Perturbative Estimates of Lepton Mixing Angles in Unified Models

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Abstract

Many unified models predict two large neutrino mixing angles, with the charged lepton mixing angles being small and quark-like, and the neutrino masses being hierarchical. Assuming this, we present simple approximate analytic formulae giving the lepton mixing angles in terms of the underlying high energy neutrino mixing angles together with small perturbations due to both charged lepton corrections and renormalisation group (RG) effects, including also the effects of third family canonical normalization (CN). We apply the perturbative formulae to the ubiquitous case of tri-bimaximal neutrino mixing at the unification scale, in order to predict the theoretical corrections to mixing angle predictions and sum rule relations, and give a general discussion of all limiting cases.
1 Introduction

It is one of the goals of theories of particle physics beyond the Standard Model to predict quark and lepton masses and mixings. Whilst the quark mixing angles are known to all be rather small [1], by contrast two of the lepton mixing angles are identified as being rather large [2]. This observation, together with the smallness of neutrino masses, provides a tantalizing clue in the search for the origin of quark and lepton flavour. One possibility, widely studied in the literature, is that these two observations are related due to the underlying nature of neutrinos, which, unlike charged lepton and quark masses, are electrically neutral and so may have Majorana masses. The origin of the neutrino Majorana masses, being different from that of the quarks and charged leptons, may then be responsible for both the smallness of neutrino masses and the largeness of two of the lepton mixing angles. Whilst not a theorem, the plausibility and attractiveness of this hypothesis makes the conclusion that the origin of the large lepton mixing lies in the neutrino sector hard to resist. This idea is reinforced in the framework of many (but not all) grand unified theories (GUTs) [3] where the quarks and leptons are treated on the same footing, resulting typically in small quark and charged lepton mixing angles (possibly related to each other) with the see-saw mechanism [4, 5, 6, 7, 8] responsible for both small neutrino masses and large neutrino mixing angles (see e.g. [9, 10, 11, 12]).

Motivated by such considerations, here we shall assume that the large lepton mixing angles originate from the neutrino sector, and that the charged lepton mixing angles are rather small, and have a similar pattern to the quark mixing angles. Indeed, in many (but not all) GUTs, the origin of the quark mixing angles derives predominantly from the down quark sector, which in turn is closely related to the charged lepton sector. In order to reconcile the down quark and charged lepton masses, simple ansatze, such as the Georgi-Jarlskog hypothesis [13], lead to very simple approximate expectations for the charged lepton mixing angles such as $\theta_{12} \approx \lambda/3$, $\theta_{23} \approx \lambda^2$, $\theta_{13} \approx \lambda^3$, where $\lambda \approx 0.22$ is the Wolfenstein parameter [1] from the quark mixing matrix. Although the charged lepton mixing angles are clearly expected to be rather small,
nevertheless it is important to take into account such charged lepton corrections in order to 
estimate reliably the physical lepton mixing angles (see for example [14]).

Another effect which must be taken into account is the renormalisation group (RG) running 
required to relate high energy (GUT scale) predictions to low energy neutrino experiments. It 
is typically calculated numerically by solving the relevant coupled system of renormalisation 
group equations including the one for the effective neutrino mass matrix [15, 16, 17, 18, 19] and 
taking into account the mass thresholds of the right-handed neutrinos [20, 21, 22]. In many 
GUT models the neutrino masses turn out to be hierarchical in nature, and in such cases the 
RG running effects are relatively small (see e.g. [23]), but none the less such effects may be 
competitive with the charged lepton corrections and so also must be taken into account before 
comparing GUT scale predictions to low energy experiment. In this case analytic approxima-
tions for the RG corrections to the neutrino parameters can be used (see e.g. [22, 24, 25, 26]).

A third class of correction, not so well studied or appreciated, but nevertheless important in 
realistic models, are the canonical normalization (CN) effects resulting from the kinetic terms 
receiving corrections from the same physics responsible for the generation of flavour. Although 
model dependent, we have shown [27, 28] that the dominant canonical normalization correction 
arising due to the physics responsible for the third family Yukawa couplings (more precisely 
from dominant 33-elements of the charged lepton and neutrino Yukawa matrices in the flavour 
basis) has the same structure as the leading logarithmic (log) RG corrections, and so both 
effects may be subsumed into a single parameter $\eta$. To be precise, $\eta = \eta^{RG} + \eta^{CN}$, where in 
the MSSM

$$\eta^{RG} \approx \frac{y_\tau^2}{8\pi^2} \ln \frac{M_{\text{GUT}}}{M_Z} + \frac{y_{\nu_3}^2}{8\pi^2} \ln \frac{M_{\text{GUT}}}{M_3}$$  \hspace{1cm} (1)$$

and $y_\tau$ is the tau Yukawa coupling, while $y_{\nu_3}$ is the largest neutrino Yukawa coupling associated 
with a heavy right-handed neutrino mass threshold $M_3$, with $M_{\text{GUT}}$ being the GUT scale (see 
also section 6.1 of [22]). The parameter $\eta^{CN}$ parametrises CN effects and is highly model de-
pendent, however it contributes in the same way as $\eta^{RG}$ to leading log, since (in supersymmetric 
theories) both effects arise from third family wavefunction corrections in the considered approx-
imation. Therefore the combined effects of RG corrections and CN effects will be approximately parametrized by a single parameter $\eta$ in the analytic estimates which follow.

In this paper we provide simple approximate analytic formulae giving the lepton mixing angles and phases in terms of the neutrino mixing angles and phases together with perturbative corrections due to non-zero charged lepton mixing angles and phases, leading log renormalisation group running corrections and third family canonical normalization effects. We derive such approximate analytic perturbative corrections to leading order in the charged lepton mixing angles $\theta^e_{ij}$ and the RG/CN universal parameter $\eta$, where these parameters are all assumed to be small as discussed above. The resulting expansions provide useful physical insight into the origin and nature of the deviations of the observable lepton mixing angles from the underlying neutrino mixing angles at the GUT scale. In addition such perturbative formulae may be useful for speeding up multi-parameter scans in particular GUT models, or simply as a means of checking the numerical results. With the additional assumption that the underlying high energy neutrino (but not the physical lepton) mixing angles have the tri-bimaximal (TB) form \cite{29}, we use the perturbative formulae to derive new relations between lepton mixing angles and the perturbative charged lepton and RG/CN corrections.

The layout of the remainder of the paper is as follows. In Section 2 we state the main conventions used in the paper. In Section 3 we present the analytic formulae for the lepton mixing angles in terms of the underlying high energy neutrino mixing angles together with small perturbations due to both charged lepton corrections and RG/CN effects. In Section 4 we give the parameterization of the lepton mixing angles in terms of parameters describing the deviations from tri-bimaximal mixing. In Section 5 we specialize to the case of tri-bimaximal neutrino mixing, and give the perturbative formulae in this case. We then go on to apply these results first to the case of GUT models, and then discuss the results for various limiting cases. Section 6 concludes the paper.
2 Conventions

The mixing matrix in the lepton sector, the PMNS matrix $U_{PMNS}$, is defined as the matrix appearing in the electroweak coupling to the $W$ bosons expressed in terms of lepton mass eigenstates. The Lagrangian is given in terms of mass matrices of charged leptons $M_e$ and neutrinos $m_{LL}$ as

$$
\mathcal{L} = -\bar{e}L M_e e_R - \frac{1}{2} \bar{\nu}_L m_{LL} \nu^c_L + H.c.
$$

(2)

The change from the flavour basis to the mass eigenbasis is achieved via

$$
V_{eL} M_e V^\dagger_{eR} = \text{diag}(m_e, m_\mu, m_\tau), \quad V_{\nu L} m_{LL} V^T_{\nu L} = \text{diag}(m_{\nu 1}, m_{\nu 2}, m_{\nu 3}),
$$

(3)

where $V_{eL}$, $V_{eR}$ and $V_{\nu L}$ are $3 \times 3$ unitary matrices. The PMNS matrix (in the “raw” form, i.e. before the “unphysical” phases were absorbed into redefinitions of the relevant lepton field operators) is then given by

$$
U_{PMNS} = V_{eL} V^\dagger_{\nu L}.
$$

(4)

We use the parameterization $U_{PMNS} = U_{23} U_{13} U_{12}$ with $U_{23}, U_{13}, U_{12}$ being defined as

$$
U_{12} = \begin{pmatrix}
    c_{12} & s_{12} e^{-i\delta_{12}} & 0 \\
    -s_{12} e^{i\delta_{12}} & c_{12} & 0 \\
    0 & 0 & 1
        \end{pmatrix}, \quad U_{13} = \begin{pmatrix}
    c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\
    0 & 1 & 0 \\
    -s_{13} e^{i\delta_{13}} & 0 & c_{13}
        \end{pmatrix},
$$

$$
U_{23} = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & c_{23} & s_{23} e^{-i\delta_{23}} \\
    0 & -s_{23} e^{i\delta_{23}} & c_{23}
        \end{pmatrix}
$$

(5)

where $s_{ij}$ and $c_{ij}$ stand for $\sin \theta_{ij}$ and $\cos \theta_{ij}$, respectively and the remaining 3 unphysical phases have been rotated away, see for instance [30] for further details. The same scheme shall be used for the individual charged lepton and neutrino sector rotations in (4), with superscripts at the relevant quantities. Recall that in the standard PDG parameterisation [1] the Dirac CP phase $\delta$ relevant for neutrino oscillations and the Majorana CP phases $\alpha_1$ and $\alpha_2$ are entering as follows:

$$
U_{PMNS} = R_{23} U_{13} R_{12} P_0,
$$

(6)
where $P_0$ is a complex diagonal matrix $P_0 = \text{diag}(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1)$ and 

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}, \quad R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7)$$

Comparing (4) and (5) with (6) one finds $\delta = \delta_{13} - \delta_{23} - \delta_{12}, \alpha_2 = 2\delta_{23}$ and $\alpha_1 = 2(\delta_{12} + \delta_{23})$ (after having performed irrelevant global rephasings to absorb the “unphysical” phases).

The mixing angles $\theta_{ij}$ are the same in both parameterizations.

## 3 Charged lepton and RG/CN perturbations

In the considered GUT motivated framework defined above, with hierarchical neutrino masses $m_\nu^i / m_\nu^1 \approx \sqrt{\Delta m^2_{\odot}} / |\Delta m^2_{\odot}| \approx 0.18$, the low energy lepton mixing angles are dominated by the high energy neutrino sector contributions which are subject to three classes of perturbations: 1) canonical normalization effects due to a would-be non-canonical structure of the kinetic terms emerging at the GUT-scale, 2) contributions from the charged lepton mixings $\theta^e_{ij}$ and 3) the RG corrections.

The goal of this section is to derive formulae for the lepton mixing angles and phases in terms of the neutrino mixing angles and phases, together with small perturbative corrections due to the above three effects.

Let us first discuss the canonical normalization effects. In order to get the correct asymptotic behaviour of the matter sector propagators one should first bring the relevant kinetic terms into the canonical form by a suitable field redefinition $\hat{L}_L \rightarrow P_L^{-1}\hat{L}_L \equiv L_L, \hat{e}_R \rightarrow P_{e_R}^{-1}\hat{e}_R \equiv e_R$ where $L_L$ stands for the $SU(2)_L$ lepton doublet $(\nu_L, e_L)$ and hats denote for the corresponding quantities in the defining basis (i.e. before canonical normalization). This, however, affects also the defining basis mass matrices $\hat{M}_e$ and $\hat{m}_{LL}$ as follows: $\hat{M}_e \rightarrow P_L^T\hat{M}_e P_e \equiv M_e$ and $\hat{m}_{LL} \rightarrow P_L^T\hat{m}_{LL} P_L \equiv m_{LL}$. The charged lepton mass matrix $M_e$ and the effective neutrino mass matrix $m_{LL}$ are subsequently evolved to the low scale my means of the renormalisation

\footnote{Above the seesaw scales, $m_{LL}$ refers to the combination of parameters $v^2\mathcal{Y}_e M_{RR}^{-1}\mathcal{Y}_\nu^T$, where $\mathcal{Y}_e$ is the running neutrino Yukawa matrix, $M_{RR}$ the running mass matrix of the right-handed neutrinos and $v$ is the low scale value of the VEV of the Higgs which is involved in the neutrino Yukawa interactions.}
group equations and the unitary transformations $V_{eL}$ and $V_{\nu L}$ entering formula (4) can be extracted.

Turning to the RG effects, as we pointed out in [27, 28], if third family contributions dominate both CN and RG corrections, the CN and RG effects can be subsumed (at leading log) into a single parameter $\eta$ denoting the non-universality in the 33 component of the $P_L$ matrix. An interested reader can find the technical details of how to obtain the low-scale diagonalisation matrices $V_{eL}$ and $V_{\nu L}$ given their GUT-scale counterparts $\hat{V}_{eL}$ and $\hat{V}_{\nu L}$ elsewhere [27, 28] (in particular see Eqs.(2.14) and (2.17) of [28]) although we emphasize that the resulting lepton mixing angles and phases were not explicitly expanded in terms of $\eta$ as we do here.

The formulae for the lepton mixing angles and phases $\theta_{ij}$, $\delta_{ij}$ in terms of the neutrino mixing angles and phases $\theta_{\nu ij}$, $\delta_{\nu ij}$, together with small perturbative corrections due to charged lepton mixing angles and phases $\theta_{ei}$, $\delta_{ei}$ have already appeared in the literature [14, 31]. The new physics that we wish to discuss here is the effect of the additional perturbative CN/RG corrections described by the universal parameter $\eta$. With $\eta$ included, using the techniques described above, the leading order expansions for the physical lepton mixing angles and phases $\theta_{\nu,e}^{ij}$, $\delta_{\nu,e}^{ij}$ become:

$$s_{23}e^{-i\delta_{23}} \approx s_{23}^{\nu} \left(1 + \frac{\eta}{2} c_{23}^{\nu} \right) e^{-i\delta_{23}^{\nu}} - \theta_{23}^{e} s_{23}^{\nu} e^{-i\delta_{23}^{e}},$$

$$s_{13}e^{-i\delta_{13}} \approx \theta_{13}^{e} e^{-i\delta_{13}^{e}} - \theta_{12}^{e} s_{23}^{\nu} e^{-i(\delta_{23}^{\nu} + \delta_{12}^{\nu})} + \frac{m_{2}^{\nu}}{m_{3}^{\nu}} \eta c_{12}^{\nu} s_{12}^{\nu} c_{23}^{\nu} s_{23}^{\nu} e^{-i(\delta_{12}^{\nu} - \delta_{23}^{\nu})},$$

$$s_{12}e^{-i\delta_{12}} \approx s_{12}^{\nu} \left(1 + \frac{\eta}{2} c_{12}^{\nu} s_{23}^{\nu} \right) e^{-i\delta_{12}^{\nu}} - \theta_{12}^{e} c_{12}^{\nu} c_{23}^{\nu} e^{-i\delta_{12}^{e}}.$$

They should be compared to the results with only charged lepton corrections included [14, 31], to which these results reduce in the limit $\eta = 0$. We have neglected $\theta_{13}^{e}$ since in GUT models we expect that $\theta_{13}^{e} \approx \lambda^{3}$ and so $\theta_{13}^{e}$ terms may be regarded as higher order. We have included terms like $\frac{m_{2}^{\nu}}{m_{3}^{\nu}} \eta$ which may compete with $\theta_{12}^{e} \approx \lambda/3$, and have also included terms like $\theta_{23}^{e} \approx \lambda^{2}$ which are not so different from $\theta_{12}^{e} \approx \lambda/3$. Terms of the order $\mathcal{O}(m_{1}/m_{2})$ and $\mathcal{O}(m_{1}/m_{3})$ have been neglected, which corresponds to the assumption of a strong hierarchy of the neutrino mass spectrum. This also implies $\frac{m_{2}^{\nu}}{m_{3}^{\nu}} \approx \sqrt{\frac{\Delta m_{2}^{\nu}}{\Delta m_{3}^{\nu}}} |\Delta_{\nu}|$ which is a quantity directly accessible in the
neutrino oscillation experiments.

We would like to remark that, in general, one of the main sources of errors in the leading log approximation for the RG corrections is associated to the fact that the 3rd family Yukawa couplings (i.e. $y_\tau$ and $y_{\nu_3}$) are themselves running quantities. However, since the running of $y_\tau$ and $y_{\nu_3}$ affects the corrections to all the mixing parameters in the same way, effectively only modifying $\eta^{RG}$, this does not introduce an additional uncertainty in our formulae as long as $\eta$ (containing $\eta^{RG}$ as well as $\eta^{CN}$) is treated as (a small but) unknown parameter. In this sense, the formula for $\eta^{RG}$ in Eq. (1) should be used only as an estimate for the approximate size of this correction parameter. The remaining leading log error stems from the running of the other parameters and is comparatively small (for hierarchical neutrinos). We also note that since our formulae only depend on the ratio $\frac{m_2^{\nu}}{m_3^{\nu}}$, part of the leading log error from the running of $m_2^{\nu}$ and $m_3^{\nu}$, due to flavour-blind interactions, cancels out.

For convenience, we summarise the conditions under which the perturbative formulae presented in Eqs. (8) - (10) can be applied:

- $\theta_{12}^\nu, \theta_{23}^\nu, \theta_{13}^\nu$ and $\eta$ are small, such that an expansion in these parameters is justified.
- $\theta_{13}^\nu$ can be neglected (which is motivated by classes of GUT models where $\theta_{13}^\nu \approx \lambda^3$).
- The light neutrino masses are hierarchical, i.e. $m_1 \ll m_2 < m_3$.
- RG and CN corrections are dominated by third family effects (which allows them to be subsumed into the single parameter $\eta = \eta^{RG} + \eta^{CN}$).

### 4 Deviation parameters

Another parametrisation of the lepton mixing matrix can be achieved by taking an expansion about the TB matrix [32, 33, 34, 35]. Following [33] three small parameters $r$, $s$ and $a$ may be introduced to describe the deviations of the reactor (r), solar (s) and atmospheric (a) angles.
from their TB values:

\[
 s_{13} = \frac{r}{\sqrt{2}}, \quad s_{12} = \frac{1}{\sqrt{3}}(1 + s), \quad s_{23} = \frac{1}{\sqrt{2}}(1 + a) .
\]  

(11)

Global fits of the conventional lepton mixing angles \cite{2} can be translated into the 2σ-ranges:\footnote{Note that $r$ must be positive definite, while $s, a$ can take either sign. Indeed there is a preference for $s$ to be negative.}

\[
 0 < r < 0.28, \quad -0.10 < s < 0.02, \quad -0.12 < a < 0.12 .
\]  

(12)

The empirical smallness of the parameters $r, s, a$ shows that this parametrisation is as general as the Wolfenstein parametrisation of the quark mixing matrix.

Without loss of generality, the perturbative formulae in Eqs. (8) - (10) may then be recast in terms of the deviation parameters as:

\[
a \approx \left| 1 + a^\nu + \frac{\eta}{4} - \theta_{23}^\nu e^{i(\delta_{23}^\nu - \delta_{23})} \right| - 1 ,
\]  

(13)

\[
r \approx \left| \theta_{12}^\nu - \frac{1}{3} m_2^\nu \eta e^{i(\delta_{12}^\nu - \delta_{12})} - r^\nu e^{i(\delta_{12}^\nu - \delta_{13}^\nu + \delta_{23}^\nu)} \right| ,
\]  

(14)

\[
s \approx \left| 1 + s^\nu + \frac{\eta}{6} - \theta_{12}^\nu e^{i(\delta_{12}^\nu - \delta_{12})} \right| - 1 ,
\]  

(15)

where the $a^\nu, s^\nu$ and $r^\nu$ factors parametrise the would-be small deviation from tri-bimaximal setting in the neutrino sector, in full analogy with Eqs. (11).

Concerning the phases, due to the strong dominance of the first terms on the RHS of Eqs. (8) and (10) the physical factors $\delta_{12}$ and $\delta_{23}$ are essentially identical to the neutrino sector ones, i.e. $\delta_{12} \approx \delta_{12}^\nu$ and $\delta_{23} \approx \delta_{23}^\nu$. Since there is not such a strongly dominant term in (9), the determination of the $\delta_{13}$ phase requires a more careful treatment.

5 Tri-bimaximal neutrino mixing and sum rules

So far the above results assume nothing about the nature of the underlying neutrino mixing angles, apart from the fact that empirically they must be close to the TB mixing values (within our GUT motivated framework). Now we shall explore the possibility that the underlying
neutrino mixing angles take TB values quite accurately, which corresponds to setting \( r^\nu = s^\nu = a^\nu = 0 \). This situation occurs for example in models based on certain family symmetries such as \( A_4 \) or \( \Delta_27 \). Although the neutrino mixing angles may very accurately have the TB form, the physical lepton mixing angles will still continue to receive charged lepton and RG/CN corrections, so the observable lepton mixing angles are not expected to be accurately of the TB form even in this case.

Setting \( r^\nu = s^\nu = a^\nu = 0 \) and neglecting \( \theta_\nu^{13} \) one can use formulae (9) and (14) to get:

\[
  r \approx \sqrt{\theta_{e12}^2 + \left( \frac{1}{3} \frac{m_2^\nu}{m_3^\nu} \eta \right)^2 - \frac{2}{3} \theta_{e12} \frac{m_2^\nu}{m_3^\nu} \eta \cos(\delta_{12}^e - \delta_{12}^\nu + 2\delta_{23}^\nu)} \tag{16}
\]

for the reactor angle deviation and

\[
  \sin \delta \approx \frac{1}{r} \left[ \theta_{e12} \sin(\delta_{12}^e - \delta_{12}^\nu + \pi) - \frac{1}{3} \frac{m_2^\nu}{m_3^\nu} \eta \sin 2\delta_{23}^\nu \right], \tag{17}
\]

\[
  \cos \delta \approx \frac{1}{r} \left[ \theta_{e12} \cos(\delta_{12}^e - \delta_{12}^\nu + \pi) + \frac{1}{3} \frac{m_2^\nu}{m_3^\nu} \eta \cos 2\delta_{23}^\nu \right], \tag{18}
\]

for the lepton sector Dirac CP phase, where \( \delta = \delta_{13} - (\delta_{12} + \delta_{23}) \) and \( \delta_{12} \approx \delta_{12}^\nu, \delta_{23} \approx \delta_{23}^\nu \) and formula (9) have been used. Recall also that with a good accuracy \( 2\delta_{23}^\nu \approx \alpha_2 \).

In the same case, Eqs. (13) and (15) become:

\[
  a \approx \frac{\eta}{4} - \Delta, \tag{19}
\]

\[
  s \approx \frac{\eta}{6} + \theta_{e12} \cos(\delta_{12}^e - \delta_{12}^\nu + \pi), \tag{20}
\]

where \( \Delta \equiv \theta_{\nu 23} \cos(\delta_{23}^\nu - \delta_{23}^e) \) has been used to parameterize the lack of information on the \( \delta_{23}^\nu - \delta_{23}^e \) phase difference.

Notice that formulae (18), (19) and (20) constitute a system of three linear equations for three a-priori unknown quantities \( \eta, \cos(\delta_{12}^e - \delta_{12}^\nu + \pi) \) and \( \Delta \) with coefficients that can be inferred from experiment. If some of these quantities happen to be negligible, the system becomes overconstrained and one can obtain non-trivial relations between the neutrino sector observables.

\footnote{Here we present formulae for both \( \sin \delta \) and \( \cos \delta \) to provide a complete information on the Dirac CP phase including the quadrant ambiguity that might arise if only the latter was present.}
5.1 Application to GUT Models

In this subsection we apply the above results to the case of a specific but well motivated class of GUT models. More general application of our results will be considered in the next subsection.

We have previously noted that the typical GUT scale expectation for the charged lepton mixing angles is $\theta_{12}^e \approx \lambda/3$, $\theta_{23}^e \approx \lambda^2$, $\theta_{13}^e \approx \lambda^3$, where $\lambda \approx 0.22$ is the Wolfenstein parameter governing the quark mixing matrix. Numerically this implies that in such a case $\theta_{12}^e \approx 0.07$, $\theta_{23}^e \approx 0.05$, $\theta_{13}^e \approx 0.01$. This provides a justification for neglecting $\theta_{13}^e$ but keeping both $\theta_{12}^e$ and $\theta_{23}^e$. The parameter $\eta$ is quite model dependent, in part because of involving the highly model dependent piece $\eta^{CN}$. Even the contribution from $\eta^{RG}$ can have quite a range of values, however for hierarchical neutrino masses, assuming the dominant contribution to arise from the first term in Eq. (1) (i.e. $\eta^{RG} = \frac{y^2}{8\pi^2} \ln \frac{M_{GUT}}{M_\mu}$), we may estimate $\eta^{RG} \approx 0.02 - 0.10$ corresponding to a range of tau Yukawa couplings of $y_\tau = 0.23 - 0.51$. In SUSY models this maps to a range of the ratio of Higgs vacuum expectation values (VEVs) $\tan \beta \approx 30 - 50$, cf. [45]. Therefore in such typical GUT models, with small CN corrections, we may expect that the quantities $\eta$, $\theta_{12}^e$ and $\Delta$ may all be of a similar order of magnitude, with none of them being negligible. In some sense this is a “worst case” situation, since it involves all three quantities, but on the other hand in this scenario there are theoretical reasons to expect that all of these quantities are quite small, each giving a correction of less than 10 per cent, which justifies our perturbative approach (e.g. the higher order corrections which we neglect would account for around a per cent or so).

In addition the ratio of assumed hierarchical neutrino masses is given by $\frac{m_2^\nu}{m_3^\nu} \approx \sqrt{\frac{\Delta m_2^2}{\Delta m_3^2}} \approx 0.18$ and the combination which enters the formulae is $\frac{1}{3} \frac{m_2^\nu}{m_3^\nu} \eta \approx 0.06$. In such a GUT motivated framework, with small CN corrections, it is seen that the second and third term in the square root on the right-hand side of Eq. (15) only give a small correction compared to the first term, since the product $\frac{1}{3} \frac{m_2^\nu}{m_3^\nu} \eta$ is much smaller than $\theta_{12}^e$, and in this case we would have the prediction:

$$r \approx \lambda/3 \approx 0.073$$

(21)
accurate to about 10 per cent assuming $\theta_{12} \approx \lambda/3$ in the considered class of GUT models. This corresponds to $s_{13} = |U_{e3}| = 0.05$ or $\theta_{13} \approx 3^\circ$ to an accuracy of about 10 per cent.

The above prediction relies on the assumption that $\theta_{12} \approx \lambda/3$. This is quite a strong assumption, since there may be alternative ways of reconciling the down quark masses and charged lepton masses other than the Georgi-Jarlskog approach. In fact, the GUT scale values of the down quark and charged lepton masses show a strong dependence on possible supersymmetric threshold corrections, as has been pointed out recently in [45, 46], which might open up new possibilities [45]. Using Eqs. (18) and (20) we may eliminate $\theta_{12} \cos(\delta_{12} - \delta_{12}^e + \pi)$ in favour of $\cos \delta$, $\eta$ and $r$ to obtain a sum rule:

$$s \approx r \cos \delta + \eta \left( \frac{1}{6} - \frac{1}{3} \frac{m_{\nu_2}}{m_{\nu_3}} \cos \alpha_2 \right).$$

Unlike the previous case we have chosen to keep the term proportional to the product $\frac{1}{3} \frac{m_{\nu_2}}{m_{\nu_3}} \eta$. Even though it is small, in this case it may conspire with the numerical value $1/6 \approx 0.17$, depending on the phases, to give a significant effect. In this case we estimate that the second term on the right-hand side proportional to $\eta$ may give a correction of up to about 0.02 which is significant compared to the first term which is governed by $r \approx 0.07$. On the other hand, the second term might involve a partial cancellation of the two terms in the bracket and consequently be much smaller than the first term, in which case we would recover the approximate relation [33] $s \approx r \cos \delta$ which is a simple expression of the well known sum rule [31, 47, 14]:

$$\theta_{12} - 35.26^\circ \approx \theta_{13} \cos \delta.$$  (23)

Finally we remark that it is difficult to predict the atmospheric deviation parameter due to the unknown phases in the quantity $\Delta \equiv \theta_{23} \cos(\delta_{23}^\nu - \delta_{23}^e)$, plus the RG correction, however in the GUT motivated cases described above we would expect typically $a \leq 0.1$.

\footnote{A similar effect has been observed in [48] where the RG stability of various lepton sector sum-rules is studied.}
5.2 Other Limits, Applications and Sum Rules

In this subsection we consider other limits which are not directly motivated by the GUT-inspired assumptions of the previous subsection. In the context of more general models of neutrino masses and mixings (satisfying the conditions of section 3) the formulae in Eqs. (13) - (15) can still be applied to analyse under which conditions a precise measurement of the leptonic mixing angles in future neutrino oscillation experiments could provide hints that the underlying neutrino mixing angles indeed satisfy the tri-bimaximal mixing pattern.

Of course, if all three deviation parameters \( \eta, \theta_{e12} \) and \( \Delta \) are negligibly small, tri-bimaximal neutrino mixing would be directly testable. However, even in the presence of corrections (a situation which is typical in GUT models of flavour) the pattern of leptonic mixing angles may point towards to tri-bimaximal neutrino mixing at some high scale (flavour scale). The simplest possibility would be that only one of the corrections is important, while the other two can be neglected. This leads to three cases:

- **Only \( \eta \) is relevant:** In this case Eqs. (13) - (15) simplify to
  \[
  a \approx \frac{\eta}{4}, \quad r \approx \frac{1}{3} |\eta| \frac{m_2^\nu}{m_3^\nu}, \quad s \approx \frac{\eta}{6}. \tag{24}
  \]

  We note that when non-zero \( \theta_{13} \) at low energy is generated only by third family RG (and CN) effects, Eqs. (16), (17) and (18) determine the Dirac CP phase \( \delta \) to be \( \delta = -\alpha_2 \) for \( \eta > 0 \) (which is the case for pure RG corrections) or \( \delta = -\alpha_2 + \pi \) for \( \eta < 0 \) (c.f. [26]).

  With three predictions of \( a, r \) and \( s \) depending only on one parameter \( \eta \), there are now two correlations which would provide a “smoking gun” signal of an underlying tri-bimaximal neutrino mixing pattern, e.g.
  \[
  s \approx \frac{2}{3} a, \quad r \approx 2 |s| \frac{m_2^\nu}{m_3^\nu} \approx \frac{4}{3} |a| \frac{m_2^\nu}{m_3^\nu}. \tag{25}
  \]

- **Only \( \theta_{e12} \) is relevant:** In this limit Eqs. (13), (14) simplify into
  \[
  a \approx 0, \quad r \approx \theta_{e12}^e. \tag{26}
  \]
Moreover, in this limit one gets $\delta_{12}^\nu - \delta_{12}^e + \pi = \delta$ from (17) and (18) which also yields:

$$s \approx \theta_{12}^e \cos \delta .$$

(27)

To start with, the first relation $a \approx 0$ (almost exactly maximal mixing $\theta_{23}$) would indicate that (barring cancellations between $\Delta$ and $\eta$ corrections) both $\Delta$ and $\eta$ are negligible [49].

The correlation between the other two corrections can be written as [33]

$$s \approx r \cos \delta .$$

(28)

which is again a compact expression of the sum rule [14, 31, 47] in Eq. (23).

- **Only $\Delta$ is relevant:** Although not a typical situation in flavour models, we mention for completeness that this case shows that there exists a possible correction to $\theta_{23}^\nu$, namely $\theta_{23}^e$ which perturbs the tri-bimaximal pattern only in $a$ while leaving $s = r = 0$ as a hint for underlying tri-bimaximality.

Let us now turn to the somewhat less simple situation that two of the corrections are important. With two unknowns and three measurements, one correlation between the observables remains to provide a possible signal of tri-bimaximal neutrino mixing. The three possible cases are as follows:

- **Only $\eta$ and $\theta_{12}^e$ are relevant, $\Delta$ is negligible:** In the limit that $\Delta = 0$, Eq. (19) yields $a \approx \frac{2}{3}$ which allows to express $\eta$ in terms of $a$ in the other two equations. Combining them we find the improved sum rule

$$s \approx r \cos \delta + \frac{2}{3} a \left( 1 - 2 \frac{m_2^\nu}{m_3^\nu} \cos \alpha_2 \right)$$

(29)

which, compared to [27, 28], includes next-to-leading correction to $s$ of the form $O(a \frac{m_2^\nu}{m_3^\nu})$. This new term, however, depends on the Majorana phase $\alpha_2$, which will be difficult to measure.

- **Only $\theta_{12}^e$ and $\Delta$ are relevant, $\eta$ is negligible:** In this case, to leading order in small parameters, we again obtain the sum rule of Eq. (28), however now with $a \neq 0$.

$^5$We remark that for this scenario it is also possible to derive a sum rule which is exact in $\theta_{23}$ [50].
- Only $\Delta$ and $\eta$ are relevant, $\theta^{e}_{12}$ is negligible: Again, this is not a typical situation in flavour models (since usually a correction $\Delta$, containing $\theta^{e}_{23}$, is accompanied by a correction $\theta^{e}_{12}$), however for completeness we mention that here the correlation

$$r = 2|s| \frac{m^{\nu}_{2}}{m^{\nu}_{3}}$$

(30)

remains as a hint for underlying tri-bimaximal neutrino mixing.

Finally, if $\eta$, $\theta^{e}_{12}$ and $\Delta$ are all important, we have as many observables as unknowns which means that predictivity is lost. Combining Eqs. (18) - (20) to eliminate $\eta$ and $\theta^{e}_{12} \cos(\delta^{e}_{12} - \delta^{e}_{12} + \pi)$ one arrives to a $\Delta$-dependent sum rule:

$$s \approx r \cos \delta + \frac{2}{3} (a + \Delta) \left( 1 - 2 \frac{m^{\nu}_{2}}{m^{\nu}_{3}} \cos \alpha_{2} \right).$$

(31)

This result may be in principle (depending again on the hard-to-measure Majorana CP phase $\alpha_{2}$) used to determine $\Delta$, which may then be compared to the theoretical expectation for $\Delta$ within specific GUT models.

Even in such a case, Eqs. (13) - (15) may be very useful. For example, they can be used to determine the possible values of the correction parameters $\eta$, $\theta^{e}_{12}$ and $\Delta$ under the assumption of underlying nearly exact tri-bimaximal neutrino mixing. This provides a useful information for model building. Eqs. (13) - (15) can also be applied in context of various models of flavour that might happen to predict some of the correction parameters.\(^6\) Then, predictivity would be regained and one could derive correlations.

6 Conclusion

Many GUTs predict two large neutrino mixing angles, with the underlying charged lepton mixing angles being small and quark-like, and the neutrino masses being hierarchical. In such frameworks we present simple approximate analytic formulae giving the lepton mixing angles in

\(^6\)Also, if models predict the Majorana CP phase $\alpha_{2}$, this would improve the predictivity and testability in some cases.
terms of the underlying high energy neutrino mixing angles together with small perturbations due to both charged lepton corrections and RG/CN effects. The resulting analytic formulae given in in Eqs. (8) - (10) (or equivalently Eqs. (13) - (15)) express the lepton mixing angles in terms of the neutrino mixing angles and the leading order corrections due to $\theta_{ij}^e$ and $\eta$, which represent the leading order terms in an expansion in powers of small parameters representing both charged lepton mixing corrections and third family RG/CN effects.

We have applied these perturbative formulae to the ubiquitous case of tri-bimaximal neutrino mixing at the unification scale, in order to predict the theoretical corrections to mixing angle predictions and sum rule relations, and have given a general discussion of all limiting cases. When applied to GUT models, we have seen that the formulae lead to a roughly 10 per cent correction to reactor angle prediction based on the Georgi-Jarlskog ansatz of $\theta_{13} \approx 3^\circ$. More generally, independently of the GJ ansatz, we have seen that the sum rule relation $s \approx r \cos \delta$ receives a correction given by Eq. (22) which we estimate to be up to about 0.02 and thus may be significant compared to $r \approx 0.07$. We have also relaxed the GUT motivated assumptions, and obtained a variety of other possible relations amongst observable parameters, which, if confirmed by experiment, could signal neutrino tri-bimaximal mixing in a more general context than the GUT paradigm. For example, if for some reason $\theta_{23}^e$ turned out to be negligible, then our perturbative formulae lead to the novel relation in Eq. (29) which, although being quite accurate, involves the Majorana phase $\alpha_2$, making it difficult to test.

In conclusion, the perturbative formulae presented here provide a useful physical insight into the origin and nature of the deviations of the observable lepton mixing angles from the underlying neutrino mixing angles at the GUT scale, thereby opening a window into the nature of the high energy GUT theory. The results may also be used to test in a more general way the hypothesis of tri-bimaximal mixing in the neutrino sector under various assumptions about the nature of the charged lepton and third family RG/CN corrections, each of which leads to a different testable relation. In this way the underlying nature of tri-bimaximal neutrino mixing (if present) may be revealed in the low energy neutrino experiments which measure only the
physical lepton mixing angles and phases. In addition, such perturbative formulae may also be useful in a more prosaic way by speeding up multi-parameter scans in particular GUT models, or simply as a means of checking the numerical results.

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