SPECIFYING A GAME-THEORETIC EXTENSIVE FORM AS AN ABSTRACT 5-ARY RELATION

Peter A. Streufert
Economics Department
Western University

Abstract. This paper specifies an extensive form as a 5-ary relation (that is, as a set of quintuples) which satisfies eight abstract axioms. Each quintuple is understood to list a player, a situation (a concept which generalizes an information set), a decision node, an action, and a successor node. Accordingly, the axioms are understood to specify abstract relationships between players, situations, nodes, and actions. Such an extensive form is called a “pentaform”. A “pentaform game” is then defined to be a pentaform together with utility functions.

The paper’s main result is to construct an intuitive bijection between pentaform games and \textbf{Gm} games (Streufert 2021Gm, arXiv:2105.11398), which are centrally located in the literature, and which encompass all finite-horizon or infinite-horizon discrete games. In this sense, pentaform games equivalently formulate almost all extensive-form games. Secondary results concern disaggregating pentaforms by subsets, constructing pentaforms by unions, and initial applications to Selten subgames and perfect-recall (an extensive application to dynamic programming is in Streufert 2023dp, arXiv:2302.03855).

1. Introduction

1.1. New concepts and main result

A 5-ary relation is merely a set of quintuples. It is like a binary relation, which is a set of pairs, and also like a ternary relation, which is a set of triples. In this paper, a quintuple is denoted \( \langle i,j,w,a,y \rangle \). The first element is understood to be a player, the second a situation (a concept which generalizes the concept of an information set), the third a decision node, the fourth an action, and the fifth a successor node.
Thereby, a set of quintuples is understood to specify relationships between players, situations, (two kinds of) nodes, and actions.

Such a quintuple set can specify a game-theoretic extensive form because an extensive form is, in essence, a collection of relationships between players, situations, nodes, and actions. More precisely, this paper specifies an extensive form as a “pentaform”, which is defined to be a quintuple set $Q$ which obeys eight axioms. These axioms are formulated in terms of various projections of $Q$. For example, let $\pi_{JI}(Q)$ denote the projection of $Q$ onto its first two coordinates (with their order reversed). Then the first axiom requires that $\pi_{JI}(Q)$ is a function. In other words, the first axiom requires that each situation $j$ (see note 2) is associated with exactly one player $i$ (casually this is the player who controls the move at the situation $j$). In a similar way, each of the other seven axioms formalizes one small independent feature of an extensive form.

A “pentaform game” is then constructed by combining a pentaform with utility functions. The main result is Theorem 5.4, which shows that there is a constructive and intuitive bijection from the collection of $G_m$ games to the collection of pentaform games with information-set situations. To see the significance of this result, note that $G_m$ games have two important features. First, $G_m$ games are defined in the standard way as a tree adorned with information sets, actions, players, and utility functions. In this and other regards, the $G_m$ formulation is centrally located among the many formulations in the game-theory literature. Second, $G_m$ games are general, in the sense that they encompass all discrete games, whether finite-horizon or infinite-horizon. Thus Theorem 5.4 indicates that pentaform games equivalently formulate almost all extensive-form games.

1.2. Motivation

Pentaforms are easy to manipulate because they are sets, and because the pentaform axioms are largely compatible with the concepts of subset and union.

More precisely, Section 4.1 shows that any subset of a pentaform satisfies six of the eight pentaform axioms (Proposition 4.1). This leads to a weak general condition under which a subset of a pentaform is itself a pentaform (Corollary 4.2). This result is a powerful tool for disaggregating a pentaform. First, Section 4.1 uses the result to characterize the subsets of a pentaform that correspond to Selten subgames (Proposition 4.3). Second, Section 4.1 explains how the same result, applied to other subsets of a pentaform, is the foundation for the generalized theory of dynamic programming in Streufert 2023dp. That sequel paper is the first paper to use value functions to characterize subgame perfection in arbitrary games.

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3Streufert 2021Gm introduces $G_m$ games as part of the category $G_m$. The role of that category in game theory is discussed tangentially in this paper’s Appendix A, for readers who may be interested.

4“Discreteness” means that each decision node has a finite number of predecessors. Non-discrete games include those in continuous time, as in Dockner, Jørgenson, Long, and Sorger 2000; and those yet more general, as in Alós-Ferrer and Ritzberger 2016, Chapters 1–5. [Discreteness is defined in terms of decision nodes because Alós-Ferrer and Ritzberger 2016, Section 6.2, admits a terminal node at the end of each infinite run (that is, each infinite play). Such terminal nodes do not appear in the present paper.]
1. Introduction

In the opposite direction, Section 4.2 defines a “block” to be a quintuple set which satisfies all but one of the axioms. Then it essentially shows that the union of a “separated” collection of blocks is itself a block (Proposition 4.4), and that the union of an expanding sequence of pentaforms is itself a pentaform (Proposition 4.5). These are convenient tools for building new pentaforms from known components, as illustrated by the finite-horizon examples of Section 4.2 and the infinite-horizon example of Streufert 2023dp Section 2.2.

In addition, pentaforms seem to have some less tangible benefits. (a) Other axiomatic foundations, such as the well-known axiomatic foundation for consumer preferences, have readily fostered new extensions and results. The same may occur with this paper’s axiomatic foundation for extensive forms. For example, the fine-grainedness of the eight axioms fostered the development of Propositions 4.1 and 4.4 for subsets and unions. (b) The pentaform notation is distinctly new. Broadly, a pentaform is one high-dimensional relation, while a standard extensive form is a list of low-dimensional relations. To suggest the value of this unification, Section 4.3 considers the concept of perfect-recall, which simultaneously involves players, situations, nodes, and actions. The pentaform version of this concept is appealing, and Section 4.3 uses it to prove in one paragraph that perfect-recall implies no-absentmindedness (Proof of Proposition 4.6).

1.3. Literature

The rough idea of expressing an extensive form as a high-dimensional relation appears in Streufert 2018. In particular, the set \( \otimes \) of triples \( \langle t,c,t^s \rangle \) defined in its Section 3.1 is very close to the set \( \pi_{WAY}(Q) \) of triples \( \langle w,a,y \rangle \) here. More broadly, the preforms of that paper supported the forms of Streufert 2020a, which in turn supported the games of Streufert 2020b. In retrospect, the Streufert 2020b specification is an uneasy compromise between the \( Gm \) specification and the pentaform specification. Incidentally, the other papers in this paragraph also define categories, and a category for pentaform games is under development. (Also, a literature related to unions is discussed in Section 4.2 footnote 17.)

1.4. Organization

Section 2 builds intuition through examples. Section 3 defines pentaform games. Section 4 discusses pentaform applications, and is unrelated to Section 5, which reviews the definition of \( Gm \) games and constructs the bijection from \( Gm \) games to pentaform games.

Appendix A verbally introduces the category \( Gm \), in case the reader is interested. Then a self-contained Appendix B develops the concept of out-tree\(^5\) which appears in Sections 3 and 5. Finally, Appendix C provides lemmas and proofs for Sections 3 and 4, while Appendix D provides lemmas and proofs for Section 5.

\(^5\)It appears that this paper’s concept of a possibly infinite out-tree is a small incidental contribution to the literature. For details see the close of Section B.2.
2. INITIAL INTUITION

This brief section builds intuition through examples. The examples suggest how an extensive form can be expressed by a quintuple set (that is, by a 5-ary relation). This section presumes familiarity with tree diagrams.

Figure 2.1. $\hat{Q}$ is defined to be the set consisting of the table’s two rows, that is, $\{\langle\text{Alex, }\{0\}, 0, \text{left}, 1\rangle, \langle\text{Alex, }\{0\}, 0, \text{right}, 2\rangle\}$. The tree diagram provides the same data. $\hat{Q}$ is a “pentaform” (Section 3.4).

Figure 2.1’s tree diagram has one player (Alex), one information set ($\{0\}$), three nodes (0, 1, and 2), and two actions (left and right). The tree has two edges (that is, “arcs” or “twigs”). These edges are (0,1) and (0,2) (tuples will be routinely bracketed by ⟨⟩ rather than () for the sake of readability). The action left labels the edge ⟨0,1⟩, while the action right labels the edge ⟨0,2⟩. This data can be encoded within the triples ⟨0, left, 1⟩ and ⟨0, right, 2⟩. Next, the node 0 is in the information set {0}. This (self-evident) fact can be encoded within the quadruples ⟨{0}, 0, left, 1⟩ and ⟨{0}, 0, right, 2⟩. Finally, the player Alex makes the decision at information set {0}. This fact can be encoded within the quintuples ⟨Alex, {0}, 0, left, 1⟩ and ⟨Alex, {0}, 0, right, 2⟩. In this sense, the set

$$\hat{Q} = \{\langle\text{Alex, }\{0\}, 0, \text{left}, 1\rangle, \langle\text{Alex, }\{0\}, 0, \text{right}, 2\rangle\}$$

expresses Figure 2.1’s tree diagram. The set’s two quintuples correspond to the two (non-header) rows in the figure’s table (eventually the information set {0} will be regarded as a special kind of situation). Finally, the dot on $\hat{Q}$ distinguishes this first example from future examples.

This process can be readily generalized. Essentially, each tree edge is changed into a quintuple. To be more specific, each tree edge is a pair ⟨w,y⟩ consisting of a decision node w and a successor node y. This pair is changed into the quintuple ⟨i,j,w,a,y⟩ in which a is the action labeling the edge ⟨w,y⟩, j is the information set containing the decision node w, and i is the player making the decision at information set j. Eventually the information set j will be regarded as a special case of a situation.

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6)[from page 5] Figures 2.2 and 2.3 correspond to Selten 1975’s well-known “horse” game. To tell a story for these figures, suppose a Kid must decide, today, between the bad action of not doing her homework (called b) and the correct action of doing her homework (called c). Next, tonight, if the homework has been finished (node 1), a Dog must decide between the dumb action of eating the homework (d) and the good action of going back to sleep (g). Finally, tomorrow, without knowing whether the kid chose bad (node 2) or the kid chose correct and the dog chose dumb (node 3), the Teacher must decide between excusing the kid (e) and failing the kid (f).
Figure 2.2. \( \bar{Q} \) is defined to be the set consisting of the table’s rows (expressed as quintuples). The tree diagram provides the same data. \( \bar{Q} \) is a “pentaform” (Section 3.4).

Figure 2.3. \( \bar{Q} \) is defined to be the set of the table’s rows (expressed as quintuples). The tree diagram provides the same data. In \( \bar{Q} \), situations are not information sets. \( \bar{Q} \) is a “pentaform” (Section 3.4).

Figure 2.2’s tree diagram is more complicated because it has a nonsingleton information set. Nonetheless it can be expressed as a quintuple set by the same process. The tree has eight edges. The edge \((3,7)\) is changed to the quintuple \((\text{Teacher}, \{2,3\}, 3, f, 7)\) to encode the facts that (i) the action \(f\) labels the edge \((3,7)\), (ii) the decision node 3 is (self-evidently) in the information set \(\{2,3\}\), and (iii) the player \text{Teacher} controls the move at the information set \(\{2,3\}\). By similarly changing the other seven edges \((w, y)\) to quintuples \((i, j, w, a, y)\), one obtains all eight rows in Figure 2.2’s table. Those rows define \(\bar{Q}\).

Figure 2.3’s tree diagram differs from the previous example to the extent that its information sets are named with the words “today”, “tonight”, and “tomorrow” (these words have meaning within the story of footnote 6). These three words are examples of situations. As footnote 2 explains, situations can be either information sets (as in the first two examples) or something else (such as the three words here). Further, this example can be expressed as a quintuple set just as the previous examples. In particular, the edge \((3,7)\) is changed to the quintuple \((\text{Teacher}, \text{tomorrow}, 3, f, 7)\) to encode the facts that (i) the action \(f\) labels the edge \((3,7)\), (ii) the decision node 3 is associated with the situation \text{tomorrow}, and (iii) the player \text{Teacher} controls the move.
in the situation tomorrow. By similarly changing the other seven edges to quintuples, one obtains all eight rows in Figure 2.3’s table. These rows define the set \( \bar{Q} \).

Finally, consider the quintuple set

\[
\bar{Q} = \{ (41,42,43,44,45), (46,47,48,49,50) \}.
\]

Obviously, an arbitrary quintuple set like \( \bar{Q} \) may or may not express a tree diagram. Relatedly, Definition 3.1 will designate which quintuple sets are to be called “pentaforms”. Next Definition 3.5 will define a “pentaform game” to be a pentaform augmented with a utility-function profile. Finally, Theorem 5.4 (the paper’s main result) will show that there is a bijection from the collection of “Gm games” to the collection of pentaform games with information-set situations. Gm games encompass almost all standard extensive-form games, and standard extensive forms are typically depicted by tree diagrams. Relatedly, the forward direction of Theorem 5.4’s bijection closely resembles this section’s informal process of expressing a tree diagram as a quintuple set.

3. Pentaform Games

This section defines pentaforms and pentaform games. The examples of Section 2 are used as illustrations.

3.1. The Components of Quintuples

An arbitrary quintuple will be denoted \( \langle i,j,w,a,y \rangle \). Call its first component \( i \) the player, call its second component \( j \) the situation, call its third component \( w \) the decision node, call its fourth component \( a \) the action, and call its fifth component \( y \) the successor node. These five terms have no formal content. They merely name the five positions in a quintuple. For example, in the quintuple \( \langle 46,47,48,49,50 \rangle \), the player is 46, the situation is 47, the decision node is 48, the action is 49, and the successor node is 50. Further, let the nodes of a quintuple be its decision node and its successor node. In other words, let the nodes of a quintuple be its third and fifth components. For example, the nodes of \( \langle 46,47,48,49,50 \rangle \) are 48 and 50.\(^7\)

This abstract notation can accommodate the literature’s wide variety of discrete (footnote 4) game notations. In particular, nodes can be specified as sequences of actions (Osborne and Rubinstein 1994), as sets of actions (Streufert 2019), as sets of outcomes (Alós-Ferrer and Ritzberger 2016 Section 6.2; see footnote 4 here), or without any special structure (as in Section 2’s examples). Meanwhile, actions can be specified as sets of nodes (van Damme 1991), as sets of edges (Selten 1975), as sets of outcomes (Alós-Ferrer and Ritzberger 2016 Section 6.2; see footnote 4 here), or without any special structure (as in Section 2’s examples). Finally, situations can be specified as sets of nodes (as in the information-set situations of Section 2’s example \( \bar{Q} \)) or without any special structure (as in Section 2’s example \( \bar{Q} \)).

\(^7\)After definition (14), \( x \) will denote a generic node. Alphabetically, \( w \) is before \( x \) is before \( y \). Similarly, decision nodes \( w \) are “early” nodes, and successor nodes \( y \) are “late” nodes.
3.2. Quintuple sets and their slices

A set of quintuples will usually be denoted by the letter $Q$. Relatedly, different quintuple sets will be distinguished from one another by means of markings around the letter $Q$. For instance, the examples in Section 2 are denoted $\tilde{Q}$, $\tilde{\tilde{Q}}$, $\bar{Q}$, and $\bar{\bar{Q}}$. Similarly, Section 4.1 will consider subsets of a quintuple set $Q$ denoted $Q' \subseteq Q$ and $'Q \subseteq Q$. Likewise, this section will consider other subsets of a quintuple set $Q$ which will be denoted $Q_j \subseteq Q$.

To do so, consider an arbitrary quintuple set $Q$, and let $J$ denote its set of situations $j$. In other words, let $J$ be the projection of $Q$ onto its second coordinate. Then, for each situation $j \in J$, define

$$Q_j = \{ \langle i, j, w, a, y \rangle \in Q \}.$$  

Thus $Q_j$ is the set of quintuples in $Q$ that have situation $j$. Call $Q_j$ the slice of $Q$ for situation $j$. By inspection, $(Q_j)_{j \in J}$ is an injectively indexed partition of $Q$. Call this the slice partition of $Q$.

For example, consider the example $\tilde{\tilde{Q}}$ in Figure 2.3, and let $\tilde{\tilde{J}}$ denote its set of situations $j$. In other words, let $\tilde{\tilde{J}}$ be the projection of $\tilde{\tilde{Q}}$ onto its second coordinate. Then the situation set $\tilde{\tilde{J}}$ is $\{\text{today}, \text{tonight}, \text{tomorrow}\}$, and the slice partition $(\tilde{\tilde{Q}}_j)_{j \in \tilde{\tilde{J}}}$ divides $\tilde{\tilde{Q}}$ into the three quintuple sets

(3a) $\tilde{\tilde{Q}}_{\text{today}} = \{ \langle \text{Kid, today, 0, c, 1} \rangle, \langle \text{Kid, today, 0, b, 2} \rangle \}$,

(3b) $\tilde{\tilde{Q}}_{\text{tonight}} = \{ \langle \text{Dog, tonight, 1, g, 8} \rangle, \langle \text{Dog, tonight, 1, d, 3} \rangle \}$, and

(3c) $\tilde{\tilde{Q}}_{\text{tomorrow}} = \{ \langle \text{Teacher, tomorrow, 2, e, 4} \rangle, \langle \text{Teacher, tomorrow, 2, f, 5} \rangle, \\
\langle \text{Teacher, tomorrow, 3, e, 6} \rangle, \langle \text{Teacher, tomorrow, 3, f, 7} \rangle \}$. 

These three sets are illustrated in both the tree diagram and the table of Figure 3.0.
3.3. Projections

Any quintuple set can be projected onto any sequence in \( \{I,J,W,A,Y\} \). For example, Figure 2.1’s table for \( \bar{Q} \) implies\(^8\)

\[
\begin{align*}
(4) & \quad \pi_Y(\bar{Q}) = \{ y \mid (\exists i,j,w,a) \langle i,j,w,a,y \rangle \in \bar{Q} \} = \{1,2\} \text{ and } \\
(5) & \quad \pi_J(\bar{Q}) = \{ \langle j,i \rangle \mid (\exists w,a,y) \langle i,j,w,a,y \rangle \in \bar{Q} \} = \{\{0\}, \text{Alex}\}.
\end{align*}
\]

For another example, equation (3c) for \( \bar{Q}_\text{tomorrow} \) implies

\[
\begin{align*}
(6) & \quad \pi_A(\bar{Q}_\text{tomorrow}) = \{ a \mid (\exists i,j,w,y) (i,j,w,a,y) \in \bar{Q}_\text{tomorrow} \} = \{e,f\} \text{ and } \\
(7) & \quad \pi_WA(\bar{Q}_\text{tomorrow}) = \{ \langle w,a \rangle \mid (\exists i,j,y) \langle i,j,w,a,y \rangle \in \bar{Q}_\text{tomorrow} \} \\
& \quad = \{\langle 2,e \rangle, \langle 2,f \rangle, \langle 3,e \rangle, \langle 3,f \rangle \}.
\end{align*}
\]

Note that projections, like \( \pi_J(\bar{Q}) \) in (5), can re-order the coordinates. Also note that projections of slices are well-defined simply because slices are quintuple sets. An example is the slice \( \bar{Q}_\text{tomorrow} \) in (3c), (6), and (7). (Slicing precedes projection in the sense that “slices of projections” are non-existent.)

Both slices and projections can be visualized by tables. Slices select rows and then projections select columns. For example, consider Figure 3.0’s table for \( \bar{Q} \). Its last four rows constitute the slice \( \bar{Q}_\text{tomorrow} \) in (3c), and this slice’s fourth column determines the projection \( \pi_A(\bar{Q}_\text{tomorrow}) \) in (6).

The notation for a single-coordinate projection will often be abbreviated by replacing the letter \( Q \) with the single coordinate. Specifically, define the five abbreviations \( I, J, W, A, \) and \( Y \) by

\[
I = \pi_I(Q), \quad J = \pi_J(Q), \quad W = \pi_W(Q), \quad A = \pi_A(Q), \quad \text{and} \quad Y = \pi_Y(Q).
\]

These abbreviations inherit any markings on the letter \( Q \). For example, (4) shows that \( Y = \pi_Y(Q) = \{1,2\} \) and (6) shows that \( A_\text{tomorrow} = \pi_A(\bar{Q}_\text{tomorrow}) \) is \( \{e,f\} \). (Two other examples are the \( J \) and \( \bar{J} \) originally defined in Section 3.2.)

One especially important projection has a special name. Consider a quintuple set \( Q \) and a situation \( j \in J \) (that is \( j \in \pi_J(Q) \)). Then \( W_j \) (that is \( \pi_W(Q_j) \)) is the situation’s decision-node set. In accord with standard terminology, \( W_j \) will henceforth be called the information set of situation \( j \). For instance, in example \( \bar{Q} \), equation (3c) or Figure 3.0 shows that the information set of the situation \( j = \text{tomorrow} \) is

\[
(9) \quad W_\text{tomorrow} = \{2,3\}.
\]

Similarly, in example \( \bar{Q} \), Figure 2.2’s table shows that the information set of the situation \( j = \{2,3\} \) is

\[
(10) \quad W_{\{2,3\}} = \{2,3\}.
\]

Thus a situation \( j \) may or may not equal its information set \( W_j \). This distinction will play a role in Sections 5.2–5.4, beginning with equation (31).

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\(^8\)When speaking aloud, it may be helpful to read \( \pi_Y(\bar{Q}) \) as “the \( Y \) of \( \bar{Q} \)” (abbreviation (8) shortens this to \( Y \)). Similarly, it may be helpful to read \( \pi_J(\bar{Q}) \) as “the \( JI \) of \( \bar{Q} \)”. 
Another important projection is the action set $A_j$ (that is $\pi_A(Q_j)$) of each situation $j \in J$. For reasons associated with Proposition 3.3, $A_j$ can also be called the feasible action set at situation $j$.

3.4. Pentaforms

For a quintuple set $Q$, let

$$p = \pi_{YW}(Q).$$

Axiom [Pw-y] in Definition 3.1 assumes that $p$ is a function (see footnote 10 for the definition of function used in this paper). Given this axiom, the statements $w = p(y)$, $\langle y, w \rangle \in \pi_{YW}(Q)$, and $\langle w, y \rangle \in \pi_{WY}(Q)$ are equivalent. Call $p$ the immediate-predecessor function.

**Definition 3.1 (Pentaform).** A pentaform is a (possibly infinite) set $Q$ of quintuples $\langle i, j, w, a, y \rangle$ such that

- $[Pi-j]^9 \pi_{H}(Q)$ is a function,
- $[Pj-w] \pi_{W}(Q)$ is function,
- $[Pwa] (\forall j \in J) \pi_{WA}(Q_j)$ is a Cartesian product,
- $[Pwa-y] \pi_{WAY}(Q)$ is a function from its first two coordinates,
- $[Pw-y] \pi_{YW}(Q)$ is a function,
- $[Pa-y] \pi_{YA}(Q)$ is a function,
- $[Py] (\forall y \in Y)(\exists m \geq 1) p^m(y) \notin Y$, and
- $[Pr] W \setminus Y$ is a singleton

(where $Q$ determines $J$, $W$, $Y$, $p$, and each $Q_j$, as summarized in Table 3.1).

The remainder of this Section 3.4 will discuss these eight axioms. En route, Section 2’s examples will be reconsidered. It will be found that examples $\tilde{Q}$, $\hat{Q}$, and $\ddot{Q}$ satisfy the axioms, and are therefore pentaforms. While these examples are finite, pentaforms can have trees with arbitrary degree (which admits decision nodes with uncountably many immediate successors), and up to countably infinite height (which admits an infinite horizon, as in the example pentaform of Streufert 2023dp Section 2.2).

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9The label $[Pi-j]$ can be read as “$i$ is a function of $j$”. The labels $[Pj-w]$, $[Pw-y]$, and $[Pa-y]$ can be read similarly. Meanwhile, the label $[Pwa-y]$ can be read as “$w$ and $a$ determine $y$”. The arrows within the labels are visual crutches in the sense that they can be removed without introducing ambiguity.

10In this paper, a function $f$ is a set of pairs $\langle x, y \rangle$ such that $(\forall x \in \pi_1 f)(\exists ! y \in \pi_2 f) \langle x, y \rangle \in f$, where $\pi_1 f$ and $\pi_2 f$ are the projections of $f$ on its first and second coordinates. Call $\pi_1 f$ the domain of $f$, and call $\pi_2 f$ the range of $f$ (this paper does not use the concept of codomain). Relatedly, a surjection from $X$ to $Y$ is a function with domain $X$ and range $Y$, and a bijection from $X$ to $Y$ is an injective surjection from $X$ to $Y$. For example, the set $g = \{(x, 3x^2) | x \in \mathbb{R}\}$ is a surjection from $\mathbb{R}$ to $\mathbb{R}_+$. Finally, “$f: X \to Z$” is occasionally used to mean “$f$ is a function such that $\pi_1 f = X$ and $\pi_2 f \subseteq Z$.”
3. Pentaform Games

Table 3.1. A pentaform is implicitly accompanied by its derivatives (\(\Delta\)). Definitions are in the sections in brackets [ ].

| Pentaform Q          | [3.4] |
|----------------------|-------|
| \(Q\)                | set of quintuples \((i,j,w,a,y)\) |
| \(I=\pi_I(Q)\)       | set of players \(i\) [3.1, 3.2] |
| \(J=\pi_J(Q)\)       | set of situations \(j\) [3.1, 3.3] |
| \(W=\pi_W(Q)\)       | set of decision nodes \(w\) [3.1, 3.3] |
| \(A=\pi_A(Q)\)       | set of actions \(a\) [3.1, 3.3] |
| \(Y=\pi_Y(Q)\)       | set of successor nodes \(y\) [3.1, 3.3] |
| \(Q_j\subseteq Q\)   | set of \(j\)'s slice of \(Q\) [3.2] |
| \(W_j=\pi_W(Q_j)\)   | set of \(j\)'s decision-node set (information set) [3.3] |
| \(A_j=\pi_A(Q_j)\)   | set of \(j\)'s (feasible) action set [3.3] |
| \(p=\pi_YW(Q)\)      | immediate-predecessor function [3.4] |
| \(F=\pi_WA(Q)\)      | feasibility correspondence [3.4] |
| \(\{r\}=W\triangle Y\) | root node \(r\) [3.4] |
| \(X=W\cup Y\)        | set of nodes \(x\) [3.5] |
| \(\pi_YW(Q)\)        | set of edges \((w,y)\) [3.5] |
| \(\preceq\)          | weak precedence order [3.5, B.4] |
| \(\prec\)            | strict precedence order [3.5, B.4] |
| \(Y\setminus W\)     | set of end nodes \(y\) [3.5, B.4] |
| \(Z\)                | collection of runs \(Z\) [3.5, B.4] |

Pentaform game \((Q,u)\)  [3.6]

\(u=\langle u_i \rangle_{i\in I}\)  profile with utility function \(u_i\) for each player \(i\) [3.6]

Axiom [Pi\(\cdot\)j] states that exactly one player \(i\) is assigned to each situation \(j\). This is interpreted to mean that exactly one player controls the move at each situation. To be clear, [Pi\(\cdot\)j] states that \(\pi_J(Q)\) is a function, which (by footnote 10) means that each \(j \in J\) is associated with exactly one \(i \in I\). For example, (5) shows \(\pi_J(\bar{Q}) = \{\{0\}, \text{Alex}\}\), and this is a (very small) function which maps \(\{0\}\) to Alex. Similarly, examples \(\bar{Q}\), \(\bar{Q}\), and \(\bar{Q}\) satisfy [Pi\(\cdot\)j].

Axiom [Pj\(\cdot\)w] states that exactly one situation \(j\) is assigned to each decision node \(w\). As Proposition 3.2(a\(\Rightarrow\)b) makes clear, this is equivalent to stating that distinct situations \(j_1\) and \(j_2\) have disjoint information sets \(W_{j_1}\) and \(W_{j_2}\). For instance, the situation set in example \(\bar{Q}\) is \(\bar{J} = \{\text{today, tonight, tomorrow}\}\), and equation (3) implies that the information sets are \(\bar{W}_{\text{today}} = \{0\}\), \(\bar{W}_{\text{tonight}} = \{1\}\), and \(\bar{W}_{\text{tomorrow}} = \{2, 3\}\) (the third already appeared in (9)). Since these are disjoint, Proposition 3.2(a\(\Leftrightarrow\)b) implies \(\bar{Q}\) satisfies [Pj\(\cdot\)w]. Similarly, examples \(\bar{Q}\), \(\bar{Q}\), and \(\bar{Q}\) satisfy [Pj\(\cdot\)w].

**Proposition 3.2.** Let \(Q\) be a quintuple set. Then the following are equivalent.

(a) \(Q\) satisfies [Pj\(\cdot\)w].

(b) (\(\forall j_1 \in J, j_2 \in J\)) \(j_1 \neq j_2\) implies \(W_{j_1} \cap W_{j_2} = \emptyset\).

(c) (\(W_{j})_{j \in J}\) is an injectively indexed partition of \(W\).11 (Proof C.2 in Appendix C.)
Axiom [Pwa] states that for each situation \( j \), the set \( \pi_{WA}(Q_j) \) is a Cartesian product. For instance, in example \( \tilde{Q} \), axiom [Pwa] is satisfied at \( j = \text{tomorrow} \) because equation (7) implies \( \pi_{WA}(\tilde{Q}_{\text{tomorrow}}) = \{2, 3\} \times \{e, f\} \). Cartesian products can also be found at the other two situations in \( \tilde{Q} \) and at all situations in \( Q, \tilde{Q} \), and \( \tilde{Q} \). Hence all four examples satisfy [Pwa].

To interpret [Pwa], define the correspondence\(^{12}\)
\[
F = \pi_{WA}(Q).
\]
Call \( F \) the feasibility correspondence, and call \( F(w) \subseteq A \) the set of feasible actions at decision node \( w \in W \). For example, Figure 2.3’s table implies \( \pi_{WA}(\tilde{Q}) = \{ (0, c), (0, b), (1, g), (1, d), (2, e), (2, f), (3, e), (3, f) \} \), which implies that \( \tilde{F}(0) = \{c, b\} \), that \( \tilde{F}(1) = \{g, d\} \), and that \( \tilde{F}(2) = \tilde{F}(3) = \{e, f\} \). Below, Proposition 3.3(a\(\leftrightarrow\)d) characterizes [Pwa] by the property that the feasible set \( F(w) \) is constant across the nodes \( w \) in an information set. For instance, in the example \( \tilde{Q} \), equation (9) implies that nodes 2 and 3 share the information set \( \tilde{W}_{\text{tomorrow}} = \{2, 3\} \), and the second-previous sentence shows that their feasible sets \( \tilde{F}(2) \) and \( \tilde{F}(3) \) are both equal to \( \{e, f\} \). Also, in accord with Proposition 3.3(b,c), this common feasible set is \( \tilde{A}_{\text{tomorrow}} = \{e, f\} \) (for verification see (6)).

**Proposition 3.3.** Suppose \( Q \) satisfies [Pj+w]. Then the following are equivalent.

(a) \( Q \) satisfies [Pwa].

(b) \( (\forall j \in J) \pi_{WA}(Q_j) = W_j \times A_j \).

(c) \( (\forall j \in J, w \in W_j) F(w) = A_j \).

(d) \( (\forall j \in J, w_1 \in W_j, w_2 \in W_j) \tilde{F}(w_1) = F(w_2) \). \((\text{Proof C.4 in Appendix C.})\)

Axiom [Pwa→y] states that each decision-node/feasible-action pair \( \langle w, a \rangle \in \pi_{WA}(Q) \) determines a successor node \( y \). To be clear, [Pwa→y] states that \( \pi_{WA}(Q) \) is a function from its first two coordinates, which means that \( \{ \langle \langle w, a \rangle, y \rangle \mid \langle w, a, y \rangle \in \pi_{WA}(Q) \} \) is a function. By projection, the domain of such a function is \( \pi_{WA}(Q) \), and this can be accurately called the set of decision-node/feasible-action pairs because the definition of \( F \) implies that \( \pi_{WA}(Q) \) is the set of pairs \( \langle w, a \rangle \) that satisfy \( a \in F(w) \). In a different direction, it is sufficient for [Pwa→y] that each decision-node/feasible-action pair appears in exactly one quintuple in \( Q \). By inspection, this holds in examples \( Q, \tilde{Q}, \text{and } \tilde{Q}. \)

Axiom [Pw→y] states that exactly one decision node is associated with each successor node (this axiom was used near definition (11) to imply that \( p = \pi_{YW}(Q) \) is a function). Similarly, [Pa→y] states that exactly one action is associated with each successor node. It is sufficient for both [Pw→y] and [Pa→y] that each successor node

\(^{11}\)from page 10\]: To be clear, Proposition 3.2(c) means that \( [1] (W_j)_{j \in J} = \{ (j, W_j) \mid j \in J \} \) is an injective function and \( [2] \{ W_j \mid j \in J \} \) partitions \( W \). This implies that \( (W_j)_{j \in J} \) is a bijection from \( J \) to \( \{ W_j \mid j \in J \} \), but the converse fails.

\(^{12}\)In this paper, a correspondence is simply a set of pairs. Occasionally, the expression “\( F: X \equiv Z \)” is used to mean “\( F \) is a correspondence such that \( \pi_1 = X \) and \( \pi_2 \subseteq Z \).” (This paper does not apply the terms “domain” and “range” to correspondences.)
\( y \in Y \) appears in exactly one quintuple in \( Q \). By inspection, this holds in examples \( Q, \bar{Q}, \ddot{Q}, \) and \( \dddot{Q} \).

Axiom [Py] can be understood in the terms of difference equations (Luenberger 1979 page 14). In particular, consider the difference equation \( y_{k-1} = p(y_k) \), where the index \( k \in \{0,-1,-2,\ldots\} \) runs backwards and \( p = \pi_{\top W}(Q) \) is the immediate-predecessor function (11). In this context, [Py] states that the path starting from any \( y_0 \in Y \) eventually leaves \( Y \). In casual terms, the set \( Y \) is a “set-source”. For instance, consider example \( \dddot{Q} \). There definition (1) implies that \( \dddot{Y} = \{45,50\} \) and that \( \dddot{p} = \pi_{\top W}(\dddot{Q}) = \{\langle 45,43\rangle,\langle 50,48\rangle\} \). Hence \( \dddot{Y} \) is a “set-source” because \( \dddot{p}(45) = 43 \notin \dddot{Y} \) and because \( \dddot{p}(50) = 48 \notin \dddot{Y} \). The examples \( \dot{Q}, \ddot{Q}, \) and \( \dddot{Q} \) also satisfy [Py], though exhaustively showing so takes longer than for \( \dddot{Q} \) because there are more successor nodes and because the paths exiting \( Y \) can be longer.

Axiom [Pr] states that there is exactly one decision node which is not also a successor node. If so, let \( r \) be this unique decision node, so that
\[
\{r\} = W\setminus Y.
\]

Call \( r \) the \textit{root} node.\(^\text{13}\) For instance, consider example \( \dddot{Q} \). There Figure 2.3’s table implies that \( \dddot{W} = \{0,1,2,3\} \) and that \( \dddot{Y} = \{1,2,3,4,5,6,7,8\} \), and these observations imply that \( \dddot{W}\setminus\dddot{Y} = \{0\} \). Thus \( \dddot{Q} \) satisfies [Pr] and \( \dddot{r} = 0 \). Examples \( \dot{Q} \) and \( \ddot{Q} \) also satisfy [Pr]. Meanwhile, example \( \dddot{Q} \) violates [Pr] because definition (1) implies \( \dddot{W} = \{43,48\}, \dddot{Y} = \{45,50\}, \) and \( \dddot{W}\setminus\dddot{Y} = \{43,48\} \).

In light of the preceding paragraphs, examples \( \dot{Q}, \ddot{Q}, \) and \( \dddot{Q} \) satisfy all eight axioms, and are consequently pentafoms. Meanwhile, example \( \dddot{Q} \) satisfies all axioms except [Pr], and is consequently not a pentaform.\(^\text{14}\) Note that \( \dddot{Q} \) proves it is possible to violate [Pr] while satisfying the other seven axioms. In fact, Table C.1 in Appendix C shows that it is essentially possible to violate any one of the eight axioms while satisfying the other seven. In this sense the eight axioms are logically independent.

To verbally summarize the definition of a pentaform, recall from Section 3.1 that the terms “player”, “situation”, “decision node”, “action”, and “successor node” are defined to mean nothing but the five positions in a quintuple. Then a pentaform is a set of quintuples which satisfies the following five properties. (1) Exactly one player is assigned to each situation. (2) Exactly one situation is assigned to each decision node. (3) The set of actions assigned to a decision node is constant across the decision nodes assigned to each situation. (4) The assignment of a decision-node/action pair to a successor node is a bijection. (5) The assignment of decision nodes to successor nodes eventually takes every successor node to the unique decision node that is not a successor node. (1) paraphrases [Pw-j], (2) paraphrases [Pj-w], (3) paraphrases [Pwa]

\(^\text{13}\)[Pw-y], [Py], and [Pr] imply that \( (\forall y \in Y)(\exists \ell \geq 1) \ p^\ell(y) = r \). To see this, take a successor node \( y \in Y \). Then [Py] implies there is \( \ell \geq 1 \) such that \( p^\ell(y) \notin Y \). Note that the range of \( p \) by \( p \)'s definition (11) is \( \pi_{\top W}(Q) \), which by abbreviation (8) is \( W \). Thus \( p^\ell(y) \in W\setminus Y \), which by [Pr] and \( r \)'s definition (13) implies \( p^\ell(y) = r \).

\(^\text{14}\)[H]ere are some other examples: [1] the empty set is not a pentaform because it violates [Pr], [2] a singleton set \( \{(i,j,w,a,y)\} \) with \( w \neq y \) is a pentaform, and [3] a singleton set \( \{(i,j,w,a,y)\} \) with \( w = y \) is not a pentaform because it violates [Pr] (and [Py]).
3. Pentaform Games

with the help of [Pj-w] and Proposition 3.3, (4) paraphrases [Pwa-y], [Pw-y], and [Pa-y], and finally, (5) is based on [Pw-y] and paraphrases [Py] and [Pr].

3.5. Out-trees

For any quintuple set $Q$, let

$$X = W \cup Y$$

be the set of $Q$’s nodes, and call $\pi_{WY}(Q)$ the set of $Q$’s edges. Proposition 3.4 shows that a certain set of three pentaform axioms is equivalent to the pair $(X, \pi_{WY}(Q))$ being a kind of graph-theoretic tree. The proposition uses three definitions from Section B.2 in Appendix B. First, an “out-tree” is defined to be the divergent orientation of a rooted tree (Definition B.4). Second, the “root” of an out-tree is defined to be the root of the rooted tree that it orients (Definition B.7). Third, an out-tree is said to be “nontrivial” iff it has at least one edge (above Proposition B.6).

**Proposition 3.4.** Suppose $Q$ is a quintuple set. Then (a) $Q$ satisfies $[Pw-y]$, $[Py]$, and $[Pr]$ iff $(X, \pi_{WY}(Q))$ is a nontrivial out-tree. Further, (b) suppose $Q$ satisfies $[Pw-y]$, $[Py]$, and $[Pr]$. Then $Q$’s root (13) equals the root (Definition B.7) of the out-tree $(X, \pi_{WY}(Q))$. (Proof C.6.)

Consider a quintuple set $Q$ which satisfies $[Pw-y]$, $[Py]$, and $[Pr]$. The proposition implies that the nodes, edges, and root of $Q$ coincide with the same entities of the out-tree $(X, \pi_{WY}(Q))$. Further, by inspection, the $W$, $Y$, and $p$ derived from $Q$ [via (8) and (11) above] coincide with the same entities derived from the out-tree $(X, \pi_{WY}(Q))$ via (33) in Section B.3.

In addition, let $Q$’s weak precedence relation $\preceq$ and $Q$’s strict precedence relation $\prec$ be the $\preceq$ and $\prec$ derived from the out-tree $(X, \pi_{WY}(Q))$ via (34) in Section B.4. Also, call $Y \setminus W$ the set of $Q$’s end nodes (or leaves), just as $Y \setminus W$ is called the set of end nodes of the out-tree $(X, \pi_{WY}(Q))$ via (35) in Section B.4. Finally, let $Q$’s run (or play) collection $Z$ be the $Z$ derived from the out-tree $(X, \pi_{WY}(Q))$ via (36) in Section B.4.

3.6. Pentaform Games

Suppose $Q$ is a pentaform with its player set $I$ (from abbreviation (8)) and its run collection $Z$ (from the previous paragraph). A utility function for player $i$ is a function of the form $u_i: Z \to \bar{R}$, where $\bar{R}$ is the extended real line $\mathbb{R} \cup \{-\infty, \infty\}$.\(^{15}\)

A utility-function profile is a $u = \langle u_i \rangle_{i \in I}$ which lists a utility function $u_i$ for each player $i \in I$. The following definition defines a pentaform game to be a pair listing a pentaform $Q$ and a utility-function profile $u$. Figures 3.1 and 3.2 provide two relatively simple examples (Streufert 2023dp Section 2.2 provides an infinite-horizon example).

---

\(^{15}\)Infinite utility numbers are included because, in economics, many popular utility functions generate $-\infty$ utility when some consumption level is zero. Such a utility function is often part of a consumer dynamic optimization problem, and such a problem can be specified as a one-player game.
Definition 3.5 (Pentaform game). A pentaform game is a pair \((Q, u)\) such that \(Q\) is a pentaform (Definition 3.1) and \(u = \langle u_i : \mathbb{Z} \rightarrow \bar{\mathbb{R}} \rangle_{i \in I}\) (where \(Q\) determines \(I\) and \(\mathbb{Z}\), as summarized in Table 3.1).

**Figure 3.1.** The pentaform game \((\dot{Q}, \dot{u})\). \(\dot{Q}\) is the set consisting of the upper table’s rows (expressed as quintuples), and \(\dot{u}_{\text{Alex}}: \dot{Z} \rightarrow \bar{\mathbb{R}}\) is defined by the lower table. The tree diagram provides the same data.

**Figure 3.2.** The pentaform game \((\ddot{Q}, \ddot{u})\). \(\ddot{Q}\) is the set consisting of the upper table’s rows, and \(\ddot{u} = \langle \ddot{u}_i : \ddot{Z} \rightarrow \bar{\mathbb{R}} \rangle_{i \in I}\) is defined by the lower table. The tree diagram provides the same data.

### 4. Some Pentaform Tools and Applications

Sections 4.1 and 4.2 develop tools for subsets and unions, and show their application to Selten subgames, dynamic programming, and pentaform construction. Separately, Section 4.3 applies pentaforms to perfect-recall and no-absentmindedness. (Section 5 will not depend on this Section 4.)

#### 4.1. Subsets of Pentaforms

Part (a) of Proposition 4.1 shows that any subset of a pentaform satisfies six of the eight pentaform axioms. This and the definition of pentaform immediately imply part (b). The proof of part (a) is intuitive. First consider the five axioms [Pi\(j\), [Pj\(w\),...
[Pwa\*y], [Pa\*y], and [Pw\*y]. Each states that some multidimensional projection of \( Q \) is a function (where a function is a special kind of set by footnote 10). If such a statement holds for \( Q \), it holds almost obviously for any \( Q' \subseteq Q \). Meanwhile, axiom [Py] means that \( Y \) is a “set-source” for the difference equation defined by \( p \), as discussed in Section 3.4. If this holds for \( Y \), it must also hold for \( Y' \subseteq Y \). Thus it is intuitive that [Py] for \( Q \) implies [Py] for any \( Q' \subseteq Q \) (the proposition’s proof addresses some details about \( p' \)).

**Proposition 4.1.** Suppose \( Q \) is a pentaform and \( Q' \subseteq Q \). Then (a) \( Q' \) satisfies [Pi\*j], [Pi\*w], [Pwa\*y], [Pa\*y], [Pw\*y], and [Py]. Thus (b) \( Q' \) is a pentaform iff it satisfies [Pwa] and [Pr]. (Proof C.7.)

As discussed in Section 3.2, each quintuple set \( Q \) is partitioned by the collection \( \{Q_j\}_{j \in J} \) of its slices \( Q_j \). The following corollary\(^{16}\) states that the union of a subcollection of a pentaform’s slice partition satisfies the first seven pentaform axioms. This follows from Proposition 4.1(a) if it can be shown that such a union of slices satisfies [Pwa]. This holds because \( Q \) satisfies [Pwa] by assumption and because [Pwa] is defined in terms of individual slices.

**Corollary 4.2.** Suppose that \( Q \) is a pentaform and that \( Q' \) is a subcollection of \( \{Q_j\}_{j \in J} \). Then (a) \( \bigcup Q \) satisfies [Pi\*j], [Pi\*w], [Pwa], [Pwa\*y], [Pa\*y], [Pw\*y], and [Py]. Hence (b) \( \bigcup Q \) is a pentaform iff it satisfies [Pr]. (Proof above.)

This remainder of this Section 4.1 uses Corollary 4.2 to construct the pentaforms for the Selten subgames of an arbitrary pentaform game. It also explains how the same Corollary 4.2 provides the basis for Streufert 2023dp’s new results on dynamic programming.

Consider a pentaform \( Q \). Then for any \( w \in W \), define
\[
\begin{align*}
{\langle i_*, j_*, w_*, a_*, y_* \rangle} & \in Q \mid w \leq w_*
\end{align*}
\]
(15)

To put this in other words, say that a quintuple is weakly after \( w \) iff its decision node weakly succeeds \( w \). Then \( {\langle i_*, j_*, w_*, a_*, y_* \rangle} \) is the set of quintuples that are weakly after \( w \). A (Selten) subroot is a member of
\[
\begin{align*}
T & \equiv \{ t \in W \mid \langle t \rangle \text{ and } \pi_J(Q \setminus \langle t \rangle) \text{ are disjoint} \}
\end{align*}
\]
(16)

where \( \langle t \rangle \) abbreviates \( \pi_J(Q \setminus \langle t \rangle) \) by the sentence following (8). In other words, a decision node \( t \in W \) is a subroot iff each situation listed in a quintuple weakly after \( t \) is not listed in a quintuple anywhere else. Lemma C.9 shows this is equivalent to \( \langle t \rangle \) being the union of a subcollection of \( Q \)’s slice partition.

Because of this, Corollary 4.2(b) can be applied to each \( \langle t \rangle \). The result is the following proposition, which shows that the \( \langle t \rangle \) associated with each Selten subroot \( t \in T \) is a pentaform. As a consequence, the pentaform \( \langle t \rangle \) can serve as the extensive form of the Selten subgame starting at \( t \) (Streufert 2023dp Section 4.2 completes the subgame by defining its utility functions).

---

\(^{16}\)Both Corollary 4.2 and Proposition 4.4 can be regarded extensions of Corollary 3.4 in the previous version (Streufert 2021v2).
Proposition 4.3. Suppose $Q$ is a pentaform and $t \in T$. Then $^tQ$ is a pentaform.

(Proof C.10.)

Corollary 4.2 also plays a central role in the dynamic-programming theory of Streufert 2023dp, which is the first paper to use value functions to characterize sub-game perfection in arbitrary games (only pure strategies are considered there). For an analogy, consider a repeated game. There, the whole-game extensive form (that is, the supergame’s extensive form) combines many replicas of a single-stage extensive form (which is typically very simple). In this paper’s terminology, the “subroots” of the whole game are the nodes starting the replicas of the stage form. Now imagine that different subroots have different stage forms. In Streufert 2023dp’s terminology, these generalized stage forms are called “piece forms”. Each of the various piece forms has some combination of [a] finite runs, each of which terminates in a subsequent subroot or whole-form endnode, and [b] infinite runs, each of which fails to reach a subsequent subroot or whole-form endnode.

Streufert 2023dp specifies an arbitrary pentaform game and shows [a] that the piece-form collection partitions the pentaform and [b] that this piece-form partition is coarser than the pentaform’s slice partition. Thus it can use Corollary 4.2(b) to show that each piece form is a pentaform. On this foundation the paper is able to build a notion of “piecewise Nashness” which generalizes dynamic-programming’s Bellman equation to arbitrary games.

4.2. Unions of blocks

This Section 4.2 shows how to construct pentaforms as unions of “blocks”.\(^{17}\) For the purposes of this section, let a (penta)block be a quintuple set $Q$ satisfying the first seven pentaform axioms, namely

\[
\text{(17)} \quad [\Pi_j], [Pj-w], [Pwa], [Pwa-y], [Pw-y], [Pwa y], \text{ and } [Py].
\]

Thus a pentaform is equivalent to a block which satisfies the final pentaform axiom $[P_r]$. To put this in other words, consider an arbitrary quintuple set $Q$ and call $W-Y$ the set of $Q$’s start nodes. Then $[P_r]$ is equivalent to $Q$ having exactly one start node. Hence a pentaform is equivalent to a block with exactly one start node.

In this section’s terminology, Corollary 4.2(a) shows that if $Q$ is a subcollection of a pentaform’s slice partition, then $\cup Q$ is a block. This implies that each slice of a pentaform is a block. For example, the slice partition of $\bar{Q}$ is the collection \{$\bar{Q}_{\text{today}}, \bar{Q}_{\text{tonight}}, \bar{Q}_{\text{tomorrow}}$\}, in equation (3) and Figure 3.0. Since $\bar{Q}$ is a pentaform, each of these three slices is a block.

\(^{17}\)This Section 4.2 seems related to the ongoing work of Ghani, Kupke, Lambert, and Nordvall Forsberg 2018, Bolt, Hedges, and Zahn 2019, and Capucci, Ghani, Ledent, and Nordvall Forsberg 2022. Both that literature and this Section 4.2 seek to construct games out of game fragments. A precise comparison is elusive because the mathematical foundations are very different. More is said there about utility. The relative advantages here include addressing infinite-horizon games, using the relatively simple operation of union, and using relatively finitely-grained axioms. (To avoid a possible confusion, note that the literature at the start of this footnote uses category theory to construct games, while Streufert 2021Gm uses essentially different parts of category theory to compare games. The two endeavours are entirely distinct.)
4. Some Pentaform Tools and Applications

Now consider an arbitrary collection \( \mathcal{Q} \) of quintuple sets \( Q \). Then \( \mathcal{Q} \) is said to be \textit{weakly separated} iff its member sets do not share situations, decision nodes, or successor nodes. In other words, \( \mathcal{Q} \) is weakly separated iff
\[
(\forall Q^1 \in \mathcal{Q}, Q^2 \in \mathcal{Q} \setminus \{Q^1\}) \ J^1 \cap J^2 = \emptyset, \ W^1 \cap W^2 = \emptyset, \text{ and } Y^1 \cap Y^2 = \emptyset.
\]
Alternatively, \( \mathcal{Q} \) is said to be \textit{strongly separated} iff its member sets do not share situations or nodes. In other words, \( \mathcal{Q} \) is strongly separated iff
\[
(\forall Q^1 \in \mathcal{Q}, Q^2 \in \mathcal{Q} \setminus \{Q^1\}) \ J^1 \cap J^2 = \emptyset \text{ and } X^1 \cap X^2 = \emptyset.
\]
Note (19) implies (18) because \( X^1 = W^1 \cup Y^1 \) and \( X^2 = W^2 \cup Y^2 \) by definition (14).

**Proposition 4.4.** (a) Suppose \( \{Q^1, Q^2\} \) is a weakly separated collection of blocks such that the start-node set \( W^1 \setminus Y^1 \) is disjoint from the end-node set \( Y^2 \setminus W^2 \). Then \( Q^1 \cup Q^2 \) is a block whose start-node set is the union of 
\[
W^1 \setminus Y^1 \text{ and } (W^2 \setminus Y^2) \setminus (Y^1 \setminus W^1)
\]
and whose end-node set is the union of 
\[
(Y^1 \setminus W^1) \setminus (W^2 \setminus Y^2) \text{ and } Y^2 \setminus W^2.
\]
(b) Suppose \( \mathcal{Q} \) is a strongly separated collection of blocks. Then \( \bigcup \mathcal{Q} \) is a block with start-node set \( \bigcup_{Q \in \mathcal{Q}} (\pi_W(Q) \setminus \pi_Y(Q)) \) and end-node set \( \bigcup_{Q \in \mathcal{Q}} (\pi_Y(Q) \setminus \pi_W(Q)) \).
(Proofs C.12 and C.13.)

To explore this proposition, consider three quintuple sets: example \( \bar{Q} \) [from Figure 2.3 with slice partition (3) in Figure 3.0],
\[
Q^{\text{guilty}} = \{\langle \text{Kid}, \text{guilty}, 4, s, 11\rangle, \langle \text{Kid}, \text{guilty}, 4, s, 12\rangle, \\
\langle \text{Kid}, \text{guilty}, 5, s, 13\rangle, \langle \text{Kid}, \text{guilty}, 5, s, 14\rangle\} \text{, and}
\]
\[
Q^{\text{innocent}} = \{\langle \text{Kid}, \text{innocent}, 6, s, 15\rangle, \langle \text{Kid}, \text{innocent}, 6, s, 16\rangle, \\
\langle \text{Kid}, \text{innocent}, 7, s, 17\rangle, \langle \text{Kid}, \text{innocent}, 7, s, 18\rangle\}.\]

These three quintuple sets are illustrated in Figure 4.1. All three are blocks. In particular, (i) \( \bar{Q} \) is a block because it is a pentaform and because every pentaform is a block by the block definition (17), (ii) \( Q^{\text{guilty}} \) is a block because it is like the slice \( \bar{Q}_{\text{tomorrow}} \) (3c) from the pentaform \( \bar{Q} \) and because every slice of every pentaform is a block by Corollary 4.2(a), and (iii) \( Q^{\text{innocent}} \) is a block because it also is like the slice \( \bar{Q}_{\text{tomorrow}} \). By inspection, the three blocks are weakly separated (18).

Proposition 4.4(a) assumes that \( Q^1 \) and \( Q^2 \) are weakly separated blocks and that no start node of \( Q^1 \) is also an end node of \( Q^2 \). This admits the possibility that an end node of \( Q^1 \) is also a start node of \( Q^2 \). Roughly, \( Q^1 \) can precede \( Q^2 \), but not vice

\[18\]Q^{\text{guilty}} \text{ and } Q^{\text{innocent}} \text{ can be interpreted as continuations of the story in footnote 6. In the guilty situation of } Q^{\text{guilty}} \text{, the kid did not do her homework, the teacher has not yet reported a verdict, and the kid must decide whether to say something to influence her parents’ reaction to the coming verdict (s denotes the action of saying something, and } \bar{s} \text{ denotes the opposite). Similarly, in the innocent situation of } Q^{\text{innocent}} \text{, the dog ate the kid’s homework, the teacher has not yet reported a verdict, and the kid must decide whether to say something to influence her parents’ reaction to the coming verdict.}
versa. For example, consider \((Q^1, Q^2) = (\bar{Q}, Q^{\text{guilty}})\) [ignore \(Q^{\text{innocent}}\)]. The only start node of \(Q^1 = \bar{Q}\) is 0, and this is not in \(\{11, 12, 13, 14\}\), which is the end-node set of \(Q^2 = Q^{\text{guilty}}\). Thus Proposition 4.4(a) implies that \(\bar{Q} \cup Q^{\text{guilty}}\) is a block whose start nodes are

\[(22a) \text{the start nodes of } Q^1 = \bar{Q} \text{ together with}
\]

\[(22b) \text{the start nodes of } Q^2 = Q^{\text{guilty}} \text{ that are not also end nodes of } Q^1 = \bar{Q}.
\]

Regarding (22b), the start nodes of \(Q^{\text{guilty}}\) are 4 and 5, which are also end nodes of \(\bar{Q}\). Thus (22b) contributes nothing, so that the start nodes of \(\bar{Q} \cup Q^{\text{guilty}}\) are (22a) the start nodes of \(\bar{Q}\). Therefore, since the only start node of \(\bar{Q}\) is 0, the block \(\bar{Q} \cup Q^{\text{guilty}}\) is a pentaform with root 0.

In a similar fashion, Proposition 4.4(a) can be applied at \((Q^1, Q^2) = (\bar{Q}, Q^{\text{innocent}})\) to show that \(\bar{Q} \cup Q^{\text{innocent}}\) is a pentaform. Further, using this technique twice shows that \(\bar{Q} \cup Q^{\text{guilty}} \cup Q^{\text{innocent}}\) is a pentaform. This can be accomplished by constructing the union as \((\bar{Q} \cup Q^{\text{guilty}}) \cup Q^{\text{innocent}}\) or as \((\bar{Q} \cup Q^{\text{innocent}}) \cup Q^{\text{guilty}}\).

Proposition 4.4(b) provides another way to prove that \(\bar{Q} \cup Q^{\text{guilty}} \cup Q^{\text{innocent}}\) is a pentaform. By inspection, \(\{Q^{\text{guilty}}, Q^{\text{innocent}}\}\) is strongly separated (19). Thus Proposition 4.4(b) implies that \(Q^{\text{guilty}} \cup Q^{\text{innocent}}\) is a block whose start-node set is

\[
\cup_{Q \in \{Q^{\text{guilty}}, Q^{\text{innocent}}\}} (\pi_W(Q) \setminus \pi_Y(Q)) = \{4, 5\} \cup \{6, 7\}
\]

and whose end-node set is

\[
\cup_{Q \in \{Q^{\text{guilty}}, Q^{\text{innocent}}\}} (\pi_Y(Q) \setminus \pi_W(Q)) = \{11, 12, 13, 14\} \cup \{15, 16, 17, 18\}.
\]
Then, as before, Proposition 4.4(a) can be applied at \((Q^1, Q^2) = (\bar{Q}, Q^{\text{guilty}} \cup Q^{\text{innocent}})\) to show that \(\bar{Q} \cup (Q^{\text{guilty}} \cup Q^{\text{innocent}})\) is a block whose only start node is 0. Thus the union is a pentaform with root 0.

Roughly, the previous paragraph showed how to augment the pentaform \(\bar{Q}\) with a “layer” \(Q = \{Q^{\text{guilty}}, Q^{\text{innocent}}\}\) of two additional blocks whose start nodes were among the end nodes of \(\bar{Q}\). This same technique can be used to augment any finite-horizon pentaform with any layer \(Q\) of additional blocks whose start nodes are among the end nodes of the original pentaform. The layer’s blocks do not need to be similar to one another. Also, the layer can have arbitrarily many (and possibly uncountably many) blocks, which freely allows arbitrarily many end nodes of the original pentaform to be connected with successor nodes in the layer’s additional blocks.

Further, layer after layer can be added to generate an infinite expanding sequence of pentaforms. Proposition 4.5 shows that the union of such an expanding sequence is a pentaform. This provides a straightforward way to construct an infinite-horizon pentaform. For instance, Streufert 2023dp’s motivating example is a partially infinitely repeated game of cry-wolf. Its pentaform is built via Propositions 4.4 and 4.5 (details in Streufert 2023dp Lemma A.6).\(^\text{19}\)

**Proposition 4.5.** Suppose \(\langle Q^n \rangle_{n \geq 0}\) is an infinite sequence of pentaforms such that \((\forall n \geq 1) Q^{n-1} \subseteq Q^n\) and \(r^n = r^0\). Then \(\bigcup_{n \geq 0} Q^n\) is a pentaform with root \(r^0\).

(\(\text{Proof C.14.}\))

4.3. Perfect-recall in terms of pentaforms

Although perfect-recall (Hart 1992, page 36) will remain a relatively subtle concept, it seems helpful to reformulate the concept in terms of a pentaform \(Q\). This section does so, and is unconnected with Sections 4.1–4.2.

To begin, the axioms \([\Pi_i]\), \([\Pi_j]\), \([\Pi_w]\), and \([\Pi_y]\) imply that each successor node \(y \in Y\) determines its entire quintuple \(\langle i, j, w, a, y \rangle \in Q\). Thus we may speak of \(y\)'s player \(i_y\), and \(y\)'s situation \(j_y\), and \(y\)'s action \(a_y\). Then a pentaform \(Q\) is said to have perfect-recall iff

\[
(\forall y_1 \in Y, y_2 \in Y, y_3 \in Y) \ y_1 \prec y_2, \ i_y_1 = i_y_2, \ j_y_2 = j_y_3 \Rightarrow (\exists y_4 \in Y) \ y_4 \prec y_3, \ j_y_4 = j_y_1, \ a_y_4 = a_y_1.
\]

(If \(y_1 \prec y_3\), the conditional’s conclusion is satisfied by \(y_4 = y_1\).) Figure 4.2 depicts the relationships between \(y_1, y_2, y_3,\) and \(y_4\). In English, suppose one successor node strictly precedes a second successor node and at the same time shares its player with the second. Further suppose the second shares its situation with a third successor

---

\(^{19}\) Lemma C.11 implies that the union of a weakly separated collection of arbitrarily many blocks satisfies all axioms except \([\Pi_y]\) and \([\Pi_r]\). This lemma is used to prove Proposition 4.4. In addition, the lemma itself can be used as a relatively unstructured alternative to the proposition. For example, the proof of Streufert 2023dp Lemma A.6 (for that paper’s infinite-horizon example) can be replaced with a relatively brief but ad hoc argument which uses only Lemma C.11 and avoids both Propositions 4.4 and 4.5.
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Figure 4.2. The definition (23) of perfect-recall assumes the solid lines and then requires the dashed lines.

Then perfect-recall requires that there is a fourth successor node which strictly precedes the third and which shares both its situation and its action with the first (it is possible that the fourth and the first coincide).

For an example, modify the \( \hat{Q} \) of Figure 2.3 (or 4.1) by replacing both the player Kid and the player Teacher with a new player named KidTeacher. Here perfect-recall fails at \( y_1 = 2 \), \( y_2 = 4 \), and \( y_3 = 6 \). To better explain this, identify each successor node \( y \) with the edge ending at \( y \). Then consider Figure 2.3 (or 4.1), or the definition (23) of perfect-recall, or Figure 4.2. From any of these perspectives, edge-to-2 precedes and shares its player (KidTeacher) with edge-to-4, and further, edge-to-4 shares its situation (tomorrow) with edge-to-6. Yet, none of the edges preceding edge-to-6 share situation and action with edge-to-2. Specifically, the edges preceding edge-to-6 are edge-to-1 and edge-to-3. Edge-to-3 does not share its situation (tonight) with edge-to-2 (whose situation is today). And finally, although edge-to-1 does share its situation (today) with edge-to-2, it does not share its action (c) with edge-to-2 (whose action is b). Thus perfect-recall fails. Casually, player KidTeacher will forget tomorrow whether she did action b today.

In accord with Piccione and Rubinstein 1997 (pages 9–10), a pentaform is said to have no-absentmindedness iff there are not \( y_1 \in Y \) and \( y_2 \in Y \) such that \( y_1 \prec y_2 \) and \( jy_1 = jy_2 \). The proof of the following proposition is new. The proposition itself is not exactly new because it is already understood in some sense. For example, Piccione and Rubinstein 1997 pages 3–5 suggest that absentmindedness occurs only if there is imperfect recall. That understanding is the contrapositive of the following proposition.

**Proposition 4.6.** If a pentaform has perfect-recall, it has no-absentmindedness.

**Proof.** Assume \( Q \) is a pentaform with absentmindedness. It will be shown that \( Q \) violates perfect-recall (23). By definition, absentmindedness means that there are successor nodes \( y_1 \) and \( y_2 \) such that \( y_1 \prec y_2 \) and \( jy_1 = jy_2 \). Note \( jy_1 = jy_2 \) and [Pi+1]

---

20To understand this sentence in terms of information sets, note that there is a bijection between situations and information sets by footnote 11. Thus the sentence essentially supposes that “the second [successor node] shares its information set with a third successor node”. However, this does not mean that the successor nodes belong to the same information set. Rather, it means that their immediate predecessors belong to the same information set. Casually, the two successor nodes “immediately follow” the same information set.
implies $i y_1 = i y_2$. Next, let

$$y_3 = \min\{ y \in Y \mid y \prec y_2, j y = j y_2 \},$$

which is well-defined because $y_2$ has finitely many predecessors, because those predecessors are linearly ordered, and because $y_1$ is in the set. Note $y_1 \prec y_2$, $i y_1 = i y_2$, and $j y_3 = j y_2$. Thus $y_1$, $y_2$, and $y_3$ satisfy the hypotheses of perfect-recall (23). Yet the minimality of $y_3$ implies there is no $y_4$ such that $y_4 \prec y_3$ and $j y_4 = j y_2$. Thus the assumption $j y_1 = j y_2$ implies there is no $y_4$ such that $y_4 \prec y_3$ and $j y_4 = j y_1$. Hence the conclusion of perfect-recall (23) is violated (there is no need to consider actions).

5. Equivalence with Gm Games

This Section 5 is unconnected with Section 4. The theorems here construct a bijection between a certain subcollection of pentaform games and the whole collection of “Gm” games. For this paper it is relatively unimportant to know that the symbol “Gm” stands for Streufert 2021Gm’s category of extensive-form games (this category is briefly introduced in Appendix A, for readers who might be interested).

Importantly for this paper, a Gm game is a tree which has been adorned with information sets, actions, players, and utility functions in a more-or-less standard way. The word “more-or-less” acknowledges that there is no standard way to define an extensive-form game. Rather, substantially different definitions appear in social science, mathematics, computer science, logic, and engineering, and further, there is often substantial variety within any one of these fields. Somewhere in the middle is the Gm definition, and unfortunately, almost every reader will be unfamiliar with some aspect of this formulation (footnotes 21–25 discuss some hazards). In developing this definition, the author’s guiding principles were to avoid specialized assumptions, to use explicit notation, and to use standard mathematics whenever possible.

5.1. Definition of Gm games

The following paragraphs define the symbols and terms that appear in Definition 5.1 below. The paragraphs are given letters to facilitate referencing. This material is summarized in the right-hand side of Table 5.1.

Paragraph a. The first component of a Gm game is a nontrivial out-tree $(X, E)$.\(^{21}\) As defined in Section B.2 of Appendix B,

\[(24) \quad \text{an out-tree } (X, E) \text{ is the divergent orientation of a rooted tree, and such an out-tree is said to be nontrivial iff } E \text{ is nonempty.} \]

$X$ is called the set of nodes, and $E$ is called the set of edges. Further, Sections B.2–B.4 derive from an

\(^{21}\)This first step may be unfamiliar for any of several reasons. First, trees can be specified as rooted trees rather than out-trees (see (24) for the connection, or Proposition B.6 for more details). Second, some popular specifications use specialized notations which implicitly impose some of the properties of trees. For example, some tree properties are implicitly imposed if nodes are specified as sequences of past actions, or sets of past actions, or sets of outcomes. Third, trees can be specified via order theory rather than graph theory. Finally, it is common to specify trees by diagrams rather than by notation.
5. Equivalence with $\text{Gm}$ Games

| Out-tree $(X, E)$ | $\bar{X} = \bar{W} \cup \bar{Y}$ | [5.1a, B.2] |
|------------------|---------------------------------|-------------|
|                  | $\pi_{\text{W}Y}(\bar{Q})$     | [5.1a, B.2] |
|                  | $\{\bar{r}\} = \bar{W} \setminus \bar{Y}$ | [5.1a, B.2] |
|                  | $\bar{W}$                       | [5.1a, B.3] |
|                  | $\bar{Y}$                       | [5.1a, B.3] |
|                  | $\bar{p} = \pi_{\text{YW}}(\bar{Q})$ | [5.1a, B.3] |
|                  | $\preceq$                       | [5.1a, B.4] |
|                  | $\prec$                        | [5.1a, B.4] |
|                  | $Z$                            | [5.1a, B.4] |

| Gm game $(X, E, \mathcal{H}, \lambda, \tau, u)$ | $J$ | [5.1b] |
|-------------------------------------------------|-----|-------|
|                                                 | $\mathcal{H}$ | collection of information sets $H$ |
|                                                 | $\lambda$    | labeling function |
|                                                 | $A$           | set of actions $a$ |
|                                                 | $F$           | feasibility correspondence |
|                                                 | $\tau$       | control-assigning function |
|                                                 | $I$           | set of players $i$ |
| $\bar{u} = \langle \bar{u}_i \rangle_{i \in I}$ | $u = \langle u_i \rangle_{i \in I}$ profile with utility function $u_i$ for each $i$ |

Table 5.1. Right-hand Side: Out-trees and Gm games are implicitly accompanied by their derivatives ($\bar{\cdot}$). Definitions are in the sections in brackets []. Left-hand Side: Pentaform equivalents in $(\bar{Q}, \bar{u}) = \mathcal{P}(X, E, \mathcal{H}, \lambda, \tau, u)$, from Theorem 5.5 (recall from (8) that $I, J, W, A,$ and $Y$ abbreviate $\pi_I(\bar{Q}), \pi_J(\bar{Q}), \pi_W(\bar{Q}), \pi_A(\bar{Q}),$ and $\pi_Y(\bar{Q}))$.

out-tree $(X, E)$ its root node $r$, its decision-node set $W = \pi_1 E$, its successor-node set $Y = \pi_2 E$, its immediate-predecessor function $p = \{\langle y, w \rangle | \langle w, y \rangle \in E\}$, its weak and strict precedence orders $\preceq$ and $\prec$, and its collection $Z$ of runs (or “plays”). In particular, $r$ is defined in Definition B.7; $W$, $Y$, and $p$ in equation (33); $\preceq$ and $\prec$ in equation (34); and $Z$ in equation (36).

Paragraph b. The next component of a Gm game is a partition $\mathcal{H}$ of the decision-node set $W$. The members of this partition are called information sets (because the controlling player will be informed that she is in the set, but not at which node in the set she is).

The following three paragraphs will build on $(X, E, \mathcal{H})$ by appending the components $\lambda, \tau,$ and $u$. These three components are independent of one another, except that the players determined by $\tau$ appear again in $u$.

Paragraph c. First, each edge of the tree is labeled with an action. Formally, this labeling is accomplished by a function $\lambda$ from the edge set $E$. Call $\lambda$ the labeling function.\(^{22}\) Then let $A$ be the range of $\lambda$ (equivalently, let $A = \{\lambda(w, y) | \langle w, y \rangle \in E\}$).

\(^{22}\)Labeling functions may be unfamiliar for any of several reasons. First, an alternative to assigning actions to edges $\langle w, y \rangle \in E$ is to assign actions to successor nodes $y \in Y$. The two are equivalent
and call an element of $A$ an action (because it says what the controlling player would “do” to choose the edge). It will be assumed that $\lambda$ is locally injective in the sense that, for any two edges of the form $\langle w, y_1 \rangle \in E$ and $\langle w, y_2 \rangle \in E$,
\[
y_1 \neq y_2 \text{ implies } \lambda(w, y_1) \neq \lambda(w, y_2). \tag{25}\]
Thus local injectivity means that two different edges from one decision node $w$ cannot be assigned the same action. Further, from the labeling function $\lambda$, derive the correspondence $F: W \rightarrow A$ by
\[
(\forall w \in W) \ F(w) = \{ a \mid (\exists y) \lambda(w, y) = a \}. \tag{26}\]
Thus each $F(w)$ is the set of actions that label the edges leaving $w$. Call $F(w)$ the set of actions that are feasible at $w$. Then let $\langle F(w) \rangle_{w \in W}: W \rightarrow \mathcal{P}(A)$ be the associated set-valued function. It will be assumed that $\langle F(w) \rangle_{w \in W}$ is measurable\textsuperscript{23} in the sense that it is measurable as a function from $W$ (endowed with the partition $\sigma$-algebra from the information-set partition $\mathcal{H}$) into the collection of the subsets of $A$ (endowed with the discrete $\sigma$-algebra). This is equivalent to
\[
(\forall H \in \mathcal{H}, w_1 \in H, w_2 \in H) \ F(w_1) = F(w_2), \tag{27}\]
which requires that two nodes in one information set have the same feasible-action set.

Paragraph d. Second, each decision node of the tree is assigned to a player. Formally, this assignment is accomplished by a function $\tau$ from the decision-node set $W$.\textsuperscript{24} Call $\tau$ the control-assigning function (because it says who will control the move where; mnemonically $\tau$ suggests control). Then let $I$ be the range of $\tau$ (equivalently, let $I = \{ \tau(w) \mid w \in W \}$), and call an element of $I$ a player. It will be assumed that $\tau$ is measurable (footnote 23) in the sense that it is measurable as a function from $W$ (endowed with the partition $\sigma$-algebra from the information-set partition $\mathcal{H}$) into the player set $I$ (endowed with the discrete $\sigma$-algebra). This is equivalent to
\[
(\forall H \in \mathcal{H}, w_1 \in H, w_2 \in H) \ \tau(w_1) = \tau(w_2), \tag{28}\]
which requires that two nodes in one information set are controlled by the same player.

Paragraph e. Third, each player is given a utility function defined over the runs of the tree. Formally, this is accomplished by a profile $u = \langle u_i \rangle_{i \in I}$ which lists a function $u_i: Z \rightarrow \bar{\mathbb{R}}$ for each player $i$. Here $Z$ is the run collection (36) of the out-tree $(X, E)$ as

---

\textsuperscript{23}Since this paper does not use measurability theory in any substantial way, the term “measurability” may be regarded as just a mathematically precise name for conditions (27) and (28). However, this measurability is equivalent to the continuity in Streufert 2021Gm [C4] and [G2], and there the general theory of measurability/continuity does play a supporting role. (The use of measurability/continuity in this context may be unfamiliar because it was new in Streufert 2021Gm.)

\textsuperscript{24}An alternative is to assign players to information sets rather than to decision nodes. This would implicitly impose the measurability of (28).
defined in paragraph a, and \( \mathbb{R} \) is the set of extended real numbers \( \mathbb{R} \cup \{ -\infty, \infty \} \) as discussed in Section 3.6. Call \( u_i \) the utility function (or “payoff function”) of player \( i \).\(^{25}\)

**Definition 5.1 (Gm game, Streufert 2021Gm).** A Gm game is a tuple \((X, E, \mathcal{H}, \lambda, \tau, u)\) such that

- \([Gm1]\) \((X, E)\) is a nontrivial out-tree (24),
- \([Gm2]\) \(\mathcal{H}\) is a partition of \( W \),
- \([Gm3]\) \(\lambda\) is a locally injective (25) function from \( E \),
- \([Gm4]\) \(\langle F(w)\rangle_{w \in W}\) is measurable (27),
- \([Gm5]\) \(\tau\) is a measurable (28) function from \( W \), and
- \([Gm6]\) \(u = \langle u_i : Z \rightarrow \mathbb{R} \rangle_{i \in I}\)

(where \((X, E)\) determines \( W = \pi_1 E \) and \( Z \), where \(\lambda\) determines \( F \) (26), and where \(\tau\) determines \( I = \{ \tau(w) | w \in W \} \), as summarized in Table 5.1).

For an example, consider Figure 3.1’s tree diagram. It has already served other purposes. Nonetheless, it also illustrates the Gm game \((X', E', \mathcal{H}', \lambda', \tau', u')\) defined by \(X' = \{0, 1, 2\} \), \(E' = \{\langle 0, 1 \rangle, \langle 0, 2 \rangle\} \), \(\mathcal{H}' = \{\{0\}\} \),

\[
\lambda' = \{ \langle \langle 0, 1 \rangle, \text{left} \rangle, \langle \langle 0, 2 \rangle, \text{right} \rangle \},
\]

\[
\tau' = \{ \langle 0, \text{Alex} \rangle \}, \text{ and }
\]

\[
u_{\text{Alex}}' = \{ \langle \{0, 1\}, 2 \rangle, \langle \{0, 2\}, 4 \rangle \}
\]

(footnote 10 explains that, in this paper, a function is a set of pairs). This definition implies that \(W' = \{0\}\), that \(Z' = \{\{0, 1\}, \{0, 2\}\}\), that \(F'(0) = \{\text{left, right}\}\), that \(A' = \{\text{left, right}\}\), and that \(I' = \{\text{Alex}\}\).

5.2. “Pentaforming” a Gm game

The remainder of this Section 5 constructs a bijection between Gm games and certain pentaform games. To begin, this Section 5.2 constructs an operator \(P\) which maps each Gm game to a pentaform game. Mnemonically, \(P\) “pentaforms” a Gm game. To be specific, let \(P\) be the operator (equivalently function) that takes a Gm game \((X, E, \mathcal{H}, \lambda, \tau, u)\) to the \(P(X, E, \mathcal{H}, \lambda, \tau, u) = (\bar{Q}, \bar{u})\) defined by

\[
\bar{Q} = \{ \langle \tau(w), H_w, w, \lambda(w, y), y \rangle | \langle w, y \rangle \in E \} \text{ and }
\]

\[
\bar{u} = u,
\]

where in (30a), for each \(w \in W\), \(H_w\) is the unique member of \(\mathcal{H}\) that contains \(w\) (each such \(H_w\) exists by \([Gm2]\)).

The range of the operator \(P\) is a proper subset of the set of pentaform games. To be more specific, recall from examples \(\bar{Q}\) and \(\bar{Q}\) in Figures 2.2 and 2.3 that a pentaform’s

\(^{25}\)This specification of utility may be slightly unfamiliar. First, it is common, in finite-horizon games, to map end nodes to utility numbers. In this special case there is a bijection from the end-node set \(Y \setminus W\) to the run collection \(Z\) (see \(Z\) in near the end of Section B.4). Second, footnote 15 explains the slight benefit of allowing utility numbers in \(\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}\) rather than the more familiar \(\mathbb{R}\).
situations may or may not be information sets. In this respect, a \textbf{Gm} game is less general because a \textbf{Gm} game has no concept of “situation” other than an information set. Relatedly, equation (30) maps a \textbf{Gm} game to a pentaform game whose situations are information sets. To be able to state this precisely, say that a quintuple set $Q$ has \textit{information-set situations} iff

$$
(\forall j \in J) \ W_j = j.
$$

For instance, equation (10) shows that example $\bar{Q}$ has information-set situations, while equation (9) shows that example $\bar{Q}$ does not.

**Theorem 5.2.** The operator $P$ takes each \textbf{Gm} game to a pentaform game with information-set situations (31). (Proof: Lemma D.1(a).)

The definition of $P$ accords with Section 2’s informal process of expanding edges into quintuples. To explore this, consider the \textbf{Gm} game $(X', E', \mathcal{H}', \lambda', \tau', u')$ defined in equation (29). It is illustrated by the tree diagram on Figure 3.1’s left-hand side. Then $P(X', E', \mathcal{H}', \lambda', \tau', u')$ is the pair $(\bar{Q}, \bar{u})$ shown in the figure’s right-hand side. Transparently, the lower table $\bar{u}$ is a mere rearrangement of the utilities in the tree diagram, in accord with (30b). Less transparently, the upper table $\bar{Q}$ is derived from the tree diagram, in accord with (30a). To see this, first note that the quintuples (rows) in the upper table can be indexed by the edges $\langle w, y \rangle$ that they contain. This indexing accords with the dummy variable $\langle w, y \rangle$ in (30a). Then consider, for example, the quintuple containing the edge $\langle w, y \rangle = \langle 0, 2 \rangle$. As in (30a), and as in Section 2’s informal process, this edge is expanded into the quintuple $\langle \text{Alex}, \{0\}, 0, \text{right}, 2 \rangle$ by including its action $\lambda'(0, 2) = \text{right}$, its decision node’s information set $H_0 = \{0\} \in \mathcal{H}'$, and its decision node’s player $\tau'(0) = \text{Alex}$. The other quintuple in the upper table is derived in the same way.

### 5.3. “Standardizing” a Pentaform Game

This Section 5.3 constructs the operator $S$ from pentaform games to \textbf{Gm} games. Mnemonically, $S$ “standardizes” a pentaform game. To be specific, let $S$ be the operator that takes a pentaform game $(Q, u)$ to the $S(Q, u) = (\hat{X}, \hat{E}, \hat{H}, \lambda, \hat{\tau}, \hat{u})$ defined by

$$
\begin{align*}
\hat{X} &= W \cup Y, \\
\hat{E} &= \pi_{WY}(Q), \\
\hat{H} &= \{ W_j \mid j \in J \}, \\
\hat{\lambda} &= \{ \langle \langle w, y \rangle, a \rangle \mid \langle w, y, a \rangle \in \pi_{WYA}(Q) \}, \\
\hat{\tau} &= \pi_{WI}(Q), \text{ and} \\
\hat{u} &= u
\end{align*}
$$

(where $I$, $J$, $W$, $A$, $Y$, and $W_j$ abbreviate $\pi_I(Q)$, $\pi_J(Q)$, $\pi_W(Q)$, $\pi_A(Q)$, $\pi_Y(Q)$, and $\pi_W(Q_j)$, in accord with (8) and the sentence thereafter).
Theorem 5.3. The operator $S$ takes each pentaform game to a Gm game. (Proof: Lemma D.2.)

Broadly speaking, a pentaform $Q$ is a high-dimensional relation, while a Gm game is a list of low-dimensional relations. Thus a pentaform describes the relationships between players, situations, nodes, and actions all at once, while a Gm game describes the same relationships one by one. In this light, it is reasonable that equations (32a)–(32e) would use various projections of a pentaform $Q$ to build the first five components of a Gm game. To be completely precise, [a] equations (32a), (32b), (32d), and (32e) use projections of $Q$ itself, while [b] each information set $W_j$ in (32c) is the projection of a slice of $Q$ (but, if $Q$ has information-set situations, then the right-hand side of (32c) reduces to $\{W_j|j\in J\} = \{j|j\in J\} = J$, which is yet another projection of $Q$ itself).

5.4. Bijection

The bijection of the following theorem is the paper’s main result. The bijection is illustrated in Figure 5.1 by the two opposing arrows between the thick vertical bars. As claimed in Section 1.1, this bijection is constructive and intuitive. The construction is provided by equations (30) and (32). The intuition is provided by Section 5.2’s last paragraph, Section 5.3’s last paragraph, and the remainder of this Section 5.4.

![Figure 5.1. The operators P and S](image)

Theorem 5.4 (Main theorem). $P$ is a bijection from the collection of Gm games to the collection of pentaform games with information-set situations (31). Its inverse is the restriction of $S$ to the collection of pentaform games with information-set situations. (Proof D.5.)

For example, consider the standard Gm game $(X', E', \mathcal{H}', \lambda', \tau', u')$ from (29). Theorem 5.4 implies $SP(X', E', \mathcal{H}', \lambda', \tau', u') = (X', E', \mathcal{H}', \lambda', \tau', u')$. In this sense, the standard game $(X', E', \mathcal{H}', \lambda', \tau', u')$ and the pentaform game $P(X', E', \mathcal{H}', \lambda', \tau', u')$ are “equivalent”. Further, Section 5.2’s last paragraph argued that this pentaform game equals the pentaform game $(\check{Q}, \check{u})$ from Figure 3.1’s tables. Thus the standard game $(X', E', \mathcal{H}', \lambda', \tau', u')$ and the pentaform game $(\check{Q}, \check{u})$ are “equivalent”. This accords with our having used Figure 3.1’s tree diagram to illustrate both games.
Appendix A. The Category Gm

For another example, consider the pentaform game \((\bar{Q}, \bar{u})\) in Figure 3.2. Since this game has information-set situations,\(^{26}\) Theorem 5.4 implies \(\text{PS}(\bar{Q}, \bar{u}) = (\bar{Q}, \bar{u})\). In this sense, the pentaform game \((\bar{Q}, \bar{u})\) and the standard game \(S(\bar{Q}, \bar{u})\) are “equivalent”. Further, this \(S(\bar{Q}, \bar{u})\) could be explicitly derived by applying the projections of definition (32) to \((\bar{Q}, \bar{u})\).

As discussed earlier, Table 5.1’s right-hand side summarizes the Gm notation and terminology. More specifically, the table’s right-hand side lists the 6 components of a Gm game, together with 10 of its derivatives. Now consider the table’s left-hand side. It shows the “pentaform equivalents” of these 16 Gm entities. In particular, the 16 rows of the table reproduce the 16 conclusions of the following theorem.

Theorem 5.5. Suppose \((X, E, H, \lambda, \tau, u)\) is a Gm game. Define \((\bar{Q}, \bar{u}) = P(X, E, H, \lambda, \tau, u)\). Then the following hold.
(a) \(\bar{Q} = Q\).
(b) \(\bar{p} = p\).
(c) \(\bar{r} = r\).
(d) \(\bar{W} = W\).
(e) \(\bar{Y} = Y\).
(f) \(\bar{\pi} = \pi\).
(g) \(\bar{\xi} = \xi\).
(h) \(\bar{\zeta} = \zeta\).
(i) \(\bar{Z} = Z\).
(j) \(\bar{J} = H\).
(k) \(\{ (\bar{w}, y, a) | (w, y, a) \in \pi_{\bar{W}YA}(\bar{Q}) \} = \lambda\).
(l) \(\bar{A} = A\).
(m) \(\bar{F} = F\).
(n) \(\pi_{\bar{W}I}(\bar{Q}) = \tau\).
(o) \(\bar{I} = I\).
(p) \(\bar{u} = u\). (Proof D.6.)

Theorem 5.4 implies that \((X, E, H, \lambda, \tau, u)\) and \((\bar{Q}, \bar{u})\) satisfy the first two sentences of Theorem 5.5 iff they satisfy the first two sentences of Corollary 5.6. Thus Theorem 5.5 implies Corollary 5.6.

Corollary 5.6. Suppose \((Q, \bar{u})\) is a pentaform game with information-set situations (31). Define \((X, E, H, \lambda, \tau, u) = S(Q, \bar{u})\). Then the conclusions of Theorem 5.5 hold. (Proof above.)

Appendix A. The Category Gm

This appendix is tangential. It briefly introduces and motivates Gm, which is the category of extensive-form games from Streufert 2021Gm. Essentially, a “category” is a kind of algebra which mathematicians have been using for some dozens of years to systematize various fields in mathematics. The category Gm is doing something similar for the field of game theory.

To motivate this systematizing, consider an analogy. Grammar involves nouns and verbs, and knowing grammar helps you to learn multiple languages like French and Swahili. Analogously, category theory involves objects and morphisms, and knowing category theory helps you to learn multiple mathematical fields like topology and graph theory. But category theory does more than assist learning. It can also guide research. For example, if a concept or theorem in one field does not have a parallel in a second field, then that parallel concept or theorem may be awaiting its discovery.

To see this more concretely, consider the apparently dissimilar mathematical fields of topology and graph theory. It turns out that a homeomorphism between two

\(^{26}\)As Figure 5.1 suggests, the composition PS changes any pentaform game into a pentaform game with information-set situations. For example, consider \((\bar{Q}, \bar{u})\), where \(\bar{Q}\) is from Figure 2.3 and \(\bar{u}\) is from Figure 3.2. This mixture of examples is a well-defined pentaform game because \(\bar{I} = I\) and \(\bar{Z} = Z\). It can be shown that \(S(\bar{Q}, \bar{u}) = S(Q, \bar{u})\), which implies \(PS(\bar{Q}, \bar{u}) = PS(Q, \bar{u})\), which by \(PS(Q, \bar{u}) = (\bar{Q}, \bar{u})\) implies \(PS(\bar{Q}, \bar{u}) = (Q, \bar{u})\). Hence the composition PS changes \((\bar{Q}, \bar{u})\), which does not have information-set situations, into \((Q, \bar{u})\), which does.
Appendix A. The Category \textbf{Gm}

topological spaces is very much like an isomorphism between two graphs. In fact, topological homeomorphisms and graph isomorphisms are now understood as examples of isomorphisms in different categories. (Incidentally, the category for topological spaces is called \textbf{Top}, the category for graphs is called \textbf{Grph}, and individual fields may or may not use special terms like “homeomorphism” in lieu of generic terms like “isomorphism”.)

Likewise, there are isomorphisms in the new category \textbf{Gm}. These new isomorphisms turn out to be powerful tools. For example, fundamental concepts such as subgame perfection are invariant to game isomorphisms (Streufert 2021Gm Theorem 4.7). Further, there are superficially small subsets, of the set of \textbf{Gm} games, which have the property that each \textbf{Gm} game is isomorphic to some game in the subset. An example is the subset consisting of those games whose nodes are sequences of past actions (Streufert 2021Gm Theorem 5.2). Theorems such as these suggest that game isomorphisms will allow researchers to formally translate results from one game to another game, and from one subset of games to another subset of games.

Appendix B. Rooted Trees, Out-trees, and Edge-trees

This appendix is self-contained. Section B.1 reviews the concept of rooted tree, which consists of a tree (a kind of undirected graph which may be infinite) and a distinguished node (called a root). Next, Section B.2 extends the concept of out-tree (a kind of directed graph) to allow for infinitely many nodes and edges, and shows that there is a bijection from rooted trees to out-trees (Proposition B.6). Finally, Section B.5 introduces the concept of edge-tree (a kind of set of ordered pairs), and shows that there is a bijection from nontrivial out-trees to edge-trees (Proposition B.17). Along the way, Sections B.3 and B.4 develop the projections and the paths of out-trees.

For general perspective, note that Section B.2’s out-trees are part of the definition of \textbf{Gm} games ([Gm1] in Definition 5.1), and that Section B.5’s edge-trees approach part of the definition of pentaform games ([Pw-y], [Py], and [Pr] in Definition 3.1). Relatedly, Proposition B.17’s equivalence between out-trees and edge-trees plays a central role in proving Proposition 3.4 and Theorem 5.4 (for details, see footnote 38 on page 43). Theorem 5.4 is the paper’s main result.

B.1. Rooted trees

As in Diestel 2010, an undirected graph is a pair \((X, \mathcal{E})\) such that \(X\) is a (possibly infinite) set and \(\mathcal{E}\) is a collection of two-element subsets of \(X\). The elements of \(X\) are called nodes (or vertices) and the elements of \(\mathcal{E}\) are called edges (or arcs or branches). A path linking \(x_0\) and \(x_\ell\) is an undirected graph \((\bar{X}, \bar{\mathcal{E}})\) of the form \(\bar{X} = \{x_0, x_1, \ldots, x_\ell\} \neq \emptyset\) and \(\bar{\mathcal{E}} = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{\ell-1}, x_\ell\}\}\) in which distinct \(i\) and \(j\) satisfy \(x_i \neq x_j\). To be clear, \((\emptyset, \emptyset)\) is not a path, a path of the form \((\{x\}, \emptyset)\) links

\footnote{Section B.1 is essentially taken from Diestel 2010, Chapter 1. Although Diestel 2010, page 2, assumes that \(X\) is finite, the relevant portions from that book do not depend upon finiteness.}

\footnote{In other words, a path linking \(x_A\) and \(x_B\) is a pair \((\bar{X}, \bar{\mathcal{E}})\) for which there are an integer \(\ell \geq 0\) and a bijection \(\langle x_m \rangle_{m=0}^{\ell}\) from \(\{0,1,\ldots,\ell\}\) to \(\bar{X}\) such that \(x_0 = x_A\), \(x_\ell = x_B\), and}
a node \( x \) with itself, and any path from a node to itself is of that form. (Infinite undirected paths will not be defined.)

One undirected graph \((X, \mathcal{E})\) is said to be in another graph \((X, \mathcal{E})\) iff \( \bar{X} \subseteq X \) and \( \mathcal{E} \subseteq \mathcal{E} \). A tree is an undirected graph \((X, \mathcal{E})\) such that every two nodes in \( X \) are linked by exactly one path in \((X, \mathcal{E})\). Relatedly, an undirected graph \((X, \mathcal{E})\) is said to be connected iff every two nodes in \( X \) are linked by at least one path in \((X, \mathcal{E})\).

Further, an undirected graph \((X, \mathcal{E})\) is said to be acyclic iff it does not contain a path \((\bar{X}, \mathcal{E})\) of the form \( \bar{X} = \{x_0, x_1, \ldots, x_\ell\} \) and \( \mathcal{E} = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{\ell-1}, x_\ell\}\} \) such that both \( \ell \geq 2 \) and \( \{x_0, x_\ell\} \in \mathcal{E} \). The following lemmas (like the other lemmas in this paper) are used in subsequent constructions and proofs.

Lemma B.1 (Diestel 2010, Theorem 1.5.1). An undirected graph is a tree iff it is connected and acyclic.

Proof. An undirected graph is not a tree iff [a] it has two nodes which are not linked or [b] it has two nodes which are connected by more than one path. [a] is equivalent to the graph not being connected, and [b] is equivalent to the graph being acyclic.

\[ \qed \]

Lemma B.2. Suppose \((X, \mathcal{E})\) is a tree. Further suppose [a] that the pair listing \( \bar{X} = \{x_0, x_1, \ldots, x_\ell\} \) and \( \mathcal{E} = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{\ell-1}, x_\ell\}\} \) is a path in \((X, \mathcal{E})\), and [b] that \( x_* \) satisfies \( \{x_\ell, x_*\} \in \mathcal{E} \).

Proof. The reverse direction holds by inspection. For the forward direction, assume \( x_* \in \bar{X} \). Then there is \( m \in \{0, 1, \ldots, \ell\} \) such that [1] \( x_m = x_* \). First note that \( m = \ell \) is impossible. In particular, \( m = \ell \) and [1] imply \( x_\ell = x_* \), which by the assumption \( \{x_\ell, x_*\} \in \mathcal{E} \) implies \( \{x_*\} \in \mathcal{E} \), which contradicts \((X, \mathcal{E})\) being an undirected graph.

Second note that \( m \leq \ell - 2 \) is impossible. In particular, \( m \leq \ell - 2 \) implies that the pair listing \( \{x_m, x_{m+1}, \ldots, x_\ell\} \) and \( \{\{x_m, x_{m+1}\}, \{x_{m+1}, x_{m+2}\}, \ldots, \{x_{\ell-1}, x_\ell\}\} \) is a path in \((X, \mathcal{E})\) with at least two edges. Further, the assumption \( \{x_\ell, x_*\} \in \mathcal{E} \) and [1] imply \( \{x_m, x_*\} \in \mathcal{E} \). The last two sentences imply that \((X, \mathcal{E})\) is not acyclic, which by Lemma B.1 contradicts \((X, \mathcal{E})\) being a tree. Since all other possibilities have been eliminated, \( m = \ell - 1 \). This implies that \( \ell \geq 1 \) and also, via [1], that \( x_{\ell-1} = x_* \).

\[ \qed \]

Definition B.3 (Rooted Tree). A rooted tree is a triple \((X, \mathcal{E}, r)\) such that \((X, \mathcal{E})\) is an (undirected) tree and \( r \in X \). The node \( r \) is called the root of the rooted tree.

B.2. Definition of Out-Trees

As in Bang-Jensen and Gutin 2009,\(^{29}\) a directed graph is a pair \((X, \mathcal{E})\) such that \( X \) is a (possibly infinite) set, \( \mathcal{E} \subseteq X^2 \), and \((\forall x \in X) (x, x) \notin \mathcal{E} \). Further, a directed graph is said to be asymmetric iff \((\forall x \in X, x_* \in X) (x, x_*) \in \mathcal{E} \) implies \( (x_*, x) \notin \mathcal{E} \). Now consider an asymmetric directed graph \((X, \mathcal{E})\) and an undirected graph \((X, \mathcal{E})\) with

\[ \{\{x_{m-1}, x_m\} | m \in \{1, 2, \ldots, \ell\}\} = \mathcal{E} \]. Since there is only one way to string a path’s edges end-to-end, the bijection is essentially unique. In particular, the only flexibility is reversing the numbering.

\(^{29}\)Sections B.2 and B.4 draw heavily from Bang-Jensen and Gutin 2009, Chapter 1. However, Bang-Jensen and Gutin 2009, page 2, assumes that \( X \) is finite. This necessitates some modifications, including that of footnote 30.
the same node set \( X \). Say that \((X,E)\) \textit{orients} \((X,\mathcal{E})\) iff \( \mathcal{E} = \{ \langle w,y \rangle \mid \langle w,y \rangle \in E \} \). Thus each asymmetric directed graph orients exactly one undirected graph, and a typical undirected graph has many different (asymmetric) orientations. To be clear, consider an undirected graph \((X,\mathcal{E})\) with a finite number of edges. It has \(2^{|E|}\) orientations because asymmetry requires that each edge in \( \mathcal{E} \) be oriented one way or the other.

Let the \textit{divergent orientation of a rooted tree} \((X,\mathcal{E},r)\) be the pair \((X,E)\) such that \([1]\) \((X,E)\) orients \((X,\mathcal{E})\) and \([2]\) for each (directed) edge \(\langle w,y \rangle \in E\), \(w\) is on the (undirected) path in \((X,\mathcal{E})\) that links \(r\) and \(y\). Casually, the divergent orientation of \((X,\mathcal{E},r)\) is the orientation of \((X,\mathcal{E})\) in which each edge is pointed away from \(r\).

\textbf{Definition B.4 (Out-Tree).} An \textit{out-tree} is the divergent orientation of a rooted tree. To be clear, a pair \((X,E)\) is an out-tree iff there is a rooted tree \((X,\mathcal{E},r)\) such that \((X,E)\) is the divergent orientation of \((X,\mathcal{E},r)\).

\textbf{Lemma B.5.} Each out-tree is the divergent orientation of exactly one rooted tree.

\textit{Proof.} By the definition of out-tree (Definition B.4), it suffices to show that distinct rooted trees have distinct divergent orientations. Toward that goal, suppose that \((X,E)\) is the divergent orientation of both the rooted tree \((X^1,\mathcal{E}^1, r^1)\) and the rooted tree \((X^2,\mathcal{E}^2, r^2)\). This implies that \((X,E)\) orients both the tree \((X^1,\mathcal{E}^1)\) and the tree \((X^2,\mathcal{E}^2)\). Thus \(X\) equals both \(X^1\) and \(X^2\), and both \(\mathcal{E}^1\) and \(\mathcal{E}^2\) equal \(\{\{w,y\} \mid \langle w,y \rangle \in E\}\). Hence \((X^1,\mathcal{E}^1) = (X^2,\mathcal{E}^2)\).

If \(r^1\) and \(r^2\) are equal, then \((X^1,\mathcal{E}^1, r^1) = (X^2,\mathcal{E}^2, r^2)\) and the proof is complete. Accordingly, assume that \(r^1\) and \(r^2\) are distinct. We will derive a contradiction. Since both \(r^1\) and \(r^2\) are in \(X^1 = X^2\), and since \((X^1,\mathcal{E}^1) = (X^2,\mathcal{E}^2)\) is a tree, there is a unique path in \((X^1,\mathcal{E}^1) = (X^2,\mathcal{E}^2)\) linking \(r^1\) and \(r^2\). Since \(r^1\) and \(r^2\) are distinct by assumption, the path has an edge \(\{w,y\}\). Assume without loss of generality that \([a]\) \(w\) is on the path linking \(r^1\) and \(y\). Then \([b]\) \(y\) is on the path linking \(r^2\) and \(w\). Since \((X,E)\) divergently orients \((X^1,\mathcal{E}^1, r^1)\), \([a]\) implies \(\langle w,y \rangle \in E\). Similarly, since \((X,E)\) divergently orients \((X^2,\mathcal{E}^2, r^2)\), \([b]\) implies \(\langle y,w \rangle \in E\). Thus \((X,E)\) violates asymmetry, in contradiction to \((X,E)\) being an orientation. \(\Box\)

Say that an undirected or directed graph is \textit{trivial} iff it has no edges. To be clear, suppose \((X,E)\) orients \((X,\mathcal{E})\). Then \(E\) is asymmetric and \(\mathcal{E} = \{\{w,y\} \mid \langle w,y \rangle \in E\}\). Thus the cardinality of \(E\) equals the cardinality of \(\mathcal{E}\), which implies that \((X,E)\) is trivial iff \((X,\mathcal{E})\) is trivial.

\textbf{Proposition B.6.} (a) The process of divergent orientation is a bijection from the collection of rooted trees to the collection of out-trees. (b) When restricted to the collection of nontrivial rooted trees, the process of divergent orientation becomes a bijection from the collection of nontrivial rooted trees to the collection of nontrivial out-trees. (Follows from Definition B.4, Lemma B.5, and the above discussion of triviality.)

\textbf{Definition B.7 (Tree and Root of an Out-Tree).} Suppose \((X,E)\) is an out-tree. By Proposition B.6(a), it is the divergent orientation of exactly one rooted tree \((X,\mathcal{E},r)\). Call \((X,\mathcal{E})\) the tree of \((X,E)\), and call \(r\) the root of \((X,E)\).
The derivation of an out-tree's tree and root is the inverse of the process of divergent orientation in Proposition B.6. To be more specific, consider an out-tree \((X,E)\). The definition of orientation implies the out-tree's tree \((X,E)\) is \((X,\{\{w,y\}|\langle w,y\rangle \in E\})\), and soon, definition (33) and Lemma B.8(c) will show that the out-tree's root \(r\) can be calculated as the sole member of \(\pi_1E \setminus \pi_2E\).

The above concept of out-tree, and its further development in Sections B.3 and B.4 below, appear to be a small contribution to the literature. In particular, the new concept generalizes\(^{30}\) the concept of finite out-tree from Bang-Jensen and Gutin 2009, page 21. (For convenience, the upper-right quadrant of Table 5.1 catalogs much of the out-tree terminology from this section and from Sections B.3 and B.4 below.)

**B.3. Projections of out-trees**

(Sections B.3 and B.4 are essentially unconnected.) Consider an out-tree \((X,E)\). Let

\[(33)\]

\[W = \pi_1E, \ Y = \pi_2E, \ \text{and} \ p = \{\langle y,w\rangle | \langle w,y\rangle \in E\},\]

where \(\pi_1E\) is the projection of \(E\) on its first coordinate, and \(\pi_2E\) the projection of \(E\) on its second coordinate. Lemma B.8 and Lemma B.9(a) are illustrated by Figure B.1 (it works best for nine-node out-trees like those within Figures 2.2 and 2.3).

**Lemma B.8.** Suppose \((X,E)\) is a nontrivial out-tree, with its \(r\) (Definition B.7), \(W\) (33), and \(Y\) (33). Then the following hold.

(a) \(W \cup Y = X\).
(b) \(\{\{r\}, Y\}\) partitions \(X\).
(c) \(W \setminus Y = X \setminus Y = \{r\}\).
(d) \(Y \setminus W = X \setminus W\).

**Proof.** Let \(E = \{\{w,y\}|\langle w,y\rangle \in E\}\) so that \((X,E)\) is the divergent orientation of the rooted tree \((X,E,r)\).

\(a)\). For the forward inclusion \(W \cup Y \subseteq X\), first note that \((X,E)\) being a directed graph implies \(E \subseteq X^2\). Thus definition (33) implies both \(W \subseteq X\) and \(Y \subseteq X\), which implies \(W \cup Y \subseteq X\).

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\(^{30}\)As discussed in footnote 29, Bang-Jensen and Gutin 2009 considers only finite graphs. On their page 21, an “out-tree” is defined to be an orientation \((X,E)\) of a (finite) tree \((X,E)\) with the property that \(X \setminus Y\) is a singleton, where \(Y\) is the projection \(\pi_2E\). Although their concept is equivalent to this paper’s concept (Definition B.4) of out-tree when \(X\) is finite, it is strictly weaker in general (this distinction was overlooked in Streufert 2021Gm and Streufert 2021v2).

In particular, an out-tree \((X,E)\) orients a tree (by Definition B.4) and has a singleton \(X \setminus Y\) (by Lemma B.8(c)). But conversely, an \((X,E)\) which orients a tree and has a singleton \(X \setminus Y\) may not be an out-tree. For example, consider the pair listing \(X = \{0,1,2,\ldots\}\) and \(E = \{\langle 0,1\rangle, \langle 2,1\rangle, \langle 3,2\rangle, \langle 4,3\rangle, \ldots\}\). This \((X,E)\) orients a tree [namely \(X,E=\{\{0,1\}, \{1,2\}, \{2,3\}, \ldots\}\)] and has a singleton \(X \setminus Y\) [namely \(\{0\}\)]. Yet it is not an out-tree. To see this, suppose it were. Then Lemma B.8(c) would imply that \(r = 0\), which by the definition of root (Definition B.7) would imply that \((X,E)\) is the divergent orientation of the rooted tree \((X,E,r=0)\). Thus \((2,1) \in E\) implies that \(2\) is on the path in \((X,E)\) that links \(r=0\) and \(1\). But the path in \((X,E)\) that links \(0\) and \(1\) is \((X,E) = \{\{0,1\}, \{\{0\}\}\}\). The previous two sentences contradict one another.
For the reverse inclusion $W \cup Y \supseteq X$, first note that the assumed nontriviality implies there is an edge $\langle w, y \rangle \in E$. Since a directed graph is irreflexive by definition, the nodes $w$ and $y$ are distinct. Thus $|X| \geq 2$. Now take $x \in X$. Since $|X| \geq 2$, there is a distinct $x_\ast \in X$. Thus since $(X, \mathcal{E})$ is a tree, there is a path linking $x$ and $x_\ast$. Thus there is an undirected edge in $\mathcal{E}$ containing $x$, which implies there is a directed edge in $E$ listing $x$, which by definition (33) implies that $x$ is in $W$ or $Y$.

(b). For nonemptiness, note (a) that the assumption of $r$’s existence implies $\{r\}$ is nonempty, and (b) that the assumption of nontriviality implies $E$ is nonempty, which by $Y$’s definition (33) implies $Y$ is nonempty.

For disjointness, it suffices to show $r \notin Y$. Toward this end, suppose $r \in Y$. Then $Y$’s definition (33) implies there is $w$ such that $\langle w, r \rangle \in E$, which by the definition of divergent orientation implies that $w$ is on the (undirected) path linking $r$ and $r$. But the only (undirected) path linking $r$ and $r$ is the trivial path $(\{r\}, \emptyset)$. Therefore $w = r$, which implies $\langle r, r \rangle \in E$, which contradicts the irreflexivity of a directed graph.

For $\{r\} \cup Y = X$, first note that $\{r\} \cup Y \subseteq X$ because $r \in X$ by the definition of rooted tree (Definition B.3), and because $Y \subseteq X$ by part (a). For the reverse inclusion $\{r\} \cup Y \supseteq X$, it suffices to show $(\forall x \in X \setminus \{r\}) \ x \in Y$. Toward that end, consider a nonroot node $x \in X \setminus \{r\}$. Since $(X, \mathcal{E})$ is a tree, we may let $(\{x_0, x_1, ... x_\ell\}, \{\{x_0, x_1\}, \{x_1, x_2\}, ... \{x_{\ell-1}, x_\ell\}\})$ be the path in $(X, \mathcal{E})$ that links $x_0 = r$ and $x_\ell = x$. Since $x \neq r$ by assumption, $\ell \geq 1$. Hence $\{x_{\ell-1}, x\} \in \mathcal{E}$ and $x_{\ell-1}$ is on the path linking $r$ and $x$. Thus $(X, E)$ being the divergent orientation of $(X, \mathcal{E}, r)$ implies $\langle x_{\ell-1}, x\rangle \in E$. Hence $Y$’s definition (33) implies $x \in Y$.

(c). Obviously $X \setminus Y = X \setminus Y$, which by part (a) implies $(W \cup Y) \setminus Y = X \setminus Y$, which by manipulation implies $W \setminus Y = X \setminus Y$. Part (b) implies $X \setminus Y = \{r\}$.

(d). Obviously $X \setminus W = X \setminus W$, which by part (a) implies $(W \cup Y) \setminus W = X \setminus W$, which by manipulation implies $Y \setminus W = X \setminus W$. □

**Lemma B.9.** Suppose $(X, E)$ is an out-tree, with its $W$, $Y$, and $p$ (33). Then
(a) $p$ is a function with domain $Y$ and range $W$, and
(b) $Y \ni y \mapsto \langle p(y), y \rangle$ is a bijection to $E$.

**Proof.** Define $\mathcal{E}$ and $r$ by $\mathcal{E} = \{\langle w, y \rangle | \langle w, y \rangle \in E\}$ and $\{r\} = W \setminus Y$ so that, by Lemma B.8(c), $(X, E)$ is the divergent orientation of the rooted tree $(X, \mathcal{E}, r)$. Part (a) holds by Claims 1 and 2. Part (b) holds by Claim 3.

**Claim 1:** $p$ is a function. In accord with $p$’s definition (33) and footnote 10 on page 9, it suffices to show that $(\forall y \in Y)(\exists! w \in W) \langle y, w \rangle \in \{\langle y_\ast, w_\ast\rangle | \langle w_\ast, y_\ast\rangle \in E\}$. By inspection, this is equivalent to showing that $(\forall y \in Y)(\exists! w \in W) \langle w, y \rangle \in E$. Toward
that end, take a successor node $y \in Y$. Then the definition of $Y$ implies there is a
decision node $w$ such that $[1] \langle w,y \rangle \in E$. Thus it remains to show that the decision
node $w$ is unique. Toward that end, suppose there were a second decision node $w_+$
such that $[2] \langle w_+,y \rangle \in E$. It will be shown that $w = w_+$.

Since $(X,\mathcal{E})$ is a tree, the path in $(X,\mathcal{E})$ that links $r$ and $y$ is of the form
$X = \{x_0,x_1,x_2,\ldots,x_\ell\}$ and $\mathcal{E} = \{(x_0,x_1),(x_1,x_2),\ldots,(x_{\ell-1},x_\ell)\}$ with $x_0 = r$ and $[3] x_\ell = y$. First consider $w$ and recall $\langle w,y \rangle \in E$ (from [1]). This implies two things. First, it implies that $w$ is on the path linking $r$ and $y$, which implies $w \in X$. Second, it implies $\{w,y\} \in \mathcal{E}$, which by $x_\ell = y$ (from [3]) implies $\{w,x_\ell\} \in \mathcal{E}$. These two facts and Lemma B.2 at $x_\ell = w$ imply $\ell \geq 1$ and $[4] w = x_{\ell-1}$. Similarly, $\{w_+,y\} \in E$ (from [2]) implies $[5] w_+ = x_{\ell-1}$ (to see this, replace [1] with [2], and $w$ with $w_+$, in the previous four sentences). Finally, [4] and [5] imply $w = w_+$.

Claim 2: $p$ has domain $Y$ and range $W$. Claim 1 implies $p$ has domain $\pi_1 p$, which
by $p$’s definition (33) equals $\pi_2 E$, which by $Y$’s definition (33) equals $Y$. Symmetrically, Claim 1 implies $p$ has range $\pi_2 p$, which by $p$’s definition (33) equals $\pi_1 E$, which by $W$’s definition (33) equals $W$.

Claim 3: $Y \ni y \mapsto \langle p(y),y \rangle$ is a bijection to $E$. Claims 1 and 2 imply that
$\{ \langle y,\langle p(y),y \rangle \rangle \mid y \in Y \}$ is a function with domain $Y$. By inspection, the function is
injective. To see that it is onto $E$, note that the image of the domain is $\{ \langle p(y),y \rangle \mid y \in Y \}$, which by (33)’s definitions for $p$ and $Y$ is $\{ \langle w,y \rangle \mid \langle w,y \rangle \in E, y \in \pi_2 E \}$, which reduces to $E$. \hfill $\Box$

B.4. Paths in out-trees

As in Bang-Jensen and Gutin 2009, Chapter 1, a (directed) path from $x_0$ to $x_\ell$ is a directed graph $(X,\mathcal{E})$ of the form $X = \{x_0,x_1,\ldots,x_\ell\} \neq \emptyset$ and $\mathcal{E} = \{\langle x_0,x_1 \rangle,\langle x_1,x_2 \rangle,\ldots,\langle x_{\ell-1},x_\ell \rangle\}$ such that distinct $i$ and $j$ satisfy $x_i \neq x_j$.\footnote{In other words, a path from $x_A$ to $x_B$ is a pair $(X,\mathcal{E})$ for which there are an integer $\ell \geq 0$ and a bijection $\langle x_m \rangle_{m=0}^{\ell}$ from $\{0,1,\ldots,\ell\}$ to $X$ such that $x_0 = x_A$, $x_\ell = x_B$, and $\{\langle x_m,x_{m+1} \rangle \mid m \in \{1,2,\ldots,\ell\}\} = \mathcal{E}$. Since there is only one way to string such a path’s directed edges end-to-end from $x_A$ to $x_B$, the bijection is unique. Similarly, an infinite path from $x_A$ is a pair $(X,\mathcal{E})$ for which there is a bijection $\langle x_m \rangle_{m=0}$ from $\{0,1,\ldots\}$ to $X$ such that $x_0 = x_A$ and $\{\langle x_{m-1},x_m \rangle \mid m \geq 1\} = \mathcal{E}$.}

Similarly, an infinite\footnote{There are other kinds of infinite directed path, such as an “infinite path to a node”. However, an “infinite path from a node” is the only kind of infinite path that can be in an out-tree, and it is the only kind of infinite path considered here.} (directed) path from $x_0$ is a directed graph $(X,\mathcal{E})$ of the form $X = \{x_0,x_1,x_2,\ldots\}$ and $\mathcal{E} = \{\langle x_0,x_1 \rangle,\langle x_1,x_2 \rangle,\ldots\}$ such that distinct $i$ and $j$ satisfy $x_i \neq x_j$. As in the previous paragraph, the word “from” indicates a directed path (here there is less potential for confusion because infinite undirected paths are left undefined).
Finally, one directed graph \((\bar{X}, \bar{E})\) is said to be in another directed graph \((X, E)\) iff \(\bar{X} \subseteq X\) and \(\bar{E} \subseteq E\). This definition of “in” can be used for both paths and out-trees, since both are special kinds of directed graphs.

**Lemma B.10.** Suppose that \((X, E)\) orients a tree, and that \(x_1\) and \(x_2\) are in \(X\). Then (a) there is no more than one path in \((X, E)\) from \(x_1\) to \(x_2\). Further, (b) if \(x_1 \neq x_2\) and there is a path in \((X, E)\) from \(x_1\) to \(x_2\), then there is no path in \((X, E)\) from \(x_2\) to \(x_1\).

**Proof.** Let \(\mathcal{E} = \{\langle w, y \rangle | \langle w, y \rangle \in E\}\) so that \((X, \mathcal{E})\) orients the tree \((X, \mathcal{E})\).

(a). Suppose both \((X_1, E_1)\) and \((X^2, E^2)\) are paths in \((X, E)\) from \(x_1\) to \(x_2\). We will see that \((X^1, E^1) = (X^2, E^2)\). Note that both \((X^1, \{\langle w, y \rangle | \langle w, y \rangle \in E_1^1\}\) and \((X^2, \{\langle w, y \rangle | \langle w, y \rangle \in E^2_2\}\) are paths linking \(x_1\) and \(x_2\). Since \((X^1, E^1)\) and \((X^2, E^2)\) are in \((X, E)\), and since \((X, E)\) orients \((X, \mathcal{E})\), these undirected paths are in \((X, \mathcal{E})\). Thus since \((X, \mathcal{E})\) is a tree, these undirected paths are equal, which is equivalent to \([a] \ X^1 = X^2 \) and \([b] \ \{\langle w, y \rangle | \langle w, y \rangle \in E_1^1\} = \{\langle w, y \rangle | \langle w, y \rangle \in E^2_2\}\).

Because of \([a]\), it suffices to show \(E^1 = E^2\). Toward that end, suppose these two sets differ. Then without loss of generality there is \(\langle w, y \rangle \in E^1 \setminus E^2\). Then \([b]\) implies \(\langle y, w \rangle \in E^2\). But both \(E^1\) and \(E^2\) are subsets of \(E\) by assumption. Thus both \(\langle w, y \rangle\) and \(\langle y, w \rangle\) belong to \(E\), which implies that \(E\) is not asymmetric, which contradicts \((X, E)\) being an orientation.

(b). Suppose that \(x_1 \neq x_2\), that \((X_1^1, E_1^1)\) is a path in \((X, E)\) from \(x_1\) to \(x_2\), and that \((X^2, E^2)\) is a path in \((X, E)\) from \(x_2\) to \(x_1\). We will find a contradiction. Note that both \((X^1, \{\langle w, y \rangle | \langle w, y \rangle \in E_1^1\}\) and \((X^2, \{\langle w, y \rangle | \langle w, y \rangle \in E^2_2\}\) are paths linking \(x_1\) and \(x_2\). Since \((X^1, E^1)\) and \((X^2, E^2)\) are in \((X, E)\), and since \((X, E)\) orients \((X, \mathcal{E})\), these undirected paths are in \((X, \mathcal{E})\). Thus since \((X, \mathcal{E})\) is a tree, these undirected paths are equal, which implies \([c]\) \ \{\langle w, y \rangle | \langle w, y \rangle \in E_1^1\} = \{\langle w, y \rangle | \langle w, y \rangle \in E^2_2\}\).

Since \(x_1 \neq x_2\) and \((X^1, E^1)\) is a path from \(x_1\) to \(x_2\), there is some \(x_\ast \in X^1\) such that \([d]\) \ \langle x_1, x_\ast \rangle \in E^1. Thus \([c]\) implies \(\langle x_1, x_\ast \rangle \in E^2\) or \(\langle x_\ast, x_1 \rangle \in E^2\). Since the former contradicts \((X^2, E^2)\) being a path from \(x_2\) to \(x_1\), assume \([e]\) \ \langle x_\ast, x_1 \rangle \in E^2. Since both \(E^1\) and \(E^2\) are subsets of \(E\) by assumption, \([d]\) and \([e]\) imply that both \(\langle x_1, x_\ast \rangle\) and \(\langle x_\ast, x_1 \rangle\) belong to \(E\), which implies that \(E\) is not asymmetric, which contradicts \((X, E)\) being an orientation. \(\square\)

**Lemma B.11.** Suppose \((X, E)\) is an out-tree with its root \(r\). Then \((\forall x \in X)\) there is a unique path in \((X, E)\) from \(r\) to \(x\).

**Proof.** Let \(\mathcal{E} = \{\langle w, y \rangle | \langle w, y \rangle \in E\}\) so that \((X, \mathcal{E})\) is the divergent orientation of the rooted tree \((X, \mathcal{E}, r)\). Consider an arbitrary node \(x \in X\). Since \((X, \mathcal{E}, r)\) is a rooted tree, there is a path \((\bar{X}, \bar{E})\) in \((X, \mathcal{E})\) of the form \(\bar{X} = \{x_0, x_1, ..., x_\ell\}\) and \(\bar{E} = \{\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, ..., \langle x_{\ell-1}, x_\ell \rangle\}\) such that \(x_0 = r\) and \(x_\ell = x\).

Let \(\bar{E} = \{\langle x_{m-1}, x_m \rangle | m \in \{1, 2, ..., \ell\}\}\). By Lemma B.10(a), it suffices to show that \((\bar{X}, \bar{E})\) is a path in \((X, E)\) from \(r\) to \(x\). By inspection, it suffices to show that \(\bar{E} \subseteq E\), and for this, it suffices to show that \((\forall m \in \{1, 2, ..., \ell\}) \ \langle x_{m-1}, x_m \rangle \in E\). Toward this end, take \(m \in \{1, 2, ..., \ell\}\). By inspection \(\{x_{m-1}, x_m\} \in \mathcal{E}\) and \(x_{m-1}\) is on the path in \((X, \mathcal{E})\) from \(r\) to \(x_m\). Thus, because \((X, E)\) is the divergent orientation of \((X, \mathcal{E}, r)\), we have \(\langle x_{m-1}, x_m \rangle \in E\). \(\square\)
Consider a nontrivial out-tree \((X, E)\). Let \(\preceq\) and \(\prec\) be the binary relations on \(X\) defined by
\[
\begin{align*}
(34a) & \quad x \preceq y \text{ iff there is a path in } (X, E) \text{ from } x \text{ to } y, \\
(34b) & \quad x \prec y \text{ iff } (x \preceq y \text{ and } x \neq y).
\end{align*}
\]
Note \(\preceq\) is reflexive because the right-hand side of (34a) admits the trivial path \((\{x\}, \emptyset)\) from a node \(x\) to itself. Call \(\preceq\) and \(\prec\) the weak and strict precedence orders, respectively.

**Lemma B.12.** Suppose that \((X, E)\) is an out-tree with its \(\preceq\) and \(\prec\). Then \((a)\) \(\preceq\) is a partial order on \(X\), and \((b)\) \(\prec\) is the asymmetric part of \(\preceq\).

**Proof.** (a). By inspection, \(\preceq\) is reflexive and transitive. For antisymmetry, suppose that \(x \in X\) and \(y \in X\) are distinct nodes such that \(x \preceq y\) and \(y \preceq x\). Then definition (34a) implies there is a path from \(x\) to \(y\) and also a path from \(y\) to \(x\). This contradicts Lemma B.10(b).

(b). It must be shown that \((\forall x \in X, y \in X)\) \(x \prec y\) iff \((x \preceq y\) and \(y \neq x\)). For the reverse direction, suppose \(x \preceq y\) and \(y \neq x\). The latter and the reflexivity of \(\preceq\) implies \(x \neq y\). Thus the former and definition (34b) imply \(x \prec y\).

For the forward direction, suppose \(x \prec y\). Then definition (34b) implies \(x \preceq y\) and \(x \neq y\). The former and definition (34a) implies there is a path from \(x\) to \(y\), which by the latter and Lemma B.10(b) implies there is no path from \(y\) to \(x\), which by definition (34a) implies \(y \neq x\). \(\square\)

If \(\bar{X}\) is an arbitrary set, there are typically many edge sets \(\bar{E}\) such that \((\bar{X}, \bar{E})\) is a path. However, if \((\bar{X}, \bar{E})\) is first assumed to be a path in some predetermined out-tree, then the node set \(\bar{X}\) determines the edge set \(\bar{E}\), as the following lemma shows.

**Lemma B.13.** Suppose that \((X, E)\) is an out-tree, and that \((\bar{X}, \bar{E})\) is a (possibly infinite) path in \((X, E)\). Then \(\bar{E} = \{\langle w, y \rangle \in E | \{w, y\} \subseteq \bar{X} \}\).

**Proof.** For the forward inclusion, take an edge \([1]\) \(\langle w, y \rangle \in \bar{E}\). First, \((\bar{X}, \bar{E})\) being in \((X, E)\) implies \(\bar{E} \subseteq E\), which by \([1]\) implies \(\langle w, y \rangle \in E\). Second, \((\bar{X}, \bar{E})\) being a directed path implies \(\bar{E} \subseteq \bar{X}^2\), which by \([1]\) implies \(\langle w, y \rangle \in \bar{X}^2\), which implies \(\{w, y\} \subseteq \bar{X}\). The conclusions of the last two sentences suffice.

For the reverse inclusion, we start with some definitions for future use. Let \([a]\) \(E = \{\{w, y\} | \langle w, y \rangle \in E\}\), so that \((X, E)\) is the tree that the out-tree \((X, E)\) orients. Similarly, let \([b]\) \(\bar{E} = \{\langle w, y \rangle | \langle w, y \rangle \in \bar{E}\}\) so that \((\bar{X}, \bar{E})\) is the (possibly infinite) undirected graph\(^{33}\) that the directed path \((\bar{X}, \bar{E})\) orients. Since the directed path \((\bar{X}, \bar{E})\) is assumed to be in the out-tree \((X, E)\), \([c]\) the undirected graph \((\bar{X}, \bar{E})\) is in the tree \((X, E)\).

Now take an edge \([i]\) \(\langle w, y \rangle \in E\) such that \([ii]\) \(\{w, y\} \subseteq \bar{X}\). Since \((\bar{X}, \bar{E})\) is a directed path, \([ii]\) implies the existence of either a (finite) directed path in \((\bar{X}, \bar{E})\) from \(w\) to \(y\), or a similar one from \(y\) to \(w\). In either case, there is a (finite) undirected path \((\bar{X}^{wy}, \bar{E}^{wy})\) in \((\bar{X}, \bar{E})\) which links \(w\) and \(y\). Thus, \((\bar{X}, \bar{E})\) being in the tree \((X, E)\)

\(^{33}\)(\(\bar{X}, \bar{E})\) cannot be called a path since infinite undirected paths are undefined.
Appendix B. Rooted Trees, Out-trees, and Edge-Trees

(by [c]) implies that \((\bar{X}^{wy}, \bar{E}^{wy})\) is a path in the tree \((X, E)\) which links \(w\) and \(y\). Meanwhile, [i] and [a] imply that \(\{(w, y), \{(w, y)\}\}\) is also a path in the tree \((X, E)\) which links \(w\) and \(y\). The definition of tree implies that these two paths are equal. Hence \(\bar{E}^{wy} = \{(w, y)\}\), which implies \(\{w, y\} \in \bar{E}^{wy}\), which by the definition of \(\bar{E}^{wy}\) (four sentences ago) implies \(w, y \in \bar{E}\), which by the definition of \(\bar{E}\) (at [b]) implies \(\langle w, y \rangle \in \bar{E}\) or \(\langle y, w \rangle \in \bar{E}\).

Thus it suffices to rule out the latter. Toward that end, suppose \(\langle y, w \rangle \in \bar{E}\). Then \((\bar{X}, \bar{E})\) being in \((X, E)\) implies \(\langle y, w \rangle \in E\). This contradicts [i] because \((X, E)\) is an orientation and every orientation is asymmetric by definition. \(\Box\)

Consider a nontrivial out-tree \((X, E)\). Henceforth represent each path \((\bar{X}, \bar{E})\) in \((X, E)\) by its node set \(\bar{X}\). This suffices because the edge set \(\bar{E}\) must be \(\{\langle w, y \rangle \in E \mid \{w, y\} \subseteq \bar{X}\}\) by Lemma B.13. Given this simpler notation, the remainder of this section defines a “run”, which is a special kind of path in \((X, E)\). Runs come in two flavours: finite and infinite.

First, let the set of end nodes (or leaves) be

\[
Y \setminus W
\]

where (33) defines \(W = \pi_1 E\) and \(Y = \pi_2 E\) (these definitions of \(W\) and \(Y\) are the only overlap between Sections B.3 and B.4). Then let a finite run (or finite play) be a path from \(r\) to an end node, and let \(Z_{ft}\) be the collection of finite runs. For instance, Figure 2.2 illustrates an out-tree \((X, E)\) with root node \(0\). One of its end nodes is \(7\), and one of its finite runs is \(\{0, 1, 3, 7\}\).

Second, let an infinite run (or infinite play) be an infinite path from \(r\), and let \(Z_{inft}\) be the collection of infinite runs. For example, the out-tree of an infinite centipede game has exactly one infinite run, and this run would be represented as an infinite set of nodes.

Finally, let

\[
Z = Z_{ft} \cup Z_{inft},
\]

and call \(Z\) the collection of runs (or plays). There are three possibilities: (i) \(Z = Z_{ft}\) and \(Z_{inft} = \emptyset\), as in the out-trees in Section 2’s figures, (ii) \(Z = Z_{inft}\) and \(Z_{ft} = \emptyset\), in which case there are no end nodes, and (iii) both \(Z_{ft}\) and \(Z_{inft}\) are nonempty, as in the out-tree of an infinite centipede game.

B.5. Edge-trees

The following definition introduces the concept of edge-tree. Then the rest of this Section B.5 will show the equivalence between nontrivial out-trees and edge-trees.

Definition B.14 (Edge-tree). Suppose \(\bar{E}\) is a (possibly infinite) set of pairs \(\langle w, y \rangle\). Let

\[
\bar{W} = \pi_1 \bar{E}, \ Y = \pi_2 \bar{E}, \ \text{and} \ \bar{p} = \{\langle y, w \rangle \mid \langle w, y \rangle \in \bar{E}\}.
\]
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Then $\hat{E}$ is an edge-tree\(^\text{34}\) iff

\[ E \text{ is a function,} \]

\[ (\forall y \in \hat{Y})(\exists \ell \geq 1) \hat{p}^\ell(y) \notin \hat{Y}, \text{ and} \]

\[ \hat{W} \setminus \hat{Y} \text{ is a singleton.} \]

Lemma B.15. Suppose $(X, E)$ is a nontrivial out-tree. Then $E$ is an edge-tree.

Proof. It suffices to show that [E1]-[E3] are satisfied at $\hat{E} = E$. To do so, define $\hat{W}, \hat{Y},$ and $\hat{p}$ by equation (37) at $\hat{E} = E$. Further, derive $\hat{W}, \hat{Y},$ and $\hat{p}$ by equation (33) applied to the out-tree $(X, E)$. By inspection, $\hat{W} = W, \hat{Y} = Y,$ and $\hat{p} = p$. Thus Lemma B.9(a) implies [E1], and Lemma B.8(c) implies [E3].

For [E2], take a node $y \in Y$. Note Lemma B.8(c) implies [1] $r \neq y,$ where $r$ is the root (Definition B.7) of the out-tree $(X, E)$. Also, Lemma B.11 implies there is a path $(\bar{X}, \bar{E})$ in $(X, E)$ of the form $\bar{X} = \{x_0, x_1, x_2, \ldots x_\ell\}$ and $\bar{E} = \{(x_0, x_1), (x_1, x_2), \ldots (x_{\ell-1}, x_\ell)\}$ with [3] $x_0 = r$ and [4] $x_\ell = y$. This path satisfies $[5] \ell \geq 1$ because of [1], [3], and [4].

In steps, $(\bar{X}, \bar{E})$ being in $(X, E)$ implies that $\bar{E} \subseteq E,$ which by [2] implies that $(\forall m \in \{1, 2, \ldots, \ell\}) x_{m-1}, x_m \in E,$ which by $p$’s definition (33) implies that $(\forall m \in \{1, 2, \ldots, \ell\}) x_{m-1} = p(x_m),$ which by substitution implies $x_0 = p^\ell(x_1),$ which by [3] and [4] implies $r = p^\ell(y),$ which by Lemma B.8(c) implies $p^\ell(y) \notin Y,$ which by [5] implies $(\exists \ell \geq 1) p^\ell(y) \notin Y.$

Lemma B.16. Suppose $\hat{E}$ is an edge-tree. Then $(\pi_1 \hat{E} \cup \pi_2 \hat{E}, \hat{E})$ is a nontrivial out-tree.

Proof. Define $\hat{W}, \hat{Y},$ and $\hat{p}$ by equation (37) in the definition of edge-tree. Further, define

\[ X = \hat{W} \cup \hat{Y}, \]

\[ \hat{E} = \{(w, y) \mid (w, y) \in \hat{E}\}, \text{ and} \]

\[ \hat{r} = \hat{W} \setminus \hat{Y}. \]

Note that $\hat{r}$ is well-defined because of [E3] (in Definition B.14). The lemma follows from Claim 14. (This claim will be reached by showing that $(\hat{X}, \hat{E}, \hat{r})$ is a rooted tree and that $(\hat{X}, \hat{E})$ is its divergent orientation.)

Claim 1: $\{\hat{r}\} = \hat{X} \setminus \hat{Y}$. The definition of $\hat{r}$ (40) states $\{\hat{r}\} = \hat{W} \setminus \hat{Y},$ which by $X$’s definition (38) suffices.

Claim 2: $(\forall x \in \hat{X}, x_\ast \in \hat{X}) \langle x, x_\ast \rangle \in \hat{E}$ implies $\langle x_\ast, x \rangle \notin \hat{E}$. Suppose nodes $x$ and $x_\ast$ were such that $\langle x, x_\ast \rangle \in \hat{E}$ and $\langle x_\ast, x \rangle \notin \hat{E}$. Then $\hat{Y}$’s definition (37) implies $x \in \hat{Y}$ and $x_\ast \in \hat{Y},$ and $\hat{p}$’s definition (37) implies both $x = \hat{p}(x_\ast)$ and $x_\ast = \hat{p}(x)$. These observations contradict [E2] at $y = x$ (they also contradict [E2] at $y = x_\ast$).

\(^{34}\)A similar concept is in Streufert 2018, equation (1). A related but less similar concept is in Knuth 1997, page 373.
Claim 3: \((\bar{X}, \bar{E})\) is an asymmetric directed graph. Definitions (37) and (38) imply \(\bar{E} \subseteq \bar{W} \times Y \subseteq (\bar{W} \cup Y)^2 = X^2\). Claim 2 implies \((\forall x \in \bar{X}) \langle x, x \rangle \notin \bar{E}\). Thus \((\bar{X}, \bar{E})\) is a directed graph. It is asymmetric by Claim 2.

Claim 4: \((\forall y \in \bar{Y})(\exists \ell \geq 1) \left[ i \right] \bar{p}^\ell(y) = \bar{r} \text{ and } [ii] \left( \forall 0 \leq n_1 < n_2 \leq \ell \right) \bar{p}^n(y) \neq \bar{p}^m(y) \). To see this, take \(y \in \bar{Y}\). Then \([E2]\) implies there is \(\ell \geq 1\) such that \([a] \bar{p}^\ell(y) \notin \bar{Y}\).

For \([i]\), note that \(\bar{p}\)'s definition (37) and \(\bar{W}\)'s definition (37) imply that the range of \(\bar{p}\) is \(\bar{W}\). Thus \([a]\) implies \(\bar{p}^\ell(y) \in \bar{W} \setminus \bar{Y}\), which by \(\bar{r}\)'s definition (40) implies \(\bar{p}^\ell(y) = \bar{r}\).

For \([ii]\), suppose there are integers \(n_1\) and \(n_2\) such that \([b] 0 \leq n_1 < n_2 \leq \ell\) and \([c] \bar{p}^n(y) = \bar{p}^m(y)\). Define the integer \(N = n_2 - n_1\). Then \([b]\) implies \(N \geq 1\), by which \([b]\) implies there is another integer \(L \geq 1\) such that \([d] LN + n_1 > \ell\). Meanwhile, the definition of \(N\) and \([c]\) imply \(\bar{p}^N(\bar{p}^n(y)) = \bar{p}^{n_2-n_1}(\bar{p}^n(y)) = \bar{p}^{n_2}(y) = \bar{p}^{n_1}(y)\) (roughly, \(\bar{p}^N\) is idempotent at \(\bar{p}^n(y)\)). Thus \(\bar{p}^{LN+n_1}(y) = [\bar{p}^N]^L(\bar{p}^n(y)) = \bar{p}^{n_1}(y)\). But, this implies that \(\bar{p}^{LN+n_1}(y)\) is well-defined. Therefore, since \(\bar{p}\)'s definition (37) and \(\bar{Y}\)'s definition (37) imply that \(\bar{p}\)'s domain is \(\bar{Y}\), we have that \((\forall 0 \leq n < LN + n_1) \bar{p}^n(y) \notin \bar{Y}\). Thus \([d]\) implies \(\bar{p}^\ell(y) \in \bar{Y}\), which contradicts \([a]\).

Claim 5: \((\bar{X}, \bar{E})\) is an undirected graph. Claim 2 implies \((\forall x \in \bar{X}) \langle x, x \rangle \notin \bar{E}\). Thus \(\bar{E}\)'s definition (39) implies that \(\bar{E}\) is a collection of two-element subsets of \(\bar{X}\). Hence \((\bar{X}, \bar{E})\) is an undirected graph.

Claim 6: \((\forall x \in \bar{X}, x_\ast \in \bar{X}) \{x, x_\ast\} \in \bar{E} \iff \{x = \bar{p}(x_\ast) \text{ or } x_\ast = \bar{p}(x)\}\). Take two nodes \(x\) and \(x_\ast\). By \(\bar{E}\)'s definition (39), \(\{x, x_\ast\} \in \bar{E}\) is equivalent to the disjunction \(\langle x, x_\ast\rangle \in \bar{E}\) or \(\langle x_\ast, x\rangle \in \bar{E}\). By two applications of \(\bar{p}\)'s definition (37), this is equivalent to the disjunction \(x = \bar{p}(x_\ast)\) or \(x_\ast = \bar{p}(x)\).

Claim 7: \((\forall y \in \bar{Y})(\exists \ell \geq 1)\) the pair listing

\[
\bar{X} = \{ \bar{p}^{\ell-m}(y) \mid m \in \{0,1,\ldots,\ell\} \} \text{ and }
\bar{\mathcal{E}} = \{ \{ \bar{p}^{\ell-(m-1)}(y), \bar{p}^{\ell-m}(y) \mid m \in \{1,2,\ldots,\ell\} \}
\]

is a path in \((\bar{X}, \bar{E})\) linking \(\bar{r}\) and \(y\). To see this, take \(y \in \bar{Y}\). Then Claim 4 implies there is \(\ell \geq 1\) such that \([a] \bar{p}^\ell(y) = \bar{r}\) and \((\forall 0 \leq n_1 < n_2 \leq \ell) \bar{p}^n(y) \neq \bar{p}^m(y)\). By inspection, the latter implies that \([b] (\forall 0 \leq m_1 < m_2 \leq \ell) \bar{p}^{\ell-m_1}(y) \neq \bar{p}^{\ell-m_2}(y)\).

Define the function \(\langle x_m \rangle_{m=0}^\ell\) by \(x_m = \bar{p}^{\ell-m}(y)\). Then \([b]\) and the claim's definition of \(\bar{X}\) implies \(\langle x_m \rangle_{m=0}^\ell\) is a bijection from \(\{0,1,\ldots,\ell\}\) to \(\bar{X}\). Further, \([a]\) implies \(x_0 = \bar{r}\); inspection shows \(x_\ell = y\); and the claim's definition of \(\bar{E}\) shows \(\{ \langle x_{m-1}, x_m \rangle \mid m \in \{1,2,\ldots,\ell\} \} = \bar{E}\). Thus footnote 28 on page 28 implies \((\bar{X}, \bar{E})\) is a path linking \(\bar{r}\) and \(y\).

Hence it remains to show that \((\bar{X}, \bar{E})\) is in \((\bar{X}, \bar{E})\). For \(\bar{X} \subseteq \bar{X}\), take \(m \in \{0,1,\ldots,\ell\}\).

If \(m = \ell\), \(\bar{p}^{\ell-m}(y) = y\), which by assumption is in \(\bar{Y}\), which by \(\bar{X}\)'s definition (38) is included in \(\bar{X}\). If \(m < \ell\), \(\bar{p}^{\ell-m}(y)\) is in the range of \(\bar{p}\), which by \(\bar{p}\)'s definition (37) and \(\bar{W}\)'s definition (37) is equal to \(\bar{W}\), which by \(\bar{X}\)'s definition (38) is included in \(\bar{X}\). For \(\bar{E} \subseteq \bar{E}\), take \(m \in \{1,2,\ldots,\ell\}\).

Then \(\{\bar{p}^{\ell-(m-1)}(y), \bar{p}^{\ell-m}(y)\}\) equals \(\{\bar{p}(\bar{p}^{\ell-m}(y)), \bar{p}^{\ell-m}(y)\}\), which by the reverse direction of Claim 6 is in \(\bar{E}\).
Claim 8: \((\hat{X}, \hat{E})\) is a connected undirected graph. It is an undirected graph by Claim 5. For connectedness, take distinct nodes \(x_1\) and \(x_2\) in \(\hat{X}\). On the one hand, suppose either \(x_1\) or \(x_2\) equals \(\hat{r}\). Without loss of generality assume \(x_1 = \hat{r}\). Then since \(x_2\) is a distinct element of \(\hat{X}\) by assumption, Claim 1 implies \(x_2 \in \hat{Y}\). Thus Claim 7 at \(y = x_2\) implies there is a path in \((\hat{X}, \hat{E})\) linking \(\hat{r} = x_1\) and \(x_2\).

On the other hand, suppose neither \(x_1\) nor \(x_2\) equals \(\hat{r}\). Then since both \(x_1\) and \(x_2\) are in \(\hat{X}\) by assumption, Claim 1 implies both \(x_1\) and \(x_2\) are in \(\hat{Y}\). So Claim 7 at \(y = x_1\) shows there is a path \((\hat{X}^1, \hat{E}^1)\) in \((\hat{X}, \hat{E})\) linking \(\hat{r}\) and \(x_1\). Similarly, Claim 7 at \(y = x_2\) shows there is a path \((\hat{X}^2, \hat{E}^2)\) in \((\hat{X}, \hat{E})\) linking \(\hat{r}\) and \(x_2\). Let \(\hat{E}\) be the symmetric difference of \(\hat{E}^1\) and \(\hat{E}^2\), and let \(\hat{X} = \hat{X}\ bào \hat{E}\). Then \((\hat{X}, \hat{E})\) is a path in \((\hat{X}, \hat{E})\) linking \(x_1\) and \(x_2\).

Claim 9: \((\hat{X}, \hat{E})\) is a tree. By Claim 8, and by Lemma B.1’s characterization of trees, it suffices to show that \((\hat{X}, \hat{E})\) is acyclic. Toward that end, suppose \((\hat{X}, \hat{E})\) were not acyclic. Then there would be a path, of length at least two, whose ends constitute a pair in \(\hat{E}\). In notation, there would be \(\ell \geq 2\) and \(\{x_0, x_1, \ldots, x_\ell\} \subseteq \hat{X}\) such that [a] distinct \(i\) and \(j\) satisfy \(x_i \neq x_j\), [b] \((\forall m \in \{1, 2, \ldots, \ell\}\} \{x_{m-1}, x_m\} \in \hat{E}\) and [c] \(\{x_0, x_\ell\} \in \hat{E}\). Note [b], [c], and \(\ell + 1\) applications of the forward direction of Claim 6 imply

\begin{equation}
(41) \quad \forall m \in \{1, 2, \ldots, \ell\} \} \{ x_{m-1} = \dot{\hat{p}}(x_m) \text{ or } x_m = \dot{\hat{p}}(x_{m-1}) \} , \quad \text{and}
\end{equation}

\begin{equation}
(42) \quad \{ x_\ell = \dot{\hat{p}}(x_0) \text{ or } x_0 = \dot{\hat{p}}(x_\ell) \} .
\end{equation}

Equation (42) defines two cases. First suppose

\begin{equation}
[d] \quad x_\ell = \dot{\hat{p}}(x_0).
\end{equation}

Since \(\ell \geq 2\), (41) at \(m = 1\) implies \(x_0 = \dot{\hat{p}}(x_1)\) or \(x_1 = \dot{\hat{p}}(x_0)\). The latter and [d] would imply \(x_1 = x_\ell\), which contradicts [a] since \(\ell \geq 2\). Hence

\begin{equation}
[d'] \quad x_0 = \dot{\hat{p}}(x_1).
\end{equation}

Next, since \(\ell \geq 2\), (41) at \(m = 2\) implies \(x_1 = \dot{\hat{p}}(x_2)\) or \(x_2 = \dot{\hat{p}}(x_1)\). The latter and [d'] imply \(x_2 = x_0\) which contradicts [a]. Hence

\begin{equation}
[d''] \quad x_1 = \dot{\hat{p}}(x_2).
\end{equation}

By [d], [d'], [d''], and \(\ell - 2\) near replicas of the argument for [d''], we have contingency (43) below. Second suppose [e] \(x_0 = \dot{\hat{p}}(x_\ell)\). Then a similar inductive argument yields contingency (44) below.

\begin{equation}
(43) \quad \{ x_\ell = \dot{\hat{p}}(x_0) \text{ and } (\forall m \in \{1, 2, \ldots, \ell\}\} \{ x_{m-1} = \dot{\hat{p}}(x_m) \} , \quad \text{or}
\end{equation}

\begin{equation}
(44) \quad \{ x_0 = \dot{\hat{p}}(x_\ell) \text{ and } (\forall m \in \{1, 2, \ldots, \ell\}\} \{ x_m = \dot{\hat{p}}(x_{m-1}) \} .
\end{equation}

Either contingency (43) or (44) implies that the domain of \(\dot{\hat{p}}\) includes \(\{x_0, x_1, \ldots, x_\ell\}\). Thus, since the domain of \(\dot{\hat{p}}\) is \(\hat{Y}\) by \(\dot{\hat{p}}\)’s definition (37) and \(\hat{Y}\)’s definition (37), we have \(\{x_0, x_1, \ldots, x_\ell\} \subseteq \hat{Y}\). Therefore either contingency (43) or (44) contradicts [E2] at \(y = x_0\) (either contingency also contradicts [E2] at any other node in \(\{x_0, x_1, \ldots, x_\ell\}\)
Claim 10: \((\hat{X}, \hat{E})\) is an orientation of \((\hat{X}, \hat{E})\). This follows from \(\hat{E}\)'s definition (39), Claim 3, and Claim 5.

Claim 11: \((\forall \langle w, y \rangle \in \hat{E})\) \(w\) is on the path in \((\hat{X}, \hat{E})\) that links \(\hat{r}\) and \(y\). Take an edge \(\langle w, y \rangle \in \hat{E}\). Then \(\hat{p}\)'s definition (37) implies \([a]\) \(w = \hat{p}(y)\). Also, \(\hat{Y}\)'s definition (37) implies \(y \in \hat{Y}\), which by Claim 7 implies there is \(\ell \geq 1\) such that the pair listing 

\[
\hat{X} = \{ \hat{p}^{\hat{f}}(y) \mid m \in \{0, 1, \ldots, \ell\} \} \text{ and }
\hat{E} = \{ \{\hat{p}^{\hat{f}}(m-1)(y), \hat{p}^{\hat{f}}(m)(y)\} \mid m \in \{1, 2, \ldots, \ell\} \}
\]

is a path in \((\hat{X}, \hat{E})\) linking \(\hat{r}\) and \(y\). Moreover, since \((\hat{X}, \hat{E})\) is a tree by Claim 9, this path \((\hat{X}, \hat{E})\) is the only path in \((\hat{X}, \hat{E})\) linking \(\hat{r}\) and \(y\). Finally, \(\ell \geq 1\) implies \(\hat{p}^{\hat{f}}(\ell-1)(y) \in \hat{X}\), which implies \(\hat{p}(y) \in \hat{X}\), which by \([a]\) implies \(w \in \hat{X}\).

Claim 12: \((\hat{X}, \hat{E})\) is an out-tree. Claim 1 implies \(\hat{r} \in \hat{X}\), which by Claim 9 implies that \((\hat{X}, \hat{E}, \hat{r})\) is a rooted tree. Further, Claims 10 and 11 imply that \((\hat{X}, \hat{E})\) is the divergent orientation of \((\hat{X}, \hat{E}, \hat{r})\). Hence \((\hat{X}, \hat{E})\) is an out-tree.

Claim 13: \((\hat{X}, \hat{E})\) is a nontrivial out-tree. \([E3]\) implies that \(W\) is nonempty, which by \(\hat{W}\)'s definition (37) implies \(\hat{E}\) is nonempty. This suffices for the claim by Claim 12 and the definition of nontriviality (before Proposition B.6).

Claim 14: \((\pi_1 \hat{E} \cup \pi_2 \hat{E}, \hat{E})\) is a nontrivial out-tree. This follows from Claim 13 because \(\hat{X}\) by definition (38) is \(\hat{W} \cup \hat{Y}\), which by definition (37) is \(\pi_1 E \cup \pi_2 E\).

Proposition B.17. \(^{35}\) The rule \((X, E) \mapsto E\) defines a bijection from the collection of nontrivial out-trees to the collection of edge-trees. Its inverse obeys the rule \((\pi_1 E \cup \pi_2 E, E) \leftrightarrow E\).

Proof. Lemma B.15 shows that \((X, E) \mapsto E\) defines a function from nontrivial out-trees to edge-trees. Conversely, Lemma B.16 shows that \(E \mapsto (\pi_1 E \cup \pi_2 E, E)\) defines a function from edge-trees to nontrivial out-trees. To show that the first followed by the second is the identity on the collection of nontrivial out-trees, take a nontrivial out-tree \((X, E)\). Note Lemma B.8(a) implies \(\hat{W} \cup \hat{Y} = \hat{X}\), which by definition (33) implies \(\pi_1 E \cup \pi_2 E = X\). Then \((X, E) \mapsto E \mapsto (\pi_1 E \cup \pi_2 E, E) = (X, E)\) where the equality holds by the previous sentence. Conversely, to show that the second followed by the first is the identity on the collection of edge-trees, take an edge-tree \(E\). Then \(E \mapsto (\pi_1 E \cup \pi_2 E, E) \mapsto E\).

Appendix C. For Pentaform Games

Lemma C.1. Suppose \(Q\) is a quintuple set. Then the following hold.

(a) \(\forall j \in J\) \(W_j = \{ w | \langle w, j \rangle \in \pi_{WJ}(Q) \}\).

(b) \(\forall j \in J\) \(Y_j = \{ y | \langle y, j \rangle \in \pi_{YJ}(Q) \}\).

(c) \(\forall j \in J\) \(\pi_{WA}(Q_j) = \{ \langle w, a \rangle | \langle j, w, a \rangle \in \pi_{JWA}(Q) \}\).

\(^{35}\)An alternative to Proposition B.17 uses a new kind of directed graph. In particular, let a domained edge-tree be a pair \((X, E)\) such that \(E\) is an edge-tree and \(X = \pi_1 E \cup \pi_2 E\). Then Lemmas B.15 and B.16 imply that a pair \((X, E)\) is a nontrivial out-tree iff it is a domained edge-tree.
Proof. (a). Consider a situation $j \in J$. Then $W_j$ by abbreviation (8) is $\pi_W(Q_j)$, which by the definition (2) of a slice is $\pi_W(\{(i_*, j, w_*, a_*, y_*) \in Q\})$, which by the general definition of projection is $\{w_* | (\exists i_*, a_*, y_*) (i_*, j, w_*, a_*, y_*) \in Q\}$, which by manipulation is $\{w_* | (w_*, j) \in \pi_W(J)\}$.

(b). Proved as part (a).

(c). Consider a situation $j \in J$. For the forward inclusion, take $(w, a) \in \pi_{WA}(Q_j)$. Then by projection there is $(i_*, j, a, y_*)$ such that $(i_*, j, w, a, y_*) \in Q_j$. This and the slice definition (2) imply $j_* = j$. Thus $(i_*, j, w, a, y_*) \in Q_j$, which by the slice definition (2) implies $(i_*, j, w, a, y_*) \in Q_j$, which implies $(j, w, a) \in \pi_{WA}(Q)$, which implies $(w, a) \in \pi_{WA}(Q_j)$.

For the reverse direction, assume $(j, w, a) \in \pi_{JWA}(Q)$. Then by projection there is $(i_*, j, w, a, y_*)$ such that $(i_*, j, w, a, y_*) \in Q_j$, which by the slice definition (2) implies $(i_*, j, w, a, y_*) \in Q_j$, which implies $(w, a) \in \pi_{WA}(Q_j)$. □

Proof C.2 (for Proposition 3.2). The proposition follows from Lemma C.1(a) and a general fact\footnote{The following holds in general. Suppose $G$ is a set of pairs $(x, y)$, let $X = \pi_1 G$, let $Y = \pi_2 G$, and define $(X^y)_{y \in Y}$ by $X^y = \{(x) | (x, y) \in G\}$. Call $X^y$ the inverse image of $y$. Then the following are equivalent. (a) $G$ is a function. (b) $(\forall y_1 \in Y, y_2 \in Y) y_1 \neq y_2$ implies $X^{y_1} \cap X^{y_2} = \emptyset$. (c) $(X^y)_{y \in Y}$ is an injectively indexed partition of $X$. (To see that (b)$\Rightarrow$(c), note that both $\cup_{y \in Y} X^y = X$ and $(\forall y \in Y) X^y \neq \emptyset$ hold by construction.)} about inverse images. In particular, apply footnote 36 with $G$, $X$, and $Y$ there equal to $\pi_{WJ}(Q)$, $W$, and $J$ here. Note that each inverse image $X^y = \{(x) | (x, y) \in G\}$ there becomes the inverse image $W^y = \{(w) | (w, y) \in \pi_{WJ}(Q)\}$, and that Lemma C.1(a) shows each such $W^y$ equals the information set $W_j$. □

Lemma C.3. Suppose $Q$ is a quintuple set which satisfies $[Pj \ast w]$ and $[Pw \ast y]$. Then $(Y_j)_{j \in J}$ is an injectively indexed partition of $Y$.

Proof. The assumptions $[Pj \ast w]$ and $[Pw \ast y]$ imply that $\pi_{YJ}(Q)$ is a function. Now apply footnote 36(a$\Rightarrow$c) with $G$, $X$, and $Y$ there equal to $\pi_{YJ}(Q)$, $Y$, and $J$ here. Note that each inverse image $X^y = \{(x) | (x, y) \in G\}$ there becomes the inverse image $Y^j = \{y | (y, j) \in \pi_{YJ}(Q)\}$ here. Thus the footnote implies that $(Y^j)_{j \in J}$ is an injectively indexed partition of $Y$. By Lemma C.1(b), each such $Y^j$ equals $Y_j$. □

Proof C.4 (for Proposition 3.3). (a$\Rightarrow$b). Assume (a). Take a situation $j \in J$. Then (a) implies $\pi_{WA}(Q_j)$ is a Cartesian product. By inspection, the projection, on $W$, of $\pi_{WA}(Q_j)$ is $\pi_W(Q_j)$, which by (8) is abbreviated $W_j$. Similarly, the projection, on $A$, of $\pi_{WA}(Q_j)$ is $\pi_A(Q_j)$, which by (8) is abbreviated $A_j$. The above imply $\pi_{WA}(Q_j) = W_j \times A_j$.

(b$\Rightarrow$c). Assume (b). Take a situation $j \in J$ and a decision node $w$ in $j$’s information set $W_j$. Note axiom $[Pj \ast w]$ implies that each quintuple listing $w$ is in $Q_j$. Then $F(w)$ by $F$’s definition (12) is $\{a | (w, a) \in \pi_{WA}(Q)\}$, which by the previous sentence is $\{a | (w, a) \in \pi_{WA}(Q_j)\}$, which by the assumption (b) is $A_j$.

(c$\Rightarrow$d). This holds by inspection.
This affects one cell in Table C.1. In that context, $[P_y]$ is taken to mean that, for each successor node $Q$.

$toward$ that end, take decision nodes $w_1 \in W_j$ and $w_2 \in W_j$. Then the assumption (d) implies $F(w_1) = F(w_2)$, which by $F$’s definition (12) implies $\{a|\langle w_1, a \rangle \in \pi_{WA}(Q)\} = \{a|\langle w_2, a \rangle \in \pi_{WA}(Q)\}$.

Meanwhile, the assumption $w_1 \in W_j$ and axiom $[P_jw]$ imply that each quintuple listing $w_1$ is in $Q_j$. Thus $\{a|\langle w_1, a \rangle \in \pi_{WA}(Q)\} = \{a|\langle w_1, a \rangle \in \pi_{WA}(Q_j)\}$ (the right-hand side differs from the left-hand side only by the subscript $j$). Similarly, $w_2 \in W_j$ and $[P_jw]$ imply $\{a|\langle w_2, a \rangle \in \pi_{WA}(Q)\} = \{a|\langle w_2, a \rangle \in \pi_{WA}(Q_j)\}$. The previous three equalities suffice. $\square$

**Lemma C.5.** Suppose $Q$ is a quintuple set. Then $\pi_{WY}(Q)$ is an edge-tree (Definition B.14) iff $Q$ satisfies $[P_{wa-y}]$, $[P_y]$, and $[P_y]$. $\square$

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&emsp;\(d \Rightarrow a\). Assume (d). Take a situation $j \in J$. By inspection, the projection, on $W$, of $\pi_{WA}(Q_j)$ is $\pi_{W}(Q_j)$, which by (8) is abbreviated $W_j$. Thus it suffices to show that  
\[(\forall w_1, w_2 \in W_j) \{a|\langle w_1, a \rangle \in \pi_{WA}(Q_j)\} = \{a|\langle w_2, a \rangle \in \pi_{WA}(Q_j)\}.\]

Toward that end, take decision nodes $w_1 \in W_j$ and $w_2 \in W_j$. Then the assumption (d) implies $F(w_1) = F(w_2)$, which by $F$’s definition (12) implies  
\[\{a|\langle w_1, a \rangle \in \pi_{WA}(Q)\} = \{a|\langle w_2, a \rangle \in \pi_{WA}(Q)\}.\]

Meanwhile, the assumption $w_1 \in W_j$ and axiom $[P_jw]$ imply that each quintuple listing $w_1$ is in $Q_j$. Thus  
\[\{a|\langle w_1, a \rangle \in \pi_{WA}(Q)\} = \{a|\langle w_1, a \rangle \in \pi_{WA}(Q_j)\}\]  
the right-hand side differs from the left-hand side only by the subscript $j$. Similarly, $w_2 \in W_j$ and $[P_jw]$ imply $\{a|\langle w_2, a \rangle \in \pi_{WA}(Q)\} = \{a|\langle w_2, a \rangle \in \pi_{WA}(Q_j)\}$. The previous three equalities suffice. $\square$

**Lemma C.5.** Suppose $Q$ is a quintuple set. Then $\pi_{WY}(Q)$ is an edge-tree (Definition B.14) iff $Q$ satisfies $[P_{wa-y}]$, $[P_y]$, and $[P_y]$. $\square$

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37When $[P_{wa-y}]$ fails, $[P_y]$ is undefined because $p = \pi_{WY}(Q)$ (equation (11)) is not a function. This affects one cell in Table C.1. In that context, $[P_y]$ is taken to mean that, for each successor node $y \in Y$, there is a path $(X, E)$ [footnote 31 on page 33] to $y$, from a non-successor node $w \in W \setminus Y$, such that $X \subseteq W \cup Y$ and $E \subseteq \pi_{WY}(Q)$. $[P_y]$ is equivalent to this condition when $[P_{wa-y}]$ holds.
Proof. To be clear, $W$, $Y$, and $p$ are implicitly derived from $Q$ (as in Table 3.1). To keep the notation clear, define
\begin{equation}
\hat{E} = \pi_{WY}(Q),
\end{equation}
and derive $W$, $Y$, and $\hat{p}$ from definition (37) in the definition of edge-tree. The lemma then follows from Claim 2 and definition (45).

Claim 1: [a] $\hat{W} = W$, [b] $\hat{Y} = Y$, and [c] $\hat{p} = p$. For [a], $\hat{W}$ by definition (37) is $\pi_1\hat{E}$, which by definition (45) is $\pi_1(\pi_{WY}(Q))$, which by inspection is $\pi_W(Q)$, which by abbreviation (8) is $W$. Similarly for [b], $\hat{Y}$ by definition (37) is $\pi_2\hat{E}$, which by definition (45) is $\pi_2(\pi_{WY}(Q))$, which by inspection is $\pi_Y(Q)$, which by abbreviation (8) is $Y$. Finally for [c], $\hat{p}$ by definition (37) is $\{(y,w)|\langle w,y\rangle \in \hat{E}\}$, which by definition (45) is $\{(y,w)|\langle w,y\rangle \in \pi_{WY}(Q)\}$, which by inspection is $\pi_{YW}(Q)$, which by definition (11) is $p$.

Claim 2: $\hat{E}$ is an edge-tree iff $Q$ satisfies [Pw-y], [Py], and [Pr]. It suffices to show that the $W$, $Y$, and $\hat{p}$ satisfy [E1], [E2], and [E3] iff $Q$ satisfies [Pw-y], [Py], and [Pr]. Note $\hat{p}$ by Claim 1[c] is $p$, which by definition (11) is $\pi_{YW}(Q)$. Thus [E1] is equivalent to [Pw-y]. Also, Claim 1[b,c] implies the equivalence of [E2] and [Py]. Finally, Claim 1[a,b] implies the equivalence of [E3] and [Pr].

Proof C.6 (for Proposition 3.4).38 (a) Suppose $(X, \pi_{WY}(Q))$ is a nontrivial out-tree. Then Proposition B.17’s forward direction implies $\pi_{WY}(Q)$ is an edge-tree, which by Lemma C.5’s forward direction implies $Q$ satisfies [Pw-y], [Py], and [Pr].

Conversely, suppose $Q$ satisfies [Pw-y], [Py], and [Pr]. Then Lemma C.5’s reverse direction implies $\pi_{WY}(Q)$ is an edge-tree, which by Proposition B.17’s reverse direction implies
\[(\pi_1(\pi_{WY}(Q)) \cup \pi_2(\pi_{WY}(Q)), \pi_{WY}(Q)) \textnormal{ is an out-tree.}\]

By manipulation this implies $(\pi_W(Q) \cup \pi_Y(Q), \pi_{WY}(Q))$ is an out-tree, which by abbreviation (8) implies $(W \cup Y, \pi_{WY}(Q))$ is an out-tree, which by definition (14) implies $(X, \pi_{WY}(Q))$ is an out-tree.

(b) Suppose $Q$ satisfies [Pw-y], [Py], and [Pr]. Then part (a) implies that $(X, \pi_{WY}(Q))$ is an out-tree. Thus Lemma B.8(c) and definition (33), applied to the out-tree $(X, \pi_{WY}(Q))$, show that the root of $(X, \pi_{WY}(Q))$ is
\[\text{the unique member of } \pi_1(\pi_{WY}(Q)) \setminus \pi_2(\pi_{WY}(Q)).\]

This by manipulation is the unique member of $\pi_W(Q) \setminus \pi_Y(Q)$, which by abbreviation (8) is the unique member of $Y \setminus W$, which by definition (13) is $Q$’s root. \hfill \Box

\[\text{\footnotesize 38It may be helpful to highlight an important chain of reasoning. Earlier, Proposition B.17 showed the equivalence between out-trees and edge-trees, and Lemma C.5 showed the (easy) equivalence between edge-trees and axioms [Pw-y], [Py], and [Pr]. Here, the proof of Proposition 3.4 uses these results to show the equivalence between out-trees and the three axioms. Later, this Proposition 3.4 will be used in the proof of Theorem 5.4’s equivalence between Gm games (which are built on out-trees) and pentaform games (which assume the three axioms). This is the paper’s main result. For details, see Claim 17 in Lemma D.1’s proof, and Claim 3 in Lemma D.2’s proof.}\]
Proof C.7 (for Proposition 4.1). (a) First consider axiom [Pi-j]. Since a function is being regarded as a set of pairs (see footnote 10 on page 9), we have that [1] any subset of a function is itself a function. Meanwhile, the assumption \( Q' \subseteq Q \) implies \( \pi_{ji}(Q') \subseteq \pi_{ji}(Q) \). Then in steps, [Pi-j] for \( Q' \) implies \( \pi_{ji}(Q) \) is a function, which by [1] and [2] implies \( \pi_{ji}(Q') \) is a function, which implies [Pi-j] for \( Q' \).

Next consider each of the axioms [Pj+w], [Pwa+y], [Pa+y], and [Pw+y]. Here the axiom for \( Q' \) implies the corresponding axiom for \( Q \) by an argument similar to that of the previous paragraph for [Pi-j].

Finally consider axiom [Py]. First note that [Pw+y] for \( Q' \) (which has already been derived) implies that the function \( p' = \pi_{YW}(Q') \) is a well-defined function with domain \( Y' \) and range \( W' \). Now take \( y \in Y' \). Then \( p'(y) \in W' \) is well-defined. [Step 1] If \( p'(y) \not\in Y' \), then [Py] holds at \( y \) and the argument is complete. Else \( p'(y) \in Y' \) so \( (p')^2(y) \in W' \) is well-defined. [Step 2] If \( (p')^2(y) \not\in Y' \), then [Py] holds at \( y \) and the argument is complete. Else \( (p')^2(y) \in Y' \) so \( (p')^3(y) \in W' \) is well-defined. By repeating this process indefinitely, either the argument finishes at some step or \((\forall m \geq 1)\ (p')^m(y) \in Y'\).

To rule out the latter contingency, suppose it held. Note \( Q' \subseteq Q \) implies \( p' = \pi_{YW}(Q') \) is a restriction of \( p = \pi_{YW}(Q) \). Hence the supposition implies that \((\forall m \geq 1)\ p^m(y) \in Y'\). Also note \( Q' \subseteq Q \) implies \( Y' \subseteq Y \). Thus the definition of \( y \) and the second-previous sentence imply \( y \in Y \) and \((\forall m \geq 1)\ p^m(y) \in Y \). This contradicts [Py] for \( Q \).

(b). This follows from part (a) and the pentaform definition (Definition 3.1).

Lemma C.8. Suppose \( Q \) is a pentaform and \( t \in T \) (where \( T \) is the subroot set (16)). Then the following hold.

(a) \( ^t W = \{ w \in W \mid t \preceq w \} \).
(b) \( ^t Y = \{ y \in Y \mid t Y \} \).
(c) \( (\forall j \in J) \ ({}^t Q)_j = Q_j \).
(d) \( {}^t Q = \bigcup \{ Q_j \mid j \in J \} \).

Proof. (a). Definition (15) implies \( {}^t Q = \{ \langle i, j, w, a, y \rangle \in Q \mid t \preceq w \} \), which by projection implies \( \pi_W({}^t Q) = \{ w \in \pi_W(Q) \mid t \preceq w \} \), which by abbreviation (8) implies part (a).

(b). Definition (15) implies \( {}^t Q = \{ \langle i, j, w, a, y \rangle \in Q \mid t \preceq w \} \), which by projection and abbreviation (8) implies \( {}^t Y = \{ y \in Y \mid (\exists w \in W) \ t \preceq w, \langle w, y \rangle \in \pi_{YW}(Q) \} \). It suffices to show that this right-hand side is equal to the right-hand side of part (b). Toward that end, consider the existence of \( w \in W \) such that \( t \preceq w \) and \( \langle w, y \rangle \in \pi_{YW}(Q) \). Since \( \langle w, y \rangle \in \pi_{YW}(Q) \) implies \( w \in W \), this is equivalent to the existence of \( w \) such that \( t \preceq w \) and \( \langle w, y \rangle \in \pi_{YW}(Q) \). By the definition of \( \preceq \) (34) and the definition of a path in \( (X, \pi_{YW}(Q)) \) (start of Section B.4), this is equivalent to the existence of a path from \( t \) to \( w \) and a nontrivial path from \( w \) to \( y \). By inspection, this is equivalent to a nontrivial path from \( t \) to \( y \), which by the definition of \( \preceq \) (34) is equivalent to \( t \preceq y \).

(c). Take a situation \( j \in J \). Simply, \( {}^t Q \subseteq Q \) implies \( ({}^t Q)_j \subseteq Q_j \). For the reverse inclusion, it suffices to show that \( Q_j \setminus ({}^t Q)_j \) is empty. Toward that end, suppose

\[{}^t Q_3 \subseteq \]
$Q_j \setminus \{t\}_j$ is nonempty. Define $F:Q \to J$ by $F((i*,j*,w*,a*,y*)) = j*$, and note by inspection that $Q_j \setminus \{t\}_j = F^{-1}(j) \setminus (F^{-1}(j) \cap \{t\}) = F^{-1}(j) \cap (Q \setminus \{t\}) = (Q \setminus \{t\})_j$. This equality and the second-previous sentence imply $(Q \setminus \{t\})_j$ is nonempty. Hence $j \in \pi_j(Q \setminus \{t\})$. This and the assumption $j \in J$ imply that $J$ and $\pi_j(Q \setminus \{t\})$ intersect. This contradicts $t \in T$ by $T$’s definition (16).

(d). For the forward inclusion, take a quintuple $(i,j,w,a,y) \in \{t\}_j$. Then projection implies $j \in \pi_j(Q_j)$, which by abbreviation (8) implies $j \in J$. Further, $(i,j,w,a,y) \in Q$ and $Q_j$’s definition (2) imply $(i,j,w,a,y) \in Q_j$. For the reverse inclusion, take a situation $j \in J$. Then part (c) implies $Q_j$ is equal to $(\{t\}_j)_j$, which by construction is a subset of $(\{t\})_j$.

Lemma C.9. Suppose $Q$ is a pentaform and $w \in W$. Then $w \in T$ iff there is $Q \subseteq \{Q_j|j \in J\}$ such that $wQ = \bigcup Q$.

Proof. For the forward direction, suppose $w \in T$ (that is, suppose $w$ is a subroot). Then Lemma C.8(d) implies $(\{t\}_j)_j = \bigcup Q$ where $Q = \{Q_j|j \in \{w\}_j\}$.

For the reverse direction, suppose $w \notin T$. Then the assumption $w \in W$ and $T$’s definition (16) imply there is a $j* \in J$ which is listed both in a quintuple of $wQ$ and in a quintuple of $Q \setminus \{w\}_j$. Thus the slice definition (2) implies that $Q_{j_*}$ intersects both $wQ$ and $Q \setminus \{w\}_j$. Thus, since $\{Q_j|j \in J\}$ is a partition, it does not have a subcollection $Q$ whose union is $wQ$.

Proof C.10 (for Proposition 4.3). The proposition follows from Claim 2.

Claim 1: $(\{t\})_j$ satisfies [Pr]. It suffices to show $(\{t\})_j \subseteq \{t\}_j$. For the forward inclusion, it suffices to show that $(\{t\})_j \subseteq \{t\}_j$. Toward that end, suppose $w \in (\{t\})_j$. That is, suppose $w$ is a decision node in $(\{t\})_j$ other than $t$ itself. Then Lemma C.8(a) implies $t \prec w$. Then $\prec$’s definition (34) implies there is a path from $t$ to $w$, which by the definition of path (start of Section B.4) implies $w \in \{t\}$. The previous two sentences and Lemma C.8(b) imply $w \in \{t\}_j$.

For the reverse inclusion, first note that $T$’s definition (16) implies $t \in W$, which by Lemma C.8(a) implies $t \in (\{t\})_j$. Thus it suffices to show $t \notin \{t\}_j$. If $t \in \{t\}_j$ did hold, then Lemma C.8(b) would imply $t \prec t$, which is impossible.

Claim 2: $(\{t\})_j$ is a pentaform. $T$’s definition (16) implies Lemma C.9’s initial assumption that $t \in W$. Thus Lemma C.9’s forward direction implies that $(\{t\})_j$ is the union of a subcollection of $\{Q_j|j \in J\}$. Thus Claim 1 and Corollary 4.2(b) suffice.

Lemma C.11. Suppose $Q$ is a weakly separated (18) collection of quintuple sets which satisfy [Pi+j], [Pj-w], [Pwa], [Pwa+y], [Pw-y], and [Pa+y]. Then $\bigcup Q$ satisfies the same six axioms.

Proof. This holds by the following claims.

Claim 1: $\bigcup Q$ satisfies [Pi+j]. By inspection, [1] $\pi_{\{j\}}(\bigcup Q) = \bigcup_{Q \in Q} \pi_{\{j\}}(Q)$. Also, each $Q \in Q$ satisfies [Pi+j] by assumption, which implies that [2] $\forall Q \in Q \pi_{\{j\}}(Q)$ is a function. Also, weak separation (18) implies that [3] the members $Q$ of $Q$ have distinct situations $j$. These three facts imply that $\pi_{\{j\}}(\bigcup Q)$ is the union of a set of functions with disjoint domains, which implies that $\pi_{\{j\}}(\bigcup Q)$ is itself a function, which implies that $\bigcup Q$ satisfies [Pi+j].
Claim 2: $\bigcup Q$ satisfies $[Pj-w]$, $[Pwa-y]$, $[Pw-y]$, and $[Pa-y]$. The arguments for these axioms mimic Claim 1’s argument for $[Pi-j]$. More precisely, the arguments for $[Pj-w]$ and $[Pwa-y]$ rely on the members of $Q$ having distinct decision nodes $w$, and the arguments for $[Pa-y]$ and $[Pw-y]$ rely on the members of $Q$ having distinct successor nodes $y$.

Claim 3: $\bigcup Q$ satisfies $[Pwa]$. It suffices to show that
\[
(\forall j \in p_j(\bigcup Q)) \pi_{WA}(\bigcup Q)_j = \pi_W(\bigcup Q)_j \times \pi_A(\bigcup Q)_j.
\]

Toward that end, take a situation $j^* \in p_j(\bigcup Q)$. Weak separation (18) implies the members of $Q$ have distinct situations, which implies there is a unique member $Q^* \in Q$ such that $j^* \in p_j(Q^*)$. Hence the union’s slice $(\bigcup Q)_{j^*}$ equals the member’s slice $Q^*_{j^*}$. Thus it suffices to show that
\[
\pi_{WA}(Q^*_{j^*}) = \pi_W(Q^*_{j^*}) \times \pi_A(Q^*_{j^*}).
\]

This holds because $Q^*$ satisfies $[Pwa]$ by assumption. \qed

Proof C.12 (for Proposition 4.4(a)). This follows from Claims 4–6.

Claim 1: $W^1$ and $Y^2$ are disjoint. Since $W^1 = (W^1 \setminus Y^1) \cup (W^1 \cap Y^1)$, it suffices to show both (a) $(W^1 \setminus Y^1) \cap Y^2 = \emptyset$ and (b) $(W^1 \cap Y^1) \cap Y^2 = \emptyset$. Note (b) holds because weak separation implies $Y^1 \cap Y^2 = \emptyset$. Now consider (a). Since $Y^2 = (Y^2 \setminus W^2) \cup (Y^2 \cap W^2)$, it suffices to show both
\[
(W^1 \setminus Y^1) \cap (Y^2 \cap W^2) = \emptyset \text{ and } (W^1 \setminus Y^1) \cap (Y^2 \cap W^2) = \emptyset.
\]
The former is assumed by part (a). The latter holds because weak separation implies $W^1 \cap W^2 = \emptyset$.

Claim 2: $Q^1 \cup Q^2$ satisfies $[Pi-j]$, $[Pj-w]$, $[Pwa]$, $[Pw-y]$, and $[Pa-y]$. This follows from Lemma C.11 and the definition (17) of block.

Claim 3: $Q^1 \cup Q^2$ satisfies $[Py]$. For notational ease, define $\bar{p} = \pi_Y W(Q^1 \cup Q^2)$. To be clear, $\bar{p}$ is a function since $Q^1 \cup Q^2$ satisfies $[Pw-y]$ by Claim 2. By inspection, it is a superset (equivalently an extension) of both $p^1 = \pi_Y W(Q^1)$ and $p^2 = \pi_Y W(Q^2)$ (these equalities are two instances of definition (11)). Also, since the range of $p^1$ is $W^1$ by definition (11), $[Py]$ for $Q^1$ implies that $(\forall y \in Y^1)(\exists \ell^1 \geq 1) (p^1)^{\ell^1}(y) \in W^1 \setminus Y^1$. Thus Claim 1 implies that
\[
(\forall y \in Y^1)(\exists \ell^1 \geq 1) (p^1)^{\ell^1}(y) \in W^1 \setminus (Y^1 \cup Y^2).
\]

To show that $Q^1 \cup Q^2$ satisfies $[Py]$, take an arbitrary successor node $y \in Y^1 \cup Y^2$. It must be shown that
\[
(\exists \ell \geq 1) \bar{p}^{\ell}(y) \notin Y^1 \cup Y^2.
\]

Obviously $y \in Y^1$ or $y \in Y^2$. First suppose $y \in Y^1$. Then (46) implies there is $\ell^1 \geq 1$ such that $(p^1)^{\ell^1}(y) \notin Y^1 \cup Y^2$, which by $p^1 \subseteq \bar{p}$ implies (47). Second suppose

---

40 For intuition, the identity $W = (W \setminus Y) \cup (W \cap Y)$ splits the decision-node set $W$ into the start-node set $W \setminus Y$ and the “middle”-node set $W \cap Y$. Later, the proof of Claim 1 uses the identity $Y = (Y \setminus W) \cup (Y \cap W)$, which splits the successor-node set $Y$ into the end-node set $Y \setminus W$ and the “middle”-node set $Y \cap W$. This intuition accords with Figure B.1.
y ∈ Y^2. Then [Py] for Q^2 implies there is ℓ^2 ≥ 1 such that (p^2)^{ℓ^2}(y) /∈ Y^2. If (p^2)^{ℓ^2}(y) /∈ Y^1, the previous sentence implies (p^2)^{ℓ^2}(y) /∈ Y^1∪Y^2, which by p^2 ⊆ ¯p implies (47). Otherwise (p^2)^{ℓ^2}(y) ∈ Y^1, which by (46) implies there is ℓ^1 ≥ 1 such that (p^1)^{ℓ^1}((p^2)^{ℓ^2}(y)) /∈ Y^1∪Y^2, which by p^1 ⊆ ¯p and p^2 ⊆ ¯p implies ¯p^{ℓ^1+ℓ^2}(y) /∈ Y^1∪Y^2, which implies (47).

Claim 4: Q^1∪Q^2 is a block. This follows from Claims 2 and 3 and the definition (17) of block.

Claim 5: Q^1∪Q^2’s start-node set is the union of

\[ W_1 \setminus Y_1 \text{ and } (W_2 \setminus Y_2) \setminus (Y_1 \setminus W_1). \]

By inspection, \( \pi_W(Q^1∪Q^2) = W_1∪W_2 \) and \( \pi_Y(Q^1∪Q^2) = Y_1∪Y_2. \) Thus Q^1∪Q^2’s start-node set is \( (W_1∪W_2) \setminus (Y_1∪Y_2), \) which by inspection is the union of

\[ W_1 \setminus (Y_1∪Y^2) \text{ and } W_2 \setminus (Y_1∪Y^2). \]

The first set is equal to W^1 \setminus Y^1 by Claim 1. The second set is equal to (W_2 \setminus Y_2) \setminus Y^1, which is equal to (W_2 \setminus Y_2) \setminus [(Y_1 \setminus W_1) \cup (Y_1 \cap W^1)], which by W_2 ∩ W^1 = ∅ (from weak separation) is equal to (W_2 \setminus Y_2) \setminus (Y_1 \setminus W^1).

Claim 6: Q^1∪Q^2’s end-node set is the union of

\[ (Y_1 \setminus W_1) \setminus (W_2 \setminus Y_2) \text{ and } Y_2 \setminus W^2. \]

By inspection, \( \pi_Y(Q^1∪Q^2) = Y_1∪Y_2 \) and \( \pi_W(Q^1∪Q^2) = W_1∪W_2. \) Thus Q^1∪Q^2’s end-node set is \( (Y_1∪Y^2) \setminus (W_1∪W^2), \) which by inspection is the union of

\[ Y_1 \setminus (W_1∪W^2) \text{ and } Y_2 \setminus (W_1∪W^2). \]

The second set is equal to Y^2 \setminus W^2 by Claim 1. The first set is equal to (Y_1 \setminus W_1) \setminus W^2, which is equal to (Y_1 \setminus W_1) \setminus [(W_2 \setminus Y_2) \cup (W_2 \setminus W^2)], which by Y_1 ∩ Y^2 = ∅ (from weak separation) is equal to (Y_1 \setminus W^1) \setminus (W_2 \setminus Y^2). □

Proof C.13 (for Proposition 4.4(b)). This follows from Claims 3–5.

Claim 1: \( \bigcup Q \) satisfies [Py-j], [Py+w], [Pwa], [Pwa+y], [Pw+y], and [P+y]. Since strong separation (19) implies weak separation (18), Q is weakly separated. Thus Lemma C.11 and the definition (17) of block implies the claim.

Claim 2: \( \bigcup Q \) satisfies [Py]. For notational ease, let \( \bar{p} = \pi_{YW}(\bigcup Q) \). To be clear, \( \bar{p} \) is a function since \( \bigcup Q \) satisfies [Pw+y] by Claim 1. Thus it suffices to show

\[ (\forall y \in \pi_Y(\bigcup Q))(\exists \ell ≥ 1) \ p^\ell(y) /∈ \pi_Y(\bigcup Q). \]

Toward that end, take an arbitrary successor node \( y \in \pi_Y(\bigcup Q) \). Since \( \pi_Y(\bigcup Q) = \bigcup_{Q ∈ Q} \pi_Y(Q) \) by inspection, there is a block \( Q^* \in Q \) such that \( y \in Y^* \). By \( p^* \)'s definition (11), \( p^* = \pi_{YW}(Q^*) \), which by \( Q^* ∈ Q \) implies \( p^* \subseteq \pi_{YW}(\bigcup Q) \), which by \( \bar{p} \)'s definition above implies \( p^* \subseteq \bar{p} \) (equivalently \( p^* \) is a restriction of \( \bar{p} \)). Further, since the range of \( p^* \) is \( W^* \) by definition (11), [Py] for \( Q^* \) implies there is \( \ell ≥ 1 \) such that \( (p^*)^{\ell}(y) \in W^* \setminus Y^* \), which by \( p^* \subseteq \bar{p} \) implies

\[ \bar{p}^\ell(y) \in W^* \setminus Y^*. \]
Since (48) implies \( p^\pi(y) \in W^* \), this \( p^\pi(y) \) is a node of \( Q^* \), which by strong separation implies that \( \bar{p}^\pi(y) \) is not a node of \( U(Q \setminus \{Q^*\}) \), which implies that \( \bar{p}^\pi(y) \) is not a successor node of \( U(Q \setminus \{Q^*\}) \), which by \( p^\pi(y) \notin Y^* \) from (48) implies that \( \bar{p}^\pi(y) \) is not a successor node of \( U Q \), which is equivalent to \( p^\pi(y) \notin \pi_y(U Q) \).

Claim 3: \( U Q \) is a block. This follows from Claims 1 and 2.

Claim 4: \( U Q \)’s start-node set is \( U Q \in Q(\pi W(Q) \setminus \pi_y(Q)) \). Since \( U Q \)’s start-node set is \( \pi W(U Q) \setminus \pi_y(U Q) \), it suffices to show

\[
\pi W(U Q) \setminus \pi_y(U Q) = U Q \in Q(\pi W(Q) \setminus \pi_y(Q))
\]

For the forward inclusion, consider one of the union’s start nodes [a] \( w \in \pi W(U Q) \setminus \pi_y(U Q) \). Note [a] implies \( w \in \pi W(U Q) \), which implies there is a block \( Q^* \in Q \) such that \( w \in \pi W(Q^*) \). Further, [a] implies \( w \notin \pi_y(U Q) \), which by \( Q^* \in Q \) implies \( w \notin \pi_y(Q^*) \). The previous two sentences imply \( w \in \pi W(Q^*) \setminus \pi_y(Q^*) \).

Conversely, for the reverse inclusion, suppose there is an individual block \( Q' \in Q \) with start node [b] \( w \in \pi W(Q') \setminus \pi_y(Q') \). Then [b] implies \( w \in \pi W(Q') \), which by \( Q' \in Q \) implies [c] \( w \in \pi W(U Q) \). The same \( w \in \pi W(Q') \) from [b] also implies that \( w \) is a node of \( Q' \), which by strong separation implies that \( w \) is not a node of \( U(Q \setminus \{Q\}) \), which implies that \( w \) is not a successor node of \( U(Q \setminus \{Q\}) \), which by \( w \notin \pi_y(Q') \) from [b] implies that \( w \) is not a successor node of \( U Q \), which is equivalent to \( w \notin \pi_y(U Q) \). This and [c] imply \( w \in \pi W(U Q) \setminus \pi_y(U Q) \).

Claim 5: \( U Q \)’s end-node set is \( U Q \in Q(\pi_y(Q) \setminus \pi W(Q)) \). This can be proved like Claim 4 was proved. Replace “start” with “end”, switch \( W \) and \( Y \), replace \( w \) with \( y \), and replace “successor node” with “decision node”.

Proof C.14 (for Proposition 4.5). For notational ease, define

\[
Q = \bigcup_{n \geq 0} Q^n.
\]

The proposition holds by Claim 8.

Claim 1: \( Q \) satisfies [Pi\( j \)]. Suppose \( Q \) violates [Pi\( j \)]. Then there exists a situation \( j \in J \) with players \( i_1 \in I \) and \( i_2 \in I \) such that \( i_1 \neq i_2 \) and both \( \langle j,i_1 \rangle \) and \( \langle j,i_2 \rangle \) and are in \( \pi_{j|}(Q) \). Note \( \langle j,i_1 \rangle \) being in \( \pi_{j|}(Q) \) implies there is \( \langle w_1,a_1,y_1 \rangle \) such that \( \langle i_1,j,w_1,a_1,y_1 \rangle \in Q \), which by \( Q \)’s definition (49) implies there is \( n_1 \geq 0 \) such that

\[
\langle i_1,j,w_1,a_1,y_1 \rangle \in Q^{n_1}.
\]

Similarly, \( \langle j,i_2 \rangle \) being in \( \pi_{j|}(Q) \) implies there is \( \langle w_2,a_2,y_2 \rangle \) such that \( \langle i_2,j,w_2,a_2,y_2 \rangle \in Q \), which by \( Q \)’s definition (49) implies there is \( n_2 \geq 0 \) such that

\[
\langle i_2,j,w_2,a_2,y_2 \rangle \in Q^{n_2}.
\]

Let \( n_* = \max\{n_1,n_2\} \). Then the assumption \((\forall n \geq 1) Q^{n-1} \subseteq Q^n \) implies that both \( \langle i_1,j,w_1,a_1,y_1 \rangle \) and \( \langle i_2,j,w_2,a_2,y_2 \rangle \) are in \( Q^{n_*} \), which implies that both \( \langle j,i_1 \rangle \) and \( \langle j,i_2 \rangle \) are in \( \pi_{j|}(Q^{n_*}) \), which by \( i_1 \neq i_2 \) implies \( Q^{n_*} \) violates [Pi\( j \)], which violates the assumption that \( Q^{n_*} \) is a pentaform.
Claim 2: $Q$ satisfies $[P_{j-w}]$, $[P_{w-a+y}]$, $[P_{w-y}]$, and $[P_{a+y}]$. Each of these four axioms is derived as Claim 1 derived $[P_{i+j}]$.

Claim 3: $Q$ satisfies $[P_{w-a}]$. Suppose $Q$ violates $[P_{wa}]$. Then there exists a situation $j \in J$ with a decision node $w_1 \in \bar{W}_j$, another decision node $w_2 \in \bar{W}_j$, and an action $a$, such that $[a] \langle w_1, a \rangle \in \pi_{WA}(\bar{Q}_j)$ and $[b] \langle w_2, a \rangle \notin \pi_{WA}(\bar{Q}_j)$. We will find a contradiction.

Note that $[a]$ and Lemma C.1(c) imply $\langle j, w_1, a \rangle \in \pi_{JWA}(Q)$, which implies there is $\langle i_1, y_1 \rangle$ such that $\langle i_1, j, w_1, a, y_1 \rangle \in \bar{Q}_j$, which by $Q$'s definition (49) implies there is $n_1 \geq 0$ such that

$$\langle i_1, j, w_1, a, y_1 \rangle \in Q^{n_1}.$$ Meanwhile, $w_2 \in \bar{W}_j$ and Lemma C.1(a) imply $\langle j, w_2 \rangle \in \pi_{JW}(Q)$, which implies there is $\langle i_2, a_2, y_2 \rangle \in Q$ such that $\langle i_2, j, w_2, a_2, y_2 \rangle \in Q$, which by $Q$'s definition implies there is $n_2 \geq 0$ such that

$$\langle i_2, j, w_2, a_2, y_2 \rangle \in Q^{n_2}.$$ Let $n_* = \max\{n_1, n_2\}$. Then the assumption $(\forall n \geq 1) Q^{n-1} \subseteq Q^n$ implies that both $\langle i_1, j, w_1, a, y_1 \rangle$ and $\langle i_2, j, w_2, a_2, y_2 \rangle$ are in $Q^{n_*}$. The first implies $\langle j, w_1, a \rangle \in \pi_{JWA}(Q^{n_*})$, which by Lemma C.1(c) implies $[c] \langle w_1, a \rangle \in \pi_{WA}(Q^{n_*})$. The second implies $\langle j, w_2 \rangle \in \pi_{JW}(Q^{n_*})$, which by Lemma C.1(a) implies $[d] \langle w_2 \rangle \in \bar{W}^{n_*}$.

By assumption, $Q^{n_*}$ is a pentaform, which implies $Q^{n_*}$ satisfies $[P_{wa}]$, which implies that $\pi_{WA}(Q^{n_*})$ is a rectangle, which by $[c]$ and $[d]$ implies $\langle w_2, a \rangle \in \pi_{WA}(Q^{n_*})$, which by Lemma C.1(c) implies $\langle j, w_2, a \rangle \in \pi_{JWA}(Q^{n_*})$, which implies there is $\langle i_2, j, w_2, a, y_2 \rangle \in Q^{n_*}$, which by $Q$'s definition implies $\langle i_2, j, w_2, a, y_2 \rangle \in Q$, which implies $\langle j, w_2, a \rangle \in \pi_{JWA}(Q)$, which by Lemma C.1(c) implies $\langle w_2, a \rangle \in \pi_{WA}(Q_j)$, which contradicts $[b]$.

Claim 4: (a) $\bar{W} = \bigcup_{n \geq 0} W^n$. (b) $\bar{Y} = \bigcup_{n \geq 0} Y^n$. First consider (a). In steps, $\bar{W}$ by abbreviation (8) is $\pi_{W}(\bar{Q})$, which by $Q$'s definition (49) is $\pi_{W}(\bigcup_{n \geq 0} Q^n)$, which by inspection equals $\bigcup_{n \geq 0} \pi_{W}(Q^n)$, which by abbreviation (8) is $\bigcup_{n \geq 0} \bar{W}^n$. A similar argument holds for (b).

Claim 5: $\{r^0\} = \bar{W} \setminus \bar{Y}$. For the forward inclusion $\{r^0\} \subseteq \bar{W} \setminus \bar{Y}$, it suffices to show $r^0 \in \bar{W}$ and $r^0 \notin \bar{Y}$. First, $r^0$'s definition (13) implies $r^0 \in W^0$, which by Claim 4(a) implies $r^0 \in \bar{W}$. Second, to show $r^0 \notin \bar{Y}$, suppose $r^0 \in \bar{Y}$. Then Claim 4(b) implies there is $n \geq 0$ such that $r^0 \in Y^n$, which implies $\{r^0\} \neq W^n \setminus Y^n$, which by $r^n$'s definition (13) implies $r^0 \neq r^n$, which contradicts the assumption that $r^n = r^0$.

For the reverse inclusion $\bar{W} \setminus \bar{Y} \subseteq \{r^0\}$, it suffices to show that $\langle \forall w \in \bar{W} \setminus \{r^0\} \rangle$ $w \in \bar{Y}$. In other words, it suffices to show that every decision node other than $r^0$ is a successor node. Toward that end, take such a decision node $w \in \bar{W} \setminus \{r^0\}$. Then Claim 4(a) implies there is $n \geq 0$ such that $[a] w \in W^n \setminus \{r^0\}$. In steps, $r^n$'s definition (13) implies $\{r^n\} = W^n \setminus Y^n$, which obviously implies $W^n \setminus Y^n \subseteq \{r^n\}$, which implies $W^n \setminus \{r^n\} \subseteq Y^n$, which by the assumption $r^n = r^0$ implies $W^n \setminus \{r^0\} \subseteq Y^n$, which by $[a]$ implies $w \in Y^n$, which by Claim 4(b) implies $w \in \bar{Y}$. 

Appendix C. For Pentaform Games 49


Claim 6: (a) \( \bar{p} \) is a function. (b) \((\forall n \geq 0) p^n \) is a function and \( p^n \subseteq \bar{p} \). For (a), note \( \bar{p} \)'s definition (11) implies \([1] \bar{p} = \pi_{WJ}(Q) \). Thus since \( Q \) satisfies \([Pj+w] \) by Claim 2, \( \bar{p} \) is a function. For (b), take \( n \geq 0 \). Note \( p^n \)'s definition (11) implies \([2] p^n = \pi_{WJ}(Q^n) \). Thus since \( Q^n \) satisfies \([Pj+w] \) by assumption, \( p^n \) is a function. Finally, [2] and \( Q \)'s definition (49) imply \( p^n \subseteq \pi_{WJ}(Q) \), which by \([1] \) implies \( p^n \subseteq \bar{p} \).

Claim 7: \( \bar{Q} \) satisfies \([Py] \). To be clear, \( \bar{p} \) is a function by Claim 6(a). Thus it suffices to show that \((\forall y \in Y)(\exists \ell \geq 1) \bar{p}^\ell(y) \notin Y \). Toward that end, take a successor node \( y \in Y \). By Claim 4(b), there is \( n \geq 0 \) such that \( y \in Y^n \). Since \( Q^n \) is a pentaform by assumption, footnote 13 on page 12 implies there is \( \ell \geq 0 \) such that \((p^n)^\ell(y) = r^n \), which by the assumption \( r^n = r^0 \) implies \( (p^n)^\ell(y) = r^0 \), which by Claim 6(b) implies \( \bar{p}^\ell(y) = r^0 \), which by Claim 5 implies \( \bar{p}^\ell(y) \notin Y \).

Claim 8: \( \bar{Q} \) is a pentaform with root \( r^0 \). Claim 5 implies that \( \bar{Q} \) satisfies \([Pr] \). Claims 1–3 and 7 show that \( Q \) satisfies every other axiom. Thus \( \bar{Q} \) is a pentaform. Finally, definition (13) states that \( \bar{Q} \)'s root is the sole member of \( \bar{W} \setminus \bar{Y} \), which by Claim 5 is \( r^0 \).



APPENDIX D. For Equivalence with \textbf{Gm} Games

Lemma D.1 (implies Theorem 5.2). Suppose \((X,E,H,\lambda,\tau,u)\) is a \textbf{Gm} game. Let \((\bar{Q},\bar{u}) = P(X,E,H,\lambda,\tau,u) \). Then (a) \((\bar{Q},\bar{u})\) is a pentaform game with information-set situations (31). Further, (b) \( \bar{I} = I \), (c) \( \bar{J} = H \), (d) \( \bar{W} = W \), (e) \( \bar{A} = A \), (f) \( \bar{Y} = Y \), (g) \( \bar{F} = F \), (h) \( \bar{X} = X \), (i) \( \pi_{WY}(\bar{Q}) = E \), (j) \( \bar{r} = r \), (k) \( \bar{p} = p \), (l) \( \bar{\tau} = \tau \), (m) \( \bar{\lambda} = \lambda \), (n) \( \bar{Z} = Z \), (o) \( \{\langle\langle w,y,a\rangle|\langle w,y,a\rangle \in \pi_{WY}(\bar{Q})\}\} = \lambda \), and (p) \( \pi_{W1}(\bar{Q}) = \tau \).

Proof. Part (a) holds by Claims 12 and 23. Parts (b)–(f) hold by Claim 7. Parts (g)–(k) hold by Claims 10, 15–16, and 20–21. Parts (m)–(n) hold by Claim 22. Parts (o)–(p) hold by Claims 24–25.

Claim 1: \( E \ni \langle w,y \rangle \mapsto w \) is a surjection to \( W \). It suffices that \( W \) is defined to be \( \pi_1E \) by Section 5.1 paragraph a.

Claim 2: \( E \ni \langle w,y \rangle \mapsto y \) is a surjection to \( Y \). It suffices that \( Y \) is defined to be \( \pi_2E \) by Section 5.1 paragraph a.

Claim 3: \( E \ni \langle w,y \rangle \mapsto \lambda(w,y) \) is a surjection to \( A \). This holds because of \([Gm3] \) (Definition 5.1) and because \( A \) is defined (Section 5.1 paragraph c) to be the range of \( \lambda \).

Claim 4: \( E \ni \langle w,y \rangle \mapsto \tau(w) \) is a surjection to \( I \). This is the composition of two surjections. First, \( E \ni \langle w,y \rangle \mapsto w \) is a surjection to \( W \) by Claim 1. Second, \( W \ni w \mapsto \tau(w) \) is a surjection to \( I \) because of \([Gm5] \) (Definition 5.1) and because \( I \) is defined (Section 5.1 paragraph d) to be the range of \( \tau \).

Claim 5: \( E \ni \langle w,y \rangle \mapsto H_w \) is a surjection to \( H \). This is the composition of two surjections. First, \( E \ni \langle w,y \rangle \mapsto w \) is a surjection to \( W \) by Claim 1. Second, \( W \ni w \mapsto H_w \) is a surjection to \( H \) by the definition of \( \langle H_w \rangle_{w \in W} \) (after definition (30)) and by \([Gm2] \).

Claim 6: \( \bar{Q} \) is well-defined. This follows from definition (30a) and Claims 3–5.
Claim 7: (a) \( \bar{I} = I \). (b) \( \bar{J} = H \). (c) \( \bar{W} = W \). (d) \( \bar{A} = A \). (e) \( \bar{Y} = Y \).

For part (a), note that \( \bar{I} \) by abbreviation (8) is equal to the projection \( \pi_I(\bar{Q}) \), which by definition (30a) and the surjection of Claim 4 is equal to \( I \).

For part (b), note that \( \bar{J} \) by abbreviation (8) is equal to the projection \( \pi_J(\bar{Q}) \), which by definition (30a) and the surjection of Claim 5 is equal to \( H \).

Similarly, parts (c), (d), and (e) follow from abbreviation (8), definition (30a), and, respectively, the surjections in Claims 1, 3, and 2.

Claim 8: \( \bar{Q} \) satisfies \([P_i\bar{j}]\). Note \( \pi_{J}(\bar{Q}) = \{ \langle H_w, \tau(w) \rangle \mid w \in W \} \) by definition (30a). Thus Claim 1 implies \( \pi_{J}(\bar{Q}) = \{ \langle H_w, \tau(w) \rangle \mid w \in W \} \). To show that this is a function, it suffices to show that \( (\forall w_1 \in W, w_2 \in W) \) the condition \( H_{w_1} = H_{w_2} \) implies \( \tau(w_1) = \tau(w_2) \). Toward that end, suppose \( w_1 \in W \) and \( w_2 \in W \) are two decision nodes such that \( H_{w_1} = H_{w_2} \). Then the definition of \( \langle H_w \rangle_{w \in W} \) (after definition (30)) implies that \( w_1 \) and \( w_2 \) belong to the same element of the partition \( H \), which by the measurability (28) of \([Gm5]\) implies \( \tau(w_1) = \tau(w_2) \).

Claim 9: \( \bar{Q} \) satisfies \([P_j\bar{w}]\). Note \( \pi_{WJ}(\bar{Q}) = \{ \langle w, H_w \rangle \mid w \in W \} \) by definition (30a). Thus Claim 1 implies \( \pi_{WJ}(\bar{Q}) = \{ \langle w, H_w \rangle \mid w \in W \} \). This is the function \( \langle H_w \rangle_{w \in W} \) (defined after definition (30)).

Claim 10: \( \bar{F} = F \). The correspondence \( F : W \rightarrow A \) is defined at equation (26) by \( F(w) = \{ a \mid (\exists y) \lambda(w,y) = a \} \). Since a correspondence is a set of pairs (footnote 12 on page 11), this implies that

\[
F = \{ \langle w, a \rangle \mid w \in W, (\exists y) \lambda(w,y) = a \}.
\]

Note [Gm3] implies that the domain of \( \lambda \) is \( E \), and that \( W = \pi_1 E \) by \( W \)’s definition (Section 5.1 paragraph a). Hence \( F = \{ \langle w, \lambda(w,y) \rangle \mid w, y \in E \} \). Thus definition (30a) implies \( F = \pi_{WA}(\bar{Q}) \), which by \( F \)’s definition (12) implies \( F = \bar{F} \).

Claim 11: \( (\forall H \in \mathcal{H}) \{ w \in W \mid H_w = H \} = H \). To see this identity, note that \( \mathcal{H} \) partitions \( W \) by \([Gm2]\), and that \( H_w \) is by definition (after (30)) the partition element that contains \( w \). Now consider a partition element \( H \in \mathcal{H} \). Then \( \{ w \in W \mid H_w = H \} \) consists of the \( w \) in the partition element \( H \), which is \( H \) itself.

Claim 12: \( \bar{Q} \) has information-set situations (31), that is, \( (\forall j \in \bar{J}) \bar{W}_j = j \). Take a situation \( j \in \bar{J} \). Note definition (30a) implies that the slice \( \bar{Q}_j \) satisfies

\[
\bar{Q}_j = \{ \langle \tau(w), H_w, w, \lambda(w,y), y \rangle \mid \langle w, y \rangle \in E, H_w = j \},
\]

which by projection implies \( \pi_W(\bar{Q}_j) = \{ w \mid \langle w, y \rangle \in E, H_w = j \} \), which by abbreviation (8) implies \( \bar{W}_j = \{ w \mid \langle w, y \rangle \in E, H_w = j \} \), which by Claim 1 implies \( \bar{W}_j = \{ w \mid w \in W, H_w = j \} \), which by manipulation implies \( \bar{W}_j = \{ w \mid w \in W, H_w = j \} \). Thus to show that \( \bar{W}_j = j \), it suffices to show that \( \{ w \in W \mid H_w = j \} = j \). So by Claim 11’s identity, it suffices to show that \( j \in \mathcal{H} \). This holds because \( j \in \bar{J} \) by assumption and because \( \bar{J} = \mathcal{H} \) by Claim 7(b).

Claim 13: \( \bar{Q} \) satisfies \([P_{wa}]\). By Claim 9 and Proposition 3.3(\( a \equiv d \)), it suffices to show that

\[
(\forall j \in \bar{J}, w_1 \in \bar{W}_j, w_2 \in \bar{W}_j) \ F(w_1) = F(w_2).
\]
Appendix D. For Equivalence with \textbf{Gm} games

By replacing \( \bar{J} \) with \( \mathcal{H} \) via Claim 7(b), by replacing the two appearances of \( \bar{W}_j \) with \( j \) via Claim 12, and by replacing \( \bar{F} \) with \( F \) via Claim 10, this is equivalent to

\[
(\forall j \in \mathcal{H}, w_1 \in j, w_2 \in j) \ F(w_1) = F(w_2).
\]

By a change of variables, this is equivalent to \( (\forall H \in \mathcal{H}, w_1 \in H, w_2 \in H) \ F(w_1) = F(w_2), \) which holds by the measurability (27) of [Gm4].

\textbf{Claim 14:} \( Q \) satisfies \( [Pw \wedge y] \). Definition (30a) implies \( \pi_{WY}(Q) = \{ \langle w, \lambda(w,y) \rangle | (w,y) \in E \} \). Thus it suffices to show, for all \( \langle w_1, y_1 \rangle \in E \) and \( \langle w_2, y_2 \rangle \in E \), that

\[
\langle w_1, \lambda(w_1,y_1) \rangle \equiv \langle w_2, \lambda(w_2,y_2) \rangle \implies y_1 = y_2.
\]

Toward that end, take \( \langle w_1, y_1 \rangle \in E \) and \( \langle w_2, y_2 \rangle \in E \) such that \( \langle w_1, \lambda(w_1,y_1) \rangle \equiv \langle w_2, \lambda(w_2,y_2) \rangle \). Since the first coordinate of the equality requires \( w_1 = w_2 \), we have two edges of the form \( \langle w, y_1 \rangle \) and \( \langle w, y_2 \rangle \) such that \( \lambda(w,y_1) = \lambda(w,y_2) \). Thus the local injectivity (25) of [Gm3] implies \( y_1 = y_2 \).

\textbf{Claim 15:} \( \bar{X} = X \). To see this, note that \( \bar{X} \) by definition (14) is equal to \( \bar{W} \cup \bar{Y} \), which by Claim 7(c,e) is equal to \( W \cup Y \), which by [Gm1] and Lemma B.8(a) is equal to \( X \).

\textbf{Claim 16:} \( \pi_{WY}(Q) = E \). In steps, \( \pi_{WY}(Q) \) by (30a) is equal to \( \{ \langle w, y \rangle | (w,y) \in E \} \), which by inspection is equal to \( E \).

\textbf{Claim 17:} \( Q \) satisfies \( [Pw \wedge y], [Py], \) and \( [Pr] \). [Gm1] states that \( (X,E) \) is a nontrivial out-tree. Thus Claims 15 and 16 imply that \( \bar{X}, \pi_{WY}(Q)) \) is a nontrivial out-tree. Thus the reverse direction of Proposition 3.4(a) implies that \( Q \) satisfies \( [Pw \wedge y] \), \([Py] \), and \([Pr] \).

\textbf{Claim 18:} \( Q \) satisfies \( [Pa \wedge y] \). Note \( \pi_{YA}(Q) = \{ \langle y, \lambda(w,y) \rangle | (w,y) \in E \} \) by definition (30a). Thus it suffices to show, for all \( \langle w_1, y_1 \rangle \in E \) and \( \langle w_2, y_2 \rangle \in E \), that

\[
y_1 = y_2 \implies \lambda(w_1,y_1) = \lambda(w_2,y_2).
\]

Toward that end, take \( \langle w_1, y_1 \rangle \in E \) and \( \langle w_2, y_2 \rangle \in E \) such that \( y_1 = y_2 \). Claim 16 implies \( \langle w_1, y_1 \rangle \in \pi_{WY}(Q) \) and \( \langle w_2, y_2 \rangle \in \pi_{WY}(Q) \). Thus, since \([Pw \wedge y] \) holds by Claim 17, the assumption \( y_1 = y_2 \) implies \( w_1 = w_2 \). Therefore \( \langle w_1, y_1 \rangle = \langle w_2, y_2 \rangle \), which implies \( \lambda(w_1,y_1) = \lambda(w_2,y_2) \).

\textbf{Claim 19:} \( Q \) is a pentaform. This follows from Claims 8, 9, 13, 14, 17, and 18.

\textbf{Claim 20:} \( \bar{r} = r \). In steps, \( \bar{r} \) by definition (13) is the root of \( Q \), which by Claim 19 and Proposition 3.4(b) is the root of \( \bar{X}, \pi_{WY}(Q) \), which by Claims 15 and 16 is the root of \( (X,E) \), which by definition (Section 5.1 paragraph a) is \( r \).

\textbf{Claim 21:} \( \bar{p} = p \). In steps, \( \bar{p} \) by definition (11) is \( \pi_{WY}(Q) \), which by manipulation is \( \{ \langle y, w \rangle | (w,y) \in \pi_{WY}(Q) \} \), which by Claim 16 is \( \{ \langle y, w \rangle | (w,y) \in E \} \), which by definition (Section 5.1 paragraph a) is \( p \).

\textbf{Claim 22:} (a) \( \bar{z} = \bar{z} \). (b) \( \bar{z} = \bar{z} \). (c) \( Z = Z \). Claims 15 and 16 imply that \( (\bar{X}, \pi_{WY}(Q)) \) equals the out-tree \( (X,E) \) from [Gm1]. This suffices since both pentaform games (end of Section 3.5) and \textbf{Gm} games (Section 5.1 paragraph a) derive
their precedence orders and run collections from their out-trees (via equations (34) and (36)).

Claim 23: $(\bar{Q}, \bar{u})$ is a pentaform game. Since $\bar{Q}$ is a pentaform by Claim 19, it suffices to show that $\bar{u}$ is a utility-function profile for $\bar{Q}$. Definition (30b) states $\bar{u} = u$, and $[\text{Gm6}]$ implies that $u$ has the form $\langle u_i: \mathbb{Z} \rightarrow \mathbb{R} \rangle_{i \in I}$. This $u$ is a utility-function profile for $\bar{Q}$ because $I = \bar{I}$ by Claim 7(a) and because $\mathbb{Z} = \bar{\mathbb{Z}}$ by Claim 22(c).

Claim 24: $\{\langle \langle w, y, a \rangle \rangle \langle w, y, a \rangle \in \pi_{WA}(Q) \} = \lambda$. In steps, the left-hand side by definition (30a) is

$$\{ \langle \langle w, y, a \rangle \rangle | \langle w, y, a \rangle \in \{ \langle w, y, \lambda(w, y) \rangle \} \},$$

which is $\{ \langle \langle w, y, a \rangle \rangle | \langle w, y \rangle \in E, a = \lambda(w, y) \}$, which is $\{ \langle w, y, \lambda(w, y) \rangle | \langle w, y \rangle \in E \}$, which by $[\text{Gm3}]$ is $\lambda$.

Claim 25: $\pi_{W}(Q) = \tau$. In steps, $\pi_{W}(Q)$ by (30a) is $\{ \langle w, \tau(w) \rangle \} | \langle w, y \rangle \in E \}$, which by Claim 1 is $\{ \langle w, \tau(w) \rangle | w \in W \}$, which by $[\text{Gm5}]$ is $\tau$.

Lemma D.2 (implies Theorem 5.3). Suppose $(Q, u)$ is a pentaform game. Let $(\hat{X}, \hat{E}, \hat{H}, \hat{\lambda}, \hat{\tau}, \hat{u}) = S(Q, u)$. Then $(\hat{X}, \hat{E}, \hat{H}, \hat{\lambda}, \hat{\tau}, \hat{u})$ is a $\text{Gm}$ game.

Proof. The lemma holds by Claim 15.

Claim 1: $\hat{X} = X$. In steps, $\hat{X}$ by definition (32a) equals $W \cup Y$, which by definition (14) equals $X$.

Claim 2: $(\hat{X}, \hat{E}) = (X, \pi_{WY}(Q))$. This follows from Claim 1 and definition (32b).

Claim 3: $[\text{Gm1}]$ holds. Proposition 3.4(a)’s forward direction shows $(X, \pi_{WY}(Q))$ is a nontrivial out-tree. Thus Claim 2 implies $(\hat{X}, \hat{E})$ is a nontrivial out-tree.

Claim 4: $\hat{Z} = Z$. Claims 2 and 3 imply that the out-trees $(\hat{X}, \hat{E})$ and $(X, \pi_{WY}(Q))$ are identical. This suffices because both pentaform games (end of Section 3.5) and $\text{Gm}$ games (Section 5.1 paragraph a) derive their run collections from their out-trees (via equation (36)).

Claim 5: $\hat{W} = W$. In steps, $\hat{W}$ by definition (Section 5.1 paragraph a, via Claim 3) is $\pi_{1}E$, which by definition (32b) is $\pi_{1}(\pi_{WY}(Q))$, which by manipulation is $\pi_{W}(Q)$, which by abbreviation (8) is $W$.

Claim 6: $[\text{Gm2}]$ holds. Axiom $[\text{Pj-w}]$ and Proposition 3.2(a$\Rightarrow$c) imply that $\{W_{j}|j \in J\}$ is a partition of $W$. Hence definition (32c) and Claim 5 imply that $\hat{H}$ is a partition of $\hat{W}$.

Claim 7: $\hat{\lambda}$ is a function from $\hat{E}$. The definition (32d) of $\hat{\lambda}$ is

$$\hat{\lambda} = \{ \langle \langle w, y, a \rangle \rangle | \langle w, y, a \rangle \in \pi_{WA}(Q) \}.$$  

By inspection $\pi_{1}\hat{\lambda} = \pi_{WY}(Q)$ and $\pi_{2}\hat{\lambda} = \pi_{A}(Q)$. Thus (by footnote 10 on page 9) it suffices to show that

$$(\forall \langle w, a \rangle \in \pi_{WY}(Q))(\exists a \in \pi_{A}(Q)) \langle w, y, a \rangle \in \pi_{WA}(Q),$$
and that $\pi_{WY}(Q) = \hat{E}$. The latter holds by $\hat{E}$’s definition (32b). For the former, take a $\langle w,a \rangle \in \pi_{WY}(Q)$. By inspection, there is an $a \in \pi_A(Q)$ such that $\langle w,y,a \rangle \in \pi_{WYA}(Q)$. Axiom [Pwa] implies it is unique.

**Claim 8:** [Gm3] holds. Because of Claim 7, it suffices to show that $\hat{\lambda}$ is locally injective (25). We will prove the contrapositive. Toward that end, consider two edges $\langle w,y_1 \rangle \in E$ and $\langle w,y_2 \rangle \in E$ from decision node $w$ which are both assigned the action $\lambda(w,y_1) = \lambda(w,y_2)$. It suffices to show $y_1 = y_2$.

Let $a$ denote the common action $\lambda(w,y_1) = \lambda(w,y_2)$. Then $\hat{\lambda}$’s definition (32d) implies that both $\langle w,y_1,a \rangle$ and $\langle w,y_2,a \rangle$ are in $\pi_{WYA}(Q)$. Thus by rearrangement, both $\langle w,a,y_1 \rangle$ and $\langle w,a,y_2 \rangle$ are in $\pi_{WAY}(Q)$. Hence axiom [Pwa] implies $y_1 = y_2$.

**Claim 9:** $\hat{F} = F$. The correspondence $\hat{F} : \hat{W} \rightarrow \hat{A}$ is defined in (26) by $\hat{F}(w) = \{ a \mid (\exists y) \lambda(w,y) = a \}$. Thus $\hat{F}$ (by footnote 12 on page 11) is equal to

$$\{ \langle w, a \rangle \mid w \in \hat{W}, (\exists y) \lambda(w,y) = a \},$$

which by Claim 5 and abbreviation (8) is equal to $\{ \langle w, a \rangle \mid w \in \pi_{W}(Q), (\exists y) \lambda(w,y) = a \}$, which by $\hat{\lambda}$’s definition (32d) is equal to $\{ \langle w, a \rangle \mid w \in \pi_{W}(Q), (\exists y) \lambda(w,y) = a \}$, which by inspection is equal to $\pi_{WA}(Q)$, which by $F$’s definition (12) is equal to $F$.

**Claim 10:** [Gm4] holds. Axiom [Pwa] and Proposition 3.3(a $\Rightarrow$ d) imply that $\forall j \in J, w_1 \in W_j, w_2 \in W_j \ F(w_1) = F(w_2)$. By inspection, this is equivalent to $\forall H \in \{ W_j \mid j \in J \}, w_1 \in H, w_2 \in H \ F(w_1) = F(w_2)$, which by $\hat{H}$’s definition (32c) and Claim 9 is equivalent to

$$(\forall H \in \hat{H}, w_1 \in H, w_2 \in H) \ \hat{F}(w_1) = \hat{F}(w_2),$$

which is the measurability (27) of [Gm4].

**Claim 11:** (a) $\pi_{WJ}(Q)$ is a surjection from $W$ to $J$. (b) $\pi_{HJ}(Q)$ is a surjection from $J$ to $I$. (c) $\hat{\tau} = \pi_{HJ}(Q) \circ \pi_{WJ}(Q)$. (a) holds because of axiom [Pj-w], and because $W = \pi_{W}(Q)$ and $J = \pi_{J}(Q)$ by abbreviation (8). Similarly (b) holds because of axiom [Pj-i], and because $J = \pi_{J}(Q)$ and $I = \pi_{I}(Q)$ by abbreviation (8). For (c), the composition is well-defined by parts (a) and (b). Further, the equality holds because $\langle w,i \rangle \in \hat{\tau}$ by definition (32e) is equivalent to $\langle w,i \rangle \in \pi_{WI}(Q)$, which by projection is equivalent to $(\exists j) \langle w,j,i \rangle \in \pi_{WJ}(Q)$, which by inspection is equivalent to $\langle w,i \rangle \in \pi_{HJ}(Q) \circ \pi_{WJ}(Q)$.

**Claim 12:** [Gm5] holds. Claim 11(a,c) implies that $\hat{\tau}$ is a function from $W$, which by Claim 5 implies that $\hat{\tau}$ is a function from $\hat{W}$. Thus it remains to show the measurability (28) of [Gm5]. In other words, it remains to show that

$$(\forall H \in \hat{H}, w_1 \in H, w_2 \in H) \ \hat{\tau}(w_1) = \hat{\tau}(w_2).$$

By $\hat{H}$’s definition (32c), and by Claim 11(c), this is equivalent to

$$(\forall H \in \{ W_j \mid j \in J \}, w_2 \in H, w_2 \in H) \ \pi_{HJ}(Q) \circ \pi_{WJ}(Q)(w_1) = \pi_{HJ}(Q) \circ \pi_{WJ}(Q)(w_2),$$

which by rearrangement is equivalent to

$$(\forall j \in J, w_1 \in W_j, w_2 \in W_j) \ \pi_{HJ}(Q) \circ \pi_{WJ}(Q)(w_1) = \pi_{HJ}(Q) \circ \pi_{WJ}(Q)(w_2).$$
Now take a situation \( j \in J \) and two decision nodes \( w_1 \) and \( w_2 \) in \( W_j \). Then \( w_1 \in W_j \) and Lemma C.1(a) imply \( \langle w_1,j \rangle \in \pi_{WJ}(Q) \), which by axiom \([PJ\cdot w]\) implies \( \pi_{WJ}(Q)(w_1) = j \). Similarly, \( \pi_{WJ}(Q)(w_2) = j \). Hence \( \pi_{WJ}(Q)(w_1) = \pi_{WJ}(Q)(w_2) \), which implies the result.

**Claim 13:** \( \hat{I} = I \). In steps, \( \hat{I} \) by definition (Section 5.1 paragraph d) is equal to the range of \( \hat{\tau} \), which by Claim 11(a–c) is equal to \( I \).

**Claim 14:** \([Gm6]\) holds. Definition (32f) implies \( \hat{u} = u \), and the definition of a pentaform game (Definition 3.5) implies \( u \) has the form \( \langle \hat{u}_i:Z \to \mathbb{R} \rangle_{i \in I} \). Thus \( \hat{u} \) has the form \( \langle \hat{u}_i:Z \to \mathbb{R} \rangle_{i \in I} \) because \( Z = \mathbb{Z} \) by Claim 4 and because \( \hat{I} = I \) by Claim 13.

**Claim 15:** \((X, \hat{E}, \hat{H}, \lambda, \hat{\tau}, \hat{u})\) is a \( Gm \) game. This follows from Claims 3, 6, 8, 10, 12, and 14.

\[ \square \]

**Lemma D.3.** Suppose \((X, E, H, \lambda, \tau, u)\) is a \( Gm \) game. Then \( SP(X, E, H, \lambda, \tau, u) = (X, E, H, \lambda, \tau, u) \).

**Proof.** Let \((\hat{Q}, \hat{u}) = P(X, E, H, \lambda, \tau, u)\), which by Theorem 5.2 is a well-defined pentaform game with information-set situations. Next let \((X, \hat{E}, \hat{H}, \lambda, \hat{\tau}, \hat{u}) = S(\hat{Q}, \hat{u})\). Since \((X, \hat{E}, \hat{H}, \lambda, \hat{\tau}, \hat{U}) = SP(X, E, H, \lambda, \tau, u)\) by inspection, it suffices to show \((X, \hat{E}, \hat{H}, \lambda, \hat{\tau}, \hat{U}) = (X, E, H, \lambda, \tau, U)\). This is done, one component at a time, by Claims 1–6.

**Claim 1:** \( \hat{X} = X \). In steps, \( \hat{X} \) by its definition (32a) is \( \hat{W} \cup \hat{Y} \), which by definition (14) is \( X \), which by Lemma D.1(h) is \( X \).

**Claim 2:** \( \hat{E} = E \). In steps, \( \hat{E} \) by its definition (32b) is \( \pi_{WY}(\hat{Q}) \), which by Lemma D.1(i) is \( E \).

**Claim 3:** \( \hat{H} = H \). By the proof’s first sentence, \((\hat{Q}, \hat{u})\) has information-set situations (31). In other words, \((\forall j \in J) \hat{W}_j = j \). Then in steps, \( \hat{H} \) by its definition (32c) is \( \{W_j|j \in J\} \), which by the previous sentence is \( \{j|j \in J\} \), which reduces to \( \hat{J} \), which by Lemma D.1(c) is \( H \).

**Claim 4:** \( \hat{\lambda} = \lambda \). Note \( \hat{\lambda} \) by its definition (32d) is \( \{\langle \langle w,y \rangle,a \rangle|\langle w,y,a \rangle \in \pi_{WYA}(\hat{Q})\} \), which by Lemma D.1(o) is \( \lambda \).

**Claim 5:** \( \hat{\tau} = \tau \). Note \( \hat{\tau} \) by its definition (32e) is \( \pi_{WI}(\hat{Q}) \), which by Lemma D.1(p) is \( \tau \).

**Claim 6:** \( \hat{u} = u \). Note \( \hat{u} \) by its definition (32f) is \( \hat{u} \), which by its definition (30b) is \( u \).

\[ \square \]

**Lemma D.4.** Suppose \((Q, u)\) is a pentaform game with information-set situations. Then \( PS(Q, u) = (Q, u) \).

**Proof.** Let \((X, \hat{E}, \hat{H}, \hat{\lambda}, \hat{\tau}, \hat{u}) = S(Q, u)\), which by Theorem 5.3 is a well-defined \( Gm \) game. Next let \((\hat{Q}, \hat{u}) = P(X, \hat{E}, \hat{H}, \hat{\lambda}, \hat{\tau}, \hat{u})\). Since \((\hat{Q}, \hat{u}) = PS(Q, u)\) by inspection, it suffices to show that \((\hat{Q}, \hat{u}) = (Q, u)\). Definitions (30b) and (32f) imply \( \hat{u} = \hat{u} = u \). Thus it suffices to show \( \hat{Q} = Q \). This is Claim 4 below.
Claim 1: \((\forall (j, w) \in \pi_{JW}(Q)) \ j = \hat{H}_w\) (where \(\hat{H}_w\) is the information set in \(\hat{H}\) that contains \(w\), as defined below (30b)). Take an original situation/decision-node pair \((j, w) \in \pi_{JW}(Q)\). Easily, projection implies \(j \in \pi_J(Q)\), which by abbreviation (8) implies \([a]\) \(j \in J\). Meanwhile, the assumption \((j, w) \in \pi_{JW}(Q)\) and Lemma C.1(a) imply \([b]\) \(w \in W_j\).

Note that \([a]\) and \(\hat{H}\)'s definition (32c) imply \(W_j \in \hat{H}\), that is, that the original pentaform information set \(W_j\) is a \(\text{Gm}\) information set in \(\hat{H}\). Thus since \(\hat{H}\) is a partition by \([\text{Gm2}]\), \([b]\) implies that \(W_j\) is the member of \(\hat{H}\) that contains \(w\). In other words, \(W_j = \hat{H}_w\). Meanwhile, since \((Q, u)\) has information-set situations (31) by assumption, \(W_j = j\). Hence \(j = \hat{H}_w\).

Claim 2: \(Q \subseteq \{ \langle \hat{\tau}(w), \hat{H}_w, w, \hat{\lambda}(w,y), y \rangle \mid \langle w,y \rangle \in \hat{E} \}\). (The following argument will unusually letter its observations.) Take an original quintuple \([q]\) \(\langle i,j,w,a,y \rangle \in Q\). We begin with four preliminary observations. First, \([q]\) implies \(\langle w,i \rangle \in \pi_{WF}(Q)\), which by \(\hat{\tau}\)'s definition (32e) implies \([i]\) \(i = \hat{\tau}(w)\). Second, \([q]\) implies \(\langle w,j \rangle \in \pi_{WF}(Q)\), which by Claim 1 implies \([j]\) \(j = \hat{H}_w\). Third, \([q]\) implies \(\langle w,y,a \rangle \in \pi_{WYA}(Q)\), which by \(\hat{\lambda}\)'s definition (32d) implies \([a]\) \(a = \hat{\lambda}(w,y)\). Fourth, \([q]\) implies \(\langle w,y \rangle \in \pi_{WY}(Q)\), which by \(\hat{E}\)'s definition (32b) implies \([e]\) \(\langle w,y \rangle \in \hat{E}\). In conclusion, \([i]\), \([j]\), and \([a]\) imply \(\langle i,j,w,a,y \rangle = \langle \hat{\tau}(w), \hat{H}_w, w, \hat{\lambda}(w,y), y \rangle\), which with \([e]\) completes the argument.

Claim 3: \(Q = \{ \langle \hat{\tau}(w), \hat{H}_w, w, \lambda(w,y), y \rangle \mid \langle w,y \rangle \in \hat{E} \}\). Claim 2 shows the forward inclusion. From another perspective, Claim 2 shows that the set \(Q\) is a subset of a function from \(\hat{E}\) (to be clear, the function’s argument \(\langle w,y \rangle \in \hat{E}\) appears in the third and fifth coordinates, and the function takes each \(\langle w,y \rangle \in \hat{E}\) to the triple \(\langle \hat{\tau}(w), \hat{H}_w, \hat{\lambda}(w,y) \rangle\) which appears in the first, second, and fourth coordinates). Thus the set \(Q\) and the function are equal if the projection \(\pi_{WY}(Q)\) is equal to the domain \(\hat{E}\) (as opposed to merely being a subset of \(\hat{E}\)). This holds by \(\hat{E}\)'s definition (32b).

Claim 4: \(Q = \bar{Q}\). The right-hand side of Claim 3’s equality is equal to \(\bar{Q}\) by \(\bar{Q}\)'s definition (30a).

Proof D.5 (for Theorem 5.4). This follows from Lemmas D.3 and D.4.

Proof D.6 (for Theorem 5.5). Parts (a)–(o) rearrange Lemma D.1(b)–(p). Part (p) holds by definition (30b).

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