TRANSCENDENTAL SERIES OF RECIPROCALS OF FIBONACCI AND LUCAS NUMBERS

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Abstract. Let $F_1 = 1, F_2 = 1, \ldots$ be the Fibonacci sequence. Motivated by the identity $\sum_{k=0}^{\infty} \frac{1}{F_{2^k}} = \frac{7 - \sqrt{5}}{2}$, Erdős and Graham asked whether $\sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$ is irrational for any sequence of positive integers $n_1, n_2, \ldots$ with $\frac{n_{k+1}}{n_k} \geq c > 1$. We resolve the transcendence counterpart of their question: as a special case of our main theorem, we have that $\sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$ is transcendental when $\frac{n_{k+1}}{n_k} \geq c > 2$. The bound $c > 2$ is best possible thanks to the identity at the beginning.

This paper provides a new way to apply the Subspace Theorem to obtain transcendence results and extends previous non-trivial results obtainable by only Mahler’s method for special sequences of the form $n_k = d^k + r$.

1. INTRODUCTION

Let $F_1 = 1, F_2 = 1, \ldots$ be the Fibonacci sequence and let $L_1 = 1, L_n = F_{n-1} + F_n$ for $n \geq 2$ be the Lucas sequence. In the chapter “Irrationality and Transcendence” of their book [EG80] p. 64–65], starting from the Millin series:

$$\sum_{k=0}^{\infty} \frac{1}{F_{2^k}} = \frac{7 - \sqrt{5}}{2},$$

Erdős and Graham asked the following:

**Question 1.1** (Erdős-Graham, 1980). Is it true that $\sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$ is irrational for any sequence $n_1 < n_2 < \ldots$ with $\frac{n_{k+1}}{n_k} \geq c > 1$?

The transcendence counterparts of this and many questions in [EG80] Chapter 7 were implicit throughout the chapter, hence its title. Indeed, this topic inspired intense research activities most of which involved the so called Mahler’s method. As a consequence of our main result (see Theorem 1.3), we resolve the transcendence version of Question 1.1 even when one is allowed to randomly mix the Fibonacci and Lucas numbers:

**Theorem 1.2.** Let $c > 2$ and let $n_1 < n_2 < \ldots$ be positive integers such that $\frac{n_{k+1}}{n_k} \geq c$ for every $k$. Then the number $\sum_{k=1}^{\infty} \frac{1}{f_k}$ is transcendental where $f_k \in \{F_{n_k}, L_{n_k}\}$ for every $k$.

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Although Fibonacci and Lucas numbers have been discovered for hundreds of years, some of their basic properties have been established only recently thanks to powerful modern methods, for example [BMS06, Ste13]. The key ingredient of the proof of our main theorems is a new application of the Subspace Theorem in treating transcendence of series in which it is hard to control the denominators of the partial sums. Before providing more details about the method, let us provide a very brief and incomplete survey of known results on irrationality and transcendence of sums like $\sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$. In a nutshell, we can divide previous results in two groups.

The first group treats series in which the denominators of the partial sums are very small compared to the reciprocals of the error terms. This includes work of Mignotte on the transcendence of $\sum_{k=1}^{\infty} \frac{1}{k!F_{2^k}}$ and $\sum_{k=1}^{\infty} \frac{2 + (-1)^k}{F_{2^k}}$ [Mig71, Mig77]. In fact, these results predate Erdős-Graham question. As another example, consider $s := \sum_{k=0}^{\infty} \frac{1}{F_{2^k+1}}$. For every sufficiently large integer $N$, thanks to divisibility properties of the Fibonacci sequence and the fact that $2^k + 1$ divides $2^{3k} + 1$, the denominator of $\sum_{k=1}^{N} \frac{1}{F_{2^k+1}}$ is at most $\prod_{k=[N/3]}^{N} F_{2^k+1}$ which is $o(F_{2N+1})$, hence $s$ must be irrational. We refer the readers to [Bad93] and the references there for similar results. In fact, if $n_1 < n_2 < \ldots$ satisfies $\frac{n_{k+1}}{n_k} \geq c > 2$ then $\sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$ is irrational since $F_{n_1} \cdots F_{n_N} = o(F_{2N+1})$ as $N \to \infty$. Likewise, if $\frac{n_{k+1}}{n_k} \geq c > 3$ then $\sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$ is transcendental by applying Roth’s theorem and the fact that $F_{n_1} \cdots F_{n_N} = O\left(F_{n_{N+1}}^{(1/2) - \epsilon}\right)$ for an appropriate $\epsilon > 0$. However, replacing those easy bounds 2 and 3 respectively by smaller numbers for irrationality and transcendence problems appears to be a very difficult task.

The second group constitutes the majority of results in this topic. In 1975, Mahler [Mah75] reproved Mignotte’s result using the method he had invented nearly 50 years earlier (see Nishioka’s notes [Nis96] for an introduction to Mahler’s method). This method is applicable when the sequence $n_k$ has the special form $n_k = d^k + r$ where $d, r \in \mathbb{Z}$ with $d \geq 2$. We refer the readers to [BT94, DKT02, KKS09] and the references there for further details. In fact, before this paper, there has not been one result establishing the transcendence of $\sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$ for an arbitrary sequence $n_k$ with $\frac{n_{k+1}}{n_k} \geq c$ for any single value $c < 3$ (as explained above, $c > 3$ is the easy bound due to an immediate application of Roth’s theorem).

From now on, let $\alpha \neq \pm 1$ be a real quadratic unit; this means $\mathbb{Q}(\alpha)$ is real quadratic and $\alpha \neq \pm 1$ a unit in the ring of algebraic integers. Let $\sigma$ be the non-trivial automorphism of $\mathbb{Q}(\alpha)$ and let $\beta = \sigma(\alpha)$. Without loss of generality, we
Let $c > f$ and assume $n_1 = a$ we define $s$ again. Let $\text{Example 1.4.}$

**Theorem 1.3.** Let $a_n, b_n, c_n$ for $n \geq 1$ be sequences of real numbers with the following properties:

- For every $n \geq 1$, $c_n \in \mathbb{Q}$, $a_n, b_n \in \mathbb{Q}(\alpha)$, and $u_n := a_n \alpha^n - b_n \beta^n \in \mathbb{Q}$.
- $\lim_{n \to \infty} \frac{h(a_n)}{n} = \lim_{n \to \infty} \frac{h(b_n)}{n} = \lim_{n \to \infty} \frac{h(c_n)}{n} = 0$.

Let $c > 2$ and let $n_1 < n_2 < \ldots$ be positive integers such that $\frac{n_{k+1}}{n_k} \geq c$, $u_{n_k} \neq 0$, and $c_{n_k} \neq 0$ for every $k$. Then the series $\sum_{k=1}^{\infty} \frac{c_{n_k}}{u_{n_k}}$ is transcendental.

**Example 1.4.** Let $n_1 < n_2 < \ldots$ be as in Theorem 1.2. Let $s = \sum_{k=1}^{\infty} \frac{1}{f_k}$ where $f_k \in \{F_{n_k}, L_{n_k}\}$ for every $k$. Note that $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ and $L_n = \alpha^n + \beta^n$ with $\alpha = \frac{1 + \sqrt{5}}{2}$ and $\beta = \frac{1 - \sqrt{5}}{2}$. We define $c_n = 1$ for every $n$. If $n \notin \{n_k : k \geq 1\}$, we define $a_n = b_n = 0$. If $n = n_k$ and $f_k = F_{n_k}$, define $a_n = b_n = \frac{1}{\sqrt{5}}$. Finally if $n = n_k$ and $f_k = L_{n_k}$, define $a_n = 1$ and $b_n = -1$. This explains why Theorem 1.2 is a special case of Theorem 1.3.

**Example 1.5.** One can consider linear recurrence sequences of rational numbers of the form $u_n = A(n)\alpha^n + B(n)\beta^n$ where $A(t), B(t) \in \mathbb{Q}(\alpha)[t]$. Then Theorem 1.3 implies that $\sum_{k=1}^{\infty} \frac{1}{u_{n_k}}$ is transcendental since we may choose $c_n = 1$, $a_n = A(n)$, and $b_n = -B(n)$. Note that $h(a_n) = O(\log n)$ and $h(b_n) = O(\log n)$ in this case.

There have been two general transcendence results using the Subspace Theorem recently and both involve values of a power series $\sum d_n z^n$ at an algebraic number $z_0$. One is a result of Adamczewski-Bugeaud [AB07] extending an earlier work of Triman \cite{1297}. In their work, the coefficients of $d_n$’s form an automatic sequence and $z_0$ is the reciprocal of a Pisot number. The authors rely on the repeating pattern of automatic sequences to apply the Subspace Theorem using linear forms in three variables. The other is a result of Corvaja-Zannier [CZ02] treating the case that $\sum d_n z^n$ is lacunary with positive real coefficients and $z_0 \in (0, 1)$. The problem considered here is different from all the above. While it is true that one can express $\sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$ as the value of a power series at $1/\alpha$ with $\alpha = \frac{1 + \sqrt{5}}{2}$, neither the coefficients are automatic nor the series is lacunary for an arbitrary choice of $n_k$ with $n_{k+1}/n_k \geq c > 2$.

The more subtle difference and key reason for the difficulty in settling our current problem are as follows. Let us consider the example $s = \sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$ and $\alpha = \frac{1 + \sqrt{5}}{2}$ again. Let $s_N = \sum_{k=1}^{N} \frac{1}{F_{n_k}}$ be the sequence of partial sums so that $|s - s_{N-1}| = \ldots$
Now the usual idea is to fix a large integer \( P \), then truncate each

\[
\frac{1}{F_{n,P,i}} = s_{N,i} + O(|\alpha|^{-nN})
\]

where \( s_{N,i} \) is a finite sum of units for \( 1 \leq i \leq P - 1 \). Then we have:

\[
|s - s_{N,P} - s_{N,1} - \ldots - s_{N,P-1}| = O(|\alpha|^{-nN})
\]

and after assuming that \( s \) is algebraic, one might attempt to apply the Subspace Theorem to this equation for \( s, s_{N,P} \) and the individual units in each \( s_{N,i} \). The difference compared to work of Adamczewski-Bugeaud or Corvaja-Zannier is that while terms in their application of the Subspace Theorem are \( S \)-integers for an appropriate choice of a finite set of places \( S \) in an appropriate number field, here we cannot find such an \( S \) so that the term \( s_{N,P} \) above is an \( S \)-integer for infinitely many \( N \). For this reason, in our situation, when applying the Subspace Theorem we may have the contribution \( H(s_{N-M})D \) where \( D \) is the number of terms. With just the constraint \( c > 2 \), it is entirely possible for the above contribution to offset the error term \( O(|\alpha|^{-nN}) \) and one fails to apply the Subspace Theorem. Therefore new ideas are needed to overcome this crucial issue. Moreover, after one applies the Subspace Theorem, it remains a highly nontrivial task to arrive at the desired conclusion from the resulting linear relation. This paper promotes the innovation that one should start with a certain “minimal expression” before applying the Subspace Theorem in order to maximize the benefit of the resulting linear relation. We refer the readers to the discussion right after Proposition 3.6 for more details.

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## 2. The Subspace Theorem

The Subspace Theorem is one of the milestones of diophantine geometry in the last 50 years. The first version was obtained by Schmidt [Sch70] and further versions were obtained by Schlickewei and Evertse [Sch92, Eve96, ES02]. This section follows the exposition in the book of Bombieri-Gubler [BG06].

Let \( M_K = M_{Q,K} \cup M_{\omega}^K \) where \( M_{Q,K}^0 \) is the set of \( p \)-adic valuations and \( M_{Q,K}^\infty \) is the singleton consisting of the usual archimedean valuation. More generally, for every number field \( K \), write \( M_K = M_{Q,K}^\infty \cup M_{\omega}^K \) where \( M_{Q,K}^\infty \) is the set of archimedean places and \( M_{\omega}^K \) is the set of finite places.

Throughout this paper, we fix an embedding of \( \mathbb{Q} \) into \( \mathbb{C} \) and let \(| \cdot |\) denote the usual absolute value on \( \mathbb{C} \). Hence for a number field \( K \), the set \( M_K^\infty \) corresponds to the set of real embeddings and pairs of complex-conjugate embeddings of \( K \) into \( \mathbb{C} \). For every \( w \in M_K \), let \( K_w \) denote the completion of \( K \) with respect to \( w \) and denote \( d(w/v) = [K_w : \mathbb{Q}_v] \) where \( v \) is the restriction of \( w \) to \( \mathbb{Q} \). Following [BG06, Chapter 1], for every \( w \in M_K \), restricting to \( v \) on \( \mathbb{Q} \), we normalize \(| \cdot |_w\) as follows:

\[
|x|_w = |N_{K_w/\mathbb{Q}_v}(x)|_{K_w}^{1/[K_w:Q]}
\]

Let \( m \in \mathbb{N} \), for every vector \( u = (u_0, \ldots, u_m) \in K^{m+1} \setminus \{0\} \) and \( w \in M_K \), let

\[
|u|_w := \max_{0 \leq i \leq m} |u_i|_w.
\]

For \( P \in \mathbb{P}^m(\mathbb{Q}) \), let \( K \) be a number field such that \( P \) has a
representative \( u \in K^{m+1} \setminus \{0\} \) and define:

\[
H(P) = \prod_{w \in M_K} |u|_w.
\]

It is an easy fact that this is independent of the choice of \( u \) and the number field \( K \). Then we define \( h(P) = \log(H(P)) \). For \( \alpha \in \bar{Q} \), write \( H(\alpha) = H([\alpha : 1]) \) and \( h(\alpha) = \log(H(\alpha)) \). Later on, we will use the classical version of Roth’s theorem \([BG06, \text{Chapter 6}]\) to give a weak upper bound on \( n_{k+1}/n_k \):

**Theorem 2.1** (Roth’s theorem). Let \( \kappa > 2 \). Let \( s \) be a real algebraic number. Then there are only finitely many rational numbers \( s' \) such that

\[
|s' - s| \leq H(s')^{-\kappa}.
\]

Let \( m \in \mathbb{N} \), for every vector \( \mathbf{x} = (x_0, \ldots, x_m) \in K^{m+1} \setminus \{0\} \), let \( \tilde{x} \) denote the corresponding point in \( \mathbb{P}^m(K) \). For every \( w \in M_K \), denote \( |\mathbf{x}|_w := \max_{0 \leq i \leq m} |x_i|_w \).

We have:

**Theorem 2.2** (Subspace Theorem). Let \( n \in \mathbb{N} \), let \( K \) be a number field, and let \( S \subset M_K \) be finite. For every \( v \in S \), let \( L_v, \ldots, L_{vn} \) be linearly independent linear forms in the variables \( X_0, \ldots, X_n \) with \( K \)-algebraic coefficients in \( K_v \). For every \( \epsilon > 0 \), the solutions \( \mathbf{x} \in K^{n+1} \setminus \{0\} \) of the inequality:

\[
\prod_{v \in S} \prod_{j=0}^n \frac{|L_j(\mathbf{x})|_v}{|\mathbf{x}|_v} \leq H(\tilde{x})^{-n-1-\epsilon}
\]

are contained in finitely many hyperplanes of \( K^{n+1} \).

3. PRELIMINARY RESULTS AND PREPARATION FOR THE PROOF OF THEOREM 1.3

Throughout this section, we assume the notation in the statement of Theorem 1.3 and put \( A_k = a_{nk}, B_k = b_{nk}, C_k = c_{nk}, U_k = u_{nk} \) for every \( k \) to simplify the notation. From now on, assume that \( s := \sum_{k=1}^{\infty} \frac{C_k}{U_k} \) is algebraic and let \( K \) be the Galois closure of \( \mathbb{Q}(\alpha, s) \). For \( m \geq 1 \), let \( s_m = \sum_{k=1}^{m} \frac{C_k}{U_k} \) be the sequence of partial sums. We will repeatedly use the following observation: if \((t_n)_n \) is a sequence in \( K^* \) such that \( h(t_n)/n \to 0 \) as \( n \to \infty \) then \( |t_n|_v = e^{o(n)} \) as \( n \to \infty \) for every \( v \in M_K \); this means for every \( \epsilon > 0 \), we have \( e^{-\epsilon n} < |t_n|_v < e^{\epsilon n} \) for all sufficiently large \( n \).

We are given that \( C_k \neq 0 \) for every \( k \). We may assume that \( A_k B_k \neq 0 \) for every \( k \), as follows. Let \( \sigma \) denote the nontrivial automorphism of \( \mathbb{Q}(\alpha) \). Suppose that \( A_k = 0 \) then \( U_k = -B_k \beta^{n_k} \in \mathbb{Q}^* \). Applying \( \sigma \) gives \( U_k = -B_k \beta^{n_k} = -\sigma(B_k)/\alpha^{n_k} \), therefore \( \sigma(B_k)/B_k = (\alpha/\beta)^{n_k} \). Since \( |\sigma(B_k)/B_k| = e^{o(n_k)} \), we conclude that \( A_k = 0 \) is possible for only finitely many \( k \). A similar conclusion holds for \( B_k = 0 \) too. By ignoring the first finitely many \( n_k \)'s, we may assume \( A_k B_k \neq 0 \) for every \( k \).

Similarly, from \( \sum_{k=m_1}^{m_2} \frac{C_k}{U_k} = \frac{C_{m_2}}{U_{m_1}} + O(C_{m+1}/U_{m+1}) \) for any \( m_1 < m_2 \leq \infty \), by ignoring the first finitely many \( n_k \)'s, we may assume:

1. The numbers \( s \) and \( s_k \)'s for \( k = 1, 2, \ldots \) are pairwise distinct and non-zero.
We start with several easy estimates:

**Lemma 3.1.**

(i) For every positive integer \( m \), we have:
\[
(c - 1)(n_1 + \ldots + n_m) < n_{m+1}.
\]

(ii) For any positive integers \( m < N \), we have:
\[
c^{N-m-1}(c - 1)(n_1 + \ldots + n_m) < n_N.
\]

*Proof.* Part (i) follows from \( \text{Lemma 3.1.} \).

Part (ii) follows from \( n_N \geq c^{N-m-1}n_{m+1} \) and part (i). \( \square \)

**Lemma 3.2.**

(i) \( H(u_n) = |\alpha|^{n+o(n)} \) and \( H(c_n/u_n) = |\alpha|^{n+o(n)} \) as \( n \to \infty \).

(ii) \( H(s_N) \leq |\alpha|^{n_1 + \ldots + n_N + o(n)} \) as \( N \to \infty \).

*Proof.* Due to the fact that \( u_n \in \mathbb{Q} \), \( \alpha \) and \( \beta \) are units, and our assumption on the \( A_n \)’s and \( B_n \)’s, we have \( |u_n| = |\alpha|^{n+o(n)} \) while the non-archimedean contribution is \( e^{o(n)} \). This proves the first assertion of part (i), the remaining one follows since \( H(c_n) = |\alpha|^{o(n)} \) as \( n \to \infty \).

For part (ii), we use the inequality:
\[
H(s_N) \leq N \prod_{i=1}^{N} H(c_{n_i}/u_{n_i}).
\]

There exists \( \delta_1 > 0 \) such that \( H(c_{n_i}/u_{n_i}) \leq |\alpha|^{n_i+\delta_1 n_i} \) for every \( i \) by part (i). Given any \( \epsilon > 0 \), part (i) also gives that \( H(u_{n_i}) \leq |\alpha|^{n_i+\epsilon n_i} \) for every sufficiently large \( i \). Choose a large integer \( M \) so that
\[
\delta_1(n_1 + \ldots + n_N - M) \leq \epsilon n_N
\]
for every \( N > M \); this is possible thanks to Lemma 3.1. Hence for all sufficiently large \( N \), we have:
\[
H(s_N) \leq N|\alpha|^{n_1 + \ldots + n_N + \epsilon n_N + (n_N - M + 1 + \ldots + n_N)} \leq |\alpha|^{n_1 + \ldots + n_N + 4 \epsilon n_N}
\]
and this finishes the proof. \( \square \)

**Corollary 3.3.** \( s \) is irrational.

*Proof.* Suppose \( s \) is rational. From:
- \( |s - s_N| = O(C_{N+1}/U_{N+1}) = O(|\alpha|^{n_{N+1}+o(n_{N+1})}) \),
- \( H(s_N) = |\alpha|^{n_1 + \ldots + n_N + o(n_N)} \),
- \( n_1 + \ldots + n_N < (c - 1)n_{N+1} \), and
- \( c > 2 \),
we have that \( s = s_N \) for all sufficiently large \( N \), contradiction. \( \square \)

**Proposition 3.4.** There are only finitely many \( k \) such that \( n_k \geq 5 \).

*Proof.* Let \( \epsilon > 0 \) that will be specified later. Suppose there are infinitely many \( k \) such that \( n_{k+1}/n_k \geq 5 \). For each such \( k \) that is sufficiently large, we have
\[
|s - s_k| = O(C_{k+1}/U_{k+1}) = O(\alpha^{n_{k+1}}) \leq O(\alpha^{-5(1-\epsilon)n_k})
\]
while $H(s_k) \leq |\alpha|^{n_1 + \ldots + n_k + \epsilon n_k}$. Note that
\[ n_1 + \ldots + n_k < \left(1 + \frac{1}{c-1}\right) n_k = \frac{c}{c-1} n_k. \]
We now require $\epsilon$ to satisfy:
\[ 5(1 - \epsilon)n_k > \left(\frac{2c}{c-1} + 3\epsilon\right)n_k; \]
this is possible since $5 > \frac{2c}{c-1}$. Then Roth’s theorem implies that the $s_k$’s take a single value for infinitely many such $k$. But this contradicts (1).

Remark 3.5. In Proposition 3.4, the same arguments can be used when we replace $5$ by any constant greater than $\frac{2c}{c-1}$. When $c > 3$, we have $c > \frac{2c}{c-1}$ and this explains the transcendence of $\sum c_n/k$ given the “easy” bound $c > 3$.

Note that $\alpha \beta = \pm 1$ since they are units. Then we can use the geometric series to express:
\begin{align*}
\frac{C_k}{U_k} &= \frac{C_k}{A_k\alpha^{n_k}(1 - (B_k/\beta^{n_k})/(A_k\alpha^{n_k}))} = \sum_{j=0}^{\infty} \frac{C_k}{A_k\alpha^{n_k}} \left(\frac{B_k(\pm 1)^{n_k}}{A_k\alpha^{2n_k}}\right)^j \\
&= \sum_{j=0}^{\infty} \frac{(\pm 1)^{n_j} C_k B_k^j}{A_k^{j+1} \alpha^{(2j+1)n_k}}
\end{align*}
which is valid when $k$ is sufficiently large so that $|B_k/A_k| < |(\alpha/\beta)^{n_k}| = |\alpha|^{2n_k}$.

Let $P$ be a large positive integers that will be specified later. In the following, $N$ denotes an arbitrarily large positive integer. In the various $O$-notations and $o$-notations, the implied constants might depend on the given data and $P$ but they are independent of $N$. We have:
\[ |s - s_{N-1}| = |\alpha|^{-n_N + o(nN)}. \]
As mentioned in the Section 1, it is typical in applications of the Subspace Theorem to break $s_{N-1}$ as $s_{N-P}$ and truncate the expression (3) for $k = N - P + 1, \ldots, N - 1$ to maintain the error term $\alpha^{-n_N + o(nN)}$.

For $1 \leq i \leq P - 1$, let
\[ D_{N,i} := \left|\frac{n_N}{2n_{N-P+i}}\right| \leq \frac{5^{P-i}}{2} \]
thanks to Proposition 3.4. The explicit upper bound here is not important: the key fact is that these $D_{N,i}$’s can be bounded from above independently of $N$.

Proposition 3.6. For all sufficiently large $N$, we have:
\[ \left|\frac{C_{N-P+i}}{U_{N-P+i}} - \sum_{j=0}^{D_{N,i}} \frac{(\pm 1)^{n_{N-P+i}} j C_{N-P+i} B_{N-P+i}^j}{A_{N-P+i}^{j+1} \alpha^{(2j+1)n_{N-P+i}}}\right| < |\alpha|^{-n_N + o(n_N)} \]
for $i = 1, \ldots, P - 1$. This means for every $\epsilon > 0$, the LHS is less than $|\alpha|^{-n_N + \epsilon n_N}$ for all sufficiently large $N$. 


**Proof.** Let $\epsilon > 0$. We have

$$\frac{C_{N-P+i}}{U_{N-P+i}} = \sum_{j=0}^{D_{N,i}} \frac{(\pm 1)^{n_N-P+i} C_{N-P+i} B_{N-P+i}^j}{A_{N-P+i}^{j+1} \alpha(2j+1)n_N-P+i}$$

$$= \sum_{j=D_{N,i}+1}^{\infty} \frac{(\pm 1)^{n_N-P+i} C_{N-P+i} B_{N-P+i}^j}{A_{N-P+i}^{j+1} \alpha(2j+1)n_N-P+i}.$$

Hence it suffices to require the first term in the RHS:

$$\frac{(\pm 1)^{n_N-P+i} C_{N-P+i} B_{N-P+i}^j}{A_{N-P+i}^{j+1} \alpha(2j+1)n_N-P+i}$$

with $j = D_{N,i} + 1$ to be $O(|\alpha|^{-(1-\epsilon/2)n_N})$. This is actually the case, as follows. First, by the definition of $D_{N,i}$ we have $(2D_{N,i} + 3)n_{N-P+i} \geq n_N$. Second, $\left| C_{N-P+i} B_{N-P+i}^{D_{N,i}+1} \right| < |\alpha|^{(\epsilon/2)n_N}$ when $N$ is sufficiently large since $D_{N,i}$ is bounded above independently of $N$ and the assumption on the sequences $(A_k)$, $(B_k)$, and $(C_k)$.

At this point, one may attempt to apply the Subspace Theorem using the inequality:

$$s - s_{N-P} - \sum_{i=1}^{P-1} \sum_{j=0}^{D_{N,i}} \frac{(\pm 1)^{n_N-P+i} C_{N-P+i} B_{N-P+i}^j}{A_{N-P+i}^{j+1} \alpha(2j+1)n_N-P+i} < |\alpha|^{-n_N+o(n_N)}$$

and linear forms in $2 + \sum_{i=1}^{P-1} (D_{N,i}+1)$ variables for the terms $s$, $s_{N-P}$, and those in the double sum in a similar manner to [CZ04] p. 180–181 or [KMN19] Proposition 3.4. However, unlike these previous papers, the term $s_{N-P}$ in our situation is not an $S$-integer (for infinitely many $N$) for any choice of a finite set $S \subset M_K$. Because of this, there is a potential contribution of $H(s_{N-P})^{2+\sum(D_{N,i}+1)}$ which could completely offset the error term $|\alpha|^{-n_N+o(n_N)}$.

Our new idea is to consider an extra “buffer zone” by specifying another positive integer $Q < P$, expressing

$$\frac{C_{N-P+1}}{U_{N-P+1}} + \ldots + \frac{C_{N-P+Q}}{U_{N-P+Q}} = \frac{x'_N}{x_N}$$

with $x_N = \prod_{i=1}^{Q} U_{N-P+i}$,

and rewriting (5) as

$$s - s_{N-P} - \frac{x'_N}{x_N} - \sum_{i=1}^{P-Q-1} \sum_{j=0}^{D_{N,Q+i}} \frac{(\pm 1)^{n_N-P+Q+i} C_{N-P+Q+i} B_{N-P+Q+i}^j}{A_{N-P+Q+i}^{j+1} \alpha(2j+1)n_N-P+Q+i}$$

$$< |\alpha|^{-n_N+o(n_N)}.$$
We then multiply both sides by \( x_N \) to get
\[
(7) \quad x NS - xNSN - P = x^\prime_N - \sum_{i=1}^{P - Q - 1} \sum_{j=0}^{D_{N, Q_i}} x_N \left( \frac{(\pm 1)^{n_{N-P+Q+1}^i C_{N-P+Q+i}^j B_{N-P+Q+i}^j}}{A_{N-P+Q+i}^j} \right) < |x_N||\alpha|^{-n_N + o(n_N)}.
\]
After that we expand \( x_N = \prod_{i=1}^{Q} U_{N-P+i} = \prod_{i=1}^{Q} (A_{N-P+i}^i \alpha^{n_{N-P+i}} - B_{N-P+i}^i \beta^{n_{N-P+i}}) \) as a linear combination of \( 2^Q \) terms, expand \( x^\prime_N \) as a linear combination of \( Q 2^{Q-1} \) terms. Note that each \( x_NS, x_N s_N - P \), as well as each individual term in the double sum now consists of \( 2^Q \) many terms. In a typical application of the Subspace Theorem, one is worse off after performing the above steps. Therefore it is amusing that in our current situation, those steps can help reduce the number of terms significantly while the new error \( |x_N||\alpha|^{-n_N + o(n_N)} \) is not too much larger than the previous \( |\alpha|^{-n_N + o(n_N)} \).

Now even if we can apply the Subspace Theorem, there remains one important technical issue to overcome. After expanding \( x^\prime_N \) and \( x_N \), it might happen that certain terms in the LHS of (7) already satisfied a linear relation and the conclusion of the Subspace Theorem trivially illustrates this fact. For instance, in the double sum in the LHS of (7), if there are two different \((i_1, j_1)\) and \((i_2, j_2)\) for which \((2j_1 + 1)n_{N-P+i_1}^i\) and \((2j_2 + 1)n_{N-P+i_2}^i\) are close (or even equal) to each other then one should “gather” the two terms corresponding to \((i_1, j_1)\) and \((i_2, j_2)\) first. So we will also need a way to efficiently “gather similar terms” so that the conclusion of the Subspace Theorem becomes helpful for our purpose. First, we expand \( x_N \) and \( x^\prime_N \):

**Lemma 3.7.**

(i) There exists \( \delta_2 > 0 \) (possibly depending on \( P \) and \( Q \)) such that for all sufficiently large \( N \), we can express:
\[
x_N = \sum_{i=1}^{2^Q} x_{N,i} \alpha^{x(N,i)}
\]
with the following properties:

(a) \( x_{N,i} \in \mathbb{Q}(\alpha)^* \) and \( x(N,i) \in \mathbb{Z} \) for every \( i \).

(b) \( x(N,1) = \sum_{i=1}^{Q} n_{N-P+i} \) and \( x(N,1) \geq x(N,j) + 2n_{N-P+1} \) for every \( j > 1 \).

(c) \( |x(N,i)| \leq x(N,1) \) for every \( i \).

(d) \( h(x_{N,i})/n_N \to 0 \) as \( N \to \infty \) for every \( i \).

(e) \( x(N,i) - x(N,j) \geq \delta_2 n_N \) for any \( 1 \leq i \neq j \leq 2^Q \).

(ii) For all sufficiently large \( N \), we can express
\[
x^\prime_N = \sum_{i=1}^{Q 2^{Q-1}} x^\prime_{N,i} \alpha^{x^\prime(N,i)}
\]
with the following properties:

(a) \( x^\prime_{N,i} \in \mathbb{Q}(\alpha)^* \) and \( x^\prime(N,i) \in \mathbb{Z} \) for every \( i \).
(b) \(|x'(N, i)| \leq n_{N-P+2} + \ldots + n_{N-P+Q}\) for every \(i\).

(c) \(h(x'_N, i)/n_N \to 0\) as \(N \to \infty\).

Proof. For part (i), let \(\mathcal{Q} = \{1, \ldots, Q\}\). For each \(T \subseteq \mathcal{Q}\), put

\[
\Sigma(N, T) = \sum_{i \in T} n_{N-P+i} - \sum_{i \in \mathcal{Q} \setminus T} n_{N-P+i}.
\]

Note that \(|\Sigma(N, T)| \leq \sum_{i=1}^Q n_{N-P+i} < 2n_{N-P+Q}\) where the last inequality follows from Lemma 3.1. We have:

\[
x_N = \prod_{j=1}^Q (A_{N-P+j}^{n_{N-P+j}} - B_{N-P+j}^{\alpha n_{N-P+j}})
\]

\[
= \prod_{j=1}^Q (A_{N-P+j}^{n_{N-P+j}} - B_{N-P+j}(\pm 1)^{n_{N-P+i}A^{-n_{N-P+j}}}).
\]

We fix once and for all a 1-1 correspondence between \(\{1, \ldots, 2Q\}\) and the set of subsets of \(\mathcal{Q}\) so that 1 corresponds to \(\mathcal{Q}\). This allows us to take the \(x(N, i)\)'s to be exactly the \(\Sigma(N, T)\)'s (with \(x(N, 1) = \Sigma(N, \mathcal{Q})\)) and the \(x_{N,i}\)'s are the corresponding products of terms among the \(A_{N-P+j}\) and \((-1)^{n_{N-P+j}}B_{N-P+j}\); this proves parts (a) and (d). The largest among the \(\Sigma(N, T)\)'s is \(\sum_{i=1}^Q n_{N-P+i}\) while the smallest is \(-\sum_{i=1}^Q n_{N-P+i}\); this proves part (c). Moreover, the second largest is

\[-n_{N-P+1} + n_{N-P+2} + \ldots + n_{N-P+Q}\]

and this proves part (b). It remains to prove part (e).

Consider two different subsets \(T\) and \(T'\) of \(\mathcal{Q}\). Let \(j^*\) be the largest element in \(T \Delta T'\), then we have:

\[
|\Sigma(N, T) - \Sigma(N, T')| \geq 2n_{N-P+j^*} - \sum_{j < j^*} 2n_{N-P+j}
\]

\[
\geq \frac{2(c-2)}{c-1} n_{N-P+j^*}
\]

\[
\geq \frac{2(c-2)}{(c-1)5^{P-1}} n_N
\]

where the last two inequalities follow from Lemma 3.1 and Proposition 3.4. We can now take \(\delta_2 = \frac{2(c-2)}{(c-1)5^{P-1}}\).

The proof of part (ii) is similar by expanding:

\[
x'_N = \sum_{i=1}^Q C_{N-P+i} \prod_{1 \leq j \leq Q, j \neq i} U_{N-P+j}
\]

\[
= \sum_{i=1}^Q C_{N-P+i} \prod_{1 \leq j \leq Q, j \neq i} (A_{N-P+j}^{n_{N-P+j}} - B_{N-P+j}(\pm 1)^{n_{N-P+j}})\]
into $Q^{2Q-1}$ many terms.

Then we expand each individual term in the double sum

$$
\sum_{i=1}^{P-Q-1} \sum_{j=0}^{D_{N,Q+i}} x_N \frac{(\pm 1)^{n_N-P+Q+i}C_{N-P+Q+i}B_{N-P+Q+i}^{j}}{A_{N-P+Q+i}^{j+1}A^{(2j+1)n_N-P+Q+i}}
$$

to get:

**Lemma 3.8.** Put $\eta = 2Q^{P-Q-1} \sum_{D_{N,Q+i}} (D_{N,Q+i} + 1)$. For all sufficiently large $N$, we can express:

$$
\sum_{i=1}^{P-Q-1} \sum_{j=0}^{D_{N,Q+i}} x_N \frac{(\pm 1)^{n_N-P+Q+i}C_{N-P+Q+i}B_{N-P+Q+i}^{j}}{A_{N-P+Q+i}^{j+1}A^{(2j+1)n_N-P+Q+i}} = \sum_{i=1}^{\eta} y_{N,i}Q^{y(N,i)}
$$

with the following properties:

(a) $y_{N,i} \in \mathbb{Q}(\alpha)^+$ and $y(N,i) \in \mathbb{Z}$ for every $i$.

(b) $h(y_{N,i})/n_N \to 0$ as $N \to \infty$.

(c) $y(N,i) \leq x(N,1) - n_{N-P+Q+1} = n_{N-P+1} + \ldots + n_{N-P+Q} - n_{N-P+Q+1}$ for every $i$.

(d) $y(N,i) > -3n_N$ for every $i$.

**Proof.** We use the expression for $x_N$ in Lemma 3.1 to expand each individual term in the double sum. This proves (b) and (b). The highest exponent of $\alpha$ in that expression for $x_N$ is $x(N,1)$ while the highest exponent of $\alpha$ among the

$$
\frac{(\pm 1)^{n_N-P+Q+i}C_{N-P+Q+i}B_{N-P+Q+i}^{j}}{A_{N-P+Q+i}^{j+1}A^{(2j+1)n_N-P+Q+i}}
$$

for $1 \leq i \leq P-Q-1$ and $0 \leq j \leq D_{N,Q+i}$ is at most $-n_{N-P+Q+1}$, this proves (c). The smallest exponent of $\alpha$ among those terms is

$$
-\max\{(2j+1)n_{N-P+Q+1} : 1 \leq i \leq P-Q-1, 0 \leq j \leq D_{N,Q+i}\} > -2n_N
$$

by definition of the $D_{N,Q+i}$'s. The smallest exponent of $\alpha$ in $x_N$ is $-n_{N-P+1} - \ldots - n_{N-P+Q} > -n_N$. This proves (d).

We also need the following:

**Lemma 3.9.** For all sufficiently large $N$:

(i) $|x_N| = |\alpha|^{n_{N-P+1} + \ldots + n_{N-P+Q} + o(n_N)}$,

(ii) $|x_N(s-s_{N-1})| = |x_N \sum_{k=N}^{\infty} \frac{C_k}{U_k}| = |\alpha|^{n_{N-P+1} + \ldots + n_{N-P+Q} - n_{N-P+Q} + o(n_N)}$.

(iii) $|x_N| < |\alpha|^{(1-o(1))n_{N-P+1} + \ldots + n_{N-P+Q} + o(n_N)}$.

(iv) $|x_N(s-s_{N-1})| < |\alpha|^{(1-o(1))n_{N-P+1} + \ldots + n_{N-P+Q} + o(n_N)}$.

**Proof.** We have $|x_N| \gg |A_{N-P+1} \cdots A_{N-P+Q}| |\alpha|^{n_{N-P+1} + \ldots + n_{N-P+Q} + o(n_N)}$ and since the $A_k$'s are nonzero with height $o(n_k)$ we have $|A_{N-P+1} \cdots A_{N-P+Q}| = |\alpha|^{o(n_N)}$. This proves part (i). Then Lemma 3.1 gives:

$$
n_{N-P+1} + \ldots + n_{N-P+Q} < \frac{n_N}{c^{P-Q-1}(c-1)}
$$
This proves part (iii).

For part (ii), we use part (i) together with:
\[ |s - s_{N-1}| \gg |C_N/U_N| = |\alpha|^{-n_N + o(n_N)}. \]
Finally part (iv) follows from part (ii) and (8).

\[ \square \]

4. The Number \( s \) is in \( \mathbb{Q}(\alpha) \)

We continue with the assumption and notation of Section 3 in particular \( s \) is algebraic. Throughout this section, let \( Q < P \) be large, yet fixed, positive integers that will be specified later and let \( N \) denote an arbitrarily large positive integer. In the various \( O \)-notations and \( o \)-notations, the implied constants might depend on the given data, \( P \), and \( Q \) but they are independent of \( N \). In this section, we finish an important step toward the proof of Theorem 1.3, namely proving that \( s \in \mathbb{Q}(\alpha) \). This conclusion is similar to the one in the paper of Adamczewski-Bugeaud \cite{AB07}, they use a result of K. Schmidt \cite{Sch80}. In this paper, we will need more sophisticated applications of the Subspace Theorem together with further combinatorial and Galois theoretic arguments in the next section to obtain the desired result.

As mentioned in the previous section, before applying the Subspace Theorem, we need to come up with an efficient way to “gather similar terms” in the LHS of (7). This is done first by proving the existence of a certain collection of data then choosing a minimal one among those collections.

**Proposition 4.1.** Recall the \( x_{N,1} \) and \( x(N,1) = \sum_{i=1}^{Q} n_{N-P+i} \) in the expression for \( x_N \) in Lemma 3.4. There exist integers \( D, E, F \geq 0 \), tuples \( (\gamma_1, \ldots, \gamma_E) \) of elements of \( K \), an infinite set \( \mathcal{N} \), tuples \( (d_{N,1}, \ldots, d_{N,D}), (d(N,1), \ldots, d(N,D)), (e_{N,1}, \ldots, e_{N,E}), (e(N,1), \ldots, e(N,E)), (f_{N,1}, \ldots, f_{N,F}), (f(N,1), \ldots, f(N,F)) \) for each \( N \in \mathcal{N} \) with the following properties:

\begin{enumerate}
  \item[(i)] \( D + E + F \leq (2^Q - 1) + (2^Q - 1) + Q 2^{Q-1} + 2Q \sum_{i=1}^{P-Q-1} (D_{N,Q+i} + 1). \)
  \item[(ii)] For every \( N \in \mathcal{N} \), the \( d(N,i) \)'s, \( e(N,j) \)'s, and \( f(N,\ell) \)'s are integers for every \( i, j, \ell \).
  \item[(iii)] For every \( N \in \mathcal{N} \), \( n_{N-P+1} + \max_{i,j,\ell} \{d(N,i), e(N,j), f(N,\ell)\} \leq x(N,1) \).
  \item[(iv)] For every \( N \in \mathcal{N} \), \( \min_{i,j,k} \{d(N,i), e(N,j), f(N,\ell)\} \geq -3n_N \).
  \item[(v)] For every \( N \in \mathcal{N} \), the \( d_{N,i} \)'s, \( e_{N,j} \)'s, and \( f_{N,\ell} \)'s are elements of \( \mathbb{Q}(\alpha) \).
  \item[(vi)] As \( N \to \infty \) we have \( h(d_{N,i})/n_N \to 0 \), \( h(e_{N,j})/n_N \to 0 \), and \( h(f_{N,\ell})/n_N \to 0 \) for every \( i, j, \ell \).
  \item[(vii)] For all sufficiently large \( N \in \mathcal{N} \), we have
\end{enumerate}

\[ |sx_{N,1}\alpha^{x(N,1)} + \sum_{j=1}^{E} \gamma_j e_{N,j} \alpha^{e(N,j)} - s_{N-P} x_{N,1}\alpha^{x(N,1)} - \sum_{i=1}^{D} s_{N-P} d_{N,i} \alpha^{d(N,i)} - \sum_{\ell=1}^{F} f_{N,\ell} \alpha^{f(N,\ell)}| < |\alpha|^{-\left(1 - \frac{1}{2^Q} - \frac{1}{2^{Q-1}}\right)n_N + o(n_N)}. \]
Proof. Recall the inequality (7):

$$sx_N - s_{N-P}x_N - x'_N - \sum_{i=1}^{P-Q-1} j \sum_{j=0}^{B_{N,Q+i}} x_N \frac{(\pm 1)^{n_{N-P+Q+i}} C_{N-P+Q+i} B_{N-P+Q+i}^{j+1}}{A_{N-P+Q+i}^{j+1} \alpha^{(2j+1)n_{N-P+Q+i}}}$$

where the last equality follows from Lemma 3.9 and holds when $N$ is sufficiently large. We now choose $\mathcal{N}$ to be the set of all sufficiently large integers, $D = E = 2^Q - 1$, and $\gamma_1 = \ldots = \gamma_E = s$. We want the sum

$$sx_{N,1} \alpha^{x(N,1)} + \sum_{j=1}^{E} \gamma_j e_{N,j} \alpha^{e(N,j)} = s \left( x_{N,1} \alpha^{x(N,1)} + \sum_{j=1}^{E} e_{N,j} \alpha^{e(N,j)} \right)$$

to be $sx_N$; therefore we simply choose the $e_{N,j}$'s and $e(N,j)$'s for $1 \leq j \leq E$ to be respectively the terms $x_{N,k}$'s and $(N,k)$'s for $2 \leq k \leq 2^Q$ in the expression for $x_N$ in Lemma 3.7.

Similarly, we want the sum

$$s_{N-P}x_{N,1} \alpha^{x(N,1)} + \sum_{i=1}^{D} s_{N-P} d_{N,i} \alpha^{d(N,i)} = s_{N-P} \left( x_{N,1} \alpha^{x(N,1)} + \sum_{i=1}^{D} d_{N,i} \alpha^{d(N,i)} \right)$$

to be $s_{N-P}x_N$; therefore we simply choose the $d_{N,i}$'s and $d(N,i)$'s for $1 \leq i \leq D$ to be respectively the terms $x_{N,k}$'s and $(N,k)$'s for $2 \leq k \leq 2^Q$ above.

Finally, choose $F = Q2^Q - 1 + \eta$ (with $\eta$ in Lemma 3.8) and we want the sum

$$\sum_{\ell=1}^{F} f_{N,\ell} \alpha^{f(n,\ell)}$$

to be

$$x'_N + \sum_{i=1}^{P-Q-1} j \sum_{j=0}^{B_{N,Q+i}} x_N \frac{(\pm 1)^{n_{N-P+Q+i}} C_{N-P+Q+i} B_{N-P+Q+i}^{j+1}}{A_{N-P+Q+i}^{j+1} \alpha^{(2j+1)n_{N-P+Q+i}}}$$

using the expressions for $x'_N$ and and the double sum given in Lemma 3.7 and Lemma 3.8. All the properties (i)-(vii) follow from our choice and properties of the expressions for $x_N$, $x'_N$, and the double sum given in the previous section. □

Remark 4.2. In the proof of Proposition 4.1 we have that $D + E + F$ is exactly the RHS of (i) and the $\gamma_i$’s are exactly $s$. However, relaxing these as in Proposition 4.1 allows us to work with more possible collections of data in order to choose a minimal one.

Remark 4.3. Note that we allow any (or even all) of the $D, E, F$ to be 0 in the statement of Proposition 4.1. For example, if $D = E = F = 0$ then all the tuples are empty, the properties (i)-(vi) are vacuously true, and property (vii) becomes:

$$\left| (s - s_{N-P})x_{N,1} \alpha^{x(N,1)} \right| < |\alpha|^{-(1 - \frac{1}{e^{2^{Q-1}}})n_N + o(n_N)}.$$

In the proof of Proposition 4.1 we prove the existence of the required data by crudely expanding out terms in $x_N$, $x'_N$, and those in the double sum without any simplification whatsoever. The *key trick* is the following:
Definition 4.4. Among all the collections of data \((D, E, F, \mathcal{N}, \ldots)\) satisfying properties (i)-(vii) in Proposition 4.4, we choose one for which \(D + E + F\) is minimal. By abusing the notation, we still use the same notation \(D\) holds for infinitely many \(x\) such that:

\[s\]

Remark 4.5. This trick is similar to the one in [KMN19, Proposition 3.4] in which the authors worked with a vector space with the minimal dimension among a certain family of finite-dimensional vector spaces so that any further non-trivial linear relation would not be possible.

Lemma 4.6. There are at most finitely many \(N\) in \(\mathcal{N}\) such that one of the terms \(s_{N-P\alpha x(N,i)}, s_{N-P\alpha x(N,j)}, s_{N-Pd_{N,i}f(N,i)}, e_{N,j}\alpha e^{(N,j)}, f_{N,\ell}\alpha f^{(N,\ell)}\) for \(1 \leq i \leq D, 1 \leq j \leq E, \) and \(1 \leq \ell \leq F\) is zero.

Proof. If there is a term that is zero for an infinite subset \(\mathcal{N}'\) of \(\mathcal{N}\), then we have a new collection of data in which \(\mathcal{N}\) is replaced by \(\mathcal{N}'\) and that zero term is removed. This violates the minimality of \(D + E + F\).

The point of Definition 4.4 is that any non-trivial linear relation among the \(x_{N,1}e_{x(N,i)}, s_{N-Px_{N,1}e^{x(N,i)}}, s_{N-Pd_{N,i}d^{N,i}}, e_{N,j}e^{e(N,j)}, f_{N,\ell}\alpha f^{(N,\ell)}\) that holds for infinitely many \(N\) must involve the first 2 terms.

Proposition 4.7. Suppose there exist an infinite subset \(\mathcal{N}'\) of \(\mathcal{N}\) and complex numbers \(\lambda_1, \lambda_2, d_i, e_j, f_\ell\) for \(1 \leq i \leq D, 1 \leq j \leq E, \) and \(1 \leq \ell \leq F\) not all of which are zero such that:

\[
\lambda_1 x_{N,1}e^{x(N,1)} + \lambda_2 s_{N-Px_{N,1}e^{x(N,1)}} + \sum_{i=1}^{D} d_i s_{N-Pd_{N,i}d^{N,i}} \\
+ \sum_{j=1}^{E} e_j s_{N,j}e^{e(N,j)} + \sum_{\ell=1}^{F} f_{N,\ell} f^{(N,\ell)} = 0
\]

(10)

for every \(N \in \mathcal{N}'\). We have:

(i) There exist \(\kappa_1, \kappa_2, \tilde{d}_i, \tilde{e}_j, \tilde{f}_\ell\) for \(1 \leq i \leq D, 1 \leq j \leq E, \) and \(1 \leq \ell \leq F\) not all of which are zero with the following properties:

(a) All the \(\kappa_1, \kappa_2, \tilde{d}_i, \tilde{e}_j, \) and \(\tilde{f}_\ell\) are in \(\mathbb{Q}(\alpha)\).

(b) For every \(N \in \mathcal{N}'\):

\[
\kappa_1 x_{N,1}e^{x(N,1)} + \kappa_2 s_{N-Px_{N,1}e^{x(N,1)}} + \sum_{i=1}^{D} \tilde{d}_i s_{N-Pd_{N,i}d^{N,i}} \\
+ \sum_{j=1}^{E} \tilde{e}_j s_{N,j}e^{e(N,j)} + \sum_{\ell=1}^{F} \tilde{f}_{N,\ell} f^{(N,\ell)} = 0
\]

(11)

(c) \(\kappa_1 \kappa_2 \neq 0\).

(ii) \(s \in \mathbb{Q}(\alpha)\).

Proof. For part (i), since the terms \(x_{N,1}e^{x(N,1)}, s_{N-Px_{N,1}e^{x(N,1)}}, s_{N-Pd_{N,i}d^{N,i}}, e_{N,j}e^{e(N,j)}, f_{N,\ell}\alpha f^{(N,\ell)}\) are in \(\mathbb{Q}(\alpha)\), this establishes the existence of the \(\lambda_1, \lambda_2, \tilde{d}_i, \tilde{e}_j, \tilde{f}_\ell\) satisfying properties (a) and (b). We now prove \(\kappa_1 \kappa_2 \neq 0\).
First, assume that $\kappa_1 = \kappa_2 = 0$. This means:

\[ (12) \quad \sum_{i=1}^{D} d_i^s s_{N,p} d_{N,i} \alpha^{(N,i)} + \sum_{j=1}^{E} d'_j e_{N,j} \alpha^{e(N,j)} + \sum_{\ell=1}^{F} f_{N,\ell} \alpha^{f(N,\ell)} = 0 \]

for $N \in \mathcal{N}'$. Now assume that $d_{i^*} \neq 0$ for some $i^*$. Then for $N \in \mathcal{N}'$, equation \(12\) allows us to express $s_{N,p} d_{N,i} \alpha^{d(N,i)}$ as a linear combination of the $s_{N,p} d_{N,i} \alpha^{d(N,i)}$ with $i \neq i^*$, the $e_{N,j} \alpha^{e(N,j)}$, and the $f_{N,\ell} \alpha^{f(N,\ell)}$ with coefficients in $\mathbb{Q}(\alpha)$. This allows us to come up with a new data satisfying the properties in Proposition 4.1 in which $\mathcal{N}$ is replaced by $\mathcal{N}'$ and $D$ is replaced by $D - 1$. This contradicts the minimality of $D + E + F$. Therefore $d_{N,i} = 0$ for $1 \leq i \leq D$ and every $N \in \mathcal{N}'$. Similarly $f_{N,\ell} = 0$ for $1 \leq \ell \leq F$ and every $N \in \mathcal{N}'$. So now \(12\) becomes:

\[ \sum_{j=1}^{E} d'_j e_{N,j} \alpha^{e(N,j)} = 0 \quad \text{for} \quad N \in \mathcal{N}'. \]

Arguing as before, we obtain a contradiction to the minimality of $D + E + F$. This proves at least $\kappa_1$ or $\kappa_2$ is non-zero.

We emphasize that the arguments should be run in the above order (i.e. obtaining $d_{N,i} = f_{N,\ell} = 0$ first). Suppose one tried to prove all the $e_{N,j} = 0$ first by using \(12\) to express some $e_{N,j} \alpha^{e(N,j)}$ as a linear combination of the $s_{N,p} d_{N,i} \alpha^{d(N,i)}$, the $e_{N,j} \alpha^{e(N,j)}$ with $j \neq j^*$, and the $f_{N,\ell} \alpha^{f(N,\ell)}$. Then due to the term $\gamma_j^* e_{N,j^*} \alpha^{e(N,j^*)}$ in the LHS of \(11\) and since at the moment we do not necessarily have $\gamma_j^* \in \mathbb{Q}(\alpha)$, the “new” $d_{N,i}$ and $f_{N,\ell}$ would not remain in $\mathbb{Q}(\alpha)$ and the new data would not satisfy all the properties in Proposition 4.1.

Suppose $\kappa_1 \neq 0$ while $\kappa_2 = 0$. We have

\[ \kappa_1 x_{N,1} \alpha^{x(N,1)} + \sum_{i=1}^{D} d_i^s s_{N,p} d_{N,i} \alpha^{d(N,i)} + \sum_{j=1}^{E} d'_j e_{N,j} \alpha^{e(N,j)} + \sum_{\ell=1}^{F} f_{N,\ell} \alpha^{f(N,\ell)} = 0 \]

for $N \in \mathcal{N}'$. We divide by $x_{N,1} \alpha^{x(N,1)}$ and note that each of $d(N,i) - x(N,1)$, $e(N,j) - x(N,1)$, and $f(N,\ell) - x(N,1)$ is at most $-n_{N,p+1} \leq -n_N/5^{p-1}$ by (iii) in Proposition 4.1 and Proposition 5.1 while each of $|s_{N,p} d_{N,i}/x_{N,1}|$, $|e_{N,j}/x_{N,1}|$, and $|f_{N,\ell}/x_{N,1}|$ is $e^{o(x)}$. Therefore taking limit as $N \to \infty$ (and $N \in \mathcal{N}'$), we get $\kappa_1 = 0$, contradiction. The case $\kappa_2 \neq 0$ and $\kappa_1 = 0$ is ruled out in a similar way. This finishes the proof of (c).

For part (ii), we use \(11\), divide by $x_{N,1} \alpha^{x(N,1)}$ and let $N \to \infty$ as above to obtain

\[ \kappa_1 + \kappa_2 s = 0 \]

and this proves $s = -\kappa_1/\kappa_2 \in \mathbb{Q}(\alpha)$. \hfill \Box

**Proposition 4.8.** The number $s$ is in $\mathbb{Q}(\alpha)$.

**Proof.** We will obtain a non-trivial linear relation as in the statement of Proposition 4.7 for infinitely many $N$ and apply part (ii).

Let $v_\infty$ be the valuation on $\mathbb{Q}(\alpha)$ corresponding to the usual $| \cdot |$ and let $w$ be the other archimedean one. Note that we follow the normalization in [BG06] Chapter 1], hence:

\[ |x|_{v_\infty} = |x|^{1/2} \quad \text{and} \quad |x|_w = |\sigma(x)|^{1/2}. \]
The archimedean valuations on $K$ are denoted as $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ where the $v_i$’s lie above $v_\infty$ and the $w_i$’s lie above $w$. They correspond to the following real or one for each pair of complex-conjugate embeddings of $K$ into $\mathbb{C}$: $\tau_1, \ldots, \tau_m$ and $\sigma_1, \ldots, \sigma_m$. In other words:

$$|x|_{v_i} = |\tau_i(x)|^{d(v_i)/[K:Q]}$$

and

$$|x|_{w_i} = |\sigma_i(x)|^{d(w_i)/[K:Q]}$$

where $d(v_i) = [K_{v_i} : \mathbb{R}] = 1$ or 2 depending on whether $v_i$ is real or complex and a similar definition for $d(w_i)$. Note that the $\tau_i$’s restrict to the identity automorphism on $Q(\alpha)$ while the $\sigma_i$’s restrict to $\sigma$ on $Q(\alpha)$. In fact, since $K/Q$ is Galois, either all archimedean valuations are real or all are complex and we simply let $\delta$ be the common value of the $d(v_i)$ and $d(w_i)$. We have:

$$\sum_{i=1}^{m} d(v_i) = \sum_{i=1}^{m} d(w_i) = m\delta = [K : Q] = \frac{[K : \mathbb{Q}]}{2}. \quad (13)$$

Our next step is to apply the Subspace Theorem using (9). Fix $\epsilon > 0$ that will be specified later. Let

$$S = M^K = \{v_1, \ldots, v_m, w_1, \ldots, w_m\}.$$ 

We will work with linear forms in variables

$$(T_1, T_2, X_1, \ldots, X_D, Y_1, \ldots, Y_E, Z_1, \ldots, Z_F)$$

and the vectors

$$\nu_N = (x_{N,1}^\alpha x(N,1), -s_{N,p} x_{N,1}, (-s_{N-p} d_N, 1 \leq i \leq D, (e_{N,N} d_{N,j} x(N,j)) 1 \leq j \leq E), (-f_{N,N} d_{N,j} x(N,j)) 1 \leq j \leq F)$$

for $N \in \mathcal{N}$.

For each $v \in S$, the linear forms are denoted: $L_{v,T,1}, L_{v,T,2}, L_{v,X,i}$ for $1 \leq i \leq D$, $L_{v,Y,j}$ for $1 \leq j \leq E$, and $L_{v,Z,\ell}$ for $1 \leq \ell \leq F$. The reason is that they will be defined as follows:

- For any $v \in S$, $L_{v,T,2} = T_2$, $L_{v,X,i} = X_i$, $L_{v,Y,j} = Y_j$, and $L_{v,Z,\ell} = Z_\ell$ for any $i, j, \ell$.
- If $v = w_k$ for some $1 \leq k \leq m$, define $L_{v,T,1} = T_1$ and we have

$$|L_{v,T,1}(\nu_N)|_v = |\sigma(x_{N,1}^\alpha x(N,1))|^{\delta/[K:Q]} = o(1)$$

since $|\sigma(x_{N,1})| = e^{\alpha n_N}$ while $|\sigma(x(N,1))| = |\alpha|^{-x(N,1)}$ and $x(N,1) \gg n_N$.
- If $v = v_k$ for some $1 \leq k \leq m$, define:

$$L_{v,T,1} = \tau_k^{-1}(s) T_1 + T_2 + X_1 + \ldots + X_D + \tau_k^{-1}(\gamma_1) Y_1 + \ldots + \tau_k^{-1}(\gamma_E) Y_E$$

$$+ Z_1 + \ldots + Z_F$$

so that $|L_{v,T,1}(\nu_N)|_v$ is exactly the LHS of (9) to the power $\delta/[K:Q]$. Therefore

$$|L_{v,T,1}(\nu_N)|_v \leq |\alpha|^{(1-\frac{1}{\mathcal{Q}(\alpha)^{\epsilon N}})^n N \cdot \delta/[K:Q]}$$

for all sufficiently large $N \in \mathcal{N}$.

Combining with (13), we have:

$$\prod_{v \in S} |L_{v,T,1}(\nu_N)|_v = O(|\alpha|^{(1-\frac{1}{\mathcal{Q}(\alpha)^{\epsilon N}})^n N \cdot \delta/[K:Q]^n}). \quad (14)$$
Now since $\alpha$ is an $S$-unit (i.e., usual algebraic integer unit), by the product formula together with the fact that $|s_{N-P}| = O(1)$ and the $x_{N,1}$, $d_{N,i}$, $e_{N,j}$, $f_{N,\ell}$ have height $o(n_N)$, we have that for all sufficiently large $N \in \mathcal{N}$:

\[
\prod_{v \in S} \prod_{L} |L(v_N)|_v = O(|\alpha|^{-(1-\epsilon_1^2)^{-\frac{1}{2}}(\epsilon_1-1)-2\epsilon_2 n_N})
\]

where $L$ ranges over all the $L_{v,T,1}$, $L_{v,T,2}$, $L_{v,X,i}$, $L_{v,Y,j}$, and $L_{v,Z,\ell}$. Recall that $\tilde{v}_N$ denotes the point in the projective space with coordinates $v_N$. Since the $|d(N,i)|$, $|e(N,j)|$, and $|f(N,\ell)|$ are less than $3n_N$, we have

\[
H(\tilde{v}_N) \leq |\alpha|^{5n_N}
\]

for every $N \in \mathcal{N}$.

We need to obtain $\epsilon' > 0$ such that:

\[
\prod_{v \in S} \prod_{L} \frac{|L(v_N)|_v}{|v_N|_v} < H(\tilde{v}_N)^{-D-E-F-2-\epsilon'}
\]

for all large $N$ in $\mathcal{N}$. We have:

\[
H(\tilde{v}_N)^{D+E+F+2} \prod_{v \in S} \prod_{L} \frac{1}{|v_N|_v} = \left( \prod_{v \in M_K \setminus S} |v_N|_v \right)^{D+E+F+2}.
\]

Since all the $x_{N,1}$, $d_{N,i}$, $e_{N,j}$, and $f_{N,\ell}$ have multiplicative height $o^o(n_N)$, we have:

\[
\left( \prod_{v \in M_K \setminus S} |v_N|_v \right)^{D+E+F+2} \leq H(s_{N-P})^{D+E+F+2}|\alpha|^{|n_N|
\]

for all sufficiently large $N \in \mathcal{N}$. We have:

\[
H(s_{N-P}) \leq |\alpha|^{n_1+\ldots+n_{N-P}+o(n_{N-P})} \leq |\alpha|^{n_{N-P+1}}
\]

for all large $N \in \mathcal{N}$. From Proposition 4.1 and the definition of the $D_{N,Q+i}$'s, we have:

\[
D + E + F + 2 \leq 2^{Q+1} + 2^{Q-1} + 2^{P-Q-1} \sum_{i=1}^{P-Q-1} (D_{N,Q+i} + 1)
\]

\[
\leq 2^{Q}(P - (Q/2) + 1) + 2^{Q} \sum_{i=1}^{P-Q-1} \frac{n_N}{2^{n_{N-P+Q+i}}}
\]

\[
\leq 2^{Q}P + 2^{Q-1} \sum_{i=1}^{P-Q-1} \frac{n_N}{n_{N-P+Q+i}}
\]

assuming $Q \geq 2$. Hence for all large $N \in \mathcal{N}$,

\[
H(s_{N-P})^{D+E+F+2} \leq |\alpha|^{|n_N|
\]
where

\[ \Omega_N := n_{N-P+1}2^Q P + 2^{Q-1} \left( \sum_{i=1}^{P-Q-1} \frac{n_{N-P+i}}{n_{N-P+Q+i}} \right) n_N \]

(20)

\leq \frac{2^Q P}{c^{P-1}} n_N + 2^{Q-1} \left( \sum_{i=1}^{P-Q-1} \frac{1}{c^{Q+i-1}} \right) n_N

\leq \left( \frac{2^Q P}{c^{P-1}} + \frac{2^{Q-1}}{c^Q} \cdot \frac{1}{1 - (1/c)} \right) n_N.

Finally, note that:

\[ H(\tilde{\nu}_N)^{-\epsilon'} \geq |\alpha|^{-5 \epsilon' n_N}. \]

In order to obtain \( \epsilon' \) satisfying (16), we combine (15) and (17)–(21) and require that:

\[ \left( \frac{2^Q P}{c^{P-1}} + \frac{2^{Q-1}}{c^Q} \cdot \frac{1}{1 - (1/c)} \right) + \epsilon - \frac{1}{2} \left( 1 - \frac{1}{c^{P-Q-1}(c-1) - 2\epsilon} \right) < -5\epsilon'. \]

Such an \( \epsilon' \) exists as long as the LHS is negative. Therefore, at the beginning of Section 5.1, we choose sufficiently small integers \( 2 \leq Q < P \) and here we choose a sufficiently small \( \epsilon > 0 \) such that:

\[ \frac{2^Q P}{c^{P-1}} + \frac{2^{Q-1}}{c^Q} \cdot \frac{1}{1 - (1/c)} + 2\epsilon - \frac{1}{2c^{P-Q-1}(c-1)} < \frac{1}{2}, \]

this is possible since \( c > 2 \). Then we can apply the Subspace Theorem to have that there exists a non-trivial linear relation satisfied by the coordinates of \( \nu_N \) for infinitely many \( N \in \mathcal{N} \). Then we use part (ii) of Proposition 4.7 to finish the proof. \( \square \)

5. THE PROOF OF THEOREM 1.3

We continue with the notations of Section 3 and ignore those in Section 4. We no longer assume the choice of \( P, Q, \) and \( \epsilon \) as in (22) and (23). However, we use the crucial result that \( s \in \mathbb{Q}(\alpha) \). While the arguments in Section 4 are valid for any sufficiently large \( Q < P \) (and sufficiently small \( \epsilon \)), those in this section require that \( Q = P - 1 \):

Assumption 5.1. From now on, \( Q = P - 1 \). Therefore \( x_N = \prod_{i=1}^{P-1} U_{N-P+i} \), the \( x(N,i) \)'s are the numbers \( \pm n_{N-P+1} \pm n_{N-P+2} \cdots \pm n_{N-1}, \)

\[ \frac{x'_N}{x_N} = \sum_{i=1}^{P-1} \frac{C_{N-P+i}}{U_{N-P+i}}, \]

and most importantly:

\[ (s - s_{N-P})x_N - x'_N = (s - s_{N-1})x_N = x_N \sum_{k=N}^{\infty} \frac{C_k}{U_k}. \]

The following numbers \( x(N,+), x(N,-), \tilde{x}(N,+), \) and \( \tilde{x}(N,-) \) will play an important role:

Lemma 5.2. \( \quad (i) \ x(N,+) := -n_{N-P+1} - \cdots - n_{N-2} + n_{N-1} \) is the smallest non-negative numbers while \( x(N,-) := -x(N,+) \) is the largest non-negative numbers among the \( x(N,i) \)'s.
(i) Write \( x(N, +) := -n_{N-P+1} - \ldots - n_{N-1} + n_N \) and write \( x(N, -) := -\tilde{x}(N, +) \). We have \( \tilde{x}(N, -) \leq x(N, -) < x(N, +) < \tilde{x}(N, +) \). Moreover:

\[
x(N, +) - x(N, -) \geq \frac{2(c-2)}{c-1} n_{N-1} \geq \frac{2(c-2)}{5(c-1)} n_N, \quad \text{and}
\]

\[
x(N, -) - \tilde{x}(N, -) = \tilde{x}(N, +) - x(N, +) = n_N - 2n_{N-1} \geq \frac{c-2}{c} n_N
\]

for all sufficiently large \( N \).

(iii) \(|(s - s_{N-1}) x_N| = |\alpha| \tilde{x}(N, -)^{\alpha(n_N)} \) for all sufficiently large \( N \).

**Proof.** We have:

\[
x(N, +) - x(N, -) = 2(n_{N-1} - n_{N-2} - \ldots - n_{N-P+1}) \geq \frac{2(c-2)}{c-1} n_{N-1} \geq \frac{2(c-2)}{5(c-1)} n_N
\]

for all large \( N \) by Lemma 3.1 and Proposition 3.4. Part (i) and the rest of part (ii) are elementary. Part (iii) is simply Lemma 3.3(ii) when \( Q = P - 1 \).

First, we prove the existence of a certain expression and then choose a minimal one:

**Proposition 5.3.** Note that \( Q = P - 1 \). There exist integers \( D, F \geq 0 \), an infinite set \( \mathcal{N} \), tuples \( (d_{N,1}, \ldots, d_{N,D}), (f_{N,1}, \ldots, f_{N,F}), (d(N,1), \ldots, d(N,D)) \) and \( f(N,1), \ldots, f(N,F) \) for each \( N \in \mathcal{N} \) with the following properties:

(i) \( D + F \leq 2^{P-1} + (P-1)2^{P-2} \).

(ii) For every \( N \in \mathcal{N} \), the \( d(N,i) \)'s and \( f(N,j) \)'s are integers for every \( i, j \).

(iii) For every \( N \in \mathcal{N} \), \( \max_i |d(N,i)| \leq n_{N-P+1} + \ldots + n_{N-1} \).

(iv) For every \( N \in \mathcal{N} \), \( \max_j |f(N,j)| \leq n_{N-P+2} + \ldots + n_{N-1} \).

(v) For every \( N \in \mathcal{N} \), the \( d_{N,i} \)'s and \( f_{N,j} \)'s are elements of \( Q(\alpha) \).

(vi) As \( N \to \infty \), we have \( h(d_{N,i})/n_N \to 0 \) and \( h(f_{N,j})/n_n \to 0 \) for every \( i, j \).

(vii) For every \( N \in \mathcal{N} \), we have

\[
\sum_{i=1}^{D} (s - s_{N-P}) d_{N,i} \alpha^{d(N,i)} - \sum_{j=1}^{F} f_{N,j} \alpha^{f(N,j)} = (s - s_{N-1}) x_N.
\]

**Proof.** The proof is very similar to the proof of Proposition 3.1. Choose \( \mathcal{N} \) to be the set of all sufficiently large integers and \( D = 2^{P-1} \). We want the sum

\[
\sum_{i=1}^{D} (s - s_{N-P}) d_{N,i} \alpha^{d(N,i)}
\]

to be \( (s - s_{N-P}) x_N \). We simply choose the \( d_{N,i} \) and \( d(N,i) \) to be the \( x_{N,i} \) and \( x(N,i) \).

Then we choose \( F = (P - 1)2^{P-2} \) and we want the sum

\[
\sum_{j=1}^{F} f_{N,j} \alpha^{f(N,j)}
\]

to be \( x_N' = \frac{(P-1)2^{P-2}}{x(N,j)} \),

so we simply take the \( f_{N,j} \) and \( f(N,j) \) to be the \( x_{N,j}' \) and \( x(N,j) \). The properties (i)-(vi) follow from (24) and Lemma 3.7. \( \square \)
Definition 5.4. Among all the collections of data \((D, F, \mathcal{N}, \ldots)\) satisfying properties (i)-(vii) in Proposition 5.3, we choose one for which \(D + F\) is minimal. By abusing the notation, we still use the same notation \(D, F, \mathcal{N}, d_{N,i}'s, d(N,i)'s, f_{N,j}'s,\) and \(f(N,j)'s\) for this chosen data with minimal \(D + F\).

Remark 5.5. As before, we allow the possibility that \(D\) or \(F\) is 0. Note that the scenario \(D = F = 0\) cannot happen since the RHS of (26) is non-zero.

Lemma 5.6. There are at most finitely many \(N\) in \(\mathcal{N}\) such that one of the terms \((s - s_{N-P})d_{N,i}\alpha^{d(N,i)}, f_{N,j}\alpha^{f(N,j)}\) for \(1 \leq i \leq D\) and \(1 \leq j \leq F\) is zero.

Proof. This is similar to the proof of Lemma 4.6.

The reason we introduce the number \(\tilde{x}(N, -)\) is that the \(d(N,i)\) for \(1 \leq i \leq D\) and \(f(N,j)\) for \(1 \leq j \leq F\) are less than \(\tilde{x}(N, -) + o(n_N)\) for an appropriate choice of \(P\). This means that for any \(\epsilon > 0\), as long as \(P\) is sufficiently large (depending on \(\epsilon\)), then the \(d(N,i)'s\) and \(f(N,j)'s\) are smaller than \(\tilde{x}(N, -) + \epsilon n_N\). For our purpose, we state and prove the next result for the specific value \(\frac{c - 2}{2c}\) for \(\epsilon\); note that \(\frac{c - 2}{2c}\) is at most half of the gap between \(\tilde{x}(N, -)\) and \(x(N, -)\) thanks to Lemma 5.2.

Proposition 5.7. Assume that \(P\) satisfies:

\[
\frac{2P-1}{eP-1} + \frac{(P-1)2^{P-2}}{e^{P-1}} < \frac{c - 2}{4e}.
\]

Then for all but finitely many \(N \in \mathcal{N}\), we have \(d(N,i) \leq \tilde{x}(N, -) + \frac{c - 2}{2c} n_N\) and \(f(N,j) \leq \tilde{x}(N, -) + \frac{c - 2}{2c} n_N\) for \(1 \leq i \leq D\) and \(1 \leq j \leq F\).

Proof. We prove by contradiction and without loss of generality, we only need to consider 2 cases:

- Case 1: there exists an infinite subset \(\mathcal{N}'\) of \(\mathcal{N}\) such that \(d(N,1) > \tilde{x}(N, -) + \frac{c - 2}{2c} n_N\) for every \(N \in \mathcal{N}'\).
- Case 2: there exists an infinite subset \(\mathcal{N}'\) of \(\mathcal{N}\) such that \(f(N,1) > \tilde{x}(N, -) + \frac{c - 2}{2c} n_N\) for every \(N \in \mathcal{N}'\).

First, we assume Case 1. Let \(\epsilon > 0\) be a small number that will be specified later. For all sufficiently large \(N \in \mathcal{N}'\), we have:

\[
|\sum_{i=1}^{D} (s - s_{N-P})d_{N,i}\alpha^{d(N,i)} - \sum_{j=1}^{F} f_{N,j}\alpha^{f(N,j)}| = |(s - s_{N-1})x_N|
\]

by (26) and Lemma 5.2. We now apply the Subspace Theorem over the field \(\mathbb{Q}(\alpha)\). Let \(S = \{v_u, w\}\) be its archimedean places as described in the proof of Proposition 4.8. We work with linear forms in the variables:

\[
(X_1, \ldots, X_D, Y_1, \ldots, Y_F)
\]

and the vectors

\[
v_N = \{(s - s_{N-P})d_{N,i}\alpha^{d(N,i)}\}_{1 \leq i \leq D}, \{-f_{N,j}\alpha^{f(N,j)}\}_{1 \leq j \leq F}
\]
for large $N \in \mathcal{N}'$. For $v \in S$, the linear forms are denoted $L_{v,X,i}$ for $1 \leq i \leq D$, and $L_{v,Y,j}$ for $1 \leq j \leq F$. They are defined as follows:

- For any $v \in S$, $L_{v,X,i} = X_i$ for $2 \leq i \leq D$ and $L_{v,Y,j} = Y_j$ for $1 \leq j \leq F$.
- If $v = w$, define $L_{v,X,1} = X_1$.
- If $v = v_\infty$, define $L_{v,X,1} = X_1 + \ldots + X_D + Y_1 + \ldots + Y_F$.

Since $s$ is irrational by Corollary 5.3 and $|s - s_{N-p}| = o(1)$, we have $|s - s_{N-p}|_w = O(1)$. Together with the fact that the $d_{N,i}$’s and $f_{N,j}$’s have multiplicative height $e^{o(nN)}$, for all large $N \in \mathcal{N}'$, we have:

$$\prod_{v \in S} |L_{v,X,1}(v_N)|_v < |\alpha|^{-(d(N,1) + d(N,-) + \epsilon nN)/2} < |\alpha|^{-(c-2)/(2c) + \epsilon} nN/2$$

where the last inequality follows from the assumption in Case 1.

As in the proof of Proposition 4.8 and using $H(s - s_{N-p}) = O(H(s_{N-p}))$, we have:

$$H(v_N)^{D+F} \prod_{v \in S} \prod_{L \mid v} \frac{1}{|v_N|_v} = \left( \prod_{v \in M \setminus S} |v_N|_v \right)^{D+F} \leq H(s_{N-p})^{D+F} |\alpha|^{\epsilon nN} \leq |\alpha|^{(D+F)n_{N-p+1} + \epsilon nN}$$

for all sufficiently large $N \in \mathcal{N}'$. Recall that $D + F \leq 2^{P-1} + (P-1)2^{P-2}$, as before, we can apply the Subspace Theorem if:

$$\frac{2^{P-1} + (P-1)2^{P-2}}{c^{P-1}} + \epsilon + \frac{1}{2} \left( -\frac{c-2}{2c} + \epsilon \right) < 0$$

or in other words

$$\frac{2^{P-1} + (P-1)2^{P-2}}{c^{P-1}} + \frac{3}{2} \epsilon < \frac{c-2}{4c}.$$

We can choose such an $\epsilon$ thanks to the given condition on $P$. Then the Subspace Theorem yields a non-trivial linear relation over $\mathbb{Q}(\alpha)$ among the $(s-s_{N-p})d_{N,i} \alpha^{d(N,i)}$’s and $f_{N,j} \alpha^{f(N,j)}$’s for $N$ in an infinite subset $\mathcal{N}''$ of $\mathcal{N}'$. This allows us to express one of them as a linear combination of the other terms and we obtain a new data satisfying the properties stated in Proposition 5.3 in which $\mathcal{N}$ is replaced by $\mathcal{N}''$ and $D + F$ is replaced by $D + F - 1$, contradicting the minimality of $D + F$. This shows that Case 1 cannot happen.

By similar arguments, we have that Case 2 cannot happen either. For Case 2, we consider the same $v_N$’s, the variables $X_1, \ldots, X_D, Y_1, \ldots, Y_F$, and $S$ as in Case 1 while the linear forms $L_{v,X,i}$’s and $L_{v,Y,j}$’s are defined as follows:

- For any $v \in S$, $L_{v,X,i} = X_i$ for $1 \leq i \leq D$ and $L_{v,Y,j} = Y_j$ for $2 \leq j \leq F$.
- If $v = w$, define $L_{v,Y,1} = Y_1$.
- If $v = v_\infty$, define $L_{v,Y,1} = X_1 + \ldots + X_D + Y_1 + \ldots + Y_F$.

Then we proceed as before and arrive at a contradiction. □

We are now at the final stage of the proof of Theorem 1.3.
Then taking the difference of the previous 2 equations, we get:

\[ (32) \]

\[
\sum_{i=1}^{D} (\sigma(s) - s_{N,i})\sigma(d_{N,i})(\pm 1)^{d_{N,i}} \alpha^{d_{N,i}} - \sum_{j=1}^{F} \sigma(f_{N,j})(\pm 1)^{f_{N,j}} \alpha^{-f_{N,j}} = (\sigma(s) - s_{N-1})x_N.
\]

Note that \(|I^-| = 2^{P-2} - 1\) and \(|I^+| = 2^{P-2}\).

We apply \(\sigma\) to both sides of:

\[ (31) \]

\[
\sum_{i=1}^{D} (s - s_{N-1})d_{N,i}\alpha^{d_{N,i}} - \sum_{j=1}^{F} f_{N,j}\alpha^{f_{N,j}} = (s - s_{N-1})x_N,
\]

and recall that \(\sigma(\alpha) = \beta = \pm \frac{1}{\alpha}\) to get:

\[ (32) \]

\[
\sum_{i=1}^{D} (\sigma(s) - s_{N-1})\sigma(d_{N,i})(\pm 1)^{d_{N,i}} \alpha^{-d_{N,i}} - \sum_{j=1}^{F} \sigma(f_{N,j})(\pm 1)^{f_{N,j}} \alpha^{-f_{N,j}} = (\sigma(s) - s_{N-1})x_N.
\]

Then taking the difference of the previous 2 equations, we get:

\[ (33) \]

\[
(\sigma(s) - s) x_N = \sum_{i=1}^{D} (\sigma(s) - s_{N-1})\sigma(d_{N,i})(\pm 1)^{d_{N,i}} \alpha^{-d_{N,i}} - \sum_{j=1}^{F} \sigma(f_{N,j})(\pm 1)^{f_{N,j}} \alpha^{-f_{N,j}}
\]

\[ \]
(i) \( A + B + C + V + W \leq 2P - 1 + 2(D + F) \leq 2P + P^{2P-1} - 1. \)

(ii) For every \( N \in \mathcal{N}' \), the \( a(N, i) \)’s, \( b(N, j) \)’s, \( c(N, k) \), \( v(N, \ell) \)’s, and \( w(N, m) \)’s are integers for every \( i, j, k, \ell, m \).

(iii) For every \( N \in \mathcal{N}' \), we have:
\[
\max_{i,j,k,\ell,m} \{|a(N, i)|, |b(N, j)|, |c(N, k)|, |v(N, \ell)|, |w(N, m)|\} < 2n_N.
\]

(iv) For every \( N \in \mathcal{N}' \), we have \( a(N, i) \leq -\delta_n N, b(N, j) \leq -\theta_n N, c(N, k) \geq \theta_n N, v(N, \ell) \leq -\theta_n N, \) and \( w(N, m) \geq \theta_n N \) for every \( i, j, k, \ell, m \).

(v) For every \( N \in \mathcal{N}' \), the \( a_{N,i}'s, b_{N,j}'s, c_{N,k}'s, v_{N,\ell}'s, \) and \( w_{N,m}'s \) are elements of \( \mathbb{Q}(\alpha) \).

(vi) As \( N \to \infty \), we have \( h(a_{N,i})/n_N \to 0, h(b_{N,j})/n_N \to 0, h(c_{N,k})/n_N \to 0, h(v_{N,\ell})/n_N \to 0, \) and \( h(w_{N,m})/n_N \to 0 \) for every \( i, j, k, \ell, m \).

(vii) For every \( N \in \mathcal{N}' \), we have:
\[
\sigma(s) - s = \sum_{i=1}^{A} a_{N,i} \alpha^{a(N,i)} + \sum_{j=1}^{B} b_{N,j} \alpha^{b(N,j)} + \sum_{k=1}^{C} c_{N,k} \alpha^{c(N,k)}
\]
\[
\sum_{i=1}^{V} (s - s_{N-p}) v_{N,i} \alpha^{v(N,\ell)} + \sum_{m=1}^{W} (\sigma(s) - s_{N-p}) w_{N,m} \alpha^{w(N,m)}.
\]

Proof. We use \((33)\) and the expression for \( x_N \) in Lemma \( 3.7 \) then divide both sides by \( x_{N,i}^{-\alpha^2(N,-)} \) to get:
\[
\sigma(s) - s = -\frac{(\sigma(s) - s)x_{N,i}^{-\alpha^{2}(N,i)-x(N,-)}}{x_{N,i}^{-}}
\]
\[
-\frac{(\sigma(s) - s)x_{N,i}^{-\alpha^{2}(N,i)-x(N,i^-)}}{x_{N,i}^{-}}
\]
\[
-\frac{\sigma(f_{N,i}) (\pm 1)^{f(N,i)} x_{N,i}^{-f(N,i)-x(N,-)}}{x_{N,i}^{-}}
\]
\[
+ \frac{\sum_{i=1}^{F} f_{N,i} x_{N,i}^{-f(N,i)-x(N,-)}}{x_{N,i}^{-}}
\]
\[
- \frac{(s - s_{N-p}) d_{N,i} x_{N,i}^{-d(N,i)-x(N,-)}}{x_{N,i}^{-}}
\]
\[
+ \frac{\sum_{i=1}^{D} (s - s_{N-p}) \sigma(d_{N,i}) x_{N,i}^{-d(N,i)-x(N,-)}}{x_{N,i}^{-}}
\]

Let \( \mathcal{N}' \) be the set of all sufficiently large \( N \in \mathcal{N}' \); in the following, \( N \) is an element of \( \mathcal{N}' \). We want \( \sum_{i=1}^{A} a_{N,i} \alpha^{a(N,i)} \) to be \( -\frac{(\sigma(s) - s)x_{N,i}^{-\alpha^{2}(N,i)-x(N,-)}}{x_{N,i}^{-}} \).

Therefore we let \( A = |I^-| = 2P - 2 - 1 \), let the \( a_{N,i} \) and \( a(N,i) \) for \( 1 \leq i \leq A \) be respectively the \( -\frac{(\sigma(s) - s)x_{N,i}^{-\alpha^{2}(N,i)-x(N,-)}}{x_{N,i}^{-}} \) and \( x(N,i) - x(N,-) \) for \( i \in I^- \). By Lemma \( 3.7 \) and the definition of \( I^- \), we have \( a(N,i) \leq -\delta_n N \).
We want $\sum_{j=1}^{B} b_{N,j} \alpha^{b(N,j)}$ to be $\sum_{i=1}^{F} \frac{f_{N,i}}{x_{N,i}} \alpha^{f(N,i) - x(N,-)}$. Therefore we let $B = F$, let the $b_{N,j}$ and $b(N,j)$ for $1 \leq j \leq B$ be respectively the $\frac{f_{N,i}}{x_{N,i}}$ and $f(N,i) - x(N,-)$ for $1 \leq i \leq F$. By Lemma 5.2 and Proposition 5.7, we have:

$$f(N,i) - x(N,-) \leq \tilde{x}(N,-) + \frac{c-2}{2c} n_N - x(N,-) \leq -\frac{c-2}{2c} n_N \leq -\theta n_N$$

for $1 \leq i \leq F$.

We want $\sum_{k=1}^{C} c_{N,k} \alpha^{c(N,k)}$ to be:

$$-\sum_{i \in I^+} (\sigma(s) - s)x_{N,i} \alpha^{\sigma(N,i) - x(N,i)} - \sum_{i = 1}^{F} \frac{\sigma(f_{N,i})}{x_{N,i}} (\pm 1)^{1} f(N,i) \alpha^{-f(N,i) - x(N,-)}.$$ 

We let $C = |I^+| + F = 2P^2 + F$ and specify the $c_{N,k}$ and $c(N,k)$ in the same manner as before. Note that $x(N,+)$ is the minimum among the $x(N,i)$ for $i \in I^+$ while Lemma 5.2 and Proposition 5.7 yields

$$-f(N,i) \geq -\tilde{x}(N,-) - \frac{c-2}{2c} n_N = \tilde{x}(N,+) - \frac{c-2}{2c} n_N > x(N,+).$$

This guarantees $c(N,k) \geq x(N,+) - x(N,-) \geq 2(c-2) n_N \geq \theta n_N$.

Finally, we want $\sum_{\ell=1}^{V} (s - s_{N,P}) v_{N,\ell} \alpha^{v(N,\ell)}$ to be

$$-\sum_{i=1}^{D} (s - s_{N,P}) d_{N,i} \alpha^{d(N,i) - x(N,-)}$$

and want $\sum_{m=1}^{W} (\sigma(s) - s_{N,P}) w_{N,m} \alpha^{w(N,m)}$ to be

$$\sum_{i=1}^{D} (\sigma(s) - s_{N,P}) \frac{\sigma(d_{N,i})}{x_{N,i}} (\pm 1)^{1} d(N,i) \alpha^{-d(N,i) - x(N,-)}.$$ 

We let $V = W = D$ and similar arguments can be used to finish the proof; note that with our choice:

$$A + B + C + V + W = 2P^1 - 1 + 2(D + F) \leq 2P + P^2P^1 - 1$$

where the last inequality follows from Proposition 5.3.

**Definition 5.10.** Among all the collections of data $(A, B, C, V, W, A', \ldots)$ satisfying properties (i)-(vii) in Proposition 5.7, we choose one for which $A + B + C + V + W$ is minimal. By abusing the notation, we still use the same notation $A, B, C, V, W, A', a_{N,i}, s, a(N,i)'s, b_{N,j}'s, b(N,j)'s, c_{N,k}'s, c(N,k)'s, v_{N,\ell}'s, v(N,\ell)'s, w_{N,m},$ and $w(N,m)'s$ for this chosen data with minimal $A + B + C + V + W$.

Another application of the Subspace Theorem yields the following:
Proposition 5.11. Recall that $\theta = \min \left\{ \frac{c-2}{2c}, \frac{2(c-2)}{5(c-1)} \right\}$. Assume that $P$ satisfies:

$$2^P + P2^{P-1} - 1 < \frac{\theta}{2}. \tag{36}$$

Then we have $B = C = V = W = 0$.

Proof. First, suppose that $B > 0$. Let $\epsilon > 0$ be a small number that will be specified later. We apply the Subspace Theorem over the field $\mathbb{Q}(\alpha)$ and let $S = \{v_\infty, w\}$ be as before. We work with linear forms in the variables:

$$(X_\ell)_{1 \leq \ell \leq A}, (Y_j)_{1 \leq j \leq B}, (Z_k)_{1 \leq k \leq C}, (R_\ell)_{1 \leq \ell \leq V}, (T_m)_{1 \leq m \leq W}$$

and the vectors

$$v_N = \left( (a_N, \alpha^{a(N,i)})_{1 \leq i \leq A}, (b_N, \alpha^{b(N,j)})_{1 \leq j \leq B}, (c_N, k \alpha^{c(N,k)})_{1 \leq k \leq C}, \right.$$ \(\left( (s - s_{N-p})^{\alpha^{a(N,i)}})_{1 \leq i \leq V}, (\sigma(s) - s_{N-p})^{\alpha^{a(N,m)}})_{1 \leq m \leq W} \right)$$

for $N \in \mathcal{N}'$. For $v \in S$, the linear forms are denoted $L_{v,X,i}$, $L_{v,Y,j}$, $L_{v,Z,k}$, $L_{v,R,\ell}$, and $L_{v,T,m}$ for $1 \leq i \leq A, 1 \leq j \leq B, 1 \leq k \leq C, 1 \leq \ell \leq V$ and $1 \leq m \leq W$ and they are defined as follows:

- For any $v \in S$, $L_{v,X,i} = X_i, L_{v,Y,j} = Y_j, L_{v,Z,k} = Z_k, L_{v,R,\ell} = R_\ell$, and $L_{v,T,m} = T_m$ for every $i, j, k, \ell, m$ except when $j = 1$.
- If $v = v_\infty$, define $L_{v,Y,1} = Y_1$.
- If $v = w$, define

$$L_{w,Y,1} = \sum_{i=1}^A X_i + \sum_{j=1}^B Y_j + \sum_{k=1}^C Z_k + \sum_{\ell=1}^V R_\ell + \sum_{m=1}^W T_m.$$ 

Therefore if $v = v_\infty$, we have $|L_{v,Y,1}(v_N)|_v = |b_N, 1|^{1/2} |\alpha|^{b(N,1)/2} \leq |\alpha|^{-\theta n_N}$. If $v = w$, we have $L_{v,Y,1}(v_N) = \sigma(s) - s$ thanks to $[34]$. Thus, arguing as before, we have:

$$\prod_{v \in S} \prod_L |L(v_N)|_v < |\alpha|^{-\theta + \epsilon} |\alpha|^{n_N/2} \tag{37}$$

for all sufficiently large $N \in \mathcal{N}'$ where $L$ ranges over all the $L_{v,X,i}$’s, $L_{v,Y,j}$’s, $L_{v,Z,k}$’s, $L_{v,R,\ell}$’s, and $L_{v,T,m}$. On the other hand,

$$H(\mathbf{v}_N)^{A+B+C+V+W} \prod_{v \in S} \prod_L \frac{1}{|v_N|_v} \left( \prod_{v \in M \setminus S} |v_N|_v \right)^{A+B+C+V+W} \leq H(s_{N-p})^{A+B+C+V+W} \alpha^{\epsilon n_N} \leq |\alpha|^{(A+B+C+V+W)n_{N-P_1} + \epsilon n_N}. \tag{38}$$

Since $A + B + C + V + W \leq 2^P + P2^{P-1} - 1$, we can apply the Subspace Theorem if:

$$2^P + P2^{P-1} - 1 < \frac{\theta}{2} + \epsilon + \frac{-\theta + \epsilon}{2} < 0.$$ 

At the beginning of the proof, can choose an $\epsilon$ satisfying the above inequality thanks to the condition on $P$. Then the Subspace Theorem implies that the coordinates of $v_N$ satisfies a non-trivial linear relation over $\mathbb{Q}(\alpha)$ for every $N$ in an infinite subset $\mathcal{N}''$ of $\mathcal{N}'$. Then we have a new data satisfying the properties in Proposition 5.9.
in which $\mathcal{M}'$ is replaced by $\mathcal{M}''$ and $A + B + C + V + W$ is replaced by $A + B + C + V + W - 1$; this contradicts the minimality of $A + B + C + V + W$.

For the case $C > 0$, $V > 0$, or $W > 0$, we use the same vectors $\mathbf{v}_N$ and the same notation for the variables and linear forms. In the case $C > 0$, the linear forms are:

- For any $v \in S$, $L_{v, X, i} = X_i$, $L_{v, Y, j} = Y_j$, $L_{v, Z, k} = Z_k$, $L_{v, R, \ell} = R_\ell$, and $L_{v, T, m} = T_m$ for every $i, j, k, \ell, m$ except when $k = 1$.
- If $v = w$, define $L_{v, Z, 1} = Z_1$.
- If $v = v_\infty$, define
  \[
  L_{v, Z, 1} = \sum_{i=1}^A X_i + \sum_{j=1}^B Y_j + \sum_{k=1}^C Z_k + \sum_{\ell=1}^V R_\ell + \sum_{m=1}^W T_m.
  \]

In the case $V > 0$, the linear forms are:

- For any $v \in S$, $L_{v, X, i} = X_i$, $L_{v, Y, j} = Y_j$, $L_{v, Z, k} = Z_k$, $L_{v, R, \ell} = R_\ell$, and $L_{v, T, m} = T_m$ for every $i, j, k, \ell, m$ except when $\ell = 1$.
- If $v = v_\infty$, define $L_{v, Z, 1} = Z_1$.
- If $v = w$, define
  \[
  L_{v, Z, 1} = \sum_{i=1}^A X_i + \sum_{j=1}^B Y_j + \sum_{k=1}^C Z_k + \sum_{\ell=1}^V R_\ell + \sum_{m=1}^W T_m.
  \]

Finally, in the case $W > 0$, the linear forms are:

- For any $v \in S$, $L_{v, X, i} = X_i$, $L_{v, Y, j} = Y_j$, $L_{v, Z, k} = Z_k$, $L_{v, R, \ell} = R_\ell$, and $L_{v, T, m} = T_m$ for every $i, j, k, \ell, m$ except when $m = 1$.
- If $v = w$, define $L_{v, Z, 1} = Z_1$.
- If $v = v_\infty$, define
  \[
  L_{v, Z, 1} = \sum_{i=1}^A X_i + \sum_{j=1}^B Y_j + \sum_{k=1}^C Z_k + \sum_{\ell=1}^V R_\ell + \sum_{m=1}^W T_m.
  \]

Then similar arguments as before lead to a contradiction. This finishes the proof.

\pagebreak

\textit{Completion of the proof of Theorem 1.3.} At the beginning of this section, we fix a sufficiently large $P$ satisfying both (27) and (36). Then the previous results show that there exist an infinite set of positive integers $\mathcal{M}'$, an integer $A \geq 0$, tuples $(a_{N,1}, \ldots, a_{N,A})$ and $(a(N,1), \ldots, a(N,A))$ satisfying the conditions of Proposition 5.9 in particular:

$$
\sigma(s) - s = a_{N,1}a^{a(N,1)} + \ldots + a_{N,A}a^{a(N,A)}
$$

for every $N \in \mathcal{M}'$. However, each $|a_{N,i}| = |a|^{a(n_N)}$ as $N \to \infty$ while each $a(N,i) < \frac{2(c - 2)}{5^p - 1(c - 1)n_N}$. Let $N \to \infty$ the we have

$$
\sigma(s) - s = 0
$$

contradicting the earlier results that $s \in \mathbb{Q}(\alpha)$ is irrational. This finishes the proof.
SÉRIE DES RECIPROCALS DE FIBONACCI ET LUCAS NUMBERS

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