Chapter 1
Regular black holes sourced by nonlinear electrodynamics

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Abstract This chapter is a brief review on the existence and basic properties of static, spherically symmetric regular black hole solutions of general relativity where the source of gravity is represented by nonlinear electromagnetic fields with the Lagrangian function $L$ depending on the single invariant $f = F_{\mu\nu}F^{\mu\nu}$ or on two variables: either $L(f,h)$, where $h = \ast F_{\mu\nu}F^{\mu\nu}$, where $\ast F_{\mu\nu}$ is the Hodge dual of $F_{\mu\nu}$, or $L(f,J)$, where $J = F_{\mu\nu}F^{\nu\rho}F^{\rho\sigma}F^{\sigma\mu}$. A number of no-go theorems are discussed, revealing the conditions under which the space-time cannot have a regular center, among which the theorems concerning $L(f,J)$ theories are probably new. These results concern both regular black holes and regular particlelike or starlike objects (solitons) without horizons. Thus, a regular center in solutions with an electric charge $q_e \neq 0$ is only possible with NED having no Maxwell weak field limit. Regular solutions with $L(f)$ and $L(f,J)$ nonlinear electrodynamics (NED), possessing a correct (Maxwell) weak-field limit, are possible if the system contains only a magnetic charge $q_m \neq 0$. It is shown, however, that in such solutions the causality and unitarity as well as dynamic stability conditions are inevitably violated in a neighborhood of the center. Some particular examples are discussed.

1.1 Introduction

Nonlinear electrodynamics (NED) as a generalization of Maxwell’s theory was proposed in the 1930s: M. Born and L. Infeld’s formulated a theory able to remove the central singularity of the electromagneticfield of a point charge as well as its en-
Another version of NED was put forward by W. Heisenberg and H. Euler while taking into consideration high-energy quantum processes with photons, such as pair creation [36]. Much later, J. Plebanski [61] developed a more general formulation of NED in special relativity, admitting an arbitrary function of the electromagnetic invariants.

More recently, the interest in NED received a new support when it was discovered that a Born-Infeld-like theory appears in the weak-field limit of some models of string theory [31, 53, 65]. It has also turned out that NED can be a material source of gravity able to lead to nonsingular geometries of interest, such as regular black holes (BHs) and solitonlike configurations without horizons in the framework of general relativity (GR) and various alternative theories. Let us also mention one more recent application of NED, namely, using it as one of the sources of gravity in Simpson-Visser-like (black-bounce) space-times [32, 50, 67] that are regular models simulating some expected effects of quantum gravity on the classical level [20, 25, 27].

This paper is devoted to NED application for obtaining regular BHs and solitons (monopoles) in GR. We will reproduce a number of well-known results in a somewhat pedagogical manner and also present some new observations. We will restrict ourselves to the simplest models assuming spherical symmetry, and also mostly focus on NED theories with Lagrangians of the form $L = L(f)$, where $f = F_{\mu\nu}F^{\mu\nu}$, and $F_{\mu\nu}$ is the electromagnetic field tensor. Then we will briefly discuss similar problems in some extensions of $L(f)$ theories: those with $L(f, h)$, where $h = \frac{1}{2}\sqrt{-\det g}\,\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the Hodge dual of $F_{\mu\nu}$, and those with $L(f, J)$, where $J$ is quartic with respect to $F_{\mu\nu}$ [28, 68]: $J = F_{\mu\nu}F^{\nu\rho}F^{\sigma\delta}F^{\rho\sigma}F^{\delta\mu}$.

When considering the NED-GR system in spherical symmetry, there are only two possible kinds of electromagnetic fields: radial electric fields and radial (monopole) magnetic ones. Two important circumstances should be taken into account. The first one is (in general) the absence of duality between electric and magnetic fields, so that solutions to the field equations containing these fields in the framework of the same NED theory will be quite different. Instead, there emerges the so-called FP duality that connects electric and magnetic solutions for different NED theories but involving the same space-time metric. The second circumstance is that it is insufficient to require finite values of the electric field itself and the electric field energy of a point charge in order to obtain a regular space-time: its regularity imposes more stringent requirements on NED, which cannot be satisfied, for example, by the Born-Infeld theory.

NED-GR solutions with electric or magnetic fields are currently widely discussed, probably beginning with finding a general form of an electric solution by Pellicer and Torrence [60]. Later on, a no-go theorem was proved [?, 23], showing that if NED is specified by a Lagrangian function $L(f)$ having a Maxwell weak-field limit ($L \sim f$ as $f \to 0$), a static, spherically symmetric solution of GR with an electric field cannot have a regular center. This theorem was extended to include static dyonic configurations, involving both electric and magnetic fields [17], and it was also proved [16, 17] that in any electric solutions describing systems with or without horizons (i.e., BH or solitonic ones), containing a regular center and a flat infinity.
with a Reissner-Nordström (RN) asymptotic behavior, different NED theories are valid at large and small $r$. The present paper describes this issue in detail.

It was also shown [17] that purely magnetic regular configurations, both BH and solitonic ones, can exist and are easily obtained if $L(f)$ tends to a finite limit as $f \to \infty$. Electric solutions with the same metric can also be found, but they suffer multivaluedness of $L(f)$ and inevitably exhibit infinite blueshifts of traveling photons on some surfaces [17].

Many further results of interest are known. In particular, the properties and examples of static, spherically symmetric dyonic NED-GR space-times were studied [18, 43–45, 54, 73]; a kind of phase transition was discussed, allowing one to circumvent the above no-go theorem on electric solutions [26]; the static, spherically symmetric solutions were extended to include a nonzero cosmological constant $\Lambda$ [52]; the thermodynamic properties of regular NED BHs were investigated (see [4, 13, 30, 41, 46] and references therein); cylindrically [24] and axially [6, 29, 34, 47, 70] symmetric (rotating) NED-GR configurations were found and studied, as well as evolving wormhole models [1, 2, 9, 19]. (Note that static wormhole models with NED as a source are impossible because this kind of matter respects the weak energy condition.) Furthermore, the stability properties of NED BHs were investigated in [14, 49, 55], and quantum effects in their fields in [8, 51]. One should also mention a number of studies of special cases of both electric and magnetic solutions, their potential observational properties like gravitational lensing, particle motion and matter accretion in the fields of NED BHs as compared to their counterparts in scalar-tensor, $f(R)$ and multidimensional theories of gravity, consideration of NED with dilaton-like interactions, non-Abelian fields, different constructions with thin shells, etc., but the corresponding list of references would be too long. For a recent brief review on NED with and without relation to gravitational theories see [69].

The most relevant to the present subject, regular BHs, are the recent results obtained by Bokulić, Smolić and Jurić [10, 11] who have proved a number of no-go theorems in NED-GR solutions with NED Lagrangians of the form $L(f,h)$. Let us mention that this wide class of theories contains, among others, the Born-Infeld and Heisenberg-Euler theories. With all these no-go theorems, it seems that regular magnetic BHs with $L = L(f)$ are the only kind of regular BHs that can be found among NED-GR solutions with an asymptotically Maxwell NED, although some opportunities are still remaining unexplored.

In this paper, we will discuss in detail the existence and main properties of static, spherically symmetric regular black holes and solitons with $L(f)$ NED theory, and more briefly consider the same with Lagrangians depending on two invariants, either $L(f,h)$ or $L(f,J)$. We begin with discussing the particular form of regularity and asymptotic conditions to be used (Section 1.2). Then, in Section 1.3, we discuss the $L(f)$ NED-GR field equations in static, spherically symmetric space-times. Section 1.4 is devoted to the regularity properties of black hole and soliton solutions to these equations, their compatibility with the known NED unitarity and causality [66] and stability [55] conditions as well as photon propagation in these space-times. Some particular examples known in the literature are also discussed. Section 1.5 presents...
some no-go theorems with \( L(f, h) \) due to \([10, 11]\) and with \( L(f, J) \), and the latter results seem to be new. Section 1.6 is a brief conclusion.

We use the following conventions: the units with \( c = 8\pi G = 1 \); the metric signature \((+−−−)\); the curvature tensor \( R^\sigma_{\mu\rho\nu} = \partial_\nu\Gamma^\sigma_{\mu\rho} = \ldots \); the Ricci tensor \( R_{\mu\nu} = R^\sigma_{\mu\sigma\nu} \), so that the Ricci scalar \( R = g^{\mu\nu}R_{\mu\nu} > 0 \) for de Sitter space-time. The Einstein equations are written in the form

\[
G^\nu_\mu \equiv R^\nu_\mu - \frac{1}{2} g^\nu_\mu R = -T^\nu_\mu, \tag{1.1}
\]

where \( T^\nu_\mu \) is the stress-energy tensor (SET) of matter, such that \( T^t_t \) is the energy density.

### 1.2 Static spherically symmetric space-times. Regularity and asymptotic conditions

Before dealing with NED-Einstein equations, it makes sense to recall the conditions to be fulfilled by the desirable solutions to these equations.

Spherical symmetry is the simplest and natural assumption for descriptions of isolated bodies when their precise shape and possible rotation are regarded insignificant. The physical fields of any island-like objects are approximately spherically symmetric far from these objects.

In the general case, one can write a spherically symmetric metric in the form (see, e.g., \([48]\))

\[
ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}dx^2 - r^2d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{1.2}
\]

In general, \( \alpha, \gamma, r \) are functions of the radial coordinate \( x \) and the time coordinate \( t \). The quantity \( r \) has the geometric meaning of the radius of a coordinate sphere \( x = \text{const}, \ t = \text{const} \), the so-called spherical radius, or it is sometimes called the areal radius since the area of a coordinate sphere is equal to \( 4\pi r^2 \). Let us note that in curved space-time this radius \( r \) has nothing to do with a distance from the center (as happens in flat space-time), and there are many spherically symmetric space-times that contain no center at all, for example, wormholes.

In what follows we restrict ourselves to static space-times, such that \( \alpha, \gamma, r \) depend on \( x \) only. There still remains the freedom of choosing the radial coordinate \( x \) and the possibility of its reparametrization by replacing \( x = x(x_{\text{new}}) \). The choice of the radial coordinate can be fixed by postulating a relation between the functions \( \alpha, \gamma, r \) or by choosing some of them (or a function of some of them) as the coordinate. For example, very often the radius \( r \) is used as a coordinate, it is then called the Schwarzschild (or curvature) radial coordinate.

The convenient “exponential” notations in the metric (1.2), which simplify the appearance of many relations without fixing the radial coordinate, assume positive values of the corresponding quantities. However, the coefficients \( g_{tt} \) and \( g_{xx} \) can
change their sign, in particular, this happens at black hole horizons. In such cases, it is helpful to use the so-called quasiglobal coordinate condition $\alpha + \gamma = 0$, and with the notation $e^{2\gamma} = e^{-2\alpha} = A(x)$, the metric is written as

$$
\text{ds}^2 = A(x)\text{dt}^2 - \frac{\text{d}x^2}{A(x)} - r^2(x)\text{d}\Omega^2.
$$

(1.3)

**Regularity.** A Riemannian space-time is generally called regular at a particular point $X$ if the Riemann tensor is well defined at $X$ (hence the metric functions must be at least twice differentiable at $X$), and all algebraic curvature invariants are finite. (Other definitions of regularity, involving differential invariants of the Riemann tensor, are sometimes used, but the above definition is sufficient for our purposes.) Hence, the metric (1.2) (or (1.3)) is manifestly regular at any point where $r \neq 0$ as long as the functions $\alpha(x)$, $\gamma(x)$ and $r(x)$ (or $A(x)$ and $r(x)$) are sufficiently smooth. A point where $r = 0$ requires special attention because the metric becomes degenerate there, hence it is a singular point of the spherical coordinate system used in (1.2) or (1.3). Furthermore, a space-time as a whole (and in particular, a black hole space-time) is called regular if all its points are regular.

Very often, to verify regularity of a particular metric of the form (1.2) or (1.3), one directly calculates its basic invariants: the scalar curvature $R$, the Ricci tensor squared $R_{\mu\nu}R^{\mu\nu}$, and the Kretschmann scalar (the Riemann tensor squared) $K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. However, for a static metric (1.2), it is quite sufficient and much easier to verify finiteness of the four independent components $R_{\alpha\beta}^{\gamma\delta}$ of the Riemann tensor with two upper and two lower indices:

$$
K_1 = -R_{01}^{\ 01} = e^{-\alpha - \gamma}(\gamma' e^{\gamma - \alpha})' = \frac{1}{2}A'',
$$

$$
K_2 = -R_{02}^{\ 02} = -R_{03}^{\ 03} = e^{-2\alpha} \frac{\gamma'}{r} = \frac{A'r'}{2r},
$$

$$
K_3 = -R_{12}^{\ 12} = -R_{13}^{\ 13} = e^{-\alpha} \frac{r'}{r} (e^{-\alpha} r')' = \frac{1}{2r} (2Ar'' - A'r'),
$$

$$
K_4 = -R_{23}^{\ 23} = \frac{1}{r^2} (1 - e^{-2\alpha} r'^2) = \frac{1}{r^2} (1 - Ar'^2).
$$

(1.4)

where the prime stands for $d/dx$ ($K_i$ in terms of the metric (1.3) are given in each line after the last equality sign). The point is that for static, spherically symmetric metrics, as well as and in many other important cases, the tensor $R_{\alpha\beta}^{\gamma\delta}$ is pairwise diagonal. Therefore, all algebraic curvature invariants are linear, quadratic, cubic, etc., combinations of $K_i$ from (1.4) and are manifestly finite if $K_i$ are finite. Moreover, the Kretschmann scalar is a sum of squares:

$$
\mathcal{K} = 4K_1^2 + 8K_2^2 + 8K_3^2 + 4K_4^2,
$$

(1.5)

hence it is finite if and only if each $K_i$ is finite. Thus finiteness of all $K_i$ is both necessary and sufficient condition of space-time regularity [22].
It is important to note that all $K_i$ in (1.4) are invariant (behave as scalars) under reparametrizations of the $x$ coordinate, and the same is true for mixed components of second-rank tensors, including the Ricci tensor $R^\nu_\mu$ and the Einstein tensor $G^\nu_\mu = R^\nu_\mu - \frac{1}{2} \delta^\nu_\mu R$. Thus the space-time regularity can be verified using $K_i$ in terms of any radial coordinate $x$.

As follows from (1.4), regularity at $r = 0$ requires not only finite values and smoothness of $\alpha$ and $\gamma$, but also, due to the expression for $K_4$,

$$e^{-2\alpha r'^2} - 1 = O(r^2) \quad \text{as} \quad r \to 0. \quad (1.6)$$

It is actually the local flatness condition, requiring a circumference to radius ratio of $2\pi$ for small circles around the center.

One more important observation follows from Eq. (1.6) for black hole space-times, in which $A(x)$ can become negative. The condition (1.6), rewritten as $Ar^2 - 1 = O(r^2)$, cannot be satisfied if $A(x) < 0$, which happens in nonstatic regions of spherically symmetric black holes (also called T-regions) beyond their horizons. We see that the metric cannot be regular in the limit $r \to 0$ in T-regions of spherically symmetric black holes. A regular center can only occur in a static region where $A > 0$.

It also follows from (1.4) that the metric (1.3) is regular at apparent horizons that correspond to regular zeros of the function $A(x)$ under the condition $r(x) > 0$.

**Asymptotics.** For an island-like system, it is natural to assume that the space-time is asymptotically flat, and far from the source of gravity there is an approximately Schwarzschild gravitational field characterized by a certain mass $m$. In terms of an arbitrary radial coordinate $x$ it means that in the metric (1.2), under the appropriate choice of the time scale,

$$e^{2\gamma(x)} = 1 - \frac{2m}{r(x)} + o(1/r) \quad \text{as} \quad r \to \infty \quad (1.7)$$

In addition, one should require a correct circumference to radius ratio for large circles around the source of gravity, which leads to a condition similar to (1.6),

$$e^{-2\alpha r'^2} \to 1 \quad \text{as} \quad r \to \infty. \quad (1.8)$$

A limit other than unity in (1.8) leads to a deficit or excess of the solid angle at infinity, characterizing a global monopole space-time [72].

In the presence of a nonzero cosmological constant $\Lambda$, the gravitational field far from its island-like source as asymptotically de Sitter (if $\Lambda > 0$) or anti-de Sitter (if $\Lambda < 0$), well described by the metric (1.3) with $r = x$ and $A = 1 - \Lambda r^2/3$. 

1.3 $L(f)$ NED coupled to general relativity. FP duality

1.3.1 Field equations

Let us now consider self-gravitating electromagnetic fields with the Lagrangian $L(f)$ in the framework of GR, so that the total action has the form

$$S = \frac{1}{2} \int \sqrt{-g} d^4x [R - L(f)],$$  \hspace{1cm} (1.9)

where $R$ is the Ricci scalar, the invariant $f$ has the standard form $f = F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2)$, where the 3-vectors $E$ and $B$ are the electric field strength and magnetic induction, and $L(f)$ is an arbitrary function. The electromagnetic tensor $F_{\mu\nu}$ obeys the Maxwell-like equations, obtained from (1.9) by variation with respect to the 4-vector potential $A_\mu$, and the Bianchi identities for the dual field $^*F_{\mu\nu}$, following from the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\nabla_\mu (L f F_{\mu\nu}) = 0, \quad \nabla_\mu *F_{\mu\nu} = 0. \hspace{1cm} (1.10)$$

The corresponding SET is given by ($L_f \equiv dL/df$)

$$T^\nu_\mu = -2L_f F_{\mu\alpha}F^{\nu\alpha} + \frac{1}{2} \delta^\nu_\mu L(f). \hspace{1cm} (1.11)$$

Let us assume spherical symmetry, with a metric of the general form (1.2). The only nonzero components of $F_{\mu\nu}$ compatible with this symmetry are $F_{tr} = -F_{rt}$, representing a radial electric field, and $F_{\theta\phi} = -F_{\phi\theta}$, corresponding to a radial magnetic field. From (1.10) it follows

$$r^2 e^{\alpha + \gamma} L_f F_{tr} = q_e, \quad F_{\theta\phi} = q_m \sin \theta, \hspace{1cm} (1.12)$$

where $q_e = \text{const}$ has the meaning of an electric charge, and $q_m = \text{const}$ is a magnetic charge. Accordingly, the only nonzero SET components have the form

$$T^r_t = T^r_t = \frac{1}{2} L + f_e L_f, \quad T^\theta_\theta = T^\phi_\phi = \frac{1}{2} L - f_m L_f, \hspace{1cm} (1.13)$$

where

$$f_e = 2E^2 = 2F_{tr}F^{tr} = \frac{2q_e^2}{L^2 r^4}, \quad f_m = 2B^2 = 2F_{\theta\phi}F^{\theta\phi} = \frac{2q_m^2}{r^4}, \hspace{1cm} (1.14)$$

so that $f = f_m - f_e$. Here, $E = |E|$ and $B = |B|$ are the absolute values of the electric field strength and magnetic induction, measured by an observer at rest in our static space-time.

The SET (1.13) has two important properties $T^r_t = 0$ and $T^t_t = T^r_r$. The first one means the absence of radial energy flows, related to the absence of monopole electromagnetic radiation. The second one, due to the Einstein equations (1.1), leads to
$G'_t = G'_r$, and this equation is easily integrated if we use the Schwarzschild radial coordinate, $x \equiv r$ (see, e.g., [48]), leading to the relation $\alpha(r) + \gamma(r) = \text{const}$. With a proper choice of the time scale, we have $\alpha + \gamma = 0$, and the metric can be rewritten as

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2 d\Omega^2.$$  \hspace{1cm} (1.15)

The other Einstein equation, $G'_t = -T'_t$, then reads

$$A + A'r = 1 - \rho r^2$$  \hspace{1cm} (1.16)

and can be rewritten in the integral form as

$$A(r) = 1 - \frac{2M(r)}{r}, \quad M(r) = \frac{1}{2} \int \rho(r) r^2 dr,$$  \hspace{1cm} (1.17)

where $\rho(r) = T'_t$ is the energy density, and $M(r)$ is called the mass function, such that $M(\infty)$ is the Schwarzschild mass in an asymptotically flat space-time. It is a solution for $A(r)$ if $\rho(r)$ is known. Note, however, that a complete solution for the system under consideration requires a knowledge of $L(f)$ and both electric and magnetic fields as functions of $r$.

### 1.3.2 FP duality

NED with a Lagrangian function $L(f)$ is known to admit an alternative representation obtained from the original one by a Legendre transformation [5, 60, 64]; to this end, the new tensor $P_{\mu\nu} = L_f F_{\mu\nu}$ is defined, with its invariant $p = P_{\mu\nu} P^{\mu\nu}$. Then one considers the Hamiltonian-like quantity

$$H(p) = 2f L_f - L = -2T'_t$$  \hspace{1cm} (1.18)

as a function of $p$. It is possible to use the function $H(p)$ to specify the whole theory. The following relations are valid:

$$L = 2pH_p - H, \quad L_f H_p = 1, \quad f = pH_p^2, \quad p = fL_f^2,$$  \hspace{1cm} (1.19)

where $H_p \equiv dH/dp$. In terms of $H$ and $P_{\mu\nu}$, the SET reads

$$T'_\mu = -2H_p P_{\mu\alpha} P^{\nu\alpha} + \delta'_\mu(pH_p - \frac{1}{2} H).$$  \hspace{1cm} (1.20)

In a spherically symmetric space-time with the metric (1.15), Eqs. (1.12) are rewritten in the P framework as

$$r^2 P^{pr} = q_e, \quad H_p P_{\theta\phi} = q_m \sin \theta.$$  \hspace{1cm} (1.21)

Let us also introduce the quantities $p_e$ and $p_m$ quite similar to $f_e$ and $f_m$:
1.4 Regular black holes with $L = L(f)$

1.4.1 Magnetic, electric and dyonic solutions

**Magnetic solutions** ($q_e = 0, q_m \neq 0$) can be found most easily. If the Lagrangian $L(f)$ is specified, then, since now $f = 2q_m^2/r^4$, the density $\rho(r) = L/2$ is known according to (1.13), and the metric function $A(r)$ is found by integration in (1.17).

If, on the contrary, we know $A(r)$ (or choose it by hand), then $\rho = L(f)/2$ is found from (1.17), leading to

$$L(f(r)) = \frac{2}{r^4}[1 - (rA)']$$

and $L(f)$ is restored since $f = 2q_m^2/r^4$.

**Electric solutions** ($q_e \neq 0, q_m = 0$) can be obtained in quite a similar manner if we use the Hamiltonian-like form of NED, see Eqs. (1.19)–(1.23). In this case, $p =
−2q_e^2/r^4, and if we specify \( H(p) = -2 \rho \), the mass function \( M(r) \) is directly found, while \( A(r) \) is obtained by integration in (1.17). If \( A(r) \) is specified, then Eq. (1.17) allows for finding \( \rho(r) = -H(p)/2 \).

However, if one starts with the Lagrangian \( L(f) \) and seeks electric solutions, a separate problem is the transition to the \( P \) framework, which is equivalent to the \( F \) framework only if \( f(p) \) is a monotonic function, or only in such ranges of \( f \) and \( p \) in which \( f(p) \) is monotonic. There is also a technical problem of expressing \( H \) as a function of \( p \) after its obtaining as a function of \( f \) according to (1.18).

For example, consider the simple rational function \[ L(f) = f \frac{1}{1 + 2\beta f}, \quad \beta = \text{const} > 0. \] (1.26)
The quantity (1.18) is easily found,
\[ H = 2fL_f - L(f) = \frac{f(1 - 2\beta f)}{(1 + 2\beta f)^2}, \] (1.27)
but finding the dependence \( f(p) \) to be substituted to (1.27) requires solving a fourth-order algebraic equation:
\[ f = p(1 + 2\beta f)^4. \] (1.28)

It is therefore not surprising that the numerous existing electric solutions either start from a specific function \( H(p) \) or postulate the metric function \( A(r) \), as is actually done in [3] and a few other papers by the same authors.

**Dyonic solutions** with both nonzero charges \( q_e \) and \( q_m \) can be obtained with more effort. Neither \( f(r) \) nor \( p(r) \) is known explicitly now. Thus, in particular,
\[ f(r) = \frac{2}{r^4} \left( q_m^2 - \frac{q_e^2}{L_f} \right). \] (1.29)
Comparing the expressions for \( \rho(r) \) from (1.13) and from (1.17), we can write
\[ \frac{1}{2} L(f) + 2q_e^2 L_f / r^4 = \frac{2M'(r)}{r^2} = \rho(r). \] (1.30)

If \( L(f) \) is known, Eq. (1.29) can be treated either (A) as an (in general, transcendental) equation for the function \( f(r) \) or (B) as an expression of \( r \) as a function of \( f \).

In case (A), if we can find explicitly \( f(r) \), integration of Eq. (1.30) gives the metric function \( A(r) \).

The scheme (B) gives a solution in quadratures expressed in terms of \( f \) that can be now chosen as a new radial coordinate. Indeed, if \( L(f) \) and \( r(f) \) are known and monotonic, so that \( L_f \neq 0 \) and \( r_f \neq 0 \), we can rearrange Eq. (1.30) as
\[ M_f = \frac{r^2}{2} \left( \frac{L}{\frac{f^2}{2L} + q_e^2} \right) \]  
\[ (1.31) \]

(as before, the subscript \( f \) denotes \( d/df \)). Since the r.h.s. of (1.31) is known, we can calculate \( M(f) \) and \( A(r) \) and also rewrite the metric in terms of the coordinate \( f \). Thus we obtain a general scheme of finding dyonic solutions under the above conditions [18].

As a trivial example of using the scheme (A), we can consider the Maxwell theory, \( L = f \). Substituting \( L = f \) and \( L_f = 1 \) to Eq. (1.30), we obtain
\[ 2M' = \frac{q_e^2 + q_m^2}{r^2}, \]
whence \( 2M(r) = 2m - \frac{(q_e^2 + q_m^2)}{r} \) and
\[ A(r) = 1 - \frac{2m}{r} + \frac{q_e^2 + q_m^2}{r^2}, \quad m = \text{const}, \]
\[ (1.32) \]

that is, the dyonic Reissner-Nordström solution, as should be the case.

Another example is obtained [18] if we assume that Eq. (1.29) is linear in \( f \). Then we have to put \( L - 2f = c_1 f + c_2 \) with \( c_1, c_2 = \text{const} \), which yields after integration
\[ L(f) = b^2 \left( -1 + \sqrt{1 + 2f/b^2} \right), \quad b = \text{const} \]
\[ (1.33) \]
the full Born-Infeld Lagrangian also involves the other electromagnetic invariant \( h^2 = (F_{\mu\nu} F^{\mu\nu})^2 \). With (1.33), we obtain
\[ f(r) = \frac{2b^2(q_m^2 - q_e^2)}{4q_e^2 + b^2 r^4}, \]
\[ \rho(r) = -\frac{b^2}{2} + \left( b^2 + 2q_e^2 \right) \sqrt{\frac{4q_m^2 + b^2 r^4}{4q_e^2 + b^2 r^4}}. \]
\[ (1.34) \]

In the special case of a self-dual electromagnetic field, \( q_e^2 = q_m^2 \), we find simply \( f = 0 \) and \( \rho(r) = 2q_e^2/r^4 \), as in the Maxwell theory, and the dyonic solution for \( A(r) \) coincides with (1.32). For arbitrary charges, Eq. (1.17) leads to a long expression with the Appel hypergeometric function \( F_1 \), not to be presented here.

Other examples of dyonic NED-GR solutions are found and discussed in [43–45, 54, 73].

1.4.2 Regularity and no-go theorems

**Magnetic solutions.** According to (1.6), a regular center requires \( A(r) = 1 + \mathcal{O}(r^2) \) at small \( r \). In magnetic solutions with \( f = 2q_m^2/r^4 \to \infty \) the metric regularity then requires \( L \to L_0 < \infty \) as \( f \to \infty \) [17] because the density that should be finite is now \( T_r^r = \rho = L/2 \). Furthermore, asymptotic flatness requires \( A(r) = 1 - 2m/r + o(1/r) \),
where \( m \) is the Schwarzschild mass. By (1.17), it is the case if \( \rho \sim r^{-4} \) or smaller as \( r \to \infty \), which happens if \( L(f) \sim f \), i.e., it has a Maxwell asymptotic behavior at small \( f \). The metric is then approximately Reissner-Nordström at large \( r \).

The infinite magnetic induction \( B \sim 1/r^2 \) at the center might cause a problem, but as discussed in [17], a correct estimate of the force applied to a charged test particle moving in the nonlinear magnetic field under consideration, obtained along the lines of Refs. [62, 63], shows that such forces are finite for both electrically and magnetically charged test particles and even vanish at \( r = 0 \).

Thus invoking a smooth function \( L(f) \) such that \( L \sim f \) as \( f \to 0 \) and \( L \to L_0 < \infty \) as \( f \to \infty \) is an easy way to obtain globally regular configurations including magnetic black holes and solitons, used in many papers, probably beginning with Ref. [17].

In all such solutions, a general feature is that \( A \to 1 \) as both \( r \to 0 \) and \( r \to \infty \). Moreover, the mass term \( -2m/r \) contributes negatively to \( A(r) \) as long as \( m > 0 \). Thus in regular solutions \( A(r) \) should inevitably have a minimum, at which the value of \( A \) depends on the mass and charge values. Their relationship determines the existence of horizons located at regular zeros of \( A(r) \). If the mass \( m > 0 \) is fixed, then at small charges (which contribute positively to \( A(r) \) at least at large \( r \)) the minimum of \( A \) is negative because the solution is close to Schwarzschild’s almost everywhere, and then any regular function \( A(r) \) has two zeros, one of which should be close to \( r = 2m \), while the other emerges since it is necessary to return to \( A(r) > 0 \) at small \( r \) to reach \( A = 1 \) at \( r = 0 \). At large charges, on the contrary, the mass term \( -2m/r \) is only significant at large \( r \), and a minimum of \( A \) should be positive, leading to a solitonic solution. Some value of \( q \) must be critical, leading to a double zero of \( A(r) \), corresponding to a single extremal horizon.

This general picture is really observed in the known examples of regular static, spherically symmetric NED-GR solutions. Let us illustrate it with the behavior of \( A(r) \) in the example from [17], where

\[
L(f) = \frac{f}{\cosh^2\left(b|f/2|^{1/4}\right)}, \quad b = \text{const} > 0. \tag{1.35}
\]

In the magnetic solution, with \( q = q_m > 0 \) (for simplicity),

\[
\rho = \frac{q^2/r^4}{\cosh^2(b\sqrt{q}/r)}, \quad A(r) = 1 - \frac{2m}{r} \left(1 - \tanh\left(q^2/2mr\right)\right), \tag{1.36}
\]

where the mass \( m \) is determined as \( M(\infty) \). The behavior of \( A(r) \) is shown in Fig. 1 for three values of \( q/m \) leading to qualitatively different geometries. The causal structures and Carter-Penrose diagrams of these space-times are the same as those for Reissner-Nordström ones, but the important difference is that now the lines \( r = 0 \) denote a regular center instead of a singularity.

Regular models with more than two horizons are also possible, see, e.g., [33, 56] for detailed studies of such solutions.
An important feature of regular solutions is that with given $L(f)$ the Schwarzschild mass $m$ is uniquely fixed by the charge $q$. Indeed, to obtain a regular center, the integration in Eq. (1.17) must be carried out from $r = 0$ (where the density $\rho$ is finite and determined by $q$) to arbitrary $r$, resulting in the Schwarzschild mass $m = M(\infty)$. It means that this mass is completely created by the electromagnetic field energy. Any additional mass $m_1$ that can appear in the solution as an integration constant in Eq. (1.17) would add the singular term $2m_1/r$ to $A(r)$.

Thus, in particular, returning to the solution (1.36) for the theory (1.35), it is easy to find that $m = q^{3/2}/(2b^{1/4})$, or on the contrary, the parameter $b$ in $L(f)$ may be expressed in terms of $m$ and $q$: $b = q^6/(16m^4)$.

**Electric solutions** with a regular center and a Reissner-Nordström asymptotic behavior can either be found in the same manner using the $P$ formulation of NED (as is done in Refs. [3, 30] and many others), or obtained directly from the magnetic ones using the FP duality. However, as we saw above, solutions with the same metric correspond to quite different NED theories than those used in magnetic solutions, and this circumstance leads to their different physical properties. First of all, let us recall a theorem proved in [17, 21, 23]:

**Theorem 1.1.** If a static, spherically symmetric electric solution ($q_e \neq 0, q_m = 0$) to the $L(f)$ NED-Einstein equation describes a space-time with a regular center, it cannot have a Maxwell behavior at small $f$ ($L \approx f, L_f \to 1$).

**Proof.** To begin with, since the Ricci tensor for our metric is diagonal, the curvature invariant $R_{\mu\nu}R^{\mu\nu} = R_{\mu}^{\mu}R_{\nu}^{\nu}$ is a sum of squares of the components $R_{\mu}^{\mu}$, hence each of them taken separately must be finite at any regular point, including a center. It then follows that each of the components of $T_{\mu}^{\nu}$ should be finite, as well as their any linear combination. In particular, by (1.13), we must have $|f_eL_f| < \infty$. But according to (1.14), $f_eL_f^2 = 2q_e^2/r^4 \to \infty$. These two conditions, taken together, lead to

$$f = -f_e \to 0, \quad L_f \to \infty \quad \text{as} \quad r \to 0.$$  \hspace{1cm} (1.37)

It means that we have a non-Maxwell function $L(f)$ at small $f$. \qed
On the other hand, regular asymptotically flat electric solutions obtained in the \( P \) formulation of NED have a correct Maxwell asymptotic behavior. How can it be combined with (1.37)?

The answer is that such solutions correspond to different Lagrangians \( L(f) \) near \( r = 0 \) and at large \( r \) \([16,17]\). Indeed, at a regular center \( r = 0 \) we have \(-p = 2q_e^2/r^4 \rightarrow \infty\) and \( f = 0 \), while at flat infinity both \( p \rightarrow 0 \) and again \( f \rightarrow 0 \). It means that \( f \) inevitably has at least one extremum at some \( p = p^* \), breaking the monotonicity of \( f(p) \), which means that on different sides of \( p^* \) we have different functions \( L(f) \) corresponding to the same \( H(p) \). As shown in \([17]\), at an extremum of \( f(p) \) the function \( L(f) \) suffers branching, at which the derivative \( L_f \) tends to the same finite limit as \( p \rightarrow p^* + 0 \) and \( p \rightarrow p^* - 0 \), while \( L_{ff} \) tends to infinities of opposite signs. This corresponds to a cusp in the plot of \( L(f) \). Another form of branching of \( L(f) \) takes place at extremum points of \( H(p) \), if any, where the monotonicity of \( f(p) \) also breaks down. The number of different Lagrangians \( L(f) \) on the way from the center to infinity is equal to the number of monotonicity ranges of \( f(p) \) \([17]\).

To illustrate this unusual behavior of \( L(f) \) let us use as an example the same metric function (1.36), where now \( q = q_e \), as a solution corresponding to \( H(p) \) dual to (1.35):

\[
H(p) = -\frac{p}{\cosh^2 \left( b|p/2|^{1/4} \right)}, \quad b = \text{const} > 0.
\]  

Calculations reveal the behavior of the corresponding functions \( f(p) \) and \( L(f) \) shown in Figs. 2, 3. It turns out that \( L(f) \) has as many as four branches, in other words, there are four NED theories acting in different parts of space.

![Fig. 1.2: The function \( f(p) \) obtained from \( H(p) \) given by Eq. (1.38) with \( b = 1 \). The points \( p_1 \) and \( p_3 \) show the maxima of \( |f(p)| \) while \( p_2 \) shows its minimum corresponding to the maximum of \( |H(p)| \). The upper inset shows the function \( H(p) \), while the lower one is an enlarged view of the neighborhood of \( p_2 \) and \( p_3 \) in the plot of \( f(p) \).](image)

**Dyonic solutions.** If there are both nonzero \( q_e \) and \( q_m \), then a combination of \( T^\nu_{\mu} \) components leads to the requirement
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Fig. 1.3: The behavior of $L(f)$ in the electric solution for $H(p)$ from Eq. (1.38) with $b = 1$. The points $p_1$, $p_2$, $p_3$ correspond to the extrema of $f(p)$, at which the function $L(f)$ passes on from one branch to another. The inset shows more clearly the range close to $p_2$ and $p_3$. Arrows on the curves show the direction of growing $|p|$.

\[(f_e + f_m)|L_f| < \infty\]  \hspace{1cm} (1.39)

that must hold at any regular point, including a regular center. Moreover, it must hold for each term separately because both $f_e$ and $f_m$ are positive. Applying it to $f_e$, we obtain, as before, that $L_f \to \infty$ at a regular center [17]. However, the inequality $f_m|L_f| < \infty$ leads to the requirement $L_f \to 0$, since $f_m = 2q_m^2/r^4 \to \infty$ as $r \to 0$. We arrive at a contradiction that leads to the general result:

**Theorem 1.2.** Static spherically symmetric dyonic solutions to the NED-Einstein equations with arbitrary $L(f)$ cannot describe space-times with a regular center.

**Inclusion of a cosmological constant.** If $\Lambda \neq 0$, asymptotically (A)dS solutions [52] are obtained by simply adding $-\Lambda r^2/3$ to $A(r)$ in (1.17). This new term does not affect the properties of the solutions near $r = 0$, therefore, all conclusions on the existence of a regular center and the necessary conditions for it, obtained with $\Lambda = 0$, remain valid with $\Lambda \neq 0$, although the latter drastically changes the global properties of space-time.

### 1.4.3 Causality and unitarity

An important viability criterion for NED theories has been suggested by A. Shabad and V. Usov [66], partly on the basis of their previous work: they have used (i) the causality principle as the requirement that elementary excitations over a background field should not have a group velocity exceeding the speed of light in vacuum and (ii) the unitarity principle formulated as the requirement that the residue of the propagator should not be negative. As a result, there emerge the following inequalities...
that should hold for a theory satisfying these principles: in our notations, for \( L(f) \) theories,

\[
L_f > 0, \quad L_{ff} \leq 0, \quad \Phi = L_f + 2fL_{ff} \geq 0. \tag{1.40}
\]

One can notice that the third condition can be rewritten as \( H_f \geq 0 \), with the Hamiltonian-like quantity \( H \) given by (1.18), but does not directly concern the derivative \( H_p \) due to a possible complexity in the dependence \( f(p) \). The quantity \( \Phi \) also plays an important role in the effective metric for photon propagation and in the stability conditions, to be considered in the next subsections.

One immediate observation can be made about magnetic solutions with a regular center, both black hole and solitonic ones:

**Theorem 1.3.** In static, spherically symmetric magnetic solutions to \( L(f) \) NED-Einstein equations, the causality and unitarity conditions (1.40) are inevitably violated in a neighborhood of a regular center.

**Proof.** A regular center requires a finite limit of \( L(f) \) as \( f \to \infty \). If \( L_f > 0 \) (as required by the first inequality in (1.40)), then, to have a convergent integral \( L = \int L_f df \), one has to require \( L_f \ll 1/f \) at large \( f \), hence the quantity \( L_f \sqrt{f} \) is decreasing as \( f \to \infty \), and its derivative in \( f \) is negative. On the other hand, we can write \( \Phi = 2\sqrt{f(L_f \sqrt{f})} \), consequently, \( \Phi < 0 \) at large \( f \), so that the first and third inequalities in (1.40) cannot hold simultaneously. \( \square \)

### 1.4.4 Light propagation and the effective metric

As we have seen, the same regular metric of the form (1.15) can be obtained with two kinds of sources, the electric and magnetic ones, described by different NED theories. It is thus natural to expect that the properties of electromagnetic fields will also be different in these two cases. Let us try to explore these differences using the effective metric formalism developed by M. Novello et al. [58] while studying the propagation of electromagnetic field discontinuities using Hadamard’s approach [35]. According to [58, 59], photons governed by NED propagate along null geodesics of the effective metric

\[
h^{\mu\nu} = g^{\mu\nu}L_f - 4L_{ff}F^\mu_aF^{\mu\nu}. \tag{1.41}
\]

**Electric solutions.** In the case of a purely electric field in the metric (1.15), \( h^{\mu\nu} \) is diagonal, and we can write the effective metric as

\[
\begin{align*}
\text{d}s^2_{\text{eff}} &= g_{\mu\nu}\text{d}x^\mu\text{d}x^\nu = \frac{1}{\Phi} \left[ A(r)\text{d}t^2 - \frac{dr^2}{A(r)} \right] - \frac{r^2}{L_f} d\Omega^2, \\
\Phi &= L_f + 2fL_{ff} = \frac{H_p}{f_p}. \tag{1.42}
\end{align*}
\]
Consider the behavior of $h_{\mu\nu}$ at branching points of $L(f)$ that are inevitable in solutions with regular $g_{\mu\nu}$. At an extremum $p = p^*$ of $f(p)$ at which $f \neq 0$ (like points $p_1$ and $p_3$ in Fig. 2), we have $\Phi \to \infty$ since $f_p = 0$ while $H_p$ is finite. This results in a curvature singularity of the effective metric due to blowing up of the quantity $K_1$ in (1.4).

Another kind of singularity of the metric (1.42) occurs at extrema of $H(P)$, those like point $p_2$ in Fig. 2: in this case, generically, $\Phi$ is finite but $L_f \to \infty$, which leads to a singular center in the auxiliary space-time with the metric $h_{\mu\nu}$ due to $h_{\theta\theta} \to 0$.

The changes in photon frequencies at their motion in space-time can be evaluated as outlined in [59]. Thus, if an emitter at rest at point $X$ sends a photon with frequency $\nu_X$, it comes to a receiver at rest at point $Y$ with frequency $\nu_Y$ related to $\nu_X$ by

$$\frac{\nu_Y}{\nu_X} = \left[ \frac{\sqrt{g_{tt}}}{h_{tt}} \right]^y \left[ \frac{\sqrt{g_{tt}}}{h_{tt}} \right]^{-1}_X = \left[ \frac{\Phi}{\sqrt{A}} \right]^y \left[ \frac{\Phi}{\sqrt{A}} \right]^{-1}_X,$$

(1.43)

where the second equality sign corresponds to the metric (1.42). If $X$ is a regular point while $Y$ is located at an inevitable branching point of $L(f)$ (like $p_1$ or $p_3$), then any photon arriving there is infinitely blueshifted, gaining an unlimited energy, which thus implies instability of the whole configuration.

The above reasoning used the assumption $A > 0$. In black hole solutions, the sphere where $L_f = 0$ may be located beyond the event horizon, where $A < 0$. In such a region, also called a T-region, $r$ is a temporal coordinate, $t$ is a spatial one, and in the redshift relation (1.43) we must replace $g_{tt}$ with $g_{rr}$, or more specifically, $\sqrt{A}$ with $1/\sqrt{-A}$. However, as long as $A$ is finite, this replacement does not affect the conclusion on an infinite blueshift on the sphere where $\Phi = \infty$.

**Magnetic solutions.** For the same metric $g_{\mu\nu}$ with a magnetic source, we get, instead of (1.42),

$$d\xi^2 = \frac{1}{L_f} \left[ A(r) dt^2 - \frac{dr^2}{A(r)} \right] - \frac{r^2}{\Phi} d\Omega^2,$$

(1.44)

where, as before, $\Phi = L_f + 2f L_{ff}$. Then, for a photon traveling from point $X$ to point $Y$, we find instead of (1.43):

$$\frac{\nu_Y}{\nu_X} = \left[ \frac{L_f}{\sqrt{A}} \right]^y \left[ \frac{L_f}{\sqrt{A}} \right]^{-1}_X.$$

(1.45)

Now $L(f)$ has no branching points, while at a regular center ($r = 0$, $A = 1$) both $L_F$ and $\Phi$ vanish, and the quantity $h_{22} \to \infty$, i.e., the spherical radius in the effective metric behaves as if in a wormhole, whereas $h_{tt} \to \infty$, which means an infinite redshift for photons. Also, all curvature invariants of the metric (1.44) vanish at $r = 0$. It is really a quiet place.

There still occurs something of interest between spatial infinity and the center of a regular magnetic model: there is necessarily a sphere $r = r'$ on which $\Phi = 0$. Indeed, $\Phi$ can be presented as $\Phi = 2\sqrt{f(\sqrt{L_f})}$. The quantity $\sqrt{L_f}$ tends to zero both at $r = 0$ (where $f \to \infty$ but $L \to \text{const}$) and in the limit $r \to \infty$. Since $\sqrt{L_f}$ is in general nonzero, it has at least one extremum at some $f \neq 0$, thus it is the value...
where $\Phi = 0$. The metric (1.44) is singular there due to $h_{22} \to \infty$, but this singularity seems to be unnoticed by the photons, as follows from an integral of their geodesic equation
\[ L_f^{-2} \dot{r}^2 + [A(r)\Phi/r^2] \ell^2 = \varepsilon^2, \tag{1.46} \]
where the overdot denotes a derivative in an affine parameter, $\varepsilon$ and $\ell$ are the photon’s constants of motion characterizing its initial energy and angular momentum. Generically we have $L_f \neq 0$ at points where $\Phi = 0$, therefore the photon frequency remains finite. However, as we will see below, the photon velocities behave there in an unusual manner.

If $L_f = 0$ at some value of $f > 0$, it leads to another kind of singularity of the metric (1.44), and this time it acts for NED photons as a potential wall, or a mirror, as is evident from (1.46) which then implies $\dot{r} = 0$. Also, from Eq. (1.45) it follows that the photons are infinitely redshifted there: $\nu_Y$ vanishes if $L_f(Y) = 0$. It means that in such a case no photon from outside can approach the center.

We thus observe a striking difference between the properties of photons moving in the same regular metric (1.15) in the cases where it is sourced by electric and magnetic fields. In the electric case, the photons inevitably “accelerate” to an infinite energy and destabilize the whole system, whereas in the magnetic case, even if they can approach the regular center (if $L_f \neq 0$, hence no mirror), they lose energy, being infinitely redshifted there.

The violent behavior of photons in electric regular black holes was discovered by Novello et al. [59] for a particular example of such a configuration. As shown in [17], it is quite a general property of NED-GR solutions.

### Photon velocities.
A question of interest is the velocity of NED photons in regular or singular space-times. From (1.44) it follows that radially moving photons have the same velocity equal to $c (=1)$ as the conventional Maxwell ones since the 2D metric of the $(t, r)$ subspace in the effective metric (1.44) is conformal to that in the space-time metric (1.15), and their 1D light cones coincide. This is true for both electric and magnetic solutions. However, the situation is different for nonradial photon paths.

Consider a photon moving instantaneously in a tangential direction. Without loss of generality we can suppose that it moves along an equator of certain radius $r$ in our coordinate system. For the corresponding null direction in terms of the effective metric we have $ds_{\text{eff}}^2 = h_{tt} dt^2 - h_{\theta\theta} d\theta^2 = 0$, and for the photon’s linear velocity $v_{ph} = r d\theta/dt$ we obtain:

\[ v_{ph}^2 = \frac{AL_f}{\Phi}, \tag{1.47} \]

in an electric solution:

in a magnetic solution:

For the Maxwell field, $L_f \equiv \Phi \equiv 1$, hence $v_{ph}^2 = A$, and it would be equal to unity if we used the local time increment $dt_{\text{local}} = \sqrt{A} dt$ instead of the coordinate time increment $dt$, and the length element equal to $dr/\sqrt{A}$ instead of $dr$. Thus, as should
be the case, Maxwell photons always travel in vacuum with the speed of light. The factor \(L_f/\Phi\) or \(\Phi/L_f\) changes the photons’ velocity, working like a refractive index.

In particular, in electric solutions, at cusplike branching points where \(\Phi \to \infty\) while \(L_f\) remains finite (like points \(p_1\) and \(p_3\) in Figs. 1.2 and 1.3), \(v_{\text{ph}} \to 0\), in other words, tangentially moving photons have zero velocity at this value of \(r\). On the contrary, at branching points like \(p_2\), where \(H_p = f_p = 0\) and \(\Phi\) is finite but \(L_f \to \infty\), we obtain \(v_{\text{ph}} \to \infty\).

In magnetic solutions, at spheres where \(\Phi = 0\) while \(L_f\) is finite, we have again \(v_{\text{ph}} = 0\), a zero velocity of tangentially moving photons.

So far we were assuming \(A(r) > 0\), while in black hole space-times there are T-regions where \(A(r)\) is negative. However, the only change in Eq. (1.47) emerging in a T-region is the simple replacement \(A \to 1/|A|\) because \(r\) is there a time coordinate instead of \(t\), and in other respects our reasoning remains unaltered.

At intermediate directions between the radial and tangential ones, the NED photon velocities will obviously have intermediate values. We conclude altogether that these velocities can be both subluminal and superluminal, varying from zero to infinity.

We also observe that the conditions under which superluminal photon velocities are avoided (\(L_f/\Phi \leq 1\) for electric solutions and \(\Phi/L_f \leq 1\) for magnetic ones) do not coincide with the causality/unitarity conditions (1.40). Even more than that: any non-Maxwell NED, in which \(\Phi/L_f \neq 1\), predicts superluminal photon motion in either electric or magnetic space-times. Actually, this observation puts to doubt either any NED theory or the described straightforward interpretation of the effective metrics.

Also, the nonlinearity of NED is acting like a highly anisotropic medium, which, if one takes into account the wave properties of photons, naturally leads to such a phenomenon as birefringence, see the relevant recent studies in [37, 38] and references therein.

### 1.4.5 Dynamic stability

Any static or stationary configuration may be regarded viable if it is stable under different kinds of perturbations, which always exist in nature, or at least if it decays slowly enough. Possible regular NED black holes do not make an exception, and their stability is discussed in a number of papers, e.g., [15, 55, 57, 71], see also references therein.

C. Moreno and O. Sarbach [55] have derived sufficient conditions for linear dynamic stability of the domain of outer communication of electric or magnetic black holes sourced by a general \(L(f)\) NED. For magnetic black holes these conditions read (in the present notations)
\[ L > 0, \quad L_y > 0, \quad L_{yy} > 0, \quad (1.48) \]

\[ 3L_y - A(r)yL_{yy} \geq 0, \quad (1.49) \]

where \( y := \sqrt{q^2 f/2} = q^2 / r^2 \), and the index “\( y \)” stands for \( d/dy \). In terms of \( f \) these conditions are rewritten as

\[ L > 0, \quad L_f > 0, \quad \Phi \equiv L_f + 2f L_{ff} > 0, \quad (1.50) \]

\[ [6 - A(r)]L_f - 2f L_{ff} \geq 0. \quad (1.51) \]

One can notice that the conditions (1.50) partly coincide with the causality and unitarity conditions (1.40). Moreover, if \( L_f > 0 \) and also the condition \( L_{ff} \leq 0 \) from (1.40) is valid, then the condition (1.51) holds automatically provided \( f > 0 \) (which is true for magnetic solutions) and \( A(r) < 6 \) (we can note that at least in regular black hole solutions, in general, \( A(r) \leq 1 \)).

Thus Eq. (1.51) is not expected to make a problem, at least for regular magnetic solutions. Unlike that, by Theorem 1.3, the condition \( \Phi > 0 \) is always violated for such solutions near a regular center. This may be important for black hole solutions only if the range of \( r \) where \( \Phi < 0 \) extends to the domain of outer communication, which must be checked for each particular black hole solution.

The sufficient stability conditions for electric black holes have a form similar to (1.48), (1.49) in terms of the P-framework of the theory [55], which could be expected due to FP duality. Their reformulation to the F-framework is not possible in a general form due to problems with a relationship between \( f \) and \( p \), see above.

More general stability conditions for NED-GR solutions involving both \( F_{\mu\nu} \) and \( \ast F_{\mu\nu} \) have been recently obtained by K. Nomura, D. Yoshida and J. Soda in [57].

\subsection*{1.4.6 Examples}

Let us enumerate some particular examples of the Lagrangians \( L(f) \) discussed in the literature, along with their basic properties at \( f > 0 \) (that is, for their magnetic solutions): the existence of a correct Maxwell weak field (MWF) limit, a finite limit as \( f \to \infty \), necessary for a regular center in magnetic solutions, and the validity of the causality, unitarity and stability conditions, (1.40) and (1.50).

It is convenient to do that in the form of a table, see Table 1. Among the conditions (1.40) and (1.50) we select there the inequalities \( L_{ff} < 0 \) and \( \Phi > 0 \) because the condition \( L > 0 \) holds in all examples, and \( L_f > 0 \) in all of them except the one with hyperbolic cosine.

The first line represents the truncated Born-Infeld Lagrangian which does not provide a regular center but satisfies the conditions (1.40) and (1.50). The next four lines correspond to different examples of NED considered by S. Kruglov, the first two of them provide regular magnetic black holes. The sixth line represents a special case from numerous examples considered by Fan and Wang in [30], selected there
Table 1.1: Some examples of $L(f)$ NED theories: properties of magnetic solutions ($f \geq 0$)

| References | Lagrangian$^a$ | Correct MWF limit | Finite as $f \to \infty$ | Condition $L_{ff} < 0$ | Condition $\Phi > 0$ |
|------------|----------------|------------------|-----------------|----------------------|---------------------|
| [12], (1.33) | $\beta^2 (-1 + \sqrt{1 + 2f/\beta^2})$ | yes | no | yes | yes |
| [41,46], (1.26) | $\frac{f}{1 + 2\beta^2 f}$ | yes | yes | yes | partly$^b$ |
| [39] | $\beta^{-1} \arctan(\beta f)$ | yes | yes | yes | partly |
| [40] | $\beta^{-1} \arcsin(\beta f)$ | yes | no | no | yes |
| [42] | $\beta^2 \log \left(1 + \frac{f}{\beta f}\right)$ | yes | no | yes | partly |
| [30] | $\frac{f}{(1 + (\beta f)^{3/4})^2}$ | yes | yes | yes | partly |
| [17], (1.35) | $\frac{f}{\cosh^2(\beta f/2)^{1/4}}$ | yes | yes | no | partly |

$^a$ In all examples, $\beta = \text{const} > 0$.

$^b$ Here and in other lines, “partly” means that $\Phi > 0$ at $f$ smaller than some critical value.

because it both has a correct MWF limit and provides a regular center. The last line is the special case of NED discussed above. The explicit form of the solutions can be found in the cited papers along with detailed discussions of their properties. This list certainly does not pretend to be complete, and many other solutions have been obtained and studied.

It can be observed from the table that in all NED theories that provide a regular center (those with “yes” in the column “finite as $f \to \infty$”), the inequality $\Phi > 0$ does not hold at sufficiently high values of $f$, in accordance with Theorem 1.3.

1.5 NED with more general Lagrangians

1.5.1 Systems with $L = L(f, h)$

Beginning with the paper by Born and Infeld [12], the researchers considered NED theories with Lagrangians more general than $L(f)$, depending on electromagnetic invariants other than $f$. The first and the most natural candidate is the pseudoscalar $h = *F_{\mu\nu}F^{\mu\nu} = 2BE$, where $E$ and $B$ are the electric field strength and magnetic induction 3-vectors, respectively. Now the total action has the form

$$S = \frac{1}{2} \int \sqrt{-g} d^4x [R - L(f, h)].$$  

(1.52)

Special cases of $L(f, h)$ are the Born-Infeld Lagrangian
\[ L^{BI} = b^2 \left( -1 + \sqrt{1 + \frac{f}{2b^2} - \frac{h^2}{16b^4}} \right), \quad b > 0, \quad (1.53) \]

and the so-called modified Maxwell (ModMax) Lagrangian \([7, 69]\)

\[ L^{MM} = \frac{1}{4} \left( f \cosh \gamma - \sqrt{f^2 + h^2} \sinh \gamma \right), \quad \gamma \in \mathbb{R}. \quad (1.54) \]

Both these models are distinguished by their symmetry properties, in particular, the ModMax NED is conformally and duality invariant.\(^1\)

The electromagnetic field equations due to (1.52) read

\[ \nabla_\mu (L F^{\mu \nu} - L^* F^{\mu \nu}) = 0, \quad \nabla_\mu F^{\mu \nu} = 0, \quad (1.55) \]

and the electromagnetic field SET has the form

\[ T^{\nu}_{\mu} = -2L F_{\mu \alpha} F^{\nu \alpha} + \frac{1}{2} \delta^{\nu}_{\mu} (L - hL_h). \quad (1.56) \]

Assuming static spherical symmetry, hence having only radial electric and magnetic fields, we are again dealing with a SET with \(T^r_r = 0\) and \(T^t_t = T^r_r\), and the metric can be written in the form (1.15). We then have according to (1.55)

\[ L F^{rr} - L^* F^{rr} = \frac{q_e}{r^2}, \quad F_{\theta \phi} = q_m \sin \theta, \quad (1.57) \]

with the corresponding charges \(q_e, q_m = \text{const.}\)

For static, spherically symmetric solutions to the NED-GR equations with \(L = L(f, h)\), a number of no-go theorems have been proved in Ref. [11]. According to these theorems, such solutions with the metric (1.15) cannot describe a geometry with a regular center under the following assumptions on the electromagnetic field:

1. \(q_e \neq 0, q_m = 0\) (electric), MWF limit.
2. \(L(f, h) = L(f)\), \(q_e \neq 0, q_m \neq 0\) (dyonic).
3. \(L(f, h) = f + \eta(h)\), with an arbitrary function \(\eta(h)\), \(q_m \neq 0\) (magnetic or dyonic).
4. \(L(f, h) = f + af^2 h^u\), with real \(a \neq 0\), positive integers \(s > 1, u > 1\), and \(q_e \neq 0, q_m \neq 0\) (dyonic).
5. \(L(f, h) = f + af^2 + bh + ch^2\), where \(a, b, c \in \mathbb{R}\), \(q_e \neq 0, q_m \neq 0\) (dyonic).
6. \(L(f, h)\) given by (1.53) or (1.54), \(q_m \neq 0\) (magnetic or dyonic).
7. \(L(f, h) = f + af^2 + bh + ch^2\), the pair \((b, c) \neq (0, 0)\), \(q_e = 0, q_m \neq 0\) (magnetic).

The numbering here corresponds to the theorem numbers in Ref. [11]. Theorem 2 from this list coincides with our Theorem 1.2 presented in Section 1.4. We can notice that only two theorems, the first and the sixth ones, use the assumption of

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\(^1\) In our notations, see (1.52), some of the signs and factors are different from those in [11] and other papers. In particular, the Maxwell theory here corresponds to \(L(f, h) = f\).
a correct MWF limit: in all other cases considered, a regular center is impossible irrespective of the weak field behavior of the theory.

On the other hand, there still remain some opportunities of obtaining regular BHs other than purely magnetic ones with \( L = L(f) \). For example, both with \( L(f) \) and \( L(f, h) \), purely electric solutions with a regular center are possible with a theory having no MWF limit. Then, assuming a regular central region governed by such a theory, one can obtain an asymptotically flat electrically charged configuration by using a kind of phase transition, such that outside a certain sphere \( r = r_{\text{crit}} \), another NED theory will be valid, having a correct MWF limit, as was suggested in [26].

1.5.2 Systems with \( L = L(f, J) \)

One more invariant, in addition to \( f \),

\[
J \equiv J_4 = F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu},
\]

has also been used for formulating an extended NED theory [28, 68]. With this invariant, the action reads

\[
S = \frac{1}{2} \int \sqrt{-g} d^4x [R - L(f, J)],
\]

the electromagnetic field equations are

\[
\nabla_\mu Q^{\mu\nu} = 0, \quad \nabla_\mu \ast F^{\mu\nu} = 0,
\]

\[
Q^{\mu\nu} := 4L_f F^{\mu\nu} + 8L_J F^{\mu\rho}F_{\rho\sigma}F^{\sigma\nu},
\]

where \( L_f = \partial L / \partial f \) and \( L_J = \partial L / \partial J \). The SET has the form

\[
T^{\nu}_{\mu} = -2(L_f + fL_J)F_{\mu\alpha}F^{\nu\alpha} + \frac{1}{2} \delta^{\nu}_{\mu}[(f^2 - 2J)L_J + L].
\]

An important subclass of the theories (1.59) is called conformal NED, or CNED, and is characterized by a zero trace of the SET (1.62) [28, 68], hence,

\[
T = 2(L - fL_f - 2JL_J) = 0.
\]

In this case, the SET as a whole is a multiple of the Maxwell field SET, and the field equations (1.60) are invariant under general conformal mappings of the metric (conformally invariant) like the Maxwell equations.

The theory has a Maxwell asymptotic behavior at small fields if \( L(f, J) \approx f \), so that \( L_f \to 1 \) and \( |L_J| < \infty \) as \( F_{\mu\nu} \to 0 \) (the latter condition takes into account that \( J \sim f^2 \) at small \( F_{\mu\nu} \)).

Assuming static spherical symmetry, we have, as before, only radial electric and magnetic fields, and the SET has again the properties \( T^r_r = 0 \) and \( T^\nu_r = T^r_\nu \), and the
metric can be written in the form (1.3). Specifically,

\[ T^r_t = T^\theta_\theta = 2(L_f + fL_J)E^2 + \frac{1}{2}L - 4B^2E^2L_J, \]

\[ T^\phi_\theta = -2(L_f + fL_J)B^2 + \frac{1}{2}L - 4B^2E^2L_J, \] (1.64)

with \( E^2 = F^r_rF_r \) and \( B^2 = F^\theta_\phi F^\phi_\theta \). The field equations (1.60) lead to

\[ Q^r = 4(L_f + fL_J)F^r_r = \frac{q_e}{r^2}, \quad F^\theta_\phi = q_m \sin \theta. \] (1.65)

Let us prove that the same no-go theorems as in \( L(f) \) theories coupled to GR, are valid in the theories (1.59).

**Theorem 1.4.** The theories (1.59) do not admit static, spherically symmetric electric solutions \((q_e \neq 0, q_m = 0)\) with a regular center and a correct MWF limit.

**Proof.** As with \( L(f) \) theories in Section 1.4, assuming regularity at \( r = 0 \), we must require that all components of \( T^\nu_\mu \) should be finite, as well as their linear combinations. In particular, we require that

\[ |T^t_t - T^\theta_\theta| = 2|L_f - 2E^2L_J|E^2 < \infty. \] (1.66)

On the other hand, from (1.65) we obtain

\[ Q^rQ^r = 16E^2(L_f - 2E^2L_J)^2 = \frac{q_e^2}{r^4} \rightarrow \infty \quad \text{as} \quad r \rightarrow 0. \] (1.67)

The conditions (1.66) and (1.67) are only compatible if \( E \rightarrow 0 \) (that is, the field becomes weak) and \( |L_f - 2E^2L_J| \rightarrow \infty \), contrary to the desirable MWF limit. This completes the proof. \( \Box \)

**Theorem 1.5.** The theories (1.59) do not admit static, spherically symmetric dyonic solutions \((q_e \neq 0, q_m \neq 0)\) with a regular center.

**Proof.** The same requirement as in the previous theorem, \(|T^t_t - T^\theta_\theta| < \infty\), necessary to be valid at a regular center, now reads

\[ |T^t_t - T^\theta_\theta| = 2|L_f + fL_J|(E^2 + B^2) < \infty. \] (1.68)

Moreover, since both \( E^2 > 0 \) and \( B^2 > 0 \), this inequality should hold with each of them taken separately. Applying it with \( E^2 \) together with the first equality (1.65) that now leads to

\[ Q^rQ^r = 16(L_f + fL_J)^2 = \frac{q_e^2}{r^4} \rightarrow \infty \quad \text{as} \quad r \rightarrow 0, \] (1.69)

we obtain, as before, \( E \rightarrow 0 \) and \( L_f + fL_J \rightarrow \infty \). The same condition (1.68) with \( B^2 \) leads to \( L_f + fL_J \rightarrow 0 \) (since \( B^2 = q_m^2/r^4 \rightarrow \infty \)). The resulting contradiction proves the theorem. \( \Box \)
It is also evident than none of the CNED theories can produce a regular BH or, more generally, a solution with a regular center. Indeed, since the SET is proportional to that of Maxwell electrodynamics, it cannot lead to any static, spherically symmetric metric other than Reissner-Nordström, though certainly the interpretation of its constants \( m \) and \( q \) will be different.

As to purely magnetic solutions \((q_c \neq 0, q_m \neq 0)\), in which \( f = 2q_m^2/r^2 \) and \( J = 2q_m^2/r^8 \), a regular center is possible only under the condition that \( L(f, J) \) tends to a finite constant as both \( f \) and \( J \) tend to infinity. The whole situation looks quite the same as with \( L(f) \) theories.

Let us give a confirming example, taking as a basis Eq. (1.26) for \( L(f) \):

\[
L(f, J) = \frac{f}{1 + af^2} + \frac{bJ}{1 + cJ^2}, \quad a, b, c = \text{const} > 0, \tag{1.70}
\]

This Lagrangian has a correct MWF limit and tends to a finite limit at large \( f \) and \( J \). Since with \( q_c = 0 \), according to (1.64), the density is simply \( \rho = L/2 \), the metric function \( A(r) \) is found as

\[
A(r) = 1 - \frac{2M(r)}{r}, \quad M(r) = M_1(r) + M_2(r),
\]

\[
M_1(r) = \frac{q^2}{2} \int \frac{r^2 dr}{r^4 + aq^2}, \quad M_2(r) = \frac{bq^4}{2} \int \frac{r^2 dr}{r^8 + cq^4}, \tag{1.71}
\]

where \( q = q_m \). Integration gives

\[
M_1(r) = \frac{q^2}{8\sqrt{2}} \left[ 2 \arctan \frac{h + \sqrt{2}r}{h} - 2 \arctan \frac{h - \sqrt{2}r}{h} + \log \frac{h^2 - \sqrt{2}hr + r^2}{h^2 + \sqrt{2}hr + r^2} \right],
\]

\[
M_2(r) = \frac{bq^4}{16j^3} \left[ 2C \left( \arctan \frac{r + jC}{jS} + \arctan \frac{r - jC}{jS} \right) \right.
\]

\[
- 2S \left( \arctan \frac{r + jS}{jC} + \arctan \frac{r - jS}{jC} \right) + S \log \frac{\sqrt{2} + 2CjS + r^2}{\sqrt{2} - 2CjS + r^2} + C \log \frac{\sqrt{2} - 2SjS + r^2}{\sqrt{2} + 2SjS + r^2} \right], \tag{1.72}
\]

where we have denoted \( h = (aq^2)^{1/4} \), \( j = (cq^4)^{1/8} \), \( S = \sin(\pi/8) \), \( C = \cos(\pi/8) \).

At \( r \to \infty \) we obtain

\[
M_1(r) \to \frac{\pi q^{3/2}}{4\sqrt{2}a^{1/4}}, \quad M_2(r) \to \frac{\pi bq^{3/2}(C - S)}{8c^{3/8}}, \quad M = \lim_{r \to \infty} (M_1 + M_2), \tag{1.73}
\]

where \( M \) is the Schwarzschild mass of completely electromagnetic origin. At the center, we have

\[
M_1(r) \approx \frac{r^3}{6a}, \quad M_2(r) \approx \frac{br^3}{6c} \quad \text{as} \quad r \to 0, \tag{1.74}
\]
which leads to $A(r) = 1 + \mathcal{O}(r^2)$, satisfying the regular center condition (1.6). We have obtained a regular asymptotically flat solution in $L(f,J)$ NED with a correct WMF limit.

Is it a black hole solution? To make it clear, let us fix the parameters $q = 1$, $a = 1$, $c = 1$, then the only remaining free parameter is $b$, and

$$M = \frac{\pi}{8}[\sqrt{2} + b(C - S)].$$

The behavior of $A(r)$ at different values of $b$ is shown in Fig. 1.4. An inspection shows that at small $b$ the solution is of solitonic nature, at $b \approx 2.436$ (corresponding to $M \approx 1.073$) there emerges a single extremal horizon, and at larger $b$ (large $M$) we obtain a regular black hole with two simple horizons.

1.6 Conclusion

We have discussed the opportunities of obtaining regular spherically symmetric black hole solutions in GR sourced by nonlinear electromagnetic fields governed by different NED theories. It happens that such NED black holes (as well as solitons) with a regular center in $L(f)$ theories can exist with pure electric or pure magnetic charges, and only systems with a magnetic charge are compatible with Lagrangians having a correct Maxwell behavior at small $f$. Dyonic configurations, with $q_e \neq 0$ and $q_m \neq 0$, cannot contain a regular center, whatever be the function $L(f)$.

In Maxwell’s electrodynamics there is the well-known symmetry (duality) between electric and magnetic fields, leading to the same symmetry between the corresponding solutions to the Einstein-Maxwell equations, at least in the absence of
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1 currents ans charges. Unlike that, in NED, we only have FP duality that connects purely electric and purely magnetic configurations with the same metric but sourced by different NED theories. Accordingly, in a theory specified by a particular function \( L(f) \), the properties of electric and magnetic solutions are quite different.

It turns out that magnetic solutions lead to completely regular configurations, while for their electric counterparts, obtained from them using FP duality and well-behaved in the framework of the “Hamiltonian” formulation of NED, the Lagrangian formulation is ill-defined, and the behavior of NED photons exhibits undesired features at some intermediate radii: they experience an infinite blueshift, indicating an instability of such a background configuration.

The dynamic stability of regular magnetic solutions is also questionable since one of the sufficient stability conditions \( (\Phi > 0) \) is inevitably violated near a regular center. Thus general stability results for regular black holes probably cannot be obtained, and stability studies of individual solutions seem to be necessary.

We here did not touch upon thermodynamic properties of NED black holes, this important issue is discussed in many papers, see, among others, [4, 13, 30, 41, 46] and references therein. Let us only remark here that a thermodynamic instability of black holes related to their negative heat capacity is implemented in the process of Hawking evaporation, which is very slow for sufficiently large black holes and can be practically ignored for black holes with stellar and larger masses, irrespective of their global regularity properties.

There are many results obtained with more general NED Lagrangians, such as \( L(f, h) \) and \( L(f, J) \). Many no-go theorems concerning possible regular black holes with \( L(f, h) \) NED have been presented in Refs. [10, 11]. As to \( L(f, J) \) NED, involving a fourth-order electromagnetic invariant, we have verified here that the restrictions on regular black hole existence obtained with \( L(f) \) are extended to this class of theories without change. In particular, regular black holes with a magnetic charge can also be obtained, as we have confirmed with an explicit example. Very probably these results can be further extended to include electromagnetic invariants of still higher orders constructed in the same manner as \( J = F_{\mu \nu} F^{\nu \rho} F_{\rho \sigma} F^{\sigma \mu} \).

Among the remaining theoretic problems deserving further studies let us mention the dynamic stability problem for regular magnetic black holes and the causality issue related to the predicted superluminal velocities of NED photons.

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