dist.M.ASCE, the D.H. Burrows Professor of Construction Engineering in the Charles Edward Via Jr. Department of Civil and Environmental Engineering at Virginia Tech. “It’s going to require a very careful forensic approach here, because I don’t think the building collapsed just because of one reason. What we tend to find in forensic investigations is that three or four things have to happen for a collapse to occur that is so catastrophic.”

Professor Leon is a widely respected authority in forensic civil engineering, and the key insight for us is his last statement that three or four things have to happen simultaneously for a catastrophic collapse.

Less dramatic examples, but quite pertinent to the recent attention being paid to the decay of infrastructure around the US, provide more examples of the utility of this approach. A first example is the Interstate 10 Twin Span Bridge over Lake Pontchartrain north of New Orleans, LA. It was rendered completely unusable by Hurricane Katrina, but the naive explanation was demonstrated false by the fact that several other bridges who had the same structural design remained intact. Upon investigation, it was determined that air trapped beneath the deck of the Interstate 10 bridges was a major contributing factor to the bridge’s collapse. While major, it was not the only contributing factor (Chen et al. [4]).

A final example is the derailment of an Amtrak train near Joplin, Montana in September, 2021. 154 people were on board the train, and 44 passengers and crew were taken to area hospitals with injuries. The train was traveling at between 75 and 78 mph, just below the speed limit of 79 mph on that section of track when its emergency brakes were activated. The two locomotives and two railcars remained on the rails and eight cars derailed. Investigations of these types of events take years, but preliminary speculation is that the accident could have been caused by problems with the railroad or track, such as a rail that buckled under high heat, or the track itself giving way when the train passed over. Both might also be possible, leading to the two stopping times \( \tau_1 \) and \( \tau_2 \), and a situation where \( P(\tau_1 = \tau_2) > 0 \). See Hanson and Brown [11].
4.3 Application to Credit Risk

In Credit Risk, there are two classes of models: structural and reduced-form models, also known as intensity or hazard models. When investors have access only to the information observed by the market, as opposed to the information held by the firm’s managers, they should use a reduced form model. Hence, hazard models are the preferred methodology in pricing and hedging (See Jarrow and Protter [14]).

The joint distribution presented in this paper falls within the framework of reduced-form models and thus it is natural to use our work to characterize, for example, the probability of a market failure caused by the simultaneous default of two globally systemically important banks (G-SIBs), i.e., financial institutions that are “too big to fail”. For a detailed discussion of this application, see Jarrow et al. [13].

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