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ON FULKERSON CONJECTURE

J.L. FOUQUET AND J.M. VANHERPE

Abstract. If $G$ is a bridgeless cubic graph, Fulkerson conjectured that we can find 6 perfect matchings (a Fulkerson covering) with the property that every edge of $G$ is contained in exactly two of them. A consequence of the Fulkerson conjecture would be that every bridgeless cubic graph has 3 perfect matchings with empty intersection (this problem is known as the Fan Raspaud Conjecture). A FR-triple is a set of 3 such perfect matchings. We show here how to derive a Fulkerson covering from two FR-triples.

Moreover, we give a simple proof that the Fulkerson conjecture holds true for some classes of well known snarks.

1. Introduction

The following conjecture is due to Fulkerson, and appears first in [4].

Conjecture 1.1. If $G$ is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of $G$ with the property that every edge of $G$ is contained in exactly two of $M_1, \ldots, M_6$.

We shall say that $\mathcal{F} = \{M_1, \ldots, M_6\}$, in the above conjecture, is a Fulkerson covering. A consequence of the Fulkerson conjecture would be that every bridgeless cubic graph has 3 perfect matchings with empty intersection (take any 3 of the 6 perfect matchings given by the conjecture). The following weakening of this (also suggested by Berge) is still open.

Conjecture 1.2. There exists a fixed integer $k$ such that every bridgeless cubic graph has a list of $k$ perfect matchings with empty intersection.

For $k = 3$ this conjecture is known as the Fan Raspaud Conjecture.

Conjecture 1.3. [2] Every bridgeless cubic graph contains perfect matching $M_1, M_2, M_3$ such that

$$M_1 \cap M_2 \cap M_3 = \emptyset$$

Let $G$ be a cubic graph with 3 perfect matchings $M_1, M_2, M_3$ having an empty intersection. Since $G$ satisfies the Fan Raspaud conjecture, when considering these perfect matchings, we shall say that $\mathcal{T} = (M_1, M_2, M_3)$ is a FR-triple. We define $T_i \in E(G)$ ($i = 0, 2$) as the set of edges of $G$ which are covered $i$ times by $\mathcal{T}$. It will be convenient to use $T'_i(i = 0, 2)$ for the FR-triple $\mathcal{T}'$.

2. FR-triples and Fulkerson covering

In this section, we are concerned with the relationship between FR-triples and Fulkerson coverings.

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2.1. On FR-triples.

**Proposition 2.1.** Let $G$ be a bridgeless cubic graph with $\mathcal{T}$ a FR-triple. Then $T_0$ and $T_2$ are disjoint matchings.

**Proof.** Let $v$ be a vertex incident to an edge of $T_0$. Since $v$ must be incident to each perfect matching of $\mathcal{T}$ and since the three perfect matchings have an empty intersection, one of the remaining edges incident to $v$ must be contained in 2 perfect matchings while the other is contained in exactly one perfect matching. The result follows. \[ \square \]

We introduce now concepts and definitions coming from [10]. Let $ab$ be an edge of bridgeless cubic graph $G$. We shall say that we have split the edge $ab$ when we have applied the operation depicted in Figure 1. The resulting graph is no longer cubic since we get 4 vertices with degree 2 instead of two vertices of degree 3. Let $A_1$ and $A_2$ be two disjoint matchings of $G$ (we insist to say that these matchings are not, necessarily, perfect matchings). For $i = 1, 2$, let $G_{A_i}$ be the graph obtained by splitting the edges of $A_i$ and let $G_{A_i}$ be the graph homeomorphic to $G_{A_i}$ when the degree 2 vertices are deleted. The connected component of $G_{A_i}$ are cubic graphs and vertexless loop graphs (graph with one edge and no vertex). We shall say that $G_{A_i}$ is 3-edge colourable whenever the cubic components are 3-edge colourable (any colour can be given to the vertexless loops).

The following Lemma can be obtained from the work of Hao and al. [10] when considering FR-triples.

**Lemma 2.2.** Let $G$ be a bridgeless cubic graph and let $\mathcal{T}$ be a FR-triple. Then $G_{T_2}$ is 3-edge colourable.

**Proof.** Assume that $T = (M_1, M_2, M_3)$ is a FR-triple. Let $ab$ be an edge of $T_2$ then the two edges of $T_1$ incident with $ab$ must be in the same perfect matching of $\mathcal{T}$. Hence, these two edges are identified in some sens. If we colour the edges of $T_1$ with 1, 2 or 3 when they are in $M_1$, $M_2$ or $M_3$ respectively, we get a natural 3-edge colouring of $G_{T_2}$. \[ \square \]
Lemma 2.3. Let $G$ be a bridgeless cubic graph containing two disjoint matchings $A_1$ and $A_2$ such that $\overline{G_{A_1}}$ is 3-edge colourable and $A_1 \cup A_2$ forms an union of disjoint cycles. Then $G$ has a FR-triple $T$ where $T_2 = A_1$ and $T_0 = A_2$.

Proof. Obviously $A_1 \cup A_2$ forms an union of disjoint even cycles in $G$. Let $C = a_0a_1 \ldots a_{2q-1}$ be an even cycle of $A_1 \cup A_2$ and assume that $a_i a_{i+1} \in A_1$ when $i \equiv 0(2)$.

Let $M_1$, $M_2$ and $M_3$ be the three matchings associated to a 3 edge-colouring of $G_{A_1}$. Thanks to the construction of $G_{A_1}$, for some $i \equiv 0[2]$, the third edge incident to $a_i$, say $e$, and the third one incident to $a_{i+1}$, say $e'$ lead to a unique edge of $G_{A_1}$. Assume that this edge of $G_{A_1}$ is in $M_1$, then $M_1$ can be extended naturally to a matching of $G$ containing $\{e, e'\}$. Moreover we add $a_i a_{i+1}$ to $M_2$ and $a_i a_{i+1}$ to $M_3$. When applying this process to all edges of $A_1$ on all cycles of $A_1 \cup A_2$ we extend the colours of $G_{A_1}$ into perfect matchings of $G$. Since every edge of $G$ belongs to at most 2 matchings in $\{M_1, M_2, M_3\}$ we have a FR-triple with $T = \{M_1, M_2, M_3\}$. By construction, we have $T_2 = A_1$ and $T_0 = A_2$, as claimed.

Proposition 2.4. Let $G$ be a bridgeless cubic graph then $G$ has a FR-triple if and only if $G$ has two disjoint matchings $A_1$ and $A_2$ such that $A_1 \cup A_2$ forms an union of disjoint cycles, moreover $\overline{G_{A_1}}$ or $\overline{G_{A_2}}$ is 3-edge colourable.

Proof. Assume that $G$ has two disjoint matchings $A_1$ and $A_2$ such that, without loss of generality, $\overline{G_{A_1}}$ is 3-edge colourable. From Lemma 2.3, $G$ has a FR-triple $T$ where $T_2 = A_1$ and $T_0 = A_2$.

Conversely, assume that $T$ is a FR-triple. From Lemma 2.2, $\overline{G_{T_2}}$ is 3-edge colourable. Let $A_1 = T_0$ and $A_2 = T_2$. Then $A_1$ and $A_2$ are two disjoint matchings and $\overline{G_{A_1}}$ is 3-edge colourable.

2.2. On compatible FR-triples. As pointed out in the introduction, any three perfect matchings in a Fulkerson covering lead to a FR-triple. Is it possible to get a Fulkerson covering when we know one or more FR-triples? In fact, we can characterize a Fulkerson covering in terms of FR-triples in the following way.

Let $G$ be a bridgeless cubic graph with $T = (M_1, M_2, M_3)$ and $T' = (M'_1, M'_2, M'_3)$ two FR-triples. We shall say that $T$ and $T'$ are compatible whenever $T_0 = T'_2$ and $T_2 = T'_0$ (and hence $T_1 = T'_1$).

Theorem 2.5. Let $G$ be a bridgeless cubic graph then $G$ can be provided with a Fulkerson covering if and only if $G$ has two compatible FR-triples.

Proof. Let $\mathcal{F} = \{M_1 \ldots M_6\}$ be a Fulkerson covering of $G$ and let $T = (M_1, M_2, M_3)$ and $T' = (M_4, M_5, M_6)$, $T$ and $T'$ are two FR-triples and we claim that they are compatible. Since each edge of $G$ is covered exactly twice by $\mathcal{F}$, $T$ the set of edges covered only once by $T$ must be covered also only once by $T'$, $T_0$ the set of edges not covered by $T$ must be covered exactly twice by $T'$ and $T_2$ the set of edges covered exactly twice by $T'$ is not covered by $T'$. Which means that $T_1 = T'_1$, $T_0 = T'_2$ and $T_2 = T'_0$, that is $T$ and $T'$ are compatible.

Conversely, assume that $T$ and $T'$ are two FR-triples compatible. Then it is an easy task to check that each edge of $G$ is contained in exactly 2 perfect matchings.
Proposition 2.6. Let $G$ be a bridgeless cubic graph then $G$ has two compatible FR-triples if and only if $G$ has two disjoint matchings $A_1$ and $A_2$ such that $A_1 \cup A_2$ forms an union of disjoint cycles and $G_{A_1}$ and $G_{A_2}$ are 3–edge colourable.

**Proof.** Let $T$ and $T'$ be 2 compatible FR-triples. From Lemma 2.2, we know that $G_T$ and $G_{T'}$ are 3–edge colourable. Since $T_0 = T'_2$ and $T'_0 = T_2$ by the compatibility of $T$ and $T'$, the result holds when we set $A_1 = T_0$ and $A_2 = T_2$.

Conversely, assume that $G$ has two disjoint matchings $A_1$ and $A_2$ such that $G_{A_1}$ and $G_{A_2}$ are 3–edge colourable. From Lemma 2.3, $G$ has a FR-triple $T$ where $T_2 = A_1$ and $T_0 = A_2$ as well as a FR-triple $T'$ where $T'_2 = A_2$ and $T'_0 = A_1$. These two FR-triples are obviously compatible.

**Proposition 2.7.** Let $G$ be a bridgeless cubic graph then $G$ can be provided with a Fulkerson covering if and only if $G$ has two disjoint matchings $A_1$ and $A_2$ such that $A_1 \cup A_2$ forms an union of disjoint cycles and $G_{A_1}$ and $G_{A_2}$ are 3–edge colourable.

**Proof.** Obvious in view of Theorem 2.5 and Proposition 2.6.

### 3. Fulkerson covering for some classical snarks

A non 3–edge colourable, bridgeless, cyclically 4–edge-connected cubic graph is called a snark.

For an odd $k \geq 3$, let $J_k$ be the cubic graph on $4k$ vertices $x_0, x_1, \ldots, x_{k-1}$, $y_0, y_1, \ldots, y_{k-1}$, $z_0, z_1, \ldots, z_{k-1}$, $t_0, t_1, \ldots, t_{k-1}$ such that $x_0 x_1 \ldots x_{k-1}$ is an induced cycle of length $k$, $y_0 y_1 \ldots y_{k-1}$ $z_0 z_1 \ldots z_{k-1}$ is an induced cycle of length $2k$ and for $i = 0 \ldots k - 1$ the vertex $t_i$ is adjacent to $x_i$, $y_i$, and $z_i$. The set $\{t_i, x_i, y_i, z_i\}$ induces the claw $C_4$. In Figure 2, we have a representation of $J_3$, the half edges (to the left and to the right in the figure) with same labels are identified. For $k \geq 5$ those graphs were introduced by Isaacs in $[6]$ under the name of flower snarks in order to provide an infinite family of snarks.

Proposition 2.7 is essentially used in $[10]$ in order to show that the so-called flower snarks and Goldberg snarks can be provided with a Fulkerson covering. We shall see, in this section, that this result can be directly obtained.

**Theorem 3.1.** For any odd $k \geq 3$, $J_k$ can be provided with a Fulkerson covering.
Proof For $k = 3$ the Fulkerson covering is given in Figure 2. We obtain a Fulkerson covering of $J_k$ by inserting a suitable number of subgraphs isomorphic to the subgraph depicted in Figure 3 when we cut $J_3$ along the dashed line of Figure 2. The labels of the edges of the two sets of three semi-edges (left and right) are identical which insures that the process can be repeated as long as necessary. These labels lead to the perfect matchings of the Fulkerson covering.

□

Let $H$ be the graph depicted in Figure 4.

Let $G_k$ ($k$ odd) be a cubic graph obtained from $k$ copies of $H$ ($H_0 \ldots H_{k-1}$ where the name of vertices are indexed by $i$) by adding edges $a_ia_{i+1}, c_ic_{i+1}, e_ie_{i+1}, f_if_{i+1}$ and $h_ih_{i+1}$ (subscripts are taken modulo $k$).

If $k = 5$, then $G_k$ is known as the Goldberg snark (see [5]). Accordingly, we refer to all graphs $G_k$ as Goldberg graphs. The graph $G_5$ is shown in Figure 5. The half edges (to the left and to the right in the figure) with same labels are identified.

Theorem 3.2. For any odd $k \geq 5$, $G_k$ can be provided with a Fulkerson covering.

Proof We give first a Fulkerson covering of $G_3$ in Figure 6(a). The reader will complete easily the matchings along the $5-$cycles by remarking that these cycles are incident to 5 edges with a common label from 1 to 6 and to exactly one edge of each remaining label. We obtain a Fulkerson covering of $G_k$ with odd $k \geq 5$ by inserting a suitable number of subgraphs isomorphic to the subgraph depicted in Figure 6(b) when we cut $G_3$ along the dashed line. The labels of the edges of the two sets of three semi-edges (left and right) are identical which insures that
the process can be repeated as long as necessary. These labels lead to the perfect matchings of the Fulkerson covering.

□

4. A technical tool

Let \( M \) be a perfect matching, a set \( A \subseteq E(G) \) is an \( M \)-balanced matching when we can find a perfect matching \( M' \) such that \( A = M \cap M' \). Assume that \( \mathcal{M} = \{A, B, C, D\} \) are 4 pairwise disjoint \( M \)-balanced matchings, we shall say that \( \mathcal{M} \) is an \( F \)-family for \( M \) whenever the three following conditions are fulfilled:

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Goldberg snark \( G_5 \)}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Fulkerson covering for the Goldberg Snarks}
\end{figure}
Every odd cycle of $G \setminus M$ has exactly one vertex incident with one edge of each matching in $M$.

Every even cycle of $G \setminus M$ incident with some matching in $M$ contains 4 vertices such that two of them are incident to one matching in $M$ while the other are incident to another matching in $M$ or the 4 vertices are incident to the same matching in $M$.

The subgraph induced by 4 vertices so determined in the previous items has a matching.

It will be convenient to denote the set of edges described in the third item by $N$.

**Theorem 4.1.** Let $G$ be a bridgeless cubic graph together with a perfect matching $M$ and an $F$-family for $M$. Then $G$ can be provided with a Fulkerson covering.

**Proof** Since $A, B, C$ and $D$ are $M$-balanced matchings, we can find 4 perfect matchings $M_A, M_B, M_C$ and $M_D$ such that

\[
\begin{align*}
M \cap M_A &= A & M \cap M_B &= B & M \cap M_C &= C & M \cap M_D &= D
\end{align*}
\]

Let $M' = M \setminus \{A, B, C, D\} \cup N$, we will prove that $F = \{M, M_A, M_B, M_C, M_D, M'\}$ is a Fulkerson covering of $G$.

**Claim 4.1.1.** $M'$ is a perfect matching

**Proof** The vertices of $G$ which are not incident with some edge in $M \setminus \{A, B, C, D\}$ are precisely those which are end vertices of edges in $M_A \cup M_B \cup M_C \cup M_D$. From the definition of an F-family, the 4 vertices defined on each cycle of $\{C_i | i = 1 \ldots k\}$ incident to edges of $M$ form a matching with two edges, which insure that $M'$ is a perfect matching.

Let $C = \{\Gamma_i | i = 1 \ldots k\}$ be the set of cycles of $G \setminus M$ and let $X$ and $Y$ be two distinct members of $M$.

**Claim 4.1.2.** Let $\Gamma \in C$ be an odd cycle. Assume that $X$ and $Y$ have ends $x$ and $y$ on $\Gamma$. Then $xy$ is the only edge of $\Gamma$ not covered by $M_X \cup M_Y$.

**Proof** Since $M_X$ ($M_Y$ respectively) is a perfect matching, the edges of $M_X$ ($M_Y$ respectively) contained in $\Gamma$ saturate every vertex of $\Gamma$ with the exception of $x$ ($y$ respectively). The result follows.

**Claim 4.1.3.** Let $\Gamma \in C$ be an even cycle. Assume that $X$ and $Y$ have ends $x_1, x_2$ and $y_1, y_2$ on $C$ with $x_1y_1 \in N$ and $x_2y_2 \in N$. Then $x_1y_1$ and $x_2y_2$ are the only edges of $\Gamma$ not covered by $M_X \cup M_Y$.

**Proof** The perfect matching $M_X$ must saturate every vertex of $\Gamma$ with the exception of $x_1$ and $x_2$. The same holds with $M_Y$ and $y_1$ and $y_2$. Since $x_1y_1$ and $x_2y_2$ are edges of $\Gamma$, these two edges are not covered by $M_X \cup M_Y$ and we can easily check that the other edges are covered.

**Claim 4.1.4.** Let $\Gamma \in C$ be an even cycle. Assume that $X$ and $Y$ have ends $x_1, x_2$ and $y_1, y_2$ on $C$ with $x_1x_2 \in N$ and $y_1y_2 \in N$. Then either $x_1x_2$ and $y_1y_2$ are the only edges of $\Gamma$ not covered by $M_X \cup M_Y$ or $M_X \cup M_Y$ induces a perfect matching.
on $\Gamma$ such that every edge in that perfect matching is covered by $M_X$ and $M_Y$ with the exception of $x_1x_2$ which belongs to $M_Y$ and $y_1y_2$ which belongs to $M_X$.

**Proof** The perfect matching $M_X$ must saturate every vertex of $\Gamma$ with the exception of the two consecutive vertices $x_1$ and $x_2$. The same holds with $M_Y$ and $y_1$ and $y_2$.

Let us recall here that, since $X$ ($Y$ respectively) is a balanced matching, the paths determined by $x_1$ and $x_2$ on $\Gamma$ have odd lengths (the paths determined by $y_1$ and $y_2$ respectively). Two cases may occur.

**case 1:** The two paths obtained on $\Gamma$ by deleting the edges $x_1x_2$ and $y_1y_2$ have odd lengths We can check that $M_X \cup M_Y$ determines a perfect matching on $\Gamma$ such that every edge in that perfect matching is covered by $M_X$ and $M_Y$ with the exception of $x_1x_2$ which belongs to $M_Y$ and $y_1y_2$ which belongs to $M_X$

**case 2:** The two paths obtained on $\Gamma$ by deleting $x_1x_2$ and $y_1y_2$ have even lengths We can check that $M_X \cup M_Y$ covers every edge of $\Gamma$ with the exception of $x_1x_2$ and $y_1y_2$.

\[ \Box \]

**Claim 4.1.5.** Let $\Gamma \in C$ be an even cycle. Assume that $X$ have ends $x_1, x_2, x_3$ and $x_4$ on $\Gamma$ with $x_1x_2 \in N$ and $x_3x_4 \in N$. Then we can choose a perfect matching $M_Y$ in such a way that $x_1x_2$ and $x_3x_4$ are the only edges of $\Gamma$ not covered by $M_X \cup M_Y$.

**Proof** Since $M_X$ is a perfect matching, the edges of $M_X$ contained in $\Gamma$ saturate every vertex of $\Gamma$ with the condition of $x_1, x_2, x_3$ and $x_4$. Since $Y$ is not incident to $\Gamma$ the perfect matching $M_Y$ can be chosen in two ways (taking one of the two perfect matchings contained in this cycle). We can see easily that we can choose $M_Y$ in such a way that every edge distinct from $x_1x_2$ and $x_3x_4$ is covered by $M_X$ or $M_Y$. \[ \Box \]

Since $\{ A, B, C, D, M' \cap M \}$ is a partition of $M$, each edge of $M$ is covered twice by some perfect matchings of $\mathcal{F}$.

Let $\Gamma \in C$ be an odd cycle, each edge of $\Gamma$ distinct from the two edges of $N$ (Claim 4.1.2) is covered twice by some perfect matchings of $\mathcal{F}$. The two edges of $N$ are covered by exactly one perfect matching belonging to $\{ M_A, M_B, M_C, M_D \}$ and by the perfect matching $M'$. Hence every edge of $\Gamma$ is covered twice by $\mathcal{F}$.

Let $\Gamma \in C$ be an even cycle. Assume first that 4 vertices of $\Gamma$ are ends of some edges in $A$ while no other set of $M$ is incident with $\Gamma$. From Claim 4.1.3 we can choose $M_B$ in such a way that every edge distinct from the two edges of $N$ is covered by $M_A$ or $M_B$. We can then choose $M_C$ in such a way that one of the two edges of $N$ belongs to $M_C$. Finally, we can choose $M_D$ in order to cover the other edge of $N$. Each edge of $\Gamma$ distinct from the two edges of $N$ (Claim 4.1.3) is covered twice by some perfect matchings of $\mathcal{F}$. The two edges of $N$ are covered by exactly one perfect matching belonging to $\{ M_A, M_B, M_C, M_D \}$ and by the perfect matching $M'$. Hence every edge of $\Gamma$ is covered twice by $\mathcal{F}$.

Assume now that 2 vertices of $\Gamma$ are ends of some edges in $A$ (say $a_1$ and $a_2$) and 2 other vertices are ends of some edges in $B$ (say $b_1$ and $b_2$).

**case 1:** $a_1b_1 \in N$ and $a_2b_2 \in N$. We can choose $M_C$ and $M_D$ in order to cover every edge of $\Gamma$. From Claim 4.1.3 every edge of $\Gamma$ is covered by $M_A \cup M_B$ with the exception of $a_1b_1$ and $a_2b_2$. Hence every edge of $\Gamma$ is covered twice by
$M_A \cup M_B \cup M_C \cup M_D$ while $a_1b_1$ and $a_2b_2$ are covered twice by $M_C \cup M_D \cup M'$. Hence every edge of $\Gamma$ is covered twice by $F$.

**case 2:** $a_1a_2 \in N$ and $b_1b_2 \in N$. Assume that $a_1a_2$ and $b_1b_2$ are the only edges of $\Gamma$ not covered by $M_A \cup M_B$ (Claim 4.1.4). Then we can choose $M_C$ and $M_D$ in such a way that every edge of $\Gamma$ is covered by $M_C \cup M_D$. In that case every edge of $\Gamma$ is covered twice by $M_A \cup M_B \cup M_C \cup M_D$ with the exception of $a_1a_2$ and $b_1b_2$ which are covered twice by $M_C \cup M_D \cup M'$.

Assume now that $M_A \cup M_B$ induces a perfect matching on $\Gamma$ where $a_1a_2 \in M_B$ and $b_1b_2 \in M_A$ while the other edges of this perfect matchings are in $M_A \cap M_B$ (Claim 4.1.4). Then we can choose $M_C$ and $M_D$ such that every edge of $\Gamma$ not contained in $M_A \cup M_B$ is covered twice by $M_C \cup M_D$ ($M_C \cup M_D$ induces a perfect matching on $\Gamma$). Hence every edge of $\Gamma$ is covered twice by $M_C \cup M_D$ or by $M_A \cup M_B$ with the exception of $a_1a_2$ which is covered twice by $M_B \cup M'$ and $b_1b_2$ which is covered twice by $M_A \cup M'$.

Finally, assume that $\Gamma$ has no vertex as end of some edge in $M$. Then we can choose easily $M_A, M_B, M_C$ and $M_D$ such that every edge of $\Gamma$ is covered twice by $M_A \cup M_B \cup M_C \cup M_D$.

Hence $F$ is a Fulkerson covering of $G$. □

**Remark 4.2.** Observe that the matchings of the Fulkerson covering described in the above proof are all distinct.

**4.1. Dot products which preserve an F-family.** In [6] Isaacs defined the dot product operation in order to describe infinite families of non trivial snarks.

Let $G_1, G_2$ be two bridgeless cubic graphs and $e_1 = u_1v_1, e_2 = u_2v_2 \in E(G_1)$ and $e_3 = x_1x_2 \in E(G_2)$ with $N_{G_2}(x_1) = \{y_1, y_2, x_2\}$ and $N_{G_2}(x_2) = \{z_1, z_2, x_1\}$.

The dot product of $G_1$ and $G_2$, denoted by $G_1 \cdot G_2$ is the bridgeless cubic graph $G$ defined as (see Figure 7):

$$G = \{G_1 \setminus \{e_1, e_2\}\} \cup \{G_2 \setminus \{x_1, x_2\}\} \cup \{u_1y_1, v_1y_2, u_2z_1, v_2z_2\}$$

It is well known that the dot product of two snarks remains to be a snark. It must be pointed out that in general the dot product operation does not permit to extend a Fulkerson covering, in other words, whenever $G_1$ and $G_2$ are snarks together with a Fulkerson covering, we do not know how to get a Fulkerson covering for $G_1 \cdot G_2$.

However, in some cases, the dot product operation can preserve, in some sense, an $F$-family, leading thus to a Fulkerson covering of the resulting graph.

**Proposition 4.3.** Let $M_1$ be a perfect matching of a snark $G_1$ such that $G_1 \setminus M_1$ contains only two (odd) cycles, namely $C$ and $C'$. Let ab be an edge of $C$ and a'b' be an edge of $C'$.

Let $M_2$ be a perfect matching of a snark $G_2$ where $\{A, B, C, D\}$ is an $F$-family for $M_2$. Let $xy$ be an edge of $G_2 \setminus \{A \cup B \cup C \cup D\}$, with $x$ and $y$ vertices of two distinct odd cycles of $G_2 \setminus M_2$.

Then $\{A, B, C, D\}$ is an $F$-family for the perfect matching $M$ of $G = G_1 \cdot G_2$ with $M = M_1 \cup M_2 \setminus \{xy\}$.

**Proof** Obvious by the definition of the $F$-family and the construction of the graph resulting of the dot product. □
Proposition 4.4. Let $M_1$ be a perfect matching of a snark $G_1$ where \{A, B, C, D\} is an $F$-family for $M_1$. Let $xy$ and $zt$ be two edges of $E(G_1)\setminus M_1$ not contained in $N$.

Let $M_2$ be a perfect matching of a snark $G_2$ such that $G_2\setminus M_2$ contains only two (odd) cycles, namely $C$ and $C'$. Let $xy \in M_2$, with $x \in V(C)$ and $y \in V(C')$.

Then \{A, B, C, D\} is an $F$-family for the perfect matching $M$ of $G = G_1 \cdot G_2$ with $M = M_1 \cup M_2 \setminus \{xy\}$.

**Proof** Obvious by the definition of the $F$-family and the construction of the graph resulting of the dot product. □

We remark that the graphs obtained via Propositions 4.3 and 4.4 can be provided with a Fulkerson covering by Theorem 4.1.

The dot product operations described in Propositions 4.3 and 4.4 will be said to preserve the F-family.

5. Applications

5.1. Fulkerson coverings, more examples. Figures 8 and 9(a) show that the Petersen Graph as well as the flower snark $J_5$ have oddness 2 and have an F-family (the dashed edges denote the related perfect matching).

Moreover, as shown in Figure 9(b) the F-family of $J_5$ can be extended by induction to all the $J_k$’s ($k$ odd).

Thus, following the above Propositions we can define a sequence $(G_n)_{n \in \mathbb{N}}$ of cubic graphs as follows:

- Let $G_0$ be the Petersen graph or the flower snark $J_k$ ($k > 3$, $k$ odd).
For $n \in \mathbb{N}^*$, $G_n = G_{n-1}.G$ where $G$ is either the Petersen graph or the flower snark $J_k$ ($k > 3$, $k$ odd) and the dot product operation preserves the F-family.

As a matter of fact this sequence of iterated dot products of the Petersen graph and/or the flower snark $J_k$ forms a family of exponentially many snarks including the Szekeres Snark (see Figure 10) as well as the two types of generalized Blanuša snarks proposed by Watkins in [9] (see Figure 11).

The family obtained when reducing the possible values of $k$ to $k = 5$ has already been defined by Skupień in [8], in order to provide a family of hypohamiltonian snarks in using the so-called Flip-flop construction introduced by Chvátal in [7].
As far as we know there is no Fulkerson family for the Golberg snark.

5.2. Graphs with a 2-factor of $C_5$'s. Let $G$ be a bridgeless cubic graph having a 2–factor where each cycle is isomorphic to a chordless $C_5$. We denote by $G^*$ the multigraph obtained from $G$ by shrinking each $C_5$ of this 2–factor in a single vertex. The graph $G^*$ is 5–regular and we can easily check that it is bridgeless.

**Theorem 5.1.** Let $G$ be a bridgeless cubic graph having a 2–factor of chordless $C_5$. Assume that $G^*$ has chromatic index 5. Then $G$ can be provided with a Fulkerson covering.
Proof Let \( M \) be the perfect matching complementary of the 2-factor of \( C_5 \). Let \( \{ A, B, C, D, E \} \) be a 5-edge colouring of \( G^* \). Each colour corresponds to a matching of \( G \) (let us denote these matchings by \( A, B, C, D \) and \( E \)). Then it is an easy task to see that \( M = \{ A, B, C, D \} \) is an F-family for \( M \) and the result follows from Theorem 4.1.

\[ \square \]

Theorem 5.2. Let \( G \) be a bridgeless cubic graph having a 2-factor of chordless \( C_5 \). Assume that \( G^* \) is bipartite. Then \( G \) can be provided with a Fulkerson covering.

Proof It is well known, in that case, the chromatic index of \( G^* \) is 5. the result follows from Theorem 5.1.

Remark that, when considering the Petersen graph \( P \), the graph associated \( P^* \) is reduced to two vertices and is thus bipartite.

We can construct cubic graphs with chromatic index 4 which are cyclically 4-edge connected (snarks in the literature) and having a 2-factor of \( C_5 \)'s. Indeed, let \( G \) be cyclically 4-edge connected snark of size \( n \) and \( M \) be a perfect matching of \( G \), \( M = \{ x_iy_i \mid i = 1 \ldots \frac{n}{2} \} \). Let \( G_1 \ldots G_\frac{n}{2} \) be \( \frac{n}{2} \) cyclically 4-edge connected snarks (each of them having a 2-factor of \( C_5 \)). For each \( G_i \) \( (i = 1 \ldots \frac{n}{2}) \) we consider two edges \( e_1 \) and \( e_2 \) of the perfect matching which is the complement of the 2-factor.

We construct then a new cyclically 4-edge connected snark \( H \) by applying the dot-product operation on \( \{ e_1, e_2 \} \) and the edge \( x_iy_i \) \( (i = 1 \ldots \frac{n}{2}) \). We remark that the vertices of \( G \) vanish in the operation and the resulting graph \( H \) has a 2 factor of \( C_5 \), which is the union of the 2-factors of \( C_5 \) of the \( G_i \). Unfortunately, when considering the graph \( H^* \), derived from \( H \), we cannot insure, in general, that \( H^* \) is 5-edge colourable in order to apply Theorem 5.1 and obtain hence a Fulkerson covering of \( H \).

An interesting case is obtained when, in the above construction of \( H \), each graph \( G_i \) is isomorphic to the Petersen graph. Indeed, the 2-factor of \( C_5 \)'s obtained then is such that we can find a partition of the vertex set of \( H \) in sets of \( 2 \) \( C_5 \) joined by \( 3 \) edges. These sets lead to pairs of vertices of \( H^* \) joined by three parallel edges. We can thus see \( H^* \) as a cubic graph where a perfect matching is taken \( 3 \) times. Let us denote by \( H \) this cubic graph (by the way \( H \) is 3-connected). It is an easy task to see that, when \( H \) is 3-edge colourable, \( H^* \) is 5-edge colourable and hence, Theorem 5.1 can be applied.

Let us consider by example the graph \( H \) obtained with 5 copies of the Petersen graph following the above construction (let us remark that the graph \( G \) involved in our construction must be isomorphic also to the Petersen graph). This graph is a snark on 50 vertices. Since \( H \) is a bridgeless cubic graph, the only case for which we cannot say whether \( H \) has a Fulkerson covering occurs when \( H \) is isomorphic to the Petersen graph and, hence \( H^* \) is isomorphic to the unslicable graph \( P(3) \) described by Rizzi [7] (see Figure 12). As a matter of fact we do not know if it is possible to construct a graph \( H \) as described above such that \( H^* \) is isomorphic to the graph \( P(3) \).

By the way, we do not know example of cyclically 5-edge connected snarks (excepted the Petersen graph) with a 2-factor of induced cycles of length 5. We have proposed in [3] the following problem.
**Problem 5.3.** Is there any 5-edge connected snark distinct from the Petersen graph with a 2-factor of $C_5$’s?

**6. On proper Fulkerson covering**

As noticed in the introduction, when a cubic graph is 3-edge colourable, we can find a Fulkerson covering by using a 3-edge colouring and considering each colour twice.

**Proposition 6.1.** Let $G$ be a bridgeless cubic graph with chromatic index 4. Assume that $G$ has a Fulkerson covering $F = \{M_1, M_2, M_3, M_4, M_5, M_6\}$ of its edge set. Then the 6 perfect matchings are distinct.

**Proof** Assume, without loss of generality that $M_1 = M_2$. Since each edge is contained in exactly 2 perfect matchings of $F$, we must have $M_3 \cap M_1 = \emptyset$. Hence $G$ is 3-edge colourable, a contradiction. $\Box$

Let us say that a Fulkerson covering is proper whenever the 6 perfect matchings involved in this covering are distinct. An interesting question is thus to determine which cubic bridgeless graph have a proper Fulkerson covering.

A 3-edge colourable graph is said to be bi-hamiltonian whenever in any 3-edge colouring, there are at least two colours whose removing leads to an hamiltonian 2-factor.

**Proposition 6.2.** Let $G$ be a bridgeless 3-edge colourable cubic graph which is not bi-hamiltonian. Then $G$ has a proper Fulkerson covering.

**Proof** Let $\Phi : E(G) \rightarrow \{\alpha, \beta, \gamma\}$ be a 3-edge colouring of $G$. When $x$ and $y$ are colours in $\{\alpha, \beta, \gamma\}$, $\Phi(x, y)$ denotes the set of disjoint even cycles induced by the two colours $x$ and $y$.

Since the graph $G$ is not bi-hamiltonian we may assume that the 2-factors $\Phi(\alpha, \beta)$ and $\Phi(\beta, \gamma)$ are not hamiltonian cycles. Let $C$ be a cycle in $\Phi(\alpha, \beta)$, we get a new 3-edge colouring $\Phi'$ by exchanging the two colours $\alpha$ and $\beta$ along $C$. We get hence a partition of $E(G)$ into 3 perfect matching $\alpha'$, $\beta'$ and $\gamma$. In the same way,
when considering a cycle $D$ in $\Phi(\beta, \gamma)$, we get a new 3-edge colouring $\Phi''$ of $G$ by exchanging $\beta$ and $\gamma$ along $D$. Let $\alpha, \beta''$ and $\gamma''$ be the 3 perfect matchings so obtained.

Since we have two distinct 3-edge colourings of $G$, $\Phi'$ and $\Phi''$, the set of 6 perfect matchings so involved \{\$, $\alpha, \beta', \beta'', \gamma, \gamma''$ \} is a Fulkerson covering. It remains to show that this set is actually a proper Fulkerson covering.

The exchange operated in order to get $\Phi'$ involve some edges in $\alpha$ and some edges in $\beta$ (those which are on $C_1$) while the other edges keep their colour. In the same way, the exchange operated in order to get $\Phi''$ involve some edges in $\beta$ and some edges in $\gamma$ (those which are on $D_1$) while the other edges keep their colour.

The 3 perfect matchings of $\Phi'$ ($\alpha$, $\beta'$ and $\gamma$) are pairwise disjoint as well as those of $\Phi''$ ($\alpha, \beta''$ and $\gamma''$). We have $\alpha \neq \alpha'$ since $\alpha'$ contains some edges of $\beta$. We have $\alpha \cap \beta'' = \emptyset$ and $\alpha \cap \gamma'' = \emptyset$ since we have exchanged $\beta$ and $\gamma$ in order to obtain $\beta''$ and $\gamma''$. We have $\beta' \neq \beta''$ since $\beta''$ contains some edges of $\alpha$ while $\beta''$ contains some edges of $\gamma$. We have $\beta' \neq \gamma''$ since $\gamma''$ contains some edges of $\alpha$ and $\gamma''$ contains only edges in $\beta$ or in $\gamma$. We have $\gamma \neq \gamma''$ since $\gamma''$ contains some edges of $\beta$.

Hence \{\$ \alpha, \alpha', \beta', \beta'', \gamma, \gamma'' \} is a proper Fulkerson covering.

The theta graph (2 vertices joined by 3 edges), $K_4$, $K_{3,3}$ are examples of small bridgeless cubic graph without proper Fulkerson covering. The infinite family of bridgeless cubic bi-hamiltonian graphs obtained by doubling the edges of a perfect matching of an even cycle has no proper Fulkerson covering. On the other hand, we can provide a bi-hamiltonian graph together with a proper Fulkerson covering. Consider for example the graph $G$ on 10 vertices which have a 2 factor of $C_5$’s, namely $abedc$ and $12345$ with the additional edges edges $a2$, $b4$, $c3$, $d5$ and $e1$, it is not difficult to check that this graph is bi-hamiltonian. Moreover since the following four balanced matchings \{\$2, \{b4\}, \{c3\} and \{d5\} form an $F$-family for the perfect matching \{\$2, b4, c3, d5, e1\}, due to Theorem 4.1 and Remark 4.2, the graph $G$ has a proper Fulkerson covering.

A challenging problem is thus to characterize those bridgeless cubic graphs having a proper Fulkerson covering.

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