Quasi 1 and 2d Dilute Bose Gas in Magnetic Traps: Existence of Off-Diagonal Order and Anomalous Quantum Fluctuations

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Abstract

Current magnetic traps can be made so anisotropic that the atomic gas trapped inside behaves like quasi one or two dimensional system. Unlike the homogeneous case, quantum phase fluctuations do not destroy macroscopic off-diagonal order of trapped Bose gases in $d \leq 2$. In the dilute limit, quantum fluctuations increase, remain constant, and decrease with size for $3, 2, 1d$ respectively. These behaviors are due to the combination of a finite gap and the universal spectrum of the collective mode.

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The technology of magnetic traps has advanced rapidly since the first series of reports of Bose-Einstein condensation in atomic gases of $^{87}$Rb \cite{1}, $^{23}$Na \cite{2}, and $^{7}$Li \cite{3}. The latest generation of magnetic traps \cite{4} \cite{5} is capable of trapping $10^6$ atoms in the ground state, three orders of magnitudes higher than that in the first Rb condensate experiment \cite{1}. A common characteristic of these traps is that they are very anisotropic. The cylindrically symmetric harmonic potentials in these traps have transverse frequency $\omega_{\perp}$ about 20 times larger than the longitudinal frequency $\omega_z$. The resulting condensate looks like a cigar aligned along the symmetry axis $z$. With current technology, there appears no difficulty in achieving even higher asymmetry, say, with $\omega_{\perp}/\omega_z \sim 100$. In such limit, one can produce atomic gases with all the atoms lying in the lowest harmonic oscillator state in the $xy$-plane, leaving the
motion along $z$ the only degrees of freedom. The system then behaves like a one dimensional ($1d$) Bose gas $[3]$. Similarly, for $\omega_\perp/\omega_z << 1$, the system behaves like a $2d$ system $[4]$. The possibility of realizing these low dimensional systems raises the question of the effects of quantum fluctuations. For homogeneous Bose systems, it is well known that phase fluctuations destroy finite (zero) temperature Bose condensation in dimension $d \leq 2 (< 1)$. The situation, however, is different for trapped Bose gases. For non-interacting bosons, finite spacing between the lowest and the next energy level allow BEC to occur at finite temperatures even in $1D$ $[3] [8]$. The absence of gapless excitations also means that BEC will not be destroyed immediately as interactions between bosons are turned on. Nevertheless, for trapped bosons, quantum fluctuation effects due to interactions do show up in an unexpected way, which is the subject of this paper. We shall show that while the gapped nature of the collective modes eliminates the usual infrared divergence in all dimensions, the number of low energy modes contributing significantly to quantum fluctuation increases with the system size. As a result, quantum fluctuations suppresses but do not destroy the condensate. They also grow with size at a rate that increases with dimensionality. More precisely, we find that

(I) The density matrix $W(\mathbf{r},\mathbf{r}')$ of a $d$-dimensional dilute Bose gas in a harmonic trap exhibits macroscopic off-diagonal order of the form $W(\mathbf{r},\mathbf{r}') \rightarrow \phi^*(\mathbf{r})\phi(\mathbf{r}')$ for $|\mathbf{r} - \mathbf{r}'|$ of the order of the system size $R$, (both $\mathbf{r}$ and $\mathbf{r}'$ are inside the atom cloud); $\rho_o$ is the density of the system, $\phi(\mathbf{r})=\sqrt{\rho_o(\mathbf{r})}e^{-\gamma_d/2}$ (up to a phase factor), and $\gamma_d$ is a constant arising from the Gaussian fluctuations about the mean field solution : $\gamma_1 \propto R^{-1}\ln(R/\sigma)$, $\gamma_d \propto (R/\sigma)^{d-2}$ for $d \geq 2$, where $\sigma$ is the width of the ground state Gaussian of the $d$-dimensional harmonic well with frequency $\omega$, $\sigma = \sqrt{\hbar/M\omega}$. These results show that within the dilute regime, the Gaussian approximation becomes more (less) accurate with increasing size in $1d$ ($3d$). This anomalous dimensionality dependence of the quantum fluctuations can be directly confirmed experimentally by studying how the momentum distribution depends on the number of particles.

(II) The behavior in (I) is due to the gapped nature as well as the universal feature of the
We shall first derive these results, and then discuss their implications for real atomic gases. Let us consider a $d$-dimensional dilute Bose gas with a partition function $Z = \text{Tr}e^{-\beta(H - \mu N)}$, where $\mu$ is the chemical potential, $K = H - \mu N = \int d^d r [(\hbar^2/2M) \nabla \hat{\psi}^+ \cdot \nabla \hat{\psi}+(U - \mu) \hat{\psi}^+ \hat{\psi} + (g_d/2) \hat{\psi}^+ \hat{\psi} \hat{\psi}^+ \hat{\psi}]$, $g_d > 0$, $U(r) = (1/2)M \omega^2 |r|^2$ is a $d$-dimensional harmonic well, and $\nabla$ is a $d$-dimensional gradient. $H$ is an effective low energy Hamiltonian where the Fourier component of $\hat{\psi}$ with wavelength shorter than an atomic scale $a_c$ has been coarse-grained out.

It is convenient to write the partition function as a coherent state path integral $Z = \int D\phi^* D\phi e^{-S/\hbar}$, where

$$ S = \int \left[ \hbar \phi^* \partial_\tau \phi + \frac{\hbar^2}{2M} |\nabla \phi|^2 + (U - \mu) |\phi|^2 + \frac{g_d}{2} |\phi|^4 \right], $$

with $\int \equiv \int d^d r \int_0^\beta d\tau$. Anticipating a condensate, we write $\phi = e^{i\theta} \sqrt{\rho}$. The stationary point is a constant phase $\theta_o$ and a density $\rho_o(r)$ satisfying the Gross-Pit'evskii (GP) equation,

$$ -\frac{\hbar^2}{2M} \nabla^2 \sqrt{\rho_o} + (U - \mu) \sqrt{\rho_o} + g_d \rho_o \sqrt{\rho_o} = 0. $$

Expanding $S$ about $(\theta_o, \rho_o(r))$ to quadratic order in phase fluctuations $\theta_1$ and density fluctuations $\rho_1$, $(\rho_1 = \rho - \rho_o, \theta_1 = \theta - \theta_o)$, we have $S = S_o[\rho_o, \theta_o] + S_1$, where

$$ S_1 = \int \left[ i\hbar \rho_1 \partial_\tau \theta_1 + \frac{1}{2} \theta_1 \cdot \hat{T} \cdot \theta_1 + \frac{1}{2} \rho_1 \cdot \hat{G} \cdot \rho_1 \right], $$

$$ \theta_1 \cdot \hat{T} \cdot \theta_1 = -\frac{\hbar^2}{2M} \nabla \left( \rho_o \left[ \frac{\hbar^2}{M} \right] \nabla \theta_o \right), $$

$$ \rho_1 \cdot \hat{G} \cdot \rho_1 = g_d \rho_1^2 - (\hbar^2/2M) [(\rho_1/\sqrt{\rho_o}) \nabla^2 (\rho_1/\sqrt{\rho_o}) - (\rho_1^2/\rho_o^{3/2} \nabla^2 \sqrt{\rho_o})] $$

One can also integrate out the density fluctuations and obtain an effective action of phase fluctuations $\theta_1$,

$$ S_{\text{phase}} = \int \left[ \frac{1}{2} \hbar^2 (\partial_\tau \theta_1) \cdot \hat{G}^{-1} \cdot (\partial_\tau \theta_1) + \frac{1}{2} \theta_1 \cdot \hat{T} \cdot \theta_1 \right]. $$

Quantum fluctuations are caused by zero point motions of the excitations. The low-lying excitations are phonons, corresponding to density oscillations inside the atomic cloud.
At high excitation energy, the period of density oscillations become comparable to cloud size, the excitations leak out of the cloud and become single-particle like, and cannot be adequately described by the phase action eq. (6). The equations of motion of the collective modes in real time \( t \) can be obtained by first analytically continuing the action eq.(3) to real time (\( \tau \to it \)) and then finding the stationary conditions. They are \( \hbar \partial_t \theta_1 = \hat{G} \rho_1 \) and \( -\hbar \partial_t \rho_1 = \hat{T} \theta_1 \). The wavefunctions and the frequencies of the collective modes (labelled by a set of quantum numbers \( \alpha \equiv \{\alpha_i\} \)) are the eigenfunctions and eigenvalues of the operator \( \hat{G} \hat{T} \), i.e.

\[
\hat{G} \hat{T} u_\alpha = \hbar^2 \Omega_\alpha^2 u_\alpha.
\]

(7)

The \( \Omega = 0 \) mode (corresponding to a constant phase) is excluded from the collective excitation spectrum because it is simply the ground state. Since all other modes are orthogonal to the \( \Omega = 0 \) mode, they have zero spatial average — a fact that shall be of use to us later.

In 1956, Penrose and Onsager [9] pointed out a unified description of Bose-Einstein condensation (BEC) for both noninteracting and interacting systems based on the general structure of the density matrix \( W(r,r') = \langle \psi^+(r) \psi(r') \rangle = \sum_\alpha \lambda_\alpha \nu_\alpha^*(r) \nu_\alpha(r') \), where \( \lambda_\alpha \) and \( \nu_\alpha \) are the eigenvalues \( \lambda_\alpha \) and normalized eigenfunctions of \( W \). BEC is characterized by the fact that below a transition temperature \( T_c \), one of the eigenvalues (say \( \lambda_0 \)) becomes order \( N \) compared with all others,

\[
W = \lambda_0 \nu_0^*(r) \nu_0(r') + K(r,r'), \quad \lambda_0 / \lambda_{i \neq 0} \sim N,
\]

(8)

where \( K \) denotes the sum of those eigenstates \( \lambda_{i \neq 0} \) and (for homogeneous systems) has the property that \( K(r,r') \to 0 \) as \( |r - r'| \to \infty \). The first term of eq.(8), which represents condensation, is referred to off-diagonal-long-range order [10]. For trapped Bose gases, however, both \( r \) and \( r' \) must remain inside the atomic cloud. The large distance behavior of \( W \) is masked by finite size effects. In spite of this, the appearance of a macroscopic eigenvalue as described in eq.(5) remains a proper characterization of BEC [8].

To demonstrate the existence of macroscopic off-diagonal order in trapped Bose gases in all dimensions at \( T = 0 \), we write
\[ W(r, r') = \frac{1}{\sqrt{\rho(r)\rho(r')}} e^{-i[\theta(r) - \theta(r')]} = \sqrt{\rho_o(r)\rho_o(r')} \left( e^{-F(r, r')} + Y(r, r') \right), \quad (9) \]

where \( F(r, r') \equiv \frac{1}{2} < \left| \theta_1(r) - \theta_1(r') \right|^2 > \), and

\[ Y(r, r') = \frac{-i}{2} \left( \frac{\rho_1(r)}{\rho_0(r)} + \frac{\rho_1(r')}{\rho_0(r')} \right) \left( \theta(r) - \theta(r') \right) \]

\[ + \frac{\langle \rho_1(r)\rho_1(r') \rangle}{4\rho_o(r)\rho_o(r')} - \frac{\langle \rho_0^2(r) \rangle}{8\rho_o^2(r)} + \frac{\langle \rho_0^2(r') \rangle}{8\rho_o^2(r')} + O \left( \left( \frac{\rho_1}{\rho_0} \right)^{3/2} \right) \quad (10) \]

As we shall see later, the term \( Y(r, r') \) is smaller than the first term in eq. (9) by at least a factor of \((\sigma/R)^4\) where \( R \) is the size of the cloud. We can then ignore the term \( Y(r, r') \) in eq. (9). Using eq. (9), it is straightforward to show that

\[ F(r, r') = \sum_\alpha \left| \langle r | \hat{G}^{1/2} | \alpha \rangle - \langle r' | \hat{G}^{1/2} | \alpha \rangle \right|^2 \frac{1}{2\hbar \Omega_\alpha}. \quad (11) \]

where \( \langle r | \alpha \rangle = u_\alpha(r) \). At first sight, the phase fluctuation appears to decrease with the size of the system \( R \) because \( u_\alpha^2 \) scales as \( R^{-d} \). This, however, is not true because the \( \alpha \)-sum (which depends on the number of excitations contributing significantly to the phase fluctuation) is also \( R \) dependent. To evaluate \( F \), we need to study the collective modes in greater detail. For systems with large number of particles, it is known that Thomas-Fermi approximation (TFA) becomes accurate [11]. TFA ignores the gradient term in eq. (2). It implies that

\[ \rho_o(r) = \frac{M \omega^2 R^2}{2g_d} \left( 1 - \frac{r^2}{R^2} \right) \Theta(R^2 - r^2), \quad \mu = \frac{1}{2} M \omega^2 R^2, \quad (12) \]

where \( \Theta(x) = 1 \) (or 0) of \( x > 0 \) (or \( < 0 \)). \( R \) is the size of the system. The number constraint \( N = \int \rho_o \) implies that the system size \( R \) is related to \( N \) as

\[ R/\sigma = \left[ g_d N (d + 2)/(\hbar \omega c_d \sigma^d) \right]^{1/(d+2)}, \quad (13) \]

where \( c_d \sigma^d \) is the volume of a \( d \)-dimensional sphere of radius \( \sigma \). The fact the \( \rho_o \propto g_d^{-1} \) in TFA means that \( \hat{T} \propto g_d^{-1} \). In addition, TFA also implies that the term \((\hbar^2/4M)\rho_o^2/\rho_o^2 \nabla^2 \sqrt{\rho_o})\) in \( \hat{G} \) (eq. (9)) is negligible compared to \( g_d \). Furthermore, for density oscillations with period small compared to cloud size (phonon regime, see below), \((\hbar^2/4M)\)
\[(\rho_1/\sqrt{\rho_o})\nabla^2(\rho_1/\sqrt{\rho_o}) \ll g_d, \text{ and can also be ignored. As a result, } \hat{G} = g_d, \text{ and the explicit } g \text{ dependence in eq.} (\tilde{G}) \text{ is cancelled out. For harmonic potentials, eq.} (\tilde{G}) \text{ becomes } \]\\(\frac{1}{2}\omega^2 \nabla([R^2 - r^2]\nabla u_\alpha) = \Omega^2_\alpha u_\alpha, \quad (14)\)

For \(d = 1\), eq. (14) is the Legendre equation. The normalized eigenfunctions are \(u_\ell(r) = (2R)^{-1/2}P_\ell(r/R)\), where \(P_\ell(x)\) are Legendre polynomials. The frequency of \(u_\ell\) is \(\Omega_\ell = (\omega/\sqrt{2})\sqrt{\ell(\ell + 1)}, \ell = 1, 2, \ldots\) In \(d = 2\), it is straightforward to show that the eigenfunctions are of the form \(u_{n,m}(r, \phi) = e^{-im\phi}\sum_{\ell=|m|}^{n} c_\ell(r/R)^\ell\) with \(n = |m| + 2p, p = 0, 1, 2, \ldots\) The corresponding frequencies are \(\Omega_{n,m} = (\omega/\sqrt{2})\sqrt{n^2 - m^2 + 2n}\). We have thus established statement (Π). The collective mode in \(d = 3\) has been derived by Stringari [12]. Because of the gapped nature of the collective mode, the phase fluctuation eq. (11) reduces its zero temperature form for \(k_B T < \bar{\hbar}\omega\). In other words, the existence of macroscopic off-diagonal order at \(T = 0\) immediately implies its existence at sufficiently low temperatures. We shall therefore from now on focus on the \(T = 0\) case.

**Dilute condition and phonon cutoff**: To evaluate eq. (Π), it is necessary to discuss a number of implicit constraints implied by the low energy model (eq. (Π)), as well as how the collective modes behave as their energy increases. First of all, to stay in the dilute limit, the density at the center of the trap must be less that the inverse volume of the atomic cutoff, \(\rho_o(0) < (c_d a^d_c)^{-1}\). This limits the size of the cloud to be smaller than \(R_m\),

\[
\left(\frac{R}{\sigma}\right)^2 < \frac{g_d}{\frac{1}{2}\hbar\omega c_d a^d_c} \equiv \left(\frac{R_m}{\sigma}\right)^2 : \text{dilute condition.} \quad (15)
\]

For a cloud of radius \(R\) satisfying eq. (13), the collective modes will be labelled by \(d\) quantum numbers \(\alpha_i, i = 1, \ldots, d\), denoting the number of nodes in \(d\) different curvilinear directions. The spatial variations of this mode along a curvilinear direction is given by an averaged wavenumber \(k_i \sim \alpha_i/R\). The corresponding gradient energy is \(\hbar^2 k_i^2/2M\). As the energy of the collective mode increases (i.e. increasing \(\alpha_i\)), \(k_i\) will increase to the point where the condition \((\hbar^2/4M) [(\rho_1/\sqrt{\rho_o})\nabla^2(\rho_1/\sqrt{\rho_o})] \ll g_d \rho_1^2\) breakdowns. This occurs when \(\hbar^2 k_i^2/2M \sim g_d \rho(0) = \frac{1}{2} M \omega^2 R^2\), which is \(k_i = R/\sigma^2\), or
Obviously, both $R_m$ and $\alpha_{ph}$ are only defined up to some numerical factor. The collective modes with quantum numbers below $\alpha_{ph}$ are described by eq.\(14\). For frequencies $\Omega > \Omega_{\alpha_{ph}}$, the excitation eq.\(14\) will not be phonon like. Instead, they leak out of the cloud and become single-particle like. The action eq.\(3\) is no longer sufficient. Eventually, the average wavevector of the collective mode will approach the “high energy” cutoff $k_i \sim 1/a_c$, corresponding to

$$\alpha_i \sim R/a_{sc} \equiv \alpha^*,$$

in which case the low energy model eq.\(1\) is no longer valid. Excitations between the phonon cutoff $\alpha_{ph}$ and the high energy cutoff $\alpha^*$ will be referred to as “high energy” modes. (Clearly we need to have $R/a_c > (R/\sigma)^2$, i.e. $\sigma/a_c > R/\sigma$) for the cutoffs in eq.\(16\) and eq.\(17\) to be consistent. For $\sigma/a_c \sim 10^2$, this condition is easily satisfied even for large clouds with $R/\sigma \sim 20$.

Now we return to eq.\(9\). We note that the first term in eq.\(10\) is odd in $\theta$ and vanishes by symmetry. For the second term in eq.\(10\), it is straightforward to show \[13\] that its contribution from a given phonon mode of average momentum $k \sim 1/R$ is down from the corresponding phase fluctuation by $\hbar^2k^2/(Mg_d\rho_0) \sim (\sigma/R)^4 \ll 1$ for large clouds. We can therefore ignore the density fluctuation term etc in eq.\(9\) as mentioned before. Writing $F(r, r') = (\sum_{\alpha_i < (R/\sigma)^2} + \sum_{(R/\sigma)^2 < \alpha_i < R/a_c})(\ldots) \equiv F_{ph} + F_{high}$, where $(\ldots)$ is the summand of eq.\(11\). The leading $R$ dependence comes from $F_{ph}$. The physics of $F_{high}$ and high energy density fluctuations is analogous to the similar high energy contributions in homogeneous systems, which are known to be less important than the phonon contributions for large $|r - r'|$ \[13\]. By noting that within the phonon regime, (i.e. for excitations with $\alpha_i < \alpha_{ph}$), the operator $\hat{G}$ reduces to the constant $g_d$, Eq.\(11\) implies that

$$F_{ph}(r, r') = \sum_{\alpha_i < (R/\sigma)^2} \frac{g_d}{2\hbar\Omega_{\alpha}} |u_{\alpha}(r) - u_{\alpha}(r')|^2.$$  \hspace{1cm} (18)
Denoting the spatial average and fluctuation of a function \( f(\mathbf{r}) \) as \( \overline{f} \) and \( \Delta(f) \), i.e. \( f = \overline{f} + \Delta(f) \), we have
\[
F_{ph}(\mathbf{r}, \mathbf{r}') = \gamma_d + \eta_d(\mathbf{r}, \mathbf{r}'), \quad \gamma_d = \frac{g_d}{c_d R^d} \sum_{\alpha < (R/\sigma)^2} \frac{1}{\hbar \Omega_\alpha},
\]
and \( \eta_d(\mathbf{r}, \mathbf{r}') = (g_d/2\hbar \Omega_\alpha) \sum_{\alpha < (R/\sigma)^2} \left[ \Delta(|u_\alpha(\mathbf{r})|^2 + |u_\alpha(\mathbf{r}')|^2) - (u_\alpha(\mathbf{r})^* u_\alpha(\mathbf{r}') + \text{c.c.}) \right] \).

To estimate \( F_{ph} \), we note that there is a one-to-one correspondence between the solutions of eq.(14) and those of the equation
\[
- \left( \omega^2 R^2 / 2 \right) \nabla^2 u = \Omega^2 u,
\]
as both of them are labelled by the same quantum numbers. Moreover, the solutions of eq.(14) resemble those of eq.(20) as the number of nodes increases so that the gradient term becomes more important. We thus approximate collective mode frequencies and wavefunctions in eq.(18) by those of eq.(20). The normalized functions \( u_\alpha(\mathbf{r}) \) will be replaced normalized plane waves \( u_\mathbf{k}(\mathbf{r}) = e^{i \mathbf{k} \cdot \mathbf{r}} / \sqrt{c_d R^d} \), where \( \mathbf{k} \) is a \( d \)-dimensional wave-vector. \( k_i R \) is the correspondence of \( \alpha_i \) as it counts the number of nodes along the \( i \)-th (linear) direction.

The frequency \( \Omega_\alpha \) in eq.(18) corresponds to \( \omega(Rk)/\sqrt{2} \) in eq.(20). With this approximation, eq.(18) becomes
\[
F_{ph}(\mathbf{r}, \mathbf{r}') \approx \frac{1}{c_d R^d} \frac{\sqrt{2} g_d}{\hbar \omega} \int_1^{(R/\sigma)^2} \frac{d^d(kR)}{(2\pi)^d} \left( \frac{1 - \cos \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}{kR} \right). \quad (21)
\]
The constant and the cosine term in the bracket in eq.(21) correspond to the average and fluctuation term (\( \gamma_d \) and \( \eta_d \) in eq.(19) respectively). The lower limit is due to the fact that the lowest collective mode has an average wavelength \( R \) (or wavevector \( 1/R \)).

Eq.(21) implies that
\[
F_{ph}^{d=1}(\mathbf{r}, \mathbf{r}') = \frac{\sqrt{2} \pi}{2} \left( \frac{g_1}{2\sigma \hbar \omega} \right) \left( \frac{\sigma}{R} \right) \left[ 2 \ln \left( \frac{R}{\sigma} \right) - \text{Ci} \left( \frac{2R^2 x}{\sigma^2} \right) + \text{Ci}(2x) \right], \quad (22)
\]

\[
F_{ph}^{d=2}(\mathbf{r}, \mathbf{r}') = \frac{\sqrt{2}}{4\pi} \left( \frac{g_2}{\hbar \omega \pi \sigma^2} \right) \left( \frac{\sigma}{R} \right)^2 \left[ \left( \frac{R}{\sigma} \right)^2 - 1 - \sqrt{\frac{\pi}{2}} \Gamma(1/2) \left[ J_0(a) H_{-1}(a) - H(a) J_{-1}(a) \right]_{a=R^2/\sigma^2} \right], \quad (23)
\]
\[ F_{ph}^{d=3}(r, r') = \frac{\sqrt{2}}{2\pi^2} \left( \frac{\sqrt{2} g_3}{\hbar \omega (4\pi/3) \sigma^2} \right) \left( \frac{\sigma}{R} \right)^3 \left[ \frac{1}{2} \left( \frac{R}{\sigma} \right)^4 - \frac{1}{2} - \frac{1}{x} (J_1 \left( \frac{R^2 x}{\sigma^2} \right) - J_1(x)) \right] \]  

(24)

where \( x = |r - r'|/R \), \( \text{Ci} \) is the Cosine integral, \( J_0, J_{-1} \) are Bessel functions, \( \Gamma \) is the Gamma function, and \( H_o, H_{-1} \) are the Struve functions. The first and second terms in these expressions are the constant \( \gamma_d \) and the function \( \eta_d \) in eq.(19). It is easy to show that \( \eta_d \to 0 \) as \( |r - r'| \to \infty \). Even for \( |r - r'|/\sigma \sim 2 \), \( \eta_d << \gamma_d \) provided \( R/\sigma > 5 \). In other words, for large clouds, we have \( F_{ph} \sim \gamma_d \) when \( |r - r'| \sim R/2 \). Eq.(9) and (11), and the fact that \( F \sim F_{ph} \) implies statement (I). Note that although the cutoff \( \alpha_{ph} \) in our derivation contains an uncertain numerical factor, it does not affect the \( R \) dependence of our results.

Our results also leads to the interesting conceptual point that for a true 1d system with a \( \delta \)-function potential of finite strength, the mean field off-diagonal form of density matrix becomes exact as \( R \to \infty \).

The anomalous \( R \) dependence of our results can be qualitatively understood simply as follows. The density of trapped bosons is not arbitrary, but increases with size \( R \) according to the scaling relation \( \rho \sim R^2 \). For a bulk homogenous system, the dimensionless coupling constant is \( \frac{mg_d \bar{\hbar}^2 \rho(d-2)/2}{2 \hbar^2 \rho(d-2)/2} \), and so increasing density implies stronger (weaker) coupling in 3D (1D). Indeed, the \( R \) dependence of the condensate depletion in \( d \)-dimension is obtained if we replace the trapped boson system with a homogenous system of size \( R \) and density \( \rho \sim R^2 \).

**Atomic gases in quasi 1d and 2d limit**: Our previous discussions show that atomic Bose condensates exist in these quasi low dimensional regimes at sufficiently low (non-zero) temperatures. These condensates can be detected through their collective modes in Statement (II). In addition, the momentum distribution \( n(k) \) (i.e. Fourier transform of \( W \)) will reflect the size dependence of the quantum fluctuation.

When only the lowest harmonic quantum state in the tightly confining direction is occupied, the coupling constant in the effective quasi 1d and 2d theory is given by

\[ g_1 = 2\hbar \omega_\perp a_{sc}, \quad g_2 = 2\sqrt{2\pi} \hbar \omega_z a_z a_{sc}, \]  

(25)

where \( \sigma_\perp \) and \( \sigma_z \) are the widths of the ground state harmonic oscillator along \( z \) and in the
$xy$-plane, and $a_{sc}$ is still the $s$-wave scattering length in three dimension. The quantum suppression parameters $\gamma_d$ read from first terms of $F_{\text{ph}}^d(r,r')$, $d = 1, 2, 3$, are then

$$
\gamma_3 = \frac{3\sqrt{2}}{8\pi^2} \left( \frac{a_{sc}}{\sigma} \right) \left( \frac{R}{\sigma} \right), \quad \gamma_2 = \frac{1}{\sqrt{\pi^3}} \left( \frac{a_{sc}}{\sigma_\perp} \right) \sqrt{\frac{\omega_\perp}{\omega_\perp}}, \quad \gamma_1 = \frac{2\sqrt{2}}{\pi} \left( \frac{\omega_\perp}{\omega_z} \right) \left( \frac{a_{sc}}{R} \right) \ln \left( \frac{R}{\sigma_z} \right). \quad (26)
$$

From eq. (13), we see that $\gamma_d \sim (a_{sc})^{(d+1)/(d+2)}$, and so these effects are particularly noticeable in systems with large $a_{sc}$.

In summary, we have shown that the quantum fluctuations in trapped boson systems exhibit anomalous dimensionality dependence. Our results may be observable in experiments on quasi-$1D$ and $2D$ systems with anisotropic traps.
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