Probabilistic pathway representation of cognitive information

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One held that psychological functions such as language or memory could never be traced to a particular region of brain. If one had to accept, reluctantly, that the brain did produce the mind, it did so as a whole and not as a collection of parts with special functions. The other camp held that, on the contrary, the brain did have specialized parts and those pars generate separate mind functions.

Antonio R. Damasio.

Abstract

We present for mental processes the program of mathematical mapping which has been successfully realized for physical processes. We emphasize that our project is not about mathematical simulation of brain’s functioning as a complex physical system, i.e., mapping of physical and chemical processes in the brain on mathematical spaces. The project is about mapping of purely mental processes on mathematical spaces. We present various arguments – philosophic, mathematical, information, and neurophysiological – in favor of the $p$-adic model of mental space. $p$-adic spaces have structures of hierarchic trees and in our model such a tree hierarchy is considered as an image of neuronal hierarchy. Hierarchic neural pathways are considered as fundamental units of information processing. As neural pathways can go

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through whole body, the mental space is produced by the whole neu-
ral system. Finally, we develop Probabilistic Neural Pathway Model
in that Mental States are represented by probability distributions on
mental space.

1 Introduction

In this paper (see also [1]-[10]) we present for mental processes the program
of mathematical mapping which has been successfully realized for physical
processes. Of course, this is the project of huge complexity and it would be
naive to expect to realize it in one paper or even in a series of papers. We
emphasize that our project is not about mathematical simulation of brain’s
functioning as a complex physical system, i.e., mapping of physical and chem-
ical processes in the brain on mathematical spaces. The project is about
mapping of purely mental processes on mathematical spaces. In particular,
the problem of the greatest importance is the creation of an adequate math-
ematical model of mental space. We agree that one can say that the project
was started two thousands years ago by famous Greek thinkers, Socrates,
Plato, Aristotle. So we can say that our paper returns us to the old question
of whether reality, either material or mental, may be fundamentally mathe-
matical. The question about material reality was answered in the positive.
The question about mental reality is still open. While harking back to Greek
thought on the possibly mathematical nature of mental reality, the project
is unusual from a contemporary point of view. These days, the more usual
role of mathematics lies in investigation of biological, chemical, and physical
bases of mental processes (even including the possibility of their quantum
origin). Our project shifts the focus to the mathematical modeling of pure
mental, rather than physical and chemical processes. It is hard to forsee
whether or not this strategy will be successful. Some form of our mathemat-
cal modeling may well prove effective in psychology, psychiatric treatment
and so forth. The crucial point is that we apply a new mathematics based on
so called $p$-adic numbers and it is too early to say whether it is adequate to
mental modeling. It is nevertheless possible that $p$-adics will open new paths
in mapping of mental processes on mathematical spaces. The main reason
for thinking that $p$-adics may be of value is that they appear better suited
than mathematics based on real or complex numbers to the heterogeneous
complexity and discontinuity of mental processes.
If \( p \) is a prime number (and this case is the most interesting from the mathematical viewpoint) then \( p \)-adic numbers, like real, form what mathematicians term a ‘field.’ They can be added, subtracted, multiplied or divided without leaving the set of \( p \)-adics (for any given \( p \)). This is the case with the sets of rational and complex numbers too, but not with all infinite sets of numbers. Integers, for example, do not form a ‘field’ since dividing one by another can lead to a fraction outside the set of integers. In such a case mathematicians use a term ‘ring.’ The set of \( p \)-adic numbers for non prime \( p \) is not a field, but just a ring. Geometrically \( p \)-adic rings are represented by hierarchic trees and this is one of the fundamental features of \( p \)-adics used in mathematical modeling of mental processes. On \( p \)-adic trees there are defined a \( p \)-adic metrics which are induced by the hierarchic structures of trees. These are so called \textit{ultra-metrics}. That \( p \)-adic spaces are metric spaces and yet are different from ‘real’ metric spaces is crucial for our project. Their metrics allow transfer of the mathematical machinery developed for ‘real’ spaces to weird-seeming spaces that are potentially better suited to describing mentality.

We recall the main steps in mathematical mapping of physical processes and then we will discuss the possibility to apply this scheme to mental processes.

2 Mapping of physical processes on mathematical spaces

\textbf{Step 1. Elaboration of the mathematical model of physical space.}

At the first stage of development of physics (which at that time was not separated from philosophy) there was created a mathematical model of physical space. This was the crucial step, because before to start the study of dynamics of material systems we should have some space which “contains” these systems.\(^1\) This stage of development took a few thousands years. First of all it should be mentioned the great contribution of ancient Greek mathematicians and philosophers. I would like to pay attention to the works of Euclid and Aristotle. Everybody knows that the Euclidean geometric model was the first axiomatic mathematical model of physical space. But not so

\(^1\)Such a point of view was dominating in physics; in philosophy it was strongly supported by Kant.
many people know about ideas of Aristotle on the structure of physical and mental (!) spaces. We shall come back to Aristotle’s ideas on geometry of mental space in section 4 and now we discuss only models of physical space. It seems that he was the first who presented a detailed discussion on continuity of physical space. For him physical space (in which material objects are represented) is infinitely divisible and continuous in the sense that it can not be represented as union of two parts which do not have common boundary.

These ideas were developed by Newton and (especially) Leibnitz and, finally, there was created (through works of 19th century mathematicians Cantor and Dedekind) the modern model of physical space - real continuum. One dimensional continuum is geometrically represented by the straight line. For further comparative analysis it is important to underline that the straight line is an ordered set: for any two points $x$ and $y$, we can say that $x \leq y$ or $y \leq x$. We also recall that continuum has the algebraic structure of the field of real numbers.

We also should mention the great invention of Descartes who introduced cartesian systems of coordinates in physical space. We present the well known story about Descartes’ discovery of systems of coordinates. This story is not just curious, but it would play an important role in the comparative analysis of mapping of physical and mental processes on mathematical spaces.

Once Descartes stayed in a hotel and he occasionally looked through a window and saw a tree in the garden nearby. There was a metal lattice on the window. And that was the point! Descartes imagined the encoding of various parts of the tree by using the lattice. He understood that by using lattices with smaller and smaller cells one could create better and better encodings of the root, the trunk, branches, and leaves of the tree.

**Step 2. Dynamical equations.** Material objects were mapped on real continuum (to be more precise: on the cartesian product of three real lines) and it became possible to describe motions of such objects in this mathematical space.

**Step 2a. Second Newton law.** I. Newton formulated the fundamental dynamical law:

$$mass \times acceleration = force$$

Since the acceleration $a = \frac{d^2x}{dt^2}(t)$, where $x(t) = (x_1(t), x_2(t), x_3(t))$ is the trajectory of a physical system, this law can be written as a differential
equation:

\[ m \frac{d^2x}{dt^2}(t) = f(x), \] (1)

where \( m \) is the mass and \( f(x) \) is the force acting to a system. By fixing the initial position and the velocity, \( x(0) = x_0, \ v(0) = \frac{dx}{dt}(0) = v_0 \), we determine the trajectory \( x(t) \) as the unique solution of (1). Therefore Newtonian mechanics is a deterministic theory: if initial conditions are fixed the trajectory is uniquely determined.

**Step 2b. Hamiltonian equations.** An important reformulation of Newton’s mechanics is given by *Hamiltonian formalism*. Let us introduce the *momentum* \( \xi = mv \equiv m\frac{dx}{dt} \) and the *energy* \( H(\xi, x) = \frac{\xi^2}{2m} + V(x) \), here \( H_{\text{Kin}} = \frac{\xi^2}{2m} \) and \( H_{\text{potential}} = V(x) \) are kinetic and potential energies. A force \( f \) is said to be potential if \( f(x) = -\frac{dV}{dx}(x) \). For potential forces \( f \), the Newton equation (1) can be rewritten in the form

\[
\frac{dx}{dt} = \frac{\partial H}{\partial \xi}, \quad \frac{d\xi}{dt} = -\frac{\partial H}{\partial x} \tag{2}
\]

The first equation is just the definition of the momentum \( \xi \) and the second coincides with (1). The system (2) of Hamiltonian equations with initial conditions \( x(0) = x_0, \ \xi(0) = \xi_0 \) determines uniquely the trajectory \( (x(t), \xi(t)) \) in the so called phase space - the cartesian product of 3 + 3 real lines. This is the classical phase space dynamics.

**Step 3: Statistical mechanics.** Consider millions of particles which motions are described by the Newton (or Hamiltonian) equations. Existence of such a model is extremely important from the philosophic viewpoint, but this deterministic model is not so useful for applications. It is meaningless to describe mathematically millions trajectories. Moreover, by investigating individual trajectories we cannot find some “collective characteristics” of an ensemble of particles such as, e.g., energy, temperature. In this situation it is natural to describe statistical behavior of ensembles of particles. Let us consider the simultaneous probability distribution of particles’ position \( x \) and momentum \( \xi \), and denote the density of this probability distribution by \( \rho(x, \xi) \). Thus the probability to find a particle in a domain \( O \) of the phase
space can be calculated as
\[ P((x, \xi) \in O) = \int \int_{O} \rho(x, \xi) \, dx \, d\xi, \]

By using the Hamiltonian equations it is easy to derive the evolution equation for the density \( \rho \). This is the Liouville equation:
\[ \frac{\partial \rho}{\partial t} = \{ \rho, H \}, \quad (3) \]

where \( \{ \rho, H \} \) is the Poisson bracket of functions \( \rho \) and \( H \):
\[ \{ \rho, H \} = \frac{\partial \rho}{\partial x} \frac{\partial H}{\partial \xi} - \frac{\partial \rho}{\partial \xi} \frac{\partial H}{\partial x}. \]

The probability distribution \( P \) on the phase-space is the basic object of statistical physics. One can forget about behavior of individual systems and investigate only probabilities.

**Step 4. Stochastic processes, Brownian motion, diffusion.** Let us consider the motion of a Brownian particle. The trajectory of such a particle can be extremely irregular. Almost all trajectories are not smooth. Thus the basic tool of Newtonian mechanics – differentials – becomes totally meaningless. It is impossible to apply the Newton (or Hamiltonian) equation to describe the dynamic of a particle. However, there were developed new mathematical methods based on *stochastic differential equations* which give the possibility to describe dynamics of Brownian-like systems:
\[ dx(t, \omega) = a(x(t, \omega)) \, dt + b(x(t, \omega)) \, d\omega(t), \quad (4) \]

where \( dt \) is the ordinary differential and \( d\omega(t, \omega) \) is the Ito differential – an “infinitesimal element of the Brownian process.” Here the parameter \( \omega \) describes “chance.” It is natural to consider \( \omega \) as the label for a particle: \( x(t, \omega) \) is the trajectory of the particle \( \omega \). This stochastic differential equation completed by the initial condition \( x(0, \omega) = x_0(\omega) \) describes the dynamics of a particle \( \omega \). At the first sight there is no difference with deterministic Newtonian mechanics. But, in fact, the difference is very large. The crucial point is that a solution of a stochastic differential equation is unique (again for Lipschitz coefficients) only up to *stochastic equivalence*. We recall that two stochastic processes \( x(t, \omega) \) and \( y(t, \omega) \) are stochastically equivalent if for any \( t \):
\[ P(\Omega_t) = 0, \text{ where } \Omega_t = \{ \omega : x(t, \omega) \neq y(t, \omega) \}. \quad (5) \]
Two solutions $x(t, \omega)$ and $y(t, \omega)$ of the same stochastic differential equation (with the same initial condition) can be different, but only with probability zero (so for negligibly small number of particles). However, these negligibly small sets of particles $\Omega_t$ depend on $t$. Thus, for a time interval $[0, \delta]$, solutions can be essentially different. Therefore in theory of stochastic differential equations people are interested not in individual solutions, but in corresponding probability distribution $P(t, U) = P(\omega : x(t, \omega) \in U)$. This is the probability that at the instant of time $t$ a particle $\omega$ belongs to a domain $U$ of physical space. The density of this probability distribution $p(t, x)$ satisfies the Kolmogorov’s forward equation:\(^2\)

$$\frac{\partial p}{\partial t}(t, x) = Lp(t, x), \lim_{\epsilon \downarrow 0} p(t, x) = p_0(x), \quad (6)$$

Here $L$ is the generator of the evolution of probability distributions. We consider a particular class of generators $L$ which in the one dimensional case have the form:

$$Lp(x) = \frac{1}{2} \frac{d^2}{dx^2} [b^2(x)p(x)] - \frac{d}{dx} [a(x)p(x)].$$

The corresponding Kolmogorov’s forward equation \(^3\) describes diffusion and the corresponding stochastic differential equation is called diffusion equation.\(^3\) In particular, if $b = 1$ and $a = 0$, i.e. $L = \frac{1}{2} \frac{d^2}{dx^2}$, the equation \(^6\) describes the evolution of the probability distribution for Brownian motion (Wiener process).

**Step 5: Quantum physics** We shall not consider quantum models in this paper, see, e.g., [10] and the extended bibliography in that paper. Quantum mechanics is also a statistical theory: all quantum experiments are about statistical behavior of huge ensembles of quantum particles. However, in the opposition of classical statistical physics, it is assumed (at least by those who use the orthodox Copenhagen interpretation) that quantum probabilities cannot be reduced to ordinary ensemble probabilities. Thus there is no any deterministic prequantum process – neither deterministic (as in

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\(^2\)This equation is known to physicists as the Fokker-Planck (or continuity equation).

\(^3\)The coefficients $a(x)$ and $b^2(x)$ are called drift and diffusion coefficients. By neglecting inertia of a particle it can be assumed that its motion consists of two components: drift induced by macroscopic velocity of flow of liquid and fluctuations induced by chaotic motion of molecules of liquid.
classical statistical physics) nor stochastic (such as the diffusion process) – which could generate quantum probabilistic distributions.4

3 Mapping of mental processes on mathematical spaces

We repeat for mental processes the scheme of mapping on mathematical spaces which has been successfully realized in physics:

Step 1: Elaboration of the notion of mental space. We should find an adequate mathematical model of mental space $M$. Points of $M$ (mental points) will represent elementary mental states.

Step 2: Dynamical equations in mental space. There should be presented dynamical equations describing trajectories $x(t)$ of elementary states in the mental space $M$.

Step 3: Mental statistical dynamics. In the same way as in classical statistical physics it is natural to study not behavior of individual states (millions of trajectories in $M$), but probabilistic behavior of huge statistical ensembles of elementary states. And this is not just the question of simplification of the mathematical description. There are numerous evidences that the mind is the product of the activity of the whole brain, see, e.g., [11]-[13]. Thus mind should be produced by ensembles and not individual states. Later we shall present some neurophysiological arguments in favor of “ensemble-mind.” At the moment we present a general argument from information theory. This is the argument of security of processing of information. For an information system producing millions of states it would be very dangerous to realize functions of large importance by operating with individual states. There should be developed the ability to take “collective decisions.” Probability is the fundamental factor determining such decisions. On the other hand, we can not exclude that some simple mental functions can be realized through dynamics of individual states.

Step 4. Mental diffusion. As we know from physics, classical statistical mechanics is still deterministic. There is the perfect mechanical order described by the Newton (or Hamilton) equations. Trajectories are smooth

4Of course, there are some attempts to construct such processes, e.g., Bohmian mechanics (deterministic prequantum process), or stochastic electrodynamics (prequantum stochastic process).
and particles do not perform actions violating Newtonian order! Intuitively the mental dynamics does not look so well ordered as the Newtonian dynamics. It may be more natural to consider in mental space $M$ the stochastic dynamics, instead of the deterministic dynamics. Later we shall present a neurophysiological argument supporting the stochastic dynamical model (in mental space!).

We emphasize that by choosing the stochastic mental dynamics, instead of the Newtonian mental dynamics, we do the step having fundamental consequences for cognitive science. Assume that the mental behavior is determined only by probabilities. Then different (stochastically equivalent) processes $x(t, \omega), y(t, \omega), \ldots$ (which satisfy the same stochastic differential equation in the mental space $M$ with the same initial condition) induce the same mental behavior (determined by the probabilities).

In our model (see section 6) elementary mental states are produced by neural structures. In this model stochastic evolution of elementary mental states implies the violation of the materialistic axiom of cognitive science. In our model:

**Different stochastic neural dynamics can produce the same mental behavior.**

It may be that the materialistic axiom is violated only for high level mental functions which operate with probabilistic distributions produced by neuronal ensembles.

This is our program of a mathematical mapping of mental processes on mathematical spaces. It seems that precisely in this form the program has never been formulated. Nevertheless, there were developed various approaches which realized some steps of this program (of course, the same step can have a few different realizations depending on choices of mathematical models for mental space and dynamical equations). We mention *Dynamical System Approach* [14]-[19] which is based on the following principles:

(a) *Embodiment* of mind;

(b) *Situatedness* of cognition.

We recall that the orthodox Dynamical System Approach emphasizes commonalties between behavior in neural and mental processes on one hand and with physiological and environmental events on the other. The most important commonality is the dimension of time shared by all these domains. In fact, sharing of time scales with physical environment implies

(c) Continuous real time evolution described by differential equations.
At the first step people working in the DS-approach decided to borrow the model of space from physics. This was an attempt to embed mind into the Euclidean physical space. This step (the special choice of space) determined the following development of the DS-approach. At the second step there were applied standard differential equations describing physical processes in the brain as a physical system. In particular, there are widely applied oscillator models.

We can also mention symbolic models [20], [21], and Neural Network Approach, [22]–[25]. In the opposition of the DS-approach, in the NN-approach physical space does not play an important role. The fundamental role is played by potentials $x_i(t)$ corresponding to neurons $i = 1, \ldots, N$. But this using of physical potentials also implies application of the standard mathematical models of physics. “Mental space” of the NN-approach – the space of potentials – is also the real space which is the mathematical basis of physics.\(^5\)

\section{p-adic hierarchic mental space}

In [1] there was proposed the $p$-adic model of mental space. We shall give the detailed description of the $p$-adic mental space later and now we discuss some general arguments in favor of this choice.

\begin{itemize}
  \item \textbf{Hierarchic structure of mental space.} In neurophysiology and psychology there is intensively discussed the idea that mental processes have hierarchic structures. I would like to say that the mental image of the physical world is a special hierarchic representation of this world. Thus a mental space should be endowed with a hierarchic structure which would adequately represent hierarchies observed in psychology and cognitive science. However, the real continuum is totally homogeneous; there is no natural hierarchic structure on the real continuous space. All physical points “have equal rights” and translation as well as rotation invariance are the fundamental features of all physical theories. This is one of reasons to reject the real continuous model as a candidate to a model of mental space. On the other hand, the $p$-adic space has a natural hierarchy which is induced by a tree structure of this space.

  \item \textbf{Tree structure, mental space as a product of neuronal activity.} It is clear that mental space should be coupled with the material\(^5\)
\end{itemize}

\(^5\)We remark that in the NN-approach there were realized main steps of our program, including stochastic neural models.
structure of the brain. This structure should be represented in some way in mental space.\textsuperscript{6} It is well known that many neuronal configurations in the brain have the tree-structure. It seems to be natural to have such a tree structure on mental space. And we remark that $p$-adic spaces have tree structures, see Figure 1 - the tree representation of the 2-adic space.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{2-adic_tree.png}
\caption{The 2-adic tree}
\end{figure}

For an arbitrary natural number $p > 1$, the $p$-adic tree is constructed in the same way: there are $p$-branches leaving each vertex. In principle, we can consider trees, where the number of branches depends on a vertex. However, mathematics on such trees is essentially more complicated, cf. [1].

We now turn back to the story about the discovery of the cartesian system of coordinates. Suppose for a moment that Descartes paid attention not to the lattice on the window of his room, but to the internal hierarchic structure of the tree: the root, the trunk, branches, subbranches, leaves. In such a case he would

\textsuperscript{6}This statement is not about \textit{reductionism}. We do not claim that mind can be directly reduced to neuronal activity. As was mentioned, in our model the materialistic axiom of cognitive science is violated. We just pay attention to the fact that physical neural and mental structures should be coupled. We even need not stay on the materialistic position. One could not totally exclude the possibility that material neuronal structures are images of mental structures which are located in mental space. However, I would not like to go deeply in such philosophic problems. This paper is simply an attempt to create a mathematical model which contains new mental coordinates which differ essentially from physical real coordinates.
discover not the ordinary orthogonal system of coordinates, but a \( p \)-adic hierarchic system on the tree.

(c.) **Absence of linear order in the space of mental states.** As we know, the real continuum is a *linearly ordered set*. Any set of points \( x_1, \ldots, x_m \) on the straight line can be ordered. On the other hand, there are no reasons to assume that such an order structure can be introduced on the space of mental states. It is impossible to order all minds, feelings, emotions, ....; for example, it is very doubtful that it would be possible to create a model of mental space such that for any two minds \( x \) and \( y \) it would be possible to say \( x \leq y \) or \( y \leq x \). Therefore the absence of a linear order on a \( p \)-adic tree is an attractive feature of the \( p \)-adic model which can be useful in mathematical modeling of mental processes.

(d.) **Link to Aristotle, ultrametricity.** As was already remarked, Aristotle discussed in detail geometries of physical and mental spaces (for him the latter was associated with the space of words), see [26]. Continuity was considered as the main distinguishing feature of physical space. For Aristotle the term continuity was related not to functions on some spaces (as mathematicians do in the contemporary analysis), but to spaces by themselves. Aristotle’s notion of *continuous space* coincides with the notion of *connected space* which is used in contemporary mathematics. A space is said to be connected if it is impossible to split it into two parts in such a way that there are no boundary points. Aristotle presented important arguments that physical space is continuous-connected. At the same time he underlined that mental space (space of words) is discontinuous (disconnected). Such a space can be split into two parts without boundary points.

By using the contemporary terminology one can say that two thousands years ago there was discussed the idea that mental spaces should be represented as disconnected mathematical spaces. The class of disconnected topological spaces is well investigated, [27], [28]. We pay attention to the important subclass of disconnected spaces – spaces which are endowed with metrics. This is a purely mathematical restriction which essentially simplifies modeling (but in principle there may exist natural mental applications of non metrizable disconnected mental spaces). Under some mathematical restrictions we get the class of so called *ultrametric spaces*. We recall that a metric is called ultrametric if it satisfies the *strong triangle inequality*:

\[
\rho(x, y) \leq \max[\rho(x, z), \rho(z, y)].
\]
The strong triangle inequality can be stated geometrically: each side of a triangle is at most as long as the longest one of the two other sides. Finally, we mention a theorem of general topology [28] which says that each ultrametric space can be represented as a tree and vice versa. So we come to the tree model of mental space which was developed in [1]-[10]. Such a model can be considered as a modern version of the Aristotle’s model of mental space. I interpret ultrametricity as the topological encoding of the hierarchic structure of a tree. By operating in an ultrametric space we can forget about the underlying hierarchic structure and use the language of analysis: neighborhoods, open and closed sets, convergence of sequences, continuity of functions (the contemporary analysis gives the possibility to consider continuous functions even on discontinuous spaces).  

(e.) $p$-adic trees and numbers. The class of general ultrametric spaces is still too large for mathematical modeling (at least at the first stage of realization of our program). In mathematical modeling of physics the important role is played not only by continuity (topology) of the space, but also by the presence of an algebraic structure on this space; elements of real line can be interpreted as numbers (real numbers). We would like to have such a structure on mental space. We need “mental numbers.” There is well defined algebraic structure on a special class of trees, so called $p$-adic trees. These are trees where $p$-branches leave each vertex and $p$ is a prime number. This number is an invariant of a tree and it does not depend on a vertex. As was already remarked, on such a tree we can introduce the structure of a field; if $p$ is not a prime number, then the $p$-adic tree is only a ring, [1], [27], [32], [33]. By using Figure 1 we can encode infinitely long branches of the 2-dic tree by strings of zeros and ones. Thus instead of branches of a tree we can work with such strings. In the general case of a $p$-adic tree these are strings of symbols belonging to the set \{0, 1, ..., $p$ - 1\}. The distance between two strings $x$ and $y$ representing branches of the $p$-adic tree is defined as

$$\rho_p(x, y) = \frac{1}{p^{l(x,y)},}$$

where $l(x, y)$ is the length of the common left-hand side part of strings $x$ and $y$. This is an ultrametric. So any $p$-adic tree is an ultrametric space. For example, let $p = 2$ and let us consider strings $x = (000...)$ and $y = (001...)$.  

In many particular cases such a relation between ultrametricity and hierarchy was used in theory of spin glasses, see, e.g., [29]-[31].
Then \( l(x, y) = 2 \) and \( \rho_2(x, y) = 1/4 \). Let \( z = (01...) \). Then \( l(x, z) = l(y, z) = 1 \) and hence \( \rho_2(x, z) = \rho_2(y, z) = 1/2 > 1/4 = \rho_2(x, y) \).

The set of \( p \)-adic numbers is denoted by the symbol \( \mathbb{Q}_p \). We choose \( \mathbb{Q}_p \) as a mathematical model of mental space. In more general models we consider mental spaces which have many \( p \)-adic coordinates: cartesian products of a few \( p \)-adic trees; or even trees corresponding to different numbers \( p_1, p_2, ..., p_n \).

In our model \( p \)-adic numbers are considered as mental numbers. At the moment we do not assign some special cognitive meaning to branches of \( p \)-adic trees. There can be various models. In a series of papers [2], [5], [6] we encoded psychological states by branches of \( p \)-adic trees. In this paper we would like to find links to neurophysiology and associate mental strings with activity of neurons, see section 6.

(f.) \( p \)-adic dynamical models for mental processes. We considered discrete dynamical systems

\[
x_{n+1} = f(x_n), \quad n = 0, 1, ...
\]

corresponding to maps \( f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p, \quad x \rightarrow f(x) \). We use the standard terminology of the theory of dynamical systems; see, for example, [1]. If \( f(a) = a \) then \( a \) is a fixed point. If \( x_n = x_0 \) for some \( n = 1, 2, ... \) we say that \( x_0 \) is a periodic point. If \( n \) is the smallest natural number with this property then \( n \) is said to be the period of \( x_0 \). A fixed point \( a \) is called an attractor if there exists a neighborhood (ball) \( V(a) \) of \( a \) such that all points \( x_0 \in V(a) \) are attracted by \( a \), i.e., \( \lim_{n \rightarrow \infty} x_n = a \).

In [1]-[8] there was proposed a dynamical model for the process of thinking which is based on hierarchic coding of mental information by \( p \)-adic numbers and processing by maps in \( p \)-adic trees. These are continuous maps with respect to tree’s ultrametric. We recall that the tree-hierarchy is encoded into the ultrametric topology. Thus such maps preserve hierarchic encoding of information.

Another basic feature of our dynamical model [2] is that the process of thinking is split into two separate (but at the same time closely connected) domains: the conscious and the unconscious. We used the following point of view of the simultaneous work of the consciousness and unconsciousness. The consciousness contains a control center \( CC \) which has functions of control. It formulates problems and sends them to the unconscious domain. The process of finding a solution is hidden in the unconscious domain. In the unconscious domain there works a gigantic dynamical system. Its work starts
with a mental state \( x_0 \) (or a group of mental states \( U_0 \)) which has been communicated by the consciousness. Mathematically it corresponds to the choice of an initial point \( x_0 \) (or a neighborhood \( U_0 \)). Starting with this initial point \( x_0 \) a thinking processor \( f \) in the unconscious domain generates at tremendous speed a huge number of new mental states:

\[
x_1 = f(x_0), \ldots, x_{n+1} = f(x_n), \ldots
\]

These mental states are not used by the consciousness. The consciousness (namely \( CC \)) controls only some exceptional moments in the work of the dynamical system in the unconscious domain. These are different regimes of stabilization. First, there are attractors which are considered by the consciousness as possible solutions of the problem. Then there are cycles \((a \rightarrow b \rightarrow \cdots \rightarrow c \rightarrow a)\) which generate signals to stop the work of the dynamical system. If the consciousness cannot take a decision then it can send a new initial mental state \( x'_0 \) to the unconscious domain or change the regime of work of a thinking processor in the unconscious domain. Mathematically the change of a regime can be described as the change of a function \( f(x) \) which determines the dynamical system. Thus we can describe the process of thinking as the work of a family of dynamical systems \( f_{\alpha}(x) \), where the parameter \( \alpha \) is controlled by the consciousness (or chance in random dynamical thinking models, [6]). This model was used for modeling of psychological behavior. In particular, in [2] we simulated sexual behavior by encoding mental states of sexual partners by hierarchic strings of information:

\[
x = (\text{sex}, \text{age}, \text{education level}, \ldots)
\]

The sexual dynamical system \( f_{\text{sex}} \) operates with such hierarchical images and produces “solutions” (or exhibit cyclic behavior).

The main problem of this model was impossibility to choose functions \( f \) generating dynamics by using purely psychological reasons. We were able only provide animation corresponding to simple functions on \( p \)-adic trees and give psychological interpretation to results of such animation. Another reason for looking for new models was that the above \( p \)-adic dynamical model was purely deterministic. As was already mentioned, it seems to be more natural to study models in which mental information is processed probabilistically; models in which mental states are represented by fields of probability on mental space and not by single points of this space. Finally, in [1]-[6] we were not able to couple hierarchic \( p \)-adic encoding of information with neurophysiology. In following sections we try to solve this problem of mind-body relation and then proceed to probabilistic processing of mental information.
5 Coupling of the $p$-adic mental space with the neural structure

In previous sections there were presented some general arguments in favor of the $p$-adic model of mental space. Now we present the neurophysiological basis of the $p$-adic mental space. Coupling between the neuronal material structures and the $p$-adic mental structures is based on the correspondence between the neuronal hierarchy (i.e., hierarchic relations between neurons and groups of neurons) and the $p$-adic tree hierarchy.

In our model each psychological function is based on a graph of neural pathways, *cognitive graph*, that is *centered* with respect to one fixed neuron. Thus basic units of processing of mental information are *centered neural pathways* and basic units of mental information are *centered strings of firings* produced by centered neural pathways.

![Centered pathway](image1)

Figure 2: Centered pathway

Centering determines a *hierarchic structure* on the cognitive graph and on the corresponding space of mental points.

![Cognitive graph](image2)

Figure 3: Cognitive graph
A centering neuron $S$ should not be considered as a kind of grandmother neuron. It simply determines a system of mental coordinates (corresponding to the concrete psychological function) on the neural system of a cognitive system. Of course, such a model with one neuron centering of a psychological function is oversimplified. Complex psychological functions should be based on a few cognitive graphs centered with respect to an ensemble of neurons. The centering hierarchic structure on a cognitive graph can be mapped on the $p$-adic hierarchic structure with the aid of the frequency coding having $p$ as the base of coding, see section 6.3, 6.4. We call this approach Neural Pathway Approach. The essence of this approach is the view to the $p$-adic mental space as the product of hierarchically ordered neural activity and frequency $p$-coding.

6 Model: thinking on a cognitive graph

6.1. Localization of psychological functions. One of the strong sides of Neural Pathway Approach is a new viewpoint to the problem of localization of psychological functions. Since an elementary unit of mental processing is represented by a centered neural pathway and a pathway can go through various domains of brain and body, there is no localization of mental functions in the Euclidean geometry of the physical space that is typically used to describe physical brain and body. On the other hand, a psychological function can be localized in the space of all pathways.  

We have to distinguish the space $\Pi$ of all centered neural pathways (hierarchic chains of neurons) in the physical brain and body and the space $M$ of all possible mental points that can be produced by elements of $\Pi$. In principle, a few distinct elements of $\Pi$, centered neural pathways, can produce (at same instant of time) the same point $x \in M$.

6.2. Body→mind field. Firings of neurons throughout centered neural pathways of the cognitive graph produce elementary mental states (points)
involved in the realization of a psychological function. We denote a psychological function by the symbol $f$ and the cognitive graph of the $f$ by the symbol $\Pi_f$. How can we describe mathematically the functioning of $f$? There are various levels of the mathematical description.

At the basic level we should provide the description of ‘body→mind’ correspondence. This correspondence is described by a function

$$\varphi : \Pi_f \rightarrow M,$$

that maps centered neural pathways into mental points produced by these pathways: $z \in \Pi_f \rightarrow x = \varphi(z) \in M$. We call the map $\varphi(z)$ body→mind field.\(^9\)

The psychological function $f$ generates the evolution of the field $\varphi$. Starting with the initial field $\varphi_0(z)$, $f$ produces time-dependent field $\varphi(t, z)$. We have to consider very important problem of interpreting of the evolution parameter, ‘time’, $t$; in particular, relation between physical and psychological time.

We shall discuss this problem in section 9. At the moment we restrict ourself to consideration of the discrete time evolution: $t = t_n = 0, 1, 2, \ldots$ By taking into account the process of wholeness of thinking we describe the functioning of $f$ by an integral operator with the kernel $K(z, y)$:

$$\varphi(t_{n+1}, z) = \int_{\Pi_f} K(z, y)\varphi(t_n, y)dy, \quad (7)$$

where integration is performed over the cognitive graph, $\Pi_f$. We notice that neither the space of pathways $\Pi_f$ nor the space of mental points $M$ have the Euclidean geometry. In particular, we cannot use the ordinary real analysis to describe this model mathematically. We should use the so called ultrametric analysis [1],[27], [32].\(^10\) Thus we propose the following model of thinking:

Each psychological function $f$ is based on a graph of neural pathways, cognitive graph $\Pi_f$, see, e.g., Figure 2. The cognitive graph has the hierarchic structure corresponding to the central neuron, $S$, of this graph. The elementary unit of mental information – elementary mental state (or mental point) – is given by the string of firings of neurons throughout a pathway, a

---

\(^9\)Of course, $\phi$ depends on the psychological function $f$: $\phi = \phi_f$.

\(^{10}\)The form of the kernel $K(z, y)$ is determined by the psychological function $f$. We notice that mental evolution \(^7\) is represented by a linear integral operator in the space of body→mind fields. In principle, we can consider more general, nonlinear models. However, the model with summation over the whole graph with a weight function $K(z, y)$ looks very natural.
branch of the graph. There can be proposed various neural pathway coding systems based on strings of firings.

6.3. **Firing/off (2-adic) coding.** For each instant of time $t$, we assign to a neuron in a pathway 1, if the neuron is firing, and 0, otherwise. Mathematically a mental point is represented by a sequence of zeros and ones. Each sequence is centered with respect to the position corresponding to firings of the central neuron $S$.

Let us consider the geometric structure of the mental space $M$ corresponding to firing/off coding. Here each centered pathway produces a centered sequence of zeros and ones. The most important digit, $x_0 = 0$ or 1, in a sequence $x \in M$ gives the state of the central neuron $S$. Hierarchy on the mental space is based on the exceptional role that is played by the central neuron. This hierarchy induces the 2-adic topology on the mental space.

In mathematical modeling it is convenient to consider infinitely long neural pathways and corresponding information strings, mental points (this is of course just a mathematical idealization). In section 4 we introduced the 2-adic metric for mental points produced by cognitive graphs having only output (with respect to the central neuron $S$) neural pathways. Now we introduce the 2-adic metric for mental points produced by general cognitive graphs which contain input as well as output neural pathways. The 2-adic distance $\rho_2$ is defined in the following way. Consider two mental points:

$$
x = (\ldots x_{-l} \ldots x_0 \ldots x_k \ldots) \quad \text{and} \quad y = (\ldots y_{-l} \ldots y_0 \ldots y_k \ldots), \quad x_{\pm j}, y_{\pm j} = 0, 1.
$$

We use index 0 for the state $x_0$ of the central neuron $S$, negative indexes for states of neurons that produce inputs propagating to the $S$ through the ordered neural chain, positive indexes – for states of neurons that are receivers of $S$-output. First suppose that all input states coincide: all $x_{-l} = y_{-l}$. Let $l \geq 0$ be the first index such that $x_l \neq y_l$. Then by definition

$$
\rho_2(x, y) = \frac{1}{2^l}.
$$

This is the distance considered in section 4. Thus if two neural pathways $z$ and $w$ produce strings $x$ and $y$ having the same input part, then $\rho_2(x, y)$ goes to zero if the length $l$ of the common output part goes to infinity.

Suppose now that input parts are different. Let $l$ be the first index (if we go from the left hand side) such that $x_{-l} \neq y_{-l}$. Then by definition

$$
\rho_2(x, y) = 2^l.
$$
Here larger common initial input part implies shorter distance between mental points. Consider two neural pathways starting at e.g. a sensory receptor. Suppose that states (e.g. on/off) of initial neurons in these pathways are distinct. Then the distance between corresponding mental points is large. It is important to remark that in such a situation longer pathways induce larger distance between mental points, see also section 9.

We remark that a cognitive graph $\Pi_f$ producing the 2-adic mental space need not have a tree-structure. Such a graph can contain numerous cycles. However, the corresponding mental space $M$ created by the 2-adic coding always has the structure of the 2-adic tree.

6.4. Frequency ($p$-adic) coding. General $p$-adic coding (where $p \geq 2$ is a natural number) may be induced by the frequency coding. We assign to each neuron in a pathway the frequency of firings. Frequencies of firing are a better basis for the description of processing of information by neurons than a simple on/off. This has been shown to be the fundamental element of neuronal communication in a huge number of experimental neurophysiological studies (see e.g. [35], [36] on mathematical modeling of brain functioning in the frequency domain approach). In the mathematical model it is convenient to consider a discrete set of frequencies: $0, 1, \ldots, p - 1$, where $p$ is some natural number. Here frequency is the number of output spikes produced by a neuron during some unit of psychological time (some period $\Delta$ of physical time, see section 6 for detailed consideration). Thus mathematically a mental point is represented by a sequence of numbers belonging to the set $\{0, 1, \ldots, p - 1\}$. Information is not homogeneously distributed throughout such sequences. The presence of the central neuron $S$ in the cognitive graph $\Pi_f$ induces a hierarchic structure for elements of an information sequence.

This system of neural pathway frequency coding should be justified by neurophysiological studies. However, at the moment there are no experimental technologies that would give the possibility to measure firings of neurons throughout even one long pathway of individual neurons. To confirm our pathway-coding hypothesis, we have to measure simultaneously firings of neurons for a huge ensemble of neural pathways.

We underline that the system of coding and not the topological structure of a cognitive graph determines the structure of the corresponding mental space, see also section 9. Totally different cognitive graphs can produce the same mental tree. For example, let us consider 2-adic coding. The graphs $a,b,c$ on Figure 4 produce the same, 2-adic, mental space.

In principle, we can consider our Neural Pathway Model an approximation
of TNGS-model, see Edelman [37] on Theory of Neural Groups Selection. To combine TNGS with our model, we should consider hierarchic chains that basis elements not singular neurons, but some groups of neurons). So it will be Neuronal Group Pathway Model. In such a model we shall use natural numbers $x_j = 0, 1, ..., p - 1$, for coding of states of basic neuronal groups.

6.5. **Main cognitive features of the model:**

a). Nonlocality (with respect to Euclidean geometry) of psychological functions.

b). Wholeness-integral evolution of body $\rightarrow$ mind field $\varphi$. \[11\]

c). Sensation-thinking. Since neural pathways go through the whole body, a part of a pathway involved in a high level psychological function can be connected to e.g. skin-sensitivity. Thus high order psychological functions also depend on various physiological stimuli.

d). Interrelation of distinct psychological functions. The central neuron $S$ of a cognitive graph plays the role of the center of the system of coordinates. Other neurons can also be considered as such centers. Therefore the same pathway contributes to distinct psychological functions. Thus evolution of various psychological functions is simultaneous evolution based on huge interrelation of corresponding cognitive graphs, see section 9.

e). Emotion based reasoning. Our pathway thinking model supports the fundamental conjecture of A. Damasio, [34], that emotions play an important role in the process of ‘reason-thinking’. Pathways going through centers creating emotions can participate in psychological function of a high order thinking process, e.g., proving mathematical theorems. On the other hand, pathways going through reasoning-centers can go through some emotional center. Thus reason participate in creation of emotions and vice versa.

\[11\] Compare to Bohmian-Hiley-Pilkänen approach [11]-[13].
7  Dynamics in the mental space.

7.1. Mental state. The body→mind field $\varphi(z)$ describes important features of functioning of the neural system (in particular, its part located in brain). However, we will not be concentrated on the study of dynamics, e.g. \[7\], of the body→mind field, since $\varphi(z)$ describes the production of information by the neural system (in particular, its part located in brain) and not the flow of mental information by itself. I would like to formulate this as

Mental Thesis: The cognitive meaning of a mental point (with respect to a psychological function $f$) does not depend on a neural pathway that produces this mental point.

Mental activity is performed not in the pathway space, cognitive graph, but in the mental space. Thus mental information does not remember its neurophysiological origin. Mental Thesis is supported (at least indirectly) by experimental evidences that functions of some damaged parts of brain can be (in some cases) taken by other parts of brain, see e.g. [34], [38], [39]. This thesis is also supported by neurophysiological evidences that very different neural structures in brains of different species (e.g. fish and rat, [40]) can fulfill the particular psychological function.\[12\]

Mental Thesis might be considered as a kind of anti-materialist thesis. We would not like to be at such a position. We understand well that the relation between the brain (in fact, in our pathway model - the whole body) and mind plays the crucial role in mental activity. Mental Thesis should be considered as directed against the individual deterministic relation between functioning physical neural pathways in the body and cognitive meaning of the corresponding mental points. The absence of such individual determinism does not contradict to statistical determinism:

Thesis of Statistical Pathway Cognition. The cognitive meaning of a mental point (with respect to a psychological function $f$) is determined by the statistical probability of realization of this point in the ensemble of pathways $\Pi_f$ (the cognitive graph corresponding to $f$).

Remark 7.1. (On the notion of probability) During many years I studied foundations of probability theory, see [41]. I know well the large diversity of

\[12\] Of course, we should recall that by choosing the central neuron $S$ we chose the concrete psychological function $f$. Thus ‘the cognitive meaning’ is related to this concrete psychological function. By choosing another psychological function (a system of coordinates) we get another cognitive meaning.
viewpoints to the notion of probability. For me probability has nothing to do with ‘potentiality’, ‘measure of belief’ and other perverse views. Probability is a statistical measure. Let $\mathcal{E}$ be a large ensemble of, e.g., physical systems. Let these physical systems have some states. These states are represented by points of the state space. The probability of a state $x$ with respect to the ensemble $\mathcal{E}$ is given by the proportion:

$$\Pr(x) = \frac{\text{the number of systems having the state } x}{\text{the total number of elements in the ensemble}}$$

In our model physical systems are centered neural pathways; states are strings of firings throughout pathways - mental points. An ensemble $\mathcal{E}$ is a cognitive graph; the state space is the mental space. We suppose that the cognitive meaning of a mental point $x \in M$ is determined by the quantity

$$\Pr(x) = \frac{\text{the number of neural pathways that produce } x}{\text{the total number of neural pathways in the cognitive graph}}$$

We call the $\Pr(x)$ (probabilistic) mental state. Here and in all following considerations it is assumed that a psychological function $f$ is fixed. In fact, $\Pr(x)$ depends on $f : \Pr(x) = \Pr_f(x)$.

### 7.2. Mental evolution

Mental processes are probability-evolution processes. These are evolutions of mental states. We have to find a mathematical model that would provide the adequate description of the evolution: $t \to \Pr(t, x)$. Consider a discrete dynamical system in the space of probability distributions:

$$\Pr(t_{n+1}, x) = L \Pr(t_n, x), \quad (10)$$

where $L$ is some operator in the space of probability distributions. The generator of evolution $L$ may be linear, may be nonlinear. There must be performed experimental studies to find the form of $L$. Of course, $L$ depends essentially on a psychological function $f$ and an individual.

Intuitively it is clear that there must be performed integration over the whole mental space. The following evolution can be considered:

$$\Pr(t_{n+1}, x) = \int_M K(t_n, x, y)\Pr(t_n, y)dy,$$

where $K(t, x, y)$ is a time-dependent kernel of evolution. For $p$-adic mental space $M = \mathbb{Q}_p$, $dy$ is the Haar measure – an analogue of the ordinary linear Lebesgue measure on the real line.
The mental state $p(t, x)$ is nothing else than the probability distribution of the body→mind field $\varphi(z)$. We can consider the cognitive graph $\Pi_f$ as a probability space $\Omega \equiv \Pi_f$ with the uniform probability measure

$$P(\omega) = \frac{1}{\text{number of elements in } \Omega},$$

for $\omega \in \Omega$. Following to the probabilistic tradition we use the symbol $\omega$ to denote a point of the probability space. The map (body→mind field) $\varphi : \Omega \to M$ is a random variable and the mental state

$$p(x) = P(\omega \in \Omega : \varphi(\omega) = x)$$

gives the probability that neural pathways in the cognitive graph represent the mental point $x$. This is intensivity of neural representation of $x$.\(^{13}\)

Thus the dynamics of the mental state, $t \to p(t, x)$, can be considered as the product of the dynamics of the corresponding stochastic process $\varphi(t, \omega)$, the process of body→mind correspondence: $P(t, U) = P(\omega \in \Omega : \varphi(t, \omega) \in U)$, where $U$ is a domain in the $Q_p$ (or in the cartesian product of $p$-adic fields). But(!) probability theory tells us that we could not reconstruct the stochastic process $\phi(t, \omega)$ as a point wise map in the unique way on the basis of corresponding probability distributions. Two stochastically equivalent body→mind fields $\varphi_1(t, \omega)$ and $\varphi_2(t, \omega)$ produce the same dynamics of probabilistic mental state $p(t, x)$, cf. section 2. This argument gives the strong support to our Mental Thesis. In our model there exists the body→mind field $\varphi(t, \omega)$, but there is no mind→body field. This probabilistic consideration can be used as the strong argument supporting nonreductionism: *Neural reduction of mental processes is impossible.*

## 8 Diffusion model for dynamics of mental state

We now consider continuous time evolution for the simplest (from the mathematical viewpoint) body→mind fields – mental diffusion.\(^ {14}\)

\(^{13}\)According to Kolmogorov’s ideology [42] (that gives the basis of the modern probability theory) the structure (e.g. topological) of a probability space (in our case a cognitive graph $\Pi_f$) does not play any role in probabilistic formalism. Probabilistic formalism depends only on the structure of the configuration space in that random variables take values. In our case this is the $p$-adic mental space $M = Q_p$.

\(^{14}\)As we have already remarked, the ultrametric topology on the mental configuration space and continuous real time dynamics are incompatible. However, we do not have such...
The evolution of the mental state \( p(t, x) \) is described by the forward Kolmogorov equation (6), where \( L \) is a differential operator on the \( p \)-adic space generating diffusion. Here the time \( t \in [0, +\infty) \) is a real parameter and the mental point \( x \in \mathbb{Q}_p \) is a \( p \)-adic parameter and the value \( p(t, x) \in [0, 1] \) is also a real number.

**Remark 4.1.** (Determinism or free will?) Our mental model combines determinism and free will. Mental determinism is a consequence of deterministic evolution equations for probability distributions. However, this determinism is determinism of probabilities. Thus probabilistic representation of the mental state gives feeling of free will. We remark that the situation has some similarities with quantum theory, see [43] and extended bibliography in this paper.

In the \( p \)-adic mental space \( M \) the simplest diffusion process is Vladimirov-Volovich diffusion. This process was intensively studied in \( p \)-adic theoretical physics, [33]. The corresponding evolution equation (\( p \)-adic heat equation) has the form:

\[
\frac{\partial p}{\partial t}(t, x) = -\frac{1}{2}D_x^2 p(t, x), \quad p(0, x) = p_0(x),
\]

where \( D_x \) is a kind of differential operator on the \( p \)-adic tree (Vladimirov’s operator, [33]). We remark that (in the opposition of the ordinary real derivative) Vladimirov’s operator is nonlocal, i.e., \( D_x p(t, x) \) contains summation over all points of the mental space. This feature of \( p \)-adic diffusion could be related to wholeness of mental processes.

We can easily find the fundamental solution \( K(t, x) \). This is the solution for the initial mental state \( p_0(x) = \delta(x) \). Here \( \delta(x) \) is well known Dirac’s \( \delta \)-function. This is the probability distribution that is concentrated in the fixed mental point. By knowing the dynamics, \( K(t, x), x \in M, \) of the mental state starting with the ‘sharp mental state’ \( \delta(x) \) (all neural pathways of the whole mental graph produce the same mental point)\(^{15} \). We can find the dynamics of the mental state \( p(t, x) \) for any initial distribution \( p_0(x) \):

\[
p(t, x) = \int_M K(t, x - y)p_0(y)dy.
\]

A more realistic model of mental evolution is based on the \( p \)-adic heat equation with a mental potential \( V \). Here \( V(x, y) \) can be chosen as e.g.

\[
V(x, y) = \rho_p(x, y)^\alpha, \quad \alpha \geq 0,
\]

where \( \rho_p(x, y) \) is the \( p \)-adic distance between mental points.

\(^{15}\)We understood that such a mental state might not be approached in reality. So \( K(t, x) \) is merely the ideal object used in the mathematical model.
9 Discussion

9.1. Can be consciousness be treated as a variable? The first part of the book [44] of B. J. Baars contains an interesting discussion on the possibility to treat consciousness as a variable and the importance of such a treatment in cognitive science. My attempt to quantify mental state was, in particular, motivated by this discussion. However, we essentially modified Baars’ idea on mental-variable. As I understood, his consciousness-variable would be a kind of Newtonian physical variable such as position, velocity, force; see p. 11 [44] on similarity of consciousness-variable to variables in Newtonian gravity. Our mental-variable, mental state, is a probability distribution. Thus there are more similarities with statistical mechanics and even with quantum mechanics. Such a variable is not a local variable on brain. This is merely a wholeness variable, compare to [10]-[13].

Starting with a mental (probabilistic) state we can define a quantitative measure of consciousness, a kind of consciousness-variable. Of course, we understood well that such a complex phenomenon as consciousness could not be completely described by probability fields on mental space. Thus we just propose a mathematical formalization of some features of consciousness. Maybe there can be found many (may be infinitely many) other quantitative measures of various features of consciousness.

First we discuss the connection between levels of neural activity and levels of consciousness. The idea that there is the direct correspondence between levels of neural activity and levels of consciousness is the very common postulate of cognitive science. An extended discussion on this problem can be found, for example, in [44], p.18-19. I disagree that there is such a direct relation between levels of neural activity and levels of consciousness. For example, let us consider the extreme case in that all possible states of brain are activated. Such a super-activation definitely would not imply a high level of consciousness. It might be that not simply the level of neural activity determines the level of consciousness.

Our conjecture is that one of quantitative measures of consciousness is determined by the variation of a mental state (compare to [10]). One of possible numerical measures of the level of consciousness is entropy of a mental state. It is well known that (for the discrete probability distribution) entropy approaches the maximal value for the uniform probability distribution. By our interpretation this is the lowest level of consciousness (so it may be better to use entropy with minus sign as a quantitative measure of consciousness).

Another possibility is to define a measure of consciousness as the variation of a mental state (with respect to the $p$-adic metric on the mental space). This is a natural quantitative measure of some features of consciousness, since the topological structure of the mental space must be taken into account. The $p$-adic variation
can be defined by using Vladimirov’s differential operator [33], $D_x$, on the $p$-adic tree:

$$C(p) = \int_M |D_x p(x)|^2 dx,$$

(12)

where $p(x)$ is the mental state of a cognitive system. $C(p)$ is always nonnegative and it takes its minimal value, $C(p) = 0$, for the uniform probability distribution. At the moment we do not know anything about mental states having the $C(p)$ of the extremely high level, namely solutions of the problem: $C(p) \to \text{max}$.

9.2. Are animals conscious? Our model strongly supports the hypothesis that animals are conscious (see e.g. [44] on the detailed neurophysiological and behavioral analysis of this problem). If one of quantitative measures of consciousness is really determined by the variation $C(p)$ of the mental state, then animals are definitely able to produce such nontrivial variations by their systems of neural pathways. Moreover, in our Neural Pathway Model pathways going throughout body play the important role in the creation of consciousness. Thus the role of differences in brain structures should not be overestimated, compare with [44]. On the other hand, animals have lower levels of $C(p)$. We can quantify this problem in the following way. There exists a threshold $C_{\text{human}}$. Animals could not produce mental states $p(x)$ such that $C(p)$ is larger than the human threshold $C_{\text{human}}$. And human beings (at least most of them) could produce (at least sometimes) mental states for that $C(p)$ is larger than this threshold.

9.3. Blindsight. Our model might be used to explain the mystery of blindsight and similar phenomena: “The mystery of blindsight is not so much that unconscious visual knowledge remains. ...The greatest puzzle seems to be that information that is not even represented in area V1 is lost to consciousness when V1 is damaged.” - [44]. However, in our Neural Pathway Model by destruction of some individual neurons (e.g. in area V1) we destroy (modify) huge ensembles of neural pathways. Of course, by our model information was never preserved in area V1 nor some other localized area. Information is preserved by ensembles of pathways and they are not located in some particular domain of brain.

However, we also have to explain the unique function of area V1 in creating of visual consciousness: “V1 is the only region whose loss abolishes our ability to consciously see objects, events... But cells in V1 respond only to a sort of pointillist level of visual perception... Thus it seems that area V1 is needed for such higher-level experiences, even though it does not contain higher-level elements! It seems like a paradox.” - [44]. Yes, area V1 is the unique region that damage destroy our ability to conscious visualization. But we recall that our model is, in fact, Centered Neural Pathway Model. The uniqueness of area V1 in conscious visualization is determined by the fact that central neurons of cognitive graphs corresponding to the psychological function of conscious visualization are located in area V1.

27
We continue citation of Baars [44]: “Cells that recognize objects, shapes, and textures appear only in much “higher” regions of cortex, strung in a series of specialized regions along the bottom of the temporal lobe.” Yes, these cells are centers of cognitive graphs corresponding to other psychological functions, e.g. object recognition.

9.4. Neural code and structure of mental space. Suppose that the coding system of a cognitive system is based on a frequency code. There exists a time interval $\Delta$ (depending on a cognitive system and a psychological function) such that a mental point produced by a centered neural pathway is a sequence with coordinates given by numbers of oscillations for corresponding neurons during the interval $\Delta$. Thus in our model the problem of the neural code is closely related to the problem of time-scaling in the neural system. For different $\Delta$, we get different coding systems, and, consequently, different structures of mental spaces. The corresponding natural number $p$ that determines the $p$-adic structure on the mental space is defined as the maximal number of oscillations that could be performed by neurons (in the cognitive graph for some fixed psychological function) for the time interval $\Delta$. The frequency coding is based on the 2-adic system induces the 2-adic mental space that differs crucially from the 5-adic (or 1997-adic) mental space induced by the 5-adic (or 1997-adic) system. As it was demonstrated in [1], by changing the $p$-adic structure we change crucially dynamics. Hence the right choice of the time scaling parameter $\Delta$ plays the important role in the creation of an adequate mathematical model for functioning of a psychological function.

9.5. Psychological time. There might be some connection between the time scale parameter $\Delta$ of neural coding and psychological time. There are strong experimental evidences, see e.g. [45], that a moment in psychological time correlates with $\approx 100$ ms of physical time for neural activity. In such a model the basic assumption is that the physical time required for the transmission of information over synapses is somehow neglected in the psychological time. In the model, the time ($\approx 100$ ms) required for the transmission of information from retina to the inferotemporal cortex (IT) through the primary visual cortex (V1) is mapped to a moment of psychological time. It might be that by using $\Delta \approx 100$ ms we shall get the right $p$-adic structure of the mental space.

Unfortunately, it seems that the situation is essentially more complicated. There are experimental evidences that the temporal structure of neural functioning is not homogeneous. The time required for completion of color information in V4 ($\approx 60$ ms) is shorter than the time for the completion of shape analysis in IT ($\approx 100$ ms). In particular it is predicted that there will be under certain conditions a rivalry between color and form perception. This rivalry in time is one of manifestations of complex level temporal structure of brain. It can be shown that at any given moment in physical time, there are neural activities in various brain
regions that correspond to a range of moments in psychological time. In turn, a moment in psychological time is subserved by neural activities in different brain regions at different physical times.

Therefore it is natural to suppose that different psychological functions have different time scales and, consequently, different mental spaces. Thus one psychological function is based on the 2-adic mental space and another on the 5-adic (or 1999-adic) mental space. This is the very delicate point and we shall try to clarify it.

There is the total space $\Pi$ of all neural pathways. The concrete psychological function $f$ is based on some centered cognitive graph $\Pi_f$ (a subset of $\Pi$). There exists the fixed time scale $\Delta = \Delta_f$ corresponding to this psychological function. Hence there exists the natural number $p$ depending on $\Delta_f$ (and hence on the $f$) determining the $p$-adic structure of the mental space for the $f$. Thus $p = p_f$. On this $p_f$-adic mental space there is defined the mental state $p_f(x)$ of the $f$. In general another psychological function $g$ has its own time scale $\Delta_g$ and corresponding $p_g$. Its mental state $p_g(x)$ is well defined on the $p_g$-adic space. If $p_f$ is not equal to $p_g$ (e.g. $p_f = 2$ and $p_g = 1997$), then dynamics of mental states corresponding to psychological functions $f$ and $g$ differs crucially – even if evolutions are described by the same e.g. diffusion equation.

Finally, we remark that many psychological functions are strongly inter related on the neural pathway level. Cognitive graphs $\Pi_f$ and $\Pi_g$ corresponding to psychological functions $f$ and $g$ can have large intersection. In the extreme case these graphs could even coincide: $\Pi_f = \Pi_g$. But the use of different time scales $\Delta_f \neq \Delta_g$ would produce totally different evolutions for corresponding mental states.

9.6. Discreteness of time. Previous considerations demonstrated that the continuous real time model for the evolution of the mental state gives only rough approximation to the really performed discrete time evolution. Moreover, discretization steps depend on corresponding psychological functions.

Thus the evolution of the mental state $p_f(t, x)$ of a psychological function $f$ is described by discrete time dynamics, where $t_{n+1} = t_n + \Delta$.

9.7. What is about $p$-adic structures of our psychological functions? If we accept Edelman’s Neural Darwinism [37], then we have to consider the possibility that $p$-adic structures of our psychological functions can depend on individuals. Thus, for an individual $I$, the basis of his/her mental space for a psychological function $f$ depends both on $f$ and $I$ : $p = p_{f,I}$. For two different individuals, for example, Ivan and Andrei, the same psychological function $f$ can be based on two different mental spaces, $M_{Ivan}$ and $M_{Andrei}$, that are the $p_{f,Ivan}$-adic tree and the $p_{f,Andrei}$-adic tree, respectively.

9.8. Does consciousness benefit from long neural pathways? Finally, we discuss one of the greatest mysteries of neuroanatomy, see, for example, [34],
[37]. It seems that in the process of neural evolution cognitive systems tried to create for each psychological function as long neural pathways as possible. This mystery might be explained on the basis of our neural pathway coding model. If such a coding be really the case, then a cognitive system $\tau$ gets great benefits by extending neural pathways for some psychological function as long as possible. For example, let the neural code be based on $p = 5$ and let a psychological function $f$ be based on very short pathways of the length $L = 2$. Then the corresponding mental space contains $N(5, 2) = 2^5 = 32$ points. Let now $p = 5$ and $L = 10000$. Then the corresponding mental space contains huge number of points $N(5, 10000) = 10^{20}$ points. On the latter (huge) mental space there can be realized mental states having essentially more complex behavior (and, consequently, higher magnitude of consciousness). This ‘mental space extending’ argument can be used to explain spatial separation of various maps in brain, see e.g. Edelman [37].

9.9. Why activity of “far away” neurons play important role? Consider a psychological function based on a cognitive graph with one central neuron $S$. Suppose that interaction between mental points (produced by the graph) depends on the $p$-adic distance between these points, e.g.

$$V(x, y) = \rho_p(x, y)^\alpha, \alpha > 0.$$ 

Then changes of states of input neurons that are located far away from the central neuron $S$ (on neural pathways belonging to the cognitive graph) play the crucial role in variation of the magnitude of the mental potential $V(x, y)$. If states (e.g. rates of firing) of initial input neurons are different, then $\rho_p(x, y)$ is very large, see (9).

10 Postulates

Our mathematical model of probabilistic thinking on $p$-adic mental spaces\textsuperscript{16} is based on following four postulates:

1. **Pr**: Mental states are determined by probability distributions on mental spaces.

   Evolution of mental states is described by classical (or may be even quantum) evolution equations for probability distributions of random processes, e.g. diffusion equations, on ultrametric $p$-adic mental spaces.

2. **NeurPath**: ‘Quant’ of mental information (point of mental space) is given by the state of a hierarchic neural pathway.

\textsuperscript{16}Such spaces are produced by cognitive graphs of hierarchic neural pathways.

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NeurGr: Each psychological function is based on a hierarchic graph of neural pathways, cognitive graph.

3. Ult: Mental topology is ultrametric.

It is supposed that mental spaces (in the opposite to spaces used in physical models) have ultrametric topology. The presence of such geometry is equivalent to a treelike representation of mental space.

4. FrCod: Mental encoding of information is performed by accounting frequencies of firings of neurons throughout hierarchic neural pathways.

This encoding of information determines the natural number $p$ (depending on a psychological function and a cognitive system).

The first postulate, Pr, determines the (probabilistic) structure of high level information processing in cognitive systems. Other postulates are related to processing of cognitive information on the primary level. In fact, Pr need not be rigidly connected with further postulates. There can be developed other cognitive models of probabilistic thinking. In particular, we need not base all models on the last postulate FrCod. There can be other models of mental coding.

Finally, we briefly discuss relation of our Probabilistic Neural Pathway Model to some traditional models of cognition. As was already remarked, we do not study neural dynamics in the physical brain. Our model is purely information model. We study flows of specially organized information. Postulates NeurPath and FrCod provide connection with neurophysiology. However, we are not interested in investigation of functioning of neural networks producing hierarchic strings of information, mental points. The only important thing is the form of probability distribution (mental state) on the space of mental points. As it was already underlined, different dynamical processes on neural level can produce the same probability distribution.

In principle, there can exist some model, for example, connectionist (neural network) model, that would describe “production” of information strings forming a mental state. However, such a generalized connectionist model should be based on new paradigm: hierarchic neural pathway as basis processing unit (not single neuron!). We even can not exclude the possibility that such an “underground model” could be some AI-model. Our experience of computer simulations shows that very complex random behaviour can be algorithmically simulated. But, in principle, we need not presuppose existence of any deterministic “underground model”.

I also would like to point to some connection with distributed representation models. We recall that A distributed representation is one in which meaning is not captured by a single symbolic unit, but rather arises from the interaction of a set of units, normally a network of some sort. If we use just the first part of this definition
(i.e., omit relation to neural networks), then we could consider Probabilistic Neural Pathway Model as a kind of model of distributed representation: mental units (points) are unified through probability distribution – mental state.

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