Weakly-coupled Hubbard chains at half-filling and Confinement

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We study two (very) weakly-coupled Hubbard chains in the half-filled case, and especially the situation where the intrachain Mott scale $m$ is much larger than the (bare) single-electron interchain hopping $t_\perp$. First, we find that the divergence of the intrachain Umklapp channel at the Mott transition results in the complete vanishing of the single-electron interchain hopping. This is significant of a strong confinement of coherence along the chains. Excitations are usual charge fermionic solitons and spinon-(anti)spinon pairs of the Heisenberg chain. Then, we show rigorously how the tunneling of spinon-(anti)spinon pairs produces an antiferromagnetic interchain exchange of the order of $J_\perp = t_\perp^2/m$. In the "confined" phase and in the far Infra Red, the system behaves as a pure spin ladder. The final result is an insulating ground state with spin-gapped excitations exactly as in the opposite "delocalized" limit (i.e. for rather large interchain hoppings) where the two-leg ladder is in the well-known insulating D-Mott phase. Unlike for materials with an infinite number of coupled chains (Bechgaard salts), the confinement/deconfinement transition at absolute zero is here a simple crossover: no metallic phase is found in undoped two-leg ladders. This statement might be generalized for N-leg ladders with N=3,4... (but not too large).

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I. INTRODUCTION

One-dimensional (1D) electron systems have attracted a great attention over the last years. The Hubbard chain, that is a nice prototype to describe 1D conductors, simply reduces to the so-called "Luttinger liquid" at sufficiently low energy $\frac{\pi v_F}{U}$ [1]. Due to the restricted motion along one direction in space, perturbations easily propagate coherently. This induces that spin and charge degrees of freedom get independent, and that all the low-energy excitations are collective modes - namely, longwavelength fluctuations of the charge- or spin density. No Landau quasiparticle type elementary excitations exist [2].

Two-coupled Hubbard chains have been studied as a basic model which includes intrachain interaction $U$, longitudinal $t$ and transverse $t_\perp$ hoppings [3]. For recent and general reviews on this subject, consult Refs. 3-5. In this paper, we study the case of a weakly coupled Hubbard ladder at (and close to) half-filling with $t_\perp \ll U \ll t$. It should be noted that this interesting situation has been rarely investigated in the litterature.

The presence of intrachain Umklapps provides two competing energy scales [3]: the Zeeman-like band splitting energy tending to delocalize (deconfine) particles in the transverse direction [3]

$$\Lambda \propto t_\perp^{1/1-\eta} \text{ with } \eta \propto U^2,$$

and the well-known single-chain Mott scale

$$m \propto \exp -\pi v_F/|U|,$$

that may rather induce confinement along the chains. Influenced by the plethora of novel phenomena predicted (postulated) at the confinement/deconfinement transition in systems with an infinite number of weakly coupled chains (Bechgaard salts) [1], it is naturally worth to consider such issues on a two-leg ladder where tractable calculations are indeed possible.

When the splitting energy dominates - i.e. when the hopping amplitude increases much faster than the Umklapp channel(s) upon renormalization - that would correspond to the deconfinement phase. The result is a two-band model with four Fermi points. Rewriting the bare interactions - Do consult Eq.(6) - in the so-called band basis, we immediately recover the model that has been previously studied by Lin, Balents and Fisher in the limit of very large interchain hoppings $\Lambda \gg U$ [1]. The renormalization group transformation scales the system to a special strong-coupling Hamiltonian with enormous symmetry - the SO(8) Gross-Neveu model, that is integrable.

The splitting of the two bands affects spin-charge separation: Cooper pairs and magnons appear as natural excitations both at the two-band Mott scale $M \approx t_\perp \exp -\pi v_F/8U$ that is of the same order as $m$. Doping such a spin liquid liberates "hole-pairs" with d-wave pairing, restoring partially spin-charge separation [3]. There is an extended massive $\pi$-mode with SO(6) symmetry, containing remnant charge and spin degrees of freedom [3].

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Then, we study this model of interacting electrons hopping on a two-leg ladder, focusing on the behavior at half-filling mainly in the opposite limit where the intrachain Umklapp vertex is flowing first to strong couplings...
excitations of Luttinger liquids are reminded. Mott scale are given. In Appendix B, “anyonic” charge definitions of spinon- and charge excitations below the isons with Bechgaard salts. In Appendix A, (Abelian) formulate a brief summary of our results and make compar- from half-filling are carefully studied. In Sec. VI, we for- effect on the confined state and incoherence of

At the opening of the Mott gap, low-lying excitations located mainly on the chains are usual empty/doubly occupied sites and spinon-(anti)spinon pairs. Then, as a result of spin-charge separation arising in the confined phase, we show rigorously how the tunneling of spinon-(anti)spinon pairs give rise to an interchain Heisenberg exchange coupling of the order of \( J_\perp = t_\perp^2/m \gg 0 \). This has been previously neglected in Ref. [19]. The result is equivalent to two weakly coupled Heisenberg chains [20]. The ground state is a disordered spin liquid with a prominent 4 \( k_F \) Charge Density Wave (CDW), like in the confined limit. It is worth noting that the spin gap \( \Delta \) rejoin the charge gap in the entrance of the D-Mott phase.

To summarize: The confinement/deconfinement transition arising in undoped two-leg ladder systems is a pure 4 \( k_F \) superfluid with a prominent 4 \( k_F \) Charge Density Wave (CDW), like in the confined limit. It is worth noting that the spin gap \( \Delta \) rejoin the charge gap in the entrance of the D-Mott phase.

We start with the following model. The kinetic energy takes the form

\[
H_{\text{kin}} = H_o + H_\perp
\]

\[
H_o = -t \sum_{j,\alpha} d_{j\alpha}^\dagger (x+1) d_{j\alpha}(x) + \text{H.c.}
\]

\[
H_\perp = -t_\perp \sum_{\alpha} d_{2\alpha}^\dagger (x) d_{1\alpha}(x) + \text{H.c.}
\]

(3)

The indices \( j = 1, 2 \) denote here the chains, and \( \alpha = \uparrow, \downarrow \). Focusing on electronic states near the Fermi points, one can expand:

\[
d_{j\alpha}(x) = d_{+j\alpha}(x)e^{ik_F x} + d_{-j\alpha}(x)e^{-ik_F x}.
\]

(4)

At half-filling, one must equate \( k_F = \pi/2 \). We put the bare short-distance cutoff, \( a = 1 \). Then, the Hubbard four-Fermion interaction can be expressed in terms of currents, defined as

\[
J_{p\alpha} = \frac{1}{2} \sum_{\alpha,\alpha'} d_{p\alpha}^\dagger d_{p\alpha'}, \quad I_{p\alpha} = \sum_{\alpha,\alpha'} d_{p\alpha}^\dagger \epsilon_{\alpha\alpha'} d_{p\alpha'},
\]

(5)

and in the following p = ± denote respectively right and left excitations, \( \sigma \) denote Pauli matrices and \( \epsilon_{\alpha\alpha'} = \text{antisymmetric: } \epsilon_{\alpha\alpha'} = -\epsilon_{\alpha'\alpha} \) and \( t_\perp = 1 \). Precisely, the set of marginal momentum (and non-)conserving four-Fermion interaction reads:

\[
H_{\text{int}} = \sum_j \left( g e J_{+jj} I_{-jj} - g_s J_{+jj} J_{-jj} + g_u I_{+jj}^\dagger I_{-jj} \right).
\]

(6)

Note that \( g_e \) and \( g_s \) describe charge- and spin backscatterings respectively, and \( g_u \) the intrachain umklapp processes. The bare interactions are of the order of \( U \).

Now, it is appropriate to use an Abelian bosonized form for fermion operators. These transform as

\[
d_{p\alpha} \to \exp \left( i \frac{\pi}{2} \left[ \rho (\Phi_{jc} + \alpha \Phi_{ja}) - (\theta_{jc} + \alpha \theta_{ja}) \right] \right).
\]

(7)

\( \alpha = \pm \) for spin up and spin down, respectively. Remarkably, electron spectrum yields spin-charge separation. Absorbing the interaction \( g_e \), which is not affected by a rescaling of the short-distance cutoff, in the charge part of \( H_o \) results in the so-called Luttinger model

\[
H^c_{\text{oj}} = \frac{u}{2\pi} \int dx \frac{1}{K} (\rho_{jc} - \rho_o)^2 + K \nabla \Theta_{jc}^2.
\]

(8)

\( \rho_{jc} \Phi_{jc} = (\rho_{jc} - \rho_o) \) measures fluctuations of charge density in each chain, and \( \nabla \Theta_{jc} \) is the conjugate momentum to \( \Phi_{jc} \). All the interaction effects are now hidden in the parameters \( u \) (the velocity of charge excitations, \( uK = v_F \) with the Fermi velocity \( v_F = 2t \sin k_F \)) and \( K \) (the
Luttinger liquid (LL) exponent controlling the decay of correlation functions, \( K = 1 - g_c/\pi v_F \). \( K < 1 \) means repulsive interactions.

The free spin Hamiltonian yields the same form replacing \( \Phi_{jc} \) by \( \Phi_{js} \) and the Luttinger exponent \( K \) by \( K_s = 1 \) due to the requirement of SU(2) invariance.

### III. CONFINEMENT DUE TO UMKLAPPS

It is first suitable to write the main Renormalization Group (RG) equations.

#### A. RG-flow analysis

By simple scaling arguments, one gets:

\[
\frac{dg_u}{dl} = (2 - 2K)g_u, \quad \frac{d\ln t_\perp}{dl} = \frac{3}{2} - \frac{1}{4}(K + \frac{1}{K}).
\]

Anisotropy between space and time induced by Umklapp scattering results in a strong renormalization of the LL exponent and the charge velocity [23]:

\[
\frac{dK}{dl} = -C_1(g_u K)^2, \quad \frac{du}{dl} = -C_2 g_u^2 u K.
\]

For a complete derivation of such equations, do consult Ref. [24]. The \( l \) describes renormalization of the short-distance cutoff \( a(l) = \exp l \). At half-filling, usual Bessel functions \( J_0 \) and \( J_2 \) are non-oscillating and have been included in the positive constants \( C_1 \) and \( C_2 \) [23]. If one is at finite temperature \( T \), then the renormalization procedure must be stopped at lengths \( a(l) \) comparable to the thermal length \( u/T \), that means \( l \sim \ln(1/T) \).

Starting with a sufficiently small (bare) interchain hopping, then the Umklapp channel \( g_u \) is flowing first to strong couplings at a critical scale \( T_c \) that is the single-chain Mott scale \( m \) given by Eq. (8). For one-loop RG equations, indeed one gets a critical driving parameter \( l_c = \alpha v_F / U \) where \( \alpha \) is of the order of unity. Although the singularity at finite \( l_c \) is an artifact of the one-loop calculations, which is actually eliminated by including higher-order terms, \( m \) always remains a meaningful characteristic energy scale for strong coupling (see Appendix A). This is significant of a Mott insulating state, with a finite charge gap equal to \( 2m \).

An important point that can be extracted from the RG-flow is that the explicit divergence of the Umklapp scattering \( g_u \) at the opening of the Mott gap results formally in:

\[
K(l > l_c) = 0.
\]

The jump of the LL exponent at finite \( T \) is significant of a Kosterlitz-Thouless transition, that is also accompanied by a sharp jump in the charge compressibility and Drude weight [8]. Likewise, we like to stress on the fact that the strong decrease of the LL exponent by Umklapps has an enormous consequence on the small single-particle interchain hopping as well.

Consulting Eqs. (1),(11), indeed we observe that this should also produce the complete vanishing of \( t_\perp \) exactly at the Mott scale (and not only a partial reduction [23]):

\[
t_\perp (l > l_c) = 0.
\]  

In agreement with Giamarchi’s assertion, we obtain that the Mott gap renders the single-particle hopping completely irrelevant [8]. We like to interpret the total vanishing of \( t_\perp \) at finite temperature as a sign of strong confinement (of coherence) along the chains. For a physical explanation, do see next subsection. The divergence of intrachain Umklapp scattering produces in that case an insulating state with two degenerate bands.

Now, we like to repeat that in absence of Umklapp scattering i.e. away from half-filling, this condition of confinement is never satisfied for the 2-leg Hubbard ladder [14]. Precisely, to make interchain hopping irrelevant away from half-filling, one should make its amplitude decreasing upon renormalization. This must be determined from inequality \( d_\perp > 2 \) with:

\[
d_\perp = \frac{1}{4}(K + \frac{1}{K}) + \frac{1}{2}.
\]

the scaling dimension of the \( t_\perp \)-perturbation. To obey this standard criterion of irrelevance [8], one should start with a very small bare LL exponent \( K < 3 - \sqrt{8} \approx 0.171... \). It is worth to note that this is never realized for purely on-site interactions and \( U \ll t \) [13]. This always produces deconfinement away from half-filling that is characterized by a finite Zeeman-like splitting of degenerate bands. The magnitude of the splitting \( \Lambda \) is defined in Eq. (8). The system behaves as a Luther-Emery liquid [29] with dominant d-wave pairing. For a brief description of the corresponding phase, see Section V.

Nonetheless, the confinement condition away from half-filling might be performed in presence of long-range interactions where the chains behave rather as Wigner crystals [13]. The underlying model in each chain is still of LL type, but the system develops a real tendency to a periodic arrangement or a \( 4k_F \) CDW: This hinders considerably the single-particle interchain transfer. It should be remarked that in that case, confinement might take place only at zero temperature.

We also like to insist on the following point. Starting with very small interchain hoppings (such as \( m \gg t_\perp \)) ensures \( g_u^* = 0 \) at zero temperature: Indeed, for the 1D Hubbard chain and repulsive interactions it is well-known that the spin backscattering is (marginally) irrelevant.
and then it can be omitted [3]. Note already the fundamental difference with the deconfined regime, where spin backscattering also flows to strong couplings at the D-Mott transition producing immediately spin-gapped excitations [4][27].

B. Breakup of electrons on the chains

The impossibility to get a finite single-particle interchain transfer at absolute zero might be naturally interpreted as follows. The system suppresses the bare-electron interchain hopping due to the strong constraint to avoid a double on-site occupancy, that is here explicitly induced by the presence of the finite Mott gap. Now, we really like to emphasize that the vanishing of $t_\perp$ immediately at the Mott transition “caches” more a marked spin-charge separation or the complete breakup of physical electrons. When $T \to 0$, one already finds that the electronic Green function (e.g for Right movers)

$$ \mathcal{G}_+(x, \tau) = \frac{\exp{i k_F x}}{\sqrt{(v_F \tau - i x)(u \tau - i x)}} \left[x^2 + (u \tau)^2\right]^{-\eta/2}, \quad (14) $$

tends to vanish for quite short time because the anomalous dimension of the electron increases vastly: $\eta = 1/4(K + K^{-1}) - 1/2 \gg 2$ through the factor $K^{-1}(t_c)$. As usual, $\tau$ denotes the Matsubara Imaginary time. Now, we show precisely that below the Mott transition there is no way to recombine a physical electron along the chains: This gives a simple explanation on the total vanishing of the bare-electron interchain transfer for $T \leq m$.

First, from Eqs. (13), one can observe that the velocity of charge excitations drastically decreases to zero. This tends to push forward the idea that charge and spin degrees of freedom get really independent. Precisely, the opening of the Mott gap transforms charge excitations on the chains into fermionic Kinks, corresponding to pairs of doubly and empty occupied sites. The spin sector is still described by spinon-(anti)spinon pairs of the Heisenberg model. Do consult Appendix A for more explanations and details.

In the following, the symbols $F_{pj}^{Q^\pm}$ and $S_{pj}^{Q^\pm}$ refer to a fermionic Kink and a (single) spinon, respectively. These are given by:

$$ F_{pj}^{Q^\pm} \approx \exp{i \sqrt{\pi} Q^\pm_c (-p\Phi_{jc} + \Theta_{jc})}, \quad (15) $$

$$ S_{pj}^{Q^\pm} = \exp{i \sqrt{\pi} Q^\pm_s (-p\Phi_{js} + \Theta_{js})}. $$

Note that $Q^\pm_c = \pm$ refer to “electron” and “hole” like excitations precisely, and the spin of the spinon obeys $S^z = Q^\pm_s/2 = \pm 1/2$. The spinon and charge objects have different wave-vectors; an empty or doubly occupied site has a (double and opposite) wave-vector $(-pQ^\pm_c)2k_F$.

Second using Eq. (16), it is advantageous to decompose the bare electron operator

$$ d_{pj\alpha}^+ = [C_{pj}^+ \sigma^\perp_{pj}], \quad (16) $$

where $C_{pj}^+$ describes the holon of the Fermi gas [28]:

$$ C_{pj}^{Q^\pm} = \exp{i \sqrt{\pi} Q^\pm_c (-p\Phi_{jc} + \Theta_{jc})}. \quad (17) $$

Now, it is sufficient to note that starting with a bare LL exponent $K \to 1$ (i.e weak U), one can equate:

$$ C_{pj}^+ = [F_{pj}^{1/\sqrt{2}}]. \quad (18) $$

This may simply explain why the recombination of electrons is definitely forbidden at the Mott transition: Indeed, this would require a fractional number of fermionic Kinks. As a natural consequence, the single-particle interchain transfer is already destroyed by the enhanced spin-charge separation arising at the Mott transition. This constitutes the main difference with the confinement phenomenon in Wigner crystals that occurs only at absolute zero.

C. Prevalent fluctuations at the Mott transition

Using known results on the Hubbard chain at half-filling and those of Appendix A, we get that the $4k_F$ CDW and spin-spin fluctuations at $q = \pi$ are then prominent around the Mott scale with the most diverging susceptibility. Here, these are described by the operator:

$$ O_{edw}^j = F_{+j}^+ F_{-j}^- + H.c., \quad (19) $$

that yields a non-zero expectation value below the Mott transition, resulting explicitly in:

$$ < O_{edw}^j(x) O_{edw}^j(0) > = < O_{edw}^j(x) >^2 = const., \quad (20) $$

and by the staggered magnetization operator:

$$ m_j = S_{+j}^z \sigma(S_{+j}^\perp)^* + H.c. \quad (21) $$

The spinon operator has the quantum scaling dimension $1/2$, that produces:

$$ < m_j(x) m_j(0) > = (-1)^z/x. \quad (22) $$

Since we have a jump in the LL exponent at the Mott transition, we recover Heisenberg correlation functions in each chain.

Again unlike in the deconfined picture [4][1], here the opening of the Mott gap does not produce any spin gap. The single chain behaves as a pure Heisenberg spin chain.
at the Mott transition. Our picture below the Mott energy, however, is still incomplete. The tendency towards suppression of single-particle transport in the transverse direction is not the only effect of $t_\perp$ [22]. The Hamiltonian must be supplemented by extra “relevant” terms which are inevitably generated in the course of renormalization, by expanding the partition function as a function of $t_\perp$ [13]. For a detailed review on this important point, do see Ref. [1].

IV. TUNNELING PROCESS

It is maybe appropriate to rewrite explicitly the bare forward and backward hoppings [18,20]:

$$\mathcal{H}_\perp = t_\perp \cos \sqrt{\pi} \Theta_{-}^c \cos \sqrt{\pi} \Theta_{-}^s \left[ \cos \sqrt{\pi} \Phi_{-}^c \cos \sqrt{\pi} \Phi_{-}^s \right]$$

combining the boson fields in the two chains into a symmetric “+” and antisymmetric “−” part. On the other hand, using Eqs (18) and (19) one can also write:

$$\mathcal{H}_\perp = \sum_{p,p'} t_\perp e^{i(p'-p)x} \left[ C^+_p C^-_{p'+2} + \text{H.c.} \right]$$

As a consequence, we get the important identifications:

$$C^+_p C^-_{p'+2} + \text{H.c.} = \cos \sqrt{\pi} \Theta_{-}^c \cos \sqrt{\pi} \Phi_{-}^c$$

$$C^+_p C^-_{p'+2} + \text{H.c.} = \cos \sqrt{\pi} \Theta_{-}^c \cos \sqrt{\pi} \Phi_{-}^c$$

and similarly for the spinon part.

A. Spinon pair hopping

As shown before, the total vanishing of $t_\perp$ for $T \ll m$ must be closely related to the breakup of electrons on the chains. On the other hand, we insist on the fact that “some” tunneling processes in $t_\perp^2$ - respecting spinon and charge fermionic Kink separation - are still allowed in the far Infra-Red for $T \leq m$. Here, we like to emphasize that in previous works spin excitations at very low temperature have not been analyzed in great details [19], or relevant tunneling processes have been completely undervalued [13].

Now, we derive these properly as follows. Note that below we implicitly drop out oscillatory terms that do not influence the physics at the fixed point.

First, it is important to mention that at the Mott transition ($l \rightarrow l_c$), the relevant contributions are essentially furnished by:

$$\left( C^+_p C^-_{p+2} + \text{H.c.} \right) \left( C^+_p C^-_{p'+2} + \text{H.c.} \right) = \cos \sqrt{4\pi} \Phi_{-}^c$$

$$\left( C^+_p C^-_{p+2} + \text{H.c.} \right) \left( C^+_p C^-_{p'+2} + \text{H.c.} \right) = \cos \sqrt{4\pi} \Phi_{+}^c$$

Such operators acquire indeed a non-zero expectation value (see Appendix A):

$$\langle \cos \sqrt{4\pi} \Phi_{-}^c \rangle > \approx \langle \cos \sqrt{4\pi} \Phi_{+}^c \rangle = m$$

whereas one still gets:

$$\langle \cos \sqrt{4\pi} \Phi_{-}^c \rangle = 0$$

These may be primarily interpreted as tunneling processes of the massive fermionic Kinks lying along the chains. For a bare LL exponent $K \rightarrow 1$ one precisely gets:

$$\left[ (F^+ \rightarrow F^-) + \text{H.c.} \right]^2 = \left[ (C^+_p C^-_{p'+2} + \text{H.c.}) \right]^2 \sqrt{2}$$

$$= m \left[ (C^+_p C^-_{p'+2} + \text{H.c.}) \right]^2$$

Approaching the Mott transition, second it is important to note that the (finite) amplitude of the relevant pair hopping(s) is given approximately by [See Appendix A and Eq. (A20)]:

$$[t_\perp (l \rightarrow l_c)]^2 / E_F^2 \approx t_\perp^2 \exp 2l_c / E_F^2 = t_\perp^2 m^{-2}$$

$E_F$ is the Fermi energy. Now, averaging on the charge sector for $l \approx l_c$, one gets the extra relevant Hamiltonian:

$$\delta \mathcal{H}_{\text{int}} = t_\perp^2 / m \left[ \sum_{p,p'} S^+_p S^-_{p'+2} + \text{H.c.} \right]^2$$

$\delta \mathcal{H}_{\text{int}}$ produces processes of coherent interchain spinon-(anti)spinon hopping triggered by the single particle one.

Using the classification scheme of particle-particle and particle-hole hoppings from Ref. [18], one can equivalently rewrite [21]:

$$\delta \mathcal{H}_{\text{int}} = g_5 \cos \sqrt{4\pi} \Phi_{-}^s + (g_1 + g_4) \cos \sqrt{4\pi} \Theta_{-}^s + g_8 \cos \sqrt{4\pi} \Phi_{+}^s$$

where we have: $g_1 = t_\perp^2 / m$. Moreover, from Appendix A, it is then clear that this generates an interchain Heisenberg interaction:

$$\delta \mathcal{H}_{\text{int}} = J_{\perp} m_1 \cdot m_2, \quad J_{\perp} = t_\perp^2 / m \ll m$$

Since the spinon spectrum (driven by the kinetic part) is known to be invariant tuning the Hubbard interaction from weak to strong interactions, therefore the low-energy model is equivalent to two weakly-coupled Heisenberg chains [20]. The intrachain Heisenberg coupling is here equal to the Fermi velocity, $v_F$. 


This proves rigorously that the interchain hopping is well sufficient to generate the antiferromagnetic spin exchange $J_1 \approx t_1^2/m > 0$, that will make all spin excitations gapful. In particular, semi-classical considerations allow to predict the pinning of the $\Theta_s^-$ and $\Phi_s^+$ spin-fields at the fixed point (in the minima of the cosine potentials). Excitations will be separated from the ground state by an energy gap $\sim J_1$. This leads to a ground state that would not have still a strong resemblance to that of the half-filled 1D chain. Magnetic excitations and doping effects are indeed very different.

### B. Effective Heisenberg ladder

Excitations of the final spin Hamiltonian $(H_s^0 + \delta H_{\text{int}})$ have been studied in great details in Refs. [31]. Rather than belabor the derivation, we only indicate the results. Rewriting the model in terms of four (real) Majorana fermions $\xi = \sum_{i=1}^3 \xi_i$ and $\rho$, one gets the following conclusions.

As for a general $S=1$ magnet, the triplet excitations in a rung namely $\xi$ will have a gap - the Haldane gap [32] - $\sim 3J_1$. For $T \ll J_1$, the asymptotic correlation function for spins on the same chain is given mainly by

$$< m_j(x)m_j(0) > = -(1)^x x^{-1/2} \exp(-m_t x). \quad (34)$$

This definitely produces a short-range Resonating Valence Bond solid exactly as in the deconfined limit.

Note that taking only into account the (bare) interchain hopping term at a short wave vector $q = 0$, some spin excitations would remain gapless at the Infra-Red fixed point:

$$\delta H_{\text{int}} = g_5 \cos \sqrt{4\pi} \Phi_x^- + g_1 \cos \sqrt{4\pi} \Theta_x^-. \quad (35)$$

The $(s^-)$ modes are protected by the duality symmetry under $\Phi_x^- \leftrightarrow \Theta_x^-$, the $(s^-)$ sector would be in fact a critical point of the two-dimensional Ising type. The resulting Hamiltonian must be assigned to the same universality class of the purely forward scattering model considered by Finkel’stein and Larkin [33] and later by Schulz [34], resulting in gapless- spinons and singlet (Majorana) fermions. These excitations can be recombined and rewritten as a massless triplet magnon excitation $\xi$ in a rung.

But, in the two-chain problem away from half-filling, again it is known that for general repulsive interactions, there is always a spin excitation gap and d-type pairing - or exceptionally orbital antiferromagnetism fluctuations. Here, the presence of such enigmatic spin liquid with remnant gapless magnon modes is definitely forbidden by the bare backward hopping at large wave vector $q = \pi$.

To summarize, we really want to emphasize that unlike in the deconfined region, the latter plays a central role in presence of strong confinement along the chains.

### C. Crossover to the D-Mott state

We can now conclude that the two weakly-coupled Hubbard chain model is in the same phase $C0S0$ for very small bare interchain hoppings (confined region) and for rather large bare interchain hoppings (deconfined phase); $CnSv$ denotes a state with $n$ massless charge and $v$ massless spin modes. This naturally demonstrates that the confinement/deconfinement transition at absolute zero in the half-filled two-chain problem is a simple crossover [21]. Now, let us compare symmetries of excited states in these two regimes.

In the confined regime, we have a total spin-charge separation on the chains. Charge and spin excitations are gapped but ruled by different energy scales. Additionally, we have shown that far below the Mott transition excitations can be classified as charge fermionic Kinks lying mainly along the chains with symmetry $U(1)$ and charge $\pm 1$, and magnon like excitations with an underlying symmetry $SU(2)_2 \times Z_2$ [34].

In the deconfined regime, the effective model yields rather an enormous global $S0(8)$ symmetry that can be briefly understood in the band basis, as follows [31]. The low-energy physics depends on a single effective coupling constant $g$ (of the order of $U$). In the band picture, the interaction part takes the specific form:

$$\mathcal{H}_{\text{int}} = -g \cos \sqrt{4\pi} \Phi_a \cos \sqrt{4\pi} \Phi_b, \quad (36)$$

with $(a,b) = 1, 2, 3, 4$ and bosonic operators are given by:

$$(\Phi, \Theta)_{1} = (\Phi, \Theta)^{+}_{\sigma}, \quad (\Phi, \Theta)_{3} = (\Phi, \Theta)^{+}_{\rho}, \quad (37)$$

$$(\Phi, \Theta)_{2} = (\Phi, \Theta)^{-}_{\sigma}, \quad (\Phi, \Theta)_{4} = (\Theta, \Phi)^{-}_{\rho}. \quad (38)$$

The indices $^{\pm}_{\rho}$ and $^{\pm}_{\sigma}$ refer to symmetric/antisymmetric charge and spin fluctuations in the band basis. It is then appropriate to use the re-Fermionization procedure (again, $Q_c^- = -$):

$$\psi_{pa} \approx \exp i\sqrt{\pi} Q_c^- (-p \Phi_a + \Theta_a). \quad (39)$$

Then, for weak $U$ the result is [31]:

$$\mathcal{H} = \sum_{a=1}^{4} \Psi_{a}^\dagger i\nu F \tau^2 \partial_x \Psi_{a} - g \bigl(\Psi_{a}^\dagger \tau^y \Psi_{a}\bigr)^2 \quad (39)$$

Pauli matrices $\tau$ act on right and left sectors and $\Psi_a = (\Psi_{+a}, \Psi_{-a})$. This is known as the $S0(8)$ Gross-Neveu model. The latter has a remarkable property of “triviality” that is useful to equate various excited states in the deconfined phase. A remarkable fact that can be
shown from integrability of the model is that lowest excited states are magnon like excitations with spin-1, and spinless charge ±2 or Cooperon and hole-pair. They are associated with two Dirac fermion excitations. This beautifully demonstrates preformed pairing with an approximate d-wave symmetry (precisely, with a relative sign change between bonding- and antibonding pairs) in the deconfined state. Unlike in the confined region, spin-charge separation is here violated.

Finally, note that in the crossover region i.e. when Λ ≈ m the spin gap J⊥ tends to rejoin the charge gap m. Both become of the same order as M exp −πvF/SU, the unique energy scale in the delocalized D-Mott phase. From the above analysis, one can furthermore predict that thermally activated fermionic Kinks on the chains with charge ±1 would then turn into excited Cooper pairs (in a rung).

V. DOPING EFFECTS

Doping the D-Mott state, it is well-known that the low energy excitations are only in the charge sector and correspond to a sound mode of the “hole pairs”. The phase is C1SO, i.e. a Luther-Emery liquid \([26]\). The resulting system behaves then as a fluid of hard-core bosons with quite long-range d-wave pairing \([\mathcal{E}]\). Very close to half-filling, note that the SO(8) symmetry still remains \([\mathcal{E}]\), but this gets easily broken down away from half-filling in SO(6) × U(1). The U(1) symmetry comes from the hole-pair bosonic field \(\Phi^+\) that becomes critical, whereas the SO(6) symmetry results from the residual massive part. This is explicitly composed of the 3 massive Dirac fermions \(\Psi_2\), \(\Psi_3\) and \(\Psi_4\) that are still coupled via a Gross-Neveu interaction. The most direct consequence is that spin-charge separation is only partially restored: There is a sixfold degenerate extended \(π\)-mode in the d-wave superconducting state that contains both massive spin and charge excitations \([\mathcal{E}]\). Note that unlike for the doped Hubbard chain, charge fluctuations in the superconducting state can only subsist at 4\(K_F\) \([\mathcal{E}]\).

It is important to remind that away from half-filling i.e. in absence of Umklapp scattering, the 2-leg Hubbard ladder is therefore described by a C1SO phase with prominent d-wave pairing and unconventional SO(6) × U(1) symmetry.

It is noteworthy that the hole-pairing phenomenon should subsist in strong magnetic field; The resulting 2-band model (of spinless fermions) predicts indeed prominent p-wave superconductivity for quite large \(t_\perp\) \([\mathcal{E}]\).

A. Hole doping effect on the confined state

For completeness, now we study light doping effect on the confined state properties.

We remind that the Umklapp interaction \(g_u\) has still to be taken into account for low dopings, \(δ < m\). It is therefore advantageous to take the half-filled picture that has been derived before, as a natural starting point.

As usual, we model the doping by adding a chemical potential \(−\mu Q_d\) with the charge operator:

\[
\hat{Q}_d = \sqrt{2 \pi} \int dx \partial_x (\Phi_{1c} + \Phi_{2c})
\]

\[
= 2\sqrt{\frac{K}{\pi}} \int dx \partial_x \Phi^+_c.
\]

Each chain is supposed to be equally doped. Note that the spin spectrum should not be affected by the low doping effect producing inevitably the pinning of the fields \(\Phi^+_s\) and \(\Theta^-\). The resulting model then reads:

\[
H_c = \frac{\mu}{2\pi} \int dx \{ (\partial_x \Phi^+_c)^2 + (\partial_x \Theta^-)^2 \}
\]

\[
- 2m \cos(\sqrt{4\pi \Phi^+_c}) - 2\mu(\sqrt{\frac{K}{\pi}} \partial_x \Phi^+_c).
\]

This Hamiltonian, describing a (purely 1D-quantum) commensurate-incommensurate transition, has been actively studied in the literature. For a review, consult p. 172 of Ref. \([\mathcal{E}]\). Formally, solutions of the equations of motion can be still written as fermionic solitons:

\[
F_{p+} = \exp i\sqrt{\pi}(\Phi^+_c - \Theta^-).
\]

The bottom of this band is typically at energy-scales close to \(\mu_c = −m\). Using Jordan-Wigner duality, one can also associate the hard-core boson field \([\mathcal{E}]\):

\[
\Delta \approx \exp i\sqrt{\pi} \Theta^+.
\]

Using the fact that the fields \(\Phi^+_c\) and \(\Theta^-\) are still pinned, then one can easily check that such object describes well a hole-pair with charge \(Q_d = −2\) and zero momentum (Consult Ref. \([\mathcal{E}]\) p. 280):

\[
\Delta = d_{+1}d_{−2} ± d_{+2}d_{−1}.
\]

Since the antisymmetric charge sector is not affected by the chemical potential, fluctuations of charge density in each chain become now strongly correlated:

\[
< \partial_x \Phi_{1c} > = < \partial_x \Phi_{2c} >
\]

inducing pairing between holes of the two chains. Physical excitations are rather described by the pairing field \(Δ\) that carries zero momentum (instead of \(F_{p+}\)). We like to emphasize that in this case, charge objects at- and close to half-filling are then different (fermionic Kinks with \(Q^\pm_c = ±\) turn into hole-pairs).

Furthermore, one gets the important equality:

\[
[\hat{Q}_d, Δ] = −2\sqrt{K} Δ.
\]
On the other hand, the (vertex) chiral operators $F_{p^+}$, or $\Delta$ must also satisfy

$$[\hat{Q}_d, \Delta] = Q_d \Delta = -2 \Delta.$$  

(47)

Therefore, at the commensurate-incommensurate transition one checks that the LL parameter $K \to 1$, exactly like for the lightly doped D-Mott state [27]. This reinforces the idea that at filling, there is no real difference between the confined- and deconfined phases (T=0): They react similarly by doping. Such a universal value of the LL exponent in the two-chain problem has been first conjectured in Ref. [38], and recently reached numerically in the strong U-limit with Density Matrix Renormalization Group approach [39] (where the confined picture can be extended). The limit $K \approx 1$ is consistent with the picture of a very dilute hole-pair gas. More generally, this describes well a hard-core boson gas in low-density limit [11]. Note the difference with the one-chain case [11].

One can also understand the preceding result as follows. The holes stay in the same rung because they do not break spin singlets. The total charge mode $\Phi^+_c$ becomes massless [18] resulting in an expected Luther-Emery liquid (C1S0) [26]. Similarly to the lightly doped-Heisenberg ladder [20] or D-Mott state [25], one immediately recovers prominent d-wave superconductivity with the pairing correlation functions:

$$< \Delta^+(x) \Delta(0) > \approx x^{-1/2}.$$  

(48)

Note that adding eventually a tiny Coulomb repulsion interaction $\hat{u}$ between the chains, the triplet mass evolves slightly as $m_\tau = J_\perp - \hat{u}$, but remains finite [4].

Finally, for larger doping i.e when $\delta > m$, the interchain Umklapp channel can be neglected resulting in total deconfinement, and the phase C1SO acquires an enlarged $S0(6) \times U(1)$ symmetry and a sixfold degenerate "$\pi$-mode" as well. Note that the hole-pair mode can be equally described in terms of the $\Theta^{\pm}_c$ or $\Theta^{\pm}_p$ phases.

B. Incoherence of $t_\perp$ away from ha-filling

Finally, we would like to discuss the fact that the Luttinger model is not a good fixed point of the 2-leg Hubbard ladder away from half-filling. This can be interpreted as an incoherent effect of $t_\perp$ because the consequent renormalization of the interchain hopping away from half-filling turns the weak-coupling LL behavior onto a strong-coupling Luther-Emery fixed point. We may understand the incoherence of $t_\perp$ mathematically, as follows. An alternative definition of incoherence of the single-particle interchain transfer in LL’s has been given in Ref. [21].

For instance, one can check that charge eigenstates of the Luttinger Hamiltonian [28] (Consult Appendix B)

$$\mathcal{L}^\pm_{pj} = \exp i \sqrt{\frac{\pi}{2}} Q^\pm_c (\sqrt{-p} \frac{\Phi_j}{K} + \Theta_{jc}),$$  

(49)

are not (exactly) equal to $\mathcal{L}^\pm_{pj}$. As a result, the consequent renormalization of $t_\perp$ progressively hinders the coherence of anyonic-type excitations lying along the chains. The resulting competition between the interchain hopping $t_\perp$ and the (bare) interaction $U$ produces inevitably a strong coupling fixed point with no charge object with charge $Q^\pm_c = \pm 1$ (but paired holes) and then no spinon (but magmons).

More generally, for doped N-leg Hubbard ladders ($N \geq 2$) this also results in a complicated strong coupling fixed point with new stable excitations [44]. A LL fixed point, however, can still arise in some specific models e.g in the 3-leg ladder (or in the N-leg ladder, with N odd) very close to half-filling [2] or for a very anisotropic and restrictive network built with an infinite number of "very strongly" coupled chains [7].

Finally, $t_\perp$ gets coherent only in the non-interacting case ($K = 1$). Spin and charge excitations have the same velocity and then eigenstates in each chain are usual Landau particles:

$$d^\dagger_{pjo} = \left[ \mathcal{L}_{pj}^+ S^\pm_c \right]_{K=1}.$$  

(50)

The ground state is simply achieved by making symmetric and antisymmetric combinations of the particle operators in the two gas. The fixed point is still a Fermi gas.

VI. CONCLUSION AND DISCUSSION

To summarize briefly, we have shown that the weakly-coupled two Hubbard chain model is in the same phase C0S0, for irrelevant interchain hoppings (confined region with $SU(2)_2 \times Z_2 \times U(1)^2$ symmetry) and for relevant interchain hoppings (deconfined region with $SO(8)$ symmetry). At absolute zero, the confinement/deconfinement transition - again, occurring when the single-chain Mott gap is of the order of the Zeeman-like band splitting energy - is a simple crossover [21]. This statement i.e the absence of a metallic phase at half-filling, can be generalized for N-leg Hubbard ladders with $N=3,4...$ (N is not too large).

First, the phase of perfect confinement is always equivalent to a N-leg Heisenberg ladder. Taking into account spinon-pair tunneling process between successive chains, it is indeed sufficient to induce a pure spin-ladder with open boundaries in the transverse direction. Based on known results on coupled spin chains, we predict an insulating C0S0 phase for $N$ even, and C0S1 for $N$ odd [8].

Second, similar conclusions can be reached for the N-leg ladder far in the deconfined regime ($t_\perp$ large), analyzing the resulting N-band model [12]. Let us briefly
insist on the main Fermi velocities. At half-filling, one gets the following Fermi velocities:

$$v_j = v_j = 2t \left(1 - \frac{t_\perp}{t} \right) \cos^2 \left(\frac{\pi j}{N + 1}\right)^{1/2},$$

(51)

with $j = N + 1 - j$. This results in:

$$v_1 = v_N < v_2 = v_{N-1} < ....$$

(52)

Integrating properly the numerous RG-equations, one finds then a decoupling into band pairs ($j, j$) and a hierarchy of energy scales

$$M_j \approx \exp -\pi v_j / U,$$

(53)

where the band pairs scale successively towards the D-Mott state of the two-leg Hubbard ladder. For $N$ even, all excitations are then gapped (the phase is C0S0). For $N$ odd, the remaining band behaves like a single chain at commensurate filling resulting in a C0S1 phase. As long as $N$ is not too large, then the deconfined region is still insulating and there is no fundamental difference with the confined part, where in contrast $t_\perp$ is strongly suppressed: The deconfinement/confinement transition is still a crossover. No metallic phase arises for large $t_\perp$ in the non-doped case.

This completely leaves open how the crossover should evolve for an an increasing number of coupled chains to give back the metal/insulating transition observed in Bechgaard materials ($N$ is very large). In particular, the decoupling of band pairs breaks down for large $N$ ($v_1 \approx v_2...$) and the analysis of the RG-flow becomes very subtle mainly due to the plethora of relevant interband coupling channels. The fixed point for quite large $t_\perp$ is not yet known. A simplified (and much less rigorous) route to describe the strange metallic state of the TMTSF salt would be to start from the confined picture (with explicit spin-charge separation) and then to interpret the relevant hopping as an induced “self-doping” on the chains, which are not obligatorily equally doped. Self-doping would appear as a possible way to mimic the small deviation of the commensurate filling due to the warping of the Fermi surface perpendicular to chain direction.

Since here chains are not equally self-doped, then one does not predict any hole-pairing effect: Charge excitations would be then those of the single-chain problem i.e. doubly and empty occupied sites. For small induced self-doping i.e. for $t_\perp$ close to $m$, the resulting system might have simulitnes with the lightly doped Hubbard chain. This simple picture could explain the very small spectral weight $\delta(w)$ (Most of spectral weight comes from charge-gapped excitations) and the unusual frequency-dependent conductivity in the finite frequency regime, observed on the basis of optical measurements.

On the other hand, it is important to remind that the TMTSF-family yields a good $T^2$-resistivity showing the relatively large importance of the transverse hopping in this system. The induced self-doping would not be so light in these materials and then the 1D LL-perscription based on a single doped Hubbard chain naturally breaks down (We remind that LL behavior is rarely stable in the N-chain model away from half-filling). A possible way to reconcile Fermi-liquid behavior with optical datas would be to study the effective infinite-band model where bands with intermediate indices would be sufficiently doped whereas bands with small or large indices would remain insulating. As soon as bands with intermediate indices are consequently doped, one indeed expects a weak-coupling fixed point of Fermi-liquid type.

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**APPENDIX A: EXCITATIONS AT THE MOTT TRANSITION**

Note that the opening of the Mott gap produces charge fermionic kinks confined to the chains.

Using the transformations ($K \approx 1$ denotes the bare LL exponent)

$$\Phi_{jc} = \sqrt{K} \Phi_{jc} \quad \Theta_{jc} = \hat{\Theta}_{jc}/\sqrt{K},$$

(A1)

the interacting part driven by umklapp in the chain $j$ is of Sine-Gordon type:

$$\mathcal{H}_{int} = -g_u \cos \sqrt{4\pi} K \tilde{\Phi}_{jc} = -m \cos \sqrt{4\pi} \Phi_{jc}.$$  

(A2)

Solutions of the underlying Sine-Gordon model are known as soliton-like; Solitons are fermionic particles:

$$\mathcal{F}^{\pm}_{pj} = \kappa_{pj} \exp i \sqrt{\pi} Q^\pm_c (-p\tilde{\Phi}_{jc} + \Theta_{jc}),$$  

(A3)

with the chosen gauge $\kappa_{+j}\kappa_{-j} = i$, and $Q^\pm_c = \pm 1$ refer to “electron” and “hole” like excitations respectively. Therefore, it is not surprising that the charge part of the Hamiltonian density can be refermionized, as

$$\mathcal{H}^c = \sum_j \pm i u \mathcal{F}^\pm_{pj} \partial_x \mathcal{F}^\mp_{pj} - im \mathcal{F}^\pm_{+j} \mathcal{F}^\mp_{-j} + H.c.$$  

(A4)

Excitations can be viewed as pairs of doubly/empty occupied sites; a Kink $\mathcal{F}^{\pm}_{pj}$ carries momentum $(-pQ^\pm_c)2k_F$. It should be noticed that fermionic (exchange) statistics means precisely:

$$\mathcal{F}^\pm_{pj}(x)\mathcal{F}^\mp_{pj}(y) = \mathcal{F}^\pm_{pj}(y)\mathcal{F}^\mp_{pj}(x) \exp -i\gamma \text{sgn}(x - y),$$  

(A5)

with
\[ \gamma = \pi. \] (A6)

Using Fourier and Bogoliubov transformations, the band structure of these solitons reads
\[ E(k) = \pm \sqrt{(uk)^2 + m^2}. \] (A7)

This results in a semi-conducting picture, where the charge gap is equal to \(2m\). At zero energy, there is no way to create pairs of doubly/empty occupied sites. The ground state yields a long-range \(4k_F\) CDW.

To study pair-hopping processes, one can rewrite:
\[ \mathcal{H}_{\text{int}} = -2g_a \cos \sqrt{4\pi\Phi^+_c} \cos \sqrt{4\pi\Phi^-_c}. \] (A8)

At the Mott transition, the pinning of these terms renders one-point correlation functions constants
\[ \langle \cos \sqrt{4\pi\Phi^+_c} \rangle = \langle \cos \sqrt{4\pi\Phi^-_c} \rangle = m. \] (A9)

We use normal ordering of phase exponentials in Sine-Gordon models:
\[ \langle \cos \beta \Phi \rangle = mK\beta^2/4\pi : \cos \beta \Phi :, \] (A10)
and the fact that the pinning phenomenon produces:
\[ : \cos \sqrt{4\pi\Phi^+_c} : = : \cos \sqrt{4\pi\Phi^-_c} : = 1. \] (A11)

Due to cluster decomposition principle, the associated two-point correlation functions become also constants at large distance. Note that the one-point correlation functions of the dual fields are still equal to zero (the 2-point correlation functions decay exponentially).

At the Mott energy, spin excitations or so-called spinons tend to be also localized on the chains. This carries the spin of a physical electron and then is described by the operator:
\[ \mathcal{S}^{\pm}_{pj} = \exp i \sqrt{2\pi} Q^\pm_s (-p\Phi_{js} + \Theta_{js}), \] (A12)

with \( Q_s^\pm /2 = \pm 1/2 \). Such objects are known to be located at wave vectors \( pk_F \), and obey a semi-ionic statistics with \( \gamma = \pi/2 \). In absence of charge fluctuations and for vanishing \( t_\perp \), spin fluctuations in each chain are produced only by spinon pairs \( \{2\} \).

As for the antiferromagnetic Heisenberg chain, it is appropriate to define the staggered spin operator as:
\[ \mathbf{m}_j = \mathcal{S}^\mu_{pj} \sigma (\mathcal{S}^{\mu j}_s)^* + \text{H.c.}, \] (A13)
resulting explicitly in:
\[ \mathbf{m}_j \propto (\cos \sqrt{2\pi}\Theta_{js}, -\sin \sqrt{2\pi}\Theta_{js}, \cos \sqrt{2\pi}\Phi_{js}). \] (A14)

Such definition of \( \mathbf{m}_j \) respects the total breakup of physical electrons below the Mott scale. Now, it is maybe important to compute explicitly the product \( \mathbf{m}_1 \cdot \mathbf{m}_2 \) (Consult subsection IV-A). One gets:
\[ \mathbf{m}_1 \cdot \mathbf{m}_2 = 2 \cos \sqrt{2\pi}(\Theta_{1s} - \Theta_{2s}) + 2 \cos \sqrt{2\pi}\Phi_{1s} \cos \sqrt{2\pi}\Phi_{2s} = 2 \cos \sqrt{4\pi}\Theta^- + \cos \sqrt{4\pi}\Phi^+_s + \cos \sqrt{4\pi}\Phi^-_s. \] (A15)

\( \xi \)From Abelian equalities:
\[ \mathcal{S}^{+\pm}_{pj} \mathcal{S}^{-\mp}_{pj} + \text{H.c.} = \cos \sqrt{\pi}\Theta^s \cos \sqrt{\pi}\Phi^-_s, \] (A16)
\[ \mathcal{S}^{+\pm}_{pj} \mathcal{S}^{-\mp}_{jq} + \text{H.c.} = \cos \sqrt{\pi}\Theta^s \cos \sqrt{\pi}\Phi^+_s, \]
this also results in:
\[ \mathbf{m}_1 \cdot \mathbf{m}_2 = \left[ \sum_{p,q} \mathcal{S}^{+\pm}_{pj} \mathcal{S}^{-\mp}_{pq} + \text{H.c.} \right]^2. \] (A17)

This gives an explicit relationship between antiferromagnetic interchain coupling and spinon-pair hopping.

Now, let us comment on the amplitude of the spinon pair hopping(s) in the entrance of the Mott transition i.e. \( l \to l_c \) [See Eq. (31) in the Section IV]. At very short distances \( l \to 0 \), pair hoppings are neglectable. Under renormalization of the short-distance cut-off, the associated amplitude is known to obey \( [18] \):
\[ \frac{d\hat{g}}{dl} = (1 - K)\hat{g} + z(1/K - K), \] (A18)
with the bare conditions for \( l \to 0 \): \( \hat{g}(0) = 0 \), \( z(0) = t_\perp^2/E_F^2 \) and \( (1/K - K) \sim U/v_F \) for small \( U \). \( E_F \) is typically the Fermi energy. See e.g. Ref. \{6\} page 226.

For \( l \leq l_c \), this can be simplified as:
\[ \frac{d\hat{g}}{dl} \approx z(t) \times U/v_F. \] (A19)

The amplitude of the pair-hopping(s) becomes finite approaching the Mott transition. Using Eq. \{1\}, we find:
\[ \hat{g}(l_c) \propto \int_0^{l_c} z(l)dl \approx \int_0^{l_c} dz(l) \approx t_\perp^2/m^2. \] (A20)

We have neglected \( z(0) \) in front of \( z(l \to l_c) = t_\perp(l \to l_c)/E_F^2 = t_\perp^2/m^2 \) because \( m \ll E_F \). Furthermore, averaging explicitly on the charge sector, we get that for \( T \to m \) the effective amplitude of the spinon pair-hopping(s) reads:
\[ g_1(l_c) = \hat{g}(l_c) < \cos \sqrt{4\pi}\Phi^+_c > \propto t_\perp^2/m. \] (A21)

Below the Mott transition \( (l > l_c) \), \( t_\perp(l) \) and \( K(l) \) are both renormalized to zero [See Part III, Eqs. \{11\} and \{12\}]. Therefore, \( g_1 \) now obeys:
\[ \frac{dg_1}{dl} = g_1, \] (A22)
with the new bare condition \( g_1(l_c) = t_\perp^2/m \). Far in the Infra-Red, the spinon pair-hopping(s) will “diverge”, producing a spin gap of the order of \( J_\perp = t_\perp^2/m \) in the spectrum (which can be exactly obtained via renormalization, see Part IV B).
APPENDIX B: CHARGE EXCITATIONS IN A LL

In a LL, the charge Hamiltonian is plasmon-like:

\[
H^c_{\pm} = \frac{u}{2\pi} \int dx \left( \frac{1}{K} (\rho_c - \rho_o)^2 + K (\nabla \Theta_c)^2 \right). \tag{B1}
\]

\(\partial_x \Phi_c = (\rho_c - \rho_o)\) measures fluctuations of charge density, and \(\nabla \Theta_c\) is the conjugate momentum to \(\Phi_c\). The equations of motion are usual d’Alembert equations.

It is appropriate to use the chiral decomposition:

\[
\Theta_{\pm} = \Theta_c \mp \frac{\Phi_c}{K}. \tag{B2}
\]

Again, \(p = \pm\) refers to the direction of propagation (right or left). The associated Hamiltonians are defined as

\[
H_{\pm} = \frac{u}{4\pi} \int dx \left( \nabla \Theta_{\pm} \right)^2. \tag{B3}
\]

The objects \(\Theta_{\pm}\) are chiral i.e. they obey \([28]\)

\[
[\Theta_{\pm}(x), \mp \frac{K}{2} \partial_y \Theta_{\pm}(y)] = i\delta(x - y). \tag{B4}
\]

Chiral vertex operators of the effective Gaussian model read

\[
\mathcal{L}^{Q^z}_{\pm} = \exp i \sqrt{\frac{\pi}{2}} Q^z_{\pm} \Theta_{\pm}. \tag{B5}
\]

They describe charge excitations of a LL. In particular, these have a charge \(Q^z_{\pm} = \pm 1\) because

\[
[Q_c, \mathcal{L}^{Q^z}_{\pm}] = Q^z_{\pm} \mathcal{L}^{Q^z}_{\pm}. \tag{B6}
\]

The charge operator is correctly normalized as:

\[
\hat{Q}_c = \sqrt{\frac{2}{\pi}} \int dx \partial_x \Phi_c. \tag{B7}
\]

Such objects correspond to anyonic excitations since they obey anyonic commutation relations with:

\[
\gamma = \frac{\pi}{2K}. \tag{B8}
\]

Note that the Hubbard interaction between physical electrons produces (at low-energy) a change in the statistics of “holons”. For the free electron gas, one gets \(K = 1\) and then holons are rather semions or half-electrons.

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