Modular Localization of Massive Particles with “Any” Spin in d=2+1

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Dedicated to Rudolf Haag on the occasion of his 80th birthday.

Abstract

We discuss a concept of particle localization which is motivated from quantum field theory, and has been proposed by Brunetti, Guido and Longo and by Schroer. It endows the single particle Hilbert space with a family of real subspaces indexed by the space-time regions, with certain specific properties reflecting the principles of locality and covariance. We show by construction that such a localization structure exists also in the case of massive anyons in $d = 2 + 1$, i.e. for particles with positive mass and with arbitrary spin $s \in \mathbb{R}$. The construction is completely intrinsic to the corresponding ray representation of the (proper orthochronous) Poincaré group. Our result is of particular interest since there are no free fields for anyons, which would fix a localization structure in a straightforward way. We present explicit formulas for the real subspaces, expected to turn out useful for the construction of a quantum field theory for anyons. In accord with well-known results, only localization in string-like, instead of pointlike or bounded, regions is achieved. We also prove a single-particle PCT theorem, exhibiting a PCT operator which acts geometrically correctly on the family of real subspaces.

1 Introduction

Following E. Wigner [27], the state space of an elementary relativistic particle corresponds to an irreducible ray representation of the Poincaré group$^1$. 

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$^1$We shall not be concerned with the concept of infra-particles [5, 23].
In three as well as in four dimensional space-time, the physically relevant representations – and hence the conceivable particle types – are classified by the mass \( m \) and the spin \( s \) which labels a representation of the covering of the rotation subgroup (if \( m > 0 \)). In three dimensional space-time the latter is isomorphic to the group of reals, hence the spin may take any real value — in contrast to the four-dimensional situation where it is quantized, \( s \in \frac{1}{2}\mathbb{N}_0 \). Thus, in three dimensional space-time there are more particle types; the exotic ones with non-half-integer spin are called *anyons*.

By modular localization of particles we mean a concept which has been advocated in recent years by Brunetti, Guido and Longo [4] and by B. Schroer [24, 9]: Suppose there is a quantum field for the particle type at hand, and consider the single particle states which are, together with a polarization cloud, created from the vacuum in a given space-time region. Thus the single particle space gets equipped with a family of subspaces indexed by the space-time regions, with certain specific properties reflecting the localization properties of the underlying quantum field, cf. Definition 2.1 below. This will be a sufficient motivation for us to call a family of subspaces of the single particle space with such properties a *localization structure* for the particle type at hand.

The question arises whether such a structure can be constructed for any given particle type \((m, s)\) intrinsically within the single particle theory – that is to say, without referring to a quantum field, but using as input only the corresponding ray representation of the Poincaré group. This has been achieved for spin zero and positive mass by P. Ramacher [20], and for all positive energy representations of the Poincaré group by Brunetti, Guido and Longo [4]. The latter analysis includes reducible representations, but restricts to proper (not ray) representations, i.e. only the case of bosons and not the case of fermions or anyons is covered.

In the present article, this construction is performed for the case of massive anyons in \( d = 3 \). The purpose of this construction is twofold: Firstly, it shows that a localization structure indeed exists for all \( m > 0, s \in \mathbb{R} \). This is of particular interest because there are no free relativistic fields for anyons\(^2\) [15], which would of course allow for a straightforward construction of the localization structure. Even worse, none of the hitherto proposed models of relativistic quantum fields for anyons in (continuous) three dimensions creating finitely many copies of the irreducible representation from the vacuum. D.R. Grigore has constructed free fields in \( d = 2 + 1 \) for any spin [12], but in contradiction to the generalized spin statistics connection holding in algebraic quantum field theory [10, 11, 6] they have bosonic statistics. Presumably, this is due to the fields having infinitely many components.

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sional space-time \[25,26,15,1,8,14,17\] has been worked out to the extent that the localization structure could be readily constructed from them. Secondly, our analysis is intended to be a step in the construction of a model which resembles as closely as possible a free field for anyons, in the sense of a “second quantization functor” from the single particle theories to field algebras. To this end it is gratifying that we have found explicit formulas for the real subspaces of localized states.

The article is organized as follows. In Section 2, we make precise our definition of a localization structure for anyons, cf. Definition 2.1. In Section 3, we construct a localization structure for any given particle type \(m > 0, s \in \mathbb{R}\), intrinsically within the corresponding Wigner space. The result is summarized in the main Theorem 3.2, which also contains a PCT theorem. All relevant properties can be shown, via modular theory along the lines of \[1\], without reference to the specific irreducible representation \((m, s)\) — except for the the so-called standard property, which guarantees that the constructed structure is non-trivial. This is the content of the last Section 4 where we explicitly exhibit sufficiently many “localized states” (Proposition 4.2). These are represented as families of functions which transform covariantly under the Poincaré group (Corollary 4.3). In Section 5, we finally prove that the Bisognano-Wichmann property essentially fixes the localization structure and also implies a single-particle version of the spin-statistics connection.

2  Definition of a Localization Structure for Anyons

Let \(\mathcal{H}\) be a Hilbert space describing anyons of the type \((m, s)\). We define a localization structure as a family of subspaces of \(\mathcal{H}\) with certain specific properties reflecting the localization properties of a hypothetical underlying quantum field.

Let us first describe the index set for this family. Each subspace is labelled by a space-time region belonging to a specific class \(\mathcal{C}\), together with some additional information, which is needed to endow the index set with a partial order relation and with a non-trivial action of the \(2\pi\)-rotation. In accord with the well-known result \[11,10\] that anyons cannot be localized in point-like, but only in string-like regions, each localization region \(C \in \mathcal{C}\) must extend to infinity in some space-like direction \(e\), \(e^2 = -1\). More specifically, we say that a space-time region \(C\) contains a space-like direction \(e\) if

\[C + e \subset C.\]  \hspace{1cm} (1)
We take $\mathcal{C}$ to be the set of convex, causally complete regions which contain some space-like direction in this sense. (A region $C$ is called causally complete if it contains all points $x$ such that every inextendible causal curve through $x$ passes through $C$.) Typical examples of regions in $\mathcal{C}$ are space-like cones and wedge regions, i.e. Poincaré transforms of the standard wedge

$$W_1 \doteq \{ x \in \mathbb{R}^3 : |x^0| < x^1 \}. \quad (2)$$

Wedges are the largest regions in the class $\mathcal{C}$, in the sense that every $C \in \mathcal{C}$ is contained in some wedge $\mathcal{W}$.

The additional information indicated above, which has to be specified along with each localization region $C \in \mathcal{C}$, is a path in the set of space-like directions. We denote the latter by

$$H \doteq \{ e \in \mathbb{R}^3 : e^2 = -1 \}, \quad (3)$$

and consider paths in $H$ starting at a reference direction $e_0$, which we fix, once and for all, to be

$$e_0 \doteq (0,0,-1). \quad (4)$$

Given a region $C \in \mathcal{C}$, we shall say that a path $\tilde{e}$ ends in $C$ if its endpoint is contained in $C$ in the sense of equation (1). Two paths $\tilde{e}_1$ and $\tilde{e}_2$ starting at $e_0$ and ending in $C$ will be called equivalent w.r.t. $C$ iff the path $\tilde{e}_1^{-1} \ast \tilde{e}_2$ (the inverse of $\tilde{e}_1$ followed by $\tilde{e}_2$) is fixed-endpoint homotopic to a path which is contained in $C$. Now the index set for our localization structure, denoted by $\hat{\mathcal{C}}$, is the set of pairs

$$(C, \tilde{e}), \quad (5)$$

where $C \in \mathcal{C}$ and $\tilde{e}$ is the equivalence class w.r.t. $C$ of a path in $H$ starting at $e_0$ and ending in $C$. For fixed $C \in \mathcal{C}$, we shall use the notation $\hat{C}$ for an element of the form $(C, \tilde{e})$. To see what is involved, suppose $C$ is a space-like cone or a wedge. Then the set of directions contained in $C$ is a connected and simply connected subset of $H$, and different elements $(C, \tilde{e}_1)$ and $(C, \tilde{e}_3)$ differ just by a winding number, cf. Figure 1. Consider now two such pairs

$$\hat{C}_1 \doteq (C_1, \tilde{e}_1) \text{ and } \hat{C}_2 \doteq (C_2, \tilde{e}_2).$$

If $C_1 \subset C_2$ and the corresponding paths $\tilde{e}_1$, $\tilde{e}_2$ are equivalent w.r.t. $C_2$, then we shall write

$$\hat{C}_1 \subset \hat{C}_2. \quad (6)$$

If $C_1$ and $C_2$ are causally separated, then $\hat{C}_1$ and $\hat{C}_2$ determine a relative winding number

$$N(\hat{C}_1, \hat{C}_2) \doteq \text{winding number of } \tilde{e}_2^{-1} \ast \tilde{e}_1 \ast \tilde{e}_{12}, \quad (7)$$
Figure 1: $C_H$ denotes the set of space-like directions contained in $C$. $(C, \tilde{e}_1)$ is equal to $(C, \tilde{e}_2)$, but different from $(C, \tilde{e}_3)$.

where $\tilde{e}_{12}$ is the “direct” path from $e_1$ to $e_2$ in clockwise direction. Finally, we note that the universal covering $\tilde{P}_+^\uparrow$ of the Poincaré group naturally acts on $\tilde{C}$ as explained in the appendix, cf. equation (83), such that a $2\pi$ rotation acts non-trivial — it maps, for example, $(C, \tilde{e}_3)$ in Figure 1 onto $(C, \tilde{e}_1)$.

We now turn to the definition of a localization structure. We admit the case of several particle species of the same type $(m, s)$, for example a particle and its anti-particle.

**Definition 2.1** Let $U$ be a finite direct sum of copies of the irreducible representation of $\tilde{P}_+^\uparrow$ for mass $m > 0$ and spin $s \in \mathbb{R}$, acting in a Hilbert space $H$. A family of closed real subspaces $K(\tilde{C})$, $\tilde{C} \in \tilde{C}$, of $H$ is called a localization structure for $(m, s)$ if it has the following properties:

1. Isotony: Let $\tilde{C}_1 \subset \tilde{C}_2$ in the sense of equation (6). Then
   \[ K(\tilde{C}_1) \subset K(\tilde{C}_2). \]

2. Twisted Locality: There is a complex number $Z$ of modulus one, such that for any pair $\tilde{C}_1, \tilde{C}_2 \in \tilde{C}$ with $C_1$ causally separated from $C_2$
   \[ Z(\tilde{C}_1, \tilde{C}_2) K(\tilde{C}_2) \subset K(\tilde{C}_1)' \]

   Here, $Z(\tilde{C}_1, \tilde{C}_2) \doteq Z^{2N+1}$, with $N = N(\tilde{C}_1, \tilde{C}_2)$, cf. (7), and the prime denotes the symplectic complement\(^3\).

\(^3\)The relevant notions referring to real subspaces of a Hilbert space are recalled in Appendix A.
(3) Poincaré covariance: For all \( \tilde{C} \in \tilde{C} \) and \( \tilde{g} \in \tilde{P}^{\uparrow}_+ \)

\[ U(\tilde{g}) K(\tilde{C}) = K(\tilde{g} \cdot \tilde{C}) . \]

(4) Standardness: \( K(\tilde{C}) \) is standard for all \( \tilde{C} \in \tilde{C} \).

Remark. (i) Covariance implies that \( K(\tilde{r}(2\pi) \cdot \tilde{C}) = e^{2\pi is} K(\tilde{C}) \), where \( \tilde{r}(\cdot) \) denotes rotation. Therefore \( K(\tilde{r}(2\pi) \cdot \tilde{C}) \) coincides with \( K(\tilde{C}) \) if, and only if, \( s \in \frac{1}{2} \mathbb{Z} \). Hence \( K(\tilde{C}) \) is independent of the path \( \tilde{e} \), but only depends on \( C \), iff \( s \in \frac{1}{2} \mathbb{Z} \). Further, it can be shown using the free field formalism (or along the same lines as in the present analysis, cf. the remark after Proposition C.2), that in this case the localization structure can be extended to bounded regions.

(ii) In the framework of algebraic quantum field theory, a field algebra for anyons \( \{ \mathcal{F}(\tilde{C}) \}_{\tilde{C} \in \tilde{C}} \) is a family of operator algebras indexed by the same class \( \tilde{C} \) (except that in general, each \( C \) must contain some space-like cone \( \mathfrak{C} \)). Suppose there are finitely many particle species of the type \( (m, s) \) and that \( \{ m \} \) is isolated from the rest of the spectrum, and denote by \( E^{m,s} \) the projection onto the corresponding single particle space. Then

\[ K(\tilde{C}) = E^{m,s} \mathcal{F}(\tilde{C})^{sa} \Omega^{-} \quad \text{(norm closure)} , \quad (9) \]

\( \tilde{C} \in \tilde{C} \), is a localization structure. This is in fact the motivation for our definition. As an illustration, we show that twisted locality \( \mathfrak{S} \) holds in the case of bosons or fermions. In these cases field operators \( \varphi_1 \) and \( \varphi_2 \) localized in causally separated regions commute or anti-commute, respectively. These relations have been shown in [6, Sect. 2] to survive the projection \( E^{m,s} \) in the sense that

\[ \langle \varphi_1 \Omega, E^{m,s} \varphi_2 \Omega \rangle = \pm \langle \varphi_2^* \Omega, E^{m,s} \varphi_1^* \Omega \rangle , \quad (10) \]

respectively. Hence, putting \( Z = 1 \) for bosons and \( Z = i \) for fermions, the imaginary part of \( (Z \varphi_1 \Omega, E^{m,s} \varphi_2 \Omega) \) is zero if \( \varphi_1 \) and \( \varphi_2 \) are self-adjoint. This is twisted locality. In the general case of anyons, analogous considerations hold, with \( Z \) being defined as a root of the statistics phase.

We finally recall the definition of a certain maximality property called twisted Haag duality. Let \( \tilde{C} = (C, \tilde{e}) \) and \( \tilde{C}' = (C', \tilde{e}') \), where \( C' \) is the causal complement of \( C \) and \( \tilde{e}' \) is the equivalence class of a path ending in \( C' \) in the same sense as in equation \( \mathfrak{H} \). If \( C \) is not a wedge, then the region \( C' \) is not contained in any wedge region. In this case we define a real subspace
corresponding to $\tilde{C}'$ via

$$K(\tilde{C}') \doteq \bigvee_{\tilde{C}_0 \subset \tilde{C}'}, \tilde{C}_0 \in \tilde{C} K(\tilde{C}_0). \quad (11)$$

**Definition 2.2** A localization structure is said to satisfy twisted Haag duality if for every pair $\tilde{C}, \tilde{C}'$ as above the identity

$$Z(\tilde{C}, \tilde{C}') K(\tilde{C}') = K(\tilde{C})' \quad (12)$$

holds.

### 3 Construction of the Localization Structure

Let $U$ be a finite direct sum of copies of the irreducible representation of $\tilde{P}_+^\uparrow$ for mass $m > 0$ and spin $s \in \mathbb{R}$. We now construct a corresponding localization structure along the same lines as in [4].

We start with the definition of the localization space associated with the standard wedge $W_1$, cf. equation (2). Associated with this wedge are the Lorentz boosts $\tilde{\lambda}_1(t)$ leaving $W_1$ invariant and acting on the coordinates $x^0, x^1$ as

$$ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}, \quad (13)$$

and the reflection $j$ about the edge of $W_1$,

$$j : (x^0, x^1, x^2) \mapsto (-x^0, -x^1, x^2). \quad (14)$$

We define $\Delta$ to be the unique positive operator satisfying

$$\Delta^\mu = U(\tilde{\lambda}_1(-2\pi t)), \quad t \in \mathbb{R}, \quad (15)$$

where $\tilde{\lambda}_1(\cdot)$ denotes the lift of $\lambda_1(\cdot)$ to the covering group $\tilde{P}_+^\uparrow$. We further pick an anti-unitary involution $J$ satisfying

$$JU(\tilde{g})J = U(\tilde{j} \tilde{g} \tilde{j}), \quad \tilde{g} \in \tilde{P}_+^\uparrow, \quad (16)$$

where $\tilde{j} \cdot \tilde{j}$ denotes the lift of the adjoint action\(^4\) of $j$ to the covering group $\tilde{P}_+^\uparrow$. Lemma [3] asserts that such an involution exists. We mention as an aside, that the localization structure which we now construct is independent

\[^4\text{See [20].}\]
of the particular choice, cf. Proposition 5.2. We then define a closed operator $S$ by

$$S = J \Delta_{1/2}.$$  

(17)

This operator is densely defined, antilinear and involutive due to the group relation $\tilde{J} \tilde{\lambda}_1(t) \tilde{J} = \tilde{\lambda}_1(t)$, cf. [4]. Hence, the eigenspace of $S$ for the eigenvalue 1 is a standard real subspace, cf. Appendix A. We take this subspace as our localization space for $\tilde{W}_1 = (W_1, \tilde{\epsilon}_{W_1})$,

$$K(\tilde{W}_1) = \{ \phi \in \text{dom} S : S\phi = \phi \}.$$  

(19)

The motivation for this definition will become clear after Definition 5.1. Covariance forces us to define the real subspaces corresponding to arbitrary wedges $\tilde{W} = \tilde{g} \cdot \tilde{W}_1$ by

$$K(\tilde{g} \cdot \tilde{W}_1) = U(\tilde{g}) K(\tilde{W}_1) \quad \text{for } \tilde{g} \in \tilde{P}^1_+.$$  

(20)

The following lemma asserts that this is well-defined.

**Lemma 3.1** Let $\tilde{g} \in \tilde{P}^1_+$ satisfy $\tilde{g} \cdot \tilde{W}_1 = \tilde{W}_1$. Then $U(\tilde{g}) K(\tilde{W}_1) = K(\tilde{W}_1)$.

*Proof.* The set of Poincaré transformations $\tilde{g} \in \tilde{P}^1_+$ which map $\tilde{W}_1$ onto itself is the Abelian group generated by the one-parameter subgroups of the translations along the 2-axes and of the 1-boosts $\tilde{\lambda}_1(t)$. Both of these subgroups commute with $\tilde{J}$ and with the 1-boosts, hence their representers commute with $S$, which implies the claim. \hfill \Box

Next we associate real closed subspaces $K(\tilde{C})$ to arbitrary regions $\tilde{C} \in \tilde{C}$ by intersections:

$$K(\tilde{C}) = \bigcap_{\tilde{W} \supset \tilde{C}} K(\tilde{W}),$$  

(21)

where the intersection goes over all wedge regions which contain $\tilde{C}$ in the sense of (6). If $C$ is a wedge, this is consistent with (20) as a consequence of the positivity of the energy [4]. Note that if $C$ is not a wedge, then (21) is the maximal subspace one can associate with $C$ in view of locality.

We now state our main result.
Theorem 3.2 The family \( \{K(\tilde{C})\}_{\tilde{C} \in \tilde{C}} \) constructed above is a localization structure for \((m, s)\) if \(Z = e^{i\pi s}\). It also satisfies twisted Haag duality, cf. equation (12). Further, the anti-unitary involution \(U(\tilde{j})\) defined by \(U(\tilde{j}) \doteq Z^{-1}J\) is a PCT operator, that is, a representer of \(\tilde{j}\) in sense of equation (16), which acts geometrically correctly on the localization structure:

\[
U(\tilde{j})K(\tilde{C}) = K(\tilde{j} \cdot \tilde{C}), \quad \tilde{C} \in \tilde{C}.
\] (22)

It is noteworthy that the “spin-statistics connection” \(Z^2 = e^{2\pi is}\) necessarily holds as a consequence of the definition (19), as we show in Proposition 5.3 below.

Proof. Isotony and Poincaré covariance, i.e. properties (1) and (3) of Definition 2.1, follow immediately by construction. We next prove equation (22). From the group relations \(\tilde{\lambda}_1(t)\tilde{j} = \tilde{j} \tilde{\lambda}_1(t)\), \(\tilde{\lambda}_1(t)\tilde{r}(\pi) = \tilde{r}(\pi)\tilde{\lambda}_1(-t)\) and \(\tilde{r}(\pi)\tilde{j} = \tilde{j} \tilde{r}(-\pi)\), and the fact that \(Z^2 = e^{2\pi is} = U(\tilde{r}(2\pi)))\), it follows that the operator \(U(\tilde{r}(\pi))U(\tilde{j})\) commutes with \(S\). But this implies that

\[
U(\tilde{j})K(\tilde{W}_1) = U(\tilde{r}(-\pi))K(\tilde{W}_1) = K(\tilde{j} \cdot \tilde{W}_1),
\] (23)

where we have used that \(\tilde{j} \cdot \tilde{W}_1 = \tilde{r}(-\pi) \cdot \tilde{W}_1\). Hence, equation (22) holds for \(\tilde{C} = \tilde{W}_1\). By covariance, it holds for all wedge regions, and by the intersection property (21) it holds for all \(\tilde{C} \in \tilde{C}\).

We next prove twisted Haag duality (12). Equation (23) implies that \(JK(\tilde{W}_1) = ZK(\tilde{j} \cdot \tilde{W}_1)\). Now according to a general result about Tomita operators, see e.g. [22, Prop. 2.3], the anti-unitary part \(J\) in the polar decomposition of \(S\) maps \(K(\tilde{W}_1)\) onto its symplectic complement:

\[
JK(\tilde{W}_1) = K(\tilde{W}_1)',
\] (24)

Further, \(Z = Z(\tilde{W}_1, \tilde{j} \cdot \tilde{W}_1)\) since the relative winding number \(N(\tilde{W}_1, \tilde{j} \cdot \tilde{W}_1)\) is zero. We therefore have

\[
Z(\tilde{W}_1, \tilde{j} \cdot \tilde{W}_1)K(\tilde{j} \cdot \tilde{W}_1) = K(\tilde{W}_1)',
\] (25)

Now any \(\tilde{W}'_1 = (W'_1, \tilde{e})\) differs from \(\tilde{j} \cdot \tilde{W}_1\) by a rotation about a multiple of \(2\pi\). Replacing \(\tilde{j} \cdot \tilde{W}_1\) by such \(\tilde{W}'_1\), the above equation is still valid because \(Z(\tilde{W}_1, \tilde{r}(2\pi N) \cdot \tilde{j} \cdot \tilde{W}_1)\) picks up a factor \(e^{-2\pi isN}\) which is compensated by the factor picked up by \(K(\tilde{r}(2\pi N) \cdot \tilde{j} \cdot \tilde{W}_1)\). By covariance and the fact that

\[5\]

The action of \(\tilde{j}\) on \(\tilde{C} \in \tilde{C}\), denoted \(\tilde{j} \cdot \tilde{C}\), is explained in the appendix, cf. [31].
\[ Z(\tilde{g} \cdot \tilde{C}_1, \tilde{g} \cdot \tilde{C}_2) \text{ is independent of } \tilde{g} \in \tilde{P}_+^1, \] for every pair \( \tilde{W}, \tilde{W}' \), we get twisted Haag duality for wedge regions, i.e. for every pair \( \tilde{W}, \tilde{W}' \) the identity
\[ Z(\tilde{W}, \tilde{W}') K(\tilde{W}') = K(\tilde{W})' \quad (26) \]
holds. For smaller regions we use a chain of equalities similar to the proof of Corollary 3.4 of [4]. Let \( \tilde{C} \) and \( \tilde{C}' \) be as in Definition 2.2. Then
\[ Z(\tilde{C}, \tilde{C}') K(\tilde{C}') = \bigvee_{\tilde{C}_0 \subset \tilde{C}'} Z(\tilde{C}, \tilde{C}_0) = \bigvee_{\tilde{W}' \subset \tilde{C}'} K(\tilde{W}') \quad (27) \]
In the second equation we have used the fact that for any pair of causally separated regions \( C, C_0 \in C \) there is a wedge \( W \) such that \( C_0 \subset W' \subset C' \), cf. [4], and also that
\[ K(\tilde{W}_1) = \bigvee_{\tilde{C} \subset \tilde{W}_1} K(\tilde{C}). \quad (28) \]
This fact is asserted by Takesaki’s theorem because the r.h.s. is a standard space contained in \( K(\tilde{W}_1) \) and is, by equation (15) and covariance, invariant under the modular group of \( K(\tilde{W}_1) \). The fourth equation follows from equation (26). We have also used the fact that \( Z(\tilde{C}_1, \tilde{C}_2) \) is insensitive to making the regions \( C_1, C_2 \) smaller. We have thus proved twisted Haag duality, which obviously implies twisted locality, so we have shown property (2) of Definition 2.1.

It remains to prove property (4) of the Definition 2.1, namely that \( K(\tilde{C}) \) is standard for each \( \tilde{C} \). The real subspace associated to \( \tilde{W}_1 \) (and hence to any other wedge region \( \tilde{W} \)) has this property by construction, cf. equation (19) and Appendix A. The property that \( K(\tilde{W}) \cap iK(\tilde{W}) = \{0\} \) transfers to the smaller spaces \( K(\tilde{C}) \). It remains to show that \( K(\tilde{C}) + iK(\tilde{C}) \) is dense for all \( \tilde{C} \). But this follows from Corollary 4.8 in the next section, bearing in mind the following consequence of the Reeh-Schlieder theorem for the free scalar massive field: Consider the set of Schwartz functions with compact support contained in a fixed open space-time region. The restrictions to the mass shell of the Fourier transforms of these functions are dense in the space of square-integrable functions on the mass shell. \( \square \)
4 Standardness of the Real Subspaces

To prove that $K(\tilde{C}) + iK(\tilde{C})$ is dense, we will explicitly exhibit sufficiently many elements in $K(\tilde{C})$. This will be the only place in our analysis where we make explicit use of the representation $U$ of $\tilde{P}^\uparrow_+$. It suffices to consider $U$ to be irreducible. For if $U$ is reducible, we may take the involution $J$, cf. equation (16), as a direct sum of suitable involutions. We then obviously end up with a localization structure which is the direct sum of irreducible localization structures.

We recall the relevant irreducible representations, starting with some notational remarks. Let $L^\uparrow_+$ be the Lorentz group in $d = 2 + 1$ and $\tilde{L}^\uparrow_+$ its universal covering group. We denote elements of $\tilde{L}^\uparrow_+$ generically by $\tilde{\lambda}$, and the covering homomorphism $\tilde{L}^\uparrow_+ \to L^\uparrow_+$ by $\tilde{\lambda} \mapsto \lambda$. (29)

We occasionally denote $(0, \tilde{\lambda})$ simply by $\tilde{\lambda}$. The irreducible representation of $\tilde{P}^\uparrow_+$ for $m > 0$ and $s \in \mathbb{R}$, denoted by $U$ in the sequel, is given as follows. Let $H_m$ denote the positive mass shell $\{p \cdot p = m^2, p_0 > 0\}$ and $d\mu$ the Lorentz invariant measure on $H_m$. Then $U$ acts on $\mathcal{H} = L^2(H_m, d\mu)$ according to

$$(U(a, \tilde{\lambda})\phi)(p) = e^{i\Omega(\lambda,p)} e^{ia \cdot p} \phi(\lambda^{-1}p),$$

where $\Omega(\lambda,p) \in \mathbb{R}$ is the Wigner rotation, cf. equation (76). To this representation a unique, up to a phase factor, anti-unitary involution $J$ can be adjoined satisfying equation (16), thus extending $U$ to $\tilde{P}^\uparrow_+$ within the same Hilbert space:

$$(J\phi)(p) = e^{i\pi s} \phi(-jp),$$

cf. Lemma 1.3. Let $\{K(\tilde{C})\}_{\tilde{C} \in \tilde{C}}$ be the resulting localization structure as in Theorem 3.2.

We now calculate elements in $K(\tilde{C})$ for given $\tilde{C} \in \tilde{C}$. By construction, $\phi \in K(\tilde{C})$ if and only if for all $\tilde{g} \in \tilde{P}^\uparrow_+$ which map $\tilde{C}$ into $\tilde{W}_1$, $^7$ the vector

\footnote{The relevant facts concerning $\tilde{L}^\uparrow_+$ and the covering $\tilde{L}^\uparrow_+ \to L^\uparrow_+$ are recalled in Appendix 18.}

\footnote{$^7\tilde{W}_1$ has been defined in equation 18.}

The relevant facts concerning $\tilde{L}^\uparrow_+$ and the covering $\tilde{L}^\uparrow_+ \to L^\uparrow_+$ are recalled in Appendix 18.
$U(\tilde{g})\phi$ is in $K(\tilde{W}_1)$. In particular, it must be in the domain of $\Delta^{\tilde{L}}$. As is well-known \[3\], this implies that the map

$$t \mapsto U(\tilde{\lambda}(t)) U(\tilde{g}) \phi \, , \ t \in \mathbb{R},$$

(33)

is the boundary value of an analytic $\mathcal{H}$-valued function on the strip $\mathbb{R} + i (0, \pi)$. But a complication arises from the Wigner rotation factor. Namely, the function $t \mapsto \exp(is\Omega(\tilde{\lambda}(t)\tilde{\lambda}, p))$ has singularities in the strip for any fixed $p \in H_m$ and $\tilde{\lambda} \in \tilde{L}^+$ in a neighbourhood of the unit, which are branch points if $s$ is not an integer (see Lemma \[6.4\]). Our strategy is to consider wave functions of the form $\phi = u \cdot \psi$ (point-wise multiplication), where $u$ is a fixed non-vanishing function on the mass shell, suitably chosen as to compensate the singularities of the Wigner rotation factor. The action of $U(\tilde{g})$, according to equation (31), on wave functions of the form $(u \cdot \psi)(p) \equiv u(p)\psi(p)$ can be written as

$$(U(a, \tilde{\lambda}) u \cdot \psi)(p) = u(p) c(\tilde{\lambda}, p) e^{ia \cdot p} \psi(\lambda^{-1} p),$$

(34)

with

$$c(\tilde{\lambda}, p) \equiv u(p)^{-1} e^{is\Omega(\tilde{\lambda}, p)} u(\tilde{\lambda}^{-1} p).$$

(35)

In group theoretical terms, the map $c(\cdot, \cdot) : \tilde{L}^+ \times H_m \to \mathbb{C} \setminus \{0\}$ is a cocycle which is equivalent to the Wigner rotation factor. As indicated above, our strategy is to choose $u$ such that $c(\tilde{\lambda}, p)$ has the desired analyticity properties. This will succeed only for certain $\tilde{\lambda} \in \tilde{L}^+$ or, differently stated, for certain $\tilde{C} \in \tilde{C}$. We shall consider, as a first step, $\tilde{C}$ of the form $(C, \tilde{e}_0)$, with $C$ containing the reference direction $e_0$, cf. equation (4), and where $\tilde{e}_0$ denotes the constant path at $e_0$. Stated differently, we consider elements $\tilde{\lambda} \in \tilde{L}^+$ which satisfy

$$\tilde{\lambda} \cdot \tilde{e}_0 \in \tilde{W}_1.$$  

(36)

By this we mean that $W_1$ contains the direction $\lambda \cdot e_0$ in the sense of equation (11), and that the paths $\tilde{\lambda} \cdot \tilde{e}_0$ and $\tilde{e}_{W_1}$, cf. equation (13), are equivalent w.r.t. $W_1$. The following function is suitable for this purpose, and in the sequel the cocycle $c$ will be defined as in equation (35) above with this choice of $u$:

$$u(p) \equiv \left(\frac{p_0 - p_1}{m} \cdot \frac{p_0 - p_1 + m - ip_2}{p_0 - p_1 + m + ip_2}\right)^s, \quad p_0 \equiv (p_1^2 + p_2^2 + m^2)^{\frac{1}{2}}.$$  

(37)

Note that $p_0 - p_1$ is strictly positive for all $p \in H_m$, hence the argument in brackets lies in the cut complex plane $\mathbb{C} \setminus \mathbb{R}_0^-$. Thus, taking it to the power
of \( s \in \mathbb{R} \) can be defined via the branch of the logarithm on \( \mathbb{C} \setminus \mathbb{R}_0^- \) with \( \ln 1 = 0 \). This will always be understood in the sequel and will be called the power of \( s \) within \( \mathbb{C} \setminus \mathbb{R}_0^- \).

**Lemma 4.1** Let \( \tilde{\lambda} \) be an element of \( \tilde{L}^1_+ \) such that \( \tilde{\lambda} \cdot \tilde{e}_0 \in \tilde{W}_1 \) in the sense of equation (36). Then for all \( p \in H_m \) the function

\[
t \mapsto c(\tilde{\lambda}_1(t)\tilde{\lambda}, p)
\]

has an analytic extension into the strip \( \mathbb{R} + i(0, \pi) \). This extension satisfies the boundary condition

\[
c(\tilde{\lambda}_1(t + i\pi)\tilde{\lambda}, p) = e^{i\pi s} c(\lambda_1(t)\lambda, -jp), \quad t \in \mathbb{R}.
\] (38)

**Proof.** As we show in Lemma 4.1 \( \tilde{\lambda} \) can be decomposed into boosts and rotations as \( \tilde{\lambda} = \tilde{\lambda}_1(t) \tilde{\lambda}_2(t') \tilde{r}(\omega) \) for some unique \( t, t', \omega \in \mathbb{R} \). We then denote \( \omega' = \omega - \frac{\pi}{2} \). Then \( \tilde{\lambda} \cdot \tilde{e}_0 \in \tilde{W}_1 \) if and only if

\[
\lambda_2(t') \tilde{r}(\omega')(0, 1, 0) \in W_1^- \quad \text{and} \quad \omega' \in (-\pi, \pi),
\] (39)

the latter condition singling out the correct leaf of the covering \( \tilde{r}(\omega') \mapsto r(\omega') \). As the vector in equation (39) is equal to

\[
(\sinh t' \sin \omega', \cos \omega', \cosh t' \sin \omega'),
\]

condition (39) is equivalent to

\[
|\sinh t' \sin \omega'| \leq \cos \omega' \quad \text{and} \quad \omega' \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right].
\] (40)

This implies condition (92) of Proposition C.2 in Appendix C which now asserts the claimed analyticity property and the correct boundary value of the cocycle. \( \square \)

We denote by \( C^\infty_0(\mathbb{R}^3) \) the \( C^\infty \)-functions on \( \mathbb{R}^3 \) with compact support, and, for \( f \in C^\infty_0(\mathbb{R}^3) \), by \( E_m f \) the restriction of the Fourier transform of \( f \) to the mass shell \( H_m \). Our main result is the following proposition.

**Proposition 4.2** Let \( C \) be a region in \( \mathcal{C} \) containing the reference direction \( e_0 \) in the sense of equation \( \Pi \), and let \( \tilde{C} = (C, \tilde{e}_0) \). Then

\[
K(\tilde{C}) \supset \{ u \cdot E_m f \mid f \in C^\infty_0(C), \text{ real valued } \}.
\]
Before proving the proposition, we point out that the local subspaces for regions containing directions other than $e_0$ are obtained via covariance, and can be nicely characterized as follows. Define, for each $\tilde{\lambda} \in \tilde{L}^+_\uparrow$, a function $u_{\tilde{\lambda}}$ on the mass shell by

$$u_{\tilde{\lambda}}(p) \doteq u(p) c(\tilde{\lambda}, p).$$

(41)

This is an “intertwiner function” for those single particle vectors which are localized in regions extending to infinity in the direction $\tilde{\lambda} \cdot \tilde{e}_0$:

**Corollary 4.3** i) Let $\tilde{\lambda} \in \tilde{L}^+_\uparrow$ and $\tilde{C} \in \tilde{C}$. If $\tilde{C}$ contains $\tilde{\lambda} \cdot \tilde{e}_0$ in the sense of equation (30), then

$$K(\tilde{C}) \supset \{u_{\tilde{\lambda}} \cdot E_m f \mid f \in C_0^\infty(C), \text{ real valued } \}.$$

ii) The wave functions $u_{\tilde{\lambda}} \cdot E_m f$ transform covariantly in the sense that

$$U(a, \tilde{\lambda}) u_{\tilde{\lambda}} \cdot E_m f = u_{\tilde{\lambda}^*} \cdot E_m(a, \lambda)_* f,$$

where the star denotes the push-forward, $(g_* f)(x) \doteq f(g^{-1} x)$.

**Proof.** i) is an immediate consequence of Proposition 4.2, and ii) follows from the cocycle relation (94) below. □

It is noteworthy that the function $u_{\tilde{\lambda}}$ only depends on the path $\tilde{\lambda} \cdot \tilde{e}_0$ up to a multiplicative constant. For the stabilizer subgroup of $\tilde{e}_0$, namely the group of 1-boosts, modifies $u_{\tilde{\lambda}}$ only by a factor $c(\tilde{\lambda}_1(t), p) = e^{st}$.

**Proof of Proposition 4.2** Let $f$ be a smooth function with compact support in $C$, and let $\tilde{g} = (a, \tilde{\lambda})$ be such that $\tilde{g} \tilde{C} \subset \tilde{W}_1$. Note that then $\tilde{\lambda} \tilde{e}_0 \in \tilde{W}_1$ and supp $g_* f \subset \tilde{W}_1$, where $g_* f$ denotes the push-forward as above. We have to show that $U(\tilde{g}) u \cdot E_m f \in K(\tilde{W}_1)$. To this end we prove that the $\mathcal{H}$-valued function

$$t \mapsto \phi(t) \doteq U(\tilde{\lambda}_1(t))U(\tilde{g}) u \cdot E_m f, \quad t \in \mathbb{R},$$

(43)

is the boundary value of an analytic function $\phi(\cdot)$ on the strip $G \supset \mathbb{R} + i (0, \pi)$ which is continuous and bounded on its closure $G^-$ and that the boundary values are related by

$$\phi(t + i\pi) = J \phi(t), \quad t \in \mathbb{R}.$$

(44)

Using the push-forward to write $e^{i\alpha p}(E_m f)(\lambda^{-1} p) = (E_m(a, \lambda)_* f)(p)$, we have

$$\phi(t) = U(\tilde{\lambda}_1(t) a, \tilde{\lambda}_1(t) \tilde{\lambda}) u \cdot E_m f = v(t) \cdot \psi(t)$$

(45)
where we have written
\[ v(t)(p) = u(p) c(\bar{\lambda}_1(t)\lambda, p), \]  \hspace{1cm} (46)
\[ \psi(t)(p) = (E_m\lambda_1(t)g_s f)(p). \]  \hspace{1cm} (47)

It follows from Lemma 4.1 that for fixed \( p \in H_m \), \( \psi(\cdot)(p) \) extends to an analytic function \( \psi(\cdot, p) \) on the strip \( G \), continuous on its closure, and that
\[ \psi(t + i\pi, p) = e^{i\pi s} u(p) c(\bar{\lambda}_1(t)\lambda, -jp). \]  \hspace{1cm} (48)

Let us discuss the analyticity properties of \( \psi(t) \). The matrix-valued function 
\[ t \mapsto \lambda_1(t) \]  extends to an entire analytic function satisfying \[ \lambda_1(t + it') = \lambda_1(t)(j\nu + i\sin(t')\sigma), \]  \hspace{1cm} (49)
where \( j\nu \) acts as multiplication by \( \cos t' \) on the coordinates \( x^0 \) and \( x^1 \) and leaves the other coordinates unchanged, and \( \sigma \) acts as the Pauli matrix \( \sigma_1 \) on \( (x^0, x^1) \) and as zero on \( x^2 \). Hence \( \psi(\cdot)(p) \) extends, for fixed \( p \in H_m \), to a function \( \psi(\cdot, p) \) on \( G^− \) as follows:
\[ \psi(t + it', p) = (2\pi)^{-3/2} \int_{W_1} d^3x e^{i\rho \lambda_1(t)j_j x} e^{-i\sin(t') \rho \lambda_1(t)\sigma x} (g_s f)(x). \]  \hspace{1cm} (50)

Now for \( x \in W_1 \), the vector \( \sigma x \) lies in the forward light cone, hence \( p \cdot \lambda_1(t)\sigma x > 0 \) for \( p \in H_m \). Thus the second exponential term in equation (50) is a damping factor, and for fixed \( \tau \in G^− \) the function \( (p_1, p_2) \mapsto \psi(\tau, p) \) is of fast decrease. Further, due to the damping factor the function \( \tau \mapsto \psi(\tau, p) \) is analytic on the strip \( G \) for fixed \( p \in H_m \). Thus our function \( t \mapsto \psi(t) = v(t)\psi(t) \) extends, point-wise in \( p \), to a function \( \phi(\tau, p) = \psi(\tau, p) \psi(\tau, p) \) on \( G^− \), analytic on the interior, and in addition \( \phi(\tau, \cdot) \in L^2(H_m, d\mu) \) for each \( \tau \in G^− \). By equations (48) and (49) the analytic continuation satisfies, since \( j_\pi = j \),
\[ \phi(t + i\pi, p) = e^{i\pi s} u(p) c(\bar{\lambda}_1(t)\lambda, -jp) (E_mj_s f_s)(p). \]  \hspace{1cm} (51)

On the other hand, using \( u(\cdot - jp) = u(\cdot) \) one calculates
\[ (J \phi(t))(p) = e^{i\pi s} u(p) c(\bar{\lambda}_1(t)\lambda, -jp) (E_mj_s f_s)(p). \]  \hspace{1cm} (52)

Thus for real valued \( f \), the Hilbert space valued function \( \tau \mapsto \phi(\tau) \) defined by \( \phi(\tau)(p) = \phi(\tau, p) \) satisfies the desired equation (44). It remains to show that \( \phi(\tau) \) is in fact analytic as a Hilbert space valued function.
To this end let, for $x \in W_1$, $t_x = \text{artanh} \frac{x_0}{x_1}$. Then $\sigma x = |\sigma x| \lambda_1(t_x)(1,0,0)$ and
\[
p \cdot \lambda_1(t)\sigma x = |\sigma x| \left\{ \cosh(t + t_x)p_0 - \sinh(t + t_x)p_1 \right\}.
\] (53)
Note that the argument in curly brackets is strictly larger than $|p_2|$ and than $|p_1|\times \exp(-|t+t_x|)$. Let $t+it'$ be contained in some compact subset $G_0 \subset G$ of the strip. Then
\[
|t| \leq T \quad \text{and} \quad \sin t' \geq \varepsilon \quad \text{for some} \quad T > 0, \varepsilon > 0.
\] (54)
Then the above estimates imply, using that $\exp(-|t_x|) = \left( \frac{x_1+|x_0|}{x_1-|x_0|} \right)^{\frac{1}{2}}$, that
\[
\sin t' p \cdot \lambda_1(t)\sigma x > \alpha_1(x)|p_1| + \alpha_2(x)|p_2| \quad \text{where}
\]
\[
\alpha_1(x) = \frac{\varepsilon}{2} e^{-T}(x_1 - |x_0|) > 0,
\]
\[
\alpha_2(x) = \frac{\varepsilon}{2}(x_1^2 - x_0^2)^{\frac{1}{2}} > 0.
\] (57)
This estimate implies that
\[
\Psi(p_1,p_2) \doteq \int_{W_1} d^3x |g_s f(x)| e^{-\alpha_1(x)|p_1|-\alpha_2(x)|p_2|}
\] (58)
is a dominating function for $\psi(\tau,\cdot)$ for all $\tau$ in the compact subset $G_0$ of the strip, in the sense that $|\psi(\tau,p)| < \Psi(p_1,p_2)$ for all $\tau \in G_0$. This function is decreasing fast enough such that
\[
\int d^2p |p_1|^{n}|p_2|^{m} |\Psi(p_1,p_2)|^2 < \infty \quad \text{for all} \quad n, m \in \mathbb{N}_0.
\] (59)
Namely, the integral coincides with $4n!m!$ times the integral of $|g_s f(x)g_s f(y)| (\alpha_1(x) + \alpha_1(y))^{-n-1} (\alpha_2(x) + \alpha_2(y))^{-m-1}$ over $x$ and $y$ in $W_1$, which is finite since $\alpha_1, \alpha_2$ are strictly positive functions on $W_1$ and $\text{supp}(g_s f)$ is compactly contained in $W_1$. By similar considerations one gets a dominating function for $\frac{d}{d\tau}\psi(\tau,p)$, which we denote by $\Psi'$ and which satisfies the analogue of equation (59).

Next we establish bounds for $v(\tau,p)$. We claim that $v(\tau,p)$ and $\frac{d}{d\tau}v(\tau,p)$ are bounded, uniformly in $\tau \in G_0$, by polynomials in $|p_1|$ and $|p_2|$ which we denote by $V$ and $V'$, respectively. We demonstrate here the case of non-negative spin $s$, the other case working analogously. One has the inequality $0 < p_0 \pm p_1 \leq 2|p_1| + |p_2| + m$ and, using the identity $\frac{p_0 + p_1 + m + i p_2}{p_0 - p_1} = i \frac{p_0 + p_1 + m + i p_2}{p_0 - p_1 + m - i p_2}$, the inequality $|(-p_2 + im)(p_0 - p_1)^{-1}| \leq 2(|p_1| + |p_2| + m)/m$. These imply, for $\tau \in G_0$, the estimate
\[
v(\tau,p) \leq c_0(2|p_1| + |p_2| + m)^n (c_1 + c_2(|p_1| + |p_2| + m))^{2n} \doteq V(|p_1|,|p_2|),
\]
where $n$ is any integer $\geq s$, $c_1 = |a-b|$ and $c_2 = \frac{2}{m}e^{T-t}|a+b|$ with $a$ and $b$ as in Proposition C.2. Similar considerations hold for $s < 0$, and for $\frac{d}{dT}v(\tau,p)$.

We have now established the following facts: $\phi(t)$ extends to a family $\phi(\tau) \in L^2(H_m,d\mu)$, $\tau \in G^-$, such that $\phi(\tau)(p)$ depends analytically on $\tau$ for each $p \in H_m$. Further, for $\tau$ in any fixed compact subset of the strip $G$, the $p$-point-wise derivative w.r.t. $\tau$ is dominated by a function $\Phi \in L^2(H_m,d\mu)$:

$$\left|\frac{d}{d\tau}\phi(\tau)(p)\right| \leq \Phi(p) = V(p)\Psi'(p) + V'(p)\Psi(p), \quad p = (\omega(p),p).$$ (60)

That $\Phi$ is in $L^2(H_m,d\mu)$ follows from equation (59) and the corresponding equation for $\Psi'$.

These facts imply, by the Lebesgue lemma on dominated convergence, that for arbitrary $\chi \in L^2(H_m,d\mu)$, the function

$$\tau \mapsto (\chi,\phi(\tau))$$

is analytic on the strip $G$, with derivative being calculated via the $p$-point-wise derivative $\frac{d}{d\tau}\phi(\tau)(p)$. Since weak and strong analyticity are equivalent, this implies that $\tau \mapsto \phi(\tau)$ is an analytic Hilbert space valued function. This concludes the proof. \hfill $\square$

5 Implications of the Bisognano-Wichmann Property

In this section we show that the Bisognano-Wichmann property, defined below, essentially fixes the localization structure, and that it implies the spin-statistics connection as mentioned after Theorem 3.2.

Given a localization structure $K(\tilde{C})$, $\tilde{C} \in \tilde{C}$, denote by $S$ the canonical involution corresponding to $K(\tilde{W}_1)$, cf. Appendix A. Since $S$ is a closed antilinear involution, it has a polar decomposition $S = J\Delta^{1/2}$ with $J$ being an anti-unitary involution and $\Delta$ a positive operator.

**Definition 5.1** A localization structure satisfies the Bisognano-Wichmann property if $\Delta^b$ and $J$ satisfy equations (15) and (16), thus representing the boosts and the reflection $\tilde{j}$, respectively.

It is noteworthy that this property in fact follows from the Definition 2.1 of a localization structure. This has been established by the author in [19] in the case of four-dimensional theories, and will be published elsewhere for anyons.
in $d=3$. Because of this fact we have been forced to take equations (15) to (19) as the starting point of our construction.

We shall now see that the Bisognano-Wichmann property fixes uniquely a certain extension of the localization structure which is maximal in the sense that it satisfies twisted Haag duality, cf. (12).

**Proposition 5.2** There is up to equivalence only one localization structure which satisfies the Bisognano-Wichmann property and twisted Haag duality.

By equivalent localization structure we mean a family $\hat{K}(\tilde{C}), \tilde{C} \in \tilde{C}$, of closed real subspaces of a Hilbert space $\hat{H}$ such that there is a unitary map $V : \mathcal{H} \to \mathcal{H}$ satisfying $\hat{K}(\tilde{C}) = V K(\tilde{C})$ for all $\tilde{C} \in \tilde{C}$.

**Proof.** Let $K(\tilde{C}), \tilde{C} \in \tilde{C}$, be a localization structure as in the Proposition. With the same argument as in the proof of Theorem 3.2, equation (28) must hold for $K(\tilde{W}_1)$. Hence, the chain of equations (27) is valid, the last equation of which shows that, under the assumption of twisted Haag duality, $K(\tilde{C})$ is maximal in the sense that it satisfies equation (21). But this implies that the localization structure is fixed by the real subspaces associated to wedge regions, which in turn are fixed, due to the Bisognano-Wichmann property and covariance, by the real subspace $K(\tilde{W}_1)$ associated to $\tilde{W}_1$ and the representation $U$. Hence the localization structure is fixed by $K(\tilde{W}_1)$ or, equivalently, by the corresponding involution $S$. The positive part of the latter is fixed by the representation $U$, cf. equation (15), hence the only remaining freedom is the anti-unitary part $J$. But it turns out that $J$, and hence the entire localization structure, is fixed up to equivalence. More precisely, let $\hat{K}(\tilde{C}), \tilde{C} \in \tilde{C}$, be another localization structure as in the Proposition, with $\hat{J}$ the anti-unitary part of the canonical involution corresponding to $\hat{K}(\tilde{W}_1)$. Then, as we show in Lemma B.3, there is a unitary $V$ commuting with the representation $U$ such that $\hat{J} = V J V^{-1}$. This implies that $\hat{K}(\tilde{C}) = V K(\tilde{C})$ for all $\tilde{C} \in \tilde{C}$, as claimed. \[ \square \]

We finally prove a single-particle version of the spin-statistics theorem:

**Proposition 5.3** Let $\{K(\tilde{C})\}_{\tilde{C} \in \tilde{C}}$ be a localization structure for $(m, s)$ satisfying the Bisognano-Wichmann property. Then the spin-statistics connection holds:

$$Z^2 = e^{2\pi is}. \quad (61)$$

**Proof.** We use the one-to-one correspondence between closed real standard subspaces $K$ and densely defined anti-linear involutive operators $S$, cf. Appendix A. Let $S'$ be the canonical involution corresponding to $K(\tilde{r}(\pi)\tilde{W}_1)$. 18
Twisted locality \(8\) implies that
\[
Z(\tilde{W}_1, \tilde{r}(\pi)\tilde{W}_1) S' Z(\tilde{W}_1, \tilde{r}(\pi)\tilde{W}_1)^* \subset S^*.
\]
Now the relative winding number \(N(\tilde{W}_1, \tilde{r}(\pi)\tilde{W}_1)\) is \(-1\), hence
\[
Z(\tilde{W}_1, \tilde{r}(\pi)\tilde{W}_1) = \overline{Z(-1)}
\]
and we have \(S' \subset Z^2 S^*\). On the other hand,
\[
S' = U(\tilde{r}(\pi)) S U(\tilde{r}(-\pi))
\]
by covariance. But the group relations imply \(4\) that \(S U(\tilde{r}(-\pi)) = U(\tilde{r}(\pi)) S\), hence \(Z^2 = U(\tilde{r}(2\pi)) \equiv e^{2\pi i}\), which proves the claim. □

A Basic Notions from the Tomita-Takesaki Theory of Real Spaces

For a review of this theory, the reader is referred to one of the articles \[16, 22, 4\]. Here we recall the relevant notions.

Let \(\mathcal{H}\) be a (complex) Hilbert space with scalar product \((\cdot, \cdot)\). If \(K\) is a real subspace of \(\mathcal{H}\), then its symplectic complement is the set of vectors \(\psi \in \mathcal{H}\) such that the imaginary part of \((\phi, \psi)\) vanishes for all \(\phi \in K\). It is a closed real subspace and is denoted by \(K'\). If \(K_\alpha, \alpha \in I\), is a family of closed real subspaces, then the closed real span of these subspaces is denoted by \(\bigvee_{\alpha \in I} K_\alpha\). Its symplectic complement is given by \((\bigvee_{\alpha \in I} K_\alpha)' = \bigcap_{\alpha \in I} K'_\alpha\).

A real closed subspace \(K\) of \(\mathcal{H}\) is called standard if \(K + iK\) is dense in \(\mathcal{H}\) and \(K \cap iK = \{0\}\). Real closed standard subspaces \(K\) of \(\mathcal{H}\) are in one-to-one correspondence with antilinear, densely defined, closed operators \(S\) acting on \(\mathcal{H}\) which are involutive (i.e., satisfy \(S^2 \subset 1\)): Given \(S\), let
\[
K = \{ \phi \in \text{dom} S : S\phi = \phi \}.
\]
Then every vector in the domain of \(S\) may be uniquely written as \(\psi = \phi_1 + i\phi_2\) with \(\phi_1, \phi_2 \in K\), namely \(\phi_1 = \frac{1}{2}(\psi + S\psi)\) and \(\phi_2 = \frac{1}{2i}(\psi - S\psi)\). Hence \(K\) is standard. It is called the real space corresponding to \(S\). Conversely, a real closed standard subspace \(K\) defines an antilinear, densely defined, closed involution \(S\), by putting \(S(\phi_1 + i\phi_2) = \phi_1 - i\phi_2\) for \(\phi_1, \phi_2 \in K\). \(S\) is then called the canonical involution corresponding to \(K\). If \(S\) corresponds to \(K\) and \(U\) is unitary, then \(USU^*\) corresponds to \(UK\), and further \(S^*\) corresponds to \(K'\).
B  The Universal Covering Group of the Poincaré Group

Covering of the Lorentz group. The universal covering group $\tilde{L}_+^\uparrow$ of the proper orthochronous Lorentz group $L_+^\uparrow$ in three dimensions can be identified with the set

$$\left\{ (\gamma, \omega) \mid \gamma \in \mathbb{C}, |\gamma| < 1, \omega \in \mathbb{R} \right\},$$

the group multiplication $(\gamma, \omega)(\gamma', \omega') = (\gamma'', \omega'')$ being given by [2, p. 594]

$$\gamma'' = (\gamma' + \gamma e^{-i\omega'})(1 + \gamma\gamma' e^{-i\omega'})^{-1},$$

$$\omega'' = \omega + \omega' + \frac{1}{i} \log \left\{ (1 + \gamma\gamma' e^{-i\omega'})(\text{c.c.})^{-1} \right\}.$$

Here (c.c.) denotes the complex conjugate of the preceding factor and log is the branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}_0^-$ with log 1 = 0.

The covering homomorphism $\tilde{L}_+^\uparrow \to L_+^\uparrow$ is conveniently described via the double covering $SU(1, 1)$ of $L_+^\uparrow$, which is the subgroup of $SL(2, \mathbb{C})$ (conjugate to $SL(2, \mathbb{R})$) consisting of elements of the form

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad \alpha\bar{\alpha} - \beta\bar{\beta} = 1.$$

The covering homomorphism $\tilde{L}_+^\uparrow \to SU(1, 1)$ associates to each $(\gamma, \omega)$ the $SU(1, 1)$-matrix

$$(1 - |\gamma|^2)^{-\frac{1}{2}} \begin{pmatrix} e^{-i\pi \gamma} & \bar{\gamma} e^{-i\pi \omega} \\ \gamma e^{i\pi \gamma} & e^{i\pi \omega} \end{pmatrix}.$$  

The double covering $SU(1, 1) \to L_+^\uparrow$ is given as follows. For $a = (a^0, a^1, a^2) \in \mathbb{R}^3$ we set

$$a = \begin{pmatrix} a^0 \\ a^1 + ia^2 \\ a^0 \end{pmatrix}.$$  

Then the double covering $SU(1, 1) \to \tilde{L}_+^\uparrow$ associates to $A \in SU(1, 1)$ the unique $\lambda \in \tilde{L}_+^\uparrow$ satisfying

$$\lambda a = A a A^*, \quad a \in \mathbb{R}^3.$$  

Let us determine the lifts of the one-parameter subgroups of boosts and rotations. Denote the boosts in $k$-direction ($k = 1, 2$) by $\lambda_k(\cdot)$ and the
rotations in the 1-2 plane by \( r(\cdot) \). Explicitly, \( \lambda_k(t) \) acts on the 0- and \( k \)-coordinates as the matrix \( \begin{pmatrix} 1 & -i \omega \\ i \omega & 1 \end{pmatrix} \), and \( r(\omega) \) acts on the 1- and 2-coordinates as
\[
\begin{pmatrix}
\cos(\omega) & -\sin(\omega) \\
\sin(\omega) & \cos(\omega)
\end{pmatrix}
\].

We denote by \( \tilde{\lambda}_1(\cdot) \), \( \tilde{\lambda}_2(\cdot) \) and \( \tilde{r}(\cdot) \) the unique lifts of these one-parameter groups to \( \tilde{L}_+^1 \).

**Lemma B.1** i) The lifts of the one-parameter groups are given by
\[
\tilde{\lambda}_1(t) = (\tanh(t/2), 0) \quad \tilde{\lambda}_2(t) = (i \tanh(t/2), 0) \quad \text{and} \quad \tilde{r}(\omega) = (0, \omega).
\]

ii) Every element \( \tilde{\lambda} \in \tilde{L}_+^1 \) has a unique decomposition \( \tilde{\lambda} = \tilde{\lambda}_1(t) \tilde{\lambda}_2(t') \tilde{r}(\omega) \) \( t, t', \omega \in \mathbb{R} \).

**Proof.** i) One verifies that the three one-parameter maps are continuous and have the correct images under the covering projection (66) and (68). ii) Consider the action of the Lorentz transformation \( \lambda \), corresponding to \( \tilde{\lambda} \), on the point \((1, 0, 0)\). Define \( t' \) as the \( \text{arsinh} \) of the 2-component of \( \lambda \cdot (1, 0, 0) \), and \( t \) as the unique real number such that \( \sinh(t) \cosh(t') \) is the 1-component of \( \lambda \cdot (1, 0, 0) \). One then checks that the actions of \( \lambda \) and \( \lambda_1(t) \lambda_2(t') \) on the point \((1, 0, 0)\) coincide. This implies that there is a unique \( \omega \in \mathbb{R} \) such that equation (71) holds. \( \square \)

**Wigner Rotation.** Let, for \( p \in H_m \),
\[
\gamma(p) = \frac{p_1 + ip_2}{p_0 + m}, \quad \tilde{h}(p) = (\gamma(p), 0),
\]
and denote by \( h(p) \) the corresponding element in \( L_+^1 \). Then
\[
h(p) : (m, 0, 0) \mapsto p.
\]

This implies that for arbitrary \( p \in H_m \) and \( \tilde{\lambda} \in \tilde{L}_3^1 \), the element
\[
t(\tilde{\lambda}, p) \doteq \tilde{h}(p)^{-1} \tilde{\lambda} \tilde{h}(\lambda^{-1} p)
\]
leaves \((m, 0, 0)\) invariant, hence is a rotation and may be written in the form
\[
t(\tilde{\lambda}, p) = (0, \Omega(\tilde{\lambda}, p)),
\]

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where $\Omega(\cdot,\cdot)$ is the so-called Wigner rotation. In fact, equations (74) and (64) imply that, for $\tilde{\lambda} = (\gamma, \omega)$,

$$\Omega(\tilde{\lambda}, p) = \omega + \frac{1}{i} \log \left\{ (1 - \gamma(p)\bar{\gamma} e^{-i\omega}) (\text{c.c.})^{-1} \right\}$$

$$+ \frac{1}{i} \log \left\{ (1 + \frac{\gamma - \gamma(p)e^{-i\omega}}{1 - \gamma(p)\bar{\gamma} e^{-i\omega}} \bar{\gamma}(\lambda^{-1}p)) (\text{c.c.})^{-1} \right\} . \quad (76)$$

Note that $\Omega((0, \omega) , p) = \omega$ for all $\omega$ and $p$, and that $\Omega$ satisfies the cocycle condition

$$\Omega(\tilde{\lambda}\tilde{\lambda}', p) = \Omega(\tilde{\lambda}, p) + \Omega(\tilde{\lambda}', \lambda^{-1}p) \quad (77)$$

for all $\tilde{\lambda}, \tilde{\lambda}' \in \tilde{L}_+^\uparrow$ and $p \in H_m$.

**Proper Poincaré Group.** The proper Poincaré group $P_+$ can be obtained from the proper orthochronous Poincaré group by adjoining the reflection $j$ at the $x^2$-axis, cf. equation (14), with the appropriate relations:

$$j^2 = 1$$

$$j(\lambda_1(t))j = \lambda_1(t)$$

$$j r(\omega) j = r(-\omega). \quad (78)$$

(Note that the last equations imply $j\lambda_2(t)j = \lambda_2(-t)$.) Correspondingly, the universal covering group $\tilde{P}_+$ of this (disconnected) group may be defined by adjoining an element $\tilde{j}$ to $\tilde{P}_+$ satisfying the relations

$$\tilde{j}^2 = 1 \quad \text{and} \quad \tilde{j}(a, (\gamma, \omega)) \tilde{j} = (ja, (\bar{\gamma}, -\omega)). \quad (79)$$

In fact, the map $\tilde{j} \mapsto j$, $\tilde{\lambda} \mapsto \lambda$ is a homomorphism and hence a covering projection. Finally, we prove an important cocycle relation of the Wigner rotation (75) with respect to $\tilde{j}$.

**Lemma B.2** For all $\tilde{\lambda} \in \tilde{L}_+^\uparrow$ and $p \in H_m$ the following relation holds:

$$\Omega(\tilde{j}\tilde{\lambda}\tilde{j}, p) = -\Omega(\tilde{\lambda}, -j\cdot p) . \quad (80)$$

**Proof.** From the definition of $\tilde{h}(p)$ via equation (12) and the group relations (79) satisfied by $\tilde{j}$ we get

$$\tilde{h}(-j\cdot p) = \tilde{j} \tilde{h}(p) \tilde{j} . \quad (81)$$

This implies $t(\tilde{\lambda}, -j\cdot p) = \tilde{j} t(j\tilde{\lambda}\tilde{j}, p) \tilde{j}$ and hence the claim. □
Lemma B.3  i) Let $U$ be the irreducible representation of $\tilde{P}_+^\uparrow$ for mass $m > 0$ and spin $s \in \mathbb{R}$ defined in equation (31), and let $J$ be the operator defined in equation (32). Then $J$ is an anti-unitary involution satisfying the representation property

$$JU(\tilde{g})J = U(\tilde{j}\tilde{g}\tilde{j}).$$

ii) Let $U$ be a finite direct sum of copies of the irreducible representation of $\tilde{P}_+^\uparrow$ for mass $m > 0$ and spin $s \in \mathbb{R}$, acting on a Hilbert space $H$. Then there is a unique, up to equivalence, extension of $U$ from $\tilde{P}_+^\uparrow$ to $\tilde{P}_+$ in $H$. Uniqueness means that if $J$ and $\hat{J}$ are anti-unitary involutions satisfying the representation property (82), then there is a unitary $V$ commuting with $U(\tilde{P}_+^\uparrow)$ and satisfying $VJ = JV$.

Proof. i) follows immediately from Lemma B.2. ii) The existence of $J$ follows from i) by taking direct sums. To see uniqueness, let $C = \hat{J}J$. It is a unitary operator commuting with the representation $U$ and satisfying $CJ = JC^{-1}$. Using spectral calculus in the same way as in the proof of Prop. 3.1 in [28], we define a unitary root $V$ of $C$, $V^2 = C$, which still commutes with the representation $U$ and satisfies $VJ = JV^{-1}$. Then $V$ has the properties claimed in the Lemma.

Action of $\tilde{P}_+$ on $\tilde{C}$. The universal covering group $\tilde{P}_+$ of the proper Poincaré group acts on $\tilde{C}$ in the following way. Let $\tilde{C} = (C, \tilde{e}) \in \tilde{C}$ where $\tilde{e}$ is the equivalence class w.r.t. $C$ of a path $t \mapsto \tilde{e}(t)$ in $H$ starting at $e_0$ and ending in $C$. Identifying $\tilde{L}_+^\uparrow$ with the set of homotopy classes of paths in $\tilde{L}_+^\uparrow$ starting at the unit, an element $\tilde{g} = (a, \tilde{\lambda}) \in \tilde{P}_+^\uparrow$ acts on $\tilde{C}$ as follows. Let $t \mapsto \tilde{\lambda}(t)$ be any path in $\tilde{L}_+^\uparrow$ which represents $\tilde{\lambda}$. Then we define

$$\tilde{g} \cdot \tilde{C} \doteq (g \cdot C, \tilde{\lambda} \cdot \tilde{e}),$$

where $\tilde{\lambda} \cdot \tilde{e}$ is the equivalence class w.r.t. $\lambda \cdot C$ of the path $t \mapsto \tilde{\lambda}(t) \cdot \tilde{e}(t)$ in $H$. Further, the element $\tilde{j} \in \tilde{P}_+$ acts on $\tilde{C}$ as

$$\tilde{j} \cdot \tilde{C} \doteq (j \cdot C, \tilde{j} \cdot \tilde{e}),$$

where $\tilde{j} \cdot \tilde{e}$ is the equivalence class w.r.t. $j \cdot C$ of the path $t \mapsto j \cdot \tilde{e}(t)$. Note that this path also starts at $e_0$ since we have chosen the reference direction $e_0$ so as to be invariant under $\tilde{j}$.
C Proof of Analyticity of the Cocycle.

We establish the required analyticity properties of the cocycle $c(\tilde{\lambda}, p)$, cf. equ. (35), starting with the Wigner rotation factor for the 1-boosts. Let

$$l(p) = p_0 - p_1 + m - ip_2 \quad \text{and} \quad v(p) = l(p) l(p)^{-1}. \quad (85)$$

Note that for all $p \in H_m$, the number $v(p)$ lies in the cut complex plane $\mathbb{C} \setminus \mathbb{R}_0^-$, allowing for our definition of the power $v(p)^s$ given before Lemma 4.1.

We have

Lemma C.1 The Wigner rotation factor for the 1-boosts is given by

$$e^{is\Omega(\tilde{\lambda}_1(t), p)} = v(p)^s v(\tilde{\lambda}_1(-t)p)^{-s}. \quad (87)$$

As a function of $t$, it has branch points in the strip $\mathbb{R} + i(0, \pi)$ for any $p \in H_m$ if $s$ is not an integer.

Proof. Equation (87) is verified by direct calculation. But using

$$(\lambda_1(t)p)_0 - (\lambda_1(t)p)_1 = e^t(p_0 - p_1), \quad (88)$$

we get

$$v(\lambda_1(-t)p) = \frac{e^t(p_0 - p_1) + m - ip_2}{e^t(p_0 - p_1) + m + ip_2}. \quad (89)$$

For any fixed $p \in H_m$, this function has zeroes in the strip, which proves the claim. □

In the next proposition, we give an explicit expression for the cocycle $c(\tilde{\lambda}, p)$, exhibiting its analyticity properties.

Proposition C.2 Let $\tilde{\lambda} = \tilde{\lambda}_1(t) \tilde{\lambda}_2(t') \tilde{r}(\omega)$, with $t, t', \omega \in \mathbb{R}$, and let $\omega' = \omega - \frac{\pi}{2}$. Let further $p \in H_m$ be arbitrary.

i) Denote by $\omega_0'$ the representant of $\omega' + 2\pi \mathbb{Z}$ in the interval $(-2\pi, 0]$. Then

$$c(\tilde{\lambda}, p) = 2^{-s} e^{s(t+t')} e^{is(\omega'-\omega_0')} \left( a - b + e^{-t}(a + b) \frac{-p_2 + im}{p_0 - p_1} \right)^{2s}, \quad \text{where} \quad \frac{\omega_0'}{2}, \quad \text{and} \quad b = e^{-t'} \sin \frac{\omega_0'}{2} \leq 0. \quad (90)$$

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The power of $2s$ is understood within $\mathbb{C} \setminus \mathbb{R}_0^-$. 

ii) Let $s \not\in \frac{1}{2}\mathbb{N}_0$. The function $\tau \mapsto c(\lambda_1(\tau)\tilde{\lambda},p)$ is analytic in the strip $\mathbb{R} + i(0,\pi)$ if and only if the parameters $t'$ and $\omega'$ satisfy the relation

$$-\sinh t' |\sin \omega'| \leq \cos \omega'.$$

(92)

In this case, the upper and lower boundary values are related by

$$c(\lambda_1(\tau)\tilde{\lambda},p) \big|_{\tau = i\pi} = e^{i\pi s} e^{4\pi isn(\omega')} c(\lambda,-jp) ,$$

(93)

where $n(\omega')$ is the unique integer such that $\omega' - 2\pi n(\omega') \in (-\pi,\pi]$.

iii) For $s \in \frac{1}{2}\mathbb{N}_0$, the function $\tau \mapsto c(\lambda_1(\tau)\tilde{\lambda},p)$ is analytic in the strip $\mathbb{R} + i(0,\pi)$ and satisfies the boundary condition (93) for all $\tilde{\lambda} \in \tilde{L}_+^\uparrow$.

Remark. From (iii) follows that for $s \in \frac{1}{2}\mathbb{N}_0$ the localization structure can be non-trivially extended to bounded regions as in Proposition 4.2. The same can be shown for $s \in -\frac{1}{2}\mathbb{N}$ if one uses, instead of our intertwining function $u =: u_s$ the function $u_s^-(p) := u|_{s}(p)$.

Proof. In the following, $p$ denotes an arbitrary point on the mass shell. We will use the cocycle identity

$$c(\tilde{\lambda}\tilde{\lambda}',p) = c(\tilde{\lambda},p) c(\tilde{\lambda}',\lambda^{-1}p) , \quad \tilde{\lambda},\tilde{\lambda}' \in \tilde{L}_+^\uparrow ,$$

(94)

satisfied by $c$ as a consequence of equation (77). Thus, we first calculate $c(\tilde{\lambda},p)$ if $\tilde{\lambda}$ is a boost in $1$-direction or a rotation, and then use the above cocycle property for a general element $\tilde{\lambda}$.

The function $v$ from Lemma C.1 is related to $u$, defined in equation (37), by

$$u(p) = \left(\frac{p_0 - p_1}{m}\right)^s v(p)^s .$$

Hence, in view of the identity (88), Lemma C.1 implies that

$$c(\tilde{\lambda}_1(t),p) = e^{st} \quad \text{for all } t \in \mathbb{R}, p \in H_m .$$

(95)

In order to calculate the cocycle for rotations, let us see how the function $u$ transforms under rotations. Writing $u$ as

$$u(p) = \left(\frac{p_0 - p_1}{m}\right)^s \left(\frac{l(p)}{|l(p)|}\right)^s$$

and using the identity

$$l(p) \cdot \overline{l(p)} = 2(p_0 + m)(p_0 - p_1) ,$$

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we get
\[ u(p) = (2m(p_0 + m))^{-s} l(p)^{2s}. \]
Here we have used the fact that \( \text{Re} l(p) > 0 \) to identify \( (l(p)^2)^s \) with \( l(p)^{2s} \).
A straightforward calculation shows that \( l \) transforms under rotations as follows: For \( \omega \in \mathbb{R} \),
\[ l(r(-\omega)p) = l(p) \cdot l_\omega(p) \quad \text{where} \]
\[ l_\omega(p) = e^{-i\omega/2 \left( \cos \frac{\omega}{2} + \sin \frac{\omega}{2} \frac{-p_2 + im}{p_0 - p_1} \right)}. \]
Note that \( l(p) \) and \( l_\omega(p) \) are, as well as the l.h.s. of equation (96), in \( \mathbb{C} \setminus \mathbb{R} \) for all \( \omega \) and \( p \). Hence we may take them to the power of \( 2s \) (within \( \mathbb{C} \setminus \mathbb{R} \)) separately, i.e. \( (l(p)l_\omega(p))^{2s} = l(p)^{2s}l_\omega(p)^{2s} \). We thus have
\[ u(r(-\omega)p) = u(p) \cdot l_\omega(p)^{2s}, \]
and hence the cocycle for rotations is given by
\[ c(\tilde{\lambda}(\omega), p) = e^{is\omega} l_\omega(p)^{2s}. \]
Now our results (95) and (99) imply, by the cocycle relation (94), that for all \( t, \omega \in \mathbb{R} \)
\[ c(\tilde{\lambda}_1(t)\tilde{\rho}(\omega), p) = e^{st} e^{is(\omega - \omega_0)} \left( \cos \frac{\omega_0}{2} + e^{-t} \sin \frac{\omega_0}{2} \frac{-p_2 + im}{p_0 - p_1} \right)^{2s}, \quad \omega_0 \in (-2\pi, 0]. \]
(For \( \omega_0 \neq 0 \) this is so because then the imaginary parts of the two factors on the r.h.s. of equation (97) have opposite sign, while for \( \omega_0 = 0 \) both factors equal one.) Using this and equation (88), we arrive at the expression
\[ c(\tilde{\lambda}_1(t)\tilde{\rho}(\omega), p) = e^{st} e^{is(\omega - \omega_0)} \left( \cos \frac{\omega_0}{2} + e^{-t} \sin \frac{\omega_0}{2} \frac{-p_2 + im}{p_0 - p_1} \right)^{2s}, \quad \omega_0 \in (-2\pi, 0]. \]
We are now prepared to prove equation \((\text{90})\). Let \(\tilde{\lambda} \in \tilde{L}_{\pm}^1\) be as in the Proposition. Using \(\tilde{\lambda}_2(t') = \tilde{r}(\frac{\pi}{2}) \tilde{\lambda}_1(t') \tilde{r}(\frac{\pi}{2})\), we rewrite \(\tilde{\lambda}\) as
\[
\tilde{\lambda} = \tilde{\lambda}_1(t) \tilde{r}(\frac{\pi}{2}) \tilde{\lambda}_1(t') \tilde{r}(\omega') \quad \text{with} \quad \omega' = \omega - \frac{\pi}{2} . \tag{103}
\]
Due to the cocycle relation \((\text{94})\), \(c(\tilde{\lambda}, p)\) consists of two factors of the form calculated in equation \((\text{102})\):
\[
c(\tilde{\lambda}, p) = c(\tilde{\lambda}_1(t) \tilde{r}(\frac{\pi}{2}), p) \cdot c(\tilde{\lambda}_1(t') \tilde{r}(\omega'), r(-\frac{\pi}{2})\lambda_1(-t) p)
\]
\[
= 2^{-s} e^{st} \left\{1 + e^{-t}\frac{p_2 + im}{p_0 - p_1}\right\}^{2s} \cdot e^{st'} e^{is(\omega' - \omega_0)} \left\{a + b \frac{-q_2 + im}{q_0 - q_1}\right\}^{2s}, \tag{104}
\]
where \(\omega_0'\) is the representant of \(\omega' + 2\pi \mathbb{Z}\) in \((-2\pi, 0]\), and we have written \(a\) and \(b\) as in equation \((\text{91})\) of the Proposition and \(q = r(-\frac{\pi}{2})\lambda_1(-t) p\).

Explicitly, \(q\) reads
\[
q = \left(\cosh t p_0 - \sinh t p_1, p_2, \sinh t p_0 - \cosh t p_1\right),
\]
and we calculate
\[
\frac{-q_2 + im}{q_0 - q_1} = \frac{-e^t p_- + e^{-t} p_+ + 2im}{e^t p_- + e^{-t} p_+ - 2p_2} = \frac{-e^t p_- + p_2 - im}{e^t p_- - p_2 + im},
\]
where \(p_{\pm} = p_0 \pm p_1\). Then the product of the two curly brackets in \((\text{104})\) yields
\[
\left\{1 + e^{-t}\frac{p_2 + im}{p_0 - p_1}\right\} \left\{a + b \frac{-q_2 + im}{q_0 - q_1}\right\} = a - b + e^{-t} (a + b) \frac{-p_2 + im}{p_0 - p_1} . \tag{105}
\]
Having chosen \(\omega_0' \in (-2\pi, 0]\), we observe that \(b \leq 0\), and equality holds only if \(\omega_0' = 0\). Hence \(a + b = 0\) implies \(a = -b = 1\). Thus the r.h.s. is in \(\mathbb{C} \setminus \mathbb{R}_0^-\). The same holds for the two factors on the l.h.s., hence we may take them to the power of \(2s\) (within \(\mathbb{C} \setminus \mathbb{R}_0^-\)) separately. We therefore have
\[
c(\tilde{\lambda}, p) = 2^{-s} e^{st(t+t')} e^{is(\omega' - \omega_0')} f(t, p)^{2s}, \quad \text{where} \quad f(t, p) = a - b + e^{-t} (a + b) \frac{-p_2 + im}{p_0 - p_1} . \tag{106}
\]
This proves part \(i)\) of the Proposition.
We now discuss the analyticity properties of the function \( c(\tilde{\lambda}_1(\cdot)\tilde{\lambda}, p) \).
If \( \lambda \) is parametrized by \( t, t', \omega \in \mathbb{R} \) as in the Proposition, then \( \lambda_1(\tau)\lambda = \tilde{\lambda}_1(\tau + t)\tilde{\lambda}_2(t')\tilde{\omega}(\omega) \) and we may write
\[
c(\tilde{\lambda}_1(\tau)\tilde{\lambda}, p) = 2^{-s} e^{s(\tau+t+t')} e^{i\omega(\omega'-\omega')} f(\tau + t, p)^{2s},
\]
with \( f(\cdot, p) \) as in equation (107). Note that \( f(\cdot, p) \) is an entire analytic function and satisfies
\[
f(t + i\pi, p) = \overline{f(t, -jp)}.
\]
For \( s \in \frac{1}{2}\mathbb{N}_0 \) (iii), the claimed analyticity and boundary conditions follow. To prove ii), let \( s \notin \frac{1}{2}\mathbb{N}_0 \). Then the function \( \tau \mapsto c(\tilde{\lambda}_1(\tau)\tilde{\lambda}, p) \) has an analytic extension into the strip \( \mathbb{C} + i(0, \pi) \) if and only if \( f(\cdot, p) \) has no zeroes in the strip. This can be decided by looking at the definition (107), taking into consideration that \((-p_2 + im)(p_0 - p_1)^{-1}\) takes all values in the upper half plane \( \mathbb{R} + i\mathbb{R}^+ \) if \( p \) runs through \( H_m \).

In the following, \( z_+^{2s} \) will denote \( z \) to the power of \( 2s \) defined via the branch of the logarithm on \( \mathbb{C} \setminus \mathbb{R}^+_0 \) satisfying \( \log(-1) = i\pi \), if \( z \in \mathbb{C} \setminus \mathbb{R}^+_0 \). For \( z \in \mathbb{C} \setminus \mathbb{R}^+_0 \), \( z \) to the power of \( 2s \) defined via the branch of the logarithm on \( \mathbb{C} \setminus \mathbb{R}^+_0 \) satisfying \( \log(1) = 0 \) will now be denoted by \( z_+^{2s} \); instead of \( z^{2s} \) as before. We will use the following rules: \( \textbf{(1)} \) If \( z \) is in the upper complex half plane, then \( z^{2s}_+ = z_+^{2s} \), while for \( z \) in the lower half plane, \( z^{2s}_- = e^{-2\pi is} z_+^{2s} \).

\( \textbf{(2)} \) Complex conjugation commutes with taking powers within \( z \in \mathbb{C} \setminus \mathbb{R}^+_0 \): \( (z)^{2s}_+ = z_+^{2s} \). \( \textbf{(3)} \) If \( f(\tau, p) \) is contained in \( \mathbb{C} \setminus \mathbb{R}^+_0 \) for all \( \tau \) in the strip \( \mathbb{R} + i[0, \pi] \), then analytic continuation in \( \tau \) commutes with taking powers within \( \mathbb{C} \setminus \mathbb{R}^+_0 \), respectively. That means in particular, \( f(\tau, p)^{2s}_+|_{\tau = i\pi} = f(i\pi, p)^{2s}_+ \), where the l.h.s. denotes the analytic continuation of \( f(\cdot, p)^{2s}_+ \) from the real line to \( i\pi \).

\textbf{Case 1:} \( |b| > |a| \). Then \( (a + b)(a - b) < 0 \), hence \( a + b \) and \( a - b \) have different sign. Then \( f(\cdot, p) \) has zeroes in the strip and hence the cocycle has, for \( s \in \frac{1}{2}\mathbb{Z} \), no analytic continuation into the strip. \textbf{Case 2:} \( |b| \leq |a| \), i.e. \( (a + b)(a - b) \geq 0 \). We observe first that \( a = 0 \) implies \( \omega'_0 = -\pi \), hence \( b = -e^{-t} < 0 \), contradicting the assumption. Hence \( a \neq 0 \) in the present case. \textbf{Case 2.1:} Both \( a + b \) and \( a - b \) are greater or equal to zero. Since \( a \neq 0 \) (as observed above), this implies that \( a > 0 \) and consequently, \( b \) being non-positive (cf. (91)) that \( a - b > 0 \). Hence \( f(\tau, p) \) is contained in \( \mathbb{C} \setminus \mathbb{R}^+_0 \) for all \( \tau \) in the strip, and our rules above, together with equation (109), imply that \( f(\tau, p)^{2s}_+|_{\tau = i\pi} = f(0, -jp)^{2s}_+ \). Hence we have
\[
c(\tilde{\lambda}_1(\tau)\tilde{\lambda}, p)|_{\tau = i\pi} = e^{i\pi s} e^{2i\pi(\omega - \omega'_0)} c(\tilde{\lambda}, -jp).
\]
In the case at hand, \( a > 0 \) and consequently \( \omega'_0 \in (-\pi, 0] \). Hence \((\omega' - \omega'_0)/2\pi\) is just the integer \(n(\omega')\) defined in the Proposition, and the above equation coincides with equation (94). **Case 2.2:** Both \( a + b \) and \( a - b \) are less or equal to zero. Similarly as in case 2.1, this implies that \(a + b < 0\). Hence \( f(\tau, p) \) is in the lower half plane for real \( \tau \), and is contained in \( \mathbb{C} \setminus \mathbb{R}_0^+ \) for all \( \tau \) in the strip. Hence our three rules above imply that \( f(\tau, p)^{2\pi}|_{\tau=i\pi} = e^{-4\pi is}\frac{f(0, -jp)^{2\pi}}{2\pi} \). We thus have

\[
c(\tilde{\lambda}_1(\tau)\tilde{\lambda}, p)|_{\tau=i\pi} = e^{i\pi s} e^{2is(\omega' - \omega'_0 - 2\pi)} c(\tilde{\lambda}, -jp) .
\]

In the case at hand, \( a < 0 \) and consequently \( \omega'_0 \in (-2\pi, -\pi) \). Hence \((\omega' - \omega'_0 - 2\pi)/2\pi\) is just the integer \(n(\omega')\) defined in the Proposition, and the above equation again coincides with equation (94).

We have now shown that the cocycle has an analytic continuation into the strip if and only if \( |b| \leq |a| \), and that the continuation satisfies equation (93). It remains to show that \( |b| \leq |a| \) is equivalent to the condition (92). Both conditions are true for \( \omega' \in 2\pi \mathbb{Z} \) and false for \( \omega' \in \pi + 2\pi \mathbb{Z} \), hence they coincide if \( \omega' \in \pi \mathbb{Z} \). If \( \omega' \notin \pi \mathbb{Z} \), then \( |b| \leq |a| \) is equivalent to

\[
e^{-t'} - e^{-t} \leq \left| \cot \frac{\omega'_0}{2} \right| \left| \tan \frac{\omega'_0}{2} \right| = 2 \cos \omega'_0 \left| \sin \omega'_0 \right|^{-1} = 2 \cos \omega' \left| \sin \omega' \right|^{-1} ,
\]

hence to condition (92). We have thus shown part ii) of the Proposition. □

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