Partial Ordering of Gauge Orbit Types for $\text{SU}_n$-Gauge Theories

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Abstract

The natural partial ordering of the orbit types of the action of the group of local gauge transformations on the space of connections in space-time dimension $d \leq 4$ is investigated. For that purpose, a description of orbit types in terms of cohomology elements of space-time, derived earlier, is used. It is shown that, on the level of these cohomology elements, the partial ordering relation is characterized by a system of algebraic equations. Moreover, operations to generate direct successors and direct predecessors are formulated. The latter allow to successively reconstruct the set of orbit types, starting from the principal type.

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1 Introduction

The study of geometrical and topological properties of classical non-abelian gauge theories turned out to be very important for our understanding of non-perturbative aspects of the corresponding quantum field theories. The configuration space of the theory is the gauge orbit space, which is obtained by factorizing the space of connections with respect to the action of the group of local gauge transformations. This space has the structure of a stratified set, because, usually, besides the principal orbit type also non-generic orbit types occur. These may give rise to singularities of the configuration space.

First, the generic, or principal, stratum was investigated – leading to a deeper understanding of the Gribov-ambiguity [15] and of anomalies in terms of index theorems [2]. In particular, one gets anomalies of purely topological type, which cannot be seen by perturbative quantum field theory [17]. Next, in a paper by Kondracki and Rogulski [12], a systematic study of the structure of the full gauge orbit space was presented. In particular, it was shown that the gauge orbit space is a stratified topological space in the ordinary sense, cf. [11] and references therein.

There are partial results and conjectures concerning the physical relevance of nongeneric strata. First of all, nongeneric gauge orbits affect the classical motion on the orbit space due to boundary conditions and, in this way, may produce nontrivial contributions to the path integral. They may also lead to localization of certain quantum states, as it was suggested by finite-dimensional examples [6]. Further, the gauge field configurations belonging to nongeneric orbits can possess a magnetic charge, i.e. they can be considered as a kind of magnetic monopole configurations, which seem to be related to the quark confinement problem in Chern-Simons theory [1]. Finally, it was suggested in [8] that nongeneric strata may lead to additional anomalies.

Most of the problems mentioned here are still awaiting a systematic investigation. In a series of papers we are going to make a new step in this direction. In [14] we have presented a complete solution to the problem of determining the strata that are present in the gauge orbit space for $SU_n$ gauge theories in compact Euclidean space-time of dimension $d = 2, 3, 4$. The basic idea behind is the 1-1-correspondence between orbit types and equivalence classes of so-called holonomy-induced Howe subbundles of the principal $SU_n$-bundle, where the gauge connections of the theory under consideration live on. It turns out that Howe subgroups of $SU_n$ as well as (holonomy-induced) Howe subbundles can be classified, leading to a classification of orbit types in terms of certain algebraical and topological data. As a first application, we have shown in [14] that within the context of Chern-Simons theory in $2 + 1$ dimensions the property of a configuration to be nodal in the sense of Asorey, see [1], is a property of strata. For a given model of this type, the nodal strata can be easily determined.

In [14] one basic problem was left open: the determination of the natural partial ordering in the set of orbit types. In the present paper we solve this problem. First, in Section 2 we recall the classification of gauge orbit types from [14]. In Section 3 we prove that the natural partial ordering is characterized by a system of algebraic equations relating the classifying data via a matrix with non-negative integer entries (inclusion matrix).
The inclusion matrix can be visualized by a Bratteli diagram, as explained in Section 4. In Sections 3 and 6 direct successors and direct predecessors are characterized. In particular, operations which generate the direct successors (splitting and merging) and the direct predecessors (inverse splitting and inverse merging) are defined. Finally, an example is discussed: For gauge group $SU(2)$ and some space-time manifolds the complete Hasse diagram of the set of orbit types is derived.

2 Classification of Gauge Orbit Types

Let $P$ be a principal $SU(n)$-bundle over a compact, connected, orientable Riemannian manifold $M$ of dimension $\dim M \leq 4$. Let $\mathcal{A}^k$ and $\mathcal{G}^k$ denote the sets of connection forms and gauge transformations, respectively, of Sobolev class $W^k$. Provided $2k > \dim M$, $\mathcal{A}^k$ is an affine Hilbert space and $\mathcal{G}^{k+1}$ is a Hilbert Lie group acting smoothly from the right on $\mathcal{A}^k$. If we view gauge transformations as equivariant maps $P \to SU(n)$ then for $A \in \mathcal{A}^k$ and $g \in \mathcal{G}^{k+1}$ the action is given by

$$A(g) = \text{Ad} (g^{-1}) A + g^{-1}dg .$$

Let $\mathcal{M}^k$ denote the quotient topological space $\mathcal{A}^k/\mathcal{G}^{k+1}$. This is the gauge orbit space, i.e., the configuration space of our gauge theory. Let $\text{OT} (\mathcal{A}^k, \mathcal{G}^{k+1})$ denote the set of orbit types of the action of $\mathcal{G}^{k+1}$ on $\mathcal{A}^k$. Recall that orbit types are given by conjugacy classes in $\mathcal{G}^{k+1}$ of stabilizer, or isotropy, subgroups of connections. The set $\text{OT} (\mathcal{A}^k, \mathcal{G}^{k+1})$ carries a natural partial ordering: Let $\tau, \tau' \in \text{OT} (\mathcal{A}^k, \mathcal{G}^{k+1})$. Then $\tau \leq \tau'$ iff there exist representatives $S, S' \subseteq \mathcal{G}^{k+1}$ of $\tau, \tau'$, respectively, such that $S \supseteq S'$. Note that this definition is consistent with [6] but not with [12] and several other authors who define the partial ordering inversely. In [12] it was shown that the family $\{ \mathcal{M}^k_\tau \mid \tau \in \text{OT} (\mathcal{A}^k, \mathcal{G}^{k+1}) \}$, where $\mathcal{M}^k_\tau$ denotes the subset of $\mathcal{M}^k$ of orbits of type $\tau$, is a stratification of $\mathcal{M}^k$ into smooth Hilbert manifolds. For the notion of stratification, see [11] or [12, §4.4]. Moreover, for any $\tau \in \text{OT} (\mathcal{A}^k, \mathcal{G}^{k+1})$, $\mathcal{M}^k_\tau$ is open and dense in the union $\bigcup_{\tau' \leq \tau} \mathcal{M}^k_{\tau'}$. In this sense, the partially ordered set $\text{OT} (\mathcal{A}^k, \mathcal{G}^{k+1})$ encodes the stratification structure of the gauge orbit space.

In [14], we have derived a description of the elements of $\text{OT} (\mathcal{A}^k, \mathcal{G}^{k+1})$ in terms of certain cohomology elements of $M$. In the present article, we are going to discuss the partial ordering. For the convenience of the reader, we begin with briefly recalling the basic results of [14].

A Howe subgroup of a group $G$ is a subgroup $H \subseteq G$ that is the centralizer $H = C_G(K)$ of some subset $K \subseteq G$. A Howe subbundle of a $G$-bundle $P$ is a reduction of $P$ to a Howe subgroup. A Howe subbundle is called holonomy-induced iff it admits a connected reduction $\hat{Q}$ to a subgroup $\hat{H} \subseteq G$ such that

$$\hat{Q} \cdot C_G \left( C_G \left( \hat{H} \right) \right) = Q .$$

Let $\text{H} \cdot (P)$ denote the set of isomorphism classes of holonomy-induced Howe subbundles of $P$ factorized by the natural action of the structure group $G$. Note that here an
Proposition 2.1 Hawe\(_\ast\)(\(P\)) is isomorphic, as a partially ordered set, to OT\(\left(\mathcal{A}^k, \mathcal{G}^{k+1}\right)\).

Proof: See [14, Thm. 3.3].

We note that in the case \(G = SU\_n\), any Howe subbundle is holonomy-induced, see [14, Thm. 6.2]. Hence, this condition is redundant here.

The following description of Howe\(_\ast\)(\(P\)) has been derived in [14]. First, the Howe subgroups of \(SU\_n\) were determined. Let \(K(n)\) denote the set of pairs of sequences of strictly positive integers \(J = (k, m) = (k_1, \ldots, k_r, m_1, \ldots, m_r)\), \(r = 1, \ldots, n\), obeying \(\sum_{i=1}^r k_i m_i = n\). Let \(g\) denote the greatest common divisor of the members of \(m\) and let \(\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_r)\) be defined by \(m_i = g\tilde{m}_i \forall i\). We shall always view \(k\) as an \((r \times 1)\)-matrix (row vector) and \(m\) as a \((1 \times r)\)-matrix (column vector). This turns out to be their natural character. Any \(J \in K(n)\) defines a decomposition\(\mathbb{C}^n = \bigoplus_{i=1}^r \mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i}\)

and an embedding

\[ M_{k_1}(\mathbb{C}) \times \cdots \times M_{k_r}(\mathbb{C}) \to M_n(\mathbb{C}), \quad (D_1, \ldots, D_r) \mapsto \bigoplus_{i=1}^r D_i \otimes 1_{m_i}. \] (1)

Here \(M_l(\mathbb{C})\) stands for the algebra of complex \((l \times l)\)-matrices. Identifying \(\mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i} \cong \mathbb{C}^{k_i m_i}\), \((c_1, \ldots, c_{k_i}) \otimes (d_1, \ldots, d_{m_i}) \mapsto (c_1 d_1, \ldots, c_{k_i} d_1, \ldots, c_1 d_{m_i}, \ldots, c_{k_i} d_{m_i})\), the tensor product \(D_i \otimes 1_{m_i}\) corresponds to the \((m_i \times m_i)\) block matrix

\[
\begin{pmatrix}
D_i & 0 & \cdots & 0 \\
0 & D_i & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_i
\end{pmatrix}.
\]

We denote the image of the embedding (1) by \(M_J(\mathbb{C})\) and its intersections with Un and SUn by \(UJ\) and \(SUJ\), respectively. Note that \(UJ\) is the image of the restriction of (1) to \(Uk_1 \times \cdots \times Uk_r\). By construction, \(M_J(\mathbb{C})\) is a unital \(*\)-subalgebra of \(M_n(\mathbb{C})\).

Proposition 2.2 Up to conjugacy, the Howe subgroups of SUn are given by SUJ, \(J \in K(n)\).
Proof: See [14, Lemma 4.1].

In order to classify principal SU\(_J\)-bundles over \(M\), the homotopy classes of maps from \(M\) to the classifying space BSU\(_J\) have to be determined. Through building the Postnikov tower of BSU\(_J\) up to the 5th stage the following was shown.

**Proposition 2.3** Let \(M\) be a manifold, \(\dim M \leq 4\) and let \(Q, Q'\) be principal SU\(_J\)-bundles over \(M\). Assume that for any characteristic class \(\alpha\) defined by an element of \(H^1(\text{BSU}_J, \mathbb{Z}_g)\), \(H^2(\text{BSU}_J, \mathbb{Z})\), or \(H^4(\text{BSU}_J, \mathbb{Z})\) there holds \(\alpha(Q) = \alpha(Q')\). Then \(Q\) and \(Q'\) are isomorphic.

Proof: See [14, Cor. 5.5].

A generating set for the characteristic classes mentioned in the proposition can be constructed as follows. Consider the natural homomorphisms

\[
\begin{align*}
  j_J & : \text{SU}_J \rightarrow \text{U}_J \quad \text{(embedding)}, \\
  \text{pr}_M^{U,J_k} & : M_J(C) \rightarrow M_{k_i}(C) \quad \text{(projection onto the } i\text{th factor)}, \\
  \text{pr}_U^{U,J_i} & : \text{U}_J \rightarrow \text{U}_{k_i} \quad \text{(ditto)}. 
\end{align*}
\]

For any positive integer \(l\), let \(\gamma_{U_l} = 1 + \gamma_{U_l}^{(2)} + \cdots + \gamma_{U_l}^{(2l)}\) denote the sum of generators of the cohomology algebra \(H^*(\text{BU}_l, \mathbb{Z})\). We assume that the generators are chosen in such a way that for the canonical blockwise embedding \(j_l : \text{U}_l \rightarrow \text{U}(l+1)\) there holds \((Bj_l)^* \gamma_{U_{l+1}} = \gamma_{U_l} \forall l\). (Recall that \(Bj_l : \text{BU}_l \rightarrow \text{BU}(l+1)\) is the map between classifying spaces associated to \(j_l\).) Then, in particular, the characteristic classes defined by the generators \(\gamma_{U_l}^{(2k)}\) are the \(k\)th Chern classes. The cohomology elements

\[
(Bj_l)^* (\text{Bpr}_U^{U,J_i})^* \gamma_{U_{k_i}}, \quad i = 1, \ldots, r,
\]

of \(H^*(\text{BSU}_J, \mathbb{Z})\) define characteristic classes

\[
\alpha_{J,i} : \text{Bun}(M, \text{SU}_J) \rightarrow H^*(M, \mathbb{Z}), \quad Q \mapsto (f_Q)^* ((Bj_l)^* (\text{Bpr}_U^{U,J_i})^* \gamma_{U_{k_i}}),
\]

where \(i = 1, \ldots, r\). Here \(f_Q : M \rightarrow \text{BSU}_J\) is the classifying map of \(Q\) and \(\text{Bun}(M, \text{SU}_J)\) stands for the set of isomorphism classes of principal SU\(_J\)-bundles over \(M\). We denote \(\alpha_J(Q) = (\alpha_{J,1}(Q), \ldots, \alpha_{J,r}(Q))\).

Next, for any positive integer \(l\), let \(j_l : \mathbb{Z}_l \rightarrow \text{U}_1\) denote the canonical embedding and let \(p_l\) denote the endomorphism \(z \mapsto z^l\) of \(\text{U}_1\). We define a homomorphism

\[
\lambda_U^{j_1} : \text{U}_J \rightarrow \text{U}_1, \quad D \mapsto \prod_{i=1}^{r} p_{\bar{m}_i} \circ \det_{U_{k_i}} \circ \text{pr}_U^{U,J_i}(D).
\]

One can check that the diagram

\[
\begin{array}{ccc}
\text{U}_J & \xrightarrow{j_J} & \text{U}_n \\
\lambda_U^{j_1} \downarrow & & \downarrow \det_{U_n} \\
\text{U}_1 & \xrightarrow{p_{g}} & \text{U}_1
\end{array}
\]
commutes. Moreover, we notice that the image of $SUJ$ under $\lambda^U_J$ is the subgroup $j_g(Z_g)$ of $U1$. Thus, $\lambda^U_J$ induces a homomorphism $\lambda^S_J : SUJ \to Z_g$ by requiring the diagram

$$
\begin{align*}
SUJ & \xrightarrow{j_J} UJ \\
\lambda^S_J & \downarrow \quad \downarrow \lambda^U_J \\
Z_g & \xrightarrow{j_g} U1 \\
\end{align*}
$$

(5)

to commute. (In fact, one can show that $\lambda^S_J$ projects to an isomorphism of the group of connected components of $SUJ$ onto $Z_g$.)

One can show that the Bockstein homomorphism $\beta_g : H^1(BZ_g, Z_g) \to H^2(BZ_g, Z)$, induced by the short exact sequence $0 \to Z \to Z \to Z_g \to 0$, is an isomorphism, see the proof of Lemma 5.9 in [14]. Thus, we can consider the cohomology element

$$(B\lambda^S_J)^* \beta_g^{-1} (Bj_g)^* \gamma^{(2)}_{U1}$$

of $H^1(\text{BSUJ}, Z_g)$. It defines a characteristic class

$$\xi_J : \text{Bun}(M, SUJ) \to H^1(M, Z_g), \quad Q \mapsto (f_Q)^* \left( (B\lambda^S_J)^* \beta_g^{-1} (Bj_g)^* \gamma^{(2)}_{U1} \right).$$

(6)

By construction, the characteristic classes $\alpha_J$ and $\xi_J$ are subject to a relation. To formulate it, let us introduce the following notation. For any $\Delta \in M_{r', r}(\mathbb{N})$ (set of $(r' \times r)$-matrices with nonnegative integer entries), we define a map

$$E_\Delta : \prod_{i=1}^r H^0_{\text{even}}(\cdot, Z) \to \prod_{i'=1}^{r'} H^0_{\text{even}}(\cdot, Z),$$

$$(\alpha_1, \ldots, \alpha_r) \mapsto \left( \alpha_1^{\Delta_{11}} \cdots \alpha_r^{\Delta_{1r}}, \ldots, \alpha_1^{\Delta_{r1}} \cdots \alpha_r^{\Delta_{rr}} \right).$$

(7)

Here powers are taken w.r.t. the cup product and $H^0_{\text{even}}(\cdot, Z)$ denotes the subset of $H^0(\cdot, Z)$ of elements of the form $\alpha = 1 + \alpha^{(2)} + \alpha^{(4)} + \cdots$. Note that $H^0_{\text{even}}$ is a semigroup w.r.t. the cup product. Let $E^{(2j)}_{\Delta, i'}(\alpha)$ denote the component of degree $2j$ of the $i'$-th member of $E_\Delta(\alpha)$.

**Proposition 2.4** The characteristic classes $\alpha_J$, $\xi_J$ are subject to the relation

$$E^{(2)}_{\tilde{m}}(\alpha_J(Q)) = \beta_g(\xi_J(Q)), \quad \forall Q \in \text{Bun}(M, SUJ),$$

(8)

(Recall that $\tilde{m}$ is viewed as a $(1 \times r)$-matrix.)

**Proof:** See [14, Theorem 5.13].
We introduce the notation
\[
H^{(J)}(\cdot, \mathbb{Z}) = \prod_{i=1}^{r} \{ \alpha_i \in H^0_{\text{even}}(\cdot, \mathbb{Z}) \mid \alpha^{(2j)}_i = 0 \text{ for } j > k_i \}.
\] (9)

Consider the following two equations in the variables \(\alpha \in H^{(J)}(M, \mathbb{Z}), \xi \in H^1(M, \mathbb{Z}_g)\):
\[
E_m^{(2)}(\alpha), = \beta_g(\xi) \quad (10)
\]
\[
E_m(\alpha) = c(P). \quad (11)
\]

Here \(c(P)\) denotes the total Chern class of \(P\).

**Proposition 2.5** If \(\dim M \leq 4\), the characteristic classes \(\alpha_J\) and \(\xi_J\) define a bijection from \(\text{Bun}(M, \text{SU}J)\) onto the set of solutions of Equation (10). By restriction, they define a bijection from the subset of \(\text{Bun}(M, \text{SU}J)\) of reductions of \(P\) onto the set of solutions of the Equations (10) and (11).

**Proof:** See [14, Thms. 5.14, 5.17].

Note that the content of Eq. (11) in degree 2 is a consequence of Eq. (10).

Let \(K(P)\) denote the disjoint union of the solution sets of Eqs. (10), (11) over all \(J \in K(n)\). We write the elements of \(K(P)\) as triples \((J; \alpha, \xi)\), where \(J \in K(n)\) and \((\alpha, \xi)\) is a solution of the corresponding equations. According to Proposition 2.5, the set \(K(P)\) classifies the Howe subbundles of \(P\) up to isomorphy.

Finally, the action of the structure group \(\text{SU}n\) on Howe subbundles of \(P\) was factored out by passing to the set \(K(P)\) that is obtained from \(K(P)\) by identifying \((J; \alpha, \xi)\) with \((\sigma J; \sigma \alpha, \sigma \xi)\) for all permutations \(\sigma\) of \(1, \ldots, r\). Here \(\sigma J\) stands for
\[
\sigma J = (\sigma k, \sigma m). \quad (12)
\]

**Theorem 2.6** The collection of characteristic classes \(\{\alpha_J, \xi_J \mid J \in K(n)\}\), defines, by passing to quotients, a bijection from \(\text{Howe}_*(P)\) onto \(K(P)\).

**Proof:** See [14, Thm. 7.2].

In the sequel, it is convenient to work with the inverse of this bijection. To construct it, for any \(L \in K(P), \ L = (J; \alpha, \xi)\), let \(Q_L\) denote the isomorphism class of SUJ-subbundles of \(P\) defined by
\[
\alpha_J(Q_L) = \alpha, \quad (13)
\]
\[
\xi_J(Q_L) = \xi. \quad (14)
\]
Then the pre-image of the element of $\hat{K}(P)$ represented by $L$ is given by the conjugacy class of $Q_L$ under SU$n$-action. The (isomorphy classes of) subbundles $Q_L$ may be viewed as some kind of standard representatives of the elements of Howe$_s(P)$.

To conclude this section, for later use, let us collect some formulae involving the function $E_\Delta$. For any $i'$, one has
\begin{align*}
E_{\Delta,r}^{(2)}(\alpha) &= \sum_{i=1}^r \Delta_{\alpha}^{(2)}_i, \\
E_{\Delta,r}^{(4)}(\alpha) &= \sum_{i=1}^r \Delta_{\alpha}^{(4)}_i + \sum_{i=1}^r \frac{\Delta_{\alpha}^{(2)}_i(\Delta_{\alpha}^{(2)}_i - 1)}{2} \alpha_i^{(2)} + \sum_{1\leq i,j\leq r} \Delta_{\alpha}^{(2)}_i \Delta_{\alpha}^{(2)}_j \alpha_i^{(2)} \alpha_j^{(2)},
\end{align*}
see [4, Lemma 5.11]. In particular, for any nonnegative integer $l$,
\begin{equation}
E_{i\Delta,r}^{(2)}(\alpha) = l E_{\Delta,r}^{(2)}(\alpha), \quad \forall \ i'.
\end{equation}
Taking into account that the cup product is commutative in even degree, one can also check that for any $\Delta \in M_{r',r}(\mathbb{N})$ and $\Delta' \in M_{r',r'}(\mathbb{N})$ there holds
\begin{equation}
E_{\Delta\Delta'} = E_{\Delta'} \circ E_\Delta.
\end{equation}

### 3 Characterization of the Partial Ordering

In this section we are going to determine the natural partial ordering of Howe$_s(P)$ on the level of the classifying set $\hat{K}(P)$.

Let $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi')$ be elements of $K(P)$. Let $[Q_L]$ and $[Q_{L'}]$ denote the conjugacy classes of $Q_L$ and $Q_{L'}$, respectively, under the action of SU$n$. The natural partial ordering on the set Howe$_s(P)$ is defined as follows:
\begin{equation}
[Q_L] \leq [Q_{L'}] \iff \exists \ D \in \text{SU}n \text{ such that } Q_L \cdot D \subseteq Q_{L'}.
\end{equation}

Here inclusion is understood up to isomorphy. We aim to express the relation (19) in terms of $L$ and $L'$.

Let $D \in \text{SU}n$ such that $D^{-1}UJ D \subseteq \text{SU}J'$. Then there also holds $D^{-1}UJ D \subseteq \text{U}J'$ and $D^{-1}M_J(\mathbb{C}) D \subseteq M_{J'}(\mathbb{C})$. We have an associated homomorphism
\[ h^M_D : M_J(\mathbb{C}) \rightarrow M_{J'}(\mathbb{C}), \ C \mapsto D^{-1}CD, \]
and, derived from that, homomorphisms $h^U_D : UJ \rightarrow UJ'$ and $h^S_D : SUJ \rightarrow SUJ'$. Due to $M_J(\mathbb{C})$ and $M_{J'}(\mathbb{C})$ being finite-dimensional unital C*-algebras, the embedding $h^M_D$ is characterized by an $(r' \times r)$-matrix $\Delta(D) \in M_{r',r}(\mathbb{N})$ (nonnegative integer entries), called inclusion matrix. The matrix $\Delta(D)$ can be constructed as follows: For $1 \leq i \leq r$ and $1 \leq i' \leq r'$, consider the homomorphism
\begin{equation}
M_{k_i}(\mathbb{C}) \rightarrow M_J(\mathbb{C}) \xrightarrow{h^M_D} M_{J'}(\mathbb{C}) \xrightarrow{\text{pr}_{J',r'}} M_{k'_{i'}}(\mathbb{C}),
\end{equation}
where the first map is canonical embedding to the \( i \)th factor of \( M_J(\mathbb{C}) \). Define \( \Delta(D)_{\nu_i} \) to be the number of fundamental irreps contained in the representation of \( M_{k_i}(\mathbb{C}) \) defined by \( \nu_i \).

Lemma 3.1 Let \( J, J' \in K(n) \). Let \( D \in \text{SU}_n \) such that \( D^{-1} \text{SU}JD \subseteq \text{SU}J' \). Then

\[
\Delta(D) k = k' \tag{21}
\]

\[
m = m' \Delta(D). \tag{22}
\]

Conversely, let \( \Delta \in M_{r',r}(\mathbb{N}) \) be a solution of \( \Delta(D) \). Then there exists \( D \in \text{SU}_n \) such that \( D^{-1} \text{SU}JD \subseteq \text{SU}J' \) and \( \Delta(D) = \Delta \).

Proof: First, let \( D \) be given as proposed. Consider the representations

\[
M_{k_i}(\mathbb{C}) \longrightarrow M_J(\mathbb{C}) \longrightarrow M_n(\mathbb{C}) \tag{23}
\]

\[
M_{k_i}(\mathbb{C}) \longrightarrow M_J(\mathbb{C}) \longrightarrow \text{SU}J'(\mathbb{C}) \longrightarrow M_n(\mathbb{C}). \tag{24}
\]

The numbers of fundamental irreps contained in \( \Delta(D) \) are \( m_i \) and \( \sum_{\nu_i = 1}^{r'} m'_{\nu_i} \Delta(D)_{\nu_i} \), respectively. Since \( \Delta(D) \) is isomorphic – a bijective intertwiner being given by \( D \) – we obtain \( \Delta(D) = \Delta \). Moreover, inserting this equation into \( m \cdot k = m' \cdot k' \) yields \( m' \cdot (k' - \Delta k) = 0 \). By construction, the members of the sequence \( k' - \Delta k \) are nonnegative. Since the members of \( m \) are strictly positive, Eq. \( \Delta(D) = \Delta \) follows.

Conversely, let \( \Delta \) be a solution of \( \Delta(D) \). Consider the decompositions

\[
\mathbb{C}^n = \bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i}, \tag{25}
\]

\[
\mathbb{C}^n = \bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i}. \tag{26}
\]

defined by \( J \) and \( J' \), respectively. Due to \( \Delta(D) \) and \( \Delta(D) \), \( \Delta(D) \) and \( \Delta(D) \) admit subdecompositions

\[
\mathbb{C}^n = \bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \bigoplus_{i=1}^{r'_{\nu_i}} \mathbb{C}^{m'_{\nu_i}}, \tag{27}
\]

\[
\mathbb{C}^n = \bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \bigoplus_{i=1}^{r'_{\nu_i}} \mathbb{C}^{m'_{\nu_i}}. \tag{28}
\]

respectively. There exists \( D \in \text{SU}_n \) transforming \( \mathbb{C}^n \) into \( \mathbb{C}^n \) by a suitable permutation of the subspaces \( \mathbb{C}^{k_i} \otimes \mathbb{C}^{\Delta_{\nu_i}} \otimes \mathbb{C}^{m'_{\nu_i}} \). One can check that \( D^{-1}M_J(\mathbb{C})D \) leaves the decomposition \( \mathbb{C}^n \) invariant. It follows \( D^{-1}M_J(\mathbb{C})D \subseteq M_{J'}(\mathbb{C}) \), hence \( D^{-1} \text{SU}JD \subseteq \text{SU}J' \). Moreover, from \( \mathbb{C}^n \) and \( \mathbb{C}^n \) one can read off that \( \Delta(D) = \Delta \).
We remark that for general inclusions of $M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_r}(\mathbb{C}) \subseteq M_{k'_1} \oplus \cdots \oplus M_{k'_{r'}}$, inclusion matrices only have to obey $\sum_{i=1}^r \Delta_{iv} k_i \leq k'_{r'}$, where the inclusion is unital if there holds equality for all $i'$.

Let us denote the set of solutions of the system of equations (21) and (22) by $N(J, J')$. We note that if $N(J, J') \neq \emptyset$, then (22) implies that $g'$ divides $g$. Hence, reduction $\varrho_{g'} : \mathbb{Z}_g \to \mathbb{Z}_g$ mod $g'$ is defined and is a ring homomorphism.

Again, let $D \in \text{SU}n$ such that $D^{-1} \text{SU}JD \subseteq \text{SU}J'$. Let $Q_L^{[h_D]} = Q_L \times_{\text{SU}J} \text{SU}J'$ denote the $\text{SU}J'$-subbundle of $P$ associated to $Q_L$ by virtue of the homomorphism $h_D : \text{SU}J \to \text{SU}J'$.

**Lemma 3.2** The characteristic classes of $Q_L^{[h_D]}$ are

\[
\alpha_{J'}(Q_L^{[h_D]}) = E_{\Delta(D)}(\alpha), \quad (29)
\]

\[
\xi_{J'}(Q_L^{[h_D]}) = \varrho_{g'}(\xi), \quad (30)
\]

**Proof:** The classifying map of $Q_L^{[h_D]}$ is

\[
f_{Q_L^{[h_D]}} = B h_D \circ f_{Q_L}. \quad (31)
\]

Hence, according to (2),

\[
\alpha_{J', \gamma}(Q_L^{[h_D]}) = \left( f_{Q_L^{[h_D]}} \right)^* \left( (B_{jJ})^* \left( B_{pr_{J', \gamma}}^U \right)^* \gamma_{UK_{\gamma'}} \right)
\]

\[
= (f_{Q_L})^* \left( (B_{jJ})^* \left( B_{pr_{J', \gamma}}^U \right)^* \gamma_{UK_{\gamma'}} \right)
\]

\[
= (f_{Q_L})^* \left( (B_{jJ})^* \left( B_{h_D}^U \right)^* \gamma_{UK_{\gamma'}} \right). \quad (32)
\]

In order to calculate $(B_{h_D}^U)^* \left( B_{pr_{J', \gamma}}^U \right)^* \gamma_{UK_{\gamma'}}$, consider the homomorphisms

\[
pr_{M_{J', \gamma}}^M \circ h_D^M : \text{M}_{J'}(\mathbb{C}) \to \text{M}_{k'_{\gamma'}}(\mathbb{C}), \quad (33)
\]

\[
pr_{U_{J', \gamma}}^U \circ h_D^U : \text{U}J \to \text{U}k'_{\gamma'}. \quad (34)
\]

Since the image of (33) is a unital $*$-subalgebra of $\text{M}_{k'_{\gamma'}}(\mathbb{C})$, the image of (34) is a Howe subgroup of $\text{U}k'_{\gamma'}$. Hence, the latter is conjugate to $\text{U}J'_{(\gamma')}$ for some $J'(\gamma') \in \text{K}(k'_{\gamma'})$. One can check that $J'(\gamma')$ is obtained from the pair of sequences $((k_1, \ldots, k_r), (\Delta_{i1}, \ldots, \Delta_{ir}))$ by deleting all pairs of entries $k_i$, $\Delta_{i1}$ for which $\Delta_{i1} = 0$. On the other hand, $UJ'_{(\gamma')}$ is the image of the homomorphism

\[
\varphi_{\gamma'} : \text{U}J \xrightarrow{d_\gamma} \Pi_{i=1}^{r'} \text{U}J \xrightarrow{\Pi_{i=1}^{r'} pr^U_j} \Pi_{i=1}^{r'} \text{U}k_i \xrightarrow{\Pi_{i=1}^{r'} d_{\Delta(D)_{i1}}} \Pi_{i=1}^{r'} \left( \Pi_{j=1}^{\Delta(D)_{i1}} \text{U}k_i \right) \xrightarrow{\iota_{(\gamma')}} \text{U}k'_{\gamma'}. \quad (35)
\]

Here $d_l$ denotes diagonal embedding into the $l$-fold product, where for $l = 0$ this product is assumed to reduce to $\{1\}$, and $\iota_{(\gamma')}$ is a standard blockwise embedding. Having conjugate
images, the homomorphisms (34) and (35) are conjugate themselves [7], i.e., there exists an inner automorphism $\psi_\nu$ of $Uk_\nu'$ such that the following diagram commutes:

$$\begin{array}{ccc}
UJ & \xrightarrow{\varphi_\nu} & \text{pr}_{J',r'} \circ h_D^U \\
\downarrow & & \downarrow \\
Uk_\nu' & \xrightarrow{\psi_\nu} & Uk_\nu'
\end{array}$$  \hspace{1cm} (36)

Since $Uk_\nu'$ is connected, $B\psi_\nu$ is null-homotopic. Thus, on the level of cohomology,

$$(Bh_D^U)^*(Bpr_{J',r'})^* \gamma_{Uk_\nu'} = (B\varphi_\nu)^* \gamma_{Uk_\nu'}.$$  \hspace{1cm} (37)

From the decomposition (33) one derives

$$(B\varphi_\nu)^* \gamma_{Uk_\nu'} = ((Bpr_{J,1})^* \gamma_{Uk_1})^{\Delta(D),i_1} \circ \cdots \circ ((Bpr_{J,r})^* \gamma_{Uk_r})^{\Delta(D),i_r},$$  \hspace{1cm} (38)

see the proof of [14, L. 5.12] for details. We remark that (38) is an analogue of the Whitney sum formula. Using (7), from (37) and (38) we deduce

$$(Bh_D^U)^*(Bpr_{J',r'})^* \gamma_{Uk_\nu'} = E_{\Delta(D),i'} ((Bpr_{J,1})^* \gamma_{Uk_1}, \ldots, (Bpr_{J,r})^* \gamma_{Uk_r}).$$  \hspace{1cm} (39)

Inserting (32) into (32) and using (2) and (13) we find

$$\alpha_{J',i'} \left( Q_L^{[h_D^S]} \right) = (f_{Q_L})^* (Bj_{i'})^* E_{\Delta(D),i'} ((Bpr_{J,1})^* \gamma_{Uk_1}, \ldots, (Bpr_{J,r})^* \gamma_{Uk_r})$$
$$= (f_{Q_L})^* E_{\Delta(D),i'} ((Bj_{i'})^* (Bpr_{J,1})^* \gamma_{Uk_1}, \ldots, (Bj_{i'})^* (Bpr_{J,r})^* \gamma_{Uk_r})$$
$$= E_{\Delta(D),i'} (\alpha_{J}(Q_L))$$
$$= E_{\Delta(D),i'} (\alpha).$$

This proves (29). Now consider (30). Using (1) and (31), we compute

$$\beta_{g'} \left( \xi_{J'} \left( Q_L^{[h_D^S]} \right) \right) = \beta_{g'} \left( f_{Q_L} \right)^* \left( (B\lambda_j^S)^* \beta_{g'}^{-1} (Bj_{g'})^* \gamma_{U1}^{(2)} \right)$$
$$= \left( f_{Q_L} \right)^* \left( B\lambda_j^S \right)^* (Bj_{g'})^* \gamma_{U1}^{(2)}$$
$$= \left( f_{Q_L} \right)^* (Bh_D^S)^* (B\lambda_j^S)^* (Bj_{g'})^* \gamma_{U1}^{(2)}.$$  \hspace{1cm} (40)

Let $l$ be such that $g = lg'$. The following relation will be proved afterwards:

$$j_{g'} \circ \lambda_j^S \circ h_D^S = p_l \circ j_g \circ \lambda_j^S.$$  \hspace{1cm} (41)

Inserting (41) into (40) yields

$$\beta_{g'} \left( \xi_{J'} \left( Q_L^{[h_D^S]} \right) \right) = \left( f_{Q_L} \right)^* (B\lambda_j^S)^* (Bj_g)^* (Bp_l)^* \gamma_{U1}^{(2)}.$$  \hspace{1cm} (42)
It is easily seen that \((p_l)_\ast : \pi_1(U1) \to \pi_1(U1)\) is multiplication by \(l\). Therefore,

\[
(Bp_l)_\ast \gamma_{U1}^{(2)} = l \gamma_{U1}^{(2)}. \tag{43}
\]

Then (12) becomes

\[
\beta_{g'} \left( \xi_{J'} \left( Q_L^{[h_D^S]} \right) \right) = l \left( f_{Q_L} \right)_\ast \left( B\lambda^S_J \right)_\ast \left( B\lambda J \right)_\ast \gamma_{U1}^{(2)} \\
= l \beta_g \left( f_{Q_L} \right)_\ast \left( B\lambda^S_J \right)_\ast \beta_g^{-1} \left( B\lambda J \right)_\ast \gamma_{U1}^{(2)} \\
= l \beta_g \left( \xi_J \left( Q_L \right) \right) \\
= l \beta_g \left( \xi \right), \tag{44}
\]

where for the last two equalities we have used (3) and (14), respectively. As a direct consequence of the definition of the Bockstein homomorphism, one has

\[
l \beta_g = \beta_{g'} \varrho_{gg'}.
\]

Thus, (44) yields

\[
\beta_{g'} \left( \xi_{J'} \left( Q_L^{[h_D^S]} \right) \right) = \beta_{g'} \varrho_{gg'} \left( \xi \right). \tag{45}
\]

Consider the following portion of the long exact sequence of coefficient homomorphisms which is induced by the short exact sequence \(0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_g \to 0\), see [5, Ch. IV, §5]:

\[
\cdots \to H^1(BSUJ, \mathbb{Z}) \to H^1(BSUJ, \mathbb{Z}_g) \xrightarrow{\beta_{g'}} H^2(BSUJ, \mathbb{Z}) \to \cdots.
\]

Since \(H^1(BSUJ, \mathbb{Z}) = 0\), see [14, Cor. 5.8], \(\beta_{g'}\) is injective here. Hence, (16) implies (30). It remains to prove the relation (41). According to (4) and (5), for any \(B \in SUJ\),

\[
\begin{align*}
\lambda_{J'}^S \circ j_{g'} \circ h_D^S(B) &= \lambda_{J'}^U \circ j_J \circ h_D^S(B) \\
&\equiv \lambda_{J'}^U \circ h_D^U \circ j_J(B) \\
&= \prod_{i'} p_{\tilde{m}_{i'}} \circ \det_{UK_{i'}} \circ \varphi_{i'} \circ j_J(B). \tag{47}
\end{align*}
\]

Using (30) to replace \(pr_{J',i'}^U \circ h_D^U\) and taking into account that an inner automorphism does not change the determinant, (47) yields

\[
\begin{align*}
\lambda_{J'}^S \circ j_{g'} \circ h_D^S(B) &= \prod_{i'} p_{\tilde{m}_{i'}} \circ \det_{UK_{i'}} \circ \varphi_{i'} \circ j_J(B). \tag{48}
\end{align*}
\]

By construction of \(\varphi_{i'}\), see (32), for any \(C \in UJ\),

\[
\det_{UK_{i'}} \circ \varphi_{i'}(C) = \prod_{i=1}^r p_{\Delta(D)_{i'}} \circ \det_{UK_i} \circ \varphi_{j}(C). \tag{49}
\]
Thus, (48) becomes

\[ j_{g'} \circ \lambda^S_{J'} \circ h^S_D(B) = \prod_{i' = 1}^{r'} \prod_{i = 1}^r p_{\tilde{m}_{i'}} \circ p_{\Delta(D)_{i'}} \circ \det U_k \circ \text{pr}^U_{J,i} \circ j_J(B) \]

\[ = \prod_{i = 1}^r p_{(\sum_{i' = 1}^{r'} \tilde{m}_{i'} \Delta(D)_{i'})} \circ \det U_k \circ \text{pr}^U_{J,i} \circ j_J(B) . \tag{49} \]

Due to (22), \( g' \sum_{i' = 1}^{r'} \tilde{m}_{i'} \Delta(D)_{i'} = \sum_{i' = 1}^{r'} m_{i'} \Delta(D)_{i'} = m_i = g \tilde{m}_i , \) hence

\[ \sum_{i' = 1}^{r'} \tilde{m}_{i'} \Delta(D)_{i'} = l \tilde{m}_i , \quad i = 1, \ldots, r . \tag{50} \]

Consequently, (49) implies

\[ j_{g'} \circ \lambda^S_{J'} \circ h^S_D(B) = \prod_{i = 1}^r p_{l \tilde{m}_i} \circ \det U_k \circ \text{pr}^U_{J,i} \circ j_J(B) \]

\[ = p_l \left( \prod_{i = 1}^r p_{\tilde{m}_i} \circ \det U_k \circ \text{pr}^U_{J,i} \circ j_J(B) \right) \]

\[ = p_l \circ \lambda^S_{J} \circ j_J(B) \]

\[ = p_l \circ j_{g} \circ \lambda^S_{J} (B) , \]

where the last two equalities are due to (3) and (5), respectively. This proves (41) and, therefore, concludes the proof of the lemma. \( \blacksquare \)

**Lemma 3.3** Let \( D \in SU_n \) such that \( D^{-1}SUJD \subseteq SUJ' \). Then \( Q_L \cdot D \) is a reduction of \( Q_L^{[h^S_D]} \) to the structure group \( D^{-1}SUJD \).

**Proof:** Define a map \( \varphi : Q_L \cdot D \rightarrow Q_L^{[h^S_D]} \), \( q \cdot D \mapsto [(q, 1)] \). This map is obviously smooth. To check equivariance, let \( C \in SUJ \). Then

\[ \varphi ((q \cdot D) \cdot D^{-1}CD) = \varphi ((q \cdot C) \cdot D) \]

\[ = [(q \cdot C, 1)] \]

\[ = [(q, h^S_D(C))] \]

\[ = [(q, 1)] \cdot h^S_D(C) \]

\[ = [(q, 1)] \cdot D^{-1}CD . \]

This proves the lemma. \( \blacksquare \)

**Theorem 3.4** Let \( L = (J; \alpha, \xi) \), \( L' = (J'; \alpha', \xi') \) be elements of \( K(P) \). Then \([Q_L] \leq [Q_L']\) if and only if

(a) \( g' \) divides \( g \) and there holds \( \xi' = \xi_{gg'}(\xi) \),

(b) there exists \( \Delta \in M_{r', r}(\mathbb{N}) \) such that

\[ \Delta k = k' \tag{51} \]

\[ m = m' \Delta \tag{52} \]

\[ E_\Delta (\alpha) = \alpha' . \tag{53} \]
Proof: To begin with, assume $[Q_L] \leq [Q_{L'}]$. Then there exists $D \in SU_n$ such that $Q_L \cdot D \subseteq Q_{L'}$. Since $Q_L \cdot D$ has structure group $D^{-1}SUJD$, $D^{-1}SUJD \subseteq SUJ'$. As a consequence, the homomorphism $h^S_D$ and the inclusion matrix $\Delta(D)$ exist. Due to Lemma 3.3, $\Delta(D) \in N(J,J')$, hence it obeys (51) and (52). The latter equation implies, in particular, that $g'$ divides $g$. Moreover, by construction, $Q_{L'}$ can be reduced to $Q_L \cdot D$. According to Lemma 3.3, so can the SU$^J$-bundle $Q_{L'}^{[h^S_D]}$. Since $Q_{L'}$ and $Q_{L'}^{[h^S_D]}$ have the same structure group, it follows $Q_{L'} \cong Q_{L'}^{[h^S_D]}$. Then Lemma 3.2 yields
$$\alpha' = \alpha_{J'}(Q_{L'}) = \alpha_{J'}(Q_{L'}^{[h^S_D]}) = E_{\Delta(D)}(\alpha).$$
Thus, $\Delta(D)$ satisfies (53). By an analogous argument, we finally find $\xi' = \varrho_{gg'}(\xi)$.

Conversely, assume that assertions (a) and (b) hold. Then, due to Lemma 3.3, there exists $D \in SU_n$ such that $D^{-1}SUJD \subseteq SUJ'$ and $\Delta(D) = \Delta$. Consider the SU$^J$-bundle $Q_{L'}^{[h^S_D]}$ associated to $Q_L$. Due to Lemma 3.2 and (53),
$$\alpha_{J'}(Q_{L'}^{[h^S_D]}) = E_{\Delta}(\alpha) = \alpha' = \alpha_{J'}(Q_{L'}).$$
Analogously, we obtain $\xi_{J'}(Q_{L'}^{[h^S_D]}) = \xi_{J'}(Q_{L'})$. Hence, $Q_{L'}$ and $Q_{L'}^{[h^S_D]}$ are isomorphic. Then Lemma 3.3 implies $Q_L \cdot D \subseteq Q_{L'}$, up to isomorphy (which is sufficient). It follows $[Q_L] \leq [Q_{L'}]$.

Let $L, L' \in K(P)$. If Condition (a) of Theorem 3.4 holds, we define $N(L,L')$ to be the set of solutions of the system of Equations (51)–(53). If this condition does not hold, we define $N(L,L') = \emptyset$. In order to be able to argue entirely on the level of $\hat{K}(P)$, we define a partial ordering on $\hat{K}(P)$ as the image of the natural partial ordering of Howe$_*(P)$ under the bijection defined by the collection of characteristic classes $\alpha_J$, $\xi_J$, $J \in K(n)$. According to Theorem 3.4, the partial ordering so defined can be characterized as follows.

**Corollary 3.5** Let $\kappa, \kappa' \in \hat{K}(P)$. Then the following assertions are equivalent:
(a) $\kappa \leq \kappa'$.
(b) There exist representatives $L, L'$ of $\kappa, \kappa'$, respectively, such that $N(L,L')$ is nonempty.
(c) For any two representatives $L, L'$ of $\kappa, \kappa'$, respectively, $N(L,L')$ is nonempty.

**Proof:**
(a) $\Rightarrow$ (c): Let $L, L'$ be given. By assumption, $[Q_L] \leq [Q_{L'}]$. Then Theorem 3.4 implies that $N(L,L')$ is nonempty.
(c) $\Rightarrow$ (b): Obvious.
(b) $\Rightarrow$ (a): Let $L, L'$ be the representatives provided by assertion (b). Since $N(L,L')$ is nonempty, assertions (a) and (b) of Theorem 3.4 hold. It follows that the subbundles $Q_L$ and $Q_{L'}$ obey $[Q_L] \leq [Q_{L'}]$. Hence, $\kappa \leq \kappa'$.
Example

Let $P = M \times SU4$. Consider elements $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi')$ of $K(P)$, where $J = ((1,1), (2,2))$ and $J' = ((2,2), (1,1))$. We remark that the subgroup $SUJ \subseteq SU4$ has connected components

$$\left\{ \begin{pmatrix} z \mathbb{1}_2 & 0 \\ 0 & z^{-1} \mathbb{1}_2 \end{pmatrix} \mid z \in U1 \right\} \cup \left\{ \begin{pmatrix} z \mathbb{1}_2 & 0 \\ 0 & -z^{-1} \mathbb{1}_2 \end{pmatrix} \mid z \in U1 \right\},$$

hence is isomorphic to the direct product $\mathbb{Z}_2 \times U1$. The subgroup $SUJ'$ can be parametrized as follows:

$$SUJ' = \left\{ \begin{pmatrix} zA & 0 \\ 0 & z^{-1}B \end{pmatrix} \mid z \in U1, A, B \in SU2 \right\}.$$

Thus, it is isomorphic to the direct product $U1 \times SU2 \times SU2$.

In order to find out whether $[Q_L] \leq [Q_{L'}]$, we are going to determine $N(L, L')$. Condition (a) of Theorem 3.4 is obviously satisfied. Thus, we can proceed as follows: First, we solve Eqs. (51) and (52), i.e., we derive $N(J, J')$. Then, for all $\Delta \in N(J, J')$, we compute $E_\Delta(\alpha)$ and compare the result with $\alpha'$. Eqs. (51) and (52) read

$$\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ \Delta_{11} & \Delta_{21} \end{pmatrix} \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = \begin{pmatrix} 2 & 2 \end{pmatrix}. $$

We extract the equations

$$\Delta_{11} + \Delta_{12} = 2, \quad \Delta_{21} + \Delta_{22} = 2, \quad \Delta_{11} + \Delta_{21} = 2, \quad \Delta_{12} + \Delta_{22} = 2. $$

The solutions are

$$\Delta^a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \Delta^b = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \Delta^c = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{54}$$

For $\alpha = (\alpha_1, \alpha_2)$, they yield

$$E_{\Delta^a}(\alpha) = (\alpha_1 \preceq \alpha_2, \alpha_1, \preceq \alpha_2),$$

$$E_{\Delta^b}(\alpha) = (\alpha_1 \preceq \alpha_1, \alpha_2, \preceq \alpha_2),$$

$$E_{\Delta^c}(\alpha) = (\alpha_2 \preceq \alpha_2, \alpha_1, \preceq \alpha_1).$$

Thus, $N(L, L') \neq \emptyset$, i.e., $[Q_L] \leq [Q_{L'}]$, if and only if $\alpha'$ coincides with one of the elements $E_{\Delta^a}(\alpha), E_{\Delta^b}(\alpha), E_{\Delta^c}(\alpha)$, listed above.

4 Bratteli Diagrams

Any $\Delta \in M_{r', r}(\mathbb{N})$ can be visualized by a diagram consisting of a series of upper vertices, labelled by $i = 1, \ldots, r$, and a series of lower vertices, labelled by $i' = 1, \ldots, r'$. For each combination of $i$ and $i'$ the corresponding vertices are connected by $\Delta_{ii'}$ edges. For example, the matrices $\Delta^a, \Delta^b,$ and $\Delta^c$ in (54) give rise to the following diagrams:
The diagrams associated in this way to the elements of $\mathcal{N}(J, J')$, $J, J' \in \mathcal{K}(n)$, are special cases of so-called Bratteli diagrams \[4\]. The latter have, in general, several stages picturing the subsequent inclusion matrices associated to an ascending sequence of finite dimensional von-Neumann algebras $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. For this reason, we refer to the diagram associated to $\Delta \in \mathcal{N}(J, J')$ as the Bratteli diagram of $\Delta$. We remark that, due to Eq. (51), $\Delta$ cannot have a zero row. Due to (52), it cannot have a zero column either. Accordingly, each vertex of the Bratteli diagram of $\Delta$ is cut by at least one edge.

Let $L = (J; \alpha, \xi)$ and $L' = (J'; \alpha', \xi')$ be elements of $\mathcal{K}(P)$. In terms of the Bratteli diagram of the variable $\Delta$, Eqs. (51), (52), and (53) can be rewritten as follows:

\begin{align*}
    k'_{i'} & = \sum_{i=1}^{r} \sum_{\text{edges from } i \text{ to } i'} k_i, \quad i' = 1, \ldots, r', \quad (55) \\
    m_i & = \sum_{i'=1}^{r'} \sum_{\text{edges from } i \text{ to } i'} m'_{i'}, \quad i = 1, \ldots, r, \quad (56) \\
    \alpha'_{i'} & = \sum_{i=1}^{r} \sum_{\text{edges from } i \text{ to } i'} \alpha_i, \quad i' = 1, \ldots, r'. \quad (57)
\end{align*}

The main use of Bratteli diagrams is to simplify calculations as, for instance, solving the equations determining $\mathcal{N}(L, L')$. Furthermore, some of the arguments in the sequel are easier to formulate on the level of these diagrams than on the level of the corresponding matrices.

## 5 Direct Successors

In this section, we are going to derive a characterization of direct successor relations in $\hat{\mathcal{K}}(P)$ and to formulate operations that generate the direct successors of any given element of $\hat{\mathcal{K}}(P)$.
5.1 The Level of an Inclusion Matrix

Let $J, J' \in K(n)$. For any $\Delta \in N(J, J')$, we define the level of $\Delta$ to be the integer

$$\ell(\Delta) = 2 \sum_{i=1}^{r} \sum_{i'=1}^{r'} \Delta_{ii'} - (r + r') \quad (58)$$

Using the quantities

$$\ell_i^+(\Delta) = \left( \sum_{i'=1}^{r'} \Delta_{ii'} \right) - 1, \quad i = 1, \ldots, r, \quad (59)$$

$$\ell_{i'}^-(\Delta) = \left( \sum_{i=1}^{r} \Delta_{ii'} \right) - 1, \quad i' = 1, \ldots, r', \quad (60)$$

we can write

$$\ell(\Delta) = \sum_{i=1}^{r} \ell_i^+(\Delta) + \sum_{i'=1}^{r'} \ell_{i'}^-(\Delta). \quad (61)$$

Due to (51) and (52), each row and each column of $\Delta$ contain at least one non-zero entry. It follows that $\ell_i^+(\Delta), \ell_{i'}^-(\Delta) \geq 0$. Hence, due to (61), $\ell(\Delta) \geq 0$.

As for the interpretation, $\ell(\Delta)$ measures, in a sense, how much $J'$ deviates from $J$ (up to permutations). On the level of the Bratteli diagram of $\Delta$, $\ell(\Delta)$ is twice the number of edges minus the number of vertices, whereas $\ell_i^+(\Delta)$ and $\ell_{i'}^-(\Delta)$ count the edges at the vertices $i$ and $i'$, respectively, minus the obligatory one edge per vertex.

For later use, we note the following formulae, which follow immediately from (61):

$$\ell(\Delta) = 2 \sum_{i=1}^{r} \ell_i^+(\Delta) + r - r' = 2 \sum_{i'=1}^{r'} \ell_{i'}^-(\Delta) + r' - r. \quad (62)$$

5.2 Lemmata about the Level

**Lemma 5.1** Let $L, L', L'' \in K(P)$ and let $\Delta \in N(L, L')$, $\Delta' \in N(L', L'')$. Then $\Delta' \Delta \in N(L, L'')$ and

$$\ell(\Delta' \Delta) \geq \ell(\Delta') + \ell(\Delta). \quad (63)$$

Moreover, $\ell(\Delta') = 0$ or $\ell(\Delta) = 0$ imply equality in (63).

**Proof:** Let $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi')$, and $L'' = (J''; \alpha'', \xi'')$. By the assumption that $N(L, L')$ and $N(L', L'')$ be nonempty, $g'$ divides $g$ and $g''$ divides $g'$, hence $g''$ divides $g$. Also by this assumption, $\xi' = \theta_{gg'}(\xi)$ and $\xi'' = \theta_{g'g''}(\xi')$, hence $\theta_{gg''}(\xi) = \theta_{g'g''} \circ \theta_{gg'}(\xi) = \theta_{g'g''}(\xi') = \xi''$. Moreover, one can check that $\Delta' \Delta$ obeys Eqs. (51), (52), and (53), where for the last one, (62) has to be used.
To prove (63), using (60), (64), and (65), we compute

$$2 \sum_{i=1}^{r'} \ell_{i}^+ (\Delta') \ell_{i}^- (\Delta) = 2 \sum_{i=1}^{r'} \left( \left( \sum_{i=1}^{r''} \Delta_{i}^{r''} \right) - 1 \right) \left( \left( \sum_{i=1}^{r'} \Delta_{i}^{r'} \right) - 1 \right)$$

$$= 2 \left( \sum_{i=1}^{r'} \sum_{i=1}^{r'} \sum_{i=1}^{r'} \Delta_{i}^{r'} \Delta_{i}^{r'} - \sum_{i=1}^{r'} \sum_{i=1}^{r'} \Delta_{i}^{r'} \right)$$

$$- \sum_{i=1}^{r'} \sum_{i=1}^{r'} \Delta_{i}^{r'} + r')$$

$$= \ell (\Delta') - \ell (\Delta) - \ell (\Delta') . \quad (64)$$

Since the lhs. of (64) is nonnegative, this yields (63). Moreover, if \(\ell (\Delta) = 0\) or \(\ell (\Delta') = 0\) then, due to (61), \(\ell_{i}^- (\Delta) = 0\) or \(\ell_{i}^+ (\Delta') = 0\), respectively, for all \(i\). Hence, the lhs. of (64) vanishes, so that equality holds in (63). \(\blacksquare\)

**Lemma 5.2** Let \(L, L' \in K(P)\) and let \(l = 0\) or 1. If \(N(L, L')\) contains an element of level \(l\) then all its elements have level \(l\).

**Proof:** Let \(L = (J; \alpha, \xi), L' = (J'; \alpha', \xi')\) and let \(\Delta \in N(L, L')\). Due to (61) and (63),

$$\sum_{i=1}^{r} k_{i} \ell_{i}^+ (\Delta) = \sum_{i=1}^{r} k_{i} \left( \left( \sum_{i=1}^{r'} \Delta_{i}^{r'} \right) - 1 \right) = \sum_{i=1}^{r} k_{i} - \sum_{i=1}^{r} k_{i} . \quad (65)$$

Since \(k_{i} > 0\) and \(\ell_{i}^+ (\Delta) \geq 0\) for all \(i\), (65) implies

$$\ell_{i}^+ (\Delta) = 0 \quad \forall i \iff \sum_{i=1}^{r} k_{i} - \sum_{i=1}^{r} k_{i} = 0 . \quad (66)$$

By a similar argument we find

$$\ell_{i}^- (\Delta) \quad \forall i = 0 \iff \sum_{i=1}^{r} m_{i} - \sum_{i=1}^{r} m_{i} = 0 . \quad (67)$$

Now assume that \(\ell (\Delta) = l\), where \(l = 0\) or 1. Then at most one of the integers \(\ell_{i}^+ (\Delta)\) or \(\ell_{i}^- (\Delta)\) can be nonzero. Thus, (66) or (67) holds. In either case, the assertion holds for any \(\Delta' \in N(L, L')\). Then (62) implies \(\ell (\Delta') = \ell (\Delta) = l\). \(\blacksquare\)

**Remarks:**

1. The proof of Lemma 5.2 shows that the lemma still holds if one replaces \(N(L, L')\) by \(N(J, J')\), for any \(J, J' \in K(n)\).

2. In general, the level function \(\ell\) may not be constant on the sets \(N(L, L')\). For example, let \(P\) be the trivial SU8-bundle over \(M\) and let \(L = (J; \alpha, \xi), L' = (J'; \alpha', \xi')\) be given by \(J = ((1, 2), (4, 2)), \alpha = 1, \xi = 0\) and \(J' = ((4, 2), (1, 2)), \alpha' = 1, \xi' = 0\). Obviously, \((\alpha, \xi) \in K(P)_J\) and \((\alpha', \xi') \in K(P)_{J'}\). One can check that \(N(L, L')\) contains the following two inclusion matrices:

\[
\Delta = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Delta' = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.
\]

One has \(\ell (\Delta) = 6\) and \(\ell (\Delta') = 4\).
Lemma 5.3 Let $L, L' \in K(P)$. The following assertions are equivalent:
(a) $L$ and $L'$ are equivalent.
(b) $N(L, L')$ contains an element of level 0.
(c) $N(L, L')$ is nonempty and all of its elements have level 0.

Proof: Due to Lemma 5.2, (b) $\iff$ (c). Hence, it suffices to prove (a) $\iff$ (b). Let $L = (J; \alpha, \xi), L' = (J'; \alpha', \xi')$. First, assume that there exist $\Delta \in N(L, L')$ such that $\ell(\Delta) = 0$. Then $\ell_i^+(\Delta) = 0$ for all $i$ and $\ell_i^-(\Delta) = 0$ for all $i'$. That means, each row and each column contains exactly one nonzero entry and this entry has value 1. It follows that $\Delta$ is square, i.e., $r' = r$, and that there exists a permutation $\sigma$ of $1, \ldots, r$ such that
\begin{equation}
\Delta_{i'i} = \delta_{\sigma(i')i}, \quad i', i = 1, \ldots, r.
\end{equation}
As an immediate consequence,
\begin{equation}
\Delta k = \sigma k, \quad m' \Delta = \sigma^{-1} m', \quad E_\Delta(\alpha) = \sigma \alpha.
\end{equation}
Since $\Delta \in N(L, L')$, (63) implies $J' = \sigma J, \alpha' = \sigma \alpha$, and $\xi' = \varrho_{gg'}(\xi)$. In particular, $m' = \sigma m$, hence $g = g'$. It follows $\xi' = \xi$. Thus, $L$ and $L'$ are equivalent.

Conversely, assume that $\xi' = \xi$ and that there exist a permutation $\sigma$ of $1, \ldots, r$ such that $J' = \sigma J$ and $\alpha' = \sigma \alpha$. Since, in particular, $m' = \sigma m$, $g'$ and $g$ coincide. Thus, trivially, $g'$ divides $g$ and $\xi' = \varrho_{gg'}(\xi)$. Hence, if we find a solution $\Delta$ of Eqs. (51), (52), and (53) then $\Delta \in N(L, L')$. Due to (63), such a solution is given by the matrix (68). By construction, it has level 0.

5.3 Splitting and Merging

Let $L = (J; \alpha, \xi) \in K(P)$. In this subsection, we are going to formulate operations that create new elements of $K(P)$ out of $L$. These operations will be used to prove a decomposition lemma in subsection 5.4 and, later on, to generate direct successors.

Splitting: Choose $1 \leq i_0 \leq r$ such that $m_{i_0} \neq 1$. Choose a decomposition $m_{i_0} = m_{i_0,1} + m_{i_0,2}$ with strictly positive integers $m_{i_0,1}, m_{i_0,2}$. Define sequences of length $(r + 1)$
\begin{align}
k^o &= (k_1, \ldots, k_{i_0-1}, k_{i_0}, k_{i_0}, k_{i_0+1}, \ldots, k_r), \quad (70) \\
m^o &= (m_1, \ldots, m_{i_0-1}, m_{i_0,1}, m_{i_0,1}, m_{i_0,2}, m_{i_0+1}, \ldots, m_r), \quad (71) \\
\alpha^o &= (\alpha_1, \ldots, \alpha_{i_0-1}, \alpha_{i_0}, \alpha_{i_0}, \alpha_{i_0+1}, \ldots, \alpha_r). \quad (72)
\end{align}
Since the greatest common divisor $g^o$ of $m^o$ divides $g$, we can furthermore define
\begin{equation}
\xi^o = \varrho_{gg^o}(\xi).
\end{equation}
Denote $J^o = (k^o, m^o)$ and $L^o = (J^o; \alpha^o, \xi^o)$.

We claim that $L^o \in K(P)$. It is easily seen that $m^o \cdot k^o = n$ and $\alpha \in H^{(J^o)}(M, \mathbb{Z})$. Consequently, it suffices to check that $\alpha^o$ and $\xi^o$ obey Eqs. (10) and (11). First, consider
Lemma 5.4  Let \( l \) be such that \( g = lg^o \). Using (15) and (17) as well as taking into account that (10) holds for \( \alpha \) and \( \xi \), we compute

\[
\beta_{g^o}(\xi^o) = \beta_{g^o} \circ \varrho_{g g^o}(\xi) = l \beta_g(\xi) = l E_{m^o}^{(2)}(\alpha) = E_{\hat{m}^o}^{(2)}(\alpha).
\]

Expanding the rhs. according to (15) yields

\[
\beta_{g^o}(\xi^o) = l \frac{m_{i_1}}{g^o} \alpha_{i_1}^{(2)} + \cdots + l \frac{m_{i_0}}{g^o} \alpha_{i_0}^{(2)} + \cdots + l \frac{m_r}{g^o} \alpha_r^{(2)}
= \frac{m_{i_1}}{g^o} \alpha_{i_1}^{(2)} + \cdots + \frac{m_{i_0}}{g^o} \alpha_{i_0}^{(2)} + \cdots + \frac{m_r}{g^o} \alpha_r^{(2)}
= \frac{m_{i_1}}{g^o} \alpha_{i_1}^{(2)} + \cdots + \frac{m_{i_0}}{g^o} \alpha_{i_0}^{(2)} + \frac{m_{i_0}}{g^o} \alpha_i^{(2)} + \cdots + \frac{m_r}{g^o} \alpha_r^{(2)}
= E_{\hat{m}^o}^{(2)}(\alpha^o),
\]

where the penultimate equality is due to the fact that \( g^o \) divides both \( m_{i_1} \) and \( m_{i_0} \).

Now consider (11). Using commutativity of the cup product in even degree, one can check that \( E_{\hat{m}^o}(\alpha^o) = E_{\hat{m}}^{(2)}(\alpha) \). Since (11) holds for \( \alpha \), it holds for \( \alpha^o \). This proves \( L^o \in K(P) \).

We say that \( L^o \) arises from \( L \) by a splitting of the \( i_0 \)th member.

**Merging:** Choose \( 1 \leq i_1 < i_2 \leq r \) such that \( m_{i_1} = m_{i_2} \). Define sequences of length \((r-1)\)

\[
\begin{align*}
k^o &= \left(k_1, \ldots, k_{i-1}, k_i + k_{i+1}, \ldots, \hat{k}_{i_2}, \ldots, k_r\right), \quad (74) \\
m^o &= (m_1, \ldots, m_{i-1}, m_{i_1}, m_{i_1+1}, \ldots, \hat{m}_{i_2}, \ldots, m_r), \quad (75) \\
\alpha^o &= (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i_1} \leftarrow \alpha_{i_2}, \alpha_{i_1+1}, \ldots, \hat{\alpha}_{i_2}, \ldots, \alpha_r), \quad (76)
\end{align*}
\]

where \( \hat{\phantom{\alpha}} \) indicates that the entry is omitted, as well as

\[
\xi^o = \xi. \quad (77)
\]

Denote \( J^o = (k^o, m^o) \) and \( L^o = (J^o; \alpha^o, \xi^o) \).
Let us show \( L^o \in K(P) \). As in the case of splitting, one can immediately verify that \( m^o \cdot k^o = n \), \( \alpha^o \in H^{(J^o)}(M, \mathbb{Z}) \), and \( E_{m^o}^{(2)}(\alpha^o) = E_{\hat{m}^o}(\alpha) \). It follows that \( \alpha^o \) obeys Eq. (11).

Due to \( g^o = g \), a similar calculation shows \( E_{\hat{m}^o}^{(2)}(\alpha^o) = E_{\hat{m}}^{(2)}(\alpha) \). Since also \( \beta_{g^o}(\xi^o) = \beta_g(\xi) \), we obtain \( \beta_{g^o}(\xi^o) = E_{\hat{m}^o}^{(2)}(\alpha^o) \). Thus, \( L^o \in K(P) \).

We say that \( L^o \) arises from \( L \) by merging the \( i_1 \)th and the \( i_2 \)th member.

**Remark:** It may happen that for certain elements of \( K(P) \) no splittings or no mergings can be applied. Amongst these elements are, for example, those with \( m_1 = \cdots = m_r = 1 \) (no splitting) and those having pairwise distinct \( m_i \) (no merging).

**Lemma 5.4** Let \( L, L^o \in K(P) \). \( L^o \) can be obtained from \( L \) by a splitting of the \( i_0 \)th
member if and only if \( N(L, L^\circ) \) contains an element with Bratteli diagram

![Bratteli diagram](image)

\( L^\circ \) can be obtained from \( L \) by merging the \( i_1 \)th and the \( i_2 \)th member if and only if \( N(L, L^\circ) \) contains an element with Bratteli diagram

![Bratteli diagram](image)

**Proof:** Assume \( L = (J; \alpha, \xi) \), \( L^\circ = (J^\circ; \alpha^\circ, \xi^\circ) \). Since the proofs for the cases of splitting and merging are completely analogous, we only give the first one. First, assume that \( L^\circ \) arises from \( L \) by a splitting of the \( i_0 \)th member. Then, by construction, \( g^\circ \) divides \( g \) and \( \xi^\circ = \varrho_{gg^\circ}(\xi) \). Hence the matrix given by the Bratteli diagram (78) belongs to \( N(L, L^\circ) \) iff it satisfies Eqs. (51), (52), and (53). By the help of Eqs. (55)–(57), this can be easily checked on diagram level. Conversely, assume that \( N(L, L^\circ) \) contains an element with Bratteli diagram (78). Then, in particular, Condition (a) of Theorem 3.4 holds, i.e., \( g^\circ \) divides \( g \) and \( \xi^\circ = \varrho_{gg^\circ}(\xi) \). An inspection of (54)–(57) shows that \( k_{i_0} = k_{i_0+1} = k_{i_0}, m_{i_0} = m_{i_0+1} = m_{i_0}, \) and \( \alpha_{i_0} = \alpha_{i_0+1} = \alpha_{i_0} \), whereas \( k_i = k_i, m_i = m_i, \alpha_i = \alpha_i \) for \( 1 \leq i < i_0 \) and \( k_{i+1} = k_i, m_{i+1} = m_i, \alpha_{i+1} = \alpha_i \) for \( i_0 < i \leq r \). Thus, \( L^\circ \) is obtained from \( L \) by a splitting of the \( i_0 \)th member according to the decomposition \( m_{i_0} = m_{i_0} + m_{i_0+1} \).

**5.4 The Decomposition Lemma**

**Lemma 5.5** Let \( L, L' \in K(P) \) and let \( \Delta \in N(L, L') \). If \( \ell(\Delta) \neq 0 \) then there exist \( L^\circ \in K(P) \) and \( \Delta^\circ \in N(L^\circ, L^\circ) \) such that \( \Delta = \Delta^\circ \Delta^\circ \) and \( \ell(\Delta^\circ) = 1 \).

**Proof:** To begin with, assume that there exist \( i_0 \) such that \( \ell_{i_0}^+(\Delta) > 0 \). Choose \( i'_0 \) such that \( \Delta_{i_i} > 0 \). We have the following estimate:

\[
m_{i_0} - m_{i_0}' = \sum_{i'=1}^{i'} m_{i'}(\Delta_{i'i_0} - \delta_{i'i_0}) \geq \sum_{i'=1}^{i'} (\Delta_{i'i_0} - \delta_{i'i_0}) = \ell_{i_0}^+(\Delta) > 0.
\]
This shows that \( m_{i_0} = (m_{i_0} - m'_{i_0}) + m'_{i_0} \) is a decomposition into strictly positive integers. We define \( L^o \) to be the element of \( K(P) \) obtained from \( L \) by the corresponding splitting operation. Furthermore, we define \( \Delta^o \) to be the \(((r + 1) \times r)\)-matrix

\[
\Delta^o = \begin{pmatrix}
1_{i_0} & 0 \\
0 \cdots 0 & 1 \cdots 0 \\
0 & 1_{r-i_0}
\end{pmatrix}
\]

(80) and \( \Delta'^o \) to be the \((r' \times (r + 1))\)-matrix

\[
\Delta'^o = \begin{pmatrix}
\Delta_{i_0} & \cdots & \Delta_{i_0} \\
\vdots & \cdots & \vdots \\
\Delta_{i_0} & \cdots & \Delta_{i_0} - 1 \\
\vdots & \cdots & \vdots \\
\Delta_{i_0} & \cdots & \Delta_{i_0} \\
\vdots & \cdots & \vdots \\
\Delta_{i_0} & \cdots & \Delta_{i_0}
\end{pmatrix}
\]

(81)

We notice that \( \Delta^o \) has Bratteli diagram (78). Hence, due to Lemma 5.4, \( \Delta^o \in N(L, L^o) \). From the diagram we read off that \( \ell(\Delta^o) = 1 \). Moreover, by means of a direct computation using (80) and (81) one can check that \( \Delta'^o \Delta^o = \Delta \). Thus, it remains to prove that \( \Delta'^o \in N(L^o, L') \). This amounts to the following items:

(a) \( g' \) divides \( g^o \): We recall from (73) that

\[
m^o = (m_1, \ldots, m_{i_0-1}, m_{i_0} - m'_{i_0}, m'_{i_0}, m_{i_0+1}, \ldots, m_r)
\]

(82)

By assumption, \( g' \) divides \( g \), hence all the \( m_i \). By definition, it also divides \( m'_{i_0} \).

(b) \( \varrho_{g^o g'}(\xi^o) = \xi' \): According to (73), \( \varrho_{g^o g'}(\xi^o) = \varrho_{g^o g'}(\xi) = \varrho_{g g'}(\xi) = \xi' \). Here the last equality holds by assumption.

(c) \( \Delta'^o k^o = k' \): Using that \( \Delta^o \in N(L, L^o) \) and \( \Delta \in N(L, L') \) we compute \( \Delta'^o k^o = \Delta'^o \Delta^o k = \Delta k = k' \).

(d) \( m^o \Delta'^o = m^o \): This has to be checked by a direct computation using (73) and (72).

(e) \( E_{\Delta'^o}(\alpha^o) = \alpha' \): Using the same arguments as for (c), as well as (18), we obtain \( E_{\Delta'^o}(\alpha^o) = E_{\Delta^o} \circ E_{\Delta^o}(\alpha) = E_{\Delta^o \Delta^o}(\alpha) = E_{\Delta}(\alpha) = \alpha' \). This proves \( \Delta'^o \in N(L^o, L') \).

Now assume that \( \ell^+(\Delta) = 0 \) for all \( i \). Then in each column of \( \Delta \) there exists exactly one nonzero entry, and this entry has value 1. On the other hand, since \( \ell(\Delta) \neq 0 \), there exists
\(i'_0\) such that \(\ell_{i'_0}^-(\Delta) > 0\). This means, the row labelled by \(i'_0\) has at least two entries of value 1. Therefore, we find two columns, labelled by \(i_1 < i_2\), such that

\[
\Delta_{i'_{ik}} = \begin{cases} 1 & i' = i'_0 \\ 0 & \text{otherwise} \end{cases}, \quad k = 1, 2.
\]

Then \(m_{i_k} = \sum_{i'_{ik}} \Delta_{i'_{ik}} m_{i'} = m_{i'_0}^\prime, k = 1, 2, \) hence \(m_{i_1} = m_{i_2}\). Thus, we can define \(L^\circ\) to be the element of \(K(P)\) obtained by merging the \(i_1\)th and the \(i_2\)th member of \(L\). Moreover, we define \(\Delta^\circ\) to be the \(((r - 1) \times r)\)-matrix

\[
\Delta^\circ = \begin{pmatrix}
1_{i_1-1} & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0 \\
\end{pmatrix},
\]

and \(\Delta'^\circ\) to be the \((r' \times (r - 1))\)-matrix

\[
\Delta'^\circ = \begin{pmatrix}
\Delta_{i_1} & \ldots & \Delta_{i_1 i_1-1} & 0 & \Delta_{i_1 i_1+1} & \ldots & \Delta_{i_1 i_2-1} & \Delta_{i_1 i_2+1} & \ldots & \Delta_{i_1 r} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Delta_{i'_0} & \ldots & \Delta_{i'_0 i_1-1} & 1 & \Delta_{i'_0 i_1+1} & \ldots & \Delta_{i'_0 i_2-1} & \Delta_{i'_0 i_2+1} & \ldots & \Delta_{i'_0 r} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Delta_{r' i_1} & \ldots & \Delta_{r' i_1 i_1-1} & 0 & \Delta_{r' i_1+1} & \ldots & \Delta_{r' i_2-1} & \Delta_{r' i_2+1} & \ldots & \Delta_{r' r} \\
\end{pmatrix}.
\]

\(\Delta^\circ\) now having Bratteli diagram \((79)\), \(\Delta^\circ \in N(L, L^\circ)\) by Lemma 5.4. Analogously to the first case one can check that \(\ell(\Delta) = 1, \Delta'^\circ \Delta^\circ = \Delta, \) and \(\Delta'^\circ \in N(L^\circ, L')\). This proves the lemma.

\section{5.5 Characterization of Direct Successors}

\textbf{Theorem 5.6} Let \(\kappa, \kappa' \in \hat{K}(P)\). The following assertions are equivalent.

(a) \(\kappa'\) is a direct successor of \(\kappa\).

(b) There exist representatives \(L\) and \(L'\) of \(\kappa\) and \(\kappa'\), respectively, such that \(N(L, L')\) contains an element of level 1.

(c) For any representatives \(L, L'\) of \(\kappa, \kappa'\), respectively, \(N(L, L')\) is nonempty and its elements have level 1.
Proof:
(a) ⇒ (c): Let \( L \) and \( L' \) be representatives of \( \kappa \) and \( \kappa' \), respectively. Since \( \kappa \leq \kappa' \), due to Corollary 5.3, there exists \( \Delta \in N(L, L') \). Since \( \kappa \neq \kappa' \), due to Lemma 5.2, \( \ell(\Delta) \neq 0 \). Then Lemma 5.3 implies that there exist \( L^0 \in \hat{K}(P) \) and \( \Delta^0 \in N(L, L^0) \), \( \Delta^{o'} \in N(L^0, L') \) such that \( \Delta = \Delta^{o'} \Delta^0 \) and \( \ell(\Delta^0) = 1 \). Let \( \kappa^o \) denote the equivalence class of \( L^0 \). We have \( \kappa \leq \kappa^o \leq \kappa' \). According to Lemma 5.3, \( \kappa \neq \kappa^o \). It follows that \( \kappa^o = \kappa' \). Hence, again due to Lemma 5.3, \( \ell(\Delta^{o'}) = 0 \). Then the sharpened version of (83) implies \( \ell(\Delta) = \ell(\Delta^0) = 1 \).

(c) ⇒ (b): Obvious.

(b) ⇒ (a): Let \( L, L' \in K(P) \) be given as assumed. Let \( \kappa^o \in \hat{K}(P) \) such that \( \kappa \leq \kappa^o \leq \kappa' \). For any representative \( L^0 \) of \( \kappa^o \), there exist \( \Delta^o \in N(L, L^0) \) and \( \Delta^{o'} \in N(L^0, L') \). Due to Lemma 5.1, \( \Delta^{o'} \Delta^o \in N(L, L') \). Since, by assumption, this set contains an element of level 1, Lemma 5.2 yields \( \ell(\Delta^{o'} \Delta^o) = 1 \). Then (83) requires that either \( \ell(\Delta^o) = 0 \) or \( \ell(\Delta^{o'}) = 0 \). According to Lemma 5.3, in the first case, \( \kappa = \kappa^o \), whereas in the second case, \( \kappa^o = \kappa' \). This shows that \( \kappa' \) is a direct successor of \( \kappa \).

The Bratteli diagram of an inclusion matrix of level 1: Let \( L, L' \in K(P) \) and let \( \Delta \in N(L, L') \). Assume that \( \ell(\Delta) = 1 \). Then either there exists \( i_0 \) such that \( \ell^+(\Delta) = 1 \) and \( \ell^i(\Delta) = 0 \) for all \( i \neq i_0 \) and \( \ell^i(\Delta) = 0 \) for all \( i' \), or there exists \( i'_0 \) such that \( \ell^i(\Delta) = 1 \) and \( \ell^i(\Delta) = 0 \) for all \( i \neq i'_0 \) and \( \ell^i(\Delta) = 0 \) for all \( i \). Accordingly, the Bratteli diagram of \( \Delta \) is given by

![Bratteli diagram](image)

for some \( 1 \leq i_1 < i_2 \leq r + 1 \), or by

![Bratteli diagram](image)

for some \( 1 \leq i_1 < i_2 \leq r \), respectively. In particular, in the first case, \( r' = r + 1 \) and in the second case, \( r' = r - 1 \).

5.6 Generation of Direct Successors
Theorem 5.7 Let $\kappa \in \hat{K}(P)$ and let $L$ be a representative of $\kappa$. Then the direct successors of $\kappa$ are obtained by applying all possible splittings and mergings to $L$ and passing to equivalence classes.

Proof: As an immediate consequence of Lemma 5.4 and Theorem 5.6, any element of $\hat{K}(P)$ generated in the way proposed is a direct successor of $\kappa$. Conversely, let $\kappa'$ be a direct successor of $\kappa$. Choose a representative $L'$ of $\kappa'$. Due to Theorem 5.6, $N(L, L')$ contains an element of level 1. As noted above, the Bratteli diagram of such an element is of the form (84) or (85). By a permutation of the lower vertices we can turn this diagram into (78) or (79), respectively. This corresponds to the passage from $L'$ to another representative $L''$ of $\kappa'$. It is immediately seen that the matrix given by the diagram with permuted lower vertices belongs to $N(L, L'')$. Then Lemma 5.4 implies that $L''$ can be obtained from $L$ by a splitting or a merging, respectively. This proves the theorem. 

5.7 Example

Let $P$ be a principal SU4-bundle. Let $L \in K(P)$, $L = (J; \alpha, \xi)$, where $J = (k, m) = ((1, 1), (2, 2))$. Then $\alpha$ has components $\alpha_i = 1 + \alpha_i^{(2)}$, $i = 1, 2$. (One may wish to recall from the example of Section 3 that SU4 has isomorphism type $Z_2 \times U1$.) We are going to determine the direct successors of the equivalence class of $L$.

Let us begin with splitting operations. For $i_0 = 1$, the only possible splitting is given by the decomposition $m_1 = 2 = 1 + 1$. It yields $L_a^o = (J_a^o; \alpha_a^o, \xi_a^o)$, where $J_a^o = ((1, 1), (1, 1, 2))$, $\alpha_a^o = (\alpha_1, \alpha_1, \alpha_2)$, and $\xi_a^o = 0$. The passage from $L$ to $L_a^o$ can be represented conveniently in a Bratteli diagram whose vertices are labelled by the respective quantities $k_i, m_i$ and $\alpha_i$ (rather than by the mere number $i$):

```
L

L_a^o

\alpha_1 \, \alpha_2 \, \xi
1,2  
1,1

\alpha_1 \, \alpha_1 \, \alpha_2
1,1  
1,2

\alpha_1 \, \alpha_2
\xi_a^o = 0
```

For $i_0 = 2$, a similar splitting operation creates $L_b^o$, given by the labelled Bratteli diagram.
As for merging operations, the only choice for $i_1, i_2$ is $i_1 = 1, i_2 = 2$. This yields $L_c^0$.

Next, we have to pass to equivalence classes. Generically, $L_a^0$, $L_b^0$, and $L_c^0$ generate their own classes. However, while $L_c^0$ can never be equivalent to $L_a^0$ or $L_b^0$, the latter are equivalent iff $\alpha_1 = \alpha_2$. In order to see for which bundles $P$ this can happen, consider Eqs. (10) and (11). The first one requires $\alpha_1^{(2)} = \alpha_2^{(2)}$ to be a torsion element. Then, due to $\alpha_1^{(4)} = \alpha_2^{(4)} = 0$, the second one implies $c_2(P) = 0$. Thus, $L_a^0$ and $L_b^0$ can be (occasionally) equivalent only if $P$ is trivial.

6 Direct Predecessors

In this section, we formulate operations to generate the direct predecessors of any given element of $\tilde{K}(P)$. Direct predecessors are, for our purposes, more interesting than direct successors for at least two reasons. First, they allow one to reconstruct the set $\tilde{K}(P)$ together with its partial ordering from the unique maximal element (which, in terms of Howe subbundles, is given by $P$ itself). Second, on the level of the stratification of the gauge orbit space, predecessors correspond to strata of higher symmetry.

In the preceding section we have been able to create all direct successors of a given element of $\tilde{K}(P)$ from one and the same representative. This was achieved by using the freedom in the choice of the representatives of the direct successors. Since here we wish to proceed likewise, we have to carry this freedom from the level of successors to that of predecessors. For this reason, the inverted operations are not just splitting and merging.
read backwards. They rather take the following form. Let $L \in K(P)$, $L = (J; \alpha, \xi)$.

**Inverse Splitting:** Choose $1 \leq i_1 < i_2 \leq r$ such that $k_{i_1} = k_{i_2}$ and $\alpha_{i_1} = \alpha_{i_2}$. Define sequences of length $(r - 1)$

$$k^o = \left(k_1, \ldots, k_{i_1-1}, k_{i_1}, k_{i_1+1}, \ldots, \tilde{k}_{i_2}, \ldots, k_r\right),$$

$$m^o = \left(m_1, \ldots, m_{i_1-1}, m_{i_1} + m_{i_2}, m_{i_1+1}, \ldots, \tilde{m}_{i_2}, \ldots, m_r\right),$$

$$\alpha^o = \left(\alpha_1, \ldots, \alpha_{i_1-1}, \alpha_{i_1}, \alpha_{i_1+1}, \ldots, \tilde{\alpha}_{i_2}, \ldots, \alpha_r\right).$$

We note that $g$ divides the greatest common divisor $g^o$ of $m^o$, so that $g^o g$ is well-defined. Choose $\xi^o \in H^1(M, \mathbb{Z}_{g^o})$ such that $\xi = g^o g(\xi^o)$ and

$$\beta_{g^o} (\xi^o) = E^{(2)}_{m^o} (\alpha^o).$$

(86)

Denote $J^o = (k^o, m^o)$ and $L^o = (J^o; \alpha^o, \xi^o)$. We check that $L^o \in K(P)$: By construction, $m^o \cdot k^o = n$ and $\alpha^o \in H^{(J^o)}(M, \mathbb{Z})$. Due to (80), $\alpha^o$ and $\xi^o$ obey Eq. (11). Using $\alpha_{i_1} = \alpha_{i_2}$ one can check that $E_{m^o}(\alpha^o) = E_m(\alpha)$. Hence, since $\alpha$ obeys Eqs. (11), so does $\alpha^o$. This proves $L^o \in K(P)$.

We say that $L^o$ arises from $L$ by an inverse splitting of the $i_1$th and the $i_2$th member.

**Inverse Merging** Choose $1 \leq i_0 \leq r$ such that $k_{i_0} \neq 1$. Choose a decomposition $k_{i_0} = k_{i_0,1} + k_{i_0,2}$ with strictly positive integers $k_{i_0,1}, k_{i_0,2}$. Choose cohomology elements $\alpha_{i_0,1}, \alpha_{i_0,2} \in H_0^{\text{even}}(M, \mathbb{Z})$ such that $\alpha_{i_0,l}^{(2j)} = 0$ for $j > k_{i_0,l}$, $l = 1, 2$, and

$$\alpha_{i_0,1} \sim \alpha_{i_0,2} = \alpha_{i_0}. \tag{87}$$

Define sequences of length $(r + 1)$

$$k^o = \left(k_1, \ldots, k_{i_0-1}, k_{i_0,1}, k_{i_0,2}, k_{i_0+1}, \ldots, k_r\right),$$

$$m^o = \left(m_1, \ldots, m_{i_0-1}, m_{i_0}, m_{i_0,1}, m_{i_0,2}, m_{i_0+1}, \ldots, m_r\right),$$

$$\alpha^o = \left(\alpha_1, \ldots, \alpha_{i_0-1}, \alpha_{i_0,1}, \alpha_{i_0,2}, \alpha_{i_0+1}, \ldots, \alpha_r\right),$$

as well as

$$\xi^o = \xi.$$

Denote $J^o = (k^o, m^o)$ and $L^o = (J^o; \alpha^o, \xi^o)$. To see that $L^o \in K(P)$, we check $m^o \cdot k^o = n$ and $\alpha^o \in H^{(J^o)}(M, \mathbb{Z})$. Using (87) one can verify that $E_{m^o}(\alpha^o) = E_m(\alpha)$. Consequently, $\alpha^o$ obeys Eq. (11). A similar calculation, using, in addition, $g^o = g$, shows that $E_{m^o}(\alpha^o) = E_m(\alpha)$. Since also $\beta_{g^o} (\xi^o) = \xi$, $\alpha^o$ and $\xi^o$ obey Eq. (11).

We say that $L^o$ arises from $L$ by an inverse merging of the $i_0$th member.

**Remark:** Like for the operations of splitting and merging, for some of the elements of $K(P)$, inverse splitting or inverse merging may not be applicable. In particular, it may happen that there does not exist a solution $\xi^o$ of Eq. (86).
**Lemma 6.1** Let $L, L^\circ \in K(P)$. $L^\circ$ arises from $L$ by an inverse splitting of the $i_1$th and the $i_2$th member if and only if $N(L, L^\circ)$ contains an element with Bratteli diagram

$$
\begin{array}{cccccccccccccccc}
1 & \cdots & i_1 - 1 & i_1 & i_1 + 1 & \cdots & i_2 - 1 & i_2 & \cdots & r \\
1 & \cdots & i_1 - 1 & i_1 & i_1 + 1 & \cdots & i_2 - 1 & i_2 & \cdots & r + 1
\end{array}
$$

(88)

$L^\circ$ arises from $L$ by an inverse merging of the $i_0$th member if and only if $N(L, L^\circ)$ contains an element with Bratteli diagram

$$
\begin{array}{cccccccccccccccc}
1 & \cdots & i_0 - 1 & i_0 & i_0 + 1 & \cdots & i_0 + 2 & r \\
1 & \cdots & i_0 - 1 & i_0 & i_0 + 1 & \cdots & r - 1 & r
\end{array}
$$

(89)

Proof: The proof is completely analogous to that of Lemma 5.4 and shall be omitted.

**Theorem 6.2** Let $\kappa \in \hat{K}(P)$ and let $L$ be a representative of $\kappa$. Then the direct predecessors of $\kappa$ are obtained by applying all possible inverse splittings and inverse mergings to $L$ and passing to equivalence classes.

Proof: The proof is completely analogous to that of Theorem 5.7. The only difference is that here we are allowed to pass to another representative of the predecessor, i.e., to permute the upper vertices in the diagrams (84) and (85), thus arriving at (88) and (89).

**Example**

As in Subsection 5.7, let $P$ be a principal SU4-bundle and let $L \in K(P)$, $L = (J; \alpha, \xi)$, where $J = ((1,1), (2,2))$. We are going to determine the direct predecessors of the equivalence class of $L$. Inverse splittings can be applied only if $\alpha_1 = \alpha_2$. In this case, for any solution $\xi^\circ \in H^1(M, \mathbb{Z}_4)$ of the system of equations

$$\begin{align*}
\xi^\circ \mod 2 &= \xi, \\
\beta_4(\xi^\circ) &= \alpha_1^{(2)},
\end{align*}$$

(90) (91)

we obtain an element $L^\circ = (J^\circ; \alpha^\circ, \xi^\circ)$, where $J^\circ = ((1),(4))$ and $\alpha^\circ = \alpha_1 = \alpha_2$. The passage from $L$ to $L^\circ$ can be summarized in the labelled Bratteli diagram
that has to be read upwards. Each \( L^\circ \) generates its own equivalence class. Due to \( k_1 = k_2 = 1 \), inverse mergings cannot be applied to \( L \). Thus, in the case \( \alpha_1 = \alpha_2 \) the direct predecessors of the equivalence class of \( L \) are labelled by the solutions of Eqs. (90) and (91), whereas in the case \( \alpha_1 \neq \alpha_2 \) direct predecessors do not exist. Recall from Subsection 5.7 that the first case can only occur if \( P \) is trivial.

As another example, consider an element \( L' \) of \( K(P) \), \( L' = (J'; \alpha', \xi') \), where \( J' = ((2), (2)) \). Inverse mergings can be applied and yield elements \( L'^\circ \) as follows:

Here \( \alpha_i' = 1 + (\alpha_i')^{(2)} \), \( i = 1, 2 \), such that \( \alpha_1'^\circ \bowtie \alpha_2'^\circ = \alpha' \). When passing to equivalence classes, elements \( L'^\circ \) with \( \alpha'^\circ = (\alpha_1'^\circ, \alpha_2'^\circ) \) and \( \alpha'^\circ = (\alpha_2'^\circ, \alpha_1'^\circ) \) have to be identified. Since \( L' \) does not allow inverse splittings, there are no more direct predecessors.

### 7 Example: Gauge Orbit Types for SU2

The gauge orbit types for SU2, i.e., the set \( \hat{K}(P) \) for a principal SU2-bundle \( P \) over \( M \), was calculated in [14] by solving Eqs. (10) and (11) for all \( J \). Here we are going to recover this result using a different technique that will also yield the partial ordering of orbit types.

A partially ordered set can be reconstructed either (a) from its minimal elements by successively determining direct successors, or (b) from its maximal elements by successively determining direct predecessors. In the case of \( \hat{K}(P) \), there exists a unique maximal element, namely the class corresponding to the bundle \( P \) itself. Minimal elements are, in
general, not unique. In fact, their number can be infinite. Thus, the preferred algorithm is (b).
The unique representative of the maximal element of $K(P)$ is $L_{\text{max}} = (J_{\text{max}}; \alpha_{\text{max}}, \xi_{\text{max}})$, where $J_{\text{max}} = ((2), (1))$, $\alpha_{\text{max}} = c(P)$, and $\xi_{\text{max}} = 0$. Inverse mergings yield elements $L^o$:

\[
\begin{array}{c}
L^o \quad \alpha_1^o \quad \alpha_2^o \quad \xi^o = 0 \\
1, 1 \quad 1, 1 \\
\end{array}
\]

where $\alpha_i^o = 1 + (\alpha_i^o)^{(2)}$ such that $\alpha_1^o \sim \alpha_2^o = c(P)$. Sorting by degree yields the equations $(\alpha_1^o)^{(2)} + (\alpha_2^o)^{(2)} = 0$ and $(\alpha_1^o)^{(2)} \sim (\alpha_2^o)^{(2)} = c_2(P)$. We parametrize

\[
(\alpha_1^o)^{(2)} = \alpha^{(2)}_2, \quad (\alpha_2^o)^{(2)} = -\alpha^{(2)},
\]

where $\alpha^{(2)} \in H^2(M, \mathbb{Z})$ has to obey

\[
-\alpha^{(2)} \sim \alpha^{(2)} = c_2(P).
\]

The passage to equivalence classes leads to an identification of solutions $\alpha^{(2)}$ and $-\alpha^{(2)}$. We note that the Howe subgroup labelled by $J = ((1, 1), (1, 1))$ is the toral subgroup $U_1$ of SU2 and that the parameter $\alpha^{(2)}$ is just the first Chern class of the corresponding reduction of $P$. By virtue of this transliteration, Eq. (92) coincides with the result given in [9].

Next, consider the direct predecessors of the classes generated by $L^o$. Inverse mergings can not be applied. Inverse splittings can be applied provided $\alpha_1^o = \alpha_2^o$, i.e., $\alpha^{(2)} = -\alpha^{(2)}$. Then for any solution $\xi^{\infty} \in H^1(M, \mathbb{Z}_2)$ of the equation

\[
\beta_2 (\xi^{\infty}) = \alpha^{(2)},
\]

we obtain an element $L^{\infty}$:

\[
\begin{array}{c}
L^{\infty} \quad \alpha_1^\infty \quad \xi^{\infty} \\
1, 2 \\
\end{array}
\]

\[
\begin{array}{c}
L^o \quad \alpha_1^o \\
1, 1 \\
\end{array}
\]

\[
\begin{array}{c}
L^{\infty} \quad \alpha_1^\infty \\
1, 1 \\
\end{array}
\]

\[
\begin{array}{c}
\xi^o = 0 \\
\end{array}
\]
Each of these elements generates its own equivalence class. Note that the Howe subgroup labelled by $J = ((1), (2))$ is the center $\mathbb{Z}_2$ of SU2 and that $\xi^\infty$ is the natural characteristic class for principal $\mathbb{Z}_2$-bundles over $M$, see [10, §13].

Now let us draw Hasse diagrams of $\hat{K}(P)$ for specific space-time manifolds $M$. In the following, vertices stand for the elements of $\hat{K}(P)$ and edges indicate the relation 'left vertex $\leq$ right vertex'. When viewing the elements of $\hat{K}(P)$ as Howe subbundles, the vertex on the rhs. represents the class corresponding to $P$ itself, the vertices in the middle and on the lhs. represent reductions of $P$ to the Howe subgroups U1 and $\mathbb{Z}_2$, respectively. When viewing the elements of $\hat{K}(P)$ as orbit types, or strata of the gauge orbit space, the vertex on the rhs. represents the generic stratum, whereas the vertices in the middle and on the lhs. represent U1-strata and SU2-strata. Here the names U1-stratum and SU2-stratum mean that the stratum consists of (orbits of) connections whose stabilizers are isomorphic to U1 or SU2, respectively.

$M = S^4$: Since $H^2(M, \mathbb{Z}) = 0$, Eq. (92) can be solved iff $c^2(P) = 0$, i.e., iff $P$ is trivial. The solution is $\alpha^{(2)} = 0$. Then Eq. (93) is trivially satisfied by $\xi^\infty = 0$. Due to $H^1(M, \mathbb{Z}_2) = 0$, there are no more solutions. Thus, in the case $c^2(P) = 0$, the Hasse diagram of $\hat{K}(P)$ is

$\bullet \quad \bullet \quad \bullet$

If $c^2(P) \neq 0$, on the other hand, $\hat{K}(P)$ is trivial, meaning that it consists only of the class corresponding to $P$ itself.

On the level of gauge orbit types, the result means that in the sector of vanishing topological charge the gauge orbit space decomposes into the generic stratum, a U1-stratum, and a SU2-stratum. If, on the other hand, a topological charge is present, only the generic stratum survives.

$M = S^2 \times S^2$: To perform the first step in the reconstruction procedure, let $1_{S^2}$ and $\gamma^{(2)}_{S^2}$ be generators of $H^0(S^2, \mathbb{Z})$ and $H^2(S^2, \mathbb{Z})$, respectively. Due to the Künneth Theorem, $H^2(M, \mathbb{Z})$ is generated by $\gamma^{(2)}_{S^2} \times 1_{S^2}$ and $1_{S^2} \times \gamma^{(2)}_{S^2}$, whereas $H^4(M, \mathbb{Z})$ is generated by $\gamma^{(2)}_{S^2} \times \gamma^{(2)}_{S^2}$. Here $\times$ denotes the cohomology cross product. Writing

$$\alpha^{(2)} = a \; \gamma^{(2)}_{S^2} \times 1_{S^2} + b \; 1_{S^2} \times \gamma^{(2)}_{S^2}$$

with $a, b \in \mathbb{Z}$, Eq. (92) becomes

$$-2ab \; \gamma^{(2)}_{S^2} \times \gamma^{(2)}_{S^2} = c^2(P).$$

If $c^2(P) = 0$, there are two series of solutions: $a = 0$ and $b \in \mathbb{Z}$ as well as $a \in \mathbb{Z}$ and $b = 0$. Due to $H^1(M, \mathbb{Z}_2) = 0$, Eq. (93) tells us that out of the elements just obtained only that labelled by $a = b = 0$ has a direct predecessor. Thus, in the case $c^2(P) = 0$ the Hasse diagram of $\hat{K}(P)$ is

$\bullet \quad \bullet \quad \bullet$
The vertices in the middle are labelled by the corresponding values of \((a,b)\). Note that passage to equivalence classes requires identification of solutions \((a,b)\) and \((-a,-b)\).

If \(c_2(P) = 2l \gamma_{S^2}^{(2)} \times \gamma_{S^2}^{(2)}, \ l \neq 0\), then the solutions of (95) are \(a = q\) and \(b = -l/q\), where \(q\) runs through the (positive and negative) divisors of \(l\). For none of these solutions, (93) is solvable. Hence, here the Hasse diagram is

where, due to the identification \((a,b) \sim (-a,-b)\), \(q\) runs through the positive divisors of \(l\) only. If \(c_2(P) = (2l + 1) \gamma_{S^2}^{(2)} \times \gamma_{S^2}^{(2)}, \) (93) has no solutions, so that \(\hat{K}(P)\) is trivial.

Finally, the interpretation of the result in terms of strata of the gauge orbit space is similar to that for space-time manifold \(M = S^4\) above.

\[ M = L^3_{2p} \times S^1: \] Recall that \(H^1(L^3_{2p}, \mathbb{Z}) = 0\) and \(H^2(L^3_{2p}, \mathbb{Z}) \cong \mathbb{Z}_{2p}\). Let \(\gamma_{L,Z}^{(2)}\) be a generator of \(H^2(L^3_{2p}, \mathbb{Z})\) and let \(1_{S^1,\mathbb{Z}}\) be a generator of \(H^0(S^1, \mathbb{Z})\). Due to the Künneth Theorem, \(H^2(M, \mathbb{Z})\) is generated by \(\gamma_{L,Z}^{(2)} \times 1_{S^1,\mathbb{Z}}\). We write

\[ \alpha^{(2)} = a \gamma_{L,Z}^{(2)} \times 1_{S^1,\mathbb{Z}}. \]  

(96)

Due to \(2p \gamma_{L,Z}^{(2)} = 0\), \(\alpha^{(2)} \sim \alpha^{(2)} = 0\). Hence, Eq. (92) is solvable iff \(c_2(P) = 0\), in which case the solutions are given by \(a \in \mathbb{Z}_{2p}\). Since when passing to equivalence classes we have to identify \(\alpha^{(2)}\) and \(-\alpha^{(2)}\), i.e., \(a\) and \(-a\), the direct predecessors are labelled by elements of \(\mathbb{Z}_p\).

Next, consider the second step of the reconstruction procedure. Let \(1_{L,\mathbb{Z}_2}\), \(\gamma_{L,\mathbb{Z}_2}^{(1)}\), and \(\gamma_{S^1,\mathbb{Z}}^{(1)}\) be generators of \(H^0(L^3_{2p}, \mathbb{Z}_2), H^1(L^3_{2p}, \mathbb{Z}_2),\) and \(H^1(S^1, \mathbb{Z})\), respectively. Then, again due to
the K"unneth Theorem, $H^1(M,\mathbb{Z}_2)$ is generated by $\gamma^{(1)}_{L,\mathbb{Z}_2} \times 1_{S^1,\mathbb{Z}}$ and $1_{L,\mathbb{Z}_2} \times \gamma^{(1)}_{S^1,\mathbb{Z}}$. Moreover, one can check that
\[
\beta_2\left(\gamma^{(1)}_{L,\mathbb{Z}_2} \times 1_{S^1,\mathbb{Z}}\right) = p \gamma^{(2)}_{L,\mathbb{Z}_2}, \quad \beta_2\left(1_{L,\mathbb{Z}_2} \times \gamma^{(1)}_{S^1,\mathbb{Z}}\right) = 0.
\] (97)

Decomposing $\xi^{\circ\circ} = a_L \gamma^{(1)}_{L,\mathbb{Z}_2} \times 1_{S^1,\mathbb{Z}} + a_S 1_{L,\mathbb{Z}_2} \times \gamma^{(1)}_{S^1,\mathbb{Z}}$ and using (96) and (97), (93) becomes
\[
p a_L = a.
\]

Thus, only the elements labelled by $a = 0$ and $a = p$ have direct predecessors. These are given by the values $a_L = 0$, $a_S = 0, 1$ and $a_L = 1$, $a_S = 0, 1$, respectively.

As a result, in the case $c_2(P) = 0$, the Hasse diagram of $\hat{K}(P)$ is

Here the vertices on the lhs. are labelled by $(a_L, a_S)$, whereas those in the middle are labelled by $a$. In the case $c_2(P) \neq 0$, $\hat{K}(P)$ is trivial. Again, the interpretation in terms of strata of the gauge orbit space goes along the lines of the case $M = S^4$ above.

To conclude, let us remark that, while for SU2 the picture is relatively simple, already for SU3 the partial ordering becomes rather involved, and the Hasse diagrams representing it are very complex.

References

[1] Asorey, M., Falceto, F., Lópex, J.L., Luzón, G.: Nodes, Monopoles, and Confinement in 2 + 1-Dimensional Gauge Theories. Phys. Lett. B 345, 125–130 (1995)

[2] Atiyah, M.F.; Singer, I.M.: Dirac Operators Coupled to Vector Fields. Proc. Nat. Acad. Sci. USA 81, 2597–2600 (1984)

[3] Bredon, G.E.: Introduction to Compact Transformation Groups. New York: Academic Press, 1972

[4] Bratteli, O.: Inductive Limits of Finite Dimensional $C^*$-Algebras. Trans. Amer. Math. Soc. 171, 195–234 (1972)

[5] Bredon, G.E.: Topology and Geometry. Berlin, Heidelberg, New York: Springer, 1993
[6] Emmrich, C., Römer, H.: Orbifolds as Configuration Spaces of Systems with Gauge Symmetries. Commun. Math. Phys. 129, 69–94 (1990)

[7] Gelbrich, G.: On Pointwise Conjugate Homomorphisms of Compact Lie Groups. Seminar Sophus Lie 1 217–223, (1991), Heldermann, 1991

[8] Heil, A., Kersch, A., Papadopoulos, N.A., Reifenhäuser, B., Scheck, F.: Anomalies from Nonfree Action of the Gauge Group. Ann. Phys. 200, 206–215 (1990)

[9] Isham, C.J.: Space-Time Topology and Spontaneous Symmetry Breaking. J. Phys. A 14, 2943–2956 (1981)

[10] Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry. Vol. I. New York: Wiley Interscience, 1963

[11] Kondracki, W., Rogulski, J.: On the Notion of Stratification. Institute of Mathematics, Polish Academy of Sciences, Preprint 281, Warszawa, 1983

[12] Kondracki, W., Rogulski, J.: On the Stratification of the Orbit Space for the Action of Automorphisms on Connections. Dissertationes Mathematicae 250. Warszawa: Panstwowe Wydawnictwo Naukowe, 1986

[13] Mitter, P.K., Viallet, C.-M.: On the Bundle of Connections and the Gauge Orbit Manifold in Yang-Mills Theory. Commun. Math. Phys. 79, 457–472 (1981)

[14] Rudolph, G., Schmidt, M., Volobuev, I.P.: Classification of Gauge Orbit Types for SU\(n\)-Gauge Theories. Preprint math-ph/0003044, submitted to Commun. Math. Phys.

[15] Singer, I.M.: Some Remarks on the Gribov Ambiguity. Commun. Math. Phys. 60, 7–12 (1978)

[16] Steenrod, N.: The Topology of Fibre Bundles. Princeton, NJ: Princeton University Press, 1951

[17] Witten, E.: An SU(2)-Anomaly. Phys. Lett. 117 B, 324–328 (1982)