A LORENTZ MODEL WITH VARIABLE DENSITY IN A GRAVITATIONAL FIELD: LIMIT THEOREMS AND RECURRENCE

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ABSTRACT. We show that with appropriate scaling of space, time and the density of obstacles, the trajectory of a tracer particle in a Markov approximation to a Lorentz gas with variable density in gravitational field converges to a diffusion with explicitly given parameters. We show how the parameters of the model determine recurrence or transience of the trajectory. Particular attention is paid to power function densities where we find two natural scalings that lead to different limiting diffusions.

1. Introduction

We consider a Lorentz gas-type model of a particle moving in a constant gravitational field with a large number of (infinitely-)small scatterers placed randomly in the space. The scatterers are assumed to be distributed according to a Poisson point process with variable intensity along the particle’s trajectory. We investigate invariance principles as well as questions of transience and recurrence. Our model is closely related, at least in the heuristic sense, to the Galton board dynamics considered in [2]. In [2] it is shown that the trajectory of a ball in a Galton board-type billiards with gravitation is recurrent and a diffusive limit for the particle trajectory is determined. One of the motivations of the present work is to investigate whether these results are robust under perturbations of the model. We establish invariance principles in several scaling regimes and determine the influence of the density of scatterers on the limiting diffusion. We also determine criteria for the recurrence or transience of the particle trajectory for particular forms of the density of scatterers. A similar model with constant scatterer density was previously considered in [12].

We consider a particle in $\mathbb{R}^d$ with $d \geq 1$ and the gravitational force acting in the $-x_d$ direction. Let $S^{d-1}$ be the unit sphere in $\mathbb{R}^d$ and let $e_1, \ldots, e_d$ be the standard basis for $\mathbb{R}^d$. Denote the path of the particle by $(X(t), t \geq 0)$. We suppose that $X(0) = (x_0^1, \ldots, x_0^d)$ with $x_0^d \leq 0$ and that the particle starts with just enough energy to hit the plane $x_d = 0$ if it travels straight up. Since we are assuming the particle is light, there is no energy transfer from reflections so this assumption on the initial energy is just a normalization assumption. We suppose that the heavy particles have a density $h = h(x_d)$ (it depends only on distance from the $x_d = 0$ plane) along the light particle’s trajectory. In our model, upon reflection at $x = (x_1, x_2, \ldots, x_d)$, the particle starts its path in a uniform direction $u = (u_1, u_2, \ldots, u_d) \in S^{d-1}$ and then travels along the gravitational parabola

$$\Lambda(x, u, t) := \sum_{i=1}^{d-1} (x_i + u_i \sqrt{2g|x_d|t}) e_i + (x_d + u_d \sqrt{2g|x_d|t} - \frac{g}{2}t^2) e_d, t \geq 0.$$

The light particle travels along this parabola until it hits a heavy particle and reflects again. The choice of a uniform scattering direction is both for mathematical convenience and physical relevance as discussed in Appendix A. We are particularly interested in the process $(Y(t), t \geq 0) := (X_d(t), t \geq 0)$ because this process captures the influence of gravitation on the path of the particle.

On the technical side, our main effort is to derive an invariance principle for the tracer particle in a Poisson cloud of obstacles with variable density. More precisely, we prove different invariance
principles for the cases when (i) the density has an arbitrary shape (but is sufficiently smooth and bounded below in an appropriate sense), and (ii) the density is a power function. It is clear that the first case requires extra care because one cannot expect the rescaled trajectory of the tracer particle to converge to any limit if the rescaled density does not have any structure. In case (ii) we find two different rescalings, which lead to two different limiting diffusions, both of which can be expressed in terms of Bessel processes. Consequently, the recurrence of the limiting process can be easily determined using classical results. In case (ii), we also consider recurrence of the trajectory without rescaling and passing to the limit.

The following theorem is our main result in case (i).

**Theorem 1.1.** Let \( h : (-\infty, 0] \to \mathbb{R}_+ \) be \( C^2 \) on \((-\infty, 0)\) and bounded away from zero on \((-\infty, a]\) for every \( a < 0 \). Fix \( g > 0 \). Let \( \{(X^n(t), \ldots, X^n_d(t), Y^n(t))\}_{t \geq 0} \) be the path of the particle started from \((x^0_1, \ldots, x^0_d)\) with \( x^0_d < 0 \), moving in gravitation \( g_n = g/\sqrt{n} \), and suppose heavy particle density is \( h_n(y) = \sqrt{n}h(y) \). Fix \( x^0_d < v < 0 \) and define \( \tau^n_{v+} = \inf\{t \geq 0 : Y^n(t) \geq v\} \). Let \( \mathcal{Y} \) be a diffusion on \((-\infty, 0)\) started at \( y_0 \), whose generator extends the operator \( A_h \) defined below, which acts on \( f \in C^2 \) with compact support in \((-\infty, 0)\) by

\[
A_h f(y) = \frac{\sqrt{2g|y|}}{dh(y)} f''(y) - \frac{\sqrt{2g|y|}}{dh(y)} \left( \frac{d-1}{2|y|} + \frac{h'(y)}{h(y)} \right) f'(y).
\]

Define \( \tau_{v+} = \inf\{t \geq 0 : \mathcal{Y}_t \geq v\} \).

As \( n \to \infty \), we have

\[
(Y^n((n^{3/4}) \wedge \tau^n_{v+}), t \geq 0) \to (\mathcal{Y}(t \wedge \tau_{v+}), t \geq 0).
\]

in distribution in the Skorokhod space \( D(\mathbb{R}_+, \mathbb{R}) \).

See Theorem 7.6 for the proof of this result. The cutoff at \( v \) is necessary because both the time between reflections and the distance between reflections may scale differently when the particle is near the \( x \)-axis. See, for example, Proposition 6.2 which shows how the time before the first reflection and the distance traveled before the first reflection for the particle started at \((0, 0)\) scale when \( h(y) = |y|^\lambda \) for some \( \lambda > 0 \). The constant density of heavy particles is a particular case of the above model and our results agree in this special case with those in [12]. Although the model considered in [2], with large periodic obstacles, is considerably different from ours (and that in [12]), our results in the case of constant obstacle density agree at the heuristic level with the results in [2].

We remark that our scaling in this invariance principle is anomalous in the sense that we have rescaled the spatial dynamics by a factor \( \sqrt{n} \), but time must be scaled by a factor of \( n^{3/4} \). This stands in contrast to typical diffusive scaling where the spatial dynamics are rescaled by a factor of \( \sqrt{n} \) and time by a factor of \( n \).

The generator \( A_h \) has a particularly nice form when \( h(y) = |y|^\lambda \).

**Proposition 1.2.** If \( h(y) = |y|^\lambda \) then \( A_h \) extends to the generator of the negative of a time-changed Bessel process of dimension \( d' = (d + 1 - 2\lambda)/2 \). More precisely, \( A_h \) acts on \( f \in C^2 \) with compact support in \((-\infty, 0)\) by

\[
A_h f(y) = \frac{2\sqrt{2g}}{d} |y|^{1/2-\lambda} \left[ \frac{1}{2} f''(y) - \left( \frac{d-1-2\lambda}{4|y|} \right) f'(y) \right].
\]

Consequently, in this case, recurrence and transience behavior of the limiting process can be easily analyzed in terms of the classical theory of Bessel processes (with possibly negative dimension).
In case (ii), when \( h(y) = c|y|^\lambda \), we can also establish recurrence and transience results for the process \((Y(t), t \geq 0)\) in addition to an invariance principle for rescaled sample paths. Our main result in this case is the following theorem. Recall that \( Y(t) := X_d(t) \).

**Theorem 1.3.** Let \( \{(X_1(t), \ldots, X_{d-1}(t), Y(t))\}_{t \geq 0} \) be the path of the particle started from \((0, \ldots, 0)\) moving in gravitation \( g \) and heavy particle density \( h(y) = c|y|^\lambda \), with \( c > 0 \) and \( \lambda \geq 0 \).

1. If \( d = 1 \) then \( (Y(t), t \geq 0) \) is recurrent, if \( d = 2 \) then \( (Y(t), t \geq 0) \) is neighborhood recurrent, and if \( d \geq 4 \) then \( (Y(t), t \geq 0) \) is transient.

2. Fix \( z < v < 0 \). Let \( T^m_z \) be the time of the first reflection at which \( Y < n^{1/(2+2\lambda)}z \) and let \( T^m_v \) be the time of the first reflection after \( T^m_z \) such that \( Y > n^{1/(2+2\lambda)}v \). Moreover, let \( \mathcal{Z} \) be a diffusion on \((-\infty, 0)\) started from \( z \) whose generator acts on \( f \in C^2 \) with compact support in \((-\infty, 0)\) by

\[
\mathcal{G} f(y) = \frac{2\sqrt{2\pi}g}{dc} |y|^{1/2-\lambda} \left[ \frac{1}{2} f''(y) - \left( \frac{d-1}{4|y|} \right) f'(y) \right].
\]

As \( n \to \infty \) we have

\[
(n^{-\frac{1}{2+2\lambda}} Y \left( \left( n^{\frac{3+2\lambda}{1+4\lambda}} t + T^m_z \right) \land T^m_v \right), t \geq 0) \to (\mathcal{Z}(t \land \tau_{v+}), t \geq 0),
\]

in distribution in the Skorokhod space \( D(R_+, R) \), where \( \tau_{v+} = \inf \{ t : \mathcal{Z}(t) > v \} \).

The recurrence and transience claims are established in Theorem 8.7 while the invariance principle is proved in Theorem 8.11. We point out that the question of whether or not \((Y(t), t \geq 0)\) is recurrent in dimension 3 remains open. There are two particularly interesting things about this invariance principle. The first is that, in contrast to case (i), here the recurrence or transience of the limiting diffusion depends only on \( d \) and not on \( \lambda \). Secondly, the scaling is again anomalous, but this time it depends on \( \lambda \) whereas in case (i), the same scaling was used for every density.

We remark that in both cases (i) and (ii), our approach bears some similarities to other work on invariance principles related to anomalous diffusions, see e.g. 10, but our situation is fundamentally different. In the current setting the anomalous scaling arises because the particle’s speed is unbounded so that the waiting time between reflections can be very small. As a result, our limiting diffusion is not anomalous. This is in contrast to 10 and other work on anomalous diffusion where the anomalous scaling arises because waiting times can be heavy tailed.

To simplify the exposition, we work primarily in dimension \( d = 2 \). In Section 7 we explain how our results extend to other dimensions and what modifications are necessary. In dimension \( d = 2 \), the primary model we consider is a particle \( \{(X(t), Y(t))\}_{t \geq 0} \) moving in \( R^2 \) with gravitation \( g \) in the negative \( y \) direction.

This article is organized as follows. Sections 2-6 focus on the case \( d = 2 \). In Section 2 we consider a simplified model where the particle travels distance exactly one between reflections. The computations in this case are simpler and the model illustrates the approach we take in the general case. In Section 3 we formally define the model in dimension \( d = 2 \) and establish the estimates on the times between reflections needed to prove the invariance principle. We also establish generator convergence for the skeleton process \((Y^m_m, m \geq 0)\) where \( Y^m_m \) is the \( y \)-coordinate of the particle at the time of the \( m \)th reflection. While \((Y^n(t), t \geq 0)\) is not a Markov process, \((Y^m_m, m \geq 0)\) is, and thus the skeleton process is easier to analyze than the full path of the particle. Section 4 contains an invariance principle for \((Y^m_m, m \geq 0)\) and Section 5 contains the invariance principle for \((Y^n(t), t \geq 0)\). In Section 6 we give a more detailed analysis of the case when \( h(y) = |y|^\lambda \), in which case the limiting diffusion is a constant multiple of a power of a Bessel process whose dimension depends on \( \lambda \). We also show that when \( \lambda = 0 \), i.e., \( h \equiv 1 \), \((Y^n(t), t \geq 0)\) is recurrent and prove a stronger invariance principle for \((Y^m_m, m \geq 0)\). Section 7 contains generalizations of the results from the first part of the paper, devoted to the 2-dimensional model, to higher dimensions. Finally,
Section 8 gives a careful analysis of the path of the particle in the special case of $h(y) = |y|^\lambda$ and, in particular, is aimed at proving Theorem 1.3.

The general literature on billiards, billiards with potential, and on Lorentz gas models is huge and we do not feel that we can do justice to this body of research. The articles [2, 12] and references therein are a good point of entry to this field.

2. Motion with deterministic distance between reflections

This section is a warm up, in the sense that we analyze a simplified model, to develop a sense for results that we can expect in a more realistic and hence more complicated situation. Specifically, we work in two dimensions and we assume that the distance between any two consecutive reflections measured along the trajectory of the particle is exactly one.

The 2-dimensional trajectory of the particle is denoted $\{(X(t), Y(t)), t \geq 0\}$ or $\{(X_t, Y_t), t \geq 0\}$. We assume that our particle starts at the origin $(0, 0)$ and its kinetic energy is always equal to the gravitational potential energy lost because of the decrease of altitude. In other words, $m|v(t)|^2/2 = mg|Y(t)|$, where $m$ is the mass of the particle, $v$ is its velocity vector, and $g > 0$ is the gravitational constant (acceleration). Hence, $|v(t)| = \sqrt{2g|Y(t)|}$ and necessarily $Y(t) \leq 0$ for all $t \geq 0$. If the particle is at $(x, y)$ at a time of reflection, we choose $\theta \in [-\pi, \pi]$, uniformly at random and independently of all previous choices, and the particle travels along the parabola

\begin{equation}
(2.1) \quad \{ (\Lambda_1(x, \theta, t), \Lambda_2(y, \theta, t)) := \left( x + \sqrt{2g|y|} \cos(\theta)t, y + \sqrt{2g|y|} \sin(\theta)t - \frac{g}{2}t^2 \right), t \geq 0 \}
\end{equation}

until it reflects again. Note that $t$ in the last equation represents time since the last reflection, not since the beginning of the motion. Let

$$
\ell(y, \theta, t) = \int_0^t \sqrt{2g|y| \cos^2(\theta) + \left( \sqrt{2g|y| \sin(\theta) - gs} \right)^2} \, ds
$$

be the length traveled along the parabola (2.1) in the first $t$ units of time after reflection; note that this does not depend on $x$. Further, let $t(y, \theta)$ be the time it takes to travel distance 1 along the parabola in (2.1). It is easy to check that we have the monotonicity relation $t(y, -\pi/2) \leq t(y, \theta) \leq t(y, \pi/2)$, which is intuitive because it takes the longest to travel straight up and the shortest to travel straight down. Moreover, assuming $y \leq -1$ as we will for the remainder, we can explicitly compute

$$
t(y, \pi/2) = \sqrt{\frac{2}{g}} \left( \sqrt{|y|} - \sqrt{|y| - 1} \right) \quad \text{and} \quad t(y, -\pi/2) = \sqrt{\frac{2}{g}} \left( \sqrt{|y| + 1} - \sqrt{|y|} \right).
$$

From this, one observes that $n^{1/4}t(\sqrt{n}y, \pm \pi/2) \to (2g|y|)^{-1/2}$ as $n \to \infty$ and, consequently,

\begin{equation}
(2.2) \quad \lim_{n \to \infty} n^{1/4}t(\sqrt{n}y, \theta) \to \frac{1}{\sqrt{2g|y|}},
\end{equation}

uniformly in $y$ and $\theta$. Let $\ell_t(y, \theta, t)$ and $\ell_{tt}(y, \theta, t)$ denote the first and second partial derivatives, resp., of $\ell(y, \theta, t)$ in the third variable. We have $\ell_t(y, \theta, 0) = \sqrt{2g|y|}$ and $\ell_{tt}(y, \theta, 0) = -g \sin(\theta)$. It follows from the definition of $t(y, \theta)$ that $\ell(\sqrt{n}y, \pm \theta, t(\sqrt{n}y, \pm \theta)) = 1$. Taylor expanding $\ell$ in the $t$ variable yields

$$
1 = \ell(\sqrt{n}y, \pm \theta, t(\sqrt{n}y, \pm \theta)) = n^{1/4} \sqrt{2g|y|} t(\sqrt{n}y, \pm \theta) + \frac{g \sin(\theta)}{2} t(\sqrt{n}y, \pm \theta)^2 + \frac{\ell_{tt}(\sqrt{n}y, \pm \theta, \alpha_n)}{6} t(\sqrt{n}y, \pm \theta)^3
$$
for some \( \alpha_n^+ \leq t(\sqrt{ny}, \pi/2) \). Rearranging, this yields the relation
\[
t(\sqrt{ny}, \pm \theta) = \frac{1}{n^{1/4} \sqrt{2g|y|}} \left( 1 \pm \frac{g \sin(\theta)}{2} t(\sqrt{ny}, \pm \theta)^2 - \frac{\ell_{\text{ut}}(\sqrt{ny}, \pm \theta, \alpha_n^+)}{6} t(\sqrt{ny}, \pm \theta)^3 \right).
\]
Using this, we find that
\[
(2.3) \quad n^{3/4}(t(\sqrt{ny}, \theta) - t(\sqrt{ny}, -\theta)) = n^{1/2} \frac{g \sin(\theta)}{2 \sqrt{2g|y|}} [t(\sqrt{ny}, \theta)^2 + t(\sqrt{ny}, -\theta)^2] \\
+ n^{1/2} \frac{\ell_{\text{ut}}(\sqrt{ny}, -\theta, \alpha_n^+)}{6 \sqrt{2g|y|}} t(\sqrt{ny}, -\theta)^3 - n^{1/2} \frac{\ell_{\text{ut}}(\sqrt{ny}, \theta, \alpha_n^+)}{6 \sqrt{2g|y|}} t(\sqrt{ny}, \theta)^3.
\]
Straightforward but tedious calculations show that \( \ell_{\text{ut}}(\sqrt{ny}, \theta, t) = O(n^{-1/4}) \) uniformly in \( y \leq -1, \theta, \) and \( 0 \leq t \leq t(\sqrt{ny}, \pi/2) \). This and (2.2) show that the last two terms on the right hand side of (2.3) vanish as \( n \to \infty \) and, computing the limit of the first term and using (2.2) once again, we find that
\[
(2.4) \quad \lim_{n \to \infty} n^{3/4}(t(\sqrt{ny}, \theta) - t(\sqrt{ny}, -\theta)) = \frac{\sin(\theta)}{\sqrt{8g|y|^3}}.
\]
Let discrete time process \( \{Y_k^*, k \in \mathbb{N}_0\} \) record the positions of \( Y \) at the reflection times. Note that this is not the same as sampling of \( Y \) at equal or identically distributed time intervals because the velocity of \( Y \) increases with \( |Y| \) and the times between scattering events become smaller on average. Recall notation from (2.1). Let \( U_n \) be the transition operator for the rescaled process \( \{n^{-1/2}Y_k^*, t \in \mathbb{N}_0/n\}, \) for \( n \in \mathbb{N} \). More formally, for a \( C^2 \) function \( f \) with compact support in \((-\infty, 0)\), we have
\[
(U_n f)(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(n^{-1/2}\Lambda_2(\sqrt{ny}, \theta, t(\sqrt{ny}, \theta))) \, d\theta.
\]
So
\[
n(U_n - I) f(y) = \frac{n}{2\pi} \int_{-\pi}^{\pi} \left( f(n^{-1/2}\Lambda_2(\sqrt{ny}, \theta, t(\sqrt{ny}, \theta))) - f(y) \right) \, d\theta \\
= \frac{n}{2\pi} \int_{0}^{\pi} \left( f(n^{-1/2}\Lambda_2(\sqrt{ny}, \theta, t(\sqrt{ny}, \theta))) + f(n^{-1/2}\Lambda_2(\sqrt{ny}, -\theta, t(\sqrt{ny}, -\theta))) - 2f(y) \right) \, d\theta.
\]
The Taylor expansion of the function \( t \to f(n^{-1/2}\Lambda_2(\sqrt{ny}, \theta, t)) \) at \( t = 0 \) combined with (2.1) yields for some \( t_1, t_2 \in [0, t(\sqrt{ny}, \pi/2)] \),
\[
n \left[ f(n^{-1/2}\Lambda_2(\sqrt{ny}, \theta, t(\sqrt{ny}, \theta))) + f(n^{-1/2}\Lambda_2(\sqrt{ny}, -\theta, t(\sqrt{ny}, -\theta))) - 2f(y) \right] \\
= f'(y) \left[ n^{3/4} \sqrt{2g|y|} \sin(\theta)(t(\sqrt{ny}, \theta) - t(\sqrt{ny}, -\theta)) \right] \\
- f'(n^{-1/2}\Lambda_2(\sqrt{ny}, \theta, t_1)) \frac{g\sqrt{n}}{2} t(\sqrt{ny}, \theta)^2 \\
- f'(n^{-1/2}\Lambda_2(\sqrt{ny}, \theta, t_2)) \frac{g\sqrt{n}}{2} t(\sqrt{ny}, \theta)^2 \\
+ f''(n^{-1/2}\Lambda_2(\sqrt{ny}, \theta, t_1)) \frac{n}{2} \left[ -n^{-1/4} \sqrt{2g|y|} \sin(\theta) - \frac{g}{\sqrt{n} t_1} \right]^2 t(\sqrt{ny}, \theta)^2 \\
+ f''(n^{-1/2}\Lambda_2(\sqrt{ny}, \theta, t_2)) \frac{n}{2} \left[ -n^{-1/4} \sqrt{2g|y|} \sin(-\theta) - \frac{g}{\sqrt{n} t_2} \right]^2 t(\sqrt{ny}, -\theta)^2.
It follows from (2.1) and (2.2) that
\[
\lim_{n \to \infty} \sup_{t \in [0, t(\sqrt{ny}, \pi/2)]} |n^{-1/2} \Lambda_2(\sqrt{ny}, \theta, t) - y| = 0,
\]
uniformly in \(\theta\) and uniformly in \(y\) on compact subsets of \((-\infty, 0)\). This, the previous calculation, (2.2) and (2.4) imply that
\[
\begin{align*}
n \left[ f(n^{-1/2} \Lambda_2(\sqrt{ny}, \theta, t(\sqrt{ny}, \theta))) + f(n^{-1/2} \Lambda_2(\sqrt{ny}, -\theta, t(\sqrt{ny}, -\theta))) - 2f(y) \right] \\
\to \left( \frac{\sin^2(\theta)}{2|y|} - \frac{1}{2|y|} \right) f'(y) + \sin^2(\theta) f''(y),
\end{align*}
\]
as \(n \to \infty\), uniformly in \(\theta\) and uniformly in \(y\) on compact subsets of \((-\infty, 0)\). Therefore
\[
\lim_{n \to \infty} n(U_n - I) f(y) = \frac{1}{2\pi} \int_0^\pi \left( \frac{\sin^2(\theta)}{2|y|} - \frac{1}{2|y|} \right) f'(y) + \sin^2(\theta) f''(y) \right] d\theta = \frac{1}{4} f''(y) - \frac{1}{8|y|} f'(y).
\]
We conclude that the rescaled processes \(\{n^{-1/2}|Y^*_t|, t \in \mathbb{N}_0/n\}\) converge, in the sense given above, to a time-change of \(3/2\)-dimensional Bessel process. We leave our claim about convergence in this weak form for the simplified model considered in this section. The point of the exercise was to obtain a good guess about a possible limit for our main model, to be discussed next.

3. Reflections from points in a Poisson cloud in dimension 2

We now formalize the two dimensional case of the model that is the subject of the rest of the paper. The particle is moving in \(\mathbb{R}^2\) in a gas with positive density \(h\), where \(h = h(y)\) depends only on the distance \(|y|\) from the \(x\)-axis. We assume that \(h\) is \(C^2\) on \((-\infty, 0)\) and that for every \(a < 0\) there is a \(\delta > 0\) such that \(\inf_{y \leq a} h(y) \geq \delta\). Given that the particle is at position \((x, y)\) with \(y \leq 0\) when it reflects in direction \(\Theta = \theta\), which is chosen uniformly on \([-\pi, \pi]\), it travels for time \(N(y, \theta)\) before reflecting again, where \(N(y, \theta)\) has distribution
\[
P(N(y, \theta) > t) = \exp \left( -\int_0^t h \left( y + \sin(\theta) \sqrt{2g|y|s - \frac{g}{2}s^2} \right) \sqrt{2g|y| \cos(\theta)^2 + (\sqrt{2g|y| \sin(\theta) - gs})^2} ds \right).
\]
The motivation for the distribution of \(N\) is as follows: Upon reflection at \(y\) in direction \(\theta\) the particle moves along the parabola \((\Lambda_1(t), \Lambda_2(t))\) defined in (2.1). On this parabola there is a Poisson point process with variable intensity \(h\) per unit length and these points represent other (heavy) particles in the gas. Upon hitting the first of these points the particle we are tracking reflects and the process begins again. We define \(N\) as the time it takes to hit the first point of this Poisson point process along the parabola \((\Lambda_1(t), \Lambda_2(t))\).

Observe that, if \(M_t(y, \theta)\) is the number of the Poisson process points on \(\{(\Lambda_1(s), \Lambda_2(s)), 0 \leq s \leq t\}\) and
\[
Z_t(y, \theta) := M_t(y, \theta) - \int_0^t h \left( y + \sin(\theta) \sqrt{2g|y|s - \frac{g}{2}s^2} \right) \sqrt{2g|y| \cos(\theta)^2 + (\sqrt{2g|y| \sin(\theta) - gs})^2} ds,
\]
for \(t \geq 0\) then \(Z_t(y, \theta)\) is a martingale. Since our assumptions on \(h\) imply that \(N(y, \theta)\) is almost surely finite, applying optional stopping with the stopping time \(N(y, \theta) \wedge r\) and using monotone convergence to let \(r \to \infty\) shows that
\[
(3.1) \quad 1 = E \left( \int_0^{N(y, \theta)} h \left( y + \sin(\theta) \sqrt{2g|y|s - \frac{g}{2}s^2} \right) \sqrt{2g|y| \cos(\theta)^2 + (\sqrt{2g|y| \sin(\theta) - gs})^2} ds \right).
\]
This equation is central to our analysis. A similar optional stopping argument using the martingale
\[
Z_t^2 - \int_0^t h \left( y + \sin(\theta) \sqrt{2g|y|s} - \frac{g}{2}s^2 \right) \sqrt{2g|y| \cos(\theta)^2 + (\sqrt{2g|y|} \sin(\theta) - gs)^2} ds, \quad t \geq 0,
\]
shows that
\[
(3.2)
2 = \mathbb{E} \left[ \left( \int_0^{N(y,\theta)} h \left( y + \sin(\theta) \sqrt{2g|y|s} - \frac{g}{2}s^2 \right) \sqrt{2g|y| \cos(\theta)^2 + (\sqrt{2g|y|} \sin(\theta) - gs)^2} ds \right)^2 \right].
\]

To obtain a diffusive limit, we scale both space and particle density. We fix gravitational acceleration \( g > 0 \) and a \( C^2 \) function \( h : (-\infty, 0) \to (0, \infty) \), such that for every \( a < 0 \) there is a \( \delta > 0 \) such that \( \inf_{y \leq a} h(y) \geq \delta \). We scale space by \( \sqrt{n} \) so that gravitation scales like \( g/\sqrt{n} \) and we scale density like \( h_n(y) = \sqrt{n}h(y) \). Let \( (Y^n_k)_{k \geq 0} \) be the Markov chain of \( y \) coordinates of jump locations for the \( n \)’th scaled process. Formally, \( Y^n \) has transition operator
\[
(3.3) \quad U^n f(y) = \mathbb{E} \left[ f \left( y + \sin(\Theta) \sqrt{2g|y|n^{-1/4}}N^n(y, \Theta) - \frac{g}{2\sqrt{n}}N^n(y, \Theta)^2 \right) \right],
\]
where \( \Theta \) is uniform on \([-\pi, \pi]\) and conditionally given \( \Theta = \theta \), \( N^n(y, \theta) \) has distribution
\[
(3.4) \quad \mathbb{P}(N^n(y, \theta) > t) = \exp \left[ -\int_0^t \sqrt{n} h \left( y + \sin(\theta) \sqrt{2g|y|n^{-1/4}}s - \frac{g}{2\sqrt{n}}s^2 \right) \right.
\]
\[
\left. \times \sqrt{2g|y|n^{-1/2}} \cos(\theta)^2 + \left( \sqrt{2g|y|} \sin(\theta)n^{-1/4} - \frac{g}{\sqrt{n}}s \right)^2 ds \right].
\]

An easy computation shows that for every \( a < 0 \),
\[
\lim_{n \to \infty} \mathbb{P}(n^{1/4}N^n(y, \theta) > t) = \exp \left( -h(y)\sqrt{2g|y|t} \right),
\]
uniformly for \( (y, \theta) \in (-\infty, a) \times [-\pi, \pi] \). Indeed, this convergence can be extended to convergence of moments as well.

**Lemma 3.1.** If \( a < 0 \) and \( 1 \leq p < \infty \) then family \( \{n^{1/4}N^n(y, \theta) : (y, \theta, n) \in (-\infty, a) \times [-\pi, \pi] \times \mathbb{N}\} \)
is bounded in \( L^p \).

**Proof.** Since \( N^n(y, \theta) \geq 0 \), an application of Fubini’s theorem shows that
\[
\mathbb{E}(N^n(y, \theta)^p) = p \int_0^\infty t^{p-1} \mathbb{P}(N^n(y, \theta) > t) dt.
\]

Hence
\[
n^{p/4} \mathbb{E}(N^n(y, \theta)^p) = p \int_0^\infty t^{p-1} \mathbb{P}(N^n(y, \theta) > n^{-1/4}t) dt.
\]

Our proof is based on estimating \( \mathbb{P}(N^n(y, \theta) > t) \).

Given \( \varepsilon > 0 \), let \( \delta > 0 \) be such that \( \inf_{y \leq \varepsilon} h(y) \geq \delta \). We assume \( a \leq -4\varepsilon \). Observe that for all \( s \geq 0 \), we have
\[
\sqrt{2g|y|n^{-1/2}} \cos(\theta)^2 + \left( \sqrt{2g|y|} \sin(\theta)n^{-1/4} - \frac{g}{\sqrt{n}}s \right)^2 \geq \sqrt{2g|y|n^{-1/4} - \frac{g}{\sqrt{n}}s}.
\]
This implies that
\begin{equation}
\mathbb{P}(N^n(y, \theta) > t) \leq \exp \left[ - \int_0^t \sqrt{n} h \left( y + \sin(\theta) \sqrt{2g|y|n^{-1/4}s} - \frac{g}{2\sqrt{n}} s^2 \right) \left| \sqrt{2g|y|n^{-1/4}} - \frac{g}{\sqrt{n}} s \right| ds \right].
\end{equation}

Next we need to estimate when
\[
\left| y + \sin(\theta) \sqrt{2g|y|n^{-1/4}s} - \frac{g}{2\sqrt{n}} s^2 \right| \leq \varepsilon.
\]
Define
\begin{equation}
s_-(\theta) = \inf \left\{ s \geq 0 : y + \sin(\theta) \sqrt{2g|y|n^{-1/4}s} - \frac{g}{2\sqrt{n}} s^2 = -\varepsilon \right\}
\end{equation}
and
\[
s_+(\theta) = \sup \left\{ s \geq 0 : y + \sin(\theta) \sqrt{2g|y|n^{-1/4}s} - \frac{g}{2\sqrt{n}} s^2 = -\varepsilon \right\}.
\]
Monotonicity arguments show that
\begin{equation}
s_-(\theta) \geq s_-(\pi/2) = \sqrt{\frac{2}{g}} n^{1/4} \left( |\sqrt{y}| - \sqrt{\varepsilon} \right)
\end{equation}
and
\[
s_+(\theta) \leq s_+(\pi/2) = \sqrt{\frac{2}{g}} n^{1/4} \left( |\sqrt{y}| + \sqrt{\varepsilon} \right).
\]
To simplify notation, let us use \( s_\pm := s_\pm(\pi/2) \). We then have
\[
\mathbb{P}(N^n(y, \theta) > t) \leq \exp \left[ - \int_0^t \sqrt{n} \delta \left| \sqrt{2g|y|n^{-1/4}} - \frac{g}{\sqrt{n}} s \right| [1(s \leq s_-) + 1(s \geq s_+)] ds \right].
\]

This implies that
\begin{equation}
\mathbb{P}(N^n(y, \theta) > t) \leq \begin{cases}
\exp \left[ -\delta t \left( \sqrt{2g|y|n^{1/4}} - \frac{g}{2} \right) \right], & t \leq s_-,
\exp \left[ -\delta s_- \left( \sqrt{2g|y|n^{1/4}} - \frac{g}{2} s_- \right) \right], & s_- \leq t \leq s_+,
\exp \left[ -\delta \left( \sqrt{2g|y|n^{1/4}}[s_- + s_+ - t] + \frac{g}{2} [t^2 - s_-^2 - s_+^2] \right) \right], & t \geq s_+.
\end{cases}
\end{equation}

Using this, we obtain the estimate
\[
n^{p/4} \mathbb{E}(N^n(y, \theta)^p) \leq p \int_0^{n^{1/4}s_-} t^{p-1} \exp \left[ -\delta t \left( \sqrt{2g|y|} - \frac{g}{2\sqrt{n}} t \right) \right] dt
\]
\begin{equation}
+ p \int_{n^{1/4}s_-}^{4n^{1/4}s_+} t^{p-1} \exp \left[ -\delta s_- \left( \sqrt{2g|y|n^{1/4}} - \frac{g}{2} s_- \right) \right] dt
+ p \int_{4n^{1/4}s_+}^{\infty} t^{p-1} \exp \left[ -\delta \left( \sqrt{2g|y|n^{1/4}}[s_- + s_+ - t] + \frac{g}{2} [t^2 - s_-^2 - s_+^2] \right) \right] dt.
\end{equation}

Let us consider the integrals on the right hand side separately. For the first integral note that
\[
\sqrt{2g|y|} - \frac{g}{2\sqrt{n}} t \geq \sqrt{2g|y|}/2 \quad \text{when} \quad 0 \leq t \leq n^{1/4}s_- \leq \sqrt{\frac{2|y|n}{g}}.
\]
It follows that
\[ \int_0^{n^{1/4}s_+} t^{p-1} \exp \left( -\delta t \left[ \sqrt{2g|y|} - \frac{g}{2\sqrt{n}}t \right] \right) dt \leq \int_0^{\frac{(2|g|/n)^{1/2}}{2}} t^{p-1} \exp \left( -\frac{\delta t \sqrt{2g|y|}}{2} \right) dt \]
(3.10)
\[ \leq \int_0^\infty t^{p-1} \exp \left( -\frac{\delta t \sqrt{2g|a|}}{2} \right) dt < \infty. \]

The second integral on the right hand side of (3.9) can be evaluated explicitly,
\[ p \int_{n^{1/4}s_-}^{4n^{1/4}s_+} t^{p-1} \exp \left[ -\delta s_- \left( \sqrt{2g|y|n^{1/4}} - \frac{g}{2}s_- \right) \right] dt \]
(3.11)
\[ = \exp \left[ -\delta s_- \left( \sqrt{2g|y|n^{1/4}} - \frac{g}{2}s_- \right) \right] n^{p/4}(4^{p}s_+^p - s_-^p), \]
which goes to 0 uniformly in \((y, \theta) \in (-\infty, a] \times [-\pi, \pi]\) as \(n \to \infty\) since \(\sqrt{2g|y|n^{1/4}} - \frac{g}{2}s_- \geq n^{1/4}\sqrt{2g|y|}/2\) and \(s_{\pm}\) grow like \(n^{1/4}\) in \(n\) and \(|y|^{1/2}\) in \(|y|\).

For the third integral on the right hand side of (3.9), note that \(\sqrt{2g|y|n^{1/4}} - \frac{g}{2}s_+ \geq n^{1/4}\sqrt{2g|y|}/4\) since \(y \leq a \leq -4\varepsilon\) and that for \(t \geq 4n^{1/4}s_+\) we have \(\frac{g}{\sqrt{n}}t - \sqrt{2g|y|} \geq \sqrt{2g|y|}/4\). Thus we have
\[ \int_{4n^{1/4}s_+}^{\infty} t^{p-1} \exp \left[ -\delta \left( \sqrt{2g|y|n^{1/4}} \left( s_+ + s_- - \frac{t}{n^{1/4}} \right) + \frac{g}{2} \left[ \frac{t^2}{\sqrt{n}} - s_-^2 - s_+^2 \right] \right) \right] dt \]
\[ \leq \exp \left( -\frac{\delta n^{1/4}\sqrt{2g|y|(s_+ + s_-)}}{4} \right) \int_{4n^{1/4}s_+}^{\infty} t^{p-1} \exp \left[ -\delta t \left( \frac{g}{2\sqrt{n}}t - \sqrt{2g|y|} \right) \right] dt \]
\[ \leq \exp \left( -\frac{\delta n^{1/4}\sqrt{2g|y|(s_+ + s_-)}}{4} \right) \int_{4n^{1/4}s_+}^{\infty} t^{p-1} \exp \left( -\delta t \sqrt{2g|y|} \right) dt, \]
which goes to 0 uniformly in \((y, \theta) \in (-\infty, a] \times [-\pi, \pi]\) as \(n \to \infty\). Thus we have shown that for each \(p \geq 1\), \(\{n^{p/4} E(N^n(y, \theta)^p)\}_{n \geq 1}\) is bounded uniformly in \((y, \theta) \in (-\infty, a] \times [-\pi, \pi]\).

**Lemma 3.2.** If \(a < 0, 1 \leq p < \infty, \) and \(r > 0\) then
\[ \lim_{n \to \infty} \sup_{(y, \theta) \in (-\infty, a] \times [-\pi, \pi]} \left| E \left( \left[ n^{p/4}N^n(y, \theta)^p \right] \wedge r^p \right) - \int_0^r pt^{p-1} \exp \left( -h(y)\sqrt{2g|y|t} \right) dt \right| = 0. \]

**Proof.** Observe that
\[ E \left( \left[ n^{p/4}N^n(y, \theta)^p \right] \wedge r^p \right) = p \int_0^\infty t^{p-1} P \left( \left[ n^{1/4}N^n(y, \theta) \right] \wedge r > t \right) dt \]
\[ = p \int_0^r t^{p-1} P \left( N^n(y, \theta) > n^{-1/4}t \right) dt. \]

Using (3.4) and the change of variables \(u = n^{1/4}s\) we compute
\[ P \left( N^n(y, \theta) > n^{-1/4}t \right) = \exp \left[ -\int_0^t \sqrt{2g|y|h(y + \sin(\theta)\sqrt{2g|y|n^{-1/2}u - \frac{g}{2n}u^2})} \right. \]
\[ \times \left. \sqrt{\cos(\theta)^2 + \left( \sin(\theta) - \frac{g}{\sqrt{2g|y|n}}u \right)^2} du \right]. \]
If $A \subset (-\infty, 0)$ is compact then the function
\[
\alpha(y, \theta, u) := n \left( y + \sin(\theta) \sqrt{2g|y|u - \frac{g}{2}u^2} \right) \sqrt{2g|y| \cos(\theta)^2 + \left( \sqrt{2g|y| \sin(\theta) - \frac{g}{\sqrt{2n}}u} \right)^2}
\]
is continuous and, therefore, uniformly continuous on the compact set $A \times [-\pi, \pi] \times [0, r]$. It follows that the functions
\[
\alpha(y, \theta, un^{-1/2}) = h \left( y + \sin(\theta) \sqrt{2g|y|n^{-1/2}u - \frac{g}{2n}u^2} \right) \sqrt{2g|y| \cos(\theta)^2 + \left( \sqrt{2g|y| \sin(\theta) - \frac{g}{\sqrt{2n}}u} \right)^2}
\]
converge uniformly to $h(y) \sqrt{2g|y|}$ on $A \times [-\pi, \pi] \times [0, r]$ as $n \to \infty$. Consequently $\mathbb{P} \left( N_n(y, \theta) > n^{-1/4} \right)$ converges uniformly to $\exp \left( -h(y) \sqrt{2g|y|} \right)$ on this set, so that
\[
\lim_{n \to \infty} \sup_{(y, \theta) \in A \times [-\pi, \pi]} \left| \mathbb{E} \left( \left[ n^{p/4} N_n^p(y, \theta)^p \right] \wedge r^p \right) - \int_0^r pt^{p-1} \exp \left( -h(y) \sqrt{2g|y|t} \right) dt \right| = 0.
\]
Since $h$ is bounded away from and above 0 for large $|y|$, there is some $\delta > 0$, independent of $n \in \mathbb{N}$ and $u \in [0, r]$, such that
\[
\lim_{y \to -\infty} \mathbb{E} \left( \left[ n^{p/4} N_n^p(y, \theta)^p \right] \wedge r^p \right) = 0.
\]
We use the fact that $h$ is bounded away from and above 0 for large $|y|$ again to see that,
\[
\lim_{y \to -\infty} \int_0^r pt^{p-1} \exp \left( -h(y) \sqrt{2g|y|t} \right) dt = 0.
\]
Thus, given $\varepsilon < 0$, we can find $b < a$ such that
\[
\sup_{y \leq b} \sup_{n \in \mathbb{N}} \mathbb{E} \left( \left[ n^{p/4} N_n^p(y, \theta)^p \right] \wedge r^p \right) + \sup_{y \leq b} \int_0^r pt^{p-1} \exp \left( -h(y) \sqrt{2g|y|t} \right) dt < \varepsilon.
\]
Since
\[
\lim_{n \to \infty} \sup_{(y, \theta) \in [b, a] \times [-\pi, \pi]} \left| \mathbb{E} \left( \left[ n^{p/4} N_n^p(y, \theta)^p \right] \wedge r^p \right) - \int_0^r pt^{p-1} \exp \left( -h(y) \sqrt{2g|y|t} \right) dt \right| = 0,
\]
the proof of the lemma is complete. \qed

**Lemma 3.3.** If $a < 0$ and $1 \leq p < \infty$ then
\[
\lim_{n \to \infty} \sup_{(y, \theta) \in (-\infty, a] \times [-\pi, \pi]} \left| \mathbb{E} \left( n^{p/4} N_n^p(y, \theta)^p \right) - \int_0^{\infty} pt^{p-1} \exp \left( -h(y) \sqrt{2g|y|t} \right) dt \right| = 0.
\]

**Proof.** Let $F(n, y, \theta, r) = \{ n^{1/4} N_n^p(y, \theta) \leq r \}$ and define $q$ by $1/q + p/(p+1) = 1$.
\[
\mathbb{E} \left( n^{p/4} N_n^p(y, \theta)^p \right) - \left[ n^{p/4} N_n^p(y, \theta)^p \right] \wedge r^p \leq \mathbb{E} \left( n^{p/4} N_n^p(y, \theta)^p \right) 1_{F^c(n, y, \theta, r)}
\]
\[
\leq \mathbb{E} \left( n^{(p+1)/4} N_n^p(y, \theta)^{p+1} \right) \left( \mathbb{E} 1_{F^c(n, y, \theta, r)} \right)^{1/q}
\]
\[
= \mathbb{E} \left( n^{(p+1)/4} N_n^p(y, \theta)^{p+1} \right) \left( \mathbb{P}(n^{1/4} N_n^p(y, \theta) > r) \right)^{1/q}
\]
\[
\leq \mathbb{E} \left( n^{(p+1)/4} N_n^p(y, \theta)^{p+1} \right) \left( \mathbb{P}(n^{1/4} N_n^p(y, \theta) > r) \right)^{1/q}.
\]
Both expectations are uniformly bounded by Lemma 3.1 so the bound goes uniformly to 0 as $r \to \infty$ and $n \to \infty$. This and Lemma 3.2 complete the proof.

Restating (3.1) for $N^n$ we see that

$$1 = \mathbb{E}\left[ \int_0^{N^n(y,\theta)} \sqrt{nh} \left( y + \sin(\theta) \sqrt{2g|y|n^{-1/4}} s - \frac{g}{2\sqrt{n}} s^2 \right) \times \left( 2g|y|n^{-1/2} \cos(\theta)^2 + \left( \sqrt{2g|y|} \sin(\theta) n^{-1/4} - \frac{g}{\sqrt{n}} s \right)^2 \right) ds \right].$$

To simplify our presentation, let us introduce the auxiliary functions

$$H_n(y,\theta,t) = h \left( y + \sin(\theta) \sqrt{2g|y|n^{-1/4}t} - \frac{g}{2\sqrt{n}} t^2 \right),$$

$$\gamma_n(y,\theta,t) = \sqrt{2g|y|n^{-1/2} \cos(\theta)^2 + \left( \sqrt{2g|y|} \sin(\theta) n^{-1/4} - gt \right)^2},$$

$$F_n(y,\theta,t) = \int_0^t H_n(y,\theta,s)\gamma_n(y,\theta,s)ds.$$

With these definitions (3.1) and (3.2) become

$$(3.13) \quad 1 = \mathbb{E}[F_n(y,\theta,N^n(y,\theta))] \quad \text{and} \quad 2 = \mathbb{E}[F_n(y,\theta,N^n(y,\theta))^2].$$

These equations can be used to find fluctuations in the convergence of Lemma 3.3. This is done by Taylor expansion in the time variable. Since we never take derivatives with respect to $y$ or $\theta$, in order to simplify notation for every differentiable function $G$ of $(y,\theta,t)$, we use $G'$ to denote the time derivative of $G$.

**Lemma 3.4.** If $A \subseteq (-\infty,0)$ is compact then

$$\lim_{n \to \infty} \sup_{(y,\theta) \in A \times [-\pi,\pi]} \left| n^{3/4} \mathbb{E} [N^n(y,\theta) - N^n(y,-\theta)] - \frac{2g \sin(\theta) (h(y) - 2|y|h'(y))}{h(y)^3 (2g|y|)^{3/2}} \right| = 0.$$

**Proof.** Taylor expanding $F_n$ in $t$ about 0, we find that for $t < n^{1/4}(2|y|/g)^{1/2}$,

$$(3.14) \quad F_n(y,\theta,t) = h(y) \sqrt{2g|y|n^{1/4}t} + F_n''(y,\theta,T(y,\theta,t))t^2/2,$$

for some $0 \leq T(y,\theta,t) \leq t$. The requirement that $t < n^{1/4}(2|y|/g)^{1/2}$ comes from the assumption that $h$ is $C^2$ on $(-\infty,0)$, but may not have a differentiable extension to the boundary. To account for this, let $B_n(y,\theta) = B_n = \{N^n(y,\theta) \leq (2|y|/g)^{1/2}\}$. Note that

$$(3.15) \quad 1 = \mathbb{E}[F_n(y,\theta,N^n(y,\theta))] = \mathbb{E}[F_n(y,\theta,N^n(y,\theta))1_{B_n}] + \mathbb{E}[F_n(y,\theta,N^n(y,\theta))1_{B_n^c}]$$

and, by the Cauchy-Schwarz inequality and (3.13),

$$\mathbb{E}[F_n(y,\theta,N^n(y,\theta))1_{B_n^c}] \leq \sqrt{2P(N^n(y,\theta) > (2|y|/g)^{1/2})}.$$

By Lemma 3.3 we see that for every $r > 0$

$$n^r P(N^n(y,\theta) > (2|y|/g)^{1/2}) \leq \left( \frac{g}{2|y|} \right)^{5r} n^r \mathbb{E} [N^n(y,\theta)^{10r}] \to 0$$

uniformly on compact subsets for $(y,\theta) \in (-\infty,0) \times [-\pi,\pi]$ as $n \to \infty$. Consequently

$$(3.16) \quad n^r \mathbb{E}[F_n(y,\theta,N^n(y,\theta))1_{B_n^c}] \to 0$$
uniformly on compact subsets for \((y, \theta) \in (-\infty, 0) \times [-\pi, \pi]\) as \(n \to \infty\). Similarly, for every \(r \geq 0\) and \(p \geq 1\) we see that

\[
(3.17)\quad n^{r}E \left[ N^{n}(y, \theta)^{p} \mathbb{1}_{B_{n}} \right] \to 0
\]

uniformly on compact subsets for \((y, \theta) \in (-\infty, 0) \times [-\pi, \pi]\) as \(n \to \infty\).

Substituting (3.14) into the first integral on the right hand side of (3.15) and solving for \(E(N^{n}(y, \theta)\mathbb{1}_{B_{n}})\) yields

\[
E(N^{n}(y, \theta)\mathbb{1}_{B_{n}}) = \frac{1}{h(y)\sqrt{2g|y|n^{1/4}}} \left( 1 - E[F_{n}(y, \theta, N^{n}(y, \theta))\mathbb{1}_{B_{n}}] \right.
\]

\[
- \frac{1}{2} E \left[ F_{n}''(y, \theta, T(y, \theta, N^{n}(y, \theta)))N^{n}(y, \theta)^{2}\mathbb{1}_{B_{n}} \right] \right).
\]

Consequently we have

\[
(3.18)\quad n^{3/4} \left[ E(N^{n}(y, \theta)\mathbb{1}_{B_{n}(\theta)}) - E(N^{n}(y, -\theta)\mathbb{1}_{B_{n}(-\theta)}) \right]
\]

\[
= \frac{\sqrt{n}}{h(y)\sqrt{2g|y|}} E[F_{n}(y, -\theta, N^{n}(y, -\theta))\mathbb{1}_{B_{n}(-\theta)}]
\]

\[
- \frac{\sqrt{n}}{h(y)\sqrt{2g|y|}} E[F_{n}(y, \theta, N^{n}(y, \theta))\mathbb{1}_{B_{n}(\theta)}]
\]

\[
- \frac{\sqrt{n}}{2h(y)\sqrt{2g|y|}} E \left[ F_{n}''(y, \theta, T(y, \theta, N^{n}(y, \theta)))N^{n}(y, \theta)^{2}\mathbb{1}_{B_{n}(\theta)} \right]
\]

\[
+ \frac{\sqrt{n}}{2h(y)\sqrt{2g|y|}} E \left[ F_{n}''(y, -\theta, T(y, -\theta, N^{n}(y, -\theta)))N^{n}(y, -\theta)^{2}\mathbb{1}_{B_{n}(-\theta)} \right].
\]

The first two terms on the right hand side vanish as \(n \to \infty\) by (3.16). For the last two terms, observe that

\[
F_{n}''(y, \theta, t) = H_{n}'(y, \theta, t)\gamma_{n}(y, \theta, t) + H_{n}(y, \theta, t)\gamma_{n}'(y, \theta, t)
\]

\[
= \left( \sin(\theta)\sqrt{2g|y|} - \frac{g}{n^{1/4}} \right) h'(y + \sin(\theta)\sqrt{2g|y|n^{-1/4}} - \frac{g}{2\sqrt{n}t}) \left( n^{-1/4}\gamma_{n}(y, \theta, t) \right)
\]

\[
- h \left( y + \sin(\theta)\sqrt{2g|y|n^{-1/4}} - \frac{g}{2\sqrt{n}t} \right) \frac{g \left( \sqrt{2g|y|\sin(\theta) - gn^{-1/4}t} \right)}{\sqrt{2g|y|\cos(\theta)^{2} + \left( \sqrt{2g|y|\sin(\theta) - gn^{-1/4}t} \right)^{2}}}
\]

Since \(h \in C^{1}(-\infty, 0)\), the function \(F_{n}''(y, \theta, t)\) is continuous, hence uniformly continuous, on \(A \times [-\pi, \pi] \times [0, (2|y|/g)^{1/2}]\). We have \(F_{n}''(y, \theta, t) = F_{1}''(y, \theta, tn^{-1/4})\). These remarks imply that

\[
\lim_{n \to \infty} \sup_{(y, \theta) \in A \times [-\pi, \pi]} \left[ \sup_{0 \leq t \leq (2|y|/g)^{1/2}} |F_{n}''(y, \theta, t) - F_{n}''(y, \theta, 0)| \right] = 0.
\]

Moreover, we see that \(F_{n}''(y, \theta, 0) = g \sin(\theta) (2|y|/h'(y) - h(y))\) and, therefore, a combination of Lemma 3.3 and (3.17) shows that these last two terms on the right hand side of (3.18) both converge to

\[
g \sin(\theta) (h(y) - 2|y|h'(y))
\]

\[
h(y)^{3}(2g|y|)^{3/2},
\]

Moreover, all of this convergence happens uniformly on compact subsets of \((y, \theta) \in (-\infty, 0) \times [-\pi, \pi]\). The lemma follows by combining this with (3.17). \(\square\)
Theorem 3.5. If \( f : (-\infty, 0) \rightarrow \mathbb{R} \) is \( C^2 \) and has compact support then

\[
\lim_{n \to \infty} \sup_{y \in (-\infty, 0)} |n(U^n - \mathcal{I})f(y) - \mathcal{A}_hf(y)| = 0,
\]

where \( U^n \) is defined in (3.3) and \( \mathcal{A}_hf \) is defined by

\[
(3.19) \quad \mathcal{A}_hf(y) = \frac{1}{2h(y)^2}f''(y) - \frac{1}{2h(y)^2} \left( \frac{1}{2} + \frac{h'(y)}{h(y)} \right) f'(y).
\]

Proof. We may clearly assume that \( f \) is not identically 0. Let \( y_- = \inf\{y : f(y) \neq 0\} \) and \( y_+ = \sup\{y : f(y) \neq 0\} \). Note that \( \mathcal{A}_hf(y) \) is 0 if \( y \notin \text{supp}(f) \). We first show that for every \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \sup_{y \leq y_- - \varepsilon} n|U^n f(y)| = 0.
\]

Let \( M = \sup_{y < 0} |f(y)| \). For \( y \leq y_- - \varepsilon \), we apply (3.6) and (3.7) with \( y_- \) in place of \( -\varepsilon \) to deduce that

\[
|U^n f(y)| \leq M \mathcal{P} \left( y + \sin(\Theta) \sqrt{2g|y|}n^{-1/4}N^n(y, \Theta) - \frac{g}{2\sqrt{n}} N^n(y, \Theta)^2 \in \text{supp}(f) \right)
\]

\[
\leq M \mathcal{P} \left( N^n(y, \Theta) > \inf \left\{ t \geq 0 : y + \sin(\Theta) \sqrt{2g|y|}n^{-1/4}t - \frac{g}{2\sqrt{n}} t^2 = y_- \right\} \right)
\]

\[
\leq M \mathcal{P} \left( N^n(y, \Theta) > \sqrt{\frac{2}{g}} n^{1/4} \left( \sqrt{|y|} - \sqrt{|y_-|} \right) \right).
\]

Now the first estimate in (3.8) applied to \( t = \sqrt{\frac{2}{g}} n^{1/4} \left( \sqrt{|y|} - \sqrt{|y_-|} \right) \) implies that there exists \( \delta_0 > 0 \) such that for \( y \leq y_- - \varepsilon \),

\[
(3.20) \quad n|U^n f(y)| \leq nM \exp \left[ -\delta_02n^{1/2} \left( \sqrt{|y|} - \sqrt{|y_-|} \right) \left( \sqrt{|y|} - \frac{1}{2} \left( \sqrt{|y|} - \sqrt{|y_-|} \right) \right) \right],
\]

which goes to 0 as \( n \to \infty \), uniformly in \( y \in (-\infty, y_- - \varepsilon] \).

Our next step is to show that

\[
\lim_{n \to \infty} \sup_{y \geq y_+ + \varepsilon} n|U^n f(y)| = 0.
\]

Suppose that \( y \geq y_+ + \varepsilon \). Again, we start with the observation that

\[
|U^n f(y)| \leq M \mathcal{P} \left( y + \sin(\Theta) \sqrt{2g|y|}n^{-1/4}N^n(y, \Theta) - \frac{g}{2\sqrt{n}} N^n(y, \Theta)^2 \in \text{supp}(f) \right)
\]

\[
\leq M \mathcal{P} \left( N^n(y, \Theta) > \inf \left\{ t \geq 0 : y + \sin(\Theta) \sqrt{2g|y|}n^{-1/4}t - \frac{g}{2\sqrt{n}} t^2 = y_+ \right\} \right)
\]

\[
= M \mathcal{P} \left( N^n(y, \Theta) > \sqrt{\frac{2}{g}} n^{1/4} \left( \sin(\Theta) \sqrt{|y|} + \sqrt{|y_+| - \cos^2(\Theta)|y|} \right) \right).
\]

In general, for \( u < v < 0 \), introduce

\[
t_\theta(v, u) := \inf \left\{ t \geq 0 : v + \sin(\theta) \sqrt{2g|v|}n^{-1/4}t - \frac{g}{2\sqrt{n}} t^2 = u \right\}
\]

\[
= \sqrt{\frac{2}{g}} n^{1/4} \left( \sin(\theta) \sqrt{|v|} + \sqrt{|u| - \cos^2(\theta)|v|} \right).
\]
Observe that for $y \geq y_+ + \varepsilon$
\[
P(N^n(y, \theta) > t_\theta(y, y_+)) = \exp \left( - \int_0^{t_\theta(y, y_+)} H_n(y, \theta, s) \gamma_n(y, \theta, s) ds \right) \leq \exp \left( - \int_0^{t_\theta(y, y_+ + \varepsilon/2)} H_n(y, \theta, s) \gamma_n(y, \theta, s) ds \right).
\]

Let $\delta = \inf_{y \leq y_+ + \varepsilon/2} h(y)$ and note that $\delta > 0$ by our assumptions on $h$. Observe that
\[
y_+ \leq y + \sin(\theta) \sqrt{2g|y|n^{-1/4}} t - \frac{g}{2\sqrt{n}} t^2 \leq y_+ + \varepsilon/2 \quad \text{when} \quad t_\theta(y, y_+ + \varepsilon/2) \leq t \leq t_\theta(y, y_+).
\]
Moreover, for $t \geq t_\theta(y, y_+ + \varepsilon/2)$ we have that
\[
\gamma_n(y, \theta, t) \geq \gamma_n(y, \theta, t_\theta(y, y_+ + \varepsilon/2)) = \sqrt{2g} \left| y_+ + \frac{\varepsilon}{2} n^{1/4}, \right.
\]
so that
\[
P(N^n(y, \theta) > t_\theta(y, y_+)) \leq \exp \left( - \delta \sqrt{2g} \left| y_+ + \frac{\varepsilon}{2} n^{1/4} \right( t_\theta(y, y_+) - t_\theta(y, y_+ + \varepsilon/2) \right) \right).
\]

Furthermore, it is easy to show that there exists $\delta_1 > 0$ such that
\[
\inf_{y \geq y_+ + \varepsilon} \inf_{\theta \in [-\pi, \pi]} |t_\theta(y, y_+) - t_\theta(y, y_+ + \varepsilon/2)| \geq n^{1/4} \delta_1,
\]
and it follows that
\[
\limsup_{n \to \infty} \sup_{y \geq y_+ + \varepsilon} n|U^n f(y)| \leq \limsup_{n \to \infty} \left( nM \exp \left( - \delta \delta_1 \sqrt{2g} \left| y_+ + \frac{\varepsilon}{2} n^{1/2} \right) \right) \right) = 0.
\]

It remains to show that if $A \subseteq (-\infty, 0)$ is a compact set that contains $\text{supp}(f)$ in its interior then
\[
\limsup_{n \to \infty} \sup_{y \in A} \left| n(U^n - I)f(y) - \bar{A}_h f(y) \right| = 0.
\]
Since $f$ is $C^2$, we can use a Taylor expansion to find that
\[
f(y + v) = f(y) + f'(y) v + \frac{f''(\kappa(y, v))}{2} v^2
\]
for some $|y - \kappa(y, v)| \leq |v|$. Define
\[
(3.21) \quad \alpha_n(y, \theta) = \sin(\theta) \sqrt{2g|y|n^{-1/4}} N^n(y, \theta) - \frac{g}{2\sqrt{n}} N^n(y, \theta)^2 \quad \text{and} \quad \beta_n(y, \theta) = \kappa(y, \alpha_n(y, \theta)).
\]
We can then write
\[
U^n f(y) = E[f(y + \alpha_n(y, \Theta))] = f(y) + f'(y) E[\alpha_n(y, \Theta)] + E \left[ \frac{f''(\beta_n(y, \Theta))}{2} \alpha_n(y, \Theta)^2 \right],
\]
so that
\[
n(U^n - I)f(y) = f'(y)n E[\alpha_n(y, \Theta)] + n E \left[ \frac{f''(\beta_n(y, \Theta))}{2} \alpha_n(y, \Theta)^2 \right].
\]
Let us consider the term $n E[\alpha_n(y, \Theta)]$. Observe that
\[
n E[\alpha_n(y, \Theta)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \sqrt{2g|y|} \sin(\theta) E \left( n^{3/4} N^n(y, \theta) \right) - \frac{g\sqrt{n}}{2} E \left( N^n(y, \theta)^2 \right) \right] d\theta.
\]
It follows from Lemma 3.3 that
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g\sqrt{n}}{2} E\left( N^n(y, \theta)^2 \right) d\theta \rightarrow \frac{1}{2h(y)^2|y|} \]
uniformly in \( y \) on \( A \). Furthermore, we see that
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(\theta) E\left( n^{3/4} N^n(y, \theta) \right) d\theta = \frac{1}{2\pi} \int_{0}^{\pi} \sin(\theta)n^{3/4} E\left[ N^n(y, \theta) - N^n(y, -\theta) \right] d\theta. \]
Therefore, by Lemma 3.4,
\[ (3.24) \]
and this convergence happens uniformly for \( y \) on \( A \). Since \( \text{supp}(f') \subseteq \text{supp}(f) \), we have that
\[ \lim_{n \rightarrow \infty} f'(y)n E[\alpha_n(y, \Theta)] = f'(y) \left( \frac{h(y) - 2|y|h(y)}{4h(y)^3|y|} - \frac{1}{2h(y)^2|y|} \right) = -f'(y) \frac{1}{2h(y)^2} \left( \frac{1}{2|y|} + \frac{h'(y)}{h(y)} \right) \]
uniformly for \( y \in (-\infty, 0) \). We now turn to
\[ (3.24) \]
\[ n E \left[ \frac{f''(\beta_n(y, \Theta))}{2} \alpha_n(y, \Theta)^2 \right] = g|y|\sqrt{n} E \left[ f''(\beta_n(y, \Theta))\sin^2(\Theta)N^n(y, \Theta)^2 \right] - \frac{\sqrt{2|y|g^3}}{2}n^{1/4} E \left[ f''(\beta_n(y, \Theta))\sin(\Theta)N^n(y, \Theta)^3 \right] + \frac{g^2}{8} E \left[ f''(\beta_n(y, \Theta))N^n(y, \Theta)^4 \right]. \]
Since \( f'' \) is bounded, Lemma 3.3 implies that the last two terms converge to 0 uniformly for \( y \in A \). Using \( B_n(y, \theta) = \{ N^n(y, \theta) \leq (2|y|/g)^{1/2} \} \) as defined above (3.15), it follows from (3.17) that
\[ \lim_{n \rightarrow \infty} \sup_{y \in A} g|y|\sqrt{n} E \left[ f''(\beta_n(y, \Theta))\sin^2(\Theta)N^n(y, \Theta)^2 \mathbb{1}_{B_n(\Theta)} \right] = 0. \]
Moreover, on \( B_n(y, \theta) \), we have
\[ |\alpha_n(y, \theta)| \leq \sqrt{2g|y|n^{-1/4}(2|y|/g)^{1/2}} + \frac{g}{2\sqrt{n}}(2|y|/g) \leq 3|y|n^{-1/4}. \]
From the definition of \( \kappa \) and the fact that \( f'' \) is uniformly continuous it follows that
\[ \lim_{n \rightarrow \infty} \sup_{y \in A} \left[ \sup_{0 \leq t \leq 3|y|n^{-1/4}} |f''(y) - f''(\kappa(y, t))| \right] = 0. \]
Lemma 3.3 and the definition of \( \beta_n \) then imply that
\[ \lim_{n \rightarrow \infty} \sup_{y \in A} g|y|\sqrt{n} E \left[ |f''(\beta_n(y, \Theta)) - f''(y)|\sin^2(\Theta)N^n(y, \Theta)^2 \mathbb{1}_{B_n(\Theta)} \right] = 0. \]
Combining Lemma 3.3 and (3.17) shows that
\[ (3.26) \]
\[ \lim_{n \rightarrow \infty} g|y|f''(y)\sqrt{n} E \left[ \sin^2(\Theta)N^n(y, \Theta)^2 \mathbb{1}_{B_n(\Theta)} \right] = \frac{1}{2h(y)^2}f''(y) \]
uniformly for \( y \in A \). This, (3.24) and (3.25) imply that
\[ (3.27) \]
\[ \lim_{n \rightarrow \infty} n E \left[ \frac{f''(\beta_n(y, \Theta))}{2} \alpha_n(y, \Theta)^2 \right] = \frac{1}{2h(y)^2}f''(y), \]
and this completes the proof of the theorem. □
Proposition 3.6. If \((Z_t, 0 \leq t < \zeta)\) is a Feller diffusion on \((-\infty, 0)\) with lifetime \(\zeta\) whose generator extends the operator \(\tilde{A}_h\) defined in (3.19) then for \(y \in (-\infty, 0)\) the scale function \(G\) and speed measure \(m\) for \(Z\) are given by
\[
G(y) = \int_{-1}^{y} \frac{h(u)}{\sqrt{|u|}} du \quad \text{and} \quad m(dy) = 2\sqrt{|y|h(y)} dy.
\]
Moreover, \(-\infty\) is inaccessible and either \(\zeta = \infty\) a.s or \(\zeta < \infty\) a.s and \(\lim_{t \uparrow \zeta} Z_t = 0\) a.s.

Proof. Since \(G\) is a continuous, increasing function such that \(\tilde{A}_h G \equiv 0\), \(G\) is a scale function for \(Z\). The speed measure can then be found using, e.g., [13] Exercise VII.3.20. For the boundary classification we use [4, Theorem VI.3.2]. Since \(h\) is bounded below, \(\lim_{y \rightarrow -\infty} G(y) = -\infty\), which shows that \(-\infty\) is inaccessible. Define
\[
\kappa(y) = \int_{-1}^{y} \left[ \int_{-1}^{u} 2\sqrt{|s|h(s)} ds \right] \frac{h(u)}{\sqrt{|u|}} du.
\]
From [4, Theorem VI.3.2], we see that if \(\kappa(0) < \infty\) then \(\zeta < \infty\) a.s. and \(\lim_{t \uparrow \zeta} Z_t = 0\) a.s., while if \(\kappa(0) = \infty\) then \(\zeta = \infty\). To see that both options are possible, observe that if \(h \equiv 1\), then \(\kappa(0) < \infty\), while if \(h(y) = |y|^{-1}\) for \(y \in (-1, 0)\) (and is continued to \((-\infty, 0)\) in a way that satisfies our assumptions), then \(\kappa(0) = \infty\).

4. Convergence of the skeleton process

In this section, we prove convergence of the skeleton process, that is, the process watched only at reflection times.

Let \((Y^n, m \geq 0)\) denote the Markov chain with transition operator \(U^n\). For \(v \in \mathbb{R}\), let \(\tau^n_{v, +} = \inf\{m \geq 0 : Y^n_m \geq v\}\).

Theorem 4.1. Consider any \(-\infty < y < v < 0\) and suppose that \((Y^n, m \geq 0)\) is the Markov chain with transition operator \(U^n\) started from \(y\). When \(n \rightarrow \infty\), the processes \((Y^n_{\lfloor nt\rfloor \wedge \tau^n_{v, +}}, t \geq 0)\) converge in distribution on the Skorokhod space to the diffusion whose generator extends the operator given in (3.19), starting at \(y\) and stopped at the hitting time of \(v\).

Proof. We now introduce an auxiliary process \(Y^{u,v,n}\), which behaves much like the original process, except that gravitation stops impacting velocity at high and low levels. Fix some \(u \in (-\infty, y)\). We will later let \(u \rightarrow \infty\). Recall \(U^n\) defined in (3.3). Let \((Y^n_{m,v,n}, m \geq 0)\) be the the Markov chain with transition operator
\[
U^n_{u,v}(y) := \begin{cases} U^n f(y) & \text{if } u < y < v, \\ \mathbb{E} \left[ f \left( y + \sin(\Theta) \sqrt{2g|v|} N^{n-1/4}(v, \Theta) - \frac{g}{\sqrt{n}} N^n(v, \Theta)^2 \right) \right] & \text{if } y \geq v, \\ \mathbb{E} \left[ f \left( y + \sin(\Theta) \sqrt{2g|u|} N^{n-1/4}(u, \Theta) - \frac{g}{\sqrt{n}} N^n(u, \Theta)^2 \right) \right] & \text{if } -\infty < y \leq u. \end{cases}
\]

More compactly, using \(\alpha_n\) as defined in (3.21), we have for all \(y\) that
\[
U^n_{u,v}(y) = \mathbb{E} \left[ f(y + \alpha_n((y \lor u) \land v, \Theta)) \right].
\]

Because of the way we extended \(U^n_{u,v} f(\cdot)\) outside of \([u,v]\), Theorem 3.5 shows that if
\[
\tilde{A}_{u,v} f(y) = \frac{1}{2h((y \lor u) \land v)^2} f''(y) - \frac{1}{2h((y \lor u) \land v)^2} \left( \frac{1}{2((y \lor u) \land v)} \frac{h'((y \lor u) \land v)}{h((y \lor u) \land v)} \right) f'(y)
\]
for \(y \in \mathbb{R}\) then
\[
\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} |n(U^n_{u,v} - I) f(y) - \tilde{A}_{u,v} f(y)| = 0.
\]
For $y < 0$, let us define
\[ b(y) = -\frac{1}{2h(y)^2} \left( \frac{1}{2|y|} + \frac{h'(y)}{h(y)} \right) \quad \text{and} \quad c(y) = \frac{1}{h(y)^2}. \]
For $y \in \mathbb{R}$, we then define $b_{u,v}(y) := b((y \vee u) \wedge v)$ and $c_{u,v}(y) := c((y \vee u) \wedge v)$.

Let $(\Gamma_t, t \geq 0)$ be a rate 1 Poisson process and define $Z^{u,v}_t = Y^{u,v,n}_t$. Then $(Z^{u,v}_t, t \geq 0)$ is a continuous time pure jump Markov process with generator $\mathcal{L}^{u,v}_{u,v}f(y) = n(U^n_{u,v} - I)f(y)$.

We now show that if $Z^{u,v}_0$ converge to $Z_0$ in distribution for some random variable $Z_0$ then $(Z^{u,v}_t, t \geq 0)$ converge weakly on the Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$ to the diffusion with the generator $\mathcal{A}_{u,v,h}$ whose initial distribution is the law of $Y_0$. We will apply [5, Theorem IX.4.21] so we have to show that the assumptions of that theorem are satisfied.

For $n \geq 1$, define the kernel
\[ K^n_{u,v}(y, \cdot) = n \, \mathbb{P}(\alpha_n((y \vee u) \wedge v, \Theta) \in \cdot), \]
so that
\[ \mathcal{L}^n_{u,v}f(y) = \int_{\mathbb{R}} (f(y + u) - f(y))K^n_{u,v}(y, du). \]
Let
\[ b^n_{u,v}(y) = \int_{\mathbb{R}} z K^n_{u,v}(y, dz) = n \, \mathbb{E}[\alpha_n((y \vee u) \wedge v, \Theta)], \]
\[ c^n_{u,v}(y) = \int_{\mathbb{R}} z^2 K^n_{u,v}(y, dz) = n \, \mathbb{E}[\alpha_n((y \vee u) \wedge v, \Theta)^2]. \]
We have already remarked that the proof of Theorem 3.5 applies in the present context. In particular, (3.23) implies that
\[ \lim_{n \to \infty} b^n_{u,v}(y) = b_{u,v}(y), \]
normally in $y \in \mathbb{R}$. It is easy to check that the argument leading to (3.27) applies equally well (in fact, it is easier) if $\beta_n(y, \Theta)$ is replaced with $y$. Hence
\[ \lim_{n \to \infty} c^n_{u,v}(y) = c_{u,v}(y), \]
normally in $y \in \mathbb{R}$. This and (4.1) show that assumption (i) of [5, Theorem IX.4.21] is satisfied.

For every $\rho > 0$,
\[ \int_{\mathbb{R}} z^2 \mathbb{1}_{(|z| > \rho)} K^n_{u,v}(y, dz) = n \, \mathbb{E}[\alpha_n((y \vee u) \wedge v, \Theta)^2 \mathbb{1}_{(|\alpha_n((y \vee u) \wedge v, \Theta)| > \rho)}]. \]
Arguing as in (3.17), we see that for every $r > 0$,
\[ \lim_{n \to \infty} \sup_{|y| \leq r} n \, \mathbb{E}[\alpha_n((y \vee u) \wedge v, \Theta)^2 \mathbb{1}_{(|\alpha_n((y \vee u) \wedge v, \Theta)| > \rho)}] = 0. \]
This verifies assumption (ii) of [5, Theorem IX.4.21]. Since $Z^{u,v,n}_0$ converge to $Z_0$ in distribution, assumption (iii) of that theorem is satisfied as well. We can now apply [5, Theorem IX.4.21] and conclude that $(Z^{u,v}_t, t \geq 0)$ converge weakly on the Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$ to the diffusion with the generator $\mathcal{A}_{u,v,h}$ whose initial distribution is the law of $Z_0$.

Suppose that $Y^{u,v,n}_0 = y$. Since $\lim_{n \to \infty} n^{-1} \Gamma_{nt} = t$ uniformly on every compact interval, a.s., the last claim implies that the process $(Y^{u,v,n}_{[nt]}, t \geq 0)$ converges in distribution on the Skorokhod space to the diffusion with the generator $\mathcal{A}_{u,v,h}$ starting at $y$. Moreover, since $c_{u,v}(y) > 0$, the diffusion with generator $\mathcal{A}_{u,v,h}$ almost surely fluctuates across fixed levels after first hitting them, so the convergence of $(Y^{u,v,n}_{[nt]}, t \geq 0)$ happens jointly with the convergence of the level crossing times of $u$ and $v$. 

For $y < 0$, let us define
\[ b(y) = -\frac{1}{2h(y)^2} \left( \frac{1}{2|y|} + \frac{h'(y)}{h(y)} \right) \quad \text{and} \quad c(y) = \frac{1}{h(y)^2}. \]
Recall from Proposition 3.6 that $-\infty$ is inaccessible for the diffusion with generator $\tilde{A}_h$. Hence, it is standard to show that we can remove the “truncation” at $u$. In other words, if $Y^{u,v,n}_0 = y$ then the process $(Y^{u,v,n}_n, t \geq 0)$ converges in distribution on the Skorokhod space to the diffusion with the generator $\tilde{A}_h$ starting at $y$. This completes the proof. \hfill \Box

5. The process on its natural time scale

In the previous section we examined convergence of the skeletal process, that is, the process watched only at reflection times. In this section we consider the process on its natural time scale. We do this in two parts. First we look at the skeletal process, but put the reflections at their real times rather than at the index of how many reflections have occurred. Second, we fill in the actual trajectory of the particle between reflections rather than only observing its location at reflection times.

We begin by looking at an augmented skeletal process where we also keep track of time. That is, we look at the Markov chain $((Y^n_m, \Delta^n_m), m \geq 0)$ with transition operator

$$\tilde{U}^n f(y, z) = E \left[ f(y + s \sqrt{2g} \left[ \sin(\Theta) \sqrt{2g} \right] | y | n^{-1/4} N^n(y, \Theta) - \frac{g}{2\sqrt{n}} N^n(y, \Theta)^2, n^{1/4} N^n(y, \Theta) \right].$$

If the process starts from $(y, 0)$, the real time of the $m$'th collision is then $T^n_m := n^{-1/4} \sum_{j=0}^m \Delta^n_m$, where we set $\Delta^n_0 = 0$ by convention. We define $T^n_s$ for all $s \in [0, \infty)$ by linear interpolation, that is, $T^n_s = T^n_{[s]} + (s - [s])(T^n_{[s]+1} - T^n_{[s]})$. Let $Z$ denote a diffusion whose generator extends the operator defined in (3.19). Recall that $\tau^n_{v+} = \inf \{ m \geq 0 : Y^n_m \geq v \}$ and let $\tau^n_{v+} = \inf \{ t \geq 0 : Z_t \geq v \}$. For $f \in D(\mathbb{R}_+, \mathbb{R})$ we define $\psi(f) \in D(\mathbb{R}_+, \mathbb{R})$ by

$$\psi_v(f)(t) = \int_0^t \frac{1}{\sqrt{2g}|f(s) \land v| h(f(s) \land v)} \, ds.$$

Theorem 5.1. Let $((Y^n_m, \Delta^n_m), m \geq 0)$ be the Markov chain with transition operator $\tilde{U}^n$ started from $(y, 0)$ and suppose that $-\infty < y < v < 0$. We then have the following convergence in distribution on $D(\mathbb{R}_+, \mathbb{R}) \times D(\mathbb{R}_+, \mathbb{R})$

$$\left( Y^n_{[t] \wedge \tau^n_{v+}}, n^{-3/4} T^n_{(t) \wedge \tau^n_{v+}} + (s - n^{-1} \tau^n_{v+})^+ \frac{1}{\sqrt{2g}|v| h(v)} \right), t, s \geq 0 \right)$$

$$\rightarrow ((Z(t \wedge \tau^n_{v+}), \psi_v(Z(\cdot \wedge \tau^n_{v+}))(s)), t, s \geq 0).$$

Proof. We again make use of the auxiliary processes $((Y^{u,v,n}_m, \Delta^{u,v,n}_m), m \geq 0)$ for $-\infty \leq u < v < 0$ with transition operators

$$\tilde{U}^n_{u,v} f(y, z) = E \left[ f(y + s \sqrt{2g} \left[ \sin(\Theta) \sqrt{2g} \right] | y | n^{-1/4} N^n((y \lor u) \land v, \Theta) - \frac{g}{2\sqrt{n}} N^n((y \lor u) \land v, \Theta)^2, n^{1/4} N^n((y \lor u) \land v, \Theta) \right].$$

Suppose that $((Y^{u,v,n}_m, \Delta^{u,v,n}_m), m \geq 0)$ starts from $(y, 0)$ and define

$$T^{u,v,n}_s = n^{-1/4} \sum_{j=1}^{[s]} \Delta^{u,v,n}_j + n^{-1/4}(s - [s]) \Delta^{u,v,n}_{[s]+1}.$$
Suppose that \((Y_t^{u,v}, t \geq 0)\) is a diffusion with generator \(\mathcal{A}_{u,v,h}\) starting at \(y\). We will show that the following convergence in distribution holds in \(D(\mathbb{R}_+, \mathbb{R})\),

\[
(n^{-3/4}T_{nt}, t \geq 0) \xrightarrow{d} \left( \int_0^t \frac{1}{\sqrt{2g|f(s) \vee u| h((f(s) \vee u) \wedge v)}} ds, t \geq 0 \right).
\]

We will show even more. For \(f \in D(\mathbb{R}_+, \mathbb{R})\) we let

\[
\psi_{u,v}(f)(t) = \int_0^t \frac{1}{\sqrt{2g|f(s) \vee u| h((f(s) \vee u) \wedge v)}} ds.
\]

We prove that the joint convergence

\[
(Y^{u,v}_{nt}, n^{-3/4}T_{nt}, t, s \geq 0) \rightarrow (Y_t^{u,v}, \psi_{u,v}(Y^{u,v})(s), t, s \geq 0),
\]

in distribution on the product space \(D(\mathbb{R}_+, \mathbb{R}) \times D(\mathbb{R}_+, \mathbb{R})\) holds.

Let \(F_m = \sigma((Y_j^{u,v}, \Delta_j^{u,v}), 0 \leq j \leq m)\) and consider the martingale with respect to the filtration \((F_m)_{m \geq 0}\) given by

\[
W_m := \sum_{j=1}^m (\Delta_j^{u,v} - \mathbb{E}[\Delta_j^{u,v} | F_{j-1}]), m \geq 0.
\]

Define \(\phi_n(y) = n^{1/4} \mathbb{E}([-Y^n((y \vee u) \wedge v, \Theta))]. By the Markov property we see that \(\mathbb{E}[\Delta_j^{u,v} | F_{j-1}] = \phi_n(Y_{j-1}^{u,v}). By Lemma 3.1 we see that \(\sup_{n,y} \phi_n(y) < \infty\) and

\[
\xi := \sup_{m,n} \mathbb{E}[(\Delta_m^{u,v} - \mathbb{E}[\Delta_m^{u,v} | F_{m-1}])^2] < \infty.
\]

By Doob’s maximal inequality we see that for every \(\varepsilon > 0\) and integer \(k \geq 1\)

\[
P\left( \sup_{1 \leq m \leq kn} |W_m^n| > n\varepsilon \right) \leq \frac{1}{n^2 \varepsilon^2} \mathbb{E}[|W_{kn}^n|^2] \leq \frac{k\xi}{n \varepsilon^2},
\]

from which it follows that \(\sup_{1 \leq m \leq kn} |n^{-1}W_m^n|\) converges to 0 in probability as \(n \rightarrow \infty\). Lemma 3.3 implies that

\[
\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \left| \phi_n(y) - \frac{1}{\sqrt{2g|Y^{u,v}_n \wedge v| h((Y^{u,v}_n \wedge u) \wedge v)}} \right| = 0,
\]

and it follows in turn that for every integer \(k \geq 1\), a.s.,

\[
\lim_{n \rightarrow \infty} \sup_{1 \leq m \leq kn} \frac{1}{n} \sum_{j=1}^m \left| \phi_n(Y^{u,v}_{j-1}) - \frac{1}{\sqrt{2g|Y^{u,v}_{j-1} \wedge v| h((Y^{u,v}_{j-1} \wedge u) \wedge v)}} \right| = 0.
\]

Therefore, using (5.3),

\[
\lim_{1 \leq m \leq kn} \frac{1}{n} \sum_{j=1}^m \left( \Delta_j^{u,v} - \frac{1}{\sqrt{2g|Y^{u,v}_{j-1} \wedge v| h((Y^{u,v}_{j-1} \wedge u) \wedge v)}} \right) \rightarrow 0,
\]

as \(n \rightarrow \infty\), in probability.

It follows from standard properties of the Skorokhod topology that the mapping \(f \rightarrow \psi_{u,v}(f)\) is continuous. Noting that

\[
\psi_{u,v}(Y^{u,v}_{[n:]})(m/n) = \frac{1}{n} \sum_{j=1}^m \sqrt{2g|Y^{u,v}_{j-1} \wedge v| h((Y^{u,v}_{j-1} \wedge u) \wedge v)},
\]
the joint convergence \((Y_{nt}^{u,v,n}, \psi_{u,v}(Y_{nt}^{u,v,n}))(s), t, s \geq 0) \rightarrow ((Y_{t}^{u,v}, \psi_{u,v}(Y_{t}^{u,v}))(s), t, s \geq 0)\) in distribution on the product space \(D(\mathbb{R}_+, R) \times D(\mathbb{R}_+, R)\) follows from the proof of Theorem 4.1. Combining this with (5.4) completes the proof of (5.2).

If \(f \in D(\mathbb{R}_+, R)\) is such that \(f \leq v\), then
\[
\psi_v(f)(t) = \int_0^t \frac{1}{\sqrt{2g[f(s)] h(f(s))}} ds
\]
so that
\[
\psi_v(Z(\cdot \land \tau_{v+}))(t) = \int_0^t \frac{1}{\sqrt{2g[Z(s \land \tau_{v+})] h(Z(s \land \tau_{v+}))}} ds.
\]

Just as at the end of the proof of Theorem 4.1, we can remove the “truncation” at \(u\) and obtain the statement of the theorem. We remark that the term \((s - n^{-1} \tau_{v+}^n)^+ + \frac{1}{\sqrt{2g(v) h(v)}}\) appears in (5.1) because, on the continuous side, \(\psi_v(Z(\cdot \land \tau_{v+}))\) increases linearly after \(Z\) hits \(v\).

The following lemma is likely to be known but we could not find a reference.

Let \(\mathbb{R}_* = \mathbb{R} \cup \{\infty\}\) and \(\mathbb{R}_+^* = \mathbb{R}_+ \cup \{\infty\}\). By convention, \(\inf \emptyset = \infty\) and for any function \(f\), \(f(\infty) = \infty\). For \(f \in D(\mathbb{R}_+, R_+)\), define \(\Psi : D(\mathbb{R}_+, R_+) \rightarrow D(\mathbb{R}_+, R_+^*)\) by \(\Psi(f)(t) = \inf \{s : f(s) > t\}\).

**Lemma 5.2.** If \(f \in D(\mathbb{R}_+, R_+)\) is continuous and strictly increasing with \(\lim_{t \to \infty} f(t) = \infty\), then \(\Psi(f) \in D(\mathbb{R}_+, R_+)\) and \(\Psi\) is continuous at \(f\).

**Proof.** First we prove that for any \(h \in D(\mathbb{R}_+, R_+)\), the function \(\Psi(h)\) is in \(D(\mathbb{R}_+, R_+^*)\). It is clear that \(\Psi(h)\) is a non-decreasing function. Since the function \(\Psi(h)\) is monotone, it has left and right limits at every point. It remains to show that it is right-continuous. Since \(\Psi(h)\) is non-decreasing, we have \(\lim_{s \uparrow t} \Psi(h)(s) \geq \Psi(h)(t)\) for every \(t\). Consider any \(t\) and an arbitrarily small \(\delta > 0\), and let \(b = \Psi(h)(t)\). If \(h(b) \leq t\) then there must exist \(b_1 \in (b, b + \delta)\) and \(t_1 > t\) such that \(h(b_1) = t_1\). This claim holds also in the case \(h(b) > t\), by the right-continuity of \(h\). For all \(s \in (t, t_1)\) we have \(\Psi(h)(s) \leq b_1 < b + \delta\). Since \(\delta > 0\) is arbitrarily small, this implies that \(\lim_{s \uparrow t} \Psi(f)(s) \leq \Psi(h)(t)\).

In view of the previously proved opposite inequality, we conclude that \(\Psi(h)\) is right continuous at \(t\). This completes the proof that \(\Psi(h) \in D(\mathbb{R}_+, R_+^*)\).

Now suppose that \(f\) satisfies the hypotheses of the lemma and that \(f_n \in D(\mathbb{R}_+, R_+)\) is a sequence converging to \(f\). Since \(f\) is continuous and strictly increasing, the function \(\Psi(f)\) is also continuous and strictly increasing. Fix any \(T < \infty\). It suffices to show that
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} |\Psi(f_n)(t) - \Psi(f)(t)| = 0.
\]

Suppose otherwise. Then there exist \(\epsilon > 0\), a subsequence \(n_k\) and a sequence \(t_{nk}\) of points in \([0,T]\), such that \(|\Psi(f_{n_k})(t_{nk}) - \Psi(f)(t_{nk})| > \epsilon\) for all \(k\). By compactness, we may suppose that \(t_{nk} \to t_\infty \in [0,T]\) as \(k \to \infty\). We will assume that \(t_\infty \in (0,T)\). The argument requires only small modifications when \(t_\infty = 0\) or \(T\).

Let \(s_\infty = \Psi(f)(t_\infty)\) and
\[
\delta = \min(f(s_\infty - \epsilon/4) - f(s_\infty - \epsilon/2), f(s_\infty + \epsilon/2) - f(s_\infty + \epsilon/4)).
\]

Since \(f(s_\infty) = t_\infty\), \(f\) is strictly increasing and \(t_{nk} \to t_\infty\), there exists \(k_1\) such that for all \(k \geq k_1\),
\[
f(s_\infty - \epsilon/4) \leq t_{nk} \leq f(s_\infty + \epsilon/4).
\]

Since \(f\) is continuous, \(f_n \to f\) uniformly on compact sets. Let \(k_2 \geq k_1\) be so large that for \(k \geq k_2\),
\[
|\Psi(f)(t_\infty) - \Psi(f)(t_{nk})| < \epsilon/4,
\]
It follows from the definition of $\delta$ and (5.7) that
\[
\sup_{t \in [0, s_{\infty} - \varepsilon / 2]} |f_{n_k}(t) - f(t)| < \delta / 4.
\]
This, (5.5), the definition of $s_{\infty}$ and (5.6) imply that
\[
\Psi(f_{n_k})(t_{n_k}) \geq \Psi(f_{n_k})(f(s_{\infty} - \varepsilon / 4)) \geq s_{\infty} - \varepsilon / 2 = \Psi(f)(t_{\infty}) - \varepsilon / 2 \geq \Psi(f)(t_{n_k}) - 3\varepsilon / 4.
\]
The following estimates can be obtained in an analogous way,
\[
\Psi(f_{n_k})(t_{n_k}) \leq \Psi(f_{n_k})(f(s_{\infty} + \varepsilon / 4)) \leq s_{\infty} + \varepsilon / 2 = \Psi(f)(t_{\infty}) + \varepsilon / 2 \leq \Psi(f)(t_{n_k}) + 3\varepsilon / 4.
\]
We combine this with (5.8) to obtain $|\Psi(f_{n_k})(t_{n_k}) - \Psi(f)(t_{n_k})| \leq 3\varepsilon / 4$. This contradicts the definition of the sequence $t_{n_k}$. This contradiction completes the proof.

Recall that $Z$ denotes the diffusion with generator (3.4). Let
\[
A_n(t) = \Psi \left( n^{-3/4} T_{(n \cdot) \wedge \tau_{v+}} + (\cdot - n^{-1} \tau_{v+})^+ \frac{1}{\sqrt{2g|v|h(v)}} \right)(t) \quad \text{and} \quad A(t) = \Psi(\psi_v(Z(\cdot \wedge \tau_{v+}))(t),
\]
and note that the dependence of these processes on $v$ is left implicit.

**Theorem 5.3.** Let $((Y^n_m, \Delta^n_m), m \geq 0)$ be the Markov chain with transition operator $\tilde{U}^n$ started from $(y, 0)$ and suppose that $-\infty < y < v < 0$. We then have the following convergence in distribution on $D(R_+, R^*)$,
\[
\left( Y^n_{[nA_n(t)] \wedge \tau_{v+}}, t \geq 0 \right) \rightarrow (Z(A(t) \wedge \tau_{v+}), t \geq 0).
\]

**Proof.** By Theorem 5.1 and the Skorokhod embedding lemma, we can construct the relevant processes on a single probability space so that for any $t_1, s_1 > 0$,
\[
\left( Y^n_{[at] \wedge \tau_{v+}}, n^{-3/4} T_{(ns) \wedge \tau_{v+}} + (s - n^{-1} \tau_{v+})^+ \frac{1}{\sqrt{2g|v|h(v)}} \right), t \in [0, t_1], s \in [0, s_1)
\]
\[
\rightarrow ((Z(t \wedge \tau_{v+}), \psi_v(Z(\cdot \wedge \tau_{v+}))(s)), t \in [0, t_1], s \in [0, s_1]),
\]
a.s., in the Skorokhod topology. Then Lemma 5.2 implies that for any $t_1, s_1 > 0$,
\[
\left( Y^n_{[at] \wedge \tau_{v+}}, \Psi \left( n^{-3/4} T_{(n \cdot) \wedge \tau_{v+}} + (\cdot - n^{-1} \tau_{v+})^+ \frac{1}{\sqrt{2g|v|h(v)}} \right)(s) \right), t \in [0, t_1], s \in [0, s_1)
\]
\[
\rightarrow ((Z(t \wedge \tau_{v+}), \Psi(\psi_v(Z(\cdot \wedge \tau_{v+}))(s)), t \in [0, t_1], s \in [0, s_1]),
\]
a.s., in the Skorokhod topology. It is easy to see that the assumptions in Lemma 5.2 apply a.s. to $f = n^{-3/4} T_{(ns) \wedge \tau_{v+}} + (\cdot - n^{-1} \tau_{v+})^+ \frac{1}{\sqrt{2g|v|h(v)}}$ and $f = \psi_v(Z(\cdot \wedge \tau_{v+}))$.

It is clear that $(Z(t \wedge \tau_{v+}), t \geq 0)$ has continuous paths and, indeed, so does $(A(t), t \geq 0)$ since $(\psi_v(Z(\cdot \wedge \tau_{v+}))(t), t \geq 0)$ is strictly increasing with derivative bounded below on compact sets. The theorem now follows from a standard result about the continuity of composition with respect to the Skorokhod topology (see e.g. [11, Section 17]).
We now turn to the full path of the particle. In addition to keeping track of time we need to keep track of the angle of reflection. That is, we consider that Markov chain \((Y^n_m, \Delta^n_m, \Theta_m), m \geq 0\) with transition operator
\[
(5.9) \quad \hat{U}^n f(y, z, w) = E \left[ f \left( y + \sin(\Theta) \sqrt{\frac{2g|y|}{n}} n^{-1/4} N^n(y, \Theta) - \frac{g}{2\sqrt{n}} N^n(y, \Theta)^2, n^{-1/4} N^n(y, \Theta), \Theta \right) \right],
\]
started from \((y, 0, 0)\). The path of the particle is then given by
\[
(5.10) \quad Y^n(t) = Y^n_{m-1} + \sin(\Theta_m) \sqrt{\frac{2g|Y^n_{m-1}|}{n}} n^{-1/4} (t-T^{n}_{m-1}) - \frac{g}{2\sqrt{n}} (t-T^{n}_{m-1})^2 \quad \text{on} \quad T^{n}_{m-1} \leq t < T^{n}_m.
\]

**Theorem 5.4.** For \((Y^n(t), t \geq 0)\) as defined in (5.10) and \(y < v < 0\) we have the following convergence in distribution on \(D(\mathbb{R}_+, \mathbb{R})\),
\[
(Y^n((n^{3/4}t) \wedge \tau^n_{\sigma^n_{\nu^+}}), t \geq 0) \to (Z(A(t) \wedge \tau_{\sigma^n_{\nu^+}}, t \geq 0).
\]

**Proof.** We need to consider the passage time of \(v\) for multiple processes, so to clarify notation, in this proof we use \(\tau^n_{\sigma^n_{\nu^+}} = \inf\{s : Y^n(s) \geq v\} \) and \(\sigma^n_{\nu^+} = \inf\{m : Y^n_m \geq v\}. \) An immediate consequence is that \(\tau^n_{\sigma^n_{\nu^+}} \leq T^n_{\sigma^n_{\nu^+}}\). Observe that for all \(m \geq \sigma^n_{\nu^+}\) we have \(nA_n(n^{-3/4} T^n_m) = m\) and, as a result, if \(T^n_{m-1} \leq n^{-3/4} t < T^n_m\) then \(m-1 \leq nA_n(t) < m\). Fix \(S > 0\) and observe that
\[
(5.11) \quad \sup_{0 \leq t \leq S} \left| Y^n((n^{3/4}t) \wedge T^n_{\sigma^n_{\nu^+}}) - Y^n_{[nA_n(t) \wedge \sigma^n_{\nu^+}]} \right| \leq \sup_{m \leq \sigma^n_{\nu^+}} \left| \sin(\Theta_m) \sqrt{\frac{2g|Y^n_{m-1}|}{n}} n^{-1/4} (T^n_m - T^n_{m-1}) + \frac{g}{2\sqrt{n}} (T^n_m - T^n_{m-1})^2 \right|.
\]

From the proof of Theorem 5.3 we know that
\[
(5.12) \quad \left( Y^n_{[nA_n(t) \wedge \sigma^n_{\nu^+}]}(A_n(t), n^{-1} \sigma^n_{\nu^+}, t \geq 0) \right) \rightarrow^d \left( (Z(A(t) \wedge \tau_{\sigma^n_{\nu^+}}), A(t), \tau_{\nu^+}, t \geq 0) \right).
\]

Let
\[
B_n = \left\{ \sup_{m \leq \sigma^n_{\nu^+}} |Y^n_m| \leq M, A_n(S \wedge T^n_{\sigma^n_{\nu^+}}) \leq M, Y^n_{\sigma^n_{\nu^+}} < v + \delta \right\}.
\]

It follows from (5.12) that for every \(p_1 < 1\) there exist \(M > 0\) and \(0 < \delta < |v|\) such that for large \(n\), \(P(B_n) > p_1\). We use (5.11) to conclude that for \(\varepsilon \in (0, 1)\) there exist \(C_1, C_2 > 0\) such that the first of the following inequalities holds and then we use Lemma 3.1 to see that
\[
(5.13) \quad P \left( \sup_{0 \leq t \leq S} \left| Y^n((n^{3/4}t) \wedge T^n_{\sigma^n_{\nu^+}}) - Y^n_{[nA_n(t) \wedge \sigma^n_{\nu^+}]} \right| > \varepsilon, B_n \right) \leq C_1 n \sup_{y \leq v + \delta, \theta \in [-\pi, \pi]} P\left( N^n(y, \theta) > C_2 \varepsilon \right) \leq C_1 n \sup_{y \leq v + \delta, \theta \in [-\pi, \pi]} \frac{\mathbb{E}(N^n(y, \theta)^5)}{(C_2 \varepsilon)^5} \leq C_1 n \frac{C_3 n^{-5/4}}{(C_2 \varepsilon)^5}.
\]

The right hand side goes to 0 as \(n \to \infty\). Recall that \(P(B_n) \to 1\). It follows from (5.12) that
\[
\left( Y^n((n^{3/4}t) \wedge T^n_{\sigma^n_{\nu^+}}), t \geq 0 \right) \rightarrow^d (Z(A(t) \wedge \tau_{\nu^+}), t \geq 0).
\]
Since \( \tau_{v+}^n \leq T_{\sigma_v^+}^n \) the continuity of \( Y^n((n^{3/4}t) \wedge \tau_{v+}^n), t \geq 0 \) and \( (Z(A(t) \wedge \tau_v^+), t \geq 0) \) implies
\[
(Y^n((n^{3/4}t) \wedge \tau_{v+}^n), t \geq 0) \xrightarrow{d} (Z(A(t) \wedge \tau_v^+), t \geq 0).
\]

6. Densities of the form \( h(y) = c|y|^\lambda \)

In this section, we assume that \( h \) has the form \( h_\lambda(y) = |y|^\lambda \) for some \( \lambda \geq 0 \). In this case, the generator of the limiting process has the particularly simple form
\[
A_{h_\lambda} f(y) = \sqrt{2g|y|^\frac{1}{2}-\lambda} \left( \frac{1}{2} f''(y) - \frac{1-2\lambda}{4|y|} f'(y) \right).
\]

Therefore \( A_{h_\lambda} \) can extend to the generator of \(-1\) times a time-changed \((3-2\lambda)/2\)-dimensional Bessel process. We let \( Y^\lambda \) be defined as \( Y \) above, but in the special case where the density is \( h_\lambda \).

**Theorem 6.1.** Suppose \( \lambda \geq 0 \) and let \( (\xi_t, 0 \leq t < \tau_0) \) be a \( \frac{4}{2\lambda+3} \)-dimensional Bessel process run until time \( \tau_0 \) when it hits the origin. If \( Y_0^\lambda = \xi_0 = y \) then we then have the following equality in distribution
\[
Y^\lambda =_d \left( -\left( \frac{(3+2\lambda)(2g)^{1/4}}{4} \xi_t \right)^{\frac{4}{3+2\lambda}}, 0 \leq t < \tau_0 \right).
\]

**Proof.** Let \( s(x) = -\left( \frac{(3+2\lambda)(2g)^{1/4}}{4} x \right)^{\frac{4}{3+2\lambda}}. \) We will show by direct computation that the generator of \( s(\xi) \) agrees with \( A_{h_\lambda} \) on functions compactly supported in \((\infty,0)\).

Let \( \xi \) be a \( \delta \)-dimensional Bessel process started from \( \xi_0 > 0 \) and run until the first time it hits \( 0 \). We see that \( \xi \) is a Feller process and its generator \( \mathcal{G} \) acts on functions on \( f \in C^2 \) with compact support in \((0,\infty)\) by
\[
\mathcal{G} f(x) = \frac{1}{2} f''(x) + \frac{\delta - 1}{2x} f'(x).
\]

For \( x > 0 \), define \( u(x) = bx^c \) where \( b, c > 0 \). We will compute the generator of \( V := u(\xi) \). Suppose that \( f \in C^2 \) has compact support in \((0,\infty)\), so that \( f \circ u \) has compact support in \((0,\infty)\) as well. Let \( E_x \) be the law of \( \xi \) started at \( x > 0 \) and \( E_{y}^V \) be the law of \( V \) started at \( y > 0 \). We then have
\[
\lim_{t \downarrow 0} \frac{E_{y}^V f(V_t) - f(y)}{t} = \lim_{t \downarrow 0} \frac{E_{u^{-1}(y)} f \circ u (\xi_t) - (f \circ u)(u^{-1}(y))}{t} = \mathcal{G}(f \circ u)(u^{-1}(y)).
\]

Noting that
\[
\mathcal{G}(f \circ u)(x) = \frac{u'(x)^2}{2} (f'' \circ u)(x) + \left[ \frac{u''(x)}{2} + \frac{(\delta - 1)u'(x)}{2x} \right] (f' \circ u)(x),
\]
we find that
\[
\mathcal{G}(f \circ u)(u^{-1}(y)) \]
\[
= b^2 c^2 \left( \frac{y}{b} \right)^{2-2c-1} \frac{1}{2} f''(y) + \left[ \frac{1}{2} bc(c-1) \left( \frac{y}{b} \right)^{2-2c-1} \left( \frac{y}{b} \right)^{-1} + \frac{\delta - 1}{2} bc \left( \frac{y}{b} \right)^{2-2c-1} \left( \frac{y}{b} \right)^{-1} \right] f'(y)
\]
\[
= b^2 c^2 \left( \frac{y}{b} \right)^{2-2c-1} \left[ \frac{1}{2} f''(y) + \frac{1}{2y} \left( \frac{c + \delta - 2}{c} \right) f'(y) \right].
\]
Defining \( \tilde{V} = -V = -u(\xi) \), we find that \( \tilde{V} \) is a Feller process on \((-\infty, 0)\) whose generator \( \tilde{G} \) acts on \( f \in C^2 \) with compact support in \((-\infty, 0)\) by

\[
\tilde{G}(f)(y) = b^2 c^2 \left( \frac{|y|}{b} \right)^{2-2c^{-1}} \left[ \frac{1}{2} f''(y) - \frac{1}{2 |y|} \left( \frac{c + \delta - 2}{c} \right) f'(y) \right].
\]

Consequently, \( \tilde{G} \) agrees with \( A_{h,\lambda} \) on smooth functions with compact support in \((-\infty, 0)\) if and only if

\[
b^2 c^2 \left( \frac{|y|}{b} \right)^{2-2c^{-1}} = \sqrt{2g} |y|^{\frac{1}{2} - \lambda} \quad \text{and} \quad \left( \frac{c + \delta - 2}{c} \right) = \frac{1 - 2\lambda}{2}.
\]

Solving for \( b, c, \) and \( \delta \) shows that the generator of \( s(\xi) \) agrees with \( A_{h,\lambda} \) on \( C^2 \) functions with compact support in \((-\infty, 0)\) and this implies that the processes have the same distribution up to the first hitting time of 0.

We remark that this theorem is very similar to [13 Proposition XI.1.11], though the parameters there are restricted because the process continuous past the hitting time of 0 in [13].

Monomial densities also illustrate the complications that arise when trying to look at the process started from 0. The next result shows that the scaling of the time (and, therefore, distance) before the first reflection depends on \( \lambda \).

Let \( N^n_\lambda(0, \theta) \) be defined as in [3.4], relative to \( h(y) = |y|^{\lambda} \). Note that in our model, \( N^n_\lambda(0, \theta) \) is necessarily equal to \( N^n_\lambda(0, -\pi/2) \) because the particle does not have energy for motion in the horizontal direction.

**Proposition 6.2.** For \( \lambda \geq 0 \),

\[
P \left( n^{-\frac{\lambda}{4(\lambda+1)}} N^n_\lambda(0, -\pi/2) > t \right) = \exp \left( -\frac{g^2}{2\lambda+1} t^{2(\lambda+1)} \right),
\]

and if \( Y^n_0 = 0 \), then

\[
E Y^n_1 = -gn^{-\frac{1}{2(\lambda+1)}} \int_0^\infty t \exp \left( -\frac{g^2}{2\lambda+1} t^{2(\lambda+1)} \right) dt.
\]

**Proof.** Computing from the definition we see that

\[
P(N^n_\lambda(0, -\pi/2) > t) = \exp \left( -\int_0^t \sqrt{n} \left| \frac{g}{2\sqrt{n}} s^2 \right|^\lambda \left( \frac{g}{\sqrt{n}} s \right) ds \right) = \exp \left( -\frac{g^2}{2\lambda+1} n^{\lambda/2} t^{2(\lambda+1)} \right),
\]

and the proposition follows.

Suppose that the particle starts from 0. This proposition implies that under our time scaling the time before the first reflection of the process started at 0 goes to \( \infty \), but slower than \( n^{1/4} \).

However, when \( \lambda = 0 \), the size of the first step is of order \( n^{-1/2} \), which is the same order as the first step of the process started at a point in \((-\infty, 0)\). This suggests that when \( \lambda = 0 \), equivalently when \( h \equiv 1 \), we may be able to find a diffusive limit on \((-\infty, 0]\), i.e., when starting at 0 is allowed. In this case, Theorems 3.5 and 4.1 show that the limiting process should be a constant time change of a 3/2-dimensional Bessel process. Indeed this is the case, but to prove it we need techniques that do not seem to extend beyond the \( h \equiv 1 \) (really, \( h \) identically some positive constant). The starting point is the scaling relation in the following proposition.

**Proposition 6.3.** When \( h \equiv 1 \), if \( Y^n_0 = y \) and \( Y^n_1 = \sqrt{ny} \) then \( Y^n_m, m \geq 0 = d (n^{-1/2} Y^1_m, m \geq 0) \) and \( N^n(y, \theta) = d N^1(\sqrt{ny}, \theta) \) for all \( y \) and \( \theta \).

**Proof.** It is easy to verify that \( N^n(y, \theta) \) and \( N^1(\sqrt{ny}, \theta) \) have the same cumulative distribution functions and, as a consequence, the two processes have the same transition operators.
This scaling relation (which fails for general \( h \)), allows us to employ results developed by Lamperti \[6\] \[7\] \[8\]. In particular, we use the following result.

**Theorem 6.4.** Let \((X_m, m \geq 0)\) be a Markov chain on \([0, \infty)\) with transition operator \(T\) and let \(\mu_k(x) = \mathbb{E}[(X_{n+1} - X_n)^k \mid X_n = x]\). Assume:

1. For each fixed \( k \in \mathbb{N} \), \(\mu_k(x)\) is uniformly bounded as a function of \( x \).
2. As \( x \to \infty \), \(x\mu_1(x) \to a\), \(\mu_2(x) \to b > 0\) with \(2a + b > 0\).
3. \(T\) maps the set \(C_0(\mathbb{R}_+, \mathbb{R})\) of continuous functions from \([0, \infty) \to \mathbb{R}\) that vanish at \(\infty\) to itself.
4. \(\mathbb{P} (\lim \sup X_n = \infty \mid X_0 = x) = 1\) for all \( x \in [0, \infty) \).

Under these conditions, regardless of the distribution of \(X_0\),

\[
\left( \frac{1}{\sqrt{n}} X_{[nt]}, t \geq 0 \right) \to_d \left( \rho_{\frac{b+2a}{b}}(bt), t \geq 0 \right)
\]

where the convergence is in distribution on \(D(\mathbb{R}_+, \mathbb{R})\) and \((\rho_{\frac{b+2a}{b}}(t), t \geq 0)\) is a \(\frac{b+2a}{b}\)-dimensional Bessel process started at 0. If \(2a < b\) the process is recurrent while if \(2a > b\) the process is transient.

If \(2a = b\), more refined analysis is needed to determine transience or recurrence.

We remark that by saying \((X_m, m \geq 0)\) is recurrent we mean that there exists \(A \geq 0\) such that \(\mathbb{P}(X_m \in [0, A] \text{ i.o.}) = 1\).

**Proof.** Since \((X_m, m \geq 0)\) is Markov, the claims of recurrence and transience are settled by \[6\] Theorem 3.2]. Assumptions 1, 2, 3, and 4 show that the hypotheses of Theorem 4.1 in \[8\] are satisfied. Combining the conclusions of \[8\] Theorem 4.1 with assumptions 1 and 4 shows that the hypotheses of Theorem 5.1 in \[7\] are satisfied so our claim follows from the conclusion of \[7\] Theorem 5.1.

We remark that the functional limit theorem of \[7\] Theorem 5.1 actually pertains to the scaled linearly interpolated process rather than the scaled step process, and convergence in distribution on \(C(\mathbb{R}_+, \mathbb{R})\), but the convergence of the scaled step process in distribution on \(D(\mathbb{R}_+, \mathbb{R})\) follows immediately.

**Theorem 6.5.** If \(h \equiv 1\) and \(Y_0^n = y\) for some \( y \in (-\infty, 0] \) then \(Y^n\) is recurrent for each \( n \) and

\[
(Y^n_{[nt]}, t \geq 0) \to_d (-\rho_{3/2}(t), t \geq 0),
\]

where the convergence is in distribution on \(D(\mathbb{R}_+, \mathbb{R})\) and \((\rho_{3/2}(t), t \geq 0)\) is a \(3/2\)-dimensional Bessel process started at \(y\).

**Proof.** We need only establish the result for \( y = 0 \). By Proposition \[6.3\], it is equivalent to show that

\[
(n^{-1/2} Y^n_{[nt]}, t \geq 0) \to_d (-\rho_{3/2}(t), t \geq 0),
\]

when \(Y_0^1 = 0\), which in turn is equivalent to showing that

\[
(n^{-1/2} |Y^n_{[nt]}|, t \geq 0) \to_d (\rho_{3/2}(t), t \geq 0).
\]

This we do by checking the conditions of Theorem \[6.4\] for \(|Y^1|\). Maintaining the notation from \[3.21\], we can write the transition operator for \(|Y^1|\) as

\[
T f(x) = \mathbb{E}[f(x - \alpha_1(-x, \Theta))].
\]
(1) Observe that
\[
|\mu_k(x)|^{1/k} = \left| E\left((|Y_{m+1}^1| - |Y_m^1|)^k \mid |Y_m^1| = x\right) \right|^{1/k} \\
\leq \left( E \left[ |\alpha_1(-x, \Theta)|^k \right] \right)^{1/k} \\
\leq \sqrt{2gx} \left( E \left[ N^1(-x, \Theta)^k \right] \right)^{1/k} + \frac{g}{2} \left( E \left[ N^1(-x, \Theta)^{2k} \right] \right)^{1/k}.
\]

Thus, to show that \( \mu_k \) is bounded for \( k \geq 2 \), it is enough that \((\sqrt{x} \lor 1) E[N^1(-x, \Theta)^r]\) is bounded in \( x \) for all \( r \geq 2 \) (the case \( k = 1 \) follows since \( \mu_1(x) \leq \sqrt{\mu_2(x)} \)). That \( E[N^1(0, \Theta)^r] \) is finite for all \( r \) follows from Proposition 5.2, so we may assume \( x > 0 \).

The proof is similar to that of Lemma 3.1 but more delicate since we need bounds for \( x \) near 0.

\[
E[N^1(-x, \Theta)^r] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^\infty rt^{-1} P(N^1(-x, \theta) > t) dt d\theta
\]
\[
\leq \int_0^\infty rt^{-1} P(N^1(-x, \pi/2) > t) dt
\]
\[
= \int_0^{\sqrt{2x/g}} rt^{-1} \exp\left[ -t \left( \sqrt{2gx} - \frac{g}{2} t \right) \right] dt + \int_{\sqrt{2x/g}}^\infty rt^{-1} \exp\left[ -t \left( \frac{g}{2} t - \sqrt{2gx} \right) - 2x \right] dt.
\]

For the first integral, we have
\[
\int_0^{\sqrt{2x/g}} rt^{-1} \exp\left[ -t \left( \sqrt{2gx} - \frac{g}{2} t \right) \right] dt \leq \left( \frac{2x}{g} \right)^{r/2} \wedge \int_0^\infty rt^{-1} \exp\left[ -t \frac{\sqrt{2gx}}{2} \right] dt
\]
\[
= \min\left\{ \left( \frac{2x}{g} \right)^{r/2}, \frac{2^{r/2}(r!)}{(gx)^{r/2}} \right\}.
\]

Since \( r \geq 2 \),
\[
(\sqrt{x} \lor 1) \min\left\{ \left( \frac{2x}{g} \right)^{r/2}, \frac{2^{r/2}(r!)}{(gx)^{r/2}} \right\}
\]

is bounded in \( x \). The second integral we further decompose
\[
\int_{\sqrt{2x/g}}^\infty rt^{-1} \exp\left[ -t \left( \frac{g}{2} t - \sqrt{2gx} \right) - 2x \right] dt = \int_{\sqrt{2x/g}}^{1/4\sqrt{2x/g}} rt^{-1} \exp\left[ -t \left( \frac{g}{2} t - \sqrt{2gx} \right) - 2x \right] dt
\]
\[
+ \int_{1/4\sqrt{2x/g}}^\infty rt^{-1} \exp\left[ -t \left( \frac{g}{2} t - \sqrt{2gx} \right) - 2x \right] dt.
\]

Since \( -t \left( \frac{g}{2} t - \sqrt{2gx} \right) - 2x \leq -x \) for all \( t \), we have
\[
\int_{1/4\sqrt{2x/g}}^{1/4\sqrt{2x/g}} rt^{-1} \exp\left[ -t \left( \frac{g}{2} t - \sqrt{2gx} \right) - 2x \right] dt \leq \exp(-x) \left[ 1 \lor \left( \frac{32x}{g} \right)^{r/2} - \left( \frac{2x}{g} \right)^{r/2} \right],
\]
from which it follows that
\[
(\sqrt{x} \lor 1) \int_{1/4\sqrt{2x/g}}^{1/4\sqrt{2x/g}} rt^{-1} \exp\left[ -t \left( \frac{g}{2} t - \sqrt{2gx} \right) - 2x \right] dt
\]
is bounded in $x$. Finally,
\[
\int_{1/4 \sqrt{2x/g}}^{\infty} t^{r-1} \exp \left[ -t \left( \frac{g}{2} t - \sqrt{2gx} \right) - 2x \right] dt \\
\leq \exp(-2x) \int_{0}^{\infty} t^{r-1} \exp \left[ -t \left( \frac{1}{2} \sqrt{2gx} - \sqrt{2gx} \right) \right] dt \\
= \exp(-2x) \frac{r!}{\left[ \frac{1}{2} \sqrt{2gx} - \sqrt{2gx} \right]^r}.
\]

Since $\left( \frac{1}{2} \sqrt{2gx} - \sqrt{2gx} \right)^r$ is bounded away from 0 for small $x$ and grows like $x^{r/2}$ as $x \to \infty$, it follows that
\[
(\sqrt{x} \lor 1) \int_{1/4 \sqrt{2x/g}}^{\infty} t^{r-1} \exp \left[ -t \left( \frac{g}{2} t - \sqrt{2gx} \right) - 2x \right] dt
\]
is bounded in $x$ for $r \geq 2$. This shows that Assumption 1 of Theorem 6.4 is satisfied.

(2) It follows from Proposition 6.3 that $\alpha_n(y, \Theta) = d \sqrt{ny, \Theta}$. Thus the results of the computation in the proofs of Theorems 3.3 and 4.1 is that
\[
\lim_{x \to \infty} |x| E \left( |Y_{m+1}^1| - |Y_m^1| \mid |Y_m^1| = x \right) = \frac{1}{4} \quad \text{and} \quad \lim_{x \to \infty} E \left( (|Y_{m+1}^1| - |Y_m^1|)^2 \mid |Y_m^1| = x \right) = 1.
\]

We remark that these two limits are easy to establish directly, but since we have already done the work for Theorems 3.3 and 4.1 we leave the direct calculation to the interested reader.

(3) It is then immediate from the cumulative distribution functions that $y \mapsto N^1(y, \Theta)$ is continuous in distribution, which shows that, if $f$ is bounded and continuous on $[0, \infty)$, then so is $Tf(x)$. The fact that $Tf(x)$ vanishes at $\infty$ if $f$ does follows from (3.20), using $x^* = \sup\{x : f(x) > \varepsilon/2\}$ in place of $y_\ast$. This shows that Assumption 3 of Theorem 6.4 is satisfied.

(4) One way to see that Assumption 4 of Theorem 6.4 is satisfied is to observe that for each $M < 0$,
\[
\inf_{M \leq y \leq 0} P(Y_1 \leq M \mid Y_0 = y) > 0. \quad \Box
\]

7. Higher Dimensions

Much of our analysis extends to the model in dimensions higher than 2 without difficulty. We consider our particle in $\mathbb{R}^d$ with $d \geq 1$ and gravitation acting in the $-x_d$ direction. Let $S^{d-1}$ be the unit sphere in $\mathbb{R}^d$ and let $e_1, \ldots, e_d$ be the standard basis for $\mathbb{R}^d$. In this model, upon reflection at $x = (x_1, x_2, \ldots, x_d)$, the particle starts its path in a uniform direction $u = (u_1, u_2, \ldots, u_d) \in S^{d-1}$ and then travels along the gravitational parabola
\[
\{ \Lambda(x, u, t) : = \sum_{i=1}^{d-1} \left( x_i + u_i \sqrt{2g|x_d|}t \right) e_i + \left( x_d + u_d \sqrt{2g|x_d|}t - \frac{gt^2}{2} \right) e_d, t \geq 0 \}.
\]

In this case, the transition operator for the Markov chain $(Y_m^1, m \geq 0)$ of $x_d$-coordinates of the particle at reflection times is given by
\[
U^n f(y) = E \left[ y + U_d \sqrt{2g|y|n^{-1/4}} N^n(y, U) - \frac{g}{2 \sqrt{n}} N^n(y, U)^2 \right]
\]
where \( U \) is uniform on \( S^{d-1} \) and conditionally given that \( U = u \),

\[
(7.2) \quad P(N^n(y, u) > t) = \exp \left[ -\int_0^t \sqrt{n} h \left( y + u_d \sqrt{2g|y|n^{-1/4} s - \frac{g}{2\sqrt{n}s^2}} \right) \right.
\]

\[
\times \sqrt{2g|y|n^{-1/2}(1 - u_d^2)} + \left( \sqrt{2g|y|u_d n^{-1/4} - \frac{g}{\sqrt{n}}} \right)^2 \right] ds .
\]

Since, given \( u \in S^{d-1} \), there exists \( \theta \in [-\pi, \pi] \) such that \( (1 - u_d^2, u_d) = (\cos(\theta)^2, \sin(\theta)) \), there exists \( \theta \) such that \( N^n(y, u) = N^n(y, \theta) \). Consequently, Lemmas 3.1, 3.2, 3.3, and 3.4 immediately extend to the current setting. The statement of Lemma 3.4 changes slightly, so we record the new statement.

**Lemma 7.1.** If \( A \subseteq (-\infty, 0) \) is compact then

\[
\lim_{n \to \infty} \sup_{(y,u) \in A \times S^{d-1}} \left| n^{3/4} \mathbb{E} \left[ N^n(y, u) - N^n(y, -u) \right] - \frac{2gu_d(h(y) - 2|y|h'(y))}{h(y)^3(2g|y|)^{3/2}} \right| = 0 .
\]

Using these extended Lemmas, we obtain an extension of Theorem 3.5. The proof is essentially the same, but the constants change because we are integrating \( u \) over \( S^{d-1} \) instead of integrating \( \theta \) over \([-\pi, \pi]\). More precisely, if \( u = (u_1, u_2, \ldots, u_d) \) is uniformly distributed on \( S^{d-1} \), then \( \mathbb{E} u_d = 1/d \). In dimension \( d = 2 \), we had parameterized \( u = (\cos(\theta), \sin(\theta)) \), so in the equations in Section 3, \( \mathbb{E} \sin(\theta)^2 \) is getting replaced by \( \mathbb{E} u_d^2 \). The calculations that are affected by this change are (3.22) and (3.23).

**Theorem 7.2.** If \( f : (-\infty, 0) \to \mathbb{R} \) is \( C^2 \) and has compact support then

\[
\lim_{n \to \infty} \sup_{y \in (-\infty, 0)} \left| n(U^n - I)f(y) - \bar{A}_{h,d}f(y) \right| = 0 ,
\]

where \( U^n \) is defined in (7.1) and \( \bar{A}_{h,d}f \) is defined by

\[
(7.3) \quad \bar{A}_{h,d}f(y) = \frac{1}{dh(y)^2} f''(y) + \frac{1}{dh(y)^2} \left( \frac{d - 1}{2|y|} + \frac{h'(y)}{h(y)} \right) f'(y) .
\]

As in the two dimensional case we can explicitly identify the scale function and speed measure of the limiting diffusion.

**Proposition 7.3.** If \( (Z_t, t \geq 0) \) is a Feller diffusion on \((\mathbb{R}^d, 0) \) whose generator extends \( \bar{A}_{h,d}f(y) \) then for \( y \in (-\infty, 0) \) the scale function \( G \) and speed measure \( m \) for \( Z \) are given by

\[
G(y) = \int_{-1}^y \frac{h(u)}{|u|^{(d-1)/2}} du \quad \text{and} \quad m(dy) = dh(y)|y|^{(d-1)/2}dy .
\]

Moreover, \(-\infty\) is inaccessible and \( P(\lim_{t \to \infty} Z_t = -\infty) = 0 \) if \( d = 3 \) while for every \( d \geq 4 \), \( P(\lim_{t \to \infty} Z_t = -\infty) \) is equal to 0 for some \( h \) and is strictly positive for some other \( h \).

**Proof.** The proof is based on results from [4] and [13] cited in the proof of Proposition 3.6. Define

\[
\kappa(y) = \int_{-1}^y \left( \int_{-1}^u \frac{h(u)}{|u|^{(d-1)/2}h(s)ds} \right) \frac{h(u)}{|u|^{(d-1)/2}} du .
\]

Observe that \( \limsup_{y \to -\infty} G(y) < 0 \). Since \( \inf_{y \leq -1} h(y) > 0 \), it follows easily that \( \lim_{y \to -\infty} \kappa(y) = \infty \). Thus \(-\infty\) is inaccessible. Note that \( \lim_{y \to -\infty} G(y) = -\infty \) if \( d \leq 3 \), which shows that \( P(\lim_{t \to \infty} Z_t = -\infty) = 0 \) in this case. For \( d \geq 4 \), observe that \( \lim_{y \to -\infty} G(y) > -\infty \) if \( h \equiv 1 \) so for this \( h \) we have \( P(\lim_{t \to \infty} Z_t = -\infty) > 0 \). However, if \( h(y) = |y|^{\beta} \) for \( \beta > (d - 3)/2 \), then \( \lim_{y \to -\infty} G(y) = -\infty \) and, consequently, \( P(\lim_{t \to \infty} Z_t = -\infty) = 0 \). □
Proposition 7.4. If \( h(y) = |y|^\lambda \) then \((Z_t, t \geq 0)\) is neighborhood recurrent if and only if \( \lambda \geq (d-3)/2 \). In particular, when \( h \equiv 1 \), \((Z_t, t \geq 0)\) is neighborhood recurrent only in dimensions 1 and 2.

Proof. If \( h(y) = |y|^\lambda \) then \((\mathbb{P}, \mathbb{F})\) becomes
\[
\mathcal{A}_{h,d} f(y) = \frac{2}{d} |y|^{-2\lambda} \left[ \frac{1}{2} f''(y) - \frac{d-1-2\lambda}{4|y|} f'(y) \right].
\]
This is the generator of time changed Bessel process with dimension \((d-1-2\lambda)/2 + 1\). Bessel process is neighborhood recurrent if and only if its dimension is less than or equal to 2. Hence, \((Z_t, t \geq 0)\) is neighborhood recurrent if and only if \( \lambda \geq (d-3)/2 \).

When \( \mathbb{P}(\lim_{t \to \infty} Z_t = -\infty) = 0 \), our limit theorems extend to the \( d \)-dimensional case without difficulty, but when \( \mathbb{P}(\lim_{t \to \infty} Z_t = -\infty) > 0 \), further analysis is needed. In particular, we need to show that the time change that transforms \( Z \) into the limiting process on the natural time scale goes to infinity, even on \( \{\lim_{t \to \infty} Z_t = -\infty\} \).

Proposition 7.5. For all \( d \) and for all \( h \) satisfying our hypotheses we have that for all \( y < v < 0 \),
\[
\mathbb{P} \left( \lim_{t \to \infty} \int_0^t \frac{1}{\sqrt{2gd}|Z(s \wedge \tau_v)|} h(Z(s \wedge \tau_v)) \, ds = \infty \right) = 1,
\]
where \( Z_0 = y \).

Proof. The result is trivial on the set where \((Z(t \wedge \tau_v), t \geq 0)\) is absorbed at \( v \). If \( G(-\infty) = -\infty \), then \((Z(t \wedge \tau_v), t \geq 0)\) is absorbed at \( v \) with probability 1, so we may assume that \( G(-\infty) \) is finite. Note that this implies that \( d > 3 \). In this case there exists \( C > 0 \) such that for all \( y < -1 \)
\[
(G(y) - G(-\infty)) \left( \frac{1}{\sqrt{2gd}|y|} \frac{dm}{dy}(y) \right) \geq C \sqrt{|y|}.
\]
Since \( \int_{-\infty}^{y} \sqrt{|u|} \, du = \infty \) for all \( y \in \mathbb{R} \), the result is a direct application of [11] Theorem 2.11. \( \square \)

Combining this with our previous analysis, we obtain the following result for the full path of the particle on its natural time scale, where the notation is the obvious extension of the notation from Section 5.

Theorem 7.6. Define \((Y^n(t), t \geq 0)\) by
\[
Y^n(t) = Y^n_{m-1} + U_d \sqrt{2gd} |Y^n_{m-1}| n^{-1/4} (t - T^n_{m-1}) - \frac{g}{2\sqrt{n}} (t - T^n_{m-1})^2 \quad \text{on} \quad T^n_{m-1} \leq t < T^n_m,
\]
and \(Y^n(0) = y < v < 0\). We then have the following convergence in distribution on \( D(\mathbb{R}_+, \mathbb{R}) \),
\[
(Y^n((n^{3/4}t) \wedge \tau^n_{\tau_v}), \ t \geq 0) \to (Z(A(t) \wedge \tau_v), \ t \geq 0).
\]
If \( \mathbb{P}(\lim_{t \to \infty} Z_t = -\infty) = 1 \), the cutoff at \( v \) can be removed.

8. A different rescaling for densities of the form \( h(y) = c|y|^\lambda \)

In this section we take \( h \) of the form \( h(y) = c|y|^\lambda \) for some \( \lambda \geq 0 \) and \( c > 0 \). We consider the process in any dimension \( d \geq 1 \). We prove some limit theorems and results on transience and recurrence. The difference with the earlier sections is that we do not “rescale h.” It is possible to avoid rescaling and nevertheless obtain limit theorems because of the special form of \( h \).

We write \( \mathcal{N}(y, u) \) instead of \( \mathcal{N}^1(y, u) \) whose distribution is defined in [7,2].
Lemma 8.1. For every $t \geq 0$ we have

$$\lim_{y \to -\infty} \sup_{u \in \mathbb{S}^d} \left| \mathbb{P} \left( \sqrt{2g|y|h(y)} N(y, u) > t \right) - e^{-t} \right| = 0.$$ 

Proof. Observe that

$$- \log \left( \mathbb{P} \left( \sqrt{2g|y|h(y)} N(y, u) > t \right) \right) = \int_0^t h \left( y + \frac{u_d}{h(y)} s - \frac{1}{4|y|h(y)^2} s^2 \right) \mathbb{E} \left[ \sqrt{2g|y|h(y)} \left( \sqrt{2g|y|h(y)} - g_s \right)^2 ds \right]$$

and the lemma follows. \hfill \square

In fact, this convergence in distribution can be extended to convergence of moments.

Lemma 8.2. For fixed $p \geq 1$,

$$\mathbb{E} \left[ \left( \max \{ \sqrt{2g|y|h(y)}, 1 \} N(y, u) \right)^p \right]$$

is bounded uniformly in $y$ and $u \in \mathbb{S}^d$.

Proof. This is similar to the proof of Lemma 3.1 but more delicate. We handle the cases $y \leq -1$ and $y > -1$ separately. For $-1 \leq y \leq 0$ there is a finite longest time for a parabolic path started with $-1 \leq y \leq 0$ to leave $[-1, 0]$. Outside this interval $h$ is bounded below by a strictly positive constant. Hence, once the particle is outside $[-1, 0]$, it will encounter a scatterer at some strictly positive rate. This implies that all of the $N(y, u)$ with $-1 \leq y \leq 0$ are stochastically dominated by a single random variable with an exponential tail. The lemma easily follows in this case.

We now turn to the case $y \leq -1$. Fix $0 < \varepsilon < 1/4$ and let $\delta = h(-\varepsilon)$ and define $s_\pm$ as in Lemma 3.1. From (3.5) and (3.9), we find that

$$\mathbb{E}(N(y, u)^p) \leq p \int_{\delta}^{s_+} t^{p-1} \exp \left[ -\int_0^t c \left( \frac{g}{2} s^2 - \sqrt{2g|y|} s - y \right)^2 \left( \sqrt{2g|y|} - g_s \right) ds \right] dt$$

and the first integral is the only one that presents new difficulties, so we take care of the second and third integrals first. Arguing as is (3.11), we have

$$\lim_{y \to -\infty} p \left( \sqrt{2g|y|h(y)} \right)^p \int_{s_+}^{4s_+} t^{p-1} \exp \left[ -\delta s_\pm \left( \sqrt{2g|y|} - \frac{g}{2} s_\pm \right) \right] dt = 0.$$
Proof. We continue with our notation from Lemma 3.4. In particular, we let

\[ H(y, u, t) = h \left( y + u_d \sqrt{2g|y|t - \frac{g}{2} t^2} \right), \]

\[ \gamma(y, u, t) = \sqrt{2g|y|(1 - u_d^2) + \left( u_d \sqrt{2g|y|} - gt \right)^2}, \]

\[ F(y, u, t) = \int_0^t H(y, u, s) \gamma(y, u, s) ds. \]

We these definitions (3.1) and (3.2) become

\begin{align*}
1 &= \mathbb{E}[F(y, u, N(y, u))], \quad 2 = \mathbb{E}[F(y, u, N^n(y, u))] \\
&= \mathbb{E}[F(y, u, N^m(y, u))] \\
&= \mathbb{E}[\delta \left( \sqrt{2g|y| \left[ s_+ + s_+ - t \right] + \frac{g}{2} \left[ t^2 - s_-^2 - s_+^2 \right] \right)] dt = 0
\end{align*}

uniformly in \( u \) since the integral term decays exponentially. Similarly we have

\[ \lim_{y \to -\infty} p \left( \sqrt{2g|y|h(y)} \right)^p \int_{4s}^{\infty} t^{p-1} \exp \left[ -\delta \left( \sqrt{2g|y| \left[ s_+ + s_+ - t \right] + \frac{g}{2} \left[ t^2 - s_-^2 - s_+^2 \right] \right) \right] dt = 0
\]

uniformly in \( u \) by (3.12). For the first integral, observe that for \( 0 \leq t \leq s_- \),

\[ -\int_0^t c \left( \frac{t^2}{2} - \sqrt{2g|y|s - y} \right)^{\lambda} \left( \sqrt{2g|y| - gs} \right) ds = \frac{c}{\lambda + 1} \left[ \left( \frac{t^2}{2} - \sqrt{2g|y|t - y} \right)^{\lambda+1} - |y|^{\lambda+1} \right] \]

\[ \leq \frac{c}{\lambda + 1} \left[ -\left( \frac{\sqrt{2g|y|}{2} - y \right)^{\lambda+1} - |y|^{\lambda+1} \right] \]

\[ \leq -\frac{c\sqrt{2g|y|}{2}}{2} |y|^\lambda t,
\]

with the last inequality being by the Mean Value Theorem. Consequently, we have

\[ p \int_{0}^{s_-} t^{p-1} \exp \left[ -\int_0^t c \left( \frac{t^2}{2} - \sqrt{2g|y|s - y} \right)^{\lambda} \left( \sqrt{2g|y| - gs} \right) ds \right] dt \]

\[ \leq p \int_{0}^{\infty} t^{p-1} \exp \left( -\frac{c\sqrt{2g|y|}{2}}{2} |y|^\lambda t \right) = \frac{2^p (p!)}{c^p (2g|y|)^{p/2} |y|^{p/2}}.
\]

which proves the result for \( y \leq -1 \).

\[ \square \]

Lemma 8.3. For every \( p \geq 1 \) we have

\[ \lim_{y \to -\infty} \sup_{u \in \mathbb{S}^d} \left| \mathbb{E} \left[ \left( \sqrt{2g|y|h(y)N(y, u)} \right)^p \right] \right| - \int_{0}^{\infty} pt^{p-1} e^{-t} dt = 0.
\]

Proof. It follows immediately from Lemma 8.1 that for every \( r \geq 0 \)

\[ \lim_{y \to -\infty} \sup_{u \in \mathbb{S}^d} \left| \mathbb{E} \left[ N(y, u) - N(y, -u) \right] - u_d (1 - 2\lambda) \right| = 0
\]

and the proof now follows as in Lemma 3.3.

\[ \square \]

Lemma 8.4.

\[ \lim_{y \to -\infty} \sup_{u \in \mathbb{S}^d} \left| (h(y)^2 \sqrt{2g|y|^2}) \mathbb{E} [N(y, u) - N(y, -u)] - u_d (1 - 2\lambda) \right| = 0.
\]

Proof. We continue with our notation from Lemma 3.4. In particular, we let

\[ \gamma(y, u, t) = \sqrt{2g|y|(1 - u_d^2) + \left( u_d \sqrt{2g|y|} - gt \right)^2}, \]

\[ F(y, u, t) = \int_0^t H(y, u, s) \gamma(y, u, s) ds. \]

Taylor expanding \( F_n \) in \( t \) about 0, we find that for \( t < (2|y|/g)^{1/2} \),

\[ F(y, u, t) = \mathbb{E}[F(y, u, N(y, u))] + 2 = \mathbb{E}[F(y, u, N^n(y, u))]^2.
\]

Taylor expanding \( F_n \) in \( t \) about 0, we find that for \( t < (2|y|/g)^{1/2} \),

\[ F(y, u, t) = h(y) \sqrt{2g|y|t} + F''(y, u, T(y, u, t)) t^2 \]

and the proof now follows as in Lemma 3.3.

\[ \square \]
for some $0 \leq T(y, u, t) \leq t$. This time, we take $B(y, u) = \{N(y, u) < 1\}$. We then have
\[
1 = E[F(y, u, N(y, u))] = E[F(y, u, N(y, u))1_B] + E[F(y, u, N(y, u))1_{B^c}]
\]
and, by the Cauchy-Schwarz inequality and (8.1),
\[
E[F(y, u, N(y, u))1_{B^c}] \leq \sqrt{2}P(N(y, u) \geq 1).
\]
By Lemma 8.2 we see that for every $r \geq 0$
\[
\lim_{y \to -\infty} \sup_{u \in S^d} |y|^r P(N(y, u) \geq 1) = 0.
\]
Consequently
\[
\lim_{y \to -\infty} \sup_{u \in S^d} |y|^r E[F(y, u, N(y, u))1_{B^c}] = 0.
\]
Similarly, for every $r \geq 0$ and $p \geq 1$ we see that
\[
\lim_{y \to -\infty} \sup_{u \in S^d} |y|^r E[N(y, u)^p1_{B^c}] = 0.
\]
For $y$ such that $y \leq -g/2$, substituting (8.2) into the first integral on the right hand side of (8.3) and solving for $E(N(y, u)1_B)$ yields
\[
E(N(y, u)1_B) = \frac{1}{h(y)\sqrt{2g|y|}} \left( 1 - E[F(y, u, N(y, u))1_{B^c}] - \frac{1}{2} E[F''(y, u, T(y, u, N(y, u)))]N(y, \theta)^2 1_B \right).
\]
Consequently we have
\[
\left( h(y)^2 \sqrt{2g|y|^2} \right) [E(N(y, u)1_{B(u)}) - E(N(y, -u)1_{B(-u)})]
\]
\[
= \frac{|y|h(y)}{2} E[F(y, -u, N(y, -u))1_{B^c(-u)}] - \frac{|y|h(y)}{2} E[F(y, u, N(y, u))1_{B^c(u)}] - \frac{|y|h(y)}{2} E[F''(y, u, T(y, u, N(y, u)))N(y, u)^2 1_{B(u)}] + \frac{|y|h(y)}{2} E[F''(y, -u, T(y, -u, N(y, -u)))N(y, -u)^2 1_{B(-u)}].
\]
The first two terms on the right hand side vanish as $y \to -\infty$ by (8.4). For the last two terms, observe that
\[
F''(y, u, t) = H'(y, u, t)\gamma(y, u, t) + H(y, u, t)\gamma'(y, u, t)
\]
\[
= \left( u_d\sqrt{2g|y|} - gt \right) h' \left( y + u_d\sqrt{2g|y|}t - \frac{g}{2} t^2 \right) \gamma(y, u, t)
\]
\[
- h \left( y + u_d\sqrt{2g|y|}t - \frac{g}{2} t^2 \right) \frac{g \left( \sqrt{2g|y|u_d - gt} \right)}{\sqrt{2g|y|(1-u_d^2) + \left( \sqrt{2g|y|u_d - gt} \right)^2}}.
\]
Elementary calculations show that
\[
\lim_{y \to -\infty} \sup_{(t, u) \in [0,1] \times S^d} \left| \frac{F''(y, u, t)}{h(y)} - gu_d(2\lambda - 1) \right| = 0.
\]
Therefore, a combination of Lemma 8.3 and (8.5) shows that these last two terms both converge to $u_d(1 - 2\lambda)/2$ uniformly in $u \in S^d$. The lemma follows by combining this with (8.5). \qed
Proposition 8.5. Let \((Y_m, m \geq 0)\) be the Markov chain corresponding to \(h(y) = c|y|^\lambda\). For \(x \geq 0\), define
\[
\mu_k(x) = \mathbb{E} \left[ (|Y_1| - x)^k \right\} Y_0 = -x]}
We then have
\[
\lim_{x \to \infty} x h(-x)^2 \mu_1(x) = \frac{d + 2\lambda - 1}{2d} \quad \text{and} \quad \lim_{x \to \infty} h(-x)^2 \mu_2(x) = \frac{2}{d}.
\]

Proof. Observe that
\[
xh(-x)^2 \mu_1(x) = \mathbb{E} \left( \frac{gxh(-x)^2}{2} N(-x, U)^2 - U_d \sqrt{2gxh(-x)^2} N(-x, U) \right).
\]
Lemma 8.3 implies that
\[
\lim_{x \to \infty} \mathbb{E} \left( \frac{gxh(-x)^2}{2} N(-x, U)^2 \right) = \frac{1}{2}.
\]
Moreover, since \(U = d - U\), using Lemma 8.4,
\[
\mathbb{E} \left( U_d \sqrt{2gxh(-x)^2} N(-x, U) \right) = \frac{1}{2} \mathbb{E} \left( U_d \sqrt{2gxh(-x)^2} (N(-x, U) - N(-x, -U)) \right) \to \frac{1 - 2\lambda}{2d}.
\]
Consequently,
\[
\lim_{x \to \infty} xh(-x)^2 \mu_1(x) = \frac{1}{2} + \frac{2\lambda - 1}{2d} = \frac{d + 2\lambda - 1}{2d}.
\]

Similarly, we see that
\[
\mu_2(x) = \mathbb{E} \left[ \left( \frac{g}{2} N(-x, U)^2 - U_d \sqrt{2gxN(-x, U)} \right)^2 \right]
= \mathbb{E} \left[ 2gxU_d^2 N(-x, U)^2 - gU_d \sqrt{2gxN(-x, U)^3} + \frac{g^2}{4} N(-x, U)^4 \right].
\]
It follows from Lemma 8.3 that \(h(-x)^2 \mu_2(x) \to \frac{2}{d}\) as \(x \to \infty\).

Proposition 8.6. Let \((Y_m, m \geq 0)\) be the Markov chain corresponding to \(h(y) = c|y|^\lambda\) in dimension \(d \geq 1\). Define \(f : (-\infty, 0] \to [0, \infty)\) by \(f(y) = c|y|^{\lambda+1}/(\lambda + 1)\) and for \(x \geq 0\), let
\[
\bar{\mu}_k(x) = \mathbb{E} \left[ (f(Y_1) - x)^k \right\} Y_0 = f^{-1}(x)]\}
We then have that \(\sup_x \bar{\mu}_k(x) < \infty\) for all \(k \geq 1\),
\[
\lim_{x \to \infty} x \bar{\mu}_1(x) = \frac{d + 2\lambda - 1}{2d(1 + \lambda)} \quad \text{and} \quad \lim_{x \to \infty} \bar{\mu}_2(x) = \frac{2}{d}.
\]

Proof. Let \(y = f^{-1}(x)\) and note that, by the Mean Value Theorem,
\[
\bar{\mu}_k(x) = \mathbb{E} \left[ \frac{c^k}{(1 + \lambda)^k} \left( \left( \frac{g}{2} N(y, U)^2 - U_d \sqrt{2g|y|N(y, U)} - y \right)^{\lambda+1} - |y|^{\lambda+1} \right) \right]^k
= c^k \mathbb{E} \left[ \left( \frac{g}{2} N(y, U)^2 - U_d \sqrt{2g|y|N(y, U)} \right)^k |y + \delta(y)|^{k\lambda} \right]
= h(y)^k \mathbb{E} \left[ \left( \frac{g}{2} N(y, U)^2 - U_d \sqrt{2g|y|N(y, U)} \right)^k \left| 1 + \frac{\delta(y)}{y} \right|^{k\lambda} \right],
\]
(8.6)
for some \(\delta(y)\) (not depending on \(k\)) with
\[
|\delta(y)| \leq \frac{g}{2} N(y, U)^2 - U_d \sqrt{2g|y|N(y, U)}.
\]
Letting $B = \{N(y, U) < 1\}$, it follows from (8.5) that for every $r \geq 0$,
\[ \lim_{y \to -\infty} |y|^r \mathbb{E} \left[ \left( \frac{g}{2} N(y, U)^2 - U_d \sqrt{2g|y|N(y, U)} \right)^k \left| 1 + \delta(y) \right|^k \mathbb{1}_{B^c} \right] = 0. \]
In particular, this is true for $r = k\lambda$. One can use the fact that
\[ \lim_{y \to -\infty} \text{ess sup} \left| 1 + \frac{\delta(y)}{y} \right| 2^\lambda \mathbb{1}_{B} - \mathbb{1}_{B} = 0, \]
and Lemma 8.2 to show that $\sup_x \tilde{\mu}_k(x) < \infty$. We use (8.5) to see that
\[ \lim_{y \to -\infty} h(y)^2 \mathbb{E} \left[ \left( \frac{g}{2} N(y, U)^2 - U_d \sqrt{2g|y|N(y, U)} \right)^2 \mathbb{1}_{B^c} \right] = 0, \]
and it follows from Proposition 8.5 and (8.6) that
\[ \lim_{x \to \infty} \tilde{\mu}_2(x) = \lim_{x \to \infty} h(-x)^2 \mu_2(x) = \frac{2}{d}. \]
Note also that $f(y) = (1 + \lambda)^{-1}|y|h(y)$, so that
\[ x\tilde{\mu}_1(x) = \frac{|y|h(y)^2}{1 + \lambda} \mathbb{E} \left[ \left( \frac{g}{2} N(y, U)^2 - U_d \sqrt{2g|y|N(y, U)} \right) \left| 1 + \frac{\delta(y)}{y} \right|^\lambda \right]. \]
The same argument handles the $|1 + \delta(y)/y|^\lambda$ factor, showing that
\[ \lim_{x \to \infty} x\tilde{\mu}_1(x) = \lim_{x \to \infty} x\tilde{\mu}_1(x) = \frac{d + 2\lambda - 1}{2d(1 + \lambda)}. \]

**Theorem 8.7.** Let $(Y_m, m \geq 0)$ be the Markov chain corresponding to $h(y) = c|y|^\lambda$, starting from $Y_0 = 0$, in dimension $d \geq 1$. The process $(Y_m, m \geq 0)$ is recurrent if $d \leq 2$ and transient if $d \geq 4$. Moreover, defining $f(y) = c|y|^{\lambda+1}/(1 + \lambda)$, we have that
\[ \left( \frac{1}{\sqrt{n}} f(Y_{[nt]}), t \geq 0 \right) \to_d \left( \rho_{d+1+4\lambda} \frac{2t}{d}, t \geq 0 \right), \]
where the convergence is in distribution on $D(\mathbb{R}_+, \mathbb{R})$ and $(\rho_{(d+1+4\lambda)/(2+2\lambda)}(t), t \geq 0)$ is a $d\frac{d+1+4\lambda}{2(1+\lambda)}$-dimensional Bessel process started from 0. Equivalently, we have
\[ \left( n^{-\frac{1}{2(1+\lambda)}} Y_{[nt]}, t \geq 0 \right) \to_d \left( \tilde{Z}^\lambda_t, t \geq 0 \right), \]
where $\tilde{Z}^\lambda_t = f^{-1} \left( \rho_{d+1+4\lambda} \frac{2t}{d} \right)$, with the convention that $f^{-1}(\cdot) \in (-\infty, 0]$.

**Proof.** By Proposition 8.6 these results are a consequence of Theorem 6.4 for the process $(f(Y_m), m \geq 0)$, and the results for $(Y_m, m \geq 0)$ follow by inverting $f$. \hfill \Box

**Remark 8.8.** (i) The fact that the results do not depend on $c$ or $g$ implies that, for fixed $n$, the results are true for the process $(Y_{[nt]}^n, m \geq 0)$ in the $n$-rescaled dynamics (i.e. with the transition operator $U^n$ in (7.1)).

(ii) The process $\rho_{(d+1+4\lambda)/(2+2\lambda)}$ is neighborhood recurrent if and only if its dimension $d\frac{d+1+4\lambda}{2(1+\lambda)}$ is less than or equal to 2, i.e., if and only if $d \leq 3$. Since $\tilde{Z}^\lambda$ is a deterministic transform of that process by the function $f^{-1}$, the process $\tilde{Z}^\lambda$ is also neighborhood recurrent if and only if $d \leq 3$. Hence the transience or recurrence of $(Y_m, m \geq 0)$ and that of its scaling limit depend only on the dimension $d$ (except possibly in the case $d = 3$). This stands in contrast to the limit of the process in rescaled dynamics (that is, the process defined in Proposition 7.3 where transience, recurrence, and the accessibility of $-\infty$ depend on both $\lambda$ and $d$ (see Propositions 7.3 and 7.4).
We now turn to the time scale. As before, it is difficult to account for how time scales when the velocity of the particle is near 0, but we can obtain results for when the velocity of the particle is large.

**Theorem 8.9.** Consider the Markov chain \(((Y_m, \Delta m), m \geq 0)\) started from \((0,0)\) with transition operator

\[
\tilde{U} f(y, z) = E \left[ f \left( y + U_d \sqrt{2g/|y|} N(y, U) - \frac{g}{2\sqrt{n}} N(y, U)^2, \ N(y, U) \right) \right]
\]

Fix \(\varepsilon > 0\) and for \(m \in \mathbb{N}\), let \(T_{m, \varepsilon} = \sum_{j=0}^{m} \Delta_j \mathbb{1}_{\{Y_j \leq -\varepsilon^n^{1/(2+2\lambda)}\}}\) be the time of the \(m\)’th collision of the particle below level \(-\varepsilon^n^{1/(2+2\lambda)}\). We extend \(T_{m, \varepsilon}\) to \(\mathbb{R}_+\) by linear interpolation. We have the joint convergence in distribution

\[
\left( \left( n^{-1/(2+2\lambda)} Y_{[sn]} - n^{-1/(2+2\lambda)} T_{m, \varepsilon}^{sn, m}, s, t \geq 0 \right) \rightarrow_d \left( \left( \tilde{Z}^\lambda_s, \Phi_\varepsilon (\tilde{Z}^\lambda)_t \right), s, t \geq 0 \right) \right)
\]

in \(D(\mathbb{R}_+, \mathbb{R}) \times D(\mathbb{R}_+, \mathbb{R})\), where \(\Phi_\varepsilon : D(\mathbb{R}_+, \mathbb{R}) \rightarrow D(\mathbb{R}_+, \mathbb{R})\) is defined by

\[
(8.7) \quad \Phi_\varepsilon(f)_t = \int_0^t \frac{1(f(s) \leq -\varepsilon)}{c\sqrt{2g/|f(s)|^{\lambda+1/2}}} ds.
\]

**Proof.** Observe that the map \(\Phi_\varepsilon\) is continuous at all continuous functions \(f\) such that \(m(\{s : f(s) = -\varepsilon\}) = 0\), where \(m\) is Lebesgue measure. In particular, it is almost surely continuous at \((\tilde{Z}^\lambda_t, t \geq 0)\), where \(\tilde{Z}^\lambda\) is as defined in Theorem 8.7. Hence, we conclude from that theorem that for every \(\varepsilon > 0\) we have the joint convergence in distribution in \(D(\mathbb{R}_+, \mathbb{R}) \times D(\mathbb{R}_+, \mathbb{R})\),

\[
(8.8) \quad \left( \left( n^{-1/(2+2\lambda)} Y_{[sn]} - n^{-1/(2+2\lambda)} T_{m, \varepsilon}^{sn, m}, s, t \geq 0 \right) \rightarrow_d \left( \left( \tilde{Z}^\lambda_s, \Phi_\varepsilon (\tilde{Z}^\lambda)_t \right), s, t \geq 0 \right) \right).
\]

Let \(\mathcal{F}_m = \sigma((Y_j, \Delta j), 0 \leq j \leq m)\) and consider the martingale with respect to the filtration \((\mathcal{F}_m)_{m \geq 0}\) given by

\[
W_m := \sum_{j=1}^{m} (\Delta_j - E[\Delta_j | \mathcal{F}_{j-1}]), \ m \geq 0.
\]

Define \(\phi(y) = E(\mathcal{N}(y, U))\). By the Markov property we see that \(E[\Delta_j | \mathcal{F}_{j-1}] = \phi(Y_{j-1})\).

By Lemma 8.2 we see that \(\sup_y \phi(y) < \infty\) and

\[
\xi := \sup_m E \left[ (\Delta_m - E[\Delta_m | \mathcal{F}_{m-1}])^2 \right] < \infty.
\]

By Doob’s maximal inequality we see that for every \(\varepsilon > 0\) and integer \(k \geq 1\)

\[
\mathbb{P} \left( \sup_{1 \leq m \leq kn} |W_m| > \varepsilon n^{4+2\lambda} \right) \leq \frac{1}{\varepsilon^2 n^{(3+2\lambda)/(2+2\lambda)}} \mathbb{E} \left[ |W_{kn}|^2 \right] \leq \frac{k\xi}{\varepsilon^2 n^{1/(2+2\lambda)}},
\]

from which it follows that \(\sup_{1 \leq m \leq kn} n^{-4+2\lambda} W_m\) converges to 0 in probability as \(n \rightarrow \infty\). Similarly, if for \(\varepsilon > 0\) we define

\[
W_{m, \varepsilon} = \sum_{j=1}^{m} (\Delta_j - E[\Delta_j | \mathcal{F}_{j-1}]) \mathbb{1}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}} = \sum_{j=1}^{m} (\Delta_j - \phi(Y_{j-1})) \mathbb{1}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}}, \ m \geq 0,
\]

we find that \(\sup_{1 \leq m \leq kn} n^{-4+2\lambda} W_{m, \varepsilon}\) converges to 0 in probability as \(n \rightarrow \infty\). We record this for future reference as

\[
(8.9) \quad \sup_{1 \leq m \leq kn} n^{-4+2\lambda} \sum_{j=1}^{m} (\Delta_j - \phi(Y_{j-1})) \mathbb{1}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}} \rightarrow 0,
\]

in probability as \(n \rightarrow \infty\).
Lemma 8.3 implies that for every \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} \sup_{y \leq -\varepsilon n^{1/(2+2\lambda)}} \left| \frac{c}{\sqrt{2g}} |y|^{\lambda+1/2} \phi(y) - 1 \right| = 0,
\]
so
\[
\lim_{n \to \infty} \sup_{y \leq -\varepsilon n^{1/(2+2\lambda)}} \left| \phi(y) - \frac{1}{\sqrt{2g} |y|^{\lambda+1/2}} \right| = 0.
\]
Combined with the fact that, a.s.,
\[
n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{kn} \frac{\mathbb{I}_{\{Y_j \leq -\varepsilon n^{1/(2+2\lambda)}\}}}{c \sqrt{2g} |Y_{j-1}|^{\lambda+1/2}} \leq \frac{k}{c \sqrt{2g} \varepsilon^{\lambda+1/2}}
\]
this implies that for every integer \( k \geq 1 \), a.s.,
\[
\lim_{n \to \infty} \sup_{1 \leq m \leq kn} n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \phi(Y_{j-1}) - \frac{1}{\sqrt{2g} |Y_{j-1}| h(Y_{j-1})} \mathbb{I}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}} = 0.
\]
Note that,
\[
\Phi_\varepsilon \left( n^{-1/(2+2\lambda)} Y_{[-n]} \right)_{m/n} = \frac{1}{n} \sum_{j=1}^{m} \frac{\mathbb{I}_{\{n^{-1/(2+2\lambda)} Y_{j-1} \leq -\varepsilon\}}}{\sqrt{2g} |n^{-1/(2+2\lambda)} Y_{j-1}| h(n^{-1/(2+2\lambda)} Y_{j-1})}
\]
\[
= n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \frac{\mathbb{I}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}}}{\sqrt{2g} |Y_{j-1}| h(Y_{j-1})}.
\]
Hence,
\[
\sup_{1 \leq m \leq kn} \left| \Phi_\varepsilon \left( n^{-1/(2+2\lambda)} Y_{[-n]} \right)_{m/n} - n^{-\frac{3+2\lambda}{4+4\lambda}} T_{m,n} \right| = \sup_{1 \leq m \leq kn} \left| \Phi_\varepsilon \left( n^{-1/(2+2\lambda)} Y_{[-n]} \right)_{m/n} - n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \mathbb{I}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}} \Delta_{j-1} \right|
\]
\[
= \sup_{1 \leq m \leq kn} \left| n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \frac{\mathbb{I}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}}}{\sqrt{2g} |Y_{j-1}| h(Y_{j-1})} - n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \mathbb{I}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}} \Delta_{j-1} \right|
\]
\[
\leq \sup_{1 \leq m \leq kn} \left| n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \left( \Delta_{j} - \phi(Y_{j-1}) \right) \mathbb{I}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}} \right|
\]
\[
+ \sup_{1 \leq m \leq kn} \left| n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \left( \Delta_{j} - \phi(Y_{j-1}) \right) \frac{1}{\sqrt{2g} |Y_{j-1}| h(Y_{j-1})} \mathbb{I}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}} \right|
\]
\[
\rightarrow_d \left( \hat{\mathcal{Z}}_s, \Phi_\varepsilon(\hat{\mathcal{Z}}^\lambda)_t \right), s, t \geq 0
\]
in \( D(\mathbb{R}_+, \mathbb{R}) \times D(\mathbb{R}_+, \mathbb{R}) \), and the result follows. \( \square \)
Remark 8.10. We conjecture that the convergence in Theorem 8.9 can be extended to include the case $\varepsilon = 0$. One reason to believe this is that the limiting process is still well defined. From the basic properties of Bessel processes it follows that for every fixed $t^* \geq 0$ we have $\lim_{\varepsilon \to 0} m(\{s \leq t^* : \tilde{Z}_s^\lambda \geq -\varepsilon\}) = 0$ almost surely. Consequently, we have that

$$
(8.11) \quad \lim_{\varepsilon \to 0} \Phi(\tilde{Z}^\lambda) = \left( \int_0^t \frac{1}{c\sqrt{2}\sqrt{|Z^\lambda|^{\lambda+1/2}}} ds, t \geq 0 \right) \equiv \Phi(\tilde{Z}^\lambda), \quad \text{a.s.}
$$

A standard occupation density computation for Bessel processes shows that $\Phi(\tilde{Z}^\lambda)_t < \infty$ a.s., for every $t \geq 0$. The problem comes in controlling the amount of time spent between collisions when $Y$ is near 0, which contribute constant order time. We remark that the same difficulty arises in the periodic Galton Board, the model is studied in [2], where the authors avoided this complication by assuming the particle had a sufficiently large initial velocity and was reflected down at the corresponding level. In [12] the authors considered a model similar to ours when $h \equiv 1$ and, in that setting, were able to overcome this difficulty through different methods.

Theorem 8.9 allows us to obtain a scaling limit for the continuous time particle path (away from 0). In addition to keeping track of time we need to keep track of the direction of reflection, as in (8.9). That is, we consider the Markov chain $((Y_m, \Delta_m, U^m), m \geq 0)$ with transition operator

$$
\tilde{U} f(y, z, w) = \mathbb{E} \left[ f \left( y + U_d \sqrt{2g|y|} N(y, U) - \frac{g}{2} N(y, U)^2, N(y, U), U \right) \right],
$$

started from $(0, 0, (0, \ldots, 0, -1))$. Let $T_m = \sum_{j=0}^m \Delta_j$. The path of the particle is then given by

$$
(8.12) \quad Y(t) = Y_{m-1} + U_m \sqrt{2g|Y_{m-1}|(t - T_{m-1}) - \frac{g}{2}(t - T_{m-1})^2} \quad \text{on} \quad T_{m-1} \leq t < T_m, m \geq 1.
$$

Recall $\Psi$ from Lemma 5.2 and $\Phi$ from (8.11).

Theorem 8.11. Fix $y < v < 0$ and define $\tau^y_{y-} = \inf\{m : Y_m \leq n^{1/(2+2\lambda)}y\}$ and $\tau^y_{y+} = \inf\{m > \tau^y_{y-} : Y_m \geq n^{1/(2+2\lambda)}v\}$. For $(Y(t), t \geq 0)$ as defined in (8.12) and $y < v < 0$ we have the following convergence in distribution on $D(\mathbb{R}_+, \mathbb{R})$,

$$
\left( n^{-\frac{1}{2+2\lambda}} Y \left( \left( \frac{3+2\lambda}{n^{1+1/2\lambda}} t + \tau^y_{y+} \right) \wedge \tau^y_{y-} \right), t \geq 0 \right) \to (\tilde{Z}^\lambda(A(t) \wedge \tau_{v+}), t \geq 0),
$$

where $\tilde{Z}^\lambda$ is the diffusion appearing in Theorem 8.7 started from $y$ and

$$
A(t) = \Psi \left( \Phi(\tilde{Z}^\lambda(\cdot \wedge \tau_{v+})) \right).
$$

Remark 8.12. (i) The theorem remains true replacing $A$ with $\Phi(\tilde{Z}^\lambda(\cdot \wedge \tau_{v+}))$ for any $0 < \varepsilon < |v|$, where $\Phi_\varepsilon$ is defined in (8.7).

(ii) If $f$ is a $C^2$ function with compact support in $(-\infty, 0)$ the generator $\mathcal{G}^\lambda$ of $(\tilde{Z}^\lambda(A(t)), t \geq 0)$ acts on $f$ by

$$
\mathcal{G}^\lambda f(y) = 2\sqrt{2g} \frac{df}{dc}|y|^{\frac{\lambda}{2+\lambda}} - \frac{1}{2}f''(y) - \frac{d-1}{4|y|}f'(y),
$$

which can be obtained via computations similar to those in the proof of Theorem 6.1.

(iii) Recall from Remark 8.8 that the process $\tilde{Z}^\lambda(t)$ is neighborhood recurrent if and only if $d \leq 3$, for any $\lambda$.

Proof of Theorem 8.11. We sketch the proof following the lines of the proofs of Theorems 5.1, 5.3 and 5.4.
Fix $0 < \varepsilon < |v|$ and define $\lambda' = (3 + 2\lambda)/(4 + 4\lambda).$ Recall $T^{m,\varepsilon}_n$ from Theorem 8.9 and let

$$A_n(t) = \Psi \left( n^{-\lambda'} \left( T^{m,\varepsilon}_{\tau_{n_+}^v} - T^{m,\varepsilon}_{\tau_{n_-}^v} \right) + \left( \cdot - n^{-1} \left( \tau_{n_+}^v - \tau_{n_-}^v \right) \right) \frac{1}{\sqrt{2g|v| h(v)}} \right)(t).$$

Arguing as in the proof of Theorem 5.3, we use Theorem 8.9 to see that

$$\left( n^{-\frac{1}{2+2\lambda}} Y_{[nA_n(t)]\wedge(\tau_{n_+}^v - \tau_{n_-}^v)}, t \geq 0 \right) \rightarrow \left( \tilde{Z}^\lambda(A(t) \wedge \tau_{v+}), t \geq 0 \right).$$

Observe that for all $0 \leq m \leq \tau_{v+}^n - \tau_{v-}^n$ we have $nA_n(n^{-\lambda'}(T^{m,\varepsilon}_{\tau_{n_+}^v} - T^{m,\varepsilon}_{\tau_{n_-}^v})) = m$ and, as a result, if $\left( T^{n,\varepsilon}_{\tau_{v+}^n-m+1} - T^{n,\varepsilon}_{\tau_{v-}^n} \right) \leq n^{\lambda'} t < \left( T^{n,\varepsilon}_{\tau_{v+}^n-m+1} - T^{n,\varepsilon}_{\tau_{v-}^n} \right)$ then $m - 1 \leq nA_n(t) < m.$

Define $\tilde{T}_m = T^{n,\varepsilon}_{\tau_{v+}^n-m}$ and fix $S > 0$ and observe that

$$\sup_{0 \leq t \leq S} \left| Y \left( \left( n^{\lambda'} t + T^{n,\varepsilon}_{\tau_{v+}^v} \right) \wedge T^{n,\varepsilon}_{\tau_{v+}^v} \right) - Y_{[nA_n(t)]\wedge(\tau_{n_+}^v - \tau_{n_-}^v)} \right| \leq \sup_{m \leq [nA_n(S \wedge n^{-\lambda'}(T^{n,\varepsilon}_{\tau_{n_+}^v} - T^{n,\varepsilon}_{\tau_{n_-}^v}))]} \sup_{\hat{T}_m \leq t \leq \tilde{T}_m} \left| U^m_d \sqrt{\frac{2g|Y^{n,\varepsilon}_{\tau_{n_+}^v-m+1}|}{n^{1/(2+2\lambda)}} n^{-\frac{1}{2+2\lambda}}(t - \hat{T}_m) - \frac{n^{\lambda'}}{2n^{1/(2+2\lambda)}}(t - \hat{T}_m)^2 \right| \leq \sup_{m \leq [nA_n(S \wedge n^{-\lambda'}(T^{n,\varepsilon}_{\tau_{n_+}^v} - T^{n,\varepsilon}_{\tau_{n_-}^v}))]} \sqrt{\frac{2g|Y^{n,\varepsilon}_{\tau_{n_+}^v-m+1}|}{n^{1/(2+2\lambda)}} n^{-\frac{1}{2+2\lambda}}(\hat{T}_m - \tilde{T}_m) \leq \frac{g}{2n^{1/(2+2\lambda)}}(\hat{T}_m - \tilde{T}_m)^2.}

As in the proof of Theorem 5.3, we see that

$$\left( n^{-\frac{1}{2+2\lambda}} Y_{[nA_n(t)]\wedge(\tau_{n_+}^v - \tau_{n_-}^v)}, A_n(t), n^{-1}((\tau_{v+}^n - \tau_{v-}^n)), t \geq 0 \right) \rightarrow \left( \tilde{Z}^\lambda(A(t) \wedge \tau_{v+}), A(t), \tau_{v+}), t \geq 0 \right).$$

Let

$$B_n = \left\{ m \leq [nA_n(S \wedge n^{-\lambda'}(T^{n,\varepsilon}_{\tau_{n_+}^v} - T^{n,\varepsilon}_{\tau_{n_-}^v}))], \frac{1}{n^{2+2\lambda}}|Y^{n,\varepsilon}_{\tau_{n_+}^v-m}| \leq M, A_n \left( S \wedge n^{-\lambda'}(T^{n,\varepsilon}_{\tau_{n_+}^v} - T^{n,\varepsilon}_{\tau_{n_-}^v}) \right) \leq M, \left| Y^{n,\varepsilon}_{\tau_{n_+}^v} - n^{1/(2+2\lambda)}(v + \delta) \right| \right\}.$$
The right hand side goes to 0 by Lemma \(8.2\) and an argument as in (5.13). Since \(\mathbb{P}(B_n) \to 1\), it follows from (8.14) that
\[
\left(\frac{1}{n^{1/2}} \mathbb{Y} \left( \left( n^{3/2} t + T_{\gamma^n} \right) \wedge T_{\gamma^n} \right), \ t \geq 0 \right) \to (\tilde{Z}^\lambda(A(t) \wedge \gamma^n), \ t \geq 0).
\]

\section{Appendix A. Reflection direction}

This short section presents an elementary fact about the classical (specular) reflection. The claim is known in dimension \(d = 3\) (see, for example, the discussion of the so-called hard-sphere scattering in [3, Sect. 4.8]) but we could not find a reference for the analogous result in all dimensions \(d \geq 2\).

Suppose that \(d \geq 2\). Let \(S^{d-1}\) be the unit sphere in \(\mathbb{R}^d\) and let \(e_1, \ldots, e_d\) be the standard basis for \(\mathbb{R}^d\). Let \(B_{d-1} = \{(0, x_2, \ldots, x_d) \in \mathbb{R}^d : x_2^2 + \cdots + x_d^2 \leq 1\}\). Let \(b\) be a random vector with the uniform distribution in \(B_{d-1}\) and let \(\mathcal{L}\) be the random straight line \(\{b + ae_1, a \in \mathbb{R}\}\). Suppose that a light ray starts from the point \(b + 2e_1\) and travels along \(\mathcal{L}\) in the direction of the point \(b - 2e_1\). Now suppose that this random light ray reflects from \(S^{d-1}\) according to the classical law of specular reflection, i.e., the angle of reflection is equal to the angle of incidence. Let \(v \in S^{d-1}\) be the vector representing the direction of the reflected ray, i.e., the reflected light ray travels along a straight line of the form \(\{w + av, a \in \mathbb{R}\}\) for some vector \(w \in \mathbb{R}^d\).

**Proposition A.1.** The distribution of \(v\) is uniform on \(S^{d-1}\) if and only if \(d = 3\).

**Proof.** Let \(n\) be the outer normal vector to the sphere \(S^{d-1}\) at the point where the light ray hits the sphere. If \(|b| = r_1\) and the angle between \(e_1\) and \(n\) is \(\alpha_1\) then \(r_1 = \sin \alpha_1\). Let \(\Theta\) be the angle between \(v\) and \(e_1\). The specular law of reflection implies that the angle between \(v\) and \(n\) is \(\alpha_1\) so \(\Theta = 2\alpha_1\). Hence, for a given \(r \in (0, 1)\), we have \(|b| \leq r\) if and only if \(\Theta \leq 2\alpha\), where \(r = \sin \alpha\). Let \(\beta = 2\alpha\) so that \(r = \sin(\beta/2)\). We obtain
\[
\mathbb{P}(\Theta \leq \beta) = \mathbb{P}(|b| \leq r) = r^{d-1} = (\sin \alpha)^{d-1} = (\sin(\beta/2))^{d-1}.
\]

Let \(A_{\beta}\) be the spherical cap with the angle \(\beta\), i.e., the set of points \(x \in S^{d-1}\) such that the angle between the vector \(\vec{0}x\) and \(e_1\) is smaller than or equal to \(\beta\). Let \(\mu\) be the uniform probability measure on \(S^{d-1}\). It suffices to show that \(\mu(A_{\beta}) = \mathbb{P}(\Theta \leq \beta)\) for all \(\beta \in (0, \pi)\) if and only if \(d = 3\).

The following formulas for the area of \(A_{\beta}\) and \(S^{d-1}\) are taken from [9]. The area of \(A_{\beta}\) is equal to \((2\pi^{(d-1)/2}/\Gamma((d-1)/2)) \int_0^\beta \sin^{d-2} \gamma d\gamma\). The area of \(S^{d-1}\) is \(2\pi^{d/2}/\Gamma(d/2)\). It follows that
\[
\mu(A_{\beta}) = \frac{\Gamma(d/2)}{\sqrt{\pi} \Gamma((d-1)/2)} \int_0^\beta \sin^{d-2} \gamma d\gamma.
\]

For \(d = 3\) and all \(\beta \in (0, \pi)\),
\[
\mathbb{P}(\Theta \leq \beta) = (\sin(\beta/2))^2 = \frac{1}{2} (1 - \cos \beta) = \frac{\Gamma(3/2)}{\sqrt{\pi} \Gamma(1)} \int_0^\beta \sin \gamma d\gamma = \mu(A_{\beta}),
\]

so the proposition is proved for \(d = 3\).

For all \(d \geq 2\) and \(\beta \in (0, \pi)\),
\[
f(\beta) := \frac{\partial}{\partial \beta} \mathbb{P}(\Theta \leq \beta) = \frac{\partial}{\partial \beta} (\sin(\beta/2))^{d-1} = \frac{d-1}{2} (\sin(\beta/2))^{d-2} \cos(\beta/2),
\]
\[
g(\beta) := \frac{\partial}{\partial \beta} \mu(A_{\beta}) = \frac{\Gamma(d/2)}{\sqrt{\pi} \Gamma((d-1)/2)} \sin^{d-2} \beta.
\]

This implies that
\[
\frac{f(\pi/2) g(\pi/4)}{g(\pi/2) f(\pi/4)} = 2^{(3/2)-d} \sec(\pi/8)(\sin(\pi/8))^{2-d} = (2 \sin(\pi/8))^{3-d}.
\]
The last quantity is not equal to 1 for $d \neq 3$ so the functions $f$ and $g$ are not identically equal to each other. Hence, for $d \neq 3$, it is not true that $P(\Theta \leq \beta) \equiv \mu(A_\beta)$.

Since $d = 3$ is the dimension of our physical space, this justifies the choice of the uniform direction of reflection in this paper. In other dimensions, we also assume that the direction of reflection is uniform, for several reasons. The first is mathematical convenience. Second, the assumption of the uniform angle of reflection allows us to use a Markov model for the process of locations of consecutive scattering events. Finally, we believe that due to mixing (in the probabilistic sense of the word), our results would remain unchanged, in the qualitative sense, if we incorporated the true distribution of reflection in dimensions $d \neq 3$.

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