Photon Propagation in Dense Media

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Abstract

Using thermal field theory, we derive simple analytic expressions for the spectral density of photons in degenerate QED plasmas, without assuming the usual non or ultra-relativistic limit. We recover the standard results in both cases. Although very similar in ultra-relativistic plasmas, transverse and longitudinal excitations behave very differently as the electron Fermi momentum decreases.
Cores of white dwarves and red giant stars are typical examples of degenerate QED plasmas, where the Fermi energy is much greater than the temperature so that the Fermi distribution function can be well approximated as a step function. The vast majority of the literature describes the particle scatterings in these systems in the non-relativistic limit [1]. But the electron Fermi momentum, \( k_F \), is typically of the same order as the electron mass. On the other hand, it is well known that the properties of light inside matter quite differ between the non and the ultra-relativistic limit [1, 2, 3]. Indeed, at low densities, longitudinal time-like excitations oscillate at fixed frequency \( \omega = \omega_0 \), where \( \omega_0 = e^2 N_e / m_e \) is the plasmon frequency [2]. This is not so at very high densities, where the dispersion relations acquire a much more complicated form [3]. Also, space-like excitations seem very different between the two cases.

Thermal Field Theory is a unique tool for describing a system of multi-interacting particles in thermal equilibrium at a fixed temperature \( T \) and chemical potential \( \mu \) [4, 5]. With the recent developments in the high temperature limit [6], this theory is now settled on a sound basis. Concerning degenerate plasmas, \( i.e., \mu \gg T \), there exists some work related with the emission of hypothetical particles such as axions, but dealing essentially with ultra-relativistic plasmas [3]. We have been able to use the same methods in the general case where no approximation is made on the electron mass. The analytic expressions turn out to be very simple and easy to manipulate, and this is the main motivation for presenting our results here, separately from the applications we have in mind, like plasmon decay and axion emission [1, 3, 7, 8, 9].

The essential mathematical object we shall manipulate is the photon polarization tensor, together with its finite density corrections, from which one can derive the photon spectral density. For illustration, we use the imaginary time formalism [4] (our results can be derived in the real-time formalism too [5]). The calculations are slightly complicated by the fact that we consider a massive fermion [10]. However it turns out, as we show below, that the so-called “hard-loop” approximation [6] can be taken in a consistent way, leading to simple expressions. This happens when the photon momentum is much smaller than the scales associated with the electron, \( i.e., \), the electron chemical potential \( \mu \) and the Fermi momentum \( k_F \) (here we adopt the notation of a relativistic chemical potential, \( \mu = \sqrt{k_F^2 + m^2} \).
The one loop contribution to the photon polarization tensor is
\[ \Pi^{\mu\nu}(q_0, q) = -e^2 T \sum_n \int \frac{d^3 p}{(2\pi)^3} \frac{\text{Tr} \gamma^\mu (P - m) \gamma^\nu (P' - m)}{P^2 - m^2 (P'^2 - m^2)}, \tag{1} \]
where \( p_0 = i \omega_n + \mu, \omega_n = \pi T (2n + 1) \) and \( P' = P - Q \).

The tensorial structure is usually decomposed into transverse and longitudinal modes \([11]\):
\[ \Pi^{\mu\nu} = \Pi^T_{\mu\nu} + \Pi^L_{\mu\nu}, \tag{2} \]
and, although thermal field theory is Lorentz-covariant, a preference is given to the plasma rest frame, where \( P^T \) and \( P^L \) are given by
\[
\begin{align*}
\Pi^{00}_T &= 0; & \Pi^{0i}_T &= 0; & \Pi^{ij}_T &= -\delta^{ij} + q_i q_j / q^2; \\
\Pi^{00}_L &= -q^2 / Q^2; & \Pi^{0i}_L &= -q_0 q_i / Q^2; & \Pi^{ij}_L &= -(q_0^2 / Q^2)(q_i q_j / q^2).
\end{align*}
\tag{3}
\]

Let us now give some details about the calculation of the transverse component. After performing the energy sums and taking the \( T = 0 \) limit we get
\[
\Pi_T(q_0, q) = -4e^2 \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_p} [1 - \theta(-\mu - E_p) - \theta(\mu - E_p)] \\
+ \frac{p^2 \sin^2 \theta - \frac{1}{2} Q^2}{4E_pE'_p} \left[ \frac{1}{E'_p - E_p + q_0} \left( \theta(\mu - E_p) - \theta(\mu - E'_p) \right) \right] \\
+ \frac{1}{E_p - E'_p + q_0} \left( \theta(-\mu - E'_p) - \theta(-\mu - E_p) \right) \\
+ \frac{1}{E'_p + E_p - q_0} \left( -1 + \theta(\mu - E_p) + \theta(-\mu - E'_p) \right) \\
- \frac{1}{E'_p + E_p + q_0} \left( 1 - \theta(\mu - E'_p) - \theta(-\mu - E_p) \right) \right\}. \tag{4}
\]

Note that in the degenerate limit \( \mu \gg T \), the Fermi-Dirac distribution functions can be approximated as step functions which greatly simplifies the energy integrations.

From now on we will ignore the \( \mu \)-independent part of the polarization tensor (absorbed in the renormalization and subleading). Also we restrict
our study to the $\mu > 0$ case. We therefore obtain
\[
\Pi_T(q_0, q) = 4e^2 \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} \theta(\mu - E_p) - \frac{p^2 \sin^2 \theta - \frac{1}{2}Q^2}{4E_p E'_p} \right. \\
\times \left[ \frac{\theta(\mu - E_p) - \theta(\mu - E'_p)}{E'_p - E_p + q_0} + \frac{\theta(\mu - E_p)}{E'_p + E_p - q_0} + \frac{\theta(\mu - E'_p)}{E'_p + E_p + q_0} \right] \right\}.
\]

We make our approximations at this level. In an equivalent way to the hard thermal loop approximation of Braaten and Pisarski, we study the case where the external momenta $q$ and $q_0$ are much smaller than the Fermi momentum. In the ultrarelativistic limit this reduces to the usual $q, q_0 \ll \mu$. The most important contribution to $\Pi$ comes from the region where $p$ is of the order of $k_F$. Then the following approximation will be valid
\[
E_p - E'_p = \sqrt{p^2 + m^2} - \sqrt{(p - q)^2 + m^2} \simeq \sqrt{p^2 + m^2} - \sqrt{p^2 + m^2 - 2pq \cos \theta} \\
\simeq \frac{pq \cos \theta}{\sqrt{p^2 + m^2}} = \frac{pq \cos \theta}{E_p}.
\]

In the same limit, we can ignore the $Q^2$ term in eq. (5) and the difference between $\theta$-functions in eq. (5) can be written as
\[
\theta(\mu - E_p) - \theta(\mu - E'_p) \simeq (E_p - E'_p) \delta(\mu - E_p).
\]

Neglecting higher orders in $q/\mu$ we obtain the final result
\[
\Pi_T(q_0, q) = \frac{e^2 k_F^3}{2\pi^2 \mu} \left[ \left( \frac{\mu q_0}{k_F q} \right)^2 + \frac{1}{2} \frac{\mu q_0}{k_F q} \left( 1 - \left( \frac{\mu q_0}{k_F q} \right)^2 \right) \ln \frac{\mu q_0 + k_F q}{\mu q_0 - k_F q} \right].
\]

The calculation of $\Pi_L$ can be performed in an identical manner. We obtain
\[
\Pi_L(q_0, q) = \frac{e^2}{\pi^2 \mu k_F} \left( 1 - \frac{q_0^2}{q^2} \right) \left[ 1 - \frac{1}{2} \frac{\mu q_0}{k_F q} \ln \frac{\mu q_0 + k_F q}{\mu q_0 - k_F q} \right].
\]

As can be seen from eqs. (8) and (9), one-loop corrections start to be important when $q, q_0 \lesssim e \sqrt{\mu k_F}$, and one should then use a resummed photon propagator [3].
The dispersion relations are simply obtained from the expressions above. In the ultrarelativistic limit, the usual relations are recovered \[3, \[10 \]. In the non-relativistic case, the dispersion relations are usually obtained from kinetic theory \[2 \] and it is interesting to see how we recover them using the methods of quantum field theory. Throughout this paper we have considered a relativistic expression for \( \mu \), that is \( \mu = m + \mu_F \) where \( \mu_F \) is the standard Fermi energy. Therefore in the non-relativistic regime, \( k_F \ll \mu \), and the mass is the highest scale we have. In a usual dispersion relation \( q \) will always be smaller than \( q_0 \), and we can thus conclude that \( qk_F \ll \mu q_0 \). This observation allows us to approximate the logarithms in eqs. (8) and (9). We find the following limiting values for the polarization tensor

\[
\Pi_T(q_0, q) = \omega_0^2 \left[ 1 + \frac{1}{5} \left( \frac{qk_F}{q_0\mu} \right)^2 \right] ;
\]
\[
\Pi_L(q_0, q) = \omega_0^2 \left( 1 - \frac{q^2}{q_0^2} \right) \left[ 1 + \frac{3}{5} \left( \frac{qk_F}{q_0\mu} \right)^2 \right],
\]

where we have introduced the plasmon frequency

\[
\omega_0^2 = \frac{e^2 k_F^3}{3\pi^2 \mu} = \frac{e^2 N_e}{\mu},
\]

which is just the relativistic generalization of the usual formula. Then, the dispersion relations are deduced by solving the pole equation \( Q^2 - \text{Re}\Pi(q_0, q) = 0 \) and we obtain

\[
q_0^2 = \omega_0^2 + \left( 1 + \frac{1}{5} v_F^2 \right) q^2 \quad \text{when} \quad q \ll \omega_0;
\]
\[
q_0^2 = \left( 1 + \frac{1}{5} v_F^2 \right) \omega_0^2 + q^2 \quad \text{when} \quad q \gg \omega_0,
\]

for the transverse oscillations and

\[
q_0^2 = \omega_0^2 + \frac{3}{5} v_F^2 q^2 \quad \text{when} \quad q \ll \omega_0,
\]

for the longitudinal. These results agree perfectly with those obtained using kinetic theory \[2 \]. We see that the transverse oscillations have particle-like
dispersion relations while the frequency of longitudinal oscillations is almost independent of the momentum.

In Fig. 1, we have plotted the dispersion relations for a typical value of the Fermi momentum encountered in the plasma core of red giant and white dwarf stars. As these two systems are just between the non and the ultra relativistic limits, the relations shown in Fig. 1 are those to be used, for instance, in the plasmon decay process \[7\].

Whether it is more convenient to use thermal field theory or kinetic theory to derive these expressions for QED is probably a matter of taste. However, this is not the case for QCD which is a non-abelian gauge theory and where one is forced to use our method. As a matter of fact, the results for the gluon dispersion relations in a degenerate quark-gluon plasma are identical to those presented here, apart for some color factors. One should remember, however, that the quarks are mostly ultra-relativistic as the critical density for the transition from hadronic matter to quark-gluon plasma is much higher than the quark masses.

In the calculation of emission rates of dense systems, not only the dispersion relations, but also the imaginary part of the polarization tensor are of interest \[3\]. This imaginary part gives rise to Landau damping like contributions \[12\]. From the general expressions given above (eqs. (8) and (9)) we see that both the longitudinal and the transverse components develop an imaginary part whenever \( q_0 \) is smaller than \( (k_F/\mu)q \):

\[
\text{Im}\Pi_T(q_0, q) = -\frac{e^2}{4\pi} k_F^2 \frac{q_0}{q} \left( 1 - \left( \frac{\mu q_0}{k_F q} \right)^2 \right) \theta(k_F q - \mu q_0);
\]

\[
\text{Im}\Pi_L(q_0, q) = \frac{e^2}{2\pi} \mu^2 \frac{q_0}{q} \left( 1 - \frac{q_0^2}{q^2} \right) \theta(k_F q - \mu q_0). \tag{16}
\]

Notice that the scales are different for the two modes.

The spectral density for space-like excitations

\[
\rho(q_0, q) = -\frac{1}{\pi} \left( \frac{\text{Im}\Pi(q_0, q)}{Q^2 - \text{Re}\Pi(q_0, q))^2 + (\text{Im}\Pi(q_0, q))^2} \right), \tag{17}
\]

is plotted in Fig. 2 for a Fermi momentum \( k_F = 400 \text{ keV} \). The static limit is worth emphasizing

\[
\text{Re}\Pi_T(q_0 = 0, q \to 0) = 0; \tag{18}
\]
\[ \text{Re} \Pi_L(q_0 = 0, q \to 0) = \frac{e^2}{\pi^2} \mu k_F = k_D^2, \]  

(19)

where \( k_D \) is the inverse Debye length corresponding to the well-known phenomenon of screening of static electric fields \([4]\). Again, the expression given here is the relativistic generalization of the usual formula. Transverse excitations are not screened for any range of \( k_F \) and \( \mu \).

Despite this screening effect, one can see in the different figures presented here that the longitudinal spectral density has a much broader spectrum than the transverse one, at least in the non-relativistic limit. This originates from the different scales associated to the two modes. As a consequence, the scattering of particles will be \emph{a priori} favored through the exchange of longitudinal photons \([13]\). Our result also illuminates the “form factor” used by several authors \([14]\). In the language of kinetic theory, this form factor is related to the electric permittivity \( \epsilon_L \) by \( F = (q^2/k_D^2)(1 - 1/\epsilon_L) \) while in the language of thermal field theory the spectral density is \( \rho_L = (1/q^2)\text{Im}(1/\epsilon_L) \) (in the static limit). Despite the fact that the calculations are very different in the two approaches, we can make some simple comparisons as the scales used in the problem are the same. In the static limit the photon propagator is just \( 1/(q^2 + k_D^2) \) and one would naively guess that the cross section of a particle interacting with a charged target through the exchange of a longitudinal photon would behave as \( 1/k_D^4 \) \([14]\). However, as \( \text{Im} \Pi_L \sim k_D^2 \) for values of \( q_0 \simeq k_F q/\mu \) (which is a static point), the overall behavior is rather like \( 1/k_D^2 \), which is the result found in \([14]\). We find a justification for another point: in his initial work \([14]\), Raffelt did not take into account the damping part of the polarization tensor into the denominator of the propagator, obtaining a form factor \( F = q^2/(q^2 + k_D^2) \). However, it is clear from eq. (17) that if \( \text{Im} \Pi_L \sim k_D^2 \) then it must be taken into account in the denominator. This point was indeed realized by Raffelt and Seckel \([15]\). The obvious conclusion is an additional suppression of the scattering rates.

In sharp contrast, these considerations do not apply in the ultra-relativistic limit. Magnetic and longitudinal scales become identical and both modes are relevant when one considers a scattering rate. Furthermore, one should not forget that static transverse photons are not screened \([3]\).

In conclusion, using thermal field theory, we have obtained simple analytic expressions for the polarization tensor at a finite chemical potential \( \mu \) for all range of the Fermi momentum. The relativistic generalization of the
plasmon frequency is $\omega^2 = e^2 N_e / \mu$ and of the Debye mass $k_D^2 = (e^2 / \pi^2) \mu k_F$ (these relations can also be used when the fermion is a proton or an ion).

In agreement with the physical picture, we have found that all collective effects (quasiparticle modes and Landau damping) occur when the photon momentum $q, q_0$ is much smaller than the Debye mass $k_D$. As we have also imposed the condition $q, q_0 \ll k_F$ (hard loop approximation), these effects are well described by our formulae only when $e \mu \ll k_F$, which is the case for densities $\rho \gtrsim 10^5 \text{ g/cm}^3$. In any case, in a degenerate plasma, the photon cannot have an energy much higher than the temperature, either because of a Boltzmann suppression factor, or because of Pauli-blocking effects which favor the soft photons exchange [3]. As $\mu, k_F \gg T$, one can see that the expressions presented here (eqs. (8) and (9)) are rather general and will be of practical use, especially in stellar systems where neither the non nor the ultra relativistic approximation is accurate.

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Figure Caption

**Fig. 1** Dispersion relations for time-like excitations for a Fermi momentum $k_F = 400$ keV. The solid line represents the transverse mode and the dashed line the longitudinal mode.

**Fig. 2a** The spectral density for transverse space-like excitations for a Fermi momentum $k_F = 400$ keV.

**Fig. 2b** The spectral density for longitudinal space-like excitations for a Fermi momentum $k_F = 400$ keV.