Properties of Clebsch-Gordan Numbers

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Abstract. Clebsch-Gordan numbers are the multiplicity numbers for the occurrence of the states of total angular momentum in the coupling of \( n \) angular momenta. These numbers are defined and completely generated by repeated application of the Clebsch-Gordan series rule. The relationship of this rule to Gaussian polynomials is presented for equal angular momenta.

1. General Setting
The Clebsch-Gordan numbers \( N_{j}(\mathbf{j}) \), \( \mathbf{j} = (j_{1}, j_{2}, \ldots, j_{n}) \), give the multiplicity of occurrence of a unitary irreducible representation \( D^{j}(U) \) (often called a Wigner [1] \( D \)-matrix) of the unitary unimodular group \( SU(2) \) in the reduction of the \( n \)-fold Kronecker product of \( n \) such unitary irreducible representations \( D^{j_{i}}(U), i = 1, 2, \ldots, n \). For each \( U \in SU(2) \), this is expressed by the unitary similarity equivalence, denoted \( \simeq \) (see Ref. [2] for details):

\[
D^{j}(U) = D^{j_{1}}(U) \otimes D^{j_{2}}(U) \otimes \cdots \otimes D^{j_{n}}(U) \simeq \bigoplus_{j=j_{\text{min}}}^{j_{\text{max}}} D^{j}(U). \quad (1)
\]

The notation \( D^{j}(U) \) denotes the direct sum of irreducible representations given by

\[
D^{j}(U) = D^{j_{1}}(U) \oplus \cdots \oplus D^{j_{n}}(U), \quad (2)
\]

in which each of the matrices \( D^{j}(U) \) is a block along the diagonal. By convention, these matrices are arranged along the diagonal from the least to the greatest value of \( j \). We refer to the special form relation (1) of arrangement of the constituents in the direct sum as the Kronecker direct sum. As noted in this relation, a given unitary irreducible representation \( D^{j}(U), j \in \{0, 1/2, 1, 3/2, \ldots\} \) occurs a number of times equal to the Clebsch-Gordan number \( N_{j}(\mathbf{j}), \mathbf{j} = (j_{1}, j_{2}, \ldots, j_{n}) \).

It is useful to point out that the matrix \( U \in SU(2) \) in (1) and (2) can be replaced by an arbitrary matrix \( Z = (z_{ij})_{1 \leq i,j \leq 2} \) of order 2, and the formula still holds. The validity of this result has been the subject of several talks by the author at previous conferences here (see, for example, Ref. [3]). The basic multiplication property has been put in a combinatorial context by Chen and Louck [4]. This property is important because we can obtain the following generating function
for the CG numbers upon specializing the matrix $Z$ in (1) to be diagonal, $Z = \text{diag}(z_1, z_2)$, taking the trace, and using the relation to Schur functions (see Ref. [5] and Stanley [6]):

$$\text{Tr} \left( D^k(\text{diag}(z_1, z_2)) \right) = s_{(2k,0)}(z_1, z_2) = \sum_{m=-k}^{k} z_1^{k+m} z_2^{-m},$$

each $k = 0, 1/2, 1, \ldots.$ (3)

The extensions of (1) to $U \rightarrow Z$ and the trace property (3) give the generating function for CG numbers as

$$\prod_{i=1}^{n} s_{(2j_i,0)}(z_1, z_2) = \sum_{j=\text{max}}^{j_{\text{min}}} N_j(j) s_{(2j,0)}(z_1, z_2).$$ (4)

When further specialized to $z_1 = z_2 = 1$, this relation also gives the equality of dimensionalities of the left- and right-hand sides of (1):

$$\prod_{i=1}^{n} (2j_i + 1) = \sum_{j=\text{max}}^{j_{\text{min}}} (2j + 1) N_j(j).$$ (5)

It is useful, before describing the properties of the Clebsch-Gordan numbers in terms of the Clebsch-Gordan series, to also give the description of the group representation relation (1) to the addition of angular momenta—the Lie algebra description, which is the more familiar context in physics.

Every isolated quantum mechanical system with $n$ constituent parts, each possessing angular momentum $J(i), i = 1, 2, \ldots, n$, has total angular momentum

$$J = J(1) + J(2) + \cdots + J(n),$$

$$J(i) = J_1(i)e_1 + J_2(i)e_2 + J_3(i)e_3,$$

where each angular momentum is referred to a common right-handed reference frame $(e_1, e_2, e_3)$ in Euclidean 3-space $\mathbb{E}^3$.

We introduce the following compact notations to describe the ket-vectors of the tensor product space:

$$j = (j_1, j_2, \ldots, j_n), \text{ each } j_i \in \{0, 1/2, 1, 3/2, \ldots\}, i = 1, 2, \ldots, n,$$

$$m = (m_1, m_2, \ldots, m_n), \text{ each } m_i \in \{j_i, j_i - 1, \ldots, -j_i\}, i = 1, 2, \ldots, n,$$

$$\mathcal{H}_j = \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \cdots \otimes \mathcal{H}_{j_n},$$

$$|j\, m\rangle = |j_1\, m_1\rangle \otimes |j_2\, m_2\rangle \otimes \cdots \otimes |j_n\, m_n\rangle,$$

$$\mathcal{C}(j) = \{m \mid m_i = j_i, j_i - 1, \ldots, -j_i; i = 1, 2, \ldots, n\}.$$

The components $J_k(i), k = 1, 2, 3,$ of $J(i)$ have the standard action in the Hilbert space $\mathcal{H}_{j_i}$ of dimension $2j_i + 1$, as given by

$$J^2(i)|j\, m\rangle = j_i(j_i + 1)|j\, m\rangle,$$

$$J_3(i)|j\, m\rangle = m_i|j\, m\rangle,$$

$$J_2(i)|j\, m\rangle = \sqrt{(j_i - m_i)(j_i + m_i + 1)}|j\, m_{i+1}(i)\rangle,$$

$$J_1(i)|j\, m\rangle = \sqrt{(j_i + m_i)(j_i - m_i + 1)}|j\, m_{i-1}(i)\rangle,$$

$$m_{i\pm 1}(i) = (m_1, \ldots, m_i \pm 1, \ldots, m_n).$$
The orthonormality of the basis functions is expressed by

$$\langle j m | j m' \rangle = \delta_{m,m'}, \text{ each pair } m, m' \in \mathbb{C}(j).$$

(9)

The components \( J(i), j = 1, 2, 3 \), of the angular momentum \( \mathbf{J}(i) \) are the generators, respectively, of rotations about axis \( e_i \); they are often called \textit{infinitesimal generators} of rotations.

The set of \( 2n \) mutually commuting Hermitian operators

$$J^2(1), J_3(1), J^2(2), J_3(2), \ldots, J^2(n), J_3(n)$$

(10)

is a complete set of operators in the tensor product space \( \mathcal{H}_j \), which means that the set of vectors \( |j m\rangle \), \( m \in \mathbb{C}(j) \) is an orthonormal basis. This basis of the space is uniquely determined by the action of the angular momentum operators \( \mathbf{J}(i), i = 1, 2, \ldots, n \), which is the standard action given by (8). Since the collection of \( 2n \) commuting Hermitian operators (10) refers to the angular momenta of the individual constituents of a physical system, and the action of the angular momentum operators is on the basis vectors of each separate space, the basis \( |j m\rangle \), \( m \in \mathbb{C}(j) \), is referred to as the \textit{uncoupled basis} of the space \( \mathcal{H}_j \).

One of the most important observables for a composite system is the total angular momentum defined by (6). A set of \( n + 2 \) mutually commuting Hermitian operators, which includes the square of the total angular momentum \( \mathbf{J} \) and \( J_3 \) is the following:

$$J^2(1), J^2(2), \ldots, J^2(n), J^2, J_3.$$  

(11)

This set of \( n + 2 \) commuting Hermitian operators is an incomplete set with respect to the construction of the states of total angular momentum; that is, the simultaneous state vectors of the \( n + 2 \) operators (11) do not determine a basis of the space \( \mathcal{H}_j \). There are many ways to complete such an incomplete basis. For example, an additional set of \( n - 2 \) independent \( SU(2) \) invariant Hermitian operators, commuting among themselves, as well as with each operator in the set (11), could serve this purpose. Other methods of labeling can also be used. We make the following assumptions:

Assumptions. The incomplete set of simultaneous eigenvectors of the \( n + 2 \) angular momentum operators (11) has been extended to a basis of the space \( \mathcal{H}_j \) with properties as follows: A basis set of vectors can be enumerated in terms of an indexing set \( \mathbb{R}(j) \) of the form

$$\mathbb{R}(j) = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, j, m) \mid j \in \mathbb{D}(j); \alpha_i \in A_i(j, j); m = j, j - 1, \ldots, -j \right\},$$

(12)

where the domains of definition \( \mathbb{D}(j) \) of \( j \) and \( A_i(j, j) \) of \( \alpha_i \) have the properties as follows. Each of the quantum labels \( \alpha_i \) belongs to a domain of definition \( A_i(j, j) \) that can depend also on \( (j, j) \). These domains of definition are to be such that for given quantum numbers \( j \) the cardinality of the set \( \mathbb{R}(j) \) is given by

$$| \mathbb{R}(j) | = | \mathbb{C}(j) | = \prod_{i=1}^{n} (2j_i + 1).$$

(13)

Moreover, these labels are to be such that the space \( \mathcal{H}_j \) has the orthonormal basis given by the ket-vectors

$$\langle j \alpha | j m \rangle, \alpha, j, m \in \mathbb{R}(j),$$

$$\langle j \alpha | j m \rangle \langle j \alpha' | j' m' \rangle = \delta_{j,j'} \delta_{m,m'} \delta_{\alpha,\alpha'},$$

(14)

$$\alpha, j, m \in \mathbb{R}(j); \alpha', j', m' \in \mathbb{R}(j).$$
Finally, the actions of the commuting angular momentum operators (11) and the total angular momentum \( \mathbf{J} \) on the orthonormal basis set (14) are given by

\[
\begin{align*}
\mathbf{J}^2(i)|j\alpha_jm\rangle &= j_i(j_i + 1)|j\alpha_jm\rangle, i = 1, 2, \ldots, n, \\
\mathbf{J}_3|j\alpha_jm\rangle &= m|j\alpha_jm\rangle, \\
J_\pm|j\alpha_jm\rangle &= \sqrt{(j - m)(j + m + 1)}|j\alpha_{j+1}m\rangle, \\
J_-|j\alpha_jm\rangle &= \sqrt{(j + m)(j - m + 1)}|j\alpha_{j-1}m\rangle.
\end{align*}
\]

It is always the case that \( D(j) \) is independent of how the extension to a basis through the parameters \( \alpha \) is effected and that for given \( j \) the domain of \( m \) is \( m = j, j - 1, \ldots, -j \).

The set \( \mathbb{R}(j) \) defined by (12) enumerates an alternative unique orthonormal basis (14) of the space \( \mathcal{H}_j \) that contains the total angular momentum quantum numbers \( j, m \); it is the analog of the set \( \mathbb{C}(j) \) given in (7) that enumerates the uncoupled orthonormal basis set (8). Any basis set with the properties (13)-(15) is called a coupled basis of \( \mathcal{H}_j \). For \( n = 2 \), the uncoupled basis set is \( |j_1 m_1\rangle \otimes |j_2 m_2\rangle \), \( \mathbb{C}(j_1j_2) = \{m_1, m_2 \mid m_i = j_i, j_i - 1, \ldots, -j_i, i = 1, 2\} \), and the coupled basis set is \( |j_1j_2j_m\rangle \), \( \mathbb{R}(j_1j_2) = \{j, m\mid j \in \{j_1 + j_2, j_1 + j_2 - 1, \ldots, j_1 - j_2\}, m = j, j - 1, \ldots, -j\} \). No extra \( \alpha \) labels are required. For \( n = 3 \), one extra label \( \alpha_1 \) is required.

Angular momentum coupling theory is about the various ways of providing the extra set of \( \alpha \) labels and their domains of definition, together with the values of the total angular momentum quantum number \( j \), such that the space \( \mathcal{H}_j \) is spanned by the vectors \( |j\alpha_jm\rangle \). It turns out, as shown below, that the set (multiset) of values that the total angular momentum quantum number \( j \) can assume is independent of the \( \alpha_i \); the values of \( j \) being \( j = j_{\text{min}}, j_{\text{min}} + 1, \ldots, j_{\text{max}} \), for well-defined minimum and maximum values of \( j \) that are expressed in terms of \( j_1, j_2, \ldots, j_n \).

Thus, the burden of completing any basis is placed on assigning the labels \( \alpha_i \) in the set

\[
\mathbb{R}(j, j) = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-2}) \mid \text{each } \alpha_i \in A_i(j, j)\}.
\]

Such an assignment is called an \( \alpha \)-coupling scheme. Since there are many ways of completing an incomplete basis of a finite vector space, there are also many coupling schemes. In this sense, the structure of the coupling scheme set \( \mathbb{R}(j, j) \) is the key object in angular momentum coupling theory; all the details of defining the coupling scheme are to be provided by the domains of definition \( A_i(j, j) \).

The cardinality of the sets \( \mathbb{R}(j) \) and \( A_i(j) \), and \( \mathbb{C}(j) \) are related by

\[
|\mathbb{R}(j)| = \sum_{j = j_{\text{min}}}^{j_{\text{max}}} (2j + 1)N_j(j) = N_j(j) = \prod_{i=1}^{n}(2j_i + 1) = |\mathbb{C}(j)|,
\]

where we have defined

\[
N_j(j) = |\mathbb{R}(j, j)|.
\]

This number is the Clebsch-Gordan number, so named because Clebsch and Gordan were the first to discover the simplest case \( N_j(j_1, j_2) \) for the coupling of two angular momenta:

\[
N_j(j_1, j_2) = \begin{cases} 
1, & \text{for } j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|, \\
0, & \text{otherwise}
\end{cases}
\]

The purpose of introducing a coupling scheme is twofold: First, a coupled basis gives a new orthonormal basis of the space \( \mathcal{H}_j \) in which the total angular momentum quantum
numbers \((j, m)\) label the basis vectors, which is often essential in physical applications. Second, the coefficients that effect the transformation between these two bases are the elements of a unitary similarity transformation that effects the reduction of the coefficients that effect the transformation between these two bases are the elements of a \(D\) matrix. It is in the second instance that the generalization of the Clebsch-Gordon numbers (19) enters, and which concerns us here. We turn next to this problem, which does not require detailed knowledge of the reduction process itself.

2. Generation of the Clebsch-Gordan Numbers

The Clebsch-Gordan numbers \(N_j(j)\) may be generated by the following recursive procedure. We first define the set \(\langle j_1, j_2 \rangle\) by

\[
\langle j_1, j_2 \rangle = \{|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2\}.
\]  

(20)

This set then contains all values that the total angular momentum \(j\) can assume in the addition of two angular momenta \(J(1) + J(2) = J\). We now define recursively the multiset \(\langle j_1, j_2, \ldots, j_i \rangle\) of total angular momenta \(j\) for the addition of \(i\) angular momenta by

\[
\langle j_1, j_2, \ldots, j_i \rangle = \{\langle k, j_i | k \in \langle j_1, j_2, \ldots, j_{i-1} \rangle\},
\]  

(21)

where \(i = 2, 3, \ldots\). Starting with \(i = 2\) and \(k = j_1\), this relation can be iterated to obtain the set of all possible values of the total angular momentum \(j\) for the addition of \(n\) angular momenta, including the multiplicity \(N_j(j)\) of each value \(j\). The cardinality of this multiset is

\[
|\langle j_1, j_2, \ldots, j_n \rangle| = \prod_{i=1}^{n}(2j_i + 1),
\]  

(22)

where repeated elements are included in the counting. The following properties of the set \(\langle j \rangle = \langle j_1, j_2, \ldots, j_n \rangle\) can be proved by induction on \(n\):

(i) The set \(j\) is invariant under all permutations of the \(j_i\):

\[
\langle j_{\pi_1}, j_{\pi_2}, \ldots, j_{\pi_n} \rangle = \langle j_1, j_2, \ldots, j_n \rangle, \quad \text{each } \pi \in S_n.
\]  

(23)

(ii) For given \(\langle j \rangle\), the least and greatest elements, \(j_{\text{min}}\) and \(j_{\text{max}}\), are given by

\[
\begin{align*}
    j_{\text{min}} &= \min\{|j_1 \pm j_2 \pm \cdots \pm j_n|\}, \\
    j_{\text{max}} &= j_1 + j_2 + \cdots + j_n.
\end{align*}
\]  

(24)

All \(2^{n-1}\) combinations of plus and minus signs in the addition of the \(j_i\) are to be considered in the definition of \(j_{\text{min}}\).

(iii) The values of the total angular momentum \(j\) corresponding to the multiset \(j\) are

\[
j = j_{\text{min}}, j_{\text{min}} + 1, j_{\text{min}} + 2, \ldots, j_{\text{max}}.
\]  

(25)

(iv) The form of the multiset \(\langle j \rangle\) is

\[
\langle j \rangle = \{j_1, j_2, \ldots, j_n\} = \left\{j_{\text{min}}^h, (j_{\text{min}} + 1)^h, \ldots, j_{\text{max}}^h\right\},
\]  

(26)

where \(h_k = N_{j_{\text{min}}+k-1}(j), k = 1, 2, \ldots, t = j_{\text{max}} - j_{\text{min}} + 1\), and the notation \(m^h\) denotes that \(m\) is repeated \(h\) times. It is always the case that \(h_1 = h_t = 1\).
Applying this relation, in turn, to (2) and taking the difference, we obtain the inverse relations

\[ N_j(j) = M_j(j) - M_{j+1}(j), \quad j = j_{\text{max}}, j_{\text{max}} - 1, \ldots, j_{\text{min}}, \]

where \( M_{j_{\text{max}}+1}(j) = 0 \). Thus, the CG numbers \( N_j(j) \) are obtained by counting the number of elements in two adjacent sets in (30) and taking the difference.

The determination of the number of elements in the set \( M_j(j) \) can be formulated as a problem in counting a certain number of compositions with certain restrictions. Here a composition...
\[ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \] of a nonnegative integer \( k \) into \( n \) parts is defined to be any sequence of nonnegative integers such that
\[ \sum_{i=1}^{n} \alpha_i = k. \] (33)

The number \( N(k, n) \) of such compositions is given (see Stanley [6], Vol. 1, p. 15) by the binomial coefficient
\[ N(k, n) = \binom{n+k-1}{k-1}. \] (34)

We now define the nonnegative integers \( \alpha_i = j_i - m_i, m_i = j_i, j_i - 1, \ldots, -j_i \), so that \( 0 \leq \alpha_i \leq 2j_i \). We also define, for each \( j \) such that \( j_{\text{min}} \leq j \leq j_{\text{max}} \), the following set, which is just the shifted version of \( M_j(j) \), defined by (30), so that their cardinalities are equal:
\[ L_j(j) = \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_n) \mid 0 \leq \alpha_i \leq 2j_i, i = 1, 2, \ldots, n; \right. \]
\[ \left. \alpha_1 + \alpha_2 + \cdots + \alpha_n = j_{\text{max}} - j \right\} \] (35)

with cardinality \( L_j(j) = |L_j(j)| = M_j(j) \). Accordingly, we have the result:
\( L_j(j) \) is the number of compositions of \( j_{\text{max}} - j \) into \( n \) nonnegative parts such that each part satisfies \( 0 \leq \alpha_i \leq 2j_i \), and the cardinality \( M_j(j) \) of the set (30) is given by
\[ M_j(j) = L_j(j), \text{ each } j = j_{\text{max}}, j_{\text{max}} - 1, \ldots, j_{\text{min}}. \] (36)

For \( j = j_{\text{max}} \), this relation gives \( M_{j_{\text{max}}}(j) = 1 \), since there is only one composition of 0 into \( n \) nonnegative parts, namely, \((0, 0, \ldots, 0)\).

All Clebsch-Gordan numbers are obtained from the difference given by (32).

**Example.** We illustrate the above relations for the case \( n = 3 \) and \((j_1,j_2,j_3) = (1/2,1,5/2)\) given by (27), hence, \( j_{\text{min}} = 1, j_{\text{max}} = 4 \). For brevity, we drop the \( j \) arguments in the above relations, noting, however, that parts 1, 2, 3 of the composition must satisfy \( 0 \leq \alpha_1 \leq 1, 0 \leq \alpha_2 \leq 2, 0 \leq \alpha_3 \leq 5 \). By direct calculation of the compositions, we obtain:
\[ L_4 = \{(0,0,0)\}, \]
\[ L_3 = \{(1,0,0), (0,1,0), (0,0,1)\}, \]
\[ L_2 = \{(0,2,0), (0,0,2), (0,1,1), (1,1,0), (1,0,1)\}, \]
\[ L_1 = \{(0,0,3), (0,1,2), (0,2,1), (1,1,1), (1,2,0), (1,0,2)\}, \]
\[ L_0 = \{(0,0,4), (0,1,3), (0,2,2), (1,0,3), (1,1,2), (1,2,1)\}. \] (37)

Thus, we have \( M_4 = 1, M_3 = 3, M_2 = 5, M_1 = 6, M_0 = 6, N_4 = 1, N_3 = 2, N_2 = 2, N_1 = 1, N_0 = 0 \), in agreement with (27). \( \Box \)

A closed formula for the general CG number is an unsolved problem, although they can always be calculated from the generating function (4) or from the multiset (21) generated by the CG series. It is quite interesting, however, that when all the angular momenta \( j_i \) are equal, the Clebsch-Gordan numbers relate to Gaussian polynomials, a result pointed out by Sunko and Svrtan [7]. We discuss this next.

Relation (31) may be rewritten in the following form for all angular momenta equal, say, \( h = 2j_i, i = 1, 2, \ldots, n \), in which case we write (31) as
\[ \mathcal{A}(n,h,k) = \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_n) \mid 0 \leq \alpha_i \leq h, i = 1, 2, \ldots, n; \right. \]
\[ \left. \alpha_1 + \alpha_2 + \cdots + \alpha_n = k \right\}. \] (38)
in which \( k = \frac{1}{2}nh - j \) with \( j \) any integer \( j = j_{\text{min}}, j_{\text{min}} + 1, \ldots, nh \), where \( j_{\text{min}} = 0 \) for \( n \) even, and \( j_{\text{min}} = \frac{1}{2}h \) for \( n \) odd. The values that \( k \) may assume are then given by

\[
 k = 0, 1, \ldots, k_{\text{max}},
\]

\[
k_{\text{max}} = \left\{ \begin{array}{ll} \\
\frac{1}{2}nh, & \text{for } n \text{ even,} \\
\frac{1}{2}(n - 1)h, & \text{for } n \text{ odd.}
\end{array} \right.
\]  

A principal property of the set \( \mathcal{A}(n,h,k) \) is that \((\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{A}(n,h,k)\) implies that \((\alpha_{\pi_1}, \alpha_{\pi_2}, \ldots, \alpha_{\pi_n}) \in \mathcal{A}(n,h,k)\) for all permutations \( \pi \in S_n \), the symmetric group. Thus, it is sufficient to determine all the partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0 \) in the set \( \mathcal{A}(n,h,k) \) to obtain all the compositions \( \alpha \). This observation reduces the problem to that of determining the set of partitions defined by

\[
\text{Par}(n,h,k) = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \mid \lambda \vdash k \text{ has at most } n \text{ nonzero parts with each part } \leq h \right\}.
\]  

When restrictions are placed on the parts of a partition, the counting and generating function problem can become quite intricate. Research on this problem has been renewed by Andrews et al. [8] based on interpretations and extensions of MacMahon’s Partition Analysis (MacMahon [9]).

The generating function problem for the cardinality of the set of partitions \( \text{Par}(n,h,k) \) goes back to Gauss. The solution may be found in Andrews [10], Chapter 3, and is summarized as follows. The generating function for \( |\text{Par}(n,h,k)| \) is the Gaussian polynomial \( G(n,h;t) = G(h,n;t) \) of degree \( nh \) in the indeterminate \( t \) :

\[
G(n,h;t) = \left[ \begin{array}{c} n + h \\ h \end{array} \right]_t = \frac{(1 - t^{n+h})(1 - t^{n+h-1}) \cdots (1 - t^{h+1})}{(1 - t^n)(1 - t^{n-1}) \cdots (1 - t)} = \sum_{k=0}^{nh} p(n,h,k)t^k,
\]

where \( G(n,h;0) = 1 \) and \( p(n,h,k) = |\text{Par}(n,h,k)| \). The coefficients \( p(n,h,k) \) satisfy the following relations for all \( n, h, k \geq 0 \):

\[
\begin{align*}
p(n,h,k) & = p(n,h-1,k) + p(n-1,h,k-h), \\
p(n,h,k) & = p(h,n,k), \\
p(n,h,k) & = p(h,n,nh-k), \\
p(n,h,k) & = p(n,h,k-1) \geq 0, \text{ for } 0 \leq k \leq nh/2, \\
p(n,0,k) & = p(0,h,k), k \geq 1, \\
p(0,0,0) & = 1.
\end{align*}
\]  

We refer to Andrews [10] for the proof of these relations and further properties of Gaussian polynomials.

The set of compositions \( \mathcal{A}(n,h,k) \) is obtained from the set of partitions \( \text{Par}(n,h,k) \) by the relation

\[
\mathcal{A}(n,h,k) = \{ \pi(\lambda) = (\lambda_{\pi_1}, \lambda_{\pi_2}, \ldots, \lambda_{\pi_n}) \mid \lambda \in \text{Par}(n,h,k), \pi \in S_n \},
\]

\[
(44)
\]
where, in applying the permutations $\pi \in S_n$ to a partition $\lambda$, only distinct compositions are included. For example, for $\lambda = (2, 1, 1)$, the symmetric group $S_3$ generates the compositions $(2, 1, 1), (1, 2, 1), (1, 1, 2)$. We now write each partition $\lambda \in \text{Par}(n, h, k)$ in the form 

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) = \mu(a) = (\mu_1^{a_1}, \mu_2^{a_2}, \ldots, \mu_q^{a_q}),$$

(45)

$$\mu_1 > \mu_2 > \cdots > \mu_q \geq 0, \ a_1 + a_2 + \cdots + a_q = n,$$

where the positive exponent $a_i$ denotes the number of repetitions of the distinct part $\mu_i$ in the partition $\lambda$. Then, the cardinality of the set of compositions $A(n, h, k)$ is given by a summation over multinomial coefficients

$$|A(n, h, k)| = \sum_{\mu(a) \in \text{Par}(n, h, k)} \binom{n}{a_1, a_2, \ldots, a_q},$$

(46)

The equality of cardinalities

$$|M_{\frac{1}{2}nh-k}(\frac{1}{2}h, \ldots, \frac{1}{2}h)| = |A(n, h, k)|$$

(47)

then gives by (36) the formula for the Clebsch-Gordan numbers:

$$N_{\frac{1}{2}nh-k}(\frac{1}{2}h, \ldots, \frac{1}{2}h) = |A(n, h, k)| - |A(n, h, k-1)|,$$

$$k = 1, 2, \ldots, k_{\text{max}}, N_{\frac{1}{2}nh}(\frac{1}{2}h, \ldots, \frac{1}{2}h) = 1.$$  

(48)

We recall that $j = \frac{1}{2}nh - k$ and $j_i = \frac{1}{2}h, i = 1, 2, \ldots, n$.

**Concluding Remarks:**
The paper of Sunko and Svrtan [7] has motivated the above presentation, but we have been unable to locate its publication journal or its authors. We thank the organizing committee of SSPCM 07 for the opportunity to present this overview on the combinatorial meaning of the Clebsch-Gordan numbers. We also thank the referee for several useful suggestions for improvement of our presentation.

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