Milnor $K$-groups attached to elliptic curves over a $p$-adic field

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May 5, 2014

We study the Galois symbol map of the Milnor $K$-group attached to elliptic curves over a $p$-adic field. As by-products, we determine the structure of the Chow group for the product of elliptic curves over a $p$-adic field under some assumptions.

2010 Mathematics Subject Classification: Primary 11G07; Secondary 11G07.

Key words and phrases: Elliptic curves, Chow groups, Local fields.

1 Introduction

K. Kato and M. Somekawa introduced in [12] the Milnor type $K$-group $K(k; G_1, \ldots, G_q)$ attached to semi-abelian varieties $G_1, \ldots, G_q$ over a field $k$ which is now called the Somekawa $K$-group. The group is defined by the quotient

$$K(k; G_1, \ldots, G_q) := \left( \bigoplus_{k'/k: \text{finite}} G_1(k') \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} G_q(k') \right)/R$$

where $k'$ runs through all finite extensions over $k$ and $R$ is the subgroup which produces “the projection formula” and “the Weil reciprocity law” as in the Milnor $K$-theory (Def. 2.3). As a special case, for the multiplicative groups $G_1 = \cdots = G_q = \mathbb{G}_m$, the group $K(k; \mathbb{G}_m, \ldots, \mathbb{G}_m)$ is isomorphic to $K^M_q(k)$ the ordinary Milnor $K$-group of the field $k$ ([12], Thm. 1.4). For any positive integer $m$ prime to the characteristic of $k$, let $G_i[m]$ be the Galois module defined by the kernel of $G_i(\bar{k}) \rightarrow G_i(\bar{k})$ the multiplication by $m$. Somekawa defined also the Galois symbol map

$$h : K(k; G_1, \ldots, G_q)/m \rightarrow H^q(k, G_1[m] \otimes \cdots \otimes G_q[m])$$

by the similar way as in the classical Galois symbol map $K^M_q(k)/m \rightarrow H^q(k, \mu^\otimes_m)$ on the Milnor $K$-group. He also presented a conjecture in which the map $h$ is injective for arbitrary field $k$. For the case $G_1 = \cdots = G_q = \mathbb{G}_m$, the conjecture holds by the Milnor-Bloch-Kato conjecture, now is a theorem of Voevodsky, Rost, and Weibel ([16]). However, it is also known that the above conjecture does not hold in general (see Conj. 2.4 and its remarks below for the other known results).

The aim of this note is to show this conjecture for elliptic curves over a local field under some assumptions.
Theorem 1.1 (Thm. 4.1 Prop. 4.3). (i) Let \( E_1, \ldots, E_q \) be elliptic curves over \( k \) with \( E_i[p] \subset E_i(k) \) \((1 \leq i \leq q)\). Assume that \( E_1 \) is not a supersingular elliptic curve. Then, for \( q \geq 3 \),
\[
K(k; E_1, \ldots, E_q)/p^n = 0
\]
(ii) Let \( E_1, E_2 \) be elliptic curves over \( k \) with \( E_i[p^n] \subset E_i(k) \) \((i = 1, 2)\). Assume that \( E_1 \) is not supersingular. Then, the Galois symbol map
\[
h^2 : K(k; E_1, E_2)/p^n \to H^2(k, E_1[p^n] \otimes E_2[p^n])
\]
is injective.

The theorem above is known when \( E_i \)'s have semi-ordinary reduction (= good ordinary or multiplicative reduction) ([17], [9], see also [8]). Hence our main interest is in supersingular elliptic curves.

In our previous paper [3], we investigate the image of the Galois symbol map \( h^2 \). As byproducts, we obtain the structure of the Chow group \( CH_0(E_1 \times E_2) \) of 0-cycles. By Corollary 2.4.1 in [9], we have
\[
CH_0(E_1 \times E_2) = \mathbb{Z} \oplus E_1(k) \oplus E_2(k) \oplus K(k; E_1, E_2).
\]
The Albanese kernel \( T(E_1 \times E_2) := \text{Ker} \text{alb} : CH_0(E_1 \times E_2) \to (E_1 \times E_2)(k) \) coincides with the Somekawa \( K \)-group \( K(k; E_1, E_2) \), where \( CH_0(E_1 \times E_2) \) is the kernel of the degree map \( CH_0(E_1 \times E_2) \to \mathbb{Z} \). Mattuck’s theorem [6] implies the following:

Corollary 1.2. Let \( E_1 \) and \( E_2 \) be elliptic curves over \( k \) with good or split multiplicative reduction. Assume that \( E_1 \) is not a supersingular elliptic curve and \( E_i[p^n] \subset E_i(k) \). Then, we have
\[
CH_0(E_1 \times E_2)/p^n \simeq \begin{cases} 
(\mathbb{Z}/p^n)^{2(k:Q_p)+6}, & \text{if } E_1 \text{ and } E_2 \text{ have a same reduction type,} \\
(\mathbb{Z}/p^n)^{2(k:Q_p)+7}, & \text{otherwise.}
\end{cases}
\]

Outline of this note. In Section 2 we recall the definition and some properties of Somekawa \( K \)-group \( K(k; G_1, \ldots, G_q) \) attached to semi-abelian varieties \( G_1, \ldots, G_q \) over a perfect field \( k \). We also introduce the Mackey product \( G_1 \otimes \cdots \otimes G_q \) which is defined as in [11] but by factoring out a relation concerning “the projection formula” only. In Section 3 we study the structure of the Mackey product \( \prod \otimes \prod \) over a \( p \)-adic field \( k \). Here, \( \prod \) is the Mackey functor defined by the higher unit groups of finite extensions over \( k \) as a sub Mackey functor of the cokernel \( \mathbb{Q}_m/p \) of the multiplication by \( p \) on \( \mathbb{Q}_m \). Tate [15], Raskind and Spieß [9] show that the Galois symbol map induces bijections
\[
h^2 : \left( \mathbb{Q}_m/p \otimes \mathbb{Q}_m/p \right)(k) \xrightarrow{\sim} K^M_2(k)/p \xrightarrow{h} H^2(k, \mu^{\otimes 2}).
\]
We further calculate the kernel and the image of the composition
\[
h^{n,a} : \left( \prod^n \otimes \prod^n \right)(k) \to \left( \mathbb{Q}_m/p \otimes \mathbb{Q}_m/p \right)(k) \xrightarrow{h^2} H^2(k, \mu^{\otimes 2})
\]
and determine the structure of \( \prod^n \otimes \prod^n \) partially. The proof of Theorem 1.1 is given in Section 4.

Throughout this note, for an abelian group \( A \) and a non-zero integer \( m \), let \( A[m] \) be the kernel and \( A/m \) the cokernel of the map \( m : A \to A \) defined by multiplication by \( m \). For a field \( k \), we denote by \( G_k := \text{Gal}(\overline{k}/k) \) the absolute Galois group of \( k \) and \( H^i(k, M) := H^i(G_k, M) \) the Galois cohomology group of \( G_k \) for some \( G_k \)-module \( M \). The tensor product \( \otimes \) means \( \otimes_{\mathbb{Z}} \).
Acknowledgements

A part of this note was written during a stay of the author at the Duisburg-Essen university. He thanks the institute for its hospitality. This work was supported by KAKENHI 30532551.

2 Somekawa $K$-groups

Throughout this section, $k$ is a perfect field.

Definition 2.1. A Mackey functor $A$ over $k$ is a contravariant functor from the category of étale schemes over $k$ to the category of abelian groups equipped with a covariant structure for finite morphisms such that $A(X_1 \sqcup X_2) = A(X_1) \oplus A(X_2)$ and if

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

is a Cartesian diagram, then the induced diagram

$$
\begin{array}{ccc}
A(X') & \xrightarrow{g'^*} & A(X) \\
\downarrow f'^* & & \downarrow f^* \\
A(Y') & \xrightarrow{g^*} & A(Y)
\end{array}
$$

commutes.

For a Mackey functor $A$, we denote by $A(K)$ its value $A(\text{Spec}(K))$ for a field extension $K$ over $k$.

Definition 2.2. For Mackey functors $A_1, \ldots, A_q$, their Mackey product $A_1 \otimes \cdots \otimes A_q$ is defined as follows: For any finite field extension $K/k$,

$$
K \mapsto \left( A_1 \otimes \cdots \otimes A_q \right)(K) := \left( \bigoplus_{L/k: \text{finite}} A_1(L) \otimes \cdots \otimes A_q(L) \right)/R,
$$

where $R$ is the subgroup generated by elements of the following form:

(PF) For any finite field extensions $K \subset K_1 \subset K_2$, and if $x_{i_0} \in A_{i_0}(K_2)$ and $x_i \in A_i(K_1)$ for all $i \neq i_0$, then

$$
\tilde{j}(x_1) \otimes \cdots \otimes x_{i_0} \otimes \cdots \otimes \tilde{j}(x_q) - x_1 \otimes \cdots \otimes j.(x_{i_0}) \otimes \cdots \otimes x_q,
$$

where $j : \text{Spec}(K_2) \to \text{Spec}(K_1)$ is the canonical map.

This product gives a tensor product in the abelian category of Mackey functors with unit $\mathbb{Z} : k' \mapsto \mathbb{Z}$. We write $\{x_1, \ldots, x_q\}_{K/k}$ for the image of $x_1 \otimes \cdots \otimes x_q \in A_1(K) \otimes \cdots \otimes A_q(K)$ in the product $\left( A_1 \otimes \cdots \otimes A_q \right)(k)$. For any field extension $K/k$, the canonical map $j = j_{K/k} : k \hookrightarrow K$ induces the pull-back

$$
\text{Res}_{K/k} := j^* : \left( A_1 \otimes \cdots \otimes A_q \right)(k) \longrightarrow \left( A_1 \otimes \cdots \otimes A_q \right)(K)
$$
which is called the restriction map. If the extension $K/k$ is finite, then the push-forward
\[ N_{K/k} := j_* : \left( A_1 \otimes \cdots \otimes A_q \right)(K) \longrightarrow \left( A_1 \otimes \cdots \otimes A_q \right)(k) \]
is given by $N_{K/k}(\{x_1, \ldots, x_q \}_L/K) = \{x_1, \ldots, x_q \}_L/k$ on symbols and is called the norm map. A smooth commutative algebraic group $G$ over $k$ forms a Mackey functor defined by $K \mapsto G(K)$. For a field extension $K_2/K_1$, the pull-back is the canonical map given by $j : K_1 \hookrightarrow K_2$ which is denoted by $j^* = \text{Res}_{K_2/K_1} : G(K_1) \hookrightarrow G(K_2)$. When the extension $K_2/K_1$ is finite, the push-forward is written as $j_* = N_{K_2/K_1} : G(K_2) \rightarrow G(K_1)$.

**Definition 2.3.** Let $G_1, \ldots, G_q$ be semi-abelian varieties over $k$. The Somekawa $K$-group attached to $G_1, \ldots, G_q$ is defined by
\[ K(k; G_1, \ldots, G_q) := \left( G_1 \otimes \cdots \otimes G_q \right)(k)/R, \]
where $R$ is the subgroup generated by the elements of the following form:

**(WR)** Let $k(C)$ be the function field of a projective smooth curve $C$ over $k$. For $g_i \in G_i(k(C))$ and $f \in k(C)\times$, assume that for each closed point $P$ in $C$ there exists $i(P)$ ($1 \leq i(P) \leq q$) such that $g_i \in G_i(O_{C,P})$ for all $i \neq i(P)$. Then
\[ \sum_{P \in C_0} g_1(P) \otimes \cdots \otimes \partial_P(g_{i(P)}, f) \otimes \cdots \otimes g_q(P) \in R. \]

Here $C_0$ is the set of closed points in $C$, $g_i(P) \in G_i(k(P))$ denotes the image of $g_i$ under the canonical map $G_i(O_{C,P}) \rightarrow G_i(k(P))$ and $\partial_P : G_i(k(C)) \times k(C)\times \rightarrow G_i(k(P))$ is the local symbol (12).

For an isogeny $\phi : G \rightarrow H$ of semi-abelian varieties, the exact sequence $0 \rightarrow G[\phi] \rightarrow G(k) \xrightarrow{\phi} H(k) \rightarrow 0$ of Galois modules gives an injection of Mackey functors
\[ h^1 : H/\phi \rightarrow H^1(-, G[\phi]), \]
where $H/\phi := \text{Coker}(\phi)$ (in the category of Mackey functors) and $H^1(-, G[\phi])$ is also the Mackey functor given by $K \mapsto H^1(K, G[\phi])$. For isogenies $\phi_i : G_i \rightarrow H_i$ ($1 \leq i \leq q$), the cup products and the norm map (= the corestriction) on the Galois cohomology groups give
\[ h^q : \left( H_1/\phi_1 \otimes \cdots \otimes H_q/\phi_q \right) \xrightarrow{\text{cup} \otimes (-) \cup (-) \cup \cdots \cup (-)} H^q(-, G[\phi_1] \otimes \cdots \otimes G_q[\phi_q]). \]

For any positive integer $m$ prime to the characteristic of $k$, we consider an isogeny $m : G_i \rightarrow G_i$ induced from the multiplication by $m$. The Galois symbol map $h^q : G_1/m \otimes \cdots \otimes G_q/m \rightarrow H^q(k, G_1[m] \otimes \cdots \otimes G_q[m])$ (12) factors through $K(k; G_1, \ldots, G_q)/m$ (12, Prop. 1.5) and the induced homomorphism
\[ h^q_m : K(k; G_1, \ldots, G_q)/m \rightarrow H^q(k, G_1[m] \otimes \cdots \otimes G_q[m]) \]
is called the Galois symbol map.

**Conjecture 2.4** (Kato-Somekawa, [12]). Let $G_1, \ldots, G_q$ be semi-abelian varieties over $k$. For any positive integer $m$ prime to the characteristic of $k$, the Galois symbol map $h^q_m$ is injective.
The surjectivity of the Galois symbol map does not hold in general (For example, see [4] in Sect. 4). The above conjecture is studied in the following special semi-abelian varieties:

(a) Case where \( G_1 = \cdots = G_q = \mathbb{G}_m \): The conjecture and more strongly the bijection of the Galois symbol map are known for the multiplicative groups \( G_1 = \cdots = G_q = \mathbb{G}_m \) by the Bloch-Kato conjecture (a theorem of Voevodsky, Rost and Weibel [16]). In fact, the group \( K(k; \mathbb{G}_m, \ldots, \mathbb{G}_m) \) coincides with the Milnor \( K \)-group \( K^M_q(k) \) ([12], Thm. 1.4) and the map \( h^q_m \) is the ordinary Galois symbol map.

(b) Case where \( G_1 = \mathbb{G}_m \) and \( G_2 = \mathbb{G}_m \): Yamazaki proved this conjecture for \( G_1 = \mathbb{G}_m \) to a torus which admits a “motivic interpretations” (e.g., \( k \) is a non-archimedean local field, [18], Thm. 3.2) and it is known for the multiplicative groups \( G_1 = \mathbb{G}_m \) (Kahn, [13], Prop. 3.1) and \( G_2 = \mathbb{G}_m \) ([18], Prop. 2.11) and disproved it for general tori with M. Spieß ([17], Prop. 7). Hence the above conjecture does not hold in general.

(c) Case where \( G_1 = \mathbb{G}_m \) and \( G_2 = J_X \): The Jacobian variety \( J_X \) is known as follows. For \( i \) such that \( \mu_p \not| \ p \) and \( \mathbb{G}_m \) divisible for \( k \geq \mathbb{Q} \) holds ([9], Rem. 4.5.8 (b), see also [17], Thm. 4.3). Note also that \( K(k; \mathbb{G}_m, \mathbb{G}_m) \) coincides with the Milnor \( K \)-group \( K^M_q(k) \) ([12], Thm. 1.4) and the map \( h^q_m \) is the ordinary Galois symbol map.

3 Higher unit groups

Let \( k \) be a finite field extension of \( \mathbb{Q}_p \) assuming \( p \neq 2 \). We denote by \( v_k \) the normalized valuation, \( m_k \) the maximal ideal of the valuation ring \( O_k \), \( O_k^* = U^0_k \) the group of units in \( O_k \) and \( \mathbb{F} = O_k / m_k \) the (finite) residue field. In this section we study the Mackey product of the Mackey functors \( \overline{U}^n \) defined by the higher unit groups. The higher unit groups \( U^n_k := 1 + m_k^n \) induce the filtration \( (U^n_k)_{m \geq 0} \) in \( k^\times / p \) which is given by \( U^n_k := \text{Im}(U^n_k \to k^\times / p) \). The structure of the graded pieces of this filtration is known as follows.

**Lemma 3.1** (cf. [5], Lem. 2.1.3). Put \( e_0 := e_0(k) := v_k(p)/(p - 1) \). Assume \( \mu_p := \mathbb{G}_m[p] \subset k \).

(a) If \( 0 \leq m < pe_0 \), then

\[
\overline{U}^m_k / \overline{U}^{m+1}_k = \begin{cases} 
\mathbb{F}, & \text{if } p \nmid m, \\
1, & \text{if } p | m. 
\end{cases}
\]

(b) If \( m = pe_0 \), then \( \overline{U}^{pe_0}_k / \overline{U}^{pe_0+1}_k \isom \mathbb{Z} / p \).

(c) If \( m > pe_0 \), then \( \overline{U}^m_k = 1 \).

We define a sub Mackey functor \( \overline{U}^n_m \) of \( \mathbb{G}_m[p] := \text{Coker}(p : \mathbb{G}_m \to \mathbb{G}_m) \) by \( K \mapsto \overline{U}^n_{k,k/k} \) for any positive integer \( m \), where \( e_{k/k} \) is the ramification index of the extension \( K/k \).

**Lemma 3.2** ([9], Lem. 4.2.1). For \( i, j \geq 0 \) with \( i + j \geq 2 \), the Galois symbol map induces

\[
\left( \overline{U}^i \right) \otimes (\mathbb{G}_m/p)^{\otimes j} \overset{M} \to \begin{cases} 
H^2(\mathbb{Z}/(p^2),), & \text{if } i + j = 2, \\
0, & \text{otherwise}. 
\end{cases}
\]
For each positive integers \( m \) and \( n \), now we define a map \( h^{m,n} \) by the composition

\[
h^{m,n} : \overline{U}^n \otimes \overline{U}^m \to \mathbb{G}_m/p \otimes \mathbb{G}_m/p \to H^2(\cdot, \mu_p^{\otimes 2}).
\]

Here, the latter \( h^2 \) is the Galois symbol map on \( \mathbb{G}_m/p \otimes \mathbb{G}_m/p \) and is bijective (Lem. 3.2). We also denote by

\[
h^{-1,m} : \mathbb{G}_m/p \otimes \overline{U}^m \to \mathbb{G}_m/p \otimes \mathbb{G}_m/p \to H^2(\cdot, \mu_p^{\otimes 2})
\]

by convention. We determine the structure of the Mackey product of these Mackey functors \( \overline{U}^m \) as follows.

**Lemma 3.3.** Let \( k \) be a \( p \)-adic field which contains \( \mu_p \). Put \( e_0 := e_0(k) := v_k(p)/(p - 1) \).

(i) For any positive integer \( m \), the map \( h^{-1,m} \) induces an isomorphism

\[
\mathbb{G}_m/p \otimes \overline{U}^m \xrightarrow{h^{-1,m}} \begin{cases} H^2(\cdot, \mu_p^{\otimes 2}), & \text{if } m \leq pe_0, \\ 0, & \text{otherwise}. \end{cases}
\]

(ii) For any positive integer \( m \), the map \( h^{0,m} \) induces an isomorphism

\[
\overline{U}^0 \otimes \overline{U}^m \xrightarrow{h^{0,m}} \begin{cases} H^2(\cdot, \mu_p^{\otimes 2}), & \text{if } m < pe_0, \\ 0, & \text{otherwise}. \end{cases}
\]

(iii) The map \( h^{pe_0,m} \) induces

\[
\overline{U}^{pe_0} \otimes \overline{U}^m \xrightarrow{h^{pe_0,m}} 0.
\]

The rest of this section is devoted to show this lemma. For any finite extension \( K/k \), the cup product \( \cup : H^1(K, \mu_p) \otimes H^1(K, \mu_p) \to H^2(K, \mu_p^{\otimes 2}) \) on the Galois cohomology groups is characterized by the Hilbert symbol \((\cdot, \cdot)_p : K^\times/p \otimes K^\times/p \to \mu_p \) as in the following commutative diagram (cf. [11], Chap. XIV):

\[
\begin{array}{ccc}
H^1(K, \mu_p) \otimes H^1(K, \mu_p) & \xrightarrow{\cup} & H^2(K, \mu_p^{\otimes 2}) \\
\downarrow & & \downarrow \\
K^\times/p \otimes K^\times/p & \xrightarrow{(\cdot, \cdot)_p} & \mu_p
\end{array}
\]

The order of the image in \( H^2(K, \mu_p^{\otimes 2}) \approx \mu_p \approx \mathbb{Z}/p \) by the Hilbert symbol are calculated by local class field theory (cf. [2], Lem. 3.1):

**Lemma 3.4.** Let \( m \) and \( n \) be positive integers.

(i) \( \#(K^\times/p, \overline{U}_K^m)_p = \begin{cases} p, & \text{if } m \leq pe_0(K), \\ 0, & \text{otherwise}. \end{cases} \)

(ii) If \( p \nmid m \) or \( p \nmid n \), then

\( \#(\overline{U}_K^m, \overline{U}_K^n)_p = \begin{cases} p, & \text{if } m + n \leq pe_0(K), \\ 0, & \text{otherwise}. \end{cases} \)

(iii) If \( p \mid m \) and \( p \mid n \), then

\( \#(\overline{U}_K^m, \overline{U}_K^n)_p = \begin{cases} p, & \text{if } m + n < pe_0(K), \\ 0, & \text{otherwise}. \end{cases} \)
From the lemma above, the image of the product does not depend on an extension $K/k$, hence we obtain the images as required in Lemma 3.3.

**Lemma 3.5.** For any symbol $\{a, b\}_{K/k}$ in $\left( \prod U^m \otimes \prod U^m \right)(K)$, if we assume $h^{0, m}(\{a, b\}_{K/k}) = 0$ then $\{a, b\}_{K/k} = 0$.

**Proof.** The symbol map is written by the Hilbert symbol $h^{0, m}(\{a, b\}_{K/k}) = (a, b)_p$ as in (3) and thus $a$ is in the image of the norm $N_{L/K} : \prod U^m \to \prod U^m$ for $L = K(\sqrt[p]{b})$ ([2], Chap. IV, Prop. 5.1). Take $\tilde{a} \in \prod U^m_L$ such that $N_{L/K}(\tilde{a}) = a$. We obtain

$$\{a, b\}_{K/k} = \{N_{L/K}\tilde{a}, b\}_{K/k} = \{\tilde{a}, N_{L/K}\tilde{b}\}_{K/k} = 0$$

by the condition (PF) in the definition of the Mackey product (Def. 2.2). \qed

**Proof of Lem. 3.3.** (iii) For any symbol $\{a, b\}_{L/k}$ in $\left( \prod U^{pe_0} \otimes \prod U^{m} \right)(K)$, we have $N_{L/k}(\{a, b\}_{L/k}) = \{a, b\}_{L/k}$. Thus it is enough to show $\{a, b\}_{K/k} = 0$ with $a \neq 1$. Put $e = e_{K/k}$. Since the extension $L = K(\sqrt[p]{a})$ is unramified of degree $p$ ([5], Lem. 2.1.5), the norm map $N_{L/K} : \prod U^m \to \prod U^{me}$ is surjective ([11], Chap. V, Sect. 2, Prop. 3). By the projection formula (PF),

$$\{a, b\}_{K/k} = \{a, N_{L/k}\tilde{b}\}_{K/k} = \{\tilde{a}, N_{L/k}(\tilde{b})\}_{L/k} = 0$$

for some $\tilde{b} \in \prod U^{me}_L$.

(i) The assertion (i) is proved by similar arguments as in (ii) below.

(ii) For $m \geq pe_0$, the assertion follows from Lemma 3.3 and Lemma 3.5. Assume $m < pe_0$. For any finite extension $K/k$ with ramification index $e$, we denote by $S(K)$ the subgroup of $\left( \prod U^{0} \otimes \prod U^{m} \right)(K)$ generated by symbols of the form $\{a, b\}_{K/k}$ ($a \in \prod U^{0}_K = \prod U^0$, $b \in \prod U^m(K) = \prod U^{me}$).

Put

$$n = \begin{cases} me + 1, & \text{if } p \mid me, \\ me, & \text{if } p \nmid me. \end{cases}$$

We also denote by $T(K) \subset S(K)$ the subgroup generated by $\{a, b\}_{K/k}$ with $a \in \prod U^{pe_0(K)-n}_K$. Fix a uniformizer $\pi$ of $K$ and let $\mathbb{F}_K = O_K/\pi O_K$ be the residue field of $K$. Define $\phi : \mathbb{F}_K \to T(K)$ by $x \mapsto \{1 + \pi^{pe_0(K)-n}, 1 + \pi^n\}_{K/k}$, where $\tilde{x} \in O_K$ is a lift of $x$. Lemma 3.3 and Lemma 3.5 imply $\{1 + a\pi^{pe_0(K)-n+1}, 1 + \pi^n\}_{K/k} = 0$ for $a \in O_K$. Thus the map $\phi$ does not depend on the choice of $\tilde{x}$. By calculations of symbols (cf. [1], Lem. 4.1), we have

$$h^{0, m}(\phi(x)) = (1 + \pi^{pe_0(K)-n}, 1 + \pi^n)_p$$

$$= (1 + \pi^{pe_0(K)-n}, 1 + (1 + \pi^{pe_0(K)-n})\pi^n)_p$$

$$= -(1 + \pi^{pe_0(K)} - \pi^n)_p$$

$$= -n(1 + \pi^{pe_0(K)}, \pi)_p.$$
we obtain
\[ h^{0,m}(\phi(x^p + ax)) = -n(1 + (\overline{x^p} + p\pi^{-\nu(K)p})\pi^{p\sigma(K)},\pi)_p \]
\[ = -n((1 + \overline{x^p})^p, \pi)_p \]
\[ = 0. \]

By Lemma 3.5, the map \( \phi \) factors through \( \mathbb{F}_K/\sigma(\mathbb{F}_K) \). On the other hand, the map \( \sigma \) is extended to \( \mathbb{F}_K \) and we have \( H^1(\mathbb{F}_K, \mathbb{F}_K) = 1 \). Since \( \text{Ker}(\sigma) \approx \mathbb{Z}/p \) as Galois modules, we obtain \( \mathbb{F}_K/\sigma(\mathbb{F}_K) \approx H^1(\mathbb{F}_K, \text{Ker}(\sigma)) \approx \mathbb{Z}/p \). Now we have the following commutative diagram
\[
\begin{array}{ccc}
\mathbb{F}_K/\sigma(\mathbb{F}_K) & \overset{\phi}{\longrightarrow} & T(K) \\
\downarrow \cong & & \downarrow h^{0,m} \\
H^2(K, \mu_p^{\otimes 2})
\end{array}
\]

Therefore, \( T(K) = \text{Im}(\phi) \oplus \text{Ker}(h^{0,m}) \). We show that for any element \( x = \sum_{i=1}^n (a_i, b_i)_{K/K} \in T(K) \), if \( h^{0,m}(x) = 0 \), then \( x = 0 \) by induction on \( n \). For any symbol \( \{a, b\}_{K/K} \in T(K) \) with \( h^{0,m}(\{a, b\}_{K/K}) = 0 \), Lemma 3.5 implies \( \{a, b\}_{K/K} = 0 \). Take an element \( x = \sum_{i=1}^n (a_i, b_i)_{K/K} \) with \( h^{0,m}(x) = 0 \). If \( \{a_i, b_i\}_{K/K} \in \text{Im}(\phi) \) for all \( i \), then \( x = \sum_{i=1}^n (a_i, b_i)_{K/K} \in \text{Im}(\phi) \). Hence \( h^{0,m}(x) = 0 \) implies \( x = 0 \) from the above diagram. When there exists \( j \) such that \( \{a_j, b_j\}_{K/K} \notin \text{Im}(\phi) \), we have \( h^{0,m}(\{a_j, b_j\}_{K/K}) = 0 \) and thus \( \{a_j, b_j\}_{K/K} = 0 \). By the induction hypothesis, \( h^{0,m}(\{a_j, b_j\}_{K/K}) = 0 \) implies \( 0 = \sum_{i\neq j}(a_i, b_i)_{K/K} = x \). By the same manner with the following commutative diagram,
\[
\begin{array}{ccc}
T(K) & \overset{\text{S}(K)}{\longrightarrow} & S(K) \\
\downarrow \cong & & \downarrow h^{0,m} \\
H^2(K, \mu_p^{\otimes 2})
\end{array}
\]
the map \( h^{0,m} \) is injective on \( S(K) \) and thus \( S(K) \approx \mathbb{Z}/p \).

Next, we show \( S(K) = \left( \overline{U}^M \otimes \overline{U}^m \right)(K) \). Take a symbol \( \{a, b\}_{L/K} \neq 0 \) and prove that it is in \( S(K) \) by induction on the exponent of \( p \) in the ramification index \( e_{L/K} \) of \( L/K \).

The case \( p \nmid e_{L/K} \). In this extension \( L/K \), the norm map \( N_{L/K} : \overline{U}_L^0 \to \overline{U}_K^0 \) is surjective. There exist \( \overline{c} \in \overline{U}_L^0(L) \) and \( d \in \overline{U}_K^0(K) \) such that \( \{N_{L/K}(\overline{c}), d\}_{K/K} \) is a generator of \( S(k) \approx \mathbb{Z}/p \). By the projection formula, we have
\[
\{N_{L/K}(\overline{c}), d\}_{K/K} = \{\overline{c}, \text{Res}_{L/K}(d)\}_{L/L} = N_{L/K}([\overline{c}, \text{Res}_{L/K}(d)]_{L/L}).
\]
Because of the symbol \( [\overline{c}, \text{Res}_{L/K}(d)]_{L/L} \) is also a generator of \( S(L) \), for some \( i \) we obtain
\[
\{a, b\}_{L/K} = N_{L/K}([\overline{c}, \text{Res}_{L/K}(d)]_{L/L}) = \{N_{L/K}(\overline{c}^i), d\}_{K/K}.
\]

The case \( p \mid e_{L/K} \). There exists a finite extension \( M/L \) of degree prime to \( p \) and an intermediate field \( M_1 \) of \( M/K \) such that \( M/M_1 \) is a cyclic and totally ramified extension of degree \( p \). The norm map \( N_{M/L} : \overline{U}_M^0/p \to \overline{U}_L^0/p \) is surjective and we have \( \{a, b\}_{L/K} = \{N_{M/L}(\overline{a}), b\}_{L/K} = \{\overline{a}, \text{Res}_{M/L}(b)\}_{M/K} \) for some \( \overline{a} \in \overline{U}_M^0 \). There exists an element \( c \in \overline{U}_M \) (\( M_1 \)) such that \( \Sigma = M_1(\sqrt[p]{c}) \) is a totally ramified nontrivial extension of \( M_1 \) and \( \Sigma \neq M \). In fact if the element
c is in \( \overline{U}_{M_1} \setminus (\overline{U}_{M_1}^{i+1}) \) (\( m_{E_1/i} < i < pe_0(M_1) = pe_0e_{M_1/k}, p \nmid i \)) then the upper ramification subgroups of \( G := \text{Gal}(\Sigma/M_1) \) ([1], Chap. IV) is known to be

\[
G = G^0 = G^1 = \cdots = G^{pe_0(M_1)-i} \supset G^{pe_0(M_1)-i+1} = 1
\]

([5], Lem. 2.1.5, see also [1], Chap. V, Sect. 3). Hence we can choose \( c \) such that the ramification break of \( G \) is different from the one of \( \text{Gal}(M/M_1) \). Thus \( N_{\Sigma/M_1}(U_0^0)^+ + N_{M_1}(U_0^0) = U_0^0 \) and we can take a symbol \( \{N_{M_1}(\overline{a}, c)\}_{M_1/M_1} \) such that it is a generator of \( S(M_1) \) for some \( \overline{a} \in \overline{U}_0^0 \) and thus \( \{d, \text{Res}_{M_1/M_1}(c)\}_{M_1/M} \) is also a generator of \( S(M) \). Therefore, for some \( i \), we have

\[
\{a, b\}_{L/K} = [\overline{a}, \text{Res}_{M/L}(b)]_{M/K} = N_{M/K}[\overline{a}, \text{Res}_{M/L}(b)]_{M/M} = N_{M/K}[\overline{d}, \text{Res}_{M_1}(c)]_{M/M} = N_{M_1/K} \odot N_{M_1/M_1}[\overline{d}, \text{Res}_{M_1/M_1}(c)]_{M/M_1} = N_{M_1/K}[\overline{d}, \text{Res}_{M_1/M_1}(c)]_{M/M_1} = \{N_{M_1/M_1}(\overline{d}), c\}_{M_1/M}.
\]

From the induction hypothesis, the symbol \( \{a, b\}_{L/K} \) is in \( S(K) \). Therefore, we obtain \( S(K) = \left(U^0 \otimes U^m\right)(K) \). Hence \( h^{0,m} : U^0 \otimes U^m \to H^2(-, \mu_{p^2}) \) is an isomorphism and the assertion follows.

\[
4 \text{ Galois symbol map for elliptic curves}
\]

We keep the notation as in the last section: \( k \) is a \( p \)-adic field assuming \( p \neq 2 \) with residue field \( \mathbb{F} = \mathcal{O}_k/m_k \) and \( e_0 = v_k(p)/(p-1) \). The main result here is the following theorem:

**Theorem 4.1.** Let \( E_1, E_2 \) be elliptic curves over \( k \) with \( E_i[p^n] \subset E_k(k) \) (\( i = 1, 2 \)). Assume that \( E_1 \) is not supersingular. Then the Galois symbol map

\[
h^2 : K(k; E_1, E_2)/p^n \to H^2(k, E_1[p^n] \otimes E_2[p^n])
\]

is injective.

**Proof.** Consider the following diagram with exact rows:

\[
\begin{array}{ccc}
K(k; E_1, E_2)/p^{n-1} & \xrightarrow{h^2_{p^{n-1}}} & K(k; E_1, E_2)/p^n & \xrightarrow{h^2_p} & K(k; E_1, E_2)/p \\
H^2(k, E_1[p^{n-1}] \otimes E_2[p^{n-1}]) & \xrightarrow{h^2_{p^{n-1}}} & H^2(k, E_1[p^n] \otimes E_2[p^n]) & \xrightarrow{h^2_p} & H^2(k, E_1[p] \otimes E_2[p]).
\end{array}
\]

The assumption \( E_i[p^n] \subset E_k(k) \) implies the injectivity of the left lower map \( H^2(k, E_1[p^{n-1}] \otimes E_2[p^{n-1}]) \to H^2(k, E_1[p^n] \otimes E_2[p^n]) \). By induction on \( n \), the assertion follows from the case of \( n = 1 \). By taking a finite field extension whose extension degree is prime to \( p \), we may assume that \( E_1 \) and \( E_2 \) do not have additive reductions. The assertion follows from the following slightly stronger claim than the required.

\[
\square
\]
Theorem 4.2. Let $E_1, E_2$ be elliptic curves over $k$ with $E_i[p] \subset E_i(k)$ ($i = 1, 2$). Assume that $E_1$ is not supersingular. Then,

$$h^2 : \left( E_1 \otimes E_2 \right)(k) / p \to H^2(k, E_1[p] \otimes E_2[p])$$

is injective.

We recall the following results on the image of the Kummer map $h^1 : E(k) \to H^1(k, E[p])$ for an elliptic curve $E$ over $k$. ([5], see also [14], Rem. 3.2). Assume $E[p] \subset E(k)$ and fix an isomorphism of the Galois modules $E[p] \cong (\mu_p)^{\otimes 2}$. From the isomorphism, we can identify $H^1(k, E[p])$ and $(k^\times / p)^{\otimes 2}$. On the latter group $k^\times / p$, the higher unit groups $U_k^m = 1 + n_k^m$ induce a filtration $U_k^m : = \text{Im}(U_k^m \to k^\times / p)$ as noted in the last section. In terms of this filtration, the image of $h_k^1 : E(k) / p \to H^1(k, E[p]) = (k^\times / p)^{\otimes 2}$ is written precisely as follows (cf. [14]):

$$\text{Im}(h_k^1) = \begin{cases} U_k^0 \oplus U_k^{p\epsilon_0} & \text{if } E: \text{ordinary}, \\ U_k^{\rho(e_0 - \epsilon_0)} \oplus U_k^{p\epsilon_0} & \text{if } E: \text{supersingular}, \\ k^\times / p \oplus 1 & \text{if } E: \text{(split) multiplicative}. \end{cases}$$

(4)

Here the invariant $t_0 := t_0(E)$ is defined by

$$t_0(E) = \max \{ i \mid P \in \widehat{E}(m_i^p) \text{ for all } P \in \widehat{E}[p] \}$$

where $\widehat{E}$ is the formal group associated to $E$. Note also the invariant $t_0$ satisfies $0 < t_0 < e_0$ and is calculated from the theory of the canonical subgroup of Katz-Lubin (cf. [3], Thm. 3.5).

Proof of Thm. 4.2. Fix isomorphisms of Galois modules $E_1[p] \cong \mu_p^{\otimes 2}$ and $E_2[p] \cong \mu_p^{\otimes 2}$. From the isomorphism we can identify $H^1(-, E_1[p]) \cong (\mathbb{G}_m / p)^{\otimes 2}$ and $H^1(-, E_2[p]) \cong (\mathbb{G}_m / p)^{\otimes 2}$. 

(a) $E_1$ has split multiplicative reduction: Consider the case that $E_1$ has split multiplicative reduction. We also assume that $E_2$ has supersingular good reduction. Other cases on $E_2$ are treated in the same way and much easier. From [4], the Kummer maps on $E_1$ and $E_2$ induces isomorphisms

$$E_1[p] \cong \mathbb{G}_m / p, \quad E_2[p] \cong U^{\rho(e_0 - \epsilon_0)} \oplus U^{p\epsilon_0}$$

where $t_0 := t_0(E_2)$. Therefore $E_1[p] \otimes E_2[p] \cong (\mathbb{G}_m / p \otimes U^{\rho(e_0 - \epsilon_0)}) \oplus (\mathbb{G}_m / p \otimes U^{p\epsilon_0})$. The Galois symbol map $h^2$ commutes with the map $h^{-1, \rho(e_0 - \epsilon_0)}$ and $h^{-1, p\epsilon_0}$ defined in the last section and the injectivity of $h^2$ follows from Lemma 3.3(i).

(b) $E_1$ has ordinary good reduction: Next we assume that $E_1$ has ordinary good reduction and $E_2$ is an supersingular elliptic curve over $k$. In this case also, by [4] we have

$$E_1[p] \cong U^{\rho(e_0)} \oplus U^{p\epsilon_0}$$

We have to show that the induced Galois symbol maps on

$$U^{\rho(e_0)} \otimes U^{p\epsilon_0}, U^{\rho(e_0 - \epsilon_0)}, U^{p\epsilon_0} \otimes U^{\rho(e_0 - \epsilon_0)}, \text{ and } U^{p\epsilon_0} \otimes U^{\rho(e_0 - \epsilon_0)}$$

are injective. However, the latter two are trivial by Lemma 3.3(iii). The rest of the assertions follow from Lemma 3.3(ii).
Proposition 4.3. Let $E_1, \ldots, E_q$ be elliptic curves over $k$ with $E_i[p] \subset E_i(k)$ $(1 \leq i \leq q)$. Assume that $E_1$ is not a supersingular elliptic curve. Then for $q \geq 3$,

$$\left( E_1^M \otimes \cdots \otimes E_q^M \right)(k)/p^n = K(k; E_1, \ldots, E_q)/p^n = 0$$

Proof. It is enough to show $(E_1^M \otimes E_2^M \otimes E_3^M)/p = 0$. We show only the case $E_1$ has ordinary reduction and $E_i$ has supersingular reduction for each $i = 2, 3$. As in the above proof, we have

$$E_1/p \rightarrow U_0^0 \oplus U_{p^0}, \quad E_i/p \rightarrow U_0^{p(e_0-h_i(E_i))} \oplus U_{p^0(E_i)} (i = 2, 3),$$

By Lemma 3.3, we have

$$U_0^0 \otimes U_{p^0(E_2)} \simeq U_0^0 \otimes U_{p^0(E_2)} \simeq \mathbb{G}_m^M \otimes \mathbb{G}_m^M.$$

Hence the assertion follows from Lemma 3.2. \hfill \Box

Remark. From the same arguments in the proof of Theorem 4.1 we also obtain the injectivity of the Galois symbol map

$$h^2 : K(k; \mathbb{G}_m, E)/p^n \rightarrow H^2(k, \mathbb{G}_m[p^n] \otimes E[p^n])$$

under the assumption $E[p^n] \subset E(k)$. As in [3] we can determine the image of the above $h^2$ and have

$$K(k; \mathbb{G}_m, E)/p^n \simeq \begin{cases} \mathbb{Z}/p^n, & \text{if } E: \text{multiplicative}, \\ (\mathbb{Z}/p^n)^{\oplus 2}, & \text{if } E: \text{good reduction}. \end{cases}$$

It is known that the Somekawa $K$-group $K(k; \mathbb{G}_m, E)$ is isomorphic to the homology group $V(E)$ of the complex

$$K_2(k(E)) \otimes \hat{\mathbb{Z}} \rightarrow \bigoplus_{P \in E: \text{closed points}} k^\times \rightarrow k^\times.$$

By the class field theory of curves over local field ([10], [19]), we have $V(E)/p^n \simeq \pi_1(E)^{ab, \text{geo}}/p^n$. Therefore, the above computations give the structure of $\pi_1(E)^{ab}$.

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