New restricted biased estimator based on modified unbiased ridge regression estimator

Bader Aboud Mohammad and Mustafa Ismaeel Naif

Department of Mathematics, College of education for pure science, University Of Anbar, Anbar, Iraq

E-mail: bad19u2003@uoanbar.edu.iq

Abstract. More researchers used biased estimators instead of the restricted least square estimator (RLS) that is unbiased to improve the efficiency of the estimator. Therefore, in this paper, we proposed new biased estimator, called restricted modified unbiased ridge regression (RMUR) based on modified unbiased ridge regression (MURR) that suggested by Batah and Gory (2009). By some theorems and a simulation study, we show that, this new estimator has desirable properties as the MUR and it is better than the RLS. A numerical example from literature is used to illustrate the results.

1. Introduction

Consider the following linear regression model

\[ Y = X\beta + \varepsilon, \]

where \( X \) is known \((n \times p)\) matrix with \( \text{rank}(X) = p \), \( Y \) is \((n \times 1)\) response vector, The error term \( \varepsilon \) is a \(n \times 1\) vector of error with \( E(\varepsilon) = 0 \) and \( \text{var}(\varepsilon) = \sigma^2 I_n \) and \( \beta \) is a \((p \times 1)\) unknown vector coefficient. Ordinary least square (OLS) estimator of \( \beta \) in model (1) is given by

\[ \hat{\beta}_L = (X'X)^{-1}X'Y \]

When there is a linear relationship between independent variables, the problem of multicollinearity will be exist. In case of multicollinearity, the (OLS) estimator of regression coefficients may be statistically significant with wrong sign and large variance. Therefore the results are far from the actual values. To overcome this problem, the biased estimator has been used. One of these estimators is ordinary ridge regression ORR which proposed by Hoerl and Kennard (1970). Hoerl and Kennard derived the ORR estimator as the solution of the following problem; minimize \( \beta^*\beta^* \) subject to \( (y - X\beta^*)'(y - X\beta^*) = c \). That means:

\[ \beta^*\beta^* + \frac{1}{k}[y - X\beta^*]'(y - X\beta^*) - c \]

Where \( \frac{1}{k} \) is a Lagrangian multiplier and \( c \) is constant. The solution (3) obtained the (ORR) estimator as the following:

\[ \hat{\beta}(k) = [I - k(X'X + kI_p)]^{-1}\hat{\beta}_{LS} \]
\[(X'X + kl_p)^{-1}X'Y, k > 0. \] (4)

Modified ridge estimator by Swindel (1976), the restricted ridge by Zhong and Yang (2007), modified new two-parameter by Lukman (2019), restricted and unrestricted by Ozkale (2007), Crouse et al. (1995) proposed the unbiased ridge regression (URR) estimator as the following:

\[\hat{\beta}(k,j) = (X'X + kl_p)^{-1}(X'Y + kf), k \geq 0\] (5)

where \(j\) is a random vector with \(j \sim N(\beta, \frac{\sigma^2}{k}I_p)\) for \(k > 0\). To improve the performance of the estimators by using biased estimation technique, Batah and Gore (2009) proposed modified unbiased ridge regression (MURR) as:

\[\hat{\beta}_j(k) = \left[I - k(X'X + kl_p)^{-1}\right]\hat{\beta}(k,j)\]
\[= \left[I - k(X'X + kl_p)^{-1}\right](X'X + kl_p)^{-1}(X'Y + kf)\] (6)

Although the (MURR) estimator is a good estimator in dealing with a problem of multicollinearity, as it is considered one of best biased estimators in dealing with this problem, but it also has negative characteristic and this characteristic arises from its dependence on the value of \(k\). As \(k \to \infty\), \(\hat{\beta}_j(k) \to 0\) a stable, but biased estimator of \(\beta\). As \(k \to 0\), \(\hat{\beta}_j(k) \to \hat{\beta}_{RLS}\), an unbiased unstable estimator of \(\beta\) where \(\hat{\beta}_{RLS}\) still dealing with the matrix \(X'X\) which is responsible for multicollinearity. That is \(\hat{\beta}_j(k)\) trace curve path the parameter space from \(\hat{\beta}_j(k)\) to zero, so the distance between \(\hat{\beta}_j(k)\) and \(\hat{\beta}_{RLS}\) decreases when \(k\) increases from 0. Therefore, we propose new biased estimator to improve the efficiency of estimation which we called restricted modified unbiased ridge regression (RMUR).

In section 2, we study the properties of new estimator. In section 3, we study the performance of the new estimator compared with some other estimators by using the simulation technique. Section 4 contains numerical example to illustrate the results. Finally, the conclusions with some remarks are given in section 5.

2. Proposed Estimator and its properties

To overcome the problem of multicollinearity we can also use prior information that can be considered as a linear restriction which is given by:

\[R\beta = r, \] (7)

where \(R\) is an \((m \times p)\) known matrix with \(p\) rank \(m > p\) and \(r\) is an \((m \times 1)\) vector.

The idea is that, we try to find an estimator of \(\beta\) that is closest to \(\hat{\beta}_j(k)\) by taking a point that is closest to \(\hat{\beta}_j(k)\) with same residual of sum square when the restrictions of \(\beta\) given by (6) are available. Thus, we derive a new estimator as the solution of the minimization problem for the function defined by

\[\Phi = (\beta - \hat{\beta}_j(k))' (\beta - \hat{\beta}_j(k)) + \frac{1}{k} [(y - X\beta)'(y - X\beta) - c] - 2\lambda' (R\beta - r)\]

where \(\frac{1}{k}\) is a Lagrangian multiplier, \(\lambda\) is a vector on Lagrangian multipliers and \(c\) is a constant. We differentiate the \(\Phi\) with respect \(\beta\) and \(\lambda\) respectively, and we obtain the following equations.
\[
(\hat{\beta} - \hat{\beta}_j(k)) + \frac{1}{k} (X'X\hat{\beta} - X'y) - R'\lambda = 0
\]  
(8)

\[
R\hat{\beta} - r = 0
\]  
(9)

From (7)

\[
\hat{\beta} = (X'X + kI)^{-1} (X'y + k\hat{\beta}_j(k)) + (X'X + k)^{-1} R'k\lambda
\]  
(10)

If we premultiply (9) by \(R\) and inserting in (8), we obtain the solution of normal equations

\[
\hat{\beta}_r^*(k) = \hat{\beta}_j(k) + S_k^{-1} R' (RS_k^{-1} R')^{-1} (r - R\hat{\beta}_j(k))
\]  
(11)

where \(S_k^{-1} = (X'X + kI)^{-1}\).

Now, we need to show that \(\hat{\beta}_r^*(k)\) satisfy the restrictions given by (6). Thus

\[
R \hat{\beta}_r^*(k) = R [\hat{\beta}_j(k) + S_k^{-1} R' (RS_k^{-1} R')^{-1} (r - R\hat{\beta}_j(k))] = R \hat{\beta}_r^*(k) = R \hat{\beta}_j(k) + r - R\hat{\beta}_j(k) = r
\]

We can rewrite \(\hat{\beta}_r^*(k)\) as follows:

Let \(\beta_1 = R'(RR')^{-1} r\) and satisfy the restriction in (7), therefore, we have

\[
\hat{\beta}_r^*(k) = \hat{\beta}_j(k) + S_k^{-1} R' (RS_k^{-1} R')^{-1} (r - R\hat{\beta}_j(k)) = S_k^{-1} S_k \hat{\beta}_j(k) + S_k^{-1} R' (RS_k^{-1} R')^{-1} (r - R\hat{\beta}_j(k)) = S_k^{-1} S_k \hat{\beta}_j(k) + S_k^{-1} R' (RS_k^{-1} R')^{-1} R S_k^{-1} S_k \hat{\beta}_j(k) = (S_k^{-1} - S_k^{-1} R' (RS_k^{-1} R')^{-1} R S_k^{-1}) S_k \hat{\beta}_j(k) + (S_k^{-1} R' (RS_k^{-1} R')^{-1} R) S_k \hat{\beta}_j(k) = N_k S_k \hat{\beta}_j(k) + [S_k^{-1} R' (RS_k^{-1} R')^{-1} R] \beta_1 = N_k S_k \hat{\beta}_j(k) + [S_k^{-1} R'(RS_k^{-1} R')^{-1} R] \beta_1 - S_k^{-1} S_k \beta_1 + \beta_1 = N_k S_k \hat{\beta}_j(k) - [S_k^{-1} - S_k^{-1} R'(RS_k^{-1} R')^{-1} R S_k^{-1}] S_k \beta_1 + \beta_1 = N_k S_k \hat{\beta}_j(k) - N_k S_k \beta_1 + \beta_1 = \hat{\beta}_r^*(k) = N_k S_k \hat{\beta}_j(k) - N_k \beta_1 + \beta_1
\]  
(12)

where \(N_k = (S_k^{-1} - S_k^{-1} R'(RS_k^{-1} R')^{-1} R S_k^{-1})\).

2.1 Matrix Mean Square Error (MMSE)

The bias and variance of an estimator are measured simultaneously by the mean square error matrix (MSE)

\[
MSE(\hat{\beta}^*) = var(\hat{\beta}^*) + Biase(\hat{\beta}^*)(Biase(\hat{\beta}^*))'
\]

For this purpose,
\[ \text{MSE}(\hat{\beta}_j(k)) = \sigma^2WS^{-1}W' + k^2S_k^{-1}\beta'\beta S_k^{-1}, \]
\[ \text{MSE}(\hat{\beta}_{RLS}) = \sigma^2N_0S^{-1}N_0, \]
\[ \text{MSE}(\hat{\beta}_k(k)) = \sigma^2N_kSN_k + k^2N_k\beta'\beta N_k', \]
where \( W = (I - kS_k^{-1}) = SS_k^{-1}, N_0 = N_k \) when \( k = 0 \) and \( S = (X'X). \)

The expected value of RMUR is given by:
\[ E(\hat{\beta}_r^*(k)) = E(N_kS_k(\hat{\beta}_j(k) - \beta_1) + \beta_1) \]

Since \( E(\hat{\beta}_j(k)) = (I - kS_k^{-1})\beta \), so
\[ E(\hat{\beta}_r^*(k)) = N_kS_k((I - kS_k^{-1})\beta - \beta_1) + \beta_1 \]
\[ = N_kS_k(\beta - \beta_1) + \beta_1 - kN_k\beta \]

Since \( R\beta = r \), from Eq(7) \( R\beta_1 = r \). So, we get that \( N_kS_k(\beta - \beta_1) = \beta - \beta_1 \)

Therefore,
\[ E(\hat{\beta}_r^*(k)) = \beta - kN_k\beta = (I - kN_k)\beta \] (16)

The variance, the bias and MMSE of RMUR are given as follows:
\[ \text{Var}(\hat{\beta}_r^*(k)) = E(\hat{\beta}_r^*(k) - E(\hat{\beta}_r^*(k))(\hat{\beta}_r^*(k) - E(\hat{\beta}_r^*(k)))' \]
\[ = \sigma^2N_kS_kWS_k^{-1}S_kW'N_k + k^2N_k\beta'\beta N_k' \]
\[ \text{bias}(\hat{\beta}_r^*(k)) = E(\hat{\beta}_r^*(k)) - \beta \]
\[ = \beta - kN_k\beta - \beta = -kN_k\beta \] (18)

So that,
\[ \text{MMSE}(\hat{\beta}_r^*(k)) = \text{Var}(\hat{\beta}_r^*(k)) + [\text{bias}(\hat{\beta}_r^*(k))][\text{bias}(\hat{\beta}_r^*(k))]' \]
\[ = \sigma^2N_kS_kWS_k^{-1}S_kW'N_k + k^2N_k\beta'\beta N_k' \] (19)

The scalar mean square error of is defined as follows:
\[ \text{mse}(\hat{\beta}_r^*(k)) = \sigma^2\text{tr}(N_kSS_k^{-1}N_k) + k^2\text{tr}(N_k\beta'\beta N_k), \]

where \( \text{tr} \) denotes trace and \( N_k = (S_k^{-1} - S_k^{-1}R'(RS_k^{-1}R')^{-1}RS_k^{-1}). \) It is well known that a linear regression model can be transformed to conical from by orthogonal transformation let \( T \) be an orthogonal matrix such that;
\[ T'X'X = \Lambda \] (20)

Where \( \Lambda \) is a \( p \times p \) diagonal matrix elements \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \) are the eigenvalues.

So, we can state the following theorem.

Theorem 2.1: The scalar mean square error of \( \hat{\beta}_r^*(k) \) is given by:
\[ \text{mse}(\hat{\beta}_r^*(k)) = \sigma^2\sum_{i=1}^{p} \frac{\lambda_i^2(\lambda_i + k - r_{ii})^2}{(\lambda_i + k)^5} + k^2\sum_{i=1}^{p} \frac{\alpha_i^2(\lambda_i + k - r_{ii})^2}{(\lambda_i + k)^4} \] (21)

Proof:
\[ N_kSS_k^{-1}N_k = (S_k^{-1} - S_k^{-1}R'(RS_k^{-1}R')^{-1}RS_k^{-1})SS_k^{-1}(S_k^{-1} - S_k^{-1}R'(RS_k^{-1}R')^{-1}RS_k^{-1}) \]
\[ = (S_k^{-1}SS_k^{-2}) - (S_k^{-1}SS_k^{-2}R'(RS_k^{-1}R')^{-1}RS_k^{-1} - S_k^{-1}R'(RS_k^{-1}R')^{-1}RS_k^{-1}SS_k^{-2}) \]
\[ + S_k^{-1}R'(RS_k^{-1}R')^{-1}RS_k^{-1}SS_k^{-2}R'(RS_k^{-1}R')^{-1}RS_k^{-1} \] (22)

We have \( S_k^{-1} = T(\Lambda + k)^{-1}T' \).
\[ \text{tr}(S_k^{-1}SSS_k^{-2}) = \text{tr}[(T(A + k)^{-1}T'\text{AT}'TAT'(T(A + k)^{-2}T')] = \text{tr}[(T(A + k)^{-3}T'\text{AT}'TAT')] \]

\[ = \sum_{i=1}^{p} \frac{\lambda_i^2}{(\lambda_i + k)^3}. \quad (23) \]

\[ \text{tr} \left( S_k^{-1}SSS_k^{-2}R'((RS_k^{-1}R')^{-1}RS_k^{-1}) \right) = \text{tr}[(T(A + k)^{-1}T'\text{AT}'TAT'(T(A + k)^{-2}T'r_{ii}'T(A + k)^{-1}T')] \]

\[ = \sum_{i=1}^{p} \frac{\lambda_i^2 r_{ii}^*}{(\lambda_i + k)^4}. \quad (24) \]

\[ R^* = TR'(RS_k^{-1}R')RT'. \text{The diag}(R^*) = r_{ii}^* \]

\[ \text{tr} \left( S_k^{-1}R'(RS_k^{-1}R')^{-1}RS_k^{-1}SSS_k^{-2} \right) = \text{tr}[(T(A + k)^{-1}T'R'T(A + k)^{-1}T'\text{AT}'T'\text{AT}'T(A + k)^{-2}T'R'T(A + k)^{-1}T')] \]

\[ = \sum_{i=1}^{p} \frac{\lambda_i^2 r_{ii}^2}{(\lambda_i + k)^5}. \quad (25) \]

From (23), (24), (25) and (26), we get the following result.

\[ \text{tr}(N_kSSS_k^{-1}N_k) = \left[ \sum_{i=1}^{p} \frac{\lambda_i^2}{(\lambda_i + k)^3} - 2 \sum_{i=1}^{p} \frac{\lambda_i^2 r_{ii}^2}{(\lambda_i + k)^4} + \sum_{i=1}^{p} \frac{\lambda_i^2 r_{ii}^2}{(\lambda_i + k)^5} \right] \]

\[ = \sum_{i=1}^{p} \frac{\lambda_i^2(\lambda_i + k - r_{ii}^*)^2}{(\lambda_i + k)^5}. \quad (27) \]

Also,

\[ \text{tr}(N_k^2) = \left( S_k^{-1} - S_k^{-1}R'(RS_k^{-1}R')^{-1}S_k^{-1} \right) \left( S_k^{-1} - S_k^{-1}R'(RS_k^{-1}R')^{-1}S_k^{-1} \right) \]

\[ = S_k^{-2} - S_k^{-2}R'(RS_k^{-1}R')S_k^{-1} - S_k^{-1}R'(RS_k^{-1}R')S_k^{-2} \]

\[ + S_k^{-1}R'(RS_k^{-1}R')^{-1}RS_k^{-2}R'(RS_k^{-1}R')^{-1} \]

\[ \text{tr}(S_k^{-2}) = \text{tr}[(T(A + k)^{-2}T')] = \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \]

\[ \text{tr}(S_k^{-2}R'(RS_k^{-1}R')S_k^{-1}) = \text{tr}[(T(A + k)^{-2}T'R'T(A + k)^{-1}T')] \]

\[ = \sum_{i=1}^{p} \frac{r_{ii}^*}{(\lambda_i + k)^3}. \quad (29) \]

Similar to (30), we get:

\[ \text{tr}(S_k^{-1}R'(RS_k^{-1}R')S_k^{-2}) = \sum_{i=1}^{p} \frac{r_{ii}^*}{(\lambda_i + k)^3}. \quad (31) \]

\[ \text{tr} \left( S_k^{-1}R'(RS_k^{-1}R')^{-1}RS_k^{-2}R'(RS_k^{-1}R')^{-1}RS_k^{-1} \right) \]
\[
\sum_{i=1}^{p} \frac{r_{ii}^{*2}}{(\lambda_i + k)^4}.
\]

From (28) to (32), we get
\[
tr(N_k^2) = \sum_{i=1}^{p} \left( \frac{1}{(\lambda_i + k)^2} - \frac{2r_{ii}^{*}}{(\lambda_i + k)^3} + \frac{r_{ii}^{*2}}{(\lambda_i + k)^4} \right)
\]
\[
= \sum_{i=1}^{p} \frac{(\lambda_i + k - r_{ii}^{*})^2}{(\lambda_i + k)^4}.
\]

Therefore,
\[
tr(\alpha^T N_k^{2} \alpha) = \sum_{i=1}^{p} \frac{\alpha_i^{2}(\lambda_i + k - r_{ii}^{*})^2}{(\lambda_i + k)^4}.
\]

Where \( \alpha = T' \beta = (\alpha_1 \ldots \ldots \alpha_p) \). Finally, we have
\[
mse(\beta^*_p(k)) = \sigma^2 \sum_{i=1}^{p} \frac{\lambda_i^2(\lambda_i + k - r_{ii}^{*})^2}{(\lambda_i + k)^8} + k^2 \sum_{i=1}^{p} \frac{\alpha_i^2(\lambda_i + k - r_{ii}^{*})^2}{(\lambda_i + k)^4}.
\]

The proof is completed.

3. Simulation Study

The purpose of this section is to make a comparison among the different biased estimators in order to identify good estimator among them. Therefore, we conduct a simulation study by using the Matlab program. This simulation is created depending on factors that affect the properties of the estimator’s due to the degree of the collinearity among several explanatory variables. Kibria (2003) was followed to generate the explanatory variables by using the equation.

\[
x_{ij} = (1 - Y^2)z_{ij} + Yz_{ip}, \quad i = 1 \ldots \ldots n, \quad j = 1 \ldots \ldots p
\]

where the \( z_{ij} \) independent standard normal pseudo-random numbers and \( Y \) represents the correlation between any two variables. These variables are standardized so that \( X'X \) is being in correlation form.

The response variable \( y \) is considered by:

\[
y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots \ldots + \beta_p x_{ip} + e_i, \quad i = 1,2,\ldots,n
\]

where \( e_i \) is i.i.d. N(0, \( \sigma^2 \)). Therefore, zero intercept for (36) will be assumed. Also the number of explanatory variables \( p = 4 \), while the values of \( \sigma \) are choose as (1, 4, 9). The correlation \( Y \) will choose as (0.90, 0.95, 0.99) and sample size \( n \) as (25, 50, 100, ).The coefficients \( \beta_1, \beta_2, \ldots, \beta_p \) are selected as the eigenvectors corresponding to the largest eigenvalue of the matrix \( X'X \) subject to constraint \( \beta' \beta = 1 \). Thus, for all \( n, \sigma, \lambda, p, \beta \) and \( Y \), sets of \( X's \) are created. The experiment was replicated 5000 times by creating new error terms. Estimated mean square error (EMSE) is calculated as follows:

\[
EMSE(\beta^*) = \frac{1}{5000} \sum_{i=1}^{5000} (\beta^* - \beta)'(\beta^* - \beta),
\]

where \( \beta^* \) would be any estimators (RLS, MURR, RRR or RMUR).

3.1 The discussion of simulation results

In this article, we present the results of the Monte Carlo experiment regarding the properties of the different methods that used to choose the ridge parameter \( k \). The simulation results are presented from Table 1 to Table 7 and we discuss the results in all cases.

Table 1 and Table 7 show the performance of the RMUR estimator compared to some other estimators as the following:

1. The RMUR estimator is the best, because the RMUR estimator has lowest MSE compared with other estimators for all different sample size \( n \) in all correlation coefficient values and \( \sigma \). However, in some cases the MURR estimator is close or similar to the RMUR estimator from these cases.

2. In case \( (n = 25, \sigma = 1, Y = .90) \) and \( (n = 50, \sigma = 1, Y = .90) \), the MUR estimator is close or similar the RMUR estimator in this cases.
4. Numerical example

In this study, the performance of the proposed estimator using a real life data is illustrated. We consider the data about the total national product, which is cited by Akdeniz (2003) for comparison of the estimators that given in this study. This data shows the relationship between the dependent variable $Y$ and the percentage that is the united states spent on four independent variables $X_1, X_2, X_3$ and $X_4$ representing the percentage spent by France, West Germany, Japan, and Soviet Union respectively. The goal is to compare the scalar mean square error $mse$ of the RLS, RRR, MURR and RMUR. The $mse$ of the MURR, RLS, RRR and RMUR are given in Eq.(3), Eq.(14), Eq.(15) and Eq.(19) respectively, which given in Table 8.

Table 1. The scalar mean square error for different estimators and different estimated ridge parameter

| $k$    | $OLS$       | $RLS$       | $RRR$       | $MURR$      | $RMUR$      |
|--------|-------------|-------------|-------------|-------------|-------------|
| 0.0161 | 602.4197    | 135.7475    | 96.4878     | 159.9901    | 83.1722     |
| 0.0243 | 602.4197    | 135.7475    | 83.0983     | 110.1283    | 67.6833     |
| 0.050  | 602.4197    | 135.7475    | 56.1576     | 54.1576     | 40.0858     |
| 0.020  | 602.4197    | 135.7475    | 89.7099     | 131.9846    | 75.1888     |
| 0.50   | 602.4197    | 135.7475    | 7.0013      | 5.9080      | 3.7911      |
| 0.10   | 602.4197    | 135.7475    | 32.7403     | 26.7126     | 20.2068     |
| 0.15   | 602.4197    | 135.7475    | 22.5322     | 17.7867     | 13.0448     |

For the linear restrictions (6), the values $R$ and $r$ are, respectively, given as follows Najarian (2013) $R = [1 1 1 1; 0 1 3 1]$, $r = [1.2170 \ 1.0904]$

In Table 8, we can observe that, the RMUR estimator has lowest $mse$ for all different estimated ridge parameter. Therefore, the performance the RMUR estimator is better than other estimators and this is clear in the Figures 1,2 and 3.

Figure 1. Estimated mse of the proposed estimator compared for different estimators and different estimated ridge parameter $k$. 
Figure 2. Estimated mse of the proposed estimator compared for different estimators and different estimated ridge parameter k.

Figure 3. Estimated mse of the proposed estimator compared for different estimators and different estimated ridge parameter k.

5. Conclusions
The simulation study in section 3 shows that, the RMOR estimator is better than of the other estimators for all values $n$, $\sigma$ and $Y$. Also, the RMUR estimator has good properties compared with other estimators. The RMUR estimator is more efficient where it has lowest mean square error compared to others, and with respect to the real data that used in the present study.

References
[1] Akdeniz F, Erol H (2003) Mean squared error matrix comparisons of some biased estimators in linear regression Communications in Statistics Theory and Methods 32:2389–2413
[2] Batah F S M & Gore S D (2009) Ridge regression estimator: Combining unbiased and ordinary ridge regression methods of estimation Surveys in Mathematics and its Applications 4 99-109
[3] Crouse R, Jin C & Hanumara R (1995) Unbiased ridge estimation with prior informatics and ridge trace Communications in Statistics – Theory and Materials 24(9) 2341-2354
[4] Hoerl A E, Kennard R W, Baldwin K F (1975) Ridge regression: some simulation Commun Statist 4:105–123
[5] Kibria B M G (2003) Performance of some new ridge regression estimators Communications in Statistics – Simulation and Computation 32(2) 419-435
[6] Lukman A F, Ayinde K, Siok Kun S & Adewuyi E T (2019) A modified new two-parameter estimator in a linear regression model Modelling and Simulation in Engineering 2019
[7] Najarian S, Arashi M and Kibria B M G (2013) A Simulation Study on Some Restricted Ridge Regression Estimators Comm. Statist. Sim. Comp. 42 871-879
[8] Ozkale M R, Kaciranlar S (2007) The restricted and unrestricted two parameter estimators Communications in Statistics - Theory and Methods 36:2707–2727
[9] Swindel B F (1976) Good ridge estimators based on prior information Communications in Statistics–Theory and Methods
[10] Zhong Z, Yang H (2007) Ridge estimation to the restricted linear model Communications in Statistics–Theory and Methods 36(11)