Faithfulness of the Lawrence representation of braid groups

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Abstract

The Lawrence representation $L_{n,m}$ is a family of homological representation of the braid group $B_n$, which specializes to the reduced Burau and the Lawrence-Krammer representation when $m$ is 1 and 2. In this article we show that the Lawrence representation is faithful for $m \geq 2$.

1 Introduction

In [2] and [4], Bigelow and Krammer proved via different approaches that the Lawrence-Krammer representation of braid groups is faithful thus the braid groups are linear. In fact, the Lawrence-Krammer representation is the only known faithful representation of the braid group $B_n$ for $n \geq 4$ till now.

In this article, by making use of a reflexive representation recently found by the author (ref. [6]), we generalize the faithfulness of the Lawrence-Krammer representation to its full family, the Lawrence representation (ref. [5]).

Theorem 1.1. The Lawrence representation is faithful for $m \geq 2$.

In the article, the Lawrence representation is defined alternatively as follows. Let $B_n$ denote the Artin’s $n$-strand braid group (ref. [3]), with standard generators $\{\sigma_1, \ldots, \sigma_{n-1}\}$, and set

$$B_{n,m} = \langle \sigma_1, \ldots, \sigma_{n-1}, \sigma_n^2, \sigma_{n+1}, \ldots, \sigma_{n+m-1} \rangle \subset B_{n+m}.$$ 

They are the fundamental groups of

$$X_n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{C}, x_i \neq x_j, \forall i \neq j \}/\Sigma_n,$$

$$X_{n,m} = \{(x_1, \ldots, x_{n+m}) \mid x_i \in \mathbb{C}, x_i \neq x_j, \forall i \neq j \}/\Sigma_n \times \Sigma_m.$$
respectively, where $\Sigma_n$ denotes the symmetric group of $n$ symbols.

Let $\xi_{n,m}$ be the reflexive representation over a free $\mathbb{Z}B_{n,m}$-module $M_{n,m}$ defined in [6] (see Section 2). Let $q, t \in \mathbb{C}$ be two algebraically independent numbers and let

$$\rho_{n,m} : \mathbb{Z}B_{n,m} \to \mathbb{C}$$

 denote the ring homomorphism given by

$$\begin{align*}
\sigma_1, \ldots, \sigma_{n-1} & \mapsto 1, \\
\sigma_n & \mapsto q, \\
\sigma_{n+1}, \ldots, \sigma_{n+m-1} & \mapsto t.
\end{align*}$$

The Lawrence representation is defined as the representation

$$L_{n,m} = \rho_{n,m} \circ \xi_{n,m}$$

over the $\mathbb{C}$-linear space

$$M_{n,m}^L = \mathbb{C} \otimes_{\rho_{n,m}} M_{n,m}.$$ 

**Remark 1.2.** It was shown in [6] that $L_{n,2}$ is precisely the Lawrence-Krammer representation and it is easily derived from the explicit matrix elements calculated in [6] that $L_{n,1}$ is precisely the reduced Burau representation (ref. [3]).

**Remark 1.3.** It is known that the reduced Burau representation is faithful for $n \leq 3$ and not faithful for $n \geq 5$ (ref. [1]), but the case $n = 4$ still remains open. Therefore, Theorem 1.1 shows that the faithfulness of Lawrence representation is only unclear for $L_{4,1}$.

Our proof essentially follows Bigelow’s approach. In Section 2 we give a quick review of the reflexive representation $\xi_{n,m}$. In Section 3 we define the pairing of noodles with multiforks and relate it to the Lawrence representation via the notion of linear function. It is the crucial part of the article. In Section 4, after some preliminary lemmas prepared, the main theorem is established.

## 2 A quick review of the representation $\xi_{n,m}$

Let $D$ be a 2-disk and $P = \{p_1, \ldots, p_n\} \subset D \setminus \partial D$ be a set of $n$ punctures. The space

$$Y_{n,m} = \{(y_1, \ldots, y_n) \mid y_i \in D \setminus P, \ y_i \neq y_j, \ \forall i \neq j\}/\Sigma_m$$

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is homotopy equivalent to the fiber of the fiber bundle $X_{n,m} \to X_n$, whose fundamental group is

$$\langle A_{1,n+1}, \ldots, A_{n,n+1}, \sigma_{n+1}, \ldots, \sigma_{n+m-1} \rangle \subset B_{n,m}$$

where $A_{i,j}$ is the standard pure braid defined by

$$A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1}^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}.$$  

Recall that an equivalent definition of $B_n$ is the mapping class group $\mathcal{M}(D, P; \partial D)$, the group of all orientation preserving homeomorphism $h : D \to D$ such that $h(P) = P$ and $h|_{\partial D} = \text{id}$, modulo isotopy relative to $P \cup \partial D$. Regarding $\mathcal{M}(D, P; \partial D)$ and $\pi_1(Y_{n,m})$ as subgroups of $B_{n,m}$ in the standard way, we have

$$\beta_*(\alpha) = \beta^{-1} \alpha \beta, \quad \forall \beta \in \mathcal{M}(D, P; \partial D), \quad \alpha \in \pi_1(Y_{n,m}).$$

Definition 2.1. A fork is a map $\phi : F \to D$ such that $\phi|_{e_{t}}$ is an embedding, $\phi(F) \cap \partial D = \phi(z_0)$ and $\phi(F) \cap P = \{\phi(z_1), \phi(z_2)\}$. A multifork with $m$ components is an $m$-tuple of forks $\Phi = (\phi_1, \ldots, \phi_m)$ such that both $\phi_1(e_t), \ldots, \phi_m(e_t)$ are disjoint and $\phi_1(e_h), \ldots, \phi_m(e_h)$ are disjoint.

Definition 2.2. Two forks $\phi$ and $\psi$ are called homotopic, denoted by $\phi \simeq \psi$, if there is a homotopy $h_t : F \to D$ such that $h_0 = \phi, h_1 = \psi, h_t(z_0)$ is independent of $t$, and $h_t$ is a fork for all $0 \leq t \leq 1$. Two multiforks $\Phi = (\phi_1, \ldots, \phi_m)$ and $\Psi = (\psi_1, \ldots, \psi_m)$ are called homotopic, also denoted by $\Phi \simeq \Psi$, if there are fork homotopies $h_{k,t} : \phi_k \simeq \psi_k$ such that $(h_{1,t}, \ldots, h_{m,t})$ is a multifork for all $0 \leq t \leq 1$. 

Figure 1: Complex $F$ and a multifork.

Let $F$ be the 1-complex shown in Fig. 2. It consists of four 0-cells $\{z, z_0, z_1, z_2\}$ and three 1-cells $\{e_0, e_1, e_2\}$. Let $e_t = e_1 \cup z \cup e_2$ and $e_h = z_0 \cup e_0 \cup z$ denote the tine edge and the handle of $F$.
Choose a base point \([b_1, \ldots, b_m] \in Y_{n,m}\) where \(b_1, \ldots, b_m \in \partial D\). Set

\[
\Gamma_{n,m} = \{(\phi_1, \ldots, \phi_m) \mid \phi_i(z_0) = b_i, \ 1 \leq i \leq m\}
\]

and denote by \(M_{n,m}^0\) the free \(\mathbb{Z}B_{n,m}\)-module generated by \(\Gamma_{n,m}\). Define four relations on \(M_{n,m}^0\) as follows.

- **\(R_H\):** \(\Phi_1 \sim \Phi_2\) if they are homotopic.
- **\(R_R\):** \((\phi_1, \ldots, \phi_k, \ldots, \phi_m) \sim -(\phi_1, \ldots, \phi_k r, \ldots, \phi_m)\) where \(r : F \rightarrow F\) denotes the cell isomorphism that swaps \(e_1\) and \(e_2\).

**Figure 2: Relation \(R_T\).**

\[
R_T: (\phi_1, \ldots, \phi_m) \sim \text{sgn } \eta \cdot \alpha \cdot (\varphi_1, \ldots, \varphi_m)\] where \(\eta \in \Sigma_m\) and \(\alpha \in \pi_1(Y_{n,m})\) if \(\phi_k|_{e_t} = \varphi_{\eta(k)}|_{e_t}\) for all \(1 \leq k \leq m\) and \(\alpha\) is represented by the loop that runs from \([b_1, \ldots, b_m]\) to \([\phi_1(z), \ldots, \phi_m(z)]\) along the curve \([\phi(t), \ldots, \phi_m(t)] \mid t \in e_h\} and backs to \([b_1, \ldots, b_m]\) along the curve \([\varphi(t), \ldots, \varphi_m(t)] \mid t \in e_h\}.

**Figure 3: Relation \(R_S\).**

\[
R_S: \Phi \sim \Phi_1 + \Phi_2\] if \(\Phi\) can be split into \(\Phi_1\) and \(\Phi_2\) by doing a surgery on the tine edge of a fork as shown in Fig. 3.

Now set \(M_{n,m} = M_{n,m}^0 / (R_H, R_R, R_T, R_S)\). It turns out that the action of \(B_n\) on \(M_{n,m}\) gives rise to a representation over a finitely generated free \(\mathbb{Z}B_{n,m}\)-module.
**Theorem 2.3.** $M_{n,m}$ is a finitely generated free $\mathbb{Z}B_{n,m}$-module. Moreover, the action
\[
\xi_{n,m}(\beta) : [\Phi] \mapsto [\beta \cdot \beta(\Phi)], \quad \forall \Phi \in \Gamma_{n,m}, \quad \beta \in B_n
\]
gives rise to a representation of $B_n$ over $M_{n,m}$.

### 3 Pairing and linear function

**Definition 3.1.** A **noodle** is an embedded oriented arc $N \subset D \setminus P$ such that $\partial D \cap N = \partial N$ and all the points $b_1, \ldots, b_m$ lies to its left.

![Diagram of a noodle and intersections](image)

It is straightforward to verify that, via the pairing, each noodle $N$ gives rise to a $\mathbb{Z}B_{n,m}$-linear function
\[
\langle N, \cdot \rangle : M_{n,m} \to \mathbb{Z}B_{n,m}.
\]
and, further, a $\mathbb{C}$-linear function
\[
\langle N, \cdot \rangle_{\rho} : M_{n,m}^L \to \mathbb{C}.
\]

Note that we have
\[
\langle N, L_{n,m}(\beta) \cdot [\Phi] \rangle_{\rho} = \langle N, [\beta(\Phi)] \rangle_{\rho}, \quad \forall \beta \in B_n, \ \Phi \in \Gamma_{n,m}.
\]

Especially, if $\beta$ is an element of the kernel of the Lawrence representation $L_{n,m}$,
\[
\langle N, [\Phi] \rangle_{\rho} = \langle N, [\beta(\Phi)] \rangle_{\rho}, \quad \forall \Phi \in \Gamma_{n,m}.
\]

**Remark 3.3.** For $m = 2$, the last equation is precisely a generalization of [2 Basis Lemma]. Here we obtain the equation via the language of representation, which makes the topological meaning much more accessible.

## 4 Proof of faithfulness

In this section, let all forks $\phi$ satisfy $\phi(z_0) = b_1$ and denote by $\phi^{(m)}$ the multifork constructed from $m$ parallel copies of $\phi$ as shown in Figure 5.

\[
\begin{align*}
\text{Figure 5: Fork to multifork.}
\end{align*}
\]

**Lemma 4.1.** Let $N$ be a noodle and $\phi$ be a fork. Suppose the tine edge of $\phi$ intersects $N$ transversely at $l$ distinct points and

\[
\langle N, \phi^{(m)} \rangle = \sum_{i_1, \ldots, i_m=1}^{l} \epsilon_{i_1} \epsilon_{i_2} \cdots \epsilon_{i_m} \alpha_{i_1, \ldots, i_m} \rho_{n,m}(\alpha_{i_1, \ldots, i_m}) = q^{\alpha_{i_1, \ldots, i_m}} (-t)^{b_{i_1, \ldots, i_m}},
\]

\[
6
\]
where \( \epsilon_i \) and \( \alpha_{i_1, \ldots, i_m} \) are same as Definition 5, Then we have

\[
\epsilon_i = (-1)^{b_{i,i}},
\]

\[
a_{i_1, \ldots, i_m} = \sum_{j=1}^{m} a_{ij},
\]

\[
b_{i_1, \ldots, i_m} = \sum_{1 \leq j < k \leq m} b_{ij,ik}.
\]

Proof. Note that for \( j > k \), \( b_{ij,ik} \) is the crossing number (define the crossing number of the generator \( \sigma_i^{\pm 1} \) to be \( \pm 1 \)) between the \( (n+j) \)-th and the \( (n+k) \)-th strand of the braid \( \alpha_{i_1, \ldots, i_m} \). \( a_{ij} \) is the linking number (half of the crossing number) of the \( (n+j) \)-th strand with the former \( n \) strands of the braid \( \alpha_{i_1, \ldots, i_m} \).

The identities follow from the facts that the crossing number between the \( (n+1) \)-th and the \( (n+2) \)-th strand of \( \alpha_{i,j} \) is even if and only if \( \epsilon_i \) is positive, \( a_{i_1, \ldots, i_m} \) is the linking number of the last \( m \) strands with the former \( n \) strands of \( \alpha_{i_1, \ldots, i_m} \), \( b_{i_1, \ldots, i_m} \) is the sum of the pairwise crossing numbers of the last \( m \) strands of \( \alpha_{i_1, \ldots, i_m} \), respectively. \( \square \)

**Lemma 4.2.** Let \( N \) be a noodle, \( \phi \) be a fork and \( m \geq 2 \) be an integer. If \( \langle N, \phi^{(m)} \rangle_\rho = 0 \) then the tine edge of \( \phi \) is isotopic to relative to \( \partial D \cup P \) to an arc which is disjoint from \( N \).

Proof. Applying a preliminary isotopy, we may assume that the tine edge of \( \phi \) intersects \( N \) transversely at \( l \) distinct points where \( l \) is minimal in possible. Suppose \( l > 0 \). In the notation of Lemma 4.1, assume \( a_1, \ldots, a_{l'} \) are all those maximal among \( a_1, \ldots, a_l \) and \( b_{i,j} \) is maximal among \( \{b_{i,j'} \mid 1 \leq i' \leq i', j' \leq l' \} \). We claim \( b_{i,i} = b_{j,j} = b_{j,j} \).

The claim implies that \( b_{i_1, \ldots, i_m} \) is maximal among \( \{b_{i_1', \ldots, i_m'} \mid 1 \leq i_1', \ldots, i_m' \leq l' \} \) if and only if \( b_{i_1,i_k} \) is maximal among \( \{b_{i',j'} \mid 1 \leq i', j' \leq l' \} \) for all \( 1 \leq j < k \leq l' \). Moreover, in this case \( \epsilon_i \cdots \epsilon_{i_m} \rho_{n,m}(\alpha_{i_1, \ldots, i_m}) \) is independent of the choice of \( i_1, \ldots, i_m \). Therefore, regarding \( \langle N, \phi^{(m)} \rangle_\rho \) as a polynomial of \( q, t \), we find the coefficient of \( q^{a_{i_1, \ldots, i_m}} t^{b_{i_1, \ldots, i_m}} \) is nonvanishing thus \( \langle N, \phi^{(m)} \rangle_\rho \neq 0 \).

Now it remains to prove the claim. Let \( \phi' \) denotes the other component of \( \phi^{(2)} \) and assume the tine edges of \( \phi \) and \( \phi' \) intersect \( N \) transversely at \( \{x_1, \ldots, x_l \} \) and \( \{x_1', \ldots, x_{l'} \} \), respectively. The rest part of the proof is copied almost word by word from the proof of [2, Claim 3.4].

Suppose, seeking a contradiction, that \( b_{i,i} < b_{i,j} \). Let \( \alpha \) be an embedded arc from \( z_i' \) to \( z_{i} ' \) along the tine edge of \( \phi' \). Let \( \beta \) be an embedded arc from \( z_{j}' \) to \( z_{j}' \) along \( N \).
If \( \beta \) does not pass through the point \( z_i, \) let \( \delta = \alpha \beta \) and let \( w \) be the winding number of \( \delta \) around \( z_i. \) Then \( b_{i,j} - b_{i,i} = 2w. \) If \( \beta \) does pass through \( z_i, \) first modify \( \beta \) in a small neighborhood of \( z_i \) so that \( z_i \) lies to its left.

Next let \( \delta = \alpha \beta \) and let \( w \) be the winding number of \( \delta \) around \( z_i. \) Then \( 1 + b_{i,j} - b_{i,i} = 2w. \) In either case, our assumption that \( b_{i,i} < b_{i,j} \) implies that \( w \) is greater than zero.

Let \( D_1 = D \setminus \{ z_1 \}. \) Let \( \pi : \tilde{D}_1 \to D_1 \) be the universal (infinite cyclic) cover. Let \( \tilde{\alpha} \) be a lift of \( \alpha \) to \( \tilde{D}_1. \) Let \( \tilde{\beta} \) be the lift of \( \beta \) to \( \tilde{D}_1 \) which starts at \( \tilde{\alpha}(1). \) Let \( \gamma \) be a loop in \( D_1 \) based at \( z'_i \) which winds \( w \) times around \( z_i \) in the clockwise (negative) direction such that \( \gamma \) is null-homotopic in \( D \setminus P. \)

Let \( \tilde{\gamma} \) be the lift of \( \gamma \) to an arc from \( \tilde{\beta}(1) \) to \( \tilde{\alpha}(0). \) Choose \( \gamma \) so that \( \tilde{\gamma} \) is an embedded arc which intersects \( \tilde{\alpha} \) and \( \tilde{\beta} \) only at its end points.

Let \( \tilde{\gamma}' \) be the first point on \( \tilde{\alpha} \) which intersects \( \tilde{\beta} \) (possibly \( \tilde{\alpha}(1) \)). Then \( \tilde{\pi}(\tilde{\gamma}') = \tilde{\gamma}' \) for some \( k = 1 \ldots l. \) Let \( \tilde{\alpha}' \) be the initial segment of \( \tilde{\alpha} \) ending at \( \tilde{\gamma}'. \) Let \( \tilde{\beta}' \) be the final segment of \( \tilde{\beta} \) starting at \( \tilde{\gamma}'. \) Let \( \tilde{\delta}' = \tilde{\alpha}' \tilde{\beta}' \tilde{\gamma}. \)

Now \( \tilde{\delta}' \) is a simple closed curve in \( \tilde{D}_1, \) so by the Jordan curve theorem it must bound a disk \( \tilde{B}. \) Since \( \gamma \) passes clockwise around \( z_i, \) there is a non-compact region to the right of \( \tilde{\delta}'. \) Thus \( \tilde{\delta}' \) must pass counterclockwise around \( \tilde{B}. \)

Let \( \alpha', \beta' \) and \( \delta' \) be the projections of \( \tilde{\alpha}', \tilde{\beta}' \) and \( \tilde{\delta}' \) to \( D_1. \) Then \( a_k - a_i \) is equal to the sum of the winding numbers of \( \delta' \) around each of the points in \( P. \) This is equal to the cardinality of \( \tilde{B} \cap \pi^{-1}(P). \) Since \( a_i \) is maximal among all integers \( a_r, \) we must have \( a_k = a_i. \) Thus \( \tilde{B} \cap \pi^{-1}(P) = \emptyset. \) It follows that the arc \( \tilde{\delta}' = \alpha' \beta' \gamma \) is null-homotopic in \( D \setminus P. \) But \( \beta' \) is homotopic relative to end points to a subarc of \( N, \) and \( \gamma \) was chosen to be null-homotopic in \( D \setminus P. \) Thus \( \alpha' \) is homotopic relative to end points to a subarc of \( N \) in \( D \setminus P. \) So \( \alpha \) and \( N \) cobound a digon in \( D \setminus P. \) But \( \alpha' \) is a subarc of the tine edge of \( \phi. \) This contradicts the fact that the tine edge of \( \phi \) intersects \( N \) a minimal number of times. Therefore our assumption that \( b_{i,j} > b_{i,i} \) must have been false, so \( b_{i,j} = b_{i,i}. \)

The proof that \( b_{i,j} = b_{j,j} \) is similar. This completes the proof of the claim, and hence of the lemma.

Now we prove the main theorem.

**Proof of Theorem 1.1.** Suppose \( \beta \in B_n \) belongs to the kernel of the Lawrence representation, i.e. \( L_{n, m}(\beta) = \text{id}. \) Then for any fork \( \phi \) and homeomorphism \( f : D \to D \) representing \( \beta, \) we have \( \langle N, \phi^{(m)} \rangle = \langle N, (f \phi)^{(m)} \rangle. \)

Choose a set of disjoint noodles \( N_1, \ldots, N_{n-1} \) and a set of forks \( \phi_1, \ldots, \phi_{n-1} \) with disjoint tine edges as shown in Figure 8. Note that \( \langle N_i, \phi_i^{(m)} \rangle = 0 \) if \( i \neq j. \) Choose a homeomorphism \( f \) representing \( \beta \) such that \( (f \phi_1)(e_l) \cup \cdots \cup \)
\[(f\phi_{n-1})(e_i)\) intersects \(N_1 \cup \cdots \cup N_{n-1}\) a minimal number of times in possible. Then, whenever \(i \neq j\), \(\langle N_j, (f\phi_i)_{\rho} \rangle = 0\) and by Lemma 4.2 \((f\phi_i)(e_i)\) is disjoint from \(N_j\); otherwise, \((f\phi_i)(e_i)\) and \(N_j\) cobound a digon in \(D \setminus P\) which contradicts the minimality of the intersections.

Therefore, we may further assume that \((f\phi_i)(e_i) = \phi_i(e_i)\) thus \(\beta\) must be a power of the full twist \(\Delta^2 = (\sigma_1 \cdots \sigma_{n-1})^n\). A straightforward calculation shows that \(L_{n,m}(\Delta^2) = q^{mn}t^m(m-1)\id\) hence we must have \(\beta = 1\).

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