PROPERTY AND NUMERICAL SIMULATION OF THE AIT-SAHALIA-RHO MODEL WITH NONLINEAR GROWTH CONDITIONS

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Abstract. The Ait-Sahalia-Rho model is an important tool to study a number of financial problems, including the term structure of interest rate. However, since the functions of this model do not satisfy the linear growth condition, we cannot study the properties for the solution of this model by using the traditional techniques. In this paper we overcome the mathematical difficulties due to the nonlinear growth condition by using numerical simulation. Thus we first discuss analytical properties of the model and the convergence property of numerical solutions in probability for the Ait–Sahalia-Rho model. Finally, an example for option pricing is given to illustrate that the numerical solution is an effective method to estimate the expected payoffs.

1. Introduction. Noise is an important phenomenon in a wide range of systems in biology, finance, physical sciences and engineering. To study the function of noise in complex systems, stochastic differential equation is a popular approach by adding a diffusion term into the existing ordinary differential equations \[3, 10, 14\]. For example, the following model
\[dY(t) = \beta(\mu - Y(t))dt + \sigma Y(t)dB(t)\] (1)
has been widely used in finance to model volatility, interest rates and other financial price. When \(\gamma = 1/2\), it is the Cox-Ingersoll-Ross (CIR) model for describing the evolution of interest rate and its solution is the well-known mean-reverting square root process. The strong convergence property of its Monte–Carlo simulation has

2010 Mathematics Subject Classification. Primary: 60H10, 65C35; Secondary: 65C05.
Key words and phrases. Ait-Sahalia-Rho model, boundedness, convergence in probability.
The work is supported by the National Natural Science Foundation of China (61304067 and 11571368), the Natural Science Foundation of Hubei Province of China (2013CFB443) and the Australian Research Council Future Fellowship (FT100100748).
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been studied in [7, 8]. When $\gamma \in [1/2, 1]$, Mao et al. [11] discussed its analytical properties and strong convergence property of numerical solutions. When $\gamma > 1$, a number of researchers estimated the parameters of the continuous-time model by using the Euler–Maruyama discrete approximation [4, 12, 15]. Moreover, Baduraliy and Mao discussed the Euler-Maruyama approximation for the mean-reverting-theta stochastic volatility model [2]. In addition, the stochastic model in [15] has been generalized to stochastic systems with jump process [9].

In the paper we consider the Ait-Sahalia-Rho model of the form

$$dY(t) = (\beta_{-1}Y^{-1}(t) - \beta_0 + \beta_1 Y(t) - \beta_2 Y^\rho(t))dt + \sigma Y^\gamma(t)dB(t)$$

with the initial value $Y(0) > 0$, where $\beta_{-1}, \beta_0, \beta_1, \beta_2$ and $\sigma$ are positive and $\gamma, \rho > 1$. Ait–Sahalia proposed this model to study the spot interest rate and discussed the property of this model when $\rho = 2$ [1]. In addition Cheng [6] discussed the boundedness and convergence properties of the solution for this model. Moreover, Szpruch et al. studied the Backward Euler-Maruyama scheme and Backward-Forward Euler-Maruyama scheme for the model when $2\gamma < \rho + 1$ [13]. However, so far there is not any result for the property of this model when $2\gamma > \rho + 1$. Since the diffusion coefficient does not satisfy the linear growth condition when $\gamma > 1$, we cannot apply the classical techniques [10] to study this model. In this case numerical simulation has become a powerful technique in studying the property of the model such as the case for the valuation of financial derivatives [5, 16, 17, 18]. If the error of numerical solutions can be controlled, numerical simulations can be used to estimate the value of certain financial options effectively. In this paper we will follow this approach by developing new techniques to overcome these difficulties and discuss boundedness property of the solution and convergence property of numerical solutions for the Ait-Sahalia-Rho model.

The rest of the paper is organized as follows. In Section 2, we study the existence and nonnegativity of the solution for Eq. (2) and then consider various boundedness property of its solution in Section 3. Section 4 discusses asymptotic pathwise estimations of Eq. (2). In Section 5 we firstly introduce the Euler–Maruyama approximations to the solution of Eq. (2) and then examine its convergence in probability. Finally, we use a single barrier call option in Section 6 to show that the numerical solution can be used to compute the expected payoffs.

### 2. Existence of positive global solutions.

Throughout this paper, let $(\mathcal{Y}, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\left\{ \mathcal{F}_t \right\}_{t \geq 0}$ satisfying the usual conditions, namely it is right continuous and increasing and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Denote $R^+ = (0, +\infty)$ and let $B(t)$ be a scalar Brownian motion defined on the probability space. In addition, we always assume that $\beta_{-1}, \beta_0, \beta_1, \beta_2$ and $\sigma$ are positive and $\gamma, \rho > 1$.

Since model (2) is used to describe the dynamics of interest rate and other quantities, the solution $Y(t)$ should not become negative. The existence of the positive solution is thus very important. Although there is a result regarding the positive solution in [13], in order to be self-contained and explain the convergence property more clearly in later sections, we give the proof of this result by using a different method. Here we first give the following theorem.

**Theorem 2.1.** For any given initial condition $Y(0) > 0$, there exists a unique positive global solution $Y(t)$ to Eq. (2) on $t \geq 0$. 

Proof. From the results in [10], we know that, for any given initial value \( Y(0) > 0 \), there exists a unique local solution \( Y(t) \in [0, \tau_e) \), where \( \tau_e \) is the stopping time of the explosion time. We now only need to claim that \( \tau_e = \infty \) a.s. For a sufficient large integer \( k > 0 \), such that \( 1/k < Y(0) < k \), define the stopping time as

\[
\tau_k = \inf\{ t \in [0, \tau_e) : Y(t) \notin (1/k, k) \},
\]

where throughout this paper we set \( \inf \emptyset = \infty \). Obviously, \( \tau_k \) is increasing as \( k \to \infty \). Set \( \tau_\infty = \lim_{k \to \infty} \tau_k \), whence \( \tau_\infty \leq \tau_e \) a.s. If we can prove \( \tau_k \to \infty \) a.s. as \( k \to \infty \), then \( \tau_e = \infty \) a.s. and \( Y(t) > 0 \) a.s. for all \( t \geq 0 \). In other words, we need to show that \( \tau_\infty = \infty \) a.s. To prove this, it is adequate to show that \( P\{\tau_k \leq T\} \to 0 \) as \( k \to \infty \) for any constant \( T \), which implies \( P\{\tau_\infty = \infty\} = 1 \) as required.

Define a function \( V \in C^2(\mathbb{R}^+, \mathbb{R}^+) \) by

\[
V(Y) = \sqrt{Y} - 1 - \frac{1}{3} \log Y.
\]

Obviously, function \( V(Y) \to \infty \) as \( Y \to \infty \) or \( Y \to 0 \). By using the Itô formula, we have that

\[
dV(Y(t)) = LV(Y(t))dt + \left[ \frac{\sigma}{3}Y^{-\frac{3}{2} + \gamma}(t) - \frac{\sigma}{3}Y^{-1 + \gamma}(t) \right] dB(t)
\]

where the operator \( LV : Y^+ \to Y \) is defined by

\[
LV(Y(t)) = \frac{1}{3}(Y^{-2/3}(t) - Y^{-1}(t)))(\beta_1Y^{-1}(t) - \beta_0 + \beta_1Y(t) - \beta_2Y^\rho(t)) + \frac{1}{6}\sigma^2Y^{2\gamma}(t)(- \frac{2}{3}Y^{-\frac{\gamma}{2}} + Y^{-2}(t)),
\]

which means that there is a constant \( K_1 \) such that

\[
LV(Y(t)) \leq K_1.
\]

From (5) and (7), for any \( t \in [0, T] \), we have that

\[
EV(Y(t \land \tau_k)) \leq V(Y(0)) + K_1T.
\]

So

\[
P(\tau_k \leq T)[V(1/k) \land V(k)] \leq EV(Y(T \land \tau_k)) \leq V(Y(0)) + K_1T,
\]

which gives

\[
P(\tau_k \leq T) \leq \frac{V(Y(0)) + K_1T}{V(1/k) \land V(k)}.
\]

Thus \( P(\tau_k \leq T) \to 0 \) since \( V(1/k) \land V(k) \to \infty \) as \( k \to \infty \). This implies \( P(\tau_\infty = \infty) = 1 \), as required. This completes the proof.

3. Boundedness. This section will give certain boundedness property for the solution of model [4], which is a practical requirement for the pricing of financial derivatives. To examine the boundedness and asymptotic behavior of the solution, we need to employ the following assumption.

(H) The parameters of Eq. (2) satisfy

\[
2\gamma > \rho + 1.
\]

Remark 1. Assumption (H) allows us to control the potential growth coming from the diffusion term. It is just converse to Assumption 2.2 in [13].

We now firstly give the boundedness property for the moment of the solution.
Theorem 3.1. If assumption (H) holds, for $0 < p \leq 1$, then the solution of model (1) satisfies

$$\sup_{0 \leq t < \infty} E|Y(t)|^p < \infty. \quad (11)$$

Proof. By using the Itô formula, we have that

$$d[e^t Y(t)] = e^t (Y^p(t) + pY^{p-1}(t)[\beta_1 Y^{-1}(t) - \beta_0 + \beta_1 Y(t) - \beta_2 Y^p(t)])$$

$$+ \frac{\sigma^2}{2} p(p-1)Y^{2\gamma+2p-2}dt + e^t p\sigma Y^{p+1}(t)dB(t). \quad (12)$$

For any sufficiently large positive number $n$, define a stopping time

$$\tau_n = \inf\{t : Y(t) \notin (\frac{1}{n}, n)\}. \quad (13)$$

Note that $p \in (0, 1]$. By using assumption (H), there exists a positive constant $K_3$ such that

$$e^t (Y^p(t) + pY^{p-1}(t)[\beta_1 Y^{-1}(t) - \beta_0 + \beta_1 Y(t) - \beta_2 Y^p(t)])$$

$$+ \frac{\sigma^2}{2} p(p-1)Y^{2\gamma+2p-2} \leq K_3 e^t.$$

Hence we have that

$$E(e^{\tau_n} Y^p(t \wedge \tau_n)) \leq Y^p(0) + K_3 e^t.$$

Letting $n \to \infty$ and using the Fatou theorem yields

$$EY^p(t) \leq Y^p(0) + K_3, \quad (14)$$

which implies the required assertion. \qed

For obtaining further results regarding boundedness, we need to the following theorem.

Theorem 3.2. Assume that $\gamma \in (1, 1.5]$ and assumption (H) holds. Then for any $t \geq 0$ and initial condition $Y(0) > 0$, there exists a positive constant $K_4$ such that the solution of model (1) satisfies

$$E(Y^{-1}(t)) \leq Y^{-1}(0) + K_4. \quad (15)$$

Proof. Set $y(t) = Y^{-1}(t)$. By using the Itô formula, we have that

$$d(e^t y(t)) = e^t [y(t) - \beta_1 y^3(t) + \beta_0 y^2(t) - \beta_1 y(t) + \beta_2 y^{-p}(t) + \sigma^2 y^{3-2\gamma}(t)]dt$$

$$- \sigma e^t y^{2-\gamma}(t)dB(t). \quad (16)$$

Obviously, there exists a positive constant $K_4$ such that

$$e^t [y(t) - \beta_1 y^3(t) + \beta_0 y^2(t) - \beta_1 y(t) + \beta_2 y^{-p}(t) + \sigma^2 y^{3-2\gamma}(t)] \leq K_4 e^t. \quad (17)$$

Hence we have

$$e^t E_y(t) \leq Y^{-1}(0) + K_4 e^t,$$

which implies the required assertion. \qed

Now from the results for the boundedness of solutions, we now show that $Y(t)$ will maintain in a belt area with a large probability.

Theorem 3.3. Assume that $\gamma \in (1, 1.5]$ and assumption (H) holds. Then for any $\varepsilon \in (0, 1)$ and $Y(0) > 0$, there exist a pair of positive constants $k_2 = k_2(\varepsilon, Y(0))$ and $k_1 = k_1(\varepsilon, Y(0))$ such that for $\forall t \geq 0$

$$P(k_1 \leq Y(t) \leq k_2) \geq 1 - \varepsilon. \quad (18)$$
Proof. For any $\epsilon > 0$, choose $k_2 = \frac{2}{\epsilon}(Y(0) + K_3)$. By the Chebyshev inequality and Theorem 3.1, we have that

$$P(Y(t) > k_2) \leq \frac{E(Y(t))}{k_2} = \frac{\epsilon}{2}.$$  

By Theorem 3.2, we know that $E(Y^{-1}(t))$ is bounded. Thus, by the Chebyshev inequality again, there exists a positive constant $k_1 = k_1(\epsilon, Y(0))$ such that

$$P(Y(t) \geq k_1) = P(Y^{-1}(t) \leq k_1^{-1}) \geq 1 - \frac{\epsilon}{2}.$$  

Therefore, we have

$$P(k_1 \leq Y(t) \leq k_2) = P(Y(t) \geq k_1) - P(Y(t) > k_2) \geq 1 - \epsilon.$$  

The proof is complete.

4. Asymptotic pathwise estimation. This section is devoted to derive the asymptotic pathwise estimation for the solutions of model (2).

Theorem 4.1. If $\gamma \in (1, 1.5]$ and assumption (H) holds, then for any initial value $Y(0) > 0$, the solution of model (2) satisfies

$$\liminf_{t \to \infty} \frac{\log Y(t)}{\log t} \geq -1 \quad a.s.$$  

Proof. Let $\eta(t) = Y^{-1}(t)$. By using the Itô formula, we have that

$$d(\eta(t)) = \left[-\beta_1 \eta^3(t) + \beta_0 \eta^2(t) - \beta_1 \eta(t) + \beta_2 \eta^{2-\rho}(t) + \sigma^2 \eta^{3-2\gamma}(t)\right]dt - \sigma \eta^{2-\gamma}(t)dB(t).$$  

The expectation of function $\eta(t)$ satisfies

$$E\eta(t + 1) + \frac{1}{2} \beta_{-1} E \int_t^{t+1} \eta^3(s)ds = E\eta(t)$$

$$+ E \int_t^{t+1} \left[- \frac{1}{2} \beta_{-1} \eta^3(t) + \beta_0 \eta^2(t) - \beta_1 \eta(t)
+ \beta_2 \eta^{2-\rho}(t) + \sigma^2 \eta^{3-2\gamma}(t)\right]ds.$$  

(21)

By using assumption (H), there exists a constant $K_5$ such that

$$-\frac{1}{2} \beta_{-1} \eta^3(t) + \beta_0 \eta^2(t) - \beta_1 \eta(t) + \beta_2 \eta^{2-\rho}(t) + \sigma^2 \eta^{3-2\gamma}(t) \leq K_5.$$  

This, together with Theorem 3.2, yields

$$\frac{1}{2} \beta_{-1} E \int_t^{t+1} \eta^3(s)ds \leq E\eta(t) + K_5$$

$$\leq Y^{-1} + K_4 + K_5.$$  

(22)

By using (17) and (21), for any $u \in [t, t + 1]$,

$$\eta(u) = \eta(t) + \int_t^u \left[-\beta_1 \eta^3(t) + \beta_0 \eta^2(t) - \beta_1 \eta(t) + \beta_2 \eta^{2-\rho}(t) + \sigma^2 \eta^{3-2\gamma}(t)\right]dt$$

$$- \int_t^u \sigma \eta^{2-\gamma}(t)dB(t)$$

$$\leq \eta(t) + K_4 - \int_t^u \sigma \eta^{2-\gamma}(t)dB(t).$$  

(23)
Thus we have that
\[
E\left(\sup_{t \leq u \leq t + 1} \eta(u)\right) \leq E\eta(t) + K_4 + \sigma E\left(\sup_{t \leq u \leq t + 1} \left| \int_t^u \eta^{2-\gamma}(s)dB(s) \right| \right).
\] (24)

By the Burkholder–Davis–Gundy inequality and Jensen inequality, we obtain
\[
E\left(\sup_{t \leq u \leq t + 1} \left| \int_t^u \eta^{2-\gamma}(s)dB(s) \right| \right) \leq 6E\left(\int_t^{t+1} \eta^{2(2-\gamma)}(s)ds\right)^{\frac{1}{2}}
\leq 6\left(\int_t^{t+1} E\eta^{2(2-\gamma)}(s)ds\right)^{\frac{1}{2}}
\leq 6\left(\int_t^{t+1} E\eta^3(s)ds\right)^{\frac{2-\gamma}{3}}.
\]

By (22) and Theorem 3.2, we therefore see from (24) there exists a constant \(K_6\) such that
\[
E\left(\sup_{t \leq u \leq t + 1} \eta(u)\right) \leq K_6.
\] (25)

Let \(\varepsilon > 0\) be arbitrary. By the Chebyshev inequality, we have
\[
P\left\{ \sup_{k \leq t \leq k+1} \eta(t) > k^{1+\varepsilon} \right\} \leq \frac{K_6}{k^{1+\varepsilon}}, \quad k = 1, 2, \ldots.
\]

Applying the Borel-Cantelli lemma, for almost all \(\omega \in \Omega\), the following result
\[
\sup_{k \leq t \leq k+1} \eta(t) \leq k^{1+\varepsilon}
\] (26)
holds for all but finitely many \(k\). Hence, there exists a \(k_0(\omega)\), for almost all \(\omega \in \Omega\), the inequality (26) holds whenever \(k \geq k_0\). Consequently, for almost all \(\omega \in \Omega\), if \(k \geq k_0\) and \(k \leq t \leq k + 1\),
\[
\log \eta(t) \leq \frac{(1 + \varepsilon) \log k}{\log k} = 1 + \varepsilon.
\] (27)

That is,
\[
\liminf_{t \to \infty} \frac{\log Y(t)}{\log t} \geq -(1 + \varepsilon).
\] (28)

Letting \(\varepsilon \to 0\), we obtain the desired assertion. The proof is therefore complete. \(\square\)

**Remark 2.** The result of the above theorem means that for any \(\varepsilon > 0\), there is a random variable \(T_{\varepsilon} > 0\) such that the following inequality holds with probability one,
\[
Y(t) \geq t^{-(1+\varepsilon)}, \quad \text{for } \forall t \geq T_{\varepsilon}.
\] (29)

That is, the solution will not decay faster than \(t^{-(1+\varepsilon)}\) with probability one.

**5. Convergence in probability.** In the section we study the convergence property in probability of the Euler–Maruyama method, which is different from the Backward Euler-Maruyama and Backward-Forward Euler-Maruyama in [13]. For simplicity, let
\[
f(Y) = \beta_{-1}Y^{-1} - \beta_0 + \beta_1 Y - \beta_2 Y^\rho, \quad g(Y) = \sigma |Y|^{\gamma}
\]
Now we define the discrete Euler-Maruyama approximate solution to (2) for a given fixed timestep \(\Delta \in (0, 1)\) and initial condition \(y_0 = Y(0)\),
\[
y_{k+1} = y_k + f(y_k)\Delta + g(y_k)\Delta B_k,
\] (30)
where $\Delta B_k = B(t_{k+1}) - B(t_k)$ is the increment of Brownian motion. Let $[t/\Delta]$ be the integer part of $t/\Delta$, we hence introduce the step process

$$
\bar{y}(t) = \sum_{k=0}^{[t/\Delta]-1} y_k 1_{[k\Delta,(k+1)\Delta)}(t)
$$

and define the continuous approximation

$$
y(t) = y_0 + \int_0^t f(\bar{y}_k)ds + \int_0^t g(\bar{y}_k)dB(s).
$$

Since $y_k = y(t_k)$, an error bound for $y(t)$ will automatically implies the error bound for $\{y_k\}_{k \geq 0}$. We then mainly discuss the error bound for $y(t)$. Throughout this section, we use $C_k$ to denote a generic positive constant that is independent of $\Delta$ but may change between occurrences. To show the convergence in probability of the Euler-Maruyama method, we firstly establish the following lemmas.

**Lemma 5.1.** Let $v_k = \inf\{t > 0 : y(t) \not\in (\frac{1}{2}, k]\}$, then for any given $\varepsilon > 0$, there exist a step index $k$ which is sufficiently large and a stepsize $\Delta$ which is sufficiently small such that

$$
P(v_k \leq T) < \varepsilon.
$$

**Proof.** Let $\varpi \in [0, t \wedge v_k]$. Let $V$ be the same function as that defined in the proof of Theorem 2.1. By using the Itô formula

$$
EV(y(\varpi)) = V(Y(0)) + E\int_0^\varpi [V_Y(y(\varpi))f(\bar{y}(\varpi)) + \frac{1}{2}V_{YY}g^2(\bar{y}(\varpi))]d\varpi.
$$

Since

$$
V_Y(y(\varpi))f(\bar{y}(\varpi)) + \frac{1}{2}V_{YY}g^2(\bar{y}(\varpi)) = LV(y(\varpi)) + V_Y(y(\varpi))(f(\bar{y}(\varpi)) - f(\bar{y}(\varpi)))
$$

$$
+ \frac{\sigma^2}{2}V_{YY}(y(\varpi))(\bar{y}(\varpi)|^2 - |y(\varpi)|^2)
$$

$$
\leq K_1 + C_k|\bar{y}(\varpi) - y(\varpi)|,
$$

therefore we have that

$$
EV(y(\varpi)) \leq V(Y(0)) + K_1 T + C_k E\int_0^{\varpi}|\bar{y}(\varpi) - y(\varpi)|d\varpi.
$$

From the solution (32), the difference between the numerical simulation and solution of model (2) is given by

$$
y(\varpi) - \bar{y}(\varpi) = f(y(\varpi/\Delta))(\varpi - [\varpi/\Delta]\Delta) + \sigma|y(\varpi/\Delta)|^\gamma (B(\varpi) - B([\varpi/\Delta]\Delta))
$$

$$
\leq C_k \Delta + C_k|B(\varpi) - B([\varpi/\Delta]\Delta)|.
$$

Consequently, the expectation of the integral of the simulation error is

$$
E\int_0^\varpi |y(\varpi) - \bar{y}(\varpi)|d\varpi \leq C_k T \Delta + C_k E\int_0^\varpi |B(\varpi) - B([\varpi/\Delta]\Delta)|d\varpi
$$

$$
\leq C_k T \Delta + C_k T \Delta^{\frac{3}{2}}
$$

$$
\leq C_k T \Delta^{\frac{3}{2}}.
$$

Hence, the bound of the simulation error is

$$
EV(y(\varpi)) \leq V(Y(0)) + K_1 T + C_k T \Delta^{\frac{3}{2}}.
$$
Therefore, we have shown that
\[
P(v_k \leq T) \leq \frac{1}{V(1/k) \wedge V(k)} [V(Y(0)) + K_1 T + C_k T \triangle ^{1/2}].
\] (39)

Choose the value of step index \( k \) which is sufficiently large such that
\[
\frac{V(Y(0)) + K_1 T}{V(1/k) \wedge V(k)} < \frac{\varepsilon}{2}
\]
and then choose the value of stepsizes \( \triangle \) which is sufficiently small such that
\[
\frac{C_k T \triangle ^{1/2}}{V(1/k) \wedge V(k)} < \frac{\varepsilon}{2}.
\]

Based on the above the above two inequalities, we obtain the required result (33) using the inequality (39).

After showing the boundness of continuous approximate solution in probability, we now give the convergence property of the continuous approximate process to the step process inside the bounded area.

**Lemma 5.2.** For any \( T > 0 \) and \( v_k \) which is be defined in Lemma 5.1, then numerical simulation is convergent to the
\[
\lim_{\Delta \to 0} E \left( \sup_{0 \leq t \leq T \wedge v_k} |y(t) - \bar{y}(t)|^2 \right) = 0.
\] (40)

**Proof.** Recalling (36), for any \( t \in [0, T \wedge v_k] \), we have
\[
E \left( \sup_{0 \leq t \leq T \wedge v_k} |y(t) - \bar{y}(t)|^2 \right) \leq C_k \Delta + C_k E \left( \sup_{0 \leq t \leq T} |B(t) - B([t/\Delta] \Delta)|^2 \right). \] (41)

By using the Doob martingale inequality, we have that
\[
E \left( \sup_{0 \leq t \leq T} |B(t) - B([t/\Delta] \Delta)|^4 \right) = E \left( \sup_{0 \leq k \leq [T/\Delta]} \sup_{k \Delta \leq t \leq (k+1)\Delta} |B(t) - B(k\Delta)|^4 \right)
\]
\[
\leq \sum_{k=0}^{[T/\Delta]-1} E \left( \sup_{k \Delta \leq t \leq (k+1)\Delta} |B(t) - B(k\Delta)|^4 \right)
\]
\[
\leq \sum_{k=0}^{[T/\Delta]-1} E |B((k + 1)\Delta) - B(k\Delta)|^4
\]
\[
\leq 3 \sum_{k=0}^{[T/\Delta]-1} \Delta^2
\]
\[
= 3T \Delta.
\]

Hence, by the Lyapunov inequality, we further have
\[
E \left( \sup_{0 \leq t \leq T} |B(t) - B([t/\Delta] \Delta)|^2 \right) \leq (3T \Delta)^{1/2}.
\]

Thus,
\[
E \left( \sup_{0 \leq t \leq T \wedge v_k} |y(t) - \bar{y}(t)|^2 \right) \leq C_k \Delta^{1/2}, \] (42)

which implies (40).
Based on the above two Lemmas, we can now prove the convergence of the step process (31) to the continuous approximation process (32) in probability.

**Theorem 5.3.** For any \( T > 0 \), the step process (31) converges to the continuous approximation process (32), which is given by

\[
\lim_{\triangle \to 0} \left( \sup_{0 \leq t \leq T} |y(t) - \bar{y}(t)|^2 \right) = 0 \text{ in probability.}
\]

**Proof.** For arbitrarily small constants \( \delta, \varepsilon \in (0, 1) \), set

\[
\tilde{\Omega} = \{ \omega : \sup_{0 \leq t \leq T} |y(t) - \bar{y}(t)|^2 \geq \delta \}.
\]

Then, by Lemma 5.2, we have

\[
\delta P(\tilde{\Omega} \cap \{ v_k \geq T \}) \leq E\left( \sup_{0 \leq t \leq T \wedge v_k} |y(t) - \bar{y}(t)|^2 \right) \leq E\left( \sup_{0 \leq t \leq T \wedge v_k} |y(t) - \bar{y}(t)|^2 \right) \leq C_k \triangle^{\frac{1}{2}}.
\]

This, together with Lemma 5.1, yields that

\[
P(\tilde{\Omega}) \leq P(\tilde{\Omega} \cap \{ v_k \geq T \}) + P(v_k \leq T) \leq \frac{C_k \delta \triangle^{\frac{1}{2}}}{\varepsilon} + \frac{C_k T \triangle^{\frac{1}{2}}}{\varepsilon},
\]

Choose \( k \) which is sufficiently large such that

\[
\frac{V(Y(0)) + K_1 T}{V(1/k) \wedge V(k)} < \frac{\varepsilon}{2},
\]

and then choose \( \triangle \) which is sufficiently small such that

\[
\frac{C_k \delta \triangle^{\frac{1}{2}}}{\varepsilon} + \frac{C_k T \triangle^{\frac{1}{2}}}{\varepsilon} < \frac{\varepsilon}{2}.
\]

Hence we have

\[
P\left( \sup_{0 \leq t \leq T} |y(t) - \bar{y}(t)|^2 \geq \delta \right) < \varepsilon,
\]

which is the desired assertion.

Finally we give the theorem for the convergence of the continuous approximation process (32) to the solution of model (2) in probability.

**Theorem 5.4.** For any \( T > 0 \),

\[
\lim_{\triangle \to 0} \left( \sup_{0 \leq t \leq T} |Y(t) - y(t)|^2 \right) = 0 \text{ in probability.}
\]

**Proof.** Let \( \theta_k = \tau_k \wedge v_k \). \( \tau_k \) and \( v_k \) are defined in the proof of Theorem 2.1 and Lemma 5.1 respectively. Firstly, we show that

\[
E\left[ \sup_{0 \leq t \leq \theta_k \wedge T} |Y(t) - y(t)|^2 \right] \leq \overline{C}_k \triangle,
\]

where \( \overline{C}_k \) is a positive constant and independent of \( \triangle \) that may change from line to line.

For any \( 0 \leq t_1 \leq T \), from (2) and (32), we have

\[
Y(t_1 \wedge \theta_k) - y(t_1 \wedge \theta_k)
\]
By using (48), we have that
\[
= \int_0^{t_1 \land \theta_k} (f(Y(\mu)) - f(\bar{y}(\mu)))d\mu + \sigma \int_0^{t_1 \land \theta_k} (Y^\gamma(\mu) - |\bar{y}(\mu)|^\gamma)dB(\mu).
\]
Therefore, for any \( t \in [0, T] \), by the Hölder inequality and the Brükholder-Davis-Gundy inequality, we have
\[
E\left( \sup_{0 \leq t_1 \leq t} |Y(t_1 \land \theta_k) - y(t_1 \land \theta_k)|^2 \right) 
\leq 2tE \int_0^{t_1 \land \theta_k} |f(Y(\mu)) - f(\bar{y}(\mu))|^2d\mu + 8\sigma^2E \int_0^{t_1 \land \theta_k} |Y^\gamma(\mu) - \bar{y}^\gamma(\mu)|^2d\mu. \tag{49}
\]
Note that \( \gamma \geq 1 \), then for \( Y(\mu), \bar{y}(\mu) \in (1/k, k) \), we have that
\[
|Y^\gamma(\mu) - \bar{y}^\gamma(\mu)|^2 
\leq \underline{C}_k|Y(\mu) - \bar{y}(\mu)|^{2\gamma} 
\leq \underline{C}_k|Y(\mu) - \bar{y}(\mu)|^2|Y(\mu) - \bar{y}(\mu)|^{2\gamma-2} 
\leq \underline{C}_k|Y(\mu) - \bar{y}(\mu)|^2 
\leq \underline{C}_k(|Y(\mu) - y(\mu)|^2 + |y(\mu) - \bar{y}(\mu)|^2). \tag{50}
\]
Similarly, we have
\[
|f(Y(\mu)) - f(\bar{y}(\mu))|^2 \leq \underline{C}_k(|Y(\mu) - y(\mu)|^2 + |y(\mu) - \bar{y}(\mu)|^2) \tag{51}
\]
Substituting (50) and (51) into (49) yields
\[
E\left( \sup_{0 \leq t_1 \leq t} |Y(t_1 \land \theta_k) - y(t_1 \land \theta_k)|^2 \right) 
\leq \underline{C}_k \left( E \int_0^{t_1 \land \theta_k} |Y(\mu) - y(\mu)|^2d\mu + E \int_0^{t_1 \land \theta_k} |y(\mu) - \bar{y}(\mu)|^2d\mu \right) 
\leq \underline{C}_k \left( \int_0^t E|Y(\mu \land \theta_k) - y(\mu \land \theta_k)|^2d\mu + E \int_0^{t_1 \land \theta_k} |y(\mu) - \bar{y}(\mu)|^2d\mu \right). \tag{52}
\]
Recalling the computation of (57), there exists a constant \( \overline{D} \) such that
\[
E \int_0^{t_1 \land \theta_k} |y(\mu) - \bar{y}(\mu)|^2d\mu \leq \overline{D}\Delta. \tag{53}
\]
Therefore, we have
\[
E\left( \sup_{0 \leq t_1 \leq t} (Y(t_1 \land \theta_k) - y(t_1 \land \theta_k))^2 \right) \leq \underline{C}_k \int_0^t E[Y(\mu \land \theta_k) - y(\mu \land \theta_k)]^2d\mu + \overline{C}_k\Delta.
\]
Using the Gronwell inequality yields the result (48).

Now let \( \varepsilon, \delta \in (0, 1) \) be arbitrarily small. Set
\[
\Omega = \{ \omega : \sup_{0 \leq t \leq T} |Y(t) - y(t)|^2 \geq \delta \}. \tag{48}
\]
By using (48), we have that
\[
\delta P(\Omega \cap \{ \theta_k \geq T \}) \leq E \left[ 1_{\{ \theta_k \geq T \}} \sup_{0 \leq t \leq T \land \theta_k} |Y(t) - y(t)|^2 \right] 
\leq E \left[ \sup_{0 \leq t \leq T \land \theta_k} |Y(t) - y(t)|^2 \right] 
\leq \underline{C}_k\Delta.
\]
This, together with Theorem 2.1 and Lemma 5.1 implies
\[
P(\Omega) \leq P(\Omega \cap \{\theta_k \geq T\}) + P(\tau_k \leq T)
\leq P(\Omega \cap \{\theta_k \geq T\}) + P(\tau_k \leq T) + P(v_k \leq T)
\leq \frac{C_k}{\delta} - \frac{2V(Y(0)) + 2K_1 T + C_k T \Delta^\frac{1}{2}}{V(1/k) \wedge V(k)},
\]
(54)
Recalling that when \( k \to \infty \), \( V(1/k) \wedge V(k) \to \infty \). Thus we can choose \( k \) which is sufficiently large for
\[
\frac{2V(Y(0)) + 2K_1 T}{V(1/k) \wedge V(k)} < \varepsilon \frac{2}{
}
and then choose \( \Delta \) which is sufficiently small for
\[
\frac{C_k}{\delta} - \frac{C_k T \Delta^\frac{1}{2}}{V(1/k) \wedge V(k)} < \varepsilon \frac{2}{
}
to obtain
\[
P(\Omega) = P\left( \sup_{0 \leq t \leq T} |Y(t) - y(t)|^2 \geq \Delta \right) < \varepsilon,
\]
(55)
which is the desired assertion.

By Theorem 5.3 and 5.4, we can easily show the convergence for the discrete-time solution to the solution of model (2).

**Corollary 1.** For any \( T > 0 \),
\[
\lim_{\Delta \to 0} \left( \sup_{0 \leq t \leq T} |Y(t) - \tilde{y}(t)| \right) = 0 \quad \text{in probability.}
\]
(56)
Finally, we give simulations of the model (2). We use part of the parameters that were estimated from financial data [1], namely \( \beta_{-1} = 1.041 \times 10^{-4}, \beta_0 = 5.652 \times 10^{-3}, \beta_1 = 9.648 \times 10^{-2}, \beta_2 = 5.349 \times 10^{-1}, \) and \( \sigma = 1.329 \times 10^{-2} \); and adjust the parameters \( \gamma \) and \( \rho \) to satisfy the condition \( 2\gamma > \rho + 1 \). Here we provide three simulations using \( (\gamma, \rho) = (1.025, 1.01), (\gamma, \rho) = (2, 2), (\gamma, \rho) = (3, 3) \). Numerical results in Figure 1A suggest that the volatility is small since the diffusion parameter is only \( \sigma = 1.329 \times 10^{-2} \). In addition, if both \( \gamma \) and \( \rho \) are smaller, the simulation converge to a small interest rate gradually. When \( \gamma = \rho = 2 \), the simulation maintains in a value which is close to the initial condition. However, when \( \gamma \) and \( \rho \) are large \( (\gamma = \rho = 3) \), the simulation is getting larger and larger. To show the fluctuations of solution, we change the diffusion parameter to \( \sigma = 0.1 \). Simulations in Figure 1B suggest that the volatility is relatively much larger.

6. **Application in finance.** The convergence theory established in the previous section guarantees that the numerical method can be used to estimate a number of financial problems of option pricing. For example, an up-and-out call option, at expiry time \( T \), pays the European option value if \( Y(t) \) never exceeded the fixed barrier \( B \), and pays zero otherwise. We suppose that the expected payoff is computed from (13). Define
\[
\Xi := E((Y(T) - K)^+1_{\{0 \leq Y(t) \leq B, 0 \leq t \leq T\}});
\]
and
\[
\hat{\Xi}_\Delta := E((\tilde{y}(T) - K)^+1_{\{0 \leq \tilde{y}(t) \leq B, 0 \leq t \leq T\}});
\]
Figure 1. Simulated discrete Euler-Maruyama approximation. (A) Parameter value $\sigma = 1.329 \times 10^{-2}$. (B) Parameter value $\sigma = 0.1$. (Solid-line: $\gamma = 1.025$, $\rho = 1.01$; dash-line: $\gamma = 2$, $\rho = 2$; dash-dot-line: $\gamma = 3$, $\rho = 3$.

where the exercise price $K$ and the barrier $B$ are constants, $Y(t)$ and $\bar{y}(t)$ be defined by (2) and (31), respectively. Hence, we can obtain

$$\lim_{\triangle \to 0} |\Xi - \hat{\Xi}_\triangle| = 0.$$

We refer to [8,11,15] for details regarding the simulation algorithms.

7. Conclusion. In this work we have studied the properties for the solution of Ait-Sahalia-Rho model that is a useful tool to study a number of financial problems, including the term structure of interest rate. Since the coefficients of this model do not satisfy the linear growth condition, we could not study the solution properties of this model by using the traditional techniques. In this paper we use numerical simulations to study this model. The major work is the theorem for the convergence property of the Euler-Maruyama method for solving the Ait-Sahalia-Rho model. We introduced the continuous approximation process and separate the proof of the theorem into two steps: the convergence of numerical solution to the continuous process and the convergence of continuous process to the solution of the model. In addition, we also studied the existence global positive solution, boundedness property of the solution and asymptotic pathway estimation of the model. However, our simulation results suggest that the stability property should also be investigated. In addition, the parameters of the model were estimated by using the financial data more than 20 years ago. More work is also needed to use the recent financial data, in particular the data after the financial crisis in 2007-2008, to estimate model parameters, which may have different indications for the model parameters.

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Received August 2015; revised April 2016.

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