Pseudo-isotopy classes of diffeomorphisms of the unknotted pairs \((S^{n+2}, S^n)\) and \((S^{2p+2}, S^p \times S^p)\)

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Abstract

We consider two pairs: the standard unknotted \(n\)-sphere in \(S^{n+2}\), and the product of two \(p\)-spheres trivially embedded in \(S^{2p+2}\), and study orientation preserving diffeomorphisms of these pairs. Pseudo-isotopy classes of such diffeomorphisms form subgroups of the mapping class groups of \(S^n\) and \(S^p \times S^p\) respectively and we determine the algebraic structure of such subgroups when \(n > 4\) and \(p > 1\).

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1 Introduction

Let \(M^n\) be a closed, oriented, smooth manifold of dimension \(n\). We denote the group of all orientation preserving diffeomorphisms of \(M\) by \(\text{Diff}(M)\). Recall that two elements of this group, say \(f\) and \(h\), are called pseudo-isotopic if there exists an orientation preserving diffeomorphism \(F \in \text{Diff}(M \times [0,1])\) such that
\( F(x,0) = f(x) \) and \( F(x,1) = h(x) \) for all \( x \in M \). The set of pseudo-isotopy classes forms a group which we will call the mapping class group of \( M \) and denote by \( \pi_0 \text{Diff}(M) \). For some special types of manifolds this group has been computed by several authors, see for example \cite{5}, \cite{15}, \cite{16}, \cite{19}, \cite{23}, \cite{26}, \cite{31}. On the contrary, the classical mapping class group \( M_g \) that is the group of the isotopy classes of orientation preserving diffeomorphisms of an orientable surface \( F^2 \) of genus \( g \) has been studied extensively and there are also results on the so-called spin mapping class group \( \left( \left[ 7 \right], \left[ 8 \right], \left[ 22 \right] \right) \). This last one is a subgroup of \( M_g \) that consists of diffeomorphisms preserving the Rokhlin quadratic form \( \rho : H_1(F^2; \mathbb{Z}_2) \to \mathbb{Z}_2 \) (see next section for the definition). This spin mapping class group has a geometrical meaning: it consists of the diffeomorphisms that extend to diffeomorphisms of the 4-sphere, if one assumes that the closed 2-surface is trivially embedded in \( S^4 \) (see \cite{8} and \cite{22}). From this point of view, there are interesting results of S. Hirose (Theorem A. in \cite{9}) and Z. Iwase (Proposition 3.1 in \cite{12}) who have shown that for a non-trivially embedded 2-torus \( T^2 \hookrightarrow S^4 \) the isotopy classes of diffeomorphisms of \( T^2 \) which extend over \( S^4 \) form a proper subgroup of the spin mapping class group of the torus. One way to obtain such a non-trivial embedding is to take a classical torus knot \( (S^3, K) \), choose a 3-disk \( D^3 \hookrightarrow S^3 \setminus (K \times D^2) \) and construct the spun \( T^2 \)-knot in \( S^4 \) using an analog of the spinning construction:

\[
(S^4, S(K)) := \left( (S^3, K) \setminus \text{int}D^3 \right) \times S^1 \bigcup_{\text{id}} S^2 \times D^2
\]

One could consider 2-surfaces not only in \( S^4 \) but in any other closed orientable 4-manifold and ask similar questions. An attractive family of 2-surfaces in a 4-manifold is the family of non-singular algebraic curves, say in \( \mathbb{C}P^2 \). Theorem 4.3 of \cite{10} claims that for the smooth algebraic curves of odd degree greater than four, the isotopy classes of diffeomorphisms extendable over \( \mathbb{C}P^2 \) form a proper subgroup of the corresponding mapping class group.

Here we are interested in a higher dimensional analog. Let \( M^n, n \geq 4 \) be a closed oriented locally unknotted submanifold of the standard sphere \( S^{n+2} \), in other words, consider an \( n \)-dimensional locally flat differentiable \( M^n \)-knot \((S^{n+2}, M^n)\). The following questions naturally rise:

- Given such a knot, which elements of the mapping class group \( \pi_0 \text{Diff}(M^n) \)
have representatives that extend to orientation-preserving diffeomorphisms of the ambient sphere $S^{n+2}$?

- How does the set of such elements depend on the knot type?

It should be mentioned that similar questions appear in the study of the homotopy type of the complement to an algebraic hypersurface in the complex space. For the details, the reader is referred to a recent work of A. Libgober (see [20], §3.2), and to §5 of [4].

In this paper we answer the first question for two “trivial” knots: $(S^{n+2}, S^n)$ with $n \geq 5$, and $(S^{2p+2}, S^p \times S^p)$ with $p \geq 2$.

Let us define first the subgroup of the mapping class group of $M$, which we will be dealing with. Suppose $M^n$, $n \geq 5$ is a closed simply-connected manifold embedded in the standard sphere $S^{n+m}$, $m \geq 1$. Consider two orientation preserving diffeomorphisms of $M$, say $\phi_1$ and $\phi_2$, which are isotopic. If we assume that $\phi_1$ extends to an orientation preserving diffeomorphism of the ambient sphere, then the isotopy extension theorem (see, for example, [11], ch. 8, Theorem 1.5) guarantees that $\phi_2$ can also be extended to a diffeomorphism of $S^{n+m}$. Moreover, it follows from the disk theorem ([11], ch. 8, Theorem 3.1) that we can always assume the extension is isotopic to the identity map of the ambient sphere (by changing the extension on a small disk embedded far away from $M$). This implies that for a $M^n$-knot $(S^{n+m}, M)$ the following subgroup of $\pi_0\text{Diff}(M)$, denoted by $\mathcal{E}(S^{n+m}, M)$, is well defined (cf. definition of $\mathcal{E}(S^4, K)$ in [8]).

$$\mathcal{E}(S^{n+m}, M) := \{ [\phi] \in \pi_0\text{Diff}(M) \mid \text{there exists } \Phi \in \text{Diff}(S^{n+m}) \text{ s.t. } \Phi|_M = \phi \}$$

When $n \geq 5$, we can replace an isotopy by pseudo-isotopy and the freedom to make such a change is guaranteed by the result of J. Cerf [3].

In the next section we will use a higher dimensional analog of the Rokhlin quadratic form introduced by J. Levine in [17] to generalize the result of J. Montesinos (see [22]) who has shown that for the trivially embedded torus $T^2 \hookrightarrow S^4$ the group $\mathcal{E}(S^4, T^2)$ consists of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ where both products $a \cdot b$ and $c \cdot d$ are even integers. In the last paragraph we study diffeomorphisms of $S^p \times S^p$ which act trivially on the homology and extend over the
ambient sphere $S^{2p+2}$. Isotopy classes of such diffeomorphisms form a subgroup of $\mathcal{E}(S^{2p+2}, S^p \times S^p)$, which will be denoted by $\mathcal{SE}(S^{2p+2}, S^p \times S^p)$. We will write $\phi \sim \varphi$, and $G \cong H$ to say that diffeomorphisms $\phi$ and $\varphi$ are (pseudo) isotopic, and groups $G$ and $H$ are isomorphic respectively. All homology and cohomology groups will have integer coefficients unless otherwise stated.

## 2 Levine - Rokhlin quadratic form.

In this part we discuss a higher dimensional analog of the Rokhlin quadratic form and show that if a diffeomorphism $f \in \text{Diff}(M^n)$ extends to a diffeomorphism of the ambient sphere $S^{n+2}$, then $f_*$ (the induced homology map) commutes with this quadratic form $q$. Definition of such a form $q$ is based on the following construction of Levine ([17], §3).

Let us denote the connected sum of $k$ copies of $S^p \times S^p$ by $M^n$ ($p \geq 3$ is to be odd in this section) and assume that $M^n$ is somehow embedded in $S^{n+2}$. Notice that since $M$ is at least 2-connected, the normal bundle $\nu(S^{n+2}, M^n) \simeq \mathbb{Z}^2$ is trivial for any embedding $M^n \hookrightarrow S^{n+2}$. Hence we can choose a framing $\mathcal{F} = (f_1, f_2)$—two orthonormal vector fields on $M$ to get a framed submanifold $(M, \mathcal{F})$ of $S^{n+2}$. Moreover, since $M$ is simply connected the set of homotopy classes $[M, SO(2)]$ consists of the trivial element only and therefore the choice of our framing $\mathcal{F}$ is unique up to homotopy. For such a framed manifold, Levine (see [17], §4 and also [6], §3.2) has defined a function

$$\phi(M, \mathcal{F}) : H_p(M; \mathbb{Z}) \longrightarrow \mathbb{Z}_2$$

which depends only on the isotopy class of the pair $(M, \mathcal{F})$ and satisfies the formula

$$\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta) + (\alpha \cdot \beta)_2$$

where $\alpha, \beta \in H_p(M; \mathbb{Z})$ and $(\alpha \cdot \beta)_2$ is the mod 2 residue of the intersection number of $\alpha$ and $\beta$. We now recall this definition of Levine in a slightly different form which will suit needs of this paper. Fix an embedding $M^n \hookrightarrow S^{n+2}$ and a framing $\mathcal{F} = (f_1, f_2)$ of the normal bundle $\nu(S^{n+2}, M^n)$. Take a point $pt$ outside a tubular neighborhood of $M$ in $S^{n+2}$ and choose a (positive) framing $\mathcal{G}$ of $\tau(S^{n+2})|_{S^{n+2} \setminus pt}$ (the tangent bundle restricted to the complement of $pt$). Let
$\mathbb{S} \hookrightarrow M$ be an embedded $p$-sphere which represents a class $[z] \in H_p(M;\mathbb{Z})$ and consider the subbundle $E$ of $\tau(S^{n+2})|_S$ which is the Whitney sum of the tangent bundle $\tau(S)$ and the line bundle $f_1|_S$. This subbundle has a canonical framing $\Upsilon_1 = (e_1,\ldots,e_p,e_{p+1})$ which comes from the standard framing of $\mathbb{R}^{p+1}$ (cf. [11], §3.2 and recall that we have fixed an embedding $M^n \hookrightarrow S^{n+2}$). This $\Upsilon_1$ together with the field $f_2|_S$ gives us a framing $\Upsilon_2$ of the sum $E \oplus f_2|_S$ which is a $(p+2)$-subbundle of $\tau(S^{n+2})|_S$. Comparing this framing $\Upsilon_2(x)$ with the framing $G(x)$ at each point $x \in \mathbb{S}$, produces a $(p+2,2p+2)$-matrix with orthonormal row vectors, i.e. an element of the Stiefel manifold $V_{p+2}(\mathbb{R}^{2p+2})$ of orthonormal $(p+2)$-frames in $\mathbb{R}^{2p+2}$. Thus we obtain an element of the homotopy group $\pi_p(V_{p+2}(\mathbb{R}^{2p+2})) \cong \mathbb{Z}_2$ (for the last isomorphism see [22], Theorem IV.1.12). In our case, this construction gives a well defined function

$$\phi(M) : H_p(M;\mathbb{Z}) \longrightarrow \mathbb{Z}_2$$

which depends only on the isotopy class of the embedding $M^n \hookrightarrow S^{n+2}$.

**Proposition 1.** (cf. [22], Proposition 4.1; [8], Theorem 1.2) Let $p = \frac{n}{2} \geq 3$ be odd and $M^n$ be a closed $(p-1)$-connected manifold embedded into $S^{n+2}$ and $[f] \in E(S^{n+2}, M^n)$. Then $f_* : H_p(M) \rightarrow H_p(M)$ commutes with the function $\phi$.

**Proof.** Let $S^p \hookrightarrow M$ be an embedded sphere representing a cycle $[S^p] \in H_p(M,\mathbb{Z})$. Since diffeomorphism $f$ extends to a diffeomorphism of the ambient sphere we could compare two $\Upsilon_2$-framings at each point of $f(S^p)$. This comparison would give us an element of $\pi_p(SO(p+2))$. Since the $(f_1,f_2)$-restrictions of these two framings are homotopic, it follows from the exact homotopy sequence of the fibration $SO(p) \hookrightarrow SO(2p+2) \twoheadrightarrow V_{p+2}(\mathbb{R}^{2p+2})$ that $\phi([f(S^p)]) = \phi([S^p])$. \hfill \Box

**Remark:** If we denote the boundary operator from this exact homotopy sequence by $\partial$, then $\partial(\phi([S^p])) \in \pi_{p-1}(SO(p))$ is the obstruction to trivializing the normal bundle of $S^p$ in $M$ (see [5], Lemma 3.4).

Let us now assume that our $2p$-manifold $M^n \simeq (S^p \times S^p)\#\ldots\#(S^p \times S^p)$ is trivially embedded in $S^{n+1}$, which is embedded in $S^{n+2}$ as an equator: $M^n \hookrightarrow S^{n+1} \hookrightarrow S^{n+2}$. In particular, we assume that $S^{n+1}$ can be presented as the union of two copies of the handlebody $\mathcal{H} \simeq (D^{p+1} \times S^p)\#\ldots\#(D^{p+1} \times S^p)$ (here #
stands for the boundary connected sum) along the boundary $M^n = \partial \mathcal{H}$:

$$S^{n+1} = \mathcal{H}_+ \bigcup_M -\mathcal{H}_-.$$ 

Taking the tensor product of $H_p(M; \mathbb{Z})$ with $\mathbb{Z}_2$ and using the function $\phi$, we obtain a quadratic form $q : H_p(M; \mathbb{Z}_2) \to \mathbb{Z}_2$ which has Arf invariant zero. For a closed orientable 4–manifold $X$ with $H_1(X; \mathbb{Z}_2) \cong 0$ and a closed orientable embedded 2-surface $F \hookrightarrow X^4$ which realizes the element of $H_2(X; \mathbb{Z}_2)$ dual to $w_2(X)$, V. Rokhlin (see [23], §3) has defined a function $\psi : H_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$ which also satisfies the formula $\psi(\alpha + \beta) = \psi(\alpha) + \psi(\beta) + (\alpha \cdot \beta)_2$. The form $\psi$ has Arf invariant zero for a trivially embedded surface too.

We now turn to the case where $M \simeq S^p \times S^p$ is standardly embedded in $S^{2p+2}$. By this we will mean that $M$ in $S^{2p+2}$ is the boundary of a tubular neighborhood of a standardly embedded $S^p \hookrightarrow S^{2p+1}$, where $S^{2p+1}$ is an equator of $S^{2p+2}$ (cf. Definition 1.6 of [21]). It then follows from our Proposition 1 and Propositions 4.2 and 4.3 of [22] that if a diffeomorphism $f \in \text{Diff}(M)$ extends to a diffeomorphism of the pair $(S^{n+2}, M)$ then the induced automorphism $f_*$ is an element of the group, which consists of the matrices $\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ where both products $d_1d_2$ and $d_3d_4$ are even integers. Clearly, any matrix of this type is congruent modulo 2 either to $\text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $V := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and we will denote such a subgroup of $\text{SL}(2, \mathbb{Z})$ by $\Gamma_V(2)$. One can use the fact that the corresponding projective group $\Gamma_V(2) / \mathbb{Z}_2 \cong \mathbb{Z}_2 \ast \mathbb{Z}$ is generated by $V$ and $T := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (cf. [27], §3) to show that $\Gamma_V(2)$ admits the following presentation $\Gamma_V(2) \cong \langle V, T \mid V^4 = \text{Id}, V^2T = TV^2 \rangle$.

Let $h : \pi_0\text{Diff}(M) \to \text{Aut}(H_p(M; \mathbb{Z}))$ be the obvious homomorphism induced by the action on the homology of $M$. We will denote the restriction of this homomorphism onto the subgroup $\mathcal{E}(S^{n+2}, M)$ by $h_\mathcal{E}$. Evidently, $\text{Im}(h_\mathcal{E}) \subset \Gamma_V(2)$. Montesinos has shown (see [22], Theorem 5.4) that if $M$ is a standardly embedded torus $S^1 \times S^1 \hookrightarrow S^3 \hookrightarrow S^4$, then $\mathcal{E}(S^4, M) \cong \Gamma_V(2)$. Here we prove the following generalization of this result.

**Lemma 1.** Let $p \geq 3$ be odd and $S^p \times S^p$ be standardly embedded in $S^{2p+2}$. Then $\text{Im}(h_\mathcal{E}) = \Gamma_V(2)$. 


Proof. We will show that there exist diffeomorphisms \( \phi_V \in \text{Diff}(S^{n+2}, M^n) \) and \( \phi_T \in \text{Diff}(S^{n+2}, M^n) \) such that \( h_{E}[\phi_V] = V \) and \( h_{E}[\phi_T] = T \). Let us start with the matrix \( T \) and consider the following commutative diagram that consists of two exact homotopy sequences of the fibration \( SO(n) \hookrightarrow SO(n+1) \twoheadrightarrow S^n \)

\[
\begin{array}{cccc}
\pi_{p+1}(S^{p+1}) & \quad & \quad & \pi_{p+1}(S^{p+1}) \\
\downarrow & \quad & \mu \times 2 & \quad \downarrow \\
\pi_p(SO(p)) & \to & \pi_p(SO(p+1)) & \to \pi_p(SO(p)) \\
\downarrow & \quad & \downarrow & \quad \downarrow \\
\pi_p(SO(p+2)) & \quad & \quad & \pi_p(SO(p+2)) \\
\end{array}
\]

Since \( p \) is odd, map \( \mu := j_p \circ \partial \) is multiplication by 2 (see [2], Lemma IV.1.9). It follows from this diagram that there exists a smooth map \( m_T : S^p \to SO(p+1) \) such that \( j_p([m_T]) = \pm 2 \) and \( i_{p+1}[m_T] = 0 \). Hence there exists a smooth map (see Smooth Approximation Theorem in §II.11 of [1]) \( \gamma : D^{p+1} \to SO(p+2) \) that extends the composition of \( m_T \) with the inclusion \( \iota : SO(p+1) \hookrightarrow SO(p+2) \), i.e. \( \gamma|_{S^p} = \iota \cdot m_T \). Now we use \( \gamma \) together with \( m_T \) to define the following self-diffeomorphism of \( S^{2p+2} = (D^{p+1} \times S^{p+1}) \cup (S^p \times D^{p+2}) \):

\[
\Phi(x,y) := \begin{cases} (x, \iota \cdot m_T(x) \circ y) & \text{if } (x,y) \in S^p \times D^{p+2} \\ (x, \gamma(x) \circ y) & \text{if } (x,y) \in D^{p+1} \times S^{p+1} \end{cases}
\]

Since the restriction of \( \Phi \) onto the product \( S^p \times S^p \hookrightarrow S^p \times S^{p+1} \) (here \( x \times S^p \) is the equator of \( x \times S^{p+1} \)) is the map \( (x,y) \mapsto (x, m_T(x) \circ y) \) and \( j_p([m_T]) = \pm 2 \), we can define \( \phi_T := \Phi \) to obtain the equality \( h_{E}[\phi_T] = \pm T \). The product \( S^p \times S^p \) bounds a \( S^p \times D^{p+1} \) smoothly embedded in \( S^{2p+2} \). Then it follows from Theorem 1.7 of [21] that the pair \((S^{2p+2}, S^p \times S^p)\) is equivalent to the standard one, where \( S^p \times S^p \) is standardly embedded in the equator \( S^{2p+1} \).

For the other case we consider \( S^p \times S^p \) standardly embedded into the unit sphere \( S^{2p+2} \subset \mathbb{R}^{2p+3} \). Using \( \{x_0, x_1, \ldots, x_{2p+2}\} \) as the coordinates in \( \mathbb{R}^{2p+3} \), we present the equator sphere \( S^{2p+1} = \{(x_0, x_1, \ldots, x_{2p+2}) \mid x_0 = 0 \& x_1^2 + \cdots + x_{2p+2}^2 = 1\} \) as the union \( S^{2p+1} = \mathcal{H}_1 \cup -\mathcal{H}_2 \) where

\[
\mathcal{H}_1 = \{(x_1, x_2, \ldots, x_{2p+2}) \in S^{2p+1} \mid x_1^2 + x_2^2 + \cdots + x_{p+1}^2 \leq \frac{1}{2}\}
\]
and 
\[ \mathcal{H}_2 = \{(x_1, x_2, \ldots, x_{2p+2}) \in S^{2p+1} \mid x_1^2 + x_2^2 + \cdots + x_{p+1}^2 \geq \frac{1}{2}\} \]

and consider a linear map \( \Omega : \mathbb{R}^{2p+3} \longrightarrow \mathbb{R}^{2p+3} \) defined by the square matrix

\[
\Omega := \begin{pmatrix}
(-1)^p & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

of size \((2p + 3) \times (2p + 3)\) (three blocks in the lowest row have sizes \((p + 1) \times 1\), \((p + 1) \times (p + 1)\), and \((p + 1) \times (p + 1)\) correspondingly). \( \Omega \) acts on vectors of \( \mathbb{R}^{2p+3} \) by multiplying a row vector by \( \Omega \) on the right.

Since \( \Omega \) is an orthogonal matrix which maps the unit disk \( D^{2p+3} \) onto itself preserving orientation (\( \det(\Omega) = (-1)^{2p+2} = 1 \)), restriction of \( \Omega \) onto the unit sphere \( S^{2p+2} \) will be an orientation preserving diffeomorphism as well (see §4.4 of [11]). If we denote a point of \( S^p \times S^p \) by \((a, b)\) with \( a = (x_1, \ldots, x_{p+1})\) and \( b = (x_{p+2}, \ldots, x_{2p+2})\), then the restriction of \( \Omega \) onto \( S^p \times S^p = \partial \mathcal{H}_i \) will map \((a, b)\) to the point \((R(b), a)\) where \( R(b) \) stands for the “reflected” point \((-x_{p+2}, x_{p+3}, \ldots, x_{2p+2})\). Such a diffeomorphism of \( S^p \times S^p \) induces in homology the automorphism of the form \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and therefore we can define a diffeomorphism \( \varphi_V \) as the restriction \( \varphi_V := \Omega|_{S^p \times S^p} \).

\(\square\)

Remark: Notice that \( \Omega^4 \), as well as \( \Omega^4|_{S^p \times S^p} \) is the identity map. We will use it later to show that certain short exact sequences split.

**Corollary 1.** Let \( 2 \leq p < q \), and \( S^p \times S^q \) be standardly embedded in \( S^{p+q+2} \). Then \( \text{Im}(h_\varepsilon) \cong \mathbb{Z}_2 \).
Proof. Indeed, it is enough to consider again the union $S^{p+q+1} = \mathcal{H}_1 \cup -\mathcal{H}_2$ where

$$\mathcal{H}_1 := \{(x_1, x_2, \ldots, x_{p+q+2}) \in S^{p+q+1} \mid x_1^2 + x_2^2 + \cdots + x_{p+1}^2 \leq \frac{1}{2}\}$$

and $\mathcal{H}_2$ is defined correspondingly and a linear map $\Omega' : \mathbb{R}^{p+q+3} \rightarrow \mathbb{R}^{p+q+3}$ defined by the obvious formula

$$\Omega'(x_0, x_1, \ldots, x_{p+q+1}, x_{p+q+2}) := (x_0, -x_1, x_2, \ldots, x_{p+1}, -x_{p+2}, x_{p+3}, \ldots, x_{p+q+2}).$$

Restriction of this map onto $S^{p+q+2}$ will be an orientation preserving diffeomorphism that restricts on $S^p \times S^q = \partial \mathcal{H}_i$ to a diffeomorphism defined by the formula $(a, b) \mapsto (R(a), R(b))$, where $a \in S^p$ and $b \in S^q$. Such a diffeomorphism generates the group $\text{Im}(h) \cong \mathbb{Z}_2$ (see Proposition 2.1 of [23]).

Corollary 2. Let $p > 2$ be even, and $S^p \times S^p$ be standardly embedded in $S^{2p+2}$. Then $\text{Im}(h_\varepsilon) = \text{Im}(h) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. This time we consider the orthogonal $(2p + 3) \times (2p + 3)$-matrix

$$\widehat{\Omega} := \begin{pmatrix}
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0
\end{pmatrix}$$

which has order two and $\det(\widehat{\Omega}) = 1$. Restriction of this linear map onto $S^{2p+2}$ will be an orientation preserving diffeomorphism, which extends the interchange of factors in $S^p \times S^p$ ($(a, b) \mapsto (b, a)$ for $(a, b) \in S^p \times S^p$, and we assumed again that the equator is $S^{2p+1} = \mathcal{H}_1 \cup -\mathcal{H}_2$, and $S^p \times S^p = \partial \mathcal{H}_i$). This interchange generates one of the $\mathbb{Z}_2$-copies of $\text{Im}(h)$, and the other copy is generated, as above, by the simultaneous orientation reversal of each factor: $(a, b) \mapsto (R(a), R(b))$ (cf. [29], §1; and [19], §1.2).
3 Trivial action on the homology

Consider a “trivial” $M^n$-knot $(S^{n+2}, M^n)$ with $n \geq 5$ (“triviality” here means that for an equator we have $S^{n+1} \cong M^n \cup W_-$ with $M^n = \partial W_\pm$). Identification of the group $\Theta_{n+1}$ of all homotopy $(n+1)$-spheres with the relative mapping class group $\pi_0 \text{Diff}(D^n, \partial \phi)$ (see [25]), gives a homomorphism $\iota : \Theta_{n+1} \to \pi_0 \text{Diff}(M^n)$ (if one presents a homotopy sphere $\Sigma\phi$ as the union of two $(n+1)$-disks glued together via a diffeomorphism $\varphi \in \text{Diff}(D^n, \partial \phi)$, then $\iota(\Sigma\phi)$ is defined to be the identity outside an embedded $n$-disk $D^n \hookrightarrow M^n$ and $\phi$ on that disk).

**Theorem 1.** If a diffeomorphism $\phi \in \text{Im}(\iota)$ extends to a diffeomorphism $\Phi$ of the trivial $M^n$-knot $(S^{n+2}, M^n)$, then $\phi$ is pseudo-isotopic to the identity.

**Proof.** Take two canonical pairs of the disks $(D_n^{n+3}, D_n^{n+3})$ and $(D_n^{n+3}, D_n^{n+3})$ with the boundary $(S^{n+2}, S^{n+2})$. Push $W_\pm$ along the normal vector field inside $D_n^{n+3}$ in such a way that we get the copy $\tilde{W}_\pm$ of $W_\pm$ embedded in $D_n^{n+3}$ so that $\tilde{W}_\pm \cap S^{n+2} = M^n$. We can assume w.l.o.g. that we have $W_+ \times I \subset D_n^{n+3}$ with $W_+ \times I \cap S^{n+2} = W_+ \times \{0\}$ and $\tilde{W}_+ = M \times I \cup W_+ \times \{1\}$. Repeat this step with $W_-$ to obtain $\tilde{W}_- \subset D_n^{n+3}$. Then we glue together two pairs $(D_+^{n+3}, \tilde{W}_+)$ and $(D_+^{n+3}, \tilde{W}_-)$ via the diffeomorphism $\Phi$ to obtain a spherical knot

$$\left( S^{n+3}, \Sigma_\phi \right) := (D_+^{n+3}, \tilde{W}_+) \cup_{\Phi}(D_-^{n+3}, \tilde{W}_-).$$

(1)

We want to show that the homotopy sphere $\Sigma_\phi$ bounds a topological disk inside $S^{n+3}$. Indeed, since $W_+ \cup W_- = S^{n+1}$ bounds a smooth disk $D^{n+2}$ in $S^{n+2}$, it is clear that the union

$$D_1^{n+2} := W_+ \times I \bigcup_{W_+ \times \{0\}} \left( D^{n+2} \subset D_+^{n+3} \right)$$

is homeomorphic to the $(n+2)$-disk with the boundary $\partial(D_1^{n+2}) = \tilde{W}_+ \cup_M W_-$. As a homeomorphism, map $\phi$ is isotopic to the identity, which implies that the union $\tilde{W}_- \cup \Phi(W_-)$ (obtained also as a result of gluing two pairs (1) together) is homeomorphic to the double $D W_-$ (for a manifold $X$ with boundary the double is defined as the boundary $\partial(X \times I) =: \partial X$). Hence, we can assume that this union bounds in the “lower” hemisphere of $S^{n+3}$ a manifold homeomorphic to
the product \( W_− × I \subset D_{n+3}^n \), and that the intersection of this product with the equator sphere \( S^{n+2} \) is \( \Phi(W_−) \). This implies that the following union

\[
D_{n+2}^n \bigcup_{\Phi(W_−)} W_− × I =: D_{n+2}^n
\]

will be a topological disk in the sphere \( S^{n+3} \) with the boundary

\[
\partial D_{n+2}^n = \tilde{W}_+ \bigcup_{\phi} W_− = \Sigma_φ.
\]

Now it follows from Corollary 4.16 of [24] that the pair \((S_{n+3}^n, \Sigma_{φ}^{n+1})\) is combinatorially unknotted and therefore the complement \( S^{n+3} \setminus \Sigma_{φ}^{n+1} \) has the homotopy type of a circle. Using Theorem III from [14] (cf. also [18]) we conclude that the exotic sphere \( \Sigma_{φ}^{n+1} \) is diffeomorphic to the standard one. This implies that \( φ \) must be isotopic to the identity (cf. [25], §4).

**Corollary 3.** If \( n ≥ 5 \) and \((S^{n+2}, S^n)\) is the unknot, then \( E(S^{n+2}, S^n) \cong Id \).

**Remark 1:** If the trivial \( M^n \)-knot is actually a spherical knot, i.e. \( M^n = S^n \), then it is not hard to see geometrically that the complement \( S^{n+3} \setminus \Sigma_{φ}^{n+1} \) constructed above is a disk bundle over a circle.

**Remark 2:** It is interesting to note that a homotopy \( n \)-sphere \( \Sigma^n \) can be smoothly embedded in \( S^{n+2} \) if and only if it bounds a parallelizable manifold (see [14], §3).

Let us continue now with the following example. Take the product \( S^{p-2} × S^{p-1} \) standardly embedded in \( S^{p-1} × S^{p-1} \) (the \((p - 1)\)-sphere), where \( p ≥ 9 \) and \( p \equiv 6 \) (mod 8), and again present the sphere \( S^{2p-1} = (D^p × S^{p-1}) \cup (S^{p-1} × D^p) \) as the union of two handlebodies. Then the group of the isotopy classes of diffeomorphisms of \( S^{p-2} × S^{p-1} \) which act trivially on the homology is isomorphic to \( \pi_{p-1}(SO(p-1)) \oplus \Theta_{2p-2} \) (see, for example, [20], Theorem 3.10). Since \( p-1 \equiv 5 \) (mod 8), we see from [13] that \( \pi_{p-1}(SO(p-1)) \cong \mathbb{Z}_2 \) and it follows also from [13] that the inclusion \( SO(p-1) \hookrightarrow SO(p) \) induces the trivial homomorphism \( \tau_\ast : \pi_{p-1}(SO(p-1)) → \pi_{p-1}(SO(p)) \). Hence there exists a smooth map \( γ : D^p → SO(p) \) which extends the composition of the generator \( g : S^{p-1} → SO(p-1) \) of \( \pi_{p-1}(SO(p-1)) \) with \( τ \), i.e. \( γ|_{S^{p-1}} = τ \circ g \). Now we
repeat the construction we used to prove Lemma 1. and define a diffeomorphism $\Phi$ of the ambient sphere $S^{2p-1}$ by the formula:

$$\Phi(x, y) := \begin{cases} (\tau(g(y)) \circ x, y) & \text{if } (x, y) \in D^p \times S^{p-1} \\ (\gamma(y) \circ x, y) & \text{if } (x, y) \in S^{p-1} \times D^p \end{cases}$$

The following lemma immediately follows from Corollary 1. and Theorem 1.

**Lemma 2.** Let $S^{p-2} \times S^{p-1}$ be standardly embedded in $S^{2p-1}$, with $p \geq 9$ and $p \equiv 6 \pmod{8}$. Then the following short exact sequence splits

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{E}(S^{2p-1}, S^{p-2} \times S^{p-1}) \overset{h_\varepsilon}{\longrightarrow} \mathbb{Z}_2 \longrightarrow 0.$$ 

Next we generalize this example and study diffeomorphisms of $S^p \times S^p$ which extend over $S^{2p+2}$ and act trivially on $H_p(S^p \times S^p; \mathbb{Z})$.

Isotopy classes of the diffeomorphisms which act trivially on $H_p(S^p \times S^p; \mathbb{Z})$ form a normal subgroup of the mapping class group and will be denoted by $\pi_0\text{SDiff}(S^p \times S^p)$. Theorem 2 of [15] gives the following description of such a subgroup for $M = S^p \times S^p$, $p \geq 3$:

$$0 \longrightarrow \Theta_{2p+1} \overset{\iota}{\longrightarrow} \pi_0\text{SDiff}(M) \overset{\chi}{\longrightarrow} \text{Hom}(H_p(M), \pi_0\text{Diff}(SO(p))) \longrightarrow 0. \quad (2)$$

Here $S\pi_p(SO(p))$ denotes image of the map $\pi_p(SO(p)) \rightarrow \pi_p(SO(p+1))$ induced by the inclusion, and the homomorphism $\chi$ is defined (for $p \geq 4$) as follows (cf. Lemma 1 of [15]): Take any $f \in \text{SDiff}(M)$, represent $x \in H_p(M)$ by a sphere $S^p \hookrightarrow M$ and use an isotopy to make $f|_{S^p} = \text{Id}$. The stable normal bundle $\nu(S^p) \oplus \varepsilon^1$ of this sphere in $M$ is trivial and therefore the differential of $f$ gives an element of $\pi_p(SO(p+1))$. It is easy to see that this element lies in the image of $S$. If $p = 6$ we have $S\pi_p(SO(p)) = 0$, and for all other $p \geq 3$ the groups $S\pi_p(SO(p))$ are given in the following table ([15], p. 644):

| $p \pmod{8}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------------|---|---|---|---|---|---|---|---|
| $S\pi_p(SO(p))$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$ |

When $p$ is odd, the map $S\pi_p(SO(p)) \hookrightarrow \pi_p(SO(p+1)) \rightarrow \pi_p(SO(p+2))$ is a monomorphism (see [30]) and therefore the induced map

$$\text{Hom}(H_p(M), S\pi_p(SO(p))) \rightarrow \text{Hom}(H_p(M), \pi_p(SO))$$
is a monomorphism too. Hence we can describe $\chi([f])[S^p]$ as the class of the stable normal bundle of $S^p \times S^1$ in the mapping torus $M_f$ of the diffeomorphism $f$ (cf. Lemma 2 of [15] and recall that $f|_{S^p} = Id$). Assuming that $f$ extends to a diffeomorphism of $S^{n+2}$, which is isotopic to the identity, we see that $S^p \times S^1 \hookrightarrow S^{2p+2} \times S^1$. Since the normal bundle of $S^p \times S^1$ in $S^{2p+2} \times S^1$ is trivial, and the stable normal bundle of $S^p \times S^1$ in $S^{2p+2} \times S^1$ is also trivial, it implies that the stable normal bundle of $S^p \times S^1$ in $M_f$ will be trivial as well. Thus the homomorphism $\chi$, restricted onto the isotopy classes of diffeomorphisms of $S^p \times S^1$, which extend over $S^{2p+2}$ and act trivially on the homology, is the trivial map for odd $p \geq 5$. This result together with Theorem 1. and the exact sequence (2) implies that in such case $\ker(h) = \{0\}$. If $p = 3$, we have $S\pi_p(SO(p)) \simeq \mathbb{Z}$ and we can identify $\text{Hom}(H_p(M), S\pi_p(SO(p)))$ with $H^p(M)$. Then one can use the Pontrjagin class of the mapping torus $M_f$ instead of $\chi$ (see [15]), to obtain the same conclusion that the kernel of the map $h$ is trivial when $p = 3$ as well.

Let us now turn to the case when $p$ is even and at least 4, and denote the kernel of $h$ by

$$\ker(h) := \mathcal{SE}(S^{2p+2}, S^p \times S^p).$$

The following computations are well known (cf. [30]):

| $p \pmod{8}$ | 0 | 2 | 4 | 6 |
|-------------|---|---|---|---|
| $S\pi_p(SO(p))$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\pi_p(SO(p+1))$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\pi_p(SO(p+2))$ | $\mathbb{Z}_2$ | 0 | 0 | 0 |

It follows from these computations and the paragraph above, that in this case, the image of the restriction of $\chi$ onto the subgroup $\mathcal{SE}(S^{2p+2}, S^p \times S^p)$ is either zero or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

**Lemma 3.** If $p$ is even and $p \geq 4$, then $\mathcal{SE}(S^{2p+2}, S^p \times S^p) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathcal{E}(S^{2p+2}, S^p \times S^p) \cong D_8 \oplus \mathbb{Z}_2$, where $D_8$ stands for the dihedral group of order 8.

**Proof.** Indeed, as we just mentioned above, the restriction of $\chi$ takes values in $\text{Hom}(H_p(M), \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and to prove that this restriction is an epimorphism, it is enough to construct a diffeomorphism of the pair $(S^{2p+2}, S^p \times S^p)$ for each of the generators of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We will use results from [24] to do that. Let
$g : S^p \to SO(p + 1)$ be a smooth map, which represents a nontrivial element of the group $\pi_p(SO(p + 1))$ and such that $\tau \circ g$ is homotopic to the identity ($\tau$ here is the canonical inclusion $SO(p + 1) \hookrightarrow SO(p + 2)$). Homotopy class of this map $g$ has order two for each even $p$, as we just saw in the table above. It then follows from Proposition 3.2 of [25] that the diffeomorphisms of $S^p \times S^p$ defined by the formulas

$$(x, y) \mapsto (x, g(x) \circ y) \quad \text{and} \quad (x, y) \mapsto (g(y) \circ x, y)$$

with $(x, y) \in S^p \times S^p$ are representatives of the generators of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $\tau \circ g \sim 1d$, there exists a smooth map $\gamma : D^{p+1} \to SO(p + 2)$ such that $\gamma |_{\partial D^{p+1}} = \tau \circ g$ and we define an orientation preserving diffeomorphism of the sphere $S^{2p+2}$ by the following formula (in the second case the formula should be modified in the obvious way):

$$\Phi(x, y) := \begin{cases} (x, \tau(g(x)) \circ y) & \text{if } (x, y) \in S^p \times D^{p+2} \\ (x, \gamma(x) \circ y) & \text{if } (x, y) \in D^{p+1} \times S^{p+1} \end{cases}$$

Thus we obtain two required diffeomorphisms of the pair $(S^{2p+2}, S^p \times S^p)$, and also exactness of the following sequence:

$$0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathcal{E}(S^{2p+2}, S^p \times S^p) \xrightarrow{h} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0$$

Recall that generators of $\text{Im}(h)$ are the simultaneous orientation reversal: $(x, y) \mapsto (R(x), R(y))$, and the interchange of factors: $u : (x, y) \mapsto (y, x)$. It is not hard to see that the orientation reversal acts trivially by conjugation on $\ker(h)$. Hence one of the $\mathbb{Z}_2$-factors of $\text{Im}(h)$ splits off. As for the conjugation by $u$, we have

$$(x, y) \mapsto u \cdot \delta_1 \cdot u : (y, x) \mapsto (y, g(y) \circ x) \mapsto u \cdot (g(y) \circ x, y),$$

that is $\delta_2 = u \cdot \delta_1 \circ u$. It implies that the factor group $\mathcal{E}(S^{2p+2}, S^p \times S^p)/\mathbb{Z}_2$ admits the following presentation: $\langle a, b, u | a^2 = b^2 = u^2 = e, ab = ba, au = ub \rangle$. We leave it as an exercise to show that this presentation gives the dihedral group of order eight.

Let us now summarize what has been proved and state
Theorem 2. Let $S^p \times S^p$ be standardly embedded in $S^{2p+2}$, and $p \geq 3$. Then

$$\mathcal{E}(S^{2p+2}, S^p \times S^p) \cong \begin{cases} 
D_8 \oplus \mathbb{Z}_2 & \text{if } p \text{ is even} \\
\Gamma_V(2) & \text{if } p \text{ is odd}
\end{cases}$$

Remark 1: The group of pseudo-isotopy classes of diffeomorphisms of $S^2 \times S^2$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (cf. [29], §2). Our proof of Corollary 2 implies that each of the generators has a representative extendable to a diffeomorphism of $S^6$.

Remark 2: It is interesting to note that the group $\mathcal{E}(S^{2p+2}, S^p \times S^p)$ is isomorphic to $\Gamma_V(2)$ for all odd $p \geq 1$ and the “Hopf invariant one” cases (i.e. when $p = 1$, 3 or 7) play no special role here (cf. [19], §1.2, for example).

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