On the Inclusion Model of Localized Heating in an Elastic Plane

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Abstract. Failure localization in a variety of mechanical structures may be ascribed to elevated temperature, which may be effectively analyzed by employing the inclusion model. This work presents an explicit solution to the plane thermal inclusion problem, based on the customized Green’s function. A contour integral representation is further developed so as to provide an effective and straightforward approach for treating an arbitrarily shaped inclusion. Several benchmark examples are examined to validate the present solution.

1. Introduction

A common challenge in engineering design and mechanical manufacturing is to understand the failure mechanisms due to elevated temperature in a localized concentration. In micromechanics, such problem may be effectively analyzed by employing the inclusion model [1, 2]. The inclusion here refers to a subdomain with identical material as the surrounding matrix, and the thermal expansion strain can be regarded as eigenstrain [2] which is a generic name for nonelastic strains such as plastic strains, misfit strains and phase transformation. Other typical eigenstrains in engineering material may exist in the forms of precipitates, twinnings and martensitic transformations [3-5], to name a few. The mechanical behavior of materials may be seriously affected especially when these inclusions undergo a localized elevated temperature.

In literature, most of the previous thermal inclusion studies are mainly focused on the strain/stress solutions. Rodin et al. [6] derived an algorithmic closed-form solution for the polygonal and polyhedral inclusions. Downes and Faux [7] obtained a unified Green’s function for interior and exterior stress fields and then proposed a simple and straightforward method [8] to evaluate the stress and strain distributions. Nozaki and Taya [9, 10] investigated the elastic fields for a polygonal or polyhedral inclusion. Jin et al. [11] developed Green’s functions for a point eigenstrain in an infinite plane and then demonstrated their applications in evaluating the interior and exterior stress fields. Trotta et al. [4, 5] developed the analytical formulae of the Eshelby tensor for an arbitrary polygonal inclusion. Xie et al. [12] reported the study of the inclusion experiment, which however is beyond the scope of this research.

In this work, an explicit solution to the displacement field caused by a point thermal eigenstrain excitation is derived in closed-form, and then utilized to evaluate the displacement field in an infinite plane. Based on the method of Green’s function, the interior and exterior displacement is represented in a unified area integral. With the assistance of contour integral, analytical solutions are derived for
arbitrary polygonal inclusion subjected to uniform thermal eigenstrains. Benchmark examples are provided to validate the effectiveness of the present method of solution. The present method provides an effective method to evaluate the displacement field produced by randomly dispersed inclusions in an infinite plane (Figure 1).

**Figure 1.** Schematic illustration of the randomly dispersed inclusions in an infinite plane.

2. **Formulation**

2.1. **Unit point eigenstrain solution**
Consider a point eigenstrain excitation located at the source point \((\xi, \eta)\) in an infinitely extended isotropic medium (Figure 2). Jin et al. [11] has obtained Green’s functions for the stresses at a field point \((x, y)\) in an infinite plane. The methodology may also be applied to solve the displacement in this study.

Firstly, the analytical solution to the displacement field associated with a uniform rectangular eigenstrain \(\varepsilon^*\) may be obtained by superposing the contributions from the primitive functions.

\[
u(x, y; \xi, \eta) = \varepsilon^* \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} (-1)^{\alpha+\beta} w(x_\alpha, y_\beta)
\]

(1)
where \( w(x_u, y_u) \) is the primitive function and may be determined from [13]. The excitation at a source point may be obtained using a limit process [11] when the dimensions of side lengths \( \Delta x \times \Delta y \) approach to zero.

\[
e^* = \frac{1}{\Delta x \Delta y}
\]

(2)

It is worthwhile to note that the eigenstrain density at \( \Delta x=0, \Delta y=0 \) may be regarded as the limit of distributed eigenstrains [11, 14]. In other words, the limit in Eq. (2) may be expressed as the delta function [14]. The point eigenstrain solution may be consequently determined as

\[
G(x, y; \xi, \eta) = \text{lim}_{\Delta x \Delta y \to 0} \frac{1}{\Delta x \Delta y} \sum_{a=-1}^{1} \sum_{b=-1}^{1} (-1)^{a+b} w(x_u, y_u)
\]

(3)

In view of the definition of the partial derivatives, Eq. (3) may be reduced to

\[
G(x, y; \xi, \eta) = w_{xy}(x-\xi, y-\eta)
\]

(4)

The corresponding displacement caused by the aforementioned point eigenstrain excitation may be expressed in a tensorial form as:

\[
u_i(x, y) = G_{ii}(x, y; \xi, \eta) \epsilon^*_i(\xi, \eta)
\]

(5)

where \( G_{ii}(x, y; \xi, \eta) \) is termed as Green’s function, and is only associated with the relative position between the field point and the location of the source excitation,

\[
G_{0}(x, y; \xi, \eta) = G_{0}(x - \xi, y - \eta)
\]

(6)

For an elastic isotropic material, the eigenstrain prescribed in the source point \( (\xi, \eta) \) may be represented as [15]

\[
\epsilon_{ij}^* = \lambda_T (T - T_0) \delta_{ij}
\]

(7)

where \( \lambda_T \) is the coefficient of linear thermal expansion; \( T \) and \( T_0 \) are the temperatures of the current and the initial state; \( \delta_{ij} \) is the Kronecker delta.

When the eigenstrain is uniformly distributed in an inclusion, i.e. \( \epsilon_{11}^* = \epsilon_{22}^* = \epsilon^*, \epsilon_{12}^* = 0 \), the displacement components for the present thermal case are formulated as:

\[
\begin{align*}
u_1 &= \frac{2\epsilon^*}{\pi(1+\kappa)} \frac{X}{X^2 + Y^2} \\
u_2 &= \frac{2\epsilon^*}{\pi(1+\kappa)} \frac{Y}{X^2 + Y^2}
\end{align*}
\]

(8)

where \( \kappa \) is Kolosov’s constant,

\[
\begin{align*}
\kappa &= 3 - 4\nu, \quad \text{in plane strain} \\
\kappa &= \frac{3-\nu}{1+\nu}, \quad \text{in plane stress}
\end{align*}
\]

(9)

with \( \nu \) being Poisson’s ratio, and \( X, Y \) depending on the relative position between the response and excitation points, i.e.
\[ X = x - \xi; \quad Y = y - \eta \]  

(10)

### 2.2. Contour integral representation

In view of the method of Green’s function, the displacement for an arbitrarily shaped inclusion \( \Omega \) subjected to any distributed eigenstrain may be evaluated as

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \iint_{\Omega} \begin{bmatrix}
  G_{111} & G_{122} & G_{112} \\
  G_{211} & G_{222} & G_{212}
\end{bmatrix} \begin{bmatrix}
  \varepsilon_{11}^* \\
  \varepsilon_{22}^* \\
  \varepsilon_{12}^*
\end{bmatrix} d\xi d\eta
\]  

(11)

It should be emphasized that the interior and exterior displacement fields can be determined simultaneously by Eq. (11). Moreover, since the eigenstrain is uniformly distributed in the present thermal case and \( \varepsilon_{11}^* = \varepsilon_{22}^* = \varepsilon^* \), \( \varepsilon_{12}^* = 0 \), the term \( \varepsilon^* \) may be moved outside the integral sign, leading to the area integral of the point Green’s function. The resulting expression for the current case may be written as

\[ u_i = \varepsilon^* \iint_{\Omega} G_i d\xi d\eta \]  

(12)

where \( G_i \) is the point eigenstrain solution for the thermal case, and their details are referred to Eq. (8).

By applying the Green’s theorem over Eq. (12), the area integral may be converted to a contour integral along the boundary \( \partial \Omega \) of the inclusion domain.

\[
\iint_{\Omega} G_i d\xi d\eta = \oint_{\partial \Omega} K_i d\xi \quad (i = 1, 2)
\]  

(13)

where \( \partial \Omega \) is a closed curve and oriented in the positive direction with the kernel function \( K_i \) being derived as

\[
\begin{cases}
  K_1 = -4 \arctan \frac{Y}{X} \\
  K_2 = -2 \ln \left( X^2 + Y^2 \right)
\end{cases}
\]  

(14)

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**Figure 3.** Schematic illustration of a polygonal domain and a line element.

A typical line element having endpoints \((x_1, y_1)\) and \((x_2, y_2)\) constituting a part of the closed boundary curve is shown in Figure. 3. The field point is represented as \((x_0, y_0)\) and the excitation is
distributed in the polygonal domain. The elementary solution may be solved in closed-form, following the strategy proposed by Downes and Faux [7]. Therefore, for a polygonal inclusion of arbitrary shape, the resultant displacement may be obtained by summing the contributions from each line element.

3. Results and discussions

Based on the contour integral solution, a computer code is programmed using the FORTRAN language with double precision calculation. The rectangular inclusion associated with the thermal expansion is examined to validate the present solution. The rectangle having dimensions $\Delta x \times \Delta y$ and centered at the origin of the Cartesian coordinate systems is demonstrated in Figure 4.

In the benchmark study, the thermal eigenstrain is chosen as $\varepsilon_{11}^* = \varepsilon_{22}^* = \varepsilon_0$, $\varepsilon_{12}^* = 0$ and the side lengths are set to be $\Delta x = \Delta y$. Moreover, $\nu = 0.3$ and a vertical target line is placed at $x = \Delta x/4$. The exact displacement may be obtained from [13] by considering the degenerate 3D case of a cuboidal inclusion. Comparisons of the normalized displacement components along the vertical line are illustrated in Figure 5, and the solution of the present method perfectly agrees with the exact solution.

![Figure 4. Schematic illustration of the rectangular inclusion and the locations of the target line.](image1)

![Figure 5. Verification of the present analytical solutions along the target line with the results in reference [13] for a rectangular thermal inclusion.](image2)

As shown in Figure 5, the displacement component $u_1$ increase moderately from a minimum value at $(\Delta x/4, -2\Delta y)$ outside the cuboid and reaches its peak at the center of the inclusion, and then decreases as $y/\Delta y$ becomes larger. It can be found that the component $u_1$ is symmetric with respect to the $y$-axis. For the displacement component $u_2$, it undergoes a fast decline and arrives at the turning point when the field point varies along the target line with $y/\Delta y$ ranging from $-2\Delta y$ to $\Delta y/2$. It can be readily observed that the component $u_2$ is linearly distributed within the cuboidal inclusion. In contrast with $u_1$, the displacement component $u_2$ is antisymmetric with respect to the $y$-axis. Furthermore, it may be verified from the calculation that the displacement components $u_1$ and $u_2$ are continuous across the interface between the inclusion and the matrix.

It should be noted that the current work provides a closed-form analytical solution for arbitrary polygonal inclusions. Since the geometrical boundary surrounded by closed curves can be approximated by polygonal shape, the present solution is also applicable to derive a highly accurate numerical solution for thermal inclusions of arbitrary shape. In order to demonstrate the effectiveness of the present method for the inclusion bounded by arbitrary closed curves, numerical benchmark for a circular dilatational expansion is performed (Figure 6).
Figure 6. Schematic illustration of the circular inclusion and the target line.

In this work, the circular inclusion is centered at \((0,0)\) with radius \(r\) and the target line is \(x = 2/r\). The circular inclusion domain is approximated by a regular polygon of 1000 sides. The problem has a closed form solution [16], which is compared with the present numerical results in Figure 7. It is demonstrated that the present solution shows satisfactory agreement with the exact solution.

Figure 7. Verification of the normalized displacements along the target line for a circular thermal inclusion.

As demonstrated in Figure 7, the pure dilatational strains in this work cause positive displacement component \(u_1\) along the target line. When the dimensionless distance \(y/r\) varies along the vertical line from \(-2\Delta y\) to the origin of the Cartesian coordinates, the displacement component \(u_1\) increases gradually and tends toward a constant value inside the inclusion domain. Also, with increasing values of \(y/r\), the component \(u_2\) drops rapidly from the ending point \((\Delta x/4, -2\Delta y)\) to the interface, and then exhibits a linear distribution inside the inclusion domain. The corresponding contour plot of the normalized displacement component \(u_2/(\varepsilon_r)\) for a regular polygonal thermal inclusion with 1000 sides is demonstrated in Figure 8.

It is worthwhile to note that the total strain is compatible and may be determined from the aforementioned displacement components. Since the above results inside the thermal domain are either constant or linear, the induced strains are distributed uniformly in the inclusion. Once the strains are obtained, the stresses can be achieved directly by taking advantage of Hook’s law. Therefore, the
stress field is also uniform inside the circular domain. This interesting phenomenon was firstly found in Eshelby’s pioneering work [1], where the induced strains inside the ellipsoidal inclusion are uniform when a uniform eigenstrain is distributed in an ellipsoidal region contained in an infinitely extended isotropic medium. The results in this study immediately verify this remarkable property.

4. Conclusions
The quantitative determination of the displacement may be of significance in many engineering designs, and the problem may be of practical interest in the presence of the localized heating. However, the analytical solution for the displacements produced by an arbitrarily shaped inclusion has not been fully examined previously. Based on the inclusion model, this work presents an effective, elegant and versatile approach to solve the thermal displacement field in an infinite plane. The point eigenstrain excitation is analyzed and then employed as Green’s function for evaluating the response field. The area integral for the present study may be converted to a contour integral by virtue of the Green’s theorem. For any polygonal inclusion subjected to uniform thermal eigenstrain, the resultant solution may be derived in closed-form via the method of contour integral. The examples of the rectangular and circular thermal inclusions are examined to illustrate the effectiveness of the present solution.

5. References
[1] Eshelby J D, P. Roy. Soc. Lond. Ser. A, 241(1226), 376-396 (1957)
[2] Mura T, Micromechanics of defects in solids. 2nd ed. (Springer, Dordrecht, The Netherlands, 1987)
[3] Nozaki H, Horibe T, Taya M, JSME Int. J. Ser. A, 44(4), 472-482 (2001)
[4] Trotta S, Marmo F, Rosati L, Compos. Part B- Eng., 106, 48-58 (2016)
[5] Trotta S, Marmo F, Rosati L, Compos. Part B- Eng., 115, 170-181 (2017)
[6] Rodin G J, J. Mech. Phys. Solids, 44(12), 1977-1995 (1996)
[7] Downes J, D. Faux A, J. appl. Phys., 77(6): 2444-2447 (1995)
[8] Downes J R, Faux D A, E. P. O’Reilly, J. appl. Phys., 81(10), 6700-6702 (1997)
[9] Nozaki H, Taya M. J. appl. Phys., 64(3), 495-502 (1997)
[10] Nozaki H, Taya M. J. appl. Phys., 68(3), 441-452 (2001)
[11] Jin X, Keer L M, Wang Q, Int. J. Solids Struct., 46(21), 3788-3798 (2009)
[12] Xie L, Zhou Q, Jin X, Wang Z, Jiang C, Lu W, Wang J, Wang Q, Int. J. Fatigue, 66(Supplement C), 127-137 (2014)
[13] Liu S, Jin X, Wang Z, Keer L M, Wang Q, Int. J. Plasticity, 35, 135-154 (2012)
[14] Stakgold I, Green’s Functions and Boundary Value Problems. 3rd ed. (Springer, Wiley, New York, 2011)
[15] Segall P, Fitzgerald S D, Tectonophysics, 289(1), 117-128 (1998)
[16] Jin X, Zhang X, Li P, Xu Z, Hu Y, Keer L M, J. Appl. Mech.- T. ASME, 84(7), 074501-074501-6 (2017)

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