Hopf structure of the Yangian $Y(sl_n)$ in the Drinfel’d realisation

N. Crampé

Laboratoire d’Annecy-le-Vieux de Physique Théorique
LAPTH, CNRS, UMR 5108, Université de Savoie
B.P. 110, F-74941 Annecy-le-Vieux Cedex, France

Abstract

The Yangian of the Lie algebra $sl_n$ is known to have different presentations, in particular the RTT realisation and the Drinfel’d realisation. Using the isomorphism between them, the explicit expressions of the comultiplication, the antipode and the counit in the Drinfel’d realisation of the Yangian $Y(sl_n)$ are given. As examples, the cases of $Y(sl_2)$ and $Y(sl_3)$ are worked out.

MSC number: 81R50, 17B37
The Yangian $Y(a)$ based on a simple Lie algebra $a$ is defined \[1, 2\] as the (unique) homogeneous quantisation of the algebra $a[u] = a \otimes \mathbb{C}[u]$ endowed with its standard bialgebra structure, where $\mathbb{C}[u]$ is the ring of polynomials in the indeterminate $u$. This algebra has a structure of a non-cocomutative Hopf algebra, which partially explains the importance of Yangians and their representations in the study of quantum inverse problem. Among the different presentations of the Yangians, the one known as the Drinfel’d realisation is well adapted for the study of their representations \[3\]. No explicit formula for the Hopf structure in this realisation was known yet, except for $sl_2$ \[4\] and for $osp(1|2)$ \[5\]. The aim of this letter is to give an explicit expression of the comultiplication, the antipode and the counit in the Drinfel’d realisation for $Y(sl_n)$. Note that partial results were given in \[6, 7\]. The comultiplication given in this letter can be extended to the double Yangian $DY(sl_n)$. One can show that the so-called Drinfel’d comultiplication defined only for the double Yangian is the twist of this extended comultiplication (see for example \[7, 8\]).

This letter is organised as follows. In the first section, the RTT formalism \[9\] and Drinfel’d realisation of $Y(sl_n)$ are presented, which allow us to give the normalisation of the generators as well as the exact form of the $R$-matrix and of the quantum determinant. In the second section, some properties about the quantum minors, needed in the following, are explained. The expressions of the isomorphism using the quantum minors or the Gauss decomposition are then presented. The main theorem of this letter, i.e. the explicit form of the Hopf structure, is exposed in the next two sections. Finally, as illustrative examples, the $Y(sl_2)$ and $Y(sl_3)$ cases are worked out.

1 Two realisations of the Yangian $Y(sl_n)$

In this section, two different realisations of the Yangian based on the Lie algebra $sl_n$ are presented: the RTT formalism and the Drinfel’d realisation \[3\].

The first realisation uses the RTT formalism \[3, 4, 9\]. Let $V^{(n)}$ denotes the $n$-dimensional fundamental vector space representation of $sl_n$. The Yang’s $R$-matrix is given by

$$R_{12}^{(n)}(u) = I \otimes I + \sum_{1 \leq i, j \leq n} \frac{E_{ij} \otimes E_{ji}}{u} \in \text{End}(V^{(n)} \otimes V^{(n)})$$ (1.1)

where $E_{ij}$ is the elementary matrix with entry 1 in row $i$ and column $j$ and 0 elsewhere. This R-matrix satisfies the following properties

$$R_{12}^{(n)}(u)R_{13}^{(n)}(u + v)R_{23}^{(n)}(v) = R_{23}^{(n)}(v)R_{13}^{(n)}(u + v)R_{12}^{(n)}(u) \quad \text{(Yang-Baxter equation)}$$ (1.2)

$$R_{12}^{(n)}(u)R_{21}^{(n)}(-u) = \frac{u^2 - 1}{u^2} (I \otimes I) \quad \text{(unitarity)}$$ (1.3)

\textbf{Theorem 1.1} The Yangian of $sl_n$, $Y(sl_n)$, is isomorphic to the associative algebra, $U(R)$, generated by the unit and the elements \{$T_{i,j}^{(k)} | 1 \leq i, j \leq n, k \in \mathbb{Z}_{>0}$\} gathered in the formal series

$$T(u) = 1 + \sum_{i,j=1}^{n} \sum_{n \in \mathbb{Z}_{>0}} T_{i,j}^{(n)} u^{-n} E_{ij} = \sum_{i,j=1}^{n} T_{i,j}(u) E_{ij}$$ (1.4)
subject to the defining relations

\[ R^{(n)}_{12}(u - v)(T(u) \otimes 1)(1 \otimes T(v)) = (1 \otimes T(v))(T(u) \otimes 1)R^{(n)}_{12}(u - v) \]  \hspace{1cm} (1.5)

and \[ qdet T(u) = 1 \]  \hspace{1cm} (1.6)

where \[ qdet T(u) = \sum_{\sigma \in S_n} sgn(\sigma) T_{\sigma(1),1}(u) \cdots T_{\sigma(n),n}(u + n - 1). \]

The defining relations (1.5) as commutators of \( T_{ij}(u) \):

\[ -(u - v) [T_{i,j}(u), T_{k,l}(v)] = T_{k,j}(u) T_{i,l}(v) - T_{k,j}(v) T_{i,l}(u). \]  \hspace{1cm} (1.7)

The map

\[ T(u) \mapsto T^{-1}(-u) \equiv T^*(u) \]  \hspace{1cm} (1.8)

defines an automorphism of \( Y(sl_n) \).

To avoid ambiguity, let us stress that \[ qdet T(-u) = \sum_{\sigma \in S_n} sgn(\sigma) T_{\sigma(1),1}(-u) \cdots T_{\sigma(n),n}(-u + n - 1) \]
is different from the quantum determinant of the matrix \( \tilde{T}(u) = T(-u) \):

\[ qdet \tilde{T}(u) = \sum_{\sigma \in S_n} sgn(\sigma) T_{\sigma(1),1}(-u) \cdots T_{\sigma(n),n}(-u - n + 1). \]  \hspace{1cm} (1.9)

The Yangian \( Y(sl_n) \) has a Hopf algebra structure and the explicit forms of comultiplication, antipode and counit are

\[ \Delta(T_{i,j}(u)) = \sum_{k=1}^{n} T_{i,k} \otimes T_{k,j}, \quad S(T(u)) = T^{-1}(u) \quad \text{and} \quad \epsilon(T_{i,j}(u)) = \delta_{ij}. \]  \hspace{1cm} (1.10)

The second realisation of the Yangian uses the so-called Drinfel’d generators. Let \( \{ \alpha_i \}_{1 \leq i \leq n - 1} \) be the set of simple roots of \( sl_n \) and \( \langle \cdot, \cdot \rangle \) be the standard non-degenerate symmetric invariant bilinear form on \( sl_n \). For each simple root \( \alpha_i, e_{\alpha_i} \) and \( f_{\alpha_i} \) are the corresponding root vectors, such that \( \langle e_{\alpha_i}, f_{\alpha_i} \rangle = 1 \), and \( h_{\alpha_i} = [e_{\alpha_i}, f_{\alpha_i}] \) are the Cartan generators. The Drinfel’d realisation of the Yangian is given by the following theorem \[ 3 \].

**Theorem 1.2** The Yangian of \( sl_n, Y(sl_n) \), is isomorphic to the associative algebra \( A \), generated by the unit and the elements \( \{ e_i^{(k)}, f_i^{(k)}, h_i^{(k)} \mid 1 \leq i \leq n - 1, k \in \mathbb{Z}_{\geq 0} \} \) subject to the defining relations

\[ [h_i^{(k)}, h_j^{(l)}] = 0, \quad [e_i^{(k)}, f_j^{(l)}] = \delta_{ij} h_i^{(k+l)}, \]  \hspace{1cm} (1.11)

\[ [h_i^{(0)}, e_j^{(0)}] = (\alpha_i, \alpha_j) e_j^{(0)}, \quad [h_i^{(0)}, f_j^{(0)}] = - (\alpha_i, \alpha_j) f_j^{(0)}, \]  \hspace{1cm} (1.12)

\[ [h_i^{(k+1)}, e_j^{(l)}] - [h_i^{(k)}, e_j^{(l+1)}] = \frac{1}{2} \ (\alpha_i, \alpha_j) \ (h_i^{(k)} e_j^{(l)} + e_j^{(l)} h_i^{(k)}), \]  \hspace{1cm} (1.13)

\[ [h_i^{(k+1)}, f_j^{(l)}] - [h_i^{(k)}, f_j^{(l+1)}] = - \frac{1}{2} \ (\alpha_i, \alpha_j) \ (h_i^{(k)} f_j^{(l)} + f_j^{(l)} h_i^{(k)}), \]  \hspace{1cm} (1.14)

\[ [e_i^{(k+1)}, e_j^{(l)}] - [e_i^{(k)}, e_j^{(l+1)}] = \frac{1}{2} \ (\alpha_i, \alpha_j) \ (e_i^{(k)} e_j^{(l)} + e_j^{(l)} e_i^{(k)}), \]  \hspace{1cm} (1.15)

\[ [f_i^{(k+1)}, f_j^{(l)}] - [f_i^{(k)}, f_j^{(l+1)}] = - \frac{1}{2} \ (\alpha_i, \alpha_j) \ (f_i^{(k)} f_j^{(l)} + f_j^{(l)} f_i^{(k)}), \]  \hspace{1cm} (1.16)
and to the Serre relations, for \(i \neq j\) and \(n_{ij} = 1 - \frac{2(\alpha, \alpha)}{\langle \alpha, \alpha \rangle}\):

\[
\sum_{\sigma \in \mathcal{S}_{n_{ij}}} [e_i^{(k_{\sigma(1)})}, \ldots, [e_i^{(k_{\sigma(n_{ij})})}, e_j^{(0)}]] = 0, \tag{1.17}
\]

\[
\sum_{\sigma \in \mathcal{S}_{n_{ij}}} [f_i^{(k_{\sigma(1)})}, \ldots, [f_i^{(k_{\sigma(n_{ij})})}, f_j^{(0)}]] = 0. \tag{1.18}
\]

For later conveniences, we define the following formal series:

\[
e_i(u) = \sum_{k=0}^{+\infty} \frac{e_i^{(k)}}{u^{k+1}}, \quad f_i(u) = \sum_{k=0}^{+\infty} \frac{f_i^{(k)}}{u^{k+1}} \quad \text{and} \quad h_i(u) = 1 + \sum_{k=0}^{+\infty} \frac{h_i^{(k)}}{u^{k+1}}, \quad \text{for} \quad 1 \leq i \leq n-1. \tag{1.19}
\]

The mapping \(e_\alpha \mapsto e_i^{(0)}\), \(f_\alpha \mapsto f_i^{(0)}\), \(h_\alpha \mapsto h_i^{(0)}\) defines an embedding \(U(sl_n) \hookrightarrow Y(sl_n)\), where \(U(sl_n)\) is the universal enveloping algebra of \(sl_n\).

## 2 Quantum minors

Before giving the expression of the isomorphism that relates the two Yangian presentations, in the next section, we introduce the notion of quantum minors and give some of their properties.

Let \(I = \{a_1, a_2, \ldots, a_m\}\) and \(J = \{b_1, b_2, \ldots, b_m\}\) such that \(I, J \subset \{1, \ldots, n\}\) and \(\text{card}(I) = \text{card}(J) = m\) with \(1 \leq m \leq n\). The set of generators \(\{T_{a_i, b_j}(u) | 1 \leq i, j \leq m\}\) defines a subalgebra of \(Y(sl_n)\) with the following commutation relations

\[
F_{12}^{(m)}(u-v) (T_{(a_1\cdots a_m)}(u) \otimes 1) (1 \otimes T_{(b_1\cdots b_m)}(v)) = (1 \otimes T_{(b_1\cdots b_m)}(v)) (T_{(a_1\cdots a_m)}(u) \otimes 1) F_{12}^{(m)}(u-v), \tag{2.1}
\]

where \(T_{(a_1\cdots a_m)}(u) = \sum_{i,j=1}^{m} T_{a_i, b_j}(u) E_{ij}\).

**Definition 2.1** The quantum minor \(t_{(a_1\cdots a_m)}^{(b_1\cdots b_m)}(u)\) of \(T(u)\) is defined by

\[
t_{(a_1\cdots a_m)}^{(b_1\cdots b_m)}(u) = q \det T_{(a_1\cdots a_m)}^{(b_1\cdots b_m)}(u) = \sum_{\sigma \in \mathcal{S}_m} \text{sgn}(\sigma) \ T_{a_{\sigma(1)}, b_1}(u) \cdots T_{a_{\sigma(m)}, b_m}(u + m - 1) \tag{2.2}
\]

One can show that

\[
t_{(a_1\cdots a_m)}^{(b_1\cdots b_m)}(u) = \sum_{\sigma \in \mathcal{S}_m} \text{sgn}(\sigma) \ T_{a_1, b_{\sigma(1)}}(u + m - 1) \cdots T_{a_m, b_{\sigma(m)}}(u). \tag{2.3}
\]

By convention, when \(m < 1\), the quantum minor is equal to one. Quantum minors satisfy some properties \(\Box\) which are analogous to those of numerical matrices minors.

**Proposition 2.2** The quantum minor \(t_{(a_1\cdots a_m)}^{(b_1\cdots b_m)}(u)\) verifies the following properties:
1. It is antisymmetric, i.e. for $\rho \in \mathfrak{S}_m$,
\[
t^{(a_1 \ldots a_m)}_{(b_1 \ldots b_m)}(u) = \text{sgn} (\rho) \ t^{(\rho(a_1) \ldots \rho(a_m))}_{(\rho(b_1) \ldots \rho(b_m))}(u) = \text{sgn} (\rho) \ t^{(a_1 \ldots a_m)}_{(\rho(b_1) \ldots \rho(b_m))}(u).
\]

2. It is alternated, i.e. if there exists $i \neq j$ such that $a_i = a_j$ or $b_i = b_j$, then $t^{(a_1 \ldots a_m)}_{(b_1 \ldots b_m)}(u) = 0$.

3. It can be expanded with respect to its last column or its last row as follows:
\[
t^{(a_1 \ldots a_m)}_{(b_1 \ldots b_m)}(u) = \sum_{k=1}^{m} (-1)^{k+m} \ t^{(a_1 \ldots a_{k-1} a_k a_{k+1} \ldots a_{m-1} a_m)}_{(b_1 \ldots b_{k-1} b_k b_{k+1} \ldots b_{m-2} b_{m-1})}(u) \ T_{a_k, b_m} (u + m - 1) (2.5)
\]
\[
= \sum_{k=1}^{m} (-1)^{k+m} \ T_{a_m, b_k} (u + m - 1) \ t^{(a_1 \ldots a_{k-1} a_k \ldots a_{m-2} a_m)}_{(b_1 \ldots b_{k-1} b_k \ldots b_{m-1} b_m)}(u). (2.6)
\]

From the defining relations \ref{1.7}, the commutation relations of the quantum minors with $T_{i,j}(u)$ can be computed:
\[
(u - v) \ [T_{i,j}(u), t^{(a_1 \ldots a_m)}_{(b_1 \ldots b_m)}(v)] = \sum_{k=1}^{m} \left( t^{(a_1 \ldots a_{k-1} a_k a_{k+1} \ldots a_m)}_{(b_1 \ldots b_{k-1} b_k \ldots b_{m-2} b_{m-1})}(v) \ T_{i,b_k}(u) - T_{a_k,j}(u) t^{(a_1 \ldots a_{k-1} a_k \ldots a_{m-2} a_m)}_{(b_1 \ldots b_{k-1} b_k b_{k+1} \ldots b_{m-1} b_m)}(v) \right). (2.7)
\]

A corollary of (2.7) is that the quantum minor $t^{(a_1 \ldots a_m)}_{(b_1 \ldots b_m)}(u)$ lies in the centre of the subalgebra generated by $\{T_{a_i, b_j}(u) | 1 \leq i, j \leq m\}$ i.e.
\[
[T_{a_i, b_j}(u), t^{(a_1 \ldots a_m)}_{(b_1 \ldots b_m)}(v)] = 0, \quad \text{for} \quad 1 \leq i, j \leq m. \quad (2.8)
\]

The map
\[
T_{i,j}(u) \mapsto t^{(1 \ldots p + i + j)}_{(1 \ldots p + j)}(u)
\]
defines an algebra homomorphism $Y(sl_{n-p}) \to Y(sl_n)$, for $1 \leq p \leq n - 1$ and $1 \leq i, j \leq n - p$. Note that this homomorphism allows us to compute a simple way the commutation relations among the $t^{(1 \ldots p + i + j)}_{(1 \ldots p + j)}(u)$ minors.

Finally, quantum minors allow us to express some elements of the inverse matrix of $T(u)$ thanks to the following proposition, proved by A.I.Molev \cite{4}:

**Proposition 2.3** For $1 \leq i, j \leq n$, the following equality holds
\[
\left( T^{-1}(u + n - 1) \right)_{i,j} = (-1)^{i+j} t^{(1 \ldots j - 1 i + 1 \ldots n)}_{(1 \ldots i - 1 j + 1 \ldots n)}(u). \quad (2.10)
\]

### 3 Isomorphisms between the two realisations of $Y(sl_n)$

For clarity purposes, the isomorphism between the two previous realisations is recalled, see e.g. \cite{3,8}.

Two presentations of this isomorphism are possible. The first one uses the quantum minors and the second one uses the Gauss decomposition.
Theorem 3.1 The map \( \phi : A \to U(R) \)

\[
e_i \left( u + i - \frac{2}{2} \right) \mapsto (t(1\ldots i)(u))^{-1} t(1\ldots i-1\ldots 1\ldots i+1)(u) \quad (3.1)
\]

\[
f_i \left( u + i - \frac{2}{2} \right) \mapsto t(1\ldots i-1\ldots i+1)(u) (t(1\ldots i)(u))^{-1} \quad (3.2)
\]

\[
h_i \left( u + i - \frac{2}{2} \right) \mapsto (t(1\ldots i)(u))^{-1} t(1\ldots i-1)(u) t(1\ldots i+1)(u-1) (t(1\ldots i)(u-1))^{-1} \quad (3.3)
\]

is an algebra isomorphism.

Note that the image of \( h_i(u) \) can be written differently as:

\[
\phi \left( h_i \left( u + i - \frac{2}{2} \right) \right) = (t(1\ldots i)(u))^{-1} t(1\ldots i-1\ldots i+1)(u) - \phi \left( f_i \left( u + i - \frac{2}{2} \right) e_i \left( u + i - \frac{2}{2} \right) \right) \quad (3.4)
\]

\[
= t(1\ldots i-1\ldots i+1)(u) (t(1\ldots i)(u))^{-1} - \phi \left( f_i \left( u + i - \frac{2}{2} \right) e_i \left( u + i - \frac{2}{2} \right) \right) \quad (3.5)
\]

The other presentation of the isomorphism \( \phi \) uses the Gauss decompositions of the matrix \( T(u) \):

\[
T(u) = \begin{pmatrix}
1 & f_{2,1}(u) & 1 & 0 \\
f_{n,1}(u) & \cdots & f_{n,n-1}(u) & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & \tilde{e}_{1,2}(u) & \cdots & \tilde{e}_{1,n}(u) \\
0 & \cdots & \tilde{e}_{n-1,n}(u) & 1
\end{pmatrix}
\begin{pmatrix}
k_1(u) & 0 & 0 \\
0 & \cdots & k_n(u) \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
1 & e_{1,2}(u) & \cdots & e_{1,n}(u) \\
0 & \cdots & 0 \\
0 & \cdots & 1
\end{pmatrix} \quad (3.6)
\]

\[
= \begin{pmatrix}
1 & \tilde{e}_{1,2}(u) & \cdots & \tilde{e}_{1,n}(u) \\
0 & \cdots & \tilde{e}_{n-1,n}(u) & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{k}_1(u) & 0 & 0 \\
0 & \cdots & \tilde{k}_n(u) \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
1 & \tilde{f}_{2,1}(u) & 1 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\tilde{f}_{n,1}(u) & \cdots & \tilde{f}_{n,n-1}(u) & 1
\end{pmatrix} \quad (3.7)
\]

The expression of the elements of the Gauss decomposition (3.6) in terms of quantum minors has been computed by K. Iohara [8]. For the alternative Gauss decomposition (3.7), the computations are similar and one obtains:

Proposition 3.2 Let \( 1 \leq i < j \leq n \) and \( 1 \leq p \leq n \). The formal series \( e_{i,j}(u), \tilde{e}_{i,j}(u), f_{j,i}(u), \tilde{f}_{j,i}(u), k_p(u) \) and \( \tilde{k}_p(u) \) in the Gauss decompositions, can be expressed in terms of quantum minors:

\[
e_{i,j}(u + i - 1) = (t(1\ldots i)(u))^{-1} t(1\ldots i-1\ldots j)(u), \quad (3.8)
\]

\[
f_{j,i}(u + i - 1) = t(1\ldots i-1\ldots j)(u) (t(1\ldots i)(u))^{-1}, \quad (3.9)
\]

\[
k_j(u + j - 1) = t(1\ldots j)(u) (t(1\ldots j-1)(u))^{-1}, \quad (3.10)
\]

\[
k_1(u) = t(1)(u) = T_{1,1}(u), \quad (3.11)
\]

and

\[
\tilde{e}_{i,j}(u + n - j) = t(j+1\ldots j-1\ldots n)(u) (t(j+1\ldots n)(u))^{-1}, \quad (3.12)
\]

\[
\tilde{f}_{j,i}(u + n - j) = (t(j+1\ldots n)(u))^{-1} t(j+1\ldots n)(u), \quad (3.13)
\]

\[
\tilde{k}_i(u + n - i) = (t(i+1\ldots n)(u))^{-1} t(i\ldots n)(u), \quad (3.14)
\]

\[
\tilde{k}_n(u) = t(n)(u) = T_{n,n}(u). \quad (3.15)
\]
Remark: Proposition 3.2 proves the existence of the two Gauss decomposition. Then, proposition 3.2 implies that the map $\tilde{\phi} : A \rightarrow U(R)$

\[
e_i(u) \mapsto e_{i,i+1} \left( u + \frac{i}{2} \right)
\]

\[
f_i(u) \mapsto f_{i+1,i} \left( u + \frac{i}{2} \right)
\]

\[
h_i(u) \mapsto k_{i+1} \left( u + \frac{i}{2} \right) k_i^{-1} \left( u + \frac{i}{2} \right)
\]

is an algebra isomorphism, for $1 \leq i \leq n - 1$.

In the following, as in equation (1.8), $T^*(u)$ denotes $T(-u)^{-1}$ and $t^*(a_1^{\ldots} a_m^{\ldots})(u)$ denotes the quantum minors of $T^*(u)$.

**Corollary 3.3** For $1 \leq m \leq n$, the following equalities hold

\[
t^*\left(\frac{1}{m} \cdot \cdot \cdot \right)(-u - n + 1) = t^{m+1 \cdot \cdot \cdot n}(u)
\]

\[
t^*\left(\frac{1}{m-1} \cdot \cdot \cdot \right)(-u - n + 1) = -t^{m+2 \cdot \cdot \cdot n}(u)
\]

\[
t^*\left(\frac{1}{m-1} \cdot \cdot \cdot \right)(-u - n + 1) = -t^{m+2 \cdot \cdot \cdot n}(u)
\]

\[
t^*\left(\frac{1}{m-1} \cdot \cdot \cdot \right)(-u - n + 1) = t^{m+2 \cdot \cdot \cdot n}(u)
\]

**Proof:** Let $T(u)$ decomposed according to \(3.7\). Then, $T^*(u)$ decomposes as in \(3.6\). Using the relation $T^*(u) = T(-u)^{-1}$, we deduce

\[
e_i^*(u) = e_{i,i+1}(-u),
\]

\[
f_i^*(u) = f_{i+1,i}(-u), \quad \text{for} \quad 1 \leq i \leq n - 1
\]

\[
k_i^*(u) = (k_i(-u))^{-1}, \quad \text{for} \quad 1 \leq i \leq n
\]

with obvious notations. Finally, using proposition 3.2, the equalities are proven.

For $1 \leq i < j \leq n$, the elements $e_i^{(0)} = T_{i,j}^{(1)}$ and $f_j^{(0)} = T_{j,i}^{(1)}$ are root generators of the algebra sl\(_n\), which can be expressed in terms of simple root generators, $e_i^{(0)}$ and $f_j^{(0)}$, as follows:

\[
e_{i,j}^{(0)} = [\cdots [e_{j-1}^{(0)}, e_{j-2}^{(0)}, e_{j-3}^{(0)}, \cdots], e_{i+1}^{(0)}, e_i^{(0)}],
\]

\[
f_{j,i}^{(0)} = [f_i^{(0)}, [f_{i+1}^{(0)}, [\cdots, f_{j-3}^{(0)}, f_{j-2}^{(0)}, f_{j-1}^{(0)}] \cdots].
\]

**Remark:** In \(3.26\) and \(3.27\), the isomorphism $\tilde{\phi}$ has been omitted for simplicity. In the following, this losely notation is always used, i.e. the isomorphisms between two realisations of the same algebra are omitted.

## 4 The Hopf structure of $Y(sl_n)$ in the Drinfel’d basis

Before giving the Hopf structure of $Y(sl_n)$ in the Drinfel’d basis, the images of any quantum minor under the coproduct, the antipode and the counit are needed. Let us recall that $T^*(u)$ denotes $T(-u)^{-1}$ and $t^*(a_1^{\ldots} a_m^{\ldots})(u)$ denotes its quantum minors.
Proposition 4.1 Let $1 \leq m \leq n$, $1 \leq a_1 < \cdots < a_m \leq n$ and $1 \leq b_1 < \cdots < b_m \leq n$. The images of a quantum minor under the coproduct, the antipode and the counit are given by:

\[
\Delta(t_{b_1 \cdots b_m}(u)) = \sum_{1 \leq c_1 < \cdots < c_m \leq n} t_{c_1 \cdots c_m}(u) \otimes t_{b_1 \cdots b_m}(u),
\]

where $[\frac{t}{2}]$ is the integer part of $\frac{t}{2}$.

Proof: Direct computation, using the definition of the Hopf structure $\phi$ and the property that the comultiplication and counit are morphisms while the antipode is an anti-morphism.

In particular, one obtains the following well-known result:

\[
\Delta(qdet(u)) = qdet(u) \otimes qdet(u).
\]

The comultiplication, the antipode and the counit are established in the Drinfel’d basis thanks to the isomorphism $\phi$ (see theorem 3.1) between $A$ and $U(R)$.

4.1 Comultiplication

The adjoint actions of the elements of the algebra $sl_n$ on $X \in Y(sl_n)$ will be denoted by, for $1 \leq i < j \leq n$:

\[
Ad_{\pm}^{\pm}(X) = \pm[e_{i,j}^{(0)}, X] \quad \text{and} \quad Ad_{\pm}^{\pm}(X) = \pm[f_{j,i}^{(0)}, X].
\]

Moreover, by convention $Ad_{\pm}^{\pm}(X) = X$ and $Ad_{\pm}^{\pm}(X) = X$. To determine the explicit form of the comultiplication, the following generalisation of the adjoint action, depending on a spectral parameter, is useful.

Definition 4.2 Let $1 \leq i \leq j \leq n$, $1 \leq \alpha \leq n$ and $X$ an element of $Y(sl_n)$. The generalised adjoint actions are defined by

\[
^\alpha \mathcal{E}_{i,j}(u)(X) = Ad_{e_{i,j}}^{\pm}(X) + \delta_{i \leq \alpha < j} Ad_{e_{i,\alpha}}^{\pm} \left(Ad_{e_{\alpha+1,j}}^{\pm}(e_{\alpha}(u))\right) X,
\]

\[
^\alpha \mathcal{E}_{i,j}(u)(X) = Ad_{e_{i,j}}^{\pm}(X) + \delta_{i \leq \alpha < j} X Ad_{e_{i,\alpha}}^{\pm} \left(Ad_{e_{\alpha+1,j}}^{\pm}(e_{\alpha}(u))\right),
\]

and

\[
^\alpha \mathcal{F}_{j,i}(u)(X) = Ad_{f_{j,i}}^{\pm}(X) + \delta_{i \leq \alpha < j} Ad_{f_{\alpha,i}}^{\pm} \left(Ad_{f_{\alpha+1,j}}^{\pm}(f_{\alpha}(u))\right) X,
\]

\[
^\alpha \mathcal{F}_{j,i}(u)(X) = Ad_{f_{j,i}}^{\pm}(X) + \delta_{i \leq \alpha < j} X Ad_{f_{\alpha,i}}^{\pm} \left(Ad_{f_{\alpha+1,j}}^{\pm}(f_{\alpha}(u))\right),
\]

where $\delta_{i \leq \alpha < j-1} = \begin{cases} 1 & \text{if } i \leq \alpha < j-1, \\ 0 & \text{otherwise}. \end{cases}$
Let $\alpha G$ be any actions on $Y(sl_n)$. Hereafter, for simplicity, the notation $\alpha G^\beta$ means either $\alpha G$ or $G^\beta$.

To compute the comultiplication, we also need:

**Definition 4.3** For $1 \leq m \leq n$, $1 \leq k_1 < k_2 < \ldots < k_m \leq n$ and $k_m \neq m$, $\alpha E_{k_1,k_2,\ldots,k_m}(u)$ and $\alpha F_{k_1,k_2,\ldots,k_m}(u)(X)$ are defined by for $X \in Y(sl_n)$:

$$
\alpha E_{k_1,k_2,\ldots,k_m}(u)(X) = \left( \prod_{1 \leq i \leq m-1} \alpha E_{i,k_i}(u) \right) \left( \alpha E_{m+1,k_m}(u)(X) \right),
$$

$$
\alpha F_{k_1,k_2,\ldots,k_m}(u)(X) = \left( \prod_{1 \leq i \leq m-1} \alpha F_{k_i,i}(u) \right) \left( \alpha F_{k_m,m+1}(u)(X) \right),
$$

where, for $\{G_p|1 \leq p \leq m-1\}$, a set of actions on $Y(sl_n)$, we denote

$$
\prod_{1 \leq i \leq m-1} G_i(X) = G_1(\cdots(G_{m-2}(G_{m-1}(X))\cdots).
$$

In particular, one gets for $k > 1$

$$
\alpha E_k(u)(X) = \alpha E_{2,k}(u)(X), \quad \alpha F_k(u)(X) = \alpha F_{k,2}(u)(X).
$$

By convention, if the set of indices $\{k_1,k_2,\ldots,k_m\}$ is empty, then $\alpha E_{k_1,k_2,\ldots,k_m}(u)(X) = 1$ and $\alpha F_{k_1,k_2,\ldots,k_m}(u)(X) = 1$. Remark that these generators can be expressed only in terms of the elements of the Drinfel’d basis, thanks to equations (3.26) and (3.27).

These generalised actions show up in the following lemma:

**Lemma 4.4** For $1 \leq i \leq n-1$, $1 \leq a_1 < \cdots < a_i \leq n$ and $a_i \neq i$, one gets

$$
(t^{(\ldots;i)}(u))^{-1} t^{(a_1,\ldots,a_i)}(u) = t^{1,\ldots;i}(u) = iE_{a_1,\ldots,a_i}(u + i) \left( e_i \left( u + \frac{i-2}{2} \right) \right)
$$

$$
th^{(\ldots;i)}(u) \left( t^{(\ldots;i)}(u) \right)^{-1} = t^{1,\ldots;i}(u) = E_{a_1,\ldots,a_i}(u + i) \left( e_i \left( u + \frac{i}{2} \right) \right)
$$

$$
(t^{(\ldots;i)}(u))^{-1} t^{(1,\ldots,i)}(u) = t^{a_1,\ldots,a_i}(u) = iF_{a_1,\ldots,a_i}(u + i) \left( f_i \left( u + \frac{i}{2} \right) \right)
$$

$$
t^{(1,\ldots;i)}(u) \left( t^{(\ldots;i)}(u) \right)^{-1} = t^{a_1,\ldots,a_i}(u) = F_{a_1,\ldots,a_i}(u + i) \left( f_i \left( u + \frac{i-2}{2} \right) \right)
$$

$$
(t^{(\ldots;i)}(u))^{-1} t^{(a_1,\ldots,a_i)}(u) = t^{1,\ldots;i}(u) = iE_{a_1,\ldots,a_i}(u + i) \left( g_i \left( u + \frac{i-2}{2} \right) \right)
$$

$$
t^{(1,\ldots;i)}(u) \left( t^{(\ldots;i)}(u) \right)^{-1} = t^{a_1,\ldots,a_i}(u) = F_{a_1,\ldots,a_i}(u + i) \left( g_i \left( u + \frac{i-2}{2} \right) \right)
$$

$$
t^{(a_1,\ldots,a_i-1,\ldots)}(u) \left( t^{(\ldots;i)}(u) \right)^{-1} = t^{a_1,\ldots,a_i}(u) = iF_{a_1,\ldots,a_i}(u + i) \left( g_i \left( u + \frac{i-2}{2} \right) \right)
$$

$$
t^{(a_1,\ldots,a_i-1,\ldots)}(u) \left( t^{(\ldots;i)}(u) \right)^{-1} = t^{a_1,\ldots,a_i}(u) = F_{a_1,\ldots,a_i}(u + i) \left( g_i \left( u + \frac{i-2}{2} \right) \right)
$$
where

\[
\begin{align*}
g_i(u) &= h_i(u) + f_i(u) \varepsilon_i(u + 1) \\
\bar{g}_i(u) &= h_i(u) + f_i(u + 1) \varepsilon_i(u)
\end{align*}
\]  

(4.22)  

(4.23)

**Proof:** The proof is only given for (4.14). Let \( i \) and \( \{a_1, \cdots, a_i\} \) fixed as in the lemma. The first step is to evaluate the quantum minor \( t^{(1 \cdots i)}_{(a_1 \cdots a_i)}(u) \) in terms of the quantum minor \( t^{(1 \cdots i_{i-1}i_i)}_{(1 \cdots i_{i-1}i_{i+1})}(u) \) and in terms of some generators of the \( sl_n \) algebra. Selecting the coefficient of \( u^0v^{-1} \) in equation (2.7), the following relation is obtained for \( 1 \leq b_1 < \cdots < b_i \leq n, 1 \leq p \leq i \) and \( 1 \leq m \leq n - b_p \):

\[
Ad^+_{e_{bp,bp+m}} \left( t^{(1 \cdots p \cdots i)}_{(b_1 \cdots b_p-b_i)}(v) \right) = \left[ T^{(1)}_{bp,bp+m} , t^{(1 \cdots p \cdots i)}_{(b_1 \cdots b_p-b_i)}(v) \right] = t^{(1 \cdots p \cdots i)}_{(b_1 \cdots b_p+m-b_i)}(v).
\]  

(4.24)

This relation allows us to increase the parameters of the quantum minor. Using this relation, the indices \( \{1, \cdots, i-1, i+1\} \) of \( t^{(1 \cdots i_{i-1}i_i)}_{(1 \cdots i_{i-1}i_{i+1})}(u) \) can be increased up to \( \{a_1, \cdots, a_i\} \):

\[
\left( \prod_{1 \leq p \leq i-1} Ad^+_{e_{p,a_p}} \right) \left( Ad^+_{e_{i+1,a_i}} \left( t^{(1 \cdots i_{i-1}i_i)}_{(1 \cdots i_{i-1}i_{i+1})}(u) \right) \right) = t^{(1 \cdots i)}_{(a_1 \cdots a_i)}(u)
\]  

(4.25)

The second step of the proof consists in determining the commutator of \( (t^{(1 \cdots i)})(u) \) with \( e^{(0)}_{j,j+1} \). The commutation relations are computed using equation (2.7):

\[
(u - v) [T_{jj+1}(u) , t^{(1 \cdots i)}(v)] = \delta_{ij} t^{(1 \cdots i_{i-1}i_i)}_{(1 \cdots i_{i-1}i_{i+1})}(v) T_{ii}(u) + O \left( \frac{1}{u} \right)
\]  

(4.26)

Multiplying by \( (t^{(1 \cdots i)})(v) \) both sides of the \( u^0 \) coefficient in (4.26), one gets

\[
\left[ e^{(0)}_{j,j+1} , (t^{(1 \cdots i)}(v))^{-1} \right] = - \delta_{ij} \varepsilon_i \left( v + \frac{i - 2}{2} \right) \left( t^{(1 \cdots i)}(v) \right)^{-1}
\]  

(4.27)

Thus, thanks to the relations (3.26) and (4.27), one obtains for \( X \in Y(sl_n) \):

\[
(t^{(1 \cdots i)})(u)^{-1} Ad^+_{e_{p,a_p}} \left( u + \frac{i - 2}{2} \right) ( (t^{(1 \cdots i)})(u)^{-1} X \right)
\]  

(4.28)

This proves the equation (4.14). Equation (4.15) is proven along the same lines, remarking that

\[
t^{(1 \cdots i_{i-1}i_i)}_{(1 \cdots i_{i-1}i_{i+1})}(u - 1) (t^{(1 \cdots i)}(u - 1))^{-1} = \left( t^{(1 \cdots i)}(u) \right)^{-1} t^{(1 \cdots i_{i-1}i_i)}_{(1 \cdots i_{i-1}i_{i+1})}(u),
\]  

(4.29)

which explains the shift in the spectral parameter between (4.14) and (4.15). For the relations (4.16)-(4.21), the proof is analogous.

Now we can state the main theorem of the letter.
Theorem 4.5 Let $1 \leq i \leq n - 1$. The comultiplication in the Drinfel’d basis is given by:

$$\Delta(e_i(u)) = \sum_{m=0}^{+\infty} (-1)^m \left( \sum_{1 \leq b_1 < \ldots < b_i \leq n}^i E_{b_1, \ldots, b_i}(e_i(u)) \otimes i F_{b_1, \ldots, b_i}(u)(f_i(u + 1)) \right)^m$$

$$\times \left(1 \otimes e_i(u) + \sum_{1 \leq a_1 < \ldots < a_i \leq n}^i E_{a_1, \ldots, a_i}(e_i(u)) \otimes i F_{a_1, \ldots, a_i}(u + 1)(\bar{g}_i(u)) \right)$$

$$\Delta(f_i(u)) = \left( f_i(u) \otimes 1 + \sum_{1 \leq a_1 < \ldots < a_i \leq n}^i E_{a_1, \ldots, a_i}(u + 1)(g_i(u)) \otimes F^i(a_1 \cdots a_i)(u)(f_i(u)) \right)^m$$

$$\times \sum_{m=0}^{+\infty} (-1)^m \left( \sum_{1 \leq b_1 < \ldots < b_i \leq n}^i E_{b_1, \ldots, b_i}(u + 1)(e_i(u + 1)) \otimes F^i_{b_1, \ldots, b_i}(u)(f_i(u)) \right)^m$$

$$\Delta(h_i(u)) = \left( f_i(u) \otimes e_i(u + 1) + \sum_{1 \leq a_1 < \ldots < a_i \leq n}^i E_{a_1, \ldots, a_i}(u + 1)(g_i(u)) \otimes F^i_{a_1, \ldots, a_i}(u)(g_i(u)) \right)^m$$

$$\times \sum_{m=0}^{+\infty} (-1)^m \left( \sum_{1 \leq b_1 < \ldots < b_i \leq n}^i E_{b_1, \ldots, b_i}(u + 1)(e_i(u + 1)) \otimes F^i_{b_1, \ldots, b_i}(u + 1)(f_i(u)) \right)^m$$

$$-\Delta(f_i(u))\Delta(e_i(u + 1))$$

**Proof:** The full proof is presented only for $e_i(u)$, the outline of proofs for $f_i(u)$ and $h_i(u)$ being similar. The comultiplication in the Drinfel’d realisation is constructed thanks to the isomorphism given in the theorem 3.1.

$$\Delta\left(e_i\left(u + \frac{i - 2}{2}\right)\right) = \Delta\left(t_{\{1, \ldots, i\}}(u)\right)^{-1} \Delta\left(t_{\{1, \ldots, i-1, i+1\}}(u)\right)$$

$$= \left( \sum_{b_1 < \ldots < b_i} t_{\{b_1, \ldots, b_i\}}(u) \otimes t_{\{1, \ldots, i\}}(u) \right)^{-1} \sum_{a_1 < \ldots < a_i} t_{\{a_1, \ldots, a_i\}}(u) \otimes t_{\{a_1, \ldots, a_i-1, a_i\}}(u)$$

$$= \sum_{m=0}^{\infty} (-1)^m \left( \sum_{b_1 < \ldots < b_i} t_{\{b_1, \ldots, b_i\}}(u) \otimes t_{\{1, \ldots, i\}}(u) \otimes t_{\{1, \ldots, i\}}(u) \right)^m$$

$$\times \sum_{a_1 < \ldots < a_i} t_{\{a_1, \ldots, a_i\}}(u) \otimes t_{\{a_1, \ldots, a_i\}}(u) \otimes t_{\{a_1, \ldots, a_i\}}(u)$$

The lemma 4.4 allows us to evaluate all the terms of equation (4.35), which proves (4.30).
4.2 Antipode and counit

As for the comultiplication, generalised adjoint actions must be introduced to express the antipode.

**Definition 4.6** Let $1 \leq i, j \leq n$, $1 \leq \alpha \leq n$. $X$ denotes an element of $Y(sl_n)$. The generalised adjoint actions are defined by

\[
\alpha \mathcal{H}_{i,j}(u)(X) = \begin{cases} 
1 & \text{if } i > j \\
X & \text{if } i = j, \\
X + \alpha \mathcal{E}_{i,j}(u)(\alpha \mathcal{F}_{j,i}(u+1)(X)) & \text{otherwise},
\end{cases}
\]

(4.36)

\[
\mathcal{H}^\alpha_{i,j}(u)(X) = \begin{cases} 
1 & \text{if } i > j \\
X & \text{if } i = j, \\
X + \mathcal{E}^\alpha_{i,j}(u+1)(\mathcal{F}^\alpha_{j,i}(u)(X)) & \text{otherwise},
\end{cases}
\]

(4.37)

Let $1 \leq m \leq n-1$ and $1 \leq k_1 < k_2 < \cdots < k_m \leq n$. Then, one has:

\[
\alpha \mathcal{H}^\beta_{k_1,k_2,\cdots,k_m}(u)(X) = \left( \prod_{1 \leq p \leq m-1} \alpha \mathcal{H}^\beta_{p,k_p}(u) \right) \left( \alpha \mathcal{H}^\beta_{m+1,k_m}(u) (X) \right).
\]

(4.38)

Similarly, $\tilde{E}_m(u)(X)$ is defined as:

\[
\begin{cases} 
\mathcal{E}^1_{2,n}(u+1)(\mathcal{F}^1_{n-1,1}(u)(X)) & \text{if } m = 1 \\
\mathcal{E}^m_{1,n-m+1}(u+1)\mathcal{F}^m_{n-m,1}(u) \left( \prod_{2 \leq p \leq m-1} \mathcal{H}^m_{p,n-m+p}(u) \right) \mathcal{E}^m_{m+1,n}(u+1)\mathcal{F}^m_{n,m}(u)(X) & \text{otherwise},
\end{cases}
\]

(4.39)

and $\tilde{F}_m(u)(X)$ as:

\[
\begin{cases} 
\mathcal{E}^1_{1,n-1}(u)(\mathcal{F}^1_{n,2}(u+1)(X)) & \text{if } m = 1 \\
\mathcal{E}^m_{1,n-m}(u+1)^m\mathcal{F}^m_{n-m+1,1}(u) \left( \prod_{2 \leq p \leq m-1} m\mathcal{H}^m_{p,n-m+p}(u) \right) m\mathcal{E}_{m,n}(u+1)^m\mathcal{F}_{m,m}(u)(X) & \text{otherwise}.
\end{cases}
\]

(4.40)

To find the image under the antipode, the following lemma is needed.

**Lemma 4.7** For $1 \leq i \leq n-1$ and $1 \leq j \leq i+1$, one has:

\[
t^{j_1\cdots j_{2\cdots m}_{i\cdots j}_{i\cdots j}}(u)(t^{1\cdots n-i}_{1\cdots n-i}(u))^{-1} = H^{n-i}_{j_1\cdots j_{2\cdots m}_{i\cdots j}_{i\cdots j}} (u + n - \frac{i - 2}{2}) (g_{n-i} (u + \frac{n - i - 2}{2})).
\]

(4.41)

and

\[
t^{i_1\cdots i_{2\cdots m}_{i\cdots j}_{i\cdots j}}(u)(t^{1\cdots n-i}_{1\cdots n-i}(u))^{-1} = \tilde{E}_{n-i} (u + n - \frac{i - 2}{2}) (g_{n-i} (u + n - \frac{i - 2}{2})).
\]

(4.42)

and

\[
t^{i_1\cdots i_{2\cdots m}_{i\cdots j}_{i\cdots j}}(u)(t^{1\cdots n-i}_{1\cdots n-i}(u))^{-1} = \tilde{F}_{n-i} (u + n - \frac{i - 2}{2}) (g_{n-i} (u + n - \frac{i - 2}{2})).
\]

(4.43)

and

\[
t^{i_1\cdots i_{2\cdots m}_{i\cdots j}_{i\cdots j}}(u)(t^{1\cdots n-i}_{1\cdots n-i}(u))^{-1} = \tilde{F}_{n-i} (u + n - \frac{i - 2}{2}) (g_{n-i} (u + n - \frac{i - 2}{2})).
\]

(4.44)
\textbf{Remark:} This Hopf structure can be extended to the double Yangian \( \text{DY}(sl_n) \). The image of the generators \( x(u) \in Y(sl_n) \subset \text{DY}(sl_n) \) under the comultiplication, the antipode or the counit are unchanged. For the dual generator \( x^*(u) \) of \( x(u) \), its image is given by the same formula where all the generators are replaced by their dual.

\section*{4.3 Examples}

We give two examples to show explicit computations using the theorems \ref{thm:4.5} and \ref{thm:4.8}. The comultiplication of \( Y(sl_2) \) is given by:

\begin{align*}
\text{Remark:} & \quad \text{This lemma is proven along the same lines as the lemma } \ref{lem:4.4} \quad \blacksquare \\
\text{Theorem 4.8} & \quad \text{The antipode and the counit in the Drinfel’d basis are given by, for } 1 \leq i \leq n - 1: \\
S(e_i(u + \frac{n}{2})) & = -\hat{E}_{n-i}(u) (e_{n-i}(u + 1)) \left( H_{i+1,\cdots,n}^{n-i} u (g_{n-i}(u)) \right)^{-1} \tag{4.45} \\
S(f_i(u + \frac{n}{2})) & = -\left( n \hat{F}_{n-i}(u) (f_{n-i}(u + 1)) \right)^{-1} \tag{4.46} \\
S(h_i(u + \frac{n}{2})) & = \left( n^{-1} H_{i+1,\cdots,n}^{n-i} u (g_{n-i}(u)) \right)^{-1} n^{-1} H_{i+2,\cdots,n}^{i} (g_{n-i}(u)) \tag{4.47}
\end{align*}

\text{Proof:} The proof is given for \( S(e_{n-i}(u)) \). \( S(h_{n-i}(u)) \) and \( S(f_{n-i}(u)) \) are proven analogously.

\begin{align*}
S(e_i(u + \frac{i-2}{2})) & = S(t(1_{i-1}i_{1_{i-1}i})u) S(t(1_{i}i_{1})u)^{-1} \tag{4.49} \\
& = t^*(1_{i-1}i_{1_{i-1}i})u (t^*(1_{i}i_{1})u)^{-1} \tag{4.50} \\
& = -t(i_{i+1 i+2 \cdots n}^{i+1}u + n + i) \left( t(1_{i_{1}i_{1}n-i})u - n + i \right)^{-1} \tag{4.51}
\end{align*}

The terms in the relation \ref{eq:4.51} can be expressed thanks to the lemma \ref{lem:4.7} which proves the relation for \( S(e_{n-i}(u)) \).

The proof for the counit is obvious. \quad \blacksquare
\[
\Delta(e_1(u)) = \sum_{m=0}^{+\infty} \left( -e_1(u) \otimes f_1(u+1) \right)^m \left( 1 \otimes e_1(u) + e_1(u) \otimes (h_1(u) + f_1(u+1)e_1(u)) \right) \\
= 1 \otimes e_1(u) + \sum_{m=0}^{+\infty} (-1)^m e_1(u)^{m+1} \otimes f_1(u+1)^m h_1(u) \\
\Delta(f_1(u)) = \left( f_1(u) \otimes 1 + \left( h_1(u) + f_1(u+1)e_1(u) \right) \otimes f_1(u) \right) \sum_{m=0}^{+\infty} \left( -e_1(u+1) \otimes f_1(u) \right)^m \\
= f_1(u) \otimes 1 + \sum_{m=0}^{+\infty} (-1)^m h_1(u)e_1(u+1)^m \otimes f_1(u)^{m+1} \\
\Delta(h_1(u)) = \left( f_1(u) \otimes e_1(u+1) + \left( h_1(u) + f_1(u)e_1(u+1) \right) \otimes \left( h_1(u) + f_1(u)e_1(u+1) \right) \right) \\
\times \sum_{m=0}^{+\infty} \left( -e_1(u+1) \otimes f_1(u) \right)^m - \Delta(f_1(u)) \Delta(e_1(u+1)) \\
= \sum_{m=0}^{+\infty} (-1)^k (k+1) h(u) e_1(u+1)^k \otimes f_1(u+1)^k h(u) \\
\] (4.52), (4.53), (4.54), (4.55), (4.56), (4.57)

The explicit forms (4.53), (4.55) and (4.57) allows us to show that the comultiplication, introduced in this letter, is the opposite of the comultiplication used by A.I. Molev\[1\]. Remark that the proof of the relation (4.57) from (4.56) is not obvious. A simpler way consists in using the form (3.5) instead of the form (3.3) in the proof of the theorem\[1,5\]. Despite its simpler form, the generalisation to \(sl_n\) of the form given by A.I. Molev does not seem possible.

The antipode and the counit of \(Y(sl_2)\) are given by:

\[
S(e_1(u+1)) = -e_1(u+1) \left( h_1(u) + f_1(u)e_1(u+1) \right)^{-1}, \\
S(f_1(u+1)) = - \left( h_1(u) + f_1(u+1)e_1(u) \right)^{-1} f_1(u+1), \\
S(h_1(u+1)) = \left( h_1(u) + f_1(u+1)e_1(u) \right)^{-1} - S(e_1(u+1)) S(f_1(u)) ,
\]

and

\[
\epsilon(e_1(u)) = 0, \quad \epsilon(f_1(u)) = 0 \quad \text{and} \quad \epsilon(h_1(u)) = 1.
\]

(4.58), (4.59), (4.60), (4.61)

For \(Y(sl_3)\), the comultiplication in the Drinfel’d basis is given by:

\[
\Delta(e_1(u)) = \sum_{m=0}^{+\infty} (-1)^m \left( e_1(u) \otimes f_1(u+1) - [e_2^0, e_1(u)] \otimes [f_2^0, f_1(u+1)] \right)^m \\
\times \left( 1 \otimes e_1(u) + e_1(u) \otimes (h_1(u) + f_1(u+1)e_1(u)) - [e_2^0, e_1(u)] \otimes [f_2^0, h_1(u) + f_1(u+1)e_1(u)] \right)
\]

(4.62)
\[ \Delta(f_1(u)) = \left( f_1(u) \otimes 1 + \left( h_1(u) + f_1(u)e_1(u+1) \right) \otimes f_1(u) - [e_2^0, h_1(u) + f_1(u)e_1(u+1)] \otimes [f_2^0, f_1(u)] \right) \]
\[ \times \sum_{m=0}^{+\infty} (-1)^m \left( e_1(u+1) \otimes f_1(u) - [e_2^0, e_1(u+1)] \otimes [f_2^0, f_1(u)] \right)^m \]  
(4.63)

\[ \Delta(h_1(u)) = \left( f_1(u) \otimes e_1(u) + \left( h_1(u) + f_1(u)e_1(u+1) \right) \otimes \left( h_1(u) + f_1(u)e_1(u+1) \right) - [e_2^0, h_1(u) + f_1(u)e_1(u+1)] \otimes [f_2^0, h_1(u) + f_1(u)e_1(u+1)] \right) \]
\[ \times \sum_{m=0}^{+\infty} (-1)^m \left( e_1(u+1) \otimes f_1(u) - [e_2^0, e_1(u+1)] \otimes [f_2^0, f_1(u)] \right)^m \]
\[ - \Delta(f_1(u)) \Delta(e_1(u+1)) \]  
(4.64)

\[ \Delta(e_2(u)) \text{ (resp. } \Delta(f_2(u)), \Delta(h_2(u)) \text{) is obtained by exchanging the subscripts 1 and 2 in equation (1.62) (resp. (1.63), (1.64)). Of course, the antipode and the counit for } Y(sl_3) \text{ can be computed thanks to theorem 4.8 however we leave it to the reader.} \]

Acknowledgements: I warmly thank D. Arnaudon, J. Avan, V. Caudrelier, L. Frappat and E. Ragoucy for discussions and advice.

References

[1] V.G. Drinfeld, *Hopf algebras and the quantum Yang–Baxter equation*, Soviet. Math. Dokl. 32 (1985) 254–258.

[2] V.G. Drinfeld, *Quantum Groups*, Proceedings Int. Cong. Math. Berkeley, California, USA (1986) 798–820.

[3] V.G. Drinfeld, *A new realization of Yangians and quantized affine algebras*, Soviet. Math. Dokl. 36 (1988) 212–216.

[4] A.I. Molev, *Yangians and their applications*, Handbook of Algebra, vol. 3, Elsevier, to appear.

[5] D. Arnaudon, N. Crampé, L. Frappat, É. Ragoucy, *Super Yangian Y(osp(1|2)) and the Universal R-matrix of its Quantum Double*, math.QA/0209167.

[6] V. Chari, A. Pressley, *Fundamental representations of Yangians and singularities of R-matrices* J. reine angew. Math. 417 (1991) 87–128.

[7] S.M. Khoroshkin, V.N. Tolstoy, *Yangian double and rational R-matrix*, hep-th/9406194.

[8] K. Iohara, *Bosonic representations of Yangian Double DY_h(g) with g = gl_N,sll_N*, J. Phys. A (Math. Gen.) 29 (1996) 4593–4621, q-alg/9603033.

[9] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. 1 (1990) 193–225.