On data reduction for dynamic vector bin packing

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Abstract

We study a dynamic vector bin packing (DVBP) problem. We show hardness for shrinking arbitrary DVBP instances to size polynomial in the number of request types or in the maximal number of requests overlapping in time. We also present a simple polynomial-time data reduction algorithm that allows to recover \((1 + \varepsilon)\)-approximate solutions for arbitrary \(\varepsilon > 0\). It shrinks instances from Microsoft Azure and Huawei Cloud by an order of magnitude for \(\varepsilon = 0.02\).

Keywords: approximation & heuristics, parameterized complexity, lossy kernelization, resource allocation

1. Introduction

Motivated by applications to computer storage allocation, Coffman et al. [7] introduced the dynamic bin packing problem (DBP). Motivated by resource allocation problems in cloud data centers, we study the multi-dimensional generalization of the problem:

**Problem 1.1** (Dynamic vector bin packing (DVBP)).

*Input:* A bin capacity \(b \in \mathbb{N}^d\), requests \(a^{(1)}, \ldots, a^{(n)} \in \mathbb{N}^d\), request start times \(s^{(1)}, \ldots, s^{(n)} \in \mathbb{N}\), request end times \(e^{(1)}, \ldots, e^{(n)} \in \mathbb{N}\).

*Find:* A partition of \([1, \ldots, n]\) into bins \(B_1, \ldots, B_k\) with minimum \(k\) such that, for any time instant \(t \in \mathbb{N}\) and each bin \(B_j\), it holds that (component-wise)

\[
\sum_{i \in B_j} a^{(i)} \leq b.
\]

**Definition 1.2** (flavor, type, height). Two requests \(a^{(i)}\) and \(a^{(j)}\) have the same flavor if \(a^{(i)} = a^{(j)}\). They are of the same type if, additionally, \(s^{(i)} = s^{(j)}\) and \(e^{(i)} = e^{(j)}\).

The height \(h\) of an instance is the maximum number of requests active at the same time. We denote the number of flavors by \(\phi\), and the number of types by \(\tau\).

**Example 1.3.** Consider a pool of identical servers of capacity \(b \in \mathbb{N}^2\), providing \(b_1\) processors and \(b_2\) units of memory. Customers make requests \(a^{(i)} \in \mathbb{N}^2\) for virtual machines with \(a_1^{(i)}\) processors and \(a_2^{(i)}\) units of memory from \(s^{(i)}\) to \(e^{(i)}\). The problem of satisfying all customer requests using a minimum number of servers is DVBP with \(d = 2\). In practice, customers usually have the choice among a few virtual machine flavors.

In the virtual machine assignment scenario, the start and end times of requests are usually not known beforehand and the problem has to be solved online: on arrival, each request is immediately assigned to a bin, reassigning requests to other bins may be allowed or not [6]. Since, in the worst case, any online algorithm for DVBP will use \(\Omega(d^{1+\varepsilon})\) times the optimal number of bins, cloud providers use various online heuristics tailored to their presumed input distributions [see, e.g., 20].

In order to empirically evaluate the quality of these heuristics, computing (close to) optimal solutions to the offline DVBP is desirable. Unfortunately, even when all requests have identical start and end times and \(d = 2\), the best known polynomial-time approximation algorithm for DVBP yields an asymptotic approximation factor of \(1.405 + \varepsilon\) [3]. Unless \(P = \text{NP}\), asymptotic \((1 + \varepsilon)\)-approximations for arbitrarily small \(\varepsilon > 0\) will require superpolynomial time [18, 21].

Data reduction has proven to be a powerful way to cope with superpolynomial problem complexity [1, 2].

**Our results and outline of this work.** We study the potential for polynomial-time data reduction for DVBP.

In Section 2, we introduce basic tools and notation. In Section 3, we show hardness results for data reduction. In Section 4, we present exact data reduction, and in Section 5 — data reduction that guarantees recoverability of \((1 + \varepsilon)\)-approximate solutions for any \(\varepsilon > 0\).

Due to our hardness results, we cannot prove bounds on the data reduction effect. For this reason, the effect is evaluated experimentally in Section 6. We were able to shrink the number of request types in real-world instances to about 2% of their initial number, and the number of requests — to 8% on one data set, and to 20% on another.

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2. Preliminaries

2.1. Parameterized optimization and decision problems

**Definition 2.1.** A decision problem is a subset $\Pi \subseteq \Sigma^*$ for some finite alphabet $\Sigma$. The task is, given an instance $x \in \Sigma^*$, determining whether $x \in \Pi$. If $x \not\in \Pi$, then $x$ is a no-instance. Otherwise, it is a yes-instance.

For optimization problems, we use the terminology of Garey and Johnson [13]. We will only consider minimization problems in our work.

**Definition 2.2.** A combinatorial optimization problem $\Pi$ is a triple $\Pi = (D_\Pi, S_\Pi, m_\Pi)$, where

1. $D_\Pi$ is a set of instances,
2. $S_\Pi$ is a function assigning to each instance $I \in D_\Pi$ a finite set $S_\Pi(I)$ of (feasible) solutions, and
3. $m_\Pi$ is a function assigning a solution cost $m_\Pi(I, \sigma)$ to each feasible solution $\sigma \in S_\Pi(I)$ of an instance $I \in D_\Pi$.

An optimal solution for an instance $I \in D_\Pi$ is a feasible solution $\sigma \in S_\Pi(I)$ minimizing $m_\Pi(I, \sigma)$. Its cost is denoted as $\text{OPT}_\Pi(I)$, where we drop the subscript $\Pi$ when the optimization problem is clear from context.

An $\alpha$-approximate solution for an instance $I$ of a combinatorial optimization problem $\Pi$ is a feasible solution of cost at most $\alpha \cdot \text{OPT}_\Pi(I)$.

Each optimization problem comes with natural associated decision versions and gap-version:

**Definition 2.3.** Let $\Pi$ be a combinatorial optimization problem and $\alpha \geq 1$.

By $\alpha$-gap $\Pi$ we denote the following decision problem: given a number $r$ and an instance $I$ of $\Pi$ such that $\text{OPT}(I) \leq r$ or $\text{OPT}(I) > \alpha r$, it is required to decide whether $\text{OPT}(I) \leq r$.

For $\alpha = 1$, we call $\alpha$-gap $\Pi$ simply the decision version of $\Pi$.

If $\alpha$-gap $\Pi$ is NP-hard, then $\Pi$ cannot be better than $\alpha$-approximated in polynomial-time, unless $P = \text{NP}$.

We use the parameterized complexity notation due to Flum and Grohe [11], as it equally well applicable to decision and optimization problems.

**Definition 2.4.** A parameterization is a polynomial-time computable mapping $\kappa : \Sigma^* \rightarrow \mathbb{N}$ of instances (of decision or optimization problems) to a parameter. For a (decision or optimization) problem $\Pi$ and parameterization $\kappa$, the pair $(\Pi, \kappa)$ is called a parameterized (decision or optimization) problem.

2.2. Data reduction with performance guarantees

A data reduction is a conversion of a problem input into a “similar” but smaller one. There are multiple ways to formalize the term. A well-known notion of data reduction with performance guarantees is kernelization [11]:

**Definition 2.5.** A kernelization for a parameterized decision problem $(\Pi, \kappa)$ is a polynomial-time algorithm that maps any instance $x \in \Sigma^*$ to an instance $x' \in \Sigma^*$ such that

(i) $x \in \Pi \iff x' \in \Pi$, and

(ii) $|x'| \leq g(\kappa(x))$ for some computable function $g$.

We call $x'$ the problem kernel and $g$ its size.

We will also consider approximate data reduction [17]:

**Definition 2.6.** An $\alpha$-approximate preprocessing for a parameterized optimization problem $(\Pi, \kappa)$ consists of two algorithms:

(i) The first algorithm reduces an instance $I$ of $\Pi$ to an instance $I'$ of $\Pi$ in polynomial time,

(ii) The second algorithm turns any $\beta$-approximate solution for $I'$ into an $\alpha\beta$-approximate solution for $I$ in polynomial time.

An $\alpha$-approximate kernelization is an $\alpha$-approximate preprocessing such that $|I'| \leq g(\kappa(I))$ for some computable function $g : \mathbb{N} \rightarrow \mathbb{N}$. We call $g$ the size of the approximate kernel $I'$.

2.3. Hardness of data reduction

To exclude problem kernels of polynomial size, we use $\text{AND-compositions}$ [5].

**Definition 2.7 (AND-composition).** An equivalence relation over $\Sigma^*$ is polynomial if

— there is an algorithm that decides $x \sim y$ in polynomial time for any two instances $x,y \in \Sigma^*$, and

— the number of equivalence classes of $\sim$ over any finite set $S \subseteq \Sigma^*$ is polynomial in $\max_{x \in S} |x|$.

A language $K \subseteq \Sigma^*$ $\text{AND-composes}$ into a parameterized language $(\Pi, \kappa)$ if there is a polynomial-time algorithm, called $\text{AND-composition}$, that, given $s$ instances $x_1, \ldots, x_s$ that are equivalent under some polynomial equivalence relation, outputs an instance $x'$ such that

— $\kappa(x') = \text{poly}(\max_{i=1}^{s} |x_i| + \log s)$,

— $x' \in \Pi$ if and only if $x_i \in K$ for all $i \in \{1, \ldots, s\}$.

**Proposition 2.8** (Drucker [9]). If an NP-hard language $K \subseteq \Sigma^*$ $\text{AND-composes}$ into a parameterized problem $(\Pi, \kappa)$, then there is no polynomial-size problem kernel for $(\Pi, \kappa)$ unless $\text{coNP} \subseteq \text{NP/poly}$.
3. Classification results

In this section, we prove that, unless the polynomial-time hierarchy collapses, DVBP has no problem kernels of size polynomial in $h + \phi$. Our reduction gives good reason to conjecture that there are no $\alpha$-approximate kernels for any $\alpha < 600/599$, either.

3.1. Existence of exponential-size kernels

Before proving the non-existence of problem kernels with size polynomial in $h + \phi$, we prove that, in principle, problem kernels (of exponential size) do exist.

A folklore result from parameterized complexity [12, Theorem 1.4] is that the following theorem gives a problem kernel of size $O(2^h) \subseteq O(h^2)$ for DVBP.

**Theorem 3.1.** The DVBP problem can be solved in $O(k^2 \cdot n \cdot n \log n)$ time.

**Proof.** In $O(n \log n)$ time, we transform the instance so that $s^{(t)}, e^{(t)} \in \{0, \ldots, 2n\}$ for all $t \in \{0, \ldots, n\}$ (see Proposition 4.2 in Section 4.2 for details). We then solve the problem using dynamic programming.

For each $t \in \{0, \ldots, n\}$, let $S_t := \{j \mid s^{(j)} \leq t < e^{(j)}\}$ be the set of requests active at time $t$. For $t \in \{0, \ldots, 2n\}$ and any partition $X_1 \cup X_2 \cup \cdots \cup X_k = S_t$ (here, $\cup$ denotes disjoint unions), consider the predicate $T[t, X_1, X_2, \ldots, X_k]$ that is true if and only if all requests $j$ with $s^{(j)} \geq t$ can be packed into $k$ bins so that requests $X_i$ are packed into bin $i$ for $i \in \{1, \ldots, k\}$. Then $T[2n + 1, X_1, X_2, \ldots, X_k]$ is true for any choice of the sets $X_1 \cup \cdots \cup X_k = S_{2n+1} = \emptyset$.

Moreover, $T[t, X_1, X_2, \ldots, X_k]$ is true if and only if there is a partition $X'_1 \cup X'_2 \cup \cdots \cup X'_k = S_{2n+1}$, such that $T[t + 1, X'_1, X'_2, \ldots, X'_k]$ is true and $X'_i \cap S_t \cap S_{n+1} = X'_i \cap S_t \cap S_{n+1}$ for each $i \in \{1, \ldots, k\}$. That is, the partitions $[X'_i]_{i=1}^k$ and $[X_i]_{i=1}^k$ put the requests in $S_{n+1}$ into the same bins.

Thus, each of the $O(k^{n})$ values of $T$ can be computed in $O(k^2)$ time by iterating over all possible partitions $[X'_i]_{i=1}^k$, yielding a total running time of $O(k^{n+2})$. The answer can be found in $T[0, 0, 0, \ldots, 0]$. □

3.2. Non-existence of polynomial-size kernels

We now prove that, unless the polynomial-time hierarchy collapses, DVBP has no problem kernels with size polynomial even in $h + \phi$. Indeed, not even 600/599-gap DVBP has such a kernel:

**Theorem 3.2.** $\alpha$-gap DVBP does not have a problem kernel of size polynomial in $h + \phi$, unless the polynomial hierarchy collapses, for any $\alpha \leq 600/599$.

Note that the statement of Theorem 3.2 holds for any parameter bounded from above by $h + \phi$. Moreover, the theorem leads us to the following conjecture.

**Conjecture 3.3.** DVBP has no $\alpha$-approximate problem kernels of size polynomial in $h + \phi$, for any $\alpha < 600/599$.

Unfortunately, we cannot formally connect this conjecture, for example, to a collapse of the polynomial-time hierarchy, the main obstacle being the final open question in the work of Lokshtanov et al. [17].

In order to prove Theorem 3.2, we prove that the following problem AND-composes into DVBP.

**Problem 3.4 (2D vector bin packing (2D-VBP)).**

**Input:** Requests $a^{(1)}, \ldots, a^{(n)} \in (0, 1]^2$.

**Find:** A partition of $[1, \ldots, n]$ into bins $B_1, \ldots, B_k$ with minimum $k$ such that, for each bin $B_j$, it holds that (component-wise)

$$\sum_{i \in B_j} a^{(i)} \leq b.$$ 

Ray [18, Theorem 6] proved that 600/599-gap 2D-VBP does not have polynomial-time algorithms with asymptotic approximation ratio better than 600/599. In fact, he shows that 600/599-gap 2D-VBP is NP-hard. We show that NP-hardness also holds for the following problem variant, in which all numbers are integer and bounded by a polynomial in the number of items.

**Problem 3.5 (Poly-weight 2D-VBP).**

**Input:** Bin capacity $b \in \mathbb{N}^2$ and requests $a^{(1)}, \ldots, a^{(n)} \in \mathbb{N}^2$, all bounded by a polynomial in $n$.

**Find:** A partition of $[1, \ldots, n]$ into bins $B_1, \ldots, B_k$ with minimum $k$ such that, for each bin $B_j$, it holds that (component-wise)

$$\sum_{i \in B_j} a^{(i)} \leq b.$$

**Lemma 3.6.** The 600/599-gap version of Problem 3.5 is NP-hard.

**Proof.** The NP-hard maximum 3-dimensional matching problem (MAX-3-DM) is, given three sets $X = \{x_1, x_2, \ldots, x_q\}$, $Y = \{y_1, y_2, \ldots, y_q\}$, and $Z = \{z_1, z_2, \ldots, z_q\}$, and a set of triples $T \subseteq X \times Y \times Z$, to find a set $T' \subseteq T$ of maximum cardinality such that each element of $X$, $Y$, and $Z$ occurs in at most one triple in $T'$. Ray [18] showed a reduction of MAX-3-DM to 600/599-gap 2D-VBP. We modify his reduction so that it outputs instances of Problem 3.5 instead, without changing optimum number of bins.

To this end, we first briefly describe the reduction, omitting the correctness proof. Let $r := 64q$, and, for each $x_i \in X$ let $x'_i := ir + 1$, for each $y_j \in Y$ let $y'_j := ir^2 + 2$, for each $z_k \in Z$ let $z'_k := ir^3 + 4$, and for each triple $(x_i, y_j, z_k) \in T$, let $r_{i, j, k} = r^3 - kr^2 - jr^2 - ir + 8$. Call the set of all these integers $U'$. The key feature of this construction is that the four numbers $x'_i, y'_j, z'_k$...
and \( t'_{i,j,k} \) sum up to exactly \( b := r^4 + 15 \). The final 2D-VBP instance consists of the requests
\[
\left( 1 + \frac{\alpha'}{5b} \cdot \frac{3}{10} - \frac{\alpha'}{5b} \right) \text{ for each } \alpha' \in U'
\]
and at most \(|T| + 3q\) (the exact number is a technical detail) additional dummy requests of the form
\[
\left( \frac{3}{5} \cdot \frac{3}{5} \right)
\]
The bin capacity, by definition of 2D-VBP, can be considered as \((1,1)\). The key observation here is that a bin can fit four requests if and only if the requests were built from the integers \( x'_i, y'_i, z'_i, \) and \( t'_{i,j,k} \), or, in other words, if the requests correspond to some triple in \( T \).

All requests are vectors of rational numbers whose denominators have a common multiple
\[
D := 10 \cdot (r^4 + 15) = 10 \cdot ((64q)^4 + 15)
\]
and whose numerators are all bounded by a polynomial of \( q \). Multiplying all requests by \( D \) and setting a bin capacity of \((D,D)\), we get an equivalent instance of Problem 3.5 in which all numbers are natural and bounded by a polynomial in \( q \). Since the number of requests in it is at least \( 3q \), all numbers are bounded polynomially in the number \( n \) of requests. Thus, Ray’s result holds for Problem 3.5.

Lemma 3.6 makes Problem 3.5 fundamentally different from standard Bin Packing, whose 3/2-gap version is NP-hard, but only weakly so — if the item sizes are bounded by a polynomial in the number of items, standard Bin Packing is polynomial-time solvable.

The polynomial bound on the request sizes plays a crucial role in bounding the number of request flavors \( \phi \), in the following result:

**Lemma 3.7.** The \( \alpha \)-gap version of Problem 3.5 AND-composes into \( \alpha \)-gap DVBP parameterized by \( h + \phi \).

**Proof.** Assume instances \( I_1, I_2, \ldots, I_s \) of the \( \alpha \)-gap version of Problem 3.5. Without loss of generality, we may assume that each of them consists of the same number \( n \) of requests, each is asking for the same number \( r \) of bins and each instance’s bin size is \( b \) (since this is a polynomial equivalence relation). As a result, the entries of all vectors are bounded by \( \text{poly}(n) \).

One now can create a DVBP instance \( I \) consisting of all requests of all instances \( I_i \) for \( i \in \{1, \ldots, s\} \), assigning the requests of instance \( I_i \) the start time \( 2i - 1 \) and the end time \( 2i \). Obviously, the created DVBP instance \( I \) can fit into \( r \) bins if and only if each of the instances \( I_i \) can fit into \( r \) bins. Moreover, if at least one of the instances \( I_i \) requires more than \( r \) bins, then it requires at least \( ar \) bins, meaning that \( I \) requires at least \( ar \) bins.

Also note that the maximum number of requests active at any time in \( I \) is \( n \) and that the number of flavors in \( I \) is bounded from above by \( \text{poly}(n) \). That is, \( h + \phi \leq \text{poly}(n) \leq \text{poly}(\max_{i=1}^s |I_i|) + \log n \); we thus built a valid AND-composition.

Finally, Theorem 3.2 now follows from Proposition 2.8 and Lemma 3.7 and the NP-hardness of the \( \alpha \)-gap variant of Problem 3.5 for all \( \alpha \leq 600/599 \) (Lemma 3.6).

### 4. Exact data reduction

Due to the results in the previous section, we do not expect \( (1 + \epsilon) \)-approximate preprocessing for DVBP for arbitrarily small \( \epsilon > 0 \) with provable a priori size bounds of the output. This, however, does not prevent us from designing data reduction algorithms that show good a posterior results in experiments.

In this section, we describe a simple data reduction rule that maintains optimality of solutions.

#### 4.1. DVBP with multiplicities

Motivated by the following theorem, which we prove in this section, our first data reduction approach will focus on lowering the number \( \tau \) of request types.

**Theorem 4.1.** DVBP with \( k \) bins and \( \tau \) request types is solvable in \( 2^{O(kr \log \tau)} \cdot \text{poly}(n) \) time.

To prove the theorem, we follow an approach of Fellows et al. [10] for deriving algorithms for packing problems parameterized by the “number of numbers”. Specifically, we modify an ILP model for DVBP of Dell’Amico et al. [8, Section 3.1], replacing Boolean variables by integer variables so as to allow for requests with multiplicities. The number of variables in the ILP will be \( O(\tau k) \), so that Theorem 4.1 immediately follows from a result of Lenstra [16] and Kannan [15].

To state the ILP, let \( T := \max_{i=1}^s |\ell^{(i)}| \) be the latest time that any request ends, \( S := \{ j \mid \ell^{(i)} \leq t < \ell^{(j)} \} \) be the requests active at time \( t \in \{0, \ldots, T\} \), and let \( n^{(i)} \) be the number of requests of type \( i, 1 \leq i \leq \tau \). The model involves the following variables:

\[ x_{ij} \text{ for } 1 \leq i \leq \tau \text{ and } 1 \leq j \leq k \]

\[ y_j \text{ for } 1 \leq j \leq k \]

\[ n^{(i)} \]

\[ n^{(i)}y_j, \text{ for } 1 \leq i \leq \tau, 1 \leq j \leq k \]

Then, DVBP is modeled using the following ILP.

\[
\min \sum_{j=1}^k y_j \quad \text{s.t.} \quad \sum_{j=1}^k x_{ij} = n^{(i)} \quad 1 \leq i \leq \tau, \quad (1)
\]

\[
x_{ij} \leq n^{(i)}y_j, \quad 1 \leq i \leq \tau, 1 \leq j \leq k \quad (2)
\]
\[ \sum_{i \in S_j} x_{ij} d^{(i)} \leq y_j b, \quad 1 \leq t \leq T, 1 \leq j \leq \tau \]  
\[ y_j \in \{0, 1\}, \quad 1 \leq j \leq k \]  
\[ x_{ij} \in \mathbb{N}, \quad 1 \leq i \leq \tau, 1 \leq j \leq j. \]

Herein, constraint (1) ensures that each request is assigned to some bin, constraint (2) makes sure that bin \( j \) is counted as “used” whenever some request \( i \) is assigned to it, and constraint (3) ensures that the bin capacity \( b \) is not exceeded at any time instant \( t \).

### 4.2. Time compression

The number of variables in the above ILP is \( O(\tau k) \), whereas the number of constraints is \( O(Tr + \tau k) \). It is therefore desirable to reduce \( \tau \) and \( T \). Since requests of the same flavor and same start and end times are of the same type, both reduction of \( \tau \) and \( T \) is achieved by the following simple approach.

**Proposition 4.2** ([4]). In \( O(n \log n) \) time, the start and end points of all requests can be moved to an interval \([1, \ldots, T]\) with minimum \( T \) maintaining all pairwise intersections of request intervals.

In Section 6, we will see that this data reduction alone reduces the number of request types by about one third in real problem instances.

### 5. \( (1 + \epsilon) \)-approximate data reduction

In this section, we show a \((1 + \epsilon)\)-approximate data reduction algorithm for DVBP, that is, the cost of solutions may increase at most \((1 + \epsilon)\) times.

In view of the hardness results in Section 3, we do not expect to prove meaningful a priori size bounds of the output. Indeed, as we will see, maximizing the data reduction effect of the algorithm is an NP-hard subproblem itself. Therefore, we first describe the high-level algorithm and then, in subsequent subsections, describe heuristics for its effective execution. The data reduction effect will be empirically evaluated in Section 6.

#### 5.1. A \((1 + \epsilon)\)-approximate data reduction rule

Let \( L \) be a lower bound on the number of bins required to accommodate all requests. For example, assuming that all request start and end times are in \([1, \ldots, 2n]\) by Proposition 4.2, one can take

\[ S_i := \{ j \mid \delta^{(i)} \leq t < \epsilon^{(i)} \}, \]

\[ L := \max_{\mathcal{R} \in \{1, \ldots, n\}} \max_{\mathcal{D} \in \{1, \ldots, d\}} \sum_{j \in S_i} \frac{d^{(i)} j}{b_j}. \]  

**Reduction Rule 5.1.** Delete an arbitrary set of requests that can be packed into \([\epsilon L]\) bins.

**Theorem 5.2.** Reduction Rule 5.1 is a \((1 + \epsilon)\)-preprocessing for DVBP.

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**Algorithm 1:** Greedy packing into one bin.

**Input:** A bin capacity \( b \in \mathbb{N}^d \), requests \( d^{(1)}, \ldots, d^{(n)} \in \mathbb{N}^d \) with start times \( s^{(1)}, \ldots, s^{(n)} \in [1, \ldots, T] \), end times \( e^{(1)}, \ldots, e^{(n)} \in [1, \ldots, T] \), and priorities \( f : [1, \ldots, n] \mapsto Q \).

**Result:** Indices of packed requests.

1. \( A \leftarrow \emptyset; \)
2. \( S \leftarrow [h, h, \ldots, h] \text{ of size } T; \)
3. \( \text{for } d^{(i)} \text{ in non-increasing order of } f(i) \text{ do} \)
4. \( \quad \text{if } d^{(i)} \leq S[t] \text{ for } t \in [s^{(i)}, \ldots, e^{(i)} - 1] \text{ then} \)
5. \( \quad \quad A \leftarrow A \cup \{i\}; \)
6. \( \quad \text{for } t \in [s^{(i)}, \ldots, e^{(i)} - 1] \text{ do} \)
7. \( \quad \quad S[t] \leftarrow S[t] - d^{(i)}; \)
8. \( \text{return } A; \)

**Proof.** Let \( I \) be the DVBP instance before and \( I' \) be the DVBP instance after applying Reduction Rule 5.1. If all requests of \( I \) can be packed into \( k \) bins, then the requests of \( I' \) can as well. Thus \( \text{OPT}(I') \leq \text{OPT}(I) \). If all requests of \( I' \) can be packed into \( k \) bins, then the requests of \( I \) can be packed into \( k + |\epsilon L| \) bins. Thus, any \( \alpha \)-approximate solution for \( I' \) can be turned into an \( \alpha(1 + \epsilon) \)-approximate solution for \( I \) since

\[ a\text{OPT}(I') + \epsilon L \leq a\text{OPT}(I) + \epsilon\text{OPT}(I) \leq a(1 + \epsilon)\text{OPT}(I). \]

In order to delete the maximum number of requests using Reduction Rule 5.1, one has to find a maximum number of requests into \([\epsilon L]\) bins, which is an NP-hard task. One possible implementation of Reduction Rule 5.1 is solving this task heuristically. Indeed, our experiments in Section 6 show that, solving the task exactly would not significantly increase the effect of Reduction Rule 5.1 on our real-world data.

### 5.2. Greedily packing \( \epsilon L \) bins

In order to pack \( \epsilon L \) bins, we repeatedly apply a greedy algorithm for packing one bin.

In order to pack one bin, we assign each request a priority, then iterate over all not yet packed requests by non-increasing priority and, if the request still fits into the bin, we pack it. In detail, the procedure is described in Algorithm 1, where the array \( S \) maintains the available bin space at each time instant and \( A \) is the set of requests packed into the bin.

The priorities are chosen as follows. Since the goal of the algorithm is to pack as many requests as possible, it makes sense to first pack those requests that require less space and block less time instants. A convenient value to use as an indicator of a request’s small size is the number of requests of the same flavor as \( d^{(i)} \) that can
fit into one bin. Thus, a canonical priority assignment to each request \( i \) is

\[
\alpha(i) := \min_{j \in \{1, \ldots, d\}} \left| \frac{b_j}{a^{(i)}_j} \right|.
\]

From \( \alpha(i) \), we also derive the following priorities:

\[
f_1(i) := \alpha(i) + \frac{1}{2(e^{(i)} - s^{(i)})},
\]

which uses the time span as a tie-breaker for \( \alpha(i) \) and places the shorter spanned requests earlier; and

\[
f_2(i) := \alpha(i) \cdot \frac{T}{e^{(i)} - s^{(i)}},
\]

which is an upper bound on the number of requests of the same flavor and life time as \( a^{(i)} \) that can be placed over all time instants.

We found that \( f_2 \) works the best among \( \alpha, f_1, f_2 \), which we evaluated as follows.

### 5.3. Quality estimation of greedy packing

In the following, we describe how to measure the effectiveness of our greedy packing algorithm, thus ultimately answering the question of how effectively we use Reduction Rule 5.1 and whether the effect of Reduction Rule 5.1 can be significantly increased by solving the problem of packing into \( eL \) in a more sophisticated way, or even optimally.

In order to estimate the effect of Reduction Rule 5.1 in comparison to its potential effect, we use two utilization values comparing the number \( n \) of requests in the input instance to the number \( n' \) of requests in the output instance (that is, \( n - n' \) requests are deleted by Reduction Rule 5.1). The first value is

\[
R := \frac{n - n'}{U} \leq 1,
\]

where \( U \) is an upper bound on the number of requests removable by Reduction Rule 5.1. The closer \( R \) is to 1, the closer the data reduction effect is to the maximum possible. The second value is

\[
K := \frac{n'}{n - U} \geq 1,
\]

where \( n - U \) is a lower bound on the number of remaining requests \( n' \). Low values of \( K \) mean that the size of the reduced instance is close to what it could be if we packed the \( \lfloor eL \rfloor \) bins optimally.

After reducing an instance, we can evaluate the effect of Reduction Rule 5.1 quantified by \( R \) and \( K \), using the upper bound \( U \). One way to obtain \( U \) is to apply Algorithm 2 with \( k = \lfloor eL \rfloor \); it works as follows.

**Algorithm 2: Computation of upper bound \( U \)**

**Input**: A bin capacity \( b \in \mathbb{N}^d \), requests \( \alpha^{(1)}, \ldots, \alpha^{(n)} \in \mathbb{N}^d \) with start times \( s^{(1)}, \ldots, s^{(n)} \in \{1, \ldots, T\} \), end times \( e^{(1)}, \ldots, e^{(n)} \in \{1, \ldots, T\} \), and the number of bins \( k \in \mathbb{N} \).

**Result**: An upper bound \( U \).

1. \( S \leftarrow \text{size-d-array of empty lists} \)
2. for any request type \((\alpha^{(i)}, s^{(i)}, e^{(i)})\) of any multiplicity \( m \) do
3. \( \ p \leftarrow \min\{m\} \cup \left\{ k : \frac{b_j}{\alpha^{(i)}_j} \right\}_{j \in \{1, \ldots, d\}} \)
4. for \( j \in \{1, \ldots, d\} \) do
5. append \( p \) times \( \alpha^{(i)} \) to \( S[j] \)
6. for \( j \in \{1, \ldots, d\} \) do
7. sort \( S[j] \) in non-decreasing order
8. \( U_j \leftarrow \max \left\{ q \leq n \sum_{i=1}^{q} S[j][i] \leq b_j \right\} \)
9. return \( \{U_j | j \in \{1, \ldots, d\}\} \)

that fit into \( k \) bins. In line 5, for each \( j \in \{1, \ldots, d\} \), it builds a list \( S[j] \) containing \( j \)-th components of this request time \( p \) times, because the other \( m - p \) requests of the same type will not fit into the bins anyway. In line 8, for each \( j \in \{1, \ldots, d\} \), it computes an upper bound \( U_j \) by greedily packing the requests with the smallest value in the \( j \)-th component first, ignoring the other components. Finally, in line 8, the smallest of the upper bounds is returned.

### 6. Experiments

We present an experimental evaluation of the effect of our data reduction on real-world instances for a virtual machine scheduling problem (Example 1.3).

**Data sets.** We evaluated our data reduction approach on six data sets with similar results. However, since four of the data sets are confidential, in detail we can present results here only for two openly available data sets.

The first one is “Huawei-East-1” released by Huawei [19]. We denote this data set \( \text{Huawei} \). In this data set, \( d = 2 \), that is, each request has two resource requirements: the number of requested CPU cores and the amount of allocated RAM. As suggested by its authors, we simulated CPU sharing for small virtual machines (with flavors 2U4G, 4U8G, 8U16G, 1U2G, 4U16G, 1U1G, 2U8G, 8U32G and 1U4G, where machine XUYG has X CPU cores and Y GB of RAM) by dividing their CPU demand by 3. We set the bin size

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[1] https://github.com/huawei-cld/VM-placement-dataset
Table 1: Data reduction effect of Proposition 4.2 and Reduction Rule 5.1 with \( \epsilon = 0.05 \). Herein, \( L \) is the lower bound (4), \( T \) is the number of time instants, \( n \) is the number of requests, \( \tau \) is the number of types, \( R \) and \( K \) are the utilization measures (5) and (6), respectively.

| data set | \( L \)   | \( T \)   | \( n \)   | \( \tau \) | \( R \) | \( K \) |
|----------|-----------|-----------|-----------|-----------|--------|--------|
| Huawei   | initial   | 814       | 125 430   | 111 050   |        |        |
|          | after Prop.| 29 347    | 125 430   | 78 010    |        |        |
|          | Rule 5.1   | 356       | 10 079    | 1 798     | 0.969  | 1.576  |
| Azure    | initial   | 116 864   | 5 559 800 | 3 792 136 |        |        |
|          | after Prop.| 695 408   | 5 559 800 | 2 271 582 |        |        |
|          | Rule 5.1   | 20 226    | 1 069 118 | 81 357    | 0.946  | 1.319  |

Figure 1: Data reduction effect of Proposition 4.2 and Reduction Rule 5.1 for \( \epsilon \in [0, 0.2] \) with step size 0.01.

to 40 CPU cores and 90 GB of RAM, following the data set authors’ experimental setup. All arithmetic calculations for this data set are carried out with 64-bit integers.

The second data set, called Azure, is released by Microsoft and is called “Azure Traces for Packing 2020” [14]. This data set has \( d = 5 \), where each request has requirements on the number of CPU cores, amount of RAM, HDD and SSD space, and network bandwidth. In the Azure dataset, each resource requirement is a fractional number in \([0, 1]\) equal to the fraction of the available resource on the servers, so the bin size is set to \((1, 1, 1, 1, 1)\). All computations for this data set are done in double precision floating point arithmetic.

Reduction Rule 5.1 with \( \epsilon = 0.05 \). Table 1 shows the results. When computing the optimal solution lower bound \( L \) according to (4), Reduction Rule 5.1 tries to pack into \( \lfloor \epsilon L \rfloor \) bins for Huawei and into \( \lfloor \epsilon L \rfloor = 40 \) bins for Huawei and into \( \lfloor \epsilon L \rfloor = 5843 \) bins for Azure.

The data reduction given by Proposition 4.2 reduces the number of time instants by about 6 times. It does not change the number of requests, yet reduces the number of request types by about one third, since after time compression, many requests of the same flavor have coinciding start and end times and are thus of the same type.

After Reduction Rule 5.1 the number of requests is decreased significantly: 5.2 times for Azure and 12.4 times for Huawei. And even more significant is the change in the number of request types, which, for each data set, fell about 50 times.

A natural question is whether the effect of Reduction Rule 5.1 can be increased by packing the \( \lfloor \epsilon L \rfloor \) bins for deletion better. The table presents the resource utilization measures \( R \) and \( K \) described in Section 5.3. The value of \( R \) shows that our implementation of Reduction Rule 5.1 removes at least 94% of all requests that it can remove in principle. However, judging by \( K \), Reduction
tion Rule 5.1 potentially leaves 57% more requests in Huawei than there have to be and 32% more requests in Azure than have to be. On the one hand, this means that there might still be a little room for improvement, yet certainly not by an order of magnitude. On the other hand, the number of requests that have to remain is only a lower bound. Thus, it might even be that the $\varepsilon L$ bins are already packed optimally.

Finally, the choice of $\varepsilon$ allows for a trade-off of solution quality versus output size of the data reduction, illustrated in Figure 1. In both cases, we can observe data reduction by an order of magnitude for $\varepsilon = 0.02$. The number of request types shrinks by another order of magnitude for $\varepsilon \in [0.08, 0.12]$.

Good candidates for compromise are $\varepsilon = 0.1$ on the Huawei data set and $\varepsilon = 0.16$ on the Azure data set, if that would allow for solving the problem and such an error is permissible.

7. Conclusion

We have studied the potential of data reduction with performance guarantees for DVBP. While we have shown obstacles for proving size bounds of instances after data reduction, we proved guarantees on the solution quality.

Despite our data reduction rules being very simple and coming without size bounds, they proved to be very effective in experiments with real data from Microsoft Azure and Huawei Cloud, shrinking problem instances by orders of magnitude.

Unfortunately, for two the two open sets, the number of request types after data reduction is still out of reach for algorithms with running time exponential in the number of request types, for example, for the ILP from Theorem 4.1.

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