Linearized Constraints in the Connection Representation:
Hamilton-Jacobi Solution.

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Abstract

Newman and Rovelli have used singular Hamilton-Jacobi transformations to reduce
the phase space of general relativity in terms of the Ashtekar variables. Their solution of the
gauge constraint cannot be inverted and indeed has no Minkowski space limit. Nonetheless,
we exhibit an explicit Hamilton-Jacobi solution of all the linearized constraints. The result
does not encourage an iterative solution, but it does indicate the origin of the singularity
of the Newman-Rovelli result.

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1. Introduction.

The new variables of Abhay Ashtekar [1,2] for canonical general relativity, introduced
new ways of thinking about the program to quantize the Einstein theory. The fact that
in terms of the new variables the constraints are polynomial gives some hope that it may
be possible to find solutions to the constraints or to implement them as operators on an
appropriate Hilbert space. Furthermore, because the configuration space variable is an
$SL(2, C)$ connection, it follows that one can work explicitly with the holonomy operators
and the geometrical concepts related to the loops on which the holonomy is defined -
namely, the knots and linkages of the loops [3]. This has enriched the conceptual arena
for the consideration of quantization of general relativity.

However, new variables do not by themselves eliminate old problems. Indeed, the
Ashtekar phase space of an $SL(2, C)$ connection, $A^i_a$, and a densitized triad, $\bar{E}_i^a$, brings
in the Gauss law constraint

$$D_a \bar{E}_i^a = 0. \quad (1.1)$$

in addition to the usual vector and scalar constraints of the theory. (Above, the index
$i = 1 - 3$ labels the bases for the $SL(2, C)$ algebra, while $a = 1 - 3$ are indices for
coordinates on a three manifold $\Sigma$.) In the quantum theory, physical state vectors are to
be independent of the mappings generated by the constraints. This means that either the
constraints are to be solved by elimination of some variables, or the constraints are to be
implemented as operators on the physical Hilbert space. The important thing is that the
state vectors should only be functionals of those quantities which represent independent
dynamical degrees of freedom of the gravitational field.

The main thrust of the research so far has been to construct functionals which are
invariant under the constraint mappings. They depend on the topological structure of
loops and not upon the location or orientation of the loops. While operators are defined
which act on these functionals, there is no clear relationship between these operators and the dynamical degrees of freedom of the gravitational field. Thus, as yet there is no understanding of how this new view of general relativity is to be connected to our space-time picture.

Recently, the Hamilton-Jacobi formalism has been considered as a way to construct the independent degrees of freedom [4,5]. In a dynamical system with first class constraints [6-9],

\[ C_r(q^a, p^a) = 0, \quad r = 1 \cdots m, \quad a = 1 \cdots n > m \]  

(1.2)

the Hamiltonian contains a linear combination of the constraints:

\[ H = H_0(q^a, p^a) + \sum_{r=1}^{m} \lambda^r C_r(q^a, p^a) \]  

(1.3)

where the \( \lambda^i \) are arbitrary functions. As a result, Hamilton’s principal function \( S(q^a, P^A, t) \) must satisfy the system of equations

\[ H_0(q^a, \frac{\partial S}{\partial q^a}) + \frac{\partial S}{\partial t} = 0, \]  

(1.4a)

\[ C_r(q^a, \frac{\partial S}{\partial q^a}) = 0. \]  

(1.4b)

In a wholly constrained theory, \( H_0 = 0 \), there are only the constraint equations to satisfy. The resulting canonical transformation is singular in that there are only \( n - m \) constants of integration \( P^A \). Inversion of the transformation thus depends on \( m \) arbitrary functions. And in the case of the wholly constrained system, there is no explicit time dependence.

This formalism is being applied by Newman, Rovelli, and co-workers [10] to the free electromagnetic field and the quantum theory is being worked out in some detail. Newman and Rovelli have also applied the formalism to canonical general relativity using the Ashtekar variables [5]. They are able to treat the Gauss law constraints and the vector constraints, but are unable to invert the canonical transformations. Therefore, they are not able to express the scalar constraint in terms of the thus reduced phase space variables.

In the following section, we shall sketch the result of Newman and Rovelli to indicate explicitly the difficulty with the inversion and the singularity of the Minkowski space limit. Then we exhibit the linearized constraints in the Ashtekar variables and show that we can carry out the Hamilton-Jacobi transformation for the whole system of constraints. However, the method leads, as expected, to non-local variables and the difficulty of constructing a systematic iteration will become clear.

2. Newman-Rovelli Transformation.

In the Ashtekar formalism, the phase space of general relativity is coordinatized by \((A^i_a, \tilde{E}^a_i)\), an \( SL(2, C) \) connection and a densitized triad on a three-manifold \( \Sigma \). Therefore, there is, in addition to the diffeomorphism invariance, the local triad invariance. As a result we have the Gauss law constraint noted above,

\[ G_i := D_a \tilde{E}^a_i = \partial_a \tilde{E}^a_i + \epsilon_{ijk} A^j_a \tilde{E}^k_a = 0, \]  

(2.1a)
as well as the vector and scalar constraints:

\[
\begin{align*}
\mathcal{H}_a & := F_{ia\tilde{b}} - E_{ib} = 0, \\
\mathcal{H}_\perp & := \epsilon_{ijk} F_{ia\tilde{b}} \tilde{E}_{ja} - E_{k\tilde{b}} = 0,
\end{align*}
\]

(2.1b)

where

\[
F_{ia\tilde{b}} = 2A_{[a,\tilde{b}]} + \epsilon^{i\tilde{j}k} A_{\tilde{j}a} A_{k\tilde{b}}.
\]

The triad rotations are generated by

\[
G(\Lambda^i) = \int_{\Sigma} d^3x \, \Lambda^i \mathcal{G}_i
\]

while the mappings of \( \Sigma \to \Sigma \) are generated by

\[
H(\tilde{N}) = \int_{\Sigma} d^3x \, N^a \mathcal{H}_a.
\]

The scalar constraint generates canonical transformations of the phase space variables which we may identify as evolution. Therefore, the Hamiltonian is just a linear combination of the constraints:

\[
H(N, \tilde{N}, \Lambda^i) = \int_{\Sigma} d^3x \, \{ N \mathcal{H}_\perp + N^a \mathcal{H}_a + \Lambda^i \mathcal{G}_i \}. 
\]

(2.2)

The singular Hamilton-Jacobi transformations discussed above, reduce the phase space variables to those which are coordinates on a constraint surface. It is possible to do this step by step, treating each constraint in turn. That is how Newman and Rovelli proceed [5]. To treat the Gauss law constraints, one needs variables which are invariant under the rotations of the triad field generated by \( \mathcal{G}_i \). These are obtained from the holonomy of the connection. The holonomy group is defined by the parallel transport around closed loops in \( \Sigma \):

\[
U[A, \alpha] = P \exp \int_{\alpha} A_{ia}^i \alpha^a \tau_i \, ds,
\]

(2.3)

where the \( \tau_i \) are the Pauli matrices and \( x^a = \alpha^a(s) \) is the closed curve. Clearly, \( U[A, \alpha] \) is gauge covariant, while its trace, is gauge invariant. Therefore, one expects the Hamilton-Jacobi functional to involve the trace. For the six new gauge invariant momenta, they choose six scalar functions \( (u^I, v^I) \), \( I = 1 - 3 \), whose level surfaces \( U^I = \lambda, \, V^I = \nu \) are assumed to intersect in closed loops, \( \alpha_I \equiv \alpha(u^I, v^I)_{\lambda, \nu} \).

The Hamilton principal functional is then written as

\[
S[A^i_{ia}, u^I, v^I] = \sum_{I=1}^{3} \int d\lambda d\nu \, Tr P \exp \int_{\alpha_I} A_{ia} \tau_i \alpha^a_i ds^I,
\]

(2.4)

where \( s^I \) is the parameter around the loop \( \alpha_I \). It follows that

\[
\tilde{E}_{i\tilde{a}} = \frac{\delta S}{\delta A^i_{ia}(x)} = \sum_I e^{abc} u^I_{,b} v^I_{,c} M_{Ii}(x),
\]

(2.5)
\[ M_{Ii} := Tr(U[A, \alpha_I] \tau_i). \]

and the new configuration variables are defined by (no sum on I)

\[ \frac{\delta S}{\delta u^I(x)} = q_{u^I} = \epsilon^{abc} v^J, a F^{i}_{bc} M_{Ii}, \]

\[ \frac{\delta S}{\delta v^I(x)} = q_{v^I} = -\epsilon^{abc} u^J, a F^{i}_{bc} M_{Ii}. \] (2.6)

Note that a general element of \( SL(2, C) \) is a linear combination of the Pauli matrices and the identity. Thus, \( (a^2_I = 1 + \sum_{i=1}^{3} (b_I^i)^2) \),

\[ U[A, \alpha_I] = a_I \mathbf{1} + b_I^i \tau_i, \]

so that

\[ M_{Ii} = 2 b_I^i. \]

Since in Minkowski space the holonomy group is the identity, it follows that \( M_{Ii} \) is identically zero in the flat space limit.

From the equations for \( \tilde{E}^a_I, q_{u^I}, \) and \( q_{v^I} \), it is evident that the vector constraint can be written in terms of these gauge reduced variables:

\[ \mathcal{H}_\alpha = F^i_{ab} \tilde{E}_i^b = -\frac{1}{2} \sum_I (q_{u^I} u^I_{,a} + q_{v^I} v^I_{,a}). \] (2.7)

To treat the vector constraint, one needs to construct 3- diffeomorphism invariant quantities. One way to do so is to introduce, on the manifold \( \Sigma \), intrinsic coordinates which are defined by an algorithm. In terms of such coordinates, the points of the manifold are uniquely labeled and other scalar functions of the intrinsic coordinates are then truly invariant. In this case there are six scalars. Choose the \( v^I \) to be the intrinsic coordinates and then

\[ U^I(v^J(x)) = u^I(x) \] (2.8)

will be the sought for invariants. The Hamilton-Jacobi equations are

\[ \sum_I (u^I_{,a} \frac{\delta S}{\delta u^I} + v^I_{,a} \frac{\delta S}{\delta v^I}) = 0 \]

which has the solution

\[ S(u^I, v^I, P_J) = \sum_I \int_{\Sigma} d^3 x \, u^I(x) \, P^I(v^J(x)) |\frac{\delta v}{\delta x}|. \] (2.9)

From this functional we find

\[ q_{u^I}(x) = \frac{\delta S}{\delta u^I(x)} = \frac{\delta v}{\delta x} |P_I(v^J(x)), \]

\[ q_{v^I}(x) = \frac{\delta S}{\delta v^I} = -\frac{\delta v}{\delta x} \sum_K P_K(v^J(x)) u^K_{,a} \frac{\partial x^a}{\partial v^I}. \] (2.10)
and for the new coordinates

\[ Q^I(v) = \frac{\delta S}{\delta P_I} = u'(x) = U'(v(x)). \quad (2.11) \]

The \((P_I(v), Q^I(v))\) are the coordinates on the gauge and 3-diffeomorphism reduced phase space. One would like to treat the scalar constraint in a similar fashion. However, we don’t know how to write the scalar constraint in terms of these reduced coordinates. It must be possible to do so, but it appears to require the inversion of the canonical transformations. The problem goes back to the Gauss law transformation. There we see that the relationships among the old and new variables involves the triad, the connection, and some information about the holonomy of the loops \(\alpha_I\). But, this appears in such a convoluted fashion that it does not even allow one to see how to construct even an implicit solution. Some deep insight is needed at this point.

We noted earlier that the flat space limit does not exist and is, in fact, singular. In flat space the triad is not zero, but constant. On the other hand, the right hand side of Eq (2.5) is zero. Thus, even a solution by iteration is precluded. However, in the next section, we shall find a solution for the linearized constraints directly and not as a limit of the Newman-Rovelli result.

3. Linearized Constraints.

The Gauss law constraint written out reads

\[ \tilde{E}^a_i, a + \epsilon_{ijk} A^j_a \tilde{E}^a_k = 0. \quad (3.1) \]

For the Minkowski space solution we choose

\[ 0\tilde{E}^a_i = \delta_i^a \text{ and } 0A^j_a = 0. \quad (3.2) \]

Therefore, we write

\[ \tilde{E}^a_i = \delta_i^a + \mathcal{E}^a_i, \quad A^i_a = \mathcal{A}^i_a, \quad (3.3) \]

where the calligraphic letter is the difference of the quantity from its Minkowski space value. Therefore, we have

\[ \mathcal{E}^a_i, a + \epsilon_{ijk} A^j_a \delta^a_k = 0, \quad (3.4a) \]

\[ \mathcal{F}^i_{ab} \delta^a_i = 0, \quad (3.4b) \]

\[ \epsilon_{ijk} \mathcal{F}^i_{ab} \delta^a_j \delta^b_k = 0. \quad (3.4c) \]

for the Gauss law, the vector constraint, and for the scalar constraint, respectively. In the above, we have used the linearization

\[ F^i_{ab} = \mathcal{F}^i_{ab} = \mathcal{A}^i_{b,a} - \mathcal{A}^i_{a,b}. \]

Although there is no need to write a transformation for the zeroth order Gauss law equation, it is convenient to express the solution in terms of the scalars \(0u^i\) and \(0v^i\). Thus, mod 3, we have

\[ 0u^i = x^{i+1}, \quad 0v^i = x^{i+2} \]
so that $\partial E_i^a = \epsilon^{abc} u^i_{\,b} \partial v^i_{\,c}$. We have used the fact that in the linearization, the index $I$ can be replaced by the group index $i$. In the following, the summation convention will not apply to the group indices $i, j, \cdots$. The summation will be explicitly exhibited.

In Eq. (3.5), we give the solution for the linearized Gauss law as the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation. It is useful to write out that solution extensively as follows ($u$ and $v$ below refer to linearized values):

\[
\begin{align*}
\mathcal{E}_1^a &= \epsilon^{abc} (u^1_{\,b} \delta^2_{\,c} + \delta^2_{\,b} v^1_{\,c}) + \\
&\quad \frac{1}{2} \{ \delta_1^a \{ \int_{-\infty}^{\infty} \mathcal{A}^3_2 \, dx^2 \} \delta_2^3 \{ \int_{-\infty}^{\infty} \} \mathcal{A}^2_3 \, dx^3 \} , \\
\mathcal{E}_2^a &= \epsilon^{abc} (u^2_{\,b} \delta^1_{\,c} + \delta^1_{\,b} v^2_{\,c}) + \\
&\quad \frac{1}{2} \{ \delta_3^a \{ \int_{-\infty}^{\infty} \mathcal{A}^3_1 \, dx^3 \} \delta_1^1 \{ \int_{-\infty}^{\infty} \} \mathcal{A}^3_1 \, dx^1 \} , \\
\mathcal{E}_3^a &= \epsilon^{abc} (u^3_{\,b} \delta^3_{\,c} + r^1_{\,b} v^3_{\,c}) + \\
&\quad \frac{1}{2} \{ \delta_1^a \{ \int_{-\infty}^{\infty} \mathcal{A}^2_1 \, dx^1 \delta_2^2 \{ \int_{-\infty}^{\infty} \} \mathcal{A}^1_2 \, dx^2 \} .
\end{align*}
\]

From the above, it is easy to show that the following Hamilton-Jacobi functional

\[
S(\mathcal{A}^i, u^i, v^i) = \sum_i \int d^3 x \epsilon^{abc} (u^i_{\,b} \delta_i^{l+2} + \delta_i^{l+1} u^i_{\,c}) \mathcal{A}^i_{\,a} + \\
\frac{1}{4} \{ \int \Sigma d^3 x \mathcal{A}^1 a(x) [\delta^2_2 \{ \int_{-\infty}^{\infty} \mathcal{A}^3_2 \, dx^2 \} \mathcal{A}^2_3 (x^1, x^2, x^3) + \\
\delta^2_3 \{ \int_{-\infty}^{\infty} \mathcal{A}^2_3 (x^1, x^2, x^3) + \\
\delta^2_2 \{ \int_{-\infty}^{\infty} \mathcal{A}^1_2 (x^1, x^2, x^3) \}
\}
\]

\[
\int \Sigma d^3 x \mathcal{A}^2 a(x) [\delta^3_3 \{ \int_{-\infty}^{\infty} \mathcal{A}^3_2 \, dx^2 \} \mathcal{A}^3_1 (x^1, x^2, x^3) + \\
\delta^3_1 \{ \int_{-\infty}^{\infty} \mathcal{A}^3_1 (x^1, x^2, x^3) + \\
\delta^3_2 \{ \int_{-\infty}^{\infty} \mathcal{A}^2_1 (x^1, x^2, x^3) \}
\]
\]

\[
\delta^2_2 \{ \int_{-\infty}^{\infty} \mathcal{A}^1_2 (x^1, x^2, x^3) \}
\]

generates a canonical transformation which gives the above solution for $\mathcal{E}_i^a$ and satisfies the Gauss law constraint,

\[
\frac{\partial}{\partial x^a} \{ \delta S \} \frac{\delta S}{\delta \mathcal{A}^i a} + \sum_{i,j=1}^3 \epsilon_{ijk} \mathcal{A}^i_{\,a} \delta^a_k = 0.
\]
We now obtain the new configuration space variables from $S(A^i, u^i, v^i)$,

$$\frac{\delta S}{\delta u^i} = q_{u^i} = B_i a^0 u^i,_a = B_i i^2, \quad (3.7a)$$

$$\frac{\delta S}{\delta v^i} = q_{v^i} = -B_i a^0 u^i, a = -B_i i^1. \quad (3.7b)$$

In the above, $B^{ia} = \frac{1}{2} \epsilon^{abc} f_{ibc}^{ac}$.

Note that the linearized vector and scalar constraints are homogeneous in $B^{ia}$, so that only the zeroth order part of $E_i^a$ appears in Eqs. (3.7). As a result, the inhomogeneous part of $E_i^a$ does not appear in the remaining canonical transformations. This result makes it easier to carry out the remaining transformations.

It is easy to see that the vector constraints take the simple form

$$\mathcal{H}_a = \sum_i (q_{u^i} \delta^i_{i+1} + q_{v^i} \delta^i_{i+2}) = 0 \quad (3.8)$$

and the scalar constraint

$$\mathcal{H}_\perp = \sum_i B_i^i = 0. \quad (3.9)$$

Note that

$$P^i := v^{i+1} u^{i+2} \quad (3.10)$$

has a vanishing Poisson bracket with $\mathcal{H}_a$ so that the Hamilton-Jacobi function

$$S(u^i, v^i, Q_i) = -\int_{\Sigma} d^3 x \sum_i Q_i (v^{i+1} u^{i+2}), \quad (3.11)$$

which leads to the transformation

$$q_{u^i} = -Q_{i+1} \quad \text{and} \quad q_{v^i} = Q_{i+2}, \quad (3.12)$$

clearly satisfies the vector constraint. The new momenta are precisely the expressions given in Eq. (3.10).

From Eqs. (3.7) above, we see that the components of $B^{ia}$ for $a \neq i$ are given in terms of $q_{u^i}$ and $q_{v^i}$, hence in terms of the reduced phase space variables $(Q_i, P^i)$. Now, from its definition, $B_i^a, a = 0$. Therefore, we can express $B_i^a$ in terms of a potential $Q_i^{ab}$ which is antisymmetric in the indices $(a, b)$. Furthermore, we find that this potential vanishes unless either $a = i$ or $b = i$ so that the non-zero components are given by (no sum on $i$)

$$Q_i^{ib} = \int_{-\infty}^x (Q_{i+2} a^{i+1} + Q_{i+1} a^{i+2})dx'. \quad (3.13)$$

The scalar constraint is now given by

$$\mathcal{H}_\perp = Q^a, a = 0, \quad \text{where} \quad Q^a := \sum_i Q_i^{ia}. \quad (3.14)$$
To find the final singular transformation which leads to the fully reduced phase space, it is convenient to carry out a non-singular canonical transformation which makes the $Q^a$ the new canonical coordinates. For this purpose, consider

$$G(Q_i, P_a) = \int_\Sigma d^3x Q^a(Q_i) P_a,$$

which leads to $Q^a$ as the new coordinates and

$$P_i = \frac{1}{2} \{ \partial_{i+2} P_{i+1} + \partial_{i+1} P_{i+2} \int_{-\infty}^{x^i} \partial_{i+1} \partial_{i+2} P_i \, dx' \}. \quad (3.15)$$

where we have used $\partial_i := \partial/\partial x^i$. We can now write the Hamilton-Jacobi functional for the scalar constraint in terms of the momenta $P_a$ and two functions $U$ and $V$ and two constant one forms $\mu_a$ and $\nu_a$,

$$S(P_a, U, V) = -\int d^3x \epsilon^{abc}(\mu_c U, b + \nu_b V, c) P_a \quad (3.16)$$

so that

$$Q^a = \epsilon^{abc}(\mu_c U, b + \nu_b V, c) \quad (3.17)$$

identically satisfies the remaining constraint. The new momenta are given by

$$\Pi_U = \epsilon^{abc} \mu_c P_{a,b} \quad \Pi_V = -\epsilon^{abc} \nu_c P_{a,b}. \quad (3.18)$$

The totally reduced phase space is described by the four functions $U, V, \Pi_U$ and $\Pi_V$. Since the Hamiltonian is now zero, there are no dynamical restrictions on these functions. They are to be defined as the quantum operators in the quantization of this linearized theory.

4. Discussion.

We have applied Hamilton-Jacobi theory to the linearized constraints of general relativity in terms of the Ashtekar variables. Unlike the treatment by Newman and Rovelli [5] of the exact formalism which has no flat space limit, we are able to show that we can use the singular Hamilton-Jacobi transformations to eliminate the constraints. In this way we arrive at a set of conjugate variables for the fully reduced phase space of the linear Einstein equations. If one could do the same for the exact theory, the inversion of these transformations would give the general solution of the Einstein equations. That is not the case for the linearized theory or any truncated version. The reason is that the linearized equations for general relativity are obtained from a quadratic Hamiltonian to which one must adjoin the linearized constraints with Lagrange multipliers. Thus, the linearized theory is not a wholly constrained theory. The Hamiltonian is not zero to first order. As a result, the reduced phase space variables are only a “gauge invariant” set of initial conditions. The quadratic Hamiltonian then generates the evolution of these variables. However, it is a “gauge invariant” set which may become the operators in a quantum theory.
One can see this clearly if one applies this technique to the ADM [11] formalism. Remember, the phase space variables are defined on a three-surface, not in space-time. So the linearized phase space variables are

\[ h_{ab} = q_{ab}\eta_{ab} \quad \text{and} \quad p^{ab}. \] (4.1)

where \( q_{ab} \) is the metric on the three-surface \( \Sigma \) and \( p^{ab} \) is essentially the linearized part of the extrinsic curvature of \( \Sigma \) when embedded in the space-time defined by the Einstein equations. The linearized constraints become

\[ \mathcal{H}_a = -2p^a_b = 0, \]
\[ \mathcal{H}_\perp = (\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd})h_{ab,cd} = 0. \] (4.2)

If one carries out a three-dimensional Fourier transform of \( h_{ab} \) and \( p^{ab} \), the Hamilton-Jacobi treatment leads to the result that the reduced phase space is defined by \((h_+, h_x, p^+, p^x)\) so that

\[ \tilde{h}_{ab}(\vec{k}) = h_+ (\hat{x}\hat{x} - \hat{y}\hat{y}) + h_x (\hat{x}\hat{y} + \hat{y}\hat{x}) \]
\[ \tilde{p}^{ab}(\vec{k}) = p^+ (\hat{x}\hat{x} - \hat{y}\hat{y}) + p^x (\hat{x}\hat{y} + \hat{y}\hat{x}) \] (4.3)

where \( \tilde{h}_{ab} \) and \( \tilde{p}^{ab} \) are the Fourier coefficients and \( \vec{k} \cdot \hat{x} = \vec{k} \cdot \hat{y} = 0 \). Of course, this is the expected result, but nothing tells us that \( \vec{k} \) is the spatial part of a null vector. That information comes from the linearized propagation equations.

The result we have suggests that for space-times which have a flat space limit, we again write \( \tilde{E}_i^a = \delta_i^a + \mathcal{E}_i^a \) so that the Gauss law constraints have the form

\[ \partial_a \mathcal{E}_i^a \epsilon_{ijk} A^j_a \mathcal{E}_k^a = \epsilon_{ijk} A^j_a \delta_k^a. \] (4.4)

The Newman-Rovelli solution solves the homogeneous equation, but the inhomogeneous equation can be solved with the help of line integrals using the parallel propagator. To proceed in this manner would lead to much more complicated expressions for the vector and scalar constraints for which we doubt that solutions can be found. But, this does exhibit in part the source of the singularity of the Newman-Rovelli solution.

It is clear that we were able to complete the transition to the reduced phase space for two reasons. One is that we only considered the linear terms of the constraints. But, the other is certainly connected with the fact that we did not need the general solution of the Einstein equations. The exact theory is wholly constrained so that solving the singular Hamilton-Jacobi problem is equivalent to finding the general solution of the Einstein equations. In principle, the work by Kozameh and Newman [12] does give this general solution in terms of data on future null infinity. However, it is not clear how to make use of this result.

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References.

1. A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986).
2. A. Ashtekar, Lectures on Non-perturbative Canonical Gravity (Notes prepared in collaboration with R. Tate), (World Scientific, Singapore, 1991).
3. C. Rovelli and L. Smolin, Nucl. Phys. B331, 80 (1990).
4. J. N. Goldberg, E. T. Newman, and C. Rovelli, J. Math. Phys. 32, 2739 (1991).
5. E. T. Newman and C. Rovelli, Phys. Rev. Lett. 69, 1300 (1992).
6. P. G. Bergmann, Phys. Rev. 144, 1078 (1966).
7. A. Komar, Phys. Rev. 170, 1175 (1968).
8. A. Komar, Phys. Rev. D1, 1521 (1970).
9. C. Teitelboim and M. Henneaux, Quantization of Gauge Systems (Princeton University Press, Princeton, N. J., 1992).
10. S. Frittelli, S. Koshti, E. T. Newman, and C. Rovelli, “Classical and Quantum Dynamics of the Faraday Lines of Force”, (University of Pittsburgh preprint, December, 1993).
11. R. Arnowitt, S. Deser, and C. Misner, in Gravitation, ed. L. Witten (John Wiley & Sons, New York, 1962).
12. C. N. Kozameh, E. T. Newman, and S. V. Iyer, J. Geom. Phys. 8, 195 (1992).