The facets of the spanning trees polytope

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Abstract
Let \( G = (V, E) \) be an undirected graph. The spanning trees polytope \( P(G) \) is the convex hull of the characteristic vectors of all spanning trees of \( G \). In this paper, we describe all facets of \( P(G) \) as a consequence of the facets of the bases polytope \( P(M) \) of a matroid \( M \), i.e., the convex hull of the characteristic vectors of all bases of \( M \).

Keywords Spanning trees · Polytope · Facets · Matroid · Bases polytope · Locked subgraphs

1 Introduction
Sets and their characteristic vectors will not be distinguished. We refer to Bondy and Murty (2008), Oxley (1992), and Schrijver (2003), respectively, about graphs, matroids, and polyhedra terminology and facts. Readers who are familiar with matroid theory can skip the next three paragraphs.

Given a finite set \( E \), a matroid \( M \) defined on \( E \), is the pair \( (E, B(M)) \) where \( B(M) \) is a nonempty class of subsets of \( E \) satisfying the following basis exchange axiom: for any pair \( (B_1, B_2) \in (B(M))^2 \) and for any \( e \in B_1 \setminus B_2 \), there exists \( f \in B_2 \setminus B_1 \), such that \( B_1 \cup \{f\} \setminus \{e\} \in B(M) \). In this case, \( E \) is the ground set of \( M \) and \( B(M) \) is the class of bases of \( M \). It follows that all bases of \( M \) have the same cardinality. We can define the dual \( M^* \) of \( M \) as the matroid \( (E, B(M^*)) \) where \( B(M^*) = \{E \setminus B \text{ such that } B \in B(M)\} \). The rank function of \( M \), denoted by \( r \), is a nonnegative integer function defined on \( 2^E \), the class of subsets of \( E \), such that \( r(X) = \max\{|X \cap B| \text{ for all } B \in B(M)\} \) for any \( X \subseteq E \). The rank of \( M \), denoted by \( r(M) \), is \( r(E) \), and it is equal to the cardinality of any basis. The rank function of \( M^* \), denoted by \( r^* \), is called the dual rank function of \( M \). It is not difficult to see that \( r^*(X) = |X| - r(E) + r(E \setminus X) \)

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for any \( X \subseteq E \). Finally, \( P(M) \) is the convex hull of the characteristic vectors of all bases of \( M \), and it is called the bases polytope of \( M \).

Given \( e \in E \), we can define two operations. The first one is called the deletion of \( e \), denoted by \( M \setminus e \), and defined by \( M \setminus e = (E \setminus \{e\}, \mathcal{B}(M \setminus e)) \), where \( \mathcal{B}(M \setminus e) = \{B \in \mathcal{B}(M) \mid e \notin B\} \). The second one is called the contraction of \( e \), denoted by \( M/e \), and defined by \( M/e = (M^*/e)^* \), i.e., the matroid dual of the deletion of \( e \) in the dual, which means that \( \mathcal{B}(M/e) = \{B \setminus \{e\} \mid B \in \mathcal{B}(M) \text{ and } e \notin B\} \). For any subset \( X \subseteq E \), we denote by \( M \setminus X \) (respectively, \( M/X \)), the matroid obtained by doing successive deletions (respectively, contractions) of all elements of \( X \). A matroid \( N \) is called a minor of \( M \) if it is obtained by successive operations of deletion and/or contraction. In this case, we denote by \( E(N) \), the ground set of \( N \). A subset \( X \subseteq E \) is called closed if \( r(X) = r(Y) + r(X \setminus Y) \). Otherwise, \( X \) is called nonseparable.

Moreover, \( M|X \) is the matroid defined by \( (X, \mathcal{B}(X)) \), where \( \mathcal{B}(X) = \{B \cap X \mid B \in \mathcal{B}(M) \text{ and } |B \cap X| = r(X)\} \). In other words, \( M|X = M \setminus (E \setminus X) \), and \( M^*|(E/X) = (M/X)^* \). A matroid \( M \) is disconnected if \( E \) is separable in \( M \). We say that \( M \) is 2-connected if it is not disconnected. It is not difficult to see that \( M \) is 2-connected if and only if \( M^* \) is too. A minor \( N \) of \( M \) is a 2-connected component of \( M \) if \( N \) is 2-connected and \( E(N) \) is maximal (by inclusion) for this property. The direct sum of two matroids \( M_1 \) and \( M_2 \), denoted by \( M_1 \oplus M_2 \), is defined by the matroid \( (E(M_1) \cup E(M_2), \mathcal{B}(M_1) \cup \mathcal{B}(M_2)) \), where \( \mathcal{B}(M_1) \cup \mathcal{B}(M_2) \) is the class of subsets \( B_1 \cup B_2 \) such that \( (B_1, B_2) \in \mathcal{B}(M_1) \times \mathcal{B}(M_2) \). It is not difficult to see that any disconnected matroid is the direct sum of its 2-connected components. We denote by \( c_2(M) \) the number of the 2-connected components of \( M \), which means that \( M \) is 2-connected if and only if \( c_2(M) = 1 \).

Suppose that \( M \) (and \( M^* \)) is 2-connected. A subset \( L \subseteq E \) is called a locked subset of \( M \) if \( M|L \) and \( M^*|(E \setminus L) \) are 2-connected, and their corresponding ranks are at least 2, i.e., \( \min\{r(L), r^*(E \setminus L)\} \geq 2 \). In other words, \( L \) is locked in \( M \) if and only if \( M|L \) and \( M/L \) are 2-connected, and \( r(L) \geq \max\{2, 2 + r(E) - |E\setminus L|\} \). It is not difficult to see that if \( L \) is locked then both \( L \) and \( E \setminus L \) are closed, respectively, in \( M \) and \( M^* \). (That is why we call it locked). For a disconnected matroid \( M \), locked subsets are those of the 2-connected components of \( M \). Locked subsets were introduced by Chaourar (2002, 2008, 2011) to solve many combinatorial problems in matroids. The class of locked subsets of \( M \) is denoted by \( L(M) \).

A parallel closure \( P \subseteq E \) of \( M \) is a closed subset with \( r(P) = 1 \) (and any \( e \in P \) satisfy \( r(e) = 1 \)). A coparallel closure \( S \subseteq E \) of \( M \) is a parallel closure of the dual \( M^* \). In the case of a 2-connected matroid \( M \), a parallel closure \( P \) (respectively, coparallel closure \( S \)) is essential if \( M/P \) (respectively, if \( M\setminus S \)) is 2-connected. The class of parallel (respectively, coparallel, essential parallel, and essential coparallel) closures of \( M \) is denoted by \( \mathcal{P}(M) \) (respectively, \( \mathcal{S}(M) \), \( \mathcal{P}_1(M) \), and \( \mathcal{S}_0(M) \)).

A subset \( L \subseteq E \) is called a general locked subset of \( M \) if \( M|L \) and \( M/L \) are both 2-connected when \( M \) is also 2-connected. In the general case, \( L \) is a general locked subset of \( M \) if \( c_2(M|L) = c_2(M/L) = c_2(M) \). Note that, in this case, \( 1 \leq r(L) \leq r(E) \), and \( L \) is not necessarily closed.
A subset $I \subseteq E$ is called independent if there exists a basis $B \in \mathcal{B}(M)$ such that $I \subseteq B$. The class of independent subsets of $M$ is denoted by $I(M)$, the convex hull of the characteristic vectors of all independent subsets by $P_I(M)$, and it is called the independence set polytope of $M$.

A minimal description of $P_I(M)$ was given by Edmonds (1970) as follows.

**Theorem 1** (Edmonds 1970) Let $M$ be a loopless matroid defined on $E$. A minimal description of $P_I(M)$ is the set of all $x \in \mathbb{R}^E$ satisfying the following constraints:

\[
\begin{align*}
  x(e) & \geq 0 \text{ for any } e \in E \quad (01) \\
  x(S) & \leq r(S) \text{ for any closed and nonseparable } S \subseteq E \quad (02)
\end{align*}
\]

Since $P(M) = P_I(M) \cap \{x \in \mathbb{R}^E \text{ such that } x(E) = r(E)\}$, one can think that the facets of the independence set polytope are kept for the bases polytope. This is not true as shown in (Grotschel 1977, Theorem 9.9, page 41) as well as in Fujishige (1984), Feichtner and Sturmfels (2005), and Hibi et al (2019) (see Kolbl 2020) as follows.

**Theorem 2** (Grötschel 1977) Let $M$ be a loopless matroid defined on $E$, and $E_i$, $i = 1, ..., c_2(M)$ be the minimal separation of $E$ in $M$, i.e., $M \setminus E_i$ are the 2-connected components of $M$. A minimal description of $P(M)$ is the set of all $x \in \mathbb{R}^E$ satisfying the following constraints:

\[
\begin{align*}
  x(e) & \geq 0 \text{ for any } e \in E \text{ with } c_2(M \setminus e) = c_2(M) \quad (1) \\
  x(L) & \leq r(L) \text{ for any closed and nonseparable } L \subseteq E \text{ with } c_2(M/L) = c_2(M) \quad (2) \\
  x(E_i) & = r(E_i), \text{ } i = 1, ..., c_2(M) \quad (3)
\end{align*}
\]

For any undirected graph $G$, and any subgraph $H$ of $G$, we denote by $V(H)$ (respectively, $E(H)$), the set of vertices (respectively, edges) of $H$. For any subset of vertices $U \subseteq V(G)$, we denote by $G(U)$ (respectively, $E(U)$) the induced subgraph of $G$ based on the vertices of $U$ (respectively, $E(G(U))$). For any subset of edges $F \subseteq E(G)$, we denote by $V(F)$ the set of vertices incident to any edge of $F$. We also use the notations: $n = |V(G)|$, $m = |E(G)|$, and $n_H = |V(H)|$, $m_H = |E(H)|$ for any subgraph or any subset of edges $H$ of $G$. For any subset $F \subseteq E$, and any $x \in \mathbb{R}^E$, $x(F) = \sum_{e \in F} x(e)$.

For an induced subgraph $H$ of $G$, $G' = (V(E(G) \setminus E(H)), E(G) \setminus E(H))$ is called the complementary subgraph of $H$ in $G$, i.e., the subgraph obtained by removing the edges of $H$ and vertices which are not incident to any edge of $E(G) \setminus E(H)$. Moreover, for any $F \subseteq E(G)$, $H \cup F$ is the subgraph $(V(H) \cup V(F), E(H) \cup F)$.

Matroids generalize graphs. Given an undirected connected graph $G$, we can define the corresponding (graphical) matroid $M(G)$ as the pair $(E(G), T(G))$, where $T(G)$ is the class of spanning trees of $G$. By analogy, $P(G) = P(M(G))$ is the spanning trees polytope, i.e., the convex hull of the characteristic vectors of all spanning trees of $G$, where $M(G)$ is the corresponding (graphical) matroid of $G$. Moreover, $P_I(G) = P_I(M(G))$ is the convex hull of the characteristic vectors of all forests of $G$. A locked subgraph $H$ of $G$ is a subgraph for which $E(H)$ is a locked subset of $M(G)$. Deletions and contractions in the corresponding (graphical) matroid $M(G)$ correspond
to classical deletions and contractions in \(G\). A minor \(N\) of \(M(G)\) correspond to the graphical matroid \(M(H)\) of a minor \(H\) of \(G\). A graphical matroid is 2-connected if and only if its corresponding graph is 2-connected. For any connected induced subgraph \(G(U)\) of \(G\), \(r(E(U)) = |U| - 1\).

In this paper, we give a minimal description of \(P(G)\) by means of graph theory. We suppose that considered matroids and graphs are 2-connected because \(P(M)\) (respectively, \(P(G)\)) is the cartesian product of the corresponding polytopes in the 2-connected components.

Schrijver claimed (Schrijver 2003, page 862, discussion after Corollary 50.7d), and referring to a result of Grotschel (1977), that the nontrivial facets of \(P(G)\) are described by induced and 2-connected subgraphs as for the forests polytope \(P_I(G)\) (a minimal description of the forests polytope was done by Pulleyblank (1989)a st h es et of all \(x \in \mathbb{R}^E\) satisfying:

\[ x(e) \geq 0 \text{ for any edge } e, \quad \text{and } x(E(U)) \leq |U| - 1 \text{ for any } U \subseteq E \text{ inducing a 2-connected subgraph with } |U| \geq 2. \]

In this paper, we show that some further assumptions are needed (see Theorem 8). We present a counterexample to Schrijver’s claim at the end of Sect. 2.

Actually this is not what Grötschel wrote in his book (Grotschel 1977, Theorem 11.2, page 49). Grötschel applied his Theorem 2 for the facets of the bases polytope to the graphical case and got less facets than those of the forest polytope.

Our main contribution in this paper is a new minimal description for the bases polytope as well as for the spanning tree polytope. Moreover we characterize by means of graph theory these facets.

The remainder of the paper is organized as follows: in Sect. 2, we give a new minimal description of \(P(M)\) and \(P(G)\), then we give alternative minimal descriptions of \(P(M)\) and \(P(G)\) in Sect. 3. Finally, we conclude in Sect. 4.

2 Facets of the spanning trees polytope

First we give an alternative minimal description of \(P(M)\) as follows.

**Theorem 3** Let \(M\) be a 2-connected matroid defined on \(E\). A minimal description of \(P(M)\) is the set of all \(x \in \mathbb{R}^E\) satisfying the following constraints:

\[ x(L) \leq r(L) \text{ for any general locked subset } L \subseteq E \]  
\[ x(E) = r(E) \]

**Proof** Since \(M\) is 2-connected, constraint (5) is equivalent to constraint (3).

Let \(L\) be a closed and nonseparable subset in \(M\) such that \(M/L\) is 2-connected. It follows that \(L\) is a general locked subset with \(1 \leq r(L) \leq r(E) - 1\). Thus constraint (4), for this case, is equivalent to constraint (2).

Now let \(e \in E\) such that \(M/e\) is 2-connected. It follows that \(L = E\setminus\{e\}\) is a general locked subset with \(r(L) = r(E)\) because \(M\) is 2-connected. This yields \(x(e) = x(E) - x(L) \geq r(E) - r(L) = 0\) because of constraints (5) and (4) for \(L\). And vice-versa, if \(x(e) \geq 0\) then \(x(L) \leq x(E) = r(E) = r(L)\) because of constraint (5). So constraint (4), for the case \(r(L) = r(E)\), is equivalent to constraint (1).
Thus the above description is equivalent to the minimal description of Theorem 2, so it is also minimal.

We need the following two lemmas for the second alternative minimal description of $P(M)$.

**Lemma 4** If $S \in \mathcal{S}(M)$ then $r(E \setminus S) = r(E) - |S| + 1$

**Proof** Since $r^*(S) = 1$ and $r(E) = |E| - r^*(E)$, we have: $r(E \setminus S) = |E| - |S| - r^*(E) + 1 = r(E) - |S| + 1$, and we are done. □

**Lemma 5** Let $M$ be a 2-connected matroid defined on $E$, and $L \subseteq E$ be a general locked subset of $M$. One of the following cases holds for $L$:

1. either $r(L) = 1$ and $L \in \mathcal{P}_1(M)$;
2. or $r^*(E \setminus L) = 1$ and $E \setminus L \in \mathcal{S}_0(M)$;
3. or $L$ is locked in $M$.

**Proof** Suppose that $L$ is not locked in $M$.

**Case 1**: $r(L) = 1$.

It follows that $L$ is an essential parallel closure by definition.

**Case 2**: $r^*(E \setminus L) = 1$

It follows that $S = E \setminus L$ is an essential coparallel closure because $M \setminus S = M|L$ is 2-connected. □

Theorem 3, Lemmas 4 and 5 imply the following second minimal description of $P(M)$.

**Corollary 6** Let $M$ be a 2-connected matroid defined on $E$. A minimal description of $P(M)$ is the set of all $x \in \mathbb{R}^E$ satisfying (5) and the following constraints:

\begin{align*}
    x(P) &\leq 1 \text{ for any } P \in \mathcal{P}_1(M) & (6) \\
    x(S) &\geq |S| - 1 \text{ for any } S \in \mathcal{S}_0(M) & (7) \\
    x(L) &\leq r(L) \text{ for any locked subset } L \subseteq E & (8)
\end{align*}

Note that our description is more natural than Grötschel’s one because it captures the duality (essential parallel and coparallel closures, locked subsets and their complements) involved in the bases polytope ($x \in P(M)$ if and only if $1_E - x \in P(M^*)$). A direct proof for this minimal description of $P(M)$ has been given by using duality by Chaourar (2018).

For the graphical case, a parallel closure is the edge set of an induced subgraph on two vertices, and a coparallel closure is a series closure, i.e., a maximal set of edges forming a simple path for which all involved vertices except its two terminals have degree 2. An essential parallel closure is the edge set of an induced subgraph on two vertices whose contraction keep the graph 2-connected. An essential coparallel closure is the edge set of a series closure whose deletion keep the graph 2-connected.

It remains to translate lockedness in graphical terms.

First, we prove the following lemma.
Lemma 7 Let $H$ be a 2-connected subgraph of $G$, and $\{L_1, L_2\}$ be a partition of $E(\overline{H})$ such that $(V(L_i), L_i)$ is connected, $i = 1, 2$. Then $G(V(G) \setminus V(H))$ is connected if and only if $n_H + n < n_{H \cup L_1} + n_{H \cup L_2}$.

Proof

\[ n_H + n < n_{H \cup L_1} + n_{H \cup L_2} \]
\[ \iff n_H + n_H + |V(\overline{H})| - |V(H) \cap V(\overline{H})| < n_H + n_L - |V(H) \cap V(L_1)| + n_H + n_L - |V(H) \cap V(L_2)| \]
\[ \iff 2n_H + |V(\overline{H})| - |V(H) \cap V(\overline{H})| < 2n_H + n_L + |V(\overline{H})| - |V(H) \cap V(L_1)| - |V(H) \cap V(L_2)| \]
\[ \iff 2n_H + |V(\overline{H})| - |V(H) \cap V(\overline{H})| < 2n_H + n_L + n_L - |V(H) \cap V(L_1)| - |V(H) \cap V(\overline{H})| - |V(H) \cap V(L_2)| \]
\[ \iff |V(\overline{H})| < n_L_1 + n_L_2 - |V(H) \cap V(L_1) \cap V(L_2)| \]
\[ \iff n_1 + n_2 - |V(L_1) \cap V(L_2)| < n_L_1 + n_L_2 - |V(H) \cap V(L_1) \cap V(L_2)| \]
\[ \iff |V(L_1) \cap V(L_2)| > |V(H) \cap V(L_1) \cap V(L_2)| \]
\[ \iff |V(L_1) \cap V(L_2)| \geq |V(H) \cap V(L_1) \cap V(L_2)| + 1 \]
\[ \iff (|V(L_1) \cap V(L_2)| \setminus V(H)) \geq 1 \]

which means that $G(V(G) \setminus V(H))$ is connected. \qed

Now we can characterize locked subgraphs by means of graphs terminology.

Theorem 8 $H$ is a locked subgraph of $G$ if and only if $H$ is an induced and 2-connected subgraph such that $3 \leq n_H \leq n - 1$, $m_{\overline{H}} \geq n_{\overline{H}}$ or $|V(H) \cap V(\overline{H})| \geq 3$, and $G(V(G) \setminus V(H))$ is a connected subgraph.

Proof Without loss of generality, we can suppose that $G$ is 2-connected.

It is not difficult to see that $E(H)$ is closed and 2-connected in $M(G)$ if and only if $H$ is an induced and 2-connected subgraph of $G$.

Now, suppose that $E(H)$ is closed and 2-connected in $M(G)$, and $E(G) \setminus E(H)$ is 2-connected in the dual matroid $M^*(G)$ (i.e., $E(H)$ is locked in $M(G)$). Let $\{L_1, L_2\}$ be a partition of $E(G) \setminus E(H)$ such that the subgraph $(V(L_i), L_i)$ is connected, $i = 1, 2$. Such partition exists because if $H$ is locked then $G/H$ is 2-connected and $|E(G) \setminus E(H)| \geq 2$, so we can choose $|L_1| = 1$, $L_2 = E(H) \setminus L_1$, and $(V(L_2), L_2)$ is still connected. It follows that $r^*(E(G) \setminus E(H)) < r^*(L_1) + r^*(L_2)$, i.e., $|E(G) \setminus E(H)| - r(E(G)) + r(E(H)) < |L_1| + |L_2| - 2r(E(G)) + r(E(H) \cup L_1) + r(E(H) \cup L_2)$. In other words, $r(E(H)) + r(E(G)) < r(E(H) \cup L_1) + r(E(H) \cup L_2)$, which is equivalent to: $n_H - 1 + n - 1 < n_{H \cup L_1} - 1 + n_{H \cup L_2} - 1$, i.e., $G(V(G) \setminus V(H))$ is connected according to Lemma 7.

Let check the condition: $\min\{r(E(H)), r^*(E(\overline{H}))\} \geq 2$. Since $r(E(H)) = n_H - 1$, we have $r(E(H)) \geq 2$ if and only if $n_H \geq 3$. Moreover, $r^*(E(G) \setminus E(H)) = m_{\overline{H}} + r(E(H)) - r(E(G)) = m_{\overline{H}} + n_H - n$ then we have $r^*(E(G) \setminus E(H)) \geq 2$ if and only if $n_H \geq 2 + n - m_{\overline{H}}$, i.e., $|V(H) \cap V(\overline{H})| \geq 2 + n - m_{\overline{H}}$ (inequality (*)). But, if $G$ is 2-connected and $G(V(G) \setminus V(H))$ is connected, then $|V(H) \cap V(\overline{H})| \geq 2$, and $m_{\overline{H}} - |V(H) \cap V(\overline{H})| \geq |V(\overline{H})| \setminus V(H)| - 1 \geq n_{\overline{H}} - |V(H) \cap V(\overline{H})| - 1$, i.e., $m_{\overline{H}} \geq n_{\overline{H}} - 1$. Thus, we have either $m_{\overline{H}} \geq n_{\overline{H}}$ or $m_{\overline{H}} = n_{\overline{H}} - 1$.\[\square\]
**Case 1:** If $m_H ≥ n_H$ then $2 + n_H - m_H ≤ 2$.

**Case 2:** If $m_H = n_H - 1$ then $H$ is a tree and $2 + n_H - m_H = 3$. If $|V(H) \cap V(\overline{H})| = 2$ then $E \setminus E(H) = E(\overline{H})$ is a series closure. Hence $E \setminus E(H)$ is a parallel closure in the dual of $M(G)$ and $r^*(E \setminus E(H)) = 1$. It follows that $H$ is not locked.

Therefore, in both cases, the inequality (*) is equivalent to $m_H ≥ n_H$ or $|V(H) \cap V(\overline{H})| ≥ 3$.

Furthermore, $E(H)$ is closed in $M(G)$ and distinct from $E$, i.e., $r(E(H)) ≤ r(E(G)) - 1$, which is equivalent to: $n_H ≤ n - 1$.

So the consequence for the spanning tree polytope is:

**Corollary 9** Let $G$ be a 2-connected graph with $n$ vertices. A minimal description of $P(G)$ is the set of all $x \in \mathbb{R}^{E(G)}$ satisfying the following constraints:

\begin{align*}
  x(P) &≤ 1 \text{ for any essential parallel closure } P \text{ of } G & (9) \\
x(S) &≥ |S| - 1 \text{ for any essential coparallel closure } S \text{ of } G & (10) \\
x(E(H)) &≤ n_H - 1 \text{ for any locked subgraph } H \text{ of } G & (11) \\
x(E(G)) = n - 1 & & (12)
\end{align*}

Note that many equivalent minimal descriptions can be given because $P(G)$ (as $P(M)$) is not full-dimensional ($\dim(P(G)) = |E(G)| - 1$ if $G$ is 2-connected). We discuss this issue in Sect. 3.

A direct consequence of Corollary 9 is the following corollary correcting the well-know idea about nontrivial facets of the spanning trees polytope.

**Corollary 10** Let $G$ be a 2-connected graph on $n$ vertices. Constraint (11) is redundant if $H$ is not locked.

**Proof** Let $H$ be an induced and 2-connected subgraph of $G$ which is not locked. It follows that $G/H$ is not 2-connected. Let $H_i$, $i = 1, \ldots, k$, be the 2-connected components of $G/H$. In this case, all subgraphs $G_i = G \setminus H_i$ are locked because, in the one hand, $G$ is 2-connected and so for all $G_i$, and in the other hand, all $G/G_i = G/(G \setminus H_i) = H_i$ are 2-connected. Let $t_{H_i}$ be the number of vertices of $H_i$ that are not common with $H$.

In this case, for $x \in P(G)$, we have: $x(E(H)) + (k-1)x(E(G)) = kx(E(G)) - x(E(G)) - x(E(H)) = kx(E(G)) - \sum_{i=1}^{k} x(E(H_i)) = \sum_{i=1}^{k} x(E(G_i)) = \sum_{i=1}^{k} x(E(G_i)) ≤ \sum_{i=1}^{k} (n_{G_i} - 1) = \sum_{i=1}^{k} (n - t_{H_i} - 1) = kn - k - \sum_{i=1}^{k} t_{H_i} = kn - k - (n - n_H) = kn - k - n + 1 + n_H - 1 = (k - 1)(n - 1) + n_H - 1.

But $x(E(G)) = n - 1$ (constraint 12), hence $x(E(H)) ≤ n_H - 1$.

Next we give here an example (see Fig.1) of an induced and 2-connected subgraph which is not locked.

Let $H = G((b, c, e, f))$, i.e., the circuit $bcebf$, $L_1 = \{ab, af\}$, and $L_2 = \{dc, de\}$. $H$ induces a 2-connected subgraph which is not locked because $n_H + n = 4 + 6 = 10 ≥ 10 = 5 + 5 = n_{H \cup L_1} + n_{H \cup L_2}$.

Note that this idea happened because it was thought that facets of the forests polytope are kept for one of its faces which is $P(G)$.

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Fig. 1 This is an example of an induced and 2-connected subgraph which is not locked

3 Alternative equivalent minimal descriptions of $P(M)$ and $P(G)$

In this section, we give alternative equivalent minimal descriptions of $P(M)$ and $P(G)$.

Now we consider the following constraints:

$$x(E \setminus P) \geq r(E) - 1 \text{ for any essential parallel closure } P \subseteq E \quad (13)$$

$$x(E \setminus S) \leq r(E \setminus S) \text{ for any essential coparallel closure } S \subseteq E \quad (14)$$

$$x(E \setminus L) \geq r(E) - r(L) \text{ for any locked subset } L \subseteq E \quad (15)$$

**Theorem 11** Let $\mathcal{P} \subseteq \mathcal{P}_1(M), \mathcal{S} \subseteq \mathcal{S}_0(M), \text{ and } \mathcal{L} \subseteq \mathcal{L}(M)$. A minimal description of $P(M)$ is the set of all $x \in \mathbb{R}^E$ satisfying the constraint (5), and the following constraints:

- Constraint (6) for any $P \in \mathcal{P}$, and constraint (13) for any $P \in \mathcal{P}_1(M) \setminus \mathcal{P}$,
- Constraint (7) for any $S \in \mathcal{S}$, and constraint (14) for any $S \in \mathcal{S}_0(M) \setminus \mathcal{S}$,
- Constraint (8) for any $L \in \mathcal{L}$, and constraint (15) for any $L \in \mathcal{L}(M) \setminus \mathcal{L}$.

**Proof** It suffices to see that the corresponding pairs of constraints ((6) and (13), (7) and (14), (8) and (15)) are equivalent if constraint (5) is satisfied ($r(E) = x(E) = x(F) + x(E \setminus F)$ for any, respectively, essential parallel closure, essential coparallel closure, or locked subset $F$), and by using Lemma 4 for the pair (7) and (14). $\square$

And similarly, we have alternative equivalent descriptions of $P(G)$ by translating matroid theory terms to graphs terminology as it was characterized in Sect. 2.

4 Conclusion

We have given new formulations of all facets of $P(G)$ correcting a wrong well-known idea about nontrivial ones of them.

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**Declarations**

**Conflict of interest** The (single) author states that there is no conflict of interest.
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