SUPPORT VARIETIES OF \((\mathfrak{g}, \mathfrak{t})\)-MODULES OF FINITE TYPE

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1. BRIEF STATEMENT OF RESULTS

Let \(\mathfrak{g}\) be a semisimple Lie algebra over an algebraically closed field \(\mathbb{F}\) of characteristic 0 and \(\mathfrak{t} \subset \mathfrak{g}\) be a reductive in \(\mathfrak{g}\) subalgebra.

**Definition 1.** A \((\mathfrak{g}, \mathfrak{t})\)-module is a \(\mathfrak{g}\)-module which after restriction to \(\mathfrak{t}\) becomes a direct sum of finite-dimensional \(\mathfrak{t}\)-modules.

**Definition 2.** A \((\mathfrak{g}, \mathfrak{t})\)-module is of finite type if it is a \(\mathfrak{t}\)-module of finite type, i.e. has finite-dimensional \(\mathfrak{t}\)-isotypic components.

Let \(Z(\mathfrak{g})\) be the center of the universal enveloping algebra \(U(\mathfrak{g})\). Let \(\chi : Z(\mathfrak{g}) \to \mathbb{F}\) be an algebra homomorphism.

**Definition 3.** We say that a \(\mathfrak{g}\)-module \(M\) affords a central character if for some homomorphism \(\chi : Z(\mathfrak{g}) \to \mathbb{F}\) we have \(zm = \chi(z)m\) for all \(z \in Z(\mathfrak{g})\) and \(m \in M\).

Any simple \(\mathfrak{g}\)-module \(M\) affords a central character \([5]\). Let \(X\) be the variety of all Borel subalgebras of \(\mathfrak{g}\). The category of \(\mathfrak{g}\)-modules which affords a central character \(\chi\) is equivalent to the category of sheaves of \(\mathcal{O}_X\)-quasicoherent modules over the sheaf of twisted differential operators \(\mathcal{D}^\lambda(X)\) for a suitable twist \(\lambda \in H^1(X, \Omega_X^{1,cl})\) \([2]\), where \(\Omega_X^{1,cl}\) is the sheaf of closed holomorphic 1-forms on \(X\). In this category there is a distinguished full subcategory of holonomic sheaves of modules. Informally, holonomic sheaves of modules are \(\mathcal{D}^\lambda(X)\)-modules of minimal growth. The simple holonomic modules \(M\) are in one-to-one correspondence with the pairs \((L, S)\), where \(L\) is an irreducible closed subvariety of \(X\) and \(S\) is a sheaf of \(\mathcal{D}^\lambda(L')\)-modules which is \(\mathcal{O}(L')\)-coherent after restriction to a suitable open subset \(L' \subset L\). Moreover, a coherent holonomic module \(S\) is locally free on \(L'\) and one could think about it as a vector bundle \(S_E\) over \(L'\) with a flat connection. Note that flat local sections of this bundle are not necessarily algebraic.

**Theorem 1.** Let \(M\) be a finitely generated \((\mathfrak{g}, \mathfrak{t})\)-module of finite type which affords a central character. Then \(\text{Ind} M\) is a holonomic \(\mathcal{D}^\lambda(X)\)-module.

We also prove the following theorem (the necessary definitions see in the following section).

**Theorem 2.** Let \(\mathcal{O} \subset \mathfrak{g}^*\) be a nilpotent coadjoint \(G\)-orbit, \(\mathfrak{t}^\perp\) be the annihilator of \(\mathfrak{t}\) in \(\mathfrak{g}^*\), and \(N_{\mathfrak{t}}\mathfrak{g}^*\) be the \(\mathfrak{t}\)-null-cone in \(\mathfrak{g}^*\). Then the irreducible components of \(\mathcal{O} \cap \mathfrak{t}^\perp \cap N_{\mathfrak{t}}\mathfrak{g}^*\) are isotropic subvarieties of \(\mathcal{O}\).

Let \(V_{\mathfrak{g}, \mathfrak{t}}\) be the set of all irreducible components of possible intersections of \(N_K \mathfrak{t}^\perp\) with the \(G\)-orbits in \(N_G \mathfrak{g}^*\). This finite set of subvarieties of \(\mathfrak{g}^*\) determines a finite set \(V_{\mathfrak{g}, \mathfrak{t}}\) of subvarieties of \(T^*X\) and a finite set \(L_{\mathfrak{g}, \mathfrak{t}}\) of subvarieties of \(X\) (see Definition \([3]\) below).

**Theorem 3.** Let \(M\) be a finitely generated \((\mathfrak{g}, \mathfrak{t})\)-module of finite type which affords a central character and \((L, S)\) be the corresponding pair consisting of a variety and a coherent sheaf as before. Then \(L\) is an element of \(L_{\mathfrak{g}, \mathfrak{t}}\).

2. PRELIMINARIES

We work in the category of algebraic varieties over \(\mathbb{F}\). By \(T^*X\) we denote the total space of the cotangent bundle of a smooth variety \(X\) and by \(T^*_xX\) the cotangent space to \(X\) at a point \(x\). By \(N^*_Y/X \subset T^*X|_Y\) we denote the conormal bundle to a smooth subvariety \(Y \subset X\).
2.1. D-modules versus g-modules. Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \) and \( \Sigma \subset \mathfrak{h} \) be the root system of \( \mathfrak{g} \) and \( \Sigma_+ \) be a set of positive roots. Denote by \( \mathfrak{h}^* \) the set of weights \( \lambda \) such that \( \alpha^\vee(\lambda) \) is not a strictly positive integer for any positive root \( \alpha^\vee \) of the dual root system \( \Sigma^\vee \subset \mathfrak{h}^* \).

For a fixed \( \lambda \) we denote by \( \mathcal{D}^\lambda(X) \) the sheaf of twisted differential operators on \( X \) and by \( \mathcal{D}^\lambda(X) \) its space of global sections. The algebras \( \mathcal{D}^\lambda(X) \) and \( \mathcal{D}^{\mu}(X) \) are naturally identified if \( \lambda \) and \( \mu \) lie in one shifted orbit of Weyl group \([8]\). Moreover, any such orbit intersects \( \mathfrak{h}^* \) \([8]\). If \( \lambda \in \mathfrak{h}^* \) the surjective homomorphism
\[
\tau : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}^\lambda(X)
\]
identifies the category of quasicoherent \( \mathcal{D}(X) \)-modules and the category of \( \mathfrak{g} \)-modules affording the central character \( \chi = \chi_\lambda \). A. Beilinson and J. Bernstein have proved that both above categories are equivalent to the category of \( \mathcal{D}^\lambda(X) \)-modules:
\[
\text{Res: } \mathcal{D}^\lambda(X) \text{-mod} \to \mathfrak{g} \text{-mod}^X \quad \text{Ind: } \mathcal{D}^\lambda(X) \text{-mod} \leftarrow \mathfrak{g} \text{-mod}^X
\]
\[
\mathcal{F} \to \Gamma(X, \mathcal{F}) \quad M \otimes (1 \otimes \tau) \mathcal{D}(X) \leftarrow M
\]
In general Res and Ind identify the category \( \mathfrak{g} \text{-mod}^X \) with a certain quotient of the category \( \mathcal{D}^\lambda(X) \text{-mod} \). For more detailed exposition of the topic see for example \([8]\).

2.2. Three faces of the support variety. The algebra \( \mathcal{U}(\mathfrak{g}) \) has a natural filtration such that \( \text{gr} \mathcal{U}(\mathfrak{g}) = \mathfrak{g}(\mathfrak{g}) \). The filtration on \( \mathcal{U}(\mathfrak{g}) \) induces a filtration on any finitely generated \( \mathfrak{g} \)-module \( M \). We denote the support of \( \text{gr} \mathcal{U}(\mathfrak{g}) \) by \( \text{gr} \mathcal{U}(\mathfrak{g}) = \mathfrak{g}^\ast \). The surjective homomorphism \( \tau: \mathcal{U}(\mathfrak{g}) \to \mathcal{D}^\lambda(X) \) induces a filtration on any finitely generated \( \mathfrak{g}(\mathfrak{g}) \)-module.

Definition 5. The singular support \( \mathcal{V}(\mathfrak{g}) \) of a simple \( \mathfrak{g} \)-module \( M \) is the cycle \( \mathcal{V}(\text{IndM}) \) in \( T^*X \).

Definition 4. The support variety \( \mathcal{L}(M) \) of a simple \( \mathfrak{g} \)-module \( M \) is the projection of \( \mathcal{V}(\mathfrak{g}) \) to \( X \).

Let \( X \) be a \( G \)-variety for a reductive group \( G \) with Lie algebra \( \mathfrak{g} \). The map \( \phi : T^*X \times \mathfrak{g} \to \mathbb{F}((l, x, g) \to l(gx), x \in X, l \in T^*_X, g \in \mathfrak{g}) \) determines a map \( \phi_X : T^*X \to \mathfrak{g}^* \) called the moment map. D.Barlet and M.Kashiwara \([1]\), have proved that \( \mathcal{V}(M) = \phi_X(\mathcal{V}(\text{IndM})) \). Therefore we have a diagram
\[
\begin{array}{ccc}
\mathcal{V}(M) \subset T^*X & \to & L(M) \subset X \\
\phi_X & & pr \\
V(M) \subset \mathfrak{g}^* & & \end{array}
\]

Lemma 1 (\([1]\)). Let \( K \) be a reductive algebraic group with a Lie algebra \( \mathfrak{k} \) and let \( X \) be an affine \( K \)-variety. Then \( \mathcal{F}[X] \) is a \( \mathfrak{k} \)-module of finite type if and only if \( X \) contains finitely many closed \( K \)-orbits. In this case any irreducible component of \( X \) contains precisely one closed \( K \)-orbit.

Lemma 2. A finitely generated \( (\mathfrak{g}, \mathfrak{k}) \)-module \( M \) is of finite type if and only if its associated variety \( \mathcal{V}(M) \) has finitely many closed \( \mathfrak{k} \)-orbits. In this case the set of closed orbits consists just of the zero orbit.

Proof. Let \( J_M \) be the annihilator of \( \mathcal{V}(M) \) in \( S(\mathfrak{g}) \). Consider the \( S(\mathfrak{g}) \)-modules
\[
J_M^{-1}\{0\} := \{m \in \text{gr} M \mid j_1...j_im = 0 \text{ for all } j_1,..., j_i \in J_M \}.
\]
One can easily see that these modules form an ascending filtration of \( \text{gr} M \) such that \( \bigcup_{i=0}^{\infty} J_M^{-1}\{0\} = \text{gr} M \). Since \( S(\mathfrak{g}) \) is a Noetherian ring, the filtration stabilizes, i.e. \( J_M^{-1}\{0\} = \text{gr} M \) for some \( i \). By \( \text{gr} M \) we denote the corresponding graded object. By definition, \( \text{gr} M \) is an \( S(\mathfrak{g}) \)-\( J_M \)-module. Suppose that \( f\text{gr} M = 0 \) for some \( f \in S(\mathfrak{g}) \). Then \( f\text{gr} M = 0 \) and hence \( f \in J_M \). This proves that the annihilator of \( \text{gr} M \) in \( S(\mathfrak{g})/J_M \) equals zero.

Suppose \( \mathcal{V}(M) \) has a unique closed \( \mathfrak{k} \)-orbit. Let \( M_0 \) be a \( \mathfrak{k} \)-stable space of generators of \( \text{gr} M \). Then there is a surjective homomorphism \( \mathcal{M}_0 \otimes S(\mathfrak{g})/J_M \to \text{gr} M \). Since \( \mathcal{V}(M) \) has finitely many closed \( \mathfrak{k} \)-orbits, \( \mathcal{M}_0 \otimes S(\mathfrak{g})/J_M \) is a \( \mathfrak{g} \)-module of finite type. Therefore \( \text{gr} M \) is of finite type, which implies that \( M \) is a \( (\mathfrak{g}, \mathfrak{k}) \)-module of finite type.

Assume now that \( M \) is of finite \( \mathfrak{k} \)-type. Set
\[
\text{Rad} M = \{m \in \text{gr} M \mid \text{there exists } f \in S(\mathfrak{g})/J_M \text{ such that } fm = 0 \text{ and } f \neq 0\}.
\]
Then $\text{Rad} \, M$ is a proper $\mathfrak{t}$-stable submodule of $\mathfrak{g} \mathfrak{g}M$. Therefore there exists a finite-dimensional $\mathfrak{t}$-subspace $M_0 \subset \mathfrak{g} \mathfrak{g}M$ such that $M_0 \cap \text{Rad} \, M = 0$. The homomorphism $M_0 \otimes_{\mathfrak{g}} (S(\mathfrak{g})/J_M) \rightarrow \mathfrak{g} \mathfrak{g}M$ induces an injective homomorphism $S(\mathfrak{g})/J_M \rightarrow M_0 \otimes_{\mathfrak{g}} \mathfrak{g} \mathfrak{g}M$. Therefore $S(\mathfrak{g})/J_M$ is of finite $\mathfrak{t}$-type and $V(M)$ has only finitely many closed $\mathfrak{t}$-orbits. As $V(M)$ is $\mathbb{F}^g$-stable, any irreducible component of it contains point 0 and this point is a closed $\mathfrak{t}$-orbit.

**Lemma 3** (S. Fernando [3]). If $M$ is a finitely generated $(\mathfrak{g}, \mathfrak{t})$-module then

$$V(M) \subset \mathfrak{t}^\perp := \{ x \in \mathfrak{g}^* : \forall k \in \mathfrak{t} \, x(k) = 0 \}.$$  

2.3. **Hilbert-Mumford criterion.** Let $K$ be a reductive group, $X$ be an affine $K$-variety, $V$ be a $K$-module.

**Theorem 4** (Hilbert-Mumford criterion). The closure of any orbit $\overline{\mathfrak{g} \mathfrak{g}x} \subset X$ contains a unique closed orbit $K \overline{\mathfrak{g} \mathfrak{g}} X$. There exists a homomorphism $\mu : F^* \rightarrow K$ such that $\lim_{t \rightarrow 0} \mu(t)x = \hat{x} \in K \mathfrak{g} \mathfrak{g}x$.

The null-cone $N_0 V := \{ x \in V : 0 \in \overline{\mathfrak{g} \mathfrak{g}x} \}$ is a closed algebraic subvariety of $V$ [10].

**Theorem 5.** Let $x \in V$ be a point. Then $0 \in \mathfrak{g} \mathfrak{g}x$ if and only if there exists a nonzero rational semisimple element $h \in \mathfrak{t}$ such that $x \in V_h^> 0$, where $V_h^> 0$ is the direct sum of $h$-eigenspaces in $V$ with positive eigenvalues.

**Corollary 1** ([10]). There exists a finite set $H$ of rational semisimple elements of $\mathfrak{t}$ such that $N_K V := \cup_{h \in H} K V_h^> 0$, where $K V_h^> 0 := \{ v \in V \mid v = k v_h \text{ for some } k \in K \text{ and } v_h \in V_h^> 0 \}$.

2.4. **Gabber’s theorem.** Let $G$ be the adjoint group of $\mathfrak{g}$ and $M$ be a finitely generated $\mathfrak{g}$-module.

**Definition 6.** Suppose $X$ is a smooth $G$-variety with a closed $G$-invariant nondegenerate 2-form $\omega$. Such a pair $(X, \omega)$ is called a symplectic $G$-variety.

**Definition 7.** Let $(X, \omega)$ be a symplectic $G$-variety. We call a subvariety $Y \subset X$

- isotropic if $\omega|_{\mathfrak{t} \mathfrak{g}Y} = 0$ for a generic point $y \in Y$;
- coisotropic if $\omega|_{(\mathfrak{t} \mathfrak{g}Y)^\perp} = 0$ for a generic point $y \in Y$;
- Lagrangian if $\mathfrak{t} \mathfrak{g}Y = (\mathfrak{t} \mathfrak{g}Y)^\perp$ for a generic point $y \in Y$ or equivalently if it is both isotropic and coisotropic.

**Theorem 6** (O. Gabber [2]). The variety $V(M) \subset N_G \mathfrak{g}^*$ is a coisotropic subvariety of $\mathfrak{g}^*$ with respect to the Kirillov symplectic structure. The variety $V(M)$ is a coisotropic subvariety of $T^* X$ with respect to the natural symplectic structure.

**Definition 8.** A finitely generated $(\mathfrak{g}, \mathfrak{t})$-module $M$ which affords a central character is called holonomic if $V$ is a Lagrangian subvariety of $G \overline{\mathfrak{g} \mathfrak{g}}$ for any irreducible component $\overline{V}$ of $V(M)$.

3. **Proofs**

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{t} \subset \mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra. Let $G$ be the adjoint group of $\mathfrak{g}$ and $K \subset G$ be a connected reductive subgroup such that lie $K = \mathfrak{t}$.

**Theorem 2.** Let $\emptyset \subset \mathfrak{g}^*$ be a nilpotent coadjoint orbit, $\mathfrak{t}^\perp$ be the annihilator of $\mathfrak{t}$ in $\mathfrak{g}^*$, and $N_0 \mathfrak{t}^\perp$ be the $\mathfrak{t}$-null-cone in $\mathfrak{g}^*$. Then the irreducible components of $\emptyset \cap \mathfrak{t}^\perp \cap N_0 \mathfrak{t}^\perp$ are isotorpic subvarieties of $\emptyset$.

**Proof.** As $\mathfrak{g}$ is semisimple, we can freely identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Let $h$ be a nonzero rational semisimple element of $\mathfrak{t}$. By definition $\mathfrak{g}_h^{> 0}$ is the direct sum of all $h$-eigenspaces in $\mathfrak{g}$ with nonnegative eigenvalues. Let $G_0^> 0 \subset G$ be the parabolic subgroup with Lie algebra $\mathfrak{g}_h^{> 0}$, and let $S_G := G/G_0^> 0$ be a quotient. In the same way we define $\mathfrak{t}_h^{> 0}, K_0^> 0, S_K$. Let $n_h$ be the nilpotent radical of $\mathfrak{g}_h^{> 0}$. Obviously $Ke \subset S_G$ is isomorphic to $S_K$. By definition,

- $G_0 := \{ x \in \mathfrak{g} \mid x = gn \text{ for some } n \in n_h, g \in G \}$,
- $K_0 := \{ x \in \mathfrak{g} \mid x = kn \text{ for some } n \in n_h, k \in K \}$,
- $K (n_h \cap \mathfrak{t}^\perp) := \{ x \in \mathfrak{g} \mid x = kn \text{ for some } n \in n_h \cap \mathfrak{t}^\perp, k \in K \}$.

Let $\phi : T^* S_G \rightarrow \mathfrak{g}^*$ be the moment map. It is a straightforward observation that $Gn_h$ coincides with $\phi(T^* S_G)$, $Kn_h$ coincides with $\phi(T^* S_G|S_K)$, $K (n_h \cap \mathfrak{t}^\perp)$ coincides with $\phi(N^*_h (S_K/S_G))$.

\[
\begin{array}{cccc}
T^* S_G \xrightarrow{\text{inclusion}} & T^* S_G|S_K \xrightarrow{\text{inclusion}} & N^*_h (S_K/S_G) \\
\phi \downarrow & \phi \downarrow & \phi \\
Gn_h \xrightarrow{\text{inclusion}} & Kn_h \xrightarrow{\text{inclusion}} & K (n_h \cap \mathfrak{t}^\perp)
\end{array}
\]
As $N_{s_K/S_G}^*$ is an isotropic subvariety of $T^*S_G$, the variety $\phi(N_{s_K/S_G}^*)$ is isotropic in $G(n_h \cap \mathfrak{k}^\perp)$ and any subvariety $\tilde{V}$ of $K(n_h \cap \mathfrak{k}^\perp)$ is isotropic in $G\tilde{V}$. Therefore by Corollary 1 any subvariety $\tilde{V}$ of $\mathfrak{t}^\perp \cap N_{\mathfrak{t}^*}\mathfrak{g}$ is isotropic in $G\tilde{V}$.

\[\square\]

**Definition 9.**
- Let $V_{g,\mathfrak{t}}$ be the set of all irreducible components of all possible intersections of $N_K\mathfrak{t}^\perp$ with a $G$-orbit in $N_G\mathfrak{g}^*$.
- Let $V_{g,\mathfrak{t}}$ be the set of all irreducible components of the preimages of $V_{g,\mathfrak{t}}$ under the moment map $T^*X \to \mathfrak{g}^*$.
- Let $L_{g,\mathfrak{t}}$ be the set of all images of elements of $V_{g,\mathfrak{t}}$ in $X$.

**Theorem 3.** Let $M$ be a finitely generated $(\mathfrak{g},\mathfrak{t})$-module of finite type which affords a central character. The irreducible components of $V(M)$ are elements of $V_{g,\mathfrak{t}}$, the irreducible components of $V(M)$ are elements of $V_{g,\mathfrak{t}}$, the irreducible components of $L(M)$ are elements of $L_{g,\mathfrak{t}}$.

**Proof.** Let $\tilde{V}$ be an irreducible component of $V(M)$ and $\mathcal{O}$ be the closure of $G\tilde{V}$. By Theorem 1 the variety $\tilde{V}$ is coisotropic. As $\tilde{V} \subset N_K\mathfrak{t}^\perp \cap \mathcal{O}$, $\tilde{V}$ is isotropic, and therefore $\tilde{V}$ is Lagrangian and is an irreducible component of $\mathcal{O} \cap N_K\mathfrak{g}^* \cap \mathfrak{t}^\perp$.

**Proof of Theorem 1.** As any irreducible component $\tilde{V}$ of $V(M)$ is Lagrangian in $T^*X$, the module $\text{Ind}M$ is holonomic.

\[\square\]

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