REMARKS ON ACTIONS ON COMPACTA BY SOME INFINITE-DIMENSIONAL GROUPS

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We discuss some techniques related to equivariant compactifications of uniform spaces and amenability of topological groups. In particular, we give a new proof of a recent result by Glasner and Weiss describing the universal minimal flow of the infinite symmetric group $S_{\infty}$ with the standard Polish topology, and extend Bekka’s concept of an amenable representation, enabling one to deduce non-amenability of the Banach–Lie groups $GL(L_p)$ and $GL(\ell_p)$, $1 \leq p < \infty$.

1 Introduction

Let a topological group act continuously by uniform isomorphisms on a uniform space $X$. (One important situation is where $X = G/H$ is a homogeneous factor-space of $G$, equipped with the right uniform structure.) A compact space $K$, equipped with a continuous action of $G$, is called an equivariant compactification of $G$ if there is a uniformly continuous mapping $i: X \to K$ with dense image, commuting with the action of $G$. Compactifications of this type always exist, moreover every such $X$ admits a maximal $G$-equivariant compactification.

Here we discuss some ways in which equivariant compactifications can be used to study minimal actions and amenability of some infinite-dimensional groups. The latter term is used in an intuitive sense, to refer to concrete topological groups of importance in mathematics, such as, for instance, the full unitary groups of the infinite-dimensional Hilbert spaces. Some of these groups form infinite-dimensional Lie groups in one or other sense.

A topological group $G$ is called amenable if every compact $G$-space admits an invariant (regular Borel) probability measure. In particular, $G$ is extremely amenable if every compact $G$-space contains a fixed point (that is, admits an invariant Dirac measure). No non-trivial locally compact group is extremely amenable but among infinite-dimensional groups extreme amenability is not uncommon.\[\text{\cite{1,17,21}}\]

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A continuous action of $G$ on a compact space $X$ is called minimal if the $G$-orbit of every point $x \in X$ is everywhere dense in $X$. Every topological group $G$ possesses the universal minimal flow ($G$-space), $\mathcal{M}(G)$, such that every other minimal $G$-flow is a factor of $\mathcal{M}(G)$. For locally compact groups the size of the universal minimal flow is so immense that no constructive description is ever likely. (Cf. e.g. [3]) It comes as a surprise then that the universal minimal flow of at least some infinite-dimensional groups is manageable.

Moreover, it turns out that extremely amenable groups can be used as a tool in order to give an explicit description of the universal minimal flow $\mathcal{M}(G)$ even in cases where the flow is nontrivial. If a topological group $G$ contains a ‘large’ extremely amenable subgroup $H$, then the universal minimal flow of $G$ is a subflow of the equivariant compactification of the homogeneous space $G/H$, which is a much smaller object than $G$ itself. In some cases, it enables one to describe $\mathcal{M}(G)$. Such a technique was first used by the present author in order to prove that the circle $\mathbb{S}^1$ forms the universal minimal flow for the group of orientation-preserving homeomorphisms of $\mathbb{S}^1$. Here we will use the argument in order to give a more transparent proof of the recent remarkable result by Glasner and Weiss, who have characterized the universal minimal flow of the infinite symmetric group $\mathfrak{S}_\infty$, equipped with the standard Polish topology, as the compact space of all linear orders on $\mathbb{N}$. (The proof proposed here has an advantage that it extends the result beyond the separable case, to groups of permutations of an arbitrary infinite rank.)

Let us get back to the concept of an amenable topological group. A finer scale of ‘shades of amenability’ is given by the following concept: say that a homogeneous factor-space $G/H$ (or just a uniform $G$-space $X$) is amenable in the sense of Eymard and Greenleaf if the maximal equivariant compactification of $G/H$ supports an invariant probability measure.

Here is an important particular case. A unitary representation $\pi$ of a group $G$ in a Hilbert space $\mathcal{H}$ is amenable in the sense of Bekker if there is a state on the von Neumann algebra of all bounded operators on $\mathcal{H}$, which is invariant under the action of $G$ by conjugations. It turns out that a representation $\pi$ is amenable if and only if the unit sphere in the Hilbert space of the representation, upon which $G$ acts by isometries, is an amenable uniform $G$-space.

In general, it is more difficult to verify amenability of infinite-dimensional groups than that of locally compact or discrete ones, because some tools present in the locally compact case are missing. For example, if a locally compact group $G$ contains a closed copy of the free non-abelian group on two generators, then $G$ is non-amenable, because amenability is inherited by closed subgroups of locally compact groups. Not so beyond the locally compact
in fact, every topological group embeds into an extremely amenable group as a topological subgroup. Another example: a locally compact group $G$ is amenable if and only if every strongly continuous unitary representation of $G$ is amenable. For infinite-dimensional groups, neither implication need hold.

Here we show that in some situations the property of amenability is, in a sense, ‘partly’ inherited by topological subgroups.

We extend Bekka’s concept as follows. Say that a representation $\pi$ of a group in a Banach space $E$ by bounded linear operators is amenable if the projective space of $E$ (upon which the group $G$ acts by isometries in a natural way) is an amenable $G$-space.

We show that every uniformly continuous representation of an amenable topological group is amenable. Since Eymard–Greenleaf amenability of an action (in particular, the Bekka amenability of a representation) of a group $G$ is clearly inherited by every subgroup $H < G$, we obtain a new possible way to prove that a topological group $G$ is non-amenable: to find a uniformly continuous representation $\pi$ of $G$ and a subgroup $H < G$ such that the restriction of $\pi$ to $H$ is apriori non-amenable.

The most natural class of infinite-dimensional groups admitting uniformly continuous representations are Banach–Lie groups and algebras of operators. As an illustration of our methods, we show that the general linear groups $\text{GL}(L_p)$ and $\text{GL}(\ell_p)$, where $1 \leq p < \infty$, are non-amenable if equipped with the uniform operator topology. Even for Hilbert spaces this seems to be a new result, answering a question that Pierre de la Harpe asked me back in 1999.

2 Some abstract nonsense

2.1 Uniformities and compactifications

For a topological group $G$, we denote by $U_r(G)$ the Bourbaki-right (= Ellis-left) uniform structure, whose entourage basis consists of the sets

$$V_r = \{(x, y) \in G \times G \mid xy^{-1} \in V\},$$

and $V$ runs over the neighbourhood filter, $\mathcal{N}_G$, of $G$ at the neutral element $e_G$. The symbol $\text{RUCB}(G)$ will denote the $C^*$-algebra of all Bourbaki right uniformly continuous bounded complex-valued functions on $G$, equipped with the supremum norm.

Denote by $S(G)$ the Samuel compactification of the uniform space $(G, U_r(G))$, that is, the maximal ideal space of $\text{RUCB}(G)$. This object (together with the distinguished point, $e = e_G$) is the well-known greatest ambit...
of $G$. In other words, $\mathcal{S}(G)$ is a $G$-ambit (a compact $G$-space with a distinguished point having dense orbit), admitting a continuous equivariant map, preserving the distinguished points, to any other $G$-ambit.

Any two minimal compact $G$-subspaces of $\mathcal{S}(G)$ (whose existence is guaranteed by Zorn’s lemma) are isomorphic as $G$-spaces. (This is a non-trivial fact, because there is, in general, no canonical isomorphism.) This unique minimal $G$-space is denoted $\mathcal{M}(G)$ and called the universal minimal $G$-space (or $G$-flow).

Let $H$ be a (closed or not) subgroup of a topological group $G$. The Bourbaki-right uniform structure $\mathcal{U}_r(G/H)$ is by definition the finest uniform structure on $G/H$ making the factor-map

$$G \ni g \mapsto gH \in G/H$$

uniformly continuous if $G$ is equipped with the uniformity $\mathcal{U}_r(G)$. In general, the uniformity $\mathcal{U}_r(G/H)$ need not be separated even if $H$ is a closed subgroup, and the topology generated on $G/H$ by $\mathcal{U}_r(G/H)$ may be coarser than the factor-topology on $G/H$.

The standard action of $G$ on $G/H$ on the left extends to the action of $G$ on the Samuel compactification $\sigma(G/H, \mathcal{U}_r(G/H))$. (Notice that the Samuel compactification is always a separated uniform space, and so the compactification map need not be an embedding.) We will denote the latter compact space by $\mathcal{S}_H(G)$.

The Banach $G$-module $C(\mathcal{S}_H(G)) \cong \text{UCB}(G/H, \mathcal{U}_r(G/H))$ embeds into the Banach $G$-module $\text{UCB}(\mathcal{S}(G))$. Since the action of $G$ on the latter is well-known to be continuous, the same is true of the action of $G$ on the former Banach space (and $C^*$-algebra), and as a corollary, the action of $G$ on the compact space $\mathcal{S}_H(G)$ is continuous. With the image of the coset $H$ as the distinguished point, $\mathcal{S}_H(G)$ is thus a $G$-ambit.

2.2 Amenable groups and homogeneous spaces

A topological group $G$ is called amenable if one of the following equivalent conditions holds. All measures are assumed to be regular Borel.

1. There is a left-invariant mean on the space $\text{RUCB}(G)$.

2. There is an invariant probability measure on the greatest ambit $\mathcal{S}(G)$.

3. There is an invariant probability measure on every compact space upon which $G$ acts continuously.

4. There is an invariant probability measure on $\mathcal{M}(G)$.
A topological group $G$ is called *extremely amenable* if one of the following equivalent conditions is true.

1. There is a multiplicative left-invariant mean on $\text{UCB}(G)$.
2. There is a fixed point in $S(G)$.
3. There is a fixed point in every compact space upon which $G$ acts continuously. (The fixed point on compacta property.)
4. The universal minimal flow $\mathcal{M}(G)$ is a singleton.

Even if this property looks exceedingly strong (in particular, no non-trivial locally compact group can possess it), now we know numerous examples and entire classes of infinite-dimensional groups that are extremely amenable. The following list is not exhaustive: the unitary group of an infinite-dimensional Hilbert space with the strong operator topology; the group of classes of measurable maps from the unit interval to the circle rotation group; or, more generally, to any amenable locally compact group equipped with the topology of convergence in measure; the group of homeomorphisms of the closed (or open) unit interval with the compact-open topology; the group of measure-preserving transformations of the standard Lebesgue measure space with the weak topology, as well as the group of measure class preserving transformations; the group of isometries of the Urysohn universal metric space; unitary groups of certain von Neumann algebras and $C^*$-algebras.

If $H$ is a subgroup of a topological group $G$, then the homogeneous space $G/H$ (or the pair $(G,H)$) is *Eymard–Greenleaf amenable* if there is a left-invariant mean on the space $\text{UCB}(G/H)$. Equivalently, there exists an invariant probability measure on the ambit $S_H(G)$.

More generally, one can talk of amenability of an action of a group $G$ on a uniform space $X$ by uniform isomorphisms. In such a situation, the topology on $G$ becomes irrelevant.

**Definition 2.1.** Let a group $G$ act by uniform isomorphisms on a uniform space $X$. Say that the action of $G$ is *Eymard–Greenleaf amenable*, or that $X$ is an *Eymard–Greenleaf amenable uniform $G$-space*, if there exists a $G$-invariant mean on the space $\text{UCB}(X)$. Equivalently (by the Riesz representation theorem), there exists an invariant probability measure on the Samuel compactification $\sigma X$.

For example, in the case $X = (G/H, \mathcal{U}(G/H))$ the above notion coincides with Eymard–Greenleaf amenability.
The following simple observation lends the concept some gravitas.

**Proposition 2.2.** Every continuous action of an amenable locally compact group $G$ on a uniform space $X$ by uniform isomorphisms is amenable.

*Proof.* Choose a point $x_0 \in X$ and set, for each $f \in \text{UCB}(X)$ and every $g \in X$,

$$\tilde{f}(g) := f(gx_0).$$

The function $\tilde{f}: G \to \mathbb{C}$ so defined is bounded (obvious) and continuous, as the composition of two continuous maps: the orbit map $g \mapsto gx_0$ and the function $f: X \to \mathbb{C}$. Also, for each $h \in G$,

$$\tilde{h}f(g) = \tilde{f}(h^{-1}g),$$

that is, the operator

$$\alpha: \text{UCB}(X) \ni f \mapsto \tilde{f} \in \text{CB}(G)$$

is $G$-equivariant. (Here $\text{CB}(G)$ denotes the $C^*$-algebra of all bounded complex-valued continuous functions on $G$.) It is also clear that $\alpha$ is positive, linear, bounded of norm one, and sends the function 1 to 1. Since $G$ is amenable and locally compact, there exists a left-invariant mean $\phi$ on the space $\text{CB}(G)$, and the composition $\phi \circ \alpha$ is a $G$-invariant mean on $\text{UCB}(X)$.

This result is no longer true for more general topological groups, cf. a discussion in Subsection 4.3.

### 2.3 More on the ambit $S_H(G)$

Let $G$ act continuously on a compact space $X$. Suppose there is a point $\xi \in X$ stabilized by $H$. The orbit map

$$G \ni g \mapsto g\xi \in X$$

then factors through the factor-space $G/H$, because for each $h \in H$ one has $(gh)\xi = g(h\xi) = g\xi$. Denote the resulting map $G/H \to X$ by $i$. Since the orbit map $G \to X$ is uniformly continuous relative to the uniformity $\mathcal{U}(G)$, the inductive definition of the uniformity $\mathcal{U}(G/H)$ implies that $i$ is uniformly continuous as well. Consequently, $i$ extends in a unique way to a
continuous equivariant map $S_H(G) \to X$. We conclude that $S_H(G)$, with the distinguished point $H$ (the coset of $e_G$), is the universal compact $G$-ambit with the property that $H$ stabilizes the distinguished point.

In general, the compact $G$-space $S_H(G)$ need not be minimal. The corresponding examples are easy to construct.

However, notice the following.

**Lemma 2.3.** Let $G$ be a topological group, and let $H$ be a closed subgroup. Suppose the topological group $H$ is extremely amenable. Then any minimal compact $G$-subspace, $M$, of $S_H(G)$ is a universal minimal compact $G$-space.

**Proof.** Let $X$ be an arbitrary minimal compact $G$-space. Because of extreme amenability of $H$, there is a point $\xi \in X$, stabilized by $H$. In view of the universality property of $S_H(G)$ described above, there is a morphism of $G$-spaces $j: S_H(G) \to X$ (taking $H$ to $\xi$). Because of minimality of $X$, the restriction of the map $j$ to $M$ is onto $X$. We are done.

**Example 2.4.** Let $G = \text{Homeo}_+(S^1)$, the group of orientation-preserving homeomorphisms of the circle with the usual topology of uniform convergence, and let $H$ be the isotropy subgroup of any chosen element $\theta \in S^1$. Then $H$ is isomorphic to the topological group $\text{Homeo}_+[0,1]$ and therefore extremely amenable. The right uniform factor-space $G/H$ is easily verified to be isomorphic to the circle $S^1$ with the unique compatible uniformity, and therefore the ambit $S_H(G)$ is $S^1$ itself with the distinguished point $\theta$. Since it is obviously a minimal $G$-space, we conclude by Lemma 2.3 that $S^1$ is the universal minimal $\text{Homeo}_+(S^1)$-space. This fact, established by the present author in (17), was probably the first instance where a non-trivial universal minimal flow of any topological group has been computed explicitly.

Here is another consequence of Lemma 2.3, showing that the class of extremely amenable group is closed under extensions, similarly to the class of amenable groups. This result was established (through a direct proof) during author’s discussion with Thierry Giordano and Pierre de la Harpe in April 1999, and is, thus, a joint result.

**Corollary 2.5.** Let $H$ be a closed normal subgroup of a topological group $G$. If topological groups $H$ and $G/H$ are extremely amenable, then so is $G$.

**Proof.** In this case, the ambit $S_H(G)$ is just the greatest ambit $S(G/H)$ of the factor-group, and it contains a fixed point since $G/H$ is extremely amenable. Now we conclude by Lemma 2.3. 

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3 The universal minimal flow of the infinite symmetric group

Here we use Lemma 2.3 in order to reprove a result by Glasner and Weiss describing the universal minimal flow of the infinite symmetric group. An idea of this new proof was briefly sketched by us in (21, Exercises 11 and 12), but appears here in any detail for the first time.

Let $X$ be an infinite set (countable or not), and let $G = \mathfrak{S}_X$ denote the full group of permutations of $X$, equipped with the topology of simple convergence on $X$ viewed as a discrete space. For countable $X$, this topology is well known to be Polish (separable completely metrizable).

Denote by $\text{LO}_X$ the set of all linear orders on $X$, equipped with the (compact) topology induced from $\{0, 1\}^{X \times X}$. (Here a linear order $\prec$ is identified with the characteristic function of the corresponding relation $\{(x, y) \in X \times X : x \prec y\}$.)

The group $\mathfrak{S}_X$ acts on $\text{LO}_X$ by double permutations:

$$(x \sigma \prec y) \Leftrightarrow (\sigma^{-1}x \prec \sigma^{-1}y)$$

for all $\prec \in \text{LO}_X$, $\sigma \in \mathfrak{S}_X$, and $x, y \in X$. This action is continuous and minimal (an easy check).

A linear order $\prec$ on $X$ is called $\omega$-homogeneous if every finite subset $A \subset X$ can be mapped onto any other subset $B \subset X$ of the same cardinality by an order-preserving bijection (order automorphism) of $(X, \prec)$. In particular, it follows that $\prec$ is a dense order without least and greatest elements. (In the case where $X$ is countable, this condition is equivalent to $\omega$-homogeneity.)

Every infinite set $X$ supports an $\omega$-homogeneous linear order. (Here is one proof: $X$ can be given the structure of an ordered field, because it has the same cardinality as $\mathbb{Q}(X)$, the purely transcendental field extension of $\mathbb{Q}$, and the field $\mathbb{Q}(X)$ is well known to be linearly orderable. And every linearly ordered field is $\omega$-homogeneous due to the existence of piecewise-linear monotone maps.) Choose an arbitrary such order on $X$, say $\prec$.

Let $H = \text{Aut} (\prec)$ be the subgroup of all permutations preserving the linear order $\prec$. The left factor-space $G/H = \mathfrak{S}_X / \text{Aut} (\prec)$ can be identified with a certain collection of linear orders on $X$, namely those obtained from $\prec$ by a permutation. Denote this collection by $\text{LO}_\prec$. Thus, $G/H \cong \text{LO}_\prec$ embeds into $\text{LO}_X$.

As every compact space, $\text{LO}_X$ supports a unique compatible uniform structure. It induces a totally bounded uniform structure on $\text{LO}_\prec$.

Lemma 3.1. The uniform structure on $G/H \cong \text{LO}_\prec$, induced from the compact space $\text{LO}_X$, coincides with the right uniform structure $\mathcal{U}_+ (\mathfrak{S}_X / \text{Aut} (\prec))$. 
Proof. We want to show that the uniform structure on $G/H$, induced from the compact space $LO_X$, is the finest uniform structure making the quotient map

$$\mathcal{S}_X \to \mathcal{S}_X/\text{Aut}(\prec) \cong LO_X$$

right uniformly continuous. The proof consists of two parts.

(1) The map $\sigma \mapsto \sigma \prec$ is uniformly continuous.

Let $F = \{x_1, \ldots, x_n\} \subset \omega$ be any finite subset, determining the following standard basic entourage of the uniformity of $LO_X$:

$$W_F := \{(<1, <2) \in LO_X \times LO_X : <1 |_F = <2 |_F\}.$$

Denote by $\text{St}_F$ the common isotropy subgroup of all $x_i \in F$, that is,

$$\text{St}_F := \{\sigma \in S_X | \sigma(x_i) = x_i, \; i = 1, 2, \ldots, n\}.$$

This $\text{St}_F$ is an open subgroup of $S_X$ and in particular a (standard) open neighbourhood of the identity. As such, it determines an element of the Bourbaki-right uniformity $U_r(S_X)$:

$$V_F := \{ (\sigma, \tau) \in S_X \times S_X | \sigma\tau^{-1} \in \text{St}_F \}.$$

In other words, $(\sigma, \tau) \in V_F$ iff for all $i = 1, 2, \ldots, n$ one has $\tau^{-1}x_i = \sigma^{-1}x_i$.

If now $(\sigma, \tau) \in V_F$, then for every $i, j = 1, 2, \ldots, n$ one has

$$x_i \sigma \prec x_j \iff \sigma^{-1}x_i \prec \sigma^{-1}x_j$$

$$\iff \tau^{-1}x_i \prec \tau^{-1}x_j$$

$$\iff x_i \tau \prec x_j,$$

meaning that the restrictions of the orders $\sigma \prec$ and $\tau \prec$ to $F$ coincide, and thus $(\sigma \prec, \tau \prec) \in W_F$.

(2) The image of the entourage $V_F$ under the (Cartesian square of) the map $\sigma \mapsto \sigma \prec$ is exactly all of $W_F \cap (LO_X \times LO_X)$.

Indeed, suppose $(<1, <2) \in W_F \cap (LO_X \times LO_X)$, that is, $<1$ and $<2$ are linear orders on $X$, obtained from $\prec$ by suitable permutations, and whose restrictions to a finite subset $F$ coincide.

Choose two permutations $\sigma, \tau \in S_X$ such that $<1 = \sigma \prec$ and $<2 = \tau \prec$. For each $i = 1, 2, \ldots, n$, one necessarily has

$$\sigma^{-1}x_i = \tau^{-1}x_i$$

(if it were not so, then the orders $\sigma \prec$ and $\tau \prec$ would differ on $F$). Consequently, $(\sigma, \tau) \in V_F$. 

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Now we conclude that every uniform structure, $\mathcal{U}$, on $\text{LO}_\prec$ that makes the map

$$(\mathfrak{S}_X, \mathcal{U}_r) \ni \sigma \mapsto \sigma \prec \in (\text{LO}_\prec, \mathcal{U})$$

uniformly continuous, must be coarser than the restriction of the uniformity of $\text{LO}_X$ to $\text{LO}_\prec$. Indeed, for every element $W \in \mathcal{U}$ there is, by the assumed uniform continuity of the above map, a finite $F \subseteq \omega$ with the image of $V_F$ contained in $W$, that is, with $W_F \subseteq W$. This accomplishes the argument.

**Lemma 3.2.** The ambit $\mathcal{S}_{\text{Aut}(\prec)}(\mathfrak{S}_X)$ is isomorphic to $\text{LO}_X$, with the distinguished element $\prec$.

**Proof.** By Lemma 3.1, $(\mathfrak{S}_X/\text{Aut}(\prec), \mathcal{U}_r(\mathfrak{S}_X/\text{Aut}(\prec)))$ embeds into $\text{LO}_X$ as a uniform subspace and an $\mathfrak{S}_X$-subspace. Also, $\text{LO}_\prec$ is everywhere dense in $\text{LO}_X$. As a consequence, the Samuel compactification of the precompact uniform space $(\mathfrak{S}_X/\text{Aut}(\prec), \mathcal{U}_r(\mathfrak{S}_X/\text{Aut}(\prec)))$ is simply its completion, that is, $\text{LO}_X$.

An application of Lemma 2.3 (bearing in mind that the topological group $H = \text{Aut}(\prec)$ is extremely amenable) yields immediately:

**Theorem 3.3 (Glasner and Weiss).** The compact space $\text{LO}_X$ forms the universal minimal $\mathfrak{S}_X$-space.

**Remark 3.4.** The original theorem by Glasner and Weiss was established in the case of countable $X$. Our proof remains true for symmetric groups of arbitrary infinite rank.

**Remark 3.5.** The group $\mathfrak{S}_X$ contains, as a dense subgroup, the union of the directed family of permutation subgroups of finite rank, and consequently it is amenable. As a result, there is an invariant probability measure on the compact set $\text{LO}_X$. Glasner and Weiss have proved that such a measure is unique, that is, the action by $\mathfrak{S}_X$ on $\mathcal{M}(\mathfrak{S}_X) \cong \text{LO}_X$ is uniquely ergodic.

Their argument can be made quite elementary (no Ergodic Theorem!) as follows. Let $\mu$ be a $\mathfrak{S}_X$-invariant probability measure on $\text{LO}_X$. If $F \subseteq X$ is a finite subset, then every linear order, $\prec$, on $F$ determines a cylindrical subset

$$C_\prec := \{ \prec \in \text{LO}_X : \prec \mid_F = \prec \} \subset \text{LO}_X.$$

Every two sets of this form, corresponding to different orders on $F$, are disjoint and can be taken to each other by a suitable permutation. As there are $n!$ of such sets, where $n = |F|$, the $\mu$-measure of each of them must equal $1/n!$. Consequently, the functional $\int d\mu$ is uniquely defined on the characteristic functions of cylinder sets $C_\prec$, which functions are continuous and separate.
points, because sets $C_<$ are open and closed and form a basis of open subsets of $LO_X$. Now the Stone–Weierstrass theorem implies uniqueness of $\int d\mu$ on all of $C(LO_X)$.

**Remark 3.6.** Every extremely amenable subgroup $H$ of $G_X$ is contained in one of the subgroups of the form $\text{Aut} (<)$. (Indeed, $H$ must possess a fixed point in the space $LO_X$, that is, preserve a linear order $<$ on $X$.)

At the same time, not every subgroup of the form $\text{Aut} (<)$ is extremely amenable. For example, if the linear order $<$ is such that for some cover of $X$ by three disjoint convex subsets $A, B, C$ one has $A < B < C$, $A$ and $C$ are densely ordered, and $B$ has type $\mathbb{Z}$, then the group $\text{Aut} (<)$ is topologically isomorphic to the product of three groups of order automorphisms, and since $\text{Aut} (B) \cong \mathbb{Z}$ is not extremely amenable, neither is $\text{Aut} (<)$.

On the other hand, a similar construction can be used to produce examples of groups of type $\text{Aut} (<)$ which are extremely amenable even if the order $<$ is not dense (admits gaps).

**Example 3.7.** The tame topology on the group $U(\infty) = \bigcup_{i=1}^{\infty} U(n)$ is the topology of simple convergence on the sphere $S(\infty) = \bigcup_{i=1}^{\infty} S^n$ (the intersection of the unit sphere of $\ell_2$ with the direct limit space $C^\infty$), viewed as discrete. Thus, $U(\infty)$ receives the subgroup topology from $G_S(\infty)$. This topology is of considerable interest in representation theory of the infinite unitary group, where unitary representations strongly continuous with regard to the tame topology are called tame representations.

As a consequence of the Remark 3.6, the group $U(\infty)$ with the tame topology is not extremely amenable: indeed, it is easy to see that no linear order on $S(\infty)$ is preserved by all operators from $U(\infty)$. Thus, the universal minimal flow $\mathcal{M}(U(\infty)_{\text{tame}})$ is nontrivial.

**Remark 3.8.** Let a group $G$ act by uniform isomorphisms on a uniform space $X$. The pair $(G, X)$ has the Ramsey–Dvoretzky–Milman property if for every bounded uniformly continuous function $f$ from $X$ to a finite-dimensional Euclidean space, every finite $F \subseteq X$, and each $\varepsilon > 0$ there is a $g \in G$ such that the oscillation of $f$ on the translate $gF$ is less than $\varepsilon$. This concept links extreme amenability with Ramsey theory, because a topological group $G$ is extremely amenable if and only if every continuous transitive action of $G$ by isometries on a metric space has the Ramsey–Dvoretzky–Milman property.

The statement in Example 3.7 can be strengthened: a result by Graham on the so-called sphere-Ramsey spaces implies that already the pair $(U(\infty), S(\infty))$, where $S(\infty)$ is equipped with the discrete ($\{0, 1\}$-valued) metric, does not have the Ramsey–Dvoretzky–Milman property.

This sort of dynamical properties, formulated for appropriate groups of
affine transformations, is linked to the central open question of Euclidean Ramsey theory: is every finite spherical metric space Ramsey?

4 Amenable representations

4.1 The projective space

Let $E$ be a (complex or real) Banach space. Denote by $\mathbb{P}_E$ the projective space of $E$. If we think of $\mathbb{P}_E$ as a factor-space of the unit sphere $S_E$ of $E$, then $\mathbb{P}_E$ becomes a metric space via the rule

$$d(x,y) = \inf \{ \| \xi - \zeta \| : \xi, \zeta \in S_E, p(\xi) = x, p(\zeta) = y \},$$

where $p: S_E \to \mathbb{P}_E$ is the canonical factor-map. Notice that the infimum in the formula above is in fact minimum. The proof of the triangle inequality is based on the invariance of the norm distance on the sphere under multiplication by scalars. The above metric on the projective space is complete.

Let $T \in \text{GL}(E)$ be a bounded linear invertible operator on a Banach space $E$. Define a mapping $\tilde{T}$ from the projective space $\mathbb{P}_E$ to itself as follows: for every $\xi \in S_E$ set

$$\tilde{T}(p(\xi)) = p \left( \frac{T(\xi)}{\|T(\xi)\|} \right).$$

The above definition is clearly independent on the choice of a representative, $\xi$, of an element of the projective space $x \in \mathbb{P}_E$.

**Lemma 4.1.** The mapping $\tilde{T}$ is a uniform isomorphism (and even a bi-Lipschitz isomorphism) of the projective space $\mathbb{P}_E$.

**Proof.** It is enough to show that $\tilde{T}$ is uniformly continuous, because $\tilde{T}\tilde{S} = \tilde{T}\tilde{S}$ and so $\tilde{T}^{-1} = \tilde{T}^{-1}$. Let $x,y \in \mathbb{P}_E$, and let $\xi, \zeta \in S_E$ be such that $p(\xi) = x$, $p(\zeta) = y$, and $\|\xi - \zeta\| = d(x,y)$. Both $\|T(\xi)\|$ and $\|T(\zeta)\|$ are bounded below by $\|T^{-1}\|^{-1}$, and therefore

$$d(\tilde{T}(x), \tilde{T}(y)) \leq \frac{\pi}{2} \|T^{-1}\| \cdot \|T(\xi) - T(\zeta)\|$$

$$\leq \frac{\pi}{2} \|T^{-1}\| \cdot \|T\| \cdot \|\xi - \zeta\|$$

$$= \frac{\pi}{2} \|T^{-1}\| \cdot \|T\| \cdot d(x,y).$$

Let us recall the following notion from theory of transformation groups.
Definition 4.2. Let a group $G$ act by uniform isomorphisms on a uniform space $X = (X, \mathcal{U}_X)$. The action is called bounded (or else motion equicontinuous) if for every $U \in \mathcal{U}_X$ there is a neighbourhood of the identity, $V \ni e_G$, such that $(x, g \cdot x) \in U$ for all $g \in V$ and $x \in X$.

Notice that every bounded action is continuous.

Example 4.3. The action of $\text{GL}(E)$ on the unit sphere $S_E$ (and moreover the unit ball) of a Banach space $E$ is bounded, by the very definition of the uniform operator topology.

Lemma 4.4. The correspondence

$$\text{GL}(E) \ni T \mapsto \tilde{T}$$

determines an action of the general linear group $\text{GL}(E)$ on the projective space $\mathbb{P}_E$ by uniform isomorphisms. With respect to the uniform operator topology on $\text{GL}(E)$, the action is bounded.

Proof. The first part of the statement is easy to check using Lemma 4.1. As to the second, if $\|T - I\| < \varepsilon$, then for every $\xi \in S_E$

$$\left\| \tilde{T}(x) - x \right\| \leq \frac{\pi}{2} \|T^{-1}\| \cdot \|T(x) - x\|$$

$$< \frac{\pi \varepsilon}{2(1 - \varepsilon)}.$$

\hfill \Box

4.2 Extension of Bekka’s amenability

We want to reformulate the concept of an amenable representation in the sense of Bekka in order to present a natural extension of it.

Let $\pi$ be a unitary representation of a group $G$ in a Hilbert space $\mathcal{H}$. One says that $\pi$ is amenable if there exists a state, $\phi$, on the von Neumann algebra $B(\mathcal{H})$ of all bounded operators on the space $\mathcal{H}$ of representation, which is invariant under the action of $G$ by inner automorphisms: $\phi(\pi(g)T\pi(g)^{-1}) = \phi(T)$ for every $T \in B(\mathcal{H})$ and every $g \in G$.

The group $G$ acts on the unit sphere $S_\mathcal{H}$ by isometries, and it was shown by the author that a unitary representation $\pi$ of a group $G$ in a Hilbert space $\mathcal{H}$ is amenable if and only if $S_\mathcal{H}$ is an amenable uniform $G$-space in the sense of our Definition 3.1, that is, there exists a $G$-invariant mean on the space $\text{UCB}(S_\mathcal{H})$ or, equivalently, an invariant probability measure on the Samuel compactification of the sphere $S_\mathcal{H}$. 

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While the implication ⇒ is based on some results obtained by Bekka using deep techniques by Connes, the implication ⇐ is elementary. We need to reproduce it here.

Let \( \psi \) be a \( G \)-invariant mean on UCB (\( \mathcal{S}_H \)). Every bounded linear operator \( T \) on \( H \) defines a bounded uniformly continuous (in fact, even Lipschitz) function \( f_T: \mathcal{S}_H \to \mathbb{C} \) by the rule

\[
\mathcal{S}_H \ni \xi \mapsto f_T(\xi) := \langle T\xi, \xi \rangle \in \mathbb{C}.
\]

Now set \( \phi(T) := \psi(f_T) \). This \( \phi \) is a \( G \)-invariant state on \( \mathcal{B}(H) \).

Notice that the function \( f_T \) in the proof above is symmetric: for every \( \lambda \in \mathbb{C}, |\lambda| = 1 \), and each \( \xi \in \mathcal{S}_\pi \), one has \( f_T(\lambda \xi) = f_T(\xi) \). In other words, \( f_T \) is constant on the preimages of \( p \). Consequently, \( f_T \) factors through a function \( \tilde{f}_T \) on the projective space \( \mathbb{P}_H \); clearly, \( \tilde{f}_T \) is also uniformly continuous and bounded. It means that the above proof only uses the existence of a \( G \)-invariant mean on the function space UCB (\( \mathbb{P}_H \)).

On the other hand, the Banach space (and \( G \)-module) UCB (\( \mathbb{P}_H \)) admits an obvious equivariant embedding into UCB (\( \mathcal{S}_H \)); namely, it can be identified with the Banach \( G \)-submodule of all functions symmetric in the above sense. The restriction of a \( G \)-invariant mean from UCB (\( \mathcal{S}_H \)) to UCB (\( \mathbb{P}_H \)) is again a \( G \)-invariant mean.

We have thus established the following.

**Theorem 4.5.** A unitary representation \( \pi \) of a group \( G \) in a Hilbert space \( H \) is amenable if and only if the projective space \( \mathbb{P}_H \) is an Eymard–Greenleaf amenable uniform \( G \)-space.

The advantage of this reformulation is that it allows for an extension of the concept of an amenable representation to group representations by bounded linear operators that are not necessarily unitary.

**Definition 4.6.** Say that a representation \( \pi \) of a group \( G \) by bounded linear operators in a normed space \( E \) is amenable if the action of \( G \) by uniform isometries on the projective space \( \mathbb{P}_E \), associated to \( \pi \) as in Lemma 4.4, is Eymard–Greenleaf amenable in the sense of Definition 2.1.

**Theorem 4.7.** Let \( G \) be a locally compact group, and let \( \mu \) be a quasi-invariant measure on \( G \). Let \( 1 \leq p < \infty \). The left quasi-regular representation of \( G \) in \( L^p(\mu) \) is amenable if and only if \( G \) is amenable.

**Proof.** The left quasi-regular representation, \( \gamma \), of \( G \) in \( L^p(\mu) \) (cf. e.g. [2]) is given by the formula

\[
\gamma f(x) = \left( \frac{d(\tau \circ g^{-1})}{dt} \right)^{\frac{1}{p}} f(g^{-1}x),
\]
where $d/d\tau$ is the Radon-Nykodim derivative. It is a strongly continuous representation by isometries. Necessity ($\Rightarrow$) thus follows at once from Proposition 2.2.

To prove sufficiency ($\Leftarrow$), assume $\gamma$ is amenable. Then there exists an invariant mean, $\phi$, on UCB($S_p$), where $S_p$ stands for the unit sphere in $L_p(\mu)$. For every Borel subset $A \subseteq G$, define a function $f_A : S_p \to \mathbb{C}$ by letting for each $\xi \in S_p$

$$f_A(\xi) = \|\xi \cdot \chi_A\|^p,$$

where $\chi_A$ is the characteristic function of $A$. The function $f_A$ is bounded and uniformly continuous on $S_p$. For every $g \in G$,

$$g f_A(\xi) = f_A(g^{-1} \xi)$$

$$= \int_A |g^{-1} \xi(x)|^p \, d\mu(x)$$

$$= \int_A \frac{d\mu \circ g}{d\mu} |\xi(gx)|^p \, d\mu(x)$$

$$= \int_{gA} |\xi(y)|^p \, d\mu(y)$$

$$= f_{gA}(\xi),$$

that is, $g f_A = f_{gA}$. It is now easily seen that $m(A) := \phi(f_A)$ is a finitely additive left-invariant measure on $G$, vanishing on locally null sets, and so $G$ is amenable. \hfill $\square$

4.3 Uniformly continuous representations

In contrast with Proposition 2.2, even a strongly continuous unitary representation of an amenable non-locally compact topological group need not be amenable. The simplest example is the standard representation of the full unitary group $U(H)_s$ of an infinite-dimensional Hilbert space, equipped with the strong topology. It is not amenable because it contains, as a subrepresentation, the left regular representation of a free nonabelian group, which is not amenable. (The left regular representation of a locally compact group $G$ is amenable if and only if $G$ is amenable, cf. also Theorem 4.7.)

A part of the story here is that when a topological group $G$ acts continuously by uniform isomorphisms on a uniform space $X$, the resulting representation of $G$ by isometries in UCB($X$) need not be continuous. (This is the
case, for instance, in the same example $G = U(H)_s$, $X = S_H$.) Equivalently, the extension of the action of $G$ to the Samuel (uniform) compactification $\sigma X$ is discontinuous, and therefore one cannot deduce the existence of an invariant measure on $\sigma X$ from the assumed amenability of $G$.

Here is a simple case where the continuity of action is assured.

**Lemma 4.8.** Suppose a topological group $G$ acts in a bounded way on a uniform space $X$. Then the resulting representation of $G$ in $UCB(X)$, as well as the action of $G$ on $\sigma X$, are both continuous.

**Proof.** Since $G$ acts on $UCB(X)$ by isometries, it is enough to show that the mapping $\forall g \ni g \mapsto gf \in UCB(X)$ is continuous at identity. By a given $\varepsilon > 0$, choose a $U \in U_X$ using the uniform continuity of $f$, and a symmetric neighbourhood $V \ni e$ so that $|f(x) - f(y)| < \varepsilon$ whenever $(x, y) \in U$ and $(x, g \cdot x) \in U$ once $g \in V$ and $x \in X$. Now for each $x \in X$, $|f(x) - f(g^{-1}x)| \leq \varepsilon$ once $g \in V$, that is, $\|f - gf\|_{sup} < \varepsilon$, and we are done.

Now recall that the Samuel (maximal uniform) compactification of $X$ is the maximal ideal space of $UCB(X)$. It is a simple observation (which was made, for instance, by Teleman) that a representation of a topological group by isomorphisms of a commutative $C^*$-algebra is strongly continuous if and only if the associated action of $G$ on the maximal ideal space is continuous.

The following three corollaries are immediate.

**Corollary 4.9.** Let a topological group $G$ act in a bounded way on a uniform space $X$. If $G$ is amenable, then $X$ is an Eymard–Greenleaf amenable uniform $G$-space.

**Corollary 4.10.** Let $\pi$ be a uniformly continuous representation of a topological group in a Banach space $E$. If $G$ is amenable, then $\pi$ is an amenable representation.

**Corollary 4.11.** Let $E$ be a Banach space, and let $G$ be a topological subgroup of $GL(E)$ (equipped with the uniform operator topology). If $H$ is a subgroup of $G$ and the restriction of the standard representation of $GL(E)$ in $E$ to $H$ is non-amenable, then $G$ is a non-amenable topological group.

4.4 **Groups of operators**

**Theorem 4.12.** The general linear groups $GL(L_p)$ and $GL(\ell_p)$, $1 \leq p < \infty$, with the uniform operator topology are non-amenable.

**Proof.** Both spaces $L_p$ and $\ell_p$ can be realized as $L_p(\mu)$, where $\mu$ is a quasi-invariant measure on a non-amenable locally compact group, $H$. (For instance, $H = SL(2, \mathbb{C})$ for the continuous case and $H = SL(2, \mathbb{Z})$ for the
purely atomic one.) Identify $H$ with an (abstract, non-topological) subgroup of $GL(L_p(\mu))$ via the left quasi-regular representation, $\gamma$. The restriction of the standard representation of $GL(L_p(\mu))$ to $H$ is $\gamma$, which is a non-amenable representation (Theorem 4.7), and Corollary 4.11 applies.

It is interesting to compare the above result with the following.

**Theorem 4.13.** The isometry group $\text{Iso}(\ell_p)$, $1 \leq p < \infty$, $p \neq 2$, equipped with the strong operator topology, is amenable, but not extremely amenable.

**Proof.** The isometry groups in question, as abstract groups, have been described by Banach in his classical 1932 treatise (Chap. XI, §5, pp. 178–179). For $p > 1$, $p \neq 2$ the group $\text{Iso}(\ell_p)$ is isomorphic to the semidirect product of the group of permutations $S_\infty$ and the countable power $U(1)^N$ (in the complex case) or $\{1, -1\}^\mathbb{N}$ (in the real case). Here the group of permutations acts on $\ell_p$ by permuting coordinates, while the group of sequences of scalars of absolute value one acts by coordinate-wise multiplication. The semidirect product is formed with regard to an obvious action of $S_\infty$ on $U(1)^N$ (in the real case, $\{1, -1\}^\mathbb{N}$).

The strong operator topology restricted to the group $S_\infty$ is the standard Polish topology, and restricted to the product group, it is the standard product topology. Thus, $\text{Iso}(\ell_p) \cong S_\infty \ltimes U(1)^N$ (correspondingly, $S_\infty \ltimes \{1, -1\}^\mathbb{N}$) is the semidirect product of a Polish group with a compact metric group. Since $S_\infty$ is an amenable topological group, so is $\text{Iso}(\ell_p)$. Since the non-extremely amenable group $S_\infty$ is a topological factor-group of $\text{Iso}(\ell_p)$, the latter group is not extremely amenable either.

**Remark 4.14.** Using the description of the universal minimal flow of $S_\infty$ due to Glasner and Weiss discussed in Section 3 as well as some standard means of uniform topology, one can show that the universal minimal flow of the topological group $\text{Iso}(\ell_p)$ is homeomorphic to the product of the compact space $\text{LO}_\infty$ of all linear orders on the natural numbers with the compact group $U(1)^N$ (complex case) or $\{1, -1\}^\mathbb{N}$ (real case). This compact space is equipped with a skew product action:

$$(\sigma, f) \cdot (<, g) = (\sigma <, f \cdot \sigma g),$$

where $\sigma g(n) = g(\sigma^{-1}n)$. Again, this action is uniquely ergodic. We leave the details out.

**Remark 4.15.** By contrast, for $p = 2$ the unitary group of an infinite-dimensional Hilbert space with the strong operator topology is extremely amenable. This is due to Gromov and Milman.
We conjecture that the groups \( \text{Iso}(L_p) \), \( 1 \leq p < \infty \), with the strong operator topology are all extremely amenable.

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