UNIQUENESS OF THE THERMAL EFFECTIVE POTENTIAL

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Abstract

We discuss the use of derivative expansion techniques for the construction of thermal effective potentials. We present a theory for which the thermal bubble is analytic at the origin of the momentum-frequency space, although the internal propagators in the loop have the same mass. This means that, for this theory, the thermal effective potential is uniquely defined. We then examine a slightly different theory for which the thermal bubble displays the usual non-analyticity at the origin and the thermal effective potential is not uniquely defined. For this latter theory we compare our results with those of other works in the literature which employ the derivative expansion but find a uniquely defined thermal effective potential. We raise several questions concerning the interchange of the order of the perturbative and the derivative expansions, the thermal generalization of some non-perturbative zero temperature methods and the use of the periodicity of the external bosonic field. Finally, we re-examine the physical interpretation given to the imaginary part of the thermal bubble in the literature.

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1 Introduction

It is well-known that for most theories at finite temperature the self-energy displays a non-analytic behaviour at the origin of the momentum-frequency space [1, 2]. This non-analyticity manifests itself in a difference between the \( \{ q_0 \to 0, \mathbf{q} \to 0 \} \) and \( \{ \mathbf{q} \to 0, q_0 \to 0 \} \) limits of the self-energy, where \( q_0 \) and \( \mathbf{q} \) are the components of the external momentum \( q_\mu = (q_0, \mathbf{q}) \) and the component listed first goes to zero first. The first limit leads to screening and the static potential whereas the second limit has been used for the calculation of the plasma frequency [3, 4]. One may argue that the two limits must differ since they refer to different physics [3]. However there is a problem with this argument. The plasma frequency does not involve the \( \{ \mathbf{q} \to 0, q_0 \to 0 \} \) limit but rather the \( q_0 \to \infty \) limit [3]. It is because the latter limit is independent of \( \mathbf{q} \) to lowest order that we appear to find the same result as with the former limit. This means that the physical significance of the \( \{ \mathbf{q} \to 0, q_0 \to 0 \} \) limit is not well-understood yet.

The non-analyticity of the self-energy at the origin of the momentum-frequency space puts in jeopardy the construction of an effective potential based on the derivative expansion technique [6, 7]. Historically, this problem was first pointed out in the BCS theory context by Abrahams and Tsuneto [8]. Later it was also seen to appear in a wide range of theories. In thermal QCD it occurs in the gluon [4, 9] and in the quark self-energy [10, 11]. Furthermore, it appears in all one-loop diagrams that have zero or two external quarks and any number of external gluons [12, 13]. The problem is also present in the graviton self-energy [14, 15] and in higher-order graviton diagrams [16]. Even in the much simpler case of interacting scalars the non-analyticity of the self-energy persists [4, 17, 18].

The reason for this behaviour is that temperature effects give rise to Landau terms and these are responsible for the development of a new branch cut in the complex plane of the external momenta with a branch point at the origin, besides the usual one which is already present at zero temperature [2, 19]. The usual branch cut exists for

\[
s = q_0^2 - |\mathbf{q}|^2 \geq 4m^2
\]

and the new one for

\[
s = q_0^2 - |\mathbf{q}|^2 \leq 0.
\]
An interesting remark is that, whenever the internal propagators in a typical loop have different masses, the self-energy is analytic at the origin \[20\]. In this non-degenerate mass case the usual branch cut is

\[ s \geq (m_1 + m_2)^2 \]

and the new one is

\[ s \leq (m_1 - m_2)^2 \]

where \( m_1 \) and \( m_2 \) are the masses of the particles in the internal loop. The new branch point is not at the origin anymore and the problem disappears from this point, allowing thus the definition of a unique effective potential. However the non-degenerate mass case is of limited physical interest \[20\].

We shall present a theory which exhibits a new and unexpected feature. The model of section 2 has self-energy which is analytic at the origin, although the mass is degenerate. In section 3 we show how subtle this new feature is and how the non-analyticity can develop, if we modify slightly our model by replacing the parity conserving interaction term with a similar but parity violating one. Sections 2 and 3 are based to a great extent on a previous work we did with M. Hott \[21\]. Finally, in section 4, we compare our results with other in the literature.

## 2 A new case

We consider the following model

\[ L[\bar{\psi}, \psi, \phi] = \bar{\psi}(i\slashed{D} - m)\psi - ig\bar{\psi}\gamma_5 \psi \phi + L_0[\phi] \]  

(1)

where \( L_0[\phi] \) is the free Klein-Gordon Lagrangian. The boson is taken to be a pseudo-scalar quantity.

We consider \( \phi(x) \) to be an external field and we want to obtain the one loop contribution to the effective action which is given by

\[ \Gamma_{\text{eff}}[\phi] = -i \ln \frac{\text{Det}[iS^{-1}[\phi]]}{\text{Det}[iS^{-1}]} \]  

(2)

where \( iS^{-1}[\phi] \) and \( iS^{-1} \) are matrices whose elements in coordinate representation are
\begin{equation}
\langle x| iS^{-1} |y \rangle = (i \varphi_x - m) \delta(x - y)
\end{equation}
\begin{equation}
\langle x| iS^{-1} [\phi] |y \rangle = (i \varphi_x - m - ig \gamma_5 \phi(x)) \delta(x - y).
\end{equation}

Since the external field depends on the coordinates, the resulting functional determinants are not straightforward to calculate. The matrices whose functional determinants we want to evaluate are not diagonal in momentum or in coordinate space. However, progress can be made, if we rewrite (2) as:

\begin{equation}
\Gamma_{\text{eff}}[\phi] = -i \text{Tr} \ln [1 - g \gamma_5 \phi(\hat{x})S(\hat{p})].
\end{equation}

Now we expand the above expression in powers of the coupling constant and show that the leading contribution to the one-loop effective action is

\begin{equation}
\Gamma^{(2)} = i \frac{g^2}{2} \int \frac{d^4 q}{(2\pi)^4} \tilde{\phi}(-q) i \Pi(q) \tilde{\phi}(q)
\end{equation}

where \( \tilde{\phi}(q) \) is the Fourier transformation of \( \phi(x) \) and

\begin{equation}
i \Pi(q) = \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ \gamma_5 \frac{1}{k + q - m} \gamma_5 \frac{1}{k - m} \right].
\end{equation}

We note that \( i \Pi(q) \) is just the self-energy bubble diagram for the boson which, after performing the trace, is given by

\begin{equation}
i \Pi(q) = -4 \int \frac{d^4 k}{(2\pi)^4} \frac{k^2 + k^\mu q_\mu - m^2}{[k + q]^2 - m^2[k^2 - m^2]}.
\end{equation}

This is one typical diagram that usually has a non-analytic behaviour in the limit of vanishing external momenta but we are going to show that this is not the case here. We keep this intermediate expression, because it will help us to show in the next section how the non-analyticity can develop in the scalar-coupling model.

Applying the usual finite temperature techniques to (4), we find the following expression for the thermal bubble diagram.

\begin{equation}
\Pi(q_0, q) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega \Omega} \left\{ 2\omega \left[ \frac{\beta(\Omega + q_0)}{2} + \frac{\beta(\Omega - q_0)}{2} \right] \right\}
\end{equation}
\[
+ \frac{1}{\Omega + \omega - q_0} \left[ [\omega q_0 + kq] \tanh \frac{\beta \omega}{2} + [q_0^2 - \Omega q_0 + kq] \tanh \frac{\beta (\Omega - q_0)}{2} \right] \\
+ \frac{1}{\Omega + \omega + q_0} \left[ -[\omega q_0 + kq] \tanh \frac{\beta \omega}{2} + [q_0^2 + \Omega q_0 + kq] \tanh \frac{\beta (\Omega + q_0)}{2} \right] \\
+ \frac{1}{\Omega - \omega + q_0} \left[ \omega q_0 + kq \tanh \frac{\beta \omega}{2} - [q_0^2 + \Omega q_0 + kq] \tanh \frac{\beta (\Omega + q_0)}{2} \right] \\
+ \frac{1}{\Omega - \omega - q_0} \left[ -[\omega q_0 + kq] \tanh \frac{\beta \omega}{2} - [q_0^2 - \Omega q_0 + kq] \tanh \frac{\beta (\Omega - q_0)}{2} \right] \right) 
\] (7)

where
\[\omega = \sqrt{k^2 + m^2} \quad \Omega = \sqrt{(k + q)^2 + m^2} \quad q_0 = \frac{2\pi n}{\beta}, \quad n = \text{integer}. \] (8)

From the above definition of \(q_0\) follows that
\[e^{\beta q_0} = 1 \] (9)

and consequently \(q_0\) disappears from all the hyperbolic tangents of (7).

\[
\Pi(q_0, q) = \int \frac{d^3k}{(2\pi)^3} 1 \left\{ 4\omega \tanh \frac{\beta \Omega}{2} + \frac{1}{2\omega \Omega} \left[ [\omega q_0 + kq] \tanh \frac{\beta \omega}{2} + [q_0^2 - \Omega q_0 + kq] \tanh \frac{\beta (\Omega - q_0)}{2} \right] \\
+ \frac{1}{\Omega + \omega - q_0} \left[ -[\omega q_0 + kq] \tanh \frac{\beta \omega}{2} + [q_0^2 + \Omega q_0 + kq] \tanh \frac{\beta (\Omega + q_0)}{2} \right] \\
+ \frac{1}{\Omega - \omega + q_0} \left[ \omega q_0 + kq \tanh \frac{\beta \omega}{2} - [q_0^2 + \Omega q_0 + kq] \tanh \frac{\beta (\Omega + q_0)}{2} \right] \\
+ \frac{1}{\Omega - \omega - q_0} \left[ -[\omega q_0 + kq] \tanh \frac{\beta \omega}{2} - [q_0^2 - \Omega q_0 + kq] \tanh \frac{\beta (\Omega - q_0)}{2} \right] \right\}. \] (10)

Even before performing the angular integration, we can have a first naive indication that the zero-momentum limit of (10) does not display the usual non-uniqueness problem. The two successive limits are

\[
\Pi(0, q) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega \Omega} \left\{ \frac{2\omega}{2} \tanh \frac{\beta \Omega}{2} + \frac{kq}{\Omega + \omega} \left[ \tanh \frac{\beta \omega}{2} + \tanh \frac{\beta \Omega}{2} \right] \\
+ \frac{kq}{\Omega - \omega} \left[ \tanh \frac{\beta \omega}{2} - \tanh \frac{\beta \Omega}{2} \right] \right\},
\]
which can be checked to give

$$\lim_{|q| \to 0} \Pi(0, q) = \frac{1}{\pi^2} \int_{|m|}^\infty d\omega \sqrt{\omega^2 - m^2} \tanh \frac{\beta \omega}{2}. \quad (11)$$

Similarly we find

$$\Pi(q_0, 0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega \Omega} \left\{ 4\omega \tanh \frac{\beta \Omega}{2} + \right.$$

$$\left. + ~ q_0^2 \left[ \frac{1}{2\omega - q_0} + \frac{1}{2\omega + q_0} \right] \tanh \frac{\beta \omega}{2} \right\}$$

$$\xrightarrow{q_0 \to 0} \frac{1}{\pi^2} \int_{|m|}^\infty d\omega \sqrt{\omega^2 - m^2} \tanh \frac{\beta \omega}{2}.$$ 

We conclude that the limits coincide. Moreover, the only term that contributes to the unique result is the first one in the integrand of equation (10) and those proportional to Landau terms - the last two terms inside the integrand - vanish in this limit.

A more general way of seeing that the limits are the same is to perform the angular integration and then use the parameterization $q_0 = a|q|$, where $a$ can be any real number, and find the limit of $\Pi(a|q|, |q|)$ as $|q| \to 0$. If the limit is independent of $a$, we have a strong indication that the function is analytic at the origin, i.e. it does not depend on the way one approaches the origin [1, 20]. Before doing so we recast equation (10) in a more convenient form by means of the transformation $k \to -(k + q)$ wherever the integrand contains $\tanh \frac{\beta \Omega}{2}$. Then we find

$$\Pi(q_0, q) = \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{2}{\omega} \tanh \frac{\beta \omega}{2} + (q_0^2 - q^2) \tanh \frac{\beta \omega}{2} \right.$$ 

$$\times \frac{1}{2\omega \Omega} \left[ \frac{1}{q_0 + \Omega + \omega} - \frac{1}{q_0 - \Omega - \omega} + \frac{1}{q_0 + \Omega - \omega} - \frac{1}{q_0 - \Omega + \omega} \right] \right\}. \quad (12)$$

One can note that at $T = 0$ the Landau terms cancel each other, as expected. We change variables from $\cos \theta$ to $\Omega$ and perform the integration over $\Omega$. The result is

$$\Pi(q_0, |q|) = \frac{1}{\pi^2} \int_{|m|}^\infty d\omega \sqrt{\omega^2 - m^2} \tanh \frac{\beta \omega}{2}$$

$$+ \frac{q_0^2 - |q|^2}{2|q|} \int_{|m|}^\infty \frac{d\omega}{(2\pi)^2} \tanh \frac{\beta \omega}{2} [L1 + L2 + L3 + L4] \quad (13)$$
where

\[
L_1(q_0, |q|) = \ln \frac{\Omega_+ + \omega + q_0}{\Omega_- + \omega + q_0} \quad L_2(q_0, |q|) = \ln \frac{\Omega_+ + \omega - q_0}{\Omega_- + \omega - q_0}
\]
\[
L_3(q_0, |q|) = \ln \frac{\Omega_+ - \omega + q_0}{\Omega_- - \omega + q_0} \quad L_4(q_0, |q|) = \ln \frac{\Omega_+ - \omega - q_0}{\Omega_- - \omega - q_0}
\]

with

\[
\Omega_+ = \sqrt{(|k| + |q|)^2 + m^2} \quad \Omega_- = \sqrt{(|k| - |q|)^2 + m^2}.
\]

If \(q_0\) is made complex and continuous, the only poles or zeros of the sum of L’s in (13) occur for \(q_0\) on the real axis. It is perfectly appropriate to have singularities on the real axis. Thus the analytic extension of (13) is trivially obtained by letting \(q_0\) be real. There are three self-energies on the real axis.

\[
\Pi_R(q_0, |q|) = \Pi(q_0 + i\epsilon, |q|) \quad \Pi_A(q_0, |q|) = \Pi(q_0 - i\epsilon, |q|) \quad \Pi_F(q_0, |q|) = \Pi(q_0 + i\epsilon q_0, |q|)
\]

where \(\epsilon \to 0^+\). The real parts of these self-energies coincide whereas the imaginary parts are related according to

\[
\text{Im}\Pi_R = -\text{Im}\Pi_A = \tanh \left( \frac{\beta q_0}{2} \right) \text{Im}\Pi_F. \quad (14)
\]

Following [1] we shall not concern ourselves with the Feynman self-energy. Using the fact that \(\epsilon\) is infinitesimal, the real part of the self-energy can be shown to be

\[
\text{Re}\Pi(q_0, |q|) = \frac{1}{\pi^2} \int_{|q|}^{\infty} d\omega \sqrt{\omega^2 - m^2} \tanh \frac{\beta \omega}{2} \]
\[
+ \frac{q_0^2 - |q|^2}{2|q|} \int_{|q|}^{\infty} \frac{d\omega}{(2\pi)^2} \tanh \frac{\beta \omega}{2} [\text{Re}L1 + \text{Re}L2 + \text{Re}L3 + \text{Re}L4] \quad (15)
\]

where

\[
\text{Re}L1(q_0, |q|) = \ln \left| \frac{\Omega_+ + \omega + q_0}{\Omega_- + \omega + q_0} \right| \quad \text{Re}L2(q_0, |q|) = \ln \left| \frac{\Omega_+ + \omega - q_0}{\Omega_- + \omega - q_0} \right|
\]
\[
\text{Re}L3(q_0, |q|) = \ln \left| \frac{\Omega_+ - \omega + q_0}{\Omega_- - \omega + q_0} \right| \quad \text{Re}L4(q_0, |q|) = \ln \left| \frac{\Omega_+ - \omega - q_0}{\Omega_- - \omega - q_0} \right|
\]

We note that the real part of the self-energy is even under \(q_0 \to -q_0\), since it can be written as a function of \(q_0^2\), if we combine the logarithms. It is also even under \(q \to -q\), since it depends only on \(|q|\).
Now we turn to the imaginary part which we calculate according to [19], making use of the following form of the delta function

$$\delta(x) = \frac{i}{2\pi} \lim_{\epsilon \to 0^+} \left[ \frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right].$$  \hspace{1cm} (16)$$

The imaginary part is

$$\text{Im}\Pi_R = -\text{Im}\Pi_A = \frac{1}{2} \int \frac{d^3k}{(2\pi)^2} \frac{1}{2\omega\Omega} \tanh\left(\frac{\beta\omega}{2}\right) \left( q_0^2 - q^2 \right) \left[ \delta(q_0 + \Omega + \omega) - \delta(q_0 - \Omega - \omega) + \delta(q_0 + \Omega - \omega) - \delta(q_0 - \Omega + \omega) \right].$$  \hspace{1cm} (17)$$

We note that it is odd under $q_0 \to -q_0$. However it is even under $q \to -q$, because we can simultaneously change the integration variable $k \to -k$. This means that, unlike the real part, the imaginary part of the retarded or advanced thermal self-energy does not contribute to the effective action. As we can see from (11), the integrand of the effective action is $\bar{\phi}(q_0, q) \bar{\phi}(-q_0, -q)$ and therefore the contribution $\bar{\phi}(q_0, q) \bar{\phi}(-q_0, -q)$ $\text{Im}\Pi(q_0, q)$ is odd under the combined transformations $q_0 \to -q_0$ and $q \to -q$ and vanishes, when integrated over $d^4q$.

We proceed to the parameterization $q_0 = a|q|$ and examine the behaviour of the real part of the self-energy as $|q| \to 0$. The limits of two of the regular terms $\text{Re}L1$ and $\text{Re}L2$ are independent of $a$, as they should be. We can see that

$$\lim_{|q| \to 0} (a^2 - 1)|q|\text{Re}L1 = 0 \quad \lim_{|q| \to 0} (a^2 - 1)|q|\text{Re}L2 = 0.$$ 

What is quite unexpected is that, for this particular model, the contributions coming from the Landau terms, $\text{Re}L3$ and $\text{Re}L4$, vanish independently of $a$, that is

$$\lim_{|q| \to 0} (a^2 - 1)|q|\text{Re}L3 = 0 \quad \lim_{|q| \to 0} (a^2 - 1)|q|\text{Re}L4 = 0.$$ 

In other words, although the Landau terms are not well-behaved at the origin of momentum space, a unique effective potential up to second order in the coupling constant can be defined here thanks to the kinetic term in the numerator of equation (12), namely $q_0^2 - q^2$. This is an interesting result but this kinetic term does not always appear in bubble diagrams as we are
going to see in the next section. In the present case the one-loop, \( g^2 \) order contribution to the effective potential is

\[
V_{\text{eff}}^{(2)} = -\frac{ig^2}{2} i \text{Re}\Pi(0, 0) \phi^2
\]

\[
\text{Re}\Pi(0, 0) = \frac{1}{\pi^2} \int_{|m|}^{\infty} d\omega \sqrt{\omega^2 - m^2} \tanh \frac{\beta \omega}{2}.
\]  

(18)

This result for the effective potential coincides with the one we would have found, had we not added to the real \( q_0 \) the infinitesimal imaginary part which corresponds to physical boundary conditions [21]. The next order in the derivative expansion is non-analytic since the derivatives of the Landau terms become dominant and the derivative expansion breaks down.

3 A usual case

In this section we shall consider a model whose only difference from the one which we examined in the previous section is that its interaction term does not contain the \( \gamma_5 \) matrix.

\[
L'[\bar{\psi}, \psi, \phi] = \bar{\psi}(i \slashed{\partial} - m)\psi - ig\bar{\psi}\psi\phi + L_0[\phi].
\]  

(19)

As we shall soon see, this simple modification of the interaction term has far-reaching consequences as far as the analytic properties of the thermal self-energy are concerned. Starting from (19) and following the procedure of the previous section we find that the one-loop effective action is

\[
\Gamma'_{\text{eff}}[\phi] = -i \text{Tr} \ln [1 - g\phi(\hat{x})S(\hat{p})].
\]  

(20)

and the self-energy bubble is given by

\[
i \Pi'(q) = 4 \int \frac{d^4k}{(2\pi)^4} \frac{k^2 + k^\mu q_\mu + m^2}{[(k + q)^2 - m^2][k^2 - m^2]}
\]  

(21)

which can be written as

\[
i \Pi'(q) = -i \Pi(q) + i \Pi''(q),
\]
where

\[ i\Pi''(q) = 4 \int \frac{d^4k}{(2\pi)^4} \frac{2m^2}{[(k + q)^2 - m^2][k^2 - m^2]} . \]

As we saw in the previous section Re\(\Pi(a|q|, |q|)\) does not depend on \(a\), when \(|q| \to 0\). We will see that Re\(\Pi''(a|q|, |q|)\) does. We have

\[ \Pi''(q_0, q) = -m^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega\Omega} \left\{ \left[ \frac{1}{\Omega + \omega - q_0} + \frac{1}{\Omega - \omega + q_0} \right] \tanh \frac{\beta\omega}{2} + \left[ \frac{1}{\Omega + \omega + q_0} + \frac{1}{\Omega - \omega - q_0} \right] \tanh \frac{\beta\omega}{2} + \left[ \frac{1}{\Omega + \omega + q_0} - \frac{1}{\Omega - \omega + q_0} \right] \tanh \frac{\beta(\Omega + q_0)}{2} + \left[ \frac{1}{\Omega + \omega - q_0} - \frac{1}{\Omega - \omega - q_0} \right] \tanh \frac{\beta(\Omega - q_0)}{2} \right\} \quad (22) \]

which after using (11) and applying the transformation \(k \to -k + q\) becomes

\[ \Pi''(q_0, q) = -2m^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega\Omega} \tan \frac{\beta\omega}{2} \left\{ \frac{1}{q_0 + \Omega + \omega} - \frac{1}{q_0 - \Omega - \omega} + \frac{1}{q_0 + \Omega - \omega} - \frac{1}{q_0 - \Omega + \omega} \right\}. \quad (23) \]

Performing the angular integration, setting \(q_0 = a|q|\) and following the steps of the previous paragraph yields the effective potential

\[ (V_{eff}^{(2)})'' = \frac{g^2}{2} \text{Re}\Pi''(0, 0) \phi^2 \]

\[ \text{Re}\Pi''(0, 0) = \frac{1}{\pi^2} \int_{|m|} d\omega \sqrt{\omega^2 - m^2} \tan \frac{\beta\omega}{2} \left\{ \frac{m^2}{\omega^2} + \frac{m^4}{(\omega^2 - m^2)^2} \right\}. \quad (24) \]

As in the previous section, the imaginary part does not contribute to the effective action.

Therefore the total effective potential for the theory of this section is

\[ (V_{eff}^{(2)})' = \frac{g^2}{2} \phi^2 \left[ -\Pi(0, 0) + \Pi''(0, 0) \right] = -\frac{g^2}{2\pi^2} \int_{|m|} d\omega \tan \frac{\beta\omega}{2} \sqrt{\omega^2 - m^2} \left\{ 1 - \frac{m^2}{\omega^2} - \frac{m^4}{(\omega^2 - m^2)^2} - 2a\omega^4 \right\}. \quad (25) \]
This result for the effective potential coincides with the one we would have found, had we not added to the real $q_0$ the infinitesimal imaginary part which corresponds to physical boundary conditions $[21]$. This effective potential is not uniquely defined, because it depends on $a$ which can take any real value. Comparing (3) to (21), we see that dropping $\gamma_5$ from the interaction resulted in changing the relative sign between the momentum terms and $m^2$ in the numerator. This slight change was enough to allow for the development of a self-energy which is non-analytic at the origin.

4 Comparison with other works

The purpose of this section is to compare our results with others in the literature.

4.1 Comparison with Dolan and Jackiw

For the theory examined in section (3) the one-loop effective potential at order $g^2$ is given by (25). In $[22]$ the same theory was considered and, setting the external field to be constant, Dolan and Jackiw obtained the following exact expression for the one-loop effective potential

$$V_{\text{eff}} = -\frac{2}{\pi^2} \int_{|m|}^\infty d\omega \omega \sqrt{\omega^2 - m^2} \left[ \frac{E}{2} + \frac{1}{\beta} \ln \left(1 + e^{-\beta E}\right)\right],$$

where

$$E = \left[\omega^2 - m^2 + (m + g\phi)^2\right]^{1/2}.$$ 

We are interested in the contribution at the second order in the coupling constant which is

$$V_{\text{eff}}^{(2)} = -\frac{g^2}{2\pi^2} \int_{|m|}^\infty d\omega \sqrt{\omega^2 - m^2} \left\{ \left(1 - \frac{m^2}{\omega^2}\right) \tanh \frac{\beta \omega}{2} + \frac{m^2 \beta}{2\omega} \cosh^{-2} \frac{\beta \omega}{2} \right\} \phi^2.$$ 

One can reproduce equation (27) by setting $(q_0, q) = (0, 0)$ in formula (21) and then performing the Matsubara sum. However, the correct thing to do is to perform the sum first so that the
explicit form of the self-energy as a function of \( q_0 \) and \(|q|\) is obtained. Then the behaviour of this function can be investigated in the limit \((q_0, q) \to (0, 0)\). We therefore conclude that the non-perturbative method employed in [22] is not generally equivalent to the perturbative calculation, because it fails to take into account the non-analyticity which appears at the origin of the space of external momenta.

It would be interesting to investigate further why our result doesn’t coincide with the one derived in [22]. Instead of just comparing the final results we shall try to find out in which stage of the calculation the difference between [22] and us arises. First we perform a perturbative expansion of the logarithm in (20) over the coupling constant:

\[
\Gamma'_\text{eff}[\phi] = i \int \frac{d^4k}{(2\pi)^4} \text{tr} \langle k | g \phi(\hat{x}) S(\hat{p}) + \frac{1}{2} g \phi(\hat{x}) S(\hat{p}) g \phi(\hat{x}) S(\hat{p}) + ... \rangle |k \rangle. \tag{28}
\]

If we wish to reproduce the result of [22], we do the derivative expansion of \( \phi(\hat{x}) \), truncate it to zeroth order and substitute the constant term \( \phi \) in (28). Each term of the expansion depends only on the momentum operator and is diagonal in momentum space. Therefore the effective action can be resummed as follows

\[
\Gamma'_\text{eff}[\phi] = i \left\{ g \int \frac{d^4k}{(2\pi)^4} \text{tr} \frac{1}{k - m} + \frac{1}{2} (g \phi)^2 \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \frac{1}{k - m} \right]^2 + ... \right\} \int d^4x
\]

\[
= -i \text{tr} \int \frac{d^4k}{(2\pi)^4} \ln [1 - g \phi S(k)] \int d^4x. \tag{29}
\]

This is the effective action of [22] which, after some differentiation trick explained therein, yields the effective potential (26).

However, if we want to find contributions to the effective action of the form (4) with the self-energy given by (21), then, before performing the derivative expansion, we should introduce complete sets of intermediate states in (28) and let the momentum and space operators act on them. This yields

\[
\Gamma'_\text{eff}[\phi] = i \left\{ g \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{tr} \langle k | \phi(\hat{x}) | x \rangle \langle x | S(\hat{p}) | k \rangle \\
+ \frac{1}{2} g^2 \int d^4x \int d^4y \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \text{tr} \langle k | \phi(\hat{x}) | x \rangle \langle x | S(\hat{p}) | q \rangle \langle q | \phi(\hat{x}) | y \rangle \langle y | S(\hat{p}) | k \rangle + ... \right\}
\]
\[
\Gamma'_{\text{eff}}[\phi] = i \left\{ g \phi \int \frac{d^4k}{(2\pi)^4} \frac{1}{k-m} \right. \\
+ \left. \frac{1}{2} g^2 \int \frac{d^4q}{(2\pi)^4} \tilde{\phi}(-q) \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k-m][k+q-m]} \tilde{\phi}(q) + \ldots \right\} 
\]

where \( \tilde{\phi}(q) \) is the Fourier transformation of \( \phi(x) \). The final step of our calculation is to perform the derivative expansion of \( \phi(x) \) and keep only the constant term. Thus we obtain,

\[
\Gamma'_{\text{eff}}[\phi] = i \left\{ g \phi \int \frac{d^4k}{(2\pi)^4} \frac{1}{k-m} \int d^4x + \frac{1}{2} (g \phi)^2 \lim_{q \to 0} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k-m][k+q-m]} \int d^4x + \ldots \right\}. 
\]

As we have already mentioned, at finite temperature, doing the integration (the Matsubara sum) first and then taking the limit in the second term of (31) is not equivalent to taking the limit of the integrand first and then performing the integration (the Matsubara sum). Comparing (29) to (31) we can clearly see that this difference is due to the interchange of the order with which we perform the derivative expansion and the action of the momentum and space operators on the bras and kets. Replacing \( \phi(\hat{x}) \) with the constant \( \phi \) in the beginning of the calculation, as Dolan and Jackiw do, means that we lose the operator behaviour of the \( \phi \) field.

Before proceeding, we shall add a further comment. If we had applied the limit before performing the angular integration in (22), we would have found

\[
(V_{\text{eff}}^{(2)})' = -\frac{g^2}{2\pi^2} \int_{|m|}^{\infty} d\omega \left\{ \tanh \frac{\beta \omega}{2} \sqrt{\omega^2 - m^2} \left[ 1 - \frac{m^2}{\omega^2} \right] \\
+ \frac{m^2 \beta}{2\omega} \cosh^{-2} \frac{\beta \omega}{2} \left[ \sqrt{\omega^2 - m^2} - \frac{\omega a}{2} \ln \frac{\omega a + \sqrt{\omega^2 - m^2}}{\omega a - \sqrt{\omega^2 - m^2}} \right] \right\} \phi^2 
\]

which depends on \( a \) and differs from the result of Dolan and Jackiw, as (23) does. This new result would have an additional property compared to (25), it would reduce to the result of Dolan and Jackiw in the \( \{q_0 \to 0, q \to 0\} \) limit or equivalently in the \( a \to 0 \) limit. This property is frequently mentioned in the literature, see for example (23). However it is an artefact of interchanging the limit with the integration and this is why we chose to perform the angular integration first.
4.2 Comparison with Gribosky and Holstein

There is another work [23] where Gribosky and Holstein claim that they have a model which doesn’t display the usual non-analyticity at the origin of the momentum-frequency space. Actually they employ two ways of proving this, a non-perturbative way and a perturbative one. However Weldon in his paper [1] proves that at least the perturbative way of Gribosky and Holstein is wrong. The reason is that Feynman parameterization at finite temperature is not as straightforward as it is at zero temperature and performing it naively, as Gribosky and Holstein did, gives misleading results. There is one issue which still remains open though; where is the mistake in the non-perturbative approach of Gribosky and Holstein? Let us examine this matter in some more detail. The model which the two authors consider is the following:

\[ L[B, \phi] = L_0[B(x)] - \frac{1}{2} \phi(x) \left( \Box + m^2 + \lambda B(x) \right) \phi(x) \]  

(33)

where \( L_0[B] \) is the free Klein-Gordon Lagrangian. The effective Lagrangian and therefore the effective action can be found from the coincident limit of the Green’s function for \( \phi(x) \), see [24].

Next they write down the differential equation which defines the Green’s function:

\[ \left( \Box + m^2 + \lambda B(x) \right) G(x, x') = -\delta(x - x'). \]  

(34)

The above equation cannot be solved for a general \( B(x) \), it is possible though to solve it, if we perform the derivative expansion of \( B(x) \) around \( x' \) and keep terms of up to second order in derivatives. This method first appeared in paper [25] by Dittrich. Our criticism is that, although at zero temperature there is no problem with this approximation, at finite temperature it amounts to replacing a periodic function with one that is not periodic. In other words \( B(x) \) obeys \( B(\tau + \beta) = B(\tau) \) but the expansion which is truncated at second or even at first order in derivatives doesn’t and consequently this non-perturbative method cannot be applied at finite temperature. The fact that this is the reason for obtaining a result which doesn’t display the usual problem of non-uniqueness can be seen from the same paper of Gribosky and Holstein [23]! In their perturbative approach they remark under their equation (3.20b) that “Thus, we see that extending \( \Pi(p) \) to continuous \( p_0 \) without ever using \( p_0 = 2\pi i l/\beta \) eliminates the non-analytic behaviour we encountered in \( \Pi_R(p) \)” where \( \Pi_R(p) \) is the self-energy.
which they found after using (9) which follows from \( p_0 = 2\pi i l / \beta \) (their \( p_0 \) is our \( q_0 \)). However, it is well-known that this expression for \( p_0 \) is a direct consequence of the periodicity of \( B(x) \) which is lost in the beginning of their non-perturbative calculation.

### 4.3 Comparison with Evans

The effects of retaining \( \exp (\beta q_0) \) after the analytic continuation where recently re-investigated in papers [26, 27] by Evans. Again the conclusion is that keeping this exponential cures the non-uniqueness of the thermal effective potential. We can also see this in our own calculation. As we have already shown in section (3), the theory examined there has the usual problems of non-uniqueness due to the \( \Pi^{\prime\prime} \) piece of the self-energy. Let us see what happens to \( \Pi^{\prime\prime} \), if we follow Evans. Letting \( q_0 \) be real we recast (22) as follows

\[
\Pi^{\prime\prime} = -2m^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\Omega \omega} \frac{1}{e^{\beta \omega} + 1} \times \left\{ \frac{e^{\beta (\Omega + \omega - q_0)} - 1}{\Omega + \omega - q_0} \frac{1}{e^{\beta (\Omega - q_0)} + 1} + \frac{e^{\beta (\Omega + \omega + q_0)} - 1}{\Omega + \omega + q_0} \frac{1}{e^{\beta (\Omega + q_0)} + 1} \right. \\
\left. \frac{1 - e^{\beta (\Omega - \omega + q_0)}}{\Omega - \omega + q_0} \frac{e^{\beta \omega}}{e^{\beta (\Omega + q_0) + 1}} + \frac{1 - e^{\beta (\Omega - \omega - q_0)}}{\Omega - \omega - q_0} \frac{e^{\beta \omega}}{e^{\beta (\Omega - q_0) + 1}} \right\}.
\]

(35)

We observe that, because we have retained \( q_0 \) in the exponentials appearing in the numerators of the four terms, the integrand of (35) does not have poles at \( q_0 = \Omega + \omega, -\Omega - \omega, -\Omega + \omega, \Omega - \omega \) anymore. These poles of the integrand were responsible for the branch points of (23) at 0 and \( 4m \) in the complex \( s = q_0^2 - |q|^2 \) plane as explained in [19] and more generally in [28]. We know, for example see [4], that the branch point at 0 is associated with the non-uniqueness of the effective potential and the elimination of this branch point leads to a unique effective potential.

However, it seems to us, that this general way out of the problem is incorrect. The argument which Evans gives for keeping \( \exp \beta q_0 \) after the analytic continuation is “The key idea is that such (derivative) expansions describe field configurations which vary slowly in time and space and hence are not thermal equilibrium configurations (i.e. are not periodic in time). By explicit calculation, I will show that the retarded thermal Green functions used in previous analyses are not relevant to this problem. These Green functions describe how the system responds to a sudden impulse as linear response theory shows.” (third paragraph of [26]). It is not clear to us what the periodicity of a field has to do with whether it is slowly or rapidly varying in time.
Our point of view is that the truncated derivative expansion to 1st or higher order in derivatives may not be periodic itself but, provided that the periodicity of the exact field configuration is not forgotten, it provides a good approximation, when we deal with slowly varying, periodic in time, fields.

Moreover, in all usual physical situations, the Landau terms are expected to cancel each other at the zero temperature limit. If \( \exp \beta q_0 \neq 1 \), \( (22) \) does not have this property unless \( |q_0| < \Omega \). This reassures us that \( \exp \beta q_0 \) must be set equal to 1.

### 4.4 Comparison with Weldon

It occurred to us that, although we are not convinced by the arguments presented in [26, 27], there may be physical situations where the calculation of Evans is correct for a different reason, namely the exact configuration itself for the external field may not be necessarily periodic. In [19], Weldon claims that, although the fields in the loop of the self-energy have to be in thermal equilibrium, it is not necessary to assume that the external field \( \phi \) is in thermal equilibrium. Initially, it should be taken to follow a non-equilibrium thermal distribution \( f_0(q_0) \). At any later time this distribution will be denoted as \( f(q_0, t) \). Changes in \( f \) result both from the processes \( \phi \to \psi \bar{\psi}, \phi \bar{\psi} \to \bar{\psi}, \phi \psi \to \psi \) which decrease the number of \( \phi \)'s with rate \( f \Gamma_d \) and from the processes \( \psi \bar{\psi} \to \phi, \psi \to \psi \phi, \bar{\psi} \to \bar{\psi} \phi \) which increase the number of \( \phi \)'s with rate \( (1 + f) \Gamma_i \). Thus \( f(q_0, t) \) satisfies:

\[
\frac{\partial f}{\partial t} = -f \Gamma_d + (1 + f) \Gamma_i.
\]

(36)

For small departures from equilibrium one finds the solution

\[
f(q_0, t) = \frac{1}{e^{\beta q_0} - 1} + c(q_0) \ e^{-\Gamma(q_0) \ t}
\]

(37)

where \( c(q_0) \) is some arbitrary function. Regardless of the distribution specified initially, \( f(q_0, t) \) inevitably approaches the equilibrium as \( t \to \infty \). The rate of approach to equilibrium \( \Gamma(q_0) \) is related to the imaginary part of the self-energy \( \text{Im} \Pi \) through the relation \( \text{Im} \Pi = -q_0 \Gamma(q_0) \).

Does this mean that the external field is non-periodic and therefore \( \exp \beta q_0 \) should be retained after the analytic continuation? At this point one realizes that the calculation of [19]...
was not performed consistently, since $\exp \beta q_0$ was set to 1. This casts doubts concerning the conclusions of this paper. Furthermore, in our calculation, it is easy to see that retaining the $\exp \beta q_0$ leads to vanishing imaginary part of the self-energy.

The imaginary part of (35) is

\[
\text{Im}\Pi''_R = -\text{Im}\Pi''_A = \frac{m^2}{2\pi q} \int_{|m|}^{\infty} \frac{d\omega}{e^{\beta \omega} + 1} \int_{\Omega_+}^{\Omega_-} d\Omega \times \\
\left\{ \left[1 - e^{\beta(\Omega+\omega-q_0)} \right] \frac{1}{1 + e^{\beta(\Omega-q_0)}} \delta(\Omega + \omega - q_0) - \left[1 - e^{\beta(\Omega+\omega+q_0)} \right] \frac{1}{1 + e^{\beta(\Omega+q_0)}} \delta(\Omega + \omega + q_0) \right. \\
\left. + \left[1 - e^{\beta(\Omega-\omega+q_0)} \right] \frac{e^{\beta\omega}}{1 + e^{\beta(\Omega+q_0)}} \delta(\Omega - \omega + q_0) - \left[1 - e^{\beta(\Omega-\omega-q_0)} \right] \frac{e^{\beta\omega}}{1 + e^{\beta(\Omega-q_0)}} \delta(\Omega - \omega - q_0) \right\} \\
= 0.
\]

Similarly we can show that, if the $\exp \beta q_0$ is retained, the imaginary part of (7) vanishes.

Consequently, if we apply the conclusions of [19] in our case, we reach a contradiction; The imaginary part of the self-energy gives the rate at which the thermal distribution of $\phi$ approaches the equilibrium as $t \rightarrow \infty$. This implies that at any finite time $\phi$ is not in thermal equilibrium and therefore not periodic. However this means that the imaginary part of the self-energy vanishes, there is no approach to equilibrium and the physical interpretation given in [19] seems to be meaningless.

5 Conclusions

We have shown that in a model, where fermions couple to a pseudo-scalar field, the effective potential for the pseudo-scalar field can be found uniquely at finite temperature. We have also shown that this is not true when the fermion couples to a scalar field, the reason for that being the non-analytic behaviour which appears in the thermal bubble diagram.

We have pointed out that truncating in the beginning of the calculation the derivative expansion either at the constant term [22] or at higher order [23, 24, 27] gives misleading results. In the former case, because the operator nature of the background field is lost, and in the latter case, because the periodicity of the background field is not taken into account.

Finally we note that the models we dealt with in sections 2 and 3 can be considered together to study chiral symmetry restoration at finite temperature for example in the Lurie model [28],
the linear $\sigma$ model \[30\] in its broken chiral symmetry phase and in the Nambu-Jona-Lasinio model \[31\] expressed in terms of auxiliary fields. In general, whenever finite temperature symmetry restoration is discussed by employing non-perturbative results for the effective potential, these may not match the perturbative results. Therefore the question of symmetry restoration at finite temperature should be reanalyzed keeping in mind the non-analyticity of some graphs. Work on this and other related issues is in progress.

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