Tomaszewski’s problem on randomly signed sums: breaking the 3/8 barrier

Ravi B. Boppana
Department of Mathematics
M. I. T.
Massachusetts, USA
rboppana@mit.edu

Ron Holzman
Department of Mathematics
Technion–Israel Institute of Technology
Israel
holzman@tx.technion.ac.il

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Abstract

Let $v_1, v_2, \ldots, v_n$ be real numbers whose squares add up to 1. Consider the $2^n$ signed sums of the form $S = \sum \pm v_i$. Holzman and Kleitman (1992) proved that at least $\frac{3}{8}$ of these sums satisfy $|S| \leq 1$. This $\frac{3}{8}$ bound seems to be the best their method can achieve. Using a different method, we improve the bound to $\frac{13}{32}$, thus breaking the $\frac{3}{8}$ barrier.

Keywords: combinatorial probability; probabilistic inequalities

1 Introduction

Let $v_1, v_2, \ldots, v_n$ be real numbers such that the sum of their squares is at most 1. Consider the $2^n$ signed sums of the form $S = \sum \pm v_i$. In 1986, B. Tomaszewski (see Guy [3]) asked the following question: is it always true that at least $\frac{1}{2}$ of these sums satisfy $|S| \leq 1$? Most examples with $n = 2$ and $v_1^2 + v_2^2 = 1$ show that $\frac{1}{2}$ can’t be replaced with a bigger number.

Holzman and Kleitman [7] proved that at least $\frac{3}{8}$ of the sums satisfy $|S| \leq 1$. This result was an immediate consequence of their main result: at least $\frac{3}{8}$ of the sums satisfy the strict inequality $|S| < 1$, provided that each $|v_i|$ is strictly less than 1. This $\frac{3}{8}$ bound for $|S| < 1$ is best possible: consider the example with $n = 4$ and $v_1 = v_2 = v_3 = v_4 = \frac{1}{2}$. So $\frac{3}{8}$ seems to be a natural barrier to their method of proof.

Using a different method, we prove that more than $\frac{13}{32}$ of the sums satisfy $|S| \leq 1$. In other words, we break the $\frac{3}{8}$ barrier. Our method, roughly speaking, goes like this. We will let the first few $\pm$ signs be arbitrary. But once the partial sum becomes near 1 in
absolute value, we will show that the final sum still has a decent chance of remaining at most 1 in absolute value.

We can actually improve the \( \frac{13}{32} \) bound a tiny bit, to \( \frac{13}{32} + 9 \times 10^{-6} \). Combining our method with other ideas, which could handle the tight cases for our analysis, may lead to further improvements of the bound. Still, the conjectured lower bound of \( \frac{1}{2} \) currently appears to be out of reach.

Ten years after Holzman and Kleitman [7] but independently, Ben-Tal, Nemirovski, and Roos [1] proved that at least \( \frac{1}{2} \) of the sums satisfy \( |S| \leq 1 \); they say that the proof is mainly due to P. van der Wal. Shnurnikov [9] refined the argument of [1] to prove a 36% bound. Even though these two bounds are weaker than that of Holzman and Kleitman, the methods used to prove them are noteworthy. In particular, we will use the conditioning argument of [1] and the fourth moment method of [9].

Let Tomaszewski’s constant be the largest constant \( c \) such that the fraction of sums that satisfy \( |S| \leq 1 \) is always at least \( c \). We now know that Tomaszewski’s constant is between \( \frac{13}{32} \) and \( \frac{1}{2} \). Both [7] and [1] conjecture that Tomaszewski’s constant is \( \frac{1}{2} \). De, Diakonikolas, and Servedio [2] developed an algorithm to approximate Tomaszewski’s constant. Specifically, given an \( \epsilon > 0 \), their algorithm will output a number that is within \( \epsilon \) of Tomaszewski’s constant. The running time of their algorithm is exponential in \( 1/\epsilon^3 \), so it’s not clear that we can run their algorithm in a reasonable amount of time to improve the known bounds on Tomaszewski’s constant.

The conjectured lower bound of \( \frac{1}{2} \) has been confirmed in some special cases. For example, von Heymann [6] and Hendriks and van Zuijlen [5] proved the conjecture when \( n \leq 9 \). Also, van Zuijlen [10] and von Heymann [6] proved the conjecture when all of the \( |v_i| \) are equal.

We will use the language of probability. Let \( \Pr[A] \) be the probability of an event \( A \). Let \( E(X) \) be the expected value of a random variable \( X \). A random sign is a random variable whose probability distribution is the uniform distribution on the set \{−1, +1\}. With this language, we can restate our main result.

**Main Theorem.** Let \( v_1, v_2, \ldots, v_n \) be real numbers such that \( \sum_{i=1}^{n} v_i^2 \) is at most 1. Let \( a_1, a_2, \ldots, a_n \) be independent random signs. Let \( S = \sum_{i=1}^{n} a_i v_i \). Then \( \Pr[|S| \leq 1] > \frac{13}{32} \).

In Section 2 of this paper, we will provide a short proof of a bound better than \( \frac{3}{8} \). In Section 3 we will refine the analysis to improve the bound to \( \frac{13}{32} \) and slightly beyond.

## 2 Beating the 3/8 bound

In this section, we will give the simplest proof we can of a bound better than \( \frac{3}{8} \). Namely, we will prove a bound of \( \frac{37}{98} \), which is a little more than 37.75%. In Section 3 we will improve the bound further.

We begin with a lemma. Roughly speaking, this lemma can be used to show that if a partial sum is a little less than 1, then the final sum has a decent chance of remaining less than 1 in absolute value.
Lemma 1. Let $x$ be a real number such that $|x| \leq 1$. Let $v_1, v_2, \ldots, v_n$ be real numbers such that
\[ \sum_{i=1}^{n} v_i^2 \leq \frac{2}{7} (1 + |x|)^2. \]
Let $a_1, a_2, \ldots, a_n$ be independent random signs. Let $Y$ be $\sum_{i=1}^{n} a_i v_i$. Then
\[ \Pr[|x + Y| \leq 1] \geq \frac{37}{98}. \]

Proof. By symmetry, we may assume that $x \geq 0$. The fourth moment of $Y$ is
\[ \mathbb{E}(Y^4) = 3 \left( \sum_{i=1}^{n} v_i^2 \right)^2 - 2 \sum_{i=1}^{n} v_i^4 \leq 3 \left( \sum_{i=1}^{n} v_i^2 \right)^2 \leq \frac{12}{49} (1 + x)^4. \]
So, by the fourth moment version of Chebyshev’s inequality\footnote{Shnurnikov [9] used the fourth moment in a similar situation.},
\[ \Pr[|Y| \geq 1 + x] \leq \frac{\mathbb{E}(Y^4)}{(1 + x)^4} \leq \frac{12}{49}. \]
Looking at the complement,
\[ \Pr[|Y| < 1 + x] \geq \frac{37}{49}. \]
Because $Y$ has a symmetric distribution,
\[ \Pr[-1 - x < Y \leq 0] \geq \frac{1}{2} \Pr[|Y| < 1 + x] \geq \frac{37}{98}. \]
Recall that $x \leq 1$. Hence if $-1 - x < Y \leq 0$, then $|x + Y| \leq 1$. Therefore
\[ \Pr[|x + Y| \leq 1] \geq \Pr[-1 - x < Y \leq 0] \geq \frac{37}{98}. \]

Next we will use Lemma\footnote{Shnurnikov [9] used the fourth moment in a similar situation.} to go beyond the $\frac{3}{8}$ bound.

Theorem 2. Let $v_1, v_2, \ldots, v_n$ be real numbers such that $\sum_{i=1}^{n} v_i^2$ is at most 1. Let $a_1, a_2, \ldots, a_n$ be independent random signs. Let $S$ be $\sum_{i=1}^{n} a_i v_i$. Then
\[ \Pr[|S| \leq 1] \geq \frac{37}{98}. \]

Proof. By inserting 0’s, we may assume that $n \geq 4$. By permuting, we may assume that the four largest $|v_i|$ are $|v_n| \geq |v_1| \geq |v_{n-1}| \geq |v_2|$. By the quadratic mean inequality,
\[ \frac{|v_1| + |v_2| + |v_{n-1}| + |v_n|}{4} \leq \sqrt{\frac{v_1^2 + v_2^2 + v_{n-1}^2 + v_n^2}{4}} \leq \sqrt{\frac{1}{4}} = \frac{1}{2}. \]
So $|v_1| + |v_2| + |v_{n-1}| + |v_n| \leq 2$. Because of our ordering,
\[
|v_1| + |v_2| \leq \frac{|v_1| + |v_n|}{2} + \frac{|v_2| + |v_{n-1}|}{2} \leq 1.
\]

Given an integer $t$ from 0 to $n$, let $X_t$ be the partial sum $\sum_{i=t+1}^{n} a_i v_i$ and let $Y_t$ be the remaining sum $\sum_{i=1}^{n-t} a_i v_i$. Let $T$ be the smallest nonnegative integer $t$ such that $t = n - 1$ or $|X_t| > 1 - |v_{t+1}|$. In a stochastic process such as ours, $T$ is called a stopping time, defined by the stopping rule in the previous sentence. Note that $T \geq 2$, since $|v_1| + |v_2| \leq 1$. By the stopping rule, $|X_{T-1}| \leq 1 - |v_T|$. Hence by the triangle inequality,
\[
|X_T| \leq |X_{T-1}| + |v_T| \leq 1 - |v_T| + |v_T| = 1.
\]

Also by the stopping rule, if $T < n - 1$, then $|X_T| > 1 - |v_{T+1}|$.

We will condition on $T$ and $X_T$. We claim that
\[
\Pr[|S| \leq 1 \mid T, X_T] \geq \frac{37}{98}.
\]

By averaging over $T$ and $X_T$, this claim implies the theorem. To prove the claim, we may assume by symmetry that $X_T \geq 0$. We will divide the proof of the claim into three cases, depending on $T$.

**Case 1:** $T = n - 1$. In this case, $|X_T| = |v_n| \leq 1$. Recall that $0 \leq X_T \leq 1$. Hence if $Y_T \leq 0$, then $|S| = |X_T + Y_T| \leq 1$. Therefore by symmetry,
\[
\Pr[|S| \leq 1 \mid T, X_T] \geq \Pr[Y_T \leq 0 \mid T, X_T] \geq \frac{1}{2}.
\]

**Case 2:** $T = n - 2$. In this case,
\[
|Y_T| \leq |v_{n-1}| + |v_n| \leq 2 - |v_1| \leq 2 - |v_{n-1}|.
\]

Recall that $1 - |v_{n-1}| < X_T \leq 1$. Hence if $Y_T \leq 0$, then $|S| = |X_T + Y_T| \leq 1$. Therefore by symmetry,
\[
\Pr[|S| \leq 1 \mid T, X_T] \geq \Pr[Y_T \leq 0 \mid T, X_T] \geq \frac{1}{2}.
\]

**Case 3:** $T \leq n - 3$. In this case, by the stopping rule,
\[
\sum_{i=T+1}^{n} v_i^2 \leq 1 - \sum_{i=1}^{T} v_i^2 \leq 1 - v_1^2 - v_2^2 \leq 1 - 2v_{T+1}^2 < 1 - 2(1 - X_T)^2.
\]

We can bound the final expression as follows:
\[
1 - 2(1 - X_T)^2 = \frac{2}{7}(1 + X_T)^2 - \frac{1}{7}(4X_T - 3)^2 \leq \frac{2}{7}(1 + X_T)^2.
\]

\[A similar stopping rule was implicitly used by Ben-Tal et al. and refined by Shnurnikov. In addition, pointed out the value of having $|v_1| + |v_2| \leq 1$.\]
Hence the hypotheses of Lemma 1 are satisfied with \( x = X_T \) and \( Y = Y_T \). By Lemma 1, we conclude that
\[
\Pr[|S| \leq 1 \mid T, X_T] = \Pr[|X_T + Y_T| \leq 1 \mid T, X_T] \geq \frac{37}{98}.
\]

3 Further improvement

In this section, we will improve the lower bound to \( \frac{13}{32} \), which is 40.625%. At the end, we will sketch how to improve the bound further, to \( \frac{13}{32} + 9 \times 10^{-6} \).

Let us examine where the proof of Theorem 2 is potentially tight. Looking at its Case 3, we see that the proof is potentially tight when \( T = 2 \) and \( |v_1| = |v_2| = |v_3| = \frac{1}{4} \). But that scenario is impossible: if \( T = 2 \), then by the stopping rule, \( |v_1| + |v_2| > 1 - |v_3| \). This suggests that we can sharpen the bound on \( \sum_{i=T+1}^n v_i^2 \) in terms of \( T \) and \( X_T \).

Another idea is that our final bound on \( \Pr[|S| \leq 1] \), instead of being the worst-case conditional bound, may be taken to be a weighted average of the conditional bounds, with weights corresponding to the distribution of \( T \).

First, we state the following generalization of Lemma 1. Given a number \( c \), define \( F(c) \) by
\[
F(c) = \frac{1}{2}(1 - 3c^2).
\]

**Lemma 3.** Let \( c \) be a nonnegative number. Let \( x \) be a real number such that \( |x| \leq 1 \). Let \( v_1, v_2, \ldots, v_n \) be real numbers such that
\[
\sum_{i=1}^n v_i^2 \leq c(1 + |x|)^2.
\]

Let \( a_1, a_2, \ldots, a_n \) be independent random signs. Let \( Y = \sum_{i=1}^n a_i v_i \). Then
\[
\Pr[|x + Y| \leq 1] \geq F(c).
\]

**Proof.** By symmetry, we may assume that \( x \geq 0 \). As in the proof of Lemma 1, the fourth moment of \( Y \) satisfies
\[
\mathbb{E}(Y^4) \leq 3 \left( \sum_{i=1}^n v_i^2 \right)^2 \leq 3c^2(1 + x)^4.
\]
So, by the fourth moment version of Chebyshev’s inequality,
\[
\Pr[|Y| \geq 1 + x] \leq \frac{\mathbb{E}(Y^4)}{(1 + x)^4} \leq 3c^2.
\]

Following the proof of Lemma 1 by taking the complement and then using the symmetry of \( Y \), we have
\[
\Pr[|x + Y| \leq 1] \geq \frac{1}{2}(1 - 3c^2) = F(c).
\]
Now we will use Lemma 3 to prove our $\frac{13}{32}$ lower bound.

**Theorem 4.** Let $v_1, v_2, \ldots, v_n$ be real numbers such that $\sum_{i=1}^{n} v_i^2$ is at most 1. Let $a_1, a_2, \ldots, a_n$ be independent random signs. Let $S$ be $\sum_{i=1}^{n} a_i v_i$. Then

$$\Pr[|S| \leq 1] > \frac{13}{32}.$$ 

**Proof.** By inserting 0’s, we may assume that $n \geq 4$. By symmetry, we may assume that each $v_i$ is nonnegative. By permuting, we may assume that the $v_i$ are ordered as follows:

$$v_n \geq v_1 \geq v_{n-1} \geq v_2 \geq v_3 \geq \ldots \geq v_{n-2}.$$ 

Except for the oddballs $v_n$ and $v_{n-1}$, the order is decreasing. As in Theorem 2 we have $v_1 + v_2 + v_{n-1} + v_n \leq 2$ and $v_1 + v_2 \leq 1$.

Given an integer $t$ from 0 to $n$, let $M_t$ be the sum $\sum_{i=1}^{t} v_i$. Let $K$ be the smallest nonnegative integer $t$ such that $t = n - 1$ or $M_t > 1 - v_{t+1}$. The parameter $K$ measures how spread out the $v_i$ are. Note that $K \geq 2$, since $v_1 + v_2 \leq 1$. By the definition of $K$, observe that $M_{K-1} \leq 1 - v_K$ and hence $M_K \leq 1$. Also, if $K < n - 1$, then $M_K > 1 - v_{K+1}$ and hence $M_{K+1} > 1$.

Given an integer $t$ from 0 to $n$, define the sums $X_t$ and $Y_t$ as in Theorem 2. Note that $|X_t| \leq M_t$. Following Theorem 2, let $T$ be the smallest nonnegative integer $t$ such that $t = n - 1$ or $|X_t| > 1 - v_{t+1}$. Note that $T \geq K$. As before, we have $|X_{T-1}| \leq 1 - v_T$ and $|X_T| \leq 1$. Also, if $T < n - 1$, then $|X_T| > 1 - v_{T+1}$.

We will bound from below the conditional probability $\Pr[|S| \leq 1 \mid T]$. Namely, we will prove the two-piece lower bound

$$\Pr[|S| \leq 1 \mid T] \geq \begin{cases} F\left(\frac{[K+1]^2-T}{(2K+1)^2}\right) & \text{if } T \leq \frac{3K+2}{2} \frac{2}{2}; \\ F\left(\frac{K}{4K+2}\right) & \text{if } T > \frac{3K+2}{2}. \end{cases}$$

We will actually prove the same lower bound on the refined conditional probability $\Pr[|S| \leq 1 \mid T, X_T]$. To prove this claim, we may assume by symmetry that $X_T \geq 0$. We will divide the proof of the claim into five cases, depending on $T$.

**Case 1:** $T = n - 1$. The proof of this case is the same as Case 1 of Theorem 2 which yields the stronger bound $\Pr[|S| \leq 1 \mid T, X_T] \geq \frac{1}{2}$.

**Case 2:** $T = n - 2$. The proof of this case is the same as Case 2 of Theorem 2 which yields the stronger bound $\Pr[|S| \leq 1 \mid T, X_T] \geq \frac{1}{2}$.

**Case 3:** $K + 1 \leq T \leq \frac{3K+2}{2}$ and $T \leq n - 3$. By the quadratic mean inequality,

$$\sum_{i=1}^{K+1} v_i^2 \geq \frac{1}{K+1} \left(\sum_{i=1}^{K+1} v_i\right)^2 = \frac{1}{K+1} M_{K+1}^2 > \frac{1}{K+1}.$$ 

Hence, by splitting our sum into two parts, we get

$$\sum_{i=1}^{T} v_i^2 > \frac{1}{K+1} + (T - K - 1)v_{T+1}^2 \geq \frac{1}{K+1} + (T - K - 1)(1 - X_T)^2.$$ 

As a simpler bound,

\[ \sum_{i=1}^{T} v_i^2 \geq T v_{T+1}^2 > T(1 - X_T)^2. \]

Multiplying the second-to-last inequality by \( \frac{2T-K-1}{2K+1} \) and the last inequality by \( \frac{3K+2-2T}{2K+1} \), both multipliers being nonnegative by the case assumption, we get

\[ \sum_{i=1}^{T} v_i^2 \geq \frac{2T-K-1}{(K+1)(2K+1)} + \frac{(K+1)^2 - T}{2K+1} (1 - X_T)^2. \]

Therefore, looking at the complementary sum, we get

\[ \sum_{i=T}^{n} v_i^2 \leq \frac{(K+1)^2 - T}{(K+1)(2K+1)} \left[ 2 - (K+1)(1 - X_T)^2 \right]. \]

We can bound the bracketed expression as follows:

\[ 2 - (K+1)(1 - X_T)^2 = \frac{K+1}{2K+1} (1 + X_T)^2 - \frac{2}{2K+1} [(K+1)X_T - K]^2 \]

\[ \leq \frac{K+1}{2K+1} (1 + X_T)^2. \]

Plugging this inequality back into the previous one, we get

\[ \sum_{i=T+1}^{n} v_i^2 \leq \frac{(K+1)^2 - T}{(2K+1)^2} (1 + X_T)^2. \]

Hence the hypotheses of Lemma 3 are satisfied with \( c = \frac{(K+1)^2 - T}{(2K+1)^2} \), \( x = X_T \), and \( Y = Y_T \).

By Lemma 3, we conclude that

\[ \Pr[|S| \leq 1 \mid T, X_T] = \Pr[|X_T + Y_T| \leq 1 \mid T, X_T] \geq F\left(\frac{(K+1)^2 - T}{(2K+1)^2}\right). \]

**Case 4:** \( \frac{3K+2}{2} \leq T \leq n - 3 \). As in Case 3, we can bound \( \sum_{i=1}^{T} v_i^2 \) as follows:

\[ \sum_{i=1}^{T} v_i^2 > \frac{1}{K+1} + (T - K - 1)(1 - X_T)^2. \]

Because \( T \geq \frac{3K+2}{2} \), this inequality implies

\[ \sum_{i=1}^{T} v_i^2 \geq \frac{1}{K+1} + \frac{K}{2} (1 - X_T)^2. \]
Compare this bound with the combined bound from Case 3:

\[
\sum_{i=1}^{T} v_i^2 \geq \frac{2T - K - 1}{(K + 1)(2K + 1)} + \frac{(K + 1)^2 - T}{2K + 1} (1 - X_T)^2.
\]

Note that our bound on \(\sum_{i=1}^{T} v_i^2\) is the same as this bound from Case 3 when \(T = \frac{3K+2}{2}\). So we can repeat the remainder of Case 3 to get the same lower bound on \(\Pr[|S| \leq 1 | T, X_T]\) when \(T = \frac{3K+2}{2}\). The bound on \(\Pr[|S| \leq 1 | T, X_T]\) in Case 3 was

\[
\Pr[|S| \leq 1 | T, X_T] \geq F\left(\frac{(K + 1)^2 - T}{(2K + 1)^2}\right).
\]

When \(T = \frac{3K+2}{2}\), this bound becomes

\[
\Pr[|S| \leq 1 | T, X_T] \geq F\left(\frac{K}{4K+2}\right).
\]

So we get the same bound in our current case.

**Case 5:** \(T = K \leq n - 3\). By the quadratic mean inequality,

\[
\sum_{i=1}^{T} v_i^2 \geq \frac{1}{T} \left(\sum_{i=1}^{T} v_i\right)^2 = \frac{1}{T} M_T^2 \geq \frac{1}{T} X_T^2 = \frac{1}{K} X_T^2.
\]

We can bound the final expression as follows:

\[
\frac{1}{K} X_T^2 = \frac{1}{K + 1} - (1 - X_T)^2 + \frac{1}{K(K + 1)} [(K + 1)X_T - K]^2
\]

\[
\geq \frac{1}{K + 1} - (1 - X_T)^2.
\]

Plugging this inequality back into the previous one, we get

\[
\sum_{i=1}^{T} v_i^2 \geq \frac{1}{K + 1} - (1 - X_T)^2 = \frac{1}{K + 1} + (T - K - 1)(1 - X_T)^2.
\]

This is the same inequality we derived at the beginning of Case 3. So we can repeat the remainder of Case 3 to get the same lower bound:

\[
\Pr[|S| \leq 1 | T, X_T] \geq F\left(\frac{(K + 1)^2 - T}{(2K + 1)^2}\right).
\]

In summary, we have proved our claim on conditional probability:

\[
\Pr[|S| \leq 1 | T] \geq \begin{cases} 
F\left(\frac{(K + 1)^2 - T}{(2K + 1)^2}\right) & \text{if } T \leq \frac{3K+2}{2} ; \\
F\left(\frac{K}{4K+2}\right) & \text{if } T \geq \frac{3K+2}{2} .
\end{cases}
\]
Next, we will use this conditional bound to derive a lower bound on the unconditional probability $\Pr[|S| \leq 1]$.

As mentioned above, we always have $T \geq K$. In fact, assuming that $K \leq n - 4$, we have $T = K$ if the signs $a_1, \ldots, a_K$ are all equal, and otherwise $T \geq K + 2$. This follows from observing that if $a_1, \ldots, a_K$ are not all equal, then $|X_K| \leq \sum_{i=1}^{K-1} v_i - v_K \leq 1 - v_{K+1}$ and $|X_{K+1}| \leq \sum_{i=1}^{K-1} v_i - v_K + v_{K+1} \leq 1 - v_{K+2}$, by the definition of $K$ and the ordering of the $v_i$.

This shows that for $K \leq n - 4$ we have $\Pr[T = K] = \frac{1}{2^{K-1}}$ and $\Pr[T \geq K + 2] = 1 - \frac{1}{2^{K-1}}$. Therefore

$$\Pr[|S| \leq 1] = \frac{1}{2^{K-1}} \Pr[|S| \leq 1 \mid T = K] + \left(1 - \frac{1}{2^{K-1}}\right) \Pr[|S| \leq 1 \mid T \geq K + 2] \geq \frac{1}{2^{K-1}} F \left(\frac{(K+1)^2 - K}{(2K+1)^2}\right) + \left(1 - \frac{1}{2^{K-1}}\right) F \left(\frac{(K+1)^2 - (K+2)}{(2K+1)^2}\right).$$

Here we have used our conditional bounds, the fact that they are nondecreasing in $T$, and the inequality $K + 2 \leq \frac{3K+2}{2}$. Note that this lower bound on $\Pr[|S| \leq 1]$ remains valid without assuming that $K \leq n - 4$. Indeed, if $K = n - 3$ it is still true that $\Pr[T = K] = \frac{1}{2^{K-1}}$, and while $T = K + 1 = n - 2$ may occur in this case, it yields a conditional bound of $\frac{1}{2}$ as shown in Case 2 above, which is even better than our stated lower bound. The values $n - 2$ and $n - 1$ for $K$ are of course covered by the conditional bound of $\frac{1}{2}$ in Cases 1 and 2 above.

Thus, to conclude our proof it suffices to show that

$$\frac{1}{2^{K-1}} F \left(\frac{(K+1)^2 - K}{(2K+1)^2}\right) + \left(1 - \frac{1}{2^{K-1}}\right) F \left(\frac{(K+1)^2 - (K+2)}{(2K+1)^2}\right) \geq \frac{13}{32}$$

holds for all $K \geq 2$. Substituting the relevant expressions into the formula for $F$ and performing routine manipulations, the latter is shown to be equivalent to

$$64(K^2 + K) < 2^{K-1}(40K^2 + 40K - 15),$$

which indeed holds for $K \geq 2$. \qed

Can we improve this $\frac{13}{32}$ lower bound? Yes, a little. The idea is to replace the fourth moment with the more flexible $p$th moment, where $p$ is a parameter to be optimized. To do so, we will need Khintchine’s inequality. This inequality was first proved by Khintchine [8] in a weaker form and later proved by Haagerup [4] with the optimal constants. Namely, given $p \geq 2$, let $B_p$ be the constant

$$B_p = \frac{2^{p/2} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}},$$

where $\Gamma$ is the gamma function. For example, $B_2 = 1$, $B_3 = 2\sqrt{2/\pi}$, and $B_4 = 3$. 9
Theorem 5 (Khintchine’s inequality). Let $p$ be a real number such that $p \geq 2$. Let $v_1, v_2, \ldots, v_n$ be real numbers. Let $a_1, a_2, \ldots, a_n$ be independent random signs. Let $S = \sum_{i=1}^{n} a_i v_i$. Then

$$\mathbb{E}(|S|^p) \leq B_p \left( \sum_{i=1}^{n} v_i^2 \right)^{p/2}.$$ 

For the improved lower bound, choose with foresight $p = 3.95937$. In Lemma 3 replace the fourth moment with the $p$th moment and apply Khintchine’s inequality (with $S = Y$), which allows us to replace the function $F$ with the function $G$ defined by $G(c) = \frac{1}{2}(1 - B_p c^{p/2})$. Use this revised lemma in Theorem 4. The resulting lower bound is $G(\frac{1}{4})$, which is bigger than $\frac{13}{32} + 9 \times 10^{-6}$. We omit the details.

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