Generalised difference sequence space of non-absolute type

A. H. Ganie, S. A. Lone and A. Afroza

1,2Basic Science Department, College of Science and Theoretical Studies, Saudi Electronic University-Abha 61421, Kingdom of Saudi Arabia
1,3Department of Applied Science and Humanities, SSM College of Engineering and Technology, Pattan (Kashmir)-India

Corresponding author: 1a.ganie@seu.edu.sa, 2s.lone@sue.edu.sa and 3hafroza@rediffmail.com

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Abstract: It was Shiue [16] who introduced the Cesàro spaces of the type Ces_p and Ces_∞. In view of Shiue, we shall introduce and study some properties of generalised Cesàro difference sequence space. We also examine some of their basic properties viz., BK property and some inclusions relations will be taken care of.

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1 Introduction

By Π we shall denote the set of all sequences (real or complex) and any subspace of it is known as the sequence space. Also, let the set of non-negative integers, the set of real numbers and the set of complex numbers be denoted respectively by N, R, and C. Let l_∞, c and c_0, respectively, denotes the space of all bounded sequences, the space of convergent sequences and the sequences converging to zero. Also, by bs, cs, l_1 and l_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively (see [1-21]).

Suppose X is a vector space (real or complex) and H : X → R. We call (X, H) a paranormed space with H a paranorm for X provided:
(i) H(θ) = 0,
(ii) H(−s) = H(s),
(iii) H(s_1 + s_2) ≤ H(s_1) + H(s_2), and
(iv) scalar multiplication is continuous, i.e., |β_n − β| → 0 and H(s_n − s) → 0 gives H(β_n s_n − β s) → 0 ∀ β ∈ R and s’s in X, where θ represent zero vector in the space X.

Suppose A = (a_mk) be an infinite matrix with X,Y ⊂ Π. Then, matrix A represents the A-transformation from X into Y, if for b = (b_k) ∈ X the sequence Ab = {(Ab)_m} is the A-transform of b exists and is in Y; where (Ab)_m = ∑_k a_mk b_k as can be seen in [24] and many more.
A $FK$ space $\mathcal{Y}$ is a complete metric sequence space with continuous coordinated $p_m : \mathcal{Y} \to \mathbb{C}$ where $p_m(u) = u_m$ for all $u \in \mathcal{Y}$ and $m \in \mathbb{N}$. A normed $FK$ space is called a BK space as defined in [26] and etc.

Let $\theta = (t_j)$ be increasing integer sequence. Then it will be called lacunary sequence if $t_0 = 0$ and $t_j = t_{j-1} - t_{j-1} \to \infty$. By $\theta$ we will denote the intervals of the form $I_j = (t_{j-1}, t_j]$ and with $q_j$ we will denote the ratio $\frac{t_j}{t_{j-1}}$ [4].

The spaces $T(\Delta)$ where

$$T(\Delta) = \{ u = (u_m) \in \Pi : (\Delta u_m) \in T \}$$

was introduced by Kizmaz [16] where $T \in \{l_\infty, c, c_0\}$ and $\Delta u_m = u_m - u_{m-1}$.

Next Tripathy and Esi [26] had studied it and considered it as follows. Consider the integer $j \geq 0$. then

$$T(\Delta^j) = \{ u = (u_k) : \Delta^j u \in T \}, \text{ for } T = l_\infty, c \text{ and } c_0,$$

where $\Delta^j u_i = u_i - u_{i+j}$.

Recently, in [27] we have the following:

$$\Delta_n^m u_k = \{ u \in \Pi : (\Delta_n^m u_k) \in Z \},$$

where

$$\Delta_n^m u_k = \sum_{\mu=0}^{n} (-1)^\mu \binom{n}{\mu} u_{k+m\mu},$$

and

$$\Delta_n^0 u_k = u_k \forall k \in \mathbb{N}.$$

The Cesàro sequence spaces $\text{Ces}_p$ and $\text{Ces}_\infty$ have been introduced by Shiue [25] and was further studied by several authors viz., Et [3], Orhan[20], Tripathy [27]. Ng and Lee [18] has introduced the Cesàro sequence spaces $X_p$ and $X_\infty$ of non-absolute type and has shown that $\text{Ces}_p \subset X_p$ is strict for $1 \leq p \leq \infty$. Our aim in this paper is to bring out the spaces $C(p)(\Delta_n^m, \theta)$ and $C(p)[\Delta_n^m, \theta]$, where $1 \leq p \leq \infty$ and study their various properties.

2 The spaces $C(p)(\Delta_n^m, \theta)$ and $C(p)[\Delta_n^m, \theta]$, $(1 \leq p \leq \infty)$.

In this section of text, we introduce the space $C(p)(\Delta_n^m, \theta)$ and $C(p)[\Delta_n^m, \theta]$, where $1 \leq p \leq \infty$ and prove that these spaces are BK.
Following Başarır [1], Sing [2], Jagers [5], Ganie [6]-[14], Karakaya [15], Mursaleen [17], Nuray [19], Savaş [21]-[23], we introduce for a sequence of strictly positive real numbers \( p = (p_i) \), the following spaces:

\[
C(p) (\Delta^m_n, \theta) = \left\{ v = (x_k) : \sum_{i=1}^{\infty} \left( \frac{1}{h_i} \sum_{k \in I_i} \Delta^m_n x_k \right)^{p_i} < \infty \right\},
\]

\[
C(p) [\Delta^m_n, \theta] = \left\{ v = (x_k) : \sum_{i=1}^{\infty} \left( \frac{1}{h_i} \sum_{k \in I_i} \Delta^m_n x_k \right)^{p_i} < \infty \right\},
\]

\[
C(\infty) (\Delta^m_n, \theta) = \left\{ v = (x_k) : \sup_i \left( \frac{1}{h_i} \sum_{k \in I_i} \Delta^m_n x_k \right)^{p_i} < \infty \right\},
\]

and

\[
C(\infty) [\Delta^m_n, \theta] = \left\{ v = (x_k) : \sup_i \left( \frac{1}{h_i} \sum_{k \in I_i} \Delta^m_n x_k \right)^{p_i} < \infty \right\}.
\]

It is obvious to see that the above spaces contain some unbounded sequences for \( m \geq 1 \). To see this, let \( \theta = (2^j) \) and \( p_j = 1 = n \ \forall j \in \mathbb{N} \), then clearly, \((j^m) \in C(\infty) (\Delta^m_n, \theta) \) but \((j^m) \notin l\infty \).

We have the following important result.

**Theorem 2.1** The spaces \( C(p) (\Delta^m_n, \theta) \), \( C(p) [\Delta^m_n, \theta] \), are linear spaces.

**Proof**: The proof is omitted, as can be proved by special well known techniques.

**Theorem 2.2** For \( 1 \leq p < \infty \), the space \( C(p) (\Delta^m_n, \theta) \) is a BK-space normed by

\[
\|x\|_{\Delta^\theta_p} = \sum_{i=1}^{m} |x_i| + \left( \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} \Delta^m_n x_k \right)^{p} \right)^{\frac{1}{p}},
\]

and the space \( C(\infty) (\Delta^m_n, \theta) \) is a BK-space normed by

\[
\|x\|_{\Delta^\theta_\infty} = \sum_{i=1}^{m} |x_i| + \sup_r \left( \frac{1}{h_r} \sum_{k \in I_r} \Delta^m_n x_k \right).
\]

**Proof**: Let \( x^j = (x^j_i) \) be any Cauchy sequence in \( C(p) (\Delta^m_n, \theta) \) for each \( j \in \mathbb{N} \). Therefore, we have

\[
\|x^i - x^j\|_{\Delta^\theta_p} \leq \sum_{i=1}^{m} |x^i_i - x^j_i| + \sup_r \left( \sum_{r=1}^{\infty} \frac{1}{h_r} \sum_{k \in I_r} \left( \Delta^m_n x^i_k - \Delta^m_n x^j_k \right) \right) \to 0, \text{ as } i, j \to \infty.
\]
Hence, \( \sum_{t=1}^{m} |x_i^t - x_j^t| \to 0 \) and \( (\Delta_n^m x_k^i - \Delta_n^m x_k^j) \to 0 \), as \( i, j \to \infty \) for each \( k \in \mathbb{N} \). Now, from,

\[
| x_{t+m}^i - x_{t+m}^j | \leq | \Delta_n^m x_k^i - \Delta_n^m x_k^j | + \left( m \atop 0 \right) | x_i^t - x_j^t | + \ldots + \left( m \atop m-1 \right) | x_{m-1}^i - x_{m-1}^j |
\]

we have \( |x_i^t - x_j^t| \to \infty \) as \( i, j \to \infty \), for each \( k \in \mathbb{N} \). Therefore, \((x_i^t)_{i=1}^\infty\) is a Cauchy sequence in \( \mathbb{C} \) and hence converges since \( \mathbb{C} \) is complete, and let \( \lim_{i} x_i^t = x_t \) for each \( t \in \mathbb{N} \). Since \( x^t \) is a Cauchy sequence, therefore for each \( \epsilon > 0 \), we can find \( n = n_0(\epsilon) \) such that

\[
|x_i^t - x_j^t| < \epsilon \quad \forall \ i, j \geq n_0.
\]

Thus, we have

\[
\lim_{j} \sum_{t=1}^{m} |x_i^t - x_j^t| = \sum_{t=1}^{m} |x_i^t - x_t| < \epsilon
\]

and

\[
\lim_{j} \frac{1}{h_r} \sum_{k \in I_r} (\Delta_n^m x_k^i - \Delta_n^m x_k^j)^p = \frac{1}{h_r} \sum_{k \in I_r} (\Delta_n^m x_k^i - \Delta_n^m x_k)^p < \epsilon^p
\]

for all \( r \in \mathbb{N} \) and \( i \geq n_0 \). This shows that \( \|x_i^t - x\|_{\Delta_n^p} < 2\epsilon \), for all \( i \geq n_0 \). Since,

\[
\left| \frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k^i \right|^p \leq 2^p \left( \left| \frac{1}{h_r} \sum_{k \in I_r} (\Delta_n^m x_n^i - x_k)^p \right| + \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k^i \right|^p \right) \to 0
\]

as \( r \to \infty \), we obtain \( x \in C(p) \left( \Delta_n^m, \theta \right) \). Therefore, \( C(p) \left( \Delta_n^m, \theta \right) \) is a Banach space. Since, \( C(p) \left( \Delta_n^m, \theta \right) \) is a Banach space with continuous co-ordinates, that is, \( \|x_i^t - x\|_{\Delta_n^p} \to 0 \) for each \( k \in \mathbb{N} \) as \( i \to \infty \), consequently, it is a BK-space. Hence, the proof of the result is complete.

**Theorem 2.3** \( C(p) \left[ \Delta_n^m, \theta \right] \) with \( 1 \leq p < \infty \) is a BK-space with norm

\[
\|x\|_{\Delta_n^p} = \sum_{i=1}^{m} |x_i| + \left( \sum_{r=1}^{\infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k \right|^p \right)^{\frac{1}{p}}
\]

and \( C(\infty) \left[ \Delta_n^m, \theta \right] \) is a BK-space normed by

\[
\|x\|_{\Delta_n^\infty} = \sum_{i=1}^{m} |x_i| + \sup_{r} \left( \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k \right| \right).
\]

**Proof:** The proof is similar to that of previous theorem and hence can be omitted.
Theorem 2.4 The spaces \( C_p(\Delta^m_n, \theta) \), \( C_p(\Delta^m_n, \theta) \), \( C_\infty(\Delta^m_n, \theta) \), and \( C_\infty(\Delta^m_n, \theta) \) are neither solid nor symmetric.

Proof: We only prove the result for \( C_\infty(\Delta^m_n, \theta) \) and rest can be proven in a similar fashion. So, to establish the result, we put \( n = p_j = 1 \) for all \( j \) and \( \theta = (2^*) \). Then, \((u_j) = (j^{m-1}) \in C_\infty(\Delta^m_n, \theta) \) but \((\alpha_j u_j) \notin C_\infty(\Delta^m_n, \theta) \) where \( \alpha_j = (-1)^j \) for all \( j \in \mathbb{N} \). Thus, \( C_\infty(\Delta^m_n, \theta) \) is not solid. This proves the result.

3 Inclusion relations

In this section, we prove some basic inclusion relations for the given spaces.

Theorem 3.1 For \( m, n \in \mathbb{N} \) with \( 1 \leq p \leq \infty \), we have

(i) \( C_p(\Delta^{m-1}_n, \theta) \subset C_p(\Delta^m_n, \theta) \),

(ii) \( C_p(\Delta^{m-1}_n, \theta) \subset C_p(\Delta^m_n, \theta) \),

(iii) \( C_p(\Delta^m_n, \theta) \subset C_q(\Delta^m_n, \theta) \),

(iv) \( C_p(\Delta^m_n, \theta) \subset C_q(\Delta^m_n, \theta) \) where \( 0 < p < q < \infty \).

Proof: We shall only prove (i). So, let \( x \in C_p(\Delta^{m-1}_n, \theta) \). Then, we have

\[
\left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^m_n x_k \right| \leq \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^{m-1}_n x_k \right| + \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^{m-1}_n x_{k+1} \right|.
\]

Hence,

\[
\left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^m_n x_k \right|^p \leq 2^p \left( \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^{m-1}_n x_k \right|^p + \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^{m-1}_n x_{k+1} \right|^p \right).
\]

Thus, for each positive integer \( r_0 \), we have

\[
\sum_{r=1}^{r_0} \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^m_n x_k \right|^p \leq 2^p \left( \sum_{r=1}^{r_0} \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^{m-1}_n x_k \right|^p + \sum_{r=1}^{r_0} \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^{m-1}_n x_{k+1} \right|^p \right).
\]

Now, taking \( r_0 \to \infty \) in the above inequality, we see that

\[
\sum_{r=1}^{\infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^m_n x_k \right|^p \leq 2^p \left( \sum_{r=1}^{\infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^{m-1}_n x_k \right|^p + \sum_{r=1}^{\infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta^{m-1}_n x_{k+1} \right|^p \right).
\]

Consequently, \( C_p(\Delta^{m-1}_n, \theta) \subset C_p(\Delta^m_n, \theta) \).
To show inclusion is proper, we see that the sequence \( x = (k^{m-1}) \) belongs to \( C_{(p)}(\Delta_{m}^{\frac{1}{m}}, \theta) \) but does not belong to \( C_{(p)}(\Delta_{m}^{\frac{1}{m-1}}, \theta) \), for \( \theta = (2^r) \). This completes the proof.

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