Quantum general relativity and the classification of smooth manifolds

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May 17, 2004

Abstract

The gauge symmetry of classical general relativity under space-time diffeomorphisms implies that any path integral quantization which can be interpreted as a sum over space-time geometries, gives rise to a formal invariant of smooth manifolds. This is an opportunity to review results on the classification of smooth, piecewise-linear and topological manifolds. It turns out that differential topology distinguishes the space-time dimension $d = 3 + 1$ from any other lower or higher dimension and relates the sought-after path integral quantization of general relativity in $d = 3 + 1$ with an open problem in topology, namely to construct non-trivial invariants of smooth manifolds using their piecewise-linear structure. In any dimension $d \leq 5 + 1$, the classification results provide us with triangulations of space-time which are not merely approximations nor introduce any physical cut-off, but which rather capture the full information about smooth manifolds up to diffeomorphism. Conditions on refinements of these triangulations reveal what replaces block-spin renormalization group transformations in theories with dynamical geometry. The classification results finally suggest that it is space-time dimension rather than absence of gravitons that renders pure gravity in $d = 2 + 1$ a ‘topological’ theory.

PACS: 04.60.-m, 04.60.Pp, 11.30.-j
keywords: General covariance, diffeomorphism, quantum gravity, spin foam model

1 Introduction

Space-time diffeomorphisms form a gauge symmetry of classical general relativity. This is an immediate consequence of the fact that the theory can be formulated in a coordinate-free fashion and that the classical observables are able to probe only the coordinate-independent aspects of space-time physics, i.e. they probe the ‘geometry’ of space-time which always means ‘geometry up to diffeomorphism’. The classical histories of the gravitational field in the covariant Lagrangian language are therefore the equivalence classes of space-time geometries modulo diffeomorphism.

In the present article, we consider path integral quantizations of classical general relativity in $d$ space-time dimensions in which the path integral is a sum over the histories of the gravitational field. For any given smooth $d$-dimensional space-time manifold $M$ with boundary...
\( \partial M \), the path integral \([1]\) with suitable boundary conditions is supposed to yield transition amplitudes between quantum states (equation (3.2) below). The states correspond to wave functionals on a suitable set of configurations that represent 3-geometries on the boundary \( \partial M \). The only background structure involved is the differentiable structure of \( M \).

It is known that path integrals of this type are closely related to Topological Quantum Field Theories (TQFTs) \([2,3]\). In order to avoid possible misunderstandings of this connection right in the beginning, we stress that in the case of general relativity in \( d = 3 + 1 \), there is no reason to require that the vector spaces [or modules] in the axioms of \([2]\) be finite-dimensional \([\text{finitely generated}]\). In fact, it is believed that TQFTs with finite-dimensional vector spaces would be insufficient and unable to capture the propagating modes of general relativity in \( d = 3 + 1 \). A similar argument is thought to apply to general relativity in \( d = 2 + 1 \) if coupled to certain matter, for example, to a scalar field.

We also stress that in the literature, the letter ‘T’ in TQFT does not necessarily refer to topological manifolds. In fact, the entire formalism of TQFTs is usually set up in the framework of smooth manifolds \([2]\), and unless \( d \leq 2 + 1 \), the structure of smooth manifolds is in general rather different from that of topological manifolds. One should therefore distinguish smooth from topological manifolds and relate the path integral of general relativity to a TQFT that uses smooth manifolds. We call such a theory a \( C^{\infty}\)-QFT. In contrast to the smooth case, we will subsequently use the term \( C^0\)-QFT for a TQFT that refers to topological manifolds. One application of the connection of general relativity with \( C^{\infty}\)-QFTs is that the partition function which can be computed from the path integral, forms \((\text{at least formally})\) an invariant of smooth manifolds \([2,3]\).

Results on the classification of topological, piecewise-linear and smooth manifolds which we review in this article, can then be used in order to narrow down some properties of the path integral. In dimension \( d \leq 2 + 1 \), smooth manifolds up to diffeomorphism are already characterized by their underlying topological manifolds up to homeomorphism. This means that in these dimensions, the \( C^{\infty}\)-QFT of general relativity is in fact a \( C^0\)-QFT which makes the colloquial assertion precise that pure general relativity in \( d = 2 + 1 \) is a ‘topological’ theory. The same is no longer true in \( d = 3 + 1 \). A given topological 4-manifold can rather admit many inequivalent differentiable structures so that in \( d = 3 + 1 \), there is a highly non-trivial difference between \( C^{\infty}\)-QFTs and \( C^0\)-QFTs. The invariants of Donaldson \([4]\) or Seiberg–Witten \([5]\) can indeed be understood as partition functions of \((\text{generalized})\) \( C^{\infty}\)-QFTs \([2]\) which are sharp enough to detect the inequivalence of differentiable structures on the same underlying topological manifold. The partition function of quantum general relativity in \( d = 3 + 1 \), if it can indeed be constructed, will offer the same potential. This relationship is the main theme of the present article.

The special role of space-time dimension \( d = 3 + 1 \) in differential topology is summarized by the following result.

**Theorem 1.1.** Let \( M \) be a compact topological \( d \)-manifold, \( d \in \mathbb{N} \) (without boundary if \( d = 5 \)). If \( M \) admits an infinite number of pairwise inequivalent differentiable structures, then \( d = 4 \).

This is a corollary of several theorems by various authors. We explain in this article why this result is related to the search for a quantum theory of general relativity.

Since in \( d \geq 3 + 1 \), smooth manifolds up to diffeomorphism are in general no longer characterized by their underlying topological manifolds up to homeomorphism, general relativity is no longer related to a \( C^0\)-QFT. There is, however, another classification result that is still
applicable in any $d \leq 5+1$: smooth manifolds up to diffeomorphism are characterized by their Whitehead triangulations up to equivalence (PL-isomorphism). We explain these concepts in greater detail below. They provide us with a very special type of triangulations that can be used in order to discretize smooth$^1$ manifolds in a way which is not merely an approximation nor introduces a physical cut-off, but which rather captures the full information about the equivalence class of differentiable structures.

This suggests that the path integral of general relativity in $d \leq 5 + 1$ admits a discrete formulation on such triangulations. General relativity in $d \leq 5 + 1$ is therefore related to what we call a PL-QFT, i.e. a TQFT based of piecewise-linear manifolds. In particular, a path integral quantization of general relativity in $d \leq 5 + 1$ is related to the construction of invariants of piecewise-linear manifolds. From the classification results, we will see that this is most interesting and in fact an unsolved problem in topology, precisely if $d = 3 + 1$. It is the decision to take the diffeomorphism gauge symmetry seriously which singles out $d = 3 + 1$ this way.

The diffeomorphism invariance of the classical observables then implies in the language of the triangulations that all physical quantities computed from the path integral, are independent of which triangulation is chosen. The discrete formulation on some particular triangulation therefore amounts to a complete fixing of the gauge freedom under space-time diffeomorphisms. The relevant triangulations can furthermore be characterized by abstract combinatorial data, and the condition of equivalence of triangulations can be stated as a local criterion, in terms of so-called Pachner moves. ‘Local’ here means that only a few neighbouring simplices of the triangulation are involved in each step. A comparison of Pachner moves with the block-spin or coarse graining renormalization group transformations in Wilson’s language reveals what renormalization means for theories with dynamical geometry for which there exists no a priori background geometry with which we could compare the dynamical scale of the theory.

We will finally see that the absence of propagating solutions to the classical field equations, for example in pure general relativity in $d = 2 + 1$, is related

1. neither to the question of whether the path integral corresponds to a $C^0$-QFT (as opposed to a $C^\infty$-QFT),
2. nor to the question of whether the vector spaces of this $C^0$-QFT or $C^\infty$-QFT are finite-dimensional,
3. nor to the question of whether the theory admits a triangulation independent discretization.

The present article is structured as follows. In Section 2 we review the classical theory and its gauge symmetry. The formal properties of path integrals and their connection with $C^\infty$-QFTs and manifold invariants are summarized in Section 3. In Section 4 we then compile the relevant results on the classification of the various types of manifolds and discuss their physical significance. In Section 5 we sketch the special case of $d = 2 + 1$. Section 6 finally contains speculations on the coincidence of open problems in physics and mathematics and on how to narrow down the path integral in $d = 3 + 1$. We try to make this article self-contained by including a rather extensive Appendix which contains all relevant definitions from topology.

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1. This might be unexpected at first sight, but in generic dimension $d \leq 5 + 1$, it is smooth rather than topological manifolds that correspond to triangulations in this way.
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\[ \frac{\partial^2}{\partial x^2} \phi^{-1} \]

\[ U_1 \cap U_2 \]

\[ \varphi_2 \circ \varphi_1^{-1} \]

Figure 1: A manifold \( M \) with two coordinate systems \((U_1, \varphi_1)\) and \((U_2, \varphi_2)\) whose patches have some overlap \( U_{12} = U_1 \cap U_2 \), and the corresponding transition function \( \varphi_2 \circ \varphi_1^{-1} \).

2 The classical theory

This section contains merely review material: Subsection 2.1 on smooth manifolds, Subsection 2.2 on the first order formulation of general relativity, Subsection 2.3 on its gauge symmetries, Subsection 2.4 on the role of differential topology in the study of general relativity, and Subsection 2.5 on space-time diffeomorphisms. Each of these subsections can be safely skipped. We nevertheless include the material here in order to fix the terminology, in particular in order to resolve the various misunderstandings that can arise in the discussion of diffeomorphisms in general relativity, just because there seems to exist no standardized terminology in the literature.

2.1 Smooth manifolds

We first review some basic facts about smooth manifolds. Detailed definitions can be found in the Appendix.

A \( d \)-dimensional manifold \( M \) is a suitable topological space which is covered by coordinate systems (Figure 1). A coordinate system \((U_i, \varphi_i)\) is a patch \( U_i \subseteq M \) together with a one-to-one map \( \varphi_i: U_i \rightarrow \varphi_i(U_i) \subseteq \mathbb{R}^d \) onto some subset of the standard space \( \mathbb{R}^d \). We can use the coordinate maps \( \varphi_i \) in order to assign \( d \) real coordinates \( \varphi_i^\mu(p), \mu = 0, \ldots, d - 1 \), with each point \( p \in U_i \). The coordinates take values in the subset \( \varphi_i(U_i) \subseteq \mathbb{R}^d \).

As soon as two coordinate systems \((U_i, \varphi_i), (U_j, \varphi_j)\) have a non-empty overlap \( U_{ij} = U_i \cap U_j \neq \emptyset \), we can change the coordinates by means of the transition function \( \varphi_{ji} := \varphi_j \circ \varphi_i^{-1}: \varphi_i(U_{ij}) \rightarrow \varphi_j(U_{ij}) \) which is a map from a subset of \( \mathbb{R}^d \) to \( \mathbb{R}^d \).

A scalar function \( \alpha: M \rightarrow \mathbb{R} \) can be described in any of the coordinate systems in terms of the functions \( \alpha \circ \varphi_i^{-1}: \varphi_i(U_i) \rightarrow \mathbb{R} \) from some subset of \( \mathbb{R}^d \) to \( \mathbb{R} \).
If we require all coordinate maps $\varphi_i$ to be homeomorphisms, i.e. continuous with continuous inverse, we obtain the definition of a topological manifold. In this case, the transition functions are homeomorphisms, too, and it makes sense to study continuous functions $\alpha: M \to \mathbb{R}$ and homeomorphisms $f: M \to N$ between manifolds.

If we want to talk about differential equations on a space-time manifold, we have to know in addition how to differentiate functions and therefore have to impose additional structure. This can be accomplished by restricting the transition functions from homeomorphisms to some special subclass of functions. A smooth manifold, for example, is a topological manifold for which all transition functions $\varphi_{ji}$ and their inverses are $C^\infty$, i.e. all partial derivatives of all orders exist and are continuous. A covering of $M$ with such coordinate systems is known as a differentiable structure. A continuous function $\alpha: M \to \mathbb{R}$ is called smooth if all its coordinate representations $\alpha \circ \varphi_i^{-1}: \varphi_i(U_i) \to \mathbb{R}$ are $C^\infty$-functions. A homeomorphism $f: M \to N$ between smooth manifolds is called a diffeomorphism if all coordinate representations $\psi_j \circ f \circ \varphi_i^{-1}: \varphi_i(U_i) \to \psi_j(V_j)$ and their inverses are $C^\infty$. Here $(V_j, \psi_j)$ denotes the coordinate systems of $N$.

A substantial part of the present article is concerned with the various structures one can impose on topological manifolds by restricting the transition functions, and with the question of how to compare these structures.

2.2 Two pictures for general relativity

For background on classical general relativity, we refer to the textbooks, for example [6, 7]. We are interested in general relativity in $d$-dimensional space-time and, for simplicity, often restrict ourselves to pure gravity without matter. Space-time is given by a smooth oriented $d$-manifold $M$.

**Second order metric picture.** We denote the (smooth) metric tensor by $g_{\mu\nu}$ from which one can calculate the unique metric compatible and torsion-free connection $\nabla_\mu$, its Riemann curvature tensor $R^{\rho}_{\nu\mu\sigma}$, the Ricci tensor $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$ and the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$. The Einstein–Hilbert action (without matter) reads,

$$S[g] = \frac{1}{16\pi G} \int_M (R - 2\Lambda)\sqrt{\det g} \, dx^0 \wedge \cdots \wedge dx^{n-1}, \quad (2.1)$$

where $G$ is the gravitational and $\Lambda$ the cosmological constant. Variation with respect to the metric yields the Einstein equations (without matter),

$$R_{\mu\nu} - \frac{1}{2}(R - 2\Lambda)g_{\mu\nu} = 0, \quad (2.2)$$

which are second order differential equations for the metric tensor $g_{\mu\nu}$.

**First order Hilbert–Palatini picture.** We sometimes contrast this second order metric picture with the first order Hilbert–Palatini formulation. For a recent review, see, for example [8]. We therefore choose local orthonormal basis vectors $e_I \in T_p M$, $I = 0, \ldots, d-1$, of the tangent spaces at each point $p \in M$, i.e. $e^I_I e^I_J g_{\mu\nu} = \eta_{IJ}$ where $\eta = \text{diag}(-1, 1, \ldots, 1)$ denotes the standard Lorentzian bilinear form. The dual basis $(e^I)$ of 1-forms $e^I = e^I_\mu dx^\mu \in T^*_p M$, i.e. $e^I(e_J) = \delta^I_J$, is called the coframe field. Denote by $A$ an $SO(1, d-1)$-connection whose
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curvature 2-form we write as an \( \mathfrak{so}(1,d-1) \)-valued 2-form
\[ F^I_J = dA^I_J + A^I_K \wedge A^K_J \] on \( M \).

Classical general relativity can be formulated as a first order theory with the Hilbert–Palatini action,
\[ S[A,e] = \frac{1}{16(d-2)! \pi G} \int_M \varepsilon_{IJK_1 \cdots K_{d-2}} e^{K_1} \wedge \cdots \wedge e^{K_{d-2}} \wedge \left( F^{IJ} - \frac{2\Lambda}{d(d-1)} e^I \wedge e^J \right), \] (2.3)

where \( F^{IJ} = F^I_K e^K_J \). Variation with respect to the connection \( A \) and the cotetrad \( e \) yields the field equations,
\[ 0 = \varepsilon^{IJK_1 \cdots K_{d-3}} e^{K_1} \wedge \cdots \wedge e^{K_{d-3}} \wedge \left( F^{IJ} - \frac{2\Lambda}{(d-1)(d-2)} e^I \wedge e^J \right), \] (2.4a)

\[ 0 = (\delta^I_L d + 2 A^I_L \wedge) \varepsilon^{IJK_1 \cdots K_{d-2}} e^{K_1} \wedge \cdots \wedge e^{K_{d-2}}, \] (2.4b)

for all \( I,J,P \). These are coupled first order differential equations for \( A \) and \( e \). Whenever the cotetrad is non-degenerate, the first of these equations is equivalent to (2.2) for the metric tensor
\[ g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}, \] (2.5)

while the second equation states that the connection is torsion-free. Note that any \( SO(1,d-1) \)-connection is always metric compatible with respect to (2.5).

One reason for introducing the Hilbert–Palatini formulation is to demonstrate that we can easily adopt a point of view in which the metric tensor is not a fundamental field of the classical theory. This illustrates once more that a generic smooth manifold \( M \) is the only input of the theory. The fields \( A \) and \( e \) and, as a consequence, the metric \( g \) are determined by the dynamics of the theory.

2.3 Gauge symmetries

Let us review the gauge symmetries of the classical theory in the first order picture.

Local Lorentz symmetry. The choice of local orthonormal bases \( (e_I) \) is unique only up to a local Lorentz transformation. Such transformations can be expressed in a coordinate system \( U \subseteq M, \varphi: U \to \mathbb{R}^d \) by some smooth Lorentz-group valued function \( \Lambda: U \to SO(1,d-1) \). We write \( x = \varphi(p), p \in U \), and denote the old variables by \( (A,e) \) and the transformed ones by \( (\tilde{A},\tilde{e}) \),
\[ \tilde{e}^I_\mu(x) = \Lambda^I_J(x) e^J_\mu(x), \] (2.6a)
\[ \tilde{A}^I_\mu J(x) = \Lambda^I_K(x) A^K_L(x) \Lambda^L_J(x) + \Lambda^I_K(x) \frac{\partial}{\partial x^\mu} \Lambda^J_K(x), \] (2.6b)

where we write \( \Lambda^I_K = \eta_{IJK} \Lambda^J_L \). The metric (2.5) is obviously invariant under (2.6a).

Space-time diffeomorphisms. Any two geometries \((M,g)\) and \((M,g')\) of \( M \) are physically identical as soon as they are related by a space-time diffeomorphism \( f: M \to M \), i.e. \( g' = f^*g \).

Let us describe the action of some diffeomorphism \( f: M \to M \) on the various fields in detail. Consider a smooth function \( \alpha \), a smooth vector field \( X = X^\mu \partial_\mu \), a smooth 1-form
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\[ \omega = \omega_\mu \, dx^\mu, \] and the coframe field \[ e^I_\mu \, dx^\mu. \] For \( p \in M, \) choose a coordinate system \((U, \varphi)\) such that \( p \in U \) and write \( x = \varphi(p) \). The diffeomorphism acts in coordinates as follows,

\[
\begin{align*}
(f^*\alpha)(x) &= \alpha(f(x)), & (2.7a) \\
(f^*X^\mu)(x) &= (Df^{-1}(f(x)))^\mu_\nu \, X^\nu(f(x)), & (2.7b) \\
(f^*\omega_\mu)(x) &= \omega_\nu(f(x)) \, (Df(x))^{\nu}_{\mu}, & (2.7c) \\
(f^*e^I_\mu)(x) &= e^I_\nu(f(x)) \, (Df(x))^{\nu}_{\mu}, & (2.7d) \\
(f^*A)^I_{\mu J}(x) &= A^I_{\nu J}(f(x)) \, (Df(x))^{\nu}_{\mu}, & (2.7e)
\end{align*}
\]

where \((Df(x))^{\mu}_{\nu} := \partial f^\mu(x) / \partial x^\nu\) denotes the Jacobi matrix of \( f \), written in coordinates, too. These rules also determine the action of the diffeomorphism on higher rank tensors\(^2\).

2.4 Differential topology versus differential geometry

The problem of classical general relativity can be summarized as follows.

**Second order picture.** Given a smooth oriented \( d \)-manifold \( M \) with boundary \( \partial M \), find a smooth metric tensor \( g_{\mu\nu} \) that satisfies the Einstein equations (2.2) in the interior of \( M \) and suitable boundary conditions on \( \partial M \). Study existence and uniqueness of the solutions. Any two solutions \( g_{\mu\nu}, g'_{\mu\nu} \) that are related by a space-time diffeomorphism \( f: M \to M \), i.e. \( g' = f^*g \), are physically identical.

**First order picture.** Given a smooth oriented \( d \)-manifold \( M \) with boundary \( \partial M \), find a smooth \( SO(1,d-1) \)-connection \( A \) and a smooth non-degenerate coframe field \( e \) that satisfy the first order field equations (2.4) in the interior of \( M \) and suitable boundary conditions on \( \partial M \). Study existence and uniqueness of the solutions. Any two solutions \( (A,e), (A',e') \) that are related by a local Lorentz transformation (2.6) or by a space-time diffeomorphism (2.7), are physically identical.

**Interpretation.** We stress that the input for classical general relativity is a smooth manifold, i.e. a topological manifold with a differentiable structure. We are therefore in the realm of differential topology. This has to be contrasted, for example, with the non-generally relativistic treatment of Yang–Mills theory in an *a priori* fixed space-time geometry. Such a background geometry is described by a Riemannian manifold \((M,g)\), i.e. by a smooth manifold \( M \) with a metric tensor \( g_{\mu\nu} \), which renders such a theory a problem of differential geometry rather than differential topology.

**Boundary conditions.** We do not comment on how the boundary and/or initial conditions affect the existence and uniqueness of the classical solutions. For simplicity, we use the following specification of boundary data in the first order formulation. In the variational principle, we fix the connection \( A|_{\partial M} \) at the boundary \( \partial M \) and assume that the variations \( \delta A \) and \( \delta e \) are supported only in the interior of \( M \). The field equations (2.4) in the interior of \( M \) can then be derived without additional boundary terms for the action.

\(^2\)We have defined the diffeomorphism action on \( e^I_\mu \) so that it acts only on the cotangent index \( \mu \), but not on the internal index \( I \).
This choice is made entirely for convenience. It is known to be satisfactory in the toy model of pure gravity in \( d = 2 + 1 \), but might have to be revised in order to find the correct path integral in \( d = 3 + 1 \).

### 2.5 Gauge symmetries and quantization

Assuming that there exists a quantization of general relativity, will the quantum theory respect the classical gauge symmetries, or not? In order to get some more insight into this question, let us recall why general relativity possesses such symmetries in the first place. The answer is quite standard \([7]\), but nevertheless worth spelling out in detail, in particular because this symmetry plays an important role in the remainder of the present article.

The entire framework of smooth manifolds has been set up in order to describe physical quantities by mathematical constructions that have a meaning independently of the coordinate systems which we choose in order to represent them.

Consider, for example, some physical field which is given by a smooth scalar function \( \alpha : M \to \mathbb{R} \). There are various possible coordinate systems \((U_i, \varphi_i)\) in order to represent this scalar field. The reader may think of flat space-time with Cartesian or polar coordinates which give rise to coordinate representations of the field, \( \alpha \circ \varphi_j^{-1} : \varphi(U_j) \to \mathbb{R} \), and whose transition functions \( \varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \subseteq \mathbb{R}^d \to \varphi_j(U_i \cap U_j) \subseteq \mathbb{R}^d \) prescribe how to change the coordinates.

One might now be tempted to think that the physical reality is described literally by the set \( M \) with its points \( p, q \in M \) which would symbolize space-time events. Starting from this ‘reality’, one would then construct various coordinate systems \( \varphi_i : U_i \subseteq M \to \mathbb{R}^d \) in order to represent this ‘reality’ in terms of coordinates. The transition functions guarantee that iterated coordinate changes always give consistent results.

Certainly, the collection of all the ranges \( \varphi_j(U_j) \subseteq \mathbb{R}^d \) of the coordinate systems together with the transition functions \( \varphi_j \circ \varphi_i^{-1} \) and the coordinate representations \( \alpha \circ \varphi_j^{-1} \) of the scalar function constitute the maximum information about our ‘reality’ that can be written down by the experimenters (who always use coordinate systems). This raises the question of whether, conversely, we can reconstruct \( M \) together with its points \( p, q \in M \) and the function \( \alpha : M \to \mathbb{R} \) from such a collection of coordinate ranges, transition functions and coordinate representations. The well-known answer to this question is ‘no’ \([7]\). One can reconstruct\(^3\) \( M \) and \( \alpha \) only up to diffeomorphism. The actual reality is therefore not given literally by the set \( M \) and the function \( \alpha : M \to \mathbb{R} \), but rather by equivalence classes \( [(M, \alpha)] \) modulo diffeomorphism. Here \((M, \alpha)\) and \((M', \alpha')\) are considered equivalent if and only if there exists a diffeomorphism \( f : M \to M' \) such that \( \alpha' = f^* \alpha \). We have called this a gauge symmetry simply because there are several different mathematical ways \((M, \alpha), (M', \alpha'), \ldots\) of specifying a single physical history.

If we now attempt to quantize such a theory, we have to remember that the classical configurations (histories) are the equivalence classes \( [(M, \alpha)] \) rather than the particular representatives \((M, \alpha)\). The above argument indicates that the quantum theory should not violate this type of gauge symmetry because otherwise the outcome of quantum experiments would depend, very roughly speaking, on whether the classical observer who performs the measurement, uses Cartesian or polar coordinates in order to write down the observations.

\(^3\)A closely related result in mathematics is the fact that fibre-bundles can be reconstructed from their transition functions, but only up to bundle automorphisms, i.e. compositions of local frame transformations and diffeomorphisms.
As classical general relativity is formulated in the language of smooth manifolds, it is automatically compatible with the action of space-time diffeomorphisms $f: \mathcal{M} \rightarrow \mathcal{M}$ and therefore well-defined as a theory on the equivalence classes of geometries. It is thus safe to choose a particular representative of the geometry whenever convenient and to perform the relevant calculations for the representative.

When one tries to apply a quantization procedure to the theory, the same is no longer obvious. For many quantization schemes, one is forced to choose representatives and to write down all the physical fields as particular functions $\alpha: \mathcal{M} \rightarrow \mathbb{R}$, i.e. as fields on space-time. The study of the equivalence classes and the central question of whether the quantization scheme was indeed compatible with space-time diffeomorphisms, are often postponed.

In the subsequent sections, we insist on working with the equivalence classes. This leads us in particular to the question of when two given smooth manifolds are diffeomorphic.

What is the difference between general relativity and other theories such as the non-generally relativistic treatment of Yang–Mills theory? That theory, too, can be formulated in a coordinate-free fashion and therefore shares the same type of gauge symmetry, i.e. any two representations $(\mathcal{M}, g, A, \psi)$ and $(\mathcal{M}', g', A', \psi')$ are physically equivalent if and only if they are related by a diffeomorphism $f: \mathcal{M} \rightarrow \mathcal{M}'$, i.e. $g' = f^*g$, $A' = f^*A$, and $\psi' = f^*\psi$. Here $(\mathcal{M}, g)$ is the underlying Riemannian manifold and $A$ and $\psi$ denote the additional fields, in the Yang–Mills case a connection and some charged fermion fields. Up to this point, there is no difference compared with general relativity at all. Even non-relativistic Newtonian mechanics can be written down in a coordinate-free fashion and shares all the properties mentioned here, see, for example p. 300 of [6].

In the non-generally relativistic treatment of field theories, for example of Yang–Mills theory, the space-time metric $g$ is, however, non-dynamical. This allows us to fix a special pair $(\mathcal{M}, g)$ forever and to study the isometries of the Riemannian manifold $(\mathcal{M}, g)$, i.e. those diffeomorphisms $f: \mathcal{M} \rightarrow \mathcal{M}$ for which $f^*g = g$. In the common non-generally relativistic terminology, these isometries are often called active transformations because they ‘actively’ move the fields $A$ and $\psi$ relative to the fixed background $(\mathcal{M}, g)$. These active transformations do not form a gauge symmetry because one can actually measure whether some object in the laboratory has been translated or not. Of course, the laboratory is here viewed as fixed with respect to the background $(\mathcal{M}, g)$. If $(\mathcal{M}, g)$ does not have enough isometries, one may resort to the study of (local) Killing vector fields, but the interpretation would not change.

The notion of active transformation requires an a priori decomposition of the physical fields into those that form the background, usually the space-time metric, which is non-dynamical, plus other fields that live on this background and which are treated as dynamical. The term passive transformation usually refers to the coordinate changes via transition functions, for example to the transition from Cartesian to polar coordinates in some flat geometry. The space-time diffeomorphisms $f: \mathcal{M} \rightarrow \mathcal{M}'$ that relate two representatives $(\mathcal{M}, g, A, \psi)$ and $(\mathcal{M}', g', A', \psi')$ of the same physical configuration and which always form a gauge symmetry of the theory, do not have any special name in the jargon of non-generally relativistic physics.

The standard treatment of non-generally relativistic field theories completely focuses on the active transformations and does not discuss the generic diffeomorphism gauge symmetry which is nevertheless present. In full general relativity, however, the metric is considered

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\footnote{Strictly speaking, the generic transformations are diffeomorphisms $f: \mathcal{M} \rightarrow \mathcal{M}'$, but there are various diffeomorphisms $f_1, f_2, \ldots: \mathcal{M} \rightarrow \mathcal{M}'$ between the same pair of manifolds, and we can use the first, $f_1$, in order to identify $\mathcal{M}' \equiv \mathcal{M}$ and then obtain a diffeomorphism $\mathcal{M} \rightarrow \mathcal{M}$ from the second, etc. One can therefore say that the symmetry is given by space-time diffeomorphisms $\mathcal{M} \rightarrow \mathcal{M}$.}
dynamical so that the separation into a Riemannian manifold plus fields that live is this geometry, is no longer available. The active transformations of non-generally relativistic physics therefore have no correspondence in the full theory of general relativity.

3 Path integrals

In this section, we review the formal properties of path integrals for general relativity (Subsection 3.1) and exhibit their close relationship with the axioms of TQFT, more precisely $C^\infty$-QFT [2] (Subsection 3.2). The connection of $C^\infty$-QFT with theories that have diffeomorphisms as a gauge symmetry, is already familiar from Witten’s work on Chern–Simons theory and knot invariants [3] although in three dimensions, the equivalence classes of smooth manifolds up to diffeomorphism are in one-to-one correspondence with those of topological manifolds up to homeomorphism. Barrett, Crane and Baez–Dolan [9–11] have explicitly proposed $C^\infty$-QFT as a framework for the quantization of general relativity in $d=3+1$. We here recall the central ideas of this connection and outline its relationship with the classification of manifolds (Subsection 3.3). In the subsequent sections, we explain why this framework is highly dimension-dependent and why the diffeomorphism gauge symmetry alone is already sufficient to single out $d = 3 + 1$. We finally speculate about an extension of the framework of $C^\infty$-QFT in order to better deal with the notion of time in general relativity (Subsection 3.4).

3.1 Formal properties

Path integral quantizations of general relativity, see, for example [1], are supposed to share the following formal properties. The subsequent discussion is purely heuristic.

Hilbert spaces. We associate Hilbert spaces $\mathcal{H}(\Sigma)$ with closed $(d-1)$-manifolds $\Sigma$. Such a $(d-1)$-manifold $\Sigma$ represents, for example, a space-like hyper-surface on which the canonical variables of the theory are defined. Choosing a polarization, we have a position representation $\mathcal{H} = L^2(\mathcal{A})$ with suitable wave functionals on some set $\mathcal{A}$ of canonical coordinates. For definiteness, let us imagine that we choose a connection representation so that $\mathcal{A}$ denotes the set of all connections $A|_{\Sigma}$.

Transition amplitudes. The path integral is supposed to describe transition amplitudes from $\mathcal{H}(\Sigma_1)$ to $\mathcal{H}(\Sigma_2)$ for hyper-surfaces $\Sigma_1$ and $\Sigma_2$, by summing over all histories of the gravitational field, i.e. over all space-time geometries, that interpolate between the ‘initial’
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\[ M = M_1 \cup \Sigma_2 \cup M_2. \]

Figure 3: Two \(d\)-manifolds \(M_1\), \(M_2\) with boundaries \(\partial M_1 = \Sigma_1^* \cup \Sigma_2\) and \(\partial M_2 = \Sigma_2^* \cup \Sigma_3\) glued together along their common boundary component \(\Sigma_2\). The result is \(M = M_1 \cup \Sigma_2 \cup M_2\).

\[ \Sigma_1 \text{ and the ‘final’ hyper-surface } \Sigma_2. \] We therefore choose a \(d\)-manifold \(M\) whose boundary \(\partial M = \Sigma_1 \cup \Sigma_2\) is the disjoint union of \(\Sigma_1\) and \(\Sigma_2\) in order to support these histories (Figure 2).

For connection eigenstates \(|A|_{\Sigma_j} \rangle \in \mathcal{H}(\Sigma_j), j = 1, 2\), the transition amplitudes are given by the (suitably normalized) path integral,

\[ \langle A|_{\Sigma_2}| T(M) |A|_{\Sigma_1} \rangle = \int_{A|_{\Sigma_1}, A|_{\Sigma_2}} DADe \exp\left(\frac{i}{\hbar} S[A, e]\right). \] (3.1)

These are the matrix elements of the linear transition map \(T(M): \mathcal{H}(\Sigma_1) \rightarrow \mathcal{H}(\Sigma_2)\). The path integration \(\int DADe\) comprises the integration over all connections \(A\) compatible with the boundary conditions, \(A|_{\Sigma_1}\) and \(A|_{\Sigma_2}\), and over all coframe fields \(e\).

Notice that the connection representation for the Hilbert spaces \(\mathcal{H}(\Sigma)\) is compatible with our choice of boundary conditions made above, namely to fix the connection \(A|_{\partial M}\) on the boundary. We work with ‘real’ time so that there is an \(i\) in the exponent. For generic states \(|\psi(A|_{\Sigma_j})\rangle \in \mathcal{H}(\Sigma_j), j = 1, 2\), the path integral reads,

\[ \langle \psi(A|_{\Sigma_2})| T(M)| \psi(A|_{\Sigma_1}) \rangle = \int DADe \overline{\psi(A|_{\Sigma_2})} \psi(A|_{\Sigma_1}) \exp\left(\frac{i}{\hbar} S[A, e]\right), \] (3.2)

where the integration over connections \(A\) is now unrestricted.

**Gauge symmetry.** All manifolds and functions are assumed to be smooth, and everything should be specified only ‘up to local Lorentz transformations’ and ‘up to diffeomorphisms’ in a suitable way in order to implement the gauge symmetry of general relativity. The Hilbert spaces therefore represent states of spatial \((d - 1)\)-geometries up to diffeomorphism, etc.

### 3.2 Axioms of \(C^\infty\)-QFT

The framework sketched above is remarkably similar to the axiomatic definition of \(C^\infty\)-QFT [2]. The main advantage of axiomatic \(C^\infty\)-QFT over Subsection 3.1, and almost the only difference, is that the diffeomorphism gauge symmetry is carefully taken into account. We give here a slightly simplified account of the original axioms. Note that the condition of
finite-dimensionality guarantees that everything is well defined, but may finally have to be
relaxed for general relativity. In the following, all manifolds are smooth and oriented, and all
diffeomorphisms are orientation preserving. A $d$-dimensional $C^\infty$-QFT,

(S1) assigns to each closed $(d - 1)$-manifold $\Sigma$ a finite-dimensional complex vector space
$\mathcal{H}(\Sigma)$, and
(S2) assigns to each compact $d$-manifold $M$ with boundary $\partial M$ a vector $T(M) \in \mathcal{H}(\partial M)$,
such that the following axioms (A1)–(A4) hold.

(A1) The vector space of some opposite oriented manifold $\Sigma^*$ is the dual vector space,
$\mathcal{H}(\Sigma^*) = \mathcal{H}(\Sigma)^*$ (Bra-vectors live in the space dual to that of ket-vectors).
(A2) The vector space associated with a disjoint union of $(d - 1)$-manifolds $\Sigma_1, \Sigma_2$ is the
tensor product, $\mathcal{H}(\Sigma_1 \cup \Sigma_2) \cong \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2)$ (The Hilbert space of a composite system
is the tensor product of the spaces of the constituents).
(A3) For each diffeomorphism $f: \Sigma_1 \rightarrow \Sigma_2$ between closed $(d - 1)$-manifolds $\Sigma_j$, there exists a
linear isomorphism $f^*: \mathcal{H}(\Sigma_1) \rightarrow \mathcal{H}(\Sigma_2)$. If $f$ and $g: \Sigma_2 \rightarrow \Sigma_3$ are diffeomorphisms, then
$(g \circ f)^* = g^* \circ f^*$. Furthermore, for each diffeomorphism $f: M_1 \rightarrow M_2$ between compact
d-manifolds $M_j$ with boundary $\partial M_j = \Sigma_j$, it is required that $T(M_2) = f^* (T(M_1))$, where
$f|_{\Sigma_j}: \Sigma_1 \rightarrow \Sigma_2$ denotes the restriction to the boundary (This is the implementation
of the diffeomorphism gauge symmetry).

The axioms (A1) and (A2) imply that any $d$-manifold $M$ whose boundary is a disjoint union of the form $\partial M = \Sigma_1^* \cup \Sigma_2$, is assigned a vector $T(M) \in \mathcal{H}(\Sigma_1)^* \otimes \mathcal{H}(\Sigma_2) \cong \text{Hom}_C(\mathcal{H}(\Sigma_1), \mathcal{H}(\Sigma_2))$, i.e. a linear map $T(M): \mathcal{H}(\Sigma_1) \rightarrow \mathcal{H}(\Sigma_2)$ (These are the desired transition maps (3.2)).

(A4) If compact $d$-manifolds $M_1, M_2$ with boundaries $\partial M_1 = \Sigma_1^* \cup \Sigma_2$ and $\partial M_2 = \Sigma_2^* \cup \Sigma_3$ are
 glued together along their common boundary component $\Sigma_2$ (Figure 3), the resulting
manifold $M = M_2 \cup_{\Sigma_2} M_1$ yields the composition of the transition maps, $T(M_2 \cup_{\Sigma_2} M_1) = T(M_2) \circ T(M_1)$ which is a linear map from $\mathcal{H}(\Sigma_1)$ to $\mathcal{H}(\Sigma_3)$.

Notice that the local Lorentz symmetry was not explicitly mentioned. It is usually taken
into account by an appropriate choice of vector spaces $\mathcal{H}(\Sigma)$. The change of spatial topology
(more precisely of the spatial smooth manifold) is possible in this framework if $\Sigma_1$ and $\Sigma_2$
in Figure 2 are not diffeomorphic. The precise form of this topology change is encoded in
the choice of $M$. The $d$-manifold $M$ itself, however, is always fixed. Generalizations that
include the ‘superposition’ of different $d$-manifolds $M$ are not covered by the correspondence
principle and would therefore require new physical assumptions.

3.3 Invariants of smooth manifolds

The axioms have some interesting consequences.

1. Axiom (A4) applied to the case of disjoint manifolds $M_1$ and $M_2$, i.e. $\partial M_1 = \Sigma_1$, $\partial M_2 = \Sigma_2$, $\Sigma_1 \cap \Sigma_2 = \emptyset$, implies that the associated vector is just the tensor product
of vectors, $T(M_1 \cup M_2) = T(M_1) \otimes T(M_2) \in \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2)$ (State of a system that is
composed from independent constituents).
2. Axiom (A2) applied to $\Sigma_2 = \emptyset$ shows that $\mathcal{H}(\emptyset) = \mathbb{C}$ or otherwise all $\mathcal{H}(\Sigma)$ are null.
3. Axiom (A4) applied to $M_1 = M_2 = \emptyset$ implies that $T(\emptyset) \in \mathcal{H}(\emptyset) = \mathbb{C}$ satisfies $T(\emptyset) = 1$
or otherwise all $T(M) = 0$. 
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4. Axiom (A4) applied to a cylinder $M = \Sigma \times I$ where $\Sigma$ is a closed $(d-1)$-manifold and $I = [0,1]$ the unit interval (Figure 4), shows that the transition map $T(M): \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma)$ is a projection operator, $T(M)^2 = T(M)$. For any $d$-manifold that can be written as a cylinder glued to something else, this projector is effective so that what matters is only its image. Therefore, one often imposes the additional condition that $T(\Sigma \times I) = \text{id}_{\mathcal{H}(\Sigma)}$.

It is well-known [2] that the appearance of these projection operators is closely related to the vanishing of the Hamiltonian, $H = 0$, in the corresponding canonical formulation of the theory. This is in fact a prominent feature of general relativity [1] as soon as the $3+1$ splitting for the Hamiltonian formulation is done with respect to coordinate time. In this sense, both axiomatic $C^\infty$-QFT and canonical general relativity are ‘non-dynamical’ theories. This is in both cases a direct consequence of the diffeomorphism gauge symmetry.

5. For closed $d$-manifolds, $\partial M = \emptyset$, so that $T(M) \in \mathcal{H}(\emptyset) = \mathbb{C}$. Therefore, $T(M)$ is just a number that depends on the diffeomorphism class of the smooth manifold $M$. It is an invariant of smooth manifolds! In analogy with statistical mechanics, this number is called the partition function of $M$ and denoted by $Z(M) := T(M)$.

6. More generally, for a compact $d$-manifold $M$ whose boundary is of the form $\partial M = \Sigma_1 \cup \Sigma_2$ where $\Sigma_1$ and $\Sigma_2$ are diffeomorphic, one can use such a diffeomorphism $f: \Sigma_2 \to \Sigma_1$ in order to glue $M$ to itself at its boundary so that one obtains a closed manifold $M_f$ (Figure 5). In this case, by axiom (A3), there is an isomorphism of vector spaces,
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$f': \mathcal{H}(\Sigma_2) \to \mathcal{H}(\Sigma_1)$ so that the partition function of $M_f$ can be written as the trace,

$$Z(M_f) = \text{tr}_{\mathcal{H}(\Sigma_1)}(f' \circ T(M)). \quad (3.3)$$

In the language of the path integral (3.1), this calculation of the trace corresponds to integrating over all possible boundary conditions on the identified $\Sigma_1 \equiv f \Sigma_2$. If actually $\Sigma_1 = \Sigma_2 = \Sigma$ and $f = \text{id}_{\mathcal{H}(\Sigma)}$,

$$Z(M_{\text{id}_\Sigma}) = \text{tr}_{\mathcal{H}(\Sigma)} T(M) = \int \mathcal{D}A \mathcal{D}e \exp \left( \frac{i}{\hbar} \mathcal{S}[A, e] \right). \quad (3.4)$$

This formula for $Z(M_{\text{id}_\Sigma})$ in terms of the unrestricted path integral (over the manifold $M_{\text{id}_\Sigma}$) motivates the term partition function.

The most important implications are the items (5.) and (6.). It is essentially a consequence of the diffeomorphism gauge symmetry that the partition function $Z(M)$ which can be computed from the path integral, is an invariant of smooth manifolds. Although the partition function itself does not have any physical meaning, it is therefore mathematically very valuable. The physically relevant objects are the matrix elements (3.2). They are more general than just the partition function which is their trace, but for the beginning, let us focus on the partition function. In the subsequent sections, we ask in which space-time dimensions we can expect interesting theories, for example,

- Are smooth manifolds up to diffeomorphism already characterized by simpler structures, for example, by their underlying topological manifolds up to homeomorphism or even up to homotopy equivalence?
- In which space-time dimensions is it possible to compute the partition function more efficiently in a purely combinatorial context using suitable triangulations of the space-time manifold? Which role do these triangulations play in general relativity?

### 3.4 Extensions of the framework of $C^\infty$-QFT

**Dimensionality.** As mentioned before, for general relativity it may be necessary to drop the words ‘finite-dimensional’ from (S1) above. In this case, one has to be more careful with the notion of dual vector space and with the construction of the traces which may become infinite. We illustrate this below in Section 5 for the example of quantum gravity in $d = 2+1$.

**Hermitean structures.** A Hermitean scalar product on the vector spaces $\mathcal{H}(\Sigma)$ gives rise to an isomorphism $\mathcal{H}(\Sigma^*) = \overline{\mathcal{H}(\Sigma)}$ where $\overline{\mathcal{H}(\Sigma)}$ denotes the vector space $\mathcal{H}(\Sigma)$ with the complex conjugate action of the scalars. In this case, it is possible to add the axiom,

(A5) For each $d$-manifold $M$, the vector $T(M) \in \mathcal{H}(\partial M)$ satisfies $T(M^*) = \overline{T(M)}$.

If $\partial M = \Sigma_1^* \cup \Sigma_2$ and $T(M): \mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$ is viewed as a linear map, this axiom relates orientation reversal of space-time with the hermitean adjoint of the transition map, $T(M^*) = (T(M))^\dagger$. 
Unitarity. Assume that the axiom (A5) is satisfied. If the transition map $T(M): \mathcal{H}(\Sigma_1) \rightarrow \mathcal{H}(\Sigma_2)$ for $\partial M = \Sigma_1 \cup \Sigma_2$ was unitary, i.e. $T(M)^\dagger = T(M)^{-1}$, orientation reversal of $M$ would just invert the transition map, $T(M^*) = T(M)^{-1}$. It is tempting to think that orientation reversal was in this way related with time reversal.

Unitarity in quantum theory, however, expresses the conservation of probability with respect to a global time parameter and therefore cannot be expected to hold in general relativity without additional assumptions.

Consider, for example, the path integral (3.1) for the $d$-manifold $M$ of Figure 2 where the boundary data $A|_{\Sigma_1}$ and $A|_{\Sigma_2}$ impose space-like geometries on $\Sigma_1$ and $\Sigma_2$, respectively. The path integral will generically contain histories in which $\Sigma_2$ is in the causal future of $\Sigma_1$ and also some in which $\Sigma_2$ is in the past of $\Sigma_1$, unless we impose additional conditions.

Following the ideas of Oeckl [12], one can consider more general manifolds of the form $M = S \times I$ where $S$ is a compact $(d-1)$-manifold with boundary $\partial S \neq \emptyset$. The boundary $\partial M$ then consists of $\partial M = (S \times \{0\}) \cup (S \times \{1\}) \cup (\partial S \times I)$. The idea is now to impose space-like geometries on $S \times \{0\}$ (initial preparation of the experiment) and $S \times \{1\}$ (final measurement of the outcome) and in addition a time-like geometry on $\partial S \times I$ which represents the clock in the classical laboratory that surrounds the quantum experiment and which ensures that $S \times \{1\}$ is in the future of $S \times \{0\}$.

Notice that this is the situation in which textbook quantum theory makes sense without conceptual extensions and in which it has been confirmed experimentally: the quantum experiment is limited both in size and duration, and all measurements are performed by classical observers who use classical clocks.

Corners and higher level. If we just plugged this choice of $M = S \times I$ into the axioms of Section 3.2, we would get a single Hilbert space associated with the boundary $\partial M$. This is not quite what we want. We would rather prefer to obtain Hilbert spaces for the boundary components $S \times \{0\}$ and $S \times \{1\}$ between which the transition map acts, but not for $\partial S \times I$. The framework of higher level TQFT or TQFT with corners, see for example [11], might be appropriate to treat this situation. We do not go into details here, but rather focus on the structure of manifolds in the subsequent sections.

4 Classification of manifolds

Given some path integral (3.1) of general relativity in $d$-dimensional space-time, let us consider the partition function $Z(M)$ of (3.4), assuming for a moment that it is well defined and can be computed for some class of space-time manifolds.

Whenever $Z(M) \neq Z(M')$, then $M$ and $M'$ are not diffeomorphic. The partition function therefore forms a tool for the classification of smooth manifolds up to diffeomorphism. Although the framework of general relativity is by definition that of smooth manifolds and smooth maps (the Einstein equations are differential equations after all), it is instructive to contrast it with other types of manifolds. Barrett [9] has already remarked that the space-time dimension $d = 3 + 1$ plays a special role. Let us explain in more detail why. The presentation in this section is a rather informal overview. The detailed definitions and theorems to which we refer, can be found in the Appendix.
Topological manifolds up to homeomorphism. Each smooth manifold has got an underlying topological manifold (Section 2.1), and any two diffeomorphic smooth manifolds have homeomorphic underlying topological manifolds. Can we use the information about topological manifolds up to homeomorphism in order to classify smooth manifolds up to diffeomorphism?

The answer is ‘yes’ in space-time dimension $d \leq 2 + 1$, but ‘no’ in general. Indeed, in $d \leq 2 + 1$, each topological manifold admits a differentiable structure, and any two homeomorphic topological manifolds have differentiable structures so that the resulting smooth manifolds are diffeomorphic.

This result is one of the explanations of why quantum general relativity in $d = 2 + 1$ is particularly simple. In fact, its path integral quantization is closely related to a $C^0$-QFT, using topological manifolds up to homeomorphism, and exploiting the fact that in $d = 2 + 1$, there is no difference between $C^0$- and $C^\infty$-QFTs. Examples of such theories are given in Section 5 below, but before we can state them, we need some more theoretical background.

In $d \geq 3 + 1$, no analogous result is available. There exist countably infinite families of (compact) smooth 4-manifolds [13, 14] which are pairwise non-diffeomorphic, but which have homeomorphic underlying topological manifolds. There is therefore a considerable discrepancy between $C^\infty$- and $C^0$-QFTs in $d = 3 + 1$ space-time dimensions.

The most striking result even concerns the standard space $\mathbb{R}^4$ [15, 16].

**Theorem 4.1.** Consider the topological manifold $\mathbb{R}^d$, $d \in \mathbb{N}$.

- If $d \neq 4$, then there exists a differentiable structure for $\mathbb{R}^d$ which is unique up to diffeomorphism.
- If $d = 4$, then there exists an uncountable family of pairwise non-diffeomorphic differentiable structure for $\mathbb{R}^d$.

The differentiable structure of $\mathbb{R}^4$ induced from $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is called standard and the others exotic.

Non-uniqueness of differentiable structures persists in higher dimensions, for example, there are 28 inequivalent differentiable structures on the topological sphere $S^7$, or 992 inequivalent differentiable structures on $S^{11}$ [17], but in dimension $d \geq 4 + 1$ ($d \geq 5 + 1$ if the manifold has a non-empty boundary), there never exists more than a finite number of non-diffeomorphic differentiable structures on the same underlying topological manifold. The space-time dimension $d = 3 + 1$ is distinguished by the feature that there can exist an infinite number of homeomorphic, but pairwise non-diffeomorphic compact smooth manifolds.

Topological manifolds up to homotopy equivalence. There is another way of classifying topological manifolds. This relation is known as homotopy equivalence. Two topological manifolds $M$, $N$ are called homotopy equivalent if there exists a pair of continuous maps $f : M \to N$ and $g : N \to M$ such that $f \circ g$ is homotopic to the identity map $\text{id}_M$ of $M$, i.e., it ‘can be continuously deformed’ to $\text{id}_M$, and $g \circ f$ is homotopic to $\text{id}_N$ (see Appendix A.7 for details).

The concept of homotopy equivalence of topological manifolds is weaker than that of homeomorphism: any two homeomorphic topological manifolds are also homotopy equivalent.

The converse implication is true, for example, in $d \leq 1 + 1$, i.e. any two compact topological manifolds that are homotopy equivalent, are also homeomorphic. But it does not hold in...
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$d = 2 + 1$: there exist compact topological 3-manifolds which are homotopy equivalent, but not homeomorphic.

In $d \leq 1 + 1$, quantum general relativity is therefore even simpler than in $d = 2 + 1$. It is not only given by a $C^0$-QFT (as opposed to a generic $C^\infty$-QFT), but even by what one could call an hQFT (TQFT up to homotopy equivalence\(^5\)). Quantum general relativity in $d = 2 + 1$, in contrast, is not an hQFT. A topological invariant closely related to its partition function has been confirmed to distinguish homotopy equivalent topological 3-manifolds that are not homeomorphic. Quantum general relativity in $d = 2 + 1$ is thus as generic as topology allows.

**Piecewise-linear manifolds up to PL-isomorphism.** Before we can cite the next relevant classification result, we have to introduce yet another type of manifolds: piecewise-linear (PL-) manifolds (see Appendix A.3 for details).

A $k$-simplex in standard space $\mathbb{R}^d$ is the smallest convex set that contains $k + 1$ points that span a $k$-dimensional hyperplane (Figure 6). A polyhedron is a locally finite union of simplices. $\mathbb{R}^d$ itself, for example, is a polyhedron. A piecewise-linear map is a map $f: P \to Q$ between polyhedra which maps simplices onto simplices. We can now obtain another type of manifold by restricting the transition functions of a topological manifold to piecewise-linear functions.

A piecewise-linear (PL-) manifold is a topological manifold such that all transition functions are piecewise-linear. This is illustrated in Figure 7. Notice that not $M$ itself is triangulated, but rather the coordinate systems are. PL-manifolds are classified up to PL-isomorphism (Appendix A.3), i.e. up to homeomorphisms that are piecewise-linear in coordinates.

**Triangulations of smooth manifolds.** If some topological manifold admits both a piecewise-linear and a smooth structure, satisfying a compatibility condition (see Appendix A.4 for details), we say that the differentiable structure is a smoothing of the piecewise-linear structure.

There is a close relationship between smooth and piecewise-linear manifolds given by Whitehead’s theorem: For each smooth manifold $M$, there exists a PL-manifold $M_{PL}$, called its Whitehead triangulation, so that $M$ is diffeomorphic to a smoothing of $M_{PL}$. $M_{PL}$ is unique up to PL-isomorphism.

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\(^5\)The name homotopy quantum field theory and the abbreviation HQFT are already gone for a different concept.
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Whitehead’s theorem is therefore a license to triangulate space-time. But does the Whitehead triangulation capture all features of the given smooth manifold? The answer is ‘yes’, at least in space-time dimension $d \leq 5 + 1$: each PL-manifold admits a smoothing, and the resulting smooth manifold is unique up to diffeomorphism. The equivalence classes of smooth manifolds up to diffeomorphism are therefore in one-to-one correspondence with those of PL-manifolds up to PL-isomorphism. We are free to choose either framework at any time.

We therefore know that the path integral of general relativity in $d \leq 5 + 1$ is closely related to a TQFT that is defined for PL-manifolds up to PL-isomorphism, i.e. to a PL-QFT. Whitehead triangulations provide us with a way of ‘discretizing’ space-time which is not merely some approximation nor introduces a physical cut-off, but which is rather exact up to diffeomorphism.

Combinatorial manifolds  Whitehead triangulations are widely used in topology because they facilitate efficient computations which are most conveniently performed in a purely combinatorial language.

It is known that each $d$-dimensional PL-manifold $M$ is PL-isomorphic to a single polyhedron $P$ in $\mathbb{R}^n$ for some $n$ (Figure 8). Such a polyhedron which itself forms a PL-manifold, is called a combinatorial manifold (see Appendix A.5 for details) or a global triangulation of $M$. If $M$ is compact, $P$ can be described in terms of a finite number of simplices. In order to characterize $M$ up to PL-isomorphism, it suffices to characterize the PL-manifold $P$ up to PL-isomorphism.

A very convenient way of stating the condition of PL-isomorphism, and for our purposes the best intuition, is provided by Pachner’s theorem. We give here the version for closed manifolds: any two closed combinatorial manifolds are PL-isomorphic if and only if they are related by a finite sequence of Pachner moves.

Pachner moves are local modifications of the triangulation by joining simplices or by
splitting some polyhedra up into smaller pieces. Figure 8 shows the Pachner moves for closed manifolds in $d = 1 + 1$ and $d = 2 + 1$. A systematic way of listing the moves in any dimension is explained in Appendix A.6. Pictures for $d = 3 + 1$ can be found in [18, 19], and the moves for manifolds with boundary in [20].

The following procedure is now available in order to decide whether two given smooth $d$-manifolds, $d \leq 5 + 1$, are diffeomorphic. Start with two closed smooth manifolds $M^{(1)}$ and $M^{(2)}$. Construct their Whitehead triangulations $M^{(1)}_{PL}$ and $M^{(2)}_{PL}$. Find combinatorial manifolds $P^{(1)}$ and $P^{(2)}$ which are PL-isomorphic to $M^{(1)}_{PL}$ and $M^{(2)}_{PL}$, respectively. $M^{(1)}$ and $M^{(2)}$ are diffeomorphic if and only if $P^{(1)}$ and $P^{(2)}$ are related by a finite sequence of Pachner moves.

**Scenario for quantum gravity.** We have reached a first goal: the diffeomorphism gauge symmetry of general relativity on a closed space-time manifold has been translated into a purely combinatorial problem involving triangulations that consist of only a finite number of simplices, and their manipulation by finite sequences of Pachner moves. If not only the partition function, but also the full path integral of general relativity in $d \leq 5 + 1$ is given by a PL-QFT, we know that all observables are invariant under Pachner moves.

The partition function of quantum general relativity is an invariant of PL-manifolds, too, and can be computed by purely combinatorial methods for any given combinatorial manifold. A generic expression of such a partition function is the state sum,

$$Z = \sum_{\{\text{colourings}\}} \prod_{\{\text{simplices}\}} (\text{amplitudes}), \quad (4.1)$$

where the sum is over all labelings of the simplices with elements of some set of colours, and the integrand is a number that can be computed for each such labelling. In Section 5 below,
we give examples and illustrate that the partition function of quantum general relativity in $d = 2 + 1$ is precisely of this form.

If quantum general relativity in $d = 3+1$ is indeed a PL-QFT, the following two statements which sound philosophically completely contrary,

- Nature is fundamentally smooth.
- Nature is fundamentally discrete.

are just two different points of view on the same underlying mathematical structure: equivalence classes of smooth manifolds up to diffeomorphism.

Further classification results which are compiled in the Appendix, indicate that the partition function in $d \geq 4 + 1$ would be much less interesting than in $d = 3 + 1$ because smooth manifolds up to diffeomorphism are already essentially classified by their underlying topological manifold up to homeomorphism, subject to only finite ambiguities in choosing a differentiable structure. In fact, one could even use the homotopy type of space-time plus some additional information about the structure of the tangent bundle. Quantum general relativity in $d \geq 4+1$ would therefore be closely related to an hQFT supplemented by additional data in order to specify the tangent bundle and to resolve the ambiguities.

On the mathematical side, the path integral quantization of general relativity is closely related to the problem of classifying smooth manifolds up to diffeomorphism by classifying their Whitehead triangulations up to PL-isomorphism. Precisely in $d = 3+1$, topology is rich enough to (potentially) provide infinitely many non-trivial partition functions that are able to distinguish non-PL-isomorphic PL-structures on the same underlying topological manifold. It is an open problem in topology to construct these invariants.

Unless $d = 2+1$ or $d = 3+1$, the problem of constructing interesting partition functions is
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\[ d \leq 2 \quad d = 3 \quad d = 4 \quad d \geq 5 \]

\begin{array}{|c|c|c|c|}
\hline
C^\infty & C^\infty & C^\infty & C^\infty \\
\hline
PL & PL & PL & PL \\
\hline
Top & Top & Top & Top \\
\hline
htpy & htpy & htpy & htpy \\
\hline
\end{array}

Table 1: The relationship between the classifications of various types of manifolds of dimension \( d \): smooth manifolds up to diffeomorphism (\( C^\infty \)), piecewise-linear manifolds up to PL-isomorphism (PL), topological manifolds up to homeomorphism (Top), and topological manifolds up to homotopy equivalence (htpy). Equivalence of manifolds of the type shown in one row implies equivalence of manifolds of the type shown in the rows below. If the rows are not separated by any line, the converse implication holds as well. If the rows are separated by a dotted line, the converse implication holds up to some obstruction and ambiguity. A solid line indicates that the converse implication is seriously violated. For details, we refer to the Appendix, in particular to Appendix A.8. This table is rather sketchy, and a number of subtleties have been suppressed, so we ask the reader not to consider this table as a theorem without actually having read the small-print in the Appendix.

Finally dominated by the study of topological manifolds up homotopy equivalence which would render general relativity 'suspiciously simple'. All these considerations apply to the partition function (3.4), but not necessarily to the matrix elements (3.2), i.e. before tracing out the boundary conditions. In the next section, we illustrate how the examples in \( d = 2 + 1 \) show that the expression for the state sum (4.1) already suggests the appropriate Hilbert spaces and boundary conditions. Finding the partition function is therefore a key step.

Notice that we are not claiming that the universe is a smooth 4-manifold \( M \) that has an 'exotic', i.e. non-standard, differentiable structure. This may or may not be true, and the answer to this question is independent of the considerations presented so far. Some consequences of exotic differentiable structures on space-time have been explored in [21, 22]. The crucial observation is rather that, just because the Einstein equations are differential equations, the path integral of general relativity ought to be sensitive to the differentiable structure of \( M \), even if it is merely the standard differentiable structure. It may eventually turn out that in many cases the smooth structure on the boundary \( \partial M \) already determines the smooth structure of the entire \( M \), i.e. that the smooth background can be viewed as part of the classical boundary data.

Table 1 finally summarizes the classification of smooth, PL- and topological manifolds in the various dimensions. Refer to the Appendix for details.

Renormalization group transformations. The \( 1 \leftrightarrow d + 1 \) Pachner move (Figure 9) always subdivides one \( d \)-simplex into \( d + 1 \) \( d \)-simplices. It obviously resembles the block spin transformations familiar from the Statistical Mechanics treatment of renormalization, but is here applied to the Whitehead triangulations of smooth manifolds as opposed to discretizations of Riemannian manifolds. In our case, the simplices do not have any metric size as there is no background geometry associated with the smooth space-time manifold \( M \).
In non-generally relativistic field theories on some given Riemannian manifold \((M,g)\), renormalization is the comparison of the dynamical scale of the theory, i.e. the relevant correlation lengths, for any given cut-off and bare parameters, with the scale of the background metric \(g\), in order to determine the relation between cut-off and bare parameters for which the physical predictions are constant.

In general relativity, there is no background metric and therefore no way of (and no need to) introduce a cut-off. The diffeomorphism gauge symmetry implies the invariance of all observables under Pachner moves so that one can say that the theory is readily renormalized or, depending on the personal taste, that there is no need to renormalize theories in which the geometry is dynamical. We stress that Whitehead triangulations neither introduce a cut-off nor break any of the symmetries. Refer to [23] for the implications on the notion of locality, on the appearance of the Planck scale and on the compatibility with ideas in the context of the holographic principle.

5 Examples

This section is a brief overview over some results on quantum general relativity in \(d = 2 + 1\) space-time dimensions. For more details, see the review articles [24–26].

Turaev–Viro invariant. We have stressed above the importance of the partition function and that it forms an invariant of smooth manifolds up to diffeomorphism. In \(d = 2 + 1\), we know that we can equivalently study an invariant of PL-manifolds up to PL-isomorphism which is given by a state sum \(\mathcal{Z} = \sum_{j: \Delta_1 \rightarrow \{0, \frac{1}{2}, 1, \ldots, \frac{k-2}{2}\}} \left( \prod_{\sigma_1 \in \Delta_1} \dim_q j(\sigma_1) \right) \left( \prod_{\sigma_3 \in \Delta_3} \{6j\}_q(\sigma_3) \right)\). Here \(k = 1, \frac{3}{2}, 2, \ldots\) is a fixed half-integer, \(\Delta_1\) and \(\Delta_3\) denote the sets of 1-simplices and 3-simplices, respectively. The sum is over all ways of colouring the 1-simplices \(\sigma_1 \in \Delta_1\) with half-integers \(j(\sigma_1) \in \{0, \frac{1}{2}, 1, \ldots, \frac{k-2}{2}\}\). The quantum dimension \(\dim_q j(\sigma_1)\) can be computed for each \(j(\sigma_1)\), and the quantum-6j-symbol \(\{6j\}_q(\sigma_3)\) depends on the labels \(j(\sigma_1)\) associated with all 1-simplices \(\sigma_1\) in the boundary of each 3-simplex \(\sigma_3\), see [27] for details. The half-integers \(j(\sigma_1)\) in fact characterize the finite-dimensional irreducible representations of the quantum group \(U_q(\mathfrak{sl}_2)\) for the root of unity \(q = e^{i\pi/k}\).

Ponzano–Regge model. The first connection with quantum gravity in \(d = 2 + 1\) can be seen in the limit \(k \to \infty\) in which \(q \to 1\) and the quantum group \(U_q(\mathfrak{sl}_2)\) is replaced by the envelope \(U(\mathfrak{sl}_2)\) which is dual to the algebra of functions on the local Lorentz group \(SU(2) = \text{Spin}(3)\) (up to complexification). The \(j(\sigma_1)\) then characterize the finite-dimensional irreducible representations of \(SU(2)\). In this limit, \(Z\) agrees with the partition function of the Ponzano–Regge model [28], a non-perturbative quantization of the toy model of general relativity in \(d = 2 + 1\) with Riemannian signature \(\eta = \text{diag}(1, 1, 1)\). In the limit, the partition function diverges and is no longer a mathematically well-defined invariant, but the Pachner move invariance of (5.1) persists formally if one accepts to divide out infinite factors.

For the model with the realistic Lorentzian signature \(\eta = (-1, 1, 1)\), \(\text{Spin}(3)\) is replaced by \(\text{Spin}(1, 2)\) so that the representation labels become continuous [29,30].
Cosmological constant. In the Ponzano–Regge model, the large spin limit of the 6j-symbols yields the connection with the classical action of general relativity [28] and shows that the labels \(j(\sigma_1)\) represent dynamically assigned lengths\(^6\) \(\hbar G(j(\sigma_1) + \frac{1}{2})\) for the 1-simplices \(\sigma_1\). The analogous argument for the Turaev–Viro invariant indicates [31,32] that the Turaev–Viro invariant is the partition function of general relativity with Riemannian signature and quantized positive cosmological constant \(\Lambda = \frac{4\pi^2}{(\hbar G^2 k)^2}\). The limit \(k \to \infty\) then sends \(\Lambda \to 0\) as expected.

TQFT. Although we have so far concentrated on the partition function, one can easily read off from the state sum (5.1) a consistent choice of boundary fields and Hilbert spaces [27]: fix the \(j(\sigma_1)\) for all 1-simplices in the boundary \(\Sigma = \partial M\) in order to characterize a state. The Hilbert space \(\mathcal{H}(\Sigma)\) then has a basis whose vectors are labelled by all these \(j(\sigma_1)\). In the limit \(k \to \infty\), the Hilbert spaces become infinite-dimensional so that the rule (S1) in Section 3.2 has to be relaxed for the Ponzano–Regge model. The Hilbert spaces admit a precise interpretation as spaces of states of 2-geometries. Although our focus on the partition function instead of the full path integral with boundary conditions (3.1), seemed to be somewhat narrow at first sight, the examples show that state sums such as the Turaev–Viro invariant (5.1) already carry the information about boundary fields and Hilbert spaces. In fact, once the set of colourings of the state sum has been specified, it automatically determines the vector spaces and boundary fields.

Of course, there can be several different formulae of state sums (4.1) with different sets of colours which yield the same invariant. This corresponds to different path integrals (3.1) with different Hilbert spaces that have the same trace (3.4). What we are looking for in the case of general relativity, is the full TQFT and not just the partition function. Since all the classification results are available for manifolds with boundary, the key step is to find the physical interpretation for the boundary fields, expressed on triangulations of the boundary, for the set of colours of the state sum (4.1).

Towards 3+1. We can now come back to the claim of the introduction that the absence of gravitons in \(d = 2 + 1\) is related

1. neither to the question of whether the path integral corresponds to a \(C^0\)-QFT (as opposed to a \(C^\infty\)-QFT),
2. nor to the question of whether the vector spaces of this \(C^0\)-QFT or \(C^\infty\)-QFT are finite-dimensional,
3. nor to the question of whether the theory admits a triangulation independent discretization.

Whereas the question of \(C^0\)-QFT versus \(C^\infty\)-QFT is related to the classification of topological versus smooth manifolds and therefore depends on the space-time dimension \(d\), the question of finite-dimensionality is presently understood only in \(d = 2 + 1\). We have seen examples for both alternatives, finite-dimensional (Turaev–Viro model) and infinite-dimensional (Ponzano–Regge model). Both are constructed from classical theories that have only constant curvature geometries as their solutions and therefore no propagating modes. The answer to question (3.) above is finally ‘yes’ in any \(d \leq 5 + 1\) from the classification results. Concrete examples

\(^6\)In \(d = 2 + 1\), \(\hbar G\) is the Planck length.
have so far been constructed in $d = 2 + 1$ for both pure general relativity (Turaev–Viro model and Ponzano–Regge model) and for a special case of general relativity with fermions [33], and in $d = 3 + 1$ only for BF-theory without [34] or with [35, 36] cosmological constant.

For general relativity in $d = 3 + 1$, the results of Section 4 still suggest that we should expect a Pachner move invariant state sum although the finite-dimensionality of the Hilbert spaces might be lost in a more drastic fashion than in $d = 2 + 1$, and there are the remarks of Section 3.4 on the role of time. A possible approach to $d = 3 + 1$ in order to narrow down the path integral is suggested by the special properties of smooth 4-manifolds as we sketch in the final section.

6 Physics meets Mathematics

Connecting the gauge symmetry of classical general relativity with results on the classification of smooth manifolds up to diffeomorphism, we have revealed the coincidence of an open problem in topology, namely to construct a non-trivial invariant of piecewise-linear 4-manifolds, with an open problem in theoretical physics, namely to find a path integral quantization of general relativity in $d = 3 + 1$ space-time dimensions.

6.1 Mathematical aspects

The mathematical question is whether one can construct invariants of piecewise-linear 4-manifolds that are non-trivial in the sense that they can distinguish inequivalent differentiable structures on the same underlying topological manifold.

From the classification results, it is known that the Whitehead triangulation of any given smooth manifold captures the full information about its differentiable structure up to diffeomorphism (Appendix A.4). This suggests that a state sum which probes the abstract combinatorial information contained in a triangulation, will be sufficient (Appendix A.6). On the other hand, there exist already way too many topological manifolds in order to extract the complete classification information by any conceivable algorithm (Appendix A.8.1). The interesting question is therefore whether one can find a suitably restricted class of piecewise-linear 4-manifolds, for example closed connected and simply connected manifolds, for which a non-trivial invariant can be constructed. The existence of the Donaldson [4] and Seiberg–Witten [5] invariants in the smooth framework is encouraging because it demonstrates that some non-trivial information can indeed be extracted.

**Question 6.1.** Do there exist state sum invariants of piecewise-linear 4-manifolds which are able to distinguish inequivalent PL-structures on the same underlying topological manifold?

Differential topologists have been considering this question for a long time, see, for example the introduction of [37]. Some state sum invariants of piecewise-linear 4-manifolds have already been constructed, for example the Crane–Yetter invariant [36] and Mackaay’s state sum [19]. So far, it has not been confirmed that any of these constructions is indeed non-trivial in the above sense. The Crane–Yetter invariant is known to depend only on the homotopy type of the underlying topological manifold [38] although it offers a novel combinatorial method for computing the signature (of the intersection form) of $M$. Mackaay’s state sum has so far been explored only for very special cases in which it, too, depends only on the homotopy type [39]. Nevertheless, in order to appreciate this state sum, a more detailed comparison with the 3-dimensional case is very instructive.
Just as the Turaev–Viro invariant [27] of 3-manifolds can be constructed for a certain class of spherical categories [40], Mackaay’s state sum [19] is defined for a class of spherical 2-categories. The situations for which Mackaay’s state sum has been carefully studied [39] involve rather special spherical 2-categories for which this state sum resembles a 4-dimensional generalization of the Dijkgraaf–Witten model [41]. It is, however, known that the Dijkgraaf–Witten invariant agrees for lens spaces as soon as they have the same homotopy type [42]. Lens spaces are the standard examples [43] of topological 3-manifolds in order to show that some invariant can distinguish manifolds of the same homotopy type that are not homeomorphic.

In order to render the Turaev–Viro invariant non-trivial, the construction of the (modular categories of representations of the) quantum groups \( U_q(\mathfrak{sl}_2) \), \( q = e^{2\pi i/\ell}, \ell \in \mathbb{N} \), seems to be essential. In this case, the Turaev–Viro invariant indeed distinguishes non-homeomorphic lens spaces of the same homotopy type\(^7\). The same cannot be accomplished with the category of representations of an ordinary group.

This comparison with the 3-dimensional case therefore suggests that Mackaay’s state sum should be studied for sufficiently sophisticated spherical 2-categories which are as ‘generic’ in the context of 2-categories as are the modular categories of representations of \( U_q(\mathfrak{sl}_2) \) in the context of 1-categories.

6.2 Physical aspects

Questions about observables of general relativity are questions about smooth manifolds and smooth functions. The physical answers are specified only up to space-time diffeomorphism. Whenever we have made use of classification results on smooth, piecewise-linear or topological manifolds, we have exploited this gauge freedom in an essential way.

The classification results show that unless \( d = 3 + 1 \), at least the partition function \( Z(M) \) of the path integral can be computed without explicitly referring to the differentiable structure of the space-time manifold \( M \).

In \( d = 2 + 1 \), \( Z(M) \) depends only on the underlying topological manifold \( M \) up to homeomorphism (Appendix A.8.3), and this is the mechanism that renders quantum general relativity in \( d = 2 + 1 \) space-time dimensions particularly simple. This indeed justifies the jargon ‘topological’. In \( d \leq 1 + 1 \) or \( d \geq 4 + 1 \), the information required in order to determine \( Z(M) \) is essentially the homotopy type of \( M \) (assuming that \( M \) admits a differentiable structure) together with information on the tangent bundle and one finite number which resolves the ambiguities in constructing first a PL-structure for the topological manifold \( M \) (Appendix A.8.3) and then a smoothing of this PL-structure (Appendix A.8.2).

Although we have defined the partition function \( Z(M) \) in (3.4) using the path integral of general relativity in a way that seems to employ the differentiable structure of space-time, the above results suggest that unless \( d = 3 + 1 \), there exists an alternative way of computing \( Z(M) \), either from the underlying topological manifold up to homeomorphism (\( d = 2 + 1 \)), or even essentially from its homotopy type together with additional choices (\( d \leq 1 + 1 \) or \( d \geq 4 + 1 \)).

It happens only in \( d = 3 + 1 \), that there can be more than finitely many pairwise inequivalent differentiable structures for the topological manifold underlying \( M \). This means that there is generically no short cut available in order to compute \( Z(M) \) from the underlying

\(^7\)By the theorem of Turaev and Walker [38], the Turaev–Viro invariant is the squared modulus of the Reshetikhin–Turaev invariant [44] which is known [45] to distinguish, for example, the lens spaces \( L(7, 1) \) and \( L(7, 2) \).
topological manifold alone. Only in \(d = 3 + 1\), differential topology is rich enough in order to provide us with a large number of non-trivial \(Z(M)\) in the context of smooth manifolds. This is very reassuring because in \(d = 3 + 1\), one might expect quite a number of quantum field theories with a gauge symmetry under space-time diffeomorphisms even though the theories other than general relativity are usually treated in a different framework.

Besides the partition function, will the entire \(C^\infty\text{-QFTs}\) in space-time dimensions other than \(d = 3 + 1\) be necessarily less interesting? This question cannot be ultimately answered yet, but one can expect that the dominance of homotopy type in this problem in \(d \leq 1 + 1\) or \(d \geq 4 + 1\) (\(d \geq 5 + 1\) if there is a non-empty boundary) would provide strong constraints. Concerning \(d = 3 + 1\), we arrive at,

**Question 6.2.** Does there exist a particular state sum invariant of piecewise-linear 4-manifolds, a degenerate limit of which yields the path integral quantization of pure general relativity in \(d = 3 + 1\) space-time dimensions (in the special case in which no boundary conditions are imposed)?

Let us again compare the situation with the toy model of general relativity in \(d = 2 + 1\). The state sum that yields the actual topological invariant, is the Turaev–Viro invariant [27]. This state sum for \(U_q(sl_2)\), \(q = e^{i\pi/k}\), \(k = 1, 3, 2, \ldots\), corresponds to the partition function of quantum gravity with Riemannian signature \(\eta = \text{diag}(1,1,1)\) and quantized positive cosmological constant \(\Lambda = 4\pi^2/(\hbar Gk)^2\) [31, 32]. This model defines [27] a proper TQFT in the strict sense [2], based on topological manifolds up to homeomorphism and involving finitely generated modules. Quantum gravity with Riemannian signature, but \(\Lambda = 0\), can then be understood as a limit of this invariant for \(k \to \infty, \Lambda \to 0, q \to 1\), in which the Hilbert spaces become infinite-dimensional and the partition function diverges. The partition function is therefore no longer a well-defined invariant of topological manifolds. Nevertheless, this degenerate limit precisely agrees with the original model of Ponzano–Regge [28] which had been invented as a non-perturbative quantization of general relativity in \(d = 2 + 1\) with Riemannian signature, well before it was realized that this framework is closely related to invariants of topological manifolds. Quantum gravity in \(d = 2 + 1\) with the realistic Lorentzian signature \(\eta = \text{diag}(-1,1,1)\) is finally even more complicated, replacing \(\text{Spin}(3)\) by \(\text{Spin}(2,1)\) and the discrete representation labels of the Ponzano–Regge model with continuous ones [29, 30].

If we try to extrapolate this experience from \(d = 2 + 1\) to \(d = 3 + 1\), on the mathematical side we may well expect that the proper invariant of piecewise-linear 4-manifolds involves some highly non-trivial 2-category which may be very difficult to guess. Its physical counterpart, the sought-after path integral of general relativity, however, needs to be ‘just’ a degenerate limit of the proper invariant.

Given the experience from \(d = 2 + 1\) and noting that the Ponzano–Regge model (although divergent and therefore not well defined in the mathematical sense) is so much easier than the Turaev–Viro invariant, it seems to be a reasonable strategy to approach the state sum invariant of piecewise-linear 4-manifolds via theoretical physics. This means to proceed in two steps: first to construct a physically motivated, but degenerate limit of the invariant which corresponds to general relativity; second to study the deformation theory of the relevant categories and to aim for the actual invariant.

The interesting perspective is here the combination of techniques developed in mathematics on how to lift combinatorial and algebraic structures from three to four dimensions, see, for example the introductory sections of [19, 39] and also [11, 18] for references, with methods
Quantum general relativity and the classification of smooth manifolds

from theoretical physics on how to construct discrete physical models related to the path integral of general relativity, see, for example, the review articles [24–26].

The framework outlined here,

- Does not involve any new physical assumptions. It just combines quantum theory (represented by the axioms (A1), (A2), (A4) of axiomatic X-QFT for \(X = C^0, C^\infty, PL, h\)) with the properties of generally relativistic theories (represented by axiom (A3) and the choice \(X = C^\infty\)).
- Singles out the space-time dimension \(d = 3 + 1\).
- Explains that, if mathematicians solve Question 6.1, this will provide physicists with (a family of) rigorously defined path integrals which have the same symmetries as general relativity.
- Explains why the diffeomorphism gauge symmetry takes care of renormalization.
- Contains the spin foam models of \((2 + 1)\)-dimensional quantum gravity as well-studied examples.
- Applies to other coordinate-free formulated field theories as well (c.f. Section 2.5), not necessarily generally relativistic.

In the search for a path integral quantization of general relativity in \(d = 3 + 1\), we are facing a coincidence of open questions in mathematics with open questions in theoretical physics. In exploring these connections, we have just scratched the surface.

Acknowledgments

The author is indebted to Marco Mackaay for explaining various results on the topology of four-manifolds. I would like to thank Louis Crane, Gary Gibbons, Robert Helling and Daniele Oriti for discussions.

A Topological, piecewise-linear and smooth manifolds

For mathematical background on topological, smooth and piecewise-linear manifolds, we refer to various textbooks [6,7,46–48] as well as to the introductory sections of [16,37]. We review here only those facts that are relevant to the classification of the various types of manifolds, hoping that this compilation of definitions and results will make the literature more accessible.

A.1 Topological manifolds

We first recall the definitions of topological and smooth manifolds.

Definition A.1. Let \(M\) be a topological space, and fix some \(d \in \mathbb{N}\), called the dimension.

1. A chart or coordinate system for \(M\) is a pair \((U, \varphi)\) of an open set \(U \subseteq M\) and a homeomorphism \(\varphi: U \rightarrow \varphi(U)\) onto an open subset \(\varphi(U) \subseteq \mathbb{R}^d_+\) of the half-space \(\mathbb{R}^d_+ := \{x \in \mathbb{R}^d : x_0 \geq 0\}\).

2. A \(\text{TOP}_d\)-atlas \(A\) for \(M\) is a family \(A = \{(U_i, \varphi_i) : i \in I\}\) of coordinate systems for \(M\) such that the \(U_i\) form an open cover of \(M\),

\[
\bigcup_{i \in I} U_i = M. \tag{A.1}
\]
Here I denotes some index set.

3. On the non-empty overlaps $U_{ij} := U_i \cap U_j \neq \emptyset$, $i,j \in I$, there are homeomorphisms,

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1}: \varphi_i(U_{ij}) \to \varphi_j(U_{ij}), \quad (A.2)$$

which are called transition functions.

4. The boundary $\partial M$ of $M$ is the set of all $p \in M$ for which $\varphi_i(p) \in \mathbb{R}_0^d$, $\mathbb{R}_0^d := \{x \in \mathbb{R}^d : x_0 = 0\}$.

5. Two atlases are called equivalent if their union is an atlas.

These are the most general atlases we are interested in. They give rise to topological manifolds.

**Definition A.2.** Fix some dimension $d \in \mathbb{N}$.

1. Let $M$ be a topological space. A $\text{TOP}_d$-structure $[\mathcal{A}]$ for $M$ is an equivalence class of $\text{TOP}_d$-atlases.

2. A topological $d$-manifold $(M, [\mathcal{A}])$ is a paracompact Hausdorff space $M$ equipped with a $\text{TOP}_d$-structure $[\mathcal{A}]$.

3. Two topological manifolds $(M, [\mathcal{A}]), (N, [\mathcal{B}])$ are called equivalent if $M$ and $N$ are homeomorphic.

4. A topological manifold $(M, [\mathcal{A}])$ is called compact if the underlying topological space $M$ is compact. It is called closed if it is compact and $\partial M = \emptyset$.

We often write just $M$ rather than $(M, [\mathcal{A}])$. The boundary $\partial M$ of any topological $d$-manifold $M$ forms a topological $(d-1)$-manifold with empty boundary. A 0-manifold is just a set of points with the discrete topology.

Due to paracompactness, each atlas of any topological manifold admits a locally finite refinement for which the $\varphi_i(U_i)$ are contained in compact subsets of $\mathbb{R}_0^d$.

The remainder of this section is a technical detail which is necessary in order to combine results from various different sources in the literature. The conditions that the underlying topological space in Definition A.2 be paracompact and Hausdorff, are the same as those used in [7] which are those that correspond to the physically relevant space-times [49]. The following alternative choices are common in the literature,

1. metrizable [50, 51],
2. second countable and Hausdorff [46],
3. separable and Hausdorff [16].

Concerning (1.), by a theorem of Stone [52], each metrizable topological space is paracompact and Hausdorff. Conversely, any paracompact Hausdorff space with a $\text{TOP}_d$-atlas inherits a metric from $\mathbb{R}^d$ on each chart and, by employing a continuous partition of unity, can be shown to be metrizable.

Concerning (2.), each topological space with a $\text{TOP}_d$-atlas is locally compact, and any locally compact and second countable Hausdorff space is metrizable [52]. Conversely, by a theorem of Alexandroff [52], each locally compact metrizable space admits a countable basis of
open sets for each connection component. As soon as the discussion is restricted to topological
spaces with a countable number of connection components, (2.) is therefore equivalent to (1.).
Concerning (3.), each second countable topological space is separable, but (3.) is in general
weaker than (2.). We note, however, that a given topological space is paracompact and locally
compact if and only if it is a free union of spaces that are σ-compact (unions of countably
many compact sets) \[53\]. The relevant examples of \[16\] are all of this type.

A.2 Smooth manifolds

We can impose additional structure on manifolds by restricting the transition functions to
appropriate sub-families of homeomorphisms. Recall that a map \( f : U \to \mathbb{R}^n \), for some open
set \( U \subseteq \mathbb{R}^m \), is called \( C^k \) if all \( k \)-th partial derivatives exist and are continuous on \( U \). A
\( C^k \)-diffeomorphism is an invertible \( C^k \)-map with a \( C^k \)-inverse.

Definition A.3. Let \( M \) be a topological \( d \)-manifold, \( d \in \mathbb{N} \), and \( k \in \mathbb{N}_0 \cup \{ \infty \} \).

1. A \( C^k \)-atlas for \( M \) is a \( \text{TOP}_d \)-atlas \( \mathcal{A} = \{(U_i, \varphi_i) : i \in I \} \) such that for all \( i, j \in I \) with
\( U_i \cap U_j \neq \emptyset \), the transition functions \( \varphi_{ji} \) are \( C^k \)-diffeomorphisms.

2. A \( C^k \)-structure \([\mathcal{A}]\) for \( M \) is an equivalence class of \( C^k \)-atlases.

This definition includes the topological case (Definition A.1) for \( k = 0 \).

Definition A.4. Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \).

1. A \( d \)-dimensional \( C^k \)-manifold \((M, [\mathcal{A}]), d \in \mathbb{N} \), is a topological \( d \)-manifold \( M \) equipped
with a \( C^k \)-structure \([\mathcal{A}]\).

2. Let \((M, [\mathcal{A}]), (N, [\mathcal{B}])\) be \( C^k \)-manifolds with atlases \( \mathcal{A} = \{(U_i, \varphi_i) : i \in I \} \) and \( \mathcal{B} = \{(V_j, \psi_j) : j \in J \} \), not necessarily of the same dimension. A map \( f : M \to N \) is called
\( C^k \) if \( f \) is continuous and if for all \( p \in M \) such that \( p \in U_i \), \( f(p) \in V_j \), for some \( i \in I, j \in J \), the map,
\[
\psi_j \circ f \circ \varphi_{ji}^{-1} : \varphi_i(U_i) \to \psi_j(V_j)
\] (A.3)
is a \( C^k \)-map.

3. A \( C^k \)-diffeomorphism \( f : M \to N \) is an invertible \( C^k \)-map whose inverse is \( C^k \).

4. Two \( C^k \)-manifolds are called equivalent if they are \( C^k \)-diffeomorphic.

The boundary \( \partial M \) of any \( d \)-dimensional \( C^k \)-manifold is a \((d-1)\)-dimensional \( C^k \)-manifold
without boundary.

Definition A.5. 1. A \( C^k \)-atlas or \( C^k \)-structure, \( k \geq 1 \), is called oriented if all transition
functions \( \varphi_{ji} \) are orientation preserving, i.e. if they have positive Jacobi determinants.

2. An oriented \( d \)-dimensional \( C^k \)-manifold is a topological \( d \)-manifold with an oriented
\( C^k \)-structure.

3. A \( C^k \)-diffeomorphism is called orientation preserving if all its coordinate representa-
tions (A.3) are orientation preserving.
The boundary $\partial M$ of any oriented $C^k$-manifold is oriented. We have defined oriented manifolds only for the cases $C^k$, $k \geq 1$ (although this can also be done for topological manifolds). If we nevertheless mention oriented topological manifolds in the following, we do this only if the topological manifold admits some oriented $C^k$-structure that is unique up to orientation preserving $C^k$-diffeomorphism.

Obviously, each $C^k$-structure is also a $C^\ell$-structure for any $0 \leq \ell < k$, including the topological case $\ell = 0$. As long as we are only interested in $C^\ell$-manifolds up to equivalence, this time excluding the topological case, i.e. $1 \leq r \leq \infty$, we can restrict ourselves to $C^\infty$-manifolds as the following theorem of Whitney shows, see, for example [48].

**Theorem A.6.** Let $M$ be a $C^\ell$-manifold, $1 \leq r \leq \infty$, with some $C^r$-structure $[A^{(r)}]$ and some $k$ with $r \leq k \leq \infty$.

1. There exists a $C^k$-structure $[A^{(k)}]$ for $M$.
2. The $C^\ell$-manifolds $(M,[A^{(r)}])$ and $(M,[A^{(k)}])$ are $C^\ell$-diffeomorphic.
3. Let $[B^{(k)}]$ denote any other $C^k$-structure for $M$. Then $(M,[A^{(k)}])$ and $(M,[B^{(k)}])$ are $C^\ell$-diffeomorphic.

Of all the $C^\ell$-manifolds, $0 \leq \ell \leq \infty$, the relevant types for the purpose of classification are therefore the topological (or $C^0$-) and the $C^\infty$-manifolds. A $C^\infty$-structure is usually called a differentiable structure. $C^\infty$-maps and $d$-dimensional $C^\infty$-manifolds are known as smooth maps and smooth $d$-manifolds, and $C^\infty$-diffeomorphisms are often called just diffeomorphisms.

### A.3 Simplices and piecewise-linear manifolds

In this section, we define another type of manifold by restricting the transition functions of topological manifolds to piecewise-linear (PL) maps. These are the maps that are compatible with triangulations of the sets $\varphi_i(U_i) \subseteq \mathbb{R}^d$ for the coordinate systems $(U_i, \varphi_i)$. Let us start with the notion of a simplex.

**Definition A.7.** Fix some dimension $d \in \mathbb{N}$.

1. The convex hull of some finite set of points $A = \{p_0, \ldots, p_k\} \subseteq \mathbb{R}^d$, $k \in \mathbb{N}_0$, is the set

   $$[A] = [p_0, \ldots, p_k] := \left\{ \sum_{j=0}^k p_j \lambda_j : \lambda_j \geq 0, \sum_{j=0}^k \lambda_j = 1 \right\} \subseteq \mathbb{R}^d. \quad (A.4)$$

   We also define $[0] := \emptyset$.

2. A finite set of points $A = \{p_0, \ldots, p_k\} \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$, is called affine independent if the set of vectors $\{p_1 - p_0, \ldots, p_k - p_{k-1}\}$ is linearly independent. Sets containing only one point are by definition affine independent.

3. A $k$-simplex in $\mathbb{R}^d$, $k \in \mathbb{N}_0$, is a set $\sigma \subseteq \mathbb{R}^d$ of the form $\sigma = [p_0, \ldots, p_k]$ for some affine independent set $\{p_0, \ldots, p_k\} \subseteq \mathbb{R}^d$. A $(-1)$-simplex is by definition the empty set.

4. Let $\sigma = [A]$ and $\tau = [B]$ be simplices in $\mathbb{R}^d$ so that $A$ and $B$ are affine independent sets. The simplex $\sigma$ is called a face of $\tau$ if $A \subseteq B$. In this case we write $\sigma \leq \tau$. 

Note that the definition of affine independence does not depend on the numbering of the points. For each simplex \( \sigma \subseteq \mathbb{R}^d \), there is a unique finite set \( A \subseteq \mathbb{R}^d \) such that \( A \) is affine independent and \( \sigma = [A] \). In this case, we call \( \text{Vert} \sigma := A \) the set of vertices of \( \sigma \). Each face of a given \( k \)-simplex, \( k \in \mathbb{N}_0 \), is an \( \ell \)-simplex for some \( \ell \leq k \). Each 0-simplex has got precisely one face, the empty set, and \( \emptyset \preceq \sigma \) for all simplices \( \sigma \). The relation ‘\( \preceq \)’ is a partial order on the set of all simplices in \( \mathbb{R}^d \).

**Definition A.8.**

1. A set \( S \) of simplices in \( \mathbb{R}^d \) is called **locally finite** if each \( p \in \mathbb{R}^d \) has got a neighbourhood \( U \) such that \( U \cap \sigma \neq \emptyset \) only for a finite number of simplices \( \sigma \in S \).
2. A polyhedron \( P \subseteq \mathbb{R}^d \) is a set of the form
   \[
   |S| := \bigcup_{\sigma \in S} \sigma, 
   \]
   for some locally finite set \( S \) of simplices in \( \mathbb{R}^d \).
3. A simplicial complex \( K \) in \( \mathbb{R}^d \) is a locally finite set of simplices in \( \mathbb{R}^d \) such that,
   (a) whenever \( \sigma \in K \) and \( \tau \preceq \sigma \) for any simplex \( \tau \subseteq \mathbb{R}^d \), then also \( \tau \in K \), and
   (b) if \( \tau, \sigma \in K \) then \( \tau \cap \sigma \preceq \tau \) and \( \tau \cap \sigma \preceq \sigma \).
4. A simplicial complex \( K \) is called **finite** if \( K \) is a finite set.
5. If \( K \) is a simplicial complex in \( \mathbb{R}^d \), we denote by \( \text{Vert} K := \{ p \in \mathbb{R}^d : \{ p \} \in K \} \) the set of its vertices.
6. If \( K \) is a simplicial complex, the set \( |K| \) is called its **underlying polyhedron**.
7. If a polyhedron \( P \) is of the form \( P = |K| \) for some simplicial complex \( K \), then \( K \) is called a triangulation of \( P \).

Each polyhedron \( P \subseteq \mathbb{R}^d \) has got a triangulation. If \( P \) is compact, then there exists a triangulation of \( P \) which is a finite simplicial complex. We can now define piecewise-linear (PL) maps to be the maps compatible with the triangulations of polyhedra.

**Definition A.9.** Let \( P \subseteq \mathbb{R}^m \), \( Q \subseteq \mathbb{R}^n \) be polyhedra and \( P = |K| \) for some triangulation \( K \).

1. A map \( f : P \to Q \) is called **piecewise-linear** (PL) if it is continuous and if the restrictions \( f|_{\sigma} \) are affine maps for each simplex \( \sigma \in K \).
2. The map \( f \) is called **piecewise differentiable** (PD) if it is continuous and if the \( f|_{\sigma} \) are \( C^\infty \)-maps of maximum rank for each \( \sigma \in K \).

The inverse of any piecewise-linear homeomorphism is also piecewise-linear. The notion of piecewise differentiability plays a role when we discuss the compatibility of piecewise-linear and smooth structures in Appendix A.4 below.

**Definition A.10.** Let \( K \) and \( L \) be simplicial complexes. A map \( f : |K| \to |L| \) is called **simplicial** is it is continuous, if \( f \) maps vertices to vertices, i.e. \( f(p) \in \text{Vert} L \) for all \( p \in \text{Vert} K \), and if for each \( k \)-simplex \( [p_0, \ldots, p_k] \in K \), the simplex generated by the images \( [f(p_0), \ldots, f(p_k)] \) is contained in \( L \). A **simplicial isomorphism** is a simplicial map that is a homeomorphism.
Let $P$ and $Q$ be polyhedra. A map $f: P \to Q$ is PL if and only if $f$ is a simplicial map for some triangulations $K$ and $L$ such that $P = |K|$ and $Q = |L|$.

**Definition A.11.**  
• An oriented $k$-simplex $(\sigma, \leq)$ is a $k$-simplex $\sigma$ together with a linear order $'\leq'$ on the set of its vertices $\text{Vert} \sigma$. In the bracket notation $[q_0, \ldots, q_k]$ for $k$-simplices, we can write for the oriented simplex,

$$\varepsilon \cdot [p_{\tau(0)}, \ldots, p_{\tau(k)}],$$

where $\tau \in S_{k+1}$ is a permutation such that $p_0 \leq \cdots \leq p_k$ and $\varepsilon := \text{sgn} \tau \in \{-1, +1\}$ is the sign of the permutation. For example, for 2-simplices,

$$[p_1, p_0, p_2] = -[p_0, p_1, p_2].$$

• The faces of an oriented $k$-simplex $\sigma = [p_0, \ldots, p_k] \subseteq \mathbb{R}^d$ are the oriented simplices with the induced orientation,

$$(-1)^j \cdot [p_0, \ldots, \hat{p}_j, \ldots, p_k],$$

where the hat denotes the omission of a vertex.

• An oriented simplicial complex $(K, \leq)$ is a simplicial complex $K$ together with a partial order $'\leq'$ on the set of vertices $\text{Vert} K$ that restricts to a linear order on $\text{Vert} \sigma$ for each $\sigma \in K$.

• A simplicial isomorphism $f: |K| \to |L|$ between oriented simplicial complexes $(K, \leq)$ and $(L, \leq)$ is called orientation preserving if it is compatible with the partial order, i.e., $p \leq q$ implies $f(p) \leq f(q)$ for all $p, q \in \text{Vert} K$.

Each $d$-simplex in $\mathbb{R}^d$, $\sigma = [p_0, \ldots, p_d]$, inherits an orientation from $\mathbb{R}^d$ such that the linear order of its vertices is $p_0 \leq \cdots \leq p_k$ up to an even permutation precisely when $\text{det}(p_0 - p_1, \ldots, p_{d-1} - p_d) > 0$. Each subset $U \subseteq \mathbb{R}^d_+$ that is open in $\mathbb{R}^d_+$ is contained in some polyhedron which has a triangulation in terms of an oriented simplicial complex, compatible with the orientation inherited from $\mathbb{R}^d_+$.

We are now ready to restrict the transition functions of manifolds to piecewise-linear maps.

**Definition A.12.** Let $M$ be a topological $d$-manifold, $d \in \mathbb{N}$.

1. A PL$_d$-atlas for $M$ is a TOP$_d$-atlas $A = \{(U_i, \varphi_i): i \in I\}$ such that for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, the transition functions $\varphi_{ji}$ are piecewise-linear homeomorphisms.

2. An oriented PL$_d$-atlas for $M$ is a PL$_d$-atlas such that the transition functions $\varphi_{ji}$ are orientation preserving simplicial maps with respect to the orientation inherited from $\mathbb{R}^d$.

3. An oriented PL$_d$-structure $[A]$ for $M$ is an equivalence class of oriented PL$_d$-atlases.

The following is as usual.

**Definition A.13.**

1. A $d$-dimensional oriented PL-manifold $(M, [A]), d \in \mathbb{N}$, is a topological $d$-manifold $M$ equipped with an oriented PL-structure $[A]$.

2. Let $(M, [A]), (N, [B])$ be PL-manifolds with atlases $A = \{(U_i, \varphi_i): i \in I\}$ and $B = \{(V_j, \psi_j): j \in J\}$. A map $f: M \to N$ is called PL if $f$ is continuous and if for any $p \in M$ such that $p \in U_i$, $f(p) \in V_j$ for some $i \in I$, $j \in J$, the map

$$\psi_j \circ f \circ \varphi_i^{-1}: \varphi_i(U_i) \to \psi_j(V_j)$$

is piecewise-linear.
3. A PL-isomorphism \( f : M \to N \) is an invertible PL-map whose inverse is PL.

4. A PL-isomorphism is called orientation preserving if all its coordinate representations are orientation preserving simplicial maps.

5. Two PL-manifolds are called equivalent if they are PL-isomorphic.

The boundary \( \partial M \) of each \( d \)-dimensional [oriented] PL-manifold is a \((d-1)\)-dimensional [oriented] PL-manifold without boundary.

### A.4 Triangulations of smooth manifolds

A smoothing of some PL-manifold is a differentiable structure that is compatible with the PL-structure in the following way.

**Definition A.14.** Let \( M \) be a PL-manifold with the PL-atlas \( \mathcal{A} = \{(V_i, \psi_i) : i \in I\} \). A differentiable structure represented by some \( C^\infty \)-atlas \( \{(U_j, \varphi_j) : j \in J\} \) on \( M \) is called a smoothing of the PL-structure \( [\mathcal{A}] \) if on each non-empty overlap \( U_j \cap V_i \neq \emptyset, i \in I, j \in J \), the homeomorphism

\[
\varphi_j \circ \psi_i^{-1}|_{\psi_i(U_j \cap V_i)} : \psi_i(U_j \cap V_i) \to \varphi_j(U_j \cap V_i) \quad (A.10)
\]

is piecewise-differentiable.

Whitehead’s theorem [54] guarantees that any smooth manifold of arbitrary dimension can be triangulated in this way.

**Theorem A.15 (Whitehead).** For each smooth manifold \( M \), there exists a PL-manifold \( M_{PL} \), called its Whitehead triangulation, such that \( M \) is diffeomorphic to a smoothing of \( M_{PL} \). Any two diffeomorphic smooth manifolds have PL-isomorphic Whitehead triangulations.

### A.5 Combinatorial manifolds

We have defined PL-manifolds above as topological manifolds subject to the additional condition that their transition functions are PL. In this section, we review the concept of combinatorial manifolds which reduces the study of an entire PL-manifold to the study of a single simplicial complex and its combinatorics. Notice that the underlying polyhedron \( |K| \subseteq \mathbb{R}^n \) of any simplicial complex \( K \) in \( \mathbb{R}^n \), forms a paracompact Hausdorff space with the relative topology induced from \( \mathbb{R}^n \).

**Definition A.16.** The underlying polyhedron \( |K| \) of a simplicial complex \( K \) in \( \mathbb{R}^n \) for some \( n \in \mathbb{N} \), is called a combinatorial \( d \)-manifold if it forms a \( d \)-dimensional PL-manifold, \( d \in \mathbb{N} \).

In order to describe the relationship between PL-manifolds and combinatorial manifolds, we need the notion of the join and the link of simplices.

**Definition A.17.** Let \( \sigma, \tau \subseteq \mathbb{R}^d \) be simplices, \( \sigma = [A], \tau = [B] \) with affine independent sets \( A, B \subseteq \mathbb{R}^d \).

1. \( \sigma \) and \( \tau \) are called joinable if the set \( A \cup B \) is affine independent.

2. If \( \sigma, \tau \) are joinable, their join is defined to be the set \( \sigma \cdot \tau := [A \cup B] \).
Definition A.18. Let $K$ be a simplicial complex and $\sigma \in K$. The link of $\sigma$ in $K$ is the following set of simplices,

$$\operatorname{lk}_K(\sigma) := \{ \tau \in K : \sigma \text{ and } \tau \text{ are joinable and } \sigma \cdot \tau \in K \}. \quad (A.11)$$

Notice that $\operatorname{lk}_K(\sigma)$ is itself a simplicial complex. We also need the definition of PL-balls and PL-spheres.

Definition A.19. Let $k \in \mathbb{N}$ and $\{p_0, \ldots, p_k\} \subseteq \mathbb{R}^d$ be affine independent.

1. A piecewise-linear $k$-ball is a polyhedron which is PL-isomorphic to the $k$-simplex,

$$B^k := [p_0, \ldots, p_k]. \quad (A.12)$$

2. For each simplex $\sigma$, we define the simplicial complex $\overline{\sigma} := \{ \tau : \tau \subseteq \sigma \}$ which contains $\sigma$ together with all its faces.

3. A piecewise-linear $(k-1)$-sphere is a polyhedron which is PL-isomorphic to the polyhedron,

$$S^{k-1} = \partial B^k := \{ \sigma \in \overline{B^k} : \sigma \neq B^k \}. \quad (A.13)$$

A 0-ball is a set containing one point, and a 0-sphere a set with precisely two points. $\ell$-balls and $\ell$-spheres for $\ell < 0$ are by definition the empty set. PL-manifolds can be characterized by combinatorial manifolds as follows.

Theorem A.20. Each PL-manifold is PL-isomorphic to some combinatorial manifold. Conversely, the underlying polyhedron $|K| \subseteq \mathbb{R}^n$ of some simplicial complex $K$ admits a PL$_d$-structure, $n, d \in \mathbb{N}$, if and only if for each $k$-simplex $\sigma \in K$, the polyhedron $|\operatorname{lk}_K(\sigma)|$ is a PL $(d-k-1)$-ball or a PL $(d-k-1)$-sphere.

This theorem implies that each PL-manifold $M$ can be described by a single simplicial complex $K$ with certain properties. If $M$ is compact, then $K$ can be chosen to be finite. Otherwise, due to paracompactness, $K$ can always be built up from only a countable number of simplices. Furthermore, $\sigma \in K$ is contained in the boundary $\partial M$ if and only if $|\operatorname{lk}_K(\sigma)|$ is a PL $(d-k-1)$-ball (as opposed to a sphere). Each $(d-1)$-simplex in $K$ is the face of at most two $d$-simplices in which it occurs with opposite induced orientations.

A.6 Abstract triangulations

It is finally even possible to forget the surrounding standard space $\mathbb{R}^n$ in which the simplices are contained, and to concentrate only on their combinatorics.

Definition A.21. An abstract simplicial complex $(K^0, K)$ is a set $K^0$, called the set of abstract vertices, together with a family $K \subseteq \mathcal{P}(K^0)$ of finite subsets of $K^0$, called the set of abstract simplices, such that

1. $\{p\} \in K$ for all $p \in K^0$, and
2. If $\sigma \in K$ and $\tau \subset \sigma$, then also $\tau \in K$.

For some subset $A \subseteq K$ of abstract simplices, we denote by $(\overline{A^0}, \overline{A})$ the smallest abstract simplicial complex that contains all simplices of $A$, i.e. $\overline{A} = \{ \sigma \in K : \sigma \subseteq \tau, \tau \in A \}$ and $\overline{A^0} = \{ p \in K^0 : \{p\} \in A \}$. We often write just $K$ rather than $(K^0, K)$. An oriented abstract simplicial complex is an abstract simplicial complex with a partial order on the set $K^0$ that restricts to a linear order on each $\sigma \in K$. 
Given some simplicial complex \( L \) in \( \mathbb{R}^n \), its abstraction is the abstract simplicial complex \((K^0,K)\) for which \( K^0 = \text{Vert} \ L \) and \( \{ p_0, \ldots, p_k \} \in K \) if and only if \( \{ p_0, \ldots, p_k \} \in L \). Conversely, given some finite abstract simplicial complex \((K^0,K)\), there exists a simplicial complex \( L \) in \( \mathbb{R}^{K^0-1} \) whose abstraction is \((K^0,K)\). \( L \) is unique up to simplicial isomorphism. If \((K^0,K)\) is a finite abstract simplicial complex, we define its underlying polyhedron as \( |K| := |L| \) which is well defined up to simplicial isomorphism. Recall the additional conditions (Theorem A.20) if this is supposed to yield a PL-manifold.

We can now study the question of whether any two given PL-manifolds are PL-isomorphic in the context of the abstractions of their global triangulations. This is accomplished in a systematic way for any dimension by Pachner’s theorem [20]. We mention here the version for closed manifolds.

**Theorem A.22 (Pachner).** Let \( K \) and \( L \) be the finite abstract simplicial complexes associated with closed oriented combinatorial \( d \)-manifolds \(|K|\) and \(|L|\), \(d \in \mathbb{N}\). The combinatorial manifolds \(|K|\) and \(|L|\) are PL-isomorphic if and only if the abstract simplicial complexes \( K \) and \( L \) are related by a finite sequence of bistellar moves, the so-called Pachner moves.

The Pachner moves in dimension \( d \in \mathbb{N} \) can be described as follows. Consider an oriented \((d+1)\)-simplex \( \sigma = [0,1,\ldots,d+1] \) where we write integer numbers for the linearly ordered vertices. Write down the set of faces of \( \sigma \). These are \( d+2 \) oriented \( d \)-simplices,

\[
\{ +[1,2,\ldots,d+1],-[0,2,3,\ldots,d+1],+[0,1,3,4,\ldots,d+1],\ldots,(-1)^d [0,1,\ldots,d] \}. \tag{A.14}
\]

For each \( \ell \in \mathbb{N}, \, 1 \leq \ell \leq d/2+1 \), partition this set into the following subsets,

\[
A_\ell = \{ +[1,2,\ldots,d+1],\ldots,(-1)^{\ell-1} [0,\ldots,\widehat{\ell-1},\ldots,d+1] \}, \quad \tag{A.15a}
B_\ell = \{ -(-1)^{\ell} [0,\ldots,\widehat{\ell},\ldots,d+1],\ldots,(-1)^d [0,1,\ldots,d] \}, \quad \tag{A.15b}
\]

where the hat denotes omission of a vertex and where we have reversed the orientation of all simplices in \( B_\ell \).

Both \( A_\ell \) and \( B_\ell \) generate oriented abstract simplicial complexes \( A_\ell \) and \( B_\ell \). Observe that both polyhedra \( \overline{A_\ell} \) and \( \overline{B_\ell} \) have PL-isomorphic boundaries. The bistellar \( \ell \)-move consists of cutting out the simplices of \( A_\ell \) from the given abstract simplicial complex and gluing in those of \( B_\ell \). The Pachner moves in dimension \( d \) are the bistellar \( \ell \)-moves, \( 1 \leq \ell \leq d/2+1 \), and their inverses. For manifolds with boundary, the Pachner moves are the so-called elementary shellings [20].

### A.7 Homotopy type

Topological manifolds can not only be compared by studying whether they are homeomorphic or not, but there is also the weaker notion of *homotopy type*.

**Definition A.23.** Let \( M, N \) be topological spaces.

1. Continuous maps \( f,g: M \to N \) are called **homotopic** \((f \simeq g)\) if there exists a continuous map \( F: [0,1] \times M \to N \) such that \( F(0,p) = f(p) \) and \( F(1,p) = g(p) \) for all \( p \in M \). The map \( F \) is called a **homotopy**.

2. \( M \) and \( N \) are called **homotopy equivalent** or of the same **homotopy type** \((M \simeq N)\) if there exist continuous maps \( f: M \to N \) and \( h: N \to M \) such that \( f \circ h \simeq \text{id}_N \) and \( h \circ f \simeq \text{id}_M \).
A.8 Comparison of structures

By Theorem A.15 any two diffeomorphic smooth manifolds have got PL-isomorphic PL-manifolds as their Whitehead triangulations. Each PL-manifold has got some underlying topological manifold, and PL-isomorphic PL-manifolds obviously have homeomorphic underlying topological manifolds. Finally, homeomorphic topological manifolds are always of the same homotopy type. We therefore have the following hierarchy,

\[
\text{diffeomorphic} \implies \text{PL-isomorphic} \implies \text{homeomorphic} \implies \text{homotopy equivalent} \quad (A.16)
\]

Which of these arrows can be reversed? It turns out that the answer to this question strongly depends on the dimension. Before we comment of the individual arrows in greater detail, we mention the following algebraic limitation to the classification of manifolds.

A.8.1 Impossibility of complete classification

A complete classification of manifolds of dimension \(d \geq 4\) is not possible, even if the question is restricted to the classification of topological manifolds up to homotopy equivalence. Manifolds \(M \simeq N\) of the same homotopy type have isomorphic fundamental groups \(\pi_1(M) \cong \pi_1(N)\). The problem is that for any \(d \geq 4\) and for any group \(G\) given by a finite presentation in terms of generators and relations, there exists a closed topological \(d\)-manifold \(M\) with \(\pi_1(M) = G\), see, for example [16]. But there exists no algorithm that decides for any two given finite group presentations whether they describe isomorphic groups or not [55].

This algebraic difficulty is the reason why some results on \(d\)-manifolds, \(d \geq 4\), are usually stated only for closed, connected and simply connected manifolds. In the case of some generic closed and connected manifold \(M\), one would therefore study its universal covering \(\hat{M}\) first, but then the covering map itself whose fibres are just \(\pi_1(M)\), can be extremely complicated.

A.8.2 PL-isomorphism versus diffeomorphism

In dimensions \(d \leq 6\), the equivalence classes of smooth manifolds up to diffeomorphism are in one-to-one correspondence with those of PL-manifolds up to PL-isomorphism as the following theorem shows.

**Theorem A.24.** Let \(d \leq 6\). Each \(d\)-dimensional PL-manifold admits a smoothing, and any two PL-isomorphic PL-manifolds of dimension \(d\) give rise to diffeomorphic smoothings.

For \(d \leq 3\), this theorem is a classical result which already holds for the underlying topological manifolds and the unique smooth structure they admit [56]. For \(d \in \{5, 6\}\), it follows from a theorem of Kirby–Siebenmann [50] whereas in \(d = 4\), it is a consequence of a result by Cerf [57] in the context of the general smoothing theory, see, for example [51, 58].

For a given \(d\)-dimensional PL-manifold, \(d \geq 7\), there can be obstructions to the existence of smoothings as well as a finite ambiguity, i.e. there can exist a finite number of pairwise non-diffeomorphic smoothings, see, for example [50]. An example for such an ambiguity is provided by the exotic smooth structures of \(S^7\) [17], and there exists an 8-dimensional PL-manifold that does not admit any differentiable structure at all [50]. Usually, the smoothings of some given PL-manifold (and also the PL-structures of some given topological manifold) are classified not up to diffeomorphism, but up to the stronger relation of isotopy [50, 51]. We have formulated the results above (Theorem A.15 and A.24) in such a way that this subtlety is
hidden: If two \(d\)-dimensional PL-manifolds, \(d \leq 6\), are PL-isomorphic, then their smoothings are diffeomorphic. The corresponding diffeomorphism, however, is in general not the same map as the PL-isomorphism with which one has started, but only isotopic to it.

### A.8.3 Homeomorphism versus PL-isomorphism

In dimension \(d \leq 3\), the equivalence classes of PL-manifolds up to PL-isomorphism are in one-to-one correspondence with those of topological manifolds up to homeomorphism because of the following theorem [56].

**Theorem A.25 (Bing, Moise).** Let \(d \leq 3\). Each topological \(d\)-manifold admits a PL-structure (proof of the triangulation conjecture). Homeomorphic topological \(d\)-manifolds give rise to PL-isomorphic PL-structures (proof of the Hauptvermutung of Steinitz).

This is no longer true if \(d = 4\). The work of Freedman [59] and Donaldson [4] has lead to the construction of families of smooth manifolds that are homeomorphic, but pairwise non-diffeomorphic. There exists, for example, an uncountable family of pairwise non-diffeomorphic differentiable structures on the topological manifold \(\mathbb{R}^4\) [15, 16], and there are countably infinite families of homeomorphic, but pairwise non-diffeomorphic compact smooth manifolds [13, 14]. Furthermore, there exist topological 4-manifolds that do not admit any differentiable structure at all [59].

Together with Theorem A.15 and Theorem A.24, this implies that in \(d = 4\), there is a substantial discrepancy between the classification of topological manifolds up to homeomorphism and that of PL-manifolds up to PL-isomorphism.

In dimension \(d \geq 5\), there can also be both obstructions to the existence of a PL-structure on some given topological manifold as well as an ambiguity, but the ambiguity leaves only a finite number of choices, see, for example [50]. Special attention has to be paid to 5-manifolds whose boundary is non-empty because of the special features in \(d = 4\).

### A.8.4 Homotopy equivalence versus homeomorphism

It is a classical result that any two closed oriented topological 2-manifolds that are homotopy equivalent, are in fact homeomorphic. The only invariant required in order to determine the homotopy type of a closed oriented topological 2-manifold, is the genus.

This correspondence disappears in dimension \(d = 3\) in view of the lens spaces \(L(p,q)\). There exist closed oriented topological 3-manifolds that are homotopy equivalent, but not homeomorphic, for example \(L(7,1)\) and \(L(7,2)\) [43].

In dimension \(d \geq 4\), however, homotopy type is again very important in the classification of topological manifolds. Very strong results are provided by the \(h\)-cobordism theorems of Smale [60] \((d \geq 5\), even in the smooth case\) and of Freedman [59] \((d = 4\), restricted to the topological case\). In particular, there is Freedman’s theorem [59].

**Theorem A.26 (Freedman).** Let \(M, N\) be connected and simply connected closed topological 4-manifolds. If \(M\) and \(N\) are homotopy equivalent, then they are homeomorphic.

Furthermore, the generalized Poincaré conjecture is true in \(d = 4\) [59] as well as in \(d \geq 5\) [60, 61].

**Theorem A.27 (generalized Poincaré conjecture).** Let \(M\) be a topological \(d\)-manifold, \(d \geq 4\). If \(M\) is homotopy equivalent to \(S^d\), then \(M\) is homeomorphic to \(S^d\).
If one wants to study smooth manifolds and start from the underlying manifold up to homotopy type, one has to supply additional data on the structure of the tangent bundle and then resolve a finite ambiguity, see, for example [62].

A.8.5 Summary (Table 1 of Section 4)

In dimension $d = 2$, all four arrows of (A.16) can be reversed, at least for closed oriented manifolds.

In dimension $d = 3$, each topological manifold still admits a differentiable structure that is unique up to diffeomorphism, but homotopy equivalence no longer implies homeomorphism.

In dimension $d = 4$, each PL-manifold still admits a smoothing that is unique up to diffeomorphism. Topological manifolds of the same homotopy type are also homeomorphic, at least for closed, connected and simply connected manifolds, but homeomorphism no longer implies diffeomorphism. There can rather exist infinitely many pairwise inequivalent differentiable structures for the same underlying topological manifold.

In dimension $d \geq 5$, there can finally be obstructions and ambiguities both for having PL-structures on a given topological manifold and for smoothing a given PL-manifold. All the ambiguities in $d \geq 5$, however, are finite [50]. What we have summarized so far, finally implies Theorem 1.1 of the Introduction.

There are therefore two very striking ‘gaps’ in which the material we have reviewed so far, does not suffice in order to reverse the arrows of (A.16),

- in $d = 3$ to study topological manifolds that are homotopy equivalent but not homeomorphic,
- in $d = 4$ to study smooth manifolds that are homeomorphic, but not diffeomorphic.

The Turaev–Viro invariant [27], closely related to the partition function $Z(M)$ of quantum gravity in $d = 2+1$, provides non-trivial topological information on the first of these questions. We speculate that quantum gravity in $d = 3 + 1$ will provide new insight into the second question.

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