DRIFT REDUCTION METHOD FOR SDES DRIVEN BY INHOMOGENEOUS SINGULAR LÉVY NOISE

TADEUSZ KULCZYCKI, OLEKSIK KULYK, AND MICHAL RYZNAR

Abstract. We study SDE
\[ dX_t = b(X_t) \, dt + A(X_{t-}) \, dZ_t, \quad X_0 = x \in \mathbb{R}^d, \quad t \geq 0 \]
where \( Z = (Z^1, \ldots, Z^d)^T \), with \( Z^i, i = 1, \ldots, d \) being independent one-dimensional symmetric jump Lévy processes, not necessarily identically distributed. In particular, we cover the case when each \( Z^i \) is one-dimensional symmetric \( \alpha \)-stable process (\( \alpha_i \in (0, 2) \) and they are not necessarily equal).

Under certain assumptions on \( b, A \) and \( Z \) we show that the weak solution to the SDE is uniquely defined and Markov, we provide a representation of the transition probability density and we establish Hölder regularity of the corresponding transition semigroup.

The method we propose is based on a reduction of an SDE with a drift term to another SDE without such a term but with coefficients depending on time variable. Such a method have the same spirit with the classic characteristic method and seems to be of independent interest.

1. Introduction

In this paper we study an SDE of the form
\[ dX_t = b(X_t) \, dt + A(X_{t-}) \, dZ_t, \quad X_0 = x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d, \quad t \geq 0, \]
where the driving process \( Z \) has the form
\[ Z = (Z^1, \ldots, Z^d)^T, \]
with \( Z^i, i = 1, \ldots, d \) being independent scalar symmetric Lévy processes. Such Lévy noise is essentially singular in the terminology of [Knopova, Kulik and Schilling2021]. Namely, the Lévy measure \( \mu \) of \( Z \) is concentrated on the collection of coordinate axes in \( \mathbb{R}^d \), which combined with a non-trivial rotation occurring due to the matrix coefficient \( A(x) \) yield lack of a single reference measure for the images of \( \mu \) under the multiplicative mappings \( z \mapsto A(x)z \) for different \( x \). Such a structure of the Lévy noise may result in a substantial novelties in a local behavior of the transition probabilities of the associated processes, e.g. [Kulczycki, Ryznar and Sztonyk2021], [Knopova, Kulik and Schilling2021]. The situation complicates even more when the components of the noise \( Z^1, \ldots, Z^d \) are allowed to have different scaling properties, e.g. in the case of \( Z^i, i = 1, \ldots, d \) being scalar (symmetric) \( \alpha \)-stable processes.

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with different indices $\alpha_i$. For a detailed discussion of the new effects which appear for such inhomogeneous singular Lévy noises we refer for to our recent article [Kulczycki, Kulik and Ryznar2022], where SDEs with such noises were studied.

The SDEs considered in [Kulczycki, Kulik and Ryznar2022] did not involve a drift term. The method used therein was essentially analytical and based on a version of the parametrix method for a construction of the heat kernel for the associated Markov process, treated as a solution to the Kolmogorov backward differential equation.

In this article we consider a further natural extension of the model, allowing the SDE (1) to have a drift term. It is known that, for the analytic methods mentioned above, to include a drift term to an SDE (or, equivalently, a gradient term in the associated PDE) is quite a non-trivial problem in the general setting, where the ‘order’ of the non-local part of the generator is allowed to be smaller than one; that is, when the gradient term is a leading one in the entire generator. One possibility to treat such models is to adapt the parametrix method for such a setting by introducing a special ‘flow corrector’ method introduced in [Knopova and Kulik2018], [Kulik2018], see also a detailed discussion in [Knopova, Kulik and Schilling2021].

In this paper we develop another possibility, which we believe to have an independent interest and be potentially useful for other types of SDEs with jumps. Namely, let $t > 0$ be fixed, and we would like to identify the law of the value $X_t$ of the solution to SDE (1) at this time moment. Denote by $\chi_t(x)$ the flow of solutions to the Cauchy problem

$$
\partial_t \chi_t(x) = b(\chi_t(x)), \quad t \in \mathbb{R}, \quad \chi_0(x) = x,
$$

and consider the process

$$
X^t_s = \chi_{t-s}(X_s), \quad s \in [0, t].
$$

Clearly, at the terminal point $s = t$ the processes $X^t$ and $X$ are equal, that is $X^t_t = X_t$. On the other hand, under proper assumptions on the coefficients and noise in (1), see Lemma 3.5 and Remark 3.6 below, we can apply the Itô formula to show that $X^t$ satisfies the SDE

$$
dX^t_s = \int_{\mathbb{R}^d} V^t_s(X^t_s, z) N(ds, dz), \quad s \in [0, t], \quad X^t_0 = \chi_t(x),
$$

where $N(ds, dz)$ is the Poisson random measure which corresponds to the Lévy process $Z$ from (1) in the usual sense that

$$
dZ_s = \int_{\mathbb{R}^d} z N(ds, dz),
$$

and the coefficient $V^t_s(x, z)$ has the form

$$
V^t_s(x, z) = \chi_{t-s}(\kappa_{t-s}(x) + A(\kappa_{t-s}(x))z) - x,
$$

here and below we denote $\kappa_t(x) = \chi_{-t}(x)$, the inverse flow for $\chi_t(x)$.

Next, the process $X^*_s = X^t_{t-s}$, $s \in [-t, 0]$ satisfies the SDE

$$
dX^*_s = \int_{\mathbb{R}^d} V_s(X^*_s, z) N^*(ds, dz), \quad s \in [-t, 0], \quad X^*_0 = \chi_t(x)
$$

with

$$
V_s(x, z) = \kappa_s(\chi_s(x) + A(\chi_s(x))z) - x,
$$
and a new Poisson point measure \( N^*(ds, dz) \) which has the same intensity measure \( ds\mu(dz) \) with the original \( N(ds, dz) \). SDE (4) does not have a drift term, and by the results of Kulczycki, Kulik and Ryznar2022 its solution is weakly unique and is a (time-inhomogeneous) Markov process with a transition probability density \( p_{t,s}(x, y) \). Then the above stochastic equivalences immediately yield the same set of results for the initial equation (1). Namely, the solution to (1) is weakly unique, and its transition probability density can be expressed as

\[
(5) \quad u_t(x, y) = p_{-1,0}(\chi_t(x), y).
\]

To summarize, the method we propose is based on a reduction, by a proper change of variables, of an SDE with a drift term to another SDE free from such a term. Such a reduction seem to have the same spirit with the classic characteristic method, which reduces a 1st order quasi-linear PDE to a family of ODEs on characteristic curves. This similarity becomes even better visible if we consider the Kolmogorov differential equation associated with (1), see Section 3.2 below. We believe that such stochastic characteristic method has an independent interest and is potentially useful for other types of SDEs with jumps.

The study of SDEs driven by singular Lévy noises have attracted attention in recent years. In Bass and Chen2006 the weak well-posedness for SDE (1) was established under the assumption that \( A(x) \) is bounded and non-degenerate for each \( x \in \mathbb{R}^d \), \( x \rightarrow A(x) \) is continuous, \( b \equiv 0 \) and \( Z \) is a cylindrical \( \alpha \)-stable process. Strong Feller property for SDEs driven by additive cylindrical Lévy processes have been studied in Priola and Zabczyk2010. The existence of densities for SDEs driven by singular Lévy processes have been studied in Debussche and Fournier2013 and Friesen, Jin, and Rüdiger2021. The strong well-posedness for SDEs driven by \( \alpha \)-stable-like Lévy processes (including cylindrical case) with Hölder drifts was established in Chen, Zhang and Zhao2021, giving an affirmative answer to an open problem proposed by Priola. In [Knopova, Kulik and Schilling2021] different stable-like models have been treated, where the stability index and the spherical kernel (i.e. the distribution of the jump direction) are space-dependent. Existence and representation of transition density and strong Feller property of the related semigroup have been established. The main idea of that paper, which can be heuristically described as a dynamical truncation of Lévy measures, is crucial for the results obtained in Kulczycki, Kulik and Ryznar2022 and indirectly for the results in our recent paper. Properties of transition densities and Hölder regularity of transition semigroups for SDEs driven by singular Lévy noises under various assumptions have been studied in Kulczycki, Ryznar and Sztonyk2021, Kulczycki and Ryznar2020, Chen, Hao and Zhang2020 and Kulczycki, Kulik and Ryznar2022.

The rest of the paper is organized as follows. In Section 2 we introduce notation, present assumptions and formulate main results. Section 3 contains proofs, its most important part is Section 3.2, where we explain the method of solution. In Section 4 we give examples of processes \( Z \), drifts \( b \) and matrices \( A \) satisfying our assumptions.

2. Main results

2.1. Notation and assumptions. In this section, we collect all the assumptions we impose on our model.

Notation:
All vectors \( x \in \mathbb{R}^d \) are column type. For any \( x, y \in \mathbb{R}^d \) by \( x \cdot y \) we denote the standard scalar product of \( x \) and \( y \).

- For a function \( f : \mathbb{R}^d \to \mathbb{R} \) its gradient is defined \( \nabla f(x) = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d} \right) \).
- For a real \( d \times d \) matrix \( A \) by \( |A| \) we denote its operator norm that is \( |A| = \sup_{|x| \leq 1} |Ax| \)
- For a function \( h = \left( \begin{array}{c} h_1 \\
\vdots \\
h_d \end{array} \right) : \mathbb{R}^d \to \mathbb{R}^d \), by 
\[
Dh = [\nabla h_1, \ldots, \nabla h_d]^T
\]
we denote its derivative, that is a \( d \times d \) matrix which rows are equal to \( [\nabla h_k]^T, 1 \leq k \leq d \). Note that, if \( g : \mathbb{R}^d \to \mathbb{R}^d \), then we have the following formula for the derivative of the composition \( g \) with \( h \), \( w(x) = g(h(x)) \),
\[
Dw(x) = Dg(h(x)) Dh(x).
\]
- Note that, if \( f : \mathbb{R}^d \to \mathbb{R} \), then we have the following formula for the gradient of the composition \( f \) with \( h \), \( w(x) = f(h(x)) \),
\[
\nabla w(x) = [Dh(x)]^T \nabla f(h(x)).
\]
- In the whole paper we assume that constants denoted by \( c, c_1, c_2, \ldots \) are positive and we adopt the convention that these constants may change their value from one use to the next.
- We denote \( a \wedge b = \min(a, b) \), \( a \vee b = \max(a, b) \).

Now we start the description of scalar Lévy processes involved, as the coordinates, in the representation \([2]\). Let the characteristic exponent \( \psi \) of a one-dimensional, symmetric Lévy process be given by \( \psi(\xi) = \int_{\mathbb{R}} (1 - \cos(\xi x)) \nu(dx) \), where \( \nu \) is a symmetric, infinite Lévy measure. The corresponding Pruitt function \( h(r) \) is given by
\[
h(r) = \int_{\mathbb{R}} (1 \wedge (|x|^2 r^{-2})) \nu(dx), \quad r > 0.
\]

We will assume the following scaling conditions for the function \( h \): for some \( 0 < \alpha \leq \beta \leq 2 \) and \( 0 < C_1 \leq 1 \leq C_2 < \infty \),
\[
C_1 \lambda^{-\alpha} h(r) \leq h(\lambda r) \leq C_2 \lambda^{-\beta} h(r), \quad 0 < r \leq 1, \quad 0 < \lambda \leq 1.
\]

It is well known (see e.g. [Kulczycki, Kulik and Ryznar2022 Section 4]) that the above assumption is equivalent to the following weak scaling property for \( \psi \): there are constants \( 0 < C_1^\ast \leq 1 \leq C_2^\ast < \infty \),
\[
C_1^\ast \lambda^{\alpha} \psi(\xi) \leq \psi(\lambda \xi) \leq C_2^\ast \lambda^{\beta} \psi(\xi), \quad |\xi| \geq 1, \quad \lambda \geq 1.
\]

Once the condition \([6]\) (or equivalently \([7]\)) is satisfied, we say that the characteristic exponent \( \psi \) (or the Lévy measure \( \nu \)) have the weak scaling property with indices \( \alpha, \beta \), and write \( \psi \in \text{WSC}(\alpha, \beta) \) (resp. \( \nu \in \text{WSC}(\alpha, \beta) \)). We also observe
that, by Bogdan, Grzywny and Ryznar 2014, Corollary 6, the Lévy measure \( \nu \) satisfies
\[
\int_{|z| \leq 1} |z|^\gamma \nu(dz) < \infty,
\]
for any \( \gamma > \beta \).

By \( \psi_i, \nu_i \) and \( h_i \), we denote corresponding characteristic exponents, Lévy measures and Pruitt functions of coordinates \( Z^i \) of the process \( Z = (Z^1, \ldots, Z^d) \); recall that these coordinates are assumed to be independent.

We will consider two cases:

(A) All characteristic exponents \( \psi_i, i = 1, \ldots, d \) are equal and \( \psi_1 \in \text{WSC}(\alpha, \beta) \).

(B) Characteristic exponents \( \psi_i, i = 1, \ldots, d \) are not all the same, i.e. there exist \( i, j \) such that \( \psi_i \neq \psi_j \), and \( \psi_i \in \text{WSC}(\alpha, \beta), i, j \in \{1, \ldots, d\} \).

In both of these cases, the process \( Z \) has the transition density \( \tilde{G}_t(x, y) = \tilde{G}_t(y-x) \), where
\[
\tilde{G}_t(w) = \prod_{i=1}^d \tilde{g}_i^t(w_i), \quad w = (w_1, \ldots, w_d)^T \in \mathbb{R}^d,
\]
and \( \tilde{g}_i^t, i = 1, \ldots, d \) are the distribution densities for the coordinates (all \( \tilde{g}_i^t \) are the same in the case (A)).

Next, we assume the following conditions on the coefficients.

(C) For any \( x \in \mathbb{R}^d \) \( A(x) = (a_{i,j}(x)) \) is a \( d \times d \) matrix and there are constants \( C_3, C_4, C_5 > 0, \eta_1 \in (0, 1] \) such that for any \( t \geq 0, x, y \in \mathbb{R}^d, i, j \in \{1, \ldots, d\} \),
\[
|A(x)| \leq C_3, \tag{9}
\]
\[
|\det(A(x))| \geq C_4, \tag{10}
\]
\[
|A(x) - A(y)| \leq C_5|x - y|^\eta_1, \tag{11}
\]
The function \( \mathbb{R}^d \ni x \to b(x) \in \mathbb{R}^d \) is differentiable and there are constants \( C_6, C_7 > 0 \) and
\[
\eta_2 > \max(0, (\beta - 1)) \tag{12}
\]
such that for any \( x, y \in \mathbb{R}^d \),
\[
|Db(x)| \leq C_6, \tag{13}
\]
\[
|Db(x) - Db(y)| \leq C_7|x - y|^{\eta_2}. \tag{14}
\]

Now we introduce functions \( \kappa_t(x), \chi_t(x) \) and a family of matrices \( A_t(x) \) which play a crucial role in our paper. Let \( \kappa, \chi \) be solutions of the following differential equations
\[
\partial_t \kappa_t(x) = -b(\kappa_t(x)), \quad \text{for } t \in \mathbb{R}, x \in \mathbb{R}^d, \tag{15}
\]
\[
\partial_t \chi_t(x) = b(\chi_t(x)), \quad \text{for } t \in \mathbb{R}, x \in \mathbb{R}^d, \tag{16}
\]
with initial conditions \( \kappa_0(x) = \chi_0(x) = x, \) for \( x \in \mathbb{R}^d \). Note that \( \chi_t(x) = \kappa_{-t}(x) \) for \( t \in \mathbb{R}, x \in \mathbb{R}^d \). It is a standard fact that if assumptions (C) are satisfied then
the above initial value problems have unique solutions. Properties of \( \kappa_t(x), \chi_t(x) \) are discussed in Section 3.3. For \( t \in \mathbb{R}, x, y \in \mathbb{R}^d \) we define

\[
A_t(x) = (D\kappa_t)(\chi_t(x))A(\chi_t(x)).
\]

We need to impose a further assumption on the coefficients, which we present in the following two slightly different forms.

(D) There is a constant \( C_8 > 0 \) such that for any \( x \in \mathbb{R}^d \)

\[
|b(x)| \leq C_8.
\]

(D) combined with (C) yields the following:

(D') There is a constant \( C_9 > 0 \) such that for any \( t, s \in \mathbb{R} \) and \( x \in \mathbb{R}^d \)

\[
|A_t(x) - A_s(x)| \leq C_9 e^{C_9(|t| + |s|)|s - t|^{\eta_1 \vee \eta_2}}.
\]

The justification that (D) combined with (C) implies (D') is contained in Section 3, see Lemma 3.11. Example 4.4 shows that (D') is strictly weaker than (D).

In the case (A), the Hölder index \( \eta_1 \) can be arbitrarily small. In the case (B), \( \alpha, \beta, \eta_1 \) and \( \eta_2 \) should satisfy certain additional assumptions. Namely, we assume the following

(E)

\[
\frac{\beta}{\alpha} < 1 + (\eta_1 \wedge \eta_2), \quad \frac{1}{\alpha} - \frac{1}{\beta} < \eta_1 \wedge \eta_2.
\]

For abbreviation, for any \( u > 0 \) we will use the notation

\[
\sigma(u) = (u, h_1(1), \ldots, h_d(1), h_1^{-1}(1), \ldots, h_d^{-1}(1), h_1^{-1}(1/u), \ldots, h_d^{-1}(1/u)).
\]

2.2. Main statements. In this section, we formulate the main statements of the paper.

For \( t > 0, x, y \in \mathbb{R}^d \) define

\[
\tilde{u}_t(x, y) = \frac{1}{|\det A(\chi_t(x))|} \tilde{G}_t(A^{-1}(\chi_t(x))(y - \chi_t(x))),
\]

where \( \tilde{G}_t(\cdot) \) is the distribution density of \( Z_t \).

**Theorem 2.1.** Assume either (A), (C), (D') or (B), (C), (D'), (E). Then for any \( x \in \mathbb{R}^d \) the SDE (1) has a unique weak solution \( X_t \). The process \( X_t \) is a Markov process which has a transition density \( u_t(x, y) \). The transition density admits a representation

\[
u_t(x, y) = \tilde{u}_t(x, y) + \tilde{q}_t(x, y), \quad x, y \in \mathbb{R}^d, \quad t > 0,
\]

where \( \tilde{u}_t(x, y) \) is given by (15) and the residual part \( \tilde{q}_t(x, y) \) satisfies

\[
\int_{\mathbb{R}^d} |\tilde{q}_t(x, y)| \, dy \leq c t^{\varepsilon_0}, \quad x \in \mathbb{R}^d,
\]

where \( \varepsilon_0 \) is defined in Remark 2.7 and the constant \( c \) depends only on \( d, \alpha, \beta, \eta_1, \eta_2, C_1, \ldots, C_7, C_9, h_1(1), \ldots, h_d(1) \).
Remark 2.2. One may think of $\tilde{u}_t(x, y)$ as the "principal part" of the transition density and $\tilde{q}_t(x, y)$ as the corresponding "residual part". If additionally (D) is satisfied, then (19) and (20) remain true with the principal part in the following simpler form

$$\tilde{u}_t(x, y) = \frac{1}{\det A(x)} \tilde{G}_t(A^{-1}(x)(y - \chi_t(x))).$$

Denote by $\mu$ the Lévy measure of the process $Z$, and define for $t \in \mathbb{R}$, $x, z \in \mathbb{R}^d$

$$T^{t,z}f(x) = f(x + V_t(x, z)),$$

where we recall that

$$V_t(x, z) = \kappa_t(\chi_t(x) + A(\chi_t(x))z) - x.$$

Assume the following.

(I) For all $t$ and $\mu$-a.a. $z$, $T^{t,z}$ is a bounded linear operator in $L_1(\mathbb{R}^d)$, and there exists $C_0 < \infty$ such that

$$\|T^{t,z}\|_{L_1 \to L_1} \leq C_0, \quad t \in \mathbb{R}, \quad z \in \text{supp } \mu.$$

We have the following representation of the transition density.

**Theorem 2.3.** Let the conditions of Theorem 2.1 and additional assumption (I) hold. Then the density $u_t(x, y)$ is bounded, that is

$$\sup_{x,y \in \mathbb{R}^d} u_t(x, y) < \infty, \quad 0 < t < \infty.$$

Moreover, for any $\tau > 0$ there exists $c > 0$, depending only on $d$, $\alpha$, $\beta$, $\eta_1$, $\eta_2$, $C_0$, $\ldots$, $C_7$, $C_9$, $\sigma(\tau)$ such that the residual term in the representation (19) satisfies

$$|\tilde{q}_t(x, y)| \leq c\tilde{G}_t(0)t^{\sigma_0}, \quad 0 < t \leq \tau, \quad x, y \in \mathbb{R}^d.$$

Define by $\{U_t\}$ the evolutionary family corresponding to the process $X$ in the usual way: for any $0 < t < T$, $x \in \mathbb{R}^d$ and a bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$,

$$U_t f(x) = \int_{\mathbb{R}^d} u_t(x, y) f(y) dy.$$

Under just the basic conditions of Theorem 2.1 we prove Hölder continuity of this evolutionary family.

**Theorem 2.4.** Assume either (A), (C), (D'), or (B), (C), (D'). For any $0 < \gamma < \gamma' < \alpha$, $\gamma \leq 1$, $0 < t < \tau$, $x, y \in \mathbb{R}^d$ and a bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$ we have

$$|U_t f(x) - U_t f(y)| \leq c|x - y|^{\gamma - \gamma'/\alpha} \|f\|_{\infty},$$

where $c$ depends only on $\gamma$, $\gamma'$, $d$, $\alpha$, $\beta$, $\eta_1$, $\eta_2$, $C_1$, $\ldots$, $C_7$, $C_9$, $\sigma(\tau)$.

**Remark 2.5.** The constant $\varepsilon_0$, which appears in above theorems is chosen in the following way. Put $\eta = \eta_1 \wedge \eta_2$. In the case (A) we set

$$\varepsilon_0 = \min\left\{ \frac{\eta}{2(d+3)\beta}, \frac{\eta}{2(d+3)}, \frac{\eta}{2 + 2/\alpha + \eta/(\beta(1+\eta))}, \frac{\eta_2}{\eta_2 + \beta(1+(d+1)/\alpha)} \right\}.$$
while in the case (B) we pick
\[ \varepsilon_0 = \min \left\{ \frac{(1+\eta)/\beta - 1/\alpha}{2(d+3)/\alpha}, \frac{\eta - (1/\alpha - 1/\beta)}{2(d+3)/\alpha}, \frac{1/\beta - 1/((1+\eta)\alpha)}{2+2/\alpha + 1/\beta - 1/((1+\eta)\alpha)}, \frac{1 + \eta_2 - \beta/\alpha}{1 + \eta_2 + \beta(1+d/\alpha)} \right\}. \]

Due to our assumptions \( \varepsilon_0 \) is positive.

Such choice of \( \varepsilon_0 \) follows from [Kulczycki, Kulik and Ryznar2022, Remark 5.1] and Lemma 3.12.

3. Proofs

3.1. Classes of functions. In this section, we define some classes of real functions which will be needed in the sequel.

For \( f \in C^1(\mathbb{R}^d) \) and \( \eta, r > 0 \) we put
\[ \rho_{\eta,r}(f) = \sup_{x,y: 0 < |x-y| < r} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\eta}. \]

By \( C_b^1(\mathbb{R}^d) \) we understand the class of functions which are in \( C^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d) \) and have all first order derivatives bounded. For \( \eta > 0 \) we define
\[ C_b^{1,\eta}(\mathbb{R}^d) = \{ f \in C_b^1(\mathbb{R}^d) : \rho_{\eta,1}(f) < \infty \}. \]

Next, we define classes of functions of two variables \( t \) and \( x \). Let \( \eta, r > 0 \) and \( I \subset \mathbb{R} \) be an interval. For \( f : I \times \mathbb{R}^d \to \mathbb{R} \) such that \( f(t,x) \) is \( C^1 \) with respect to \( t \) in \( I \) we put
\[ D_1 f(t,x) = \partial_t f(t,x). \]
For \( f : I \times \mathbb{R}^d \to \mathbb{R} \) such that \( f(t,x) \) is \( C^1 \) with respect to \( x \in \mathbb{R}^d \) we put
\[ D_2 f(t,x) = \nabla_x f(t,x) \]
and
\[ \rho_{\eta,r}^f(f) = \sup_{t,x,y: 0 < |x-y| < r, t \in I} \frac{|D_2 f(t,x) - D_2 f(t,y)|}{|x-y|^\eta}. \]

\( C_b^{1,(1,\eta)}(I \times \mathbb{R}^d) \) is the class of functions \( f(t,x) \), \( f : I \times \mathbb{R}^d \to \mathbb{R} \), such that \( f \) is \( C^1 \) in \( t \), \( C^1 \) in \( x \), has its first order derivatives continuous on \( I \times \mathbb{R}^d \), for any fixed \( t \in I \) the function and the derivatives are in \( C_b(\mathbb{R}^d) \) and \( \rho_{\eta,1}^t(f) < \infty \).

Remark 3.1. It is easy to note that in the above definitions the function \( \rho_{\eta,1} \) can be replaced by \( \rho_{\eta,r} \) for some \( r > 0 \).

Remark 3.2. Suppose that \( f \in C^{1,\eta}(\mathbb{R}^d) \). If \( |x-y| \leq r \), then
\[ |f(y) - f(x) - \nabla f(x) \cdot (y-x)| \leq \rho_{\eta,r}(f)|y-x|^{1+\eta}. \]

Proof. By Taylor’s formula with the remainder of order 1 we have
\[ f(y) - f(x) = \int_0^1 \nabla f(x+t(y-x)) \cdot (y-x) dt. \]
Hence
\[ f(y) - f(x) - \nabla f(x) \cdot (y-x) = \int_0^1 [\nabla f(x+t(y-x)) - \nabla f(x)] \cdot (y-x) dt. \]
This yields
\[
|f(y) - f(x) - \nabla f(x) \cdot (y - x)| = \left| \int_0^1 \left( \nabla f(x + t(y - x)) - \nabla f(x) \right) \cdot (y - x) dt \right| \\
\leq |y - x|^{1+\eta} \int_0^1 \frac{|\nabla f(x + t(y - x)) - \nabla f(x)|}{|y - x|^\eta} dt,
\]
which completes the proof. \[\square\]

3.2. Main idea of the proof. In this section we will explain the method of the solution. In principal, the idea is to transform our SDE (1) (with a drift) to the auxiliary time-inhomogeneous SDE (without a drift).

Let us define the operator \( Q \)
\[
Qf(x) = \nabla f(x) \cdot b(x) + Q^{(\text{jump})} f(x),
\]
where \( \mu \) is the Lévy measure of the process \( Z \), \( \nu_k \) is the Lévy measure of the \( k \)-th component \( Z^k, k = 1, \ldots, d \), and P.V. means that the first integral is taken in the principal value sense. One can easily show, using (8) and (22), that for each Markov process defined by (4).

As we have outlined in the Introduction, we would like to avoid solving the above Cauchy problem directly and to use already available results from [Kulczycki, Kulik and Ryznar2022] instead. For that, we consider the (time-dependent) operator family \( \{L_t\} \)
\[
L_t f(x) = \text{P.V.} \int_{\mathbb{R}^d} \left( f(x + V_t(x, z)) - f(x) \right) \mu(dz), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d,
\]
call that \( V_t(x, z) = \kappa_t(x) + A(x)z - x \) for \( t \in \mathbb{R}, x, z \in \mathbb{R}^d \). One can easily show, using (8) and (22), that \( L_t f \in C_0(\mathbb{R}^d) \). It is clear (e.g. by the virtue of the Itô formula) that the operator \( L_t \) is a generator for the (time-inhomogeneous) Markov process defined by (1).

The following lemma relates the new operator family \( \{L_t\} \) with the original generator \( Q \).
Lemma 3.3. Let $f \in C^{1,1}(I \times \mathbb{R}^d)$, where $I \subset \mathbb{R}$ is an interval. Put $g(t,x) = f(t,\kappa_t(x))$. Then for any $t \in I$ and $x \in \mathbb{R}^d$ we have

\[ \partial_t g(t,x) + Q_x g(t,x) = (D_1 f)(t,\kappa_t(x)) + (L_{t,x} f)(t,\kappa_t(x)). \]

Proof. We have

\[
(L_{t,x} f)(t,\kappa_t(x)) = \text{P.V.} \int_{\mathbb{R}^d} \left( f(t,\kappa_t(x) + V_t(\kappa_t(x), z)) - f(t,\kappa_t(x)) \right) \mu(dz)
\]

\[
= \text{P.V.} \int_{\mathbb{R}^d} \left( f(t,\kappa_t(x)) + \kappa_t(\chi_t(\kappa_t(x))) + A(\chi_t(\kappa_t(x))) z - \kappa_t(\kappa_t(x)) \right) - f(t,\kappa_t(x)) \mu(dz)
\]

\[
= \text{P.V.} \int_{\mathbb{R}^d} (f(t,\kappa_t(x) + A(x)z)) - f(t,\kappa_t(x)) \mu(dz)
\]

\[
= \text{P.V.} \int_{\mathbb{R}^d} (g(t, x + A(x)z) - g(t, x)) \mu(dz)
\]

\[
= (\mathcal{Q}^{\text{jump}}_x) g(t,x).
\]

By Lemma 3.8 we get

\[
Q_x g(t,x) = Q^{\text{jump}}_x g(t,x) + [D\kappa_t(x)]^T D_2 f(t,\kappa_t(x)) \cdot b(x)
\]

\[
= Q^{\text{jump}}_x g(t,x) + D_2 f(t,\kappa_t(x)) \cdot D\kappa_t(x) b(x)
\]

\[
= Q^{\text{jump}}_x g(t,x) - D_2 f(t,\kappa_t(x)) \cdot \partial_t \kappa_t(x).
\]

Also note that

\[
\partial_t g(t,x) = D_1 f(t,\kappa_t(x)) + D_2 f(t,\kappa_t(x)) \cdot \partial_t \kappa_t(x).
\]

Applying (26) we end the proof. \qed

Remark 3.4. We can interpret the statement of the lemma as follows: by changing the variables

\[(t,x) \sim (t^*,x^*) = (t,\kappa_t(x)),\]

we transform the operator $\partial_t + Q$ into the operator $\partial_t + L_t$, where the (time-dependent) PDO $L_t$, on the contrary to the original PDO $Q$, does not contain a gradient term. This change of variables well corresponds to the classical characteristic method for 1st order PDEs.

Using Lemma 3.3 one can try construct a fundamental solution to the parabolic Cauchy problem for the operator $Q$ once such a solution for $L_t$ is available. Namely, the function $u_t(x,y)$ given by (3) is a natural candidate for solving (23), (24) since $p_{t,x}(x,y)$ is the heat kernel corresponding to $\partial_t + L_t$. However, there are two substantial difficulties for using such analytic approach directly in order to identify $u_t(x,y)$ as the transition distribution density for (1). First, the transition density $p_{t,x}(x,y)$ constructed in [Kulczycki, Kulik and Ryznar2022] is proved to satisfy Kolmogorov’s backward differential equation only in a certain approximate sense, see [Kulczycki, Kulik and Ryznar2022] Lemma 3.3, and may fail have enough regularity in $x$ for Lemma 3.3 to be applicable. Second, even if one manages to show that the formula (5) gives some fundamental solution, this is not yet enough to identify the law of the solution to (1) until uniqueness of such a solution is proved. We avoid these substantial analytical complications by using stochastic calculus arguments outlined in the Introduction. Namely, under the assumptions of Theorem 2.1, we have the following
Lemma 3.5. Let $X$ be a weak solution to SDE (1) and $t > 0$ be fixed. Then the Itô formula is applicable to this process and the function $f(s, x) = \chi_{t-s}(x)$, implying that the process $\chi_{t-s}(X_s)$, $s \in [0, t]$ satisfies

$$
\text{P.V.} \int_{|z| \leq 1} A_{t-s}(X^t_s) z \mu(dz) = 0.
$$

Using this, we write (27) in the following more cumbersome but completely rigorous form, which will be actually proved below:

$$
\begin{align*}
\frac{dX^t_s}{ds} = & \int_{|z| \leq 1} V^t_s(X^t_s, z) \tilde{N}(ds, dz) \\
& + \int_{|z| < 1} \left( V^t_s(X^t_s, z) - A_{t-s}(X^t_s) z \right) ds \mu(dz) \\
& + \int_{|z| > 1} V^t_s(X^t_s, z) N(ds, dz), & s \in [0, t], \quad X^t_0 = \chi_t(x),
\end{align*}
$$

where $\tilde{N}$ is the compensated random measure corresponding to $N$.

Proof of Lemma 3.5. First, we recall that by the standard Itô formula for any $\mathbb{R}^d$-valued function $f(s, x) = (f^1(s, x), \ldots, f^d(s, x))$ such that all its coordinates are from the class $C^{1,2}([0, t] \times \mathbb{R}^d)$, bounded with their derivatives, the process

$$
Y_s = f(s, X_s), \quad s \in [0, t]
$$

satisfies

$$
Y_s - Y_0 = \int_0^s \left( (D_1 f)(r, X_r) + (D_2 f)(r, X_r)b(X_r) \right) dr \\
+ \int_0^s \int_{|z| \leq 1} \left( f(r, X_{r-} + A(X_{r-}) z) - f(r, X_{r-}) \right) \tilde{N}(dr, dz) \\
+ \int_0^s \int_{|z| > 1} \left( f(r, X_{r-} + A(X_{r-}) z) - f(r, X_{r-}) \right) N(dr, dz) \\
+ \int_0^s \int_{|z| \leq 1} \left( f(r, X_r + A(X_r) z) - f(r, X_r) - (D_2 f)(r, X_r) A(X_r) z \right) \mu(dz) dr,
$$

where $(D_2 f)(r, x) = [\langle D_2 f^1 \rangle(r, x), \ldots, \langle D_2 f^d \rangle(r, x)]^T$ for $r \in \mathbb{R}, x \in \mathbb{R}^d$.

Here we have used that, by the symmetry of the Lévy measure $\mu(dz)$, for any $x \in \mathbb{R}^d, r \in [0, t]$,

$$
\text{P.V.} \int_{|z| \leq 1} \left[ f(r, x + A(x) z) - f(r, x) \right] \mu(dz)
= \int_{|z| \leq 1} \left[ f(r, x + A(x) z) - f(r, x) - (D_2 f)(r, X_r) A(x) z \right] \mu(dz).
$$

Next, we observe that the formula (28) remains true for any $f = (f^1, \ldots, f^d)$ such that $f^k \in C^1_b(1, \eta_2)([0, t] \times \mathbb{R}^d)$, $k = 1, \ldots, d$. Indeed, we can approximate $f$ in a standard way, e.g. by taking convolutions in spatial variable Gaussian kernels,
by a sequence of functions \( f_n = (f_1^n, \ldots, f_d^n) \) with \( f_k^n \in C^{1,2}([0,t] \times \mathbb{R}^d) \), \( k = 1, \ldots, d, n \geq 1 \) such that
\[
\sup_{n \geq 1} \rho_{0,t}^{[0,1]}(f_k^n) < \infty, \quad k = 1, \ldots, d
\]
and \( f^k, D_1f^k, D_2f^k \) are approximated uniformly on compact sets by \( f_k^n, D_1f_k^n, D_2f_k^n \), respectively. The formula (28) holds true for every \( f_n, n \geq 1 \), and both sides of this formula converge in probability as \( n \to \infty \) to the corresponding terms in (28) for \( f \). For the LHS and the first (deterministic) integral in the RHS this follows easily from the uniform convergence of \( f_k^n, D_1f_k^n, D_2f_k^n \) on compacts. For the second (stochastic) integral this follows by the usual Itô isometry arguments, convergence \( f_n \to f \) and uniform bound on the increments \( |f(s,x) - f(s,y)| \leq C|x-y| \), valid on every compact set. The third integral in the RHS is just an a.s. finite sum over the set of large jumps for the process \( Z \), and thus the required converges follows simply from \( f_n \to f \). Finally, for the fourth integral the required convergence follows by the dominated convergence theorem because, by (22), for \( n \geq 1 \), we have
\[
|f_n(r,x+A(x)z) - f_n(r,x) - (D_2f)(r,X_r)A(x)z| \leq C|z|^{1+\eta_2}, \quad r \in [0,t], x \in \mathbb{R}^d, |z| \leq 1,
\]
and, by (8),
\[
\int_{|z| \leq 1} |z|^{1+\eta_2} \mu(dz) < \infty
\]
since \( 1 + \eta_2 > \beta \).

To conclude the proof, we simply take \( f(s,x) = \chi_{t-s}(x) \). By Lemma 3.8 and Remark 3.9 below, for any \( t > 0 \) this is a function with all coordinates of the class \( C^{1,1,\eta_2}([0,t] \times \mathbb{R}^d) \) which satisfies
\[
D_1f(s,x) = -\partial_x \chi_t(x) \bigg|_{r=t-s} = -\partial_x \chi_t(x)b(x) \bigg|_{r=t-s} = -D_2f(s,x)b(x).
\]
Thus the formula (28) holds true for this function, and the first (deterministic) integral at its RHS equals zero. The 2nd, 3rd, and 4th integrals at the RHS of (28) coincide with the 2nd, 4th and 3rd terms at the RHS of (27), respectively, integrated in \( s \).

**Corollary 3.7.** The SDE (1) has a unique weak solution \( X \). This solution is a Markov process with the transition probability density \( u_t(x,y) \) which satisfies (5).

**Proof.** By Lemma 3.5 for any weak solution \( X_s, s \in [0,t] \) to (1) with \( X_0 = x \), the process
\[
X^*_s = \chi_{t-s}(X_{s+t}) = \kappa_s(X_{s+t}), \quad s \in [-t,0]
\]
is a weak solution to SDE (1) with \( X^*_t = \chi_t(x) \). For the latter equation the weak uniqueness follows by [Kulczycki, Kulik and Ryznar 2022, Theorem 2.1]: we show in Lemma 3.12 below that all the assumptions of this theorem are satisfied, see also Remark 3.13. This gives weak uniqueness for (1).

Existence of a weak solution to (1) can be either derived from the similar result for (4), or proved directly by standard weak compactness arguments; we omit the details.

Finally, we have
\[
X_0^* = \kappa_0(X_t) = X_t \implies \text{Law} (X_t | X_0 = x) = \text{Law} (X_0^* | X^*_t = \chi_t(x)),
\]
which proves (5).
3.3. Properties of the flow. In this Section we prove some properties $\kappa_t$ and $\chi_t$ needed in the sequel. Such properties seem to be standard in the theory of ODE, but we present them here for the convenience of the reader.

Lemma 3.8. If assumptions (C) are satisfied then for any $t, s \in \mathbb{R}, x, y \in \mathbb{R}^d$,

\begin{align}
|\kappa_t(x) - \kappa_s(x)| &\leq c|t - s|e^{c|t|}\|x\|, \\
|\kappa_t(x) - \kappa_t(y)| &\leq c|x - y|e^{c|t|}, \\
|D\kappa_t(x)| &\leq ce^{c|t|}, \\
|D\kappa_t(x) - D\kappa_t(y)| &\leq c|x - y|^{\eta_2}e^{c|t|} 
\end{align}

and

\begin{equation}
\partial_t \kappa_t(x) = -D\kappa_t(x)b(x).
\end{equation}

Remark 3.9. The lemma is true if we replace $\kappa_t(x)$ by $\chi_t(x)$ and $b(x)$ by $-b(x)$.

Proof. It is clear that it is enough to prove it for $t, s \geq 0$. Note that

\begin{equation}
\kappa_t(x) = x - \int_0^t b(\kappa_s(x))ds.
\end{equation}

We observe that from the general theory $D\kappa_t(x)$ exists for any $x \in \mathbb{R}^d$. We have

\begin{align}
|\kappa_t(x) - \kappa_t(y)| &= |x - y - \int_0^t (b(\kappa_s(x)) - b(\kappa_s(y)))ds| \\
&\leq |x - y| + \int_0^t |b(\kappa_s(x)) - b(\kappa_s(y))|ds \\
&\leq |x - y| + C_7 \int_0^t |\kappa_s(x) - \kappa_s(y)|ds.
\end{align}

Next we use Gronwall Lemma to get (30). As a consequence of this and the existence of $D\kappa_t(x)$ we obtain

\begin{equation}
|D\kappa_t(x)| \leq ce^{ct}.
\end{equation}

Applying $D$ to the equality (34), then using (35) and (13) we have

\begin{equation}
D\kappa_t(x) = I - \int_0^t Db(\kappa_s(x))D\kappa_s(x)ds,
\end{equation}

where $I$ is identity matrix. This representation, the estimate (30) and the arguments based on Gronwall Lemma leads to (32).

Using the continuity of the function $s \mapsto Db(\kappa_s(x))D\kappa_s(x)$ we have the following matrix differential equation for $y_t(x) = D\kappa_t(x)$:

\begin{equation}
\frac{\partial}{\partial t} D\kappa_t(x) = -Db(\kappa_t(x))D\kappa_t(x), \quad y_0(x) = D\kappa_0(x) = I,
\end{equation}

which can be written, denoting $J_t(x) = -Db(\kappa_t(x))$, as

\begin{equation}
\frac{\partial}{\partial t} y_t(x) = J_t(x)y_t(x), \quad y_0(x) = I.
\end{equation}

Now, apply to both sides the vector $b(x)$ to obtain

\begin{equation}
\frac{\partial}{\partial t} y_t(x)b(x) = J_t(x)y_t(x)b(x), \quad y_0(x)b(x) = b(x).
\end{equation}
On the other hand
\[ \frac{\partial}{\partial t} b(\kappa_t(x)) = \partial_t(\kappa_t(x)) b(\kappa_t(x)) = J_t(x) b(\kappa_t(x)), \quad b(\kappa_0(x)) = b(x). \]
Finally, we observe that \( z_1(t) = b(\kappa_t(x)) \) and \( z_2(t) = y_t(x)b(x) \) satisfy the same differential equations with the same initial condition, hence they are equal. Therefore we have
\[ \partial_t \kappa_t(x) = -b(\kappa_t(x)) = -y_t(x)b(x) = -D\kappa_t(x)b(x), \]
which ends the proof.

By (13) for any \( x \in \mathbb{R}^d \) we have \( |b(x)| \leq c|x| \). Using this, (31) and (33) we get (38).

**Lemma 3.10.** If assumptions (C) are satisfied then there is \( c \) such that for any \( t \in \mathbb{R} \) and \( x, y \in \mathbb{R}^d \) we have
\[ |A_t(x) - A_t(y)| \leq ce^{c|t|} |x - y|^{\eta_1 \wedge \eta_2}. \]

**Proof.** We recall that
\[ A_t(x) = D\kappa_t(\chi_t(x))A(\chi_t(x)). \]
Applying (32) and (30) we get
\[ |D\kappa_t(\chi_t(x)) - D\kappa_t(\chi_t(y))| \leq ce^{c|t|} |x - y|^{\eta_2}. \]
By (11) and (30),
\[ |A(\chi_t(x)) - A(\chi_t(y))| \leq ce^{c|t|} |x - y|^{\eta_1}. \]
Since
\[ \sup_{x \in \mathbb{R}^d} (|D\kappa_t(x)| + |A(x)|) \leq ce^{c|t|} \]
we get (38).

**Lemma 3.11.** If conditions (C) and (D) hold then condition (D') is satisfied, with a constant \( C_9 \) depending only on \( d, \eta_1, \eta_2, C_1, \ldots, C_8 \).

**Proof.** By (34) we obtain
\[ \kappa_t(x) - \kappa_s(x) = -\int_s^t b(\kappa_u(x)) du \]
and then
\[ D\kappa_t(x) - D\kappa_s(x) = -\int_s^t Db(\kappa_u(x)) D\kappa_u(x) du. \]
Since
\[ |D\kappa_u(x)| \leq ce^{c|u|} \]
we arrive at
\[ |\kappa_t(x) - \kappa_s(x)| \leq \|b\|_\infty |t - s| \]
and
\[ |D\kappa_t(x) - D\kappa_s(x)| \leq ce^{c(|s| + |t|)} |t - s|. \]
These yield
\[ |D\kappa_t(\chi_t(x)) - D\kappa_s(\chi_s(x))| \leq ce^{c(|s| + |t|)} \kappa_t(x) - \kappa_s(x) + ce^{c(|s| + |t|)} |t - s| \leq ce^{c(|s| + |t|)} \|b\|_\infty \|b\|_\infty |t - s| + |t - s|. \]
Moreover,
\[ |A(\chi_t(x)) - A(\chi_s(x))| \leq C_5|\chi_t(x) - \chi_s(x)|^{\eta_1} \leq C_5||b||^2|t-s|^{\eta_1}. \]

Hence we obtain (16). \(\square\)

Let \(a_{t,i,j}(x)\) be elements of \(A_t(x)\) (where \(t \in \mathbb{R}, i, j \in \{1, \ldots, d\}\)).

**Lemma 3.12.** Put \(\gamma_1 = \gamma_2 = \min(\eta_1, \eta_2), \gamma_3 = 1 + \eta_2\). If assumption \((C)\) is satisfied then for any \(x, y \in \mathbb{R}^d, t, s \in \mathbb{R}\) we have

\[
\begin{align*}
|a_{t,i,j}(x)| &\leq cc^{-[t]}, \\
|\det(A_t(x))| &\geq cc^{-[t]}, \\
|a_{t,i,j}(x) - a_{s,i,j}(y)| &\leq cc^{[t]|x - y|^{\gamma_1}}.
\end{align*}
\]

We also have

\[
\begin{align*}
\gamma_3 &> \max(1, \beta), \\
|V_t(x, z) - A_t(x)z| &\leq cc^{[t]|z|^{\gamma_3}}.
\end{align*}
\]

If \((17)\) is satisfied then

\[
\frac{\beta}{\alpha} < 1 + \gamma_1, \quad \frac{1}{\alpha} - \frac{1}{\beta} < \gamma_2, \quad \frac{\beta}{\alpha} < \gamma_3.
\]

**Remark 3.13.** Note that \((D')\) clearly implies that for any \(x, y \in \mathbb{R}^d, t, s \in \mathbb{R}\) we have

\[ |a_{t,i,j}(x) - a_{s,i,j}(x)| \leq cc^{\max([t], [s])}|t - s|^{\gamma_2}. \]

This and the above lemma shows that assumptions on \(A_t(x)\) and \(V_t(x, z)\) stated in (8-15) in Kulczycki, Kulik and Ryznar 2022 are satisfied (locally in \(t\)).

**Proof of Lemma 3.12.** By (9) and (31) we obtain (39).

Put \(J_t(x) = -Db(\kappa_t(x))\). For any \(t \in \mathbb{R}, x \in \mathbb{R}^d\) by (13) we obtain \(|\text{tr} J_t(x)| \leq C_0d^4\), which implies \(\int_0^t \text{tr} J_s(x)\, ds \leq C_0d^4|t|\). By (57) and Coddington and Levinson 1953 Theorem 7.3 we obtain

\[ |\det(D\kappa_t(x))| = \exp\left(\int_0^t \text{tr} J_s(x)\, ds\right) \geq e^{-C_0d^4[t]}. \]

Using this and (10) we get (10).

The estimate (11) follows from (38), while (12) implies (12).

Recall that

\[ V_t(x, z) = \kappa_t(\chi_t(x) + A(\chi_t(x))z) - x = \kappa_t(\chi_t(x) + A(\chi_t(x))z) - \kappa_t(\chi_t(x)). \]

We apply Remark 3.2 to the coordinates of the vector functions \(\kappa_t\) with \(y\) replaced by \(\chi_t(x) + A(\chi_t(x))z\) and \(x\) replaced by \(\chi_t(x)\), hence we get, by (14),

\[ |V_t(x, z) - A_t(x)z| \leq cc^{[t]|A(\chi_t(x))z|^{1+\eta_2}}. \]

Since \(|A(\chi_t(x))z| \leq C_3|z|\) we get (13).

Finally, (14) follows from elementary calculations. \(\square\)
3.4. Proofs of the main results. In the whole section we assume either (A), (C), (D') or (B), (C), (D'), (E).

Proof of Theorem 2.1 and Remark 2.2. By Corollary 3.7, for any $t > 0$, $x, y \in \mathbb{R}^d$ we have

$$u_t(x, y) = p_{-t,0}(\chi_t(x), y).$$

Note that for any $x \in \mathbb{R}^d$ and $t > 0$ we have

$$\int_{\mathbb{R}^d} u_t(x, y) \, dy = \int_{\mathbb{R}^d} \tilde{u}_t(x, y) \, dy = \int_{\mathbb{R}^d} \tilde{u}_t(x, y) \, dy = 1,$$

so it is enough to show (20) for $t > 0$. We need to recall some notation from Kulczycki, Kulik and Ryznar2022. Let $G_t$ be the density defined in Section 5 in (roughly speaking, it is a modified version of the density $G_1$). For $t, s \in \mathbb{R}$, $s > t$, $x, w \in \mathbb{R}^d$ put (cf. Kulczycki, Kulik and Ryznar2022 (5.3))

$$p_{t,s}^x(w) = \frac{1}{|\det(A_s(x))|} G_{s-t}(A_s^{-1}(x)w).$$

We also define, for $t, s \in \mathbb{R}$, $s > t$, $x, w \in \mathbb{R}^d$,

$$\overline{p}_{t,s}^x(w) = \frac{1}{|\det(A_s(x))|} \tilde{G}_{s-t}(A_s^{-1}(x)w).$$

To the end of the proof we assume that $|s| \leq 1$, $0 < s - t \leq 1$ and $x \in \mathbb{R}^d$. By Kulczycki, Kulik and Ryznar2022 (3.13) and (5.4), we have

$$\int_{\mathbb{R}^d} |p_{t,s}(x, y) - p_{t,s}^y(x - y)| \, dy \leq c(s - t)^{\sigma_0}. \tag{46}$$

By Lemma 5.8 from Kulczycki, Kulik and Ryznar2022 we obtain

$$\int_{\mathbb{R}^d} |p_{t,s}^y(x - y) - p_{t,s}^x(x - y)| \, dy \leq c(s - t)^{\sigma_0}, \tag{47}$$

while Lemma 4.8 from Kulczycki, Kulik and Ryznar2022 yields

$$\int_{\mathbb{R}^d} |p_{t,s}^x(w) - \overline{p}_{t,s}^x(w)| \, dw \leq c(s - t)^{\sigma_0}. \tag{48}$$

Note that

$$\overline{p}_{t,s}^x(\chi_t(x) - y) = \frac{1}{|\det(A(\chi_t(x)))|} \tilde{G}_t(A^{-1}(\chi_t(x))(y - \chi_t(x))) = \tilde{u}_t(x, y)$$

for any $t > 0$, $x, y \in \mathbb{R}^d$. Using this, (45) and (46, 47, 48) we get (20).

Now assume that (D) holds. Then one can easily prove

$$\int_{\mathbb{R}^d} |p_{t,s}^x(w) - \overline{p}_{t,s}^{\chi_t(x)}(w)| \, dw \leq c(s - t)^{\sigma_0}. \tag{50}$$

The above estimate can be obtained by slight modifications in the proof of Lemma 5.9 from Kulczycki, Kulik and Ryznar2022. Here we also need the bound $|\chi_{t-s}(x) - x| \leq c|t - s|$, which follows from (54) and is a consequence of the assumption (D).

Combining the estimates (46, 47, 48, 50) we obtain

$$\int_{\mathbb{R}^d} |p_{t,s}(x, y) - \overline{p}_{t,s}^{\chi_{t-s}(x)}(x - y)| \, dy \leq c(s - t)^{\sigma_0}. \tag{51}$$
Note that
\[
\overline{p}_{t-s}^{-1}(\chi_t(x) - y) = \overline{p}_{t-0}^{-1}(\chi_t(x) - y) = \frac{1}{|\det(A(x))|} \overline{G}_t(A^{-1}(x) y - \chi_t(x)).
\]
Using this, (46) and (51) (with \(x\) replaced by \(\chi_t(x)\)) we get (20) in this case with \(\overline{u}_t(x, y)\) replaced by \(\overline{u}_t(x, y)\).

\[\square\]

**Proof of Theorem 2.3.** Fix \(\tau > 0\).

By (45) and Theorem [Kulczycki, Kulik and Ryznar2022, Theorem 2.2] we obtain
\[
\sup_{x,y \in \mathbb{R}^d} |u_t(x, y)| = \sup_{x,y \in \mathbb{R}^d} |p_{t-0}(\chi_t(x), y)| < \infty, \quad 0 < t < \infty.
\]
For \(t, s \in \mathbb{R}, 0 < s - t \leq \tau, x, y \in \mathbb{R}^d\) we have
\[
|p_{t,s}(x, y) - \overline{p}_{t,s}(x, y)| \leq |p_{t,s}(x, y) - p_{t,0}^\alpha(x, y)| + |p_{t,0}^\alpha(x, y) - \overline{p}_{t,0}(x, y)| + |\overline{p}_{t,0}(x, y) - \overline{p}_{t,s}(x, y)|.
\]
Using [Kulczycki, Kulik and Ryznar2022, Lemmas 4.8 and 5.8] and some arguments from Section 3.5 from [Kulczycki, Kulik and Ryznar2022] all these terms are bounded by \(c(s - t)^{\alpha_\mathbb{R}}\), where \(c\) depends only on \(d, \alpha, \beta, \eta_1, \eta_2, C_1, \ldots, C_8, \sigma(\tau)\). Using this, (46) and (49) we get (21).

\[\square\]

**Proof of Theorem 2.4.** Fix \(\tau > 0, \gamma \in (0, \alpha)\) such that \(\gamma \leq 1\) and \(\gamma' \in (\gamma, \alpha)\). Let \(f: \mathbb{R}^d \to \mathbb{R}\) be a bounded Borel function. Then for \(t > 0, x \in \mathbb{R}^d\) we have
\[
U_t f(x) = \int_{\mathbb{R}^d} u_t(x, z)f(z) dz = \int_{\mathbb{R}^d} p_{t,0}(\chi_t(x), z)f(z) dz = P_{t,0} f(\chi_t(x)).
\]
Using this and [Kulczycki, Kulik and Ryznar2022, Theorem 2.3] we obtain for \(x, y \in \mathbb{R}^d, t \in (0, \tau]\)
\[
|U_t f(x) - U_t f(y)| = |P_{t,0} f(\chi_t(x)) - P_{t,0} f(\chi_t(y))| \leq c|\chi_t(x) - \chi_t(y)|^{\gamma t - \gamma' \frac{\alpha}{2}} \|f\|_{\infty},
\]
where \(c\) depends on \(\gamma, \gamma', d, \alpha, \beta, \eta_1, \eta_2, C_1, \ldots, C_7, C_9, \sigma(\tau)\).

By Lemma 3.8 we get \(|\chi_t(x) - \chi_t(y)| \leq ce^{\tau\gamma'}|x - y|\), which implies the assertion of the theorem.

\[\square\]

4. **Examples**

Let us give several examples illustrating various specific issues of the model.

Note that a simplest example of a Lévy measure \(\nu \in WSC(\alpha, \beta)\) is a symmetric \(\alpha_0\)-stable Lévy measure

\[
\nu(dx) = c \frac{dx}{|x|^{\alpha_0 + 1}},
\]
for which
\[
h(r) = \frac{4c}{\alpha_0(2 - \alpha_0)} r^{-\alpha_0}
\]
and thus (10) holds true with \(\beta = \alpha = \alpha_0\) and \(C_1 = C_2 = 1\).

**Example 4.1.** Let \(Z_i = (Z_i^1, \ldots, Z_i^d)^T\) be such that \(Z_i^1, \ldots, Z_i^d\) are independent and for each \(i \in \{1, \ldots, d\}\) \(Z_i^j\) is a one-dimensional, symmetric \(\alpha_0\)-stable process, where \(\alpha_0 \in (0, 2)\). Put \(\alpha = \beta = \alpha_0\). Then assumptions (A) are satisfied.

One can compare the results in such case with results from [Chen, Hao and Zhang2020]. In general, their results do not imply ours and our results do not imply theirs. On
one hand the results from [Chen, Hao and Zhang2020] allow to consider matrices $A$ and drifts $b$ to be dependent on time and the drifts can be Hölder continuous with respect to $x$ variable. These assumptions are less restrictive than ours. However, it is assumed in [Chen, Hao and Zhang2020] that matrices must be differentiable in $x$ variable and $\alpha \in (1/2, 2)$ (when drift is nontrivial). These assumptions are more restrictive than ours. One has also to add that the assertions on the transition density and the corresponding semigroup in [Chen, Hao and Zhang2020] are stronger than ours.

Example 4.2. Let $Z_t = (Z^1_t, \ldots, Z^d_t)^T$ be such that $Z^1_t, \ldots, Z^d_t$ are independent and for each $i \in \{1, \ldots, d\}$ $Z^i_t$ is a one-dimensional, symmetric $\alpha_i$-stable process ($\alpha_i \in (0, 2)$ and they are not all equal). Put $\alpha = \min(\alpha_1, \ldots, \alpha_d)$ and $\beta = \max(\alpha_1, \ldots, \alpha_d)$. Then assumptions (B) are satisfied. The SDE (1) driven by such process $Z$ and the corresponding anisotropic generator are of great interest see e.g. [Chaker2020], [Chaker2019], [Chaker and Kassmann2020], [Friesen, Jin, and Rüdiger2021], example (Z2) on page 2). Note that results from [Chen, Hao and Zhang2020] do not allow to consider SDE driven by such process.

The next example shows that under just the basic assumptions of Theorem 2.1 the transition probability density may be locally unbounded.

Example 4.3. (See [Kulczycki, Ryznar and Sztonyk2021], Remark 4.23). Let $Z_t = (Z^1_t, \ldots, Z^d_t)^T$ be such that $Z^1_t, \ldots, Z^d_t$ are independent and for each $i \in \{1, \ldots, d\}$ $Z^i_t$ is a one-dimensional, symmetric $\alpha_0$-stable process, where $\alpha_0 \in (0, 2)$, $b \equiv 0$ and matrices $A(x)$ are Hölder continuous in $x$, for each $x \in \mathbb{R}^d$ the matrix $A(x)$ is a rotation (hence, an isometry) and for any $x$ in some open cone with vertex at 0, which satisfies $|x| \geq 1$ we have $A(x)e_1 = x/|x|$. Then assumptions (B), (C), (D) are satisfied, so by Theorem 2.1 the transition density $u_t(x, y)$ exists. However, for $\alpha + 1 \leq d$, for any $x \in \mathbb{R}^d$ the transition density $u_t(x, y)$ is unbounded at any neighbourhood of the point $y = 0$.

Now, we present example of matrices $A$ and a drift $b$ for which assumption (D) is not satisfied (the drift is unbounded) but for which assumptions (C) and (D') hold.

Example 4.4. Assume that $b(x) = Bx$ for some constant $d \times d$ matrix $B$ and any $x \in \mathbb{R}^d$. Assume $d \times d$ matrices $A(x)$ satisfy (9), (10), (11) with $\eta_1 = 1$ and additionally $A(x)$ is constant when $|x|$ is large. More precisely, there exists a $d \times d$ matrix $J$ and $p_0 > 0$ such that for any $x \in \mathbb{R}^d$ if $|x| \geq p_0$ then $A(x) = J$.

Assume also that $\beta \in (0, 2)$. Then one can show that conditions (C) (with $\eta_2 = 1$) and (D') are satisfied.

Indeed, $\eta_2 = 1 > \min(0, \beta - 1)$, $Db(x) = B$ so (12), (13), (14) are satisfied. This implies that conditions (C) holds. We also have $\kappa_t(x) = e^{-Btx}$, $\chi_t(x) = e^{Btx}$. One can easily show that there exists $p_1 > 0$ such that for any $t \in \mathbb{R}$, $x \in \mathbb{R}^d$ we have

$$|e^{Btx}| \geq \frac{1}{p_1}e^{-p_1|t||x|}.
$$

We may assume that $|t - s| \leq 1$ and $|s| < |t|$.

We have $A_t(x) = e^{-Btx}A(e^{Bsx})$. Note that

$$|A(e^{Btx}) - A(e^{Bsx})| \leq c|e^{Btx} - e^{Bsx}| \leq c|t - s|e^{C|t||x|}.
$$
If \(|x| \geq p_0 p_1 e^{p_1 |t|}\) then
\[
|e^{B t} x| \geq \frac{1}{p_1} e^{-p_1 |t|} |x| \geq p_0, \quad \text{and} \quad |e^{B s} x| \geq \frac{1}{p_1} e^{-p_1 |t|} |x| \geq p_0,
\]
so \(|A(e^{B t} x) - A(e^{B s} x)| = 0\). On the other hand, if \(|x| \leq p_0 p_1 e^{p_1 |t|}\) then
\[
|A(e^{B t} x) - A(e^{B s} x)| \leq c p_1 p_0 |t - s| e^{(c+1)p_1 |t|}.
\]
This clearly implies (16).

Our last example explains why in the case (B), i.e. for a cylindrical noise which has different scaling indices of the coordinates, additional assumption (E) on the Hölder indices of the coefficients should be made. This example is similar to Example 2.7 from Kulczycki, Kulik and Ryznar2022.

**Example 4.5.** Let \(Z^i, i = 1, 2\) be independent, symmetric \(\alpha_i\)-stable processes with \(0 < \alpha_1 < \alpha_2 \leq 1\). The process \(Z = (Z^1, Z^2)^T\) fits to our case (B) with \(\alpha = \alpha_1, \beta = \alpha_2\). In this example, we show that in such – extremely spatially non-homogeneous – setting the additional assumption (E) is crucial in the sense that, without this condition, the structure of the transition density can be quite different.

For any \(x \in \mathbb{R}^2\) let \(A(x)\) be the identity matrix and \(b(x) = b \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}\). Then for any \(x \in \mathbb{R}^2\) and \(t \in \mathbb{R}\) we have
\[
\kappa_t(x) = \begin{pmatrix} x_1 \cos t + x_2 \sin t \\ -x_1 \sin t + x_2 \cos t \end{pmatrix}.
\]
It follows that
\[
A_t(x) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}
\]
Note that \(A_t(x)\) does not depend on \(x\) so we will write \(A_t\) instead of \(A_t(x)\). We have \(\kappa_t(x) = A_t x\) and \(\chi_t(x) = \kappa_{-t}(x) = A_{-t} x\).

Note that assumptions (B), (C), (D') are satisfied with \(\eta_1 = \eta_2 = 1\). For any \(x, z \in \mathbb{R}^2\) and \(t \in \mathbb{R}\) we have
\[
V_t(x, z) = \kappa_t(\chi_t(x) + A(\chi_t(x)) z) - x = \kappa_t(A_{-t} x + z) - x = A_t z
\]
Therefore the additional assumption (I) holds true since each operator \(T_{t}^{x, z}\) is just an isometry which corresponds to the shift of the variable \(x \mapsto x + A_t z\). The assumption (E) is equivalent to
\[
\frac{1}{\alpha} - \frac{1}{\beta} < 1.
\]
If this condition is satisfied then assertions of Theorems 2.1 and 2.3 hold. Then using (13) and these theorems one has for any \(t \in (0, 1]\) and \(x \in \mathbb{R}^d\)
\[
\overline{u_t}(0, 0) = \overline{G_t}(0) = c t^{-\frac{1}{\alpha} - \frac{1}{\beta}},
\]

\[
c t^{-\frac{1}{\alpha} - \frac{1}{\beta}} - c_1 t^{-\frac{1}{\alpha} - \frac{1}{\beta} + \epsilon_0} \leq u_t(0, 0) \leq c t^{-\frac{1}{\alpha} - \frac{1}{\beta} + \epsilon_0} + c_1 t^{-\frac{1}{\alpha} - \frac{1}{\beta} + \epsilon_0},
\]
which implies that there exists sufficiently small \(\tau \in (0, 1]\) and \(c_2 > 1\) such that for any \(t \in (0, \tau]\) we have
\[
\left(\frac{1}{t} \right)^{-\frac{1}{\beta} - \frac{1}{\alpha}} \overline{u_t}(0, 0) \leq c_2.
\]
For $\frac{1}{\alpha} - \frac{1}{\beta} > 1$ the situation changes drastically; namely, we have for $t \in (0, \pi/6]$

$$u_t(0,0) \leq ct^{\frac{1}{\alpha} - \frac{1}{\beta} - 1} \to 0 \quad \text{as} \quad t \to 0^+,$$

so (54) does not hold.

Now we will prove (55). The justification is similar to the proof of (19) in Appendix A [Kulczycki, Kulik and Ryznar 2022]. We have $u_t(0,0) = p_{t,0}(\kappa_{t}(0),0) = p_{t,0}(0,0)$. Recall that for $-\infty < t < s < \infty$ and $x \in \mathbb{R}^d$ $p_{t,s}(x,\cdot)$ is the transition density of the solution of the SDE

$$dX_r = A_r dZ_r, \quad X_t = x, \quad r \geq t.$$  

Hence, for $t > 0$ $p_{t,0}(0,\cdot)$ is the transition density of the solution of the SDE

$$dX_r = A_r dZ_r, \quad X_{-t} = 0, \quad r \geq -t.$$  

Let $X_r$ be the solution of (56) and $Z_r = (Z^1_r, Z^2_r)$. We have

$$X_0 = \int_{-t}^{0} A_r dZ_r = \left( \int_{-t}^{0} \cos r dZ^1_r + \int_{-t}^{0} \sin r dZ^2_r \right) \quad \left( \int_{-t}^{0} \sin r dZ^1_r + \int_{-t}^{0} \cos r dZ^2_r \right).$$

The characteristic function of $X_0$ has the form

$$\phi(z) = \exp \left\{ - \int_{-t}^{0} (|A_r z|_{\alpha_1} + |A_r z|_{\alpha_2}) dr \right\},$$

where $(A_r z)_{\alpha_1} = z_1 \cos r - z_2 \sin r$, $(A_r z)_{\alpha_2} = z_1 \sin r + z_2 \cos r$. Thus the distribution density equals

$$p(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp \left\{ - i z \cdot z - \int_{-t}^{0} (|A(r) z|_{\alpha_1} + |A(r) z|_{\alpha_2}) dr \right\} dz.$$  

In particular,

$$p(0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp \left\{ - \int_{-t}^{0} (|A(r) z|_{\alpha_1} + |A(r) z|_{\alpha_2}) dr \right\} dz.$$  

For $t \in [0, \pi/6]$ let us estimate from below

$$\int_{-t}^{0} (|A_r z|_{\alpha_1} + |A_r z|_{\alpha_2}) dr \geq \int_{-t}^{0} |A_r z|_{\alpha_2} dr.$$  

We will consider 2 cases.

Case 1: $|z_2| > \frac{2}{3} |z_1|$.  
Then for $r \in [-t/8, 0]$ we have

$$|A_r z|_{\alpha_2} = |z_1 \sin r + z_2 \cos r| \geq |z_2 \cos r| - |z_1 \sin r|.$$  

We have $|z_1 \sin r| \leq |r||z_1| \leq t|z_1|/8 \leq |z_2|/4$ and $|z_2 \cos r| \geq \cos(\pi/6)|z_2| \geq |z_2|/2$.  
This gives

$$\int_{-t}^{0} |A_r z|_{\alpha_2} dr \geq ct|z_2|^{\alpha_2}.$$  

Case 2: $|z_2| \leq \frac{2}{3} |z_1|$.  
For $r \in [-t, -3t/4]$ we have

$$|A_r z|_{\alpha_2} = |z_1 \sin r + z_2 \cos r| \geq |z_1 \sin r| - |z_2 \cos r| \geq \sin \left( \frac{3t}{4} \right) |z_1| - \frac{t|z_1|}{2} \geq \left( \frac{9}{4\pi} - \frac{1}{2} \right) t|z_1|.$$
which gives
\[ \int_{-t}^{0} |(A_r z)_2|^{\alpha_2} dr \geq c t^{1+\alpha_2}|z_1|^{\alpha_2}. \]

Now we can complete the estimate of \( p(0) \). We have
\[ p(0) \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp \left\{ - \int_{-t}^{0} |(A_r z)_2|^{\alpha_2} dr \right\} dz \]
\[ \leq \frac{1}{(2\pi)^2} \int_{|z_2| > \frac{1}{2}|z_1|} \exp \{-ct|z_2|^{\alpha_2}\} dz + \frac{1}{(2\pi)^2} \int_{|z_2| \leq \frac{1}{2}|z_1|} \exp \{-ct^{1+\alpha_2}|z_1|^{\alpha_2}\} dz =: I_1 + I_2. \]

Since
\[ I_1 = \frac{4}{(2\pi)^2 t} \int_{\mathbb{R}} |z_2| \exp \{-ct|z_2|^{\alpha_2}\} dz = ct^{-1-2/\alpha_2}, \]
\[ I_2 = \frac{t}{(2\pi)^2} \int_{\mathbb{R}} |z_1| \exp \{-ct^{1+\alpha_2}|z_1|^{\alpha_2}\} dz = ct^{-1-2/\alpha_2}, \]
this completes the proof of (55).

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