ON THE MATHEMATICAL CHARACTER OF THE
RELATIVISTIC TRANSFER MOMENT EQUATIONS

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ABSTRACT

General-relativistic, frequency-dependent radiative transfer in spherical, differentially-moving media is considered. In particular we investigate the character of the differential operator defined by the first two moment equations in the stationary case. We prove that the moment equations form a hyperbolic system when the logarithmic velocity gradient is positive, provided that a reasonable condition on the Eddington factors is met. The operator, however, may become elliptic in accretion flows and, in general, when gravity is taken into account. Finally we show that, in an optically thick medium, one of the characteristics becomes infinite when the flow velocity equals \( \pm c/\sqrt{3} \). Both high-speed, stationary inflows and outflows may therefore contain regions which are “causally” disconnected.

Key Words: Radiative Transfer – Relativity

1. INTRODUCTION

Radiative transfer in differentially-moving media has been extensively investigated in the past and a large body of literature is available on this subject (see Mihalas & Mihalas 1984, and references therein). Despite the large efforts, however, works dealing with the transfer of radiation through media moving at relativistic speeds are comparatively few. The special relativistic transfer equation was firstly derived by Thomas (1930) and, including Thomson scattering, by Simon (1963) and Castor (1972); a thorough derivation can be found in the monograph by Mihalas & Mihalas. Stationary solutions in spherical symmetry were discussed by Mihalas (1980), Mihalas, Winkler & Norman (1984) and Hauschildt & Wehrse (1991). Radiative transfer in curved spacetimes was investigated by Lindquist (1966), Anderson & Spiegel (1972), Thorne (1981), Schinder (1988), Schinder & Bludman (1989), Anile
& Romano (1992) and Nobili, Turolla & Zampieri (1993). All the solutions
to the relativistic transfer problem found up to now were obtained using
essentially two different approaches: either the transfer equation is directly
solved for the radiation intensity $I_{\nu}(r, \mu)$ or the angular dependence is removed
by introducing the moments of $I_{\nu}(r, \mu)$ and the moment equations are then
integrated. Each method has both advantages and disadvantages. The solution
of the transfer equation gives directly $I_{\nu}(r, \mu)$ but is a quite formidable numerical
task and requires special techniques, like the DOME method introduced by
Hauschildt & Wehrse. On the other hand, the moment equations, being of lower
dimensionality, are easier to handle numerically but their solution alone does
not specify completely $I_{\nu}(r, \mu)$. Moreover, an exact solution for the moments
themselves can be obtained only if the Eddington factors $f_E = K_{\nu}/J_{\nu} - 1/3$
and $g_E = N_{\nu}/H_{\nu} - 3/5$ are computed self-consistently (these are the definitions
of $f_E$ and $g_E$ appropriate to the PSTF moment formalism, see below). An
exact determination of $f_E$ and $g_E$ was obtained by means of the Tangent
Ray Method (TRM, originally due to Mihalas, Kunasz & Hummer 1975) by
Mihalas (1980) in the special relativistic case and by Schinder & Bludman (1989)
for a spherically-symmetric, static atmosphere in a Schwarzschild spacetime.
While it is not entirely obvious that TRM could be fruitfully applied to more
complex situations in which dynamics and gravity are both present, the moment
method can still be used to get, at least, an approximate solution for $I_{\nu}(r, \mu)$ by
introducing “a priori”, reasonable expressions for the Eddington factors (see
e.g. Minerbo 1978, Nobili et al. 1993).

In this paper we present an analysis of the stationary, spherically-
symmetric relativistic moment equations placing particular emphasis on the
character of the second order differential operator implicitly defined by the
zero-th and first equations. This point never received proper attention in the
past, despite the fact that it appears to be of non-negligible relevance since the choice of the boundary conditions is crucially related to the character of the operator. Mihalas, Kusnaz & Hummer (1976) discussed to some extent the problem of frequency conditions in the non-relativistic case, concluding that for accelerated winds and decelerated inflows the transfer equation is of the Feautrier type, although they stressed that different velocity laws may produce anomalous behaviours. Here we prove that the special relativistic moment equations form a hyperbolic system for positive logarithmic velocity gradients at least if $f_E - g_E - 4/15 < 0$, so that they should be solved as a two-point boundary value problem in space and an initial value problem in frequency. In converging flows, however, advection and aberration effects may produce a region of ellipticity. Moreover, when gravity is taken into account, the operator may become elliptic even in the wind case. We point out that one of the two characteristics, when they exist, diverges when the flow velocity equals a “sound” speed $v_s$, $1/3 \leq (v_s/c)^2 \leq 1$ depending on optical depth. The sphere of radius $r_s$ is completely analogous to a sound horizon and behaves like a one-way membrane as far as the propagation of boundary data is concerned. Finally the connection between the “pathologies” in the moment equations and the vanishing of the coefficients of $I_\nu$-derivatives in the transfer equation is discussed.
2. MOMENT EQUATIONS

General relativistic moment equations were derived by Thorne (1981) who introduced Projected Symmetric Trace Free tensors to describe the moments of the radiation intensity. In spherical symmetry the $k$–th PSTF moment has just one independent component, the radial one denoted by $w_k$, and the formalism greatly simplifies. The first two radiation moments are the radiation energy density and flux measured by a comoving observer; the third PSTF moment is the radiative shear, $w_2 = 4\pi(K_\nu - J_\nu/3)$. An alternative form of the moment equations in a static, spherical spacetime, making use of a Lagrangian comoving coordinate system, has been presented by Schinder and Schinder & Bludman.

Using Thorne’s notation, with all moments in ergs cm$^{-3}$, the first two stationary moment equations are (see also Nobili et al.)

\[
\frac{\partial w_1}{\partial \ln r} + 2w_1 + \frac{y'}{y} \left( w_1 - \frac{1}{3} \frac{\partial w_0}{\partial \ln \nu} \right) + \frac{v}{c} \left[ - \left( \frac{u'}{u} - 1 \right) \frac{\partial w_2}{\partial \ln \nu} \right] = \frac{s_0^\nu r}{y}, \tag{1a}
\]

\[
\frac{1}{3} \frac{\partial w_0}{\partial \ln r} + \frac{y'}{y} \left( w_0 - \frac{1}{3} \frac{\partial w_0}{\partial \ln \nu} \right) + \frac{\partial w_2}{\partial \ln r} + 3w_2 - \frac{y'}{y} \frac{\partial w_2}{\partial \ln \nu} \nonumber \\
+ \frac{v}{c} \left[ \frac{\partial w_1}{\partial \ln r} + \frac{1}{5} \left( \frac{7}{u} + 8 \right) w_1 - \frac{1}{5} \left( \frac{3}{u} + 2 \right) \frac{\partial w_1}{\partial \ln \nu} \right] - \frac{\left( \frac{u'}{u} - 1 \right) \left( w_3 + \frac{\partial w_3}{\partial \ln \nu} \right)}{y} = \frac{s_1^\nu r}{y}. \tag{1b}
\]

Here $y = \gamma \sqrt{1 - r_g/r}$ is the total energy per unit mass, $u = yv/c$ is the radial component of the fluid 4–velocity and $s_0^\nu$, $s_1^\nu$ are the source moments; a prime denotes the total derivative in the $r$–direction and $r_g$ is the gravitational radius.

An analysis on the nature of the various dynamical terms appearing in the moment equations was presented by Buchler (1983); a similar discussion
for the transfer equation can be found in Castor (1972), Mihalas et al. (1976), Hauschildt & Wehrse (1991). Terms of order $v/c$ in equations (1) account both for the local Doppler shift of photons and for advection and aberration. Mihalas et al. have shown that advection and aberration produce a fractional variation on the solution which is $\sim 5v/c$ and can be safely neglected for small velocities. In some astrophysical situations however, like photospheric supernovae expansion, jets, accretion onto black holes and neutron stars, velocities $\gtrsim 0.1c$ are expected and such effects cannot be ignored. In the next sections we show that, apart from obvious quantitative effects, advection/aberration terms may change substantially the mathematical properties of the moment equations in spherical inflows.

3. CHARACTERISTIC ANALYSIS

In the following we investigate the mathematical character of the second order differential operator defined implicitly by equations (1). For the sake of simplicity we shall assume that source moments contain no derivatives of the radiation moments; an extension of the present analysis will be needed to include radiative processes like non–conservative electron scattering treated in the Fokker–Planck approximation which depends on both first and second $\nu$–derivatives of $w_0$ (see discussion at the end of this section).

The characteristic analysis of a generic, linear system of first order partial differential equations can be easily performed once the system is brought into the form (see e.g. Whitham 1974)

$$\frac{\partial u_i}{\partial t} + a_{ij} \frac{\partial u_j}{\partial x} + b_i(x, t; u_j) = 0 \quad i, j = 1, n. \quad (2)$$

In this case the characteristic velocities are the roots of the equation

$$|a_{ij} - \lambda \delta_{ij}| = 0 \quad (3)$$
and the system is hyperbolic if equation (3) has \( n \) different real roots. Rewriting the moment equations in the form (2) and introducing the Eddington factors 
\[ f_E = \frac{w_2}{w_0} = \frac{K_\nu}{J_\nu} - 1/3, \quad g_E = \frac{w_3}{w_1} = \frac{N_\nu}{H_\nu} - 3/5, \]
we obtain, after some manipulations,

\[
\frac{\partial w_0}{\partial \ln r} + \frac{1}{f_E + 1/3 - v^2/c^2} \left\{ \left[ \frac{v^2}{c^2} F - \frac{y'}{y} \left( f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) \right] \frac{\partial w_0}{\partial \ln \nu} \right. \\
\left. - \frac{v}{c} G \frac{\partial w_1}{\partial \ln \nu} \right\} + C_1 = 0 \tag{4a}
\]

\[
\frac{\partial w_1}{\partial \ln r} + \frac{1}{f_E + 1/3 - v^2/c^2} \left\{ - \frac{v}{c} \left( f_E + \frac{1}{3} \right) F \frac{\partial w_0}{\partial \ln \nu} + \right. \\
\left. \left[ \frac{v^2}{c^2} G - \frac{y'}{y} \left( f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) \right] \frac{\partial w_1}{\partial \ln \nu} \right\} + C_2 = 0. \tag{4b}
\]

Here \( F = (\beta - 1)f_E + (2 + \beta)/3 - y'/y, \ G = (\beta - 1)g_E + (2 + 3\beta)/5 - y'/y, \)
\( \beta = u'/u, \ y'/y = \beta v^2/c^2 + r_g/2y^2r \) and all terms not containing derivatives of the moments are grouped together into \( C_1 \) and \( C_2 \). Actually \( C_1 \) and \( C_2 \) do contain derivatives of both \( f_E \) and \( g_E \) but the Eddington factors are to be regarded as known functions either coming from the solution of the transfer equation, as in the TRM, or being specified “a priori” if a closure is assumed.

The Eddington factors used here differ from the usual ones inasmuch PSTF moments are originated by a Legendre polynomial expansion of the intensity; in particular \( f_E \) and \( g_E \) are restricted in the ranges \( 0 \leq f_E \leq 2/3 \) and \( 0 \leq g_E \leq 2/5 \). We note that the matrix of the coefficients \( a_{ij} \), defined in equation (2), is symmetric and then its eigenvalues are real, if \( f_E = 2/3 \) and \( g_E = 2/5 \). This means that the moment equations are always hyperbolic in the streaming limit, for any value of \( v \) and \( \beta \).
The equation for the characteristic velocities is

\[
\lambda^2 + \frac{1}{f_E + 1/3 - v^2/c^2} \left[ 2 \frac{y'}{y} \left( f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) - \frac{v^2}{c^2} (F + G) \right] \lambda
\]

\[+ \frac{1}{f_E + 1/3 - v^2/c^2} \left[ \left( \frac{y'}{y} \right)^2 \left( f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) - \frac{y'}{y} \frac{v^2}{c^2} (F + G) - \frac{v^2}{c^2} FG \right] = 0.
\]

(5)

By introducing \( U^2 = (v^2/c^2)/(f_E + 1/3) \), the discriminant of equation (5) can be written as

\[
\Delta = \frac{U^2}{(U^2 - 1)^2} \left[ (U^2 - 1)(F - G)^2 + (F + G)^2 \right]
\]

\[= \frac{U^2}{(U^2 - 1)^2} \left[ U^2(F - G)^2 + 4FG \right].
\]

(6)

From equation (6) it follows that the sign of \( \Delta \) depends only on the sign of the term in square brackets and it is \( \Delta > 0 \) if either \( U^2 > 1 \), that is to say \( v^2/c^2 > f_E + 1/3 \), or \( FG > 0 \), regardless of the values of the Eddington factors and of the velocity gradient. In order to make the analytical treatment affordable in the following we shall neglect gravity, so that \( y'/y = \beta v^2/c^2 \). In this case it is easy to see that \( F \) and \( G \) are opposite in sign and, consequently, \( \Delta \) may become negative for flow velocities in the range \( a < v^2/c^2 < b \), where

\[
a = \min \left[ f_E + \frac{1}{3} + \frac{1}{\beta} \left( \frac{2}{3} - f_E \right), g_E + \frac{3}{5} + \frac{1}{\beta} \left( \frac{2}{5} - g_E \right) \right],
\]

\[
b = \max \left[ f_E + \frac{1}{3} + \frac{1}{\beta} \left( \frac{2}{3} - f_E \right), g_E + \frac{3}{5} + \frac{1}{\beta} \left( \frac{2}{5} - g_E \right) \right].
\]

(7)

Let us consider the case \( \beta > 0 \) first. Since we have already shown that \( \Delta > 0 \) if \( v^2/c^2 > f_E + 1/3 \), it follows that only the velocity interval

\[
g_E + \frac{3}{5} + \frac{1}{\beta} \left( \frac{2}{5} - g_E \right) < \frac{v^2}{c^2} < f_E + \frac{1}{3}
\]

(8)
needs to be considered. It can be easily checked that the above conditions are never fulfilled if $0 < \beta \leq 1$. For $\beta > 1$, a sufficient condition for the positiveness of $\Delta$ can be obtained imposing that

$$g_E + \frac{3}{5} + \frac{1}{\beta} \left( \frac{2}{5} - g_E \right) > f_E + \frac{1}{3},$$

which is equivalent to

$$(\beta - 1)(g_E - f_E + \frac{4}{15}) + \frac{2}{3} - f_E > 0.$$ 

In order for the left hand side of the previous inequality to be positive, it is enough to ask that

$$f_E - g_E - \frac{4}{15} < 0 \quad (9)$$

which, we stress again, gives only a sufficient conditions for the hyperbolicity of the moment equations for $\beta > 1$. On the other hand, we note that if condition (9) is violated there always exists a value of $\beta$, $\beta = 1 + (2/3 - f_E)/(f_E - g_E - 4/15) > 1$, beyond which $\Delta$ may become negative.

Condition (9) can not be proved to hold in full generality and should be verified case by case. It should be taken into account, however, that the first two Eddington factors are not independent from each other, although we avoided up to now to specify any relation between them. In order to check if condition (9) can be physically acceptable, we compare it with the results obtained by Fu (1987a, b). Using a statistical formalism to approximate the radiation intensity at all depths, he was able to derive constraints on the Eddington factors, showing that the values of $K/J$ are bounded by two curves, the “logarithmic” (upper) and “hyperbolic” (lower) limits, in the $H/J \times K/J$ plane. Expressed in terms of the more conventional Eddington factors $K/J$ and $N/H$, condition (9) reduces to $K/J < N/H$. We have computed the
logarithmic and hyperbolic limits of the second Eddington factor and verified that it is \((K/J)_{\text{log}} \leq (N/H)_{\text{log}}\), \((K/J)_{\text{hyp}} \leq (N/H)_{\text{hyp}}\) for \(H/J \leq 1\), although the inequality \((K/J)_{\text{log}} \leq (N/H)_{\text{hyp}}\) is not satisfied for \(H/J > 0.67\). On the other hand, since the hyperbolic (logarithmic) limit should be attained in the streaming (diffusion) regime, it seems more meaningful to compare values of \(K/J\) and \(N/H\) that describe statistical properties of the radiation field in the same physical conditions; so our request that \(K/J < N/H\) at all depths seems indeed compatible with the results of Fu’s analysis.

Unfortunately the study of the limits given by equation (7) is not so straightforward if \(\beta < 0\) and when gravity is taken into account. However, if the gravitational field is strong enough and/or the gas flow is almost in free–fall, the existence of velocity ranges where \(\Delta\) changes sign, and the operator becomes elliptic, is certainly possible, even if \(v/c\) is small.

Up to now we have discussed the conditions for the existence of real characteristics without considering the actual behaviour of the characteristics themselves. Assuming \(\beta > 0\), \(f_E - g_E - 4/15 < 0\) and neglecting gravity, equation (5) can be used to analyze how the characteristics change varying \(v/c\). In the limit of vanishing velocity there is just the double root \(\lambda = 0\) which indicates that the two moment equations decouple (no “frequency mixing” between the moments). As \(v/c\) increases the characteristics become distinct. It is possible to prove that the solutions of equation (5) are opposite in sign, but not equal in magnitude, if \((v/c)^2 < f_E + 1/3\). From equation (5), in fact, it follows that the product of the roots, \(\lambda_1 \lambda_2\), is

\[
\lambda_1 \lambda_2 = \frac{v^2/c^2}{f_E + 1/3 - v^2/c^2} \left[ \beta^2 \left( f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) \frac{v^2}{c^2} - \beta (F + G) \frac{v^2}{c^2} - FG \right].
\]

The term in square brackets can be written as
\[ \beta \left[ \beta \left( f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) - F \right] \frac{v^2}{c^2} - G \left( \beta \frac{v^2}{c^2} + F \right) = \]
\[ - \left( \beta \frac{v^2}{c^2} + G \right) \left( \frac{2}{3} - f_E \right) - \beta \left( f_E + \frac{1}{3} \right) G . \]

Since \( G > 0 \) for \((v/c)^2 \leq f_E + 1/3\) (see condition [9]), the previous expression is always negative and we can conclude that \( \lambda_1 \lambda_2 < 0 \) for \((v/c)^2 < f_E + 1/3\).

As equation (5) shows, one of the characteristics switches from positive to negative through a pole at \((v/c)^2 = f_E + 1/3\). The existence of a diverging characteristic implies that the two spatial regions separated by the line \( r = r_s \), where \((v/c)^2 = f_E + 1/3\), are causally disconnected in the sense that the behaviour of the solution for \((v/c)^2 < f_E + 1/3\) is not influenced by what happens for \((v/c)^2 > f_E + 1/3\). The surface of radius \( r_s \) behaves like a one–way membrane in the same way as the sound horizon does in transsonic flows. As a consequence, if the flow velocity equals the “sound” speed \( v_s = (f_E + 1/3)^{1/2} c \), the moment equations are not to be solved as a two–point boundary value problem in space and an initial value problem in frequency, contrary to the case in which \( v < v_s \) everywhere. Now the solution depends only on the data assigned on the spatial boundary of the “subsonic” region plus the two initial frequency conditions.

The presence of a “sound” horizon is not an artifact introduced by the moment expansion as can be seen examining the special relativistic form of the transfer equation, see e.g. Hauschildt & Wehrse equation (1). The \( r–\)derivative of the radiation intensity is, in fact, multiplied by the factor \( e = \gamma(\mu + v/c) \) which is zero at \( v/c = -\mu \). This means that if a non–vanishing velocity field is present, the transfer equation becomes necessary singular on the surface \( v(r)/c = -\mu \) in the \( r \times \mu \times \nu \) space, where the dimensionality of the equation lowers from 3 to 2. Moment equations contain the same kind of
pathology but, being obtained by angle averaging the transfer equation, the coefficients of the space derivatives vanish at a fixed value of $\mu$ which is just $\sqrt{\langle \mu^2 \rangle} = 1/\sqrt{3}$ in the Eddington approximation. We note that the transfer equation does not exhibit any singularity when written in its characteristic form in the $r \times \mu$ plane, as in the Tangent Ray Method, because this amounts to use a coordinate system which establishes a one–to–one, regular map between the integral surface and the integration domain.

The same kind of considerations can be used to relate the possible ellipticity of the moment equations to the vanishing of the coefficient of the $\nu$–derivative in the transfer equation,

$$g = \frac{\gamma v}{r c} [1 - \mu^2 + \mu(\mu + v/c)\beta],$$

see equation (3c) of Hauschildt & Wehrse where our definition of $\beta$ was used. It can be shown that $g$ is always non-negative for $-1 \leq \mu \leq 1$ only if $v > 0$ and $0 \leq \beta \leq 2$; for all other values of the velocity and of the velocity gradient there exist a value of $\mu$ at which $g$ vanishes. The transfer equation may, therefore, become singular even in the outflow case and this agrees with the fact that the moment equations can be proved to be hyperbolic without any additional constraint only for $0 \leq \beta \leq 1$. In general the degeneracy occurs along certain lines in the $r \times \mu$ plane. Actually, in the moment equations (4), the coefficients of the $\nu$–derivatives, that are obviously related to $g$, contain some averaged value of $\mu$ and they can change sign at a certain value of $r$ in the integration domain. In particular, since these coefficients depend on the two Eddington factors $f_E$ and $g_E$, they can change sign at two different radii, say $r_1$ and $r_2$. This kind of pathology manifests through the appearance of an interval $(r_1, r_2)$, in which the system of differential equations becomes elliptic.

As stressed by Mihalas et al. (1976) and Hauschildt & Wehrse, both $e$ and
depend on the flow velocity only if advection and aberration are taken into account, even to first order in $v/c$. In this respect it is interesting to note that the moment equations reduce to a parabolic system in the diffusion limit for any given $\beta$ if only local Doppler shift of photons is retained (Nobili et al., see also Blandford & Payne 1981 a, b, Payne & Blandford 1981). It is, therefore, the inclusion of advection and aberration terms, which act as singular perturbations, that introduces pathologies either in the transfer equation or in the system of the moment equations. A similar conclusion, although in a different context, was reached recently by Gombosi et al. (1993) who studied energetic particle transport by means of a moment expansion of the distribution function which is very similar to the one used here for the radiation intensity. A situation like this arises also when non–conservative scattering is included in the source term. Assuming that it can be treated in the Fokker–Planck approximation, the presence of $\nu$–derivatives of the radiation intensity produces an effect analogous to advection/aberration. In this case, see Colpi (1988), it can be shown that the transfer equation, in the diffusion limit and retaining only local Doppler shift, is always of the elliptic type and it must be integrated giving suitable conditions on all the boundary of the integration domain. Actually, we want to stress that a general analysis of the mathematical character of the transfer equation is not possible “a priori”, depending on the input physics included in the source term. In the present study, we dealt with the more complete form of the moment equations in dynamical flows, but assuming that only conservative scattering and isotropic true emission–absorption processes are present. Even under these assumptions, we have shown that in accretion flows the operator defined by equations (1) may become of the mixed type, switching from hyperbolic to elliptic. The presence of a spatially–limited elliptic region around $\tau \approx 1$ implies that, there, conditions must be specified at both the frequency boundaries,
although the problem remains two-point boundary valued in space.

As a final point, let us briefly discuss the effects induced by the presence of a gravitational field on the existence of real characteristic and, consequently, on the character of the operator defined by the moment equations. Both the expressions for $F$ and $G$ contain, now, an extra term $-r_g/2y^2r$ with respect to the special relativistic case and, even if $0 \leq \beta \leq 1$, it could be $\Delta > 0$ or $\Delta < 0$, depending on the sign of $F = F_{SR} - r_g/2y^2r$ and $G = G_{SR} - r_g/2y^2r$. This leads to the conclusion that, irrespective of the sign of $\beta$, the presence of a gravitational field can induce a change in the character of the moment equations; in particular, if $0 \leq \beta \leq 1$ the possible appearance of regions of ellipticity is completely due to gravity.

4. CONCLUSIONS

We have analyzed the mathematical character of the system formed by the first two relativistic transfer moment equations. It has been shown that, similarly to the non-relativistic case, the differential operator is of the hyperbolic type when the flow velocity is a monotonically increasing function of the radial coordinate. On the contrary, in converging flows and when gravity is taken into account, the character of the operator is much more complex and the system of equations may become of the mixed type. This result can be of interest in connection with models of spherical accretion onto compact objects and seems to be originated by advection and aberration effects.
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