A SHARP LOWER BOUND ON THE POLYGONAL ISOPERIMETRIC DEFICIT

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ABSTRACT. A sharp quantitative polygonal isoperimetric inequality is obtained.

1. INTRODUCTION

The polygonal isoperimetric inequality states that if \( n \geq 3 \) and \( P \) is an \( n \)-gon with area \( |P| \) and perimeter \( L(P) \), then the deficit is nonnegative,

\[
\delta(P) := L^2(P) - 4n \tan \frac{\pi}{n} |P| \geq 0,
\]

and uniquely minimized when \( P \) is convex and regular. A full stability result for this classical inequality has recently been obtained in [IN14] via a novel approach involving a functional minimization problem on a compact manifold and the spectral theory for circulant matrices. The heart of the matter is a quantitative polygonal isoperimetric inequality for convex polygons which states that

\[
\sigma_s^2(P) + \sigma_r^2(P) \lesssim \delta(P),
\]

where \( \sigma_s^2(P) \) is the variance of the side lengths of \( P \) and \( \sigma_r^2(P) \) is the variance of its radii (i.e. the distances between the vertices and their barycenter).

The starting point of the proof is the following inequality [FRS85, pg. 35] which holds for any \( n \)-gon:

\[
8n^2 \sin^2 \frac{\pi}{n} \sigma_r^2(P) \leq nS(P) - 4n \tan \frac{\pi}{n} |P|,
\]

where \( S(P) \) is the sum of the squares of the side lengths of \( P \). Since \( n^2 \sigma_s^2(P) = nS(P) - L^2(P) \), it follows that (1.2) is equivalent to

\[
8n^2 \sin^2 \frac{\pi}{n} \sigma_r^2(P) \leq \delta(P) + n^2 \sigma_s^2(P).
\]

In order to establish (1.1), it is shown in [IN14] that

\[
\sigma_s^2(P) \lesssim \delta(P)
\]

whenever \( P \) is a convex \( n \)-gon; thereafter, a general stability result is deduced via a version of the Erdős-Nagy theorem which states that a polygon may be convexified in a finite number of “flips” while keeping the perimeter invariant. The method of

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proof of \((1.2)\) given in \[FRS85\] is based on a polygonal Fourier decomposition, whereas
the technique in \[IN14\] is based on a third order Taylor expansion of the deficit (in
a suitable sense) and as mentioned above involves circulant matrix theory and an
optimization problem on a compact manifold. It is natural to wonder whether one can
directly deduce \((1.1)\) via the method in \[IN14\] without relying on \[FRS85\]. A positive
answer is given in this paper. In fact, a new inequality is established which combined
with \((1.4)\) improves \((1.1)\).

Let \(\sigma^2_a(P)\) denote the variance of the central angles of \(P\) (i.e. the angles generated
by the vertices and barycenter of the vertices of \(P\), see \[2\]). Then the following is true.

**Theorem 1.1.** Let \(n \geq 3\) and \(P\) be a convex \(n\)-gon. There exists \(c_n > 0\) such that
\[
c_n \delta(P) \geq \sigma^2_a(P) + |P|\sigma^2_a(P),
\]
and the exponent on the deficit is sharp.

This result directly combines with \((1.4)\) and yields:

**Corollary 1.2.** Let \(n \geq 3\) and \(P\) be a convex \(n\)-gon. There exists \(c_n > 0\) such that
\[
c_n \delta(P) \geq \sigma^2_a(P) + \sigma^2_r(P) + |P|\sigma^2_a(P).
\]

**Remark 1.3.** The theorem holds for a more general class of polygons. The only re-
quirement in the proof is that the central angles of \(P\) sum to \(2\pi\).

**Remark 1.4.** An inequality of the form
\[\sigma^2_a(P) \leq c_n \delta(P)\]
cannot hold in general. One can see this by a simple scaling consideration: let \(P\) be a
convex polygon and \(P_\alpha\) be the convex polygon obtained by dilating the radii of \(P\) by
\(\alpha > 0\). Then \(\delta(P_\alpha) = \alpha^2 \delta(P)\), but \(\sigma^2_a(P_\alpha) = \sigma^2_a(P)\).

Quantitative polygonal isoperimetric inequalities turn out to be useful tools in geo-
metric problems. For instance \((1.1)\) was recently utilized in \[CM14\] to improve a result
of Hales which showed up in his proof of the honeycomb conjecture \[Hal01\]. More-
over, \[IN14\] has also been employed in \[CN14\] to prove a quantitative version of a
Faber-Krahn inequality for the Cheeger constant of \(n\)-gons obtained in \[BFar\]. Related
stability results for the isotropic, anisotropic, and relative isoperimetric inequalities
have been obtained in \[FMP08, FMP10, FI13\], respectively.

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2. Preliminaries

Let \( n \geq 3 \) and \( P \subset \mathbb{R}^2 \) be an \( n \)-gon with vertices \( \{A_1, A_2, \ldots, A_n\} \subset \mathbb{R}^2 \) and center of mass \( O \) which is taken to be the origin. For \( i \in \{1, 2, \ldots, n\} \), the \( i \)-th side length of \( P \) is \( l_i := A_iA_{i+1} \), where \( A_i = A_j \) if and only if \( i = j \) (mod \( n \)); \( \{r_i := OA_i\}_{i=1}^n \) is the set of radii. Furthermore, \( x_i \) denotes the angle between \( \overrightarrow{OA_i} \) and \( \overrightarrow{OA_{i+1}} \).

The circulant matrix method introduced in [IN14] is based on the idea that a large class of polygons can be viewed as points in \( \mathbb{R}^{2n} \) satisfying some constraints. More precisely, consider

\[
M := \left\{ (x; r) \in \mathbb{R}^{2n} : x_i, r_i \geq 0, \ (2.1), \ (2.2), \ (2.3) \ right\} ,
\]

where

\[
(2.1) \quad \sum_{i=1}^{n} x_i = 2\pi,
\]

\[
(2.2) \quad \sum_{i=1}^{n} r_i = n.
\]

\[
(2.3) \quad \begin{cases}
\sum_{i=1}^{n} r_i \cos \left( \sum_{k=1}^{i-1} x_k \right) = 0, \\
\sum_{i=1}^{n} r_i \sin \left( \sum_{k=1}^{i-1} x_k \right) = 0.
\end{cases}
\]

Note that \( M \) is a compact \( 2n - 4 \) dimensional manifold and each point \( (x; r) \in \mathcal{M} \) represents a polygon centered at the origin with central angles \( x \) and radii \( r \); therefore, it is appropriate to name such objects \textit{polygonal manifolds}. Indeed, a point \( O \) is the barycenter if and only if

\[
\sum_{i=1}^{n} \overrightarrow{OA_i} = 0,
\]

which is equivalent to saying that the projections of \( \sum_{i=1}^{n} \overrightarrow{OA_i} \) onto \( \overrightarrow{OA_1} \) and \( \overrightarrow{OA_1} \perp \) vanish; in other words, \( (x; r) \) satisfies \( (2.3) \). Furthermore, \( (2.1) \) is satisfied by all convex polygons (also many nonconvex ones) and \( (2.2) \) is a convenient technical assumption which derives from scaling considerations. Note that the convex regular \( n \)-gon corresponds to the point \( (x_\ast; r_\ast) = \left( \frac{2\pi}{n}, \ldots, \frac{2\pi}{n}; 1, \ldots, 1 \right) \). With this in mind, the variance of the interior angles and radii of \( P \) are represented, respectively, by the quantities

\[
\sigma_a^2(P) = \sigma_a^2(x; r) := \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \frac{1}{n^2} \left( \sum_{i=1}^{n} x_i \right)^2 ,
\]
\[ \sigma^2(P) = \sigma^2_P(x; r) := \frac{1}{n} \sum_{i=1}^{n} r_i^2 - \frac{1}{n^2} \left( \sum_{i=1}^{n} r_i \right)^2. \]

Moreover, in \((x; r)\) coordinates, the deficit is given by the formula

\[ \delta(P) = \delta(x; r) := \left( \sum_{i=1}^{n} \left( r_{i+1}^2 + r_i^2 - 2r_{i+1}r_i \cos x_i \right)^{1/2} \right)^2 - 2n \tan \frac{\pi}{n} \sum_{i=1}^{n} r_i r_{i+1} \sin x_i. \]

### 3. Proof of Theorem 1.1

By a simple reduction argument, it suffices to prove the inequality on \(M\): let \(P\) be a convex \(n\)-gon and note that it is represented by \((x; r) \in \mathbb{R}^{2n}\), where \(x \in \mathbb{R}^n\) denotes its interior angles and \(r \in \mathbb{R}^n\) its radii. Convexity implies (2.1), and (2.3) follows from the definition of barycenter. If \(\sum_{i=1}^{n} r_i = s \neq n\), consider (by a slight abuse of notation) the polygon \(P_s = (x; s r)\) obtained by scaling the radii of \(P\). Evidently \(\sigma^2(P_s) = \sigma^2_s(P), |P_s| = (n/s)^2 |P|, \sigma^2_P(P) = (n/s)^2 \sigma^2_s(P), \delta(P_s) = (n/s)^2 \delta(P)\). Hence if the inequality stated in the theorem holds for \(P_s \in M\), then it also holds for \(P\). Now let

\[ \phi(x; r) := n^2(|P| \sigma^2_a + \sigma^2_r) \]

\[ = \frac{1}{2} \left( \sum_{i=1}^{n} r_i r_{i+1} \sin x_i \right) \left( n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2 \right) + n \sum_{i=1}^{n} r_i^2 - \left( \sum_{i=1}^{n} r_i \right)^2, \]

and note that it suffices to show

(3.1) \[ \phi(x; r) \leq c \delta(x; r) \]

for all \((x; r) \in M\). The polygonal isoperimetric inequality implies \(\delta(x; r) \geq 0\) for every \((x; r) \in M\) with \(\delta(x; r) = 0\) if and only if \((x; r) = z_* := (x_*; r_*)\). Since \(M\) is compact and \(\delta\) is continuous it follows that

\[ \inf_{M \setminus B_\delta(z_*)} \delta > 0, \]

and so (3.1) follows easily on \(M \setminus B_\delta(z_*)\). Thus it suffices to prove (3.1) for some neighborhood \(B_\delta\) of the point \(z_*\). Direct calculations imply (recall that the notation is periodic mod \(n\))

(3.2) \[ D\phi(z_*) := (D_x \phi(z_*), D_r \phi(z_*)) = 0, \]

\[ D_{x_k} \phi(z_*) = \begin{cases} n(n - 1) \sin \frac{2\pi}{n}, & k = l, \\ -n \sin \frac{2\pi}{n}, & k \neq l, \end{cases} \]
\[ D_{r_k r_l} \phi(z_*) = \begin{cases} 2(n-1), & k = l, \\ -2, & k \neq l, \end{cases} \]

and \( D_{r_k x_l} \phi(z_*) = 0 \). Thus by letting \( \Phi := D^2 \phi(z_*) \) it follows that

\[ \Phi = \left( \begin{array}{cc} n \sin \frac{2\pi}{n} C & 0_{n \times n} \\ 0_{n \times n} & 2C \end{array} \right), \]

where \( 0_{n \times n} \) is the \( n \times n \) zero matrix and

\[ C = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots \\ \vdots & -1 & n-1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}_{n \times n}. \]

Moreover, \( D\delta(z_*) \) is given by

\[
\begin{aligned}
D_{x_k} \delta(z_*) &= 2n \tan \frac{\pi}{n}, \\
D_{r_k} \delta(z_*) &= 0;
\end{aligned}
\]

hence, \((2.1)\) implies

\[
\left\langle D\delta(z_*), (x-x_*; r-r_*) \right\rangle = \left\langle D_x \delta(z_*), x-x_* \right\rangle + \left\langle D_r \delta(z_*), r-r_* \right\rangle 
= 2n \tan \frac{\pi}{n} \sum_{i=1}^{n} (x_i - (x_*)_i) = 0.
\]

(3.3)

Since \( \phi(z_*) = \delta(z_*) = 0 \), by utilizing \((3.2)\) and \((3.3)\) and performing a third order Taylor expansion it follows that for \( z \) close enough to \( z_* \),

\[
\left| \phi(z) - \frac{1}{2} \langle D^2 \phi(z_*)(z-z_*), (z-z_*) \rangle \right| \leq C |z-z_*|^3,
\]

(3.4)

and

\[
\left| \delta(z) - \frac{1}{2} \langle D^2 \delta(z_*)(z-z_*), (z-z_*) \rangle \right| \leq C |z-z_*|^3,
\]

(3.5)

where \( C > 0 \). In particular, there exists \( \eta = \eta(n) \) such that

\[
\phi(z) \leq \frac{1}{2} \| \Phi \|_2 |z-z_*|^2 + C |z-z_*|^3
\]

(3.6)
for all \( z \in B_n(z_*) \). By the results of [14], see (iv) in §3, it follows that
\[
\inf_{w \in S_H} \langle D^2 \delta(z_*) w, w \rangle =: \sigma > 0 \tag{1}
\]
where \( H \) is the tangent space of \( M \) at \( z_* \) and \( S_H \) is the unit sphere in \( H \) with center \( z_* \). Moreover, by continuity there exists a neighborhood \( U \subset \mathbb{R}^{2n} \) of \( S_H \) such that
\[
\langle D^2 \delta(z_*) w, w \rangle \geq \frac{\sigma}{2},
\]
for all \( w \in U \). Note that \( \frac{z - z_*}{|z - z_*|} \in U \) for \( z \in M \) sufficiently close to \( z_* \). Hence, there exists \( \mu = \mu(\eta, \sigma) \in (0, \eta] \) such that
\[
\langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle \geq \frac{\sigma}{2} |z - z_*|^2
\]
for \( z \in B_\mu(z_*) \). In particular, for \( \tilde{\mu} := \min\{\mu, \frac{\sigma}{8C}\} \) and \( z \in B_{\tilde{\mu}}(z_*) \),
\[
\delta(z) \geq \frac{1}{4} \langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle;
\]
thus, recalling (3.6),
\[
\phi(z) \leq \left( \frac{1}{\sigma} \|\Phi\|_2 + \frac{2C}{\sigma} |z - z_*| \right) \langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle \leq c_n \delta(z),
\]
where \( c_n := \frac{4}{\sigma} \|\Phi\|_2 + \frac{8C}{\sigma} \tilde{\mu} \). To achieve the second part of the theorem, it suffices to prove the existence of \( c > 0 \) such that
\[
\langle \Phi(x; r), (x; r) \rangle \geq c |(x; r)|^2,
\]
for
\[
(x; r) \in \mathcal{Z} := \left\{ (x; r) : \sum_{i=1}^n x_i = 0, \sum_{i=1}^n r_i = 0 \right\}.
\]
Indeed, if (3.7) holds, let \( \omega : [0, \infty] \to [0, \infty] \) be any modulus of continuity (i.e. \( \omega(0^+) = 0 \)) such that
\[
\phi(z) \leq c_n \omega(\delta(z)).
\]
Then for \( z \in M \) close to \( z_* \), (3.5) implies
\[
\delta(z) \leq c_0 |z - z_*|^2,
\]
for some \( c_0 > 0 \). Moreover, \( z - z_* \in \mathcal{Z} \) since \( z \in M \), and by combining (3.4) with (3.7) it follows that
\[
\delta(z) \leq c_0 |z - z_*|^2 \leq c_1 \langle \Phi(z - z_*), (z - z_*) \rangle \leq c_2 \phi(z) \leq \tilde{c} \omega(\delta(z)),
\]
In fact, something stronger is proved: namely that \( \inf_{w \in S_H} \langle D^2 f(z_*) w, w \rangle =: \sigma > 0 \) where \( f \) is an explicit function for which \( D^2 f \leq D^2 \delta \). This is achieved via the spectral theory for circulant matrices and an analysis involving the tangent space of \( M \) at \( z_* \) and the identification of a suitable coordinate system in which calculations can be performed efficiently. The barycentric condition (2.3) built into the definition of \( M \) comes up in this analysis.
for some $\tilde{c} > 0$ provided $z$ is close to $z^*$; however, since $\delta(z) \to 0$ as $z \to z^*$ and $\delta(z) > 0$ for $z \neq z^*$, (3.8) leads to a contradiction if

$$\liminf_{t \to 0^+} \frac{\omega(t)}{t} = 0.$$  

Thus the lim inf is strictly greater than zero and this implies $\omega$ is at most linear at zero. To verify (3.7), note first that $C$ is a real, symmetric, circulant matrix generated by the vector $(n - 1, -1, \ldots, -1)$. A calculation shows that the eigenvalues of $C$, say $\lambda_k$, are given by

(3.9) \quad \lambda_0 = 0 \quad \text{and} \quad \lambda_k = n \quad \text{for} \quad k = 1, \ldots, n - 1.

Moreover, let $v_0 := (1, \ldots, 1)$, and for $l \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \}$ define

$$v_{2l-1} := \left(1, \cos \frac{2\pi l}{n}, \cos \frac{4\pi l}{n}, \ldots, \cos \frac{2\pi l(n-1)}{n}\right),$$

$$v_{2l} := \left(0, \sin \frac{2\pi l}{n}, \sin \frac{4\pi l}{n}, \ldots, \sin \frac{2\pi l(n-1)}{n}\right).$$

One can readily check that $v_k$ is an eigenvector of $C$ corresponding to the eigenvalue $\lambda_{\lfloor \frac{k}{2} \rfloor}$, and that the set $\{v_0, v_1, \ldots, v_{n-1}\}$ forms a real orthogonal basis of $\mathbb{R}^n$ (see e.g. Proposition 2.1 in [IN14]). For $k = 1, 2, \ldots, n$, define $b_k := (v_{k-1}; 0, \ldots, 0) \in \mathbb{R}^{2n}$ and $b_k := (0, \ldots, 0; v_{k-n-1}) \in \mathbb{R}^{2n}$ for $k = n + 1, \ldots, 2n$. Since the set $\{b_k\}_{k=1}^{2n}$ forms a real orthogonal basis of $\mathbb{R}^{2n}$, given $(x; r) \in \mathbb{R}^{2n}$ there exist unique coefficients $\alpha_k \in \mathbb{R}$ such that

$$(x; r) = \sum_{k=1}^{2n} \alpha_k b_k.$$  

Thus, by utilizing (3.9) it follows that

$$\langle \Phi(x; r), (x; r) \rangle = \sum_{k,k'=1}^{2n} \alpha_k \alpha_{k'} \langle \Phi b_k, b_{k'} \rangle$$

$$= n \sin \frac{2\pi}{n} \sum_{k=1}^{n} \alpha_k^2 \lambda_{\lfloor \frac{k}{2} \rfloor} |b_k|^2 + 2 \sum_{k=n+1}^{2n} \alpha_k^2 \lambda_{\lfloor \frac{k-n-1}{2} \rfloor} |b_k|^2$$

$$= n^2 \sin \frac{2\pi}{n} \sum_{k=2}^{n} \alpha_k^2 |b_k|^2 + 2n \sum_{k=n+2}^{2n} \alpha_k^2 |b_k|^2.$$  

Furthermore, if $(x; r) \in Z$,

$$\alpha_1 = \frac{\langle (x; r), b_1 \rangle}{|b_1|^2} = \sum_{i=1}^{n} x_i = 0,$$  

\[ \alpha_{n+1} = \frac{\langle (x; r), b_{n+1} \rangle}{|b_1|^2} = \sum_{i=1}^{n} r_i = 0; \]

hence,

\[
\langle \Phi (x; r), (x; r) \rangle = n^2 \sin \frac{2\pi}{n} \sum_{k=1}^{n} \alpha_k^2 |b_k|^2 + 2n \sum_{k=n+1}^{2n} \alpha_k^2 |b_k|^2 \geq 2n \sum_{k=1}^{2n} \alpha_k^2 |b_k|^2;
\]

and this concludes the proof.

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