Renormalization group approach to layered superconductors

C. Timm

Universität Hamburg, I. Institut für Theoretische Physik, Jungiusstrasse 9, D-20355 Hamburg, Germany

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A renormalization group theory for a system consisting of coupled superconducting layers as a model for typical high-temperature superconductors is developed. In a first step the electromagnetic interaction over infinitely many layers is taken into account, but the Josephson coupling is neglected. In this case the corrections to two-dimensional behavior due to the presence of the other layers are very small. Next, renormalization group equations for a layered system with very strong Josephson coupling are derived, taking into account only the smallest possible Josephson vortex loops. The applicability of these two limiting cases to typical high-temperature superconductors is discussed. Finally, it is argued that the original renormalization group approach by Kosterlitz is not applicable to a layered system with intermediate Josephson coupling.

I. INTRODUCTION

Since the highly anisotropic high-temperature superconductors (HTSC’s) consist of weakly coupled superconducting layers one expects two-dimensional effects to be important. Quasi-two-dimensional superconducting films near the critical temperature and in the absence of an external magnetic field are well described by the Berezinskii-Kosterlitz-Thouless (BKT) theory. The BKT theory states that the unbinding of spontaneously created vortex-antivortex pairs is responsible for the phase transition of these films. A rigorous renormalization group formulation of the BKT theory was given by Kosterlitz. The question arises of whether the BKT theory can be extended to describe layered superconductors.

In this paper we first consider a system of superconducting layers with exclusively electromagnetic coupling and than extend our approach to include the Josephson coupling between the layers. The main consequence of the Josephson coupling is the appearance of Josephson vortices (JV’s), i.e. vortex strings that reside between the superconducting layers (for a review see Blatter et al.).

The layered system without Josephson coupling has been treated by Scheidl and Hackenbroich within a self-consistent linear response theory. The authors describe the renormalization of the interaction between vortices residing in the same layer, in neighboring layers, etc., by a single equation. It is shown below, however, that the screening cannot be described in such a simple way.

Several authors have considered the Josephson coupled case. Horovitz has studied the phase transitions due to a) the unbinding of vortex-antivortex pairs with vanishing Josephson coupling and b) JV loops that reside entirely between the layers within a sine-Gordon renormalization group approach. The competition between these two mechanisms is then discussed in a partly heuristic
manner. The role of JV loops that cross the layers does not become clear.

Fischer has assumed that the renormalization of the linear term in the interaction of pancake vortices, which is due to the presence of JV’s, and the logarithmic, electromagnetic term can be described by the same screening factor. This is not the case.

Pierson has used a renormalization group approach similar to Ref. 2 to treat the Josephson coupled system. However, the partition function given by Pierson does not correctly describe a layered superconductor because it contains a linear term in the vortex interaction for any two pancake vortices in the same layer or in neighboring layers, whereas such a term should only be present for two pancakes that are actually connected by a JV.

Friesen has considered the further simplified model of one superconducting layer between two thick superconducting slabs, using an expression for the vortex interaction which is correct for small separations only. This approximation is not sufficient since in the renormalization procedure the interaction between widely separated vortices must be taken into account.

A different approach to the same problem has been used by Chattopadhyay and Shenoy. They start from the anisotropic three-dimensional (3D) XY model on a cubic lattice. There are two reasons, however, why we prefer to use a different model: 1. The HTSC’s are essentially discrete in one direction, but continuous in the other directions, whereas the XY model of Ref. 9 is discrete in all directions. Universality ensures that both systems behave similarly within the 3D critical region, but the present paper is concerned with the 2D behavior outside that region, where universality does not hold. 2. The XY model on a cubic lattice contains three independent parameters (two spin couplings and a lattice constant), whereas the model considered here contains five parameters in the case of non-vanishing Josephson coupling. Experiments suggest that there are no universal relations between these parameters, i.e. they are truly independent. Thus the 3D XY model appears to be insufficient to describe the HTSC’s. Furthermore, like Pierson and Friesen, the authors make the approximation of a linear term in the interaction between any two pancake vortices, irrespective of the positions of the JV’s.

The present paper is organized as follows: In Sec. II a rigorous renormalization group theory for the exclusively electromagnetically coupled system is developed and the quantitative predictions of this theory are discussed for a typical HTSC. This theory then serves as a reference frame for the further discussion. In Sec. III the consequences of Josephson coupling are investigated. The paper concludes with a brief summary.

II. THE ELECTROMAGNETIC APPROXIMATION
A. Derivation of the renormalization group equations

Let us now turn to the layered system with negligible Josephson coupling. This approximation may be applicable to artificially grown HTSC superlattices since these structures are characterized by a large layer separation compared to the coherence length perpendicular to the layers.

The unrenormalized interaction between two vortices separated by a distance $r$ within the same layer is given by

$$U_0(r) = \pm \frac{\phi_0^2 s}{8\pi^2 \lambda_{ab}} \ln \frac{r}{\tau}, \quad (2.1)$$

where the upper (lower) sign is for vortices of the same (opposite) vorticity, $\phi_0 = h c / 2e$ is the flux quantum, $s$ is the layer separation, $\lambda_{ab}$ is the magnetic penetration depth for fields applied perpendicular to the layers, and $\tau$ is the minimum vortex separation. The interaction between vortices that are $n \neq 0$ layers apart is

$$U_n(r) = \pm \frac{\phi_0^2 s^2}{16\pi^2 \lambda_{ab}} \exp \left( -\frac{|n| s}{\lambda_{ab}} \right) \ln \frac{r}{\tau}, \quad (2.2)$$

Note the opposite sign and the reduction by the small factor $s/2\lambda_{ab}$ compared to the interaction within the same layer. Note further that the interaction of vortices that are, say, 2 or 3 layers apart is of the same order of magnitude as the interaction for $n = 1$. Thus it would not be justified to include only the interaction between vortices in neighboring layers and neglect terms with $|n| > 1$.

Defining the vortex “charge”

$$q = \pm \sqrt{\frac{\phi_0^2 s}{8\pi^2 \lambda_{ab}}} \quad (2.3)$$

and coupling parameters

$$\alpha_n = \begin{cases} -1 & \text{for } n = 0 \\ \frac{s}{2\lambda_{ab}} \exp \left( -\frac{|n| s}{\lambda_{ab}} \right) & \text{for } n \neq 0 \end{cases} \quad (2.4)$$

we can write down the grand canonical partition function,

$$Z = \sum_N \frac{1}{N!^2} \left( \frac{z}{\tau^2} \right)^{2N} \int_{D_1} d^2 r_1 \cdots \int_{D_{2N}} d^2 r_{2N} \exp \left( -\frac{\beta}{2} \sum_{i \neq j} q_i q_j \alpha_{n_i - n_j} \ln \frac{|r_i - r_j|}{\tau} \right). \quad (2.5)$$

Here, $N$ is the total number of vortex-antivortex pairs in the system, $z = \exp(\beta \mu)$ is the vortex fugacity, $\beta = 1/k_B T$ is the inverse temperature, $n_i$ and $r_i$ are the number of the layer and the position within the layer of vortex $i$, respectively, and the range of integration $D_i$ is the whole of the layer $n_i$ except discs of radius $\tau$ centered...
at the vortices $j < i$. Since the external magnetic field vanishes the sums over the layer indices $n_i$ are subject to the constraint that the total vorticity in every layer be zero.

Following the program of the renormalization group approach\textsuperscript{[3]} we now proceed to integrate out the smallest vortex-antivortex pairs (of size between $\tau$ and $\tau + d\tau$). The central concept is to rewrite the resulting expression in such a way that it has the same functional form as the original partition function, but with renormalized parameters. In our case, these parameters are the fugacity $z$ and the coupling parameters $\alpha_n$. As more and more pairs are integrated out, the parameters are renormalized to incorporate the effect of these pairs on the system. The renormalization is described by differential equations for the parameters as functions of the smallest pair size $\tau$. The properties of the macroscopic system are governed by the limiting behavior of these recursion relations for large $\tau$.

In the case of a single layer\textsuperscript{[3]} only neutral pairs are integrated out so that the total vorticity remains zero. A change in the total vorticity would result in an infinite contribution to the energy and could not be compensated for by suitable renormalization of parameters. Similarly, in the present case the vorticity in every layer must remain zero because removal of any configuration of vortices that does not consist of pairs in the layers would make the energy of the remaining system infinite. (This observation is related to the fact that the interaction is screened by vortices which move freely within the planes, but not perpendicular to the planes.) Thus only pairs which reside in the same layer can be integrated out. This limitation of the present approach is expected to have unphysical consequences, especially concerning the correlations between pancakes in different layers, which are essential within the (narrow) 3D critical region.

The integration procedure is similar to the case of a single layer.\textsuperscript{[3]} It may be written as the following prescription:

\[
\sum_{n_1} \int_{D_i} d^2r_1 \cdots \sum_{n_{2N}} \int_{D_{2N}} d^2r_{2N} \equiv \sum_{n_1} \int_{D_i'} d^2r_1 \cdots \sum_{n_{2N}} \int_{D_{2N}'} d^2r_{2N} \\
+ \frac{1}{2} \sum_{i \neq j} \sum_{n_1} \int_{D_i'} d^2r_1 \cdots \sum_{n_{i-1}} \int_{D_{i-1}'} d^2r_{i-1} \sum_{n_{i+1}} \int_{D_{i+1}'} d^2r_{i+1} \cdots \sum_{n_{j-1}} \int_{D_{j-1}'} d^2r_{j-1} \sum_{n_{j+1}} \int_{D_{j+1}'} d^2r_{j+1} \\
\times \sum_{n_{2N}} \int_{D_{2N}'} d^2r_{2N} \sum_{n_j} \int_{D_j} d^2r_j \sum_{n_i} \int_{D_i} d^2r_i \sum_{\tau \leq |r_i - r_j| < \tau + d\tau} d^2r_i \delta_{n_i, n_j} \delta_{q_i, -q_j}. \tag{2.6}
\]

Here, the $D_i'$ are the same as the $D_i$ above, but with $\tau$ replaced by $\tau + d\tau$, and $D_j'$ is the whole layer except discs of radius $\tau$ centered at the vortices $k \neq i,j$. The sum over $n_i$ can be performed trivially. The right-hand side of Eq. (2.6) consists of two summands: The first one is similar to the whole left-hand side, but with minimum pair size $\tau + d\tau$ instead of $\tau$. The second one gives the approximate correction. The last integral in Eq. (2.6) integrates over the separation vector of a small pair. The integral over $r_j$ and the sum over $n_j$ take the pair over the whole layer and all the layers, respectively. The sum $1/2 \sum_{i \neq j}$ selects every possible pair just once.

We now proceed to apply the prescription (2.6) to the partition function (2.3). The mathematical procedure is similar to the one of Ref.\textsuperscript{[3]} taking into account the additional sums over layer indices. The result is

\[
Z = \exp \left[ 2\pi (\frac{z}{\tau^2})^2 \tau d\tau MF \right] \sum_{n_1}^{1} \frac{1}{N^{12}} \left(\frac{z}{\tau^2}\right)^{2N} \sum_{n_1} d^2r_1 \cdots \sum_{n_{2N}} d^2r_{2N} 
\]
\[
\times \exp \left[ -\beta \sum_{i \neq j} \left( \alpha_{n_i - n_j} + 2\pi^2 z^2 \frac{d\tau}{\tau} \beta q^2 \tilde{\alpha}_{n_i - n_j} \right) q_i q_j \ln \frac{|r_i - r_j|}{\tau} \right],
\]

where \( \tilde{\alpha}_n = \sum_{m} \alpha_{m-n} \alpha_m \) and \( F \) and \( M \) are the size and the number of the layers, respectively. Up to this point, \( \tau \) has been replaced by \( \tau + d\tau \) only in the ranges of integration \( D'_i \). However, we have to rescale \( \tau \) everywhere to be consistent. To order \( d\tau \) we eventually obtain

\[
Z = \exp \left[ 2\pi \left( \frac{z}{\tau^2} \right)^2 \tau d\tau MF \right] \sum_{N} \frac{1}{N!^2} \left( \frac{z}{(\tau + d\tau)^2} \right)^{2N} \left[ 1 + \left( 2 + \frac{\beta^2 q^2}{2} \alpha_0 \right) \frac{d\tau}{\tau} \right]^{2N} \times \sum_{n_1} \int_{D'_1} d^2r_1 \cdots \sum_{n_{2N}} \int_{D'_{2N}} d^2r_{2N} \exp \left[ -\beta \sum_{i \neq j} \left( \alpha_{n_i - n_j} + 2\pi^2 z^2 \frac{d\tau}{\tau} \beta q^2 \tilde{\alpha}_{n_i - n_j} \right) q_i q_j \ln \frac{|r_i - r_j|}{\tau + d\tau} \right].
\]

Note that no length parallel to the \( z \) axis is rescaled, in contrast to the length scale in the planes, \( \tau \). Indeed, the partition function (2.5) does not contain any length scale in the \( z \) direction. We can interpret the layer index as an internal, discrete degree of freedom. The system we get in this way is equivalent to the layered model since both have the same partition function, but it is two-dimensional. Thus, rescaling of lengths in the \( z \) direction is meaningless for both systems.

If we compare Eq. (2.8) with the original partition function (2.5), we find that both have indeed the same functional form. Dropping the irrelevant factor \( \exp(2\pi(z/\tau^2)^2\tau d\tau MF) \), the partition function takes the form of the original partition function if we set

\[
z \to \left[ 1 + \left( 2 + \frac{\beta^2 q^2}{2} \alpha_0 \right) \frac{d\tau}{\tau} \right] z\quad (2.9)
\]

and

\[
\alpha_n \to \alpha_n + 2\pi^2 z^2 \frac{d\tau}{\tau} \beta q^2 \tilde{\alpha}_n.
\]

The same information can be expressed as a system of coupled differential equations:

\[
\frac{dz}{dl} = z \left( 2 + \frac{\beta^2 q^2}{2} \alpha_0 \right),
\]
\[
\frac{d\alpha_n}{dl} = 2\pi^2 z^2 \beta q^2 \tilde{\alpha}_n,
\]

where a logarithmic length scale \( l = \ln \tau/\tau_0 \) is introduced. Here, \( \tau_0 \) is the unrenormalized minimum pair separation. These equations may be rewritten as

\[
\frac{dz^2}{dl} = z^2(4 + \beta q^2 \alpha_0),
\]
\[
\frac{d\alpha_n}{dl} = 2\pi^2 z^2 \beta q^2 \sum_m \alpha_{m-n} \alpha_m.
\]

This infinite set of equations replaces the Kosterlitz recursion relations in the case of an electromagnetically coupled layered system. It is subject to the boundary condition that \( z^2 \) and \( \alpha_n \) take on their bare values for \( l = 0 \). Note that by letting \( \alpha_0 = -1 \) and \( \alpha_n = 0 \) for \( n \neq 0 \) we regain the Kosterlitz recursion relations.
B. Weak electromagnetic coupling

The macroscopic properties of the layered superconductor are controlled by the behavior of the solution of Eqs. (2.13) and (2.14) for large length scales \( l \). These equations can be simplified by means of an expansion for small electromagnetic interaction between vortices in different layers.

To this end we define the quantity

\[
A^2 = \sum_{n \neq 0} \frac{\alpha^2_n}{\alpha^2_0},
\]

(2.15)

which is a measure for the coupling between different layers as compared with the coupling within the same layer. From the definition (2.4) it follows that in the unrenormalized case \( A^2 \sim \frac{s}{4 \lambda_{ab}} \ll 1 \) for all HTSC’s. We will see below that this inequality holds for the renormalized quantities also.

If we further define the usual stiffness constant

\[
K(l) = -\frac{\alpha_0(l) \beta q^2}{2\pi},
\]

(2.16)

we find, to linear order in \( A^2 \),

\[
\frac{dz^2}{dl} = 2 z^2 (2 - \pi K),
\]

(2.17)

\[
\frac{dK}{dl} = -4 \pi^3 z^2 K^2 (1 + A^2),
\]

(2.18)

\[
\frac{dA^2}{dl} = -8 \pi^3 z^2 K A^2.
\]

(2.19)

Formal integration of the last equation yields

\[
A^2(l) = A^2_0 \exp \left[ -8 \pi^3 \int_0^l dl' z^2(l') K(l') \right],
\]

(2.20)

where \( A^2_0 = A^2(l = 0) \).

Since \( A^2_0 \ll 1 \) we may expand the square of the fugacity and the stiffness constant to linear order in \( A^2_0 \),

\[
z^2 = z^2_0 + \Delta z^2 A^2_0,
\]

(2.21)

\[
K = K_0 + \Delta K A^2_0.
\]

(2.22)

To the same order, \( z^2 \) and \( K \) in Eq. (2.20) can be replaced by \( z^2_0 \) and \( K_0 \). Inserting the expansions into Eqs. (2.17) and (2.18), we obtain a set of four coupled equations,

\[
\frac{dz^2_0}{dl} = 2 z^2_0 (2 - \pi K_0),
\]

(2.23)

\[
\frac{d\Delta z^2}{dl} = 2 (2 - \pi K_0) \Delta z^2 - 2 \pi z^2_0 \Delta K,
\]

(2.24)

\[
\frac{dK_0}{dl} = -4 \pi^3 z^2_0 K^2_0,
\]

(2.25)

\[
\frac{d\Delta K}{dl} = -4 \pi^3 K^2_0 \Delta z^2 - 8 \pi^3 z^2_0 K_0 \Delta K - 4 \pi^3 z^2_0 K^2_0
\]

\[\times \exp \left[ -8 \pi^3 \int_0^l dl' z^2_0(l') K_0(l') \right].\]

(2.26)
These equations are subject to the boundary conditions
\[ z_0^2(0) = z_0^2(0) = \exp(2\beta \mu), \quad \Delta z^2(0) = 0, \quad K_0(0) = K(0) = -\alpha_0(0)\beta q^2/2\pi, \text{ and } \Delta K(0) = 0. \]
With the help of Eq. (2.25), the integral (2.20) is found to be
\[ A^2(l) = A_0^2 \left( \frac{K_0(l)}{K(l)} \right)^2. \tag{2.27} \]

Thus we obtain a purely differential equation for \( \Delta K \):
\[ \frac{d\Delta K}{dl} = -4\pi^2 K_0^2 \Delta z^2 - 8\pi^3 z_0^2 K_0 \Delta K - 4\pi^3 z_0^2 \frac{K_0^4}{K^2(l)}. \tag{2.28} \]

Equations (2.23) to (2.25) and Eq. (2.28) describe the layered system for small \( s/4\lambda_{ab} \). Note that the number of equations has been reduced from infinity to four.

Note further that Eqs. (2.23) and (2.25) are the original Kosterlitz recursion relations, which describe the uncoupled layers and can be solved by themselves.

The solution of the Kosterlitz recursion relations is well known. For \( T \) smaller than a critical temperature \( T_c \), the fugacity \( z_0 \) converges exponentially to zero for large \( l \), while the stiffness goes to a finite value \( K_0 \geq 2/\pi \). Thus the vortices are bound in small vortex-antivortex pairs. For \( T > T_c \) the fugacity diverges exponentially and \( K_0 \) goes to zero for \( l \rightarrow \infty \). Here, very many unbound vortices exist. At \( T_c \) the stiffness constant \( K_0 \) jumps from \( 2/\pi \) to zero; this is the famous “universal jump”.

Let us now take the electromagnetic coupling between the layers into account. The asymptotic behavior of the quantity \( A^2 \) is given by Eq. (2.27), whereas the asymptotic forms of \( \Delta z^2 \) and \( \Delta K \) can be read off from Eqs. (2.24) and (2.28) in connection with the known forms of \( z_0^2 \) and \( K_0 \).

We first consider the case \( T > T_c \). The quantity \( \Delta z^2 \) diverges exponentially for large \( l \). Thus the fugacity \( z = \sqrt{z_0^2 + \Delta z^2} \) also diverges exponentially, whereas \( \Delta K, K, \text{ and the coupling parameter } \alpha_0 \) vanish. Thus the fugacity and the stiffness constant qualitatively behave as in the case of a single layer. Note that \( d/dl(\Delta z^2/z_0^2) = -2\pi \Delta K \) so that \( \Delta z^2/z_0^2 \) approaches a constant for \( l \rightarrow \infty \) and the expansion (2.21) remains valid for arbitrarily large \( l \). From Eq. (2.27) it follows that \( A^2 \) goes to zero in this regime. Since \( A^2 \) is a measure of the electromagnetic coupling between the layers this means that the renormalized interaction between vortices in different layers vanishes faster than the interaction within the same layer. In this sense the layers decouple and the system becomes two-dimensional for \( T > T_c \).

For \( T < T_c \), on the other hand, the quantity \( \Delta z^2 \) and, thus, the fugacity \( z \) converge towards zero. The correction to the stiffness, \( \Delta K \), approaches a finite and negative value. Therefore the coupling parameter \( \alpha_0 \) of vortices within the same layer is reduced by the presence of other layers, the correction being proportional to \( s/\lambda_{ab} \). As long as this correction is small, the pairs are still bound, though less tightly. The quantitative significance of this
correction is discussed below. The quantity $A^2$ is reduced by the renormalization, but remains finite. Thus the interaction between vortices in different layers remains finite, and the system is three-dimensional for $T < T_c$.

Whereas the above results are obtained by analytical study of the asymptotic behavior of the recursion relations, numerical integration is necessary to produce trajectories in the $\Delta K - \Delta z^2$ plane. Trajectories for $z(0) = 0.03$ and several starting values $K(0)$ are shown in Fig. [3]. The value $z(0) = \exp(\beta \mu) = 0.03$ seems reasonable for typical HTSC’s (Ref. [8] gives the value $\mu = -E_c = -0.72q^2$ and $\beta q^2$ must be larger than 4).

We now turn to the question of the absolute size of the correction $\Delta K$ to the stiffness constant below the critical temperature. Numerical studies of Eqs. (2.24) and (2.28) show that

$$\Delta K(\infty) \approx \frac{2\pi^3 z^2(0) K_0^2(\infty)}{2 - \pi K_0(\infty)}$$  \hspace{1cm} (2.29)

is a very good approximation for small $z(0)$ and small $2 - \pi K_0(\infty)$. The temperature dependence of $K_0$ is known to be

$$K_0(\infty) \approx \frac{2}{\pi} \left(1 + \sqrt{2B} \sqrt{\frac{T_c - T}{T_c}} \right),$$  \hspace{1cm} (2.30)

where $B$ is a nonuniversal constant. Thus

$$\Delta K(\infty) \approx -2\pi z^2(0) \sqrt{\frac{2T_c}{B}} (T_c - T)^{-1/2}$$  \hspace{1cm} (2.31)

to leading order in $T_c - T$.

If the critical temperature is approached from below, the correction to the stiffness constant thus diverges with an exponent of $-1/2$. Very near to the transition temperature of a single layer, $T_c$, this divergence causes the stiffness constant $K(\infty)$ to become negative, which is physically impossible. In this region higher order terms neglected here should render $K \geq 0$. We expect that $K$ vanishes and the pairs start to break up at a temperature $T_c^{3D} < T_c$. $T_c^{3D}$ should be larger than the temperature $T^*$ at which $K$ vanishes in the linear approximation (which thus definitely fails at $T^*$). $T^*$ can be estimated from Eq. (2.23). For Bi-2212 the parameters are $z(0) \approx 0.03$, $s/4\lambda_{ab} \approx 0.0019$, and $\sqrt{2B} \approx 2.91$. We thus obtain $T^*/T_c = 1 - 1.3 \cdot 10^{-10}$. Obviously, $T^*$ is indistinguishable from $T_c$, the linear approximation is always valid in practice, and the transition temperature $T_c^{3D}$ is not significantly shifted by the electromagnetic coupling.

The foregoing discussion indicates that the effect of the other layers is very small. Indeed, even if we could measure a correction of $\Delta K/K_0 = 1\%$ in Bi-2212, we would still need to resolve a temperature range of the order of $10^{-6} T_c$ to see it, which is impossible. In practice the effect of the electromagnetic coupling will be overruled by the Josephson coupling, see below.

By using HTSC superlattices it may be possible to increase the ratio $s/\lambda_{ab}$. An additional advantage of superlattices is that the electromagnetic approximation is
more appropriate in this case. However, even then the effect will be very small. More promising systems are superlattices fabricated from conventional superconductors.

III. LAYERED SUPERCONDUCTORS WITH JOSEPHSON COUPLING

A. General consequences of Josephson coupling

Since Josephson tunneling between the layers has been found experimentally in Bi-2212 and other HTSC’s we are faced with the question how to incorporate the Josephson coupling into the renormalization group theory presented above.

The Josephson coupling between the layers leads to the appearance of JV’s. Because of the smoothness of the phase of the order parameter in the layers and the conservation of magnetic flux every two-dimensional vortex (i.e. pancake vortex) in a layer must be connected to two JV strings and vice versa. Since the external magnetic field vanishes and surface effects are neglected the JV’s form vortex loops. On these loops pancake vortices sit, where ever the loops penetrate a layer. It is clear from this picture that a JV connects either a vortex and an antivortex within the same layer or two vortices of the same vorticity in neighboring layers.

The program of the renormalization group approach to this particular system is to derive an expression for the energy of any possible configuration of JV’s, add this expression to the Hamiltonian, insert the new Hamiltonian into the partition function, integrate out small vortex-antivortex pairs, and derive a set of recursion relations. To obtain a tractable expression for the energy of JV’s we introduce several approximations.

The energy of JV strings is approximately proportional to their length $L$ for $L \gtrsim \lambda_J$, where $\lambda_J$ is the Josephson length. If two pancakes are separated by less than $\lambda_J$, no full JV develops and the energy due to the Josephson coupling is not simply linear. However, for small separations the electromagnetic contribution to the interaction dominates and the form for the Josephson term is unimportant. Furthermore, as the interaction is renormalized, small pancake pairs are integrated out and only large pairs remain, for which the linear approximation is correct. Thus we assume that the energy of a JV of length $L$ is given by

$$U_J = \kappa L. \quad (3.1)$$

It is further assumed that the JV’s form straight lines between the pancakes. Thus we may take the JV’s into account by adding a term of the form (3.1) to the Hamiltonian for any two vortices that are connected by a JV. The Hamiltonian takes the form
\[
\mathcal{H} = \frac{1}{2} \sum_{i \neq j} q_i q_j \alpha_{n_i, n_j} \ln \frac{|r_i - r_j|}{\tau} + \sum_{(p, p') \in \mathcal{C}_j} \kappa |r_{p'} - r_p|,
\]

(3.2)

where \( \mathcal{C}_j \) denotes a given configuration of JV’s connecting the pancakes \( i = 1, \ldots, 2N \) and \( (p, p') \) is a single JV string characterized by the pancakes \( p \) and \( p' \) at its ends.

The constant \( \kappa \) in Eq. (3.2) can be evaluated for small \( s/\lambda_{ab} \), the unrenormalized value being \( \kappa(l = 0) \equiv q^2 \lambda_{ab}/s\lambda_c \), where \( \lambda_c \) is the penetration depth for magnetic fields applied parallel to the layers. Note that \( \lambda_{ab}/s\lambda_c \approx 10^{-3} \ \text{Å}^{-1} \) for Bi-2212. We can now see that the electromagnetic approximation is not sufficient for Bi-2212: The average distance between neighboring pairs and the separation on which the linear term in the interaction becomes important are of the same order of magnitude (1000 Å).

Since the partition function is a sum over all possible configurations, we must, in addition to the summations and integrations in Eq. (2.5), sum over all configurations of JV’s for given pancake positions. Thus the grand canonical partition function may be written as

\[
\mathcal{Z} = \frac{1}{N!^2} \left( \frac{z}{\tau^2} \right)^{2N} \sum_{n_1} \int_{D_1} d^2 r_1 \cdots \sum_{n_{2N}} \int_{D_{2N}} d^2 r_{2N} \sum_{\mathcal{C}_j} \exp(-\beta \mathcal{H})
\]

(3.3)

with \( \mathcal{H} \) given by Eq. (3.2). The last sum in Eq. (3.3) is over all configurations \( \mathcal{C}_j \) of JV’s.

We are now faced with the task to rewrite the prescription (2.6), which tells us how to integrate out small pairs, for the partition function (3.3). The new aspect here is that we must take all possibilities into account to insert the small pair into the JV loops. These possible insertions are of three topologically different types: 1. the small pair can form a vortex loop by itself, as shown in Fig. 2(a). 2. the pair can be part of a larger vortex loop, but still be directly connected by one JV, see Fig. 2(b). 3. the pair is not directly connected by a JV, see Fig. 2(c). Since the present approach neglects some contributions to the interlayer vortex correlations as discussed in subsection (3.4), we may expect unphysical results if vortex loops across more than one layer are essential. On the other hand, the approach is consistent if only small loops of type 1 are important.

If the sum over all possible insertions of the small pair \((i, j)\) into the configuration \( \mathcal{C}_j' \) of JV’s between the other pancakes is denoted by

\[
\sum_{(i, j) \in \mathcal{C}_j'}
\]

(3.4)

the prescription for integrating out small pairs is given by

\[
\sum_{n_1} \int_{D_1} d^2 r_1 \cdots \sum_{n_{2N}} \int_{D_{2N}} \sum_{\mathcal{C}_j} \exp(-\beta \mathcal{H}) \approx \sum_{n_1} \int_{D_1} d^2 r_1 \cdots \sum_{n_{2N}} \int_{D_{2N}} \sum_{\mathcal{C}_j} \exp(-\beta \mathcal{H})
\]

(3.5)

Here, \( \mathcal{C}_j' \) denotes a configuration of JV’s if the pancakes \( i \) and \( j \) are absent. Note that the sum (3.4) contains one summand for the first type of vortex loops and many summands for the second and the third type. Before the general case is discussed, we turn to a simple limiting case.

**B. Very strong Josephson coupling**

Let us now consider a layered superconductor with very strong Josephson coupling between the layers. In this case the energy of the JV strings is generally large compared to the other energy scales. Fluctuations of the JV’s cost
much energy and the system prefers states with small total JV length for any given configuration of pancake vortices. Although states with perfectly aligned pancakes in all layers have the lowest Josephson energy, such states are not created by thermal fluctuations since their total energy is infinite. The elementary excitations are small vortex loops across one layer (see Fig. 2(a)). Larger, more complicated loops will not form, except within a narrow 3D region, since the energy of JV strings is large. Thus we consider only loops of the first type, and the sum (3.4) is reduced to one term.

The limit of large $\kappa$ is not physically accessible since for a layered superconductor one always finds $\lambda_c > \lambda_{ab}$. Nevertheless it is considered here because it forms the opposite limiting case as compared with the system without any Josephson coupling and should allow one to estimate the maximum effect Josephson coupling can have.

The additional energy resulting from the two JV's is $2U_J = 2\kappa |r_i - r_j| = 2\kappa \tau$. This is a constant term in the Hamiltonian. Therefore, a constant factor $e^{-2\beta \kappa \tau}$ appears in the integrand.

For the same reasons as in the exclusively electromagnetic case only pairs within the same layer can be integrated out. The calculations are similar. The constant factor mentioned above is just dragged through. After the integration and the replacement of $\tau$ by $\tau + d\tau$, we obtain the partition function

$$Z = \exp \left[ 2\pi \left( \frac{z}{\sqrt{\tau}} \right)^2 \tau d\tau MFe^{-2\beta \kappa \tau} \right] \sum_N \frac{1}{N!} \left( \frac{z}{(\tau + d\tau)^2} \right)^{2N} \times \left[ 1 + \left( 2 + \frac{\beta}{2} q^2 \alpha_0 \right) \frac{d\tau}{\tau} \right]^{2N} \sum_{n_1} \int_{D_1} d^2r_1 \cdots \sum_{n_2N} \int_{D_2} d^2r_{2N} \sum_{C,J} \exp \left[ -\frac{\beta}{2} \sum_{i \neq j} (\alpha_{n_i-n_j} + 2\pi^2 z^2 \frac{d\tau}{\tau} \beta q^2 \alpha_{n_i-n_j} e^{-2\beta \kappa \tau}) q_i q_j \ln \frac{|r_i - r_j|}{\tau + d\tau} - 2\beta \kappa \sum_{(p,p') \in C,J} |r_{p'} - r_p| \right]. \quad (3.6)$$

Again, there is no rescaling of lengths in the $z$ direction, in contrast to Refs. 7 and 8. However, the arguments given in subsection II A remain valid in the presence of Josephson coupling and, therefore, the system is still essentially two-dimensional.

Comparison with the original partition function (3.3) yields the recursion relations

$$\frac{dz^2}{dl} = z^2 (4 + \beta q^2 \alpha_0), \quad (3.7)$$

$$\frac{d\alpha_n}{dl} = 2\pi^2 z^2 \beta q^2 \exp(-2\beta \kappa \tau_0 \delta^l) \sum_m \alpha_{m-n} \alpha_m, \quad (3.8)$$

$$\frac{d\kappa}{dl} = 0, \quad (3.9)$$

where again $l = \ln \tau/\tau_0$. We thus find that the linear coupling constant $\kappa$ is not renormalized due to screening if we consider only loops of the first type. (Neither is there renormalization due to rescaling of lengths, as noted above.) These equations, especially Eq. (3.9), will break down very near to the phase transition due to the appearance of larger, three-dimensional vortex loops.

The additional factor $\exp(-2\beta \kappa \tau_0 \delta^l)$ in the equation for $\alpha_n$ approaches zero very rapidly. The change of the electromagnetic coupling parameters $\alpha_n$ with the scale $l$ and, therefore, the screening of the electromagnetic interaction are strongly suppressed. In connection with the observation that the linear interaction due to JV's is not screened at all within this approximation, this result indicates that the low-temperature phase of bound pairs is stabilized and $T_c$ is higher than in the electromagnetic case.

To address this issue the recursion relations (3.7) and (3.8) are expanded for small $s/4\lambda_{ab}$. The procedure is analogous to the one employed above and is not repeated.
It is seen that the electromagnetic coupling between the layers brings about a very small correction to the renormalized electromagnetic coupling within the same layer. Thus it suffices to consider the vortex fugacity $z_0$ and the stiffness constant $K_0$ to order zero. The renormalization of these quantities is described by the equations

\[
\frac{dz_0^2}{dl} = 2z_0^2(2 - \pi K_0),
\]

\[
\frac{dK_0}{dl} = -4\pi^2 z_0^2 K_0^2 \exp(-2\beta\tau_0 e^l).
\]

Note that $\beta q^2 = 2\pi K(l = 0)$. Numerical integration for $z(0) = 0.03$, $\kappa = 10^{-3} \text{Å}^{-1} q^2$, $\tau_0 = 21.5 \text{Å}$, and several values of $K(0)$ yields the trajectories shown in Fig. 3.

By studying the asymptotic behavior of Eqs. (3.10) and (3.11) two temperature regimes can be identified. In this section the temperature which divides these two regimes is denoted by $T_c$, whereas the original BKT transition temperature is $T_{\text{BKT}}$. As is shown in Fig. 3, $T_c$ is significantly higher than $T_{\text{BKT}}$. For Bi-2212, experiments on ultra-thin films have found a transition temperature $T_{\text{BKT}} = 35 \text{K}$ and a mean-field temperature $T_{c0} = 86.8 \text{K}$. From the above results, the transition temperature for bulk Bi-2212 should be $T_c = 37.4 \text{K}$, but the experimental value is $84.7 \text{K}$. Taking into account that a more realistic description of the Josephson coupling (inclusion of larger loops) would probably yield an even lower $T_c$, the main effect must be brought about by other mechanisms, e.g. change of the hole density in the CuO$_2$ planes.

For $T < T_c$ the fugacity goes to zero and the stiffness constant approaches a finite value $K_0(\infty) > 2/\pi$ as $l \to \infty$. The linear coupling due to JV’s is not changed. The vortices are bound in small vortex-antivortex pairs.

For $T > T_c$ the fugacity diverges exponentially, but the stiffness constant $K_0$ still approaches a non-vanishing value $K_0(\infty) < 2/\pi$. Very many vortices exist, but they are still bound in pairs. This result is different from the exclusively electromagnetic case. Also, the electromagnetic interaction between vortices in different layers does not vanish and the system remains three-dimensional even above $T_c$.

Furthermore, numerical studies indicate that there is no special feature in $K_0(\infty)$ at $T_c$ (see Fig. 3). Nor do we expect the appearance of ohmic resistance at $T_c$ since the vortices are bound in pairs below as well as above this temperature. Thus the question arises of whether there really is a phase transition at $T_c$. However, since the true transition is governed by effects not included here (larger, 3D loops), this result should be interpreted with caution.

C. Intermediate Josephson coupling—failure of the renormalization group approach

Let us finally consider a layered superconductor with general Josephson coupling. In this case we must take
all the possible configurations of JV’s into account (see Figs. 2). This is especially necessary in the vicinity of $T_c$. Here, we consider only one term that shows in what way the approach fails.

It suffices to investigate one summand of the sum (3.4) of the second type (see Fig. 2(b)). Let $r_{i0}$ ($r_{j1}$) be the position of the neighbor of vortex $i$ ($j$) other than vortex $j$ ($i$), see Fig. 4. The energy of these three JV’s is given by $\kappa |r_i - r_{i0}| + \kappa \tau + \kappa |r_j - r_{j1}|$. Furthermore, to insert the new pair we must remove the JV from $r_{i0}$ to $r_{j1}$. Thus we have to consider an additional factor

$$\exp\left(-\beta\kappa |r_i - r_{i0}| - \beta\kappa \tau - \beta\kappa |r_j - r_{j1}| + \beta\kappa |r_{j1} - r_{i0}|\right)$$

in the integrand. The integral over the lowest order term in $\beta$ of the factor containing the electromagnetic interaction—this term is simply unity—times the above factor is evaluated in the appendix. After integration over $r_i$ and $r_j$ we obtain the expression

$$2\pi \tau d\tau \exp\left(-\beta\kappa \tau + \beta\kappa |r_{j1} - r_{i0}|\right) I_0(\beta\kappa \tau)$$

$$\times |r_{j1} - r_{i0}|^2 K_2(\beta\kappa |r_{j1} - r_{i0}|),$$

where $I_0$ and $K_2$ are Bessel functions. Expansion for large separations $r = |r_{j1} - r_{i0}|$ yields

$$2\pi \tau d\tau e^{-\beta\kappa \tau} I_0(\beta\kappa \tau) \sqrt{\frac{\pi}{2\beta\kappa}} r^{3/2}.$$ (3.14)

After performing the sums over the layer indices $n_i$ and $n_j$ and the vortex indices $i$ and $j$ we end up with a partition function based on a Hamiltonian that contains a term proportional to $|r_{p' - p}|^{3/2}$ for any two pancakes $p$ and $p'$ that are connected by a JV.

It can be shown that this term cannot be canceled by any other terms: All other terms in the integrand either contain factors $\beta q^2$ or $\beta^2 q^4$ (as opposed to the term discussed above) or have the same sign.

But no term proportional to the vortex separation to the power $3/2$ is present in the original Hamiltonian! Thus it is impossible to renormalize the parameters in the partition function in such a way as to regain the original partition function (3.3). The program of the renormalization group approach thus fails for the layered superconductor with (general) Josephson coupling, at least within the present model. We speculate that the reason for this failure lies in the one-dimensional nature of the JV strings as opposed to the point-like pancake vortices.

IV. CONCLUSIONS

The application of the renormalization group formulation of the BKT theory to layered superconductors has been investigated in detail. This program has successfully been carried out for a layered system without
Josephson coupling. It has been found that the transition temperature $T_c$ is not affected by the presence of other layers and that the correction to the stiffness constant $K$ is significant only in a inaccessibly narrow temperature range below $T_c$ for typical HTSC’s, but may be observable for layered structures made of conventional superconductors. The opposite limiting case of very strong Josephson coupling has also been investigated. A significant upward shift of the transition temperature has been found, but even using approximations valid only for unphysically strong Josephson coupling the shift is still too small to account for the difference in the transition temperatures between Bi-2212 bulk and thin film samples.

Finally we have shown that the renormalization group approach fails for intermediate Josephson coupling. The reason for this failure has been argued to lie in the one-dimensional nature of the Josephson vortex strings. One would welcome a renormalization group theory for this system, but if and how it may be constructed remains a question for the future.

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APPENDIX A:

As discussed in Sec. [11] we have to integrate the expression $\exp(-\beta \kappa |r_i - r_0^i| - \beta \kappa |r_i^j - r_j^i|)$ over $r_i$ and $r_j$. The integral over $r_i$ is given by

$$I_1 = \int_{\tau \leq |r_i - r_j| < \tau + d\tau} d^2 r_i \exp \left( -\beta \kappa |r_i - r_0^i| - \beta \kappa \tau - \beta \kappa |r_i^j - r_j^i| + \beta \kappa |r_i^j - r_0^i| \right)$$

$$= \tau d\tau \exp \left( -\beta \kappa \tau - \beta \kappa |r_j - r_0^j| + \beta \kappa |r_i^j - r_0^i| \right)$$

$$\times \int_{0}^{2\pi} d\theta \exp \left( -\beta \kappa |r_j - r_0^j| \sqrt{1 + 2 \frac{\tau}{|r_j - r_0^j|} \cos \theta} \right).$$

Expansion for small $\tau/|r_j - r_0^j|$ yields

$$I_1 \approx \tau d\tau \exp \left( -\beta \kappa \tau - \beta \kappa |r_j - r_0^j| + \beta \kappa |r_i^j - r_0^i| \right) \int_{0}^{2\pi} d\theta \exp \left[ -\beta \kappa |r_j - r_0^j| \left( 1 + \frac{\tau}{|r_j - r_0^j|} \cos \theta \right) \right]$$

$$= 2\pi \tau d\tau \exp \left( -\beta \kappa |r_j - r_0^j| - \beta \kappa \tau - \beta \kappa |r_j - r_0^j| + \beta \kappa |r_i^j - r_0^i| \right) I_0(\beta \kappa),$$

and integration over $r_j$ finally gives

$$\int_{r_j} d^2 r_j \tilde{I}_1 \approx 2\pi \tau d\tau \exp \left( -\beta \kappa \tau + \beta \kappa |r_i^j - r_0^j| \right) I_0(\beta \kappa) |r_i^j - r_0^j|^2 K_2 \left( \beta \kappa |r_i^j - r_0^j| \right).$$

This is the result stated above.
FIG. 1. The trajectories of Eqs. (2.24) and (2.28), obtained by numerical integration for $z(0) = 0.03$ and several values of $K(0)$. The flow is to the left.

FIG. 2. The three distinct configuration of a small vortex-antivortex pair with respect to the JV’s. The arrows denote pancake vortices and the thick gray lines JV’s. See text.

FIG. 3. The trajectories for a system with strong Josephson coupling, but vanishing electromagnetic coupling between the layers. The flow is to the left. The dashed curve would be the critical trajectory in the original BKT case.

FIG. 4. Definition of vortex coordinates. The circles denote pancake vortices and the dashed line is a JV. The arrows point in the direction of the magnetic field within the JV.