Manifestation of Hamiltonian chaos in an open quantum system with ballistic atoms in an optical lattice

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Manifestation of dynamical instability and Hamiltonian chaos in the fundamental near-resonant matter-radiation interaction has been found analytically and in a Monte Carlo simulation in the behavior of atoms moving in a rigid optical lattice. Character of diffusion of spontaneously emitting atoms changes abruptly in the range of the values of parameters and initial conditions where their Hamiltonian dynamics is shown to be chaotic.

Atoms in an optical lattice, formed by a laser standing wave, is an ideal system for studying quantum nonlinear dynamics. Operating at low temperatures and controlling the lattice parameters, experimentalists now are able to tailor practically one-dimensional potentials and manipulate with internal and external degrees of freedom of atoms. Experimental study of quantum chaos has been carried out with ultracold atoms interacting with a periodically modulated optical lattice [1]. To suppress spontaneous emission and provide Hamiltonian quantum dynamics atoms are detuned far from the optical resonance. Adiabatic elimination of the excited state amplitude leads to an effective Hamiltonian for the external motion [2] corresponding to a 3/2 degree-of-freedom classical system which has a mixed phase space with regular islands embedded in a chaotic sea. De Broglie waves of ultracold atoms have been shown to demonstrate under appropriate conditions the effect of dynamical localization which means quantum suppression of chaotic diffusion [1,2]. Decoherence due to spontaneous emission tends to suppress this quantum effect and restore classical-like dynamics [3].

A new arena of quantum nonlinear dynamics with atoms in optical lattices is opened if we work near the optical resonance and take the internal dynamics into account. In the Hamiltonian approximation, when one neglects spontaneous emission (SE), the coupling of internal and external atomic degrees of freedom has been shown to produce a number of nonlinear effects in rigid (i.e. without any modulation) optical lattices: chaotic Rabi oscillations, chaotic atomic transport, dynamical fractals, and Lévy flights [4,5]. In real life the dynamics of atoms in near-resonant laser fields is not deterministic because of SE. The problem of interrelation between deterministic chaos and noise is rather general. Natural systems are subject to noise which, usually, acts continuously. If noise is practically continuous and comparatively weak we can study in which way it affects chaotic deterministic evolution of the system under consideration. Spontaneous emission is a kind of a shot noise which is not small because SE recoils may change the internal state significantly. In this Letter we demonstrate analytically and in a Monte Carlo simulation that manifestation of dynamical instability and Hamiltonian chaos in the fundamental near-resonant matter-radiation interaction can be found in the behavior of atoms moving in a rigid optical lattice.

Since we study manifestation of quantum nonlinear effects in ballistic transport of atoms, when the average atomic momentum is very large as compared with the photon momentum $\hbar k_f$, the translational motion is described classically by Hamilton equations. We start with the Hamilton-Schrödinger equations of motion for a two-level atom in a standing light wave which have been derived in Refs. [1,7]:

$$\dot{x} = \omega_r p, \quad \dot{p} = -u \sin x + \sum_{j=1}^{\infty} p_j \delta(\tau - \tau_j),$$

$$\dot{u} = \Delta v + \frac{\gamma}{2} u z - u \sum_{j=1}^{\infty} \delta(\tau - \tau_j),$$

$$\dot{v} = -\Delta u + 2z \cos x + \frac{\gamma}{2} v z - v \sum_{j=1}^{\infty} \delta(\tau - \tau_j),$$

$$\dot{z} = -2v \cos x - \frac{\gamma}{2} (u^2 + v^2) - (z + 1) \sum_{j=1}^{\infty} \delta(\tau - \tau_j),$$

(1)

where $x \equiv k_f X$ and $p \equiv P / \hbar k_f$ are normalized atomic center-of-mass position and momentum, $u$, $v$, and $z$ are synphase and quadrature components of the atomic electric dipole moment and the population inversion, respectively. The length of the Bloch vector, $u^2 + v^2 + z^2 = 1$, is conserved. The dot denotes differentiation with respect to the normalized time $\tau = \Omega t$. The values of the normalized decay rate $\gamma \equiv \Gamma / \Omega$ and the recoil frequency $\omega_r \equiv \hbar k_f^2 / m_a \Omega$ are chosen to be $\gamma = 3.3 \cdot 10^{-3}$ and $\omega_r = 10^{-5}$ and correspond to a cesium atom ($\lambda_a = 852.1$ nm and $\Gamma = 3.2 \cdot 10^7$ s$^{-1}$) in a strong field with the Rabi frequency $\Omega = 10^{10}$ s$^{-1}$. So, the normalized detuning between the field and atomic...
frequencies, $\Delta \equiv (\omega_f - \omega_a)/\Omega$, is a single variable parameter. In Eqs. (1), $\tau_j$ are random time moments of SE events and $p_j$ are random recoil momenta with the values between $\pm 1$ (1D case). In terms of the normalized time $\tau$ the mean frequency of SE events is $\gamma(z+1)/2$. At $\tau = \tau_j$, the atomic variables change as follows: $p \rightarrow p + p_j, u \rightarrow 0, v \rightarrow 0, z \rightarrow -1$.

Equations (1) with $\gamma = 0$ and without the terms containing delta-functions describe Hamiltonian coherent evolution of the internal and external degrees of freedom of an atom that has been shown [4] to be chaotic (in the sense of exponential sensitivity to small changes in initial conditions) in certain ranges of values of the parameters $\omega_r$ and $\Delta$ and initial momenta. With comparatively small values of the initial atomic momentum $p_0$, atoms may wander in an optical lattice with alternating trapping in the wells of the optical potential and flights over its hills. It is a kind of a random walking that may occur without any modulation of the lattice parameters and/or any noise like SE [4, 5].

In this work we consider only fast ballistic atoms which never change the direction of motion. There is a range of large values of initial momentum $p_0$ where the maximal Lyapunov exponent $\lambda$ of the Hamiltonian equations of motion has been computed to be positive [5]. It means that the momentum of a ballistic atom without SE may oscillate in a deterministic way around a mean value $\langle p \rangle$. The central question of the present study is the following. In which way the Lyapunov instability and Hamiltonian chaos, that may occur between SE events, manifest itself in ballistic atomic transport which is a stochastic process due to SE?

To answer the question we simulate Eqs. (1) by a Monte Carlo method (for details see [6]) and compute atomic trajectories in the momentum space to find the momentum diffusion coefficient $D_p$, as a function of the momentum $p$. The results are compared with the maximal Lyapunov exponent $\lambda$ computed with the Hamiltonian analogue of the set (1) (without SE). More exactly, as a measure of Hamiltonian chaos, we compute chaos probability $\Lambda \equiv \langle \theta(\lambda - 1) \rangle$, where $\theta(\lambda)$ is a Heaviside function, which is equal to 0 for $\lambda < 0$, 1/2 for $\lambda = 0$, and 1 for $\lambda > 0$. The values of $\Lambda$ in Fig. 1 have been computed by averaging over many atomic trajectories with close values of $p$. If $\lambda > 0$ with all those atoms, then $\Lambda = 1$, and we have Hamiltonian chaos with probability 1. If $\lambda = 0$ then $\Lambda = 0$, and the motion is regular with the probability 1. The values in the range $0 < \Lambda < 1$ mean that chaotic and regular trajectories are mixed in a small range of values of initial momenta, and chaos probability is proportional to the fraction of atoms with positive $\Lambda$. Fig. 1 demonstrates a correlation between the regimes of chaotic (regular) Hamiltonian transport and the behavior of the momentum diffusion coefficient $D_p$, for spontaneously emitting atoms. Beginning with those values of the momentum $p$, for which the probability of Hamiltonian chaos becomes smaller than 1 (see the lower panels in Fig. 1), one observes an abrupt transition to a more regular regime of motion with another law of decay of $D_p$ (see the upper panels in Fig. 1).

We stress that the atomic transport in reality is stochastic with all the values of $p$ due to SE, but the measure of its stochasticity, $D_p$, decays rapidly in the same range of the momenta where the Hamiltonian analogue of the system demonstrates a transition from chaos to order. It is more important that this difference could be measured in real experiments and would provide us with direct signatures of atomic Hamiltonian chaos in terms of transport characteristics which are more easy to measure than the Rabi oscillations.

In what follows we estimate the diffusion coefficient $D_p$ when the corresponding Hamiltonian ballistic transport is chaotic and regular. In the weak Raman-Nath approximation, $\omega_r p^2/2 \gg |u \cos x + \Delta z/2|$, when the atomic kinetic energy is not strictly a constant, but much larger than the potential one, the momentum fluctuations between SE are small. In Ref. [7] we have shown that at small detunings, $|\Delta| < 1$, the evolution of the total energy $H \equiv \omega_r p^2/2 - u \cos x - \Delta z/2$ (which is a constant in the absence of dissipation) is a quasi-random process with sudden changes in $H$, when SE occurs, and a slight linear drift in between. We can treat the evolution of $H$ as the following mapping:

$$H_j = H_{j-1} + \omega_r p(\tau_j)p_j + \frac{\omega_r}{2}p_j^2 + \frac{\lambda}{2} + u(\tau_j) \cos x(\tau_j) + \frac{\Delta}{2} \omega_r z(\tau_j) + \frac{\Delta \gamma}{4} (1 - |z|^2)(\tau_j - \tau_{j-1}),$$

where $H_j$ is a value of the energy just after $j$-th SE, $x(\tau_j), u(\tau_j), z(\tau_j)$, and $p(\tau_j)$ are values of the corresponding variables just before $j$-th SE which are determined by the evolution between SE events. The last term with the averaging over a time exceeding the period of the Rabi oscillations is a result of an energy drift between SE events. In general, this random walk is asymmetric. There is a friction force $F \equiv \langle \dot{p} \rangle$ which can accelerate or decelerate atoms in average. The measure of momentum fluctuations in an atomic flight (duration of which exceeds largely the range of the momenta where the Hamiltonian ballistic transport is chaotic) becomes smaller than 1 (see the range of the momenta where the Hamiltonian ballistic transport is chaotic) when

$$D_p \approx \frac{\langle (H_j - H_{j-1})^2 \rangle - \langle H_j - H_{j-1} \rangle^2}{2\omega_r p^2(\tau_j - \tau_{j-1})^2},$$

where the average value of the momentum in the weak Raman-Nath approximation is $p \approx \sqrt{2H/\omega_r}$. Using the largest second and fifth terms in Eq. (2), we can
estimate $D_p$ as follows:

$$D_p \simeq \frac{\gamma}{12} + \frac{\langle u^2(\tau_j) \rangle \gamma}{8\omega^2 p^2},$$

(4)

All the other terms in Eq. (2) are small since $|\Delta| \ll 1$, $|z| \sim 1$, and $|u| \gg |\Delta|$. In deriving Eq. (4), we put $\langle u \cos x \rangle \simeq 0$, $\langle \tau_j - \tau_{j-1} \rangle \simeq 2/\gamma$, and $\langle p_j^2 \rangle = 1/3$.

To estimate the value of $\langle u^2(\tau_j) \rangle$ in Eq. (4) we use the results of our theory [3] where we have shown, that in the absence of SE the variable $u$ can be approximated by a constant when atoms move between nodes of a standing wave (it fact, it performs shallow oscillations) which changes suddenly its value when they cross any node at $\cos x = 0$. Spontaneous emission results in additional jumps, $u \rightarrow 0$, but between SE events one can approximately describe the dynamics using the Hamiltonian theory. In the chaotic case, the evolution of $u$ can be approximated as a stochastic mapping

$$u_m \simeq |\Delta| \sqrt{\frac{\pi}{\omega r p}} \sin \phi_m + u_{m-1},$$

(5)

where $u_m$ are values of $u$ after $m$-th node crossing (starting with the latest SE event), $\phi_m$ are random phases in the range $[0, \pi]$. The index $m$ increases by 1 just after each node crossing and jumps to zero just after SE event. This map is obtained from the expression (11) in Ref. [3] in the limit $|u| \ll 1$ which is valid because we have $u = 0$ after any act of SE, and the values of $u$ never go far away from zero due to small magnitudes of jumps. Between the acts of SE, a sequence of values of $u$ looks like a Markov chain of random jumps where the next state depends only on the previous one. In the weak Raman-Nath approximation, the number of node crossings between SE can be estimated in the average to be $\langle M \rangle = 2\omega r p/(\gamma \pi)$. Now we can estimate the value of $u(\tau_j) \simeq u_M$ and, using Eq. (4), get the following formula for the momentum diffusion coefficient in the regime of Hamiltonian chaos

$$D_{ch} \simeq \frac{\gamma}{12} + \frac{\Delta^2}{8\omega^2 p^2}.$$  

(6)

In Fig. 1 this function (6) is shown by the dashed lines in a log-log scale. It fits well numerical data in the range of atomic momenta where Hamiltonian dynamics is fully chaotic, i.e. at $\Lambda = 1$. The formula (6) is valid in a wide range of moderately small detunings, but it does not work in the Hamiltonian mixing and regular regimes.

At very small detunings, we may estimate $D_p$ both in the fully chaotic ($\Lambda = 1$) and regular ($\Lambda = 0$) regimes. With fast atoms at $|\Delta| \ll 1$, we can use the strong Raman-Nath approximation (neglecting the momentum fluctuations between SE at all) and adopt the simple linear law of motion $x = \omega r p t$. 

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In Ref. 5 we have derived a formula (see Eq. (A3) therein) for the value of $u$ after crossing the first node. So, we get for $u$ after crossing the $m$-th node

$$ u_m \simeq \Delta \left[ \frac{\pi}{\omega_r p} \left( v_0 \cos \left( \frac{2}{\omega_r p} - \frac{\pi}{4} \right) + (-1)^m \right) \times z_0 \sin \left( \frac{2}{\omega_r p} - \frac{\pi}{4} \right) + (-1)^m z_0 \right] + u_{m-1}, $$(7)

where $v_0$ and $z_0$ are constant values of $v$ and $z$ at $x = \pi k$, $k = 1, 2, \ldots$. Now jumps of $u$ are not random, and in the range between two SE events a trajectory looks like a ladder with odd and even jumps of different size with the total number of steps $M$. It can be shown that $u(t_j) \simeq u_M \simeq M \Delta v_0 \sqrt{\pi/\omega_r p \cos(2/\omega_r p - \pi/4)}$. Since $v_0$ differs after different SE events, we put $\langle v_0^2 \rangle \simeq M$. At very small detunings, the diffusion coefficient, corresponding to Hamiltonian regular motion ($\Lambda = 0$), is

$$ D_{reg} \simeq \frac{\gamma}{12} + \frac{\Delta^2}{4\gamma \omega_r p \pi} \cos^2 \left( \frac{2}{\omega_r p} - \frac{\pi}{4} \right) \simeq \frac{\gamma}{12} + \frac{\Delta^2}{8\gamma \omega_r p \pi}. $$

Thus, we have the analytic expressions for $D_p$ in the regimes of Hamiltonian chaos ($\Lambda = 1$) and Hamiltonian order ($\Lambda = 0$). In general case, $0 \leq \Lambda \leq 1$, we suppose a linear law for the momentum diffusion:

$$ D_p \simeq (1 - \Lambda) D_{reg} + \Lambda D_{ch} \simeq \frac{\gamma}{12} + \frac{\Delta^2}{8\gamma \omega_r p} \left( \frac{1 - \Lambda}{\gamma \pi} + \frac{\Lambda}{\omega_r p} \right). $$

This function is shown by the solid line in the upper right panel in Fig. 4.

Let us consider a small cloud of atomic gas moving in one direction with the mean momentum $\langle p \rangle$. Initial position and momentum distributions are supposed to be Gaussian with the standard deviations $\sigma_r^2 \equiv \langle (x - x) \rangle^2$ and $\sigma_p^2 \equiv \langle (p - \langle p \rangle) \rangle^2$. The momentum diffusion coefficient is $D_p = d(\sigma_p^2)/(2d\tau)$. The temperature of atomic gas and its heating speed (in units K/s) are

$$ T \equiv \frac{2 \langle E_k \rangle}{k_B} = \frac{h^2 k_f^2 \sigma_r^2}{m_a k_B}, \quad \frac{dT}{dt} = \frac{2h^2 k_f^2 \Omega D_p}{m_a k_B}, $$

where $E_k$ is the atomic kinetic energy (in J) in the center-of-mass moving frame. The heating speed is proportional to $D_p$ which has been shown in this Letter to demonstrate different behavior in the regimes of regular and chaotic Hamiltonian dynamics.

In real experiments the measurable quantity is a linear cloud size $L \equiv \sigma_x/k_f$ (in meters). At small observation times $\tau < p/F$ and small temperatures $\sigma_x \ll (p)$, we can approximate $D_p$ by a constant for all the atoms in a cloud which does not change significantly under the action of the force $F$. In this approximation we find: $\sigma_r^2 \approx \sigma_p^2(0) + (1/2)\omega_r^2 \sigma_r^2(0) \tau^2 + (2/3)D_p \omega_r^2 \tau^3$. We have computed $L$ with Eqs. 4 and with that formula and found a good correlation between the results and a strict difference in extension of atomic clouds in the presence and absence of Hamiltonian chaos.

In conclusion, we have found in numerical experiments the manifestation of dynamical instability and Hamiltonian chaos in ballistic motion of two-level atoms in a near-resonance standing-wave field. The effect of dynamical chaos in the fundamental atom-light interaction is masked by random events of SE. Nevertheless, we proved analytically and numerically that, under certain conditions, there exists a clear correlation between the behavior of the momentum diffusion coefficient $D_p$ and chaos probability $A$. To detect and quantify this effect in a real experiment, we propose to measure linear extensions $L$ of atomic clouds with different values of the mean atomic momentum $\langle p \rangle$. We predict that beginning with those values of $\langle p \rangle$, for which Hamiltonian chaos probability becomes to be 1, the value of $L$ for the corresponding atomic clouds should increase sharply.

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