Random Time Forward Starting Options

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March 2015

Abstract

We introduce a natural generalization of the forward-starting options, first discussed by M. Rubinstein ([28]). The main feature of the contract presented here is that the strike-determination time is not fixed ex-ante, but allowed to be random, usually related to the occurrence of some event, either of financial nature or not. We will call these options Random Time Forward Starting (RTFS).

We show that, under an appropriate “martingale preserving” hypothesis, we can exhibit arbitrage free prices, which can be explicitly computed in many classical market models, at least under independence between the random time and the assets’ prices. Practical implementations of the pricing methodologies are also provided. Finally a credit value adjustment formula for these OTC options is computed for the unilateral counterparty credit risk.

Keywords: Random times, forward-starting options, CVA.

JEL Classification: G13

1 Introduction

Forward-starting options are path dependent put/call financial contracts characterized by having a strike price expressed in terms of a pre-specified percentage $\alpha$ of the asset price taken at a contractually fixed intermediate date $u \in (t, T]$, $T$ being the option maturity. The time $u$ is known as strike-determination time. The payoff of a forward starting call is therefore

$$(S_T - \alpha S_u)^+.$$ (1)

These products represent the fundamental component of the so-called cliquets (see [18]), which are indeed equivalent to a series of forward starting at-the-money options, activated along a sequence of intermediate dates, upon payment of an initial premium. Cliquets are often employed to buy protection against downside risk, though preserving an upside potential, for instance in pension plans in order to hedge the guarantees attached to embedded equity linked products.

Wilmott in [29] showed that these products are particularly sensitive to the model that one chooses for the dynamics of the underlying’s price.

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In this paper we study a generalization of forward starting options allowing for random strike-determination times. That is to say, we hypothesize that we are making the forward starting option worth, if some future random event happens during the life of the contract with positive probability. The value of the asset at this random time is then compared at maturity with the final value of the underlying, so the call’s payoff (1) is replaced by

$$(S_T - \alpha S_{\tau \wedge T})^+, \quad (2)$$

for some appropriate random time $\tau$, where we denoted by $x \wedge y = \min(x, y)$. The pricing methodology we will present is an extension of Rubinstein’s approach (see [28]), and it can be applied to several existing option models.

It is easy to imagine a promising application of this type of contracts in several situations, as we show with the following examples.

- A first example with a speculative goal, could be represented by a call written (with $\alpha = 1$) on the Manchester United stock (NYSE:MANU), starting the beginning day of the Premier League (PL) soccer tournament and ending the closing day of the tournament. The call is triggered by the first time, $\tau$, the United will eventually lead the tournament with at least three points more than all other teams. If this event happens, then the United will increase its chances to take part to the next Champions League (CL) tournament, an event which generates substantial additional cash-flows. If, at the end of PL tournament, the United will get the right to play CL, the share value $S_T$ of MANU at that date will probably be larger than $S_\tau$ and an investor in the call contract could pocket the difference.

- Another possible context we may imagine is when a firm expects an adverse market scenario that might cause a downgrading of the firm’s rating in the near future. So the firm’s owner offers a contract of this type to a smart manager (hired in the past), as an incentive to convince him/her to remain within the firm and to work to improve the firm’s position before maturity. As an example, the firm’s owner offers this contract to the manager on January, 1st 2015 with maturity $T = 1$ year (with $\alpha = 1$). If the event $\{\tau < T\}$ happens, for instance on June, 6th 2015, then the manager has time until maturity to make the firm’s value rise again above $S_\tau$ and pocket the payoff, otherwise no reward is gained. The manager is therefore interested in assessing the correct current value of the incentive.

In this context, it is also interesting to consider the counterparty credit risk aspects of this type of contract from the point of view of the manager: if the firm (the counterparty) does not recover from the adverse market scenario, it might default a short time after the downgrading event or default could even happen before downgrading. This leads to the problem of computing unilateral Credit Value Adjustment (CVA) for this kind of option, since the manager needs to evaluate this further risk of the contract in order to assess correctly the value of the firm’s offer.

- Lastly, it is also possible to consider contracts depending on more than one single random event: for example a cliquet call option could be triggered by the first time the underlying asset reaches a pre-specified upper barrier $K_1$ until the asset reaches, if before maturity $T$, a new barrier level $K_2 > K_1$, at that time a second call option with strike $K_2$ and expiring at maturity gets activated. In general, depending on the characteristics of the financial
product, one can imagine to construct cliquet type options along some increasing sequence of random times.

The literature on classical forward starting options is wide in terms of underlying’s dynamics model and evaluation techniques. We shortly review the main contributions, since we will use these results as starting point in our analysis.

Rubinstein (1991) (see [28]) was the first to consider the pricing problem of a forward-start option in the Black and Scholes model.

Later, valuation of forward starting options in stochastic volatility models was addressed by several authors. Lucic (2003) (see [24]) wrote an evaluation formula by employing the asset’s price at determination time as numeraire, transforming those options in plain vanilla ones, giving a closed formula for the Heston’s model. Kruse and Nögel (2005) (see [25]), followed the same approach, but they developed a pricing formula exploiting the distributional properties of the Heston’s model; they remarked the importance of such an extension (w.r.t. the Black-Scholes environment) since forward starting options are particularly sensitive to volatility changes.

Further, Guo and Hung (2008) (see [19]) derived quasi-analytic pricing formulas in general stochastic volatility/stochastic interest rates models, presenting jumps in both the asset’s price and the volatility. This was important, since, in contrast to the plain vanilla case, the value of a forward-starting option may not always increase with maturity. It depends on the current term structure of interest rates. In the same context, Ahlip and Rutkowski (2009) (see [1]) chose a Heston’s stochastic volatility model and a CIR dynamics for the interest rates, with volatility and short rate both correlated with the dynamics of the stock returns. The main result is an analytic formula for the call case, derived by using the probabilistic approach combined with the Fourier inversion technique, as developed in Carr and Madan (1999) (see [10]). Van Haastrecht and Pelsser (2011), ([20]), exploiting Fourier inversion of characteristic functions, developed a quantitative analysis of the pricing of forward starting options under stochastic volatility and stochastic interest rates (with Ornstein-Uhlenbeck dynamics), confirming that these not only depend on future smiles, but are also very sensitive to model specifications such as volatility, interest rate and correlation movements, concluding that it is of crucial importance to take all these factors into account for a correct valuation and risk management of these securities. More recently, Ramponi (2012) (see [27]) extended this Fourier analysis to regime switching models for the asset price.

Finally, in 2006 Chicago Board Options Exchange introduced options on its implied volatility index (VIX), enhancing the interest in forward starting options to forecast future volatility. Albanese, Lo and Mijatovic ([2]) proposed a spectral method to evaluate numerically the joint distribution between the stock price and its realized variance, which gave a way of pricing consistently European, general accrued variance payoffs, forward-starting and VIX options.

To evaluate our generalization of FS options to random determination time, we exploit the techniques developed in a number of papers (refer to [5] and to [6] for a general exposition) to deal with default time in credit derivatives. The main tool we employ is the so called assumption (H), that guarantees that all martingales remain as such when adding the information generated by the default time. In other words, this means that the no-arbitrage structure is not altered by the introduction of the random time. This leads to the so called key lemma, that allows to switch from the global information (generated by the asset’s price and the random time) to the information generated only by the observable prices.
Following the intensity based approach, we give a general expression of the pricing formula either in conditions of independence or not. Under independence between the asset’s price and the random time, we specialize and explicit our prices in several computable models (Black & Scholes, Heston, Variance Gamma) and we suggest an affine dependency structure for the intensity when the independence assumption is dropped, showing it is still possible to recover a partially easily computable formula.

Finally, keeping in mind the second example presented above, we focus on computing the Credit Value Adjustment, due to counterparty risk, required by these products, when in presence of a further default time (for a general reference see Brigo and Chourdakis (2009), [7]).

The paper is organized as follows. Section 2 gives the set up, in Section 3 we present a general approach to the evaluation of this kind of product, in Section 4 we develop some explicitly computable cases under independence. Section 5 provides some numerical implementation of the derived pricing formulas, finally Section 6 treats briefly the counterparty risk implied by this product, when a default time for the issuer might occur.

2 The set up

We consider a time interval $[0, T]$, with $T$ denoting the maturity of the contracts. We assume that the market is made up of a single asset and a bond living in a probability space $(\tilde{\Omega}, \mathcal{F}, \mathbb{P})$, with a filtration $\{\mathcal{F}_s\}_{s \geq 0}$, made complete and right continuous to satisfy the usual hypotheses (see [26]). Moreover, we assume we are in absence of arbitrage opportunity and that $\mathbb{P}$ is a risk neutral measure selected by some criterion.

Before formalizing mathematically the new contract previously described, we need to recall briefly the evaluation of classical forward-start options.

Denoting by $B(t, T)$ the discounting process, the price of a forward-start call with percentage $\alpha$ and strike-determination time $u > t$ is given by

$$c(t, u, T) = \mathbb{E}^\mathbb{P}[B(t, T)(S_T - \alpha S_u)^+]_\mathcal{F}_t, \quad 0 \leq t \leq T.$$  \hfill (3)

In market models where the price of a plain vanilla call option is represented by a deterministic homogeneous function of degree 1 in two spatial variables $(x, y)$, that is

$$call(t, \gamma x, \gamma y, T) = \gamma call(t, x, y, T), \quad \forall \gamma, x, y > 0$$

applied to the given pair $(S_t, K)$, the price of a classical (call) forward-start option is

$$c(t, u, T) = \mathbb{E}^\mathbb{P}[B(t, u)\mathbb{E}^\mathbb{P}[B(u, T)(S_T - \alpha S_u)^+]_\mathcal{F}_u]_\mathcal{F}_t = \mathbb{E}^\mathbb{P}[B(t, u)call(u, S_u, S_u\alpha, T)]_\mathcal{F}_t$$

$$= \mathbb{E}^\mathbb{P}[B(t, u)call(u, \gamma 1, \gamma \alpha, T)]_\gamma=S_u|_\mathcal{F}_u = \mathbb{E}^\mathbb{P}[B(t, u)(\gamma call(u, 1, \alpha, T))]_\gamma=S_u|_\mathcal{F}_u$$

$$= \mathbb{E}^\mathbb{P}[B(t, u)S_u call(u, 1, \alpha, T)]_\mathcal{F}_t.$$ \hfill (4)

Scale-invariant models (see [3]) certainly hold this property. Furthermore, if the call price is a deterministic function of the current price of the underlying and it does not include any additional stochastic factor, we may conclude that the price of the forward-start option verifies

$$\mathbb{E}^\mathbb{P}[B(t, u)S_u call(u, 1, \alpha, T)]_\mathcal{F}_t = \mathbb{E}^\mathbb{P}[B(t, u)S_u|_\mathcal{F}_t] call(u, 1, \alpha, T) = S_t call(u, 1, \alpha, T),$$ \hfill (5)
where in the last equality we used the martingale property of the discounted asset price. Notice that in such a case the portfolio having \( \text{call}(u, 1, \alpha, T) \) units of the stock “replicates” the \( t \)-value of the forward-start option.

Sometimes an analogous result may be achieved also in presence of stochastic factors via change of numeraire arriving at a formula of the type

\[
E^P[B(t, u)S_u \text{call}(u, 1, \alpha, T)|\mathcal{F}_t] = S_t E^P[\text{call}(u, 1, \alpha, T)|\mathcal{F}_t]
\]

for some proper equivalent measure \( \bar{P} \).

We want to generalize these contracts allowing the strike-determination time to be random, thus we need to consider a market model (and hence a probability space) where the asset price \( S \) lives together with a random time \( \tau \). We therefore assume that there exists a \( \sigma \)-algebra \( \mathcal{G} \) containing \( \mathcal{F} \) and a filtration \( \{\mathcal{G}_s\}_{s \geq 0} \subseteq \mathcal{G} \), satisfying the usual hypotheses, that makes \( S_t \) and \( H_t = 1_{\{\tau \leq t\}} \) adapted processes (that is to say that \( \tau \) is stopping time w.r.t. \( \{\mathcal{G}_s\}_{s \geq 0} \)) and that we may extend the probability \( P \) to a new probability measure \( Q \) on the measure space \((\Omega, \mathcal{G})\). Hence, referring to the previous notation, we have \( \mathcal{F}_t \subseteq \mathcal{G}_t \), for all \( t \geq 0 \) and from now on we will assume \( \mathcal{G}_t = \mathcal{F}_t \lor \mathcal{H}_t \), where \( \mathcal{H}_t = \sigma(H_s : s \leq t) \). We recall that the filtration stopped at the stopping time \( \tau \) will be denoted \( \mathcal{G}_\tau \) and it means

\[
\mathcal{G}_\tau = \{A \in \mathcal{G} : A \cap \{\tau \leq t\} \in \mathcal{G}_t, \forall t \geq 0\}.
\]

Besides we introduce the classical and fundamental hypothesis

\[\text{(H)}\]

Every \( \mathcal{F}_t \)-martingale remains a \( \mathcal{G}_t \)-martingale.

This assumption, known as the H-hypothesis, is widely used in the credit risk literature (see e.g. [17] and references therein). With this hypothesis, we may conclude that \( S_u \) remains in general a \( \mathcal{G}_u \)-semimartingale and \( B(t, u)S_u \) a martingale under the extended probability \( Q \). So we can replace the arbitrage free pricing formula (3) by the more general expression

\[
c(t, T) = E^Q[B(t, T)(S_T - \alpha S_{\tau \wedge T})^+|\mathcal{G}_t].
\]

From now on, this product will be called a **Random Time Forward Starting** option and we will refer to it as \( \text{RTFS} \) options. Notice that the payoff remains \( \mathcal{G}_T \)-measurable. In what follows we treat only the call case as, under hypothesis (H), a natural extension of the put-call parity holds for these products

\[
E^Q[B(t, T)(S_T - \alpha S_{\tau \wedge T})^+|\mathcal{G}_t] - E^Q[B(t, T)(\alpha S_{\tau \wedge T} - S_T)^+|\mathcal{G}_t] = S_t - \alpha S_{\tau \wedge t} B(t, \tau \wedge t).
\]

Lastly we remark that for \( \alpha = 1 \), these options are worth something only if the random occurrence happens before maturity, in a sense we could say they are triggered by the random time.

### 3 Pricing formulas

In this section we want to understand how to make formula (7) more explicitly computable.
As a general consideration, let us notice that

\[ c(t, T) = \mathbb{E}^Q[B(t, T) (S_T - \alpha S_{\tau \wedge T})^+] 1_{\{0 < \tau \leq T\}} + 1_{\{\tau > T\}} | \mathcal{G}_t] \]

\[ = \mathbb{E}^Q[B(t, T) (S_T - \alpha S_{\tau})^+] 1_{\{0 < \tau \leq T\}} | \mathcal{G}_t] + (1 - \alpha) \mathbb{E}^Q[B(t, T) S_T 1_{\{\tau > T\}} | \mathcal{G}_t] \]

\[ = \mathbb{E}^Q[B(t, T) (S_T - \alpha S_{\tau})^+] | \mathcal{G}_t] 1_{\{0 < \tau \leq T\}} + \mathbb{E}^Q[B(t, T) (S_T - \alpha S_{\tau})^+] 1_{\{\tau > T\}} | \mathcal{G}_t] \]

\[ + (1 - \alpha) S_t 1_{\{t < \tau\}} - (1 - \alpha) \mathbb{E}^Q[B(t, T) S_T 1_{\{t < \tau \leq T\}} | \mathcal{G}_t]. \]  

(8)

having written \( 1_{\{\tau > T\}} \) as \( 1 - 1_{\{0 < \tau \leq T\}} \).

For the first term there is not much to say, there \( \tau \leq t \) and it behaves as a call price with strike price \( \alpha S_{\tau} \), which is completely known at time \( t \). The third term represents the guaranteed payoff of this contract, so we have to study the remaining terms. Hypothesis (H) is going to help us for the last one. We know that \( B(t, u) S_u \) is a martingale under \( \mathcal{G} \) and that the event \( \{ t < \tau \leq T \} \in \mathcal{G}_t \) (since for any \( t \leq u \leq T \), \( \{ t < \tau \leq T \} \cap \{ \tau \leq u \} = \{ t < \tau \leq u \} \in \mathcal{G}_u \), so conditioning internally w.r.t. \( \mathcal{G}_t \), we obtain

\[ \mathbb{E}^Q[ B(t, T) S_T 1_{\{t < \tau \leq T\}} | \mathcal{G}_t] = \mathbb{E}^Q[ \mathbb{E}^Q( B(t, T) S_T 1_{\{t < \tau \leq T\}} | \mathcal{G}_r) | \mathcal{G}_t] \]

\[ = \mathbb{E}^Q[ \mathbb{E}^Q( B(t, T) S_T | \mathcal{G}_r) 1_{\{t < \tau \leq T\}} | \mathcal{G}_t] = \mathbb{E}^Q( B(t, \tau) S_T 1_{\{t < \tau \leq T\}} | \mathcal{G}_t], \]

where we used the optional sampling theorem, assuming enough integrability of the asset price process, as \( \tau \leq T \) almost surely.

The second term may be rewritten as

\[ \mathbb{E}^Q[ B(t, T) (S_T - \alpha S_{\tau})^+] 1_{\{t < \tau \leq T\}} | \mathcal{G}_t] = \mathbb{E}^Q[ \mathbb{E}^Q( B(t, T) (S_T - \alpha S_{\tau})^+] 1_{\{t < \tau \leq T\}} | \mathcal{G}_r) | \mathcal{G}_t] \]

\[ = \mathbb{E}^Q[ B(t, \tau) \mathbb{E}^Q( B(\tau, T) (S_T - \alpha S_{\tau})^+] | \mathcal{G}_r) 1_{\{t < \tau \leq T\}} | \mathcal{G}_t] = \mathbb{E}^Q[ B(t, \tau) c(\tau, \tau, T) 1_{\{t < \tau \leq T\}} | \mathcal{G}_t], \]

with \( c \) defined w.r.t. \( \mathcal{G}_t \) as in (3), where we extended the definition to include also \( u = t \).

To proceed with our evaluation we have two possibilities. Either \( \mathcal{G}_t \equiv \mathcal{F}_t \), i.e. \( \tau \) is an \( \mathcal{F}_t \)-stopping time, or not. In the first case, some computable situations may occur. For instance, if \( \tau = \inf \{ u > 0 : S_u \geq H \} \) is a hitting time (choosing the barrier level \( H > S_0 \)), we have

\[ c(t, T) = \mathbb{E}^Q[ B(t, T) (S_T - \alpha H)^+] | \mathcal{F}_t] 1_{\{0 < \tau \leq t\}} + \mathbb{E}^Q[ B(t, T) (S_T - \alpha H)^+] 1_{\{t < \tau \leq T\}} | \mathcal{F}_t] \]

\[ + (1 - \alpha) S_t 1_{\{t < \tau\}} - (1 - \alpha) \mathbb{E}^Q[B(t, T) S_T 1_{\{t < \tau \leq T\}} | \mathcal{F}_t]. \]

Summarizing

\[ c(t, T) = \text{call price}(t, T, S_t, \alpha H) 1_{\{0 < \tau \leq t\}} + \text{barrier price}^{up/in}(t, T, S_t, \alpha H, H) \]

\[ + (1 - \alpha) S_t 1_{\{t < \tau\}} - (1 - \alpha) \mathbb{E}^Q[B(t, T) S_T 1_{\{t < \tau \leq T\}} | \mathcal{F}_t] + (1 - \alpha) S_t 1_{\{t < \tau\}}, \]

where “barrier price” denotes the price of a barrier call option with strike price \( \alpha H \) which is activated as soon as the threshold \( H \) is reached by the asset price. If this event does not happen before maturity \( T \), at maturity the contract is not going to pay zero to the holder as in standard barrier contracts, but \( (1 - \alpha) \)-percent of the terminal asset price \( S_T \), since the first three pieces of the formula are zero. Therefore, if \( \tau > T \), this random starting forward option differs from a plain vanilla barrier option by a specific non negative value, never exceeding \((1 - \alpha)H\).
Other situations may be more or less complex to evaluate, depending upon the definition of the stopping time \( \tau \).

In general, since the observable is the asset’s price process, it would be interesting to be able to write the pricing formula in terms of \( \mathcal{F}_t \), rather than \( \mathcal{G}_t = \mathcal{F}_t \lor \mathcal{H}_t \). For that we have the following key Lemma (see [5], [4]).

**Lemma 3.1** For any integrable \( \mathcal{G} \)-measurable r.v. \( Y \), the following equality holds

\[
\mathbb{E}^Q\left[ 1_{\{\tau > t\}} Y | \mathcal{G}_t \right] = Q(\tau > t | \mathcal{G}_t) \frac{\mathbb{E}^Q\left[ 1_{\{\tau > t\}} Y | \mathcal{F}_t \right]}{Q(\tau > t | \mathcal{F}_t)}.
\] (10)

Applying this lemma to the second and fourth term of (8), respectively with

\[
Y = B(t, \tau)c(\tau, \tau, T)1_{\{t < \tau \leq T\}} \quad \text{and} \quad Y = B(t, \tau)S_\tau 1_{\{t < \tau \leq T\}}
\]

and remembering that \( 1 - H_t = 1_{\{\tau > t\}} \) is \( \mathcal{G}_t \)-measurable, we obtain

\[
\mathbb{E}^Q[B(t, \tau)c(\tau, \tau, T)1_{\{t < \tau \leq T\}} | \mathcal{G}_t] = 1_{\{\tau > t\}} \frac{\mathbb{E}^Q[B(t, \tau)c(\tau, \tau, T)1_{\{t < \tau \leq T\}} | \mathcal{F}_t]}{Q(\tau > t | \mathcal{F}_t)} \] (11)
\[
\mathbb{E}^Q[B(t, \tau)S_\tau 1_{\{t < \tau \leq T\}} | \mathcal{G}_t] = 1_{\{\tau > t\}} \frac{\mathbb{E}^Q[B(t, \tau)S_\tau 1_{\{t < \tau \leq T\}} | \mathcal{F}_t]}{Q(\tau > t | \mathcal{F}_t)}.
\] (12)

The above may be rewritten following the hazard process approach. Let us denote the conditional distribution of the default time \( \tau \) given \( \mathcal{F}_t \) as

\[
F_t = Q(\tau \leq t | \mathcal{F}_t), \quad \forall \ t \geq 0
\]

and let us notice that for \( u \geq t \), \( Q(\tau \leq u | \mathcal{F}_t) = \mathbb{E}^Q(Q(\tau \leq u | \mathcal{F}_u) | \mathcal{F}_t) = \mathbb{E}^Q(F_u | \mathcal{F}_t) \).

In order to apply the so called intensity based approach, we need to assume that \( F_t(\omega) < 1 \) for all \( t > 0 \) (which automatically excludes that \( \mathcal{G}_t \equiv \mathcal{F}_t \) to well define the so called risk or hazard process

\[
\Gamma_t := -\ln(1 - F_t) \quad \Rightarrow \quad F_t = 1 - e^{-\Gamma_t} \quad \forall \ t > 0, \quad \Gamma_0 = 0.
\] (13)

With this notation, we rewrite (11) and (12) as

\[
\mathbb{E}^Q[B(t, \tau)c(\tau, \tau, T)1_{\{t < \tau \leq T\}} | \mathcal{G}_t] = (1 - H_t)\mathbb{E}^Q[B(t, \tau)c(\tau, \tau, T)(H_T - H_t)e^{\Gamma_t} | \mathcal{F}_t]
\] (14)

\[
\mathbb{E}^Q[B(t, \tau)S_\tau 1_{\{t < \tau \leq T\}} | \mathcal{G}_t] = (1 - H_t)\mathbb{E}^Q[B(t, \tau)S_\tau (H_T - H_t)e^{\Gamma_t} | \mathcal{F}_t],
\] (15)

but by proposition 5.1.1 (ii) page 147 of [6] these conditional expectations may be written as

\[
\mathbb{E}^Q[B(t, \tau)c(\tau, \tau, T)(H_T - H_t) | \mathcal{F}_t] = \mathbb{E}^Q[\int_t^TB(t, u)c(u, u, T)dF_u | \mathcal{F}_t]
\] (16)

\[
\mathbb{E}^Q[B(t, \tau)S_\tau (H_T - H_t) | \mathcal{F}_t] = \mathbb{E}^Q[\int_t^TB(t, u)S_u dF_u | \mathcal{F}_t].
\] (17)

If we have the additional hypotheses \( \mathcal{G}_t \equiv \mathcal{F}_t \lor \mathcal{H}_t = \mathcal{F}_t \otimes \mathcal{H}_t \) for all \( t \) and


(IND) \( \mathcal{F} \) and \( \mathcal{H} \) are independent,

then \( \tau \) is independent of \( \mathcal{F}_t \), hypothesis (H) is automatically satisfied and \( F_t = Q(\tau \leq t) \) is deterministic, so the previous formula becomes

\[
E^Q[\mathcal{B}(t, \tau) e_{(\tau \leq t)} | \mathcal{F}_t] = E^Q[\int_t^T \mathcal{B}(t, u) e_{(\tau \leq t)} d\mathcal{F}_t| \mathcal{F}_t]
\]

and the other is treated similarly.

We can summarize the above results in the following

**Proposition 3.1** With the above notation, we have

(a) Under hypothesis (H) the price of a RTFS call option is given by

\[
c(t, T) = E^Q[\mathcal{B}(t, T)(S_T - \alpha S_\tau)^+ | \mathcal{G}_t] 1_{\{0 < \tau \leq t\}} + (1 - \alpha) S_t 1_{\{\tau > t\}}
\]

(b) Under hypothesis (IND) the price of a RTFS call option is given by

\[
c(t, T) = E^Q[\mathcal{B}(t, T)(S_T - \alpha S_\tau)^+ | \mathcal{G}_t] 1_{\{0 < \tau \leq t\}}
\]

\[
+ 1_{\{\tau > t\}} \left[ \int_t^T c(t, u, T) e^{-(\Gamma_u - \Gamma_t)} d\Gamma_u + (1 - \alpha) S_t e^{-(\Gamma_T - \Gamma_t)} \right]
\]

**Proof:** Part (a) is only the assembling of the various pieces previously presented. As for part (b), because of independence, \( \Gamma \) is deterministic and it gets pulled out of the expectation. Since \( Q \) is a martingale measure, the integrands of the last two terms verify the martingale property w.r.t. \( \mathcal{F}_t \) under \( Q \).

**Remark 3.1** If we assume that \( \Gamma_t \) is absolutely continuous w.r.t. the Lebesgue measure, that is to say

\[
\Gamma_t = \int_0^t \lambda_s ds \quad \forall t > 0
\]

for some \( \mathcal{F}_t \)-adapted and non negative process \( \lambda_t \) called intensity (see [5]), then also

\[
F_t = 1 - e^{-\Gamma_t} = 1 - e^{-\int_0^t \lambda_s ds}
\]

is absolutely continuous and if we denote by \( f_t \) its density, necessarily we have \( \lambda_t = \frac{f_t}{1 - F_t} \)

In this case we have
(a) Under hypothesis (H) the price of a RTFS call option is given by
\begin{align}
c(t, T) &= E^Q[B(t, T)(S_T - \alpha S_\tau)^+|\mathcal{G}_t]1_{\{0 < \tau \leq t\}} + (1 - \alpha)S_t 1_{\{\tau > t\}} \\
&+ 1_{\{\tau > t\}} E^Q[\int_t^T B(t, u)c(u, u, T)\lambda_u e^{-\int_u^T \lambda_s ds} du | \mathcal{F}_t] \\
&- (1 - \alpha)1_{\{\tau > t\}} E^Q[\int_t^T B(t, u)S_u \lambda_u e^{-\int_u^T \lambda_s ds} du | \mathcal{F}_t]
\end{align}

(b) Under hypothesis (IND) the price of a RTFS call option is given by
\begin{align}
c(t, T) &= E^Q[B(t, T)(S_T - \alpha S_\tau)^+|\mathcal{G}_t]1_{\{0 < \tau \leq t\}} \\
&+ 1_{\{\tau > t\}} \left[ \int_t^T c(t, u, T)\lambda_u e^{-\int_u^T \lambda_s ds} du + (1 - \alpha)S_t e^{-\int_t^T \lambda_s ds} \right]
\end{align}

The above formulas, either in presence of independence or not, rely on the knowledge of the model and of the conditional distribution of the default time. Hence, deriving an explicit or implementable formula for the pricing will depend heavily on the modelling choices we are going to make.

Of course, the independent case is much easier than the other and we focus on it in the next section, to show that there are some interesting treatable cases. We will always take \( \alpha = 1 \).

**Remark 3.2 (Affine hazard rates)**

When we no longer assume independence of \( \tau \) and \( S \), we need to resort to formula (21) and a reasonable choice is to make the hazard rate depend also on the price process. As a first attempt one may try an affine model, so we decide that
\begin{equation}
\lambda_u = a(u)S_u + b(u)Z_u,
\end{equation}
where \( a, b \) are deterministic, positive and bounded functions, while \( Z \) is a positive \( \mathcal{G}_t \)-adapted process independent of \( \mathcal{F}_t \), \( 0 \leq t \leq T \).

By using the optional projection theorem (see [15] theorem VI.57), we may arrive at the pricing formula
\begin{align}
c(t, T) &= E^Q[B(t, T)(S_T - \alpha S_\tau)^+|\mathcal{G}_t]1_{\{0 < \tau \leq t\}} \\
&+ \int_t^T a(u)\text{call}(u, 1, \alpha, T)E^Q[B(t, u)S_u^2 e^{-\int_u^T a(s)S_s ds}|\mathcal{F}_t]E^Q[e^{-\int_u^T b(s)Z_s ds}] du \\
&+ \int_t^T b(u)\text{call}(u, 1, \alpha, T)E^Q[B(t, u)S_u e^{-\int_u^T a(s)S_s ds}|\mathcal{F}_t]E^Q[Z_u e^{-\int_u^T b(s)Z_s ds}] du \\
&+ (1 - \alpha)E^Q[B(t, T)S_T e^{-\int_t^T a(s)S_s ds}|\mathcal{F}_t]E[e^{-\int_t^T b(s)Z_s ds}],
\end{align}

where all the above expectations are of the same type. If not explicitly computable, the expectations can be evaluated via Monte Carlo or transform methods, once a grid of points \( t = u_0 < u_1 < \cdots < u_N = T \) in order to approximate the time integrals has been fixed. More precisely, one applies the Monte Carlo approach to the inner expectations by generating \( M \) independent paths of the processes \( S \) and \( Z \) at the grid points.
4 Some computable models

We now show some examples where the above formulas for the RTFS options can be explicitly computed, of course the first case to analyze is the Black and Scholes model.

Pricing in the Black & Scholes market

In the classical BS market with constant risk-free rate $r$ and volatility $\sigma$, the price of a plain vanilla European call with maturity $T$ and strike $K$ will be denoted by $c_{BS}(t, S_t, K, T)$, while the price of a forward-start option with maturity $T$, strike-determination time $u$ and percentage $\alpha = 1$ (see [28]) is simply

$$c_{BS}(t, u, T) = S_t c_{BS}(u, 1, 1, r, T), \quad u \geq t$$

(24)

where

$$c_{BS}(t, S_t, K; \sigma, r, T) = S_t \mathcal{N}(d_1) - e^{-r(T-t)} K \mathcal{N}(d_2)$$

with

$$d_1 = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \quad d_2 = \frac{\log(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}},$$

hence in formula (24) we have $d_i = c_i \sqrt{T-u}$, $i = 1, 2$, $c_1 = \left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)$, $c_2 = c_1 - \sigma$. Let us notice that $c_2^2 > r$ is always true.

Henceforth, under (IND), the price of a (RTFS) option is given by

$$c(t, T) = 1_{\{0<\tau\leq t\}}c_{BS}(t, S_t, S_\tau, r, T) + 1_{\{\tau>t\}}S_t \int_t^T c_{BS}(u, 1, 1, r, T) \lambda_u e^{-\int_u^T \lambda_d ds} du.$$  (25)

The valuation of (25) can be found in closed form in some simple cases. If $\tau \sim \exp(\lambda)$, $\lambda > 0$ under $Q$, then for the second term we obtain

$$1_{\{\tau>t\}}S_t \int_t^T c_{BS}(u, 1, 1, r, T) \lambda_u e^{-\int_u^T \lambda_d ds} du = 1_{\{\tau>t\}}S_t \left[ A_1(t, T) - A_2(t, T) \right].$$  (26)

where

$$A_1(t, T) = \lambda e^{\lambda t} \int_t^T \mathcal{N}(c_1 \sqrt{T-u}) e^{-\lambda u} du, \quad A_2(t, T) = \lambda e^{\lambda t} \int_t^T e^{-\tau(T-u)} \mathcal{N}(c_2 \sqrt{T-u}) e^{-\lambda u} du$$  (27)

and it is clear that $A_1 > A_2$ for all values of the parameters.

The above integrals specialize depending on the choice of the parameters. If $r \neq \lambda$ we have three cases.

If $c_1^2 > 2\lambda$ then these integrals can be explicitly computed exploiting integration by parts and the Gaussian density, obtaining (keeping in mind that $c_2^2 - 2(\lambda - r) = c_1^2 - 2\lambda$)

$$A_1(t, T) = \mathcal{N}(c_1 \sqrt{T-t}) - \frac{c_1 e^{-\lambda(T-t)}}{\sqrt{c_1^2 - 2\lambda}} \mathcal{N}\left(\sqrt{(c_1^2 - 2\lambda)(T-t)}\right) + \frac{e^{-\lambda(T-t)}}{2} \left( \frac{c_1}{\sqrt{c_1^2 - 2\lambda}} - 1 \right)$$  (28)

$$A_2(t, T) = \frac{\lambda}{\lambda - r} \left[ e^{-\tau(T-t)} \mathcal{N}(c_2 \sqrt{T-t}) - \frac{c_2 e^{-\lambda(T-t)}}{\sqrt{c_2^2 - 2\lambda}} \mathcal{N}\left(\sqrt{(c_2^2 - 2\lambda)(T-t)}\right) \right]$$

$$+ \frac{e^{-\lambda(T-t)}}{2} \left( \frac{c_2}{\sqrt{c_2^2 - 2\lambda}} - 1 \right)$$  (29)

and it is clear that $A_1 > A_2$ for all values of the parameters.
When $c_1^2 = 2\lambda$ (hence $c_1^2 = 2(\lambda - r)$), then the above formulas reduce to

$$A_1(t, T) = N(c_1\sqrt{T-t}) - e^{-\lambda(T-t)}\left(\frac{1}{2} + \frac{c_1\sqrt{T-t}}{\sqrt{2\pi}}\right)$$ (30)

$$A_2(t, T) = \frac{\lambda}{\lambda - r} \left[ e^{-r(T-t)} N(c_2\sqrt{T-t}) - e^{-\lambda(T-t)}\left(\frac{1}{2} + \frac{c_2\sqrt{T-t}}{\sqrt{2\pi}}\right) \right]$$ (31)

and for $c_1^2 < 2\lambda$ we lose the Gaussian integral and we arrive at

$$A_1(t, T) = N(c_1\sqrt{T-t}) - e^{-\lambda(T-t)}\left[\frac{1}{2} + \frac{c_1\sqrt{T-t}}{\sqrt{2\lambda - c_1^2}} \int_0^{\sqrt{(2\lambda - c_1^2)(T-t)}} \frac{e^{z^2}}{\sqrt{2\pi}} dz\right]$$ (32)

$$A_2(t, T) = \frac{\lambda}{\lambda - r} \left\{ e^{-r(T-t)} N(c_2\sqrt{T-t}) - e^{-\lambda(T-t)}\left[\frac{1}{2} + \frac{c_2\sqrt{T-t}}{\sqrt{2\lambda - c_1^2}} \int_0^{\sqrt{(2\lambda - c_1^2)(T-t)}} \frac{e^{z^2}}{\sqrt{2\pi}} dz\right] \right\}. \quad (33)$$

If instead condition $\lambda = r$ holds, we arrive in any case to an explicit formula with $A_1(t, T)$ the same as in (28) and $A_2(t, T)$ given by

$$A_2(t, T) = \lambda e^{-\lambda(T-t)} \left[ (T-t)N(c_2\sqrt{T-t}) + \sqrt{\frac{T-t}{2\pi c_2}} e^{-\frac{c_2^2}{2}(T-t)} - \frac{1}{c_2^2} \left( N(c_2\sqrt{T-t}) - \frac{1}{2} \right) \right]. \quad (34)$$

**Remark 4.1 (The Merton market model)**

The above results can be extended with moderate computational effort to the Merton jump-diffusion market model (see e.g. [12])

$$\frac{dS_t}{S_t} = (r - \nu \kappa) dt + \sigma dW_t + (e^J - 1) dN_t,$$

where $W_t$ is a Brownian motion, $N_t$ a Poisson process with jump-arrival intensity $\nu$, $J \sim N(\mu, \delta^2)$ all independent of each other, $\kappa = E^Q[e^J] - 1$ is the jump-size expectation, $\sigma$ is the volatility and $r$ is the risk-free rate.

Under (IND), for $\alpha = 1$ and under the martingale $Q$, if $\tau \sim \exp(\lambda)$, with $\lambda > 0$, the price of a (RTFS) option is given by

$$c(t, T) = 1_{\{0 < \tau \leq t\}} \text{call}^M(t, S_t, S_\tau, T) + 1_{\{\tau > t\}} \sum_{n=0}^{+\infty} \frac{\nu^n}{n!} \left( A_{1,n}(t, T) - A_{2,n}(t, T) \right), \quad (35)$$

where $\nu = \nu(1 + \kappa)$, $A_{1,n}(t, T)$ and $A_{2,n}(t, T)$ are given as in (27) for appropriate sets of parameters varying with $n$. The explicit formula is available from the authors, upon request.

**Pricing with Fourier Transform: the variance Gamma model**

Here we assume that the price dynamics is of the form $S_T = S_u e^{X_T-X_u}, u \leq T$, where $X_u$ is a Lévy process. Fourier transform is a well established computational tool for derivative pricing, first introduced in [21] and then further developed by several authors (e.g. see [22] or [16] and
the references therein). Different ways of applying this technique lead to different representation formulas for the risk neutral price of a plain vanilla call option, we choose to consider

\[
call(u, S_u, K, T) = e^{-r(T-u)}K \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-iz(\log(S_u/K)+r(T-u))} \frac{\phi_{X_T-X_u}(z)}{iz-z^2} dz,
\]

for \( z \in \mathbb{C} \) and \( \nu = \text{Im}(z) > 1 \), with \( \phi_{X_T-X_u}(z) = E[e^{iz(X_T-X_u)}] \) the generalized characteristic function of \( X_T - X_u \). Because of the independence of the increments, \( \phi_{X_T-X_u} \) does not depend on \( S_u \), so the model is scale-invariant and, applying (5), we may conclude that the price of the standard Forward Starting option with determination time \( u \) is

\[
c(t, u, T) = S_t \text{call}(u, 1, \alpha, T) = S_t e^{-r(T-u)} \frac{\alpha}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-iz(-\log(\alpha)+r(T-u))} \frac{\phi_{X_T-X_u}(-z)}{iz-z^2} dz.
\]

By inserting this Fourier representation of the Forward Starting call price into the formulas of Proposition 1.1 or Remark 3.1, we get a double integral representation of our RTFS call price.

As usual, under hypothesis (IND), setting \( \alpha = 1 \), we obtain

\[
c(t, T) = E^Q[B(t, T)(S_T-S_T)^+|\mathcal{G}_t]1_{\{0<r\leq t\}}
\]

\[+ \ 1_{\{r>t\}} \frac{S_t}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} \frac{1}{iz-z^2} \left[ \int_t^T e^{-r(T-u)}(1+i\nu)\phi_{X_T-X_u}(-z)\lambda_t e^{-r\lambda_s ds} du \right] dz.
\]

In some cases the inner integral can be solved in closed form. As a matter of fact, if we choose a Variance Gamma model for the price dynamics

\[X_t = bY_t + cW_t, \quad S_T = S_u e^{r+\omega(T-u)+X_T-X_u},\]

with \( W_t \) a standard Brownian motion and \( Y_t \) a Gamma process independent of \( W_t \), with parameters 1 and \( \mu \) and \( \omega = 1/\mu \log(1-b\mu - \mu c^2/2) \) (see [11]), we have

\[\phi_{X_T-X_u}(z) = \left( \frac{1}{1-b\mu z + c^2\mu^2 z^2} \right)^{T-u} = e^{-\frac{1}{\mu}(T-u)\ln(1-b\mu z + c^2\mu^2 z^2)},\]

where \( \ln \eta = \ln |\eta| + i \text{arg}(\eta) \), with \( \eta \in \mathbb{C} \) and \( -\pi \leq \text{arg}(\eta) \leq \pi \), denotes the principal complex logarithm. Taking also (for ease of computation) a constant hazard rate \( \lambda \), we may conclude

\[
c(t, T) = E^Q[B(t, T)(S_T-S_T)^+|\mathcal{G}_t]1_{\{0<r\leq t\}}
\]

\[+ \ 1_{\{r>t\}} \frac{S_t}{2\pi} \lambda e^{-\lambda(T-t)} \int_{i\nu-\infty}^{i\nu+\infty} \frac{1}{iz-z^2} \left( 1 - e^{-r(1+i\nu) + \frac{1}{\mu} \ln(1-b\mu z + c^2\mu^2 z^2) - \lambda(T-t)} \right) \frac{1}{r(1+i\nu) + \frac{1}{\mu} \ln(1 - b\mu z + c^2\mu^2 z^2) - \lambda} dz, \quad (37)
\]

leading to a computable formula. It is reasonable that the same arguments might be extended to tempered stable distributions as well as to the Carr-Geman-Madan-Yor model (see [9]).

**Pricing in the Heston stochastic volatility market.**
Assuming independence, we can arrive to an explicit formula also in the Heston stochastic volatility model, given (under the risk neutral measure) by

\[ dS_t = S_t(r + \sqrt{\sigma_t}dW^1_t) \]
\[ d\sigma_t = \kappa(\theta - \sigma_t)dt + c\sqrt{\sigma_t}(\rho dW^1_t + \sqrt{1-\rho^2}dW^2_t), \]

where \( r \) is a constant interest rate. Here \( \mathcal{F}_t \) has to be the natural filtration generated by the couple \((S, \sigma)\), which is jointly a bidimensional Markov process, hence the final pricing formula will depend only upon the initial values of both \( S \) and \( \sigma \).

As before, we start from the expression

\[ c_H(t,T) = \mathbb{E}^Q[B(t,T)(S_T - \alpha S_T^\tau)]1_{\{0 < \tau \leq t\}} + 1_{\{\tau > t\}}\int_t^T c_H(t,u,T,S_t,\sigma_t)^\tau e^{-\int_t^u \lambda_s ds} du + (1 - \alpha)S_t e^{-\int_T^t \lambda_s ds}. \] (38)

By employing the results in [25] (or [24]), we can find an explicit formula for the forward start option. Indeed by virtue of the scale invariant property verified also by this model, we have

\[ c_H(t,u,T,S_t,\sigma_t) = \int_0^{+\infty} S_t call_H(u,1,\alpha,\sigma,T) f_{\sigma|\sigma_t}(\sigma) d\sigma, \] (39)

where \( call_H(u,1,\alpha,\sigma,T) \) is the standard price of a call option in the Heston model (for details see for instance [25] Lemma 2.1) and

\[ f_{\sigma|\sigma_t}(\sigma) = \frac{B(u)e^{-\frac{B(u)\sigma + \Lambda(u)}{2}(R/2-1)/2}}{2\Lambda(u)B(u)\sigma}I_{R/2-1}(\sqrt{\Lambda(u)B(u)}\sigma) \] (40)

\[ B(u) = \frac{4(\kappa - \rho c)}{c^2}(1 - e^{-(\kappa - \rho c)(u-t)})^{-1} \] (41)

\[ \Lambda(u) = B(u)e^{-(\kappa - \rho c)(u-t)}\sigma_t, \quad R = \frac{4\kappa \theta}{c^2}, \] (42)

and \( I \) is a modified Bessel function of the first kind.

\textbf{Remark 4.2} The previous method can be adapted to cover also the case of affine stochastic interest rate market model proposed in [20], where the interest rate follows a Hull and White model and the stochastic volatility is a Ornstein-Uhlenbeck process.

\section{Numerical results}

In this section we report the results of the numerical implementation of some of the models presented before: the Black & Scholes, the Variance Gamma and the Heston models, with a random time \( \tau \sim \exp(\lambda) \) with parameter \( \lambda > 0 \). In all experiments we set \( \alpha = 1, t = 0 \) and we assume the independent hypothesis (IND). Analytical or closed-form prices, eventually requiring numerical quadrature, are compared with standard Monte Carlo approximations of the arbitrage free pricing formula (7). Sample paths of the underlying dynamics are evaluated at the
random time \( \tau \) and the corresponding payoffs are collected to estimate (7). In our experiments we simulated \( M = 1000000 \) samples/paths\(^1\).

Adaptive Gauss-Lobatto quadrature was used to approximate representation integrals (VG and Heston models), which required the evaluation of extended Fourier transforms involving multivalued functions, such as the complex logarithm. To avoid the well-known numerical instabilities due to a wrong formulation of the characteristic function, we used the so-called rotation count algorithm by Kahl and Jäckel (see e.g. [23]). All routines were coded and implemented in MatLab\(^\circledR\), version 8.0 (R2012b) on an Intel Core i7 2.40 GHz machine running under Windows 7 with 8 GB physical memory.

For each model we report the closed-form (CF) and the MC prices for different values of \( \lambda \), fixing the other model parameters. In parenthesis we report the absolute error w.r.t. the CF price, followed by the corresponding length of the 95\% confidence interval. Without loss of generality we set \( S_0 = 100, \, T = 2 \) and \( r = 0 \).

In the Black & Scholes model the RTFS price is given by formulas (25)-(34). Exact simulations of the geometrical Brownian motion with \( \sigma = 0.2 \) at the random time \( \tau \) are used to estimate the pricing formulas.

| \( \lambda \) | CF Price | MC | C.I. Length |
|-------------|---------|----|-------------|
| 0.25        | 3.0989  | 3.0998 (8.4 \times 10^{-4}) | 3.7 \times 10^{-2} |
| 0.75        | 6.6457  | 6.6512 (5.4 \times 10^{-3}) | 5.5 \times 10^{-2} |
| 1.25        | 8.3710  | 8.3699 (1.1 \times 10^{-3}) | 6.1 \times 10^{-2} |
| 1.75        | 9.2709  | 9.2872 (1.6 \times 10^{-2}) | 6.5 \times 10^{-2} |

Table 1: Black & Scholes model for different values of \( \lambda \).

The RTFS prices in the VG model are given by formula (37). In this example we consider the same parameters as in [11], \( b = -0.1463, \, c = 0.1213 \) and \( \mu = 0.1686 \). Sample paths are simulated as a Gamma time-changed Brownian motion with a discretization step equal to \( 1/\sqrt{M} \).

| \( \lambda \) | CF Price | MC | C.I. Length |
|-------------|---------|----|-------------|
| 0.25        | 2.0159  | 2.0163 (4.1 \times 10^{-4}) | 5.5 \times 10^{-2} |
| 0.75        | 4.3394  | 4.3426 (3.2 \times 10^{-3}) | 4.1 \times 10^{-2} |
| 1.25        | 5.4801  | 5.4748 (5.3 \times 10^{-3}) | 3.9 \times 10^{-2} |
| 1.75        | 6.0796  | 6.0857 (6.1 \times 10^{-3}) | 5.8 \times 10^{-2} |

Table 2: Variance Gamma model for different values of \( \lambda \).

In Heston’s stochastic volatility model parameters are chosen as in [25]: \( \sigma_0 = 0.09, \, \kappa = 4, \, \theta = 0.06, \, c = 0.65 \) and \( \rho = -0.9 \). We need to compute numerically formula (38), which requires evaluating the function given by formula (39). This function, though theoretically well defined for all values of \( u \), generates a numerical singularity in the computations, that we treat by approximating \( c_H(0, u, T, S_0, \sigma_0) \) with \( S_0 call_H(u, 1, \alpha, E(\sigma_u), T) \). This gives some

\(^1\)Even though we are not employing them here, we remark that variance reduction techniques (control variates, antithetic variates) or more efficient simulation algorithms (e.g. the Broadie and Kaya exact simulation, as proposed in [8]) could improve the MC estimates. Furthermore, alternative simulation schemes exploiting Proposition 3.1 and Remark 3.1 can be easily designed.
advantage since we explicitly have $\mathbb{E}(\sigma_u) = \sigma_0 e^{-\kappa u} + \theta (1 - e^{-\kappa u})$. Comparison with Monte Carlo approximations is run with sample paths simulated by the basic Euler scheme with a step length equal to $\sqrt{M}$.

| $\lambda$  | CF price | MC            | C.I. Length |
|------------|----------|---------------|-------------|
| 0.25       | 3.4907   | 3.4903 (3.7 \times 10^{-4}) | 3.6 \times 10^{-2} |
| 0.75       | 7.5290   | 7.5227 (6.2 \times 10^{-3}) | 5.2 \times 10^{-2} |
| 1.25       | 9.5307   | 9.5157 (1.4 \times 10^{-2}) | 5.7 \times 10^{-2} |
| 1.75       | 10.5988  | 10.5920 (6.8 \times 10^{-3}) | 6.0 \times 10^{-2} |

Table 3: **Heston** model for different values of $\lambda$.

Lastly, we choose the random time $\tau \sim \exp(\lambda)$ with varying $\lambda \geq \frac{1}{2}$ in order to compare at time $t = 0$, the RTFS price and the price of a FS option with determination time $u = \mathbb{E}(\tau) = \frac{1}{\lambda} \leq T$. In other words we compare $\mathbb{E}((S_T - S_{\tau\wedge T})^+) + \mathbb{E}((S_T - S_{E(\tau)})^+)$, see Figure (1). In all the instances prices decrease as $u$ increases, moreover we notice that: i) for $u \to 0$ (i.e. $\lambda \to +\infty$) FS and RTFS prices both tend to the price of an ATM plain vanilla call; ii) for $u \to T$ (i.e. $\lambda \to 1/2$) the FS value trivially goes to 0, while the value of a RTFS option stays positive since $Q(\tau \leq T) \to 1 - 1/e \approx 63\%$.

6 Counterparty credit risk for RTFS options

In this section we are interested in computing the credit value adjustment (CVA) for a RTFS option, when one of the parties is subject to a possibly additional default time.

The interest lies in the fact that those products would be OTC and knowing that the CVA is computable might be convenient.
We first recall what the CVA is for an additive cashflow. Let us denote by $\Pi(t, T)$ the discounted value of the cashflow between time $t$ and $T$; this value has to be additive, which means that for times $t \leq s \leq u$ it holds

$$\Pi(t, s) + B(t, s)\Pi(s, u) = \Pi(t, u). \quad (43)$$

All European contingent claims with integrable, $\mathcal{G}_T$–measurable payoff $h$, trivially verify (43) since $\Pi(t, s) = h\mathbf{1}_{\{s = T\}}.$ We call Net Present Value of $\Pi$

$$NPV(t) = E(\Pi(t, T) | \mathcal{G}_t). \quad (44)$$

We now look into unilateral counterparty credit risk. Let us suppose that the cashflow is between two parties $B$ (buyer) and the counterparty $C$, that is subject to default, with default time $\tau_C$. Then, the discounted value of the cashflow has to be adjusted by subtracting from the defaultless case the quantity

$$CVA(t) = (1 - R)E(\mathbf{1}_{\{t < \tau_C \leq T\}} B(t, \tau_C) NPV(\tau_C)^+ | \mathcal{G}_t) \quad (45)$$

where $0 \leq R \leq 1$ represents a deterministic recovery rate (see [7]).

In the case of classical Forward Start options with fixed strike-determination time $u$, it is possible to compute directly the CVA. We denote by $\{\mathcal{H}_t^C\}$ the filtration generated by the process $H_t^C := \mathbf{1}_{\{\tau_C \leq t\}}$ and by $\mathcal{G}_t^C = \mathcal{F}_t \vee \mathcal{H}_t^C$. Under independence between $\{\mathcal{F}_t\}$ and $\{\mathcal{H}_t^C\}$, taking for granted that $B(t, u)S_u$ remains a martingale in the enlarged filtration $\{\mathcal{G}_t^C\}$ w.r.t. a risk neutral probability measure $Q^C$ which extends the risk neutral probability $P$, we have

$$CVA^{FS}(t) = (1 - R)Q^C(t < \tau_C < T)E^{Q^C}(B(t, T)(S_T - \alpha S_u)^+ | \mathcal{F}_t), \quad (46)$$

where the key Lemma and the same notation as in section 1 were used, under the assumption that $Q^C(\tau_C > t) = 1$.

To extend the previous formula to the case of the RTFS options we need to consider two random times: the strike-determination time $\tau$ and the counterparty default time $\tau_C$.

Hence we need to extend the original filtration $\{\mathcal{F}_t\}$ with both $\{\mathcal{H}_t\}$ and $\{\mathcal{H}_t^C\}$, obtaining

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \vee \mathcal{H}_t^C.$$ 

Accordingly, we assume there exists an extension of the risk neutral probability $P$ to a common probability space $(\Omega, \mathcal{G}_T)$ that we still denote by $Q$ and we set the corresponding key assumption

**\(HC\)**

Every $\mathcal{F}_t$ martingale remains a $\mathcal{G}_t$ martingale.

Taking $\alpha = 1$ for simplicity, we have that the price of a defaultable RTFS call option is

$$\bar{c}(t, T) := E^Q[\mathbf{1}_{\{\tau_C > T\}} B(t, T)(S_T - S_{\tau \wedge T})^+ | \mathcal{G}_t].$$

So, in presence of a recovery rate $R$, we obtain that the CVA for this product has to be

$$CVA(t, T) := (1 - R)[c(t, T) - \bar{c}(t, T)] = (1 - R)E^Q[\mathbf{1}_{\{t < \tau_C \leq T\}} B(t, T)(S_T - S_{\tau \wedge T})^+ | \mathcal{G}_t]. \quad (47)$$
We may decompose the expectation as

\[ E^Q[1_{\{t<\tau_C \leq T\}} B(t, T)(S_T - S_{\tau_C T})^+|\mathcal{G}_t] \]

\[ = E^Q[1_{\{t<\tau_C \leq T\}} 1\{\tau > T\} B(t, T)(S_T - S_{\tau_C T})^+|\mathcal{G}_t] + E^Q[1_{\{t<\tau_C \leq T\}} 1\{\tau \leq T\} B(t, T)(S_T - S_{\tau_C T})^+|\mathcal{G}_t] \]

\[ = E^Q[1_{\{t<\tau_C \leq T\}} 1\{t<\tau \leq T\} B(t, T)(S_T - S_{\tau})^+|\mathcal{G}_t] + 1\{\tau \leq T\} E^Q[1_{\{t<\tau_C \leq T\}} B(t, T)(S_T - S_{\tau})^+|\mathcal{G}_t], \]

where the last equality is justified by the fact that the first term in the first passage is equal to 0. In the second equality the crucial term is the first, as the second reduces to the CVA of a standard forward start option, for which formula (46) applies. We may handle the first term under condition of independence between \( \tau_C \) and \((\tau, \{S_t\})\). Denoting by \( \mathcal{F}_t^H = \mathcal{F}_t \vee \mathcal{H}_t \) and applying the key lemma to \( \mathcal{G}_t \) and \( \mathcal{F}_t^H \) we obtain

\[ E^Q[1_{\{t<\tau_C \leq T\}} 1\{t<\tau \leq T\} B(t, T)(S_T - S_{\tau})^+|\mathcal{G}_t] \]

\[ = Q(\tau_C > t|\mathcal{G}_t) \frac{E^Q[1_{\{t<\tau_C \leq T\}} 1\{t<\tau \leq T\} B(t, T)(S_T - S_{\tau})^+|\mathcal{F}_t^H]}{Q(\tau_C > t|\mathcal{F}_t^H)}. \]

The independence of \( 1\{t<\tau \leq T\} \) from the remaining factors gives

\[ E^Q[1_{\{t<\tau_C \leq T\}} 1\{t<\tau \leq T\} B(t, T)(S_T - S_{\tau})^+|\mathcal{G}_t] \]

\[ = Q(\tau_C > t|\mathcal{G}_t) \frac{Q(t < \tau \leq T)}{Q(\tau_C > t)} E^Q[1_{\{t<\tau \leq T\}} B(t, T)(S_T - S_{\tau})^+|\mathcal{F}_t^H] \]

and if \( Q(\tau_C > t) = 1, \) the above reduces to

\[ E^Q[1_{\{t<\tau_C \leq T\}} 1\{t<\tau \leq T\} B(t, T)(S_T - S_{\tau})^+|\mathcal{G}_t] = Q(t < \tau_C \leq T) E^Q[1_{\{t<\tau \leq T\}} B(t, T)(S_T - S_{\tau})^+|\mathcal{F}_t^H] \]

which is the weighted value of a RTFS option, exactly as it happened with the standard FS option. Summarizing we have

**Proposition 6.1** Under hypothesis (HC) and the condition that \( Q(\tau_C > t) = 1, \) if \( \tau_C \) is independent of \((\tau, \{S_t\})\), then the CVA of a defaultable RTFS call option of price \( c(t, T) \), with recovery rate \( R \in [0, 1] \), is given by \( CVA(t, T) = (1 - R)Q(t < \tau_C \leq T)c(t, T). \)

Lastly we remark that if the two random times coincide, \( Q(\tau = \tau_C) = 1, \) then we may conclude that

\[ CVA(t, T) = (1 - R)c(t, T), \]

which implies that the defaultable price is \( \bar{c}(t, T) = Rc(t, T). \)

**Acknowledgements**

The authors thank Prof. C. Chiarella for useful suggestions at an early stage of this work.
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