Sanderson, E., & Windmeijer, F. (2016). A Weak Instrument F-Test in Linear IV Models with Multiple Endogenous Variables. *Journal of Econometrics, 190*(2), 212-221. https://doi.org/10.1016/j.jeconom.2015.06.004

Publisher's PDF, also known as Version of record

Link to published version (if available): 10.1016/j.jeconom.2015.06.004

Link to publication record in Explore Bristol Research
PDF-document

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A weak instrument $F$-test in linear IV models with multiple endogenous variables

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Abstract

We consider testing for weak instruments in a model with multiple endogenous variables. Unlike Stock and Yogo (2005), who considered a weak instruments problem where the rank of the matrix of reduced form parameters is near zero, here we consider a weak instruments problem of a near rank reduction of one in the matrix of reduced form parameters. For example, in a two-variable model, we consider weak instrument asymptotics of the form $\pi_1 = \delta \pi_2 + c/\sqrt{n}$ where $\pi_1$ and $\pi_2$ are the parameters in the two reduced-form equations, $c$ is a vector of constants and $n$ is the sample size. We investigate the use of a conditional first-stage $F$-statistic along the lines of the proposal by Angrist and Pischke (2009) and show that, unless $\delta = 0$, the variance in the denominator of their $F$-statistic needs to be adjusted in order to get a correct asymptotic distribution when testing the hypothesis $H_0 : \pi_1 = \delta \pi_2$. We show that a corrected conditional $F$-statistic is equivalent to the Cragg and Donald (1993) minimum eigenvalue rank test statistic, and is informative about the maximum total relative bias of the 2SLS estimator and the Wald tests size distortions. When $\delta = 0$ in the two-variable model, or when there are more than two endogenous variables, further information over and above the Cragg–Donald statistic can be obtained about the nature of the weak instrument problem by computing the conditional first-stage $F$-statistics.

1. Introduction

Following the work of Staiger and Stock (1997) and Stock and Yogo (2005), testing for weak instruments is now commonplace. For a single endogenous variable model, the standard first-stage $F$-statistic can be used to test for weakness of instruments, where weakness is expressed in terms of the size of the bias of the IV estimator relative to that of the OLS estimator, or in terms of the magnitude of the size distortion of the Wald test for parameter hypotheses. Stock and Yogo (2005) tabulated critical values for the standard $F$-statistic that have been incorporated in software packages.

For multiple endogenous variables, inspection of the individual first-stage $F$-statistics is no longer sufficient. The Cragg and Donald (1993) statistic can be used to evaluate the overall strength of the instruments in this case, and Stock and Yogo (2005) have tabulated critical values of the minimum eigenvalue of the Cragg–Donald statistic for testing weakness of instruments. They derive the limiting distributions under weak instrument asymptotics where the reduced form parameters are local to zero in each reduced form equation, and obtain critical values that are conservative in the sense that they are rejecting the null of weak instruments too infrequently when the null is true.

In this paper, we are interested in analysing tests for weak instruments in a model with multiple endogenous variables in a setting where the reduced form parameters are not local to zero, but where the reduced form parameter matrix is local to a rank reduction of one. In this case, the values of the $F$-statistics in each of the first-stage equations can be high, but the identification of (some of) the model parameters is weak. We will focus initially on a model with two endogenous variables. The weak instrument asymptotics we consider are local to a rank reduction of one, of the form $\pi_1 = \delta \pi_2 + c/\sqrt{n}$.

http://dx.doi.org/10.1016/j.jeconom.2015.06.004
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where $\pi_1$ and $\pi_2$ are the parameters in the two reduced-form equations, $c$ is a vector of constants and $n$ is the sample size. We call these asymptotics LRR1 weak instrument asymptotics.

We will focus solely on the properties of the 2SLS estimator. We investigate the use of a conditional first-stage $F$-statistic along the lines of the proposal by Angrist and Pischke (2009) and show that the variance formula in the denominator of their $F$-statistic needs to be adjusted in order to get a correct asymptotic distribution when testing the null hypothesis, in the two-variable model, $H_0 : \pi_1 = \delta \pi_2$. We further show that the resulting new conditional $F$-statistic is equivalent to the Cragg–Donald minimum eigenvalue statistic. Using our weak instrument asymptotics we show that this conditional $F$-statistic cannot be used in the same way as the Stock and Yogo (2005) procedure for a single endogenous variable to assess the magnitude of the relative bias of the 2SLS estimator of an individual structural parameter. This is because the OLS bias expression contains additional terms such that the expression for the bias of the 2SLS estimator relative to that of the OLS estimator does not have the simple expression as in the one-variable case. However, the total relative bias can be bounded as can the size distortions of Wald tests on the structural parameters.

In a two-endogenous-variable model the conditional $F$-statistics for each reduced form are equivalent to each other and to the Cragg–Donald minimum eigenvalue statistic under our LRR1 weak instrument asymptotics. This holds unless $\delta = 0$, in which case the local rank reduction is due to the fact that $\pi_1$ is local to zero and the first-stage $F$-statistic for $x_1$ will be small and that for $x_2$ will be large. In this case, both the Angrist–Pischke $F$-statistic and our conditional $F$-statistic for $x_1$ can be assessed against the Stock–Yogo critical value, and the 2SLS estimator for the structural parameter on $x_2$ is consistent. Additional information can also be obtained from our conditional $F$-statistics when there are more than two endogenous variables, as they will identify which variables cause the near rank reduction. For example, if in a three variable model the near rank reduction is due to the reduced form parameters on two variables only, the conditional $F$-statistic for the third variable will remain large giving the researcher valuable information about the nature of the problem and directions for solving it. We also show that the 2SLS estimator for the structural parameter of the third variable is consistent in that case.

The paper is organised as follows. In Section 2 we introduce the linear model with one endogenous variable and summarise the Staiger and Stock (1997) and Stock and Yogo (2005) results for testing weak instruments. Section 3 considers weak instrument test statistics for the linear model with two endogenous explanatory variables and introduces the new conditional $F$-tests. Section 4 considers the relative bias and Wald test size distortions for the 2SLS estimator under the LRR1 weak instrument asymptotics as outlined above and presents some Monte Carlo results for the two-variable model. Section 4 also shows the usefulness of the conditional $F$-test statistics in a model with more than two endogenous variables. Finally, Section 5 concludes.

### 2. Weak instrument asymptotics in one-variable model

In this section we follow the basic Staiger and Stock (1997) and Stock and Yogo (2005) setup. The developments of the weak instrument setup and concepts for the one-variable model play an important role when we expand the model to multiple endogenous variables in the next section. The simple model is

\[
y = x \beta + u,
\]

where $y,$ $x,$ and $u$ are $n \times 1$ vectors, with $n$ the number of observations. There is endogeneity, such that $E(u|x) \neq 0$. The reduced form for $x$ is

\[
x = Z \pi + v,
\]

where $Z$ is a $n \times k_z$ matrix of instruments and $v$ is $n \times 1$. For individual $u_i$ and $v_i$ we assume,

\[
\begin{bmatrix} u_i \\ v_i \end{bmatrix} \sim (0, \Sigma) \quad \Sigma = \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix}.
\]

The 2SLS estimator is given by

\[
\hat{\beta}_{2SLS} = \frac{x' P_y y}{x' P_y X} = \beta_0 + \frac{x' P_u u}{x' P_u X},
\]

where $P_y = Z (Z'Z)^{-1} Z'$. The concentration parameter is given by

\[
CP = \frac{\pi ' Z' Z \pi}{\sigma^2_c}.
\]

and is a measure of the strength of the instruments, see Rothenberg (1984). A small concentration parameter is associated with a bias of the 2SLS estimator and deviations from its asymptotic normal distribution.

A simple test whether the instruments are related to $x$ is of course a Wald or $F$-test for the hypothesis $H_0 : \pi = 0$. The Wald test is given by

\[
W_{\pi} = \frac{\hat{\pi}' Z' Z \hat{\pi}}{\sigma^2_c} = \frac{x' Z (Z'Z)^{-1} Z' x}{\sigma^2_c}.
\]

where $\hat{\pi} = (Z'Z)^{-1} Z' x$ is the first-stage OLS estimator, and $\sigma^2_c = x' M_Z x / n$, where $M_Z = I - P_z$. Under the null, $W_{\pi} \overset{d}{\rightarrow} \chi^2_{\lambda}$. The $F$-test is given by $F = W_{\pi} / k_z$. Note that we refrain from a degrees of freedom correction in the variance estimate. Also, note that the analyses here and further below extend to a model with additional exogenous regressors, as we can replace $y$, $x$ and $Z$ everywhere by their residuals from regressions on those exogenous regressors.

Staiger and Stock (1997) introduce weak instrument asymptotics as a local to zero alternative, $\pi = c / \sqrt{n}$, which ensures that the concentration parameter does not increase with the sample size

\[
CP = \frac{\pi ' Z' Z \pi}{\sigma^2_c} \overset{p}{\rightarrow} c' Q_{ZZ} c
\]

where $Q_{ZZ} = \text{plim} (n^{-1} Z' Z)$.

Assuming that conditions are fulfilled, such that

\[
\begin{align*}
\left( \frac{1}{\sqrt{n}} Z' u \\
\frac{1}{\sqrt{n}} Z' v
\end{align*} \overset{d}{\rightarrow} \begin{pmatrix} \psi_{2u} \\ \psi_{2v} \end{pmatrix} \sim N \left( 0, \Sigma \otimes Q_{ZZ} \right),
\end{align*}
\]

and $k_z \geq 3$ when assessing relative bias. Then under weak instrument asymptotics,

\[
\tilde{\beta}_{2SLS} - \beta = \frac{x' Z (Z'Z)^{-1} Z' u}{x' Z (Z'Z)^{-1} Z' x} \overset{d}{\rightarrow} \frac{\sigma_u}{\sigma_v} (\lambda + z_v) z_u
\]

where

\[
\lambda = \sigma^{-1}_{v} Q_{ZZ}^{1/2}; \quad z_v = \sigma^{-1}_{v} Q_{ZZ}^{-1/2} \psi_{2v}; \quad z_u = \sigma^{-1}_{u} Q_{ZZ}^{-1/2} \psi_{2u}.
\]

The bias of the OLS estimator is given by

\[
\tilde{\beta}_{OLS} - \beta = \frac{x' u}{x' x} = \frac{(Z \pi + v)' u}{(Z \pi + v)' (Z \pi + v)} \overset{p}{\rightarrow} \frac{n^{-1} (n^{-1/2} c' Z' u + u' u)}{n^{-1} (n^{-1/2} c' Z' Z c + 2n^{-1/2} c' Z' v + v' v)} \overset{p}{\rightarrow} \text{plim} n^{-1} u' u = \frac{\sigma_u}{\sigma_v} \overset{p}{\rightarrow} \rho,
\]

where $\rho = \frac{\sigma_{uv}}{\sigma_u \sigma_v}$. 

As a measure of relative bias, Stock and Yogo (2005) propose

$$B_n^2 = \left( \frac{E[\hat{\beta}_{SLS} - \beta]}{E[\hat{\beta}_{OLS} - \beta]} \right)^2.$$  

From the derivations above, and as $E[z_i | z_e] = \rho z_e$, it follows that

$$B_n^2 = \left( \frac{E\left[ (\lambda + z_e) z_e \right]}{E\left[ (\lambda + z_e) (\lambda + z_e) \right]} \right)^2,$$

or

$$B_n = \left| \frac{E\left[ (\lambda + z_e) z_e \right]}{E\left[ (\lambda + z_e) (\lambda + z_e) \right]} \right|.$$

Using weak instrument asymptotics, Stock and Yogo (2005) are therefore able to assess the size of the relative bias in relation to the first-stage $F$-statistic. As $z_e \sim N(0, I_{k_e})$, $B_n$ is determined by the values of $l$ and $k_e$. Let

$$l = \lambda' k_e / k_e = c(\lambda' / k_e) \sigma_u^2,$$

then using Monte Carlo simulation, i.e. draws of $z_e \sim N(0, I_{k_e})$, Stock and Yogo (2005) find the values of $l$ such that $B_n$ is a certain value, say 0.1, for different values of $k_e$. For example, when $k_e = 4$ and using 100,000 Monte Carlo draws, we obtain a relative expected bias $E\left[ (\lambda + z_e) z_e / (\lambda + z_e) (\lambda + z_e) \right] = 0.1$ for $l = 4.98$. When $k_e = 8$, we find $l = 7$, again for $B_n = 0.1$.

Using weak instrument asymptotics, Staiger and Stock (1997) derive the asymptotic distribution for the first-stage $F$-statistic, which is given by

$$F \xrightarrow{d} \chi^2_1 (k_e l) / k_e,$$

where $\chi^2_1 (a)$ is the non-central chi-squared distribution with non-centrality parameter $a$. The $F$-test statistic can therefore be used to test the hypothesis

$$H_0 : CP / k_e \leq l_0 \quad \text{vs} \quad H_1 : CP / k_e > l_0,$$

where $l_0$ is the value for $l$ determined above such that the $B_n = b$. For $b = 0.1$, we find from the scaled non-central chi-squared distribution a critical values of 10.20 when $k_e = 4$ and 11.38 when $k_e = 8$. In comparison, Stock and Yogo (2005), henceforth SY, find very similar critical values of 10.27 and 11.39 for these two cases respectively.

As an illustration, we performed a small simulation. The model is as in (1) and (2), with $\beta = 1$;

$$\left( \begin{array}{c} u_i \\ v_i \end{array} \right) \sim N \left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0.5 \\ 0.5 & 1 \end{array} \right) \right);$$

the instruments in $Z$ are four independent standard normally distributed random variables and $\pi = (c \ c \ c \ c)' / \sqrt{n}$, with $c$ chosen such that the relative bias $B_n$ for $n \to \infty$ is equal to 0.1, or 10%. We set the sample size $n = 10,000$ and show the results in Table 1 for 10,000 Monte Carlo replications. The results are clearly in line with the theory. The observed relative bias is just over 10% and the rejection frequency of the $F$-test using the SY weak instrument critical value is 5% at the 5% nominal level.

| $\beta_{SLS}$ | $\beta_{OLS}$ | $\rho$ | $\sigma_u^2$ | $\sigma_v^2$ | $\nu = 5$ | $\nu = 10$ |
|----------------|----------------|--------|---------------|--------------|------------|------------|
| 1.9935         | 1.0318         | 0.0008 | 0.1184        | 0.0994       | 0.0501     | 0.0501     |

Notes: Sample size 10,000; 10,000 MC replications; $\beta = 1$; $F$ is the first-stage $F$-statistic for $x_i$; rel bias is the relative bias of the 2SLS estimator, relative to that of the OLS estimator; SY rej freq uses the 5% Stock–Yogo critical value for the $F$-test for a 10% relative bias.

Table 2

| $\beta_{SLS}$ | $\beta_{OLS}$ | $\rho$ | $\sigma_u^2$ | $\sigma_v^2$ | $\nu = 5$ | $\nu = 10$ |
|----------------|----------------|--------|---------------|--------------|------------|------------|
| 1.9935         | 1.0318         | 0.0008 | 0.1184        | 0.0994       | 0.0501     | 0.0501     |

Notes: Sample size 10,000; 10,000 MC replications; $\beta = 1$, $\rho = 1$; $W$ is the Wald test for testing $H_0 : \beta = 1$; rej freq uses 5% critical value of $\chi^2_1$; SY rej freq uses the 5% Stock–Yogo critical value for the $F$-test, for a maximal 10% size of $W$.

where

$$v_i = (\lambda + z_e)' (\lambda + z_e); \quad v_2 = (\lambda + z_e)' z_u.$$

The Wald size distortion is maximised for $\rho = 1$, and SY find the critical values for the $F$-test such that the maximal size of the Wald test is a certain value, say 10%, at a nominal 5% level. For the Monte Carlo example above, we set $\rho = 1$ and choose $c$ such that the maximal size distortion of the Wald test is 10%, in which case the value of $l$ is given by 16.415. The SY critical value in this case is given by 24.58. The results are given in Table 2, and confirm that the size of the Wald test is 10% and the rejection frequency of the $F$-test using the SY critical values is indeed 5%.

3.2. Variable model

Following the exposition in Expos and Pischke (2009), we first consider the following two-variable model

$$y = x_1 \beta_1 + x_2 \beta_2 + u \quad (3)$$

$$x_1 = Z \pi_1 + v_1$$

$$x_2 = Z \pi_2 + v_2$$

where $y, x_1, x_2, u, v_1$ and $v_2$ are $n \times 1$ vectors, with $n$ the number of observations. $Z$ is an $n \times k_2$ matrix of instruments, with $k_2 \geq 2$ ($k_2 \geq 4$ when assessing relative bias), and $\pi_1$ and $\pi_2$ are $k_2 \times 1$ vectors. For an individual observation $i$,

$$\left( \begin{array}{c} u_i \\ v_{1i} \\ v_{2i} \end{array} \right) \sim N \left( \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{ccc} \sigma_u^2 & \sigma_{u1} & \sigma_{u2} \\ \sigma_{1u} & \sigma_{11} & \sigma_{12} \\ \sigma_{2u} & \sigma_{21} & \sigma_{22} \end{array} \right) \right); \quad \Sigma_v = \left( \begin{array}{cc} \sigma_u^2 & \sigma_{u1} \\ \sigma_{1u} & \sigma_{11} \end{array} \right).$$

Equivalently, we can write

$$y = X \beta + u$$

$$X = Z \Pi + V$$

where $\beta = (\beta_1, \beta_2)'$; $X = (x_1 \ x_2); \Pi = (\pi_1 \ \pi_2)$ and $V = (v_1 \ v_2)$. Further, let $x = vec(X), \pi = vec(\Pi)$ and $v = vec(V)$.

The OLS estimates for $\pi_1$ are denoted $\hat{\pi}_1 = (Z'Z)^{-1} Z' x_i, j = 1, 2,$ and the estimated variances are given by

$$\hat{\Sigma}_v = \left( \begin{array}{cc} \hat{\sigma}_u^2 & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_{12} \end{array} \right) = \frac{1}{n} \hat{\nabla} \hat{\nabla} = \frac{1}{n} \left( \begin{array}{ccc} \hat{v}_1^2 & \hat{v}_1 \hat{v}_2 & \hat{v}_2 \hat{v}_2 \end{array} \right),$$

where $\hat{\nabla} = X - \hat{\Pi}$. 
The first-stage F-statistics are given by

\[ F_j = \frac{\hat{\pi}_j'Z'\hat{\pi}_j}{\kappa_j \hat{\sigma}_j^2} = \frac{x_j'Z'(Z'Z)^{-1}Z'x_j}{\kappa_j \hat{\sigma}_j^2}; \quad j = 1, 2, \]

and \( k_j F_j \) converges in distribution to a \( \chi^2 \) distribution under the null \( H_0 : \tau_j = 0 \). Significant first-stage F-statistics are clearly necessary, but not sufficient, for identification of \( \beta \). For example, if \( \tau_1 = \delta \tau_2 \neq 0 \), both first-stage F-statistics will reject their null in large samples, but the model is clearly unidentified.

Staiger and Stock (1997) and Stock and Yogo (2005) consider weak instrument asymptotics where all reduced form parameters are local to zero, i.e. \( \Pi = C/\sqrt{n} \). The Wald test for \( H_0 : \pi = 0 \) is given by

\[ W_n = \hat{\pi} (\hat{\Sigma}^{-1} \otimes Z'Z) \hat{\pi}, \]

which is identical to the trace of the Cragg and Donald (1993) statistic

\[ CD = \hat{\Sigma}^{-1/2} \hat{\pi} Z' \hat{\pi} \hat{\Sigma}^{-1/2}. \]

However, this Wald test statistic on the reduced form cannot be used in an equivalent way to assess relative bias and 2SLS Wald test size distortions as in the one-variable model above, because these are determined largely by the minimum eigenvalue of CD, \( \tau_{\text{min}} \). In other words, relative bias and Wald size distortions can be large if \( \text{tr} (CD) \) is large but \( \tau_{\text{min}} \) is small. In a general setting with \( g \) endogenous explanatory variables, \( W_n = \text{tr} (CD) \) is a test for \( H_0 : \text{rank} (\Pi) = 0 \), whereas \( \tau_{\text{min}} \) is a test for \( H_0 : \text{rank} (\Pi) = g - 1 \). SY derive critical values for \( \tau_{\text{min}} / \kappa_j \) under the local to zero weak instrument asymptotics for maximal total relative bias and Wald test distortions, where the total relative bias is given by

\[ B^2 = \left( \frac{\Sigma_x}{\Sigma_\epsilon} \frac{\hat{\beta}_{\text{OLS}} - \beta}{\hat{\beta}_{\text{2SLS}} - \beta} \right)' \Sigma_x \left( \frac{\hat{\beta}_{\text{OLS}} - \beta}{\hat{\beta}_{\text{2SLS}} - \beta} \right), \]

(4)

with \( \Sigma_x = \text{plim} (n^{-1}XX) \). In this case, as \( \tau_{\text{min}} \) is not the test statistic for \( H_0 : \pi = 0 \), unlike in the case of one endogenous variable, the correspondence is not exact and use of the SY critical values results in a conservative test in the sense that the null of weak instruments is rejected too infrequently when the null is true. This is not altogether an undesirable feature of the test, as a researcher can be quite confident that instruments are not weak when \( \tau_{\text{min}} / \kappa_j > 1 \) is larger than the SY critical value.

The Staiger and Stock (1997) and Stock and Yogo (2005) results for the F-test and Cragg–Donald statistic in the one-variable and multiple-variable model respectively in relation to the relative bias and Wald test size distortions hold under the stated assumptions of the model and the reduced form equations for the endogenous variables. When the variances in the reduced forms are conditionally heteroskedastic, then one can compute robust F-statistics and the Kleibergen and Paap (2006) robust version of the Cragg–Donald statistic. These test statistics are then valid tests for underidentification as they have correct size under the null that the instruments are not informative, i.e. for testing that \( \text{rank} (\Pi) = 0 \). But the documented relationship of the weak-instrument critical values and the sizes of the relative bias and Wald-test size distortion no longer holds, see for example Bune and De Haan (2010). This limits the exact use of the weak-instrument tests, as for example binary endogenous explanatory variables automatically produce a conditionally heteroskedastic reduced form. Also, this relationship breaks down in simple panel data models, when there is serial correlation in the reduced form errors, or indeed in simple time-series models with serial correlation. In our development of the conditional F-statistics for models with multiple endogenous variables, we maintain the same assumptions as Staiger and Stock (1997) and Stock and Yogo (2005), and hence the same limitations. Olea and Pflueger (2013) have recently proposed an alternative robust F-test type procedure for weak instruments, but thus far it can only be applied to the one-endogenous variable model.

### 3.1. Conditional F-test

Angrist and Pischke (2009) propose an alternative conditional first-stage F-statistic for the case of multiple endogenous variables by reformulating the estimation problem to a one-variable model after replacing the other endogenous variables with their reduced form predictions. For instance, for the two-variable model, the 2SLS estimator for \( \beta_1 \) is obtained by 2SLS in the model

\[ y = x_1 \beta_1 + \hat{x}_2 \beta_2 + u^*, \]

where \( \hat{x}_2 = Z \hat{\pi}_2 = P_Z x_2 \), using \( Z \) as the instruments, and hence

\[ \hat{\beta}_1 = (\hat{x}_2 M_2 P_Z M_2 x_1)^{-1} \hat{x}_2 M_2 P_Z y. \]

Therefore, \( \hat{\beta}_1 \) can be seen as the 2SLS estimator in the one-variable model

\[ y = M_2 x_1 \beta_1 + \xi, \]

where the residual \( M_2 x_1 = x_1 - \hat{x}_2 \delta \), with \( \delta = (\hat{x}_2 \hat{x}_2)^{-1} \hat{x}_2 x_1 \), is instrumented by \( Z \). The reduced form is then

\[ M_2 x_1 = Z \kappa + \varepsilon \]

and the Angrist–Pischke F-statistic is testing the hypothesis \( H_0 : \kappa = 0 \), given by

\[ F_{\text{AP}} = \frac{\hat{\kappa}' Z' \hat{\kappa}}{(k_1 - 1) \hat{\sigma}_{\hat{\kappa}}^2} = \frac{x_1' M_2 P_Z M_2 x_1}{(k_1 - 1) \hat{\sigma}_{\hat{\kappa}}^2}. \]

(8)

where \( \hat{\kappa} \) is the OLS estimator of \( \kappa \).

\( \hat{\kappa} = (Z'Z)^{-1} Z'M_2 x_1 = (Z'Z)^{-1} Z'(x_1 - \hat{x}_2 \delta) = \hat{\pi}_1 - \hat{\pi}_2 \delta; \)

and \( \sigma_{\hat{\kappa}}^2 = \hat{\sigma}^2 / n \), with \( \hat{\sigma} = M_2 x_1 - Z \kappa \). The degrees of freedom correction follows because \( \hat{x}_2 \) has been predicted using the same instruments \( Z \). If we partition \( Z = [z_1 \ z_2] \) with \( z_2 (k_1 - 1) \times n \), matrix, then the instrument set for (5) could equivalently be written as \([\hat{x}_2 \ z_2] \).

As the problem seems to have been reduced to a one-endogenous variable case, \( F_{\text{AP}} \) has been proposed to determine instrument strength for identification of individual structural parameters, like \( \beta_1 \) in the above derivation, and Stock and Yogo (2005) weak instrument critical values used to determine maximum relative bias of the IV estimator, relative to the OLS estimator for the single parameter. There are some issues with this, however, that seem to invalidate such an approach. Under the null that \( \kappa = 0 \), \( (k_1 - 1) F_{\text{AP}} \) does not follow an asymptotic \( \chi^2_{k_1 - 1} \) distribution, unless \( \tau_1 = 0 \). An alternative F-statistic is easily derived that corrects for this, but the relative bias results as described in the previous section for the one-variable model do not carry over to the individual parameters in this multiple endogenous variables model.

To consider the asymptotic distribution, for any given value of \( \delta \) we have that

\[ x_1 - \hat{x}_2 \delta = x_1 - x_2 \delta + (x_2 - \hat{x}_2) \delta = Z (\tau_1 - \delta \tau_2) + v_1 - \delta v_2 + \delta M_2 v_Z \]

\[ = Z (\tau_1 - \delta \tau_2) + v_1 - \delta M_2 v_Z. \]

Clearly, the OLS estimator for \( \kappa_0 \) in the model

\[ x_1 - \hat{x}_2 \delta = Z \kappa_0 + \varepsilon^* \]

is given by

\[ \hat{\kappa}_0 = (Z'Z)^{-1} Z' (x_1 - \hat{x}_2 \delta) = (Z'Z)^{-1} Z' (x_1 - \hat{x}_2 \delta) = \hat{\pi}_1 - \delta \hat{\pi}_2 + (Z'Z)^{-1} Z' (v_1 - \delta v_2) \]
and hence the variance of the OLS estimator is given by
\[
\text{Var}(\hat{\beta}_3) = \left(\sigma_1^2 + 2\delta\sigma_{12} + \delta^2\sigma_2^2\right)(Z'Z)^{-1}. \tag{10}
\]
The $F$-statistic for testing $H_0 : \kappa_3 = 0$ in (9) is
\[
F_{3} = \frac{\hat{\beta}_3^2}{k_2(\hat{\sigma}_1^2 + 2\hat{\sigma}_{12} + \hat{\delta}^2\hat{\sigma}_2^2)},
\]
and $k_2F_3$ converges in distribution to a $\chi^2_{k_2}$ distribution under the null that $\kappa_3 = 0$, or $\pi_1 = \delta\pi_2$. However, computing the standard $F$-test statistic in (9) as
\[
F_{3} = \frac{\hat{\beta}_3^2}{k_2\hat{\sigma}_2^2},
\]
does not result in $F_3$ as
\[
\tilde{\text{F}} \equiv \frac{\hat{\beta}_3^2}{k_2\hat{\sigma}_2^2} Z \tilde{\kappa}_3 \tilde{\kappa}_3 \Rightarrow \text{F}_{k_2, k_2 - 1}
\]
and hence
\[
\text{F}_{k_2, k_2 - 1} \sim \text{F}_{k_2, k_2 - 1}
\]
and the denominator of $F_3$ does not estimate the variance as in (10) correctly and $k_2F_3$ does not converge to a $\chi^2_{k_2}$ distribution under the null, unless $\delta = 0$. The correct $F$-statistic would be obtained by the standard $F$-test if the dependent variable in (9) was $x_1 - \delta x_2$ instead of $x_1 - \delta x_2$.

The Angrist–Pischke approach does replace $\delta$ by an estimate $\hat{\delta}$. By developing a formal testing framework we show that the same issues arise and that $(k_1 - 1)F_{AP}$ does not have an asymptotic $\chi^2_{k_2 - 1}$ distribution under the null that $\pi_1 = \delta\pi_2$, unless $\delta = 0$.

Partition $Z = [Z_1, Z_2]$. We can write the reduced form for $x_1$ as
\[
x_1 = Z_1\pi_1 + v_1 = Z\pi_2 + Z(\pi_1 - \pi_2) + v_1 = Z\pi_1 + Z_2(\pi_1 - \pi_2) + \pi_2 v_1 - \delta v_2 \tag{11}
\]
where $\pi_1$ and $\pi_2$ are partitioned as $[\pi_1, \pi_2]$ and $[\pi_2, 1]$, respectively; $\delta = \pi_1/\pi_2$, implicitly assuming that $\pi_2 \neq 0$. Hence a test for underidentification for $H_0 : \gamma = 0$, in the model
\[
x_1 = x_2\delta + Z_1\gamma + v^*, \tag{12}
\]
where $v^* = v_1 - \delta v_2$. Clearly, $x_2$ is an endogenous variable in (12), but we can estimate the parameters $\delta$ and $\gamma$ by IV, using $Z_2$ as instruments. The 2SLS estimators for $\delta$ and $\gamma$ are given by
\[
\hat{\delta} = \left(\tilde{Z}_2'\tilde{Z}_2\right)^{-1}\tilde{Z}_2\tilde{v}_1,
\]
and
\[
\text{Var}(\hat{\delta}) = \sigma_2^2\left(\tilde{Z}_2'\tilde{Z}_2\right)^{-1},
\]
with $\sigma^2_2 = \sigma_2^2 + 2\delta\sigma_{12} + \delta^2\sigma^2_2$. The $F$-test statistic for testing $H_0 : \gamma = 0$ is therefore given by
\[
F_{\gamma} = \frac{\hat{\gamma}^2}{k_2 - 1} \frac{(\hat{\sigma}^2_2 - \hat{\sigma}_2^2)}{(\hat{\sigma}^2_2 - \hat{\sigma}_2^2)} \tag{13}
\]
with $\hat{\gamma} = \hat{\gamma}_{12} - \hat{\gamma}_{22},$ as the IV estimates are given by
\[
\hat{\delta} = \hat{\gamma}_{11} \hat{\gamma}_{21},
\]
and hence
\[
\hat{\sigma}_2^2 = \frac{1}{n}(\hat{\delta}^2 + \hat{\gamma}^2) = \hat{\sigma}_2^2 - \hat{\delta}^2\sigma_{12} + \hat{\delta}^2\sigma^2_2
\]
is a consistent estimator of $\sigma^2_2$.

The Angrist and Pischke (2009) $F$-statistic as described above is related to $F_{\gamma}$, as
\[
F_{\gamma} = \frac{\hat{\gamma}^2}{(k_1 - 1) \frac{\hat{\sigma}^2_2 - \hat{\sigma}_2^2}{\hat{\sigma}^2_2 - \hat{\sigma}_2^2}} = \frac{\hat{\gamma}^2}{(k_1 - 1) \frac{\hat{\sigma}^2_2 - \hat{\sigma}_2^2}{\hat{\sigma}^2_2 - \hat{\sigma}_2^2}},
\]
because
\[
\hat{\gamma} = \tilde{Z}_2'\tilde{Z}_2\tilde{v}_1 = \hat{\gamma}_{12} - \hat{\gamma}_{22} \tilde{v}_2 = \hat{\gamma}_{12} - \hat{\gamma}_{22} \tilde{v}_2
\]
and the sum of squared residuals is given by
\[
\hat{\gamma}^2 = \frac{\hat{\gamma}^2}{(k_1 - 1) \frac{\hat{\sigma}^2_2 - \hat{\sigma}_2^2}{\hat{\sigma}^2_2 - \hat{\sigma}_2^2}} = \frac{\hat{\gamma}^2}{(k_1 - 1) \frac{\hat{\sigma}^2_2 - \hat{\sigma}_2^2}{\hat{\sigma}^2_2 - \hat{\sigma}_2^2}}.
\]

3.2. Relationship with Cragg–Donald statistic

With $g$ endogenous variables, the minimum eigenvalue of the Cragg–Donald statistic, $\tau_{mm}$, is a test for $H_0 : \text{rank}(T) = g - 1$ against the alternative $H_1 : \text{rank}(T) = g$. For the two-variable model, this null is of course equivalent to $H_0 : \pi_1 = \delta\pi_2$. The Cragg–Donald test is based on the restricted estimates under the null, using the minimum-distance criterion
\[
(\hat{\delta}, \hat{\pi}_2) = \arg \min_{\delta, \pi_2} H(\delta, \pi_2)
\]
with
\[ H(\delta, \pi_2) = \left( \left( \hat{\pi}_1 \ - \ \frac{\delta \pi_2}{\pi_2} \right) \right)' \left( \Sigma^{-1} \otimes Z'Z \right) \times \left( \left( \hat{\pi}_1 \ - \ \frac{\delta \pi_2}{\pi_2} \right) \right). \]

The Cragg–Donald test statistic is then
\[ \tau_{\text{min}} = H \left( \hat{\delta}, \pi_2 \right) \xrightarrow{d} \chi^2_{k-1} \]
under the null. We show in the Appendix that
\[ H \left( \hat{\delta}, \pi_2 \right) = \left( \hat{\pi}_1 - \frac{\delta \pi_2}{\pi_2} \right)' Z'Z \left( \hat{\pi}_1 - \frac{\delta \pi_2}{\pi_2} \right) \]
and hence the only difference between \( F_{1,2} \) and \( \tau_{\text{min}} \) is the estimate for \( \delta \). Unlike the \( F \)-statistics, \( \tau_{\text{min}} \) is invariant to normalisation, as \( H \left( \hat{\delta}, \pi_1 \right) = H \left( \hat{\delta}, \pi_2 \right) \). Because of this, computation of both \( F_{1,2} \) and \( \tau_{\text{min}} \) can provide further information about the nature of the weakness of the instruments, as their values can indicate whether the rank reduction is due to e.g. \( \pi_1 = 0 \) \( (\delta = 0) \), which \( \tau_{\text{min}} \) cannot distinguish, as we will show below. We will also present a three-endogenous variables example in Section 4.3 which further highlights the additional information about instrument strength revealed by the three conditional \( F \)-statistics relative to that of the Cragg–Donald statistic.

4. Local to rank one weak instrument asymptotics in the two-variable model

In the previous section, we have shown that \((k_2 - 1) F_r \) has a limiting \( \chi^2_{k-1} \) distribution under the null that \( \gamma = 0 \) in (12). We next investigate whether \( F_r \) can be used to assess whether instruments are weak for individual parameters as described in Section 2. We focus in the derivation below on \( F_r \), as the setup for this test is easier to use with our weak instruments asymptotics, but results of course carry over directly to \( F_{1,2} \) and \( \tau_{\text{min}} \).

We are interested in the case that the instruments are not weak for each equation, but where the rank of \( IT \) approaches a rank reduction of one. We specify LRR1 weak instrument asymptotics as \( \gamma = c/\sqrt{n} \), or
\[ \pi_{12} = \delta \pi_{22} + c/\sqrt{n}. \]

We can then write the reduced form of \( x_1 \) as
\[ x_1 = Z \pi_1 \delta + Z_2 (\pi_{12} - \delta \pi_{22}) + v_1 = \tilde{X}_2 \delta + Z c/\sqrt{n} + (v_1 - \delta \tilde{P}_2 v_2). \]
The IV estimator for \( \beta_1 \) is given by
\[ \tilde{\beta}_{1,\text{LS}} = \frac{x_1' M_{z2} Z_3 (Z_3' M_{z2} Z_3)^{-1} Z_3' M_{z2} y}{x_1' M_{z2} Z_3 (Z_3' M_{z2} Z_3)^{-1} Z_3' M_{z2} x_1} \]
and it follows that
\[ \tilde{\beta}_{1,\text{LS}} - \beta_1 = \frac{x_1' M_{z2} Z_3 (Z_3' M_{z2} Z_3)^{-1} Z_3' M_{z2} u}{x_1' M_{z2} Z_3 (Z_3' M_{z2} Z_3)^{-1} Z_3' M_{z2} x_1} \]
as \( M_{z2} \tilde{X}_2 = 0, M_{z2} M_{z2} = (I - P_2 X_2) (x_1' P_2 x_2)^{-1} x_1' P_2 )M_{z2} = M_{z2} \), and hence \( Z_3' M_{z2} M_{z2} v_3 = Z_3' M_{z2} v_2 = 0 \).

We assume that
\[ \left( \frac{1}{\sqrt{n}} Z_2' M_{z2} \right) \xrightarrow{d} \left( \psi_2' \right) = N \left( 0, \Omega \otimes Q \right), \]
where
\[ \Omega = \left( \begin{array}{cc} \sigma_u^2 & \sigma_u (\delta_{u1} - \delta_{u2}) \\ \sigma_u (\delta_{u1} - \delta_{u2}) & \sigma^2 + \delta^2 \sigma^2 - 2 \delta \sigma_{u12} \end{array} \right) ; \]
\[ Q = \text{plim} \left( n^{-1} Z_2' Z_2 \right). \]

It is then easily shown that
\[ x_1' M_{z2} Z_3 (Z_3' M_{z2} Z_3)^{-1} Z_3' M_{z2} u \xrightarrow{d} \sigma_u \sigma_{u1} - \delta \sigma_{u2} \]
and
\[ x_1' M_{z2} Z_3 (Z_3' M_{z2} Z_3)^{-1} Z_3' M_{z2} x_1 \xrightarrow{d} \sigma_{u1} - \delta \sigma_{u2} \]
where
\[ \sigma_{u1} - \delta \sigma_{u2} = \sqrt{\sigma_u^2 + \delta^2 \sigma_u^2 - 2 \delta \sigma_{u12}}. \]

We are therefore in the same setup as Staiger and Stock (1997) and Stock and Yogo (2005), and the distribution of the bias of \( \beta_1,\text{LS} \) is given by
\[ \left( \tilde{\beta}_{1,\text{LS}} - \beta_1 \right) = \frac{\sigma_u}{\sigma_{u1} - \delta \sigma_{u2}} \left( \sigma_{u1} - \delta \sigma_{u2} \right) \frac{1}{c' Q c} \]
and the critical values from the non-central chi-squared distribution applied to
\[ F_r = \frac{x_1' M_{z2} Z_3 (Z_3' M_{z2} Z_3)^{-1} Z_3' M_{z2} x_1}{k_2 - 1 (\sigma^2 + \delta^2 \sigma^2 - 2 \delta \sigma_{u12})} \]
\( (k_2 - 1) \left( \sigma^2 + \delta^2 \sigma^2 - 2 \delta \sigma_{u12} \right) \)

However, in this case the bias of the OLS estimator of \( \beta_1 \) in the model
\[ y = x_1 \beta_1 + x_2 \beta_2 + u \]
is given by
\[ \tilde{\beta}_{1,\text{OLS}} - \beta_1 = \frac{x_1' M_{z2} u}{x_1' M_{z2} x_1}. \]

As
\[ x_1 = x_2 \delta + Z c/\sqrt{n} + (v_1 - \delta v_2), \]
we get that
\[ \text{plim} \left( n^{-1} (x_1' M_{z2} u) \right) = \text{plim} \left( n^{-1} \left( \frac{c}{\sqrt{n}} Z_2 M_{z2} u + (v_1 - \delta v_2) M_{z2} u \right) \right). \]

Further,
\[ \text{plim} \left( n^{-1} (x_1' M_{z2} x_1) \right) = \text{plim} \left( n^{-1} \left( \frac{c}{\sqrt{n}} Z_2' M_{z2} Z_2 + 2 \frac{c}{\sqrt{n}} Z_2 M_{z2} (v_1 - \delta v_2) + (v_1 - \delta v_2) M_{z2} (v_1 - \delta v_2) \right) \right). \]
From these results we find that the bias of the OLS estimator converges to
\[
\text{plim} (\hat{\beta}_{1,\text{OLS}} - \beta_1) = \frac{\text{plim} \left( n^{-1} (v_1 - \delta v_2) M_{x_z u} \right)}{\text{plim} \left( n^{-1} (v_1 - \delta v_2) M_{x_z (v_1 - \delta v_2)} \right)} = \frac{\sigma_u - \delta \sigma_u}{\sigma_u^2 + \delta^2 \sigma_u^2 - 2 \delta \sigma_1^2 - \frac{\sigma_{\epsilon_1} - \delta \sigma_{\epsilon_1}}{\sigma_{\epsilon_1}^2 + \delta^2 \sigma_{\epsilon_1}^2}}
\]
and therefore we have that
\[
B_{n,1} = \frac{E \left[ \hat{\beta}_{\text{OLS}} - \beta_1 \right]}{E \left[ \hat{\beta}_{\text{OLS}} - \beta_1 \right]} = \left( \frac{\hat{\lambda} + \hat{z}_1}{\hat{\lambda} + \hat{z}_1} \right)
\]
and so the direct relationship between the relative bias of the individual parameter and the value of the concentration parameter does not hold in this setting. However, we can get a result for the total relative bias. First of all, we show in the Appendix that for the 2SLS estimator for \( \beta_2 \) the following holds,
\[
\hat{\beta}_{2,\text{2SLS}} - \beta_2 \overset{d}{\to} -\frac{\sigma_u}{\sigma_{\epsilon_1 - \delta \sigma_{\epsilon_1}}} \left( \frac{\hat{\lambda} + \hat{z}_1}{\hat{\lambda} + \hat{z}_1} \right)
\]
and hence, asymptotically,
\[
E \left[ \hat{\beta}_{\text{2SLS}} - \beta_2 \right] = -\delta E \left[ \hat{\beta}_{\text{OLS}} - \beta_1 \right]
\]
From this it follows that \( \hat{\beta}_{2,\text{2SLS}} \) is consistent when \( \delta = 0 \), that is in the situation where the instruments are strong for \( x_2 \), but weak for \( x_1 \) in the sense that \( \sigma_1 \) is local to zero.

We then show in the Appendix that
\[
\frac{\sigma_u^2}{\sigma_{\epsilon_1}^2 + \delta^2 \sigma_{\epsilon_1}^2 - 2 \delta \sigma_1^2 - \frac{\sigma_{\epsilon_1} - \delta \sigma_{\epsilon_1}}{\sigma_{\epsilon_1}^2 + \delta^2 \sigma_{\epsilon_1}^2}}
\]
and hence, asymptotically,
\[
E \left[ \hat{\beta}_{\text{2SLS}} - \beta_2 \right] = -\delta E \left[ \hat{\beta}_{\text{OLS}} - \beta_1 \right]
\]

This design is such that the weak instrument relative bias can apply to the OLS estimator in this model instead. The OLS estimator is given by
\[
\hat{\beta}_{\text{OLS}} = \frac{x_{1z} y}{x_{1z} x_{1z} x_{1z}} = \hat{\beta}_1 + \frac{\hat{\beta}_1 x_{1z} x_{1z} + \hat{\beta}_1 x_{1z} u}{x_{1z} x_{1z} x_{1z}}
\]
and therefore
\[
\text{plim} \left( \hat{\beta}_{\text{OLS}} - \beta_1 \right) = \frac{\beta_1 \sigma_{\epsilon_1} + \sigma_{\epsilon_1}}{\sigma_{\epsilon_1}^2}
\]

and hence, again
\[
B_{n,1} = \frac{E \left[ \hat{\beta}_{\text{OLS}} - \beta_1 \right]}{E \left[ \hat{\beta}_{\text{OLS}} - \beta_1 \right]} = \left( \frac{\hat{\lambda} + \hat{z}_1}{\hat{\lambda} + \hat{z}_1} \right)
\]

where
\[
\tilde{\sigma}_u^2 = \left( y - x_{1z} \hat{\beta}_{1,\text{2SLS}} - x_{1z} \hat{\beta}_{2,\text{2SLS}} \right) / \left( y - x_1 \hat{\beta}_{1,\text{2SLS}} - x_{1z} \hat{\beta}_{2,\text{2SLS}} \right)
\]

We find that
\[
\tilde{\sigma}_u^2 \overset{d}{\to} \frac{\sigma_u^2}{\sigma_{\epsilon_1}^2 + \delta^2 \sigma_{\epsilon_1}^2 - 2 \delta \sigma_1^2 - \frac{\sigma_{\epsilon_1} - \delta \sigma_{\epsilon_1}}{\sigma_{\epsilon_1}^2 + \delta^2 \sigma_{\epsilon_1}^2}}
\]

and so we can again use the SY critical values for the F-statistic for maximal size of the Wald-test, achieved when \( \tilde{\theta} = 1 \). Clearly, we get the same results for \( \tilde{W}_2 \), the Wald test for \( H_0 : \beta_2 = \beta_0^d \).

4.1. Monte Carlo Illustration

To illustrate, we generate data from the model as specified above, with
\[
\begin{pmatrix} u_1 \\ v_{1z} \\ v_{2z} \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{u1} & \sigma_{u2} \\ \sigma_{u1} & \sigma_{u1}^2 & \sigma_{u12} \\ \sigma_{u2} & \sigma_{u12} & \sigma_{u2}^2 \end{pmatrix} \right)
\]

The instruments are drawn independently from the standard normal distribution, with \( k_u = 4 \), and hence \( Q_{2z} = I_k \). We set \( \pi_2 = (-0.5, 0.5, -0.5, 0.5)' \) and \( \pi_1 = \delta \pi_2 + (c, c, c)' / \sqrt{n} \). We have
\[
Q = \text{plim} - Z_{1z} Z_{2z} = k_{z-1} - \frac{\pi_{2z} \pi_{2z}'}{\pi_{2z} \pi_{2z}}
\]

where \( \pi_2 = [\pi_{21} \pi_{22}]' \) is partitioned commensurate with \( Z = [Z_1 Z_2] \).

The limit of the concentration parameter for this specific configuration is given by
\[
CP1 = \frac{c' Q c}{\sigma_1^2 + \delta^2 \sigma_1^2 - 2 \delta \sigma_1^2}
\]

We choose \( c \) such that the concentration parameter has the value for which the IV estimator for \( \beta \) has a maximal total relative bias of 10%. We have further set the parameters as follows: \( \beta_1 = 0.5; \beta_2 = -0.3; \sigma_1^2 = \sigma_2^2 = 1; \sigma_1 \sigma_2 = -0.7; \sigma_{1z} = -0.7; \delta = 0.7 \). This design is such that the additional terms in the OLS bias are important, with
\[
\text{bias} \overset{d}{\to} -\frac{\sigma_1 \sigma_2}{\sigma_1^2 + \delta^2 \sigma_1^2 - 2 \delta \sigma_1^2}
\]

i.e. the OLS bias for \( \beta_1 \) is much smaller than \( -\frac{\sigma_1 \sigma_2}{\sigma_1^2 + \delta^2 \sigma_1^2 - 2 \delta \sigma_1^2} \). The results are given in Table 3 for a sample size of 10,000 observations. The individual standard F-statistics are very large. As expected, the IV estimator for \( \beta_1 \) has a large relative bias of 0.3441, approximately equal to 3.56 * 0.1, but the relative bias of \( \beta_2 \) is much smaller at 0.0498. The distributions of \( F_{1z} F_{2z} \) and \( v_{zmin} / (k_z - 1) \) are virtually identical, each with a mean of 4.7 and rejection frequency of 4.6% at the 5% nominal level using the weak instrument critical value.
Table 3
Estimation results and relative bias for two-variable model.

| Mean | St dev | Rel bias | SY rej freq |
|------|--------|----------|-------------|
| β1,0.05 | 0.5695 | 0.0070 |             |
| β2,0.05 | −0.6506 | 0.0062 |             |
| β1,2SLS | 0.5239 | 0.1979 | 0.3441      |
| β2,2SLS | −0.3174 | 0.1419 | 0.0498      |

Notes: Sample size 10,000; 10,000 MC replications; values for a maximum 10% total relative bias.

γ is relative to that of the OLS estimator; SY rej freq uses the 5% Stock–Yogo critical values for a maximum 10% total relative bias.

Table 4
Estimation and Wald tests results for two-variable model.

| Mean | St dev | Rej freq | SY rej freq |
|------|--------|----------|-------------|
| β1,0.05 | 1.4990 | 0.0007 |             |
| β2,0.05 | 0.3899 | 0.0006 |             |
| β1,2SLS | 0.5257 | 0.1565 |             |
| β2,2SLS | −0.2827 | 0.1071 |             |
| W1 | 1.47 | 2.86 | 0.1016       |
| W2 | 1.46 | 2.87 | 0.1017       |
| W12 | 2.61 | 3.58 | 0.1080       |
| F1,12 | 14.85 | 4.40 | 0.0548       |
| F2,1 | 14.93 | 4.45 | 0.0585       |
| τmin/(k1 − 1) | 0.6744 | 4.07 | 0.0517 |
| τmin/k2 | 11.13 | 3.30 | 0.0549       |

Notes: Sample size 10,000; 10,000 MC replications; β1 = 0.5; β2 = −0.3; Fj is the first-stage F-statistic for x1, j = 1, 2; Fj,k is the Angrist–Pischke F-statistic and Fj,k1 and Fj,k2 are the conditional F-statistics as in [14]; τmin is the Cragg–Donald minimum eigenvalue statistic; rej freq uses Wald tests 5% critical value; SY rej freq uses the 5% Stock–Yogo critical values for a maximal 10% total relative bias.

In comparison, the AP F-statistics are much larger in this case with the mean of Fj,k1 equal to 11.82, and that of Fj,k2 equal to 22.93.

The total relative bias in this design is found to be equal to 7.6%, which is less than 10%, as predicted by the theory above. The SY test for weak instruments for 1j local to 0 is conservative and has a rejection frequency of 2.6%. This test is given by τmin/k2, and the weak instrument critical value is derived for two endogenous variables with k2 instruments. In contrast, the weak instrument critical values for Fj,k1 and τmin/(k1 − 1) are those for one endogenous variable with k1 − 1 instruments. From Table 1 in SY, it is easily established that when τmin/k2 is larger than its associated tabulated critical value, then τmin/(k1 − 1) is also larger than its weak instrument critical value, so we would always reject LRR1 weak instrument problems whenever we reject rank zero weak instrument problems.

In Table 4 we present results for the Wald test statistics in a design with 1, by changing the variance parameters to σ1 = 0.755, σ2 = 0.35 and − = −0.35, again choosing c such that the size of the Wald tests is 10% at the 5% level. The simulations confirm the analytical results. The rejection frequencies of the Wald tests are just over 10% and the rejection frequencies of Fj,k1 and τmin/(k2 − 1) just over 5%. In this case, the SY weak instrument test τmin/k2 using the tabulated critical value for two endogenous variables and four instruments is also just over 5%.

4.2. The case δ = 0

When δ = 0, we have in the process above that π1 is local to zero, and hence the instruments for x1 are weak, but not for x2. As shown above, β2,2SLS is in this case consistent for β2, but β1,2SLS will suffer from a weak instrument bias. This situation may actually be of interest if the main research focus is on the effect of x2 on y. If the instruments used are then strong for x1 but weakly or not informative for x1, the IV estimator for β2 will be well behaved. In Table 5, we show the results for the bias of the 2SLS estimates, for when δ = 0 and where we have further set σ1 = 0.8. All other parameters remain the same as for the results presented in Table 3, and we have set the value of c again such that the maximum total relative bias is 10%. As can be seen from the table, the results are as expected. The value of the first-stage F-statistic for x1, F1, is now small, whilst that of F2, is large. The behaviour of Fj,k1 is now the same as that of Fj,k2, both rejecting the null of weak instruments 5% of the time using the SY critical values for k2 − 1 instruments. β2,2SLS is consistent, but the total relative bias is at 9.7% only just below 10%.

4.3. More than two endogenous variables

As is clear from the analyses above for the two-variable model, the use of F1,2 and F1,j under our LRR1 weak instrument asymptotics do not reveal more information than the Cragg–Donald statistic τmin/(k2 − 1), unless δ = 0 and hence π1 is local to zero. The derivations for the two-variable model easily extend to the general case of several endogenous variables. The computation of the individual conditional F-statistics could then reveal further interesting patterns that the Cragg–Donald statistic will not be able to. For example, consider a three-variable model, which has a local rank reduction of one, of the form

$$\pi_1 = \delta_2\pi_2 + \delta_3\pi_3 + c/\sqrt{n}$$

but with δ3 = 0. The conditional F-statistics are in this case computed from

$$x_j - X_j\hat{\beta} = Zj + \xi,$$

where Xj is the matrix of endogenous variables with xj excluded and

$$\hat{\beta} = (X'\hat{\beta}X)^{-1}X'\hat{\beta}X.$$

The conditional F-statistics are then

$$F_{j|x,j} = \frac{\hat{k}'Z\hat{k}}{(k2 - 2)(\hat{\xi}'\hat{\xi}/n)},$$

(15)

see the Appendix for simple Stata code to calculate F_{j|x,j}.

Table 6 presents some simulation results for this particular case for the following design

$$\begin{pmatrix}
 u_1 \\
 v_1 \\
 v_2 \\
 v_3
\end{pmatrix} \sim N
\begin{pmatrix}
 0 & 1 & 0.8 & 0.3 & 0.6 \\
 0 & 0.8 & 1 & 0.3 & 0.5 \\
 0 & 0.3 & 0.1 & 0.4 \\
 0 & 0.6 & 0.5 & 0.4 & 1
\end{pmatrix}.$$
that of the OLS estimator; SY rejt frequ se est the 5% Stock–Yogo critical values for a total relative bias and Wald tests sized is distortions for individual parameters. Our weak instrument asymptotics is defined as local to a rank reduction of one of this matrix. We find that the conditional $F$-statistics in a two-endogenous variables model provide the same information as the Cragg–Donald test statistic for testing a rank reduction of one, unless the rank reduction is due to the fact that the instruments are uninformative for one of the endogenous variables. The conditional $F$-statistics are informative for total relative bias and Wald test size distortions for individual structural parameters. With more than two endogenous variables, the conditional $F$-statistics can provide additional information regarding the strength of the instruments for the different reduced forms. We therefore recommend in applied work that researchers report standard first-stage $F$-statistics, the Cragg–Donald statistic and the conditional $F$-statistics in order to gauge the nature of the weak instrument problem, if any. The Stock and Yogo (2005) weak instrument critical values can be used for the Cragg–Donald and conditional $F$-statistics. When reduced form errors are conditionally heteroskedastic and/or serially correlated, robust conditional $F$-statistics can be computed and used as tests for underidentification. However, the exact link of the Stock and Yogo (2005) critical values with the magnitude of the relative bias and Wald-test size distortions no longer holds for the robust statistics and is therefore an important avenue for future research.

### Appendix

#### A.1. Cragg–Donald statistic

The Cragg–Donald statistic in the two-variable model is obtained as

$$
\tau_{\min} = \min H (\delta, \pi_2) = \left( \frac{(\pi_1 - \pi_2)}{\delta^2} \right)^{\frac{1}{2}} \left( \frac{\pi_1}{\delta^2} \right) - \left( \frac{\pi_2}{\delta^2} \right)
$$

The first-order condition is given by, writing $\hat{\Sigma}^{-1} = \left( \hat{\sigma}^{11}_{12} \hat{\sigma}^{12}_{22} \right)$.

$$
-1 \frac{\partial H (\delta, \pi_2)}{\partial \pi_2} = \left( \frac{\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12}}{\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12} + \hat{\sigma}^{22}} \right) \frac{Z'Z}{(\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12} + \hat{\sigma}^{22})} \left( \frac{\pi_1}{\delta^2} - \frac{\pi_2}{\delta^2} \right) = 0,
$$

resulting in

$$
(\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12}) Z'Z \hat{\pi}_1 + (\delta \hat{\sigma}^{12} + \hat{\sigma}^{22}) Z'Z \hat{\pi}_2
$$

$$
= \delta (\delta \hat{\sigma}^{11} + \hat{\sigma}^{12}) Z'Z \pi_2 + (\delta \hat{\sigma}^{12} + \hat{\sigma}^{22}) Z'Z \pi_2.
$$

Hence,

$$
\pi_2 = \frac{(\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12}) \hat{\pi}_1 + (\delta \hat{\sigma}^{12} + \hat{\sigma}^{22}) \hat{\pi}_2}{\delta (\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12} + \hat{\sigma}^{22})},
$$

$$
\hat{\pi}_1 = \frac{(\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12}) \hat{\pi}_1 + (\delta \hat{\sigma}^{12} + \hat{\sigma}^{22}) \hat{\pi}_2}{\delta (\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12} + \hat{\sigma}^{22})},
$$

and

$$
\hat{\pi}_2 - \hat{\pi}_1 = -\frac{(\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12})}{\delta (\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12} + \hat{\sigma}^{22})} \left( \frac{1}{\delta} \right) \hat{\pi}_1 - \frac{(\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12})}{\delta (\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12} + \hat{\sigma}^{22})} \left( \frac{1}{\delta} \right) \hat{\pi}_2.
$$

As

$$
\left( \frac{(\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12})}{(\delta \hat{\sigma}^{12} + \delta \hat{\sigma}^{12})} \right)^{\frac{1}{2}} \hat{\pi}_1 - \frac{(\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12})}{(\delta \hat{\sigma}^{12} + \delta \hat{\sigma}^{12})} \hat{\pi}_2 = \frac{1}{\delta} \hat{\pi}_1 - \frac{1}{\delta} \hat{\pi}_2,
$$

it follows that

$$
H (\delta, \pi_2) = \frac{(\hat{\pi}_1 - \hat{\pi}_2)^{\prime} Z'Z (\hat{\pi}_1 - \hat{\pi}_2)}{\delta^2 (\delta \hat{\sigma}^{11} + \delta \hat{\sigma}^{12} + \hat{\sigma}^{22})}.
$$

#### A.2. Total relative bias

Equivalently to (11) we can write

$$
x_2 = x_1 \delta^* + Z_2 (\pi_{22} - \pi_{12} \delta^*) + v_2 - \delta^* v_1
$$

where $\delta^* = \pi_{21}/\pi_{11} = \delta^{-1}$. Hence, under LRR1 weak instrument asymptotics, we have

$$
x_2 = x_1 \delta^* - Z_2 c \delta^*/\sqrt{n} + v_2 - \delta^* v_1.
$$

### Table 6

|        | Mean | St dev | Rel bias | SY rej freq |
|--------|------|--------|----------|-------------|
| $\hat{\beta}_{1,015}$ | 1.1337 | 0.0068 |          |             |
| $\hat{\beta}_{1,025}$ | -0.4581 | 0.0050 |          |             |
| $\hat{\beta}_{1,035}$ | 0.9526 | 0.0055 |          |             |
| $\hat{\beta}_{1,045}$ | 0.5709 | 0.3086 | 0.1120   |             |
| $\hat{\beta}_{1,055}$ | -0.3361 | 0.1575 | 0.2285   |             |
| $\hat{\beta}_{1,065}$ | 0.6990 | 0.0161 | -0.0400  |             |
| $F_1$     | 650  | 26     |          |             |
| $F_2$     | 2504 | 67     |          |             |
| $F_3$     | 902  | 32     |          |             |
| $F_{1,2}$ | 4.82 | 2.38   | 0.0514   |             |
| $F_{2,3}$ | 4.84 | 2.41   | 0.0531   |             |
| $F_{3,2}$ | 198.21 | 329.06 | 0.8779   |             |
| $\tau_{\min} (k_2 - 2)$ | 4.82 | 2.38 | 0.0513 |             |
| $\tau_{\min}/k_2$ | 2.89 | 1.43 | 0.0156 |             |

Notes: Sample size 10,000; 10,000 MC replications; $\delta_1 = 0.5; \delta_2 = -0.3; \beta_1 = 0.7$. The simulations are again drawn independently from the standard normal distribution, with $k_2 = 5$, and $c$ is again chosen such that the total relative bias is less than 10%.

5. Conclusions

We have shown that a conditional first-stage $F$-test statistic can be informative about the information that instruments provide for models with multiple endogenous variables. The conditional $F$-test is similar to the one proposed by Angrist and Pischke (2009), but takes the variance of the multiple equations into account for testing a rank reduction of one of the matrix of reduced parameters. Our weak instrument asymptotics is defined as local to a rank reduction of one of this matrix. We find that the conditional $F$-statistics in a two-endogenous variables model provide the same information as the Cragg–Donald test statistic for testing a rank reduction of one, unless the rank reduction is due to the fact that the instruments are uninformative for one of the endogenous variables. The conditional $F$-statistics are informative for total relative bias and Wald test size distortions for individual structural parameters. With more than two endogenous variables, the conditional $F$-statistics can provide additional information regarding the strength of the instruments for the different reduced forms. We therefore recommend in applied work that researchers report standard first-stage $F$-statistics, the Cragg–Donald statistic and the conditional $F$-statistics in order to gauge the nature of the weak instrument problem, if any. The Stock and Yogo (2005) weak instrument critical values can be used for the Cragg–Donald and conditional $F$-statistics. When reduced form errors are conditionally heteroskedastic and/or serially correlated, robust conditional $F$-statistics can be computed and used as tests for underidentification. However, the exact link of the Stock and Yogo (2005) critical values with the magnitude of the relative bias and Wald-test size distortions no longer holds for the robust statistics and is therefore an important avenue for future research.

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As
\[
\left( \frac{1}{\sqrt{n}} Z_{2}' M_{31} u \right) \xrightarrow{d} \left( \psi_{Z_{2}Z_{1}} \right);
\]
\[\Omega^* = \left( \sigma_{u2} - \delta \sigma_{u1} \quad \sigma_{u2} - \delta \sigma_{u1} \right) ; \quad Z_{2}^{**} = M_{31} Z_{2}; \]
\[Q^* = \text{plim} \left( n^{-1} Z_{2}^{**} Z_{2}^{**} \right) \]
\[Q^* = \text{plim} \left( \frac{1}{n} (Z_{2}' Z_{2} - Z_{2}' \delta x_{2} (Z'Z) Z' \delta x_{2}) \right) = Q. \]
It follows that \( \psi_{Z_{2}Z_{1}} = \psi_{Z_{2}u} \) and \( \psi_{Z_{2}(v_{2}-\delta v_{1})} = \psi_{Z_{2}(v_{2}-\delta v_{1})} = -\frac{1}{n} \psi_{Z_{2}(v_{2}-\delta v_{1})} \), as e.g.
\[Z_{2}' M_{31} u = Z_{2}' u - Z_{2}' x_{1} (X_{P} x_{2})^{-1} X_{P} P_{u} \]
\[= Z_{2}' u - Z_{2}' x_{1} (X_{P} (Z'Z)^{-1} Z_{1}) X_{P} (Z'Z)^{-1} Z_{1} u. \]
Further,
\[\text{plim} \left( n^{-1} Z_{1}' x_{1} \right) = \delta \text{plim} \left( Z_{1}' x_{2} \right), \]
\[\tilde{\lambda}^{*} = \sigma_{v_{2} - \delta v_{1}} Q \delta^{*} = (\delta^{*} \sigma_{v_{1} - \delta v_{1}}) Q \delta^{*} = \tilde{\lambda}, \]
so we get that
\[\tilde{\beta}_{LS,LS} - \beta_{2} \xrightarrow{d} \delta \sigma_{u} \quad \sigma_{u} \]
and hence, asymptotically,
\[E \left( \tilde{\beta}_{LS,LS} - \beta_{2} \right) = -\delta E \left( \tilde{\beta}_{LS,LS} \right) - \beta_{1}. \]
Using this, we can express the total relative bias \( b^{2} \) as
\[b^{2} = b^{2} \left( \Sigma_{X} \frac{\Sigma_{X}^{-1}}{\Sigma_{X}} D \Sigma_{D} \Sigma_{X}^{-1} \Sigma_{X} \right) \frac{\Sigma_{X}^{-1}}{\Sigma_{X}} \Sigma_{X} \]
where
\[b = E \left[ \left( \frac{\tilde{\lambda} + \tilde{\lambda}_{x}}{\tilde{\lambda} + \tilde{\lambda}_{x}} \right) \right] ; \quad \Sigma_{X} = \left( \sigma_{u1} / \sigma_{u2} \right) ; \quad D = \frac{1}{\sigma_{u1}^{2} + \delta^{2} \sigma_{u2}^{2}} - 2 \delta \sigma_{u1} \left( \begin{array}{c}
1 \\
-\delta \\
\delta^{2}
\end{array} \right) \]
Hence
\[b^{2} = b^{2} \text{ max} \left( \Sigma_{X} \frac{\Sigma_{X}^{-1}}{\Sigma_{X}} D \Sigma_{X} \right) \frac{\Sigma_{X}^{-1}}{\Sigma_{X}}. \]
As \( \Sigma_{X}^{1/2} D \Sigma_{D} \Sigma_{X}^{1/2} \) is a symmetric idempotent matrix, we get that \( B^{2} \leq b^{2} \). To show this, note that
\[\Sigma_{X} = \text{plim} \frac{1}{n} \left( \begin{array}{ccc}
X' x_{1} & X' x_{2} \\
X' x_{2} & X' x_{2}
\end{array} \right) = \pi_{x} P \pi_{x} \left( \begin{array}{ccc}
\delta^{2} & \delta & 1 \\
\delta & 1 & 1 \\
1 & 1 & 1
\end{array} \right) + \Sigma \]
and hence \( \Sigma_{X} D = \Sigma_{D} D. \) Let
\[d = \frac{1}{\sqrt{\sigma_{u1}^{2} + \delta^{2} \sigma_{u2}^{2}} - 2 \delta \sigma_{u1}} \left( \begin{array}{c}
1 \\
-\delta \\
\delta^{2}
\end{array} \right) \]
so that \( D = dd' \), then \( d \Sigma_{D} d = 1 \) and hence
\[\Sigma_{X} D \Sigma_{X} D = \Sigma_{X} D \Sigma_{X} D = \Sigma_{V} d d' \Sigma_{X} d d' = \Sigma_{X} d d' = \Sigma_{X} D = \Sigma_{X} D \]
and therefore
\[\Sigma_{X}^{1/2} D \Sigma_{X}^{1/2} D \Sigma_{X} D \Sigma_{X}^{1/2} = \Sigma_{X}^{1/2} D \Sigma_{X} D \Sigma_{X} D \Sigma_{X}^{1/2} = \Sigma_{X}^{1/2} D \Sigma_{X} D \Sigma_{X}^{1/2}. \]

A.3. Stata code

Simple Stata code to calculate the conditional F-statistic \( F_{1,2,3} \) for the case of 3 endogenous variables, \( x_{1}, x_{2} \) and \( x_{3} \), with 5 instruments \( z_{1}, \ldots, z_{5} \) and 2 other exogenous variables \( w_{1} \) and \( w_{2} \) is as follows:
\[\text{ivregress 2sls x1 (x2 x3 = z1 z2 z3 z4 z5) w1 w2}
\text{predict res123, r}
\text{reg res123 z1 z2 z3 z4 z5 w1 w2}
\text{test z1 z2 z3 z4 z5}
\text{scalar Fsw = r(F)^r(df)/(r(df)-2)}
\text{di Fsw}
\text{References}

Angrist, J., Pischke, J.-S., 2009. Mostly Harmless Econometrics: An Empiricist’s Companion. Princeton University Press, Princeton.
Bun, M., de Haan, M., 2010. Weak instruments and the First Stage F-Statistic in IV Models with a Nonscalar Error Covariance Structure. Discussion Paper 2010/02, UVa Econometrics, University of Amsterdam.
Cragg, J.G., Donald, S.G., 1993. Testing identifiability and specification in instrumental variables models. Econometric Theory 9, 222–240.
Kleibergen, F., Paap, R., 2006. Generalized reduced rank tests using the singular value decomposition. J. Econometrics 133, 97–126.
Olfa, J.L.M., Pflueger, C., 2013. A robust test for weak instruments. J. Bus. Econom. Statist. 31, 358–368.
Rothenberg, T.J., 1984. Approximating the distributions of econometric estimators and test statistics. In: Griliches, Z., Intriligator, M.D. (Eds.), Handbook of Econometrics, Vol. 2. North Holland, Amsterdam, pp. 881–935.
Staiger, D., Stock, J.H., 1997. Instrumental variables regression with weak instruments. Econometrica 65, 557–586.
Stock, J.H., Yogo, M., 2005. Testing for weak instruments in linear IV regression.
In: Andrews, D.W.K., Stock, J.H. (Eds.), Identification and Inference for Econometric Models, Essays in Honor of Thomas Rothenberg. Cambridge University Press, New York, pp. 80–108.