Numerical verification of universality for the Anderson transition

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We analyze the scaling behavior of the higher Lyapunov exponents at the Anderson transition. We estimate the critical exponent and verify its universality and that of the critical conductance distribution for box, Gaussian and Lorentzian distributions of the random potential.

I. INTRODUCTION

Universality is a concept that is central to the theory of critical phenomena and continuous phase transitions. The idea is that critical phenomena are independent of the “details” of the physical system depending only on its symmetry and dimensionality. All systems in the same “universality class” should exhibit the same critical phenomena.

The Anderson transition is a continuous phase transition and so the concept of universality should apply. In this case, the important symmetries are those of time reversal and spin rotation. Three universality classes are expected: orthogonal, unitary, and symplectic. In this paper we are concerned with the orthogonal universality class, which is comprised of systems that have both time reversal and spin rotation symmetry. This corresponds to an electron in a random potential in the absence of a magnetic field and with negligible spin-orbit interaction.

One of the simplest ways to test the validity of the concept of universality is to change the distribution of the random potential in which the electron moves. Since this does not change the universality class, the critical exponents for different distributions of the random potential should be the same. In a recent paper we presented numerical estimates of the critical exponent for the Anderson transition for three different distributions of the random potential: box, Gaussian and Lorentzian. The estimates were made by analyzing the finite size scaling of the localization length of electrons on long wires, and the results support the contention that the critical exponent has a universal value independent of the distribution of the random potential.

In this paper, we analyze the scaling of the higher Lyapunov exponents for the box, Gaussian and Lorentzian distributions of the random potential. We give a clear demonstration of single parameter scaling for these higher Lyapunovs, a clear verification of the universality of this scaling with respect to these three distributions of random potential, and show that this scaling yields estimates of the critical exponent that are in almost exact agreement with our previous estimates based on the scaling of only the lowest Lyapunov exponent. (The lowest Lyapunov exponent is the inverse of the localization length.)

We also present an estimation of the critical conductance distribution \( p_c(g) \). This is expected to take a size independent universal form at the critical point. The size independence of \( p_c(g) \) has been verified previously. Here, we verify that the form of \( p_c(g) \) is independent of the distribution of the random potential for the three distributions mentioned.

II. SCALING OF THE HIGHER LYAPUNOV EXPONENTS

A. Model and method

The motion of the electrons is described by the Anderson model

\[
H = V \sum_{<i,j>} C_i^\dagger C_j + \sum_i W_i C_i^\dagger C_i,
\]

where \( C_i^\dagger \), \( C_i \) denotes the creation (annihilation) operator of an electron at the site \( i \) of a three dimensional cubic lattice. The value of the potential at site \( i \) is \( W_i \). hopping is restricted to nearest neighbors and its amplitude is taken as the unit of energy, \( V = 1 \).

We have examined three different distributions of the random potential.

1. A box distribution with each \( W_i \) uniformly distributed on the interval \([-W/2, W/2]\).
2. A Gaussian distribution with mean zero and variance \( W^2/12 \).
3. A Lorentzian distribution of the form

\[
p(W_i) = \frac{W}{\pi (W_i^2 + W^2)},
\]

for which the variance diverges.

The standard method used in transfer matrix studies of the Anderson transition involves rewriting the
Schrödinger equation for electrons on a long bar with dimensions $L \times L \times L_z$ as a product of transfer matrices

$$M = \prod_{n=1}^{L_z} M_n,$$

where $M_n$ is the transfer matrix. From $M$ we can define a matrix $\Omega$ by

$$\Omega = \ln MM^\dagger,$$

with eigenvalues $\{-\nu_N, \ldots, -\nu_1, +\nu_1, \ldots, +\nu_N\}$ where $N = L^2$ and the $\nu_i$'s are arranged in increasing order. The eigenvalues occur in pairs of opposite sign as a consequence of current conservation and therefore we need only consider the positive $\nu_i$'s. The Lyapunov exponents $\{\alpha_n\}$ for the random matrix product (3) can be defined as

$$\alpha_n = \lim_{L_z \to \infty} \frac{\nu_n}{2L_z}.$$

These were calculated to within a specified accuracy using the method described in (Ref. 5). The localization length $\lambda$ for electrons on the bar is the reciprocal of the smallest positive Lyapunov exponent $\lambda = 1/\alpha_1$. In our previous work, we examined only the scaling of the quantity $\Lambda = \lambda/L = 1/\alpha_1 L$. Here we have examined the scaling of the quantities $z_n$ defined by

$$z_n = \alpha_n L,$$

for higher values of $n$. In performing the scaling analysis corrections to scaling were taken into account using the method described in (Ref. 5). We assumed that the data obey the following scaling law

$$z_n = F_0^{(n)}(\psi L^{1/\nu}, \phi L^y),$$

where $\psi$ is a relevant scaling variable and $\phi$ is an irrelevant scaling variable. We approximated this scaling function by its first order expansion in the irrelevant scaling variable and fitted the data for each $z_n$ to the form

$$z_n = F_0^{(n)}(\psi L^{1/\nu}) + \phi L^y F_1^{(n)}(\psi L^{1/\nu}).$$

The scaling variables were approximated by expansions in terms of the dimensionless disorder $w = (W_c - W)/W_c$ where $W_c$ is the critical disorder separating the insulating and metallic phases,

$$\psi = \psi_1 w + \psi_2 w^2, \quad \phi = \phi_0.$$

The critical exponent $\nu$ describes the divergence of the localization length (correlation length on the metallic side) as the transition is approached

$$\xi = \xi_0 |\psi|^{-\nu}.$$ 

Note that the absolute scale of the localization length $\xi_0$ cannot be determined with the method we have used here. The decay of the irrelevant scaling variable with system size is described by an exponent $y < 0$.

The functions $F_0^{(n)}$ and $F_1^{(n)}$ were expanded either to second or third order in $w$. Also, when possible, the expansion for $\psi$ was truncated at the first order in $w$. The decision at which order to truncate these various series was based on the criteria of obtaining an acceptable goodness of fit with as few parameters as possible.

### B. Results

We simulated systems with sizes $L = 4, 5, 6, 8, 10, 12, 14$ and 16 and Fermi energy $E_F = 0$ calculating data to 0.1% accuracy with the exception of some data very close to the critical point that were calculated to 0.05% accuracy. We found that the $z_n$'s obey single parameter scaling provided that the system sizes considered are large enough. The minimum system size required increased with $n$ so that, for example, we could fit only the data with $L \geq 10$ when analyzing $z_0$ for the box distribution. In practice, we could demonstrate scaling for the first ten exponents for the box and Gaussian distributions. For the Lorentzian distribution the required system sizes seem to be larger and we could demonstrate scaling only for the first three exponents.

The results of the scaling analysis are presented in Tables I-III and some of the data are plotted in Figs. 1-4. The estimates of the critical disorder obtained by scaling different Lyapunov exponents for a given distribution of the random potential are in very close agreement. The same can also be said for the critical values of the parameter $z_n$ for various $n$ between different distributions of the random potential. The critical values $z_c^{(n)}$ for a given $n$ is defined by

$$z_c^{(n)} = F_0^{(n)}(0),$$

and is expected to be universal if the scaling function for each Lyapunov is universal. The data clearly support this idea. Turning to the critical exponent $\nu$, we find estimates that are remarkably consistent across all the Lyapunov exponents and disorder distributions we considered.

### III. THE ZERO TEMPERATURE CONDUCTANCE DISTRIBUTION

Next we considered the zero temperature conductance distribution when $W = W_c$, i.e. the critical conductance distribution at the Anderson transition. We set the Fermi energy $E_F = 0$ and, based on the results of Section II, we used $W_c = 16.54$ for the box distribution, $W_c = 21.3$ for the Gaussian distribution, and $W_c = 4.26$ for the Lorentzian distribution when simulating the conductance distribution. An ensemble of samples of cubic shape was
generated, and for each sample, a Green’s function iteration technique was used to determine the transmission matrix $t$ in a two probe measuring geometry. The dimensionless conductance $g$ was calculated using the Landauer formula
\[ g = 2\pi t t^t, \]  
where the factor of two takes account of spin degeneracy. (We would like to emphasize that the $t$ matrix that appears in this formula is an $N \times N$ matrix, not an $L^2 \times L^2$ matrix, where $N$ is the number of propagating modes at the Fermi energy in the contacts attached at the left and right of the sample.)

In Fig. 5 and 6, the conductance distribution for ensembles of $12 \times 12 \times 12$ cubic samples for the three distributions of random potential are presented. Fixed boundary conditions were imposed on the wavefunction in the direction transverse to current flow so that the system size $L = 12$ is large enough that the distribution shown are a good approximation to the true critical distributions $p_c(g)$ and $p_c(\ln g)$ obtained in the asymptotic limit $L \to \infty$. (Larger system sizes are required if periodic boundary conditions are considered and the asymptotic form of the distribution is different.) Note that despite the appearance of Fig. 5, $p_c(g) \to 0$ as $g \to 0$. This can be deduced from the form of $p_c(\ln g)$ in Fig. 6 or a more detailed plot of $p_c(g)$ near $g = 0$.

Looking at Figs. 5 and 6 we see that critical distributions obtained for the three different distributions of the random potential are almost identical. This is consistent with the assertion that the critical conductance distribution is universal.

**IV. SUMMARY AND DISCUSSION**

The critical exponent for the Anderson model with random hopping but no random potential has been estimated to be $\nu = 1.61 \pm 0.07$. Since models with random hopping and random potentials are in the same universality class, we should, and do, obtain estimates for the critical exponent that are consistent with Ref. 18.

The computer time required to estimate the Lyapunov exponents with the method used here scales as $L^7$ for a given accuracy. Thus, it is unlikely that this method can be extended to much larger systems sizes in the near future. The critical exponent of the Anderson transition has also been estimated by analyzing the scaling of energy level statistics (ELS). The most recent, and in our opinion the most accurate, estimate of $\nu$ obtained with ELS is $\nu = 1.45 \pm 0.03$. This is consistent with our results but the estimated uncertainty is an order of magnitude worse than we have achieved here.

In the simulations whose results we have presented here, the Fermi energy is always set at the band center $E_F = 0$. For one dimensional systems it has been demonstrated that single parameter scaling does not hold for states with energies in a band gap of the corresponding ordered (i.e. zero disorder $W = 0$) system. In that case, single parameter scaling is only observed for sufficiently strong disorder. We suspect that this might also hold for the higher Lyapunov exponents in the three dimensional systems we have studied here for energies outside the band.

To conclude, we have demonstrated that the higher Lyapunov exponents obey single parameter scaling at the Anderson transition and presented evidence that this scaling behavior is universal and independent of the choice of distribution of the random potential. Analysis of this scaling behavior yields estimates of the critical exponent that are remarkably consistent between different Lyapunov exponents and different distributions of the random potential. Finally, we presented evidence that the critical conductance distribution is also universal and independent of the choice of the distribution of the random potential.

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TABLE I. The results of the scaling analysis for the box distributed random potential. Here \( n \) indicates which Lyapunov exponent is being considered, \( N_d \) the number of data available, \( N_p \) the number of parameters used to fit the data, \( W_c \) is the estimate of the critical disorder, \( z_{\nu}^{(n)} \) is defined in (11) and \( \nu \) is the estimated critical exponent. The figures in brackets are 95% confidence intervals for the corresponding critical parameter estimated using the bootstrap method (Ref. 14).

| \( n \) | \( N_d \) | \( N_p \) | \( W_c \) | \( z_{\nu}^{(n)} \) | \( \nu \) |
|---|---|---|---|---|---|
| 1 | 336 | 12 | 16.54(53,55) | 1.737(733,742) | 1.56(55,58) |
| 2 | 336 | 12 | 16.55(54,55) | 2.801(797,804) | 1.57(56,58) |
| 3 | 292 | 12 | 16.55(55,56) | 3.596(591,602) | 1.57(56,58) |
| 4 | 250 | 10 | 16.56(55,57) | 4.25(24,26) | 1.58(57,59) |
| 5 | 250 | 10 | 16.57(56,59) | 4.83(82,84) | 1.58(57,59) |
| 10 | 162 | 10 | 16.59(51,71) | 7.0(6.9,7.2) | 1.55(52,59) |

TABLE II. The results of the scaling analysis for the Gaussian distributed random potential. The notation is explained in the caption for Table 1.

| \( n \) | \( N_d \) | \( N_p \) | \( W_c \) | \( z_{\nu}^{(n)} \) | \( \nu \) |
|---|---|---|---|---|---|
| 1 | 200 | 10 | 21.30(28,31) | 1.737(733,742) | 1.56(56,61) |
| 2 | 200 | 9 | 21.29(28,30) | 2.796(794,799) | 1.57(56,58) |
| 3 | 175 | 9 | 21.31(30,32) | 3.593(588,597) | 1.58(57,59) |
| 4 | 150 | 9 | 21.31(29,33) | 4.244(236,252) | 1.58(56,60) |
| 5 | 125 | 9 | 21.29(27,33) | 4.80(79,82) | 1.57(55,60) |
| 10 | 100 | 9 | 21.40(30,57) | 7.05(6.98,7.18) | 1.55(50,62) |

TABLE III. The results of the scaling analysis for the Lorentzian distributed random potential. The notation is explained in the caption for Table 1. (We could not execute the bootstrap method for \( n = 3 \) so no confidence intervals are available.)

| \( n \) | \( N_d \) | \( N_p \) | \( W_c \) | \( z_{\nu}^{(n)} \) | \( \nu \) |
|---|---|---|---|---|---|
| 1 | 256 | 10 | 4.25(24,26) | 1.70(68,71) | 1.57(52,61) |
| 2 | 192 | 10 | 4.24(23,26) | 2.71(64,75) | 1.58(53,65) |
| 3 | 127 | 10 | 4.29 | 3.62 | 1.58 |
FIG. 1. $z_2$ vs disorder $W$ for the box distributed random potential.

FIG. 2. The same data as in Figure 1 after corrections to scaling have been subtracted and plotted versus $L/\xi$ where $\xi$ is the localization (correlation) length.
FIG. 3. $z_5$ vs disorder $W$ for the Gaussian distributed random potential.

FIG. 4. The same data as in Figure 3 after corrections to scaling have been subtracted and plotted versus $L/\xi$. 
FIG. 5. The distribution of the conductance for ensembles of 100,000 $12 \times 12 \times 12$ cubes for box (solid line), Gaussian (dotted line) and Lorentzian (dashed line) distributed random potentials.

FIG. 6. The distribution of the logarithm of the conductance for ensembles of 100,000 $12 \times 12 \times 12$ cubes for box (solid line), Gaussian (dotted line) and Lorentzian (dashed line) distributed random potentials.