REES ALGEBRAS AND $p_g$-IDEALS IN A TWO-DIMENSIONAL NORMAL LOCAL DOMAIN

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Abstract. The authors previously introduced the notion of $p_g$-ideals for two-dimensional excellent normal local domain over an algebraically closed field in terms of resolution of singularities. In this note, we give several ring-theoretic characterizations of $p_g$-ideals. For instance, an $m$-primary ideal $I \subset A$ is a $p_g$-ideal if and only if the Rees algebra $R(I)$ is a Cohen-Macaulay normal domain.

1. Introduction

In [9, Sect. 7], Lipman proved that for any integrally closed $m$-primary ideal $I$ in a rational singularity of dimension 2, $I^2 = QI$ holds for every minimal reduction $Q$ of $I$ and that all powers of $I$ are integrally closed. This implies that the Rees algebra $R(I) = \bigoplus_{n \geq 0} I^n$ is a Cohen-Macaulay normal domain. Moreover, for any two integrally closed $m$-primary ideals $I, J$ in a two-dimensional rational singularity, one can choose general elements $a \in I$ and $b \in J$ so that $IJ = aJ + bI$. This fact implies that the bigraded Rees algebra $R(I, J)$ is a Cohen-Macaulay ring. An ideal theory in a two-dimensional rational singularity is established based upon these facts.

In [12], the authors introduced the notion of $p_g$-ideals for two-dimensional normal local domains using a resolution of singularities; see Section 2 for the definition and basic properties. Notice that the notion of $p_g$-ideals is a natural generalization of an integrally closed $m$-primary ideal in a two-dimensional rational singularity; see [12].

The main purpose of this note is to give several ring-theoretic characterizations of $p_g$-ideals. Namely, we prove the following theorem.

**Theorem 1.1** (see Corollary 3.3 and Theorem 4.1). Let $(A, m)$ be a two-dimensional excellent normal local domain over an algebraically closed field. Let $I \subset A$ be an $m$-primary ideal, and let $Q$ be a minimal reduction of $I$. Then the following conditions are equivalent:

1. $I$ is a $p_g$-ideal.
2. $I^2 = QI$ and $\overline{I^n} = \overline{I^n}$ for every $n \geq 1$, where $\overline{J}$ denotes the integral closure of an ideal $J$. 

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(3) The Rees algebra $R(I)$ is a Cohen-Macaulay normal domain.
(4) $\overline{e}_2(I) = 0$.

Let us explain the organization of the paper. In Section 2, we recall the definition and several basic properties for $p_g$-ideals. For instance, $IJ = aJ + bI$ holds true for any two $p_g$-ideals $I, J$ and general elements $a \in I, b \in J$. In Section 3, we give a characterization of $p_g$-ideals in terms of normal Hilbert polynomials. Namely, the vanishing of the second normal Hilbert coefficient of $I$ yields that the ideal is a $p_g$-ideal (see Theorem [3.2]). In Section 4, we give a characterization of $p_g$-ideals in terms of Rees algebras. Namely, an ideal $I$ is a $p_g$-ideal if and only if the Rees algebra $R(I)$ is a Cohen-Macaulay normal domain. Applying these results, one can find some examples of $p_g$-ideals.

2. Basic results

Throughout this paper, let $(A, m)$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k$ and $f : X \to \text{Spec} A$ a resolution of singularities with exceptional divisor $E := f^{-1}(m)$ unless otherwise specified. Let $E = \bigcup_{i=1}^{r} E_i$ be the decomposition into irreducible components of $E$.

First, we recall the definition of $p_g$-ideals. For the definition of the integral closure and the reduction of ideals, refer to the textbook [14]. An $m$-primary ideal $I$ is said to be represented on $X$ if the ideal sheaf $I\mathcal{O}_X$ is invertible and $I = H^0(X, I\mathcal{O}_X)$. If $I$ is represented on $X$, then there exists an anti-nef cycle $Z$ such that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$; $I$ is also said to be represented by $Z$ and written as $I = I_Z$. Note that such an ideal $I$ is integrally closed in $A$. See [9 Sect. 18]; note that an ideal $I$ is integrally closed if and only if it is complete ([9 Sect. 5]).

We say that $\mathcal{O}_X(-Z)$ has no fixed component if

$$H^0(\mathcal{O}_X(-Z)) \neq H^0(\mathcal{O}_X(-Z - E_i))$$

for every $E_i \subset E$, i.e., the base locus of the linear system $H^0(\mathcal{O}_X(-Z))$ does not contain any component of $E$.

We denote by $h^1(\mathcal{O}_X(-Z))$ the length $\ell_A(H^1(\mathcal{O}_X(-Z)))$. It is known that $h^1(\mathcal{O}_X)$ is independent of the choice of the resolution of singularities. The invariant $p_g(A) := h^1(\mathcal{O}_X)$ is called the geometric genus of $A$.

In [12 Theorem 3.1], the authors proved $h^1(\mathcal{O}_X(-Z)) \leq p_g(A)$ if $\mathcal{O}_X(-Z)$ has no fixed component. Based upon this result, they introduced the notion of $p_g$-ideals. The definition of $p_g$-ideal is independent of the choice of the resolution of singularities ([12 Lemma 3.4]).

Definition 2.1 ($p_g$-ideals, $p_g$-cycles). A cycle $Z > 0$ is called a $p_g$-cycle if $\mathcal{O}_X(-Z)$ is generated and $h^1(\mathcal{O}_X(-Z)) = p_g(A)$. An $m$-primary ideal $I$ is called a $p_g$-ideal if $I$ is represented by a $p_g$-cycle on some resolution.

Assume that $p_g(A) = 0$. Such a ring $A$ is called a rational singularity. Then every anti-nef cycle is a $p_g$-cycle (Lipman [9 Theorem 12.1]). See [12 Proposition 3.10] for another characterization of $p_g$-ideals in the case of $p_g(A) > 0$.

In what follows, let us discuss whether $Z + Z'$ is a $p_g$-cycle.
Proposition 2.2 (see [12, Theorem 3.5]). Let \( Z, Z' \) be anti-nef cycles on the resolution \( X \to \text{Spec} \ A \) such that \( \mathcal{O}_X(-Z) \) and \( \mathcal{O}_X(-Z') \) are generated. Take general elements \( a \in I_Z, b \in I_{Z'} \), so that the natural homomorphism \( b\mathcal{O}_X(-Z) \oplus a\mathcal{O}_X(-Z') \to \mathcal{O}_X(-Z - Z') \) is surjective, and put
\[
\varepsilon(Z, Z') := \ell_A(I_{Z+Z'}/aI_{Z'} + bI_Z)
\]
\[
= p_g(A) - h^1(\mathcal{O}_X(-Z)) - h^1(\mathcal{O}_X(-Z')) + h^1(\mathcal{O}_X(-Z - Z')).
\]
Then:

1. If \( Z \) is a \( p_g \)-cycle on \( X \), then \( \varepsilon(Z, Z') = 0 \) for any \( Z' \). In particular, if \( a \in I_Z \) and \( b \in I_{Z'} \) are general elements, then
\[
I_{Z+Z'} = aI_Z + bI_Z.
\]

2. Assume that \( Z \) is a \( p_g \)-cycle. Then \( Z' \) is a \( p_g \)-cycle if and only if so is \( Z + Z' \).

3. If \( Z + Z' \) is a \( p_g \)-cycle for some cycle \( Z' \), then so is \( Z \).

Proof. (1), (2) It follows from [12, Theorem 3.5].
(3) Let \( \alpha \in H^0(\mathcal{O}_X(-Z')) \) be a general element. Then \( \text{div}_X(\alpha) = Z' + H \), where \( H \) is the proper transform of \( \text{div}_{\text{Spec} \ A}(\alpha) \). From the exact sequence
\[
0 \to \mathcal{O}_X(-Z) \xrightarrow{\times \alpha} \mathcal{O}_X(-Z - Z') \to C \to 0
\]
we obtain \( h^1(\mathcal{O}_X(-Z)) \geq h^1(\mathcal{O}_X(-Z - Z')) = p_g(A) \). Hence \( h^1(\mathcal{O}_X(-Z)) = p_g(A) \) by [12, Theorem 3.10].

The following corollary immediately follows from Proposition 2.2.

Corollary 2.3 ([12, Corollary 3.6]). Let \( I, J \) be \( \mathfrak{m} \)-primary integrally closed ideals.

1. Assume that \( I \) is a \( p_g \)-ideal. For general elements \( a \in I, b \in J \), we have \( \text{IJ} = aJ + bI \).

2. If \( I \) and \( J \) are \( p_g \)-ideals, then \( \text{IJ} \) is also a \( p_g \)-ideal.

3. If \( \text{IJ} \) is a \( p_g \)-ideal, then so are \( I \) and \( J \).

3. The normal Hilbert polynomials

For an \( \mathfrak{m} \)-primary ideal \( I \subset A \), there exist integers \( \tau_0(I), \tau_1(I), \tau_2(I) \) such that
\[
\ell_A(A/I^{n+1}) = \tilde{e}_0(I) \begin{pmatrix} n+2 \\ 2 \end{pmatrix} - \tilde{e}_1(I) \begin{pmatrix} n+1 \\ 1 \end{pmatrix} + \tilde{e}_2(I) \text{ for large enough } n \gg 0.
\]
Then
\[
P_I(n) = \tilde{e}_0(I) \begin{pmatrix} n+2 \\ 2 \end{pmatrix} - \tilde{e}_1(I) \begin{pmatrix} n+1 \\ 1 \end{pmatrix} + \tilde{e}_2(I)
\]
is called the normal Hilbert polynomial of \( I \). See e.g. [7].

Lemma 3.1. Let \( Z > 0 \) be a cycle such that \( \mathcal{O}_X(-Z) \) has no fixed component. Then:

1. \( h^1(\mathcal{O}_X(-nZ)) \geq h^1(\mathcal{O}_X(-(n+1)Z)) \) for \( n \geq 0 \).

2. If we put \( n_0 = \min \{ n \in \mathbb{Z}_{\geq 0} \mid h^1(\mathcal{O}_X(-nZ)) = h^1(\mathcal{O}_X(-(n+1)Z)) \} \), then \( n_0 \leq p_g(A) \) and \( h^1(\mathcal{O}_X(-nZ)) = h^1(\mathcal{O}_X(-n_0Z)) \) for all \( n \geq n_0 \).
Proof. (1) follows from the argument of Proposition 2.2.
(2) From the exact sequence
\[ 0 \to O_X(-nZ) \to O_X(-(n+1)Z)^{\oplus 2} \to O_X(-(n+2)Z) \to 0, \]
we obtain that \( h^1(O_X(-nZ)) \geq 2 \cdot h^1(O_X(-(n+1)Z)) - h^1(O_X(-(n+2)Z)). \) Thus if \( h^1(O_X(-nZ)) = h^1(O_X(-(n+1)Z)) \) is satisfied, then \( h^1(O_X(-(n+1)Z)) = h^1(O_X(-(n+2)Z)) \) holds true. □

The following result, the so-called Kato’s Riemann-Roch formula (3.1), plays an important role in the next theorem. For an anti-nef cycle on \( Z \) on \( X \) and \( I_Z = H^0(O_X(-Z)) \), we have
\[ (3.1) \quad \ell_A(A/I_Z) + h^1(O_X(-Z)) = -\frac{Z^2 + K_XZ}{2} + p_g(A), \]
where \( K_X \) denotes the canonical divisor of \( X \).

**Theorem 3.2.** Assume that \( I \) is represented by a cycle \( Z > 0 \). Let \( P_I(n) \) be a normal Hilbert polynomial of \( I \). Then
(1) \( P_I(n) = \ell_A(A/I^{n+1}) \) for all \( n \geq p_g(A) - 1 \).
(2) \( \bar{e}_0(I) = e_0(I) = -Z^2 \).
(3) \( \bar{e}_1(I) = e_0(I) - \ell_A(A/I) + (p_g(A) - h^1(O_X(-Z))) = -\frac{Z^2 + ZK_X}{2} \).
(4) \( \bar{e}_2(I) = p_g(A) - h^1(O_X(-nZ)) \) for all \( n \geq p_g(A) \).

Proof. It follows from the Riemann-Roch formula (3.1) that
\[ \ell_A(A/I^{n+1}) = -\frac{(n+1)^2Z^2 + (n+1)ZK_X}{2} + p_g(A) - h^1(O_X(-(n+1)Z)) \]
\[ = -Z^2 \left( \frac{n+2}{2} \right) - \frac{Z^2 + ZK_X}{2} \left( \frac{n+1}{1} \right) + p_g(A) - h^1(O_X(-(n+1)Z)). \]

Since \( h^1(O_X(-(nZ)) \) is stable for \( n \geq p_g(A) \) by Lemma 3.1, we obtain the required assertions. □

As a corollary, we obtain a simple characterization of \( p_g \)-ideals in terms of normal Hilbert coefficients.

**Corollary 3.3.** The following conditions are equivalent:
(1) \( I \) is a \( p_g \)-ideal.
(2) \( \bar{e}_1(I) = e_0(I) - \ell_A(A/I) \).
(3) \( \bar{e}_2(I) = 0 \).

Proof. (1) \( \implies \) (2) follows from Theorem 3.2.
(2) \( \implies \) (3): By assumption, \( I = I_Z \) is a \( p_g \)-ideal. Hence \( I_nZ = I^n \) is a \( p_g \)-ideal by Corollary 2.3(2), and thus \( \bar{e}_2(I) = 0 \) by Theorem 3.2.
(3) \( \implies \) (1): Theorem 3.2 yields that \( h^1(O_X(-(n+1)Z)) = p_g(A) \) for \( n \gg 0 \) and thus \( I_nZ = I^n \) is a \( p_g \)-ideal. By Corollary 2.3(3), we obtain that \( I_Z \) is also a \( p_g \)-ideal. □

For any cycle \( Z \) on \( X \), we put \( Z^\perp = \sum_{E_i=0} Z_{E_i}E_i \).
**Proposition 3.4.** Let $Z > 0$ be a cycle such that $\mathcal{O}_X(-Z)$ has no fixed component. If $C$ is the cohomological cycle on $Z^\perp$, i.e., the smallest cycle with

$$h^1(\mathcal{O}_C) = \max_{D > 0, D \cdot e \leq Z^\perp} h^1(\mathcal{O}_D),$$

then $\mathcal{O}_C \cong \mathcal{O}_C(-n_0Z)$ and $h^1(\mathcal{O}_C) = h^1(\mathcal{O}_X(-n_0Z)) = \bar{p}_g(A) - \bar{e}_2(I_Z)$, where $n_0$ is an integer given by Lemma 3.1.

**Proof.** Let $D > 0$ satisfy that $\text{Supp}(D) = Z^\perp$ and $DE_i < 0$ for all $E_i \leq Z^\perp$. Then $H^1(\mathcal{O}_X(-nD - mZ)) = 0$ (cf. the proof of Proposition 3.10). Then $H^1(\mathcal{O}_X(-mZ)) = H^1(\mathcal{O}_{nD}(-mZ))$. Since $\mathcal{O}_{nD}(-mZ) \cong \mathcal{O}_{nD}$, $h^1(\mathcal{O}_X(-mZ)) = h^1(\mathcal{O}_C)$ for sufficiently large $m$. \qed

**Remark 3.5.** Assume that $\mathcal{O}_X(-Z)$ is generated. Let $E^{(1)}, \ldots, E^{(k)}$ be the connected components of $Z^\perp$ and assume that each $E^{(i)}$ contracts to a normal surface singularity isomorphic to $(A_i, m_i)$. Then we have $p_g(A) = \bar{e}_2(I_Z) + \sum_{i=1}^{k} p_g(A_i)$ (cf. [11, Corollary 4.5]).

**Example 3.6.** Let $e \geq 2$ be an integer, and let $A = k[[x, y, z]]/(x^e + y^e + z^e)$. Then the Poincaré series of $k[x, y, z]/(x^e + y^e + z^e)$ is equal to

$$\sum_{k \geq 0} \ell_A(m^k/m^{k+1})t^k = \frac{1 - t^e}{(1-t)^3} = \frac{1 + t + t^2 + \cdots + t^{e-1}}{(1-t)^2}.$$

It follows that

$$\ell_A(A/m^{n+1}) = \begin{cases} e \binom{n+2}{2} - \frac{e(e-1)}{2} \binom{n+1}{1} + \frac{e(e-1)(e-2)}{6} & (n \geq e), \\ \frac{(n+1)(n+2)(n+3)}{6} & (n \leq e-1). \end{cases}$$

Hence

$$\begin{aligned} e_0(m) &= e_0(m) = e, \\
e_1(m) &= e_1(m) = \frac{e(e-1)}{2}, \\
e_2(m) &= e_2(m) = \frac{e(e-1)(e-2)}{6} = p_g(A); \text{ see [15] (4.11)}. \end{aligned}$$

Write $m = I_Z$ for some anti-nef cycle $Z$ on some resolution $X \to \text{Spec} A$. Then $h^1(\mathcal{O}_X(-kZ)) = 0$ for every $k \geq e$. On the other hand,

$$\frac{e(e-1)}{2} = e_1(m) = e_0(m) - \ell_A(A/m) + p_g(A) - h^1(\mathcal{O}_X(-Z))$$

yields $h^1(\mathcal{O}_X(-Z)) = \frac{(e-1)(e-2)(e-3)}{6}$. 

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Furthermore, since \( ZK = 2 \cdot \mathfrak{e}_1(m) - (-Z^2) = e(e - 2) \) and \( e_0(m^k) = k^2e \), we have

\[
h^1(\mathcal{O}_X(-kZ)) = e_0(m^k) - \ell_A(A/m^k) + p_g(A) - \frac{-(kZ^2 + (kZ)K)}{2}
\]

\[
= k^2e - \binom{k + 2}{3} + \binom{e}{3} - \frac{k^2e + ke(e - 2)}{2}
\]

\[
= \frac{(e - k)(e - k - 1)(e - k - 2)}{6} = \binom{e - k}{3}
\]

for each \( k = 1, 2, \ldots, e - 1 \). In particular, we get

\[
h^1(\mathcal{O}_X(-(e - 3)Z)) = 1 \quad \text{and} \quad h^1(\mathcal{O}_X(-(e - 2)Z)) = 0,
\]

and thus \( n_0 = e - 2 \) in Lemma 3.1.

4. The Rees Algebra

Let \((A, \mathfrak{m})\) be a Cohen-Macaulay local ring of dimension \( d \), and let \( I \) be an ideal of \( A \). Now consider three \( A \)-algebras, which are called blow-up algebras,

\[
\mathcal{R}(I) := A[It] = \bigoplus_{n \geq 0} I^n t^n \subset A[t],
\]

\[
\mathcal{R}'(I) := A[It, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} I^n t^n \subset A[t, t^{-1}],
\]

\[
G(I) := \mathcal{R}(I)/I\mathcal{R}(I) \cong \mathcal{R}'(I)/t^{-1}\mathcal{R}'(I).
\]

The algebra \( \mathcal{R}(I) \) (resp. \( \mathcal{R}'(I), G(I) \)) is called the Rees algebra (resp. the extended Rees algebra, the associated graded ring) of \( I \).

The main purpose of this section is to characterize \( p_g \)-ideals in terms of blow-up algebras.

**Theorem 4.1.** Let \((A, \mathfrak{m})\) be a two-dimensional excellent normal local domain over an algebraically closed field, and let \( I \subset A \) be an \( \mathfrak{m} \)-primary ideal. Then the following conditions are equivalent:

1. \( I \) is a \( p_g \)-ideal in the sense of Definition 2.1.
2. \( I^2 = QI \) for some minimal reduction \( Q \) of \( I \), and \( \overline{I^n} = I^n \) holds true for every \( n \geq 1 \).
3. \( \mathcal{R}(I) \) is a Cohen-Macaulay normal domain.
4. \( \mathcal{R}'(I) \) is a Cohen-Macaulay normal domain with \( a(G(I)) < 0 \), where \( a(G(I)) \) denotes the \( a \)-invariant of the graded ring \( G(I) \); see [3, Definition 3.14].

**Proof.** (1) \( \implies \) (2): It follows from Corollary 2.3.

(2) \( \implies \) (3): Since \( I^2 = QI \) for some minimal reduction \( Q \) of \( I \), \( \mathcal{R}(I) \) is Cohen-Macaulay by Valabrega–Valla [16] and Goto–Shimoda [1]. Moreover, since \( A \) is normal and \( \overline{I^n} = I^n \) for every \( n \geq 1 \), \( \mathcal{R}(I) \) is a normal domain.

(4) \( \iff \) (3) \( \implies \) (2) follows from Goto–Shimoda [11] and Herzog et al. [5, Proposition 2.1.2].

(2) \( \implies \) (1): Assume that \( I^n \) is integrally closed for \( n \geq 1 \) and that \( I^2 = QI \) for a minimal reduction \( Q \) of \( I \). Suppose that \( I \) is represented by a cycle \( Z \) on \( X \).
Consider the following exact sequence given by general elements of $I = I_Z$ and $I_nZ$ (see [12, (2.3)]):

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(-Z) \oplus \mathcal{O}_X(-nZ) \to \mathcal{O}_X(-(n+1)Z) \to 0.$$ 

Since $QI^n = I^{n+1} = I^{n+1}$, we obtain that $\varepsilon(Z, nZ) = 0$ for $n \geq 1$. Therefore, $p_g(A) = h^1(\mathcal{O}_X(-Z))$ because $h^1(\mathcal{O}_X(-nZ))$ is stable for $n \gg 0$.

The following two examples are known.

**Example 4.2 (cf. Lipman [10] Example 3).** Let $A$ be a two-dimensional rational singularity. Then any integrally closed $m$-primary ideal $I$ is a $p_g$-ideal and $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.

**Example 4.3.** Let $A$ be a complete Gorenstein local ring with $p_g(A) > 0$. If $m$ is a $p_g$-ideal of $A$, then $m$ is stable, that is, $m^2 = Qm$ for some minimal reduction $Q$ of $m$. Since $A$ is Gorenstein, we obtain that $A$ is a hypersurface of degree 2. So we may assume that $A = K[[x, y, z]]/(f)$, where $f = x^2 + g(y, z)$. As $A$ is not rational, $g(y, z) \in \langle y, z \rangle^3$. Moreover, since $R(m)$ is normal, we have $g(y, z) \notin \langle y, z \rangle^4$.

Conversely, if $A = K[[x, y, z]]/(x^2 + g(y, z))$, where $g(y, z) \in \langle y, z \rangle^3 \setminus \langle y, z \rangle^4$, then for every $n$, $m^n = (y, z)^n + x(y, z)^n-1$ and is integrally closed. Then, since $m$ is stable and $m^n$ is integrally closed for every $n \geq 1$, $m$ is a $p_g$-ideal.

The next example gives a hypersurface local ring $A$ whose maximal ideal is a $p_g$-ideal and $p_g(A) = p$ for a given integer $p \geq 1$.

**Example 4.4.** Let $p \geq 1$ be an integer, and let $k$ be an algebraically closed field. Let $B = k[x, y, z]/(x^2 + y^3 + z^{6p+1})$. If we put $\deg x = 3(6p + 1)$, $\deg y = 2(6p + 1)$ and $\deg z = 6$, then $A$ can be regarded as a quasi-homogeneous $k$-algebra with $a(A) = 6p - 5$. In particular,

$$p_g(B) = \sum_{i=0}^{6p-5} \dim_k B_i = p; \quad \text{(cf. [13][19]).}$$

Moreover, if we put $X = xt$, $Y = yt$, $Z = zt$ and $U = t^{-1}$, then the extended Rees algebra of $m = (x, y, z)$ is

$$\mathcal{R}'(m) \cong k[X, Y, Z, U]/(F),$$

where $F = X^2 + Y^3U + Z^{6p+1}U^{6p-1}$. Since the Jacobian ideal is

$$\left( \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}, \frac{\partial F}{\partial U} \right) = (X, Y^2U, Z^{6p+1}U^{6p-1}, Y^3 + (6p - 1)Z^{6p+1}U^{6p-2}),$$

one can check the $(R_1)$-condition of $\mathcal{R}'(m)$. Thus $\mathcal{R}'(m)$ is a normal domain because it is Cohen-Macaulay.

Now let us put $A = B_{(x, y, z)}$ and $m = (x, y, z)A$. Then we can conclude that $A$ is a two-dimensional normal hypersurface with $p_g(A) = p$ and that $m$ is a $p_g$-ideal by applying the theorem above.

Similarly, if we consider $I_k = (x, y, z^k)A$ and $Q_k = (y, z^k)$ for $k = 2, 3, \ldots, 3p$, then $I_k^2 = Q_kI_k$ and $\mathcal{R}'(I_k)$ is a normal domain. Hence $I_k$ is a $p_g$-ideal.

The next example gives a hypersurface local ring $A$ whose maximal ideal is not a $p_g$-ideal and $p_g(A) = p$ for a given integer $p \geq 1$. 

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Example 4.5. Let \( p \geq 1 \) be an integer. Let \( A = k[x, y, z]/(x^2 + y^4 + z^{4p+1}) \). Then \( A \) is a two-dimensional normal hypersurface with \( p_g(A) = p \). Then \( \mathfrak{m} = (x, y, z) \) is not a \( p_g \)-ideal and \( I_k = (x, y, z^k) \) is a \( p_g \)-ideal for every \( k = 2, 3, \ldots, 2p \) because \( \mathcal{R}'(I_k) \) is normal but \( \mathcal{R}'(\mathfrak{m}) \) is not.

Furthermore, \( \mathfrak{m}^k \) is not a \( p_g \)-ideal for every \( k \geq 1 \) by Corollary 2.3.

It is not so difficult to extend our result to the case of bigraded Rees algebras. Let \( I, J \subset A \) be ideals. Then

\[
\mathcal{R}(I, J) := A[I_{t_1}, J_{t_2}] = \bigoplus_{n=1}^{\infty} \bigoplus_{m=1}^{\infty} I^m J^n t_1^n t_2^m \subset A[t_1, t_2]
\]

is called the multi-Rees algebra of \( I \) and \( J \).

Corollary 4.6. Let \((A, \mathfrak{m})\) be a two-dimensional excellent normal local domain over an algebraically closed field, and let \( I, J \) be \( \mathfrak{m} \)-primary ideals. Then the following conditions are equivalent:

1. \( I \) and \( J \) are \( p_g \)-ideals.
2. \( \mathcal{R}(I, J) \) is a Cohen-Macaulay normal domain.
3. \( I, J \) are integrally closed and \( \mathcal{R}(I, J) \) is a Cohen-Macaulay normal domain.

Proof. \((1) \implies (2)\): Since \( I \) and \( J \) are \( p_g \)-ideals, \( \mathcal{R}(I) \) and \( \mathcal{R}(J) \) are Cohen-Macaulay and \( IJ = aI + bJ \) for some joint reduction \((a, b)\) of \((I, J)\); see [14 Sect. 17]. Hence \( \mathcal{R}(I, J) \) is Cohen-Macaulay by [6 Corollary 3.5] (see also e.g. [17,18]). Since \( S = \mathcal{R}(I) \) is a normal domain and \( JJ^k \) is integrally closed for every \( k \geq 1 \), \( \mathcal{R}(I, J) \) is normal.

\((2) \implies (1)\): Since \( \mathcal{R}(I, J) \) is Cohen-Macaulay, \( \mathcal{R}(I) \) and \( \mathcal{R}(J) \) are Cohen-Macaulay by [6 Corollary 3.5]. Since \( \mathcal{R}(I) \) and \( \mathcal{R}(J) \) are pure subrings of \( \mathcal{R}(I, J) \), they are normal domains. Hence \( I \) and \( J \) are \( p_g \)-ideals by Theorem 4.1.

\((1) \iff (3)\): It follows from Theorem 4.1 and Corollary 2.3. \( \square \)

Remark 4.7. By a similar argument as in the proof of \((1) \implies (2)\), we can obtain that the multi-Rees algebra \( \mathcal{R}(I_1, \ldots, I_r) \) is a Cohen-Macaulay normal domain for every \( p_g \)-ideal \( I_1, \ldots, I_r \).

Remark 4.8. Assume that \( A \) is a rational singularity. Let \( I \) and \( J \) be \( \mathfrak{m} \)-primary integrally closed ideals of \( A \). Then \( I \) and \( J \) are \( p_g \)-ideals and thus \( \mathcal{R}(I) \), \( \mathcal{R}(J) \) and \( \mathcal{R}(I, J) \) are Cohen-Macaulay normal domains. In fact, S. Goto, N. Matsuoka, N. Taniguchi and the third author [2] prove that \( \mathcal{R}(I) \) and \( \mathcal{R}(J) \) are almost Gorenstein. Moreover, Verma [18] proved that they admit minimal multiplicities.

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