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 Modeling the dynamics of coronavirus with super-spreader class: A fractal-fractional approach  
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A R T I C L E I N F O

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A B S T R A C T

Super-spreaders of the novel coronavirus disease (or COVID-19) are those with greater potential for disease transmission to infect other people. Understanding and isolating the super-spreaders are important for controlling the COVID-19 incidence as well as future infectious disease outbreaks. Many scientific evidences can be found in the literature on reporting and impact of super-spreaders and super-spreading events on the COVID-19 dynamics. This paper deals with the formulation and simulation of a new epidemic model addressing the dynamics of COVID-19 with the presence of super-spreader individuals. In the first step, we formulate the model using classical integer order nonlinear differential system composed of six equations. The individuals responsible for the disease transmission are further categorized into three sub-classes, i.e., the symptomatic, super-spreader and asymptomatic. The model is parameterized using the actual infected cases reported in the kingdom of Saudi Arabia in order to enhance the biological suitability of the study. Moreover, to analyze the impact of memory index, we extend the model to fractional case using the well-known Caputo–Fabrizio derivative. By making use of the Picard–Lindelöf theorem and fixed point approach, we establish the existence and uniqueness criteria for the fractional-order model. Furthermore, we applied the novel fractal-fractional operator in Caputo–Fabrizio sense to obtain a more generalized model. Finally, to simulate the models in both fractional and fractal-fractional cases, efficient iterative schemes are utilized in order to present the impact of the fractional and fractal orders coupled with the key parameters (including transmission rate due to super-spreaders) on the pandemic peaks.

Introduction

COVID-19 is a newly emerged respiratory infection caused by the SARS-CoV-2 virus. The first case confirmed with this infection was reported in the population of Wuhan, China in December 2019. After the first outbreak in Wuhan, the infection spread to the whole mainland of China within few days. This novel infection has been reached to almost every country on the globe leading to the ongoing COVID-19 pandemic. The incubation period after exposure to the virus ranges from 2 to 14 days. A person who developed noticeable disease symptoms is categorized as a symptomatic case of COVID-19. On the other hand, a COVID-19 patient who has not developed or has mild disease symptoms is classified as an asymptomatic case. Mostly, (up to 80%) infected cases develop mild or moderate diseases signs while 14% of infected cases were found with severe symptoms. The common symptoms noticed in COVID-19 patients include muscle and joint pain, headache, fever, fatigue, vomiting, diarrhea, loss of taste combined with loss of smell and in some severe cases, it may lead to shortness of breath [1,2]. Initially, the transmission of COVID-19 was considered from animal to human and the first case was found to catch virus from an animal. But later on it was revealed that transmission of the infection can be transmitted from human to human. The disease is mainly transmitted via the respiratory route. The people closed to the infected individuals are more likely to catch the virus through the droplets and small airborne particles released by infectious person when they breathe.
cough, sneeze or even talk [1,2]. Super-spreader of COVID-19 are those infectious individuals who have a high ability of virus transmission and can and can infect multiple people [3,4]. A super-spreader may be active in social activities and may not be a severe case and have chances to contact many persons within a short period of time. They have a higher viral load and longer virus shedding time. Timely identification and isolation of such people are very important to combat the disease incidence [3].

The key challenge to overcome the ongoing COVID-19 pandemic is to understand its complex dynamics. Various approaches have been employed for this purpose. Mathematical modeling is one of the considerable tools in this regard and has been utilized effectively so far. Mathematical models in epidemiology often result in nonlinear differential systems with either ordinary, partial, stochastic or delay types [5–7]. Mathematical models addressing the dynamics of COVID-19 with aforementioned differential systems are formulated and simulated in the recent literature as we refer to and reference therein [8–10]. It is noticed that the epidemic models with classical differential operators rather than fractional cases are local in nature and do not have memory effects. Therefore, due to the non-local nature of the fractional derivatives, the mathematical model formulated via these operators describes the disease dynamics in a better way. The fractional order derivatives gain considerable attention and are employed for this purpose. Mathematical modeling is one of the concerned methods to understand its complex dynamics. Various approaches have been proposed to model the transmission dynamics of COVID-19. The dynamics of COVID-19 in Wuhan, China. The application of various fractal-fractional operators in epidemiology has been addressed with the help of ABC fractional operator in [22]. The application of various fractal-fractional operators in epidemiology can be found in [23,24].

Keeping the above fact in mind, the present study focuses on the formulation and simulation of a new compartmental model addressing the dynamics of COVID-19. Based on the evidence of super-spreaders of COVID-19, the infected individuals causing the disease transmission are further categorized into three sub-classes i.e., symptomatic, asymptomatic and super-spreaders. The model is parameterized from the actual infected cases reported in the Kingdom of Saudi Arabia for a specific period of time. The basic mathematical analysis of the proposed model is performed. Moreover, the model is extended to fractional case using the Caputo–Fabrizio derivative. Finally, the model in integer formulation and simulation of a new compartmental model addressing the dynamics of COVID-19 with aforementioned differential systems are formulated and simulated in the recent literature as we refer to and reference therein [23,24].

The application of various fractal-fractional operators in epidemiology have been addressed with the help of ABC fractional operator in [22].

Basics on fractional calculus

We recall some necessary definitions and results on the fractional and fractal operators [25–27].

Definition 1. Let \( Z(t) \in \mathbb{C}^n \) be function then the Caputo operator with order \( \sigma \) such that \( n-1 < \sigma < n \in \mathbb{N} \), is defined as follows:

\[
\begin{align*}
C^D_\sigma^n(Z(t)) &= \frac{1}{\Gamma(n-\sigma)} \int_0^t Z^\sigma(\zeta) \left( \frac{t-\zeta}{\Gamma(n-\sigma)} \right)^{n-\sigma-1} d\zeta.
\end{align*}
\]

Clearly, \( C^D_\sigma^n(Z(t)) \to Z'(t) \) as \( \sigma \to 1 \).

Definition 2. Consider \( Z \in H^1(a_1, a_2) \), \( a_2 > a_1 \), \( \sigma \in [0, 1] \), then the Caputo–Fabrizio derivatives [25] is defined as:

\[
\begin{align*}
C^F_\sigma D^\sigma_1 Z(t) &= M(\sigma) \int_{a_1}^t Z'(\phi) \exp \left( -\frac{(t-\phi)}{1-\sigma} \right) d\phi.
\end{align*}
\]

Remark 3. If \( \sigma = \frac{1-\mu}{1-\alpha} \in [0, \infty) \) then, \( \sigma = \frac{1}{1+\mu} \in [0, 1] \) and hence eq: (2) can be reconstructed as:

\[
\begin{align*}
C^F_\sigma D^\sigma_1 Z(t) &= N(\mu) \int_{a_1}^t Z'(\phi) \exp \left( -\frac{t-\phi}{\mu} \right) d\phi.
\end{align*}
\]
where, the normalization part satisfies the condition $N(0) = N(\infty) = 1$. Further,

$$\lim_{\nu \to 0} \frac{1}{\nu} \exp \left( -\frac{t}{\nu} \right) = \delta(t) = 1.$$  

(5)

**Definition 4.** The fractional integral corresponds to (2) is given by [28]:

\[
\begin{align*}
\mathcal{C}_0^\nu I_s^\mu Z(t) &= \frac{2(1-\nu)}{M(\nu)(2-\nu)} Z(t) \\
&+ \frac{2\nu}{M(\nu)(2-\nu)} \int_0^t Z(\tau) d\tau, \quad t > 0.
\end{align*}
\]

(6)

Consider the function $X(t)$ which is continuous as well as fractal differentiable over the interval $(a, b)$ with fractal order $\beta$ and fractional order $\nu$ then we recalled the following definitions [27]:

**Definition 5.** The fractal-fractional derivative of $W$ in the RL case and with exponential kernel is as follows:

\[
\begin{align*}
^{F^E}_{D_{0+}}D_{\nu}^\sigma (W(t)) &= \frac{M(\nu)}{1-\nu} \int_0^t \exp \left( -\frac{\nu}{1-\nu}(t-\zeta) \right) W(\zeta) d\zeta, \\
\end{align*}
\]

with $n-1 < \nu, \beta \leq n \in \mathbb{N}$ and moreover $M(0) = M(1) = 1$.

**Definition 6.** The corresponding FF integral operator is defined as follows:

\[
\begin{align*}
^{F^E}_{D_{0+}}J_{\nu}^\sigma (W(t)) &= \frac{\nu\beta}{M(\nu)} \int_0^t \exp \left( -\frac{\nu}{1-\nu}(t-\zeta) \right) W(\zeta) d\zeta \\
&+ \frac{\beta(1-\nu)\nu^{\beta-1}W(t)}{M(\nu)}. \\
\end{align*}
\]

(8)
In this section, we present the formulation of the proposed COVID-19 epidemic model using the classical integer order nonlinear differential system. The infectious individuals (who can transmit the infection) are further categorized into three different subgroups i.e., the symptomatic, asymptomatic and super-spreader individuals symbolized by $I(t)$, $A(t)$, and $P(t)$ respectively at any instance of time $t$. The super-spreaders of COVID-19 were considered to have efficient transmission ability of the infection. Moreover, the rest of population subgroups are the susceptible denoted by $S(t)$ who are at risk to catch infection, the exposed individuals denoted by $E(t)$ who are infected but are unable to transmit the infection and finally the fully recovered from the infection is denoted by $R(t)$. Thus, we divided the whole population $N(t)$ into six sub-classes so that

$$N(t) = S(t) + E(t) + I(t) + P(t) + A(t) + R(t).$$

We constructed the following system of nonlinear differential equations in order to describe the dynamics of the COVID-19 infection.

$$\begin{align*}
\frac{dS}{dt} &= A - \beta_1 \frac{(1 + \omega) AS}{N} - \beta_2 \frac{PS}{N} - \mu_S, \\
\frac{dE}{dt} &= \beta_1 \frac{(1 + \omega) AS}{N} + \beta_2 \frac{PS}{N} - (\theta + \mu) E, \\
\frac{dI}{dt} &= \theta \omega_1 E - (\tau_1 + \mu + \zeta_1) I, \\
\frac{dP}{dt} &= \theta \omega_2 E - (\tau_2 + \mu + \zeta_2) P, \\
\frac{dA}{dt} &= \theta (1 - \omega_1 - \omega_2) E - (\tau_3 + \mu) A, \\
\frac{dR}{dt} &= \tau_3 A + \tau_2 P + \tau_1 I - \mu R.
\end{align*}$$

The initial conditions (ICs) for the problem described in (9) are as follows:

$$\begin{align*}
S(0) &= S_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, P(0) = P_0 \geq 0, \\
A(0) &= A_0 \geq 0, R(0) = R_0 \geq 0.
\end{align*}$$

The detail of parameters used in the COVID-19 model with super-spreader class is as: $\alpha$ shows the recruitment rate, $\mu$ used for the natural mortality rate for all subgroups of population. The symbol $\beta_1$ shows the disease transmission rates of symptomatic and asymptomatic COVID-19 individuals and $\psi_1$ denotes the relative infectiousness rate corresponding $A(t)$. The parameter $\beta_2$ denotes the disease transmission rate of super-spreader individuals where it is natural to note that ($\psi_1 < \beta_1 < \beta_2$). The incubation period is denoted by $\theta$. The exposed individuals move to three different infectious compartments after the incubation period. A fraction $\omega_1$ of exposed individuals develop the disease signs (or symptoms) and thus join symptomatic compartment and a fractional $\omega_2$ (with or without disease signs) join the super-spreader class while the remaining with no or mild disease signs move to the asymptomatic class $A(t)$. The symptomatic, super-spreader and
Fig. 4. Simulation for super-spreaders $P(t)$ with different rates of $\beta_i$. Further, in the subplots we have (a) $\sigma = 1$, (b) $\sigma = 0.90$, (c) $\sigma = 0.80$.

| Table 1 | System (9) state variables. |
|---------|-----------------------------|
| $S$     | Susceptible individuals     |
| $E$     | Exposed individuals         |
| $I$     | Symptomatic individuals     |
| $P$     | Super-spreader individuals  |
| $A$     | Asymptomatic individuals    |
| $R$     | Recovered individuals       |

The biologically feasible region for the problem design in (9) with super-spreader class is given by $\mathcal{Z} \subset \mathbb{R}_+^6$, where, $\Omega = \{(S, E, I, P, A, R) \in \mathbb{R}_+^6: S + E + I + P + A + R \leq \frac{A}{\mu}\}$.

The basic reproductive number

In order to compute the crucial biological parameter known as the basic reproductive number, we need the disease free equilibrium of the COVID-19 model with super-spreader class (9). The $DFE$ say $W_0$ and is described as follows:

$$D_0 = \left(\frac{A}{\mu}, 0, 0, 0, 0, 0\right).$$

Furthermore, the reproductive number $R_0$ is obtained by utilizing the well-known next generation matrix approach. To proceed for the calculation of $R_0$, the required Jacobian matrices obtained from the problem (9) are as follows:

$$F = \begin{pmatrix}
0 & \beta_1 & \psi_1 & \beta_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

$$V = \begin{pmatrix}
(\theta + \mu) & 0 & 0 & 0 \\
-\theta \omega_1 & (r_1 + \mu + \zeta_1) & 0 & 0 \\
-\theta \omega_2 & 0 & (r_2 + \mu + \zeta_2) & 0 \\
-\theta (1 - \omega_1 - \omega_2) & 0 & 0 & (r_3 + \mu)
\end{pmatrix}.$$

Thus, with the help of aforementioned approach, the basic reproductive number is given by the following expression:

$$R_0 = \frac{\theta (\omega_1 (r_1 + \mu) + \psi_1 (r_1 + \mu + \zeta_1) (1 - \omega_1 - \omega_2)) \beta_1}{(\theta + \mu) (r_1 + \mu + \zeta_1) (r_2 + \mu) + \theta \omega_2 \beta_2 (\theta + \mu) (r_2 + \mu + \zeta_2)}.$$

The estimation procedure and data fitting

This part of the study addresses the data fitting analysis and the extension procedure of the model parameters from the actual infected
cases reported in the Kingdom of Saudi Arabia (KSA) from the start of pandemic. The resulting values of the parameters will be utilized in the model simulation. The estimation from the incidence data has biological importance and enhances the suitability of the study. The parametrization process of the proposed COVID-19 model with super-spreaders (9) is conducted via two approaches. Firstly, the values of $\lambda$ and $\mu$ are estimated from the population of KSA as tabulated below [2].

The second step is to make use of the well-known statistical method based on minimizing the square of residuals to evaluate the rest of biological parameters involved in the proposed epidemic model from actual infected cases in the selected region. The main steps that take place in this procedure can be described as:

- Since, in KSA the average lifespan in years is 74.7 [2] therefore, the value of $\mu = 1/(74.7 \times 365)$ per day.
- The value of $A$ is obtained as $A = 1254.9$ per day under the assumption that the present total population of KSA is susceptible to the infection.
- The objective function taken into the account to estimate the rest of the involved parameters is presented in the following Eq. (11):

$$ \phi = \sum_{i=1}^{k} (x_{ti} - \hat{x}_{ti})^2. \quad (11) $$

In (11) $x_{ti}$ and $\hat{x}_{ti}$ denote respectively the actual and predicted cases at time $t_i$, while $k$ is the actual data points considered in the estimation procedure.

- The prediction of model (9) (shown by solid curve) to the reported cases (shown by red circles) is illustrated in Fig. 1. It shows better agreement to the real data curve. The estimated, as well as fitted model parameters values, are presented in Table 2. The initial conditions are taken as $S(0) = 34806870, E(0) = 20000, I(0) = 2, A(0) = 100, P(0) = R(0) = 0$. The resulting estimated value of $R_0$ based on the estimated parameters values is approximately 1.4945.

### The fractional COVID-19 model in Caputo–Fabrizio case

The epidemic transmission models formulated using fractional operators give a deeper information about the disease dynamics. The fractional epidemic models capture the memory effect on the disease dynamics. This effect is important and is not captured by classic models. This part of the paper presents the formulation of fractional epidemic
Fig. 6. Graphical interpretation of exposed individuals $E(t)$ for reduction in $\beta_1$ with different rates. In the subplots (a) $\sigma = 1$, (b) $\sigma = 0.90$, (c) $\sigma = 0.80$.

model for the COVID-19 dynamics with the super-spreader class. The Caputo–Fabrizio operator based on exponential type kernel is utilized for this purpose. In order to reformulate the epidemic model (9) in fractional environment, the classical integer-derivative is replaced with a Caputo–Fabrizio derivative. So, we obtained the following fractional problem describing the COVID-19 dynamics with super-spreader class

$$\begin{aligned}
\frac{C F D^\sigma_0 S}{0} &= A - \beta_1 \frac{(I + \psi_1) A S}{N} - \beta_2 \frac{P S}{N} - \mu S, \\
\frac{C F D^\sigma_0 E}{0} &= \beta_1 \frac{(I + \psi_1) A S}{N} + \beta_2 \frac{P S}{N} - (\theta + \mu) E, \\
\frac{C F D^\sigma_0 I}{0} &= \theta \omega_1 E - (\tau_1 + \mu + \zeta_1) I, \\
\frac{C F D^\sigma_0 P}{0} &= \theta \omega_2 E - (\tau_2 + \mu + \zeta_2) P, \\
\frac{C F D^\sigma_0 A}{0} &= \theta (1 - \omega_1 - \omega_2) E - (\tau_1 + \mu) A \\
\frac{C F D^\sigma_0 R}{0} &= \tau_3 A + \tau_2 P + \tau_1 I - \mu R.
\end{aligned}$$

(12)

The expression $\frac{C F D^\sigma_0}{0}$ symbolized the Caputo–Fabrizio derivative with exponential kernel such that $0 < \sigma \leq 1$. The problem (12) can be described alternatively in simpler structure, by letting

$b_1 = (\theta + \mu), \quad b_2 = (\tau_1 + \mu + \zeta_1), \quad b_3 = (\tau_2 + \mu + \zeta_2), \quad b_4 = (\tau_3 + \mu).

The existence and uniqueness of the fractional problem

$\frac{C F D^\sigma_0 S}{0} = A - \beta_1 \frac{(I + \psi_1) A S}{N} - \beta_2 \frac{P S}{N} - \mu S, \\
\frac{C F D^\sigma_0 E}{0} = \beta_1 \frac{(I + \psi_1) A S}{N} + \beta_2 \frac{P S}{N} - b_1 E, \\
\frac{C F D^\sigma_0 I}{0} = \theta \omega_1 E - b_2 I, \\
\frac{C F D^\sigma_0 P}{0} = \theta \omega_2 E - b_3 P, \\
\frac{C F D^\sigma_0 A}{0} = \theta (1 - \omega_1 - \omega_2) E - b_4 A \\
\frac{C F D^\sigma_0 R}{0} = \tau_3 A + \tau_2 P + \tau_1 I - \mu R.$

(13)

The existence and uniqueness of the model (13) along with ICs (10) are necessary to confirm before further analysis. In this section, we prove the aforementioned criterion for the fractional problem. The fixed point theory and the Picard–Lindelöf concepts are applied to obtain the desired results. The utilization of Caputo–Fabrizio derivative stated in (2) over the problem (13) and further using the relation in (6),

\[ ... \]
the following system is obtained

\[ S(t) - S(0) = \frac{CF}{\theta} \left( A - \beta_1 \frac{(I + \psi A S)}{N} - \frac{PS}{2} \right), \]
\[ E(t) - E(0) = \frac{CF}{\theta} \left( \beta_1 \frac{(I + \psi A S)}{N} + \frac{PS}{2} - \beta_1 E \right), \]
\[ I(t) - I(0) = \frac{CF}{\theta} \left( \theta \omega_1 E - \beta_1 I \right), \]
\[ P(t) - P(0) = \frac{CF}{\theta} \left( \beta_1 E - \gamma_1 P \right), \]
\[ A(t) - A(0) = \frac{CF}{\theta} \left( \theta (1 - \omega_1 - \omega_2) E - \gamma_1 A \right), \]
\[ R(t) - R(0) = \frac{CF}{\theta} \left( \gamma_2 A + \gamma_2 P + \gamma_1 I - \mu R \right). \]

The above system (14) can be simplified into the following form

\[ S(t) - S(0) = 2 \frac{(1 - \sigma t)}{(2 - \sigma t)} K_1(t, S) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_1(r, S)dr, \]
\[ E(t) - E(0) = 2 \frac{(1 - \sigma t)}{(2 - \sigma t)} K_2(t, E) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_2(r, E)dr, \]
\[ I(t) - I(0) = 2 \frac{(1 - \sigma t)}{(2 - \sigma t)} K_3(t, I) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_3(r, I)dr, \]
\[ P(t) - P(0) = 2 \frac{(1 - \sigma t)}{(2 - \sigma t)} K_4(t, P) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_4(r, P)dr, \]
\[ A(t) - A(0) = 2 \frac{(1 - \sigma t)}{(2 - \sigma t)} K_5(t, A) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_5(r, A)dr, \]
\[ R(t) - R(0) = 2 \frac{(1 - \sigma t)}{(2 - \sigma t)} K_6(t, R) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_6(r, R)dr, \]

where, the symbols \( K_i(t, x) \) are used for the kernels with \( i = 1, \ldots, 6 \) and \( x \) represents the respective state variables of the model which are given as follows:

\[ K_1(t, S) = A - \beta_1 \frac{(I + \psi A S)}{N} - \frac{PS}{2} - \mu S, \]
\[ K_2(t, E) = \beta_1 \frac{(I + \psi A S)}{N} + \frac{PS}{2} - \beta_1 E, \]
\[ K_3(t, I) = \theta \omega_1 E - \beta_1 I, \]
\[ K_4(t, P) = \theta \omega_2 E - \beta_1 P, \]
\[ K_5(t, A) = \theta (1 - \omega_1 - \omega_2) E - \beta_1 A, \]
\[ K_6(t, R) = \gamma_2 A + \gamma_2 P + \gamma_1 I - \mu R. \]

After using the Picard iterations, the recursive equations given below are obtained:

\[ S^{n+1}(t) = \frac{2(1 - \sigma)}{(2 - \sigma t)} K_1(t, S^n) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_1(r, S^n)dr, \]
\[ E^{n+1}(t) = \frac{2(1 - \sigma)}{(2 - \sigma t)} K_2(t, E^n) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_2(r, E^n)dr, \]
\[ I^{n+1}(t) = \frac{2(1 - \sigma)}{(2 - \sigma t)} K_3(t, I^n) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_3(r, I^n)dr, \]
\[ P^{n+1}(t) = \frac{2(1 - \sigma)}{(2 - \sigma t)} K_4(t, P^n) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_4(r, P^n)dr, \]
\[ A^{n+1}(t) = \frac{2(1 - \sigma)}{(2 - \sigma t)} K_5(t, A^n) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_5(r, A^n)dr, \]
\[ R^{n+1}(t) = \frac{2(1 - \sigma)}{(2 - \sigma t)} K_6(t, R^n) + \frac{2 \sigma}{H(\sigma)(2 - \sigma t)} \int_0^t K_6(r, R^n)dr. \]
Further, to complete the desired prove, the problem formulated in (13) can be express as follows:

\[
\begin{align*}
C^T \mathcal{D}_t^\alpha G(t) &= \mathcal{K}(t, G(t)), \\
\mathcal{K}(0, t) &= 0, \\
G(0) &= \mathcal{G}_0.
\end{align*}
\] (18)

In problem (18), \( \mathcal{K}(t, G(t)) = (S(t), E(t), I(t), R(t)) \) and \( \mathcal{K}(t, G(t)) \) are respectively defined by

\[
\mathcal{K}(t, G(t)) = \begin{cases}
A - \beta_1 \frac{I(t)}{N} - \beta_2 \frac{E(t)}{N} - \mu S(t), \\
\beta_1 \frac{I(t)}{N} + \beta_2 \frac{E(t)}{N} - \beta_1 E(t), \\
\theta \omega_1 E - \beta_2 I(t), \\
\theta (1 - \omega_1 - \omega_2) E - \beta_4 A, \\
\tau_1 A + \tau_2 P + \tau_1 I - \mu R(t).
\end{cases}
\] (19)

and the ICs are \( \mathcal{G}_0 = (S(0), E(0), I(0), R(0)) \).

The system (18), in the integral form can be written as

\[
\mathcal{G}(t) = \mathcal{G}(0) + \phi_1(t) \mathcal{K}(t, \mathcal{G}(t)) + \phi_2(t) \int_0^t \mathcal{K}(s, \mathcal{G}(s)) ds,
\] (20)

where, \( \phi_1(t) = \frac{2(1-t^\alpha)}{(2-\alpha)(1-t^\alpha)} \) and \( \phi_2(t) = \frac{2t^\alpha}{(2-\alpha)(1-t^\alpha)} \).

Proposition 7. Let \( \Sigma = [0, T] \), and \( \mathcal{G}(t) \in \mathcal{C}(\Sigma, \mathbb{R}^6) \), then we can find \( \delta > 0 \) such that

\[
\|\mathcal{K}(t, \mathcal{G}_1(t)) - \mathcal{K}(t, \mathcal{G}_2(t))\| \leq \delta \|\mathcal{G}_1(t) - \mathcal{G}_2(t)\|
\] (21)

for all \( \mathcal{G}_1(t), \mathcal{G}_2(t) \in \mathcal{C}(\Sigma, \mathbb{R}^6) \), and \( t \in \Sigma \).

Proof. The utilization of kernels defined in (16) leads to the following:

\[
\|\mathcal{K}(t, \mathcal{G}_1(t)) - \mathcal{K}(t, \mathcal{G}_2(t))\| = \begin{cases}
\|K_1(t, S_1) - K_1(t, S_2)\| \\
\|K_2(t, E_1) - K_2(t, E_2)\| \\
\|K_3(t, I_1) - K_3(t, I_2)\| \\
\|K_4(t, P_1) - K_4(t, P_2)\| \\
\|K_5(t, A_1) - K_5(t, A_2)\| \\
\|K_6(t, R_1) - K_6(t, R_2)\|
\end{cases}
\]

\[
= \begin{cases}
\|\mathcal{K}(t, G(t)) - \|\mathcal{K}(t, G(t))\| \leq \delta \|\mathcal{G}(t) - \mathcal{G}(t)\|
\end{cases}
\]

\[
= \begin{cases}
D_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
D_2 & 0 & 0 & 0 & 0 & 0 \\
D_3 & 0 & 0 & 0 & 0 & 0 \\
D_4 & 0 & 0 & 0 & 0 & 0 \\
D_5 & 0 & 0 & 0 & 0 & 0 \\
D_6 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\end{cases}
\] (22)

where,

\[
D_1 = \{\|\mathcal{K}(t, G(t))\| + \mu\}. \\
D_2 = D_3 = D_4 = D_5 = D_6 = \mu.
\]

Further we have

\[
\|\mathcal{K}(t, \mathcal{G}_1(t)) - \mathcal{K}(t, \mathcal{G}_2(t))\| \leq \sup_{||\mathcal{G}||<\delta} ||\mathcal{D}|| \|\mathcal{G}_1(t) - \mathcal{G}_2(t)\|.
\] (23)

were,

\[
\delta = \max \left(\sup_{1 \leq i \leq 6} D_i, \sup_{1 \leq i \leq 6} \sup_{1 \leq i \leq 6} D_i, \sup_{1 \leq i \leq 6} \sup_{1 \leq i \leq 6} D_i, \sup_{1 \leq i \leq 6} \sup_{1 \leq i \leq 6} D_i, \sup_{1 \leq i \leq 6} \sup_{1 \leq i \leq 6} D_i \right).
\] (24)

Theorem 8. A unique solution for the problem described in (13) will exists, under the following condition holds

\[
(\phi_1(\alpha) + \phi_2(\alpha)T)\delta < 1.
\] (25)

Proof. Letting the map denoted \( \mathcal{J} : \mathcal{C}(\Sigma, \mathbb{R}^6) \rightarrow \mathcal{C}(\Sigma, \mathbb{R}^6) \) and defined by

\[
\mathcal{J}(G(t)) = \mathcal{G}_0 + \phi_1(t) \mathcal{K}(t, \mathcal{G}(t)) + \phi_2(t) \int_0^t \mathcal{K}(s, \mathcal{G}(s)) ds,
\] (28)

Eq. (20) simplifies to

\[
\mathcal{G}(t) = \mathcal{J}(\mathcal{G}(t)).
\] (29)

The space \( \mathcal{C}(\Sigma, \mathbb{R}^6) \) coupled with the norm \( ||\mathcal{G}||_C = \sup_{t \in \Sigma} ||\mathcal{G}(t)|| \) is a Banach space.

Now with the help of relation defined in (20), we lead to

\[
||\mathcal{J}(\mathcal{G}_1(t)) - \mathcal{J}(\mathcal{G}_2(t))|| \leq \phi_1(t) ||\mathcal{K}(t, \mathcal{G}_1(t)) - \mathcal{K}(t, \mathcal{G}_2(t))||_C
\]

\[
+ \phi_1(t) \int_0^t ||\mathcal{K}(s, \mathcal{G}_1(s)) - \mathcal{K}(s, \mathcal{G}_2(s))||_C ds
\]

\[
\leq \phi_1(t)||\mathcal{K}(t, \mathcal{G}_1(t)) - (\mathcal{K}(t, \mathcal{G}_2(t)))||_C
\]

\[
+ \phi_2(t) \int_0^t ||\mathcal{K}(s, \mathcal{G}_1(s)) - (\mathcal{K}(s, \mathcal{G}_2(s)))||_C ds
\]

\[
\leq \phi_1(t)||\mathcal{G}_1(t) - \mathcal{G}_2(t)||_C
\]

\[
+ \phi_2(t)\delta \int_0^t ||\mathcal{G}_1(s) - \mathcal{G}_2(s)||_C ds
\]

\[
\leq \phi_1(t)||\mathcal{G}_1(t) - \mathcal{G}_2(t)||_C
\]

\[
+ \phi_2(t)\delta T ||\mathcal{G}_1(t) - \mathcal{G}_2(t)||_C
\]

\[
\leq (\phi_1(t) + \phi_2(t)T)\delta ||\mathcal{G}_1(t) - \mathcal{G}_2(t)||_C.
\]

Under the assumption \( (\phi_1(t)) + (\phi_2(t)T)\delta < 1 \), the mapping \( \mathcal{J} \) is a contraction therefore, the fractional problem (13) possess a unique solution.

Numerical treatment of the problem

This section presents the iterative scheme and simulations of the fractional problem describes the COVID-19 dynamics. The aim of simulation results is to analyze the influence of \( \alpha \), i.e., the fractional order and some of the biologically important parameters on the pandemic incidence. The brief procedure for the derivation of iterative scheme is as follows.

Iterative scheme

Firstly, we solve the proposed fractional problem (13) in order to obtain an iterative scheme. Then utilizing this iterative scheme we will depict the simulation results. The fractional Adams–Bashforth approach is taken into the account for the solution of fractional problem in the Caputo–Fabrizio case [30,31]. The implementation of the fundamental
principle of integration on the first equation of the system (13), we lead to the following expression

\[ S(t) - S(0) = \left(1 - \frac{\sigma}{M(\sigma)} \right)K_1(t, S) + \frac{\sigma}{M(\sigma)} \int_0^t K_1(\zeta, S) d\zeta. \]  

(30)

For setting \( t = t_{n+1} \) it further leads to

\[ S(t_{n+1}) - S(0) = \left(1 - \frac{\sigma}{M(\sigma)} \right)K_1(t_{n+1}, S_n) + \frac{\sigma}{M(\sigma)} \int_0^{t_{n+1}} K_1(t, S) dt. \]  

(31)

Eqs. (31) and (32) gives

\[ S_{n+1} - S_n = \left(1 - \frac{\sigma}{M(\sigma)} \right) \left[ K_1(t_{n+1}, S_n) - K_1(t_{n-1}, S_{n-1}) \right] + \frac{\sigma}{M(\sigma)} \int_{t_n}^{t_{n+1}} K_1(t, S) dt. \]  

(33)

Using Lagrange interpolation approximation for the kernel \( K_1(t, S) \) on the interval \([t_n, t_{n+1}]\) and then evaluating the integral involved (33) as follows:

\[ \int_{t_n}^{t_{n+1}} K_1(t, S) dt = \frac{1}{h} \left[ \frac{K_1(t_n, S_n)}{h} (t - t_{n-1}) + \frac{K_1(t_{n+1}, S_{n+1})}{h} (t - t_n) \right] dt \]

\[ = \frac{1}{2} K_1(t_n, S_n) - \frac{h}{2} K_1(t_{n+1}, S_{n+1}). \]  

(34)

By putting (34), in Eq. (33) finally we obtained

\[ S_{n+1} = S_n + \left(1 - \frac{\sigma}{M(\sigma)} \right) \left\{ K_1(t_n, S_n) + \frac{3h}{2M(\sigma)} K_1(t_{n+1}, S_{n+1}) \right\} - \left(1 - \frac{\sigma}{M(\sigma)} + \frac{\sigma h}{2M(\sigma)} \right) K_1(t_{n-1}, S_{n-1}). \]  

(35)

In similar approach, the iterative formulae obtained for rest of equation of the fractional problem (13) are as follows:

\[ E^{n+1} = E^n + \left(1 - \frac{\sigma}{M(\sigma)} + \frac{3h}{2M(\sigma)} \right) K_2(t_n, E^n) - \left(1 - \frac{\sigma}{M(\sigma)} + \frac{\sigma h}{2M(\sigma)} \right) K_2(t_{n+1}, E_{n+1}), \]  

\[ I^{n+1} = I^n + \left(1 - \frac{\sigma}{M(\sigma)} + \frac{3h}{2M(\sigma)} \right) K_3(t_n, I^n) - \left(1 - \frac{\sigma}{M(\sigma)} + \frac{\sigma h}{2M(\sigma)} \right) K_3(t_{n+1}, I_{n+1}), \]  

\[ P^{n+1} = P^n + \left(1 - \frac{\sigma}{M(\sigma)} + \frac{3h}{2M(\sigma)} \right) K_4(t_n, P^n) - \left(1 - \frac{\sigma}{M(\sigma)} + \frac{\sigma h}{2M(\sigma)} \right) K_4(t_{n+1}, P_{n+1}), \]  

\[ A^{n+1} = A^n + \left(1 - \frac{\sigma}{M(\sigma)} + \frac{3h}{2M(\sigma)} \right) K_5(t_n, A^n) - \left(1 - \frac{\sigma}{M(\sigma)} + \frac{\sigma h}{2M(\sigma)} \right) K_5(t_{n+1}, A_{n+1}), \]  

\[ R^{n+1} = R^n + \left(1 - \frac{\sigma}{M(\sigma)} + \frac{3h}{2M(\sigma)} \right) K_6(t_n, R^n) - \left(1 - \frac{\sigma}{M(\sigma)} + \frac{\sigma h}{2M(\sigma)} \right) K_6(t_{n+1}, R_{n+1}). \]  

(36)
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Simulation and discussion

This part presents the graphical interpretation of the COVID-19 fractional problem (13) in order to investigate the dynamics of the model population classes for various values of $\sigma \in (0, 1]$. Moreover, the impact of variation in $\beta_1$ and $\beta_2$ (the disease transmission rates) is shown graphically. The iterative formulae derived in (35) and (36) are utilized while the numerical values of the model parameters are estimated from the actual incidence data and presented in Table 2.

Initially, we demonstrate the dynamics of the fractional problem (13) for five values of $\sigma \in (0, 1]$, where the value of time interval is considered as $t = 200$ days. The simulation results of the model in this case are shown in Fig. 2 with subplots (a–f). The dynamics of $S(t)$, $E(t)$, $I(t)$, $P(t)$ and asymptotically infected individuals are explored in Figs. 2(a), 2(b), 2(c), 2(d) and 2(e) respectively. The dynamics of fully recovered population are shown in 2(f). It is observed from these plots that all population curves converge to the equilibrium point.

Secondly, we make variation in the important parameter $\beta_1$ which shows the disease transmission rate (or effective contacts) due to symptomatic infected individuals. The values of $\beta_1$ are reduced at 10%, 20% and 30% to its baseline value. The resulting impact is shown for the asymptomatic, symptomatic, super-spreader and exposed individuals in Figs. 3–6 respectively. The subplots (b) and (c) of each Figs. 3–6 demonstrate the simulation of the model (13) for fractional case $\sigma = 0.90$. From these graphical results, it is revealed that peaks of curves in each infected population class reduce significantly with reduction in the effective contact rate $\beta_1$. Although, the impact of fractional order $\sigma$ is not significant, but for the smaller values of $\sigma$ the peaks slightly appear over a long period of time.

Finally, we analyzed the impact of the disease transmission rate due to super-spreader individuals denoted by $\beta_2$. We make reduction in the baseline value of $\beta_2$ with the same rate as in $\beta_1$. The resulting impact is shown for the asymptomatic, symptomatic, super-spreader and exposed individuals in Figs. 7–10 respectively. This interpretation is graphically depicted for fractional values of $\sigma$ in subplots (b) and (c) of each Figs. 7–10 showing no significant impact. It is found that the reduction in $\beta_2$ reduces the peaks in all infective classes with a comparatively faster rate as with a reduction in $\beta_1$. Therefore, the identification and isolation of super-spreader individuals to reduce their efficient transmission ability are important in order to mitigate the ongoing COVID-19 pandemic.

A fractal-fractional approach

A fractal-fractional operator is a novel approach in formulating the epidemic models. The concept of fractal-fractional operators has been introduced by Atangana in [27]. These differential operators have been successfully utilized to model various scientific problems including infectious diseases. The current section addresses the formulation of the COVID-19 model via the fractal-fractional operator with an exponential
where $\psi$ values of fractional order dynamical aspects of the model showing the influence of variation in account for this purpose. The resulting scheme is applied to depict the fractal-fractional case COVID-19 model (37) is addressed in order Numerical scheme and simulation of fractal-fractional model

$$
\begin{align*}
\mathcal{C}^{\psi\beta}_{0,t} S &= A - \beta_1 \frac{(1 + \psi_{2\psi}) S}{N} - \beta_2 \frac{S S}{N} - \mu S, \\
\mathcal{C}^{\psi\beta}_{0,t} E &= \beta_1 \frac{(1 + \psi_{2\psi}) A S}{N} + \beta_2 \frac{S S}{N} - b_1 E, \\
\mathcal{C}^{\psi\beta}_{0,t} I &= \theta_0 E - b_2 I, \\
\mathcal{C}^{\psi\beta}_{0,t} P &= \theta_0 E - b_3 P, \\
\mathcal{C}^{\psi\beta}_{0,t} A &= \theta(1 - \omega_1 - \omega_2) E - b_1 A, \\
\mathcal{C}^{\psi\beta}_{0,t} R &= \tau_1 A + \tau_2 P + \tau_1 I - \mu R,
\end{align*}
$$

(37)

where $\mathcal{C}^{\psi\beta}_{0,t}$ with $t > 0$, shows the fractal-fractional operator with fractional order $\psi \in (0, 1]$ and fractal order $\beta \in (0, 1]$. Numerical scheme and simulation of fractal-fractional model

In the present section, a brief derivation of the numerical solution of the fractal-fractional case COVID-19 model (37) is addressed in order to simulate the model. The procedure studied in [32] is taken into account for this purpose. The resulting scheme is applied to depict the dynamical aspects of the model showing the influence of variation in values of fractional order $\psi$ and fractal order $\beta$. Initially, the system (37) can be comprehensively described as follows:

$$
\mathcal{C}^{\psi\beta}_{0,t} (g(t)) = \beta t^{\psi-1} \left[ G(t, g(t)) \right].
$$

(38)

Utilizing the Caputo–Fabrizio integral, we lead to the following expression

$$
g(t) = g(0) + \beta t^{\psi-1} \frac{1 - \omega}{M(\psi)} G(t, g(t)) + \frac{\omega \beta}{M(\psi)} \int_0^t t^{\psi-1} G(\lambda, g(\lambda)) d\lambda.
$$

(39)

Setting $t = t_{m+1}$, and (39) leads the form

$$
g_{m+1} = g_0 + \frac{\beta t_{m+1}^{\psi-1} (1 - \omega)}{M(\psi)} G(t_m, g(t_m)) + \frac{\omega \beta}{M(\psi)} \int_{t_m}^{t_{m+1}} t^{\psi-1} G(\lambda, g(\lambda)) d\lambda + \frac{\omega \beta}{M(\psi)} \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} t^{\psi-1} G(\lambda, g(\lambda)) d\lambda.
$$

(40)

Taking the difference between the consecutive terms, we obtain

$$
g_{m+1} = g_m + \frac{\beta t_{m+1}^{\psi-1} (1 - \omega)}{M(\psi)} G(t_m, g(t_m)) - \frac{\beta t_{m+1}^{\psi-1} (1 - \omega)}{M(\psi)} G(t_{m-1}, g(t_{m-1})) + \frac{\omega \beta}{M(\psi)} \int_{t_m}^{t_{m+1}} t^{\psi-1} G(\lambda, g(\lambda)) d\lambda.
$$

(41)

The Lagrangian piece-wise interpolation on the interval $[t_m, t_{m+1}]$ is applied to approximate the integral involved in (41). The function $t^{\psi-1} G(\lambda, g(\lambda))$ in above Eq. (40) is approximated by the following polynomial

$$
P_m(\lambda) = \frac{\lambda - t_m}{t_{m+1} - t_m} G(\lambda, g(t_m)) - \frac{\lambda - t_{m+1}}{t_{m+1} - t_m} G(\lambda, g(t_{m+1})).
$$

(42)
Using (42) we finally derived the following iterative scheme for the problem under consideration.

\[
S^{m+1} = S^m + \frac{\omega \beta^m}{M(\omega)} \left( 3 \frac{h}{2} G_{m-1}(t_m, g(t_m)) - G_{m-1}(t_{m-1}, g(t_{m-1})) \right)
\]

\[
E^{m+1} = E^m + \frac{\omega \beta^m}{M(\omega)} \left( 3 \frac{h}{2} G_{m-1}(t_m, g(t_m)) - G_{m-1}(t_{m-1}, g(t_{m-1})) \right)
\]

Hence, we have the following recursive formulae for the COVID-19 epidemic model with fractal-fractional case

\[
I^{m+1} = I^m + \frac{\beta^m}{M(\omega)} \left[ G_{m-1}(t_m, g(t_m)) - G_{m-1}(t_{m-1}, g(t_{m-1})) \right]
\]

\[
P^{m+1} = P^m + \frac{\beta^m}{M(\omega)} \left[ G_{m-1}(t_m, g(t_m)) - G_{m-1}(t_{m-1}, g(t_{m-1})) \right]
\]

\[
A^{m+1} = A^m + \frac{\beta^m}{M(\omega)} \left[ G_{m-1}(t_m, g(t_m)) - G_{m-1}(t_{m-1}, g(t_{m-1})) \right]
\]

\[
R^{m+1} = R^m + \frac{\beta^m}{M(\omega)} \left[ G_{m-1}(t_m, g(t_m)) - G_{m-1}(t_{m-1}, g(t_{m-1})) \right]
\]

Fig. 11. Dynamics of the fractional-fractal COVID-19 epidemic model (37) when \( \omega = 1 \) and \( \beta = 1, 0.90, 0.80, 0.70 \).
Based on the numerical scheme discussed in (44), we simulate the COVID-19 model in fractal-fractional case (37). The same estimated values of model parameters and initial conditions are used in the simulation. The dynamics of the model are shown in Figs. 11 and 12 by considering specific values of fractional and fractal orders. Fig. 11 depicts the impact of variation in the fractal order \( \beta = 1, 0.9, 0.80, 0.70 \) and fixed the fractional order \( \sigma = 1 \). Fig. 12, shows the model dynamics when fractal order is fixed i.e., \( \beta = 1 \) and varies the fractional order \( \sigma = 1, 0.9, 0.80, 0.70, 0.60 \), fractal order.

Conclusion

In this study, we addressed the dynamics of the ongoing novel coronavirus pandemic using fractional and fractal-fractional order mathematical modeling approaches. The infectious population is sub-divided into three sub-classes namely the symptomatic, super-spreaders and asymptomatic individuals. The super-spreaders of COVID-19 have a significant role in the disease incidence. They have efficient transmission ability and can infect multiple susceptible people. The proposed model is formulated using the integer-order differential system composed of six equations. In order to parameterize the model, the actual incidence data from the Kingdom of Saudi Arabia is used. The nonlinear least square approach is applied for the estimation procedure. The Caputo–Fabrizio fractional operator is used in order to extend the integer case epidemic model to the fractional case to gain deeper insights into the pandemic. The important mathematical assessments including the existence and uniqueness of the epidemic model in the fractional case have been proven with the help of fixed point theory. The modified fractional Adams–Bashforth scheme is utilized to obtain an iterative solution for the fractional epidemic model. Finally, extensive simulation results are performed to demonstrate the impact memory index \( \sigma \) and important model parameters (i.e., disease transmission rate \( \beta_1 \) and \( \beta_2 \)) on the pandemic peaks. Moreover, the model is reformulated.
using a novel fractal-fractional approach. The mathematical model in the fractal-fractional case is simulated for different values of fractional order \(a\) and fractal order \(\beta\) in order to analyze the dynamics of the pandemic. The simulation results show that the reduction in the disease transmission rate relative to super-spreaders of COVID-19 \(R_t\), significantly decreases the pandemic peaks. Therefore, the identification and isolation of infectious individuals and especially of super-spreaders individuals to reduce their efficient transmission ability are important to curb the ongoing COVID-19 pandemic.

**CRediT authorship contribution statement**

Xiao-Ping Li: Conceptualization, Methodology, Supervision, Formal analysis, Validation. Saif Ullah: Conceptualization, Investigation, Validation, Software, Writing – original draft. Hina Zahir: Formal analysis, Methodology, Data analysis, Supervision, Reviewing. Ahmed Alshehri: Formal analysis, Methodology, Supervision, Simulation. Muhammad Bilal Riaz: Formal analysis, Methodology, Data analysis, Funding acquisition, Supervision, Reviewing. Basem Al Alwan: Supervision, Simulation, Reviewing, Data analysis, Funding acquisition.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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