ON THE ANNIHILATOR OF A DOLBEAULT GROUP

Imre Patyi

ABSTRACT. We show that any Dolbeault cohomology group $H^{p,q}(D)$, $p \geq 0$, $q \geq 1$, of an open subset $D$ of a closed finite codimensional complex Hilbert submanifold of $\ell_2$ is either zero or infinite dimensional. We also show that any continuous character of the algebra of holomorphic functions of a closed complex Hilbert submanifold $M$ of $\ell_2$ is induced by evaluation at a point of $M$. Lastly, we prove that any closed split infinite dimensional complex Banach submanifold of $\ell_1$ admits a nowhere critical holomorphic function.

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1. INTRODUCTION.

In this paper we study ideals of holomorphic functions and their connections with holomorphic vector fields and Lie derivatives. We generalize in Theorem 6.2 a theorem of Laufer on the infinite dimensionality of certain Dolbeault groups. We also generalize in Theorem 4.5 a theorem of Schottenloher on continuous characters of the algebra of holomorphic functions of complex Banach manifolds. We show in Theorem 4.4 that ideals of holomorphic functions over a complex Banach manifold without common zeros are often sequentially dense. We prove that certain complex Banach manifolds admit nowhere critical holomorphic functions; see Theorems 3.4, 3.5, 7.4, Propositions 3.6 and 3.7.

2. BACKGROUND.

In this section we collect some definitions and theorems that are useful for this paper. Some good sources of information on complex analysis on Banach spaces are [D, M, L1].

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Put $B_X$ for the open unit ball of a Banach space $(X, \| \cdot \|)$, $\text{End}(X)$ for the Banach space of bounded linear operators $T: X \to X$ endowed with the operator norm, and $X^*$ for the dual space of $X$. A **complex Banach manifold** $M$ modelled on a complex Banach space $X$ is a paracompact Hausdorff space $M$ with an atlas of biholomorphically related charts onto open subsets of $X$. A subset $N \subset M$ is called a **closed complex Banach submanifold** of $M$ if $N$ is a closed subset of $M$ and for each point $x_0 \in N$ there are an open neighborhood $U$ of $x_0$ in $M$ and a biholomorphic map $\varphi: U \to B_X$ onto the unit ball $B_X$ of $X$ that maps the pair $(U, U \cap N)$ to a pair $(B_X, B_X \cap Y)$ for a closed complex linear subspace $Y$ of $X$. The submanifold $N$ is called a **split** or **direct** Banach submanifold of $M$ if at each point $x_0 \in N$ the corresponding subspace $Y$ has a direct complement in $X$. Following [L2] by Lempert we say that **plurisubharmonic domination** is possible in a complex Banach manifold $M$ if for any $u: M \to \mathbb{R}$ locally upper bounded there is a $\psi: M \to \mathbb{R}$ continuous and plurisubharmonic such that $u(x) < \psi(x)$ for all $x \in M$. This is a kind of holomorphic convexity property of $M$.

**Theorem 2.1.** (Lempert, [L2]) If $X$ is a Banach space with an unconditional basis and $\Omega \subset X$ is pseudoconvex open, then plurisubharmonic domination is possible in $\Omega$.

We make use of the following vanishing Theorem 2.2.

**Theorem 2.2.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $M \subset \Omega$ a closed split complex Banach submanifold of $\Omega$, and $E \to M$ a holomorphic Banach vector bundle. Suppose that plurisubharmonic domination is possible in every pseudoconvex open subset of $\Omega$. Then the following hold.

(a) Let $\mathcal{O}^E \to M$ be the sheaf of germs of holomorphic sections of $E \to M$. Then the sheaf cohomology group $H^q(M, \mathcal{O}^E)$ vanishes for all $q \geq 1$.

(b) Any holomorphic function $f \in \mathcal{O}(M, Z)$ into any Banach space $Z$ can be extended to an $\tilde{f} \in \mathcal{O}(\Omega, Z)$ with $\tilde{f}(x) = f(x)$ for all $x \in M$.

(c) If $0 \to E' \to E \to E'' \to 0$ is a pointwise split short exact sequence of holomorphic Banach vector bundles over $M$, then it admits a holomorphic global splitting over $M$.

(d) If $(f_n) \in \mathcal{O}(M, \ell_2)$ is nowhere zero on $M$, then there is a $(g_n) \in \mathcal{O}(M, \ell_2)$ with $\sum_{n=1}^{\infty} f_n(x)g_n(x) = 1$ for all $x \in M$, where the series converges absolutely and uniformly on every compact subset of $M$.

**Proof.** These are special cases of the vanishing theorem of [LP]; for (d) see also [DPV].

**3. RECIPROCAL PAIRS.**

In this section we look at the following simple notion. Let $M$ be a complex
Banach manifold, \( f \in \mathcal{O}(M) \) a holomorphic function, and \( v \in \mathcal{O}(M, T^{1,0}M) \) a holomorphic (tangent) vector field on \( M \). We call \( f, v \) a reciprocal pair if the Lie derivative \( (\mathcal{L}_v f)(x) = (vf)(x) = (df)(x)v(x) = 1 \) for all \( x \in M \). If \( f, v \) is a reciprocal pair on \( M \), then the Fréchet differential \( df \) and \( v \) do not have zeros on \( M \); in particular, \( f \) is nowhere critical on \( M \), i.e., it has no points \( x \in M \) with \( (df)(x) = 0 \). A useful weakening of the notion of a reciprocal pair is a generalized reciprocal pair defined as follows. Let \( n \geq 1 \), \( f_i \in \mathcal{O}(M) \) and \( v_i \in \mathcal{O}(M, T^{1,0}M) \) for \( i = 1, \ldots, n \). We call \( f_1, \ldots, f_n, v_1, \ldots, v_n \) a generalized reciprocal pair if \( \sum_{i=1}^n v_if_i = 1 \) on \( M \). The simple proof of Proposition 3.1 below is omitted.

**Proposition 3.1.** Let \( M \) be a complex Banach manifold.

(a) If \( M \) admits a generalized reciprocal pair, and \( M' \) is biholomorphic to \( M \), then \( M' \) also admits an analogous generalized reciprocal pair.

(b) If \( D \subset M \) is open, and \( M \) admits a generalized reciprocal pair \( f_i, v_i \) for \( i = 1, \ldots, n \), then \( D \) admits the generalized reciprocal pair \( f_i|D, v_i|D \) of restrictions.

(c) Suppose that \( M \) admits functions \( f_i \in \mathcal{O}(M) \) for \( i = 1, \ldots, n \) without common critical points, and look at the short exact sequence

\[
0 \to K \to (T^{1,0}M)^n \to M \times \mathbb{C} \to 0
\]

of holomorphic Banach vector bundles over \( M \), where the third mapping is \((\xi_i) \mapsto \sum_{i=1}^n (df_i)\xi_i\), and the second mapping is inclusion; so \( K \) is the kernel of the third mapping. Then there are holomorphic vector fields \( v_i \in \mathcal{O}(M, T^{1,0}M) \) for \( i = 1, \ldots, n \) with \( \sum_{i=1}^n v_if_i = 1 \) on \( M \) if and only if the above short exact sequence of holomorphic Banach vector bundles splits holomorphically over \( M \). The latter is the case if the sheaf cohomology group \( H^1(M, \mathcal{O}^K) \) vanishes, where \( \mathcal{O}^K \to M \) is the sheaf of germs of holomorphic sections of the holomorphic Banach vector bundle \( K \to M \).

(d) If \( M' \) is a complex Banach manifold that admits a generalized reciprocal pair \( f'_i, v'_i \) for \( i = 1, \ldots, n \), and \( M'' \) is any complex Banach manifold, then \( M = M' \times M'' \) also admits an analogous reciprocal pair \( f_i, v_i \) given by \( f_i(x', x'') = f'_i(x') \) and \( v_i(x', x'') = (v'_i(x'), 0) \) for \( i = 1, \ldots, n \).

**Theorem 3.2.** Let \( X \) be a Banach space with a Schauder basis, \( \Omega \subset X \) pseudoconvex open, \( M \subset \Omega \) a closed split complex Banach submanifold of \( \Omega \), and suppose that plurisubharmonic domination is possible in every pseudoconvex open subset of \( \Omega \) (the last is guaranteed by Lempert’s Theorem 2.1 if \( X \) has an unconditional basis). If \( f_i \in \mathcal{O}(M), i = 1, \ldots, n \), have no common critical points in \( M \), then there are \( v_i \in \mathcal{O}(M, T^{1,0}M), i = 1, \ldots, n \), such that \( \sum_{i=1}^n v_if_i = 1 \) on \( M \).

**Proof.** Theorem 2.2(a) implies that Proposition 3.1(c) applies, completing the proof of Theorem 3.2.
We recall a deep theorem of Forstnerič.

**Theorem 3.3.** (a) (Forstnerič, [F]) *Every Stein manifold admits a nowhere critical holomorphic function.*

(b) *Every Stein manifold without isolated points admits a reciprocal pair.*

**Proof of (b).** As Theorem B of Cartan, Oka, Serre implies the vanishing of the relevant cohomology group, (b) follows from (a) via Proposition 3.1(c).

**Theorem 3.4.** If $M$ is a complex Banach manifold satisfying (a) or (b) below, and $D \subset M$ is any open subset, then $D$ admits a reciprocal pair.

(a) $M$ is any Banach space.

(b) $M$ is the $W_2^{(k)}$-Sobolev space of mappings $x: K \to N$ that have $k$ derivatives in $L_2$, where $K$ is any compact smooth manifold, $N$ is any Stein manifold, and $k$ is any integer with $2k > \dim \mathbb{R}(K)$.

**Proof.** (a) Choose a linear functional $f \in M^*$ and a vector $v \in M$ such that $f(v) = 1$. Regard $v$ as a constant vector field on $M$. (b) Forstnerič’s Theorem 3.3(a) gives a nowhere critical $g \in \mathcal{O}(N)$, choose any probability Radon measure $\mu$ on $K$, e.g., $\mu = \delta_{t_0}$ the Dirac delta measure concentrated at a point $t_0 \in K$, and define $f \in \mathcal{O}(M)$ by $f(x) = \int_{t \in K} g(x(t)) \, d\mu(t)$ for $x \in M$. It is easy to check that $f$ is indeed a holomorphic function on $M$, and that it has no critical points in $M$. Theorem 2.2(a) implies that Proposition 3.1(c) applies and gives us a vector field $v \in \mathcal{O}(M, T_{1,0}^1 M)$ with $vf = 1$. Proposition 3.1(b) concludes the proof of Theorem 3.4.

We omit the simple proof of the following Theorem 3.5, cf. Theorem 7.4.

**Theorem 3.5.** If $X$ is a separable Banach space, its dual $X^*$ is nonseparable, $\Omega \subset X$ open, and $M \subset \Omega$ a closed complex Banach submanifold of $\Omega$ of finite codimension, then there is a linear functional $\xi \in X^*$ whose restriction $f = \xi|_M$ is a nowhere critical function $f \in \mathcal{O}(M)$ on $M$. Further, if $X$ has an unconditional basis and $\Omega$ is pseudoconvex, then there is a $v \in \mathcal{O}(M, T^{1,0} M)$ with $vf = 1$ on $M$.

In Theorem 3.5 the Banach space $X$ can be $X = \ell_1 \times Y$, where $Y$ is any Banach space with an unconditional basis. Sometimes a reciprocal pair can be constructed with explicit computation as in Proposition 3.6 below, whose easy proof we omit.

**Proposition 3.6.** Let $X$ be a finite dimensional or separable Hilbert space of dimension at least two with standard coordinate functions $x_1, x_2, \ldots$, and $M \subset X$ the smooth hypersurface defined by $g(x) = 0$, where $g(x) = -1 + \sum x_j^2$. Then a reciprocal pair $f, v$ on $M$ is given by $f(x) = x_1 + ix_2$, and $v(x) = -ix_1^2 D_2 + ix_1 x_2 D_1 + D_1 - x_1 E$, where $i = \sqrt{-1}$, $D_j = \frac{\partial}{\partial x_j}$ is the
usual Wirtinger derivative with respect to $x_j$, and $E$ is the Euler derivative $(Eh)(x) = (dh)(x)x$ for $h \in \mathcal{O}(X)$, i.e., $E = \sum_j x_j D_j$.

A generalization of part of Proposition 3.6 is given below in Proposition 3.7, whose proof is clear from the definitions.

**Proposition 3.7.** Let $X = X' \times X''$ be a direct decomposition of Banach spaces, $g_1 \in \mathcal{O}(X')$, $g_2 \in \mathcal{O}(X'')$ entire functions, and suppose that the hypersurface $M' \subset X'$ defined by $0 = g_1(x')$ is smooth, $(dg_1)(x') \neq 0$ for $x' \in M'$, there is an $f_1 \in \mathcal{O}(X')$ such that $f_1$ is nowhere critical on $X'$ and so is $f_1|M'$ on $M'$, $g_2(0) = 0$, and the only critical point of $g_2$ in $X''$, if any, is $x'' = 0$. Then the hypersurface $M \subset X$ defined by $0 = g(x', x'')$, where $g(x', x'') = g_1(x') + g_2(x'')$, is smooth and the function $f \in \mathcal{O}(X)$ defined by $f(x', x'') = f_1(x')$ is nowhere critical on $M$.

Proposition 3.6 (without the vector field $v$) is a special case of Proposition 3.7, where $X' = \mathbb{C}^2, g_1(x_1', x_2') = -1+(x_1')^2+(x_2')^2, f_1(x_1', x_2') = x_1'+ix_2'$, and $g_2(x''_2, x''_3, \ldots) = \sum_j (x''_j)^2$.

We can also make various other special cases of Proposition 3.7, e.g., $X', g_1, f_1$ as above, $X'' = \ell_p, 1 \leq p \leq \infty$, and $g_2(x'') = \sum_{j=1}^{\infty} a_j (x''_j)^{r_j}$, where the $r_j \geq 1$ are integers, and the coefficients $0 \neq a_j \in \mathbb{C}$ are such that $g_2 \in \mathcal{O}(X'')$, e.g., $a_j \rightarrow 0$ fast enough as $j \rightarrow \infty$.

Proposition 3.8 below is obvious from elementary linear algebra.

**Proposition 3.8.** Let $M'$ be a complex Banach manifold, $M \subset M'$ a closed complex Banach submanifold of $M'$ of finite codimension $k \geq 1$. If $f_\kappa \in \mathcal{O}(M')$ for $\kappa = 0, \ldots, k$ satisfy that $(df_\kappa)(x)$ for $\kappa = 0, \ldots, k$ are linearly independent at each point $x \in M$, then the restrictions $f_\kappa|M \in \mathcal{O}(M)$ for $\kappa = 0, \ldots, k$ have no common critical points in $M$.

If $M'$ is a Banach space of dimension at least $k + 1$, then we can choose the $f_\kappa \in (M')^\ast$ to be any $k + 1$ linearly independent linear functionals in Proposition 3.8.

### 4. Families of Functions.

In this section we look at the following notion. Let $M$ be a topological space, $f_n: M \rightarrow \mathbb{C}, n \geq 1$, a sequence of numerical functions. We say that the sequence $(f_n)$ has a **uniform local rate of growth** (as $n \rightarrow \infty$) if there are a sequence of constants $L_n > 0, n \geq 1$, an open covering $\mathcal{U}$ of $M$, and a function $C: \mathcal{U} \rightarrow (0, \infty)$ such that if $U \in \mathcal{U}, x \in U$, and $n \geq 1$, then $|f_n(x)| \leq C(U)L_n$.

In other words, near each point $x_0 \in M$ our functions $f_n(x)$ are bounded, and they grow no faster than $(\text{const})L_n$ for a sequence of constants $L_n, n \geq 1$. 

Proposition 4.1. (a) If $M$ is a Lindelöf space, then any sequence of locally bounded functions $f_n: M \to \mathbb{C}$, $n \geq 1$, has a uniform local rate of growth.

(b) Let $M$ be a paracompact Hausdorff space, and $f_n: M \to \mathbb{C}$, $n \geq 1$, a sequence of functions. Then $(f_n)$ has a uniform local rate of growth $(L_n)$ if and only if there is a continuous function $\gamma: M \to (0, \infty)$ with $|f_n(x)| \leq \gamma(x)L_n$ for all $x \in M$ and $n \geq 1$.

On a finite dimensional complex manifold any sequence of holomorphic functions has a uniform local rate of growth by Proposition 4.1(a), unlike on an infinite dimensional one. We now show that certain natural sequences do.

Proposition 4.2. (a) Let $X$ be a complex Banach space, $B_X$ its open unit ball, and $f \in \mathcal{O}(B_X)$. If there is a bound $0 \leq M < \infty$ such that $|f(x)| \leq M$ for $\|x\| < 1$, then $|f^{(n)}(x)\xi_1 \ldots \xi_n| \leq \frac{Mn^n}{(1-\|x\|)^n}\|\xi_1\| \ldots \|\xi_n\|$ for $x \in B_X$, $\xi_1, \ldots, \xi_n \in X$, and $n \geq 0$.

(b) Let $X$ be a complex Banach space, $\Omega \subset X$ open, $f \in \mathcal{O}(\Omega)$, and $\xi^{(n)}_j \in B_X$ for $1 \leq j \leq n$, $n \geq 0$. Then the sequence of functions $f^{(n)}(x)\xi^{(n)}_1 \ldots \xi^{(n)}_n$ for $x \in \Omega$, $n \geq 0$, has a uniform local rate of growth $n^2n$.

Proof. Part (a) follows from the usual Cauchy estimate for a polydisc. Part (b) follows on applying (a) locally on balls in $\Omega$ on which $f$ is bounded.

Proposition 4.3. If $X$ is a Banach space, $\Omega \subset X$ open, and $f \in \mathcal{O}(\Omega)$, then the following hold.

(a) If $v_n \in \mathcal{O}(\Omega, X)$, $n \geq 1$, is a sequence of holomorphic vector fields that has a uniform local rate of growth of 1, then the sequence of Lie derivatives $f_n = \mathcal{L}_{v_n}\mathcal{L}_{v_{n-1}} \ldots \mathcal{L}_{v_1}f \in \mathcal{O}(\Omega)$ has a uniform local rate of growth of $n^5n^2$.

(b) Let $P \in \mathcal{O}(\Omega, \text{End}(X))$ be a holomorphic function with operator values, $\xi_n \in B_X$, $n \geq 1$, and $N$ the set of all finite sequences $n = (n_1, \ldots, n_s)$ of natural numbers $n_j \geq 1$ for $s \geq 1$. Define $f_n \in \mathcal{O}(\Omega)$, $n \in N$, by $f_{n_1 \ldots n_s} = \mathcal{L}_{P\xi_{n_1}} \ldots \mathcal{L}_{P\xi_{n_s}}f$. Then $(f_n)$ has a uniform local rate of growth of $s^5n^2$, where $s$ is the length of $n \in N$, i.e., there are an open covering $\mathcal{U}$ of $\Omega$, and a function $C: \mathcal{U} \to (0, \infty)$ such that if $U \in \mathcal{U}$, $x \in U$, and $n \in N$, then $|f_{n_1 \ldots n_s}(x)| \leq C(U)s^5n^2$.

Proof. (a) The function $f_n$ is the sum of $n!$ products, whose $n+1$ factors of each are at most $n$th derivatives of $f$ and at most $(n-1)st$ derivatives of the $v_j$. Each factor has a uniform local rate of growth of $n^{2n}$ by Proposition 4.2(b), each product of $n+1$ of them $(n^{2n})^{n+1}$, and the sum $f_n$ of the $n!$ such products $n!(n^{2n})^{n+1} \leq n^{5n^2}$. (The accumulation of the constants is much less severe than the growth of $n^{5n^2}$, hence it can be incorporated in the said growth rate.)
As (b) follows just in the same way as (a) does, we omit the rest of the proof of Proposition 4.3.

**Theorem 4.4.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $M \subset \Omega$ a closed split complex Banach submanifold of $\Omega$, $N$ a countable set, and $f_n \in \mathcal{O}(M)$, $n \in N$. Suppose that plurisubharmonic domination is possible in every pseudoconvex open subset of $\Omega$. If the functions $f_n$, $n \in N$, have no common zeros, and have a uniform local rate of growth, then there are holomorphic functions $g_n \in \mathcal{O}(M)$, $n \in N$, such that $\sum_{n \in N} f_n(x)g_n(x) = 1$ for all $x \in M$, where the series converges absolutely and uniformly on every compact subset of $M$.

**Proof.** There are an open covering $\mathcal{U}$ of $M$, a function $C: \mathcal{U} \to (0, \infty)$, and constants $L_n > 1$, $n \in N$, such that if $U \in \mathcal{U}$, $x \in U$, and $n \in N$, then $|f_n(x)| \leq C(U)L_n$. Let $i: N \to \{1, 2, 3, \ldots \}$ be an injection, $L'_n = 2^{i(n)}L_n$, $n \in N$, $H = \{z = (z_n)_{n \in N}: z_n \in \mathbb{C}, \|z\| = (\sum_{n \in N} |z_n|^2)^{1/2} < \infty\}$ our Hilbert space, and $F \in \mathcal{O}(M, H)$ defined by $F(x) = (F_n(x))$, where $F_n(x) = f_n(x)/L'_n$, $n \in N$. Then $F$ is indeed a holomorphic function $F: M \to H$, and $F(x) \neq 0$ for every $x \in M$. Theorem 2.2(d) applies and gives a holomorphic $G = (G_n) \in \mathcal{O}(M, H)$ with $1 = F(x) \cdot G(x) = \sum_{n \in N} F_n(x)G_n(x)$ for $x \in M$, where the convergence is absolute and uniform on every compact subset of $M$. Letting $g_n = G_n/L'_n$, $n \in N$, completes the proof of Theorem 4.4.

In the setting of Theorem 4.4 suppose that $I \subset \mathcal{O}(M)$ is an ideal of $\mathcal{O}(M)$, and $I$ is sequentially closed in the sense that if $f(x) = \sum_{n=1}^{\infty} f_n(x)g_n(x)$ for $x \in M$, where $f_n \in I$, $g_n \in \mathcal{O}(M)$, $n \geq 1$, and the series converges absolutely and uniformly on every compact subset of $M$, then $f \in \mathcal{O}(M)$ also lies in the ideal $I$. Then $I = (1)$ is the unit ideal if and only if $I$ admits a sequence $f_n \in I$, $n \geq 1$, without common zeros and with a uniform local rate of growth.

**Theorem 4.5.** (a) Let $X$ be a Banach space with a Schauder basis, $M \subset X$ a closed split complex Banach submanifold of $X$, and suppose that plurisubharmonic domination is possible in every pseudoconvex open subset of $X$. If $\chi: \mathcal{O}(M) \to \mathbb{C}$ is a continuous character of the algebra $\mathcal{O}(M)$ (i.e., $\chi$ is a multiplicative linear functional, $\chi(1) = 1$, and there is a compact set $K \subset X$ with $|\chi(f)| \leq \sup_{x \in K} |f(x)|$ for all $f \in \mathcal{O}(M)$), then there is a point $x_0 \in M$ with $\chi(f) = f(x_0)$ for all $f \in \mathcal{O}(M)$.

(b) Let $X$ be a Banach space with an unconditional basis, $\Omega \subset X$ pseudoconvex open, $M \subset \Omega$ a closed split complex Banach submanifold of $\Omega$, and $\chi: \mathcal{O}(M) \to \mathbb{C}$ a continuous character, then there is a point $x_0 \in M$ with $\chi(f) = f(x_0)$ for all $f \in \mathcal{O}(M)$.

**Proof.** (a) Let $\xi_n \in X^*$, $n \geq 1$, be the coordinate functionals of a bimo-
ton Schauder basis of $X$. Thus there is a bound $0 \leq B < \infty$ with $|\xi_n(x)| \leq B\|x\|$ for $x \in X$ and $n \geq 1$. Look at the functions $f_n \in \mathcal{O}(X)$ defined by $f_n = \xi_n - \chi(\xi_n|\mathcal{M})$ for $n \geq 1$. Note that $|f_n(x)| \leq B\|x\| + B\text{diam}(K \cup \{0\})$ for $x \in X$ and $n \geq 1$. Hence the sequence $f_n, n \geq 1$, has a uniform local rate of growth. Let $I = \text{Ker} \chi$, and note that $f_n|\mathcal{M}, n \geq 1$, have no common zeros in $\mathcal{M}$, then by Theorem 4.4 and the remark following its proof we find that $I = (1)$, i.e., $\chi = 0$, which contradicts that $\chi(1) = 1$.

Hence there is a point $x_0 \in \mathcal{M}$ with $f_n(x_0) = 0$, i.e.,

$$f(x_0) = \chi(f|\mathcal{M})$$

for $f = \xi_n$ for all $n \geq 1$. As (4.1) subsists for any polynomial of finitely many $\xi_n$, and as any $f \in \mathcal{O}(X)$ is the limit of a sequence of such polynomials uniformly on $K \cup \{x_0\}$, we see that (4.1) holds for all $f \in \mathcal{O}(X)$. As any $f \in \mathcal{O}(\mathcal{M})$ can be extended to a holomorphic $\tilde{f} \in \mathcal{O}(X)$ with $f = \tilde{f}|\mathcal{M}$ by Theorem 2.2(b), the proof of (a) is complete. Since (b) follows from (a) and from Zerhusen’s embedding theorem [Z] by embedding $\mathcal{M}$ as a closed split complex Banach submanifold $\mathcal{M}'$ of a Banach space $X'$ with an unconditional basis, the proof of Theorem 4.5 is complete.

It was shown much earlier by Schottenloher [S], see also [M], that a continuous character of $\mathcal{O}(\mathcal{M})$ is a point evaluation of $\mathcal{M}$ if $\mathcal{M}$ is a Riemann domain spread over a pseudoconvex open subset of a Banach space with a Schauder basis. It is unclear whether the statement of Theorem 4.5 follows from the above mentioned result of Schottenloher. It is, however, possible to replace a part of the proof of Theorem 4.5(a) by an application of his result. Indeed, one can consider the character $\chi'$ of $\mathcal{O}(X)$ defined by $\chi'(f) = \chi(f|\mathcal{M})$, apply his result to find a point $x_0 \in X$ with (4.1) for $f \in \mathcal{O}(X)$, and conclude as above by invoking the extension Theorem 2.2(b) from $\mathcal{M}$ to $X$ and from $\mathcal{M} \cup \{x_0\}$ to $X$, should $x_0$ lie outside $\mathcal{M}$.

Note that if $\mathcal{M}'$ is a holomorphic covering Banach manifold with countably many leaves of an $\mathcal{M}$ as in Theorem 4.5(b), then $\mathcal{M}'$ is biholomorphic to a closed split complex Banach submanifold $\mathcal{M}''$ of a Banach space with an unconditional basis, hence any continuous character of $\mathcal{O}(\mathcal{M}')$ is gotten by evaluation at a point of $\mathcal{M}'$.

Theorem 4.6 below follows from standard linear algebra and Theorem 4.5.

**Proposition 4.6.** Let $\mathcal{M}$ be a complex Banach manifold as in Theorem 4.5, $E = \text{End}(\mathbb{C}^n)$ the algebra of complex $n$ by $n$ matrices, $n \geq 1$, $A: \mathcal{O}(\mathcal{M}) \rightarrow E$ a complex algebra homomorphism with $A(1) = 1$, and $I = \text{Ker} A$. Choose a basis of $\mathbb{C}^n$ so that the commuting matrices $A(f), f \in \mathcal{O}(\mathcal{M})$, are simultaneously upper triangular with respect to the chosen basis. So $A(f) =$
\[ [a_{ij}(f)]_{ij=1}^n, \text{ and } a_{ij}(f) = 0 \text{ for } i > j. \] If at least one of the characters \( a_{ii}: \mathcal{O}(M) \to \mathbb{C}, i = 1, \ldots, n, \) is continuous, then the ideal \( I \) has a common zero.

5. LIE DERIVATIVES AND IDEALS.

In this section we look at ideals of holomorphic functions that relate to Lie derivatives.

**Proposition 5.1.** Let \( X \) be a Banach space, \( \Omega \subset X \) open, \( f \in \mathcal{O}(\Omega), v_1, \ldots, v_n \in \mathcal{O}(\Omega, X) \) holomorphic vector fields, \( n \geq 1 \), and \( x_0 \in \Omega \). If \( f \) has a zero at least of order \( n \) at the point \( x_0 \) (i.e., \( f^{(i)}(x_0) = 0 \) for \( i = 0, \ldots, n-1 \)), then \( (\mathcal{L}_{v_1} \ldots \mathcal{L}_{v_n} f)(x_0) = f^{(n)}(x_0)v_1(x_0)\ldots v_n(x_0) \) holds for the iterated Lie derivative.

**Proof.** This follows from the rules of differentiation such as the product rule.

**Proposition 5.2.** Let \( N \) be as in Proposition 4.3(b), \( M \) a connected complex Banach manifold, \( f \in \mathcal{O}(M), v_n \in \mathcal{O}(M, T^{1,0}M) \) holomorphic vector fields for \( n \geq 1 \). If the set of values \( v_n(x_0) \in T^{1,0}M, n \geq 1 \), is dense in a neighborhood of \( 0 \) in the Banach space \( T^{1,0}M \) at a point \( x_0 \in M \), and \( (\mathcal{L}_{v_n} \ldots \mathcal{L}_{v_n} f)(x_0) = 0 \) for all \( n \in N \), then \( f \) is a constant \( f(x_0) \) on \( M \).

**Proof.** Let \( s \) be the vanishing order of \( f - f(x_0) \) at \( x_0 \). If \( s = \infty \), then we are done. If \( 1 \leq s < \infty \), then we conclude the proof of Proposition 5.2 by an application of Proposition 5.1 to the restrictions to an open neighborhood of \( x_0 \) biholomorphic to an open set \( \Omega \) in a Banach space \( X \).

**Proposition 5.3.** (a) Let \( X \) be a Banach space, and \( P \in \text{End}(X) \) a projection, i.e., \( P^2 = P \). If \( x_n \in X, n \geq 1 \), is dense in the unit ball \( B_X \), then \( \{P x_n : n \geq 1\} \) has a subset contained and dense in the ball \( \frac{1}{\|P\|}B_{PX} \) of the image Banach space \( PX \).

(b) Let \( N \) be as in Proposition 4.3(b), \( X \) a separable Banach space, \( \xi_n \in X, n \geq 1 \), dense in \( B_X, \Omega \subset X \) open, \( M \subset \Omega \) a connected closed split complex Banach submanifold of \( \Omega, P \in \mathcal{O}(M, \text{End}(X)) \) a holomorphic operator function with \( P(x)P(x) = P(x) \) and \( \text{Im} P(x) = P(x)X = T_xM \) for all \( x \in M, f \in \mathcal{O}(M), \) and \( f_n = \mathcal{L}_{P_1}\ldots \mathcal{L}_{P_{\xi_n}}f \in \mathcal{O}(M) \) for \( n \in N \). If the functions \( f_n, n \in N, \) have a common zero \( x_0 \in M \), then \( f \) is a constant \( f(x_0) \).

**Proof.** (a) Fix \( \varepsilon > 0 \), \( x_0 \in PX \) with \( \|x_0\| < \frac{1}{\|P\|} \), and choose an \( \eta > 0 \) so small that \( \|x_0\| + \eta\|P\| < \frac{1}{\|P\|} \) and \( \|P\|\eta < \varepsilon \). There is an \( n \geq 1 \) with \( \|x_n - x_0\| < \eta \). Then \( Px_0 = x_0, \|Px_n - Px_0\| < \|P\|\eta < \varepsilon \), and \( \|Px_n\| \leq \|x_0\| + \|P\|\eta < \frac{1}{\|P\|} \).

(b) As the vector fields \( v_n(x) = P(x)\xi_n, n \geq 1 \), have values dense near zero
in $T_xM$ by (a) for each $x \in M$, an application of Proposition 5.2 completes the proof of Proposition 5.3.

**Theorem 5.4.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $M \subset \Omega$ a connected closed split complex Banach submanifold of $\Omega$, and $f_0 \in \mathcal{O}(M)$. Suppose that plurisubharmonic domination is possible in every pseudoconvex open subset of $\Omega$. If $f_0$ is not constant zero on $M$, then there are iterated Lie derivatives $f_n \in \mathcal{O}(M)$, $n \geq 1$, of $f_0$, and holomorphic functions $g_n \in \mathcal{O}(M)$, $n \geq 0$, with $\sum_{n=0}^{\infty} f_n(x)g_n(x) = 1$ for all $x \in M$, where the series converges absolutely and uniformly on every compact subset of $M$. Further, if $I \subset \mathcal{O}(M)$ is a nonzero ideal that is closed under Lie derivation (i.e., $vf \in I$ for $f \in I$ and $v \in \mathcal{O}(M; T^{1,0}M)$), then $I$ is sequentially dense in $\mathcal{O}(M)$.

**Proof.** If $f_0$ is constant on $M$, then letting $g_0 = 1/f_0$ we are done. Suppose now that $f_0$ is not constant on $M$, and extend $f_0$ from $M$ to $f_0 \in \mathcal{O}(\Omega)$ by Theorem 2.2(b). As $M$ is a split Banach submanifold of $\Omega$, locally the trivial bundle $M \times X$ splits as $TM \oplus E$. By Theorem 2.2(c) there is a global splitting, i.e., we can write $M \times X = TM \oplus E$, where $E \to M$ is a holomorphic Banach vector subbundle of $M \times X$. Define $P' \in \mathcal{O}(M, \text{End}(X))$ by projecting $(x, \xi) \in M \times X$ to $P'(x)\xi \in T_xM$ in the above global direct decomposition of $M \times X$. Theorem 2.2(b) gives us a holomorphic extension $P \in \mathcal{O}(\Omega, \text{End}(X))$ with $P|_M = P'$. Choose a sequence $\xi_n \in X$, $n \geq 1$, dense in $B_X$, and define $f_n \in \mathcal{O}(\Omega)$ for $n \in N$, where $N$ is as in Proposition 4.3(b), by $f_n = L_{P\xi_n} \ldots L_{P\xi_1} f_0$. Let $N' = N \cup \{0\}$, and note that the functions $f_n \in \mathcal{O}(\Omega)$, $n \in N'$, have a uniform local rate of growth by Proposition 4.3(b). Hence $f_n|_M \in \mathcal{O}(M)$, $n \in N'$, also has a uniform local rate of growth on $M$, and no common zeros in $M$ by Proposition 5.3(b). Theorem 4.4 applies and completes the proof of Theorem 5.4.

An example of a proper dense ideal $I \subset \mathcal{O}(\mathbb{C})$ that is closed under (Lie) derivation is $I = \{f \in \mathcal{O}(\mathbb{C}) : \text{ord}_n(f) \to \infty \text{ as } n \to \infty \text{ in } \mathbb{N}\}$, where $\text{ord}_{z_0}(f)$ is the vanishing order of $f$ at the point $z_0 \in \mathbb{C}$.

**6. THE ANNIHILATOR OF A DOLBEAULT GROUP.**

In this section we generalize to certain infinite dimensional complex Banach manifolds the following Theorem 6.1 of Laufer.

**Theorem 6.1.** (Laufer, [L]) Let $M$ be a Stein manifold, $D \subset M$ open, and $H = H^{p,q}(D)$, $p \geq 0$, $q \geq 1$, a Dolbeault cohomology group of $D$. Then either $H = 0$ or $\dim_\mathbb{C} H = \infty$.

In the remainder of this section we adopt the following. Let $M$ a complex Banach manifold, $D \subset M$ open, and $H = H^{p,q}(D)$, $p \geq 0$, $q \geq 1$, a Dolbeault cohomology group, or $H = H^q(D, \mathcal{O}^\Lambda_n)$ a sheaf cohomology group with val-
ues in the sheaf $O^A_p$ of germs of holomorphic sections of the Banach vector bundle $\Lambda_p \to M$ of $(p,0)$-forms. (In finite dimensions the above-mentioned Dolbeault and sheaf cohomology groups are naturally isomorphic by the Dolbeault isomorphism theorem. In infinite dimensions the analog of the Dolbeault isomorphism is not yet proved except in very special cases, and sometimes may in fact fail.) If $f \in O(D)$, $v \in O(D, T^{1,0}D)$, then $f$ and $v$ both act naturally on $H$ as linear operators. We can set up these actions of multiplication and Lie derivation as follows. We take the case of the Dolbeault group; the argument for the sheaf cohomology group is similar, only simpler. If $\alpha$ is a smooth $(p,q)$-form on $D$, then the product $f\alpha$ and the Lie derivative $L_v\alpha$ are smooth $(p,q)$-forms on $D$, and the commutation relations $\partial(f\alpha) = f\partial\alpha$ for the product and $\partial L_v\alpha = L_v\partial\alpha$ for the Lie derivative show that the actions $M_f$ of multiplication by $f$ and $L_v$ of Lie derivation by $v$ descend to complex linear operators $[M_f], [L_v] \colon H \to H$. As $L_v(f\alpha) - fL_v\alpha = (vf)\alpha$, the commutation relation $[L_v,M_f] = L_v M_f - M_f L_v = M_vf$ shows that $[[L_v],[M_f]] = \text{ad}_{[L_v]} [M_f] = [M_vf]$. Put $[f] = [M_f], [v] = [L_v]$ for short.

Let $I \subset O(M)$ be the kernel of the representation $O(M) \to E = \text{End}(H)$ given by $f \mapsto [f|D]$; hence $f \in I$ if and only if $[f|D] = 0$. Henceforth we drop the restrictions from $M$ to $D$ from the notation. Note that if $v \in O(M, T^{1,0}M)$ and $f \in I$, then $[vf] = \text{ad}_{[v]} [f] = \text{ad}_{[v]} 0 = 0$, i.e., $vf \in I$, so $I$ is closed under Lie derivation. We call $I$ the annihilator (ideal) of the group $H$ with respect to $M$.

**Theorem 6.2.** Let $X$ be a Banach space, $\Omega \subset X$ pseudoconvex open, $M \subset \Omega$ a closed split complex Banach submanifold of $\Omega$, $D \subset M$ open, and $H$ a cohomology group and $I$ its annihilator ideal as above. If (a), (b), or (c) below holds, then either $H = 0$ or $\dim_{\mathbb{C}} H = \infty$.

(a) $D$ (or $M$) admits a reciprocal pair $f, v$.

(b) $X$ is infinite dimensional and has a Schauder basis, plurisubharmonic domination is possible in every pseudoconvex open subset of $\Omega$, and $M \subset \Omega$ is of a finite codimension $k \geq 1$.

(c) $X$ has an unconditional basis, $M$ is connected, and $I$ has a common zero or $I$ is sequentially closed.

**Proof.** Suppose for a contradiction that $1 \leq n = \dim_{\mathbb{C}} H < \infty$. If $I = 0$, then the infinite dimensional vector space $O(M)$ (or $O(D)$) is embedded in the finite dimensional vector space $E = \text{End}(H)$ by the injective representation $O(M) \to E$ induced by multiplication $f \mapsto [f|D]$. Hence $I \neq 0$. Below in each case (a), (b), (c) we find a contradiction or we show that $I = (1)$. Then $H = 1H = 0$ is a contradiction that proves Theorem 6.2.

As mentioned in Proposition 4.6 we may and do choose a basis of $H$ so that each matrix $[f]$ is upper triangular, and denote its diagonal characters
by $\chi_i$: $O(M) \to \mathbb{C}$ for $i = 1, \ldots, n$. Let $K$ be the joint kernel of these characters, i.e., $K = \{f \in O(M): \chi_i(f) = 0, i = 1, \ldots, n\}$.

The Cayley–Hamilton theorem of linear algebra tells us that for every matrix $[f] \in E$, $f \in O(M)$, there is a one-variable polynomial $p \in \mathbb{C}[z]$ monic of degree $n$ such that $0 = p([f]) = |p(f)|$, i.e., $p(f) \in I$.

Similarly, for every linear operator $ad_{[v]} \in \text{End}(E)$ there is a one-variable polynomial $p \in \mathbb{C}[z]$ monic of degree $m = n^2$ such that $0 = p(ad_{[v]}), i.e., if $f \in O(M)$, then $0 = p(ad_{[v]})(f) = |p(v)f|$. So $p(v)f = p(L_v)f$ belongs to the annihilator $I$ for all $f \in O(M)$, where $p(L_v)f$ is the resulting function obtained by applying the differential operator $p(L_v)$ to the function $f$.

(a) Somewhat more generally, (a) is valid if $D$ is any complex Banach manifold with a reciprocal pair $f, v$. Indeed, there is a one-variable polynomial $p \in \mathbb{C}[z]$ monic of degree $n$ such that $p(f) \in I$. As $I$ is closed under Lie derivation, $1 = \frac{1}{m!}v^np(f) \in I$ as well.

(b) Consider the intersection $K \cap X^*$, where $X^*$ is the dual space of $X$. The vector space $K \cap X^* = \{\xi \in X^*: \chi_i(\xi|M) = 0, i = 1, \ldots, n\}$ is of codimension at most $n$ in the infinite dimensional vector space $X^*$, hence $K \cap X^*$ itself is infinite dimensional. There are $k + 1$ linearly independent $\xi_\kappa \in K \cap X^*$ for $\kappa = 0, \ldots, k$. Letting $f_\kappa = \xi_\kappa|M$ we find by Proposition 3.8 that $f_0, \ldots, f_k$ have no common critical points in $M$. Moreover, $f_k^m$ belongs to $I$ for $\kappa = 0, \ldots, k$. Theorem 2.2 gives us vector fields $v_\kappa \in O(M, TM)$ for $\kappa = 0, \ldots, k$ such that $\sum_{\kappa=0}^k(v_\kappa f_\kappa)(x) = 1$ for all $x \in M$. Let $p_\kappa \in \mathbb{C}[z]$ be a one-variable polynomial monic of degree $m = n^2$ such that $p(ad_{[v_\kappa]}) = 0$, and define $g_\kappa \in O(M)$ by $g_\kappa = p_\kappa(v_\kappa)f_\kappa^m = v_\kappa^m f_\kappa^m + \ldots$ for $\kappa = 0, \ldots, k$. Note that $g_\kappa$ belongs to $I$ for all $\kappa$, and if $x_0 \in M$ and $f_\kappa(x_0) = 0$, then $g_\kappa(x_0) = m!(v_\kappa f_\kappa)(x_0)^m$.

Consider the functions $f_\kappa^n, g_\kappa$ for $\kappa = 0, \ldots, k$. They all belong to $I$, and we claim that they have no common zeros in $M$. Indeed, let $x_0 \in M$ be any point and suppose that $f_\kappa(x_0)^n = g_\kappa(x_0) = 0$ for all $\kappa = 0, \ldots, k$. Then $f_\kappa(x_0) = 0$, and so the equality $g_\kappa(x_0) = m!(v_\kappa f_\kappa)(x_0)^m = 0$ implies that $(v_\kappa f_\kappa)(x_0) = 0$. Now $\sum_{\kappa=0}^k(v_\kappa f_\kappa)(x_0)$ equals both 0 and 1; thus our members $f_\kappa^n, g_\kappa$, $\kappa = 0, \ldots, k$, of the ideal $I$ have no common zeros in $M$. Theorem 2.2(d) gives us $a_\kappa, b_\kappa \in O(M)$ for $\kappa = 0, \ldots, k$ with $1 = \sum_{\kappa=0}^k(a_\kappa f_\kappa^n + b_\kappa g_\kappa) \in I$ on $M$.

(c) If $\chi_i: O(M) \to \mathbb{C}$ is a continuous character, then $I$ has a common zero by Proposition 4.6. (Nobody has ever seen a discontinuous character of an $O(M)$; see Michael’s problem in [Mc] or [M]..) Suppose that $x_0 \in M$ is a common zero of $I$. As $I \neq 0$ and $M$ is connected, there is an $f \in I$ with vanishing order $1 \leq s < \infty$ at $x_0$. Nonzero is the Fréchet derivative $f^{(s)}(x_0)$
relative to a biholomorphism of an open neighborhood $U \subset M$ of $x_0$ onto an open subset $V$ of a Banach space. There are vectors $\xi_1, \ldots, \xi_s \in T_{x_0}M$ with $f^{(s)}(x_0)\xi_1 \ldots \xi_s = 1$. There are vector fields $v_i \in \mathcal{O}(M, TM)$ with $v_i(x_0) = \xi_i$ for $i = 1, \ldots, s$, e.g., of the form $v_i(x) = P'(x)\xi_i$, where $P' \in \mathcal{O}(M, \text{End}(X))$ is as in the proof of Theorem 5.4. The function $g = \mathcal{L}_{v_1} \ldots \mathcal{L}_{v_s}f \in \mathcal{O}(M)$ belongs to $I$, hence $g(x_0) = 0$ on the one hand. On the other hand $g(x_0) = f^{(s)}(x_0)\xi_1 \ldots \xi_s = 1$ by Proposition 5.1; a contradiction.

If $I$ is sequentially closed and has no common zeros in $M$, then $1 \in I$ by Theorem 5.4. The proof of Theorem 6.2 is complete.

Theorem 6.2 applies to all finite codimensional closed complex Hilbert submanifolds of $\ell_2$, and also to some infinite dimensional ones such as those in Theorem 3.4(b) (mapping spaces). It seems likely (but currently unknown) that the conclusion of Theorem 6.2 also holds for every closed complex Hilbert submanifold $M$ of $\ell_2$, because $M$ might always admit a reciprocal pair $f, v$. It is already known, see [Pt], that there are a nowhere zero vector field $v \in \mathcal{O}(M, TM)$ and a nowhere zero covector field $\omega \in \mathcal{O}(M, T^*M)$, but it seems unknown whether such an $\omega$ can be chosen to be exact or even closed on $M$.

The proof of Theorem 6.2 has much in common with that of Laufer’s for Theorem 6.1, but unlike in finite dimensions finitely many numerical functions do not separate the points of an infinite dimensional Banach manifold. To overcome this difficulty, we work harder with the vector fields $v$.

### 7. NOWHERE CRITICAL HOLOMORPHIC FUNCTIONS.

In this section we look at a simple mechanism that extends the special case $X = \ell_1$ of Theorem 3.5 to all closed infinite dimensional split complex Banach submanifolds of $\ell_1$.

**Proposition 7.1.** Let $T$ be a separable topological space, $X, Y$ Banach spaces, $Y$ nonseparable, $Z = X \times Y$ with the product norm $\|z\| = \max\{\|x\|, \|y\|\}$ for $z = (x, y) \in Z$, $H = \text{Hom}(X, Y)$ with the operator norm, and $A \in C(T, H)$ a bounded continuous function with operator values. Denote for a map $f : X \to Y$ its graph by $\Gamma(f) = \{(x, y) \in Z : y = f(x)\}$. We then have for the set $E = \bigcup_{t \in T} \Gamma(A(t))$ that

(a) $\overline{E} \cap \{(x_0) \times Y\} = F_{x_0}$ is separable for every $x_0 \in X$, and

(b) the closure $\overline{E}$ is nowhere dense in $Z$.

**Proof.** As it implies (b) let us prove (a). Let $T' \subset T$ be countable and dense in $T$, and define the countable subset $F' = \{(x_0, A(t)x_0) : t \in T'\}$ of $F = F_{x_0}$. As $z_0 = (x_0, y_0) \in \overline{E}$ there is a sequence $z_n = (x_n, A(t_n)x_n) \in \Gamma(A(t_n))$, $t_n \in T$, $n \geq 1$, with $z_n \to z_0$ as $n \to \infty$; in particular, $\lim_{n \to \infty} x_n = x_0$. Number 13.
Our $A$ being continuous at each $t_n$, $n \geq 1$, there is a $t'_n \in T'$ so close to $t_n$ in $T$ that $\|A(t_n) - A(t'_n)\| < 1/n$. The bound $M = \sup_{t \in T} \|A(t)\|$ is finite by assumption. Look at $z'_n = (x_0, A(t'_n)x_0) \in F'$ for $n \geq 1$. As $\|z_n - z_0\| = \max\{\|x_n - x_0\|, \|A(t_n)x_n - A(t'_n)x_0\|\} \leq \|x_n - x_0\| + \|A(t_n)x_n - A(t'_n)x_0\| \leq \|x_n - x_0\| + \|A(t_n)x_n - A(t'_n)x_0\| + \|x_0\|/n \to 0$ as $n \to \infty$, we find that $\lim_{n \to \infty} z'_n = z_0$, i.e., the countable subset $F'$ of $F$ is dense in $F$. The proof of Proposition 7.1 is complete.

**Proposition 7.2.** Let $X, Y$ be Banach spaces, $Z = X \times Y$, $\Omega \subset X$ open, $m: \Omega \to Y$ with continuous Fréchet derivative $m'$ bounded on $\Omega$, and $M = \Gamma(m) \subset \Omega \times Y$ the graph of $m$. If $X$ is separable but its dual space $X^*$ is nonseparable, then there is a nowhere dense closed subset $B \subset Z^*$ such that for $f \in Z^* \setminus B$ the restriction $f|_{M}$ is nowhere critical on $M$.

**Proof.** As $Z^* = X^* \times Y^*$, we may write any $f \in Z^*$ uniquely as $f(x, y) = \xi(x) + \eta(y)$ with $\xi \in X^*$ and $\eta \in Y^*$. A point $z_0 = (x_0, y_0)$ with $y_0 = m(x_0)$ is a critical point of $f|_{M}$ if and only if $x = x_0$ is a critical point of $f(x, m(x))$, i.e., $\xi + \eta m'(x_0) = 0$ in $Z^*$. In other words, $f|_{M}$ is critical at $z_0$ if and only if $\xi = A(x_0)\eta$, where $A(x) \in \text{Hom}(Y^*, X^*)$ is given for $x \in \Omega$ by the transpose of $-m'(x) \in \text{Hom}(X, Y)$. The set of ‘exceptional’ functions $f$ with $f|_{M}$ critical at some point of $M$ is thus of the form $E = \bigcup_{x \in \Omega} \Gamma(A(x))$ in $Z^* = X^* \times Y^*$. Proposition 7.1 applies and shows that $B = E$ is nowhere dense and does the job.

**Proposition 7.3.** Let $Z$ be a separable Banach space, $\Omega \subset Z$ open, and $M$ a closed $C^1$-smooth split Banach submanifold of $\Omega$. If the cotangent space $T^*_x M$ is nonseparable for all $x \in M$, then there is a dense $G_δ$ subset $H$ of the dual space $Z^*$ of $Z$ such that of each $f \in H$ the restriction $f|_{M}$ is nowhere critical on $M$.

**Proof.** As locally the Lindelöf space $M$ is given by a graph $y = m(x)$ of a function $m$ bounded in the $C^1$-norm, our claim follows from Proposition 7.2 and the Baire category theorem applied to the Banach space $Z^*$.

**Theorem 7.4.** Let $\Omega \subset \ell_1$ be open and $M$ a closed split complex Banach submanifold of $\Omega$. If $M$ is infinite dimensional at each of its points, then $M$ admits a nowhere critical holomorphic function $f \in \mathcal{O}(M)$ that can be extended to a nowhere critical holomorphic function $\tilde{f} \in \mathcal{O}(\ell_1)$ on $\ell_1$. In fact, $\tilde{f}$ can be taken linear and to be an arbitrary member of a dense $G_δ$ subset of the dual space $\ell_1^* = \ell_\infty$. Further, if $\Omega$ is pseudoconvex, then there is a $v \in \mathcal{O}(M, T^{1,0}M)$ with $vf = 1$ on $M$.

**Proof.** It is a famous theorem of Pełczyński’s, see [P] or [LT, Thm. 2.a.3], that any closed complemented infinite dimensional linear subspace (such as the tangent space $T_x M$ for $x \in M$) of $\ell_1$ is isomorphic to $\ell_1$, hence it is
separable but its dual (such as the cotangent space $T^*_x M$ for $x \in M$) is not, being isomorphic to $\ell_\infty$. Proposition 7.3 yields the $f$, and Theorem 3.2 a $v$ for $f$, completing the proof of Theorem 7.4.

It would be interesting to know whether one could use in a similar manner Banach spaces of polynomials of higher degree. Note that many of the standard separable Banach spaces $Z$ (e.g., $Z = \ell_2$) admit nonseparable Banach spaces of polynomials over them (e.g., quadratic forms over $\ell_2$) even if their duals $Z^*$ themselves are still separable. Another question is whether a complex algebraic submanifold $M$ of $\mathbb{C}^n$ admits a holomorphic polynomial $f$ nowhere critical on $\mathbb{C}^n$ whose restriction $f|M$ is also nowhere critical on $M$.

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IMRE PATYI, DEPARTMENT OF MATHEMATICS AND STATISTICS, GEORGIA STATE UNIVERSITY, ATLANTA, GA 30303-3083, USA, matixp@langate.gsu.edu