Compressibility enhancement in an almost staggered interacting Harper model

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We discuss the compressibility in the almost staggered fermionic Harper model with repulsive interactions in the vicinity of half-filling. It has been shown by Kraus et al. [33] that for spinless electrons and nearest neighbors electron-electron interactions the compressibility in the central band is enhanced by repulsive interactions. Here we would like to investigate the sensitivity of this conclusion to the spin degree of freedom and longer range interactions. We use the Hartree-Fock (HF) approximation, as well as density matrix renormalization group (DMRG) calculation to evaluate the compressibility. In the almost staggered Harper model, the central energy band is essentially flat and separated from the other bands by a large gap and therefore, the HF approximation is rather accurate. In both cases the compressibility of the system is enhanced compared to the non-interacting case, although the enhancement is weaker due to the inclusion of Hubbard and longer ranged interactions.

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INTRODUCTION

The interplay between electron-electron (e-e) interactions and quasi-disorder has drawn much excitement since the discovery of quasi-crystals [1, 2]. Much of the work has focused on a specific model of a one-dimensional (1D) quasi-crystal, namely the Harper (or Aubry-André) model [3, 4]. One of the main attractions of this model is that contrary to conventional 1D disordered systems which are localized for any amount of disorder [5], the Harper model exhibits a metal-insulator transition as function of the quasi-disordered potential strength, even in the absence of interactions [4, 6–11]. The influence of e-e interactions on the metal-insulator transition of the Harper model was studied in several publications [12–14]. Interest in the Harper model has lately peaked after it has been shown that for an irrational modulation, the Harper model may be a 1D topologically nontrivial system, and have topological boundary states [15–23]. This property, coupled with the fact that the Harper model may be realized in the context of cold atoms and molecules [24, 25] added to the excitement surrounding the Harper model.

Recently, an additional aspect of the model has been investigated, namely the inverse compressibility, which measures the change in the chemical potential when an electron is added to the system. In the context of disordered quantum dots this has become a very popular measurement to extract information on the role of e-e interactions, particularly the inverse compressibility. In the almost staggered Harper model, the central energy band is essentially flat and separated from the other bands by a large gap and therefore, the HF approximation is rather accurate. In both cases the compressibility of the system is enhanced compared to the non-interacting case, although the enhancement is weaker due to the inclusion of Hubbard and longer ranged interactions.

For non-interacting systems at zero temperature,

\[ \Delta_2(N) = E_N - E_{N-1} = \Delta(N), \]  

(2)

where \( E_N \) is the \( N^{th} \) single-particle eigenenergy and \( \Delta(N) \) is the single-particle level spacing.

How do the e-e interactions affect the inverse compressibility? The conventional wisdom leads to the constant interaction (CI) model [28, 29], which essentially assumes that the interactions between the electrons are long range interactions. Here we would like to investigate the sensitivity of this conclusion to the spin degree of freedom and longer range interactions. We use the Hartree-Fock (HF) approximation, as well as density matrix renormalization group (DMRG) calculation to evaluate the compressibility. In the almost staggered Harper model, the central energy band is essentially flat and separated from the other bands by a large gap and therefore, the HF approximation is rather accurate. In both cases the compressibility of the system is enhanced compared to the non-interacting case, although the enhancement is weaker due to the inclusion of Hubbard and longer ranged interactions.

\[ \Delta_2(N) = \varepsilon(N) - 2\varepsilon(N-1) + \varepsilon(N-2), \]  

(1)

where \( \varepsilon(N) \) is the single-particle level spacing.

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\[ \Delta_2(N) = \varepsilon(N) - 2\varepsilon(N-1) + \varepsilon(N-2), \]  

(1)
the central band due to the interaction with the lower band electronic density results in an increase of the compressibility.

In this paper we address the question whether this increase of the compressibility is the result of the particular model studied in Ref. [33]. Specifically, we shall see what happens to the compressibility when the spin degree of freedom is taken into account, or equivalently when considering a spinless two legged ladder. Another case which we explore is when next nearest neighbors interactions are included. To study the compressibility we mainly rely on the HF approximation, which has been shown to be extremely accurate for this model [33] due to the large gap between the flat central band and the narrow band. We will also compare some of these results to density matrix renormalization group (DMRG) numerical calculations, which for these 1D systems are essentially exact [34, 35], and describe very well the dependence of the ground state energy on the number of particles [36].

HUBBARD INTERACTION

In this section we discuss the influence of the spin degree of freedom on the compressibility in the staggered Harper model close to half-filling. The clearest difference between spin-polarized (spinless) and non-polarized electron is the fact that for non-polarized (spinfull) electrons there are Hubbard interactions. The on-site potential is spatially modulated with a frequency of almost the total electron density between spin-polarized (spinless) and non-polarized particles [36].

\[ H = \sum_{s \neq s'} \sum_{j=1}^{L} \left[ t(c_{j,s}^{\dagger} c_{j+1,s} + h.c.) + t' c_{j,s}^{\dagger} c_{j,s'}^{\dagger} n_{j,s} + U n_{j,s} n_{j+1,s} + U' n_{j,s} n_{j,s'} \right]. \]

where \( c_{j,s} \) is the single particle annihilation operator at site \( j \) with spin \( s \) and \( n_{j,s} = c_{j,s}^{\dagger} c_{j,s} \) is the number operator. \( t, t' \in \mathbb{R} \) are the site hopping and spin flipping amplitudes, respectively. \( \lambda > 0 \) controls the on-site potential amplitude. The potential is a cosine modulated in space with frequency \( b \) and a phase factor \( \phi \). \( U > 0 \) and \( U' > 0 \) are the strengths of the repulsive n.n. Hubbard interactions, respectively. We discuss the region \( \lambda < 2t \), which is the metallic regime [4]. We further assume that \( b \mod 1 = 1/2 + \epsilon, \epsilon < 1/2 \) corresponding to an almost staggered case. \( \epsilon \in \mathbb{R} \) is non-rational so that this system is disordered.

Let us first discuss the non-interacting Hamiltonian, i.e., set \( U, U' = 0 \) in Eq. (4). A numerical solution in this case reveals the existence of an almost flat central energy band (see Fig. 1), splitted due to the spin flip matrix element to a lower and higher central band. We are mostly interested in the central band energy spectrum, and since these energy states which are close to zero minimize both kinetic and potential energy, we conclude that the most important contribution comes from states localized in the potential valleys, i.e. states localized around the position \( l_z \) corresponding to \( 2\pi e l_z + \phi = (Z + \frac{1}{2})\pi \) [33]. In the valley, we can approximate \( \cos(2\pi e l_z + \phi) \approx 2\pi \epsilon (|j - l_z|) s_z \), and \( s_z = -\text{sign}(\sin(2\pi e l_z + \phi)) \) \( \pm 1 \). The effective Hamiltonian describing the central band is

\[ H^{\text{val}} = \sum_{s \neq s'} \sum_{j=1}^{L} \left[ t(c_{j,s}^{\dagger} c_{j+1,s} + h.c.) + t' c_{j,s}^{\dagger} c_{j,s'}^{\dagger} n_{j,s} + U n_{j,s} n_{j+1,s} + U' n_{j,s} n_{j,s'} \right]. \]

where

\[ \psi_{k,s} = \begin{pmatrix} c_{ek,s} \\ c_{ok,s} \end{pmatrix}, \]

is the sub-lattice pseudo-spinor that splits the lattice into even and odd sites, according to \( c_{ek,s} = \frac{1}{2} \sum_{j=1}^{L/2} e^{ik(2j-1)} c_{2j,s} \) and \( c_{ok,s} = \frac{1}{2} \sum_{j=1}^{L/2} e^{ik(2j-1)} c_{2j-1,s} \). \( \hat{p}_k \equiv i\partial_k \) and \( \sigma_x, \sigma_z \) are the \( 2 \times 2 \) Pauli matrices. Diagonalizing the spin degrees of freedom (which are independent of \( k \)-space), we get

\[ \psi_{k,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} c_{ek,\uparrow} + c_{ek,\downarrow} \\ c_{ok,\uparrow} + c_{ok,\downarrow} \end{pmatrix}. \]
The central band is splitted due to the gap created by the spin flipping amplitude \( t' \). The parameters used through the figures are \( t = 1; t' = 0.05; \lambda = 0.7; \phi = 0.7\pi; b = \sqrt{30}; \epsilon = -0.0228; L = 200 \). The isolated points correspond to protected edge (topological) states in the Harper model and are not discussed in this paper.

This representation allows us to write the Hamiltonian as a sum of two distinct subspaces, each relates to a different spin eigenstate. The subspaces depend only on the momentum \( k \), and therefore can be solved using the same methods used for spinless fermions \[33\]. Thus, the eigenergies for the Hamiltonian of the potential valley are \( E_{1,1}^{val} = \pm \sqrt{8m_t^2 \xi + t'} \), and \( E_{2,2}^{val} = \pm \sqrt{8m_t^2 \xi - t'} \), where \( m, n \in \{0, 1, 2...\} \), and \( \xi^2 = \frac{t}{\pi \lambda |\epsilon|} \). \( E_{1,1}^{val}, E_{2,2}^{val} \) correspond to the spin states 1,2 respectively. The central zero-energy band splits due to the spin flip, resulting in an energy splitting between the two bands equal to \( 2t' \).

The eigenfunctions for the states belonging to the splitted central band are:

\[
|l_{z,i} \rangle \approx (\pi \xi^2)^{-\frac{1}{4}} \sum_{j=1}^{L} (s_j) L S_j e^{-\frac{|l_{z,j}|^2}{2b^2}} |j, i \rangle, \quad (5)
\]

where \( |j, i \rangle = \frac{1}{\sqrt{2}} (c_{j,\uparrow} + c_{j,\downarrow}) |0 \rangle \), where \( |0 \rangle \) is the vacuum state. These wavefunctions are Gaussians of width \( \xi \) around \( l_z \). In the limit of small \( t' \) our assumptions hold and this result is a good approximation of the real ground state.

These states form a basis for the central band, defined by \( m, n = 0 \), since \( <l_{z,i}|l_{z\pm 1,i}|> = 0, <l_{z,1}|l_{z,2}| > = 0 \), and \( |<l_{z,i}|l_{z',i}|>| \leq e^{-\frac{|l_{z,i}-l_{z',i}|^2}{4b^2}} \ll 1 \).

Let us now consider the contribution of the overlap between the localized states in the central band. The Gaussian decay of the localized states implies that the Hamiltonian matrix elements, \( <l_{z,i}|H|l_{z',j}> \), are not negligible only between nearest neighbors states \( |z - z'| = 1 \). Thus, the central band states follow an effective Hamiltonian:

\[
H^{central} = -U \sum_{z=1}^{L} \sum_{i=1,2} (-1)^z c_{l_z,i}^{\dagger} c_{l_{z+1,i}}^{\dagger} + h.c. + t' c_{l_z,i}^{\dagger} c_{l_{z+1,i}}^{\dagger} \quad (6)
\]

Diagonalizing this Hamiltonian yields the eigenstates

\[
|k, i \rangle = L_z^{-1/2} \sum_{z=1}^{L_z} S_z e^{i k z} |l_{z,i} \rangle \quad (7)
\]

with eigenvalues \( E^{central}(k) = -2t \cos(k) \pm t' \).

Now, let us focus on the case where the Hubbard interactions in the Hamiltonian Eq. \[3\] are turned on (\( U' \neq 0 \)), but no longer range interactions are yet considered (\( U = 0 \)). For \( U' \rightarrow \infty \) the model can be solved analytically. In that limit only the interaction term is important. The eigenenergies are therefore \( E = 0 \) and \( E = U' \). The latter case occurs when two particles with opposite spins occupy the same site. This will cost infinite energy and therefore such states are decoupled from the theory. The remaining states contain a single particle per site.

Next, we consider the case where \( U' \) is much bigger than the other energy scales in the theory, i.e. \( U' \gg t, t', \lambda \). Using perturbation theory with \( t \) as the perturbation parameter on the Hubbard model reveals that ferromagnetism is the lowest energy state. Adding \( t' \) to the theory will not change the ground state, since the correction in \( t' \) will be of at least third order in perturbation theory.

As is discussed in Ref. \[33\] because the central band is essentially protected by the large gaps to the other bands, the HF approximation results are very accurate. Therefore, we approximate the Hubbard interaction using the HF method for interaction strength values smaller than these gaps \( U' \ll \sqrt{\frac{8}{\xi}} \).}

\[
\sum_{j} n_{j,\uparrow} n_{j,\downarrow} \approx \sum_{j} [n_{j,\uparrow} > n_{j,\downarrow} + n_{j,\downarrow} > n_{j,\uparrow}] - [n_{j,\uparrow} < n_{j,\downarrow}] \quad (8)
\]

Rewriting the Hamiltonian in Eq. \[3\] with \( U = 0 \), and ignoring the constant term which is simply a shift in the energy, results in

\[
H = \sum_{s \neq s'} \sum_{j=1}^{L} \sum_{s=1,2} \left[ t(c_{j,s}^{\dagger} c_{j+1,s} + h.c.) + t' c_{j,s}^{\dagger} c_{j,s'}^{\dagger} \right] (\lambda \cos(2 \pi bj + \phi) + U' < n_{j,s'} > ) n_{j,s}. \quad (9)
\]
We find that the averaged electronic density between the valleys of potential is \( <n_{j,s}> \approx \frac{1}{x} - \frac{1}{2}(1-n_{\frac{3}{4}})\cos(2\pi ej + \phi) \), with \( \tilde{n}(x) = \frac{1}{\pi \sqrt{1 + x^2}} \), and \( K \) is the complete elliptic integral of the first kind. Hence

\[
H^{HF} = \sum_{s \neq s' = \pm} \sum_{j=1}^{L} \left[ t|c_j^\dagger c_{j+1,s} + h.c. \right]
\]
\[
+ t'|c_{j,s}c_{j,s'} + (\lambda_{eff} \cos(2\pi bj + \phi) + \frac{1}{4} U')n_{j,s},
\]
where \( \lambda_{eff} = \lambda - U'n_{\frac{3}{4}} \).

The solutions of \( H^{HF} \) are closely related to the solutions of \( H \) in the non-interacting case. Yet, the width of the valley states, \( \lambda \), has changed due to the change in \( \lambda \).

Moreover, for n.n.-interactions \( (U \neq 0) \) it is possible to use the HF approximation, and obtain the HF eigenstates and eigenvalues, which are identical to the non-interacting solutions, up to the modified parameters \( t \), and \( \lambda \). The many-body density and the exchange terms are proportional to those obtained already for the spinless case up to a proportionality constant of 1/2, due to the spin degrees of freedom. Therefore, \( <p_{j,s}> \approx \frac{1}{2}\tilde{p}(\lambda \frac{3}{4}) \).

We can now write the HF. Hamiltonian with both Hubbard and n.n. interactions:

\[
H^{HF} = \sum_{s \neq s' = \pm} \sum_{j=1}^{L} \left[ t_{eff}|c_j^\dagger c_{j+1,s} + h.c. \right]
\]
\[
+ t'|c_{j,s}c_{j,s'} + (\lambda_{eff} \cos(2\pi bj + \phi) + \frac{1}{2} U + \frac{1}{4} U')n_{j,s},
\]
with \( t_{eff} = t + \frac{1}{2} U\tilde{p}(\lambda \frac{3}{4}) \) and \( \lambda_{eff} = \lambda + (2U - U')n_{\frac{3}{4}} \).

We again can solve the system with the modified parameters, and obtain the HF eigenvalues and eigenstates,

\[
E^{HF}_{val} = \pm \sqrt{8\xi t_{eff}} \xi \pm s' + \frac{1}{2} (U + \frac{1}{2} U')
\]
\[
|z;i > \approx (\pi \xi)^{-\frac{1}{2}} \sum_{j=1}^{L} (s_z) |S| e^{-\frac{(j-\xi)^2}{\xi^2}} |j;i >,
\]
where the Gaussian decay parameter \( \xi = \xi(\frac{L_{eff}}{\xi_{eff}}) \) is modified due to the effective values taken by \( \lambda \) and \( t \). \( \xi^2 \) is multiplied by a numerical constant equal to 1.16 as in 33.

Projecting the HF Hamiltonian on the central band yields

\[
H^{HF}_{central} = -\tilde{t}^{HF} \sum_{z=1}^{L_z} \sum_{s \neq s' = \pm} (-1)^{s} \xi_{lz,s}^\dagger \xi_{lz+1,s} + h.c.
\]
\[
+ \frac{1}{4} (2U + U') \xi_{lz,s} \xi_{lz,s'.}
\]
The eigenvalues and the eigenstates of the central band are then given by:

\[
E^{central}(k) = (-1)^{n+1} 2\tilde{t}^{HF} \cos(k) + \frac{1}{4} (2U + U'),
\]
\[
|k,i > = L_z^{-1/2} \sum_{z=1}^{L_z} (\xi_{lz}) |l_{z,i} >, k = \frac{2\pi n}{L_z}, n = 1,..,L_z,
\]
with \( L_z = 2|e|L \) the number of valley states. The hopping amplitude \( \tilde{t}^{HF} \) is given by

\[
\tilde{t}^{HF} \approx e^{-\frac{2\pi |e|}{4\xi^2}} \sinh\left( \frac{1}{4\xi^2} \right) - \lambda_{eff} e^{-\frac{2\pi |e|^2}{4\xi^2}}.
\]
Thus the inverse compressibility \( \Delta_2(N) \) can be calculated using (9) and the eigenvalues are presented in Eq. (14).

As shown in Fig. 3, \( \Delta_2(N) \) decreases with the n.n.-interaction \( U \), in agreement with the case of spinless fermions 33. However, the Hubbard interaction \( U' \) enhances \( \Delta_2(N) \). As was shown in Eq. (10), the Hubbard interaction reduces the value of the effective Harper potential amplitude, \( \lambda_{eff} \). The decrease in \( \lambda_{eff} \) increase the width of the Gaussian wavefunctions. Thus, there is more overlap between different states and therefore any change of configuration in the system, such as adding another particle, requires more energy. For \( U = 2U' \) the system returns to the non-interacting Hamiltonian value of \( \Delta_2(N) \). The interplay between \( U \) and \( U' \) determines whether \( \Delta_2(N) \) will be larger \( (U < 2U') \) than its non-interacting value or smaller \( (U > 2U') \) than it.

For an intuitive understanding let us revisit Fig. 1. The states which occupy the lowest energy band reside in the valleys of potential. When the Hubbard interaction is turned on, occupying these states become too costly in energy for some of the spins. In order to avoid the Hubbard interaction they tend to occupy the surroundings of potential peaks, where there are less spins to interact with. This tendency delocalizes the Gaussian wavefunctions. However, since only half of the particles participate in the interaction between the opposite spins it is less significant (by a factor of \( \frac{1}{2} \)) than \( U \).

An exception to this behavior is found for the state at edge of the lower splitted band. As detailed earlier, due to the spin flipping amplitude \( t' \), a gap of size \( 2U' \) opens between the lower central band occupied by \( \frac{1}{2\sqrt{2}} (\uparrow + \downarrow) \) states and the higher central band with states corresponding to \( \frac{1}{2\sqrt{2}} (\uparrow - \downarrow) \). \( \Delta_2(N) \) decreases with \( U' \) and increases with \( U \) at the edge, similar to the behavior
and the Hubbard interaction ($U$) and the next-nearest neighbors interaction ($U'$). $\Delta_2(N)$ decreases with $U$, which is in line with the results of Ref. 33 and increases with $U'$. Thus, the Hubbard interaction delocalizes the particles, smearing their wave functions and increasing the amount of energy needed for adding another particle to the system.

![FIG. 2: The variation of the inverse compressibility $\Delta_2(N)$ of the lower central band states with the n.n. interaction ($U$) and the Hubbard interaction ($U'$). $\Delta_2(N)$ decreases with $U$, which is in line with the results of Ref. 33 and increases with $U'$.](image)

Next-nearest neighbors interactions

In order to understand the behavior of the compressibility for a system with long range interactions, we consider here the influence of next nearest neighbors interaction. For simplicity, we discuss spinless fermions. The results of this section can be easily extended for fermions with spin using the methods described in the previous section.

The Hamiltonian is given by:

$$H = \sum_{j=1}^{L} \left[ t(c_j^\dagger c_{j+1} + h.c.) + \lambda \cos(2\pi bj + \phi) n_j \right] + U n_j n_{j+1} + U_2 n_j n_{j+2}], \quad (16)$$

and the mean-field approximation yields

$$\sum_{j=1}^{L} n_{j+2} n_j \approx \sum_{j=1}^{L} (\langle n_{j+2} \rangle + \langle n_{j-2} \rangle) n_j$$

$$- \langle n_j \rangle \langle n_{j+2} \rangle - \langle \tilde{\rho}_j \rangle c_{j+2}^\dagger c_j + h.c. + |\langle \tilde{\rho}_j \rangle|^2, \quad (17)$$

where $\langle n_j \rangle$ is the (already known) background density. The background exchange energy is $\langle \tilde{\rho}_j \rangle = \langle c_j^\dagger c_{j+2} \rangle$.

Here we ignore constant terms, since they do not contribute to $\Delta_2$. Using the known value of $\langle n_j \rangle_{\varepsilon=0}$,

$$\sum_{j=1}^{L} (\langle n_{j+2} \rangle + \langle n_{j-2} \rangle) = \sum_{j} (1 - 2\bar{n}(\frac{\lambda}{2l}) \cos(2\pi bj + \phi)).$$

Interestingly, the exchange term disappears (the calculation appears in the appendix) resulting in

$$\langle \tilde{\rho}_j \rangle = 0. \quad (19)$$

This structural robustness can be attributed to the symmetry of the non-interacting Hamiltonian’s wave-functions used in the calculation. Thus, the additional interaction only changes the value of $\lambda_{eff}$ without changing the structure of the HF Hamiltonian. The effective Hamiltonian becomes

$$H_{HF}^{central} = \sum_{z=1}^{N} -t^{HF} (-1)^z c_{z+1}^\dagger c_{z+1} + h.c., \quad (20)$$

where $t^{HF}$ given by Eq. (15) with $t_{eff} = t + \frac{1}{2}U\bar{n}(\frac{\lambda}{2l})$ and $\lambda_{eff} = \lambda + (2U - 2U_2)\bar{n}(\frac{\lambda}{2l})$. Here we ignored on-site terms, which just lead to an overall energy shift.

We also calculate $\Delta_2(N)$ using DMRG [34, 35], for the following parameters: $b = \sqrt{30}$ (corresponding to $\epsilon \approx -0.023$) and $\phi = 0.7\pi$. The length of the system is $L = 200$, and we calculated the ground state energy $E(N)$ for each number of electrons $N = 91, 92, \ldots, 108$. For $t = 1$, the potential amplitude was chosen as $\lambda = 0.7$, which results in a flat central band, with the typical $\Delta_2$ greater than the numerical accuracy. Interaction strengths of $U = 0, U_2 = 0$ and $U = 0.1$ with $U_2 = 0, 0.025, 0.05, 0.075$ are considered. The boundary conditions are open, since it significantly improves accuracy [34] and we retain 384 target states. The accuracy of $\Delta_2$ is about $\pm 1 \cdot 10^{-4}t$ and the discarded weigh is $\sim 10^{-7}$.

The resulting change in the compressibility can be viewed in Fig. 3. Comparing the analytic values to the results obtained using the numerical DMRG results, we find good agreement between the two methods. Here the Gaussian decay parameter $\xi^2$ is modified according to $\xi^2 \rightarrow 1.16\xi^2(1-0.4U_2)$. The 1.16 factor arise from using the linear approximation of the potential also between the valleys, leading to a too-fast decay of the wave function as was discussed for the n.n interactions [33]. For the n.n interaction an additional linear dependence of $\xi$ on $U_2$ is needed. It seems that the longer-range interaction results in an additional correction of the wave function behavior in the valleys.

With the additional interactions the compressibility $(1/\Delta_2)$ decreases. Intuitively, the increase in the value of
The different role played by the Hubbard interactions for the clean 1D Hubbard model and the Harper model. For the clean Hubbard model close to to the metal-insulator phase transition at half-filling of 1D systems, the Hubbard interaction enhance compressibility [30]. This behavior is also manifested for the Harper model close to the edge of the lower central band. On the other hand, for the rest of the central band, the Hubbard term effectively reduces the strength of the on-site potential in the system $(\lambda_{eff} < \lambda)$ and thus the energy gaps become smaller, weakening the enhancement of compressibility. An open question is the classification of interaction terms – which terms lead to delocalization and decrease in $\Delta_2(N)$, and which localize the wavefunctions and increase $\Delta_2(N)$.

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APPENDIX

For the Hamiltonian with $U = \epsilon = 0$, the energy spectrum of the central band is $E_{k, \pm} = \pm \sqrt{4t^2 \cos^2(k) + \lambda^2 \cos^2 \phi}$. The corresponding eigenstates are

$$\chi_{k, \pm}^\dagger = \sqrt{\frac{1}{2}} \left( \begin{array}{c} c_{ek, \pm}^\dagger \\ c_{ok, \pm}^\dagger \end{array} \right) \left( \begin{array}{c} \chi_{ek, \pm} \\ \chi_{ok, \pm} \end{array} \right)$$

where

$$\left( \begin{array}{c} \chi_{ek, \pm} \\ \chi_{ok, \pm} \end{array} \right) = \frac{1}{\sqrt{2E_{k, \pm}(E_{k, \pm} - \lambda \cos \phi)}} \left( \begin{array}{c} 2t \cos(k) \\ E_{k, \pm} - \lambda \cos \phi \end{array} \right)$$

The exchange energy is given by

$$\chi_{k, \pm}|c_{j+2}^\dagger c_j|\chi_{k, \pm} > = e^{-2ik}(\chi_{ek, \pm}^2 + \chi_{ok, \pm}^2).$$

Normalizing yields

$$\chi_{ek, \pm}^2 + \chi_{ok, \pm}^2 = 1.$$

Assuming the lower band is fully occupied,

$$< c_{j+2}^\dagger c_j > |_{\kappa = 0} = \int_{-\pi/2}^{\pi/2} \frac{dk}{\pi} < \chi_{k, -}|c_{j+2}^\dagger c_j|\chi_{k, -} > = 0.$$

DISCUSSION

In this paper we considered the variation of the inverse compressibility $\Delta_2(N)$ with respect to repulsive Hubbard interaction and next n.n.-interaction in the central band of the almost staggered fermionic Harper model in the vicinity of half-filling. The behavior of the central band states is studied using the HF approximation, justified by the flatness of this band and its isolation from the other bands. For the next n.n.-interaction we also calculated $\Delta_2(N)$ using DMRG. The comparison between the two methods promise reliable results. We found both for the Hubbard interaction and for the next n.n. interactions an increase in $\Delta_2(N)$, which corresponds to a decrease in the compressibility of the system. Thus, the increase in the compressibility due to the n.n. interactions is somewhat suppressed once Hubbard or next n.n. interactions are considered. It is interesting to note

\[ \chi_{k, \pm} = \frac{1}{\sqrt{E_{k, \pm}(E_{k, \pm} - \lambda \cos \phi)}} \left( \begin{array}{c} 2t \cos(k) \\ E_{k, \pm} - \lambda \cos \phi \end{array} \right) \]

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