Non-colliding Jacobi processes as limits of Markov chains on Gelfand-Tsetlin graph.

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Abstract
We introduce a stochastic dynamics related to the measures that arise in harmonic analysis on the infinite-dimensional unitary group. Our dynamics is obtained as a limit of a sequence of natural Markov chains on Gelfand-Tsetlin graph.

We compute finite-dimensional distributions of the limit Markov process, the generator and eigenfunctions of the semigroup related to this process.

The limit process can be identified with Doob h–transform of a family of independent diffusions. Space-time correlation functions of the limit process have a determinantal form.

Introduction
The present paper originated from harmonic analysis on the infinite-dimensional unitary group. Decomposition of natural representations of $U(\infty)$ into irreducible ones leads to a family \( \{P_{z,w}\} \) of probability measures, which depend on the two parameters \( z \) and \( w \). These measures live on the infinite-dimensional domain \( \Omega \). The definition of the measures can be extended to even larger set of values of the parameters. According to this, we use four subscripts \( z, w, z', w' \), instead of two.

The measures \( P_{z,w} \) and \( P_{z,w,z',w'} \) were introduced in the papers [Olsh] and [BO]. Our goal is to construct and study stochastic dynamics related to the measures \( P_{z,w,z',w'} \).

The structure of measures \( P_{z,w,z',w'} \) substantially depends on whether parameters are integers or not. In the present paper we consider the former case, the parameters \( z \) and \( w \) will be integers. Denote these integers by \( p \) and \( q \), respectively. In our case the support of the measures is a finite-dimensional subset of \( \Omega \) and can be identified with \( X = [0,1]^{p+q+1} \). The probability distributions \( P_{p,q,z',w'} \) were explicitly computed in [BO]. They are given by the Jacobi orthogonal polynomial ensemble.

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We construct a family $J_{p,q,z',w'}(t)$ of stationary Markov processes in $\mathcal{X}$. Each $P_{p,q,z',w'}$ serves as an invariant distribution of the corresponding $J_{p,q,z',w'}(t)$. The processes $J_{p,q,z',w'}(t)$ are obtained as limits of certain Markov chains on the Gelfand-Tsetlin graph. These Markov chains arise in a natural way, due to the approximation of the infinite-dimensional unitary group by the increasing chain of the groups $U(N)$. We call these chains “up–down” chains. Similar Markov chains have already appeared earlier, they were studied by Fulman [Fu1], [Fu2], Borodin and Olshanski [BO2] and Petrov [Pe].

The proof of the convergence of our Markov chains on the Gelfand-Tsetlin graph is based on the special determinantal form of the transition probabilities of these Markov chains. We reduce convergence of the transition probabilities to convergence of certain matrices. Passing to the limit in these matrices is simplified by the fact that we are able to diagonalize them.

The limit Markov processes $J_{p,q,z',w'}(t)$ turn out to have some interesting properties. One proves that $J_{p,q,z',w'}(t)$ is a time-dependent determinantal point process. It means that its dynamical (space-time) correlation functions have determinantal form and can be expressed through the minors of certain extended kernel.

We also explain that $J_{p,q,z',w'}(t)$ can be identified with Doob $h$–transform of $p + q + 1$ independent random motions in $[0,1]$.

We fully describe processes $J_{p,q,z',w'}(t)$, i.e. we compute their transition probabilities and write down generators and eigenfunctions of the Markov semi-groups corresponding to the processes.

The study of the harmonic analysis on the infinite-dimensional unitary group shows numerous connections with infinite symmetric group $S(\infty)$ (see [KOV] and [BO, Part (m) in Introduction]). The constructions of the present paper are similar to the constructions of [BO2], where the dynamics related to $S(\infty)$ were studied.

It turned out that Markov chains of the present paper have lots of similarities with the ones considered in [Gor]. We use some ideas and formulas of [Gor].

We want to emphasize that while in our case of integral parameters all processes live in a finite-dimensional space, in the case of arbitrary parameters the space becomes infinite-dimensional. Thus, our case can be viewed as a degeneration of the general case. The problem of constructing dynamics for non-integral parameters remains open, it seems like one has to use different arguments for that case.

The paper is organized in the following way. In the first four sections we do some preparatory work: In Section 1 we introduce Markov chains related to the Gelfand-Tsetlin graph. In Section 2 we present the measures $P_{z,w,z',w'}$ and formulate the problem, solved in the present paper. In Section 3 we study basic properties of the Markov chains under consideration. Finally, In Section 4 we express transition probabilities of the Markov chains in the determinantal form, which is convenient for limit transitions.

In Section 5 we state and prove the main results of the paper. We prove the existence of the limit process and compute its one-dimensional distributions and transition probabilities (see Theorem 5.1). Determinantal form of transition
probabilities imply the determinantal property of the limit Markov processes (see Proposition 5.5).

In Section 6 we study Markov semigroup related to the constructed process. We compute eigenfunctions and eigenvalues of the semigroup (see Theorem 6.1); then we find a simple expression for the generator of the semigroup (see Theorem 6.4).

In Section 7 we explain the connection between our process and Doob $h$-transform.

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1 General Markov chains on the Gelfand-Tsetlin graph

In this section we introduce Markov chains studied in the paper.

The Gelfand-Tsetlin graph $\mathbb{GT}$ (also known as the graph of signatures) is a graded graph, whose vertices are so-called signatures. The $N$-th level of the graph, denoted by $\mathbb{GT}_N$, consists of $N$-tuples of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$ which are are called signatures of the order $N$. We join two signatures $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_{N+1}$ by an edge and write $\lambda \prec \mu$ if

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \lambda_N \geq \mu_{N+1}.$$ 

We agree that $\mathbb{GT}_0$ consists of a single element, the empty signature $\emptyset$. $\emptyset$ is joined by an edge with every signature from $\mathbb{GT}_1$.

By a path in the Gelfand-Tsetlin graph we mean a sequence of vertices

$$\lambda(n) \prec \lambda(n + 1) \prec \cdots \prec \lambda(m), \quad \lambda_i \in \mathbb{GT}_i.$$

Denote by $\text{Dim}(\lambda)$ the number of paths going from $\emptyset$ to $\lambda \in \mathbb{GT}_N$. For $\mu \in \mathbb{GT}_{N+1}$ and $\lambda \in \mathbb{GT}_N$ set

$$p^\downarrow(\lambda | \mu) = \begin{cases} \frac{\text{Dim}(\lambda)}{\text{Dim}(\mu)}, & \text{if } \lambda \prec \mu \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that

$$\sum_{\lambda \in \mathbb{GT}_N : \lambda \prec \mu} p^\downarrow(\lambda | \mu) = 1.$$

The numbers $p^\downarrow(\lambda | \mu)$ are called cotransition probabilities or “down” transition function.

A sequence $\{M_N\}_{N=0,1,\ldots}$, where $M_N$ is a probability measure on $\mathbb{GT}_N$, is called a coherent system of distributions provided that for any $N \geq 0$ the
measures $M_{N+1}$ and $M_N$ are consistent with the cotransition probabilities from $G\Gamma_{N+1}$ to $G\Gamma_N$, i.e.

$$\sum_{\mu \in G\Gamma_{N+1}} p^\uparrow(\lambda \mid \mu)M_{N+1}(\mu) = M_N(\lambda) \quad \text{for any } \lambda \in G\Gamma_N.$$  

Define the support of a coherent system $M$ as the subset

$$\text{supp}(M) = \{ \lambda \in G\Gamma : M(\lambda) \neq 0 \} \subset G\Gamma,$$

where $M(\lambda)$ means the measure $M_N(\lambda)$ of the singleton $\{ \lambda \} \subset G\Gamma_N$.

Given a coherent system of distributions we may speak about transition probabilities or “up” transition functions. The transition probabilities $p^\uparrow(\mu \mid \lambda)$ are defined for all $\lambda \in \text{supp}(M)$ by

$$p^\uparrow(\mu \mid \lambda) = \frac{M_{N+1}(\mu)}{M_N(\lambda)} \cdot p^\downarrow(\lambda \mid \mu), \quad \lambda \in G\Gamma_N, \mu \in G\Gamma_{N+1}, M_N(\lambda) \neq 0.$$

Any coherent system of distributions $\{M_N\}$ defines a Markov chain $H(t), t = 0, 1, \ldots$ on the state set $\text{supp}(M)$. $H(t)$ takes values in $L_t = G\Gamma_t \cap \text{supp}(M)$ and its one-dimensional distributions are given by $M_t$. The transition probabilities of $H(t)$ are precisely the numbers $p^\uparrow(\mu \mid \lambda)$:

$$\text{Prob}\{H(t+1) = \mu \mid H(t) = \lambda\} = p^\uparrow(\mu \mid \lambda).$$

Note that, while the transition probabilities of $H(t)$ depend on the coherent system, the cotransition probabilities $\text{Prob}\{H(t) = \lambda \mid H(t+1) = \mu\}$ are nothing but the quantities $p^\downarrow(\lambda \mid \mu)$, and they depend solely on the structure of the Gelfand-Tsetlin graph.

We call $H(t)$ the up chain, corresponding to $\{M_N\}$.

We also define a family of stationary Markov chains $T_N(t), \; N = 0, 1, \ldots$. The state space of $T_N(t)$ is $L_N = G\Gamma_N \cap \text{supp}(M)$. The distribution of $T_N(t)$ is given by $M_N$. The transition probabilities are given by the composition of “up” and “down” transition functions, from $L_N$ to $L_{N+1}$ and then back to $L_N$. Denote by $p^{\uparrow\downarrow}_N(\lambda \mid \lambda')$ the transition probabilities of $T_N$. We have

$$p^{\uparrow\downarrow}_N(\lambda \mid \lambda') = \sum_{\mu \in L_{N+1}} p^\downarrow(\lambda \mid \mu)p^\uparrow(\mu \mid \lambda').$$

We call $T_N(t)$ the $N$-th level up-down chain, corresponding to $\{M_N\}$.

In the present paper we study the behavior of the up-down chains, corresponding to certain coherent systems, as $N \to \infty$.

**Remarks.** Our definitions make sense not only for the Gelfand-Tsetlin graph $G\Gamma$. In their papers Kerov and Vershik introduced transition and cotransition probabilities for arbitrary branching graphs (see e.g. [KO87, Section 9] and references therein). Thus, we can define Markov chains similar to $H(t)$ and $T_N(t)$ for an arbitrary branching graph. The up-down chains for the Young graph were recently studied by Borodin and Olshanski in [BO2]. Our work was influenced by that paper.
2 Measures $P_{z,w,z',w'}$ and construction of the limit process

In the paper [Olsh] a 4-parameter family of coherent systems $M_{N}^{z,w,z',w'}$ was introduced. These measures appear in a natural way in harmonic analysis on the infinite-dimensional unitary group. They are given by

$$M_{N}^{z,w,z',w'}(\lambda) = (S_{N}(z, w, z', w'))^{-1} \cdot \text{Dim}^2(\lambda) \times \prod_{i=1}^{N} \frac{1}{\Gamma(z - \lambda_{i} + i)\Gamma(w + N + 1 + \lambda_{i} - i)\Gamma(z' - \lambda_{i} + i)\Gamma(w' + N' + 1 + \lambda_{i} - i)},$$

where $z, w, z', w'$ are complex numbers, the quadruple $(z, w, z', w')$ belongs to the set of admissible values (see [Olsh] Definition 7.6) and $S_{N}(z, w, z', w')$ is a normalization constant

$$S_{N}(z, w, z', w') = \prod_{i=1}^{N} \frac{\Gamma(z + w + z' + w' + i)}{\Gamma(z + w + i)\Gamma(z' + w' + i)\Gamma(z' + w' + i)\Gamma(z + w + i)\Gamma(z' + w + i)\Gamma(z + w + i)}.$$

Let us assume that $z = k$ and $w = l$, where $k, l \in \mathbb{Z}, k + l \geq 0$. It was shown in [Olsh] that $(k, l, z', w')$ forms an admissible quadruple of parameters if $z'$ and $w'$ are real and $z' - k > -1$, $w' - l > -1$.

Note that for any integer $n$ the shift

$$k \mapsto k + n, \quad l \mapsto l - n, \quad z' \mapsto z' + n, \quad w' \mapsto w' - n \quad \lambda_{i} \mapsto \lambda_{i} + n$$

leaves the probability distributions $M_{N}$ invariant. This means that essentially the coherent system $\{M_{N}^{k,l,z',w'}\}$ depends on three, not four parameters. From now on we assume that $k = p \geq 0$ and $l = 0; z > p - 1$ and $w > -1$.

It is easily seen from the definition of $M_{N}^{p,0,z',w'}$ that supp$(M)$ consists of the signatures $\lambda \in \mathbb{GT}_{N}$ such that

$$p \geq \lambda_{1} \geq \cdots \geq \lambda_{N} \geq 0.$$

We always assume below that $\lambda$ satisfies these inequalities.

Set $X_{N,p} = \{0, 1, \ldots, N + p - 1\}$. Observe that $0 \leq \lambda_{i} - i + N \leq N + p - 1$.

Let us associate to $\lambda$ a collection $X(\lambda)$ of distinct points in $X_{N,p}$ as follows

$$X(\lambda) = X_{N,p} \setminus \{\lambda_{1} - 1 + N, \lambda_{2} - 2 + N, \ldots, \lambda_{N} - N + N\}.$$ 

It is clear that $|X(\lambda)| = p$ and $X(\lambda)$ defines $\lambda$ uniquely. Let

$$X(\lambda) = \{0 \leq x_{1} < x_{2} < \cdots < x_{p} \leq N + p - 1\}.$$ 

Remark. Equivalently $\lambda \mapsto X(\lambda)$ can be described in the following way. First, we identify $\lambda$ with a Young diagram. Row lengths of this diagram are $\lambda_{i}$. Then we consider the transposed diagram and its row lengths (in other words,
we consider heights of the columns of our diagram). We arrive to a collection of numbers

\[ 0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_p \leq N. \]

Finally, we set \( x_i = \mu_i + i - 1 \). Equivalence of this definition with the one given above follows, e.g. from [Mac (1.7)].

We work with \( X(\lambda) \) instead of \( \lambda \) by the following reasons. First, \( |X(\lambda)| = p \) and, thus, it does not depend on \( N \). Consequently, we can identify \( X(\lambda) \) corresponding to different \( N \) with elements of one fixed space. Second, transition to the next level of the Gelfand-Tsetlin graph is very simple in the \( X(\lambda) \) interpretation. We will provide more details later.

Denote by \( P_{N, p, z'}^w \) the pushforward of the probability measure \( M_{N, p, 0, z', w'}^0 \) under the map \( \lambda \mapsto X(\lambda) \). It is clear that \( P_N \) is a probability measure on \( (X_N, p) \) and its support is

\[ \text{supp}(P_N) = \{(x_1, \ldots, x_p) \in (X_N, p)^p : x_1 < x_2 < \cdots < x_p \}. \]

For any \( N = 0, 1, 2 \ldots \) we embed the set \((X_N, p)\) into \( X = [0, 1]^p \) as follows:

\[ \pi_N : (x_1, \ldots, x_p) \mapsto \left( \frac{x_1}{N + p - 1}, \ldots, \frac{x_p}{N + p - 1} \right). \]

Denote by \( \tilde{P}_{N, p, z', w'}^w \) the pushforward of the measure \( P_{N, p, z', w'}^0 \) under the embedding \( \pi_N \).

**Proposition 2.1.** As \( N \to \infty \) the measures \( \tilde{P}_{N, p, z', w'}^w \) weakly converge to a measure \( \mu_{p, z', w'} \).

The measure \( \mu_{p, z', w'} \) is given by its density function \( \rho_{p, z', w'} \)

\[
\rho_{p, z', w'}(x_1, \ldots, x_p) = \begin{cases} 
B_{p, z', w'} \cdot \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^p x_i^{w'}(1 - x_i)^{z'-p}, & x_1 < x_2 < \ldots < x_p, \\
0, & \text{otherwise},
\end{cases}
\]

where \( B_{p, z', w'} \) is a normalization constant.

**Proof.** A slightly different version of this proposition was proved in [BO, Theorem 11.6]. We will also verify this proposition by straightforward computations later (see Proposition 5.2). \( \square \)

The aim of the present paper is to introduce a stationary stochastic process, which has \( \mu_{p, z', w'} \) as an equilibrium measure. We construct this process as a limit of the up-down Markov chains on the Gelfand-Tsetlin graph.

Denote by \( X_{p, z', w'}(t) \) the image of the up Markov process \( H(t) \), corresponding to the coherent system \( \{M_{N, p, 0, z', w'}^0\} \), under the map \( \lambda \mapsto X(\lambda) \).

Denote by \( U_{p, z', w'}(t) \) the image of the up-down Markov process \( T_N(t) \), corresponding to the coherent system \( \{M_{N, p, 0, z', w'}^0\} \), under the map \( \lambda \mapsto X(\lambda) \).
Finally, set
\[ J_{p,z,w'}(t) = \pi_N \left( U_{p,z',w'}^N \left( \left\lfloor t \cdot N^2 \right\rfloor \right) \right) \]

Below we prove the following theorem:

**Theorem 2.2.** There exists a limit stationary Markov process \( J_{p,z',w'}(t) \) on \( X \). The finite-dimensional distributions of \( J_{p,z',w'}^N(t) \) converge as \( N \to \infty \) to the corresponding finite-dimensional distributions of \( J_{p,z',w'}(t) \).

We will compute the transition probabilities of \( J_{p,z',w'}(t) \) and the generator of \( J_{p,z',w'}(t) \).

### 3 Properties of Markov chain \( X_{p,z',w'}(t) \)

In this section we compute one-dimensional distributions and transition probabilities of the Markov chain \( X_{p,z',w'}(t) \). In what follows we omit indices and write \( X(t) \).

Recall that one-dimensional distributions of \( X(N) \) coincide with \( P_{p,z',w'}^N \). \( P_{p,z',w'}^N \) is a pushforward of the measure \( M_{p,0,z',w'}^N \) under the map \( \lambda \mapsto X(\lambda) \), and

\[
M_{N}^{z,w,z',w'}(\lambda) = \left( S_N(z, w, z', w') \right)^{-1} \cdot \text{Dim}^2(\lambda)
\]

\[
\times \prod_{i=1}^N \frac{1}{\Gamma(z + \lambda_i) \Gamma(w + N + 1 + \lambda_i) \Gamma(z' - \lambda_i) \Gamma(w' + N + 1 + \lambda_i)}
\]

By Weyl’s dimension formula (see e.g. [Zh]):

\[
\text{Dim}(\lambda) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}
\]

\[= \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - i + N) - (\lambda_j - j + N)}{j - i}, \quad \lambda \in \mathbb{GT}_N.\]

**Lemma 3.1.** Let \( A = \{0, 1, \ldots, k\} \), \( X \subset A, \overline{X} = A \setminus X \). Denote

\[
V(X) = \prod_{x \in X, y \notin X} (y - x).
\]

Then we have

\[
V(X) = V(\overline{X}) \prod_{x \in X} \frac{1}{x!(k - x)!} \cdot \prod_{i=1}^k i!
\]
Proof. It is clear that

\[ V(X) \prod_{x \in X} \left( \prod_{i \neq x} (x - i) \right) = V(A) \cdot V(\overline{X}). \]

To complete the proof we observe that

\[ V(A) = \prod_{i=1}^{k} i! \]

and

\[ \prod_{i \neq x} (x - i) = x!(k - x)!. \]
Now suppose that $\lambda \in \mathcal{G}_N$, $\mu \in \mathcal{G}_{N+1}$, $X = (x_1 < \cdots < x_p) = X(\lambda)$, $X' = (x_1' < \cdots < x_p') = X(\mu)$. We write $X \prec X'$, provided that $\lambda \prec \mu$. Note that $X \prec X'$ if and only if for every $i$, $x_i' = x_i$ or $x_i' = x_i + 1$.

**Remark.** Consider an arbitrary path $\tau = (\tau(0), \tau(1), \ldots)$ in $\mathcal{G}$. Let $X(N) = (x_1 \cdots x_N') = (x_1^N < \cdots < x_p^N)$. For every $1 \leq i \leq p$ and $N \geq 0$ we put a point on the plane $\mathbb{Z}^2$ with coordinates $(N, x_i^N)$ and draw a segment connecting $(N, x_i^N)$ with $(N + 1, x_i^{N+1})$. In this way every path $\tau$ corresponds to a collection of $p$ non-intersecting paths on the plane $\mathbb{Z}^2$. This interpretation shows deep similarities between the objects of the present paper and the objects of the paper [Gor] where collections of non-intersecting paths inside a hexagon were studied.

Our next aim is to obtain explicit formulas for the transition probabilities of the process $X_{p,z',w'}(t)$.

**Proposition 3.4.** Suppose $X = (x_1 < \cdots < x_p) \in (X_{N,p})^p$ and $X' = (x_1' < \cdots < x'_p) \in (X_{N+1,p})^p$, then

$$\text{Prob}\{X(N + 1) = X' | X(N) = X\} = \begin{cases} \frac{1}{(z + w' + N + 1) p} V(X') \prod_{i, x_i' = x_i} (z' + N - x_i) \prod_{i, x_i' = x_i + 1} (w' + 1 + x_i), & X \prec X' \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** By the definition $\text{Prob}\{X(N + 1) = X' | X(N) = X\}$ is transition function $p^1(\mu | \lambda)$, where $X' = X(\mu)$ and $X = X(\lambda)$. Lemma 3.2 and Proposition 3.3 imply that

$$\text{Prob}\{X(N + 1) = X' | X(N) = X\} = p^1(\mu | \lambda) = \frac{M_{N+1}(\mu)}{M_N(\lambda)} \cdot \frac{\text{Dim}(\lambda)}{\text{Dim}(\mu)} = \frac{Z_N^{p,z',w'} \cdot V^2(X') \prod_{i=1}^p w_{N+1}(x_i') V(X) \cdot \prod_{i=1}^p \frac{1}{x_i!(N+p-1-x_i)!} \cdot \prod_{i=1}^p (N + i - 1)!}{Z_N^{p,z',w'} \cdot V^2(X) \prod_{i=1}^p w_N(x_i) V(X') \cdot \prod_{i=1}^p \frac{1}{x_i!(N+p-x_i)!} \cdot \prod_{i=1}^p (N + i)!}.$$

Using $\square$ we obtain the desired formula.

4 Determinantal form of transition probabilities

In this section we introduce determinantal formulas for the transition probabilities of the Markov chains under consideration. These formulas are very important for our arguments.

Denote

$$c_i^N = \sqrt{\left(1 - \frac{i}{p+N}\right) \left(1 + \frac{i}{w'+z'+N+1}\right)}.$$
Let \( v_N(x, y) \) be \((N+p) \times (N+p-1)\) matrix \((0 \leq x \leq N+p-1, 0 \leq y \leq N+p)\) given by

\[
v_N(x, y) = \begin{cases} 
\sqrt{\frac{(w'+x+1)(x+1)}{(x+N)(w'+z'+N+1)}}, & y = x + 1 \\
\frac{\sqrt{\frac{(z'+N-x)(p+N-z)}{(x+N)(w'+z'+N+1)}}}{\sqrt{\frac{\sqrt{\frac{(z'+N-x)(p+N-z)}{(x+N)(w'+z'+N+1)}}}{\sqrt{\frac{(z'+N-x)(p+N-z)}{(x+N)(w'+z'+N+1)}}}}}, & y = x \\
0, & \text{otherwise}.
\end{cases}
\]

### Proposition 4.1.

\[
\text{Prob}\{X(N+1) = X' \mid X(N) = X\} = \frac{\sqrt{P_{N+1}^{z',w'}(X')}}{\sqrt{P_N^{z',w'}(X)}} \cdot \frac{\det[v_N(x_i, x'_j)_{i,j=1,\ldots,p}]}{\prod_{i=0}^{p-1} c'_i} \cdot \frac{1}{\prod_{i=0}^{p-1} c_i} \cdot \frac{V(X')}{V(X)} \prod_{i=1}^{p} \frac{w_{N+1}(x'_i)}{w_N(x_i)}
\]

where \([v_N(x_i, x'_j)]_{i,j=1,\ldots,p}\) is a submatrix of the matrix \(v_N(x, y)\).

**Proof.** Observe that \(v_N(x, y)\) is a two-diagonal matrix. Any submatrix of a two-diagonal matrix, which has non-zero determinant, is block-diagonal, where each block is either upper or lower triangular matrix. Thus, any non-zero minor is a product of suitable matrix elements. Consequently, if \(X \prec X'\), then

\[
\frac{\sqrt{P_{N+1}^{z',w'}(X')}}{\sqrt{P_N^{z',w'}(X)}} \cdot \frac{\det[v_N(x_i, x'_j)_{i,j=1,\ldots,p}]}{\prod_{i=0}^{p-1} c'_i} \cdot \frac{1}{\prod_{i=0}^{p-1} c_i} \cdot \frac{V(X')}{V(X)} \prod_{i=1}^{p} \frac{w_{N+1}(x'_i)}{w_N(x_i)}
\]

\[
\times \prod_{i:x'_i=x_i} \sqrt{\frac{(z'+N-x_i)(p+N-x_i)}{(p+x_i)(w'+z'+N+1)}} \prod_{i:x'_i=x_i+1} \sqrt{\frac{(w'+x+1)(x+1)}{(x+N)(w'+z'+N+1)}}
\]

\[
= \frac{\Gamma(z'+w'+N+1)}{\Gamma(z'+w'+p+N+1)} \frac{V(X')}{V(X)} \prod_{i:x'_i=x_i} (z'+N-x_i) \prod_{i:x'_i=x_i+1} (w'+1+x_i)
\]

\[
= \text{Prob}\{X(N+1) = X' \mid X(N) = X\},
\]

and both sides of \((\text{2})\) are equal to zero if \(X \not\prec X'\). \(\square\)

Let \(Q_{w',z'-p,N+p-1}^k(x)\) be the Hahn polynomial of degree \(k\). These polynomials are orthogonal with respect to the weight \(w_N(x)\) (see [KS]).

Denote

\[
f_N^k(x) = \frac{Q_{w',z'-p,N+p-1}^k(x)}{\sqrt{(Q_{w',z'-p,N+p-1}^k, Q_{w',z'-p,N+p-1}^k)}}
\]

where \((Q_{w',z'-p,N+p-1}^k, Q_{w',z'-p,N+p-1}^k)\) is the squared norm of \(Q_{w',z'-p,N+p-1}^k\) in \(L_2([0,1,\ldots,N+p-1], w_N(x))\). We provide the exact value of this norm in Appendix.
Functions $f_k^N$, $k = 0, 1, \ldots, N + p - 1$ form an orthonormal bases in $L_2(\{0, 1, \ldots, N + p - 1\})$ (this $L_2$ is with respect to the uniform measure).

Remark. Here and below we use definitions of classical orthogonal polynomials given in [KS]. Thus

$$Q_{x, \alpha, \beta, M}^k = 3F2\left(-k, -x, k + \alpha + \beta + 1 \atop -M, \alpha + 1\right) \quad (1).$$

We also use the book [NSU], which uses slightly different definition of these polynomials (two definitions differ by a factor). All formulas in the present paper are written according to the definitions of [KS].

Proposition 4.2. We have

$$v_N(x, y) = \sum_{i=0}^{N+p-1} c_i^N f_N^i(x) f_{N+1}^i(y).$$

Equivalently,

$$v_N = (F_N)^T \cdot C_N \cdot (F_{N+1}),$$

where $v_N$ is $(N + p) \times (N + p + 1)$ matrix $v_N(x, y)$, $F_m$ is $(m + p) \times (m + p)$ matrix $f_m^i(j)_{i,j=0,\ldots,m+p-1}$ and $C_N$ is $(N + p) \times (N + p + 1)$ matrix given by

$$(C_N)_{ij} = \begin{cases} c_i^N, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The argument repeats [Gor, Lemma 8]. More details are given in Appendix.

Our next aim is to obtain a determinantal representation for the transition probabilities of the Markov chain $U_{p,z',w'}^N(t)$.

Lemma 4.3. If $X' \prec X$ then

$$\text{Prob}\{X(N) = X' \mid X(N+1) = X\} = \frac{\sqrt{P_{p,z',w'}^N(X')}}{\sqrt{P_{p,z',w'}^{N+1}(X)}} \cdot \det[v_N(x'_i, x_j)]_{i,j=1,\ldots,p} \cdot \frac{1}{\prod_{i=0}^{p-1} c_i^N}.$$
Thus,

\[
\begin{align*}
\text{Proposition 4.4. Transition probabilities of the process } & U_{p,z',w'}^{N}(t) \text{ are given by} \\
\text{Prob}\{X(N) = X' \mid X(N+1) = X\} &= \frac{\text{Prob}\{X(N+1) = X \mid X(N+1) = X'\}}{\text{Prob}\{X(N+1) = X\}} \frac{\text{Prob}\{X(N) = X'\}}{\text{Prob}\{X(N+1) = X\}} \\
&= \frac{\sqrt{p_{N+1}^{p,z',w'}(X')}}{\sqrt{p_{N}^{p,z',w'}(X')}} \cdot \det[v_{N}(x'_i, x'_j)]_{i,j=1,...,p} \cdot \frac{1}{\Pi_{i=0}^{p-1} c_i} \frac{p_{N}^{p,z',w'}(X')}{p_{N+1}^{p,z',w'}(X)} \\
&= \frac{\sqrt{p_{N}^{p,z',w'}(X')}}{\sqrt{p_{N+1}^{p,z',w'}(X')}} \cdot \det[v_{N}(x'_i, x'_j)]_{i,j=1,...,p} \cdot \frac{1}{\Pi_{i=0}^{p-1} c_i} \frac{p_{N}^{p,z',w'}(X')} {p_{N+1}^{p,z',w'}(X)} \\
\end{align*}
\]

Proof. By the definition of \( U_{p,z',w'}^{N}(t) \) we have

\[
\text{Prob}\{X(N) = X' \mid U_{p,z',w'}^{N}(t) = X\} = \sum_{Y} \text{Prob}\{X(N) = X' \mid X(N+1) = Y\} \cdot \text{Prob}\{X(N+1) = Y \mid X(N) = X\}.
\]

Thus,

\[
\text{Prob}\{X(N) = X' \mid U_{p,z',w'}^{N}(t) = X\} = \frac{\sqrt{p_{N}^{p,z',w'}(X')}}{\sqrt{p_{N+1}^{p,z',w'}(X')}} \cdot \sum_{y_1 < y_2 < \cdots < y_p} \det[v_{N}(x'_i, y_j)]_{i,j=1,...,p} \cdot \frac{\det[v_{N}(x'_i, y_j)]_{i,j=1,...,p}}{\Pi_{i=0}^{p-1} (c_i)^2}.
\]

Let \( A \) be \( p \times (N + p + 1) \) matrix given by

\[
A_{ij} = v_{N}(x_i, j - 1), \quad i = 1, \ldots, p, \quad j = 1, \ldots, N + p + 1.
\]

Let \( B \) be \( (N + p + 1) \times p \) matrix given by

\[
B_{ij} = v_{N}(i - 1, x'_j), \quad i = 1, \ldots, N + p + 1, \quad j = 1, \ldots, p.
\]
Denote by $A_{j_1 j_2 \cdots j_p}$ the square submatrix of the matrix $A$ consisting of the columns $j_1, j_2, \ldots, j_p$; denote by $B_{j_1 j_2 \cdots j_p}$ the square submatrix of the matrix $B$ consisting of the rows $j_1, j_2, \ldots, j_p$. By Cauchy-Binet identity

$$\det(AB) = \sum_{1 \leq j_1 < j_2 < \cdots < j_p \leq N+p+1} \det(A_{j_1 j_2 \cdots j_p}) \cdot \det(B_{j_1 j_2 \cdots j_p}).$$

Observe that $(AB)_{ij} = u_N(x_i, x_j')$. Consequently,

$$\Prob[U_{p,z',w'}^N(t + 1) = X' \mid U_{p,z',w'}^N(t) = X] = \frac{\sqrt{\det(P_{p,w'})} \det[u_N(x_i, x_j')]_{i,j=1,\ldots,p}}{\sqrt{\det(P_{p,w'})} \prod_{c=0}^{p-1} (c_N)^{2k}},$$

(3)

\textbf{Proposition 4.5.} Let $k \in \mathbb{Z}^+$, then

$$\Prob[U_{p,z',w'}^N(t + k) = X' \mid U_{p,z',w'}^N(t) = X] = \sum_{i=0}^{N+p-1} (c_N^{i})^{2k} f_{N}^{i}(x) f_{N}^{i}(x').$$

\textbf{Proof.} We have

$$\Prob[U_{p,z',w'}^N(t + k) = X' \mid U_{p,z',w'}^N(t) = X] = \sum_{Y} \Prob[U_{p,z',w'}^N(t + k) = X' \mid U_{p,z',w'}^N(t + 1) = Y] \cdot \Prob[U_{p,z',w'}^N(t + 1) = Y \mid U_{p,z',w'}^N(t) = X]$$

$$= \sum_{Y} \Prob[U_{p,z',w'}^N(t + k - 1) = X' \mid U_{p,z',w'}^N(t) = Y] \cdot \Prob[U_{p,z',w'}^N(t + 1) = Y \mid U_{p,z',w'}^N(t) = X].$$

Applying Proposition 4.4 and Cauchy-Binet identity by induction we obtain the formula (3), where $[w_{N,k}(x_i, x_j')]_{i,j=1,\ldots,p}$ is a submatrix of the matrix $w_{N,k} = (u_N)^k$. Consequently,

$$w_{N,k} = (v_N(v_N)^T)^k = ((F_N)^T \cdot C_N \cdot F_{N+1} \cdot (F_N+1)^T \cdot (C_N)^T \cdot F_N)^k.$$

Recall that

$$(F_n)_{ij} = f_{n}^{i}(j), \quad i, j = 0, \ldots, m + p - 1.$$
Since functions $f_i^j$ form an orthonormal basis, $F_m \cdot (F_m)^T = Id$, where $Id$ means the identity matrix. Thus,

$$w_{N,k} = (F_N)^T \cdot (C_N (C_N)^T)^k \cdot F_N.$$ 

This equality implies Proposition 4.5. □

## 5 Limit process

Recall that we are interested in studying the stationary process

$$J_{p,z',w'}(t) = \lim_{N \to \infty} J_{p,z',w'}^N(t),$$

where

$$J_{p,z',w'}^N(t) = \pi_N \left( U_{p,z',w'}^N \left( \left\lfloor t \cdot N^2 \right\rfloor \right) \right)$$

and

$$\pi_N : (x_1, x_2, \ldots, x_p) \mapsto \left( \frac{x_1}{N + p - 1}, \frac{x_2}{N + p - 1}, \ldots, \frac{x_p}{N + p - 1} \right).$$

Let us introduce some notation. Denote by $w_{p,z',w'}(x)$ the density function of $B$-distribution:

$$w_{p,z',w'}(x) = x^{w'}(1-x)^{z'-p}, \quad 0 < x < 1.$$ 

Let $Jac_{z'-p,w'}^k(x)$ be the orthogonal Jacobi polynomial of the power $k$. These polynomials are defined for $x \in (0,1)$ and are orthogonal with respect to the weight function $w_{p,z',w'}$. According to [KS]

$$Jac_{\alpha,\beta}^n(x) = \frac{(\alpha + 1)_n}{n!} F_1 \left( -n, n + \alpha + \beta + 1 \atop \alpha + 1 \right) (1-x).$$

Note that while Jacobi polynomials are often defined for $x \in (-1,1)$, in the present paper we scale and shift the domain of definition, and our polynomials live on $(0,1)$.

Denote

$$j_{p,z',w'}^k(x) = \frac{Jac_{z'-p,w'}^k(x) \sqrt{w_{p,z',w'}(x)}}{\sqrt{(Jac_{z'-p,w'}^k, Jac_{z'-p,w'}^k)}}.$$ 

where $(Jac_{z'-p,w'}^k, Jac_{z'-p,w'}^k)$ is the squared norm of $Jac_{z'-p,w'}^k$ in $L_2([0,1], w_{p,z',w'})$. We provide the exact value of this norm in Appendix.

It is clear that functions $j_{p,z',w'}^k$, $k = 0, 1, \ldots$ form an orthonormal system in $L_2([0,1])$ with Lebesgue measure.

Finally, set

$$K_{p,z',w'}(i) = i(i + w' + z' + 1 - p).$$

In this section we prove the existence of the limit process $J_{p,z',w'}(t)$ (Theorem 2.2) and compute its one-dimensional distributions and transition probabilities. Our main result is the following.
Theorem 5.1. Finite-dimensional distributions of $J_{N}^{p,z',w'}(t)$ weakly converge as $N \to \infty$ to the corresponding finite-dimensional distributions of a limit process $J_{p,z',w'}(t)$. $J_{p,z',w'}(t)$ is a stationary Markov process. Its initial distribution is given by the density

$$\rho_{p,z',w'}(X) = B_{p,z',w'} \prod_{1<j} (x_i - x_j)^2 \prod_{i=1}^{p} w_{p,z',w'}(x_i).$$

The transition probabilities are given by the density

$$\mathcal{P}_{p,z',w'}^{t}(Y \mid X) = \frac{\sqrt{\rho_{p,z',w'}(Y)}}{\sqrt{\rho_{p,z',w'}(X)}} \cdot e^{t K_{p,z',w'}} \cdot \det \left[ \mathcal{J}_{p,z',w'}^{t}(x_i, y_j) \right]_{i,j=1,2,\ldots,p},$$

where

$$\mathcal{J}_{p,z',w'}^{t}(x, y) = \sum_{i=0}^{\infty} e^{-tK_{p,z',w'}(i)} j_{p,z',w'}^{t}(x) j_{p,z',w'}^{t}(y)$$

and

$$K_{p,z',w'} = \sum_{i=0}^{p-1} K_{p,z',w'}(i).$$

Remark. Determinantal form of transition probabilities implies that $J_{p,z',w'}(t)$ is space-time (dynamical) determinantal process. See Proposition 5.5 for the exact statement.

Let us begin the proof.

In the first place we want to recompute the density of the one-dimensional distribution of $J_{p,z',w'}(t)$ (it was originally computed in [BO]).

Proposition 5.2. We have

$$\text{Prob}\left\{ J_{p,z',w'}(t) \in dX \right\} = \rho_{p,z',w'}(X) dx_1 \ldots dx_p

= B_{p,z',w'} \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^{p} x_i^{w'} (1 - x_i)^{z'-p} dx_i,$$

where $X = (x_1 < x_2 < \cdots < x_p)$

Proof. Clearly, one-dimensional distributions of $J_{N}^{p,z',w'}(t)$ are

$$\text{Prob}\left\{ J_{N}^{p,z',w'}(t) \in X \right\} = \mathcal{P}_{N}^{p,z',w'}((N + p - 1) \cdot X),$$

where $X = (x_1 < x_2 < \cdots < x_p)$ and $(N + p - 1) \cdot X = ((N + p - 1)x_1 < (N + p - 1)x_2 < \cdots < (N + p - 1)x_p) \in \{0,1,\ldots,N + p - 1\}^p$.

If the limit below exist, then the following equality holds

$$\text{Prob}\left\{ J_{p,z',w'}(t) \in dX \right\} = \lim_{N \to \infty} \mathcal{P}_{N}^{p,z',w'}((N + p - 1)^p \cdot X) \cdot \text{Prob}\left\{ J_{N}^{p,z',w'}(t) = X \right\}. \quad (4)$$
Applying the formula

\[ \frac{\Gamma(M + a)}{\Gamma(M + b)} \sim M^{a-b}, \quad M \to \infty, \]

(here \( A \sim B \) means \( \lim \frac{A}{B} = 1 \)) we get:

\[
\text{Prob}\{J_{p,z',w'}^N(t) = X\} = Z_{N}^{p,z',w'} \cdot \prod_{i<j}(N + p - 1)x_i - (N + p - 1)x_j)^2
\]

\[
\cdot \prod_{i=1}^{p} \frac{\Gamma(z' + N - (N + p - 1)x_i)\Gamma(w' + (N + p - 1)x_i + 1)}{\Gamma(N + p - (N + p - 1)x_i)\Gamma((N + p - 1)x_i + 1)}
\]

\[
\sim Z_{N}^{p,z',w'} \cdot \prod_{i<j} (x_i - x_j)^2 \cdot (N + p - 1)^{(p-1)}
\]

\[
\cdot \prod_{i=1}^{p} ((N + p - 1) - (N + p - 1)x_i)^{z'} \cdot (N + p - 1)^{p(z'+w')-p} \cdot \prod_{i<j} (1 - x_i)^{z'-p} x_i^{w'}
\]

Substituting \( Z_{N}^{p,z',w'} \) from Proposition 5.3 we obtain

\[
\text{Prob}\{J_{p,z',w'}^N(t) = X\} = \frac{B_{p,z',w'}}{(N + p - 1)^p} \prod_{i<j} (x_i - x_j)^2 \cdot \prod_{i=1}^{p} (1 - x_i)^{z'-p} x_i^{w'}.
\]

\[\square\]

**Remark.** Note that the convergence in (1) is uniform on any compact set

\[ D \subset \{ (x_1, x_2, \ldots, x_p) : 0 < x_1 \leq x_2 \leq \cdots \leq x_p < 1 \}. \]

Next we concentrate on the multidimensional distributions of \( J_{p,z',w'}^N(t) \).

**Proposition 5.3.** Consider \( n \) distinct times

\[ 0 \leq t_1 < t_2 < \cdots < t_n. \]

As \( N \) tends to infinity, \( n \)-dimensional distribution of \( J_{p,z',w'}^N(t) \) corresponding to \( t = t_1, \ldots, t_n \) converges. The limit distribution is given by its density

\[
\rho(t_1, 0 < x_1^1 < \cdots < x_1^p < 1; \ldots; t_n, 0 < x_n^1 < \cdots < x_n^p < 1)
\]

\[
= \sqrt{\rho_{p,z',w'}(x_1^1, \ldots, x_p^1)} \times \frac{\det[J_{p,z',w'}^{t_n-t_1}(x_1^1, x_1^2)]_{i,j=1,2,\ldots,p}}{e^{-(t_2-t_1)K_{p,z',w'}}} \cdots \frac{\det[J_{p,z',w'}^{t_n-t_{n-1}}(x_{n-1}^1, x_{n-1}^2)]_{i,j=1,2,\ldots,p}}{e^{-(t_n-t_{n-1})K_{p,z',w'}}} \times \sqrt{\rho_{p,z',w'}(x_1^n, \ldots, x_p^n)}, \quad (5)
\]

where \( J_{p,z',w'}^N(x,y) \) and \( K_{p,z',w'} \) are the same as in Theorem 5.1.
Proof. Recall that \( J_{p,z',w'}^N(t) \) is a Markov process. Proposition 3.3 gives its one-dimensional distributions while Proposition 4.5 provides transition probabilities of \( J_{p,z',w'}^N(t) \).

Consequently, the joint distribution of \( J_{p,z',w'}^N(t_1), \ldots, J_{p,z',w'}^N(t_n) \) is given by

\[
\text{Prob}\{ J_{p,z',w'}^N(t_1) = X^1, \ldots, J_{p,z',w'}^N(t_n) = X^n \} = \sqrt{P_{N}}^{p,z',w'}((N + p - 1)X^1) \times \det[w_{n,[t_2-N^2]-(t_1,N^2)](N + p - 1)x^1, (N + p - 1)x^2)]_{i,j=1,...,p} \times \ldots \times \det[w_{n,[t_{n-1}-N^2]-(t_{n-2},N^2)](N + p - 1)x^{n-1}, (N + p - 1)x^n)]_{i,j=1,...,p} \times \sqrt{P_{N}}^{p,z',w'}((N + p - 1)X^n)
\]

Proposition 5.2 implies that as \( N \to \infty \),

\[
\sqrt{P_{N}}^{p,z',w'}((N + p - 1)X) \cdot (N + p - 1)^p \to \rho_{p,z',w'}(X).
\]

Next observe that

\[
\lim_{N \to \infty} (c_i^N)^{2[t-N^2]} = \lim_{N \to \infty} \left( \left( 1 - \frac{i}{p + N} \right) \left( 1 + \frac{i}{w' + z' + N + 1} \right) \right)^{[t-N^2]} = \left( 1 + \frac{i(p - w' - z' - 1) - i^2}{(p + N)(w' + z' + N + 1)} \right)^{[t-N^2]} = e^{-i(i + w' + z' + 1 - p)t}
\]

It remains to prove that for \( t > s \), as \( N \to \infty \)

\[
(N + p - 1)w_{N,[t-N^2]-[s,N^2]}((N + p - 1)x, (N + p - 1)y) \to J_{p,z',w'}^{t-s}(x, y).
\]

In the following computations we omit parameters and write \( Q^x \) instead of \( Q_{w',z'-p,N+p-1}^x \). Also we write \( \tilde{x} \) instead of \( (N + p - 1)x \) and \( \tilde{y} \) instead of \( (N + p - 1)y \). We have

\[
w_{N,[t-N^2]-[s,N^2]}(\tilde{x}, \tilde{y}) = \sum_{i=0}^{N+p-1} (c_i^N)^{2[t-N^2]-2[s,N^2]} f_N^i(\tilde{x}) f_N^i(\tilde{y})
\]

\[
= \sqrt{w_N(\tilde{x})w_N(\tilde{y})} \sum_{i=0}^{N+p-1} (c_i^N)^{2[t-N^2]-2[s,N^2]} \frac{Q^i(\tilde{x})Q^i(\tilde{y})}{(Q^x, Q^y)}
\]

\[
\sim \sqrt{w_{p,z',w'}(x)w_{p,z',w'}(y)(N + p - 1)^{w' + z' - p}} \sum_{i=0}^{N+p-1} (c_i^N)^{2[t-N^2]-2[s,N^2]} \frac{Q^i(\tilde{x})Q^i(\tilde{y})}{(Q^x, Q^y)}
\]

\[
= \sqrt{w_{p,z',w'}(x)w_{p,z',w'}(y)(N + p - 1)^{w' + z' - p}(A + B)},
\]

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where
\[ A = \sum_{i=0}^{l} (c_i)^{2(tN^2 - 2sN^2)} \frac{Q^i(\bar{x})Q^i(\bar{x})}{(Q^i, Q^i)} \]

and
\[ B = \sum_{i=l+1}^{N+p-1} (c_i)^{2(tN^2 - 2sN^2)} \frac{Q^i(\bar{x})Q^i(\bar{x})}{(Q^i, Q^i)}. \]

Now fix large enough \( l \) and send \( N \) to \( \infty \). First, let us examine \( A \). We have (see [KS, Section 2.5])
\[ Q^i_{w', z'-p, N+p-1}((N + p - 1)x) \to \frac{Jac^i_{w', z'-p}(1-x)}{Jac^i_{w', z'-p}(0)} = \frac{Jac^i_{z'-p, w'}(x)}{Jac^i_{z'-p, w'}(1)}, \]
and the convergence is uniform in \( x \) belonging to any compact subset of \((0, 1)\).

Here we used the relation
\[ Jac^i_{\alpha, \beta}(1-x) = (-1)^i Jac^i_{\beta, \alpha}(x). \]

By straightforward computation one proves that
\[ \lim_{N \to \infty} (N + p - 1)^{w' + z' + 1-p} = \frac{(Jac^i_{z'-p, w'}(1))^2}{(Jac^i_{z'-p, w'}(x), Jac^i_{z'-p, w'}(x))}. \]

in slightly different form this claim was also proved in [NSU]. Consequently,
\[ \lim_{N \to \infty} (N + p - 1) \cdot w_{p, z', w'}(x)w_{p, z', w'}(y)(N + p - 1)^{w' + z' - p} A \]
\[ = \sum_{i=0}^{l} e^{-(t-s)K_{p, z', w'}(i)} j^i_{p, z', w'}(x)j^i_{p, z', w'}(y) \]
\[ = \mathcal{J}^i_{p, z', w'}(x, y) - \sum_{i=l+1}^{\infty} e^{-(t-s)K_{p, z', w'}(i)} j^i_{p, z', w'}(x)j^i_{p, z', w'}(y) \]

Observe that for an arbitrary \( \varepsilon > 0 \) we have
\[ \left| \sum_{i=l+1}^{\infty} e^{-(t-s)K_{p, z', w'}(i)} j^i_{p, z', w'}(x)j^i_{p, z', w'}(y) \right| < \varepsilon, \]
if \( l \) is large enough. Indeed,
\[ e^{-(t-s)K_{p, z', w'}(i)} < e^{-(t-s)i(i-1)} \]
and
\[ |j^i_{p, z', w'}(x)| < e^{c(x)i}, \]
where $c(x)$ is a bounded function for $x$ belonging to any compact subset of $(0, 1)$. This estimate follows, for instance, from the recurrence relations on Jacobi polynomials.

Thus, to finish the proof we should show that
\[
\limsup_{N \to \infty} \sqrt{w_{p,z',w}(x)w_{p,z',w}(y)(N + p - 1)^{w' + z' + 1 - p}} |B| < \varepsilon(l),
\]
and $\varepsilon(l)$ tends to zero when $l$ tends to $\infty$.

We need the following lemma:

**Lemma 5.4.** For large enough $N$ and any $i < N + p$ we have
\[
(c_i^N)^{2 \lfloor t \cdot N^2 \rfloor - \lfloor s \cdot N^2 \rfloor} < e^{-c \cdot i \ln^2(i)},
\]
where constant $c$ does not depend on $N$.

**Proof.**
\[
(c_i^N)^2 = 1 - \frac{i(i + w' + z' - p + 1)}{(p + N)(w' + z' + N + 1)} < 1 - c_1 \frac{i^2}{N^2}.
\]

If $i < \frac{N}{\ln(N)}$ and $N$ is large enough, then
\[
\left(1 - c_1 \frac{i^2}{N^2}\right)^{2 \lfloor t \cdot N^2 \rfloor - \lfloor s \cdot N^2 \rfloor} < \left(1 - c_1 \frac{i^2}{N^2}\right)^{c_2 \cdot N^2} = \left(1 - c_1 \frac{i^2}{N^2}\right)^{\frac{N^2}{c_1 \ln^2(i)}} < (1 - \varepsilon)^{c_1 c_2 i^2} < e^{c_3 i \ln^2 i}.
\]

If $i \geq \frac{N}{\ln(N)}$ then
\[
\left(1 - c_1 \frac{i^2}{N^2}\right)^{2 \lfloor t \cdot N^2 \rfloor - \lfloor s \cdot N^2 \rfloor} \leq \left(1 - c_1 \frac{i^2}{N^2}\right)^{c_2 \cdot N^2} \leq (1 - c_1 \frac{N/\ln(N)^2}{N^2})^{c_2 \cdot N^2} = \left(1 - \varepsilon\right)^{c_1 c_2 \frac{N^2}{\ln^2(N)}} < (1 - \varepsilon)^{c_1 c_2 \frac{1}{\ln^4}} < e^{c_4 i \ln^2 i}.
\]

Recurrence relations on Hahn polynomials (see e.g. [KS] (1.5.3)) imply that
\[
|Q_{w',z'-p,N+p-1}(x)| < e^{c' \cdot i \ln(i)},
\]
where constant $c'$ does not depend on either $N$ or $x$.

Next note that
\[
\frac{(N + p - 1)^{w' + z' + 1 - p}}{(Q_{w',z'-p,N+p-1}^i, Q_{w',z'-p,N+p-1}^i)}
\]

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is bounded from above by $e^{c'' \cdot i \ln(i)}$.

For large enough $N$ and $l$ we obtain the following estimate

$$(N + p - 1)^{w' + z' + 1 - p} |B| = \frac{1}{(Q_{w',z'-p,N+p-1}|(N+p-1)x)\times(N+p-1)^{w'+z'-p}}$$

$$\times\left|\frac{(N + p - 1)^{w' + z' + 1 - p}}{(Q_{w',z'-p,N+p-1},Q_{w',z'-p,N+p-1})}\right| < \sum_{i=l+1}^{N+p-1} e^{-cl \cdot \ln(i)} < \sum_{i=l+1}^{N+p-1} e^{-cl \cdot \ln(i)} < \sum_{i=l}^{\infty} e^{-cl} = \frac{e^{-cl}}{1 - e^{cl}}.$$

Observing that $\frac{e^{-cl}}{1 - e^{cl}} \to 0$ when $l \to \infty$, completes the proof of Proposition 5.3.

Since multidimensional distributions of the process $J_{p,z',w'}(t)$ converge, we define $J_{p,z',w'}(t)$ as a limit process. Formulas 5 for the multidimensional distributions imply that $J_{p,z',w'}(t)$ is a stationary Markov process with initial distribution and transition probabilities as in Theorem 5.1. Thus, Theorems 2.2 and 5.1 are proved.

The Markov process $J_{p,z',w'}(t)$ has one interesting feature. Its dynamical (space-time) correlation functions can be expressed as minors of a certain extended kernel.

Let $\rho_n(x_1, t_1; x_2, t_2; \ldots; x_n, t_n)$ be the $n$th correlation function of $J_{p,z',w'}(t)$. Informally $\rho_n$ can be defined by

$\text{Prob}\{x_1 \in J_{p,z',w'}(t_1), \ldots, x_n \in J_{p,z',w'}(t_n)\} = \rho_n(x_1, t_1; \ldots; x_n, t_n) \cdot dx_1 \ldots dx_n$.

**Proposition 5.5.** Consider $n$ distinct points $(x_1, t_1), \ldots, (x_n, t_n)$. We have

$$\rho_n(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = \det[K_{p,z',w'}(x_i, t_i; x_j, t_j)]_{i,j=1,\ldots,n},$$

where

$$K_{p,z',w'}(x, t; y, s) = \begin{cases} 
\sum_{i=0}^{p-1} e^{(t-s)K_{p,z',w'}(i)} j_{p,z',w'}(x) j_{p,z',w'}(y), & \text{if } t \geq s, \\
-\sum_{i=p}^{\infty} e^{(t-s)K_{p,z',w'}(i)} j_{p,z',w'}(x) j_{p,z',w'}(y), & \text{if } t < s.
\end{cases}$$

We are not going to give the proof of this claim here. It follows from Theorem 5.1 and Eynard-Metha theorem. See [EM] and [BO3 Section 7.4].

[S] is a good survey on determinantal point processes. Additional information about extended kernels can be found in e.g. [EM], [J], [TW].
6 Markov semigroup of the limit process

Let \( W_p \) be the Weyl chamber.

\[
W_p = \{(x_1, \ldots, x_p) \in [0,1]^p : x_1 \leq x_2 \leq \cdots \leq x_p\}.
\]

Clearly, \( W_p \) is a state space of the process \( J_{p,z',w'}(t) \). Recall that \( \mu_{p,z',w'} \) is the one-dimensional distribution of \( J_{p,z',w'}(t) \). According to Theorem 5.1, \( \mu_{p,z',w'} \) is a probability measure on \( W_p \) given by its density \( \rho_{p,z',w'} \).

Denote by \( M^t_{p,z',w'} \) the Markov semigroup of the process \( J_{p,z',w'}(t) \) on \( L_2(W_p, \mu_{p,z',w'}) \). \( M^t_{p,z',w'} \) acts on a function \( f(X) \in L_2(W_p, \mu_{p,z',w'}) \) as follows

\[
(M^t_{p,z',w'}f)(X) = \int_{W_p} F(Y) \tilde{P}^t_{p,z',w'}(Y | X) dY,
\]

where the integration is performed with respect to the Lebesgue measure on \( W_p \).

In this section we are going to compute the eigenfunctions and the generator of \( M^t_{p,z',w'} \).

Recall that \( \text{Jac}^{\lambda}_{z'-p,w'}(x) \) is Jacobi polynomial of degree \( k \) on \((0,1)\). Now we want to introduce multi-dimensional Jacobi polynomials. Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0) \) be a partition of \( n \). Denote

\[
\text{Jac}^{\lambda}_{z'-p,w'}(x_1, \ldots, x_p) = \frac{\det[\text{Jac}^{\lambda}_{z'-p,w'}(x_j)]_{i,j=1,\ldots,p}}{\prod_{i>j}(x_i - x_j)}.
\]

It is clear that \( \text{Jac}^{\lambda}_{z'-p,w'}(x_1, \ldots, x_p) \) is a symmetric polynomial in \( p \) variables of degree \( n = \lambda_1 + \cdots + \lambda_p \).

We may view \( \text{Jac}^{\lambda}_{z'-p,w'}(x_1, \ldots, x_p) \) as a function on \( W_p \). In what follows we use the same notation for these functions.

It is well-known that the polynomials \( \text{Jac}^{\lambda}_{z'-p,w'}(x_1, \ldots, x_p) \) form an orthogonal basis in \( L_2(W_p, \mu_{p,z',w'}) \).

**Theorem 6.1.** Polynomials \( \text{Jac}^{\lambda}_{z'-p,w'}(x_1, \ldots, x_p) \) are eigenfunctions of the Markov semigroup \( M^t_{p,z',w'} \).

\[
M^t_{p,z',w'}(\text{Jac}^{\lambda}_{z'-p,w'}) = c(\lambda, t) \text{Jac}^{\lambda}_{p,z',w'},
\]

where

\[
c(\lambda, t) = e^{t \left( \sum_{i=1}^p K_{p,z',w'}(p-i) - K_{p,z',w'}(\lambda_i + p-i) \right)}.
\]
Proof.

\[ (M_{p,z',w'}^t \lambda \text{Jac}_t^{\lambda,-p,w'})(x_1,x_2,\ldots,x_p) \]

\[ = \int_{y_1<\cdots<y_p} \text{Jac}_t^{\lambda,-p,w'}(y_1,\ldots,y_p) \sqrt{\rho_{p,z',w'}(y_1,\ldots,y_p)} \sqrt{\rho_{p,z',w'}(x_1,\ldots,x_p)} \]

\[ \times e^{tK_{p,z',w'}} \cdot \det \{ \text{Jac}_t^{\lambda,-p,w'}(x_i,y_j) \}_{i,j=1,2,\ldots,p} dy_1 \ldots dy_p \]

\[ = \frac{e^{tK_{p,z',w'}}}{p!} \prod_{i>j} (x_i - x_j) \int_{0<y_1<1} \cdots \int_{0<y_p<1} \frac{\det \{ \text{Jac}_t^{\lambda,-p,w'}(y) \}_{i,j=1,2,\ldots,p}}{\prod_{i=1,p} \sqrt{w_{p,z',w'}'(y_i)}} \]

\[ \times \prod_{i,j=1,2,\ldots,p} \frac{\text{Jac}_t^{\lambda,-p,w'}(x_i,y_j)}{\text{Jac}_t^{\lambda,-p,w'}(y_i)\text{Jac}_t^{\lambda,-p,w'}(x_i)} dy_1 \ldots dy_p. \]

Note that the integrand is a symmetric function in \(y_1,\ldots,y_p\). Thus, we can integrate over \(p\)-dimensional cube \((0,1)^p\), instead of the simplex \(0 < y_1 < \cdots < y_p < 1\). We arrive to the following expression

\[ \frac{e^{tK_{p,z',w'}}}{p!} \prod_{i>j} (x_i - x_j) \int_{0<y_1<1} \cdots \int_{0<y_p<1} \frac{\det \{ \text{Jac}_t^{\lambda,-p,w'}(y) \}_{i,j=1,2,\ldots,p}}{\prod_{i=1,p} \sqrt{w_{p,z',w'}'(y_i)}} \]

\[ \times \prod_{i,j=1,2,\ldots,p} \frac{\text{Jac}_t^{\lambda,-p,w'}(x_i,y_j)}{\text{Jac}_t^{\lambda,-p,w'}(y_i)\text{Jac}_t^{\lambda,-p,w'}(x_i)} dy_1 \ldots dy_p. \]

Multiplying two matrices under determinants and using orthogonality relations for the Jacobi polynomials we obtain the desired formula. \(\square\)

Let \(G_{p,z',w'}\) be the infinitesimal generator of the Markov semigroup \(M_{p,z',w'}^t\). Theorem 6.1 implies the following proposition.

**Proposition 6.2.** Polynomials \(\text{Jac}_t^{\lambda,-p,w'}(x_1,\ldots,x_p)\) are eigenfunctions of \(G_{p,z',w'}\).

\[ G_{p,z',w'}(\text{Jac}_t^{\lambda,-p,w'}) = \tilde{c}(\lambda) \text{Jac}_t^{\lambda,-p,w'}, \]

where

\[ \tilde{c}(\lambda) = \sum_{i=1}^p K_{p,z',w'}(p-i) - K_{p,z',w'}(\lambda_i + p - i) \]

Our next aim is to obtain explicit formulas for the generators \(G_{p,z',w'}\). Observe that the numbers \(-K_{p,z',w'}(i) = -i(i+w'+z'+1-p)\) are eigenvalues of the Jacobi differential operator corresponding to the eigenvectors \(\text{Jac}_t^{\lambda,-p,w'}(x)\). Namely, denote

\[ D_{\text{Jac}}^{\lambda,-p,w'} = x(1-x) \frac{d^2}{dx^2} + (w' + 1 - (w' + z' - p + 2)x) \frac{d}{dx}. \]

The following proposition holds (see e.g. [KS (1.8.5)])

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Proposition 6.3. We have
\[ D_{z' - p, w'}(\text{Jac}_z^{-}) = -K_{p, z', w'}(i)\text{Jac}_z^{-} \]

Set
\[ D_{z', w', p} = \sum_{i=1}^{p} x_i(1 - x_i) \frac{\partial^2}{\partial x_i^2} + (w' + 1 - (w' + z' - p + 2)x_i) \frac{\partial}{\partial x_i}. \]

Recall that if \( X = (x_1, \ldots, x_p) \) then \( V(X) = \prod_{i > j} (x_i - x_j) \).

Theorem 6.4. We have
\[ G_{p, z', w'}(f) = \frac{1}{V(X)} \cdot D_{z', w', p}(V(X) \cdot f) + K_{p, z', w'} f, \]

and the domain of definition of \( G_{p, z', w'} \) consists of all functions \( f \in L_2(W_p, \mu_{p, z', w'}) \) such that
\[ \frac{1}{V(X)} \cdot D_{z', w', p}(V(x) \cdot f) + K_{p, z', w'} f \in L_2(W_p, \mu_{p, z', w'}). \]

Remark. After some simplifications the generator \( G_{p, z', w'} \) can be alternatively expressed as
\[ G_{p, z', w'}(f) = D_{z', w', p}(f) + \sum_{i=1}^{p} \left( x_i(1 - x_i) \sum_{j \neq i} \frac{1}{x_i - x_j} \right) \frac{\partial f}{\partial x_i} \]

Proof. It is sufficient to verify that
\[ G_{p, z', w'}(\text{Jac}_z^{\lambda}) = \frac{1}{V(X)} \cdot D_{z', w', p}(V(x) \cdot \text{Jac}_z^{\lambda}) + K_{p, z', w'} \text{Jac}_z^{\lambda}. \]

We have
\[ \frac{1}{V(X)} \cdot D_{z', w', p}(V(x) \cdot \text{Jac}_z^{\lambda}) + K_{p, z', w'} \text{Jac}_z^{\lambda} = \frac{D_{z', w', p} \det[\text{Jac}_z^{\lambda+p-i}(x_j)]_{i,j=1,\ldots,p}}{V(X)} + K_{p, z', w'} \text{Jac}_z^{\lambda}. \]

Let us write the determinant as the alternating sum of products
\[ A_{\sigma} = \prod_{j=1}^{p} \text{Jac}_z^{\lambda+p-j}(x_{\sigma(j)}). \]

(Here \( \sigma \in S_p \) is an arbitrary permutation.)
Proposition 6.2 implies that

\[ D_{z',w',p} A_\sigma = \left( -\sum_{j=1}^{p} K_{p,z',w'}(\lambda_j + p - j) \right) A_\sigma. \]

Consequently,

\[
\frac{D_{z',w',p} \det [Jac_{z'+p,w'}^\lambda (x_j)]_{i,j=1,\ldots,p} + K_{p,z',w'} Jac_{z'-p',w'}^\lambda}{V(X)}
\]

\[
= \left( K_{p,z',w'} - \sum_{j=1}^{p} K_{p,z',w'}(\lambda_j + p - j) \right) Jac_{z'-p',w'}^\lambda = G_{p,z',w'}(Jac_{z'-p',w'}^\lambda)
\]

\[ \square \]

7 Doob h-transform interpretation

In this section we explain that the process \( J_{p,z',w'}(t) \) can be viewed as \( p \) independent identically distributed diffusions conditioned (in the sense of Doob) never to collide, i.e. it is the Doob h-transform of \( p \) independent processes.

**Proposition 7.1.** We have

\[ D_{z',w',p} V(X) = -K_{p,z',w'} V(X). \]

**Proof.** This claim follows from Proposition 6.2 and Theorem 6.4, but we also present an independent proof.

**Lemma 7.2.** For arbitrary \( a, b, c \in \mathbb{R} \) set

\[(Gf)(X) = \sum_{i=1}^{p} (x^2 + ax + b) \frac{\partial^2}{\partial x_i^2} + \left( -\frac{2}{3} (p-2)x_i + c \right) \frac{\partial}{\partial x_i}.\]

Then \( GV(X) = 0 \).

**Proof.** See [KO, Lemma 3.1]. Note that originally there was a misprint in [KO], the minus sign was missing. \( \square \)

Lemma 7.2 implies that

\[
D_{z',w',p} V(X) = \left( -\frac{2}{3} (p-2) - (w' + z' - p + 2) \right) \sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i} V(X)
\]

\[
= -\frac{p(p-1)}{2} \left( -\frac{1}{3} (p-2) + w' + z' \right) V(X).
\]
To finish the proof let us compute explicitly $K_{p,z',w'}$:

$$K_{p,z',w'} = \sum_{i=0}^{p-1} K_{p,z',w'}(i) = \sum_{i=0}^{p-1} (i^2 + i(w' + z' - p + 1))$$

$$= \frac{p(p-1)(2p-1)}{6} + \frac{p(p-1)}{2}(w' + z' - p + 1)$$

$$= \frac{p(p-1)}{2} \left( \frac{2p-1}{3} - p + 1 + w' + z' \right) = \frac{p(p-1)}{2} \left( -\frac{1}{3}(p-2) + w' + z' \right).$$

Now consider $p$ independent diffusions with generators $D_{z',w'}$. We call these diffusions Jacobi processes; they can be viewed as solutions of certain stochastic differential equations.

The infinitesimal generator of these $p$ diffusions is precisely $D_{z',w'}$. Proposition 7.1 implies that we can construct new Markov process, which is a variant of Doob $h$–transform of $p$ diffusions. Here function $h$ is $V(X)$. The infinitesimal generator of Doob $h$–transform with $h = V(X)$ is precisely $G_{p,z',w'}$. The Doob $h$–transforms and its properties were studied in numerous papers, see e.g. [D, Kon].

Thus, our process $J_{p,z',w'}(t)$ can be interpreted as Doob $h$–transform of $p$ independent diffusions. In many examples Doob $h$–transform coincides with the conditional process, given that there is no collision of the components. In our case, clearly, there are no collisions, trajectories of $J_{p,z',w'}(t)$ are collections of $p$ nonintersecting paths. However the question whether $J_{p,z',w'}(t)$ is precisely $p$ independent diffusions conditioned on never having a collision between any two of its components, remains open.

8 Appendix. Some formulas for Hahn and Jacobi polynomials

The following formulas can be found in e.g. [KS].

**Lemma 8.1.** For $\alpha > -1, \beta > -1$,

$$(Q_{\alpha,\beta,M}^k, Q_{\alpha,\beta,M}^l) = \sum_{x=0}^{M} \frac{\Gamma(\alpha + x + 1)\Gamma(\beta + N - x + 1)}{\Gamma(x+1)\Gamma(N-x+1)} Q_{\alpha,\beta,M}^k(x)Q_{\alpha,\beta,M}^l(x)$$

$$= (-1)^k(k + \alpha + \beta + 1)_{M+1}(\beta + 1)_{k!}\Gamma(\alpha + 1)\Gamma(\beta + 1)$$

$$\frac{1}{(2k + \alpha + \beta + 1)(\alpha + 1)k(-M)_{k!}} \delta_{kl}.$$
Lemma 8.2. For \( x, y \in \{0, 1, \ldots, M\} \),

\[
\sum_{k=0}^{M} \frac{(2k + \alpha + \beta + 1)(\alpha + 1)_{k}(-M)_{k}M!}{(-1)^{k}(k + \alpha + \beta + 1)_{M+1}(\beta + 1)_{k}k!}Q_{k}(x, \alpha, \beta, M)Q_{k}(y, \alpha, \beta, M) = \delta_{xy} \frac{\Gamma(\alpha + 1)}{x!(\beta + 1)_{M-x}!}.
\]

The following Lemma was proved in [Gor, Lemma 8]

Lemma 8.3.

\[ xQ_{k}(x-1; \alpha, \beta, M-1) + (M-x)Q_{k}(x; \alpha, \beta, M-1) = MQ_{k}(x; \alpha, \beta, M). \]

Proof of Proposition 4.2. Recall that our goal is to prove the following equality

\[
\sum_{i=0}^{N+p-1} c_{i}^{N} f_{N}^{k}(x) f_{N+1}^{k}(y) = \begin{cases} \frac{(w'+x+1)(x+1)}{(x+N)(w'+z'+N+1)}, & y = x+1 \\ \frac{(z'+N-z)(p+N-n)}{(p+x)(w'+z'+N+1)}, & y = x \\ 0, & \text{otherwise.} \end{cases} \tag{6}
\]

First, we express left-hand side of (6) through the polynomials \( Q_{\alpha,\beta,M}^{k} \). By the definition

\[ f_{N}^{k}(x) = \frac{Q_{w',z'-p,N+p-1}^{k}(x)\sqrt{w_{N}(x)}}{(Q_{w',z'-p,N+p-1}^{k}, Q_{w',z'-p,N+p-1}^{k})} \]

Lemma 8.1 provides the value of \( (Q_{w',z'-p,N+p-1}^{k}, Q_{w',z'-p,N+p-1}^{k}) \). Next, we express \( Q_{w',z'-p,N+p}^{k} \) through \( Q_{w',z'-p,N+p-1}^{k} \) using Lemma 8.3. Applying Lemma 8.2 completes the proof.

The following lemma in slightly different form can be found in [KS].

Lemma 8.4. For \( \alpha > -1, \beta > -1 \),

\[
\int_{0}^{1} (1-x)^{\alpha}x^{\beta}J_{\alpha}^{k}(x)J_{\alpha}^{l}(x) = \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)!\Gamma(k+\alpha+\beta+1)}\delta_{kl}.
\]

References

[BO] A. Borodin, G. Olshanski, Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes, Ann. of Math. 161 (2005), no. 3, 1319–1422, arXiv:math/0109194

[BO2] A. Borodin, G. Olshanski, Infinite dimensional diffusions as limits of random walks on partitions. To appear in Probability Theory and Related Field. arXiv:0706.1034

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[BO3] A. Borodin, G. Olshanski, Markov processes on partitions. Probab. Theory and Related Fields 135 (2006), no. 1, 84–152. arXiv: math-ph/0409075

[D] J. L. Doob, Classical potential theory and its probabilistic counterpart. Grundlehren der mathematischen Wissenschaften, vol. 262. Springer-Verlag, New York 1984

[EM] B. Eynard and M. L. Mehta, Matrices coupled in a chain. I. Eigenvalue correlations. J. Phys. A: Math. Gen. 31(1998), 4449–4456

[Fu1] J. Fulman, Stein’s method and Plancherel measure of the symmetric group. Trans. Amer. Math. Soc. 357 (2005), 555–570. arXiv: math.RT/0505423

[Fu2] J. Fulman, Commutation relations and Markov chains. To appear in Probability Theory and Related Fields. arXiv:0712.1375

[Gor] V. Gorin, Non-intersecting paths and Hahn orthogonal polynomial ensemble. To appear in Functional Analysis and its Applications. arXiv:0708.2349

[J] K. Johansson. Discrete polynuclear growth and determinantal processes. Comm. Math. Phys. 242(2003), 277–329, arXiv:math/0206208

[KO] W. König, N. O’Connell, Eigenvalues of the Laguerre process as non-colliding squared Bessel processes, Elec. Comm. Probab. 6 (2001), no. 11, 107–114

[Kön] W. D. König, Orthogonal polynomial ensembles in probability theory, Probability Surveys 2, (2005), 385-447

[KOV] S. Kerov, G. Olshanski, A. Vershik, Harmonic Analysis on the infinite symmetric group. Invent. Math. 158 (2004), no. 3, 551–642, arXiv:math/0312270.

[KS] R. Koekoek and R. F. Swarttouw, The Askey–scheme of hypergeometric orthogonal polynomials and its q-analogue, http://aw.twi.tudelft.nl/~koekoek/reports.html

[M] M. L. Mehta, Random matrices, 2nd edition, Academic Press, Boston, MA, 1991.

[Mac] I. Macdonald, Symmetric Functions and Hall Polynomials, Clarendon Press Oxford,1979.

[NSU] A. F. Nikiforov, S. K. Suslov and V. B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer Series in Computational Physics, Sprnger-Verlag, New York, 1991.
[Olsh] G. Olshanski, The problem of harmonic analysis on the infinite-dimensional unitary group, J. Funct. Anal. 205 (2003), no. 2, 464–524, arXiv:math/0109193.

[Pe] L. Petrov, A Two-Parameter Family of Infinite-Dimensional Diffusions in the Kingman Simplex. arXiv:0708.1930

[S] A. Soshnikov, Determinantal random point fields, Russian Math. Surveys 55 (2000), no. 5, 923-975

[TW] C. A. Tracy, H. Widom, Differential equations for Dyson processes. Comm. Math. Phys. 252(2004), 7–41, arXiv:math/0309082

[Zh] D. P. Zhelobenko, Compact Lie Groups and their Representations, Nauka, Moscow, 1970 (Russian); English translation: Transl. Math. Monographs 40, A. M. S., Providence, RI, 1973.