On the growth of torsion in the cohomology of arithmetic groups

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Abstract In this paper we consider certain families of arithmetic subgroups of $SO_0(p, q)$ and $SL_3(\mathbb{R})$, respectively. We study the cohomology of such arithmetic groups with coefficients in arithmetically defined modules. We show that for natural sequences of such modules the torsion in the cohomology grows exponentially.

1 Introduction

Let $G$ be a semi-simple connected algebraic group over $\mathbb{Q}$, $K$ a maximal compact subgroup of its group of real points $G(\mathbb{R})$. Let $\widetilde{X} = G(\mathbb{R})/K$ be the associated Riemannian symmetric space. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G(\mathbb{R})$ and $K$, respectively. Put $\delta(\widetilde{X}) = \text{rank}(\mathfrak{g}_\mathbb{C}) - \text{rank}(\mathfrak{k}_\mathbb{C})$. Sometimes $\delta(\widetilde{X})$ is called the fundamental rank.

Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup and $X = \Gamma \backslash \widetilde{X}$ the corresponding locally symmetric space. We assume that $G$ is anisotropic over $\mathbb{Q}$, which implies that $\Gamma$ is cocompact in $G(\mathbb{R})$. Let $M$ be an arithmetic $\Gamma$-module, which means that $M$ is a finite rank free $\mathbb{Z}$-module, and there exists an algebraic representation of $G$ on $M \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\Gamma$ preserves $M$. Then the cohomology $H^*(\Gamma, M)$ is a finite rank $\mathbb{Z}$-module. Note that if $\Gamma$ is torsion free, then $H^*(\Gamma, M) \cong H^*(X, \mathcal{M})$, where $\mathcal{M}$ is the local system of free $\mathbb{Z}$-modules associated to $M$.

For arithmetic reasons, one expects that if $\delta(\widetilde{X}) = 0$, there is little torsion in $H^*(\Gamma, M)$ and the free part dominates the cohomology. On the other hand, if $\delta(\widetilde{X}) = 1$, one expects a lot of torsion in the cohomology and the free part to be small. This has been substantiated by Bergeron and Venkatesh [5], who studied the growth of the
torsion if $\Gamma$ varies through a sequence of congruence subgroups $\Gamma_n$ for which the injectivity radius of $\Gamma_n \backslash \tilde{X}$ goes to infinity. They showed that if $\delta(\tilde{X}) = 1$ and $M$ is strongly acyclic, the torsion grows exponentially proportional to the volume of $\Gamma_n \backslash \tilde{X}$. Furthermore, for compact oriented hyperbolic 3-manifolds, in [8] the growth of the torsion has been studied if $\Gamma$ is fixed but the $\Gamma$-module $M$ grows. More precisely, let $X = \Gamma \backslash \mathbb{H}^3$ be a compact, oriented hyperbolic 3-manifold with $\Gamma \subset \text{SL}(2, \mathbb{C})$. Let $V_m$ be the holomorphic irreducible representation of $\text{SL}(2, \mathbb{C})$ of dimension $m + 1$. By [6] one has $H^*(\Gamma, V_m) = 0$. It was proved in [8] that for each even $k \in \mathbb{N}$ there exists a $\Gamma$-invariant lattice $M_k \subset V_k$. Then $H^p(\Gamma, M_k)$ is a finite abelian group for all $p$, and the main result of [8] is the following asymptotic formula

$$\lim_{k \to \infty} \frac{\log |H^2(\Gamma, M_{2k})|}{k^2} = \frac{2}{\pi} \text{vol}(X),$$

and the estimation

$$\log |H^p(\Gamma, M_{2k})| \ll k \log k, \quad p = 1, 3.$$  

Note that $H^0(\Gamma, M_{2k}) = 0$.

The goal of the present paper is to study the growth of the torsion if $M$ varies, for all compact arithmetic quotients $\Gamma \backslash \tilde{X}$ of irreducible symmetric spaces $\tilde{X}$ with $\delta(\tilde{X}) = 1$. By the classification of simple Lie groups, the irreducible symmetric spaces with $\delta(\tilde{X}) = 1$ are $\tilde{X} = \text{SO}^0(p, q)/\text{SO}(p) \times \text{SO}(q)$, for $p, q$ odd, and $\tilde{X} = \text{SL}(3, \mathbb{R})/\text{SO}(3)$.

The first family of arithmetic groups that we consider are cocompact arithmetic subgroups of $\text{SO}^0(p, q)$ that arise from quadratic forms over totally real number fields. More precisely, let $F$ be a totally real finite Galois extension of $\mathbb{Q}$ of degree $d > 1$. We fix an embedding $F \subset \mathbb{R}$. Let $Q : \mathbb{R}^{p+q} \to \mathbb{R}$ be a non-degenerate quadratic form defined over $F$ of signature $(p, q)$. Assume that all non-trivial Galois conjugates of $Q$ are positive definite. Let $G := \text{SO}_Q \subset \text{GL}_{p+q}$ be the special orthogonal group of $Q$, i.e., the subgroup of all elements of determinant one leaving $Q$ invariant. This is a connected algebraic group over $F$ and its group of real points $G(\mathbb{R})$ is isomorphic to $\text{SO}(p, q)$.

Let $\mathcal{O}_F$ be the ring of algebraic integers of $F$, and let $G_{\mathcal{O}_F}$ be the group of $\mathcal{O}_F$-valued points of $G$. Then $G_{\mathcal{O}_F}$ is a discrete cocompact subgroup of $G(\mathbb{R})$. Via the isomorphism $G(\mathbb{R}) \cong \text{SO}(p, q)$, it corresponds to a discrete cocompact subgroup $\Gamma_0$ of $\text{SO}(p, q)$ (see Sect. 3). If we pass to an appropriate subgroup of finite index $\Gamma \subset \Gamma_0$, we may assume that $\Gamma$ is torsion free and that it is contained in $\text{SO}^0(p, q)$.

Since $G$ is only defined over $F$, we need to generalize the notion of an arithmetic $\Gamma$-module. Let $G' = \text{Res}_{F/\mathbb{Q}}(G)$ be the algebraic $\mathbb{Q}$-group obtained from $G$ by restriction of scalars. Let $\Delta : G \to G'$ be the diagonal embedding. Consider $\Gamma$ as a subgroup of $G_{\mathcal{O}_F}$. Let $\Gamma' = \Delta(\Gamma)$. Then $\Gamma' \subset G'(\mathbb{Q})$ is an arithmetic subgroup. Let $M$ be an arithmetic $\Gamma'$-module. Since $\Gamma \cong \Gamma'$, it becomes a $\Gamma$-module and we also call it an arithmetic $\Gamma$-module.

To state our main result, we need to introduce some notation. Let $\tilde{X}^d$ be the compact dual symmetric space of $\tilde{X}$. We chose an $\text{SO}^0(p, q)$-invariant metric on $\tilde{X}$ and equip
X and $\tilde{X}^d$ with the induced metrics. Assume that $p, q$ are odd, $p \geq q$, $p > 1$ and let $n := (p + q)/2 - 1$. We let $\epsilon(q) := 0$ for $q = 1$ and $\epsilon(q) := 1$ for $q > 1$ and we put
\[
C_{p, q} = \frac{(-1)^{pq-1}2^{\epsilon(q)\pi}}{\text{vol}(\tilde{X}^d)}\left(\frac{n}{2}\right)^p.
\]
(1.3)

Then our first main result is the following theorem.

**Theorem 1.1** Let $F$ be a totally real Galois extension of $\mathbb{Q}$ of degree $d > 1$. Let $\Gamma$ be a torsion free cocompact arithmetic subgroup of $\text{SO}^0(p, q)$ derived from a quadratic form $Q$ over $F$ as above. Then there exists a sequence of arithmetic $\Gamma$-modules $M_m$, $m \in \mathbb{N}$, with the following properties. The rank $\text{rk}_\mathbb{Z}(M_m)$ is a polynomial in $m$ and there exists $C > 0$, which depends only on $n$, such that
\[
\text{rk}_\mathbb{Z}(M_m) = C d m^{dn(n+1)/2} + O(m^{dn(n+1)/2-1})
\]
as $m \to \infty$. Furthermore each cohomology group $H^j(\Gamma, M_m)$ is finite and
\[
\sum_{j \geq 0} (-1)^j \log |H^j(\Gamma, M_m)| = -C_{p, q} \text{vol}(\Gamma\backslash\tilde{X}) m \text{rk}_\mathbb{Z}(M_m) + O(\text{rk}_\mathbb{Z}(M_m))
\]
as $m \to \infty$.

Let $k = (\dim(\tilde{X}) + 1)/2$. Then it follows from Theorem 1.1 that there exists a constant $\tilde{C}_{p, q} > 0$, which depends only on $p, q$, such that
\[
\liminf_{m \to \infty} \sum_{j=0}^{k(2)} \frac{\log |H^j(\Gamma, M_m)|}{m^{dn(n+1)/2+1}} \geq \tilde{C}_{p, q} d \text{vol}(\Gamma\backslash\tilde{X}).
\]
(1.4)

Thus there is at least one $j$ for which $|H^j(\Gamma, M_m)|$ grows exponentially in $m$. Given (1.1) and (1.2), one is tempted to pose the following conjecture.

**Conjecture 1.2** Let $\Gamma$ and $M_m$, $m \in \mathbb{N}$, be as above. Then
\[
\lim_{m \to \infty} \frac{\log |H^j(\Gamma, M_m)|}{m^{dn(n+1)/2+1}} = \begin{cases} 
\tilde{C}_{p, q} d \text{vol}(\Gamma\backslash\tilde{X}), & j = (\dim(\tilde{X}) + 1)/2, \\
0, & \text{else}.
\end{cases}
\]

The next case that we consider are arithmetic subgroups of $\text{SL}_3(\mathbb{R})$ which arise from 9-dimensional division algebras $D$ over $\mathbb{Q}$. Let $\mathfrak{o}$ be an order in $D$. Then $\mathfrak{o}$ induces an arithmetic subgroup $\Gamma$ of $\text{SL}_3(\mathbb{R})$ which is cocompact (see Sect. 4). After passing to a subgroup of finite index, we may assume that $\Gamma$ is torsion-free.

Let $t_3^\mathbb{C}$ be the standard complexified Cartan-subalgebra of the Lie algebra of $\text{SL}_3(\mathbb{R})$ equipped with the standard ordering of the roots and let $\omega_1, \omega_2 \in t_3^\mathbb{C}$ be the corresponding fundamental weights, see (4.2). If $\Lambda = \tau_1 \omega_1 + \tau_2 \omega_2 \in t_3^\mathbb{C}$, $\tau_1, \tau_2 \in \mathbb{N}$, is a dominant weight and $\tau_\Lambda$ is the corresponding irreducible finite-dimensional representation of $\text{SL}_3(\mathbb{R})$, then $\tau_\Lambda$ is defined over $\mathbb{R}$. Let $\theta: G \to G$ be the Cartan involution with
respect to $K$. We let $\Lambda_\theta$ be the highest weight of $\tau_\Lambda \circ \theta$. One has $\Lambda_\theta = \tau_2 \omega_1 + \tau_1 \omega_2$. Moreover, the representation of $SL_3(\mathbb{C})$ corresponding to $\tau$ is trivial on the center of $SL_3(\mathbb{C})$ if and only if $\tau_1 + 2 \tau_2 \equiv 0 \mod 3$. For $m \in \mathbb{N}$ we let $\tau_\Lambda(m)$ be the irreducible representation of $SL_3(\mathbb{R})$ on a real vector space $V_\Lambda(m)$ with highest weight $m\Lambda$. Let $\widetilde{X} = SL_3(\mathbb{R})/SO(3)$, let $\widetilde{X}_d$ be the compact dual of $\widetilde{X}$ and let $X = \Gamma \backslash \widetilde{X}$. We fix a $G$-invariant metric on $\widetilde{X}$ which induces metrics on $X$ and on $\widetilde{X}_d$. Then our main result for the $SL_3(\mathbb{R})$-case is the following theorem.

**Theorem 1.3** Let $\Gamma$ be an arithmetic subgroup of $SL_3(\mathbb{R})$ which arises from a 9-dimensional division algebra over $\mathbb{Q}$. Let $\Lambda \in \Gamma_\Lambda^+$ be a highest weight with $\Lambda_\theta \neq \Lambda$ and such that the corresponding representation of $SL_3(\mathbb{C})$ is trivial on the center of $SL_3(\mathbb{C})$. Then for each $m$ there exists a lattice $M_\Lambda(m)$ in $V_\Lambda(m)$ which is stable under $\Gamma$. Moreover, each cohomology group $H^p(\Gamma, M_\Lambda(m))$ is finite and one has

$$
\sum_{p=0}^{5} (-1)^p \log \left| H^p(\Gamma, M_\Lambda(m)) \right| = -\frac{\pi \operatorname{vol}(X)}{\operatorname{vol}(\widetilde{X}_d)} C(\Lambda) m \cdot \operatorname{rk}_\mathbb{Z} M_\Lambda(m) + O(\operatorname{rk}_\mathbb{Z} M_\Lambda(m)),
$$

(1.5)
as $m \to \infty$, where $C(\Lambda) > 0$ is a constant which depends only on $\Lambda$. If $\Lambda$ equals $3 \omega_f$, where $\omega_f$ is one of the fundamental weights, then $C(\Lambda) = 4/3$.

The rank of $M_\Lambda(m)$ can be computed explicitly as follows. Firstly, if $\Lambda$ is equal to $3\omega_1$ or $3\omega_2$, then $\Lambda_\theta \neq \Lambda$ and the corresponding representation of $SL_3(\mathbb{C})$ is trivial on the center of $SL_3(\mathbb{C})$. Weyl’s dimension formula gives

$$\operatorname{rk}_\mathbb{Z} M_\Lambda(m) = \dim_{\mathbb{R}}(V_\Lambda(m)) = \frac{9m^2}{2} + O(m),$$

as $m \to \infty$. Secondly, if $\Lambda = \tau_1 \omega_1 + \tau_2 \omega_2, \tau_1, \tau_2 \in \mathbb{N}, \tau_1 \tau_2 \neq 0, \tau_1 + 2 \tau_2 \equiv 0 \mod 3$, then the condition $\Lambda \neq \Lambda_\theta$ is equivalent to $\tau_1 \neq \tau_2$ and again by Weyl’s dimension formula one has

$$\operatorname{rk}_\mathbb{Z} M_\Lambda(m) = \dim_{\mathbb{R}}(V_\Lambda(m)) = \frac{\tau_1^2 \tau_2 + \tau_2^2 \tau_1}{2} m^3 + O(m^2),$$

as $m \to \infty$. Let $M_{i,m} := M_{3\omega_i}(m), i = 1, 2$. Then it follows that

$$\liminf_m \sum_{j=0}^{2} \frac{\log |H^{2j+1}(\Gamma, M_{i,m})|}{m^3} \geq \frac{6\pi}{\operatorname{vol}(\widetilde{X}_d)} \operatorname{vol}(X).$$

(1.6)

Again, by (1.1) and (1.2), one is led to the following conjecture.

**Conjecture 1.4** Let $\Gamma$ and $M_{i,m}, m \in \mathbb{N}$, be as above. Then one has

$$\lim_{m \to \infty} \frac{\log |H^3(\Gamma, M_{i,m})|}{m^3} = \frac{6\pi}{\operatorname{vol}(\widetilde{X}_d)} \operatorname{vol}(X)$$

(1.7)
and

\[ \log |H^j(\Gamma, M_{i,m})| = o(m^3), \quad j \neq 3. \]  

(1.8)

There are similar statements for highest weights \( \Lambda = \tau_1 \omega_1 + \tau_2 \omega_2 \) with \( \tau_1 \tau_2 \neq 0 \).

Next we describe our approach to prove the main results. As in [8], it is based on the study of the analytic torsion. To begin with we consider an arbitrary connected semi-simple algebraic group \( G \) over \( \mathbb{Q} \). Let the notation be as at the beginning of the introduction. Assume that \( \delta(\tilde{X}) = 1 \). Choose a \( G \)-invariant Riemannian metric \( g \) on \( \tilde{X} \). Assume that \( \Gamma \subset G(\mathbb{Q}) \) is torsion free. Let \( X := \Gamma \backslash \tilde{X} \) equipped with the metric induced by \( G \). Then we have \( H^*(\Gamma, M) = H^*(X, M) \). Let \( V := M \otimes \mathbb{Z} \mathbb{R} \) and let \( \rho: G(\mathbb{R}) \to \text{GL}(V) \) be the representation associated to the arithmetic \( \Gamma \)-module \( M \).

Let \( E \to X \) be the flat vector bundle associated to \( \rho|_{\Gamma} \). Choose a Hermitian fibre metric in \( E \). Let \( T_X(\rho) \in \mathbb{R}^+ \) be the analytic torsion of \( X \) and \( E \). Recall that

\[ \log T_X(\rho) = \frac{1}{2} \sum_{p=1}^{n} (-1)^p p \frac{d}{ds} \xi_p(s; \rho) \bigg|_{s=0}, \]  

(1.9)

where \( \xi_p(s; \rho) \) is the zeta function of the Laplace operator \( \Delta_p(\rho) \) on \( E \)-valued \( p \)-forms and \( n = \dim X \) (see [11]). Assume that \( \rho \) is acyclic, that is \( H^*(X, E) = 0 \).

Then \( T_X(\rho) \) is metric independent [11, Corollary 2.7] and equals the Reidemeister torsion \( \tau_X(\rho) \) [11, Theorem 1]. Moreover, \( H^*(\Gamma, M) \) is a finite abelian group. Using Proposition 2.1, we get

\[ \log T_X(\rho) = \sum_{q=0}^{n} (-1)^{q+1} \log |H^q(\Gamma, M)|. \]  

(1.10)

This is the key equality which we apply to prove Theorems 1.1 and 1.3. In [10] we studied the asymptotic behavior of \( T_X(\tau_m) \) for certain sequences of irreducible representations of \( G(\mathbb{R})^0 \). We will apply the results of [10] to our case. The main issue is to construct appropriate arithmetic \( \Gamma \)-modules.

We start with the case of \( \tilde{X} = SO^0(p, q)/ SO(p) \times SO(q), \; p, q \text{ odd} \). Let \( G = SO_Q \) be the special orthogonal group of a quadratic form \( Q \) over a totally real number field \( F \) as defined above. Then \( G \) is a connected algebraic group over \( F \). Let \( G' = \text{Res}_{F/\mathbb{Q}}(G) \) be the algebraic \( \mathbb{Q} \)-group obtained from \( G \) by restriction of scalars [17]. The group of real points \( G'(\mathbb{R}) \) is given by

\[ G'(\mathbb{R}) \cong SO(p, q) \times K_1, \]

where \( K_1 \) is the product of \( d - 1 \) copies of \( SO(p + q) \). Then we construct a sequence \( \rho(m): G' \to GL(V_m), \; m \in \mathbb{N}, \) of \( \mathbb{Q} \)-rational representations such that the irreducible components of \( \rho(m)(\mathbb{R}): G'(\mathbb{R})^0 \to GL(V_m \otimes \mathbb{Q} \mathbb{R}) \) are of the form considered in [10, Theorem 1.1]. Let \( \Delta: G \to G' \) be the diagonal embedding. Let \( \Gamma' = \Delta(\Gamma) \). Then \( \Gamma' \) is an arithmetic subgroup of \( G'(\mathbb{Q}) \). Therefore, \( V_m \) contains a lattice \( M_m \), which is invariant under \( \rho(m)(\Gamma') \). Through the isomorphism \( \Gamma \cong \Gamma' \), \( M_m \) becomes
a $\Gamma$-module. This is our arithmetic $\Gamma$-module. By construction we have $H^*(\Gamma, M_m) \cong H^*(\Gamma', M_m)$. Thus it suffices to prove the statement of Theorem 1.1 for $\Gamma'$.

Let $K' = SO(p) \times SO(q) \times K_1$. Then $K'$ is a maximal compact subgroup of $G'(\mathbb{R})^0$. Let $\tilde{X}' = G'(\mathbb{R})^0/K'$ and $X' := \Gamma'\backslash \tilde{X}'$. Now we apply [10, Propositions 1.2, 1.3] to determine the asymptotic behavior of $T_{X'}(\rho(m))$ as $m \to \infty$. Finally we use (1.10) to establish Theorem 1.1.

The proof of Theorem 1.3 uses similar arguments, which are also based on (1.10) and [10].

The paper is organized as follows. In Sect. 2 we collect some facts about cohomology of fundamental groups of manifolds with coefficients in a free $\mathbb{Z}$-module. We also recall some elementary facts about algebraic groups. In Sect. 3 we consider arithmetic subgroups of $SO^0(p, q)$ and prove Theorem 1.1. The prove of Theorem 1.3 is the content of the final Sect. 4.

## 2 Preliminaries

Let $X$ be a closed connected smooth manifold of dimension $d$. Let $\Gamma := \pi_1(X, x_0)$ be the fundamental group of $X$ with respect to some base point $x_0$ and let $\tilde{X}$ be the corresponding universal covering. Thus $\Gamma$ acts properly discontinuously and fixed point free on $\tilde{X}$ and $X = \Gamma \backslash \tilde{X}$. Assume that $\tilde{X}$ is contractible.

Let $M$ be a free finite-rank $\mathbb{Z}$-module and let $\rho$ be a representation of $\Gamma$ on $M$. Let $H^q(\Gamma, M)$ be the $q$-th cohomology group of $\Gamma$ with coefficients in $M$, see [4]. These groups can be computed as follows. Let $L$ be a smooth triangulation of $X$ and let $\tilde{L}$ be the lift of $L$ to a triangulation of $\tilde{X}$. Let $C_q(\tilde{L}; \mathbb{Z})$ be the free abelian group generated by the $q$-chains of $\tilde{L}$, let $C^q(\tilde{L}; \mathbb{Z}) := \text{Hom}_{\mathbb{Z}}(C_q(\tilde{L}, \mathbb{Z}); \mathbb{Z})$ and let $C^*_s(\tilde{L}; \mathbb{Z})$ resp. $C^*(\tilde{L}; \mathbb{Z})$ be the associated simplicial chain- resp. cochain complexes. Each $C_q(\tilde{L}; \mathbb{Z})$ is a free $\mathbb{Z}[\Gamma]$ module and if one fixes an embedding of $L$ into $\tilde{L}$, then the $q$-cells of $L$ form a base of $C_q(\tilde{L}; \mathbb{Z})$ over $\mathbb{Z}[\Gamma]$. Let

$$C^q(L, M) := C^q(\tilde{L}; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} M.$$ 

Then the $C^q(L, M)$ form again a cochain complex $C^*(L, M)$ and the corresponding cohomology groups will be denoted by $H^q(L, M)$. There is an isomorphism

$$C^q(L, M) \cong \text{Hom}_{\mathbb{Z}[\Gamma]}(C_q(\tilde{L}, \mathbb{Z}), M),$$

which induces an isomorphism of the corresponding cochain complexes. Since $\tilde{L}$ is contractible, the complex $C^*_s(\tilde{L})$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$ and thus one has

$$H^q(\Gamma, M) \cong H^q(\text{Hom}_{\mathbb{Z}[\Gamma]}(C_s(\tilde{L}), M)) \cong H^q(L, M). \quad (2.1)$$

Each cohomology group $H^q(L, M)$ is a finitely generated abelian group. Let $H^q(\Gamma, M)_{\text{tors}}$ be the torsion subgroup of $H^q(\Gamma, M)$ and let $H^q(\Gamma, M)_{\text{free}} =$
\[ H^q(\Gamma, M)/H^q(\Gamma, M)_{\text{tors}} \] be the free part. Then one has
\[ H^q(\Gamma, M) = H^q(\Gamma, M)_{\text{free}} \oplus H^q(\Gamma, M)_{\text{tors}}. \]

Now let \( V := M \otimes \mathbb{C} \) and \( V_\mathbb{R} := M \otimes \mathbb{R} \). Then \( V \) is a finite-dimensional complex vector space, \( V_\mathbb{R} \subset V \) is a real structure on \( V \) and \( M \) is a lattice in \( V_\mathbb{R} \). We regard \( \rho \) as a representation of \( \Gamma \) on \( V \). Then \( \rho \) is unimodular, i.e., \( |\det \rho(\gamma)| = 1 \) for all \( \gamma \in \Gamma \).

Let \( C^q(L, V) := C^q(\widetilde L; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} V \).

The \( C^q(L, V) \)'s form a chain complex \( C^*(L, V) \) of finite-dimensional \( \mathbb{C} \)-vector spaces and one has
\[ C^q(L, V) = C^q(L, M) \otimes_{\mathbb{Z}} \mathbb{C}. \] (2.2)

Let \( E := \tilde X \times_{\rho} V \) be the flat vector bundle over \( X \) associated to \( \rho \). Then by the de Rham isomorphism, the cohomology groups \( H^q(L, V) \) of the complex \( C^*(L, V) \) are canonically isomorphic to the cohomology groups \( H^q(X, E) \) of the complex of \( E \)-valued differential forms on \( X \). By Hodge theory they are canonically isomorphic to the space of \( E \)-valued harmonic forms for any choice of metrics on \( X \) and \( E \), respectively.

We assume that the bundle \( E \) is acyclic i.e. that \( H^q(L; V) = H^q(X; E) = 0 \) for all \( q \). This holds in all cases that we study in this paper. Let \( \sigma_j^q, j = 1, \ldots, r_q \), be the oriented \( q \)-simplices of \( L \) considered as a preferred basis of the \( \mathbb{Z}[\Gamma] \)-module \( C^q(\widetilde L; \mathbb{Z}) \). Let \( e_1, \ldots, e_m \) be a basis of \( M \). Then \( \{ \sigma_j^q \otimes e_k : j = 1, \ldots, r_q, k = 1, \ldots, m \} \) is a preferred basis of \( C^q(L; M) \) and also of \( C^q(L; V) \). Let \( \tau_X(\rho) \in \mathbb{R}^+ \) be the Reidemeister torsion with respect to these volume elements (see [8,11]). Note that \( \tau_X(\rho) = |\tau_X^\mathbb{C}(\rho)| \), where \( \tau_X^\mathbb{C}(\rho) \in \mathbb{C}^\times \) is the complex Reidemeister torsion. Since \( \rho \) is acyclic, \( \tau_X(\rho) \) is a combinatorial invariant of \( X \) and \( \rho \) which is independent of the choices that we made (see [11, section 1]). Moreover, each cohomology group \( H^q(\Gamma, M) \) is finite, i.e., \( H^q(\Gamma, M) = H^q(\Gamma, M)_{\text{tors}} \) and the order \( |H^q(\Gamma, M)| \) of these groups is related to the Reidemeister torsion as follows.

**Proposition 2.1** Assume that \( H^q(X, E) = 0 \) for all \( q \). Then one has
\[ \sum_{q=0}^d (-1)^{q+1} \log |H^q(\Gamma, M)| = \log \tau_X(\rho). \]

**Proof** Let \( C^*(L, V_\mathbb{R}) \) be the chain complex of the finite-dimensional real vector spaces
\[ C^q(L, V_\mathbb{R}) := C^q(\widetilde L; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} V_\mathbb{R}, \quad q = 0, \ldots, d. \]

We have
\[ C^q(L, V_\mathbb{R}) = C^q(L, M) \otimes_{\mathbb{Z}} \mathbb{R}. \]
Let $\rho_R : \Gamma \to \text{GL}(V_R)$ be the representation induced by $\rho$ and let $E_R := \tilde{X} \times_{\rho_R} V_R$ be the associated flat real vector bundle. Then $H^*(X; E_R) = 0$. The basis of the free $\mathbb{Z}$-module $C^q(L; M)$, described above, gives rise to a distinguished basis of $C^q(L; V_R)$. Let $\tau_X(\rho_R)$ be the Reidemeister torsion of the complex $C^*(L; V_R)$ with respect to volume elements defined by these bases. Then it follows from (2.1) and (2.2) as in [5, section 2.2] that

$$\log \tau_X(\rho_R) = \sum_{q=0}^{d} (-1)^{q+1} \log |H^q(\Gamma, M)|. \quad (2.3)$$

See also [8] and [16, Lemma 2.1.1]. Since the coboundary operators of the complexes $C^*(L; V_R)$ and $C^*(L; V)$, respectively, are induced by the coboundary operators of $C^*(L; M)$, it follows from the definition of the Reidemeister torsion that $\tau_X(\rho_R) = \tau_X(\rho)$. Combined with (2.3) the proposition follows.

Finally we recall some facts concerning linear algebraic groups. For all details we refer to [2]. Let $F$ be a finite Galois extension of $\mathbb{Q}$ with Galois group $\Sigma := \text{Gal}(F/\mathbb{Q})$. For $\sigma \in \Sigma$ and $x \in F$ let $x^\sigma$ denote the image of $x$ under $\sigma$. If $G$ is a linear algebraic group over $F$ with coordinate algebra $F[G]$, let $G^\sigma$ denote the linear algebraic group conjugate by $\sigma$, see [2]. If $G$ is realized as the zero set in some $F^n$ of an ideal $I$ in $F[x_1, \ldots, x_n]$, then $G^\sigma$ is the zero set of the ideal $I^\sigma$, where $I^\sigma$ is obtained from $I$ by applying $\sigma$ to each polynomial coefficient. Each $F$-rational homomorphism $\rho : G \to H$ of linear algebraic groups over $F$ induces canonically an $F$-rational homomorphism $\rho^\sigma : G^\sigma \to H^\sigma$.

If $G$ is an algebraic group defined over $F$, an algebraic group $G'$ defined over $\mathbb{Q}$ together with an $F$-rational isomorphism $\mu : G' \times_\mathbb{Q} F \to G$ is called a $\mathbb{Q}$-structure of $G$. The $\mathbb{Q}$-structure canonically induces an action of $\Sigma$ on the coordinate algebra of $G$ and thus on $G$ itself.

Let $V$ be a finite-dimensional $F$-vector space. A $\mathbb{Q}$-structure $V_0$ of $V$ is a $\mathbb{Q}$-subspace $V_0$ of $V$ such that the embedding $V_0 \hookrightarrow V$ induces an isomorphism $V_0 \otimes_\mathbb{Q} F \cong V$ of $F$-vector spaces. For $\sigma \in \Sigma$ a $\mathbb{Q}$-linear automorphism $A$ of $V$ is called $\sigma$-linear if $A(\lambda v) = \sigma(\lambda) A(v)$, $\lambda \in F$, $v \in V$. Then a semi-linear action of $\Sigma$ on $V$ is given by a family $\{f_\sigma\}_{\sigma \in \Sigma}$ of $\sigma$-linear automorphisms $f_\sigma$ of $V$, satisfying $f_{\sigma \tau} = f_\sigma \circ f_\tau$, $\sigma, \tau \in \Sigma$. Given a semi-linear action of $\Sigma$ on $V$, the set $V^\Sigma := \{v \in V : f_\sigma(v) = v, \forall \sigma \in \Sigma\}$ is a $\mathbb{Q}$-structure of $V$ and every $\mathbb{Q}$-structure is of this form (see [2, AG.14.2]). If $V_0$ is a $\mathbb{Q}$-structure of $V$, then $\text{GL}(V_0)$ is a $\mathbb{Q}$-structure of $\text{GL}(V)$ and the corresponding action of $\Sigma$ on $\text{GL}(V)$ is given by $\sigma \cdot g := f_\sigma \circ g \circ f_\sigma^{-1}$, $g \in \text{GL}(V)$.

3 Arithmetic subgroups of $SO^0(p, q)$

Let $p, q \in \mathbb{N}$ be odd. Put

$$p_1 = (p - 1)/2, \quad q_1 = (q - 1)/2, \quad n := p_1 + q_1.$$
We denote by $\text{SO}(p, q)$ the group of isometries of the standard quadratic form of signature $(p, q)$ on $\mathbb{R}^{p+q}$ with determinant 1. Let $\text{SO}^0(p, q)$ denote the identity component of $\text{SO}(p, q)$. The group $\text{SO}^0(p, q)$ is of fundamental rank one. Let $g$ be the Lie algebra of $\text{SO}^0(p, q)$. We choose the fundamental Cartan subalgebra as follows. Let $E_{i, j}$ be the $(p + q) \times (p + q)$-matrix which is one at the $i$-th row and $j$-th column and which is zero elsewhere. Let $H_1 := E_{p, p+1} + E_{p+1, p}$. (3.1)

and let

$$H_i := \begin{cases} \sqrt{-1}(E_{2i-3,2i-2} - E_{2i-2,2i-3}), & 2 \leq i \leq p_1 + 1 \\ \sqrt{-1}(E_{2i-1,2i} - E_{2i,2i-1}), & p_1 + 1 < i \leq n + 1. \end{cases}$$

(3.2)

Then

$$\mathfrak{h} := H_1 \oplus \bigoplus_{i=2}^{n+1} \sqrt{-1}H_i$$

is a Cartan subalgebra of $g$. Define $e_i \in \mathfrak{h}^*, \ i = 1, \ldots, n + 1$, by

$$e_i(H_j) = \delta_{i,j}, \ 1 \leq i, j \leq n + 1.$$

The finite-dimensional irreducible complex representations $\tau$ of $\text{SO}^0(p, q)$ are parametrized by their highest weights $\Lambda(\tau) \in \mathfrak{h}^*_\mathbb{C}$ given by

$$\Lambda(\tau) = k_1(\tau)e_1 + \cdots + k_{n+1}(\tau)e_{n+1}, \ (k_1(\tau), \ldots, k_{n+1}(\tau)) \in \mathbb{Z}^{n+1}, \ k_1(\tau) \geq k_2(\tau) \geq \cdots \geq k_n(\tau) \geq |k_{n+1}(\tau)|. \ (3.3)$$

For $\Lambda(\tau)$ a weight as in (3.3), the highest weight $\Lambda(\tau_\theta)$ of the representation $\tau \circ \theta$ is

$$\Lambda(\tau_\theta) = k_1(\tau)e_1 + \cdots + k_n(\tau)e_n - k_{n+1}(\tau)e_{n+1}. \ (3.4)$$

If we let

$$\omega_{j,n}^+ := \sum_{j=1}^{n+1} e_j, \ \omega_{f,n}^- := (\omega_{f,n}^+)_\theta = \sum_{j=1}^{n} e_j - e_{n+1}, \ (3.5)$$

then $\frac{1}{2}\omega_{f,n}^\pm$ are the fundamental weights which are not invariant under $\theta$. We now recall the construction of certain arithmetically defined cocompact subgroups of $\text{SO}^0(p, q)$. For more details see [14, section 3.2, Appendix B] and for the $\text{SO}^0(p, 1)$-case [9].
Let $F$ be a totally real number field of degree $d = [F : \mathbb{Q}] > 1$. Let $\Sigma$ be the Galois group of $F$ over $\mathbb{Q}$. We fix an embedding $F \subset \mathbb{R}$. Let $1 \in \Sigma$ be the identity. Let $\alpha_j \in F^\times, \ j = 1, \ldots, p + q$, be such that
\[
\text{sign}(\alpha_j) = \begin{cases} +1, & \text{if } j \leq p, \\ -1, & \text{if } p < j \leq p + q, \end{cases}
\]
and
\[
\text{sign}(\sigma(\alpha_j)) = +1, \quad \sigma \in \Sigma \setminus \{1\}, \ j = 1, \ldots, p + q.
\]
For $\sigma \in \Sigma$ let $Q^\sigma$ be the quadratic form on $\mathbb{R}^{p+q}$ defined by
\[
Q^\sigma(x) = \sum_{j=1}^{p+q} \sigma(\alpha_j)x_j^2.
\]
Then $Q := Q^1$ is a non-degenerate quadratic form of signature $(p, q)$ and $Q^\sigma, \ \sigma \neq 1$, is positive definite.

Let $G := \text{SO}_Q \subset \text{GL}_{p+q}$ be the special orthogonal group of $Q$, i.e., the subgroup of all elements of determinant one leaving $Q$ invariant. Then $G$ is a connected algebraic group defined over $F$. Let $J \in \text{GL}_{p+q}(\mathbb{R})$ be defined by
\[
J := \text{diag} \left( \sqrt{\alpha_1}, \ldots, \sqrt{\alpha_p}, \sqrt{-\alpha_{p+1}}, \ldots, \sqrt{-\alpha_{p+q}} \right).
\]
Then the map $g \mapsto JgJ^{-1}$ establishes an isomorphism $G(\mathbb{R}) \cong \text{SO}(p, q)$. Similarly, we have $G^\sigma(\mathbb{R}) \cong \text{SO}(p + q)$, if $\sigma \neq 1$. Let
\[
G' \cong \text{Res}_{F/\mathbb{Q}}(G)
\]
be the algebraic $\mathbb{Q}$-group obtained by restriction of scalars. There is a canonical isomorphism of algebraic groups over $F$
\[
\alpha : G' \times_{\mathbb{Q}} F \cong \prod_{\sigma \in \Sigma} G^\sigma,
\]
and the group of real points $G'(\mathbb{R})$ satisfies
\[
G'(\mathbb{R}) \cong \text{SO}(p, q) \times \prod_{\sigma \in \Sigma \setminus \{1\}} \text{SO}(p + q).
\]
Denote by
\[
\Delta : G \to \prod_{\sigma \in \Sigma} G^\sigma
\]
the diagonal embedding given by $\Delta(g) = (g^\sigma)_{\sigma \in \Sigma}$. 

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Let $O_F$ be the ring of integers of $F$ and let $G_{O_F}$ be the group of $O_F$-units of $G$. An arithmetic subgroups of $G(F)$ is by definition a subgroup which is commensurable with $G_{O_F}$. Let $\Gamma_0 := J G_{O_F} J^{-1}$. Then $\Gamma_0$ is a subgroup of $SO(p, q)$.

**Lemma 3.1** $\Gamma_0$ is a discrete, cocompact subgroup of $SO(p, q)$.

**Proof** For $\sigma \in \Sigma \setminus \{1\}$, the group $G^\sigma(\mathbb{R})$ is isomorphic to $SO(p + q)$. Thus by [3, p. 530], $\Gamma_0$ is discrete in $SO(p, q)$. Since all quadratic forms $Q^\sigma$, $\sigma \neq 1$, are positive definite, the form $Q$ is anisotropic over $F$. Thus, by [2, page 256] $G$ is anisotropic over $F$. Therefore, $G_{O_F}$ contains no non-trivial unipotent elements. Using [3, Lemma 11.4, Theorem 12.3], it follows that the diagonal image of $G_{O_F}$ in $\prod_{\sigma \in \Sigma} G^\sigma(\mathbb{R})$ is cocompact and since $G^\sigma(\mathbb{R})$ is compact for $\sigma \neq 1$, $\Gamma_0$ is also cocompact in $SO(p, q)$. 

**Remark 3.2** If $F = Q[\sqrt{v}]$ is a real quadratic field, then putting $\alpha_1 = \cdots = \alpha_p = 1$, $\alpha_{p+1} = \cdots = \alpha_{p+q} = -\sqrt{v}$ the above construction has already been given in [1, section 4.3].

Now we let $B$ be the symmetric bilinear form on $F^{p+q}$ given by

$$B(e_i, e_j) = \begin{cases} 1, & i + j = p + q + 1 \\ 0, & i + j \neq p + q + 1 \end{cases},$$

for $e_1, \ldots, e_{p+q}$ the standard base of $F^{p+q}$. Let $O_B$ be the orthogonal group of $B$ and let $SO_B$ be the elements of $O_B$ of determinant one. Then $O_B$ and $SO_B$ are algebraic groups defined over $F$ and there exists an isomorphism $\mu : G(\bar{F}) \to SO_B(\bar{F})$, i.e. $G$ is a form of $SO_B$ over $F$.

Let $T$ be the maximal torus of $SO_B(\bar{F})$ given by

$$T = \{ \text{diag}(\lambda_1, \ldots, \lambda_{n+1}, \lambda_1^{-1}, \ldots, \lambda_{n+1}^{-1}), \lambda_1, \ldots, \lambda_{n+1} \in \bar{F}^* \},$$

where $n = (p + q)/2 - 1$. Then $T$ is defined over $F$. Let $X(T)$ be the character group of $T$, written additively. Then a base of $X(T)$ is given by the $f_i : T \to \bar{F}$, $f_i(\text{diag}(\lambda_1, \ldots, \lambda_{n+1}, \lambda_1^{-1}, \ldots, \lambda_{n+1}^{-1})) = \lambda_i$, where $1 \leq i \leq n + 1$.

By $\text{Rep}(SO_B(\bar{F}))$ we denote the finite-dimensional representations of $SO_B(\bar{F})$ which are irreducible. Then the elements of $\text{Rep}(SO_B(\bar{F}))$ correspond bijectively to their highest weights $\lambda_\tau := m_1 f_1 + \cdots + m_{n+1} f_{n+1}$, where $m_1, \ldots, m_{n+1} \in \mathbb{Z}$, $m_1 \geq m_2 \geq \cdots \geq m_n \geq |m_{n+1}|$. Since $T$ is split over $F$, every finite-dimensional irreducible representation of $SO_B(\bar{F})$ is defined over $F$. [15, Proposition 2.3].

For $\tau \in \text{Rep}(SO_B(\bar{F}))$ with highest weight $\lambda_\tau = m_1(\tau) f_1 + \cdots + m_{n+1}(\tau) f_{n+1}$ let $\tau' \in \text{Rep}(SO_B(\bar{F}))$ be the element with highest weight $\lambda_{\tau'} = m_1(\tau) f_1 + \cdots + m_n f_n - m_{n+1}(\tau) f_{n+1}$. Then the following lemma holds.

**Lemma 3.3** For every $\tau \in \text{Rep}(SO_B(\bar{F}))$ there exists a representation $\tilde{\tau}$ of $O_B(\bar{F})$ that restricts to $\tau + \tau'$ on $SO_B(\bar{F})$. 

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Proof The proof of the corresponding proposition for $SO_B(\mathbb{C})$ given in [7, Theorem 5.22] extends without difficulty to any algebraically closed field of characteristic zero.

Remark 3.4 If $\tau$ satisfies $\tau = \tau'$ then there exists in fact a representation of $O_B$ that restricts to $\tau$. However, we are only interested in the case $\tau \neq \tau'$ and in this case the representation $\tilde{\tau}$ from the previous lemma is irreducible.

Now we let $PSO_B := SO_B /\{ \pm \text{Id} \}$. Then an element $\tau \in \text{Rep}(SO_B(\tilde{F}))$ of highest weight $\lambda_{\tau} = m_1 f_1 + \cdots + m_{n+1} f_n$ descends to a representation of $PSO_B(\tilde{F})$ if and only $m_1 + \cdots + m_{n+1}$ is even.

**Lemma 3.5** Let $\tau$ be a representation of $SO_B$ over $\tilde{F}$ which descends to a representation of $PSO_B$ over $\tilde{F}$. Then there exists an $F$-rational representation of $G$ which over $\tilde{F}$ is equivalent to $(\tau + \tau') \circ \mu$.

Proof For $\sigma \in \text{Gal}(\tilde{F}/F)$ define an automorphism $\phi_{\sigma}$ of $SO_B(\tilde{F})$ by $\phi_{\sigma} := \mu \circ \sigma \circ \mu^{-1} \circ \sigma^{-1}$. Since the Dynkin diagram $D_{n+1}$ has exactly one non-trivial automorphism, there is a natural isomorphism $\text{Aut}(SO_B(\tilde{F})) \cong PO_B(\tilde{F})$, where $PO_B(\tilde{F})$ acts on $SO_B(\tilde{F})$ by conjugation, and thus there exists a unique $a_{\sigma} \in PO_B(\tilde{F})$ such that for each $g \in SO_B(\tilde{F})$ one has $\phi_{\sigma}(g) = a_{\sigma} g a_{\sigma}^{-1}$. Thus one has

$$\mu(\sigma(g)) = a_{\sigma} \sigma(\mu(g)) a_{\sigma}^{-1}. \quad (3.9)$$

We can regard the assignment $\sigma \mapsto a_{\sigma}$ as an element of the first Galois-cohomology set $H^1(\text{Gal}(\tilde{F}/F), PO_B(\tilde{F}))$. By Lemma 3.3 there exist a representation $\tilde{\tau}$ of $PO_B(\tilde{F})$ on $V_{\tilde{\tau}} = V_\tau \oplus V_{\tau'}$ which restricts to $\tau \oplus \tau'$. Then $\sigma \mapsto \tilde{\tau}(a_{\sigma})$ is an element of $H^1(\text{Gal}(\tilde{F}/F), GL(V_{\tilde{\tau}}))$ and since this set is trivial by Hilbert’s theorem 90, there exists an $x \in GL(V_{\tilde{\tau}})$ such that

$$\tilde{\tau}(a_{\sigma}) = x^{-1} \sigma(x) \quad \forall \sigma \in \text{Gal}(\tilde{F}/F). \quad (3.10)$$

Now define a representation $\rho$ of $G(\tilde{F})$ by $\rho := \text{Int}(x) \circ (\tau + \tau') \circ \mu$. Then $\rho$ is equivalent to $(\tau + \tau') \circ \mu$. Applying (3.9) and (3.10) it follows that for $\sigma \in \text{Gal}(\tilde{F}/F)$ and $g \in G(\tilde{F})$ one has

$$\rho(\sigma(g)) = \alpha \tilde{\tau}(a_{\sigma})(\tau + \tau')(\sigma(\mu(g)))\tilde{\tau}(a_{\sigma}^{-1}) \alpha^{-1}$$

$$= \sigma(x) \sigma((\tau + \tau')(\mu(g))) \alpha(x)^{-1} = \sigma(\rho(g)),$$

where we used that $\tau + \tau'$ is defined over $F$ and hence commutes with $\text{Gal}(\tilde{F}/F)$. Thus $\rho$ commutes with $\text{Gal}(\tilde{F}/F)$ and thus it is defined over $F$. \qed

Now we may fix an embedding of $SO^0(p, q)$ into $SO_B(\mathbb{C})$ such that the representations of $SO^0(p, q)$ with highest weight $m_1 e_1 + \cdots + m_{n+1} e_{n+1}$ are the restrictions to $SO^0(p, q)$ of the representation of $SO_B(\mathbb{C})$ with highest weight $m_1 f_1 + \cdots + m_{n+1} f_{n+1}$.

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The following proposition is certainly well known and was used already by Bergeron and Venkatesh [5, section 8.1]. However, for the sake of completeness we include a proof here. If $V$ is a finite-dimensional $F$-vector space, let $V^\sigma$ be the $F$-vector space with scalar-multiplication $a \cdot v := \sigma(a)v$, $a \in F$, $v \in V$.

**Lemma 3.6** Let $G'$ be an algebraic group defined over $\mathbb{Q}$. Let $V$ be a finite-dimensional $F$-vector space and let $\rho : G' \to \text{GL}(V)$ be a representation defined over $F$. Then $\tilde{\rho} := \prod_{\sigma \in \Sigma} \rho^{-1}$ is defined over $\mathbb{Q}$, where $\rho^{-1}$ is regarded as an $F$-rational representation of $G'$ on $V^\sigma$.

**Proof** Each $\sigma \in \Sigma$ acts on $\prod_{\sigma \in \Sigma} V^\sigma$ as a $\sigma$-linear automorphism by permuting the factors. The corresponding $\mathbb{Q}$-structure of $\prod_{\sigma \in \Sigma} V^\sigma$ is $V$, regarded as a $\mathbb{Q}$-vector space and embedded diagonally into $\prod_{\sigma \in \Sigma} V^\sigma$. Now it is easy to see that $\tilde{\rho}$ commutes with the action of $\Sigma$ on $G'$ and the action of $\Sigma$ on $\text{GL}(\prod_{\sigma \in \Sigma} V^\sigma)$ associated to this $\mathbb{Q}$-structure. Thus $\tilde{\rho}$ is defined over $\mathbb{Q}$. $\square$

Let $G'(\mathbb{R})^0$ be the connected component of $1 \in G'(\mathbb{R})$ and $G'(\mathbb{R})^c := \prod_{\sigma \in \Sigma \setminus \{1\}} \text{SO}(p + q)$. Then we have

$$G'(\mathbb{R})^0 \cong \text{SO}^0(p, q) \times G'(\mathbb{R})^c.$$ 

Let $\theta$ be the standard Cartan-involution of $\text{SO}^0(p, q)$. Then $\theta \otimes \text{Id}_{G'(\mathbb{R})^c}$ is a Cartan involution of $G'(\mathbb{R})^0$ which we continue to denote by $\theta$. By $\text{Rep}(G'(\mathbb{R})^0)$ we denote the finite-dimensional irreducible complex representations of $G'(\mathbb{R})^0$. For $\tau \in \text{Rep}(G'(\mathbb{R})^0)$, let $\tau_\theta$ be the element of $\text{Rep}(G'(\mathbb{R})^0)$ defined by $\tau_\theta := \tau \circ \theta$.

**Proposition 3.7** There exists a sequence $\rho(m)$ of $\mathbb{Q}$-rational representations of $G'$ on finite-dimensional $\mathbb{Q}$-vector spaces $V_{\rho(m)}$ such that

1. For the decomposition

$$\rho(m) = \bigoplus_{\tau \in \text{Rep}(G'(\mathbb{R})^0)} [\rho(m) : \tau] \tau,$$

$$[\rho(m) : \tau] \in \mathbb{N}^0 \text{ of } \rho(m), \text{ regarded as a complex representation of } G'(\mathbb{R})^0 \text{ on the vector space } V_{\rho(m)} \otimes \mathbb{Q} \mathbb{C}, \text{ into irreducible representations of } G'(\mathbb{R})^0 \text{ one has } \tau \neq \tau_\theta \text{ for each } \tau \in \text{Rep}(G'(\mathbb{R})^0) \text{ with } [\rho(m) : \tau] \neq 0.$$

2. The dimension $\dim(V_{\rho(m)})$ is a polynomial in $m$ with leading term

$$\dim(V_{\rho(m)}) = C \ d \ m^{d(n+1)/2} + O(m^{d(n+1)/2-1}),$$

as $m \to \infty$, where $C > 0$ is a constant which depends only on $n$.

**Proof** The Galois group $\Sigma$ acts on $\prod_{\sigma \in \Sigma} G^\sigma$ as follows. For $g \in \prod_{\sigma \in \Sigma} G^\sigma$ and $\sigma \in \Sigma$ we denote the projection of $g$ to $G^\sigma$ by $g_\sigma$. Then for $\sigma, \sigma' \in \Sigma$ one has

$$(\sigma g)_{\sigma'} = \sigma(g_{\sigma^{-1} \sigma'}).$$
Now assume that for each $\sigma \in \Sigma$ we are given a finite-dimensional $F$-vector space $V_{\rho(\sigma)}$ and a representation $\rho(\sigma)$ of $G^\sigma$ on $V_{\rho(\sigma)}$, defined over $F$. Then the tensor-product

$$\rho := \bigotimes_{\sigma \in \Sigma} \rho(\sigma)$$

is a representation of $\prod_{\sigma \in \Sigma} G^\sigma$ on $\bigotimes_{\sigma \in \Sigma} V_{\rho(\sigma)}$ and it follows that for $\sigma' \in \Sigma$ one has

$$\rho^{\sigma'} = \bigotimes_{\sigma \in \Sigma} \rho((\sigma')^{-1}\sigma)^{\sigma'}.$$  \hfill (3.12)

Now if $n$ is even, for $m \in \mathbb{N}$ we let $\tau(m)$ be the representation of $G$ over $\tilde{F}$ of highest weight $2m e_1 + \cdots + 2m e_{n+1}$. If $n$ is odd, we let $\tau(m)$ be the representation of highest weight $me_1 + \cdots + me_{n+1}$. Then $\tau(m)$ and $\tau(m)_{\theta}$ descend to representations of $PG$. Thus by Lemma 3.5 there exists a representation of $G$ over $F$ which over $\tilde{F}$ is equivalent to $\tau(m)+\tau(m)_{\theta}$. Thus if we define $\rho_0(m)$ by

$$\rho_0(m) := \bigotimes_{\sigma \in \Sigma} (\tau(m)+\tau(m)_{\theta})^{\sigma},$$ \hfill (3.13)

then $\rho_0(m)$ is defined over $F$ and by (3.12), $\rho_0(m)$ is equivalent to $\rho_0(\sigma)^{\sigma}$ for each $\sigma \in \Sigma$. Hence if we let $\rho(m)$ be the direct sum of $d$ copies of $\rho_0(m)$ then by Lemma 3.6 $\rho(m)$ is defined over $\mathbb{Q}$. Each irreducible component of $\rho(m)|_{G'(\mathbb{R})^0}$, regarded as a complex representation of $G'(\mathbb{R})^0$ on $V_{\rho(m)} \otimes_{\mathbb{Q}} \mathbb{C}$, is of the form $\tau(m) \otimes \pi$, or $\tau(m)_{\theta} \otimes \pi'$, where $\pi$ and $\pi'$ are irreducible representations of $G'(\mathbb{R})^c$. Since $\tau(m)$ and $\tau(m)_{\theta}$ are not $\theta$-invariant, the same holds for each irreducible component of $\rho(m)|_{G'(\mathbb{R})^0}$. This proves the first statement. The second statement follows from Weyl’s dimension formula.

We can now turn to the proof of Theorem 1.1. Let $\Delta$ be the diagonal embedding of $G$ into $\prod_{\sigma \in \Sigma} G^\sigma$. Then we can choose the isomorphism $\alpha$ in (3.7) such that $\alpha(G_{\Sigma}^*) = \Delta(G_{O_F})$. Let $\Gamma \subset G_{O_F}$ be a subgroup of finite index. Via the isomorphism $G'(\mathbb{R}) \cong SO(p,q)$ we identify $\Gamma$ with a subgroup of $SO(p,q)$. We choose $\Gamma$ such that it is torsion free and is contained in $SO^0(p,q)$. By Lemma 3.1, $\Gamma$ is a cocompact lattice in $SO^0(p,q)$. Let $\Gamma' = \Delta(\Gamma)$. Since $\Gamma$ and $\Gamma'$ are isomorphic, it suffices to prove the statements of Theorem 1.1 for $\Gamma'$.

The group $K_0 := SO(p) \times SO(q)$ is a a maximal compact subgroup of $SO^0(p,q)$. Put

$$K' := K_0 \times \prod_{\sigma \notin \Sigma} SO(p+q).$$

Then $K'$ is a maximal compact subgroup of $G'(\mathbb{R})^0$. Put $\tilde{X}' := G'(\mathbb{R})^0/K'$ and $X' := \Gamma' \backslash \tilde{X}'$. Let $(\rho(m), V_{\rho(m)})$ be the sequence of $\mathbb{Q}$-rational representations of $G'$.
of Proposition 3.7. Since each \( \rho(m) \) is defined over \( \mathbb{Q} \), there exists a free \( \mathbb{Z} \)-module \( M_{\rho(m)} \) in \( V_{\rho(m)} \) which is stable under \( \rho(m)(\Gamma') \) and such that \( M_{\rho(m)} \otimes_{\mathbb{Z}} \mathbb{Q} \cong V_{\rho(m)} \), see for example [12, page 173]. Let \( V_{\rho(m)}^\mathbb{C} := V_{\rho(m)} \otimes_{\mathbb{Q}} \mathbb{C} \). Then the restriction of \( \rho(m) \) to \( \Gamma' \) induces the flat complex vector bundle

\[
E_{\rho(m)} := \tilde{X}' \times_{\Gamma'} V_{\rho(m)}^\mathbb{C}
\]

over \( X' \). The decomposition (3.11) of \( \rho(m) \) induces a corresponding decomposition of \( E_{\rho(m)} \) into the direct sum of complex vector bundles associated to the restriction to \( \Gamma' \) of irreducible finite-dimensional representations \( \tau \) of \( G'(\mathbb{R})^0 \). By Proposition 3.7, each \( \tau \) with \([\rho(m) : \tau] \neq 0\) satisfies \( \tau \neq \tau_\theta \) and thus by [6, Chapter VII, Theorem 6.7] one has

\[
H^*(X'; E_{\rho(m)}) = 0,
\]

(3.14)

where \( H^*(X'; E_{\rho(m)}) \) denotes the de Rham cohomology with coefficients in \( E_{\rho(m)} \). Chose a Hermitian fibre metric in \( E_{\rho(m)} \). Let \( TX'(\rho(m)) \) be the analytic torsion of \( X' \) and \( \rho(m) \) [see (1.9)]. It follows from (3.14) that \( TX'(\rho(m)) \) is metric independent [11, Corollary 2.7]. Moreover \( H^*(\Gamma', M_{\rho(m)}) \) is a finite abelian group and by (1.10) we have

\[
\log TX'(\rho(m)) = \sum_{q=0}^{n} (-1)^{q+1} \log |H^q(\Gamma', M_{\rho(m)})|.
\]

(3.15)

This equality reduces the proof of Theorem 1.1 to the study of the asymptotic behavior of \( TX'(\rho(m)) \) as \( m \to \infty \), which is exactly the problem that has been dealt with in [10]. Since \( \{\rho(m)\} \) is not a sequence of representations that has been considered in [10], we cannot apply the results of [10] directly. We first need to reduce it to a case to which [10] can be applied. Let \( \rho_0(m) \) be defined by (3.13) and let \( TX'(\rho_0(m)) \) be the corresponding analytic torsion. Since \( \rho(m) \) is the direct sum of \( d \) copies of \( \rho_0(m) \), we get

\[
\log TX'(\rho(m)) = d \log TX'(\rho_0(m))
\]

Now let \( TX'(\rho_0(m)) \) be the \( L^2 \)-torsion with respect to \( \rho_0(m) \) (see [10, section 5]). If we apply [10, Proposition 1.2] to the irreducible components of \( \rho_0(m) \), it follows that there exists \( c > 0 \) such that

\[
\log TX'(\rho_0(m)) = \log TX'(\rho_0(m)) + O(e^{-cm}),
\]

as \( m \to \infty \). Using the definition of \( \rho_0(m) \) by (3.13), [10, (5.21)] and [10, Proposition 5.3], it follows that

\[
\log TX'(\rho_0(m)) = \left( \log TX'(\tau(m)) + \log TX'(\tau(m)_\theta) \right) (2 \dim \tau(m))^{d-1}.
\]
If \( C_{p,q} \) is as in (1.3), then by [10, Proposition 6.7] one has
\[
\log T_{X'}^{(2)}(\tau(m)) = \log T_{X'}^{(2)}(\tau(m)\theta) = C_{p,q} \operatorname{vol}(X') m \dim \tau(m) + O(\dim(\tau(m))),
\]
as \( m \to \infty \). Thus putting everything together we obtain
\[
\log T_{X'}(\rho(m)) = C_{p,q} \operatorname{vol}(X') m \dim(\rho(m)) + O(\dim(\rho(m))),
\]
as \( m \to \infty \). Since \( X \cong X' \) and \( H^*(\Gamma, M_{\rho(m)}) \cong H^*(\Gamma', M_{\rho(m)}) \), Theorem 1.1 follows from (3.15) and the second statement of Proposition 3.7.

4 Arithmetic subgroups of \( \text{SL}_3(\mathbb{R}) \)

Let \( D \) be a nine-dimensional division algebra over \( \mathbb{Q} \). Then by the Brauer-Hasse-Noether theorem [13], \( D \) is a cyclic algebra for a cubic extension \( L \) of \( \mathbb{Q} \). Moreover, \( L \) splits \( D \), i.e. there exists an isomorphism of \( L \)-algebras
\[
\phi : D \otimes \mathbb{Q} L \cong \text{Mat}_{3 \times 3}(L). \tag{4.1}
\]
Thus for \( x \in D \) the reduced norm \( N(x) \) is given by \( N(x) := \det(\phi(x \otimes 1)) \). Now let \( G := \text{SL}_1(D) \), where
\[
\text{SL}_1(D) := \{ x \in D : N(x) = 1 \}.
\]
Then by [12, 2.3.1], \( G \) has a canonical structure of an algebraic group defined over \( \mathbb{Q} \). We regard \( \text{SL}_3 \) as an algebraic group over \( \mathbb{Q} \). The isomorphism \( \phi \) from (4.1) induces an isomorphism
\[
\phi : G(L) \cong \text{SL}_3(L),
\]
i.e. \( G \) is a form of \( \text{SL}_3 \) over \( L \). Moreover, the following Lemma holds.

**Lemma 4.1** Let \( \rho \) be a \( \mathbb{Q} \)-rational representation of \( \text{SL}_3 \) which is trivial on the center of \( \text{SL}_3 \). Then there exists a \( \mathbb{Q} \)-rational representation of \( G \) which over \( L \) is equivalent to \( \rho \circ \phi \).

**Proof** By [12, Proposition 2.17], \( G \) is an inner form of \( \text{SL}_3 \). Thus the proof of Lemma 3.1 in [8] can be generalized without difficulties to prove the Lemma.  \( \square \)

Let \( \mathfrak{o} \) be an order in \( D \), i.e., \( \mathfrak{o} \) is a free \( \mathbb{Z} \)-submodule of \( D \) which is generated by a \( \mathbb{Z} \)-base of \( D \) and which is also a subring of \( D \). Put
\[
\mathfrak{o}^1 := \{ x \in \mathfrak{o} : N(x) = 1 \}.
\]
The left regular representation of $D$ on itself induces a $\mathbb{Q}$-rational representation of $G$ on $D$, see [12, 2.3.1] and the stabilizer of $\sigma$ is $\mathfrak{o}^1$. Thus $\mathfrak{o}^1$ is arithmetic subgroup of $G(\mathbb{Q})$. Put

$$\Gamma := \phi(\mathfrak{o}^1).$$

Then $\Gamma$ is an arithmetic subgroup of $\text{SL}_3(\mathbb{R})$. Moreover, the following lemma holds.

**Lemma 4.2** The group $\Gamma$ is a cocompact subgroup of $\text{SL}_3(\mathbb{R})$.

**Proof** By [12, Proposition 2.12], $G$ is anisotropic over $\mathbb{Q}$. Thus the proposition follows from [3, Lemma 11.4, Theorem 11.8]. $\square$

Let $T$ be the standard maximal torus in $\text{SL}_3$ consisting of the diagonal matrices of determinant 1. Then $T$ is defined over $\mathbb{Q}$ and is $\mathbb{Q}$-split. Let $t$ be the Lie-algebra of $T(\mathbb{R})$ consisting of all diagonal matrices of trace 0. Let $e_i \in t^*$ be defined by $e_i(\text{diag}(t_1, t_2, t_3)) := \sum_{j=1}^{3} \delta_{i,j}t_j$. Then with respect to the standard ordering of the roots of $t_3$ in $\text{sl}_3(\mathbb{C})$ the fundamental weights $\omega_1, \omega_2 \in t_3^*$ are given by

$$\omega_1 = \frac{2}{3}(e_1 - e_2) + \frac{1}{3}(e_2 - e_3); \quad \omega_2 = \frac{1}{3}(e_1 - e_2) + \frac{2}{3}(e_2 - e_3). \quad (4.2)$$

The finite-dimensional irreducible representations $\tau$ of $\text{SL}_3(\mathbb{R})$ correspond to the finite dimensional irreducible holomorphic representations of $\text{SL}_3(\mathbb{C})$ and are parametrized by their highest weights $\Lambda_\tau = \tau_1 \omega_1 + \tau_2 \omega_2$. Each such representation of $\text{SL}_3(\mathbb{R})$ is defined over $\mathbb{R}$. If $\theta$ is the standard Cartan involution of $\text{SL}_3(\mathbb{R})$, then the highest weight of the representation $\tau_\theta := \tau \circ \theta$ is given by $\Lambda_{\tau_\theta} = \tau_2 \omega_1 + \tau_1 \omega_2$. The center of $\text{SL}_3(\mathbb{C})$ has order 3 and is generated by $\exp(X)$, where $X \in t_3$ is given by $X := \frac{2\pi i}{3} \text{diag}(1, 1, -2)$. Thus $\tau$, regarded as a representation of $\text{SL}_3(\mathbb{C})$, is trivial on the center of $\text{SL}_3(\mathbb{C})$ if and only if $\tau_1 + 2\tau_2 \equiv 0 \bmod 3$.

**Proposition 4.3** Let $\tau$ be a finite-dimensional irreducible representation of $\text{SL}_3(\mathbb{R})$ on a finite-dimensional real vector space $V_\tau$ and assume that the corresponding representation of $\text{SL}_3(\mathbb{C})$ is trivial on the center of $\text{SL}_3(\mathbb{C})$. Then there exists a lattice $M$ in $V_\tau$ which is invariant under $\tau(\Gamma)$.

**Proof** Since $T$ is $\mathbb{Q}$-split $\tau$ is defined over $\mathbb{Q}$ [15, Proposition 2.3]. By Lemma 4.1, there exists a rational representation $\tau'$ of $G$ on a finite-dimensional $\mathbb{Q}$-vector space $V(\tau')$ which over $L$ is equivalent to $\tau \circ \phi$. Since $\mathfrak{o}^1$ is an arithmetic subgroup of $G(\mathbb{Q})$, there exists a lattice in $V(\tau')$ which is stable under $\tau'(\mathfrak{o}^1)$, see for example [12, page 173]. Since $\Gamma = \phi(\mathfrak{o}^1)$, the Proposition follows. $\square$

We can now turn to the proof of Theorem 1.3. Let $\tilde{X} = \text{SL}_3(\mathbb{R})/\text{SO}(3)$ and $X = \Gamma \backslash \tilde{X}$, where $\Gamma \subset \text{SL}_3(\mathbb{R})$ is an arithmetic subgroup as above. Chose a $\text{SL}_3(\mathbb{R})$-invariant metric on $\tilde{X}$ and equip $X$ with the induced metric. Let $\Lambda \in \mathfrak{h}^*_C$ be a highest weight. Assume that $\Lambda$ satisfies $\Lambda \neq \Lambda_\theta$ and that the representation of $\text{SL}_3(\mathbb{C})$ of highest weight $\Lambda$ is trivial on the center of $\text{SL}_3(\mathbb{C})$. Then the same holds for each weight $m\Lambda$, $m \in \mathbb{N}$. Let $\tau_\Lambda(m)$ be the irreducible finite-dimensional representation on $V_\Lambda(m)$.
with highest weight $m \Lambda$. Let $E_{\tau_\Lambda(m)}$ be the flat vector bundle over $X$ associated to $\tau_\Lambda(m)$. By [6, Chapter VII, Theorem 6.7] we have

$$H^*(X, E_{\tau_\Lambda(m)}) = 0. \quad (4.3)$$

Let $T_X(\tau_\Lambda(m))$ be the analytic torsion with respect to any Hermitian fibre metric in $E_{\tau_\Lambda(m)}$. By (4.3) and [11, Corollary 2.7], $T_X(\tau_\Lambda(m))$ is independent of the choice of metrics on $X$ and in $E_{\tau_\Lambda(m)}$. Let $M_\Lambda(m) \subset V_\Lambda(m)$ be an arithmetic $\Gamma$-module, which exists by Proposition 4.3. By (4.3), $H^*(X, M_\Lambda(m))$ is a finite abelian group and by (1.10) we have

$$\log T_X(\tau_\Lambda(m)) = \sum_{q=0}^{5} (-1)^{q+1} \log |H^q(X, M_\Lambda(m))|.$$  

(4.4)

Using Theorem 1.1 and Corollary 1.5 of [10], the proof of Theorem 1.3 follows.

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