Critical exponents of a multicomponent anisotropic \( t - J \) model in one dimension

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A recently presented anisotropic generalization of the multicomponent supersymmetric \( t - J \) model in one dimension is investigated. This model of fermions with general spin-\( S \) is solved by Bethe ansatz for the ground state and the low-lying excitations. Due to the anisotropy of the interaction the model possesses \( 2S \) massive modes and one single gapless excitation. The physical properties indicate the existence of Cooper-type multiplets of \( 2S + 1 \) fermions with finite binding energy. The critical behaviour is described by a \( c = 1 \) conformal field theory with continuously varying exponents depending on the particle density. There are two distinct regimes of the phase diagram with dominating density-density and multiplet-multiplet correlations, respectively. The effective mass of the charge carriers is calculated. In comparison to the limit of isotropic interactions the mass is strongly enhanced in general.

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1 Introduction

The $t-J$ model is one of the most fundamental models of strongly correlated electron systems. It has been studied intensively in recent years since its connection with high $T_c$ superconductivity was pointed out [1, 2]. It has been conjectured [3] that one and two-dimensional variants of this model have properties in common. Therefore it is important to have exact results available in one dimension for integrable coupling parameters.

The $t-J$ model describes electrons with nearest-neighbour hopping and spin-exchange interaction with the constraint that no site is allowed to be occupied by more than one particle. The one-dimensional model for spin-1/2 particles was found to be integrable at the supersymmetric point [4–6] and the critical exponents for the long-distance asymptotics of its correlation functions were calculated [7]. These results were extended to a $t-J$ model of fermions of arbitrary spin $S$ with supersymmetric interaction [8–10]. Recently, the integrability of an anisotropic generalization of the multicomponent supersymmetric $t-J$ model has been shown [11, 12]. The Hamiltonian reads

$$H = - \sum_{j,s} \mathcal{P} (c^+_{js} c_{j+1s} + c^+_j c_{js}) \mathcal{P} + \sum_{j,s,s'} \{c^+_{js} c^+_{j+1s'} c_{j+1s} - \exp[\text{sign}(s'-s)\eta]n_{js}n_{j+1s'}\},$$

where $c_{js}$ annihilates a fermion at site $j$ with spin component $s$. We assume $s = 1, 2, ..., 2S+1$ for convenience. $n_{js}$ denotes the number operator and $\mathcal{P}$ the projection operator onto the subspace of at most singly occupied sites. For $\eta = 0$ the interaction is supersymmetric. In the general case the Hamiltonian is $\text{sp}_q(2S+1,1)$ invariant and $\eta > 0$ is a measure of the anisotropy.

In this paper we calculate the critical exponents of the correlation functions for the integrable anisotropic $t-J$ model of fermions with arbitrary spin $S$. For $S = 1/2$ some of these results were presented in [13]. The paper is organized as follows. In section 2 the Bethe ansatz solution is presented and the ground state and excitations are discussed. In section 3 the density-density and superconducting correlation functions are investigated, and the transport masses are calculated. The paper ends with a discussion of the results.

2 The Bethe Ansatz

The general Bethe ansatz solution of the eigenvalue problem for Hamiltonian (1) has been obtained on the basis of the Perk-Schultz model [14–16] in [11] from which we quote the relevant equations. The eigenstates are characterized by sets of rapidities $\lambda_{j}^{(0)} (j = 1, 2, ..., N)$ for the electrons which total number is $N$, and by spin rapidities $\lambda_{\alpha}^{(l)} (l = 1, 2, ..., 2S; \alpha = 1, 2, ..., M_l)$. The eigenvalue problem reduces to
the task of solving the following Bethe ansatz equations

\[
\begin{align*}
\frac{\sin \left( \lambda_j^{(0)} - \eta \right)}{\sin \left( \lambda_j^{(0)} + \eta \right)} L &= \prod_{\alpha=1}^{M_1} \frac{\sin \left( \lambda_\alpha^{(0)} - \lambda_\alpha^{(1)} - \eta \right)}{\sin \left( \lambda_\alpha^{(0)} - \lambda_\alpha^{(1)} + \eta \right)}, \\
\frac{\prod_{\beta=1}^{M_l} \sin \left( \lambda_\alpha^{(l)} - \lambda_\beta^{(l)} - \eta \right)}{\prod_{\beta=1}^{M_{l+1}} \sin \left( \lambda_\alpha^{(l)} - \lambda_\beta^{(l)} + \eta \right)} &= -\prod_{\beta=1}^{M_{l-1}} \frac{\sin \left( \lambda_\alpha^{(l)} - \lambda_\beta^{(l-1)} - \eta \right)}{\sin \left( \lambda_\alpha^{(l)} - \lambda_\beta^{(l-1)} + \eta \right)}. \tag{2}
\end{align*}
\]

where \( L \) is the number of lattice sites and we have set \( M_0 = N \) and \( M_{2S+1} = 0 \). (The number of particles with spin component \( m \) is conserved and equal to \( M_{2S+1-m} - M_{2S+2-m} \).) The total energy and momentum of the system are given in terms of the electron rapidities \( \lambda_j^{(0)} \)

\[
\begin{align*}
E &= -2N \cosh \eta + 2 \sinh^2 \eta \sum_{j=1}^{N} \frac{1}{\cosh \eta - \cos 2\lambda_j^{(0)}}, \\
P &= N\pi - \sum_{j=1}^{N} \Theta \left( 2\lambda_j^{(0)}; \frac{\eta}{2} \right), \tag{3}
\end{align*}
\]

where we have used the definition

\[
\Theta(v; \eta) := -i \ln \frac{\sin (i\eta - v/2)}{\sin (i\eta + v/2)} = 2 \arctan \left( \coth \eta \tan \frac{v}{2} \right). \tag{4}
\]

We now discuss the patterns of distributions of rapidities satisfying (2). Due to the attractive nature of the interaction the ground state is described by bound complexes of rapidities \( \lambda^{(l)} \) \( (l = 0, 1, \ldots, 2S - 1) \) which can be parametrized conveniently by the (real) rapidities \( \lambda^{(2S)} \). (In this paragraph subscripts are dropped). In order to accommodate for the groundstate we assume that the number of electrons \( N \) is a multiple of \( 2S + 1 \), \( N = M(2S + 1) \). Then we have \( M_{2S} = M \) real rapidities \( \lambda^{(2S)} \) and for each such \( \lambda^{(2S)} \) there are \( 2S + 1 - l \) complex spin rapidities \( \lambda^{(l)} \) given by

\[
\lambda^{(2S)} + i\frac{p}{2} \eta, \quad p = -(2S - l), -(2S - l - 2), \ldots, (2S - l - 2), (2S - l). \tag{5}
\]

In particular we have \( M_l = M(2S + 1 - l) \) from which we conclude that the particles are equally distributed over all spin components, i.e. the state is a singlet.

Using (3) the equations (2) are straightforwardly reduced to one set of \( M \) equations for the parameters \( v_\alpha = 2\lambda_\alpha^{(2S)} \)

\[
L \Theta \left( v_\alpha; \frac{2S + 1}{2} \eta \right) = 2\pi I_\alpha + \sum_{\beta=1}^{M} \theta(v_\alpha - v_\beta),
\]

\[2\]
\[ \theta(v) := \sum_{l=1}^{2S} \Theta(v; l\eta). \quad (6) \]

The numbers \( I_\alpha \) are integer (half-integer) for even (odd) \( 2S(M-1) \). In the ground-state these numbers are arranged symmetrically around zero. The energy and momentum are given by

\[ E = \sum_{\alpha=1}^{M} e(v_\alpha), \quad P = \sum_{\alpha=1}^{M} p(v_\alpha), \quad (7) \]

with the following abbreviations

\[ e(v) := 2 \left\{ \frac{\sinh(2S+1)\eta \sinh \eta}{\cosh(2S+1)\eta - \cos v} - (2S+1) \cosh \eta \right\} \]

\[ p(v) := \pi - \Theta \left(v; \frac{2S+1}{2}\eta\right). \quad (8) \]

In the thermodynamic limit \( L, M \to \infty \) with finite \( M/L \) the parameters \( v_\alpha \) fill an interval \([v_0, 2\pi - v_0]\) uniformly with density \( \sigma(v) \). From (6) we obtain an integral equation of Fredholm type for the distribution function \( \sigma(v) \)

\[ 2\pi \sigma(v) = \Theta' \left(v; \frac{2S+1}{2}\eta\right) - \int_{v_0}^{2\pi - v_0} \theta'(v-w)\sigma(w)dw, \quad (9) \]

with \( \Theta'(v, \eta) = \sinh 2\eta/(\cosh 2\eta - \cos v) \) and subsidiary condition

\[ \int_{v_0}^{2\pi - v_0} \sigma(v)dv = \frac{M}{L} = \frac{\rho}{2S+1}, \quad (10) \]

where \( \rho \) is the particle density. The solution to these equations \( \sigma(v) \) yields the ground state energy per site

\[ \frac{E_0}{L} = \int_{v_0}^{2\pi - v_0} e(v)\sigma(v)dv. \quad (11) \]

A gapless mode of excitations is given by a redistribution of the \( v_\alpha \) parameters in (6) by keeping the string structure (5). These excitations are of particle-hole type with quasilinear dispersion. There are \( 2S \) massive modes of excitations corresponding to the breaking of some complexes (5). These energy states are not considered here in detail as they are irrelevant for the behaviour of the algebraically decaying correlations. The integral equations defining these excitations as well as analytic results for the gaps in various limits of \( \rho \) and \( \eta \) can be found in the appendix.

### 3 Critical exponents of the correlation functions and masses of charge carriers

In this section we present the results for the correlation functions which are obtained from applications of conformal field theory to finite-size calculations performed as in
In the anisotropic model we have only one gapless excitation in contrast to the isotropic model with \(2S + 1\) such modes. As a result our model is described by a \(c = 1\) conformal field theory and the scaling dimensions of the primary fields are uniquely expressed in terms of the dressed charge [18–21] as

\[
x = \left[ \frac{\Delta M}{2\xi(v_0)} \right]^2 + [\xi(v_0)d]^2, \tag{12}
\]

where \(\Delta M\) and \(d\) label the quantum numbers which specify the massless excitations. \(\Delta M\) is the change in the number of complexes (5) compared to the groundstate, \(d\) is the number of complexes excited from the left Fermi point to the right one. All other excitations correspond to broken complexes (5) and have a gap. Therefore these excitations do not affect the critical properties.

The dressed charge function \(\xi(v)\) is given by the solution of the integral equation associated with (9)

\[
2\pi \xi(v) = 2\pi - \int_{v_0}^{2\pi-v_0} \theta'(v-w)\xi(w)dw. \tag{13}
\]

According to conformal field theory the long-distance asymptotics of the equal-time correlation functions of primary scaling fields \(\Phi(r)\) are determined by the critical exponents \(2x\)

\[
\langle \Phi(r)\Phi(0) \rangle \sim \exp\left(-2idk_FRr\right), \quad k_F = \frac{\rho}{2S+1}\pi. \tag{14}
\]

The dimensions of descendent fields differ from \(x\) by integers \(N^\pm\).

We first consider the density-density correlations. The leading contributions to the algebraic decay of this correlator are given by formula (12) for \(d = 0\), \((N^+, N^-) = (1,0)\) or \((0,1)\), and \(d = 1\), \(N^\pm = 0\), respectively. This leads to the asymptotic form

\[
\langle \rho(r)\rho(0) \rangle \approx \rho^2 + A_1r^{-2} + A_2r^{-\alpha}\cos(2k_FRr), \tag{15}
\]

where

\[
\rho(r) = \sum_{s=1}^{2S+1} n_{rs}, \quad \alpha = 2[\xi(v_0)]^2. \tag{16}
\]

Turning to the superconducting correlation functions we discuss the correlations of complexes of \(2S+1\) particles

\[
C(r) = \prod_{s=1}^{2S+1} c_{rs}. \tag{17}
\]

In this case the operator changes the number of complexes by \(\Delta M = 1\) with \(d = 0\) or \(d = 1/2\) for half-integer or integer spin \(S\), respectively. We thus obtain

\[
\langle C^+(r)C(0) \rangle \approx Br^{-\beta}, \quad \beta = \frac{1}{\alpha}, \quad \text{for half-integer } S, \tag{18}
\]
and
\[
\langle C^+(r)C(0) \rangle \simeq Br^{-\beta} \cos(k_F r), \quad \beta = \frac{1}{\alpha} + \frac{\alpha}{4}, \quad \text{for integer } S. \tag{19}
\]

In our one-dimensional system we have no superconductivity in the literal sense. Our model does not have finite off-diagonal long-range order. However, the superconducting correlations have a longer range than the density-density correlations provided \( \beta < \alpha \). Analytically we find \( \alpha(\rho = 0) = 2 \) and \( \alpha(\rho_{\text{max}} = 1) = 2/(2S + 1)^2 \). This implies that for all \( S \) there is a density regime \([0, \rho_c]\) where the system has dominating superconducting correlations, see Fig. 1. This property of the anisotropic model differs from the behaviour of the isotropic \( t - J \) model where spin and charge rapidities are also classified as bound states, however, with arbitrarily small binding energy. Therefore, the latter system does not manifest superconducting properties.

The formation of complexes of \( 2S + 1 \) bound particles can be substantiated through a study of the conduction properties, notably the transport masses of the charge carriers. Following [22, 23, 17] we determine the charge stiffness (Drude weight) \( D_c \) from the dressed charge
\[
D_c = \frac{(2S + 1)^2}{2\pi} v_F \xi^2(v_0). \tag{20}
\]

\( v_F \) is the velocity of the gapless excitations
\[
v_F = \frac{\left| \varepsilon'(v_0) \right|}{2\pi \sigma(v_0)}, \tag{21}
\]

and \( \varepsilon(v) \) is the dressed energy defined in (A.1) in the appendix. Figs. 2 and 3 show numerical results for the charge stiffness and for the effective mass \( m \) of the charge carriers per bare mass \( m_e \) of the particles
\[
\frac{m}{m_e} = \frac{D_c^0}{D_c}, \tag{22}
\]

where \( D_c^0 = \frac{2S + 1}{\pi} \sin \pi \frac{\rho}{2S + 1} \) is the charge stiffness of free fermions with spin \( S \). In the strong-coupling limit the charge stiffness is of order \( e^{-2S\eta} \)
\[
D_c = \frac{(2S + 1)^2}{\pi} \left( 1 - \frac{2S}{2S + 1} \rho \right) \sin \frac{\pi \rho}{2S + 1} \frac{\pi \rho}{2S + 1 - 2S \rho} e^{-2S\eta}, \tag{23}
\]

and therefore the transport masses are of order \( e^{2S\eta} \). The reason for these large masses is the binding of the particles in complexes. The hopping of particles from one lattice site to another is always accompanied by the breaking of the complex to which it belongs, at the expense of the binding energy of order \( e^{2S\eta} \).

In summary, we have obtained the spectral gaps, the transport masses, and the exact critical exponents of the long-distance asymptotics of the correlation functions of an anisotropic multicomponent \( t - J \) model. In particular the algebraically decaying
density-density and superconducting correlations have been analyzed and superconducting properties have been found. The one-particle Green’s functions $\langle c_{r\alpha}^+ c_{0\alpha} \rangle$ on the other hand show exponential decay, because the fermion operators $c_{r\alpha}$ change the total number of particles by 1 corresponding to an excitation with gap. As a consequence the distribution function $\langle n_{k\alpha} \rangle$ is non-singular. Therefore the model under consideration is characterized as a (marginal) Luttinger liquid with a single mode of gapless charge excitations and $2S$ spin excitations with gap.

**Appendix**

Here we give a brief account of the elementary excitations over the ground state which is characterized by a distribution of the $\lambda_j^{(0)}$ parameters in strings of length $2S + 1$.

The first kind of excitations consists of a redistribution of these strings and is of particle-hole type. The relevant equation governing the distribution of the string parameters $v_{\alpha}$ is (6) from which we derive the integral equation for the dressed energy $\varepsilon(v)$ in standard way

$$
\varepsilon(v) + \frac{1}{2\pi} \int_{v_0}^{2\pi-v_0} \theta'(v-w)\varepsilon(w)dw = e(v) + (2S + 1)\mu,
$$

where $e(v)$ is the bare energy of $(2S + 1)$-strings with rapidity $v$ and we have introduced the chemical potential $\mu$ per particle. The chemical potential is a function of the particle density $\rho (= \rho(v_0))$ via the condition

$$
\varepsilon(v_0) = \varepsilon(2\pi - v_0) = 0.
$$

The dressed energy $\varepsilon(v)$ is negative for $v$ inside the interval $[v_0, 2\pi-v_0]$ and positive for $v$ outside, thus corresponding to excitations of hole- and particle-type, respectively. The general particle-hole type excitations read

$$
E - E_0 = \sum_{v_p} \varepsilon(v_p) - \sum_{v_n} \varepsilon(v_n),
$$

$$
P = \sum_{v_p} \pi(v_p) - \sum_{v_n} \pi(v_n),
$$

where the momentum $\pi(v)$ is defined by

$$
\pi(v) = -\frac{1}{2 \sinh \eta} \int_{\pi}^{v} [\varepsilon(w) + (2S + 1)(2 \cosh \eta - \mu) \xi(w)] dw.
$$

There are $2S$ massive modes of excitations corresponding to the breaking of some strings (8). In the general situation the $\lambda_j^{(0)}$ are distributed in additional strings of length $m = 1, 2, \ldots, 2S$. Instead of (6) we obtain

$$
L \Theta \left( v_\alpha; \frac{2S + 1}{2} \eta \right) = 2\pi I_\alpha + \sum_{\beta=1}^{M'} \theta(v_\alpha - v_\beta) + \sum_{m=1}^{2S} \sum_{\beta} \theta_m (v_\alpha - v_{\beta}^{(m)}),
$$

(A.5)
where the parameters $v_{\beta}^{(m)}$ are the rapidities of the strings of length $m$ and we have used the definition

$$\theta_m(v) = \sum_{t=(2S+1-m)/2}^{(2S-1+m)/2} \Theta(v; l\eta). \quad (A.6)$$

Applying standard transformations to (A.5) we arrive at the energy-momentum dispersions of the massive modes in terms of the dressed energy $\varepsilon(v)$

$$\varepsilon_m(v) = 2 \left( \frac{\sinh m\eta \sinh \eta}{\cosh m\eta - \cos v} - m \cosh \eta \right) + m\mu - \frac{1}{2\pi} \int_{v_0}^{2\pi-v_0} \theta'_m(v-w)\varepsilon(w)dw,$$

$$p_m(v) = \pi - \Theta(v, m\eta) + \frac{1}{4\pi \sinh \eta} \int_{v_0}^{2\pi-v_0} \theta_m(v-w)\varepsilon(w) + (2S+1)(2 \cosh \eta - \mu)\xi(w)]dw. \quad (A.7)$$

The gap $\Delta_m = \varepsilon_m(\pi)$ of the $m$th excitation is found to be $0$ and $[1 - m/(2S+1)]e^{\eta}$ in the limits of the anisotropy parameter $\eta \to 0$ and $\eta \to \infty$, respectively. For density $\rho \to 0$ we find

$$\Delta_m = 2 \sinh \eta \left( \frac{\sinh m\eta}{\cosh m\eta + 1} - \frac{m}{2S+1} \frac{\sinh(2S+1)\eta}{\cosh(2S+1)\eta + 1} \right), \quad (A.8)$$

and for maximum density $\rho \to 1$

$$\Delta_m = 2 \sinh \eta \left( \frac{\sinh m\eta}{\cosh m\eta + 1} - \frac{m}{2S+1} - 2 \sum_{k=1}^{\infty} (-1)^k e^{-(2S+1)\eta k} \frac{\sinh m\eta k}{\sinh(2S+1)\eta k} \right), \quad (A.9)$$

in the latter limit the dressed energy is

$$\varepsilon(v) = 4 \sinh \eta \sum_{k=1}^{\infty} e^{-\eta k} \frac{\sinh \eta k}{\sinh(2S+1)\eta k}(\cos kv - 1). \quad (A.10)$$
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Captions

Figure 1: (a) Depiction of the critical exponent $\alpha$ for $S = 1/2$ and different values of $\eta = 0.1, 0.5, 1, 2, 10$. For values of $\alpha$ above the broken line we have dominating superconducting correlations. (b) The same for $S = 1$.

Figure 2: (a) The charge stiffness $D_c$ for $S = 1/2$ and different anisotropies $\eta = 0.1, 0.5, 1, 2, 5$. (b) The same for $S = 1$. Note that the values for $\eta = 5$ are zero within the graphical resolution.

Figure 3: (a) The effective mass $m$ of the charge carriers for $S = 1/2$ and different anisotropies $\eta = 0.1, 0.5, 1, 2$. (b) The same for $S = 1$. 
$S = 1/2$

$S = 1$
