Staggered Fermion, its Symmetry and Ichimatsu-Patterned Lattice

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We investigate exact symmetries of a staggered fermion in D dimensions. The Dirac operator is reformulated by SO(2D) Clifford algebra. The chiral symmetry, rotational invariance and parity symmetries are clarified in any dimension. Local scalar and pseudo-scalar modes are definitely determined, in which we find non-standard modes. The relation to Ichimatsu-patterned lattice approach is discussed.

1. INTRODUCTION

When we consider super Yang-Mills theory on a lattice, gauge fields are put on links and fermi fields are put on sites on the staggered way [12]. The staggered fermion is formulated for the purpose of solving doubling phenomena in lattice fermion [34]. Although the number of doubling is translated into that of flavor, it is not clear whether each flavor is a spinor or not. Discrete space-time symmetries such as P, C of the fermion have not been defined. Specially, for the chiral symmetry, nobody answers why there exist a mass-protect symmetry for odd dimensions and what is the symmetry.

In this talk, we present the new formulation of a staggered fermion based on cell’s idea in our Ichimatsu-patterned lattice and SO(2D) Clifford algebra in a D-dimensional lattice. The transformation of a staggered fermion for the space-time rotation is clarified. The associated symmetries(chiral and parity) are also investigated for the staggered fermion. The bi-linear operators for scalar and pseudo-scalar modes are found owing to the rotational transformation. Further details will be presented in [5].

2. FORMULATION

A staggered fermion, $\xi_n$, has been formulated by putting a Grassmann variable on a site, $n$. For fermion, we need to obtain the spinorization for the staggered Grassmann variable. Usually, a sign factor,

$$\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu},$$

appears in the staggered fermion [34]. But the validity is held only when the spinor has $2^D$-components on a D-dimensional lattice. It must be shown that single component staggered fermion is reconstructed as the Dirac spinor.

The factor is defined on a link, $(n, \mu)$ on the lattice and has a property of modulo-2 translational invariance,

$$\eta_\mu(n + 2\hat{\nu}) = \eta_\mu(n).$$

This means a just Ichimatsu-pattern [12]. See Fig. 1. In the pattern, a minimal unit is so-called a cell, i.e. a fundamental lattice and its coordinates, $(N, r)$, are expressed as

$$n_\mu = r_\mu + 2N_\mu,$$

where $r_\mu$ has only value of 0 or 1 and $N_\mu$ implies the position of the cell with double lattice constant $2a$. 

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*Talk presented by H. So. This work was supported in part by Grants-in-Aid for Scientific Research No. 13135209 from the Japan Society for the Promotion of Science.
and $\vec{\epsilon}$ that vector along $\vec{e}$ whose components are only 0 or 1. 

3. SYMMETRIES IN A STAGGERED FERMI ON

3.1. Chiral symmetry

For the staggered fermion, we can consider the exact chiral symmetry, 

$$\psi' = \exp(i\theta \Gamma_{2D+1})\psi, \quad \bar{\psi}' = \bar{\psi} \exp(i\theta \Gamma_{2D+1}).$$

Under the transformation, it can be shown that our Dirac action is invariant by the following relation,

$$\{\Gamma_{2D+1}, \Gamma_{\mu, \vec{e}}\} = 0,$$

where

$$\Gamma_{2D+1}(r, r') = (-i)^D(\gamma_1 \gamma_1 \cdots \gamma_D \gamma_D)(r, r')$$

This is equivalent to a well-known even-odd transformation,

$$\xi_n = (-1)^{|n|}, \quad \bar{\xi}_n = -(-1)^{|n|} \bar{\xi}_n,$$

and explains why there exists a mass-protect symmetry in a staggered fermion on an odd-dimensional lattice because $2D$ is always even.

3.2. Rotation

For cubic lattice, we can rotate by the angle $\pi/2$ with respect to $\mu\nu$-plane as

$$R_{\mu\nu} [\psi(N)] = a_{\mu\nu} \psi(R_{\mu\nu}[N]),$$

$$R_{\mu\nu} [\bar{\psi}(N)] = \bar{\psi}(R_{\mu\nu}[N]) a_{\mu\nu}^\dagger,$$

where the prefactor of $\psi$,

$$a_{\mu\nu} = \frac{i}{2} \bar{\Gamma}_{2D+1}(\gamma_\mu - \gamma_\nu)(1 + \gamma_\mu \gamma_\nu),$$

is determined by the invariance of our Dirac action and the discrete rotation for a vector is

$$(R_{\mu\nu}[N])_\rho = \begin{cases} N_\rho & \rho \neq \mu, \nu \\ -N_\nu & \rho = \mu \\ N_\mu & \rho = \nu. \end{cases}$$

The character of this rotation is that $\gamma_\mu$ is transformed as a vector,

$$a_{\mu\nu} \gamma_\rho a_{\mu\nu}^\dagger = \begin{cases} \gamma_\rho & \rho \neq \mu, \nu \\ -\gamma_\nu & \rho = \mu \\ \gamma_\mu & \rho = \nu. \end{cases}$$

but $\bar{\gamma}_\mu$ is done under permutation,

$$a_{\mu\nu} \bar{\gamma}_\rho a_{\mu\nu}^\dagger = \begin{cases} \bar{\gamma}_\rho & \rho \neq \mu, \nu \\ \bar{\gamma}_\nu & \rho = \mu \\ \bar{\gamma}_\mu & \rho = \nu. \end{cases}$$

**Fig. 1. Ichimatsu-patterned lattice in two dimensions.**

We define a 'fundamental' spinor on a cell, 

$$\psi_r(N) \equiv \xi_{r+2N}, \quad \bar{\psi}_r(N) \equiv \bar{\xi}_{r+2N}. \quad (4)$$

The staggered Dirac operator is written as

$$D_{st}(n, n') = \sum_{\mu} \eta_{\mu}(n) \frac{\delta_{n, n'+\vec{\mu}} U^r_{n, \mu} - \delta_{n, n'-\vec{\mu}} U^r_{n, \mu}}{2a}$$

where $d_{\mu, \vec{e}}$ is a generalized difference operator defined as 

$$d_{\mu, \vec{e}} = \frac{1}{2d} \sum_{\vec{e}} (-1)^{\vec{e}} \vec{e} \nabla_{\mu}, \quad (6)$$

and $\nabla_{\mu}$ is a backward difference operator along $\mu$-direction at the place, $2N + \vec{e}$. The coefficient matrices in eq. (5),

$$(\Gamma_{\mu, \vec{e}}) \equiv ((\sigma_3 \varepsilon_1 \otimes \cdots \otimes \sigma_3 \varepsilon_D)$$

$$\times (\sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1 \otimes 1 \otimes \cdots \otimes 1)), \quad (7)$$

are base of SO(2D) Clifford algebra, $\gamma_\mu = \Gamma_{\mu, \vec{e}}$ and $\bar{\gamma}_\mu = -i \Gamma_{\mu, \vec{e}}$, and their odd products. Note that $\vec{e}$ and $\vec{e}$ are bit-valued D-dimensional vectors whose components are only 0 or 1. $\vec{e}_\mu$ is a unit vector along $\mu$-direction.
3.3. Parity

For the odd-dimensional space, a usual parity transformation does not imply disconnected one. Instead, to discuss general dimensions, we adopt \( \mu \)-directional reflection,

\[
P_\mu[\psi(N)] = \Gamma_{2D+1}\gamma_\mu\psi(P_\mu[N]),
\]
\[
P_\mu[\bar{\psi}(N)] = \bar{\psi}(P_\mu[N])\gamma_\mu\Gamma_{2D+1},
\]

where \((P_\mu[N])_\rho = \begin{cases} N_\rho, & \rho \neq \mu \\ -N_\rho, & \rho = \mu. \end{cases}\)

(16)

It is noted that under the transformation, (14), our action is invariant.

3.4. Scalar and pseudo-scalar modes

Local meson operators are defined as

\[
\bar{\psi}_r(N)\Gamma_{\tau,\tau'}\psi_{\tau'}(N).
\]

(17)

We concentrate on scalar and pseudo-scalar modes. As discussion of sect. 3.2 and 3.3, only two scalar meson operators,

\[
M_1 = \bar{\psi}_r(N)\delta_{\tau,\tau'}\psi_{\tau'}(N),
\]

(18)

and

\[
M_2 = \bar{\psi}_r(N)(\sum_\mu \tilde{\zeta}_\mu)_{\tau,\tau'}\psi_{\tau'}(N),
\]

(19)

are permitted. For pseudo-scalar meson operators, we find only the following two modes,

\[
M_3 = \bar{\psi}_r(N)(\gamma_1\gamma_2\cdots\gamma_D)_{\tau,\tau'}\psi_{\tau'}(N),
\]

(20)

and

\[
M_4 = \bar{\psi}_r(N)(\gamma_1\gamma_2\cdots\gamma_D)\sum_\mu \tilde{\zeta}_\mu)_{\tau,\tau'}\psi_{\tau'}(N). \]

(21)

4. DISCUSSIONS

On a D-dimensional cubic lattice, a naive discretized fermion has \(2^D - 1\) doublers per one mode. The number of sites in a cell just corresponds to that of an original mode plus doublers which is exactly dimension of irreducible representation of SO(2D) Clifford algebra. It leads to uniqueness of rotational, chiral and parity transformations.

For our staggered Dirac spinor, we should formulate the fermion gauge-covariantly. It is formally possible if one connects the site of the fermion with the origin of the cell by a product of some link variables.

Although we find only symmetric charge conjugation matrices, \(C\), we can show the invariance under \(C\), PT, CPT. That means that our staggered fermion is a vector theory.

By using our formulation, we find an scalar mode and a pseudo-scalar one in addition to well-known scalar and pseudo-scalar modes. With a bare mass, operators are mixed between two scalar modes and between pseudo-scalar ones. To get the lowest meson mass, the mixing effect should be solved before taking the continuum limit. In a cell, we may write down scalar modes by link variables. Our first clue to supersymmetric extension is the yukawa coupling term. From the discussion in sect. 3 and the charge conjugation matrix, we have only one mode for the scalar in our action with nearest neighbor interactions. Exceptionally, for \(D=2\) case, there are two candidates. Further investigation is expected.

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