The Classification of Rank 4 Locally Projective Polytopes and Their Quotients *

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In Memory of H. S. M. "Donald" Coxeter, 1907–2003

Abstract

This article announces the completion of the classification of rank 4 locally projective polytopes and their quotients. There are seventeen universal locally projective polytopes (nine nondegenerate). Amongst their 441 quotients are a further four (nonuniversal) regular polytopes, and 152 nonregular but section regular polytopes. All 156 of the latter have hemidodecahedral facets or hemi-icosahedral vertex figures. It is noted that, remarkably, every rank 4 locally projective section regular polytope is finite. The article gives a survey of the literature of locally projective polytopes and their quotients, and fills one small gap in the classification in rank 4.

1 Introduction

The classical study of polytopes concentrated on what are now called “spherical” polytopes. The modern “abstract” polytopes include tesselations of euclidian and hyperbolic space and other spaceforms, as well as objects are best described by their “local” topology, that is, the topology of their facets and vertex figures, or smaller sections. Good accounts of the classical theory and of the emergence of the theory of the abstract polytopes are to be found in [1] or [2].

Extensive study has been made by McMullen, Schulte and others of the so called “locally toroidal” polytopes, that is, polytopes whose minimal

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nonspherical sections are toroids. See for example Chapters 10 to 12 of [11], which bring together and update earlier work on the topic (such as [13]). Comparatively little has been done, until recently, on polytopes of other local topological types. The next three paragraphs summarise briefly the literature on locally projective polytopes.

There are two non-equivalent definitions used in the literature for a “locally $X$” polytope. The broader definition is that the minimal nonspherical sections of the polytope must be of topological type $X$ (see for example [11]). The narrower definition is that these minimal nonspherical sections must actually be the facets or vertex figures (or both) of the polytope (see for example [13]). For rank 4 polytopes, both definitions are equivalent: there do not exist nonspherical polytopes of rank less than 3, so the minimal nonspherical sections of a rank 4 polytope must be of rank 3, if they exist at all. The terminology “locally $X$” is normally restricted to regular polytopes, or at least to section regular polytopes. (A section regular polytope is one for which the isomorphism type of its sections $F/G$ depends only on the ranks of $F$ and $G$.)

In the late 1970’s and early 1980’s, Grünbaum and Coxeter independently discovered a self-dual polytope with 11 hemi-icosahedral facets (see [3] and [4]). During the same period, (see [2]), Coxeter discovered another self-dual polytope with 57 hemidodecahedral facets.

As the theory of abstract polytopes developed, these sporadic examples were joined by others that arose out of more general considerations. Schulte, in [12], noted some examples of locally projective polytopes of type $\{4, 3, 4\}$, that is, with hemicubes for facets, and octahedra or hemioctahedra for vertex figures. Some years later, in 1994, McMullen in [10] examined locally projective polytopes of type $\{3^{k-2}, 4, 3, 3, 4, 3^{n-k-3}\}$ (that is, $\{3, 4, 3^{n-3}\}$, $\{3, 3, 4, 3^{n-4}\}$, $\{4, 3, 3, 4, 3^{n-5}\}$ respectively for $k = 0, 1, 2$), where $n \geq 5$. This was the first paper to give an infinite family of locally projective polytopes. As was shown in [7], the theory developed by Schulte in [12] can be easily applied to produce another infinite family. However, Schulte had chosen a different focus for that paper.

The most recent work on the problem has been a series of articles by the author and others (see [6], [7] and [9]) which focused on specific examples of locally projective polytopes and their quotients. That work almost completed the classification of locally projective rank 4 polytopes and their quotients. This article serves mainly as a survey of the classification (see Section 4), but also (in Section 3) closes the one small gap that remains. Section 2 contains observations that show that the cases examined are indeed all that can occur.

The most remarkable feature of the classification is the fact that all such polytopes are finite. This is in contrast with the rank 5 case, where infinite
locally projective polytopes exist (an example being \(\{\{4,3,3\},\{3,3,5\}/2\}\) (see Theorem 5.2 of [7]).

2 The Form and Scope of the Classification.

The usual way to search for polytopes with given (regular) facets \(\mathcal{K}\) and vertex figures \(\mathcal{L}\) is to find the universal such polytope \(\mathcal{P} = \{\mathcal{L}, \mathcal{K}\}\), and then seek its quotients. The universal polytope is characterised by its automorphism group \(\text{Aut}(\mathcal{P}) = \Gamma(\mathcal{P}) = [\mathcal{L}, \mathcal{K}]\). By definition, all polytopes of the desired form will be quotients of \(\mathcal{P}\). It is guaranteed that if any polytope exists with the desired facets and vertex figures, then the universal polytope \(\mathcal{P}\) does (see Theorem 4A2 of [11] or Theorem 2.5 of [8]).

A locally projective section regular polytope of rank 4 must have spherical or projective facets and vertex figures. Therefore, the most straightforward strategy for classifying these polytopes is as follows:

- Identify the projective and spherical rank 3 polytopes.
- For each pair of spherical or projective \(\mathcal{K}\) and \(\mathcal{L}\), not both spherical, identify the universal \(\{\mathcal{L}, \mathcal{K}\}\), if it exists.
- Enumerate the quotients of each \(\{\mathcal{L}, \mathcal{K}\}\), and identify those which are section regular.

Note for the second step, if \(\mathcal{L}\) is of type \(\{p, q\}\) and \(\mathcal{K}\) of type \(\{q', r\}\), a simple necessary condition that \(\{\mathcal{L}, \mathcal{K}\}\) exist is that \(q\) must equal \(q'\). Then \(\{\mathcal{L}, \mathcal{K}\}\) will be of type \(\{p, q, r\}\). As we shall see, this necessary condition is not sufficient. The case \(q = 2\) cannot yield locally projective polytopes, since the polytopes of type \(\{p, 2\}\) are “dihedra”, with \(p\)-gonal faces, and in particular are spherical (along with their duals). There do exist locally projective polytopes with \(p = 2\) or \(r = 2\). The only cases for \(r = 2\) are “ditopes” \(\{p, q, 2\}\) with projective facets of type \(\{p, q\}\). Their enumeration is easy, and since the projective \(\{p, q\}\) have no proper quotients, neither do the ditopes. If \(p = 2\) we obtain the duals of the ditopes. For most of the rest of this article, we shall ignore these degenerate cases and assume \(p, q, r \geq 3\).

The spherical rank 3 polytopes have been known since the time of the ancient Greeks. They are the tetrahedron \(\{3,3\}\), the cube \(\{4,3\}\), the octahedron or cross \(\{3,4\}\), the dodecahedron \(\{5,3\}\) and the icosahedron \(\{3,5\}\). Except for the tetrahedron, the groups of each of these has a central inversion \(\omega\). Taking the quotient of these polytopes by the group \(\langle \omega \rangle\) yields four projective polytopes: the hemicube \(\{4,3\}_3 = \{4,3\}/2\), its dual the hemicross, the hemidodecahedron \(\{5,3\}_5 = \{5,3\}/2\), and its dual the hemiicosahedron. As noted in [5], these are the only proper quotients of the
platonic solids, and therefore, the only rank 3 section regular projective polytopes.

Ignoring duality leads to 22 cases that need to be analysed. These are listed in Table 1.

| #  | Facets | V. Figures | Type    | #  | Facets | V. Figures | Type    |
|----|--------|------------|---------|----|--------|------------|---------|
| 1  | \{3,3\} | \{3,4\}_3  | \{3,3,4\} | 2  | \{3,3\} | \{3,5\}_5  | \{3,3,5\} |
| 3  | \{3,4\} | \{4,3\}_3  | \{3,4,3\} | 4  | \{3,4\}_3 | \{4,3\}_3  | \{3,4,3\} |
| 5  | \{3,4\}_3 | \{4,3\}   | \{3,4,3\} | 6  | \{3,5\}_5 | \{5,3\}_5  | \{3,5,3\} |
| 7  | \{3,5\}_5 | \{5,3\}_5  | \{3,5,3\} | 8  | \{3,5\}_5 | \{5,3\}   | \{3,5,3\} |
| 9  | \{4,3\}_3 | \{3,3\}   | \{4,3,3\} | 10 | \{4,3\}_3 | \{3,4\}_3  | \{4,3,4\} |
| 11 | \{4,3\}_3 | \{3,4\}_3  | \{4,3,4\} | 12 | \{4,3\}_3 | \{3,4\}   | \{4,3,4\} |
| 13 | \{4,3\}_3 | \{3,5\}_5  | \{4,3,5\} | 14 | \{4,3\}_3 | \{3,5\}_5  | \{4,3,5\} |
| 15 | \{4,3\}_3 | \{3,5\}   | \{4,3,5\} | 16 | \{5,3\}_5 | \{3,3\}   | \{5,3,3\} |
| 17 | \{5,3\}_5 | \{3,4\}_3  | \{5,3,4\} | 18 | \{5,3\}_5 | \{3,4\}_3  | \{5,3,4\} |
| 19 | \{5,3\}_5 | \{3,4\}   | \{5,3,4\} | 20 | \{5,3\}_5 | \{3,5\}_5  | \{5,3,5\} |
| 21 | \{5,3\}_5 | \{3,5\}_5  | \{5,3,5\} | 22 | \{5,3\}_5 | \{3,5\}   | \{5,3,5\} |

Table 1: The Scope of the Classification Problem.

These cases have all been completely analysed, and the results reported in the literature, with the exception of cases 10 to 12, for which a small amount of additional work remains to be done.

3 The Unfinished Cases.

In this short section, we consider cases 10 and 11. Case 12 is dual to case 10, and therefore does not warrant separate consideration. As mentioned earlier, polytopes of these types exist (see [12]). Furthermore, the only polytopes with the given facets and vertex figures are in fact the universal polytopes, as was shown in [7]. However, a full analysis of the quotients of these polytopes has not been done. It will be useful at this point to review the classification of the quotients of the cube (and therefore the cross), given in [5]. Let the group of the cube be \( W' = \langle s_0, s_1, s_2 \rangle \), and let \( x = s_0, y = s_1xs_1 \) and \( z = s_2ys_2 \). The semisparse subgroups of \( W' \) are the trivial subgroup (leading to the cube itself), the group \( \langle xy \rangle \) and its conjugates \( \langle yz \rangle \) and \( \langle xz \rangle \) (leading to the digonal prism), the group \( \langle xyz \rangle \) (leading to the hemicube), and the group \( \langle xy, yz \rangle \) (leading to the polytope \{2,3\}).

In particular, note that the 3-hemicube (and therefore the 3-hemicross) has no proper quotients (this is not true in higher ranks). Since the vertex figures of \{\{4,3\}, \{3,4\}/2\} have no proper quotients, Theorem 2.7 of [8] may
be applied to case 10, so that semisparse subgroups of \( W = \langle s_0, s_1, s_2, s_3 \rangle = \langle \{4,3\}, \{3,4\}/2 \rangle \) are characterised by the property that all their conjugates intersect \( \langle s_0, s_1, s_2 \rangle \langle s_1, s_2, s_3 \rangle \) in a semisparse subgroup of \( W' \). It is simple enough (although a little unsatisfying) to do a computer search of the subgroups of \( W \) to find all those which satisfy the required property. This leads to the following theorem and corollary.

3.1 Theorem The universal polytope \( P = \{\{4,3\}, \{3,4\}_3\} \) has four quotients. These are \( P \) itself, \( \{\{4,3\}_3, \{3,4\}_3\} \), \( \{\{2,3\}, \{3,4\}_3\} \), and a nonregular polytope whose facets are all digons.

Note that the four quotients of \( P \) correspond very naturally to the four quotients of the cube. Note also the following Corollary.

3.2 Corollary The polytope \( Q = \{\{4,3\}_3, \{3,4\}_3\} \) has no proper quotients.

The author notes that it would be a worthwhile endeavour to characterize the quotients of \( \{\{4,3^n-3\}, \{3^n-3,4\}/2\} \) for general \( n \geq 4 \), but feels that such a characterization falls beyond the scope of this article.

4 The Classification.

This final section of the article will go through the cases of Table 1 one by one, giving a description of the universal polytope and its quotients, with references to the literature.

- Cases 1 to 5: The results of [5], where polytopes of “finite type” were studied (that is, polytopes with the same Schlafli symbols as the classical spherical polytopes) show that there are no locally projective polytopes of types \( \{3,3,4\} \), \( \{3,3,5\} \) or \( \{3,4,3\} \). This likewise eliminates cases 9 and 16, as is noted below.

- Case 6: There is no polytope with icosahedral facets and hemidodecahedral vertex figures. See the dual Case 8 for an explanation.

- Case 7 is Coxeter and Grünbaum’s 11-cell (see [2] or [4]). The polytope has a group of order 660, isomorphic to the projective special linear group \( L_2(11) \). It was noted in [6] that this polytope has no proper quotients.
• Case 8: There is no polytope with hemiicosahedral facets and dodecagonal vertex figures. If such a polytope existed, it could be constructed (in principle) by arranging hemiicosahedra face to face, with three around each edge, either until the polytope closed up, or ad infinitum. However, an attempt to perform this construction results in the 11-cell of Case 7, as noted in [4].

• Case 9 is dual to case 1, thus yields no polytopes.

• Case 10: The universal polytope \{\{4, 3\}, \{3, 4\}\} appeared in [12]. A simple proof that the universal polytope is the only polytope of type \{\{4, 3\}, \{3, 4\}\} appeared in [7]. The polytope may be constructed from its vertex figures via the “twisting” operation developed by Schulte and McMullen and described in Chapter 9 of [11]. Here, \{\{4, 3\}, \{3, 4\}\} is \(2^H\), where \(H = \{3, 4\}\) is the hemicross. Its group is the semidirect product \(2^3 \rtimes S_4\), with \(S_4\) (the group of \(H\)) acting by conjugation on the generators of \(2^3\) in the same way that it acts on the vertices of the hemicross. The quotients of this polytope were described in Theorem 3.1.

• Case 11: The universal polytope \{\{4, 3\}_3, \{3, 4\}_3\} is a quotient of \{\{4, 3\}, \{3, 4\}\} and has itself no proper quotients. This was shown in Theorem 3.2 but could easily have been noted from the results of [7]. Its group is the quotient of \(2^3 \rtimes S_4\) by the (normal) subgroup generated by the product of the three generators of \(2^3\).

• Case 12 is dual to case 10.

• Case 13 was studied in depth in [6], where it was shown that the universal \{\{4, 3\}, \{3, 5\}\} exists and is finite. Its existence and finiteness are also noted in Section 8E of [11]. It appears there as \(2^I\), where \(I\) is the hemi-icosahedron (whose group is \(A_5\)). Its group is therefore \(2^6 \rtimes A_5\), with \(A_5\) acting by conjugation on the generators of \(2^6\) in the same way that it acts on the vertices of \(I\). The universal polytope therefore has 80 cubes as facets, and 64 vertices. In [6] it was shown, via a computer search, that the polytope has 70 quotients. Of these, three are regular, these being \{\{2, 3\}, \{3, 5\}\}, the universal polytope, and a quotient of the universal polytope by a group of order 2. Besides the latter two, there are another nine which are nonregular polytopes of type \{\{4, 3\}, \{3, 5\}\}, and besides these another eight of type \{4, 3, 5\}. The latter eight each have, as facets, some cubes and some hemicubes, hence they are not section regular. For more details on the quotients of this polytope, the reader is referred to [6].
• Cases 14 and 15: It was shown in Theorem 3.6 of [7] that a polytope with hemicubes for facets cannot be of type \( \{4, 3, \ldots, 3, p\} \) for odd \( p \). Thus cases 14 and 15 do not yield examples of locally projective polytopes.

• Case 16 is dual to case 2.

• Cases 17, 18 and 19 are dual to cases 15, 14 and 13 respectively.

• Case 20 was examined in [9]. The polytope \( \{\{5, 3\}, \{3, 5\}_5\}\) exists and is finite, with group \( J_1 \times L_2(19) \) of order 600415200. It therefore has 10006920 vertices and half that many facets. Here, \( L_2(19) \) is a projective special linear group of order 3420, and \( J_1 \) is the first Janko group, of order 175560. The quotients of this polytope are 145 in number, including as regular quotients Coxeter’s 57-cell \( \{\{5, 3\}_5, \{3, 5\}_5\} \) (see below), and a polytope of type \( \{\{5, 3\}, \{3, 5\}_5\} \) with group \( J_1 \). There are also 67 nonregular quotients of this type. The remaining 75 proper quotients (all of type \( \{5, 3\}_5 \)) each have, as facets, some dodecahedra and some hemidodecahedra. For more details, see [9].

• Case 21 was discovered by Coxeter in [2]. The universal polytope has 57 facets, and its group is \( L_2(19) \) of order 3420. It was shown in [6] that it has no proper quotients. It is itself a quotient of the universal polytope of case 20.

• Case 22 is dual to case 20, and therefore does not warrant separate discussion.

To summarize, there are nine nondegenerate universal locally projective regular polytopes of rank 4, namely \( \{\{3, 5\}_5, \{5, 3\}_5\}, \{\{4, 3\}, \{3, 4\}_3\} \) and its dual, \( \{\{4, 3\}_3, \{3, 4\}_3\}, \{\{4, 3\}, \{3, 5\}_5\} \) and its dual, \( \{\{5, 3\}_5, \{3, 5\}_5\} \) and \( \{\{5, 3\}, \{3, 5\}_5\} \) and its dual. There are 13 regular polytopes of these types, since each of those with hemi-icosahedral vertex figures (or hemidodecahedral facets) has a proper regular quotient of the same type.

These universal polytopes have a total of 437 quotients, 17 regular, and 169 section regular. The extra four regular quotients are the quotients \( \{\{2, 3\}, \mathcal{K}\} \) of locally projective \( \{\{4, 3\}, \mathcal{K}\} \) and their duals. These are also counted amongst the section regular quotients. The 152 nonregular section regular quotients (or their duals) all have hemi-icosahedral vertex figures and spherical (cubic or dodecahedral) facets. Besides these, there are a further four degenerate locally projective polytopes, namely \( \{\{2, 4\}, \{4, 3\}_3\} \) and \( \{\{2, 5\}, \{5, 3\}_5\} \) and their duals. These four are not quotients of any nondegenerate locally projective polytopes.

The most remarkable feature of the classification is the following result.
4.1 Theorem \textit{All locally projective rank 4 polytopes are finite.}

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