The Camassa-Holm equation as a geodesic flow for the $H^1$ right-invariant metric

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Abstract

The fundamental role played by the Lie groups in mechanics, and especially by the dual space of the Lie algebra of the group and the coadjoint action are illustrated through the Camassa-Holm equation (CH). In 1996 Misiołek observed that CH is a geodesic flow equation on the group of diffeomorphisms, preserving the $H^1$ metric. This example is analogous to the Euler equations in hydrodynamics, which describe geodesic flow for a right-invariant metric on the infinite-dimensional group of diffeomorphisms preserving the volume element of the domain of fluid flow and to the Euler equations of rigid body with a fixed point, describing geodesics for a left-invariant metric on $SO(3)$.

The momentum map and an explicit parametrization of the Virasoro group, related to recently obtained solutions for the CH equation are presented.

Key words: Euler top, Sobolev inner product, coadjoint action, Lie group, Virasoro group, group of diffeomorphisms

1 Motion of a rigid body with a fixed point – the $SO(3)$ example

Let us start with a very familiar example – the Euler top. Consider an orthogonal basis $\tilde{e}_k(t)$, $k = 1, 2, 3$, rotating about a fixed basis $e_k$. Both bases share the same origin. We can think about the moving frame as a rigid body, moving about the origin.

The relation between the two frames is given by an orthogonal transformation: $\tilde{e}_k(t) = g_{kj}(t)e_j$, where $g_{kj} = \tilde{e}_k.e_j$, $g^T = g^{-1}$, i.e. $g$ belongs to the group $G \equiv SO(3)$, the corresponding algebra $\mathfrak{g}$ being

$$\mathfrak{g}\equiv \mathfrak{so}(3) : \quad x \in \mathfrak{g} \iff x = -x^T.$$  \hfill (1)

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Figure 1: Quantities and operators related to the so(3) algebra and its dual (Euler top case).

All quantities in the moving frame $\tilde{e}_k(t)$ (related to the body) will be marked by subscript ‘$L$’, the ones in the fixed basis - by subscript ‘$R$’. Let us take the following explicit parametrization for the angular velocity:

$$\omega_L = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathfrak{g} \leftrightarrow \vec{\omega}_L = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \in \mathbb{R}^3$$

(2)

The quantities related to the Euler top are schematically presented at Fig. 1 (the dot denotes the time derivative) [1, 18, 21]. The identification between the algebra $\mathfrak{g}$ and its dual is given by the inertia operator, see Fig. 2:

$$m_L = J(\omega_L) \equiv A\omega_L + \omega_L A,$$

(3)

where $A = \text{diag}(a_1, a_2, a_3)$ is a constant symmetric matrix.

The Hamiltonian is the kinetic energy $H(m_L, \omega_L) = \frac{1}{2} \text{tr}(m_L \omega_L^T)$, given by a left-invariant quadratic form: $H(m_L, \omega_L) = H(m, \omega)$.

This invariance by the virtue of Noether’s Theorem leads to the momentum conservation: $\frac{d}{dt}m_R = 0$. This defines a momentum map $TG \to \mathfrak{g}^*$, constant along the geodesics. Furthermore, since $\omega_L = g^{-1}\dot{g}$ we have:

$$m_L = \text{Ad}_g^* m_R = g^{-1}m_R g, \quad \dot{m}_L = \text{ad}_{\omega_L}^* m_L = -[\omega_L, m_L].$$

Finally we obtain the equations of motion (the Euler top equations):

$$\frac{d}{dt} J(\omega_L) = [J(\omega_L), \omega_L] \quad \text{or} \quad \dot{\omega}_1 = \frac{a_2 - a_3}{a_2 + a_3} \omega_2 \omega_3,$$

etc. (4)

2 Camassa-Holm equation – right invariant metric on the diffeomorphism group

The construction described briefly in the previous section can be easily generalized in cases where the Hamiltonian is a left- or right-invariant bilinear form.
Such an interesting example is the Camassa-Holm (CH) equation [2, 12, 17]. This geometric interpretation of CH was noticed firstly by Misioślejk [23] and developed further by several other authors [7–9, 13, 14, 18]. Let us introduce the notation \( u(g(x)) \equiv u \circ g \) and let us consider the \( H^1 \) Sobolev inner product

\[
H(u, v) \equiv \frac{1}{2} \int_{\mathcal{M}} (uv + u_x v_x) d\mu(x), \quad \text{with} \quad \mu(x) = x
\] (5)

The manifold \( \mathcal{M} \) is \( S^1 \) or in the case when the class of smooth functions vanishing rapidly at \( \pm \infty \) is considered, we will allow \( \mathcal{M} \equiv \mathbb{R} \).

Suppose \( g(x) \in G \), where \( G \equiv \text{Diff}(\mathcal{M}) \). Then \( H(u, v) = H(u \circ g, v \circ g) \) is a right-invariant \( H^1 \) metric.

Let us define \( g(x,t) \) as

\[ \dot{g} = u(g(x,t), t), \quad g(x, 0) = x, \quad \text{i.e.} \quad \dot{g} = u \circ g \in T_g G; \] (6)

\[ u = \dot{g} \circ g^{-1} = R_{g^{-1}} \dot{g} \in \mathfrak{g}, \] where \( \mathfrak{g} \) is \( \text{Vect}(\mathcal{M}) \). Now we recall the following result:

**Theorem 1 (A. Kirillov, 1980)** [19, 20] The dual space of \( \mathfrak{g} \) is a space of distributions but the subspace of local functionals, called the regular dual \( \mathfrak{g}^* \), is naturally identified with the space of quadratic differentials \( m(x)dx^2 \) on \( \mathcal{M} \). The pairing is given for any vector field \( u \partial_x \in \text{Vect}(\mathcal{M}) \) by

\[ \langle m dx^2, u \partial_x \rangle = \int_{\mathcal{M}} m(x)u(x) dx \]

The coadjoint action coincides with the action of a diffeomorphism on the quadratic differential:

\[ \text{Ad}^*_{g}: \quad m dx^2 \mapsto m(g)g_x^2 dx^2 \]

If \( m(x) > 0 \) for all \( x \in \mathcal{M} \), then the square root \( \sqrt{m(x)}dx^2 \) transforms under \( G \) as a 1-form. This means that \( C = \int_{\mathcal{M}} \sqrt{m(x)} dx \) is a Casimir function, i.e. an invariant of the coadjoint action.
Let us now allow the above pairing to be the $H^1$ right-invariant metric, mentioned earlier. This is possible by choosing the inertia operator $J = 1 - \partial_x^2$, i.e. by taking $m = u - u_{xx}$, see Fig. 3. Again, for the Hamiltonian $H = \frac{1}{2} \int_M m dx$, given by the $H^1$ right-invariant metric, Noether's Theorem yields [8] the conservation of $m_L \equiv g_x^2 m(g(x,t), t)$, i.e. $g_x^2 m(g(x,t), t) = m(x,0)$.

We have a momentum map $TG \to g^*$, constant along the geodesics:

$$0 = \dot{m}_L = g_x^2(2u_x m + um_x + m_t) \circ g,$$

iff $m$ satisfies the Camassa-Holm equation

$$m_t + 2u_x m + um_x = 0.$$

Similarly to the Euler top [11], CH can be written also in a Hamiltonian form $\dot{m} = -ad^*_\omega m$. Indeed,

$$\langle ad^*_\omega m dx^2, v \partial_x \rangle = \langle m dx^2, [u \partial_x, v \partial_x] \rangle = \int_M m(u_x v - v_x u) dx =$$

$$\int_M v(2mu_x + um_x) dx = \langle (2mu_x + um_x) dx^2, v \partial_x \rangle,$$

i.e. $ad^*_\omega m = 2u_x m + um_x$.

### 3 Inverse Scattering for the CH equation

In this section we consider the Camassa-Holm equation (CH) in the form

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_uxx - uu_{xxx} = 0,$$

which depends on an arbitrary parameter $\omega$ (which is not an angular velocity!). The traveling wave solutions of (9) are smooth solitons if $\omega > 0$, and peaked solitons (peakons) if $\omega = 0$ [2, 10, 11, 17, 22].

If $\omega \neq 0$ the invariance group of the Hamiltonian is the Virasoro group, $Vir = Diff(S_1) \times \mathbb{R}$ and the central extension of the corresponding Virasoro
algebra is proportional to ω [7, 15]. Thus, for ω ≠ 0, CH has various conformal properties [15]. CH is also completely integrable, possesses bi-Hamiltonian form and infinite sequence of conservation laws [2, 6, 12, 16, 24]. The Lax pair is

$$\Psi_{xx} = \left( \frac{1}{4} + \lambda(m + \omega) \right) \Psi$$  \hspace{1cm} (10) \\
$$\Psi_t = \left( \frac{1}{2\lambda} - u \right) \Psi_x + \frac{u}{2} \Psi + \gamma \Psi$$  \hspace{1cm} (11)

where γ is an arbitrary constant (for a given eigenfunction). CH is obtained from the compatibility condition $\Psi_{xxt} = \Psi_{txx}$. Let us introduce a new spectral parameter $k$ such that $\lambda(k) = -\frac{1}{2} \left( k^2 + \frac{1}{4} \right)$. From now on we consider the case where $m$ is a Schwartz class function, and $m(x, 0) + \omega > 0$. Then $m(x, t) + \omega > 0$ for all $t$ [3]. The spectral picture of (10) is [3]: continuous spectrum: $k$ real; discrete spectrum: finitely many points $k_n = \pm i\kappa_n$, $n = 1, \ldots, N$ where $\kappa_n$ is real and $0 < \kappa_n < 1/2$. Eigenfunctions: for all real $k ≠ 0$ a basis in the space of solutions can be introduced, fixed by its asymptotic when $x \to \infty$: $\psi(x, k)$ and $\bar{\psi}(x, k)$, such that

$$\psi(x, k) = e^{-ikx} + o(1), \hspace{1cm} x \to \infty.$$  \hspace{1cm} (12)

Another basis can be introduced, fixed by its asymptotic when $x \to -\infty$: $\varphi(x, k)$ and $\bar{\varphi}(x, k)$ such that

$$\varphi(x, k) = e^{-ikx} + o(1), \hspace{1cm} x \to -\infty.$$  \hspace{1cm} (13)

The relation between the two bases is

$$\varphi(x, k) = a(k) \psi(x, k) + b(k) \bar{\psi}(x, k),$$  \hspace{1cm} (14)

where [6]

$$|a(k)|^2 - |b(k)|^2 = 1.$$  \hspace{1cm} (15)

Further, one can define transmission and reflection coefficients: $T(k) = a^{-1}(k)$ and $R(k) = b(k)/a(k)$ correspondingly. According to (15)

$$|T(k)|^2 + |R(k)|^2 = 1.$$  

The entire information about these two coefficients is provided by $R(k)$ for $k > 0$. It is sufficient to know $R(k)$ only on the half line $k > 0$, since $\tilde{a}(k) = a(-k)$, $\tilde{b}(k) = b(-k)$ and therefore $R(-k) = \overline{R(k)}$. At the points of the discrete spectrum, $a(k)$ has simple zeroes, $\varphi$ and $\bar{\psi}$ are linearly dependent: $\varphi(x, i\kappa_n) = b_n \bar{\psi}(x, -i\kappa_n)$. In other words, the discrete spectrum is simple with eigenfunctions $\varphi^{(n)}(x) \equiv \varphi(x, i\kappa_n)$. The asymptotic behavior of $\varphi^{(n)}$ is

$$\varphi^{(n)}(x) = e^{\kappa_n x} + o(e^{\kappa_n x}), \hspace{1cm} x \to -\infty;$$  \\
$$\varphi^{(n)}(x) = b_n e^{-\kappa_n x} + o(e^{-\kappa_n x}), \hspace{1cm} x \to \infty.$$  \hspace{1cm} (16)

The sign of $b_n$ obviously depends on the number of the zeroes of $\varphi^{(n)}$. Suppose that $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_N < 1/2$. Then from the oscillation theorem for the Sturm-Liouville problem $\varphi^{(n)}$ has exactly $n - 1$ zeroes, i.e. $b_n = (-1)^{n-1}|b_n|$. The set

$$\mathcal{S} \equiv \{ R(k) \mid k > 0, \hspace{1cm} \kappa_n, \hspace{1cm} |b_n|, \hspace{1cm} n = 1, \ldots, N \}$$  \hspace{1cm} (17)
is called scattering data. The time evolution of the scattering data can be obtained from (11) with the choice $\gamma = \frac{i k}{\chi}$ for the eigenfunction $\varphi(k, x)$ and $x \to \infty$: $a(k, t) = 0$, $b(k, t) = \frac{ik}{\chi} b(k, t)$, or

$$a(k, t) = a(k, 0), \quad b(k, t) = b(k, 0) \exp \left(\frac{ik}{\chi} t\right); \quad (18)$$

In other words, $a(k)$ is independent on $t$ and can serve as a generating function of the conservation laws [6].

The time evolution of the data on the discrete spectrum is obtained as follows: $i \kappa_n$ are zeroes of $a(k)$, which does not depend on $t$, and therefore $\kappa_n = 0$. From (11) and (16) in a similar fashion

$$b_n = \frac{4\omega \kappa_n}{1 - 4\kappa_n^2} b_n, \quad b_n(t) = b_n(0) \exp \left(\frac{4\omega \kappa_n}{1 - 4\kappa_n^2} t\right). \quad (19)$$

4 Soliton solutions and the diffeomorphisms

The inverse scattering is simplified in the important case of the so-called reflectionless potentials, when the scattering data is confined to the case $R(k) = 0$ for all real $k$. This class of potentials corresponds to the $N$-soliton solutions of the CH equation. In this case $b(k) = 0$ and $|a(k)| = 1$ and $ia'(i\kappa_p)$ is real:

$$ia'(i\kappa_p) = \frac{1}{2\kappa_p} e^{\alpha \kappa_p} \prod_{n \neq p} \frac{\kappa_p - \kappa_n}{\kappa_p + \kappa_n}, \quad \text{where} \quad \alpha = \sum_{n=1}^{N} \ln \left(1 + \frac{2\kappa_n}{1 - 2\kappa_n}\right)^2.$$ 

Thus, $ia'(i\kappa_p)$ has the same sign as $b_n$, and therefore $c_n \equiv \frac{b_n}{ia'(i\kappa_p)} > 0$. The time evolution of $c_n$ is $c_n(t) = c_n(0) \exp \left(\frac{4\omega \kappa_n}{1 - 4\kappa_n^2} t\right)$ in the view of (19).

The $N$-soliton solution is [5]

$$u(x, t) = \frac{1}{2} \int_{-\infty}^{\infty} \exp \left(-|x - g(\xi, t)|\right) p(\xi, t) d\xi - \omega, \quad (20)$$

where $g(\xi, t), p(\xi, t)$ can be expressed through the scattering data as:

$$g(\xi, t) \equiv \ln \int_{0}^{\xi} \left(1 - \sum_{n,p} c_n(t) \frac{\xi - 2\kappa_n}{\kappa_n + 1/2} A_{np}^{-1}(\xi, t)\right)^{-2} d\xi; \quad (21)$$

$$p(\xi, t) = \omega \xi^{-2} g_{\xi}^{-1}(\xi, t), \quad \text{where} \quad (22)$$

$$A_{pn}[\xi, t] \equiv \delta_{pn} + \frac{c_n(t) \xi^{-2\kappa_n}}{\kappa_p + \kappa_n}. \quad (23)$$

Then the computation of $m = u - u_{xx}$ gives

$$m(x, t) = \int_{-\infty}^{\infty} \delta(x - g(\xi, t)) p(\xi, t) d\xi - \omega. \quad (24)$$

From the CH equation $m_t + um_x = -2(m + \omega) u_x$, (20) and (24) it follows

$$\dot{g}(\xi, t) = \frac{1}{2} \int_{0}^{\infty} e^{-|g(\xi, t) - g(\xi, t)|} p(\xi, t) d\xi - \omega, \quad \dot{g}(\xi, t) = u(g(\xi, t), t),$$
therefore $g(x,t)$ in (21) is the diffeomorphism (Virasoro group element) in the purely solitonic case. The situation when the condition $m(x,0) + \omega > 0$ on the initial data does not hold is more complicated and requires separate analysis (if $m(x,0) + \omega$ changes sign there are infinitely many positive eigenvalues accumulating at infinity and singularities might appear in finite time [3, 4]).

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