A NOTE ON OVERCONVERGENCE OF HECKE ACTION

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Abstract. We show that the action of Hecke operators away from $p$ on space of $(p$-adic) overconvergent modular forms is overconvergent in certain sense. As a corollary, the action of Hecke algebra can be extended naturally to an action of rigid functions on its generic fiber. This directly determines the Hodge-Tate-Sen weights of Galois representation associated to an overconvergent eigenform.

1. Introduction

The notion of $p$-adic modular forms was introduced by Serre in the study of congruences between modular forms. It is well-known that to get a better spectral theory of the $U_p$-operator, one should consider the subspace of overconvergent modular forms, on which $U_p$ acts completely continuously. In this short note, we will show that Hecke operators away from $p$ also have better convergence when acting on overconvergent modular forms. As a consequence, we deduce that the action of (big) Hecke algebra $T$ naturally extends to an action of the rigid functions on its generic fiber (denoted by $T^\text{rig}$ by some people). Since having a Hodge-Tate-Sen weight 0 is a Zariski-closed property on $\text{Spec} T^\text{rig}$, the density of classical points implies directly that

Theorem 1.1 (Corollary 4.6). The two dimensional semi-simple Galois representation associated to an overconvergent eigenform of weight $k \in \mathbb{Z}$ has Hodge-Tate-Sen weights $0, k-1$.

This result was recently obtained by myself in [Pan20] and by Sean Howe independently in [How20] (when $k \neq 1$), by relating overconvergent modular forms with completed cohomology. Our method here is more straightforward. Hopefully it will be clear to the readers that the argument can be easily generalized to other contexts.

This note is organized as follows. We will first introduce a class of operators acting on a $p$-adic Banach space called overconvergent operators and give several (simple) examples. Then using fake-Hasse invariants introduced by Scholze [Sch15], we show that Hecke operators are overconvergent on space of overconvergent modular forms (with fixed radius). As suggested by Matthew Emerton, this also reproves a result of Calegari-Emerton. At the end, we also discuss a similar phenomenon in the context of locally analytic vectors of completed cohomology.

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2. Overconvergent operator

Definition 2.1. Let $W$ be a $p$-adic Banach space over $\mathbb{Q}_p$. A continuous operator $T \in \text{End}(W)$ is called overconvergent if it has norm $\leq 1$ and there exists a monic polynomial
f(X) ∈ Z_p[X] such that f(T) ∈ End(W) has norm strictly less than 1, or equivalently 
f(T)(W^o) ⊆ pW^o, where W^o denotes the unit ball of W.

Note that for an overconvergent operator T, the image of T inside End(W^o/pW^o) generates a finite F_p-algebra.

**Example 2.2.** Suppose W is a finite dimensional vector space over Q_p. Then any operator of norm ≤ 1 is overconvergent by examining its characteristic polynomial.

**Example 2.3.** Suppose W = Q_p(X), the (p-adic) completion of Q_p[X] with respect to unit ball Z_p[X]. Let T ∈ End(Q_p(X)) be the translation X → X + 1. It is overconvergent because (T^p − 1) · F(X) = F(X + p) − F(X) ∈ pZ_p[X] for any F(X) ∈ Z_p[X].

We can also generalize this notion to representations of algebras.

**Definition 2.4.** Suppose A is a Z_p-algebra acting on a p-adic Banach space W. We say this action is **overconvergent** if A(W^o) ⊆ W^o and the image of A → End(W^o/pW^o) is a finite F_p-algebra.

**Example 2.5.** Suppose A = Z_p[[X]] acting continuously on a p-adic Banach space W. Then the overconvergence of this action is equivalent with W being an overconvergent operator on W. When this is the case, the image of A in End(W^o/pW^o) has the form F_p[X]/(X^n) for some n ≥ 1. Hence X^n(W^o) ⊆ pW^o, or equivalently \( \frac{X^n}{p} \) has norm ≤ 1. This implies that the action of Z_p[[X]] on W can be extended to Q_p(X^o, X), rigid analytic functions on a closed disc with radius \( p^{-1/n} \) strictly less than 1. This justifies the name “overconvergent”.

**3. Fake-Hasse invariants**

In order to study the Hecke action on overconvergent modular forms, we need fake-Hasse invariants and strange formal integral models of the modular curve constructed by Scholze in Chapt. IV of [Sch15]. The overconvergence of Hecke action will be a formal consequence of the existence of these Hecke-invariant sections (of \( \omega \) modulo \( p \)).

Our setup is as follows. Let \( C = \mathbb{Q}_p \) be the p-adic completion of \( \mathbb{Q}_p \) with ring of integers \( \mathcal{O}_C \). For an open compact subgroup \( K \) of GL_2(\( \mathbb{A}_f \)), we denote by \( \mathcal{X}_K^{*} \) the complete Adelic modular curve over \( C \) of level \( K \) and by \( \mathcal{X}_K \) its associated rigid analytic space. We will always assume \( K \) is sufficiently small so that \( \mathcal{X}_K^{*} \) is a variety. On \( \mathcal{X}_K \), we have the usual automorphic line bundle \( \omega_{KpK_o} \). Fix an open compact subgroup \( K^p \subset \text{GL}_2(\mathbb{A}_f^p) \) contained in the level-N-congruence subgroup for some \( N \geq 3 \) prime to \( p \). For a sufficiently small open subgroup \( K_p \subset \text{GL}_2(\mathbb{Q}_p) \), in the proof of Theorem IV.3.1. of [Sch15], Scholze constructed

- a formal integral model \( \mathcal{X}_{K^pK_o}^{*} \) of \( \mathcal{X}_{K^pK_o}^{*} \) equipped with an affine open cover \( \mathfrak{V}_1, \mathfrak{V}_2 \);
- an ample line bundle \( \omega_{K^pK_o}^{\text{int}} \) on \( \mathcal{X}_{K^pK_o}^{*} \) whose generic fiber is \( \omega_{KpK_o} \);
- global sections \( \bar{s}_1, \bar{s}_2 \in H^0(\mathcal{X}_{K^pK_o}^{*}, \omega_{K^pK_o}^{\text{int}}/p) \) such that \( \mathfrak{V}_i \) is the locus where \( \bar{s}_i \) is invertible for \( i = 1, 2 \). In particular, \( \bar{s}_1, \bar{s}_2 \) generate \( \omega_{K^pK_o}^{\text{int}}/p \).

Moreover, all \( \mathcal{X}_{K^pK_o}^{*}, \mathfrak{V}_1, \mathfrak{V}_2 \) and \( \omega_{K^pK_o}^{\text{int}} \) are functorial in \( K^p \), hence GL_2(\( \mathbb{A}_f^p \)) acts on the tower of \( (\mathcal{X}_{K^pK_o}^{*}, \omega_{K^pK_o}^{\text{int}}) \). Both sections \( \bar{s}_1, \bar{s}_2 \) are invariant under this action.

Let \( \mathcal{T} = \mathcal{T}_{K^p} = \mathbb{Z}_p[\text{GL}_2(\mathbb{A}_f^p)//K^p] \) be the abstract Hecke algebra of \( K^p \)-biinvariant compactly supported functions on \( \text{GL}_2(\mathbb{A}_f^p) \), where the Haar measure gives \( K^p \) measure 1. It
follows from the functorial properties of $\mathfrak{M}_i, \mathfrak{M}_2$ that $H^0(\mathfrak{M}_i, (\omega^\text{int}_{KpK_p})^{\otimes k}), i = 1, 2, k \in \mathbb{Z}$ admits a natural action of $\mathbb{T}$. Denote by $V_i \subset X_{KpK_p}^*$ the generic fiber of $\mathfrak{M}_i$. Then $H^0(V_i, \omega^{\otimes k})$ is a $p$-adic Banach space with unit ball $H^0(\mathfrak{M}_i, (\omega^\text{int}_{KpK_p})^{\otimes k})$. Our main result here is

**Theorem 3.1.** The Hecke actions of $\mathbb{T}$ on $H^0(V_i, \omega^{\otimes k}_{KpK_p}), i = 1, 2, k \in \mathbb{Z}$ are overconvergent.

**Remark 3.2.** We will relate $H^0(V_i, \omega^{\otimes k}_{KpK_p})$ with classical overconvergent modular forms later in next section. See the proof of Corollary 1.6

Since $\mathfrak{M}_i$ is affine, $H^0(\mathfrak{M}_i, (\omega^\text{int}_{KpK_p})^{\otimes k})/p = H^0(\mathfrak{M}_i, (\omega^\text{int}_{KpK_p})^{\otimes k}/p)$. Note that $\bar{s}_i$ is an invertible section on $\mathfrak{M}_i$. There are isomorphisms:

$$H^0(\mathfrak{M}_i, \mathcal{O}_{X_{KpK_p}}^*/p) \simeq H^0(\mathfrak{M}_i, (\omega^\text{int}_{KpK_p})^{\otimes k}/p)$$

which is $\mathbb{T}$-equivariant because $\bar{s}_i$ commutes with Hecke actions. Hence $H^0(\mathfrak{M}_i, (\omega^\text{int}_{KpK_p})^{\otimes k})/p$ is independent of $k$ as a Hecke module. Thus it suffices to prove Theorem 3.1 for $k = 0$ and we have the following corollary.

**Corollary 3.3.** The Hecke actions of $\mathbb{T}$ on

$$(\prod_{k \in \mathbb{Z}} H^0(\mathfrak{M}_i, (\omega^\text{int}_{KpK_p})^{\otimes k})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, i = 1, 2$$

are overconvergent.

**Proof of Theorem 3.1.** By definition, we need to show the image of

$$\mathbb{T} \to \text{End}(H^0(\mathfrak{M}_i, \mathcal{O}_{X_{KpK_p}}^*/p))$$

is a finite $\mathbb{F}_p$-algebra. Since $\mathfrak{M}_i$ is the locus where $\bar{s}_i$ is invertible, we may write

$$H^0(\mathfrak{M}_i, \mathcal{O}_{X_{KpK_p}}^*/p) = \lim_{\bar{s}_i} H^0(X^*_{KpK_p}, (\omega^\text{int}_{KpK_p})^{\otimes k}/p).$$

Hence it suffices to show the image of

$$\mathbb{T} \to \text{End}(H^0(X^*_{KpK_p}, (\omega^\text{int}_{KpK_p})^{\otimes k}/p))$$

stabilizes (as a quotient of $\mathbb{T}$) when $k$ is sufficiently large. Consider the exact sequence

$$0 \to (\omega^\text{int}_{KpK_p})^{\otimes k-1}/p \xrightarrow{\bar{s}_1 \bar{s}_2} (\omega^\text{int}_{KpK_p})^{\otimes k}/p \xrightarrow{\bar{s}_2 \bar{s}_1} (\omega^\text{int}_{KpK_p})^{\otimes k+1}/p \to 0.$$

(This essentially comes from the non-split sequence $0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1) \to 0$ on $\mathbb{P}^1$.) When $k$ is sufficiently large, taking global sections of this exact sequence remains exact as $\omega^\text{int}_{KpK_p}$ is ample. Thus the Hecke action of $\mathbb{T}$ on $H^0(X^*_{KpK_p}, (\omega^\text{int}_{KpK_p})^{\otimes k+1}/p)$ factors through $H^0(X^*_{KpK_p}, (\omega^\text{int}_{KpK_p})^{\otimes k}/p)^{\otimes 2}$. This gives exactly what we need.

\[\square\]

4. **Hodge-Tate-Sen weights**

In this section, we study Galois representations associated to eigenforms in $H^0(V_i, \omega^{\otimes k}_{KpK_p})$. Let me introduce some (standard) notation first. For simplicity, from now on we assume $K^p \subset \text{GL}_2(A_f^\mathbb{A}_f^\text{int})$ is of the form $\prod_{l \neq p} K_l$. Let $S$ be a finite set of rational primes containing $p$ such that $K_l \cong \text{GL}_2(\mathbb{Z}_l), l \notin S$. Denote by $\mathbb{T}_S = \mathbb{Z}_p[\text{GL}_2(A_f^\mathbb{A}_f^\text{int})]/\prod_{l \notin S} K_l] \subset \mathbb{T}$ the subalgebra generated by spherical Hecke operators. Consider the image $\mathbb{T}_{i,1}$ of $\mathbb{T}_S \to$
End($H^0(\mathfrak{W}_i, \mathcal{O}_{\mathfrak{X}_{K^p,K_p}}/p)$). By Theorem 5.31, this is a finite $\mathbb{F}_p$-algebra. Moreover, by Corollary V.1.11 of [Sch15], there is a continuous 2-dimensional determinant $D$ of $G_{Q,S}$ valued in $\mathbb{T}_{i,1}$ in the sense of Chenevier [Che14] satisfying the following property: for any $l \notin S$, the characteristic polynomial of $D(\text{Frob}_l)$ is

$$X^2 - l^{-1}T_lX + l^{-1}S_l.$$ 

Here $G_{Q,S}$ denotes the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside of $S$ and infinity, $\text{Frob}_l \in G_{Q,S}$ denotes a geometric Frobenius element at $l$ and $T_l, S_l$ denote the usual Hecke operators

$$[K_l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}], [K_l \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}].$$

Let $\mathbb{F}$ be a finite field so that all residue fields of $\mathbb{T}_{i,1}$ can be embedded into $\mathbb{F}$. Fix an embedding of $\mathbb{F}(\mathbb{F}[\frac{1}{p}])$ into $\mathbb{Q}_p$, or equivalently an embedding $\mathbb{F} \to \mathcal{O}_C/p$. Then $\mathbb{T}_{i,1} \otimes_{\mathbb{F}_p} \mathbb{F}$ acts on $H^0(\mathfrak{W}_i, \mathcal{O}_{\mathfrak{X}_{K^p,K_p}}/p)$ and we denote by $\mathbb{T}_i$ its image in $\text{End}(H^0(\mathfrak{W}_i, \mathcal{O}_{\mathfrak{X}_{K^p,K_p}}/p))$. Finally, for any maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{i,1}$, we have a continuous 2-dimensional determinant $D_{\mathfrak{m}}$ of $G_{Q,S}$ valued in $\mathbb{T}_{i,1}/\mathfrak{m} = \mathbb{F}$. Let $R_{\mathfrak{m}}^{\text{ps}}$ be the universal formal $W(\mathbb{F})$-algebra parametrizing all liftings of $D_{\mathfrak{m}}$. This is a noetherian ring. Denote the product over all $\mathfrak{m}$ by

$$R^{\text{ps}} = \prod_{\mathfrak{m} \in \text{Spec} \mathbb{T}_i} R_{\mathfrak{m}}^{\text{ps}}.$$ 

Now for any $k \in \mathbb{Z}, n > 0$, by Corollary V.1.11 of [Sch15], there is a lifting of $\prod_{\mathfrak{m} \in \text{Spec} \mathbb{T}_i} D_{\mathfrak{m}}$ valued in the image of $\mathbb{T}_S \otimes_{\mathbb{Z}_p} W(\mathbb{F}) \to \text{End}(H^0(\mathfrak{X}_{K^p,K_p}^\times, (\omega_{K^p,K_p}^\text{int})^\otimes k/p^n))$. By the universal property, this image receives a map from $R_{\mathfrak{m}}^{\text{ps}}$. Hence we obtain an action of $R_{\mathfrak{m}}^{\text{ps}}$ on $H^0(V_i, \omega_{K^p,K_p}^\otimes k)$ factoring through the Hecke action. In particular, by Corollary 5.50

**Corollary 4.1.** The action of $R_{\mathfrak{m}}^{\text{ps}}$ on $(\prod_{k \in \mathbb{Z}} H^0(\mathfrak{W}_i, (\omega_{K^p,K_p}^\text{int})^\otimes k)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is overconvergent.

Concretely, since each $R_{\mathfrak{m}}^{\text{ps}}$ is a noetherian local formal $W(\mathbb{F})$-algebra, it can be written as a quotient of $W(\mathbb{F})[[x_1, \ldots, x_g]]$ for some $g$. As explained in Example 2.93 there exists an integer $n > 0$ such that $x_j^p \equiv n \equiv 1$ acting on $(\prod_{k \in \mathbb{Z}} H^0(\mathfrak{W}_i, (\omega_{K^p,K_p}^\text{int})^\otimes k)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for any $j = 1, \ldots, g$. Therefore, let $E \subset \mathbb{Q}_p$ be a finite extension of $W(\mathbb{F})[\frac{1}{p}]$ containing a $n$-th root of $p$ and fix such a root $p^{1/n} \in E$. We can extend the action of $W(\mathbb{F})[[x_1, \ldots, x_g]]$ to an $E$-linear action of $E(\frac{x_1}{p^{1/n}}, \ldots, \frac{x_g}{p^{1/n}})$ on $(\prod_{k \in \mathbb{Z}} H^0(\mathfrak{W}_i, (\omega_{K^p,K_p}^\text{int})^\otimes k)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Geometrically, the generic fiber of $W(\mathbb{F})[[x_1, \ldots, x_g]]$ is an open ball and $E(\frac{x_1}{p^{1/n}}, \ldots, \frac{x_g}{p^{1/n}})$ corresponds to a closed polydisc inside. Roughly speaking, this means the spectrum of $H^0(V_i, \omega_{K^p,K_p}^\otimes k)$ is in a bounded region with radius strictly less than 1.

We make such a choice for each $\mathfrak{m}$. As a consequence, the action of $R_{\mathfrak{m}}^{\text{ps}}$ can be extended to an action of a topologically finitely generated Banach $E$-algebra. We denote its image in

$$\text{End}((\prod_{k \in \mathbb{Z}} H^0(\mathfrak{W}_i, (\omega_{K^p,K_p}^\text{int})^\otimes k)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

by $\mathcal{R}$. There is a natural map $R_{\mathfrak{m}}^{\text{ps}} \to \mathcal{R}$. Hence we have a 2-dimensional determinant $D_{\mathcal{R}}$ of $G_{Q,S}$ valued in $\mathcal{R}$ which is continuous with respect to the $p$-adic topology on $\mathcal{R}$. The whole point of showing the overconvergence of Hecke action is to improve the continuity of determinant on $R_{\mathfrak{m}}^{\text{ps}}$ from the rad($R_{\mathfrak{m}}^{\text{ps}}$)-adic topology to a $p$-adic topology.
The Hecke action on $H^0(\mathcal{X}_{K^p K_p}^*, \omega_{K^p K_p}^{\otimes k}), k \geq 0$ extends naturally to $\mathcal{R}$. In fact, the image of $T_S \otimes E$ in $\text{End}(H^0(\mathcal{X}_{K^p K_p}^*, \omega_{K^p K_p}^{\otimes k}))$ agrees with the image of $\mathcal{R}$. In particular, the action of $\mathcal{R}$ on $H^0(\mathcal{X}_{K^p K_p}^*, \omega_{K^p K_p}^{\otimes k})$ is semi-simple.

**Lemma 4.2.** The kernel of 
\[
\mathcal{R} \rightarrow \text{End}\left(\prod_{k \geq 0} H^0(\mathcal{X}_{K^p K_p}^*, \omega_{K^p K_p}^{\otimes k})\right)
\]
is trivial.

**Proof.** This is a standard application of fake Hasse invariants. See the proof of Theorem IV.4.1. of [Sch15]. We give a sketch here. Suppose $f \in \mathcal{R}$ is a non-zero element in the kernel of the above map. We may assume it has norm $\leq 1$ acting on $\prod_{k \in \mathbb{Z}} H^0(\mathfrak{M}_i, (\omega_{K^p K_p}^{\text{int}})^{\otimes k})$ and its image in $\text{End}(\prod_{k \in \mathbb{Z}} H^0(\mathfrak{M}_i, (\omega_{K^p K_p}^{\text{int}})^{\otimes k}/p))$ is non-zero. Now since $R^{\text{int}} \otimes_{W(\mathbb{F})} E \rightarrow \mathcal{R}$ has dense image, the action of $f$ on $H^0(\mathfrak{M}_i, (\omega_{K^p K_p}^{\text{int}})^{\otimes k}/p)$ commutes with $\lambda_i$ if $n$ is sufficiently divisible by $p$. This means $f$ acts non-trivially on $H^0(\mathcal{X}_{K^p K_p}^*, (\omega_{K^p K_p}^{\text{int}})^{\otimes k}/p)$ for some sufficiently large $k$ by the same argument as in the proof of Corollary 3.2. In this case, $H^0(\mathcal{X}_{K^p K_p}^*, (\omega_{K^p K_p}^{\text{int}})^{\otimes k}/p) = H^0(\mathcal{X}_{K^p K_p}^*, (\omega_{K^p K_p}^{\text{int}})^{\otimes k}/p)$ because $(\omega_{K^p K_p}^{\text{int}})$ is ample. But this contradicts our assumption on $f$. \hfill $\square$

Recall that there is a determinant $D_{\mathcal{R}}$ of $G_{Q,S}$ valued in $\mathcal{R}$. Since $\mathcal{R}$ is over a characteristic zero field, one can also view this as a function $T : G_{Q,S} \rightarrow \mathcal{R}$, which behaves like the trace of a two-dimensional representation, i.e. a pseudo-representation. For any non-zero $E$-algebra homomorphism $\lambda : \mathcal{R} \rightarrow \overline{\mathbb{Q}}_p$, we can associate a two-dimensional semi-simple continuous representation $\rho_\lambda : G_{Q,S} \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$, well-defined up to conjugation, whose trace is given by $\lambda \circ T$. Moreover, if $\lambda$ arises from an eigenform in $H^0(\mathcal{X}_{K^p K_p}^*, \omega_{K^p K_p}^{\text{int}})$, then by Faltings’s result [Fal87], $\rho_\lambda|_{G_{Q,p}}$ has Hodge-Tate weights $0, k - 1$. Our convention is that cyclotomic character has Hodge-Tate weight $-1$. The density result [CLO8] has the following consequence.

**Theorem 4.3.** For any $\lambda : \mathcal{R} \rightarrow \overline{\mathbb{Q}}_p$, one of the Hodge-Tate-Sen weights of $\rho_{\lambda}|_{G_{Q,p}}$ is $0$, i.e. $(\rho_{\lambda} \otimes_{\overline{\mathbb{Q}}_p} C)^{G_{Q,p}} \neq 0$.

**Proof.** Recall that given a continuous representation of $G_{Q_p} \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$, Sen constructs a monic polynomial $P_{\text{sen}, \rho}$ of degree $n$ with coefficients in $\overline{\mathbb{Q}}_p \otimes_{\overline{\mathbb{Q}}_p} \mathbb{Q}_p(\mu_{p^n})$. It is called the Sen polynomial of $\rho$ and only depends on the semi-simplification of $\rho$. Its roots are called the Hodge-Tate-Sen weights of $\rho$ (up to a sign depending on the normalization). Moreover, Sen shows that this polynomial varies analytically in family. See [Sen88, Sen93] and also Théorème 5.1.4. of [BC08]. We are going to apply Sen’s theory in our context.

First, suppose that there exists a continuous Galois representation $\rho_\mathcal{R} : G_{Q,S} \rightarrow \text{GL}_2(\mathcal{R})$ whose trace is $T$. Then by Sen’s result, we can find a polynomial $P_{\text{sen}, \rho_{\mathcal{R}}}$ with coefficients in $\mathcal{R} \otimes_{\overline{\mathbb{Q}}_p} \mathbb{Q}_p(\mu_{p^n})$, such that for any $\lambda : \mathcal{R} \rightarrow \overline{\mathbb{Q}}_p$, the Sen polynomial of $\rho_\lambda$ is given by $\lambda(P_{\text{sen}, \rho_{\mathcal{R}}})$. By Lemma 4.2 and Faltings’s result, the constant term of $P_{\text{sen}, \rho_{\mathcal{R}}}$ vanishes as it vanishes after composing with any $\lambda$ arisen from an eigenform in $H^0(\mathcal{X}_{K^p K_p}^*, \omega_{K^p K_p}^{\text{int}})$. (Implicitly we using $\mathbb{Q}_p(\mu_{p^n})$ is flat over $\mathbb{Q}_p$) This immediately implies our claim.

In general, we may assume $\mathcal{R}$ is an integral domain. We are going to use the following lemma, whose proof will be given later.
Lemma 4.4. Assume \( R \) is normal. There exists a polynomial \( P \in R \otimes_{Q_p} Q_p(\mu_{p^\infty})[X] \) such that for any \( \lambda : R \to \bar{Q}_p \), the Sen polynomial of \( \rho_\lambda \) is \( \lambda(P) \).

Let \( R' \) be the normal closure of \( R \) in its fraction field. Note that \( R \) is a quotient of products of \( E(x_1, \ldots, x_k) \). Hence it is excellent because the Tate algebra \( E(x_1, \ldots, x_k) \) is excellent by the weak Jacobian condition (Matsumura Theorem 102). In particular, \( R \) is a Nagata ring and \( R' \) is a finite \( R \)-algebra. Thus \( R' \) is a Banach \( E \)-algebra.

Now consider the pseudo-representation \( G_{Q,S} \overset{\epsilon}{\to} R \to R' \). Note that by the going-up property of integral extension, any \( \lambda : R \to \bar{Q}_p \) can be extended to a map \( \lambda' : R' \to \bar{Q}_p \) and \( \rho_\lambda \cong \rho_{\lambda'} \). In particular, it’s enough to show \( \rho_\lambda \) has a Hodge-Tate-Sen weight zero for any \( \lambda' : R' \to \bar{Q}_p \). Applying previous lemma to \( R' \), we get a universal Sen polynomial \( P \) with coefficients in \( R' \otimes_{Q_p} Q_p(\mu_{p^\infty}) \). Again it suffices to show the constant term of \( P \) vanishes. Write the constant term of \( P \) as \( \sum_{i=1}^n a_i \otimes b_i \) with \( a_i \in R', b_i \in Q_p(\mu_{p^\infty}) \) and \( b_i \) are linearly independent over \( Q_p \). If one of \( a_i \) is non-zero, say \( a_1 \), we can find a monic polynomial \( Q(X) \in R'[X] \) with constant term \( Q(0) \neq 0 \) and \( Q(a_1) = 0 \). By Lemma 4.12 there exists a \( \lambda : R \to \bar{Q}_p \) arisen from an eigenform in \( H^0(X_{K^p,K_p}, \omega_{K^p/K_p}^{\otimes a_1}) \) and \( \lambda(Q(0)) \neq 0 \). Let \( \lambda' : R' \to \bar{Q}_p \) be a map extending \( \lambda \). By Faltings’s result, \( \lambda'(a_1) = 0 \). But \( 0 = \lambda'(Q(a_1)) = \lambda(Q(0)) \neq 0 \). Contradiction. Thus we prove \( P(0) = 0 \).

Proof of Lemma 4.4. First let me recall some standard constructions in the theory of pseudo-representations. Fix a complex conjugation \( \sigma^* \in G_{Q,S} \). Our pseudo-representation \( T \) is odd in that \( T(\sigma^*) = 0 \). For any \( \sigma, \tau \in G_{Q,S} \), let

\[
\begin{align*}
\bullet \quad a(\sigma) &= T(\sigma^* \sigma) + T(\sigma), \\
\bullet \quad d(\sigma) &= T(\sigma) - a(\sigma), \\
\bullet \quad x(\sigma, \tau) &= a(\sigma \tau) - a(\sigma)a(\tau).
\end{align*}
\]

We denote by \( \mathcal{I} \) the ideal of \( R \) generated by all \( x(\sigma, \tau) \). It is called the ideal of reducibility as \( \rho_\lambda \) is reducible if and only if \( \lambda(\mathcal{I}) = 0 \). If \( \mathcal{I} \) is generated by some \( x(\sigma_0, \tau_0) \neq 0 \). Then

\[
\sigma \in G_{Q,S} \mapsto \left( \begin{array}{c} a(\sigma) \\ x(\sigma, \tau_0) \\ x(\sigma_0, \sigma) \\ d(\sigma) \end{array} \right)
\]

defines a representation \( G_{Q,S} \to \text{GL}_2(R) \) whose trace is \( T \). In this case, our claim follows from Sen’s result directly.

In general, \( \mathcal{I} \) might not even be principal. Here is a sketch of what we are going to do. \( \mathcal{X} := \text{Spm} \, R \) is viewed as an affinoid rigid analytic variety. Consider the blowup \( \bar{\mathcal{X}} \) of \( \mathcal{X} \) along the ideal sheaf defined by \( \mathcal{I} \). Then \( \bar{\mathcal{X}} \) becomes an invertible sheaf on \( \bar{\mathcal{X}} \) and we can apply the GIT construction glue a polynomial on \( \bar{\mathcal{X}} \) interpolating Sen polynomial at each point. Now the normal assumption guarantees that the coefficients of this polynomial actually belong to \( R \). This gives the polynomial we are looking for. Since everything is relatively simple here, the blowup process will be replaced by explicit construction below. But it seems helpful to keep this blowup picture in mind.

If \( \mathcal{I} = 0 \), then \( a, d \) are characters and our claim is clear. So we may assume \( \mathcal{I} \neq 0 \) from now on. Let \( x_1 = x(\sigma_1, \tau_1), \ldots, x_r = x(\sigma_r, \tau_r) \) be a set of non-zero generators of \( \mathcal{I} \). Denote by \( R^+ \) the unit ball of \( R \) and by \( K \) the fraction field of \( R \). For each \( i \in \{ 1, \ldots, r \} \), we define \( R_i^+ \) as the \( p \)-adic completion of \( R^+[\frac{x_1}{x_1}, \ldots, \frac{x_i}{x_i}] \subset K \) and \( R_i = R_i^+[\frac{1}{p}] \). Consider the
pseudo-representation $G_{\mathbb{Q},S} \xrightarrow{T} \mathcal{R} \rightarrow \mathcal{R}_i$. The ideal of reducibility in this case is generated by $x_i$. Hence we have a polynomial $P_i \in \mathcal{R}_i \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})[X]$ interpolating Sen polynomial at each point of $\text{Spm} \mathcal{R}_i$.

Denote by $\mathcal{Y}_i \subset \text{Spm} \mathcal{R}_i$ the open subvariety defined by $x_i \neq 0$ and by $\mathcal{X}_i \subset \mathcal{X}$ the open subvariety defined by $x_i \neq 0$, $\|x_j\| \leq \|x_i\|$, $j = 1, \ldots, r$. It is easy to see $\mathcal{Y}_i$ maps isomorphically onto $\mathcal{X}_i$ under the natural map $\pi_i : \text{Spm} \mathcal{R}_i \rightarrow \mathcal{X}$. Hence we may view $P_i|_{\mathcal{Y}_i}$ as a polynomial on $\mathcal{X}_i$. Clearly, it interpolates Sen polynomial at each point in $\mathcal{X}_i$. Hence we can glue all $P_i$ and get a polynomial $P$ on $\mathcal{X}' := \mathcal{X} \setminus V(\mathcal{I})$, the locus of irreducible representations. (Here we are using $\mathcal{R}$ is reduced.) Since $\mathcal{R}$ is normal and the coefficients of $P$ are bounded functions, by Bartenwerfer’s result [Bar76 §3], the coefficients of $P$ can be extended to functions defined everywhere on $\mathcal{X}$, i.e. $P \in \mathcal{R} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})[X]$.

We claim this polynomial $P$ interpolates Sen polynomial at each point in $\mathcal{X}$. By construction, this is true for points in $\mathcal{X}'$. It rests to verify points in $V(\mathcal{I})$. Let $\lambda : \mathcal{R} \rightarrow \overline{\mathbb{Q}_p}$ be a non-zero map whose kernel contains $\mathcal{I}$. Note that there exists $i \in \{1, \ldots, r\}$ such that $\lambda$ can be extended to a map $\lambda' : \mathcal{R}_i \rightarrow \overline{\mathbb{Q}_p}$. This is because the usual blowup (in algebraic geometry) of $\text{Spec} \mathcal{R}$ along $\mathcal{I}$ maps surjectively onto $\text{Spec} \mathcal{R}$. Fix an integer $n$ so that $\lambda'(\mathcal{R}^+[\frac{p^n x_i}{x_i}, \ldots, \frac{p^n x_r}{x_i}])$ is inside the ring of integers of $\overline{\mathbb{Q}_p}$. We define $\mathcal{R}_i^+$ as the $p$-adic completion of $\mathcal{R}^+[\frac{p^n x_i}{x_i}, \ldots, \frac{p^n x_r}{x_i}]$ and $\mathcal{R}_i' = \mathcal{R}_i^+[\frac{1}{p}]$. Then $\lambda'$ extends to $\mathcal{R}_i'$ naturally. Again there is a polynomial $P_i' \in \mathcal{R}_i' \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})[X]$ interpolating Sen polynomial at each point of $\text{Spm} \mathcal{R}_i'$. It suffices to prove that $P$, considered as an element of $(\mathcal{R}_i')^{\text{red}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})[X]$, agrees with $P_i'$. Clearly this is true for points in the irreducible locus $\text{Spm} \mathcal{R}_i' \setminus V(x_i)$. But this also implies points in $V(x_i)$ as $x_i$ is not a zero-divisor in $\mathcal{R}_i'$ by the flatness of $\mathcal{R}_i'$ over $\mathcal{R}^+[\frac{p^n x_i}{x_i}, \ldots, \frac{p^n x_r}{x_i}] \subset \mathcal{K}$. This finishes the proof. \hfill $\Box$

**Remark 4.5.** It is natural to ask whether the normal assumption in Lemma 4.4 is really necessary here. Note that the local universal deformation ring at $p$ of a pseudo-representation is often normal, for example, this is always true when $p \geq 5$. So in these cases, one does not need this normal assumption.

**Corollary 4.6.** The two dimensional semi-simple Galois representation associated to an overconvergent eigenform of weight $k \in \mathbb{Z}$ has Hodge-Tate-Sen weights $0, k - 1$.

**Proof.** We use the (generalized) notion of overconvergent modular forms introduced in [Pan20, Definition 5.2.5.]. See also discussion there for its relation with classical overconvergent modular forms. More precisely, for $K_p$ of the form $\Gamma(p^n) = 1 + p^n M_2(\mathbb{Z}_p)$, we defined an open subset $\mathcal{X}_{K_p}^{\ast \ast}$ of $\mathcal{X}_{K_p}$ and an overconvergent form of weight $k$ is a section of $\omega^k$ defined in a strict neighborhood of $\mathcal{X}_{K_p}^{\ast \ast}$. (There is no superscript $\ast$ in *ibid.*)

Fix an open set $V$ strictly containing $\mathcal{X}_{K_p}$, $\mathcal{C}$. We are going to find an element $g \in \text{GL}_2(\mathbb{Q}_p)$, such that under the natural isomorphism $\mathcal{X}_{K_p}^{\ast \ast} \xrightarrow{\sim} \mathcal{X}_{K_p}^{\ast \ast}$ induced by $g$, the image of $V$ contains $V_1$. Hence sections on $V$ map Hecke-equivariantly to sections on $V_1$. Our claim then follows from Theorem 1.3 as the (usual) determinant of Galois representation associated to an overconvergent eigenform of weight $k$ has Hodge-Tate weight $k - 1$. To produce such an element $g$, we work with the preimage $V_\infty$ of $V$ in the infinite level $\mathcal{X}_{K_p}$. Note that by the argument in second to the last paragraph in Proof of [Pan21 Proposition 5.2.6], $V_\infty$ contains the preimage of an open subset of $\mathcal{H}$ under the Hodge-Tate period map
\[ \pi_{HT} : X_{K^p}^\times \to \mathcal{H}. \] But by construction, the preimage of \( V_1 \) in \( X_{K^p}^\times \) is of the form \( \pi_{HT}^{-1}(V_1) \) for some open subset \( V_1 \) of \( \mathcal{H} \). The existence of \( g \) follows from the geometry of \( \mathcal{H} \).

\[ \square \]

5. A result of Calegari-Emerton

Matthew Emerton pointed out the following consequence of Corollary 3.3 which reproves a result of Calegari-Emerton [CE04, Theorem 2.2] and can be viewed as some evidence towards a question of Buzzard [Buz05, Question 4.4] asking whether for a fixed level, all Hecke eigenvalues of arbitrary weights lie in a finite extension of \( \mathbb{Q}_p \). We denote by \( \overline{\mathbb{Q}}_p \) the ring of integers of \( \overline{\mathbb{Q}}_p \) and by \( \mathfrak{m} \) its maximal ideal.

**Theorem 5.1.** Let \( S \) be a finite set of rational primes containing \( p \) and \( K = \prod_l K_l \) be an open compact subgroup of \( \text{GL}_2(\mathbb{A}_f) \) with \( K_l \cong \text{GL}_2(\mathbb{Z}_l), l \notin S \). There exists a rational number \( \kappa = \kappa(K, p) \) such that for any \( \lambda : T_S \to \mathbb{Z}_p \) appearing in \( H^0(V_i, \omega_{K^p}^{\otimes k}) \) and \( \lambda' : T_S \to \mathbb{Z}_p \) appearing in \( H^0(V_i, \omega_{K^p}^{\otimes k'}) \) for some integers \( k, k' \), \( (\lambda, \lambda' \text{ may come from classical forms for example}) \) if \( \lambda \equiv \lambda' \mod \mathfrak{m} \), then

\[ \lambda \equiv \lambda' \mod p^\kappa \mathbb{Z}_p. \]

**Proof.** Clear from the definition of overconvergence. \( \square \)

6. Hecke action on locally analytic vectors

In this last section, we provide another example of overconvergence of Hecke action: Hecke action on locally analytic vectors of completed cohomology. Again we restrict ourselves to a simple setting, but the argument should work in general.

Let \( D \) be a definite quaternion algebra over \( \mathbb{Q} \). Let \( K^p \) be a compact open subgroup of \( (D \otimes \mathbb{Q} \mathbb{A}_f^p)^\times \). Then the double coset

\[ X_{K^p} = D^\times \setminus (D \otimes \mathbb{Q} \mathbb{A}_f^p)^\times / K^p \]

has a natural profinite topology and admits a continuous action of \( D_p^\times := (D \otimes \mathbb{Q} \mathbb{Q}_p)^\times \) by right translation. The \( (\mathbb{Q}_p, \mathbb{Q}) \)-valued completed cohomology of \( D \) with tame level \( K^p \) is defined as the space of continuous functions on \( X_{K^p} \):

\[ \mathcal{C}(X_{K^p}, \mathbb{Q}_p) \]

Hecke operators away from \( p \) and \( D_p^\times \) naturally act on this space.

Let \( K_p \) be an open pro-\( p \) subgroup of \( D_p^\times \) which is sufficiently small so that it makes sense to talk about analytic functions on it. For example, we can take \( K_p \) to be principal congruence subgroup with level \( p^n, n \geq 2 \). We have the subspace \( \mathcal{C}(X_{K^p}, \mathbb{Q}_p)^{K_p-an} \) of \( K_p \)-analytic vectors in \( \mathcal{C}(X_{K^p}, \mathbb{Q}_p) \). Concretely, this can be identified with continuous functions

\[ f : X_{K^p} \to \mathcal{C}^{an}(K_p, \mathbb{Q}_p) \]

which are \( K_p \)-equivariant. Here \( \mathcal{C}^{an}(K_p, \mathbb{Q}_p) \) denotes the space of analytic functions on \( K_p \) equipped with left translation action of \( K_p \). See for example [Pan20, §2] for more details. Any such \( f \) defines an element in \( \mathcal{C}(X_{K^p}, \mathbb{Q}_p) \) by composing with \( \mathcal{C}^{an}(K_p, \mathbb{Q}_p) \to \mathbb{Q}_p \), the evaluation map at identity element of \( K_p \). There is a natural \( p \)-adic Banach space structure on \( \mathcal{C}(X_{K^p}, \mathbb{Q}_p)^{K_p-an} \) coming from the norm on \( \mathcal{C}^{an}(K_p, \mathbb{Q}_p) \). Clearly Hecke operators away from \( p \) act on this Banach space \( \mathcal{C}(X_{K^p}, \mathbb{Q}_p)^{K_p-an} \).
Theorem 6.1. The Hecke action on $\mathcal{C}(X_{K^p}, \mathbb{Q}_p)^{K_p-\text{an}}$ is overconvergent.

Proof. We denote the open ball of $\mathcal{C}^\text{an}(K_p, \mathbb{Q}_p)$ by $\mathcal{C}^\text{an}(K_p, \mathbb{Q}_p)^o$ and $\mathcal{C}(X_{K^p}, \mathbb{Q}_p)^{K_p-\text{an}, o}$ is defined similarly. Then elements of $\mathcal{C}(X_{K^p}, \mathbb{Q}_p)^{K_p-\text{an}, o}/p$ can be identified with $K_p$-equivariant functions

$$\tilde{f}: X_{K^p} \to \mathcal{C}^\text{an}(K_p, \mathbb{Q}_p)^o/p.$$  

Note that $\mathcal{C}^\text{an}(K_p, \mathbb{Q}_p)^o/p$ is fixed by an open subgroup $K'_p$ of $K_p$ by [Pan20, Lemma 2.1.2.], i.e. any such $\tilde{f}$ factors through $X_{K^p}/K'_p$, which is a finite set. In particular, the action of Hecke operators away from $p$ necessarily factors through finite $\mathbb{F}_p$-algebra $\text{End}(\mathcal{C}(X_{K^p}/K'_p, \mathbb{F}_p))$. This verifies the definition of overconvergence. □

Remark 6.2. As in the case of overconvergent modular forms, this result has the following consequence in terms of spectral theory. Again the Hecke action induces an action of (product of) deformation rings $R^\text{an}$ of determinant of some $G_{Q,S}$ on everything. The generic fiber of $R^\text{an}$ is a (product of) Zariski-closed subset of some open unit polydisc. Theorem 6.1 implies that Hecke-eigenvalues appearing in space of $K_p$-analytic vectors are in a bounded region with radius strictly less than 1. The radius of this bounded region is clearly related to size of $K_p$, or equivalently, analyticity of $D_p^\ast$-action, and plays the role of conductor here.

This phenomenon is not new and in fact very classical in the case of $\text{GL}_1$: the moduli space of characters of $G_{Q,(p)}^{ab} \cong \mathbb{Z}_p^\times$ (for simplicity $S = \{p\}$ here) can be identified with several copies of open unit disc around 1 by looking at the image of $1 + p \in \mathbb{Z}_p^\times$ (or $1 + p^2$ if $p = 2$). The relation between the analyticity of a character $\chi$ and $\chi(1+p)−1$ (or $\chi(1+p^2)−1$ if $p = 2$) goes back to the work of Amice [Ami64].

References

[Ami64] Yvette Amice, Interpolation $p$-adique, Bull. Soc. Math. France 92 (1964), 117–180. MR 188199

[Bar76] Wolfgang Bartenwerfer, Der erste Riemannsche Hebbarkeitssatz im nichtarchimedischen Fall, J. Reine Angew. Math. 286(287) (1976), 144–163. MR 422680

[BC08] Laurent Berger and Pierre Colmez, Familles de représentations de de Rham et monodromie $p$-adique, no. 319, 2008, Repr´ esentations $p$-adiques de groupes $p$-adiques. I. Repr´ esentations galoisiennes et $(\phi, \Gamma)$-modules, pp. 303–337. MR 2493221

[Buz05] Kevin Buzzard, Questions about slopes of modular forms, no. 298, 2005, Automorphic forms, I, pp. 1–15. MR 2141701

[CE04] Frank Calegari and Matthew Emerton, The Hecke algebra $T_k$ has large index, Math. Res. Lett. 11 (2004), no. 1, 125–137. MR 2046205

[Che14] Gaëtan Chenevier, The $p$-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 221–285. MR 3444227

[Fal87] Gerd Faltings, Hodge-Tate structures and modular forms, Math. Ann. 278 (1987), no. 1-4, 133–149. MR 909221

[How20] Sean Howe, Overconvergent modular forms are highest weight vectors in the hodge-tate weight zero part of completed cohomology, 2020.

[Mat80] Hideyuki Matsumura, Commutative algebra, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980. MR 575344

[Pan20] Lue Pan, On locally analytic vectors of the completed cohomology of modular curves, 2020.

[Sch15] Peter Scholze, On torsion in the cohomology of locally symmetric varieties, Ann. of Math. (2) 182 (2015), no. 3, 945–1066. MR 3418533
[Sen88] Shankar Sen, *The analytic variation of $p$-adic Hodge structure*, Ann. of Math. (2) **127** (1988), no. 3, 647–661. MR 942523

[Sen93] ———, *An infinite-dimensional Hodge-Tate theory*, Bull. Soc. Math. France **121** (1993), no. 1, 13–34. MR 1207243

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