Fixed-Point Design of Generalized Comb Filters: A Statistical Approach

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Abstract—This paper is concerned with the problem of designing computationally efficient Generalized Comb Filters (GCF). Basically, GCF filters are anti-aliasing filters that guarantee superior performance in terms of selectivity and quantization noise rejection compared to classical comb filters, when used as decimation filters in multistage architectures.

Upon employing a partial polyphase (PP) architecture proposed in a companion paper, we develop a sensitivity analysis in order to investigate the effects of the coefficients’ quantization on the frequency response of the designed filters.

We show that the sensitivity of the filter response to errors in the coefficients is dependent on the particular split of the decimation factor between the two sub-filters constituting the PP architecture. The sensitivity analysis is then used for developing a fixed-point implementation of a sample filter from the class of GCF filters, used as reference filter throughout the paper.

Finally, we present computer simulations in order to evaluate the performance of the designed fixed-point filters.

Index Terms—CIC-filters, comb, decimation, decimation filter, delta, delta-sigma, fixed-point, GCF, generalized comb filter, partial polyphase, polyphase, ΣΔ, sigma, sigma-delta, sinc filters.

I. INTRODUCTION AND PROBLEM FORMULATION

The design of computationally efficient decimation filters for oversampled ΣΔ A/D converters [1]-[3], as well as for classical oversampled A/D converters, has received a renewed interest recently, spurred by intense research activities in connection to the design of digital front-ends for both wideband digital receivers and Software Defined Radio receivers [4]-[6].

Consider a base-band analog signal \( x(t) \) (with bandwidth \([-f_s, +f_s]\)) sampled by an Analog-to-Digital (A/D) converter at rate \( f_s = 1/T_s = 2\rho f_x \gg 2f_x \), where \( \rho \geq 1 \) is the so-called oversampling ratio. If \( \rho \) is close to unity, the A/D converter operates at the Nyquist frequency, whereas for \( \rho \gg 1 \) we are referring to oversampled A/D converter. When \( \rho \gg 1 \), the decimation of the oversampled discrete-time signal \( x(n/f_s) \) is usually accomplished by cascading two (or more) decimation stages, followed by a FIR filter that provides the required selectivity on the discrete-time signal \( x(nT_N) \) at baseband.

Fig. 1 shows a multistage decimation architecture composed by \( m \) decimation stages operating on the oversampled signal \( x(nT_s) \). Also shown are the data rates of the sampled data at the input, as well as at the output, of each decimation stage in the multistage decimation architecture.

For the sake to contain the computational complexity of the overall architecture, the first decimation stage usually employs a multiplier-less filter [7]-[8]. A widely used filter featuring this property is the comb filter [2], [3], [9], which provides an intrinsic anti-aliasing effect by placing its zeros in the middle of each folding band, i.e., in the integer multiples of the digital frequency \( 1/D \) (\( D \) is the decimation factor). The transfer function of a \( N_c \)-th-order comb filter is defined as [2]:

\[
H_C(z) = \left( \frac{1}{D} \frac{1}{1 - z^{-D}} \right)^{N_c} = \frac{1}{D^{N_c}} \prod_{i=1}^{D-1} \left( 1 - z^{-1} e^{i2\pi i/D} \right)^{N_c},
\]

(1)

where \( D \) is the decimation factor.

Unless the design of classical FIR filters, the design of a decimation filter embedded in a multistage architecture imposes stringent constraints on proper bandwidths. Let us elaborate.

Consider the architecture shown in Fig. 1 where the oversampling ratio \( \rho \) is factorized as

\[
\rho = \prod_{i=1}^{m} P_i.
\]

In the previous relation, each \( P_i \) is a proper positive integer. The sampling rates at the input and output of the \( i \)-th stage are, respectively,

\[
f_{i-1} = f_i \cdot P_i, \quad \forall i = 1, \ldots, m,
\]

with \( f_0 = f_s \), and

\[
f_i = \frac{f_s}{\prod_{k=i}^{m} P_k}, \quad \forall i = 1, \ldots, m.
\]

With this setup, consider the frequency response \( H_i(f_d) \) of the \( i \)-th decimation filter pictorially shown in Fig. 2. The digital frequency \( f_d^{-1} \) is the normalized signal bandwidth at the input of the \( i \)-th decimation filter. Notice that, for \( i = 1 \), it is \( f_d^1 = f_c/f_s = (2\rho)^{-1} \); this is the normalized bandwidth of the signal sampled by the A/D converter at rate \( f_s \). For any other \( i \), the relation between \( f_d^i \) and \( f_d^0 \) is

\[
f_d^i = f_d^{i-1} P_{i-1}, \quad \forall i = 1, \ldots, m.
\]

Given this setup, the frequency response \( H_i(f_d) \) has to attenuate the quantization noise (QN) within the frequency bands

\[
\left[ \frac{k}{P_k} - f_d^{i-1}, \frac{k}{P_k} + f_d^{i-1} \right], \quad k = 1, \ldots, k_M
\]

\[
k_M = \left\lfloor \frac{f_d^1}{2} \right\rfloor,
\]

\[
k_M = \left\lfloor \frac{f_d^1 \cdot P_{i-1}}{2} \right\rfloor,
\]

(2)

because the QN falling inside these frequency bands, will fold down to baseband (i.e., within the useful signal bandwidth \([-f_c^{-1}, +f_c^{-1}]\)) due to the sampling rate reduction by \( P_i \) in the \( i \)-th decimation stage [10]. Such a QN will irremediably affect the signal resolution after the multistage decimation.
architecture. On the other hand, the frequency ranges labeled as \textit{don’t care} bands in Fig. 2 do not require stringent selectivity, since the QN within these bands will be rejected by the subsequent anti-aliasing filters in the multistage chain.

With this background, let us provide a survey of the recent literature related to the problem addressed in this paper. Tutorials on the design of multirate filters can be found in [11], [12], while essential books on this topic are [7]-[8]. The design of optimized multistage decimation and interpolation filters has been recently addressed by Coffey in [13]-[14], while the design of multistage decimation architectures relying on constituent cyclotomic polynomial filters has been presented in [10]. A 3rd-order modified decimation sinc filter was proposed in [15], and developed in [16]. The class of comb filters was generalized in [17], whereby the author proposed an optimization framework for deriving the optimal zero rotations of GCFs for any filter order and decimation factor \( D \).

Other works somewhat related to the topic addressed in this paper are [18]-[24]. In [18] and [19], the authors proposed computational efficient decimation filter architectures for implementing non recursive classical comb filters. In [20], the authors proposed the use of decimation sharpened filters embedding comb filters, whereas in [21]-[22] the authors addressed the design of a novel two-stage sharpened comb decimator. In [23], the authors proposed novel decimation schemes for \( \Sigma \Delta \) A/D converters based on Kaiser and Hamming sharpened filters, then generalized in [24] for higher order decimation filters. Papers [25]-[27] focus on the design of decimation filters with improved frequency responses.

The main aim of this paper is to develop a mathematical framework for the design of fixed-point GCF decimation filters relying on a partial polyphase FIR architecture proposed in the companion paper [28]. To this end, we develop a sensitivity analysis in order to investigate the effects of the coefficients’ quantization on the frequency response of the designed filters.

For conciseness, we focus on the design of the first decimation filter in the multistage architecture in Fig. 1 even though the proposed analysis can be easily extended to the design of the other anti-aliasing filters in the cascade.

The sensitivity analysis paves the way to a statistical approach useful to identify the coefficient word lengths of the proposed fixed-point architecture. Moreover, we show that the proposed analysis highlights some key issues in connection to the choice of the proper split of the decimation factor between the polyphase stage and the cascaded FIR sections of the employed partial polyphase architecture.

The rest of the paper is organized as follows. In Section III we briefly review the transfer functions of GCF filters, as well as the partial polyphase architecture employed throughout the paper, and outline the key advantages that these filters feature with respect to classical comb filters. Section IV presents a mathematical framework for evaluating the sensitivity of the frequency response of GCF filters to the quantization of the coefficients. In Section V we discuss general guidelines for the design of the proposed filters, and present some simulation results. Finally, Section VI draws the conclusions.

II. OVERVIEW OF GCF FILTERS: THE PARTIAL POLYPHASE ARCHITECTURE

Our objective in this section is to recall the fundamental concepts for analyzing both the time-domain behaviour and the frequency response of GCF filters, as well as to highlight the main differences between GCF and classical comb filters.

For conciseness, our discussion will be restricted to a 3rd-order GCF filter, which will be used as a reference scheme throughout the paper, and we will present the non-recursive, partial polyphase architecture developed in the companion paper [28].

1Even though both recursive and non recursive implementations can be devised for GCF filters, the non-recursive architecture does not present any instability problem deriving from the quantization of the coefficients. We notice in passing that recursive GCF filter realizations rely on zero-pole cancellations, which can be impaired by the quantization of the coefficients. This is the basic reason for the use of this FIR architecture in the developments that follow.
Let us focus on the design of the i-th decimation filter, $H_i(z)$, in Fig. 1 and, for ease of notation, assume $P_i = D$ and omit the subscript $i$ in $H_i(z)$. Given $D$, and recalling the definition of the folding bands in (2), a classical 3rd-order comb filter (see (1) with $N_c = 3$) presents 3rd-order zeros in the complex locations

$z_k = e^{j2\pi k}, \forall k = 1, \ldots, D - 1,$

or, equivalently, in the digital frequencies $f_{zh} = \frac{k}{D}, k \in \{1, \ldots, k_M\}$. Therefore, a 3rd-order zero is placed in the middle of each folding band. This idea is illustrated in Fig. 3, where the k-th folding band is shown in the z-plane, as well as in the frequency domain.

On the other hand, a 3rd-order GCF filter places, in the k-th folding band, 3 zeros in the digital frequencies $\frac{k}{D} - \frac{p}{D}$, $\frac{k}{D} - \frac{p}{D} + \frac{1}{2}$, whereas the edges of the k-th folding band are $\frac{k}{D} - \frac{p}{D} + \frac{1}{2}$ and $\frac{k}{D} + \frac{1}{2}$. Therefore, as shown in Fig. 3, the choice $\alpha = q2\pi f_{ci}^{-1}$, with $q \in [0, +1]$, allows a better distribution of the three zeros within the k-th folding band, whose width is strictly related to the bandwidth $f_{ci}^{-1}$ of the useful discrete-time signal.

The optimal parameter $q$ has been found in [17], and we will employ such a value throughout this work. As an example, the optimal value $q = 0.79$ is such that a 3rd-order GCF filter features an additional 8dB of QN rejection over a classical 3rd-order comb filter around the folding bands.

Once again, let us focus our attention on the 3rd-order GCF filter, and consider a decimation factor $D$ that can be expressed as the p-th power-of-two, i.e., $D = 2^p$, where $p$ is a suitable integer greater than zero. Moreover, let us factorize the decimation factor $D$ as $D = D_1 \cdot D_2$, whereby $D_1 = 2^{p_1}$, $D_2 = 2^{p_2}$, and $p_p$ can take on any integer value in the set $\{-1, \ldots, p-1\}$. With this setup, the $z$-transfer function of a third-order GCF filter realized with the partial polyphase FIR architecture in Fig. 4 can be defined as follows:

$$H(z) = H_o \cdot H_P(z) \cdot H_N(z),$$

whereby $H_o$ is a constant term ensuring unity gain at baseband. The function $H_P(z)$ is the $z$-transfer function of the polyphase section decimating by $D_1$, whereas $H_N(z)$ is the $z$-transfer function of the non recursive filter decimating by $D_2$. The latter function is defined as

$$H_N(z) = \prod_{i=p_p+1}^{p-1} \left[1 + r_i \cdot (z^{-2^i} + z^{-2^{i-1}}) + z^{-3 \cdot 2^i}\right]$$

whereby the coefficients $r_i$ are defined as

$$r_i = 1 + 2 \cos \left(2^i \alpha\right), \quad \forall i = p_p + 1, \ldots, p - 1,$$

whereas $\alpha = 2 \cdot 0.79 \cdot \pi f_{ci}^{-1}$. We notice in passing that according to the commutative property in [9], the filter $H_N(z)$ can be realized with the cascade of $p - p_p - 1$ stages, each one decimating by 2. The integer $p_p$ can take on any value in the set $\{-1, \ldots, p-1\}$.

After some algebra, the frequency response of the filter $H_N(z)$ can be evaluated by substituting $z = e^{j\omega}$ in (4):

$$H_N(e^{j\omega}) = 2 \prod_{i=p_p+1}^{p-1} e^{-j2\pi(2^{i-1})\omega} \cdot \left[\cos \left(3 \cdot 2^{i-1}\omega\right) + r_i \cos \left(2^{i-1}\omega\right)\right],$$

whereby $\omega = 2\pi f_d$, and $r_i$ is defined in (5).

The impulse response $h_P(n), \forall n \in [0, 3D_1 - 3]$, whose $z$-transfer function is identified by $H_P(z)$, is defined as [28]:

$$h_P(n) = e^{j\alpha n} \sum_{k_0=0}^{n} e^{-2j\pi k_3/k_2} \sum_{k_2=0}^{p} e^{j\pi k_2} \sum_{k_1=0}^{p} x_1(k_1).$$

The definition of the sequence $x_1(n)$ in (7) is

$$x_1(n) = \delta(n) - r \delta(n - D_1) + r \delta(n - 2D_1) - \delta(n - 3D_1),$$

whereby $r = 1 + \cos(\alpha D_1)$ and $\alpha = q2\pi f_{ci}^{-1}$.

The impulse response in (7) is used to obtain the polyphase components

$$e_k(n) = h_P(D_1 n + k), \quad \forall k \in [0, D_1 - 1]$$

of the filters $E_k(z)$ in the architecture shown in Fig. 4.

Let us spend a few words about the parameters noticed in $H(z)$. The choice $p_p = p - 1$ allows the GCF filter $H(z)$ to be fully realized in polyphase form, whereas the value $p_p = -1$ is such that the filter is realized as the cascade of $p$ non recursive decimation stages, each one decimating by 2. Any intermediate value of $p_p \in \{0, \ldots, p-2\}$ yields the partial polyphase decomposition depicted in Fig. 4.
The first polyphase decimation stage allows the reduction of the sampling rate by $D_1$, thus reducing the operating rate of the subsequent decimation stages belonging to $H_N(z)$. Any stage of $H_N(z)$ in Fig. 4 is constituted by a simple FIR filter operating at a different data rate. Such an example, the $i$-th stage, with $i \in \{0, \ldots, p - p_p - 2\}$, is characterized by the transfer function $\left[1 + r_i \left( z^{-1} + z^{-2} + z^{-3}\right) \right]$ operating at rate $f_s / (D_1 \cdot 2^i)$, where $f_s$ is the sampling frequency of the A/D converter.

III. DESIGN OF FIXED-POINT GCF FILTERS

In this section we consider the problem of evaluating the sensitivity of the filter $H(z)$ in (3) with respect to the coefficients enclosed in both $H_P(z)$ and $H_N(z)$. Then, the sensitivity function is employed in a design algorithm that defines statistically the size of the registers in the fixed-point implementation of the filter in such a way that the error function in the frequency domain between the filter $H(e^{j\omega})$ and the filter $H_q(e^{j\omega})$, which employs quantized coefficients, is within given bounds with a preassigned probability.

We show that the proposed framework gives a precise answer on the choice of the proper split of the decimation factor $D$ between the two substages $H_P(z)$ and $H_N(z)$. It is anticipated that the best solution from a sensitivity point of view consists in implementing the filter $H(z)$ without the polyphase stage $H_P(z)$, i.e., with $D = D_2$ and $D_1 = 1$.

Before proceeding further, let us derive some observations on the sensitivity function employed throughout this section. Given a frequency response $H(e^{j\omega})$, the sensitivity analysis is usually accomplished on the magnitude of $H(e^{j\omega})$ with respect to its coefficients. However, the sensitivity analysis based on the use of the frequency response $H(e^{j\omega})$ can be derived much easier than the one that employs the function $\left| H(e^{j\omega})\right|$ [29]. Moreover, the sensitivity function related to $H(e^{j\omega})$ provides an upper bound to the one related to $\left| H(e^{j\omega})\right|$. Consider the frequency response

$$H(e^{j\omega}) = \left| H(e^{j\omega})\right| e^{j\varphi(H(e^{j\omega}))},$$

and a tagged multiplier $m$ belonging to $H(e^{j\omega})$. Then, the derivative of $H(e^{j\omega})$ with respect to $m$ can be evaluated as follows:

$$\frac{\partial H(e^{j\omega})}{\partial m} = \frac{\partial \left| H(e^{j\omega})\right|}{\partial m} e^{j\varphi(H(e^{j\omega}))} + j \left| H(e^{j\omega})\right| e^{j\varphi(H(e^{j\omega}))} \frac{\partial \varphi(H)}{\partial m}.$$  

Upon observing that both $\left| H(e^{j\omega})\right|$ and $\varphi(H(e^{j\omega}))$ are real functions of $\omega$, the bound

$$\left| \frac{\partial H(e^{j\omega})}{\partial m} \right| \geq \left| \frac{\partial \left| H(e^{j\omega})\right|}{\partial m} \right|$$

easily follows. Owing to this result, the sensitivity function used in this work relies on the derivatives of the frequency response $H(e^{j\omega})$ with respect to its multipliers.

When the multipliers belonging to $H(e^{j\omega})$ are quantized by employing rounding, the magnitude of the frequency response becomes:

$$\left| H_q(e^{j\omega})\right| = \left| H(e^{j\omega})\right| + \Delta \left| H(e^{j\omega})\right|,$$

whereby $\Delta \left| H(e^{j\omega})\right|$ is an error function that measures the distortion of the ideal frequency response $\left| H(e^{j\omega})\right|$ from the one employing quantized coefficients. Recalling the definition of the folding bands given in (4), a key observation in the proposed framework is that the error function $\Delta \left| H(e^{j\omega})\right|$ must be properly bounded only within the folding bands, concisely identified by FB. Therefore, care must be devoted to the sensitivity analysis only within the folding bands, while the behaviour of the error function outside the FB does not affect the proposed fixed-point design. Notice that these considerations only hold for the design of decimation filters in multistage architectures, and cannot be extended to the design of classical FIR filter.

The aforementioned considerations can be formalized as follow:

$$\left| \Delta \left| H(e^{j\omega})\right|\right| \leq \chi(\omega), \forall \omega \in \text{FB},$$

whereby $\chi(\omega)$ is a suitable–positively defined– tolerance function. Even though we can theoretically choose to differentiate the behaviour of the tolerance function $\chi(\omega)$ among the various folding bands, we do not pursue this approach in this

![Fig. 4. Architecture of the partial polyphase implementation of the decimation filter $H(z)$.](image-url)
work. Therefore, the functions \( \chi(\omega) \) used in the following will be constant functions across the folding bands.

Next line of pursuit consists in investigating a statistical technique in order to identify the word-lengths of the filter coefficients. To this end, assume that the frequency response \( H(e^{j\omega}) \) contains \( N \) coefficients rounded by employing the same fixed-point resolution \( \Delta \) (membering the relation (11), the variance of the error function \( \Delta |H(e^{j\omega})| \) can be evaluated as follows:

\[
\sigma_{\Delta m}^2 \approx \sigma_{2\Delta m}^2 \sum_{i=1}^{N} \left| \frac{\partial H(e^{j\omega})}{\partial m_i} \right|^2 = \sigma_{2\Delta m}^2 S_T(e^{j\omega}), \tag{13}
\]

whereby \( \sigma_{2\Delta m}^2 = 2^{-2b}/12 \) is the variance of the random variable \( \Delta m \) under the hypothesis to employ rounding to the nearest quantization level.

Owing to the condition \( N \gg 1 \), the error function \( \Delta |H(e^{j\omega})| \) can be modeled as a zero-mean Gaussian random variable with variance given by (13) [31]-[32]. Therefore, we can estimate the probability

\[
p = P \left[ |\Delta |H| \leq y\sigma_{\Delta |H(e^{j\omega})|} \right]
\]

that \( |H(e^{j\omega})| \) falls within a proper interval, say from \(-y\sigma_{\Delta |H(e^{j\omega})|}\) to \(+y\sigma_{\Delta |H(e^{j\omega})|}\), as follows:

\[
\frac{1}{\sqrt{2\pi}\sigma_{\Delta |H(e^{j\omega})|}} \int_{-y\sigma_{\Delta |H(e^{j\omega})|}}^{+y\sigma_{\Delta |H(e^{j\omega})|}} e^{-\frac{z^2}{2\sigma_{\Delta |H(e^{j\omega})|}^2}} dz. \tag{14}
\]

It is convenient to employ the new variable

\[
z = \frac{x}{\sqrt{2}\sigma_{\Delta |H(e^{j\omega})|}}
\]

in (14), thus obtaining

\[
p = P \left[ |\Delta |H| \leq y\sigma_{\Delta |H(e^{j\omega})|} \right] = \frac{2}{\sqrt{\pi}} \int_{0}^{+y\sigma_{\Delta |H(e^{j\omega})|}} e^{-z^2} dz. \tag{15}
\]

Let us spend few words about the result (15). The term \( p \) is the probability that the magnitude of the error function \( |\Delta |H(e^{j\omega})| \) in (12) is upper-bounded by \( y\sigma_{\Delta |H(e^{j\omega})|} \).

The relation between \( p \) and \( y \) in (15) is illustrated in Fig. 5.

As an instance, the value \( y = 2 \) has to be chosen in order to guarantee with a probability equal to 95\% that the error function is bounded by \( 2\sigma_{\Delta |H(e^{j\omega})|} \) in the frequency domain.

How can we employ this result in a practical design? Upon recalling (12), and given a proper \( p \), we choose \( y \) in such a way that the following relation holds:

\[
y \cdot \sigma_{\Delta |H(e^{j\omega})|} \approx \chi(\omega) \Rightarrow y \cdot \sigma_{\Delta m} \sqrt{S_T(e^{j\omega})} \approx \chi(\omega).
\]

This relation can be rewritten as

\[
y \cdot 2^{-b} \sqrt{S_T(e^{j\omega})} \approx \chi(\omega). \tag{16}
\]

By doing so, we guarantee that (12) is verified with probability \( p \) given by (15).

Given this statistical framework, let us focus on the fixed-point design of the considered GCF filter.

Assume that the filter coefficients are represented with the following fixed-point notation: \( I_n \) bits are devoted to the integer part of the coefficients, while \( F_n \) is the number of bits devoted to the fractional part. Therefore, the size of the filter coefficients is equal to \( I_n + F_n \) bits, accounting for the sign of the number.

The next two subsections derive the sizes of both \( I_n \) and \( F_n \) in the fixed-point implementation.

A. Evaluation of the fractional size, \( F_n \)

Considering \( b = F_n \), and solving (16) for \( F_n \), we can obtain the size \( F_n \) of the fractional part in order for (12) to hold with a probability \( p \) given by (15):

\[
F_n = \left\lfloor -\log_2 \left[ \sqrt{\frac{12}{\pi}} \min_{\omega \in \text{FB}} \frac{\chi(\omega)}{\sqrt{S_T(e^{j\omega})}} \right] \right\rfloor, \tag{17}
\]

whereby the minimum is taken only over the folding bands derived in (2), and \( \lfloor \cdot \rfloor \) is the ceil of the underlined number.

The evaluation of the fractional part \( F_n \) in (17) relies on the sensitivity function \( S_T(e^{j\omega}) \). To keep the presentation concise, the derivation of the sensitivity function is reported in the Appendix.

The behaviour of \( F_n \) in (17) as a function of the decimation factor \( D_n \) of the polyphase stage is illustrated in Fig. 6 for various values of the function \( \chi(\omega) \). The setup for deriving the results in Fig. 6 is as follows. We considered a two-stage decimating architecture \( (m = 2 \text{ in Fig. 1}) \) and an oversampling ratio \( \rho = 4 \cdot D \), where \( D \) is the other parameter associated to each subplot in Fig. 6. The first decimation filter in the 2-stage architecture is the investigated GCF filter, thus \( H_c(z) \mid_{z=1} = H(z) \) in Fig. 1.

The normalized digital bandwidth of the useful signal at the input of the GCF decimation filter is \( f_c \mid_{z=1} = f_c^0 = \frac{1}{2\rho} \).

This is also half the width of the folding bands seen by the first decimation filter in the two-stage architecture. Therefore, the value of \( \alpha \) appearing in the definition of filters \( H_p(z) \) and...
Fig. 6. Behaviour of $F_n$ in (17) as a function of the decimation factor $D_1$ of the polyphase stage for three values of the function $\chi(\omega)$, assumed constant across the folding bands. Left subplots are associated to the value $y = 2$ in (15) corresponding to the probability $p = 95\%$, whereas the rightmost subplots are associated to the value $y = 1.63$, which corresponds to the probability $p = 90\%$. The three curves in each subplot are parameterized with respect to the value of the function $\chi(\omega)$ as follows: curves labeled with the mark $\cdots$ are for $\chi(\omega) = 5 \times 10^{-3}$, the mark $\ast$ is associated to $\chi(\omega) = 10^{-3}$, and mark $\circ$ is used for $\chi(\omega) = 10^{-4}$.

$H_N(z)$ in (4) and (7) is $0.79 \cdot 2\pi f_c = 0.79\pi / \rho$. As a note aside, notice that the value of $\alpha$ would be different if the GCF filter were used in the second stage of the multistage chain in Fig. 1 due to the different value of $f_c^{-1}$.

We considered three different constant functions $\chi(\omega)$ in order to draw (17), namely $\chi(\omega) = 5 \times 10^{-3}$, $\chi(\omega) = 10^{-3}$, and $\chi(\omega) = 10^{-4}$.

Each subplot is associated to a specific decimation factor $D = D_1 \cdot D_2$. Therefore, given the constant $D$ noticed in the ordinate of each subplot, the number of stages belonging to $H_N(z)$ is reduced as long as $D_1$ increases. In particular, the abscissa $D_1 = 1$ in each subplot is associated to the case $D = D_2$, which means that the GCF filter is implemented without the polyphase stage.

Moreover, we consider two different values of $y$ related to the probabilities $p = 90\%$ and $p = 95\%$ illustrated in Fig. 5.

Some observations are in order.

- A comparison among the leftmost and the rightmost subplots in Fig. 6 reveals the need of one additional bit for the fractional part in order to guarantee that the constraint on the tolerance function is attained in the frequency domain with probability 95% with respect to the case 90%.
- For given $y$, $D$, and $D_1$ in the abscissa, the number $F_n$ of fractional bits increases as long as a lower tolerance function $\chi(\omega)$ is desired.
- Given $D$, the size $F_n$ of the fractional part of the fixed-point implementation increases as long as the number of cascaded cells in $H_N(z)$ decreases. This is equivalent to say that $F_n$ increases as long as $D_1$ does. This observation suggests that the GCF filter $H(z)$ implemented as $H_N(z)$, i.e., without the polyphase stage, allows to contain the computational complexity of the GCF filter. The latter observation above suggests that an effective implementation of the GCF filter is $H(z) = H_N(z)$. Therefore, the 3rd-order GCF filter is realized with the cascaded architecture shown in Fig. 7.

This architecture follows from (4) upon setting $p_r = -1$:

$$H_N(z) = \prod_{k=0}^{p-1} \left[ 1 + r_k \cdot \left( z^{-2^k} + z^{-2^{k+1}} \right) + z^{-3 \cdot 2^k} \right]$$

(18)
whereby the coefficients \( r_k \) are defined as
\[
   r_k = 1 + 2 \cos (2^k \alpha), \quad \forall k = 0, \ldots, p - 1
\]
\[
   \alpha = 2 \cdot 0.79 \cdot \pi f_c^{i-1}.
\]
(19)

Applying the commutative property [9], the cascaded implementation shown in Fig. 7 easily follows.

B. Evaluation of the integer part, \( I_n \)

This section is focused on the evaluation of the size of the integer part \( I_n \) in the fixed-point representation of the filter coefficients. To this goal, consider the architecture in Fig. 7 and focus on the \( k \)-th decimation stage. Let \( I_n^k \) be the size of the integer part in the \( k \)-th decimation stage.

The impulse response associated to the transfer function
\[
1 + r_k (z^{-1} + z^{-2}) + z^{-3}
\]
is
\[
h_k(n) = \delta(n) + r_k \delta(n-1) + r_k \delta(n-2) + \delta(n-3).
\]

Upon relying on general considerations about dynamic range overflow, it is simple to observe that the worst-case dynamic range growth \( G_k \) of the \( k \)-th stage is
\[
G_k \leq \log_2 \left( \sum_{n=0}^{3} |h_k(n)| \right) = \log_2 (2 + 2r_k) \leq 3,
\]
where the last inequality stems from the observation
\[
r_k = 1 + 2 \cos (2^k \alpha) \leq 3, \forall k = 0, \ldots, p - 1.
\]
Therefore, the size of the integer part \( I_n^k \) (in bits) that avoids overflow, is equal to the sum between the width of the input word (in bits) and \( G_k \).

IV. SIMULATION RESULTS

In this section, we compare some GCF filters designed with the framework proposed in the previous section, with classical comb filters. We also provide a set of simulation results obtained by employing a fixed-point realization of a 3rd-order GCF filter for decimating a discrete-time signal oversampled by a \( \Sigma \Delta \) A/D converter.

The first set of results is proposed to compare the frequency response of a 3rd-order GCF filter \( H(e^{j\omega}) \) with the one obtained with a fixed-point implementation. Let us summarize the setup. We consider a two-stage decimating architecture \((m = 2\) in Fig. 1\), whereby the first stage employs a GCF filter decimating by \( D = D_2 = 16 \) (i.e., the GCF filter is implemented with the architecture shown in Fig. 7 with \( p = 4 \)), whereas the second stage presents a decimation factor equal to 4. With this setup, the oversampling ratio is \( \rho = 64 \).

From the upper-leftmost subplot in Fig. 6 we notice that \( F_n = 7 \) in order to satisfy the bound
\[
|\Delta |H(e^{j\omega})|| \leq 10^{-4}, \forall \omega \in \text{FB},
\]
with a probability equal to 95%.

The normalized digital bandwidth of the useful signal at the input of the GCF decimation filter is
\[
f_c^{i-1} = f_c^p = \frac{1}{2\rho} = \frac{1}{128}.
\]

The magnitude of the frequency response (dotted-line curve) of the 3rd-order filter \( H(e^{j\omega}) \) without coefficients’ quantization is shown in Fig. 8 for \( D = 16 \). From (2), we notice the presence of the following folding bands:
\[
\left[ \frac{k}{16} \pm \frac{128}{128} \cdot \frac{k}{16} \right], \quad k = 1, \ldots, k_M = 8,
\]
some of which have been highlighted in Fig. 8. In the same figure, we show for comparison the frequency response (continuous curve) of the GCF filter whereby the coefficients have been quantized with \( F_n = 7 \), as discussed above. Notice that the two frequency responses are mostly superimposed, thus confirming the effectiveness of the proposed design framework.

The frequency response \( H_q(e^{j\omega}) \) is compared with the one of a classical 3rd-order comb filter in Fig. 9. The figure clearly highlights the behavior of the GCF filter across the folding bands: unless a classical 3rd-order comb that places a 3rd-order zero in the frequencies \( \pm \frac{1}{16} \cdot \forall k = 1, \ldots, k_M = 8 \), the

Fig. 7. Architecture of a non-recursive implementation of the decimation filter \( H(z) \). Index \( k \), which identifies the decimation stage, belongs to the range \( 0, 1, \ldots, \log_2 D - 1 = p - 1 \). The figure also shows an effective implementation of the \( k \)-th filter with transfer function \( 1 + r_k (z^{-1} + z^{-2}) + z^{-3} \).
Employing Matlab, we simulated a 2nd-order \( \Sigma \Delta \) converter with a 2-level quantizer and a sampling frequency \( f_s = 25.6 \) kHz. The input signal is a band-limited random signal with bandwidth \( f_r = 100 \) Hz. From the values of \( f_r \) and \( f_s \), it is \( \rho = 128 \), while the normalized digital bandwidth of the sampled signal is \( f^0 = \frac{1}{2^7} = \frac{1}{128} \).

The oversampled signal is then decimated by \( D = 16 \) employing a 3rd-order GCF filter. The power spectrum of the digital signal at the output of the \( \Sigma \Delta \) A/D converter is shown in the upper subplot of Fig. 10. Notice that, as expected, the useful signal with bandwidth \( f^0_c = 1/256 \) is shrunken at baseband, while the \( \Sigma \Delta \) A/D converter has pushed the noise power spectrum outside the useful signal bandwidth \([0, f^0_c]\).

The power spectrum of the decimated signal is shown in the lower subplot of Fig. 10. Notice that the useful signal bandwidth is now \( f^0_d = f^0_c; D = \frac{16}{256} \approx 0.063 \).

V. CONCLUSIONS

This paper focused on the design of computationally efficient Generalized Comb Filters (GCF), i.e., anti-aliasing filters that, employed as decimation filters in multistage architectures, guarantee superior performance in terms of selectivity and quantization noise rejection compared to classical comb filters. GCF filters can be realized by relying on both IIR and FIR architectures, even though FIR schemes do not present instability problems stemming from coefficients’ quantization.

As a reference filter in the class of GCF filters, a third order FIR architecture, realized by employing a partial polyphase architecture, was used throughout the paper. We proposed a sensitivity analysis in order to first investigate the effects of the filters’ quantization on the frequency response of the designed filters, and, then, to define the registers’ lengths in the proposed fixed-point implementation.

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APPENDIX

In this Appendix we derive the sensitivity function $S_T(e^{j\omega})$ needed in the evaluation of the fractional part $F_n$ in (17). Let us evaluate the function $S_T(e^{j\omega})$ in (15) for the considered partial polyphase architecture. To this end, we consider three different cases depending on the value of $p_p$.

First Case: $p_p = -1$. This is the case in which $D_1 = 1$ and $D = D_2$. The frequency response of the GCF filter is $H(e^{j\omega}) = H_N(e^{j\omega})$, and the filter is implemented without the polyphase stage. Therefore, the sensitivity function $S_T(e^{j\omega})$ can be evaluated as follows:

$$S_T(e^{j\omega}) = |H_N(e^{j\omega})|^2 \sum_{i=0}^{L} \frac{(3 - 2^{-1-i})}{(2^{-1-i})} + r_i|^{-2}$$

The previous equation stems from (6) upon noting that, for $p_p = -1$, the derivative of $H_N(e^{j\omega})$ with respect to $r_u$, $\forall u = 0, \ldots, p-1$, can be evaluated as:

$$\frac{dH_N(e^{j\omega})}{dr_u} = 2e^{-j3\omega} - \cos(2\omega) \prod_{m=0, m\neq u}^{p-1} e^{-j3\cdot2^{-m-1}\omega} \cos(3 \cdot 2^{m-1}\omega) + r_m \cdot \cos(2^{m-1}\omega)$$

By multiplying and dividing for the function

$$\cos(3 \cdot 2^{-1}\omega) + r_u \cdot \cos(2\omega),$$

and recalling (6), (21) can be rewritten as follows:

$$\frac{\partial H_N(e^{j\omega})}{\partial r_u} = H_N(e^{j\omega}) \cdot \frac{\cos(2\omega)}{\cos(3 \cdot 2^{-1}\omega) + r_u \cdot \cos(2\omega)} \quad (22)$$

Second Case: $p_p = p - 1$. This is the case in which $D_2 = 1$ and $D = D_1$. The frequency response of the GCF filter is $H(e^{j\omega}) = H_P(e^{j\omega})$, and the filter is fully implemented with a polyphase architecture.

The polyphase decomposition of the $z$-transfer function $H_P(z)$ is defined as follows:

$$H_P(z) = \sum_{k=0}^{D_1-1} z^{-k} E_k(z^{D_1}) \quad (23)$$

From (23) and (9), $H_P(e^{j\omega})$ can be rewritten as:

$$H_P(e^{j\omega}) = \sum_{k=0}^{D_1-1} \sum_{n=0}^{D_2} h_P(D_1 \cdot n + k) e^{-j\omega(D_1 \cdot n + k)} \quad (24)$$

which is valid $\forall n, k$ such that $0 \leq D_1 \cdot n + k < L$.

Upon observing that

$$\frac{\partial H_P(e^{j\omega})}{\partial h_P} = e^{-j\omega(D_1 \cdot n + k)}, \forall n, k,$$

the sensitivity $S_T(e^{j\omega})$ reduces to:

$$S_T(e^{j\omega}) = \sum_{i=1}^{L} \frac{\partial H_P}{\partial h_P(i)} = L = 3D_1 - 2$$

which corresponds to the number of multipliers in $H_P(e^{j\omega})$. 

...
**Intermediate Case:** $p_p = 0, \ldots, p - 2$. This is the case where the GCF filter is implemented with the partial polyphase architecture discussed above. Let $N = L + p - p_p - 1$ be the number of multipliers belonging to $H(e^{j\omega})$ ($L = 3D_1 - 2$ is the number of multipliers belonging to $H_P(e^{j\omega})$, while $p_p - 1$ is the number of coefficients belonging to $H_N(e^{j\omega})$).

The sensitivity function $S_T(e^{j\omega})$ assumes the following expression:

$$
\sum_{i=1}^{N} \left| \frac{\partial H(e^{j\omega})}{\partial m_i} \right|^2 = \sum_{i=1}^{L} |H_N|^2 \left| \frac{\partial H_P}{\partial m_i} \right|^2 + \sum_{i=1}^{p_p - 1} |H_P|^2 \left| \frac{\partial H_N}{\partial m_i} \right|^2. \quad (25)
$$

After some algebra, the previous relation can be rewritten as:

$$
S_T(e^{j\omega}) = L \cdot |H_N(e^{j\omega})|^2 + |H_P(e^{j\omega})|^2 \\
\cdot \sum_{i=1}^{p_p - 1} \frac{\cos (3 \cdot 2p_p + i - 1 \cdot \omega)}{\cos (2p_p + i - 1 \cdot \omega)} + r_{p_p+i} \left| \frac{\partial H_P}{\partial m_i} \right|^2 \\
= L |H_N(e^{j\omega})|^2 + |H(e^{j\omega})|^2 \sum_{i=1}^{p_p - 1} \frac{\cos (3 \cdot 2p_p + i - 1 \cdot \omega)}{\cos (2p_p + i - 1 \cdot \omega)} + r_{p_p+i} \left| \frac{\partial H_N}{\partial m_i} \right|^2, \quad (26)
$$

whereby

$$
r_{p_p+i} = 1 + 2 \cdot \cos (2p_p + i \alpha) .
$$