N=1 M-theory-Heterotic Duality in Three Dimensions and Joyce Manifolds.

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Abstract

It is argued that $M$-theory compactifications on any of Joyce’s $Spin(7)$ holonomy 8-manifolds are dual to compactifications of heterotic string theory on Joyce 7-manifolds of $G_2$ holonomy.
1 Introduction.

Over the past year or so, our perception of string theory has dramatically altered [1, 2]. The emergence of the new dimension which has opened up in our understanding, can perhaps be attributed to the magical $M$-theory [3, 4]. One virtue of this viewpoint is that one can “understand” connections between string theories [1, 2, 5, 6, 7, 8, 9] from an eleven dimensional perspective [3, 4].

In [11], we presented an ansatz to construct dual $M$-theory/heterotic compactifications, starting from $M$-theory/heterotic duality in seven dimensions. In [11] we focussed on dual compactifications with $N = 1$ supersymmetry in four dimensions. This paper is devoted to constructing $M$-theory/heterotic duals with $N = 1$ supersymmetry in three dimensions, by applying the same ansatz as in [11]. In fact, we will construct heterotic duals (on Joyce 7-manifolds of $G_2$ holonomy) for $M$-theory compactifications on all known Joyce 8-manifolds of $Spin(7)$ holonomy.

It turns out, following the recent work of Sen [17], that one can explain the ansatz presented in [11] in the following way. Consider $M$-theory on an eight-torus, $T^8$, with coordinate labels $x_1, x_2, \ldots x_8$. This is equivalent [2] to Type IIA string theory on $T^7$, where the coordinate labels of $T^7$ are any seven element subset of $x_1, x_2, \ldots x_8$. For definiteness, we choose $T^7$ to be labelled by $x_1, x_2, x_3, x_5, x_6, x_7, x_8$. Take a $Z_2$ orbifold of the $M$-theory compactification, where the $Z_2$ acts on $x_1, x_2, x_3, x_4$ as reflection. This gives $M$-theory on an orbifold limit of $K3 \times T^4$. In [17], it was argued that this orbifold commutes with Type IIA/$M$-theory duality and is equivalent to Type IIA theory on $T^7/Z_2'$, where $Z_2'$ acts as reflection on $x_1, x_2, x_3$ combined with world-sheet parity and the element $(-1)^{F_L}$, where $F_L$ is the left moving fermion number operator in the IIA theory. One can then make a $T$-duality transformation on the three circles labelled by $x_1, x_2, x_3$, which inverts their radii. This gives the Type IIB theory on a $T^7/\Omega$ orbifold, where $\Omega$ is the world-sheet parity operator. This is by definition Type I theory on a $T^7$ which shares the same coordinate labels as its equivalent Type IIA compactification above. One can then use Type I-heterotic duality [13] to map this to the heterotic string compactified on $T^7$. The crucial point, for what follows, is that the seven coordinate labels of the heterotic string on $T^7$ are a subset of the eight coordinate labels of the $M$-theory compactification on $T^4/Z_2 \times T^4$.

The analysis presented above is an explanation of the first part of the
ansatz presented in [11]. As in [11], we wish to consider taking further $Z_2$ orbifolds of the three dimensional $N = 8$ $M$-theory compactification we are considering. In this paper we will consider orbifolds which break the supersymmetry to $N = 1$. We will then resolve all the orbifold singularities, which will give $M$-theory compactifications on smooth Joyce 8-manifolds of $Spin(7)$ holonomy [14]. As in [4, 17, 19, 20], and by analogy with string theory, we can assume that $M$-theory on an orbifold has twisted sectors which are a necessary requirement for consistency of the theory. We will further assume, again by analogy with string theory, that the massless fields associated with the resolution of orbifold singularities constitute precisely these twisted sectors of $M$-theory. It is possible that these twisted sectors will have a $p$-brane interpretation [19, 17, 20] in future.

A crucial point is the following: because the coordinate labels of the $T^7$ compactified heterotic string are a subset of the dual $T^4/Z_2 \times T^4$ $M$-theory compactification, these further orbifolds, which break $N = 8$ to $N = 1$, also define orbifolds of the heterotic string on $T^7$. To this end, these further orbifolds must only act on these seven “common” coordinates, if the orbifolding is to commute with the original duality. This was the case for the dual pairs conjectured in [11]. Further, because we resolve all singularities in the $M$-theory compactification, it is natural to do this in the heterotic compactification. We thus have the following picture, which summarises the construction of the dual pairs in the following sections:

$M$-theory on $\frac{S^1 \times T^7}{Z_2 \times \Theta}$ is dual to the heterotic string theory on $\frac{T^7}{\Theta}$.

Here, $Z_2$ is the original $Z_2$ which acts as reflection on a $T^4$ factor, giving $M$-theory on an orbifold limit of $K3 \times T^4$, which is the $N = 8$ theory dual to the heterotic string on $T^7$. This $Z_2$ is the only generator of the orbifold group which acts on the $S^1$ whose coordinate label is not common to both compactifications. $\Theta$ denotes the orbifold group which breaks supersymmetry from $N = 8$ to $N = 1$. The definition of $\Theta$ on the $M$-theory coordinates then defines its action on the $T^7$ coordinates of the heterotic theory. We will see in the next section that the heterotic compactification also has $N = 1$ supersymmetry on general grounds.

In the next section we consider a simple example in some detail. In section three we show that our ansatz allows one to construct heterotic dual compactifications for $M$-theory compactifications on all known, to date, Joyce 8-manifolds of $Spin(7)$ holonomy [14].
2 Construction.

The long wavelength dynamics of $M$-theory is effectively described by eleven dimensional supergravity. This has a bosonic massless field content consisting solely of the metric and three-form potential. Compactification of the eleven dimensional theory on 8-manifolds of $\text{Spin}(7)$ holonomy was considered in [15]. This gives a three-dimensional supergravity theory with one supersymmetry in the vacuum. The non-trivial Betti numbers of a $\text{Spin}(7)$ Joyce manifold are $b^2$, $b^3$ and $b^4$; with $b_1 = 0$, $b_4 = b_4^+ + b_4^-$, where $b_4^+$ and $b_4^-$ are the dimensions of the self-dual and anti-self-dual pieces of $H^4(X)$ respectively, where $X$ is the Joyce manifold. The dimension of the moduli space of such a Joyce manifold is $b_4^- + 1$ [14]. Thus, compactification of the eleven dimensional theory on a Joyce $\text{Spin}(7)$ 8-manifold leads to a three dimensional $\mathcal{N} = 1$ vacuum with $b_2 + b_3 + b_4^- + 1$ massless scalar supermultiplets [14]. In this paper we will apply our ansatz to $M$-theory compactified on Joyce 8-manifolds of $\text{Spin}(7)$ holonomy. We will restrict our attention to the study of the simplest example of such a manifold [14], and present our full results in the next section.

In [14], Joyce constructed many examples of $\text{Spin}(7)$ 8-manifolds as blown up orbifolds of the eight torus, $T^8$. As far as we are aware, these are, to date, the only known manifolds of this type. As in [14], let us denote the finite group, by which one is modding out, by $\Gamma$;ie in the notation of the introduction $\Gamma \equiv \mathbb{Z}_2 \times \Theta$. It was realised in [14] that there arise essentially five types of singularity in the space $T^8/\Gamma$ which one has to blow up to construct a Joyce 8-manifold of $\text{Spin}(7)$ holonomy. Each of these different blow ups contribute different numbers of massless scalars to the $M$-theory compactification. We will label these blow ups by (i) - (v). Of the five types, three have a unique resolution. They make the following contribution to the Betti numbers of the manifold:

\[
\text{Type}(i) : \text{ adds } 1 \text{ to } b_2, \ 4 \text{ to } b_3, \ 3 \text{ to } b_4^+, \ 3 \text{ to } b_4^- . \quad (1)
\]
\[
\text{Type}(ii) : \text{ adds } 1 \text{ to } b_2, \ 3 \text{ to } b_4^+, \ 3 \text{ to } b_4^- . \quad (2)
\]
\[
\text{Type}(iii) : \text{ adds } 1 \text{ to } b_4^+ . \quad (3)
\]

\footnote{The definition of these singularities will be irrelevant to this paper. The interested reader may consult [14] for details.}
For each of the other two types there exist two topologically distinct resolutions of the singularity:

**Type (iv) resolution (A)**: adds 1 to $b_2$, 2 to $b_3$, 1 to $b_4^+$, 1 to $b_4^-$.  (4)

**Type (iv) resolution (B)**: adds 2 to each of $b_3$, $b_4^+$, $b_4^-$.  (5)

**Type (v) resolution (A)**: adds 1 to each of $b_2$, $b_4^+$, $b_4^-$.  (6)

**Type (v) resolution (B)**: adds 2 to both $b_4^+$, $b_4^-$.  (7)

Begin with $M$-theory on $T^8$. Orbifold this theory by the $Z_2$ isometry denoted by $\alpha$, defined as follows:

$$\alpha(x_1, x_2, \ldots, x_8) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8)$$  (8)

where $(x_1, \ldots, x_8)$ are the coordinates of $T^8$. Resolving the sixteen singularities associated with $\alpha$ gives $M$-theory on $K3 \times T^4$, which we expect to be equivalent \cite{2} to the heterotic string on $T^7$.

The crux of the ansatz \cite{11} which we explained in the introduction following Sen \cite{17}, is that the $T^7$ coordinates are labelled by a subset of $(x_1, \ldots x_8)$. For definiteness take the labels of $T^7$ as\footnote{According to our discussion in the introduction, the $T^7$ coordinate labels must contain $x_5, x_6, x_7, x_8$ and any three of $x_1, x_2, x_3, x_4$. We choose these to be $x_1, x_2, x_3$ for ease of compatibility with \cite{14}. However, we are of course free to choose any three labels. This only means that the group $\Theta$ which we further mod out by must be suitably modified so that it acts solely on the chosen seven labels.}

$$(x_1, x_2, x_3, x_5, x_6, x_7, x_8)$$  (9)

Any further orbifolds of the $M$-theory geometry will then also be orbifolds of the heterotic geometry. In order to avoid confusion between the heterotic and $M$-theory geometries later in the paper, we will re-label the $T^7$ coordinates of the heterotic string as follows:

$$(x_1, x_2, x_3, x_5, x_6, x_7, x_8) \equiv (y_1, y_2, y_3, y_4, y_5, y_6, y_7)$$  (10)

so that from now on $x_i$ will label coordinates of the $M$-theory background and $y_i$ those of the heterotic background. We therefore have $M$-theory on a $K3 \times T^4$ background specified by $\alpha$ and $(x_1, \ldots x_8)$ and the heterotic string on a
seven torus labelled by \((y_1, \ldots y_7)\). When we speak of \(T^8\), it is implied that we are discussing the \(M\)-theory compactification; similarly, when we speak of \(T^7\), we are implicitly discussing the heterotic compactification. In what follows we will take further orbifolds of these two \(d = 3, N = 8\) theories (which will give rise to \(N = 1\) vacua in three dimensions), resolve all singularities and consider the massless spectra.

Consider then the following \(\Gamma \equiv (\mathbb{Z}_2)^4 \equiv (\alpha, \Theta)\) orbifold of \(T^8\) generated by the following isometries:

\[
\alpha(x_1, \ldots x_8) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8) \quad (11)
\]

\[
\beta(x_1, \ldots x_8) = (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8) \quad (12)
\]

\[
\gamma(x_1, \ldots x_8) = (c_1 - x_1, c_2 - x_2, x_3, x_4, c_5 - x_5, c_6 - x_6, x_7, x_8) \quad (13)
\]

\[
\delta(x_1, \ldots x_8) = (d_1 - x_1, x_2, d_3 - x_3, x_4, d_5 - x_5, x_6, d_7 - x_7, x_8) \quad (14)
\]

where the \(c_i\) and \(d_i\) are constants which remain to be specified and \(\alpha\) is precisely the \(\mathbb{Z}_2\) element which defines the \(K3 \times T^4\) background dual to the heterotic string on \(T^7\). According to our ansatz then, \(\Gamma\) also defines the action on the heterotic string toroidally compactified to three dimensions. This is given by \(\Theta \equiv (\mathbb{Z}_2)^3\) generated by \((\beta, \gamma, \delta)\). Thus, the heterotic dual compactification according to our ansatz, will be on a blown up orbifold of \(T^7\), with orbifold group defined by \(\Theta\).

It can be checked [14] that \(\Gamma\) preserves the torsion free \(Spin(7)\) structure defineable on \(T^8\). This orbifold therefore has discrete holonomy contained in \(Spin(7)\). By considering specific values for the constants \(c_i\) and \(d_i\) Joyce [14] encountered the five singularity types mentioned above. Blowing up all of these singularities in each case leads to a smooth compact 8-manifold of \(Spin(7)\) holonomy.

Let us consider the “untwisted” sector of \(M\)-theory on the orbifold defined by equations (11)-(14). To compute the massless spectrum, it suffices to calculate the Betti numbers of \(T^8/\Gamma\). These are computed to be: \(b_1 = b_2 = b_3 = 0\) and \(b_4^+ = b_4^- = 7\). This is always the case for any choice of the constants \(c_i\) and \(d_i\). This implies that \(M\)-theory on the orbifold defined by equations (11)-(14) has an untwisted sector consisting of \(N = 1\) three-dimensional supergravity coupled to eight scalar multiplets. We will see shortly the interpretation of this fact in terms of the dual heterotic theory.
The first example considered in [14] is the following. Set

\[(c_1, c_2, c_5, c_6) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\]  (15)

and

\[(d_1, d_3, d_5, d_7) = (0, 1/2, 1/2, 1/2)\]  (16)

In this example, it may be checked [14] that the singular set of \(T^8/\Gamma\) contains four singularities of Type \((i)\), eight of Type \((ii)\) and 64 of Type \((iii)\). Some simple arithmetic shows that the non-zero Betti numbers of the smooth \(Spin(7)\) manifold are

\[b_2 = 12, b_3 = 16, b_4^- = 43, b_4^+ = 107\]  (17)

We can, by analogy with string theory, consider the massless fields coming from the blowing up modes of the \(Spin(7)\) manifold, as constituting twisted sector states of the \(M\)-theory background; ie in addition to the “universal” eight scalar multiplets from the untwisted sector, we have 64 from the twisted sector. This means that \(M\)-theory compactified on such a manifold gives a three-dimensional \(N = 1\) theory with 72 scalar multiplets.

Let us now see what this implies for the heterotic theory. According to our ansatz, the finite group \(\Theta \equiv \mathbb{Z}_2^3\) will act on the heterotic \(T^7\) coordinates as follows:

\[\beta (y_1...y_7) = (y_1, y_2, y_3, -y_4, -y_5, -y_6, -y_7)\]  (18)

\[\gamma (y_1, ...y_7) = (c_1 - y_1, c_2 - y_2, y_3, c_5 - y_4, c_6 - y_5, y_6, y_7)\]  (19)

\[\delta (y_1, ...y_8) = (d_1 - y_1, y_2, d_3 - y_3, d_5 - y_4, y_5, d_7 - y_6, y_7)\]  (20)

It is not too difficult to show [12, 13] that, in general the resolution of the singularities in \(T^7/\Theta\) will always give rise to a Joyce 7-manifold of \(G_2\) holonomy, for any choice of the constants, \(c_i\) and \(d_i\). This will give rise to a heterotic background in three dimensions which always has \(N = 1\) supersymmetry. This is consistent with the fact that the dual \(M\)-theory compactification also has one supersymmetry in three dimensions. This fact is a highly non-trivial statement because the isometry group defined by equations (18)-(20) is essentially the only \((\mathbb{Z}_2)^3\) group which gives rise to the correct holonomy (ie \(G_2\)) for the heterotic compactification space, for a specific choice of \(G_2\) structure [14].
Let us consider, as we did above for the $M$-theory compactification, the “orbifold interpretation”. In the untwisted sector of the heterotic string on the orbifold defined by equations (18)-(20), one finds for the massless modes three dimensional supergravity coupled to eight scalar multiplets. This is precisely in accord with what we found for the dual $M$-theory result above\footnote{4}. However, different choices of the constants $c_i$ and $d_i$ will lead to different Betti numbers for the $M$-theory and heterotic backgrounds respectively and it would be truly remarkable if the massless spectra in the two theories are the same. We will demonstrate that this is indeed the case for all the $Spin(7)$ 8-manifolds constructed by Joyce \cite{Joyce}. Let us consider our example.

It can be checked \cite{Joyce} that for the specific choice of constants $c_i$ and $d_i$ that we are considering that the resolution of orbifold singularities on the heterotic side of the duality map gives the heterotic string on a Joyce $G_2$ manifold with Betti numbers $b_2=12$ and $b_3=43$. In order to specify the heterotic background we must specify the expectation value of the gauge fields of the heterotic string on the Joyce manifold. In this paper, we will limit ourselves to an abelian embedding of the spin connection in the gauge connection, such that the heterotic gauge group, $SO(32)$ or $E_8 \times E_8$ is broken to its maximal abelian subgroup, as in \cite{Witten}. With this choice of embedding, the massless spectrum of the heterotic compactification is an $N=1$ theory in three dimensions with 72 scalar multiplets, precisely what is expected by duality!

\footnote{The heterotic theory also has 16 vector multiplets which are dual to scalar multiplets in three dimensions. However, the $M$-theory duals of these vectors have in all examples to date arisen from the twisted sector of the $M$-theory compactification. So, it suffices to consider the “matter” sector.}

\footnote{This choice of embedding is possible because the Joyce manifolds of $G_2$ holonomy which we consider in this paper contain $K3$ submanifolds \cite{Joyce} (or see \cite{Dolphin}). If we restrict the gauge fields to take expectation values on a $K3$ submanifold, then it is known that abelian embeddings are consistent \cite{Bogomolov}. It is of course possible that abelian embeddings are consistent for Joyce manifolds in much more generality.}
3 A Heterotic Dual for All M-theory Compactifications on Joyce Spin(7) Manifolds.

The Spin(7) manifold constructed in the previous section is the simplest example of [14]. This was because the singular set of $T^8/\Gamma$ contained elements of Type (i)-(iii), for each of which there exists a unique resolution. All the other examples in [14] contain, in addition, singular elements of Type (iv) and (v), for each of which there exist two topologically distinct choices of resolution. It follows that these examples will consist of finite families of Joyce Spin(7) manifolds, labelled by an integer parametrising the choice of resolution made for each singularity for which a choice exists. As mentioned, the precise details of the construction of the manifolds is not of immediate interest to the present paper. Therefore, we will now present our results, which are documented in the two tables below.

| Example | $c_i$ (c_1...c_4) | $d_i$ (d_1...d_4) | $Spin(7)$ Betti numbers | $G_2$ Betti numbers | Scalars |
|---------|------------------|------------------|------------------------|------------------|---------|
|         |                  |                  | $b_2$ | $b_3$ | $b_4^+$ | $b_4^-$ | $b_2$ | $b_3$ | n |
| 1       | $((1/2)^4)$      | $(0,(1/2)^4)$    | 12   | 16   | 107    | 43     | 12   | 43   | 72 |
| 2       | $(1/2,0,1/2,0)$  | $(0,(1/2)^3)$    | 10+j | 16   | 109-j  | 45-j   | 12   | 43   | 72 |
| 3       | $((1/2)^3,0)$    | $(0,1/2,1/2,0)$  | 8+k  | 16   | 111-k  | 47-k   | 8+l  | 47-l | 72 |
| 4       | $(1/2,0,1/2,0)$  | $(0,1/2,1/2,0)$  | 6+m  | 16   | 113-m  | 49-m   | 8+l  | 47-l | 72 |

Table 1: Examples of dual $N=1$ M-theory and heterotic compactifications to three dimensions.

In Table 1, the columns labelled $c_i$ and $d_i$ specify the orbifold isometry groups $\Gamma$ for M-theory and $\Theta$ for the heterotic string, according to equations (11)-(14) and (18)-(20) respectively. The column labelled $Spin(7)$ gives the Betti numbers of the smooth Joyce 8-manifold of Spin(7) holonomy on which M-theory is compactified. The column labelled $G_2$ gives the Betti numbers of the smooth Joyce 7-manifold of $G_2$ holonomy on which the heterotic theory is compactified. The integers $j, k, l, m$ range from 0 to 4, 8, 8, 12 respectively. The last column gives the number, $n$ of massless $N=1$ scalar multiplets in three dimensions. For the M-theory compactification $n = b_2 + b_3 + b_4^+ + 1$, whereas for the heterotic compactification, $n = b_2 + b_3 + 17$. It is remarkable that the number of such multiplets agrees for both theories.

In [14] Joyce goes on to consider a further $Z_2$ orbifold of some of the above Spin(7) manifolds and produces further examples of such manifolds.
This further $Z_2$ orbifold is generated by the following isometry:

$$\epsilon(x_1, \ldots, x_8) = (c_1 + x_1, c_2 + x_2, x_3, 1/2 + x_4, c_5 + x_5, c_6 + x_6, x_7, 1/2 + x_8) \quad (21)$$

The constants $c_i$ appearing in this equation are the same constants which appeared in equation (13). Thus, if we consider our first example again, which had $c_i = ((1/2)^4)$, then we can take a further orbifold of this manifold, with isometry generated by $\epsilon$ and resolve all singularities. Because the element $\epsilon$ acts freely on all coordinates, the torsion free $Spin(7)$ structure is left invariant. Thus the manifolds produced this way will also be Joyce $Spin(7)$ manifolds. Because of our ansatz, the isometry $\epsilon$ will also define a $Z_2$ orbifold of the heterotic string on the Joyce $G_2$ manifolds in Table 1 above. It is easily seen along similar lines that the action of $\epsilon$ preserves the $G_2$ structure of the heterotic geometry as well. We document the results of two additional examples considered in [14] in Table 2 below.

| Example | $Spin(7)$ | $G_2$ | Scalars |
|---------|-----------|-------|---------|
|         | Ex.+\(\epsilon\) | \(b_2\) | \(b_3\) | \(b_4\) | \(b_2\) | \(b_3\) | \(n\) |
| 5       | 1 + \(\epsilon\) | 9     | 4      | 98   | 34   | 6     | 25   | 48   |
| 6       | 4 + \(\epsilon\) | 4+n   | 4      | 103-n| 39-n | 2+l   | 29-l | 48   |

Table 2: Further examples of dual M-theory/heterotic compactifications.

In Table 2 the second column denotes which manifold in Table 1 is being further orbifolded by $\epsilon$. In this table, the integers $n, l$ range from 0 to 10, 4 respectively. Again we see that both theories have the same massless spectra. An interesting illustration of the remarkable nature of these results comes from example 6, Table 2. In this example the orbifold action on the heterotic geometry given in our ansatz was such that the resulting smooth Joyce manifold of $G_2$ holonomy was not one which has appeared in [12, 13]. Our ansatz nevertheless succeeded in not only producing a new family of Joyce manifolds of $G_2$ holonomy, but precisely a family which gives the correct massless heterotic spectrum as required by duality.

Finally, there was one more family of Joyce $Spin(7)$ manifolds constructed in [14]. The orbifold group in this example was the ($Z_2^5$) group generated by $(\alpha, \beta, \gamma, \delta, \epsilon)$, as defined above. $c_i$ and $d_i$ were chosen for this example to be:

$$(c_1, \ldots, c_4) = (0, 1/2, 0, 1/2) \quad (22)$$

$$(d_1, \ldots, d_4) = (1/2, 0, 0, 1/2) \quad (23)$$
This example leads to a compactification of $M$-theory on a Joyce 8-manifold of $\text{Spin}(7)$ holonomy with Betti numbers given by:

$$b_2 = 8 + j; b_3 = 8; b_4^+ = 103 - j; b_4^- = 39 - j,$$

for $j = 0, .. 4$.

This example leads to a three-dimensional theory with 56 scalar multiplets. Applying our ansatz to construct the heterotic dual, we find the heterotic string compactified on a Joyce 7-manifold of $G_2$ holonomy, with betti numbers $b_2 = 4 + l$ and $b_3 = 35 - l$, for $l = 0, .. 8$. This also leads to an $N = 1$ theory in three dimensions with 56 scalar multiplets.

We have thus demonstrated the consistency of our ansatz, and constructed heterotic dual compactifications for all known $M$-theory compactifications on Joyce 8-manifolds of $\text{Spin}(7)$ holonomy.

4 Discussion and Comments.

We have presented strong evidence for the existence of heterotic duals for all known $M$-theory compactifications on Joyce manifolds of $\text{Spin}(7)$ holonomy. The heterotic duals are compactifications on Joyce manifolds of $G_2$ holonomy. All Joyce manifolds constructed to date [12, 13, 14] are based on the blown up orbifold construction. This fact was utilised in [21], where string compactifications on these spaces were considered as orbifold conformal field theories. This of course applies to the heterotic compactifications considered here. Most of the Joyce orbifolds we have considered here possess orbifold singularities which admit more than one resolution. This leads to the families of manifolds that we have discussed.

It was found in [21] that string theories compactified on different Joyce manifolds from the same family give equivalent conformal field theories up to deformations in the moduli space. Specifically, for string compactification on Joyce $G_2$ orbifolds, equivalent conformal field theories have the same $b_2 + b_3$. This means that all the heterotic compactifications in Table 1 are equivalent up to moduli deformation. This in turn implies that the four families of $M$-theory compactifications are also equivalent up to deformation. An argument which supports this statement is the following: Compactify the theories in Table 1 on an $S^1$. These should then be equivalent [4] to $\text{Spin}(7)$ compactifications of Type IIa string theory. However, from [21], one learns that all these string compactifications are equivalent as conformal field theories, up to changes in the moduli. Then take the strong coupling limit of the
Type IIa theory, and we will recover three dimensional Lorentz invariance 

In this limit the Type IIa compactification is described by the weakly coupled \( M \)-theory compactification in Table 1. This is just a particular limit in the moduli space of the Type IIa theory, in which the moduli of the Joyce manifold are not varied. Hence, the \( M \)-theory compactifications of Table 1 should also be equivalent. The same reasoning of course applies to all the compactifications in Table 2. In fact, if the above were not true, then the duality proposed in this paper would be less concrete. For example, consider Table 1, row 4. Here we propose 13 (labelled by \( k \)) compactifications of \( M \)-theory dual to 9 (labelled by \( l \)) compactifications of heterotic string theory. If it were not for the preceding comments, one would be left wondering, how is \( k \) related to \( l \)? However, given our comments, we do not have to worry about such a question as these parameters do not have any bearing on the moduli space of these theories. This, presumably, also follows from the number of scalar fields present in three dimensions, together with information concerning the structure of the moduli spaces of Joyce manifolds. Unfortunately such knowledge is not yet available to check this.

Much progress has recently been made in discussing \( M \)-theory on orbifolds in dimensions where anomaly cancellation arguments are useful [4, 19, 17, 20]. In particular, in [4, 17, 20] and the first reference of [19], \( p \)-branes played a crucial role in determining the full twisted sector spectrum. An important open question is do \( p \)-branes in \( M \)-theory play an analogous role here. It is certainly natural to speculate that they do.

Finally, we wish to comment on the relationship between this work and possible physics in twelve dimensions. Recently [22, 23] (see also [24] and references therein), the existence of a mysterious theory in twelve dimensions has been speculated upon. It is not yet clear what the relationships between these various ideas is, but in [22, 23] it was conjectured that compactification of a twelve dimensional theory on a circle is equivalent to \( M \)-theory in eleven dimensions. Using this fact, it was realised in [23] that compactification of the twelve dimensional theory on a Joyce manifold of Spin(7) holonomy apparently gives a theory with no supersymmetry in four dimensions. This is because further \( S^1 \) compactification gives the compactifications considered.
in this paper. It was pointed out in [23] that this could be an explicit realisation of the ideas of Witten [25], which may solve the cosmological constant problem together with the problem of bose-fermi mass degeneracy, in a supersymmetric context. If this is indeed the case, then the compactifications considered in this paper deserve yet further study. One possible avenue for this is to use Type I - heterotic duality [18] and consider the resulting Type I compactifications on the same Joyce $G_2$ manifolds that we have considered here. One advantage of this approach is that one can study non-perturbative Type I physics in these backgrounds along the lines of [26]. We hope to report on these issues in the near future.

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