EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A NONLINEAR EQUATION WITH CONVECTION TERM

MUSTAPHA AIT HAMMOU*

ABSTRACT. In this paper, we consider the existence and uniqueness of weak solutions of a nonlinear elliptic equation with a variable exponent, a monotonic type operator and a convection term. With the topological degree theory, we prove the existence of at least one weak solution under some Leray-Lions and growth conditions. Moreover, we obtain the uniqueness of the solution of the problem under some additional assumptions. Our results generalize and improve existing results with another approach.

1. INTRODUCTION

In the present paper, we focus on the existence and uniqueness of a weak solution for a class of nonlinear elliptic boundary value problems in the framework of the Lebesgue and Sobolev spaces with variable exponent. The use of such spaces is justified by the study of several materials that present inhomogeneities and for which the context given by classical spaces is not adequate. Indeed, for such materials, the exponents involved in the constitutive law could be variable. The great attention on this topic is due to the many and various applications concerning thermorheological fluids [5], image restoration [7], electrorheological fluids [15, 16] and elastic materials [20].

Consider the following problem with a Neuman boundary condition

\[
\begin{cases}
\text{div} \ a(x, \nabla u) + \lambda |u|^{q(x)-2}u = f(x, u), & x \in \Omega, \\
a(x, \nabla u).\eta = 0, & x \in \partial \Omega,
\end{cases}
\]

where $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $a$ generates a nonlinear operator of Leray-Lions $A$ from $W^{1, p(.)}_0(\Omega)$ to its dual $W^{-1, p'(\cdot)}(\Omega)$ which is defined by

$A(u) = -\text{div} a(x, \nabla u),$

$p(.)$ and $q(.)$ are two variable exponents satisfying some conditions to be seen in the paper suite, $f$ is a Carathéodory function satisfying a growth condition with a variable exponent that is suitably controlled by $p(.)$ and $\eta$ is the outward unit normal to $\partial \Omega$.

In [4], the authors showed the existence and uniqueness of a weak solution to the problem

$-\Delta_p u + m(x)|u|^{p-2}u = f(x, u)$ in $\mathbb{R}^N$

2010 Mathematics Subject Classification. 35J60, 47H11, 35D30, 35A01, 35A02.

Key words and phrases. Nonlinear elliptic equations, Topological degree, Weak solution, Existence and Uniqueness.
which involves the $p$-Laplacian through Browder’s theorem, where $1 < p < N$, $N \geq 3$ and under some conditions for the functions $m$ and $f$. When function $f$ is null and $a(x,\nabla u) = |\nabla u|^{q(x)-2}\nabla u$, the problem (1.1) has been treated as an eigenvalue and eigenvector problem [13].

Zhao et al. [21] have established the existence and the multiplicity of weak solutions of the boundary-value problem

$$
\begin{align*}
-\text{div } a(x,\nabla u) + |u|^{p-2}u = \lambda f(x,u), & \quad x \in \Omega, \\
u(x) = \text{constant}, & \quad x \in \partial\Omega, \\
\int_{\partial\Omega} a(x,\nabla u).n \, ds = 0,
\end{align*}
$$

They found one nontrivial solution by the mountain pass lemma in [14], when the nonlinearity has a $(p - 1)$-superlinear growth at infinity, and two nontrivial solutions by minimization and mountain pass when the nonlinear term has a $(p - 1)$-sublinear growth at infinity.

For $\lambda = 0$ and $a(x,\nabla u) = |\nabla u|^{q(x)} - 2\nabla u$, P.S. Ilias in [11] and Fan and Zhang in [9] gave several sufficient conditions for the existence of weak solutions for that problem under Dirichlet’s boundary condition.

In this paper, we study problem (1.1), on the one hand as a kind of generalisation of the few previous works, and on the other hand, using another method based on the Topological Degree Theory developed by Berkovits [6] for some classes of operators in Banach reflexive spaces (For some applications of this degree, the reader can see [1, 2, 3, 13]).

This document is organised as follows: Section 2 is reserved for some mathematical preliminaries. In section 3, we give our basic assumptions, some technical lemmas and we give and prove our results of existence and uniqueness.

2. Mathematical Preliminaries

2.1. Definitions and proposition. Let us start with a short reminder of the classes of operators mentioned in the introduction and of an important proposition which will be the key to proving the existence of at least one weak solution of problem (1.1). Let $X$ be a real separable reflexive Banach space with dual $X^*$ and with continuous pairing $\langle \cdot, \cdot \rangle$ and let $\Omega$ be a nonempty subset of $X$. The symbol $\rightarrow$ (\rightharpoonup) stands for strong (weak) convergence.

Let $Y$ be a real Banach space. We recall that a mapping $F : \Omega \subset X \rightarrow Y$ is bounded, if it takes any bounded set into a bounded set. $F$ is said to be demicontinuous, if for any $(u_n) \subset \Omega$, $u_n \rightarrow u$ implies $F(u_n) \rightarrow F(u)$. $F$ is said to be compact if it is continuous and the image of any bounded set is relatively compact. A mapping $F : \Omega \subset X \rightarrow X^*$ is said to be of class $(S_+)$, if for any $(u_n) \subset \Omega$ with $u_n \rightarrow u$ and $\limsup \langle Fu_n, u_n - u \rangle \leq 0$, it follows that $u_n \rightarrow u$. $F$ is said to be quasimonotone, if for any $(u_n) \subset \Omega$ with $u_n \rightarrow u$, it follows that $\limsup \langle Fu_n, u_n - u \rangle \geq 0$.

For any operator $F : \Omega \subset X \rightarrow X$ and any bounded operator $T : \Omega_1 \subset X \rightarrow X^*$ such that $\Omega \subset \Omega_1$, we say that $F$ satisfies condition $(S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightarrow u$, $y_n := Tu_n \rightarrow y$ and $\limsup \langle Fu_n, y_n - y \rangle \leq 0$, we have $u_n \rightarrow u$. For any $\Omega \subset X$, we consider
the following classes of operators:

\( \mathcal{F}_1(\Omega) := \{ F : \Omega \rightarrow X^* \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+) \} \),

\( \mathcal{F}_T(\Omega) := \{ F : \Omega \rightarrow X \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)_T \} \),

\( \mathcal{F}_T, B(\Omega) := \{ F : \Omega \rightarrow X \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)_T \} \).

**Proposition 2.1.** Let \( S : X \rightarrow X^* \) and \( T : X^* \rightarrow X \) be two operators bounded and contin-
uous such that \( S \) is quasimonotone and \( T \) is of class \( (S_+) \). If

\[ \Lambda := \{ v \in X^* \mid v + tSoT v = 0 \text{ for some } t \in [0, 1] \} \]

is bounded in \( X^* \), then the equation

\[ v + SoT v = 0 \]

admits at least one solution in \( X^* \).

**Proof.** Since \( \Lambda \) is bounded in \( X^* \), there exists \( R > 0 \) such that

\[ \| v \|_{X^*} < R \text{ for all } v \in \Lambda. \]

This says that

\[ v + tSoT v \neq 0 \text{ for all } v \in \partial B_R(0) \text{ and all } t \in [0, 1] \]

where \( B_R(0) \) is the ball of center 0 and radius \( R \) in \( X^* \).

From [6, Lemma 2.2 and 2.4] it follows that

\[ I + SoT \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = LoT \in \mathcal{F}_T(\overline{B_R(0)}). \]

Since the operators \( I, S \) and \( T \) are bounded, \( I + SoT \) is also bounded. We conclude that

\[ I + SoT \in \mathcal{F}_{T,B}'(\overline{B_R(0)}) \text{ and } I \in \mathcal{F}_{T,B}'(\overline{B_R(0)}). \]

Consider a homotopy \( H : [0, 1] \times \overline{B_R(0)} \rightarrow X^* \) given by

\[ H(t, v) := v + tSoT v \text{ for } (t, v) \in [0, 1] \times \overline{B_R(0)}. \]

Let us apply the homotopy invariance and normalization property of the Berkovits degree (which we note \( d \)) introduced in [6], we get

\[ d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1, \]

and hence there exists a point \( v \in B_R(0) \) such that

\[ v + SoT v = 0. \]

\( \square \)
2.2. Functional framework. In the sequel, Ω is a smooth bounded domain in \(\mathbb{R}^N\) (\(N \geq 2\)).

In order to discuss problem (1.1), we start with the definition of the variable exponent Lebesgue spaces \(L^{p(\cdot)}(\Omega)\) and the variable exponent Sobolev spaces \(W_0^{1,p(\cdot)}(\Omega)\), and some properties of them; for more details, see [10, 12].

Let us denote
\[
C_+(\Omega) = \{ h \in C(\Omega) : h(x) > 1 \text{ for every } x \in \Omega \}.
\]

For any \(h \in C_+(\Omega)\), we write
\[
h^- := \min_{x \in \Omega} h(x) , \quad h^+ := \max_{x \in \Omega} h(x).
\]

For any \(p \in C_+(\Omega)\), we define the variable exponent Lebesgue space by
\[
L^{p(\cdot)}(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable and } \rho_{p(\cdot)}(u) < \infty \},
\]
where
\[
\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx.
\]
We consider this space to be endowed with the so-called Luxemburg norm:
\[
\|u\|_{p(\cdot)} = \inf \{ \lambda > 0 : \rho_{p(\cdot)} \left( \frac{u}{\lambda} \right) \leq 1 \}.
\]

We define the variable exponent Sobolev spaces \(W^{1,p(\cdot)}(\Omega)\) by
\[
W^{1,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \}
\]
equipped with the norm
\[
\|u\|_{W^{1,p(\cdot)}} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.
\]

The space \(W_0^{1,p(\cdot)}(\Omega)\) is defined by the closure of \(C_0^\infty(\Omega)\) in \(W^{1,p(\cdot)}(\Omega)\). With these norms, the spaces \(L^{p(\cdot)}(\Omega), W^{1,p(\cdot)}(\Omega)\) and \(W_0^{1,p(\cdot)}(\Omega)\) are separable reflexive Banach spaces.

The conjugate space of \(L^{p(\cdot)}(\Omega)\) is \(L^{\overline{p}(\cdot)}(\Omega)\) where \(\frac{1}{p(x)} + \frac{1}{\overline{p}(x)} = 1\).

For any \(u \in L^{p(\cdot)}(\Omega)\) and \(v \in L^{\overline{p}(\cdot)}(\Omega)\), Hölder inequality holds [12, Theorem 2.1]:
\[
(2.1) \quad \left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p(x)} + \frac{1}{\overline{p}(x)} \right) \|u\|_{p(x)} \|v\|_{\overline{p}(x)} \leq 2 \|u\|_{p(\cdot)} \|v\|_{\overline{p}(\cdot)}.
\]

If \(p(\cdot), q(\cdot) \in C_+(\Omega)\), \(q(\cdot) \leq p(\cdot)\) a.e. in \(\Omega\) then there exists the continuous embedding \(L^{p(\cdot)}(\Omega) \hookrightarrow L^{\overline{p}(\cdot)}(\Omega)\).

In this paper, we suppose that such that \(p(\cdot)\) satisfies the log-Hölder continuity condition, i.e. there exists \(C > 0\) such that for all \(x, y \in \Omega, x \neq y\), one has
\[
(2.2) \quad |p(x) - p(y)| \log \left( e + \frac{1}{|x - y|} \right) \leq C
\]

An interesting feature of generalized variable exponent Sobolev space is that smooth functions are not dense in it without additional assumptions on the exponent \(p(\cdot)\). However, when the exponent satisfies the log-Hölder condition (2.2), we recall the Poincaré
inequality (see [8, Theorem 8.2.4] and [10, Theorem 2.7]): there exists a constant $C > 0$ depending only on $\Omega$ and the function $p$ such that

$$\|u\|_{p()} \leq C\|\nabla u\|_{p()}, \quad \forall u \in W_0^{1,p()}(\Omega),$$

In particular, the space $W_0^{1,p()}(\Omega)$ has a norm given by

$$\|u\|_{1,p()} = \|\nabla u\|_{p()},$$

which is equivalent to the norm $\|\cdot\|_{W_0^{1,p()}}$.

Moreover, the embedding $W_0^{1,p()}(\Omega) \hookrightarrow L^{p'}(\Omega)$ is compact (see [12]). The space $(W_0^{1,p()}(\Omega), \|\cdot\|_{1,p()})$ is also a Banach space separable and reflexive.

The dual space of $W_0^{1,p()}(\Omega)$, denoted $W^{-1,p'}(\Omega)$, is equipped with the norm

$$\|v\|_{-1,p'} = \inf \{ v_0 \mid \|v_0\|_{p'} + \sum_{i=1}^N \|v_i\|_{p} \},$$

where the infimum is taken on all possible decompositions $v = v_0 - divF$ with $v_0 \in L^{p'}(\Omega)$ and $F = (v_1, \ldots, v_N) \in (L^{p'}(\Omega))^N$.

**Proposition 2.2.** [10] Let $(u_n) \subset L^{p'}(\Omega)$ and $u \in L^{p'}(\Omega)$, we have

1. $\|u\|_{p()} > 1 \Rightarrow \|u\|_{p()}^p \leq \rho_{p()}(u) \leq \|u\|_{p()}^p$,
2. $\|u\|_{p()} < 1 \Rightarrow \|u\|_{p()}^p \leq \rho_{p()}(u) \leq \|u\|_{p()}^p$,
3. $\lim_{n \to \infty} \|u_n - u\|_{p()} = 0$ if and only if $\lim_{n \to \infty} \rho_{p()}(u_n - u) = 0$,
4. $\rho_{p()}(u) = \|u\|_{p()} + 1$,
5. $\rho_{p()}(u) \leq \|u\|_{p()}^p + \|u\|_{p()}^p$.

3. **Basic Assumptions and Main Results**

Let $q \in C_+((\Omega))$, $1 < q^- \leq q(x) \leq q^+ < p^+ \leq p(x) \leq p^+ < \infty$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a real-valued function such that:

(f1): $f$ satisfies the Carathéodory condition, i.e. $f(., \eta)$ is measurable on $\Omega$ for all $\eta \in \mathbb{R}$ and $f(x, .)$ is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$.

(f2): $f$ has the growth condition

$$|f(x, \eta)| \leq c_1(k(x) + |\eta|^{r(x)-1})$$

for a.e. $x \in \Omega$ and all $\eta \in \mathbb{R}$, where $c_1$ is a positive constant, $k \in L^{p'(x)}(\Omega)$, $k(x) \geq 0$ and $r \in C_+((\Omega))$ with $2 < r^- \leq r(x) \leq r^+ < p^-$.

Let $A : W_0^{1,p()}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear operator of Leray-Lions, which is defined by

$$A(u) = -diva(x, \nabla u).$$
Lemma 3.2. \( S \) is compact.

Then \( u \) is a weak solution of problem (1.1).

Proof. Thanks to the assumption \( (A_3) \) and the properties of operator \( A \) seen in Lemma 3.2 and in view of Minty-Browder Theorem [19, Theorem 26A], the inverse operator \( T := A^{-1} : W^{-1,q'(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\Omega) \) is bounded, continuous and of type \((S_+)\). Moreover, note from Lemma 3.1 that the operator \( S \) is bounded, continuous and quasismonotone. Therefore, equation (3.2) is equivalent to

\[
(3.3) 
A u = - S u 
\]

To solve equation (3.3), we will apply the Proposition 2.1. It is sufficient to show that the set

\[
(3.4) 
\Lambda := \{ v \in W^{-1,q'(\cdot)}(\Omega) | v + t S o T v = 0 \text{ for some } t \in [0, 1] \} 
\]
is bounded.

Indeed, let \( v \in \Lambda \) and set \( u := Tv \), then \( \|Tv\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)} \).

If \( \|\nabla u\|_{p(\cdot)} \leq 1 \), then \( \|Tv\|_{1,p(\cdot)} \) is bounded.

If \( \|\nabla u\|_{p(\cdot)} > 1 \), then we have by Proposition 2.2

\[
\|Tv\|_{1,p(\cdot)}^{p^-} = \|\nabla u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(\nabla u).
\]

By the assumption \((A_4)\), we have

\[
a(x, \nabla u) \cdot \nabla u \geq c|\nabla u|^{p(x)}.
\]

Then

\[
\rho_{p(\cdot)}(\nabla u) = \int_{\Omega} |\nabla u|^{p(x)} dx \\
\leq \frac{1}{c} \int_{\Omega} a(x, \nabla u) \cdot \nabla u \\
= \frac{1}{c} \langle \Lambda u, u \rangle \\
= \frac{1}{c} \langle v, Tv \rangle \\
= -\frac{t}{c} \langle Sv, Tv \rangle.
\]

This implies that

\[
\rho_{p(\cdot)}(\nabla u) \leq \frac{t}{c} \int_{\Omega} -\lambda |u|^{q(\cdot) - 2} u + f(x, u) u dx.
\]

We get, by the inequalities \((3.3), (3.5)\), the growth condition the growth condition \((f_2)\),
the Hölder inequality \((2.1)\), the inequality \((5)\) of Proposition \((2.2)\) and the Young inequality
the estimate

\[
\|Tv\|_{1,p(\cdot)}^{p^-} \leq \text{const} \lambda \rho_{q(\cdot)}(u) + \int_{\Omega} |k(x) u(x)| dx + \rho_{r(\cdot)}(u) \\
\leq \text{const} (\|u\|_{q(\cdot)}^q + \|u\|_{q(\cdot)}^q + \|k\|_{p(\cdot)}^p \|u\|_{p(\cdot)} + \|u\|_{r(\cdot)}^r + \|u\|_{r(\cdot)}^r) \\
\leq \text{const} (\|u\|_{q(\cdot)}^q + \|u\|_{q(\cdot)}^q + \|u\|_{p(\cdot)} + \|u\|_{r(\cdot)}^r + \|u\|_{r(\cdot)}^r).
\]

From the Poincaré inequality \((2.3)\) and the continuous embedding \(L^{p(x)} \hookrightarrow L^{q(x)}\) and
\(L^{p(x)} \hookrightarrow L^{r(x)}\), we can deduct the estimate

\[
\|Tv\|_{1,p(\cdot)}^{p^-} \leq \text{const} \|Tv\|_{1,p(\cdot)}^{q^+} + \|Tv\|_{1,p(\cdot)} + \|Tv\|_{1,p(\cdot)}^{r^+}.
\]

It follows that \( \{Tv|v \in \Lambda\} \) is bounded.

Since the operator \( S \) is bounded, it is obvious from \((3.3)\) that the set \( \Lambda \) is bounded in
\(W^{-1,p'(\cdot)}(\Omega)\).

Hence, in virtu of Proposition \((2.1)\), the equation \( v + Sv \) have at lest one non trivial
solution \( \bar{v} \) in \(W^{-1,p'(\cdot)}(\Omega)\). So,

\[
\bar{u} = T\bar{v}
\]

is a weak solution of \((1.1)\). □
Next, we consider the uniqueness of solutions of \((1.1)\). To this end, we also need the following hypothesis on the convection term:

\((f_3)\): There exists \(c_2 \geq 0\) such that
\[
(f(x,t) - f(x,s))(t - s) \leq c_2|t - s|^{q(x)}
\]
for a.e. \(x \in \Omega\) and all \(t, s \in \mathbb{R}\).

Our uniqueness result reads as follows.

**Theorem 3.5.** Assume that \((f_1) - (f_3)\) and \((A_1) - (A_4)\) hold. The weak solution of \((1.1)\) is unique provided
\[
\frac{2q^+ c_2}{\lambda} < 1.
\]

**Proof.** Let \(u_1, u_2 \in W_0^{1, p}(\Omega)\) be two weak solutions of \((1.1)\). by choosing \(v = u_1 - u_2\) in the Definition 3.3, we have
\[
\int_{\Omega} a(x, \nabla u_1) \nabla (u_1 - u_2) \, dx + \int_{\Omega} \lambda |u_1|^{q(x)-2}u_1 (u_1 - u_2) \, dx = \int_{\Omega} f(x, u_1)(u_1 - u_2) \, dx
\]
and
\[
\int_{\Omega} a(x, \nabla u_2) \nabla (u_1 - u_2) \, dx + \int_{\Omega} \lambda |u_2|^{q(x)-2}u_2 (u_1 - u_2) \, dx = \int_{\Omega} f(x, u_2)(u_1 - u_2) \, dx
\]
Subtracting the above two equations, we have
\[
\int_{\Omega} (a(x, \nabla u_1) - a(x, \nabla u_2)) \nabla (u_1 - u_2) \, dx + \int_{\Omega} \lambda (|u_1|^{q(x)-2}u_1 - |u_2|^{q(x)-2}u_2)(u_1 - u_2) \, dx
\]
= \(\int_{\Omega} (f(x, u_1) - f(x, u_2))(u_1 - u_2) \, dx\).

By assumption \((A_3)\), we have
\[
(a(x, \nabla u_1) - a(x, \nabla u_2)) \nabla (u_1 - u_2) > 0.
\]

Moreover, since \(q(x) \geq 2\), then we have the following inequality (see [17]):
\[
(|u_1|^{q(x)-2}u_1 - |u_2|^{q(x)-2}u_2)(u_1 - u_2) \geq \left(\frac{1}{2}\right)^{q(x)}|u_1 - u_2|^{q(x)}.
\]

So, by using assumption \((f_3)\), we have
\[
\lambda \left(\frac{1}{2}\right)^{q(x)} \int_{\Omega} |u_1 - u_2|^{q(x)} \, dx \leq \int_{\Omega} (f(x, u_1) - f(x, u_2))(u_1 - u_2) \, dx
\]
\[
\leq c_2 \int_{\Omega} |u_1 - u_2|^{q(x)} \, dx.
\]

Consequently, when \(\frac{2q^+ c_2}{\lambda} < 1\), it follows that \(u_1 = u_2\) and so the solution of \((1.1)\) is unique. □
EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A NONLINEAR EQUATION WITH CONVECTION TERM

REFERENCES

[1] M. Ait Hammou and E. Azroul, Nonlinear Elliptic Problems in Weighted Variable Exponent Sobolev Spaces by Topological Degree, Proyecciones, 38(4) (2019), 733–751.

[2] M. Ait Hammou, E. Azroul, and B. Lahmi, Topological degree methods for a Strongly nonlinear p(x)-elliptic problem, Rev. Colomb. de Mat., 53(1) (2019), 27–39.

[3] A. Abbassi, C. Allalou and A. Kassidi, Topological degree methods for a Neumann problem governed by nonlinear elliptic equation, Moroccan J. of Pure and Appl. Anal., 6(2) (2020), 231–242.

[4] M. V. Abdelkadery and A. Ourraouiz, Existence And Uniqueness Of Weak Solution For p-Laplacian Problem In $\mathbb{R}^N$, Appl. Math. E-Notes, 13 (2013), 228–233.

[5] S.N. Antontsev and J.F. Rodrigues, On stationary thermo-rheological viscous flows, Ann. Univ. Ferrara Sez. VII Sci. Mat., 52 (2006), 19–36.

[6] J. Berkovits, Extension of the Leray-Schauder degree for abstract Hammersteins type mappings, J. Differential Equations, 234 (2007), 289–310.

[7] Y. Chen, S. Levine, and R. Rao, Variable Exponent, Linear Growth Functionals in Image Restoration, SIAM Journal of Applied Mathematics, 66 (2006), 1383–1406.

[8] L. Dingien, P. Harjulehto, P. Hästö, and M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, Springer, Berlin, 2011.

[9] X.L. Fan and Q.H. Zhang, Existence of Solutions for $p(x)$-Laplacian Dirichlet Problem, Nonlinear Anal., 52 (2003) 1843–1852.

[10] N. S. Papageorgiou, E. M. Rocha and V. Staicu, A multiplicity theorem for hemivariational inequalities with a $p$-Laplacian-like differential operator, Nonlinear Anal. TMA, 69 (2008), 1150–1163.

[11] P.S. Iliaş, Dirichlet problem with $p(x)$-Laplacian, Math. Report, 10(60),1 (200)8, 43–56.

[12] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J., 41(1991), 592–618.

[13] D. Nabab and J. Vélin, On a nonlinear elliptic system involving the $(p(x),q(x))$-Laplacian operator with gradient dependence, Complex Variables and Elliptic Equations (2021), DOI: 10.1080/17476933.2021.1885385

[14] F. Thelin, Local regularity properties for the solutions of a nonlinear partial differential equation, Nonlinear Anal. 6 (1982), 839–844.

[15] X. Fan, Q. Zhang and D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, J. Math. Anal. Appl., 302 (2005) 306–317.

[16] E. Zeidler, Nonlinear Functional Analysis and its Applications, II/B: Nonlinear monotone Operators, Springer-Verlag, New York, (1985).

[17] V.V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, Math. USSR Izv., 29 (1987), 33–66.

[18] L. Zhao, P. Zhao and X. Xie, Existence and multiplicity of solutions for divergence type elliptic equation, Electron. J. Differential Equations, 2011(43) (2011), 1–9.

*LABORATORY LAMA
DEPARTMENT OF MATHEMATICS
SIDI MOHAMED BEN ABDELLAH UNIVERSITY
FÉZ
MOROCCO
Email address: mustapha.aithammou@usmba.ac.ma