Gravitation and Electromagnetism

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Abstract
The realms of gravitation, belonging to Classical Physics, and Electromagnetism, belonging to the Theory of the Electron and Quantum Mechanics have remained apart as two separate pillars, inspite of a century of effort by Physicists to reconcile them. In this paper it is argued that if we extend ideas of Classical spacetime to include in addition to non integrability non commutativity also, then such a reconciliation is possible.

1 Introduction
A problem that has troubled Physicists throughout the twentieth century is that of a reconciliation of electromagnetism and gravitation into a single theory. As Einstein noted in the Stafford Little Lectures in 1921[1], ”...if we introduce the energy tensor of the electromagnetic field into the right hand side (of the gravitational field equation) we obtain (the first of Maxwell’s systems of equations in tensor density form), for the special case \( \sqrt{-g} \frac{\partial \varphi}{\partial x^\nu}, \equiv \tau^\mu = 0, \cdots \) This inclusion of the theory of electricity in the scheme of General Relativity has been considered arbitrary and unsatisfactory... a theory in which the gravitational field and the electromagnetic field do not enter as logically distinct structures would be much preferable...”

As we are beginning to realise now, and as we will see in the sequel, the problem is, more generally, that of reconciling the spacetime used in Classical Physics including General Relativity with the spacetime of Quantum Theory, that is the difference between what Witten calls Bosonic spacetime and Fermionic spacetime. In the words of J.A. Wheeler[2], "the most evident shortcoming of the geometrodynamic model as it stands is this, that it
fails to supply any completely natural place for spin 1/2 in general and for the neutrino in particular”, while ”it is impossible to accept any description of elementary particles that does not have a place for spin half.”

It should also be borne in mind that Classical spacetime which is also the spacetime of General Relativity is not only deterministic, but it is also meaningful to speak in terms of definite spacetime points. The Uncertainty Principle on the other hand forbids such descriptions in a Quantum Mechanical context. Indeed as Wheeler has noted[2] four dimensional spacetime exists only as a classical approximation.

We will now argue that if in addition to non integrability non commutativity is also introduced into Classical spacetime, then it is possible to recover the Quantum Mechanical description: this is the divide which keeps the two apart.

2 Non Integrability and Non Commutativity

We start with the effect of an infinitessimal parallel displacement of a vector[3].

\[
\delta a^\sigma = -\Gamma^\sigma_{\mu\nu} a^\mu dx^\nu
\] (1)

As is well known, (1) represents the extra effect in displacements, due to the curvature of space - in a flat space, the right side would vanish. Considering partial derivatives with respect to the \(\mu^{th}\) coordinate, this would mean that, due to (1)

\[
\frac{\partial a^\sigma}{\partial x^\mu} \rightarrow \frac{\partial a^\sigma}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} a^\nu
\] (2)

where the \(\Gamma\)s are the Christoffel symbols. The second term on the right side of (2) can be written as:

\[-\Gamma^\lambda_{\mu\nu} g^\lambda a^\sigma = -\Gamma^\nu_{\mu\nu} a^\sigma\]

where we have utilized the linearity property that in the above formulation

\[g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\]

\(\eta_{\mu\nu}\) being the Minkowski metric and \(h_{\mu\nu}\) a small correction whose square is neglected.

That is, (2) becomes,

\[
\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu\nu}
\] (3)
From (3) we get
\[ \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\lambda} \rightarrow \frac{\partial}{\partial x^\lambda} \Gamma^{v}_{\mu v} - \frac{\partial}{\partial x^\mu} \Gamma^{v}_{\lambda v} \] (4)

If we now impose the condition that the right hand side in (4) does not vanish, then we have a non commutativity of the momentum components in Quantum Theory. Indeed the left side of (4) can be written as
\[ \frac{1}{\hbar^2} [p_\lambda, p_\mu] \approx 0(1) \]
where \( l \) is the Compton wavelength and \( \hbar \) the reduced planck length wherein we have utilised the fact that at the extreme scale of the Compton wavelength, the Planck scale being a special case, the momentum is \( mc \).

If we write the right side of (4) as
\[ \frac{e}{c \hbar} F^{\mu \nu} \]
then (4) can be written as
\[ Bl^2 \sim \frac{\hbar c}{e} \] (5)
where \( B \) is the magnetic field, if we identify \( F^{\mu \nu} \) with the electromagnetic field tensor.

Equation (5) is the well known equation for the magnetic monopole. Indeed it has been shown by Saito and the author[4, 5] that a non commutative spacetime at the extreme scale throws up the monopole.

What we have shown here is that once we consider the non commutativity of the right or left side of (4), then it is meaningful to identify
\[ A^\mu = \hbar \Gamma^{\mu \nu}_v \] (6)
with the electromagnetic four potential, thus leading to a unification of electromagnetism with gravitation theory. This unification is not possible in the usual commutative spacetime, that is when the right or left side of (4) vanish.

We now make a number of remarks which corroborate the above deductions. The identification of (6) with the electromagnetic vector potential was deduced and discussed at length though from a completely different and infact Quantum Mechanical point of view[6, 7, 8]. There the spinorial or pseudo
vector property of the Dirac four spinor was used, in a purely Quantum Mechanical derivation. This has been discussed at length in the references cited. But briefly, if the Dirac bispinor is written as \( \begin{pmatrix} \Theta \\ \phi \end{pmatrix} \), then at the Compton scale, it is the spinor \( \phi \) which predominates and moreover, under reflection,

\[ \phi \rightarrow -\phi \]

This was shown to immediately lead to (3) or (6). It was pointed out there that interestingly (6) was mathematically identified to Weyl’s original formulation except that Weyl had introduced it ad hoc, in fact as an external element, without any internal derivation. This was why Weyl’s formulation was rejected\[1, 3\]. The above considerations were shown to be a manifestation of the non point like structure of spacetime, in fact a non commutative spacetime\[9\]. We have in fact the non commutative geometry

\[ [x, y] = 0(l^2) \]
\[ [x, p_x] = i\hbar [1 + (a/\hbar)^2 p_x^2]; \]
\[ [t, p_t] = i\hbar [1 - (a/\hbar c)^2 p_t^2]; \]
\[ [x, p_y] = [y, p_x] = i\hbar (a/\hbar)^2 p_x p_y; \]
\[ [x, p_t] = c^2 [p_x, t] = i\hbar (a/\hbar)^2 p_x p_t; \text{ etc.} \]

whence, not only can the Dirac equation be deduced therefrom, but as already pointed out, also the existence of the magnetic monopole at extreme Compton scales can be deduced\[5\]. It must be mentioned that such strong magnetic fields can be shown to be associated with a non commutative geometry from an alternative point of view\[4\]. Interestingly the relations (7) reappear in Quantum SuperString Theory, and as pointed out earlier are a manifestation of what Witten has called Fermionic spacetime, as against the usual commutative geometry of Classical spacetime\[10, 11, 12\]. This non commutativity is at the root of the emergence of electromagnetism and spin as can be seen from (5) or (6) and subsequent remarks, and as can be seen from (7), this is an \( 0(l^2) \) effect.
References

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