On deformations of standard $R$-matrices for integrable infinite-dimensional systems

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Abstract

Simple deformations, with a parameter $\epsilon$, of classical $R$-matrices which follow from decomposition of appropriate Lie algebras, are considered. As a result nonstandard Lax representations for some well known integrable systems are presented as well as new integrable evolution equations are constructed.

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1 Introduction

In the theory of nonlinear evolutionary systems one of the most important problems is construction of integrable systems. By integrable systems we understand those which have infinite hierarchy of commuting symmetries. It is well known that a very powerful tool, called the classical $R$-matrix formalism, can be used for systematic construction of field and lattice integrable dispersive systems (soliton systems) as well as dispersionless integrable field systems (see [1]-[10] and the references there).

The crucial point of the formalism is the observation that integrable systems can be obtained from Lax equations. Let $\mathfrak{g}$ be a Lie algebra, equipped with the Lie bracket $[\cdot,\cdot]$. A linear map $R : \mathfrak{g} \to \mathfrak{g}$, such that the bracket $[a, b]_R := [Ra, b] + [a, Rb]$ is a second Lie product on $\mathfrak{g}$, is called the classical $R$-matrix. Assume that $R$ satisfies an Yang-Baxter equation $\text{YB}(\alpha)$: $[Ra, Rb] - R[a,b]_R + \alpha[a,b] = 0$, which is a sufficient condition for $R$ to be an $R$-matrix. Then, powers of $L$ generate mutually commuting vector fields

$$L_{tn} = [R(L^n), L].$$

For fixed $n$ the remaining systems are considered as its symmetries. In this sense (1) represents a hierarchy of integrable dynamical systems.

In this article the deformation method for systems (1), preserving the integrability, is presented. It has been done on the level of their Lax representations through simple deformations, with parameter $\epsilon$, of classical $R$-matrices. It is shown that such a procedure leads to

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the construction of nonstandard Lax representations for some well known integrable systems as well as to the construction of new integrable evolution equations.

2 Deformations of standard $R$-matrices

To construct the simplest $R$-structure let us assume that the Lie algebra $\mathfrak{g}$ can be split into a direct sum of Lie subalgebras $\mathfrak{g}_+\oplus \mathfrak{g}_-$, i.e. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $[\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm$. Denoting the projections onto these subalgebras by $P_\pm$, we define the $R$-matrix as

$$ R = \frac{1}{2}(P_+ - P_-). \quad (2) $$

Straightforward calculation shows that (2) solves YB($\frac{1}{4}$). The classical $R$-matrices constructed in this way we understand as standard ones.

Let us consider the following deformation of (2)

$$ R'(a) = R(a) + \epsilon r(a) \quad (3) $$

where $\epsilon$ is an arbitrary constant playing the role of a deformation parameter and $r$ is a linear deformation operator. First, assume that $r$ satisfies the following two relations

$$ [ra, b_+] \in \mathfrak{g}_+ \quad [ra, b_-] \in \mathfrak{g}_- \quad a \in \mathfrak{g}, \ b_+ \in \mathfrak{g}_+, \ b_- \in \mathfrak{g}_-. \quad (4) $$

So, the question arises when the deformed $R$ preserves the property of being $R$-matrix. Once again, straightforward calculation shows that (3) solves YB($\frac{1}{4}$) when the following condition is fulfilled

$$ r [a, b]_R + \epsilon r [a, b]_r - \epsilon [ra, rb] = 0 \quad (5) $$

where $[a, b]_r = [ra, b] + [a, rb]$.

3 Dispersionless systems

Let $A$ be the algebra of formal Laurent series (Lax polynomials) in $p$ [6]

$$ A = \left\{ L = \sum_{i \in \mathbb{Z}} u_i(x)p^i \right\} \quad (6) $$

where the coefficients $u_i(x)$ are smooth functions of $x$. Poisson brackets on $A$ can be introduced in infinitely many ways as

$$ [f, g] \equiv \{f, g\}_s := p^s \left( \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \right) \quad s \in \mathbb{Z}. \quad (7) $$

Then, fixing $s$, $A$ is the Poisson algebra with an appropriate bracket [7]. We construct the standard $R$-matrix, through a decomposition of $A$ into a direct sum of Lie subalgebras. For a fixed $s$ let $A_{\geq -s+k} = \{\sum_{i \geq -s+k} u_i(x)p^i\}$ and $A_{< -s+k} = \{\sum_{i < -s+k} u_i(x)p^i\}$. Then, $A_{\geq -s+k}, A_{< -s+k}$ are Lie subalgebras in the following cases:
1. \( s = 0, \ k = 0, \)
2. \( s \in \mathbb{Z}, \ k = 1, 2, \)
3. \( s = 2, \ k = 3. \)

So, fixing \( s \) we fix the Lie algebra structure with \( k \) numbering the standard \( R \)-matrices given in the following form

\[
R = \frac{1}{2}(P_{\geq-s+k} - P_{\leq-s+k}) = \frac{1}{2} - \frac{1}{2} P_{\leq-s+k}
\]

where \( P \) are appropriate projections. The Lax hierarchy \( (1) \) can be represented by two equivalent representations

\[
L_{t_q} = \{(L^q)_{\geq-s+k}, L\}_s = -\{(L^q)_{\leq-s+k}, L\}_s.
\]

Notice that different schemes are interrelated. Under the transformation

\[
x' = x \quad p' = p^{-1} \quad t' = t
\]

the Lax hierarchy \( (2) \) defined by \( k, s \) and \( L \) transforms into the Lax hierarchy \( (3) \) defined by \( k' = 3 - k, s' = 2 - s \) and \( L' = L \), i.e.

\[
L \text{ for } k, \ s \Leftrightarrow L' = L \text{ for } k' = 3 - k, \ s' = 2 - s.
\]

In such a situation it is enough to consider the cases of \( k = 0 \) and \( k = 1 \).

We are interested in extracting closed systems for a finite number of fields. To obtain a consistent Lax equation, the Lax operator \( L \) has to form a proper submanifold of the full Poisson algebra \( A \), i.e. the left and right-hand sides of expression \( (9) \) have to coincide. They are given in the form

\[
s = 0, \ k = 0 \ : \ L = p^N + u_{N-2} p^{N-2} + u_{N-3} p^{N-3} + \ldots + u_1 p + u_0
\]
\[
s \in \mathbb{Z}, \ k = 1 \ : \ L = p^N + u_{N-1} p^{N-1} + \ldots + u_{1-m} p^{1-m} + u_{-m} p^{-m}
\]

where \( u_i \) are dynamical fields. Notice, that powers of \( L \), in general fractional, can be calculated by expanding them around poles, for \( (12) \) around \( \infty \) and for \( (13) \) around \( \infty \) and 0. So, for \( k = 0 \) we construct one Lax hierarchy and for \( k = 1 \) we construct, in general, two mutually commuting Lax hierarchies.

We are looking for a simple deformation of \( (8) \) in the form

\[
r = \alpha P_\beta \quad P_\beta(L) = [L]_\beta = u_\beta
\]

which will satisfy \( (4) \) and \( (5) \) for arbitrary \( \epsilon \). By some straightforward calculations, we find them in the form

\[
r = p^{-s+k} P_{k-1} \quad \text{for} \begin{cases} 
1. & s = 0, \ k = 0 \\
2. & s = 0, \ k = 1 \\
3. & s = 1, \ k = 1
\end{cases}
\]
and
\[ r = p^{-s+k-1}P_{k-2} \]
for
\[
\begin{align*}
4. & \quad s = 1, \quad k = 2 \\
5. & \quad s = 2, \quad k = 2 \\
6. & \quad s = 2, \quad k = 3
\end{align*}
\] (16)

We see that deformations of (8) given by the form (3), (14) exist only for distinguish values of \( s \) and \( k \). Nevertheless, for particular fixed values of \( \epsilon \) and fixed \( s \) (i.e. fixed Lie algebra) there exist other deformations of the form (14), but they are trivial in the sense that they relate standard \( R \)-matrices (8) with different \( k \). Moreover, deformations (16) are constructed from (15) by using transformation (10-11). Hence, the only relevant deformations are (15) and so further we will consider only them. The deformed \( R \)-matrices for the cases in (15) take the form
\[
R' = P_{\geq-s+k} - \frac{1}{2} + \epsilon p^{-s+k}P_{k-1} = \frac{1}{2} - P_{<-s+k} + \epsilon p^{-s+k}P_{k-1}. \tag{17}
\]

The case: \( s = 0, \ k = 0 \).

The Lax hierarchy for a deformed \( R \)-matrix is
\[
L_{t_q} = \{(L^q)_{\geq 0} + \epsilon [L^q]_{-1}, L\}_0 = -\{(L^q)_{< 0} - \epsilon [L^q]_{-1}, L\}_0. \tag{18}
\]

Consistent Lax equations are obtained for Lax operators of the form
\[
L = p^N + u_{N-1}p^{N-1} + u_{N-2}p^{N-2} + \ldots + u_1 p + u_0 \tag{19}
\]
where \( u_i \) are dynamical fields. From (18) it follows that
\[
\begin{align*}
(u_{N-1})_{t_q} &= -\epsilon N([L^q]_{-1})_x \\
(u_{N-2})_{t_q} &= N([L^q]_{-1})_x - \epsilon (N-1)u_{N-1}([L^q]_{-1})_x \\
\vdots 
\end{align*}
\] (20)

So, for \( \epsilon = 0 \) the field \( u_{N-1} \) becomes time-independent and without loosing generality we can assume that it is zero. Then, Lax operator becomes a standard Lax operator (12) for the case with non-deformed \( R \)-matrix. From (20) the following relation between \( u_{N-1} \) and \( u_{N-2} \) results
\[
u_{N-2} = -\frac{1}{\epsilon}u_{N-1} + \frac{N-1}{2N}(u_{N-1})^2 \tag{21}
\]
so we can eliminate one of them. Eliminating \( u_{N-2} \) we will consider constrained Lax operator in the form
\[
L = p^N + u_{N-1}p^{N-1} + \left(-\frac{1}{\epsilon}u_{N-1} + \frac{N-1}{2N}(u_{N-1})^2\right)p^{N-2} + u_{N-3}p^{N-3} + \ldots + u_1 p + u_0. \tag{22}
\]

Reparameterizing (22): \( u_{N-1} \mapsto -\epsilon u_{N-2} \) and then taking the limit \( \epsilon \to 0 \) it becomes the standard Lax operator (12).
Lemma 3.1 For arbitrary $\epsilon$ the Lax hierarchy \((18),(22)\) is equivalent to the Lax hierarchy \((9),(12)\) with $s = k = 0$.

The sketch of the proof is as follows. We are looking for transformations that will relate fields from (22) and fields from (12). We postulate the following form of these relations:

$$
u_{N-1} \mapsto -\epsilon \nu_{N-2}$$

$$\nu_i \mapsto \nu_i + f_i(\nu_{N-2}, \nu_{N-3}, ..., \nu_{i+1}) \quad \text{for} \quad N - 3 \geq i \geq 0. \quad (23)$$

Then, we construct functions $f_i$ in such a way that hierarchy (18) will lead to the same evolution system as (9) for $s = k = 0$. We compare the first non-trivial systems from these hierarchies. Functions $f_i$ are recursively constructed comparing evolution expressions for $u_{i+1}$. Such a procedure guarantees that the expressions for the fields will be the same only for components $u_{N-2i}, ..., u_i$. So, the equality between both evolution expressions for $u_0$ has to be argued. The systems for Lax operators (12) and (22) both can be understood as the reduction of infinite-field systems for Lax operators of the form $L' = a_1p + a_0 + a_{-1}p^{-1} + ...$, given by constraint $L = L^N$. The equivalence between the hierarchies considered, constructed from $L'$, can be shown by explicit infinite recurrence form (23). Now, reducing them to finite-field systems one finds the appropriate transformation between finite-field systems, including the evolution for $u_0$. So, the Lax hierarchy (18), (22) is a new representation of well known integrable dispersionless hierarchies. The form of transformation (23), related both systems, guarantees that it is an invertible transformation.

Example 3.2 Dispersionless KdV: $N = 2$.

For $L = p^2 + up + v$ and $L_{t_1} = \{ (L^4)_{\geq 0} + \epsilon[L^4]_{-1}, L \}$ we find

$$
\begin{align*}
(u)_{t_1} &= \left( \frac{1}{2} \epsilon uu_x - ev_x \right) \\
(v)_{t_1} &= \left( \frac{1}{4}(\epsilon u - 2)(uu_x - 2v_x) \right)
\end{align*}
$$

$$
\begin{align*}
(u)_{t_3} &= \left( -\frac{3}{16} \epsilon (u^2 - 4v)(uu_x - 2v_x) \right) \\
(v)_{t_3} &= \left( -\frac{3}{32} \epsilon (\epsilon u - 2)(u^2 - 4v)(uu_x - 2v_x) \right)
\end{align*}
$$

$$
\vdots
$$

In the limit $\epsilon \rightarrow 0$ and $u = 0$ it becomes the standard dispersionless KdV hierarchy. Notice, that fields $u$ and $v$ are not independent. According to (21) $v = -\frac{1}{\epsilon} u + \frac{1}{4} u^2$ and the hierarchy (21) is equivalent to the one

$$
\begin{align*}
u_{t_1} &= u_x \\
\nu_{t_3} &= \frac{3}{2\epsilon} uu_x \\
\vdots
\end{align*}
$$

i.e. reparameterized dispersionless KdV. The transformation to the standard form of dispersionless KdV is given by $u \mapsto -\epsilon u$. The hierarchy (23) is generated from $L = p^2 + up + \frac{1}{4} u^2 - \frac{1}{\epsilon} u$.

Example 3.3 Dispersionless Boussinesq: $N = 3$.

Here we present only the result for the constrained Lax operator (22) $L = p^3 + up^2 + (\frac{1}{3} u^2 - \frac{1}{\epsilon} u) p + w$. Then, the first nontrivial system from the hierarchy is

$$
L_{t_2} = \left\{ \left( L^4 \right)_{\geq 0} + \epsilon \left[ L^4 \right]_{-1}, L \right\} \leftrightarrow \left\{ \begin{array}{l}
\left( u \right)_{t_2} = \left( \frac{2}{9}(\epsilon u - 6)uu_x - 2\epsilon w_x \right) \\
\left( w \right)_{t_2} = \left( \frac{2}{81\epsilon^2}(\epsilon u - 3)(\epsilon u - 9)uu_x - \frac{2}{3}(\epsilon u - 6)w_x \right) \end{array} \right.
$$
Eliminating the field $w$ we obtain the reparameterized dispersionless Boussinesq: $u_t = \frac{2}{3\epsilon}(u^2)_{xx}$.

The transformation \cite{28} to the standard form of the dispersionless Boussinesq system is given by: $u \mapsto -\epsilon u$, $w \mapsto w - \frac{1}{3\epsilon}u^2 - \frac{1}{27\epsilon^3}u^3$. 

**The case:** $s = 0$, $k = 1$.

The Lax hierarchy for the deformed $R$-matrix \cite{17} is

$$L_{t_0} = \{(L^q)_{\geq 1} + \epsilon [L^q]_0 p, L\}_0 = -\{(L^q)_{< 1} - \epsilon [L^q]_0 p, L\}_0.$$  \hspace{1cm} (26)

Appropriate Lax operators are of the form

$$L = u_N p^N + u_{N-1} p^{N-1} + \ldots + u_1 p^{1-m} + u_m p^{-m}. \hspace{1cm} (27)$$

From (26) one finds that $(u_N)_t = \epsilon (u_N)_x [L^q]_0 - \epsilon N u_N ([L^q]_0)_x$. Hence, in the limit of $\epsilon = 0$ the field $u_N$ becomes a time-independent field $c_N$. Fixing $c_N = 1$ the Lax operator becomes a standard Lax operator \cite{13} for $s = 0$, $k = 1$. Moreover, there is no constraint contrary to the previous case. Hence, the Lax hierarchy \cite{27}, \cite{26} leads to new integrable dispersionless systems, at least to the best of our knowledge. Notice that the zero power of $L$ always leads to the space translation symmetry: $L_{t_0} = \epsilon L_x$.

**Example 3.4** Extended dispersionless Benney: $N = m = 1$.

Let $L = up + v + wp^{-1}$, then for $L_{t_i} = \{(L^i)_{\geq 1} + \epsilon [L^i]_0 p, L\}_0$ we find

$$
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}_{t_0} = \epsilon 
\begin{bmatrix}
  u_x \\
  v_x \\
  w_x
\end{bmatrix}, \hspace{1cm}
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}_{t_1} = 
\begin{bmatrix}
  \epsilon u_x v - \epsilon w_x \\
  w_x + \epsilon v_x \\
  u_x w + uw_x + \epsilon v_x w + \epsilon w_x
\end{bmatrix}, \hspace{1cm}
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}_{t_2} = 
\begin{bmatrix}
  \epsilon u_x v^2 - 2\epsilon uv w_x - 2\epsilon u^2 w_x \\
  2uv w_x + 2v w_x + 2u^2 w_x + \epsilon v^2 v_x + 2\epsilon w x v \\
  2u x v w + 2u v w_x + 2uv w_x + 2\epsilon u x w^2 + 2\epsilon v w w_x + \epsilon v^2 w_x + 4\epsilon uv w_x
\end{bmatrix}
\vdots
$$

In the limit $\epsilon \to 0$ and $u = 1$ we obtain the standard dispersionless Benney system.

**Example 3.5** Two field system: $N = 0, m = 1$.

Consider $L = v + wp^{-1}$. Then $[L^i]_0 = v^i$ and $(L^i)_{\geq 1} = 0$ for $i = 0, 1, 2,...$. Hence we obtain the system

$$L_{t_i} = \{\epsilon [L^i]_0 p, L\}_0 \iff \begin{bmatrix}
  v \\
  w
\end{bmatrix}_{t_i} = \left( \begin{array}{c}
  \epsilon v^i v_x \\
  \epsilon (v^i w)_x
\end{array} \right).$$  \hspace{1cm} (28)

which does not have any standard counterpart.
The case: \( s = 1, k = 1 \).

The Lax hierarchy is
\[
L_t = \{ (L^q)_{>0} + \epsilon [L^q]_0, L \}_1 = - \{ (L^q)_{<0} - \epsilon [L^q]_0, L \}_1
\]
and appropriate Lax operators take the form
\[
L = u_N p^N + u_{N-1} p^{N-1} + \ldots + u_{1-m} p^{1-m} + u_{-m} p^{-m}.
\]
From (29) it follows that
\[
(u_N)_t = -\epsilon N u_N ([L^q]_0)_x \quad \ldots \quad (u_{-m})_t = (1 + \epsilon) m u_{-m} ([L^q]_0)_x.
\]
So, we find that the highest and lowest fields are related by
\[
u_{(1+\epsilon)m} = u_{-m}^{-\epsilon}.
\]
For \( \epsilon = 0 \) the field \( u_N \) becomes a time-independent field \( c_N \) (let \( c_N = 1 \)), then the Lax operator \( 30 \) becomes a standard Lax operator \( 13 \) for \( s = k = 1 \). For \( \epsilon = -1 \) the field \( u_{-m} \) becomes time-independent and the Lax operator becomes a standard Lax operator for \( s = 1, k = 2 \). This last case follows from the fact that for \( s = 1, k = 1 \) and \( \epsilon = -1 \) the deformed R-matrix \( 17 \) becomes the standard R-matrix \( 8 \) for \( s = 1, k = 2 \). Eliminating \( u_N \) field the Lax operator takes the form
\[
L = u_{-m}^{-1} p^N + u_{N-1} p^{N-1} + \ldots + u_{1-m} p^{1-m} + u_{-m} p^{-m}
\]
In the limit \( \epsilon \to 0 \) it becomes the standard Lax operator \( 13 \) for \( s = k = 1 \).

**Lemma 3.6** For arbitrary \( \epsilon \), the Lax hierarchy (29), (30) is equivalent to the Lax hierarchy (9), (13) with \( s = k = 1 \).

To show this let us make the following transformation
\[
u_i \mapsto u_i^{\frac{1}{N}} u_N^{\frac{1}{N}} \quad p \mapsto u_N^{-\frac{1}{N}} p \quad \text{for} \quad N - 1 \geq i > -m.
\]
The Poisson bracket (7) for \( s = 1 \) is invariant under (33). Moreover, the Lax operators (30) transform into (13) one. Then, after the transformation of coordinates (33) we get
\[
L_t \mapsto L_t + \frac{1}{N} u_{(N)}^{-1} (u_N)_t p L_p \quad \text{by (33)} \quad L_t - \epsilon ([L^q]_0)_x p L_p = L_t + \{ \epsilon [L^q]_0, L \}_1.
\]
Hence, the hierarchy (29) turns into (9) one with \( s = k = 1 \).

**Example 3.7** Extended dispersionless Toda: \( N = m = 1 \).

For Lax operator \( L = up + v + wp^{-1} \) from \( L_{t_1} = \{ (L)_{>0} + \epsilon [L]_0, L \}_0 \) we find
\[
\begin{pmatrix}
u \\ v \\ w
\end{pmatrix}_{t_1} = \begin{pmatrix}
-\epsilon u_x v \\ u_x w + uw_x \\ (1 + \epsilon)v_x w
\end{pmatrix}.
\]
In the limit $\epsilon = 0$ and $u = 1$ we obtain the standard dispersionless Toda system. In the limit $\epsilon = -1$ and $w = 1$ we obtain the reparameterized dispersionless Toda system. The transformation (33) to the standard case is given by: $v \mapsto v$, $w \mapsto u - 1 - w$. Eliminating field $u$, for $L = w^{-\frac{1}{1+\epsilon}} p + v + w p^{-1}$, we get

$$
\left( \begin{array}{c}
v \\
w
\end{array} \right)_{t_1} = \left( \frac{v}{(1 + \epsilon) v x w} \right).
$$

For $\epsilon = 0$ or by the transformation: $v \mapsto v$, $w \mapsto w^{1+\epsilon}$ it becomes the standard dispersionless Toda system.

Notice that for some dispersionless systems it is possible to construct their integrable dispersive counterparts: field and lattice soliton systems. Actually, one can do it on the level of their Lax representation through Weyl-Moyal like deformation quantization procedure [10] of dispersionless case. The idea relies on the deformation of the usual multiplication in $A$ (6) to the new associative but non-commutative product

$$
f \ast g = f \exp (\hbar p^s \partial_p \otimes \partial_x) g = \sum_{i \geq 0} \frac{\hbar^i}{i!} (p^s \partial_p)^i f \cdot \partial_x^i g \quad f, g \in A
$$

called $\ast$-product. It depends on the formal deformation parameter $\hbar$. The Lie algebra structure is defined by the commutator $\{ f, g \}^\ast = \frac{1}{\hbar} (f \ast g - g \ast f)$. Then, the $\ast$-product (34) in the limit $\hbar \to 0$ reduces to the standard multiplication and the commutator reduces to the Poisson bracket (7) for fixed $s$. To construct integrable dispersive systems one has to split the algebra $A$ with the $\ast$-product into a direct sum of its Lie subalgebras and then construct the standard $R$-matrices. It can be done only for $s = 0, 1, 2$. But, the case $s = 2$ is equivalent to the case $s = 0$. The algebra $A$ with $\ast$-product (34) for $s = 0$ is isomorphic to the Lie algebra of pseudo-differentials operators (35), while for $s = 1$ is isomorphic to the Lie algebra of shift operators (47) ($\mathcal{E} = \exp \hbar \partial_x$). The first case leads to the construction of field soliton systems, and the second one leads to the construction of lattice soliton systems. Obviously, integrable dispersionless systems can be constructed from integrable dispersive systems in the so-called quasi-classical (dispersionless) limit: $\partial_t \mapsto \hbar \partial_t$, $\partial_x \mapsto \hbar \partial_x$ and $\hbar \to 0$.

4 Field soliton systems

Let $\mathfrak{g}$ be the algebra of pseudo-differential operators [2]

$$
\mathfrak{g} = \left\{ L = \sum_{i \in \mathbb{Z}} u_i(x) \partial_x^i \right\}
$$

where the multiplication of two such operators uses the generalized Leibniz rule $\partial^m u = \sum_{s \geq 0} \binom{m}{s} u_{sx} \partial_x^{m-s}$. The Lie algebra structure of $\mathfrak{g}$ is given by the commutator $[L_1, L_2] = L_1 L_2 - L_2 L_1$. We consider decomposition of $\mathfrak{g}$ in the form $A_{\geq k} = \left\{ \sum_{i \geq k} u_i(x) \partial_x^i \right\}$ and $A_{< k} = \left\{ L = \sum_{i < k} u_i(x) \partial_x^i \right\}$, which are Lie subalgebras for $k = 0, 1, 2$. In this cases the standard $R$-matrices are given by

$$
R = \frac{1}{2} (P_{\geq k} - P_{< k}) = P_{\geq k} - \frac{1}{2} = \frac{1}{2} - P_{< k}.
$$
So, the Lax hierarchy has the form
\[ L_{tq} = [(L^q)_{\geq k}, L] = - [(L^q)_{< k}, L]. \] (36)

Consistent Lax equations are obtained for Lax operators of the form [3]
\[ k = 0 : \quad L = \partial_x^N + u_{N-2}\partial_x^{N-2} + \ldots + u_1\partial_x + u_0 \] (37)
\[ k = 1 : \quad L = \partial_x^N + u_{N-1}\partial_x^{N-1} + \ldots + u_0 + \partial_x^{-1}u_{-1} \] (38)
\[ k = 2 : \quad L = u_N\partial_x^N + u_{N-1}\partial_x^{N-1} + \ldots + u_0 + \partial_x^{-1}u_{-1} + \partial_x^{-2}u_{-2}. \] (39)

where \( u_i \) are dynamical fields. Comparing Lax operators [87, 39] with those for the dispersionless case [12, 13] for \( s = 0 \) we see that not all dispersionless systems have dispersive counterparts.

The simple deformations satisfying [11, 5] are the following ones
\[ r = P_{k-1}(\cdot)\partial_x^k \quad \text{for} \quad k = 0, 1. \]

Note, that the first case has been considered, in little bit different manner, earlier in [4]. Hence, the deformed \( R \)-matrices have the form
\[ R' = P_{\geq k} - \frac{1}{2} + \epsilon P_{k-1}(\cdot)\partial_x^k = \frac{1}{2} - P_{< k} + \epsilon P_{k-1}(\cdot)\partial_x^k. \]

The case: \( k = 0. \)

The Lax hierarchy is
\[ L_{tq} = [(L^q)_{\geq 0} + \epsilon [L^q]_{-1}, L] = - [(L^q)_{< 0} - \epsilon [L^q]_{-1}, L] \] (40)
and the appropriate Lax operators are given in the form
\[ L = \partial_x^N + u_{N-1}\partial_x^{N-1} + u_{N-2}\partial_x^{N-2} + \ldots + u_1\partial_x + u_0. \] (41)

From (40) one finds
\[ (u_{N-1})_{tq} = -\epsilon\partial_v_0([L^q]_{-1})_x \]
\[ (u_{N-2})_{tq} = \partial_v_0([L^q]_{-1})_x - \epsilon(N - 1)u_{N-1}([L^q]_{-1})_x - \epsilon(N - 1)/2([L^q]_{-1})_x. \] (42)

Hence, for \( \epsilon = 0 \) the field \( u_{N-1} \) becomes time-independent \( c_{N-1} \) one (let \( c_{N-1} = 0 \)), then Lax operator becomes a standard Lax operator [37]. Expression (42) implies the relation between fields \( u_{N-1}, u_{N-2} \)
\[ u_{N-2} = \frac{1}{\epsilon}u_{N-1} + \frac{N-1}{2N}(u_{N-1})^2 + \frac{N-1}{2}(u_{N-1})_x. \]

We eliminate the field \( u_{N-2} \) and as a result the Lax operators take the form
\[ L = \partial_x^N + u_{N-1}\partial_x^{N-1} + \left(-\frac{1}{\epsilon}u_{N-1} + \frac{N-1}{2N}(u_{N-1})^2 + \frac{N-1}{2}(u_{N-1})_x\right)\partial_x^{-2} + \ldots + u_0. \] (43)

In the dispersionless limit [13] reduces to (22). Reparameterizing (43): \( u_{N-1} \mapsto -\epsilon u_{N-2} \) and then taking limit \( \epsilon \to 0 \) we obtain the standard Lax operator [37].
Lemma 4.1 For arbitrary $\epsilon$ the Lax hierarchy (40), (43) is equivalent to the Lax hierarchy (36), (37).

We are looking for relations between fields from Lax operators (43) and (37), respectively. They are given in the following form:

$$u_{N-1} \mapsto -\epsilon u_{N-2}$$
$$u_i \mapsto u_i + f_i(u_{N-2}, u_{N-3}, \ldots, u_{i+1}) \quad \text{for} \quad N-3 \geq i \geq 0. \quad (44)$$

The square brackets in (44) mean that functions $f_i$, in opposite to the case (23), depend not only on $u_i$, but also on the derivatives $(u_i)_x, (u_i)_{xx}, \ldots$. Functions $f_i$ are constructed in such a way that hierarchy (40) will lead to the same evolution system as hierarchy (36) for $k = 0$. Argumentation that this equality indeed holds is of the same nature as in Section 3 the paragraph $s = k = 0$.

Example 4.2 KdV: $N = 2$.

For the constrained Lax operator (43) of the form $L = \partial_x^2 + u \partial_x + \frac{1}{4} u^2 - \frac{1}{\epsilon} u + \frac{1}{2} u_x$ we find reparameterized KdV:

$$[L^2_{t3}]_{\geq 0} + \epsilon \left[ L^2_{t3} \right]_{-1}, L ] \quad \iff \quad u_{t3} = \frac{1}{4} u_{3x} - \frac{3}{2\epsilon} uu_x.$$

The transformation to the standard form of KdV is given by $u \mapsto -\epsilon u$.

Example 4.3 Deformed Boussinesq: $N = 3$.

Let $L = \partial_x^3 + u \partial_x^2 + \left(\frac{1}{3} u^2 - \frac{1}{\epsilon} u + u_x\right) \partial_x + w$ then

$$[L^2_{t2}]_{\geq 0} + \epsilon \left[ L^2_{t2} \right]_{-1}, L ] \iff \quad u_{t2} = -\frac{4}{3} uu_x - u_{2x} + \frac{2}{9} \epsilon u^2 u_x + \frac{2}{3} \epsilon u_x^2 + \frac{2}{3} \epsilon uu_{2x} + \frac{2}{3} \epsilon uu_{3x} - 2\epsilon w_x$$
$$w_{t2} = -\frac{2}{3} uu_x - \frac{8}{9} u^2 u_x + \frac{2}{3} \epsilon u_x^2 + \frac{2}{3} \epsilon uu_{2x} + \frac{2}{3} \epsilon uu_{3x} - \frac{8}{27} u^3 u_x - \frac{14}{9} uu_{2x}$$
$$+ \frac{4}{3} uu_x - \frac{10}{9} u^2 u_x - \frac{8}{3} uu_{2x} + w_{2x} - \frac{14}{9} uu_{3x} - \frac{2}{3} u_{4x} + \frac{2}{81} \epsilon u^4 u_x$$
$$+ \frac{8}{27} uu_{3x} - \frac{2}{3} uu_{2x} + \frac{10}{27} \epsilon u_x^3 - \frac{2}{3} \epsilon u_x^2 w_x - \frac{2}{3} \epsilon uu_{2x} + \frac{4}{27} \epsilon uu_{2x} + \frac{4}{3} uu_{2x} u_{2x}$$
$$+ \frac{2}{3} uu_{2x} + \frac{2}{3} uu_{3x} + \frac{10}{27} \epsilon u_x^2 u_{3x} + \frac{10}{9} \epsilon uu_{3x} u_{3x} - \frac{2}{3} \epsilon uu_{3x} - \frac{4}{9} uu_{4x} + \frac{2}{9} \epsilon uu_{5x}.$$  

Eliminating the field $w$ we obtain reparameterized Boussinesq: $u_{t4} = (\frac{2}{3} u^2 - \frac{1}{3} u_{xx})_{xx}$. The transformation (44) to the standard form of the Boussinesq system is given by: $u \mapsto -\epsilon u, w \mapsto w - \frac{1}{3} \epsilon u^2 - \frac{1}{27} \epsilon^3 u^3 + \frac{1}{3} \epsilon^2 uu_x - \frac{1}{3} \epsilon u_{2x}$.

The case: $k = 1$.

The Lax hierarchy becomes

$$L_{tq} = [(L^q)_{\geq 1} + \epsilon [L^q]_0 \partial_x, L] = -[(L^q)_{<1} - \epsilon [L^q]_0 \partial_x, L] \quad (45)$$
and the appropriate Lax operators have the form
\[ L = u_N \partial_x^N + u_{N-1} \partial_x^{N-1} + \ldots + u_0 + \partial_x^{-1}u_1. \] (46)

From (45) one finds that \( (u_N)_t = \epsilon (u_N)_x [L^q]_0 - \epsilon N u_N([L^q]_0)_x. \) Hence in the limit \( \epsilon = 0 \) the field \( u_N \) becomes a time-independent \( c_N \) one, (let \( c_N = 1 \)), then Lax operator becomes a standard Lax operator (46). There is no constraint contrary to the previous case. The Lax operators (45) with (46) lead to the construction of new integrable soliton systems, at least to the best of our knowledge. Again the zero power of \( L \) always leads to the space translation symmetry: \( L_{t_0} = \epsilon L_x. \)

**Example 4.4** Extended Kaup-Broer: \( N = 1. \)

Let \( L = u \partial_x + v + \partial_x^{-1}w, \) then for \( L_{t_i} = [(L^i)_{\geq 1} + \epsilon [L^i]_0 \partial_x, L] \) we find

\[
\begin{pmatrix}
  u \\
  v \\
  w
\end{pmatrix}
t_0 = \epsilon
\begin{pmatrix}
  u_x \\
  v_x \\
  w_x
\end{pmatrix}
\]

\[
\begin{pmatrix}
  u \\
  v \\
  w
\end{pmatrix}
t_1 = \epsilon
\begin{pmatrix}
  \epsilon u_x v - \epsilon w_x \\
  w_x + \epsilon v_x \\
  u_x w + uw_x + \epsilon v_x w + \epsilon vw_x
\end{pmatrix}
\]

\[
\begin{align*}
  u_{t_2} &= \epsilon u_x v^2 - 2\epsilon uvv_x + 2\epsilon v^2 w_x - \epsilon^2 v_{2x} \\
  v_{t_2} &= 2u_x w + 2uvx_x + 2u^2 w_x + uw_x + u^2 v_{2x} + \epsilon v^2 v_x + 2\epsilon uvw_x + \epsilon vw_x \\
  w_{t_2} &= 2u_x vw + 2uv_x w + 2uvw_x - u_x^2 w - 3u_x w_x - uw_{2x} - u^2 w_{2x} + 2\epsilon u_x w^2 \\
  &\quad + 2\epsilonuvw_x + \epsilon uv_x v_x + \epsilon v^2 w_x + 4\epsilon uvw_x + \epsilon v_x w_x + \epsilon uv_{2x} w
\end{align*}
\]

\[ \vdots \]

It is dispersive counterpart of the hierarchy from Example 3.4. In the limit \( \epsilon \to 0 \) and \( u = 1 \) we obtain the standard Kaup-Broer system.

**Example 4.5** Two field system: \( N = 0. \)

For \( L = \partial_x + w \partial_x^{-1} \) we have \( [L^i]_0 = v^i \) and \( (L^i)_{\geq 1} = 0, \) where \( i = 0, 1, 2, \ldots \). Then, we obtain for \( L_{t_i} = [\epsilon [L^i]_0 \partial_x, L] \) again the dispersionless hierarchy (28).

## 5 Lattice soliton systems

Let \( \mathfrak{g} \) be the algebra of shift operators \( \mathfrak{g} \)

\[ \mathfrak{g} = \left\{ L = \sum_{i \in \mathbb{Z}} u_i(x) \mathcal{E}^i \right\} \] (47)

where \( \mathcal{E} \) is the shift operator such that \( \mathcal{E}^m u(x) = u(x + m) \mathcal{E}^m. \) The Lie algebra structure of \( \mathfrak{g} \) is given by the commutator \( [L_1, L_2] = L_1 L_2 - L_2 L_1. \) We consider simple decomposition of \( \mathfrak{g} \) in the form \( A_{\geq k} = \{ \sum_{i \geq k} u_i \mathcal{E}^i \} \) and \( A_{< k} = \{ \sum_{i < k} u_i \mathcal{E}^i \}, \) which are Lie subalgebras for \( k = 0, 1. \) In these cases the standard \( R \)-matrix is given by

\[ R = \frac{1}{2} (P_{\geq k} - P_{< k}) = P_{\geq k} - \frac{1}{2} = \frac{1}{2} - P_{< k}. \]
The Lax hierarchy is

\[ L_{tq} = [(L^q)_{\geq k}, L] = - [(L^q)_{< k}, L], \quad k = 0, 1. \quad (48) \]

Notice that these two cases are related by simple transformation \( \mathcal{E} \mapsto \mathcal{E}^{-1} \) and \( u_i(x-m) \mapsto u_{-i}(x+m) \). Then, \( k = 0 \) goes to \( k = 1 \) and vice versa. So, it is enough to consider only the first case. For \( k = 0 \), the appropriate Lax operators are of the form

\[ L = \mathcal{E}^N + u_{N-1}(x)\mathcal{E}^{N-1} + \ldots + u_{1-m}(x)\mathcal{E}^{1-m} + u_{-m}(x)\mathcal{E}^{-m}. \quad (49) \]

The powers of \( L \) are in general fractional and can be constructed in two ways: for \( L^\frac{1}{N} = a_1\mathcal{E} + a_0 + a_{-1}\mathcal{E}^{-1} + \ldots \) by requiring \( (L^\frac{1}{N})^N = L \) and for \( L^\frac{1}{m} = \ldots + a_1\mathcal{E} + a_0 + a_{-1}\mathcal{E}^{-1} \) by requiring \( (L^\frac{1}{m})^m = L \). Then, in (48) we use \( L^\frac{1}{N} \) and \( L^\frac{1}{m} \) for \( i = 0, 1, 2 \ldots \). The Lax hierarchies (48) for \( k = 0 \) are dispersive counterparts of the dispersionless hierarchies (49) for \( s = 1, k = 1 \) and \( s = 1, k = 2 \), respectively.

The simple deformations satisfying (50) are of the form

\[ k = 0, 1: \quad r = P_q. \]

But for the same reason as above it is enough to consider the case \( k = 0 \).

The case: \( k = 0 \).

The deformed \( R \)-matrix is given by

\[ R = P_{\geq 0} - \frac{1}{2} + \epsilon P_0 = \frac{1}{2} - P_{< 0} + \epsilon P_0. \]

Hence

\[ L_{tq} = [(L^q)_{\geq 0} + \epsilon [L^q]_0, L] = - [(L^q)_{< 0} + \epsilon [L^q]_0, L]. \quad (50) \]

The appropriate Lax operators are of the form

\[ L = u_N(x)\mathcal{E}^N + u_{N-1}(x)\mathcal{E}^{N-1} + \ldots + u_{1-m}(x)\mathcal{E}^{1-m} + u_{-m}(x)\mathcal{E}^{-m}. \quad (51) \]

From (50) it follows that

\[ u_N(x)_t = \epsilon u_N(x) \left( 1 - E^N \right) [L^q]_0 \quad \ldots \quad u_{-m}(x)_t = (1 + \epsilon)u_{-m}(x) \left( 1 - E^{-m} \right) [L^q]_0. \quad (52) \]

As a result we find that the highest and lowest field are interrelated in the following way

\[ \left( \frac{u_N(x)}{u_N(x-m)} \right)^{1+\epsilon} = \left( \frac{u_{-m}(x)}{u_{-m}(x+N)} \right)^\epsilon. \quad (53) \]

For \( \epsilon = 0 \) the field \( u_N \) becomes a time-independent \( c_N \) one, (let \( c_N = 1 \)), then the Lax operator (51) becomes a standard Lax operator (49) for \( k = 0 \). For \( \epsilon = -1 \) the field \( u_{-m} \) becomes time-independent and the Lax operator becomes a standard Lax operator for \( k = 1 \). It is so, because for \( k = 0 \) and \( \epsilon = -1 \) the deformed \( R \)-matrix becomes the standard \( R \)-matrix for \( k = 1 \).
Lemma 5.1 For arbitrary $\epsilon$, the Lax hierarchy (50), (51) is equivalent to the Lax hierarchy (48), (49) for $k = 0$.

Consider the following transformations:

\[
\mathcal{E}'^{N} = u_{N}(x)\mathcal{E}^{N} \iff \mathcal{E}' = a(x)\mathcal{E}
\]

\[
u'_{i}(x) = \begin{cases} 
\frac{a(x)a(x+1)\cdots a(x+i-1)}{a(x+1)a(x+2)\cdots a(x+i-1)} & \text{for } N-1 \geq i > 0 \\
u_{0}(x) & \text{for } i = 0 \\
u_{i}(x)a(x-1)a(x-2)\cdots a(x-i) & \text{for } 0 > i > -m 
\end{cases}
\]  

(54)

and $t' = t$, where $a(x)$ is given by the following relation

\[
u_{N}(x) = a(x)a(x+1)\cdots a(x+N-1).
\]  

(55)

It transforms the Lax operators (51) into the Lax operators (49) with $u'_{i}(x)$ components. From (54) it follows that

\[
\left(\mathcal{E}'^{N}\right)_{t} = \Pi_{i} \left(\ln a(x)\right)_{t} \mathcal{E}^{ni}
\]  

(56)

where

\[
\Pi_{i} = \begin{cases} 
1 + E + \cdots + E^{i-1} & \text{for } i \geq 1 \\
0 & \text{for } i = 0 \\
-E^{-1} - E^{-2} - \cdots - E^i & \text{for } i \leq -1 
\end{cases}
\]

One finds also from (55), that

\[
\left(\ln a(x)\right)_{t} = (\Pi_{N})^{-1} \left(\ln \nu_{N}(x)\right)_{t} \text{ by } (56) \epsilon (\Pi_{N})^{-1} (1 - E^{N}) [L^{q}]_{0}.
\]

Then, using relation $(1 - E^{N})\Pi_{i} = (1 - E^{i})\Pi_{N}$ which is valid for arbitrary $N, i > 0$, we have

\[
L_{t} = L_{\nu} + \sum_{i=-m}^{N} \nu'_{i}(x) \left(\mathcal{E}'^{N}\right)_{t} \text{ by } (56) L_{\nu} + \epsilon \sum_{i=-m}^{N} \nu'_{i}(x) (1 - E^{N}) \Pi_{i} (\Pi_{N})^{-1} [L^{q}]_{0}\mathcal{E}^{ni} = L_{\nu} + \epsilon [L^{q}]_{0}, L
\]

where $u'_{N}(x) = 1$. Hence, the hierarchy (50) becomes (48) one with $k = 0$.

Example 5.2 Extended Toda: $k = 1$.

For the Lax operator $L = u(x)\mathcal{E} + v(x) + w(x)\mathcal{E}^{-1}$ and $L_{t_{1}} = [(L)_{>0} + \epsilon[L]_{0}, L]$ we find

\[
\begin{pmatrix} u(x) \\ v(x) \\ w(x) \end{pmatrix} = \begin{pmatrix} -\epsilon u(x) [v(x+1) - v(x)] \\ u(x)w(x+1) - u(x-1)w(x) \\ (1 + \epsilon) [v(x) - v(x-1)] w(x) \end{pmatrix}.
\]  

(57)

Again, in the limit $\epsilon \to 0$ and $u(x) = 1$ or by the transformation (54): $v(x) \mapsto v(x), w(x) \mapsto \frac{w(x)}{u(x)-1}$ we obtain the standard Toda system. In the limit $\epsilon = -1$ and $w(x) = 1$ we obtain the reparameterized Toda system. The fields $u(x)$ and $v(x)$ in (57) according to (53) are related by $u(x-1)^{1+\epsilon} = w(x)^{-\epsilon}$.
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