On the number of maximal independent sets: From Moon–Moser to Hujter–Tuza

Cory Palmer1 | Balázs Patkós2

1Department of Mathematical Sciences, University of Montana, Missoula, Montana, USA
2Department of Combinatorics and its Applications, Alfréd Rényi Institute of Mathematics, Budapest, Hungary

Correspondence
Balázs Patkós, Department of Combinatorics and its Applications, Alfréd Rényi Institute of Mathematics, Budapest, Hungary.
Email: patkos@renyi.hu

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Abstract
We connect two classical results in extremal graph theory concerning the number of maximal independent sets. The maximum number $\text{mis}(n)$ of maximal independent sets in an $n$-vertex graph was determined by Miller and Muller and independently by Moon and Moser. The maximum number $\text{mis}_\Delta(n)$ of maximal independent sets in an $n$-vertex triangle-free graph was determined by Hujter and Tuza. We give a common generalization of these results by determining the maximum number $\text{mis}_t(n)$ of maximal independent sets in an $n$-vertex graph containing no induced triangle matching of size $t + 1$. This also improves a stability result of Kahn and Park on $\text{mis}_\Delta(n)$. Our second result is a new (short) proof of a second stability result of Kahn and Park on the maximum number $\text{mis}_{\Delta^*}(n)$ of maximal independent sets in $n$-vertex triangle-free graphs containing no induced matching of size $t + 1$.

KEYWORDS
extremal graph theory, induced (triangle) matchings, maximal independent sets, stability results

1 | INTRODUCTION

Recall that a set of vertices in a graph is independent if no two are adjacent. An independent set is maximal if it is not a proper subset of another independent set. Let $\text{mis}(G)$ denote the number of maximal independent sets in a graph $G$. The classic result determining $\text{mis}(n)$, the maximum of $\text{mis}(G)$ over all graphs $G$ on $n$ vertices is:
**Theorem 1** (Miller and Muller [9] and Moon and Moser [10]). For \( n \geq 3 \), we have

\[
\text{mis}(n) = \begin{cases} 
3^{n/3} & \text{if } n \text{ is divisible by 3,} \\
4 \cdot 3^{(n-4)/3} & \text{if } n \equiv 1 \pmod{3}, \\
2 \cdot 3^{(n-2)/3} & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

For graphs \( F, G \) and positive integers \( a, b \), we denote by \( aF + bG \) the vertex-disjoint union of \( a \) copies of \( F \) and \( b \) copies of \( G \). Then the constructions giving the lower bound in Theorem 1 are \( \frac{n}{3}K_3, \frac{n-4}{3}K_3 + K_4 \) or \( \frac{n-4}{3}K_3 + 2K_2 \), and \( \frac{n-2}{3}K_3 + K_2 \) in the three respective cases. As these constructions contain many triangles, one can ask the natural question to maximize \( \text{mis}(G) \) over all triangle-free graphs. The maximum over all such \( n \)-vertex graphs, denoted by \( \text{mis}_\triangle(n) \), was determined by Hujter and Tuza [7].

**Theorem 2** (Hujter and Tuza [7]). For \( n \geq 4 \), we have

\[
\text{mis}_\triangle(n) = \begin{cases} 
2^{n/2} & \text{if } n \text{ is even,} \\
5 \cdot 2^{(n-5)/2} & \text{if } n \text{ is odd.}
\end{cases}
\]

The maximum of the parameter \( \text{mis}(G) \) has been determined for several graph classes including \( n \)-vertex connected graphs (see [5, 6]) and \( n \)-vertex trees (see [11, 13]). The value of \( \text{mis}(n) \) has implications for the runtime of various graph-coloring algorithms (see [14] for several references). Bounds on \( \text{mis}(G) \) for various classes of graphs are given in [1, 2, 4] to apply the container method [3, 12] to count maximal sum-free subsets and maximal triangle-free graphs.

Answering a question of Rabinovich, Kahn and Park [8] proved stability versions of both Theorems 1 and 2. An **induced triangle matching** is an induced subgraph that is a vertex-disjoint union of triangles; its **size** is the number of its triangles.

**Theorem 3** (Kahn and Park [8]). For any \( \varepsilon > 0 \), there is a \( \delta = \delta(\varepsilon) = \Omega(\varepsilon) \) such that for any \( n \)-vertex graph \( G \) that does not contain an induced triangle matching of size \( (1 - \varepsilon) \frac{n}{3} \), we have \( \log \text{mis}(G) < \left( \frac{1}{3} \log 3 - \delta \right)n \).

An **induced matching** is an induced subgraph that is a matching; its **size** is the number of its edges.

**Theorem 4** (Kahn and Park [8]). For any \( \varepsilon > 0 \), there is a \( \delta = \delta(\varepsilon) = \Omega(\varepsilon) \) such that for any \( n \)-vertex triangle-free graph \( G \) that does not contain an induced matching of size \( (1 - \varepsilon) \frac{n}{2} \), we have \( \log \text{mis}(G) < \left( \frac{1}{2} - \delta \right)n \).

Let \( \text{mis}_t(n) \) denote the maximum number of maximal independent sets in an \( n \)-vertex graph that does not contain an induced triangle matching of size \( t + 1 \). With this notation we have \( \text{mis}_0(n) = \text{mis}_\triangle(n) \) and Theorem 3 gives \( \text{mis}_t(n) < 3^{(1/3-\delta \log_3 \varepsilon)n} \) when \( t + 1 = (1 - \varepsilon) \frac{n}{3} \).
The first result of this note is the following common generalization of Theorems 1 and 2 which gives a strengthening of Theorem 3 as it determines $\text{mis}_t(n)$ for all $n$ and $t \leq n/3$. Observe that the case $n \leq 3t$ has no restrictions on $G$ as the number of vertex-disjoint triangles is clearly no more than $n/3$. So in this case, Theorem 1 gives the value of $\text{mis}_t(n)$.

**Theorem 5.** For any $0 \leq t \leq n/3$ put $m = n - 3t$. Then

$$\text{mis}_t(n) = \begin{cases} 3^t \cdot 2^{m/2} & \text{if } m \text{ is even}, \\ 3^{t-1} \cdot 2^{(m+3)/2} & \text{if } m \text{ is odd and } t > 0, \\ 5 \cdot 2^{(n-5)/2} & \text{if } m \text{ is odd and } t = 0. \end{cases}$$

Constructions showing the lower bounds are $tK_3 + \frac{m}{2}K_2, (t - 1)K_3 + \frac{m+3}{2}K_2$, and $C_5 + \frac{n-5}{2}K_2$, respectively.

Let $\text{mis}_{\Delta, t}(n)$ denote the maximum of $\text{mis}(G)$ over all $n$-vertex triangle-free graphs that do not contain an induced matching of size $t + 1$. The second result of this note is a short proof of the following version of Theorem 4.

**Theorem 6.** Let $c$ denote the largest real root of the equation $x^6 - 2x^2 - 2x - 1 = 0$, $c = 1.40759 \ldots < \sqrt{2}$. Then $\text{mis}_{\Delta, t}(n) \leq 2^t c^{n-2t}$.

## PROOFS

In our proofs we shall use an observation due to Wood [14]. It follows from the fact that any maximal independent set in $G$ must meet the closed neighborhood $N[v] = N(v) \cup \{v\}$ of any vertex $v$ of $G$.

**Observation 7 (Wood [14]).** For any graph $G$ and vertex $v \in V(G)$, we have

$$\text{mis}(G) \leq \sum_{w \in N[v]} \text{mis}(G \setminus N[w]).$$

We begin with several inequalities involving the bound, denoted $g_t(n)$, from the statement of Theorem 5.

**Fact 8.** For $t > 0, k \geq 4$, and $n \geq 3t + k$, we have $\frac{g_t(n-3)}{g_t(n)} \leq 3/8$, $\frac{g_t(n-2)}{g_t(n)} = 1/2$, $\frac{g_t(n-4)}{g_t(n)} = 1/4$, and $k \cdot g_t(n - k) \leq g_t(n)$.

**Proof.** The first three statements follow from definition by considering two cases according to the parity of $n$. To see $k \cdot g_t(n - k) \leq g_t(n)$, we distinguish two cases. If $k = 2\ell$ is even, then applying $2g_t(n - 2) = g_t(n) \ell$ times gives $k \cdot g_t(n - k) \leq 2^\ell g_t(n - k) = g_t(n)$. If $k = 2\ell + 1$, then applying $\frac{8}{3}g_t(n - 3) \leq g_t(n)$ once and $2g_t(n - 2) = g_t(n) \ell - 1$ times gives $k \cdot g_t(n - k) \leq \frac{8}{3} 2^{\ell-1} g_t(n - k) \leq g_t(n)$. □
Observe that if \( n \) is odd and \( t = 0 \), then the bounds in Fact 8 may not hold and, in particular, Case III of the following argument will not work. Fortunately, we may assume that \( t > 0 \) as the \( t = 0 \) case is exactly Theorem 2.

In the proof, we will always compare \( g_t(n - k) \) to \( g_t(n) \), and it might happen that \( n - k \) drops below \( 3t \). In this case, we consider \( g_t(n - k) \) to be \( g_t(n - 3) \). Fortunately, each (in) equality in Fact 8 continues to hold as \( g_t(n) \) is nondecreasing in \( t \).

**Proof of Theorem 5.** By the discussion above we may assume \( t > 0 \). We proceed by induction on \( m \). Observe that the cases \( m = 0, 1, 2 \) are covered by Theorem 1. Let \( G \) be a graph on \( n \) vertices not containing an induced triangle matching of size \( t + 1 \). We distinguish cases according to the minimum degree of \( G \).

**Case I.** \( G \) has a vertex \( x \) of degree 1.
Applying Observation 7 with \( v = x \) and Fact 8 yields
\[
\text{mis}(G) \leq 2\text{mis}(n - 2) \leq 2g_t(n - 2) \leq g_t(n).
\]

**Case II.** \( G \) has a component \( C \) of minimum degree \( d \geq 3 \).
Applying Observation 7 to any \( v \in C \) and Fact 8 yields
\[
\text{mis}(G) \leq (d + 1)\text{mis}(n - d - 1) \leq g_t(n).
\]

**Case III.** \( G \) has a component \( C \) with a vertex \( x \) of degree 2 and a vertex of degree at least 3.
We may assume that \( x \) is adjacent to a vertex \( y \) of degree \( d(y) \geq 3 \). Applying Observation 7 with \( v = x \) and Fact 8, we obtain
\[
\text{mis}(G) \leq 2\text{mis}(n - 3) + \text{mis}(n - 4) \leq 2g_t(n - 3) + g_t(n - 4) \leq g_t(n).
\]

**Case IV.** \( G \) is 2-regular, that is, a cycle factor.
It is not hard to verify (see, e.g., [5]) that \( \text{mis}(C_3) = 3 \), \( \text{mis}(C_4) = 2 \), \( \text{mis}(C_5) = 5 \), and \( \text{mis}(C_n) = \text{mis}(C_{n-2}) + \text{mis}(C_{n-3}) \). In particular, if \( n \neq 3 \), then \( \text{mis}(C_n)^{1/n} \) is maximized for \( n = 5 \) with value \( 5^{1/5} \). Thus, for cycle factors containing at most \( t \) triangles, we have
\[
\text{mis}(G) \leq 3^t \cdot 5^{(n-3t)/5} \leq g_t(n).)
\]

Before the proof of Theorem 6, we gather facts about the bound \( h_t(n) := 2^t \cdot c^{n-2t} \).

**Fact 9.** For the largest real root \( c = 1.40759 \ldots < \sqrt{2} \) of \( x^6 - 2x^2 - 2x - 1 = 0 \), we have

1. \( 2 + c \leq 2c^2 \) and so \( h_t(n - 2) + h_{t-1}(n - 3) \leq h_t(n) \),
2. for \( d \geq 4 \), we have \( (d + 1) \leq c^{d+1} \) and so \( (d + 1)h_t(n - d - 1) \leq h_t(n) \),
3. \( 3c + 1 \leq c^3 \) and so \( 3h_t(n - 4) + h_t(n - 5) \leq h_t(n) \),
4. \( 2c + 1 \leq c^4 \) and so \( 2h_t(n - 3) + h_t(n - 4) \leq h_t(n) \),
5. \( 2c^2 + 2c + 1 = c^6 \) and so \( 2h_t(n - 4) + 2h_t(n - 5) + h_t(n - 6) = h_t(n) \).

Note that (5) yields the equation defining \( c \).
Proof. For (1), it is easy to confirm $2 + c \leq 2c^2$ numerically. Then the inequality involving $h_t(n)$ follows by its definition. Indeed, $h_t(n) = 2^t \cdot c^{n-2t} = 2c^2(2^{t-1} \cdot c^{n-2-2t}) \geq (c + 2)(2^{t-1} \cdot c^{n-2-2t}) = 2^{t-1} \cdot c^{n-3-2(t-1)} + 2^t \cdot c^{n-2t-2} = h_t(n-2) + h_{t-1}(n-3)$. A similar argument establishes (3)–(5). For (2), it is easy to check the statement for $d = 4$ and then observe that when $(d + 1)$ increases by 1, then $c^{d+1}$ increases by more than 1.

The (in)equalities in Fact 9 correspond to distinct cases in the inductive proof of Theorem 6. Just as in the proof of Theorem 5, $n - k$ might drop below $2t$, in which case we consider $h_t(n - k)$ to be $h_{t+k}(n - k)$ and again each (in)equality in Fact 9 continues to hold.

Proof of Theorem 6. We proceed by double induction. First on $m := n - 2t$ and then on $t$. Theorem 2 yields the statement when $m = 0$, 1. For any $m$, if $t = 0$, then the only graph $G$ on $m - 2t = m$ vertices that does not contain an induced matching of size one is the empty graph and in this case $\text{mis}(G) = 1 \leq h_0(m) = c^m$. Let $G$ be an $n$-vertex triangle-free graph that contains no induced matching of size $t + 1$. We again distinguish cases according to the minimum degree of $G$.

Case I. $G$ has a vertex $x$ of degree 1.

Let $y$ be the neighbor of $x$. If $xy$ is an isolated edge, then $G \backslash \{x, y\}$ does not contain induced matchings of size $t$. Applying Observation 7 yields $\text{mis}(G) \leq 2\text{mis}_{\Delta,t-1}(n - 2) \leq 2h_{t-1}(n - 2) = h_t(n)$. If $y$ has further neighbors, then $G \backslash N[y]$ does not contain induced matchings of size $t$. Applying Observation 7 and Fact 9 yields $\text{mis}(G) \leq \text{mis}_{\Delta,t}(n - 2) + \text{mis}_{\Delta,t-1}(n - 3) \leq h_t(n - 2) + h_{t-1}(n - 3) \leq h_t(n)$.

Case II. $G$ has minimum degree $d \geq 4$.

Applying Observation 7 and Fact 9 yields $\text{mis}(G) \leq (d + 1)h_t(n - d - 1) \leq h_t(n)$.

Case III. $G$ has a component $C$ of minimum degree 3 with a vertex of degree at least 4.

Let $x$ be a vertex of $C$ of degree 3 and $y \in N(x)$ of degree at least 4. Applying Observation 7 and Fact 9 yields $\text{mis}(G) \leq 3h_t(n - 4) + h_t(n - 5) \leq h_t(n)$.

Case IV. $G$ is 3-regular.

Suppose first that there exist vertices $x, y$ with $N(x) = N(y)$. Immediately, $x$ and $y$ cannot be adjacent. Moreover, for any maximal independent set $X$, we have $x \in X$ if and only if $y \in X$. Therefore, when applying Observation 7, the number of maximal independent sets containing $x$ can be bounded by $\text{mis}(G \backslash (N[x] \cup \{y\}))$ instead of $\text{mis}(G \backslash N[x])$. We obtain $\text{mis}(G) \leq 3h_t(n - 4) + h_t(n - 5) \leq h_t(n)$.

Suppose next that $G$ does not contain two vertices $x, y$ with $N(x) = N(y)$. Let $v$ be an arbitrary vertex of $G$ and $N(v) = \{a, b, c\}$. We apply Observation 7 in a slightly modified form: we keep $\text{mis}(G \backslash N[v])$ and $\text{mis}(G \backslash N[a])$, but to bound the number of maximal independent sets $X$ that contain $b$ or $c$, we use $\text{mis}(G \backslash (N[b] \cup \{c\})) + \text{mis}(G \backslash (N[c] \cup \{b\})) + \text{mis}(G \backslash (N[b] \cup N[c]))$. The three terms bound the number of maximal independent sets $X$ that contain only $b$, only $c$ or both $b$ and $c$ (from $\{b, c\}$), respectively. Observe that by the triangle-free property, $b$ and $c$ are not adjacent. Also, as $N(b) \neq N(c)$, we have $|N[b] \cup N[c]| \geq 6$. Therefore, we
\begin{align*}
\text{Case V.} & \quad G \text{ has a component } C \text{ of minimum degree } 2 \text{ with a vertex of degree at least } 3. \\
& \text{Let } x \text{ be a vertex of } C \text{ of degree } 2, \text{ and } y \in N(x) \text{ of degree at least } 3. \text{ By applying Observation 7 and Fact 9, we obtain }
\text{mis}(G) \leq 2h_t(n - 3) + h_t(n - 4) \leq h_t(n). \\
\text{Case VI.} & \quad G \text{ is 2-regular, that is, a cycle factor.} \\
& \text{As in Case IV of Theorem 5, we have mis}(G) \leq 5^{n/5} \leq h_t(n). \quad \Box
\end{align*}

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**ORCID**

\begin{itemize}
\item Cory Palmer  \hspace{1cm} \url{http://orcid.org/0000-0002-6718-5762}
\item Balázs Patkós  \hspace{1cm} \url{http://orcid.org/0000-0002-1651-2487}
\end{itemize}

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