REAL ANALYTIC EXPANSION OF SPECTRAL PROJECTION
AND EXTENSION OF HECKE-BOCHNER IDENTITY

RAJESH K. SRIVASTAVA

Abstract. In this article, we review the Weyl correspondence of bigraded spherical harmonics and use it to extend the Hecke-Bochner identities for the spectral projections $f \times \phi_k^{-1}$ for function $f \in L^p(\mathbb{C}^n)$ with $1 \leq p \leq \infty$. We prove that spheres are sets of injectivity for the twisted spherical means with real analytic weight. Then, we derive a real analytic expansion for the spectral projections $f \times \phi_k^{-1}$ for function $f \in L^2(\mathbb{C}^n)$.

1. Introduction

Weyl correspondence is a natural question of asking about an operator analogue of the non-commutative polynomials on $\mathbb{C}^n$. Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $\mathbb{H}$ be a separable Hilbert space. Let $W_1, \ldots, W_n, W_1^+, \ldots, W_n^+$ be unbounded operators on $\mathbb{H}$ satisfying

$$W_j^+ = W_j^* \text{ and } [W_j^+, -W_j] = \frac{\lambda}{2} I, \ j = 1, 2, \ldots, n,$$

on a dense subspace $\mathcal{D}$ of $\mathbb{H}$ and all other commutators are zero. Consider a polynomial $P(z) = z_1^2 \bar{z}_1$. In general, the question (asked by Weyl) about the possible expression for $P(W, W^+)$, is still open. In 1984, Geller has partially answered Weyl’s question about operator analogue of the harmonic polynomials. Let $z \in \mathbb{C}^n$. For $\alpha, \beta \in \mathbb{N}^n$ and $P(z) = z^\alpha \bar{z}^\beta$, define

$$\tau(P) = (W^+)^\beta W^\alpha \text{ and } \tau'(P) = W^\alpha (W^+)^\beta.$$

The maps $\tau$ and $\tau'$ can be linearly extended to any polynomial on $\mathbb{C}^n$. In a long paper $[5]$, Geller prove that for any harmonic polynomial $P$, $\tau(P) = \tau'(P)$. Using this, an analogue of Hecke-Bochner identity for the Weyl transform has been derived, (see $[5]$, p.645, Theorem 4.2).

A continuous function $f$ on $\mathbb{R}^n$ can be decomposed in terms of spherical harmonics as

$$f(x) = \sum_{k=0}^{\infty} a_{k,j}(\rho) Y_{k,j}(\omega),$$

where $x = \rho \omega, \ \rho = |x|, \ \omega \in S^{n-1}$ and $\{Y_{k,j}(\omega) : 1, 2, \ldots, d_k\}$ is an orthonormal basis for the space $V_k$ of the homogeneous harmonic polynomials in $n$ variables.

\textbf{Date:} May 10, 2014.

\textbf{2000 Mathematics Subject Classification.} Primary 43A85; Secondary 44A35.

\textbf{Key words and phrases.} Coxeter group, Hecke-Bochner identity, Heisenberg group, Laguerre polynomials, spherical harmonics, twisted convolution.
of degree \( k \), restricted to the unit sphere \( S^{n-1} \) with the series in the right-hand side converges locally uniformly to \( f \). However, for more details, see [13].

The well know Heche-Bochner identity says that the Fourier transform of any piece in the above decomposition (1.1) is preserved, (see [13]). That is, the Fourier transform of \( \tilde{a}P_k \in L^1(\text{or } L^2) \), satisfies \( \hat{\tilde{a}P_k}(x) = \tilde{b}(|x|)P_k(x) \), where \( P_k \) is a solid spherical harmonic of degree \( k \). An analogue of the Hecke-Bochner identity for the spectral projections \( f \times \varphi^{n-1}_k \) for function \( f \in L^1(\text{or } L^2) \), satisfies

\[
\hat{\tilde{f}}(x) = \tilde{b}(|x|)\hat{f}(|x|),
\]

where \( \varphi^{n-1}_k \) is a solid spherical harmonic of degree \( k \).

An analogue of the Hecke-Bochner identity for the spectral projections \( f \times \varphi^{n-1}_k \) for function \( f \in L^1(\text{or } L^2) \), has been obtained, (see [14], p.70). Using the Weyl correspondence of the spherical harmonics, we give a much simpler proof of this result. Further, we extend this result for \( f \in L^p(\mathbb{C}^n) \) with \( 1 \leq p \leq \infty \).

As another application of the Weyl correspondence of the spherical harmonics, we prove that sphere \( S_R(o) = \{ z \in \mathbb{C}^n : |z| = R \} \) is a set of injectivity for the twisted spherical means (TSM) with real analytic weight, for the radial functions on \( \mathbb{C}^n \).

Since Laguerre function \( \varphi^{n-1}_k \) is an eigenfunction of the special Hermite operator \( A = -\Delta_z + \frac{1}{4}|z|^2 \) with eigenvalue \( 2k + n \), the projection \( f \times \varphi^{n-1}_k \) is also an eigenfunction of \( A \) with eigenvalue \( 2k + n \). As \( A \) is an elliptic operator and eigenfunction of an elliptic operator is real analytic [6], the projection \( f \times \varphi^{n-1}_k \) must be a real analytic function on \( \mathbb{C}^n \). We derive an important real analytic expansion for the spectral projections \( f \times \varphi^{n-1}_k \)'s for function \( f \in L^2(\mathbb{C}^n) \), which we call Hecke-Bochner-Laguerre series for spectral projection. In the complex plane, it is much simpler and it has been used in [10], for proving a result that any Coxeter system of even number of lines is a set of injectivity for the twisted spherical means for \( L^p(\mathbb{C}) \) with \( 1 \leq p \leq 2 \).

As an application of expansion for the spectral projections \( f \times \varphi^{n-1}_k \), we have shown that any complex cone is a set of injectivity for the twisted spherical means for \( L^p(\mathbb{C}^n) \), \( 1 \leq p \leq 2 \) as long as it does not lay on the level surface of any bi-graded homogeneous harmonic polynomial on \( \mathbb{C}^n \). An analogous result has been proved for the Euclidean spherical mean over the class of continuous functions by Zalcman et al., see [2].

2. NOTATION AND PRELIMINARIES

We define the twisted convolution which arises in the study of group convolution on Heisenberg group. The group \( \mathbb{H}^n \), as a manifold, is \( \mathbb{C}^n \times \mathbb{R} \) with the group law

\[
(z, t)(w, s) = (z + w, t + s + \frac{1}{2}\text{Im}(z, \bar{w})), \quad z, w \in \mathbb{C}^n \text{ and } t, s \in \mathbb{R}.
\]

The group convolution of function \( f, g \in L^1(\mathbb{H}^n) \) is defined by

\[
(2.1) \quad f * g(z, t) = \int_{\mathbb{H}^n} f((z, t)(-w, -s))g(w, s) \, dwds.
\]

An important technique in many problem on \( \mathbb{H}^n \) is to take partial Fourier transform in the \( t \)-variable to reduce matters to \( \mathbb{C}^n \). Let

\[
f^\lambda(z) = \int_{\mathbb{R}} f(z, t)e^{i\lambda t} \, dt
\]
be the inverse Fourier transform of \( f \) in the \( t \)-variable. Then a simple calculation shows that
\[
(f \ast g)_{\lambda}(z) = \int_{-\infty}^{\infty} f \ast g(z, t) e^{i\lambda t} \, dt
\]
\[
= \int_{\mathbb{C}^n} f^\lambda (z - w) g^\lambda(w) e^{\frac{i\lambda}{2} \text{Im}(z, w)} \, dw
\]
\[
= f^\lambda \times g^\lambda(z).
\]

Thus the group convolution \( f \ast g \) on the Heisenberg group can be studied using the \( \lambda \)-twisted convolution \( f^\lambda \times g^\lambda \) on \( \mathbb{C}^n \). For \( \lambda \neq 0 \), a further scaling argument shows that it is enough to study the twisted convolution for the case of \( \lambda = 1 \).

We need the following basic facts from the theory of bigraded spherical harmonics (see [14], p.12 for details). We shall use the notation \( K = U(n) \) and \( M = U(n - 1) \). Then, \( S^{2n-1} \cong K/M \) under the map \( kM \rightarrow k.e_n, k \in U(n) \) and \( e_n = (0, \ldots, 1) \in \mathbb{C}^n \). Let \( \hat{K}_M \) denote the set of all equivalence classes of irreducible unitary representations of \( K \) which have a nonzero \( M \)-fixed vector. It is known that for each representation in \( \hat{K}_M \) there is a unique nonzero \( M \)-fixed vector, up to a scalar multiple.

For a \( \delta \in \hat{K}_M \), which is realized on \( V_{\delta} \), let \( \{e_1, \ldots, e_{d(\delta)}\} \) be an orthonormal basis of \( V_{\delta} \) with \( e_1 \) as the \( M \)-fixed vector. Let \( t_{ij}^\delta(k) = \langle e_i, \delta(k)e_j \rangle, k \in K \) and \( \langle , \rangle \) stand for the inner product on \( V_{\delta} \). By Peter-Weyl theorem, it follows that \( \{\sqrt{d(\delta)} t_{ij}^\delta : 1 \leq j \leq d(\delta), \delta \in \hat{K}_M \} \) is an orthonormal basis of \( L^2(K/M) \) (see [14], p.14 for details). Define \( Y_{j}^\delta(\omega) = \sqrt{d(\delta)} t_{j1}^\delta(k) \), where \( \omega = k.e_n \in S^{2n-1}, k \in K \). It then follows that \( \{Y_{j}^\delta : 1 \leq j \leq d(\delta), \delta \in \hat{K}_M, \} \) forms an orthonormal basis for \( L^2(S^{2n-1}) \).

For our purpose, we need a concrete realization of the representations in \( \hat{K}_M \), which can be done in the following way. See [8], p.253, for details. For \( p, q \in \mathbb{Z}_+ \), let \( P_{p,q} \) denote the space of all polynomials \( P \) in \( z \) and \( \bar{z} \) of the form
\[
P(z) = \sum_{|\alpha| = p} \sum_{|\beta| = q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.
\]

Let \( H_{p,q} = \{P \in P_{p,q} : \Delta P = 0\} \). The elements of \( H_{p,q} \) are called the bigraded solid harmonics on \( \mathbb{C}^n \). The group \( K \) acts on \( H_{p,q} \) in a natural way. It is easy to see that the space \( H_{p,q} \) is \( K \)-invariant. Let \( \pi_{p,q} \) denote the corresponding representation of \( K \) on \( H_{p,q} \). Then representations in \( \hat{K}_M \) can be identified, up to unitary equivalence, with the collection \( \{\pi_{p,q} : p, q \in \mathbb{Z}_+\} \).

Define the bigraded spherical harmonic by \( Y_{j}^{p,q}(\omega) = \sqrt{d(p, q)} t_{j1}^{p,q}(k) \). Then \( \{Y_{j}^{p,q} : 1 \leq j \leq d(p, q), p, q \in \mathbb{Z}_+\} \) forms an orthonormal basis for \( L^2(S^{2n-1}) \). Therefore, for a continuous function \( f \) on \( \mathbb{C}^n \), writing \( z = \rho \omega \), where \( \rho > 0 \) and \( \omega \in S^{2n-1} \), we can expand the function \( f \) in terms of spherical harmonics as
\[
(2.2) \quad f(\rho \omega) = \sum_{p, q \geq 0} \sum_{j=1}^{d(p, q)} a_{j}^{p,q}(\rho) Y_{j}^{p,q}(\omega),
\]
where the series on the right-hand side converges uniformly on every compact set \( K \subseteq \mathbb{C}^n \). The functions \( a_j^{p,q} \) are called the spherical harmonic coefficients of \( f \).

3. A review on the Weyl correspondence of spherical harmonics

In this section, we revisit the Weyl correspondence of the spherical harmonics. Then, we derive some important identities which describe the action of Weyl correspondence of a harmonic polynomial to the Laguerre functions \( \varphi_k^{n-1} \)'s. We use these identities to give a very short and simple proof of the Hecke-Bochner identities for the spectral projections. We would like to collect some of the preliminaries from the work of Geller [5], where he has established the Weyl correspondence of the spherical harmonics.

Define the symplectic Fourier transform \( \mathcal{F} \) on Schwartz space \( \mathcal{S}(\mathbb{C}^n) \) by

\[
\mathcal{F}(f)(z) = \int_{\mathbb{C}^n} e^{-\frac{i}{4} \text{Im}(z.\bar{w})} f(w) dw
\]

Let \( \mathcal{B}(\mathbb{H}) \) be the space of all bounded linear operators on \( \mathbb{H} \). Since the operator \(-i(-z.W^* + \bar{z}.W)\) is essentially self-adjoint, therefore, we can define the operator analogue of function \( \mathcal{F}^{-1}(f) \) by an operator \( \mathcal{G} : \mathcal{S}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{H}) \), which is given by

\[
\mathcal{G}f = \int_{\mathbb{C}^n} \exp\left(\frac{-z.W^* + \bar{z}.W}{4}\right) f(z) dz.
\]

The operator \( \mathcal{G} \) is known as the Weyl transform. A composite operator \( \mathcal{W} : \mathcal{S}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{H}) \) which is defined by

\[
\mathcal{W}(f) = \mathcal{G} \circ \mathcal{F}^{-1}(f),
\]

is the Weyl correspondence of \( f \). We state the following results which are proved for the Weyl correspondence \( \mathcal{W} \) of the spherical harmonics, (see [5]). We use these results for proving our main results.

Lemma 3.1. [5] If \( P \) is a harmonic polynomial then

\[
\mathcal{W}(P) = \tau(P) = \tau'(P).
\]

Lemma 3.2. [5] Let \( P, P_1 \in \mathcal{P}_{p,q} \). Suppose there exist \( \sigma \in U(n) \) such that \( P = \pi(\sigma)P_1 \), where \( \pi \) is an unitary representation of \( U(n) \). Then

\[
\mathcal{W}(P) = \begin{cases} 
\pi(\bar{\sigma})\mathcal{W}(P_1)\pi(\bar{\sigma})^*, & \text{if } \lambda > 0; \\
\pi(\sigma)\mathcal{W}(P_1)\pi(\sigma)^*, & \text{if } \lambda < 0,
\end{cases}
\]

on the dense subspace \( \mathcal{D} \) of \( \mathbb{H} \).

Let us consider the following invariant differential operators which arises in study of the twisted convolution on \( \mathbb{C}^n \):

\[
W_j = \tilde{Z}_{j,\lambda} = \frac{\partial}{\partial z_j} - \frac{\lambda}{4} \bar{z}_j \text{ and } W_j^* = \tilde{Z}_{j,\lambda}^* = \frac{\partial}{\partial \bar{z}_j} + \frac{\lambda}{4} z_j, \ j = 1, 2, \ldots, n.
\]
Let $P$ be a non-commutative homogeneous harmonic polynomial on $\mathbb{C}^n$ with expression

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.$$ 

Since $P$ is harmonic, by Lemma 3.1, the operator $P(\tilde{Z})$ is the Weyl correspondence $\mathcal{W}(P)$ of $P$. Hence the operator $P(\tilde{Z})$ can be expressed as

$$P(\tilde{Z}) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} \tilde{Z}^\alpha \tilde{Z}^\beta.$$ 

Next, we describe the action of operator $P(\tilde{Z})$ to the Laguerre functions. For $k \in \mathbb{Z}_+$, the Laguerre functions $\varphi_{n-1}^k$ are defined by

$$\varphi_{n-1}^k(z) = L_{n-1}^k(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2},$$

where $L_{n-1}^k$'s are the Laguerre polynomials of degree $k$ and order $n-1$. In [5], Geller had proved an analogue of the Hecke-Bochner identity for the Weyl transform of the type functions $\tilde{a}P \in L^1(\mathbb{C}^n)$, which in turn gives an analogous result for the spectral projections, (see [7]). That is, spectral projection of any type function is a type function. An application of the following identities gives a very simple proof of the Hecke-Bochner identity for the spectral projections.

**Lemma 3.3.** [9] For $P_1(z) = z^p \bar{z}^q \in H_{p,q}$, we have

$$P_1(\tilde{Z}) \varphi_{n-1}^k(z) = (-2)^{-p-q} P_1(z) \varphi_{n+p+q-1}^{n+p+q-1}(z),$$

if $k \geq p$ and 0 otherwise.

**Proof.** We have

$$\tilde{Z}_1 \varphi_{n-1}^k(z) = \left( \frac{\partial}{\partial \tilde{z}_1} + \frac{1}{4} z_1 \right) \varphi_{n-1}^k(z)$$

For $z \in \mathbb{C}^n$, let $z.\bar{z} = 2t$. By chain rule $\frac{\partial}{\partial z_1} = \frac{1}{2} z_1 \frac{\partial}{\partial t}$. Therefore,

$$\tilde{Z}_1 \varphi_{n-1}^k(z) = \left( \frac{1}{2} z_1 \frac{\partial}{\partial t} + \frac{1}{4} z_1 \right) \left( L_{n-1}^k(t) e^{-\frac{1}{2} t} \right)$$

$$= \frac{1}{2} z_1 \left( \frac{\partial}{\partial t} L_{n-1}^k(t) - \frac{1}{2} L_{n-1}^k(t) + \frac{1}{2} L_{n-1}^k(t) \right) e^{-\frac{1}{2} t}.$$ 

Since, the Laguerre’s polynomials $L_n^k$’s satisfy the recursion relations

$$\frac{d}{dx} L_n^k(x) = -L_{n+1}^k(x) \text{ and } L_{n+1}^k(x) + L_n^k(x) = L_{n+1}^k(x).$$
Therefore, we can write $\bar{Z}_1 \varphi_k^{n-1}(z) = -\frac{i}{2} z_1 \varphi_k^{n-1}(z)$. Similarly
\[
\bar{Z}_2 \varphi_k^{n-1}(z) = \left( \frac{1}{2} \bar{z}_2 \frac{\partial}{\partial t} - \frac{1}{4} \bar{z}_2 \right) \left( L_k^{n-1}(t)e^{-\frac{t}{2}} \right) 
= \frac{1}{2} \bar{z}_2 \left( \frac{\partial}{\partial t} L_k^{n-1}(t) - \frac{1}{2} L_k^{n-1}(t) - \frac{1}{2} L_k^{n-1}(t) \right) e^{-\frac{t}{2}} 
= -\frac{1}{2} \bar{z}_2 \varphi_k^{n}(z).
\]
Thus, we have $\bar{Z}_1 \bar{Z}_2 \varphi_k^{n-1}(z) = 2^{-2} z_1 \bar{z}_2 \varphi_k^{n+1}(z)$. Hence, by induction, we can conclude that
\[
\bar{Z}_1 \bar{Z}_2 \varphi_k^{n-1}(z) = (-2)^{-p-q} z_1^p \bar{z}_2^q \varphi_k^{n+1}(z).
\]

The identities (3.11) has been used in an article [9] for proving a result that any sphere centered at the origin is a set of injectivity for the weighted twisted spherical means on $\mathbb{C}^n$. In this article, we generalize the identities (3.11) for an arbitrary bigraded spherical harmonic $P \in H_{p,q}$. We further extend the identities (3.11) for generalized Laguerre functions.

We use Lemma 3.3 to deduce the following identities which have been used frequently. For $\lambda \in \mathbb{R} \setminus \{0\}$, let $\varphi_{k,\lambda}^{n}(z) = \varphi_k^{n-1}(|\lambda|z)$.

**Theorem 3.4.** Let $P \in H_{p,q}$. Then
\[
P(\bar{Z}_1 \varphi_k^{n-1}(z)) = \begin{cases} (-2\lambda)^{-p-q} P \varphi_k^{n+p+q-1}(z), & \text{if } \lambda < 0, \ k \geq p; \\ (-2\lambda)^{-p-q} P \varphi_k^{n+p+q-1}(z), & \text{if } \lambda > 0, \ k \geq q. \end{cases}
\]

**Proof.** Using the scaling argument, it is enough to prove these identities for the cases $\lambda = \pm 1$. For $\lambda = 1$, we claim
\[
P(\bar{Z}_1 \varphi_k^{n-1}(z)) = (-2)^{-p-q} P \varphi_k^{n+p+q-1}(z), \quad \text{if } k \geq p.
\]

We use the fact that any irreducible representation of a group is cyclic [11]. Since $H_{p,q}$ is an irreducible representation of $U(n)$, it follows that every non-zero element of $H_{p,q}$ is a cyclic vector. Let $P_1(z) = z_1^p z_2^q \in H_{p,q}$. Then any $P \in H_{p,q}$ can be written as
\[
P = \sum_{i=1}^{m} \pi(\sigma_i) P_1.
\]
Since $W$ is a linear operator, without loss of generality, we can assume that $P = \pi(\sigma) P_1$. In view of Lemma 3.2 we can write
\[
W(P) = \pi(\sigma) W(P_1) \pi(\sigma)^*.
\]
Since $\varphi_k^{n-1}$ is $U(n)$-invariant, by Lemma 3.1 we have
\[
P(\bar{Z}_1 \varphi_k^{n-1}) = \pi(\sigma) P_1(\bar{Z}) \varphi_k^{n-1}.
\]
An application of Lemma 3.3 gives

$$P(\tilde{Z})\varphi_{k-1}^{n-1} = (-2)^{-p-q+1} \pi(\sigma) (P_1\varphi_{k-1}^{n+p+q-1})$$

$$= (-2)^{-p-q+1} \pi(\sigma) P_1\varphi_{k-1}^{n+p+q-1}$$

$$= (-2)^{-p-q} P\varphi_{k-1}^{n+p+q-1}.$$  

It is clear from the above computation and Lemma 3.3 that the proof for the case $\lambda = -1$ is similar and hence we omit it here.

**Alternative proof:** In the above proof, Lemma 3.2 played a crucial role. However, we give an alternative proof of the identities (3.3) without using Lemma 3.2. Since the symplectic Fourier transform $F$ is invariant under $U(n)$ action, it follows that

$$F^{-1}(P) = F^{-1}(\pi(\sigma)P_1) = \pi(\sigma)F^{-1}(P_1).$$

That is,

$$P \left( \frac{\partial}{\partial \tilde{z}} \cdot \frac{\partial}{\partial \tilde{\bar{z}}} \right) \delta = \pi(\sigma)P_1 \left( \frac{\partial}{\partial \tilde{z}} \cdot \frac{\partial}{\partial \tilde{\bar{z}}} \right) \delta = \pi(\sigma)P_1(\tilde{Z})\delta,$$

where $\delta$ is Dirac distribution at the origin. Since, we know that $z_j\delta = 0 = \bar{z}_j\delta$, we can write $P(\tilde{Z})\delta = \pi(\sigma)P_1(\tilde{Z})\delta$. For more detail, see Geller [5], p. 625. Denote

$$T\delta = \left( P(\tilde{Z}) - \pi(\sigma)P_1(\tilde{Z}) \right) \delta = 0.$$

For $z, w \in \mathbb{C}^n$, define $\tau_z(w) = (w - z)e^{\frac{i}{2}\text{Im}(z,w)}$. Then, it follows that $\tau_z T\delta = 0$ on $S(\mathbb{C}^n)$. For $\varphi = \varphi_k^{n-1}$, we have

$$\tau_z T\delta(\varphi) = \delta(\tau_z T\varphi) = \tau_z (T\varphi) (0) = T\varphi(z) = 0.$$

Hence

$$P(\tilde{Z})\varphi_{k-1}^{n-1} = \pi(\sigma)P_1(\tilde{Z})\varphi_{k-1}^{n-1}$$

$$= (-2)^{-p-q+1} \pi(\sigma) P_1\varphi_{k-1}^{n+p+q-1}$$

$$= (-2)^{-p-q} P\varphi_{k-1}^{n+p+q-1}.$$  

These identities are quite useful and lead to a very short and simple proof of the Hecke-Bochner identity for the spectral projections $f \times \varphi_k^{n-1}$ for function $f \in L^2(\mathbb{C}^n)$, (see [14], p.70). Using the identities (3.3), we prove that the boundary of any bounded domain is a set of injectivity for certain weighted twisted spherical means for the radial functions on $\mathbb{C}^n$.

**Lemma 3.5.** Let $\hat{a}P \in L^2(\mathbb{C}^n)$, where $\hat{a}$ is a radial function and $P \in H_{p,q}$. Then

$$(3.5) \quad \hat{a}P \times \varphi_{k-1}^{n-1}(z) = (2\pi)^{-n} P(z) \hat{a} \times \varphi_{k-1}^{n+p+q-1}(z),$$

if $k \geq p$ and 0 otherwise. The convolution in the right hand side is on the space $\mathbb{C}^{n+p+q}$.  

Proof. Since the Laguerre functions \( \{ \varphi^{n+p+q-1}_{k-p}(r) : k \geq p \} \) forms an orthonormal basis for \( L^2(\mathbb{R}^+, r^{2(n+p+q)-1}dr) \), therefore, we can express \( \hat{a} \) as
\[
\hat{a} = \sum_{k \geq p} C_{k-p} \varphi^{n+p+q-1}_{k-p}.
\]

An application of the identities (3.3) for \( \lambda = 1 \) gives
\[
\hat{a} P = (-2)^{p+q} \sum_{k \geq p} C_{k-p} P(\tilde{Z}) \varphi^{n-1}_{k}.
\]

Let \( j \geq p \). Convolve both sides of the above equation by \( \varphi^{n-1}_{j} \). Then, using the fact that \( P(\tilde{Z}) \) is a left invariant operator, we can write
\[
\hat{a} P \times \varphi^{n-1}_{j} = (-2)^{p+q} \sum_{k \geq p} C_{k-p} P(\tilde{Z})(\varphi^{n-1}_{k} \times \varphi^{n-1}_{j}).
\]

Since the Laguerre functions satisfy the orthogonality conditions \( \varphi^{n-1}_{k} \times \varphi^{n-1}_{j} = (2\pi)^n \delta_{jk} \varphi^{n-1}_{k} \). Therefore,
\[
\hat{a} P \times \varphi^{n-1}_{j}(z) = (2\pi)^n (-2)^{p+q} C_{j-p} P(\tilde{Z}) \varphi^{n-1}_{j}(z) = (2\pi)^n \langle \hat{a}, \varphi^{n+p+q-1}_{j-p} \rangle P(z) \varphi^{n+p+q-1}_{j-p}(z) = (2\pi)^{-n} P(z) \hat{a} \times \varphi^{n+p+q-1}_{j-p}(z).
\]

Since the subspace \( L^2 \cap L^r(\mathbb{C}^n) \) is dense in \( L^r(\mathbb{C}^n) \) for \( 1 \leq r < \infty \), therefore, the identities (3.3) can be extended for \( f \in L^r(\mathbb{C}^n) \). However, for functions in \( L^\infty(\mathbb{C}^n) \), we use the weighted functional equations for spherical function \( \varphi^{n-1}_{k} \). The weighted functional equations can be obtained by considering the Hecke-Bochner identity for the spectral projection of compactly supported functions. For more details, see [14], p. 98.

Lemma 3.6. [14] For \( z \in \mathbb{C}^n \), let \( P \in H_{p,q} \) and \( dv_t = Pd\mu_t \). Then
\[
\varphi^{n-1}_{k} \times \nu_t(z) = (2\pi)^{-n} C(k, n+p+q)r^{2(p+q)}\varphi^{n+p+q-1}_{k}(t)P(z)\varphi^{n+p+q-1}_{k}(z),
\]
if \( k \geq q \) and 0 otherwise.

Following result is an extension of the Hecke-Bochner identities for the spectral projections for function \( f \in L^r(\mathbb{C}^n) \) with \( 1 \leq r \leq \infty \).

Theorem 3.7. Let \( \hat{a} P \in L^r(\mathbb{C}^n), 1 \leq r \leq \infty \), where \( \hat{a} \) is a radial function and \( P \in H_{p,q} \). Then
\[
\hat{a} P \times \varphi^{n-1}_{k}(z) = (2\pi)^{-n} C(k, n+p+q)\varphi^{n+p+q-1}_{k-p}(z)P(z)\varphi^{n+p+q-1}_{k-p}(z),
\]
if \( k \geq p \) and 0 otherwise.
Proof. Since $\tilde{a}P \in L^r(\mathbb{C}^n)$, for $1 \leq r \leq \infty$. By Young’s inequality, $\tilde{a}P \times \varphi_k^{n-1}$ exists and smooth. Consider the equation $\tilde{a}P \times \varphi_k^{n-1}(z) = \varphi_k^{n-1} \times \tilde{a}P(z)$. Let $\nu \mu = \bar{P} \mu$. By polar decomposition, we can write

$$\tilde{a}P \times \varphi_k^{n-1}(z) = \int_0^\infty d\nu \mu \tilde{a}_P(t) \varphi_k^{n-1}(z) \bar{a}(t) t^{2n-1} dt.$$ 

In view of Lemma 3.6, the above equation can be written as

$$\tilde{a}P \times \varphi_k^{n-1}(z) = (2\pi)^{-n} C(k, \gamma) \bar{P}(z) \varphi_k^{2n-1}(z) \int_0^\infty \varphi_k^{2n-1}(z) \tilde{a}(t) t^{2n-1} dt,$$

where $\gamma = n + p + q$. By taking the complex conjugate of both sides, we get

(3.6) $\tilde{a}P \times \varphi_k^{n-1}(z) = (2\pi)^{-n} C(k, \gamma) P(z) \varphi_k^{2n-1}(z) \int_0^\infty \tilde{a}(t) \varphi_k^{2n-1}(t) t^{2n-1} dt.$

This completes the proof of the theorem.

Remark 3.8. From (3.6), we observe that the identities (3.5) can be extended for more general class of functions. An easy example is when $\tilde{a}$ to be a polynomial. Since the integral in the right-hand side of (3.6) exists, even when $\tilde{a}$ is a radial tempered distribution, therefore, it is a feasible question to prove the identities (3.5) for $f$ to be a tempered distribution, which we leave open for the time being.

As another application of the identities (3.3), we prove that the boundary of any bounded domain is a set of injectivity for certain weighted twisted spherical means, for the radial functions on $\mathbb{C}^n$. The weights here are spherical harmonics on $\mathbb{S}^{2n-1}$. We use this to prove that sphere $S_R(o)$ is a set injectivity for the twisted spherical means with real analytic weight, for the radial functions on $\mathbb{C}^n$. Since this article is more concerned about Weyl correspondence and its applications, therefore, we skip here to write more histories of sets of injectivity for the twisted spherical means. We would like to refer [1, 7, 9, 10, 12].

In order to prove these results, we need the following result by Filaseta and Lam [11], about the irreducibility of the Laguerre polynomials. Define the Laguerre polynomials by

$$L_k^\alpha(x) = \sum_{i=0}^{k} (-1)^i \binom{\alpha + k}{k - i} \frac{x^i}{i!},$$

where $k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{C}$.

Theorem 3.9. [11] Let $\alpha$ be a rational number, which is not a negative integer. Then for all but finitely many $k \in \mathbb{Z}_+$, the polynomial $L_k^\alpha(x)$ is irreducible over the field of rationals.

Using Theorem 3.9, we have obtained the following corollary about the zeros of the Laguerre polynomials.

Corollary 3.10. Let $k \in \mathbb{Z}_+$. Then for all but finitely many $k$, the Laguerre polynomials $L_k^{n-1}$’s have distinct zeros over the reals.
Proof. By Theorem 3.9, there exists \( k_0 \in \mathbb{Z}_+ \) such that \( L_{k_0}^{n-1} \)'s are irreducible over \( \mathbb{Q} \) whenever \( k \geq k_0 \). Therefore, we can find polynomials \( P_1, P_2 \in \mathbb{Q}[x] \) such that \( P_1 L_{k_1}^{n-1} + P_2 L_{k_2}^{n-1} = 1 \), over \( \mathbb{Q} \) with \( k_1, k_2 \geq k_0 \). Since this identity continue to hold on \( \mathbb{R} \), it follows that \( L_{k_1}^{n-1} \) and \( L_{k_2}^{n-1} \) have no common zero over \( \mathbb{R} \). \( \square \)

In order to prove Lemma 3.12, we also need the following lemma from [14] about the spectral projections of a radial function.

Lemma 3.11. [14] Let \( f \) be a radial function in \( L^2(\mathbb{C}^n) \). Then

\[
 f \times \varphi_k^{n-1}(z) = B_k^n \langle f, \varphi_k^{n-1} \rangle \varphi_k^{n-1} \quad \text{where} \quad B_k^n = \frac{k!(n-1)!}{(n+k-1)!}.
\]

Thus \( f \) can expressed as

\[
 f = (2\pi)^{-n} \sum_{k=0}^{\infty} B_k^n \langle f, \varphi_k^{n-1} \rangle \varphi_k^{n-1}.
\]

Let \( \partial \Omega \) be the boundary of a bounded domain \( \Omega \) in \( \mathbb{C}^n \). Let \( \mu_r \) be the normalized surface measure on the sphere \( S_r(o) = \{ z \in \mathbb{C}^n : |z| = r \} \) in \( \mathbb{C}^n \). For \( P \in H_{p,q} \), write \( d\nu_r = Pd\mu_r \).

Lemma 3.12. Let \( f \) be a radial function on \( \mathbb{C}^n \) such that \( e^{\frac{1}{4}|z|^2} f(z) \in L^p(\mathbb{C}^n) \), for \( 1 \leq p \leq \infty \). Suppose \( f \times \nu_r(z) = 0, \forall \; z \in \partial \Omega \) and \( \forall \; r > 0 \). Then \( f = 0 \) a.e. on \( \mathbb{C}^n \).

Proof. Given that \( f \times \nu_r(z) = 0, \forall \; z \in \partial \Omega \) and \( \forall \; r > 0 \). By polar decomposition, we can write

\[
 f \times P \varphi_k^{n+p+q-1}(z) = \int_{r=0}^{\infty} f \times \nu_r(z) \varphi_k^{n+p+q-1}(r) r^{2n-1} dr = 0,
\]

whenever \( z \in \partial \Omega \) and \( k \geq q \). Let \( \varphi_\epsilon \) be a smooth, radial compactly supported approximate identity on \( \mathbb{C}^n \). Then \( f \times \varphi_\epsilon \in L^1 \cap L^\infty(\mathbb{C}^n) \) and in particular \( f \times \varphi_\epsilon \in L^2(\mathbb{C}^n) \). Since \( \varphi_\epsilon \) is radial, by Lemma 3.11, we can write

\[
 f \times \varphi_\epsilon \times \nu_r(z) = \sum_{k \geq 0} B_k^n \langle \varphi_\epsilon, \varphi_k^{n-1} \rangle f \times (\varphi_k^{n-1} \times \nu_r)(z).
\]

ByLemma 3.6, it follows that \( f \times \varphi_\epsilon \times \nu_r(z) = 0, \forall \; k \geq q \) and \( \forall \; z \in \partial \Omega \). Thus, without loss of generality, we can assume \( f \in L^2(\mathbb{C}^n) \). We use the identities (3.3) for \( \lambda = -1 \). Let

\[
 \tilde{A}_j = \tilde{Z}_{j,-1} = \frac{\partial}{\partial z_j} + \frac{1}{4} \tilde{z}_j \quad \text{and} \quad \tilde{A}_j^* = \tilde{Z}_{j,-1}^* = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4} \tilde{z}_j, \; j = 1, 2, \ldots, n.
\]

Then the set \( \{ I, \tilde{A}_j, \tilde{A}_j^* : \; j = 1, 2, \ldots, n \} \) generates the space of all the right invariant differential operators for twisted convolution on \( \mathbb{C}^n \). Using Theorem 3.4 for \( \lambda = -1 \), we get

\[
 P(\tilde{A}) \varphi_k^{n-1}(z) = (-2)^{-p-q} P(z) \varphi_k^{n+p+q-1}(z),
\]
if $k \geq q$ and 0 otherwise. In view of the identities (3.8) and the fact that $P(\hat{A})$ is a right invariant operator, equation (3.7) gives

$$P(\hat{A}) (f \times \varphi_k^{n-1})(z) = 0.$$ 

Since $f$ is radial, by Lemma 3.11 $f \times \varphi_k^{n-1} = B_k^n \langle f, \varphi_k^{n-1} \rangle \varphi_k^{n-1}$. Therefore,

$$B_k^n \langle f, \varphi_k^{n-1} \rangle P(\hat{A})(\varphi_k^{n-1})(z) = 0.$$ 

Once again using the identities (3.8), we can write

$$\langle f, \varphi_k^{n-1} \rangle P(z) \varphi_{k-q}^{n+p+q-1}(z) = 0,$$

whenever $z \in \partial \Omega$ and $k \geq q$. Since $P$ is a non zero harmonic polynomial, therefore, by the maximal principle for harmonic function, $P$ can not vanish on $\partial \Omega$. Thus,

$$\langle f, \varphi_k^{n-1} \rangle \varphi_{k-q}^{n+p+q-1}(z) = 0,$$

whenever $z \in \partial \Omega$ and $k \geq q$. Since $\varphi_{k-q}^{n+p+q-1}$ is radial, we have two cases. First, when $\partial \Omega$ is a sphere centered at the origin. Let $\partial \Omega = \{ z \in \mathbb{C}^n : |z| = R \}$. In view of Corollary 3.10, the functions $\varphi_{k-q}^{n+p+q-1}$ can not have a common zero except for finitely many $k \in \mathbb{Z}_+$ with $k \geq q$. Thus, there exists some $k_o \in \mathbb{Z}_+$ such that $\langle f, \varphi_{k}^{n-1} \rangle = 0$, $\forall k \geq k_o \geq q$. Hence $f$ is a finite linear combination of $\varphi_{k}^{n-1}$’s. By the given decay condition on $f$, we conclude that $f = 0$ a.e. on $\mathbb{C}^n$. On the other hand, when $\partial \Omega$ is of non-constant curvature (or sphere off centered the origin), we reach to the conclusion that

$$\langle f, \varphi_{k}^{n-1} \rangle \varphi_{k-q}^{n+p+q-1}(R) = 0,$$

whenever $R \in (a, b)$ with $a < b$ and $k \geq q$. This is possible only if $\langle f, \varphi_{k}^{n-1} \rangle = 0$, $\forall k \geq q$. Thus, we infer that $f = 0$ a.e. on $\mathbb{C}^n$. □

Using Lemma 3.12 we prove the following partial result for injectivity of the TSM with real analytic weight. Let $g$ be a real analytic function on $\mathbb{C}^n$. Let $d\nu_r = gd\mu_r$ and $S_R(o) = \{ z \in \mathbb{C}^n : |z| = R \}$.

Theorem 3.13. Let $f$ be a radial function on $\mathbb{C}^n$ such that $e^{\frac{1}{2}|z|^2} f(z) \in L^p(\mathbb{C}^n)$, for $1 \leq p \leq \infty$. Suppose $f \times \nu_r(z) = 0$, $\forall z \in S_R(o)$ and $\forall r > 0$. Then $f = 0$ a.e. on $\mathbb{C}^n$.

Proof. By the given conditions, we have $f \times g \mu_r(z) = 0$, $\forall z \in S_R(o)$ and $\forall r > 0$. Since $g$ is continuous, its spherical harmonic components can be expressed as

$$g_{lm}(z) = d(p, q) \int_{U(n)} g(\sigma^{-1}z) \mu_{lm}^{p,q}(\sigma)d\sigma,$$
Lemma 3.14. Let $\mu_\nu(z)$ hold for the generalized Laguerre functions $M_{\nu}(z)$. The following properties are similar to the Laguerre polynomial $L_{\nu}(z)$:

We have $f \times g_{\nu m}(z) = d(p, q) \int_{U(n)} \mu_{\nu m}^{p,q}(\sigma) \int_{S_{\nu}(\sigma)} f(z - w)g(\sigma^{-1}w)e^{2\text{Im}(\sigma^{-1}z, w)}d\mu_\nu(w)d\sigma$

for all $z \in S_\nu(\sigma)$ and $\forall r > 0$. Since, each of the component $g_{\nu m}(z)$ can also be expressed as $g_{\nu m}(z) = \tilde{\mu}_{\nu m}(\sigma)P(z)$, where $P \in H_{p,q}$. Therefore, it follows that $\tilde{\mu}_{\nu m}(r) f \times P_{\nu m}(z) = 0$, $\forall z \in S_\nu(\sigma)$ and $\forall r > 0$. Since $\tilde{\mu}_{\nu m}$ is real analytic, by Lemma 3.12 we conclude that $f = 0$ a.e. on $\mathbb{C}^n$.

We observe that the similar identities as to the identities continue to work for the generalized Laguerre functions, (see [3]). These are the eigenfunctions of the special Hermite operator, (see [1]). Let $\mathbb{C}_4 = \{a \in \mathbb{C} : -a \notin \mathbb{Z}_+, \Re(a) < 1\}$. The formal expansion of the generalized Laguerre function is

$$M_{\nu}^{n-1}(x) = \frac{\Gamma(n - a)}{\Gamma(1 - a)\Gamma(n)} \sum_{s=0}^{\infty} \frac{a_s x^s}{(n + s - 1)!}!; \ x, a \in \mathbb{C},$$

where $a_s = a(a + 1)(a + 2) \cdots (a + s - 1)$. For $a = -k$, $k \in \mathbb{Z}_+$, the function $M_{-k}^{n-1}$ is the usual Laguerre polynomial $L_k^{n-1}$ of degree $k$. For $a \in \mathbb{C}_4$, the function $M_{a}^{n-1}$ has another representation

$$M_{a}^{n-1}(x) = e^z \sum_{i=0}^{\infty} (-1)^i \binom{n + a + i - 1}{a} \frac{x^i}{i!}.$$ 

The following properties as similar to the Laguerre polynomial $L_k^{n-1}$, continue to hold for the generalized Laguerre functions $M_{a}^{n-1}$'s.

Lemma 3.14. Let $a \in \mathbb{C}_4$. Then

$$\frac{d}{dx}M_{a}^{n}(x) = -M_{a-1}^{n+1}(x) \quad \text{and} \quad M_{a-1}^{n+1}(x) + M_{a}^{n}(x) = M_{a+1}^{n}(x).$$

Proof. We have

$$\frac{d}{dx}M_{a}^{n}(x) = M_{a}^{n}(x) + e^z \sum_{i=1}^{\infty} (-1)^i \binom{n + a + i}{a} \frac{x^{i-1}}{(i-1)!}.$$ 

Put $i = j + 1$. Then

$$\frac{d}{dx}M_{a}^{n}(x) = M_{a}^{n}(x) - M_{a-1}^{n+1}(x)$$

$$= -M_{a+1}^{n}(x).$$

\[\square\]
We use the above lemma to prove an analogue of the identities (3.3) for the generalized Laguerre functions $\varphi_a^{-1}(z) = M_a^{-1} \left( \frac{1}{2} |z|^2 \right) e^{-\frac{1}{2} |z|^2}$. These $\varphi_a^{-1}$s are precisely the eigenfunctions of the special Hermite Operator $A = -\Delta_z + \frac{1}{2} |z|^2$ with eigenvalue $-2a + n$. For the sake of simplicity, we prove the result for the case $\lambda = 1$.

**Lemma 3.15.** Let $a \in \mathbb{C}_+$ and $P_1(z) = z_1^p z_2^q \in H_{p,q}$. Then

$$P_1(\bar{Z})\varphi_a^{-1}(z) = (-2)^{-p-q}P_1(z)\varphi_a^{-1}(z).$$

**Proof.** The proof of Lemma 3.15 is quite similar to the proof of Lemma 3.3 and hence we omit it here. \qed

**Remark 3.16.** (a). For $a \in \mathbb{C}_+$, the function $\varphi_a^{-1}$ satisfies the growth condition $\varphi_a^{-1}(z) \approx B_a |z|^{2(a-n)} e^{\frac{1}{2} |z|^2}$ as $|z| \to \infty$, (see [1]). This shows that we can not use Lemma 3.2 to generalize Lemma 3.15 for an arbitrary $P \in H_{p,q}$. However, as the proof is based on the process of point wise differentiation and multiplication, it is enough to prove the identity (3.9) for each compact set in $\mathbb{C}^n$.

(b). Let $f(z) = \tilde{a}(|z|)P(z)$, where $P \in H_{p,q}$. Suppose $f$ satisfies the condition $f(z)e^{\frac{1}{2} |z|^2} \in L^p(\mathbb{C}^n)$, for some $\epsilon > 0$ and $1 \leq p \leq \infty$. Then, it is a feasible question to ask for an analogue of the Hecke-Bochner identity for $f \times \varphi_a^{-1}$.

Since $\varphi_a^{-1}$ is an eigenfunction of the operator $A$ with eigenvalue $-2a + n$, the function $f \times \varphi_a^{-1}$ is also an eigenfunction of $A$ with eigenvalue $-2a + n$. As $A$ is an elliptic operator and eigenfunction of an elliptic operator is real analytic [6], the function $f \times \varphi_a^{-1}$ must be a real analytic function on $\mathbb{C}^n$.

4. A REAL ANALYTIC EXPANSION OF THE SPECTRAL PROJECTIONS

In this section, we derive a real analytic expansion for the spectral projections $f \times \varphi_k^{-1}$ for function $f \in L^2(\mathbb{C}^n)$. For $n = 1$, we have given its proof in [10], which is much simpler and used for proving some injectivity results for the twisted spherical means on $\mathbb{C}$. A non-empty set $C \subset \mathbb{C}^n (n \geq 2)$ satisfying $\lambda C \subseteq C$, for all $\lambda \in \mathbb{C}$ is called a complex cone. Using expansion of the spectral projections, we have shown that any complex cone $C$ is a set of injectivity for the twisted spherical means for $L^p(\mathbb{C}^n)$, $1 \leq p \leq 2$ if and only if $C \not\subseteq P_{s,t}^{-1}(0)$, $\forall s, t \in \mathbb{Z}_+$.

We need the following result about an estimate of the bi-graded spherical harmonics.

**Lemma 4.1.** Let $Y_{p,q} \in H_{p,q}$. Then $\|Y_{p,q}\|_{L^\infty} \leq C(p + q + 1)^{n-1}\|Y_{p,q}\|_{L^2}$.

**Proof.** Let $\omega \in S^{2n-1}$. Then $Y_{p,q}$ can be expressed as

$$Y_{p,q}(\omega) = \sum_{|\alpha| = p \atop |\beta| = q} \sum c_{\alpha\beta} \omega^\alpha \bar{\omega}^\beta.$$

It has been shown in [14], p.66 that

$$\|Y_{p,q}\|_{L^2} = \left( \sum_{|\alpha| = p \atop |\beta| = q} |c_{\alpha\beta}|^2 \alpha! \beta! \right)^{1/2}.$$
where \( \dim H_{p,q} = d(p,q) \) is given by
\[
d(p,q) = \frac{(p + n - 2)!(q + n - 2)!((p + n - 1)(q + n - 1) - pq)}{p!q!(n-1)!}.
\]

Since \( H_{p,q} \) is a finite dimensional space and the fact that all norms on a finite dimension space are equivalent, we get the following estimate for the bigraded spherical harmonics.
\[
\|Y_{p,q}\|_{\infty} \leq \sup_{\omega \in S^{2n-1}} \sum_{|\alpha|=p,|\beta|=q} |c_{\alpha\beta}\omega^{\alpha}\bar{\omega}^{\beta}| \leq \sum_{|\alpha|=p,|\beta|=q} |c_{\alpha\beta}|
\]
\[
\leq \sqrt{d(p,q)} \left( \sum_{|\alpha|=p,|\beta|=q} |c_{\alpha\beta}|^2 \right)^{\frac{1}{2}} \leq \sqrt{d(p,q)} \|Y_{p,q}\|_{L^2}.
\]

By a simple computation, it can be shown that \( \sqrt{d(p,q)} \leq C(p+q+1)^{n-1} \), which gives the result.

**Theorem 4.2.** Let \( f \in L^2(\mathbb{C}^n) \). Then \( Q_k(z) = f \times \varphi_k^{n-1}(z) \) is real analytic and its real analytic expansion can be written as
\[
Q_k(z) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} P^k_{p,q}(z)\varphi_{k-p}^{n+p+q-1}(z).
\]

**Proof.** We can write the spherical harmonic decomposition of function \( f \) as
\[
f(z) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=1}^{d(p,q)} \tilde{a}_{j}^{p,q}(|z|)P^j_{p,q}(z).
\]

Since \( f \in L^2(\mathbb{C}^n) \) and set \( \{\varphi_{k-p}^{n+p+q-1} : k \geq p \} \) form an orthonormal basis for \( L^2(r^{2(n+p+q)-1}dr) \), it follows that
\[
\tilde{a}_{j}^{p,q} = \sum_{k \geq p} C_{k-p,j}^{p,q} \varphi_{k-p}^{n+p+q-1}.
\]

In view of the identities \([3.3]\) for \( \lambda = 1 \), we have
\[
\tilde{a}_{j}^{p,q}P^j_{p,q} = \sum_{k \geq p} (-2)^{p+q}C_{k-p,j}^{p,q} P^j_{p,q}(\bar{Z})\varphi_{k}^{n-1}.
\]

From \([4.2]\) and orthogonality relations \( \varphi_{k}^{n-1} \times \varphi_{j}^{n-1} = (2\pi)^n \delta_{jk} \varphi_{k}^{n-1} \), we conclude that
\[
f \times \varphi_{k_0}^{n-1} = \sum_{p=0}^{k_0} \sum_{q=0}^{\infty} \sum_{j=1}^{d(p,q)} (2\pi)^n C_{k_0-p,j}^{p,q} P^j_{p,q} \varphi_{k_0-p}^{n+p+q-1}
\]
\[
= \sum_{p=0}^{k_0} \sum_{q=0}^{\infty} P^{k_0}_{p,q} \varphi_{k_0-p}^{n+p+q-1}, \text{ where } P^{k_0}_{p,q} \in H_{p,q}.
\]
Now, look at the following concrete expression for the spectral projections as

\begin{equation}
Q_k(z) = \sum_{p=0}^{k} \sum_{q=0}^{\infty} P_{p,q}^k(z) \varphi_{k-p}^{n+p+q-1}(z),
\end{equation}

where the series on the right-hand side converges to $Q_k$ in $L^2(\mathbb{C}^n)$. Since spaces $H_{p,q}$'s are orthogonal among themselves, it follows that

\begin{equation}
\|Q_k\|_2^2 = \sum_{p=0}^{k} \sum_{q=0}^{\infty} \|Y_{p,q}^{k}\|_2^2 \|\varphi_{k-p}^{n+1}\|_2^2 < \infty,
\end{equation}

where

\[
\left\|\varphi_{k-p}^{n+1}\right\|_2^2 = \int_0^{\infty} \left|\varphi_{k-p}^{n+1}(r)\right|^2 r^{2n+1}dr = 2^{n+1} \frac{(k-p+\gamma-1)!}{(k-p)!}.
\]

Thus from (4.4), we get an estimate

\[
\|Y_{p,q}^{k}\|_2 \leq C \left( \frac{(k-p)!}{(2^{n+1}(k-p+\gamma-1))^{\frac{1}{2}}} \right).
\]

Using Lemma 4.1 we can write

\begin{equation}
\|Y_{p,q}^{k}\|_{L^\infty} \leq C \frac{(p+q+1)^{n-1}}{(2^{n+1}(k-p+\gamma-1))^{\frac{1}{2}}}.
\end{equation}

In order to prove that the series (4.1) converges uniformly on each compact set $K \subseteq \mathbb{C}^n$, it is enough to prove that the series

\[
h_p(z) = \sum_{q=m}^{\infty} P_{p,q}^k(z) \varphi_{k-p}^{n+1}(z),
\]

converges uniformly over each ball $B_R(o) \subseteq \mathbb{C}^n$, where $m = n + k - 2p + 2$ and $\gamma = n + p + q$. Since $q \geq m$, it follow that $\gamma - 1 \geq k - p + 1$. Hence

\[
|\varphi_{k-p}^{n+1}(z)| = \left| \sum_{j=0}^{k} (-1)^j \left( \frac{k-p+\gamma-1}{k-p-j} \right)^{\frac{1}{2}} \left| z^j \right|^2 j! e^{-\frac{1}{4}|z|^2} \right|
\]

\[
\leq \left( \frac{k-p+\gamma-1}{k-p} \right)^{\frac{1}{2}} \left| \sum_{j=0}^{k} \left( \frac{1}{2} \left| z^j \right|^2 j! e^{-\frac{1}{4}|z|^2} \right| \right|
\]

\[
\leq \frac{(k-p+\gamma-1)!}{(k-p)! (\gamma-1)!} e^{\frac{1}{4}|z|^2}.
\]
Let $|z| \leq R$. Then we can write

$$|h_p(z)| \leq \sum_{q=m}^{\infty} \|Y_{p,q}\|_\infty |z|^{p+q} |\varphi_k^{-1} - p(z)|$$

$$\leq C|R|^p e^{\frac{1}{2}|R|^2} \sum_{q=m}^{\infty} \frac{(p + q + 1)^{n-1}(k - p + \gamma - 1)!}{(2\gamma-1)(k - p + \gamma - 1)!} |R|^q$$

$$= C|R|^p e^{\frac{1}{2}|R|^2} \sum_{q=m}^{\infty} b_q |R|^q.$$ 

By a straightforward calculation we get $\lim_{q \to \infty} \frac{b_{q+1}}{b_q} = 0 < 1$. Hence each of the function $h_p$ is real analytic on $\mathbb{C}^n$. That is, the right-hand side of (4.1) is a real analytic function which agreeing to the real analytic function $Q_k$ a.e. on $\mathbb{C}^n$. Hence (4.1) is a real analytic expansion of $Q_k$.

**Remark 4.3.** (a). Since the spectral projections $Q_k = f \times \varphi_k^{-1}$ is a real analytic function for $f \in L^r(\mathbb{C}^n)$, $1 \leq r \leq \infty$, therefore, it is feasible to ask the question of finding a real analytic expansion for $Q_k$ as similar to (4.1).

(b). Let $f \in L^2(\mathbb{C}^n)$ and for each $k \in \mathbb{Z}_+$, the projection $e^{\frac{1}{2}|z|^2}Q_k(z)$ is a polynomial. Then there exits $m = m(k) \in \mathbb{Z}_+$ such that

$$Q_k(z) = \sum_{p=0}^{k} \sum_{q=0}^{m} P_{p,q}(z) \varphi_{k-p}^{n+p+q-1}(z), \quad P_{p,q} \in H_{p,q}.$$ 

The space consists of functions $f$ is much larger than $U(n)$-finite functions in $L^2(\mathbb{C}^n)$. For this class of functions, we can ask the following question. Does $Q_k(z) = 0$, for all $z \in \mathbb{C}^{n-1} \times \mathbb{R}$ and $\forall k \in \mathbb{Z}_+$, implies $f = 0$ a.e. on $\mathbb{C}^n$? This question for the case $n = 1$, is simpler and has been done by equating the highest degree term of $e^{\frac{1}{2}|z|^2}Q_k$ to zero, (see [10], Theorem 3.4).

Using expansion (4.1) for the spectral projections, we deduce the following result.

**Proposition 4.4.** Let $f \in L^p(\mathbb{C}^n)$, $1 \leq p \leq 2$. Suppose $f \times \mu_r(z) = 0$, $\forall z \in C$ and $\forall r > 0$. Then $f = 0$ a.e. on $\mathbb{C}^n$ if and only if $C \subseteq P_{s,t}^{-1}(0)$, $\forall s, t \in \mathbb{Z}_+$.

**Proof.** Since $f \in L^p(\mathbb{C}^n)$, $1 \leq p \leq 2$, therefore, by convolving $f$ with a right and radial compactly supported smooth approximate identity, we can assume $f \in L^2(\mathbb{C}^n)$. Given that $f \times \mu_r(z) = 0$, $\forall z \in C$ and $\forall r > 0$. By polar decomposition, we can write

(4.6) 

$$Q_k(z) = f \times \varphi_k^{-1}(z) = \int_{r=0}^{\infty} f \times \mu_r(z) \varphi_k^{-1}(r)r^{2n-1}dr = 0,$$

$\forall z \in C$ and $\forall k \in \mathbb{Z}_+$. Therefore, by Theorem 4.2 we get

$$Q_k(z) = \sum_{s=0}^{k} \sum_{t=0}^{\infty} P_{s,t}(z) \varphi_{k-s-t}^{n+s+t-1}(z) = 0,$$
$\forall z \in C$ and $\forall k \in \mathbb{Z}_+$. Since cone $C$ is closed under complex scaling, therefore, for $z \in C$, it implies $r e^{i\theta}z \in C$, for all $r > 0$ and $\theta \in \mathbb{R}$. Thus,

$$Q_k(r e^{i\theta}z) = \sum_{s=0}^{k} \sum_{t=0}^{\infty} r^{s+t} e^{i(s-t)\theta} P_{s,t}^k(z) \varphi_{k-s}^{n+s+t-1}(rz) = 0,$$

$\forall r > 0$ and $\forall \theta \in \mathbb{R}$. Since $Q_k$ is real analytic in $r$, therefore by equating the coefficients of $1, r, r^2, \ldots$ to zero, we conclude that

$$(4.7) \sum_{s+t=\alpha, s \leq k} \left(\frac{n+k+t-1}{k-s}\right) e^{i(s-t)\theta} P_{s,t}^k(z) = 0,$$

$\forall \alpha \in \mathbb{Z}_+$. Using the fact that set $\{e^{i\beta\theta} : \beta \in \mathbb{Z}\}$ form an orthogonal set and the sum vanishes over each of the diagonal $s + t = \alpha, s \leq k$, it follows that $P_{s,t}^k(z) = 0$, for all $s \leq k$ and $t \in \mathbb{Z}_+$. Since the equation (4.7) is valid for each $k \in \mathbb{Z}_+$. Therefore, we infer that for $z \in C$, the projections $Q_k(z) = 0, \forall k \in \mathbb{Z}_+$ if and only if $P_{s,t}(z) = 0$, for all $s, t \in \mathbb{Z}_+$. This forces $Q_k \equiv 0, \forall k \in \mathbb{Z}_+$ if and only if $C \not\subseteq P_{s,t}^{-1}(0), \forall s, t \in \mathbb{Z}_+$. As $f \in L^2(\mathbb{C}^n)$, we can expand $f$ in terms of its spectral projections as

$$(4.8) f = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_{k}^{n-1},$$

where series in the right hand side converges in $L^2(\mathbb{C}^n)$. This is known as the special Hermite expansion, (see [14], p.58). Hence in view of (4.8), we reach to the conclusion that $f = 0$ a.e. on $\mathbb{C}^n$ if and only if $C \not\subseteq P_{s,t}^{-1}(0), \forall s, t \in \mathbb{Z}_+$. \hfill $\square$

**Remark 4.5.** We have observed that Proposition [14] works for the class of all continuous functions on $\mathbb{C}^n$. However, proof for the class of continuous functions is different than that of $L^p(\mathbb{C}^n), 1 \leq p \leq 2$, because the expansion (4.4) is valid only for the functions in $L^2(\mathbb{C}^n)$. Hence, its proof would be presented in full detail elsewhere.

**Acknowledgements:** The author wishes to thank E. K. Narayanan for several fruitful discussions during my visit to IISc, Bangalore. The author would also like to gratefully acknowledge the support provided by the University Grant Commission and the Department of Atomic Energy, Government of India.

**References**

[1] M. L. Agranovsky and R. Rawat, *Injectivity sets for spherical means on the Heisenberg group*, J. Fourier Anal. Appl., 5 (1999), no. 4, 363–372.

[2] M. L. Agranovsky, V. V. Volchkov and L. A. Zalcman, *Conical uniqueness sets for the spherical Radon transform*, Bull. London Math. Soc. 31 (1999), no. 2, 231236.

[3] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tricomi and Francesco G. Higher transcendental functions* (Based on notes left by Harry Bateman), Vol. I. McGraw-Hill, New York, 1953.

[4] M. Filaseta and T-Y. Lam, *On the irreducibility of the generalized Laguerre polynomials*, Acta Arith. 105 (2002), no. 2, 177–182.

[5] D. Geller, *Spherical harmonics, the Weyl transform and the Fourier transform on the Heisenberg group*, Canad. J. Math. 36 (1984), no. 4, 615–684.
[6] R. Narasimhan, *Analysis on real and complex manifolds*, North-Holland Publishing Co., Amsterdam, 1985.

[7] E. K. Narayanan and S. Thangavelu, *Injectivity sets for spherical means on the Heisenberg group*, J. Math. Anal. Appl. 263 (2001), no. 2, 565-579.

[8] W. Rudin, *Function theory in the unit ball of \( \mathbb{C}^n \)*, Springer-Verlag, New York-Berlin, 1980.

[9] R. K. Srivastava, *Sets of injectivity for weighted twisted spherical means and support theorems*, J. Fourier Anal. Appl., 18 (2012), no. 3, 592-608.

[10] R. K. Srivastava, *Coxeter system of lines are sets of injectivity for the twisted spherical means on \( \mathbb{C} \)*, (communicated). DOI: [arXiv:1103.4571v1](https://arxiv.org/abs/1103.4571).

[11] M. Sugiura, *Unitary representations and harmonic analysis*, North-Holland Mathematical Library, 44. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1990.

[12] G. Sajith and S. Thangavelu, *On the injectivity of twisted spherical means on \( \mathbb{C}^n \)*, Israel J. Math. 122 (2001), 7992.

[13] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971.

[14] S. Thangavelu, *An introduction to the uncertainty principle*, Prog. Math. 217, Birkhauser, Boston (2004).

School of Mathematics, Harish-Chandra Research Institute, Allahabad, India 211019.

E-mail address: rksri@hri.res.in