Abstract. We show that any $m \times m$ matrix $M$ with integer entries and $\det M = \Delta \neq 0$ can be equipped by a finite digit set $D \subset \mathbb{Z}^m$ such that any integer $m$-dimensional vector belongs to the set

$$\text{Fin}_D(M) = \left\{ \sum_{k \in I} M^k d_k : \emptyset \neq I \text{ finite subset of } \mathbb{Z} \text{ and } d_k \in D \text{ for each } k \in I \right\} \subset \bigcup_{k \in \mathbb{N}} \frac{1}{\Delta^k} \mathbb{Z}^m.$$ 

We also characterize the matrices $M$ for which the sets $\text{Fin}_D(M)$ and $\bigcup_{k \in \mathbb{N}} \frac{1}{\Delta^k} \mathbb{Z}^m$ coincide.

Keywords: vector representation, number system, Jordan form

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1. Introduction

The idea to represent $m$-dimensional vectors by a single string of digits can be traced back to the work of A. Vince [18] and [19] who showed that for any expansive matrix $M \in \mathbb{Z}^{m \times m}$ there exists a digit set $D \subset \mathbb{Z}^m$ such that any integer vector $x \in \mathbb{Z}^m$ can be written in the form $x = \sum_{k=0}^{n} M^k d_k$, where $d_k \in D$. In other words, the whole $\mathbb{Z}^m$ is representable in the matrix numeration system $(M, D)$. On the other hand, if a matrix $M$ has an eigenvalue inside the unit circle, no choice of the digit set $D \subset \mathbb{Z}^m$ allows to represent all integer vectors as a combination of non-negative powers of $M$ only.

Many works devoted to positional representations of elements of a commutative finitely generated ring can be interpreted as a special case of the matrix numeration systems. From this point of view, the history of the matrix numeration systems had started several decades before the year 1993, the year Vince’s results have been published. Positional number systems used to represent the Gaussian integers or, more general, elements of a ring of integers in a quadratic number field can be viewed as a matrix number system given by a $2 \times 2$ matrix. Number systems of this type were studied by Penney [15], Kátai and Szabó [10], Kátai and B. Kovács, [8], [9], Gilbert [5]. This concept was extended to algebraic fields of higher order by B. Kovács [11] and B. Kovács and Pethő [12]. A number system with an algebraic base $\beta$ can be interpreted as a matrix number system with the base being the companion matrix of the minimal polynomial of $\beta$. Hence the characteristic polynomial of the matrix is irreducible over $\mathbb{Q}$. The concept of the so-called canonical number system for irreducible polynomials was further generalized to arbitrary polynomials from $\mathbb{Z}[x]$ by Pethő [16]. The dynamic properties of canonical number systems can be well studied in the formalism of the shift radix systems introduced in [1]. A very general setting of the canonical number systems was recently considered in [2].

Our aim is to study the matrix numeration systems. The most important property of a square matrix can be deduced from its Jordan form. Let us point out that the Jordan form of the companion matrix associated to a polynomial has a specific property: only one Jordan block corresponds to each eigenvalue. Therefore the study of matrix numeration systems can display new phenomena and is of its own interest. The matrix formalism for numeration systems – under the name numeration systems in lattices – was systematically used by A. Kovács in [13]. Of course, A. Kovács, just like Vince, considers integer matrices as they map a lattice into itself. The Vince’s results on integer matrices were recently generalized by J.
Jankauskas and J. Thuswaldner to matrices $M \in \mathbb{Q}^{m \times m}$ with rational entries and without eigenvalues in modulus strictly smaller than 1, see [7].

In this paper, we generalize the Vince’s result in another direction: our representation of an integer vector can use both positive and negative powers of a matrix $M \in \mathbb{Z}^{m \times m}$ with $\det M = \Delta \neq 0$. Our aim is to study the set

$$\text{Fin}_D(M) = \left\{ \sum_{k \in I} M^k d_k : I \text{ finite non-empty subset of } \mathbb{Z} \text{ and } d_k \in D \text{ for each } k \in I \right\} \subset \bigcup_{k \in \mathbb{N}} \frac{1}{\Delta^k} \mathbb{Z}^m.$$\[Notice that because the set of positions $I$ is finite, we can treat the position not in $I$ as being occupied by the zero digit. Therefore we always assume that zero is in the digit set $D$. We show that for any non-singular matrix $M \in \mathbb{Z}^{m \times m}$ there exists a finite set $D \subset \mathbb{Z}^m$ such that $\mathbb{Z}^m \subset \text{Fin}_D(M)$. For such $D$, we also characterise the matrices $M$ for which the sets $\text{Fin}_D(M)$ and $\bigcup_{k \in \mathbb{N}} \frac{1}{\Delta^k} \mathbb{Z}^m$ coincide.

2. The main result

This section contains the main result and its proof that relies heavily on Proposition 4. The proof of this proposition is, however, quite technical, therefore we provide it in its own Section 3.

The method of proving the main theorem is distinct from the one used in [7]. Their setting allowed the problem to be reduced to a problem in canonical number systems. This approach does not work in our case because we allow eigenvalues that are smaller than one in absolute value.

**Theorem 1.** Let $M \in \mathbb{Z}^{m \times m}$ be a non-singular matrix. Then there exists a finite digit set $A \subset \mathbb{Z}^m$ such that any vector $z \in \mathbb{Z}^m$ can be written in the form

$$z = \sum_{k \in I} M^k d_k, \quad \text{where } \emptyset \neq I \subset \mathbb{Z}, I \text{ finite and } d_k \in A \text{ for each } k \in I.$$ \[The property $\mathbb{Z}^m \subset \text{Fin}_D(M)$ easily implies a stronger property. Because $\text{Fin}_D(M)$ is closed under multiplication by $M^i$ for $i \in \mathbb{Z}$, we easily obtain the following corollary.

**Corollary 2.** Let $\mathbb{Z}^m \subset \text{Fin}_D(M)$, then we have $\bigcup_{n \in \mathbb{Z}} M^n \mathbb{Z}^m = \text{Fin}_D(M)$.

In particular, for each $M \in \mathbb{Z}^{m \times m}$ non-singular, there exists a finite digit set $D \subset \mathbb{Z}^m$, such that $\text{Fin}_D(M) = \bigcup_{n \in \mathbb{Z}} M^n \mathbb{Z}^m$.

The full proof of Theorem 1 is rather technical. In order to simplify readability, we put the technical part into Proposition 4 whose proof will be the content of Section 3. Nevertheless, we can now present the main idea of our work.

For $D \subset \mathbb{Z}^n$ finite, define $T_d(x) = Mx - d$ and $x_k = T_d(x_{k-1})$ with $x_0 = x$. We directly obtain

$$x = M^{-1}d_1 + M^{-2}d_2 + \cdots + M^{-k}d_k + M^{-k}x_k. \tag{1}$$

Assume that $S \subset \mathbb{Z}^m$ consists of vectors with the property that there exist $d_1, d_2, \ldots$ such that $\|x_i\| < C$ eventually (with the constant being universal for the whole $S$). Then the digit set $A = D + \{d \in \mathbb{Z}^n : ||d|| < C\}$ suffices for (1) to be a finite representation of members of $S$ over $A$. It then suffices to show that for any $x \in \mathbb{Z}^m$, $M^{-n}x \in S$ for some $n \in \mathbb{N}$. Indeed, we would have $M^{-n}x = \sum_{i=1}^k M^{-i}d_i$, i.e. $x = \sum_{i=1}^k M^{n-i}d_i$.

In the following, we will show that an appropriate choice of $S$ is all the elements of $\mathbb{Z}^m$ whose projection into the expansive eigenspace is bounded by a constant depending on $M$ only. Such a choice ensures that $M^{-n}x \in S$ is achievable for all $x \in \mathbb{Z}^m$. It then remains to be shown that with a suitable digit set, a sequence of iterations $T_{d_n}T_{d_{n-1}} \cdots T_{d_1}(x)$ will have a bounded norm eventually for any $x \in S$. The main obstacle are unimodular eigenvalues of $M$, in particular those with different algebraic and geometric multiplicity.

Our proof uses the real Jordan form $J$ of the matrix $M$. If $M = P^{-1}JP$,

$$z = \sum_{k \in I} M^k d_k \quad \text{gives} \quad Pz = \sum_{k \in I} J^k Pd_k. \tag{2}$$
Instead of looking for a finite digit set from the lattice \( \mathbb{Z}^m \) which is suitable for the matrix \( M \) and representation of integer vectors, we look for a digit set from the lattice \( P\mathbb{Z}^m \), suitable for \( J \) and we represent lattice points of \( P\mathbb{Z}^m \). The matrix \( P \) transforming \( M \) to its Jordan form is not determined uniquely. We choose it carefully to approximate in some sense the lattice \( \mathbb{Z}^m \). For this purpose, we introduce the perturbation set

\[
\mathcal{E} = \{ \varepsilon \in \mathbb{R}^m : \| \varepsilon \|_\infty < \frac{1}{4} \}.
\]

In the article we work with two norms of \( \mathbb{R}^m \):

- \( \| \cdot \|_\infty \) denotes the norm defined by \( \| (x_1, x_2, \ldots, x_m) \|_\infty = \max_i |x_i| \).
- \( \| \cdot \|_2 \) denotes the Euclidean norm, i.e. \( \| (x_1, x_2, \ldots, x_m) \|_2 = \sqrt{\sum_{j=1}^m |x_j|^2} \).

**Definition 3.** We say that a lattice \( L \subset \mathbb{R}^m \) is close to the lattice \( \mathbb{Z}^m \), if \( \mathbb{Z}^m \subset L + \mathcal{E} \).

For every \( M \in \mathbb{R}^{m \times m} \), there exists a non-singular \( P \in \mathbb{R}^{m \times m} \) such that \( PMP^{-1}M = J = \bigoplus_k J_k \) where \( J_k \) is a real Jordan block. Let us recall that the real Jordan block to \( \lambda \in \mathbb{C} \) is a matrix in the form

\[
\begin{pmatrix}
R & I \\
\vdots & \ddots & I \\
& \ddots & I \\
& & R
\end{pmatrix}
\]

with \( R = \begin{cases} 
(\lambda) & \text{if } \lambda \in \mathbb{R}, \\
(a, b) & \text{if } \lambda = a + ib \in \mathbb{C} \setminus \mathbb{R}
\end{cases} \).

Note that \( I \) is a unit matrix of order 1 or 2, according to the order of \( R \).

The vector space \( \mathbb{R}^m \) can be decomposed into invariant subspaces \( \mathbb{R}^m = V_e \oplus V_u \oplus V_c \) of the matrix \( J \) such that \( J \) restricted to \( V_e \) is expansive, \( J \) restricted to \( V_u \) is contractive and \( J \) restricted to \( V_c \) is orthogonal. In other words, all eigenvalues of \( J \) restricted to \( V_u \) are in modulus \( > 1 \), all eigenvalues of \( J \) restricted to \( V_u \) are in modulus \( < 1 \), and all eigenvalues of \( J \) restricted to \( V_c \) are in modulus \( = 1 \). The subspaces \( V_e, V_u \) and \( V_c \) may be trivial. Note that \( \dim V_e + \dim V_u > 0 \) as \( M \) is an integer non-singular matrix. Any \( x \in \mathbb{R}^m \) can be uniquely written as \( x = x_e + x_u + x_c \) with \( x_e \in V_e, x_u \in V_u \) and \( x_c \in V_c \).

We will denote \( x_e = \pi_e(x), x_u = \pi_u(x), \) and \( x_c = \pi_c(x) \).

The proof of the main theorem will be a consequence of the following proposition.

**Proposition 4.** Let \( J \in \mathbb{R}^{m \times m} \) be a non-singular matrix in the real Jordan form and \( \pi_e : \mathbb{R}^m \to V_e \) be the projection into the expansive invariant subspace \( V_e \) of \( J \). Let \( L \subset \mathbb{R}^m \) be a lattice close to \( \mathbb{Z}^m \). Then there exists a finite set \( D \subset L \) and a constant \( C \) such that for any \( x \in \mathbb{R}^m \) with \( \| \pi_e(x) \|_\infty \leq 1 \) and for any sufficiently large \( N \in \mathbb{N} \) we can write

\[
x = J^{-1}d_1 + J^{-2}d_2 + \cdots + J^{-N}d_N + J^{-N}y, \quad \text{for some } d_1, d_2, \ldots, d_N \in D
\]

and \( y \in \mathbb{R}^m, \) with \( \| y \|_\infty \leq C \).

**Proof of Theorem 1.** We use the statement and notation of Proposition 4. Let \( M = P^{-1}JP \), where \( J \) is the real Jordan canonical form of \( M \). The matrix \( P \) is not given uniquely. For example, any matrix \( \alpha P \) with a non-zero \( \alpha \) transforms \( M \) to its Jordan form, as well. Therefore we can assume without loss of generality that the lattice \( L = P\mathbb{Z}^m \) is close to the lattice \( \mathbb{Z}^m \) in the sense of Definition 3. As \( P^{-1}JP = M \), we get \( JP\mathbb{Z}^m = PM\mathbb{Z}^m \subset P\mathbb{Z}^m \), i.e., \( J \) maps the lattice \( L \) into \( L \). For such \( J \) and \( L \), we find by Proposition 4, the digit set \( D \subset L \).

Due to (2), we have to show that there exists a finite digit set \( A \subset L \) such that

\[
z \in L \quad \implies \quad z = \sum_{k \in I} J^k d_k, \quad \text{where } I \subset \mathbb{Z} \text{ is finite and } d_k \in A \text{ for each } k \in I.
\]

Define

\[
A := D + \mathcal{B}, \quad \text{where } \mathcal{B} := \{ d \in L : \| d \|_\infty \leq C \}.
\]

Obviously, \( \mathcal{B} \) is finite and \( 0 \in \mathcal{B} \). Therefore \( A \) is finite and \( D \subset A \subset L \). Let us show that our choice of the digit set \( A \) has the property stated in (4).
Let $z \in \mathbb{L}$. Since $V_\varepsilon$ is a contractive subspace of the matrix $J^{-1}$, there exists $j \in \mathbb{N}$ such that $\|\pi_e(J^{-j}z)\|_\infty < 1$. Applying (3) to $x = J^{-j}z$ with $N \geq j$, we find $d_1, \ldots, d_n \in D \subset \mathbb{A}$ and $y$ with $\|y\|_\infty \leq C$ such that

$$J^{-j}z = J^{-1}d_1 + J^{-2}d_2 + \cdots + J^{-N}d_n + J^{-N}y.$$  

Multiplying (5) by $J^N$, we deduce $y + d_N = J^{-j}z - \sum_{k=1}^{N} J^{N-k}d_k$. As $J$ maps the lattice $L$ into $L$, we have $y \in L$ and $y \in B$. Obviously, $y + d_n \in \mathbb{A}$. Altogether, $z = \sum_{k=1}^{N} J^{-k}d_k + Mj^{-N}(d_N + y)$, and all the coefficients $d_1d_2, \ldots, d_{N-1}$ and $d_N + y$ belong to $\mathbb{A}$.

\[ \square \]

3. Proof of Proposition 4

The proof of Proposition 4 will be done in following way. To the matrix $J$ in the real Jordan form, we find a finite digit set $D \subset \mathbb{R}$, such that the iterations of the transformations $Jx - d$ for some $d \in D$, have small norm eventually.

First we find a digit set in $\mathbb{Z}^m$ and then we replace each integer digit by a close element from the lattice $L$. When working with the matrix $J$, the integer lattice has an important advantage. It can be decomposed into the direct sum $\mathbb{Z}^m = \oplus_i \mathbb{Z}^{m_i}$ where each lattice $\mathbb{Z}^{m_i}$ is contained in an invariant subspace of a real Jordan block. Therefore, one can treat each Jordan block separately.

3.1. A real Jordan block to an eigenvalue $\lambda$ on the unit circle. Let us assume first that $\lambda$ is not real. The real Jordan block to such a $\lambda$ has an even size, say $2\ell$, and the form

$$J = \begin{pmatrix} R & I & \cdots & \cdots & \cdots \\ R & R & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ I & \cdots & \cdots & \cdots & R \end{pmatrix} \in \mathbb{R}^{2\ell \times 2\ell} \quad \text{with} \quad R = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition 5. Let $K_1, K_2, \ldots, K_\ell$ be positive constants and $J$ the matrix given in (6). We define the index $\text{ind} : \mathbb{R}^{2\ell} \to \{0, 1, \ldots, \ell\}$ in the following way. For $x = (x_1, x_2, \ldots, x_{2\ell-1}, x_{2\ell})^T$, we put

- $\text{ind}(x) = 0$, if $\|(x_{2i-1}, x_{2i})\|_2 < K_i$ for all $i = 1, 2, \ldots, \ell$;
- otherwise, $\text{ind}(x)$ is the maximal index $j$ such that $\|(x_{2j-1}, x_{2j})\|_2 \geq K_j$.

First we show two simple claims.

Claim 6. Given $q \in \mathbb{N}$, denote $B = \{(0, 0)^T, (3q, 0)^T, (0, 3q)^T, (-3q, 0)^T, (0, -3q)^T\} \subset \mathbb{R}^2$. Then for each $z \in \mathbb{R}^2$ there exists $b \in B$ such that

$$\|z - b\|_2 \leq 6q \quad \text{or} \quad \|z - b\|_2 \leq \|z\|_2 - q.$$

Proof. Because of the symmetry of the digit set $B$, it is enough to consider $z = (z_1, z_2)^T \in \mathbb{R}^2$ with $z_1 \geq z_2 \geq 1$.

If $z_1 \geq 3q$, put $b = (3q, 0)^T$. Then $\|z - b\|_2 = \sqrt{(z_1 - 3q)^2 + z_2^2}$. The inequality $\|z - b\|_2 \leq \|z\|_2 - q$ we want to show is equivalent to $4q + \sqrt{z_1^2 + z_2^2} < 3z_1$. The last inequality can be easily checked, since $4q < (3 - \sqrt{2})z_1$ and $\sqrt{z_1^2 + z_2^2} \leq \sqrt{2}z_1$.

If $z_1 < 3q$, put $b = (0, 0)^T$. Then $\|z - b\|_2 = \sqrt{z_1^2 + z_2^2} \leq \sqrt{(3q)^2 + (3q)^2} \leq 6q.$

Claim 7. Let a constant $c_1 \geq 0$ and $R = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ be given. Then there exist a constant $c_2 > 0$ and $B \subset \mathbb{Z}^2$ with $\#B = 5$ such that for any $u, v \in \mathbb{R}^2$, $\|u\|_2 \leq c_1$, there exists $b \in B$ such that

$$\|Ru + v - b\|_2 < c_2 - \frac{1}{2} \quad \text{or} \quad \|Ru + v - b\|_2 < \|u\|_2 - 1.$$
Lemma 8. Let $J$ be the matrix given in (6). There exist a finite digit set $D \subset \mathbb{Z}^{2\ell}$ and constants $K_1, \ldots, K_{\ell}$, such that the function $\text{ind}$ defined by $K_1, \ldots, K_{\ell}$ has the following property.

For each $x \in \mathbb{R}^{2\ell}$ there exists a digit $d \in D$ such that for all $\varepsilon \in \mathcal{E}$, the vector $y = Jx - d + \varepsilon$ satisfies

- $\text{ind}(y) \leq \text{ind}(x)$;
- If $\text{ind}(y) = \text{ind}(x) = j$, then $\|(y_{2j-1}, y_{2j})\|_2 \leq \|(x_{2j-1}, x_{2j})\|_2 - \frac{1}{2}$.

Proof. Consider the matrix $J \in \mathbb{R}^{2\ell \times 2\ell}$ given in (6). Using $\ell$ times Claim 7, we will construct $\ell$ constants $K_1, \ldots, K_{\ell}$ to determine the function $\text{ind}$:

- by $K_{\ell}$ and $\mathcal{B}_{\ell}$ we denote the constant $c_2$ and the digit set found by Claim 7 for $c_1 = 0$;
- by $K_{\ell-1}$ and $\mathcal{B}_{\ell-1}$ we denote the constant $c_2$ and the digit set found by Claim 7 for $c_1 = K_{\ell}$;
- by $K_{\ell-2}$ and $\mathcal{B}_{\ell-2}$ we denote the constant $c_2$ and the digit set found by Claim 7 for $c_1 = K_{\ell-1}$;
- etc.

The digit set $D$ is defined by

$$
(7) \quad d \in D \text{ if and only if } d = \begin{pmatrix} b^{(1)}_1 \\ b^{(2)}_1 \\ \vdots \\ b^{(\ell)}_1 \end{pmatrix} \in \mathbb{Z}^{2\ell}, \text{ where } b^{(k)} \in \mathcal{B}_k, \text{ for } k = 1, 2, \ldots, \ell.
$$

We show that the function $\text{ind}$ defined by the constants $K_1, \ldots, K_{\ell}$ and the digit set $D$ have the property declared in the statement of the lemma.

Let $x \in \mathbb{R}^{2\ell}$. Then

$$
\begin{pmatrix} (Jx)_{2i-1} \\ (Jx)_{2i} \end{pmatrix} = R \begin{pmatrix} x_{2i-1} \\ x_{2i} \end{pmatrix} + \begin{pmatrix} x_{2i+1} \\ x_{2i+2} \end{pmatrix}, \text{ for } i = 1, 2, \ldots, \ell, \text{ where we put } \begin{pmatrix} x_{2\ell+1} \\ x_{2\ell+2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

To this $x$, we define $d \in D$ by determining all its components $b^{(i)}$.

- If $i < \text{ind}(x)$, then we choose $b^{(i)} \in \mathcal{B}_i$ randomly.
- If $i \geq \text{ind}(x)$, then $\|(x_{2i+1}, x_{2i+2})\|_2 < K_{i+1}$ and $b^{(i)} \in \mathcal{B}_i$ is determined by Claim 7 for $u = (x_{2i-1}, x_{2i})^T$ and $v = (x_{2i+1}, x_{2i+2})^T$. Recall that we used Claim 7 to find $K_i := c_2$ for $c_1 := K_{i+1}$.

Therefore,

$$
(8) \quad \left\| \begin{pmatrix} (Jx)_{2i-1} \\ (Jx)_{2i} \end{pmatrix} - b^{(i)} \right\|_2 < K_i - \frac{1}{2} \quad \text{or} \quad \left\| \begin{pmatrix} (Jx)_{2i-1} \\ (Jx)_{2i} \end{pmatrix} - b^{(i)} \right\|_2 < \left\| \begin{pmatrix} x_{2i-1} \\ x_{2i} \end{pmatrix} \right\|_2 - 1.
$$

For the digit $d$ we have described above and an $\varepsilon \in \mathcal{E}$, we focus on $y = Jx - d + \varepsilon$. Let us realize that the inequality $\|\varepsilon\|_\infty < \frac{1}{4}$ means $|\varepsilon_j| \leq \frac{1}{4}$ for each coordinate $j$ and thus $\|(\varepsilon_{2i-1}, \varepsilon_{2i})\|_2 \leq \frac{1}{2}$.

For each $i > \text{ind}(x)$, we have $\|(x_{2i-1}, x_{2i})\|_2 < K_i$, and thus both inequalities in (8) imply

$$
\left\| \begin{pmatrix} (Jx)_{2i-1} \\ (Jx)_{2i} \end{pmatrix} - b^{(i)} \right\|_2 + \left\| \begin{pmatrix} \varepsilon_{2i-1} \\ \varepsilon_{2i} \end{pmatrix} \right\|_2 < K_i.
$$

Consequently, $\text{ind}(y) \leq \text{ind}(x)$.

If $\text{ind}(y) = \text{ind}(x) =: j$, then necessarily $\|(y_{2j-1}, y_{2j})\|_2 = \left\| \begin{pmatrix} (Jx)_{2j-1} \\ (Jx)_{2j} \end{pmatrix} - b^{(j)} \right\|_2 + \left\| \begin{pmatrix} \varepsilon_{2j-1} \\ \varepsilon_{2j} \end{pmatrix} \right\|_2 > K_j$ and the left most inequality in (8) cannot hold true. The validity of the right most inequality implies $\|(y_{2j-1}, y_{2j})\|_2 \leq \|(x_{2j-1}, x_{2j})\|_2 - \frac{1}{2}$.

In case of real $\lambda$, that is, $\lambda = \pm 1$, the approach is analogous, only easier. The digit set $\{(a_1, \ldots, a_m)^T : a_i \in \{-1, 0, 1\}\}$ now suffices for the analogy Lemma 8. \qed
Corollary 9. Let \( J = J_m(\lambda) \) with \(|\lambda| = 1\). Then there exists \( \mathcal{D} \subset \mathbb{Z}^m \) finite with the following property. For each \( x \in \mathbb{R}^m \) there exists a sequence \((d_{i+1})_{i \in \mathbb{N}}\) of digits from \( \mathcal{D} \) such that for any sequence \((\varepsilon_{i+1})_{i \in \mathbb{N}}\) from the perturbation set \( \mathcal{E} \), the sequence defined recursively
\[
  x^{(0)} = x \quad \text{and} \quad x^{(n+1)} := Jx^{(n)} - d_{n+1} + \varepsilon_{n+1} \quad \text{for } n \in \mathbb{N}
\]
satisfies \( \text{ind}(x^{(N)}) = 0 \) for all sufficiently large \( N \in \mathbb{N} \). In particular, there exists a constant \( C \) such that \( ||x^{(N)}||_\infty < C \) for all sufficiently large \( N \in \mathbb{N} \).

3.2. A real Jordan block to an eigenvalue \( \lambda \) strictly inside the unit circle.

Lemma 10. Let \( J \in \mathbb{R}^{m \times m} \) be the real Jordan block to \( \lambda \) of modulus \( < 1 \) and \( \mathcal{E} \) be the perturbation set. Then there exist a norm \( ||.||_c \) of \( \mathbb{R}^m \) and a constant \( \gamma \) such that

for any \( x \in \mathbb{R}^m \) and any \( \varepsilon \in \mathcal{E} \), the vector \( y = Jx + \varepsilon \) satisfies:

- If \( ||x||_c \leq \gamma \), then \( ||y||_c \leq \gamma \);
- If \( ||x||_c \geq \gamma \), then \( ||y||_c \leq ||x||_c - \frac{1}{2} \).

Proof. Let us choose \( \beta \) such that \( 1 > \beta > |\lambda| \). By Theorem 3 from [6], there exists a norm \( ||x||_c \) of \( \mathbb{R}^m \) such that \( ||Jz||_c \leq \beta ||z||_c \) for all \( z \in \mathbb{R}^m \). Then
\[
  ||Jx + \varepsilon||_c \leq ||Jx||_c + ||\varepsilon||_c \leq \beta ||x||_c + E \quad \text{where } E = \max\{||\varepsilon||_c : \varepsilon \in \mathcal{E}\}.
\]

Now, it is enough to check that
\[
  ||x||_c \leq \frac{1}{1-\beta} \left( \frac{1}{2} + E \right), \text{ implies } \beta ||x||_c + E \leq \frac{1}{1-\beta} \left( \frac{1}{2} + E \right)
\]
and
\[
  ||x||_c \geq \frac{1}{1-\beta} \left( \frac{1}{2} + E \right), \text{ implies } \beta ||x||_c + E \leq ||x||_c - \frac{1}{2}.
\]

Therefore, we put \( \gamma = \frac{1}{1-\beta} \left( \frac{1}{2} + E \right) \).

Corollary 11. Let \( J \in \mathbb{R}^{m \times m} \) be the real Jordan block to \( \lambda \) of modulus \( < 1 \). Then there exists a constant \( \gamma \) and a norm \( ||x||_c \) of \( \mathbb{R}^m \) such that for each \( x \in \mathbb{R}^m \) and for any sequence \((\varepsilon_{i+1})_{i \in \mathbb{N}}\) from \( \mathcal{E} \), the sequence defined recursively
\[
  x^{(0)} = x \quad \text{and} \quad x^{(n+1)} := Jx^{(n)} + \varepsilon_{n+1} \quad \text{for } n \in \mathbb{N}
\]
satisfies \( ||x^{(N)}||_c < \gamma \) for all sufficiently large \( N \in \mathbb{N} \).

3.3. A real Jordan block to an eigenvalue \( \lambda \) strictly outside the unit circle.

Lemma 12. Let \( J \in \mathbb{R}^{m \times m} \) be the real Jordan block to \( \lambda \) of modulus \( > 1 \) and \( \mathcal{E} \) be the perturbation set. Then there exists a digit set \( \mathcal{D} \subset \mathbb{Z}^m \) with the property:

for each \( x \in \mathbb{R}^m \) with \( ||x||_\infty \leq 1 \) there exists a digit \( d \in \mathcal{D} \) such that for all \( \varepsilon \in \mathcal{E} \), the vector \( y = Jx - d + \varepsilon \) satisfies \( ||y||_\infty \leq 1 \).

Proof. Since \( ||Jx||_\infty \leq (2|\lambda| + 1)||x||_\infty \) for each \( x \in \mathbb{R}^m \), we put \( \mathcal{D} = \{d \in \mathbb{Z}^m : ||d||_\infty \leq 2|\lambda| + 2 + E\} \), where \( E = \max\{||\varepsilon||_\infty : \varepsilon \in \mathcal{E}\} \).

Corollary 13. Let \( J \in \mathbb{R}^{m \times m} \) be the real Jordan block to \( \lambda \) of modulus \( > 1 \), and \( \mathcal{D} \) and \( \mathcal{E} \) be as in Lemma 12. Then for each \( x \in \mathbb{R}^m \) there exists a sequence \((d_{i+1})_{i \in \mathbb{N}}\) of digits from \( \mathcal{D} \) such that for any sequence \((\varepsilon_{i+1})_{i \in \mathbb{N}}\) from \( \mathcal{E} \) the sequence defined recursively
\[
  x^{(0)} = x \quad \text{and} \quad x^{(n+1)} := Jx^{(n)} - d_{n+1} + \varepsilon_{n+1} \quad \text{for } n \in \mathbb{N},
\]
satisfies \( ||x^{(N)}||_\infty < 1 \) for all sufficiently large \( N \in \mathbb{N} \).
3.4. Completion of proof of Proposition 4. Now we combine properties of all types of the real Jordan blocks.

Proof of Proposition 4. Let $J_1, J_2, \ldots, J_s$ be the real Jordan blocks of the size $m_1, \ldots, m_s$ respectively, such that $J = J_1 \oplus J_2 \oplus \cdots \oplus J_s$.

To each block $J_k \in \mathbb{R}^{m_k}$ corresponding to $\lambda$ with $|\lambda| \geq 1$ we find a constant $C_k$ and a finite digit set $\mathcal{D}_k \subset \mathbb{Z}^{m_k}$ with the properties described in Corollaries 13 and 9.

To each block $J_k$ corresponding to $\lambda$ with $|\lambda| < 1$, we assign the constant $\gamma_k$ found in Corollary 11 for the norm $\| \cdot \|_c$. As two norms on a finite dimensional space are equivalent, there exists a constant $C_k$ such that $\| x \|_c < \gamma_k$ implies $\| x \|_< < C_k$. We put $\mathcal{D}_k = \{0\} \subset \mathbb{R}^{m_k}$.

We use the digit sets $\mathcal{D}_k$ for construction of a new digit set $\tilde{\mathcal{D}} \subset \mathbb{Z}^m$

\begin{equation}
\tilde{d} \in \tilde{\mathcal{D}} \quad \text{if and only if} \quad \tilde{d} = \left( \begin{array}{c}
d^{(1)}_1 \\
d^{(2)}_2 \\
\vdots \\
d^{(s)}_s 
\end{array} \right) \in \mathbb{Z}^m, \quad \text{where } d^{(k)} \in \mathcal{D}_k, \text{ for } k = 1, 2, \ldots, s.
\end{equation}

Put $C = \max_k C_k$. In this notation, we get the following.

Claim A: for each $x \in \mathbb{R}^m$ with $\| \pi_c(x) \|_\infty \leq 1$ there exists a sequence $(\tilde{d}_i)_{i \in \mathbb{N}}$ of digits from $\tilde{\mathcal{D}}$ such that for any sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ from $\mathcal{E}$ the sequence defined recursively

\begin{equation}
x^{(0)} = x \quad \text{and} \quad x^{(n+1)} := Jx^{(n)} - \tilde{d}_{n+1} + \varepsilon_{n+1} \quad \text{for } n \in \mathbb{N},
\end{equation}

satisfies $\| x^{(N)} \|_\infty < C$ for all sufficiently large $N \in \mathbb{N}$.

Since $L$ is close to $\mathbb{Z}^m$, we can assign to each $\tilde{d} \in \tilde{\mathcal{D}}$ a lattice point $d \in L$ such that $d = \tilde{d} + \mathcal{E}$. We obtain a new digit set $\mathcal{D} \subset L$ and the size of $\mathcal{D}$ does not exceed the size of $\tilde{\mathcal{D}}$. We rewrite Equation (10) for the specific choice of the sequence $(\varepsilon_i)_{i \in \mathbb{N}}$, namely for $\varepsilon_i := \tilde{d}_i - d_i \in \mathcal{E}$. We get a modification of the previous claim

Claim B: for each $x \in \mathbb{R}^m$ with $\| \pi_c(x) \|_\infty \leq 1$ there exists a sequence $(d_i)_{i \in \mathbb{N}}$ of digits from $\mathcal{D}$ such that the sequence defined recursively

\begin{equation}
x^{(0)} = x \quad \text{and} \quad x^{(n+1)} := Jx^{(n)} - d_{n+1} \quad \text{for } n \in \mathbb{N},
\end{equation}

satisfies $\| x^{(N)} \|_\infty < C$ for all sufficiently large $N \in \mathbb{N}$.

Multiplying the equality $x^{(n+1)} = Jx^{(n)} - d_{n+1}$ by $J^{-n-1}$ and summing up for $n = 0, 1, \ldots, N-1$ we get

\begin{equation}
\sum_{n=0}^{N-1} J^{-n-1}x^{(n+1)} = \sum_{n=0}^{N-1} J^{-n}x^{(n)} - \sum_{n=0}^{N-1} J^{-n}d_{n+1} \quad \text{and thus} \quad \sum_{n=1}^{N} J^{-n}x^{(n)} = \sum_{n=0}^{N-1} J^{-n}x^{(n)} - \sum_{n=1}^{N} J^{-n}d_{n}.
\end{equation}

It implies $J^{-N}x^{(N)} = x^{(0)} - \sum_{n=1}^{N} J^{-n}d_{n}$ with $\| x^{(N)} \|_\infty < C$, as required in the statement of the proposition.

\[ \Box \]

4. $\operatorname{Fin}_D(M)$ VERSUS $\bigcup_{k \in \mathbb{N}} \frac{1}{\Delta} \mathbb{Z}^m$

In general, we have that $\operatorname{Fin}_D(M) \subseteq \bigcup_{k \in \mathbb{N}} \frac{1}{\Delta} \mathbb{Z}^m$. In this section we describe when the inclusion turns into an equality. The inverse to a matrix $M \in \mathbb{Z}^{m \times m}$ with the determinant $\Delta \neq 0$ belongs to $\frac{1}{\Delta} \mathbb{Z}^{m \times m}$. Therefore, the definition of the set $\operatorname{Fin}_D(M)$ directly implies the following properties.

Lemma 14. Let $M \in \mathbb{Z}^{m \times m}$ with $\det M = \Delta \neq 0$ and $\mathcal{D} \subset \mathbb{Z}^m$, $\mathcal{D}$ finite. Then

(1) $M^k(\operatorname{Fin}_D(M)) = \operatorname{Fin}_D(M)$ for each $k \in \mathbb{Z}$.

(2) $\operatorname{Fin}_D(M) \subseteq \bigcup_{k \in \mathbb{N}} \frac{1}{\Delta} \mathbb{Z}^m$.

(3) If $\mathbb{Z}^m \subset \operatorname{Fin}_D(M)$, then $\operatorname{Fin}_D(M)$ is closed under addition and subtraction.
Any classical $b$-ary numeration system, with base $b \in \mathbb{N}, b \geq 2$ and the canonical digit set $\mathcal{D} = \{0, 1, \ldots, b-1\}$ can be considered as a matrix system with $M = b \in \mathbb{Z}^{1 \times 1}$ and the determinant $\Delta = b$. The canonical digit set allows to represent only non-negative numbers and obviously $\text{Fin}_\mathcal{D}(M) = \bigcup_{k \in \mathbb{N}} \frac{1}{k} \mathbb{Z}^n$.

In the next auxiliary claim we denote by $e_i \in \mathbb{Z}^m$ the vector $(\delta_{i1}, \delta_{i2}, \ldots, \delta_{im})^T$, where $\delta_{ij}$ is the Kronecker symbol, i.e. $\delta_{ij} = 0$, if $i \neq j$ and $\delta_{ii} = 1$.

**Claim 15.** $\text{Fin}_\mathcal{D}(M) = \bigcup_{k \in \mathbb{N}} \frac{1}{k} \mathbb{Z}^m$ if and only if for each $i \in \{1, 2, \ldots, m\}$ there exists $\ell_i \in \mathbb{N}$ such that $\frac{1}{k} e_i \in M^{-\ell_i}(\mathbb{Z}^m)$.

**Proof.** $(\Leftarrow)$ Due to Item (2) of Lemma 14, it is enough to show that $\frac{1}{k} \mathbb{Z}^m \subset \text{Fin}_\mathcal{D}(M)$ for each $k \in \mathbb{N}$. We show it by induction on $k \in \mathbb{N}$.

The assumption $\frac{1}{k} e_i \in M^{-\ell_i}(\mathbb{Z}^m)$ gives $\frac{1}{k} \mathbb{Z}^m \subset \sum_{i=1}^{m} M^{-\ell_i}(\mathbb{Z}^m)$. Multiplying this inclusion by $\frac{1}{k}$ and using the induction hypothesis we obtain
\[
\frac{1}{k^2} \mathbb{Z}^m \subset \sum_{i=1}^{m} M^{-\ell_i}(\frac{1}{k} \mathbb{Z}^m) \subset \sum_{i=1}^{m} M^{-\ell_i}(\text{Fin}_\mathcal{D}(M)) = \text{Fin}_\mathcal{D}(M).
\]
The last equality follows from Items (1) and (3) of Lemma 14.

$(\Rightarrow)$ This implication is obvious. \hfill $\square$

**Theorem 16.** Let $M \in \mathbb{Z}^{m \times m}$, $\det M = \Delta \neq 0$. Let $\mathcal{D} \subset \mathbb{Z}^m$ be a finite digit set such that $\mathbb{Z}^m \subset \text{Fin}_\mathcal{D}(M)$. Then
\[
\text{Fin}_\mathcal{D}(M) = \bigcup_{k \in \mathbb{N}} \frac{1}{k} \mathbb{Z}^m \iff \exists \ell \in \mathbb{N} \text{ such that } M^\ell = \Theta \mod \Delta.
\]
In particular, if $\det M = \pm 1$, then $\mathbb{Z}^m = \text{Fin}_\mathcal{D}(M)$.

**Proof.** By Claim 15 we have to study when $\frac{1}{k} e_i \in M^{-\ell_i}(\mathbb{Z}^m)$ for some $\ell_i \in \mathbb{N}$. It can be rewritten into $M^{\ell_i} e_i \in \Delta \mathbb{Z}^m$, or equivalently, $M^{\ell_i} e_i = 0 \mod \Delta$. We denote the $i^{th}$ column of a matrix $A$ by $A_{*i}$. Obviously, $M^{\ell_i} e_i = (M^{\ell_i})_{*i}$.

Let us realize that if $(M^{\ell_i})_{*i} = 0 \mod \Delta$, then $(M^{\ell_i})_{*i} = 0 \mod \Delta$ for all $\ell \in \mathbb{N}, \ell > \ell_i$. Indeed, $(M^{\ell_i})_{*i} = M^{\ell-i}(M^{\ell_i})_{*i} = M^{\ell-i} 0 \mod \Delta$. Therefore, we can look for a common exponent $\ell$ for all columns of the matrix $M$. \hfill $\square$

**Example 17.** Let $M_1 = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$. Then $\det M_1 = 2$ and $\det M_2 = -2$.

\[
M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mod \Delta, \quad M_1^2 = \Theta \mod \Delta \quad \text{and thus} \quad \text{Fin}_\mathcal{D}(M) = \bigcup_{k \in \mathbb{N}} \frac{1}{k} \mathbb{Z}^2
\]

for a suitable $\mathcal{D} \subset \mathbb{Z}^2$.

On the other hand,

\[
M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mod \Delta, \quad \text{and thus} \quad M_2^\ell \neq \Theta \mod \Delta \text{ for all } \ell \in \mathbb{N}.
\]

In particular, $\frac{1}{2} \notin \text{Fin}_\mathcal{D}(M_2)$ for any choice $\mathcal{D} \subset \mathbb{Z}^2$. 
The main idea of this paper comes from [17] concerning number systems with algebraic base without any restrictions on its Galois conjugates. We do not give an explicit bound on the size of the digit set, even though our proof is a constructive one. For matrices without eigenvalues on the unit circle, the method used in [4],[3] can be used to obtain explicit digit sets, as well as arithmetic algorithms.

To find a digit set $\mathcal{D} \subset \mathbb{Z}^n$ of the minimal cardinality may be a hard problem. It is known that if $M$ is expanding, then we need a digit set of size at least $|\det M|$ to be able to represent integer vectors in the form $\sum_{k=0}^{n} M^k d_k$. In our case it seems that an appropriate lower bound would be around the product of absolute values of the eigenvalues outside the unit circle.

In any case, the digit set needs to have at least one digit beside zero. Two digits might be already enough for some matrices of dimension two or more, as is illustrated by the following examples.

The first example is a case of a matrix having only one eigenvalue outside the unit circle. It can be seen from the proof of Proposition 4 that this dominant eigenvalue will determine the size of the digit set. We will, however, utilize the connection between matrix numeration systems and those with real base.

Example 18. Let $T = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$. As $|\det T| = -1$, we have $\text{Fin}_\mathcal{D}(T) = \mathbb{Z}^3$ for a suitable digit set guaranteed by Theorem 1. The characteristic polynomial of $T$ is $x^3 + x^2 - x + 1$. $T$ has one negative eigenvalue $\beta = -1.839 \ldots$ and a pair of complex conjugate eigenvalues in modulus smaller than 1. As shown in [14], any element of the ring $\mathbb{Z}[\beta] = \{a + b\beta + c\beta^2 : a, b, c \in \mathbb{Z}\}$ can be written as

$$a + b\beta + c\beta^2 = \sum_{i=k}^{n} a_i \beta^i,$$

where $k, n \in \mathbb{Z}, k \leq n$ and $a_i \in \{0, 1\}$.

We use the isomorphism $\psi : \mathbb{Z}[\beta] \to \mathbb{Z}^3$ of two additive groups given by $\psi(a + b\beta + c\beta^2) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Since $\beta(a + b\beta + c\beta^2) = -c + (a + c)\beta + (b - c)\beta^2$, we have

$$\psi(\beta(a + b\beta + c\beta^2)) = \begin{pmatrix} c \\ a + c \\ b - c \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = T \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Applying this rule to Equation (11) we deduce that any element of $\mathbb{Z}^3$ can be expressed as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \psi(a + b\beta + c\beta^2) = \sum_{i=k}^{n} \psi(\beta^i a_i) = \sum_{i=k}^{n} T^i \psi(a_i) = \sum_{i=k}^{n} T^i \begin{pmatrix} a_i \\ 0 \\ 0 \end{pmatrix}.$$

In other words, $\text{Fin}_\mathcal{A}(T) = \mathbb{Z}^3$ already for the small digit set $\mathcal{A} = \{(0, 0, 0)^T, (1, 0, 0)^T\}$.

The second example is a rotation matrix in $\mathbb{R}^2$. Notice that in this case the link to real/complex base numeration systems is missing, for there is no eigenvalue of modulus $> 1$. Moreover, the actual minimal digit set is significantly smaller than the one ensured by the the proof of Proposition 4.

Example 19. Consider $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with the digit set $\mathcal{D} = \{(0, 0)^T, (1, 0)^T\}$.

We have the identities

$$M^{4k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M^{4k+1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M^{4k+2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M^{4k+3} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence

$$M^{4k} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad M^{4k+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad M^{4k+2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad M^{4k+3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
It is now easy to see how the representations are created. For instance, if $a, b > 0$, then

\[
\begin{pmatrix} a \\ b \end{pmatrix} = \sum_{k=1}^{a} M^{4k} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k=1}^{b} M^{4k+3} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

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