Escape Probabilities for Branching Brownian Motion Among Soft Obstacles

Jean-François Le Gall · Amandine Véber

Received: 18 March 2010 / Revised: 26 October 2010 / Published online: 9 February 2011
© Springer Science+Business Media, LLC 2011

Abstract We derive asymptotics for the quenched probability that a critical branching Brownian motion killed at a small rate $\varepsilon$ in Poissonian obstacles exits from a large domain. Results are formulated in terms of the solution to a semilinear partial differential equation with singular boundary conditions. The proofs depend on a quenched homogenization theorem for branching Brownian motion among soft obstacles.

Keywords Branching Brownian motion · Poissonian obstacles · Super-Brownian motion · Escape probability · Homogenization · Semilinear partial differential equation

Mathematics Subject Classification (2000) Primary 60K37 · 60J80 · Secondary 60J68

1 Introduction

In the present work, we are interested in the long-term behavior of branching Brownian motion killed in Poissonian obstacles. Let us start by describing a simple special case of our results. We consider a critical branching Brownian motion in $\mathbb{R}^d$ ($d \geq 1$), where all initial particles start at the origin. We assume that particles are killed at a (small) rate $\varepsilon > 0$ within random balls of fixed radius, whose centers are distributed according to a homogeneous Poisson point process on $\mathbb{R}^d$. Then, how many initial particles do we need so that, with high probability, one of their descendants reaches...
distance $R$ from the origin? Let $p_\varepsilon(R)$ be the (quenched) probability for our randomly killed branching Brownian motion starting with a single particle at 0 to visit the complement of a large ball of radius $R$ centered at the origin. The preceding question is equivalent to determining the limiting behavior of $p_\varepsilon(R)$ when $\varepsilon$ tends to 0 and simultaneously $R$ tends to infinity.

The answer involves several regimes depending on the respective values of $\varepsilon$ and $R$. If $\varepsilon$ is small in comparison with $1/R^2$, the killing phenomenon does not matter and the result is the same as if there were no killing: $p_\varepsilon(R)$ behaves like a constant times $1/R^2$ (informally, the branching process must survive up to a time of order $R^2$ so that at least one of the particles travels a distance $R$, and well-known estimates for critical branching processes then lead to the correct asymptotics). On the other hand, if $\varepsilon$ is large in comparison with $1/R^2$, then the probability $p_\varepsilon(R)$ decreases exponentially fast as a function of $R\sqrt{\varepsilon}$; see Proposition 1 below.

Our main results focus on the critical regime where $\varepsilon R^2$ converges to a constant $a > 0$. We show that the probability $p_\varepsilon(R)$ behaves like $R^{-2}$, as in the case without killing, but with a multiplicative constant which depends on $a$ and can be identified as the value at the origin of the solution of a semilinear partial differential equation with singular boundary conditions. A key tool to derive these asymptotics is a quenched homogenization theorem which shows that our branching Brownian motions among obstacles, suitably rescaled, are close to super-Brownian motion killed at a certain rate depending on $a$.

Let us formulate our assumptions more precisely in order to state our results. First, let us define the collection of obstacles. We denote the set of all compact subsets of $\mathbb{R}^d$ by $\mathcal{K}$. This set is equipped with the usual Hausdorff metric $d_H$. Recall that $(\mathcal{K}, d_H)$ is a Polish space. For every $r > 0$, $\mathcal{K}_r$ denotes the subset of $\mathcal{K}$ which consists of all compact sets that are contained in the closed ball of radius $r$ centered at the origin. Let $\Theta$ be a finite measure on $\mathcal{K}$, and assume that $\Theta$ is supported on $\mathcal{K}_{r_0}$ for some $r_0 > 0$. Let

$$N = \sum_{i \in I} \delta_{(x_i, K_i)}$$

be a Poisson point measure on $\mathbb{R}^d \times \mathcal{K}$ with intensity $\lambda_d \otimes \Theta$, where $\lambda_d$ stands for Lebesgue measure on $\mathbb{R}^d$. We assume that this point measure is defined on a probability space $(\Omega, \mathcal{F}, P)$ and we denote the generic element of $\Omega$ by $\omega$. Our set of obstacles is then defined by

$$\Gamma_\omega = \bigcup_{i \in I} (x_i + K_i), \quad (1)$$

where obviously $x_i + K_i = \{z = x_i + y : y \in K_i\}$. Note that we use the notation $\Gamma_\omega$ to emphasize that the set of obstacles depends on the variable $\omega$ representing the environment. Let us also define a constant $\kappa$ by

$$\kappa = P(0 \in \Gamma_\omega) = 1 - \exp \left( -\int_{\mathcal{K}} \Theta(dK) \lambda_d(K) \right).$$

To avoid trivial cases, we assume that $\kappa > 0$, or equivalently, $\Theta(\lambda_d(K) > 0) > 0$. By translation invariance, we also have $P(x \in \Gamma_\omega) = \kappa$ for every $x \in \mathbb{R}^d$. 

$\mathbb{C}$ Springer