Continuous approximation of binomial lattices

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Abstract

A systematic analysis of a continuous version of a binomial lattice, containing a real parameter $\gamma$ and covering the Toda field equation as $\gamma \to \infty$, is carried out in the framework of group theory. The symmetry algebra of the equation is derived. Reductions by one-dimensional and two-dimensional subalgebras of the symmetry algebra and their corresponding subgroups, yield notable field equations in lower dimensions whose solutions allow to find exact solutions to the original equation. Some reduced equations turn out to be related to potentials of physical interest, such as the Fermi–Pasta–Ulam and the Killingbeck potentials, and others. An instanton–like approximate solution is also obtained which reproduces the Eguchi–Hanson instanton configuration for $\gamma \to \infty$. Furthermore, the equation under consideration is extended to $(n+1)$-dimensions. A spherically symmetric form of this equation, studied by means of the symmetry approach, provides conformally invariant classes of field equations comprising remarkable special cases. One of these ($n=4$) enables us to establish a connection with the Euclidean Yang–Mills equations, another appears in the context of Differential Geometry in relation to the so-called Yamabe problem. All the properties of the reduced equations are shared by the spherically symmetric generalized field equation.
1 Introduction

The continuous (or long-wave) approximations of chains of particles can provide a fruitful theoretical framework to be used as a guide in the study of the corresponding original discrete systems. These models are generally formulated as nonlinear field equations. When such equations are integrable, exact solutions can be found as, for example, solitons, vortices and shock-waves.

Continuous limits of lattice systems may be considered as phenomenological models with a proper identity, and their study could be interesting apart from the reduction procedure applied in passing to the continuous representation [1]. This feature is shared by different models, among which the well-known continuous form

$$u_{xx} + u_{yy} - k(e^u)_{zz} = 0 \quad (1.1)$$

of the two-dimensional discrete Toda equation, where $u = u(x, y, z)$, plays a basic role [2]. (Subscripts denote partial derivatives and $k = \pm 1$).

Indeed, Eq. (1.1) appears in a variety of physical areas, running from the theory of Hamiltonian systems to general relativity [3], [4]. In the latter context, Eq. (1.1) occurs in the theory of self-dual Einstein spaces, where it is rechristened heavenly equation [4], [5]. This emerges as a limit case from the Toda molecule equation [5], which is exactly integrable [6].

Another nonlinear field equation associated with a nonlinear lattice which deserves to be investigated is

$$\Delta \equiv u_{xx} + u_{yy} - k \left[ \left( 1 + \frac{u}{\gamma} \right)^{\gamma-1} \right]_{zz} = 0, \quad (1.2)$$

where $\gamma$ is a (real) parameter such that $\gamma \neq 0, 1$.

Really, after suitable rescalings, Eq. (1.2) can be interpreted as the continuous limit of a uniform two-dimensional nonlinear lattice of $N$ particles interacting through the nearest-neighbor potential [7]

$$\phi(r_n) = \frac{a_n}{b_n} \left[ \left( 1 + \frac{b_n r_n}{\gamma} \right)^{\gamma} - (1 + b_n r_n) \right], \quad (1.3)$$

where $a_n$ and $b_n$ are constants of the $n$-th nonlinear spring, $r_n = y_n - y_{n+1}$, and $y_n$ is the displacement of the $n$-particle from its equilibrium position.

We shall call "binomial lattice" the chain described by the potential (1.3).

We notice that the function (1.3) covers the Toda potential ($\gamma \to \infty$) [8], the harmonic potential ($\gamma = 2$), and the potential used by Fermi, Pasta and Ulam in their computer experiment in the early’s 1950 ($\gamma = 3$) [9].

As one expects, for $\gamma \to \infty$ Eq. (1.2) becomes Eq. (1.1).

The purpose of this paper is both to find the symmetry structure and exact solutions of Eq. (1.2). An effective method to get insight into the symmetry properties of Eq. (1.2) and, consequently, to construct explicit configurations,
is based on the reduction approach \[1\], which exploits group-theoretical techniques. Applying this procedure, we obtain the symmetry algebra of Eq. (1.2). The generator of the corresponding symmetry group has been determined in part with the help of a computer, by means of the symbolic language MAPLE V Release 4 \[11\]. Our result is that Eq. (1.2) admits a finite-dimensional Lie group of symmetries, i.e. a local group \(G\) of transformations acting on the independent variables \((x, y, z)\) and the dependent variable \(u\) with the property that whenever \(u(x, y, z)\) is a solution of Eq. (1.2), then \(u' = (g \circ u)(x', y', z')\) is also a solution for any \(g \in G\).

We remind the reader that in contrast with what happens for Eq. (1.2), the Toda equation (1.1), handled within the group theory \[12\], allows an infinite-dimensional symmetry algebra, a realization of which is given by generators obeying a Virasoro algebra without a central charge (Witt algebra). Furthermore, certain reduced equations give rise to instanton and meron-like solutions endowed with integer and fractional topological numbers, respectively.

Thus, the comparative analysis of the properties of Eqs. (1.1) and (1.2) is of particular relevance, keeping in mind also the fact that both the equations come from the continuous approximation of physically significant lattice systems.

Another important characteristic common to Eqs. (1.1) and (1.2) is that they are related to nonlinear systems of the hydrodynamical type \[3\]. Although interesting, here this aspect will not be treated.

The main results achieved in this article are presented in the Sections in which the paper is organized.

In Sec. 2 we find the symmetry algebra of Eq. (1.2). It turns out that the related independent infinitesimal operators obey a closed set of commutation rules. Starting from special linear combinations of these operators, in Sec. 3 the corresponding symmetry group transformations are derived. These provide reduced differential equations leading to exact solutions to Eq. (1.2). Notable cases arise for \(\gamma = 3, -2, \frac{5}{2}, \frac{5}{3}\). This choice of values is motivated by the fact that, in correspondence, the function (1.3) coincides with potentials of physical interest, such as the Fermi-Pasta-Ulam (\(\gamma = 3\)) \[9\] and a potential involved in the Thomas-Fermi model of an atom (\(\gamma = 5/2\)) \[13\], p. 116). We point out that for \(\gamma = 3\), a remarkable representation of the inverse Weierstrass function is given in terms of the Gauss hypergeometric function. In Sec. 4, we obtain an approximate instanton-like configuration, holding for large values of the parameter \(\gamma\), which reproduces the Eguchi-Hanson instanton solution of Eq. (1.1) in the limit \(\gamma \to \infty\) \[14\].

In Sec. 5 we extend Eq. (1.2) to the \((n + 1)\)-dimensional case. The exploration of the symmetry properties of this more general nonlinear field equation, that is Eq. (5.1), has been suggested by the purpose of ascertaining whether a link might exist between Eq. (5.1) and certain problems inherent in Yang-Mills theory and Differential Geometry \[15\], \[16\], \[17\].

Precisely, we show that applying the reduction technique to the spherically symmetric version of Eq. (5.1), a connection can be established between so-
olutions of Eq. (1.2), and solutions of a reduced nonlinear ordinary differential equation, i.e., Eq. (5.15), which is invariant under conformal transformations. Equation (5.15) comprises interesting cases, which come from $n = 1, 3, 4, 6$ and $10$ respectively. For instance, for $n = 10$, Eq. (5.15) can be interpreted as an extended (elliptic) form of the equation governing the Thomas–Fermi model of an atom, while for $n = 4$, Eq. (5.15) turns out to be related to Euclidean Yang–Mills equations via ’t Hooft’s ansatz [17]. The properties of Eq. (5.15) can be reflected on those of Eq. (5.1). This is illustrated in Sec. 6.

Finally, Sec. 7 contains some concluding remarks, while in Appendixes A, B, C and D details of the calculation are reported.

\section{Group analysis}

To the aim of looking for the symmetry algebra of Eq. (1.2), we shall apply the standard procedure described in [10]. In doing so, let us introduce the vector field

$$V = \xi \partial_x + \eta \partial_y + \zeta \partial_z + \phi \partial_u,$$

where $\xi, \eta, \zeta$ and $\phi$ are functions of $x, y, z, u$, and $\partial_x = \frac{\partial}{\partial x}$, and so on.

A local group of transformations is a symmetry group for Eq. (1.2), if and only if

$$pr^{(2)} V[\Delta] = 0,$$

whenever $\Delta = 0$ for every generator $V$ of $G$, where $pr^{(2)}V$ is the second prolongation of $V$ [10].

Equation (2.2) give rise to a set of constraints in the form of partial differential equations which enable us to determine the coefficients $\xi, \eta, \zeta$ and $\phi$. This has been carried out in part using a computer, by means of the symbolic language of [11]. However, to facilitate the understanding of the method to non-specialist readers, in Appendix A we write $pr^{(2)}V$ explicitly.

A general element of the symmetry algebra of Eq. (1.2) is

$$V = (c_1 x + c_2 y + c_3) \partial_x + (-c_2 x + c_1 y + c_4) \partial_y + (c_5 + c_6 z) \partial_z + 2 \frac{\gamma + u}{\gamma - 2} (c_6 - c_1) \partial_u,$$

where $c_1, c_2, ..., c_6$ are arbitrary constants. The expression (2.3) holds for any allowed value of $\gamma$ ($\gamma \neq 0, 1$), except for $\gamma = \frac{2}{3}, 2$. We shall come back to these cases later.

From (2.3) we obtain the following independent generators of the symmetries of Eq. (1.2):

$$V_1 = x \partial_x + y \partial_y - 2 \frac{\gamma + u}{\gamma - 2} \partial_u,$$

$$V_2 = y \partial_x - x \partial_y,$$

$$V_3 = \partial_x,$$
\[ V_4 = \partial_y, \quad (2.7) \]
\[ V_5 = \partial_z, \quad (2.8) \]
\[ V_6 = z\partial_z + 2\frac{\gamma + u}{\gamma - 2}\partial_u. \quad (2.9) \]

These operators satisfy the commutation relations

\[ [V_1, V_2] = [V_1, V_5] = [V_1, V_6] = [V_2, V_5] = [V_2, V_6] = [V_3, V_4] = [V_3, V_5] = [V_3, V_6] = [V_4, V_5] = [V_4, V_6] = 0, \quad (2.10) \]
\[ [V_1, V_3] = -V_3, \quad (2.11) \]
\[ [V_1, V_4] = -V_4, \quad (2.12) \]
\[ [V_2, V_3] = V_4, \quad (2.13) \]
\[ [V_5, V_6] = V_5. \quad (2.14) \]

We deduce that the operators (2.4)–(2.9) form the basis of a six–dimensional solvable Lie algebra \( \mathcal{L} \) containing a three–dimensional ideal \( \{V_2, V_3, V_4\} \) which is isomorphic to the algebra of \( \mathbb{E}_2 \), the Euclidean group in the plane. The center of \( \mathcal{L} \) is zero, while its derived algebra is \( \{V_2, V_3, V_4\} \). Furthermore, \( \mathcal{L} \) admits the Levi decomposition \( \mathcal{L} = \mathcal{L}_1 \triangleright \mathcal{L}_2 \), where \( \mathcal{L}_1 = \{V_2, V_3, V_4\} \) and \( \mathcal{L}_2 = \{V_1, V_2, V_6\} \). This decomposition is not unique. We can easily write other decompositions of this kind. The symbol \( \triangleright \) stands for the operation of semidirect sum. (To keep the length of the paper reasonable, here we shall not explain the algebraic terminology used above. Anyway, the reader unfamiliar with the previous mathematical concepts, could address, for instance, to [18]).

From the elements \( V_1 \) and \( V_6 \) we give the generator of the coordinate scale transformation, that is \( V_6 = V_1 + V_6 = x\partial_x + y\partial_y + z\partial_z; V_3, V_4, V_5 \) generate \( x, y \) and \( z \)-translations, \( V_2 \) is a rotation symmetry operator, and \( V_1 \) and \( V_6 \) have the meaning of generators of two dilations together with a translation of \( 2\frac{\gamma + u}{\gamma - 2} \) along the \( \mp u \) directions, respectively.

As we have already mentioned, the symmetry generator (2.3) does not include the value \( \gamma = \frac{2}{3} \). This is due to the fact that a special case emerges, just for \( \gamma = \frac{2}{3} \), in solving the equations determining the coefficients involved in (2.1). In this case, to the generators (2.4)–(2.9) of the symmetries of Eq. (1.2), one needs to add another operator, namely

\[ V_7 = z^2\partial_z - z(2 + 3u)\partial_u, \quad (2.15) \]

which commutes with \( V_1, V_2, V_3, V_4 \), while

\[ [V_6, V_7] = V_7, \quad [V_5, V_7] = 2V_6. \quad (2.16) \]

A second special case to be dealt with separately is \( \gamma = 2 \), in correspondence of which Eq. (1.2) becomes the linear wave equation

\[ u_{xx} + u_{yy} - \frac{k}{2}u_{zz} = 0. \quad (2.17) \]
This equation is the continuous approximation of the lattice system arising from the harmonic potential \( \phi(r_n) = \frac{1}{2}a_n b_n r_n^2 \) (see (1.3)). The symmetry algebra related to Eq. (2.17) is represented by the vector fields

\[
G_1 = x(z \partial_z + y \partial_y) + \frac{1}{2}(x^2 - y^2 + 2kz^2) \partial_x - \frac{1}{2}xy \partial_u,
\]

\[
G_2 = x \partial_x + y \partial_y + z \partial_z,
\]

\[
G_3 = x \partial_z + 2kz \partial_x,
\]

\[
G_4 = -x \partial_y + y \partial_x,
\]

\[
G_5 = \partial_y,
\]

\[
G_6 = \partial_z,
\]

\[
G_7 = \partial_x,
\]

\[
G_8 = u \partial_u,
\]

\[
G_9 = f(x, y, z) \partial_u,
\]

where \( f \) is an arbitrary solution of the wave equation \( f_{xx} + f_{yy} - 2kf_{zz} = 0 \). The commutation rules fulfilled by \( G_1, \ldots, G_9 \) can be easily found.

For brevity, the case \( \gamma = \frac{2}{3} \) and \( \gamma = 2 \) will not be further discussed. We remind only the reader interested in going deep into the case \( \gamma = 2 \), that an exhaustive analysis of an equation of the type (2.17) is performed in [10].

### 3 Exact solutions from reduced equations

The technique of symmetry reduction of a field equation amounts essentially to finding the invariants (symmetry variables) of a given subgroup of the symmetry group allowed by the equation under consideration [10]. This method is usually preceded by the classification of the subalgebras of the symmetry algebra. The classification scheme is based on the adjoint action of the symmetry group [15]. However, because of the finite–dimensionality of the symmetry algebra (2.10)–(2.14), we shall adopt a more heuristic procedure. In other words, since for practical purposes generally one confines oneself to handle only low–dimensional symmetry subalgebras, which can be recognized in our case directly by inspection, we do not need to employ the adjoint subgroup classification method.

For each symmetry group generator \( V \), we can obtain a basis set of invariants \( I(x, y, z, u) \) by solving the first order partial differential equation

\[
VI(x, y, z, u) = 0.
\]

In the following, we shall build up the reduction of Eq. (1.2) by the one–dimensional subalgebras \( \{V_1\}, \{V_2\}, \{V_6\} \), and by the two–dimensional subalgebras \( \{V_1, V_2\}, \{V_1, V_5\}, \{V_2, V_6\} \).
3.1 Reductions by two–dimensional subalgebras

The reduced equations associated with the two–dimensional subalgebras

\[ \{ V_1, V_2 \}, \]
\[ \{ V_1, V_5 \} \]
\[ \{ V_2, V_6 \} \]

turn out to be nonlinear ordinary differential equations which can be solved to give exact solutions to Eq. (1.2).

\textit{Case i)}

A basis of invariants of the subgroup of the subalgebra \( \{ V_1, V_2 \} \) is furnished by the system of equations \( V_1 I = 0, \ V_2 I = 0 \), which have to be satisfied simultaneously. We obtain

\[ I_1 = z, \ I_2 = A(z) = \left( x^2 + y^2 \right)^{1/2} \left( 1 + \frac{u}{\gamma} \right)^{\frac{n-2}{2}} \],

(3.2)

from which the reduced equation

\[ AA'' - \frac{2}{2\gamma} A'^2 - \frac{2k\gamma}{(\gamma - 1)(\gamma - 2)} = 0, \]

(3.3)

with \( A' = \frac{dA}{dz} \), is derived.

Putting

\[ A = \theta^{\frac{\gamma-2}{\gamma}} \]

(3.4)

into Eq. (3.3), we get

\[ \theta'' = \frac{2k}{\gamma - 1} \left( \frac{\gamma}{2 - \gamma} \right)^2 \theta^{\frac{4-\gamma}{\gamma}}, \]

(3.5)

yielding

\[ \theta'^2 = \frac{k\gamma^3}{(\gamma - 1)(\gamma - 2)^2} \theta^4 + c, \]

(3.6)

where \( c \) is a constant of integration.

Equation (3.6) leads to the relation

\[ \int_0^\theta \left( 1 + b\theta^{\frac{4}{\gamma}} \right)^{-\frac{1}{4}} d\theta' = \pm \sqrt{c(z - z_0)}, \]

(3.7)

\( z_0 \) being an arbitrary constant, and

\[ b = \frac{k}{c} \left( \frac{\gamma^3}{(\gamma - 1)(\gamma - 2)^2} \right). \]

(3.8)

The integral at the left–hand side of (3.7) can be calculated resorting to the formula (see [19], p. 284)

\[ \int_0^X \frac{X'^{-\mu-1} dX'}{(1 + bX)^{\nu}} = \frac{X^\mu}{\mu} 2F_1(\nu, \mu; \mu + 1; -bX), \]

(3.9)
where \( \arg(1 + bX) < \pi \), \( \Re \mu > 0 \) and \( \, _2F_1 \) denotes the Gauss hypergeometric function. Indeed, setting in (3.7) \( X = \theta \frac{\gamma}{4} \) and using (3.9) with \( \mu = \frac{\gamma}{4} \) and \( \nu = \frac{1}{2} \), we find

\[
\theta \, _2F_1 \left( \frac{1}{2}, \frac{\gamma}{4}; \frac{\gamma}{4} + 1; -b \theta^\frac{4}{4} \right) = \pm \sqrt{c} (z - z_0),
\]

(3.10)

where \( \theta = \theta(\pm \sqrt{c}(z - z_0)) \) is explicitly known whenever Eq. (3.10) is invertible. When this occurs, Eq. (3.2) provides an exact solution to Eq. (1.2). For example, for \( \gamma = 4 \), \( k = -1 \) and \( z_0 = -\frac{3}{8} \sqrt{c} \), we have

\[
\, _2F_1 \left( \frac{1}{2}, 1; 2; -b \theta \right) = 2 \left( -1 + \sqrt{1 + b \theta} \right),
\]

(3.11)

with \( b = -\frac{16}{3} \).

Then, combining together Eqs. (3.11) and (3.2), we obtain the solution

\[
\eta = 4 \left[ \frac{3c}{16} \frac{1}{x^2 + y^2} \left( 1 - \frac{64}{9} z^2 \right)^{\frac{1}{2}} - 1 \right] - 4
\]

(3.12)

to the equation

\[
u_{xx} + \nu_{yy} + \left[ (1 + \frac{\nu}{4})^3 \right]_{zz} = 0.
\]

(3.13)

Case ii)

The subgroup of the subalgebra \( \{ V_1, V_2 \} \) admits the basis of invariants

\[
I_1 \equiv \eta = \frac{x}{y}, \quad I_2 \equiv B(\eta) = \frac{1}{z} \left( 1 + \frac{u}{\gamma} \right)^{\frac{2 - \gamma}{2}}.
\]

(3.14)

Then, the reduced equation of Eq. (1.2) coming from the symmetry variables \( \eta \) and \( B(\eta) \) takes the form

\[
\eta^2 (\eta^2 + 1) \frac{B''}{B} + 2 \eta \left( \eta^2 + \frac{2}{2 - \gamma} \right) \frac{B'}{B} + \frac{\gamma}{2 - \gamma} \left[ \eta^2 (\eta^2 + 1) \left( \frac{B'}{B} \right)^2 + 1 \right] = 0,
\]

(3.15)

with \( B' = \frac{dB}{d\eta} \).

By introducing the change of variable

\[
\frac{B'}{B} = \frac{2 - \gamma}{2} \frac{Q'}{Q},
\]

(3.16)

Eq. (3.15) can be cast into the linear differential equation

\[
\eta^2 (\eta^2 + 1) Q'' + 2 \eta \left( \eta^2 + \frac{2}{2 - \gamma} \right) Q' + \frac{2\gamma}{(2 - \gamma)^2} Q = 0.
\]

(3.17)
Resorting to the MATHEMATICA symbolic language, we find the general solution
\[ Q(\eta) = \left( \frac{1 + \eta^2}{\eta^2} \right)^{\frac{1}{2\gamma}} \left[ c_1 \sin \left( \frac{2}{2 - \gamma} \arctan \eta \right) + c_2 \cos \left( \frac{2}{2 - \gamma} \arctan \eta \right) \right] \]

of Eq. (3.17), where \( c_1 \) and \( c_2 \) are arbitrary constants.

Thus, by virtue of (3.18) and (3.16), an exact solution of Eq. (1.2) is determined, namely
\[ u = \gamma \left( z^{\frac{2}{\gamma}} Q - 1 \right). \]

**Case iii)**

A set of basis invariants for the subgroup of the subalgebra \( \{ V_2, V_6 \} \) is
\[ I_1 \equiv \tau = (x^2 + y^2)^{\frac{1}{2}}, \quad I_2 \equiv G(\tau) = z \left( 1 + \frac{u}{\gamma} \right)^{\frac{2}{\gamma}}. \]

These give rise to the reduced equation
\[ \frac{\gamma}{2 - \gamma} G^2 + \frac{GG'}{\tau} + GG'' = k \frac{\gamma - 1}{2 - \gamma} \]

of Eq. (1.2), with \( G' = \frac{dG}{d\tau} \).

Equation (3.21) can be written as
\[ \theta'' + \frac{1}{\tau} \theta' = 2k \frac{\gamma - 1}{(2 - \gamma)^2} \theta^{\gamma - 1} \]

via the transformation
\[ G = \theta^{\frac{2}{\gamma} - \gamma}. \]

Thus, a particular solution of Eq. (3.22) is
\[ \theta = \left( \frac{2k}{\gamma - 1} \frac{1}{\tau^2} \right)^{\frac{\gamma - 2}{\gamma - 1}}, \]

which yields (see (3.20) and (3.24))
\[ u = \gamma \left[ \left( \frac{2k}{\gamma - 1} \frac{\tau}{z} \right)^{\frac{2}{\gamma} - 1} - 1 \right]. \]

### 3.2 Reductions by one–dimensional subalgebras

In opposition to what happens in the case of two–dimensional (symmetry) subalgebras, the one–dimensional subalgebras provide reduced equations of (1.2) given by partial differential equations in two independent variables. Hence, in
order to arrive at reduced ordinary differential equations, we have to apply again
the reduction procedure. This emerges in dealing with the one–dimensional sub-
algebras i) \{V_1\}, ii) \{V_2\} and iii) \{V_6\}.

Case i)

Associated with the symmetry vector field \(V_1\), the basis of invariants
\[I_1 = x, \ I_2 \equiv \eta = \frac{x}{y}, \ I_3 \equiv A(\eta, z) = \frac{1}{x}\left(1 + \frac{u}{2}\right)^{\frac{2}{a-2}}, \ (3.26)\]
can be constructed. Then, substitution from the variables \(3.26\) into Eq. \(1.2\) yields the reduced equation
\[
\eta^2(\eta^2 + 1)AA'' + 2\eta(\eta^2 + a)AA' + (a - 1)\eta^2(\eta^2 + 1)A^2 + (a - 1)A^2 = \frac{k}{2} (a - 2) \left(\frac{a}{2}\right)^2 + \frac{A''}{A}, \ (3.27)
\]
where \(A' = \frac{\partial A}{\partial \eta}, \ A_x = \frac{\partial A}{\partial x}\) and \(a = \frac{2}{a-2}\).

The symmetry algebra of Eq. \(3.27\) is expressed by the generators
\[
T_1 = z\partial_x - u\partial_u, \quad (3.28)
T_2 = (\eta^2 + 1)\partial_\eta - \frac{z}{a}\partial_u, \quad (3.29)
T_3 = \partial_z. \quad (3.30)
\]
The reduction by the vector field \(T_1\) leads to the invariants
\[I_1 = \eta, \ I_2 \equiv U(\eta) = zA(\eta, z), \ (3.31)\]
which once replaced into Eq. \(3.27\) provide the ordinary differential equation
\[
\eta^2(\eta^2 + 1)UU'' + 2\eta(\eta^2 + a)UU' + (a - 1)\eta^2(\eta^2 + 1)U^2 + (a - 1)U^2 = \frac{k}{2} (a - 2), \ (3.32)
\]
with \(U' = \frac{dU}{d\eta}\). This equation can be simplified by putting
\[U = \theta^\frac{1}{2}. \quad (3.33)\]
Indeed, we have
\[
\eta^2(\eta^2 + 1)\theta'' + 2\eta(\eta^2 + a)\theta' + a(a - 1)\theta = \frac{k}{2} a(a - 2)\theta^{\frac{a-2}{2}}, \ (3.34)
\]
where the following
\[
\theta = \left(\frac{k a - 2}{2 a - 1}\right)^{\frac{2}{a}} \frac{1}{\eta^a} \quad (3.35)
\]
is a special solution. Now, taking account of (3.35), (3.33), (3.31) and (3.26), we find
\[ u = \gamma \left[ \left( \frac{\gamma - 1}{\gamma} \frac{y^2}{z^2} \right)^{\frac{1}{\gamma - 1}} - 1 \right]. \]  
(3.36)
Furthermore, we observe that (3.32) allows the constant solution
\[ U = \sqrt{k a - 2}, \]
to which the solution (symmetric of (3.36) with respect to the exchange \( y \rightarrow x \)):
\[ u = \gamma \left[ \left( \frac{\gamma - 1}{\gamma} \frac{x^2}{z^2} \right)^{\frac{1}{\gamma - 1}} - 1 \right] \]  
(3.37)
to Eq. (1.2) corresponds.
Another interesting reduced equation can be derived from the symmetry generator \( T_2 \). The related invariants are
\[ I_1 = z, \quad I_2 \equiv W(z) = A(x, \eta) \eta \sqrt{\eta^2 + 1}. \]  
(3.38)
The introduction of (3.38) into Eq. (3.27) gives
\[ W'' + (a - 3)W' = 2k \frac{a(a - 1)}{a - 2} W^3, \]  
(3.39)
with \( W' = \frac{dW}{dz} \).
For \( a = 3 \left( \gamma = \frac{4}{3} \right) \), Eq. (3.39) becomes
\[ W'' = 12k W^3, \]  
(3.40)
which affords the solution
\[ W = \frac{1}{\sqrt{6 z - z_0}} \]  
(3.41)
for \( k = -1 \), \( z_0 \) being an arbitrary constant.
Another solution of Eq. (3.40) can be obtained carrying out a first integration, which furnishes
\[ W' = 6k W^4 + \text{const}. \]  
(3.42)
A second integration provides a solution expressed in terms of elliptic functions \( \text{[2]}, \text{p.543}) \).
In general, by setting \( W = \xi^{3-a} \sigma(\xi), \xi = e^z \), Eq. (3.39) can be written as
\[ \xi \sigma_{\xi \xi} + (4 - a) \sigma_{\xi} - 2k \frac{a(a - 1)}{a - 2} \xi^{-2a+5} \sigma^3 = 0, \]  
(3.43)
that is an equation belonging to the class
\[ \xi \sigma_{\xi \xi} + a_1 \sigma_{\xi} + a_2 \xi \sigma^2 = 0 \]  
(3.44)
(\(a_1, a_2\) are constants) studied by Flower and by other authors, the references of them are quoted in [\(20\), p. 560).

**Case ii)**

The symmetry subalgebra \(\{V_2\}\) gives rise to the reduced equation

\[
u_{\tau\tau} + \frac{1}{\tau}u_\tau = k \left[ \left( 1 + \frac{u}{\gamma} \right) \gamma^{-1} \right]_{zz}, \tag{3.45}\]

where the basis of invariants

\[I_1 \equiv \tau = \sqrt{x^2 + y^2}, \quad I_2 \equiv u(\tau, z) \tag{3.46}\]

has been used.

A group analysis of Eq. (3.45) provides the vector fields

\[
S_1 = z \partial_z + \tau \partial_{\tau}, \tag{3.47}
\]

\[
S_2 = \frac{\gamma-2}{2} z \partial_z + (\gamma + u) \partial_u, \tag{3.48}
\]

\[
S_3 = \partial_z. \tag{3.49}
\]

We note that for \(\gamma = \frac{2}{3}\), in addition to \(S_1, S_2, S_3\) a further generator exists, i.e.

\[
S_4 = \left( \frac{2}{3} + u \right) z \partial_u - \frac{1}{3} z^{-2} \partial_z. \tag{3.50}
\]

The reduced equation coming from (3.47) reads

\[
\rho^2 V'' + \rho V' = k \frac{d^2}{d \rho^2} V^{\gamma-1}, \tag{3.51}
\]

where

\[
V = 1 + \frac{u}{\gamma}, \quad \rho = \frac{z}{\tau}, \quad u = u(\rho). \tag{3.52}
\]

A particular solution of Eq. (3.51) is

\[
V = \left( \frac{2k}{\gamma - 1} \right)^{\frac{1}{\gamma-2}} \rho^{\frac{2}{\gamma-2}}, \tag{3.53}
\]

which yields

\[
u = \gamma \left[ \left( \frac{2k}{\gamma - 1} \right)^{\frac{1}{\gamma-2}} \left( \frac{z}{\tau} \right)^{\frac{2}{\gamma-2}} - 1 \right] \tag{3.54}\]

from (3.52), with \(\tau = \sqrt{x^2 + y^2}\).

On the other hand, the generator (3.48) leads to the reduced equation

\[
W'' + \frac{1}{\tau} W' = \lambda W^{\gamma-1}, \tag{3.55}\]
where \( w(\tau) = \frac{\gamma + u}{\sqrt{\tau}} \), \( \tau = \sqrt{x^2 + y^2} \), \( W' = \frac{dW}{d\tau} \) and \( \lambda = \frac{2k}{\gamma - 2} \frac{3 - 1}{\gamma - 2} \).

A solution of Eq. (3.55) is

\[
W = W_0 \tau - \frac{2}{\gamma - 2},
\]

(3.56)

with \( W_0 = \gamma \left( \frac{2k}{\gamma - 1} \right) \). In correspondence of (3.56), the following solution

\[
u = W_0 \left( \frac{z^2}{x^2 + y^2} \right)^{\frac{1}{\gamma - 2}} - \gamma
\]

(3.57)

to Eq. (1.2) is found.

Case iii)

By solving the equation \( V_6 I = 0 \), we get the set of basis invariants

\[
I_1 = x, \quad I_2 = y, \quad I_3 = W(x, y) = \ln \left[ \frac{1}{z^2} \left( 1 + \frac{u}{\gamma} \right)^{\gamma - 2} \right].
\]

(3.58)

Then, from (1.2) we have the reduced equation

\[
W_{xx} + W_{yy} + \frac{1}{\gamma - 2} (W_x^2 + W_y^2) = \frac{2k(\gamma - 1)}{\gamma - 2} e^W.
\]

(3.59)

The group technique can be applied to obtain a further reduction of Eq. (3.59). In doing so, we find the symmetry generator

\[
N = (-c_1 x + c_2 y + c_3) \partial_x + (-c_2 x - c_1 y + c_4) \partial_y + 2c_1 \partial_W.
\]

(3.60)

From (3.60) we deduce the independent vector fields

\[
N_1 = -x \partial_x - y \partial_y + 2 \partial_W,
\]

(3.61)

\[
N_2 = y \partial_x - x \partial_y,
\]

(3.62)

\[
N_3 = \partial_x,
\]

(3.63)

\[
N_4 = \partial_y,
\]

(3.64)

which fulfill the commutation relations

\[
[N_1, N_2] = 0, \quad [N_1, N_3] = N_3, \quad [N_1, N_4] = N_4,
\]

(3.65)

\[
[N_2, N_3] = N_4, \quad [N_2, N_4] = -N_3, \quad [N_3, N_4] = 0.
\]

(3.66)

It is noteworthy that Eqs. (3.65)–(3.66) define a Lie algebra isomorphic to the algebra \( \text{so}(2) \) of the similitude group in \( R^2 \), governed by the rules (21):

\[
[X_1, X_2] = -X_3, \quad [X_1, X_3] = X_2, \quad [X_1, X_3] = 0, \quad [X_2, X_4] = X_2, \quad [X_3, X_4] = X_3,
\]

\[
[X_1, X_4] = 0. \quad \text{This can be checked by identifying} \quad N_1, N_2, N_3, N_4 \text{ with} \quad -X_4, -X_1, X_2, X_3, \text{ respectively.}
\]
Now let us examine the generator $N_1$, whose a set of symmetry variables is

$$I_1 \equiv t = \frac{y}{x}, \quad I_2 \equiv \sigma = 2 \ln x + W(x, y). \quad (3.67)$$

The reduced equation of (3.59) emerging from (3.67) reads

$$(t^2 + 1)\sigma_{tt} + \frac{1}{\gamma - 2}(t^2 + 1)\sigma_t^2 + \frac{2\gamma}{\gamma - 2}t\sigma_t + \frac{2\gamma}{\gamma - 2} = \frac{2k(\gamma - 1)}{\gamma - 2}, \quad (3.68)$$

which is solved by

$$\sigma = \ln \left[ \frac{k\gamma(a^2 + b^2)}{\gamma - 1} \right]^{\frac{1}{\gamma - 2}} \left( \gamma \right) - 1, \quad (3.69)$$

where $a$ and $b$ are constants of integration.

Combining together (3.69), (3.67) and (3.58), we obtain the class of $\infty^2$-solutions

$$u = \gamma \left\{ \frac{k\gamma(a^2 + b^2)}{(ax + by)^2} \right\}^{\frac{1}{\gamma - 2}}. \quad (3.70)$$

At this stage it is instructive to look for other nontrivial solutions of (1.2). To this aim, let us set

$$W = (\gamma - 2) \ln \psi \quad (3.71)$$

into Eq. (3.59). Then, this equation takes the form

$$\psi_{xx} + \psi_{yy} = 2k\frac{\gamma - 1}{(\gamma - 2)^2} \psi^{\gamma - 1}. \quad (3.72)$$

By way of example, we seek a solution of the type $\psi = \psi(\xi)$, where $\xi = x - vy$ and $v$ is a (real) constant. In such a manner Eq. (3.72) transforms into the ( nonlinear) ordinary differential equation

$$\psi_{\xi\xi} = 2k\frac{\gamma - 1}{1 + v^2(\gamma - 2)^2} \psi^{\gamma - 1}, \quad (3.73)$$

from which

$$\psi^2 = a\psi^\gamma + c, \quad (3.74)$$

where $c$ is a constant of integration and $a = \frac{4k}{1 + v^2(\gamma - 2)^2}$. For $c = 0$ Eq. (3.74) provides

$$\psi = \pm \left[ a^{\frac{1}{2}} \left( \frac{2 - \gamma}{2} \right) \right]^{\frac{\gamma - 1}{2(\gamma - 2)}} (\xi - \xi_0)^{\frac{1}{\gamma - 2}}, \quad (3.75)$$

$\xi_0$ being a constant of integration.

On the other hand, if $c \neq 0$ we obtain

$$\int_0^\psi \frac{d\psi'}{\sqrt{1 + b\psi'^2}} = \pm \sqrt{c}(\xi - \xi_0), \quad (3.76)$$
with \( b = \frac{\gamma}{2} \).

The integral at the left-hand side of (3.76) can be evaluated by Eq. (3.9). Indeed, putting in (3.76) \( X = \psi^{\gamma} \), from Eq. (3.9) we find

\[
\gamma \psi^2 F_1 \left( \frac{1}{2}; \frac{1}{2}; \frac{1+\gamma}{2}; -b \psi^\gamma \right) = \pm \sqrt{c} (\xi - \xi_0),
\]

with \( \mu = \frac{1}{\gamma} \) and \( \nu = \frac{1}{2} \).

With the help of (3.58) and (3.71), we get the exact solution

\[
u = \gamma \left( z^{\frac{2}{1+\gamma}} \psi - 1 \right)
\]

(3.78) to Eq. (1.2), where \( \psi = \psi(x - vy) \) is explicitly known if one can invert Eq. (3.77).

Equation (3.77) covers some cases related to potentials of physical interest, arising for a) \( \gamma = 3 \), b) \( \gamma = -2 \), c) \( \gamma = \frac{5}{2} \), d) \( \gamma = \frac{5}{3} \), and e) \( \gamma = -1 \) (see (1.3)).

The first two choices correspond, respectively, to the Fermi–Pastur–Ulam potential [9] and to a potential whose nonlinear part, \( 1/(1 - u^2)^2 \), finds applications in the treatment of the scattering states in conformally invariant Quantum Mechanics [22]. In cylindric coordinates, case c) leads to an equation involved in the Thomas–Fermi model of an atom ([12] p. 116), while in the case d) Eq. (3.74) appears in the theory of the white dwarf stars ([13], p. 113). Finally, for \( \gamma = -1 \) (case e)) the potential (1.3) takes the form \( \phi \sim (1 - u)^{-1} - (1 + u) \), which mimics a special case of the Killingbeck potential \( V = -\frac{A}{2} + Br + Cr^2 \) [23]. We recall that a Coulomb potential perturbed by a second degree polynomial can be used to study the ground state properties of a hydrogen atom.

For brevity, below we handle in detail only the cases a) and b).

Case a)

For \( \gamma = 3 \), Eq. (3.74) can be integrated straightforwardly. Hence, first we shall evaluate \( \psi \) explicitly from (3.74).

Subsequently, we shall compare the expression of \( \psi \) determined in such a way with that coming from (3.77). In doing so, let us reshape Eq. (3.74) into

\[
\psi^2 \eta = 4 \psi^3 - g_3,
\]

(3.79)

after the rescaling \( \eta = \frac{\sqrt{-2}}{3(1+\psi^2)} \xi \), with \( c = -\frac{2}{3(1+\psi^2)} g_3 \) and \( k = 1 \).

Equation (3.73) is a special version of the equation

\[
\psi^2 \eta = 4 \psi^3 - g_2 \psi - g_3,
\]

(3.80)

which is satisfied by the Weierstrass elliptic function \( \wp(\eta; g_2, g_3) \), where \( g_2 \) and \( g_3 \) are the invariants of \( \wp \) ([24] p. 629).

For \( g_3 = 0 \) we have

\[
\psi(\eta) = \wp(\eta; 0, 0) = \frac{1}{\eta^2},
\]

(3.81)
that is the first term of the series representation of \( \wp(\eta; g_2, g_3) \) \([24]\) p. 635.

Then, \((3.78)\) gives
\[
 u = 3 \left( \frac{z^2}{\eta^2} - 1 \right) .
\] (3.82)

For \( g_3 \neq 0, \) Eq. \((3.80)\) yields
\[
 \int_\psi^{\infty} \frac{d\psi'}{\sqrt{4\psi'^3 - g_3}} = \eta ,
\] (3.83)
namely
\[
 \psi = \wp(\eta; 0, g_3) .
\] (3.84)

Thus, we infer that
\[
 u = 3 \left[ z^2 \wp(\eta; 0, g_3) - 1 \right] .
\] (3.85)

from \((3.78)\).

Combining \((3.84)\) together with \((3.77)\), it can be shown (see Appendix B) that the remarkable formula
\[
 \wp^{-1} = \sigma_2 F_1 \left( \frac{1}{2}, \frac{1}{3}; \frac{4}{3}; -4\sigma^3 \right) - \frac{4^{1/3}B}{3} \left( \frac{1}{3}, \frac{1}{6} \right)
\] (3.86)
holds, where \( \wp^{-1} \) stands for the inverse Weierstrass function \((g_2 = 0), \) \( \sigma = -\frac{\psi}{g_3^{1/3}}, \)
and \( B(p, q) \) is the beta function (Euler’s integral of the first kind) \(\) \([19]\) p. 948. The relation \((3.86)\) establishes a link between the Gauss hypergeometric function and the inverse Weierstrass function \((g_2 = 0).\)

Case \( b)\)

If \( \gamma = -2, \) Eq. \((3.77)\) furnishes
\[
 -2\psi^2 F_1 \left( \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; -b\psi^{-2} \right) = \pm \sqrt{c(\xi - \xi_0)} ,
\] (3.87)
where \( b = \frac{8k}{\pi(1+\gamma)}, \) and
\[
 F_1 \left( \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; -b\psi^{-2} \right) = \sqrt{1 + b\psi^{-2}}
\] (3.88)
(see \([19]\) p. 1042).

Consequently,
\[
 u = -2 \left( \sqrt{c(\xi - \xi_0)^2 - 4b} \right) \frac{4}{4z} - 1
\] (3.89)
from \((3.78)\).
4 Instanton–like approximate solution

Equation (3.59) can be exploited as well to give approximate solutions to Eq. (1.2) of the instanton–like type. To this purpose, it is convenient to write Eq. (3.59) in the form

\[ \partial_\eta \partial_\eta W \frac{1}{\gamma - 2} W + \frac{1}{2} \gamma - 1 \gamma ^{- 2} e^W = \frac{k}{2} \gamma - 1 \gamma ^{- 2} e^W, \] (4.1)

where \( \eta = x + iy, \bar{\eta} = x - iy. \)

For \( \gamma \to \infty, \) Eq. (4.1) reproduces the Liouville equation

\[ \partial_\eta \partial_\eta W \frac{1}{2} e^W, \] (4.2)

which is conformally invariant, in the sense that if \( W(\eta, \bar{\eta}) \) is a solution of Eq. (4.2), the function \( \tilde{W} = \tilde{W}(\tilde{\eta}, \bar{\tilde{\eta}}) \) given by

\[ W = \tilde{W} - \ln f' \bar{f}' \] (4.3)

is also a solution, where \( \eta = f(\tilde{\eta}), \bar{\eta} = f(\bar{\tilde{\eta}}), \) and \( f' = f'_\tilde{\eta}, \bar{f}' = f'_{\bar{\tilde{\eta}}}. \)

We notice that the Liouville equation emerges as a symmetry reductio n of the continuous Toda equation (1.1) [12] corresponding to the gene rator \( V_6(\infty) = z \partial_z + 2 \partial_u, \) coming from (2.9) for \( \gamma \to \infty. \) Equation (4.2) affords the general solution

\[ e^W = \frac{4 f_\tilde{\eta}(\eta) \bar{f}'(\eta)}{1 + f(\eta) f'(\eta)^2} \] (4.4)

for \( k = -1, \) where \( f(\eta) \) denotes an arbitrary holomorphic function. This expression can be exploited to get (exact) instanton solutions to Eq. (1.1), [12], [14]. Conversely, we are not able to obtain exact instanton solutions to Eq. (1.1). However, we can find (heuristically) an approximate instanton-like solution, which arises from the assumption

\[ W = W_0 + \epsilon \phi, \] (4.5)

where \( W_0 \) satisfies (4.3), \( \epsilon = \frac{1}{\gamma - 2} \) is such that \( |\epsilon| \ll 1, \) and \( \phi \) is a function of \( \eta, \bar{\eta} \) to be determined.

Substitution from (4.5) into (4.1) gives

\[ \epsilon^0 : W_0 \eta \bar{\eta} = \frac{k}{2} e^{W_0} \] (4.6)

\[ \epsilon : \phi \eta \bar{\eta} + W_0 \eta \bar{\eta} W (\eta) = \frac{k}{2} e^{W_0} (1 + \phi), \] (4.7)
for large values of $\gamma$. At this point we choose $k = -1, f = \eta, \bar{f} = \overline{\eta}$ in (4.4). Then, $W_0$ becomes the instanton configuration

$$e^{W_0} = \frac{4}{(1 + \eta\overline{\eta})^2}. \quad (4.8)$$

With the help of (4.8), Eq. (4.7) can be written as

$$g_{rr} + \frac{1}{r}g_r + \frac{1}{r^2}g_{\theta\theta} + \frac{8g}{(1 + r^2)^2} = 8 \quad (4.9)$$

in polar coordinates ($x = r \cos \theta, y = r \sin \theta$), where

$$g(r, \theta) = \phi(r, \theta) + 2r^2 + 1. \quad (4.10)$$

We are interested in solutions of (4.9) of the type $g = g(r, 0)$. Therefore, by introducing the change of variable $r = \left(\frac{1 + t^2}{1 - t^2}\right)^{1/2}$, Eq. (4.9) takes the form

$$(1 - t^2)g_{tt} - 2tg_t + 2g = \frac{8}{(1 - t^2)^2}. \quad (4.11)$$

It is noteworthy that, in general, this equation can be solved exactly. In fact, its homogeneous part coincides with a special case of the hypergeometric equation defining the Legendre functions (24, p. 331). Precisely, the homogeneous part of (4.11) allows the independent integrals

$$g_1(t) = P_1(t) \equiv t, \quad (4.12)$$

$$g_2(t) = Q_1(t) \equiv \frac{1}{2}t \ln \frac{t + 1}{t - 1} - 1, \quad (4.13)$$

where $P_1$ and $Q_1$ are the Legendre polynomial and the Legendre function of the second kind, respectively, corresponding to $n = 1$.

In terms of $r^2$, we have

$$g_1 = \frac{r^2 - 1}{r^2 + 1}, \quad (4.14)$$

$$g_2 = \frac{1}{2} \frac{r^2 - 1}{r^2 + 1} \ln r^2 - 1 + \frac{i\pi}{2}g_1. \quad (4.15)$$

Furthermore, a particular solution to Eq. (4.11) is

$$g_0 = 2 \left\{ 4 \ln (1 + r^2) - 5 + r^2 + \frac{r^2 - 1}{r^2 + 1} \left[2\text{dilog}(1 + r^2) + \ln r^2\right] \right\}, \quad (4.16)$$

where

$$\text{dilog}(x) = \int_1^x \frac{\ln t}{1 - t} \, dt \quad (4.17)$$
is the dilogarithm function ($[24]$ p. 1004).

Now coming back to the original function $\phi$ (see (4.10)), we are led to the following instanton-like approximate solution to Eq. (4.1) (see (4.5), (4.8)):

$$W = \ln \frac{4}{(1 + r^2)^2} + \epsilon \left\{ c_1 \frac{r^2 - 1}{r^2 + 1} + c_2 \left( \frac{1 + r^2}{2} - \ln r^2 - 1 \right) + 8 \ln (1 + r^2) - 10 + 2 \frac{r^2 - 1}{r^2 + 1} \left[ \text{dilog}(1 + r^2) + \ln r^2 \right] - 1 \right\}, \tag{4.18}$$

$c_1, c_2$ being arbitrary constants. This formula is meaningful for small $\epsilon$ (large $\gamma$).

In order to confer an instanton character to the solution (4.18), we need to choose $c_2 = -4$. Thus, the function $W$ becomes singularity free, precisely:

$$W = \ln \frac{4}{(1 + r^2)^2} + \epsilon \left\{ \frac{r^2 - 1}{r^2 + 1} \left[ c_1 + 4 \text{dilog}(1 + r^2) \right] + 8 \ln (1 + r^2) - 7 \right\}. \tag{4.19}$$

For small values of $\epsilon$, the function (4.19) can be used as a good approximation of the regular solution $W_I(r)$ of Eq. (3.59) expressed in terms of $r$. In Appendix C we report two Tables, Ia and Ib, containing values of $W(r)$ and $W_I(r)$ for some $r$ with $\gamma$ fixed. The initial conditions have been chosen in such a way that $c_1 = -7$ and $c_2 = -4$. From the Tables one argues that $W(r)$ and $W_I(r)$ are quite closed already for $\gamma = 6$ (i.e. $\epsilon = 0.25$). The quantity $|W(r) - W_I(r)|$ tends to zero as $\gamma$ increases.

## 5 Extension of Eq. (1.2) to $(n + 1)$-dimensions

Let us consider the generalized version

$$\left( \partial^2_{x_1} + \cdots + \partial^2_{x_n} \right) u = k \left[ \frac{1 + u}{\gamma} \right]^{\gamma-1} \tag{5.1}$$

of Eq. (1.2), where $u = u(x_1, \ldots, x_n, z)$, with $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Assuming that $u = u(r, z)$, where $r = \sqrt{x_1^2 + \cdots + x_n^2}$, Eq. (5.1) can be written as

$$u_{rr} + \frac{n - 1}{r} u_r = k \left[ \frac{1 + u}{\gamma} \right]^{\gamma-1}. \tag{5.2}$$

The application of the symmetry approach to Eq. (5.2) provides the symmetry generator

$$G = k_1 r \partial_r + \left[ k_1 + \frac{\gamma - 2}{2} k_2 \right] \partial_z + k_2 (u + \gamma) \partial_u, \tag{5.3}$$

with $k_1, k_2, k_3$ arbitrary constants.
From (5.3) we get the operators
\[ G_1 = r \partial_r + z \partial_z, \]  
\[ G_2 = \frac{\gamma - 2}{2} z \partial_z + (\gamma + u) \partial_u, \]  
\[ G_3 = \partial_z, \]  
which satisfy the commutation relations
\[ [G_1, G_2] = 0, \quad [G_1, G_3] = -G_3, \quad [G_2, G_3] = -\frac{\gamma - 2}{2} G_3. \]  
(5.7)

The symmetry operator \( G_2 \) is of particular interest. In fact, it gives rise to the invariants \( r' = r, \ W'(r') = W(r) \), with
\[ W(r) = z^{-\frac{\gamma - 2}{2}} (\gamma + u), \]  
(5.8)

which leads to the ordinary differential equation
\[ W_{rr} + \frac{n - 1}{r} W_r = \frac{2(\gamma - 1)k}{\gamma^{-2}(\gamma - 2)^2} W^{\gamma - 1}. \]  
(5.9)

Equation (5.9) becomes
\[ \psi_{rr} + \frac{n - 1}{r} \psi_r = g \psi^{\gamma - 1}, \]  
(5.10)

with \( g = \frac{2(\gamma - 1)k}{(\gamma - 2)^2} \), by setting \( W = \gamma \psi \). The link between \( u \) and \( \psi \) is given by (see (5.8))
\[ u = \gamma \left( z^{\frac{\gamma - 2}{2}} \psi - 1 \right). \]  
(5.11)

Now we look for a transformation of the type
\[ r' = \frac{1}{r}, \quad \psi = f(r') \tilde{\psi}(r'), \]  
(5.12)

which leaves Equation (5.10) invariant, in the sense that
\[ \tilde{\psi}_{r' r'} + \frac{n - 1}{r'} \tilde{\psi}' = g \tilde{\psi}^{\gamma - 1}. \]  
(5.13)

A straightforward calculation shows that this can be accomplished if
\[ f(r') = r'^{n - 2}, \quad \gamma = \frac{2n}{n - 2}. \]  
(5.14)

In this case Eq. (5.10) takes the form
\[ \psi_{rr} + \frac{n - 1}{r} \psi_r = g_0 \psi^{\frac{n - 2}{n - 2}}, \]  
(5.15)
with
\[ g_0 = \frac{k}{8}(n^2 - 4) \quad (n \neq 2). \] (5.16)

Furthermore, it is easy to see that if \( \psi(\tilde{r}) \) is a solution of Eq. (5.15), i.e.
\[ \psi_{\tilde{r}\tilde{r}} + \frac{n - 1}{\tilde{r}} \psi_{\tilde{r}} = g_0 \psi^{\frac{n+2}{n-2}}, \] (5.17)

then \( \psi(r) = \lambda^{\frac{n-2}{2}} \psi(\tilde{r}) \), where \( \tilde{r} = \lambda r \) and \( \lambda \) is a real parameter, is also a solution.

To summarize, we have

**Proposition.** Equation (5.15) is invariant under the conformal transformation
\[ r' = \frac{1}{\lambda r}, \quad \psi(r) = r'^{n-2} \lambda^{\frac{n-2}{2}} \psi(r'). \] (5.18)

The positive (spherically symmetric) regular solution to Eq. (5.15) is given by
\[ \psi(r) = \frac{a}{(1 + r^2)^{\frac{n-2}{2}}}, \] (5.19)

with \( a = \left( -\frac{8k}{n+2} \right)^{\frac{n-2}{2}} \), where \( k \) and \( n \) are to be chosen in such a way that \( a > 0 \).

We notice that Eq. (5.15) comprises remarkable special cases, which are listed in the following Table:

| Case | \( n \) | \( \gamma \) | Equation (5.15) | Regular solution \( \psi(r) \) \( (k = -1) \) |
|------|--------|--------|------------------|----------------------------------|
| a    | 1      | -2     | \( \psi_{rr} = -\frac{3}{2}k\psi^{-3} \) | \( (-\frac{8k}{3})^{-1/4} (1 + r^2)^{1/2} \) |
| b    | 3      | 6      | \( \psi_{rr} + \frac{2}{r} \psi_r = \frac{2}{k} \psi^{5} \) | \( (-\frac{16}{k})^{1/4} (1 + r^2)^{-1/2} \) |
| c    | 4      | 4      | \( \psi_{rr} + \frac{4}{r} \psi_r = \frac{2}{k} \psi^{3} \) | \( (-\frac{16}{k})^{1/2} (1 + r^2)^{-1} \) |
| d    | 6      | 3      | \( \psi_{rr} + \frac{4}{r} \psi_r = 4k \psi^{3} \) | \( -6k(1 + r^2)^{-2} \) |
| e    | 10/2   | 2      | \( \psi_{rr} + \frac{4}{r} \psi_r = 12k \psi^{3/2} \) | \( \frac{10}{9} (1 + r^2)^{-4} \) |

The solutions of the original Equation (5.9), related to cases a), ..., e), arise from
\[ u = \frac{2n}{n - 2} \left( z^{\frac{n-2}{2}} \psi - 1 \right) \] (5.20)

(see (5.11)).

It is noteworthy that Eq. (5.15) appears in the context of Differential Geometry [12], [13], in particular it is related to the so-called Yamabe problem, which consists essentially in establishing when a Riemannian metric can be changed by a conformal (length) factor to have constant scalar curvature [25].
Moreover, some special cases of Eq. (5.15) play a basic role in certain branches of physics. Precisely, the equation of case b) of the Table II constitutes a static and spherically symmetric version of a nonlinear wave equation discussed by Rosen some years ago [26]. Equation b), which is of the Emden type, finds application in Astrophysics.

Equation e) can be interpreted as an extended (elliptic) conformal invariant version of the equation

\[ \Phi_{rr} + \frac{2}{r} \Phi_r = C \Phi^{3/2} \]  

(5.21)
governing the Thomas–Fermi model of an atom, where \( C > 0 \) is a certain constant and \( \Phi(r) \) is the potential field originated by \( Z - 1 \) electrons, acting on the \( Z^{1b} \) one ([...], p. 125).

Equation a) corresponds to a potential whose nonlinear part, \( 1/(1 - \frac{u^2}{r^2}) \), mimics the inverse square potential appearing in the treatment of the scattering states in conformally invariant Quantum Mechanics [22]. Finally, Eq. d) is associated with the Fermi–Pasta–Ulam potential.

To conclude the group analysis of Eq. (5.2), we mention another non trivial reduction, which can be written down starting from the generator \( G_0 = G_1 + G_2 + G_3 \). It reads

\[ X^2W_{XX} + (n - 1)(W - WX_X) = \frac{4(n-1)}{\gamma^2-1} \frac{1}{X^{\gamma-2}} W^{\gamma-2} W_X X + \]

\[ \frac{2n-2}{\gamma^2-1} \frac{W^{\gamma-3} W_X}{X^{\gamma-1}} (2WX_X + W), \]  

(5.22)

where \( X = \frac{(\gamma+2)^{\frac{2}{\gamma}}}{r} \) and \( W = W(X) = \frac{u+\gamma}{r} \) are two invariants. Equation (5.22) will not be discussed here.

6 Relation to the Yang–Mills theory

Equation c) of Table II can be considered as a reduction form of the Yang–Mills equations (see later). This is a well–known result [16], [17]. What is new, is the link between special solutions of the Yang–Mills equations and special solutions of the field equation (5.1) for \( n = 4 \). To this aim, let us start from Eq. (5.10), which can be written as

\[ \Omega_{\tau\tau} + \frac{n-2}{\alpha-1} \left( \frac{n-2}{n-2} - \alpha \right) \Omega_{\tau} - 2 \frac{n-2}{(\alpha-1)^2} \left( \alpha - \frac{n}{n-2} \right) \Omega + \Omega^\alpha = 0, \]  

(6.1)

via the transformations

\[ r = e^{-\tau}, \quad \Omega = \frac{1}{\mu} \frac{d}{dr} \psi(r), \]  

(6.2)

with \( \mu = \left( \frac{(\alpha-1)^2}{2\alpha} \right)^{\frac{1}{\alpha-1}} \), and \( \alpha = \gamma - 1 \). We notice that Eq. (6.1) is the same as Eq. (1.6) contained in [16]. Therefore, all the results achieved about this
equation can be transferred to the field equation (5.1) through the formula (5.11), namely
\[
 u = (\alpha + 1) \left[ \frac{(\alpha - 1 \sqrt{2 \alpha} z)}{r} \right]^{\frac{1}{\alpha - 1}} \Omega - 1, \quad (6.3)
\]
where \( \Omega \) and \( r \) are expressed by (6.2).

In general, Eq. (6.1) may be investigated using phase–space variables. Here we restrict ourselves to the case \( \alpha = \frac{n+2}{n-2} \), in correspondence of which Eq. (6.1) reads
\[
 \Omega_{\tau\tau} - \left( \frac{n - 2}{2} \right)^2 \Omega + \Omega^{\frac{n+2}{n-2}} \Omega^{n-2} = 0. \quad (6.4)
\]
Equation (6.4) can be solved exactly. Indeed, we easily find
\[
 \Omega_{\tau}^2 - \left( \frac{n - 2}{2} \right)^2 \Omega^2 + \frac{n - 2}{n} \Omega^{\frac{n+2}{n-2}} = c, \quad (6.5)
\]
where \( c \) is a constant of integration.

By choosing for example \( c = 0 \), a simple calculation gives
\[
 \Omega(r) = \left[ \lambda \sqrt{\frac{n(n - 2)}{r^2 + \lambda^2}} \right]^{\frac{n-2}{2}}, \quad (6.6)
\]
or, in the variable \( \psi(r) \) (see (6.2)):
\[
 \psi(r) = \left[ 4\lambda \sqrt{\frac{n}{2(n + 2)} r^2 + \lambda^2} \right]^{\frac{n-2}{2}}. \quad (6.7)
\]
This function represents a regular solution to the equation
\[
 \psi_{rr} + \frac{n - 1}{r} \psi_r + \frac{n^2 - 4}{8} \psi^{\frac{n+2}{n-2}} = 0, \quad (6.8)
\]
which comes from Eq. (5.10) for \( \alpha = \gamma - 1 = \frac{n+2}{n-2} \). On the other hand, replacing (6.6) into (6.3) provides the exact regular solution
\[
 u = \frac{2n}{n - 2} \left\{ \left[ 4\lambda \sqrt{\frac{n}{2(n + 2)} \frac{z}{z^2 + \lambda^2}} \right]^{\frac{n-2}{2}} - 1 \right\}, \quad (6.9)
\]
to the field equation (5.1) \( (\gamma = \frac{2n}{n-2}) \).

Equation (6.4) admits also singular solutions \( \{16\} \). The link between this kind of solutions and the corresponding ones of Eq. (5.1) can be obtained via (6.3). At this stage, we point out that for \( n = 4 \), a connection can be established
between solutions of Eq. (5.1) and solutions of the Euclidean Yang–Mills (YM) equations. To pursue this goal, let us introduce some preliminaries.

The pure YM equations (i.e., in absence of matter fields) are

$$
\partial_\mu F^{\mu\nu}(x) + [A_\mu(x), F^{\mu\nu}(x)] = 0
$$

(6.10)

$$(\mu, \nu = 0, 1, 2, 3),$$

where the summation convention is understood.

The fields $A_\mu(x)$ take values on the Lie algebra $G$ of a compact semisimple Lie group $G$ (the gauge group). By choosing SU(2) as the gauge group, we can write

$$
A_\mu(x) = \frac{1}{2i} A^a_\mu(x) \sigma^a
$$

(6.11)

$$(a = 1, 2, 3),$$

where $\sigma^a$ are the Pauli matrices and $A^a_\mu = i \text{Tr}\{A_\mu(x) \sigma^a\}$.

On the other hand, the tensor $F^{\mu\nu}$ associated with $A_\mu(x)$, defined by

$$
F^{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],
$$

(6.12)

is expressed by

$$
F^{\mu\nu}(x) = \frac{1}{2i} F^a_{\mu\nu}(x) \sigma^a,
$$

(6.13)

$$_a = i \text{Tr}\{F_{\mu\nu}(x) \sigma^a\}.
$$

Now, inserting the 't Hooft ansatz [17]

$$
A^a_\mu = -\eta_{a\mu\nu} \partial_\nu \ln \psi(x)
$$

(6.14)

into Eq. (6.10), where the tensor $\eta_{a\mu\nu}$ will be specified later (see Appendix D), we obtain the equation

$$
\nabla^2 \psi + K \psi^3 = 0,
$$

(6.15)

$K$ being a constant. By writing $\nabla^2$ in spherical coordinates and putting $K = \frac{2}{3}$, Eq. (6.15) becomes just Eq. (6.8) for $n = 4$. Then, by virtue of (6.7), (6.2), (6.6) and (6.3), from (6.14) we find the solution to the Yang–Mills equations (6.12):

$$
A^a_\mu = -\eta_{a\mu\nu} \partial_\nu \ln \left[ \frac{1}{z} \left( 1 + \frac{u}{4} \right) \right],
$$

(6.16)

where $u$ is the regular solution

$$
u = 4 \left( \frac{4\lambda z}{\sqrt{3} r^2 + \lambda^2} - 1 \right)
$$

(6.17)

to the nonlinear field equation (see (5.2))

$$
u_{rr} + \frac{3}{r} \nu_r = - \left[ \left( 1 + \frac{u}{4} \right)^3 \right]_{zzz},
$$

(6.18)

with $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$.

For completeness, in Appendix D a detailed proof of the correspondence between Eq. (6.13) and (6.12) is presented.
7 Conclusion

Using a group–theoretical approach, we have investigated the nonlinear field equation (1.2), arising from a lattice of the binomial type as a continuous approximation, and its extension (5.1) to $(n + 1)$–dimensions. For $\gamma \to \infty$, both equations take the form of Toda field equations. We have determined the symmetry algebra (2.10)–(2.14) admitted by Eq. (2.2). This algebra is finite–dimensional, while that allowed by the Toda field equation (1.1) is infinite–dimensional. This result constitutes a first discrimination between the two equations. Another important difference is that Eq. (1.1) enjoys the conformal invariance property. On the contrary, this property is not valid for Eq. (1.2), but it is regained through Eq. (5.15) $(n \neq 2)$. In other words, the conformal invariance property holds again within the generalized equation (5.1). This feature enables us to build up conformal versions of physically interesting models, listed in Table II. These conformal models afford both regular and singular solutions reflecting on the solutions of Eq. (5.1) via the transformation (5.20), which comes from the reduction procedure applied to Eq. (5.2) (the spherically symmetric form of Eq. (5.1)). The introduction of the generalized equation (5.1) leads to a scenario where several nonlinear field equations appearing in different physical and differential geometrical contexts can be dealt with in a unifying manner. This is the spirit of the present work, which has essentially a speculative character.

Appendix A: explicit form of $pr^{(2)}V$

The second prolongation at the left–hand side of Eq. (2.3) is given by

$$pr^{(2)}V = pr^{(1)}V + \phi^{xx} \frac{\partial}{\partial u_x} + \phi^{xy} \frac{\partial}{\partial u_y} + \phi^{xz} \frac{\partial}{\partial u_z} + \phi^{yy} \frac{\partial}{\partial u_y} + \phi^{yz} \frac{\partial}{\partial u_z} + \phi^{zz} \frac{\partial}{\partial u_z}$$

**(A.1)**

where

$$pr^{(1)}V = V + \phi^{x} \frac{\partial}{\partial u_x} + \phi^{y} \frac{\partial}{\partial u_y} + \phi^{z} \frac{\partial}{\partial u_z}$$

**(A.2)**

$$\phi^{x} = D_x (\phi - \xi u_x - \eta u_y - \zeta u_z) + \xi u_{xx} + \eta u_{xy} + \zeta u_{xz}$$

**(A.3)**

$$\phi^{y} = D_y (\phi - \xi u_x - \eta u_y - \zeta u_z) + \xi u_{xy} + \eta u_{yy} + \zeta u_{yz}$$

**(A.4)**

$$\phi^{z} = D_z (\phi - \xi u_x - \eta u_y - \zeta u_z) + \xi u_{xz} + \eta u_{yz} + \zeta u_{zz}$$

**(A.5)**

$$\phi^{xx} = D_x^2 (\phi - \xi u_x - \eta u_y - \zeta u_z) + \xi u_{xxx} + \eta u_{xyy} + \zeta u_{xzz}$$

**(A.6)**

$$\phi^{xy} = D_x D_y (\phi - \xi u_x - \eta u_y - \zeta u_z) + \xi u_{xyy} + \eta u_{xyy} + \zeta u_{xyy}$$

**(A.7)**

$$\phi^{xz} = D_x D_z (\phi - \xi u_x - \eta u_y - \zeta u_z) + \xi u_{xzz} + \eta u_{yzz} + \zeta u_{yzz}$$

**(A.8)**

$$\phi^{yy} = D_y^2 (\phi - \xi u_x - \eta u_y - \zeta u_z) + \xi u_{yyy} + \eta u_{yyy} + \zeta u_{yyy}$$

**(A.9)**

$$\phi^{yz} = D_y D_z (\phi - \xi u_x - \eta u_y - \zeta u_z) + \xi u_{yzz} + \eta u_{yzz} + \zeta u_{yzz}$$

**(A.10)**

$$\phi^{zz} = D_z^2 (\phi - \xi u_x - \eta u_y - \zeta u_z) + \xi u_{zzz} + \eta u_{zzz} + \zeta u_{zzz}$$

**(A.11)**
and $D_x$, $D_y$, $D_z$ denote the total derivative operators

$$
D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u}, \quad D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u}, \quad D_z = \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u}.
$$

(A.12)

**Appendix B: derivation of Eq. (3.86)**

Let us start from (3.79), which can be written as

$$
\frac{d\sigma}{\sqrt{1 + 4\sigma^3}} = ig^\frac{1}{3} d\eta \tag{B.1}
$$

via the substitution $\psi = -g^\frac{1}{3} \sigma$. Then

$$
\int_{\sigma}^{\infty} \frac{d\sigma'}{\sqrt{1 + 4\sigma'^3}} = \frac{1}{3} \int_{0}^{\infty} X^{-2/3} dX - \frac{1}{3} \int_{0}^{\sigma^3} X^{-2/3} dX = ig^\frac{1}{3} (\eta - \eta_\infty), \tag{B.2}
$$

where $X = \sigma^3$ and $\eta$ is a constant.

Now, we recall that ([19], p. 285, formula 3):

$$
\int_{0}^{1} t^{\mu-1}(1-t)^{\nu-y} dt = b^{-\mu} B(\mu, \nu - \mu) \tag{B.3}
$$

with $|\arg b| < \pi$, $Re \nu > Re \mu > 0$, where $B(x, y)$ is the beta function (Euler’s integral of the first kind) defined by ([19], p. 948)

$$
B(x, y) = \int_{0}^{1} t^{x-1}(1-t)^{y-1} dt. \tag{B.4}
$$

Taking account of (3.9) and (B.4) with $b = 4$, $\mu = \frac{1}{3}$ and $\nu = \frac{1}{2}$, Eq. (B.2) implies

$$
\int_{\infty}^{\sigma} \frac{d\sigma'}{\sqrt{1 + 4\sigma'^3}} = -\frac{4^{-1/3}}{3} B\left(\frac{1}{3}, \frac{1}{6}\right) + \sigma \mathcal{F}_1\left(\frac{1}{2}, \frac{1}{3}; \frac{4}{3}; -4\sigma^3\right) = ig^\frac{1}{3} (\eta_\infty - \eta). \tag{B.5}
$$

Since $\sigma = \varphi(ig^\frac{1}{3} (\eta_\infty - \eta); 0, -1)$ (see (3.83) and (3.84)), we obtain $ig^\frac{1}{3} (\eta_\infty - \eta) = \varphi^{-1}(\sigma)$ and, therefore, formula (3.86).

With the help of the property $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, the quantity $B\left(\frac{1}{3}, \frac{1}{6}\right)$ can be calculated explicitly. We have

$$
B\left(\frac{1}{3}, \frac{1}{6}\right) = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{2}\right)} = 2^{\frac{2}{3}} \frac{\Gamma^2\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \simeq 8.413 \tag{B.6}
$$

where the duplication formula for the gamma function, $\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-\frac{1}{2}} \Gamma(z)\Gamma\left(z + \frac{1}{2}\right)$, has been applied to $\left(z = \frac{1}{6}\right)$ ([24], p. 255).
Appendix C: some values of $W(r)$ and $W_I(r)$ for fixed $\gamma$

Table Ia

| $\gamma$ | $W_I(0.1)$ | $W(0.1)$ | $W_I(0.3)$ | $W(0.3)$ | $W_I(0.5)$ | $W(0.5)$ |
|----------|------------|----------|------------|----------|------------|----------|
| 4        | 1.100      | 0.763    | 0.597      | 0.406    | 0.163      | 0.069    |
| 6        | 1.148      | 1.048    | 0.729      | 0.678    | 0.369      | 0.345    |
| 8        | 1.164      | 1.117    | 0.773      | 0.750    | 0.438      | 0.427    |
| 10       | 1.172      | 1.145    | 0.795      | 0.782    | 0.572      | 0.466    |
| 12       | 1.177      | 1.159    | 0.809      | 0.800    | 0.493      | 0.489    |
| $\infty$ | 1.196      | 1.196    | 0.862      | 0.862    | 0.575      | 0.575    |

Table Ib

| $\gamma$ | $W_I(1)$  | $W(1)$   | $W_I(2)$  | $W(2)$   | $W_I(3)$  | $W(3)$   |
|----------|-----------|----------|-----------|----------|-----------|----------|
| 4        | -0.727    | -0.727   | -2.041    | -2.438   | -3.030    | -5.840   |
| 6        | -0.364    | -0.364   | -1.426    | -1.487   | -2.208    | -2.418   |
| 8        | -0.242    | -0.242   | -1.221    | -1.245   | -1.934    | -2.012   |
| 10       | -0.182    | -0.182   | -1.118    | -1.131   | -1.797    | -1.837   |
| 12       | -0.145    | -0.145   | -1.057    | -1.065   | -1.715    | -1.740   |
| $\infty$ | 0         | 0        | -0.811    | -0.811   | -1.386    | -1.386   |

Appendix D: solutions of Eq. (6.10) via solutions of Eq. (6.15)

The tensor $\eta_{\mu\nu}$ appearing at the right–hand side of (6.14) is defined as

$$\eta_{\mu ij} = \delta_{\mu ij}, \quad \eta_{\mu 0} = -\eta_{0\nu} = \delta_{\mu \nu}, \quad \eta_{00} = 0,$$

where $\epsilon_{\mu ij}$ denotes the Ricci tensor. Other properties of the symbol $\eta$ are reported in [17]. It is also convenient to use the matrices

$$\sigma_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu} \sigma^a,$$

(here $a, i, j, \ldots = 1, 2, 3$, and $\mu, \nu, \lambda, \ldots = 0, 1, 2, 3$).

In order to write $F_{\mu\nu}$ in terms of $\sigma_{\mu\nu}$, we shall evaluate the commutator $[\sigma_{\mu\nu}, \sigma_{\rho\lambda}]$. Taking into account (D.1) and (D.2), we find

$$[\sigma_{\mu\nu}, \sigma_{\rho\lambda}] = \frac{i}{2} \eta_{\mu\nu\rho} \eta_{\rho\lambda} \epsilon_{abc} \sigma^c,$$

(6.15)
where the commutation rule \([\sigma^a, \sigma^b] = 2i\sigma^c\) has been employed. On the other hand, we have \[17\]

\[
\eta_{\alpha\mu\nu}\eta_{\beta\rho\lambda} \epsilon_{abc} = \\
\delta_{\mu\rho} \eta_{\lambda\nu\rho} - \delta_{\mu\lambda} \eta_{\nu\mu\rho} - \delta_{\nu\lambda} \eta_{\mu\mu\rho} + \delta_{\nu\lambda} \eta_{\mu\mu\rho},
\]

(D.4)

Substituting (D.4) into (D.3) yields

\[
[\sigma_{\mu\nu}, \sigma_{\rho\lambda}] = i(\delta_{\mu\rho}\sigma_{\nu\lambda} - \delta_{\mu\lambda}\sigma_{\nu\rho} - \delta_{\nu\lambda}\sigma_{\mu\rho} + \delta_{\nu\rho}\sigma_{\mu\lambda}),
\]

(D.5)

with the help of (D.2).

Now, the YM field (6.11) can be written as (see (6.14) and (D.2))

\[
A_\mu = i\sigma_{\mu\nu} \partial_\nu \ln \psi.
\]

(D.6)

Hence, the tensor \(F_{\mu\nu}\) (see (6.12)) takes the form

\[
F_{\mu\nu} = i\sigma_{\nu\rho}(\partial_\mu \ln \psi) - i\sigma_{\mu\rho}(\partial_\nu \ln \psi) - i(\partial_\rho \ln \psi)(\partial_\lambda \ln \psi)[\sigma_{\mu\rho}, \sigma_{\nu\lambda}] = \\
i(\Delta_{\mu\rho} - \Delta_{\mu}\Delta_\rho)\sigma_{\nu\rho} - i(\Delta_{\nu\rho} - \Delta_{\nu}\Delta_\rho)\sigma_{\mu\rho} - i(\Delta_\rho)^2 \sigma_{\mu\nu},
\]

(D.7)

where the shorthand notation

\[
\Delta_\nu = \partial_\nu \ln \psi, \quad \Delta_{\nu\rho} = \partial_\nu \partial_\rho \ln \psi, \quad \Delta_\mu = \partial_\mu \ln \psi, \quad (\Delta_\nu)^2 = (\partial_\nu \ln \psi)^2, ...
\]

(D.8)

is used.

Keeping in mind (D.7) and (D.6), the quantities \(\partial_\mu F_{\mu\nu}\) and \([A_\mu, F_{\mu\nu}]\) (see Eq. (6.12)) can be elaborated as follows:

\[
\partial_\mu F_{\mu\nu} = i(\Delta_{\mu\rho\nu} - \Delta_{\mu\rho}^2 \Delta_\nu - \Delta_\mu \Delta_{\rho\nu})\sigma_{\nu\rho} - i(\Delta_{\nu\rho\mu} - \Delta_{\nu\rho}^2 \Delta_\mu - \Delta_\nu \Delta_{\rho\mu})\sigma_{\mu\rho} - 2i\Delta_\rho \Delta_{\mu\nu}\sigma_{\mu\nu},
\]

(D.9)

\[
[A_\mu, F_{\mu\nu}] = i\Delta_\rho[\sigma_{\mu\rho}, F_{\mu\nu}] = \\
-(\Delta_{\rho\mu\nu} - \Delta_{\rho\nu\mu})[\sigma_{\mu\rho}, \sigma_{\nu\lambda}] + (\Delta_{\rho\nu\mu} - \Delta_{\rho\nu\mu})[\sigma_{\mu\rho}, \sigma_{\mu\lambda}] - \Delta_\rho(\Delta_\lambda)^2[\sigma_{\mu\rho}, \sigma_{\mu\nu}],
\]

(D.10)

Then, putting (D.9) and (D.10) into the YM equations (6.10) and exploiting the commutation rule (D.5), after some manipulations we obtain

\[
\sigma_{\nu\rho}[\Delta_{\mu\rho\nu} - \Delta_{\mu\rho}^2 \Delta_\rho + 2\Delta_\lambda \Delta_{\rho\lambda} - \Delta_\rho \Delta_\lambda^2] - 2\Delta_\rho(\Delta_\lambda)^2 = 0.
\]

(D.11)

Since the tensor multiplying \(\sigma_{\mu\rho}\) is symmetric under the exchange \(\mu \leftrightarrow \rho\), while \(\sigma_{\mu\rho}\) is antisymmetric, Eq. (D.11) becomes

\[
\Delta_{\mu\rho\nu} + 2\Delta_\mu \Delta_{\rho\nu} - 2\Delta_\rho[\Delta_{\mu\rho}^2 + (\Delta_\nu)^2] = 0,
\]

(D.12)
which can be reduced to the form
\[ \partial_\rho \left( \ln \frac{\partial^2 \psi}{\psi} - \ln \psi^2 \right) = 0. \]  
(D.13)

Equation (D.13) tells us that the function in the bracket is independent from \( x_\rho \) for any value of \( \rho = 0, 1, 2, 3 \). Consequently, Eq. (6.15) arises \( (\partial^2_\mu = \nabla^2) \), where \( K \) is a constant of integration.

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