A REMARK ON THE ASYMPTOTIC BEHAVIOR OF THE EXTERIOR SOLUTIONS TO MONGE-AMPERE EQUATION

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Abstract. We improve the result of Caffarelli-Li [CL03] on the asymptotic behavior at infinity of the exterior solution $u$ to Monge-Ampère equation $\det(D^2u) = 1$ on $\mathbb{R}^n \setminus K$ for $n \geq 3$. We prove that the error term $O(|x|^{2-n})$ can be refined to $d(\sqrt{x'Ax})^{2-n} + O(|x|^{1-n})$ with $d = \text{Res}[u]$ the residue of $u$.

1. Introduction

The seminal results of Jörgens ($n = 2$ [Jo54]), Calabi ($n \geq 5$ [Ca58]), and Pogorelov (all $n$ [Po72]) state that the classical convex solution to $\det(D^2u) = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. Caffarelli extended the result for classical solutions to viscosity solutions (see [CL03]).

Let $\mathcal{A} = \{A : A$ is $n \times n$ symmetric positive definite matrix with $\det(A) = 1\}$. Caffarelli and Yanyan Li [CL03] proved the following results.

Theorem 1.1. Let $f \in C^0(\mathbb{R}^n)$ satisfies $0 < \inf_{\mathbb{R}^n} f \leq \sup_{\mathbb{R}^n} f < \infty$ and the support of $f - 1$ is bounded. Assume that $u$ is a convex viscosity solution of $\det(D^2u) = f$ in $\mathbb{R}^n$. Then $u$ is $C^\infty$ in the complement of the support of $(f - 1)$ and there exist some $A \in \mathcal{A}, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, such that

(i) for $n \geq 3$, 
$$u(x) = \frac{1}{2} x'A x + b \cdot x + c + O_k(|x|^{2-n}) \quad \text{as } x \to \infty;$$

(ii) for $n = 2$, 
$$u(x) = \frac{1}{2} x'A x + b \cdot x + d \log \sqrt{x'Ax} + c + O_k(|x|^{-1}) \quad \text{as } x \to \infty$$

with $d = \frac{1}{2\pi} \int_{S^2} (f - 1)$. The notation $\varphi(x) = O_k(|x|^m)$ means that $|D^k \varphi(x)| = O(|x|^{m-k})$ for all $k = 0, 1, 2, \cdots$.

Theorem 1.2. Let $K$ be a bounded closed convex subset of $\mathbb{R}^n$, and let $u \in C^0(\mathbb{R}^n \setminus K)$ be a locally convex viscosity solution of $\det(D^2u) = 1$ in $\mathbb{R}^n \setminus K$. Then $u \in C^\infty(\mathbb{R}^n \setminus K)$ and there exist some $A \in \mathcal{A}, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, such that

(i) for $n \geq 3$, (1.1) holds;

(ii) for $n = 2$, (1.2) holds for some $d \in \mathbb{R}$.

Key words and phrases. Monge-Ampère equation, exterior domain, asymptotic behavior.
We denote \( u_i = \frac{∂u}{∂x_i} \), \( u_{ij} = \frac{∂^2 u}{∂x_i∂x_j} \), and \( \tilde{u}_{ij} \) is the cofactor of \( u_{ij} \). It is well known that the Monge-Ampère operator has divergence structure

\[
det(D^2u) = \sum_{j=1}^{n} \partial_j(u_1\tilde{u}_{ij}) := \text{div}(\psi(u))
\]

since the vector field \((\tilde{u}_{11}, \tilde{u}_{12}, \cdots, \tilde{u}_{1n})\) is divergence free (see e.g. [BNST08]). Let \( \xi(x) \) be any vector field in \( \mathbb{R}^n \) satisfying \( \text{div}\xi = 1 \), say \( \xi(x) = x_1e_1 \) or \( \frac{ξ}{n} \). Let \( U \) be a bounded domain with smooth boundary satisfying \( U \supset \text{supp}(f - 1) \) (in case of Theorem 1.1) or \( U \supset K \) (in case of Theorem 1.2). Then the integral

\[
\int_{\partial U}(\psi(u) - \xi) \cdot \tilde{n}dσ = \int_{\partial U}(\psi(u) - \xi) \cdot \tilde{n}dσ - |U|
\]

is independent of the specific choice of \( U \). In case of Theorem 1.1, it is \( \int_{\mathbb{R}^n}(f - 1) \).

We define

\[
\text{Res}[u] = \frac{1}{2\pi} \int_{\partial U}(\psi(u) - \xi) \cdot \tilde{n}dσ \quad \text{for} \quad n = 2,
\]

and

\[
\text{Res}[u] = \frac{1}{(n - 2)n\omega_n} \int_{\partial U}(\psi(u) - \xi) \cdot \tilde{n}dσ \quad \text{for} \quad n \geq 3,
\]

where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \).

Therefore, in our notation, \( d = \text{Res}[u] \) in (ii) of Theorem 1.1. Using the same method (p. 570 in [CL03]), one can also confirm that \( d = \text{Res}[u] \) in (ii) of Theorem 1.2.

The residue \( \text{Res}[u] \) is an essential quantity for \( u \), so it is natural to expect that it also appears in the expansion of \( u \) at infinity for \( n \geq 3 \). The purpose of this paper is to prove the following refined version of (1.1).

**Theorem 1.3.** Under the conditions of Theorem 1.1 or Theorem 1.2, for \( n \geq 3 \), we have for some \( A \in \mathcal{A} \), \( b \in \mathbb{R}^n \) and \( c \in \mathbb{R} \)

\[
u(x) = \frac{1}{2}x'Ax + b \cdot x + c - \text{Res}[u](\sqrt{x'Ax})^{2-n} + O_k(|x|^{1-n}) \quad \text{as} \quad x \to \infty.
\]

2. **Proof of Theorem 1.3**

We prove Theorem 1.3 by an argument that we have used in [HY20] (see Step 3 in §6.1).

**Proof.** Without loss of generality, we assume \( A = I \), \( b = 0 \) and \( c = 0 \) because otherwise we can make some affine transformation as in [CL03]. Denote

\[
E(x) := u(x) - \frac{1}{2}|x|^2.
\]

By (1.1), we have

\[
E(x) = O(|x|^{2-n}), \quad DE(x) = O(|x|^{1-n}), \quad D^2E(x) = O(|x|^{-n}). \tag{2.1}
\]

We use the notation \( F(\xi) := \det(\xi_{ij})^{1/n} \). Then \( E(x) \) satisfies the equation

\[
\sum_{ij} a_{ij}(x)D_{ij}E(x) = F(I + D^2E) - F(I) = 0 \tag{2.2}
\]

in \( \mathbb{R}^n \setminus B_{R_0} \) for some \( R_0 > 0 \), where

\[
a_{ij}(x) = \int_{0}^{1} F_{\xi_{ij}}(I + sD^2E(x))ds = \delta_{ij} + O(|x|^{-n}).
\]
We write (2.2) as
\[ \triangle E(x) = \sum_{ij} (\delta_{ij} - a_{ij}(x)) D_{ij} E(x) := g(x) = O(|x|^{-2n}). \] (2.3)

We use Kelvin transformation. Define \( K[E](x) := |x|^{2-\eta} E(\frac{x}{|x|^2}) \) for \( x \in B_{\frac{1}{\eta_0}} \setminus \{0\} \). Then
\[ \triangle K[E] = |x|^{-2-\eta} g \left( \frac{x}{|x|^2} \right) := \tilde{g}(x), \quad \text{in} \ B_{\frac{1}{\eta_0}} \setminus \{0\}. \] (2.4)

From (2.3) and (2.4), we see \( \tilde{g}(x) = O(|x|^{-2}) \) as \( x \to 0 \), so \( \tilde{g}(x) \in L^\infty(B_{\frac{1}{\eta_0}}) \). Let \( N[\tilde{g}] \) be the Newtonian potential of \( \tilde{g} \) in \( B_{\frac{1}{\eta_0}} \). Since \( \tilde{g} \) is in \( L^p(B_{\frac{1}{\eta_0}}) \) for any \( p > 0 \), \( N[\tilde{g}] \) is \( W^{2,p} \) for any \( p \) and hence is \( C^{1,\alpha} \) for any \( 0 < \alpha < 1 \). Now \( K[E] - N[\tilde{g}] \) is harmonic in \( B_{\frac{1}{\eta_0}} \setminus \{0\} \). From (2.1), we know \( K[E] \) is bounded. So \( K[E] - N[\tilde{g}] \) is bounded and hence \( \{0\} \) is its removable singularity. That is \( K[E] - N[\tilde{g}] \) is harmonic in \( B_{\frac{1}{\eta_0}} \). So \( K[E](x) \) is a \( C^{1,\alpha} \) function in \( B_{\frac{1}{\eta_0}} \). Fix an \( \alpha \in (0,1) \), for some affine function \( \tilde{c} + \tilde{b} \cdot x \), we have
\[ |K[E](x) - \tilde{c} - \tilde{b} \cdot x| \leq C|x|^{1+\alpha} \quad \text{in} \ B_{\frac{1}{\eta_0}}. \]

Going back to \( E \), we have
\[ |E(x) - \tilde{c}| x|^{2-\eta} - \tilde{b} \cdot x| x|^{-n} \leq C|x|^{1-n-\alpha} \quad \text{for} \ |x| \geq R_0. \]

That is
\[ E(x) = \tilde{c}|x|^{2-\eta} + O(|x|^{1-n}) \quad \text{as} \ x \to \infty. \]

By Lemma 3.5 in [CL03], one can improve the above \( O(|x|^{1-n}) \) to \( O_k(|x|^{1-n}) \).

The remaining thing is to confirm that \( \tilde{c} = -\text{Res}[u] \). This can be done in the same way as in the 2 dimensional case (p.570 in [CL03]). We give the details in the following.

For \( r > R_0 \),
\[ \text{Res}[u] = \frac{1}{(n-2)n\omega_n} \int_{\partial B_r} (\psi(u) - x_i e_i) \cdot \tilde{u} d\sigma \]
\[ = \frac{1}{(n-2)n\omega_n r} \int_{\partial B_r} \left( \sum_{j=1}^n u_1 \tilde{u}_{1j} x_j - x_1^2 \right) d\sigma. \] (2.5)

We write \( u = w + \eta + O_k(|x|^{1-n}) \), where \( w = \frac{|x|^2}{2} \) and \( \eta = \tilde{c}|x|^{2-n} \). By computation, we have
\[ u_1 = w_1 + \eta_1 + O(r^{-n}) \]
\[ = x_1 + \tilde{c}(2-n)r^{-n}x_1 + O(r^{-n}), \]
\[ \tilde{u}_{11} = 1 + \sum_{k=2}^n \eta_{kk} + O(r^{-2n}) \]
\[ = 1 - \eta_{11} + O(r^{-2n}) \quad \text{(since} \ \Delta \eta = 0) \]
\[ = 1 - \tilde{c}(2-n)r^{-n} - \tilde{c}(2-n)(-n)r^{-n-2}x_1^2 + O(r^{-2n}) \]

and for \( j \geq 2 \),
\[ \tilde{u}_{1j} = -\eta_{j1} + O(r^{-2n}) \]
\[ = -\tilde{c}(2-n)(-n)r^{-n-2}x_1 x_j + O(r^{-2n}). \]
So
\[\sum_{j=1}^{n} u_1 \tilde{u}_j x_j = x_1^2 + x_1 \eta_1 - \eta_1 x_1^2 - \sum_{j=2}^{n} x_1 \eta_j x_j + O(r^{1-n})\]
\[= x_1^2 - \tilde{c} n (n-2) r^{-n} x_1^2 + O(r^{1-n}). \quad (2.6)\]

Inserting (2.6) into (2.5), we have
\[\text{Res}[u] = -\tilde{c} + O(r^{-1}).\]

Letting \( r \to \infty \), we get
\[\text{Res}[u] = -\tilde{c}. \]
\[\square\]

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