Analyzing Three Factor Experiments using Partitioned Design Matrices

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Abstract. When analyzing factorial experiments, especially from just looking at the data layout, we could end up at wrong conclusion. Several possibilities of the models using three factors are being discussed. The models are characterized by experimental unit conditions and the way the treatments are allocated to the experimental units. In some experimental situation, it also depends on sampling units. Those models are: factorial experiments in completely randomized design, factorial experiments in randomized complete block design, split plot design with one factor allocated in the main plot, split plot design with two factors allocated in the main plot, and split-split plot design. Partitioned Design Matrices approach could be an alternative solution in analyzing experimental design models instead of using usual sigma notation for summation. Kronecker product is used to build the design matrix for each source of variation components used in the model. This approach is much easier and simpler to be used in explaining on how the partitioned design matrices for each source of variation being built, to calculate the degrees of freedom and sum of squares, to show that each source of variation has a chi-square distribution, and to show the independence between two sum of squares.

1. Introduction
Sum of squares calculation for each source of variation is one step in the design analysis. In general, total sum of squares consists of model sum of square and experimental error sum of square [1].

The aspect of being studied in a single experiment is called a factor. In addition, different categories in a factor are called levels. When an experiment is designed / conducted using more than one factor, then it is called a factorial experiment [2].

All treatment levels used in one replication are cross combination from all treatment levels used in the experiment. When each treatment level is replicated r times, then the number experimental units in the experiment should be r times the number of all treatment levels [3].

Classical formulas that commonly used in most literatures to calculate sum of squares for each source of variation are using algebraic notation. Moreover, several summations have to be used depending the number of subscripts representing the number of factors and replication [4]. Special care must be taken into account in order to avoid wrong calculation. Using matrices notation could be a better alternative to calculate sum of squares for each of source of variation in the model. Since the design matrix in an Experimental Design Model is not a full rank matrix, the generalized inverse has to be used for the calculation of the model sum of squares [5].

Partitioned design matrices to analyze split-plot experimental design has been used by [6]. This article discusses the analysis of three factor experiments using partitioned design matrices regarding several possibilities of experimental units conditions and treatment allocations.
2. Notation
Consider several models that can be used for three factor experiments, such as Three factor experiments in a Completely Randomized Design Model; Three factor experiments in a Randomized Complete Block Design Model; Split-Plot with One Factor in the Main-Plot; Split-Plot with Two-Factor in the Main Plot; and Split-Split Plot Model. In all models, we only consider the balanced models.

Three factor experiments in a Completely Randomized Design Model could be written as
\[
Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_ij + (\alpha\gamma)_ik + (\beta\gamma)_jk + e_{ijkl}
\]
\[
i = 1,2,\ldots,a \quad j = 1,2,\ldots,b \quad k = 1,2,\ldots,c \quad l = 1,2,\ldots,r
\]  
with \( Y_{ijkl} \) \( l \)-th observation receiving \( i \)-th level of factor A, \( j \)-th level of factor B and \( k \)-th level of factor C; \( \mu \) general mean; \( \alpha_i \) the \( i \)-th level of factor A effect; \( \beta_j \) the \( j \)-th level of factor B effect; \( \gamma_k \) the \( k \)-th level of factor C effect; \( (\alpha\beta)_ij \) the \( i \)-th level of factor A and \( j \)-th level of factor B interaction effect; \( (\alpha\gamma)_ik \) the \( i \)-th level of factor A and \( k \)-th level of factor C interaction effect; \( (\beta\gamma)_jk \) the \( j \)-th level of factor B and \( k \)-th level of factor C interaction effect; \( e_{ijkl} \) the \( i \)-th level of factor A, \( j \)-th level of factor B and \( k \)-th level of factor C interaction effect; \( e_{ijkl} \) the \( i \)-th level of factor A, \( j \)-th level of factor B and \( k \)-th level of factor C interaction effect; \( e_{ijkl} \) the \( i \)-th level of factor A, \( j \)-th level of factor B and \( k \)-th level of factor C interaction effect. By assuming all treatment combinations levels are repeated \( r \) times, the model stated in eq. (1) is also called Balanced Completely Randomized Design Model. The model is characterized by the randomization where all the \( abc \) treatment combinations replicated \( r \) times are allocated randomly throughout the homogeneous as possible \( rabc \) experimental units.

Three factor experiments in a Randomized Complete Block Design Model could be written as
\[
Y_{ijkl} = \mu + \rho_i + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_ij + (\alpha\gamma)_ik + (\beta\gamma)_jk + e_{ijkl}
\]
\[
i = 1,2,\ldots,a \quad j = 1,2,\ldots,b \quad k = 1,2,\ldots,c \quad l = 1,2,\ldots,r
\]  
with \( \rho_i \) the \( l \)-th level of the block, \( \alpha_i \) the \( i \)-th level of factor A effect; \( \beta_j \) the \( j \)-th level of factor B effect; \( \gamma_k \) the \( k \)-th level of factor C effect; \( (\alpha\beta)_ij \) the \( i \)-th level of factor A and \( j \)-th level of factor B interaction effect; \( (\alpha\gamma)_ik \) the \( i \)-th level of factor A and \( k \)-th level of factor C interaction effect; \( (\beta\gamma)_jk \) the \( j \)-th level of factor B and \( k \)-th level of factor C interaction effect; \( e_{ijkl} \) the \( i \)-th level of factor A, \( j \)-th level of factor B and \( k \)-th level of factor C interaction effect; \( e_{ijkl} \) the \( i \)-th level of factor A, \( j \)-th level of factor B and \( k \)-th level of factor C interaction effect. The \( abc \) homogeneous experimental units are provided in each and every blocks, while the homogeneity from one block to another blocks are not the same. A number of \( r \) blocks should be provided as replications.

Three Factor Experiment in Split-Plot with One Factor in the Main-Plot Model (A is in the main plot)
\[
Y_{ijkl} = \mu + \rho_i + \alpha_i + \delta_{ij} + \beta_j + \gamma_k + (\alpha\beta)_ij + (\alpha\gamma)_ik + (\beta\gamma)_jk + e_{ijkl}
\]
\[
i = 1,2,\ldots,a \quad j = 1,2,\ldots,b \quad k = 1,2,\ldots,c \quad l = 1,2,\ldots,r
\]  
with \( \delta_{ij} \) the main plot error, which is actually equal to the interaction between \( l \)-th level of the block and the \( i \)-th level of factor A, while the other notations are the same as in eq. (2). In each replication, the main plot is divided into \( a \) sub plot, where the levels of factor A are randomly allocated first. Then for each level of A in each sub plot, \( bc \) treatment combinations of factor B and C are randomly assigned. Such, randomization has to be done for each block or replication.

Three Factor Experiment in Split-Plot with Two-Factor in the Main Plot Model (A and B are in the main plot)
\[
Y_{ijkl} = \mu + \rho_i + \alpha_i + \beta_j + (\alpha\beta)_ij + \varphi_{ij} + \gamma_k + (\alpha\gamma)_ik + (\beta\gamma)_jk + e_{ijkl}
\]
\[
i = 1,2,\ldots,a \quad j = 1,2,\ldots,b \quad k = 1,2,\ldots,c \quad l = 1,2,\ldots,r
\]  
with \( \varphi_{ij} \) the main plot error or the whole plot error, while the other notations are the same as in eq. (3). In each replication, the main plot is divided into \( ab \) sub plots, where the levels treatment combination of factor A and B are randomly allocated first. Then for each levels treatment combination of A and B in each sub plot, \( c \) treatment levels of factor C are randomly assigned. Such, randomization has to be done for each blocks or replications.

Three Factor Experiment in Split-Split-Plot Model.
\[ Y_{ijl} = \mu + \rho_i + \alpha_j + \beta_k + (\alpha\beta)_ij + \gamma_l + (\alpha\gamma)_jl + (\beta\gamma)_lk + (\alpha\beta\gamma)_{ijl} + \epsilon_{ijl} \]

with \( \delta_i \) the main plot error or whole plot error and \( \eta_{jl} \) the sub plot error, while the other notations are the same as in eq. (4). In each replication, the main plot is divided into \( a \) sub plots, where the levels of factor A are allocated first. Then, for each level of A in each sub plot, it is divided into \( b \) sub-sub plots to assign randomly the levels of factor B. Finally, in each sub-sub plot to where each levels of factor B is assigned, \( c \) levels of factor C are randomly allocated. This basic Split-Split Plot Model stated as in eq. (5) can be generalized or expanded into more complex models whenever the experiment uses more than three factors.

Using matrix notation, in general, the model could be written as \( Y = X\beta + \epsilon \) with \( Y \) is the vector of observation having size of \( abcr \) x \( l \), while the design matrix \( X \) has the size of \( abcr \times (1+a+b+c+ab+ac+bc+abc) \) for the model as described in eq. (1), and is partitioned as \([X_{\mu},X_{\alpha},X_{\beta},X_{\gamma},X_{\alpha\beta},X_{\alpha\gamma},X_{\beta\gamma},X_{\alpha\beta\gamma}]\) with

\[
X_{\mu} = I_{cel} \otimes I_{cdl} \otimes I_{crd} \otimes I_{crl} \\
X_{\alpha} = I_{cve} \otimes I_{cel} \otimes I_{crd} \otimes I_{crl} \\
X_{\beta} = I_{cve} \otimes I_{cel} \otimes I_{crd} \otimes I_{crl} \\
X_{\gamma} = I_{cve} \otimes I_{cel} \otimes I_{crd} \otimes I_{crl} \\
X_{ab} = I_{cve} \otimes I_{veh} \otimes I_{cre} \otimes I_{crl} \\
X_{ac} = I_{cve} \otimes I_{veh} \otimes I_{cre} \otimes I_{crl} \\
X_{bc} = I_{cve} \otimes I_{veh} \otimes I_{cre} \otimes I_{crl} \\
X_{abc} = I_{cve} \otimes I_{veh} \otimes I_{cre} \otimes I_{crl}
\]

and \( \beta \) is the vector of parameters having size of \((1+a+b+c+ab+ac+bc+abc) \times l\),

\( \beta^T = (\mu, \alpha_1, \alpha_2, …, \alpha_{ab}, \beta_1, \beta_2, …, \beta_{ab}; \gamma_1, …, \gamma_{abc}; \gamma_1, \gamma_2, …, \gamma_{abc}; \gamma_1, \gamma_2, …, \gamma_{abc}; \gamma_1, \gamma_2, …, \gamma_{abc}; \gamma_1, \gamma_2, …, \gamma_{abc}; \gamma_1, \gamma_2, …, \gamma_{abc}; \gamma_1, \gamma_2, …, \gamma_{abc}) \) and \( \epsilon \) is the vector of experimental error with the size of \( abcr \times l \). The number of columns of the design matrices for other models are the same as the number of parameters used.

3. Sum of Squares and Degrees of Freedom

Before calculating the Sum of Squares for each component, we calculate the projection matrix in the form of

\[ M_s = X'(X'X)^{-1}X' \]

Therefore, \( M_\mu, M_\alpha, M_\beta, M_\gamma, M_{ab}, M_{ac}, M_{bc}, \) and \( M_{abc} \) are representing the projection matrices for the constant, main effect (A, B, and C), and interaction (AB, AC, BC, and ABC) stated in the source of variation of the model in the eq. (1). Using the properties of Kronecker Product on transpose and multiplication, it can be shown the following projection matrices:

\[
M_\mu = (rac)^{-1} J_{cve} \otimes J_{veh} \otimes J_{cre} \otimes J_{crl} \\
M_\alpha = (rbc)^{-1} I_{cve} \otimes J_{veh} \otimes J_{cre} \otimes J_{crl} \\
M_\beta = (rac)^{-1} J_{cve} \otimes I_{veh} \otimes J_{cre} \otimes J_{crl} \\
M_\gamma = (rab)^{-1} J_{cve} \otimes J_{veh} \otimes I_{cre} \otimes J_{crl}
\]
The sum of squares for each source of variation component having quadratic form of $Y^tZY$, where $Z$ has the following pattern $s$:

- $I - M_\mu$ for the Total,
- $M_X - M_\mu$ for the main effects,
- $M_{YZ} - M_Y - M_Z + M_\mu$ for the interaction between two factors, and
- $M_{WYZ} - M_W - M_{YZ} - M_{WY} + M_W + M_Y + M_Z - M_\mu$ for the interaction involving three factors; while the $Z$ for the experimental error could be determined by subtracting from the Total by all source of variation components.

Next, consider any vector of random variables $Y$ having size $n \times 1$ that is distributed normally multivariate $N(y : \mu, I)$, then a quadratic random variable $U = Y^tZY$ is distributed as $\chi^2(u;K;\lambda)$ with $K$ degrees of freedom, and the noncentrality parameter $\lambda$, with $\lambda = \mu'Z\mu/2$, if and only if $Z$ is an idempotent matrix having rank $K$ [7].

For the balanced design models, the $Z$ matrices and the corresponding Expected Mean Squares are tabulated from Table 1 for Three Factor Experiments in a Completely Randomized Design, Table 2 for Three Factor Experiments in a Randomized Complete Block Design, Table 3 for Three Factor Experiments in a Split Plot Design with One Factor in the Main Plot, Table 4 for Three Factor Experiments in a Split Plot Design with Two Factors in the Main Plot, and Table 5 for Three Factor Experiments in a Split-Split-Plot Design.

**Table 1. The $Z$ Matrices and Expected Means Squares for Three Factor Experiments in a Completely Randomized Design (Fixed Model)**

| Source of Variation | $Z$ Matrices | Expected Mean Square |
|---------------------|--------------|---------------------|
| A                   | $M_a - M_\mu$ | $\sigma^2 + 2 \mu^T \kappa_a$ |
| B                   | $M_b - M_\mu$ | $\sigma^2 + 2 \mu^T \kappa_b$ |
| C                   | $M_c - M_\mu$ | $\sigma^2 + 2 \mu^T \kappa_c$ |
| AB                  | $M_{ab} - M_a - M_b + M_\mu$ | $\sigma^2 + 2 \mu^T \kappa_{ab}$ |
| AC                  | $M_{ac} - M_a - M_c + M_\mu$ | $\sigma^2 + 2 \mu^T \kappa_{ac}$ |
| BC                  | $M_{bc} - M_b - M_c + M_\mu$ | $\sigma^2 + 2 \mu^T \kappa_{bc}$ |
| ABC                 | $M_{abc} - M_{ab} - M_{ac} - M_{bc} + M_a + M_b + M_c - M_\mu$ | $\sigma^2 + 2 \mu^T \kappa_{abc}$ |
| Exp Error           | $I - M_{abc}$ | $\sigma^2$ |
Table 2. The $Z$ Matrices and Expected Means Squares for Three Factor Experiments in a Randomized Complete Block Design (Fixed Model)

| Source of Variation | $Z$ Matrices | Expected Mean Square |
|---------------------|--------------|----------------------|
| Blocks/Reps         | $M_i - M_\mu$ | $\sigma_i^2 + abc \kappa_{abc}^i$ |
| A                   | $M_a - M_\mu$ | $\sigma_i^2 + rbc \kappa_a^i$ |
| B                   | $M_b - M_\mu$ | $\sigma_i^2 + rac \kappa_a^i$ |
| C                   | $M_c - M_\mu$ | $\sigma_i^2 + rbc \kappa_c^i$ |
| AB                  | $M_{ab} - M_a - M_b + M_\mu$ | $\sigma_i^2 + abc \kappa_{abc}^i$ |
| AC                  | $M_{ac} - M_a - M_c + M_\mu$ | $\sigma_i^2 + rb \kappa_{ac}^i$ |
| BC                  | $M_{bc} - M_b - M_c + M_\mu$ | $\sigma_i^2 + ra \kappa_{bc}^i$ |
| ABC                 | $M_{abc} - M_a - M_b - M_c + M_\mu$ | $\sigma_i^2 + r \kappa_{abc}^i$ |

Table 3. The $Z$ Matrices and Expected Means Squares for Three Factor Experiments in a Split Plot Design with One Factor in the Main Plot (Fixed Model)

| Source of Variation | $Z$ Matrices | Expected Mean Square |
|---------------------|--------------|----------------------|
| Blocks/Reps         | $M_i - M_\mu$ | $\sigma_i^2 + bc \sigma_i^2 + abc \kappa_{abc}^i$ |
| A                   | $M_a - M_\mu$ | $\sigma_i^2 + bc \sigma_i^2 + rbc \kappa_a^i$ |
| Main Plot Error     | $M_{ia} - M_i - M_a + M_\mu$ | $\sigma_i^2 + bc \sigma_i^2$ |
| B                   | $M_b - M_\mu$ | $\sigma_i^2 + rac \kappa_a^i$ |
| C                   | $M_c - M_\mu$ | $\sigma_i^2 + rbc \kappa_c^i$ |
| AB                  | $M_{ab} - M_a - M_b + M_\mu$ | $\sigma_i^2 + rc \kappa_{ac}^i$ |
| AC                  | $M_{ac} - M_a - M_c + M_\mu$ | $\sigma_i^2 + rb \kappa_{ac}^i$ |
| BC                  | $M_{bc} - M_b - M_c + M_\mu$ | $\sigma_i^2 + ra \kappa_{bc}^i$ |
| ABC                 | $M_{abc} - M_a - M_b - M_c + M_\mu$ | $\sigma_i^2 + r \kappa_{abc}^i$ |
| Exp Error           | $I - M_{ia} - M_a + M_\mu$ | $\sigma_i^2$ |

Table 4. The $Z$ Matrices and Expected Means Squares for Three Factor Experiments in a Split Plot Design with Two Factors in the Main Plot (Fixed Model)

| Source of Variation | $Z$ Matrices | Expected Mean Square |
|---------------------|--------------|----------------------|
| Blocks/Reps         | $M_i - M_\mu$ | $\sigma_i^2 + c \sigma_i^2 + abc \kappa_{abc}^i$ |
| A                   | $M_a - M_\mu$ | $\sigma_i^2 + c \sigma_i^2 + rbc \kappa_a^i$ |
| Main Plot Error     | $M_{rab} - M_r - M_{ab} + M_\mu$ | $\sigma_i^2 + c \sigma_i^2$ |
| B                   | $M_b - M_\mu$ | $\sigma_i^2 + rac \kappa_a^i$ |
| C                   | $M_c - M_\mu$ | $\sigma_i^2 + rbc \kappa_c^i$ |
| AB                  | $M_{abc} - M_a - M_b - M_c + M_\mu$ | $\sigma_i^2 + c \kappa_{abc}^i$ |
| AC                  | $M_{ac} - M_a - M_c + M_\mu$ | $\sigma_i^2 + rb \kappa_{ac}^i$ |
| BC                  | $M_{bc} - M_b - M_c + M_\mu$ | $\sigma_i^2 + ra \kappa_{bc}^i$ |
| ABC                 | $M_{abc} - M_a - M_b - M_c + M_\mu$ | $\sigma_i^2 + r \kappa_{abc}^i$ |
| Exp Error           | $I - M_{rab} - M_{abc} + M_{ab}$ | $\sigma_i^2$ |
Table 5. The Z Matrices and Expected Means Squares for Three Factor Experiments in a Split Split Plot Design (Fixed Model)

| Source of Variation | Z Matrices                      | Expected Mean Square          |
|---------------------|---------------------------------|-------------------------------|
| Blocks/Reps         | $M_i - M_\mu$                   | $\sigma_i^2 + c \sigma_\epsilon^2 + c \sigma_\delta^2 + abc \kappa_{abc}^2$ |
| A                   | $M_a - M_\mu$                   | $\sigma_i^2 + c \sigma_\epsilon^2 + b c \sigma_\delta^2 + r b c \kappa_{abc}^2$ |
| Main Plot Error     | $M_{ra} - M_i - M_a + M_\mu$    | $\sigma_i^2 + c \sigma_\epsilon^2 + b c \sigma_\delta^2$ |
| B                   | $M_b - M_\mu$                   | $\sigma_i^2 + c \sigma_\epsilon^2 + r c \kappa_{abc}^2$ |
| AB                  | $M_{ab} - M_a - M_b + M_\mu$    | $\sigma_i^2 + c \sigma_\epsilon^2 + r c \kappa_{abc}^2$ |
| Sub Plot Error      | $M_{rab} - M_{ra} - M_{ab} + M_\mu$ | $\sigma_i^2 + c \sigma_\epsilon^2 + r c \kappa_{abc}^2$ |
| C                   | $M_c - M_\mu$                   | $\sigma_i^2 + r a b \kappa_{abc}^2$ |
| AC                  | $M_{ac} - M_a - M_c + M_\mu$    | $\sigma_i^2 + r b c \kappa_{abc}^2$ |
| BC                  | $M_{bc} - M_b - M_c + M_\mu$    | $\sigma_i^2 + r a c \kappa_{abc}^2$ |
| ABC                 | $M_{abc} - M_{ab} - M_{ac} - M_{bc} + M_a + M_b + M_c - M_\mu$ | $\sigma_i^2 + r c \kappa_{abc}^2$ |
| Exp Error           | $I - M_{rab} - M_{abc} + M_{ab}$ | $\sigma_i^2$ |

It can be easily verified using the properties of Kronecker product on transpose and multiplication, the Z matrices for each source of variation are symmetric and idempotent. Thus, the rank for each of the matrices just mentioned is equal to the trace of the corresponding matrices [8]. As an example from the model stated in eq. (1), these properties are actually the corresponding degrees of freedom in the analysis of variance for the respected source of variation: main effect of A, B, and C; the interactions of: AB, AC, BC, and ABC; and also Experimental Error and Total:

$$db(A) = tr(M_a - M_\mu) = a - 1$$
$$db(B) = tr(M_b - M_\mu) = b - 1$$
$$db(C) = tr(M_c - M_\mu) = c - 1$$
$$db(AB) = tr(M_{ab} - M_a - M_b + M_\mu) = (a - 1)(b - 1)$$
$$db(AC) = tr(M_{ac} - M_a - M_c + M_\mu) = (a - 1)(c - 1)$$
$$db(BC) = tr(M_{bc} - M_b - M_c + M_\mu) = (b - 1)(c - 1)$$
$$db(ABC) = tr(M_{abc} - M_a - M_b - M_c + M_a + M_b + M_c - M_\mu) = (a - 1)(b - 1)(c - 1)$$
$$db(Exp Err) = tr(I - M_{abc}) = (r - 1)abc$$
$$db(Total) = tr(I - M_\mu) = rabc$$

Using the above information, without loss of generality, whenever $Y$ is distributed as $N(\mu, I)$, then for each source of variation in the model, the sum of squares is distributed as chi-squared with the respective degrees of freedom.

4. Hypothesis Testing
The ratio between sum of squares with its degrees of freedom, for each source of variation, is defined as Mean squares, $MS[*] = SS[*]/[df[*]]$. Furthermore, if $W$ is distributed as chi-square with $dbw$ degrees of freedom and $Z$ is distributed as chi-square with $dbz$ degrees of freedom, and also $W$ and $Z$ are independent, then $(W/dbw)/(Z/dbz)$ has an F distribution with degrees of freedom $dbw$ and $dbz$. 
Two quadratic form random variables $Y_A Y$ and $Y_B Y$ are said to be independent if and only if $AB = O$. It can be easily verified that all sum of squares of the source of variation are mutually independent.

Based on the expected means of squares for the model as stated in eq. (1) we do have the following results:

a. To test the main effect of A factor, reject the null hypothesis whenever $MS[A]/MS[Exp Err]$ is large enough. $MS[A]/MS[Exp Err]$ having $F$ distribution with degrees of freedom $a-1$ and $(r-1)abc$.

b. To test the main effect of B factor, reject the null hypothesis whenever $MS[B]/MS[Exp Err]$ is large enough. $MS[B]/MS[Exp Err]$ having $F$ distribution with degrees of freedom $b-1$ and $(r-1)abc$.

c. To test the main effect of C factor, reject the null hypothesis whenever $MS[C]/MS[Exp Err]$ is large enough. $MS[C]/MS[Exp Err]$ having $F$ distribution with degrees of freedom $c-1$ and $(r-1)abc$.

d. To test the $AB$ interaction, reject the null hypothesis whenever $MS[AB]/MS[Exp Err]$ is large enough. $MS[AB]/MS[Exp Err]$ having $F$ distribution with degrees of freedom $(a-1)(b-1)$ and $(r-1)abc$.

e. To test the $AC$ interaction, reject the null hypothesis whenever $MS[AC]/MS[Exp Err]$ is large enough. $MS[AC]/MS[Exp Err]$ having $F$ distribution with degrees of freedom $(a-1)(b-1)$ and $(r-1)abc$.

f. To test the $BC$ interaction, reject the null hypothesis whenever $MS[BC]/MS[Exp Err]$ is large enough. $MS[BC]/MS[Exp Err]$ having $F$ distribution with degrees of freedom $(b-1)(c-1)$ and $(r-1)abc$.

g. To test the $ABC$ interaction, reject the null hypothesis whenever $MS[ABC]/MS[Exp Err]$ is large enough. $MS[ABC]/MS[Exp Err]$ having $F$ distribution with degrees of freedom $(a-1)(b-1)(c-1)$ and $(r-1)abc$.

h. Similar rules could be used to make such proper hypothesis testing for the other models by considering the expected means squares.

5. Example

Consider we have the layout data of an experiment involving three factors where each factor is fixed and consists of two levels. Four replications in each level treatment combinations are used. The data in Table 6 are taken from [9].

Table 6. The Effect of Tool Type, Angel of Bevel, and Type of Cut on Power Consumption for Ceramic Tool Cutting. Code : 2(X-28).

| Type of Cut | Tool Type | 1 | 2 |
|-------------|-----------|---|---|
|              | Bevel Angle | Bevel Angle |
|              | 15° | 30° | 15° | 30° |
| Continuous  | 2   | 1   | 0   | 3   |
|            | -3  | 1   | 1   | 8   |
|            | 5   | 4   | 0   | 2   |
|            | -2  | 9   | -6  | 0   |
| Interrupted| 0   | -2  | -7  | -1  |
|            | -6  | 2   | -6  | 0   |
|            | -3  | -1  | 0   | -2  |
|            | -3  | -1  | -4  | -4  |
The sum of squares of the experimental error in the Balanced Completely Randomized Design listed in Table 7 is divided into the sum of squares of the Block and the sum of squares of the experimental error in the Randomized Complete Block Design listed in Table 8. And so does the degrees of freedom.

Table 9. The R Output of Analysis of Variance of Three Factor Experiments in Split-Plot Design with One Factor in the Main Plot.

| .[1] | .[2] | .[3] | .[4] | .[5] |
|------|------|------|------|------|
| "Source" | "Deg Frdm" | "SS" | "MS" | "F" |
| "Block" | "3" | "16.094" | "5.365" | "0.570" |
| "A" | "1" | "11.281" | "11.281" | "1.199" |
| "B" | "1" | "81.281" | "81.281" | "8.636" |
| "C" | "1" | "124.031" | "124.031" | "13.178" |
| "AB" | "1" | "0.781" | "0.781" | "0.003" |
| "AC" | "1" | "0.031" | "0.031" | "0.083" |
| "BC" | "1" | "3.781" | "3.781" | "0.083" |
| "Exp Error" | "21" | "197.656" | "9.412" | " " |
| "Total" | "31" | "435.719" | " " | " " |
The sum of squares of the experimental error in the Randomized Complete Block Design listed in Table 8 is divided into the sum of squares of the Main Plot Error and the sum of squares of the experimental error in the Split-Plot Design with One Factor in the Main Plot listed in Table 9. And so does the degrees of freedom.

Table 10. The R Output of Analysis of Variance of Three Factor Experiments in Split-Plot Design with Two Factors in the Main Plot.

| Source   | Deg Frdm | SS       | MS       | F     |
|----------|----------|----------|----------|-------|
| Block    | 3        | 16.094   | 5.365    | 0.428 |
| A        | 1        | 11.281   | 11.281   | 0.900 |
| B        | 1        | 81.281   | 81.281   | 6.486 |
| AB       | 1        | 0.781    | 0.781    | 0.062 |
| Main Plot Err | 9       | 112.781  | 12.531   | 1.112 |
| C        | 1        | 124.031  | 124.031  | 17.536 |
| AC       | 1        | 0.031    | 0.031    | 0.004 |
| BC       | 1        | 3.781    | 3.781    | 0.535 |
| ABC      | 1        | 0.781    | 0.781    | 0.110 |
| Exp Error | 12      | 84.875   | 7.073    |       |
| Total    | 31       | 435.719  |          |       |

The sum of squares of the experimental error in the Randomized Complete Block Design listed in Table 8 is divided into the sum of squares of the Main Plot Error and the sum of squares of the experimental error in the Split-Plot Design with Two Factors in the Main Plot listed in Table 10. And so does the degrees of freedom.

Table 11. The R Output of Analysis of Variance of Three Factor Experiments in Split-Split Plot Design

| Source   | Deg Frdm | SS       | MS       | F     |
|----------|----------|----------|----------|-------|
| Block    | 3        | 16.094   | 5.365    | 0.378 |
| A        | 1        | 11.281   | 11.281   | 0.795 |
| B        | 1        | 81.281   | 81.281   | 6.948 |
| AB       | 1        | 0.781    | 0.781    | 0.067 |
| Sub Plot Err | 6       | 70.188   | 11.698   | 1.654 |
| C        | 1        | 124.031  | 124.031  | 17.536 |
| AC       | 1        | 0.031    | 0.031    | 0.004 |
| BC       | 1        | 3.781    | 3.781    | 0.535 |
| ABC      | 1        | 0.781    | 0.781    | 0.110 |
| Exp Error | 12      | 84.875   | 7.073    |       |
| Total    | 31       | 435.719  |          |       |

The sum of squares of the Main Plot Error in the Split-Plot Design with Two Factors in the Main Plot listed in Table 10 is divided into the sum of squares of the Main Plot Error and the sum of squares of the Sub Plot error in the Split-Split-Plot Design listed in Table 11. And so does the degrees of freedom.
6. Conclusion and Suggestion
Since the design matrix in the experimental design model is not a full rank matrix, therefore, generalized inverse has to be used to perform the analysis of variance. Partitioned designed matrices method is used as an alternative to calculate the degrees of freedom and sum of squares for each source of variation components. By partitioning the design matrix with respect to the source of variation of the model, the generalized inverse are avoided. This method is much easier and simpler than using conventional sigma notation for summation, especially for calculating the sum of square. It also gives insight for explaining where the degrees of freedom come from for each source of variation. By looking closely the pattern of the partitioned for each component, programming using R for analyzing each model gets easier and easier.

When we look at the data only, we could end up at wrong conclusion. Therefore, we need to know how the experiment was conducted, that is the condition of the experimental units and also the way the treatment combinations are allocated to the experimental units. Sometimes, we also have to consider the sampling units.

Since this article only discusses the analysis of three factor balanced experiments, the reader may consider other experimental setups, the case for unbalanced models or even generalization for the case have just been discussed.

7. Acknowledgement
I would like to thank to all my colleagues in the Department of Statistics, University of Bengkulu for their hospitality and sincerity in giving the input in the discussion especially in the field of statistics. We often discuss something new to study. Last but not least, my special thank to the Faculty of Mathematics and Natural Sciences, University of Bengkulu for giving me the financial support so that this article was disseminated through paper presentation and journal publication.

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