THE FOURTH SKEIN MODULE AND THE MONTESINOS-NAKANISHI CONJECTURE FOR 3-ALGEBRAIC LINKS

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Abstract. We study the concept of the fourth skein module of 3-manifolds, that is a skein module based on the skein relation $b_0L_0 + b_1L_1 + b_2L_2 + b_3L_3 = 0$ and a framing relation $L^{(1)} = aL$ (a, $b_0$, $b_3$ invertible). We give necessary conditions for trivial links to be linearly independent in the module. We investigate the behavior of elements of the skein module under the $n$-move and compute the values for $(2, n)$-torus links and twist knots as elements of the skein module. Using the idea of mutants and rotors, we show that there are different links representing the same element in the skein module. We also show that algebraic links (in the sense of Conway) and closed 3-braids are linear combinations of trivial links. We introduce the concept of $n$-algebraic tangles (and links) and analyze the skein module for 3-algebraic links. As a byproduct we prove the Montesinos-Nakanishi 3-moves conjecture for 3-algebraic links (including 3-bridge links). In the case of classical links (i.e. links in $S^3$) our skein module suggests three polynomial invariants of unoriented framed (or unframed) links. One of them generalizes the Kauffman polynomial of links and another one can be used to analyze amphicheirality of links (and may work better than the Kauffman polynomial). In the end, we speculate about the meaning and importance of our new knot invariants.

1. Introduction

To understand the structure of 3-manifolds and links inside them, the first author introduced the concept of the skein module [Pr-2] motivated by the Conway idea of “linear skein”. Skein modules are quotients of a free module of formal linear combinations of links in a 3-manifold by properly chosen (local) skein relations. In search of appropriate skein relations, we analyze deformations of moves on links previously studied in knot theory. The simplest of such moves are $n$ (twist) moves (Figure 1). The simplest skein modules are deformations of 1-moves, [Pr-3] (a linear skein relation). The Conway type skein modules are deformations of 2-moves (a quadratic skein relation) and they are studied extensively (e.g. [Pr-2, Tu, H-P, Bu, P-S]). In this paper we start a systematic study of skein modules based on a deformation of 3-moves (a cubic skein relation). Since there are four terms involved in the relation, we call such a skein module the fourth skein module and denote it by $S_4(M)$, following the notation of [Pr-2]. In Section 3, we discuss the Montesinos-Nakanishi conjecture on 3-moves. This conjecture, and its partial solutions, make the study of fourth skein modules feasible. In Section 3, we define the fourth skein module and we make the general conjecture about generators of the fourth skein module in $S^3$ and $B^3$ with $n$ boundary points (n-tangles). In Section 4, we discuss the

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necessary conditions for coefficient ring so that trivial links are linearly independent in the fourth skein module. We prove the independence in two, rather degenerated cases. In Section 4, we consider the case in which trivial links differ only by a multiplicative constant. We show that after proper substitution the skein module reduces to the Kauffman skein module, at least for links generated by trivial links. Generally we have always epimorphism into the Kauffman skein module. In Section 5, we give an example of computation, finding eigenvectors of Dehn twists on 2-tangles and applying this to describe the effect of $n$-moves on elements of the skein module. As a corollary we compute the values for $(2,n)$-torus links and twist knots. In Section 4, we use the idea of mutants and rotors to construct different links representing the same element in the fourth skein module. In Section 6, we give an example of computation, finding eigenvectors of Dehn twists on 2-tangles and applying this to describe the effect of $n$-moves on elements of the skein module. As a corollary we compute the values for $(2,n)$-torus links and twist knots. In Section 7, we use the idea of mutants and rotors to construct different links representing the same element in the fourth skein module. In Section 8, we introduce a notion of an $n$-algebraic tangle and link, generalizing the notion of an algebraic link in the sense of Conway. We show that $n$-bridge links and closed $n$-braids are $n$-algebraic. We prove Conjectures 2.1 and 3.2 for 2- and 3-algebraic tangles. In the last section, we speculate on existence of “fourth” polynomial link invariants of classical links (in $S^3$) and show connection of our work with investigation of Coxeter, Assion and Wajnryb on finite quotients of braid groups.

2. THE MONTESINOS-NAKANISHI CONJECTURE

Let $L$ be a link (possibly a relative link) in a 3-manifold $M$ (considered up to ambient isotopies). An $n$-move is a local change of a link which adds $n$ positive or negative half-twists to $L$ ($L_0 \leftrightarrow ^n$-move $L_n$). In an oriented 3-manifold we may distinguish a move $L_0 \to L_n$ from a move $L_n \to L_0$. We occasionally use the notation $+n$-move and $-n$-move in these cases (see Figure 1). The first part of Conjecture 2.1 (Montesinos-Nakanishi conjecture) motivated, in part, this work.

Conjecture 2.1. (i) [Montesinos — Nakanishi conjecture; [K]] Every link is 3-equivalent to a trivial link (i.e. it can be converted to a trivial link by 3-moves).

(ii) ([K]) Every 2-tangle is 3-equivalent to a tangle with no more than one crossing (equivalently to one of the four “basic” 2-tangles, with possible trivial components); see Figure 2.

(iii) Every 3-tangle is 3-equivalent to one of the 40 “basic” 3-tangles, with possible trivial components; see Figure 3. Note that every basic 3-tangle has no more than four crossings.

(iv) There is a finite number, $g(n)$, of “basic” $n$-tangles, such that every $n$-tangle is 3-equivalent to one of the $g(n)$ “basic” $n$-tangles, with possible trivial components.

(v) $g(n) = \prod_{i=1}^{n-1} (3^i + 1)$ (e.g. $g(4) = 1120$).
3. The fourth skein module, \( \mathcal{S}_4(M) \)

In this section we start a systematic study of the fourth skein module based on a linear relation among links \( L_0, L_1, L_2 \) and \( L_3 \) of Fig.4 (the module was only shortly mentioned in [Pr-6]).

**Definition 3.1.** Let \( M \) be an oriented 3-manifold, \( R \) a commutative ring with identity, and \( \mathcal{L}_{fr} \) the set of ambient isotopy classes of unoriented framed links in \( M \). Let \( R\mathcal{L}_{fr} \) be the free \( R \) module generated by \( \mathcal{L}_{fr} \) and \( a, b_0, b_1, b_2, b_3 \), elements in \( R \) such that \( a, b_0 \) and \( b_3 \) are invertible. We define the fourth skein module, \( \mathcal{S}_4(M; R, a, b_0, b_1, b_2, b_3) \), or shortly \( \mathcal{S}_4(M) \) as the quotient of \( \mathcal{L}_{fr} \) by a submodule generated by the framing relations \( L^{(1)} = aL \) and the fourth skein expression: \( b_0L_0 + b_1L_1 + b_2L_2 + b_3L_3 \). Similarly, we can define a relative fourth skein module starting from an oriented manifold \( M \) with \( 2n \) points \( x_1, x_2, \ldots, x_{2n} \) chosen on \( \partial M \) and the set of ambient isotopy classes of unoriented relative framed links in \( (M, \partial M) \) such that \( L \cap \partial M = \partial L = \{x_i\} \) for each relative link \( L \).

Note that the following conjecture is a generalization of Conjecture 2.1. In fact, Conjecture 3.2 reduces to Conjecture 2.1 for \( b_1 = b_2 = 0 \) and \( b_0 = -b_3 \).

**Conjecture 3.2.**

(i) \( \mathcal{S}_4(S^3) \) is generated by trivial links.

(ii) The fourth skein module of 2-tangles in a disk is generated by tangles with no more than one crossing.

(iii) The fourth skein module of 3-tangles in a disk is generated by the 40 basic 3-tangles described in Figure 3 with possibly trivial components.

(iv) There is a function \( h(n) \) such that the fourth skein module of \( n \)-tangles in a disk is generated by tangles with no more than \( h(n) \) crossings.

In this paper, we prove the conjecture for 3-algebraic tangles and 3-algebraic links (including 3-bridge links and closures of balanced 3-tangles). The notion of an \( n \)-algebraic tangle (and link) is a new concept defined for the first time in this paper, Section 8. We end this section by listing several useful properties of the fourth skein module, including the Universal Coefficient Property, which our skein module shares with other skein modules [Pr-2, Pr-6].

**Theorem 3.3.**

1. An orientation preserving embedding of 3-manifolds \( i : M \to N \) yields a homomorphism of skein modules \( i_* : \mathcal{S}_4(M) \to \mathcal{S}_4(N) \). The above correspondence leads to a functor from the category of 3-manifolds and orientation preserving embeddings (up to ambient isotopy) to the category of \( R \)-modules (with specified elements \( a, b_0, b_1, b_2, b_3 \) in \( R \), \( a, b_0, b_3 \) invertible).

2. (Universal Coefficient Property)

Let \( r : R \to R' \) be a homomorphism of rings (commutative with 1). We can think of \( R' \) as an \( R \) module. Then the identity map on \( \mathcal{L}_{fr} \) induces the isomorphism of \( R' \) (and \( R \)) modules:

\[
\tilde{r} : \mathcal{S}_4(M; R, a, b_0, b_1, b_2, b_3) \otimes_R R' \to \mathcal{S}_4(M; R', r(a), r(b_0), r(b_1), r(b_2), r(b_3)).
\]
In particular $S_4(M; Z, 1, 0, 0, -1) = S_4(M; Z[x], 1, x, -x, -1) \otimes_{Z[x]} Z$, where $r(x) = 0$.

(3) Let $M = F \times I$ where $F$ is an oriented surface. Then $S_4(M)$ is an algebra, where $L_1 \cdot L_2$ is obtained by placing $L_1$ above $L_2$ with respect to the product structure. The empty link $T_0$ is the neutral element of the multiplication. Every embedding $i : F' \to F$ yields an algebra homomorphism $i_* : S_4(F' \times I) \to S_4(F \times I)$. □

4. Linear independence of trivial links

We analyze the general question: for which $a$ and $b_i$ are trivial links in $S^3$ linearly independent in $S_4(S^3)$? We give necessary conditions and conjecture that they are also sufficient. From the relation $b_0L_0 + b_1L_1 + b_2L_2 + b_3L_3 = 0$ (Figure 5), we get the relation $(b_0 + a^{-1}b_1 + a^{-2}b_2 + a^{-3}b_3)L_0 = 0$ for any framed link $L_0$ (including the trivial knot). Thus to have the linear independence of trivial links we should assume:

**Condition 4.1.** $b_0 + a^{-1}b_1 + a^{-2}b_2 + a^{-3}b_3 = 0$

![Figure 5](image)

Now consider two ambient isotopic diagrams $P_1$ and $P_2$ (Figure 6). Let $T_n$ be the trivial $n$ component link. For $P_1$ we compute: $P_1 = -b_3^{-1}(ab_2 + a^{-1}b_0 + b_1T_1)P$ and for $P_2$ we have: $P_2 = -b_0^{-1}(a^{-1}b_1 + ab_3 + b_2T_1)P$. Since $P_1 = P_2$ and $P$ can be any link, therefore in order to have linear independence we require:

**Condition 4.2.** (a) $b_0b_1 = b_2b_3$, and
(b) $ab_0b_2 + a^{-1}b_0^2 = a^{-1}b_1b_3 + ab_3^2$

![Figure 6](image)

Condition 4.1 can be rewritten as (using Condition 4.2 (a) to eliminate $b_1$):

$$a^3 + \frac{b_3}{b_0} + \frac{b_2}{b_0} (a \frac{b_3}{b_0} + a) = 0$$

and Condition 4.2 (b) can be written as (again using Condition 4.2 (a) to eliminate $b_1$):

$$1 - a^2(\frac{b_3}{b_0})^2 + \frac{b_2}{b_0} (a^2 - (\frac{b_3}{b_0})^2) = 0$$

If we eliminate $b_2$, then we get:

**Condition 4.3.** $(a^4 - 1)(b_3^3 + ab_3^3) = 0$

It is convenient to work with a ring without complicated zero divisors so we consider two cases of Condition 4.3 separately.

(1) Assume $a^4 = 1$. Our conditions reduce now to $(b_3 + a^3b_0)(b_2 + a^2b_0) = 0$ which again leads to two cases:
(i) Let \( b_3 = -a^3b_0 \). Then \( b_2 \) is a free variable, and \( b_1 = -a^{-1}b_2 \). As \( b_0 \) is invertible and relations are homogeneous, we can put \( b_0 = 1 \). Let also write \( b_1 = ax \); then \( b_2 = -a^2x \), \( b_3 = -a^4 \). We work in this case with a ring \( R = \mathbb{Z}[x][a]/(a^4 - 1) \). In particular we measure framing only mod 4. Our skein relation has a form: \( L_0 + axL_1 - a^2xL_2 - a^3xL_3 = 0 \).

(ii) Let \( b_2 = -a^2b_0 \). Then \( b_3 = -a^2b_1 \) and after substituting \( b_0 = 1, b_1 = ax \) one obtains the skein relation: \( L_0 + axL_1 - a^2L_2 - a^3xL_3 = 0 \).

(2) Assume \( b_3^3 + ab_0^3 = 0 \). This leads to the following solution (for simplicity we put \( b_0 = 1 \) and \( b_3 = \hat{b} \)): \( b_1 = b(b^2 + b^{-2}), b_2 = b^2 + b^{-2}, \) and \( a = -b^3 \). The skein relation has now the form \( L_0 + b(b^2 + b^{-2})L_1 + (b^2 + b^{-2})L_2 + bL_3 \), and the framing relation \( L^{(1)} = -b^3L \). \( R = \mathbb{Z}[\hat{b}] \).

We have considered conditions which are necessary for trivial links to be linearly independent in the fourth skein module, \( S_4(M) \). We conjecture that these conditions are also sufficient. In the case of Condition \( \text{Conjecture 4.3} \) (1)(ii) we are able to prove it, and the skein module of \( S^3 \) seems to be rather trivial.

**Theorem 4.4.** (i) If \( a^4 = 1 \), \( b_2 = -a^2b_0 \), \( b_3 = -a^2b_1 \) then trivial links, \( T_i \), are linearly independent in \( S_4(S^3) \).

(ii) Consider a homomorphism \( h : \mathcal{RL}_f \to \mathbb{Z}[x,t,a]/(a^4 - 1) \) given by \( h(L) = a^{fr(L)} t^{com(L)} \) where \( com(L) \) is the number of components of a link \( L \) and \( fr(L) \) is for oriented framed link a difference between framing of \( L \) and 0-framing in \( S^3 \). For unoriented links \( fr(L) \) is well defined mod(4). Then \( h \) yields a homomorphism \( \hat{h} \) from the fourth skein module described by Conjecture \( \text{Conjecture 4.3} \) (1)(ii) (i.e. \( b_0 = 1, b_1 = ax, b_2 = -a^2, b_3 = -a^3x \)) to \( \mathbb{Z}[x,t,a]/(a^4 - 1) \).

If we allow the empty link, \( T_0 \), in \( \mathcal{L}_f \) and Conjecture \( \text{Conjecture 3.2} \) (i) holds then \( \hat{h} \) is an algebra isomorphism, where product of two links is defined to be their disjoint sum.

**Proof.** To show (ii) it suffices to show that \( h \) sends the skein relation \( L_0 + axL_1 - a^2L_2 - a^3xL_3 \) to 0. We have \( h(L_0 + axL_1 - a^2L_2 - a^3xL_3) = h(L_0 - a^2L_2) + axh(L_1 - a^2L_3) = 0 \) because \( com(L_i) = com(L_{i+2}) \) and \( fr(L_i) \equiv fr(L_{i+2}) + 2 \mod(4) \). The last statement follows from the fact that if \( L \) is an oriented framed link and \( D_L \) its diagram such that the framing of \( L \) is the flat framing of \( D_L \) then \( fr(L) = Tait(D_L) = \sum_p sgn(p) \) where the sum is taken over all crossings of \( D_L \). \( h \) and \( \hat{h} \) are clearly algebra epimorphisms and \( h^{-1} \) and \( \hat{h}^{-1} \), where \( h^{-1}(t) = T_1 \) are left inverses of \( h \) and \( \hat{h} \) respectively. Thus \( \hat{h} \) is an isomorphism on the subspace of \( S_4(S^4) \) generated by trivial links. In fact our proof works for any rational homology sphere, and assuming \( a^2 = 1 \), for any 3-manifold. (i) follows immediately from (ii), because \( t^i \) are linearly independent in the ring of polynomials.

**Conjecture 4.5.** The trivial links, \( T_i \), are linearly independent in \( S_4(S^3) \) in the following cases:

(i) \( a^4 = 1, b_3 = -a^3b_0 \) and \( b_1 = -a^3b_2 \)

(ii) \( ab_0^3 = -b_3^3, b_2b_0^{-1} = b_3^2b_0^{-2} + b_3^{-2}b_0^2, b_0b_1 = b_2b_3 \)

Notice that in all cases of Conjecture 4.4 and in Theorem 4.4, we assumed that \( b_0b_1 = b_2b_3 \) (compare Section 3). Conjecture 4.3 (1)(i) leads to the polynomial invariant of unframed links.
(a = 1) in $S^3$ (for links generated by trivial links), $S_4(L)(x, t)$. If $L = \sum_i w_i(x) T_i$ in our skein module, then $S_4(L)(x, t) = \sum_i w_i(x)t^i$. Conjecture 4.6(ii) leads to the polynomial invariant of framed links in $S^3$ (for links generated by trivial links), $S_4(L)(b, t)$. If $L = \sum_i v_i(b) T_i$ in our skein module, then $S_4(L)(b, t) = \sum_i v_i(b)t^i$. For $n$-tangles the analogue of Conjecture 1.3 is more involved but very interesting. We will consider below the case (i).

**Conjecture 4.6.** (1) Consider the skein module $S_4(D(n), x)$ with $a = 1$ and the skein relation $L_0 + x L_1 - x L_2 - L_3$ where $D(n)$ is a disk with $2n$ boundary points. Let $B(n)$ be a set of $n$-tangles, one from each 3-move equivalence class. Then $B(n)$ is a base for the fourth skein module $S_4(D(n), x)$, for $x \neq 1$.

(2) Consider the $Z_3$-linear space of all 3-colorings of a tangle, $b$, denoted by $\text{Tri}(b)$ and the homomorphism $\phi_b : \text{Tri}(b) \to Z_3^{2n}$ where $Z_3^{2n}$ is the space of all colorings of boundary points $\overline{b}$. Then different elements of $B(n)$ can be distinguished by their 3-colorings. Precisely, if $b_1 \neq b_2$ then either $\text{rank}(\text{Tri}(b_1)) = \text{rank}(\text{Tri}(b_2))$ or ranks are the same but $\phi_b(\text{Tri}(b_1)) \neq \phi_b(\text{Tri}(b_2))$.

By definition, (1) holds for $x = 0$, and so the theorem says here that the deformation $(0 \to x)$ does not change the module. We exclude the case $x = 1$ in the conjecture because we know that our skein module of $S^3$ (or of $D(0)$) behave differently for $x = 0$ and $x = 1$; in the first case we identify 3-move equivalence classes and in the second case (as far as we know, compare Theorem 4.4) links with the same number of components. It is known that each 3-move is preserving a 3-coloring of a tangle and we conjecture the inverse holds, that is, $n$-tangles with the same 3-coloring structure are related by 3-moves. Conjecture 4.6 suggests the importance of analyzing $\text{Tri}(b)$ and $\phi_b(\text{Tri}(b))$. We have proved recently that $\phi_b(\text{Tri}(b))$ are Lagrangians in the space of colorings of the boundary of the tangle $b$ with respect to the properly chosen symplectic structure $[\text{Pr}-7]$.

5. **Generic case of the fourth skein module; $b_0 b_1 \neq b_2 b_3$.**

In the previous section we showed that in all cases in which $T_i$ are (conjecturally) linearly independent, one has $b_0 b_1 = b_2 b_3$. We show that if $b_0 b_1 - b_2 b_3$ is invertible then one has:

5.1.

$$L \sqcup T_1 = \frac{a^{-1} b_1 b_3 - ab_0 b_2 + ab_2^2 - a^{-1} b_0^2}{b_0 b_1 - b_2 b_3} L$$

In particular:

5.2.

$$T_n = (\frac{a^{-1} b_1 b_3 - ab_0 b_2 + ab_2^2 - a^{-1} b_0^2}{b_0 b_1 - b_2 b_3})^{n-1} T_1.$$
the substitution $b_0 = 1, b_1 = -(z + a), b_2 = za + 1, b_3 = -a$ our skein module (polynomial) reduces to the Kauffman polynomial for links generated by trivial links (we get in particular $T_2 = \frac{a + a^{-1} - z}{z} T_1$). More precisely, our substitutions give the skein relations:

5.3. $L_0 - (z + a) L_1 + (za + 1) L_2 - a L_3 = 0$, $L^{(1)} = aL$.

On the other hand the Kauffman relations are:

5.4. $L_- + L_+ = z(L_0 + L_\infty)$, $L^{(1)} = aL$.

We work then with the coefficients in a commutative ring with identity, $R$, where $z$ and $a$ are chosen invertible elements in $R$. The Kauffman skein module $S_{3,\infty}(M; R, z, a)$ is defined to be $R\mathcal{L}_{fr}/(\text{Kauffman relations})$. As a consequence of Kauffman relations we have $L \sqcup T_1 = \frac{a + a^{-1} - z}{z} L$ in $S_{3,\infty}(M; R, z, a)$. By applying 5.3 twice we get:

$$L_0 + L_2 = z(L_1 + a^{-1} L_\infty),$$
$$L_1 + L_3 = z(L_2 + a^{-2} L_\infty).$$

From this we get the relation:

$$L_0 + L_2 - a(L_1 + L_3) = z(L_1 - aL_2),$$

which is equivalent to 5.3. Thus we have an epimorphism from the fourth skein module onto the Kauffman skein module (Proposition 5.5).

**Proposition 5.5.** Let $M$ be an oriented 3-manifold, $R$ a commutative ring with identity, and $a, z$ invertible elements in $R$. Then

(a) We have an epimorphism

$$\phi : S_{4}(M; \text{relations 5.3}) \to S_{3,\infty}(M; R, z, a)$$

(b) For $a^2 - 1$ invertible in $R$ and $M = S^3$, the epimorphism $\phi$ restricted to the subspace generated by trivial links is a monomorphism. If Conjecture 3.2 (i) holds then $\phi$ is an isomorphism.

**Proof.** Part (a) follows from the fact that relations 5.3 follow from the Kauffman relations, 5.4. If the trivial link $T_1$ is linearly independent in $S_{3,\infty}(M; R, z, a)$ as is the case for $M = S^3$ or more generally for $M = F \times I$ (i.e. the product of a surface and the interval), and as is conjectured for any 3-dimensional manifold, then we have a monomorphism $\psi$ from $RT_1$, the submodule generated by $T_1$ to $S_4(M; \text{relations 5.3})$, which is left inverse to $\phi$ ($\psi \phi = \text{Id}_{RT_1}$). As observed before, for $a^2 - 1$ invertible $L \sqcup T_1 = \frac{a + a^{-1} - z}{z} L$ in the fourth skein module, thus (b) of the proposition follows. \qed
6. The fourth skein module for \((2, n)\)-torus links and twist knots

We analyze in this section the behavior of the value of a link as an element of the fourth skein module \(S_4(S^3)\) under \(n\)-moves and as a corollary compute the values of \((2, n)\)-torus links and of twist knots. We show an example in the case of the skein relation \(L_0 + axL_1 - a^2xL_2 - a^3L_3 = 0\) (we discussed the importance of this relation, for \(a^4 = 1\), in Section 3).

Let \(\tau\) denote the positive half twist on a tangle \(L_0\), taking framing into account we can write \(\tau(L_0) = aL_1\), or generally \(\tau^k(L_i) = a^kL_{i+k}\). We will find the expression for \(a^nL_n\) in the basis \(L_{-1}, L_0, L_1\). Of course there is nothing sophisticated in our computation but it is useful to be able to evaluate our invariants for several classes of links. Because \(\tau(L_{-1}) = aL_0, \tau(L_0) = aL_1,\) and \(\tau(L_1) = aL_2 = a^3(-a^2xL_1 + axL_0 + L_{-1}) = a^2L_{-1} + a^{-1}xL_0 - xL_1\), therefore the matrix, \(A_\tau\) of our linear map has the form:

\[
A_\tau = \begin{pmatrix}
0 & 0 & a^{-2} \\
a & 0 & a^{-1}x \\
a & a & -x
\end{pmatrix}
\]

The characteristic polynomial of \(\tau\) is therefore \(\chi(A_\tau) = -\lambda^3 - x\lambda^2 + x\lambda + 1 = (1 - \lambda)(\lambda^2 + (x + 1)\lambda + 1)\). As \(\Delta = (x + 1)^2 - 4\), it is convenient to put \(x + 1 = -s^2 - s^{-2}\). Thus 1 is an eigenvalue of \(\tau\) and two other eigenvalues are \(\lambda_\pm = \frac{s^2 + s^{-2} \pm (s^2 - s^{-2})}{2}\), which gives \(\lambda_+ = s^2\) and \(\lambda_- = s^{-2}\). From this we get the eigenvectors of \(\tau\) to be:

\[
E(1) = a^{-1}L_{-1} - (s^2 + s^{-2})L_0 + aL_1 \\
E(s^2) = a^{-1}s^{-1}L_{-1} - (s + s^{-1})L_0 + asL_1 \\
E(s^{-2}) = a^{-1}sL_{-1} - (s + s^{-1})L_0 + as^{-1}L_1
\]

From these we obtain \(L_0 = \frac{E(s^2) - (s + s^{-1})E(1) + E(s^{-2})}{(s + s^{-1})(s - s^{-1})^2}\), and in general, we have the formula for \(a^nL_n\).

6.1.

\[
a^nL_n = \tau^n(L_0) = \frac{s^{-2n}E(s^{-2}) - (s + s^{-1})E(1) + s^{2n}E(s^2)}{(s + s^{-1})(s - s^{-1})^2} = \frac{s^{-2n}(a^{-1}sL_{-1} - (s + s^{-1})L_0 + as^{-1}L_1)}{(s + s^{-1})(s - s^{-1})^2} - \frac{(s + s^{-1})(a^{-1}L_{-1} - (s^2 + s^{-2})L_0 + aL_1)}{(s + s^{-1})(s - s^{-1})^2}
\]

\[
+ \frac{s^{2n}(a^{-1}s^{-1}L_{-1} - (s + s^{-1})L_0 + asL_1)}{(s + s^{-1})(s - s^{-1})^2} = \frac{(s^{2n-1} - (s + s^{-1}) + s^{-2n+1})a^{-1}L_{-1}}{(s + s^{-1})(s - s^{-1})^2} - \frac{(s^{2n+1} - (s + s^{-1}) + s^{-2n-1})aL_1}{(s + s^{-1})(s - s^{-1})^2}
\]

\[
= \frac{(s^n - s^{-n})(sn-1 - s^{-1})a^{-1}L_{-1}}{(s + s^{-1})(s - s^{-1})^2} - \frac{(s + s^{-1})(s^n - s^{-1})(s^{n+1} - s^{-n-1})L_0}{(s + s^{-1})(s - s^{-1})^2}
\]

\[
+ \frac{(s^{n+1} - s^{-n-1})(s^n - s^{-n})aL_1}{(s + s^{-1})(s - s^{-1})^2}.
\]
Corollary 6.2. Consider the Dehn twist $\tau^2$ on the tangle. It has eigenvalues 1, $s^4$, and $s^{-4}$ and the same eigenvectors as $\tau$.

One can use this observation to construct a torsion in a manifold with an incompressible $S^2$ (e.g. $S^1 \times S^2$). In the spirit of [Pr-1] we can check the condition at which an n-move changes our invariant only by a constant.

Corollary 6.3. If $s^{2n} = 1$ and $s^4 \neq 1$ for $n$ even and $s^2 \neq 1$ for $n$ odd then $a^n L_n = \tau^n (L_0) = L_0$. In particular, as we know, if $x = -s^2 - s^{-2} - 1 = -\frac{s^3 - s^{-3}}{s - s^{-1}} = 0$, $\tau^3 (L_0) = a^3 L_3 = L_0$.

To find the expression for torus links, $T_{2,n}$ (Figure 7 (a)), in the fourth skein module, we observe that $aT_{2,-1} = a^{-1}T_{2,1} = T_1$ and $T_{2,0} = T_2$.

Figure 7

Corollary 6.4. The invariant of the torus link of type $(2, n)$ is given by:

$$a^n T_{2,n} = \tau^n (T_{2,0}) = \frac{(s^n - s^{-n})(a^{-2}(s^{n-1} - s^{-n+1}) + a^2(s^{n+1} - s^{-n-1}))(s + s^{-1})(s - s^{-1})^2}{(s + s^{-1})(s - s^{-1})^2}T_1$$

$$- \frac{(s + s^{-1})(s^{n-1} - s^{-n+1})(s^{n+1} - s^{-n-1})}{(s + s^{-1})(s - s^{-1})^2}T_2.$$  

Notice that for unoriented links $T_{2,2} = T_{2, -2}$ which gives $(a^2 - a^{-2})((s^2 + 1 + s^{-2})T_2 + (-a^2 - a^{-2})T_1) = 0$. We have already discussed the condition $a^4 = 1$ which is necessary if we want trivial links to be linearly independent. The condition $(s^2 + 1 + s^{-2})T_2 = (a^2 + a^{-2})T_1$ leads to $T_2 = \frac{a^2 + a^{-2}}{s^2 + 1 + s^{-2}}T_1 = -\frac{a^2 + a^{-2}}{x}T_1$ (this agrees with [5.2]).

A similar calculation for twist knots, $a^n T[2, n]$ (Figure 7 (b)) uses the Formula 6.1 and the initial conditions $T[2, 0] = a^2 T_1$, $T[2, 1] = a^{-1}T_1$, and $T[2, 1] = T_{2,3} = a^{-3}(s^3 - s^{-3})T_1 = \frac{a^{-3}(s^3 - s^{-3})(a^{-2}(s^2 - s^{-2}) + a^2(s^2 - s^{-2}))}{(s + s^{-1})(s - s^{-1})^2}T_2$.  

More generally we can write a straightforward formula for the pretzel link $P[n_1, n_2, \ldots n_k]$, Figure 8 by first using Formula 6.1 for each column to get a linear combination generated by torus links and then using the formula again for these torus links to get a linear combination generated by trivial links. To write a formula for a 2-bridge link (and tangle) or for an algebraic tangle, we should first decompose an algebraic tangle into n-twist tangles and then, from the table of multiplication of the four basic 2-tangles, we can read off the result. Using the table and this formula, we can also figure out the automorphism of the “rotation”. A similar formula can be built for 3-algebraic tangles (with knowledge of the $40 \times 40$ multiplication table, and the “rotation” formula; see Section 8).

Figure 8
7. Mutations and rotations in the fourth skein module

Using the idea of mutants and rotors, we show that there are different links representing the same element in the fourth skein module. If we have a 2-tangle, $P$, we consider three involutions $m_x$, $m_y$, and $m_z$ on it which denote the rotation of the tangle along the axis denoted in the subscript, Figure 9. If we have a link diagram with a 2-tangle in it, a mutation of the diagram along the tangle is obtained by performing one of the described involutions on the tangle (see Figure 9).

![Figure 9](image)

![Figure 10](image)

**Proposition 7.1.** Consider a 2-tangle part of a link diagram. If it is generated by tangles of Figure 2 (with possible trivial components), then any mutation of the link preserves the element of the fourth skein module.

**Proof.** Any tangle of Figure 2 is symmetric with respect to $x$, $y$, and $z$ axes. □

Consider a tangle $R$ in a regular $n$-gon $D$ in $S^2$ ($n \geq 3$). We call $R$ an $(n)$-rotor if $R$ is invariant under the rigid rotation $\rho : D \to D$ by angle $2\pi/n$ around the center of $D$. Take a link $L$ and let $L$ have a tangle decomposition $R \cup S$, i.e., $L$ has a projection which intersects $\partial D$ transversely in 2 points on each edge of $D$. If $R$ is an $n$-rotor, then we call $S$ a stator. Consider a line which passes through the center of $D$ and either a corner or middle point of an edge. Let $\mu : D \to D$ be a $\pi$-rotation through the third dimension with the line as an axis. Then we call $L' = \mu(R) \cup S$ a rotant of $L$. Here note that we do not assume that $R$ is invariant under the operation $\mu$. If we consider a similar rotation $\mu'$ for a disk $S^2 - D$, then we have the same result $L' = R \cup \mu'(S)$. See [A-P-R] for more details.

Regard each tangle of Figure 3 as a stator of a 3-rotor. Then 34 of the 40 basic 3-tangles are invariant under an operation $\mu'$ for an axis of a symmetry of the polygon. Exceptions are $\sigma_1 s_2$, $\sigma_1 U_1 \sigma_1^{-1}$, and their mirror images as they have only plane symmetries. Thus we have the following proposition.

**Proposition 7.2.** Assume that a link $L$ has a tangle decomposition $R \cup S$, where $R$ is a 3-rotor and $S$ is a 3-stator. If $S$ is generated by 34 tangles in Figure 3 as described above (with possible trivial components), then its rotant and $L$ are the same in the fourth skein module. □

8. $n$-algebraic tangles and links

In this section, we study the fourth skein module of the 3-sphere and the relative skein module of the 3-dimensional disk. To understand the structure better, we also study fourth skein algebra for $n$-braid groups. Here note that each skein module in this section has its tangle elements (including links as 0-tangles) in $S^3$ or in $D^3$. Therefore we can analyze their diagrams...
on $\mathbb{R}^2$ ($\subset S^2$) or $D^2$. and count the number of crossings of these diagrams. We use $\sigma_i$ for a crossing of a braid, as usual and use $\hat{\sigma}_i$ in the case we do not specify whether the crossing is positive or negative. We use three kinds of equality symbols to denote various relations between a pair of n-tangle diagrams (including link diagrams) $a$ and $b$. We say that $a$ and $b$ are skein module equivalent, denoted by $a = b$, if they are equal in the skein module. We say that $a$ and $b$ are strongly equivalent, denoted by $a \equiv b$, if they are ambient isotopic. Finally, we say that $a$ and $b$ are dot equivalent, denoted by $a \doteq b$, if the difference of $a$ and $b$ is presented in the skein module as a linear combination of elements with fewer crossings than $a$ and $b$ have. If every element of the linear combination has no more than $n$ crossings, then we denote the linear combination by $O(n)$.

8.1. $n$-algebraic tangles and $n$-algebraic links. Let $\mathcal{T}$ denote the category of unoriented tangles and $\mathcal{T}(n)$ a semigroup of $n$-tangles. We “read” morphisms from left to right and compose tangles in the same manner (it corresponds to the notation $P_1P_2$ denoting composition of tangles $P_1$ and $P_2$). $\mathcal{T}(n)$ allows the natural action by the $D_{2n} \oplus \mathbb{Z}_2$ group, where $D_{2n}$ is the action of the dihedral group (preserving orientation) of $4n$ elements, generated by the rotation $r$ along the $z$-axis by the angle $2\pi/2n$, and the rotation $\pi$ along the $y$-axis denoted by $m_y$ and often called a mutation. The factor $\mathbb{Z}_2$ corresponds to the mirror image map with respect to the $x, y$ plane. The semigroup of $n$-tangles has two important subsemigroups: the braid group $B_n$, generated by crossings (tangles) $\sigma_i$, and a semigroup of balanced $n$-tangles, $W(n)$, generated by $\sigma_i$ and tangles $U_i$ creating maxima and minima (for a visual notation, $(\sigma_i)$ might be good. See Figure 3 for an example). In fact $U_1$ would suffice, as $U_{i+1} = \sigma_i U_{i+1} \sigma_{i+1}^{-1} \sigma_{i}^{-1}$. Representations of $W(n)$ (their finite dimensional quotients) were studied by B. Westbury. We are introducing also another semigroup, generalizing Conway’s algebraic tangles and allowing successful induction.

Definition 8.1. (1) We define an $n$-algebraic tangle inductively as follows:

(i) An $n$-tangle with no more than one crossing is $n$-algebraic.

(ii) Inductive step: If $P_1$ and $P_2$ are two $n$-algebraic tangles then their twisted sum, $r^i(P_1)r^j(P_2)$ is also $n$-algebraic.

(2) In a more restrictive way, we define $(n, k)$-algebraic tangles if in step (ii) $P_2$ has no more than $k$ crossings.

(3) An $n$-algebraic link (resp. $(n, k)$-algebraic link) is a link with a diagram obtained by closing an $n$-algebraic (resp. $(n, k)$-algebraic) tangle by pairwise disjoint arcs.

We denote by $A_n$ the monoid generated by $n$-algebraic tangles.

Lemma 8.2. 1. (a) Tangles in $B_n$, $W_n$ and $n$-bridge tangles are $(n, 1)$-algebraic.

2. (b) $n$-bridge links and closed $n$-braids are $(n, 1)$-algebraic links.

Proof. The fact that elements of $B_n$ and $W_n$ are $(n, 1)$-algebraic follows from the definition. For an $n$-bridge tangle, we start from the $n$-tangle which is equal to $U_1U_3...U_{2k-1}$ for $n = 2k$ or $n = 2k + 1$, and then we add crossings one by one (compare Figure 11, the figure is drawn to stress the fact that we deal with a 3-bridge tangle. In the algebra of 3-tangles we would describe it as $r^{-1}(r(U_1\sigma_2)\sigma_2))$. 

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Our main result in this section is the following.

**Theorem 8.3.** Any 3-algebraic link can be generated as a linear combination of trivial links in $S_4(S^3; R)$.

Our proof works for any coefficients thus as a corollary we have:

**Theorem 8.4.** The Montesinos-Nakanishi conjecture is true for 3-algebraic links. □

**Remark.** What concerns the Montesinos-Nakanishi conjecture, we have two candidates for a possible counterexample. One is the double of Borromean rings proposed by Y. Nakanishi [K1]; it has 24 crossings. The other is proposed by Q. Chen in [Ch] and it has 20 crossings. Chen’s link is a reduction of the closure of the 5-braid $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$ by 3-moves. In [Ts-1, Ts-2], it is shown that the double of Borromean rings and Chen’s link are 4-algebraic (in fact, $(4, 6)$-algebraic) up to 3-moves, which tell us that it is hard to prove Theorems 8.3 and 8.4 for 4-algebraic links (it is shown in [Ts-2] that these theorems hold for $(4, 5)$-algebraic links).

8.2. **The fourth skein module for the 3-algebraic links.** Each of the propositions and theorems presented in this section holds also for $n = 2$ (we take $\{Id, \sigma_i^\pm\}$ and $\{U_i\}$ instead of $B_3$ and $C_3$, respectively). Since proofs are much simpler in that case, we omit them. First of all, let us define the fourth skein algebras for $A_n$ and $B_n$.

**Definition 8.5.** Let $R$ be commutative ring with identity. Let $RB_n$ be the $R$-module with the basis $B_n$. Furthermore, let $I_4$ be the ideal of $RB_n$ generated by the skein relation $b_0\sigma_i^0 + b_1\sigma_i + b_2\sigma_i^2 + b_3\sigma_i^3$, where $b_0$ and $b_3$ are invertible in $R$. We define the fourth skein module $S_4(B_n; R)$ for $B_n$ as the quotient $RB_n/I_4$. The product for elements of $B_n$ induces a bilinear map $S_4(B_n; R) \times S_4(B_n; R) \to S_4(B_n; R)$ so that, with respect to this product, $S_4(B_n; R)$ becomes an algebra. Thus we call $S_4(B_n; R)$ the fourth skein algebra for $B_n$. We define the forth skein algebra $S_4(A_n; R)$ for $A_n$ as a subalgebra of the algebra of $n$-tangles modulo the fourth skein relations and the framing relations generated by $n$-algebraic tangles.

Note that in $S_4(B_3; R)$, any braid which contains $\sigma_i^2$ or any non-alternating expression of $(\sigma_1\sigma_2)^2$ or $(\sigma_2\sigma_1)^2$ can be generated as a linear combination of elements with fewer crossings than those we start from. From the following proposition, we also see that $(\sigma_1\sigma_2)^2\sigma_1$ and $(\sigma_2\sigma_1)^2\sigma_2$ can be generated as a linear combination of elements with fewer crossings than those we start from. We call a braid in $S_4(B_3; R)$ reducible if the braid contains one of those configurations. Otherwise, we call the braid irreducible.

**Proposition 8.6.** Four configurations $(\sigma_1\sigma_2^{-1})^2$, $(\sigma_1^{-1}\sigma_2)^2$, $(\sigma_2\sigma_1^{-1})^2$, and $(\sigma_2^{-1}\sigma_1)^2$ are dot equivalent to each other in $S_4(B_3; R)$. 


Proof. It is sufficient to show that $(\sigma_1^{-1}\sigma_2)^2 \equiv (\sigma_1\sigma_2^{-1})^2$, $(\sigma_2^{-1}\sigma_1)^2 \equiv (\sigma_1\sigma_2^{-1})^2$, and $(\sigma_2\sigma_1^{-1})^2 \equiv (\sigma_2^{-1}\sigma_1)^2$ using the fourth skein relations.

\[(\sigma_1^{-1}\sigma_2)^2 \equiv (\sigma_2^{-1}\sigma_1)^2 - (\sigma_1^{-1}\sigma_2)^2 = (-b_0^{-1}b_3)(-b_0b_3^{-1}) \sigma_1\sigma_2^{-1}\sigma_1\sigma_2 + (-b_0^{-1}b_3)(-b_2b_3^{-1}) \sigma_1\sigma_2^{-1}\sigma_1\sigma_2 + \mathcal{O}(3)\]

\[= (-b_0^{-1}b_3)(-b_0b_3^{-1}) \sigma_1\sigma_2^{-1}\sigma_1\sigma_2 + \mathcal{O}(3)\]

\[(\sigma_2^{-1}\sigma_1)^2 \equiv (\sigma_2^{-1}\sigma_1)^2 - (\sigma_2^{-1}\sigma_1)^2 = (-b_0^{-1}b_3)(-b_0b_3^{-1}) \sigma_1\sigma_2^{-1}\sigma_1\sigma_2 + (-b_0^{-1}b_3)(-b_2b_3^{-1}) \sigma_1\sigma_2^{-1}\sigma_1\sigma_2 + \mathcal{O}(3)\]

\[= (-b_0^{-1}b_3)(-b_0b_3^{-1}) \sigma_1\sigma_2^{-1}\sigma_1\sigma_2 + \mathcal{O}(3)\]

\[(\sigma_2\sigma_1^{-1})^2 \equiv (\sigma_2\sigma_1^{-1})^2 - (\sigma_2\sigma_1^{-1})^2 = (-b_0^{-1}b_3)(-b_0b_3^{-1}) \sigma_1\sigma_2^{-1}\sigma_1\sigma_2 + (-b_0^{-1}b_3)(-b_2b_3^{-1}) \sigma_1\sigma_2^{-1}\sigma_1\sigma_2 + \mathcal{O}(3)\]

\[= (-b_0^{-1}b_3)(-b_0b_3^{-1}) \sigma_1\sigma_2^{-1}\sigma_1\sigma_2 + \mathcal{O}(3)\]

Define $B_3$ as the set of 24 invertible (braid type) basic tangles in Figure 3. Then we have the following proposition.

**Proposition 8.7.** For any irreducible element $b$ of $B_3$, there exists an element $b'$ of $B_3$ such that $b'$ is dot equivalent to $b$.

Proof. Let $n$ be the number of crossings of $b$. If $n \leq 3$, then there exists an element $b'$ of $B_3$ such that $b'$ is strongly equivalent to $b$. If $n = 4$, then $b$ is an alternating expression of $(\hat{\sigma}_1\hat{\sigma}_2)^2$ or $(\hat{\sigma}_2\hat{\sigma}_1)^2$. Then, $b$ is dot equivalent to $(\sigma_1\sigma_2^{-1})^2$ from Proposition 8.6. There is no irreducible element of $B_3$ with more than four crossings.

Using this proposition, we obtain the following theorem.

**Theorem 8.8.** Any element of $S_4(B_3; R)$ can be generated as a linear combination of elements of $B_3$.

Proof. We prove the theorem by an induction on the number of crossings of the element. If $n = 0$, then clearly the statement holds. Assume that the statement holds in the case $n$ fewer than $k$. Consider the case $n = k$. If the element is reducible, then it is generated by elements with fewer crossings than $k$, which can be generated by elements of $B_3$ from the assumption of the induction. If the element is irreducible, then it is generated by an element of $B_3$ and elements with fewer crossings than $k$ from Proposition 8.7. This induces that the irreducible element is also generated by elements of $B_3$.

Define $C_3$ as the set of 16 non-invertible basic tangles in Figure 3. Then we have the following.

**Proposition 8.9.** A product of any pair of elements of $B_3$ and $C_3$ can be generated as a linear combination of elements of $B_3 \cup C_3$ (with possible trivial components).
Proof. If both elements of the pair are 3-braids, then the statement follows from Theorem 8.8. The other cases are easy to be checked. For simplicity, we ignore trivial components in the consideration.

**Proposition 8.10.** Any rotation of elements of $B_3$ and $C_3$ can be generated as a linear combination of elements of $B_3 \cup C_3$.

**Proof.** We need to show only for $r^2((\sigma_1\sigma_2^{-1})^2)$ and $r^{-1}((\sigma_1\sigma_2^{-1})^2)$. From Figure 13, we have

\[
\begin{align*}
 r^2((\sigma_1\sigma_2^{-1})^2) &= (\sigma_1\sigma_2^{-1})^2 + b_0^{-1}b_2(\sigma_2^{-1}\sigma_1^{-1} - \sigma_1U_2\sigma_1^{-1}) + b_1b_3^{-1}(\sigma_2\sigma_1 - \sigma_1^{-1}U_2\sigma_1) \\
 &\quad + b_0^{-1}b_1b_3^{-1}(\sigma_2^{-1}\sigma_1 - \sigma_1U_2\sigma_1) \\
 r^{-1}((\sigma_1\sigma_2^{-1})^2) &= (\sigma_1^{-1}\sigma_2)^2 + b_0^{-1}b_2(\sigma_2^{-1}\sigma_1^{-1} - \sigma_1U_2\sigma_1^{-1}) + b_1b_3^{-1}(\sigma_2\sigma_1 - \sigma_1^{-1}U_2\sigma_1) \\
 &\quad + b_0^{-1}b_1b_3^{-1}(\sigma_2\sigma_1^{-1} - \sigma_1^{-1}U_2\sigma_1^{-1})
\end{align*}
\]

Figure 13

Since any 3-tangle with no more than one crossing can be expressed as $a^it$, where $i = 0, \pm 1$, $t$ is an element of $B_3 \cup C_3$, and $a$ reflects a possible framing change yielded by a “kink”, we obtain the following theorem from Propositions 8.9 and 8.10.

**Theorem 8.11.** Any element of $S_4(A_3; R)$ can be generated as a linear combination of elements of $B_3 \cup C_3$ (with possible trivial components).

**Proof of Theorem 8.11.** From Theorem 8.11, we know that every 3-algebraic link can be generated as a linear combination of 3-algebraic links obtained from elements of $B_3 \cup C_3$. For each of these links, it is easy to see that it can be generated as a linear combination of trivial links.

9. Speculations

In this section we present results of calculations that suggest that even for $S^3$ the fourth skein module is more powerful (can distinguish more links) than the third (Jones-Conway) and Kauffman skein modules. In particular, if the fourth skein polynomial exists (trivial links are linearly independent), then the polynomial is (at least sometimes) more powerful than the Jones-Conway (Homflypt) and Kauffman polynomials. Our calculation shows that under these assumptions we can distinguish the $9_{42}$-knot from its mirror image $\bar{9}_{42}$ ([Ko]) and some 2-bridge links that share the same Jones-Conway and Kauffman polynomials ([Ka-1, Ka-2]). In the second part of this section we present general ideas that can lead to a solution of Montesinos-Nakanishi conjecture and its generalizations. In particular we speculate about the geometrical meaning of the finite quotients of the braid group described by J.Assion [A-1, B-W, Wa] (Symplectic and Unitary cases).

**Conjecture 9.1.** (1) There is a polynomial invariant of unoriented links, $P_1(L) \in Z[x, t]$ that satisfies:
(i) Initial conditions: $P_1(T_n) = t^n$, where $T_n$ is the trivial link of $n$ components.
(ii) Skein relation, $P_1(L_0) + xP_1(L_1) - xP_1(L_2) - P_1(L_3) = 0$, where $L_0, L_1, L_2, L_3$ is a standard, unoriented skein quadruple $(L_{i+1})$ obtained from $L_i$ by a right-handed half twist on two arcs involved in $L_i$.

(2) There is a polynomial invariant of unoriented framed links, $P_2(L) \in \mathbb{Z}[b^{\pm 1}, t]$ which satisfies:
(i) Initial conditions: $P_2(T_n) = t^n$,
(ii) Framing relation: $P_2(L^{(1)}) = -b^3P_2(L)$, where $L^{(1)}$ is obtained from a framed link $L$ by a positive half twist on its framing.
(iii) Skein relation: $P_2(L_0) + b(b^2 + b^{-2})P_2(L_1) + (b^2 + b^{-2})P_2(L_2) + bP_2(L_3) = 0$.

(3) In each case the polynomial is uniquely defined.

**Example 9.2.** Consider the 9_{42}-knot, in notation of \([\text{Ro}]\). Then the polynomial $P_2(b, t)$ distinguishes $9_{42}$ from its mirror image $9_{42}$. In fact, we obtain the following for $9_{42}$. We use the fact that we have the formula: $P_2(L)(b, t) = P_2(\bar{L})(b^{-1}, t)$.

\[
P_2(9_{42})(b, t) = (3b^{11} + 7b^7 + 9b^3 + 8b^{-1} + 6b^{-5} + 4b^{-9} + 3b^{-13} + b^{-17}) t - (b^{13} + 6b^9 + 14b^5 + 20b^{-3} + 12b^{-7} + 5b^{-11} + b^{-15}) t^2 + (b^{11} + 4b^7 + 8b^3 + 10b^{-1} + 8b^{-5} + 4b^{-9} + b^{-13}) t^3.
\]

**Figure 14**

**Example 9.3.** Consider two 2-bridge knots $K_1$ and $K_2$ as shown in the following figure. According to Kanenobu \([\text{Ka-1, Ka-2, K-S}]\), they share the Jones-Conway (Homflypt) and Kauffman polynomials. We can, however, distinguish them by $P_1(x, t)$ polynomial. In fact, we obtain the following for $x = -2$: $P_1(K_1)(-2, t) = 49t - 48t^2$ and $P_1(K_2)(-2, t) = 28t - 27t^2$.

**Figure 15**

We speculate that Conjectures \([\text{Pr-1, Pr-3, Ki}]\) can be approached by using results of Coxeter and of Assion and by interpreting them using some elementary moves on oriented links. Coxeter showed that $C_n = B_n/(\sigma_i)^3$ is finite if $n \leq 5$, \([\text{Co, A-2, M}]\). Assion found two basic cases in which $C_n/(\text{Ideal})$ is finite: the “symplectic” and “unitary” cases \([\text{A-1, B-W, M}]\). Let $\Delta^5 = (\sigma_1\sigma_2\sigma_3\sigma_4)^5$ be a generator of the center of the braid group, $B_5$. It is easy to check that Assion’s ideals are generated by $\Delta^{10}$ and $\Delta^{15}$, respectively (see also \([\text{M}]\) Appendix III Exercise 1.3).

(1) “Symplectic case”. $C_n/(\Delta^{10})$ is a finite group.
(2) “Unitary case”. $C_n/(\Delta^{15})$ is a finite group.

We should remark that $\Delta^{30} = 1$ in $C_5$, and $C_5/\Delta^5$ is a simple group $PSp(4, 3)$ (projective symplectic group). One could try to incorporate deformation of relations $\Delta^{10} = 1$ or $\Delta^{15} = 1$ into skein module relations, but we would speculate that another approach may give appropriate skein modules. We will work with oriented links and the following useful definition and conjecture \([\text{Pr-1, Pr-3, Ki}]\).
Definition 9.4.  (i) The $t_n$-move is a local change of an oriented link which adds $n$ positive half-twists to $L$ ($L_0 \to L_n$) as in Figure 16 (i).
(ii) The $\bar{t}_k$-move is a local change of an oriented link which adds $k$ right-handed half-twists to $L$ ($L_0 \to L_k$) as in Figure 16 (ii), where $k$ is an even integer (notice anti-parallel orientation of the strings involved in the move).
(iii) Two links are $t_n, \bar{t}_k$ equivalent if one can start from the first one and reach the second one by using $t_n$-moves, $\bar{t}_k$-moves, and their inverses.

Figure 16

Conjecture 9.5.  (i) Every oriented link in $S^3$ is $t_3, \bar{t}_6$ equivalent to a trivial link.
(ii) Every oriented link in $S^3$ is $t_3, t_4$ equivalent to a trivial link.

We speculate that there is the following connection between our moves and Assion ideals.

Problem 9.6.  (i) The 5-tangle associated with the 5-braid $\Delta^{10}$ is 3-move equivalent to the trivial 5-braid tangle.
(ii) The 5-tangle associated with the 5-braid $\Delta^{15}$ is $t_3, \bar{t}_4$ equivalent to the trivial 5-braid tangle.

Note that from (i) (and [Ch]) it follows immediately that closed 5-braids are 3-equivalent to trivial links. B.Wajnryb proved (see [Wa], p.694) that every link can be reduced to a trivial link by $t_3$ moves and $\Delta^{10}$-moves (Wajnryb’s move $\xi$ is $t_3$-equivalent to $\Delta^{10}$-move). Thus the positive answer to (i) yields the positive answer to the Montesinos-Nakanishi conjecture. Generally the positive solution to Problem 9.6 would allow partial solution to Conjectures 2.1 and 3.2 One should comment, at least shortly, on the background of Problem 9.6. We follow approach and notation of [Pr-5] (compare [C-F], [F-R]). Let $D$ be an oriented link diagram. We associate $D$ with the Alexander-Burau module over the ring $R$ (with invertible element $t$), as follows. To every arc of the diagram we associate a generator (variable) $y_i$ and every crossing, $p$, of the diagram yields the relation $(1 - t^\epsilon)y_i + t^\epsilon y_k - y_j = 0$ where $\epsilon = \pm 1$ is the sign of the crossing $p$, see Figure 17. If we think of a neighborhood of a crossing as a 2-tangle (read from left to right), our relation yields a $2 \times 2$ matrix of a linear map (we can say that we consider a contra-variant functor from the category of 2-tangles to $R$-modules category). For tangles of Figure 17 matrices are:

$$A_1 = \begin{bmatrix} 1 - t^{-1} & t^{-1} \\ 1 & 0 \end{bmatrix}, \quad A_2 = A_1^{-1} = \begin{bmatrix} 0 & 1 \\ t & 1 - t \end{bmatrix}$$

Figure 17

In particular the matrix corresponding to the move of Figure 18 is:

$$A_1^3 = \begin{bmatrix} 1 - t^{-1} & t^{-1} \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 - t^{-3}(t^2 - t + 1) & t^{-3}(t^2 - t + 1) \\ t^{-2}(t^2 - t + 1) & 1 - t^{-2}(t^2 - t + 1) \end{bmatrix}$$
Generally we get that the \( t_3 \)-move preserves the Alexander-Burau module iff \( t^2 - t + 1 = 0 \) in the ring \( R \). We can repeat our analysis for \( \bar{t}_2 \)-moves. In particular the Figure 19 illustrates an example of \( \bar{t}_2 \)-moves, with the matrix

\[
B_1 = \begin{bmatrix}
1 - t & t \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 - t^{-1} & t^{-1} \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
2 - t^{-1} & t^{-1} - 1 \\
1 - t^{-1} & t^{-1}
\end{bmatrix}
\]

We can now analyze the equivalences \( B_1^3 \equiv Id \) and \( B_1^2 \equiv Id \) noting that

\[
B_1^k = \begin{bmatrix}
1 + k(1 - t^{-1}) & k(t^{-1} - 1) \\
k(1 - t^{-1}) & 1 - k(1 - t^{-1})
\end{bmatrix}
\]

Our results are summarized in the following lemma (compare [Pr-1]).

**Lemma 9.7.**  
(1) The Alexander-Burau module (of an oriented link or a tangle) is unchanged by a \( t_3 \)-move if and only if \( t^2 - t + 1 = 0 \) in the ring \( R \).

(2) The Alexander-Burau module is unchanged by \( t_3 \) and \( \bar{t}_6 \)-moves iff \( t^2 - t + 1 = 0 \) and \( 3 = 0 \) in \( R \). Notice that \( t^2 - t + 1 = (t + 1)^2 \) follows from \( 3 = 0 \).

(3) The Alexander-Burau module is unchanged by \( t_3 \) and \( \bar{t}_4 \)-moves iff \( t^2 - t + 1 = 0 \) and \( 2 = 0 \) in \( R \).

**Corollary 9.8.** Let \( M_L^{(k)} \) denote the \( k \)-fold cyclic branched cover of \( S^3 \) (resp. \( D^3 \)) with the branching set the link (resp. tangle) \( L \). Then:

(1) \( H_1(M_L^{(2)}, \mathbb{Z}_3) \) is unchanged by a \( t_3 \) and \( \bar{t}_6 \)-moves (more generally by \( 3 \)-moves).

(2) \( H_1(M_L^{(3)}, \mathbb{Z}_2) \) is unchanged by a \( t_3 \) and \( \bar{t}_4 \)-moves.

For our heuristic argument for a positive answer to Conjecture 9.5, it remains to show that \( \Delta^{10} \)-move preserves Alexander-Burau matrix modulo \( (t + 1, 3) \) and \( \Delta^{15} \)-move preserves Alexander-Burau matrix modulo \( (t^2 - t + 1, 2) \). It is a tedious but easy task. In particular Figure 20 illustrates the fact that the matrix \( M(\Delta^{-1}) \) for \( \Delta^{-1} = \sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1} \) is

\[
M(\Delta^{-1}) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
t & 0 & 0 & 1 - t \\
0 & t & 0 & 1 - t \\
0 & 0 & t & 1 - t
\end{bmatrix}
\]
We get modulo \((t^2 - t + 1)\):

\[
M(\Delta^{-10}) \equiv \begin{bmatrix}
-2 & -1 + 2t & 1 + t & 2 - t & 1 - 2t \\
-2 + t & -1 + t & 1 + t & 2 - t & 1 - 2t \\
-2 + t & -1 + 2t & 1 + t & 2 - 2t & 1 - 2t \\
-2 + t & -1 + 2t & 1 + t & 2 - t & 1 - 3t \\
\end{bmatrix}
\]

which is the identity matrix iff \(3 \equiv 0\) and \(t + 1 \equiv 0\). In fact we have more generally: \(M(\Delta^{-10}) \equiv \text{Id mod}(t + 1)\). One should stress here that the \(\Delta^{-10}\) move is not preserving the Alexander-Burau module modulo \((t^2 - t + 1, 3)\) therefore this move is not a combination of \(t_3\) and \(t_6\) moves (we conjecture only that it is a combination of 3-moves). We have also (still modulo \((t^2 - t + 1)\)):

\[
M(\Delta^{-15}) \equiv \begin{bmatrix}
-3 + 2t & 2t & 2 & 2 - 2t & -2t \\
-2 + 2t & -1 + 2t & 2 & 2 - 2t & -2t \\
-2 + 2t & 2t & 1 & 2 - 2t & -2t \\
-2 + 2t & 2t & 2 & 1 - 2t & -2t \\
-2 + 2t & 2t & 2 & 2 - 2t & -1 - 2t \\
\end{bmatrix}
\]

which is the identity matrix iff \(2 \equiv 0\). It follows from Coxeter work that \(M(\Delta^{-30}) \equiv \text{Id mod}(t^2 - t + 1)\). We have more generally: \(M(\Delta^{-30}) \equiv \text{Id mod}(t^3 + 1)\).

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move
move
move
\[ n \text{ half twists} \]

\[ +3\text{-move} \quad -3\text{-move} \quad n\text{-move} \]

Figure 1

\[
L_0 = \text{Id} \quad L_+ = s_1 \quad L_- = s_1^{-1} \quad L_\infty = U_1
\]

Figure 2

Invertible (braid type) basic tangles

\[
s_1 \quad s_1^{-1} \quad s_2 \quad s_2^{-1} \quad s_1 s_2 \quad s_1 s_2^{-1} \quad s_1^{-1} s_2
\]

Non-invertible basic tangles

\[
U_1 \quad U_2 \quad U_1 U_2 \quad U_2 U_1 \quad S_1 U_2 \quad S_1^{-1} U_2 \quad S_2 U_1 \quad S_2^{-1} U_1
\]

\[
U_2 S_1 \quad U_2 S_1^{-1} \quad U_1 S_2 \quad U_1 S_2^{-1} \quad S_1 U_2 S_1 \quad S_1 U_2 S_1^{-1} \quad S_1^{-1} U_2 S_1 \quad S_1^{-1} U_2 S_1^{-1}
\]

Figure 3
Figure 4

Figure 5

Figure 6

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Figure 7

Figure 8

Figure 9
The Conway knot  The Kinoshita-Terasaka knot

Figure 10

Constructing a 3-bridge tangle

Figure 11

The double of Borromean rings  Chen’s link

Figure 12
\[
\begin{align*}
\text{Figure 13} & \\
\text{Figure 14} & \\
\end{align*}
\]
Figure 15

(i) \( \cdots \rightarrow XXX \cdots \ \rightarrow (1-t^{-1})a + t^{-1}b \)

(ii) \( \cdots \rightarrow XXX \cdots \ \rightarrow ta + (1-t)b \)

Figure 16

Figure 17
\[
(1-t^{-1}+t^{-2}+t^{-3})a + (t^{-1}-t^{-2}+t^{-3})b
\]

Figure 18

\[
(1-t^{-1}+t^{-2})a + (t^{-1}-t^{-2})b
\]

Figure 19

\[
(2-t^{-1})a + (t^{-1}-1)b
\]

\[
(1-t^{-1})a + t^{-1}b
\]

Figure 20