GLOBAL DYNAMICS ABOVE THE GROUND STATE ENERGY
FOR THE ENERGY-CRITICAL KLEIN-GORDON EQUATION

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ABSTRACT. Consider the focusing energy-critical Klein-Gordon equation in dimension \(d \in \{3, 4, 5\}\)

\[
\begin{cases}
\partial_{tt} u - \Delta u + u = |u|^{\frac{4}{d-2}} u \\
u(0, x) := f_0(x) \\
\partial_t u(0, x) := f_1(x)
\end{cases}
\]

with data \((f_0, f_1) \in \mathcal{H} := H^1 \times L^2\). We describe the global dynamics of real-valued solutions of which the energy is slightly larger than that of the ground states. We classify the flows of the solutions that are ejected from a small neighborhood of the ground states or that are away from them. The classification relies upon a modification of the arguments in [29] to prove blow-up in finite time, and a modification of the arguments in [10, 17, 21, 18] to prove scattering as \(t \to \pm \infty\). There are three main differences between this paper and [18]. The first one is the lack of scaling symmetry. The second one appears in the proof of the ejection lemma: one has to control the mass in the ejection process. The third one appears in the proof of the one-pass lemma: in the worst scenario, one cannot use the equipartition of energy and therefore one has to prove a decay estimate which allows to use an argument in [5].

1. Introduction

In this paper we consider the energy-critical Klein-Gordon equation on \(\mathbb{R}^d\)

\[
\partial_{tt} u - \Delta u + u = |u|^{2^* - 2} u
\]

with data \((u(0), \partial_t u(0)) := (f_0, f_1)\) with \(\mathcal{H} := H^1 \times L^2\). Here \(2^* := \frac{2d}{d-2}\) denotes the critical Sobolev exponent (with \(d\) the number of dimensions of the space), and \(H^1\) denotes the standard inhomogeneous Sobolev space, i.e \(H^1\) is the completion of the Schwartz space with respect to the norm \(\|f\|_{H^1} := \|f\|_{L^2} + \|\nabla f\|_{L^2}\).

In this paper we restrict our attention to (strong) real-valued solutions of (1.1), i.e real-valued functions \(\tilde{u} := (u, \partial_t u) \in \mathcal{C} (I, H^1) \times \mathcal{C} (I, L^2)\) that satisfy the integral equation below on an interval \(I\) containing 0

\[
u(t) = \cos (t \langle D \rangle) f_0 - \frac{\sin (t \langle D \rangle)}{\langle D \rangle} f_1 - \int_0^t \sin \left( (t-t') \langle D \rangle \right) \left( |u|^{2^*-2} (t') u(t') \right) dt'.
\]

It is well-known that (strong) real-valued solutions \(\tilde{u}\) of (1.1) enjoy the following energy conservation law
The energy-critical Klein-Gordon equation (1.1) is closely related to the energy-critical wave equation, that is
\[ \partial_t u - \Delta u = |u|^{2^*-2} u. \]

We recall some properties of (1.4). It is well-known that of (strong) real-valued solutions \( \bar{u} := (u, \partial_t u) \) of (1.4) satisfy the following energy conservation law
\[ E_{wa}(\bar{u}) := \frac{1}{2} \int_{\mathbb{R}^d} |\partial_t u(t, x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^d} |u(t, x)|^{2^*} \, dx. \]
We can write (1.4) in the Hamiltonian form
\[ \partial_t \bar{u} = \mathcal{J} E'_{wa}(\bar{u}), \text{ with } \mathcal{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
Here \( E'_{wa}(\bar{v}) \) defined in the following fashion\(^1\)
\[ \langle E'_{wa}(\bar{v}), \bar{h} \rangle := \left( \partial_\lambda E_{wa}(\bar{v} + \lambda(h_1, 0))|_{\lambda=0}, \partial_\lambda E_{wa}(\bar{v} + \lambda(0, h_2))|_{\lambda=0} \right) \]
\[ = \langle (-\Delta v_1 + |v_1|^{2^*-2} v_1, v_2), \bar{h} \rangle. \]
The symplectic form \( \omega \) associated to this Hamiltonian system is defined by
\[ \omega(u, v) := \langle \mathcal{J} u, v \rangle, \]
We now recall the definition of some object that appear in the statement of our theorem. The equation (1.4) admits a family of stationary solutions \( \pm S \) (or, equivalently, ground states) described by the translation parameter \( c \in \mathbb{R}^d \) and the scaling parameter \( \sigma \in \mathbb{R}^\ast \)
\[ \pm S := \pm \left\{ T^c S^\sigma \bar{W}, \bar{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \right\} \]
Here \( T^c \) and \( S^\sigma \) denote the following operators
\[ T^c(\bar{f}) := \bar{f}(\cdot - c), \]
\[ S^\sigma(\bar{f}) := (S^\sigma_1 f_1, S^\sigma_0 f_2) := \left( e^{\sigma(\frac{d}{2} - 1)} f_1(e^{\sigma \cdot}), e^{\sigma \cdot} f_2(e^{\sigma \cdot}) \right), \]
with \( \bar{f} := \mathbb{R}^d \to \mathbb{R}^2 : x \to (f_1(x), f_2(x)) \) a vector-valued function. Here \( W \) satisfies
\[ -\Delta W = W^{2^*-1}. \]
It is well-known that \( W(x) = \frac{1}{(1 + |x|^{2^*})^{\frac{d-2}{4}}} \).

The local well-posedness theory of (1.1) is well-known. A consequence of this theory is that one can construct a maximal time interval of existence \( I(u) := \frac{1}{|x|^{2^*}} \).

\(^{1}\)i.e real-valued functions \( \bar{u} := (u, \partial_t u) \in \mathcal{C}(I, \mathcal{H}) \times \mathcal{C}(I, L^2) \) that satisfy on an interval \( I \) containing 0 (\( I \))

\(^{2}\) Recall that the natural dot product on \( L^2 \) is given by \( \langle f, g \rangle := \int_{\mathbb{R}^d} f g \, dx \)
Remark 1.2. Observe that if \( \vec{\varphi} \) is close to \( \vec{f} \) in some sense, then the solutions exist globally in time and scatter. The classification of the global behavior of solutions of \((1.1)\) for energies that are slightly above the ground states’ energy was proved in \([13]\) for energies that are slightly above the ground states’ energy. To this end we define for \( \epsilon > 0 \)

\[
\mathcal{H}^\epsilon := \left\{ \vec{\varphi} \in \mathcal{H}, \ E(\vec{\varphi}) < E_{\text{wca}}(\vec{W}) + \epsilon^2 \right\}.
\]

We also define for \( \mu > 0 \) the set

\[
\mathcal{B}^\mu := \left\{ \vec{\varphi} \in \mathcal{H}, \ d_S(\vec{\varphi}) \leq \mu \right\}.
\]

Here \( d_S(\vec{f}) \) denotes the distance of \( \vec{f} \) to \( \mathcal{S} \), i.e.

\[
d_S(\vec{f}) := \inf_{(\sigma, c) \in \mathbb{R} \times \mathbb{R}^N} \left\| \vec{f} - T_c \vec{S}_\sigma(\vec{W}) \right\|_{\mathcal{H}},
\]

with \( \mathcal{H} := \dot{H} \times L^2 \).

We now state the main result of this paper:

**Theorem 1.1.** Let \( d \in \{3, 4, 5\} \). There exist \( 0 < \epsilon_* \ll 1 \), \( 0 < \mu = O(\epsilon_*) \), a non-empty set \( \mathcal{T} \subset \mathcal{B}^\mu \cap \mathcal{H}^\epsilon \), and a continuous functional \( \Theta : \mathcal{H}^\epsilon \setminus \mathcal{T} \to \{\pm 1\} \) such that for any solution \( u \) of \((1.1)\) with real data \( \vec{\varphi} \in \mathcal{H}^\epsilon \) that has maximal time interval of existence \( I(u) \) the following properties hold:

1. \( I_0(u) := \{ t \in I(u) : \vec{u}(t) \in \mathcal{T} \} \) is an interval, \( I_+(u) := \{ t \in I(u) : \vec{u}(t) \notin \mathcal{T}, \ \Theta(\vec{u}(t)) = 1 \} \) consists of at most two infinite intervals, and \( I_-(u) := \{ t \in I(u) : \vec{u}(t) \notin \mathcal{T}, \ \Theta(\vec{u}(t)) = -1 \} \) consists of at most two finite intervals;

2. \( u \) scatters to a solution of the linear Klein-Gordon equation as \( t \to \pm \infty \) if and only if \( \pm t \in I_+(u) \) for large \( t > 0 \), and, moreover,

\[
\| u \|_{L^\infty L^{2(d+1)}} < \infty;
\]

3. For each \( \sigma_1, \sigma_2 \in \{\pm\} \), let \( A_{\sigma_1, \sigma_2} \) be the collection of initial data \((u_0, u_1)\) in \( \mathcal{H}^\epsilon \) such that for some \( t_1 < 0 < t_2 \)

\[
(-\infty, t_1) \cap I(u) \subset I_{\sigma_1}(u), \quad (t_2, \infty) \cap I(u) \subset I_{\sigma_2}(u).
\]

Then \( A_{\sigma_1, \sigma_2} \neq \emptyset \).

**Remark 1.2.** Observe that if \( \vec{\varphi} := (\phi_1, \phi_2) \in \mathcal{T} \) then not only \( d_S(\vec{\varphi}) \lesssim \epsilon_* \) (in other words \( \vec{\varphi} \) is close to \( \mathcal{S} \) with \( O(\epsilon_*) \) distance in \( \mathcal{H} \)) but also the mass is small: more precisely \( \| \phi_1 \|_{L^2} \lesssim \epsilon_* \).

\(^3\)For a definition of \( E(\vec{\varphi}) \), see Subsection [2.7]
Now we explain the main ideas of this paper and how it is organized.

In order to prove Theorem 1.1 we use a strategy based upon two lemmas: the ejection lemma and the one-pass lemma. This strategy was designed by the Nakanishi and Schlag in [24] in the study of the global dynamics above the ground states of solutions of the focusing Klein-Gordon equation with a focusing and energy-subcritical power-type nonlinearity (namely the cubic nonlinearity). It was successfully applied to a focusing Schrödinger equation with a energy-subcritical cubic nonlinearity [26], to the focusing energy-critical wave equation with radial data [19] and nonradial data [18], and to the focusing energy-critical Schrödinger equation with radial data [27].

The ejection lemma (see Proposition 3.11) aims at describing the dynamics of the solution if it is close to \( S \) in \( \dot{H} \) and if it moves away from it. In the ejection mode, we would like the dynamics be ruled by the unstable eigenmode of the spectral decomposition of the remainder resulting from the linearization around \( S \). But this can only be done if we can control the dynamics of the orthogonal component \( \vec{\gamma} \). The dynamics of \( \vec{\gamma} \) is estimated by that of the quadratic expansion of the energy along the orthogonal direction of the remainder, provided that some orthogonality conditions are satisfied. In [18], the authors perform the decomposition (3.1) in order to deal with the solutions of the energy-critical wave equation close to \( S \). The decomposition involves two parameters (the translation parameter \( c \) and the scaling parameter \( \sigma \)) that evolve as time goes by so that two orthogonality conditions hold, namely (3.2) and (3.3). In our work, one has to control extra terms (such as \( \text{Res}(t) \)) that appear due to the “\( u \)” term in (1.1). We perform this task by relating the dynamics of these terms to that of the mass of the solution, then by controlling the growth of the mass through again the variation of the quadratic expansion of the energy along the orthogonal component. At the end of the ejection, we prove that the dynamics of the solution is dominated by the exponential growth of the unstable eigenmode; moreover, a relevant functional (denoted by \( K \)) grows exponentially and its sign eventually becomes opposite to that of the eigenmode.

The one-pass lemma (see Proposition 3.14) aims at classifying the flow of the solution as the nonlinear distance is in the ejection mode. It shows that the nonlinear distance cannot be at two different times (say \( t_2 \) and \( t_3 \)) too small and in the ejection mode. The proof is by contradiction. Assuming that it is false, then we can apply the ejection lemma whenever the nonlinear distance is small and then variational estimates (see Proposition 3.6) whenever the nonlinear distance is large. The contradiction appears when we integrate by part the virial identity (8.5). The left-hand side is much smaller than the right-hand side thanks to the exponential growth of the functional \( K \) in the ejection mode and variational estimates far from \( S \). The process involves a parameter \( m \). We mention the main differences between our work and the previous works. First notice that unlike the energy-subcritical focusing cubic Klein-Gordon equation there is an additional parameter, i.e the scaling parameter. Notice also that, unlike the energy-critical wave equation (see [19] [18]), the energy-critical Klein-Gordon equation does not have any scaling property. Because of this lack of symmetry one cannot restrict our analysis to a narrower range.
of values of this parameter. Consequently one has to perform an analysis taking into account all the possible values of this parameter. In the case where \( \Theta(u) = 1 \), the worst scenario is the following: one has degeneracy of \( K \), i.e. the kinetic part is small on average in the region where the nonlinear distance is large. Notice that for the energy-critical wave equation (see [18]) a localized version of the equipartition of energy allowed to prove that this scenario never occurs, more precisely we always have

\[
\int_{t_2}^{t_3} \int_{\mathbb{R}^N} |\nabla u(t)|^2 \, dx \, dt \geq t_3 - t_2.
\]

Notice also that if we were to apply this strategy to the energy-critical Klein-Gordon equation, the same argument would yield

\[
\int_{t_2}^{t_3} \int_{\mathbb{R}^d} |\nabla u(t)|^2 + |u(t)|^2 \, dx \, dt \geq t_3 - t_3,
\]

which would not be sufficient to prevent this scenario from occurring. Instead, we proceed as follows: we prove a decay estimate (see (8.15)) that allows to use an argument in [5]. More precisely, using to our advantage this decay estimate, we prove a propagation estimate involving a frequency localized part of the solution and dispersive estimates to construct a solution \( \tilde{w} \) that is initially close to our solution and such that its energy is smaller than the energy of the ground states; then, applying the theory for energies below that the ground states (see [10]), one can prove that \( \tilde{w} \) is far from \( S \) and, applying a perturbation argument, we see that the nonlinear distance cannot be too small at \( t_2 \) or \( t_3 \), which is a contradiction.

The fate of the solution depends on the value of \( \Theta(u) \) once it is ejected. In the case where \( \Theta(u) = -1 \) after ejection, we prove the existence of blow-up in finite time by modifying an argument due to Payne-Sattinger [29]. If \( \Theta(u) = 1 \), then we prove that there is scattering: see Section 5.2. The proof is proved by a modification of an approach from [16]. Assuming that scattering fails, then one can find a critical level of energy above which scattering does not hold for solutions that satisfy \( \Theta(u) = 1 \) and such that the nonlinear distance is large. But this means that there exists a sequence of solutions \( (u_n)_{n \geq 1} \) of the energy-critical Klein-Gordon equation (1.1) that satisfy the properties that we have just mentioned (in fact, the nonlinear distance can be upgraded from large to very large, by applying to the ejection lemma). Next we use a linear profile decomposition at an appropriate time (see Proposition 9.1) that is proved in [11]. The process involves a list of sequences of scaling parameters \( \{h_n^j\}_{n \in \mathbb{N}, j \in \mathbb{N}} \) that are either equal to one or tend to zero as \( n \) goes to infinity. We then construct for each \( j \) a nonlinear profile depending on the sequence \( \{h_n^j\}_{n \in \mathbb{N}} \). If \( h_n^j = 1 \) then the nonlinear profile is obviously a solution of (1.1); if \( h_n^j \) approaches zero as \( n \) grows, then we prove that the nonlinear profile is a good approximation of a solution of the energy-critical wave equation. Again, in the latter case, one has to deal with the lack of scaling property for the energy-critical Klein-Gordon equation. We manage to overcome this issue by proving some estimates that are uniform throughout the process. Finally, the theory below the ground states for the energy-critical wave equation [17], that for the energy-critical Klein-Gordon equation [11], and that just above the ground states for the energy-critical wave equation [18], one can construct after some work a critical element \( U_c \) that is a

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4 Here \( t_2 \) and \( t_3 \) are defined in Proposition 3.14 and its proof.
solution of \([\text{11}]\), that does not scatter, has an energy equal to the critical level of energy, satisfies \(\Theta(U_c) > 0\), and such that its nonlinear distance is large. Moreover its flow is precompact modulo some translation parameter. Finally, the arguments in \([\text{11}]\) (see also \([\text{17}]\)) allow to prove that this critical element does not exist.

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2. **Notation**

In this section, we set up some notation that appear in this paper.

2.1. **General Notation.** If \(y\) is a real number, then \(y^+\), \(y^-\) is a slightly larger, smaller number respectively. If \(x \in \mathbb{R}\) then \(\text{sign}(x) := 1\) if \(x \geq 0\) and \(\text{sign}(x) := -1\) if \(x < 0\). Let \(\vec{f} := \left(f_1, f_2\right)\) be a vector-valued function that is made of two real-valued functions: \(f_1\) and \(f_2\). If \(\vec{u}\) is a real-valued solution of \([\text{11}]\) or a solution of (1.4) then we write \(u_1 := u\), \(u_2 := \partial_t u\), and \(\vec{u} := (u, \partial_t u)\). Unless otherwise specified, we do not mention specify in the sequel the spaces to which the functions belong.

2.2. **Paley-Littlewood decomposition.** Some estimates that we establish throughout this paper require the Paley-Littlewood technology. We set it up now. Let \(f\) be a function defined on \(\mathbb{R}^d\). Let \(\phi(\xi)\) be a real, radial, nonincreasing function that is equal to 1 on the unit ball \(\{\xi \in \mathbb{R}^N : |\xi| \leq 1\}\) and that is supported on \(\{\xi \in \mathbb{R}^N : |\xi| \leq 2\}\). Let \(\psi\) denote the function \(\phi(\xi) := \phi(\xi) - \phi(2\xi)\). If \(\alpha \in \mathbb{R}^+\) then we define the Paley-Littlewood operators \(P_\alpha f\), \(P_{<\alpha} f\), and \(P_{\geq\alpha} f\) in the Fourier domain by

\[
\begin{align*}
P_\alpha f(\xi) & := \psi(\frac{\xi}{\alpha}) \hat{f}(\xi), \\
P_{<\alpha} f(\xi) & := \phi(\frac{\xi}{\alpha}) \hat{f}(\xi), \text{ and} \\
P_{\geq\alpha} f(\xi) & := \hat{f}(\xi) - P_{<\alpha} f(\xi).
\end{align*}
\]

We also define \(P_0 f\) by the formula \(P_0 f(\xi) := \phi(\xi) \hat{f}(\xi)\). Observe that we can write \(f = P_0 f + \sum_{M \in 2^n} P_M f\): this decomposition is referred to as the Paley-Littlewood decomposition of \(f\).

It is well-known that there exists \(\bar{C} > 0\) such that \(\|P_0 f\|_{L_2^*} \leq \bar{C}\|f\|_{L_2^*}\) and \(\|P_M f\|_{L_2^*} \leq \bar{C}\|f\|_{L_2^*}\). Hence, by defining the “normalized” Paley-Littlewood projectors \(\tilde{P}_0 f := \bar{C}^{-1} P_0 f\) and \(\tilde{P}_M f := \bar{C}^{-1} P_M f\), we get \(\|\tilde{P}_0 f\|_{L_2^*} \leq \|f\|_{L_2^*}\) and \(\|\tilde{P}_M f\|_{L_2^*} \leq \|f\|_{L_2^*}\). We also have \(f = \bar{C} \left(\tilde{P}_0 f + \sum_{M \in 2^n} \tilde{P}_M f\right)\): this decomposition is referred to as the “normalized” Paley-Littlewood decomposition of \(f\).

\(^5\)We will use these inequalities in Subsection 8.3: see the estimate \(I\left(\chi_R(0) P_{<M_0^*} u(0)\right) \leq I(u(0))\)


2.3. **Operators.** We define some operators that we use throughout this paper.

Let $f$ be a function defined on $\mathbb{R}^d$. Let $(\sigma, a) \in \mathbb{R}^2$ and $S_a^\sigma$ be the operator defined as follows

$$S_a^\sigma f(x) := e^{(\frac{d}{2} + a)\sigma} f(e^{\sigma}x)$$

Let $S_a := \partial_a S_a^\sigma (\sigma = 0)$. We define the operator $\tilde{A} := \left( S_1', S_0' \right)$. Let $(S_a^\sigma)^*$ be the adjoint of $S_a^\sigma$. It is easy to see that $(S_a^\sigma)^* = S_{-a}^\sigma$. By differentiating $\langle S_a^\sigma \phi_1, \phi_2 \rangle = \langle \phi_1, (S_a^\sigma)^* \phi_2 \rangle$ with respect to $\sigma$ at $\sigma = 0$ we see that

$$\langle S_a^\sigma \rangle = -S_{-a}^\sigma$$

Let $\tilde{A}^\ast := \left( S_{-1}', S_0' \right)$.

We will work with families of operators. Given $(h_n, a_n) \in (0, 1] \times \mathbb{R}^d$ let $\tau_n^j$, $T_n^j$ and $\langle D \rangle_n^j$ denote the scaled time shift, the unitary and self-adjoint operators, that is

$$\tau_n^j := \frac{t_j}{h_n}, \quad T_n^j f(x) := (h_n)^{-\frac{d}{2}} f \left( \frac{x - x_j}{h_n} \right), \quad \text{and} \quad \langle D \rangle_n^j := \sqrt{-\Delta + (h_n)^2}.$$

2.4. **Norms and Function spaces.** With $h_n^j$ defined above let $H_{h_n,j} := H_{h_n,j} \times L^2$ where $H_{h_n}$ is the closure of the Schwartz space with respect to the norm

$$\|f\|_{H_{h_n}} := \|\langle D \rangle_n f\|_{L^2}.$$  

Let $\sigma \in \mathbb{R} - \{0\}$ and $\bar{q} \geq 1$. Let $M_n^j$ be the smallest dyadic number such that $M_n^j \geq h_n^j$. With $h_n^j$ defined above we define the following quantity

$$\|f\|_{B_{\bar{q},2}^{\sigma,h_n^j}} := (h_n^j)^{-\sigma} \left\| P_{\leq h_n^j} f \right\|_{L^q} + \left( \sum_{M \in \mathbb{Z}^2 : M \geq M_n^j} M^{2\sigma} \|P_M f\|_{L^q}^2 \right)^{\frac{1}{2}}.$$  

Observe that if $h_{n,j} = 1$ then $\|f\|_{B_{\bar{q},2}^{\sigma,h_{n,j}}} = \|f\|_{B_{\bar{q},2}^{\sigma}},$ with $B_{\bar{q},2}^{\sigma}$ belonging to the class of well-known inhomogeneous Besov spaces.

2.5. **Sobolev-type embedding.** We recall some Sobolev-type embeddings that we use throughout this paper.

Let $f$ be a function depending on $\mathbb{R}^d$. Then we have

$$\|f\|_{L^2} \lesssim C\|f\|_{\dot{H}^1}.$$  

It is known (see [11, 34]) that $W$ is an extremizer for the inequality (2.2), i.e.

$$C_* := \sup \left\{ \frac{\|f\|_{L^2}}{\|f\|_{\dot{H}^1}} : f \in \dot{H}^1, f \neq 0 \right\} = \frac{\|W\|_{L^2}}{\|W\|_{\dot{H}^1}}.$$
2.6. Dispersive estimates and nonlinear estimates. We recall dispersive estimates and nonlinear estimates that we will use in this paper.

Let $f$ and $g$ be two functions depending on $\mathbb{R}^d$. Let $F(f) := |f|^{2^*-2}$. Let $(R, \bar{R}) \in \left( B_{\frac{1}{2(d+1)}}, B_{\frac{1}{d+1}}, J \right)^2 \cap B_{\frac{1}{2(d+1)}}, B_{\frac{1}{d+1}}, J \right)$. Then the nonlinear estimates read as follows

\[
\|F(f)\|_R \lesssim \|f\|_{\bar{R}} \|f\|^{2^*-2}_{L^{\frac{2}{2^*-2}}(\mathbb{R}^d)},
\]

and

\[
\|F(f) - F(g)\|_R \lesssim \|f - g\|_{\bar{R}} \left( \|f\|^{2^*-2}_{L^{\frac{2}{2^*-2}}(\mathbb{R}^d)} + \|g\|^{2^*-2}_{L^{\frac{2}{2^*-2}}(\mathbb{R}^d)} \right) + \|f - g\|_{L^{\frac{2d+1}{d+1}}(\mathbb{R}^d)} \left( \|f\|^{2^*-3}_{L^{\frac{2d+1}{d+1}}(\mathbb{R}^d)} + \|g\|^{2^*-3}_{L^{\frac{2d+1}{d+1}}(\mathbb{R}^d)} \right)
\]

We will combine the estimates above with the Strichartz-type estimates below with $(q, r)$ two numbers that satisfy $q > 2$ and $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - 1$.

Let $w$ be a solution of $\partial_t w - \triangle w + w = F + G$ on $J = [a, \cdot]$. Recall the Duhamel formula, that is $w(t) = w_{l,a}(t) + w_{nl,a}(t)$ with $w_{l,a}(t) := \cos ((t - a)(D))w(a) + \frac{\sin ((t - a)(D))}{D} \partial_t w(a)$ and $w_{nl,a}(t) := -\int_a^t \sin \left( (t - \xi)(D) \right) \left( F(\xi') + G(\xi') \right) d\xi'$. $w_{l,a}$ (resp. $w_{nl,a}$) denotes the linear part (resp. the nonlinear part of $w$ starting from $a$; moreover the following wave-type Strichartz estimate holds

\[
\left\| (w, \partial_t w) \right\|_{L_t^q H^r(J)} + \left\| w \right\|_{L_t^q L_x^r(J)} + \left\| \partial_t w \right\|_{L_t^q L_x^r(J)} \lesssim \left\| (w(a), \partial_t w(a)) \right\|_{H^r} + \left\| F \right\|_{L_t^q L_x^r(J)} + \left\| G \right\|_{L_t^q L_x^r(J)}
\]

Let $w$ be a solution of $\partial_t w - \triangle w + w = F + G$ on $J = [a, \cdot]$. Recall the Duhamel formula that is $w(t) = w_{l,a}(t) + w_{nl,a}(t)$ with $w_{l,a}(t) := \cos ((t - a)(D))w(a) + \frac{\sin ((t - a)(D))}{D} \partial_t w(a)$ and $w_{nl,a}(t) := -\int_a^t \sin \left( (t - \xi)(D) \right) \left( F(\xi') + G(\xi') \right) d\xi'$. $w_{l,a}$ (resp. $w_{nl,a}$) denotes the linear part (resp. the nonlinear part of $w$ starting from $a$; moreover the following Strichartz-type estimate holds

\[
\left\| (w(t), \partial_t w(t)) \right\|_{L_t^q H^r(J)} + \left\| (w(t), \partial_t w(t)) \right\|_{L_t^q L_x^r(J)} + \left\| (w, \partial_t w) \right\|_{L_t^q L_x^r(J)} \lesssim \left\| (w(a), \partial_t w(a)) \right\|_{H^r} + \left\| F \right\|_{L_t^q L_x^r(J)} + \left\| G \right\|_{L_t^q L_x^r(J)}
\]

If $w$ satisfies $\partial_t w - \triangle w + (h_{n,j})^2 w = F + G$ on $J = [a, \cdot]$ then we can write similarly $w(t) = w_{l,a}(t) + w_{nl,a}(t)$ with $w_{l,a}(t) := \cos ((t - a)(D)^{\frac{1}{2}})w(a) + \frac{\sin ((t - a)(D)^{\frac{1}{2}})}{(D)^{\frac{1}{2}}} \partial_t w(a)$ and $w_{nl,a}(t) := -\int_a^t \sin \left( (t - \xi)(D) \right) \left( F(\xi') + G(\xi') \right) d\xi'$. Moreover we have
Observe that

(2.8)

\[ \| (w(t), \partial_t w) \|_{L_t^\infty H_{\tilde{h}_n}^s(J)} + \| w \|_{L_t^2 L_x^4(J)} + \| (w, \partial_t w) \|_{L_t^{\frac{2(d+1)}{B^2_2(h_i^2, 2)}} H_{\tilde{h}_n}^{\frac{2}{B^2_2(h_i^2, 2)}}(J) \times L_t^{\frac{2(d+1)}{B^2_2(h_i^2, 2)}}(J) \] \leq \| (w(a), \partial_t w(a)) \|_{H_{\tilde{h}_n}^s} + \| F \|_{L_t^4 L_x^8(J)} + \| G \|_{L_t^{\frac{2(d+1)}{B^2_2(h_i^2, 2)}} H_{\tilde{h}_n}^{\frac{2}{B^2_2(h_i^2, 2)}}(J)}.

Remark 2.1. The estimates \((2.4)\) and \((2.6)\) for \((R, \tilde{R}) \neq (\tilde{B}^2_2(h_i^2, 2), \tilde{B}^2_2(h_i^2, 2))\) can be proved by using finite differences characterizations of homogeneous Besov spaces: see e.g. \([6]\). In order to prove \((2.4)\) and \((2.5)\), we apply \((2.4)\) and \((2.6)\) for \((R, \tilde{R}) = (\tilde{B}^2_2(h_i^2, 2), \tilde{B}^2_2(h_i^2, 2))\) to \(h_i(T^2_{f} f, T^2_{g} g)\) instead of \((f, g)\), taking into account \((\ast)\): \(P_M(f(x)) = P_M(f(x))\) (resp. \(P_{\leq M}(f(x)) = P_{\leq M}(f(x))\)). The estimates \((2.6)\) and \((2.7)\) are well-known. For papers dealing with Strichartz-type estimates for the wave equation and the Klein-Gordon equation see e.g. \([13, 18, 20, 11]\) and references therein. In order to prove \((2.8)\) observe that if \(w\) satisfies \(\partial_{tt} w - \Delta w + (h_i^2)^2 w = H\), then \(w(t, x) := w(\frac{t}{h_i^2}, \frac{x}{h_i})\) is a solution of \(\partial_{tt} \bar{w} - \Delta \bar{w} + \bar{w} = (h_i^2)^2 H(\frac{t}{h_i}, \frac{x}{h_i})\); then use \((2.6)\) and \((\ast)\) to get \((2.8)\).

We will mostly use \((2.6)\), \((2.7)\), and \((2.8)\) with \((q, r) = \frac{2(d+1)}{d-2} (1, 1)\) or \((q, r) = \frac{2(d+1)}{d-2} (1, 2)\).

2.7. Functionals. In this paper we will often use the functionals defined below.

Let \(\bar{g} := (g_1, g_2)\) with \(g_1\) and \(g_2\) two real-valued functions defined on \(\mathbb{R}^d\). We denote by \(E(\bar{g})\), \(E_{wa}(\bar{g})\) the following expressions (for \(\bar{g}\) and \((g_1, 0)\) lying in the appropriate spaces)

\[ E(\bar{g}) := \frac{1}{2} \int_{\mathbb{R}^d} g_1^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g_1|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} g_2^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^d} |g_1|^{2^*} \, dx \]

\[ E_{wa}(\bar{g}) := \frac{1}{2} \int_{\mathbb{R}^d} g_1^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g_1|^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^d} |g_1|^{2^*} \, dx. \]

More generally, given \(h\) a function on \(\mathbb{R}^d\), \(R > 0\) and \(c \in \mathbb{R}^d\), let

\[ E(h, \bar{g}) := \frac{1}{2} \int_{\mathbb{R}^d} h|g_2|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} h|\nabla g_1|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} h|g_1|^2 \, dx - \frac{1}{2R} \int_{\mathbb{R}^d} h|g_1|^{2^*} \, dx, \]

\[ E_R(c, \bar{g}) := \frac{1}{2} \int_{|x-c| \geq R} g_2^2 + \frac{1}{2} \int_{|x-c| \geq R} |\nabla g_1|^2 + \frac{1}{2} \int_{|x-c| \geq R} |g_1|^2 \, dx - \frac{1}{2R} \int_{|x-c| \geq R} |g_1|^{2^*} \, dx. \]

Let \(f\) be a function defined on \(\mathbb{R}^d\). We denote by \(K(f)\), \(I(f)\), and \(G(f)\) the following numbers

\[ K(f) := \|\nabla f\|_{L^2}^2 - \|f\|_{L^{2^*}}^2, \quad I(f) := \frac{1}{2} \|f\|_{L^{2^*}}^2, \quad G(f) := \frac{1}{2} \|f\|_{H^1}^2. \]

Observe that
\[ E(\vec{g}) - \frac{K(g_1)}{2} = \frac{1}{2}\|g_2\|_{L^2}^2 + \frac{1}{2}\|g_1\|_{L^2}^2 + I(g_1), \quad E_{wa}(\vec{g}) - \frac{K(g_1)}{2} = \frac{1}{2}\|g_2\|_{L^2}^2 + I(g_1), \]

(2.9) \[ E(\vec{g}) - \frac{K(g_1)}{2} = \frac{1}{2}\|g_2\|_{L^2}^2 + \|g_1\|_{L^2}^2 + G(g_1), \quad E(\vec{g}) - \frac{K^2(g_1)}{2} = \frac{1}{2}\|g_2\|_{L^2}^2 + G(g_1). \]

(2.10) \[ E_{wa}(\vec{W}) = \inf\{E_{wa}((g_1,0)), g_1 \neq 0, g_1 \in \dot{H}^1, K(g_1) = 0\} \]

We also have

(2.11) \[ E_{wa}(\vec{W}) = \inf\{G(f), f \neq 0, f \in \dot{H}^1, K(f) \leq 0\} \]

and

(2.12) \[ E_{wa}(\vec{W}) = \inf\{I(f), f \neq 0, f \in \dot{H}^1, K(f) \leq 0\} \]

2.8. **Linearized operators.** In this paper, we constantly use the linearized operator \( L_+ \) defined by

\[ L_+ := -\Delta - (2^* - 1)W^{2^*-2} \]

We recall some spectral properties of \( L_+ \):

- the discrete spectrum consists of a unique negative eigenvalue (denoted by \(-k^2\)) and there exists a unique smooth, positive, exponentially decaying eigenfunction \( \rho \) such that

(2.14) \[ L_+ \rho = -k^2 \rho, \quad \rho > 0, \quad \|\rho\|_{L^2} = 1 \]

- the essential spectrum is \([0, \infty)\)

- the threshold 0 of the essential spectrum of \( L_+ \) is an eigenvalue: indeed \( L_+ (\partial_x, W) = 0 \) for all \( 1 \leq i \leq d \). Moreover \( L_+ (\Lambda_{-1} W) = 0 \).

3. **Proof of Theorem 1.1**

The proof of Theorem 1.1 relies upon some propositions that we state below. In the first proposition, proved in [18], we perform a decomposition of \( u \) close to the ground states, taking into account the symmetries of the equation and some constraints (the so called orthogonality conditions)

**Proposition 3.1.** There is a small constant \( 0 < \delta_* \ll 1 \) such that for all \( \vec{f} \in \mathcal{V} := \{ \vec{g} \in \dot{H} : d_S(\vec{g}) < \delta_* \} \) there exists a \( C^1 \)- function

\[ \mathcal{V} \to \mathbb{R}^d \times \mathbb{R} \times \dot{H} \]

\[ \vec{f} \to (c, \sigma, \vec{v}) := (c(\vec{f}), \sigma(\vec{f}), \vec{v}(f)) \]

such that
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(3.1) \[ \vec{f} := T^c \vec{S} \sigma (\vec{W} + \vec{v}), \]
and such that \( \vec{v} \) satisfies two orthogonality conditions, namely

(3.2) \[ \langle v_1, S'_0 \rho \rangle = 0, \quad \text{and} \]

(3.3) \[ \langle v_1, \nabla \rho \rangle = 0. \]

Remark 3.2. Let \( \mathcal{L} := \begin{pmatrix} L_+ & 0 \\ 0 & 1 \end{pmatrix} \). We can decompose the vector \( \vec{v} \) as a linear combination of (generalized) eigenvectors of \( \mathcal{J} \mathcal{L} \) plus a remainder term. More precisely let \( \vec{g}^\pm := \begin{pmatrix} 1 \\ \pm k \end{pmatrix} \rho \sqrt{k} \). Then \( \mathcal{J} \mathcal{L} \vec{g}^\pm = \pm k \vec{g}^\pm \). Let \( \vec{v} \) be such that

(3.4) \[ \vec{v} = \lambda_+ \vec{g}^+ + \lambda_- \vec{g}^- + \vec{\gamma}, \]

with \( \lambda_+ := \omega(\vec{v}, \vec{g}^-) \) and \( \lambda_- := -\omega(\vec{v}, \vec{g}^+) \). Let \( \lambda_1 := \frac{\lambda_+ + \lambda_-}{\sqrt{2k}} \) and \( \lambda_2 := \frac{\sqrt{2k} \lambda_+ - \lambda_-}{2} \). Then

(3.5) \[ v_i = \lambda_i \rho + \gamma_i, \quad i \in [1..2]. \]

Remark 3.3. We will apply the decomposition \((3.4)\) to \( \vec{u} \), solution of \((1.1)\) on an interval \( J \) such that \( \vec{u}(t) \in V, \ t \in J \). From Proposition 3.1 we see that there exists a \( C^1(J) \)– function \( t \to (c(t), \sigma(t), \vec{v}(t)) \) such that for all \( t \in J \)

(3.6) \[ \vec{u}(t) = T^{c(t)} \vec{S} \sigma(t) \left( \vec{W} + \vec{v}(t) \right), \]
\[ \langle v_1(t), S'_0 \rho \rangle = 0, \quad \text{and} \quad \langle v_1(t), \nabla \rho \rangle = 0. \]

The next proposition, proved in Section 4, aims at describing the dynamics of the solution near the ground states, using the decomposition in Proposition 3.1

Proposition 3.4. Let \( \vec{u} \) be a solution of \((1.1)\) on an interval \( I := [0, \] \( \) such that for all \( t \in I \) we have \( \vec{u}(t) \in V \). Consider the decomposition \((3.6)\). Let \( \tau \) be such that \( \partial_t \tau := e^{\sigma(t)} \) and \( \tau(0) = 0 \). Then

(1) \[ \partial_t \vec{v} = \mathcal{J} \mathcal{L} \vec{v} + \left( e^\sigma \partial_t e \cdot \nabla - \partial_t \sigma \vec{X} \right) (\vec{W} + \vec{v}) + N(\vec{v}) - e^{-2\sigma} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
\[ = \begin{pmatrix} 0 \\ -N(v_1) \end{pmatrix} \]
\[ \text{with } N(f) := |W + f|^{2\ast - 2} (W + f) - W^{2\ast - 1} - (2\ast - 1)W^{2\ast - 1} f. \]
(2) We now apply the decomposition \((3.4)\) to \(v(t)\). We have

\[
\partial_t \lambda_\pm = \pm k \lambda_\pm \mp \omega \left( \vec{v}, \nabla_\perp \bar{g}^- \right) \partial_t \sigma + \partial_t \omega \left( \vec{v}, \bar{S}_e \bar{g}^- \right) + e^{-2\sigma} \left( \left( \begin{array}{c} 0 \\ W + v_1 \end{array} \right), \bar{g}^- \right),
\]

\[
\partial_t \lambda_1 = \lambda_2 + \frac{1}{\sqrt{2k}} e^{-2\sigma} \omega \left( \left( \begin{array}{c} 0 \\ W + v_1 \end{array} \right), \bar{g}^- + \bar{g}^\perp \right),
\]

(3.9)

\[
\partial_t \lambda_2 = k^2 \lambda_1 - \partial_t \sigma \cdot (v_2, \nabla \rho) + \partial_t \sigma (v_2, \bar{S}_e \rho) + \langle N(v_1), \rho \rangle, \quad \text{and}
\]

(3.10)

\[
|\partial_t \sigma| + |\partial_t c| e^\sigma \lesssim \|\bar{\gamma}\|_{\dot{H}^1}.
\]

The next proposition proved in Section 5 shows that a solution \(\bar{u}\) of \((1.1)\) with data \(\vec{u}(0)\) that is closed enough to \(S\) has a maximal time interval of existence is larger or equal to an interval of size roughly equal to \(e^{-\sigma(0)}\); moreover, on this interval, the distance to \(S\) does not vary much:

**Proposition 3.5.** Let \(0 \leq \delta_1 \ll \delta \) and \(0 < c_1 \ll 1\). Let \(\bar{u}\) be a solution of \((1.1)\) such that \(E(\bar{u}) \leq E_\text{wa}(\bar{W}) + c_1 d_S^2 (\bar{u}(0))\) and such that \(d_S (\bar{u}(0)) \leq \delta_1\). There exists \(1 \gg c_1 > 0\) such that \([-c_1 e^{-\sigma(0)}, c_1 e^{-\sigma(0)}] \subset I(\bar{u})\) and there exists a constant \(C_1 \gg 1\) such that

\[
t \in [-c_1 e^{-\sigma(0)}, c_1 e^{-\sigma(0)}] : \frac{1}{c_1} d_S (\bar{u}(0)) \leq d_S (\bar{u}(t)) \leq C_1 d_S (\bar{u}(0)).
\]

The next proposition, proved in e.g \([18, 28]\), establishes some variational estimates:

**Proposition 3.6.** Let \(\delta > 0\). Then there exist \(\bar{\epsilon}_v := \epsilon_v(\delta) > 0\), \(k := k(\delta) > 0\) and an absolute constant \(c > 0\) such that if \(d_S (\bar{f}) > \delta, \bar{\epsilon} \leq \bar{\epsilon}_v, \) and \(E_\text{wa}(\bar{f}) < E_\text{wa}(\bar{W}) + \bar{\epsilon}\), then

\[
K(f_1) \geq \min \left( k, c \|\nabla f_1\|_{L^2}^2 \right)
\]

or else

\[
K(f_1) \leq -k
\]

Here \(d_S (\bar{f}) := \inf_{(\sigma, c) \in \mathbb{R} \times \mathbb{R}^N} \|\bar{f} - T_c \tilde{S}\bar{W}\|_{\dot{H}^1}\).

The next proposition (see \([18]\) ) shows that the orthogonal direction \(\tilde{\gamma}\) of \(\vec{v}\) in \((3.4)\) can be controlled by the linearized energy \(\langle L^{\tilde{\gamma}}, \tilde{\gamma} \rangle\):

**Proposition 3.7.** For any \(g \in \dot{H}^1\) satisfying \(\langle g, \rho \rangle = 0\), we have

\[
\|\nabla g\|_{L^2}^2 \sim (L_+ g, g) + |\langle g, \bar{S}_0 \rho \rangle|^2 + |\langle g, \nabla \rho \rangle|^2
\]

(3.11)

In particular let \(\bar{v} \in \dot{H}^1\). Then \(\|\tilde{\gamma}\|_{\dot{H}^1} \sim \langle L^{\tilde{\gamma}}, \tilde{\gamma} \rangle\), with \(\tilde{\gamma}\) defined in the decomposition \((3.5)\).

In the next proposition, we recall the definition and some properties of the functions \(d_0\) that was used in the study of the energy-critical focusing wave equation (see \([18]\) ):
Proposition 3.8. Let $\vec{f} \in \mathcal{V}$. Consider the decomposition (3.4) followed by the decomposition (3.1). Let $d_0$ denote the following number:

$$d_0^2(\vec{f}) := E_{wa}(\vec{f}) - E_{wa}(\vec{W}) + k^2 \lambda_1^2.$$  

Then

$$d_0^2(\vec{f}) \sim \|\vec{v}\|_E^2 \sim \|\vec{v}\|_{\mathcal{H}}^2 \sim d_0^2(\vec{f}),$$

with $\|\vec{v}\|_E$ denoting the linearized energy norm, i.e

$$\|\vec{v}\|_E^2 := \frac{1}{2}(k^2 \lambda_1^2 + \lambda_2^2) + \frac{1}{2} \langle \mathcal{L}_0 \vec{v}, \vec{v} \rangle.$$

We now define $\tilde{d}_S$ for the equation (1.1) and we recall that for $\vec{\phi}$ such that (3.14) holds, it behaves like the unstable mode:

**Proposition 3.9.** Let $0 < \delta_s \ll \delta_l$. Let $\chi$ be a smooth, decreasing, and nonnegative function such that $\chi(r) = 1$ if $r \leq 1$ and $\chi(r) = 0$ if $r \geq 2$. Let $\vec{\phi} \in \mathcal{H}^l$. Then the nonlinear distance function $\tilde{d}_S$ is defined as follows:

$$\tilde{d}_S(\vec{\phi}) := \chi \left( \frac{d_S(\vec{\phi})}{\delta_s} \right) d_0(\vec{\phi}) + \left( 1 - \chi \left( \frac{d_S(\vec{\phi})}{\delta_s} \right) \right) d_S(\vec{\phi}).$$

Hence if $d_S(\vec{\phi}) \ll \delta_s$ and

$$E(\vec{\phi}) < E_{wa}(\vec{W}) + \frac{\tilde{d}_S^2(\vec{\phi})}{2},$$

then

$$\tilde{d}_S^2(\vec{\phi}) \sim \lambda_1^2(\vec{\phi}).$$

The proof of (3.15) is short: it follows from (3.14) and the definition of $d_0$.

**Remark 3.10.** The same conclusion holds if $\vec{\phi} \in \mathcal{H}^1$, $E_{wa}(\vec{\phi}) < E_{wa}(\vec{W}) + \bar{\varepsilon}^2$, and $E_{wa}(\vec{\phi}) < E_{wa}(\vec{W}) + \frac{\tilde{d}_S^2(\vec{\phi})}{2}$.

In the sixth proposition, proved in Section 6, we study the dynamics of the solution $u$ close to the ground states in the ejection mode:

**Proposition 3.11.** Let $\vec{u}$ be a solution of (1.1). Let $t_0 \in I(\vec{u})$. Let $c_d$ be such that $0 < c_d \ll c_l$. Let $R := \tilde{d}_S(\vec{u}(t_0))$ and $\delta_f$ be a positive constant such that $R \ll \delta_f \ll \delta_s$. Assume that $\vec{u}$ satisfies the energy estimate

$$E(\vec{u}) \leq E_{wa}(\vec{W}) + c_d R^2,$$

and the ejection scenario, i.e

$$\partial_t \tilde{d}_S^2(\vec{u})(t_0) \geq 0.\footnote{Here $k$ is the number defined in (2.14)}$$
Then there exists $t_f$ such that the following properties hold: $[t_0, t_f] \subset I(u); \tilde{d}_S(\tilde{u}(t_f)) = \delta_f; \tilde{d}_S(\tilde{u})$ is increasing on $[t_0, t_f]; \text{sign}(\lambda_1)$ is constant on $[t_0, t_f];$

\[(3.18) \quad t \in [t_0, t_f]: \tilde{d}_S(\tilde{u}(t)) \sim |\lambda_1(t)| \sim Re^{k\tau};\]

\[(3.19) \quad t \in [t_0, t_f]: \|\tilde{\gamma}(t)\|_{\mathcal{H}} + \|u(t)\|_{L^2} \lesssim R + R^2 e^{2k\tau};\]

\[(3.20) \quad t \in [t_0, t_f]: \left|\left(\lambda_1(t) - \lambda_1(t_0), e^{\sigma(t_0)}(c(t) - c(t_0))\right)\right| \ll 1; \text{ and}\]

there exists a constant $C_K > 0$ such that

\[(3.21) \quad t \in [t_0, t_f]: -\text{sign}(\lambda_1(t))K(u(t)) \gtrsim (e^{k\tau} - C_K)R.\]

Here we consider the decomposition \[(3.6)\] followed by the decomposition \[(3.7)\] applied to $\tilde{v}(t)$; we denote by $\tilde{d}_S(\tilde{u})$ the map $d_S(\tilde{u}) := t \to \tilde{d}_S(\tilde{u}(t));$ we define $\tau := \tau(t)$ by $\frac{d\tau}{dt} = e^{\sigma(t)}$ and $\tau(t_0) := 0.$

**Remark 3.12.** By time-reversal invariance \footnote{\textbf{i.e} $(t, x) \to \tilde{u}(t, x)$ is a solution of (1.1) then $(t, x) \to \tilde{u}(-t, x)$} and time-translation invariance \footnote{\textbf{i.e} if $a \in \mathbb{R}$ and $(t, x) \to \tilde{u}(t, x)$ is a solution of (1.1) then $(t, x) \to \tilde{u}(t - a, x)$} a similar result holds if

\[(3.22) \quad \partial_t \tilde{d}_S(\tilde{u})(t_0) \leq 0\]

More precisely three exists $t_f \leq t_0$ such that the following properties hold: $[t_0, t_f] \subset I(u); \tilde{d}_S(\tilde{u}(t_f)) = \delta_f; \tilde{d}_S(\tilde{u})$ is decreasing on $[t_f, t_0]; \text{sign}(\lambda_1)$ is constant on $[t_f, t_0];\ (3.18), (3.19), (3.20),$ and (3.21) hold with “$\tau(t)$” (resp. “$t$, $t_0$”) substituted with “$-\tau(t)$” (resp. “$t_f, t_0$”).

In the next proposition, we define a continuous sign function that will decide the fate of the solution of (1.1) as $t \to T_+(u), t < T_+(u),$ or $t \to T_-(u), t > T_-(u)$

**Proposition 3.13.** Let $\delta_b > 0$ be a positive number such that $\delta_b \ll \delta_f.$ Let $0 < \epsilon \ll \epsilon_n(\bar{b}_0).$ Let

$\tilde{u}(t) := \tilde{d}_S(\tilde{u})(t) \leq \delta_f$

Then $\Theta$ is continuous. Here in the region $\left\{\tilde{d}_S(\tilde{u}) \leq \delta_f\right\}$ we apply the decomposition \[(3.3)\] to $\tilde{u}.$
Proposition 3.14. Let \( u \) be a solution of (1.1) such that \( \bar{u}(0) \in H^\epsilon \) with

\[
\bar{\epsilon} \ll \min(\delta_\epsilon, \epsilon_\epsilon, k(\delta_\epsilon)) \cdot
\]

Assume that there exist \( t_1, t_2 \in I(u) \) such that \( t_1 < t_2 \), \( \tilde{d}_S(\bar{u}(t_1)) < \delta_b = \tilde{d}_S(\bar{u}(t_2)) \). Then for all \( t \in I(u) \cap [t_2, \infty) \) we have \( \tilde{d}_S(\bar{u}(t)) \geq \delta_b \).

Remark 3.15. By time-reversal symmetry the following statement is also true: if there exist \( t_1, t_2 \in I(u) \) such that \( t_2 < t_1 \) and \( \tilde{d}_S(\bar{u}(t_1)) < \delta_b = \tilde{d}_S(\bar{u}(t_2)) \), then for all \( t \in I(u) \cap (-\infty, t_2] \) we have \( \tilde{d}_S(\bar{u}(t)) \geq \delta_b \).

Proposition 3.16. Let \( u \) be a solution of (1.1) such that \( \bar{u}(0) \in H^\epsilon \) with \( \bar{\epsilon} \) satisfying (3.23) and \( \bar{\epsilon} \ll \epsilon_\epsilon(\delta_b) \). Assume that there exists \( t_0 \in I(u) \) such that \( \tilde{d}_S(\bar{u}(t)) \geq \delta_b \) for all \( t \in [t_0, \infty) \cap I(u) \). If \( \Theta(u(t_0)) = 1 \) then \( T_+(u) = \infty \) and \( u \) scatters to a free solution as \( t \to \infty \); if \( \Theta(u(t_0)) = -1 \) then \( T_-(u) = \infty \).

Remark 3.17. By time-reversal symmetry the following statement is also true: assume that there exists \( t_0 \in I(u) \) such that \( \tilde{d}_S(\bar{u}(t)) \geq \delta_b \) for \( t \in (-\infty, t_0] \cap I(u) \); if \( \Theta(u(t_0)) = 1 \) then \( T_+(u) = \infty \) and \( u \) scatters to a free solution as \( t \to -\infty \); if \( \Theta(u(t_0)) = -1 \) then \( T_-(u) = \infty \).

With these propositions in mind, we can now prove Theorem 1.1.

Let \( 0 < \epsilon_\epsilon \ll \min(\delta_b, k(\delta_\epsilon), \epsilon_\epsilon(\delta_b)) \).

Let \( \mathcal{T} := \{ \bar{\psi} \in H^{\epsilon_\epsilon} : E(\bar{\psi}) \geq E_{wa}(\bar{W}) + c_d \tilde{d}_S^2(\bar{\psi}) \} \).

Proposition 3.8 implies that \( \tilde{d}_S(\bar{\psi}) \sim \tilde{d}_S^2(\bar{\psi}) \); hence \( \mathcal{T} \subset B^{0(\epsilon_\epsilon)} \cap H^{\epsilon_\epsilon} \).

If \( \bar{u}(t) \in \mathcal{T} \) for all \( t \in I(u) \) then the conclusions of Theorem 1.1 (1) and (2), hold. If not there exists \( \bar{t} \in I(u) \) such that \( E(\bar{u}) \leq E_{wa}(\bar{W}) + c_d \tilde{d}_S^2(\bar{\bar{u}(\bar{t})}) \). Assume that \( \partial_t \tilde{d}_S(\bar{u}(\bar{t})) \geq 0 \) (resp. \( \partial_t \tilde{d}_S(\bar{u}(\bar{t})) < 0 \)); then we see by applying Proposition 3.14 that \( \tilde{d}_S(\bar{u}) \) increases (resp. decreases) from \( \bar{t} \) (resp. \( \bar{t}'' \)) to \( \bar{t}''\) (resp. \( \bar{t}' \)) with \( \bar{t}'' \) such that \( \tilde{d}_S(\bar{u}(\bar{t}'')) = \delta_b \); hence by applying Proposition 3.14 and Proposition 3.8 we see that either \( (f+) \) or \( (f-) \) holds, with \( (f\pm) : \Theta(\bar{u}(\bar{t})) = \pm 1 \) for all \( t \in [\bar{t}', \infty) \cap I(u) \) (resp. \( (b+) \) or \( (b-) \) holds, with \( (b\pm) : \Theta(\bar{u}(\bar{t})) = \pm 1 \) for all \( t \in (-\infty, \bar{t}] \cap I(u) \)); moreover, by Proposition 3.10 we see that if \( (f+) \) holds (resp. \( (f-) \) holds) then \( u \) scatters to a free solution as \( t \to \infty \) (resp. \( t \to -\infty \)) and if \( (b+) \) holds (resp. \( (b-) \) holds) then \( t'' \cap I(u) \) (resp. \( (-\infty, \bar{t}') \cap I(u) \) is finite. Hence the conclusions of Theorem 1.1 (1) and (2), also hold.

Theorem 1.1 (3) is proved in Section 11.

---

Footnote: There are other choices for \( \mathcal{T} \): one could also have chosen for example \( \mathcal{T} := \{ \bar{\psi} \in H^{\epsilon_\epsilon} : \tilde{d}_S(\bar{\psi}) \leq (c_d)^{-\frac{1}{2} \epsilon_\epsilon} \} \).
4. Proof Of Proposition \ref{prop4}

In this section we prove Proposition \ref{prop4}

\[ \partial_t \vec{u} = J D \vec{u} + \left( \begin{array}{c} 0 \\ -u \end{array} \right) + \left( \begin{array}{c} 0 \\ |u|^2 - 2u \end{array} \right) \]

Noting that (with $\vec{h} := (h_1, h_2)$)

\[ \partial_t (T, \vec{h}) = T_c (\partial_t \vec{h} - \partial_t c \cdot \nabla \vec{h}), \quad \partial_t (\vec{S} \vec{h}) = \vec{S} \partial_t \vec{h} + \partial_t \vec{S} \vec{h} \]

\[ \triangle S_{-1} h_1 = e^{2\sigma} S_{-1} \triangle h_1, \quad |S_{-1} h_1|^2 = e^{2\sigma} S_{-1} (|h_1|^2 - 2h_1) \]

we see (after composition by $T^{-c} \vec{S}^{-\sigma}$)

\[ \partial_t \vec{v} = \left( e^\sigma \partial_t c \cdot \nabla - \partial_t \vec{S} \vec{h} \right) (\vec{W} + \vec{v}) + e^\sigma \left( J L \vec{v} + \nabla (\vec{v}) \right) - e^{-\sigma} \left( \begin{array}{c} 0 \\ W + v_1 \end{array} \right) \]

Therefore \ref{prop4} holds.

Hence, differentiating with respect to $\tau$ the relation $\lambda_{\pm} := \omega \left( \vec{v}, \vec{g}^\tau \right)$, one sees that \ref{prop5} holds. Hence, differentiating with respect to $\tau$ \ref{prop6} and \ref{prop7}, we get, by integration by part and \ref{prop8}, \ref{prop9}: see \ref{18} for more details.

5. Proof of Proposition \ref{prop5}

In this section we prove Proposition \ref{prop5}

By using the time-reversal invariance \ref{14} it suffices to prove the proposition by replacing $\left[ -\vec{c} e^{-\sigma(\theta)}, \vec{c} e^{-\sigma(\theta)} \right]$ with $\left[ 0, \vec{c} e^{-\sigma(\theta)} \right]$ in its statement.

Let $O$ be the largest interval of the form $[0, a)$ such that $\vec{u}(t) \in V$, $t \in [0, a)$. We apply the decomposition \ref{15} for all $t \in O$. Let $C_1 \gtrsim 1$ be a constant such that all the statements below and in this section are true. Let $\vec{u}(t) := T_{c(t)} \vec{S}(\vec{u}(t)) = \vec{u}(t) - T_{c(t)} \vec{S}(\vec{u}(t)) \vec{W}$. We write $\vec{u}(t) := (w_1(t), w_2(t))$. If $J$ is an interval let $Q(J) := ||\vec{w}||_{L^\infty \vec{H}(J)} + ||w_1||_{L^2 \vec{L}^2(J)} + 2 \frac{d_{\vec{w}} (J)}{d_{\vec{w}} (J)}$. Let $\mu_{\text{max}} := \max \left\{ \mu \in \mathbb{R}^+: Q \left( \left[ 0, \mu e^{-\sigma(\theta)} \right] \right) \leq C_1^2 d_{\vec{w}} (J) \right\}$.

Assume that $\mu_{\text{max}} < \vec{c}_1$. We will show that it is impossible.

Let $J := [0, \cdot) \subset \left[ 0, \mu_{\text{max}} e^{-\sigma(\theta)} \right]$. Then $\vec{v}$ satisfies

\[ \partial_t w_1 - \triangle w_1 = -u + \left| \vec{v} + T_{c(t)} \vec{S}(\vec{w}^t) W \right|^2 - \left( w_1 + T_{c(t)} \vec{S}(\vec{w}^t) W \right) - ||T_{c(t)} \vec{S}(\vec{w}^t) W ||^2 - 2T_{c(t)} \vec{S}(\vec{w}^t) W . \]

We have

\[ Q(J) \lesssim \| \vec{v} \|_{\vec{H}} + ||u||_{L^2_1 \vec{L}^2_1 \vec{L}^2_1(J)} + ||X||_{L^1_1 \vec{L}^1_2 \vec{L}^2_1(J)}, \quad \text{with} \quad X := ||\vec{v}^t + \vec{S}(\vec{w}^t) W ||^2 - \left( \vec{v}(t) + \vec{S}(\vec{w}^t) W \right) - ||\vec{S}(\vec{w}^t) W ||^2 - 2 \vec{S}(\vec{w}^t) W . \]

10\text{.e if } u \text{ is a solution of } \ref{13} \text{ then } (t, x) \rightarrow u(-t, x) \text{ is also a solution of } \ref{13}.
The fundamental theorem of calculus yields $|X| \lesssim \left( |S_{\sigma(t)}^\sigma W|^{2^*-2} + |\bar{v}(t)|^{2^*-2} \right) |\bar{v}(t)|$.

Hence $\|X\|_{L^2_1 L^2_2(J)} \lesssim \|A\|_{L^1_1 L^2_2(J)} + \|B\|_{L^1_1 L^2_2(J)}$, with

$(A, B) := \left( |\bar{v}(t)|^{2^*-2} \bar{v}(t), S_{\sigma(t)}^\sigma W |\bar{v}(t)|^{2^*-2} \right)$. Clearly $\|A\|_{L^1_1 L^2_2(J)} \lesssim \|\bar{v}\|^{2^*-1}_{L^2_1 \frac{2d+1}{2d+2}} \lesssim d^2_{\gamma} (\bar{u}(0))$. We also get from (3.10)

$$\|B\|_{L^1_1 L^2_2(J)} \lesssim \left\| S_{\sigma_1}^\sigma W \right\|_{L^2_1 \frac{2d+1}{2d+2}} \lesssim \|\bar{v}\|_{L^2_1 \frac{2d+1}{2d+2}} \lesssim d^2_{\gamma} (\bar{u}(0)) .$$

We then estimate $\|u\|_{L^1_1 L^2_2(J)}$. Let $t \in J$. We have $E_{wa}(\bar{u}(t)) = E_{wa}(T_{\sigma(t)}(\bar{W} + \bar{v}(t))) = E_{wa}(\bar{W} + \bar{v}(t))$. Hence, expanding around $W$, we get from (2.14)

$$E_{wa}(\bar{u}(t)) = E_{wa}(\bar{W}) + \frac{1}{2} (\mathcal{L} \bar{u}(t), \bar{v}(t)) - C(\bar{v}(t))
= E_{wa}(\bar{W}) + \frac{1}{2} \left( \lambda_3^2 - k^2 \lambda_3^2 \right) + \frac{1}{2} (\mathcal{L} \bar{\gamma}(t), \bar{\gamma}(t)) - C(\bar{v}(t)) .$$

Here

$$C(\bar{f}) := \int_{\mathbb{R}^N} \frac{|W + f_1|^2 - W_2 - W^{2^*-1} f_1 - \frac{2^*-1}{2} W^{2^*-2} f_1^2}{2^*} dx .$$

Applying elementary estimates for $|v_1(t)| \gtrsim |W|$, the Taylor formula for $|W| \ll |v_1(t)|$, and the embedding $\dot{H}^1 \hookrightarrow L^{2^*}$, we get $|C(\bar{v}(t))| \lesssim ||\bar{v}(t)||^{3}_{\dot{H}} + ||\bar{v}(t)||^{2^*}_{\dot{H}} \lesssim d^3_{\gamma}(\bar{u}(0))$. Hence $E_{wa}(\bar{u}(t)) = E_{wa}(\bar{W}) + O \left( d^3_{\gamma}(\bar{u}(0)) \right)$. Since $E(\bar{u}) \leq E_{wa}(\bar{W}) + c_1 d^3_{\gamma}(\bar{u}(0))$, we deduce that

$$\|u(t)\|_{L^2} \lesssim d_{\gamma}(\bar{u}(0)) .$$

Let $\chi$ be a radial function such that $\chi(r) \geq 0$, $\chi(r) = 1$ if $0 \leq r \leq 1$, and $\chi(r) = 0$ if $r \geq 2$. We get from Hölder inequality

$$(\int_{\mathbb{R}^d} |W + v_1(t)|^2 dx)^{\frac{1}{2}} \gtrsim \int_{\mathbb{R}^d} (W + v_1(t)) \chi(r) dx \gtrsim 1 ,$$

using again at the last step Hölder inequality followed by $|v_1(t)|_{L^{2^*}} \lesssim Q(J)$. The above estimates and (3.10) yield $d_{\gamma}(\bar{u}(0)) \gtrsim \|u(t)\|_{L^2} \sim e^{-\sigma(0)} \left( \int_{\mathbb{R}^d} |W + v_1(t)|^2 dx \right)^{\frac{1}{2}} .$$

Hence $e^{-\sigma(0)} \lesssim d_{\gamma}(\bar{u}(0))$ and

$$\|u\|_{L^1_1 L^2_2(J)} \lesssim e^{-\sigma(0)} \|u\|_{L^\infty_1 L^2_2(J)} \lesssim d_{\gamma}(\bar{u}(0)) .$$

The estimates above and a continuity argument show that $\left\| \mathcal{D}(\mu_{max} e^{-\sigma(0)}) \right\|_{\dot{H}} \leq 2 d_{\gamma}(\bar{u}(0))$, which contradicts $\left\| \mathcal{D}(\mu_{max} e^{-\sigma(0)}) \right\|_{\dot{H}} = C_{\dot{H}} d_{\gamma}(\bar{u}(0))$. Hence using Proposition 3.8, we see that $d_{\gamma}(\bar{u}(t)) \lesssim d_{\gamma}(\bar{u}(0))$.

Let $t_0 \in [0, c_0 e^{-\sigma(0)}]$. Since $v(t) := u(t_0 - t)$ is a solution of (1.1) that satisfies $d_{\gamma}(\bar{v}(0)) \leq C_0 d_{\gamma}$ we see from applying the above result to $v$ that $d_{\gamma}(\bar{v}(t_0)) \lesssim d_{\gamma}(\bar{u}(0))$. 

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6. Proof of Proposition 3.11

In this section we prove Proposition 3.11.

Let \( \tilde{C} > 0 \) be a fixed constant large enough such that all the statements below are true.

Let

\[
\frac{1}{C} \lambda_1(t_0) e^{k \tau} \leq \lambda_1(t) \leq \tilde{C} \lambda_1(t_0) e^{k \tau} \quad (A),
\]

\[\bar{\tau} := \sup \left\{ t \geq t_0 : \bar{d}_\tau^2(\bar{u}) \text{ is } \not\equiv 0 \text{ on } [t_0, \bar{\tau}] \right\} \quad (B),\]

\[
\|u(t)\|_{L^2} + \|\bar{\gamma}(t)\|_{\dot{H}} \leq \tilde{C} \left( R + R^2 e^{2k \tau} \right) \quad (C).
\]

Let \( \bar{\tau} \) be such that \( \delta_s' \gg Re^{C^{-1}e^{\tau(t_0)}(\bar{\tau} - t_0)} \gg \delta_f \).

**Claim:** \( \bar{\tau} > t_0 + \bar{\tau} \).

Assuming that the claim holds, then by Proposition 3.8 and 3.10, we see that 3.18, 3.19, and 3.20 hold. From the invariance of \( S^{\alpha(\tau)}_1 \) and by the translation transform \( T^{\varepsilon(\tau)} \) we get from plugging 3.9 and Taylor expansion in the region \( \{|v_1(t)| \ll |W|\} \)

\[
K(u(t)) = -(2^* - 2)(W^{2^*-1}, v_1(t)) + O(\|v_1(t)\|_{L^1}^2)
\]

Since \( v_1(t) = \lambda_1(t) \rho + \gamma_1(t) \), we see from 3.18 and 3.19 that 3.21 holds.

It remains to prove the claim. Let \( \bar{\tau} \) be such that \( \bar{\tau} := \int_{t_0}^\tau e^{\tau(t)} dt \). Assume that it fails. Let \( \text{Res}(t) := \frac{1}{\sqrt{2k}} e^{-2\tau(t)} \omega \left( \begin{array}{cc} 0 & W + v_1 \\ W & v_1 \end{array} \right), \tilde{g}^+ + \tilde{g}^- \) \( \in [t_0, \bar{\tau}] \). Observe by positivity of \( \rho \) and \( W \), by 3.19, and the embedding \( H^1 \hookrightarrow L^{2^*} \), that

\[
\int_{\mathbb{R}^d} |W + v_1(t)| \rho \, dx \gtrsim \int_{\mathbb{R}^d} W \rho \, dx - \|v_1(t)\|_{L^{2^*}} \|\rho\|_{L^{2^*}} \gtrsim 1.
\]

Hence, using also the Cauchy-Schwarz inequality and 3.6 we get

\[
|\text{Res}(t)| \lesssim e^{-2\tau(t)} \left( \int_{\mathbb{R}^d} |W + v_1(t)| \rho \, dx \right)^2 \lesssim \|u(t)\|_{L^2}^2 \lesssim R^2 + R^4 e^{4k \tau}.
\]

We also see from Proposition 3.8 and Proposition 3.9 that \( \bar{d}_S(\bar{u})(\tau) \sim |\lambda_1(t_0)| \sim \|\bar{\gamma}(\tau)\|_{\dot{H}} \). A computation of the first derivative and the second derivative of \( \tau \to \bar{d}_S^2(\bar{u})(\tau) \) combined with 3.7, 3.9, 3.10, and the above estimates shows that

\[
\partial^2_{\tau} \bar{d}_S^2(\bar{u}) = k^2 \lambda_1^2 + O(\lambda_1^2).
\]

In the expression above we also used the estimate \( |N(f)| \lesssim |f|^2 + |f|^{2^*-1} \) combined with the Hölder inequality and the embedding \( \dot{H} \hookrightarrow L^{2^*} \). Hence we get for \( \tau \in [0, \bar{\tau}] \) \( \partial^2_{\tau} \bar{d}_S(\bar{u}(t)) > 0 \).

We see from 3.17 that \( \text{sign}(\lambda_1) \lambda_2(0) \gtrsim -\lambda_1^2(0) \). Consequently \( \lambda_2(0) \sim \lambda_1(0) \).

\[\text{in the region } \{|f| \ll |W|\} \text{ we use the Taylor-Lagrange formula}\]
Integrating (3.8) and using the above estimates combined with the Hölder inequality and the embedding $H^1 \hookrightarrow L^2$ , we get

$$|\lambda_+ (\tau) - e^{\pm k\tau} \lambda_+ (0)| \lesssim \int_\tau^T e^{k(\tau-s)} R^2 e^{2us} \, ds \lesssim R^2 e^{2k\tau}$$

Collecting the estimates above one sees that $\text{sign}(\lambda_1)$ is constant on $[0, \bar{\tau})$ and $(A')$ holds with $(A') : \lambda_1 (\tau) \sim \lambda_1 (t_0) e^{k\tau}$.

Next we modify an argument used in Lemma 4.3 of [28] (see also [24], [18]). More precisely we consider the expansion of $\bar{v}$, ignoring the $\gamma$ term, i.e.

$$\bar{v}_\lambda := \lambda_+ \bar{g}^+ + \lambda_- \bar{g}^- = \bar{v} - \bar{\gamma}$$

Expanding around $\bar{W}$ we get

$$E_{wa}(\bar{W} + \bar{v}_\lambda) = E_{wa}(\bar{W}) + \frac{1}{2} (\lambda_1^2 - k^2 \lambda_1^2) - C(\bar{v}_\lambda)$$

with $C(\bar{f})$ defined in (5.2). Hence using also (5.1)

(6.3) $$E(\bar{u}) - E_{wa}(\bar{W} + \bar{v}_\lambda) = \frac{1}{2} \|u\|_2^2 + \frac{1}{2} (\|\bar{\gamma}\|_{L^\infty}^2 + C(\bar{v}_\lambda) - C(\bar{v})$$

Let $X := \|\bar{\gamma}\|_{L^\infty} + \|u\|_{L^2}$. We compute (using (3.9) and (3.10))

(6.4) $$\partial_\tau E_{wa}(\bar{W} + \bar{v}_\lambda) = \lambda_2 \partial_\tau \lambda_2 - k^2 \lambda_1 \partial_\tau \lambda_1 - N(v_{\lambda,1})\partial_\tau v_{\lambda,1}$$

$$= \lambda_2 \left( k^2 \lambda_2 + O(\|\bar{\gamma}\|_{L^\infty}^2) + O(N(v_{1,1})) \right) - k^2 \lambda_1 (\lambda_2 + Re)$$

$$= O \left( R^{k\tau} X + Re^{k\tau} X^2 \right)$$

Hence, using also Proposition 3.7 and (6.3), we see that for $t' \in [t_0, t]$

$$|E(\bar{u}(t')) - E_{wa}(\bar{W} + \bar{v}_\lambda(t'))| \lesssim |E(\bar{u}(t_0)) - E_{wa}(\bar{W} + \bar{v}_\lambda(t_0))| + |E_{wa}(\bar{W} + \bar{v}_\lambda(t')) - E_{wa}(\bar{W} + \bar{v}_\lambda(t_0))|$$

$$\lesssim R^2 + (Re^{k\tau})^2 \|X\|_{L^\infty([t_0, t])} + Re^{k\tau} \|X\|_{L^\infty([t_0, t])}^2$$

Therefore

$$\|X\|_{L^\infty([t_0, t])}^2 \lesssim \sup_{t' \in [t_0, t]} \left( |E(\bar{u}(t')) - E_{wa}(\bar{W} + \bar{v}_\lambda(t'))| + |C(\bar{v}_\lambda(t')) - C(\bar{v}(t'))| \right)$$

$$\lesssim R^2 + (Re^{k\tau})^2 \|X\|_{L^\infty([t_0, t])} + Re^{k\tau} \|X\|_{L^\infty([t_0, t])}^2$$

Hence $(B')$ holds with $(B') : \|u(t)\|_{L^2} + \|\bar{\gamma}(t)\|_{H^1} \lesssim R + R^2 e^{2k\tau}$.

Now by applying Proposition 5.5 close (from the left) to $\bar{t}$ we see that $\bar{t} \in I(u)$. Hence $\bar{t}$ is in fact the max such that $(A)$, $(B)$, and $(C)$ hold; moreover we see from (5.2) that $\partial_\tau d_S(\bar{u}(\bar{t})) > 0$. Hence contradiction.
7. Proof of Proposition 3.13

In this section we prove Proposition 3.13.

We claim that \( \text{sign}(\lambda_1) \) is continuous at \( \vec{\phi} \in \mathcal{H}^c \) such that \( \hat{d}_S(\vec{\phi}) \leq \delta_f \). Indeed \( \lambda_1 \) is clearly continuous at \( \vec{\phi} \): this follows from Proposition 3.1 and the formula of \( \lambda_1 \) in Remark 3.2. Moreover \( \lambda_1(\vec{\phi}) \neq 0 \): if not Proposition 3.9 implies that \( \hat{d}_S(\vec{\phi}) = 0 \) (and hence \( E(\vec{\phi}) = E_{\text{wa}}(W) \): see Proposition 3.3), which contradicts \( \vec{\phi} \in \mathcal{H}^c \).

We also claim that \( \text{sign}(K \circ P) \) is continuous at \( \vec{\phi} \in \mathcal{H}^c \) such that \( \hat{d}_S(\vec{\phi}) \geq \delta_b \). Indeed \( K \) is continuous at \( \vec{\phi} \in \mathcal{H}^c \) such that \( \hat{d}_S(\vec{\phi}) \geq \delta_b \). Moreover \( K(\phi_1) \neq 0 \): this follows from Proposition 3.6 and Proposition 3.8.

It remains to show that \( \text{sign} \left( \lambda_1(\vec{\phi}) \right) = -\text{sign}(K(\phi_1)) \) for all \( \vec{\phi} \in \mathcal{H}^c \) such that \( \delta_b \leq \hat{d}_S(\vec{\phi}) \leq \delta_f \). Let \( \mathcal{O} \) be the connected component of \( \left\{ \vec{\psi} \in \mathcal{H}^c : \delta_b \leq \hat{d}_S(\vec{\psi}) \leq \delta_f \right\} \) that contains \( \vec{\phi} \). By continuity \( \text{sign}(\lambda_1) \) and \( \text{sign}(K \circ P) \) are constant on \( \mathcal{O} \). Let \( \vec{u} \) be a solution of (1.11) such that \( \vec{u}(0) := \vec{\phi} \) and such that (3.17) holds. Then we can apply Proposition 3.11 to get \( \text{sign}(K(u(t))) = -\text{sign}(\lambda_1(t)) \) for \( t \approx e^{-\sigma_0(t)} \). We have \( \vec{u}(t) \in \mathcal{O} \) since \( \{\vec{u}(t'), t' \in [0, t]\} \) is connected as the image of a connected set by the continuous function \( \vec{u} \). Hence \( \text{sign} \left( \lambda_1(\vec{\phi}) \right) = -\text{sign}(K(\phi_1)) \).

8. Proof of Proposition 3.14

In this section we prove Proposition 3.14.

Setting

By decreasing the value of \( t_2 \) if necessary we may assume WLOG that \( \hat{d}_t \hat{d}_S(\vec{u}(t_2)) \geq 0 \). Assume towards a contradiction that \( \hat{d}_t \hat{d}_S(\vec{u}(t)) \geq \delta_b \) does not hold for all \( t \in I(u) \cap [t_2, \infty) \). This means that there exists \( t_3 > t_2 \) such that \( \hat{d}_S(\vec{u}(t_3)) = \delta_b \), \( \hat{d}_t \hat{d}_S \vec{u}(t_3) \leq 0 \), and \( \hat{d}_S(\vec{u}(t)) \geq \delta_b \) for all \( t \in [t_2, t_3] \).

By space translation invariance, we may assume WLOG that

\[
\begin{align*}
c(t_2) &= 0
\end{align*}
\]

Hyperbolic region, Variational region

We now define the hyperbolic region and the variational region.

Let \( \delta_b \ll \delta_V \ll \delta_m \ll \delta_f \). Let \( s \) be a local minimizer of the function \( t \rightarrow \hat{d}(\vec{u}(t)) \) on \( \left\{ t \in [t_2, t_3] : \hat{d}_S(\vec{u}(t)) < \delta_V \right\} \). By Proposition 3.11 there exists an interval \( I[s] \subset [t_2, t_3] \) such that \( ^{13} \)

1. \( \hat{d}_S(\vec{u}(t)) \sim \hat{d}_S(\vec{u}(t_3)) e^{k \tau} \)
2. \( -\text{sign}(\lambda_1(\tau)) K(u(t)) \gtrsim (e^{k \tau} - C_K) \delta_b \)
3. \( \hat{d}_S(\vec{u}(\partial I[s] \cap (t_2, t_3))) = \delta_m \).

\( ^{12} \)Here \( P \) is the projection defined by \( P(\vec{\phi}) = \phi_1 \)

\( ^{13} \)Here \( \tau \) is defined by \( \frac{d\tau}{dt} := e^{\sigma(t)} \) and \( \tau(s) := 0 \)
Let $\mathcal{L}$ be the set of these $s$. Note that $\mathcal{L}$ is finite: this follows from the uniform continuity of $\tilde{d}_S(\tilde{u})$ on $[t_2, t_3]$, along with the fact that $[\delta I, \delta]$ is included in the range of $\tilde{d}_S(\tilde{u})$ between two consecutive $s$. The hyperbolic interval $I_H$ and the variational interval are defined by the formulas below:

$$ I_H := \bigcup_{s \in \mathcal{L}} I[s], \quad I_V := [t_2, t_3]/I_H. $$

Taking into account (3.10) we see that

$$ s \neq \{t_2, t_3\} : |I[s]| \geq e^{-\sigma(s)} \log \left( \frac{\delta}{\delta m} \right) $$

$$ s \in \{t_2, t_3\} : |I[s]| \approx e^{-\sigma(s)} \log \left( \frac{\delta}{\delta m} \right). $$

Let $m_H := \max_{s \in \mathcal{L}} e^{-\sigma(s)}$: this number will be used in the sequel when we use the virial identity.

Let $s \neq \{t_2, t_3\}$. By still applying Proposition 3.11 we see that there exist intervals $I_I[s] \subset I_V$ and $I_r[s] \subset I_V$ such that $I_I[s], I_r[s] \cap \partial I[s] \neq \emptyset$, $I_I[s]$ (resp. $I_r[s]$) is located at the left (resp. right) of $I[s]$ and such that

$$ s \neq \{t_2, t_3\} : |I_I[s]| + |I_r[s]| \approx e^{-\sigma(s)} \log \left( \frac{\delta}{\delta m} \right) $$

$$ s = t_2 : |I_r[s]| \approx e^{-\sigma(s)} \log \left( \frac{\delta}{\delta m} \right) $$

$$ s = t_3 : |I_I[s]| \approx e^{-\sigma(s)} \log \left( \frac{\delta}{\delta m} \right) $$

We prove the following estimate:

**Result:** Define $\bar{I}[s]$ as follows: $\bar{I}[s] \in \{I_I[s], I_r[s]\}$ if $s \neq \{t_2, t_3\}$; $\bar{I}[s] := I_r[s]$ if $s = t_2$, and $\bar{I}[s] := I_I[s]$ if $s = t_3$. Then

$$ \|u\|_{L^2_t L^\frac{d+1}{d-2} (\bar{I}[s])} \geq 1 $$

(8.1)

$$ \|u\|_{L^2_t L^\frac{d+1}{d-2} (\bar{I}[s])} \geq 1: \text{the other cases are} \text{ treated similarly. Let } 0 < \delta \ll \hat{c}_1 \text{ be a small constant. Let } \hat{t} \in I_r[t_2], \text{ Observe from (3.10) that } \sigma(\hat{t}) \approx \sigma(t_2). \text{ Hence, by (2.7), (3.6), and (3.10) we see that there exists a constant } C > 0 \text{ such that} \]$$

$$ \left\| \left( e^{(t-\hat{t})D} u(\hat{t}), \frac{e^{(t-\hat{t})D}}{D} \partial_t u(\hat{t}) \right) \right\|_{L^2_t L^\frac{d+1}{d-2} (\{[\hat{t}, \hat{t}+\delta e^{-\sigma(t_2)}]\})} \lesssim \|e^{itD}W\|_{L^2_t L^\frac{d+1}{d-2} (\{[0, \delta]\})} + \|\tilde{v}(\hat{t})\|_{\mathcal{H}} \ll 1, $$

14Here we also use the fact that $e^{itD} (S_{-1}^{\mu} f) = S_{-1}^{\mu} (e^{itD} f)$ for $\mu \in \mathbb{R}$ and that $e^{itD} (T^\mu f) = T^\mu (e^{itD} f)$ for $\mu \in \mathbb{R}^d$
Moreover, since \( \tilde{u}(t) \in H^\ell \) for \( t \in [\tilde{t}, \tilde{t} + \delta e^{-\sigma(t_2)}] \), we see (using Proposition 3.5 and Proposition 3.8) that \( E(\tilde{u}) \leq E_{u_0}(\tilde{W}) + c_d \delta^2 \). Hence by using similar arguments as those from Section 5.1 until the end of Section 5, we see that \( e^{-\sigma(t_2)} \sup_{t \in [\tilde{t}, \tilde{t} + \delta e^{-\sigma(t_2)}]} \| u(t) \|_{L^2} \lesssim \delta_f \). Hence

(8.3) \[ \| u \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim e^{-\sigma(t_2)} \log \left( \frac{\delta_f}{\delta_m} \right) \| u \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \leq \delta_f^0 + \]

Let \( I := [\tilde{t}, \cdot] \subset [\tilde{t}, \tilde{t} + \delta e^{-\sigma(t_2)}] \). By (2.7) and (2.4) we get

\[ \| u \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim \| u \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \]

Moreover \( \| u \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim \| W \|_{H^1} \lesssim 1 \): this follows from Proposition 3.8. A continuity argument shows that \( \| u \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim 1 \) and that \( \| u \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim \)

1. Iterating we get

(8.4) \[ \| u \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim \delta_f \],

Computations show that \( \| T^{c(t)}S_{-1}^\sigma W \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim \log \left( \frac{\delta_f}{\delta_m} \right) \). Hence

(8.5) \[ \| \tilde{v}_1 \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim \log \left( \frac{\delta_f}{\delta_m} \right) \], \[ \tilde{v}_1 := T^{c(t)}S_{-1}^\sigma v_1 \],

Interpolating with

\[ \| \tilde{v}_1 \|_{L_t^{\infty}L_x^{rac{4}{d-2}}(I)} \lesssim \| v_1 \|_{L_t^\infty H^1(I) \lesssim \delta_f}, \]

we see that \( \| \tilde{v}_1 \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim \delta_f^0 \). Computations show that \( \| T^{c(t)}S_{-1}^\sigma W \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \lesssim \log \left( \frac{\delta_f}{\delta_m} \right) \). Hence (8.2) holds.

The result allows to divide \( IV \) into subintervals into subintervals \( (I_j = [a_j, b_j]) \) such that

(8.4) \[ \| u \|_{L_t^{2(d+1)}L_x^{rac{2(d+1)}{d-2}}(I)} \sim \eta \],

with \( \eta \) a constant such that \( 0 < \eta \ll 1 \).
\textbf{Virial identity}

Before proceeding, we recall the well-known virial identity

\begin{equation}
\partial_t (x \cdot \nabla u + \frac{d}{2} u, w \partial_t u) = -K(u(t)) + E_{ext}(u(t)) = RHS,
\end{equation}

with \(w(t,x) := \chi \left( \frac{|x|}{t-t_2+m} \right)\) and

\[
E_{ext}(\bar{u}(t)) = \int_{|x| \geq t-t_2+m} |\partial_t u(t)|^2 + |\nabla u(t)|^2 + |u(t)|^2 + |u(t)|^2 \, dx.
\]

Here \(m > 0\) is to be determined; \(\chi\) is a smooth, decreasing, and nonnegative function defined on \(\mathbb{R}^+\) such that \(\chi(r) = 1\) if \(r \leq 1\) and \(\chi(r) = 0\) if \(r \geq 2\). Let \(t' \in \{t_2, t_3\}\). We place \(\nabla \left( T^{(t')} S_{\sigma_1}^\sigma \right) W\), \(\nabla \left( T^{(t')} S_{\sigma_1}^\sigma v_1 \right)\), and \(T^{(t')} S_{\sigma_1}^\sigma v_2\) in \(L^2\). We also place \(T^{(t')} S_{\sigma_1}^\sigma v_1\) in \(L^2\). Since \(\bar{u}(t') \in \mathcal{H}\), we see that \(\|u(t')\|_{L^2} \leq \delta_6\). Hence, using \(\text{(5.6)}\) and \(\text{(5.11)}\) we get

\begin{equation}
\left| \left[ w \partial_t u, x \cdot \nabla u + \frac{d}{2} u \right] \right|_{t_2}^{t_3} \leq \delta_6 (t_3 - t_2 + m)
\end{equation}

and

\[
E_{ext}(\bar{u}(t_2)) \lesssim (me^{\sigma(t_2)})^{-1} + \delta_6^2
\]

Then by finite speed propagation and the local well-posedness theory for small data we see that if \(me^{\sigma(t_2)} \gg 1\) then

\begin{equation}
E_{ext}(\bar{u}(t)) \lesssim E_{ext}(\bar{u}(t_2)) \lesssim (me^{\sigma(t_2)})^{-1} + \delta_6^2
\end{equation}

By \(\text{(3.21)}\),

\begin{equation}
\left| \int_{t_2} K(u) + E_{ext}(\bar{u}) \, dt \right| \geq e^{-\sigma(t_2)} \left( \delta_m - \log \left( \frac{\delta_m}{\delta_6} \right) \left( (me^{\sigma(t_2)})^{-1} + \delta_6^2 \right) \right) = e^{-\sigma(t_2)} \delta_m, \quad me^{\sigma(t_2)} \gg \delta_m \left( \frac{\delta_m}{\delta_6} \right)^{0+}
\end{equation}

Similarly if \(s \notin \{t_2, t_3\}\) then

\begin{equation}
\left| \int_{t} K(u) + E_{ext}(\bar{u}) \, dt \right| \geq e^{-\sigma(t)} \delta_m, \quad me^{\sigma(s)} \gg \delta_m^{-1} \left( \frac{\delta_m}{\delta_6,s} \right)^{0+}
\end{equation}

(Here \(\delta_{6,s} := \tilde{d}_S(\bar{u}(s))\)). We also have

\begin{equation}
\left| \int_{t} K(\bar{u}) + E_{ext}(\bar{u}) \, dt \right| \geq e^{-\sigma(t_3)} \delta_m, \quad me^{\sigma(t_3)} \gg \delta_m^{-1} \left( \frac{\delta_m}{\delta_6} \right)^{0+}
\end{equation}
8.1. $\Theta(u) = -1$. Choose $m$ such that

$$\frac{\delta_m}{\delta_b} m_H \gg m \gg \delta_m^{-1} \left( \frac{\delta_m}{\delta_b} \right)^{0+} m_H$$

From Proposition 3.6 we get for $t \in I_V$ (with $k := k(\delta_V)$)

$$K(u(t)) \leq -k,$$

Hence

$$\left| \int_{t_2}^{t_3} -K(u) + E_{ext}(\vec{u}) \, dt \right| \gtrsim \frac{1}{\delta_b} \delta_b |I_V| + \sum_{s \in L, s \neq t_2, t_3} \frac{\delta_m}{\log \left( \frac{\delta_m}{\delta_b} \right)} |I[s]| + \frac{\delta_m}{\log \left( \frac{\delta_m}{\delta_b} \right)} (|I[t_1]| + |I[t_2]|)$$

$$+ \delta_m m_H$$

$$\gg \min \left( \frac{\delta_m}{\log \frac{\delta_m}{\delta_b}}, \frac{1}{\delta_b} \right) (t_3 - t_2) \delta_b + \delta_m m_H$$

(8.12)

$$\gg (t_3 - t_2 + m) \delta_b,$$

This is a contradiction.

8.2. $\Theta(u) = 1$. By (2.10)

$$\|\vec{u}(t)\|_H^2 + \|u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^\ast \lesssim 1$$

Let $0 < \beta \ll 1$ be a small constant. There are two cases:

**Case 1:**

$$\int_{I_V} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \, dt \geq \beta |I_V|$$

Choose $m$ that satisfies (8.11). From Proposition 3.6 and (8.14) we have for $t \in I_V$ (with $k := k(\delta_V)$)

$$K(u) \gtrsim \min (k, \epsilon \|\nabla u\|_{L^2}^2) \gtrsim \|\nabla u\|_{L^2}^2$$

Hence by using (8.7)

$$\int_{I_V} -K(u) + E_{ext}(\vec{u}) \, dt \lesssim -\beta |I_V|$$

Hence by a similar argument used in (8.13)

$$\left| \int_{t_2}^{t_3} -K(u) + E_{ext}(\vec{u}) \, dt \right| \gg (t_3 - t_2 + m) \delta_b$$
This is a contradiction.

Case 2:

\[ \int_{I_V} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \, dt \leq \beta |I_V| \]

Since \( K \geq 0 \), we can upgrade this inequality to

\[ (8.15) \int_{I_V} \int_{\mathbb{R}^N} |u|^2 \, dx \, dt \leq \beta |I_V| \]

This allows to use an argument in [5]. Following [23], we see from (8.4) that there exists \( M_j \in 2^N \) such that for \( M \geq M_j \) there exist \( J_j \subset I_j \) and \( R_j \sim \frac{1}{M_j} \) that satisfy \(|J_j| \sim \frac{1}{M_j}\)

\[ (8.16) \int_{|x-c_j| \leq R_j} |\nabla \tilde{P}_M u(t)|^2 + |\tilde{P}_M u(t)|^2 \, dx \geq \eta^C, \]

and

\[ (8.17) \int_{|x-c_j| \leq R_j} |\tilde{P}_M u(t)|^2 \, dx \geq \eta^C \]

for \( t \in J_j = \{ \hat{t}_j, \ldots \} \). Here \( C \gg 1 \) is a well-chosen large positive constant. Hence using (8.15) we see that there exists \( j_0 \in \{1, \ldots, J\} \) such that

\[ (8.18) |J_{j_0}| \lesssim \beta |I_{j_0}| \]

Without loss of generality we may assume that \( \hat{t}_{j_0} - a_{j_0} \leq b_{j_0} - \hat{t}_{j_0} \). Let \( K := [a_j, t_1] \subset I_j \). Notice that on each \( K \), we have, by (8.14), (2.4), and (2.14)

\[
\begin{align*}
\|[\tilde{u}]\|_{L^{\frac{2(d+1)}{d-1}} B_{2(d+1)}^2(K)} & \lesssim \|\tilde{u}(a_j)\|_{H^1} + \|u\|_{L^{\frac{2(d+1)}{d-1}} B_{2(d+1)}^2(K)} \|u\|_{H^{2-\frac{d}{2(d+1)}} L^{\frac{2(d+1)}{d+1}}(K)} \\
& \lesssim 1 + \eta^{2-\frac{d}{2(d+1)}} \|u\|_{L^{\frac{2(d+1)}{d-1}} B_{2(d+1)}^2(K)} \lesssim 1.
\end{align*}
\]

Hence a continuity argument shows that

\[ (8.19) \|[\tilde{u}]\|_{L^{\frac{2(d+1)}{d-1}} B_{2(d+1)}^2(I_j) \times L^{\frac{2(d+1)}{d+1}} B_{2(d+1)}^2(I_j)} \lesssim 1. \]

Hence

\[ \text{Let } C' \text{ be a large constant such that all the statements below are true. It is shown in [23] that there exist } M_j \in 2^N \text{ such that (say) } \|\tilde{P}_{M_j} u\|_{L^\infty L^{\frac{2(d+1)}{d-1}}(I_j)} \geq \eta^{C'} \text{ with } M_j |I_j| \lesssim 1. \]

Once this observation made, the arguments to get to get (8.19) and (8.17) are almost the same as those in [23]; therefore they are omitted.
\[
\|u\|^{2^* - 2} \leq \frac{2(4d+1)}{L_t^2 2^{(d+1)/2} \chi} (I_j) \lesssim \eta^{2^* - 2} \|u\| \leq \frac{2(4d+1)}{L_t^2 2^{(d+1)/2} \chi} (I_j) \lesssim 1
\]

Next we prove the following estimates:

**Lemma 8.1.** Let \(0 < \eta_2 \ll \eta^C\) (with \(C\) defined just below (8.10)) and let \(B \gg 1\). Then there exist \(M'_{j_0} \sim M_{j_0}\) and \(\alpha := \alpha(B) \gg 1\) such that for some \(R' \leq \frac{\alpha(B)}{M_{j_0}}\), \(\tilde{t}_{j_0} \in I_{j_0}\), and \(\tilde{t}'_{j_0} \in I_{j_0}\) we have

\[
(A_1) : \quad |\tilde{t}_{j_0} - \tilde{t}'_{j_0}| \geq B_0 B(B\tilde{R}')^2 ; \quad (A_2) : \quad \|\tilde{P}_{< M_{j_0}' \tilde{L}(\tilde{t}_{j_0})}\|_{\mathcal{H}(|x-c_{j_0}| < R')} \gtrsim \eta^C ; \quad (A_3) : \quad \|u\|_{\frac{d+1}{2} L^2(\tilde{t}_{j_0}, \tilde{t}'_{j_0})} \leq \eta_2 ; \quad (A_4) : \quad \|\tilde{P}_{< M_{j_0}' \tilde{L}(\tilde{t}_{j_0})} \tilde{u}(\tilde{t}_{j_0})\|_{L^2(\tilde{t}_{j_0}, \tilde{t}'_{j_0})} \ll \eta^C .
\]

The proof of Lemma 8.1 is postponed to Subsection 8.3. We define \(\tilde{v} := (v, \partial_t v)\) solution of the free Klein-Gordon equation with initial data

\[
\tilde{v}(\tilde{t}_{j_0}) := \chi_{\tilde{R}'} \tilde{P}_{< M_{j_0}' \tilde{L}(\tilde{t}_{j_0})}, \quad \chi_{\tilde{R}'}(x) := \chi\left(\frac{x - c_{j_0}}{\tilde{R}'}\right)
\]

(Recall that \(\chi\) is defined below (8.5)). Assume that \(\tilde{t}_{j_0} > \tilde{t}_{j_0}\) (resp. \(\tilde{t}_{j_0} < \tilde{t}_{j_0}\))

Using the well-known dispersive estimate (see e.g [8]) we see that there exists \(\gamma > 0\) such that for \(t \geq \tilde{t}_{j_0}\) (resp. \(t \leq \tilde{t}_{j_0}\))

\[
(8.21) \quad \left\| \cos (\tilde{t}_{j_0}) (D) v(\tilde{t}_{j_0}) + \frac{\sin (\tilde{t}_{j_0}) (D)}{(D)} \partial_t v(\tilde{t}_{j_0}) \right\|_{B^{0^*}_{2^*, 2}} \lesssim \frac{1}{|t - \tilde{t}_{j_0}|^{\frac{d+1}{2}}} \left( \left\| v(\tilde{t}_{j_0}) \right\|_{\frac{d+1}{2} L^2(\tilde{t}_{j_0}, \tilde{t}'_{j_0})} + \left\| \partial_t v(\tilde{t}_{j_0}) \right\|_{\frac{d+1}{2} L^2(\tilde{t}_{j_0}, \tilde{t}'_{j_0})} \right)
\]

\[
\lesssim \frac{1}{|t - \tilde{t}_{j_0}|^{\frac{d+1}{2}}} \left( \left\| \chi_{\tilde{R}'} \tilde{P}_{< M_{j_0}' \tilde{L}(\tilde{t}_{j_0})} \tilde{u}(\tilde{t}_{j_0}) \right\|_{L^2(\tilde{t}_{j_0}, \tilde{t}'_{j_0})} + \left\| \chi_{\tilde{R}'} \tilde{P}_{< M_{j_0}' \tilde{L}(\tilde{t}_{j_0})} \partial_t \tilde{u}(\tilde{t}_{j_0}) \right\|_{L^2(\tilde{t}_{j_0}, \tilde{t}'_{j_0})} + \left\| \chi_{\tilde{R}'} \tilde{P}_{< M_{j_0}' \tilde{L}(\tilde{t}_{j_0})} \tilde{u}(\tilde{t}_{j_0}) \right\|_{L^2(\tilde{t}_{j_0}, \tilde{t}'_{j_0})} \right)
\]

\[
\lesssim B^{-\gamma} .
\]

Here at the last step of (8.21) we used the definition of the Besov norms that uses norms of the normalized Paley-Littlewood projectors and we combined (8.14) with the estimates below (derived from Bernstein-type inequalities)

\[
M \in \{2^N, 0\}, M \lesssim \left(\tilde{R}'\right)^{-1} : \quad \left\| \tilde{P}_M \chi_{\tilde{R}'} \right\|_{B^{0^*}_{2^*, 2}} \lesssim M'' \chi_{\tilde{R}'} \left\|_{L^2} \lesssim M'' \left(\tilde{R}'\right)^2 ,
\]

\[
(M, r) \in 2^N \times \left\{ \frac{d+1}{2}, \frac{d}{2} \right\} : \quad M \gg \left(\tilde{R}'\right)^{-1} : \quad \left\| \tilde{P}_M \chi_{\tilde{R}'} \right\|_{B^{0^*}_{2^*, 2}} \lesssim M'' \left\| \nabla^3 \chi_{\tilde{R}'} \right\|_{L^2} \lesssim M'' \left(\tilde{R}'\right)^{-1},
\]

\[
\left\| \tilde{P}_{< M'_{j_0} \tilde{L}(\tilde{t}_{j_0})} \tilde{u}(\tilde{t}_{j_0}) \right\|_{L^2} \lesssim \left\| \tilde{u}(\tilde{t}_{j_0}) \right\|_{H^1_1} , \quad \left\| \tilde{P}_{< M'_{j_0} \tilde{L}(\tilde{t}_{j_0})} \tilde{u}(\tilde{t}_{j_0}) \right\|_{B^{0^*}_{2^*, 2}} \lesssim M_{j_0} \left\| \tilde{u}(\tilde{t}_{j_0}) \right\|_{H^1_1} , \quad \left\| \tilde{P}_{< M'_{j_0} \tilde{L}(\tilde{t}_{j_0})} \tilde{u}(\tilde{t}_{j_0}) \right\|_{L^2} \lesssim M_{j_0} \left\| \tilde{u}(\tilde{t}_{j_0}) \right\|_{L^2} .
\]
Assume that $\bar{T}_{j_0} > \bar{T}_{j_0}$ (resp. $\bar{T}_{j_0} < \bar{T}_{j_0}$). A similar argument that uses again well-known dispersive estimates (see e.g. [8]) and integration w.r.t. time shows that if $J \subset [\bar{T}_{j_0}, \infty)$ (resp. $J \subset (-\infty, \bar{T}_{j_0}]$) then there exists a constant (that we still denote by $\gamma$) such that for $t \geq \bar{T}_{j_0}$ (resp. $t \leq \bar{T}_{j_0}$)

\begin{equation}
\left\| \cos\left((t - \bar{T}_{j_0})(D)\right)v(\bar{T}_{j_0}) + \frac{\sin\left((t - \bar{T}_{j_0})(D)\right)}{(D)} \partial_t v(\bar{T}_{j_0}) \right\|_{A(J)} \lesssim B^{-\gamma}
\end{equation}

Here $A(J) \in \left\{ L_t^{\frac{2(d+1)}{p-2}} L_x^{\frac{2(d+1)}{d-2}} (J), L_t^{\frac{2(d+1)}{d-2}} B^{\frac{d}{(d+1)} \frac{2}{2}} (J) \right\}$. Let $\bar{w} := \bar{u} - \bar{v}$. By (2.6) there exists $c > 0$ such that

$$
\left\| \bar{w}(\bar{T}_{j_0}) \right\|_{H^s} \lesssim \left\| \bar{w}(\bar{T}_{j_0}) \right\|_{H^s} + \| u \|^2_{L^2_x} \| u \|_{L^\infty_t B^{\frac{d}{(d+1)} \frac{2}{2}} (\bar{T}_{j_0}, \bar{T}_{j_0})} \lesssim \| \bar{w}(\bar{T}_{j_0}) \|_{H^s} + \eta^c_2
$$

The Plancherel theorem\footnote{More precisely we use $\int_{\mathbb{R}^d} f(x) \hat{g}(x) \, dx = \int_{\mathbb{R}^d} f(\xi) \hat{\psi}(\xi) \, d\xi$ with $f := |u(t)|^2 - 2 u(t)$ and $g := \partial_t u(t)$} and a Paley-Littlewood decomposition for $f$ and $g$ show that

$$
\begin{align*}
\left\| \frac{1}{2} \| u(\bar{T}_{j_0}) \|^2_{L^2_x} - \| u(\bar{T}_{j_0}) \|_{L^2_x}^2 \right\|_{H^s} & = 2^s \int_{T_{j_0}}^{\bar{T}_{j_0}} \int_{\mathbb{R}^d} |u(t)|^2 \cdot |u(t)| \partial_t u \, dx \, dt \\
& \lesssim \left\| \| u \|^2_{L^2_x} \right\|_{L^\infty_t B^{\frac{d}{(d+1)} \frac{2}{2}} (\bar{T}_{j_0}, \bar{T}_{j_0})} \left\| \| u \|_{L^\infty_t B^{\frac{d}{d+1} \frac{d}{d+1}} (\bar{T}_{j_0}, \bar{T}_{j_0})} \right\|_{H^s} \lesssim \eta^c_2.
\end{align*}
$$

Hence using also (A2), (A4), and (8.21) we get

\begin{equation}
E(\tilde{w}(\bar{T}_{j_0})) \leq \frac{1}{2} \left( \left\| \tilde{w}(\bar{T}_{j_0}) \right\|_{H^s}^2 - \frac{1}{2^s} \| u(\bar{T}_{j_0}) \|_{L^2_x}^2 \right) + o(\eta^C)
\end{equation}

\begin{equation}
\leq \frac{1}{2} \left( \left\| \tilde{w}(\bar{T}_{j_0}) \right\|_{H^s}^2 - \frac{1}{2^s} \| u(\bar{T}_{j_0}) \|_{L^2_x}^2 \right) + o(\eta^C)
\end{equation}

\begin{equation}
\leq E(\tilde{u} - \frac{\tilde{u}^*}{2^s}).
\end{equation}

Let $\tilde{w}$ be the solution of (1.1) with data $\tilde{w}(\bar{T}_{j_0}) := \tilde{w}(\bar{T}_{j_0})$. We claim that $K(\tilde{w}(\bar{T}_{j_0})) = K(u(\bar{T}_{j_0})) > 0$. Clearly from (A2) and (A4) we see that there exists $c > 0$ such that $\| \tilde{w}(\bar{T}_{j_0}) \|_{H^s}^2 \leq \| u(\bar{T}_{j_0}) \|_{H^s}^2 - c \eta^C$. Then, from $\tilde{u}(0) \in H^s$, the conservation of energy, $K(\tilde{u}(\bar{T}_{j_0})) \geq 0$ and (2.10), we see that $\| \tilde{u}(\bar{T}_{j_0}) \|_{H^s}^2 < \| \nabla W \|_{H^s}^2 + \bar{c}^2$. Therefore using also (8.23) we get $\| \tilde{w}(\bar{T}_{j_0+1}) \|_{H^s}^2 < \| \nabla W \|_{H^s}^2$ and the claim holds by (2.12). Next we write a perturbation lemma:

**Lemma 8.2.** Let $\bar{w}^1$ and $\bar{w}^2$ solutions of (1.1). Let $a \in (-T_-(w_1), T_+(w_1))$ and $I := [a, b] \subset (-T_-(w_2), T_+(w_2))$. Let $\nu > 0$. Assume that there exists a constant $C_0 < \infty$ such that

\begin{equation}
\| \bar{w}^1 \|_{H^s} + \| \bar{w}^2 \|_{L^\infty_t B^{\frac{d}{(d+1)} \frac{2}{2}} (I)} \leq C_0.
\end{equation}

There exists $0 < \epsilon_0 := \epsilon_0(C_0, \nu) \ll 1$ such that if for all $j \in \{1, 2, 3\}$

\begin{equation}
\left\| \cos ((t - a)(D)) (w_1(a) - w_2(a)) + \frac{\sin ((t - a)(D))}{(D)} (\partial_tw_1(a) - \partial_tw_2(a)) \right\|_{X_j(I)} \leq \epsilon_0,
\end{equation}

then $I \subset (-T_-(w_1), T_+(w_1))$ and

\begin{equation}
\|w_1 - w_2\|_{L_t^\infty L_x^2(I)} \leq \nu.
\end{equation}

Here $X_1(I) := L_t^{\frac{2(4+\alpha)}{4+\alpha}} L_x^{\frac{2(d+1)}{d+1}} (I)$, $X_2(I) := L_t^{\frac{2(4+\alpha)}{4+\alpha}} B^{\frac{1}{2}}_{\frac{2(d+1)}{d+1}, 2}(I)$, and $X_3(I) := L_t^\infty B^{\frac{3}{2}}_{2, 2}(I)$.

The proof of Lemma 8.2 is postponed to Subsection 8.4.

Let $\tilde{w}_2 := \tilde{w}$. By (8.23), $K(w_2(\tilde{t}_j)) = K(w(\tilde{t}_j)) > 0$, we see from [11] that $\|w_2\|_{L_t^{\frac{2(d+\alpha)}{4+\alpha}} L_x^{\frac{2(d+1)}{d+1}} (\mathbb{R})} < \infty$. Hence, using also (2.20), the Payne-Sattinger argument [11] and (8.23) that there exists $c > 0$ such that

\begin{equation}
\|w_2(t)\|_{L_t^\infty L_x^2} \geq \|W\|_{L_t^\infty L_x^2} - ct^3.
\end{equation}

Notice also that (8.25) is satisfied on $I := [\tilde{t}_j, t_3]$ by (8.21) and (8.22). Consequently (8.26) holds. But then we see from (8.27), (2.2), and Proposition 3.8 that it contradicts $\tilde{d}_S(\tilde{u}(t_3)) = \delta_b$.

8.3. Proof of Lemma 8.1. In this subsection we prove Lemma 8.1

We may assume WLOG that $c_{j_0} = 0$ and $\tilde{t}_{j_0} = 0$.

Given $R' > 0$, let $\chi_{R'}(t, x) := \chi \left( \frac{\sqrt{|t|}}{t + R} \right)$ with $\chi$ defined below (8.5). Let $\alpha \geq 0$ and let $M_0 := 2^\alpha M_{j_0}$. Let $C' \gg 1$ be a constant large enough such that all the estimates in the sequel of this proof are true.

We first prove the following result:

Result: Let

\[ X_1(M_\alpha) := \int_0^{b_{j_0}} \int |\partial_t \tilde{P}_{\leq M_\alpha} u| |u|^{2^* - 2} u - \tilde{P}_{\leq M_\alpha} |u|^{2^* - 2} u| \ dx \ dt, \]

\[ X_2(M_\alpha) := \int_0^{b_{j_0}} \int |\partial_t \tilde{P}_{\leq M_\alpha} u| |u|^{2^* - 2} u - |\tilde{P}_{\leq M_\alpha} u|^{2^* - 2} \tilde{P}_{\leq M_\alpha} u| \ dx \ dt. \]

Then there exists $2^{C' \eta - C} M_{j_0} \geq M_{j_0}' \geq M_{j_0}$ such that

\begin{equation}
X_1(M_{j_0}') + X_2(M_{j_0}') \ll \eta^C.
\end{equation}

\[ \text{[17] that is } K(w_2(t)) > 0, t \in I(w_2) = \mathbb{R} \]
Here we also applied the Young inequality for sequences at the last line of (8.29) and \(M\) such that (8.28) holds with \(\alpha \geq 1\). Then (8.19), (8.24), and the fundamental theorem of calculus yield

\[
\sum_{\alpha=1}^\tilde{\alpha} \bar{X}_1 (M_\alpha)
\]

\[
\leq \sum_{\alpha=1}^\tilde{\alpha} \left\| \frac{\partial_t \bar{P}_{<M_\alpha} u}{L_t^{\frac{d+1}{2d}} L_x^{\frac{d+1}{2d}} (\bar{0},b_{\alpha0})} \right\| \| \bar{P}_{\geq M_\alpha} (|u|^{2^* - 2} u) \|_{L_t^2 \frac{d+1}{d+3} L_x^{\frac{d+1}{d+3}} (\bar{0},b_{\alpha0})}
\]

\[
\leq \sum_{\alpha=1}^\tilde{\alpha} \sum_{L_1 < M_\alpha \leq L_2} \left( \frac{(L_1)}{(L_2)} \right)^\frac{1}{2^*} \left\| \frac{\partial_t \bar{P}_{<M_\alpha} u}{L_t^{\frac{d+1}{2d}} L_x^{\frac{d+1}{2d}} (\bar{0},b_{\alpha0})} \right\| \| \bar{P}_{\geq M_\alpha} (|u|^{2^* - 2} u) \|_{L_t^2 \frac{d+1}{d+3} L_x^{\frac{d+1}{d+3}} (\bar{0},b_{\alpha0})}
\]

\[
\leq \sum_{L_1 \leq L_2} \left( \frac{(L_1)}{(L_2)} \right)^\frac{1}{2^*} \left\| \frac{\partial_t \bar{P}_{<M_\alpha} u}{L_t^{\frac{d+1}{2d}} L_x^{\frac{d+1}{2d}} (\bar{0},b_{\alpha0})} \right\| \| \bar{P}_{\geq M_\alpha} (|u|^{2^* - 2} u) \|_{L_t^2 \frac{d+1}{d+3} L_x^{\frac{d+1}{d+3}} (\bar{0},b_{\alpha0})}
\]

\[
\leq 1,
\]

and

\[
\sum_{\alpha=1}^\tilde{\alpha} \bar{X}_2 (M_\alpha)
\]

\[
\leq \sum_{\alpha=1}^\tilde{\alpha} \left\| \frac{\partial_t \bar{P}_{<M_\alpha} u}{L_t^{\frac{d+1}{2d}} L_x^{\frac{d+1}{2d}} (\bar{0},b_{\alpha0})} \right\| \| u |^{2^* - 2} u - |P_{<M_\alpha} u|^{2^* - 2} \bar{P}_{<M_\alpha} u \|_{L_t^2 \frac{d+1}{d+3} L_x^{\frac{d+1}{d+3}} (\bar{0},b_{\alpha0})}
\]

\[
\leq \sum_{\alpha=1}^\tilde{\alpha} \sum_{L_1 < M_\alpha \leq L_2} \left( \frac{(L_1)}{(L_2)} \right)^\frac{1}{2^*} \left\| \frac{\partial_t \bar{P}_{<M_\alpha} u}{L_t^{\frac{d+1}{2d}} L_x^{\frac{d+1}{2d}} (\bar{0},b_{\alpha0})} \right\| \| u |^{2^* - 2} \bar{P}_{<M_\alpha} u \|_{L_t^2 \frac{d+1}{d+3} L_x^{\frac{d+1}{d+3}} (\bar{0},b_{\alpha0})}
\]

\[
\leq \sum_{L_1 \leq L_2} \left( \frac{(L_1)}{(L_2)} \right)^\frac{1}{2^*} \left\| \frac{\partial_t \bar{P}_{<M_\alpha} u}{L_t^{\frac{d+1}{2d}} L_x^{\frac{d+1}{2d}} (\bar{0},b_{\alpha0})} \right\| \| u |^{2^* - 2} \bar{P}_{<M_\alpha} u \|_{L_t^2 \frac{d+1}{d+3} L_x^{\frac{d+1}{d+3}} (\bar{0},b_{\alpha0})}
\]

\[
\leq 1.
\]

Here we also applied the Young inequality for sequences at the last line of (8.29) and (8.30). The pigeonhole principle implies that there exists \(2^{\tilde{C} \eta^C} M_{\bar{J}_0} \geq M_\alpha \geq M_{J_0}\) such that (8.28) holds with \(M'_{J_0} := M_\alpha\).
Next, given $0 \leq T_1 \leq b_j$, we prove some estimates, namely (8.31).

First we show that $K \left( \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \right) \geq 0$.
Indeed we may assume WLOG that $k \ll 1$. First assume that $\| \nabla u(0) \|_{L^2} \ll k$ with $k$ defined just below (8.11). Then by expansion of the gradient, the Hardy inequality, and Plancherel equality we get $\| \nabla (\chi_{R'}(0) \bar{P}_{<M_j'} u(0)) \|_{L^2} \lesssim \frac{\| \bar{P}_{<M_j'} u(0) \|_{L^2}}{R'}$.

Hence we see from (2.2) that $K \left( \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \right) \geq k \| \nabla (\chi_{R'}(0) \bar{P}_{<M_j'} u(0)) \|_{L^2}^2$. Now assume that $\| \nabla u(0) \|_{L^2} \gtrsim k$. The estimate $K (u(0)) \geq k$ that follows from Proposition 3.6 (2.9), the equality $\frac{1}{2} \int_{R'} |\nabla W|^2 \ dx = \frac{1}{2} \int_{R'} |W|^2 \ dx$ that comes from $-\Delta W = W^{2^* - 1}$, and the estimate $I \left( \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \right) \leq I (u(0))$ yield $I \left( \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \right) < I(W) - \frac{k}{2}$. Hence, using also (2.3), we see that there exists $c := c(\delta, \nu) > 0$ such that

$$K \left( \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \right) \geq \| \nabla \left( \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \right) \|_{L^2}^2 \left( 1 - C_2^2 \| \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \|_{L^2}^{2^* - 2} \right) \geq \| \nabla \left( \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \right) \|_{L^2}^2 \left( 1 - C_2^2 \left( \frac{\| W \|_{L^2}^{2^*} - \frac{M_j}{k} \right)^{2^* - 2} \right) \geq c \| \nabla \left( \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \right) \|_{L^2}^2.$$

Hence in both cases $K \left( \chi_{R'}(0) \bar{P}_{<M_j'} u(0) \right) \geq 0$.
Then observe from (2.10) that

$$E(\chi_{R'}(0), \bar{P}_{<M_j'} \bar{u}(0)) \geq \left( \frac{1}{2} - \frac{1}{2^*} \right) \| \chi_{R'}(0) \bar{P}_{<M_j'} \bar{u}(0) \|_{L^2}^2$$

and consequently

$$E \left( \chi_{R'}(0), \bar{P}_{<M_j'} \bar{u}(0) \right) \geq \| \bar{P}_{<M_j'} u(0) \|_{L^2}^2 + O \left( \int_{|x| \approx R'} |\nabla \bar{P}_{<M_j'} u(0)|^2 \ dx \right)$$

Since $\sum_{Q \in 2^n} \int_{|x| \approx R'} |\nabla \bar{P}_{<M_j'} u(0)|^2 \ dx \lesssim \| u(0) \|_{L^2}^2$, one can find a sequence $\{ Q_{1,n} \}_{n \geq 1}$ such that $Q_{1,n} \leq 2C' \eta^{-C} \nu$ and $\int_{|x| \approx Q_{1,n} R_j} |\nabla \bar{P}_{<M_j'} u(0)|^2 \ dx \ll \eta^C$. Hence by (8.16)

$$E \left( \chi_{R_n}(0), \bar{P}_{<M_j'} \bar{u}(0) \right) \gtrsim \eta^C, \ R_n := Q_{1,n} R_j$$

Applying $\bar{P}_{<M_j'}$ to (1.1), multiplying the result by $\chi_{R_n} \partial_x \bar{P}_{<M_j'} u$ and integrating on the region $[0, T_1] \times R^d$ we get

$$E \left( \chi_{R_n}(T_1), \bar{P}_{<M_j'} \bar{u}(T_1) \right) = E \left( \chi_{R_n}(0), \bar{P}_{<M_j'} \bar{u}(0) \right) + O (X_1(T_1)) + O (X_2(T_1)) + O (X_3(T_1, R_n)).$$
with
\[ X_3 \left( T_1, R' \right) := \int_0^{T_1} \int_{|x| < R'_0} \frac{1}{R'_0} \left( |\partial_t \tilde{P}_{<M'_0} u|^2 + \|
abla \tilde{P}_{<M'_0} u|^2 + |\tilde{P}_{<M'_0} u|^2 + |\tilde{P}_{<M'_0} u|^2 \right) \, dx \, dt. \]

Since \( \sum_{n \geq 1} X_3 \left( T_1, R_n \right) \lesssim \frac{T}{R_{00}} \) one can then find a subsequence \( \{Q_{2,n} := Q_{2,n}(T_1)\}_{n \geq 1} \) of \( \{Q_{1,n}\}_{n \geq 1} \) and a sequence \( \{b_n\}_{n \geq 1} \) such that \( 1 \leq b_n \leq 2 C' \eta^{-C} \frac{T}{R_{00}} \), \( Q_{1, b_n} \leq Q_{2,n} \leq Q_{1,b_{n+1}} \), and \( X_3 \left( T_1, Q_{2,n} M_{j_0}' \right) \ll \eta^C. \)

Hence, using also (8.3) we get
\[ \left\| \tilde{P}_{<M'_0} \tilde{u}(T_1) \right\|_{H(\{ |x| \leq Q_{2,n} R_{00} + T_1 \})} \gtrsim \eta^C. \]

The Hardy inequality and the Plancherel theorem show that \( \sum_{Q_{2,n} \in 2^{Q_{2,n}}} \int_{|x| < R_{00} = Q_{2,n}} \frac{|\tilde{P}_{<M'_0} u(T_1)|^2}{|x|^2} \, dx \lesssim \| u(T_1) \|_{L^2}^2 \). Hence one can find a subsequence \( \{Q_{3,n} := Q_{3,n}(T_1)\}_{n \geq 1} \) of \( \{Q_{2,n}\}_{n \geq 1} \) and a sequence \( \{b_n\}_{n \geq 1} \) such that \( 1 \leq b_n \leq 2 C' \eta^{-C} \), \( Q_{2,b_n} \leq Q_{3,n} \leq Q_{2,b_{n+1}} \), and
\[ \left\| \frac{P_{<M'_0} u(T_1)}{|x|} \right\|_{L^2(\{ |x| \leq Q_{3,n} R_{00} \})} \ll \eta^C. \]

Hence there exists \( \tilde{Q}_1 \) (it suffices to choose \( \tilde{Q}_1 := Q_{3,1}(T_1) \)) such that \( \tilde{Q}_1 \leq (C') C' \eta^{-C} \), \( \tilde{Q}_1 \ll \frac{T}{R_{00}} (C') C' \eta^{-C} \), and
\[ (8.31) \]
\[ \left\| \tilde{P}_{<M'_0} \tilde{u}(T_1) \right\|_{H(\{ |x| \leq \tilde{Q}_1 R_{00} + T_1 \})} \gtrsim \eta^C, \text{ and } \left\| \frac{\tilde{P}_{<M'_0} u(T_1)}{|x|} \right\|_{L^2(\{ |x| \leq \tilde{Q}_1 R_{00} \})} \ll \eta^C. \]

Next, given \( B \gg 1 \), we write an algorithm that will allow us to define \( \tilde{I}_{j_0}, \tilde{I}_{j_0}, \alpha := \alpha(B), \text{ and } \tilde{R} \).

Let \( S := 0, R := R_{j_0}, \text{ and } T := B(M_{j_0} R)^B R. \) Then, as long as \( \|u\|_{L^\infty_t (S,T)} > \eta_2, \) do the following: \( R := \tilde{Q}(T) R_{j_0} + T, S := T, \text{ and } T := S + B(M_{j_0} R)^B R. \) Observe that algorithm has a finite number of steps in view of (8.13), Moreover \( (A1), (A2), (A3), \text{ and } (A4) \) hold with \( \tilde{I}_{j_0} := S, \tilde{I}_{j_0} := T, \text{ and } \tilde{R} := \tilde{Q}(T) R_{j_0} + T. \)

8.4. Proof of Lemma \( 8.26 \)
Let \( F(v) := \|v\|_{L^2}^{-2} v \) and \( e := w_1 - w_2. \) It is made of four steps:

1. \( \|w_2\|_{L^\infty_t H(I)} + \|\tilde{u}_2\|_{L^\infty_t H(I)} \lesssim C_0 1. \)

Indeed let \( 1 \gg \alpha > 0. \) We can find a partition of \( I \) into subintervals \( (J_l := [a_l, b_l])_{1 \leq l \leq \tilde{l}} \) such that \( \|w_2\|_{L^\infty_t H(I)} \ll \alpha \) for all \( 1 \leq l < \tilde{l} \)
and \(\|w_2\|_{L^{\frac{2(d+1)}{d-2}}_t L^{\frac{2(d+1)}{d-2}}_x} \leq \alpha\). Let \(K_1 := [a_1, b] \subset J_1\) and \(Y(K_1) := \|w_2\|_{L^{\frac{2(d+1)}{d-2}}_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (K_1)\) + \(\|\bar{w}_2\|_{L^\infty_t L^\infty_x} (K_1)\). By (2.6) and (2.4) we see that there exists \(C > 0\) such that

\[
Y(K_1) \leq C(C_0 + \alpha^{2^* - 2} \|w_2\|_{L^\infty_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (K_1)) \\
\leq C(C_0 + \alpha^{2^* - 2} Y(K_1)).
\]

Hence a continuity argument shows that \(Y(J_1) \leq 2C C_0\). Iterating over \(l\) we get \(Y(J_l) \leq (2C C_0)^l\). Hence by summing over \(l\) we get \(\lesssim C_0\).

(2) Short-time perturbation argument.

Let \(1 \gg \alpha > 0\) and \(J = [\bar{a}, \bar{b}] \subset I\) such that \(\|w_2\|_{L^{\frac{2(d+1)}{d-2}}_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (J) \leq \alpha\). Then there exists \(1 \gg \mu > 0\) such that if \(\epsilon' \leq \mu\) and for all \(j \in \{1, 2, 3\}\)

\[
\left\|\cos \left((t - \bar{a}) \langle D\rangle \right) e(\bar{a}) + \frac{\sin \left((t - \bar{a}) \langle D\rangle \right)}{(D)} \partial_t e(\bar{a})\right\|_{L^1_t(J)} \leq \epsilon',
\]

(*) \(\|F(w_1) - F(w_2)\|_{L^{\frac{2(d+1)}{d-2}}_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (J) \leq \epsilon'\).

Indeed let \(Z(J) := \|e\|_{L^{\frac{2(d+1)}{d-2}}_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (J) + \|e\|_{L^{\frac{2(d+1)}{d-2}}_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (J)\). We get from (8.25) and (2.6) followed by (2.5)

\[
Z(J) \lesssim \epsilon' + \left(\|w_1\|_{L^{\frac{2(d+1)}{d-2}}_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (J) + \|w_2\|_{L^{\frac{2(d+1)}{d-2}}_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (J)\right)^rac{2^* - 3}{2^* - 2}
\]

\[
+ \left(\|w_1\|_{L^{\frac{2(d+1)}{d-2}}_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (J) + \|w_2\|_{L^{\frac{2(d+1)}{d-2}}_t B^{\frac{1}{d-2}}_2 \frac{2(d+1)}{d-2} L^{\frac{2(d+1)}{d-2}}_x} (J)\right)^rac{2^* - 3}{2^* - 2}
\]

\[
\lesssim \epsilon' + Z^{2^* - 1}(J) + \alpha Z^{2^* - 2}(J) + \alpha^{2^* - 3} Z^2(J) + \alpha^{2^* - 2} Z(J)
\]

Hence a continuity argument shows that \(Z(J) \lesssim \epsilon'\). Hence (*) holds.

(3) Long-time perturbation argument.
Observe from the first step and \[\text{(8.24)}\] that we can partition \(I\) into subintervals \((J_j := [a_j, b_j])_{1 \leq j \leq \bar{j}}\) such that \(\bar{j} \lesssim C_0 1,\)

\[
\max \left( \left\| w_2 \right\|_{L_t^{2(d+1)/d - 1} B_{2(d+1)/d - 2}^{1/2}(J_j)}, \left\| w_2 \right\|_{L_t^{2(d+1)/d - 1} B_{2(d+1)/d - 2}^{1/2}(J_j)} \right) = \alpha \quad \text{for} \quad 1 \leq j \leq \bar{j} - 1 \text{ and max}\left( \left\| w_2 \right\|_{L_t^{2(d+1)/d - 1} B_{2(d+1)/d - 2}^{1/2}(J_j)}, \left\| w_2 \right\|_{L_t^{2(d+1)/d - 1} B_{2(d+1)/d - 2}^{1/2}(J_j)} \right) \leq \alpha. \]

We claim that for all \(1 \leq j \leq \bar{j}\) there exists a positive constant \(C(j)\) such that (with \(F(x) := |x|^{2 - 2x}\))

\[
\left\| F(w_2) - F(w_1) \right\|_{L_t^{2(d+1)/d - 1} B_{2(d+1)/d - 2}^{1/2}(J_j)} \leq C(j)\epsilon_0.
\]

Indeed by induction assume that \[\text{(8.32)}\] holds for all \(1 \leq j \leq k\) with \(k \in \{1, ..., \bar{j} - 1\}\). Then from the Duhamel formula above \[\text{(2.6)}\] we see that

\[
\cos \left( (t - a_k + 1)(D) \right) e(a_k + 1) + \frac{\sin \left( (t - a_k + 1)(D) \right)}{(D)} \partial_t e(a_k + 1)
\]

\[
= \cos \left( (t - a)(D) \right) e(a) - \frac{\sin \left( (t - a)(D) \right)}{(D)} \partial_t e(a) - \sum_{j=1}^k b_j \sin \left( (t - t')(D) \right) (F(w_1(t')) - F(w_2(t'))) \ dt'
\]

Hence we get from \[\text{(2.6)}\]

\[
\left\| \cos \left( (t - a_k + 1)(D) \right) e(a_k + 1) + \frac{\sin \left( (t - a_k + 1)(D) \right)}{(D)} \partial_t e(a_k + 1) \right\|_{X_j(J_{k+1})} \leq \epsilon_0 + \sum_{j=1}^k C(j)\epsilon_0 \leq \mu.
\]

Hence an application of the short-time perturbation argument shows that \[\text{(8.32)}\] also holds for \(j = k + 1\). This proves the claim.

Hence by summation

\[
\text{(8.33)} \quad \left\| F(w_1) - F(w_2) \right\|_{L_t^{2(d+1)/d - 1} B_{2(d+1)/d - 2}^{1/2}(I)} \lesssim \sum_{j=1}^{\bar{j}} C(j)\epsilon_0.
\]

(4) Estimate \[\text{(8.26)}\] : this follows from \[\text{(8.25)}\], the embeddings \(B_{2^r, 2}^0 \hookrightarrow L^{2^r}\) and \(\dot{H}^1 \hookrightarrow L^{2^r}\), and \[\text{(8.33)}\].

9. Proof of Proposition \[\text{3.16}\]

In this section we prove Proposition \[\text{3.16}\].

By using time translation symmetry of \[\text{(1.1)}\] if necessary we may assume WLOG that \(t_0 = 0\). The continuity of the flow, Proposition \[\text{3.6}\] and Proposition \[\text{3.8}\] show that \(\Theta (u([0, T_+(u)])) = \pm 1\) if \(\Theta (u(0)) = \pm 1\).
9.1. \( \Theta(u(t_0)) = -1 \). We prove that \( T_+(u) < \infty \). We adapt an argument of Payne-Sattinger [29] to energies slightly above that of the ground states. Let \( y := \langle u, u \rangle \). Elementary computations show that

\[
\partial_t y = 2 \partial_t u, u \tag{9.1}
\]

and

\[
\partial_t^2 y = 2 \partial_t \| u \|_{L^2}^2 + 2 \partial_t \langle u, u \rangle = 2 \| \partial_t u \|_{L^2}^2 - 2 K(u) - 2 \| u \|_{L^2}^2 \tag{9.2}
\]

Since \( \tilde{d}_S(\tilde{u}(t)) \geq \delta_b, t \geq t_2 \), we get from Proposition [3, 22] (with \( k := k(\delta_b) \))

\[
\partial_t \tilde{u}, \tilde{u} \geq 2 \| \partial_t u \|_{L^2}^2 + 2k - 2 \| u \|_{L^2}^2 \tag{9.3}
\]

We have

\[
K(\tilde{u}) + \| u \|_{L^2}^2 = 2^* E(\tilde{u}) - 2^* \| \partial_t u \|_{L^2}^2 - 2^* \left( \frac{1}{2} - \frac{1}{2^*} \right) (\| \nabla u \|_{L^2}^2 + \| u \|_{L^2}^2) \]

From \( \tilde{u}(0) \in \mathcal{H} \), the conservation of energy, and (2.12) we get

\[
E(\tilde{u}) \leq \left( \frac{1}{2} - \frac{1}{2^*} \right) \| \nabla u \|_{L^2}^2 + \epsilon^2 \]

Hence collecting these estimates and (9.1)

\[
\partial_t^2 y \geq 2 \left( 1 + \frac{2^*}{2} \right) \| \partial_t u \|_{L^2}^2 - 2 \times 2^* \epsilon^2 + 2 \times 2^* \left( \frac{1}{2} - \frac{1}{2^*} \right) \| u \|_{L^2}^2 \tag{9.4}
\]

From (9.2) and (9.3) we see that \( \partial_t^2 y \geq \epsilon^2 \). Hence, assuming that \( T_+ = \infty \) this leads to

\[
y(t) \to \infty, \text{ as } t \to \infty \tag{9.5}
\]

Since \( |\partial_t y| \leq 2 \| \partial_t u \|_{L^2} \| u \|_{L^2} \) we get for \( t \gg 1 \)

\[
\partial_{tt} y \geq 2 \left( 1 + \frac{2^*}{2} \right) \left( \frac{\partial_t y}{2y} \right) \tag{9.6}
\]

Hence \( \partial_t^2 y^{-\alpha} \leq 0 \) with \( \alpha := \frac{1}{2} \left( \frac{2^*}{2} - 1 \right) \). This contradicts (9.3).

9.2. \( \Theta(u(t_0)) = 1 \). Throughout this section, we work with complex-valued functions so that we can use the results in [11]. Unless previously defined we denote by \( \tilde{f} = f_1 + i f_2 \) a complex-valued function. Let \( E(\tilde{f}) := \frac{1}{2} \| \tilde{f} \|_{L^2}^2 - \frac{1}{2^*} \| \text{Re}(D^{-1} \tilde{f}) \|_{L^{2^*}}^2 \)

and \( E_{wo}(\tilde{f}) := \frac{1}{2} \| \tilde{f} \|_{L^2}^2 - \frac{1}{2^*} \| \text{Re}(D^{-1} \tilde{f}) \|_{L^{2^*}}^2 \). If \( \tilde{f} := q f_1 + i f_2 \) with \( q \in \{ D, \langle D \rangle \} \) and \( f_1, f_2 \) real-valued functions then \( \tilde{d}_S(\tilde{f}) := \tilde{d}_S((f_1, f_2)) \).

We recall the following result proved in [11] and obtained by a concentration compactness procedure (see e.g [3, 22], relying on [21])
Proposition 9.1. Let \( (v_n(0), \partial_t v_n(0)) \) be a bounded sequence in \( H \). Let \( \bar{v}_n(0) := \langle D \rangle v_n(0) - i \partial_t v_n(0) \). Let \( \{\hat{v}_n(t) := e^{i(D)t} \bar{v}_n(0)\}_{n \in \mathbb{N}} \) be a sequence that is bounded in \( L^2 \). Then there exist \( \hat{\varphi} \in L^2(\mathbb{R}^d) \) such that, up to a subsequence, \( \hat{v}_n \to \hat{\varphi} \) in \( L^2 \). Clearly from \([11]\), we know that we may assume that

\[
\lim_{n \to \infty} \|e^{i(D)t} \hat{v}_n(t)\|_{L^\infty B_{\infty \cdot \infty}} = 0,
\]

for \( j \neq l \) we have

\[
\lim_{n \to \infty} \langle \hat{v}_n^j(t), \hat{v}_n^l(t) \rangle = 0,
\]

and

\[
\lim_{n \to \infty} |\langle \hat{v}_n^j(t), e^{i(D)t} \hat{v}_n^k(t) \rangle| = 0.
\]

In particular

\[
\lim_{n \to \infty} \|\hat{v}_n(t)\|_{L^2}^2 - \sum_{j=0}^k \|\hat{v}_n^j(t)\|_{L^2}^2 - \|e^{i(D)t} \hat{v}_n^k\|_{L^2}^2 = 0.
\]

We define the following statement \( \mathcal{P}(E) \): if \( \bar{u} \) is a solution of \([11]\) with data \((u_0, u_1) \in H\), forward maximal time of existence \( T_+(\bar{u}) \), and energy \( E \) that satisfies

1. \( \delta \bar{S}(\bar{u}(t)) \geq \delta_b, t \in [0, T_+(\bar{u})) \)
2. \( \Theta(u(t)) = 1, t \in [0, T_+(u)) \)
3. \( E(\bar{u}) \leq E_{wa}(\bar{W}) + \bar{c}^2 \)

Then \( T_+(u) = \infty \) and \( \|u\|_{L_t^{2(d+1)} L_x^{\frac{2(d+1)}{d-2}}(\mathbb{R}^+)} < \infty \). We define

\[
E_c = \sup \{E > 0, \mathcal{P}(E) \text{ holds} \}
\]

Clearly from \([11]\), we know that we may assume that \( E_c \geq E_{wa}(\bar{W}) \). Assume toward a contradiction that \( E_c = E_{wa}(\bar{W}) + \bar{c}^2 \) with \( \bar{c} < c \). We prove the following claim:

Claim 9.2. There exists a critical element \( U_c \), i.e a solution of \([11]\), that satisfies \( E(U_c) = E_c \), (1) and (2). Here \( \tilde{U}_c := \langle D \rangle U_c - i \partial_t U_c \). Moreover \( U_c \) does not scatter, i.e \( \|U_c\|_{L_t^{2(d+1)} L_x^{\frac{2(d+1)}{d-2}}(\mathbb{R}^+) \to \infty} \)

By definition of \( E_c \) there exists a sequence \( \{\bar{u}_n\}_{n \geq 1} \) of solutions \( \bar{u}_n := \langle D \rangle u_n - i \partial_t u_n \) with forward maximal time of existence \( T_+(u_n) \) that satisfy (1), (2) and (3), \( E(\bar{u}_n) \to E_c \) from above and \( \|u_n\|_{L_t^{2(d+1)} L_x^{\frac{2(d+1)}{d-2}}([0, T_+(u_n))]} \to \infty \).

We may assume WLOG that for all \( t \in [0, T_+(u_n)) \), \( \delta \bar{S}(\bar{u}_n(t)) \geq \delta_b^* \) with \( \delta_b^* \gg \delta_b \). If not we may find \( \bar{u}_n \in I(u) \) such that \( \delta_b \leq \delta \bar{S}(\bar{u}_n(t_n)) \leq \delta_b^* \). Hence, if \( \delta \partial \bar{S}(\delta_{u_n}(\bar{u}_n)(t_n)) \geq 0 \) (resp. \( \delta \partial \bar{S}(\bar{u}_n(t_n)) < 0 \)) then we can apply Proposition 3.11 to

\[\text{[18]}\] This implies scattering, by a standard procedure
We claim that there exists $t_n \in [0, T_+ (u_n))$ such that $\| \nabla u_n (t_n) \|_{L^2} \geq 1$. If not (*) would hold, with (*) : $\| \nabla u_n (t) \|_{L^2} \ll 1$, $t \in [0, T_+ (u_n))$. Now defining $X_n (t) := \| u_n \|_{L_t^2 \cdot L_x^{\frac{2(d+1)}{d-2}}} (0, T_+ (u_n))$, $t \in [0, T_+ (u_n))$ and $r$ such that $\frac{1}{r} = \frac{d-1}{2(d+1)} - \frac{1}{2d}$ we get for some $0 < c < 1$

$$X_n (t) \lesssim \| u_n (0) \|_{\mathcal{H}} + \| u_n \|_{L_t^2 \cdot L_x^{\frac{2(d+1)}{d-2}}} (0, T_+ (u_n)) \| u_n \|_{L_t^{2(d+1)} \cdot L_x^{\frac{2(d+1)}{d-2}}} (0, T_+ (u_n))$$

$$\lesssim \| u_n (0) \|_{\mathcal{H}} + \| u_n \|_{L_t^2 \cdot L_x^{\frac{2(d+1)}{d-2}}} (0, T_+ (u_n)) \| u_n \|_{L_t^{2(d+1)} \cdot L_x^{\frac{2(d+1)}{d-2}}} (0, T_+ (u_n))$$

$$\lesssim \| u_n (0) \|_{\mathcal{H}} + o (X_n^{1+(2^* - 2) (1-c)} (t)),$$

where at the third line we used (2.2), (*), and $H^{\frac{2(d+1)}{d-1} \to L^r}$, followed by a continuity argument at the fourth line. This is a contradiction. Hence, by letting $\tilde{u}_n (t) := \tilde{u}_n (t + t_n)$, we may assume that

$$\| \nabla u_n (0) \|_{L^2} \geq 1$$

Unless otherwise specified, let $0 < \alpha < 1$, $k \gg 1$, and $n \gg 1$ such that all the statements below are true. Applying Proposition 9.1 to $\tilde{\varphi}_n (0) := u_n (0) := (D) u_n (0) - i \partial_t u_n (0)$, we get

$$\tilde{u}_n (0) = \sum_{j=0}^k e^{-i(D) t \frac{t}{h_n}} T_n \tilde{\varphi}_n + u_n^k.$$ 

We define the nonlinear concentrating wave associated with $\left\{ e^{-i(D) t n} T_n \tilde{\varphi}_n \right\}$ in the following fashion

$$\tilde{u}_n^j := T_n \tilde{U}_n^j \left( \frac{-t_n}{h_n} \right)$$

with

$$\tilde{U}_n^j = e^{i t \left( D \frac{t}{h_n} \right)} \tilde{\varphi}_n - i \int_{-\infty}^t e^{i \left( t - s \right) \left( D \frac{t}{h_n} \right)} F \left( U_n^j (s) \right) ds,$$

$F (x) := |x|^{\alpha - 2} x$, $\tau_{h_n} := \lim_{n \to \infty} \tau_{h_n} := \lim_{n \to \infty} \frac{-t_n}{h_n}$ (up to a subsequence) and $U_n^j := \Re \left( (D \frac{t}{h_n})^{-1} \tilde{U}_n^j \right)$. Observe that $\tilde{U}_n^j$ satisfies $(i \partial_t + (D) \tilde{U}_n^j = F (U_n^j)$ and, consequently

$$\tilde{U}_n^j = (D) t \tilde{U}_n^j \left( \frac{-t_n}{h_n} \right) - i \partial_t \tilde{U}_n^j$$

$$\left( \partial_t - \Delta + \left( \frac{h_n}{h_n} \right)^2 \right) U_n^j = F (U_n^j)$$

Let $u_n^j := \Re \left( (D)^{-1} \tilde{U}_n^j \right)$. Then observe that $u_n^j = \frac{h_n}{h_n} \tilde{U}_n^j \left( \frac{-t_n}{h_n} \right)$ and that $u_n^j$ is a solution of (1.1). We will use $u_n^j$ when we apply Result 9.10.

We now turn to the local existence of $U_n^j$ around $\tau_{h_n}$. We claim the following:
Claim: Let $0 < \delta \ll 1$. Then there exists a small open interval $J$ containing $\tau_{J,\infty}^1$ and that does not depend on $n$ such that

$$
(9.11) \quad \left\| \cos\left( t (D_n^J)^{\frac{1}{2}} \right) \Re\left( \langle (D_n^J) \rangle^{-\frac{1}{2}} \phi \right) - \frac{\sin\left( t (D_n^J)^{\frac{1}{2}} \right) \Im\left( \phi \right)}{t \frac{2(d+1)}{d-2} \frac{2(d+1)}{d-2} (J) \langle (D_n^J) \rangle} \right\| \leq \delta
$$

Proof. Assume now that $h_{\delta,\infty}^1 = 1$. Then (9.11) clearly follows from the dominated convergence theorem and (2.6).

Assume now that $h_{\delta,\infty}^1 = 0$.

First assume that $\tau_{J,\infty}^1 \in \mathbb{R}$. Let $\tilde{\phi}_0^J$ be a Schwartz function such that $\supp \left( \tilde{\phi}_0^J \right) \subset \mathbb{R}^d - \{(0,\ldots,0)\}$ and $\| \tilde{\phi}_0^J - \phi \| \ll \delta$. Let $\left( q(x), X_{n,j}(\vec{f}), X_j(\vec{f}) \right) := \left( \cos(x), \Re\left( \langle (D_n^J) \rangle^{-\frac{1}{2}} \vec{f} \right), \Re\left( D^{-\frac{1}{2}} \vec{f} \right) \right)$ or $\left( q(x), X_{n,j}(\vec{f}), X_j(\vec{f}) \right) := \left( \cos(x), \Im(\vec{f}), \Im(\vec{f}) \right)$. Then integration by parts (using in particular the formula $e^{iyx} = \frac{x \cdot \nabla(e^{iyx})}{|x|^2}$) and the Schwartz nature of $\tilde{\phi}_0^J$ allow to use the dominated convergence theorem to conclude that

$$
\left\| q(t(D_n^J)^{\frac{1}{2}}) X_n^J (\tilde{\phi}_0^J) - q(t\vec{f}) X_n^J (\phi) \right\| \ll \delta.
$$

and (2.8) that

$$
\left\| q(t(D_n^J)^{\frac{1}{2}}) X_n^J (\tilde{\phi}_0^J) \right\| \ll \delta.
$$

Now assume that $\tau_{J,\infty}^1 \in (-\infty, \infty)$. We only prove the claim for $\tau_{J,\infty}^1 = \infty$: the proof for $\tau_{J,\infty}^1 = -\infty$ is a straightforward modification of the proof for $\tau_{J,\infty}^1 = \infty$. It suffices to prove that for $S > 1$, $\| q(t(D_n^J)^{\frac{1}{2}}) \phi \| _{L^2_t L^2_x} \leq \delta$. Following a similar scheme as the one from (10.20) to (10.23), we get the former estimate, taking into account that $\| e^{\pm t(D_n^J)^{\frac{1}{2}}} \phi \| _{B_{t}^{0} L_{x}^{2}} \leq \frac{1}{|t|^{2(d+1)}} \| \phi \| _{B_{t}^{0} L_{x}^{2}} \left( \frac{2(d+1)}{d-2} \right)^{\frac{1}{4}}$.

Then, by (2.8), (2.24), and (2.25), we get

$$
(9.12) \quad \left\| w \right\| _{L^2_t L^2_x} \ll \left\| \cos\left( t (D_n^J)^{\frac{1}{2}} \right) \Re\left( \langle (D_n^J) \rangle^{-\frac{1}{2}} \phi \right) - \frac{\sin\left( t (D_n^J)^{\frac{1}{2}} \right) \Im\left( \phi \right)}{t \frac{2(d+1)}{d-2} \frac{2(d+1)}{d-2} (J) \langle (D_n^J) \rangle} \right\| + \left\| w \right\| _{L^2_t L^2_x} \left\| \frac{\sin\left( t (D_n^J)^{\frac{1}{2}} \right) \Im\left( \phi \right)}{t \frac{2(d+1)}{d-2} \frac{2(d+1)}{d-2} (J) \langle (D_n^J) \rangle} \right\| + \left\| w \right\| _{L^2_t L^2_x} \left\| \frac{\sin\left( t (D_n^J)^{\frac{1}{2}} \right) \Im\left( \phi \right)}{t \frac{2(d+1)}{d-2} \frac{2(d+1)}{d-2} (J) \langle (D_n^J) \rangle} \right\| \ll \left\| \phi \right\| _{L^2} + \left\| w \right\| _{L^2_t L^2_x} \left\| \frac{\sin\left( t (D_n^J)^{\frac{1}{2}} \right) \Im\left( \phi \right)}{t \frac{2(d+1)}{d-2} \frac{2(d+1)}{d-2} (J) \langle (D_n^J) \rangle} \right\| \ll \left\| \phi \right\| _{L^2} + \left\| w \right\| _{L^2_t L^2_x} \left\| \frac{\sin\left( t (D_n^J)^{\frac{1}{2}} \right) \Im\left( \phi \right)}{t \frac{2(d+1)}{d-2} \frac{2(d+1)}{d-2} (J) \langle (D_n^J) \rangle} \right\| ,
$$

and
Let $\vec{U}$ denote the solution on $J$. By construction we get the following estimates

$$
\|w - \tilde{w}\|_{C^0_{L^2}(J)} \lesssim \|w - \tilde{w}\|_{C^0_{L^2}(J)} \left( \left\|w\right\|_{L^p_x(L^2_t)}^{2^* - 2} + \left\|\tilde{w}\right\|_{L^p_x(L^2_t)}^{2^* - 2} \right)
$$

From the above claim, (9.12), and (9.13), we see (see e.g., [16]) that we can construct a solution on $J$ by using a standard fixed point argument.

Let $h_{\infty} := \lim_{n \to \infty} h_i^n$.

**Remark 9.3.** Assume that $h_{\infty} = 1$. If $\tau_{\infty} \notin \{-\infty, \infty\}$, then

$$
\left\|U^i_{\infty}(t) - e^{it(D)} \tilde{\phi}\right\|_{L^2} \lesssim \left\|U^i_{\infty}(t) - e^{it(D)} \tilde{\phi}\right\|_{L^2} + \left\|e^{it(D)} - e^{it\tau_{\infty}(D)}\right\|_{L^2} \to_{t \to \tau_{\infty}} 0,
$$

since, by construction of $\tilde{U}^i_{1}$,

$$
\left\|U^i_{\infty}(t) - e^{it(D)} \tilde{\phi}\right\|_{L^2} \lesssim \left\|U^i_{\infty}(t) - e^{it(D)} \tilde{\phi}\right\|_{L^2} + \left\|e^{it(D)} - e^{it\tau_{\infty}(D)}\right\|_{L^2} \to_{t \to \tau_{\infty}} 0.
$$

If $\tau_{\infty} \in \{-\infty, \infty\}$ then

$$
\left\|U^i_{\infty}(t) - e^{it(D)} \tilde{\phi}\right\|_{L^2} \to_{t \to \tau_{\infty}} 0
$$

**Remark 9.4.** Notice that the existence of $\tilde{U}^i_{\infty}$ can be proved using a similar scheme to prove the existence of $U^i_{\infty}$ replacing (2.8) with (2.7). This allows to define $I_{\text{max}}(\tilde{U}^i_{\infty})$. By construction we get the following estimates

$$
\tau_{\infty} \notin \{-\infty, \infty\} : \left\|U^i_{\infty}(t) - e^{it\tau_{\infty}(D)}\phi\right\|_{L^2} \to_{t \to \tau_{\infty}} 0
$$

$$
\tau_{\infty} \in \{-\infty, \infty\} : \left\|U^i_{\infty}(t) - e^{it\tau_{\infty}(D)}\phi\right\|_{L^2} \to_{t \to \infty} 0
$$

Next we write the following theorem that allows to approximate well $\tilde{U}^i_{\infty}$ with $\tilde{U}^i_{\infty}$ in the most difficult case, i.e $h_{\infty} = 0$. The proof is essentially well-known: see Lemma 4.2 in [12]. We write down a proof for convenience of the reader in Section [10].

**Lemma 9.5.** (see also Lemma 4.2 in [12]) Assume that $h_{\infty} = 0$. Let $\tilde{U}^i_{\infty}$ be defined by the following

$$
\tilde{U}^i_{\infty} := e^{itD} \tilde{\phi}^i - i \int_{\tau_{\infty}}^{t} e^{i(t-s)D} F \left( U^i_{\infty}(s) \right) ds,
$$
with \( U_j^t := \mathbb{R}\left(D^{-1}U_j^t(\infty)\right) \).

Assume that \( \|U_j^t\|_{L^{\frac{d+2}{d-2}}L^{\frac{2(d+2)}{d-2}}(\mathbb{R})} \) is finite. Then

\[
(9.16)
\]

\[
K \in \mathbb{R} \text{ bounded} : \lim_{n \to \infty} \left\| U_n^t - U_\infty^t \right\|_{L^\infty L_2^2(K)} + \left\| U_n^t - U_\infty^t \right\|_{L^\infty L_2^2(K)} \to 0, \text{ and}
\]

\[
\lim_{n \to \infty} \left\| U_n^t - U_\infty^t \right\|_{L^\infty L_2^2(\mathbb{R})} = 0.
\]

**Remark 9.6.** Notice that if \( h_\infty^j = 1 \) then all the estimates in this lemma clearly hold, with \( U_j^t \) substituted for \( U_\infty^t \).

**Remark 9.7.** Observe from (2.8), Lemma 9.3 and the previous remark that

\[
\left\| U_j^t(\tau_n^j) - e^{it\Delta(D)^{\frac{1}{4}}}\phi^j \right\|_{L_2^2} \leq \left\| U_j^t(\infty) \right\|_{L^{\frac{d+2}{d-2}}L^{\frac{2(d+2)}{d-2}}(\mathbb{R})} \to_{n \to \infty} 0
\]

Since \( T_n^k e^{it\Delta(D)^{\frac{1}{4}}} = e^{-it\Delta(D^j_t)T_n^j} \) and the \( L^2 \) norm is invariant by application of \( T_n^j \) we get

\[
(9.17)
\]

\[
\left\| \tilde{\omega}_n^j(0) - e^{-it\Delta(D)^{\frac{1}{4}}}T_n^j\phi^j \right\|_{L_2^2} \to_{n \to \infty} 0
\]

Let \( A_1 := \{ j \in [0..k] : h_\infty^j = 0 \} \) and \( A_2 := \{ j \in [0..k] : h_\infty^j = 1 \} \).

Next we prove the following result:

**Result 9.8.** If \( h_\infty^j = 1 \) (resp. \( h_\infty^j = 0 \)) then \( K(U_j^t(t)) > 0 \) (resp. \( K(U_j^t(t)) > 0 \)) in a neighborhood of \( \tau_\infty^j \). We also have \( K(w_n^k) \geq 0 \) (with \( w_n^k := \mathbb{R}\left(D^{1/4}e^{it\Delta(D)^{\frac{1}{4}}}\phi^j\right) \)).

**Proof.** Assume that \( \tau_\infty^j \neq \{-\infty, \infty\} \). By (2.10), (9.10), Proposition 6.6 and (9.9)

\[
E_{\omega_0}(\tilde{W}) - \tilde{c}^2 \geq E(\hat{\omega}_n(0)) - \frac{1}{2}K(\tilde{u}_n(0)) + 2\tilde{c}^2
\]

\[
\geq \sum_{j=0}^k \frac{\|\hat{\omega}_n(0)\|^2_d}{d} + 2\tilde{c}^2
\]

\[
\geq \sum_{j=0}^k \frac{\|\hat{\omega}_n(0)\|^2_d}{d} + \|\tilde{w}_n^{0,k}\|^2_d + \tilde{c}^2
\]

\[
\geq \sum_{j=0}^k \frac{e^{it\Delta(D)^{\frac{1}{4}}}\phi^j}d\|\tilde{w}_n^{0,k}\|^2_d + \|\tilde{w}_n^{0,k}\|^2_d + \tilde{c}^2
\]

\[
\geq \sum_{j \in A_1} G\left(\mathbb{R}\left(D^{-1}e^{it\Delta(D)^{\frac{1}{4}}}\phi^j\right)\right) + \sum_{j \in A_2} G\left(\mathbb{R}\left(D^{-1}e^{it\Delta(D)^{\frac{1}{4}}}\phi^j\right)\right) + G(w_n^k)
\]

By (2.12) we see that \( K(w_n^k) \geq 0 \).

Therefore, if \( \mathbb{R}\left(D^{-1}e^{it\Delta(D)^{\frac{1}{4}}}\phi^j\right) \neq 0 \) (resp. \( \mathbb{R}\left(D^{1/4}e^{it\Delta(D)^{\frac{1}{4}}}\phi^j\right) \neq 0 \)), then

\[
K\left(\mathbb{R}\left(D^{-1}e^{it\Delta(D)^{\frac{1}{4}}}\phi^j\right)\right) > 0 \text{ (resp. } K\left(\mathbb{R}\left(D^{-1}e^{it\Delta(D)^{\frac{1}{4}}}\phi^j\right)\right) > 0 \text{ ) by (2.12).}
\]

Hence we see from Remark 9.3, Remark 6.4 and (9.14), that \( K(U_j^t(t)) > 0 \) (resp. \( K(U_j^t(t)) > 0 \)) if \( \mathbb{R}\left(D^{-1}e^{it\Delta(D^j_t)^{\frac{1}{4}}}\phi^j\right) = 0 \) (resp. \( \mathbb{R}\left(D^{-1}e^{it\Delta(D)^{\frac{1}{4}}}\phi^j\right) = 0 \)).

\[
\left\| U_j^t(t) \right\|_{H^1} \to_{t \to \tau_\infty^j} 0 \text{ (resp. } \left\| U_j^t(t) \right\|_{H^1} \to_{t \to \tau_\infty^j} 0 \text{ ) and, consequently, we see, by (2.22), that the results also holds.}
\]

If \( \tau_\infty^j = \pm \infty \) then we claim that
\[ h^j \infty = 1 : \left\| \Re \left( \langle D \rangle^{-1} e^{it(D)} \phi^j \right) \right\|_{L^2} \to_{t \to \pm \infty} 0 \]  
(9.18) 
\[ h^j \infty = 0 : \left\| \Re \left( D^{-1} e^{itD} \phi^j \right) \right\|_{L^2} \to_{t \to \pm \infty} 0 \]

Assume that \( h^j \infty = 1 \) (resp. \( h^j \infty = 0 \)). Then the claim easily follows for a smooth function \( g \): it is enough to use the standard dispersive estimate (see for example [8])

\[
\left\| \langle D \rangle^{-1} e^{it(D)} g \right\|_{B^{s}_{2,2}} \lesssim \frac{1}{|t|^{1/2}} \| g \|_{B^{s}_{2,2}},
\]

(9.19)

( resp. \( \| D^{-1} e^{itD} g \|_{B^{s}_{2,2}} \lesssim \frac{1}{|t|^{1/2}} \| g \|_{B^{s}_{2,2}} \)) and the embedding \( B^{s}_{2,2} \hookrightarrow L^2 \) (resp. \( B^{s}_{2,2} \hookrightarrow L^2 \)). The claim for \( \phi^j \in L^2 \) follows from \( L^2 \) approximation by a smooth function and (2.2), using a standard procedure.

From (9.18, Remark 9.3 and Remark 9.4) we see that the result holds.

We prove the following result:

**Result 9.9.**

\[
E(\bar{u}_n) = \sum_{j \in A_1} E_{w\alpha}(\bar{U}_{\infty}^j) + \sum_{j \in A_2} E(\bar{U}_n^j) + E(\bar{\omega}_n^k) + O(\alpha)
\]

(9.20)

**Proof.** From (9.9), Remark 9.3, and Remark 9.4 we see that

\[
\frac{1}{2} \| \bar{u}_n(0) \|_{L^2}^2 = \frac{1}{2} \sum_{j=0}^{k} \| e^{-i\langle D \rangle t_j} \bar{T}_j \phi_j^j \|_{L^2}^2 + \frac{1}{2} \| \bar{u}_n^k \|_{L^2}^2 + O(\alpha)
\]

(9.21)

\[
= \frac{1}{2} \sum_{j=0}^{k} \left( \| e^{i\tau_j \langle D \rangle} \bar{T}_j \phi_j^j \|_{L^2}^2 + \frac{1}{2} \| \bar{u}_n^k \|_{L^2}^2 + O(\alpha) \right)
\]

\[
= \frac{1}{2} \sum_{j \in A_1} \left( \| \bar{T}_j^j(\tau_j^j) \|_{L^2}^2 + \frac{1}{2} \sum_{j \in A_2} \| \bar{U}_n^j(\tau_n^j) \|_{L^2}^2 + \frac{1}{2} \sum_{j \in A_2} \| \bar{\omega}_n^k \|_{L^2}^2 + O(\alpha) \right).
\]

It remains to prove the decoupling of the potential part of the energy, i.e.

\[
\tilde{F}(u_n(0)) = \sum_{j \in A_1} \tilde{F}(U_n^j(\tau_n^j)) + \sum_{j \in A_2} \tilde{F}(\bar{U}_n^j(\tau_n^j)) + \tilde{F}(\bar{\omega}_n^k) + O(\alpha),
\]

with \( \tilde{F}(g) := \frac{\| g \|_{L^2}^2}{2} \).

From \( B^s_{\infty,2} \cap B^s_{1,2} \hookrightarrow B^s_{2,1} \) (see e.g. Proposition 2.22 in [2]), \( B^0_{2,1} \hookrightarrow L^2 \), \( \| \cdot \|_{L^2} \), and (9.9) we see that \( \tilde{F}(\bar{w}_n^k) = O(\alpha) \).

Assume that \( \tau^j = \pm \infty \). Then in view of (9.18, Remark 9.3 and Remark 9.4) we get

\[10\]
Indeed, first observe that if $\vec{\phi}$, we also get from (9.24)

\[
\| \Re \left( (\langle D \rangle)^{-1} e^{i\tau_n^D \phi} \right) \|_{L^2} = O(\alpha).
\]

Assume that $h_\infty = 0$. We claim that

(9.22) \[ \frac{1}{h_n \tau_{n,\pm}^2} \| (\langle D \rangle)^{-1} e^{-i\tau_n^D \phi} \|_{L^2} = O(\alpha). \]

Indeed, observe first that if $\vec{\phi}_0$ is a Schwartz function such that $\| \vec{\phi}_0 - \vec{\phi}_0 \|_{L^2} = O(\alpha)$, then we get from (2.2) and Plancherel theorem that

\[ \left\| \Re \left( (\langle D \rangle)^{-1} e^{-i\tau_n^D \phi} \right) \right\|_{L^2} \lesssim \| \vec{\phi}_0 - \vec{\phi}_0 \|_{L^2} = o(\alpha). \]

Hence it suffices to prove that (9.22) with $\vec{\phi}_0$ Schwartz. But then, we get from (9.19) and $L^{(2')}$, it follows that

\[
\frac{1}{h_n \tau_{n,\pm}^2} \left\| (\langle D \rangle)^{-1} e^{i\tau_n^D \phi} \right\|_{L^2} = o(\alpha).
\]

Now assume that $h_\infty = 1$. Then (9.22) also holds from (9.18).

Hence in view of (9.23), it is enough to prove that

\[
\left| \hat{F} \left( \sum_{j \in \mathbb{Z}} \Re \left( (\langle D \rangle)^{-1} T_n^j e^{i\tau_n^D \phi} \right) \right) - \sum_{j \in \mathbb{Z}} \hat{F} \left( \Re (D^{-1} T_n^j e^{i\tau_n^D \phi}) \right) \right| = O(\alpha).
\]

Assume that $h_\infty = 0$. We see from the Plancherel theorem, the equality $\langle D \rangle^{-1} T_n^j = h_n T_n^j (\langle D \rangle)^{-1}$, (2.2), and the estimate $\| (\langle D \rangle)^{-1} g - D^{-1} g \|_{L^2} \lesssim \| (\langle D \rangle)^{-2} D^{-2} g \|_{L^2} \rightarrow 0$, that follows from the Hormander-Mikhlin multiplier that

(9.23) \[
\hat{F} \left( \Re \left( (\langle D \rangle)^{-1} T_n^j e^{i\tau_n^D \phi} \right) \right) = \hat{F} \left( \Re (D^{-1} T_n^j e^{i\tau_n^D \phi}) \right) \approx \hat{F} \left( \Re (D^{-1} T_n^j e^{i\tau_n^D \phi}) \right) \approx O(\alpha).
\]

We also get from $D^{-1} T_n^j = h_n T_n^j D^{-1}$ the estimate below

(9.24) \[
\left| \hat{F} \left( \Re (D^{-1} T_n^j e^{i\tau_n^D \phi}) \right) - \hat{F} \left( \Re (D^{-1} T_n^j e^{i\tau_n^D \phi}) \right) \right| \approx \hat{F} \left( h_n T_n^j D^{-1} \Re (e^{i\tau_n^D \phi}) \right) \approx O(\alpha).
\]
If \( h_\infty^j = 1 \), then it is easier to see from \([2.2]\) that \([9.23]\) and \([9.24]\) hold: the proof is left to the reader.

Hence letting

\[
\hat{\phi}^j := \begin{cases} 
\Re \left( \langle D \rangle^{-1} e^{i \tau_\infty^j(D)} \hat{\phi}^j \right), & \hbar_\infty^j = 1 \\
\Re \left( D^{-1} e^{i \tau_\infty^j D} \hat{\phi}^j \right), & \hbar_\infty^j = 0,
\end{cases}
\]

it is enough to prove

\[
F \left( \sum_{j \in [0, k]} \hbar_n^j T_n^j \hat{\phi}^j \right) - \sum_{j \in [0, k]} F \left( \hbar_n^j T_n^j \hat{\phi}^j \right) = O(\alpha)
\]

We may assume WLOG that \( \hat{\phi}^j \) is compactly supported. Let \( \hbar_n^j := \frac{h_n^j}{\hbar_n^j} \) and \( x_n^j := \frac{x_n^j - x_n^l}{h_n^l} \). From \([9.8]\) we see that

\[
h_n^j \to 0, \text{ or } h_n^j \to \infty, \text{ or } |x_n^j| \to \infty,
\]

From the elementary estimate

\[
\left| \sum_{i=0}^{m} x_i^j \right|^2 - \sum_{i=0}^{m} |x_i^j|^2 \lesssim \sum_{i,j} \sum_{|x_i^j|^2 \neq 1} |x_i^j|^2 - 1 |x_i^j|
\]

(that can be proved by induction on \( m \)) we see that

\[
F \left( \sum_{j \in [0, k]} \hbar_n^j T_n^j \hat{\phi}^j \right) - \sum_{j \in [0, k]} F \left( \hbar_n^j T_n^j \hat{\phi}^j \right) \lesssim \sum_{j \neq l} \int_{\mathbb{R}^4} (\hbar_n^j)^{2 - 1} |T_n^j \hat{\phi}^j|^2 - 1 |T_n^j \hat{\phi}^j| dx
\]

\[
\lesssim \sum_{j \neq l} \int_{\mathbb{R}^4} \left| \phi^j \right|^2 - 1 (x) \left| \phi^j \right| \left( \frac{x - x_n^l}{h_n^l} \right) dx.
\]

Clearly the last quantity goes to zero as \( n \to \infty \), in view of \([9.26]\).

\[
\square
\]

Letting \( \alpha \to 0 \) (hence \( k \to \infty \) and \( n \to \infty \)) in \([9.20]\) we see that

\[
h_\infty^j = 1 : E(\bar{U}_n^j) \leq E_c \\
h_\infty^j = 0 : E_{\text{wa}}(\bar{U}_n^j) \leq E_c
\]

If \( E(\bar{U}_\infty^j) < E_{\text{wa}}(\bar{W}) \) for all \( j \) such that \( h_\infty^j = 0 \) and \( E(\bar{U}_\infty^j) < E_{\text{wa}}(\bar{W}) \) for all \( j \) such that \( h_\infty^j = 1 \) then we conclude from \([17]\) and \([11]\) that \( U_n^j \) and \( \bar{U}_n^j \) exist globally in time and scatters, i.e \( \| U_n^j \|_{L^2_{\infty} \text{ to } L^{2(d+1)}_{\infty} (\mathbb{R})} < \infty \) and \( \| U_n^j \|_{L^2_{\infty} \text{ to } L^{2(d+1)}_{\infty} (\mathbb{R})} < \infty \).

Next we prove a perturbation result:

**Result 9.10.** Let \( w \) be a solution of \([1.1]\) and \( \bar{w} \) be a solution of

\[
\partial_t \bar{w} - \triangle \bar{w} + \bar{w} = |\bar{w}|^{2-\alpha} \bar{w} + eq(\bar{w})
\]
Let $t_0 \in (-T_0(w), T_0(w))$ and $I := [t_0, \cdot]$. Assume that there exists $C_1$ such that 
\[ \| \tilde{w} \|_{L_t^{2(d+1)} L_x^{d+2}} \leq C_1. \]
Then there exists $\epsilon_0 := \epsilon_0(C_1)$ such that if $\epsilon \leq \epsilon_0$ and (a) (resp. (a')) hold, with
\[
(9.30) \quad (a) : \| \cos ((t-t_0)(D)) (w(t_0) - \tilde{w}(t_0)) + \frac{\sin((t-t_0)(D))}{(D)} (\partial_t w(t_0) - \partial_t \tilde{w}(t_0)) \|_{L_t^{\frac{d+2}{d}} L_x^{\frac{d+2}{d}}} \leq \epsilon \quad \text{and} \quad \| \text{eq}(\tilde{w}) \|_{L_t^1 L_x^2(I)} \leq \epsilon'.
\]
\[
(9.31) \quad (a') : \| \tilde{w}(t_0) - \tilde{w}(t_0) \|_{L^2} \leq \epsilon' \quad \text{and} \quad \| \text{eq}(\tilde{w}) \|_{L_t^1 L_x^2(I)} \leq \epsilon'.
\]
then $I \subset (-T(w), T(w))$ and (b) (resp. (b')) hold, with
\[
(9.28) \quad (b) : \| w - \tilde{w} \|_{L_t^{2(d+1)} L_x^{d+2}} = O(\epsilon'),
\]
\[
(9.29) \quad (b') : \| w - \tilde{w} \|_{L_t^{2(d+1)} L_x^{d+2}} = O(\epsilon'), \quad \text{and} \quad \| w - \tilde{w} \|_{L_t^\infty L_x^2(I)} = O(\epsilon').
\]
Here $\tilde{w} := (D)w - i\partial_t w$ and $\tilde{w} := (D)\tilde{w} - i\partial_t \tilde{w}$.

**Proof.** We only prove the result if (b) holds. Indeed, if (a) holds, then the proof of the result is a slight modification of that if (b) holds.

Let $1 \gg \epsilon_0 > 0$ small enough such that all the statements below are true. Let $F(v) := |v|^{\frac{d+2}{d}} v$ and $e := w - \tilde{w}$ and $0 < \alpha \ll 1$. The proof is made of three steps:

- $\| \tilde{w} \|_{L_t^{\frac{d+2}{d}} L_x^{\frac{d+2}{d}}} (R) < \infty$.

Indeed, dividing $I$ into subintervals $J = [a, \cdot)$ such that $\| \tilde{w} \|_{L_t^{\frac{d+2}{d}} L_x^{\frac{d+2}{d}}} (J) \approx \alpha$, we get from (2.1) and (2.1)

\[
\max \left( \| \tilde{w} \|_{L_t^{\frac{d+2}{d}} L_x^{\frac{d+2}{d}}} (J), \| \tilde{w} \|_{L_t^{\frac{d+2}{d}} L_x^{\frac{d+2}{d}}} (J), \| \tilde{w} \|_{L_t^{\frac{d+2}{d}} L_x^{\frac{d+2}{d}}} (J) \right) \leq \| \tilde{w} \|_{L_t^{\frac{d+2}{d}} L_x^{\frac{d+2}{d}}} (J) \right) \leq \| \tilde{w} \|_{L_t^{\frac{d+2}{d}} L_x^{\frac{d+2}{d}}} (J) \right)
\]

By a continuity argument applied to (9.30) on each $J$ and by iteration, we see that the claim holds.

- short-time perturbation argument

Let $1 \gg \alpha > 0$ and $J = [a, \cdot) \subset I$ such that $\| \tilde{w} \|_{L_t^{\frac{d+2}{d}} L_x^{\frac{d+2}{d}}} (J) \leq \alpha$. There exists $1 \gg \epsilon'' > 0$ such that if $\| \text{eq}(\tilde{w}) \|_{L_t^1 L_x^2(J)} \leq \epsilon''$, $\| \tilde{w}(a) - \tilde{w}(a) \|_{L^2} \leq \epsilon''$, and $\epsilon'' \leq \epsilon_0''$, then

\[
(9.31) \quad \| F(w) - F(\tilde{w}) \|_{L_t^1 L_x^2(J)} \leq \epsilon''.
\]

Indeed we see from (2.6) that
\[ \|e\|_{L_t^{\frac{d+2}{d+2}} L_x^{\frac{2(d+2)}{d+2}}}^{J} \lesssim \varepsilon'' + \|F(w) - F(\tilde{w})\|_{L_t^{1} L_x^{\frac{2}{d}}}(J) \lesssim \varepsilon'' + \|e\|_{L_t^{\frac{d+2}{d+2}} L_x^{\frac{2(d+2)}{d+2}}}^{J} \left(\|\tilde{w} + e\|_{L_t^{\frac{2-2}{d+2}} L_x^{\frac{2(d+2)}{d+2}}}^{J} + \|\tilde{w}\|_{L_t^{\frac{2-2}{d+2}} L_x^{\frac{2(d+2)}{d+2}}}^{J}\right) \lesssim \varepsilon'', \]

applying a continuity argument at the last line. Hence (9.31) holds.

- Long-time perturbation argument

We divide \(I\) into subintervals \((J_j)\) such that \(\|\tilde{w}\|_{L_t^{\frac{d+2}{d+2}} L_x^{\frac{2(d+2)}{d+2}}}^{J_j} \leq \alpha\) for \(1 \leq j \leq \tilde{j} - 1\) and \(\|\tilde{w}\|_{L_t^{\frac{d+2}{d+2}} L_x^{\frac{2(d+2)}{d+2}}}^{J_{\tilde{j}}} \leq \alpha\). We then prove by induction that if \(\varepsilon_0 \ll 1\) then there exists \(0 \leq C(j) < \infty\) such that \((*)\) holds for all \(1 \leq j \leq k\) with \(k \in \{\tilde{j}, \tilde{j} - 1\}\). Then by iterating (2.6) on \(j\) we get

\[ \|w - \tilde{w}\|_{L_t^{-\infty}(J_j)} \lesssim \varepsilon' + \sum_{j=1}^{k} \|F(w) - F(\tilde{w})\|_{L_t^{1} L_x^{\frac{2}{d}}}(J_j) \lesssim \varepsilon' + \sum_{j=1}^{k} C(j) \varepsilon' \lesssim \varepsilon''. \]

Hence by applying the short-perturbation argument we see that \((*)\) holds for \(j = k + 1\). Hence \(\|F(w) - F(\tilde{w})\|_{L_t^{1} L_x^{\frac{2}{d}}}(J) = O(\varepsilon)\) and by (2.6) we get (9.30).

\[ \square \]

Let \(w := u_n\) and \(\tilde{w} := \sum_{j=0}^{k} u_n^j\).

We first estimate \(\|\tilde{w}\|_{L_t^{\frac{2(d+1)}{d+2}} L_x^{\frac{2(d+1)}{d+2}}}^{J} \). In view of (9.9), and the global theory for small data we see that there exists finite set \(J\) for which

\[ \|\tilde{w}\|_{L_t^{\frac{2(d+1)}{d+2}} L_x^{\frac{2(d+1)}{d+2}}}^{J} \lesssim \left(\sum_{i=0}^{m} \frac{2^{d+1}}{2}\right)^{\frac{1}{2}} \lesssim \|\tilde{w}\|_{L_t^{\frac{2(d+1)}{d+2}} L_x^{\frac{2(d+1)}{d+2}}}^{J} \]

(that can be proved by induction on \(m\)). We see that \(\|\tilde{w}\|_{L_t^{\frac{2(d+1)}{d+2}} L_x^{\frac{2(d+1)}{d+2}}}^{J} \lesssim 1\). Hence, using also Lemma 9.5 we get \(X_3 \lesssim 1\). We also have \(X_2 \lesssim 1\). We get from (9.9) \(X_3 \lesssim \sum_{j=0}^{k} \|u_n^j(0)\|^{2}_{L_x^{\frac{2}{d}}} \ll 1\). We have
\begin{equation}
X_n \lesssim \sum_{(j,l) \in [0, \ldots, k]^2} \left\| u_n^{j,l} \right\|^2_{L^{d+1}_t L^\infty_x} \max_{(j,l) \in A_1 \times A_2} \left( \left\| U_n^{j,l} \right\|_{L^{2(d+1)-2}_t L^{2(d+1)}_x}^{2(d+1) - 2}, \left\| U_n^{j,l} \right\|_{L^{2(d+1)-2}_t L^{2(d+1)}_x}^{2(d+1)} \right)
\end{equation}

\begin{equation}
\lesssim \sum_{(j,l) \in [0, \ldots, k]^2} (h_n^{j,l})^\alpha \left( \int_{\mathbb{R}^{d+1}} \left| U_n^{j,l}(t,x) \right|^\frac{d+1}{2} \left\| U_n^{j,l} \left( \frac{x-x^{j,l}}{h_n^{j,l}} \right) \right\|_{L^\infty_x} \frac{d+1}{2} dx \right) \frac{1}{2(d+2)}
\end{equation}

In order to prove that the summand approaches zero as \( n \to \infty \), we see from the dominated convergence theorem that we may replace WLOG \( U_n^{j,l}, \chi RU_n^{j,l} \) respectively for \( R >> 1 \) (Here \( \chi(t,x) := \chi \left( \frac{\left\| (t,x) \right\|}{R} \right) \) with \( \chi : \mathbb{R}^+ \to \mathbb{R} \) is a smooth function that satisfies \( \chi(s) = 1 \) if \( s \leq 1 \) and \( \chi(s) = 0 \) if \( s \geq 2 \) then we see from (9.20) that the integral involving \( \chi RU_n^{j,l} \) and \( \chi RU_n^{l,j} \) goes to zero as \( n \to \infty \).

Hence \( \left\| \tilde{w} \right\|_{L^{\frac{4(d+1)}{d-2}}(\mathbb{R})} \lesssim 1. \)

Observe from the embedding \( L^{\frac{4(d+1)}{d-2}} \to B^{0,\frac{2(d+1)}{d-2}} \) and Proposition 2.22. in [2] \( \oplus \) and (9.6) that

\begin{equation}
\left\| \langle D \rangle^{-1} e^{\pm it(D)} u_{n}^{j,l} \right\|_{L^{\frac{4(d+1)}{d-2}}(\mathbb{R})} \ll \varepsilon_0.
\end{equation}

Hence, using also (9.5) and (2.6), we see that the first estimate of (9.28).

From the elementary estimate above we get for some \( \alpha > 0 \)

\begin{equation}
\left\| F \left( \sum_{j=0}^{k} u_n^j - \sum_{j=0}^{k} F(u_n^j) \right) \right\|_{L^1 \left( \mathbb{R} \right)} \ll \varepsilon_0.
\end{equation}

( In order to prove the last estimate we proceed as follows. First observe that \( \left\| U_n^{j} \right\|_{L^{\frac{d+2}{d-2}} L^{\infty}_{x}} \ll \varepsilon_0 \) if \( j \in A_1 \) and that \( \left\| U_n^{j} \right\|_{L^{\frac{d+2}{d-2}} L^{\infty}_{x}} \ll \varepsilon_0 \) if \( j \in A_2 \) by proceeding similarly as the first stage of the proof of Result 9.10. Then follow the same steps as those to estimate \( X_n \).) Hence the second estimate of (9.28) holds.

Hence applying Result 9.10 we see that there is contradiction with the properties of \( u_n^j \), defined just below Claim 9.2.

This means that at least one of the \( j \) satisfies \( E(\tilde{U}_n^j) \geq E_{\text{wca}}(\tilde{W}) \) if \( h_n^{j} = 0 \) (or \( E(\tilde{U}_n^j) \geq E_{\text{wca}}(\tilde{W}) \) if \( h_n^{j} = 1 \). Assume without loss of generality that this \( j \) is equal
to 0. By Result $9.8$ Result $9.9$ Result $9.20$, the local well-posedness theory for small data we see that for the other $j$s $T_+(U^j_\infty) = T_+(U^j_\infty) = \infty$, $T_-(U^j_\infty) = T_-(U^j_\infty) = -\infty$ and

$$
\sum_{j \in A_1, j \neq 0} \|\bar{U}^j_\infty\|^2_{L^1_{t \infty} L^2_x} + \sum_{j \in A_2, j \neq 0} \|\bar{U}^j_\infty\|^2_{L^1_{t \infty} L^2_x} + \|\check{u}^n_n\|^2_{L^2} \lesssim \varepsilon^2
$$

(9.33)

Hence, applying Result $9.10$ with $\check{w} := u^0_n$ and $w := u_n$, we see that

$$
\left\| (u_n, \partial_t u_n) - (u^0_n, \partial_t u^0_n) \right\|_{L^\infty H((0,T_+(u^0_n)))} = O_H(\varepsilon),
$$

(9.34)

Notice that $\tau_0^0 \neq \infty$. If not, if $h^0_\infty = 1$ (resp. $h^0_\infty = 0$) then $U^0_\infty$ (resp. $U^0_\infty$) would scatter by construction in a neighborhood of $\infty$: see $[9.15]$ and Remark $9.3$. But then, applying again Remark $9.3$ and Result $9.10$, we see that there is contradiction. Notice also that if $h^0_\infty = 1$ (resp. $h^0_\infty = 0$) then $\|U^0_n\|_{L^\infty L^2} < \infty$ (resp. $\|U^0_0\|_{L^\infty L^2} < \infty$) if not this would imply by a standard blow-up criterion (the proof is similar to that in $[10]$) that $T_+(U^0_0) = \infty$ if $h^0_\infty = 1$ (or $T_+(U^0_0) = \infty$ if $h^0_\infty = 0$). But we have already seen that this is not possible. We claim that for $t > \tau_0^0$ we have

$$
\tau^0_0 = 0 : \check{d}_S(\bar{U}^0_\infty(t)) \geq \delta_b; \ h^0_\infty = 1 : \check{d}_S(\bar{U}^0_\infty) \geq \delta_b
$$

(9.35)

Assume that $\tau_0^0 = -\infty$. Assume also that $h^0_\infty = 0$. If the claim were wrong, then there would exist $t_* \in \text{I}(U^0_\infty)$ such that $\check{d}_S(\bar{U}^0_\infty(t_*)) \leq \delta_b$. But then, taking into account $9.34$ and Lemma $9.5$ we see that

$$
\check{d}_S(\bar{u}(h^0_\infty(t_* - \tau^0_n))) \lesssim \delta_b,
$$

which is a contradiction, since $t_* > \tau^0_n$. The other case $h^0_\infty = 1$ is easier and therefore left to the reader. The case $\tau_0^0 \in \mathbb{R}$ is treated similarly to the case $\tau_0^0 = -\infty$ and therefore the proof is omitted.

Hence, using also Result $9.8$ and Proposition $8.18$ (or Proposition $8.0$) we see that $K(U^0(t)) \geq \min(\|\nabla U^0_\infty(t)\|^2_{L^2}, k(\delta_b))$.

Define $\bar{U}(t) := \bar{U}(t+c)$ if $h^0_\infty = 1$ (resp. $\bar{U}(t) := \bar{U}^0_\infty(t+c)$ if $h^0_\infty = 0$ with $c > \tau^0_n$). Notice from the estimates above, the results in $[18]$ for $d \in \{3,4\}$, or the arguments in $[17]$ (see Section 5) for $d \in \{3,4,5\}$, that we cannot have $h^0_\infty = 0$. If $E(\bar{U}_c) < E_c$ then, by definition of $E_c$, we have $\|U_c\|_{L^{2(d+1)}_t L^{d+1}_x L^{\frac{2(d+1)}{d+2}}_{t \infty} (c, \infty)} < \infty$ or, equivalently, $\|U^0_\infty\|_{L^{2(d+1)}_t L^{d+1}_x L^{\frac{2(d+1)}{d+2}}_{t \infty} (c, \infty)} < \infty$. From the construction of the nonlinear profile, we also know that $\|U^0_\infty\|_{L^{2(d+1)}_t L^{d+1}_x L^{\frac{2(d+1)}{d+2}}_{t \infty} (\tau_0^0, c)) < \infty$. Taking also into account $9.33$, this leads to a contradiction by applying again Result $9.10$ with $\check{w} := u^0_n$ and $w := u_n$. Hence $E(\bar{U}_c) = E_c$. 


Claim 9.12. \textit{There exists }c(t) \in \mathbb{R}^d, \ t \in [0, T_+(U_c)) \text{ such that}

\begin{equation}
K = \left\{ v(t) := \tilde{U}_c(x - c(t), t), t \in [0, T_+(U_c)) \right\}
\end{equation}

is precompact in }L^2\text{. In fact we have }T_+(U_c) = \infty.\\n
\textbf{Proof.} If }K \text{ were not precompact, then there would exist }\eta > 0 \text{ and a sequence }\{t_n \in [0, T_+(U_c))\} \text{ such that}

\begin{equation}
\|\tilde{U}_c(t_n, \cdot) - c_0\|_{L^2} \geq \eta
\end{equation}

for }n \neq n'\text{ and for all }c_0 \in \mathbb{R}^d. \text{ In view of the continuity of the flow, we must have }t_n \rightarrow T_+(U_c) \text{ in order not to violate }\text{(9.37)}. \text{ Applying }\text{(9.35)} \text{ to }\tilde{v}_n(0) := \tilde{U}_c(t_n) \text{ and following the same steps down to Remark 9.11 we see that there exists }\tilde{t}_n, \tilde{\phi}_0 \in \mathbb{R} \times L^2 \text{ such that}

\begin{equation}
\tilde{U}_c(t_n) = e^{-it_n^0(D)}\tilde{\phi}_n + o_{L^2}(1),
\end{equation}

with }\tilde{\phi}_n := \tilde{\phi}_0(\cdot - \tilde{t}_n^0) \text{ for some }\tilde{t}_n^0 \in \mathbb{R}^d. \text{ Assume that (up to a subsequence) that }\tilde{t}_n \rightarrow -\infty. \text{ Then}

\begin{align*}
&\|\mathbb{R} \left( \langle D \rangle^{-1}e^{it(D)}\tilde{U}_c(t_n) \right)\|_{L_t^1 L_x^{\frac{d+1}{d-2}}} \lesssim \|\mathbb{R} \left( \langle D \rangle^{-1}e^{it(D)}\tilde{\phi}_0 \right)\|_{L_t^1 L_x^{\frac{d+1}{d-2}}} + o_{H^1}(1) \\
&\ll 1.
\end{align*}

But then, by the local well-posedness theory, this would imply that }\|U_c(t+t_n)\|_{L_t^\frac{2(d+1)}{d-2} L_x^{\frac{2(d+1)}{d-2}}((0, \infty))} = \|U_c\|_{L_t^\frac{2(d+1)}{d-2} L_x^{\frac{2(d+1)}{d-2}}((t_n, \infty))} < \infty, \text{ which is a contradiction.}

Assume that (up to a subsequence) that }\tilde{t}_n \rightarrow \infty. \text{ Then}

\begin{align*}
&\|\mathbb{R} \left( \langle D \rangle^{-1}e^{it(D)}\tilde{U}_c(t_n) \right)\|_{L_t^1 L_x^{\frac{d+1}{d-2}}} \lesssim \|\mathbb{R} \left( \langle D \rangle^{-1}e^{it(D)}\tilde{\phi}_0 \right)\|_{L_t^1 L_x^{\frac{d+1}{d-2}}} + o_{H^1}(1) \\
&\ll 1.
\end{align*}

But then, by the local well-posedness theory, this would imply that }\|U_c(t+t_n)\|_{L_t^\frac{2(d+1)}{d-2} L_x^{\frac{2(d+1)}{d-2}}((-\infty, 0))} = \|U_c\|_{L_t^\frac{2(d+1)}{d-2} L_x^{\frac{2(d+1)}{d-2}}((-\infty, t_n))} < 1, \text{ which is a contradiction since }t_n \rightarrow T_+(U_c).
Assume that (up to a subsequence) $\tilde{U}_n \to \tilde{U}$ for some $\tilde{t} \in \mathbb{R}$. Then, in view of (9.37) and (9.38), we get after an appropriate change of variable

$$
\left\| e^{-it\mathcal{H}(D)} \tilde{\phi}_0(x + x_0^0 - \tilde{t}_l - c_0) - e^{-it\mathcal{H}(D)} \tilde{\phi}_0(x) \right\|_{L^2} \geq \frac{\eta}{2}.
$$

But then, choosing $c_0 := x_0^0 - \tilde{t}_l$ we see that the estimate above cannot hold for $l \gg 1$ and $l' \geq l$.

Next we claim that $T_+(U_c) = \infty$. If not let $(t_l)_{l \geq 1}$ be a sequence such that $t_l \to T_+(U_c)$. By precompactness there exists $\tilde{V} \in L^2$ such that $\tilde{U}_c(t_l, \cdot - c(t_l)) \to \tilde{V}$ in $L^2$. Now let $\tilde{H}$ (resp. $\tilde{H}_1$) be the solution of $(i\partial_t + \langle D \rangle) \tilde{H} = F(H)$ (resp. $(i\partial_t + \langle D \rangle) \tilde{H}_1 = F(H_1)$) with $H := \Re \left( \langle D \rangle^{-1} \tilde{H} \right)$ (resp. $H_1 := \Re \left( \langle D \rangle^{-1} \tilde{H}_1 \right)$) that satisfies $\tilde{H}(T_+(U_c)) = \tilde{V}$ (resp. $\tilde{H}_1(T_+(U_c)) = \tilde{V}(t_l, \cdot - c(t_l))$). Then, by (2.4) and the dominated convergence theorem

$$
\left\| \Re \left( \langle D \rangle^{-1} e^{i(t - T_+(U_c)) \langle D \rangle} \tilde{H}_1(T_+(U_c)) \right) \right\|_{L^\infty} \leq \frac{2(d+1)}{L_1} \left( T_+(U_c), T_+(U_c) + \delta \right) \\
\left\| \tilde{H}_1(T_+(U_c)) - \tilde{H}(T_+(U_c)) \right\|_{L^2} \\
+ \left\| \Re \left( \langle D \rangle^{-1} e^{i(t - T_+(U_c)) \langle D \rangle} \tilde{H}(T_+(U_c)) \right) \right\|_{L^\infty} \leq 1,
$$

for some $\delta \ll 1$. Consequently

$$
\left\| \cos \left( (t - T_+(U_c)) \langle D \rangle \right) H_1(T_+(U_c)) + \frac{\sin \left( (t - T_+(U_c)) \langle D \rangle \right)}{\langle D \rangle} \partial_t H_1(T_+(U_c)) \right\|_{L^\infty} \leq 1.
$$

Hence by the local well-posedness theory, we see that we can extend $\tilde{H}_1$ to $T_+(U_c) + \delta$. Since $\tilde{H}_1(t) = \tilde{U}_c(\cdot - c(t_l), t + t_l - T_+(U_c))$, this is a contradiction.

Next we prove the following claim:

Claim 9.13. $\tilde{U}_c$ does not exist.

Proof. We may assume without loss of generality that $c(0) = 0$. A straightforward modification of an argument using the Lorentz transform in (11) (the argument is also used in Section 2.4 in [28] for the cubic focusing nonlinear Klein-Gordon equation) shows that $|\tilde{P}(\tilde{U}_c)| \leq \tilde{c}$, with $\tilde{P}(\tilde{U}_c) := \int_{\mathbb{R}^d} \nabla \tilde{U}_c \partial_t \tilde{U}_c \, dx$ being the conserved-in-time momentum. By the previous claim, we see that there exists $R_0 := R_0(\tilde{c}) \gg 1$ such that

$$
E_{c(t), R_0}(\tilde{U}_c(t)) \leq \tilde{c} E(\tilde{U}_c), \quad t \in [0, \infty)
$$

From the identity\footnote{Here $X_R(\tilde{U}_c(t))$ denoting the localized center of energy, i.e $X_R(\tilde{U}_c(t)) := \int \chi_R(x) \, dx \tilde{U}_c(t, x)$ $dx$, with $\chi_R(\tilde{U}_c(t, x))$ such that $E(\tilde{U}_c(t)) = \int \chi_R(\tilde{U}_c(t, x)) \, dx$} $\partial_t X_R(\tilde{U}_c) = -d \times \tilde{P}(\tilde{U}_c) + \int_{|x| \geq R} c(\tilde{U}_c(t)) \, dx,$
and a slight modification of an argument in [11], we see that if $R \gg R_0 := R_0(\bar{c})$, then

$$|c(t) - c(0)| \leq R,$$

for $0 < t < t_0 \approx \frac{R}{\bar{c}}$. Hence

$$E_{2R,0}(\bar{U}_c(t)) \leq \bar{c} E(\bar{U}_c), \quad t \in [0, t_0]$$

By using the previous claim, we can prove (see [11]) that

$$\int_0^{t_0} \| \partial_t U_c(t) \|_{L^2}^2 + \| U_c(t) \|_{L^2}^2 \, dt \lesssim E(\bar{U}_c) + \int_0^{t_0} \| \nabla U_c(t) \|_{L^2}^2 \, dt$$

and, consequently

$$t_0 E(\bar{U}_c) \lesssim E(\bar{U}_c) + \int_0^{t_0} \| \nabla U_c(t) \|_{L^2}^2 \, dt$$

Next we apply (8.5) to $U_c$ and $w(x) := \chi \left( \frac{|x|}{2\pi} \right)$; we integrate (8.5) on $[0, t_0)$ and we get (using Proposition 3.6)

$$R \gtrsim \left( \| w \bar{d}_t U_c \cdot x \cdot \nabla U_c + \frac{d}{2} U_c \|_0 \right) \gtrsim t_0 - O(1) - \bar{c} t_0 \gtrsim \frac{R}{\bar{c}}$$

This is a contradiction.

\[ \square \]

10. Proof of Lemma 9.5

First observe that the finiteness of $\| U_{\infty}^J \|_{L^{\frac{d+2}{2}} L^{\frac{d+2}{d+2}}(\mathbb{R}^d)}$ implies that $\| \bar{U}_{\infty}^J \|_{L^{d+2}_{t \in [a,b]} L^{d+2}_{x \in \mathbb{R}^d}} < \infty$: indeed, partitioning $\mathbb{R}^+$ into a finite number of subintervals $(J_k := [a_k, b_k])_{1 \leq k \leq l}$ such that $\| U_{\infty}^J \|_{L^{\frac{d+2}{2}} L^{\frac{d+2}{d+2}}(J_k)} = \eta$, $1 \leq k < l$, and $\| U_{\infty}^J \|_{L^{\frac{d+2}{d+2}} L^{\frac{d+2}{d+2}}(J_l)} \leq \eta$, we see from (2.7) and Hölder inequality that for all $K_k := [a_k, \ldots] \subset J_k$

$$\| \bar{U}_{\infty}^J \|_{L^{d+2}_{t \in \mathbb{R}} L^{d+2}_{x \in \mathbb{R}^d}(K_k)} + \| U_{\infty}^J \|_{L^{\frac{d+2}{d+2}} L^{\frac{d+2}{d+2}}(K_k)} \leq \| \bar{U}_{\infty}^J(a_k) \|_{L^2_{t \in \mathbb{R}}} + \| U_{\infty}^J \|_{L^{\frac{2^{d+2}}{d+2}} L^{\frac{2^{d+2}}{d+2}}(K_k)}$$

Hence a continuity argument and an iteration over $k$ show that $\| \bar{U}_{\infty}^J \|_{L^{d+2}_{t \in \mathbb{R}} L^{d+2}_{x \in \mathbb{R}^d}} < \infty$. Proceeding similarly we have $\| \bar{U}_{\infty}^J \|_{L^{d+2}_{t \in \mathbb{R}^d} L^{d+2}_{x \in \mathbb{R}^d}} < \infty$.

If $J$ is an interval let $X(J) := \| U_{\infty}^J - \bar{U}_{\infty}^J \|_{L^{d+2}_{t \in \mathbb{R}} L^{d+2}_{x \in \mathbb{R}^d}(J)} + \| U_{\infty}^J - \bar{U}_{\infty}^J \|_{L^{\frac{d+2}{2}} L^{\frac{d+2}{d+2}} (J)}$.

Let $0 < \epsilon < 1$ and $S \gg 1$ be a large constant.

We first estimate $X([-S, S])$. We have

$$\partial_t (U_{\infty}^J - U_{\infty}^J) - \triangle (U_{\infty}^J - U_{\infty}^J) = F(U_{\infty}^J) - F(U_{\infty}^J) - (h^J_k)^2 U_{\infty}^J.$$

Let $0 < \eta \ll 1$ be a constant small enough such that all the estimate below are true. We partition $[-S, S]$ into a finite number of subintervals $(J_k := [a_k, b_k])_{1 \leq k \leq l}$ such that $\| U_{\infty}^J \|_{L^{\frac{d+2}{d+2}} L^{\frac{d+2}{d+2}}(J_k)} = \eta$, $1 \leq k < l$, and $\| U_{\infty}^J \|_{L^{\frac{2^{d+2}}{d+2}} L^{\frac{2^{d+2}}{d+2}}(J_l)} \leq \eta$. We have

\[ \text{with } t - t_2 + m \text{ substituted with } 2R \]
The Hölder-in-time inequality yields $\|(h_n^j)^2 U_n^j\|_{L^1_t L^2_x(J)} \lesssim (h_n^j)^2 S \|\vec{U}_n^j\|_{L^\infty_t L^2_x(J)}$.

Let $M := \|U^j_n\|_{L^2_t L^\infty_x}$. We see from (2.7) that

$$\|U^j_n - U^j_{n+1}\|_{L^2_t L^\infty_x(J)} \lesssim \|\vec{U}_n^j(a_k) - \vec{U}_{n+1}^j(a_k)\|_{L^2} + \|(h_n^j)^2 U_n^j\|_{L^1_t L^2_x(J)} + \|F(U_n^j) - F(U_{n+1}^j)\|_{L^1_t L^2_x(J)}.$$  

Hence, combining the estimates above, iterating over $k$, and recalling that $\vec{U}_0^j(a_1) = \vec{U}_{\infty}^j(a_1)$, we get $X([-T, S]) \lesssim \epsilon$ for $n \gg 1$.

We then have to prove that $\|U^j_n - U^j_\infty\|_{L^{\frac{d+2}{d}}_t L^{\frac{2(d+2)}{d+2}}_x((S, \infty))} \lesssim \epsilon$ for $n \gg 1$. We get from (2.7) $\|\cos(tD)\mathcal{R}(D^{-1}\vec{\phi}) + \frac{\sin((tD)\mathcal{R}(\vec{\phi}))}{D}\|_{L^{\frac{d+2}{d}}_t L^{\frac{2(d+2)}{d+2}}_x(-\infty, \infty)} \lesssim \|\vec{\phi}\|_{L^2}$.

This estimate, combined with the dominated convergence theorem, shows that $\|\cos(tD)\mathcal{R}(D^{-1}\vec{\phi}) + \frac{\sin((tD)\mathcal{R}(\vec{\phi}))}{D}\|_{L^{\frac{d+2}{d}}_t L^{\frac{2(d+2)}{d+2}}_x((S, \infty))} \ll \epsilon$. Let $J := [S, b]$ be an interval. Applying (2.7) again, we get

$$\|U^j_n\|_{L^{\frac{d+2}{d}}_t L^{\frac{2(d+2)}{d+2}}_x(J)} \lesssim \epsilon + \|F(U^j_n)\|_{L^1_t L^2_x(J)} \lesssim \epsilon + \|U^j_n\|_{L^{\frac{d+2}{d}}_t L^{\frac{2(d+2)}{d+2}}_x(J)}.$$  

Hence a continuity argument shows that $\|U^j_n\|_{L^{\frac{d+2}{d}}_t L^{\frac{2(d+2)}{d+2}}_x((S, \infty))} \lesssim \epsilon$. Let $\phi \in L^2(\mathbb{R}^d)$. We claim that for $n \gg 1$

$$\left\|\frac{e^{itD}U_n^j}{(D)^{\frac{1}{2}}}\phi\right\|_{L^{\frac{d+2}{d}}_t L^{\frac{2(d+2)}{d+2}}_x((S, \infty))} \ll \epsilon.$$  

Indeed let $\phi_\epsilon$ be a Schwartz function such that $\|\phi_\epsilon - \phi\|_{L^2} \ll \epsilon$. From (2.3) we see that $\left\|\frac{e^{itD}U_n^j}{(D)^{\frac{1}{2}}}\phi_\epsilon - \phi\right\|_{L^{\frac{d+2}{d}}_t L^{\frac{2(d+2)}{d+2}}_x((S, \infty))} \ll \|\phi_\epsilon - \phi\|_{L^2}$. Hence it suffice to prove that (10.2) holds with $\phi$ a Schwartz function.

Recall that $e^{itD}U^j_n = T^j_n e^{itD}U^j_n$ and that $\langle D \rangle^{-m} T^j_n = (h_n^j)^m T^j_n$ $\langle (D)^j_n \rangle^{-m}$. Hence $\frac{e^{itD}U^j_n}{(D)^{\frac{1}{2}}} = (h_n^j)^{-1} T^j_n$ $\langle (D)^j_n \rangle^{-1} e^{itD}U^j_n$ and, denoting by $\left\langle \frac{2(d+1)}{d-2}\right\rangle$ the number such that $\left\langle \frac{2(d+1)}{d-2}\right\rangle = 1$, we get from the estimate $\|\langle D \rangle^{-m} \phi \|_{B^{\frac{2}{2}}_\alpha \langle \frac{2(d+2)}{d-2}\rangle}$ $\ll$ $\left\langle \frac{2(d+1)}{d-2}\right\rangle$ and the embedding $B^{\frac{2}{2}}_{\frac{2}{2}} \hookrightarrow H^{\frac{2}{2}}_{\langle \frac{2(d+2)}{d-2}\rangle}$.
\[
\left\| \frac{e^{\pm K(t)}}{(D)_{\delta}}^{h} \right\|_{L^2_t L^2_x} \leq \left\| \frac{e^{\pm K(t)}}{(D)_{\delta}}^{h} \right\|_{L^2_t L^2_x},
\]

Hence we get (10.1) from (2.8) after replacing “\(U^j_n\)” with “\(U^j_n\)”. Hence \(\|U^j_n\|_{L^2_t L^2_x} \lesssim \epsilon\) for \(n \gg 1\). Hence (10.10) holds.

11. PROOF OF THEOREM 1.1 (3).

We apply the decomposition \(\tilde{u}(0)\) to the data \(\bar{u}(0)\). Hence \(\bar{u}(0) = \tilde{S}^{\geq 0}(0) \left( \tilde{W} + \bar{v}(0) \right)\) with \(\sigma(0)\) to be chosen and \(\bar{v}(0) = \tilde{S}^{\leq -\sigma(0)} \bar{u}(0) - \tilde{W}\). Let \(0 < \beta \ll \epsilon, \lambda\). Let \(R \gg 1\) and \(\sigma(0) \gg 1\) such that

\[
d = 3: \begin{cases} 
\int_{\mathbb{R}^d} \left| \nabla \left( \frac{1 - \chi \left( \frac{x}{R} \right) }{R} \right) W \right|^2 \, dx \leq \beta^2 \\
\int_{\mathbb{R}^d} \left| \chi \left( \frac{x}{R} \right) W \right|^2 + \rho^2 \, dx \leq \beta^2
\end{cases}
\]

\[
d = 4, 5: e^{-2\sigma(0)} \int_{\mathbb{R}^d} W^2 \, dx \ll \beta^2
\]

We then apply the decomposition \(\tilde{u}(0)\) to \(v(0)\). Hence \(\bar{v}(0) = \hat{v}(0) + \hat{\sigma}(0)\) with

\[
\hat{\sigma}(0) := \beta(\pm 1, 0), \quad \beta(0, \pm 1); d = 4, 5: \hat{\sigma}(0) = 0;
\]

\[
d = 3: \hat{\sigma}(0) = -(1 - \chi \left( \frac{x}{R} \right) ) \tilde{W} + \left( (1 - \chi \left( \frac{x}{R} \right) ) \tilde{W} \right) \hat{\rho}
\]

(Recall that \(\chi\) is defined below (8.3)). Here \(\hat{\rho} := \rho(1, 1)\). We have \(\|\hat{\sigma}(0)\|_{H^2} + \|\hat{\sigma}(0)\|_{L^2} \ll \beta^2\). From (3.12) we see that (with \(\tau(0) = 0, \partial_t \tau = e^{\sigma(t)}\))

\[
X^2(\tau) \lesssim o(\beta^2) + \int_0^\tau \left( \tilde{d}_S^2(\bar{u}(\tau')) + \tilde{d}_S^2(\bar{u}(\tau')) \right) X(\tau') + \left( \tilde{d}_S(\bar{u}(\tau')) + \tilde{d}_S(\bar{u}(\tau')) \right) X^2(\tau') \, d\tau',
\]

with

\[
X(\tau) := \|\bar{v}(\tau)\|_{L^2} + \|\bar{u}(\tau)\|_{L^2}.
\]

Hence (using also (3.18), (3.19) and (6.11)) we see from a bootstrap argument that for \(|\tau| \in [0, \tau_f]\) with \(\tau_f\) such that \(\beta e^{k \tau_f} \approx \delta_f, \beta e^{k \tau_f} \leq \delta_f\),

\[
(\lambda_1(0), \lambda_2(0)) = (\beta(\pm 1, 0) \Rightarrow (\lambda_1, \lambda_2) \approx \beta(\pm \cosh(\kappa \tau), \pm \sinh(\kappa \tau)),
\]

and \(|X(\tau)\|_{L^\infty[-\tau_f, \tau_f]} \lesssim \beta^2 e^{2k|\tau|} .
\]
Hence there exists $\tau_{+, \text{med}}$ (resp. $\tau_{-, \text{med}}$) such that $\beta e^{k \tau_{+, \text{med}}} \sim \delta_b$ (resp. $\beta e^{-k \tau_{-, \text{med}}} \sim \delta_b$) and $\hat{d}_S(\vec{u}(\pm \tau_{, \text{med}})) \geq \delta_b$ with $\tau_{+, \text{med}} := \tau^{-1}(\pm \tau_{\text{med}})$. Moreover $E(\vec{u}) < E_{wa}(\vec{W}) + c_d \hat{d}_S^{2}(\vec{u}(\pm \tau_{, \text{med}}))$. Hence from Proposition 3.13, Proposition 3.16 and (6.1), we see that the following holds

\[ (\lambda(1), \lambda(2)) = \beta(1, 0), \Rightarrow \]
\[ E(\vec{u}) < E_{wa}(\vec{W}), \vec{u}(0) \in A_{-}, t \to \pm T_{\pm}(u) : \text{u blows - up} \]
\[ (\lambda(1), \lambda(2)(0)) = \beta(-1, 0), \Rightarrow \]
\[ E(\vec{u}) < E_{wa}(\vec{W}), \vec{u}(0) \in A_{+, \pm}, t \to \pm \infty : \text{u scatters} \]
\[ (\lambda(1), \lambda(2)(0)) = \beta(0, 1), \Rightarrow \]
\[ E_{wa}(\vec{W}) < E(\vec{u}) < E_{wa}(\vec{W}) + c^2_*, \vec{u}(0) \in A_{-}, t \to -\infty : \text{u scatters, t} \to T_{+}(u) : \text{u blows - up} \]
\[ (\lambda(1), \lambda(2)(0)) = \beta(0, -1), \Rightarrow \]
\[ E_{wa}(\vec{W}) < E(\vec{u}) < E_{wa}(\vec{W}) + c^2_*, \vec{u}(0) \in A_{+, \pm}, t \to -T_{-}(u) : \text{u blows - up, t} \to \infty : \text{u scatters} \]

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