Lagrangian transport in two-dimensional time-periodic cavity flow

Lukas Babor\textsuperscript{1,*} and Hendrik C. Kuhlmann\textsuperscript{1}

\textsuperscript{1} Institute of Fluid Mechanics and Heat Transfer, TU Wien, Getreidemarkt 9, 1060 Vienna, Austria

The Lagrangian transport in a laminar incompressible flow in a two-dimensional square cavity driven by a harmonic tangential oscillation of a single cavity wall is investigated numerically for a range of Reynolds (Re) and Strouhal (Str) numbers. The topology of fluid trajectories is described by means of stroboscopic projections, which reveal the co-existence of chaotic trajectories and regular Kolmogorov-Arnold-Moser (KAM) tori. The higher the frequency of the lid oscillation the more regular the fluid motion becomes and the size of the KAM tori increases. For low frequencies the KAM tori are strongly stretched along instantaneous streamlines of the flow, while for high frequencies they resemble streamlines of a mean flow.

1 Introduction

Lagrangian transport in laminar flows is crucial for the optimization of mixing in a wide range of industrial processes, as well as in microfluidic devices. Owing to its simple geometry and importance as a benchmark \cite{1, 2}, we consider the motion of an incompressible fluid with density $\rho$ and kinematic viscosity $\nu$ in a two-dimensional square cavity with side length $L$. The flow is driven by a lid which moves tangentially in $x$ direction with a time-dependent velocity $u(t) = U \sin \phi(t)$, where $\phi = 2\pi t/T$ is the phase while $U$ and $T$ are the amplitude and period of oscillation, respectively (Fig. 1a). Since the mean lid velocity is zero, any net transport of the fluid after one period of oscillation is due to inertia. The Navier–Stokes equation and boundary conditions exhibit the shift–invert symmetry $(t, x) \rightarrow (t + T/2, -x)$.

2 Method of solution

The flow is obtained by numerically solving the dimensionless Navier-Stokes equations, where lengths, velocities, time and pressure are scaled with $L$, $U$, $L/U$ and $\rho U^2$, respectively. The solution depends on the Reynolds number $\text{Re} = UL/\nu$ and the Strouhal number $\text{Str} = L/UT$. The spectral-element solver Nek5000 is employed, discretizing the problem in space on $20^2$ uniform square elements of polynomial orders $P_7$ and $P_5$ for velocity and pressure, respectively. A third-order semi-implicit scheme is used for the temporal discretization. For a given flow field trajectories of fluid elements are computed with the adaptive Dormand-Prince pair of Runge-Kutta 4-5 formulae \cite{3}.

3 Results

Time-periodic vortex structures of the Eulerian velocity field typically become more complex the higher the oscillation frequency of the lid \cite{4}. The reverse applies to the Lagrangian topology, primarily characterized by the structure and location of the KAM tori which must arise as shift-inverted pairs to the symmetry of the equations (Fig. 1b).

Fig. 1: (a) Sketch of the problem. The coordinate origin is the center of the cavity. Time is visualized as a third dimension. The plane of stroboscopic projection is shown in grey. (b) KAM tori (opaque) and isosurfaces of the instantaneous stream function (transparent) over one period of driving. $\text{Re} = 500$, $\text{Str} = 0.05$.

\* Corresponding author: e-mail lukas.babor@tuwien.ac.at, phone +43 1 58801 32214
For \( \text{Re} = 1 \) and \( 0.01 \leq \text{Str} \leq 1 \) all Lagrangian trajectories are found to be regular, within numerical resolution, and nearly symmetric to \( x = 0 \). For high-frequency oscillations (\( \text{Re} = 1, \text{Str} = 1 \), Fig. 2a) most of the domain is occupied by two large regular regions, containing nested KAM tori and resembling the streamlines of the time-averaged (mean) flow (red). Smaller regular regions exist near the bottom corners and immediately below the lid, separated by heteroclinic connections between hyperbolic fixed points (separation points on the wall or free stagnation points). As the oscillation frequency is decreased, at \( \text{Re} = 1 \), the regular regions next to the lid vanish and a new pair of regular regions is created near the point \((x, y) \approx (0, 0.3)\) in the Poincaré plane \( t = T/4 + nT, n \in \mathbb{Z} \). The new KAM tori are bounded by separatrices including stagnation points near the line \( x = 0 \) (Fig. 2b). As the frequency is further decreased the existing KAM tori are displaced, in the Poincaré plane, towards the side walls \( y = \pm 0.5 \) and the creation process of new KAM tori repeats itself (Fig. 2c). The closer the KAM tori are to the cavity walls at \( y = \pm 0.5 \), the more they are stretched along the instantaneous streamlines of the flow (blue).

As the Reynolds number is increased inertia becomes more important and the mostly-stretched regular KAM tori with very low winding frequencies (\( T \gg 1 \)) break up due to the resonance phenomenon. With increasing Re also tori with higher and higher winding frequency start to break. An example is shown in Fig. 2d in which KAM tori with period 3 and 4 are visible (pink). Along with the increase of Re the periodic orbits in the centers of the KAM tori are displaced towards the singular corners of the cavity at \((x, y) = (\pm 0.5, 0.5)\).

The Reynolds numbers at which the resonances arise depends on the oscillation frequency. Typically, breakup occurs at lower Reynolds numbers the lower the oscillation frequency is (the lower Str). For \( \text{Str} = 0.1 \) the largest tori break up at \( \text{Re} \approx 10 \), creating subharmonic islands of period \( 12T \). These vanish with a further increase of Re, while smaller and smaller tori break up. At this Strouhal number most of the cavity is occupied by chaotic trajectories already when \( \text{Re} \gtrsim 50 \). But there always remain small regular elliptic regions in the bottom corners at \((x, y) = (\pm 0.5, -0.5)\). For \( \text{Str} \leq 0.02 \) we find all fluid elements to move chaotically when the Reynolds number exceeds \( \text{Re} \approx 200 \). But there is also a reverse trend: For \( \text{Str} = 0.05 \) the KAM tori first shrink upon increasing Re and then, for \( \text{Re} > 200 \) the remaining tori in the upper part of the cavity grow in size again. At the highest frequency considered, \( \text{Str} = 1 \), the topology remains mainly regular up to \( \text{Re} = 100 \), except for small chaotic regions which can be detected near the singular corners.

4 Conclusions

For the range of Strouhal numbers investigated, the Lagrangian topology sensitively depend on Str. At the lowest Strouhal number, \( \text{Str} = 10^{-2} \), the topology is made of multiple KAM tori, stretched along the streamlines of the instantaneous flow field. The most stretched KAM tori break upon an increase of the Reynolds number and chaotic trajectories emerge from the side walls of the cavity. The mixing potential of the two-dimensional flow, i.e. the existence of chaotic streamlines, does not directly correlate with the complexity of the instantaneous Eulerian velocity field which can be found as the quasi-steady limit \( \text{Str} \rightarrow 0 \) is approached.

References

[1] J. M. Ottino, The kinematics of mixing: stretching, chaos, and transport (Cambridge University Press, Cambridge, 1989).
[2] P. D. Anderson, O. S. Galaktinov, G. W. M. Peters, F. N. van de Vosse, and H. E. H. Meijer, Int. J. Heat and Fluid Flow 21, 176-185 (2000).
[3] J. R. Dormand and P. J. Prince, J. Comput. Appl. Math. 6, 19-26 (1980).
[4] J. Zhu, L. E. Holmedal, H. Wang, and D. Myrhaug Eur. J. Mech. B/Fluids 79, 255-269 (2020).