Moving Quark in a General Fluid Dynamical Flow

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Abstract: We determine the most general form of the covariant drag force exerted on a quark moving through a fluid dynamical flow. Up to first order in derivative expansion, our general formula requires the specification of seven coefficient functions. We use the perturbative method introduced in arXiv:1202.2737 and find all these coefficients in the hydrodynamic regime of a $\mathcal{N} = 4$ SYM plasma. Having this general formula, we can obtain the rate of the energy and momentum loss of a quark, namely the drag force, in a general flow. This result makes it possible to perturbatively study the motion of heavy quarks moving through the Bjorken flow up to first order in derivative expansion.

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Contents

1 Introduction

The only thing that we obtain from heavy ion collisions experiment is the spectrum of the hadrons coming out of a fireball. As it is well known, after such collision, a new phase of system is produced which is the most strongly coupled matter known in nature [1]. Having the spectrum of hadrons, we may phenomenologically find several properties of the initial state and evolution of this hot and dense matter. Although the right place to explore these phenomena is QCD, due to failure of perturbative methods in the strong coupling regime, progress looks formidable.

As a well known replaced tool, the AdS/CFT duality [2, 3] has been trying to investigate some aspects of the problem during recent years. In one of the preliminary attempts in this direction, [4] and [5] computed the drag force exerted on a heavy quark moving through a $N = 4$ SYM thermal plasma. Their computations are based on studying of a classical string dynamics in a five dimensional AdS black brane dual to the four dimensional boundary thermal plasma through which the quark is moving. Based on their result, Horowitz and Gyulassy showed that the AdS/CFT duality predicts a decreasing nuclear modification factor of bottom jets, $R_{AA}^{\text{AdS}}(p_T)$, while the pQCD energy loss gives an increasing $R_{AA}^{\text{PQCD}}(p_T)$ [6].

On the other hand, there are some evidence confirming that just shortly after the scattering, the evolution of the QGP is described with hydrodynamic approximation (See [7] and references therein.). So to theoretically produce a more precise hadron spectrum compatible with the one obtained from data, it seems necessary to take the hydrodynamical evolution of the QGP into account. As it is well known, a relativistic fluid dynamical flow consistent with the final detected particles in the QGP experiment is the Bjorken flow [8]. So our interesting problem would be computing the rate of the energy loss of a heavy quark moving through such flow. To this end, we must study the dynamics of a classical string in the gravity background dual to a boundary Bjorken flow\footnote{It should be noted that in [6], the authors have used the AdS/CFT drag force replacing the temperature with the zero order temperature of the Bjorken flow therein.}.

Relatedly, the Fluid/Gravity correspondence [9] provides our desired dual background. Fluid/Gravity duality is able to perturbatively constructs the asymptotically AdS5 background dual to every boundary fluid dynamical flow, order by order, in a boundary derivative expansion. In [10] we explained ”how” to use this duality and to compute the drag force in a general flow perturbatively. Then as an example we computed the first order corrected drag force exerted on specific quarks in Bjorken flow, i.e. the quarks constrained to move in the zero rapidity plane. More importantly, it should be noted that our method in [10] was not able to give the drag force
exerted on a quark with non-vanishing initial rapidity. In this paper however, we give a solution for the "covariant drag force" in a general fluid dynamical flow, completely answering the questions remained unresolved about a general quark moving through The Bjorken flow.

Before proceeding further, let us exactly explain what we mean by the "drag force" and "covariant drag force". When an external quark moves through the medium, the energy-momentum tensor of the medium is not conserved as before, instead we have

\[
\partial_\nu T^{\mu\nu}(x) = -f^\mu \delta^3(\vec{x} - \vec{x}_0(t)).
\]

As expected, the right hand side determines the volume density of the energy and momentum pumped from the quark into the medium per unit time. \(^2\) \(^3\) \(f^\mu\) is the drag force and gives the rate of the energy and momentum loss of the quark. It is clear that the drag force does not transform under Lorentz transformations as a four vector. On the other hand, what we will refer to as the covariant drag force is:

\[
F^\mu = \frac{dp^\mu}{d\tau} = \gamma \frac{dp^\mu}{dt} = \gamma f^\mu = \gamma \left( \frac{dE}{dt} \frac{d\vec{p}}{dt} \right).
\]

What we want to do in this paper is to compute \(F^\mu\) covariantly in a general fluid dynamical flow and then to find the drag force \(f^\mu\). To proceed, we first determine the most general form of the covariant drag force up to first order in derivative expansion, independent of AdS/CFT methods. This general four vector is given in terms of the quark velocity, local fluid dynamical variables and their derivatives. At zeroth order, namely ideal fluid, there is only one coefficient function which has to be specified. In \(\mathcal{N} = 4\) SYM field theory, this coefficient has been already found \(^3\). Rigorously speaking, covariant drag force to this order may be given by:

\[
F^\mu_{\text{ideal}}(x) = \frac{1}{2\pi\alpha b(x)^2} \left((\hat{u} \cdot u(x))\hat{u}^\mu + u^\mu(x)\right).
\]

At first order however, this formula acquires derivative corrections constructed from fluid dynamical variables.\(^4\) Formally, the most general corrections may appear as the following:

\[
F^\mu(x) = (1 + \sum_i \alpha_i s_i) F^\mu_{\text{ideal}}(x) + \sum_i \beta_i v_i^\mu
\]

where \(s_i(v_i^\mu)\) is a basis of on shell inequivalent one derivative scalars (vectors) and \(\alpha_i\) and \(\beta_i\) are unknown coefficient functions. As it will be discussed in detail throughout the next section, there are three (three) inequivalent one derivative scalars (vectors)

---

\(^2\)\(\vec{x} = \vec{x}_0(t)\) is the quark’s path in the medium.

\(^3\)\(\hat{u}^\mu\) stands for the quark velocity. \(u^\mu(x)\) and \(b(x) = \frac{1}{\sqrt{\epsilon}}\) define a dynamical flow compatible with fluid dynamics equations. In addition, \(x\) refers to all space-time points.

\(^4\)Recall that in\([4],[5]\) and \([10]\) and this work, it has been assumed that quark is dragging with constant velocity through the medium.
respecting symmetry considerations. We use the method we developed in [10] to specify these six coefficient functions. We consider a quark moving through a general fluid flow and compute the drag force in its rest frame (RF) at one instant of time. Comparing the result with the covariant formula evaluated in the same situation, we find six equations giving exactly these six coefficient functions. We therefore fully compute the covariant drag force in a general fluid flow of a $\mathcal{N} = 4$ SYM field theory, up to first order in derivative expansion.

It is interesting to note that all $\beta_i$ functions turn out to be zero, meaning that the vectorial structure of the drag force remains unchanged up to first order and just the scalar factor is corrected\(^5\). This result is not predictable in the field theory side and is specifically obtained in the $\mathcal{N} = 4$ SYM field theory via holographic computations.

Considering the Bjorken flow as the relevant fluid flow to the QGP experiments, we successfully reproduce the results of [10] by use of our general covariant formula.\(^6\) In addition, having obtained the covariant drag force, one can simply find the drag force exerted on a general heavy quark with arbitrary initial conditions in the Bjorken flow; what we were not able to obtain in [10].

The paper is organized as it follows; in section (2) we determine the general form of covariant drag force up to first order in derivative expansion. In section (3) we first briefly review how to compute the drag force via AdS/CFT and then apply it to the case of a globally boosted thermal plasma. In section (4) we first glimpse at the Fluid/Gravity duality then we outline in detail the structure of our perturbative computations. Finally we will compute different components of covariant drag force at one instant of time. In section (5), we proceed to exploit the results of previous sections to fix the unknown coefficients in the general covariant drag force. In addition we argue on the importance of such covariant formula from the viewpoint of experiment. In section (6) we end with pointing the open questions.

## 2 The most general form of the covariant drag force

Let us recall the relativistic constraint on the proper force exerted on a particle\(^7\)

\[
\hat{u}_\mu F^\mu = 0. \quad (2.1)
\]

This condition forces the covariant drag force to belong to the vector representation of the $SO(3)$ group orthogonal to the quark velocity $\hat{u}$.\(^8\) In Table 1 we have listed

\(^5\)Notice that at zero order the there exists just one independent vector term.

\(^6\)In [10], the corrections have been computed through a different manner.

\(^7\)This constraint was truly met at ideal order in (1.3).

\(^8\)This idea is reminiscent of the analogue idea used in preparing the first order energy-momentum tensor of relativistic fluid dynamics in Landau frame[12].
| Type    | Data                  | Evaluated in RF |
|---------|-----------------------|-----------------|
| Scalars | $S_1 = \tilde{u}.u(x)$ | $u_0(x)$        |
|         | $S_2 = b(x)$          | $b(x)$          |
| Vectors | $V_1^\mu = \tilde{p}^{\mu\nu}u_\nu(x)$ | $u^\mu(x)$ |

Table 1. Zero order data. $SO(3)$ scalars and vectors. Note that $\tilde{p}^{\mu\nu} = \tilde{u}^\mu\tilde{u}^\nu + \eta^{\mu\nu}$ is the projection tensor on the three dimensional space-time orthogonal to $\tilde{u}^\mu$. All scalars and vectors constructed by $u^\mu(x)$, $b(x)$ and $\tilde{u}^\mu$ compatible with (2.1). As it is clearly seen, there is only one appropriate vector term at ideal order. So the covariant drag force at ideal order must be as the following:

$$F_{\text{ideal}}^\mu(x) = \xi(S_1, S_2) \tilde{p}^{\mu\nu}u_\nu(x)$$

(2.2)

with $\xi(S_1, S_2)$ the unknown scalar coefficient function.

In Table 2 we have organized one derivative data. There exist both scalar and vector derivative terms contributing to first order. Scalar one derivative terms can contribute to the scalar factor of $F_{\text{ideal}}^\mu(x)$, while every vector one derivative term may separately contribute to drag force as an additive term. So to first order in derivative correction, the most general covariant drag force is constructed by adding a linear combination of $s_i$ to $\xi(S_1, S_2)$ accompanied by adding a linear combination of $v^\mu_i$ to $\xi(S_1, S_2)$-corrected drag force as the following:

$$F^\mu(x) = \left(1 + 3 \sum_{i=1}^3 \alpha_i(S_1, S_2) s_i\right) F_{\text{ideal}}^\mu(x) + \sum_{i=1}^3 \beta_i(S_1, S_2) v^\mu_i.$$  

(2.3)

Our goal is to determine the six coefficient functions, $\{\alpha_i(S_1, S_2)\}$ and $\{\beta_i(S_1, S_2)\}$. It may be interesting to note that the condition (2.1) is automatically satisfied once we use the $SO(3)$ decomposition. It is because both $F_{\text{ideal}}^\mu(x)$ and $v^\mu_i$ are transverse to $\tilde{u}^\mu$, by construction.

Before ending this section let us explain how we have arranged derivative terms in different sectors of $SO(3)$ group in Table 2. Recall that the basic one derivative data are $\partial_\mu b$ and $\partial_\mu u_\nu$. Derivative terms in the scalar sector may be constructed by contracting these basic data with each of $u^\mu$, $\tilde{u}^\mu$ or $\eta^{\mu\nu}$. The reason is that scalar terms are supposed to multiply by $F_{\text{ideal}}^\mu(x)$ which has been originally constructed compatible with (2.1). In the case of vector terms however, we must contract $\tilde{p}^{\mu\nu}$ with one-free-index derivative data.

### 3 Drag force in a thermal state of $\mathcal{N} = 4$ SYM plasma

The starting point of the computations in hydrodynamic regime is to study the problem in a thermal state. In this section we will investigate the motion of a quark
in a strongly coupled thermal field theory through studying its dual gravity picture. The problem in gravity side is to study the dynamics of a classical string with a specific boundary condition in an asymptotically AdS5 background. Like [4] and [5], we demand one end point of the string is to be constrained to move with a constant velocity on the boundary. It is also required that the string to be trailed into the bulk all the way from boundary to the horizon.

### 3.1 Dynamics of a classical string and drag force

In order to study the dynamics of string, we have to solve the equations of motion (EoM) related to string’s degrees of freedom. Consider the Nambu-Goto action:

\[ S = - \frac{1}{2\pi\alpha'} \int d\tau d\sigma \mathcal{L} = - \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-g} \quad \sigma \in [0, \sigma_1] \]  

with \( g = \det g_{\alpha\beta} \) and \( g_{\alpha\beta} = G_{MN} \partial_\alpha x^M \partial_\beta x^N \) the induced metric on the world-sheet. \(^9\)

We choose \( \tau = t \) and \( \sigma = r \) to fix the reparametrization freedom on the world-sheet. So the most general embedding of the string in the mentioned background may be given in the following form:

\[ x^M_{\text{sol}}(r, t) = (r, t, X_x(r, t), X_y(r, t), X_z(r, t)) \]  

where \( X_i(r, t), i = x, y, z \) are the string’s degrees of freedom. Throughout this paper, we always prefer to write EoM in the form of world-sheet currents conservation law:

\[ \partial_\alpha \Pi^\alpha_M = f_M \quad \alpha = t, r \]

\(^9\)Latin indices \( \{M, N, \cdots \} \) have been used to denote bulk directions \( (r, t, \vec{x}) \), while \( \alpha \) and \( \beta \) indices refer to world-sheet coordinates \( (\tau, \sigma) \).
where
\[
\Pi^\alpha_M = \frac{\partial \mathcal{L}}{\partial (\partial_\alpha x^M)} = -\frac{1}{2\pi \alpha'} \sqrt{-g} \, P^\alpha_M, \quad f_M = \frac{\partial \mathcal{L}}{\partial x^M}.
\] (3.4)

Let us quickly recall that in our picture, the end point of the string on the boundary is identified with the quark in the plasma. So in order to compute the drag force we have to compute the momentum current going down into the bulk at this end point. It would be nothing except the ingoing momentum flux crossing the time-like boundary of world-sheet, \( \sigma = \sigma_1 \text{ path} \). Assuming \( n^\alpha \) as the inward normal vector to this path, it turns out that:
\[
f^\mu = (\frac{dE}{dt}, d\vec{p} dt) = -\Pi^\mu_\alpha n^\alpha. \tag{3.5}
\]

Since we are interested in firstly computing \( F^\mu \), we need to an object that behaves covariantly under the Lorentz transformations on the boundary. From (3.4) and \( P^M_\alpha = \partial_\alpha x^M \), it is clear that \( \Pi^\mu_\alpha n^\alpha \) has not our desired property. Instead, \( P^\mu_\alpha n^\alpha \) transforms covariantly and we can write:
\[
F^\mu = \gamma (\frac{dE}{dt}, d\vec{p} dt) = \frac{1}{2\pi \alpha'} P^\mu_\alpha n^\alpha. \tag{3.6}
\]

Our gauge choice explained above (3.2) leads to \( n_\alpha = (0, -1) \). So the only thing which we need to compute is the value of \( P^r_\mu \) on the boundary.

Our strategy is to solve (3.3) for three spacial boundary directions \( i \in \{x, y, z\} \). Since the Lagrangian of the string includes \( G_{MN}, X_i'(r, t) \) and \( \dot{X}_i(r, t) \) \(11\) (see (A.1)), we can choose one of the spacial directions, i.e. \( m \), and write the Lagrangian in the following simple form:
\[
\mathcal{L} = \sqrt{A_m X_m'^2 + 2B_m X_m' + C_m}.
\] (3.7)

Expectedly, \( A_m, B_m \) and \( C_m \) are expressed in terms of \( \dot{X}_m(r, t), \dot{X}_i(r, t) \) and \( X_i'(r, t) \) where \( \{i, j, ...\} \) indices refer to the other two spacial directions (A.2). This choice of notation extremely simplifies our next computations. Instead of solving EoM for each of \( m \) directions, we derive \( X_m' \) from the \( \Pi^r_m = \partial \mathcal{L}/\partial x'_m \):
\[
X'_m = -\frac{B_m}{A_m} \pm \frac{\pi_m}{A_m} \sqrt{\frac{B^2_m - A_mC_m}{\pi^2_m - A_m}}, \quad \pi_m = -2\pi \alpha' \Pi^r_m. \tag{3.8}
\]

Then we demand the right hand side to be defined in the whole range of the radial coordinate from the boundary to the horizon. This requirement fixes the value of \( \Pi^r_m \) at just one depth in the bulk, namely the world-sheet horizon. Having the value of \( \Pi^r_m \) at the world-sheet horizon, one can integrate equation (3.3) to find \( \Pi^r_m \) and thereby \( P^r_\mu \) on the boundary.

---

10 Greek indices \( \mu \in \{t, x, y, z\} \) refer to boundary directions.
11 In this note, prime and dot stand for the derivative with respect to \( r \) and \( t \).
| Direction | Components of $F^\mu$ | The string profile |
|-----------|-------------------|------------------|
| transverse | $F^x = F^y = 0$ | $X_x(r) = X_y(r) = 0$ |
| boost | $F^z = \frac{\pi}{\sqrt{3}} \frac{\pi}{\sqrt{3}}$ | $X_z(r) = u_z b \left( \arctan(br) - \frac{\pi}{2} \right)$ |

**Table 3.** Review of the results for a quark in its RF in a thermal boosted plasma (in the Eddington-Finkelstein coordinates).

In the next subsection we briefly explain how to compute the drag force in a thermal plasma. Such computation will be essentially the same as computing drag in the fluid dynamics regime at zero order in derivative expansion.

### 3.2 Quark in a globally boosted thermal plasma

Consider a thermal state in a $\mathcal{N} = 4$ SYM plasma which has been boosted with a global velocity $u^\mu$. The gravity dual to this state in AdS side may be written as:

$$
\begin{align*}
    ds^2 &= G_{MN} dx^M dx^N = \frac{dr^2}{r^2 f(br)} + r^2 (P_{\mu\nu} - f(br) u_\mu u_\nu) dx^\mu dx^\nu, \\
    f(r) &= 1 - \frac{1}{r^4}, \quad b = \frac{1}{\pi R^2 T}
\end{align*}
$$

(3.9)

where, $p_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}$ is the projector tensor orthogonal to $u^\mu$ and $\eta_{\mu\nu} = (-1, 1, 1, 1)$ is the flat boundary metric. In a Lorentz frame where plasma is at rest, $u^\mu = (1, 0, 0, 0)$ and so (3.7) changes to the familiar form of an AdS5 black brane metric [4, 5]. It should be noted that the metric given above is a four parameter family of solutions for the Einstein equations.

For our next requirements, it is necessary to change the coordinates to the Eddington-Finkelstein coordinates and implement the computations therein. The metric (3.9) in the Eddington-Finkelstein coordinates takes the following form:

$$
\begin{align*}
    ds^2 &= G_{MN} dx^M dx^N = -2 u_\mu dx^\mu dr + r^2 (P_{\mu\nu} - f(br) u_\mu u_\nu) dx^\mu dx^\nu.
\end{align*}
$$

(3.10)

We are interested in studying a quark in its RF where $\tilde{u}^\mu = (1, 0, 0, 0)$. So $u^\mu$ is the global velocity of the plasma in the RF of the quark. For the sake of simplicity we also assume $u^\mu = (u^0, 0, 0, u^z)$. In order to solve the EoMs, we should attend to this point that under transforming (3.9) to (3.10), all boundary coordinates remain unchanged and only the time coordinate in the bulk transforms. Accordingly, we must change the world-sheet time to Eddington-Finkelstein time as well.

In (A.3) we have discussed in detail how to compute the components of the drag force in a globally boosted thermal plasma. We have presented the results in Table 3.

---

12 $T$ is the Hawking temperature of the AdS black brane and $R$ is the radius of AdS which from now on, we get it equal to one in our formulas.

13 In [10], it has been discussed in detail the importance of choosing this coordinates.

14 We have assumed that quark is located at $\vec{r} = 0$ on the boundary.
4 Drag force in the hydrodynamic regime of $\mathcal{N} = 4$ SYM plasma

After determining the most general form of the covariant drag force in section (2), our next goal is to determine the unknown coefficient functions in (2.3) in the $\mathcal{N} = 4$ SYM field theory. On the other hand, Fluid/Gravity duality provides the dual gravity of a general fluid dynamical flow in this field theory. So in this section we first briefly review the the Fluid/Gravity duality and then explain in detail how to implement perturbative computations on a classical string in gravity side.

4.1 Review of the Fluid/Gravity duality

Fluid dynamics studies long-wavelength perturbations in a thermal system. This feature can be captured in local derivative expansion of physical quantities, i.e. energy density, pressure, etc.. The order of every term in this expansion is determined by number of its derivatives. Roughly speaking, the ratio of every term in a definite order to the terms in one lower order, is of the order of $1/LT$, with $L$ the scale over which the thermodynamic variables vary significantly. The system will be in local equilibrium state if

$$\frac{1}{LT} \ll 1.$$  \hspace{1cm} (4.1)

In this limit the scale of variations is so long that in each of fluid patches\textsuperscript{15} thermodynamic is dominant. So it makes sense to promote the equilibrium degrees of freedom\textsuperscript{16} to become local functions of space and time. Local conservation of energy-momentum tensor and thermodynamic equation of state will completely describe the evolution of the system in the limit (4.1). This is ideal fluid description or zero order fluid dynamics.

The greater the above ratio is, the more deviation from local equilibrium would arise. So it will be needed to at least take the one derivative terms into account in the expansion. In more quickly varying flows where $1/LT$ becomes greater, one also might have to go beyond the first order and correct the expansion by adding terms with more than one derivative.

In [9] it has been shown that there is a one to one map between fluid dynamical flows of a $\mathcal{N} = 4$ SYM field theory on the boundary and long-wavelength perturbations of an AdS5 black brane in the bulk. This duality, i.e. the Fluid/Gravity duality, is able to construct the five dimensional gravitational background dual to every given four dimensional boundary flow, perturbatively order by order in a boundary derivative expansion.

\textsuperscript{15}A region with the size of effective $l_{mfp} \sim 1/T$.

\textsuperscript{16}In 4-dim there are four of them, $T$ and three of $u^\mu$, $(u^\mu u_\mu = -1)$. 
4.1.1 Dual gravity of a boundary flow

Fluid/Gravity duality has been originally constructed for a general fluid dynamical flow on the boundary without using its profile. The idea is to restrict the computations to be implemented within just one boundary patch. Fixing the value of the temperature and velocity at one point, one can find the fluid profile over the whole patch points via Taylor expansion. On the other hand using Eddington-Finkelstein coordinates allows to extend this boundary patch into the bulk through a tube-wise region and so all computations in gravity side reduces to solving Einstein equations in this tube perturbatively, for full range of the radial coordinate. The final step is to covariantize the obtained metric in boundary directions.

Requiring the slowly varying condition on $u^\mu(x)$ and $T(x)$, one may expect every tube-wise region to become a local black brane solution (3.10), at zero order:

$$ds^2(0) = G_{MN}^{(0)} dx^M dx^N = -2u_\mu(x^\alpha)dr dx^\nu + \frac{r^2}{R^2}(P_{\mu\nu}(x^\alpha) - f(b(x^\alpha)r)u_\mu(x^\alpha)u_\nu(x^\alpha)) dx^\mu dx^\nu$$

(4.2)

where $P_{\mu\nu}(x^\alpha) = u_\mu(x^\alpha)u_\nu(x^\alpha) + \eta_{\mu\nu}$.

Going to next orders in perturbation, $u^\mu(x^\alpha)$ and $T(x^\alpha)$ will not be constant over the patch points though. In any order of perturbation, we have to add some appropriate derivative terms to the metric to make it a tube-wise solution in the bulk to the given order. Note that added terms are generically constructed by the derivatives of $u^\mu$ and $T$. The first order corrections to the metric, $G_{MN}^{(1)}$ elements, are given by:

$$ds^2(1) = r^2 b F(r) \sigma_{\mu\nu} dx^\mu dx^\nu + \frac{2}{3} r u_\mu u_\nu \partial_\lambda u^\lambda dx^\mu dx^\nu - r u^\lambda \partial_\lambda (u_\mu u_\nu) dx^\mu dx^\nu$$

(4.3)

where

$$F(r) = \frac{1}{4} \left[ \ln \left( \frac{(1 + r)^2 (1 + r^2)}{r^4} \right) - 2 \arctan(r) + \pi \right]$$

(4.4)

and

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \partial_{(\alpha} u_{\beta)} - \frac{1}{3} P^{\mu\nu} \partial_\alpha u^\alpha.$$  

(4.5)

In next subsection we argue how to use (4.2) and (4.3) to implement perturbative computations in a patch.

4.1.2 Nature of derivative expansion

Let us restate that in hydrodynamic expansion, the order of every term is determined by number of its derivatives. In order to explicitly observe the order of different terms, it would be useful to regard $u$ and $b$ as functions of $\epsilon x^\alpha$ where $\epsilon$ is a parameter that will be finally set to be one. Since every derivative of $u$ and $b$ produces a power of $\epsilon$, powers of $\epsilon$ count the order of terms in the expansion.
According to our discussion in previous subsection, we have to perform our computations within one patch. The reason for this is that we are studying the drag force in a general flow with unknown dynamical profile. Practically we must express every dynamical term in a Taylor expansion around an arbitrary point in the patch. Let us clarify it by an example. Suppose we want to expand a metric element, $H$, in derivative expansion up to first order. In contrast to the AdS black brane case, $H$ is not constant here and generically is a function of boundary coordinates through fluid variables, $\lambda_p(t, \vec{x}) = \{u^\mu(x^\alpha), b(x^\alpha)\}$, and also of radial coordinate, $r$. So $H$ may be expanded in a patch as following:\[\]

\[
H(\lambda_p(t, \vec{x}), r) = H^{(0)}(\lambda_p(0, 0), r) + \epsilon (\vec{x} \partial_x + t \partial_t)\lambda_p(0, 0) \frac{\partial H^{(0)}}{\partial \lambda_p}(\lambda_p(0, 0), r) + \epsilon H^{(1)}(\lambda_p(0, 0), r) + O(\epsilon^2).\]

In the first line we have demonstrated the zero order part of $H$ that is exactly the same as $H$ in an AdS black brane metric with $\lambda_p = \lambda_p(0, 0)$. This may simply recall that zero order fluid dynamics in a patch is nothing except thermodynamics. The first order corrections of $H$ have been written in the second line; while the first term comes from the Taylor expanding, the second one is a non trivial pure hydrodynamic correction. Such non trivial correction is a combination of $\partial \lambda_p$ terms which can be specified via the Fluid/Gravity duality.

We can resume Taylor expansion and write $H$ in the following compact form:

\[
H(\lambda_p(t, \vec{x}), r) = H^{(0)}(\lambda_p(t, \vec{x}), r) + \epsilon H^{(1)}(\lambda_p(0, 0), r) + O(\epsilon^2) \quad (4.6)
\]

The really important point with this statement is that the first term does not have to be taken into account as a fully local term. It must be only regarded as a local function of $x^\alpha = (t, \vec{x})$ up to first order in Taylor expansion. So from now on, we make the following convention: We mean by every local function, the summation of zero and first order terms in its Taylor expansion.

### 4.2 Quark in the hydrodynamic regime of $\mathcal{N} = 4$ SYM plasma

Let us recall that as far as the string profile in the bulk does not extend beyond a tube, one can proceed to perturbatively solve the string EoMs. In \cite{10} we showed that using Eddington-Finkelstein coordinates, the string always lies in one tube and perturbative computations is meaningful.

\[17\] We have labelled different variables with $p$.

\[18\] Notice that we have expanded $H$ around the point $(t, \vec{x}) = (0, 0)$. 

10
4.2.1 Expanding the metric to first order

Now we exploit what we have explained in previous subsection to rewrite the first order corrected metric. Suppose at $t = 0$ at the quark position, $\mathbf{x} = 0$, in its RF, fluid velocity is given by:

$$u^\mu(0,0) = (u^0, 0, 0, u^z);$$ (4.7)

so the corrected metric in the neighbourhood of the quark can be written as:

$$ds^2 = -2u_0 \, dr \, dt - 2u_z \, dr \, dz + \frac{u_0^2}{r^2 b^4} \, dt^2 + \frac{u_z^2}{r^2 b^4} \, dz^2 + \frac{u_0 u_z}{r^2 b^4} \, dt \, dz + r^2 \eta_{\mu \nu} \, dx^\mu \, dx^\nu + G^{(Tayl)}_{MN} \, dx^M \, dx^N + G^{(1)}_{MN} \, dx^M \, dx^N.$$ (4.8)

As a useful point, in next subsection we will show in detail that it will be possible to implement the computations without dealing with “Taylor” corrections. Before proceeding we discuss about $G^{(1)}_{MN}$ elements now. These are pure hydrodynamic corrections that can be simply computed via (4.3) with applying the useful expressions given below:

$$\sigma_{tz} = \frac{u_z}{u_0} \, \partial_z u_z + u_z \, \partial_t \, \ln b$$
$$\sigma_{tt} = \frac{u_z^2}{u_0^2} \, \partial_z u_z - \frac{u_z^2}{u_0} \, \partial_t \, \ln b.$$ (4.9)

There is a delicate point with the expressions given above. In the Fluid/Gravity duality, derivative corrections at first order have been introduced in terms of velocity derivatives. The temperature derivatives have been eliminated in favour of them via using fluid dynamics EoMs. However as it is clear in Table 2 we may deal with temperature derivative terms throughout our computations, i.e. $s^2$. Hence we must reuse EoMs to restore such terms in all possible expressions. Using

$$\partial_\mu T^{\mu \alpha}_{(0)} = 0, \quad \partial_\mu T^{\mu z}_{(0)} = 0$$ (4.10)

one can simply show: 19

$$\partial_z b = \frac{b}{u_0^2} \, u^\alpha \partial_\alpha u_z + \frac{u_z}{u_0} \, \partial_t b$$
$$\partial_\mu u^\mu = \frac{3}{u_0} \left( \frac{u_z}{u_0} \, u^\alpha \partial_\alpha u_z - \partial_t \, \ln b \right).$$ (4.11)

This is why we see only $\partial_t b$ and $\partial_\alpha u_z$ in (4.9). Respecting all considerations mentioned above, we will compute the drag force in the background with metric (4.8).

Before start trying to compute the drag force, we will briefly discuss about structure of our perturbative method in next subsection.

---

19 Notice that $\sigma_{tz}$ and $\sigma_{tt}$ are not involved with the derivative terms in the other two equations $\partial_\mu T^{\mu x}_{(0)} = 0$ and $\partial_\mu T^{\mu y}_{(0)} = 0$. 

11
4.2.2 Structure of the perturbative method

In order to compute the drag force in the hydrodynamic regime we should first expand $A_m$, $B_m$ and $C_m$ in (3.8) up to first order in derivative. This expanding includes both metric elements and components of the string profile, see (A.2). In (4.1.2) we explained how to expand metric elements in a patch. However the situation is a little bit different in the case of the string profile. Let us clarify this point now.

We may naturally expect:

$$X_i(r, t) = X_i^{(0)}(r, t) + \epsilon X_i^{(1)}(r, t) + O(\epsilon^2)$$  \tag{4.12}

where the process of generating zeroth order solution of the string is the same as producing zero order metric (4.2). We must get the solution of the string in a thermal system and replace thermodynamic variables with local hydrodynamic functions in it, i.e.:

$$X_x^{(0)}(r, t) = 0$$
$$X_y^{(0)}(r, t) = 0$$
$$X_z^{(0)}(r, t) = u_z(t, X_i) b(t, X_i) \left[\arctan(b(t, X_i)r) - \frac{\pi}{2}\right].$$  \tag{4.13}

The very important point with this expression is the presence of $X_z(r, t)$ not only in left hand side, but also in the argument of the hydrodynamic variables in right hand side! Notice that $X_i(r, t)$ in right hand side is a function of $u_z(t, X_i)$ and $b(t, X_i)$; $X_i$ itself; in the argument of these variables is also function of $u_z(t, X_i)$ and $b(t, X_i)$ and so on. Hence this equation behaves as a recursive formula for $X_z(r, t)$.

According to discussion below (4.6), the locality of (4.13) is valid up to the first order in Taylor expansion around every arbitrary point. It means that whenever (4.13) is considered in a patch, it will also include a first order part. So up to first order in derivative expansion we may write:

$$X_z(r, t) = u_z(0, 0) b(0, 0) \left[\arctan(b(0, 0)r) - \frac{\pi}{2}\right] + \epsilon X_z^{(Taylor)}(r, t) + \epsilon X_z^{(1)}(r, t) + O(\epsilon^2)$$  \tag{4.14}

In both one derivative corrections above, the hydrodynamic variables and their derivative can be computed at any arbitrary point, for instance at $(0, 0)$. Our strategy is to perturbatively compute $X_z^{(1)}(r, t)$ via correcting the world-sheet horizon location, i.e. $r^*$. As we described in section(3.2), the expression $B_m^2 - A_m C_m$ has a simple root, $r^*$, in the bulk of AdS black brane. Our main idea in hydrodynamic regime is that $r^*$ is perturbatively corrected order by order in derivative expansion. In what
follows we try to clarify some important points with perturbatively solving $E_m = B^2_m - A_mC_m = 0$.

In the language of section (4.1.2), we are interested in finding $r^*$, the root of $E_m$. So we must compute $r^*(t)$ in the following equation up to first order:

$$E_m \left( \lambda_p(t, X_i^* (r^*(t), t)) , r^*(t) \right) = 0. \quad (4.15)$$

It should be noted that the time dependency of $r^*$ is originated from its dependence on fluid variables, i.e. $\lambda_p(t, \vec{x})$. Carefully rewriting:

$$r^*(t) := r^* \left( \lambda_p(t, X_i^* (r^*(t), t)) \right) \quad (4.16)$$

where $r^*(t)$ is clearly present in both sides. As a result, this equation only expresses $r^*(t)$ implicitly. However up to first order in derivative expansion we can derive it as:

$$r^*(t) = r^*_{(0)}(\lambda_p(0,0)) + \epsilon \left( X_i^* \partial X_i^* + t \partial t \right) \lambda_p(0,0) \frac{dr^*_{(0)}}{d\lambda_p}(\lambda_p(0,0)) + \epsilon r^*_{(1)}(\lambda_p(0,0)) + O(\epsilon^2).$$

In this equation $r^*_{(0)}(\lambda_p(0,0)) := r^*_{(0)}$ is nothing except the $r^*$ introduced below (A.6) whose $\lambda_p$ terms have been replaced with $\lambda_p(0,0)$. Now we plague $r^*(t)$ in (4.15) and expand $E_z$ up to first order:

$$E_z = E_z^{(0)}(\lambda_p(0,0), r^*_{(0)}(0)) + \epsilon \left( X_i^* \partial X_i^* + t \partial t \right) \lambda_p(0,0) \frac{\partial E_z^{(0)}}{\partial \lambda_p}(\lambda_p(0,0), r^*_{(0)}(0)) + \epsilon r^*_{(1)}(0) \frac{\partial E_z^{(0)}}{\partial r^*}(\lambda_p(0,0)) + \epsilon E_z^{(1)}(\lambda_p(0,0), r^*_{(0)}(0)) + O(\epsilon^2). \quad (4.17)$$

As it is clear, the second term in the first line and the term in the second line are first order Taylor terms. Interestingly, they cancel out each other. $E_{(0)}$ as the only zero order term in $E_z$ gives $r^*_{(0)} = r^*_{(0)}(\lambda_p(0,0)) = constant$. The remaining two terms in the third line determine $r^*_{(1)}$.

In summary, we need only to work out the value of $B^2_m - A_mC_m$ and its pure hydrodynamic correction at the position of the quark. Setting $E_z$ to zero, the former one gives $r^*_{(0)}$ and $r^*_{(1)}$ can be computed via hydrodynamic correction part.

### 4.2.3 Drag force in the boost direction

Let us denote that by the boost direction here, we mean the direction of the fluid velocity at the quark position in its RF. Having used (3.4) and (3.6), it is clear that

$$F^z = F_z = \lim_{r \to \infty} P^r_z = \lim_{r \to \infty} \left( \frac{1}{\sqrt{-g}} \Pi_z^r \right). \quad (4.18)$$

We have used the standard relation $y_z' = - \partial_z F(x, y) / \partial_y F(x, y)$. 

---

13
To compute the covariant drag in the boost direction we only need to know the value of $\Pi^r_z$ on the boundary.

Following our explanations in last subsection, we try to specify zero and first order parts of $A_z$, $B_z$ and $C_z$ with $\lambda_p$ terms computed at the quark position. Instead of listing these coefficients here we directly proceed to compute $B_z^2 - A_z C_z$ with $u$ and $b$ computed at the position of the quark (see (A.4.1) for more details). Setting (4.17) to zero, $r^*_0$ and $r^*_1$ turn out to be as the following:

\[
 r^*_0 = \frac{\sqrt{u^0/b}}{b}, \quad r^*_1 = \frac{1}{4} r^3(z_0) \left[ \frac{2 u_0 u_z}{b^4} X_z^0(r^*_0) + \frac{u^0}{b^2} C_{tt}^{(1)}(r^*_0) \right]. \tag{4.19}
\]

Now we demand (3.8) to be well-defined for the whole range of the radial coordinate. It simply results in fixing the value of $\pi_z$ at the corrected world-sheet horizon, $r^*_0(0, t)$, up to first order. Considering the true direction for the energy and momentum flow on the world-sheet $[10]$, we may write:

\[
 \pi_z \left( r^*(t), \lambda_p(t, X^*_i) \right) = -\sqrt{A_z^0(r^*_0)} - \epsilon \frac{A_z^{(0)}(r^*_0) + r^*_1 A_z^{(0)'}(r^*_0) + A_z^{(1)}(r^*_0)}{2 \sqrt{A_z^0(r^*_0)}} \tag{4.20}
\]

where

\[
 A_{z(Taylor)}(r^*_0) = 4 r^3(z_0) \left( X^*_i \partial X_i + t \partial_t \right) \lambda_p \frac{dr^*_0}{d\lambda_p} + \frac{4}{b^3} \left( X^*_i \partial X_i + t \partial_t \right) b \tag{4.21}
\]

is the only time dependent term in (4.20). According to (4.16), $\pi(r^*(t))$ is a function of the fluid variables computed at the world-sheet horizon on the boundary, i.e. $(t, X^*_i)$. However the drag force is related to the fluid variables computed at the quark position. To find $\pi_z$ at the desired point we should integrate (3.3) for $M = z$ from the corrected world-sheet horizon to the boundary:

\[
 \Pi^r_z \left( r, \lambda_p(t, X_i) \right) = \Pi^r_z \left( r^*(t), \lambda_p(t, X^*_i) \right) + \int_{r^*(t)}^{r} (f_z - \partial_t \Pi^r_z) \, dr. \tag{4.22}
\]

There are two important points regarding this equation. Firstly, the integral term in right hand side appears at first order. The reason for this is that both terms in the integrand are boundary derivative terms. So we must also take into account just the zero order part of $r^*$ in the lower limit of the integral. Secondly, computing $\Pi^r_z$ at the the quark position, $X_i = 0$ at $t = 0$, removes explicit time dependency from (4.22) (see (4.21)).

After rather tedious simplifications one reaches the following formula:

\[
 F^r_z(\lambda_p(0, 0)) = F^r_{(0)} + \epsilon F^r_{(1)} + O(\epsilon^2)
 = -\frac{1}{2\pi \alpha'} \left( -\frac{u_z}{b^2} + A \partial_t u_z + B \partial_z u_z + C \partial_t b \right). \tag{4.23}
\]
with \( u_0, u_z \) and \( b \) the fluid variables at \((t, \vec{x}_i^j) = (0, 0)\) and

\[
\begin{align*}
A &= \frac{\sqrt{u^0}}{b} - \mathcal{U} \frac{u_z^2}{u_0}, \\
B &= \frac{1}{b} \frac{u_z}{\sqrt{u^0}} + \mathcal{U} \frac{1 + u_z^2}{u_0^2} - \frac{F(\sqrt{u^0}) (1 + u_0^2)}{b u_0^2} u_z, \\
C &= -\frac{\sqrt{u^0}}{b^2} u_z - 2 \mathcal{U} \frac{1}{b^2} \frac{u_z}{u_0} + \frac{F(\sqrt{u^0}) 1 + u_0^2}{b^2} u_0 u_z.
\end{align*}
\]

(4.24)

One might worry about the absence of the other derivative terms in (4.23), i.e. \( \partial_x u_z \). The point is that we have used the fluid dynamics equation of motion to eliminate such terms. In our computations, the only source of these terms is \( \partial_\mu u^\mu \) appearing in metric corrections. It is pretty simple to replace such term with the above-mentioned derivatives in (4.23) by use of (4.11).

In (A.4), we have listed all intermediate steps related to the computations of this subsection.

### 4.2.4 Drag force in the transverse directions

Notice that by the transverse directions here, we mean the directions orthogonal to the fluid velocity at the quark position in its RF. Similar to the boost direction, the only thing that we need to compute related to these directions is the value of \( \Pi^r_{x,y} \) on the boundary. We have to firstly know the zero and first order parts of \( A_T, B_T \) and \( C_T \) with fluid variables computed at \((t, \vec{x}_i^j) = (0, 0)\) (T stands for the transverse directions; i.e. x,y.). The crucial point that extremely simplifies the expansion of these coefficients is vanishing of the string profile in the transverse directions at zero order (See (A.5) for more details.). Consequently it can be shown that the drag force in transverse directions can be computed by use of just zero order parts of these coefficients listed below:

\[
\begin{align*}
A^{(0)}_T (\lambda_p(0,0)) &= r^4 f(br) - \frac{u_z^2}{b^4}, \\
B^{(0)}_T (\lambda_p(0,0)) &= 0, \\
C^{(0)}_T (\lambda_p(0,0)) &= 1.
\end{align*}
\]

(4.25)

Like the z direction case, we assume \( B^2_T - A_T C_T \) vanishes at \( r^*_T = r^*_T(0) + \epsilon r^*_T(1) \). It is simple to find \( r^*_T(0) \) and \( r^*_T(1) \) via expanding \( B^2_T - A_T C_T \) in the form of (4.17). As it was explained below (4.17), Taylor contributions would precisely cancel. Solving \( E_T = 0 \) at zero and first order separately, one simply reaches to:

\[
\begin{align*}
&\ r^*_T(0) = \frac{\sqrt{u^0}}{b}, \\
&\ r^*_T(1) = -\frac{A_T^{(1)}}{4 r^*_T(0)^3}.
\end{align*}
\]

(4.26)
Interestingly, by use of (A.18), it is pretty simple to show that
\[ r^*_T(1) = r^*_T(1). \]  
(4.27)

Undoubtedly, this equality can be regarded as a very good check of our computations. In analogy with [13], we have determined the location of the world-sheet horizon as a local function of the fluid dynamical variables, perturbatively up to first order in derivative expansion.

Now we must try to compute \( \pi_T \left( \tilde{r}^*_T(t), \lambda_p(t, X^*_i) \right) \) via expanding \( A_T \), similar to what we performed in (4.20). Due to presence of \( u_z \) in \( A_T^{(0)} \), we expect \( A_T^{(0)}(\tilde{r}^*_T(0)) \) not to be the same as \( A_z^{(0)}(\tilde{r}^*_z(0)) \) in (4.21). Instead it turns out to be:
\[
A_T^{(0)}(\tilde{r}^*_T(0)) = \frac{\partial A_T^{(0)}(r^*_T(0))}{\partial r^*_T(0)} (X^*_i \partial X_i + t \partial t) \lambda_p \frac{dr^*_T(0)}{d\lambda_p} + (X^*_i \partial X_i + t \partial t) \lambda_p \frac{\partial A_T^{(0)}(r^*_T(0))}{\partial \lambda_p} = 0.
\]  
(4.28)

Using (4.25), (4.26) and (4.28) one can simply show that:
\[
\pi_T \left( r^*_T(t), \lambda_p(t, X^*_i) \right) = O(\epsilon^2).
\]  
(4.29)

The final step is to derive \( \Pi_T \) at the position of the quark at \( t = 0 \). To this end, we must solve EoM for \( T \) directions; equations similar to (4.22) with quantities whose subscripts \( z \) has has been changed to \( T \). It can be simply seen that \( f_T \) and \( \partial_t \Pi_T \) do not contribute to EoM up to first order. Consequently, \( \Pi_T \left( r, \lambda_p(t, X_i) \right) \) turns out to be equal to \( \Pi_T \left( r^*_T(t), \lambda_p(t, X^*_i) \right) \) and as a result the transverse components of the drag force vanish to this order (See (??)):
\[
F^x(\lambda_p(0,0)) = F^y(\lambda_p(0,0)) = 0.
\]  
(4.30)

In (A.5) we have given intermediate steps in detail.

Let us summarise. In this section we studied a quark moving through a general flow of \( \mathcal{N} = 4 \) SYM field theory. Our main achievement was computing the components of the drag force exerted on quark, perturbatively, up to first order in derivative expansion. Our results were specially obtained in the quark’s RF at \( t = 0 \) once the fluid velocity at the quark position directed along \( z \) direction. Interestingly, the drag force in directions orthogonal to the fluid velocity vanished. This can be regarded as a prediction of the holography. In order to derive the drag force in next times, we must covariantize our results; just like deriving the global metric in gravity side dual to a boundary flow in the Fluid/Gravity duality.
| SO(3) Type | In dependent data | Evaluated in RF |
|------------|------------------|------------------|
| Scalars    |                  |                  |
| $s_1 = \tilde{u}^\mu \tilde{u}^\nu \partial_\mu u_\nu$ | $u_0^{-1} u_z \partial_t u_z$ |
| $s_2 = \tilde{u}^\mu \partial_\mu b$ | $\partial_t b$ |
| $s_3 = u^\mu \tilde{u}^\nu \partial_\mu u_\nu$ | $u_0^{-1} u_z (-u_0 \partial_t u_z + u_z \partial_z u_z)$ |
| Vectors    |                  |                  |
| $v^\mu_1 = \tilde{p}^{\mu\alpha} \tilde{u}^\beta \partial_\alpha u_\beta$ | $\tilde{p}^{\mu\alpha} \partial_\alpha u_0$ |
| $v^\mu_2 = \tilde{p}^{\mu\alpha} \tilde{u}^\beta \partial_\beta u_\alpha$ | $\tilde{p}^{\mu\alpha} \partial_\alpha u_\alpha$ |
| $v^\mu_3 = \tilde{p}^{\mu\alpha} u^\beta \partial_\beta u_\alpha$ | $\tilde{p}^{\mu\alpha} (u_\alpha \partial_z u_\alpha - u_0 \partial_\alpha u_\alpha)$ |

Table 4. First order derivative data evaluated in RF.

5 Covariant drag force in $\mathcal{N} = 4$ SYM field theory

In section(2), we presented the most general form of first order corrected covariant drag force in a general fluid flow with six unknown coefficient functions. In this section we find these coefficient functions in the $\mathcal{N} = 4$ SYM field theory, by use of the results obtained in last section.

5.1 Specification of the coefficient functions in $F^\mu$

In order to compare the general covariant formula with the results of section(4), we have to first rewrite the components of general $F^\mu$ under the same conditions where we obtained (4.23) and (4.30). Therefore we require components of $\tilde{F}^\mu$ in the quark RF at $(t, \vec{x}) = (0, 0)$, assuming $u^{\mu(0,0)} = (u^0, 0, 0, w^z)$ and $\tilde{u}^\mu = (1, 0, 0, 0)$. Denoting all these consideration and information listed in Table 4, we write spacial components of general $F^\mu$ at the quark position at $t = 0$

\[
F^x(\lambda_p(0,0)) = \sigma_1 \partial_x u_0 + \sigma_2 \partial_t u_x + \sigma_3 \partial_z u_x, \quad (5.1a)
\]

\[
F^z(\lambda_p(0,0)) = \frac{1}{2\pi\alpha'} \left( 1 + \kappa_1 \partial_t u_z + \kappa_2 \partial_z u_z + \kappa_3 \partial_\alpha b \right) \frac{u_z}{b^2}, \quad (5.1b)
\]

with

\[
\kappa_1 = -\alpha_3 u_z + \alpha_1 u_z u_0^{-1}, \quad \sigma_1 = \beta_1, \quad (5.2a)
\]

\[
\kappa_2 = \alpha_3 u_z^2 u_0^{-1}, \quad \sigma_2 = \beta_2 - u_0 \beta_3, \quad (5.2b)
\]

\[
\kappa_3 = \alpha_2, \quad \sigma_3 = \beta_3 u_z. \quad (5.2c)
\]

Let us first compare $F^{x,y}$ obtained from two sources explained above. Since we have used fluid dynamics EoMs to obtain the components of the general drag, all derivative terms appeared in (5.1a) amount to independent fluid data. In the $\mathcal{N} = 4$ SYM field theory, our results given in (4.30), forces $\sigma_i$ function to vanish. Clearly $\beta_i$ vanishes as well:

\[
\beta_1 = \beta_2 = \beta_3 = 0. \quad (5.3)
\]
In order to find $\alpha_i$ function, we should compare $F^z$ presented in (4.23) with (5.1b). The argument in the last paragraph of (4.2.3) assures us that the three different derivative terms appeared in (4.23) are fully independent. Correspondingly, the same derivative terms have been appeared in (5.1b). So the comparison results in three equations giving exactly four unknown $\alpha_i$ function as the following:

$$
\alpha_1 = -\frac{u_0^2}{u_z^3} b^2 \left( A \frac{u_z}{u_0} + B \right), \quad \alpha_2 = -\frac{1}{u_z^2} b^2 C, \quad \alpha_3 = -\frac{u_0}{u_z^2} b^2 B.
$$

(5.4)

Notice that $\alpha_i$ and $\beta_i$ functions are Lorentz scalar and it is sufficient to specify them in just one specific frame. We have chosen the quark RF. Having computed these coefficient functions in the quark RF, one may use Table 1 to restate them in the following fully covariant form:

$$
\alpha_1 = -\frac{S_1^2}{(S_1^2 - 1)^{3/2}} S_2^2 \left( \hat{A} \frac{(S_1^2 - 1)^{1/2}}{S_1} + \hat{B} \right),
$$

$$
\alpha_2 = -\frac{1}{(S_1^2 - 1)^{1/2}} S_2^2 \hat{C},
$$

$$
\alpha_3 = -\frac{S_1}{(S_1^2 - 1)^{3/2}} S_2^2 \hat{B},
$$

(5.5)

where $\{\hat{A}, \hat{B}, \hat{C}\}$ are the same as $\{A, B, C\}$ introduced in (4.24) with $u_0$ and $b$ replaced with $S_1$ and $S_2$ respectively.

5.2 Reproducing the result of arXiv:1202.2737 [10]

In [10] we have computed the drag force exerted on a transverse quark with zero rapidity in Bjorken flow. In equation (6.15) and (6.16) therein, we have presented the drag force in the quark RF. Although both [10] and the present work have common origins, there is a subtle difference which distinguishes them from each other. The point is that the procedure used in [10] was clearly dependent on the fluid profile. Due to the analogy between the boundary metric in the RF of transverse quarks and metric (7.10) in [14], we used the Forced-Fluid/Gravity duality to absorb the velocity of the Bjorken flow into the boundary metric. As a result we dealt only with local temperature as the only fluid variable in [10].

Now we want to reproduce those results by use of our general covariant drag formula obtained in this paper. Before proceeding, let us fix the notation: Consider the one dimensional Bjorken flow in the x direction:

$$
u^\mu(t, x, y, z) = \frac{1}{\sqrt{t^2 - x^2}}(t, x, 0, 0), \quad T(t, x, y, z) = \frac{T_0}{(t^2 - x^2)^{1/6}}.
$$

(5.6)
In the RF\textsuperscript{21} of a quark moving along the z direction with $\tilde{u}^{\mu} = (\tilde{u}^{0}, 0, 0, \tilde{u}^{z})$ the fluid velocity and temperature turn out to be
\begin{equation}
\begin{aligned}
b_{RF}(T, X, Y, Z) &= b_0 \left( \sqrt{(\tilde{u}^{0}T + \tilde{u}^{z}Z)^2 - X^2} \right)^{1/3}, \\
u_{RF}^{\mu}(T, X, Y, Z) &= (U^{0}, 0, 0, U^{Z}) \\
&= \frac{\tilde{u}^{0}T + \tilde{u}^{z}Z}{\sqrt{(\tilde{u}^{0}T + \tilde{u}^{z}Z)^2 - X^2}}\left( \tilde{u}^{0}, \frac{X}{\sqrt{(\tilde{u}^{0}T + \tilde{u}^{z}Z)^2 - X^2}}, 0, -\tilde{u}^{z} \right).
\end{aligned}
\tag{5.7}
\end{equation}

In order to use (4.23) and (4.24) we require compute three independent derivative terms. One can simply show that in the quark position, $X = 0$,
\begin{equation}
\partial_{T} b_{RF} = \dot{b} \text{ is the only derivative term contributing to the drag in RF. Now we can rewrite (4.23) and (4.24) as following:}
\end{equation}
\begin{equation}
\begin{aligned}
F_{Z}^{1} &= \frac{1}{2\pi\alpha'} \frac{U^{Z}}{b_{RF}^{2}} \left[ (2 \arctan \sqrt{U^{0}} - \pi) \frac{1}{U^{0}} \dot{b} \\
&+ \frac{1}{2\pi\alpha'} \frac{U^{z}}{b_{RF}^{2}} \left[ F(\sqrt{U^{0}}) \left( \frac{1 + U_{0}^{2}}{U^{0}} \right) - \sqrt{U^{0}} \right] \dot{b} \right].
\end{aligned}
\tag{5.9}
\end{equation}

In order to reproduce the drag force in desired form we first use equations (6.20) in [10] to simplify the first line of (5.9):
\begin{equation}
\frac{1}{U^{0}} \dot{b}_{RF} = \frac{U_{0}^{2} - U_{z}^{2}}{U_{0}} \dot{b}_{RF} = (-U_{0} \dot{b}_{RF} + U_{z} \dot{b}_{RF}'). \tag{5.10}
\end{equation}

In second line we only need to change $\dot{b}$. By use of (5.6) it can be replaced with:
\begin{equation}
\dot{b} = U^{0} \frac{b}{3T}. \tag{5.11}
\end{equation}

Plaguing (5.10) and (5.11) in (5.9), $F_{Z}^{1}$ will be turned out to be in complete agreement with (6.15) and (6.16) in [10]. In the convention of [10]:
\begin{equation}
F^{Z} = F_{(0)}^{Z} + F_{(1)}^{Z} = \frac{1}{2\pi\alpha'} \frac{U^{Z}}{b_{RF}^{2}} \left( 1 + A U_{0} \dot{b} + B U_{Z} \dot{b}' + C \frac{b_{RF}^{2}}{T} \right), \tag{5.12}
\end{equation}

with
\begin{equation}
\begin{aligned}
B &= -A = 2 \mathcal{U}, \text{ } \\
C &= \frac{1}{3}(U^{0}\sqrt{U^{0}} - (1 + U_{0}^{2})F(\sqrt{U^{0}})).
\end{aligned}
\tag{5.13}
\end{equation}

Finally, it would be interesting to discuss a little bit more about the nature of the correction terms in (5.12). As we saw in (5.8), there can not be present any correction terms with derivative of the velocity in RF. One does not have to think that

\textsuperscript{21}Let us take the coordinates in this frame as $(T, X, Y, Z)$. 

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this is the only representation of derivative terms for Bjorken flow. Fluid dynamics equations of motion allows us to present derivative terms in some various combinations. For instance the second line of (5.9) which seems as a $b$ derivative term here, has appeared from $\partial u$ term in [10]:

$$\partial_\mu u^\mu = \partial_\mu \left( \frac{t}{\sqrt{t^2 - x^2}}, \frac{x}{\sqrt{t^2 - x^2}}, 0, 0 \right) = \frac{1}{\sqrt{t^2 - x^2}} = \frac{1}{\tau}. \quad (5.14)$$

Such freedom in presenting the correction terms can be used to exchange the derivative terms in same order with each other. As we have shown in (4.11), we have always replaced the divergence of the velocity with some other derivative terms throughout our computations\textsuperscript{22}.

6 Open questions

Our result reported in this paper may be encountered with several follow-up questions. We list some of them below and leave the answers to our future work.

Our main result in this paper is finding the general form of the covariant drag force exerted on a quark moving through a general fluid flow perturbatively, up to first order in derivative expansion. As the first extension one may be interested in finding the second order derivative corrections. Basically it would be a solvable problem. To proceed, firstly the most general two derivative terms contributing to (2.3) have to be determined. These terms can be generally classified in two sets. The first set of terms are independent two derivative data, $I_2$ data, that have been listed in Table 5\textsuperscript{23}. The second set, namely the $I_{1,1}$ data, contains the products of one derivative terms. This set of two derivative terms has been organized in table 6.

Accordingly, there are 17 scalar terms and 14 vector terms which may contribute to the covariant drag force at second order. In order to specify the 31 unknown coefficient functions corresponding to these two derivative terms, we must implement the second order analogue of our computations in section(4)\textsuperscript{24}. Will the pure vectorial corrections be absent at this order as well as their absence at first order(See (5.3))? If so, What can be the reason behind it? Any answer to this question will be a prediction in the framework of holography, something unpredictable in a general field theory. This is a specific of the $\mathcal{N} = 4$ SYM field theory.

Let us recall that the common point in [4], [5] and present work is that in all of them a quark is restricted to move uniformly through the medium. The problem

\textsuperscript{22}In our present case, $u^\alpha \partial_\alpha u_2$ vanishes.

\textsuperscript{23}Considering the five(four) scalar(vector) equations of motion, one can drop every arbitrary five-term(four-term) set of scalar(vector) data to introduce an inequivalent basis of scalar(vector) terms, namely the $s_\alpha(\nu_\beta)$ terms.

\textsuperscript{24}Throughout such computations, we will deal with $X^{(1)}_i$ function. Using the results obtained in this paper, we can integrate (3.8) to find them.
of shooting a quark through the plasma and investigating its energy loss has not been generally resolved yet and is remained as an open question. Although some prescriptions have been suggested to compute the drag force at late times in the plasma [4], there does not exist a clear picture explaining the whole-time dynamics. It would be interesting to explore the problem firstly in the equilibrated plasma and then in the hydrodynamic regime. In the latter case, if analytically possible, some new derivative terms may appear due to the quark acceleration.

In another direction, our covariant formula might be used to study heavy quarks moving through the Bjorken flow in the QGP experiment. Having the phenomenological rate of the energy loss of probe quarks, one may proceed to produce the $R_{AA}$ plot analogue of what has been obtained in [15]. Such plot would be the full prediction of AdS/CFT duality up to first order in derivative expansion. We leave more study on the issue to our future work.
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A Appendices

A.1 Lagrangian of the string

According to our gauge choice,

\[ \mathcal{L} = \sqrt{g_{rt}^2 - g_{rr}g_{tt}} \]  

(A.1)

with

\[ g_{rt} = G_{tt} + 2 G_{ti} \dot{X}_i + G_{ij} \dot{X}_i \dot{X}_j \]
\[ g_{rr} = G_{rr} + 2 G_{rj} \dot{X}_j + G_{ij} \dot{X}_i \dot{X}_j \]  

(A.2)

where for sake of brevity, we have dropped the argument of derivative terms.

A.2 Coefficient functions in the Lagrangian (3.7)

\[ A_m = G_{tm}^2 - G_{mm}G_{tt} + 2 (G_{tm}G_{km} - G_{mn}G_{tk}) \dot{X}_k \]
\[ + (G_{im}G_{km} - G_{mm}G_{ik}) \dot{X}_i \dot{X}_k \]  

(A.3)

\[ B_m = G_{rt}G_{tm} - G_{rm}G_{tt} + (G_{ti}G_{tm} - G_{im}G_{tt}) \dot{X}_i \dot{X}_t + (G_{rt}G_{tm} + G_{ri}G_{tm} - 2G_{rm}G_{ti}) \dot{X}_i \]
\[ + (G_{ri}G_{jm} - G_{rm}G_{ij}) \dot{X}_i \dot{X}_j + (G_{ti}G_{jm} + G_{rm}G_{ij} - 2G_{im}G_{ij}) \dot{X}_i \dot{X}_j \]
\[ + (G_{ij}G_{km} - G_{im}G_{ik}) \dot{X}_i \dot{X}_j \dot{X}_k \dot{X}_l \]  

(A.4)

\[ C_m = G_{rt}^2 - G_{rr}G_{tt} + 2(G_{rt}G_{rt} - G_{rr}G_{ti}) \dot{X}_i + 2(G_{rt}G_{ti} - G_{rt}G_{tt}) \dot{X}_i' \]
\[ + (G_{ti}G_{jt} - G_{tt}G_{ij}) \dot{X}_i' \dot{X}_j' + (G_{rt}G_{rj} - G_{rr}G_{ij}) \dot{X}_i \dot{X}_j \]
\[ + 2(G_{rt}G_{ij} + G_{ri}G_{ij} - 2G_{ti}G_{ij}) \dot{X}_i \dot{X}_j' \]
\[ + 2(G_{ij}G_{tk} - G_{ti}G_{jk}) \dot{X}_i \dot{X}_j' \dot{X}_k' + (G_{ik}G_{jk} - G_{ij}G_{kl}) \dot{X}_i \dot{X}_j' \dot{X}_k' \dot{X}_l' \]  

(A.5)

A.3 Quark in a globally boosted thermal plasma

Since the thermal state and its dual gravity are time independent, the \( X_i(r, t) \) in (3.2) will be just functions of \( r \) coordinate. Let us start by the boost direction, considering \( X_m(r) \) for the case of \( m = z \). Using (A.2) it is straightforward to obtain

\[ X'_z(r) = \frac{r^2 u_z}{r^4 f(br)} \pm \frac{\pi_z(r)}{r^4 f(br)} \sqrt{\frac{r^4 u_z^2 - r^4 f(br)u_0^2}{\pi_z(r)^2 - r^4 f(br)}}. \]  

(A.6)
The numerator of the square root in the second term has a simple root at \( r^* = \sqrt{u^0/b} \). In order to string to be trailed from boundary to the horizon, the denominator must vanish at the same point. This requirement fixes the value of \( \pi_z \) at \( r^* \):

\[
\pi_z(r^*) = -\frac{u_z}{b^2}. \tag{A.7}
\]

Integrating (3.3) gives exactly the same constant value for \( \pi_z(r) \) at any other point on the string. So, the \( z \) component of the covariant drag force and the \( z \)-component of the embedding are determined as we have demonstrated in table 3.

In the case of transverse directions we again utilize (A.2) to rewrite equation (3.8) for \( m = x, y \) directions:

\[
X'_{x,y}(r) = \pm \frac{\pi_{x,y}(r)}{r^4 f(b^r)} \sqrt{- \frac{(r^4 f(b^r) - u_z^2/b^4)}{\pi_{x,y}(r)^2 - (r^4 f(b^r) - u_z^2/b^4)}}. \tag{A.8}
\]

Just like in the boost direction the numerator vanishes at \( r^* \). But in contrast to (A.6) there is no similar way here to remove the sign change of the numerator. The only possibility is to force \( \pi_{x,y}(r) \) to vanish everywhere on the world-sheet. As a result, there would be no drag force in these directions (see (3.5),(3.6)). In addition (A.8) directly gives the \( x \) and \( y \) components of the embedding as it is shown in table 3.

### A.4 Intermediate steps of computation in \( m = z \) direction

#### A.4.1 The coefficient functions

Notice that the first term in each of following three expressions is a "local" zero order term which according to convention below (4.6), has a first derivative Taylor part contributing to first order corrections in the patch. However we have argued that in order to determine \( r^* \) through computing \( B^2_z - A_z C_z \), we only need to write down these coefficients at \((t, \vec{x}) = (0, 0)\).

\[
A_z = r^4 f(b^r) + \frac{2u_0 u_z}{r^4 f(b^r)} G^{(1)}_{tz} - r^2 (\frac{u_z^2}{r^4 f(b^r)} - 1) G^{(1)}_{zz} - r^2 (\frac{u_z^2}{r^4 f(b^r)} + 1) G^{(1)}_{tt} \tag{A.9}
\]

\[
B_z = -r^2 u_z + 2(u_z G^{(1)}_{tt} - u_0 G^{(1)}_{tz}) - u_0 r^2 \hat{X}^{(0)}_z \tag{A.10}
\]

\[
C_z = u_0^2 + 2u_0 u_z \hat{X}^{(0)}_z \tag{A.11}
\]

As we have seen in the text, the correction part of these coefficients are always evaluated at \( r = r^*_{(0)} \). In the following we only give those corrections that contribute to the next expressions, evaluated at \( r = r^*_{(0)} \):

\[
G^{(1)}_{tz}(r^*_{(0)}) = \frac{1}{b} \sqrt{u^0} \partial_t u_z - \frac{1}{b} (2u_0^2 F(\sqrt{u^0}) - \frac{u_z}{\sqrt{u^0}}) \partial_z u_z - \frac{2u_z}{b^2} (u_0 F(\sqrt{u^0}) + \sqrt{u^0}) \partial_b b
\]

\[
G^{(1)}_{tt}(r^*_{(0)}) = -\frac{2u_z}{u_0 b} \partial_z u - \frac{2u_z}{b^2} (u_z F(\sqrt{u^0}) - (\sqrt{u^0})^3 \frac{u_z}{u_z}) \partial_b b. \tag{A.12}
\]
Notice that since in (A.9) the term containing $G_{zz}^{(1)}$ vanishes at $r_{(0)}^*$, $G_{zz}^{(1)}$ does not arise in our computations and we do not need to know its expression.

Another useful expression is:

$$X_z^{(0)}(r_{(0)}^*) = \left( \arctan(\sqrt{u^0}) - \frac{\pi}{2} \right) \partial_t(u_z b) + \frac{u_z r b}{1 + r^2 b^2} \partial_t b \quad \text{(A.13)}$$

### A.4.2 EoM in detail to first order

$$f_z = \frac{\partial L}{\partial z} = \epsilon \frac{-1}{2\pi \alpha'} \frac{1}{2} \partial_z \left( A^{(0)}_z (X_z^{(0)})' + 2B_z^{(0)} (X_z^{(0)})' + C_z^{(0)} \right) + O(\epsilon^2)$$

$$= 0 + O(\epsilon^2) \quad \text{(A.14)}$$

$$\Pi_z^I(r, t) = \frac{\partial L}{\partial X_z} = \frac{-1}{2\pi \alpha'} \left( u_0^2 - 2u_z r^2 (X_z^{(0)})' + r^4 f(rb) (X_z^{(0)})' \right) + O(\epsilon) \quad \text{(A.15)}$$

Notice that in the equation above, we have written only zero order part of the $\Pi_z^I(r, t)$. The reason for this is that EoM contains time derivative of $\Pi_z^I(r, t)$ which is one upper order than $\Pi_z^I(r, t)$ itself.

In the expression below we give the manipulated integral term of (4.22):

$$(-2\pi \alpha') \int_{r_{(t)}}^{\infty} (f_z - \partial_t \Pi_z^I) \, dr = -\epsilon \, U \, \frac{1}{b} \left( u_0 + \frac{u_z^2}{u_0} \right) \partial_t u_z$$

$$+ \epsilon \, V \, \frac{1}{b^2} u_0 u_z \partial_t b + O(\epsilon^2) \quad \text{(A.16)}$$

with

$$U = \frac{\pi}{2} - \arctan(\sqrt{u^0}),$$

$$V = -U + \frac{\sqrt{u^0}}{1 + u^0}. \quad \text{(A.17)}$$

### A.5 Intermediate steps of computation in $m = \{x, y\} = T$ directions

$$A_T = -r^4 \left( \frac{u_0^2}{r^4 b^4} - 1 \right) - r^2 \left( G_{tt}^{(1)} + \left( \frac{u_0^2}{r^4 b^4} - 1 \right) G_{TT}^{(1)} \right) - 2r^4 \frac{u_0 u_z}{r^4 b^4} \dot{X}_z^{(0)} \quad \text{(A.18)}$$

$$B_T = -u_0 G_{tt}^{(1)} + r^2 \left( \frac{u_0 u_z}{r^4 b^4} G_{tt}^{(1)} - \left( \frac{u_0^2}{r^4 b^4} - 1 \right) G_{TT}^{(1)} \right) \dot{X}_z^{(0)} \quad \text{(A.19)}$$

$$C_T = 1 + 2 \left( u_z G_{tt}^{(1)} - u_0 G_{tt}^{(1)} \right) (X_z^{(0)})' - 2r^2 u_z (X_z^{(1)})'$$

$$+ r^2 \left( 2 \left( \frac{u_0 u_z}{r^4 b^4} \right) G_{tt}^{(1)} - \left( \frac{u_0^2}{r^4 b^4} + 1 \right) G_{tt}^{(1)} + \left( \frac{u_0^2}{r^4 b^4} - 1 \right) G_{zz}^{(1)} \right) (X_z^{(0)})' \quad \text{(A.20)}$$

$$+ 2 f(rb) (X_z^{(0)})' (X_z^{(1)})' + 2 u_0 u_z (1 - (X_z^{(0)})') \dot{X}_z^{(0)}$$

Notice that in (4.25) we have written the zero order part of these coefficients.
Recall that for $T$ directions, Lagrangian is given by:

$$\mathcal{L} = \sqrt{A_T X_T^2 + 2B_T X_T'} + C_T.$$  \hspace{1cm} (A.21)

Considering the coefficients given in above and that the $X_T$ appears at first order in gradient expansion, we understand that $A_T X_T^2 + 2B_T X_T'$ can not contribute to $f_T$ up to first order. Notice that derivative with respect to $T$ increases the order of every term present in $\mathcal{L}$ by one. So the only term whose $T$ derivative might survive to first order is the zero order part of $C$ which is itself a constant independent of all coordinates. Consequently,

$$f_T = 0 + O(\epsilon^2)$$  \hspace{1cm} (A.22)

The other necessary quantity is $\Pi_T^T = \partial \mathcal{L} / \partial \dot{X}_T$. As it was noted above there do not exist any $T$ derivative terms in $\mathcal{L}$ up to first order. It means that:

$$\Pi_T^T = 0 + O(\epsilon^2)$$  \hspace{1cm} (A.23)

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