Nonminimal coupling and the cosmological constant problem

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(Dated: April 6, 2015)

Abstract

We consider a universe with a positive effective cosmological constant and a nonminimally coupled scalar field. When the coupling constant is negative, the scalar field exhibits linear growth at asymptotically late times, resulting in a decaying effective cosmological constant. The Hubble rate in the Jordan frame reaches a self-similar solution, $H = 1/(\epsilon t)$, where the principal slow roll parameter $\epsilon$ depends on $\xi$, reaching maximally $\epsilon = 2$ (radiation era scaling) in the limit when $\xi \to -\infty$. Similar results are found in the Einstein frame (E), with $H_E = 1/(\epsilon_E t)$, but now $\epsilon_E \to 4/3$ as $\xi \to -\infty$. Therefore in the presence of a nonminimally coupled scalar de Sitter is not any more an attractor, but instead (when $\xi < -1/2$) the Universe settles in a decelerating phase. Next we show that, when the scalar field $\phi$ decays to matter with $\epsilon_m > 4/3$ at a rate $\Gamma \gg H$, the scaling changes to that of matter, $\epsilon \to \epsilon_m$, and the energy density in the effective cosmological becomes a fixed fraction of the matter energy density, $M_P^2\Lambda_{\text{Eff}}/\rho_m = \text{constant}$, exhibiting thus an attractor behavior. While this may solve the (old) cosmological constant problem, it does not explain dark energy. Provided one accepts tuning at the 1% level, the vacuum energy of neutrinos can explain the observed dark energy.

PACS numbers: 98.80.-k, 98.80.Qc, 04.62.+v

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I. INTRODUCTION

Here we consider a simple tensor-scalar theory of gravity, originally considered by Jordan, Brans and Dicke \[1, 2\] and generalized by Bergmann \[3\] to what we today refer as tensor-scalar (TeS) theory of gravity. The JBD – and more generally TeS – theories were used as a testing ground for simplest extensions of general relativity (for a review see Ref. \[4\]).

In cosmology the more general class of TeS theories (that includes a potential) has been used to formulate the Higgs field driven inflationary models \[5–8\], and to build models that explain late time dark energy from inflationary fluctuations \[9, 10\]. Here we investigate how a nonminimally coupled scalar can help in resolving the cosmological constant problem.

The cosmological constant problem regards the huge discrepancy between natural theoretical value and the observed value. While quantum field theory predicts a huge value (of the order of the Planck scale), the observed value is consistent with that of the dark energy, which is more than 120 orders of magnitude smaller.

This problem has already been discussed in literature in the context of a nonminimally coupled scalar field, and it was observed that the mechanism works, but the price to pay is that the effective gravitational coupling constant goes to zero, which is not acceptable \[11–13\]. In this we point out that this is indeed the case when one considers the problem in the Jordan frame. However, the analysis in the Einstein frame warrants further investigation, and that is precisely what we do here.

In section II we define the model and perform the analysis in the Jordan frame. Section III is devoted to the corresponding analysis in the Einstein frame. In section ?? we extend the analysis of section III by coupling the scalar field to matter and observe that in the tight coupling regime the effective cosmological constant in the Einstein frame scales away as matter. Finally, in section V we discuss the results and address possible issues and shortcomings.

II. JORDAN FRAME ANALYSIS

The scalar-tensor model we consider here is defined by the following action in the Jordan frame,

\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2} F(\phi) R - M_p^2 \Lambda - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right),
\]  

(1)
where \( g = \text{det}[g_{\mu\nu}] \), \( g^{\mu\nu} \) is the inverse of the metric tensor \( g_{\mu\nu} \). For simplicity we take,

\[
F(\phi) = M_P^2 - \xi \phi^2, \quad \text{and} \quad V = 0, \tag{2}
\]

where \( \xi \) is the nonminimal coupling. In our sign conventions conformal coupling corresponds to \( \xi_c = 1/6, M_P^2 = 1/(8\pi G_N) \) and we work with natural units in which \( \hbar = 1 = c \). For the metric we take a cosmological, spatially flat, background,

\[
g_{\mu\nu} = \text{diag}[-1, a^2(t), a^2(t), a^2(t)]. \tag{3}
\]

The action (1) implies the following equations of motion (see e.g. Refs. [14, 15]),

\[
\ddot{\phi} + 3H\dot{\phi} + 6\xi(2H^2 + \dot{H})\phi = 0 \tag{4}
\]

\[
\dot{H} = -\frac{1}{M_P^2 - \xi(1 - 6\xi)\phi^2} \left[ \frac{1}{2} (1 - 2\xi)\dot{\phi}^2 + 4\xi H\dot{\phi}\phi + 12\xi^2 H^2\phi^2 \right] \tag{5}
\]

\[
H^2 = \frac{1}{3(M_P^2 - \xi\phi^2)} \left[ M_P^2 \Lambda + \frac{1}{2}\dot{\phi}^2 + 6\xi H\dot{\phi} \right], \tag{6}
\]

where \( H = \dot{a}/a \) is the Jordan frame Hubble rate. Equation (6) is the constraint equation, and taking its time derivative gives a combination of the first two equations (4–5), representing a non-trivial validity check of the solutions of Eqs. (4–6). The cosmological constant \( \Lambda \) appears only in the constraint equation (6) and in that respect does not directly affect the dynamical equations (4–5). The information about \( \Lambda \) is introduced by the initial conditions.

We assume that the field is initially in a homogeneous state with a small expectation value, \( \phi_0 \sim \sqrt{\Lambda} \). A detailed analysis is required to properly answer the question how much the subsequent analysis depends on the homogeneous initial conditions. The following heuristic arguments suggest that our results do not depend on the details of initial conditions. Since the early period is de Sitter, during which fluctuations tend to be exponentially damped with time, \(^1\) we expect that our results remain robust for a rather broad set of initial conditions. In order to properly understand the dynamics and duration of the initial inflationary period, it is important to include the backreaction of created scalar (and gravitational) particles, which is work in progress. We expect that the backreaction from quantum fluctuations will

\(^1\) This of course does not hold for very long wavelength fluctuations, for which the physical momentum, \( k/a < \sqrt{-12\xi H} \). These fluctuations will also grow exponentially, but slower than the homogeneous field mode, and hence our results can be understood as a lower bound on the duration of the initial inflationary period.
shorten the initial inflationary period [9] [16]. To make a more quantitative statement a (pertubative) quantum analysis is required. In this work we examine the problem without including quantum backreaction effects.

At early stages of the evolution the cosmological constant contribution dominates the right hand side of Eq. (6), and $\epsilon = -\dot{H}/H^2 \approx 0$, such that we are in an approximately de Sitter space. From Eq. (4) we see that, for a negative $\xi$, $\phi$ is tachyonic and therefore it exhibits exponential growth,

$$
\phi(t) \approx \phi_0 \exp(-4\xi H t), \quad a(t) \approx a_0 \exp(\epsilon t), \quad H \approx \sqrt{\frac{\Lambda}{3}} = \text{const.}
$$

This stage ends at the time $t_{\text{end}} \approx N_{\text{end}}/H$ when the scalar field develops the energy density comparable to that of the cosmological constant (see Eq. (6)). This happens when the number of e-folds of inflation $N(t) = \ln(a(t)/a_0)$ is about,

$$
N_{\text{end}} \approx -\frac{1}{8\xi} \ln \left( \frac{3M_P^2}{16\xi^2 \phi_0^2} \right),
$$

such that one can get a large number of e-folds either by choosing $\xi$ or $\phi_0$ very small. For example, when $\xi = -1$ one gets $N_{\text{end}} \sim 50$ when $\phi_0 = (\sqrt{3}/4) \exp(-240) M_P \sim 5 \times 10^{-60}$ eV, an extremely small field value, implying that this choice of parameters does not result in a good model for primordial inflation. In this initial period, the principal slow roll parameter $\epsilon$ is exponentially close to zero, hence this period is, to an excellent approximation, a de Sitter epoch.

After this initial quasi-de Sitter stage ends, a transitory stage sets in, after which the system enters an asymptotic regime. We shall now show that there is an attractor solution in the asymptotic regime which is not de Sitter, i.e. during which the (effective) cosmological constant relaxes rapidly to zero. To show that, let us make the following scaling Ansatz which holds at late times,

$$
\phi(t) \to \dot{\phi}_0 t, \quad H(t) \to \frac{1}{\epsilon t},
$$

where $\dot{\phi}_0$ and $\epsilon$ are constants. This Ansatz corresponds to a universe dominated by a perfect fluid with a constant equation of state parameter $w = p/\rho = -1 + 2\epsilon/3$, for which the scale factor, $a(t) \to a_0(t/t_0)^{1/\epsilon}$. Indeed, inserting (9) into (4) one obtains,

$$
\epsilon = \frac{4\xi}{1 - 2\xi} \left( 1 - 2\xi \right) \epsilon^2 + 2\xi (5 - 6\xi) \epsilon + 24\xi^2 = 0.
$$

4
From the two solutions to the latter equation, $\epsilon = -4\xi/(1 - 2\xi)$ and $\epsilon = -6\xi$, it is the former that is consistent with the slow roll parameter implied by the scalar equation of motion. Numerical investigations, an example of which is shown in figure 1, shows that, independently on the initial field value $\phi_0$, the late time solution is given by (9), implying that (9) is an attractor. We conclude that, in a model with a scalar field with a negative nonminimal coupling, a non-vanishing cosmological constant gets dynamically compensated by the field, and such a universe settles in a power law expansion. From Eq. (10) we see that the late time solution is accelerating ($\epsilon < 1$) when $0 > \xi > -1/2$ and decelerating ($\epsilon > 1$) when $\xi < -1/2$. In the limit when $\xi \rightarrow -\infty$, $\epsilon \rightarrow 2$, which corresponds to a conformally coupled fluid (radiation). This conclusion holds also in general $D$ space-time dimensions. Inserting the Ansatz (9) into the constraint equation (6) determines the late time rate of the scalar field growth,

$$\dot{\phi}_0^2 = \frac{-8\xi}{(1-6\xi)(3-10\xi)} M_P^2 \Lambda$$

In both periods scalar field grows exponentially with the number of e-foldings. Indeed, at early times, $\phi \propto \exp[-4\xi N]$, while at late times, $\phi \propto \exp(\epsilon N) = \exp[-4\xi N/(1-2\xi)]$, as can be seen from the left panel of figure 1.

The above analysis shows that a negatively coupled scalar relaxes a cosmological constant at a rate,

$$\Lambda_{\text{eff}}(t) = \Lambda_0 a^{-2\epsilon}, \quad \epsilon = -\frac{4\xi}{1-2\xi},$$

The question worth investigating is whether this can resolve the old cosmological constant problem: Why is the (observed) cosmological constant so small when compared with its natural value suggested by quantum field theory? Here we assume that the cosmological constant generated by the vacuum fluctuations of quantum fields and by symmetry breaking (such as the BEH mass generation mechanism) is of the order the electroweak

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2 That this is not a coincidence shows the analysis in $D$ spacetime dimensions, in which case the two solutions are, $\xi = -D\xi/(1-2\xi)$ and $\xi = -2(D-1)\xi$, the former solution being the attractor in general $D$ spacetime dimensions.
FIG. 1: Left panel: $\ln(3\phi^2/\Lambda)$ as a function of the number of e-foldings. Both in the quasi-de Sitter stage as well as in the late power-law expansion stage the field $\phi(t)$ grows exponentially with the number of e-foldings, as explained in the main text. We show results for $\xi = -10, -1$ and $\xi = -0.4$ (curves from left to right). Right panel: $\epsilon = -\dot{H}/H^2$ as a function of the number of e-foldings. $\epsilon$ stays close to zero during the early quasi-de Sitter, and transits to a constant value given in (10) during the late-time attractor regime. For $\xi = -10, -1$ and $\xi = -0.4$ (curves from left to right).

On the other hand, the cosmological constant corresponding to the observed dark energy is

$$\rho_{\Lambda \text{EW}} = M^2_{\text{P}}\Lambda_{\text{EW}} \sim [2 \times 10^2 \text{ GeV}]^4 \implies \Lambda_{\text{EW}} \sim 4 \times 10^{-28} \text{ GeV}^2. \quad (13)$$

It is often assumed that the natural value from quantum field fluctuations is given by the Planck scale. However, this cannot be so. Consider for simplicity the flat space case (since we are primarily discussing ultraviolet issues, due to the adiabaticity of the ultraviolet, the conclusions reached here are easily carried over to expanding backgrounds). The Planckian value for the cosmological constant is obtained by setting the (physical) UV cutoff, $\Lambda_{\text{UV}}$ at the Planck scale, $\Lambda_{\text{UV}} \sim m_{\text{P}}$, $m_{\text{P}} = \sqrt{8\pi}M_{\text{P}}$. In this case the one-loop contribution to the stress energy tensor can be described by an ideal fluid with the energy density and pressure $[20]$, $\rho_{\text{UV}} = \Lambda_{\text{UV}}^2/(16\pi^2) = 3p_{\text{UV}}$, implying an equation of state parameter of radiation, $w_{\text{UV}} = 1/3$. Then the covariant energy conservation in a (homogenous) Universe dominated by such a vacuum energy, $\rho_{\text{UV}} + 3H(\rho_{\text{UV}} + p_{\text{UV}}) \simeq 0$ tells us that $\rho_{\text{UV}} \propto 1/a^4$. That then implies that one either has to give up imposing a physical momentum cutoff, or cutoff regularization altogether. Here we assume that cutoff regularization is incorrect, since it violates the (observed) Lorentz symmetry of the (quantum) vacuum. When a Lorentz symmetric regularization is used $[20]$, one gets a universal (regularization independent) result for the vacuum energy and pressure induced by (one-loop) vacuum fluctuations that is of the order the electroweak scale, which we assume here to be the physical contribution to $\Lambda$ from (the vacuum fluctuations of) quantum fields. Moreover, this result depends only logarithmically on the regularization energy scale $[20]$, rendering this contribution stable under a change of the renormalization scale.
about $\Lambda_{\text{DE}} \simeq 4 \times 10^{-84}$ GeV$^2$, which is about $10^{56}$ times smaller. This discrepancy between the natural (expected) value of the cosmological constant and the observed one constitutes the cosmological constant problem. Based on the above analysis, one would be tempted to conclude that the answer is positive. This is, however, not so for the following reason. Even though vacuum fluctuations of matter fields that couple minimally (canonically) to gravity in the Jordan frame will provide a large and approximately constant contribution to the cosmological constant, they will also feel a time dependent effective Newton constant, $G_{\text{eff}}(t) = G_N/[1 - 8\pi G_N \xi \phi^2(t)]$ [13], and no such time dependence of the gravitational coupling strength has been observed. This observation implies that the Jordan frame analysis cannot solve the (old) cosmological constant problem. Let us, therefore, repeat the analysis in the Einstein frame.

III. EINSTEIN FRAME ANALYSIS

In this section we discuss the model defined in Eq. (1) in the Einstein frame. To get to the Einstein frame, one ought to perform the following frame (conformal) transformations,

$$a_E^2 = \frac{F(\phi)}{M_P^2} a^2, \quad d\phi_E = \sqrt{\frac{M_P^2}{F(\phi)}} \left(1 + \frac{3}{2} \frac{[dF(\phi)/d\phi]^2}{F(\phi)}\right) d\phi,$$

(14)

where the index $E$ denotes the Einstein frame. The cosmological constant transforms as the (constant part of the) corresponding Jordan frame potential, $V_0 = M_P^2 \Lambda$,

$$V_E(\phi_E) = \frac{M_P^6 \Lambda}{F^2(\phi(\phi_E))}.$$

(15)

This can be easily seen by requiring that $\sqrt{-g} \Lambda$ must be invariant under frame transformations, from which it follows, $\sqrt{-g} \Lambda = \sqrt{-g_E} \Lambda M_P^4 / F^2(\phi)$, where we made use of $\sqrt{-g} = a^4$. When the above transformations are exacted, the scalar-tensor action [1] in the Einstein frame becomes simply,

$$S_E = \int d^4 x \sqrt{-g_E} \left(\frac{M_P^2}{2} R_E - \frac{1}{2} g_E^{\mu\nu} \partial_\mu \phi_E \partial_\nu \phi_E - V_E(\phi_E)\right),$$

(16)

where in the case when $V(\phi) = 0$, $V_E(\phi_E)$ is given in (15), making the (effective) cosmological constant field dependent. To get an insight into how $V_E(\phi_E)$ depends on $\phi_E$, it is worth devoting some attention to the form of this Einstein frame ‘potential.’ Assuming $F(\phi)$ is
given by Eq. (2), Eq. (14) for $\phi_E$ can be integrated, to give,

$$
\frac{\phi_E(\phi)}{M_P} = \sqrt{\frac{1-6\xi}{-\xi}} \text{Arcsinh} \left( \frac{\sqrt{-\xi (1-6\xi)\phi}}{M_P} \right) - \sqrt{6} \text{Arctanh} \left( \frac{\sqrt{6 (-\xi) \phi}}{\sqrt{M_P^2 - (1-6\xi)\xi\phi^2}} \right),
$$

(17)

where the integration constant is chosen such to get $\phi_E(0) = 0$. The Einstein frame potential is shown in figure 2 (for several values of nonminimal coupling, $\xi = -10, -1, -0.1$). The effective potential $V_E$ is a monotonically decreasing function of $\phi_E$. For small values of the field, $\phi_E \ll M_P$, the potential [15] can be approximated by a constant plus a negative mass term,

$$
V_E(\phi_E) \simeq \Lambda \left[M_P^2 + 2\xi\phi_E^2\right] + O(\phi_E^4),
$$

(18)

while for $\phi_E \gg M_P$, the potential decays exponentially with the field,

$$
V_E(\phi_E) \simeq V_{E0} \exp \left(-\lambda_E \frac{\phi_E}{M_P}\right), \quad V_{E0} = 16M_P^2 \Lambda (1-6\xi)^2 \left[\sqrt{1-6\xi} - \sqrt{-6\xi}\right] \frac{\sqrt{-\xi}}{\sqrt{1-6\xi}}.
$$

(19)

where $\lambda_E = 4\sqrt{-\xi/(1-6\xi)}$. The relevant equations of motion in the Einstein frame are,

$$
\ddot{\phi}_E + 3H_E \dot{\phi}_E + V'_E(\phi_E) = 0
$$

$$
H_E^2 = \frac{1}{3M_P^2} \left( \frac{\dot{\phi}_E^2}{2} + V_E(\phi_E) \right)
$$

$$
\dot{H}_E = -\frac{\dot{\phi}_E^2}{2M_P^2},
$$

(20)
If the initial field value $\phi_E$ is homogeneous and close to zero, then from Eqs. (18) we see that the Universe undergoes a relatively brief period of inflation, followed by a period of a slow roll parameter $\epsilon_E$ that asymptotes to a constant (see e.g. Refs. [21–23], in which attractors in exponential and power law potentials were considered),

$$\epsilon_E = \frac{\lambda_E^2}{2} = \frac{-8\xi}{1-6\xi}. \quad (21)$$

For a large and negative $\xi$, $\epsilon_E \rightarrow 4/3$, which is a decelerating epoch. The limiting case (between acceleration and deceleration), $\epsilon_E = 1$, is reached when $\xi = -1/2$ (this is the same value as in the Jordan frame) in which case the Universe behaves as if it were spatial curvature dominated. Numerical solutions of Eqs. (20) – shown in figure 3 – confirm this simple picture. For large and negative $\xi$ and when $\phi_E \gg M_P$ the Einstein frame potential $V_E$ induced by a Jordan frame cosmological constant $\Lambda$ behaves as $V_E \propto 1/t^2$, approaching

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Left panel: $\ln(3\phi_E^2/\Lambda)$ as a function of the number of e-foldings in the Einstein frame. Just as in the Jordan frame, shown in figure 1 both in the quasi-de Sitter stage and in the late power-law expansion stage the field $\phi_E(t)$ grows exponentially with the number of e-foldings. The curves (from left to right) correspond to $\xi = -10, -1$ and $\xi = -0.1$, respectively. Right panel: The Einstein frame principal slow roll parameter, $\epsilon_E = -\dot{H}_E/H_E^2$ as a function of the number of e-foldings. $\epsilon_E$ stays close to zero during the quasi-de Sitter stage and transits to a constant value $\epsilon_E = -8\xi/(1-6\xi)$ during the late-time scaling regime. From left to right: $\xi = -10, -1$ and $\xi = -0.1$. In all cases the initial value of the field is $\phi_0 = 10^{-6} M_P$, which is the typical size of quantum fluctuations during primordial inflation.}
\end{figure}
zero at asymptotically late times. From figure [3] we see that the Universe enters a late time power law expansion corresponding to the slow roll parameter $\epsilon_E$ given in (21). $\epsilon_E$ increases monotonically as $-\xi$ increases, reaching asymptotically $4/3$ as $-\xi \rightarrow \infty$. Since during radiation and matter eras $\rho_m \propto a_E^{-2\epsilon_m}$, with $\epsilon_m = 2$ and $3/2$, respectively, the asymptotic scaling $\Lambda_{\text{Eff}} \propto a_E^{-2\epsilon_m} \approx a_E^{-8/3}$ (cf. Eq. (12)) is not enough to solve the cosmological constant problem. An important question is whether this scaling can be improved by suitably changing the nonminimal coupling function $F = F(\phi)$ in Eq. (2). We have the following

**Conjecture:** For an arbitrary positive nonminimal coupling function, $F(\phi) > 0$, the fastest scaling of the effective cosmological constant in Einstein frame is $\Lambda_{\text{Eff}} \propto a_E^{-8/3}$, i.e. $\epsilon_E \leq 4/3$.

We are unable to rigorously prove this conjecture. However, we have collected strong evidence – which we summarize in Appendix A – that supports it.

Therefore, if we want to have a viable solution to the cosmological constant problem, we have to add matter and, as we argue below, the field $\phi_E$ must sufficiently quickly decay into matter, which is what we discuss next.

### IV. ADDING MATTER FIELDS

The simplest way to include matter fields is to add a homogeneous, time dependent perfect fluid, which is in the fluid rest frame characterized by energy density $\rho_m(t)$, pressure $p_m(t)$ and an equation of state, which is in cosmological space-times typically of the form, $p_m = w_m \rho_m$, where $w_m$ is a constant (for a relativistic fluid, $\epsilon_m = 2$, for a non-relativistic fluid, $\epsilon_m = 3/2$). In the regime when matter energy density is much smaller than the scalar field energy density the interaction of matter with the TeS gravitational sector can be neglected and the energy density and pressure will scale as,

$$\rho_m = \frac{\rho_{m0}}{a_E^{2\epsilon_m}}, \quad p_m = w_m \frac{\rho_{m0}}{a_E^{2\epsilon_m}}, \quad \epsilon_m = \frac{3}{2}(1+w_m),$$

(22)

where $a_E = a_E(t) = e^{N_E(t)}$ is the Einstein frame scale factor and $N_E$ denotes the number of e-folds in the Einstein frame.
In presence of matter, Eqs. (20) generalise to

\[
\dot{\rho}_{\phi E} + 3H_{E}(\rho_{\phi E} + p_{\phi E}) = -\Gamma(\rho_{\phi E} - g\rho_m), \quad \rho_{\phi E} + p_{\phi E} = \dot{\phi}_{E}^2 \tag{23}
\]

\[
\dot{\rho}_m + 3H_{E}(\rho_m + p_m) = \Gamma(\rho_{\phi E} - g\rho_m) \tag{24}
\]

\[
H_{E}^2 = \frac{1}{3M_P^2}(\rho_{\phi E} + \rho_m), \quad \rho_{\phi E} = \frac{\dot{\phi}_{E}^2}{2} + V_E(\phi_{E}) \tag{25}
\]

\[
\dot{H}_{E} = -\frac{1}{2M_P^2}(\rho_{\phi E} + p_{\phi E} + \rho_m + p_m), \tag{26}
\]

where \( g = g_{\phi}/g_{m} \) (not to be confused with \( \det[g_{\mu\nu}] \)) is the ratio of the number of relativistic degrees of freedom in the field and in matter (it is reasonable to take \( g_{\phi} = 1 \) and \( g_{m} \simeq 100 \)) and \( \Gamma \) is the decay rate at which the field decays into matter.

The rate \( \Gamma \) can be a true constant, or it can be time dependent, and its time dependence is frame dependent. The details of this frame dependence in some field theoretical models are discussed in Appendix B; here we discuss phenomenological models for \( \Gamma \) in the Einstein frame. Plausible time dependences can be parametrized in terms of the Hubble rate as, \( \Gamma = \gamma_{\theta}(H_{E}/H_0)^{\theta}H_0 \), where \( H_0 = \sqrt{\Lambda/3} \). We shall assume that, at early times (when \( H \approx H_0 \)) \( \Gamma \ll H_{E} \), while at sufficiently late times, \( \Gamma \gg H_{E} \), which can be realised when \( 0 \leq \theta \leq 1 \). In this paper we consider in detail two cases:

\begin{itemize}
  \item **Case A:** \( \Gamma = \gamma_{0}H_0 \) is constant, and
  \item **Case B:** \( \Gamma = \gamma_{1}H_{E} \), with \( \gamma_{1} = \mathcal{O}(1) \).
\end{itemize}

Below we argue that late time results for the more general case when \( 0 < \theta < 1 \) can be subsumed in Case A.

Assuming that initially \( \rho_m \ll \Lambda M_P^2 \) (which is typically the case in the early Universe setting, even during phase transitions induced by a Higgs-like field), from Eqs. (23, 26) one can study the time dependence of the Hubble parameter \( H_{E} \) and the corresponding principal slow roll parameter \( \epsilon_{E} \).

\begin{itemize}
  \item **Case A:** \( \Gamma = \gamma_{0}H_0 \), such that at late times, \( \Gamma \gg H \) (tight coupling limit), which is implied by the scaling of the Hubble parameter: it scales at least as \( H \propto 1/t \). In this case, at sufficiently late times, \( \Gamma \gg H_{E} \) enforces,
  \[
  \rho_{\phi E} = g\rho_m, \tag{27}
  \]
  plus small corrections. The condition [27] becomes exact in thermal equilibrium. Note that
Eq. (27) simplifies Eq. (24) to
\[ \dot{\rho}_m + 2\epsilon_m H_E \rho_m \simeq 0, \quad \epsilon_m = \frac{3}{2}(1+w_m), \] (28)
which is solved by, \( \rho_m = \rho_{m0}/a_{2\epsilon_m}^2 \). The condition (27) implies that the scalar field energy scales the same way, i.e. \( \rho_{\phi E} = \rho_{\phi E0}/a_{2\epsilon_{\phi}}^2 \), with
\[ \epsilon_{\phi} = \epsilon_m, \quad \epsilon_{\phi} = \frac{3}{2} \frac{\rho_{\phi E} + p_{\phi E}}{\rho_{\phi E}} = \frac{3}{2} \frac{\dot{\phi}_{\phi}^2}{\rho_{\phi E}}, \] (29)
and \( \rho_{\phi E0} = g\rho_{m0} \). The condition (29) must be consistent with the solution of the Friedmann equations (25-26), from which we extract,
\[ \epsilon_E = -\frac{\dot{H}_E}{H_E^2} = \frac{3}{2} \frac{\dot{\phi}_{\phi}^2}{\rho_{\phi E}} + \frac{2}{3} \epsilon_m \rho_m = \frac{g\epsilon_{\phi} + \epsilon_m}{g + 1} = \epsilon_m, \] (30)
which is consistent with (29). We have thus proved that
\[ \rho_{\phi E} = g\rho_m = g \frac{\rho_{m0}}{a_{2\epsilon_m}^2}; \quad p_{\phi E} = \left( \frac{2}{3} \epsilon_m - 1 \right) \rho_{\phi E}; \quad p_m = \left( \frac{2}{3} \epsilon_m - 1 \right) \rho_m. \] (31)
We think that this solution is the late time attractor, but we were unable to prove it (an attempt to numerically solve Eqs. (23–26) failed because these equations become stiff at late times).

**Case B:** \( \Gamma = \gamma_1 H_E \), with \( \gamma_1 = O(1) \). Making the constant late time \( \epsilon_E \) Ansätze,
\[ \rho_m = \frac{\rho_{m0}}{a_{2\epsilon_m}^2}, \quad \rho_{\phi E} = \frac{\rho_{\phi E0}}{a_{2\epsilon_{\phi}}^2}, \quad r = \frac{\rho_{\phi E}}{\rho_m} = \text{const.} \] (32)
and inserting them into Eqs. (23–24) yields,
\[ -\epsilon_E + \epsilon_{\phi} = \frac{\gamma_1}{2r} (g-r) \] (33)
\[ \delta \epsilon \equiv \epsilon_E - \epsilon_m = \frac{\gamma_1}{2} (g-r) \] (34)
where \( \epsilon_{\phi} = (3/2)(\dot{\phi}_{\phi}^2/\rho_{\phi E}) \). Next, by combining Eqs. (25–26) one finds,
\[ \frac{V_{E0}l_0^2}{M_P^3} = \frac{3}{\epsilon_E^2} - \frac{2}{\lambda_E^2} - \frac{3}{\epsilon_E \epsilon_m \lambda_E^2} + \frac{\epsilon_E}{\epsilon_m \lambda_E^2}, \quad \frac{\rho_m t^2}{M_P^2} = \frac{3}{\epsilon_m} \left( \frac{1}{\epsilon_E} - \frac{2}{\lambda_E^2} \right) \] (35)
\[ \epsilon_{\phi} = \frac{1 + \frac{\lambda_E^2}{2} \epsilon_E - \epsilon_m \epsilon_{\phi}}{\epsilon_E}. \] (36)
Inserting the latter equation into Eq. (33) results in the following quartic equation for \( \delta \epsilon \equiv \epsilon_E - \epsilon_m = (\gamma_1/2)(g-r) \),
\[ \delta \epsilon \left\{ \delta \epsilon^3 + \left[ 2\epsilon_m - \frac{\lambda_E^2}{2} - \frac{\gamma_1}{2} (g+1) \right] \delta \epsilon^2 \right. \]
\[ \left. + \left[ \epsilon_m \left( \epsilon_m - \frac{\lambda_E^2}{2} \right) - \frac{\gamma_1}{2} (g+1) \left( 2\epsilon_m - \frac{\lambda_E^2}{2} \right) \right] \delta \epsilon - \frac{\gamma_1}{2} \epsilon_m \left[ \epsilon_m + g \left( \epsilon_m - \frac{\lambda_E^2}{2} \right) \right] \right\} = 0. \] (37)
This equation has two real solutions and two complex solutions, which we immediately discard. The first real solution is $\epsilon_E = \epsilon_m$, $r = g$, and the second is plotted in figure 4 for typical choices of the parameters. Numerical investigation of these solutions shows that, for positive $\Gamma$, $\epsilon_E > \epsilon_m$ and $r = \rho_{\phi E}/\rho_m < 0$ (see figure 4), rendering the second solution unphysical. The first solution is also unphysical because $\epsilon_E = \epsilon_m > \lambda_E^2/2$ is only possible if $\rho_m < 0$, cf. Eq. (35). This forces us to conclude that, in the case when $\Gamma = \gamma_1 H_E$, there are no asymptotic solutions for which both $\rho_m$ and $\rho_{\phi E}$ are positive and $\rho_{\phi E}/\rho_m = \text{constant}$.

We have thus found out that, to get an effective decay of $V_E$, $\Gamma = \gamma_0 (H_E/H_0)^\theta$ has to grow with respect to $H_E$, i.e. $\theta < 1$.

In Appendix B we consider some simple (tree-level) decay channels, in which the fields couple canonically in the Jordan frame. These include a Yukawa interaction, a fermionic mass term, a second scalar ($\chi$-)field mass term and a cubic (and quartic) term of the type, $\sim (m^2_{\chi} + \sigma \phi) \chi^2$ and a canonical coupling to photons. Surprisingly, we find that (at tree level) none of them produces a viable decay channel of the type considered in Case A above.
The processes that work are a non-canonical coupling of photons to the gravitational scalar \( \phi \) (examples of \( G(\phi) \) in Eq. (47) that work are \( G(\phi) \propto 1/\phi^2 \) and \( G(\phi) \propto 1/\phi^4 \)) and a decay of \( \phi \) into a massless scalar \( \chi \) (induced by a bilinear coupling term \( \propto \phi \chi \)).

V. DISCUSSION AND CONCLUSIONS

We have constructed a simple model of tensor-scalar gravity in which the cosmological constant generated in the Jordan frame decays sufficiently fast to be suppressed during radiation and matter eras. In order for this to represent a satisfactory solution to the cosmological constant problem, one must not mess up with any of the processes in the early universe that are known to work. These include: nucleosynthesis, photon decoupling and the growth of large scale structure (not enough is known about inflation, baryogenesis, dark matter production, neutrino decoupling, and cosmic neutrino background to place meaningful constraints on our model).

Nucleosynthesis and photon decoupling remains unaffected by the scenario considered here, because in the tight coupling limit discussed in section IV, Case A, both the effective cosmological constant \( V_E \) and matter density scale in the Einstein frame as free matter does; for example \( \epsilon_E = 2 \) and \( \epsilon_E = 3/2 \) for relativistic and nonrelativistic fluids, respectively.

The tight coupling regime in section IV means that the cosmological constant contribution \( V_E \) and matter \( \rho_m \) remain in balance throughout the history of the Universe (from the moment the tight coupling approximation holds). During that time the scalar field \( \phi \) incessantly decays into matter. The decay channel must remain open up to late times, which means that \( \phi \) must decay (not only) into baryons and leptons of the standard model, but also to (almost) massless particles such as photons and massless scalars (which must be therefore chosen outside the standard model). If one produces primarily photons, their energy will redshift fine (during matter era), but one will be able to see this as a faint diffuse background source of very long wavelength stochastic photon radiation which, when included, will increase the photon-to-baryon ratio, leading thus to a potentially observable phenomena. If the decaying product is an ultralight scalar outside the standard model, its energy density will redshift as radiation, and at the moment we do not see how one would observe it.

As regards the growth of structure, since background matter evolves according to the
standard law, \( \rho_m \propto 1/a^{2\epsilon_m} \), where \( 2\epsilon_m = 3(1+w_m) \) and \( w_m = p_m/\rho_m \), matter perturbations will perceive the Einstein frame Hubble rate, \( H_E = 1/(\epsilon_m t) \), and thus grow in the same way as in the standard (minimally coupled) cosmology. At late times (after nucleosynthesis, at which \( H_E \sim 10^{-17} \) eV) the production of electrons, baryons or even neutrinos is kinematically forbidden, and hence the effective cosmological constant decays cannot affect the growth of structure. These remarks imply that in our model we expect the same growth of structure as in the standard ΛCDM model.

Note that our model does not violate the Weinberg’s no go theorem \[13\] for the following reasons. Strictly speaking, Weinberg’s theorem applies to static situations, in which one seeks a solution to the cosmological constant by adjusting the scalar field value to a suitable constant, and the corresponding metric is also time independent, while in our model both the scalar field and the metric are dynamical. Further criticism of the scalar field adjustment mechanisms in \[13\] refers to the observation that any adjustment mechanism implies in the Jordan frame an effective Newton constant that asymptotically vanishes in time. We address this issue by postulating that the observers (us) perceive the Universe expansion in the Einstein frame, in which the effective Newton constant does not change in time.

At the end we briefly discuss possible dark energy candidates. One possibility is the contribution from (light, minimally or non-minimally coupled scalar field) inflationary fluctuations; the details are discussed in Refs. \[9, 10, 24–26\] and the contribution from neutrinos and photons (assuming they couple minimally in the Einstein frame such that their contribution remains constant in the physical Einstein frame). Other proposals involve modified gravity or scalar field (quintessence) models, see e.g. Ref. \[27\].

Here we will briefly consider another possibility. Since neutrino masses cannot be explained within the standard model, it is quite natural to postulate that they couple minimally in the Einstein frame. If so, their contribution to the cosmological constant does not scale away with time. \(^4\) Let us calculate it. Imposing a Lorentz invariant regularization, the cosmological constant produced by three light Majorana neutrinos is (the fermionic contribution per relativistic degree of freedom is the same as that of a massive scalar field in

\(^{4}\) This contribution comes from the virtual vacuum fluctuations which dominates at late times, and should not be confused with the classical contribution which scales as \( \propto 1/a^3 \) at late times, and thence becomes eventually subdominant.
Eq. (11) of Ref. [20], but with an opposite sign),

\[ \langle \hat{\rho}^{\text{ren}}_{\nu} \rangle = -\langle \hat{p}^{\text{ren}}_{\nu} \rangle = -2 \sum_{i=1}^{3} \frac{m_i^4}{64\pi^2} \ln \left( \frac{m_i^2}{\mu^2} \right) + C(\mu), \tag{38} \]

(here \( C(\mu) \) is an arbitrary constant to be fixed by measurements and which runs logarithmically with \( \mu \) while the photons do not contribute because they are massless (the contributions proportional to the curvature invariants squared are small and can be neglected at late times). While we do not know what are neutrino masses, from the results of the MINOS experiment [28] we know that the neutrino mass difference squared between the second and third generation of neutrinos is about \( |\Delta m_{23}^2| \sim 0.0023 \text{ eV}^2 \), which implies that at least one of the neutrinos is \( m_i \sim 4 \times 10^{-11} \text{ GeV} \). Inserting this into (38) gives

\[ \langle \hat{\rho}^{\text{ren}}_{\nu} \rangle \sim -8 \times 10^{-45} \text{ GeV}^4 \times \ln \left( \frac{2 \times 10^{-21} \text{ GeV}^2}{\mu^2} \right) + C(\mu). \tag{39} \]

This needs to be compared with the dark energy density today, \( \rho_{\text{DE}} \sim 2 \times 10^{-47} \text{ GeV}^4 \), which means that one ought to tune \( C(\mu) \) in (39) at a 1% level if the neutrino contribution is to explain the dark energy today.

Acknowledgements

This work is part of the D-IITP consortium, a program of the Netherlands Organization for Scientific Research (NWO) that is funded by the Dutch Ministry of Education, Culture and Science (OCW).

Appendix A: Evidence for the Conjecture in section III

In section III we stated the following

**Conjecture:** For an arbitrary positive nonminimal coupling function, \( F(\phi) > 0 \), the fastest scaling of the effective cosmological constant in the Einstein frame is \( \Lambda_{E\text{ff}} \propto a_E^{-8/3} \), i.e. \( \epsilon_E \leq 4/3 \).

Here we present evidence that supports it.

It is instructive to divide \( F \) in three classes: (A) \( F \)'s that asymptotically (for large \( \phi \)) grow faster than \( \phi^2 \); (B) \( F \)'s that grow slower than \( \phi^2 \) (the limiting case, \( F \propto \phi^2 \), has already
been discussed in section III and (C) $F$’s that decrease asymptotically as $\phi$ increases. Below we discuss the limit $\phi \to \infty$ because we are interested in late time behavior.

In Class A, $F' \gg F$ and Eq. (14) can be integrated to give,

$$F(\phi) \simeq \sqrt{\frac{2}{3} \phi E M_P},$$

such that,

$$V_E \simeq \frac{\Lambda M_P^2}{\phi_E^2}.$$ (41)

In this case asymptotically $\epsilon_E \to t^{-2/3} \to 0$ as $\phi_E \to \infty$, and one gets asymptotically de Sitter space.

Class B is more difficult to prove. Here $F \gg F'$, in which case (14) reduces to,

$$\phi_E = \int \frac{d\phi}{\sqrt{F(\phi)}}.$$ (42)

This cannot be solved for general $F$. Let us therefore consider the following simple case, $F \propto \phi^{2\omega}$, where $0 < \omega < 1$. In this case (43) can be integrated to give, $\phi_E \propto \phi^{1-\omega}$, $F \propto \phi_E^{2\omega/(1-\omega)}$ such that

$$V_E \propto \phi_E^{-4\omega/(1-\omega)},$$ (43)

which is a negative power potential. We know that a negative power potential, $V_E \propto \phi_E^{-n}$ ($n > 0$) gives at asymptotically late times, $\epsilon_E(t) \propto t^{-4/(n+4)} = t^{-(1-\omega)} \to 0$. In the special case, when $n = 0$ ($\omega = 0$), one gets a logarithmic potential, $F \propto \ln(\phi_E/M_P)$, in which case $\epsilon \propto 1/t \to 0$.

Class C problems can be analysed by considering $F$’s that take the following form, $F \propto \phi^{2\omega}$ (for large $\phi$ and $\omega < 0$). As above, this then implies $V_E \propto \phi_E^{-4\omega/(1-\omega)}$. One might hope that, when $\omega \to -\infty$, $V_E \propto \phi_E^4$ and one could get $\epsilon_E \to 2$. However, the hope is shattered when one realises that the condition, $F'' \ll F$ implies $\phi, \phi_E \gg M_P$ and in that regime the quartic potential gives an inflationary slow roll parameter, $0 < \epsilon_E < 1$.

To gain more confidence that our conjecture holds, it is instructive to consider the inverse problem. Namely, we know that the Einstein frame potentials, (1) $V_E(\phi_E) = V_{E0} \exp(-\lambda_E \phi_E/M_P)$ with $\lambda_E \geq \sqrt{4}$ and $V_E(\phi_E) = \lambda_n \phi_E^n/M_P^{n-4}$ with $n \geq 4$ could be used to get a sufficiently fast scaling of $V_E$ with time to suppress the effective Einstein frame cosmological constant. Indeed, in the former case we have $\epsilon_E \to \lambda_E^2/2 \geq 2$ and in the latter case the energy density averaged over an oscillation cycle will scale as, $\epsilon_E \simeq 3n/(n+2)$.
such that $\epsilon_E \geq 2$ when $n \geq 4$. This of course will be true if the field starts at a large (super-Planckian) value, and ends up oscillating around the origin.

The inverse reconstruction problem can be exacted by firstly observing that Eq. (14) can be rewritten as,

$$d\phi_E \sqrt{F - \frac{3}{2} \left[\frac{dF/d\phi_E}{F}\right]^2} = M_P d\phi. \tag{44}$$

In the first case, $F = F_0 \exp[(\lambda_E/2)\phi_E/M_P]$ and Eq. (44) reduces to $d\phi_E \sqrt{F(1 - 3\lambda_E^2/8)} = M_P d\phi$ which then implies,

$$F(\phi) = \frac{\lambda_E^2}{8 - 3\lambda_E^2} \phi^2, \quad \lambda_E^2 < \frac{8}{3} \iff \epsilon_E < \frac{4}{3}. \tag{45}$$

The first inequality comes from the requirement $F > 0$ (which must be the case if we demand attractive gravity, i.e. a positive effective Newton constant), which then turns into the condition $\epsilon_E < \frac{4}{3}$, which does not give a fast enough scaling of the effective cosmological constant. One can try to repair the problem by adding a constant to $F$ and still keeping the coefficient of the $\phi^2$ term negative. This does not work, because that would change the scaling for small field value such that one would enter an accelerating stage with $\epsilon_E < 1$.

In the second case, $V_E(\phi_E) = \lambda_n \phi_E^n/M_P^{n-4}$ and $\epsilon_E \rightarrow 3n/(n+2)$. The corresponding equation for $F$ is

$$d\phi_E \sqrt{F \left(1 - \frac{3n^2 M_P^2}{8 \phi_E^2}\right)} = M_P d\phi, \tag{46}$$

which admits real solutions only when $\phi_E > M_P \sqrt{3n^2/8}$, in which the potential $V_E$ drives an accelerating expansion with $0 < \epsilon_E < 1$.

All of these cases present a sufficient evidence to support the above conjecture.

**Appendix B: Decay rate estimates for simple tree-level decay channels**

In this Appendix we consider some simple decay channel candidates, to see if one can get a growing $\Gamma/H_E$. Arguably, the simplest decay channels are the ones included by a Yukawa coupling to fermions $\psi$, a cubic coupling to some scalar $\chi$ and a (non-)canonical coupling to photons $A_\mu$. The corresponding interaction Lagrangian (in the Jordan frame) is,

$$\sqrt{-\tilde{g}} \mathcal{L}_{\text{int}} = \sqrt{-\tilde{g}} \left( - (m_\psi + y\phi)\bar{\psi}\psi - \frac{1}{2}(m_\chi^2 + \sigma\phi)\chi^2 - \frac{1}{4}G(\phi)F_{\mu\nu}F^{\mu\nu}\right), \tag{47}$$

where $y$ denotes a Yukawa coupling, $m_\psi$ a fermion mass, $m_\chi$ is a mass of $\chi$, $\sigma$ is a cubic coupling to $\chi$ and $G(\phi) = 1$ in the case of a canonical coupling to photons. To transform this
Lagrangian into the Einstein frame, recall that the fields transform as, $\chi_E = \chi / \sqrt{F(\phi)/M_P^2}$, $A_E^\mu = A_\mu$ and $\psi_E = \psi / [F(\phi)/M_P^2]^{3/4}$, such that we get in the Einstein frame,

$$\sqrt{-g_E} \mathcal{L}_{\text{intE}} = \sqrt{-g_E} \left( -\frac{m_\psi + y\phi(\phi_E)}{\sqrt{F/M_P^2}} \bar{\psi}_E \psi_E - \frac{1}{2} \left( m_\chi^2 + \sigma\phi(\phi_E) \right) \chi_E^2 - \frac{1}{4} G(\phi(\phi_E)) F_{\mu\nu}^E F^{E\mu\nu} \right),$$

(48)

which, in the asymptotic regime when $\phi \gg M_P$, simplifies to,

$$\mathcal{L}_{\text{intE}} \to \left( -m_\psi e^{-\lambda E\phi_E/4M_P} + \bar{y} M_P \right) \bar{\psi}_E \psi_E - \frac{1}{2} \left( m_\chi^2 e^{-\lambda E\phi_E/2M_P} + \bar{\sigma} M_P e^{-\lambda E\phi_E/4M_P} \right) \chi_E^2$$

$$- \frac{1}{4} G(\phi(\phi_E)) F_{\mu\nu}^E F^{E\mu\nu}. \tag{49}$$

The corresponding tree level decay rates for the processes $\phi_E \to 2$ particles are given by

$$\Gamma \sim \frac{y_{\text{eff}}^2}{4\pi} \omega_E + \frac{\sigma_{\text{eff}}^2}{4\pi\omega_E}, \tag{50}$$

where $y_{\text{eff}}$ and $\sigma_{\text{eff}}$ are the effective Yukawa and cubic couplings, respectively, and $\omega_E$ is the energy of $\phi_E$ excitations, which is for the above case given by,

$$\omega_E^2 = \frac{d^2V_E}{d\phi_E^2} \simeq \frac{\lambda_E^2 V_E}{M_P^2} \sim H_E^2. \tag{51}$$

From (49) we see that, $y_{\text{eff}} \sim (m_\psi/M_P)(H_E/H_0)^{1/2}$ and $\sigma_{\text{eff}} \sim (m_\chi^2/M_P)(H_E/H_0) + \bar{\sigma}(H_E/H_0)^{1/2}$ (the contribution from the photons vanishes when $G = 1$). Thence, we get the following estimate for the decay rate (50),

$$\Gamma \sim \frac{m_\psi^2 H_E^2}{M_P^4 H_0} + \frac{m_\chi^4 H_E}{M_P^2 H_0^2} + \frac{\bar{\sigma}^2}{H_0}. \tag{52}$$

This formula implies that only the last contribution grows with respect to the Hubble rate, and hence it is a potential candidate for the decay channel. For this decay to be kinematically allowed, $\omega_E > 2m_\chi_{\text{eff}}$, where $m_\chi_{\text{eff}} \simeq \sqrt{\bar{\sigma} M_P (H_E/H_0)^{1/4}}$ is the effective $\chi$ mass that corresponds to that decay channel. Thus we have the following conditions,

$$\frac{\bar{\sigma}^2}{H_0} \gg H_E, \quad H_E \gg \sqrt{\bar{\sigma} M_P (H_E/H_0)^{1/4}} \Rightarrow \frac{H_E}{H_0} \gg \left( \frac{M_P}{H_0} \right)^2. \tag{53}$$

which cannot be satisfied since $H_E \ll H_0$ (due to the evolution) and $H_0 \ll M_P$ (because gravity must be in its perturbative regime).

Therefore, we need to look harder for an interaction that works. One possible way out is to admit non-canonical couplings. Taking, for example, $G(\phi) = (M_P/\phi)^{2n}$ in (49) results in
the photon effective cubic coupling, \( \sigma_{\text{eff}} \sim k^2/M_P(H_0/H_E)^n \). This process is kinematically allowed when the photon momenta \( k < \omega_E \sim H_E \), and therefore, \( \sigma_{\text{eff}} \sim H_0^3/(M_PH_E^{n-2}) \).

The corresponding decay rate is then, \( \Gamma \sim H_0^{2n}/(M_P^2H_E^{2n-3}) \). The rate must grow with respect to \( H_E \), such that eventually, \( \Gamma/H_E \sim H_0^{2n}/(M_P^2H_E^{2(n-1)}) \gg 1 \), implying that \( n > 1 \); the simplest case when this is realised is when \( n = 3/2 \) and \( n = 2 \), for which we require \( H_E/H_0 \ll (H_0/M_P)^2 \) and \( H_E/H_0 \ll (H_0/M_P)^2 \), respectively, which can be satisfied for a sufficiently small \( H_E \). One can show that attempting to introduce a power-law nonminimal coupling into the Yukawa and \( \phi - \chi \) interaction does not open any kinematically allowed decay channel.

But can we construct a decay with canonical couplings that works? Let us consider the following simple bilinear Lagrangian density,

\[
\sqrt{-g}L'_{\text{int}} = -\frac{1}{2}m^2_{\phi\chi}\phi\chi \quad \Rightarrow \quad L'_{\text{Eint}} = -\frac{M_P}{2}m^2_{\phi\chi}\phi(\phi_E)\chi_E \rightarrow \frac{1}{2}m^2_{\phi\chi}\exp\left(-\frac{\lambda E\phi_E}{2M_P}\right)\chi_E \cdot
\]

When the cubic interaction is extracted, one gets a decay, \( \phi \rightarrow \phi\chi \), but this channel has a very small (classically zero) phase space, and hence we shall neglect it. On the other hand, \( \phi \) will constantly convert into \( \chi \) through the bi-linear coupling term (mass oscillations). In flat space these (neutrino-like) mass oscillations do not induce any decays, because they are reversible (formally, the decay rate for that process is zero). However, in an expanding universe, the created \( \chi \) particles are massless, and their energy density redshifts as \( \rho_\chi \propto 1/a^\epsilon \) (the corresponding \( \epsilon = 2 \)), such that the process is not any more reversible: more energy gets transferred from \( \phi \) into \( \chi \) than \( v.v \), and one effectively gets a decay. The decay is for sure kinematically allowed (because \( \chi \) is massless); the question is whether this decay is sufficiently effective. The decay rate can be estimated as follows, \( \Gamma \sim m^4_{\phi\chi}(H_E/H_0)^2/||\vec{k}||^2H_E \gg H_E \).

With \( ||\vec{k}||_{\text{max}} \sim H_E \) one gets that the process becomes fast when \( H_E/H_0 \ll (m_{\phi\chi}/H_0)^2 \).

Note that, when one includes gravity, this process can be viewed as a perturbative decay of \( \phi \) into one \( \chi \) and one graviton.

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