Critical String from Non-Abelian Vortex in Four Dimensions

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In a class of non-Abelian solitonic vortex strings supported in certain four-dimensional super-Yang-Mills theories we search for the vortex which can behave as a critical fundamental string. We use the Polchinski-Strominger criterion of the ultraviolet completeness. We identify an appropriate four-dimensional bulk theory: it has the \( U(2) \) gauge group, the Fayet-Iliopoulos term and four flavor hypermultiplets. It supports semilocal vortices with the world-sheet theory for orientational (size) moduli described by the weighted \( CP(2,2) \) model. The latter is superconformal. Its target space is six-dimensional. The overall Virasoro central charge is critical. We show that the world-sheet theory on the vortex supported in this bulk model is the \textit{bona fide} critical string.

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\[ \text{Introduction.} — \text{Since 2003 a large variety of non-Abelian solitonic vortices supported in certain four-dimensional super-Yang-Mills theories were discovered} \] 1. \text{Such vortices contain extra non-Abelian moduli (the so-called orientational moduli), in addition to the conventional translational moduli. The low-energy theory for the orientational moduli fields on the vortex world sheet is usually a nonlinear sigma model, typically \( CP(N-1) \), with different degrees of supersymmetry. The primary purpose of the non-Abelian vortex explorations is modeling confinement and related phenomena in QCD-like theories.}

\text{These vortex strings are not similar to the critical strings of the fundamental string theory, and cannot be treated as such. The most clear-cut distinction is the fact that the world sheet theory is not conformal. In the terminology of Ref.} 2 \text{ such world sheet theories are not ultraviolet (UV) complete: higher derivative terms are needed to make them consistent in the UV.}

\text{In this Letter we report the following finding. A semilocal} \( N = 2 \) \text{vortex with the orientational moduli described by the weighted} \( CP(2,2) \text{model is UV complete, and reduces to a critical string in four dimensions.}

\text{This vortex is supported in four-dimensional} \( N = 2 \) \text{super-Yang-Mills with the} \( U(2) \text{gauge group, the Fayet-Iliopoulos term, and four flavor hypermultiplets. The target space metric in the world sheet theory has a block form: four-by-four block in the upper left corner (corresponding to flat metric for translational moduli) and six-by-six block in the lower right corner (corresponding to a Calabi-Yau metric with the vanishing Ricci tensor for orientational moduli). This metric can be read off from Eq.} 3 \text{ The world-sheet theory is conformally invariant.}

\text{General considerations.} — \text{It is known that the hadron spectrum is well described by linear Regge trajectories. In the early days of string theory this fact motivated people to consider dual resonance models as a theory of hadrons. It is believed that confinement in QCD is due formation of confining strings. In all known examples in which the confining strings are formed in a controllable way, say, the Abrikosov-Nielsen-Olesen (ANO) string \( 4 \text{ in the weakly coupled Abelian-Higgs model or the Seiberg-Witten strings in slightly deformed} \( N = 2 \) \text{super-Yang-Mills theory} 5 \text{, the Regge trajectories will show linear behavior only at asymptotically large spins} 6 \text{.}

\text{Indeed, consider an open string rotating with the spin} \( J \). \text{In the (semi)classical approximation its length is determined by the relation} \( L^2 \sim J/T \text{ where} \( T \text{is the string tension. Its transverse size is given by the inverse mass} m \text{of the bulk fields forming the string, say, for the ANO string the masses of the gauge and Higgs fields} 25 \text{. At weak coupling these masses typically scale as} m \sim g \sqrt{T} \text{ where} g \text{is a small gauge coupling constant.}

\text{Clearly the string excitation spectrum can form linear Regge trajectories only if the string length is much larger than its transverse size. This gives the condition}

\[ mL \gg 1 \text{, or} J \gg 1/g^2 \text{.} \]

\text{At weak coupling} g \text{is small so spin} J \text{should be large.}

\text{At} J \sim 1 \text{ the condition} 1 \text{ is not met at weak coupling: the string is not developed. Rather, we deal with a sausage-like field configuration. The quark-antiquark mesons formed are closer to spherical symmetry. No linear Regge trajectory apply at} J \sim 1 \text{.}

\text{Empirically in the real world QCD we have practically linear Regge trajectories at} J \sim 1 \text{. Can we find any example of a four-dimensional bulk theory where confining string remains thin at} J \sim 1 \text{? If so, the string must satisfy the condition}

\[ T \ll m^2 \text{,} \]

\text{to be referred to as the thin string condition. This condition cannot be met at weak coupling.}
**Strong coupling.**—We have to find an appropriate strongly coupled four-dimensional bulk theory. We will find super-Yang-Mills theory which supports vortices similar to critical strings (e.g., with conformal world sheet theory) and then formulate a necessary conditions for the existence of the thin string regime.

**Thin string regime.**—In the effective two-dimensional theory on the string world sheet the problem can be understood as follows. For the ANO string the effective theory on the string world sheet is given by the Nambu-Goto action plus higher derivative corrections. Higher derivative terms are needed to make the world sheet theory UV complete [2]. This requirement can be used to constrain higher derivative terms order by order in the derivative expansion, see e.g. [7] and references therein.

Higher derivative corrections run in powers of the ratio $\partial^2/m^2$ where the mass of the bulk fields $m$ is given by $m \sim g\sqrt{T}$ and typical energy in the numerator at $J \sim 1$ is determined by the string tension. Thus, higher derivative corrections materialize as powers of $T/m^2$. Obviously they all blow up at weak coupling – the string surface become “crumpled” [3]. This is the world-sheet implementation of the bulk picture of a short and thick “string.”

We want to find a regime in which the string remains thin, see [2]. This means that the higher derivative corrections should be parametrically small. In other words, the low-energy world-sheet theory [27] should be UV complete. This leads us to the following necessary conditions to have such a regime:

(i) The low-energy world-sheet theory on the string must be conformally invariant;

(ii) It must have the critical value of the Virasoro central charge.

These are the famous conditions satisfied by the fundamental string. In particular, the bosonic fundamental string becomes critical in $D = 26$, while the superstring becomes critical in $D = 10$. The low energy world sheet theory for the ANO string is not critical in four dimensions. We will show below that the above conditions are met in a class of the non-Abelian vortices [10, 12]. In particular, the solitonic vortex in question must have six orientational moduli, which, together with four translational moduli, will form a ten-dimensional space.

**Non-Abelian vortices.**—Non-Abelian vortices are supported in a large class of supersymmetric and non-supersymmetric gauge theories. We will focus on the bulk four-dimensional theories in which the non-Abelian vortices were first found: $\mathcal{N} = 2$ supersymmetric QCD with the $U(N)$ gauge group, $N_f$ quark flavor multiplets ($N_f \geq N$) and the Fayet-Iliopoulos (FI) parameter $\xi$ of the $U(1)$ factor of the gauge group. In this theory the vortices under consideration are BPS-saturated and preserve half of the bulk supersymmetry. Thus, they possess $\mathcal{N} = (2, 2)$ supersymmetry on the world sheet. The string tension is determined exactly by

$$T_P = 2\pi\xi. \quad (3)$$

These strings are formed due to the (s)quark condensation; therefore, they confine monopoles. More precisely, in the $U(N_f)$ gauge theories the confined monopoles are implemented as junctions of two vortices of different kinds [12, 14].

Dynamics of the translational modes in the Polyakov formulation [9] can be described by the action

$$S_{\text{tr}} = \frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu, \quad (4)$$

where $h^{\alpha\beta}$ (indices $\alpha = 1, 2$) are the world-sheet coordinates, $x^\mu$ ($\mu = 1, ..., 4$) describes the string world sheet and $h = \det(h_{\alpha\beta})$ where $h_{\alpha\beta}$ is the world-sheet metric which is understood as an independent variable [27].

If $N_f = N$ the dynamics of the orientational zero modes of the vortex, which become orientational moduli fields on the world sheet, is described by two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric $CP(N - 1)$ model. If one adds extra quark flavors, non-Abelian vortices become semilocal. They acquire size moduli [12].

Non-Abelian semilocal vortices in $\mathcal{N} = 2$ SQCD with $N_f > N$ were studied in [10, 13, 16, 18]. The world-sheet theory for the orientational moduli of the semilocal vortex is given by the weighted $CP(N, N)$ sigma model where $\tilde{N} = (N_f - N)$ [28]. Its gauged formulation is as follows [19]. One introduces two types of complex fields, with the $U(1)$ charges $\pm 1$: $n^P$ ($P = 1, ..., N$) and $\rho^K$ ($K = N + 1, ..., N_f$). The orientational moduli are described by the $N$-plets $n^P$ while the size moduli are parametrized by the $\tilde{N}$-plet $\rho^K$.

The effective two-dimensional theory on the world sheet has the action

$$S_{\text{tr}} = \int d^2\sigma \sqrt{h} \left\{ \nabla_\alpha n_P \nabla_\beta n^P + \nabla_\alpha \rho_K \nabla_\beta \rho^K \right\} + \text{fermions} , \quad (5)$$

where $P = 1, ..., N$ and $K = N + 1, ..., N_f$. The fields $n^P$ and $\rho^K$ have charges $+1$ and $-1$ with respect to the auxiliary $U(1)$ gauge field; hence,

$$\nabla_\alpha = \partial_\alpha - iA_\alpha \, , \quad \nabla_\alpha = \partial_\alpha + iA_\alpha \, .$$

The limit $\epsilon^2 \to \infty$ is implied. The coupling constant $\beta$ in (5) is related to the bulk coupling via

$$\beta = 2\pi/g^2. \quad (6)$$

Note that the first (and the only) coefficient of the $\beta$ function $b = N - \tilde{N}$ is the same for the bulk and world-sheet theories.
The bosonic part of the total string action for the non-Abelian vortex under consideration is the sum of \( S_{\text{tr}} \) and \( S_{\text{or}} \),

\[
S = S_{\text{tr}} + S_{\text{or}}.
\]

The target space of \( S_{\text{tr}} \) has dimension \( 2(N + \tilde{N}) - 2 = 2(N_f - 1) \). If \( N = \tilde{N} = 2 \) the target space is six-dimensional.

From vortices to thin non-Abelian strings.---We can ask whether the above vortex can satisfy two necessary conditions to become the thin string. In other words, can the string theory \( S_{\text{tr}} \) be UV complete in the same way as the fundamental string theory?

In the conformal gauge the translational part of the action is a free theory and therefore conformal, while the orientational part’s \( \beta \) function is proportional to \( b = N - \tilde{N} \). Thus, the condition of the conformal invariance \( b = 0 \) implies

\[ N = \tilde{N}, \text{ or } N_f = 2N. \]

The total Virasoro central charge for the string theory \( S_{\text{tr}} \) is given by

\[
c_{\text{tot}} = \frac{3}{2} \left( D + \frac{2}{3} c_{\text{wcp}} - 10 \right),
\]

where the first term is the contribution of the translational sector with \( D = 4 \), \( c_{\text{wcp}} \) is the Virasoro central charge of the weighted \( CP(N, \tilde{N}) \) model \( S_{\text{tr}} \) in the conformal gauge, and the last term is the ghost contribution. Our world-sheet theory has \( N = (2,2) \) world sheet supersymmetry, and we make half of it local. This gives the last term in \( c_{\text{tot}} \), much in the same way as for the fundamental string.

To calculate the central charge of the weighted \( CP(N, \tilde{N}) \) model \( S_{\text{tr}} \) we use the formula \( c_{\text{wcp}} = 3 \sum_i (1 - 2q_i) \)

\[
(10)
\]

which relates the central charge of \( N = (2,2) \) sigma model to the \( R \) charges. Here the sum runs over the chiral multiplets and \( q_i \) is the \( R \) charge of the given multiplet. In our theory \( c_{\text{wcp}} \) the \( R \) charges of the fields \( n^P \) and \( \rho^K \) vanish. Thus, Eq. \( (10) \) just counts the number of degrees of freedom, namely

\[
c_{\text{wcp}} = 3(N + \tilde{N} - 1).
\]

Now the condition of criticality takes the form

\[
c_{\text{tot}} = \frac{3}{2} \left[ D + 2(2N - 1) - 10 \right] = 0, \quad (12)
\]

\[
N = (12 - D)/4. \quad (13)
\]

where we used \( D = 4 \). For \( D = 4 \) this condition has a solution

\[ N = \tilde{N} = 2, \quad N_f = 4. \]

For these values of \( N \) and \( \tilde{N} \) the target space of the weighted \( CP(N, \tilde{N}) \) model \( S_{\text{tr}} \) is a noncompact Calabi-Yau manifold studied in \( \text{[19]} \).

Thus, the non-Abelian string can potentially become thin in a particular four-dimensional theory: \( N = 2 \) supersymmetric QCD with the \( U(2) \) gauge group and \( N_f = 4 \) flavors of quarks. Given the necessary conditions are met we conjecture that the thin-string condition \( (2) \) is actually satisfied in this theory in the strong coupling limit.

In fact, the bulk theory at zero \( \xi \) possess a strong-weak duality \( \tau \to -1/\tau \), where \( \tau = i 8\pi/g^2 + \theta/\pi \) and \( \theta \) is the \( CP \) angle \( \text{[21]} \). Therefore, even at non-zero \( \xi \) the region of \( g^2 \gg 1 \) can be described in terms of the weakly coupled \( U(N) \) dual gauge theory \( \text{[24]} \). Thus, our conjecture above is equivalent to the assumption that the mass of quarks and gauge bosons \( m \) has singularities as a function of \( g^2 \) at \( g^2 \sim 1 \) in the bulk theory.

The global symmetry of the world-sheet theory we obtained is

\[
SO(3,1) \times SU(2) \times SU(2) \times U(1), \quad (15)
\]

where the first factor is the Lorentz group, while the other factors represent global internal symmetries of the weighted \( CP(2,2) \) model \( (N = \tilde{N} = 2) \).

Gauge-string duality.---Abstracting ourselves from the solitonic origin of our model \( S_{\text{tr}} \) and \( S_{\text{or}} \) we can view it as a perfectly legitimate critical string theory, with all ensuing consequences. In particular, leaving aside a possible subtle structure for operators of vanishing dimension \( \text{[30]} \) in \( \text{[15]} \), one could arrive at the conclusion that a massless spin-2 four-dimensional state is present in the spectrum. Such states are forbidden in our bulk Yang-Mills theory because of the Weinberg-Witten argument \( \text{[24]} \).

To describe physics at weak coupling we use the bulk theory in its original formulation in terms of quarks and gauge bosons. As was already mentioned, the quarks and gauge bosons have masses of the order of \( m \sim g\sqrt{\tau} \). At \( g^2 \ll 1 \) they are light, while stringy states with masses of the order of \( \sqrt{\tau} \) are heavy.

Since it is impossible at the moment to determine the behavior of \( m \) versus the bulk coupling \( g \) in the strong coupling regime it seems reasonable to conjecture that at strong coupling the condition \( (2) \) is satisfied – quarks and gauge bosons are heavy while the stringy states are light. To describe physics in the regime of large but finite \( m \) we should use string theory \( S_{\text{or}} \) supplemented by higher derivative terms which can be considered as small corrections. We call this gauge-string duality.

Strings in the \( U(N) \) theories are stable; they cannot be broken. Thus, we deal with the closed string. If in the limit of the infinitely thin string the four-dimensional spin-2 closed string state is massless, corrections to the thin-string limit will lift this mass from zero; it will become of the order of \( T/m \). Therefore, we expect that the spin-2 closed string state is the lightest state.
The ten-dimensional spin-2 representation splits into two-index representations of the global group $\mathfrak{su}(2)$ factors in $\mathfrak{su}(4)$ states are formed by monopole-antimonopole pairs connected by two confining non-Abelian strings (Fig. 1), see [1] for details [31]. At strong coupling the monopole-antimonopole meson becomes stable because it is light and cannot decay into screened quarks and gauge bosons. In fact, these mesons are formed in the so-called “instead-of-confinement” phase, recently reviewed in [23]. They are quite similar to mesons in real-world QCD: they have the “correct” (adjoint or singlet) quantum numbers with respect to the unbroken global group. We expect them to lie on the linear Regge trajectories even at small spins.

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[25] For BPS-saturated strings these masses are equal.
[26] By low energy-theory we mean a theory with no more than two derivatives in the Polyakov formulation [6], see Eq. (4).
[27] Effective world-sheet theories for both translational and orientational moduli are derived in the quasiclassical approximation. In this approximation two alternative string theory formulations – in terms of the induced metric and in terms of the metric $h_{\alpha\beta}$ as an independent variable – are equivalent.
[28] In fact, the world-sheet theory on the semilocal non-Abelian string is not exactly the weighted $CP(N,\bar{N})$ model. The actual theory is called $zn$ model [18]; however, here we ignore this difference.
[29] Although the gauge coupling constant does not run in the bulk theory at $N_f = 2N$ the conformal invariance in the bulk is broken by the (s)quark vacuum expectation values proportional to $\sqrt{\xi}$ (the Fayet-Iliopoulos term).
[30] This is a subject for future studies.
[31] This picture assumes an infrared regularization such that the vacua of the world-sheet theory are discrete.