Some characteristics of three exact solutions of Einstein equations minimally coupled to a Quintessence field

Xiao-hua Zhou

Department of Mathematics and Physics, The Fourth Military Medical University, Xi’an 710032, People’s Republic of China

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We show some characteristics of three exact solutions to the Einstein’s gravity minimally coupled to a Quintessence field. Besides eternal inflation, several other interesting inflationary processes, such as transitory inflation, are attained in these solutions. Singularity is avoided in some special cases.

I. INTRODUCTION

Since it has been found that the universe is undergoing an accelerated and expanding process,[1–4] which is named inflation, many methods have been put forward to explain this phenomenon. Typical methods, the so-called dark energy models which can be described as the Quintessence,[5] Phantom,[6] K-essence[7] and so on (see review in Ref.[8]), are being used to achieve an inflationary universe. The Lagrangian of a scalar field is minimally coupled to gravity and named Quintessence,

\[ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \]

where \( V \) is a suitable potential. Under the assumption that the universe is homogeneous and isotropic, the corresponding pressure and density of the scalar field can be written as

\[ p = \frac{1}{2} \dot{\phi}^2 - V(\phi), \quad \rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \]

where an overdot denotes differential with respect to time. We choose metric as the spatially flat Friedmann-Lemaître-Robertson-Walker form, \( ds^2 = dt^2 - a^2(t)dx^2 \), where the 3-D lien-element \( dx^2 \) is a flat 3-D space, and take a perfect fluid energy-momentum tensor \( T^{\mu \nu} = (\rho_M + P_M)U^\mu U^\nu - P_M g^{\mu \nu} \), for which the curvature scalar is

\[ R = 6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] = 6 \left( \dot{H} + 2H^2 \right), \]

where the Hubble parameter \( H = \frac{\dot{a}}{a} \), and \( a(t) \) is the scale factor. The cosmological Einstein equations are

\[ 3M_p^2 H^2 - \frac{1}{2} \dot{\phi}^2 - V = 0, \]

\[ \ddot{\phi} + 3H \dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \]

where \( M_p = (8\pi G)^{-1/2} \). The above two equations are complicated and difficult to extract details about cosmological evolution by solving them generally. The de Sitter solution: \( a(t) \propto \exp(Ht) \) (\( H > 0 \)) and power-law solution \( a(t) \propto t^A (A > 1) \) have been investigated extensively by Barrow,[9–12] Besides, a complex solution with a sixth-degree potential was obtained by Islam,[13] and other interesting solutions were shown in Ref.[14–17]

Generally speaking, inflationary solution, which is important to avoid the flatness, horizon and isotropic problems, means the accelerated and expanding universe, in which, we need

\[ \ddot{a}(t) > 0, \]

where one should note that the expanding universe condition \( H > 0 \) should be satisfied in the meantime. This condition is easily to be satisfied with the de Sitter solution and power-law solution. We suppose that condition (6) is satisfied in a region:

\[ 0 \leq t_b \leq t \leq t_e, \]

In inequality (7), if \( t_b > 0 \), we name this process the transitory inflation; if \( t_b = 0 \), we name it the immediate inflation because inflation occurs with the beginning of the universe immediately; and if \( t_b = 0 \) and \( t_e \to \infty \), we name it the eternal inflation. In this letter, we discuss three exact solutions which contain several interesting inflationary processes.

II. THREE EXACT SOLUTIONS

A. Solution one

\[ a(t) = a_0 \exp \left[ -\alpha t \ln \left( \frac{t}{t_0} \right) + \beta t \right], \]

\[ \phi = \pm 2\sqrt{2\alpha} M_p \sqrt{t} + \phi_0, \]

\[ V(\phi) = M_p^2 \alpha^2 \left\{ 3 \left( 1 - \frac{\beta}{\alpha} + 2 \ln \left( \frac{\phi - \phi_0}{\phi_p} \right) \right)^2 - \frac{8M_p^2}{(\phi - \phi_0)^2} \right\}, \]
where $a_0 > 0, \alpha > 0, t_0 > 0$ and $\beta, \phi_0$ are constants and $\phi_p = 2M_p \sqrt{2\alpha t_0}$. A simple method to prove that solution one and the following two solutions are indeed solutions of Eqs. (4) and (5) is shown in appendix A. If $\alpha = 0$, this solution is reduced to be de Sitter (or anti-de Sitter) solution. Moreover, several characteristics of this solution can be found.

(1). Expanding process. The Hubble parameter is
\[
H = \beta - \alpha \left[ 1 + \ln \left( \frac{t}{t_0} \right) \right].
\]
(11)
Let $\xi = e^{1+\beta/\alpha}$, it is easy to find that expanding universe condition $H > 0$ induces
\[
t < \xi t_0.
\]
(12)
At $t = \xi t_0$, the universe reaches its biggest extension $a_{\text{max}} = a_0 e^{\beta t_0} \xi - \alpha t_0 \xi$.

(2). Inflationary process. Inflationary condition $\ddot{a} > 0$ leads to
\[
t \ln^2 \left( \frac{t}{\xi t_0} \right) - \frac{1}{\alpha} > 0.
\]
(13)
Now, we introduce two new parameters $x = t(\xi t_0)^{-1}$ and $\eta = (\alpha \xi t_0)^{-1}$, then one can find that condition (13) will be reduced to a simple state in which there are only two parameters
\[
x \ln^2(x) > \eta.
\]
(14)
Two parameters make sure that a general picture of inequality (14) can be shown in a planar figure. We let $f(x) = x \ln^2(x)$ and show it in Fig. 1. It is easy to find that $f(0) = f(1) = 0$ and $f(x)$ reaches its local maximum $f_{\text{max}} = 4e^{-2}$ at $x = e^{-2}$. Noting that expanding condition $H > 0$ satisfies $t < \xi t_0 (x < 1)$ and $f(x)$ reaches its minimum at $t = \xi t_0 (x = 1)$, thus the existence of inflation requires $\eta < 4e^{-2}$. Let $x_1, x_2, x_3 (x_1 < x_2 < x_3)$ be the three solutions of $f(x) = \eta$ with $\eta < 4e^{-2}$, transitory inflation occurs only in the range $x_1 < x < x_2$ ($x_1 \xi t_0 < t < x_2 \xi t_0$). Whether $\eta < 4e^{-2}$ or $\eta \geq 4e^{-2}$, the universe will change from expanding to collapsing when it crosses $t = \xi t_0$. Therefore, this solution provides us with an interesting evolution process which is different to de Sitter solution and power-law solution, in which inflation occurs immediately and forever.

(3). Singularity. Submitting (9) into (3), the curvature scalar is
\[
R = 6\alpha \left[ 2\alpha \ln \left( \frac{t}{\xi t_0} \right) - \frac{1}{t} \right].
\]
(15)
To locate singularities in this solution, we should calculate the Kretschmann scalar $K = R_{abcd}R^{abcd}$, which is
\[
K = 24 \left\{ \left[ \alpha - \beta + \alpha \ln \left( \frac{t}{t_0} \right) \right]^4 \\
- \alpha \left[ \alpha - \beta + \alpha \ln \left( \frac{t}{t_0} \right) \right]^2 + \frac{\alpha^2}{2t^2} \right\}.
\]
(16)
It indicates that there are two singularities in this solution. One is at the initial time $t = 0$ and the other will occur in the infinite future.

(4). State parameter. The state parameter
\[
\omega = \frac{P}{\rho} = -1 + \frac{2\eta}{3x \ln^2 x},
\]
(17)
it reaches its minimum
\[
\omega_{\text{min}} = -1 + \frac{e^2 \eta}{6}
\]
(18)
at $x = e^{-2}$. Nearby $x = e^{-2}$, it is just the inflation process (see Fig. 1).

B. Solution two

\[
a(t) = a_0 \exp \left( \mu t - \lambda t^n \right),
\]
(19)
\[
\phi = \frac{1}{n} \sqrt{2n(n-1)\lambda M_p t^n} + \phi_0,
\]
(20)
\[
V(\phi) = 3M_p^2 \left\{ \mu - 2\frac{\dot{\phi}}{\phi} - 3n\lambda \left[ \frac{n(\phi - \phi_0)}{\phi_p} \right]^{2-\frac{4}{n}} \right\}^2
\]
\[
-2\mu^2 (n-1)n\lambda M_p^2 \left[ \frac{n(\phi - \phi_0)}{\phi_p} \right]^{2-\frac{4}{n}},
\]
(21)
where $a_0 > 0, \mu, \lambda, \phi_0$ are constants, and where $\phi_p = 4M_p \sqrt{(n-1)n\lambda}$. This solution is given by Barrow and Liddle in Ref.[18]. In (20), we need $(n-1)n\lambda \geq 0$, which leads to the following conditions
\[
\lambda > 0, n < 0;
\]
(22)
\[
\lambda > 0, n > 1;
\]
(23)
\[
\lambda < 0, 0 < n < 1;
\]
(24)
\[
(n-1)n\lambda = 0.
\]
(25)
Several characteristics of this solution are shown as below.

(1) De Sitter solution and anti-de Sitter solution. This solution can also be reduced to de Sitter solution in the following two cases
\[
\begin{align*}
&\mu > 0, \ n\lambda = 0; \\
&\mu > \lambda, \ n = 1.
\end{align*}
\]
\hfill (26)
\hfill (27)

Anti-de Sitter solution can be attained in two states
\[
\begin{align*}
&\mu < 0, \ n\lambda = 0; \\
&\mu < \lambda, \ n = 1.
\end{align*}
\]
\hfill (28)
\hfill (29)

(2) Expanding process. The Hubble parameter is
\[
H = -n\lambda t^{n-1} + \mu.
\]
\hfill (30)

Considering conditions in (22)-(25), expanding universe condition \(H > 0\) is valid in the following cases
\[
\begin{align*}
t \geq 0 \quad &\text{for } \lambda > 0, n < 0, \mu \geq 0; \\
t < \left(\frac{\mu}{n\lambda}\right)^{\frac{1}{n-1}} \quad &\text{for } \lambda > 0, n < 0, \mu < 0; \\
t < \left(\frac{\mu}{n\lambda}\right)^{\frac{1}{n-2}} \quad &\text{for } \lambda > 0, n > 1, \mu > 0; \\
t \geq 0 \quad &\text{for } \lambda < 0, 0 < n < 1, \mu \geq 0; \\
t < \left(\frac{\mu}{n\lambda}\right)^{\frac{1}{n-1}} \quad &\text{for } \lambda < 0, 0 < n < 1, \mu < 0; \\
t \geq 0 \quad &\text{for } \mu > 0, n\lambda = 0; \\
t \geq 0 \quad &\text{for } \mu > \lambda, n = 1.
\end{align*}
\]
\hfill (31)
\hfill (32)
\hfill (33)
\hfill (34)
\hfill (35)
\hfill (36)
\hfill (37)

In the above cases, (31), (34), (36) and (37) are the eternal expanding states, and the universe will turn from expanding to collapsing at \(t = \left(\frac{\mu}{n\lambda}\right)^{\frac{1}{n-1}}\) in the other states.

(3) Inflationary process. Inflationary condition \(\ddot{a} > 0\) is reduced to
\[
n^2\lambda^2t^{2n} + \mu^2t^2 - n\lambda t^n(2\mu t + n - 1) > 0.
\]
\hfill (38)

Unlike solution one, many constants make it difficult to give a general picture of the valid region of condition (38). We will give an example in the later part of this section, in which inflation condition is expressed in apparent state.

(4) Singularity. The curvature scalar is
\[
R = 6\left[(1 - n)n\lambda t^{n-2} + 2(\mu - n\lambda t^{n-1})\right],
\]
\hfill (39)
and the Kretschmann scalar is
\[
K = 12\left\{\left[(n\lambda t^{n-1})^2 + \mu^2 - n\lambda t^{n-2}(2\mu t + n - 1)\right]^2 + (\mu - n\lambda t^{n-1})^4\right\}.
\]
\hfill (40)

If \(n < 1\), the universe begins with an initial singularity and it will evolve to de Sitter (or anti-de Sitter) phase as \(H \approx \mu (R \approx 12\mu)\) when \(t \to \infty\). Specially, \(n = 1\) is the de Sitter (or anti-de Sitter) spacetime. Else if \(1 < n < 2\) the universe begins with an initial singularity and it will evolve to another end-singularity. If \(n \geq 2\), the universe begins with de Sitter spacetime and it will evolve to a singularity in the infinite future.

Now, let’s see an example for \(n = -1\). Then this solution can be written as
\[
a(t) = a_0 \exp\left(\frac{\mu t}{\lambda} - \frac{\lambda}{t}\right),
\]
\hfill (41)
\[
\phi = \pm 4M_p\sqrt{\lambda/\mu} t + \phi_0,
\]
\hfill (42)
\[
V(\phi) = M_p^2\left[3\left[\mu + \left(\frac{\phi - \phi_0}{\phi_p}\right)^4\right]^2 - \left(\frac{\phi - \phi_0}{\phi_p}\right)^6\right],
\]
\hfill (43)
where \(\lambda \geq 0\). Several characteristics of this solution can be found as follows.

(1) Expanding process. The Hubble parameter is
\[
H = \mu + \frac{\lambda}{t^2}.
\]
\hfill (44)

For \(n = -1\), valid expanding universe conditions (31), (32) and (36) can be simplified as
\[
\begin{align*}
t \geq 0 \quad &\text{for } \lambda > 0, \mu \geq 0; \\
t < \sqrt{-\lambda/\mu} \quad &\text{for } \lambda > 0, \mu < 0; \\
t \geq 0 \quad &\text{for } \lambda = 0, \mu > 0.
\end{align*}
\]
\hfill (45)
\hfill (46)
\hfill (47)

(2) Inflationary process. Condition \(\ddot{a} > 0\) in (38) can be simplified as
\[
(\mu^2 + \lambda)^2 - 2\lambda t > 0.
\]
\hfill (48)

The left hand of inequality (48) is a high-order function of \(t\), and which is difficult to be solved generally. However, by letting \(x = \mu^2/\lambda\) and \(\delta = \frac{\mu}{\lambda}\) (\(\mu\lambda \neq 0\)), condition (48) can be reduced to the following simple forms in which there are only two parameters
\[
\begin{align*}
(1 + x)^{4/\delta} > \delta &\text{ for } \mu > 0; \\
(1 + x)^{4/\delta} < \delta &\text{ for } \mu < 0.
\end{align*}
\]
\hfill (49)
\hfill (50)

Specially, when \(\mu\lambda = 0\), we have
\[
\begin{align*}
t < \lambda/2 &\text{ for } \mu = 0, \lambda > 0; \\
t \geq 0 &\text{ for } \mu > 0, \lambda = 0.
\end{align*}
\]
\hfill (51)
\hfill (52)

Noting that expanding conditions in (45)-(47) should be satisfied in the meantime, allying the above conditions (45)-(47) with (49)-(52), one gets the valid inflationary regions
\[
\begin{align*}
t < \lambda/2 &\text{ for } \mu = 0, \lambda > 0; \\
t \geq 0 &\text{ for } \mu > 0, \lambda = 0; \\
(1 + x)^{4/\delta} > \delta &\text{ for } \mu > 0, \lambda > 0; \\
(1 + x)^{4/\delta} < \delta &\text{ and } x > -1 \text{ for } \mu < 0, \lambda > 0.
\end{align*}
\]
\hfill (53)
\hfill (54)
\hfill (55)
\hfill (56)
In the above cases, (53) is immediate inflationary state and (54) is de Sitter inflationary process. Let \( f(x) = (1 + x^4)/x \), a general picture of conditions (55) and (56) can be obtained, which is shown in Fig. 2. One can find that \( f(x) \) has a local minimum \( f_{\min} = \frac{256}{27} \) at \( x = \frac{1}{4} \) and a local maximum \( f_{\max} = 0 \) at \( x = -1 \). When \( \delta > 0 \) (\( \mu > 0 \)), if \( \delta < \frac{256}{27} (\mu \lambda > \frac{27}{64}) \) condition (55) is always satisfied and inflation will undergo forever; if \( \delta \geq \frac{256}{27} (\mu \lambda \leq \frac{27}{64}) \), there are two inflationary processes: the immediate inflation when \( x < x_1 (t < \sqrt{\lambda x_3/\mu}) \) and the late-time inflation when \( x > x_2 (t > \sqrt{\lambda x_2/\mu}) \), where \( x_1, x_2 \) are the two solutions for \( f(x) = \delta \). When \( \delta < 0 \) (\( \mu < 0 \)), letting \( x_3, x_4 \) be the two solutions of \( f(x) = \delta \), considering condition (56), one finds that there is only one inflationary process: the immediate inflation when \( x > x_4 (t < \sqrt{\lambda x_2/\mu}) \).

(3). State parameter. The corresponding state parameter is

\[
\omega = -1 + \frac{2\sqrt{3\delta}}{3(1 + x)^2}.
\]

If \( \delta > 0 \) (\( \mu > 0 \)), one finds that \( \omega \) reaches its maximum

\[
\omega_{\text{max}} = -1 + \frac{\sqrt{3\delta}}{8}
\]

at \( x = \frac{1}{4} \) \((t = \sqrt{\lambda/(3\mu)})\). If \( \delta < 0 \) (\( \mu < 0 \)), one gets the state parameter \( \omega \rightarrow \infty \) when \( x \rightarrow -1 \) \((t \rightarrow -\sqrt{-\lambda/\mu})\). Specially when \( \mu = 0 \), one can easily find that \( \omega \) increases linearly with the evolvement of \( t \). We now turn to investigate the evolvement of the potential in Eq.(43). By solving \( \frac{dv}{dx} = 0 \), we obtain

\[
\phi = \phi_0, \phi = \phi_0 \pm 2M_p \sqrt{1 + \sqrt{1 - 16\lambda \mu}}.
\]

Choosing suitable constants, we will get several different evolvement of the potential. Fig. 3 shows an example of \( V(\phi) \) in Eq.(43) with \( 0 < 16\lambda \mu < 1 \). From it We can see that the curve has three turning points at \( \phi = \phi_0, \phi = \phi_1 \) and \( \phi = \phi_2 \). The universe begins with \( \phi = \phi_0 \) and clams up to the local maximum of \( V(\phi) \) at which \( \phi \) reaches \( \phi_1 \). After that, the universe will roll down from the maximum of \( V(\phi) \) to the minimum of \( V(\phi) \) at \( \phi = \phi_2 \) or it will roll back to \( \phi_0 \). This process is close to the one shown in Ref.[13].

C. Solution three

\[
a(t) = \sigma (e^{\mu t} - \tau e^{-\mu t})^n,
\]

where \( \sigma > 0, \tau, \mu, n \) are suitable constants, and we need

\[
\mu = \text{any value}, \tau \leq 0, n \leq 0;
\]

\[
\mu > 0, 0 < \tau \leq 1, n > 0.
\]

Here, \((\mu = \text{any value}, \tau \leq 0)\) or \( (\mu > 0, 0 < \tau \leq 1) \) will make sure \( e^{\mu t} - \tau e^{-\mu t} \geq 0 \) in (60) in the region \( 0 \leq t < \infty \). For \( 0 < \tau \leq 1 \) and \( n > 0 \), we have

\[
\phi = \pm \sqrt{2nM_p} \log \left( \frac{e^{\mu t} - \sqrt{\tau}}{e^{\mu t} + \sqrt{\tau}} \right) + \phi_0,
\]

\[
V(\phi) = \frac{n}{2} \mu^2 M_p^2 \left[ 1 + 3n + (3n - 1) \cosh \left( \frac{\phi - \phi_0}{\phi_p} \right) \right].
\]

In this case, the state parameter

\[
\omega = -1 + \frac{8\tau e^{2\mu t}}{3n(\tau + e^{2\mu t})^2}.
\]

It reaches its maximum \( \omega_{\text{max}} = -1 + \frac{2}{3n} \) at \( t = \frac{\ln \tau}{2\mu} \). For \( \tau < 0 \) and \( n < 0 \), we attain

\[
\phi = \pm 2\sqrt{-2nM_p} \arctan \left( \frac{e^{\mu t}}{\sqrt{-\tau}} \right) + \phi_0.
\]
The universe will expand forever. Inflationary condition (74) can be written as
\[ (e^{\mu t} - \sqrt{\tau})^2 > 0. \] (83)
Clearly, inflation will continue forever in this case.

III. CONCLUSIONS

In summary, the non-linear flow equation is a widely-used approach to explore the inflationary behavior of the universe, and our discussion provide us with several interesting evolutionary models of the universe. By our discussion, the fate of the universe depends strongly on those constants, different constants are corresponding to different cosmological evolvements. The most prominent characteristics of those solutions are that inflation will not undergo forever in some spacial cases, which makes those solutions that need to be studied deeply because current investigation do not support typical interne inflation.\(^{[19]}\)

In Ref.[20], Parsons and Barrow give a way to generate a family of new exact solutions. In their method,
if there is a solution \( a(t) = \exp[f(t)] \) (\( f(t) \) is an arbitrary function), we can get another exact solution with the form \( a(t) = \exp[At + f(t)] \). Specifically, by choosing \( f(t) = -\alpha t \ln(t/t_0) \), (note that \( a(t) = \exp[-\alpha t \ln(t/t_0)] \) is an exact solution), we obtain solution one. For solution two, there is \( f(t) = -\lambda t^n \), which is corresponding to the intermediate inflationary: \( a(t) = \exp(At^n) \), \( 0 < n < 1 \) in Refs.[21-23]. In this letter, we present the full valid regions of those constants and give some useful information about this solution. However, solution three cannot be obtained by this way. It can be written as \( a(t) = \exp[\ln \delta + \eta \mu t + \eta \ln(1 - \tau e^{-2\mu t})] \), and it is easy to find that \( a(t) = \exp[\eta \ln(1 - \tau e^{-2\mu t})] \) is not an exact solution.

Recently, Spalinski finds a way,[16] which is different to the standard truncating method,[24] to attain exact solution. In his solutions, the Hubble parameter \( H \) and potential \( V \) are the function of \( \phi \), but it is difficult to get the scalar factor \( a(t) \) with \( t \) as a variable. In this paper, solution three gives us an analytical function of \( a(t) \) in (60) and the corresponding \( H(t) \) in (69). Moreover, in this solution, the periodic potential in (67) is unlike the other potentials in (10), (21) and in (64) which will go to infinite value when \( \phi = \phi_0 \) or \( \phi \to \infty \), and singularity is avoided in some special cases, all of which leads it to be a recommendable solution.

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**APPENDIX A**

In this appendix, we give a simple way to prove that solution one is indeed a solution of Eqs.(4) and (5). The other two solutions in this paper can be tested by this way.

Taking Eq.(4) with a differential of \( t \), allying Eq.(5), one gets

\[
\phi = \pm M_p \int \sqrt{-2Hdt}, \quad (84)
\]

\[
V = M_p^2 (3H + \dot{H}), \quad (85)
\]

where one should note \( H = \frac{\dot{a}}{a} \), and \( \dot{H} = \frac{\dot{a}a - \ddot{a}a^2}{a^2} \). In general cases, we suppose that the primitive function of the right side of Eq.(84) is existent for a given function \( a(t) \), and we get an analytical function \( \phi = f(t) \) with \( t \) as the variable. Then we need to attain the inverse solution of \( \phi = f(t) \):

\[
t = f^{-1}(\phi), \quad (86)
\]

where the superscript \(-1\) denotes to the corresponding inverse function. Submitting (86) into (85), one will get \( V \) with \( \phi \) as the variable. The above process can be used to validate the three solutions in this paper. Let’s see an example. Inserting (8) into Eq. (84), one attains Eq.(9). Then one gets the inverse solution of Eq.(9)

\[
t = \frac{1}{8\alpha} M_p^{-2}(\phi - \phi_0)^2. \quad (87)
\]

Inserting (8) into Eq.(85), one has

\[
V = M_p^2 \left\{ 3 \left[ \alpha - \beta + \alpha \ln \left( \frac{t}{t_0} \right) \right]^2 - \frac{\alpha}{t} \right\}. \quad (88)
\]

Submitting (87) into (88), one will attain \( V \) with \( \phi \) as the variable in (10).