SOBOLEV SPACES AND ELLIPTIC THEORY ON UNBOUNDED DOMAINS IN $\mathbb{R}^n$

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Abstract. In this article, we develop the theory of weighted $L^2$ Sobolev spaces on unbounded domains in $\mathbb{R}^n$. As an application, we establish the elliptic theory for elliptic operators and prove trace and extension results analogous to the bounded, unweighted case.

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In this article, we develop weighted $L^2$-Sobolev spaces and elliptic theory on unbounded domains in $\mathbb{R}^n$. For spaces of functions with suitable regularity and domains with regular boundaries, we show that traces exist and functions defined on the boundary extend in a bounded manner. In the second part of the paper, we show that elliptic equations gain the full number of derivatives up to the boundary and satisfying an elliptic equation is sufficient for taking traces for functions as rough as $L^2$.

With this article, we are laying the groundwork to develop the $L^2$-theory for the $\bar{\partial}$ and $\bar{\partial}_b$ equations on unbounded domains and their boundaries in $\mathbb{C}^n$. Weighted $L^2$ spaces are instrumental tools in several complex variables, and the analysis cannot proceed without them.

Unlike the bounded case, however, unweighted Sobolev spaces on unbounded domains fail to have many critical features, such as the Rellich identity, and so we develop spaces with these properties in mind. One method to solve $\bar{\partial}_b$-problem involves using extension and trace operators, so we need fractional Besov and Sobolev spaces. As a result of our several complex

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variables considerations (to be developed in later papers), both the types of weighted Sobolev spaces we study and the types of results we prove are quite different than what appears in the literature. See, for example [Kuf85]. The weights that authors typically study involve powers of the distance to the boundary [MT06, Can03]. Although these weights are quite natural and reflect the geometry of the boundary, they are not the only useful weights in several complex variables (see, e.g., [Har09, Hör65, Sha85, Koh86, HR11, HRa, Rai10, Str10]). With respect to the literature on elliptic theory on unbounded domains, authors seem to be less concerned with proving trace results for solutions to elliptic equations and more interested in solvability, typically for the Dirichlet problem (e.g., see [BMT08] and the references contained within). Even when the author solves an elliptic equation with a nonzero boundary condition (e.g., [Kim08]), the derivatives are the standard derivatives, and not the weighted derivatives that we consider. Consequently, we must build the theory from the most basic building blocks.

Our Sobolev space techniques mainly involve real interpolation, so they define Besov spaces. The fractional Sobolev spaces and Besov spaces agree (see, for example, [LM72] or [BL76]), and in the elliptic regularity section of the paper, we use the fractional Sobolev and Besov spaces interchangeably. In fact, even at the integer levels, the main results hold for the interpolated spaces $B^{k,2}(\Omega; X)$ and Sobolev spaces $W^{k,2}(\Omega; X)$ (the former by interpolation and the latter by direct proof).

1. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\varphi : \Omega \to \mathbb{R}$ be $C^\infty$. Define the weighted $L^p$-space

$$L^p(\Omega, \varphi) = \{ f : \Omega \to \mathbb{C} : \int_\Omega |f|^p e^{-\varphi} \, dV < \infty \}$$

where $dV$ is Lebesgue measure on $\mathbb{C}^n$. Let $b\Omega$ be the boundary of $\Omega$. We will always assume that $b\Omega$ is at least Lipschitz, so that integration by parts is always justified. For most results we will need additional boundary regularity, as indicated below.

1.1. Hypotheses on $\Omega$, $\varphi$, and $\rho$. Let $A \subset \mathbb{R}^n$.

Let $\delta_A$ be the distance function from $A$, i.e., $\delta_A(x) = \inf_{y \in A} |x - y|$. Let $U_A = \{ x \in \mathbb{R}^n : \text{there exists a unique point } y \in A \text{ such that } \delta_A(x) = |y - x| \}$. Define $\pi_A : U_A \to A$ by $\pi_A(x) = y$. The following concepts were introduced in [Fed59].

**Definition 1.1.** If $y \in A$, then define the **reach** of $A$ at $y$ by

$$\text{Reach}(A, y) = \sup \{ r \geq 0 : B(y, r) \subset U_A \}$$

and the **reach** of $A$ to be

$$\text{Reach}(A) = \inf \{ \text{Reach}(A, y) : y \in A \}.$$  

The majority of our results use a subset of the following hypotheses. Fix $m \in \mathbb{N}$, $m \geq 2$.

HI. The domain $\Omega$ has a $C^m$ boundary with positive reach. Moreover, there exists $\epsilon > 0$ and a defining function $\rho$ so that on $\Omega' = \{ y : \delta_{b\Omega}(y) < \epsilon \}$, $\|\rho\|_{C^m(\Omega')} < \infty$ (i.e., $\Omega$ is uniformly $C^m$ in the sense of [HR55]).

HII. There exists $\theta \in (0, 1)$ so that

$$\lim_{|x| \to \infty} \left( \theta |\nabla \varphi|^2 + \Delta \varphi \right) = \infty$$
where
\[ |\nabla \varphi|^2 = \sum_{j=1}^{n} |\frac{\partial \varphi}{\partial x_j}|^2. \]

HIII. There exists \( \theta \in (0, 1) \) so that
\[ \lim_{|x| \to \infty} (\theta |\nabla \varphi|^2 - \Delta \varphi) = \infty \]

HIV. There exists a constant \( C_m > 0 \) so that
\[ |\nabla^k \varphi| \leq C_m (1 + |\nabla \varphi|) \]
for \( 1 \leq k \leq m \) and \( x \in \Omega \).

HV. Hypotheses (HII)-(HIV) can be extended to \( \mathbb{R}^n \).

HVI. If \( \frac{\partial}{\partial \nu} \) denotes the outward unit normal to \( \partial \Omega \), we have
\[ \inf_{r > 0} \sup_{|x| > r, x \in \partial \Omega} |\nabla \varphi|^{-1} |\frac{\partial \varphi}{\partial \nu}| < 1. \]

(HII) and (HIII) have their origin in [Gan, GH10] (who in turn adapt the ideas in [KM94]).

The family of examples *par excellence* of weight functions is
\[ \varphi(x) = t|x|^2 \]
for any nonzero \( t \in \mathbb{R} \). Such functions always satisfy (HII)-(HV) ((HI) is examined in detail in [HRb]). It is possible to construct domains for which (HVI) fails for this choice of \( \varphi \), but observe that if \( \Omega \) satisfies (HVI) for \( \varphi(x) = t|x|^2 \), then any isometry of \( \mathbb{R}^n \) will map \( \Omega \) to another domain which also satisfies (HVI) (because the composition of \( |x|^2 \) with any isometry will equal \( |x|^2 \) plus lower order terms).

1.2. **Weighted Sobolev spaces.** Set \( D_j = \frac{\partial}{\partial x_j} \) and define the weighted differential operators
\[ X_j = \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} = e^{\varphi} \frac{\partial}{\partial x_j} e^{-\varphi}, \quad 1 \leq j \leq n \]
and
\[ \nabla_X = (X_1, \ldots, X_n). \]

**Definition 1.2.** Let \( Y_j = X_j \) or \( D_j \), \( 1 \leq j \leq n \). For a nonnegative \( k \in \mathbb{Z} \), let the weighted Sobolev space
\[ W^{k,p}(\Omega, \varphi; Y) = \{ f \in L^p(\Omega, \varphi) : Y^\alpha f \in L^p(\Omega, \varphi) \text{ for } |\alpha| \leq k \} \]
where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is an \( n \)-tuple of nonnegative integers and \( Y^\alpha = Y_1^{\alpha_1} \cdots Y_n^{\alpha_n} \). The space \( W^{k,p}(\Omega, \varphi; Y) \) has norm
\[ \|f\|_{W^{k,p}(\Omega, \varphi; Y)}^p = \sum_{|\alpha| \leq k} \|Y^\alpha f\|_{L^p(\Omega, \varphi)}^p. \]

Also, let
\[ W^{k,p}_0(\Omega, \varphi; Y) = \{ g \in W^{k,p}(\Omega, \varphi; Y) : \text{there exists } \psi_\ell \in C^\infty_c(\Omega) \text{ satisfying } \|g - \psi_\ell\|_{W^{k,p}(\Omega, \varphi; Y)} \to 0 \text{ as } \ell \to \infty \}. \]

In other words, \( W^{k,p}_0(\Omega, \varphi; Y) \) is the closure of \( C^\infty_c(\Omega) \) in the \( W^{k,p}(\Omega, \varphi; Y) \)-norm.
Remark 1.3. Our analysis focuses on the weighted spaces $W^{k,p}(\Omega, \varphi; X)$ and we prove results on the spaces $W^{k,p}(\Omega, \varphi; D)$ only where necessary. The choice of which space to focus on is not central to the theory. We could have written the arguments with the roles of the two spaces reversed.

1.3. Weighted Sobolev spaces on $b\Omega$. Let $\epsilon > 0$ and set $M = b\Omega$. Recall that

$$\Omega' = \{ x \in \mathbb{R}^n : \text{dist}(x, M) < \epsilon \}$$

For discussions involving $M$, we always assume $(HI)$ and $m \geq 2$. Therefore, by [HRb], there exists $\epsilon > 0$ and a defining function $\rho$ so that $\|\rho\|_{C^m(\Omega')} < \infty$ and $|d\rho| = 1$ on $b\Omega$. Let $Z_1, \ldots, Z_{n-1} \in TM$ be an orthonormal basis near a point $x \in M$ and let $Z_n = \frac{\partial}{\partial \nu}$ be the unit outward normal to $\Omega$. Moreover, $Z_n$ is also the unit normal to the level curves of $\rho$ (pointing in the direction in which $\rho$ increases). For $1 \leq j \leq n$, set

$$T_j = Z_j - Z_j(\varphi).$$

We call a first order differential operator $T$ tangential if the first order component of $T$ is tangential. For $1 \leq j \leq n - 1$, $Z_j$ is defined locally, and if $U$ is a neighborhood on which $Z_1, \ldots, Z_{n-1}$ form a basis of $T(M \cap U)$, we denote $Z_j$ by $Z_j^U$ to emphasize the dependence on $U$. In analogy to $\nabla_X$, we define $\nabla_U = (T_1, \ldots, T_n)$,

$$\nabla^\text{tan}_T = (T_1, \ldots, T_{n-1}) \quad \text{and} \quad \nabla^\text{tan}_Z = (Z_1, \ldots, Z_{n-1})$$

By $(HI)$, we can construct an open cover $\{U_j\}$ of $\Omega'$ where $U_j$ are of comparable surface area and admit local coordinates $Z_1, \ldots, Z_{n-1}$ with coefficients bounded uniformly in $C^{m-1}$. Let $\chi_j$ be a $C^m$ partition of unity subordinate to $\{U_j\}$ where $\chi_j$ are uniformly bounded in the $C^m$ norm. With $\chi_j$ in hand, we set $v_j = v\chi_j$, so $v = \sum_{j=1}^{\infty} v_j$. Observe that we have the following equivalent norms on $W^{k,2}(\Omega', \varphi; X)$ and $W^{k,2}(\Omega', \varphi; D)$, respectively:

$$\|v\|_{W^{k,2}(\Omega', \varphi; X)} \sim \sum_{j=1}^{\infty} \sum_{|\alpha| \leq k} \|T^0_{U_j} v_j\|_{L^2(\Omega', \varphi)} \quad \text{and} \quad \|v\|_{W^{k,2}(\Omega', \varphi; D)} \sim \sum_{j=1}^{\infty} \sum_{|\alpha| \leq k} \|Z^0_{U_j} v_j\|_{L^2(\Omega', \varphi)}$$

where $T^0_j = Z^0_j - Z^0_j(\varphi)$ and $T^0_\alpha = T^0_{\alpha_1} \cdots T^0_{\alpha_\alpha_j}$ (and similarly for $Z^0$).

For the boundary Sobolev space, set

$$W^{k,p}(M, \varphi; T) = \{ f \in L^p(M, \varphi) : T^\alpha f \in L^p(M, \varphi), \ |\alpha| \leq k \text{ and } T_{\alpha_j} \text{ is tangential for } 1 \leq j \leq k \}.$$

1.4. Notation for differential operators. In the second part of the paper, we establish the elliptic theory for strongly elliptic operators $\Omega \subset \mathbb{R}^n$ that satisfy $(HI)$-$(HV)$ (and sometimes $(HVI)$ as well). Much of our development follows the outline in [Fol95]. Let $L$ be a second order operator of the form

$$L = \sum_{j,k=1}^{n} X_j^* a_{jk} X_k + \sum_{j=1}^{n} (b_j X_j + X_j^* b_j') + b$$

where $a_{jk}$ and $b_j'$ are functions on a neighborhood of $\bar{\Omega}$ that are bounded in the $C^1$ norm, and $b_j$ and $b$ are bounded functions on a neighborhood of $\bar{\Omega}$.

Note that the formal adjoint $(X^\alpha)^* = (-1)^{|\alpha|} D^\alpha$. 

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The formal adjoint of $L$ is the operator given by the formula

$$(L^* v, u)_\varphi = (v, Lu)_\varphi$$

so integration by parts yields that

$$L^* = \sum_{j,k=1}^n X_k^* a_{jk} X_j + \sum_{j=1}^n (\bar{b}'_j X_j + X_j^* b_j) + \bar{b}.$$

We say that the operator $L$ is strongly elliptic on $\bar{\Omega}$ if there exists a constant $\theta > 0$ so that

$$\Re \left( \sum_{j,k=1}^n a_{jk} \xi_j \bar{\xi}_k \right) \geq \theta |\xi|^2.$$  

(2)

Associated to $L$ is a (nonunique) sesquilinear form $\mathcal{D}$ called a Dirichlet form given by

$$\mathcal{D}(v, u) = \sum_{j,k=1}^n (X_j v, a_{jk} X_k u)_\varphi + \sum_{j=1}^n (v, b'_j X_j u)_\varphi + \sum_{j=1}^n (X_j v, b'_j u)_\varphi + (v, bu)_\varphi.$$  

(3)

$\mathcal{D}$ is called a Dirichlet form for the operator $L$ if

$$\mathcal{D}(v, u) = (v, Lu)_\varphi \quad \text{for all } v, u \in C^\infty_c(\Omega).$$

The Dirichlet form $\mathcal{D}$ given by (3) is called strongly elliptic on $\bar{\Omega}$ if (2) holds.

Definition 1.4. The Dirichlet form $\mathcal{D}$ on $\Omega$ is called coercive over $X$ if $W^{1,2}(\Omega, \varphi; X) \subset X \subset W^{1,2}(\Omega, \varphi; X)$, $X$ is closed in $W^{1,2}(\Omega, \varphi; X)$, and there exist $C > 0$ and $\lambda \geq 0$ such that

$$\Re \mathcal{D}(u, u) \geq C\|u\|^2_{W^{1,2}(\Omega, \varphi; X)} - \lambda\|u\|^2_{L^2(\Omega, \varphi)} \quad \text{for all } u \in X.$$  

(4)

$\mathcal{D}$ is called strictly coercive if we can take $\lambda = 0$.

If $\mathcal{D}$ is coercive, then $\mathcal{D}'(v, u) = \mathcal{D}(v, u) + \lambda(v, u)_\varphi$ is strictly coercive.

We can also consider the adjoint Dirichlet form

$$\mathcal{D}^*(v, u) = \overline{\mathcal{D}(u, v)} \quad \text{for all } v, u \in W^{1,2}(\Omega, \varphi; X).$$

The form $\mathcal{D}$ is called self-adjoint if $\mathcal{D} = \mathcal{D}^*$.

2. Main Results

2.1. Sobolev space and trace theorems. Let $\epsilon > 0$ and set $M = b\Omega$ and

$$\Omega'_\epsilon = \{x \in \mathbb{R}^n : \text{dist}(x, M) < \epsilon\}.$$  

In Section 5.1 we will use interpolation to define the Besov space $B^{s,p,q}$. The following theorem is the analog of the Trace Theorem [AF03, Theorem 7.39]

Theorem 2.1. Let $m \geq 2$ and assume that $\Omega \subset \mathbb{R}^n$ satisfies (HI)-(HVI). If $1 \leq k \leq m - 1$, then the following two conditions on a measurable function $u$ on $M$ are equivalent:

(a) There exists $U \in W^{k,2}(\Omega'_\epsilon, \varphi; X)$ supported in $\Omega'_\epsilon$ so that $u = \text{Tr} U$;

(b) $u \in B^{k-\frac{1}{2},2}(M, \varphi; T)$.

The proof of Theorem 2.1 is divided into two results, each of which is more general than one direction of Theorem 2.1. In Section 5.2 we will show
Lemma 2.2. \( \text{Given the hypotheses of Theorem 2.7, if } U \in W^{k,2}(\Omega', \varphi; X), \text{ then Tr} U \in B^{k-\frac{1}{2},2}(M, \varphi; T) \text{ and there exists a constant } K \text{ independent of } U \text{ so that} \)

\[
\| \text{Tr} U \|_{B^{k-\frac{1}{2},2}(M, \varphi; T)} \leq K \| U \|_{W^{k,2}(\Omega', \varphi; X)}.
\]

Remark 2.3. The result also holds (by the same proof) if we replace \( \Omega' \cap \Omega \) with \( \Omega' \cap \Omega^c \).

The second half of the proof of Theorem 2.1 is proven in Section 5.3, as part of the general result:

Theorem 2.4. Let \( \ell, \ell' \) be integers so that \( 0 \leq \ell + \ell' \leq m - 2 \). If \( u \in B^{\ell+\frac{1}{2},2}(M, \varphi; T) \), then

\[
u = \text{Tr} \frac{\partial^{\ell'} U}{\partial \nu^{\ell'}}
\]

for some \( U \in W^{\ell+\ell'+1,2}(\Omega', \varphi; X) \) supported in \( \Omega' \) satisfying

\[
\text{Tr} U = \cdots = \text{Tr} \frac{\partial^{\ell'-1} U}{\partial u^{\ell'-1}} = 0
\]

and

\[
\| U \|_{W^{\ell+\ell'+1,2}(\Omega', \varphi; X)} \leq C \| u \|_B
\]

for some \( C \) independent of \( U \) and \( u \).

The trace and extension theorems above allow us to prove the following result concerning the equality of the spaces with weighted and unweighted derivatives. We also prove the following Rellich identity in Section 5.5.

Proposition 2.5. Let \( \Omega \) satisfy (HI)-(HVI). Then for \( 0 \leq k \leq m \), \( W^{k,2}(\Omega, \varphi; X) = W^{k,2}(\Omega, \varphi; D) \). Furthermore, if \( m = 2 \) and \( 1 \leq k \leq m - 1 \), then \( W^{k,2}(\Omega, \varphi; X) \) embeds compactly in \( W^{k-1,2}(\Omega, \varphi; X) \).

The analog of Proposition 2.5 for \( M \) is contained in Corollary 4.6. It is easier in this case since \( C_\infty^c(M) \) is dense in \( W^{\ell,2}(M, \varphi; \cdot) \) where \( \cdot \) is either \( Z \) or \( T \). Likewise, for \( W^{1,2}(\Omega, \varphi; X) \), the result is easier (though not easy) and is contained Proposition 3.3 and its corollaries.

A useful application of the trace and extension theorems is the construction of a simple \((k,2)\)-extension operator for each \( k \), \( 1 \leq k \leq m - 1 \). Recall that a simple \((k,2)\)-extension operator \( E : W^{k,2}(\Omega, \varphi; X) \to W^{k,2}(\mathbb{R}^n, \varphi; X) \) is one that satisfies \( Eu(x) = u(x) \) for a.e. \( x \in \Omega \) and there exists a constant \( C = C(k) \) so that \( \| Eu \|_{W^{k,2}(\mathbb{R}^n, \varphi; X)} \leq C \| u \|_{W^{k,2}(\Omega, \varphi; X)} \). In Section 5.3 we will show:

Theorem 2.6. Let \( \Omega \) satisfy (HI)-(HVI). Then for \( 1 \leq k \leq m - 1 \), there exists a simple \((k,2)\)-extension operator.

Our final embedding result is proven in Section 5.5.

Theorem 2.7. Let \( M, \Omega \), and \( \varphi \) satisfy the hypotheses of Theorem 2.4. If \( s > \frac{1}{2} \) and \( 1 \leq q \leq \infty \), then

\[
B^{s,2,q}(\Omega, \varphi; X) \hookrightarrow B^{s-\frac{1}{2},2,q}(M, \varphi; T).
\]
2.2. Elliptic regularity – solvability. The Sobolev space theory that we develop is powerful enough that it allows us to adapt the proofs in the bounded, unweighted setting in a straightforward manner and establish the following theorems, see [Pol95, Chapter 7]. In particular, we can establish that strong ellipticity is equivalent to Gårding’s inequality and solve the \((\mathcal{X}, \mathcal{D})\) Boundary Value Problem (BVP): namely, for a closed subspace \(\mathcal{X}\) satisfying \(W^{1,2}_0(\Omega, \varphi; X) \subset \mathcal{X} \subset W^{1,2}(\Omega, \varphi; X)\) and \(f \in L^2(\Omega)\), find \(u \in \mathcal{X}\) so that \(\mathcal{D}(v, u) = (v, f)_{\varphi}\) for all \(v \in \mathcal{X}\). The case \(\mathcal{X} = W^{1,2}_0(\Omega, \varphi; X)\) is the classical Dirichlet problem, but we also want to include the case \(\mathcal{X} = W^{1,2}(\Omega, \varphi; X)\).

Note that \(C_\infty^\infty(\Omega) \subset \mathcal{X}\), so a solution of the \((\mathcal{X}, \mathcal{D})\) BVP will satisfy \((Lu, u)_{\varphi} = (v, f)_{\varphi}\) for all \(v \in C_\infty^\infty(\Omega)\) and hence will be a distributional solution to \(Lu = f\). Furthermore, the requirement that \(\mathcal{D}(v, u) = (v, f)_{\varphi}\) for all \(v \in \mathcal{X}\) leads to a free boundary condition, i.e., integration by parts imposes a boundary condition on \(u\).

**Theorem 2.8** (Gårding’s inequality). Let
\[
\mathcal{D}(v, u) = \sum_{j,k=1}^n (X_j v, a_{jk} X_k u)_{\varphi} + \sum_{j=1}^n (v, b_j X_j u)_{\varphi} + \sum_{j=1}^n (X_j v, b_j u)_{\varphi} + (v, bu)_{\varphi}
\]
be a strongly elliptic Dirichlet form on \(\Omega\) and suppose that \(a_{jk}, b_j, b_j^*, b^*\) are bounded on \(\Omega\). Then \(\mathcal{D}\) is coercive over \(W^{1,2}(\Omega, \varphi; X)\) (and hence over any \(\mathcal{X} \subset W^{1,2}(\Omega, \varphi; X)\) that contains \(W^{1,2}_0(\Omega, \varphi; X)\)).

The converse to Gårding’s inequality holds as well.

**Theorem 2.9.** If the Dirichlet form \(\mathcal{D}\) is coercive over \(W^{1,2}_0(\Omega, \varphi; X)\) and \(a_{jk} \in C(\bar{\Omega})\), then \(\mathcal{D}\) is strongly elliptic.

We can prove existence and uniqueness of weak solutions for operators giving rise to strictly coercive Dirichlet forms.

**Theorem 2.10.** Let \(\mathcal{X}\) be a closed subspace of \(W^{1,2}(\Omega, \varphi; X)\) that contains \(W^{1,2}_0(\Omega, \varphi; X)\) and let \(\mathcal{D}\) be a Dirichlet form that is strictly coercive over \(\mathcal{X}\). There is a bounded, injective operator \(A : L^2(\Omega, \varphi) \to \mathcal{X}\) that solves the \((\mathcal{X}, \mathcal{D})\) BVP, that is, \(\mathcal{D}(v, Af) = (v, f)_{\varphi}\) for all \(v \in \mathcal{X}\) and \(f \in L^2(\Omega, \varphi)\).

Even in the case \(\mathcal{D}\) is not strictly coercive, we can still gain information regarding weak solutions.

**Theorem 2.11.** Let \(\mathcal{X}\) be a closed subspace of \(W^{1,2}(\Omega, \varphi; X)\) that contains \(W^{1,2}_0(\Omega, \varphi; X)\). Let \(\mathcal{D}\) be a Dirichlet form that is coercive over \(\mathcal{X}\). Define
\[
V = \{u \in \mathcal{X} : \mathcal{D}(v, u) = 0 \text{ for all } v \in \mathcal{X}\}
\]
and
\[
W = \{u \in \mathcal{X} : \mathcal{D}(u, v) = 0 \text{ for all } v \in \mathcal{X}\}.
\]
Then \(\dim V = \dim W < \infty\). Moreover, if \(f \in L^2(\Omega, \varphi)\), there exists \(u \in \mathcal{X}\) so that \(\mathcal{D}(v, u) = (v, f)_{\varphi}\) for all \(v \in \mathcal{X}\) if and only if \(f\) is orthogonal to \(W\) in \(L^2(\Omega, \varphi)\) in which case the solution is unique modulo \(V\). In particular, if \(V = W = \{0\}\), the solution always exists and is unique.

In the case that \(\mathcal{D}\) is self-adjoint, we can prove that \(L^2(\Omega, \varphi)\) has a basis of eigenvectors. We will see in Proposition 2.23 that \(W^{1,2}(\Omega, \varphi; X)\) embeds compactly in \(L^2(\Omega, \varphi)\). Thus,
Theorem 2.12. Let \( \mathcal{X} \) be a closed subspace of \( W^{1,2}(\Omega, \varphi; X) \) that contains \( W^{1,2}_0(\Omega, \varphi; X) \). Suppose that \( \mathcal{D} \) is a Dirichlet form that is coercive over \( \mathcal{X} \) and satisfies \( \mathcal{D} = \mathcal{D}^* \). There exists an orthonormal basis \( \{u_j\} \) of \( L^2(\Omega, \varphi) \) consisting of eigenfunctions for the \((\mathcal{X}, \mathcal{D})\) BVP; that is, for each \( j \), there exists \( u_j \in \mathcal{X} \) and a constant \( \mu_j \in \mathbb{R} \) so that \( \mathcal{D}(v, u_j) = \mu_j(v, u_j) \varphi \) for all \( v \in \mathcal{X} \). Moreover, \( \mu_j > -\lambda \) for all \( j \) where \( \lambda \) is the constant in the coercive estimate \([4]\), \( \lim_{j \to \infty} \mu_j = \infty \), and \( u_j \in C^\infty(\Omega) \) for all \( j \).

2.3. Elliptic regularity – estimates. We can prove elliptic regularity in the interior of \( \Omega \) in Section 6.

Theorem 2.13. Let \( \Omega \subset \mathbb{R}^n \) satisfy (HI)-(HV) for some \( m \geq 2 \). Let \( L \) be defined by \([1]\) and \( a_{jk}, b_j' \in C^{\ell+1}(\Omega) \cap W^{\ell+1,\infty}(\Omega) \) and \( b_j \in C^{\ell}(\Omega) \cap W^{\ell,\infty}(\Omega) \) for some \( 0 \leq \ell \leq m \). Assume that \( f \in W^{\ell,2}(\Omega, \varphi; X) \) and \( L \) is strongly elliptic. Suppose that \( u \in W^{1,2}(\Omega, \varphi; X) \) is a weak solution (i.e., \( \mathcal{D}(v, u) = (v, f)_\varphi \) for all \( v \in W^{1,2}_0(\Omega, \varphi; X) \)) of the elliptic PDE

\[
Lu = f \quad \text{in } \Omega.
\]

Then \( u \in W^{\ell+2,2}_0(\Omega, \varphi; X) \) and if \( V \subset \Omega \) is open and satisfies \( \text{dist}(V, \partial \Omega) > 0 \),

\[
\|u\|_{W^{\ell+2,2}(V, \varphi; X)} \leq C(\|f\|_{W^{\ell}(\varphi; X)} + \|u\|_{L^2(\varphi; \varphi)})
\]

where \( C = C(\text{dist}(V, \partial \Omega), C_{a,\ell}, \|a_{jk}\|_{C^{\ell+1}(\Omega)}, \|b_j\|_{C^\ell(\Omega)}, \|b_j'\|_{C^{\ell+1}(\Omega)}, \|b\|_{C^\ell(\Omega)}, n, \theta, \ell) \).

Note that the inequality in Theorem 2.13 is not an \textit{a priori} inequality. The meaning of \( \mathcal{D} \) is that if the right-hand side is finite, then \( u \in W^{\ell+2,2}(V, \varphi; X) \).

The case that \( \Omega \) is bounded is not the only case for which we know the hypothesis that \( u \in W^{1,2}(\Omega, \varphi; X) \) is satisfied. Indeed, we if combine Theorem 2.10 and Theorem 2.11 with Theorem 2.13 for \( \ell = 0 \), we have the following corollary.

Corollary 2.14. Let \( L, V, \) and \( \Omega \) be as in Theorem 2.13. Let \( \mathcal{X} \) be a closed subspace of \( W^{1,2}(\Omega, \varphi; X) \) that contains \( W^{1,2}_0(\Omega, \varphi; X) \). Let \( \mathcal{D} \) be the Dirichlet form corresponding to \( L \) in \( \mathcal{X} \). If any of the following conditions hold:

(i) \( \mathcal{D} \) is strictly coercive,

(ii) \( f \perp W = \{w \in \mathcal{X} : \mathcal{D}(w, v) = 0 \text{ for all } v \in \mathcal{X}\} \) and \( u \) is the weak solution that is orthogonal to \( V \),

then \( u \in \mathcal{X} \) and hence in \( W^{2,2}(V, \varphi; X) \).

We can also prove that elliptic regularity holds near the boundary for weak solutions of the partial differential equation \( Lu = f \) in Sections 7.2 and 7.3.

Theorem 2.15. Let \( \Omega \subset \mathbb{R}^n \) satisfy (HI)-(HVI) with \( m \geq 3 \). Let \( 0 \leq \ell \leq m - 3 \) and the operator \( L \) be defined by \([1]\) where \( a_{jk}, b_j' \in C^{\ell+1}(\Omega) \cap W^{\ell+1,\infty}(\Omega) \) and \( b_j \in W^{\ell,\infty}(\Omega) \). Assume that \( f \in W^{\ell,2}(\Omega, \varphi; X) \) and \( L \) is strongly elliptic. Let \( \mathcal{X} \) be a closed subspace of \( W^{1,2}(\Omega, \varphi; X) \) that contains \( W^{1,2}_0(\Omega, \varphi; X) \). Suppose that \( u \in \mathcal{X} \) is a weak solution (i.e., \( \mathcal{D}(v, u) = (v, f)_\varphi \) for all \( v \in \mathcal{X} \)) of the elliptic PDE

\[
Lu = f \quad \text{in } \Omega.
\]

Then \( u \in W^{\ell+2,2}(\Omega, \varphi; X) \) and

\[
\|u\|_{W^{\ell+2,2}(\Omega, \varphi; X)} \leq C(\|f\|_{W^{\ell}(\varphi; X)} + \|u\|_{L^2(\varphi; \varphi)})
\]

where \( C = C(M_{\ell}, \|a_{jk}\|_{C^{\ell+1}(\Omega)}, \|b_j\|_{C^{\ell+1}(\Omega)}, \|b_j'\|_{C^{\ell+1}(\Omega)}, \|b\|_{C^\ell(\Omega)}, n, \theta, \|\rho\|_{C^{\ell+3}(\Omega)}). \)
2.4. Boundary values of \( L \)-harmonic functions. We conclude the paper with a study of the boundary values of \( L \)-harmonic functions. The goal is to show that \( L \)-harmonic functions (i.e., functions \( u \) satisfying \( Lu = 0 \)) have unique boundary values in \( W^{s-1/2,2}(b\Omega, \varphi; T) \) when \( u \in W^{s,2}(\Omega, \varphi; X) \) and \( s \geq 0 \).

We first establish a simple but easily applicable uniqueness condition. Let \( L \) be a strongly elliptic second order operator. We would like to understand conditions on \( L \) so that if \( Lu = 0 \) and \( u|_{M} = 0 \), then \( u = 0 \). Theorem 2.10 present one condition, and we will show the following in Section 8.

**Lemma 2.16.** Let \( \Omega \subset \mathbb{R}^{n} \) be a domain that satisfies (HII). Let \( L \) be a strongly elliptic operator that has a Dirichlet form \( \mathcal{D} \) so that for all \( u \in W^{1,2}_{0}(\Omega, \varphi; X) \) there exists a constant \( c \) satisfying

\[
\text{Re} \mathcal{D}(u, u) \geq c \| \nabla_X u \|_{L^2(\Omega, \varphi)}.
\]

If \( Lu = 0 \) and \( u \in W^{1,2}_{0}(\Omega, \varphi; X) \), then \( u \equiv 0 \).

**Remark 2.17.** If the operator \( L \) is of the form

\[
L = \sum_{j,k=1}^{n} X^*_j a_{jk} X_k + b
\]

where \( b > 0 \), then \( L \) satisfies (7).

With this restriction on \( \mathcal{D} \), we can prove in Sections 8.1 and 8.2:

**Theorem 2.18.** Let \( \Omega \subset \mathbb{R}^{n} \) be a domain that satisfies (HI)-(HVI) for \( m = 2 \). Let \( L \) be a strongly elliptic operator that has a Dirichlet form \( \mathcal{D} \) which satisfies (7). The map sending \( u \mapsto (Lu, \text{Tr} u) \)

is an isomorphism from

\[
W^{s,2}(\Omega, \varphi; X) \to W^{s-2,2}(\Omega, \varphi; X) \times W^{s-1/2,2}(b\Omega, \varphi; T)
\]

for \( 1 \leq s \leq m - 1 \).

With an additional restriction on \( L \), we can prove that \( L \)-harmonic functions in \( L^2(\Omega) \) have boundary values if \( s \geq 0 \) in Section 8.3.

**Theorem 2.19.** Let \( \Omega \subset \mathbb{R}^{n} \) be a domain that satisfies (HI)-(HVI) for \( m \geq 3 \). Let \( L \) be of the form

\[
L = \sum_{j=1}^{n} \left( X^*_j X_j + b_j X_j + X^*_j b'_j \right) + b,
\]

If \( f \in W^{s,2}(\Omega, \varphi; X) \) for \( s \geq 0 \) and \( Lf = 0 \), then \( \text{Tr} f \) is well-defined and an element of \( W^{s-1/2,2}(b\Omega, \varphi; T) \).

3. Facts for \( W^{m,p}(\Omega, \varphi; X) \), \( 1 < p < \infty \)

3.1. The spaces \( W^{-k,q}(\Omega, \varphi; X) \) and \( W^{k,p}(\Omega, \varphi; X)^* \). Let \( 1 \leq p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Fix \( k \in \mathbb{N} \) and let \( N(k) \) be the number of multiindices \( \alpha \) where \( |\alpha| \leq k \). As \( k \) is fixed, we suppress the argument of \( N \). Let \( \alpha^1, \ldots, \alpha^N \) be an enumeration of such multiindices. For a vector \( g \),
we write $g = (g_1, \ldots, g_N) = (g_\alpha)$ interchangeably. For functions $g_1, \ldots, g_N \in L^q(\Omega, \varphi)$, there exists a bounded linear functional $T_{g_1, \ldots, g_N}$ on $W^{k,p}(\Omega, \varphi; X)$ defined by

$$T_{g_1, \ldots, g_N} f = \int_\Omega \left( \sum_{j=1}^N X^{\alpha j} f_{\alpha j} \right) e^{-\varphi} dV.$$ 

We can show that every functional on $W^{k,p}(\Omega, \varphi; X)$ arises in this way.

**Proposition 3.1.** For $\Omega \subset \mathbb{R}^n$, let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(i) The dual space $W^{-k,q}(\Omega, \varphi; X) := W^{k,p}_0(\Omega, \varphi; X)^*$ is the set $W^{-k,q}(\Omega, \varphi; X) = \{ u \in \mathcal{D}'(\Omega) : u = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha g_\alpha, \ g_\alpha \in L^q(\Omega, \varphi) \mbox{ for all } \alpha \}$. Moreover, the norm on $W^{-k,q}(\Omega, \varphi; X)$ is given by

$$\|u\|_{W^{-k,q}(\Omega, \varphi; X)} := \sup\{|u(f)| : f \in W^{k,p}_0(\Omega, \varphi; X), \|f\|_{W^{k,p}(\Omega, \varphi; X)} = 1\}$$

$$= \inf_F \left\{ \sum_{|\alpha| \leq k} \|g_\alpha\|_{L^q(\Omega, \varphi)}^q \right\}$$

where $F$ is the set of $N$-tuples $(g_1, \ldots, g_N) \in L^q(\Omega, \varphi)^N$ representing the functional $u$.

(ii) The dual space $W^{k,p}(\Omega, \varphi; X)^*$ consists of $u \in \mathcal{D}'(\Omega)$ for which there exists a vector $g = (g_\alpha) \in (L^q(\Omega, \varphi))^N$ so that for all $f \in W^{k,p}(\Omega, \varphi; X)$,

$$u(f) = \sum_{|\alpha| \leq k} (X^\alpha f, g_\alpha)_{\varphi}.$$ 

Moreover, the norm on $W^{k,p}(\Omega, \varphi; X)^*$

$$\|u\|_{W^{k,p}(\Omega, \varphi; X)^*} := \sup\{|u(f)| : f \in W^{k,p}(\Omega, \varphi; X), \|f\|_{W^{k,p}(\Omega, \varphi; X)} = 1\}.$$ 

**Proof.** The proof is standard. See, for example, [AF03] Sections 3.9, 3.12, 3.13]



3.2. Approximation by $W^{k,p}(\Omega, \varphi; X)$.

**Proposition 3.2.** Let $1 \leq p < \infty$ and assume that $b\Omega$ satisfies (HI) for some $m \geq 1$. Let $1 \leq \ell \leq m$ be an integer. Then $C_c^\infty(\mathbb{R}^n)$ is dense in both $W^{\ell,p}(\Omega, \varphi; X)$ and $W^{\ell,p}(\Omega, \varphi; D)$ in the sense that if $\epsilon > 0$ and $f \in W^{\ell,p}(\Omega, \varphi; X)$, then there exists $\xi \in C_c^\infty(\mathbb{R}^n)$ so that

$$\|\xi - f\|_{W^{\ell,p}(\Omega, \varphi; X)} \leq \epsilon$$

where $C$ is independent of $f$, $\varphi$, and $\ell$. Similarly, if $\epsilon > 0$ and $f \in W^{\ell,p}(\Omega, \varphi; D)$, then there exists $\psi \in C_c^\infty(\mathbb{R}^n)$ so that

$$\|\psi - f\|_{W^{\ell,p}(\Omega, \varphi; D)} \leq \epsilon$$

where $C$ is independent of $f$, $\varphi$, and $\ell$.

**Proof.** Let $\epsilon > 0$ and $\chi_R$ be a smooth, nonnegative cut-off function so that $\chi_R \equiv 1$ on $B(0, R)$, $\chi_R \equiv 0$ off $B(0, 2R)$, and $|D^\alpha \chi_R| \leq C_{|\alpha|} |R|^{|\alpha|}$ for $|\alpha| \geq 0$. For $R$ sufficiently large, it follows that

$$\|(1 - \chi_R) f\|_{W^{\ell,p}(\Omega, \varphi; X)} \leq \epsilon.$$
Let $g = \chi_R f$, and extend $g$ to be zero outside of $\Omega$. It is enough to prove the result for $g$. Let $\Omega'$ be a $C^m$ domain satisfying

$$B(0, 2R) \cap \Omega \subset \Omega' \subset B(0, 3R) \cap \Omega.$$ 

Since supp $\chi_R \subset B(0, 2R)$, $g = g_1\Omega'$. Since $\Omega'$ is bounded, there exists $C_{\Omega'} > 0$ so that

$$\|h\|_{W^{\ell,p}(\Omega', \varphi; X)} \leq C_{\Omega'} \|h\|_{W^{\ell,p}(\Omega')}$$

for any $h \in W^{\ell,p}(\Omega')$. The function $x$ is constructed in the following manner. Extend $g$ to $\tilde{g} \in W^{\ell,p}(\mathbb{R}^n)$ following the technique of \cite[Theorem 5.22]{AF03}. Since $g$ is identically zero in a neighborhood of $\partial\Omega' \cap \Omega$, this construction can be used to guarantee that $\tilde{g}$ is identically zero on $\Omega \backslash \Omega'$. Form $x$ by mollifying $\tilde{g}$ in such a way that $x$ is also identically zero on $\Omega \backslash \Omega'$. Since $x$ is constructed so that $\|g - x\|_{W^{\ell,p}(\Omega')} < \epsilon/C_{\Omega'}$, we have

$$\|g - x\|_{W^{\ell,p}(\Omega', \varphi; X)} \leq \|g - x\|_{W^{\ell,p}(\Omega', \varphi; X)} < \epsilon.$$

\[\square\]

### 3.3. Embeddings and compactness for $p = 2$

**Proposition 3.3.** Let $\Omega \subset \mathbb{R}^n$ satisfy (HII). Then the embedding of $W_0^{1,2}(\Omega, \varphi; X) \hookrightarrow L^2(\Omega, \varphi)$ is compact.

**Proof.** We start by making a number of preliminary calculations. Note that the formal adjoint $X_j^* = -\frac{\partial}{\partial x_j}$. This means

$$\begin{equation}
X_j + X_j^* f = -\frac{\partial \varphi}{\partial x_j} f
\end{equation}$$

and

$$\begin{equation}
[X_j, X_j^*] f = -\frac{\partial^2 \varphi}{\partial x_j^2} f.
\end{equation}$$

Note that for $f \in C_c^\infty(\Omega)$,

$$\begin{equation}
([X_j, X_j^*] f, f)_{\varphi} = (X_j X_j^* f, f)_{\varphi} - (X_j^* X_j f, f)_{\varphi} = \|X_j^* f\|_{L^2(\Omega, \varphi)}^2 - \|X_j f\|_{L^2(\Omega, \varphi)}^2.
\end{equation}$$

Next, we see that for $\epsilon > 0$, a small constant/large constant argument yields

$$\|X_j + X_j^* f\|_{L^2(\Omega, \varphi)}^2 \leq \left(1 + \frac{1}{2\epsilon}\right) \|X_j f\|_{L^2(\Omega, \varphi)}^2 + (1 + \epsilon) \|X_j^* f\|_{L^2(\Omega, \varphi)}^2.$$

Now set

$$\Psi(x) = |\nabla \varphi(x)|^2 + (1 + \epsilon) \Delta \varphi(x).$$

Consequently,

$$\begin{equation}
\Psi f, f)_{\varphi} = \sum_{j=1}^n \left(\|X_j + X_j^* f\|_{L^2(\Omega, \varphi)}^2 - (1 + \epsilon)([X_j, X_j^*] f, f)_{\varphi}\right) \leq \left(2 + \epsilon + \frac{1}{2\epsilon}\right) \sum_{j=1}^n \|X_j f\|_{L^2(\Omega, \varphi)}^2.
\end{equation}$$

Since $C^\infty_c(\Omega)$ is dense in $W_0^{1,2}(\Omega, \varphi; X)$, this inequality holds for all $f \in W_0^{1,2}(\Omega, \varphi; X)$. 


Let \( \{ f_k \} \subset W^{1,2}_0(\Omega, \varphi; X) \) be a bounded sequence and set \( M = \max_k \| f_k \|_{W^{1,2}(\Omega, \varphi; X)}^2 \). Set \( I(R) = \inf_{x \in \Omega \setminus B(0, R)} |\Psi(x)| \). Since \( I(R) > 0 \) for \( R \) sufficiently large,

\[
\| f_k - f_j \|_{L^2(\Omega, \varphi)}^2 \leq \int_{B(0, R) \cap \Omega} |f_k - f_j(x)|^2 e^{-\varphi} dV + \int_{\Omega \setminus B(0, R)} \frac{\Psi(x)}{I(R)} |f_k - f_j(x)|^2 e^{-\varphi} dV
\]

\[
\leq C_{\varphi, R} \| f_k - f_j \|_{L^2(B(0, R))}^2 + \frac{C \| f_k - f_j \|_{W^{1,2}(\Omega, \varphi; X)}}{I(R)}
\]

(13)

\[
\leq C_{\varphi, R} \| f_k - f_j \|_{L^2(B(0, R))}^2 + C \frac{M}{I(R)}.
\]

Fix an increasing sequence \( R_j \to \infty \), so that \( R_j \) satisfies \( M/I(R_j) \leq 1/j \). We may inductively construct a sequence of subsequences \( f_{k_j} \) so that

(i) \( f_{k_{j+1}} \) is a subsequence of \( f_{k_j} \),

(ii) \( \lim_{j \to \infty} f_{k_j} = f_k \) in \( L^2(B(0, R_m) \cap \Omega, \varphi) \), and

(iii) \( f_{k_m} \lvert_{B(0, R_k)} = f_k \) if \( \ell \leq m \).

It is now easy to see from (13) that \( f_{k_j} \) is a Cauchy sequence in \( L^2(\Omega, \varphi) \) and hence converges in \( L^2(\Omega, \varphi) \). Thus, \( W^{1,2}_0(\Omega, \varphi; X) \) embeds compactly in \( L^2(\Omega, \varphi) \). 

\[ \square \]

**Corollary 3.4.** The embedding \( L^2(\Omega, \varphi) \hookrightarrow W^{-1,2}(\Omega, \varphi; X) \) is compact.

**Corollary 3.5.** Let \( \Omega \subset \mathbb{R}^n \) satisfy (HIII). There exists a constant \( C > 0 \) so that

\[ \| \nabla f \|_{L^2(\Omega, \varphi)}^2 \leq C \| \nabla x f \|_{L^2(\Omega, \varphi)}^2 \]

for \( f \in W^{1,2}_0(\Omega, \varphi; X) \).

**Proof.** By (11) and (12),

\[
\| \nabla f \|_{L^2(\Omega, \varphi)}^2 = \| \nabla x f \|_{L^2(\Omega, \varphi)}^2 - \sum_{j=1}^n ([X_j, X_j^*] f, f)_{\varphi} \leq \| \nabla x f \|_{L^2(\Omega, \varphi)}^2 + \frac{1}{1 + \epsilon} (\Psi f, f)_{\varphi}
\]

\[
\leq C \| \nabla x f \|_{L^2(\Omega, \varphi)}^2.
\]

\[ \square \]

**Proposition 3.6.** Let \( \Omega \subset \mathbb{R}^n \) satisfy (HIII). Then the embedding of \( W^{1,2}_0(\Omega, \varphi; D) \hookrightarrow L^2(\Omega, \varphi) \) is compact.

**Proof.** The proof follows the lines of the proof of Proposition 3.3 with

\[ \Theta(x) = |\nabla \varphi(x)|^2 - (1 + \epsilon) \Delta \varphi(x). \]

replacing \( \Psi(x) \).

\[ \square \]

**Corollary 3.7.** Let \( \Omega \subset \mathbb{R}^n \) satisfy (HIII). Then the embedding \( L^2(\Omega, \varphi) \hookrightarrow W^{-1,2}(\Omega, \varphi; D) \) is compact.

**Corollary 3.8.** Let \( \Omega \subset \mathbb{R}^n \) satisfy (HIII). Then there exists a constant \( C > 0 \) so that

\[ \| \nabla x f \|_{L^2(\Omega, \varphi)}^2 \leq C \| \nabla f \|_{L^2(\Omega, \varphi)}^2 \]

for all \( f \in W^{1,2}_0(\Omega, \varphi; D) \).
Remark 3.9. Proposition 3.2 and Corollaries 3.5 and 3.8 allow us to define a number of equivalent ways to measure the $W^{k,2}_0(\Omega, \varphi; X)$ norm (which we will use later on $\Omega'$). Let $\psi \in W^{k,2}_0(\Omega, \varphi; X)$ and

$$Y_j = \frac{1}{2}(X_j - X_j^*) = \frac{\partial}{\partial x_j} - \frac{1}{2} \frac{\partial \varphi}{\partial x_j} = X_j + \frac{1}{2} \frac{\partial \varphi}{\partial x_j}$$

and consequently (HIV)

$$\nabla Y \psi = (Y_1 \psi, \ldots, Y_n \psi).$$

Note that

$$\|Y_j \psi\|_{L^2(\Omega, \varphi)}^2 = \left(\frac{\partial \psi}{\partial x_j} - \frac{1}{2} \frac{\partial \varphi}{\partial x_j}, \frac{\partial \psi}{\partial x_j} - \frac{1}{2} \frac{\partial \varphi}{\partial x_j}\right) \varphi$$

$$= \left\|\frac{\partial \psi}{\partial x_j}\right\|^2_{L^2(\Omega, \varphi)} + \frac{1}{4} \left\|\frac{\partial \varphi}{\partial x_j}\right\|^2_{L^2(\Omega, \varphi)} - \Re \left(\frac{\partial \psi}{\partial x_j}, \frac{\partial \varphi}{\partial x_j}\right) \varphi$$

and

$$\|Y_j \psi\|_{L^2(\Omega, \varphi)}^2 = \left(\frac{1}{2} \frac{\partial \varphi}{\partial x_j}, \frac{1}{2} \frac{\partial \varphi}{\partial x_j}\right) \varphi$$

$$= \left\|X_j \psi\right\|^2_{L^2(\Omega, \varphi)} + \frac{1}{4} \left\|\frac{\partial \varphi}{\partial x_j}\right\|^2_{L^2(\Omega, \varphi)} + \Re \left(\frac{\partial \psi}{\partial x_j}, \frac{\partial \varphi}{\partial x_j}\right) \varphi.$$
where $T^\alpha$ is a tangential operator of order $|\alpha|$. Moreover, the norm on $W^{-k,q}(M, \varphi; T) := W^{k,p}(M, \varphi; T)^*$

$$\|u\|_{W^{-k,q}(M,\varphi; T)} := \sup \{|u(f)| : f \in W^{k,p}(M, \varphi; T), \|f\|_{W^{k,p}(M, \varphi; T)} = 1\}$$

$$= \inf_{\mathcal{F}} \left\{ \sum_{|\alpha| \leq k} \|g_\alpha\|_{L^q(M, \varphi)}^q \right\}$$

where $\mathcal{F}$ is the set of $N$-tuples $(g_1, \ldots, g_N) \in L^q(M, \varphi)^N$ representing the functional $u$.

4.1. **Approximations and embeddings for $W^{k,p}(M, \varphi; T)$**. When considering results on the boundary, we will generally need both (HII) and (HIII). Adding these, it is helpful to observe that we have

(15) \[ \lim_{|x| \to 0} |\nabla \varphi| = \infty. \]

Conversely, (HIV) and (15) imply both (HII) and (HIII), since (HIV) with $k = 2$ implies

(16) \[ |\Delta \varphi| \leq n|\nabla^2 \varphi| \leq nC_2(1 + |\nabla \varphi|). \]

By classical results, we know that $C^m_c(M)$ is dense in $W^{k,2}(M, \varphi; T)$.

Let $B = (\tau_{j\ell})$ be the matrix with bounded $C^{m-1}$ coefficients so that

$$Z_j = \sum_{\ell=1}^n \tau_{j\ell} \frac{\partial}{\partial x_\ell}. $$

Since $T_j = Z_j - Z_j \varphi$, $T_j = (B\nabla X)_j$. Then $T_\ell = \sum_{\ell'=1}^n \tau_{\ell\ell'} X_{\ell'}$ implies

$$T_\ell^* = \sum_{\ell'=1}^n \left( \tau_{\ell\ell'} X_{\ell'}^* - \frac{\partial \tau_{\ell\ell'}}{\partial x_{\ell'}} \right). $$

Using the formula for $T_\ell^*$ and [9], we observe that

$$T_j + T_j^* = -\sum_{\ell=1}^n \left(\tau_{j\ell} \frac{\partial \varphi}{\partial x_\ell} + \frac{\partial \tau_{j\ell}}{\partial x_\ell}\right) = -(B\nabla \varphi)_j - \sum_{\ell=1}^n \frac{\partial \tau_{j\ell}}{\partial x_\ell}. $$

If $H \varphi$ is the Hessian of $\varphi$, then from (10) it follows that

$$[T_j, T_j^*] = \sum_{\ell,\ell'=1}^n \left( -\tau_{j\ell} \tau_{j\ell'} \frac{\partial^2 \varphi}{\partial x_\ell \partial x_{\ell'}} + \tau_{j\ell} \frac{\partial \tau_{j\ell'}}{\partial x_\ell} X_{\ell'} + \tau_{j\ell'} \frac{\partial \tau_{j\ell}}{\partial x_{\ell'}} X_\ell - \tau_{j\ell} \frac{\partial^2 \tau_{j\ell'}}{\partial x_\ell \partial x_{\ell'}} \right)$$

$$= \sum_{\ell,\ell'=1}^n \left( -\tau_{j\ell} \tau_{j\ell'} \frac{\partial^2 \varphi}{\partial x_\ell \partial x_{\ell'}} - \tau_{j\ell} \frac{\partial \tau_{j\ell'}}{\partial x_\ell} \frac{\partial \varphi}{\partial x_{\ell'}} - \tau_{j\ell'} \frac{\partial \tau_{j\ell}}{\partial x_{\ell'}} \frac{\partial \varphi}{\partial x_\ell} - \tau_{j\ell} \frac{\partial^2 \tau_{j\ell'}}{\partial x_\ell \partial x_{\ell'}} \right)$$

$$= -(B(H \varphi) B^T)_{jj} - \sum_{\ell,\ell'=1}^n \left[ \tau_{j\ell} \frac{\partial \tau_{j\ell'}}{\partial x_\ell} \frac{\partial \varphi}{\partial x_{\ell'}} + \tau_{j\ell'} \frac{\partial \tau_{j\ell}}{\partial x_{\ell'}} \frac{\partial \varphi}{\partial x_\ell} \right]. $$

The key to the proof of Proposition 3.3 was the construction of $\Psi$, an unbounded function so that $(\Psi f, f)_\varphi$ could be written in terms of inner products involving $\|\nabla X f\|_{L^2(\Omega, \varphi)}$ and
Adapting the heuristic of Proposition 3.3, we compute

\[
\sum_{j=1}^{n-1} \left[ \| (T_j^* + T_j) f \|_{L^2(M,\varphi)}^2 - (1+\epsilon)([T_j, T_j^*]f, f)_{M,\varphi} \right]
\]

\[
= \sum_{j=1}^{n-1} \left[ \| - (B\nabla \varphi)_j f - \sum_{\ell=1}^{n} \frac{\partial \tau_{\ell j'}}{\partial x_{\ell'}} f \|_{L^2(M,\varphi)}^2 \right]
\]

\[
+ (1+\epsilon) \left( \left[ \text{Tr} (B(H\varphi)B^T) \right]_{jj} f + \sum_{\ell,\ell'=1}^{n} \left[ \tau_{\ell j'} \frac{\partial \tau_{\ell j'}}{\partial x_{\ell'}} \frac{\partial \varphi}{\partial x_{\ell'}} + \tau_{\ell j'} \frac{\partial^2 \tau_{\ell j'}}{\partial x_{\ell}\partial x_{\ell'}} \right] f, f \right)_{M,\varphi} \right].
\]

Therefore, the analog of \( \Psi \) in Proposition 3.3 is

\[
(17) \quad \Psi_M(x) = \sum_{j=1}^{n-1} \left[ (B\nabla \varphi)_j + \sum_{\ell=1}^{n} \frac{\partial \tau_{\ell j}}{\partial x_{\ell}} \right]^2
\]

\[
+ (1+\epsilon) \left( \left[ \text{Tr} (B(H\varphi)B^T) + \sum_{\ell,\ell'=1}^{n} \left[ \tau_{\ell j'} \frac{\partial \tau_{\ell j'}}{\partial x_{\ell'}} \frac{\partial \varphi}{\partial x_{\ell'}} + \tau_{\ell j'} \frac{\partial^2 \tau_{\ell j'}}{\partial x_{\ell}\partial x_{\ell'}} \right] \right] f, f \right)_{M,\varphi} \right].
\]

The matrix \( B \) plays a critical role here. We observe that

\[
B\nabla = \begin{pmatrix} \nabla_{Z_{\tan}}^\top \\ \partial/\partial \nu \end{pmatrix}
\]

where \( \frac{\partial}{\partial \nu} = Z_n \) is the unit outward pointing normal. Now, (HVI) and (H5) tell us

\[
\lim_{|x| \to \infty} |\nabla_{Z_{\tan}}^\top \varphi| = \infty,
\]

as well. Using (HIV) and (HVI) to bound \( H\varphi \), we have

\[
\Psi_M(x) \geq |\nabla_{Z_{\tan}}^\top \varphi|^2 - O(|\nabla_{Z_{\tan}}^\top \varphi| + 1).
\]

Hence, we have the following analogue of (HIII):

BI. There exists \( \epsilon > 0 \) so that \( \Psi_M \) defined by (17) satisfies

\[
\lim_{|x| \to \infty} \Psi_M(x) = \infty.
\]

Proposition 4.2. Let \( \Omega \subset \mathbb{R}^n \) satisfy (HI)-(HK) and \( b\Omega \) satisfy (BI). Then the embedding

\[
W^{1,2}(M,\varphi;T) \hookrightarrow L^2(M,\varphi)
\]

is compact.

Proof. The proof follows the argument of Proposition 3.3.

As earlier, we have the following corollary.

Corollary 4.3. Under the assumptions and notation of Proposition 4.2

\[
\| \nabla_{T^*} f \|_{L^2(M,\varphi)}^2 \leq C \| \nabla_{T} f \|_{L^2(M,\varphi)}^2
\]

for some constant \( C \) independent of \( f \).
A similar argument shows the following Rellich identity. Set
\[(18) \quad \Theta_M = \sum_{j=1}^{n-1} \left[ \left( B\nabla \varphi \right)_j - \sum_{\ell=1}^{n} \frac{\partial \tau_{j\ell}}{\partial x_\ell} \right]^2 \]
\[- (1 + \epsilon) \left( \text{Tr} \left( B(\mathcal{H}\varphi)B^T \right) + \sum_{\ell,\ell'=1}^{n} \left[ \tau_{j\ell} \frac{\partial \varphi}{\partial x_{\ell'}} + \tau_{j\ell'} \frac{\partial^2 \tau_{j\ell}}{\partial x_{\ell}\partial x_{\ell'}} \right] \right) \]

As before, (HIII)-(HIV) and (HVI) can be used to prove an analogue to (HIII):

BII. There exists \( \epsilon > 0 \) so that \( \Theta_M \) satisfies
\[ \lim_{|x| \to \infty, x \in M} \Theta_M(x) = \infty. \]

**Proposition 4.4.** Let \( \Omega \subset \mathbb{R}^n \) satisfy (HI)-(HV) and \( b\Omega \) satisfy (BII). Then the embedding \( W^{1,2}(M, \varphi; L) \hookrightarrow L^2(M, \varphi) \) is compact.

**Corollary 4.5.** Under the assumptions and notation of Proposition 3.3
\[ \| \nabla^\tan f \|_{L^2(M, \varphi)}^2 \leq C \| f \|_{W^{1,2}(M, \varphi; L)}^2 \]
for some constant \( C \) independent of \( f \).

Our final comment on the consequences of (HIV) and (HVI) is the following:

BIII. There exist constants \( C_k \) so that
\[ \left| (\nabla^\tan Z)^k \varphi \right| \leq C_k (1 + |\nabla^\tan Z\varphi|) \]
for all \( x \in M \) and \( 1 \leq k \leq m \).

Since we have shown that (BI)-(BIII) follow from (HI)-(HVI), we will suppress the individual boundary hypotheses and assume only (HVI) in the following.

**Corollary 4.6.** Suppose that \( \Omega \subset \mathbb{R}^n \) satisfies (HI)-(HVI). Then if \( 0 < k \leq m \),
\[ \| f \|_{W^k(M, \varphi; T)} \sim \| f \|_{W^k(M, \varphi; Z)}. \]

**Proof.** The proof goes by induction. The \( k = 1 \) case is the content of Corollary 4.3 and Corollary 4.5. The higher \( k \) follow from the \( k = 1 \) case, the inductive hypothesis and the fact that \([T_j, T^*_j]\) is a function bounded by a multiple of \((1 + |\nabla^\tan Z\varphi|)\). \( \square \)

**Proposition 4.7.** Let \( \Omega \subset \mathbb{R}^n \) satisfy (HI)-(HVI). Then there exists \( K, K' > 0 \) depending on \( n, m \) so that for any \( \delta > 0 \), \( u \in W^{m,2}(M, \varphi; T) \), and \( 0 < j < m \),
\[(19) \quad \sum_{|\alpha| = j} \| T^\alpha u \|_{L^2(M, \varphi)}^2 \leq K \left( \delta \sum_{|\beta| = m} \| T^{\beta} u \|_{L^2(M, \varphi)}^2 + \delta^{-j/(m-j)} \| u \|_{L^2(M, \varphi)}^2 \right) \]
\[(20) \quad \| u \|_{W^{j,2}(M, \varphi; T)} \leq K' \left( \delta \| u \|_{W^{m,2}(M, \varphi; T)} + \delta^{-j/(m-j)} \| u \|_{L^2(M, \varphi; T)} \right) \]
\[(21) \quad \| u \|_{W^{j/m,2}(M, \varphi; T)} \leq 2K' \| u \|_{W^{j/m,2}(M, \varphi; T)}^j \| u \|_{L^2(M, \varphi; T)}^{(m-j)/m} \]

**Proof.** Note that (20) follows from repeated applications of (19). Equation (21) follows from (20) by choosing \( \epsilon \) so that the two terms on the right-hand side are equal.
We first prove the result for \( m = 2, j = 1 \). In this case,
\[
\|T_j u\|_{L^2(M,\varphi)}^2 = (T_j u, T_j u)_\varphi = (T^*_j T_j u, u)_\varphi
\]
\[
\leq \|T^*_j T_j u\|_{L^2(M,\varphi)} \|u\|_{L^2(M,\varphi)} \leq \epsilon \|T^*_j T_j u\|^2_{L^2(M,\varphi)} + \frac{1}{4\epsilon} \|u\|^2_{L^2(M,\varphi)}.
\]
However, since \( \lim_{|x| \to \infty} (\theta|\nabla \varphi|^2 + \triangle \varphi) = \infty \), it follows by Corollary 4.3 that
\[
\|T^*_j T_j u\|_{L^2(M,\varphi)}^2 \leq C \|T_j u\|_{W^{1,2}(M,\varphi;T)}^2.
\]
This proves the result for the case \( m = 2, j = 1 \). We can follow the argument of [AF03 Theorem 5.2] to finish proof.

4.2. **Approximation.** We can also prove a boundary version of the \( L^2 \) analog to [AF03 Theorem 5.33], the Approximation Theorem for \( \mathbb{R}^n \).

**Proposition 4.8.** Let \( \Omega \subset \mathbb{R}^n \) satisfy (HI)-(HVI). There exists a constant \( C = C(m,n) \) so that for \( 0 < k \leq m \), \( v \in W^{k,2}(M,\varphi;T) \), and \( 0 < \delta \leq 1 \), there exists \( v_\delta \in C^m(M) \) so that:
\[
\|v - v_\delta\|_{L^2(M,\varphi)} \leq C\delta^k \sum_{|\alpha|=k} \|T^\alpha u\|_{L^2(M,\varphi)}
\]
and
\[
\|v_\delta\|_{W^{j,2}(M,\varphi;T)} \leq C \begin{cases} \|v\|_{W^{k,2}(M,\varphi;T)} & \text{if } j \leq k - 1 \\ \delta^{k-j} \|v\|_{W^{k,2}(M,\varphi;T)} & \text{if } k \leq j \leq m. \end{cases}
\]

Proposition 4.8 means that \( M \) has the approximation property.

**Proof.** In this proof, we work locally and use the boundary operators \( Y_j^\beta = Z_j - \frac{1}{2}Z_j(\varphi) \). It follows from Corollary 4.6 that
\[
\|ve^{-\frac{1}{2}\varphi}\|_{W^{k,2}(M)} = \sum_{|\alpha| \leq k} \|(Y_j^\beta)^\alpha v\|_{L^2(M,\varphi)} \sim \|v\|_{W^{k,2}(M,\varphi;T)}
\]
for \( 0 \leq k \leq m \). Then
\[
\|ve^{-\frac{1}{2}\varphi}\|_{W^{k,2}(M)} \sim \sum_{j=1}^\infty \|v_j e^{-\frac{1}{2}\varphi}\|_{W^{k,2}(M)}
\]
where \( v_j = \chi_{U_j} v \) and \( \{\chi_{U_j}\} \) is a partition of unity subordinate to \( \{U_j\} \). By the classical theory, there exists \( \psi_{\delta,j} \in C^m(M \cap U_j) \) so that
\[
\|v_j e^{-\frac{1}{2}\varphi} - \psi_{\delta,j} e^{-\frac{1}{2}\varphi}\|_{L^2(M,\varphi)} \leq C\delta^k \sum_{|\alpha|=k} \|Z^\alpha(v_j e^{-\frac{1}{2}\varphi})\|_{L^2(M,\varphi)}
\]
and
\[
\|\psi_{\delta,j} e^{-\frac{1}{2}\varphi}\|_{W^{\ell,2}(M)} \leq C \begin{cases} \|v_j e^{-\frac{1}{2}\varphi}\|_{W^{k,2}(M \cap U_j)} & \text{if } \ell \leq k - 1 \\ \delta^{k-j} \|\psi\|_{W^{k,2}(M \cap U_j,\varphi;X)} & \text{if } k \leq \ell \leq m. \end{cases}
\]
Since the \( M \cap U_j \) are of comparable surface area, the constant \( C \) arising from the classical Approximation Theorem can be taken independent of \( j \). Thus, the result follows by summing in \( j \) and observing that the decomposition \( v = \sum_{j=1}^\infty v_j \) is locally finite. \( \square \)
5. Weighted Besov spaces on $\Omega$ and $M$

We start with the following proposition. We initially prove a boundary version because we need to strengthen Proposition 3.2 before we can prove an analog for $\Omega$. This is an $L^2$ adaptation of the Approximation Theorem, [AF03, Theorem 7.31].

**Proposition 5.1.** Let $\Omega \subset \mathbb{R}^n$ satisfy (HI)-(HVI). If $0 < k < m$, then
\[ W^{k,2}(M, \varphi; T) \in \mathcal{H}(k/m; L^2(M, \varphi), W^{m,2}(M, \varphi; T)) \]

**Proof.** The argument is the same as [AF03, Theorem 7.31] with Proposition 4.7 filling in for Proposition 5.1.

The importance of Proposition 4.8 is that the Reiteration Theorem (see Theorem A.10) holds for interpolation spaces generated from the weighted $L^2$-Sobolev spaces.

### 5.1. Real interpolation of boundary Sobolev spaces

We are now ready to define our weighted Besov spaces.

**Definition 5.2.** Let $0 < s < \infty$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $m$ be the smallest integer larger than $s$. We define the Besov space $B^{s,p,q}(M, \varphi; T)$ to be the intermediate spaces between $L^p(M, \varphi)$ and $W^{m,p}(M, \varphi; T)$ corresponding to $\theta = s/m$, i.e.,
\[ B^{s,p,q}(M, \varphi; T) = \left( L^p(M), W^{m,p}(M, \varphi; T) \right)_{s/m,q,J}. \]

We define the Besov space $B^{s,p,q}(M, \varphi; Z)$ to be the intermediate spaces between $L^p(M, \varphi)$ and $W^{m,p}(M, \varphi; Z)$ corresponding to $\theta = s/m$, i.e.,
\[ B^{s,p,q}(M, \varphi; Z) = \left( L^p(M, \varphi), W^{m,p}(M, \varphi; Z) \right)_{s/m,q,J}. \]

We will focus on the case $p = 2$ since we only proved an $L^2$ Approximation Theorem. By Theorem [A.4] $B^{s,2,q}(M, \varphi; T)$ is a Banach space with interpolation norm
\[ \|u\|_{B^{s,2,q}(M, \varphi; T)} = \|u; (L^2(M, \varphi), W^{m,2}(M, \varphi; T))_{s/m,q,J}\|. \]

Also, $B^{s,2,q}(M, \varphi; T)$ inherits density and approximation properties from $W^{m,2}(M, \varphi; T)$. For example, $\{\psi \in C^\infty(M) : \|\psi\|_{W^{m,2}(M, \varphi; T)} < \infty\}$ is dense in $B^{s,2,q}(M, \varphi; T)$.

Let $\Omega$ satisfy (HI)-(HVI). Proposition 5.1 and the Reiteration Theorem imply that if $0 \leq k < s < m$ and $s = (1 - \theta)k + \theta m$, then
\[ B^{s,2,q}(M, \varphi; T) = \left( W^{k,2}(M, \varphi; T), W^{m,2}(M, \varphi; T) \right)_{\theta,q,J}. \]

More generally, if $0 \leq k < s < m$ and $s = (1 - \theta)s_1 + \theta s_2$ and $1 \leq q_1, q_2 \leq \infty$, then
\[ B^{s,2,q}(M, \varphi; T) = \left( B^{s_1,2,q_1}(M, \varphi; T), B^{s_2,2,q_2}(M, \varphi; T) \right)_{\theta,q,J}. \]

The following corollary is an immediate consequence of Proposition 5.1 and Lemma A.8.

**Corollary 5.3.**
\[ B^{m,2,1}(M, \varphi; T) \hookrightarrow W^{m,2}(M, \varphi; T) \hookrightarrow B^{m,2,\infty}(M, \varphi; T). \]
5.2. Proof of Lemma 2.2

Proof of Lemma 2.2. We follow the outline of [AF03, Lemma 7.40]. We may apply the Reiteration Theorem to obtain

\[ B := B^{k-\frac{1}{2},2}(M,\varphi; T) = (W^{k-1,2}(M,\varphi; T), W^{k,2}(M,\varphi; T))_{\theta,2,J} \]

where \( \theta = 1 - \frac{1}{2} = \frac{1}{2} \). From Theorem A.3 we can apply the discrete version of the J-method and obtain that \( u \in B \) if and only if there exist \( u_i \in W^{k-1,2}(M,\varphi; T) \cap W^{k,2}(M,\varphi; T) \) for \( i \in \mathbb{Z} \) so that

\[ \sum_{i \in \mathbb{Z}} u_i = u \]

in \( W^{k-1,2}(M,\varphi; T) + W^{k,2}(M,\varphi; T) = W^{k-1,2}(M,\varphi; T) \) and such that

\[ \{ 2^{-i/2} \| u_i \|_{W^{k-1,2}(M,\varphi; T)} \}, \{ 2^{i/2} \| u_i \|_{W^{k,2}(M,\varphi; T)} \} \in C^2. \]

Let \( \tilde{\pi} : \Omega' \to M \) be the map that sends \( x \in \Omega' \) to the unique point \( \tilde{\pi}(x) \in M \) obtained by flowing along \( Z_n \). That is, there exists \( t = t_x \) such that \( x = e^{tZ_n}(\tilde{\pi}(x)) \). The constant \( \epsilon > 0 \) is small enough so that each point \( x \in \Omega' \) can be uniquely represented by \( x = (\tilde{\pi}(x), t_x) \). In this way, if \( U \in C^\infty_c(\Omega') \) and \( x \in \Omega_e \), then

\[ U(x) = \int_{-\infty}^{t_x} \frac{d}{dt} U(e^{tZ_n}(\tilde{\pi}(x))) dt = \int_{-\infty}^{t_x} Z_n U(e^{tZ_n}(\tilde{\pi}(x))) dt. \]

Let \( \tilde{\psi} \in C^\infty_c(\mathbb{R}) \) be so that

(i) \( \tilde{\psi}(t) = 1 \) on \([-1, 1]\),

(ii) \( \tilde{\psi}(t) = 0 \) if \( |t| \geq 2 \),

(iii) \( 0 \leq \tilde{\psi}(t) \leq 1 \) for all \( t \in \mathbb{R} \),

(iv) and there exists \( c_j \geq 0 \) so that \( |\tilde{\psi}^{(j)}(t)| \leq c_j \) for all \( j \geq 1 \) and \( t \in \mathbb{R} \).

Define \( \psi_i(t) = \tilde{\psi}(t/2^i) \) and \( \psi_i = \psi_{i+1} - \psi_i \). Then \( \psi_i \) vanishes outside \((2^i, 2^{i+2}) \cup (-2^{i+2}, -2^i)\) (and at the endpoints in particular). Also, \( \| \psi_i \|_{L^\infty(\mathbb{R})} = 1 \) and \( \| \psi_i' \|_{L^\infty(\mathbb{R})} \leq 2^{-i}c_1 \).

Let \( U_i(x) \in C^\infty_c(\Omega_e') \). Define \( U_i(x) \) by

\[ U_i(x) = U_i(\tilde{\pi}(x), t_x) = e^{\frac{\phi(x)}{2}} \int_{-\infty}^{t_x} \psi_i(t) Z_n(Ue^{-\frac{\theta}{2}}) \big|_{e^{tZ_n}(\tilde{\pi}(x))} dt \]

\[ = e^{\frac{\phi(x)}{2}} \int_{-\infty}^{t_x} \psi_i(t) \left. \frac{d}{dt} (Ue^{-\frac{\theta}{2}}) \right|_{e^{tZ_n}(\tilde{\pi}(x))} dt. \]

Next, for \( x \in M \), define \( u_i \) by \( u_i(x) = U_i(x) \). Then

\[ u_i(x) = e^{\frac{\phi(x)}{2}} \int_{-\infty}^{t_x} \psi_i(t) \left. \frac{d}{dt} (Ue^{-\frac{\theta}{2}}) \right|_{e^{tZ_n}(\tilde{\pi}(x))} dt. \]

By the support condition on \( \psi_i \) and the Fundamental Theorem of Calculus,

\[ u_i(x)e^{-\frac{\phi(x)}{2}} = \int_{-2^{i+2}}^{-2^i} \psi_i(t) \left. \frac{d}{dt} (Ue^{-\frac{\theta}{2}}) \right|_{e^{tZ_n}(\tilde{\pi}(x))} dt = -\int_{-2^{i+2}}^{-2^i} \psi_i(t) (Ue^{-\frac{\theta}{2}}) \big|_{e^{tZ_n}(\tilde{\pi}(x))} dt. \]
Since $U$ has compact support, $U_i$ and consequently $u_i$ vanish for all $i \in \mathbb{Z}$ when $|x|$ is sufficiently large. Therefore, the support of $\text{Tr}\ U$ is a compact set on which $\sum_{i \in \mathbb{Z}} u_i$ converges uniformly to $u = \text{Tr}\ U$. Also, if $|\alpha| \leq k - 1$, then

$$Z_\alpha^\alpha(u_i(x)e^{-\frac{\varphi(x)}{2}}) = \int_{-2i}^{2i} \psi_i(t) Z_\alpha \Big|_{x=\tau} \Big| Z_{\tau}(Ue^{-\varphi}) \Big|_{e^{t\tau}} dt$$

Recall that

$$Z_j \bigg|_{x} = \sum_{i=1}^{n} \tau_j'(x) \frac{\partial}{\partial x_j}.$$ 

On Cauchy-Schwarz, $\Omega'_\epsilon$, $\rho$ is bounded in the $C^{k+1}$ norm, so $\tau_j'$ is bounded in the $C^k$ norm. Consequently, by Cauchy-Schwarz,

$$|Z_\alpha u_i(x)e^{-\frac{\varphi(x)}{2}}| \leq (2^{i+2})^{1/2} C\|\tau_j\|_C^{k+1} \Bigg( \int_{-2i}^{2i} \bigg| \nabla|\alpha|+1 \bigg| Ue^{-\varphi} \bigg|_{e^{t\tau}} \bigg|^{2} dt \Bigg)^{1/2}.$$ 

For a fixed $t > 0$, $t \in [2^i, 2^{i+2})$ for exactly two (adjacent) $i$. Set $Y_j f = e^{\frac{\varphi}{2}} Z_j (e^{-\frac{\varphi}{2}}) = \frac{1}{2}(T_j + Z_j)$. By Corollary 4.6, $\|f\|_{W^{k,2}(\Omega,\varphi;T)} \sim \sum_{|\alpha| \leq j} \|Y_{\tau} \|_{M,\varphi}$. Moreover, the paths $e^{t\tau}$ foliate $\Omega_\epsilon$, so multiplying by $2^{-1/2}$, squaring, summing over $i$, and integrating yields

$$\sum_{i \in \mathbb{Z}} 2^{-i} \|u_i\|^2_{W^{k,2}(\Omega,\varphi;T)} \leq C \sum_{i \in \mathbb{Z}} 2^{-i} \|Y_{\tau} u_i\|^2_{L^2(M,\varphi)} = C \sum_{i \in \mathbb{Z}} 2^{-i} \|Z_{\tau} u_i e^{-\frac{\varphi(x)}{2}}\|^2_{L^2(M)}$$

$$= C \sum_{|\alpha| \leq k-1} \int_{\Omega_\epsilon} \bigg| \nabla|\alpha|+1 (U(x)e^{-\frac{\varphi(x)}{2}}) \bigg|^2 dx \leq C \|U\|_{W^{k,2}(\Omega_\epsilon,\varphi;X)}$$

where the last inequality follows from (14).

Using the second equality in (23) and Cauchy-Schwarz, we have

$$|Z_\alpha u_i(x) e^{-\frac{\varphi(x)}{2}}| \leq 2^{-i} 2^{(i+2)/2} C \left( \int_{-2i}^{2i} \bigg| Z_{\tau} \bigg| \bigg| U(e^{t\tau}) \bigg|_{e^{t\tau}} \bigg|^{2} dt \right)^{1/2}$$

$$\leq 2^{-i} 2^{(i+2)/2} C \|\tau_j\|_{C^{k+1}(\Omega_\epsilon)} \left( \int_{-2i}^{2i} \bigg| \nabla|\alpha| \bigg| U \bigg|_{e^{t\tau}} \bigg|^{2} dt \right)^{1/2}.$$ 

Therefore,

$$\sum_{i \in \mathbb{Z}} 2^{i} \|u_i\|^2_{W^{k,2}(\Omega,\varphi;T)} \leq C \sum_{i \in \mathbb{Z}} 2^{i} \|Y_{\tau} u_i\|^2_{L^2(M,\varphi)} = C \sum_{i \in \mathbb{Z}} 2^{i} \|Z_{\tau} u_i e^{-\frac{\varphi(x)}{2}}\|^2_{L^2(M,\varphi)}$$

$$= C \|\tau_j\|_{C^{k+1}(\Omega_\epsilon)} \sum_{|\alpha| \leq k} \int_{\Omega_\epsilon} \bigg| \nabla|\alpha| (U(x)e^{-\frac{\varphi(x)}{2}}) \bigg|^2 dx \leq C \|U\|_{W^{k,2}(\Omega_\epsilon,\varphi;X)}.$$ 

where the final inequality follows from (14).

Together, these inequalities show that $\|u\|_{W^{k-1/2,2}(\Omega,\varphi;T)} \leq C \|U\|_{W^{k,2}(\Omega_\epsilon,\varphi;X)}$ when $U \in C^\infty_c(\Omega'_\epsilon)$. Since $C^\infty_c(\Omega'_\epsilon)$ is dense in $W^{k,2}_0(\Omega'_\epsilon, \varphi; X)$, the proof is complete. □
5.3. Proof of Theorem 2.4

Proof of Theorem 2.4. Let \(\ell' = \ell + \ell' + 1\). Set \(B = B^{\ell+\frac{\ell}{2}+\ell'}(M, \varphi; T)\). By definition,
\[
B = \left( L^2(M, \varphi), W^{\ell', 2}(M, \varphi; T) \right)_\theta,
\]
where \(\theta = \frac{\ell + \frac{\ell}{2}}{\ell'}\).

From the discrete J-method (Theorem A.3), \(u \in L^2(M, \varphi)\) belongs to \(B\) if and only if there exist \(\{u_j\}_{j \in \mathbb{Z}} \subset W^{\ell', 2}(M, \varphi; T)\) so that \(u = \sum_{j \in \mathbb{Z}} u_j\) where the sum converges in \(L^2(M, \varphi)\) and \(\{2^{-j\theta}J(2^j; u_j)\} \in \ell^2\). The latter condition means that there exists \(K > 0\) so that
\[
\sum_{j \in \mathbb{Z}} 2^{-\frac{2j+1}{\ell' + 1}} \|u_j\|_{L^2(M, \varphi)}^2 \leq K^2\|u\|_B^2 \quad \text{and} \quad \sum_{j \in \mathbb{Z}} 2^{-\frac{2(\ell' + 1)}{\ell'}} \|u_j\|_{W^{\ell', 2}(M, \varphi; T)}^2 \leq K^2\|u\|_B^2.
\]

Let \(\tilde{\psi} \in C^\infty_c(\mathbb{R})\) be the bump function from the proof of Lemma 2.2. Set
\[
\psi_j(t) = \tilde{\psi}(t/\delta^j)
\]
for \(j \in \mathbb{Z}\) and \(\delta > 0\) to be decided later. Set \(\eta(t) = \tilde{\psi}(2t/\epsilon)\). It follows that \(|\psi_j^{(k)}(t)| \leq c_k \delta^{-j^k}\).

Also, for \(k \geq 1\),
\[
\text{supp} \psi_j^{(k)} \subset [-2\delta^j, -\delta^j] \cup [\delta^j, 2\delta^j].
\]

For \(y \in \Omega'\), there exists a unique \(x \in M\) and \(t \in [-\epsilon, \epsilon]\) so that \(y = e^t Z_n(x)\). Set \(\tilde{\pi}(y) = x\). Since \(\|\rho\|_{C^m(\Omega')} < \infty\), it follows that the projection \(\|\tilde{\pi}\|_{C^{m-1}(\Omega')} < \infty\). Set \(U_j(y) = \frac{1}{\ell!} e^{\frac{1}{2} \varphi(y)} \eta(\rho(y)) \psi_j(\rho(y)) \left( \rho(y) \right)^\epsilon u_j(\tilde{\pi}(y)) e^{-\frac{1}{2} \varphi(\tilde{\pi}(y))}\).

Since \(\frac{\partial}{\partial \nu} = Z_n\), it is immediate that
\[
\text{Tr} U_j = \text{Tr} \frac{\partial U_j}{\partial \nu} = \cdots = \text{Tr} \frac{\partial^{\ell' - 1} U_j}{\partial \nu^{\ell' - 1}} = 0
\]
and
\[
\text{Tr} \frac{\partial^{\ell'} U_j}{\partial \nu^{\ell'}} = u_j
\]
for all \(j \in \mathbb{Z}\).

Thus, we only need to show that \(U \in W^{\ell''}(\Omega', \varphi; X)\) and is supported in \(\Omega'\). Since \(\text{supp} U_j \subset \Omega'\) for all \(j\), it follows that \(U\) is supported in \(\Omega'\). Note that if \(f\) is a smooth function on \(M\), then there exist functions \(c_{\alpha_1, \alpha_2}, 1 \leq \alpha_1, \alpha_2 \leq n - 1\), that are bounded in \(C^{m-2}(\Omega')\) so that
\[
Z_\alpha(f \circ \tilde{\pi})(y) = \sum_{\alpha_2} c_{\alpha_1, \alpha_2} Z_{\alpha_2} f(\tilde{\pi}(y)).
\]

Also, by construction, \(Z_n(f \circ \tilde{\pi})(y) = 0\). Since \(U_j\) has support in \(\Omega'\), (14) shows that we may use the \(Y_k\) operators (instead of the \(X_k\)'s) for differentiation. Let \(\gamma = (\gamma_1, \ldots, \gamma_{\ell''})\) be a multiindex of length \(\ell''\). Set
\[
\gamma_T = \{\alpha \in \gamma : \gamma_\alpha \text{ is tangential}\} \quad \text{and} \quad \gamma_N = \{\alpha \in \gamma : \gamma_\alpha = n\}.
\]

Set
\[
f_j(t) = \eta(t) \psi_j(t) t^\ell\quad \text{and} \quad g_j(x) = u_j(x) e^{-\frac{1}{2} \varphi(x)}
\]
for \(x \in M\). Observe that
\[
|f_j^{(\alpha_1)}(t)| \leq c_{\alpha_1} \delta^{(\ell' - \alpha_1)}.
\]
The function $\eta$ does not affect the estimates – derivatives of $\eta$ are supported where $|s| \in [\epsilon, 2\epsilon]$ and the support of derivatives of $\eta$ and derivatives of $\psi_j$ cause $\eta \sim \delta^j$ (or else the particular combination $\psi_j'\eta'$ is identically zero). By (14) and the fact that $U$ is supported in $\Omega'_\epsilon$, it is enough to bound $\sum_{j \in \mathbb{Z}} \|Y^n U_j(y)\|_{L^2(\Omega'_\epsilon, \varphi)}$ to show that $U \in W^{\ell, 2}(\Omega'_\epsilon, \varphi; X)$.

Since $\rho$ is bounded in the $C^m$ norm and $c_{\alpha_1, \alpha_2}$ are bounded in the $C^{m-2}$ norm, there exists functions $\sigma_{\alpha, \beta}$ that are bounded on $\Omega'_\epsilon$ so that

$$Y^n (U_j(y)) e^{-\frac{1}{2} \varphi(y)} = Z^n (U_j(y)) e^{-\frac{1}{2} \varphi(y)} = \sum_{\alpha = 0}^{\gamma_N} \sum_{|\beta| \leq \gamma_T} \sigma_{\alpha, \beta}^\gamma f_j^{(\alpha)}(\rho(y)) Z^\beta g_j(\tilde{\eta}(y)).$$

Thus,

$$\|Y^n U_j(y)\|_{L^2(\Omega'_\epsilon, \varphi)} \leq C \sum_{\alpha = 1}^{\gamma_N} \sum_{|\beta| \leq \gamma_T} \int_{\Omega'_\epsilon} \left| f_j^{(\alpha)}(\rho(y)) \right|^2 \left| Y^\beta u_j(\tilde{\eta}(y)) \right|^2 e^{-\varphi(\tilde{\eta}(y))} \, dy$$

$$= C \sum_{\alpha = 1}^{\gamma_N} \sum_{|\beta| \leq \gamma_T} \int_{\Omega'_\epsilon} \int_{-\delta^j}^{\delta^j} \left| f_j^{(\alpha)}(t) \right|^2 \left| Y^\beta u_j(x) \right|^2 e^{-\varphi(x)} \, dt \, d\sigma(x)$$

$$\leq C \sum_{\alpha = 1}^{\gamma_N} \sum_{|\beta| \leq \gamma_T} \int_{\Omega'_\epsilon} \delta^{2(\ell'-2\alpha+1)} \left| u_j \right|_{W^{\gamma_T, 2}(b\Omega, \varphi; T)}$$

$$\leq C \left( \delta^{j(2\ell'+1)} + \delta^{j(2\ell'-2\gamma_N+1)} \right) \left| u_j \right|_{W^{\gamma_T, 2}(b\Omega, \varphi; T)}.$$

where $C$ is independent of $j$. Set $\delta = 2^{\frac{j'}{m}}$. This means

$$\|Y^n U(y)\|_{L^2(\Omega'_\epsilon, \varphi)} \leq \sum_{j \in \mathbb{Z}} \|Y^n U(y)\|_{L^2(\Omega'_\epsilon, \varphi)}$$

$$\leq C \sum_{j \in \mathbb{Z}} \left( \delta^{j(2\ell'+1)} + \delta^{j(2\ell'-2\gamma_N+1)} \right) \left| u_j \right|_{W^{\gamma_T, 2}(b\Omega, \varphi; T)}.$$

To check that the sum on the right hand side of (24) is finite, observe that

$$\sum_{j \in \mathbb{Z}} \delta^{j(2\ell'+1)} \left| u_j \right|^2_{W^{\gamma_T, 2}(b\Omega, \varphi; T)} = \sum_{j \in \mathbb{Z}} 2^{j(2\ell'+1)} \left| u_j \right|^2_{W^{\gamma_T, 2}(b\Omega, \varphi; T)}$$

$$\leq \sum_{j \in \mathbb{Z}} 2^{j(2\ell'+1)} \left| u_j \right|^2_{W^{\ell', 2}(b\Omega, \varphi; T)} \leq K \left| u \right|^2_{B}.$$

To bound the remaining term in (24), we use (20) to bound

$$\delta^{j(2\ell'-2\gamma_N+1)} \left| u_j \right|^2_{W^{\gamma_T, 2}(b\Omega, \varphi; T)} \leq K' \delta^{j(2\ell'-2\gamma_N+1)} \left( \epsilon^2 \left| u_j \right|^2_{W^{\ell', 2}(b\Omega, \varphi; T)} + \epsilon^{-2\frac{m}{m'-\gamma_T}} \left| u_j \right|^2_{L^2(b\Omega, \varphi)} \right).$$
We require $\delta j(2\ell' - 2\gamma N + 1)\epsilon^2 = \delta j(2\ell' + 1)$. This means $\epsilon^2 = \delta 2\gamma N j$. Since $\gamma T + \gamma N = \ell''$, it follows that $\frac{\gamma T}{\ell'' - \gamma T} = \frac{\ell'' - \gamma N}{\gamma N}$ and

$$\delta j(2\ell' - 2\gamma N + 1)\delta - 2\gamma N j\frac{\ell'' - \gamma N}{\gamma N} = \delta - 2\ell'$$

since $\ell + \ell' + 1 = \ell''$. Thus, since $\delta = 2\mathbb{I}$,

$$\delta j(2\ell' - 2\gamma N + 1)\|u_j\|_{W^{r+2}(\Omega; T)}^2 \leq K' \sum_{j \in \mathbb{Z}} \left(2^{2j}\|u_j\|_{W^{r+2}(\Omega; T)}^2 + 2^{-2j}\|u_j\|_{L^2(\Omega; \psi)}^2\right).$$

Thus, $U \in W^r(\Omega, \varphi; X)$ and the proof is complete. \hfill $\Box$

The proof of Theorem 2.1 is now complete.

**Remark 5.4.** If (for example) $\ell' = 0$, then the formula for $U_j$ is

$$U_j(y) = e^{\frac{1}{2} \varphi(y)} \eta(\rho(y)) \psi_j(\rho(y)) u_j(\pi(y)) e^{-\frac{1}{2} \varphi(\pi(y))}$$

Since $Z_n(\pi(y)) = 0$, we can compute

$$(Y_n U_j(y)) e^{-\frac{1}{2} \varphi(y)} = Z_n(U_j(y)) e^{-\frac{1}{2} \varphi(y)} = Z_n \rho(y) (\eta \psi_j' (\rho(y)) u_j(\pi(y)) e^{-\frac{1}{2} \varphi(\pi(y))})$$

It follows from the support conditions on $\eta$ and $\psi$ that $\text{supp}(\eta \psi_j') \cap \left(-\delta j, \delta j\right) \cap (-\epsilon, \epsilon) = \emptyset$. Therefore

$$\text{Tr}(Y_n U_j) = 0$$

for all $j$. Similarly, $\text{Tr}(Y_k^* U_j) = 0$ for all $k$ for which $Y_n^* k$ is defined. A similar result also holds if $\ell' > 0$ since $\text{Tr} \rho = 0$.

Now that Theorem 2.4 is proven, we can apply our trace and extension theorems to prove that a simple $(k, 2)$-extension operator exists.

**Proof of Theorem 2.6.** Let $1 \leq k \leq m - 1$ and $f \in W^{k,2}(\Omega, \varphi; X)$. We begin by constructing functions $u_1, \ldots, u_k$ recursively. By Theorem 2.1, $\text{Tr} f \in B^{k-\frac{3}{2}, 2}(M, \varphi; T)$ and

$$\| \text{Tr} f \|_{B^{k-\frac{3}{2}, 2}(M, \varphi; T)} \leq K \| f \|_{W^{k,2}(\Omega, \varphi; X)}.$$  

Next, by Theorem 2.4 there exists a function $u_1 \in W^{k,2}(\Omega', \varphi; X)$ supported in $\Omega'$ and so that $\text{Tr} u_1 = \text{Tr} f$ and $\| u_1 \|_{W^{k,2}(\Omega', \varphi; X)} \leq K \| f \|_{B^{k-\frac{3}{2}, 2}(M, \varphi; T)}$. Thus,

$$\| u_1 \|_{W^{k,2}(\Omega, \varphi; X)} \leq K \| f \|_{W^{k,2}(\Omega, \varphi; X)}.$$  

It now follows that $(f - u_1) \in W^{k,2}(\Omega, \varphi; X) \cap W^{1,2}_0(\Omega, \varphi; X)$.

Next, $T_n (f - u_1) \in W^{k-1,2}(\Omega, \varphi; X)$ so by Theorem 2.1, $\text{Tr}(T_n (f - u_1)) \in B^{k-3/2, 2}(M, \varphi; T)$ and

$$\| \text{Tr}(T_n (f - u_1)) \|_{B^{k-\frac{3}{2}, 2}(M, \varphi; T)} \leq K \| T_n (f - u_1) \|_{W^{k-1,2}(\Omega, \varphi; X)}.$$  

By Theorem 2.3 there exists a function $u_2 \in W^{k,2}(\Omega', \varphi; X)$ supported in $\Omega'$ and so that $\text{Tr} u_2 = 0$ and $\text{Tr} \left(\frac{\partial u_2}{\partial x_1}\right) = \text{Tr}(T_n u_2) = \text{Tr}(T_n (f - u_1))$ and

$$\| u_2 \|_{W^{k,2}(\Omega', \varphi; X)} \leq K \| \text{Tr}(T_n (f - u_1)) \|_{B^{k-\frac{3}{2}, 2}(M, \varphi; T)}.$$  

Thus, $(f - u_1 - u_2) \in W^{k,2}(\Omega, \varphi; X) \cap W^2_0(\Omega, \varphi; X)$ and

$$\| f - u_1 - u_2 \|_{W^{k,2}(\Omega, \varphi; X)} \leq K \| f \|_{W^{k,2}(\Omega, \varphi; X)}.$$  

$\Box$
Iterating this process, we can show that there exist functions
\[ u_1, \ldots, u_k \in W^{k,2}(\Omega', \varphi; X) \]
supported in \( \Omega' \) and so that \( (f - u_1 - \cdots - u_j) \in W^{k,2}(\Omega, \varphi; X) \cap W^j_0(\Omega, \varphi; X) \) and
\[
\|f - u_1 - \cdots - u_j\|_{W^{k,2}(\Omega, \varphi; X)} \leq K\|f\|_{W^{k,2}(\Omega, \varphi; X)}.
\]
Thus, \( f - (u_1 + \cdots + u_k) \in W^{k,2}_0(\Omega, \varphi; X) \).

Next, we can write
\[
f = (u_1 + \cdots + u_k) + (f - u_1 - \cdots - u_k).
\]
The function \( (f - u_1 - \cdots - u_k) \) can be extended by 0 to produce a function in \( W^{k,2}(\mathbb{R}^n, \varphi; X) \) and \( (u_1 + \cdots + u_k) \in W^{k,2}(\mathbb{R}^n, \varphi; X) \). Thus, we define
\[
Ef(x) = \begin{cases} f(x) & x \in \Omega \\ u_1(x) + \cdots + u_k(x) & x \in \Omega'. \end{cases}
\]

\[ \blacksquare \]

5.4. **Approximation of functions in** \( W^{k,2}(\Omega, \varphi; X) \). Using the Trace Theorem \([23]\), we are now in a position to improve Proposition \([5,2]\) for \( p = 2 \) and relax the condition that \( f \in W^{k,2}_0(\Omega, \varphi; X) \).

**Proposition 5.5.** Assume that \( b\Omega \) satisfies (HI)-(HVI) for some \( m \geq 2 \). Let \( 1 \leq \ell \leq m - 1 \) be an integer. Then \( C^\infty_c(\mathbb{R}^n) \) is dense \( W^{\ell,2}(\Omega, \varphi; X) \) in the sense that if \( \epsilon > 0 \), then there exist \( \psi \in C^\infty_c(\mathbb{R}^n) \) so that
\[
\|\psi - f\|_{W^{\ell,2}(\Omega, \varphi; X)} \leq \epsilon
\]
and
\[
\|\psi\|_{W^{\ell,2}(\mathbb{R}^n, \varphi; X)} \leq C\|f\|_{W^{\ell,2}(\Omega, \varphi; X)}
\]
where \( C \) is independent of \( f, \varphi, \) and \( \epsilon \).

The function \( \psi \) that we construct will actually satisfy \( \text{supp } \psi \subset \Omega' \cup \Omega \).

**Proof.** The proof is a consequence of Theorem \([23]\). Let \( \epsilon > 0 \). Constructing the functions \( u_1, \ldots, u_\ell \) as in the proof of Theorem \([23]\) then for any \( 1 \leq j \leq \ell \), \( (f - u_1 - \cdots - u_j) \in W^{\ell,2}(\Omega, \varphi; X) \cap W^j_0(\Omega, \varphi; X) \) and
\[
\|f - u_1 - \cdots - u_j\|_{W^{\ell,2}(\Omega, \varphi; X)} \leq K\|f\|_{W^{\ell,2}(\Omega, \varphi; X)}.
\]
Thus, \( f - (u_1 + \cdots + u_\ell) \in W^{\ell,2}_0(\Omega, \varphi; X) \) so there exists \( \xi \in C^\infty_c(\Omega) \) so that
\[
\|f - (u_1 + \cdots + u_\ell + \xi)\|_{W^{\ell,2}(\Omega, \varphi; X)} < \min\{\epsilon, \epsilon\|f\|_{W^{\ell,2}(\Omega, \varphi; X)}\}.
\]
We can now take \( \psi = u_1 + \cdots + u_\ell + \xi \).

\[ \blacksquare \]

**Proposition 5.6.** Let \( \Omega \subset \mathbb{R}^n \) have a \( C^m \) boundary and satisfy (HI)-(HVI). There exist \( K, K' > 0 \) depending on \( n, m \) so that for each \( \epsilon > 0 \), \( u \in W^{m,2}(\Omega, \varphi; X) \), and \( 0 < j < m \),
\[
\sum_{|\alpha| = j} \|X^\alpha u\|_{L^2(\Omega, \varphi)}^2 \leq K\left( \epsilon \sum_{|\beta| = m} \|X^\beta u\|_{L^2(\Omega, \varphi)}^2 + \epsilon^{-j/(m-j)}\|u\|_{L^2(\Omega, \varphi)}^2 \right) \tag{25}
\]
\[
\|u\|_{W^{\ell,2}(\Omega, \varphi; X)} \leq K'\left( \epsilon\|u\|_{W^{m,2}(\Omega, \varphi; X)} + \epsilon^{-j/(m-j)}\|u\|_{L^2(\Omega, \varphi)} \right) \tag{26}
\]
\[
\|u\|_{W^{\ell,2}(\Omega, \varphi; X)} \leq 2K'\|u\|_{W^{m,2}(\Omega, \varphi; X)}^m \|u\|_{L^2(\Omega, \varphi)}^{(m-j)/m} \tag{27}
\]
\[ 24 \]
Definition 5.9. Let $\Omega \subset \mathbb{R}^n$ satisfy (HI)-(HVI) for some $m \in \mathbb{N}$. If $0 < k \leq m$ then there exists a constant $C = C(m,n)$ so that for $v \in W^{k,2}(\Omega, \varphi; X)$ and $0 < \epsilon \leq 1$, there exists $v_\epsilon \in C^\infty_c(\Omega)$ so that:

$$\|v - v_\epsilon\|_{L^2(\Omega, \varphi)} \leq C\epsilon^k \sum_{|\alpha| = k} \|X^\alpha v\|_{L^2(\Omega, \varphi)}$$

and

$$\|v_\epsilon\|_{W^{j,2}(\Omega, \varphi; X)} \leq C \left\{ \begin{array}{ll}
\|v\|_{W^{k,2}(\Omega, \varphi; X)} & \text{if } j \leq k - 1 \\
\epsilon^{k-j}\|v\|_{W^{k,2}(\Omega, \varphi; X)} & \text{if } k \leq j \leq m.
\end{array} \right.$$

Proof. The proof uses the same argument as the proof of Proposition 4.8. □

We can also prove an $L^2$ analog to [AF03, Theorem 5.33], the Approximation Theorem for $\mathbb{R}^n$.

Proposition 5.7. Let $\Omega \subset \mathbb{R}^n$ satisfy (HI)-(HVI) for some $m \in \mathbb{N}$. If $0 < k \leq m$ then there exists a constant $C = C(m,n)$ so that for $v \in W^{k,2}(\Omega, \varphi; X)$ and $0 < \epsilon \leq 1$, there exists $v_\epsilon \in C^\infty_c(\Omega)$ so that:

$$\|v - v_\epsilon\|_{L^2(\Omega, \varphi)} \leq C\epsilon^k \sum_{|\alpha| = k} \|X^\alpha v\|_{L^2(\Omega, \varphi)}$$

and

$$\|v_\epsilon\|_{W^{j,2}(\Omega, \varphi; X)} \leq C \left\{ \begin{array}{ll}
\|v\|_{W^{k,2}(\Omega, \varphi; X)} & \text{if } j \leq k - 1 \\
\epsilon^{k-j}\|v\|_{W^{k,2}(\Omega, \varphi; X)} & \text{if } k \leq j \leq m.
\end{array} \right.$$

Proof. The proof uses the same argument as the proof of Proposition 4.8. □

Proposition 5.7 means that $\Omega$ has the approximation property in the sense of [AF03]. As a consequence of our improved approximation results, following the proof of Proposition 5.1, we can prove

Proposition 5.8. If $\Omega \subset \mathbb{R}^n$ satisfies (HI)-(HVI), then

$$W^{k,2}(\Omega, \varphi; X) \cap L^p(\Omega) \subset \mathcal{S}(k/m; L^2(\Omega, \varphi), W^{m,2}(\Omega, \varphi; X)).$$

5.5. Besov spaces on $\Omega$ and Additional Trace Results. We are now ready to define weighted Besov spaces on $\Omega$.

Definition 5.9. Let $0 < s < \infty$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $\ell$ be the smallest integer larger than $s$. We define the Besov space $B^{s,p,q}(\Omega, \varphi; X)$ to be the intermediate space between $L^p(\Omega)$ and $W^{\ell,p}(\Omega, \varphi; X)$ corresponding to $\theta = s/\ell$, i.e.,

$$B^{s,p,q}(\Omega, \varphi; X) = \left( L^p(\Omega, \varphi), W^{\ell,p}(\Omega, \varphi; X) \right)_{s/\ell,q,J}.$$

We will focus on the case $p = 2$ since we only proved an $L^2$ Approximation Theorem. By Theorem 4.4, $B^{s,p,q}(\Omega, \varphi; X)$ is a Banach space with interpolation norm

$$\|u\|_{B^{s,p,q}(\Omega, \varphi; X)} = \|u; \left( L^p(\Omega, \varphi), W^{\ell,p}(\Omega, \varphi; X) \right)_{s/\ell,q,J}\|.$$

Also, $B^{s,p,q}(\Omega, \varphi; X)$ inherits density and approximation properties from $W^{\ell,p}(\Omega, \varphi; X)$. For example, $\{ \psi \in C^\infty(\Omega) : \|\psi\|_{W^{\ell,p}(\Omega, \varphi; X)} < \infty \}$ is dense in $B^{s,p,q}(\Omega, \varphi; X)$.

For $\Omega$ that satisfies (HI)-(HVI), Proposition 5.7 and the Reiteration Theorem (Theorem A.10), if $0 \leq k \leq s < \ell$ and $s = (1 - \theta)k + \theta \ell$, then

$$B^{s,2,q}(\Omega, \varphi; X) = \left( W^{k,2}(\Omega, \varphi; X), W^{\ell,2}(\Omega, \varphi; X) \right)_{\theta,q,J}.$$

More generally, if $0 \leq k \leq s < \ell$, $s = (1 - \theta)s_1 + \theta s_2$, and $1 \leq q_1, q_2 \leq \infty$, then

$$B^{s,2,q}(\Omega, \varphi; X) = \left( B^{s_1,2,q_1}(\Omega, \varphi; X), B^{s_2,2,q_2}(\Omega, \varphi; X) \right)_{\theta,q,J}.$$

We are now in a position to prove the following Trace Lemma.

Lemma 5.10. The trace operator $\text{Tr}$ embeds $B^{1/2,2,1}(\Omega' \cap \Omega, \varphi; X)$ into $L^2(M)$.
Proof. Let $U$ be an element in $B = B^{1/2,1}(\Omega' \cap \Omega, \varphi; X)$. Without loss of generality, we may assume that $\|U\|_B \leq 1$. By the discrete $J$-interpolation method, there exist functions $U_i, i \in \mathbb{Z}$, so that $U = \sum_{i \in \mathbb{Z}} U_i$ and

$$\sum_{i \in \mathbb{Z}} 2^{-i/2}\|U_i\|_{L^2(\Omega_0 \cap \Omega, \varphi)} \leq C$$

and

$$\sum_{i \in \mathbb{Z}} 2^{i/2}\|U_i\|_{W^{1,2}(\Omega_0 \cap \Omega, \varphi; X)} \leq C$$

for some constant $C$. As in the proof of Lemma 2.2, we may assume that the functions $U_i$ are smooth and at most finitely many are not identically zero. For any of these functions, we have, for $2^i \leq t \leq 2^{i+1}$ and $x \in M$,

$$|e^{-\frac{\varphi(x)}{2}}U_i(x)| \leq \int_0^t \left| \frac{d}{ds} e^{-\frac{\varphi(sZ_n(x))}{2}} U_i(e^{sZ_n(x)}) \right| ds + \left| e^{-\frac{\varphi(tZ_n(x))}{2}} U_i(e^{tZ_n(x)}) \right| ds \leq \int_0^{2^{i+1}} \left| Z_n(e^{-\frac{\varphi(sZ_n(x))}{2}} U_i(e^{sZ_n(x)})) \right| ds + \left| e^{-\frac{\varphi(tZ_n(x))}{2}} U_i(e^{tZ_n(x)}) \right| ds + \left| e^{-\frac{\varphi(tZ_n(x))}{2}} U_i(e^{tZ_n(x)}) \right| ds.$$

Averaging $t$ over $[2^i, 2^{i+1}]$, we now have the estimate

$$|e^{-\frac{\varphi(x)}{2}}U_i(x)| \leq \int_0^{2^{i+1}} \left| Z_n(e^{-\frac{\varphi(sZ_n(x))}{2}} U_i(e^{sZ_n(x)})) \right| ds + \frac{1}{2i} \int_{2^i}^{2^{i+1}} \left| e^{-\frac{\varphi(tZ_n(x))}{2}} U_i(e^{tZ_n(x)}) \right| dt.$$

By Cauchy-Schwarz,

$$|e^{-\frac{\varphi(x)}{2}}U_i(x)| \leq 2^{(i+1)/2} \left( \int_0^{2^{i+1}} \left| Y_n U_i(e^{tZ_n(x)}) \right|^2 e^{-\varphi(e^{tZ_n(x)})} dt \right)^{1/2} + 2^{-i/2} \left( \int_{2^i}^{2^{i+1}} \left| U_i(e^{tZ_n(x)}) \right|^2 e^{-\varphi(e^{tZ_n(x)})} dt \right)^{1/2} := a_i(x) + b_i(x).$$

Observe that $\|a_i\|_{L^2(M)} \leq C 2^{i/2}\|U_i\|_{W^{1,2}(\Omega_0 \cap \Omega, \varphi; X)}$ and $\|b_i\|_{L^2(M)} \leq 2^{i/2}\|U_i\|_{L^2(\Omega_0 \cap \Omega, \varphi)}$. Summing in $i$, we have

$$\|U\|_{L^2(M, \varphi)} \leq \sum_{i \in \mathbb{Z}} \|e^{-\frac{\varphi(x)}{2}}U_i\|_{L^2(M)} \leq C \left( \sum_{i \in \mathbb{Z}} 2^{i/2}\|U_i\|_{W^{1,2}(\Omega_0 \cap \Omega, \varphi; X)} + 2^{-i/2}\|U_i\|_{L^2(\Omega_0 \cap \Omega, \varphi)} \right) \leq C.$$

As a consequence of Theorem 2.1 and Lemma 5.10, we can now prove our Trace Theorem for $L^2$ Besov spaces, Theorem 2.7. Theorem 2.7 is an $L^2$-analog of [AF03, Theorem 7.43].

Proof of Theorem 2.7. The proof of Theorem 2.7 follows from (22), Theorem 2.1, Lemma 5.10 and the Exact Interpolation Theorem.

We conclude our discussion of Sobolev space results with an extension of Proposition 3.3 and Corollary 3.5, namely the proof of Proposition 2.5.

Proof of Proposition 2.5. We will first show that $\|v\|_{W^{1,2}(\Omega, \varphi; D)} \leq C \|v\|_{W^{1,2}(\Omega, \varphi; X)}$ for some $C$ independent of $v$. By Theorem 2.1, $\text{Tr} v \in B^{1/2,2}(M, \varphi; T)$, and there exists $v' \in W^{1,2}(\Omega', \varphi; X)$ with support in $\Omega'$ so that

$$\text{Tr} v' = \text{Tr} v.$$
Thus, the second term is also controlled by $\|v\|_{W^{1,2}(\Omega, \varphi; X)}$. Writing $v = (v - v') + v'$, we see that $v - v' \in W_0^{1,2}(\Omega, \varphi; X)$. Since $W_0^{1,2}(\Omega, \varphi; X) = W_0^{1,2}(\Omega, \varphi; D)$ by Corollary 3.3 and Corollary 3.8, we estimate
\[
\|v - v'\|_{W^{1,2}(\Omega, \varphi; D)} \leq C \|v - v'\|_{W^{1,2}(\Omega, \varphi; X)} \leq C \|v\|_{W^{1,2}(\Omega, \varphi; X)}.
\]
Thus, we need only to prove the result for $v'$. However, since $v'$ has compact support in $\Omega'_\ell$, we can use Corollary 3.3 to bound
\[
\|v'\|_{W^{1,2}(\Omega', \varphi; D)} \leq \|v'\|_{W^{1,2}(\Omega', \varphi; X)} \leq \|\nabla v'\|_{L^2(\Omega, \varphi)} \leq \|\nabla v'\|_{B^{1/2,2}(M, \varphi, T)} \leq C \|v\|_{W^{1,2}(\Omega, \varphi; X)},
\]
and the result is proved for $k = 1$.

To show that $W^{k,2}(\Omega, \varphi; X) = W^{k,2}(\Omega, \varphi; D)$ for $k \geq 2$, we induct. $k = 1$ is the base case. If we assume the norms are equivalent up to order $k - 1$, then let $|\alpha| = k$ and $|\beta| = k - 1$ so that $X^\alpha = X^\beta X_j$ for some $j$. Then
\[
\|D^\alpha v\|_{L^2(\Omega, \varphi)} = \|D^\beta D_j v\|_{L^2(\Omega, \varphi)} \leq C \sum_{|\gamma| = k - 1} \|X^\gamma D_j v\|_{L^2(\Omega, \varphi)} \leq C \sum_{|\gamma| = k - 1} \left( \|X^\gamma X_j v\|_{L^2(\Omega, \varphi)} + \|X^\gamma \frac{\partial \varphi}{\partial x_j}\|_{L^2(\Omega, \varphi)} \right).
\]
The first term is bounded by $\|v\|_{W^{k,2}(\Omega, \varphi; X)}$. The second term can be estimated as follows:
\[
\left\|X^\gamma \frac{\partial \varphi}{\partial x_j}\right\|_{L^2(\Omega, \varphi)} \leq \sum_{\gamma_1 + \gamma_2 = \gamma} C \left( \left\|D^{\gamma_1} \frac{\partial \varphi}{\partial x_j}\right\|_{L^2(\Omega, \varphi)} \right) X^{\gamma_2} v_{x_j}.
\]
By (HIV), the derivatives of $\varphi$ are controlled by $|\nabla \varphi|$ which in turn is controlled by $X\ell + X^\ell$. Thus, the second term is also controlled by $\|v\|_{W^{k,2}(\Omega, \varphi; X)}$.

The proof that $W_0^{1,2}(\Omega, \varphi; X)$ embeds compactly in $L^2(\Omega, \varphi; X)$ is contained in Proposition 3.3 and its corollaries. We will show that this implies that $W_0^{k,2}(\Omega, \varphi; X)$ embeds compactly in $W_0^{k-1,2}(\Omega, \varphi; X)$ by induction.

Assume that for some $2 \leq k \leq m - 1$, the result holds for $1 \leq j \leq k - 1$. Let $\{\psi_\ell\}$ be a bounded sequence of functions in $W_0^{k,2}(\Omega, \varphi; X)$. Then $\{\psi_\ell\}$ is a bounded sequence of functions in $W_0^{k-1,2}(\Omega, \varphi; X)$ and hence there exists a subsequence (renamed to be $\psi_\ell$) converging in $W_0^{k-2,2}(\Omega, \varphi; X)$. Moreover, $\{\nabla_X \psi_\ell\}$ is a sequence of bounded vectors whose components are in $W_0^{k-1,2}(\Omega, \varphi; X)$, so there exists a further subsequence, renamed $\psi_\ell$, so that $\{\nabla_X \psi_\ell\}$ is Cauchy in $W_0^{k-2,2}(\Omega, \varphi; X)$. It now follows that $\{\psi_\ell\}$ is Cauchy in $W_0^{k-1,2}(\Omega, \varphi; X)$.

Next, let $\{f_\ell\}$ be a bounded sequence of functions in $W^{k,2}(\Omega, \varphi; X)$. Using the simple $(k, 2)$-extension operator $E$ from Theorem 2.6, we extend $f_\ell$ to $Ef_\ell \in W^{k,2}(\mathbb{R}^n, \varphi; X)$. By the previous paragraph with $\mathbb{R}^n$ playing the role of $\Omega$, there exists a subsequence $Ef_\ell$ that converges in $W_0^{k-1,2}(\mathbb{R}^n, \varphi; X)$. Since $Ef_\ell|_\Omega = f_\ell$, it follows that $f_\ell$ converges in $W_0^{k-1,2}(\Omega, \varphi; X)$.

6. Interior estimates – the proof of Theorem 2.13

6.1. The $\ell = 0$ case.
Proof of Theorem 2.13 for $\ell = 0$. We follow the outline of [Eva10, §6.3, Theorem 1]. Choose $W \subset \Omega$ so that $V \subset W$, $\text{dist}(V, bW) > 0$, and $\text{dist}(W, b\Omega) > 0$. We first assume that $V$ and $W$ are bounded. Let $\zeta \in C^\infty(\Omega)$ be a smooth cutoff so that $\zeta|_V = 1$ and $\text{supp} \zeta \subset W$. Since $L$ is elliptic, by the classical theory $u \in W_{\text{loc}}^{2,2}(\Omega, \varphi; X)$.

Since $u$ is a weak solution of $Lu = f$, we have $\mathcal{D}(v, u) = (v, f)_\varphi$ for all $v \in W_0^{1,2}(\Omega, \varphi; X)$. Thus

\begin{equation}
\sum_{j,k=1}^n \int_\Omega X_j v a_{jk} X_k u e^{-\varphi} \, dx = \int_\Omega v \bar{g} e^{-\varphi} \, dx
\end{equation}

where

\begin{equation}
g = f - \sum_{j=1}^n \left( b_j X_j u + b_j^* X_j^* u + (X_j^* b_j^*) u - bu \right).
\end{equation}

We would like to use (29) substituting $v = X^*_\ell (\zeta^2 X_\ell u)$. This is problematic as $u \in W_0^{2,2}(W, \varphi; X)$ and not thrice-differentiable. Instead, we let $u_\epsilon \in C^\infty(\Omega)$ be so that $u_\epsilon \to u$ in $W^{2,2}(W, \varphi; X)$. We set $v_\epsilon = X^*_\ell (\zeta^2 X_\ell u_\epsilon)$. In this case, the left-hand side of (29) becomes

\begin{equation}
A_\epsilon = \sum_{j,k=1}^n \left( X_j (X^*_\ell \zeta^2 X_\ell u_\epsilon), a_{jk} X_k u \right)_\varphi,
\end{equation}

and the right-hand side becomes

\begin{equation}
B_\epsilon = \int_\Omega v_\epsilon \bar{g} e^{-\varphi} \, dx = \left( X^*_\ell \zeta^2 X_\ell u_\epsilon, f - \sum_{j=1}^n \left( b_j X_j u + b_j^* X_j^* u - \frac{\partial b_j^*}{\partial x_j} u - bu \right) \right)_\varphi.
\end{equation}

Equation (29) now says that $A_\epsilon = B_\epsilon$.

Observe that

\begin{align}
A_\epsilon &= \sum_{j,k=1}^n \left( X^*_\ell X_j \zeta^2 X_\ell u_\epsilon, a_{jk} X_k u \right)_\varphi + \sum_{j,k=1}^n \left( [X_j, X^*_\ell] \zeta^2 X_\ell u, a_{jk} X_k u \right)_\varphi \\
&= \sum_{j,k=1}^n \left( X_j \zeta^2 X_\ell u_\epsilon, a_{jk} X_k X_\ell u \right)_\varphi + \sum_{j,k=1}^n \left( [X_j, X^*_\ell] \zeta^2 X_\ell u, a_{jk} X_k u \right)_\varphi \\
&\quad + \left( X_j \zeta^2 X_\ell u_\epsilon, \frac{\partial a_{jk}}{\partial x_\ell} X_k u \right)_\varphi.
\end{align}

Since no more than two derivatives of $u_\epsilon$ are taken in $A_\epsilon$, we can let $\epsilon \to 0$ and observe that $A = B$ where

\begin{equation}
A = \sum_{j,k=1}^n \left( X_j \zeta^2 X_\ell u, a_{jk} X_k X_\ell u \right)_\varphi + \sum_{j,k=1}^n \left( [X_j, X^*_\ell] \zeta^2 X_\ell u, a_{jk} X_k u \right)_\varphi + \left( X_j \zeta^2 X_\ell u, \frac{\partial a_{jk}}{\partial x_\ell} X_k u \right)_\varphi
\end{equation}

and

\begin{equation}
B = \left( X^*_\ell \zeta^2 X_\ell u, f - \sum_{j=1}^n \left( b_j X_j u + b_j^* X_j^* u - \frac{\partial b_j^*}{\partial x_j} u - bu \right) \right)_\varphi.
\end{equation}
We continue our investigation of $A$. Observe that

$$[X_j, X^*_\ell] = -\frac{\partial \varphi}{\partial x_j \partial x_\ell} \quad \text{and} \quad [X_\ell, X_k] = ([X_\ell, X_k]^*)_\ell = [X^*_k, X^*_\ell]_\ell = 0.$$  

We have

$$A = \sum_{j,k=1}^n \left( X_j \zeta^2 X_\ell u, a_{jk} X_k X_\ell u \right) \varphi + \sum_{j,k=1}^n \left( [X_j, X^*_\ell] \zeta^2 X_\ell u, a_{jk} X_k u \right) \varphi$$

$$+ \left( X_j \zeta^2 X_\ell u, \frac{\partial a_{jk}}{\partial x_\ell} X_k u \right) \varphi + \left( X_j \zeta^2 X_\ell u, a_{jk} \left[ X_\ell, X_k \right] u \right) \varphi \right|_{\ell=0}$$

$$= \sum_{j,k=1}^n \left( \zeta^2 X_j X_\ell u, a_{jk} X_k X_\ell u \right) \varphi + \sum_{j,k=1}^n \left( [X_j, X^*_\ell] \zeta^2 X_\ell u, a_{jk} X_k u \right) \varphi$$

$$+ \left( X_j \zeta^2 X_\ell u, \frac{\partial a_{jk}}{\partial x_\ell} X_k u \right) \varphi + \left( 2 \zeta \frac{\partial \zeta}{\partial x_j} X_\ell u, a_{jk} X_k X_\ell u \right) \varphi \right].$$

The strong ellipticity condition implies that

$$\sum_{j,k=1}^n \left( \zeta^2 X_j X_\ell u, a_{jk} X_k X_\ell u \right) \varphi \geq \theta \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)}.$$  

The remaining terms we bound as follows:

$$\left| \left( X_j \zeta^2 X_\ell u, \frac{\partial a_{jk}}{\partial x_\ell} X_k u \right) \varphi + \left( 2 \zeta \frac{\partial \zeta}{\partial x_j} X_\ell u, a_{jk} X_k X_\ell u \right) \varphi \right| \leq C_1 \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)} \left\| \nabla X_\ell u \right\|_{L^2(w, \varphi)} ,$$

where $C_1$ depends on $\|a_{jk}\|_{C^1(\Omega)}$ and $\|\zeta\|_{C^1(\Omega)}$. In particular, $C_1$ does not depend on $|\text{supp } \zeta|$. Next, using (HIV), Corollary 3.5 and the fact that $\frac{\partial a_{jk}}{\partial x_j} = -X^*_j - X_j$, we have

$$\left| ([X_j, X^*_\ell] \zeta^2 X_\ell u, a_{jk} X_k u) \varphi \right| \leq C_2 \left( (1 + |\nabla \varphi|) \zeta^2 |X_\ell u|, |a_{jk}| |X_k u| \right) \varphi$$

$$\leq C_2' \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)} \left( \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)} + \left\| \nabla X_\ell u \right\|_{L^2(w, \varphi)} \right),$$

where $C_2'$ depends on $C_2$ and $\|a_{jk}\|_{L^\infty(\Omega)}$. Thus, using (35) and the bounds on the error terms, we can bound (with $C = n^2(C_1 + C_2')$)

$$|A| \geq \theta \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)}^2 - C \left( \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)} \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)} + \left\| \nabla X_\ell u \right\|_{L^2(w, \varphi)}^2 \right)$$

$$\geq \frac{\theta}{2} \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)}^2 - C_3 \left\| \nabla X_\ell u \right\|_{L^2(w, \varphi)}^2 ,$$

where $C_3 = C_3(\|a_{jk}\|_{C^1(\Omega)}, \|\zeta\|_{C^1(\Omega)}, \theta)$. We can bound $B$ with Cauchy-Schwarz and the small constant/large constant inequality. In particular, we can use Corollary 3.4 to show that for some constant $C_4 > 0$ where $C_4 = C_4(||b_j||_{L^\infty(\Omega)}, ||b'_j||_{C^1(\Omega)}, ||b||_{L^\infty(\Omega)}, \|\zeta\|_{C^1(\Omega)}, n, \theta)$, we have the estimate

$$|B| \leq C_4 \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)} \left[ ||f||_{L^2(w, \varphi)} + ||u||_{W^{1,2}(w, \varphi, X)} \right]$$

$$\leq \frac{\theta}{4} \left\| \zeta \nabla X_\ell u \right\|_{L^2(w, \varphi)}^2 + C_5 \left( ||f||_{L^2(w, \varphi)}^2 + ||u||_{W^{1,2}(w, \varphi, X)}^2 \right),$$

where $C_5 = C_5(||b_j||_{L^\infty(\Omega)}, ||b'_j||_{C^1(\Omega)}, ||b||_{L^\infty(\Omega)}, n, \theta)$.  

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Combining (36) and (37), we have shown that

\[ \|u\|_{W^{2,2}(V,\varphi;X)} \leq C_6 \left( \|f\|_{L^2(W,\varphi)}^2 + \|u\|_{W^{1,2}(W,\varphi;X)}^2 \right), \]

where \( C_6 = C_6(\|a_{jk}\|_{C^1(\Omega)}, \|\zeta\|_{C^1(\Omega)}, \|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n, \theta) \). We can improve the estimate (38) and replace \( \|u\|_{W^{1,2}(W,\varphi;X)} \) with \( \|u\|_{L^2(W,\varphi)} \). Let \( \eta \in C_c^\infty(\Omega) \) be a cutoff so that \( \eta|_W = 1 \). Using (29) and strong ellipticity, we estimate

\[ \|\eta \nabla_X u\|_{L^2(\Omega,\varphi)}^2 \leq \frac{1}{\theta} \sum_{j,k=1}^n (\eta^2 X_j u, a_{jk} X_k u)_\varphi = \frac{1}{\theta} \|\eta^2 u\|_{L^2(\Omega,\varphi)}, \]

where \( C_7 = C_7(\|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n, \theta) \). Using a small constant/large constant argument, we have

\[ \|\nabla_X u\|_{L^2(W,\varphi)} \leq \|\eta \nabla_X u\|_{L^2(\Omega,\varphi)} \leq C_8(\|f\|_{L^2(\Omega,\varphi)} + \|u\|_{L^2(\Omega,\varphi)}), \]

where \( C_8 = C_8(\|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n, \theta) \). Thus, we can refine (38) by

\[ \|u\|_{W^{2,2}(V,\varphi;X)} \leq C(\|f\|_{L^2(\Omega,\varphi)} + \|u\|_{L^2(\Omega,\varphi)}), \]

where \( C = C(\text{dist}(V,b\Omega), \|a_{jk}\|_{C^1(\Omega)}, \|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n, \theta) \). \( \zeta \) and \( \eta \) have disappeared from \( C \) as \( \|\zeta\|_{C^1(\Omega)} \) depends only on \( \text{dist}(V,b\Omega) \). Thus, we can relax the boundedness condition on \( V \) and let \( V \) be as in the statement of the theorem.

6.2. \( \ell \geq 1 \) case.

**Proof of Theorem 2.** \( \ell \geq 1 \). As with the \( \ell = 0 \) case, that \( u \in W^{\ell+2,2}_{\text{loc}}(\Omega,\varphi;X) \) follows from the classical theory. We will establish (5) by induction. The \( \ell = 0 \) case has already been established.

Let \( V \subset W \subset \Omega \) so that \( \text{dist}(V,bW) > 0 \) and \( \text{dist}(W,b\Omega) > 0 \).

Assume that (5) holds for a nonnegative integer \( \ell \), for \( a_{jk}, b'_j \in C^{\ell+2}(\Omega) \cap W^{\ell+2,\infty}(\Omega), \) for \( b_j, b \in C^{\ell+1}(\Omega) \cap W^{\ell+1,\infty}(\Omega), \) and for \( f \in W^{\ell+1,2}(\Omega,\varphi;X) \). Assume further that \( u \in W^{1,2}(\Omega,\varphi;X) \) is a weak solution of \( Lu = f \) in \( \Omega \). By the induction hypothesis, we have the estimate

\[ \|u\|_{W^{\ell+2}(W,\varphi;X)} \leq C(\|f\|_{W^{\ell}(\Omega,\varphi;X)} + \|u\|_{L^2(\Omega,\varphi)}), \]

where \( C = C(\text{dist}(W,b\Omega), \|a_{jk}\|_{C^{\ell+1}(\Omega)}, \|b_j\|_{C^\infty(\Omega)}, \|b'_j\|_{C^{\ell+1}(\Omega)}, \|b\|_{C^\infty(\Omega)}, n, \theta, \ell) \). Let \( \alpha \) be a multiindex of length \( |\alpha| = \ell + 1 \). Let \( \tilde{v} \in C^\infty_c(W) \) and set

\[ v = (X^\alpha)^* \tilde{v} \quad \text{and} \quad \tilde{u} = X^\alpha u. \]

Since \( D(v,u) = (v,f) \), we plug in \( v = (X^\alpha)^* \tilde{v} \) and compute

\[
D(v,u) = \sum_{j,k=1}^n (X_j(X^\alpha)^* \tilde{v}, a_{jk} X_k u)_\varphi + \sum_{j=1}^n \left[ ((X^\alpha)^* \tilde{v}, b_j X_j u)_\varphi + (X_j(X^\alpha)^* \tilde{v}, b'_j u)_\varphi \right] + ((X^\alpha)^* \tilde{v}, bu)_\varphi.
\]
Integrating by parts gives us

\[ \mathcal{D}(v, u) = \sum_{j,k=1}^{n} (X_j \delta, a_{jk} X_k X^\alpha u) \phi + \sum_{j=1}^{n} \left( (\delta, b_j X_j X^\alpha u) \phi + (\delta, X_j^\phi (b_j X^\alpha) u) \phi + (\delta, b X^\alpha u) \phi \right) \]

+ \sum_{j,k=1}^{n} \left( (\delta, [X_j, (X^\alpha)^t] a_{jk} X_k u) \phi + \sum_{\beta \subset \alpha \atop \beta \neq \alpha} \left( (\delta, (D^{\alpha-\beta} a_{jk}) X_k X^\beta u) \phi \right) \right)

+ \sum_{j=1}^{n} \left( (\delta, [X_j, (X^\alpha)^t] b_j u) \phi \right)

+ \sum_{\beta \subset \alpha \atop \beta \neq \alpha} \left( (\delta, (D^{\alpha-\beta} b_j) X^\beta u) \phi + \sum_{j=1}^{n} \left( (\delta, (D^{\alpha-\beta} b_j) X^\beta X_j u) \phi + (\delta, X_j^\phi ((D^{\alpha-\beta} b_j) X^\beta u) \phi \right) \right) \},

so

\[ \mathcal{D}(v, u) = \mathcal{D}(\delta, \delta) + (\delta, g) \phi, \]

where

\[ g = \sum_{j,k=1}^{n} \left( [X_j, (X^\alpha)^t] a_{jk} X_k u + \sum_{\beta \subset \alpha \atop \beta \neq \alpha} (D^{\alpha-\beta} a_{jk}) X_k X^\beta u \right) + \sum_{j=1}^{n} \left( [X_j, (X^\alpha)^t] b_j u \right) \]

+ \sum_{\beta \subset \alpha \atop \beta \neq \alpha} \left( (D^{\alpha-\beta} b_j) X^\beta u + \sum_{j=1}^{n} \left( (D^{\alpha-\beta} b_j) X^\beta X_j u + X_j^\phi ((D^{\alpha-\beta} b_j) X^\beta u) \right) \right). \]

To compute \([X_j, (X^\alpha)^t] \), observe that

\[ X_j (X^\alpha)^t = X_j X^\alpha \cdots X^\alpha_{\ell+1} = X^\alpha_{\ell+1} X_j X^\alpha_{\ell} \cdots X^\alpha_1 + [X_j, X^\alpha_{\ell+1}] X^\alpha \cdots X^\alpha_1 \]

\[ = \cdots = (X^\alpha)^t X_j + [X_j, X^\alpha_{\ell+1}] X^\alpha \cdots X^\alpha_1 + \cdots + X^\alpha_{\ell+1} X_j [X_j, X^\alpha_{\ell+1}] \]

\[ = (X^\alpha)^t X_j - \frac{\partial \phi}{\partial x_j \partial x_{\alpha_1}} X^\alpha_{\ell+1} \cdots X^\alpha_{\ell+1} \cdots - X^\alpha_{\ell+1} \cdots X^\alpha_{\ell} \frac{\partial \phi}{\partial x_j \partial x_{\alpha_{\ell+1}}} \]

\[ = (X^\alpha)^t X_j + \sum_{\gamma \subset \alpha \atop \gamma \neq \alpha} c_{\alpha \gamma} (X^\gamma)^t D^{\alpha-\gamma} \frac{\partial \phi}{\partial x_j} \]

for some constant \(c_{\alpha \gamma}\). Thus,

\[ [X_j, (X^\alpha)^t] = \sum_{\gamma \subset \alpha \atop \gamma \neq \alpha} c_{\alpha \gamma} \left( D^{\alpha-\gamma} \frac{\partial \phi}{\partial x_j} \right) X^\gamma \]
and we can rewrite
\[
g = \sum_{\gamma \subset \alpha \atop \gamma \neq \alpha} c_{\alpha \gamma} \left[ \sum_{j,k=1}^{n} \left( D^{\alpha-\gamma} \frac{\partial \varphi}{\partial x_j} \right) X^{\gamma}(a_{jk} X_k u) + \sum_{j=1}^{n} \left( D^{\alpha-\gamma} \frac{\partial \varphi}{\partial x_j} \right) X^{\gamma}(b_j u) \right] + \sum_{\beta \subset \alpha \atop \beta \neq \alpha} c_{\alpha \beta} \left[ \sum_{j,k=1}^{n} \left( D^{\alpha-\beta} b_{jk} X^\beta u + \sum_{j=1}^{n} \left( D^{\alpha-\beta} b_j X^\beta u + X^* (D^{\alpha-\beta} b_j X^\beta u) \right) \right] \right].
\]

Thus, if \( \tilde{f} = X^\alpha f - g \), then we can express \( \mathcal{D}(v, u) = (v, f)_{\varphi} = (\tilde{v}, X^\alpha f)_{\varphi} \) as \( \mathcal{D}(\tilde{v}, \tilde{u}) = (\tilde{v}, \tilde{f})_{\varphi} \). Since \( \tilde{v} \in C^\infty_c(\Omega) \) is arbitrary, \( \tilde{u} \) is a weak solution for \( Lu = f \). Since
\[
\| \tilde{f} \|_{L^2(W, \varphi)} \leq C' \left( \| f \|_{W^{\ell+1,2(\Omega, \varphi; X)} + \| g \|_{L^2(\Omega, \varphi)} \right)
\]
it follows from the induction hypothesis that \( \tilde{f} \in L^2(W) \) with
\[
\| \tilde{f} \|_{L^2(W, \varphi)} \leq C \left( \| f \|_{W^{\ell+1,2(\Omega, \varphi; X)} + \| u \|_{L^2(\Omega, \varphi)} \right)
\]
As a consequence of the \( \ell = 0 \) case of Theorem 2.13
\[
\| \tilde{u} \|_{W^{2,2}(V, \varphi; X)} \leq C \left( \| f \|_{L^2(W, \varphi)} + \| \tilde{u} \|_{L^2(W, \varphi)} \right) \leq C \left( \| f \|_{W^{\ell+1,2(\Omega, \varphi; X)} + \| u \|_{L^2(\Omega, \varphi)} \right).
\]
Since this inequality holds for any \( \alpha \) of length \( \ell + 1 \), it follows that \( u \in W^{\ell+3,2}(V, \varphi; X) \) and
\[
\| u \|_{W^{\ell+3,2}(V, \varphi; X)} \leq C \left( \| f \|_{W^{\ell+1,2(\Omega, \varphi; X)} + \| u \|_{L^2(\Omega, \varphi)} \right).
\]

7. Elliptic regularity at the boundary

The standard technique to prove elliptic regularity at the boundary is to work locally, rotate, and flatten the domain. Working on a weighted \( L^2 \) space complicates the matter and we instead work with tangential and normal derivatives.

7.1. Tangential Operators. Recall that a first order differential operator \( T \) is a tangential operator if the first order component of \( T \) annihilates \( \rho \). By the hypotheses on \( \Omega \), there exists \( \epsilon > 0 \) so that on \( \Omega'_\epsilon \), there exist first order differential operators \( T_1, \ldots, T_n \) so that
\[
T_j = \sum_{\ell=1}^{n} \tau_{j\ell} X_\ell,
\]
where \( (\tau_{j\ell}) \) is an orthogonal matrix, the components \( \tau_{j\ell} \) are bounded in \( C^{m-1}(\Omega'_\epsilon) \), \( T_j \) is tangential for \( 1 \leq j \leq n - 1 \), the first order part of \( T_n \) is the unit outward normal to the
level curve of $\rho$ and

\begin{equation}
\sum_{j=1}^{n} |T_j f|^2 = |\nabla X f|^2.
\end{equation}

From (40), it is clear that if $k \leq m$, then

\begin{equation}
\|u\|_{W^{k,2}(\Omega, \varphi; X)} \sim \sum_{|\alpha| \leq k} \|T^\alpha u\|_{L^2(\Omega, \varphi)} + \|u\|_{W^{k,2}(\Omega \setminus \Omega_\epsilon, \varphi; X)}.
\end{equation}

By assumption $|d\rho| = 1$ on $b\Omega$ so that if $\nu_\epsilon$ is the unit (outward) normal to $\{ x \in \mathbb{C}^n : \rho(x) = -\epsilon \}$, then $\nu_\epsilon = \frac{\partial \rho}{\partial x_i}$.

As an immediate consequence of the Divergence Theorem,

\begin{equation}
\int_{\Omega} \frac{\partial f}{\partial x_j} \, dx = \int_{\partial \Omega} f \frac{\partial \rho}{\partial x_j} \, d\sigma
\end{equation}

where $d\sigma$ is the surface area measure on $b\Omega$.

7.2. **Proof of Theorem 2.15, $\ell = 0$ case.** We are now ready to prove the regularity of solutions of $Lu = f$ near $b\Omega$.

**Proof of Theorem 2.15, $\ell = 0$ case.** Given Theorem 2.13, it is enough to show that $u \in W^{2,2}(\Omega, \varphi; X)$. Let $V, W \subset \Omega'_\epsilon$ be smooth, bounded domains so that $V \subset W$ and $\text{dist}(V, bW) > 0$. Let $\zeta \in C^\infty(\mathbb{R}^n)$ be a smooth cutoff so that $\zeta|_V = 1$ and supp $\zeta \subset W$. Since $L$ is elliptic and $W$ is bounded, the classical theory yields $u \in W^{2,2}(\Omega_\epsilon \cap W, \varphi; X)$. By (HI), it is enough to work locally, i.e., we can assume that supp $u$ is small enough that $T_1, \ldots, T_n$ are well-defined on supp $u$.

The function $u$ is a weak solution of $Lu = f$, so we have $\mathcal{D}(v, u) = (v, f)_\varphi$ for all $v \in \mathcal{X}$. Thus $u$ satisfies the free boundary condition for $\mathcal{X}$ and equations (29) and (30) hold. As in the proof of Theorem 2.13, we would like to use (29) substituting $v = X_k^*(\zeta^2 X_k u)$. This is problematic as $u \in W^{2,2}(W, \varphi; X)$ and not thrice-differentiable. Instead, we use Proposition 3.2 which constructs $u_\epsilon \in C^\infty_c(\mathbb{R}^n)$ so that $u_\epsilon \to u$ in $W^{2,2}(W \cap \Omega_\epsilon, \varphi; X)$. Let $1 \leq k \leq n - 1$. Then set $v_\epsilon = T_k^*(\zeta^2 T_k u_\epsilon)$. In this case, the left-hand side of (29) becomes

$$
A_\epsilon := \sum_{j,j'=1}^{n} \left( X_j T_k^*(\zeta^2 T_k u_\epsilon), a_{jj'} X_{j'} u_\epsilon \right)_\varphi,
$$

and the right-hand side becomes

$$
B_\epsilon := \int_{\Omega} v_\epsilon \bar{g} e^{-\varphi} \, dx = \left( T_k^*(\zeta^2 T_k u_\epsilon), f - \sum_{j=1}^{n} \left( b_j X_j u + b_j' X_j u - \frac{\partial b_{j}'}{\partial x_j} u \right) - bu \right)_\varphi.
$$
Equation (29) now says that $A_\epsilon = B_\epsilon$. Since $T_k$ is tangential, $T_k^*$ is also tangential and we compute

$$A_\epsilon = \sum_{j,j'=1}^{n} \left( T_k^* X_j (\zeta^2 T_k u_\epsilon), a_{jj'}X_{j'} u \right) \phi + \sum_{j,j'=1}^{n} \left( [X_j, T_k^*] (\zeta^2 T_k u_\epsilon), a_{jj'}X_{j'} u \right) \phi$$

$$= \sum_{j,j'=1}^{n} \left( X_j (\zeta^2 T_k u_\epsilon), a_{jj'}T_k X_{j'} u \right) \phi$$

$$+ \sum_{j,j'=1}^{n} \left( [X_j, T_k^*] (\zeta^2 T_k u_\epsilon), a_{jj'}X_{j'} u \right) + \sum_{k'=1}^{n} \left( X_j (\zeta^2 T_k u_\epsilon), \tau_{kk'} \frac{\partial a_{jj'}}{\partial x_{k'}} X_{j'} u \right) \phi \right].$$

Since no more than two derivatives of $u_\epsilon$ are taken in $A_\epsilon$, we can let $\epsilon \to 0$ and observe that $A = B$ where

$$A = \sum_{j,j'=1}^{n} \left( X_j (\zeta^2 T_k u), a_{jj'}T_k X_{j'} u \right) \phi$$

$$+ \sum_{j,j'=1}^{n} \left( [X_j, T_k^*] (\zeta^2 T_k u), a_{jj'}X_{j'} u \right) + \sum_{k'=1}^{n} \left( X_j (\zeta^2 T_k u), \tau_{kk'} \frac{\partial a_{jj'}}{\partial x_{k'}} X_{j'} u \right) \phi \right]=$$

and

$$B = \left( T_k^* (\zeta^2 T_k u), f - \sum_{j=1}^{n} \left( b_j X_j u + b_j^* X_j^* u - \frac{\partial b_j^*}{\partial x_j} u \right) - bu \right) \phi.$$  

We continue our investigation of $A$. Observe that $T_k^* = \sum_{k'=1}^{n} \left( \tau_{kk'} X_{k'}^* - \frac{\partial a_{kk'}}{\partial x_{k'}} \right)$, so

$$[X_j, T_k^*] = \sum_{k'=1}^{n} -\tau_{kk'} \frac{\partial^2 \phi}{\partial x_j \partial x_{k'}} + \frac{\partial \tau_{kk'}}{\partial x_j} X_{k'}^* - \frac{\partial^2 \tau_{kk'}}{\partial x_{k'} \partial x_j}$$

and

$$[T_k, X_{j'}] = -\sum_{k'=1}^{n} \frac{\partial \tau_{kk'}}{\partial x_{j'}} X_{k'}.$$  

We have

$$A = \sum_{j,j'=1}^{n} \left( X_j (\zeta^2 T_k u), a_{jj'}X_{j'} T_k u \right) \phi + \sum_{j,j'=1}^{n} \left( [X_j, T_k^*] (\zeta^2 T_k u), a_{jj'}X_{j'} u \right) \phi$$

$$+ \left( X_j (\zeta^2 T_k u), \frac{\partial a_{jj'}}{\partial x_k} X_{j'} u \right) + \left( X_j (\zeta^2 T_k u), a_{jj'} [T_k, X_{j'}] u \right) \phi$$

$$= \sum_{j,j'=1}^{n} \left( \zeta^2 X_j T_k u, a_{jj'}X_{j'} T_k u \right) \phi + \sum_{j,j'=1}^{n} \left( [X_j, T_k^*] (\zeta^2 T_k u), a_{jj'}X_{j'} u \right) \phi$$

$$+ \left( X_j (\zeta^2 T_k u), \frac{\partial a_{jj'}}{\partial x_k} X_{j'} u \right) + \left( X_j (\zeta^2 T_k u), a_{jj'} [T_k, X_{j'}] u \right) \phi + \left( 2 \zeta \frac{\partial \zeta}{\partial x_j} T_k u, a_{jj'} X_{j'} T_k u \right) \phi \right].$$
As in the proof of Theorem 2.13, the remaining terms of (45) are bounded by
\[ C_2(\|\zeta\nabla x T_k u\|_{L^2(W\cap \Omega, \varphi)} \|\nabla x u\|_{L^2(W\cap \Omega, \varphi)} + \|\nabla x u\|_{L^2(W\cap \Omega, \varphi)}^2) \]
where \( C_2 \) depends on \( \|a_{jj'}\|_{C^1(\Omega)}, \|\rho\|_{C^3(\Omega)} \) and \( \|\zeta\|_{C^1(\Omega)} \). In particular, \( C \) does not depend on the size \( \text{supp} \zeta \). Thus, using (46) and the bounds on the error terms, we can bound
\[ |A| \geq \frac{\theta}{2} \|\nabla x T_k u\|_{L^2(W\cap \Omega, \varphi)}^2 \]
where \( C_3 = C_3(\|a_{jj'}\|_{C^1(\Omega)}, \|\zeta\|_{C^1(\Omega)}, n, \theta, \|\rho\|_{C^3(\Omega)}) \). We can bound \( B \) with Cauchy-Schwarz and the small constant/large constant inequality. In particular, for a constant \( C_4 = C_4(\|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n, \|\rho\|_{C^2(\Omega)}) \), we have the estimate
\[ |B| \leq C_4 \|\nabla x T_k u\|_{L^2(W\cap \Omega, \varphi)} [\|f\|_{L^2(W\cap \Omega, \varphi)} + \|u\|_{W^{1,2}(W\cap \Omega, \varphi, x)} ] \]
(48)

where \( C_5 = C_5(\|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n, \theta, \|\rho\|_{C^2(\Omega)}) \). Combining (47) and (48), it follows that
\[ \|\nabla^\tan u\|_{W^{1,2}(V, \cap \Omega, \varphi, x)}^2 \leq C_6(\|f\|_{L^2(W\cap \Omega, \varphi)} + \|u\|_{W^{1,2}(W\cap \Omega, \varphi, x)}) \]
where \( C_6 = C_6(\|a_{jj'}\|_{C^1(\Omega)}, \|\zeta\|_{C^1(\Omega)}, \|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n, \theta, \|\rho\|_{C^3(\Omega)}) \).

We can improve the estimate (49) and replace \( \|u\|_{L^2(W\cap \Omega, \varphi)} \) with \( \|u\|_{L^2(W, \varphi)} \). Let \( \eta \in C^\infty_c(\Omega') \) be a cutoff so that \( \eta|_{\cap \Omega} = 1 \). Using (29) and strong ellipticity, we estimate
\[ \|\eta \nabla x u\|_{L^2(\Omega, \varphi)}^2 \leq C_7(\|f\|_{L^2(\Omega, \varphi)} + \|\eta \nabla x u\|_{L^2(\Omega, \varphi)} + \|u\|_{L^2(\Omega, \varphi)}) \]
where \( C_7 = C_7(\|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n, \theta) \).

Using a small constant/large constant argument, we have
\[ \|\nabla x u\|_{L^2(W\cap \Omega, \varphi)} \leq \|\eta \nabla x u\|_{L^2(\Omega, \varphi)} \leq C_8(\|f\|_{L^2(\Omega, \varphi)} + \|u\|_{L^2(\Omega, \varphi)}) \]
where \( C_8 = C_8(\|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n) \). Thus, we can refine (49) by
\[ \|\nabla^\tan u\|_{W^{1,2}(V, \cap \Omega, \varphi, x)}^2 \leq C(\|f\|_{L^2(\Omega, \varphi)} + \|u\|_{L^2(\Omega, \varphi)}) \]
where \( C = C(\|a_{jj'}\|_{C^1(\Omega)}, \|b_j\|_{L^\infty(\Omega)}, \|b'_j\|_{C^1(\Omega)}, \|b\|_{L^\infty(\Omega)}, n, \theta, \|\rho\|_{C^3(\Omega)}) \). \( \zeta \) and \( \eta \) have disappeared from \( C \) as the bound depended on \( \|\zeta\|_{C^1(\Omega)} \) but that bound depends on \( \text{dist}(V, W) \). Thus, we can relax the boundedness condition on \( V \) and let \( V \) be as in the statement of the theorem.
Lemma 7.1. Let \( T_j = \sum_{j,j'=1}^n \tau_{jj'} X_{j'} \) where \((\tau_{jj'})\) is an orthogonal matrix. Therefore, if \((\tau_{jj'})^{-1} = (\tau_{jj'})\), then \( X_{j'} = \sum_{k'=1}^n \tau_{j'k'} T_{k'} \). Therefore,

\[
\sum_{j,j'=1}^n (X_j)^{a_{jj'}} X_{j'} = \sum_{j,j',k,k'=1}^n \left( T_k^* \tau_{jk} a_{jj'} \tau_{j'k'} T_{k'} \right).
\]

Let \( a_{\tan}^{kk} = \sum_{j,j'=1}^n \tau_{jk} a_{jj'} \tau_{j'k'} \). Since \((\tau_{jk})\) is an orthogonal matrix and the smallest eigenvalue of \( a_{jj'} \) is \( \theta \), it follows that

\[
\sum_{k,k'=1}^n a_{\tan}^{kk} \xi_k \xi_{k'} \geq \theta |\xi|^2.
\]

Therefore, if \( \xi = (0, \ldots, 1) \), then we see that \( a_{\tan}^{nn} \geq \theta \). Consequently, using the fact that \( Lu = f \), we see

\[
|T_n^* T_n u| \leq C_9 \left( |\nabla_X \nabla_T u| + |\nabla_X u| + \sum_{j=1}^n \left( |b_j X_j u| + |X_j^* (b_j^* u)| \right) + |b u| + |f| \right)
\]

where \( C_9 = C_9(\|\rho\|_{C^2(\Omega)}, \theta) \). Since the right-hand side is bounded in \( L^2(\Omega, \varphi) \), \( T_n^* T_n u \in L^2(\Omega, \varphi) \). Finally, since \( \|\nabla_X^* T_n u\|_{L^2(\Omega, \varphi)} \geq C \|T_n^* u\|_{L^2(\Omega, \varphi)} \), the proof of Theorem 2.15 for the case \( \ell = 0 \) is complete.

7.3. The \( \ell \geq 1 \) case. Before proving the higher order case, we perform a quick computation regarding tangential and nontangential operators.

**Lemma 7.1.** Let \( X \) be a first order differential operator with coefficients bounded by \( \|\rho\|_{C^k(\Omega'_c)} \) for \( k \geq 1 \) and let \( T_{\alpha_1}, \ldots, T_{\alpha_\ell} \) be tangential operators with coefficients bounded by \( \|\rho\|_{C^k(\Omega'_c)} \). If \( T^\alpha = T_{\alpha_1} \cdots T_{\alpha_\ell} \), then

i. \n
\[
XT^\alpha = \sum_{\beta \subseteq \alpha} T^\beta X_\beta
\]

for first order operators \( X_\beta \) with coefficients bounded by \( \|\rho\|_{C^{k-1-|\beta|}(\Omega'_c)} \).

ii. With \( X_\beta \) as in i.,

\[
[X, T^\alpha] = \sum_{\beta \subseteq \alpha} T^\beta X_\beta
\]

**Proof.** The proof is by induction on \( \ell \). When \( \ell = 1 \) this is self-evident for any \( k \geq 1 \), since \( XT = [X, T] + TX \). Observe that for \( 2 \leq j \leq \ell \),

\[
XT_{\alpha_1} \cdots T_{\alpha_j} = T_{\alpha_1} XT_{\alpha_2} \cdots T_{\alpha_j} + [X, T_{\alpha_1}] T_{\alpha_2} \cdots T_{\alpha_j}.
\]

If \( X \) has coefficients bounded by \( \|\rho\|_{C^k(\Omega'_c)} \), then the commutator \([X, T_{\alpha_1}]\) is a first order differential operator with coefficients bounded by \( \|\rho\|_{C^{k+1}(\Omega'_c)} \). If we apply the induction hypothesis with \( \ell = j - 1 \) to both terms, then (i) is proved.

The proof of (ii) follows from proof of (i). \( \square \)

**Proof of Theorem 2.15.** \( \ell \geq 1 \). This proof is loosely based on the proof of [Fol95, Theorem 7.29]. By Theorem 2.13 and the classical theory, we know that if \( f \in W^{l,2}(\Omega, \varphi; X) \) and \( Lu = f \), then \( u \in W^{l+2,2}_{loc}(\Omega, \varphi; X) \). As with the \( \ell = 0 \) case, we can restrict ourselves to \( \Omega \) for
\( \epsilon > 0 \) suitably small. Let \( V, W \subseteq \Omega' \) be bounded subsets and satisfy \( V \subset W \), \( \text{dist}(V, bW) > 0 \) and \( \text{dist}(W, b\Omega') > 0 \). Choose \( \zeta \in C_c^\infty(W) \) with \( \zeta|_V = 1 \).

We first indct on the number of tangential derivatives. The base case is already done. The induction hypothesis is that if \( \ell \geq 1 \) and \( |\beta| \leq \ell \), then there exists a constant \( C \) that does not depend on \( V \), the size of the support of \( \zeta \), or \( W \) so that

\[
\|T^\beta u\|_{W^{1,2}(W;\partial\Omega, \varphi; X)} \leq C (\|f\|_{W^{1-1,2}(W;\partial\Omega, \varphi; X)} + \|u\|_{L^2(\Omega, \varphi)}).
\]

Let \( \alpha \) be a multiindex of length \( \ell + 1 \). Let \( v \in W^{1,2}(\Omega, \varphi; X) \). We start by showing

\[
|\mathcal{D}(v, T^\alpha \zeta u)| \leq C_1 \|v\|_{W^{1,2}(\Omega, \varphi; X)} (\|f\|_{W^{1-1,2}(\Omega, \varphi; X)} + \|u\|_{L^2(\Omega, \varphi)})
\]

where \( C_1 = C_1 (\|a_{jk}\|_{C^{\ell+1}(\Omega)}, \|b\|_{C^{\ell+1}(\Omega)}, \|b\|_{C^{\ell+1}(\Omega)}, \|\zeta\|_{C^\ell(\Omega)}, \|\rho\|_{C^{\ell+2}(\Omega)}) \).

By Proposition 3.2 there exist \( \psi \in C^\infty_c(\mathbb{R}^n) \) so that \( \psi \to v \) in \( W^{1,2}(\Omega, \varphi; X) \). Therefore, since \( \mathcal{D} \) involves at most first order derivatives,

\[
\lim_{\delta \to 0} \mathcal{D}(v_\delta, T^\alpha \zeta u) = \mathcal{D}(v, T^\alpha \zeta u).
\]

We compute

\[
\mathcal{D}(v_\delta, T^\alpha \zeta u) = \sum_{j,k=1}^n (X_j v_\delta, a_{jk} X_k (T^\alpha u)) \varphi
\]

\[
+ \sum_{j=1}^n \left[ (v_\delta, b_j X_j (T^\alpha \zeta u)) \varphi + (X_j v_\delta, b_j' (T^\alpha \zeta u)) \varphi \right] + (v_\delta, b T^\alpha (\zeta u)) \varphi.
\]

We examine each term separately

\[
(X_j v_\delta, a_{jk} X_k (T^\alpha \zeta u)) \varphi = ((T^\alpha)^* X_j v_\delta, a_{jk} X_k (T^\alpha \zeta u)) \varphi + (X_j v_\delta, [a_{jk} X_k, T^\alpha \zeta u]) \varphi
\]

\[
= (\zeta (T^\alpha)^* X_j v_\delta, a_{jk} X_k u) \varphi + (X_j (T^\alpha)^* v_\delta, \frac{\partial}{\partial x_k} a_{jk} u) \varphi + (X_j v_\delta, [a_{jk} X_k, T^\alpha \zeta u]) \varphi
\]

\[
= (X_j (\zeta (T^\alpha)^* v_\delta), a_{jk} X_k u) \varphi + ([\zeta (T^\alpha)^*, X_j] v_\delta, a_{jk} X_k u) \varphi + (X_j (T^\alpha)^* v_\delta, \frac{\partial}{\partial x_k} a_{jk} u) \varphi
\]

\[
+ (X_j v_\delta, [a_{jk} X_k, T^\alpha \zeta u]) \varphi.
\]

Next,

\[
(v_\delta, b_j X_j (T^\alpha \zeta u)) \varphi = ((T^\alpha)^* v_\delta, b_j X_j (T^\alpha \zeta u)) \varphi + (v_\delta, [b_j X_j, T^\alpha \zeta u]) \varphi
\]

\[
= (\zeta (T^\alpha)^* v_\delta, b_j X_j u) \varphi + (T^\alpha)^* v_\delta, b_j \frac{\partial}{\partial x_j} u \varphi + (v_\delta, [b_j X_j, T^\alpha \zeta u]) \varphi.
\]

Also,

\[
(X_j v_\delta, b_j' T^\alpha (\zeta u)) \varphi = (\zeta (T^\alpha)^* X_j v_\delta, b_j' T^\alpha (\zeta u)) \varphi + (X_j v_\delta, [b_j', T^\alpha \zeta u]) \varphi
\]

\[
= (X_j (\zeta (T^\alpha)^* v_\delta), b_j') \varphi + ([\zeta (T^\alpha)^*, X_j] v_\delta, b'_j u) \varphi + (X_j v_\delta, [b_j', T^\alpha \zeta u]) \varphi.
\]

Plugging (53), (54) and (55) into (52), we see

\[
\mathcal{D}(v_\delta, T^\alpha \zeta u) = \mathcal{D}(\zeta (T^\alpha)^* v_\delta, u) + E = (\zeta (T^\alpha)^* v_\delta, f) \varphi + E
\]

\[
= (T^\alpha v_\delta, T_{\alpha_2} \cdots T_{\alpha_{\ell+1}} (\zeta f)) \varphi + E
\]
where
\[
E = \sum_{j,k=1}^{n} \left[ \left( [\zeta(T^\alpha)^*, X_j]v_\delta, a_{jk}X_k u \right)_\varphi + \left( X_j(T^\alpha)^* v_\delta, \frac{\partial \zeta}{\partial x_k} a_{jk} u \right)_\varphi + \left( X_j v_\delta, [a_{jk}X_k, T^\alpha]\zeta(u) \right) \right] \\
+ \sum_{j=1}^{n} \left[ \left( (T^\alpha)^* v_\delta, b_j \frac{\partial \zeta}{\partial x_j} u \right)_\varphi + \left( v_\delta, [b_jX_j, T^\alpha]\zeta(u) \right) \right] + \left( \left[ (T^\alpha)^* X_j v_\delta, b_j' u \right) \varphi \right] \\
+ \sum_{j=1}^{n} \left( X_j v_\delta, [b_j, T^\alpha]\zeta(u) \right) \varphi + \left( v_\delta, [b, T^\alpha]\zeta(u) \right) \varphi.
\]

Since \([\zeta, (T^\alpha)^*] = [T^\alpha, \zeta]\) is a differential operator of order \(\ell\), by the induction hypothesis
\[
|\mathcal{D}(v_\delta, T^\alpha \zeta u)| \leq C_2 \|v_\delta\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; \ell)} \|f\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; \ell)} + |E| \tag{56}
\]
where \(C_2 = C_2(\|\zeta\|_{C^\ell(\Omega) \cap \|\rho\|_{C^\ell(\Omega')}})\). We turn our attention to \(E\). Using integration by parts, \([T^\alpha)^*, X]\)\(=[X, T^\alpha]\) (formally), and Lemma [7.1] (with \(k = 1\), since the result is not improved when the coefficients of \(X\) are smooth), it follows from the induction hypothesis that
\[
|E| \leq C_3 \|v_\delta\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; \ell)} \sum_{|\beta| \leq \ell} \|T^\beta u\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; \ell)} \\
\leq C_4 \|v_\delta\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; \ell)} (\|f\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; \ell)} + \|u\|_{L^2(\Omega \cap X_\varphi; \phi)}) \tag{57}
\]
where
\[
C_4 = C_4(\|a_{jk}\|_{C^{\ell+1}(\Omega)}, \|b_j\|_{C^{\ell+1}(\Omega)}, \|b'_j\|_{C^{\ell+1}(\Omega)}, \|b\|_{C^{\ell+1}(\Omega)}, n, \|\zeta\|_{C^{\ell+1}(\Omega)}, \|\rho\|_{C^{\ell+1}(\Omega)}).
\]
By plugging (57) into (56) and letting \(\delta \to 0\), we observe that (51) has been verified.

Since \(T^\alpha(\zeta u) \in X\), we can set \(v = T^\alpha(\zeta u)\) in (51) and use the coercive estimate (4) to obtain
\[
\|T^\alpha(\zeta u)\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; X)} \leq C_5 \|\mathcal{D}(T^\alpha(\zeta u), T^\alpha(\zeta u))\| + \|T^\alpha(\zeta u)\|_{L^2(\Omega \cap X_\varphi; \phi; X)} \\
\leq C_6 \|T^\alpha(\zeta u)\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; X)} (\|f\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; X)} + \|T^\alpha(\zeta u)\|_{L^2(\Omega \cap X_\varphi; \phi; X)}) \\
+ \|T^\alpha(\zeta u)\|_{L^2(\Omega \cap X_\varphi; \phi; \phi)} \tag{58}
\]
where
\[
C_6 = C_6(\|a_{jk}\|_{C^{\ell+1}(\Omega)}, \|b_j\|_{C^{\ell+1}(\Omega)}, \|b'_j\|_{C^{\ell+1}(\Omega)}, \|b\|_{C^{\ell+1}(\Omega)}, n, \theta, \|\zeta\|_{C^{\ell}(\Omega)}, \|\rho\|_{C^{\ell+1}(\Omega)}).
\]
Applying a small constant/large constant argument and the induction hypothesis (50) for \(|\alpha| = \ell + 1\), we can finish the proof of (50) for \(|\alpha| = \ell + 1\).

We now need to lift the restriction that \(T^\alpha\) is tangential. Without loss of generality, we may assume that \(|\alpha| = \ell + 2\) and \(T^\alpha = T^\beta T^n\) where \(T^\beta\) is tangential. We will show that there exists a constant \(C_7\) so that
\[
\|T^\alpha u\|_{L^2(\Omega \cap X_\varphi; \phi)} \leq C_7 (\|f\|_{W^{1,2}(\Omega \cap X_\varphi; \phi; X)} + \|u\|_{L^2(\Omega \cap X_\varphi; \phi; X)}) \tag{58}
\]
where
\[
C_7 = C_7(\|a_{jk}\|_{C^{\ell+1}(\Omega)}, \|b_j\|_{C^{\ell+1}(\Omega)}, \|b'_j\|_{C^{\ell+1}(\Omega)}, \|b\|_{C^{\ell+1}(\Omega)}, n, \|\zeta\|_{C^{\ell+1}(\Omega)}, \|\rho\|_{C^{\ell+1}(\Omega)}).
\]
The $\gamma = 0$ case follows from (50). Similarly, since the commutator $[T_j, T_n]$ is a first-order operator, we can write the $\gamma = 1$ case as

$$T^\alpha = T_nT^\beta + \text{lower order tangential terms}$$

and the estimate again follows from (50). We prove the $\gamma \geq 2$ case with an induction argument. Assume now that (58) holds for $\gamma = 0, \ldots, J - 1$ with $J \geq 2$. Assume that $|\gamma| = J$. Redefine $\beta$ so that $T^\alpha = T^\beta T_n^2$. Note that $T^\beta$ contains at most $(J - 1)$ occurrences of $T_n$. Since $u \in W^{\ell+2}_{\text{loc}}(\Omega)$ and $Lu = f$ in $\Omega$, we have $T^\beta Lu = T^\beta f$ a.e. in $\Omega$. We can write

$$T^\beta f = T^\beta Lu = a_{nn}T^\alpha u + \text{terms involving } T_n \text{ at most } J - 1 \text{ times and of order at most } \ell + 2.$$

Since $a_{nn} \geq \theta > 0$, by the induction hypothesis and (58), it follows that

$$\|T^\gamma u\|_{L^2(\nu \cap \Omega, \varphi)} \leq C_8\left(\|f\|_{W^{\ell+2}(\nu \cap \Omega, \varphi)} + \|u\|_{L^2(\nu \cap \Omega, \varphi)}\right).$$

where $C_8 = C_8(|a_{jk}|_{C^{k+1}(\Omega)}, |b_1|_{C^{k+1}((\Omega)}, |b_2|_{C^{k+1}(\Omega)}, |b_3|_{C^{k+1}(\Omega)}, n, \theta, \|\rho\|_{C^{k+1}(\Omega)})$.

Since the constant $C_8$ does not depend on the size of $V$, the estimate holds for all $V$ and hence

$$\|u\|_{W^{\ell+2}(\nu, \varphi)} \leq C_8\left(\|f\|_{W^{\ell+2}(\nu, \varphi)} + \|u\|_{W^1(\nu, \varphi)}\right).$$

\vspace{1cm}

8. TRACES OF $L$-HARMONIC FUNCTIONS

In this section, we wish to show that $L$-harmonic functions (i.e., functions $u$ so that $Lu = 0$) have unique boundary values in $W^{s-1/2,2}(b\Omega, \varphi; T)$ when $u \in W^{s,2}(\Omega, \varphi; X)$ and $s \geq 0$.

We first establish a simple but easily applicable uniqueness condition by proving Lemma 2.16.

**Proof Lemma 2.16.** Since $Lu = 0$ and $u \in W^{1,2}_0(\Omega, \varphi; X)$, it follows that

$$\text{Re} \mathcal{D}(u, u) = \text{Re}(u, Lu, \varphi) = 0.$$

Since $\text{Re} \mathcal{D}(u, u) \geq c\|\nabla_X u\|_{L^2(\Omega, \varphi)}$, it follows that $\nabla_X u = 0$. By Corollary 3.5, $\|\nabla u\|_{L^2(\Omega, \varphi)} \lesssim \|\nabla_X u\|_{L^2(\Omega, \varphi)} = 0$. Therefore, $\nabla u = 0$ and $u$ is constant (on each component of $\Omega$). Since $u|_M = 0$, $u \equiv 0$. \qed

8.1. The $s \geq 2$ case in Theorem 2.18

**Lemma 8.1.** Let $\Omega \subset \mathbb{R}^n$ be a domain that satisfies (HI)-(HVI) with $m \geq 3$. Let $L$ be a strongly elliptic operator that has a Dirichlet form $\mathcal{D}$ that satisfies (7) for all $u \in W^{1,2}_0(\Omega, \varphi; X)$. Let $2 \leq k \leq m - 1$ be an integer. Then there is a one-to-one correspondence between $B^{k-1/2,2}(M, \varphi; T)$ and $W^{k,2}(\Omega, \varphi; X) \cap \ker L$ with norm equivalence.

**Proof.** Assume that $U \in W^{k,2}(\Omega, \varphi; X)$ and $LU = 0$. Since $U \in W^{k,2}(\Omega, \varphi; X)$, Theorem 2.1 implies that $\text{Tr} U \in B^{k-1/2,2}(M, \varphi; T)$. Since $L$ satisfies the hypotheses of Lemma 2.16, $U$ is the unique function in $W^{k,2}(\Omega, \varphi; X) \cap \ker L$ with boundary value $\text{Tr} U$.

Now assume that $u \in B^{k-1/2,2}(M, \varphi; T)$. By Theorem 2.1, there exists a function $\tilde{U} \in W^{k,2}(\Omega, \varphi; X)$ with boundary value $u$ and

$$\|\tilde{U}\|_{W^{k,2}(\Omega, \varphi; X)} \leq C\|u\|_{B^{k-1/2,2}(M, \varphi; T)}.$$
Since \( k \geq 2 \), \( L\tilde{U} \in W^{k-2,2}(\Omega, \varphi; X) \). By Theorem 2.11, there exists \( U_0 \in W^{1,2}_0(\Omega, \varphi; X) \) so that \( \mathfrak{D}(v, U_0) = (v, L\tilde{U})_\varphi \) for all \( v \in W^{1,2}_0(\Omega, \varphi; X) \). Since \( L \) satisfies (7), \( U_0 \) is unique. By Theorem 2.15 \( U_0 \in W^{k,2}(\Omega, \varphi; X) \). Moreover, the mapping

\[
L : W^{k,2}(\Omega, \varphi; X) \cap W^{1,2}_0(\Omega, \varphi; X) \to W^{k-2,2}(\Omega, \varphi; X)
\]
is a bijective linear mapping, so the Open Mapping Theorem (or, more directly, its corollary the Bounded Inverse Theorem) proves that its inverse is continuous, i.e.,

\[
\|U_0\|_{W^{k,2}(\Omega, \varphi; X)} \leq C\|L\tilde{U}\|_{W^{k-2,2}(\Omega, \varphi; X)} \leq C\|\tilde{U}\|_{W^{k,2}(\Omega, \varphi; X)} \leq C\|u\|_{B^{k-1/2,2,2}(M, \varphi; T)}.
\]

Let \( U = \tilde{U} - U_0 \). Then \( LU = 0 \) and \( \text{Tr} U = \text{Tr} \tilde{U} = u \) and

\[
\|U\|_{W^{k,2}(\Omega, \varphi; X)} \leq C\|u\|_{B^{k-1/2,2,2}(M, \varphi; T)}.
\]

\[
\square
\]

8.2. The case \( s = 1 \) in Theorem 2.18. We use the arguments in [Tay96] for the following.

**Theorem 8.2.** Let \( L \) be a strongly elliptic operator and \( S \) be a first order operator with bounded coefficients. Set

\[
Au = Lu + Su.
\]

There exists a constant \( C > 0 \) so that for all \( u \in W^{1,2}_0(\Omega, \varphi; X) \),

\[
\|u\|_{W^{1,2}(\Omega, \varphi; X)}^2 \leq C\|Au\|_{W^{-1,2}(\Omega, \varphi; X)}^2 + C\|u\|_{L^2(\Omega, \varphi)}^2.
\]

**Proof.** Observe that for any \( \varepsilon > 0 \)

\[
|(u, Su)_\varphi| \leq C\|u\|_{L^2(\Omega, \varphi)}\|u\|_{W^{1,2}(\Omega, \varphi; X)} \leq \frac{C}{2} \left( \|u\|_{W^{1,2}(\Omega, \varphi; X)}^2 + \varepsilon \|u\|_{L^2(\Omega, \varphi)}^2 \right).
\]

Therefore,

\[
\text{Re}(u, Au)_\varphi \geq \frac{\theta}{2} \|u\|_{W^{1,2}(\Omega, \varphi; X)}^2 - C'\|u\|_{L^2(\Omega, \varphi)}^2 \text{ for all } u \in W^{1,2}_0(\Omega, \varphi; X),
\]

so

\[
\|u\|_{W^{1,2}(\Omega, \varphi; X)}^2 \leq C\text{Re}(u, Au)_\varphi + C'\|u\|_{L^2(\Omega, \varphi)}^2.
\]

Also,

\[
\text{Re}(u, Au)_\varphi \leq C\|Au\|_{W^{-1,2}(\Omega, \varphi; X)}\|u\|_{W^{1,2}(\Omega, \varphi; X)} \leq \frac{C}{2}\|u\|_{W^{1,2}(\Omega, \varphi; X)}^2 + \frac{C}{2\varepsilon} \|Au\|_{W^{-1,2}(\Omega, \varphi; X)}^2.
\]

Putting our inequalities together and choosing \( \varepsilon > 0 \) small enough so that we can absorb the \( C\varepsilon\|u\|_{W^{1,2}(\Omega, \varphi; X)}^2 \) term, we see that

\[
\|u\|_{W^{1,2}(\Omega, \varphi; X)}^2 \leq C\|Au\|_{W^{-1,2}(\Omega, \varphi; X)}^2 + C\|u\|_{L^2(\Omega, \varphi)}^2.
\]

\[
\square
\]

We next show that \( L : W^{1,2}_0(\Omega, \varphi; X) \to W^{-1,2}(\Omega, \varphi; X) \) is continuous, injective and has a bounded inverse.

We first assume that \( L \) gives rise to a strictly elliptic Dirichlet form over \( W^{1,2}_0(\Omega, \varphi; X) \). Then

\[
\text{Re}(v, Lu)_\varphi = \text{Re} \mathfrak{D}(v, u) \geq C\|v\|_{W^{1,2}(\Omega, \varphi; X)}\|u\|_{W^{1,2}(\Omega, \varphi; X)}.
\]

Consequently, \( L : W^{1,2}_0(\Omega, \varphi; X) \to W^{-1,2}(\Omega, \varphi; X) \) and

\[
\|Lu\|_{W^{-1,2}(\Omega, \varphi; X)} \geq C\|u\|_{W^{1,2}(\Omega, \varphi; X)}.
\]
Therefore, \( L : W^{1,2}_0(\Omega, \varphi; X) \to W^{-1,2}(\Omega, \varphi; X) \) has closed range. If \( L \) is not surjective, there exists a nonzero \( v^* \in W^{-1,2}(\Omega, \varphi; X) \) so that \( v^* \perp \text{Range}(L) \). By the Riesz Representation Theorem, we can therefore choose \( v \in W^{1,2}_0(\Omega, \varphi; X) \) so that \( v^*(v) \neq 0 \) and \( w^*(v) = 0 \) for \( w^* \in \text{Range}(L) \). In this case

\[ 0 = (v, Lu)_\varphi \text{ for all } u \in W^{1,2}_0(\Omega, \varphi; X). \]

Setting \( u = v \) forces \( v = 0 \) (and hence \( v^* = 0 \) as well). Therefore, \( L \) is surjective. We also know that \( L \) is injective as a consequence of Lemma 2.16. Consequently, the inverse to \( L \) exists, call it \( G \). Then \( G : W^{-1,2}(\Omega, \varphi; X) \to W^{1,2}_0(\Omega, \varphi; X) \). As \( L^2(\Omega, \varphi) \hookrightarrow W^{-1,2}(\Omega, \varphi; X) \) compactly, \( G : L^2(\Omega, \varphi) \to W^{1,2}_0(\Omega, \varphi; X) \) compactly.

We now investigate the equation

\[ Au = f \]

where \( f \in W^{-1,2}(\Omega, \varphi; X) \), \( u \in W^{1,2}_0(\Omega, \varphi; X) \), and \( A = L + S \) as in Theorem 8.2. We continue to assume that \( L \) has a strictly elliptic Dirichlet form over \( W^{1,2}_0(\Omega, \varphi; X) \). If \( u \in W^{1,2}_0(\Omega, \varphi; X) \), then there exists \( v \in W^{-1,2}(\Omega, \varphi; X) \) so that

\[ u = Gv \]

If \( Au = f \), then

\[ f = AGv = (L + S)Gv = (I + SG)v. \]

We know that \( SG : W^{-1,2}(\Omega, \varphi; X) \to L^2(\Omega, \varphi) \) and \( L^2(\Omega, \varphi) \hookrightarrow W^{-1,2}(\Omega, \varphi; X) \) is compact. Therefore \( I + SG : W^{-1,2}(\Omega, \varphi; X) \to W^{-1,2}(\Omega, \varphi; X) \) is a compact perturbation of the identity. The Fredholm alternative implies that the map \( I + SG \) is therefore surjective if and only if it is injective. Lemma 2.16 supplies a condition that guarantees injectivity.

Since the difference between a strongly elliptic operator and a strongly elliptic operator that gives rise to a strictly elliptic Dirichlet form is the addition of a multiple of the identity, the case of relevance is \( S = \lambda I \) for some \( \lambda \in \mathbb{R} \). If \( Lu = v \neq 0 \), then

\[ (L + \lambda I)u = (L + \lambda I)Gv = (I + \lambda G)v \neq 0 \]

since \( I + \lambda G \) is injective. We have therefore proved the following.

**Proposition 8.3.** Let \( L \) be a strongly elliptic operator that has a Dirichlet form that satisfies \( (7) \). Then the map

\[ L : W^{1,2}_0(\Omega, \varphi; X) \to W^{-1,2}(\Omega, \varphi; X) \]

is an isomorphism with norm equivalence.

With regard to the norm equivalence, it follows immediately that \( \|Lu\|_{W^{-1,2}(\Omega, \varphi; X)} \leq \|u\|_{W^{1,2}(\Omega, \varphi; X)} \). The reverse inequality follows from the Bounded Inverse Theorem. We are now in a position to improve Lemma 8.1.

**Lemma 8.4.** Let \( \Omega \subset \mathbb{R}^n \) be a domain that satisfies \( (HI)-(HVI) \) for \( m = 2 \). Let \( L \) be a strongly elliptic operator that has a Dirichlet form \( \mathcal{D} \) which satisfies \( (7) \). There is a one-to-one correspondence between \( B^{1/2,2,2}(M, \varphi; T) \) and \( W^{1,2}(\Omega, \varphi; X) \cap \ker L \) with norm equivalence.

**Proof.** We already know that \( \text{Tr} : W^{1,2}(\Omega, \varphi; X) \to B^{1/2,2,2}(M, \varphi; T) \) is continuous. Now let \( f \in B^{1/2,2,2}(M, \varphi; T) \). By Theorem 2.1, there exists \( F \in W^{1,2}(\Omega, \varphi; X) \) so that \( \text{Tr} F = f \) and

\[ \|F\|_{W^{1,2}(\Omega, \varphi; X)} \leq C\|f\|_{B^{1/2,2,2}(M, \varphi; T)}. \]
Solving \(Lu = 0\) in \(\Omega\) and \(\text{Tr } u = f\) is equivalent to finding \(v \in W^{1,2}_0(\Omega, \varphi; X)\) where \(Lv = -LF\) because we could then set \(u = F + v\) and it would follow from Proposition 8.3 that
\[
\|u\|_{W^{1,2}(\Omega, \varphi; X)} \leq \|F\|_{W^{1,2}(\Omega, \varphi; X)} + \|v\|_{W^{1,2}(\Omega, \varphi; X)} \leq C \left( \|f\|_{B^{1/2,2}(\Omega, \varphi; X)} + \|Lv\|_{W^{-1,2}(\Omega, \varphi; X)} \right) 
\leq C \|f\|_{B^{1/2,2}(\Omega, \varphi; X)}.
\]
However, \(-LF \in W^{-1,2}(\Omega, \varphi; X)\) so such a \(v\) exists by Proposition 8.3 \(\square\)

Combining our results, we can prove Theorem 2.18.

**Proof of Theorem 2.18.** Let \(f \in W^{s-2,2}(\Omega, \varphi; X)\) and \(g \in W^{s-1/2,2}(b\Omega, \varphi; T)\). By Theorem 2.11 and Theorem 2.15 there exists a unique \(u_1 \in W^{s,2}(\Omega, \varphi; X) \cap W^{1,2}_0(\Omega, \varphi; X)\) so that \(Lu_1 = f\). If \(G : W^{s-2,2}(\Omega, \varphi; X) \to W^{s,2}(\Omega, \varphi; X) \cap W^{1,2}_0(\Omega, \varphi; X)\) is the inverse to \(L\), then \(G\) is continuous, i.e., there exists a constant \(C\) so that \(\|Gf\|_{W^{s-2,2}(\Omega, \varphi; X)} \leq C \|f\|_{W^{s,2}(\Omega, \varphi; X)}\). Plugging in \(f = Lu_1\), we see that \(\|u_1\|_{W^{s,2}(\Omega, \varphi; X)} \lesssim \|f\|_{W^{s-2,2}(\Omega, \varphi; X)}\). Also, by Lemma 8.4 and Lemma 8.1 there exists a unique \(u_2 \in W^{s,2}(\Omega, \varphi; X)\) so that \(Lu_2 = 0\) and \(\text{Tr } u_2 = g\). Also, \(u_2\) satisfies \(\|u_2\|_{W^{s,2}(\Omega, \varphi; X)} \lesssim \|g\|_{W^{s-1/2}(b\Omega, \varphi; T)}\). Thus, \(u = u_1 + u_2\) is the unique function in \(W^{s,2}(\Omega, \varphi; X)\) so that
\[
\begin{cases}
Lu = f & \text{in } \Omega \\
\text{Tr } u = g & \text{on } b\Omega
\end{cases}
\]
and
\[
\|u\|_{W^{s,2}(\Omega, \varphi; X)} \leq C \left( \|f\|_{W^{s,2}(\Omega, \varphi; X)} + \|g\|_{W^{s-1/2}(b\Omega, \varphi; T)} \right)
\]
for a constant \(C\) independent of \(u\), \(f\), and \(g\).

In the reverse direction, let \(u \in W^{s,2}(\Omega, \varphi; X)\). There exists a unique \(u_1\) so that \(u_1 \in W^{s,2}(\Omega, \varphi; X) \cap W^{1,2}_0(\Omega, \varphi; X)\), \(Lu = Lu_1\), and
\[
\|u_1\|_{W^{s,2}(\Omega, \varphi; X)} \lesssim \|Lu_1\|_{W^{s-2,2}(\Omega, \varphi; X)} \leq \|u\|_{W^{s,2}(\Omega, \varphi; X)}.
\]
If \(u_2 = u - u_1\), then \(u = u_1 + u_2\), \(Lu_2 = 0\), and we have already established that \(\|u_2\|_{W^{s,2}(\Omega, \varphi; X)} \lesssim \|\text{Tr } u_2\|_{W^{s-1/2}(\Omega, \varphi; X)}\). Thus, we have a unique decomposition \(u = u_1 + u_2\) and
\[
\|u\|_{W^{s,2}(\Omega, \varphi; X)} \leq \|u_1\|_{W^{s,2}(\Omega, \varphi; X)} + \|u_2\|_{W^{s,2}(\Omega, \varphi; X)} \lesssim \|Lu_1\|_{W^{s-2,2}(\Omega, \varphi; X)} + \|\text{Tr } u_2\|_{W^{s-1/2}(\Omega, \varphi; X)} 
\lesssim \|u_1\|_{W^{s,2}(\Omega, \varphi; X)} + \|u_2\|_{W^{s,2}(\Omega, \varphi; X)} \lesssim \|u\|_{W^{s,2}(\Omega, \varphi; X)}
\]
where the last inequality uses the fact that \(u_2 = u - u_1\) \(\square\).

8.3. **Proof of Theorem 2.19.** In this subsection, we prove that functions \(f \in L^2(\Omega, \varphi)\) that are \(L\)-harmonic have traces in \(B^{-1/2,2}(\Omega, \varphi; X)\). Our motivation for the trace definition is from [BC]. If we define the operator \(S\) by
\[
S = -\sum_{j,k=1}^{n} a_{jk} \frac{\partial \rho}{\partial x_k} X_j,
\]
then for \(v \in W^{2,2}(\Omega, \varphi; X) \cap W^{1,2}_0(\Omega, \varphi; X)\) and \(\psi \in W^{2,2}(\Omega, \varphi; X)\)
\[
(L^*v, \psi)_{\varphi} = \int_{b\Omega} \text{Tr } Sv \overline{\psi} e^{-\varphi} d\sigma + \mathcal{D}(v, \psi) = \int_{b\Omega} \text{Tr } Sv \overline{\psi} e^{-\varphi} d\sigma + (v, L\psi)_{\varphi}.
\]
If $L\psi = 0$, then

\begin{equation}
(L^* v, \psi) = \int_{b\Omega} T v \overline{\psi} e^{-\varphi} d\sigma.
\end{equation}

Since $v \in W^{2,2}(\Omega, \varphi; X)$, we have $Sv \in W^{1,2}(\Omega, \varphi; X)$ and $T v \in B^{1/2,2,2}(b\Omega, \varphi; T)$.

We would like to show a partial converse to the argument, i.e., that if $\varphi \in B^{1/2,2,2}(\Omega, \varphi; X)$, then $\varphi = T v$ for some $v \in W^{1,2}(\Omega, \varphi; X) \cap W^{2,2}(\Omega, \varphi; X)$.

Our goal is to show that if $f \in L^2(\Omega, \varphi)$ and $Lf = 0$, then there exists a well-defined $g \in B^{-1/2,2,2}(b\Omega, \varphi; T)$ so that $Tr f = g$. Equation (59) is the key. Motivated by Theorem 2.4, we investigate operators $L$ of the form in [3]. To define an element $g \in B^{-1/2,2,2}(b\Omega, \varphi; T)$, it suffices to determine the action of $g$ on elements $\psi \in B^{1/2,2,2}(b\Omega, \varphi; T)$. Let $f \in L^2(\Omega, \varphi)$ satisfy $Lf = 0$, and let $\psi \in B^{1/2,2,2}(b\Omega, \varphi; T)$. From Theorem 2.4, there exists a (nonunique) element $v \in W^{2,2}(\Omega, \varphi; X) \cap W^{1,2}(\Omega, \varphi; X)$ so that

\[ \frac{\partial v}{\partial \nu} = \psi \text{ on } b\Omega. \]

Define $Tr f$ by

\begin{equation}
(Tr f, \psi) := (L^* v, f)\varphi.
\end{equation}

Observe that

\[ |(L^* v, f)\varphi| \lesssim \|v\|_{W^{2,2}(\Omega, \varphi; X)} \|f\|_{L^2(\Omega, \varphi)} \lesssim \|\psi\|_{B^{1/2,2,2}(b\Omega, \varphi; T)} \|f\|_{L^2(\Omega, \varphi)}. \]

That $Tr f$ is well-defined follows from approximating $f$ by functions in $W^{2,2}(\Omega, \varphi; X)$ and following the argument that leads to (59). In particular, if $\eta_j \to f$ in $L^2(\Omega, \varphi)$ and $\eta_j \in W^{2,2}(\Omega, \varphi; X)$, then $L\eta_j \to Lf = 0$ in $W^{-2,2}(\Omega, \varphi; X)$. We need to show that $L\eta_j \to 0$ in $L^2(\Omega, \varphi)$ so we can achieve (59). $C^\infty_c(\Omega)$ is dense in $L^2(\Omega, \varphi)$, and if $\zeta \in C^\infty_c(\Omega)$, then

\[ (L\eta_j, \zeta)\varphi = (\eta_j, L^* \zeta)\varphi \longrightarrow (f, L^* \zeta)\varphi = (Lf, \zeta)\varphi \]

where the last equality follows from the pairing of $f$ as a distribution against the test function $\zeta$.

Thus $Tr f$ is a well-defined element of $B^{-1/2,2,2}(b\Omega, \varphi; T)$. The use of the name trace is appropriate because if $f \in L^2(\Omega, \varphi) \cap \ker L$ and has enough regularity so that (59) applies, then the two definitions of $Tr f$ agree. Thus we have proven Theorem 2.19.

**Appendix A. Background on interpolation – the real method**

A.1. **The Bochner Integral.** Our discussion of the real interpolation method closely follows [AF03].

A.2. **$L^q$-spaces.** Let $X$ be $\mathbb{R}$ or $\mathbb{C}$. For $1 \leq q \leq \infty$, let $L^q(a, b; d\mu(t))$ be the space of functions $f : (a, b) \to X$ such that the norm

\[ \|f; L^q(a, b; d\mu(t), X)\| = \begin{cases} \left( \int_a^b \|f(t)\|_X^q \, d\mu(t) \right)^{1/q} & 1 \leq q < \infty \\ \text{ess sup}_{a < t < b} \|f(t)\|_X & q = \infty \end{cases} \]

is finite.

We focus on the special case where $d\mu = dt/t$. We denote $L^q(a, b; d\mu) = L^q_t$. 


Let $X_0$ and $X_1$ be two Banach spaces that are continuously imbedded on a Hausdorff topological vector space $X$ and whose intersection is nontrivial. Such a pair of Banach spaces $\{X_0, X_1\}$ is called an **interpolation pair**, and we now turn to the construction of Banach spaces $X$ suitably intermediate between $X_0$ and $X_1$. It is often the case that $X_1 \hookrightarrow X_0$, e.g., $X_0 = L^p(\Omega, \varphi)$ and $X_1 = W^{m,p}(\Omega, \varphi; X)$.

Let $\| \cdot \|_{X_j}$ denote the norm in $X_j$, $j = 0, 1$. The spaces $X_0 \cap X_1$ and $X_0 + X_1 = \{u = u_0 + u_1 : u_0 \in X_0, u_1 \in X_1\}$ are Banach spaces with norms

$$\|u\|_{X_0 \cap X_1} = \max\{\|u_0\|_{X_0}, \|u_1\|_{X_1}\}$$

and

$$\|u\|_{X_0 + X_1} = \inf\{\|u_0\|_{X_0} + \|u_1\|_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\},$$

respectively. Note that $X_0 \cap X_1 \hookrightarrow X_j \hookrightarrow X_0 + X_1$. We say that a Banach space $X$ is **intermediate** between $X_0$ and $X_1$ if

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1.$$  

### A.3. The $J$ and $K$ norms.

For a fixed $t > 0$, set

$$J(t; u) = \max\{\|u\|_{X_0}, t\|u\|_{X_1}\}$$

and

$$K(t; u) = \inf\{\|u_0\|_{X_0} + t\|u_1\|_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}.$$  

**Definition A.1** (The $K$-method). If $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$, then we define

$$(X_0, X_1)_{\theta, q; K} = \{u \in X_0 + X_1 : t^{-\theta}K(t; u) \in L^q_t = L^q(0, \infty; dt/t)\}.$$  

In fact,

**Theorem A.2** (Theorem 7.10, [AF03]). If and only if either $1 \leq q < \infty$ and $0 \leq \theta < 1$ or $q = \infty$ and $0 \leq \theta \leq 1$, then the space $(X_0, X_1)_{\theta, q; K}$ is a nontrivial Banach space with norm

$$\|u\|_{\theta; q; K} = \|t^{-\theta}K(t; u) : L^q_t\|.$$  

Furthermore,

$$\|u\|_{X_0 + X_1} \leq \frac{\|u\|_{\theta; q; K}}{t^{-\theta} \min\{1, t\}; L^q_t} \leq \|u\|_{X_0 \cap X_1},$$

and there hold the embeddings

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta, q; K} \hookrightarrow X_0 + X_1,$$

and $(X_0, X_1)_{\theta, q; K}$ is an intermediate space between $X_0$ and $X_1$.

**Definition A.3** (The $J$-method). If $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$, then we define

$$(X_0, X_1)_{\theta, q; J} = \left\{ u \in X_0 + X_1 : u = \int_0^{\infty} f(t) \frac{dt}{t}, f \in L^1(0, \infty; dt/t, X_0 + X_1) \right\}$$

having values in $X_0 \cap X_1$ and such that $t^{-\theta}J(t; f) \in L^q_t = L^q(0, \infty; dt/t)$.

In fact,
Theorem A.4 (Theorem 7.13 [AF03]). If either \(1 \leq q < \infty\) and \(0 < \theta < 1\) or \(q = \infty\) and \(0 \leq \theta \leq 1\), then the space \((X_0, X_1)_{\theta,q,J}\) is a Banach space with norm
\[
\|u\|_{\theta,q,J} = \inf_{f \in S(u)} \|t^{-\theta}J(t; f(t)) : L_\theta^q\|
\]
where
\[
S(u) = \left\{ f \in L^1(0, \infty; dt/t, X_0 + X_1) : u = \int_0^\infty f(t) \frac{dt}{t} \right\}.
\]
Furthermore,
\[
\|u\|_{X_0 + X_1} \leq \|t^{-\theta} \min\{1, t\} ; L_\theta^q\| \|u\|_{\theta,q,J} \leq \|u\|_{X_0 \oplus X_1},
\]
where \(\frac{1}{q} + \frac{1}{q'} = 1\). Consequently, there hold the embeddings
\[
X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,q,J} \hookrightarrow X_0 + X_1,
\]
and \((X_0, X_1)_{\theta,q,J}\) is an intermediate space between \(X_0\) and \(X_1\).

It is very useful to have a discrete version of the \(J\) method.

Theorem A.5 (Theorem 7.15, [AF03]). An element \(u \in X_0 + X_1\) belongs to \((X_0, X_1)_{\theta,q,J}\) if and only if \(u = \sum_{j=-\infty}^{\infty} u_j\) where the series converges in \(X_0 + X_1\) and the sequence \(\{2^{-j\theta}J(2^j; u_j)\} \in \ell^q\). In this case,
\[
\inf \left\{ \|2^{-j\theta}J(2^j; u_j) : \ell^q\| : u = \sum_{j=-\infty}^{\infty} u_j \right\}
\]
is a norm on \((X_0, X_1)_{\theta,q,J}\) equivalent to \(\|u\|_{\theta,q,J}\).

If \(0 < \theta < 1\), the \(J\) and \(K\) interpolations are equivalent. In fact,

Theorem A.6 (Theorem 7.16, [AF03]). If \(0 < \theta < 1\) and \(1 \leq q \leq \infty\), then
\[
(X_0, X_1)_{\theta,q,J} = (X_0, X_1)_{\theta,q,K},
\]
the two spaces having equivalent norms.

A.4. An important class of intermediate spaces.

Definition A.7. Let \(\{X_0, X_1\}\) be an interpolation pair of Banach spaces. We say that \(X \in \mathcal{J}(\theta; X_0, X_1)\) if there exist constants \(C_1, C_2 > 0\) so that for all \(u \in X\) and \(t > 0\),
\[
C_1 K(t; u) \leq t^\theta \|u\|_X \leq C_2 J(t; u)
\]

Lemma A.8. Let \(0 \leq \theta \leq 1\) and let \(X\) be an intermediate space between \(X_0\) and \(X_1\). Then \(X \in \mathcal{J}(\theta; X_0, X_1)\) if and only if \((X_0, X_1)_{\theta,1,J} \hookrightarrow X \hookrightarrow (X_0, X_1)_{\theta,\infty,K}\).

Corollary A.9 (Corollary 7.20, [AF03]). If \(0 < \theta < 1\) and \(1 \leq q \leq \infty\), then
\[
(X_0, X_1)_{\theta,q,J} = (X_0, X_1)_{\theta,q,K} \in \mathcal{J}(\theta; X_0, X_1).
\]
Moreover, \(X_0 \in \mathcal{J}(0; X_0, X_1)\) and \(X_1 \in \mathcal{J}(1; X_0, X_1)\).

The importance of the class \(\mathcal{J}(\theta; X_0, X_1)\) is made clear from the following theorem (which is part of Theorem 7.21, [AF03]).
Theorem A.10 (The Reiteration Theorem). Let $0 \leq \theta_0 < \theta_1 \leq 1$ and let $X_{\theta_0}$ and $X_{\theta_1}$ be intermediate spaces between $X_0$ and $X_1$. For $0 \leq \lambda \leq 1$, let $\theta = (1 - \lambda)\theta_0 + \lambda\theta_1$. If $X_{\theta_i} \in \mathcal{S}(\theta_i; X_0, X_1)$ for $i = 0, 1$ and if either $0 < \lambda < 1$ and $1 \leq q \leq \infty$ or $0 \leq \lambda \leq 1$ and $q = \infty$, then
\[
(X_0, X_1)_{\theta,q;J} = (X_{\theta_0}, X_{\theta_1})_{\lambda,q;J} = (X_{\theta_0}, X_{\theta_1})_{\lambda,q;K} = (X_0, X_1)_{\theta,q;K}.
\]

A.5. Interpolation Spaces. Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs of Banach spaces. Let $T \in \mathcal{B}(X_0 + X_1, Y_0 + Y_1)$ satisfy $T \in \mathcal{B}(X_i, Y_i)$, $i = 1, 2$, with norm at most $M_i$. That is,
\[
\|Tu_i\|_{Y_i} \leq M_i\|u_i\|_{X_i}
\]
for all $u_i \in X_i$, $i = 1, 2$.

If $X$ and $Y$ are intermediate spaces for $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$, respectively, then we call $X$ and $Y$ interpolation spaces of type $\theta$ for $P$ and $Q$, where $0 \leq \theta \leq 1$ if every such linear operator $T$ maps $X$ to $Y$ with norm $M$ satisfying
\[
M \leq CM_0^{1-\theta}M_1^\theta
\]
where $C$ is independent of $T$ and $C \geq 1$. If we can take $C = 1$ in (61), then we say that the interpolation spaces $X$ and $Y$ are exact.

Theorem A.11 (The Exact Interpolation Theorem, Theorem 7.23 [AF03]). Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs.

(i) If either $0 < \theta < 1$ and $1 \leq q \leq \infty$ or $0 \leq \theta \leq 1$ and $q = \infty$, then the intermediate spaces $(X_0, X_1)_{\theta,q;K}$ and $(Y_0, Y_1)_{\theta,q;K}$ are exact interpolation spaces of type $\theta$ for $P$ and $Q$.

(ii) If either $0 < \theta < 1$ and $1 \leq q \leq \infty$ or $0 \leq \theta \leq 1$ and $q = \infty$, then the intermediate spaces $(X_0, X_1)_{\theta,q;J}$ and $(Y_0, Y_1)_{\theta,q;J}$ are exact interpolation spaces of type $\theta$ for $P$ and $Q$.

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