RATIONALITY OF QUOTIENTS BY FINITE HEISENBERG GROUPS

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Abstract. We prove rationality of the quotient $\mathbb{C}^n/H_n$ for the finite Heisenberg group $H_n$, any $n \geq 1$, acting on $\mathbb{C}^n$ via its irreducible representation.

1. Introduction

1.1. In the present paper, we study rationality of the quotient $\mathbb{C}^n/G$ (Noether’s problem) for the affine space $\mathbb{C}^n$, $n \geq 1$, equipped with a linear action of an algebraic group $G$. Recall that for finite $G$ variety $\mathbb{C}^n/G$ can be non-rational (e.g. this is the case for certain $p$-groups in [22]). At the same time, for connected $G$ the quotient $\mathbb{C}^n/G$ is typically stably rational, that is the product $\mathbb{C}^k \times (\mathbb{C}^n/G)$ is rational for some $k$ (see [3, Theorem 2.1]).

Note that variety $\mathbb{C}^n/G$ is rational when $G$ is Abelian (see [7]). Some rationality constructions for $\mathbb{C}^n/G$ with non-Abelian $G$ can be found in [19] (see also [12]). In the present paper, we consider a particular case of the Heisenberg group $G := H_n$ generated by two elements $\xi, \eta$, which act on $\mathbb{C}^n$ as follows (Schrödinger representation):

$$
\xi : x_i \mapsto \omega^{-i} x_i, \quad \eta : x_i \mapsto x_{i+1} \quad (i \in \mathbb{Z}/n, \ \omega := e^{\frac{2\pi i}{n}}),
$$

where $x_1, \ldots, x_n$ form a basis in $\mathbb{C}^n$ (up to a choice of $\omega$ this is the only irreducible linear representation of $H_n$).

When studying rationality problem for $\mathbb{C}^n/H_n$ it is reasonable to pass to the projectivization and consider the quotient $X := \mathbb{P}^{n-1}/H_n$ (cf. [19, Proposition 1.2]). Here is our main result:

Theorem 1.2. Variety $X$ is rational for every $n$.

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The group $H_n$ is a central extension of $\mathbb{Z}/n \oplus \mathbb{Z}/n$ by $\mathbb{Z}/n \cong [\xi, \eta]$ and so the action of $H_n$ on $\mathbb{P}^{n-1}$ factors through that of $\mathbb{Z}/n \oplus \mathbb{Z}/n$. Thus Theorem 1.2 is a natural generalization of linear Abelian case mentioned above. Let us also point out that the case of central extensions of the cyclic groups has been treated in [23].

Our result confirms in addition (a stronger version of) Conjecture 15 in [5]. Actually, stable rationality of $X$ can be proved via a direct argument by considering diagonal action of $H_n$ on $V \times V$, with linear action of $\mathbb{Z}/n \oplus \mathbb{Z}/n$ on the second factor. Note also that Theorem 1.2 is evident when $n \leq 3$ and the case $n = 4$ has been treated in [19, Theorem 5.2] (compare with [3, Lemma 3.1]).

1.3. Let us outline our approach towards the proof of Theorem 1.2. One may observe that the quotient $\mathbb{C}^n/G$ is a toric variety for Abelian group $G$. In our case of $G = H_n$, its action on $\mathbb{P}^{n-1}$ is also Abelian, and so it is reasonable to expect that $X$ is toric as well. This turns out to be (almost) so.

Namely, one employs an instance of the toric conjecture after V. V. Shokurov, characterizing toric varieties (with Picard number 1) in terms of the log pairs: we construct a $\mathbb{Q}$-divisor $D$ on $X$ satisfying the assumptions of Proposition 2.2 below and reduce rationality problem for $X$ to that for a cyclic quotient of $\mathbb{P}^{n-1}$ (the latter is rational by the discussion in 1.1). In turn, the explicit action of $H_n$ on $\mathbb{P}^{n-1}$ allows one to find appropriate invariant divisors descending to the components of $D$, which is done in 2.3.

Our point was, more generally, to develop a geometric approach to the Noether’s problem for central extensions of Abelian groups (cf. 3.1 below). Thus the case of $\mathbb{C}^n/H_n$ is a special corollary of this approach. On the other hand, after our paper appeared online, Professor Ming-chang Kang has kindly communicated to us an algebraic proof of Theorem 1.2 (see Appendix after Section 3).

Remark 1.4. We show in Proposition 2.2 that $X$ is actually a cyclic quotient of $\mathbb{P}^{n-1}/\widetilde{G}$ for a linearized Abelian group $\widetilde{G}$. Thus $X$ resembles the so-called fake weighted projective space (see [13]). Note however that $X$ need not be toric. Let us consider the first non-trivial case $n = 3$. Here the group $H_3$ acts on $\mathbb{P}^2$ preserving the Hesse pencil $\{ E_t : x^3 + y^3 + z^3 + txyz = 0 \mid t \in \mathbb{P}^1 \}$ and on the smooth cubic $E_t$ the $H_3$-action coincides with the one of the group of 3-torsion points $E_t[3]$ (see [1]). The quotient surface $X = \mathbb{P}^2/H_3$ has 4 singular points of type $A_2$ and so can not be toric. One may also observe that the algebra of invariants of $H_3$ in
\(\mathbb{C}[x, y, z]\) is generated by polynomials \(xyz, x^3 + y^3 + z^3, x^3y^3 + y^3z^3 + z^3x^3\) and \(x^3y^6 + y^3z^6 + z^3x^6\) (cf. [1] Section 6).

2. Proof of Theorem [1.2]

2.1. We will be using freely standard notions and facts about the singularities of pairs (see e.g. [15] Chapter 5). All varieties are assumed to be normal, projective, over \(\mathbb{C}\), and all divisors are \(\mathbb{Q}\)-Cartier with rational coefficients.

Our proof of Theorem [1.2] is based on the following:

Proposition 2.2 (cf. [14], [20], [10]). Let \(V\) be a \(d\)-dimensional variety with a boundary divisor \(D = \sum_{i=1}^{d+1} d_i D_i\), where \(D_i\) are prime Weil divisors, such that the following holds:

- the Picard number of \(V\) is 1,
- the log pair \((V, D)\) is log canonical,
- \(K_V + D \sim_{\mathbb{Q}} 0\),
- \(d_i D_i \sim_{\mathbb{Q}} d_j D_j\) for all \(1 \leq i, j \leq d+1\),
- there exists a finite, étale in codimension 1 cyclic cover \(p: V' \rightarrow V\) such that \(p^*(d_i D_i) \sim_{\mathbb{Q}} W_i\), \(1 \leq i \leq d+1\), where \(W_i\) are distinct Weil divisors on \(V'\).

Then \(V'\) is a toric quotient \(\mathbb{P}^d/\tilde{G}\) for a finite Abelian group \(\tilde{G}\) with linearized action on \(\mathbb{P}^d\). In particular, if \(\Gamma \simeq \mathbb{Z}/m\mathbb{Z}\) is the Galois group of \(p\), then \(V = V'/\Gamma\) is birational to \(\mathbb{P}^d/\Gamma\) (hence \(V\) is rational).

Proof. We follow the proof of Lemma 3.1 in [20]. Namely, after repeated finite, étale in codimension 1 cyclic covers \(V \leftarrow V' \leftarrow \ldots \leftarrow \tilde{V}\) we obtain a new log pair \((\tilde{V}, \tilde{D} = \varphi^*(D))\), where \(\varphi : \tilde{V} \rightarrow V'\) is the resulting morphism, such that all \(\varphi^*p^*(W_i)\) are Cartier. Furthermore, we have

\[K_{\tilde{V}} + \tilde{D} \sim_{\mathbb{Q}} \varphi^*p^*(K_V + D) \sim_{\mathbb{Q}} 0\]

and \((\tilde{V}, \tilde{D})\) is log canonical, i.e. \(\tilde{V}\) is a log Fano (note that \(\varphi^*p^*(D)\) is ample).

The Fano index of \(\tilde{V}\) is \(\geq d + 1\), since \(-K_{\tilde{V}} \sim_{\mathbb{Q}} \tilde{D}\) and \(\varphi^*p^*(d_i D_i) \sim_{\mathbb{Q}} \varphi^*p^*(d_i D_j)\) for all \(1 \leq i, j \leq d+1\). This implies that \(\tilde{V} = \mathbb{P}^d\) and \(\varphi\) coincides with the quotient morphism by some finite group \(\tilde{G}\) (the Galois group of the field
extension \(\mathbb{C}(\tilde{V})/\varphi^*\mathbb{C}(V'))\). Also, by construction \(\tilde{G}\) leaves invariant \(d + 1\) hyperplanes \(\varphi^*p^*(W_i)\) in \(\mathbb{P}^d\), whence it is Abelian.

Further, if \(T := (\mathbb{C}^\ast)^d \subset V'\) is the open torus with coordinates \(z_1, \ldots, z_d\), then for some \(N \in \mathbb{N}\) we have: \(Np^*(d_iD_i) \sim \) the closure \(\tilde{W}_i\) of \((z_i = 0) \subset V', 1 \leq i \leq d\), and \(Np^*(d_{d+1}D_{d+1}) \sim \) the closure \(\tilde{W}_{d+1}\) of \((z_{d+1} := (z_1 \ldots z_d)^{-1} = 0) \subset V'\). In particular, since each \(p^*(d_iD_i)\) generates the \(\mathbb{Q}\)-Picard group of \(V'\) and \(V' \setminus T = \bigcup_{i=1}^{d+1} \tilde{W}_i\) on the toric variety \(V'\), up to twist by a character we may assume that \(\Gamma\) either preserves all the \(z_i\) or permutes them cyclicly. In both cases, compactifying \(T\) by \(\mathbb{P}^d\), we obtain that \(T/\Gamma\) is birational to the rational variety \(\mathbb{P}^d/\Gamma\). \(\square\)

2.3. We now turn to the variety \(X = \mathbb{P}^{n-1}/H_n\) from Theorem 1.2. Let \(\pi : \mathbb{P}^{n-1} \rightarrow X\) be the quotient morphism.

**Lemma 2.4.** \(\pi\) is étale in codimension 1 and \(K_{\mathbb{P}^{n-1}} \sim_\mathbb{Q} \pi^*(K_X)\).

**Proof.** The first assertion follows from the fact that every \(\neq 1\) element in \(H_n\) has non-multiple spectrum (see 1.1). Then the equivalence \(K_{\mathbb{P}^{n-1}} \sim_\mathbb{Q} \pi^*(K_X)\) is the usual Hurwitz formula. \(\square\)

Identify \(x_0, \ldots, x_{n-1}\) from 1.1 with projective coordinates on \(\mathbb{P}^{n-1}\). Put \(f_k := \sum_{i \in \mathbb{Z}/n} x_i^k x_{i+1}^{-k}\) for \(1 \leq k \leq n\). We have \(\xi^* f_k = \omega^k f_k\) and \(\eta^* f_k = f_k\). Hence polynomials \(f_k^n\) are \(H_n\)-invariant.

**Lemma 2.5.** The linear system \(\mathcal{L} \subset |O_{\mathbb{P}^{n-1}}(n^2)|\) spanned by \(f_1^n, \ldots, f^n_n\) and \((x_0 \ldots x_{n-1})^n\) is basepoint-free.

**Proof.** It suffices to show that \(f_1, \ldots, f_n\) and \(x_0 \ldots x_{n-1}\) span a basepoint-free linear system. Fix an arbitrary \(m \geq n\) and consider the polynomials \(f_k^{(m)} := \sum_{i \in \mathbb{Z}/n} x_i^k x_{i+1}^{-k}\) for various \(1 \leq k \leq m\). Let \(\mathcal{L}^{(m)}\) be the linear system spanned by \(f_1^{(m)}, \ldots, f_m^{(m)}\) and \(x_0, x_1, \ldots, x_{n-1}^{m-n+1}\). Then we claim that \(\mathcal{L}^{(m)}\) is basepoint-free (note that \(m = n\) corresponds to our case). Indeed, for \(n = 2\) this is trivially true, whereas for \(n > 2\) we restrict to the hyperplanes \((x_i = 0)\) and argue by induction. \(\square\)

Let \(B_1, \ldots, B_n\) be generic elements in the linear system \(\mathcal{L}\) from Lemma 2.5. We may assume the pair \((\mathbb{P}^{n-1}, \sum_{i=1}^n B_i)\) is log canonical.
Further, put \( D_i := \pi(B_i), 1 \leq i \leq n \), so that \( B_i = \pi^*(D_i) \),

\[
K_{\mathbb{P}^{n-1}} + \sum_{i=1}^{n} B_i \sim_{\mathbb{Q}} \pi^*(K_X + \sum_{i=1}^{n} D_i)
\]

(cf. Lemma 2.4) and the pair \((X, \sum_{i=1}^{n} D_i)\) is also log canonical.

**Lemma 2.7.** \( \pi \) factorizes as \( \mathbb{P}^{n-1} \xrightarrow{q} X' \xrightarrow{p} X \), where \( q, p \) are both degree \( n \), étale in codimension 1 cyclic covers, \( p^*(d_iD_i) \sim_{\mathbb{Q}} W_i, 1 \leq i \leq n, \) for \( d_i := 1/n^2 \) and some distinct Weil divisors \( W_i \).

**Proof.** Note that the field extension \( \mathbb{C}(\mathbb{P}^{n-1})/\pi^*\mathbb{C}(X) \) is Galois with the group \( S := \mathbb{Z}/n \oplus \mathbb{Z}/n \). Restricting to the field of \( \xi \)-invariants yields an intermediate field \( \pi^*\mathbb{C}(X) \subset F \subset \mathbb{C}(\mathbb{P}^{n-1}) \). Note that extension \( \mathbb{C}(\mathbb{P}^{n-1})/F \) corresponds to the quotient morphism \( q : \mathbb{P}^{n-1} \longrightarrow X' \) for \( X' = \mathbb{P}^{n-1}/\langle \xi \rangle \) and the cyclic subgroup \( \langle \xi \rangle \subset S \). Finally, \( \mathbb{C}(X)/F \) is also Galois, corresponding to the quotient morphism \( p : X' \longrightarrow X = X'/\langle \eta \rangle \).

Further, consider the divisors \( B_0 := ((x_0 \ldots x_{n-1})^n = 0) \) and \( H_i := (x_i = 0), 0 \leq i \leq n - 1 \), so that \( B_0 = n \sum_{i=0}^{n-1} H_i \). We have \( \frac{1}{n^2} B_0 \sim_{\mathbb{Q}} H_i \) for all \( i \) and hence

\[
q_* \left( \frac{1}{n^2} B_0 \right) \sim_{\mathbb{Q}} q_* (H_i) \sim_{\mathbb{Q}} n q(H_i)
\]

because \( q_*(H_i) = nq(H_i) \) for \( H_i \) being \( \xi \)-invariant hyperplanes. This implies that

\[
p^*(\frac{1}{n^2} D_i) = \frac{1}{n^2} q(B_i) = \frac{1}{n^2} q_*(B_i) \sim_{\mathbb{Q}} q_*(\frac{1}{n^2} B_0) \sim_{\mathbb{Q}} q(H_{i-1}) =: W_i
\]

for all \( 1 \leq i \leq n \). \( \square \)

Put \( D := \frac{1}{n^2} \sum_{i=1}^{n} D_i \). Then it follows immediately from (2.6) and Lemma 2.7 that the log pair \((X, D)\) satisfies all the assumptions in Proposition 2.2 (for \( V := X \)). Thus \( X \) is rational and the proof of Theorem 1.2 is complete.

### 3. Miscellany

**3.1.** It would be interesting to extend the technique presented in Section 2 to the case of quotients \( \mathbb{P}^{n-1}/G \) by other finite central extensions of Abelian groups. This requires, however, analogs of technical lemmas from 2.3 where we have crucially used that \( G = H_n \).
More generally, it would be interesting to give a characterization of those finite groups $G$, for which $\mathbb{P}^{n-1}/G$ is a cyclic quotient of toric variety (cf. Remark 1.4). Observe at this point that the singularities of $X = \mathbb{P}^{n-1}/H_n$ are non-exceptional (cf. [18] Proposition 3.4), and one might try to look for a similar property, distinguishing (cyclic quotients of) toric $\mathbb{P}^{n-1}/G$.

3.2. Initially, our interest was in constructing a mirror dual $Y^+$ for Calabi–Yau threefolds $Y$, studied in [8]. Recall that $Y$ is a small resolution of a nodal Calabi–Yau $V \subset \mathbb{P}^{n-1}$, invariant under $H_n$, such that there is a pencil of $(1,n)$-polarized Abelian surfaces $A \subset V$. The action of $H_n$ extends to a free one on $Y$ and it is expected that $Y^+ = Y/H_n$. Indeed, when $n = 8$ the derived equivalence between $Y$ and $Y/H_n$ was established in [21], which on the level of Abelian surfaces is the Mukai equivalence between $A$ and $\text{Pic}^0(A) = A/H_n$ (note that $H_n$ acts on $A$ via shifts by $n$-torsion points).

In particular, when $n = 5$ and $V$ is the Horrocks–Mumford quintic (see [8] Section 3)), $V/H_5$ is a Calabi–Yau hypersurface in (almost) toric variety $\mathbb{P}^4/H_5$. This brings in a possibility for applying Batyrev’s construction of mirror pairs (see [2]) as well as other explicit methods: matrix factorizations, period integrals, etc. (see e.g. [9], [17]). We plan to return to this subject elsewhere.

3.3. As a complement to 3.1, one may try to attack (stable) rationality problem for various quotients $\mathbb{P}^{n-1}/G$ by considering their classes $[\mathbb{P}^{n-1}/G]$ in $K_0(\text{Var})$, the Grothendieck ring of complex algebraic varieties, and applying [16] Corollary 2.6 to them. It is important however that $[\mathbb{P}^{n-1}/G]$ be non-zero modulo $L := [\mathbb{A}^1]$ (see [11] for some examples of varieties $Z$ with $[Z] = 0 \mod L$).

It is trivially true that $[\mathbb{P}^{n-1}/H_n]$ is not divisible by $L$ (for it equals $[\mathbb{P}^{n-1}]$ modulo $L$ by Theorem 1.2) and it would be interesting to find out whether this is always the case for all quotients $\mathbb{P}^{n-1}/G$. Perhaps the fact that any such variety is stably b-inf. trans. (see [4] Corollary 3.2)) might be of some use here.

Appendix by Ming-chang Kang

The algebraic proof of Theorem 1.2 follows the same lines as in [6]. Namely, put $\lambda := [\xi, \eta]$ and $y_0 := x_0^n$, $y_i := x_i/x_{i-1}$, $1 \leq i \leq n-1$ (see 1.1). Then we have $C(x_0, \ldots, x_{n-1})^{(\lambda)} = C(y_0, \ldots, y_{n-1})$. By [6] Theorem 4.1 it suffices to prove
rationality of $\mathbb{C}(y_1, \ldots, y_{n-1})^{(\xi, \eta)}$. Note that the action of $\xi$ on $y_i$, $1 \leq i \leq n - 1$, is given by $\xi : y_i \mapsto \omega y_i$.

Define $z_1 := y_1^n$, $z_i := y_i/y_{i-1}$, $2 \leq i \leq n - 1$. Then we have $\mathbb{C}(y_1, \ldots, y_{n-1})^{(\xi)} = \mathbb{C}(z_1, \ldots, z_{n-1})$. Note that the action of $\eta$ on $z_i$ is the same as the action of $\tau$ in $[6, \text{p. 686}]$ (by replacing $p$ with $n$ everywhere).

Now define $w_1 := z_2$, $w_i := \eta^{-1}(z_2)$, $2 \leq i \leq n - 1$. Then we have $\mathbb{C}(z_1, \ldots, z_{n-1}) = \mathbb{C}(w_1, \ldots, w_{n-1})$ and the action of $\eta$ is as follows:

$$\eta : w_1 \mapsto w_2 \mapsto w_3 \mapsto \ldots \mapsto w_{n-1} \mapsto \frac{1}{w_1 \ldots w_{n-1}}.$$ 

The latter action can be linearized exactly as in the middle of $[6, \text{p. 687}]$. Hence we can apply Fischer’s Theorem (see $[7]$).

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