Regular Totally Separable Sphere Packings

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Abstract

The topic of totally separable sphere packings is surveyed with a focus on regular constructions, uniform tilings, and contact number problems. An enumeration of all regular totally separable sphere packings in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$ which are based on convex uniform tessellations, honeycombs, and tetracombs, respectively, is presented, as well as a construction of a family of regular totally separable sphere packings in $\mathbb{R}^d$ that is not based on a convex uniform $d$-honeycomb for $d \geq 3$.

Keywords: sphere packings, hyperplane arrangements, contact numbers, separability.

MSC 2010 Subject Classifications: Primary 52B20, Secondary 14H52.

1 Introduction

In the 1940s, P. Erdős introduced the notion of a separable set of domains in the plane, which gained the attention of H. Hadwiger in [1]. G.F. Tóth and L.F. Tóth extended this notion to totally separable domains and proved the densest totally separable arrangement of congruent copies of a domain is given by a lattice packing of the domains generated by the side-vectors of a parallelogram of least area containing a domain [2]. Totally separable domains are also mentioned by G. Kertész in [3], where it is proved that a cube of volume $V$ contains a totally separable set of $N$ balls of radius $r$ with $V \geq 8Nr^3$. Further results and references regarding separability can be found in a manuscript of J. Pach and G. Tardos [4].

This manuscript continues the study of separability in the context of regular unit sphere packings, i.e., infinite sets of unit spheres

$$\mathcal{P} = \bigcup_{i=1}^{\infty} \left( x_i + S^{d-1} \right)$$

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in $\mathbb{R}^d$ with $\|x_i - x_j\| \geq 2$, whose contact graphs $G_P = (V, E)$, where $V = \{x_i \mid i \in \mathbb{N}\}$ and

$$E = \{(i, j) \mid (x_i + \mathbb{S}^{d-1}) \cap (x_j + \mathbb{S}^{d-1}) \neq \emptyset\},$$

are regular (every vertex has equal degree); this means that every sphere in the packing touches the same number of spheres.

Let $C(P_n)$ be the contact number of a unit sphere packing $P_n$ with $n$ spheres, i.e., the cardinality of the edge set of the contact graph $G_{P_n}$. Determining the maximum contact number of a unit sphere packing with $n$ spheres is known as the contact number problem. The contact number problem for circle packings in $\mathbb{R}^2$ was solved exactly in 1974 by H. Harborth in [5] to be $\lfloor 3n - \sqrt{12n - 3} \rfloor$. Upper and lower bounds on the contact number problem for finite packing of unit balls in $\mathbb{R}^3$ were provided by K. Bezdek and the author in [6] and studied in detail up to $n = 18$ by M. Holmes-Cerfon in [7] improving the lower bounds for some values. Consult [8] and references therein for more information regarding contact numbers of unit sphere packings and arrangements of spheres in higher dimensions.

**Definition 1.** A sphere packing $\mathcal{P}$ is totally separable if every tangent hyperplane to a pair of touching spheres has an empty intersection with the interior of all spheres in $\mathcal{P}$.

The contact number problem for totally separable sphere packings is studied and all regular totally separable sphere packings in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$ based on convex uniform tessellations (classified in an unpublished manuscript of G. Olshevsky [10]) are constructed. Now, let

$$c(n, d) = \max_{\text{sep}(\mathcal{P}_n) = 1} C(\mathcal{P}_n),$$

where sep(·) is a measure on sphere packings called the separability of the packing which is defined formally in the appendix; intuitively, the separability of a packing is 0 if the packing is inseparable and 1 if it is totally separable. The theory of minimal area polyominoes developed in [9] is used with Euler’s formula to provide a proof of the contact number problem for totally separable circle packings:

$$c(n, 2) = \left\lfloor 2(n - \sqrt{n}) \right\rfloor.$$

Furthermore, heuristics are provided for the upper bound on the contact number problem for totally separable sphere packings in $\mathbb{R}^d$ which is based on the number of edges of polyominoes over the cubic $d$-honeycomb and hence exact when $\sqrt[n]{n} \in \mathbb{N}$:

$$c(n, d) \leq \left\lfloor d \left( n - n^{\frac{d-1}{d}} \right) \right\rfloor.$$
Lemma 1. If the contact graph $G_P$ of a sphere packing $P$ in $\mathbb{R}^d$ contains a $k$-simplex for $2 \leq k \leq d$, then $P$ is not totally separable.

Proof. First consider the case where $G_P$ contains a 2-simplex and observe that it violates total separability. For, the tangent line generated by the touching circles associated with an edge $e$ of the 2-simplex intersects the interior of the circle associated with the vertex which is not an endpoint of $e$. Proceed by induction, observing from the base case $d = 2$ that any $k$-simplex with $3 \leq k \leq d$ in $G_P$ violates total separability as that $k$-simplex contains a 2-simplex somewhere in its flag, thus proving the lemma.

This lemma will be used extensively for classifying totally separable sphere packings based on convex uniform tesselations of $\mathbb{R}^d$, also known as tilings or honeycombs.

2 Regular Totally Separable Circle Packings in $\mathbb{R}^2$

Regular totally separable circle packings in $\mathbb{R}^2$ which are based on convex uniform tilings are classified by the following theorem.

Theorem 1. There are exactly 4 convex uniform tilings in $\mathbb{R}^2$ which generate totally separable circle packings:

1. $\mathcal{P}_1$ - Square tiling, $\{4,4,4\}$

2. $\mathcal{P}_3$ - Hexagonal tiling, $\{6,6,6\}$

3. $\mathcal{K}_6$ - Truncated square tiling, $\{4,8,8\}$

4. $\mathcal{K}_9$ - Omninetruncated trihexagonal tiling, $\{4,6,12\}$

Proof. Apply Lemma 1 to the list of 11 convex uniform tilings of $\mathbb{R}^2$; three Pythagorean tilings and eight Keplerian tilings [10]. Clearly, if $P$ is a 4-regular totally separable packing of unit circles in $\mathbb{R}^2$ generated by a convex uniform tiling, then $P$ is congruent to $\mathcal{P}_1$. If $P$ is a 3-regular totally separable packing of unit circles in $\mathbb{R}^2$ generated by a convex uniform tiling, then $P$ is congruent to $\mathcal{P}_3$, $\mathcal{K}_6$, $\mathcal{K}_9$ or a subset of $\mathcal{P}_1$. If $P$ is a 2-regular totally separable packing of unit circles in $\mathbb{R}^2$ generated by a convex uniform tiling, then $P$ is congruent to a subset of either $\mathcal{P}_1$, $\mathcal{P}_3$, $\mathcal{K}_6$, or $\mathcal{K}_9$.

The theory of minimal area polyominoes and Euler’s formula is used to provide an exact solution to the contact number problem for totally separable circle packings; an alternative explicit proof, not relying on the results of [9], which extends a proof technique of H. Harborth [5] appears in [11].

Theorem 2. Given $n \in \mathbb{N}$, there exists a totally separable circle packing $\mathcal{P}_n$ in $\mathbb{R}^2$ with contact number

$$C(\mathcal{P}_n) = \left\lfloor 2(n - \sqrt{n}) \right\rfloor.$$

Furthermore, no totally separable circle packing in $\mathbb{R}^2$ has a larger contact number.
Proof. By Euler’s formula, \( n - (|E| + P(c)) + a = 2 \), where \(|E|\) is the cardinality of the edge set of the contact graph \( G_{P_n} \), \( P(c) \) is the perimeter of the polyomino \( c \) with area \( a \) generated by placing \( n \) unit 2-cubes so that elements of \( P_n \) are incircles. Interpolate the piece-wise defined function from Corollary 2.5 of [9] which provides the minimal perimeter of a polyomino of area \( a \) in order to obtain the desired formula.

Figure 1: A finite part of the contact graph, convex uniform tiling, and 3-regular totally separable circle packing generated by the truncated square tiling.
3 Regular Totally Separable Sphere Packings in $\mathbb{R}^3$

Regular totally separable sphere packings in $\mathbb{R}^3$ which are based on convex uniform honeycombs are classified by the following theorem.

**Theorem 3.** There are exactly 7 convex uniform honeycombs in $\mathbb{R}^3$ which generate totally separable sphere packings in $\mathbb{R}^3$:

1. $J_1$ - Cubic honeycomb
2. $J_3$ - Hexagonal prismatic honeycomb
3. $J_6$ - Truncated square prismatic honeycomb
4. $J_9$ - Omnitruncated trihexagonal prismatic honeycomb
5. $J_{16}$ - Bitruncated cubic honeycomb
6. $J_{18}$ - Cantitruncated cubic honeycomb
7. $J_{20}$ - Omnitruncated cubic honeycomb

**Proof.** Apply Lemma [1] to N. Johnson’s list of 28 convex uniform honeycombs [12]. Clearly, if $\mathcal{P}$ is a 6-regular totally separable packing of unit spheres in $\mathbb{R}^3$ generated by a convex uniform honeycomb, then $\mathcal{P}$ is congruent to $J_1$. If $\mathcal{P}$ is a 5-regular totally separable packing of unit spheres in $\mathbb{R}^3$ generated by a convex uniform honeycomb, then $\mathcal{P}$ is congruent to $J_3$, $J_6$, $J_9$, or a subset of $J_1$. If $\mathcal{P}$ is a 4-regular totally separable packing of unit spheres in $\mathbb{R}^3$ generated by a convex uniform honeycomb, then $\mathcal{P}$ is congruent to $J_{16}$, $J_{18}$, $J_{20}$, or a subset of either $J_1$, $J_3$, $J_6$, $J_9$. If $\mathcal{P}$ is a 3-regular, or 2-regular totally separable packing of unit spheres in $\mathbb{R}^3$ generated by a convex uniform honeycomb, then $\mathcal{P}$ is congruent to a subset of either $J_1$, $J_3$, $J_6$, $J_9$, $J_{16}$, $J_{18}$, or $J_{20}$. $\square$

4 Regular Totally Separable Sphere Packings in $\mathbb{R}^4$

Regular totally separable sphere packings in $\mathbb{R}^4$ based on convex uniform 4-honeycombs are classified by the following theorem.

**Theorem 4.** There are exactly 18 convex uniform tetracombs in $\mathbb{R}^4$ which generate totally separable sphere packings in $\mathbb{R}^4$:

1. $O_1$ - Tesseractic tetracomb
2. $O_3$ - Square-hexagonal duoprismatic tetracomb
3. $O_6$ - Tomosquare-square duoprismatic tetracomb
4. $O_9$ - Omnitruncated-trihexagonal-square duoprismatic tetracomb
Proof. Apply Lemma 1 to G. Olshevsky’s list of 143 convex uniform 4-honeycombs [10].

Clearly, if \( P \) is a 8-regular totally separable packing of unit spheres in \( \mathbb{R}^4 \) generated by a convex uniform tetrambock, then \( P \) is congruent to \( O_1 \). If \( P \) is a 7-regular totally separable packing of unit spheres in \( \mathbb{R}^4 \) generated by a convex uniform tetrambock, then \( P \) is congruent to \( O_3, O_6, O_9, \) or a subset of \( O_1 \). If \( P \) is a 6-regular totally separable packing of unit spheres in \( \mathbb{R}^4 \) generated by a convex uniform tetrambock, then \( P \) is congruent to \( O_{16}, O_{18}, O_{20}, O_{39}, O_{42}, O_{45}, O_{63}, O_{66}, O_{78}, \) or a subset of either \( O_1, O_3, O_6, \) or \( O_9 \). If \( P \) is a 5-regular, 3-regular, or 2-regular totally separable packing of unit spheres in \( \mathbb{R}^4 \) generated by a convex uniform tetrambock, then \( P \) is congruent to a subset of either \( O_1, O_3, O_6, O_9, O_{16}, O_{18}, O_{20}, O_{39}, O_{42}, O_{45}, O_{63}, O_{66}, O_{78}, \) or \( O_{99}, O_{100}, O_{103}, O_{132}, O_{140}, \) or a subset of either \( O_1, O_3, O_6, O_9, O_{16}, O_{18}, O_{20}, O_{39}, O_{42}, O_{45}, O_{63}, O_{66}, O_{78}, O_{99}, O_{100}, O_{103}, O_{132}, \) or \( O_{140} \).

The regularity of each 4-honeycomb is determined by inspecting the number of vertices of the vertex figure associated with the honeycomb, e.g., the vertex figure of \( O_{100} \) is an irregular pentachoron, implying that the 4-dimensional sphere packing generated by the great diprismatotessseractic tetrambock is 5-regular.
Totally Separable Sphere Packings in $\mathbb{R}^d$

Totally separable sphere packings in $\mathbb{R}^d$ are studied and future research directions are outlined. The following heuristics for the upper bound to the contact number problem for totally separable sphere packings in $\mathbb{R}^d$ provides a reasonable intuitive explanation of the following theorem.

From the formula for the number of $m$-cubes on the boundary of a $d$-cube for $m = 1$

$$2^{d-1} \binom{d}{1} = \left\lfloor d \left( 2^d - (2^d)^{\frac{d-1}{d}} \right) \right\rfloor = \left\lfloor d \left( n - n^{\frac{d-1}{d}} \right) \right\rfloor$$

for $n = 2^d$. Similarly, for any $n = \sqrt[d]{k} \in \mathbb{N}$ there is a $k \times k \times \cdots \times k$ $d$-cube with

$$\left\lfloor d \left( k^d - (k^d)^{\frac{d-1}{d}} \right) \right\rfloor$$

edges, implying that the upper bound in the following theorem is an equality. Assume that $k^d < n < (k + 1)^d$ and observe that the upper bound on $c(n, d)$ overestimates the supremum over edge cardinalities of $(k + \delta_1) \times (k + \delta_2) \times \cdots \times (k + \delta_d)$ unit polyominoes with $n$ cells, where $\delta_i \in \{0, 1\}$.

**Theorem 5.** For $n \in \mathbb{N}$,

$$c(n, d) \leq \left\lfloor d \left( n - n^{\frac{d-1}{d}} \right) \right\rfloor,$$

with equality when $\sqrt[d]{n} \in \mathbb{N}$.

**Proof.** Improving upon an earlier and lengthier unpublished case analytic proof, K. Bezdek, B. Szalkai, and I. Szalkai provide an elegant proof using box-polytopes and the isoperimetric inequality [11].

The classification of uniform $d$-honeycombs is incomplete, leading to great difficulty in establishing the above characterizations of totally separable sphere packings in $d = 2, 3, 4$ for $d \geq 5$. The ongoing work by J. Bowers, G. Olshevsky, N. Johnson, and others of classifying uniform polytopes will soon result in the complete classification of uniform 5-honeycombs, and the study of uniform polytopes generating uniform 6-honeycombs has only recently begun. For $d \geq 7$ there appears to be no significant work on uniform honeycombs; although R. Klitzing has classified certain uniform polytopes up to $d = 8$ [13]. Future research on the topic of regular totally separable sphere packings should include a comprehensive construction of families of $k$-regular totally separable sphere packings in $\mathbb{R}^d$ for $3 \leq k \leq 2d-1$ and $d \geq 5$. These are the unknown bounds on $k$-regularity because for $k = 2$ spheres can be placed along an apeirogon (infinite line with evenly spaced points) and for $k = 2d$ spheres can be placed on the cubic $d$-honeycomb. For an example to motivate future research in this direction, a construction in $\mathbb{R}^d$ of a $(d+1)$-regular totally separable sphere packing which is not based on a convex uniform $d$-honeycomb for $d \geq 3$ is presented. A similar construction would be desired for $3 \leq k \leq d$ and $d + 2 \leq k \leq 2d - 1$; regardless of whether or not it is based on a convex uniform $d$-honeycomb.
**Theorem 6.** There exists a \((d + 1)\)-regular totally separable sphere packing in \(\mathbb{R}^d\) for \(d \geq 3\) which is not based on a convex uniform \(d\)-honeycomb.

**Proof.** Let \(Q_0^d = \text{conv} \{x_{0,1}, \ldots, x_{0,2^d}\}\) be a unit \(d\)-cube in \(\mathbb{R}^d\) and place \(2^d\) unit \(d\)-cubes

\[
Q_1^d = \text{conv} \{x_{1,1}, \ldots, x_{1,2^d}\} \\
\vdots \\
Q_{2^d}^d = \text{conv} \{x_{2^d,1}, \ldots, x_{2^d,2^d}\}
\]

so that \(\|x_{0,1} - x_{1,1}\| = 2, \ldots, \|x_{0,2^d} - x_{2^d,1}\| = 2\) with \(x_{i,1}\) lying outside \(Q_0^d\) along a line emanating from the centroid of \(Q_0^d\) through \(x_{0,i}\) for \(1 \leq i \leq 2^d\). Now construct

\[
\mathcal{P}_{2^d + 4^d} = \bigcup_{i=1}^{2^d + 4^d} \bigcup_{j=1}^{2^d} (x_{i,j} + \mathbb{S}^{d-1})
\]

and iteratively place \(2^d - 1\) unit \(d\)-cubes diagonally out of each existing unit \(d\)-cube \(Q_1^d, \ldots, Q_{2^d}^d\) as above so that spheres may be placed around their vertices which generate a packing congruent to \(\mathcal{P}_{2^d + 4^d}\). Indefinitely extending this procedure leads to an infinite totally separable sphere packing which is \((d + 1)\)-regular. For, let \(x + \mathbb{S}^{d-1}\) be an arbitrary sphere in this packing and observe that it touches \(d\) other spheres placed on adjacent vertices of the unit \(d\)-cube which \(x\) is a vertex of, and also touches \(1\) other sphere which is diagonally outward as in the construction. Furthermore, for \(d = 2\) this construction corresponds to the truncated square tiling \(K_6\) and for \(d \geq 3\) this construction corresponds to a scaliform which contains an elongated cubic bifrustum. 

The classification of regular totally separable sphere packings which are not based on convex uniform 3-honeycombs is then a sub-problem of classifying all scaliforms (vertex-transitive honeycombs) in \(\mathbb{R}^3\); from a simplex-free scaliform in \(\mathbb{R}^3\) one can construct a totally separable sphere packing by placing equal size spheres at the vertices. The questionable existence of aperiodic totally separable sphere packings in any dimension remains unexplored.

**Conjecture 1.** No aperiodic totally separable sphere packing exists in any dimension.

**Appendix: Separability as a Geometric Measure**

Separability is introduced as a geometric measure where inseparable sphere packings have a separability of 0 and totally separable sphere packings have a separability of 1. Let \(H_e\) denote the tangent hyperplane to a pair of touching spheres in \(\mathbb{R}^d\) associated with edge \(e\) of the contact graph \(G_P = (V, E)\). First define the separability measure for finite sphere packings \(\mathcal{P}_n\) with \(G_{\mathcal{P}_n} = (V_n, E_n)\) by

\[
\text{sep}(\mathcal{P}_n) = \sum_{e \in E_n} \left| \left\{ H_e \mid H_e \cap \text{int} (x_i + \mathbb{S}^{d-1}) = \emptyset, 1 \leq i \leq n \right\} \right| / |E_n|.
\]
If a sphere packing $P \hookrightarrow \mathbb{R}^d$ can be constructed so that $P = \lim_{n \to \infty} P_n$ for some sequence of finite sphere packings $P_n$, then

$$\text{sep}(P) = \lim_{n \to \infty} \sum_{e \in E_n} \left| \left\{ H_e \mid H_e \cap \text{int} \left( x_i + S_{\mathbb{R}} - 1 \right) = \emptyset, 1 \leq i \leq n \right\} \right| / |E_n|.$$  

Observe that if every tangent hyperplane $H_e$ at a contact point associated with the edge $e$ intersects the interior of another sphere in the packing $P$ then $\text{sep}(P) = 0$ and similarly if none intersect the interior of a sphere in the packing then $\text{sep}(P) = 1$; in the former case $P$ is called inseparable and in the latter case $P$ is called totally separable.

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