ON CONNECTIONS ON PRINCIPAL BUNDLES

INDRANIL BISWAS

ABSTRACT. A new construction of a universal connection was given in \cite{BHS}. The main aim here is to explain this construction. A theorem of Atiyah and Weil says that a holomorphic vector bundle $E$ over a compact Riemann surface admits a holomorphic connection if and only if the degree of every direct summand of $E$ is degree. In \cite{AB}, this criterion was generalized to principal bundles on compact Riemann surfaces. This criterion for principal bundles is also explained.

1. Introduction

A connection $\nabla^0$ on a $C^\infty$ principal $G$–bundle $E_G \rightarrow X$ is called universal if given any $C^\infty$ principal $G$–bundle $E_G$ on a finite dimensional $C^\infty$ manifold $M$, and any connection $\nabla$ on $E_G$, there is a $C^\infty$ map

$$\xi : M \rightarrow X$$

such that

- the pulled back principal $G$–bundle $\xi^*E_G$ is isomorphic to $E_G$, and
- the isomorphism between $\xi^*E_G$ and $E_G$ can be so chosen that it takes the pulled back connection $\xi^*\nabla^0$ on $\xi^*E_G$ to the connection $\nabla$ on $E_G$.

In \cite{NR} and \cite{Sc} universal connections were constructed. In \cite{BHS} a very simple, in fact quite tautological, universal connection was constructed.

2. Atiyah bundle

All manifolds considered here will be $C^\infty$, second countable and Hausdorff. Later we will impose further conditions such as complex structure.

Let $G$ be a finite dimensional Lie group. Take a connected $C^\infty$ manifold $M$. A principal $G$–bundle over $M$ is a triple of the form

$$(E_G, p, \psi),$$

(2.1)

where

1. $E_G$ is a $C^\infty$ manifold manifold,
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(2) \[ p : E_G \longrightarrow M \] is a $C^\infty$ surjective submersion, and

(3) \[ \psi : E_G \times G \longrightarrow E_G \] is a $C^\infty$ map defining a right action of $G$ on $E_G$, such that the following two conditions hold:

- the two maps $p \circ \psi$ and $p \circ p_1$ from $E_G \times G$ to $M$ coincide, where $p_1$ is the natural projection of $E_G \times G$ to $E_G$, and
- the map to the fiber product

\[ \text{Id}_{E_G} \times \psi : E_G \times G \longrightarrow E_G \times_M E_G \]

is a diffeomorphism; note that the first condition $p \circ \psi = p \circ p_1$ implies that the image of $\text{Id}_{E_G} \times \psi$ is contained in the submanifold $E_G \times_M E_G \subset E_G \times E_G$ consisting of all point $(z_1, z_2) \in E_G \times E_G$ such that $p(z_1) = p(z_2)$.

Therefore, the first condition implies that $G$ acts on $E_G$ along the fibers of $p$, while the second condition implies that the action of $G$ on each fiber of $p$ is both free and transitive.

Take a $C^\infty$ principal $G$–bundle $(E_G, p, \psi)$ over $M$. The tangent bundle of the manifold $E_G$ will be denoted by $TE_G$. Take a point $x \in M$. Let

\[ (TE_G)^x := (TE_G)|_{p^{-1}(x)} \longrightarrow p^{-1}(x) \]

be the restriction of the vector bundle $TE_G$ to the fiber $p^{-1}(x)$ of $p$ over the point $x$. As noted above, the action $\psi$ of $G$ on $E_G$ preserves $p^{-1}(x)$, and the resulting action of $G$ on $p^{-1}(x)$ is free and transitive. Therefore, the action of $G$ on $TE_G$ given by $\psi$ restricts to an action of $G$ on $(TE_G)^x$. Let $\operatorname{At}(E_G)_x$ be the space of all $G$–invariant sections of $(TE_G)^x$. Since the action of $G$ on the fiber $p^{-1}(x)$ is transitive, it follows that any $G$–invariant section of $(TE_G)^x$ is automatically smooth. More precisely, any $G$–invariant sections of $(TE_G)^x$ is uniquely determined by its evaluation of some fixed point of $p^{-1}(x)$. Therefore, $\operatorname{At}(E_G)_x$ is a real vector space whose dimension coincides with the dimension of $E_G$.

There is a natural vector bundle over $M$, which was introduced in [At], whose fiber over any $x \in M$ is $\operatorname{At}(E_G)_x$. This vector bundle is known as the Atiyah bundle, and it is denoted by $\operatorname{At}(E_G)$. We now recall the construction of $\operatorname{At}(E_G)$.

As before, consider the action of $G$ to $TE_G$ given by the action $\psi$ of $G$ on $E_G$. Since the action of $G$ is free and transitive on each fiber of $p$, it follows that this action of $G$ on $TE_G$ free and proper. Therefore, we have a quotient manifold

\[ \operatorname{At}(E_G) := (TE_G)/G \] (2.4)

for this action of $G$ on $TE_G$. Since the natural projection $TE_G \longrightarrow E_G$ is $G$–equivariant, it produces a projection

\[ \operatorname{At}(E_G) := (TE_G)/G \longrightarrow E_G/G = M . \] (2.5)
This projection in (2.5) is clearly surjective. Furthermore, it is a submersion because the projection $T\mathcal{E} \rightarrow \mathcal{E}$ is so. It is now straightforward to check that the projection in (2.5) makes $\text{At}(\mathcal{E})$ a $C^\infty$ vector bundle over $\mathcal{E}$. Its rank coincides with the rank of the tangent bundle $T\mathcal{E}$, so its rank is $\dim \mathcal{E} + \dim \mathcal{M}$. From (2.4) it follows immediately that we have a natural diffeomorphism

$$\mu : p^*\text{At}(\mathcal{E}) \rightarrow T\mathcal{E}. \quad (2.6)$$

It is straightforward to check that $\mu$ is a $C^\infty$ isomorphism of vector bundles over $\mathcal{E}$.

Let $$dp : T\mathcal{E} \rightarrow p^*\mathcal{M} \quad (2.7)$$ be the differential of the projection $p$ in (2.2). Consider the surjective $C^\infty$ homomorphism of vector bundles $$dp \circ \mu : p^*\text{At}(\mathcal{E}) \rightarrow p^*\mathcal{M}, \quad (2.8)$$ where $\mu$ is constructed in (2.6). Since $p^*\text{At}(\mathcal{E})$ and $p^*\mathcal{M}$ are pulled back to $\mathcal{E}$ from $\mathcal{M} = \mathcal{E}/\mathcal{G}$, they are naturally equipped with an action of $\mathcal{G}$. The homomorphism $dp \circ \mu$ in (2.8) is clearly $\mathcal{G}$-equivariant. Therefore, it descends to a surjective $C^\infty$ homomorphism of vector bundles

$$\eta : \text{At}(\mathcal{E}) \rightarrow \mathcal{M}. \quad (2.9)$$

The kernel of the differential $dp$ in (2.7) is clearly preserved by the action of $\mathcal{G}$ on $T\mathcal{E}$. The quotient kernel($dp$)/$\mathcal{G}$ will be denoted by $\text{ad}(\mathcal{E})$. It is a $C^\infty$ vector bundle on $\mathcal{M}$ whose rank is $\dim \mathcal{G}$. The inclusion of kernel($dp$) in $T\mathcal{E}$ produces a fiberwise injective $C^\infty$ homomorphism of vector bundles

$$\iota_0 : \text{ad}(\mathcal{E}) \rightarrow \text{At}(\mathcal{E}).$$

The kernel of the homomorphism $\eta$ in (2.9) coincides with the image of $\iota_0$. Therefore, we have a short exact sequence of $C^\infty$ vector bundles over $\mathcal{M}$

$$0 \rightarrow \text{ad}(\mathcal{E}) \xrightarrow{\iota_0} \text{At}(\mathcal{E}) \xrightarrow{\eta} \mathcal{M} \rightarrow 0, \quad (2.10)$$

which is known as the Atiyah exact sequence for $\mathcal{E}$. Using the Lie bracket operation of vector fields on $\mathcal{E}$, the fibers of $\text{ad}(\mathcal{E})$ are Lie algebras; this will be elaborated below.

The Lie algebra of $\mathcal{G}$ will be denoted by $\mathfrak{g}$. Consider the action of $\mathcal{G}$ on itself defined by $\text{Ad}(g)(h) = g^{-1}hg$. This action defines an action of $\mathcal{G}$ on $\mathfrak{g}$, which is known as the adjoint action; this adjoint action of $\mathcal{G}$ on $\mathfrak{g}$ will also be denoted by $\text{Ad}$. Consider the quotient of $\mathcal{E} \times \mathfrak{g}$ where two points $(z, v), (z', v') \in \mathcal{E} \times \mathfrak{g}$ are identified if there is some $g_0 \in \mathcal{G}$ such that $z' = zg_0$ and $v' = \text{Ad}(g_0^{-1})(v)$. This quotient space coincides with the total space of the adjoint vector bundle $\text{ad}(\mathcal{E})$ in (2.10). Note that the projection

$$\text{ad}(\mathcal{E}) \rightarrow \mathcal{M}. \quad (2.11)$$

sends the equivalence class of any $(z, v) \in \mathcal{E} \times \mathfrak{g}$ to $p(z)$ (it is clearly independent of the choice of the element in the equivalence class). The fibers of $\text{ad}(\mathcal{E})$ are identified with $\mathfrak{g}$ up to conjugation. Since the adjoint action of $\mathcal{G}$ on $\mathfrak{g}$ preserves its Lie algebra
structure, the fibers of $\text{ad}(E_G)$ are in fact Lie algebras isomorphic to $\mathfrak{g}$. This Lie algebra structure of a fiber of $\text{ad}(E_G)$ coincides with the one constructed earlier using the Lie bracket operation of vector fields. The pulled back vector bundle $p^*\text{ad}(E_G)$ on $E_G$ is identified with the trivial vector bundle $E_G \times \mathfrak{g}$ with fiber $\mathfrak{g}$. This identification sends any vector $(z, v) \in (p^*\text{ad}(E_G))_z$ in the fiber over $z$ of the pulled back bundle to the element $(z, v)$ of the trivial vector bundle $E_G \times \mathfrak{g}$.

A connection on $E_G$ is a $C^\infty$ splitting of the Atiyah exact sequence for $E_G$ [At]. In other words, a connection on $E_G$ is a $C^\infty$ homomorphism of vector bundles

$$D : TM \longrightarrow \text{At}(E_G)$$

such that $\eta \circ D = \text{Id}_{TM}$, where $\eta$ is the projection in (2.9).

Let

$$D : TM \longrightarrow \text{At}(E_G)$$

be a homomorphism defining a connection on $E_G$. Consider the composition homomorphism

$$p^*TM \xrightarrow{p^*D} p^*\text{At}(E_G) \xrightarrow{\mu} TEG,$$

where $\mu$ is the isomorphism in (2.6). Its image

$$\mathcal{H}(D) := (\mu \circ p^*D)(p^*TM) \subset TEG$$

is known as the horizontal subbundle of $TEG$ for the connection $D$. Since $\mu$ is an isomorphism, and the splitting homomorphism $D$ in (2.13) is uniquely determined by its image $D(TM) \subset \text{At}(E_G)$, it follows immediately that the horizontal subbundle $\mathcal{H}(D)$ determines the connection $D$ uniquely.

The composition

$$\text{kernel}(dp) \hookrightarrow TEG \longrightarrow TEG/\mathcal{H}(D)$$

is an isomorphism. Hence we have

$$TEG = \mathcal{H}(D) \oplus (E_G \times \mathfrak{g});$$

it was noted earlier that $p^*\text{ad}(E_G)$ is identified with the trivial vector bundle $E_G \times \mathfrak{g}$. The projection of $TEG$ to the second factor of the above direct sum decomposition defines a $\mathfrak{g}$-valued smooth one-form on $E_G$. The connection $D$ is clearly determined uniquely by this $\mathfrak{g}$-valued one-form on $E_G$.

See [BHS, p. 370, Lemma 2.2] for a proof of the following lemma:

**Lemma 2.1.** Any principal $G$-bundle $E_G \longrightarrow M$ admits a connection.

The space of all connections on a principal $G$-bundle $E_G$ is an affine space for the vector space $C^\infty(M; \text{Hom}(TM, \text{ad}(E_G)))$. 
3. A universal connection

3.1. A tautological connection. As before, let $p : E_G \to M$ be a $C^\infty$ principal $G$–bundle. Consider the Atiyah exact sequence in (2.10). Tensoring it with the cotangent bundle $T^*M = (TM)^*$ we get the following short exact sequence of vector bundles on $M$

$$0 \to \text{ad}(E_G) \otimes T^*M \to \text{At}(E_G) \otimes T^*M \xrightarrow{\eta \otimes \text{Id}_{T^*M}} TM \otimes T^*M \to \text{End}(TM) \to 0.$$  

(3.1)

Let $\text{Id}_{TM}$ denote the identity automorphism of $TM$. It defines a $C^\infty$ section of the endomorphism bundle $\text{End}(TM)$. Let

$$\delta : \mathcal{C}(E_G) := (\eta \otimes \text{Id}_{T^*M})^{-1}(\text{Id}_{TM}) \subset \text{At}(E_G) \otimes T^*M \to M$$

be the fiber bundle over $M$, where $\eta \otimes \text{Id}_{T^*M}$ is the surjective homomorphism in (3.1).

We recall that a connection on $E_G$ is a $C^\infty$ splitting of the Atiyah exact sequence. See [BHS, p. 371, Lemma 3.1] for a proof of the following:

**Lemma 3.1.** The space of all connections on $E_G$ is in bijective correspondence with the space of all smooth sections of the fiber bundle

$$\delta : \mathcal{C}(E_G) \to M$$

constructed in (3.2).

Combining Lemma 2.1 with Lemma 3.1, the following is obtained.

**Corollary 3.2.** The fiber bundle $\delta$ in (3.2) is an affine bundle over $M$ for the vector bundle $\text{Hom}(TM, \text{ad}(E_G))$. In particular, if we fix a connection on $E_G$ (which exists by Lemma 2.1), then the fiber bundle in (3.2) gets identified with the total space of the vector bundle $\text{Hom}(TM, \text{ad}(E_G))$.

See [BHS, p. 372, Proposition 3.3] for a proof of the following:

**Proposition 3.3.** There is a tautological connection on the principal $G$–bundle $\delta^*E_G$ over $\mathcal{C}(E_G)$.

The key observations in the construction of the tautological connection in Proposition 3.3 are the following:

There is a tautological homomorphism

$$\beta : \delta^*\text{At}(E_G) \to \delta^*\text{ad}(E_G) = \text{ad}(\delta^*E_G).$$

On the other hand, there is a tautological projection

$$\beta' : \text{At}(\delta^*E_G) \to \delta^*\text{At}(E_G)$$
such that the diagram
\[
\begin{array}{ccc}
\text{At}(\delta^*E_G) & \xrightarrow{\beta'} & \delta^*\text{At}(E_G) \\
\downarrow & & \downarrow \\
T\mathcal{C}(E_G) & \xrightarrow{\delta \iota} & \delta^*\mathcal{T}M
\end{array}
\]
where the projection \(\text{At}(\delta^*E_G) \to T\mathcal{C}(E_G)\) is constructed as in (2.9) for the principal \(G\)-bundle \(\delta^*E_G\). Finally, the composition
\[
\beta \circ \beta' : \text{At}(\delta^*E_G) \to \text{ad}(\delta^*E_G)
\]
gives a splitting of the Atiyah exact sequence for \(\delta^*E_G\). This splitting \(\beta \circ \beta'\) defines the tautological connection on \(\delta^*E_G\).

The above tautological connection on the principal \(G\)-bundle \(\delta^*E_G\) will be denoted by \(D_0\).

In Lemma 3.1 we noted that the connections on \(E_G\) are in bijective correspondence with the smooth sections of \(\mathcal{C}(E_G)\). Take any smooth section
\[
\sigma : M \to \mathcal{C}(E_G)
\]
of the fiber bundle \(\mathcal{C}(E_G) \to M\). Let \(D(\sigma)\) be the corresponding connection on the principal \(G\)-bundle \(E_G\). We note that \(\sigma^*\delta^*E_G = E_G\) because \(\delta \circ \sigma = \text{Id}_M\).

The following lemma is a consequence of the construction of the tautological connection \(D_0\).

**Lemma 3.4.** The connection \(D(\sigma)\) on \(E_G\) coincides with the pulled back connection \(\sigma^*D_0\) on the principal \(G\)-bundle \(\sigma^*\delta^*E_G = E_G\).

### 3.2. Construction of universal connection

All infinite dimensional manifolds will be modeled on the direct limit \(\mathbb{R}\) of the sequence of vector spaces \(\{\mathbb{R}^n\}_{n>0}\) with natural inclusions \(\mathbb{R}^i \hookrightarrow \mathbb{R}^{i+1}\).

Let
\[
p_0 : E_G \to B_G
\]
be a universal principal \(G\)-bundle in the \(C^\infty\) category; see [Mi] for the construction of a universal principal \(G\)-bundle. So, \(B_G\) is a \(C^\infty\) manifold, the projection \(p_0\) is smooth, and \(E_G\) is contractible. Define
\[
B_G := B_G \times \mathbb{R}\.
\]
Define
\[
\mathcal{E}_G := p_{BG}^*E_G = E_G \times \mathbb{R}\,
\]
where \(p_{BG} : B_G \times \mathbb{R} \to B_G\) is the natural projection.

See [BHS, p. 374, Lemma 4.1] for a proof of the following:

**Lemma 3.5.** the principal \(G\)-bundle
\[
p := p_0 \times \text{Id}_{\mathbb{R}^\infty} : \mathcal{E}_G \to B_G
\]
is universal.

Set the principal $G$–bundle $E_G \rightarrow M$ in Section 3.1 to be $\mathcal{E}_G \rightarrow \mathcal{B}_G$. Construct $\mathcal{C}(\mathcal{E}_G)$ as in (3.2). Let

$$\delta : \mathcal{C}(\mathcal{E}_G) \rightarrow \mathcal{B}_G \quad (3.5)$$

be the natural projection (see Lemma 3.1). Let $D_0$ be the tautological connection on $\delta^* \mathcal{E}_G$ constructed in Proposition 3.3.

The following theorem is proved in [BHS, p. 375, Lemma 4.2].

**Theorem 3.6.** The connection $D_0$ on the principal $G$–bundle $\delta^* \mathcal{E}_G$ is universal.

In Theorem 3.6, we took a special type of universal $G$–bundle, namely we took the Cartesian product of a universal $G$–bundle with $\mathbb{R}^\infty$. It should be mentioned that Theorem 3.6 is not valid if we do not take this Cartesian product. For example, take $G$ to be the additive group $\mathbb{R}^n$. Since $\mathbb{R}^n$ is contractible, the projection $\mathbb{R}^n \rightarrow \{\text{point}\}$ is a universal $\mathbb{R}^n$–bundle. Note that $\mathcal{C}(\mathbb{R}^n)$ is a point. But the trivial principal $\mathbb{R}^n$ bundle on any manifold $X$ of dimension at least two admits connections with nonzero curvature.

### 4. Holomorphic connections

Assume that $M$ is a complex manifold and $G$ is a complex Lie group. A *holomorphic* principal $G$–bundle on $M$ is a triple $(E_G, p, \psi)$ as in (2.1) such that $E_G$ is a complex manifold, and both the maps $p$ and $\psi$ are holomorphic.

Let $(E_G, p, \psi)$ be a holomorphic principal $G$–bundle on $M$. Consider the holomorphic tangent bundle $T^{1,0}E_G$, which is a holomorphic vector bundle on $E_G$. The real tangent bundle $TE_G$ gets identified with $T^{1,0}E_G$ in the obvious way. More precisely, the isomorphism $T^{1,0}E_G \rightarrow TE_G$ sends a tangent vector to its real part. Using this identification between $T^{1,0}E_G$ and $TE_G$, the complex structure on the total space of $T^{1,0}E_G$ produces a complex structure on the total space of $TE_G$. This complex structure on $TE_G$ produces a complex structure on the quotient $\text{At}(E_G)$ in (2.4), because the action of $G$ on $TE_G$ is holomorphic.

The differential $dp$ in (2.7) is holomorphic, which makes the projection $\eta$ in (2.9) holomorphic. The exact sequence in (2.10) becomes an exact sequence of holomorphic vector bundles. The holomorphic structure on $E_G$ produces a holomorphic structure on any fiber bundle associated to $E_G$ for a holomorphic action of $G$. In particular, the adjoint vector bundle $\text{ad}(E_G)$ has a holomorphic structure, because the adjoint action of $G$ on $\mathfrak{g}$ is holomorphic. The homomorphism $\iota_0$ in (2.10) is holomorphic with respect to this holomorphic structure on $\text{ad}(E_G)$.

A connection

$$D : TM \rightarrow \text{At}(E_G)$$

on $E_G$ as in (2.12) is called *holomorphic* if the homomorphism $D$ is holomorphic.
4.1. Holomorphic connection on principal bundles over a compact Riemann surface. Now take $M$ to be a compact connected Riemann surface. It is natural to ask the question when a holomorphic vector bundle on $M$ admits a holomorphic connection. Note that any holomorphic connection on a Riemann surface is automatically flat because there are no nonzero $(2, 0)$ forms on a Riemann surface. A well-known theorem of Atiyah and Weil says that a holomorphic vector bundle $E$ over $M$ admits a holomorphic connection if and only if each direct summand of $E$ is of degree zero (see [At], [We]). We will describe a generalization of it to principal bundles.

Let $G$ be a complex connected reductive affine algebraic group. A parabolic subgroup of $G$ is a Zariski closed connected subgroup $P \subset G$ such that the quotient $G/P$ is compact. A Levi subgroup of $G$ is a Zariski closed connected subgroup $L \subset G$ such that there is a parabolic subgroup $P \subset G$ containing $L$ that satisfies the following condition: $L$ contains a maximal torus of $P$, and moreover $L$ is a maximal reductive subgroup of $P$. Given a holomorphic principal $G$–bundle $E_G$ on $M$ and a complex Lie subgroup $H \subset G$, a holomorphic reduction of $E_G$ to $H$ is given by a holomorphic section of the holomorphic fiber bundle $E_G/H$ over $M$. Let $q_H : E_G \twoheadrightarrow E_G/H$ be the quotient map. If $\nu : M \rightarrow E_G/H$ is a holomorphic section of the fiber bundle $E_G/H$, then note that $q_H^{-1}(\nu(M)) \subset E_G$ is a holomorphic principal $H$–bundle on $M$. If $E_H$ is a holomorphic principal $H$–bundle over $M$, and $\chi$ is a holomorphic character of $H$, then the associated holomorphic line bundle $E_H(\lambda) = (E_H \times \mathbb{C})/H$ is the quotient of $E_H \times \mathbb{C}$, where $(z_1, c_1), (z_2, c_2) \in E_H \times \mathbb{C}$ are identified if there is an element $g \in H$ such that

- $z_2 = z_1 g$, and
- $c_2 = c_1 \chi(g)$.

The following theorem is proved in [AB] (see [AB, Theorem 4.1]).

**Theorem 4.1.** A holomorphic $G$–bundle $E_G$ over $M$ admits a holomorphic connection if and only if for every triple of the form $(H, E_H, \lambda)$, where

1. $H$ is a Levi subgroup of $G$,
2. $E_H \subset E_G$ is a holomorphic reduction of structure group to $H$, and
3. $\lambda$ is a holomorphic character of $H$,

the associated line bundle $E_H(\lambda) = (E_H \times \mathbb{C})/H$ over $M$ is of degree zero.

Note that setting $G = \text{GL}(n, \mathbb{C})$ in Theorem 4.1 the above mentioned criterion of Atiyah and Weil is recovered.

We will describe a sketch of the proof of Theorem 4.1.
Let $E_G$ be a holomorphic $G$–bundle over $M$ equipped with a holomorphic connection $\nabla$. Take any triple $(H, E_H, \lambda)$ as in Theorem 4.1. We will first show that the connection $\nabla$ produces a holomorphic connection on the principal $H$–bundle $E_H$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of $G$ and $H$ respectively. The group $H$ has adjoint actions on both $\mathfrak{h}$ and $\mathfrak{g}$. To construct the connection on $E_H$, fix a splitting of the injective homomorphism of $H$–modules

$$0 \to \mathfrak{h} \to \mathfrak{g}.$$ 

Since a holomorphic connection on $E_G$ is a given by a holomorphic splitting of the Atiyah exact sequence for $E_G$, a holomorphic connection $\nabla$ on $E_G$ produces a $\mathfrak{g}$–valued holomorphic 1–form $\omega$ on $E_G$ satisfying the following two conditions:

- $\omega$ is $G$–equivariant ($G$ acts on $\mathfrak{g}$ by inner automorphism), and
- the restriction of $\omega$ to any fiber of $E_G$ is the Maurer–Cartan form on the fiber.

Using the chosen splitting homomorphism

$$\mathfrak{g} \to \mathfrak{h} \to 0,$$ 

the connection form $\omega$ on $E_G$ defines a $\mathfrak{h}$–valued holomorphic one–form $\omega'$ on $E_G$. The restriction of $\omega'$ to the complex submanifold $E_H \subset E_G$ satisfies the two conditions needed for a holomorphic $\mathfrak{h}$–valued 1–form on $E_H$ to define a holomorphic connection on $E_H$.

Therefore, $E_H$ admits a holomorphic connection. A holomorphic connection on $E_H$ induces a holomorphic connection on the associated line bundle $E_H(\lambda)$. Any line bundle admitting a holomorphic connection must be of degree zero [At]. Therefore, if $E_G$ admits a holomorphic connection then we know that the degree of $E_H(\lambda)$ is zero.

To prove the converse, let $E_G$ be a holomorphic $G$–bundle over $M$ such that

$$\text{degree}(E_H(\lambda)) = 0$$ 

for all triples $(H, E_H, \lambda)$ of the above type. We need to show that the Atiyah exact sequence for $E_G$ in (2.10) splits holomorphically.

As the first step, in [AB] the following is proved: it is enough to prove that the Atiyah exact sequence for $E_G$ splits holomorphically under the assumption that $E_G$ does not admit any holomorphic reduction of structure group to any proper Levi subgroup of $G$. Therefore, we assume that $E_G$ does not admit any holomorphic reduction of structure group to any proper Levi subgroup of $G$.

Let $\Omega^1_M$ denote the holomorphic cotangent bundle of $M$. The obstruction for splitting of the Atiyah exact sequence for $E_G$ is an element

$$\tau(E_G) \in H^1(M, \Omega^1_M \otimes \text{ad}(E_G)).$$ 

By Serre duality,

$$H^1(M, \Omega^1_M \otimes \text{ad}(E_G)) = H^0(M, \text{ad}(E_G))^*.$$
So we have

\[ \tau(E_G) \in H^1(M, \text{ad}(E_G))^*. \]  

(4.1)

Any homomorphic section \( f \) of \( \text{ad}(E_G) \) has a Jordan decomposition

\[ f = f_s + f_n, \]

where \( f_s \) is pointwise semisimple and \( f_n \) is pointwise nilpotent. From the assumption that \( E_G \) does not admit any holomorphic reduction of structure group to any proper Levi subgroup of \( G \) it follows that the semisimple section \( f_s \) is given by some element of the center of \( \mathfrak{g} \). Using this, from the assumption on \( E_G \) it can be deduced that

\[ \tau(E_G)(f_s) = 0, \]

where \( \tau(E_G) \) is the element in (4.1).

The nilpotent section \( f_n \) of \( \text{ad}(E_G) \) gives a holomorphic reduction of structure group \( E_P \subset E_G \) of \( E_G \) to a proper parabolic subgroup \( P \) of \( G \). This reduction \( E_P \) has the property that \( f_n \) lies in the image

\[ H^0(M, \text{ad}(E_P)) \hookrightarrow H^0(M, \text{ad}(E_G)), \]

where \( \text{ad}(E_P) \) is the adjoint bundle of \( E_P \). Using this reduction it can be shown that

\[ \tau(E_G)(f_n) = 0. \]

Hence \( \tau(E_G)(f) = 0 \) for all \( f \), which implies that \( \tau(E_G) = 0 \). Therefore, the Atiyah exact sequence for \( E_G \) splits holomorphically, implying that \( E_G \) admits a holomorphic connection.

5. Real Higgs bundles

As before, let \( M \) be a compact connected Riemann surface. Let

\[ \sigma : M \rightarrow M \]

be an anti-holomorphic automorphism of order two. Take a holomorphic vector bundle \( E \) on \( M \) of rank \( r \). Let \( \overline{E} \) denote the \( C^\infty \mathbb{C} \)-vector bundle on \( M \) of rank \( r \) whose underlying \( C^\infty \mathbb{R} \)-vector bundle is the \( \mathbb{R} \)-vector bundle underlying \( E \), while the multiplication by \( \sqrt{-1} \) on the fibers of \( \overline{E} \) coincides with the multiplication by \( -\sqrt{-1} \) on the fibers of \( E \). We note that the pullback \( \sigma^*\overline{E} \) has a natural structure of a holomorphic vector bundle. Indeed, a \( C^\infty \) section \( s \) of \( \sigma^*\overline{E} \) defined over an open subset \( U \subset M \) is holomorphic if the section \( \sigma^*s \) of \( E \) over \( \sigma(U) \) is holomorphic; this condition uniquely defines the holomorphic structure on \( \sigma^*\overline{E} \). We use the terminology “\( \mathbb{R} \)-vector bundles” because the terminology “real vector bundles” will be used for something else.

If \( \alpha : A \rightarrow B \) is a \( C^\infty \) homomorphism of holomorphic vector bundles on \( M \), then \( \overline{\alpha} \) will denote the homomorphism \( \overline{A} \rightarrow \overline{B} \) defined by \( \alpha \) using the identifications of \( A \) and \( B \) with \( \overline{A} \) and \( \overline{B} \) respectively. A real structure on \( E \) is a holomorphic isomorphism of vector bundles

\[ \phi : E \rightarrow \sigma^*\overline{E} \]
over the identity map of $M$ such that the composition
$$E \xrightarrow{\phi} \sigma^*E \xrightarrow{\sigma^*\overline{\sigma}} \sigma^*\sigma^*E = E$$
(5.1)
is the identity map of $E$.

A quaternionic structure on $E$ is a holomorphic isomorphism of vector bundles
$$\phi : E \rightarrow \sigma^*E$$
over the identity map of $M$ such that the composition $E \rightarrow E$ in (5.1) is $-\text{Id}_E$.

A real vector bundle on $(M, \sigma)$ is a pair of the form $(E, \phi)$, where $E$ is a holomorphic vector bundle on $M$ and $\phi$ is a real structure on $E$.

A quaternionic vector bundle on $(M, \sigma)$ is a pair of the form $(E, \phi)$, where $E$ is a holomorphic vector bundle on $M$ and $\phi$ is a quaternionic structure on $E$.

Consider the differential $d\sigma : T^{\mathbb{R}}M \rightarrow \sigma^*T^{\mathbb{R}}M$ of the automorphism $\sigma$. Since $\sigma$ is anti-holomorphic, it produces an isomorphism
$$\sigma'' : T^{1,0}M \rightarrow \sigma^*T^{0,1}M = \sigma^*T^{1,0}M.$$ It is easy to check that $\sigma''$ is holomorphic and it is a real structure on the holomorphic tangent bundle $T^{1,0}M$. Let
$$\sigma' : K_M := (T^{1,0}M)^* \rightarrow \sigma^*\overline{K_M}$$
be the real structure on the holomorphic cotangent bundle $K_M$ obtained from $\sigma''$.

We recall that a Higgs field on $E$ is a holomorphic section of $\text{Hom}(E, E \otimes K_M) = \text{End}(E) \otimes K_M$ [Hi, Si]. A Higgs field $\theta$ on a real or quaternionic vector bundle $(E, \phi)$ is called real if the following diagram is commutative:

$$
\begin{array}{ccc}
E & \xrightarrow{\theta} & E \otimes K_M \\
\downarrow{\phi} & & \downarrow{\phi \otimes \sigma'} \\
\sigma^*E & \xrightarrow{\sigma^*\overline{\sigma}} & \sigma^*E \otimes K_M = \sigma^*E \otimes \sigma^*\overline{K_M}
\end{array}
$$

where $\sigma'$ is the isomorphism in (5.2). A real (respectively, quaternionic) Higgs bundle on $(M, \sigma)$ is a triple of the form $((E, \phi), \theta)$, where $(E, \phi)$ is a real (respectively, quaternionic) vector bundle on $(M, \sigma)$ and $\theta$ is a real Higgs field on $(E, \phi)$.

We recall that the slope of a holomorphic vector bundle $W$ on $M$ is the rational number
$$\mu(W) := \frac{\text{deg}(W)}{\text{rank}(W)} := \mu(W).$$ A real or quaternionic Higgs bundle $((E, \phi), \theta)$ on $(M, \sigma)$ is called semistable (respectively, stable) if for all nonzero holomorphic subbundle $F \subset E$ with

1. $\phi(F) \subset \sigma^*F \subset \sigma^*E$, and
2. $\theta(F) \subset F \otimes K_M$, we have $\mu(F) \leq \mu(E)$ (respectively, $\mu(F) < \mu(E)$). A semistable real (respectively, quaternionic) Higgs bundle is called polystable if it is a direct sum of stable real (respectively, quaternionic) Higgs bundles.
It is known that a real Higgs bundle \((E, \phi), \theta\) is semistable (respectively, polystable) if and only if the Higgs bundle \((E, \theta)\) is semistable (respectively, polystable) \([BGH, \text{p. 2555, Lemma 5.3}]\). Similarly, a quaternionic Higgs bundle \((E, \phi), \theta\) is semistable (respectively, polystable) if and only if the Higgs bundle \((E, \theta)\) is semistable (respectively, polystable).

A polystable Higgs vector bundle \((E, \theta)\) of degree zero on \(M\) admits a harmonic metric \(h\) that satisfies the Yang–Mills–Higgs equation \([Si], [Do], [Hi]\). If \((E, \phi), \theta\) is real or quaternionic polystable of degree zero, then \(E\) admits a harmonic metric \(h\) because \((E, \theta)\) is polystable of degree zero. The harmonic metric \(h\) on \(E\) can be so chosen that the isomorphism \(\phi\) is an isometry (note that \(h\) induces a Hermitian structure on \(\overline{E}\)) \([BGH, \text{p. 2557, Proposition 5.5}]\).

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\end{align*}\]

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

\textit{E-mail address: indranil@math.tifr.res.in}