Spectral Properties of the Symmetry Generators of Conformal Quantum Mechanics: A Path-Integral Approach

H. E. Camblong, A. Chakraborty, P. Lopez-Duque, and C. R. Ordóñez

1Department of Physics and Astronomy, University of San Francisco, San Francisco, California 94117-1080, USA
2Department of Physics, University of Houston, Houston, Texas 77024-5005, USA

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Abstract

A path-integral approach is used to study the spectral properties of the generators of the SO(2,1) symmetry of conformal quantum mechanics (CQM). In particular, we consider the CQM version that corresponds to the weak-coupling regime of the inverse square potential. We develop a general framework to characterize a generic symmetry generator $G$ (linear combinations of the Hamiltonian $H$, special conformal operator $K$, and dilation operator $D$), from which the path-integral propagators follow, leading to a complete spectral decomposition. This is done for the three classes of operators: elliptic, parabolic, and hyperbolic. We also highlight novel results for the hyperbolic operators, with a continuous spectrum, and their quantum-mechanical interpretation. The spectral technique developed for the eigensystem of continuous-spectrum operators can be generalized to other operator problems.
I. INTRODUCTION AND CONTEXT

The path-integral approach, along with associated functional techniques, is a powerful methodology that provides a complete characterization of quantum-mechanical systems. For ordinary quantum mechanics problems, great advances in finding useful solutions have been made in recent decades—and exhaustive lists of path-integral solutions can be found in [1–3]. In this work, we use the path-integral approach to derive the eigenvectors and eigenvalues of some of the operators relevant to conformal quantum mechanics.

Conformal quantum mechanics (CQM) has attracted considerable attention following its initial formulation in the 1970s, when it was first proposed as an example of a scale invariant theory [4], and analyzed in detail by de Alfaro, Fubini, and Furlan (dAFF) [5] as a (0+1)-dimensional form of conformal field theory. The dAFF model corresponds to the $D = 1$ limit of the spacetime $D$-dimensional conformally invariant Lagrangian [5]

$$L = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - g \phi^{2/(D-2)}.$$  \hspace{1cm} (1)

In the transition to $D = 1$, one can interpret the resulting theory as standard quantum mechanics, with a “field” $Q(t)$ described as a configuration or position variable subject to an inverse square potential,

$$L = \frac{1}{2} \dot{Q}^2 - \frac{g}{2Q^2},$$  \hspace{1cm} (2)

where the dot represents the usual time derivative. This original form of CQM, centered on its SO(2,1) conformal symmetry structure, was subsequently used in seminal papers by Jackiw [6, 7], also including an analysis of a related CQM system of contact interactions [8]. Systems exhibiting this kind of SO(2,1) conformal symmetry have seen renewed interest in a broad range of physical applications over the years. This is primarily due to its simplicity as a model of conformal field theory and its remarkably wide range of applicability to established physical problems that exhibit approximate conformal symmetry in a window of physical scales. While not included in the original dAFF model formulation, outstanding realizations of this type of inverse-square-potential systems have been found where this symmetry generates a quantum anomaly. A list of noteworthy applications of such systems includes molecular physics [9], black hole thermodynamics [10–12] and acceleration radiation [13–16], the Efimov effect [17–20], and graphene [21, 22], among several others [20]. It is noteworthy that, in addition to the quantum symmetry breaking based on a strong-coupling
version of Lagrangian (2), another class of quantum anomalies have been found in systems
with SO(2,1) conformal symmetry with contact interactions—they involve 1D three-body
interactions in 1D and 2D two-body interactions in Fermi systems of ultracold atoms [23–27].

The previous list emphasizes those cases where the CQM interaction is attractive and
sufficiently strong, where it leads to a quantum anomaly [9, 28–30] or some kind of renor-
malization [31–36], or to the fall-to-the-center phenomenon [37, 38]—this regime is called
“strong coupling” for short. On the other hand, the form of CQM often discussed in the
context of conformal symmetry analyses is based on the dAFF model [5]. The latter strictly
applies to the case when the conformal potential is repulsive (or sufficiently weak, even if
attractive) to avert the pathologies inherent for strong coupling $g$ in Eq. (2), and the con-
formal symmetry is maintained. For the sake of simplicity, in this work we solely focus on
this version of CQM (in a slightly generalized format), for which we analyze the spectral
properties of some of its symmetry generators. We will address elsewhere the more general
case, including the presence of anomalies.

A. Scope and Main Results of This Paper

Of particular interest for the present work is the fact that the dAFF model (in the original
weak-coupling formulation of the CQM symmetry generators) has been recently found to
be a CFT$_1$ realization of the AdS/CFT correspondence [39, 40], leading to renewed interest
in this topic [41, 42]. In addition, using the operators we discuss in this paper, the dynamic
evolution within the dAFF model has been used to study causal diamonds in Minkowski
spacetime [47, 48]. The associated physics of finite-lifetime observers, often called diamond
observers, was first addressed in Ref. [49] in a remarkable finding that generalizes a similar
thermalization of the vacuum for accelerated observers (the celebrated Unruh effect [50]).
Accordingly, these observers have access to only a limited region of spacetime, known as
the conformal (causal) diamond; and the vacuum they perceive has a diamond tempera-
ture inversely proportional to their lifetime. This insightful result on the thermalization
of the vacuum of diamond observers has been further confirmed in Refs. [51–56]. As there
are a number of parallels with the thermodynamic behavior of black hole horizons, where
CQM has been successfully used, it is not surprising that that the dAFF model may also
be relevant for causal diamonds. Specifically, in Refs. [47, 48], the dynamic evolution within
the dAFF model was shown to be in correspondence with the time evolution of Minkowski observers with a finite lifetime, which is described by the radial conformal Killing fields (RCKF) previously developed in Ref. [57]. This problem is of interest in further characterizing the causal structure of spacetime, and CQM appears to be a promising tool for this purpose. The prototypical CQM generators of generalized time evolution can be taken as the operators $H$, $R$, and $S$, where $H$ is the quantum-mechanical Hamiltonian associated with the Lagrangian (2), and $R$ and $S$ are the linear combinations of $H$ and the special conformal generator $K$ involved in setting up the Cartan-Weyl basis. In particular, the spectral decomposition of the operator $S$, which plays a significant role in causal diamonds [47, 48, 57], is a novel result of our paper. More generally, the dynamical evolution can be described by any of the generalized generators $G$ defined as linear combinations of $H$, $D$, and $K$, where $D$ is the dilation operator.

While the usual operator properties of the set \{$H$, $R$, $S$\} and the generalized generators $G$ have been considered in the literature [5], a path-integral treatment is lacking. It is the purpose of this paper to develop such functional integral approach, with which we compute the propagators and the spectral properties of the generators \{$H$, $R$, $S$\}, as well as those of the linear combinations that define a generalized generator $G$, falling under the three possible classes: elliptic ($R$-like), parabolic ($H$-like), and hyperbolic ($S$-like).

The main results of the paper are summarized below in a convenient format that can help identify the key ideas within the extensive mathematical properties being discussed.

- We have developed a complete Hamiltonian framework for all the generalized generators $G$ of CQM (which proves essential for the path-integral approach).

- We have derived the spectral properties, including the eigenvalues and eigenstates for the three classes of CQM generators $G$, viz., elliptic ($R$), parabolic ($H$), and hyperbolic ($S$), using a path-integral approach. This involves using different limits of the propagator for the generalized radial harmonic oscillator $K_{t+\nu}^{(\text{RHO})}$ given in Eq. (45). Depending on the classes of the operators, we have used different methods to extract the spectral properties of the operator $G$ from the propagator. The table below acts a pointer to the equations describing the spectral properties found in this paper.

- We have used a novel method based on Fourier transforms (which we call the Fourier Method) to find the eigenstates of the noncompact CQM operators from the propa-
TABLE I: References to the key results about the spectral properties of the CQM generators.

| CQM Generator | Propagator | Eigenvalues | Eigenstates |
|---------------|------------|-------------|-------------|
| Elliptic $(R)$ | $K^{(RHO)}_{l+\nu}$ – Eqs. (45), (49) | Discrete – Eq. (52) | Eq. (54) |
| Parabolic $(H)$ | $\lim_{\omega \to 0} K^{(RHO)}_{l+\nu}$ – Eq. (57) | Continuous in $\mathbb{R}^+$ | Eq. (61) |
| Hyperbolic $(S)$ | $\lim_{\omega \to -i\omega} K^{(RHO)}_{l+\nu}$ – Eq. (76) | Continuous in $\mathbb{R}$ | Eqs. (91) & (92) |

• In particular, we have derived a complete spectral characterization of the operator $S$ and all hyperbolic generators—a result that has been surprisingly lacking in the literature.

• As a bonus, our detailed analysis of the spectral properties of the propagators via different approaches has uncovered additional mathematical connections and identities, as mentioned at the end of Subsec. V A and in Appendix C.

B. Organization of This Paper

In Section II we summarize the symmetry properties of CQM and the specifics of the dAFF model; we define the operators $R$, $H$, and $S$ that generate the conformal symmetry group, as well as the generic conformal generator $G$; and we develop the framework that classifies the possible types of $G$ and their role in the dynamical time evolution via their Hamiltonian representation. In Sec. III we review the path-integral approach for radial problems in quantum mechanics and specifically use it to study a generalized radial harmonic oscillator relevant to CQM. In Sec. IV we fully characterize the spectral properties of the operator $R$ and its class (elliptic). In Sec. V we analyze the spectral properties of the operator $H$ and its class (parabolic); and we develop a novel procedure for spectral properties of operators with continuous spectra. In Sec. VI we apply this method to the conformal operator $S$ and its class of non-compact hyperbolic generators to fully characterize their spectral data and Green’s functions. We conclude the paper in Sec. VII with a brief
summary and directions for future work. The appendices cover an alternative dimensional framework for the Hamiltonian description of generators; detailed properties of the relevant path-integral framework; evaluation of the inversion integrals for the spectral decomposition of parabolic and hyperbolic generators; and further analysis of the conformal generators within a differential-equation approach.

The logic of the paper organization is outlined in Fig. 1. The flowchart stresses the need to use an appropriate, alternative Hamiltonian formulation for the path-integral framework before the spectral analysis of the CQM generators is performed; then, from both the Hamiltonian formulation and the path-integral treatment, all the properties of the CQM generators follow systematically, with additional technical details provided by the appendices.
II. SYMMETRY PROPERTIES AND GENERATORS OF CONFORMAL QUANTUM MECHANICS (CQM): HAMILTONIAN FORMULATION

As outlined in Sec. I, the dAFF model is the limiting $D = 1$ field theory of Eq. (1). While Eq. (2) is the simplest form of CQM, in this paper, we consider the more general multicomponent case, where the $d$ components correspond to the $d$ spatial dimensions of the position coordinates $Q(t)$ in quantum mechanics, evolving dynamically with respect to the ordinary time $t$. This physical system, with Lagrangian

$$L = \frac{M}{2} \dot{Q}^2 - \frac{\lambda}{2Q^2},$$

(3)

can be described in polar hyperspherical coordinates, with the radial variable $Q = |Q| \geq 0$ defined in the nonnegative half-line. In the path-integral framework, a system with Lagrangian (3) can be more naturally described within a class of radial functional problems, as we will see in Sec. III. In Eqs. (2) and (3), we are adopting a definition of the coupling constant with an additional factor of 1/2 for convenience, consistent with the numerical factors of Ref. [5], and rescaled in the transition to the quantum theory, with $\lambda = \hbar^2 g/M$, with $g$ dimensionless. In addition, we are inserting an extra dimensional parameter $M$ that can be interpreted as the mass of a nonrelativistic particle in the conventional form of quantum mechanics. Often, the choices $M = 1$ and $\hbar = 1$ are made for the sake of simplicity (including in Ref. [5]), but we will keep conventional dimensional parameters to help in some of the derivations with the use of appropriate analytic continuations, limiting procedures, and familiar physical interpretations for both the Hamiltonian operator and the other symmetry generators.

A. Conformal Symmetries

Rather than starting directly from the Lagrangian (3), one can begin the problem looking at the more general conditions that define conformal invariance for a generic Lagrangian

$$L = \frac{M}{2} \dot{Q}^2 - V(Q).$$

(4)

From Noether’s theorem, for systems with Lagrangians of the form (4), the necessary conditions to yield an invariant action under general time reparametrizations [3,7] (with time
transformations such that the Lagrangian itself is not necessarily invariant) are: (i) the potential function \( V(Q) \) is a homogeneous function of degree \(-2\); (ii) the time transformations appear restricted to any of three independent building blocks: time translations, time dilatations, and inverse time translations. In what follows, \( P \equiv P_Q \) is the momentum conjugate to \( Q \). Then, the transformation algebra consists of the three generators (including operator ordering at the quantum level):

- The Hamiltonian
  \[
  H = \frac{P^2}{2M} + V(Q),
  \]  
  associated with time translations (with respect to \( t \)).

- The dilation generator
  \[
  D = tH - \frac{1}{4}(Q \cdot P + P \cdot Q),
  \]
  which enforces scale transformations.

- The special conformal generator
  \[
  K = Ht^2 - \frac{1}{2}(Q \cdot P + P \cdot Q)t + \frac{1}{2}MQ^2
  \]
  that enforces inverse time translations.

This class of conformal Hamiltonians includes both the inverse square potential (ISP) \( V(Q) \propto Q^{-2} \) as well as contact interactions—most notably the two-dimensional delta [8], in addition to anisotropic versions of the ISP and the derivative-delta interaction in one spatial dimension [26]. While the symmetry properties of these realizations are generically the same for the whole conformal class, the analytical details of the solutions are specifically dependent on the chosen model. In this paper, we only consider the dAFF model of non-contact interactions that corresponds to a (0+1)-dimensional conformal field theory; in this setting, this amounts to the Lagrangian of Eq. (3).

The composition of the three basic types of transformations (5)–(7) generates the linear fractional transformation

\[
\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta},
\]

governed by a matrix in the special linear group \( SL(2, \mathbb{R}) \) (with \( \alpha \delta - \beta \gamma = 1 \)), and with transformed field variable

\[
\tilde{Q}(\tilde{t}) = \frac{Q(t)}{\gamma t + \delta}.
\]
At the quantum level, Eq. (9) is written in the Heisenberg picture, and has the same form as its classical counterpart. The associated symmetry group is $SL(2, \mathbb{R})$, which is homomorphic to $SO(2, 1)$, with the generators $\{H, D, K\}$ satisfying the commutator relations

$$[D, H] = -i\hbar H, \quad [D, K] = i\hbar K, \quad [H, K] = 2i\hbar D,$$

which correspond to the one-dimensional case of the algebra of the generic conformal group.

To introduce an appropriate Cartan-Weyl basis, one can replace $H$ and $K$ by the operators $R$ and $S$ defined by the linear combinations

$$R = \frac{1}{2} \left( \frac{1}{a} K + aH \right), \quad (11)$$
$$S \equiv -S' = \frac{1}{2} \left( \frac{1}{a} K - aH \right), \quad (12)$$

where $a$ is an arbitrary parameter with dimensions of time (with $R$ and $S$, as well as $D$, defined with dimensions of action). The commutators of the Cartan-basis generators are $[S, D] = -i\hbar R$, $[D, R] = i\hbar S$, and $[R, S] = i\hbar D$. The $so(2, 1)$ algebra has rank one and dimension 3, with the Cartan generator $R$ and the ladder operators $L_{\pm}$ defined below, along with their corresponding commutator relations:

$$L_{\pm} = S \pm iD \quad (13)$$
$$[\hbar^{-1} R, L_{\pm}] = \pm L_{\pm}, \quad [L_{+}, L_{-}] = -2\hbar R. \quad (14)$$

In this paper, we will use the operator $S' = -S$ (with the sign reversed, as defined above), so that a more transparent physical interpretation can be given to the corresponding effective potential, as in Sec. VI—this choice can be found in the literature, for example, in Ref. [47].

**B. Generalized Conformal Generators: Definition, Dynamics, and Alternative Hamiltonian Formulation**

The properties of the conformal generators are well-known from the exhaustive analysis of Ref. [5]. The operator $R$ generates a compact subgroup [two-dimensional rotations as the $O(2)$ subgroup of $SO(2, 1)$], and $S$ and $D$ generate non-compact boosts. These distinct behaviors can be characterized by considering the generalized generator

$$G = uH + vD + wK,$$  

(15)
with discriminant

$$\Delta = v^2 - 4uw$$  \hspace{1cm} (16)$$

that determines its nature: rotations (elliptic type), for $\Delta < 0$, which include $R$; boosts (hyperbolic type), for $\Delta > 0$, which include both $S$ and $D$; and parabolic (“lightlike”) operators, for $\Delta = 0$, which include the original $H$ and $K$ operators.

A complete analysis [5] in terms of the operators (15) shows that $G$ can be regarded as the generator of time evolution with respect to a modified, effective time $\tau$. For the $d$-component field $Q(t)$, from the definitions (5)–(7) and (15), the basic equation satisfied in the Heisenberg picture by the action of the operator $G$ on the field variables $Q(t)$ is

$$\frac{1}{i\hbar} [Q(t), G] = f_G(t) \frac{dQ(t)}{dt} - \frac{1}{2} \frac{df_G(t)}{dt} Q(t), \hspace{1cm} (17)$$

where

$$f_G(t) = u + vt + wt^2 = \sigma |u + vt + wt^2|, \hspace{1cm} (18)$$

with the sign $\sigma \equiv \sigma_G = \text{sgn}[f_G(t)]$ allowing for arbitrary linear combinations of the generalized generator (15). The dynamical equation (17), just as in the $d = 1$ case [5], can be simplified to a more standard form by redefining the dynamical time to $\tau$ according to

$$d\tau = \frac{dt}{(u + vt + wt^2)}, \hspace{1cm} (19)$$

and the field variables to

$$q(\tau) = \frac{Q(t)}{|u + vt + wt^2|^{1/2}}. \hspace{1cm} (20)$$

In Eq. (20), the absolute value is needed to reproduce Eq. (17), with the auxiliary Eq. (18). Equation (19) can be easily integrated; particular cases (including $R$, $S$, $D$, and $K$) are listed in Ref. [5]. Thus, the dynamics in the Heisenberg picture is fully described by the equation

$$\frac{1}{i\hbar} [q(\tau), G] = \frac{dq(\tau)}{d\tau}, \hspace{1cm} (21)$$

which shows that $G$ acts as an effective Hamiltonian for time evolution. One can alternatively describe the dynamics in a generalized Schrödinger picture with respect to $G$, in terms of evolving state vectors $|\Psi(\tau)\rangle_s \equiv |\Psi(\tau)\rangle = U_G(T) \cdot |\Psi(\tau_0)\rangle$, where $|\Psi(\tau_0)\rangle$ is the original state in the Heisenberg picture and

$$U_G(T) \equiv U_G(\tau; \tau_0) = \exp \left[-\frac{i}{\hbar} G (\tau - \tau_0)\right] \hspace{1cm} (22)$$

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acts as an effective evolution operator; thus, the state vector $|\Psi(\tau)\rangle$ satisfies the generalized Schrödinger equation

$$G |\Psi(\tau)\rangle = i\hbar \frac{d|\Psi(\tau)\rangle}{d\tau}. \quad (23)$$

Then, Eqs. (21) and (23) describe the evolution of the field variables and states under the action of $G$ in the Heisenberg and Schrödinger pictures respectively, with the new effective time $\tau$; in other words, $G$ formally acts as $i\hbar \partial_\tau$. In these transformed variables, the time evolution only depends on the effective time difference $T = \tau - \tau_0$. It should be noted that this is not the original Schrödinger picture associated with the Hamiltonian $H$, and it is $G$-specific.

In principle, the analysis leading to Eqs. (19)–(23) allows a reformulation of the theory in terms of a new “Hamiltonian” $H_G(q, p) \equiv G$. This assignment $H_G \equiv G$, by abuse of notation, simply states that the value of $G$ is to be rewritten in terms of the transformed canonical variables $(q, p)$. By construction, $q$ is given by Eq. (20), but $p$ has to be defined consistently in such a way that $|f_G|^{1/2}dq/d\tau = f_G dQ/dt - \dot{f}_G Q/2$ (with the dot notation for derivatives with respect to the original time $t$)—an expression that can be read off by comparison of Eqs. (17) and (21) or from Eqs. (19)–(20). An obvious choice for $p$ is the one that satisfies $p = M\dot{q}$, whence the transformed momentum is given by

$$p = \sigma |f_G|^{1/2} \left( P - \frac{\dot{f}_G}{2f_G} MQ \right). \quad (24)$$

From Eq. (15), this implies that the operator $G = uH + v(tH + D_0) + w(t^2H + 2tD_0 + K_0) = f_G H + \dot{f}_GD_0 + wK_0$ [where $D_0$ and $K_0$ are the time independent parts in Eqs. (6) and (7)] takes the “normal” Hamiltonian form

$$G \equiv H_G(q, p) = \sigma \tilde{H}_G(q, p), \quad (25)$$

where

$$\tilde{H}_G(q, p) = \frac{1}{2M}P^2 + \frac{1}{2} \frac{\lambda}{q^2} + \frac{M}{2} \left( -\frac{\Delta}{4} \right) q^2, \quad (26)$$

and the discriminant (16) is shown to be the same as $\Delta = \dot{f}_G^2 - 4wf_G$. The generalized momentum $p$ in Eq. (24) and the functional form of $H_G(q, p)$ in Eqs. (25) and (26) can also be verified operationally at the classical level by a straightforward computation of the Hamiltonian from the transformed Lagrangian $L_G$, using Eqs. (3) and (19)–(20), as shown in Ref. [5].
C. Equivalence Classes of the Generalized Generators and Hamiltonian $H_G$

An important property follows from the functional form of Eqs. (25)–(26): for each given value of $\Delta$, there is a continuous range of linear combinations $G$ with values of $u$, $v$, and $w$ that give rise to the same effective Hamiltonian $H_G(q, p)$. For all generators $G$ that have the same value of $\Delta$, the corresponding effective Hamiltonians are identical and yield equivalent models within the theory, with identical dynamical evolutions with respect to $\tau$, as well as equivalent spectral properties (up to a possible rescaling of the spectrum). Thus, the theory is organized in equivalence classes labeled by the value of the discriminant $\Delta$. Moreover, these classes are further categorized in the three types of operator behavior: $\Delta \leq 0$ and $\Delta = 0$.

Regarding the spectral characterization of operators that we are addressing in this paper, the primary goal is to fully determine the eigenstates and eigenvalues of the generalized conformal generator $G$. We denote its eigenvalues as $\hbar g$, where $g$ is dimensionless; then, from Eq. (25), it follows that

$$\hbar g = E_G = \sigma \tilde{E}_G,$$

in terms of the eigenvalues of the associated $\tilde{H}_G(q, p)$. (It should be noted that, in Ref. [5], the eigenvalues of $G$ are denoted by $G'$. In light of Eq. (27), the eigenstates will be labeled interchangeably with $g$ or $E_G$, and the latter may be simplified as $E$ if there is no obvious conflict.

For the remainder of the paper, we will use the notation $q \equiv r$ along with the Hamiltonian $\tilde{H}_G(r, p)$. Equation (26) is an exact analog of a standard quantum-mechanical central problem with Hamiltonian and potential

$$\tilde{H}(r, p) = \frac{p^2}{2M} + \tilde{V}(r), \quad \tilde{V}(r) = \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2M} \frac{g}{r^2},$$

which can be interpreted as representing a harmonic oscillator of squared “frequency”

$$\omega^2 = -\frac{\Delta}{4}$$

(including possibly imaginary frequencies) superimposed with the original inverse square potential. Even though our conformal problem has a different meaning, this analog parametrization is insightful and the “CQM frequency” from Eq. (29) will play an important role in describing the spectral solutions. In what follows, we will also use the functional
notation \( \tilde{H}[M, g, \omega] \equiv \tilde{H}_{r,p}[M, g, \omega] \) for the corresponding parameter dependence, where \( \tilde{H}_{r,p} \equiv \tilde{H}(r, p) \), with the appropriate canonical variables. A summary of the different Hamiltonian notations used in the main text is given in Table II.

TABLE II: Here we tabulate the different notations that we have used for specific Hamiltonians. All the expressions are written with position and momentum arguments denoted by \( r \) and \( p \), respectively.

| Notation | Expression |
|----------|------------|
| \( H(r, p) \) | \( \frac{p^2}{2M} + \frac{g}{r^2} \) |
| \( \tilde{H}_G(r, p) \) | \( \frac{p^2}{2M} + \frac{\lambda}{r^2} + \frac{M}{r^2} (-\Delta) r^2 \) |
| \( G \equiv H_G \) | \( \sigma \tilde{H}_G(r, p) \) |
| \( \tilde{H}(r, p) \equiv \tilde{H}_{r,p} \equiv \tilde{H}[M, g, \omega] \) | \( \frac{p^2}{2M} + \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2M} \frac{g}{r^2} \) |
| \( \tilde{H} \) | Generic quantum mechanical Hamiltonian |

The Hamiltonian function of Eq. (26) exhibits a scaling property that is a consequence of the definition (15), which is linear in the coefficients \( u, v, w \), and with \( \Delta \) given as a quadratic function (16). If this Hamiltonian function is rewritten in the generic analog form of Eq. (28), with frequency (29), then it satisfies the scaling

\[
\tilde{H}[M, g, c\omega] = c\tilde{H}[M, g, \omega] ,
\]

where \( c > 0 \) is an arbitrary real and positive scaling constant. In other words, the Hamiltonian is a homogenous function of first degree with respect to the frequency \( \omega \). An outline of the proof is as follows: if \( G \) is rescaled with the coefficients \( u' = cu, v' = cv, w' = cw \), then it takes the value \( G' = cG = \sigma c\tilde{H}[M, g, \omega] \), without a change in \( \sigma \), but with the discriminant changing as \( \Delta' = c^2 \Delta \). At the same time, Eq. (29) gives \( \omega' = c\omega \) and this yields \( G' = \sigma \tilde{H}[M, g, c\omega] \), whence Eq. (30) follows by comparing the two expressions for \( G' \). This can be also be shown more generally directly at the level of Eq. (28) due to the general scaling properties of this Hamiltonian (as in ordinary dimensional analysis), in terms of the frequency parameter, with \( \tilde{H} \) and \( \omega \) having the same scaling.

From Eq. (26), the spectral properties of the different types of conformal generators can be immediately understood. In this procedure, we can choose the prototypical generators \( R \),
\( H \), and \( S' \) as representatives of the three main classes of operator behaviors: their general properties are shared by the generators \( G \) with \( \Delta < 0 \), \( \Delta = 0 \), and \( \Delta > 0 \), respectively. Then, \( \omega = \sqrt{|\Delta|}/2 \) for \( \Delta < 0 \) and \( \omega = -i\sqrt{\Delta}/2 \) for \( \Delta > 0 \) (the choice of sign for the latter is discussed in Sec. VI), with \( \omega = 0 \) for \( \Delta = 0 \). Moreover, with the same notation as in Eq. (30), we can characterize the parameter dependence of all the operators, starting with the prototypical operators \( R \), \( H \), and \( S' \), whose equivalence classes have discriminants \( \Delta = -1 \), \( \Delta = 0 \), and \( \Delta = 1 \), respectively. Thus,

\[
\begin{align*}
R &= \tilde{H} [M, g, \omega = 1/2], \\
H &= \tilde{H} [M, g, \omega = 0], \\
S' &= \tilde{H} [M, g, \omega = -i/2].
\end{align*}
\]

In addition, as in Ref. [5], the parameters \( \hbar \) and \( M \) could be set equal to unity (though we will mostly keep the general parametrization for convenience). Finally, applying the scaling (30), it follows that \( \tilde{H}_G = \sqrt{|\Delta|} \tilde{H} [M, g, \omega = 1/2] \) for elliptic operators, and \( \tilde{H}_G = \sqrt{\Delta} \tilde{H} [M, g, \omega = -i/2] \) for hyperbolic operators. Consequently, the three categories or classes of operators admit the following characterization.

- **Elliptic Generators**, with \( \Delta < 0 \). These are “generalized \( R \) operators,” of the form

\[
G = \sigma \sqrt{|\Delta|} \tilde{H} [M, g, \omega = 1/2] \\
\approx \sigma \sqrt{|\Delta|} R,
\]

where \( R \) is the prototypical elliptic operator (\( \Delta = -1 \)) of Eqs. (11) and (31).

- **Parabolic Generators**, with \( \Delta = 0 \). These are “generalized \( H \) operators,” of the form

\[
G = \sigma \tilde{H} [M, g, \omega = 0] \\
\approx \sigma H,
\]

where \( H \) is the prototypical parabolic operator of Eqs. (5) and (32).

- **Hyperbolic Generators**, with \( \Delta > 0 \). These are “generalized \( S \) operators,” of the form

\[
G = \sigma \sqrt{\Delta} \tilde{H} [M, g, \omega = -i/2] \\
\approx \sigma \sqrt{\Delta} S',
\]

where \( S' \) is the prototypical hyperbolic operator (\( \Delta = 1 \)) of Eqs. (12) and (33).
TABLE III: In this table, the CQM generators are summarized according to their characteristics.

| Operator Class | Sign of $\Delta$ | Prototypical Definition | Generalized Generators |
|----------------|------------------|-------------------------|------------------------|
| Elliptic       | $\Delta < 0$    | $R \equiv \tilde{H}_G[M, g, \omega = 1/2]$ | $\sigma \sqrt{|\Delta|} R$ |
| Parabolic      | $\Delta = 0$    | $H \equiv \tilde{H}_G[M, g, \omega = 0]$ | $\sigma H$            |
| Hyperbolic     | $\Delta > 0$    | $S' \equiv \tilde{H}_G[M, g, \omega = -i/2]$ | $\sigma \sqrt{|\Delta|} S'$ |

A summary of the different generalized generators $G$, Eqs. (31)–(36), is given in Table III. In Eqs. (31)–(36), the notation $\approx$ signifies that the given operators are in the same equivalence class, with a proportionality constant that correspondingly rescales their eigenvalues. This analysis verifies that the theory is organized in equivalence classes defined by the value of $\Delta$, and where $\sigma = \text{sgn}(f_G)$ just gives the “orientation” of the operator spectrum. Within each class, regardless of the values of $u$, $v$, and $w$, the effective Hamiltonian, dynamical evolution, and spectral properties are identical (except for a possible rescaling of the eigenvalues)—but the form of the effective time $\tau$ (in terms of $t$) is specific to each operator. It is also noteworthy that the results of Eqs. (34)–(36) are completely general, and apply even for the parabolic operator $K$ (which, according to this, gives the same propagator as $H$) and for the hyperbolic operator $D$ (having the same propagator as $S'$). Incidentally, an alternative form of this theory is presented in Appendix A which is restricted to the cases with $u \neq 0$ only, but has some advantages in terms of dimensional analysis.

In this section, we have established the general framework for the generators $G$. Next, we will use the parametrization of Eqs. (25)–(26) for a complete path-integral analysis of the spectral properties of all the conformal operators, for three associated classes (elliptic, parabolic and hyperbolic).

III. PATH-INTEGRAL FRAMEWORK: BASIC SETUP FOR CONFORMAL GENERATORS

All properties of a physical system can be completely characterized by using path-integral methods. This includes the spectral properties of operators relevant to the system. We will show how to derive such properties for the CQM operators $R$, $H$, and $S$, and, by extension, according to Eqs. (34)–(36), for the generalized generators $G$ of Eq. (15). Due to the
form of the Lagrangian (3) of the dAFF model and the special conformal operator (7), leading to the effective Hamiltonian (26), we need the general $d$-dimensional path-integral solution for the radial harmonic oscillator, along with the inverse square potential, i.e., for a central problem with Hamiltonian and potential given by Eq. (28). Of course, this is strictly needed for the multicomponent case of CQM, but it is also a useful approach to properly incorporate the inverse square potential with an appropriate analytic continuation (see below and Appendix B), even in the simplest case $d = 1$.

Remarkably, we are computing the path-integral propagators associated with the operators $R, H, S$, as well as their generalized forms $G$, as generators of dynamical evolution. The specific functional form of these generators involves the generalized canonical variables defined in Eqs. (20) and (24) (or rescalings thereof). For $H$ itself, the effective propagator is indeed the generator of dynamics in ordinary time $t$, but all the other generators involve dynamical evolutions with respect to their “natural” effective times $\tau$. (In Appendix A an alternative, dimensional framework is defined for the cases with $u \neq 0$, in terms of a rescaled time $\tilde{\tau}$, which pays a similar role.) Moreover, in addition to describing the dynamics, the propagators play the role of generating functions for the spectral decomposition and will allow us to find the spectral data, namely, the eigenvalues and eigenstates for the family of conformal generators.

A. Path-Integral Framework

We begin setting up the framework with some notational remarks. In the theory outline of this subsection and in Appendix B we typically denote by $t$ the dynamical time, with $\hat{H}$ describing a generic Hamiltonian, as in the usual quantum-mechanical applications. Within this general framework, we use a radial harmonic oscillator (properly generalized to include inverted oscillators and any number of dimensions) as an analog system for a generic conformal generator $G$. In this sense, when describing the spectral properties and dynamics generated by $G$, the general framework involves replacing $\hat{H}$ by $G$ or $\tilde{H}$, and $t$ by $\tau$.

We begin our construction by considering the quantum-mechanical propagator for a particle of mass $M$ subject to an interaction potential $V(r, t)$ in $d$ spatial dimensions,

$$K_{(d)}(r'', r'; t'', t') = \left\langle r'' \bigg| \hat{T} \exp \left[ -\frac{i}{\hbar} \int_{t'}^{t''} \hat{H}dt \right] \bigg| r' \right\rangle,$$

(37)
where $\hat{T}$ is the time-ordering operator and $\hat{H}$ the Hamiltonian. The corresponding path-integral expression

$$K(d)(r'', r'; t'', t') = \int_{r(t')=r''}^{r(t''=r''')} D\mathbf{r}(t) \exp \left\{ \frac{i}{\hbar} S[r(t)](r'', r'; t'', t') \right\}$$

(38)

involves the classical action functional $S[r(t)](r'', r'; t'', t')$ for “paths” $\mathbf{r}(t)$ connecting the end points $\mathbf{r}(t') = \mathbf{r}', \mathbf{r}(t'') = \mathbf{r}''$. For time-independent potentials, the propagator $K(d)(r'', r'; t'', t') \equiv K(d)(r'', r'; T)$ is a function of $T = t'' - t'$ alone and not of the individual endpoint times; we will use this assumption for the remainder of the paper. Specifically, for the generic conformal generator $G$, the associated propagator is given by $K^{(G)}(r'', r'; T) = \langle r'' | U_G(T) | r' \rangle$, with $U_G(T)$ as in Eq. (22), in terms of the effective time $\tau$.

In Appendix B, we review the required construction for central potentials in hyperspherical coordinates, and specifically for the radial harmonic oscillator. In general, the propagator $K(d)(r'', r'; t'', t')$ can be formally rewritten in hyperspherical polar coordinates [33, 34, 39], which yield a complete set of angular functions, the $d$-dimensional hyperspherical harmonics $Y_{lm}(\Omega)$, where the quantum labels correspond to the $d$-dimensional angular momentum with numbers $l$ and $m$, and such that the set $m$ takes $g_l = (2l + d - 2)!(l + d - 3)!/l!(d - 2)!$ values as multiplicity for a given $l$ (Chap. XI in Ref. [59]). The corresponding partial wave expansions of $K(d)(r'', r'; t'', t')$ read

$$K(d)(r'', r'; T) = \langle r'' | r' \rangle^{-(d-1)/2} \sum_{l=0}^{\infty} \sum_{m=1}^{g_l} Y_{lm}(\Omega'')Y_{lm}^*(\Omega') K_{l+\nu}(r'', r'; T)$$

(39)

$$= \frac{\Gamma(\nu)}{2\pi^{d/2}} (r'' r')^{-(d-1)/2} \sum_{l=0}^{\infty} (l + \nu) C_{l}^{(\nu)}(\cos \psi_{\Omega'' \Omega'}) K_{l+\nu}(r'', r'; T) ,$$

(40)

where $\nu = d/2 - 1$, $\cos \psi_{\Omega'' \Omega'} = \mathbf{r''} \cdot \mathbf{r}'$ (with $\mathbf{r} = r/r$), and the addition theorem for hyperspherical harmonics provides the expression in terms of Gegenbauer polynomials $C_{l}^{(\nu)}(x)$. For the important case of central potentials, in Eqs. (39)–(40), the radial propagator $K_{l+\nu}(r'', r'; T)$ is independent of the angular coordinates and quantum numbers $m$. The chosen normalization of radial prefactors is such that the radial propagator $K_{l+\nu}(r'', r'; T)$ satisfies the composition property

$$K_{l+\nu}(r'', r'; t'' - t') = \int_{0}^{\infty} dr K_{l+\nu}(r'', r; t'' - t)K_{l+\nu}(r, r'; t - t') .$$

(41)

For the radial propagator associated with $G$ as generalized Hamiltonian with effective time $\tau$, the notation $K^{(G)}_{l+\nu}(r'', r'; T)$ will be used for the remainder of the paper.
An explicit expression for the radial propagator in Eqs. (39)–(40) can be derived in the time-sliced path integral. In short, for a time lattice $t_j = t' + j\epsilon$, for the time interval $T = t'' - t'$ corresponding to the end points $r_0 \equiv r'$ and $r_N \equiv r''$; in this lattice, $\epsilon = T/N$, with $j = 0, \cdots, N$, with $t_0 \equiv t'$ and $t_N \equiv t''$, such that $r_j = r(t_j)$. Then, with the restriction to the half-line $r(t) \geq 0$, the radial propagator admits a formal continuum-limit representation [1–3, 60, 61]

$$K_{l+\nu}(r'', r'; T) = \int D\mathbf{r}(t) \ w_{l+\nu}[r^2] \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} dt \left[ \frac{M}{2} \dot{r}^2 - V(r) \right] \right\}, \quad (42)$$

in terms of the usual one-dimensional path-integral measure $D\mathbf{r}(t)$ (as in Cartesian coordinates), but with a nontrivial radial functional weight $w_{l+\nu}[r^2] = \lim_{N\to\infty} w_{l+\nu}^{(N)}[r^2]$ [see Eq. (B5)], which involves Bessel functions of order $l + \nu$. This has been called a Besselian path integral [1, 2].

The general expressions for Besselian path integrals are given in Appendix B, including statements on the associated property of interdimensional dependence [62, 63] and the theorem for the insertion of inverse square potential terms as generalized angular momenta [61, 63, 64]: any extra inverse square potential term added to $V(r)$ gets combined with the centrifugal potential and leads to an effective angular momentum as a result of asymptotic recombination. More precisely:

Given a potential $\tilde{V}(r) = V(r) + (\hbar^2/2M)g r^{-2}$, the propagator $\tilde{K}_{l+\nu} \equiv K_{l+\nu}[\tilde{V}]$ associated with a potential $\tilde{V}(r)$ in Eq. (42) is equivalent to the propagator $K_{\mu}[V]$ with the reduced potential $V(r)$ and an effective, typically non-integer, angular momentum variable

$$\mu \equiv l_{\text{eff}} + \nu = \sqrt{(l + \nu)^2 + g}. \quad (43)$$

In short,

$$\tilde{K}_{l+\nu} \equiv K_{l+\nu}[\tilde{V}] = K_{\mu}[V]. \quad (44)$$

Thus, this procedure expresses a practical rule for the insertion of inverse square potential terms as generalized angular momenta by the straightforward analytic continuation (43).
B. Propagator for Generalized Radial Harmonic Oscillator

The derivation of the general path integral for the radial propagator of the \(d\)-dimensional harmonic oscillator involves explicit use of the radial path integral (42). With the inclusion of an additional inverse square potential, i.e., for a Hamiltonian \(\tilde{H}(r, p)\) with an effective potential of the form (28), the propagator takes the form

\[
K_{l+\nu}^{(RHO)}(r'', r'; T) = \frac{M\omega}{i\hbar \sin \omega T} \sqrt{r'r''} \exp \left[ \frac{iM\omega}{2\hbar} \left( r'^2 + r''^2 \right) \cot \omega T \right] I_\mu \left( \frac{M\omega r'r''}{i\hbar \sin \omega T} \right), \tag{45}
\]

with a conformal parameter index

\[
\mu = \sqrt{g + (l + \nu)^2}. \tag{46}
\]

The path-integral result (45) for the “radial harmonic oscillator” was first derived by Peak and Inomata [64], and has a broad range of applications; it is established by using Weber’s second exponential integral for Bessel functions [65] combined with appropriate recursion relations [1–3, 64]. This is briefly reviewed and discussed in Appendix B. Incidentally, while Eq. (45) refers to the problem with an inverse square potential, i.e., of the form (44), we will keep the notation without the tilde for the sake of simplicity.

We will apply the path integral (45) to the characterization of the spectral properties of the generator \(G\) of Eqs. (25)–(26), where \(\tilde{H}_G\), given in Eq. (26), is equivalent to Eqs. (28)–(29). This applies to the operator \(H\) as well as to \(R\) and \(S' = -S\), defined in Eqs. (11) and (12), and related to the generic Hamiltonian of Eq. (28) with the assignments of Eqs. (31) and (33). Moreover, from the analysis of Eqs. (34)–(36), these assignments allow for a thorough characterization of the three operator classes: parabolic, elliptic, and hyperbolic. In this generic framework, the CQM frequency \(\omega\) to use in Eq. (45) is given through Eq. (29), and \(r \equiv q\) is the configuration variable of Eq. (20).

Additionally, for some of the applications below, we will compute the path integrals with appropriate analytic continuations. As will be further analyzed in Sec. VI in Eqs. (31)–(33), the operator \(S\) is obtained from \(R\) by an analytic continuation \(\omega \to -i\omega\) that completely changes the nature of the spectrum (both \(\omega \to \mp i\omega\) are equally valid, but the negative sign has operational advantages discussed in Sec. VI). Another such extension is the commonly used continuation to Euclidean time; in this context, for the propagator \(K_{l+\nu}(r'', r'; T)\), the following formal replacement is made: \(T \to -iT\). For the propagator of Eq. (45), the
corresponding Euclidean-time path integral reads

$$K_{l+\nu}^{(RHO)}(r'', r'; -iT) = \frac{M\omega}{\hbar \sinh \omega T} \sqrt{r''} \exp \left[ -\frac{M\omega}{2\hbar} (r'^2 + r''^2) \coth \omega T \right] I_{\mu} \left( \frac{M\omega r' r''}{\hbar \sinh \omega T} \right).$$

(47)

It should be stressed that the propagators (45)–(47) correspond specifically to the Hamiltonian $\tilde{H}$ of the form of Eq. (28); for CQM, Eq. (25) provides the connection with the generic Hamiltonian $H_G$ associated with the generator $G$ (with possibly an extra sign $\sigma$).

In the next sections, for our general description of the generators $G$, we will use the notations $K_{l+\nu}^{(ell)}$, $K_{l+\nu}^{(par)}$, and $K_{l+\nu}^{(hyp)}$ for the generic propagators of the three classes of generators.

IV. PATH-INTEGRAL SPECTRAL ANALYSIS OF THE GENERIC ELLIPTIC (R-LIKE) GENERATORS

In this section, we analyze the propagators for generic elliptic generators, including the prototypical operator $R$. The analysis of the spectral decomposition is straightforward for these operators, as it consists of a purely discrete spectrum.

A. Elliptic Generators: Properties and Path-Integral Propagator

From Eq. (31), it is clear that $R$ can be viewed as an analog harmonic oscillator with an inverse square potential; thus, the propagator is given in Eq. (45). Moreover, a generic elliptic operator $G$, with a discriminant $\Delta < 0$, inherits the basic spectral properties of $R$, according to Eq. (34), i.e., $G \approx \sigma \sqrt{|\Delta|} R$. Thus, it follows that the eigenstates of an elliptic operator $G$ are the same as those of $R$, with a spectrum of eigenvalues $\hbar g$ such that

$$g = \sigma \sqrt{|\Delta|} r$$

(48)

is rescaled from the spectrum $\hbar r$ of $R$. In addition, the functional form of the elliptic-generator propagator

$$K_{l+\nu}^{(ell)}(r'', r'; T) = K_{l+\nu}^{(RHO)}(r'', r'; T),$$

(49)

where $K_{l+\nu}^{(RHO)}$ is given in Eq. (45), applies to this entire family, with a CQM frequency $\omega = \sqrt{|\Delta|}/2$, according to Eq. (29). We can expect the spectrum to be discrete, and we will confirm this below using the propagator.
B. Elliptic Generators: Spectral Analysis

We will proceed as follows. A systematic technique to derive the eigenvalues and eigenvectors of an operator $A$ is the spectral decomposition $f(A) = \sum_n f(a_n)P_n$, where $a_n$ are its eigenvalues, $f$ is a function of the operator, and $P_n = |n\rangle \langle n|$ is the orthogonal projector—this is for a discrete spectrum, with similar expressions with integrals to include continuous parts of the spectrum. The most basic expansion of this kind in the path-integral approach is provided by the propagator itself, due to its basic definition (37) in terms of the time evolution operator. Thus, for operators with a discrete spectrum, like is the case for $R$ and all the elliptic generators, the propagator admits a spectral decomposition in the form of a series expansion,

$$K_{l+\nu}(r'', r'; T) = \sum_n e^{-i\tilde{E}_G,n T/\hbar} U_{n,l}(r'') U^*_{n,l}(r').$$

Therefore, Eq. (50) provides the expansion for the radial propagator in terms of reduced radial wave functions $U_{n,l}(r)$, and a similar expansion involves the $d$-dimensional wave functions $\psi_{n,l,m}(r)$ with the full-fledged propagator of Eq. (39). We will rewrite the spectral series in the Euclidean-time framework with the analytic continuation $T \to -iT$, so that

$$K_{l+\nu}(r'', r'; -iT) = \sum_n e^{-E_{l+\nu} T/\hbar} U_{n,l}(r'') U^*_{n,l}(r'),$$

This form of the spectral expansion provides better convergence properties and permits an unambiguous identification of the corresponding eigenstates and eigenvalues.

The spectral decomposition (50), with $T \to -iT$, can be made explicit with the bi-linear generating function of associated Laguerre polynomials known as the Hille-Hardy formula (pp. 189-190 in Ref. [59]),

$$\frac{1}{(xyz)^{\mu/2}(1-z)} \exp \left[ -z \frac{(x+y)}{(1-z)} \right] I_{\mu} \left( \frac{2\sqrt{xyz}}{1-z} \right) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\mu+1)} L^{(\mu)}_n(x) L^{(\mu)}_n(y) z^n,$$

with $|z| < 1$. The definition of normalization of the generalized Laguerre polynomials used in Eq. (51) is that of Ref. [59], corresponding to its relation $L^{(\mu)}_n(x) = \binom{n+\mu}{n} {_1F_1}(-n, \mu+1; x)$ to the confluent hypergeometric function. Then, with the substitutions $z = e^{-2\omega T}$, $x = M\omega r'^2/\hbar$, $y = M\omega r''^2/\hbar$, and rewriting the exponent in the exponential with $2z/(1-z) = (1+z)/(1-z) - 1 = \coth \omega T - 1$, the propagator $K^{(ell)}_{l+\nu}(r'', r'; T)$ can be recast in the form of Eq. (50), with $T \to -iT$. By direct inspection, the eigenvalues $\tilde{E}_{G,n}$ corresponding to $\tilde{H}_G$ are of the form $\tilde{E}_{G,n} = \hbar \omega (1 + \mu + 2n)$; thus, from Eqs. (27), (29), and (48), and using the
eigenvalues $\hbar r_n$ of $R$,

$$E_{G,n} = \sigma \tilde{E}_{G,n} = \hbar \sigma_n = \sigma \sqrt{\Delta} \, r_n = \sigma \hbar \omega (1 + \mu + 2n) \, , \quad (52)$$

where

$$r_n = r_0 + n \, , \quad r_0 = \frac{1}{2} (1 + \mu) \, , \quad (53)$$

and the parameter $\mu$ is specified by Eq. (46); in Ref. [5] this is explicitly written in the form $\mu = 2r_0 - 1$. The spectrum (53) is the discrete series $D^+_0$ bounded below of the unitary, irreducible representations of the group $SO(2,1)$ [58]. Furthermore, their common eigenstates $U_{n,l}(r)$ are identified as

$$U_{n,l}(r) = \sqrt{\frac{2 \Gamma(n+1)}{\Gamma(1+\mu+n)}} \frac{1}{r^{1/2}} \left( \sqrt{\frac{M\omega}{\hbar}} r \right)^{\mu+1} \exp \left( -\frac{M\omega}{2\hbar} r^2 \right) I_n^{(\mu)} \left( \frac{M\omega}{\hbar} r^2 \right) \, . \quad (54)$$

### C. Elliptic Generators: Conclusions

These results agree with the corresponding expressions in Sec. 4 of the original dAFF-model of Ref. [5] for $d = 1$. It should be noted that, for $d > 1$, the value of $r_0$ is also dependent on the angular momentum $l$. The $d$-dimensional wave functions $\psi_n(r)$ can be assembled by considering the corresponding spectral expansion of the propagator of Eq. (39), whence

$$\psi_{n,l,m}(r) = r^{-(d-1)/2} U_{n,l}(r) Y_{lm}(\Omega) \, , \quad (55)$$

which only changes the factor $r^{-1/2}$ in Eq. (54) to $r^{-d/2}$, and inserts the angular dependence.

As expected, the derived results can be interpreted as a rescaled, ISP-extended version of those for the standard isotropic oscillator in quantum mechanics. For example, the energy levels of the isotropic oscillator are given by $E_n = \hbar \omega (N + d/2)$, with principal quantum number $N = 2n + l + \nu + 1$ (and $n$ being the radial quantum number), which is extended with the rule (43), so that $\tilde{E}_n = \hbar \omega (2n + \mu + 1)$, as in Eq. (52). A similar replacement (43) for the wave functions confirms this interpretation for Eq. (54).
V. PATH-INTEGRAL SPECTRAL ANALYSIS OF THE GENERIC PARABOLIC (H-LIKE) GENERATORS AND FOURIER METHOD FOR CONTINUOUS-SPECTRUM OPERATORS

In this section, we analyze the propagators for generic parabolic generators, including the prototypical operator $H$. The case of $H$ (and parabolic operators by extension) is particularly interesting because it describes the initial Hamiltonian, and a number of approaches can be successfully applied, including a novel technique to derive the eigenfunctions of operators with a continuum spectrum. In addition, this novel technique is further expanded, with a discussion of its general properties and its relationship to Green’s functions. (This topic is then used in the next section to study the spectral properties of the hyperbolic generators, including $S$.)

A. Spectral Properties of the Hamiltonian $H$ and the Generic Parabolic Generators

A generic parabolic operator $G$, with a discriminant $\Delta = 0$, inherits the basic spectral properties of the original Hamiltonian $H$, according to Eq. (35), i.e., $G \approx \sigma H$. Its eigenstates are the same as those of $H$, with a rescaled spectrum of eigenvalues $\hbar \sigma$ such that $g = \sigma E/\hbar$ is determined from the spectrum $E$ of $H$. In particular, the functional form of the propagator $K^{(H)}_{l+\nu}$ applies without any modifications to this family of parabolic generators, with a CQM frequency $\omega = 0$.

Clearly, the Hamiltonian operator $H$ corresponds to a pure inverse square potential. The corresponding path integral could be computed separately (see Appendix B), but it is straightforward to derive it by taking the limit $\omega \to 0$ of the generalized harmonic oscillator propagator of Eq. (45). Thus, the propagator corresponding to the operator $H$ and other members of its parabolic family is given by

$$K^{(\text{par})}_{l+\nu}(r'', r'; T) = K^{(H)}_{l+\nu}(r'', r'; T) = \lim_{\omega \to 0} K^{(\text{RHO})}_{l+\nu}(r'', r'; T)$$

$$= \frac{M}{i\hbar T} \sqrt{r'' r'} \exp \left[ \frac{iM}{2\hbar T} (r'^2 + r''^2) \right] I_{\mu} \left( \frac{Mr''}{i\hbar T} \right).$$

The spectral decomposition and spectral properties of this Hamiltonian model can be analyzed in several ways. We will consider the following 3 methods: (i) the zero-frequency
limit of the spectral decomposition of the radial harmonic oscillator, using the Mehler-
Heine formula for generalized Laguerre polynomials; (ii) direct evaluation of the spectral
decomposition from the propagator (45) using Weber’s second exponential integral; (iii)
computation of the eigenfunctions with the Fourier transform of the propagator (45), using
an integral representation of the product of Bessel functions.

1. Parabolic Generators: Zero-Frequency Limit of the Propagator—Mehler-Heine Formula

The first method uses the Mehler-Heine limit formula for the Laguerre polynomials (p.
191 in Ref. [59]),
\[ \lim_{n \to \infty} n^{-\mu} L_n^{(\mu)} \left( \frac{x}{n} \right) = x^{-\mu/2} J_{\mu}(2 \sqrt{x}) . \] (58)
Starting from the spectral decomposition expansion (50), with Eqs. (52) and (54), and \( \sigma = 1 \),
we can turn the series into an integral by realizing that the eigenvalue level spacing \( E_{n+1} - E_n = 2 \hbar \omega \)
asymptotically approaches zero. Thus, the energy values \( E_n \) approach the zero limit
except when \( n \to \infty \), so that the actual energy levels \( E \) of the continuum satisfy
the asymptotic approximate replacement \( E \approx 2 \hbar \omega n \). Correspondingly, the
asymptotic conversion into an integral follows the rule \( \sum_n \approx \int dE/(2 \hbar \omega) \),
which amounts to \( \sum_n 1/n \approx \int dE/E \). In addition, defining \( x/n = Mr^2 \omega/\hbar \) to match the argument of the
Laguerre polynomials, this gives \( x = k^2 r^2/4 \), with \( k \) such that \( E = \hbar^2 k^2/2M \), and the
argument of the Bessel function in Eq. (58) becomes \( 2 \sqrt{x} = kr \). Then, approximating
the eigenfunction coefficient in Eq. (54) as \( c_{n,1} = \sqrt{2 \Gamma(n+1)/\Gamma(1+\mu+n)} \approx \sqrt{2}/n^{\mu} \), the
asymptotic discrete eigenfunctions become
\[ U_{n,1}(r) \approx \sqrt{2} n^{-1/2} (x/r)^{1/2} J_{\mu}(kr) , \]
where \( x/r = k^2 r/4 \). Thus, Eq. (50) turns into
\[ K_{l+\nu}^{(\text{par})}(r'', r', T) \approx \int_{0}^{\infty} dE \frac{E}{\hbar} n e^{-iET/\hbar} U_{n,1}(r'') U_{n,1}^*(r') \]
\[ \approx \int_{0}^{\infty} dE e^{-iET/\hbar} \left[ \sqrt{\frac{M}{\hbar^2}} \sqrt{\tau} J_{\mu}(kr'') \right] \left[ \sqrt{\frac{M}{\hbar^2}} \sqrt{\tau} J_{\mu}(kr') \right] , \] (59)
which takes the form of the continuous spectral decomposition
\[ K_{l+\nu}^{(\text{par})}(r'', r'; T) = \int_{0}^{\infty} dE e^{-iET/\hbar} U_{E,1}(r'') U_{E,1}^*(r') , \] (60)
where the continuous energy eigenfunctions are given by

\[ U_{E,l}(r) = \sqrt{\frac{M}{\hbar^2}} \sqrt{\tau} J_\mu(kr) , \quad (61) \]

with \( k = \sqrt{2ME/\hbar^2} \). By the normalized form of the expansion (60), the eigenstates (61) are Dirac-normalized with respect to the variable \( E \). Therefore, through the zero-frequency limit, we have derived the continuous analog of the spectral decomposition series (50), for a spectrum with values restricted to the half-energy line \( E \in [0, \infty) \) and eigenstates (61).

2. Parabolic Generators: Explicit Continuous Spectral Decomposition of the Propagator—Weber’s Second Exponential Integral

The second method involves writing the spectral decomposition directly in its continuous spectral form (60), which will be expressed in the Euclidean-time version,

\[ K^{(\text{par})}_{l+\nu}(r'',r';-iT) = \int_0^\infty dE \exp\left(-\frac{ET}{\hbar}\right) U_{E,l}(r'') U^{*}_{E,l}(r') , \quad (62) \]

where \( K^{(\text{par})}_{l+\nu} \) is given by Eq. (57), and the corresponding real-time propagator involves the replacement \( T \to iT \). No additional assumptions are needed as the propagator is expanded in this form by construction from a well-established integral identity for Bessel functions (Sec. 13.31 of Ref. [65] and 10.22.67 of Ref. [66]): Weber’s second exponential integral (B11). With the substitutions \( E = x^2, T/\hbar = c^2, \sqrt{2M} r'/\hbar = a, \sqrt{2M} r''/\hbar = b \) in Eq. (B11), and insertion of an additional factor \( 2M\sqrt{r'r''}/\hbar^2 \), the identity becomes straightforwardly the desired Eq. (62), with the same continuous energy eigenfunctions as displayed in Eq. (61).

3. Parabolic Generators: Spectral Decomposition via a Fourier Method

The third method is based on a simple restatement of the spectral expansion of an operator with a continuous spectrum. From the continuous spectral decomposition (60), the inverse Fourier transform with respect to the variable \( E \) converts the propagator \( K^{(\text{par})}_{l+\nu}(r'',r';T) \) into the wave-function product

\[ U_{E,l}(r'') U^{*}_{E,l}(r') = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dT \exp\left(\frac{iET}{\hbar}\right) K^{(\text{par})}_{l+\nu}(r'',r';T) . \quad (63) \]
This yields an explicit Fourier-integral representation of the spectral-decomposition wavefunction product. Evaluation of this spectral integral can be established from the integral representation (C1) of the product of Bessel functions (Sec. 13.7 of Ref. [65]). In practice, we are interested in the limit \( c \to 0^+ \), for which the integral (C1), with \( s = c + 2it \), becomes

\[
J_{\mu}(z') J_{\mu}(z'') = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp(it) \exp \left[ \frac{i(z'^2 + z''^2)}{4t} \right] I_{\mu} \left( \frac{z'z''}{2it} \right) \frac{dt}{t}.
\]

In Eq. (64), the following substitutions are made on the right-hand side: \( z' = kr', z'' = kr'' \), and \( t = ET/\hbar \), with the time interval \( T \) as integration variable. Then, the exponential factors become \( \exp(i(z'^2 + z''^2)/4t) = \exp(iM(r'^2 + r''^2)/(2\hbar T)) \), with the argument of the Bessel function being \( z'z''/2it = Mr'r''/i\hbar T \). This shows that the integrand of Eq. (C1) is proportional to the propagator \( K_{l+\nu}^{(par)} \) of Eq. (57) after inserting an additional factor \( M \sqrt{r'r''}/\hbar^2 \), so that

\[
\left[ \sqrt{\frac{M}{\hbar^2}} \sqrt{r} J_{\nu}(kr') \right] \left[ \sqrt{\frac{M}{\hbar^2}} \sqrt{r} J_{\nu}(kr'') \right] = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dT \exp \left( \frac{iET}{\hbar} \right) K_{l+\nu}^{(par)} (r', r'', T) .
\]

Comparison of Eq. (65) with Eq. (63) shows that the energy eigenfunctions are given by Eq. (61).

Incidentally, the explicit form of the wave function product in (63) can also be established via the zero-frequency limit of the radial harmonic oscillator propagator. In this limit, the generalized Laguerre polynomials turn into Bessel functions, and the coefficients of the Fourier series (50) (where \( E_n \) is linear in \( n \)) turn into the required Fourier integral.

4. Parabolic Generators: Conclusions

In conclusion, we have shown by several techniques that the Dirac-normalized continuous eigenstates are given by Eq. (61). All the methods yield the same result, with a purely continuous spectrum restricted to the non-negative energy half-line \( E \in [0, \infty) \). This result agrees with the value in Eq. (A.18) of the original dAFF-model of Ref. [3] for \( d = 1 \), and can be interpreted with the physical insights known for analog elementary problems (e.g., reduces to the free particle when \( g = 0 \)). For the \( d \)-dimensional model, the radial wave function is \( U_{E,l}(r)/r^{(d-1)/2} \).

Interestingly, the derivations of this section show that there are simple connections between the known mathematical theorems used. Specifically, Weber’s second exponential
integral has the following two properties: (i) it is a restatement of the continuous limit of
the Hille-Hardy formula; (ii) it can be interpreted as a Fourier transform of the integral rep-
resentation (C1). To our knowledge, these statements are not spelled out in the mathematics
literature.

A number of final remarks are in order. The first method shows consistency with the more
general result for elliptic operators; while these operators are of a different nature, they are
still related by a continuous transformation with the parameter \( \omega \) enforcing the asymptotic
transition. The second method is constructive and explicitly yields the spectral decompo-
sition; however, this requires guessing or finding the correct expression that mimics this
resolution. By contrast, the third method inverts the process and introduces a systematic
calculational procedure whereby the wave functions are obtained by a direct computation,
provided that the integrals can be performed. In the next subsection, we elaborate on this
novel result, stating it in more general terms and addressing its basic properties.

**B. Fourier Method for Operators with a Purely Continuous Spectrum and Rela-
tion to Green’s Functions**

1. **Fourier Method**

For any operator \( \tilde{H} \) with a purely continuous spectrum, the spectral decomposition takes
the form

\[
K_{l+\nu}(r'', r'; T) = \int_S \delta E e^{-iET/\hbar} U_{E,l}(r'') U^*_{E,l}(r'),
\]

where the “energy” values \( E \equiv \tilde{E} \) of \( \tilde{H} \) extend over the set \( S \). We assume that \( \tilde{H} \) has a purely
continuous spectrum, and can be regarded as a sort of Hamiltonian operator associated with
an effective time evolution, i.e., defining \( K_{(d)}(r'', r'; T) = \langle r'' | e^{-i\tilde{H}T} | r' \rangle \) and extracting the
radial counterpart according to the rules of Sec. III.

Equation (66) has the form of a Fourier transform restricted to the set \( S \). Its inverse
Fourier transform can be obtained by performing the integral with respect to the variable
\( T \in (-\infty, \infty) \), with the kernel \( \exp(iET/\hbar) \). This general inverse Fourier integral converts
the propagator \( K_{l+\nu}(r'', r'; T) \) into the wave-function product \( U_{E,l}(r'') U^*_{E,l}(r') \), i.e.,

\[
F(E; r'', r') \equiv U_{E,l}(r'') U^*_{E,l}(r') = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dT \exp \left( \frac{iET}{\hbar} \right) K_{l+\nu}(r'', r'; T),
\]
where the values of the physical energy on the real axis are restricted to the original set \( E \in S \). Indeed, this general theorem simply follows by performing the inverse Fourier integral on the right-hand side of Eq. (66) written in terms of a variable \( E' \), with the familiar auxiliary identity \( \int_{-\infty}^{\infty} dT e^{i(E-E')T/\hbar} = 2\pi\hbar\delta(E-E) \). Moreover, Eqs. (66) and (67) provide the broader context and justification for the results of the previous Subsec. V A 3, which were specific to the proper Hamiltonian operator \( H \) and parabolic operators.

This “Fourier method” is remarkably simple and elegant, but it has not been explicitly used in the literature. This is partly due to the fact that operators with a purely continuous spectrum are not common (other than the trivial case of a free particle), and partly because it is customary to use Green’s functions techniques, which are related but not identical to the result of Eq. (67).

2. Green’s Functions: Definitions and Relation to the Fourier Method

In general, the retarded/advanced Green’s functions or resolvents associated with \( \hat{H} \) are defined from

\[
G_{l+\nu}^{(\pm)}(r'', r'; E) = \pm \frac{1}{i\hbar} \int_{-\infty}^{\infty} dT \theta(\pm T) \exp \left( \frac{iET}{\hbar} \right) K_{l+\nu}(r'', r'; T),
\]

where \( \theta \) stands for the Heaviside function, and with the replacement \( E \to E \pm i\epsilon \) that guarantees convergence. Unlike Eq. (67), the Green’s function technique does not give a direct result for the wave function product, though this product can be extracted via an additional step as the residue of the energy poles.

The definitions of Eq. (68) correspond to the Fourier transform of the (retarded/advanced) Green operators

\[
G^{(\pm)}(T) = \theta(\pm T) e^{-i\hat{H}T/\hbar}.
\]

This should be contrasted with the propagator counterparts that involve the time evolution operator \( U = e^{-i\hat{H}T/\hbar} \) of Eq. (37) without the Heaviside cutoff. Equation (69) yields the Fourier-transformed or energy Green operators

\[
G^{(\pm)}(E) = \left( E - \hat{H} \pm i\epsilon \right)^{-1},
\]

where the \( i\epsilon \) prescription provides convergence for each case. The corresponding coordinate-space representations give the associated time and energy Green’s functions \( G_{(d)}^{(\pm)}(r'', r'; T) \)
and $G_{(d)}^{(±)}(r'', r'; E)$; in particular,

$$G_{(d)}^{(±)}(r'', r'; E) = \left\langle r'' \left| (E - \hat{H} \pm i\epsilon)^{-1} \right| r' \right\rangle .$$  \hspace{1cm} (71)

Equations (69)–(70) can be applied either to the full-fledged multidimensional quantities, or their reduced radial counterparts, which follow from the usual hyperspherical expansion; specifically, the radial energy Green’s functions $G^{(±)}_{l+\nu}(r'', r'; E)$ in Eq. (68) are defined from

$$G_{(d)}^{(±)}(r'', r'; E) = (r'' r')^{- (D-1)/2} \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} Y_{lm}(\Omega'') Y_{lm}^*(\Omega') G^{(±)}_{l+\nu}(r'', r'; E) .$$  \hspace{1cm} (72)

Moreover, with the usual distributional expansion $A^{-1} = \mathcal{P}(A^{-1}) \mp i\pi \delta(A)$ (where $\mathcal{P}$ is the Cauchy principal value), the ratio $- (G^{(+)}(E) - G^{(-)}(E)) \mp (2\pi i) = \delta(E - H)$ is a state density operator. Thus, for a continuous-energy spectral expansion,

$$F(r'', r'; E) \equiv U_{E,\nu}^E (r'') U_{E,\nu}^* (r') = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dT \exp \left( \frac{iET}{\hbar} \right) \frac{K_{t+\nu}(r'', r'; T)}{2\pi i} \left[ G^{(+)\nu}_{t+\nu}(r'', r'; E) - G^{(-\nu}_{t+\nu}(r'', r'; E) \right] ,$$  \hspace{1cm} (73)

where $\text{disc}[G_{t+\nu}(E)] = G^{(+)\nu}_{t+\nu}(E) - G^{(-\nu}_{t+\nu}(E)$ measures the discontinuity of the Green’s functions across the branch cut in the complex energy plane. In short, Equation (73) summarizes the connection between the straightforward Fourier transform of the propagator and the Green’s functions; for the former, the time domain of the Fourier integrals involves the whole time axis, while for the Green’s functions, the time domain is restricted to the positive and negative half-axes.

3. Green’s Functions—Example: Parabolic Generators

As an example of the relations above, we can revisit the conformal operator $H$ and its parabolic class. The whole time-axis Fourier integral gives directly the wave function product (65), which we derived from Eq. (C1). If we instead use the related integrals of Eq. (C2), the Green’s functions become

$$G^{(±)}_{t+\nu}(r'', r'; E) = \mp \pi i \left( M / \hbar^2 \right) \sqrt{r'' r'} J_\nu (kr_<) H^{(1,2)}_\mu (kr_<) ,$$  \hspace{1cm} (74)

whence

$$G^{(±)}_{t+\nu}(r'', r'; E) = - \pi i \left( M / \hbar^2 \right) \sqrt{r'' r'} J_\nu (kr_<) \left[ H^{(1)}_\mu (kr_<) + H^{(2)}_\mu (kr_<) \right] .$$  \hspace{1cm} (74)
which, with $H^{(1)}_\mu(z) + H^{(2)}_\mu(z) = 2J_\mu(z)$, gives $-2\pi i$ times the wave function product displayed in Eq. (65), in agreement with the relation (73).

In the next section, we will apply the technique of Eq. (67) to derive the spectral decomposition associated with the more involved conformal operator $S$, and will display the corresponding network of relations developed here.

VI. PATH-INTEGRAL SPECTRAL ANALYSIS OF THE GENERIC HYPERBOLIC (S-LIKE) GENERATORS

A generic hyperbolic operator $G$, with a discriminant $\Delta > 0$, inherits the basic spectral properties of $S$ or $S'$, according to Eq. (36), i.e., $G \approx \sigma \sqrt{\Delta} S'$. Thus, we can proceed as for the other families of operators: the eigenstates of a hyperbolic operator $G$ are the same as those of $S'$, with a spectrum of eigenvalues $\hbar g$ such that

$$g = \sigma \sqrt{\Delta} s'$$

is rescaled from the spectrum $\hbar s'$ of $S'$.

A. Hyperbolic Generators: Path-Integral Propagator

The behavior of the hyperbolic generators falls within the analog model of the generic extended radial harmonic oscillator Hamiltonian $\tilde{H}$ of Eq. (28). Therefore, the functional form of the propagator $K^{(hyp)}_{l+\nu}$ applies to the whole family of hyperbolic generators, with a CQM frequency $\omega = -i\sqrt{\Delta}/2$, from Eq. (39). The imaginary frequency can be obtained with the prescription to analytically continue the potential according to $\omega \to -i\omega$, with $\omega$ a real “frequency.” The result of the analytic continuation $\omega \to -i\omega$ is to generate an inverted harmonic oscillator that overlaps with the inverse square potential. By the form of this potential, we can predict that all values of the “energy” are possible, forming a continuum from minus infinity to plus infinity. This physical statement will be verified with the path-integral calculation that follows. It should be noted that systems with such a continuous spectrum, unbounded from below and above, are not common and represent idealized models (as actual physical systems exhibit energy bounds). But the treatment of such systems with a general path-integral technique is of theoretical and practical interest,
and the results of the previous subsection can be applied directly. For example, the operator \( S' \) (being an inverted radial oscillator) is related to the one-dimensional inverted harmonic oscillator, which is a system that is related to a variety of physical problems of current relevance \cite{67, 68}.

By comparison with the propagator equation (45), which is completely general for complex values of the parameter \( \omega \), and with the replacement \( \omega \rightarrow -i\omega \), the propagator associated with \( S' \) and all hyperbolic operators is given by

\[
K_{l+\nu}^{(\text{hyp})}(r'', r'; T) = \frac{M\omega}{i\hbar \sinh(\omega T)} \sqrt{r' r''} \exp \left[ \frac{iM\omega}{2\hbar} (r'^2 + r''^2) \coth(\omega T) \right] I_{\mu} \left( \frac{M\omega r' r''}{i\hbar \sinh(\omega T)} \right).
\] (76)

Incidentally, either extension \( \omega \rightarrow \pm i\omega \) is equally acceptable. This can be seen directly from the Hamiltonian, or explicitly from the propagator. However, the choice \( \omega \rightarrow -i\omega \) has the operational advantage that it preserves the boundary condition at infinity corresponding to retarded Green’s functions solutions (either bound states or outgoing waves). This statement can be verified in an explorative manner by a simple WKB evaluation, which is asymptotically exact and yields a solution function \( \mathcal{U} \sim r^{-1/2 + 2\kappa} \exp(-M\omega r^2/2\hbar) \xrightarrow{(\omega \rightarrow -i\omega)} \mathcal{U} \sim r^{-1/2 + 2\kappa} \exp(iM\omega r^2/\hbar) \), where \( \kappa = \bar{E}_G/(2\hbar\omega) \) (see Appendix D); and it is further confirmed by the exact solution in terms of Whittaker functions (see final results in this subsection). The advantage of this rule is that one can directly extrapolate by analytic continuation the correct solutions, including Green’s functions, from one sector of the theory to another (for example, as shown below, from the inverted to the regular harmonic oscillator).

**B. Hyperbolic Generators: Spectral Analysis**

The spectral decomposition can be obtained from the Fourier integral \cite{67}, which, with the substitution \( \zeta = \omega T \) takes the explicit form

\[
\mathcal{U}_{E,i}(r'') \mathcal{U}_{E,i}(r') = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dT \exp \left( \frac{iET}{\hbar} \right) K_{l+\nu}^{(\text{hyp})}(r'', r'; T)
\]

\[
= \frac{1}{2\pi i} \frac{M}{\hbar^2} \sqrt{r' r''} \frac{1}{i} \int_{-\infty}^{\infty} \frac{d\zeta}{\sinh(2\kappa \zeta)} \exp(2i\kappa \zeta) \exp(i\beta \coth \zeta) I_{\mu} \left( \frac{\alpha}{i \sinh(\zeta)} \right),
\] (77)
where
\[ \kappa = \frac{\tilde{E}}{2\hbar\omega}, \quad \beta = \frac{1}{2} (x' + x'') \quad \alpha = \sqrt{x'x''}, \]
with \( x' \equiv \tilde{r}'^2 = \frac{M\omega}{\hbar} r'^2 \) and \( x'' \equiv \tilde{r}''^2 = \frac{M\omega}{\hbar} r''^2 \).

In all the ensuing equations below, the dimensionless coordinate \( \tilde{r} = \sqrt{M\omega/\hbar} r \) is used.

The integral \( \mathcal{I} \) in Eq. (77) can be evaluated by using the approach of Ref. [69] (Sec. 6.1). This involves developing integral representations for the product of two Whittaker functions \( M_{\kappa,\mu/2}(z) \) and \( W_{\kappa,\mu/2}(z) \) (in different combinations of \( M \) and \( W \) as well as function indices); the regularized Whittaker function \( \mathcal{M}_{\kappa,\mu/2}(z) = M_{\kappa,\mu/2}(z)/\Gamma(1+\mu) \) of Ref. [69] (that removes singular behavior at negative integer values of \( \mu \)) is also used in the equations below. In Appendix C we briefly review the basics of these functions, which are related to the confluent hypergeometric functions \( M(a,b,z) \) and \( U(a,b,z) \), and consider the operational procedure of Ref. [69] that defines the integral representations, with miscellaneous substitutions. For the calculation of \( \mathcal{I} \) in Eq. (77), we are going to use two such specific representations.

1. **Evaluation of the Spectral Decomposition of Hyperbolic Operators by the Fourier Method—Whole-Line Integral**

The first computation of \( \mathcal{I} \) is based on the direct evaluation of the integral for the entire real line according to the novel identity (C6), which implies
\[ \mathcal{I} = e^{\pi \kappa} \Gamma \left( \frac{1+\mu}{2} + i\kappa \right) \Gamma \left( \frac{1+\mu}{2} - i\kappa \right) \left( x'x'' \right)^{1/2} M_{\kappa,\mu/2}(ix'') M_{\kappa,\mu/2}(-ix'), \]
where the limit \( c \to 0 \) is taken, as in discussed Appendix C. This approach has the distinct advantage of being direct and straightforward [provided that Eq. (C6) is established]. Moreover, due to the symmetry of the propagator with respect to an exchange of \( r' \) and \( r'' \) [thus, an exchange of \( x' \) and \( x'' \), according to Eq. (78)], the products can be written in either equivalent form
\[ M_{\kappa,\mu/2}(ix'') M_{\kappa,\mu/2}(-ix') = M_{\kappa,\mu/2}(ix'') M_{\kappa,\mu/2}(-ix'), \]
which can also be explicitly verified via a double application of the Whittaker identity of Eq. (84) below.
2. Evaluation of the Spectral Decomposition of Hyperbolic Operators by the Fourier Method—Half-Line Integrals

There is a second computation of $I$, which uses an alternative (though related) representation from Ref. [69]: Eqs. (C7), and (C8), where the latter is another novel identity. This second approach, while more involved, provides a separate check and additional information, showing the different roles played by the positive and negative energy values, as well as their relationship to the Green’s functions. Specifically, in Eq. (77), the integral $I$ is split into the two contributions from the half-axes (of positive and negative times):

$$I = I_+ + I_-,$$

where $I_+$ and $I_-$ are defined by

$$I_{\pm} = \int_{L_{\pm}} \frac{d\zeta}{\sinh \zeta} \exp (2i\kappa \zeta) \exp (i\beta \coth \zeta) I_\mu \left( \frac{\alpha}{i \sinh \zeta} \right),$$

(81)

where $L_+ = [0, \infty)$ and $L_- = (-\infty, 0]$. The integrals (81) are explicitly functions of the given parameters: (i) the indices $\kappa$ and $\mu$; and (ii) the variables $\alpha$ and $\beta$, or alternatively $x'$ and $x''$. The reversal of the sign of $\zeta$ in the integrand of $L_-$ compared to $L_+$ implies the existence of straightforward symmetries with respect to the parameters of the integrand, such that $I_{\pm}$ are not independent but related by

$$I_- (\kappa, \mu; x', x'') = -I_+ (-\kappa, \mu; -x', -x''),$$

(82)

[or, by abuse of notation, also $I_- (\kappa, \mu; \alpha, \beta) = -I_+ (-\kappa, \mu; -\alpha, -\beta)$]. In Eq. (82), as well as all the equations considered in this paper, the final expressions should be evaluated with the principal values of the multivalued functions involved. With the results of Eq. (C8), we get

$$I_{\pm} = \frac{i \Gamma_{\pm}}{\sqrt{x'x''}} W_{\pm i\kappa, \mu/2} (\mp ix_>) \mathcal{M}_{\pm i\kappa, \mu/2} (\mp ix_<),$$

(83)

with the shorthand $\Gamma_{\pm} = \Gamma ((1 + \mu)/2 \pm i\kappa)$. When adding the integrals $I = I_+ + I_-$, the following two relations are applied: (i) the semi-circuital analytic continuation

$$\mathcal{M}_{\lambda, \mu/2} (e^{\pm i z}) = e^{\pm (\mu+1)i/2} \mathcal{M}_{-\lambda, \mu/2} (z),$$

(84)

and (ii) the connection formula

$$\mathcal{M}_{\lambda, \mu/2} (z) = e^{\pm i \lambda z} \left[ \frac{1}{\Gamma_+} e^{\mp i(\mu+1)i/2} W_{\lambda, \mu/2} (z) + \frac{1}{\Gamma_-} W_{-\lambda, \mu/2} (e^{\mp i z}) \right].$$

(85)

The Whittaker functions $P_{\lambda, \mu/2} (z) = \{ \mathcal{M}_{\lambda, \mu/2} (z), W_{\lambda, \mu/2} (z) \}$ have to be examined to guarantee results valid within their principal branches, where the branch cut is conventionally
taken as the negative real half-axis (thus, with the arguments $-\pi < \text{ph}(z) \leq \pi$). For the present calculation, this involves using the lower and upper signs of the identities (84) and (85) (with $\lambda = i\kappa$) respectively, as applied to $I = I_+ + I_+$ in that order, so that

$$I = i \frac{\Gamma_+ \Gamma_-}{\sqrt{x'x''}} \left[ \frac{1}{\Gamma_+} W_{ik,\mu/2}(-ix_>) M_{ik,\mu/2}(-ix<) + \frac{1}{\Gamma_-} W_{-ik,\mu/2}(ix>) M_{-ik,\mu/2}(ix<) \right] e^{-(\mu+1)i/2 M_{-ik,\mu/2}(ix<)} e^{\pi\kappa M_{ik,\mu/2}(-ix>) M_{-ik,\mu/2}(ix<)}, \quad (86)$$

which reduces to the same value as before, Eq. (79), for $x' > x''$; however, due to the symmetry of Eq. (80), this is also true for $x' < x''$, showing the equality of both results without restriction.

3. Evaluation of the Spectral Decomposition of Hyperbolic Operators—Green’s Functions

Finally, the integrals $I_\pm$ in Eq. (83) give the retarded/advanced Green’s functions

$$G_{i+\nu}(r'', r'; E) = \mp i \frac{\Gamma((1 + \mu)/2 \mp i\kappa)}{\Gamma(1 + \mu)} \frac{1}{\sqrt{r'r''}} W_{\pm k,\mu/2}(\mp i\kappa^2) M_{\pm k,\mu/2}(\mp i\kappa^2), \quad (87)$$

where the jump in the Green’s functions according to Eq. (73) provides another, related proof of Eq. (77). Incidentally, if an analytic continuation back to real frequencies is enforced via $\omega \to i\omega$, i.e., $i\kappa \to \kappa$ [with sign reversal compared to the analytic continuation that gave Eq. (76)], the Green’s functions $G_{i+\nu}(r'', r'; E) = \mp (\hbar\omega)^{-1} \frac{1}{\Gamma_+} W_{\pm k,\mu/2}(\mp i\kappa^2) M_{\pm k,\mu/2}(\mp i\kappa^2)/\sqrt{r'r''}$ for the elliptic generators (including the operator $R$) or ordinary radial harmonic oscillator are obtained, a familiar result that also allows a rederivation of their spectrum, Eqs. (52)–(53). The critical difference in the spectral behaviors of hyperbolic generators (continuous operators) versus elliptic generators (discrete operators) arises from the asymptotic behavior of $M_{\pm k,\mu/2}(\mp i\kappa^2)$ versus $M_{\pm k,\mu/2}(\mp i\kappa^2)$ (both with $\kappa \in \mathbb{R}$), when $r \to \infty$, which forces the reduction of the latter to generalized Laguerre polynomials, with a discrete quantization of the spectrum.

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4. Spectral Decomposition of Hyperbolic Operators: Eigenvalues and Eigenfunctions—

Conclusions

As a consequence of Eqs. (77), (79) [or Eq. (86)], and (80), the final result of this calculation is the wave function product

$$U_{E,l}(r'')U_{E,l}^*(r') = \frac{1}{2\pi\hbar\omega}\Gamma_+e^{\pi\kappa/2}\frac{M_{i\kappa,\mu/2}(-i\tilde{r}'')}{\sqrt{r''}}$$

$$= \frac{1}{2\pi\hbar\omega}\Gamma_+e^{\pi\kappa/2}\frac{M_{-i\kappa,\mu/2}(i\tilde{r}'')}{\sqrt{r'}}$$

(88)

(89)

Again, the equivalence of Eqs. (88) and (89) is due to the symmetry of the propagator with respect to the end points, and enforced via the analytic continuation of Eq. (84).

Two important conclusions follow from this result.

(i) The spectrum is indeed a continuum, from minus to plus infinity, as the solution above applies to all such values of the effective energy $\tilde{E}_G \equiv \sigma E_G$, or [from Eqs. (36) and (78)] the parameter

$$\kappa = \frac{\sigma E_G}{2\hbar\omega} = \sigma\frac{g}{\sqrt{\Delta}} = \sigma' \in (-\infty, \infty)$$

(90)

which is a dimensionless value covering the whole real axis and equal to the first Whittaker index $\kappa$, and such that $\sigma' = \kappa$ for the eigenvalues $\hbar\sigma'$ of the operator $S' = -S$.

(ii) The wave functions can be read off from either one of Eqs. (88) or (89), by comparison with Eq. (67), and are given by

$$U_{E,l}(r) \equiv U_{\kappa,\mu}(r) = \frac{e^{\pi\kappa/2}}{\sqrt{2\pi\hbar\omega}}\frac{\Gamma((1 + \mu)/2 + i\kappa)}{\Gamma(1 + \mu)}\frac{M_{i\kappa,\mu/2}(-i\tilde{r}'')}{\sqrt{r''}}e^{i\chi}$$

$$= \frac{e^{\pi\kappa/2}}{\sqrt{2\pi\hbar\omega}}\frac{\Gamma((1 + \mu)/2 - i\kappa)}{\Gamma(1 + \mu)}\frac{M_{-i\kappa,\mu/2}(i\tilde{r}'')}{\sqrt{r'}}e^{i\chi'},$$

(91)

(92)

where we have reverted back to the ordinary Whittaker functions $M_{\pm i\kappa,\mu/2}$, and $\chi$ and $\chi'$ are arbitrary phase factors. The simplest choices are arguably $\chi = 0$ or $\chi' = 0$, but these values are undetermined by the nature of the products (88)–(89); in particular, the choice of gamma function factors is also arbitrary, as $\Gamma_+ = \Gamma^*_-$. It should be noted that, in the derivation above, $M_{-i\kappa,\mu/2}(i\tilde{r}'') \propto M_{i\kappa,\mu/2}(-i\tilde{r}'')$, according to Eq. (84); thus, up to a phase factor, the Whittaker functions in Eq. (91)–(92) can take either equivalent form $M_{\pm i\kappa,\mu/2}(\pm i\tilde{r}'')$.  

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Moreover, the basic results of Eqs. (88)–(89) and (91)–(92) are also verified by the analysis of Appendix D, based on the associated differential equation.

This concludes our detailed analysis of the most relevant spectral properties of the symmetry operators of CQM, with the added bonus of having established appropriate analytic continuation techniques to compare the different types.

VII. CONCLUSIONS

In this work, we have derived a comprehensive path-integral treatment of the symmetry generators of conformal quantum mechanics (CQM). This analysis is relevant in the context of CQM as a one-dimensional conformal field theory \[39–46\]. Moreover, it is noteworthy that the symmetry operators in the weak-coupling regime of CQM are of current interest in understanding the nature of spacetime causal structure in the context of causal diamonds and thermal properties of the vacuum \[47, 49, 51–57\].

The main results of our analysis include a complete characterization via path integrals of all the operators from the three families (elliptic, parabolic, and hyperbolic), with distinctly different spectral and time-evolution properties. In addition to establishing appropriate analytic continuations of the path integral for the analog system of a radial harmonic oscillator, we have derived novel expressions and a simple general technique to deal with the spectral characterization of continuous operators. These results can be used to provide further insight into the physical and mathematical properties of CQM, including, inter alia, the physical meaning of this form of SO(2,1) conformal symmetry for near-horizon physics \[10, 70\] and applications in quantum cosmology \[71–73\]. Moreover, the general methodology and spectral properties we established for hyperbolic operators are of potential interest in related applications of the inverted harmonic oscillator \[67, 68\], which we are currently investigating, with impact on problems from the quantum Hall effect to black holes, including issues of thermality and complexity \[74, 77\].

Another line of work for which the CQM generators, as considered in this paper, are relevant is the recent development of Schwarzian mechanics \[78–83\], which is related to the dAFF model. Most importantly, Schwarzian mechanics is relevant to the low-energy limit of the Sachdev-Ye-Kitaev model \[84\], as was shown in Refs. \[78, 79\].

Moreover, the use of the symmetry generators \(G\) as alternative Hamiltonians with trans-
formed times $\tau$ is still an open question, though some special cases have been suggested in the literature, for example, for the magnetic monopole 6 and the magnetic vortex 7. In Ref. [85], the roles played by the generators $H$, $R$, and $S'$ is considered within a CFT-inspired sine-square deformation, with an interpretation that can be further reexamined within our generalized framework. In addition, the Niederer-Takagi time transformation 86–89 involved in the elliptic generators has been used in the physics of cold atoms 90, 91. However, the interpretation of the time variable is a subtle notion that finds a more natural context in general-relativistic applications, as has been suggested for matrix models 92. This interpretation has been partly implemented for the dynamics of harmonic-oscillator-type Unruh-DeWitt detectors in curved spacetimes 93. More generally, this is an issue of relevance in the near-horizon version of CQM for black hole thermodynamics 10 and causal diamonds 47–49, 51–57, which deserves further investigation.

Finally, the current presentation has been limited to the weak-coupling regime of CQM, as described in Sec. I. An extension of this framework to the strong coupling regime, with additional implications for renormalization and quantum anomalies, is in progress, and will be reported elsewhere.

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Appendix A: A Dimensional Form of the Theory of Symmetry Generators and Their Effective Hamiltonian Representation

The framework of Subsec. II B is based on the generators $G$ leading to the effective Hamiltonian (25). It has the definite advantage of being completely general, but the Hamiltonian involves canonical variables and time with reduced dimensions, which are different from those of the original CQM Hamiltonian (5) (for example, $\tau$ is dimensionless and $[q] = [Q]/[f]^{1/2}$, with $[f] = \text{time}$). While this poses no serious technical challenges, it may be desirable to develop an alternative method where the relevant variables have their “usual” dimensions,
meaning those of the original Hamiltonian \( H \). Specifically, dimensions can be restored to those of \( H \) via a characteristic time scale; here, this is naturally provided by the factor \( u \) of the generalized generator, Eq. (15), provided that \( u \neq 0 \).

In principle, under the assumption \( u \neq 0 \), the rescaling \( G = u \hat{H}_G(\hat{r}, \hat{p}) \), rather than \( G = \sigma \hat{H}_G(\hat{r}, \hat{p}) \), allows the theory to be redefined consistently in terms of the effective time \( \hat{\tau} = u \tau \) and canonical variables \( \hat{q}, \hat{p} \), such that \( \hat{q}^2 = uq^2 \) and \( \hat{p}^2 = u^{-1}p^2 \). Then, the reduced effective Hamiltonian becomes

\[
\frac{1}{u}G \equiv \hat{H}_G(\hat{r}, \hat{p}) = \frac{\hat{p}^2}{2M} + \frac{\hbar^2}{2M} \frac{g}{\hat{q}^2} + \frac{M}{2} \left( -\frac{\Delta}{4u^2} \right) \hat{q}^2
\]

(A1)

(with the additional assignments \( \lambda = \hbar^2 g/M \) and \( \hat{p}^2 = \hat{p}^2 \) to be used below). The evolution operator (22) admits the alternative form \( U_{\hat{H}}(T) = e^{-i\hat{H}(\hat{\tau} - \hat{\tau}_0)/\hbar} \), and the CQM frequency becomes

\[
\hat{\omega}^2 = -\frac{\Delta}{4u^2}.
\]

(A2)

With these redefinitions, the time \( \hat{\tau} \), the Hamiltonian \( \hat{H}_G(\hat{r}, \hat{p}) \), and the frequency in Eq. (A2) have the usual dimensions.

In addition, it may prove useful to define the real and positive time-scale parameter

\[
a = 2 \frac{|u|}{\sqrt{\Delta}},
\]

(A3)

which generalizes the original parameter \( a \) of Ref. [5] in Eqs. (11) and (12); then, \( \hat{\omega}^2 = -\text{sgn}(\Delta)/a^2 \), which leads to \( \hat{\omega} = 1/a \) for \( \Delta < 0 \) and \( \hat{\omega} = -i/a \) for \( \Delta > 0 \) (the choice of sign for the latter is discussed in Sec. VII), with \( \hat{\omega} = 0 \) and \( a = \infty \) for \( \Delta = 0 \). As in Subsec. II B, denoting the corresponding parameter dependence in Eq. (A1) with \( \hat{H}[M, g, \hat{\omega}] \) (where the subscript \( G \) is removed for simplicity), we can make the assignments

\[
R = \frac{a}{2} \hat{H}[M, g, \hat{\omega} = 1/a],
\]

(A4)

\[
H = \hat{H}[M, g, \hat{\omega} = 0],
\]

(A5)

\[
S' \equiv \frac{a}{2} \hat{H}[M, g, \hat{\omega} = -i/a],
\]

(A6)

as \( u = a/2 \) for both \( R \) and \( S' \). The corresponding description of the three classes of operators follows from \( G = u \hat{H} [M, g, \hat{\omega}] \), which also implies that \( G = \text{sgn}(u)\sqrt{|\Delta|}(a/2) \hat{H} [M, g, \hat{\omega}] \)
for elliptic and hyperbolic operators; consequently,

\[ G \approx \text{sgn}(u)\sqrt{|\Delta|} \, R \quad \text{for elliptic generators}, \quad (A7) \]

\[ G \approx uH \quad \text{for parabolic generators}, \quad (A8) \]

and

\[ G \approx \text{sgn}(u)\sqrt{\Delta} \, S' \quad \text{for hyperbolic generators}. \quad (A9) \]

This restricted framework agrees with the more general characterization of Eqs. (34)–(36), where \( \sigma \) is the more general sign, which can be evaluated from \( \text{sgn}(u) \) when \( \text{sgn}(u) \neq 0 \). Despite being somewhat restricted, this approach has some appealing features, i.e., its dimensional-analysis structure, and includes the all-important operators \( R, H, \) and \( S' \).

A final remark is in order regarding the restrictions of this particular approach. We have assumed that \( u \neq 0 \), which basically modifies the original inverse-square-potential Hamiltonian \( H \) with the extension \((A1)\). This excludes the family of Hamiltonians \( H_G \) with \( u = 0 \), for which the discriminant is \( \Delta = v^2 \), and which encompasses two subclasses: (i) hyperbolic operators with \( v \neq 0 \), including \( D \) by itself, as well as linear combinations of \( D \) and \( K \); (ii) parabolic operators with \( v = 0 \), which reduce to simply \( K \) (up to an arbitrary multiplication constant \( w \)). However, we have seen in Subsec. II B that these operators can be handled with the more general framework, where \( K \) and \( D \) are described as equivalent to \( H \) and \( S' \) respectively.

**Appendix B: Path-Integral Framework—Summary of Basic Results**

In this appendix, we summarize the main results on path integration needed to understand and compute the spectral properties of the operators discussed in the main text. These include the setup of the time-sliced path integral, the transformation to hyperspherical coordinates, and the evaluation of the propagator for the radial harmonic oscillator.

The basic setup starts, as in Sec. III, with the path-integral expression \((B1)\) for the propagator, which can be evaluated as the limit of a properly time-sliced integral in Cartesian coordinates,

\[ K_{(d)}(\mathbf{r}''; \mathbf{r}'; t''; t') = \lim_{N \to \infty} \left( \frac{M}{2\pi i\hbar} \right)^{dN/2} \left[ \prod_{k=1}^{N-1} \int_{\mathbb{R}^d} d^d \mathbf{r}_k \right] e^{iS(N)/\hbar}. \quad (B1) \]

As in Subsec. III A, we use \( t \) for the dynamical time and \( \hat{H} \) for a generic Hamiltonian. The Cartesian form of Eq. \((B1)\) involves a time lattice \( t_j = t' + j\epsilon \), for the time interval
\[ T = t'' - t' \] corresponding to the end points \( r_0 \equiv r' \) and \( r_N \equiv r'' \); in this lattice, \( \epsilon = T/N \), with \( j = 0, \ldots, N \), with \( t_0 \equiv t' \) and \( t_N \equiv t'' \), such that \( r_j = r(t_j) \). The discrete action in Eq. (B1) is \( S^{(N)} = \sum_{j=0}^{N-1} S_j^{(N)} \), with \( S_j^{(N)} = M (r_{j+1} - r_j)^2 / 2\epsilon - \epsilon V(r_j, t_j) \), where \( V(r, t) \) is the potential—if assumed to be time-independent, as is the case for the computation of all the CQM operators, the result of this path integral is only a function of \( T \).

The propagator \( K_{(d)}(r'', r'; t'', t') \) can be formally rewritten in non-Cartesian coordinate systems. It is well-known that caution must be exercised when evaluating Eq. (38) in non-Cartesian coordinates, as this typically leads to extra terms of order \( h^2 \) in the action. These arise when nonlinear transformations are performed, both in quantum mechanics [60, 94–97], and quantum field theory [98]. In general, coordinate transformations of Eq. (B1) can be applied before taking the continuum limit, but prescriptions for the choice of position values of the potential are dictated by operator ordering.

For the important case of hyperspherical polar coordinates [33, 34, 59], the \( d \)-dimensional hyperspherical harmonics \( Y_{lm}(\Omega) \) and Gegenbauer polynomials \( C_l^{(\nu)}(x) \) provide the necessary framework for separation of variables; see Sec. III and Eqs. (39)-(40) for the partial wave expansions. Then, an explicit expression for the propagator can be derived in the time-sliced path integral by transforming Eq. (38) into hyperspherical coordinates before taking the \( N \to \infty \) limit. The critical step in this derivation is rewriting the elements of the discretized action \( S_j^{(N)} = M (r_{j+1} - r_j)^2 / 2\epsilon - \epsilon V(r_j, t_j) \) by separating the angles \( \cos \psi_{j+1,j} = \hat{r}_{j+1} \cdot \hat{r}_j \) (with \( \hat{r} = r/r \)) in a radial-angular resolution \( \exp \left( i S_j^{(N)}/\hbar \right) = \exp \left[ \frac{iM}{\hbar \epsilon} (r_{j+1}^2 + r_j^2) - \frac{i\epsilon V_j}{\hbar} \right] \exp (z_j \cos \psi_{j+1,j}) \), where \( z_j = \frac{Mr_j r_{j+1}}{i\epsilon \hbar} \). The product of the last factor for all the intervals \( (j = 0, \ldots N - 1) \) can be evaluated in the angular integrals using the degenerate form of Gegenbauer’s addition theorem on partial waves (generalization of the 3D Rayleigh expansion of a plane wave; Sec. 11.5 of Ref. [65]):

\[
\exp (i z \cos \psi) = (i z/2)^{\nu} \pi T(\nu) \sum_{l=0}^{\infty} (l + \nu) I_{l+\nu}(iz) C_l^{(\nu)}(\cos \psi), \tag{B2}
\]

where \( I_p(x) \) is the modified Bessel function of the first kind and order \( p \). This resolution yields the path integral for the radial propagator in Eq. (40) as

\[
K_{l+\nu}(r'', r'; T) = \lim_{N \to \infty} \left( \frac{M}{2\pi i \epsilon \hbar} \right)^{N/2} \prod_{k=1}^{N-1} \left[ \int_0^\infty dr_k \right] \frac{w^{(N)}_{l+\nu}(r^2)}{\hbar} \times \exp \left\{ \frac{i \hat{P}^{(N)}}{\hbar} [r_1, \ldots, r_{N-1}] (r'', r'; T) \right\}, \tag{B3}
\]
where the radial action is

\[ R^{(N)}[r_1, \ldots, r_{N-1}] (r'', r'; T) = \sum_{j=0}^{N-1} \left[ \frac{M (r_{j+1} - r_j)^2}{2\epsilon} - eV(r_j) \right] . \]  

(B4)

In Eq. (B3) a radial functional weight

\[ w_{l+\nu}^{(N)}[r^2] = \prod_{j=0}^{N-1} \left[ \sqrt{2\pi z_j} e^{-z_j} I_{l+\nu}(z_j) \right] \]  

(B5)

has been properly defined with the radial variables appearing through the characteristic dimensionless ratio \( z_j = \frac{Mr_j r_{j+1}}{i\hbar} \). Equation (B3) [with the restriction to the half-line \( r(t) \geq 0 \)] admits the formal Besselian path-integral representation (42), i.e.,

\[ K_{l+\nu}(r'', r'; T) = \int \mathcal{D}r(t) \ w_{l+\nu}[r^2] \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} dt \left[ \frac{M}{2} \dot{r}^2 - \frac{\hbar^2}{2M} \frac{(l+\nu)^2}{r^2} - V(r) \right] \right\} , \]  

(B6)

with the angular part of the problem contributing through the nontrivial radial functional weight \( w_{l+\nu}[r^2] \).

As a result of this hyperspherical resolution, the property known as interdimensional dependence [62, 63] is exhibited by Eqs. (B3)–(B6): \( d \) and \( l \) appear in the combination \( l + \nu \). The associated radial action \( R[r(t)] \) in Eq. (B3) only includes the radial kinetic energy and the interaction potential, and excludes the centrifugal potential, whose role is played instead by the functional weight. The connection with the usual formulation of the classical radial action can be shown using the asymptotic form of the Bessel function \( \sqrt{2\pi z} e^{-z} I_{\mu}(z) \sim e^{-(\mu^2-1/4)z+O(1/z^2)} \) (for \( |z| \to \infty \)) in Eq. (B5), where \( z \sim 1/\epsilon \), thus leading to the alternative expression

\[ K_{l+\nu}(r'', r'; T) = \int \mathcal{D}r(t) \ \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} dt \left[ \frac{M}{2} \dot{r}^2 - \frac{\hbar^2}{2M} \frac{(l+\nu)^2}{r^2} - V(r) \right] \right\} . \]  

(B7)

where the action does include the centrifugal term. This procedure has been called asymptotic recombination [61].

The equivalence of Eq. (B6) and (B7) leads to the important theorem relating angular momentum and inverse square potential terms. The latter can be inserted in the path integral (as can be seen in Eq. (B7) by absorbing the inverse-square coupling \( g \) as part of an effective angular momentum with \( \mu = \sqrt{(l+\nu)^2 + g} \); see Eqs. (43) and (44) and associated statement in Subsec. IIIA). This was originally shown as above in Ref. [64] for \( d = 3 \), with a rigorous proof in Refs. [61, 63].
The generic radial path integral \( \text{[B3]} \) can be used to evaluate the propagator for the inverse square potential and radial harmonic oscillator, by rewriting it in a more convenient form,

\[
K_{l+\nu}(r'', r'; T) = \lim_{N \to \infty} \left( \frac{\alpha}{l} \right)^N \prod_{k=1}^{N-1} \left[ \int_0^\infty dr_k r_k \right] \prod_{j=0}^{N-1} \left[ I_{l+\nu}(z_j) e^{i A_j / \hbar} \right] 
\]

where \( \alpha = M/(\epsilon \hbar) \) and \( A_j = \frac{M}{2\epsilon} \left( r_{j+1}^2 + r_j^2 \right) - \epsilon V(r_j) \). The limit in Eq. \( \text{[B8]} \), \( K_{l+\nu}(r'', r'; T) = \lim_{N \to \infty} K_{l+\nu}^{(N)}(r'', r'; T) \), can be taken by evaluating the \( N \)-th order term \( K_{l+\nu}^{(N)}(r'', r'; T) \) recursively \([1,3,64]\). Transforming the path integral into its Euclidean-time form (with the replacements \( \epsilon \to -i\epsilon, \alpha \to i\alpha \), and \( z_j \to \zeta_j = iz_j \)), and defining the parameter \( \beta = \alpha(1 + \epsilon^2 \omega^2/2) \), if follows that

\[
K_{l+\nu}^{(N)}(r'', r'; T) = \exp \left[ -\frac{\alpha}{2} \left( r''^2 + r'^2 \right) \right] \frac{\alpha}{\gamma_N I_\mu} \left( \frac{\alpha}{\gamma_N} r_{N}r_0 \right) \exp \left[ \frac{\alpha}{2\lambda_N} \left( r_0^2 + r_N^2 \right) \right] P_N(r'', r'; \alpha, \beta), \tag{B9} \]

which has a functional form governed by

\[
P_N(r_N, r_0; \alpha, \beta) = \frac{\alpha}{\gamma_N I_\mu} \left( \frac{\alpha}{\gamma_N} r_{N}r_0 \right) \exp \left[ \frac{\alpha}{2\lambda_N} \left( r_0^2 + r_N^2 \right) \right]. \tag{B10} \]

This can be established, along with the recursion relations: \( \lambda_{N+1} = \lambda_N = 2\eta, \lambda_N = \gamma_N/\gamma_{N-1} \), and \( 1 + \gamma_N \gamma_{N-2} = \gamma_N^2 \) (with \( \eta = \beta/\alpha \)), by repeated application of Weber’s second exponential integral for Bessel functions (Sec. 13.31 of Ref. \[65\] and 10.22.67 of Ref. \[66\]),

\[
\int_0^\infty \exp \left( -c^2 x^2 \right) J_\mu(ax) J_\mu(bx) x dx = \frac{1}{2c^2} \exp \left( -\frac{a^2 + b^2}{4c^2} \right) I_\mu \left( \frac{ab}{2c^2} \right) \] \( \text{[B11]} \)

[Re(\mu) > -1 and |arg(c)| < \pi/4]. The consistency of the recursion relations and generic patterns can be proved by mathematical induction, leading to a closed solution in the continuum limit \( N \to \infty \). An efficient approach \([99]\) to derive the expressions in this limit is via a function \( \Phi(T) = \lim_{N \to \infty} \Phi_N \), where \( \Phi_N = \epsilon \gamma_N \), with finite differences \( \Phi_N = (\Phi_{N+1} - \Phi_N)/\epsilon = (\gamma_{N+1} - \gamma_N) \), and \( \Phi_N = (\Phi_{N+1} + \Phi_{N-1} - 2\Phi_N)/\epsilon^2 = (\gamma_{N+1} + \gamma_{N-1} - 2\gamma_N)/\epsilon \). This shows, from Eqs. \( \text{[B9]} \) and \( \text{[B10]} \) that

\[
K_{l+\nu}(r'', r'; T) = \frac{M}{\hbar \Phi(T)} \sqrt{r''r'} \exp \left[ -\frac{M}{2\hbar} \frac{\Phi(T)}{\Phi(T)} \left( r''^2 + r'^2 \right) \right] I_\mu \left( \frac{M}{\hbar \Phi(T)} r'r'' \right), \tag{B12} \]

where, from the recursion relations and comparison with the free particle, the function

\[
\Phi = \frac{1}{\omega} \sinh(\omega T) \tag{B13} \]
(Euclidean-time version) can be found as the solution to the differential equation \( \ddot{\Phi}(T) = \omega^2 \Phi(T) \) with the initial conditions \( \Phi(0) = 0 \) and \( \dot{\Phi}(0) = 1 \). In conclusion, these equations, after conversion to real time, show that the path integral is given by Eq. (45). The propagator (B12) can also be generalized to account for possible time dependence of the oscillator parameters [99].

For the case of the inverse square potential alone, an exact derivation from a perturbative series is possible [100], including a renormalized version for the strong-coupling regime [101] (similar to the case of the renormalized path integral for the two-dimensional delta interaction [102]).

Appendix C: Relevant Integral Representations for the Product of Bessel and Whittaker Functions as Fourier Transforms of Propagators

In this appendix, we summarize the key results on integral representations of products of the special functions relevant for the propagators discussed in this paper. We will first identify the identities relevant for the operator \( H \) and related parabolic generators, i.e., generally for the pure inverse square potential, using products of Bessel functions; and we will then explore a general radial and appropriate identities for the hyperbolic operators, corresponding to an inverted radial harmonic oscillator, using products of Whittaker functions. The identification of the relevant identities is a useful addition to the literature of path integrals; and, in the case of the latter class, we establish, inter alia, a new integral identity.

1. Integral Representations for the Product of Bessel Functions

Let us consider the following integral representation of the product of Bessel functions (Sec. 13.7 of Ref. [65]):

\[
J_\mu(z')J_\mu(z'') = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp \left( \frac{s}{2} - \frac{z'^2 + z''^2}{2s} \right) I_\mu \left( \frac{z'z''}{s} \right) \frac{ds}{s},
\]

(C1)

where \( c \) is a real positive constant and \( \text{Re}(\mu) > -1 \). As shown in Ref. [65]), this representation (C1) can be established by combining an appropriate form of Bessel function Gegenbauer addition theorems with Schlüfli’s integral \( J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{2\pi i} \int_{-\infty}^{(0+)} \exp \left( t - \frac{z^2}{4t} \right) \frac{dt}{t^{\nu+1}} \). As
shown in Subsec. V A, we consider the limit \( c \to 0^+ \), that reduces the integral (C1), with \( s = c + 2it \), to the form (64), where \( t \to t - i0^+ \). Its relevance for our study stems from the fact that the substitution \( t \equiv ET/\hbar \) makes it directly applicable to Eq. (63) for the operator \( H \) (inverse square potential) as a Fourier transform, leading to Eq. (65).

In a similar way, the counterparts of Eq. (C1) for the half-axis intervals can be found using Hankel functions (for which the corresponding Schl"afli’s integrals only involve half-axes), yielding

\[
H_{\mu}^{(1,2)}(z_>) J_{\mu}(z_<) = \pm \frac{1}{\pi i} \int_{0}^{\infty} \exp \left[ \frac{s}{2} - \frac{(z'_2 + z''^2)}{2s} \right] I_{\mu} \left( \frac{z' z''}{s} \right) ds
\]

where \( c \) is again a real positive constant and \( \text{Re}(\mu) > -1 \); and \( z_> \) and \( z_< \) are the greater and lesser of the set \( \{z', z''\} \) respectively. (A related integral is listed in Ref. [103], 6.653-1.)

2. Integral Representations for the Product of Whittaker Functions

We will begin by summarizing and adapting the technique defined in Ref. [69] (Section 6) to set up integral representations of products of Whittaker functions. The method involves writing a single Whittaker function as a confluent hypergeometric function that is related to a Bessel function in integral form. Writing this integral twice for a product of two Whittaker functions gives

\[
\mathcal{M}_{\lambda_1,\mu_1/2}(z_1) \mathcal{M}_{\lambda_2,\mu_2/2}(z_2) = \frac{4}{\Gamma \left( \frac{1+\mu_1}{2} + \lambda_1, \frac{1+\mu_2}{2} - \lambda_2 \right)} \times \int_0^\infty dt \int_0^\infty du e^{-t^2-u^2} t^{2\lambda_1} u^{2\lambda_2} J_{\mu}(2t \sqrt{z_1}) J_{\mu}(2u \sqrt{z_2})
\]

whence a large class of representations can be developed by appropriate variable substitutions. For the important case of products of functions with the same indices, choosing \( \lambda_1 = -\lambda_2 \equiv \lambda \) and \( \mu_1 = \mu_2 \equiv \mu \), the following representation is obtained by going to “polar coordinates” via \( t = \rho \cos \phi \) and \( u = \rho \sin \phi \), and evaluating the integral with respect to \( \rho \) via Weber’s second exponential integral (BII),

\[
\mathcal{M}_{i\kappa,\mu/2}(-ix') \mathcal{M}_{-i\kappa,\mu/2}(ix'') = \frac{2}{\Gamma \left( \frac{1+\mu}{2} + i\kappa \right) \Gamma \left( \frac{1+\mu}{2} - i\kappa \right)} \times \int_0^{\pi} d\phi \left( \cot \phi \right)^{2i\kappa} e^{i(x' \cos^2 \phi - x'' \sin^2 \phi)} I_{\mu} \left( \sqrt{x' x''} \sin 2\phi \right)
\]
where we are also making the assignments \( \lambda = i\kappa \), \( z_1 = -ix' \), and \( z_2 = ix'' \), tailored specifically to the Whittaker functions with imaginary first index needed for the radial inverted oscillator or the conformal operator \( S \).

From Eq. (C4), a whole class of integrals can be obtained by appropriate substitutions. In particular, if \( \zeta \) is defined such that \( \sin 2\phi = 1/(i\sinh \zeta) \), then Eq. (C4) turns into a form that appears to match the propagator of Eq. (76), with \( \zeta = \omega t \). However, this is a nontrivial substitution that takes the variable \( \zeta \) into the complex plane and requires further analysis. We will first implement this substitution in two steps: (i) defining a real variable \( s \) such that \( \sin 2\phi = 1/\cosh s \); (ii) replacing \( s \) by \( \zeta \) via \( \cosh s = i\sinh \zeta \) in the complex plane. Moreover, there are issues with the interpretation of the integral in the complex plane that require an appropriate deformation of the integration contour. This is shown next.

First, from the range of the integral (C4), the substitution \( \sin 2\phi = 1/\cosh s \) establishes a one-to-one correspondence that maps \( 2\phi \in [0, \pi] \) into \( s \in (-\infty, \infty) \), only involving real variables. By straightforward algebra, \( d\phi = ds/(2\cosh s) \), \( \cos 2\phi = \tanh s \), and \( \cot \phi = e^s \), which directly yields an integral of the well-known form of Eq. (3a) in Sec. 6.1 of Ref. [69] in the specific variant

\[
\mathcal{M}_{ik,\mu/2}(-ix') \mathcal{M}_{-ik,\mu/2}(ix'') = \frac{(x'x'')^{1/2}}{\Gamma \left( \frac{1+\mu}{2} + i\kappa \right) \Gamma \left( \frac{1+\mu}{2} - i\kappa \right)}
\times \int_{-\infty}^{\infty} \frac{ds}{\cosh s} \exp(2i\kappa s) \exp \left[ \frac{i}{2} (x' + x'') \tanh s \right] I_\mu \left( \sqrt{x'x''} \cosh s \right)
\]  
(C5)

(which agrees with Ref. [103], 6.669-5).

Second, the transformation \( \cosh s = i\sinh \zeta \) is evidently a translation \( \zeta = s + \sigma \) along the imaginary axis in the complex plane, with displacement \( \sigma \). The general solution of this equation is \( \zeta = \pm s - i\pi/2 + 2\pi n \), with \( n \in \mathbb{Z} \), which follows from the \( 2\pi i \) periodicity of the \( \cosh \) function (or simply by finding the solution via inversion of the hyperbolic or exponential functions). Moreover, the negative sign can be ignored if we only consider the case of a path without inversion. This shows that the translation displacement is \( \sigma = \zeta_n \equiv \zeta_0 + 2\pi n \), with the “principal value” \( \zeta = ic \), with \( c = -\pi/2 \), which we will use below. With this transformation, the following relations are immediately satisfied: \( ds = d\zeta \), \( \sinh s = i\cosh \zeta \), \( \tanh s = \coth \zeta \), \( \sinh s = i\cosh \zeta \), and \( e^s = e^{i\pi/2}e^\zeta \). As a result, the integration path for the transformed version of the integral (C5) is an infinite straight line \( L \) parallel to the real \( \zeta \)
axis, with $\text{Im} \zeta = \text{Im} \sigma = c$. With this procedure, we have established a novel identity

$$M_{i\kappa,\mu/2}(-ix') M_{-i\kappa,\mu/2}(ix'') = e^{-\pi \kappa} \frac{(x'x'')^{1/2}}{\Gamma \left(\frac{1+\mu}{2} + i\kappa\right) \Gamma \left(\frac{1+\mu}{2} - i\kappa\right)} \cdot$$

$$\times \int_{i\zeta - \infty}^{i\zeta + \infty} \frac{d\zeta}{i \sinh \zeta} \exp \left(2i\kappa \zeta\right) \exp \left[\frac{i}{2} (x' + x'') \coth \zeta\right] I_{\mu} \left(\frac{\sqrt{x'x''}}{i \sinh \zeta}\right). \tag{C6}$$

The integral of direct applicability for the required form of the propagator (76) involves using a path with $c \to 0$. This can be justified from the following properties: (i) the only singularities of the integrand occur at $\zeta = 2\pi n i$, which we will circumvent with appropriate integration contours; and (ii) the behavior is regular at infinity along curves asymptotically parallel to the real axis, with the integral vanishing exponentially for $|\text{Re} \zeta| \to \infty$. Thus, $\oint_C F(\zeta) d\zeta = 0$, for the integrand $F(\zeta)$ in Eq. (C6) with a closed contour $C$ consisting of the line $L$ and a parallel line $L'$ with $c \in (-\pi/2, 0)$. This proves Eq. (C6) for arbitrary $c$ with $-\pi/2 < c < 0$. Moreover, if the restriction to straight lines is relaxed, the line $L'$ can further deformed into any other path with asymptotic limits $ic' - i\infty$ and $ic'' + i\infty$, where $c'$ and $c''$ are arbitrary. For the propagator (76), it suffices to take the limit $c \to 0$. It should be noted that this is an analog for Whittaker functions of the Bessel-function identity (C1). With this auxiliary integral (C6), making the replacements $\zeta = \omega T$, $x' = M \omega r'^2 / \hbar$, $x'' = M \omega r''^2 / \hbar$, $\kappa = E/(2\hbar \omega)$, and $c = 0^\mp$, the main result of Eqs. (88)–(89) in Sec. VI is straightforwardly derived.

An alternative set of representations can be established by performing integrals of the forms (C5) and (C6), but restricted to the half-axis intervals. For example, Eq. (5b) in Sec. 6.1 of Ref. [69] reads

$$W_{\kappa,\mu/2}(a_1 t) M_{\kappa,\mu/2}(a_2 t) = \frac{t \sqrt{a_1 a_2}}{\Gamma \left((1+\mu)/2 - \kappa\right)} \cdot$$

$$\times \int_0^\infty d\xi \exp \left[-\frac{1}{2} (a_1 + a_2) t \cosh \xi\right] I_{\mu} \left(t \sqrt{a_1 a_2} \sinh \xi\right) \left[\coth \left(\frac{\xi}{2}\right)\right]^{2\kappa} \tag{C7}$$

(which agrees with Ref. [103], 6.669-4), where $a_{1,2}$ are real parameters satisfying the critical inequality $a_1 > a_2$, and $\text{Re}((1+\mu)/2 - \kappa) > 0$. The variables $t$, $a_1$, and $a_2$ in Eq. (C7), provide some flexibility of choices; however, as shown in the steps leading to their derivation, with the notation used in Eq. (C3), one can identify $z_1 = ta_1$ and $z_2 = ta_2$. Then, as before, one can make the additional replacements $\kappa \to i\kappa$ and $z_{1,2} = -ix', -ix''$. (This could be done most easily with $t = -i$ and $a_{1,2}$ chosen from the set $x', x''$.) Finally, with the substitution...
\[
\sinh \xi = \frac{1}{\sinh \zeta} \quad \text{(which implies \cosh \xi = \coth \zeta, \ coth(\xi/2) = e^\xi, and \ d\xi = -d\zeta/\sinh \zeta)},
\]
Eq. (C7) turns into
\[
W_{i\kappa/2}(ix_>) \mathcal{M}_{i\kappa/2}(ix_<) = \frac{\sqrt{x'x''}}{\Gamma((1 + \mu)/2 - i\kappa)} \times \int_0^\infty \frac{d\zeta}{i \sinh \zeta} \exp(2i\kappa\zeta) \exp\left[\frac{i}{2}(x' + x'') \coth \zeta \right] I_\mu \left(\frac{\sqrt{x'x''}}{i \sinh \zeta}\right), \tag{C8}
\]
where \(x_>\) and \(x_<\) are the greater and lesser of the set \(\{x', x''\}\), respectively.

Incidentally, the known identity of Eq. (C7) is proved, as stated in Ref. [69], by using appropriate substitutions combined with identities (semi-circuital and connection) among Whittaker functions. In fact, this is a reversal of the procedure we used in the main text; in other words, we could just as well use the symmetry property (82), along with the Whittaker identities (84) and (85), and with appropriate limits, to justify the half-axis integrals(s) (C8) from the full integral (C6); and from Eq. (C8) rederive Eq. (C7) via \(\sinh \xi = 1/\sinh \zeta\).

**Appendix D: Generalized Symmetry Generators of CQM: Differential Equation and Consistency Checks**

In this appendix, we consider another aspect of the CQM generalized generator \(G = uH + vD + wK\): its behavior and spectral properties using a differential equation approach. These properties can be deduced by going to the Schrödinger picture for the general multi-component case of the dAFF model we introduced in Sec. II (\(d\) components as \(d\)-dimensional position coordinates in quantum mechanics). This setup for \(d\) dimensions and for the generic operator \(G\) extends the particular results of Ref. [5].

Then, in the Schrödinger picture, with the Hamiltonian \(\tilde{H}_G(r, p) \equiv G/\sigma\) introduced in Eq. (A1), the states \(\Psi(r, \tau)\) evolve according to
\[
i\hbar \frac{\partial \Psi(r, \tau)}{\partial \tau} = \tilde{H}_G(r, p) \Psi(r, \tau). \tag{D1}
\]
From Eq. (D1), “stationary states” can be defined with respect to \(\tau\), such that the eigenvalue equation \(\tilde{H}_G |\psi\rangle = \tilde{E}_G |\psi\rangle\) is satisfied; this leads to the radial-coordinate differential equation
\[
\frac{1}{2} \frac{\hbar^2}{M} \left[ -\frac{d^2}{dr^2} + g \left( \frac{l + \nu}{2} \right)^2 - \frac{1}{4} + \frac{M^2}{\hbar^2} \left( -\frac{\Delta}{4} \right) r^2 \right] U_{q,l}(r) = \tilde{E}_G U_{q,l}(r), \tag{D2}
\]
with the eigenvalues of the operator $G$ being $\hbar g = \sigma \tilde{E}_G$—see Eq. (27) and the discussion therein. In Eq. (D2), the reduced function $U_{g,l}(y)$ corresponds to the $d$-dimensional wave function $\psi(r) = r^{-(d-1)/2} U_{g,l}(r) Y_{lm}(\Omega)$, following the usual separation of angular variables.

With the substitutions

$$A = -i(M/\hbar)\sqrt{\Delta}/2, \quad \lambda = i\sigma g/\sqrt{\Delta}, \quad \mu = g + (l + \nu)^2,$$

Eq. (D2) is of the known form (Ref. [69], p. 34)

$$\left[ -\frac{d^2}{dr^2} + \frac{\mu^2 - 1/4}{r^2} + A^2 r^2 \right] \Phi(r) = 4\lambda A \Phi(r),$$

which can be solved exactly in terms of Whittaker functions. Specifically, the standard Whittaker differential equation for a function $P(z)$ can be transformed into Eq. (D3) with the substitution $z = Ar^2$, such that the solutions are

$$\Phi(r) = r^{-1/2} P_{\lambda,\mu/2}(Ar^2),$$

where, with the notation of Ref. [69], $P_{\lambda,\mu/2}(z)$ stands for either one of the Whittaker functions $\{ \mathcal{M}_{\lambda,\mu/2}(z), W_{\lambda,\mu/2}(z) \}$, or a linear combination thereof. The substitutions used in Eq. (D3), which involve the scale $A$ and the index $\lambda$, are completely general, covering all classes of operators (elliptic, parabolic, and hyperbolic), according to the value of $\Delta$.

Furthermore, enforcing the regular behavior at the origin, the function $W_{\lambda,\mu/2}(z)$ should be excluded for the wave function solutions—though it is still relevant for the Green’s functions (see below). Then, using the values for $A$, $\lambda$, and $\mu$, the ensuing regular choice of the solution (D4) to Eq. (D2) becomes

$$U(r) = r^{-1/2} \mathcal{M}_{i\sigma g/\sqrt{\Delta},\mu/2}(-i\tilde{r}^2),$$

where

$$\tilde{r}^2 = \frac{\sqrt{\Delta}}{2} \left( \frac{M}{\hbar} \right) r^2 \equiv \frac{M|\omega|}{\hbar} r^2.$$  

The variable $\tilde{r}$ in Eq. (D6) is written in terms of the general CQM frequency of Eq. (29), i.e., $\omega = -i|\omega|$, which is in agreement with the definitions in Sec. VI in Eq. (78) and with the analytic extension $\omega \rightarrow -i\omega = -i\sqrt{\Delta}/2$ used therein. Moreover, the solution (D5) gives a continuous spectrum with $g \in (-\infty, \infty)$. This set of results reproduces the path-integral treatment leading to Eq. (91) [with Eq. (78)] for the hyperbolic-operator eigenfunctions. Most importantly, Eq. (D5) is completely general; when $\Delta < 0$, it can also be applied to the elliptic case—this amounts to the analytic continuation $\kappa = i\sigma g/\sqrt{\Delta} \rightarrow \sigma g/\sqrt{\Delta}$, which (from the asymptotics) requires eigenvalue “quantization” leading to the generalized
Laguerre polynomials, as in Eq. (54). For the parabolic case, the limit \( \Delta = 0 \) leads to Bessel functions \( J_\mu(kr) \) as regular solutions, as can also be verified directly from Eq. (D2).

In the latter case, there is a known subtlety or ambiguity in the choice of the sign of the regular solution from the set \( J_{\pm \mu}(kr) \), with a physical argument \[37\] selecting the positive sign—this is related to the inequivalence of the self-adjoint extension method with physical regularization techniques \[104\]. In short, the differential-equation approach fully agrees with the path-integral results of the main text for all the conformal generators.

In addition, the Green’s functions can be derived from the corresponding differential equation (D2), using the general solutions we already found, Eq. (D4). The usual notational simplification \( E \equiv \tilde{E}_G \) is used below [or else, this symbol could be replaced in favor of \( \mathfrak{g} \), according to Eq. (27)]. The radial energy Green’s functions \( G_{l+i\nu}^{(\pm)}(r'', r'; E) \) are defined in Eq. (10), which we will rescale as \( G_{l+i\nu}^{(\pm)}(r'', r'; E) = (\hbar^2/2M) G_{l+i\nu}^{(\pm)}(r'', r'; E) \). As is well-known, because of the specific form of Eqs. (70)–(71), which effectively invert the Schrödinger eigenvalue equation, these coincide with the Green’s functions used for the solution of linear differential equations. Then, the radial differential equation for \( G_{l+i\nu}^{(\pm)}(r'', r'; E) \) reads

\[
\left\{ \frac{d^2}{dr^2} + \frac{2M}{\hbar^2} [E - V(r')] - \frac{(l + \nu)^2 - 1/4}{r^2} \right\} G_{l+i\nu}^{(\pm)}(r'', r'; E) = \delta(r'' - r') , \tag{D7}
\]

which is a particular case of a one-dimensional Sturm-Liouville problem (with constant coefficient \( p(r) = 1 \) for the second-order derivative) \[105\]. Equation (D2) is the homogeneous form of Eq. (D7), with the obvious identifications; then, defining the the functions \( U_{(\leq)}^{(\pm)}(r) \) and \( U_{(>)}^{(\pm)}(r) \) that satisfy boundary conditions at the left boundary (here: \( r = 0 \)) and right boundary (here: \( r = \infty \)), then \[105\]

\[
G_{l+i\nu}^{(\pm)}(r'', r'; E) = \frac{U_{(\leq)}^{(\pm)}(r_<) U_{(>)}^{(\pm)}(r_>) - U_{(\leq)}^{(\pm)}(r_<) U_{(>)}^{(\pm)}(r_>)}{p(r') \{ U_{(\leq)}^{(\pm)}, U_{(>)}^{(\pm)} \}(r')} , \tag{D8}
\]

where \( r_< (r_>\) is the lesser (greater) of \( r' \) and \( r'' \) and \( \{ U_{(\leq)}^{(\pm)}, U_{(>)}^{(\pm)} \} \) is the Wronskian of \( U_{(\leq)}^{(\pm)}(r) \) and \( U_{(>)}^{(\pm)}(r) \). This Green’s function technique has also been used for the study of the related strong-coupling inverse square potential regularization \[106\].

For Eq. (D2), from the general solution (D4), we identify

\[
U_{(\leq)}^{(\pm)}(r) = r^{-1/2} M_{\pm i \sigma g/\sqrt{\Delta}, \mu/2} (\mp i \tilde{r}^2) \tag{D9}
\]

as the solutions that satisfy a regular boundary condition \( U(r)|_{r=0} = 0 \) at the origin; and

\[
U_{(>)}^{(\pm)}(r) = r^{-1/2} W_{\pm i \sigma g/\sqrt{\Delta}, \mu/2} (\mp i \tilde{r}^2) \tag{D10}
\]
as the solutions that satisfy the appropriate asymptotic boundary condition at infinity (outgoing/incoming waves for $G^{(\pm)}$). The identity [84] shows that $U^{(\pm)}(\langle) \propto M^{\pm i\sigma\sqrt{\Delta,\mu}/2}$ are actually the same function, up to a proportionality constant; however, $U^{(\pm)}(\rangle) \propto M^{\pm i\sigma\sqrt{\Delta,\mu}/2}$ are distinctly different. The Wronskian can be computed with the identity $\mathfrak{W}\{M_{\lambda,\mu/2},W_{\lambda,\mu/2}\}(z) = -[\Gamma((1+\mu)/2-\lambda)]^{-1}$; using Eqs. (D9) and (D10), along with $p(r) = 1$, and applying the chain rule for $z = \mp i\tilde{r}^2$, the final result for $G^{(\pm)}_{\ell_0}(r''', r'; E)$ is identical to Eq. (87). This again verifies the equivalence of the path-integral and differential-equation approaches, and provides additional consistency checks for the network of relations defined in this paper.

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