I. INTRODUCTION

The successful detection of gravitational waves from the inspiral and merger of binary black holes by the LIGO-Virgo interferometers [1–5] was made possible not only by technological advancements in instrumentation, but also by substantial improvements in theoretical modeling that furnished the gravitational wave templates necessary for performing matched filtering [6–10]. To generate a bank of complete template signals, the equations of general relativity have to be solved during the late compact binary inspiral, merger and post-merger phases, because these events involve extreme gravitational fields, whose description with post-Newtonian methods is not accurate. Obtaining an analytic solution to describe these systems during these dynamic stages is not possible. Therefore, numerical integration of the full Einstein equations provides the only viable avenue to understanding such relativistic astrophysical systems from first principles, and for helping to build gravitational wave templates during the most dynamical phases of their evolution.

Assuming that general relativity is the correct theory of gravity, the problem of two black holes is solved by integrating Einstein’s equations is vacuum. Despite the simpler description of black hole spacetimes compared to spacetimes with matter, it took decades for the field of numerical relativity to mature enough to be able to stably evolve two black holes until merger [11–13]. Some of the issues that hindered the development were due to the highly non-linear character of the Einstein equations, the coordinate freedom of general relativity, and the intrinsically singular nature of black holes. However, since the 2005 breakthrough, numerical relativity has advanced considerably with state-of-the-art codes that can simulate the inspiral and merger of uncharged binary black holes, and extract gravitational waves with high precision (see e.g. [14–20] and references therein). Numerical relativity furnishes invaluable information for gravitational-wave detection and analysis, which includes the development of templates (see e.g. [9, 10, 21, 22]), and the accurate parameter estimation of already detected events [23].

Apart from binary black holes, binary neutron stars and binary black hole-neutron stars are also most promising gravitational wave sources for currently operating interferometers [24]. In fact, among the eleven confirmed detections of gravitational waves so far [5], event GW170817 is attributed to the inspiral and merger of a binary neutron star [25] (although a binary black hole–neutron star cannot be ruled out [26] as a possibility). A complete simulation of compact binaries with matter requires the evolution of the spacetime coupled to matter, radiation/neutrinos, electromagnetic fields in conjunction with detailed microphysics. A full solution including radiation/neutrinos without approximation is impossible at this time, and even with approximation, evolution of perfect fluids with existing numerical schemes involves density floors and other ad hoc prescriptions that are necessary to stabilize the calculations (see e.g. [30, 31]), but are designed such that their impact on the global solution is minimal. However, this means that in a sense,
simulations involving perfect fluids are not as “clean” as the ones in vacuum, which do not require ad hoc prescriptions. Nevertheless, many important results have been obtained through binary neutron star and binary black hole–neutron star simulations in full general relativity, see [32–38] for reviews (see also [39] for other applications of numerical relativity).

Interesting spacetimes that are as “clean” as vacuum spacetimes, but have received little attention in numerical relativity, are those described by Einstein-Maxwell’s theory. This theory involves only gravitational and electromagnetic fields, and the corresponding spacetimes are referred to as electrovacuums or electrovacs. However, force-free electrodynamics has received some attention [10–17], but those simulations are not “clean”, in the sense that when the force-free conditions are violated during the evolution (typically in current sheets), one must interfere and enforce them to continue the calculations. On the other hand, electrovacuum spacetimes can be solved without physical approximations or ad hoc prescriptions, as the only assumption here is that electromagnetism and gravitation are described by the source-free Einstein-Maxwell equations. This simplification is the reason why these spacetimes have attracted numerous theoretical, analytic investigations for a long time, including the celebrated Kaluza-Klein theory [18–21] unifying gravity and electromagnetism.

Examples of interesting electrovacuum spacetimes are those with electrically charged black holes [22]. The case of a single charged non-rotating black hole is analytically solved by the Reissner-Nordström metric [50, 51]. This solution has been extended to non-vanishing angular momentum in the Kerr-Newman spacetime [52], which generalizes the uncharged rotating black hole solution found by Kerr [53]. Another interesting class of solutions with multiple black holes is the static Majumdar-Papapetrou solution [54, 55] that describes non-spinning black holes whose electric repulsion and gravitational attraction balance, producing a zero net force condition, and thus equilibrium. The hypothesis of staticity was relaxed to simple stationarity by [56–58]. This list summarizes the known analytical solutions of the source-free Einstein-Maxwell equations in four-dimensional asymptotically-flat spacetimes.

A reason why the source-free Einstein-Maxwell theory has been primarily confined to the realm of theoretical explorations is the fact that astrophysically relevant black holes are not believed to be electrically charged, as the charge would be neutralized by the surrounding plasma [59] or as result of a pair-production through a Schwinger-like process [60]. Nonetheless, there are some viable mechanisms to have a black hole with non-zero charge.

One example is the model proposed by [61], where the charge is retained due to the presence of an external magnetic field. This is known as the “Wald mechanism”. It was shown in [61] that if an asymptotically uniform magnetic field $B_0$ can be sustained, a black hole with mass $M$ spinning with angular momentum $J$ would acquire an electric charge $Q = 2B_0J$ (measured in geometrized units $\text{M}^{-1}\text{G}^{-1}$), which we can rewrite as $Q/M = 2B_0\chi M$ with $\chi = J/M^2$ the black hole dimensionless spin parameter. Since for black holes $\chi^2 \leq 1$, there exists a maximum possible charge-to-mass ratio in the Wald mechanism: $(Q/M) \leq (Q/M)_{\text{max}} = 2B_0 M$ [61]. In the case of a solar mass black hole in the galactic magnetic field $B_2$, the ratio has to be $Q/M \leq 10^{-24}$. The charge-to-mass ratio quantifies the deformation of the spacetime due to electromagnetism, so if it is very small it means that the spacetime is well-described by a vacuum (uncharged) black hole. Black holes with mass $M \gtrsim 10^9M_\odot$ immersed in a magnetic field of order $10^{11}$ G would be needed to reach values of $Q/M$ large enough to be relevant for the spacetime structure. Fields of such strength are expected to be found only in neutron stars. Based on the Wald mechanism, it has been recently proposed that a binary black hole – neutron star could provide a suitable environment to charge the black hole itself [63]. A second case in which charged black holes might occur in the Universe is immediately after the collapse of a compact star where the resulting hole might briefly retain some charge [63]. A similar scenario is the collapse of magnetized stars [64], which was also considered as candidate for fast-radio bursts [65]. Finally, charged black holes can emerge in more exotic theories associated with “hidden” gauge fields and elementary particles whose charge is a fraction of the electron charge [60].

In spite of the apparently compelling reasons to believe that astrophysical black holes have practically zero net charge compared to their mass, it is still worth studying the source-free Einstein-Maxwell system to advance our comprehension of strong-field gravitation and electromagnetism in this largely unexplored territory. The interplay between electromagnetism and gravity in a highly dynamical spacetime, which can be probed only with numerical investigations, and it is likely to offer a unique laboratory for both theoretical and more exotic astrophysical studies. The inclusion of charge in highly relativistic collisions of black holes (see for example [68–71] for such studies with zero charge) would advance our understanding in a new direction never explored before. Another interesting application of dynamical electrovacuums is related to cosmic censorship. In a recent series of papers, it was argued that strong cosmic censorship can be violated by electrovacuums with a positive cos-

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1 It is also possible to include magnetic charges. This will not be done in the study presented in this paper, so we always take the term charge to mean electric charge. We note the extension of the work to include magnetic charges would be straightforward.

2 The conversion factor between our units and the SI is $c^2G^{-\frac{1}{2}}(4\pi\varepsilon_0)^{\frac{1}{2}} = 1.16 \times 10^{20}\text{Ckm}^{-1}$, so $1M_\odot = 1.71 \times 10^{33}\text{C}$, with $c$ speed of light in vacuum, $G$ gravitational constant, $\varepsilon_0$ vacuum permittivity and $M_\odot$ solar mass.
We note that the formalism outlined in this paper applies not only to electromagnetism, but for any U(1) charge (such as the one described in [62]).

Notation and conventions

We assume that gravity and electromagnetism are described by Einstein-Maxwell’s theory [59] and we follow the same notation as in [59]. In particular, we use Einstein’s summation convention and the signature of the metric is $(-,+,+,+)$. We use geometrized units with $G = c = 1$, where $c$ is the speed of light in vacuum and $G$ the gravitational constant. The unit of charge is defined so that the proportionality constant in Coulomb’s law is 1 (for more details, see [62]). Indices $a, b, c$ and $d$ run in the set $\{0,1,2,3\}$, whereas the other Latin letters, such as $i, j$ or $k$ in the set $\{1,2,3\}$ and are referred to as spatial components. Parentheses and brackets in the indices
mean symmetrization and anti-symmetrization, respectively. We also use the abstract index notation \[59\]. We reserve the symbol \(\nabla\) for the four-dimensional covariant derivative associated with the spacetime metric \(g_{ab}\) and \(D\) for the three-dimensional covariant derivative, compatible with the spatial metric \(\gamma_{ij}\). We denote the determinant of these metrics as \(g = \det g_{ab}\) and \(\gamma = \det \gamma_{ij}\). We prepend the symbol “(4)” to all the four-dimensional tensors, with exception of the metric \(g_{ab}\). For the completely antisymmetric Levi-Civita tensor we use the convention that \(\epsilon_{1230} = -\sqrt{-g}\), and \(\epsilon_{123} = \sqrt{\gamma}\), and denote the Levi-Civita symbol with \(\bar{\epsilon}_{ijk}\) or \(\bar{\epsilon}^{ijk}\.

II. FORMALISM

In this Section we describe the theoretical tools that we use later to generate initial data for arbitrary configurations of charged black holes. Specifically, in Section II.A we survey the 3 + 1 decomposition of Einstein-Maxwell’s equations, focusing on the constraint equations. Section II.B reviews the Reissner-Nordström solution for a single charged stationary black hole in isotropic coordinates. Section II.C summarizes the theory of isolated horizons, which we employ to assign the black hole physical properties: mass, charge and angular momentum.

A. 3 + 1 decomposition of Einstein-Maxwell

In this paper we study systems described by the source-free Einstein-Maxwell equations \[59\]

\[
\begin{align*}
(4)R_{ab} - \frac{1}{2}g_{ab}(4)R &= 8\pi(4)T_{ab}^{\text{EM}}, \\
\nabla_a (4)F^{ab} &= 0, \\
\nabla_a (4)\star F^{ab} &= 0,
\end{align*}
\]

where \(\nabla\) is Ricci tensor associated with the metric \(g_{ab}\), \(\nabla^{(4)}R = \nabla^{(4)}R_{a}^{\alpha}(4)F_{ab} = (4)A_{[a,b]}\) is the Maxwell field-strength tensor, with \(4)A_{a}\) the four-potential, and \(\nabla^{(4)}\star F_{ab}\) is its Hodge dual, defined by

\[
\begin{align*}
(4)\star F_{ab} = \frac{i}{2}c_{abcd} (4)F_{cd}.
\end{align*}
\]

The electromagnetic stress-energy tensor is

\[
4\pi(4)T_{ab}^{\text{EM}} = (4)F_{ac} (4)F_{bd}g^{cd} - \frac{1}{4}g_{ab}(4)F_{cd}(4)F^{cd}.
\]

Solving the coupled Einstein-Maxwell equations in four dimensions is a challenging task. In particular, the form of Equations (1) is not suitable for a numerical solution. Therefore, we adopt the standard 3 + 1 decomposition to express the equations as a Cauchy problem, and cast them in a form amenable for numerical integration \[97\].

Assuming that the spacetime is described by a globally hyperbolic Lorentzian manifold \(M\) with metric tensor \(g_{ab}\), \(M\) can be foliated by a family of spacelike non-intersecting hypersurfaces \(\Sigma_t\), taken as level surfaces of a time function \(t\). Let \(n^a\) be the future-directed, timelike unit normal vector to \(\Sigma_t\). The projection operator along this vector is \(n^a n_b\), whereas the one onto \(\Sigma_t\) is

\[
\gamma^a_b = \delta^a_b + n^a n_b.
\]

The induced metric on \(\Sigma_t\), is derived by applying twice the projection operator on \(g_{ab}\), which yields

\[
\gamma_{ab} = g_{ab} + n_a n_b.
\]

The induced metric is purely spatial \((\gamma_{ab} n^b = 0)\), it encodes the intrinsic curvature of the hypersurfaces \(\Sigma_t\) and can be used to defined a spatial covariant derivative \(D_t\) on \(\Sigma_t\).

Instead of working with the normal vector \(n^a\), it is convenient to use the normalized time vector

\[
t^a = \alpha n^a + \beta^a,
\]

where \(\alpha\) and \(\beta^a\) are the lapse function and shift vector. With these quantities, the spacetime metric assumes the Arnowitt-Deser-Misner (ADM) form \[81\ 98\]

\[
d\sigma^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).
\]

The spatial metric is not sufficient to fully describe the curvature properties of the four-dimensional spacetime. The extrinsic curvature \(K_{ab}\) supplies the missing information by expressing how \(\Sigma_t\) is embedded in \(M\), and is defined as

\[
K_{ab} = -\gamma^{c}c_{abcd}\nabla_c n_d.
\]
equations, precisely because our goal is the generation of valid initial data for general relativistic simulations in Einstein-Maxwell theory.

Let \( T_{\alpha \beta}^{\text{EM}} \) be the stress-energy tensor, and define

\[
\mathcal{E} = n_{\alpha} n_{\beta} T_{\alpha \beta}^{\text{EM}}, \\
S^{i} = -\gamma^{ij} n^{a} T_{\alpha \beta}^{\text{EM}} n_{\alpha} n_{\beta}.
\]

The Einstein constraints then become

\[
\mathcal{R} + K^{2} - K_{ij} K^{ij} = 16\pi \mathcal{E}, \\
D_{j}(K^{ij} - \gamma^{ij} K) = 8\pi S^{i},
\]

with \( \mathcal{R} \) three-dimensional Ricci scalar associated with \( \gamma_{ij} \), and \( K \) the trace of the extrinsic curvature. Equation \( (10a) \) is known as the Hamiltonian constraint, Equations \( (10b) \) as the momentum constraints.

Equations \( (10) \) are not the only constraints in Einstein-Maxwell’s theory. As for Einstein’s equations, a \( 3+1 \) split of Maxwell’s equations must be performed. First, we introduce the electric and magnetic fields as seen by normal observers with four-velocity \( n^{a} \)

\[
E^{a} = (4) F^{ab} n_{b}, \tag{11a}
\]

\[
B^{a} = (4) * F^{ab} n_{b} = \frac{1}{2} \epsilon^{abcd} n_{b} (4) F_{cd}, \tag{11b}
\]

which are both purely spatial \( (n_{a} E^{a} = n_{a} B^{a} = 0) \). The electromagnetic tensor becomes

\[
(4) F_{ab} = n_{a} E_{b} - n_{b} E_{a} + \epsilon_{abcd} B^{c} n^{d}, \tag{12}
\]

and its dual is

\[
(4) * F_{ab} = n_{a} B_{b} - n_{b} B_{a} - \epsilon_{abcd} E^{c} n^{d}. \tag{13}
\]

With these decompositions, Maxwell’s equations can be expressed in terms of \( 3+1 \) quantities. As in the case of the Einstein equations, the \( 3+1 \) split leads to evolution and constraint equations. In particular, the electromagnetic constraints are

\[
D_{a} E^{a} = 0, \tag{14a}
\]

\[
D_{a} B^{a} = 0. \tag{14b}
\]

The electromagnetic sector couples with the spacetime through the stress-energy tensor \( T_{\alpha \beta}^{\text{EM}} \) which is re-written in terms of the \( 3+1 \) variables as

\[
4\pi T_{\alpha \beta}^{\text{EM}} = \frac{1}{2} (n^{a} n^{b} + \gamma^{ab}) (E_{c} E^{c} + B_{c} B^{c}) + 2n^{(a} \epsilon^{bc)} E_{c} B_{d} - (E^{a} E^{b} + B^{a} B^{b}), \tag{15}
\]

where \( \epsilon^{abcd} = n_{a} \epsilon^{abcd} \). Plugging Equation \( (15) \) in the source terms of Equations \( (9) \), we find

\[
4\pi \mathcal{E} = \frac{1}{2} (E_{i} E^{i} + B_{i} B^{i}), \tag{16a}
\]

\[
4\pi S^{i} = \epsilon^{ijkl} E_{j} B_{l}, \tag{16b}
\]

that are the familiar electromagnetic energy density and Poynting vector.

### B. The Reissner-Nordstrm spacetime

The Reissner-Nordstrm spacetime \( \text{[50 51]} \) describes an isolated non-rotating black hole with electric charge \( q \) and mass \( m \text{[59]} \). This solution will be the base of our generalization to charged black hole systems. In Boyer-Lindquist coordinates \( (t, r, \theta, \phi) \) the Reissner-Nordstrm metric is given by

\[
ds^{2} = -\left(1 - \frac{2m}{r} + \frac{q^{2}}{r^{2}}\right) dt^{2} + \left(1 - \frac{2m}{r} + \frac{q^{2}}{r^{2}}\right)^{-1} dr^{2} + r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2}), \tag{17}
\]

and the electromagnetic potential of the solution is

\[
(4) A = -\frac{q}{r} dt. \tag{18}
\]

In the following Sections we will adopt the puncture approach, so we transform the Boyer-Lindquist coordinates to isotropic ones. In order to do so, we define a new radial coordinate \( R \) satisfying

\[
r = R \left(1 + \frac{m}{R} + \frac{R_{H}^{2}}{R^{2}}\right), \tag{19}
\]

with \( R_{H} = \frac{1}{2}\sqrt{m^{2} - q^{2}} \) the radius of the black hole horizon in isotropic coordinates. The metric then assumes the following form

\[
ds^{2} = -\Psi^{-4} dt^{2} + \Psi^{4} \delta_{lk} dx^{l} dx^{k}, \tag{20}
\]

with \( \delta_{lk} \) the flat Euclidean metric, and \( \Psi \) the conformal factor defined as

\[
\Psi = \sqrt{1 + \frac{m}{R} + \frac{R_{H}^{2}}{R^{2}}} = \sqrt{\left(1 + \frac{m}{2R}\right)^{2} - \left(\frac{q}{2R}\right)^{2}}. \tag{21}
\]

As is clear from Equation \( (20) \), the spatial metric is manifestly conformally flat in isotropic coordinates. Moreover, there is no magnetic field and the electric field has only an \( R \) component

\[
E^{R} = \Psi^{-6} \frac{q}{R^{2}}. \tag{22}
\]

As a result, the Poynting vector defined in Equation \( (16b) \) is identically zero everywhere.
C. Isolated horizons

Once the constraint equations are solved, it is important to interpret the physical configuration the initial data correspond to. This can be achieved by locating the black hole apparent horizons and applying the theory of isolated horizons [93] (see [100] for a review). Isolated horizons provide a quasi-local notion of the black hole physical properties. In this Section we review basic identities of the formalism, including, in particular, the electric charge of the horizon, and the electromagnetic field contribution to angular momentum, elements that have not received much attention in numerical relativity applications [94] [101].

Isolated horizons have several desirable features. For instance, they always lie inside the event horizon, to which they reduce for stationary spacetimes, and they imply the existence of a future singularity [102] [103]. Most relevant for our purpose, they provide well-defined notions of mass, charge and angular momentum. For spacetimes with suitable symmetries, these quasi-local definitions and those at infinity differ [93]. Furthermore, the formalism does not provide a quasi-local definition of linear momentum due to the lack of a meaningful notion of space-translational symmetry in curved space-time [100] [104].

Here, we follow closely [94] in using isolated horizons to assign black hole physical parameters. Given a spatial section of an isolated horizon, the variables we are interested in are defined as follows. First, the areal radius is given by

\[ R_S = \left( \frac{1}{4\pi} \int_S \epsilon \right)^{\frac{1}{2}}, \]

where \( \epsilon \) is the area two-form on the 2-surface, given by \( \epsilon = \frac{1}{2} \sqrt{q_{ab}} \, dx^a \wedge dx^b \), where \( q_{ab} \) induced metric on the horizon and \( q = \det q_{ab} \). \( \int_S \epsilon \) is the surface area of the horizon.

Next, the definition of the angular momentum is based on an approximate rotational killing vector field \( \varphi^a \) on the 2-surface [93]

\[ J_S = -\frac{1}{8\pi} \int_S (\varphi \cdot \omega) \epsilon + 2(\varphi \cdot (4) A) (4) \star F, \]

where \( \omega \) is defined by the condition \( t^a \nabla_a k^b = t^a \omega_a k^b \) for any vector \( t^a \) tangent to \( S \) and for \( k^b \) the outgoing future-directed vector normal to \( S \) (by construction of \( k^b \), \( \omega \) always exists [93]). The two terms in the right-hand-side of Equation (24) are the gravitational and electromagnetic contribution to the horizon angular momentum.

The charge is defined by means of Gauss’s law

\[ Q_S = \frac{1}{4\pi} \int_S (4) \star F, \]

and finally, the gravitational mass of the isolated horizon is given by

\[ M_S = \frac{1}{2R_S} \left[ (R_S^2 + Q_S^2)^2 + 4J_S^2 \right]^{\frac{1}{2}}. \] (26)

For Kerr-Newman black holes, this formula perfectly reduces to the equation that relates total mass, irreducible mass, charge and angular momentum [105].

The definitions of angular momentum and charge involve four-dimensional quantities, but during simulations with the 3 + 1 formalism it is more convenient to use 3 + 1 variables. In [94] it was shown that the gravitational contribution to the horizon angular momentum can be computed using an ADM-like formula

\[ J_S^{GR} = -\frac{1}{8\pi} \int_S (\varphi \cdot \omega) \epsilon = \frac{1}{8\pi} \int_S \varphi^a R^b K_{ab} \epsilon, \] (27)

where \( R^a \) is the spatial unit vector normal to \( S \). The electromagnetic component of the angular momentum \( J_S^{EM} = J_S - J_S^{GR} \) depends both on (4) \( A \) and (4) \( \star F \). The first is directly accessible if instead of the electric and magnetic fields one evolves the vector potential [106] [107], whereas the second is decomposed as

\[ (4) \star F_{ab} = (2n_{[a} B_{b]} - \epsilon_{abc} E^c). \] (28)

When integrated over a spatial 2-surface the term \( 2n_{[a} B_{b]} \) does not contribute because \( n_a = (-\alpha, 0, 0, 0) \). Therefore, the electromagnetic contribution to the horizon angular momentum becomes

\[ J_S^{EM} = -\frac{1}{4\pi} \int_S (\varphi \cdot (4) A) \frac{1}{2!} \epsilon_{abc} E^c dx^a \wedge dx^b, \] (29)

where \( \epsilon_{abc} = n^d \epsilon_{abcd} \). By use of Equation (28), Equation (25) for the charge becomes

\[ Q_S = \frac{1}{4\pi} \int_S \frac{1}{2!} \epsilon_{abc} E^c dx^a \wedge dx^b. \] (30)

These definitions provide a complete characterization of black holes during a general relativistic simulation with the 3 + 1 decomposition. An example of how the integrations above are performed is in Appendix C.

III. SOLVING THE CONSTRAINT EQUATIONS

To solve the constraint equations we adopt the conformal transverse-traceless approach, also referred to as Bowen-York technique [91]. The goal of this method is to expose and specify degrees of freedom containing physical information about the system by applying conformal transformations on the spatial quantities, and working directly on the conformal variables instead of the physical ones.

The first step in the method is to conformally decompose \( \gamma_{ij} \) by introducing the conformal factor \( \psi \) and metric \( \tilde{\gamma}_{ij} \)

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}. \] (31)
In the following we use an overlayed bar to indicate conformal quantities.

A common assumption when generating multiple black hole initial data is that the spatial metric is conformally flat $\bar{g}_{ij} = \bar{\gamma}_{ij} \Omega^2$. In other words, we fix the conformal three-dimensional metric $\bar{\gamma}_{ij}$ to be the flat Euclidean metric $\delta_{ij}$ (in Cartesian coordinates). This choice greatly simplifies computations and it is a good approximation for the systems we are interested in studying, in spite of the fact that conformally flat spatial slices of the Kerr metric do not exist \cite{113}. Conformal flatness limits the maximum equilibrium value that the black hole dimensionless spin can attain \cite{15, 112}, but values of order 0.9 are completely achievable. Thus, we do not anticipate this approximation to impose severe constraints on the equilibrium values of the black hole spin and charge. Considering what happens in the uncharged case \cite{114, 115}, we expect that conformal flatness will generate initial data with spurious gravitational radiation in the charged black hole cases, too. Nonetheless, this is not a major concern since in dynamical simulations the system is evolved until this “junk” radiation propagates away, and the fields relax to their quasi-equilibrium values.

In addition to the conformal decomposition of the metric, it is also useful to transform the extrinsic curvature $K_{ij}$ by separating it into its traceless $A_{ij}$ and trace $(K = K^i_i)$ parts

$$K_{ij} = A_{ij} + \frac{1}{3} \gamma_{ij} K.$$

Following standard practice, we adopt the maximal slicing condition $K = 0$ \cite{110}, and introduce a conformal, traceless extrinsic curvature $\bar{A}_{ij}$ as

$$K_{ij} = A_{ij} = \psi^{-2} \bar{A}_{ij}.$$  

Then, $A_{ij}$ can be split in a transverse-traceless and a longitudinal part

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + \bar{A}^{ij}_L.$$  

We set $\bar{A}^{ij}_{TT} = 0$, which corresponds to suppressing the radiative degrees of freedom, so

$$\bar{A}^{ij} = \bar{A}^{ij}_L.$$  

The longitudinal part can always be expressed in terms of a vector $V^i$ as

$$\bar{A}^{ij} = \bar{A}^{ij}_L = 2 \delta^{ik} \delta^{jh} V_{(h,k)} - \frac{2}{3} \delta^{ij} \partial_k V^k,$$

where Cartesian coordinates are adopted. Going back to Equation \eqref{eq:33}, the extrinsic curvature is given by

$$K_{ij} = \psi^{-2} \left( 2 V_{(i,j)} - \frac{2}{3} \delta_{ij} \partial_k V^k \right).$$  

We already exploited much of the freedom we had in specifying variables during the previous steps. Under these assumptions, we just need the vector $V^i$ and the conformal factor $\psi$ to fully determine $\gamma_{ij}$ and $K_{ij}$, and the constraint Equations \eqref{eq:10} take the form

$$\nabla^2 \psi + \frac{1}{8} \psi^{-7} A_{ij} A^{ij} + 2 \pi \psi^3 E = 0,$$  

$$\nabla^2 V^i + \frac{1}{3} \delta^{ij} \partial_j (\partial_k V^k) - 8 \pi \psi^{10} S^i = 0,$$

where $\nabla^2 = \partial_k \partial^k$.

Next, we turn to the electromagnetic sector of the problem. We rescale the electromagnetic fields as in \cite{85}

$$\bar{E}^i = \psi^6 E^i,$$  

$$\bar{B}^i = \psi^6 B^i.$$  

The factor $\psi^6$ is chosen in order to have $D_i \bar{E}^i = \psi^{-6} \partial_i \bar{E}^i$, where we used the fact that for any vector $v^i$ it holds true that $D_i v^i = \gamma^{-1/2} \partial_i (\sqrt{\gamma} v^i)$. The Maxwell constraints \cite{14} read

$$\partial_i \bar{E}^i = 0,$$  

$$\partial_i \bar{B}^i = 0.$$  

These equations do not depend on the conformal factor $\psi$, and so the electromagnetic constraints can be solved independently from the spacetime ones. Moreover, the equations are linear hence we can superpose solutions.

Having fixed the conformal scalings of the $E^i$ and $B^i$ fields, the source terms $E$ and $S^i$ of the Einstein constraints conformally transform as

$$E = \psi^{-8} \bar{E},$$  

$$S^i = \psi^{-10} \bar{S}^i,$$

where

$$4 \pi \bar{E} = \frac{1}{2} (\bar{E}_i \bar{E}^i + \bar{B}_i \bar{B}^i),$$  

$$4 \pi \bar{S}^i = \frac{1}{2} \delta^{ij} \bar{E}_i \bar{B}^j.$$  

With these redefinitions, the Einstein constraints become

$$\nabla^2 \psi + \frac{1}{8} \psi^{-7} A_{ij} A^{ij} + 2 \pi \psi^3 \bar{E} = 0,$$  

$$\nabla^2 V^i + \frac{1}{3} \delta^{ij} \partial_j (\partial_k V^k) - 8 \pi \psi^{10} \bar{S}^i = 0.$$  

The problem is now greatly simplified because the momentum constraints do not depend on $\psi$, are linear in $V^i$, and along with the Hamiltonian constraint they have decoupled from the Maxwell constraints.

Next, we exploit the linearity of Equation \eqref{eq:43b} by decomposing $V^i$ as

$$V^i = V^i_{0,GR} + V^i_{EM},$$

where $V^i_{0,GR}$ solves the homogeneous Equation \eqref{eq:10} (when $\bar{S}^i = 0$), and $V^i_{EM}$ the inhomogeneous one. The subscript 0 does not indicate any component, but it reminds that the field is solution of the homogeneous equation.

\textsuperscript{6}
first term does not contain any reference to the electromagnetic sector of the problem. Thus, as in [92], we choose
\[
V_{0,GR}^i = \sum_{n=1}^{N_p} \left( -\frac{7}{4} \frac{P_n^i}{R_n^3} - \frac{1}{4} \delta_{jk} x_n^j P_n^k \frac{x_n^i}{R_n^3} + \frac{\bar{e}^{ij} x_n^j S_n^k}{R_n^3} \right),
\]
with \( R_n = |x - x_n| \) the Euclidean coordinate distance from puncture \( n \), where \( x_n \) is the location of the \( n \)-th puncture, and \( P_n^i \) and \( S_n^k \) are its linear and angular momenta, respectively. Equation (45) solves the homogeneous version of Equation (43b) and it is known that for suitable single black hole solutions \( P_{ADM}^i = P^i \) and \( J_{ADM}^i = S^i \), with \( P_{ADM} \) and \( J_{ADM} \) being the ADM linear and angular momenta evaluated at infinity \([91] [92]\), respectively. By use of the decomposition (44), the momentum constraints further reduce to three decoupled linear equations for \( V_{EM}^i \), effectively replacing Equation (43b) with
\[
\nabla^2 V_{EM}^i + \frac{1}{3} \delta^{ij} \partial_j (\partial_k V_{EM}^k) - 8\pi S^i = 0.
\]
(46)
We also manipulate the Hamiltonian constraint \([43a]\) further by separating the singular part of the conformal factor from the finite one \( u \), motivating our ansatz based on the conformal factor of the Reissner-Nordström spacetime in Equation (21)
\[
\psi = \left[ \left( 1 + u + \sum_{n=1}^{N_p} \frac{M_n}{2R_n^2} \right) - \left( \sum_{n=1}^{N_p} \frac{Q_n}{2R_n^2} \right)^2 \right]^{1/2}.
\]
(47)
We introduce the following abbreviations for compactness
\[
\eta = \sum_{n=1}^{N_p} \frac{M_n}{2R_n^2}, \quad \varphi = \sum_{n=1}^{N_p} \frac{Q_n}{2R_n^2}, \quad \kappa = 1 + u + \eta.
\]
(48)
Therefore, the conformal factor becomes
\[
\psi = \sqrt{\kappa^2 - \varphi^2}.
\]
(49)
Equation (47) is essentially an ansatz that states that our solution is a superposition of Reissner-Nordström black holes plus corrections (in \( u \)), which parallels what is performed in the uncharged case \([117]\).
Expanding Equation (43a), we reach
\[
\kappa \nabla^2 u + \partial_k \kappa \partial^k \kappa - \partial_k \varphi \partial^k \varphi - \partial_k \psi \partial^k \psi + \frac{1}{8} \psi^{-6} \bar{A}_{ij} \bar{A}^{ij} + 2\pi \psi^{-2} \bar{E} = 0.
\]
(50)
In deriving the last expression, we used the fact that the Laplacian of \( \eta \) and \( \varphi \) is zero. Equation (50) is a second order, non-linear elliptic partial differential equation in \( u \) that depends on \( V_{EM}^i \) through the term \( \bar{A}_{ij} \bar{A}^{ij} \). Now, the momentum and Hamiltonian constraints \([10]\) have been re-expressed as elliptic equations (46), (50) for \( V_{EM}^i \) and \( u \). The associated boundary conditions are found from the assumption of asymptotic flatness so that \( u \) and \( V_{EM}^i \) have to go to zero at spatial infinity.

The problem of generating valid initial data for multiple charged black holes is now reduced to solving Equations (40) and (50), which is done once Maxwell-compliant electromagnetic fields are found. In this paper, we assume that each puncture is endowed with a Reissner-Nordström electromagnetic field, and hence the total conformal electric field is a superposition of Reissner-Nordström electromagnetic fields in isotropic coordinates, i.e.,
\[
\bar{E}^i = \sum_{n=1}^{N_p} \frac{Q_n}{2R_n^2} \bar{R}_n^i,
\]
(51)
where \( \bar{R}_n \) is the radial unit vector centered on the \( n \)-th puncture. In the case of a single, non-rotating black hole with zero linear momentum, our choice of Reissner-Nordström fields exactly produces a spatial slice of that solution, since the constraints are solved by \( V_{GR} = V_{EM} = 0 \), and \( u = 0 \) (so \( \psi = \Psi \), where \( \Psi \) is given in Equation (21)). For systems of spinning black holes with linear momenta, the superposition of Reissner-Nordström fields are first approximations to the equilibrium electromagnetic fields generated by these systems. As for the gravitational fields generated in the puncture approach (and the gauge fields), we expect that the time evolution will relax our electromagnetic-field initial data to their quasi-equilibrium values on a light-crossing timescale. An advantage of choosing Reissner-Nordström electromagnetic fields is that they allow for a clear description of each black hole in the system with a specific charge, whose isolated horizon value \( Q_S \) equals the “bare” charge entering Equation (51). In addition, this choice ensures that there are no electromagnetic contributions to the extrinsic curvature, implying that the parameters entering Equation (45) can be still interpreted as \( P_{ADM}^i = P^i \) and \( J_{ADM}^i = S^i \).

The choice of Reissner-Nordström electromagnetic fields is by no means unique. Another possibility is Kerr-Newman fields in quasi-isotropic coordinates. We present a detailed discussion of this case and the complexities associated with it in Appendix D.

IV. NUMERICAL IMPLEMENTATION

We implement the formalism outlined in the previous Sections by modifying the TwoPunctures [92] and QuasiLocalMeasures open-source codes [91]. The software is run within the Cactus infrastructure [118] and all physical variables are interpolated on a Carpet grid [119] [120]. Black hole apparent horizons are found with AhFinderDirect [121].

The main component in our software stack is TwoChargedPunctures, which is used to generate initial
data for two punctures located at $(\pm b, 0, 0)$ given the bare black hole properties $(M_0, Q_0, P_0, S_0)$. This code implements a pseudo-spectral collocation method that solves the constraint equations (46) and (50) to find $u$ and $V_{EM}^i$.

In what follows, we adopt Reissner-Nordstrm electromagnetic fields. Since there is only an electric field, $S^i = 0$ in Equation (46), and the momentum constraint is trivially satisfied by $V_{EM}^i = 0$. Hence, we only need to solve the Hamiltonian constraint (50).

TwoChargedPunctures implements a single domain pseudo-spectral method that covers all $\mathbb{R}^3$ with spatial infinity on the grid. This region is parametrized by the coordinates $(A, B, \phi)$, with $A, B \in [-1, 1]$ and $\phi \in [0, 2\pi]$. To be more specific, the code uses a system of bispherical coordinates that transform to the usual Cartesian ones with the law\footnote{This parametrization is slightly different compared to what is done in [92]. The spectral expansion used here treats $A$ and $B$ on equal footing, i.e., the spectral decomposition in $A$ and $B$ uses the same Chebyshev polynomial basis, unlike what is reported in [92].}

\[
\begin{align*}
x & = b (1 + A)^2 + 4 \frac{2B}{(1 + A)^2 - 4} \frac{1 + B^2}{1 - B^2} \cos \phi, \\
y & = b \frac{4(1 + A)}{4(1 + A)^2} \frac{1 - B^2}{1 + B^2} \cos \phi, \\
z & = b \frac{4(1 + A)}{4(1 + A)^2} \frac{1 - B^2}{1 + B^2} \sin \phi,
\end{align*}
\]

where the $x$ axis is along the line connecting the two punctures. Equations (52) describe a set of cylindrical-like coordinates around the $x$ axis with a radius that depends on both $A$ and $B$.

The coordinates $(A, B, \phi)$ live on a compact grid where spatial infinity corresponds to $A = 1$, which makes it straightforward to impose the desired outer boundary conditions ($u \to 0$ at infinity). This condition is enforced by solving the equations for an auxiliary variable $U$ defined as $u = (A - 1) \cdot U$. The code expands $U$ in Chebyshev polynomials along $A$ and $B$, and adopts a Fourier basis along $\phi$. The coordinates are discretized with $n_A$, $n_B$ and $n_\phi$ grid points chosen as the zeros of Chebyshev polynomials $T_{n_A}(x)$, $T_{n_B}(x)$ and of the sine function $\sin(n_\phi \phi)$. The coefficients of the spectral expansion are found by evaluating the relevant equation on the collocations points and solving the corresponding multi-dimensional non-linear system with a modified Newton-Raphson method [122] (more details on how this is done can be found in Section II of the original paper [92]). We consider the equations to be solved, when the residuals are smaller than a threshold value. To choose this threshold value, we solve for increasingly smaller values of this threshold, and compute the ADM and the horizon masses. When these masses have converged to within one part in $10^6$ we consider the solution converged.

With the equations solved and $u$ known, TwoChargedPunctures reverts back to the physical fields using Equations (37), (39), (41), (45) and (47). We then spectrally interpolate the physical fields on a Carpet grid where AHFinderDirect is subsequently run to locate the apparent horizons. Once the horizons are found, we compute mass, charge and angular momentum of each black hole with our version of QuasiLocalMeasures, which we call QuasiLocalMeasuresEM, and which implements the formalism of isolated horizons for the full Einstein-Maxwell theory as reviewed in Section IIC. Moreover, having the spectral expansion of the fields we can interpolate them at a very large radius to compute the ADM mass, linear and angular momentum.

### A. Code validation

We validate our approach and numerical implementation with a series of tests that are presented in the section.

We report our results in terms of the input bare mass $M$ of the punctures, which is the only mass known a priori. In all the runs we confine the black hole in a region where the Carpet grid resolution is $\Delta r_i = 0.0078 M$, which usually guarantees that the diameter of the horizon is resolved by about 100 points, making it easily found by AHFinderDirect. We also fix the resolution of the AHFinderDirect grid to be 79 points in the azimuthal direction, and 39 in the meridional direction. We have confirmed that the resolution on the AHFinderDirect grid has negligible impact in our results. In the cases presented here, doubling the AHFinderDirect grid resolution introduces a variation in the computed parameters of order 0.01%. We compute ADM integrals by spectrally interpolating our fields on a sphere of radius 10000 $M$, and discretized with 256 points in both the meridional and azimuthal directions.

As first test, we made sure that our modified code with zero charge, TwoChargedPunctures, produces the same output as the standard open-source TwoPunctures code. This is not a trivial test because the equations used in our code and in the original one are different, having different numerical properties, even though they are mathematically equivalent. In particular, our formulation is more susceptible to numerical instabilities due to the finite-arithmetic error in regions close to the puncture. The reason for this is that our equations have terms that are not present in the original code, but that should perfectly cancel out when $Q = 0$. Such a numerical cancelation near the punctures is not trivial. However, the result of the test with different spectral resolutions shows that the two implementations agree at the round-off-error level for punctures with no charge.

Another key test that our code successfully passes consists in recovering the only conformally flat analytical solutions known: the Reissner-Nordstrm and the case of two black holes with the same charge-to-mass-ratio (see
Appendix [E] for more details), both of which are found with $u = 0$. We find the solution $u = 0$ is recovered to machine precision everywhere outside the horizons, and it is non-identically zero only very close to the punctures, again due to numerical precision.

The next test for TwoChargedPunctures is reproducing the numerical solution found by [85] for two non-rotating black holes with opposite charge-to-mass ratio starting at rest. Figure [1] reports the value of $u$ along the $x, y$ and $z$ axes for a system of two punctures with the same mass but opposite charge ($Q_1 = -Q_2 = 0.5 \, M$). We graphically superposed our plot with Figure 1 in [85], finding perfect agreement.

Continuing the progression of complexity in the considered systems, we generate a single puncture with angular momentum, but no linear momentum, and one with linear momentum but no angular momentum (Figures [2] and [3], respectively). In these single-black hole cases we compare the horizon mass with the ADM mass measured at infinity and we find agreement of order $0.1\%$ even with resolution as low as $n = 16$. The same is true for the ADM angular momentum and the horizon spin, as computed with QuasiLocalMeasuresEM. We repeated these two tests by aligning the linear and angular momentum vectors once along the $x$ and once along the $z$ direction to ensure that the built-in asymmetry in the coordinates [Equations (52)] does not spoil expected symmetries in symmetric configurations. By doing this, we find that the solutions are rotationally invariant to better than 1 part in $10^6$ for a resolution $n = 32$ or higher.

Finally, we considered the generic system shown in Figure [1]. This is formed by two equal-mass black holes with charge $Q_1 = -0.3 \, M$ and $Q_2 = 0.5 \, M$. Both black holes are spinning with angular momentum $S_1^z = S_2^z = 0.5 \, M^2$. The black holes also have linear momentum $P_1^x = P_2^x = 0.5 \, M,$ and $P_1^y = P_2^y = 0.5 \, M,$ and are indicated by the vertical dotted lines. The horizon has radius $R_S = 0.433 \, M$. The different style curves have the same meaning as in Figure [1].

**FIG. 2.** $u$ along the coordinate axes for a single puncture with charge $Q = 0.5 \, M$ rotating around the $z$ axis with angular momentum $S^z = 0.5 \, M^2$. The plot corresponds to spectral grid resolution $n_A = n_B = n_\phi = 64$. The horizon has radius $R_S = 0.433 \, M$. The different style curves have the same meaning as in Figure [1].

**FIG. 3.** $u$ along the coordinate axes for a single puncture with charge $Q = 0.5 \, M$ with linear momentum $P^x = 0.5 \, M$. The plot corresponds to spectral grid resolution $n_A = n_B = n_\phi = 64$. The horizon has radius $R_S = 0.421 \, M$. The different style curves have the same meaning as in Figure [1].

**B. Convergence**

We graphically superposed our plot with Figure 1 in [85], starting at rest. Figure 1 reports the value of $u$ along the different coordinate axes (solid line for the $x$ axis, dotted and dashed for the $y$ and $z$, respectively) for two punctures with equal mass and opposite charge ($Q_1 = -Q_2 = 0.5 \, M$) located on the $x$ axis at $\pm 2 \, M$. This configuration is generated with a spectral grid resolution $n_A = n_B = n_\phi = 64$. We graphically compared our solution with the solution in [85], and find that the curves shown here perfectly match the solution of [85]. The horizons have radius $R_S = 0.387 \, M$ and are indicated by the vertical dotted lines.
The solution for $u$ for this system is depicted in Figure 5. With QuasiLocalMeasuresEM, we find that the quasi-local angular momenta (charges) agree with their bare counterparts to within 1 part in $10^4$ ($10^6$). We find that the mass of the first horizon is $1.187 M$ and the second is $1.202 M$. The total (ADM) mass of the system is $2.337 M$, the difference between this value and the sum of the individual masses is the binding energy.

This system is used to study the self-convergence properties of the code. In particular, we consider the maximum relative error of $u$ with respect to a reference solution at high resolution $N$. For this, we sampled $u$ on a set of points $\mathcal{T}$, and computed the infinity norm

$$\|\Delta_N^T u\|_\infty = \max_{x \in \mathcal{T}} \left| \frac{u^N(x) - u^N(x)}{u^N(x)} \right|.$$  

We choose $\mathcal{T}$ as the set of points where spheres of radii $1 M, 2 M, 5 M, 10 M, 100 M$ and $1000 M$ intersect the coordinate axes for $x > 0, y > 0, z > 0$.

We set as reference solution ($N$) one obtained at high-resolution with $n_A = n_B = n_\phi = n = 64$, which is between $n = 50$ and $n = 70$ that were used for self-convergence tests in the original TwoPunctures code [92]. Here, we simply choose resolutions which are multiples of 4, but our results do not depend on this choice. Our convergence test (Figure 5) shows that the algorithm is robust; $u$ quickly converges to its high-resolution value. The code converges approximately at sixth-order. We also verified that the code exhibits the same convergence properties when we repeat the convergence test with $Q_1 = Q_2 = 0$, which also agree with the convergence properties of the original TwoPunctures code [92]. The convergence of $u$ also results in excellent convergent behavior for both the ADM mass and momenta and the horizon properties as computed by QuasiLocalMeasuresEM.

V. CONCLUSIONS AND FUTURE WORK

Gravitational waves offer new opportunities to study the Universe that are not accessible with electromagnetic or neutrino astronomy. In this landscape, numerical-relativity simulations are a powerful tool to gain insight into the properties and the characteristics of both the waves and their sources. The majority of numerical-relativity simulations of black holes to-date do not treat the electric charge. This is because it is believed that astrophysically relevant black holes should have a charge which is negligibly small compared to the mass. For this reason, there are no studies of highly dynamical
electrovacuum spacetimes that involve the inspiral and merger of binary black holes with charge and spin. Nevertheless, evovacuum spacetimes are of great interest, having both a theoretical appeal and exotic astrophysical applications.

In this paper, we initiated an effort towards solving the coupled Einstein-Maxwell equations in a dynamical and fully general relativistic regime. The first step to perform this type of simulations is the generation of valid initial data. Here, we employed the conformal transverse-traceless approach to build a formalism for generating initial data for multiple black holes with charge, angular momentum, and linear momenta. Moreover, we applied the theory of isolated horizons to attribute the physical mass, charge, and angular momentum to the horizon, providing a solid understanding on the physical content of our initial data. We implemented the formalism in a software based on the TwoPuncture and the QuasiLocalMeasures open-source codes, verifying our implementation with a series of test involving analytical or previously-known results. The algorithm was found to recover the expected solutions and showed excellent convergence properties.

With the valid initial data for charged, rotating and moving punctures it is now possible to simulate dynamical evolution of several systems that have never been taken in consideration, such as ultra-relativistic head-on collision, and the quasi-circular or eccentric inspiral and merger of two black holes. As first application of the formalism outlined in this paper we plan to study in the near-future the case of charged and spinning black holes in quasi-circular orbit. Some of these simulations are already underway, and will be presented in forthcoming work.

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Appendix A: Algorithm and important equations

In this Appendix we sketch the algorithm and summarize the important equations to generate initial data for 3 + 1 evolutions of arbitrary systems of N black holes with electric charge, linear and angular momenta using the conformal transverse-traceless decomposition. In the following n is used to index the n-th black hole in the system that is \( n \in \{1, \ldots, N\} \). Unless otherwise specified sums in this Appendix are over all punctures. We also assume that each black hole is endowed with Reissner-Nordstrom electromagnetic fields \((\tilde{E}^i, \tilde{B}^j)\) associated with electric charge \( Q_n \). The steps in generating the initial data are as follows:

1. Choose the bare parameters \( M_n, Q_n, S^i_n, P^i_n, x_n \) for each black hole, respectively representing mass, charge, angular momentum, linear momentum, and position.

2. Compute the conformal electromagnetic fields \((\tilde{E}^i_n, \tilde{B}^j_n)\) for each black hole. Under the assumption of Reissner-Nordstrom fields we obtain

\[
\tilde{E}^i_n = \frac{Q_n}{R_n^2} \hat{R}_n^i, \quad (A1)
\]

\[
\tilde{B}^j_n = 0, \quad (A2)
\]

with \( R_n = |x - x_n| \) the Euclidean coordinate distance from puncture \( n \), and \( \hat{R}_n^i \) the corresponding unit vector. Then, superpose the conformal electromagnetic fields of all black holes

\[
\tilde{E}^i = \sum_n \tilde{E}^i_n (Q_n, x_n), \quad (A3)
\]

\[
\tilde{B}^j = \sum_n \tilde{B}^j_n (Q_n, x_n). \quad (A4)
\]

3. Solve the inhomogeneous momentum constraint for \( V_{EM} \)

\[
(\nabla^2 V_{EM})^j + \frac{1}{3} \delta^{ij} \partial_j (\partial_k V_{EM}^k) - 8\pi \tilde{S}^i = 0, \quad (A5)
\]

with

\[
4\pi \tilde{S}^i = \varepsilon^{ijk} \tilde{E}_i \tilde{B}_j, \quad (A6)
\]
4. Compute the total auxiliary vector \( V^i \)

\[
V^i = V^i_{GR} + V^i_{EM},
\]

with

\[
V^i_{GR} = \sum \left( -\frac{7}{4} \frac{P^i_n}{R_n} - \frac{1}{4} \delta_{jk}^i \frac{P^k_n x^j_n}{R^3_n} + \frac{\tilde{e}^i_{jk} x^j_n x^k_n}{R^3_n} \right).
\]

(A8)

5. Solve the Hamiltonian constraint for \( u \), imposing \( u \to 0 \) at spatial infinity

\[
\kappa \nabla^2 u + \partial_a \kappa \partial^a \kappa - \partial_a \varphi \partial^a \varphi - \partial_a \psi \partial^a \psi + \frac{1}{8} \psi^{-6} \tilde{A}_{ij} \tilde{A}^{ij} + 2\pi \psi^{-2} \tilde{E} = 0.
\]

(A9)

6. With \( \psi \) now known, compute the physical fields that are necessary for the evolution

\[
E^i = \psi^6 \tilde{E}^i,
\]

\[
B^i = \psi^6 \tilde{B}^i,
\]

\[
\gamma_{ij} = \psi^4 \delta_{ij},
\]

\[
K_{ij} = \psi^{-2} \left( 2V_{(ij)} - \frac{2}{3} \delta_{ij} \partial_k V^k \right).
\]

(A10)\(\quad\)\(\quad\)\(\quad\)\(\quad\)\(\quad\)\(\quad\)\(\quad\)\(\quad\)

7. Find the isolated horizons \( S_n \) and compute the associated physical properties

\[
Q_{S_n} = \frac{1}{4\pi} \int_{S_n} (4^i)^* F,
\]

\[
R_{S_n} = \left( \frac{1}{4\pi} \int_{S_n} \epsilon \right)^{\frac{1}{2}},
\]

\[
J_{S_n} = -\frac{1}{8\pi} \int_{S_n} (\varphi \cdot \omega) \epsilon + 2(\varphi \cdot (4^i)^* F),
\]

\[
M_{S_n} = \frac{1}{2R_{S_n}} \left[ (R_{S_n}^2 + Q_{S_n}^2)^2 + 4J_{S_n}^2 \right]^{\frac{1}{2}},
\]

(A20)\(\quad\)\(\quad\)\(\quad\)\(\quad\)\(\quad\)\(\quad\)\(\quad\)

where \((4^i)^* F\) is the dual of the electromagnetic tensor, \( \epsilon \) is the horizon surface 2-form, \((4^i) A\) is the electromagnetic vector potential, \( \varphi \) the approximate rotational Killing vector on \( S_n \), and \( \omega \) is the electromagnetic potential, \( \approx \) the approximate rotational Killing vector on \( S_n \). \( \approx \) is defined in the main text. \( Q_{S_n}, R_{S_n}, J_{S_n} \) and \( M_{S_n} \) are respectively the charge, radius, angular momentum and mass of the \( n \)-th horizon.

### Appendix B: Isolated horizon in the Reissner-Nordström solution

The goal of this Appendix is to show that the formalism of isolated horizons produces the expected black hole properties in the case of the Reissner-Nordström solution. This can be proven starting from metric (17), which we rewrite here for convenience

\[
ds^2 = - \left( 1 - \frac{2m}{r} + \frac{q}{r^2} \right) dt^2 + \left( 1 - \frac{2m}{r} + \frac{q}{r^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

(B1)

with electromagnetic potential

\[
(4^i) A = - \frac{q}{r} dt.
\]

(B2)

In this case, a spherical surface with coordinate radius \( r_+ = m + \sqrt{m^2 - q^2} \) is a Killing horizon, which implies that it is an isolated horizon. This is because every Killing horizon which is topologically \( S^2 \times \mathbb{R} \) is an isolated horizon [93]. Therefore, the metric \( q_{ab} \) induced on the horizon is simply the metric on a sphere and the value of \( R_S \) defined by Equation (23) coincides with \( r_+ \) itself, since the radial coordinate in Equation (B1) is the areal radius. In this case, the rotational vector \( \varphi \) in (24) is taken to be the generator of the azimuthal symmetry on the sphere, which is also a Killing vector of the entire spacetime. Hence, we find that \( \varphi \cdot (4^i) A = 0 \) as \( \approx \) has only temporal component and \( \varphi \) only spatial. Moreover, since the future-directed vector \( k^a \) orthogonal to \( S \) has only radial and temporal component, and any \( t^a \) tangent to \( S \) has only azimuthal and meridional components, \( t^a \nabla_a k^b = 0 \). By construction, we also have \( t^a \nabla_a k^b = t^a \omega_a k^b = 0 \), which implies that \( \omega_a = 0 \), because the equation is zero for each \( t^a \). Hence, by use of Equation (24), we conclude that \( J_S = 0 \).

To compute charge and mass we need the electromagnetic tensor, which is given by

\[
(4^i)^* F = d(4^i) A = - \frac{q}{r^2} dt \wedge dr = \frac{q}{r^2} dt \wedge dr,
\]

(B3)

and its dual

\[
(4^i)^* F = \sqrt{-g} \frac{q}{r^2} d\theta \wedge d\varphi = q \sin \theta d\theta \wedge d\varphi.
\]

(B4)

The integration of \( (4^i)^* F/4\pi \) over any sphere of coordinate radius \( r \) results in exactly \( q \), so Equation (25) implies \( Q_S = q \).
Finally, from Equation (26) the horizon mass is

\[ M_S = \frac{(R_s^2 + q^2)}{2R_s} = \frac{2m(m + \sqrt{m^2 - q^2})}{2(m + \sqrt{m^2 - q^2})} = m. \quad (B5) \]

For a Reissner-Nordström black hole, \( m, q \) are interpreted as the spacetime total energy and electric charge, respectively [39]. Therefore, in this case the bare mass (charge), the isolated horizon mass (charge), and the physical mass (charge) all coincide.

**Appendix C: Computing the charge of an isolated horizon**

In this Appendix we discuss how we perform the computation of the horizon charge. To compute the charge of the horizon, we need to perform the following integration (see Section [HC])

\[ Q_S = \frac{1}{4\pi} \int_S \frac{1}{2!} \epsilon^{abc} E^c \, dx^a \wedge dx^b. \quad (C1) \]

This quantity is coordinate-independent, so choosing Cartesian coordinates \((x^a) = (x, y, z)\), we can write

\[ Q_S = \frac{1}{4\pi} \int_S (E_x \, dx \wedge dy + E_y \, dy \wedge dz - E_z \, dx \wedge dz). \quad (C2) \]

We introduce a parametrization of \( S \) with polar coordinates \((\theta, \phi)\) around the origin \((x_0, y_0, z_0)\)

\[ \begin{cases} x(\theta, \phi) = x_0 + s(\theta, \phi) \sin \theta \cos \phi \\ y(\theta, \phi) = y_0 + s(\theta, \phi) \sin \theta \sin \phi \\ z(\theta, \phi) = z_0 + s(\theta, \phi) \cos \theta \end{cases}, \quad (C3) \]

with \( s(\theta, \phi) \) a suitable smooth function. This is always possible since by hypothesis \( S \) has spherical topology and by construction \( Q_S \) does not depend on the parametrization. Then, the first term in Equation (C2) can be written as

\[ \int_S E_x(x, y, z) \, dx \wedge dy = \int_0^\pi \int_0^{2\pi} E_x(\theta, \phi) \det J_{xy}(\theta, \phi) \, d\theta \, d\phi, \quad (C4) \]

where \( J_{xy}(\theta, \phi) \) is the Jacobian of the transformation (C3) involving the coordinates \( x \) and \( y \)

\[ J_{xy}(\theta, \phi) = \begin{pmatrix} \partial_x x(\theta, \phi) & \partial_y x(\theta, \phi) \\ \partial_x y(\theta, \phi) & \partial_y y(\theta, \phi) \end{pmatrix}. \]

The remaining terms in Equation (C2) are dealt with accordingly.

In **QuasiLocalMeasuresEM**, we use the parametrization \( s(\theta, \phi) \) provided by **AHFinderDirect**, and we compute the derivatives in the Jacobians using a centered, second-order accurate finite-difference scheme.

**Appendix D: Kerr-Newman spacetime**

In this Appendix we review the Kerr-Newman spacetime and discuss challenges associated with using the Kerr-Newman electromagnetic fields as source terms in the Hamiltonian and momentum constraints.

The Kerr-Newman black hole with mass \( m \), electric charge \( q \) and angular momentum \( am \) in Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) is [59]

\[
\begin{align*}
    ds^2 &= -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} \, dt^2 + \frac{\rho^2}{\Delta} \, dr^2 + r^2 \, d\theta^2 \\
    &\quad - 2a \sin^2 \theta \frac{(r^2 + a^2 - \Delta)}{\rho^2} \, dt \, d\phi \\
    &\quad + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\rho^2} \sin^2 \theta \, d\phi^2,
\end{align*}
\]

with

\[
\begin{align*}
    \rho^2 &= r^2 + a^2 \cos^2 \theta, \\
    \Delta &= r^2 - 2mr + a^2 + q^2.
\end{align*}
\]

The electromagnetic vector potential is

\[
(4) \ A = -\frac{q}{\rho^2} (dt - a \sin^2 \theta \, d\phi).
\]

Following the usual procedure for generating puncture initial data, we transform to quasi-isotropic coordinates by introducing a new radial coordinate \( R \) as in [59]

\[ r = R \left( 1 + \frac{m}{R} + \frac{R_H}{R} \right), \quad (D4) \]

with \( R_H = \frac{1}{2} \sqrt{m^2 - a^2 - q^2} \) radius of the black hole horizon in the new coordinate system. The metric takes now the form

\[ ds^2 = (-\alpha^2 + \beta_\phi \beta^\phi) \, dt^2 + 2\beta_\phi \, d\phi \, dt + \gamma_{lk} \, dx^l \, dx^k, \quad (D5) \]

where

\[
\begin{align*}
    \gamma_{lk} \, dx^l \, dx^k &= \Psi^4 [dR^2 + R^2 \, d\theta^2 + R^2 \sin^2 \theta \, d\phi^2] \\
    &\quad + a^2 \rho^4 \sin^4 \theta \, d\phi^2],
\end{align*}
\]

where \( \Psi \) is the conformal factor, \( l, k \in \{R, \theta, \phi\} \), and \( \alpha, \beta, \gamma \) and \( h \) functions of \((R, \theta, \phi)\), with

\[
\begin{align*}
    \Psi^4 &= \rho^2 / R^2, \\
    \alpha &= \rho^6 (R + R_H)(R - R_H), \\
    \beta_\phi &= -a \alpha \sin^2 \theta, \\
    \beta^\phi &= \beta_\phi / \gamma_{\phi\phi}, \\
    h &= (1 + \sigma) / (\rho^2 R^2), \\
    \sigma &= \frac{2mr - q^2}{\rho^2}, \\
    \Upsilon &= \rho^6 \sqrt{r^2 + a^2 (1 + \sigma \sin^2 \theta)}, \\
    \gamma_{\phi\phi} &= \sin^2 \theta \left( r^2 + a^2 (1 + \sigma \sin^2 \theta) \right).
\end{align*}
\]
The non-zero components of the electromagnetic fields are\[^{[3]}\]

\[
\begin{align*}
E^R &= \frac{q(R(2r^2 - \rho^2)(r^2 + a^2))}{\mathcal{Y}}, \\
E^\theta &= -\frac{2aq^2(R - R_H)(R + R_H)r \cos \theta \sin \theta}{RT}, \\
B^R &= \frac{2aqR(r^2 + a^2) \cos \theta}{\mathcal{Y}}, \\
B^\theta &= \frac{aq(R - R_H)(R + R_H)(2r^2 - \rho^2) \sin \theta}{RT}.
\end{align*}
\] (D8a, D8b, D8c, D8d)

The conformal fields are obtained by scaling by \(\sqrt{\mathcal{Y}} = \mathcal{Y} \rho^{-4} R^{-1} \sin \theta\)

\[
\begin{align*}
\tilde{E}^R &= \frac{q(2r^2 - \rho^2)(r^2 + a^2) \sin \theta}{\rho^4}, \\
\tilde{E}^\theta &= -\frac{2aq^2(R - R_H)(R + R_H)r \cos \theta \sin^2 \theta}{\rho^4 R^2}, \\
\tilde{B}^R &= \frac{2aqr(r^2 + a^2) \cos \theta \sin \theta}{\rho^4}, \\
\tilde{B}^\theta &= \frac{aq(R - R_H)(R + R_H)(2r^2 - \rho^2) \sin^2 \theta}{\rho^4 R^2}.
\end{align*}
\] (D9a, D9b, D9c, D9d)

In these coordinates, the conformal fields are regular for \(R \to 0\) (in this limit \(\rho \sim r \sim 1/R\)).

However, Equations \([46]\) and \([50]\) are in Cartesian coordinates. Transforming to Cartesian coordinates as in flat spacetime, the conformal fields are obtained as

\[
\begin{align*}
\tilde{E}^i &= \frac{\partial x^i}{\partial R} \tilde{E}^R + \frac{\partial x^i}{\partial \theta} \tilde{E}^\theta, \\
\tilde{B}^i &= \frac{\partial x^i}{\partial R} \tilde{B}^R + \frac{\partial x^i}{\partial \theta} \tilde{B}^\theta,
\end{align*}
\] (D10a, D10b)

where here \(i \in \{x, y, z\}\) and the factor of \(R^2 \sin \theta\) is the determinant of the Jacobian of the transformation and ensures that the resulting fields \(\tilde{E}^i\) and \(\tilde{B}^i\) satisfy the Maxwell constraints

\[
\begin{align*}
\partial_i \tilde{E}^i &= 0, \\
\partial_i \tilde{B}^i &= 0.
\end{align*}
\] (D11a, D11b)

In these coordinates, the fields are singular when \(x, y, z \to 0\). Given this singular behavior, \(V_{EM}\) is expected to be singular as well near the punctures because the source of the momentum constraint \([46]\) diverges with a high power of \(R\). This is precisely what we find when we implement our algorithm with the Kerr-Newman electromagnetic fields. In particular, for a single Kerr-Newman black hole without linear momentum, the singular source terms are \(\bar{S}^x\) and \(\bar{S}^y\), which at leading order for \(x, y, z \to 0\) scale as

\[
\begin{align*}
\bar{S}^x &\sim \frac{aq^2 y}{R_H(x^2 + y^2 + z^2)^{3/2}}, \\
\bar{S}^y &\sim -\frac{aq^2 x}{R_H(x^2 + y^2 + z^2)^{3/2}}.
\end{align*}
\] (D12a, D12b)

A possible approach to deal with the singular source would be to separate the singular part of the solution from the regular one, as is done for the Hamiltonian constraint. However, this approach typically requires a known analytic solution, and this does not seem possible within the approach of conformal flatness, because the Kerr-Newman solution does not admit conformally flat spatial slices. In future work, we will explore potential solutions to these challenges by lifting the conformal flatness approximation.

**Appendix E: Generalized Majumdar-Papapetrou**

Here we show that our formalism recovers spatial slices of a generalized Majumdar-Papapetrou’s solution found by \([85]\) when each black hole is at rest, non-spinning and all black holes have the same charge-to-mass ratio. This happens because under these assumptions the momentum constraint is trivially satisfied, and the Hamiltonian one is solved by \(u = 0\), as we verify in what follows.

Given our definitions of \(\eta\) and \(\varphi\) \([\text{Equations} \ (45)\]\), if the charge-to-mass ratio is fixed to \(\lambda\) for every black hole, then \(\varphi = \lambda \eta\). Moreover, with our choice of Reissner-Nordström fields, there are no magnetic fields, so the electromagnetic energy is \(8\pi \mathcal{E} = 4\partial_a \varphi \partial^a \varphi\), where the factor of 4 arises from the fact that \(\varphi\) is not the electrostatic potential but it is half of it. Plugging the ansatz \(u = 0\) in the Hamiltonian constraint \([\text{Equation} \ (50)\]\) yields

\[
\partial_a \kappa \partial^a \kappa - \partial_a \varphi \partial^a \varphi - \partial_a \psi \partial^a \psi + \psi^{-2} \partial_a \varphi \partial^a \varphi = 0.
\] (E1)

But, \(\psi = \sqrt{k^2 - \varphi^2}\), thus, multiplying the last equation by \(\psi^2\), and expressing the derivatives of \(\psi\) in terms of \(\kappa, \phi\) and their derivatives yields

\[
(1 - \kappa^2)\partial_a \varphi \partial^a \varphi - \varphi^2 \partial_a \kappa \partial^a \kappa + 2\kappa \varphi \partial_a \kappa \partial^a \varphi = 0.
\] (E2)

Plugging \(\kappa = 1 + \eta = 1 + \varphi/\lambda\), and \(\partial_a \kappa = \partial_a \varphi/\lambda\) in this last expression, after some algebra we find that the Hamiltonian constraint is satisfied. If we choose \(\lambda = 1\) we find

\[
\mathcal{E}_{ij} = \left(1 + \sum_{n=1}^{N_e} \frac{M_n}{R_n}\right) \delta_{ij},
\] (E3)

which describes a spatial slice of the Majumdar-Papapetrou spacetime with \(N\) extremal black holes \([54, 55]\).
We conclude that yields the correct electric charge even on non-spherical surfaces. We have checked that our electric field components listed in [89] do not satisfy Maxwell's equations, and, as a result, Gauss's law yields a value for the charge that is correct for spherical surfaces, but the value is different on non-spherical surfaces, e.g. ellipsoidal ones. Of course, Equation (3.5) of [89] by a factor of $E^\theta$ in [89] has a typo.

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Our expression for $E^\theta$ differs from the corresponding one in Equation (3.5) of [89] by a factor of $r/R$. We find that the electric field components listed in [89] do not satisfy Maxwell's equations, and that Gauss's law yields a value for the charge that is correct for spherical surfaces, but the value is different on non-spherical surfaces, e.g. ellipsoidal ones. We have checked that our electric fields satisfy Maxwell's equations, and, as a result, Gauss's law yields the correct electric charge even on non-spherical surfaces. We conclude that $E^\theta$ in [89] has a typo.
