Hyperbolic inflationary model with nonzero curvature

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We consider a cosmological model consisting of two scalar fields defined in the hyperbolic plane known as hyperbolic inflation. For the background space, we consider a homogeneous and isotropic spacetime with nonzero curvature. We study the asymptotic behaviour of solutions and we search for attractors in the expanding regime. We prove that two hyperbolic inflationary stages are stable solutions that can solve the flatness problem and describe acceleration for both open and closed models, and additionally we obtain a Milne-like attractor solution for the open model. We also investigate the contracting branch obtaining mirror solutions with the opposite dynamical behaviours.

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1. INTRODUCTION

A simple mechanism that has been proposed to solve the homogeneity, isotropy, and flatness problems are the so-called cosmic inflation \cite{1,2}. In the inflation, the universe has gone through a rapid expansion which has been driven by an exotic matter source with a negative pressure component known as inflation. The importance of rapid expansion is that

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the size of the universe increases so fast that it loses its memory of the initial conditions. The source and the nature of the inflationary mechanism are still unknown. There are various approaches which are based on scalar fields [3–8], on Chaplygin Gas [9–11] or on the modification of the Einstein-Hilbert Action Integral [12–15].

An inflationary model which has drawn the attention of cosmologists in recent years is the so-called hyperbolic inflation [18]. It is a two-scalar field inflationary model in which the kinetic part of the scalar fields lies on a hyperbolic plane. The model is inspired by the $\sigma$-theory and this kind of model has been widely studied as a dark energy alternative in the past [19–23]. In hyperbolic inflation, for the exponential potential, the inflationary era is attributed to a scaling attractor in which the inflation consists of the two-scalar fields and it does not slow-roll [24]. Furthermore, because the scalar fields do not need to have the same values at the beginning and the end of inflation, that means that there are no observable non-Gaussianities in the power spectrum [25, 26]. The analytic solution for the cosmological field equations of the hyperbolic inflation in a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) background space was determined in [27]. Other extensions were proposed recently in [28, 29]. Recently, the scalar hyperbolic inflationary model was investigated in the case of anisotropic spacetimes, in which analytic solutions were determined [29], while the dynamics were investigated in [30].

In this study, we investigate the dynamics of the hyperbolic inflation model in the case of FLRW spacetime with nonzero curvature. Specifically, we are interested to investigate if the given model can drive the dynamics so that a future attractor be the inflationary solution or another spatially flat FLRW universe. With this analysis, we shall understand further if this specific multi-field model can solve the flatness problem. The dynamical analysis is an essential approach for the study of physical viability for given gravitational theories. See for instance [31–33] and references therein.

The plan of the paper is as follows. In Section 2 we present the model of our consideration and we define the field equations. Section 3 includes the main analysis of this study in which we investigate the asymptotic dynamics for the field equations. Finally, in Section 5 we discuss our results.
2. HYPERBOLIC INFLATION

We consider the two-scalar field model \[18\]
\[
S = \int \sqrt{-g} dx^4 \left( R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g^{\mu\nu} e^{2\kappa \phi} \nabla_\mu \psi \nabla_\nu \psi - V(\phi) \right),
\]
where the two scalar fields $\phi(x^\mu)$ and $\psi(x^\nu)$ have kinetic terms which lie on a two-dimensional hyperbolic manifold.

For the background space we assume the FLRW universe
\[
ds^2 = -dt^2 + a^2(t) \left( \frac{dv^2}{1 - K r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right),
\]
where $K$ is the spatial curvature for the three-dimensional hypersurface. For $K = 0$, we have a spatially flat universe, for $K = 1$ we have a closed universe and for $K = -1$ the line element \((2)\) describes an open universe. In previous studies the cosmological model with Action Integral \((1)\) has been investigated for $K = 0$ \[24\]. In the following we consider the case $K \neq 0$. Moreover, we assume that the scalar fields inherit the symmetries of the spacetime \((2)\) which means $\phi(x^\mu) = \phi(t)$ and $\psi(x^\mu) = \psi(t)$.

For the line element \((2)\) and the Action Integral \((1)\) we end with the gravitational field equations
\[
-3H^2 + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} e^{2\kappa \phi} \dot{\psi}^2 + V(\phi) - 3Ka^{-2} = 0 ,
\]
\[
2\dot{H} + 3H^2 + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} e^{2\kappa \phi} \dot{\psi}^2 - V(\phi) + Ka^{-2} = 0 ,
\]
\[
\ddot{\phi} + 3H \dot{\phi} - \kappa e^{2\kappa \phi} \dot{\psi}^2 + V_{,\phi}(\phi) = 0 ,
\]
\[
\ddot{\psi} + 3H \dot{\psi} + 2\kappa \dot{\phi} \dot{\psi} = 0 ,
\]
where $H = \frac{\dot{a}}{a}$ is the Hubble function.

We can define the effective energy density, $\rho_{\text{eff}}$, and the effective pressure, $p_{\text{eff}}$, as
\[
\rho_{\text{eff}} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} e^{2\kappa \phi} \dot{\psi}^2 + V(\phi) ,
\]
\[
p_{\text{eff}} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} e^{2\kappa \phi} \dot{\psi}^2 - V(\phi) .
\]
Thus, the parameter for the equation of state for the effective cosmological fluid is
\[
w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} e^{2\kappa \phi} \dot{\psi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} e^{2\kappa \phi} \dot{\psi}^2 + V(\phi)} ,
\]
while the deceleration parameter in the presence of spatial curvature is $q = -1 - \dot{H}/H^2$ and $\Omega_K = K(aH)^{-2}$. 
3. DYNAMICAL ANALYSIS

In order to investigate the dynamics and the asymptotic behaviour of the field equations we define new dimensionless variables. Indeed, we assume the new independent variable,
\[ d\tau = \sqrt{H^2 + |K|a^{-2}}dt, \]
and the new dependent variables
\[ \lambda = \frac{V'(\phi)}{V(\phi)}, x = \frac{\dot{\phi}}{\sqrt{6\sqrt{H^2 + |K|a^{-2}}}}, y^2 = \frac{V(\phi)}{3(H^2 + |K|a^{-2})}, \]
\[ z = e^{\kappa \phi} \frac{\dot{\psi}}{\sqrt{6\sqrt{H^2 + |K|a^{-2}}}}, \eta = \frac{H}{\sqrt{H^2 + |K|a^{-2}}}. \]

The functional forms of the field equations depend upon the sign of curvature \( K \). That is,
\[ \frac{d\lambda}{d\tau} = \sqrt{6} x \lambda^2 (\Gamma(\lambda) - 1), \quad \Gamma(\lambda) = \frac{1}{\lambda} V_{\phi\phi}, \]
\[ \frac{dx}{d\tau} = \frac{1}{2} \left( \eta x \left( \text{sgn}(K) + 3x^2 - 3y^2 + 3z^2 - 4 \right) - \eta^3 x (\text{sgn}(K) - 1) + \sqrt{6} (2\kappa z^2 - \lambda y^2) \right), \]
\[ \frac{dy}{d\tau} = \frac{y}{2|K|} \left( \left| K \right| \left( \eta^3 + \eta \left( 3x^2 - 3y^2 + 3z^2 + 2 \right) + \sqrt{6}\lambda x \right) + \eta \left( 1 - \eta^2 \right) K \right), \]
\[ \frac{dz}{d\tau} = \frac{1}{2} \left( - \left( \eta^3 (\text{sgn}(K) - 1) \right) - 2\sqrt{6}\kappa x + \eta \left( \text{sgn}(K) + 3x^2 - 3y^2 + 3z^2 - 4 \right) \right), \]
\[ \frac{d\eta}{d\tau} = \frac{(\eta^2 - 1) \left( |K| \left( \eta^2 + 3 \left( x^2 - y^2 + z^2 \right) \right) + \left( 1 - \eta^2 \right) K \right)}{2|K|}, \]

with the Friedman constraint
\[ \eta^2 - (\eta^2 - 1) \text{sgn}(K) - x^2 - y^2 - z^2 = 0. \]

For the scalar field potential we assume the exponential function, \( V(\phi) = V_0 e^{\lambda \phi} \). In this case, \( \lambda \) is a constant parameter. Hence, we omit the additional equation (12) that is trivially satisfied.

The deceleration parameter can be expressed in terms of the normalized variables as
\[ q = \frac{\eta^2 - \eta^2 \text{sgn}(K) + \text{sgn}(K) + 3x^2 - 3y^2 + 3z^2}{2\eta^2} = \left( 1 - \frac{1}{\eta^2} \right) \text{sgn}(K) + \frac{3(x^2 + z^2)}{\eta^2} - 1, \]

and the effective equation of state parameter as
\[ w_{\text{eff}} = 1 - \frac{2y^2}{x^2 + y^2 + z^2} = \frac{2(x^2 + z^2)}{\eta^2 + (1 - \eta^2)\text{sgn}(K)} - 1. \]

Now, we investigate separately the two cases, \( K = 1 \) and \( K = -1 \).


| Label | $x$ | $z$ | $\eta$ | $e_1(P)$ | $e_2(P)$ | $e_3(P)$ |
|-------|-----|-----|--------|----------|----------|----------|
| $A_1^+$ | 1   | 0   | 1      | 4        | $-\sqrt{6}\kappa$ | $6 + \sqrt{6}\lambda$ |
| $A_1^-$ | $-1$ | 0   | 1      | 4        | $\sqrt{6}\kappa$ | $6 - \sqrt{6}\lambda$ |
| $\bar{A}_1^+$ | 1   | 0   | $-1$   | $-4$     | $-\sqrt{6}\kappa$ | $-(6 - \sqrt{6}\lambda)$ |
| $\bar{A}_1^-$ | $-1$ | 0   | $-1$   | $-4$     | $\sqrt{6}\kappa$ | $-(6 + \sqrt{6}\lambda)$ |
| $A_2$ | $\frac{1}{\sqrt{6}}$ | 0   | 1      | $\frac{1}{2}(\lambda^2 - 6)$ | $\lambda^2 - 2$ | $-3 + \frac{1}{2}\lambda(2\kappa + \lambda)$ |
| $\bar{A}_2$ | $\frac{1}{\sqrt{6}}$ | 0   | $-1$   | $-\frac{1}{2}(\lambda^2 - 6)$ | $-(\lambda^2 - 2)$ | $3 - \frac{1}{2}\lambda(2\kappa + \lambda)$ |
| $A_3$ | $-\frac{\sqrt{6}}{2\kappa + \lambda}$ | $\frac{\sqrt{2\kappa + \lambda - \sqrt{6}}}{\sqrt{2\kappa + \lambda}^2}$ | 1      | $4 - \frac{12\kappa}{2\kappa + \lambda}$ | $3\kappa + i\sqrt{8\kappa^3 \lambda + \kappa(8\lambda^2 - 27) + 2\lambda(\lambda^2 - 6)}$ | $6 + \sqrt{6}\lambda$ |
| $A_3$ | $-\frac{\sqrt{6}}{2\kappa + \lambda}$ | $\frac{\sqrt{2\kappa + \lambda - \sqrt{6}}}{\sqrt{2\kappa + \lambda}^2}$ | 1      | $4 - \frac{12\kappa}{2\kappa + \lambda}$ | $3\kappa + i\sqrt{8\kappa^3 \lambda + \kappa(8\lambda^2 - 27) + 2\lambda(\lambda^2 - 6)}$ | $6 - \sqrt{6}\lambda$ |
| $\bar{A}_3$ | $\frac{\sqrt{6}}{2\kappa + \lambda}$ | $\frac{\sqrt{2\kappa + \lambda + \sqrt{6}}}{\sqrt{2\kappa + \lambda}^2}$ | $-1$   | $-4 + \frac{12\kappa}{2\kappa + \lambda}$ | $3\kappa + i\sqrt{8\kappa^3 \lambda + \kappa(8\lambda^2 - 27) + 2\lambda(\lambda^2 - 6)}$ | $-(6 - \sqrt{6}\lambda)$ |
| $\bar{A}_3$ | $\frac{\sqrt{6}}{2\kappa + \lambda}$ | $\frac{\sqrt{2\kappa + \lambda + \sqrt{6}}}{\sqrt{2\kappa + \lambda}^2}$ | $-1$   | $-4 + \frac{12\kappa}{2\kappa + \lambda}$ | $3\kappa + i\sqrt{8\kappa^3 \lambda + \kappa(8\lambda^2 - 27) + 2\lambda(\lambda^2 - 6)}$ | $6 + \sqrt{6}\lambda$ |
| $A_4$ | $-\frac{1}{\sqrt{3}}$ | 0   | $\frac{1}{\sqrt{6}}$ | $\sqrt{2}(\kappa - \lambda)$ | $\frac{\lambda + \sqrt{8 - 3\lambda^2}}{\sqrt{2}}$ | $\frac{\lambda - \sqrt{8 - 3\lambda^2}}{\sqrt{2}}$ |
| $A_4$ | $-\frac{1}{\sqrt{3}}$ | 0   | $-\frac{1}{\sqrt{6}}$ | $-\sqrt{2}(\kappa - \lambda)$ | $\frac{\lambda - \sqrt{8 - 3\lambda^2}}{\sqrt{2}}$ | $\frac{\lambda + \sqrt{8 - 3\lambda^2}}{\sqrt{2}}$ |

**TABLE I:** The stationary points $A = (x(A), z(A), \eta(A))$ and $\bar{A} = (x(\bar{A}), z(\bar{A}), \eta(\bar{A}))$ of the dynamical system (21), (22), (23).

### 3.1. Closed Universe

For $K = 1$ the Friedmann constraint (17) reduces to

$$1 - x^2 - y^2 - z^2 = 0. \quad (20)$$

We determine the stationary points of the system (12), (13), (14), (15) and (16) for $K = 1$ and we study their stability properties. Each stationary point corresponds to a specific epoch of the cosmological evolution. It is important to mention that from (20) parameters $x, y, z$ are constraints in a unitary sphere. Moreover, we can use the constraint equation (20) to reduce the dimension of the dynamical system. Indeed, we assume $y = \sqrt{1 - x^2 - z^2}$

We remark that $y \to -y$ is a discrete symmetry for the field equations. In addition, by definition parameter $\eta$ is constrained as $|\eta| \leq 1$. The case $|\eta| = 1$ corresponds to spatially flat universes. A positive value of $\eta$ denotes an expanding universe, i.e. $H > 0$. Thus, in the following we focus on the expansion era that is we assume $\eta > 0$. Hence, we obtain the
reduced system:

\[
\begin{align*}
\frac{dx}{d\tau} &= 3\eta x (x^2 + z^2 - 1) + \sqrt{3} \left( \lambda (x^2 - 1) + z^2 (2\kappa + \lambda) \right), \\
\frac{dz}{d\tau} &= z \left( 3\eta (x^2 + z^2 - 1) - \sqrt{6}\kappa x \right), \\
\frac{d\eta}{d\tau} &= (\eta^2 - 1) (3x^2 + 3z^2 - 1).
\end{align*}
\]

The stationary points \( A = (x(A), z(A), \eta(A)) \) and \( \bar{A} = (x(\bar{A}), z(\bar{A}), \eta(\bar{A})) \) of the dynamical system (21), (22), (23) are represented in table I. Since the points \( \bar{A} \) have the time-reversal dynamical behaviour of the related points \( A \) under the change \((\tau, x, \eta) \mapsto (-\tau, -x, -\eta)\). \( A_1^+ \) maps onto \( \bar{A}_1^- \), \( A_1^- \) maps onto \( \bar{A}_1^+ \), \( A_2 \) maps onto \( \bar{A}_2 \), \( A_3^+ \) maps onto \( \bar{A}_3^- \), \( A_3^- \) maps onto \( \bar{A}_3^+ \) and \( A_4^- \) maps onto \( A_4^+ \) (where we have suppressed the bar). Then, we focus on points \( A \)'s:

\[
\begin{align*}
A_1^+ &= (\pm 1, 0, 1), \\
A_2 &= \left( -\frac{\lambda}{\sqrt{6}}, 0, 1 \right), \\
A_3^+ &= \left( -\frac{\sqrt{6}}{2\kappa + \lambda}, \pm \sqrt{\frac{\lambda^2 + 2\kappa\lambda - 6}{(2\kappa + \lambda)^2}}, 1 \right), \\
A_4^\pm &= \left( \mp \frac{1}{\sqrt{3}}, 0, \pm \frac{\lambda}{\sqrt{2}} \right). 
\end{align*}
\]

Stationary points, \( A_1^+ \), \( A_2 \) and \( A_3^\pm \) describe spatially flat universes. For the asymptotic solution at point \( A_1^+ \) we derive \( w_{\text{eff}}(A_1^+) = 1 \) and \( q(A_1^+) = 2 \), which means that the universe is dominated by the kinetic part of the scalar field \( \phi(t) \). Point \( A_2 \) provides \( w_{\text{eff}}(A_2) = \frac{\lambda^2 - 3}{3} \), \( q(A_2) = \frac{\lambda^2 - 2}{2} \), which is nothing other than the quintessence scaling solution of \[31\]. Acceleration occurs for \(|\lambda| < \sqrt{2}\).

Points \( A_3^\pm \) describe the hyperbolic inflationary solutions, \( w_{\text{eff}}(A_3^\pm) = 1 - \frac{4\kappa}{2\kappa + \lambda} \), \( q(A_3^\pm) = 2 - \frac{6\kappa}{2\kappa + \lambda} \). Points are real and physically acceptable when \( 2\kappa + \lambda \neq 0 \) and \( \lambda^2 + 2\kappa\lambda - 6 > 0 \). Acceleration occurs when \( \{ \lambda \leq -\sqrt{2}, \kappa < \lambda \} \), \( \{ -\sqrt{2} < \lambda < 0, \kappa < \frac{6 - \lambda^2}{2\lambda} \} \), or \( \{ 0 < \lambda < \sqrt{2}, \kappa > \frac{6 - \lambda^2}{2\lambda} \} \), or \( \{ \lambda \geq \sqrt{2}, \kappa > \lambda \} \).

Furthermore, points \( A_4^+ \) exist for \( \lambda^2 < 2 \) and describe Milne-like solutions with \( a(t) = a_0 t \) and \( w_{\text{eff}}(A_4^+) = -\frac{1}{3} \), \( q(A_4^+) = 0 \). Point \( A_4^- \) describes an expanding universe for \( \lambda > 0 \), while \( A_4^- \) corresponds to an expanding universe for \( \lambda < 0 \).

To infer the stability properties of the stationary points we derive the eigenvalues of the linearised system around the stationary points. Indeed, for the points \( A_1^+ \) we find the eigenvalues \( e_1(A_1^+) = 4 \), \( e_2(A_1^+) = \mp \sqrt{6}\kappa \), \( e_3(A_1^+) = 6 \pm \sqrt{6}\lambda \). Thus, points \( A_1^\pm \) are nonhyperbolic for \( \kappa = 0 \) or \( \lambda = \mp \sqrt{6} \). Point \( A_1^+ \) is a source for \( \lambda > -\sqrt{6} \) and \( \kappa < 0 \). Point \( A_1^- \) is a source for \( \lambda < \sqrt{6} \) and \( \kappa > 0 \). Otherwise, points \( A_1^\pm \) are saddle points.
$A_2$ exists for $-\sqrt{6} \leq \lambda \leq \sqrt{6}$. We derive the eigenvalues $e_1 (A_2) = -3 + \frac{\lambda^2}{2}$, $e_2 (A_2) = \lambda^2 - 2$ and $e_3 (A_2) = -3 + \frac{1}{2} \lambda (2 \kappa + \lambda)$. Thus, point $A_2$ is nonhyperbolic for $\lambda^2 = 2$ or $\lambda^2 = 6$ or $\lambda (2 \kappa + \lambda) = 6$. Point $A_2$ is a sink when $\lambda^2 < 2$ and $\lambda (2 \kappa + \lambda) < 6$, otherwise point $A_2$ is a saddle point.

Points $A^\pm_3$ exist when $2 \kappa + \lambda \neq 0$ and $\lambda^2 + 2 \kappa \lambda - 6 > 0$. The eigenvalues are $e_1 (A^\pm_3) = 4 (1 - 3 \frac{\kappa}{2 \kappa + \lambda})$, $e_{2,3} (A^\pm_3) = -\frac{3 \kappa \pm i \sqrt{3 \kappa (8 \kappa^2 + 2 \lambda (\lambda^2 - 6) + \kappa (8 \lambda^2 - 27)}}{2 \kappa + \lambda}$.

We define $\kappa_\pm = \frac{27 - 8 \lambda^2}{16 \lambda} \pm \frac{1}{16} \sqrt{3 - \frac{16 \lambda^2 - 243}{\lambda^2}}$. When hyperbolic, $A^\pm_3$ can be sink for $\left\{ -\sqrt{\frac{13}{6}} < \lambda \leq -\sqrt{2}, \kappa_- \leq \kappa < \lambda \right\}$, or $\left\{ -\sqrt{2} < \lambda < 0, \kappa_- \leq \kappa < \frac{6 - \lambda^2}{2 \lambda} \right\}$, or $\left\{ 0 < \lambda < \sqrt{2}, \frac{6 - \lambda^2}{2 \lambda} < \kappa \leq \kappa_+ \right\}$, or $\left\{ \sqrt{2} \leq \lambda < \sqrt{\frac{13}{6}}, \lambda \leq \kappa \leq \kappa_+ \right\}$, or $\left\{ \lambda \leq -\sqrt{\frac{13}{6}}, \kappa < \lambda \right\}$, or $\left\{ -\sqrt{\frac{13}{6}} < \lambda < 0, \kappa < \kappa_- \right\}$, or $\left\{ 0 < \lambda < \sqrt{\frac{13}{6}}, \kappa > \kappa_+ \right\}$, or $\left\{ \lambda \geq \sqrt{\frac{13}{6}}, \kappa > \lambda \right\}$. They can be a saddle otherwise.

Finally, $A^\pm_4$ exists when $|\lambda| \leq \sqrt{2}$ with eigenvalues $e_1 (A^\pm_4) = \pm \sqrt{2} (\kappa - \lambda)$, $e_2 (A^\pm_4) = \mp \frac{\lambda - \sqrt{8 - 3 \lambda^2}}{\sqrt{2}}$ and $e_3 (A^\pm_4) = \mp \frac{\lambda + \sqrt{8 - 3 \lambda^2}}{\sqrt{2}}$ from which we conclude that in the expanding region the stationary points are saddle points.

We conclude that there is an attractor in the expanding branch for the field equations in the presence of positive spatial curvature dominated by quintessence $A_2$, that is a sink when $\lambda^2 < 2$ and $\lambda (2 \kappa + \lambda) < 6$. Moreover, the point $A^\pm_3$, that exist when $2 \kappa + \lambda \neq 0$ and $\lambda^2 + 2 \kappa \lambda - 6 > 0$ and correspond to inflationary power-law solutions can be attractors. We continue by considering the negative curvature.

### 3.2. Open Universe

For an open FLRW universe and $K = -1$ the constraint (17) is

$$x^2 + y^2 + z^2 = 2 \eta^2 - 1.$$  \hspace{1cm} (24)

With the use of the constraint equation (24) the reduced dynamical system lies on the three-dimensional surface. By definition $\eta^2 \leq 1$. Consequently, parameters $\{x, y, z\}$ are constrained, that is, $\frac{1}{2} \leq \eta^2 \leq 1$. To compare with the closed FLRW case, we assume $y = \sqrt{2 \eta^2 - 1 - x^2 - z^2}$. We remark that $y \rightarrow -y$ is a discrete symmetry for the field equations. A positive value of $\eta$ denotes an expanding universe, i.e. $H > 0$. Thus in the following we focus on the expansion era that is we assume $\eta > 0$. 
TABLE II: The stationary points $B = (x(B), z(B), \eta(B))$ and $\bar{B} = (x(\bar{B}), z(\bar{B}), \eta(\bar{B}))$ of the dynamical system (25), (26), (27).

| Label | $x$ | $z$ | $\eta$ | $e_1(P)$ | $e_2(P)$ | $e_3(P)$ |
|-------|----|----|-------|---------|---------|---------|
| $B_1^+$ | 1 | 0 | 1 | 4 | $-\sqrt{6}\kappa$ | $6 + \sqrt{6}\lambda$ |
| $B_1^-$ | -1 | 0 | 1 | 4 | $\sqrt{6}\kappa$ | $-6 + \sqrt{6}\lambda$ |
| $B_2^+$ | 1 | 0 | -1 | -4 | $\sqrt{6}\kappa$ | $-6 - \sqrt{6}\lambda$ |
| $B_2^-$ | -1 | 0 | -1 | -4 | $\sqrt{6}\kappa$ | $-6 + \sqrt{6}\lambda$ |
| $B_3^+$ | $\frac{\lambda^2}{\sqrt{6} + \lambda^2}$ | 0 | -1 | $\frac{1}{2} (\lambda^2 - 6)$ | $\lambda^2 - 2$ | $\kappa \lambda + \frac{\lambda^2}{2} - 3$ |
| $B_3^-$ | $\frac{\lambda^2}{\sqrt{6} + \lambda^2}$ | 0 | -1 | $\frac{1}{2} (\lambda^2 - 6)$ | $\lambda^2 - 2$ | $\kappa \lambda + \frac{\lambda^2}{2} - 3$ |

Hence, the field equations are

$$\frac{dx}{d\tau} = -\sqrt{6}\eta^2 \lambda + \eta x (3x^2 + 3z^2 - 1) + \sqrt{\frac{3}{2}} (\lambda (x^2 + 1) + z^2 (2\kappa + \lambda)) - 2\eta^3 x, \tag{25}$$

$$\frac{dz}{d\tau} = \eta \left(-2\eta^3 + \eta (3x^2 + 3z^2 - 1) - \sqrt{6}\kappa x\right), \tag{26}$$

$$\frac{d\eta}{d\tau} = (\eta^2 - 1) (-2\eta^2 + 3x^2 + 3z^2 + 1). \tag{27}$$

The stationary points $B = (x(B), z(B), \eta(B))$ and $\bar{B} = (x(\bar{B}), z(\bar{B}), \eta(\bar{B}))$ of the dynamical system (25), (26), (27) are presented in table II. Since the points $B$ have the time-reversal dynamical behaviour of the related points $\bar{B}$ under the change $(\tau, x, \eta) \mapsto (-\tau, -x, -\eta)$. $B_1^+$ maps onto $\bar{B}_1^-$, $B_1^-$ maps onto $\bar{B}_1^+$, $B_2$ maps onto $\bar{B}_2$, $B_3^+$ maps onto $\bar{B}_3^-$, $B_3^-$ maps onto $\bar{B}_3^+$, $B_4^+$ maps onto $\bar{B}_4^-$ (where we have suppressed the bar), $B_5$ maps onto $\bar{B}_5$. Then, we focus on points $B$’s:

$$B_1^\pm = (\pm 1, 0, 1), B_2 = \left(-\frac{\lambda}{\sqrt{6}}, 0, 1 \right),$$

$$B_3^\pm = \left(-\frac{\sqrt{6}}{2\kappa + \lambda}, \pm \frac{\sqrt{\lambda^2 + 2\kappa \lambda - 6}}{2\kappa + \lambda} \right),$$

$$B_5 = (0, 0, 0).$$
\[ B_4^\pm = \left( \pm \frac{1}{\sqrt{3} (\lambda^2 - 1)}, 0, \mp \sqrt{\frac{\lambda^2}{2(\lambda^2 - 1)}} \right), \quad B_5 = \left( 0, 0, \frac{1}{\sqrt{2}} \right). \]

Stationary points \( B_1^\pm, B_2 \) and \( B_3^\pm \) describe spatially flat FLRW asymptotic solutions with physical properties similar with that of points \( A_1^\pm, A_2 \) and \( A_3^\pm \), respectively.

The eigenvalues of the linearised system around the stationary points \( B_1^\pm \) and \( B_2 \) are derived to be \( e_1 (B_1^\pm) = 4, e_2 (B_1^\pm) = \mp \sqrt{6}\kappa, e_3 (B_1^\pm) = 6 \pm \sqrt{6}\lambda \) and \( e_1 (B_2) = -3 + \frac{\lambda^2}{2}, e_2 (B_2) = -2 + \lambda^2, e_3 (B_2) = -3 + \frac{1}{2}\lambda (2\kappa + \lambda) \). Thus, points \( B_1^\pm \) can be a source when \( \{ \pm \kappa < 0, 6 \pm \sqrt{6}\lambda > 0 \} \). \( B_2 \) is real only for \( \lambda^2 < 6 \) and can be an attractor for \( \lambda^2 < 2 \) and \( \lambda (2\kappa + \lambda) < 6 \). Consequently this hyperbolic inflationary solution is a future attractor.

As far as points \( B_3^\pm \) are concerned, they exist when \( 2\kappa + \lambda \neq 0 \) and \( \lambda^2 + 2\kappa\lambda - 6 > 0 \), and the eigenvalues are \( e_1 (B_3^\pm) = 4 - \frac{12\kappa}{2\kappa + \lambda}, e_2 (B_3^\pm) = -\frac{3\kappa + i \sqrt{3 [8\kappa^2 \lambda + \kappa (8\lambda^2 - 27) + 2\lambda (\lambda^2 - 6)]}}{2\kappa + \lambda} \) and \( e_3 (B_3^\pm) = -\frac{3\kappa - i \sqrt{3 [8\kappa^2 \lambda + \kappa (8\lambda^2 - 27) + 2\lambda (\lambda^2 - 6)]}}{2\kappa + \lambda} \). Hence, when hyperbolic, \( B_3^\pm \) can be sink for \( \left\{ \begin{array}{l} -\sqrt{\frac{13}{6}} \leq \lambda \leq -\frac{\sqrt{2}}{\kappa}, \kappa \leq \kappa < \lambda \end{array} \right\} \), or \( \left\{ \begin{array}{l} -\sqrt{2} < \lambda < 0, \kappa_- \leq \kappa < \kappa < \frac{6 - \lambda^2}{2\kappa} \end{array} \right\} \), or \( \left\{ \begin{array}{l} 0 < \lambda < \sqrt{\frac{2}{6}}, \kappa < \kappa_+ \end{array} \right\} \), or \( \left\{ \begin{array}{l} \sqrt{\frac{2}{6}} \leq \lambda < \sqrt{\frac{13}{6}}, \lambda < \kappa \leq \kappa_+ \end{array} \right\} \), or \( \left\{ \begin{array}{l} \sqrt{\frac{13}{6}} \leq \kappa < \lambda \end{array} \right\} \), or \( \left\{ \begin{array}{l} \kappa > \sqrt{\frac{13}{6}}, \kappa > \lambda \end{array} \right\} \), where we have defined \( \kappa_+ = \frac{27 - 8\lambda^2}{16\lambda} \pm \frac{1}{16} \sqrt{3} \sqrt{-\frac{16\lambda^2 - 243}{\lambda^2}} \). They can be a saddle otherwise. Therefore, these are also hyperbolic inflationary solutions which can be a future attractor.

Points \( B_4^\pm \) exist for \( \lambda^2 > 2 \) and describe Milne-like solutions, \( w_{eff} (B_4^\pm) = -\frac{1}{3} \) and \( q (B_4^\pm) = 0 \) with \( \eta (B_4^\pm) = \pm \sqrt{\frac{\lambda^2}{2(\lambda^2 - 1)}} \). The eigenvalues are \( e_1 (B_4^\pm) = \pm \sqrt{\frac{2}{\lambda^2 - 1}} (\lambda - \kappa), e_2 (B_4^\pm) = \pm \sqrt{\frac{(8 - 3\lambda^2)}{2(\lambda^2 - 1)}} \) and \( e_3 (B_4^\pm) = \pm \sqrt{\frac{(8 - 3\lambda^2)}{2(\lambda^2 - 1)}} \). When hyperbolic, \( B_4^+ \) (respectively \( B_4^- \)) can be a sink (respectively a source) for \( \left\{ \begin{array}{l} -2\sqrt{\frac{2}{3}} < \lambda < -\sqrt{\frac{2}{3}}, \kappa > \lambda \end{array} \right\} \), or \( \left\{ \begin{array}{l} \lambda \leq -2\sqrt{\frac{2}{3}}, \kappa > \lambda \end{array} \right\} \). \( B_4^+ \) (respectively \( B_4^- \)) can be a source (respectively a sink) for \( \left\{ \begin{array}{l} \sqrt{2} < \lambda < 2\sqrt{\frac{2}{3}}, \kappa < \lambda \end{array} \right\} \), or \( \left\{ \begin{array}{l} \lambda \geq 2\sqrt{\frac{2}{3}}, \kappa < \lambda \end{array} \right\} \). They are saddle otherwise.

Finally, \( B_5 \) is the vacuum solution with eigenvalues \( e_1 (B_5) = -\sqrt{2}, e_2 (B_5) = -\sqrt{2} \) and \( e_3 (B_5) = \sqrt{2} \) that is always a saddle.
4. HAMILTONIAN ANALYSIS AND ANALYTIC SOLUTION

We proceed our analysis with the construction of analytic solutions for the field equations. Indeed, the field equations form a Hamiltonian system with Hamiltonian function

\[ \mathcal{H} = -\frac{p_a^2}{12a^2} + \frac{1}{2a^4} \left( p_\phi^2 + e^{-2\kappa \phi} p_\psi^2 \right) + a^2 V(\phi) - K, \]  

where \( p_a = -6a^2 a' \), \( p_\phi = a^4 \phi' \) and \( p_\psi = a^4 e^{2\kappa \phi} \phi' \),

and \( a' = \frac{da}{d\xi} \) such that \( d\xi = adt \).

For the exponential potential \( V(\phi) = V_0 e^{\lambda \phi} \) and for the free parameters \((\kappa, \lambda) = \left(-\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}\right)\), the field equations admits the conservation law

\[ I_0 = (\sqrt{6}p_a + ap_\phi) a e^{\frac{\sqrt{6}}{3} \phi}. \]  

Moreover, an additional conservation law is

\[ I_1 = a^4 e^{\frac{2\sqrt{6}}{3} \phi} p_\psi. \]  

Consequently, the conservation laws \( \{\mathcal{H}, I_0, I_1\} \) are independent and in involution, which means that the field equations form a Liouville integrable dynamical system.

We follow the procedure presented earlier in [27] and we consider the new variables

\[ a = \left( \frac{8}{3} \left( xz - y^2 \right) \right)^{\frac{1}{4}}, \phi = \sqrt{\frac{3}{2}} \ln \left( \sqrt{\frac{2}{3}} \frac{xz - y^2}{x} \right), z = \frac{y}{x}, \]  

such that the Hamiltonian function to become

\[ \mathcal{H} = \frac{1}{4} p_y - p_x p_z + V_0 x - K \]  

and the conservation laws

\[ p_x = \Phi_x, \quad p_y = \Phi_y. \]  

Hence, the analytic solution of the field equations is

\[ x = x_1 \xi + x_0, \quad y = y_1 \xi + y_0, \]  

and

\[ z = \frac{\xi^2}{2} V_0 + z_1 \xi + z_0. \]
Finally, the constraint equation gives
\[ K = y_1^2 - z_1 x_1 + x_0 V_0. \] (37)

Thus for \( y_1^2 - z_1 x_1 + x_0 V_0 > 0 \) the exact solution is valid for a closed universe while when \( y_1^2 - z_1 x_1 + x_0 V_0 < 0 \) the exact solution describes an open universe, while the solution is real in the original variables when \( y_1^2 - z_1 x_1 < 0 \). At this point it is important to mention that the same solution is recovered under the change of variable \( \phi \rightarrow -\phi \) if initially we assume \((\kappa, \lambda) = \left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)\).

Hence, the scale factor for large values of \( \xi \), is determined as \( a(\xi) \sim \xi^\frac{1}{4} \), that is, the Hubble function is \( H = \frac{1}{a^2} \frac{da}{d\xi} = \frac{1}{\xi^\frac{1}{4}} \), that is, \( (H(a))^2 = \frac{1}{4} a^{-6} \), and the deceleration parameter is derived to be \( q(\xi) = 2 \). That is nothing else than the asymptotic solution described by points \( A_1^\pm \) and \( B_1^\pm \). This means that the results derived from the analysis of the asymptotic are confirmed by the analytical solution.

5. CONCLUSIONS

In this study, we investigated the dynamics of hyperbolic inflation in the presence of curvature in a homogeneous and isotropic universe. The question we wanted to answer is if hyperbolic inflation solves the flatness problem. Indeed, in an FLRW background space with nonzero curvature, we found as stationary points the exact solutions which describe the hyperbolic inflation. We focused on the expanding regime and in the case of a closed \((K = +1)\) universe, we found two types of attractors. Say, in the presence of positive spatial curvature we have the solution dominated by quintessence \( A_2 \) (respectively \( \bar{A}_2 \)) that is a sink (respectively a source) when \( \lambda^2 < 2 \) and \( \lambda(2\kappa + \lambda) < 6 \). Similar conditions were found in the quintom context in [35] (under the parameter re-scaling \( 2\kappa \mapsto \kappa \), \( A_2 \) corresponds to \( \bar{E} \), \( \bar{A}_2 \) corresponds to \( \bar{F} \)). Moreover, the point \( A_3^\pm \), that exist when \( 2\kappa + \lambda \neq 0 \) and \( \lambda^2 + 2\kappa\lambda - 6 > 0 \) and correspond to inflationary power law solutions can be attractors. \( A_3^\pm \) (respectively \( \bar{A}_3^\pm \)) are related to the point \( G \) (respectively \( H \)) studied in quintom context in [35].

On the other hand, in the case of an open \((K = -1)\) universe, we find, as before, a solution dominated by quintessence \( B_2 \) (respectively \( \bar{B}_2 \)), that is a sink (respectively a source) when \( \lambda^2 < 2 \) and \( \lambda(2\kappa + \lambda) < 6 \), and the point \( B_3^\pm \), that exist when \( 2\kappa + \lambda \neq 0 \) and \( \lambda^2 + 2\kappa\lambda - 6 > 0 \)
and correspond to inflationary power law solutions which can be attractors. Additionally we have the Milne-like (late- and early time-) attractor solutions. Say, $B_4^+$ (respectively $B_4^-$) can be a sink (respectively a source) for $\left\{-2\sqrt{\frac{2}{3}} < \lambda < -\sqrt{2}, \kappa > \lambda \right\}$, or $\left\{\lambda \leq -2\sqrt{\frac{2}{3}}, \kappa > \lambda \right\}$. $B_4^+$ (respectively $B_4^-$) can be a source (respectively a sink) for $\left\{\sqrt{2} < \lambda < 2\sqrt{\frac{2}{3}}, \kappa < \lambda \right\}$, or $\left\{\lambda \geq 2\sqrt{\frac{2}{3}}, \kappa < \lambda \right\}$.

From the above analysis, it is clear that in a universe with initial conditions of nonzero curvature, the hyperbolic inflationary solution exists. We have proved that two hyperbolic inflationary stages are stable solutions that can solve the flatness problem and describe acceleration for both open and closed models, and additionally a Milne-like solution for the open model. We also investigate the contracting branch obtaining mirror solutions with the opposite dynamical behaviours. That result was also found to be valid with the derivation of the analytic solution for the problem.

For a more general scalar field potential, it was found that this cosmological model can be seen as a unified dark energy model [36] because it provides various epochs of the cosmological evolution. We note that a similar result will follow in the presence of curvature.

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