SETS OF HILBERT SERIES AND THEIR APPLICATIONS

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Abstract. We consider graded finitely presented algebras and modules over a field. Under some restrictions, the set of Hilbert series of such algebras (or modules) becomes finite. Claims of that type imply rationality of Hilbert and Poincare series of some algebras and modules, including periodicity of Hilbert functions of common (e.g., Noetherian) modules and algebras of linear growth.

1. Introduction

We consider graded finitely presented algebras and modules over a fixed basic field \( k \). The set of Hilbert series of such algebras (or modules), that satisfy some additional restrictions, becomes finite. We give several applications of the claims of that type: the rational dependence of Hilbert series and rationality of Poincare series of ideals in finitely presented algebras, and the periodicity of Hilbert functions of common finitely presented algebras and modules of linear growth.

The paper is organized as follows. In subsection 1.2 we introduce some notations. Then, in section 2, we present our key results about sets of Hilbert series. In particular, we prove here the following

**Theorem 1.1** (Theorem 2.4, (a)). Given 4 positive integers \( n, a, b, c \), let \( D(n, a, b, c) \) denote the set of all connected graded algebras \( A \) over a fixed field \( k \) with at most \( n \) generators such that \( m_1(A) \leq a, m_2(A) \leq b \), and \( m_3(A) \leq c \). Then the set of Hilbert series of algebras from \( D(n, a, b, c) \) is finite.

Here \( m_i(A) = \sup \{ j | \text{Tor}^i_A(k, k)_j \neq 0 \} \) (if \( \text{Tor}^i_A(k, k) = 0 \), we put \( m_i(A) = 0 \)); in particular, \( m_1(A) \) is the exact bound for degrees of generators of the algebra \( A \), and \( m_2(A) \) is the exact bound for degrees of relations of \( A \). For example, an algebra \( A \) is Koszul iff \( m_i(A) \leq i \) for all \( i \geq 0 \).

This Theorem has been also proved in [Pi1] (a version for Koszul algebras had been early proved in [PP]), but here we give another proof. This new proof seems more clear and elementary. It is based on the following

**Theorem 1.2** (Theorem 2.2). Let \( D \geq D_0 \) be two integers. Consider a set \( E(D_0, D) \) of all graded (bi)modules \( M \) over connected graded algebras \( A \) such that \( A \) and \( M \) are generated by \( \leq D \) elements, the generators of \( M \) have degrees at least \( D_0 \), and the generators and relations of both \( A \) and \( M \) have degrees at most \( D \). Then the set of all Hilbert series of (bi)modules from \( E(D_0, D) \) has no infinite ascending chains (with respect to the lexicographical order on Hilbert series).

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It is a generalization of a result of D. Anick \[\text{An}\], where \(M = A\).

The above theorems gives the following new

**Corollary 1.3** (Corollary \[\text{Coroll. 2.7}\]). Let \(D > 0\) be an integer, and let \(A\) be a finitely presented algebra. Then the set of Hilbert series of right-sided ideals in \(A\) having generators and relations in degrees at most \(D\) is finite.

As before, here we put \(m_i(M) = \sup \{j \mid \text{Tor}_i^A(M, k)_j \neq 0\}\) (with \(m_0(M) = 0\) if \(\text{Tor}_i^A(M, k) = 0\)); in particular, \(m_0(M)\) is the exact bound for degrees of generators of \(M\), and \(m_1(M)\) is the exact bound for degrees of relations of \(M\).

Our results on sets of Hilbert series gives some interesting applications \[\text{[11]}\]. A finitely presented graded module \(M\) over a connected graded algebra \(A\) is called 

**effectively coherent**

if there is a function \(D_M : \mathbb{N} \to \mathbb{N}\) such that, whenever a graded submodule \(L \subset M\) is generated in degrees \(\leq d\), the relations of \(L\) are concentrated in degrees at most \(D(d)\). A module \(M\) is called effective for series if for every integer \(d\) there is only a finite number of possibilities for Hilbert series of submodules of \(M\) generated in degrees at most \(d\). Theorem 1.1 has been essentially used to establish the following

**Theorem 1.4** (\[\text{[11]}\]). (a) Every strongly Noetherian connected algebra over an algebraically closed field is effectively coherent.

(b) Every effectively coherent algebra is effective for series.

Recall that an algebra \(A\) is called strongly Noetherian \[\text{ASZ}\] if an algebra \(A \otimes C\) is Noetherian for every Noetherian commutative \(k\)-algebra \(C\); in particular, the most of common rings of non-commutative projective geometry are strongly Noetherian \[\text{ASZ}\].

Here we consider other applications of sets of Hilbert series. First, we consider (in section \[\text{8}\]) the following question: when Hilbert function \(f_V(n) := \dim V_n\) of a graded algebra or a module \(V\) is periodic? An obvious necessary condition is that the algebra (module) \(V\) must have linear growth, that is, \(\text{GK–dim } V \leq 1\). It happens that in some common cases this condition is sufficient.

**Theorem 1.5** (\[\text{Theorem 3.1}\]). Let \(M\) be a finitely presented graded module over a connected finitely presented algebra \(A\). Suppose that \(\text{GK–dim } M = 1\) and at least one of the following conditions holds:

(a) the field \(k\) is finite;

(b) the vector spaces \(\text{Tor}_1^A(M, k)\) and \(\text{Tor}_3^A(k, k)\) have finite dimensions.

Then the Hilbert series \(M(z)\) is rational, that is, the Hilbert function \(f_M(n) = \dim M_n\) is periodic.

**Corollary 1.6.** Let \(M\) be a graded finitely generated (finitely presented) right module over a right Noetherian (respectively, right coherent) connected graded algebra \(A\). If \(\text{GK–dim } M \leq 1\), then the Hilbert function of \(M\) is periodic.

Notice that, in non-commutative projective geometry, critical modules of linear growth are in one-to-one correspondence of closed points to \(q\)-coh \(A\). So, a period of the Hilbert function becomes a numerical invariant of such point.

There exist finitely generated algebras with non-periodic but bounded Hilbert function, e.g., an algebra \(k\langle x, y \mid yxy, x^2, xy^2, x, n \geq 1\rangle\); if the field \(k\) is finite, this algebra is even effective for series.

We do not know if any finitely presented algebra of linear growth has periodic Hilbert function. However, we establish the periodicity in several important cases.
Corollary 1.7 (Corollary 3.2). Let $A$ be a finitely generated connected algebra of Gelfand–Kirillov dimension one. Suppose that $A$ satisfies at least one of the following properties:

(i) two-sided or right Noetherian;
(ii) (semi)prime;
(iii) coherent;
(iv) finitely presented over a finite field;
(v) Koszul;
(vi) $A$ has finite Bakelin’s rate.

Then the Hilbert function of $A$ is periodic.

Here an algebra $A$ is said to be of finite Backelin’s rate if there is a number $r$ such that every space $\text{Tor}^A_{i}(k, k)$ is concentrated in degrees at most $ri$ [Ba].

An algebra $A$ is said to be (right graded) coherent if every finitely generated homogeneous right–sided ideal is finitely presented, or, equivalently, a kernel of any homogeneous map $F_1 \to F_2$ of two finitely generated free modules $F_1, F_2$ is finitely generated [Bu, F].

An ideal in non-coherent algebras may also be finitely presented and, moreover, it may admit a free resolution of finite type; to describe some of such ideals, we introduce (in section 4) the following concept.

Definition 1.8. Let $A$ be a connected graded algebra, and let $\mathbf{F}$ be a set of finitely generated homogeneous right-sided ideals in $A$. $\mathbf{F}$ is said to be quasi-coherent family of ideals if $0 \in \mathbf{F}$ and for every $0 \neq I \in \mathbf{F}$ there are $J_1, J_2 \in \mathbf{F}$ such that $J_1 \neq I, m_0(J_1) \leq m_0(I)$, and $I/J_1 \cong A/J_2[-t]$ for some $t \in \mathbb{Z}^+$. A quasi-coherent family $\mathbf{F}$ is said to be of degree $d$ if $m_0(I) \leq d$ for all $I \in \mathbf{F}$.

If the maximal ideal $\mathfrak{A} := A_1 \oplus A_2 \oplus \cdots \in \mathbf{F}$, a quasi-coherent family $\mathbf{F}$ is called coherent [Pi1]; for example, all finitely presented monomial algebras admits coherent families of finite degree [Pi1], as well as some homogeneous coordinate rings [CNR]. A coherent family of degree 1 is called Koszul filtration; Koszul filtrations have been studied in a number of papers [Bl, Co1, Co2, CRV, CTV, Pi2]. A quasi-coherent family of degree one is called Koszul family: such families exist, e. g., in homogeneous coordinate rings of some finite sets of points in projective spaces [Po].

It is not hard to see that every ideal in a quasi-coherent family admits free resolution of finite type. Using the above results on the sets of Hilbert series, we deduce

Proposition 1.9 (Corollary 4.3). Let $\mathbf{F}$ be a quasi-coherent family of degree $d$ in a finitely presented algebra $A$. Then the set of Hilbert series of ideals $I \in \mathbf{F}$ is finite.

A version of Proposition 1.9 for coherent families has been proved in [Pi1]; here we just generalize it to quasi-coherent families using Corollary 1.3.

It is proved in [Pi1, Pi2] that every ideal in a coherent family of finite degree has rational Hilbert series, and that every ideal in a Koszul filtration has also rational Poincare series. Proposition 1.9 allows us to establish similar properties in quasi-coherent case.

Corollary 1.10 (Corollaries 4.4, 4.5). Let $\mathbf{F}$ be a quasi-coherent family of degree $d$ in a finitely presented algebra $A$.

(i) For every ideal $I \in \mathbf{F}$ there are two polynomials $p(z), q(z)$ with integer coefficients such that $I(z) = A(z)^{p(z)} / q(z)$.
(ii) Assume that $d = 1$, i.e. $F$ is a Koszul family. Then for every ideal $I \in F$ its Poincaré series $P_I(z) := \sum_{i \geq 0}(\dim \text{Tor}_i(I, k))z^i$ is a rational function.

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1.2. Notations and assumptions. We will deal with $\mathbb{Z}_+$-graded connected associative algebras over a fixed field $k$, that is, algebras of the form $R = \bigoplus_{i \geq 0}R_i$ with $R_0 = k$. By assumption, all our modules and ideals are graded and right-sided.

For an $R$–module $M$, we will denote by $H_iM$ the graded vector space $\text{Tor}_i^R(M, k)$. By $H_iR$ we will denote the graded vector space $\text{Tor}_i^R(k, k) = H_i(k_A)$. In particular, the vector space $H_1R$ is isomorphic to the $k$–span of a minimal set of homogeneous generators of $R$, and $H_2R$ is isomorphic to the $k$–span of a minimal set of its homogeneous relations. Analogously, the space $H_0M$ is the span of generators of $M$, and $H_1M$ is the span of its relations.

Let $m(M) = m_0(M)$ denote the supremum of degrees of minimal homogeneous generators of $M$: if $M$ is just a vector space with trivial module structure, it is simply the supremum of degrees of elements of $M$. For $i \geq 0$, let us also put $m_i(M) := m(H_iM) = \sup\{j | \text{Tor}_j^R(M, k) \neq 0\}$. Similarly, let us put $m_i(R) = m(H_iR) = m_i(k_R)$. For example, $m(R) = m_0(R)$ is the supremum of degrees of the generators of $R$, and $m_1(R)$ (respectively, $m_1(M)$) is the supremum of degrees of the relations of $R$ (resp., of $M$).

Note that the symbols $H_iR$ and $m_iR$ for an algebra have different meaning that the respective symbols $H_iR_R$ and $m_iR_R$ for $R$ considered as a module over itself; however, the homologies $H_iR_R$ are trivial, so that there is no place for confusion.

For a graded locally finite vector space (algebra, module...) $V$, its Hilbert series is defined as the formal power series $V(z) = \sum_{i \in \mathbb{Z}}(\dim V_i)z^i$. For example, the Euler characteristics of a minimal free resolution of the trivial module $k_R$ leads to the formula

$$(1.1) \quad R(z)^{-1} = \sum_{i \geq 0}(-1)^iH_iR(z).$$

As usual, we write $\sum_{i \geq 0}a_i z^i = o(z^n)$ iff $a_i = 0$ for $i \leq n$.

Let us introduce a lexicographical total order on the set of all power series with integer coefficients, i.e., we put $\sum_{i \geq 0}a_i z^i >_{l.e.x.} \sum_{i \geq 0}b_i z^i$ iff there is $q \geq 0$ such that $a_i = b_i$ for $i < q$ and $a_q > b_q$. This order extends the coefficient-wise partial order given by $\sum_{i \geq 0}a_i z^i \geq \sum_{i \geq 0}b_i z^i$ iff $a_i \geq b_i$ for all $i \geq 0$.

2. Properties of sets of Hilbert series

The following theorem of Anick shows that the set of Hilbert series of algebras of bounded (by degrees and numbers) generators and relations is well-ordered.

**Theorem 2.1** ([An] Theorem 4.3). Given three integers $n, a, b$, let $C(n, a, b)$ be the set of all $n$–generated connected algebras $R$ with $m_1(R) \leq a$ and $m_2(R) \leq b$ and let $H(n, a, b)$ be the set of Hilbert series of such algebras. Then the ordered set $(H(n, a, b), >_{l.e.x.})$ admits no infinite ascending chains.

The example of an infinite descending chain of Hilbert series in the set $C(7, 1, 2)$ is constructed in [An] Example 7.7.

We will prove this theorem in a more general form, with (bi)modules instead of algebras. The proof is based on the same idea as the original proof in [An].
Theorem 2.2. Let \( D \geq D_0 \) be two integers. Consider a set \( E(D_0, D) \) of all graded (bi)modules \( M \) over connected graded algebras \( A \) such that \( A \) and \( M \) are generated by \( \leq D \) elements, the generators of \( M \) have degrees at least \( D_0 \), and the generators and relations of both \( A \) and \( M \) have degrees at most \( D \). Then the set of all Hilbert series of (bi)modules from \( E(D_0, D) \) has no infinite ascending chains.

Let us first introduce an additional notation.

Given four formal power series \( V(z), R(z), W(z), S(z) \), let us denote by \( \mathcal{M} = \mathcal{M}(V(z), R(z), W(z), S(z)) \) the set of all modules \( M \) over algebras \( A \) such that \( H_1A(z) = V(z), H_2A(z) = R(z), H_0M(z) = W(z) \), and \( H_1M(z) = S(z) \).

We may assume that all these algebras are generated by the same vector space \( V \) and all our modules are generated by the same vector space \( W \). Indeed, because the relations of a right module \( A \) are minimal sets of relations of \( A \) and \( A \) is algebraic as well. Then the set \( \mathcal{M}(V(z), R(z), W(z), S(z)) \) is algebraic. Since \( \dim N \leq h_i \) for every \( i \geq 0 \), the last condition means that the rank of the vectors generating the vector space \( N_i \) is bounded above by \( h_i \). Obviously, this condition is algebraic for \( u \), because it simply means suitable minor determinants vanish. Therefore, the set \( L_{\geq h}(h(z)) \) is a countable intersection of closed subsets, hence it is closed.

Now, the condition \( u \in L_{> h}(h(z)) \) means that \( N(z) \leq \hat{h}(z) \), that is, \( \dim N_j < h_j \) for every \( i \geq 0 \). For every \( i \geq 0 \), the last condition means that the rank of the vectors generating the vector space \( N_i \) is bounded above by \( h_i \). Obviously, this condition is algebraic for \( u \), because it simply means suitable minor determinants vanish. Therefore, the set \( L_{> h}(h(z)) \) is a countable intersection of closed subsets, hence it is closed.

Proof of Theorem 2.2. First, up to a shift of grading we may assume that \( D_0 = 0 \). Second, every bimodule over an algebra \( A \) may also be considered as a right module over the algebra \( B = A \otimes A^{op} \). Since \( m_1(B) = m_1(A) \leq D \) and \( m_2(B) \leq \max\{m_1(A)^2, m_2(A)\} \leq D^2 \), it is sufficient to prove Theorem 2.2 for every \( D \) for a subset \( E'(0, D) \subset E(0, D) \) which consists of right modules (that is, of bimodules with zero left multiplication). Indeed, because the relations of a right module \( M \) as a bimodule may have degrees at most \( \max\{m_0(M) + m_1(A), m_1(M)\} \leq 2D \),
the statement for the set $E(0, D)$ will follow from the same statement for the set $E'(0, \max\{2D, D^2\})$.

Assume that there is an infinite ascending chain

$$M_1(z) <_{\text{lex}} M_2(z) <_{\text{lex}} \ldots$$

in $E'(0, D)$ of modules over some algebras $A_1, A_1, \ldots \in C(D, D, D)$. We may assume that all these algebras are generated by the same finite-dimensional graded vector space $V$ and that their minimal vector spaces of relations $R_1, R_2, \ldots \subset T(V)$ have the same Hilbert series (polynomial) $R_i(z) = R(z)$. Analogously, we assume that all modules $M_i$ are generated by the same finite-dimensional graded vector space $W$ and has the minimal vector spaces of relations $S_1, S_2, \ldots$ of the same finite Hilbert series $S_i(z) = S(z)$.

Thus we obtain an infinite descending set of closed subsets in $\mathcal{M}(V(z), R(z), W(z), S(z))$:

$$L_{>_{\text{lex}}} (M_1(z)) \supset L_{>_{\text{lex}}} (M_2(z)) \supset \ldots,$$

a contradiction.

The following theorem is the main result of this section. Another its proof may be found in [311].

**Theorem 2.4.** Let $n, a, b, c, m, p_1, p_2, q, r$ be 9 integers.

(a) Let $D(n, a, b, c)$ denote the set of all connected algebras $A$ over a fixed field $k$ with at most $n$ generators such that $m_1(A) \leq a, m_2(A) \leq b, \text{ and } m_3(A) \leq c$. Then the set of Hilbert series of algebras from $D(n, a, b, c)$ is finite.

(b) Let $DM = DM(n, a, b, c, m, p_1, p_2, q, r)$ denote the set of all graded right modules over algebras from $D(n, a, b, c)$ with at most $m$ generators such that $M_i = 0$ for $i < p_1$, $m_0(M) \leq p_2, m_1(M) \leq q$, and $m_2(M) \leq r$. Then the set $HDM$ of Hilbert series of modules from $DM$ is finite.

For this statement, we need the following standard version of Koenig lemma.

**Lemma 2.5.** Let $P$ be a totally ordered set satisfying both ACC and DCC. Then $P$ is finite.

**Proof of Theorem 2.4.** Up to a shift of grading, we may assume that $p_1 = 0$. Let $M \in DM$ is a module over an algebra $A \in D(n, a, b, c)$. Consider the first terms of the minimal free resolution of $M$:

$$0 \to \Omega \to H_0(M) \otimes A \to M \to 0.$$ 

Here the syzygy module $\Omega$ has generators in degrees at most $m_1(M) \leq q$ and relations in degrees at most $m_2(M) \leq r$. Since $H_0(\Omega) = H_1(M)$, the number dim $H_0(\Omega)$ of its generators is not greater than dim $(H_0(M) \otimes A)_{\leq q} \leq m(1 + n + \cdots + n^q) =: D'$. We have that $\Omega \in E(0, D)$ for $D = \max\{D', q, r\}$, so, the set of all possible Hilbert series for $\Omega$ satisfies ACC.

Assume for a moment that $M = A_1 \oplus A_2 \oplus \ldots$ (to connect the cases (a) and (b), we take here $m = n, p_2 = a, q = b$, and $r = c$). We have here $M(z) = A(z) - 1$ and $H_0(M) = H_1(A)$. Taking Eulerian characteristics, we obtain $\Omega(z) - H_1(A)(z)A(z) + M(z) = 0$, or

$$A(z) = (1 - \Omega(z))(1 - H_1(A)(z))^{-1} = (1 - \Omega(z))(1 + H_1(A)(z) + H_1(A)(z)^2 + \ldots).$$

Notice that the order $<_{\text{lex}}$ is compatible with the multiplication by a formal power series with positive coefficients. Because there is only finite number of possibilities
for $H_1(A)(z)$, the set of the Hilbert series $A(z)$ of that type satisfies DCC. By Theorem 2.1, this set also satisfies ACC. In the view of Lemma 2.5, the statement (a) follows.

Now, return to the general case (b). We have $M(z) = H_0(M)(z)A(z) - \Omega(z)$. Here there is only finite number of possibilities for $H_0(M)(z)$ because of dimension arguments; also, there is only finite number of possibilities for $A(z)$ by part (a). It follows that the set $HDM$ of all such Hilbert series $M(z)$ satisfies DCC. Because $HDM \subset E(0, \max\{n, a, b, m, p_2, q\})$, it also satisfies ACC by Theorem 2.2. By Lemma 2.5, it if finite.

If we restrict our consideration to the modules over a single algebra $A$, the additional condition $m_3(A) < \infty$ may be omitted.

**Corollary 2.6.** Let $A$ be a finitely presented algebra. Given five integers $m, p_1, p_2, q, r$, consider a set $DM_A = DM_A(m, p_1, p_2, q, r)$ of finitely presented $A$-modules $M$ with at most $m$ generators such that $M_i = 0$ for $i < p_1$, $m_0(M) \leq p_2, m_1(M) \leq q$, and $m_2(M) \leq r$. Then the set of Hilbert series of modules from $DM_A$ is finite.

**Proof.** Like the proof of Theorem 2.4, we may assume that $p_1 = 0$ and consider the exact sequence

$$0 \to \Omega \to H_0(M) \otimes A \to M \to 0.$$ 

As before, we see that the set of all possible Hilbert series for $\Omega$ satisfies ACC, so, the set $HDM_A$ of Hilbert series $M(z) = H_0(M)(z)A(z) - \Omega(z)$ satisfies DCC. Because $DM_A \subset E(0, D)$ for $D = \max\{m_1(A), m_2(A), \dim H_0(A), p_2, q, r\}$, Lemma 2.5 implies that the set of Hilbert series of modules from $DM_A$ is finite.

**Corollary 2.7.** Let $D > 0$ be an integer, and let $A$ be a finitely presented algebra.

(a) The set of Hilbert series of (two-sided or right-sided) ideals in $A$ generated in degrees at most $D$ satisfies DCC.

(b) The set of Hilbert series of right-sided ideals in $A$ having generators and relations in degrees at most $D$ is finite.

**Proof.** Just apply Theorem 2.2 and Corollary 2.6 to the modules $A/I$, where $I$ is an ideal.

### 3. Periodic Hilbert Functions

**Theorem 3.1.** Let $M$ be a finitely presented graded module over a connected finitely presented algebra $A$. Suppose that $GK$–dim $M = 1$ and at least one of the following conditions holds:

(a) the field $k$ is finite;

(b) the vector spaces $\text{Tor}_2^A(M, k)$ and $\text{Tor}_3^A(k, k)$ have finite dimensions.

Then the Hilbert series $M(z)$ is rational, that is, the Hilbert function $f_M(n) = \dim M_n$ is periodic.

**Corollary 3.2.** Let $A$ be a finitely generated algebra of Gelfand–Kirillov dimension one. Suppose that $A$ satisfies at least one of the following properties:

(i) two-sided or right Noetherian;

(ii) (semi)prime;

(iii) coherent;

(iv) finitely presented over a finite field;

(v) Koszul.
(vi) of finite Bakelin’s rate.
Then the Hilbert function of $A$ is periodic.

Proof of Corollary $\text{(3.4)}$. Recall that any affine algebra $A$ with $\text{GK–dim} A = 1$ is PI $\text{SSW}$, hence it has rational Hilbert series provided that it is Noetherian $\text{L}$. Moreover, it is shown in $\text{L}$ that every Noetherian module over a PI algebra has rational Hilbert series; since every Noetherian $A$–bimodule is a Noetherian module over the algebra $A^{op} \otimes A$, it follows that any weak Noetherian (i. e., satisfying ACC for two-sided ideals) PI algebra has rational Hilbert series. This proves the case $(i)$.

Any semiprime algebra of Gelfand–Kirillov dimension one is a finite module over its Noetherian center $\text{SSW}$, hence it has periodic Hilbert function. This proves $(ii)$.

The rest 5 cases follow from Theorem $\text{(3.1)}$. Notice that the case $(v)$ has been also proved by L. Positselski (unpublished). $\square$

Lemma 3.3. Let $M$ be an infinite–dimensional graded (by nonnegative integers) module over a connected graded algebra $A$. Let $M^n$ denote the right $A$–module $\bigoplus_{t \geq n} M^t$ with degree shifted by $n$, i.e., $(M^n)_t = M_{n+t}$ for all $t \geq 0$. Then the following conditions are equivalent:

(a) $\text{GK–dim} M = 1$ and $M(z)$ is a rational function;
(b) for some $i \neq j$, we have $M^i(z) = M^j(z)$;
(c) the set of Hilbert series $\mathcal{H}_M = \{M^n(z) | n \geq 0\}$ is finite.

In this case, the sequence $\{\dim M^n\}$ is periodic with a period $d$ such that $d \leq |\mathcal{H}_M|$ and $d \leq |i-j|$.

Proof. If $\text{GK–dim} M = 1$, the rationality of $M(z)$ means that there are two positive integers $d, D$ such that $\dim M_n = \dim M_{n+d}$ for all $n \geq D$, that is, $M^n(z) = M^{n+d}(z)$ for all $n \geq D$. In this case the set $\mathcal{H}_M = \{M^n(z) | n < D + d\}$ is finite.

On the other hand, if $M^n(z) = M^{n+d}(z)$ for some $n \geq 0, d > 0$, then $\dim M_i = \dim M_{i+d}$ for all $i \geq n$, so, $\dim M_i \leq \max_{j \leq n+d} M_j$. In particular, it follows that $\text{GK–dim} M = 1$.

Finally, if the set $\mathcal{H}_M$ is finite, then $(b)$ obviously holds. $\square$

Remark 3.4. By the same way, it may be shown more: if $\text{GK–dim} M \leq 1$ and $M^i(z) \leq M^j(z)$ for some $i \neq j$, then the Hilbert function of $M$ is periodic.

Lemma 3.5. Let $M$ be a nonegatively graded module over a connected graded algebra $A$, and let $M^n$ be as in Lemma $\text{SSW}$. Then $m_i(M^n) < \max\{m_i(M), m_{i+1}(A)\}$ for all $n \geq 1, i \geq 0$.

Proof. By definition, $m_i(M^0) = m_i(M)$ for all $i \geq 0$. Let us prove by induction on $n \geq 1$ that $\text{Tor}_i^A(M^{n+1}, k)_{-1} = 0$ provided that $\text{Tor}_i^A(M^n, k)_j = \text{Tor}_{i+1}^A(k, k)_j = 0$.

Let $s = \dim A_n$. The exact triple

$$0 \to M^{n+1}[-1] \to M^n \to k^s \to 0$$

leads to the triple

$$\text{Tor}_{i+1}^A(k^s, k) \to \text{Tor}_i^A(M^{n+1}, k)[-1] \to \text{Tor}_i^A(M^n, k)$$

for every $i \geq 0$. It remains to notice that $\text{Tor}_{i+1}^A(k^s, k) = \bigoplus_{t} \text{Tor}_{i+1}^A(k, k)$. $\square$
Proof of Theorem 3.1. Up to the shift of grading, we may assume that \( M_i = 0 \) for \( i < 0 \). Put \( g_A = m(A), g_M = m(M), r_A = m_2(A) \), and \( r_M = m_1(M) \). Let \( N \) be a number such that \( \dim M_n \leq N \) for all \( n \geq 0 \). By Lemma 8.5, every module \( M^n \) is isomorphic to a quotient of a free module \( F = V \oplus A \) by a submodule generated by a homogeneous subspace \( W \subset F_{[1..r_M]} = F_1 \oplus \cdots \oplus F_{r_M} \), where \( V = V_0 \oplus \cdots \oplus V_{g_M} \) is a graded vector space with \( \dim V_i = 0, 0 \leq i \leq g_M \).

Case (a). Let \( q \) be the cardinality of \( k \). Since \( \dim F_{[1..r_M]} \) is finite (namely, \( \dim F_{[1..r_M]} < g_A r_M N : T \), so \( \{F_{[1..r_M]}\} < q^r \)), then there is only finite number of possibilities for its subset \( W \) (namely, there are at most \( 2^{q^r} \) proper subsets of \( F_{[1..r_M]} \)). Then there is a finite number of the isomorphism types of the modules \( M^n \), and so, by Lemma 8.3, the sequence \( \{\dim M_n\} \) is periodic with a period \( d < 2^{q^r} \).

Case (b). By Lemma 8.5 for every \( n \geq 2 \) we have \( m_2(M^n) = m_3(A) - 1 \). By Corollary 2.6 there is only finite number of possibilities for Hilbert series \( L(z) \) of modules \( L \) with bounded \( \dim H_0(L) \) and \( m_i(L) \) for \( i \leq 3 \). Thus we conclude that the set of all Hilbert series of the modules \( M^i \) is finite. It remains to apply Lemma 8.3.

4. Quasi-coherent families of ideals

Definition 4.1. Let \( A \) be a connected graded algebra, and let \( F \) be a set of finitely generated homogeneous right-sided ideals in \( A \). \( F \) is said to be quasi-coherent family of ideals if \( 0 \in F \) and for every \( 0 \neq I \in F \) there are \( J_1, J_2 \in F \) such that \( J_1 \neq I, m_0(J_1) \leq m_0(I) \), and \( I/J_1 \cong A/J_2[-t] \) for some \( t \in \mathbb{Z}_+ \).

A quasi-coherent family \( F \) is called of degree \( d \) if \( m_0(I) \leq d \) for all \( I \in F \).

A quasi-coherent family of degree 1 is called Koszul family of ideals. It was introduced by Polishchuk [22] in order to prove that homogenous coordinate rings of some sets of points in projective spaces are Koszul. If \( A \geq 1 \in F \), then a quasi-coherent family is called coherent [41]. The term "quasi-coherent family" appears because all ideals in such a family are finitely presented, like finitely generated submodules in a quasi-coherent module. Moreover, there is

Proposition 4.2. Let \( F \) be a quasi-coherent family in an algebra \( A \). Then every ideal \( I \in F \) has free resolution of finite type. If \( F \) has degree \( d \), then for every \( I \in F \) we have \( m_i(I) \leq m_0(I) + id \) for all \( i \geq 0 \).

The proof is the same as for coherent families in [41].

Proof. We proceed by induction in \( i \) and in \( I \) (by inclusion of ideals \( J_1 \) with \( m_0(J_1) \leq m_0(I) \)). Let \( J_1, J_2, t \) be as in Definition 4.1. In particular, \( t \leq m_0(I) \leq d \). The exact sequence

\[ 0 \rightarrow J_1 \rightarrow I \rightarrow A/J_2[-t] \rightarrow 0 \]

leads for every \( i \geq 1 \) to the following fragment of the long exact sequence of Tor’s:

\[ \cdots \rightarrow H_i(J_1) \rightarrow H_i(I) \rightarrow H_i(J_2) \rightarrow \cdots \]

By induction, we have \( m_i(I) \leq \max\{m_i(J_1), m_{i-1}(J_2) + t\} < \infty \). If \( t \leq d \), we have also \( m_i(I) \leq t + id \).

Corollary 4.3. Let \( F \) be a quasi-coherent family of degree \( d \) in a finitely presented algebra \( A \). Then the set of Hilbert series of ideals \( I \in F \) is finite.
Proof. In the view of Proposition 4.2, we can apply Corollary 2.7.

**Corollary 4.4.** Let $F$ be a quasi-coherent family of degree $d$ in a finitely presented algebra $A$. Then for every ideal $I \in F$ there are two polynomials $p(z), q(z)$ with integer coefficients such that $I(z) = A(z)\frac{p(z)}{q(z)}$.

Proof. Because of Corollary 4.3, we can apply exactly the same arguments as for coherent families, see [111] Theorem 4.5.

**Corollary 4.5.** Let $F$ be a Koszul family (that is, a quasi-coherent family of degree 1) in a finitely presented algebra $A$. Then for every ideal $I \in F$ its Poincare series $P_I(z) := \sum_{i \geq 0} (\dim \text{Tor}_i(I, k))z^i$ is a rational function.

Proof. Every ideal in a Koszul family is a Koszul module (see [Po] or Proposition 4.2). By Corollary 4.4, we have $I(z) = p(z)/q(z)$. By Koszulity, we have $I(z) = P_I(-z)A(z)$, so that $P(z) = p(-z)/q(-z)$.

**References**

[An] D. Anick, *Generic algebras and CW–complexes*, Proc. of 1983 Conf. on algebra, topol. and K–theory in honor of John Moore. Princeton Univ., 1988, p. 247–331

[ASZ] M. Artin, L. W. Small, J. J. Zhang, *Generic flatness for strongly Noetherian algebras*, J. Algebra, 221 (1999), 2, p. 579–610

[Ba] J. Backelin, *On the rates of growth of homologies of Veronese subrings*, Lecture Notes Math., 1183 (1986), p. 79–100

[Bi] S. Blum, *Initially Koszul algebras*, Beiträge Algebra Geom., 41 (2000), 2, p. 455–467

[Bu] N. Burbaki, *Algèbre, Ch.10. Algèbre homologique*, Masson, Paris–NY–Barcelona–Milan, 1980

[Co1] A. Conca, *Universally Koszul algebras*, Math. Ann., 317 (2000), 2, p. 329–346

[Co2] A. Conca, *Universally Koszul algebras defined by monomials*, Rendiconti del seminario matematico dell’Università di Padova, 107 (2002), p. 1–5

[CNR] A. Conca, E. de Negri, and M. E. Rossi, *On the rate of points in projective spaces*, Israel J. Math., 124 (2001), p. 253–265

[CRV] A. Conca, M. E. Rossi, and G. Valla, *Groebner flags and Gorenstein algebras*, Compositio Math., 129 (2001), p. 95–121

[CTV] A. Conca A., N. V. Trung, and G. Valla, *Koszul property for points in projective spaces*, Math. Scand., 89 (2001), 2, p. 201–216

[F] K. Faith, *Algebra: rings, modules, and categories. V. I*, Corrected reprint. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 190. Springer-Verlag, Berlin-New York, 1981

[L] M. Lorenz, *On Gelfand-Kirillov dimension and related topics*, J. Algebra, 118 (1988), 2, p. 423–437

[Pi1] D. Piontkovski, *Linear equations over noncommutative graded rings*, to appear in J. Algebra; preprint math.RA/0404119 (2004)

[Pi2] D. Piontkovski, *Koszul algebras and their ideals*, to appear in Funct. Anal. Appl., 2, 2004; an extended version is available as Noncommutative Koszul filtrations, preprint math.RA/0301233

[Po] A. Polishchuk, *Koszul configurations of points in projective spaces*, preprint math.AG/0412441 (2004)

[PP] A. Polishchuk and L. Positselski, *Quadratic algebras*, preprint (1994–2000)

[SSW] L. W. Small, J. T. Stafford, R. B. Warfield (Jr), *Affine algebras of Gelfand-Kirillov dimension one are PI*, Math. Proc. Cambridge Philos. Soc., 97 (1985), 3, p. 407–414

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