Fourier’s Quantum Information Processing

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Abstract
We demonstrate that quantum information processing (QIP) completely rests on quantum Fourier transform (QFT), and 190 years after his death, the work of Jean-Baptiste Joseph Fourier is more present than ever in one of the most important pillars of physics: QIP. Specifically, this study shows the impact of QFT on the main tool of quantum communication: entanglement, with a particular emphasis on its spectral nature, whose study is completed when entanglement is used in teleportation, and quantum secret sharing. Finally, a better understanding of the spectral nature of quantum entanglement will lead to better protocols for quantum communications, in general, and quantum cryptography, in particular.

Keywords Entanglement · Quantum communications · Quantum Fourier transform · Quantum information processing · Quantum internet · Teleportation

Introduction
The fact that 190 years after his death, the work of Jean-Baptiste Joseph Fourier (21 March 1768–16 May 1830) continues to be the center of the following scientific disciplines: all heat physics; optics; circuit analysis in electricity; pulse and wideband analysis in electronics; computing; a type of digital transmission and a method of encoding digital data on multiple carrier frequencies called orthogonal frequency-division multiplexing (OFDM) in telecommunications, which has developed into a popular scheme for wideband digital communication used in applications such as digital television and audio broadcasting, digital subscriber line (DSL) internet access, wireless networks, power line networks, and 4G/5G mobile communications; automatic control; sound propagation in solid materials; convolution in synthetic aperture radar via Schönhage–Strassen algorithm; filtering and compression in digital signal, image and video processing; protein docking, Fourier transform ion cyclotron resonance, correlation in genomic, proteomic, and metabolomics in molecular biology, as well as, high speed multiple sequence alignment, and fast Fourier transform-based correlation of deoxyribonucleic acid (DNA) sequences using complex plane encoding in bioinformatics; spot intensities measurement and image generation by Fourier transformation of the intensities in X-ray crystallography; and so on, does not represent anything new to anyone. What is surprising is its timeless projection on the most disruptive of all technologies currently under scrutiny by the scientific community, i.e., quantum information processing (QIP) [1–3]. In fact, the inexcusable dependency that QIP has on quantum Fourier transform (QFT) [1, 4–6] is clearly highlighted in this work. This dependence will be evident both for individual gates of one to four qubits, as well as in those cases where more complex configurations must be implemented, such as the case of the most notable and at the same time strange effect of the QIP, i.e., the entanglement [7–9] between two, three, four or more qubits, as well as the projection of this effect on its most outstanding application, known as quantum teleportation [10–13].

On the other hand, it is well known that QFT is key in the phase detection of innumerable quantum algorithms like that of Shor [14]; however, it is not so well known that QFT is behind the nature and consequence of the flip errors and noises present in every QIP experiment [1]. Quantum gates, particularly those optically implemented [15], inherently generate a series of noises, specifically, three types: bit flip, phase flip (or phase damping), and bit–phase flip, where undoubtedly, the influence of these noises on total process
performance is in direct proportion to the number of gates used by the quantum algorithm on every qubit.

In the same way that the work of Prof. Y.S. Weinstein [5, 6] finished cementing the link between the discrete Fourier transform and the QFT with a clear projection on several of the main quantum algorithms currently in use: Shor’s algorithm, quantum phase estimation, quantum counting; this work establishes a clear relationship between QFT [1, 4–6] and QIP [1–3], in general, and entanglement, in particular, with a notable conceptual impact on quantum computing and communications, respectively.

To summarize, this work allows us to establish that all relative aspects of QIP are covered by QFT, representing its most important tool for its finer analysis and future expansion.

We hope this work is of interest to theorists, experimentalists, and others who might want to study the underlying relationships behind all quantum algorithms, their respective implementation on a physical machine (based on supericonductor technology [16, 17]) or an optical circuit [15], and an in-depth analysis of the respective outcomes. In “Fourier’s Quantum Information Processing”, all equivalences between every quantum gate, from one to four qubits, and their respective representation via QFT are developed.

“Quantum Entanglement, Teleportation, and Secret Sharing” is especially dedicated to entanglement [7–9] for Greenberger–Horne–Zeilinger (GHZ) [1] states of two (GHZ₂), three (GHZ₃), and four (GHZ₄) entangled particles; and quantum teleportation [10–13]. Finally, the last section has our conclusions.

Fourier’s Quantum Information Processing

In this section, we will establish the equivalences between quantum Fourier transform (QFT) and all the gates involved in quantum information processing (QIP). In fact, we will analyze these equivalences from least to greatest, i.e., for gates from one to four qubits, for which, we will start with a brief description of the QFT and its inverse.

The QFT represents an important family of quantum operations over the ring $\mathbb{Z}_2^n$. The n-qubit QFT coherently transforms an input state $|x\rangle$ in the computational basis as follows [18]:

$$|x\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{v=0}^{2^n-1} \omega_{2^n}^{x \cdot u} |y\rangle, \quad u = 0, 1, 2, \ldots, 2^n - 1. \quad (1)$$

where $|x\rangle$ represents a qubit-string $|x_1 \ldots x_n\rangle$, and $\omega_{2^n} = e^{2\pi i / 2^n}$ is the $2^n$ root of unity, such that $\omega_{2^n}^{2^n} = 1$, while the inverse QFT is:

$$|y\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{u=0}^{2^n-1} \omega_{2^n}^{-x \cdot u} |x\rangle, \quad v = 0, 1, 2, \ldots, 2^n - 1. \quad (2)$$

Then, if $F_{2^n}$ represents the matrix of QFT [1] and $F_{2^n}^{-1}$ its reverse, the simple number 1 (unity), i.e., QFT for 0-qubits, $F_{0} = F_{2}^{-1} = 1$, will be present, per se, inside all the gates used in QIP [1].

**Equivalences for 1-Qubit Gates**

It is known from literature [19, 20], that Hadamard matrix $H$ is equivalent to the 2-qubit QFT and its inverse, that is,

$$F_{2} = H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H^{-1} = F_{2}^{-1}. \quad (3)$$

What is not so present in the literature is that both the Pauli’s and phase matrices [1] are derived from the Hadamard matrix $H$ through simple arithmetic and flipping operations. In fact, if we flip over the matrix $H$ with respect to an imaginary horizontal axis that crosses it in the middle, as shown in Fig. 1, we will obtain the splitting operator in its matrix form [21]. Then, the splitting operator together with the recombining operator, which arises from doing the same thing that we have done in Fig. 1 but with respect to an imaginary vertical axis that crosses halfway through the matrix $H$, constitute the pair of operators used in the interference experiments [21].

1. Pauli’s matrices: Next, we will establish the existing relations between the Pauli’s matrices [1] ($I$: identity, $X$: inverter, $Y$, and $Z$) and the Hadamard matrix $H$ from simple arithmetic and flipping operations, thus establishing the underlying relation between the Pauli’s matrices and the QFT.

$$I = HH = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = XX, \quad (4a)$$

$$X = I_{\text{flipped}} = XI = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (4b)$$

Fig. 1 Flipping the Hadamard matrix $H$ with respect to an imaginary horizontal axis that crosses it in the middle, where $H_{\text{flipped}} = XIH$, and $X$ is the inverter Pauli matrix, or flipping matrix.
\[ Z = X(H_{\text{flipped}} H_{\text{flipped}}) = X((XH) (XH)) = H X H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (4c) \]

\[ Y = i Z_{\text{flipped}} = i X Z = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (4d) \]

where \( H_{\text{flipped}} = X H \) is the operation shown in Fig. 1 that flips the matrix \( H \) with respect to an imaginary horizontal axis that crosses it in the middle, \( Z_{\text{flipped}} = X Z \), \( i = \sqrt{-1} \), and \( H = H^{-1} = (X + Z) / \sqrt{2} \).

2. Phase matrices [1]: These matrices are derived from the \( Z \) matrix of Eq. (4c), where

\[ S = \sqrt{Z} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad (5a) \]

\[ S^{-1} = \text{conj}(S) = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}, \quad (5b) \]

\[ T = \sqrt{S} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i \pi} \end{bmatrix}, \quad (5c) \]

\[ T^{-1} = \text{conj}(T) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i \pi} \end{bmatrix}, \quad (5d) \]

\[ U = \sqrt{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i \pi} \end{bmatrix}, \quad (5e) \]

\[ U^{-1} = \text{conj}(U) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i \pi} \end{bmatrix}, \quad (5f) \]

where \( \text{conj}(\cdot) \) means conjugate of “.”.

3. Square-root-of-Not matrices: These matrices are derived from the \( X \) matrix of Eq. (4b), where

\[ V = \sqrt{X} = HTTH = \frac{(1 + i)}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}, \quad (6a) \]

\[ V^{-1} = \text{conj}(V) = HT^{-1}T^{-1}H = \frac{(1 + i)}{2} \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}. \quad (6b) \]

Therefore, we have been able to verify here that all the 1-qubit matrices used in QIP are derived from the Hadamard gate \( H \), or QFT \( F_2 \), and will be used below for gates of more than 1-qubit.

### Equivalences for 2-Qubit Gates

In the same way as in the previous subsection where we established a relationship between the QFT \( F_2 \) and the Hadamard matrix \( H \) for 1-qubit, in this section, we will do the same between QFT of 2-qubits \( F_2 \), and the CNOT gate [1]. Then, being,

\[ F_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -i & 1 \end{bmatrix}, \quad (7a) \]

and

\[ F_2^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -i & 1 \\ 1 & -i & 1 \end{bmatrix}, \quad (7b) \]

it is evident that QFT and its inverse do not coincide for the 2-qubit case \( F_2 \neq F_2^{-1} \) as in the previous cases. Therefore, we will proceed to represent both of them in Fig. 2 by means of \( H \), Controlled-\( S \), and SWAP gates for the case of QFT, and \( H \), Controlled-\( S^{-1} \), and SWAP gates for the case of QFT\(^{-1} \). The difference is that Controlled-\( S \) is used in QFT, while Controlled-\( S^{-1} \) is used in QFT\(^{-1} \).

Regardless of what was said above, it is necessary to establish with which of the two criteria present in Fig. 3 we will work from here on, mainly in relation to the Feynman’s gate [1–3], also known as Controlled-NOT, CNOT, Controlled-X, or simply CX. Figure 3 shows two associated matrices with CNOT gate, depending on which qubit the control is, as well as, which qubit the target is [22]. Different books, papers, and quantum platforms [16, 17] order
their qubits differently. In this paper, the option of Fig. 3b is chosen, precisely, because it is closer to that used in the main quantum platforms [16, 17].

Incorporating the option selected above, if we multiply $F_{22}$ by itself, and we do the same with $F_{-1}^{-1}22$, then both multiplications result identical and also equal to the CNOT gate of Fig. 3b [1], given that, multiplying $F_{22} \times F_{22} = F_{-1}^{-1}22 \times F_{-1}^{-1}22 = CNOT$ [1],

$$F_{22} \times F_{22} = F_{-1}^{-1}22 \times F_{-1}^{-1}22 = CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (8)$$

given that, multiplying $F_{22} \times F_{22}$ by $F_{-1}^{-1}22 \times F_{-1}^{-1}22$ and regrouping, yields,

$$(F_{22} \times F_{22}) \times (F_{-1}^{-1}22 \times F_{-1}^{-1}22) = F_{22} \times (F_{22} \times F_{-1}^{-1}22) \times F_{-1}^{-1}22 = F_{22} \times I \times F_{-1}^{-1}22 = F_{22} \times F_{-1}^{-1}22 = I. \quad (9)$$

However,

$$\sqrt{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (1+i)/2 & 0 & (1-i)/2 \\ 0 & 0 & 1 & 0 \\ 0 & (1-i)/2 & 0 & (1+i)/2 \end{bmatrix}, \quad (10)$$

therefore, considering Eqs. (7a, 7b), it is evident that $\sqrt{CNOT} \neq F_{22}$ and $\sqrt{CNOT} \neq F_{-1}^{-1}22$.
the case of the Controlled-$S$ and Controlled-$S^{-1}$ gates, where Fig. 7 shows their equivalences as a function of QFT$^{-1}$.

In Fig. 8, we implement the anti-control (or zero control) condition on a qubit being OFF for the CNOT gate [1–3], which constitutes a complement of the control condition on a qubit being ON that represents the original CNOT gate of Fig. 3.

Considering Fig. 4 and Eq. (8), we deduce that the CNOT gate has the equivalences present in Fig. 9.

From here on, and since each implementation that involves CNOT can be carried out both through QFT and its inverse, with the same criteria regarding the phase matrices \{S, T, and so on\}, then we will make the implementation of each case, exclusively based on QFT$^{-1}$. Then, we will implement the Controlled-Z gate, or $CZ$, which can be observed in Fig. 10 which its equivalences are in terms of \{H, CNOT\}, \{QFT, QFT$^{-1}$\}, \{H, Controlled-$S^{-1}$\}, and simply Controlled-$S^{-1}$. All these options will have an important role in the next section.

Figure 11 shows the Controlled-$Y$ gate with its equivalences. The first one in terms of \{S$^{-1}$, CNOT, S\}, while the second one is obtained replacing CNOT with QFT$^{-1} \times$ QFT$^{-1}$.
The Controlled-T gate is implemented in Fig. 12 based on its equivalences and depends on Eq. (3). It could also be called Controlled-$F_2^1$, and can be represented via \{S, H, T, CNOT, $T^{-1}$, $S^{-1}$\} and \{S, H, T, $F_2^{-1} \times F_2^{-1}$, $T^{-1}$, $S^{-1}$\}, as we can see in Fig. 12.

The Controlled-H gate is implemented in Fig. 13 based on its equivalences and depends on Eq. (3). It could also be called Controlled-$F_2^1$, and can be represented via \{S, H, T, CNOT, $T^{-1}$, $S^{-1}$\} and \{S, H, T, $F_2^{-1} \times F_2^{-1}$, $T^{-1}$, $S^{-1}$\}, as we can see in Fig. 13.

Finally, Fig. 14 shows the implementation of the Controlled-V and Controlled-V$^{-1}$ gates. a by \{H, Controlled-$T$\}, or \{H, $T$, $T^{-1}$, flipped CNOT\}. b by \{H, Controlled-$T^{-1}$\}, or \{H, $T$, $T^{-1}$, flipped CNOT\}.

**Equivalences for 3-Qubit Gates**

With the same criteria used in Fig. 2 for the representation of QFT and QFT$^{-1}$ in the case of 2-qubits, Fig. 15 shows the equivalence between the QFT of 3-qubits and the gates \{H, Controlled-S, Controlled-T, SWAP\}, and between QFT$^{-1}$ and the gates \{H, Controlled-$S^{-1}$, Controlled-$T^{-1}$, SWAP\}.

The implementation of $F_2^{-1} \times F_2^{-1}$, i.e., the 3-qubits QFT$^{-1} \times$ QFT$^{-1}$ versions are represented in Fig. 16, where the first and second equivalences use \{H, Controlled-$S^{-1}$, Controlled-$T^{-1}$, SWAP\}, while the last one does not need the SWAP gate.

Figure 17 shows the techniques used to step over an intermediate qubit in a quantum circuit using the CNOT gate. All versions of Fig. 17 are equivalent, with more or less intervention of QFT$^{-1}$. In practice, the most recommendable
of all the equivalences is undoubtedly the first one; however, all the versions together allow us to appreciate in detail the intervention of $\text{QFT}^{-1}$ behind a simple step over on a specific qubit. We must bear in mind, in this case, as in all the preceding ones, and in all those that follow, that the idea of presenting such an equivalence number does not have to do with a desperate search for alternatives, which is absurd since when we observe Fig. 17 in detail, it is evident that if we want to step over using $\text{QFT}^{-1}\times\text{QFT}^{-1}$, with the first option is more than enough. The idea behind the presentation of all the alternatives has to do with the fact that whatever option is chosen, behind it will inevitably be Fourier. Moreover, as we have seen in “Equivalences for 1-qubit gates”, all the phase matrices with which both $\text{QFT}^{-1}$ (Fig. 15) and $\text{QFT}^{-1}\times\text{QFT}^{-1}$ (Fig. 16) are implemented are derived from Hadamard matrix $H$, which thanks to Eq. (3), is $\text{QFT} = \text{QFT}^{-1}$ for 1-qubit case.

Figure 18 represents the double Feynman gate with its equivalences. This gate is fundamental for the generation of the Greenberger–Horne–Zeilinger state (GHZ state) [1–3], in this particular case, GHZ$_3$, which will be used to the maximum in the next section, where we will implement different configurations about entanglement [7–9] and teleportation [10–13]. The second line of Fig. 18 shows an implementation of the double Feynman gate in terms of one CNOT gate and one step over CNOT gate. Finally, in the same line, we have another implementation of it based on the equivalence of Fig. 9b.

Figure 19 presents the first set of implementations of the 3-qubits gates: Toffoli, Fredkin, Peres, and Miller [23]. Fredkin, Peres, and Miller gates [23] are implemented based on the Toffoli, the step over CNOT (Fig. 17), and the flipped CNOT (Fig. 5) gates.

In Fig. 20, we have the second set of implementations of the 3-qubits gates: Toffoli, Fredkin, Peres, and Miller [23]. In this case, we replace the two known configurations of the Toffoli gate with the intervention of $\text{QFT}^{-1}\times\text{QFT}^{-1}$.

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**Fig. 15** a QFT, $F_{2^3} \in \mathbb{C}^{2^3 \times 2^3}$ in terms of the \{H, Controlled-S, Controlled-T, SWAP\} gates. b Inverse of QFT, $F_{2^3}^{-1} \in \mathbb{C}^{2^3 \times 2^3}$ in terms of the \{H, Controlled-S$^{-1}$, Controlled-T$^{-1}$, SWAP\} gates

**Fig. 16** Implementation of $F_{2^3}^{-1}\times F_{2^3}^{-1}$. The first and second equivalences use \{H, Controlled-S$^{-1}$, Controlled-T$^{-1}$, SWAP\}, while the last one only uses \{H, Controlled-S$^{-1}$, Controlled-T$^{-1}$\}
Instead, in the case of the Fredkin, Peres, and Miller gates, we incorporate.

Toffoli gate in their interior as a known module. This greatly simplifies the implementation of these gates. Finally, all gates are fully expressed by QFT$^{-1} \times$ QFT$^{-1}$, i.e., Fourier.

Figure 21 shows the implementation of the Toffoli gate incorporating the version of QFT$^{-1} \times$ QFT$^{-1}$ in terms of the equivalence seen so far, i.e., based on \{H, Controlled-S$^{-1}$\} gates, while Fredkin, Peres, and Miller in terms of \{H, Controlled-S$^{-1}$, Toffoli as a module\} gates.

Figure 22 shows the Toffoli gate in terms of the gates $V = \sqrt{X}$ and $V^{-1}$ [1–3], their replacements according to Eqs. (6a, 6b), their simplifications, and, the replacement of the CNOT gate based on Fig. 9b.

Figure 23 represents another implementation of the Toffoli gate in terms of the \{H, T, T$^{-1}$, and CNOT\} gates, and \{QFT$_{2 \times 2}$, T, T$^{-1}$, and QFT$_{4 \times 4}^{-1} \times$ QFT$_{4 \times 4}^{-1}$\} gates. In fact, it is the simplest of all. Moreover, we can resort to the existing equivalences between Toffoli and the Fredkin, Peres, and Miller gates of Fig. 20 to obtain these three last gates thanks to this version of the Toffoli gate.

Finally, Fig. 24 represents the implementation of the Toffoli, Fredkin, Peres, and Miller gates thanks to a double Controlled-QFT$^{-1}$ gate, which notably simplifies the implementation of all the gates, since, here too the last three gates use the Toffoli gate like a module. We must take into account that each QFT$^{-1}$ matrix used in every Controlled-QFT$^{-1}$ gate is of the type $F_{2 \times 2}^{-1}$, and no $F_{2 \times 2}^{-1}$, then if $I_{2 \times 2}$ is a $2 \times 2$ identity matrix, and $0_{2 \times 2}$ is a $2 \times 2$ zero matrix, the Controlled-QFT$^{-1}$ gate will be,
### Equivalences for 4-Qubit Gates

As we have done in the previous cases, we present QFT and QFT\(^{-1}\) for 4-qubits in Fig. 25, where the equivalences for QFT are based on the \(\{H, \text{Controlled-}S, \text{Controlled-}T, \text{Controlled-}U, \text{SWAP}\}\) gates, while, the equivalences for QFT\(^{-1}\) depends on the \(\{H, \text{Controlled-}S^{-1}, \text{Controlled-}T^{-1}, \text{Controlled-}U^{-1}, \text{SWAP}\}\) gates.

In Fig. 26, we use the configuration of Fig. 25b to build QFT\(^{-1}\)×QFT\(^{-1}\) for the 4-qubits case. These types of equivalences are very useful when we have to implement QFT, QFT\(^{-1}\), and QFT\(^{-1}\)×QFT\(^{-1}\) on platforms such as IBM Q Experience [16] or Rigetti [17] that do not have built-in versions of these tools in their respective toolboxes.

The triple Feynman gate is presented in Fig. 27. The first equivalence is based on one CNOT gate, and two step over CNOT gates, while the second one is obtained using the equivalence of Fig. 9b.

#### Quantum Entanglement, Teleportation, and Secret Sharing

We will focus our analysis on the intervention of Fourier in quantum communications [24–26], by demonstrating its presence in quantum entanglement [7–9], quantum teleportation [10–13], and quantum secret sharing [27]. Therefore, this section will make it clear that the two pillars of quantum information processing (QIP): superposition and entanglement, are two attributes of unequivocally spectral characteristics.

### Quantum Entanglement

In this section, we will start with the so-called Bell states [1–3], which result from the combination of the Hadamard \((H)\), and CNOT (Eq. 8) matrices, where,

\[
\text{Controlled-QFT}^{-1} = \begin{bmatrix}
I_{2 \times 2} & 0_{2 \times 2} \\
0_{2 \times 2} & F_{2}^{-1}
\end{bmatrix},
\]  

\((11)\)
Fig. 20 Implementation of the Toffoli, Fredkin, Peres, and Miller gates thanks to $QFT^{-1} \times QFT^{-1}$. Additionally, the Fredkin, Peres, and Miller gates are implemented thanks to the Toffoli gate as a module.

Fig. 21 Implementation of the Toffoli gate based on the \{H, Controlled-T, Controlled-S^{-1}, Controlled-T^{-1}\} gates, while the rest in terms of \{H, Controlled-S^{-1}, Toffoli as a module\} gates.
Fig. 22 Toffoli gate, where the first equivalence is implemented thanks to the $V$ and $V^{-1}$ gates. The second one in terms of the replacement of both gates according to Eqs. (6a, 6b). The second one corresponds to simplifications, while the last one has to do with the replacement of CNOT based on Fig. 9b.

Fig. 23 Implementation of the Toffoli gate based on the \{$H, T, T^{-1}, \text{and CNOT}$\} gates, and \{$QFT_{2\times2}, T, T^{-1}, \text{and QFT}^{-1}_{4\times4} \times \text{QFT}^{-1}_{4\times4}$\} gates.

Fig. 24 The Toffoli, Fredkin, Peres, and Miller gates thanks to a simplification of the Toffoli gate based on a double Controlled-QFT$^{-1}$ gate.

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = CX((I \otimes H)|00\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \quad (12)$$

The configuration of matrices of Eq. (12), which in optical
circuits [15] is known as beamsplitter, are represented in Fig. 28 thanks to \{H, \text{CNOT}\}, \{\text{QFT},\ \text{QFT}^{-1} \times \text{QFT}^{-1}\}, and \{H, \text{Controlled-S}^{-1}\}. The latter allows us to obtain Bell states in terms of Fourier on platforms like IBM Q Experience [16], and Rigetti [17] which do not have QFT or QFT$^{-1}$ as built-in structures.
Figure 28 represents the first link between an Einstein–Podolsky–Rosen (EPR) state and QFT. However, a strict way of discovering this link consists of passing through all the intermediate instances, where said route will begin with Fig. 28a. From Fig. 14, we know that the Controlled-V gate is equivalent to a square root of CNOT gate, i.e., Fig. 14 establishes the link between Fig. 29a, b, and c. Thanks to Eq. (5c), Fig. 29 experiments an evolution from (c) to (d). In

![Fig. 29](image)

**Fig. 29** Complete transition between CNOT gate and QFT × QFT

![Fig. 30](image)

**Fig. 30** GHZ$_3$ in terms of: a \{H, CNOT, step over CNOT\}, b \{QFT, QFT$^{-1}$×QFT$^{-1}$\}, and c \{H, Controlled-S$^{-1}$\}

![Fig. 31](image)

**Fig. 31** GHZ$_4$ in terms of: a \{H, CNOT, step over CNOT\}, b \{QFT, QFT$^{-1}$×QFT$^{-1}$\}, and c \{H, Controlled-S$^{-1}$\}
(e), we apply the left side of Eq. (4a), and we apply it again in the upper qubit in (f). In (g), we apply two SWAP gates, which cancel each other without altering the settings. From Fig. 4a, we know that both sides of (g) represent a QFT, in fact, flipping the second one, we again have two QFTs in (h), which shows the total equivalence between Fig. 28a and b.

Finally, Fig. 30 shows GHZ₃ in terms of \{H, CNOT, step over CNOT\}, \{QFT, QFT⁻¹×QFT⁻¹\}, and \{H, Controlled-S⁻¹\}, while Fig. 31 does the same for GHZ₄. Both configurations (Figs. 30, 31) will be of vital importance in the implementation of the quantum secret sharing [27] protocol.

Quantum Teleportation

Using the equivalences of the immediately previous figures, we can implement the famous quantum teleportation protocol [10–13] in terms of Fourier. Figure 32 presents four options to carry out this task. The first is the one that we can find in all the literature that deals with quantum teleportation [10–13], which is based on \{H, CNOT, quantum measurement (QuMe), CZ\}. In the second one, we replace CZ by \{H, CNOT\}. In the third one, we replace all the gates of the latter by their representation in terms of QFT, and QFT⁻¹, while in the last one, we implement QFT and QFT⁻¹×QFT⁻¹ via the discrete gates that we can find in any commercial platform like IBM Q Experience [16], and Rigetti [17]. Finally, the pair \{H, CNOT\} is known as beamsplitter, and the group \{H, CNOT, quantum measurement (QuMe)\} is called bell state measurement (BSM), where the quantum measurement is carried out by single-photon detectors (SPD) [11–13].

Quantum Secret Sharing

Among other things, the quantum secret sharing (QSS) protocol [27] is a generalization of the quantum teleportation protocol previously seen for the case where we worked with GHZ [1–3] instead of Bell states. Figure 33 shows five different implementations of QSS for GHZ₃ thanks to \{H, CNOT, step over CNOT, QuMe, CZ\}, \{H, CNOT, step over CNOT, QuMe\}, \{QFT, QuMe, QFT⁻¹×QFT⁻¹\}, and \{H, Controlled-S⁻¹, QuMe\}.

This protocol, as well as many others like those from quantum key distribution (QKD), with entangled [28, 29] or polarized photons [30, 31], and a relatively new family of protocols for quantum data security called quantum secure direct communication (QSDC) [32–42] are of main importance in a critical branch of quantum communications [24–26] called quantum cryptography [43, 44]. Success of the latter has an obvious impact on a new version of the internet, named quantum internet [45–58]. In fact, a spectral

![Fig. 32](image-url) Quantum teleportation protocol: a the traditional one, b replacing CZ by its equivalence based on \{H, CNOT\}, c in terms of \{QFT, QFT⁻¹×QFT⁻¹, QuMe\}, and d via \{H, Controlled-S⁻¹, QuMe\)
Fig. 33 Quantum secret sharing protocol for GHZ. \(\textbf{a}\) The traditional one. \(\textbf{b}\) \(\text{CZ}\) is replaced by \{\(H, \text{CNOT}\)\}. \(\textbf{c}\) Simplification of the last case. \(\textbf{d}\) Replacement of each gate of the previous case with \{\(\text{QFT, QFT}^{-1}\times \text{QFT}^{-1}\)\}. \(\textbf{e}\) Implementation of \{\(\text{QFT, QFT}^{-1}\times \text{QFT}^{-1}\)\} thanks to \{\(H, \text{Controlled-}S^{-1}\)\}. 
Fig. 34  Quantum secret sharing protocol for $GHZ_4$. 

a The traditional one. 

b $CZ$ is replaced by $\{H, CNOT\}$. 

c Simplification of the last case. 

d Replacement of each gate of the previous case with $\{QFT, QFT^{-1} \times QFT^{-1}\}$. 

e Implementation of $\{QFT, QFT^{-1} \times QFT^{-1}\}$ thanks to $\{H, \text{Controlled-}S^{-1}\}$.
analysis of this new network is of fundamental importance to increase its performance, given that the information exchange that will take place in quantum internet is based on signal traffic and processing, as well as on the use of entangled pairs in the transmitter, receiver, terrestrial quantum repeaters [59–63], and satellite quantum repeaters [64, 65].

With the same criteria established for GHZ$_2$, Fig. 34 shows the representation of GHZ$_4$ in terms of QFT. Note that when we replace Controlled-$Z$ in Fig. 34 by its alternative version of Fig. 10, we go from (a) to (b). Furthermore, the Hadamard matrices ($H$) cancel each other out by the application of Eq. (4a), so we go from (b) to (c) simplifying the circuit even more. Then, we replace each $H$ and CNOT gates in (d) with its equivalent implementation using QFT. Finally, we replace the two-qubit QFTs by their implementation through Controlled-$S$ type gates. All these alternatives represent a complete arsenal of options for the implementation of these tools on any platform [16, 17, 66–68].

Projection on Quantum Internet

Based on the equivalences established in “Fourier’s Quantum Information Processing” and “Quantum Entanglement, Teleportation, and Secret Sharing” of this work between QFT and quantum gates and circuits, as well as, QFT and quantum entanglement, respectively, it is possible to evaluate the evident projection of this study on the future quantum internet [45–58]. Specifically, and in relation to the leading literature on quantum internet [45–53], we can say that the equivalences presented in this work for gates of 1, 2 and more qubits are fundamental to improve the performance of the protocols used for entanglement swapping and quantum repeaters [45, 47], as well as the entanglement analysis of qubits over longer links [45], quantum entanglement distillation and quantum teleportation [46], high and low-level quantum teleportation schemes [47], noisy and decoherence analysis and single-qubit error-detection using a single noisy Einstein–Podolsky–Rosen (EPR) pair [46, 48], entanglement distribution via quantum switch [49], including analysis of quantum teleportation process in terms of density matrices [49]. Moreover, the understanding obtained from this work about the existing bridge between quantum Fourier transform (QFT) and the gates used in quantum information processing (QIP) will allow a better approach to quantum internet from the point of view of distributed computing [50, 51]. Finally, a better understanding of the spectral aspect behind quantum internet will lead to better distributed routing algorithms [52], and at the same time, it will result in the development of better cryptographic protocols, which will make it possible to mark the difference between the current internet and the future network of networks [53] (Fig. 35).

On the other hand, Fig. 36 represents a sketching of such projection, where the equivalences demonstrated in the previous sections show a capillarity in all its branches which inexorably leads to the new version of the network of networks.

While QFT intervenes directly or explicitly in quantum algorithms such as those of Shor, quantum phase detection, and quantum counting, it also intervenes indirectly or implicitly in all existing quantum algorithms based on the equivalences established in this work. In other words, QFT is the common building block on which the entire quantum information processing building rests, including of course quantum computing and quantum communications, which

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Fig. 35 Sketch about the projection of quantum Fourier transform on quantum internet through this study, where: QKD means quantum key distribution [28–31], QSS is quantum secret sharing [27], and QSDC is quantum secure direct communications [32–42]
inevitably projects onto quantum internet. A clear example of this is shown in Fig. 36, where an essential tool of quantum internet called quantum repeater is represented [59–63], which must be used every 50 km on a terrestrial laid of optical fiber due to a propagation speed \( v \) equal to 2/3 of the speed of light \( c \) in vacuum, losses and an attenuation in the material.

Specifically, Fig. 36a shows a typical quantum repeater [59–63] based on the gates known in quantum information processing: Hadamard \( (H) \), Controlled-X or CNOT, Controlled-Z, and quantum measurement. Using the equivalences developed in Fig. 10, Fig. 36b shows the same quantum repeater but in terms of QFT modules of 2 \( \times \) 2 and 4 \( \times \) 4. Finally, Fig. 36c represents the same quantum repeaters but replacing each QFT modules for its respective representation in terms of gate \( H \) for the QFT \( 2 \times 2 \) case, and Controlled-\( S^{-1} \) and \( H \) for the QFT \( 4 \times 4 \) case.

On the other hand, at the input and output of each quantum repeater, we will need quantum memories [69, 70] since quantum repeaters require classical communication between the points of the connecting channel, i.e., the receiving part should work as a quantum memory to keep the quantum state until at least the end of the classical communication (double line in Fig. 36). This is important as it inherently implies that quantum repeaters require the use of quantum memories [69, 70]. A significant problem of the Bell states generated from an entanglement distribution scheme between the two remote nodes is that they are not perfect [69]. While losses can be mitigated by repeating the scheme many times, other errors will occur in such systems [69]. If the entangled states are stored in matter qubits of a quantum memory, they become highly prone to dephasing [69].

**Conclusions**

We have demonstrated that quantum information processing (QIP) completely rests on quantum Fourier transform (QFT). In fact, every gate, circuit, algorithm, procedure, and QIP protocol can be expressed thanks to QFT and its inverse. On the other hand, we know that the Pauli matrices are the fundamental building blocks of QIP [1–3], given that they form a complete basis for any Hermitian operator. In consequence, any observable can be expanded into strings of Pauli operators, the expectation values of which we can measure efficiently with a quantum computer. “Equivalences for
1-qubit gates” highlighted the close link between the Pauli and Hadamard matrices. Therefore, this demonstrates the unequivocal presence of Fourier, 190 years after his death, as a substrate for this novel discipline as well as all its current and future technological derivations.

Finally, this study makes clear the spectral nature of quantum entanglement [7–9], quantum teleportation [10–13], and therefore quantum secret sharing [27], with an obvious projection on quantum communications [24–26], in general, and quantum cryptography [43, 44] as well as quantum internet [45–58], in particular.

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