Effects of anisotropic spin-exchange interactions in spin ladders

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We investigate the effects of the Dzialoshinskii-Moriya (DM) and Kaplan-Shekhtman-Entin-Wohlman-Aharony (KSEA) interactions on various thermodynamic and magnetic properties of a spin 1/2 ladder. Using the Majorana fermion representation, we derive the spectrum of low energy excitations for a pure DM interaction and in presence of a superimposed KSEA interaction. We calculate the various correlation functions for both cases and discuss how they are modified with respect to the case of an isotropic ladder. We also discuss the electron spin resonance (ESR) spectrum of the system and show that it is strongly influenced by the orientation of the magnetic field with respect to the Dzialoshinskii-Moriya vector. Implications of our calculations for NMR and ESR experiments on ladder systems are discussed.

I. INTRODUCTION

In recent years low dimensional quantum spin systems have attracted great interest due to their fascinating properties originating from low dimensionality and quantum fluctuations. Quantum effects are particularly relevant in $S = 1$ systems. The quasi-one-dimensional antiferromagnetic $S = 1/2$ ladders have been extensively investigated both theoretically and experimentally. Several experimental realizations of such systems are available. In particular $\text{Sr}_{n-1}\text{Cu}_{n+1}\text{O}_2$ and $\text{Sr}_{14}\text{Cu}_{24}\text{O}_{41}$ seem to be quite well described by a simple isotropic Heisenberg ladder model. However, countless experiments on higher dimensional compounds indicate that in many systems the isotropic exchange interaction is not sufficient to describe magnetic properties such as weak ferromagnetism. From the theoretical point of view, more than thirty years ago Dzialoshinskii pointed out that a term (allowed by symmetry in non centrosymmetric structures) of the form $\mathbf{D} \cdot (\mathbf{S}_1 \times \mathbf{S}_2)$, where $\mathbf{D}$ is the constant Dzialoshinskii vector and $\mathbf{S}_{1,2}$ are the sublattice magnetizations, favors a canted spin arrangement over the antiferromagnetic one and a weak ferromagnetic moment. This term was later derived by Moriya from a microscopic Hubbard-type Hamiltonian by including spin-orbit coupling in the Anderson superexchange calculation. Actually, the calculation of Moriya showed that besides the Dzialoshinskii-Moriya (DM) term, a symmetric anisotropy term of the form $\mathbf{S} \cdot \mathbf{A} \cdot \mathbf{S}$ was also obtained, but was assumed by Moriya to be negligible compared to the antisymmetric one. However, it has been realized recently that this assumption is incorrect if the underlying microscopic model from which spin-spin interactions are derived has an SU(2) spin symmetry. In that case, the effective spin-spin interactions have the same SU(2) invariance as the microscopic model. As the DM term alone breaks SU(2) symmetry, the symmetric term has to be such that it compensates exactly the spin anisotropy induced by the DM term. For this class of systems, the symmetric anisotropy term is called the Kaplan-Shekhtman-Entin-Wohlman-Aharony (KSEA) and it is tuned in such way that SU(2) symmetry is recovered. A derivation of the DM and the KSEA term starting from a microscopic SU(2) invariant Hubbard model with spin-orbit coupling can be found in [1].

The main objective of this paper is to study the influence of the anisotropic (DM and KSEA) spin-exchange interactions in a spin 1/2 two leg ladder. We will consider a uniform Dzialoshinskii-Moriya interaction in contrast with the staggered interaction discussed in [12]. We mainly concentrate on a DM interaction along the rung of the ladder, but we discuss also briefly the case of a uniform DM interaction along the chains. Our motivation for restricting mainly to the case of interactions along the rungs is the situation in $\text{CaCu}_2\text{O}_3$ in which the Cu–O–Cu bond angle in the ladder rungs is 123° indicating that in addition to the Heisenberg coupling a Dzialoshinskii-Moriya interaction is also present in the rungs of the ladder. Note that a model of coupled chains with Dzialoshinskii-Moriya interactions, motivated by experiments on $\text{Cs}_2\text{CuCl}_4$ has also been considered in Ref. [13]. However, in Ref. [13] the authors are considering an two dimensional array of coupled chains using bosonization and RPA rather than an isolated ladder. The paper is organized as follows. We first outline the bosonization procedure of a weakly coupled two leg spin ladder in presence of DM and KSEA interactions along the rungs. The derivation of the low energy theory is now quite standard, and the reader already familiar with bosonization techniques should read it only to get acquainted with
our notations. Expressions for the staggered correlation functions can then be obtained using a mapping onto four non-critical Ising models. Then we study all uniform magnetic correlation functions and calculate the temperature dependence of the spin susceptibility and the NMR relaxation rate. Finally we discuss the very interesting effects of the anisotropic spin interactions on the electron spin resonance (ESR) spectrum, where the DM interaction triggers a phase with gapped but incommensurate correlation functions with applied magnetic field perpendicular to the DM vector, while the effect of a superimposed KSEA interaction is the closure of the gap with a restoration of the SU(2) symmetry.

II. GENERAL CONSIDERATIONS AND BOSONIZATION

A. General considerations

The Hamiltonian of an antiferromagnetic two leg ladder with both Dzialoshinskii-Moriya (DM) and Kaplan-Shekhtman-Entin-Wohlman-Aharony (KSEA) interactions along the rungs is:

\[ H = J_{\parallel} \sum_i (S_{1,i}S_{1,i+1} + S_{2,i}S_{2,i+1}) + J_{\perp} \sum_i S_{1,i}S_{2,i} + D \cdot \sum_i (S_{1,i} \times S_{2,i}) + \sum_i S_{1,i} \hat{A} S_{2,i} \]  

(1)

where \( D \) is the uniform Dzialoshinskii vector associated with the bond between the spins on different chains. The last term in the Hamiltonian describes the KSEA interaction, where \( \hat{A} \) is the anisotropy tensor. Generally, in a system having full SU(2) symmetry, \( D \) and \( \hat{A} \) are not independent; they are related to each other in such a way that a rotation of the spin axes maps the single-bond Hamiltonian onto an isotropic one with no preferred direction of the staggered magnetization. If we use the quantization axis \( \hat{z} \) for the spins such that \( D = D \hat{z} \), the anisotropy tensor reduces to \( A_{zz} = A \) and all other components become zero. The Hamiltonian reads:

\[ H = \sum_i J_{\parallel} (S_{1,i}S_{1,i+1} + S_{2,i}S_{2,i+1}) + \frac{J_{\perp} + iD}{2} S^+_{i,1}S^-_{i,2} + \frac{J_{\perp} - iD}{2} S^-_{i,1}S^+_{i,2} + (J_{\perp} + A)S^z_{i,1}S^z_{i,2} \]  

(2)

This Hamiltonian can be simplified by the gauge transformation:

\[ S^+_{i,1} = e^{-i\alpha} S^+_{i,1}; \quad S^+_{i,2} = e^{i\alpha} S^+_{i,2} \]  

(3)

where \( 2\alpha = \arctan(D/J_{\perp}) \), and brought to the form:

\[ H = \sum_i J_{\parallel} (\tilde{S}_{1,i} \tilde{S}_{1,i+1} + \tilde{S}_{2,i} \tilde{S}_{2,i+1}) + \tilde{J}_{\perp} (\tilde{S}^x_{i,1} \tilde{S}^x_{i,2} + \tilde{S}^y_{i,1} \tilde{S}^y_{i,2}) + (\tilde{J}_{\perp} + A)\tilde{S}^z_{i,1}\tilde{S}^z_{i,2} \]  

(4)

where \( \tilde{J}_{\perp} = \sqrt{J_{\perp}^2 + D^2} \text{sign}(J_{\perp}) \). The KSEA interaction is given by \( A = \sqrt{J_{\perp}^2 + D^2} \text{sign}(J_{\perp}) - J_{\perp} \) leading to the restoration of SU(2) symmetry. For \( D \ll J_{\perp} \), one recovers the approximate expression \( A \sim D^2/(2J_{\perp}) \) (see Eq. (2.19) of Ref. [13]). Let us mention that a Dzialoshinskii-Moriya interaction along the chain direction:

\[ H_{DM}^{||} = D_{||} \cdot \sum_{n,p} S_{n,p} \times S_{n+1,p} \]  

(5)

can also be removed by another gauge transformation:

\[ S^+_{n,p} = e^{i\delta n} \tilde{S}^+_{n,p}, \]  

(6)

where \( \delta = \arctan(D_{\parallel,\text{parallel}}/J) \). This gauge transformation has no effect on the interchain coupling. However, in the absence of KSEA interaction, it can induce an anisotropic interaction along the chain direction. Further, this term induces incommensurability in the correlation functions. Let us discuss briefly the limit \( J_{\parallel} = 0 \). With Dzialoshinskii-Moriya interactions only, and for classical spins, the spins \( \tilde{S}_{1,2} \) would lie in the ground state forming an angle of \( \pi \) with each other. Thus, the real spins would make an angle of \( \pi - 2\alpha \) with each other. In that case, a weak ferromagnetic moment proportional to \( \sin \alpha \) would appear in the ground state. In the case of quantum spins 1/2, we can rewrite the interchain interaction up to an unimportant constant as:

\[ H_{\text{interchain}} = \frac{\tilde{J}_{\perp}}{2} (\tilde{S}^1 + \tilde{S}^2)^2 + \frac{J_{\perp} - \tilde{J}_{\perp}}{2} (\tilde{S}^z_1 + \tilde{S}^z_2)^2 \]  

(7)
As a result, the total spin $\tilde{S} = \tilde{S}_1 + \tilde{S}_2$ and the spin projection along the z-axis $\tilde{S}^z = \tilde{S}_1^z + \tilde{S}_2^z$ are good quantum numbers. In the case of an antiferromagnetic interaction, the ground state is the singlet state with $|\tilde{S} = 0\rangle$ and the excited states are $|\tilde{S} = 1; \tilde{S}^z = \pm 1\rangle$ with energy $E(\tilde{S} = 1, \tilde{S}^z = \pm 1) = \frac{J_1 + J_2}{2}$, and $|\tilde{S} = 1; \tilde{S}^z = 0\rangle = \tilde{J}_\perp$. We notice that the state with $\tilde{S}^z = 0$ is the highest excited state in this case.

For a ferromagnetic interaction, the state $|\tilde{S} = 1; \tilde{S}^z = 0\rangle$ becomes the state of lowest energy. The excited states are then $|\tilde{S} = 1; \tilde{S}^z = \pm 1\rangle$ and $|\tilde{S} = 0\rangle$, the latter one being the highest excited state. Clearly, in the case of ferromagnetic interactions, the system becomes equivalent to a XXZ spin 1 chain with easy plane interaction $-K(S^z)^2$. This could lead to the observation of an XY2 phase.

**B. Bosonization treatment**

For weak interchain coupling $J_\perp$ and $D, \tilde{A} \ll J_\parallel$, the Hamiltonian (4) can be bosonized following Ref. 19.

For the moment let us consider $\tilde{A} = 0$. For $J_\perp = D = 0$, the low energy properties of the Hamiltonian (4) are described in terms of the boson operators $\phi_\alpha$ ($\alpha = 1, 2$) and their conjugate momenta $\pi \Pi = \partial_x \theta_\alpha$. The Hamiltonian of chain $\alpha$ is:

$$H_\alpha = \int \frac{dx}{2\pi} u \left[ (\pi \Pi_\alpha)^2 + (\partial_x \phi_\alpha)^2 \right]$$

where $u = \frac{\pi}{2} J_\parallel a$ is the “spinon” velocity and $a$ is the lattice spacing. Let us now turn a weak non zero $J_\perp, D$ in the Hamiltonian (4). By using the continuum limit of spin operators, we can rewrite (4) as:

$$\tilde{S}_\alpha^z (x = na) = \frac{\tilde{S}_{\pi,\alpha}^z}{a} = -\frac{\partial_x \phi_\alpha}{\pi \sqrt{2}} + \frac{e^{i \frac{\pi}{\lambda}}}{\pi a} \lambda \sin \sqrt{2} \phi_\alpha$$

$$\tilde{S}_\alpha^z (x = na) = \frac{\tilde{S}_{\pi,\alpha}^z + i \tilde{S}_{\pi,\alpha}^x}{a} = \frac{e^{i \sqrt{2} \theta_\alpha}}{\pi a} \left[ e^{i \frac{\pi}{\lambda}} \lambda + \cos \sqrt{2} \phi_\alpha \right],$$

we can rewrite (4) as:

$$H = H_s + H_a$$

$$H_s = \int \frac{dx}{2\pi} u [ (\pi \Pi_\alpha)^2 + (\partial_x \phi_\alpha)^2 ] - \frac{J_\perp}{2 \pi a} \int dx \cos 2\phi_\alpha$$

$$H_a = \int \frac{dx}{2\pi} u [ (\pi \Pi_\alpha)^2 + (\partial_x \phi_\alpha)^2 ] + \frac{J_\perp}{2 \pi a} \int dx \cos 2\phi_\alpha + \frac{\tilde{J}_\perp}{\pi \alpha} \int dx \cos 2\theta_\alpha,$$

where we have introduced the symmetric and antisymmetric combinations:

$$\phi_\alpha = \frac{\phi_1 + \phi_2}{\sqrt{2}}, \quad \phi_\alpha = \frac{\phi_1 - \phi_2}{\sqrt{2}},$$

and similar combinations for $\Pi$ and $\theta$.[3][2][1]. $\lambda \simeq 1$ is the expectation value of the charge operator[4]. Note that the Hamiltonian $H_s, H_a$ can also be obtained by bosonizing directly the Hamiltonian (4) and then performing a shift of the field $\theta_\alpha$ which is equivalent to the gauge transformation Eq. (3). In the presence of a Dzialoshinskii-Moriya interaction along the chains, the effects are two-fold. First, we have to make a shift $\theta_p \rightarrow \theta_p + \delta \xi_\alpha$, i.e. $\theta_s + \sqrt{2} \beta \xi_\alpha$. Since in the Hamiltonian (4), only the term $\cos 2\phi_\alpha$ is present, such shift does not affect the spectrum of the ladder. However, it induces incommensurability. The spin-spin correlation function $\langle S^+(q)S^-(q) \rangle$ will present divergences for $q = \delta/a$ and $q = \pm \pi/a + \delta/a$ rather than respectively $q = 0$ and $q = \pi/a$. The second effect of the longitudinal Dzialoshinskii-Moriya interaction is to induce an anisotropic interaction $\delta J_\parallel \sum_{i,p} S_i^z S_{i+p}^z$. Such interaction gives marginally relevant interactions upon bosonization. On the other hand, the interchain coupling gives relevant interactions of dimension 1. Thus, for not too weak interchain coupling, the extra anisotropic interaction will not affect the physical properties of the ladder, and the only effect of the longitudinal Dzialoshinskii-Moriya interaction will be a shift of the correlation functions. The Hamiltonians $H_a, H_s$ can be further rewritten in terms of non-interacting Majorana fermions[4]:

$$\xi_{R,1} = \frac{\cos(\theta_\alpha - \phi_\alpha)}{\sqrt{\pi a}}, \quad \xi_{R,2} = \frac{\sin(\theta_\alpha - \phi_\alpha)}{\sqrt{\pi a}};$$

$$\xi_{R,3} = \frac{\cos(\theta_\alpha - \phi_\alpha)}{\sqrt{\pi a}}, \quad \xi_{R,4} = \frac{\sin(\theta_\alpha - \phi_\alpha)}{\sqrt{\pi a}},$$

$$\xi_{R,5} = \frac{\cos(\theta_\alpha - \phi_\alpha)}{\sqrt{\pi a}}, \quad \xi_{R,6} = \frac{\sin(\theta_\alpha - \phi_\alpha)}{\sqrt{\pi a}}.$$
and similar definitions for the $\xi_L$’s with $-\phi_{s,a} \to \phi_{s,a}$. Reexpressed in terms of Majorana fermions the Hamiltonian reads:

$$H = \sum_{a=1}^{4} \int dx \left\{-\frac{iu}{2} (\xi_R^a \partial_x \xi_R^a - \xi_L^a \partial_x \xi_L^a) - im_a \xi_R^a \xi_L^a \right\}.$$  \hspace{1cm} (14)

The spectrum is thus composed of a Majorana doublet $(\xi_L^1, \xi_L^2)$, $(\nu = L, R)$, with mass $m_{1,2} = m = \lambda^2 J_\perp / 2\pi$ and two singlets $\xi_3^\pm$, of respective masses $m_3 = \lambda^2 (\frac{1}{2} - \frac{t}{\pi u}), m_4 = -\lambda^2 (\frac{1}{2} + \frac{t}{\pi u})$. Without the DM interaction the spectrum would be formed of a triplet $\xi_a^\nu$, $a = 1, 2, 3$ with mass $m$ and a singlet $\xi_4^\nu$ with a larger modulus mass $3|m|$.

The loss of SU(2) symmetry in the presence of a DM interaction thus translates into the lifting of the degeneracy of the triplet. Let us note that the Majorana doublet describes the excitations of spin $\tilde{S}^z = \pm 1$, whereas the singlet corresponds to the excitations of spin $\tilde{S}^z = 0$. We notice that the hierarchy of energy scales as well as the degeneracy of excitations with $S^z = \pm 1$ that we had obtained in the strong rung coupling limit are preserved in the limit of a small rung coupling. In the presence of a KSEA interaction, the bosonization treatment of (1) is identical to the one in [19]. We find that $m_{1,2,3} = \lambda^2 J_\perp / 2\pi$, thus recovering the triplet of the isotropic ladder.

From the Hamiltonian Eq. (14), the specific heat of the system at low temperature is obtained in the form:

$$C_v \sim \left( \frac{m_1}{T} \right)^{3/2} \frac{m_1}{\sqrt{2\pi u}} e^{\frac{m_2}{T}} + \left( \frac{m_2}{T} \right)^{3/2} \frac{m_2}{\sqrt{2\pi u}} e^{\frac{m_3}{T}} + \left( \frac{m_3}{T} \right)^{3/2} \frac{m_3}{\sqrt{2\pi u}} e^{\frac{m_4}{T}},$$  \hspace{1cm} (15)

If the KSEA interaction is different from zero, the SU(2) symmetry is restored and the specific heat becomes, $C_v \sim \left( \frac{\Delta''}{T} \right)^{3/2} \frac{\Delta''}{\sqrt{2\pi u}} e^{-\frac{\Delta''}{T}}$, where $\Delta''$ is the triplet mass $\lambda^2 J_\perp / 2\pi$. The difference between the regular ladder and the ladder with only DM interaction is the contribution from the singlet excitation with $S^z = 0$. The specific heat as a function of temperature is plotted on figure [1]. A remarkable feature is that at low temperature, the specific heat of the ladder with DM interaction is lower while it is higher at high temperature.

### III. Staggered Correlation Functions

To calculate the two-point correlation functions of the staggered magnetization, we use the mapping onto a pair of non-critical 2d Ising model introduced in Refs. [19] [24]. This mapping permits to express the staggered correlation functions in terms of the correlation functions of the order and disorder parameters of the Ising model away from criticality [24]. To precise the correspondence, we will use the convention of Ref. [19] that $J_\perp / t > 0$, where $t$ is the reduced temperature of the corresponding noncritical Ising model, $t = (T - T_c) / T_c$. With this convention, $J_\perp > 0$ corresponds to the disordered phase and we have the following bosonization formulas for order and disorder parameters $\sigma_i$ and $\mu_i$ ($i = 1, \ldots, 4)$:

$$\begin{align*}
\cos(\phi_s) &= \mu_1 \mu_2, \quad \sin(\phi_s) = \sigma_1 \sigma_2, \\
\cos(\theta_s) &= \mu_3 \mu_4, \quad \sin(\theta_s) = \sigma_3 \mu_4, \\
\cos(\phi_a) &= \sigma_1 \sigma_2 \sigma_3 \sigma_4, \quad \sin(\phi_a) = \sigma_3 \mu_4. \\
\cos(\theta_a) &= \sigma_3 \sigma_4, \quad \sin(\theta_a) = \mu_3 \mu_4.
\end{align*}$$  \hspace{1cm} (16)

Using formulas (16) along with the gauge transformation (3), we obtain the components of the total $\mathbf{M}$ and relative $\mathbf{m}$ staggered magnetization as:

$$\begin{align*}
M_x &\sim (\cos \sigma_1 \mu_2 \sigma_3 \mu_4 - \sin \alpha \sigma_1 \mu_2 \mu_3 \mu_4) \quad m_x \sim (\sin \alpha \mu_1 \sigma_2 \sigma_3 \mu_4 + \cos \alpha \mu_1 \sigma_2 \mu_3 \mu_4) \\
M_y &\sim (\cos \sigma_1 \mu_2 \sigma_3 \mu_4 - \sin \alpha \sigma_1 \mu_2 \mu_3 \mu_4) \quad m_y \sim (\sin \alpha \sigma_1 \mu_2 \sigma_3 \mu_4 + \cos \alpha \sigma_1 \mu_2 \mu_3 \mu_4) \\
M_z &\sim \sigma_1 \mu_2 \sigma_3 \mu_4 \quad m_z \sim \mu_1 \mu_2 \sigma_3 \mu_4.
\end{align*}$$  \hspace{1cm} (17)

Comparing these expressions with those obtained by Shelton et al [19], we see that the DM interaction mixes the total and relative magnetization of the isotropic case in the $x - y$ plane, while the $z$ components are unchanged. This is a consequence of the easy-plane effect induced by the DM interaction. From Eqs. (13) the two-point correlation functions perpendicular and along the DM vector, are expressed in term of correlation functions of the order and disorder parameters.
with account the KSEA interaction, the of the dynamical transverse spin susceptibility for very small momentum. We find:

\[ \langle \chi^{xx}(\mathbf{q}, \omega) \rangle = \frac{A^4}{\pi^2} \frac{\sin^2 \alpha}{\sqrt{(k \omega)^2 + m^2}} \left[ \delta(\omega - \sqrt{(k \omega)^2 + m^2}) - \delta(\omega + \sqrt{(k \omega)^2 + m^2}) \right] \]

where \( r = \sqrt{x^2 + (\tau r)^2} \) and \( G_\alpha(m_\pi r/u) \) are the two-point correlation functions given by Eqs. (38) and (39) of Ref. [3]. In order to determine explicitly the correlations length in the direction orthogonal and along the DM vector, we use the asymptotic expansions of the functions \( K_0(v) \) and \( K_1(v) \) for \( v \to \infty \):

\[ K_0(v) = \sqrt{\pi/(2v)} e^{-(1 - 1/(8v) + o(1/v))} \]
\[ K_1(v) = \sqrt{\pi/(2v)} e^{-(1 + 3/(8v) + o(1/v) \ldots)} \]

Substituting these expressions in Eq. (13) we get, after some cancellations:

\[ \langle T_{xx}(x, \tau) M_x(0, 0) \rangle \sim \frac{A^4}{\pi^2} \frac{\sin^2 \alpha}{\sqrt{(k \omega)^2 + m^2}} \left[ \delta(\omega - \sqrt{(k \omega)^2 + m^2}) - \delta(\omega + \sqrt{(k \omega)^2 + m^2}) \right] \]

We see that with only Dzialoshinskii-Moriya interactions, the in-plane and out of plane correlation functions have a different correlation length in contrast to the case of the regular lattice. This difference of correlation lengths should be observable in neutron scattering experiments. The ratio of the correlation lengths should give access to the intensity of the Dzialoshinskii-Moriya interactions, since it is equal to \( m_\pi/m = 2\sqrt{1 + (D/J_\perp)^2} - 1 \). Another difference with the regular lattice is that \( \langle M_x M_z \rangle \) decays as slowly as \( \langle m_x m_z \rangle \) with DM interactions whereas its decay is much faster without them. The ratio \( \langle m_x m_z \rangle/\langle M_x M_z \rangle \) provides a second measurement of \( D/J_\perp \). When we take into account the KSEA interaction, the symmetric is restored and correlation functions have the same decay with \( r \) in all directions with \( \xi = (\frac{A^2}{2\pi u})^{-1} \). Nevertheless, in presence of a KSEA interaction the decay of \( \langle M_x M_z \rangle \) remains not faster than the decay of \( \langle m_x m_z \rangle \), and the ratio of these correlation functions gives access to the strength of the Dzialoshinskii-Moriya interaction. Using the above expressions (18) we can easily calculate the imaginary part of the dynamical transverse spin susceptibility for very small momentum. We find:

\[ \text{Im} \chi^{xx}(q_{||}, q_{\perp} = 0, \omega) = \frac{A^4}{\pi} \frac{\sin^2 \alpha q_{\perp}^2}{(u q_{\perp})^2 + m^2} \left[ \delta(\omega - \sqrt{(u q_{\perp})^2 + m^2}) - \delta(\omega + \sqrt{(u q_{\perp})^2 + m^2}) \right] \]

\[ \text{Im} \chi^{xx}(q_{||}, q_{\perp} = \pi, \omega) = \frac{A^4}{\pi} \frac{\sin^2 \alpha q_{\perp}^2}{(u q_{\perp})^2 + m^2} \left[ \delta(\omega - \sqrt{(u q_{\perp})^2 + m^2}) - \delta(\omega + \sqrt{(u q_{\perp})^2 + m^2}) \right] \]

In presence of a KSEA interaction the asymmetry disappears and the same optical magnon peak is obtained in both directions at \( \omega = \sqrt{(u q_{\perp})^2 + (\lambda^2 J_\perp/2\pi)^2} \). Thus the analysis reveals that there is net way of determining the DM interaction and the interplay between DM and KSEA interactions into experiments.
IV. MAGNETIC SUSCEPTIBILITY AND NMR RELAXATION RATE

A. General formalism

To determine the temperature dependence of the magnetic susceptibility and of the NMR relaxation rate \(1/T_1\), we only need the slowly varying part of the spin density. That component of the spin density can be expressed in terms of Majorana fermions. The corresponding correlators can be obtained by the approach of Ref. [30]. The following relations hold for the sum and the difference of the spin density currents of chains 1 and 2 along the spatial directions (1,2,3) (see also Appendix A):

\[
(J_{\nu 1} + J_{\nu 2})^1 = -i (\cos \alpha \xi_{\nu \nu}^2 \xi_{\mu \nu}^4 - \sin \alpha \xi_{\nu \nu}^2 \xi_{\mu \nu}^3)
\]

\[
(J_{\nu 1} - J_{\nu 2})^1 = -i (\cos \alpha \xi_{\nu \nu}^2 \xi_{\mu \nu}^4 + \sin \alpha \xi_{\nu \nu}^2 \xi_{\mu \nu}^3)
\]

\[
(J_{\nu 1} + J_{\nu 2})^2 = -i (\cos \alpha \xi_{\nu \nu}^2 \xi_{\mu \nu}^4 + \sin \alpha \xi_{\nu \nu}^2 \xi_{\mu \nu}^3)
\]

\[
(J_{\nu 1} - J_{\nu 2})^2 = -i (\sin \alpha \xi_{\nu \nu}^2 \xi_{\mu \nu}^4 + \cos \alpha \xi_{\nu \nu}^2 \xi_{\mu \nu}^3)
\]

\[
(J_{\nu 1} + J_{\nu 2})^3 = -i \xi_{\nu \nu}^1 \xi_{\mu \nu}^2
\]

\[
(J_{\nu 1} - J_{\nu 2})^3 = i \xi_{\nu \nu}^3 \xi_{\mu \nu}^4,
\]

where \(\nu = L, R\). From the previous expressions we evaluate the slowly varying component of the Matsubara spin-spin correlation functions which are useful for neutron scattering experiments and NMR relaxation rate:

\[
\chi^{ab}(q, i\omega_n) = \int_0^\beta d\tau dxe^{i(q \cdot \tau - q x)} \langle T_\tau J^a(x, \tau) J^b(0, 0) \rangle_T\]

(a, b = 1, 2, 3). \(\beta\)

The finite-temperature correlations are obtained by the analytic continuation \(i\omega_n \rightarrow \omega + i0_+\). Note that expression \(\chi^{ab}\) corresponds to a tensor susceptibility, and \(J^{ab}\) is the total spin current.

\[
J^{a(b)} = \sum_{\nu = R, L} (J^{a(b)}_{\nu 1} + J^{a(b)}_{\nu 2}).
\]

The explicit expression for the correlation functions of the uniform currents in real space is:

\[
\langle T_\tau J^1(x, \tau) J^1(0, 0) \rangle = -\sum_{\nu = L, R} \left[ \cos^2 \alpha \langle T_\tau \xi_{\nu \nu}^2 \xi_{\mu \nu}^4(0, 0) \rangle + \sin^2 \alpha \langle T_\tau \xi_{\nu \nu}^2 \xi_{\mu \nu}^4(0, 0) \rangle \right]
\]

\[
\langle T_\tau J^2(x, \tau) J^2(0, 0) \rangle = \langle T_\tau J^1(x, \tau) J^1(0, 0) \rangle
\]

\[
\langle T_\tau J^3(x, \tau) J^3(0, 0) \rangle = -\sum_{\nu = L, R} \left[ \langle T_\tau \xi_{\nu \nu}^2 \xi_{\mu \nu}^4(0, 0) \rangle \right],
\]

the other contributions being zero.

From [32] we can evaluate the uniform susceptibility by applying Wick’s theorem. The corresponding expressions can be written in compact form introducing the thermal Green’s function [34] for left- and right-moving triplet and singlet Majorana fermions:

\[
G_{RR}^{\alpha}(k, i\omega_n) = G_{LL}^{\alpha}(-k, i\omega_n) = -\frac{i\omega_n + uk}{\omega_n^2 + u^2k^2 + m^2},
\]

\[
G_{RL}^{\alpha}(k, i\omega_n) = G_{LR}^{\alpha}(k, i\omega_n) = -\frac{im_n}{\omega_n^2 + u^2k^2 + m^2},
\]

where \(\alpha\) stands for (1,2) in the case of the doublet excitation or for (3,4) in case of a singlet excitation, and the "polarization" function:

\[
\Gamma^{\alpha\beta}(q, i\omega_n) = -\beta^{-1} \sum_{\omega_m} \int \frac{dk}{2\pi} G_{RR}^{\alpha}(k, i\omega_m)G_{RR}^{\beta}(q - k, i\omega_n - i\omega_m) + G_{RL}^{\alpha}(k, i\omega_m)G_{RL}^{\beta}(q - k, i\omega_n - i\omega_m)
\]

\[
+ G_{LR}^{\alpha}(k, i\omega_m)G_{LL}^{\beta}(q - k, i\omega_n - i\omega_m) + G_{LL}^{\alpha}(k, i\omega_m)G_{LL}^{\beta}(q - k, i\omega_n - i\omega_m).
\]

We find:

\[
\chi^{11}(i\omega_n) = \chi^{11}(i\omega_n),
\]

\[
\chi^{22}(i\omega_n) = \chi^{22}(i\omega_n),
\]

\[
\chi^{33}(i\omega_n) = \Gamma^{12}(q, i\omega_n),
\]

\[
\chi^{12}(i\omega_n) = \chi^{21}(i\omega_n) = 0.
\]
B. Calculation of the magnetic susceptibility

To obtain the static susceptibility, we need to take the limit $\omega \to 0$, and the $k \to 0$ in $\chi(k, \omega)$. The evaluation of the Matsubara frequency sum for each term of type (34), in the limit $q \to 0$, leads to the following expression in the static limit. If $m_a = m_\beta$:

$$\Gamma^{\alpha\alpha}(\omega \to 0) = \frac{1}{T} \int_0^\infty \frac{dk}{2\pi} \text{sech}^2 \left(\frac{\beta \epsilon_{\alpha k}}{2}\right),$$

where $\epsilon_{\alpha(\beta)k} = \sqrt{u^2 k^2 + m^2_{\alpha(\beta)}}$. For $m_\alpha \neq m_\beta$, a more complicated expression is obtained (see Appendix B). In this expression, a new contribution can be isolated:

$$\Gamma^0_{\alpha\beta}(q \to 0, \omega = 0) = \int \frac{dk}{2\pi} \left(1 - \frac{(uk)^2 + m_\alpha m_\beta}{\epsilon_\alpha(k)\epsilon_\beta(k)}\right) \frac{1 - n_F(\epsilon_\alpha(k)) - n_F(\epsilon_\beta(k))}{\epsilon_\alpha(k) + \epsilon_\beta(k)}.$$  

This contribution does not vanish for $T \to 0$, but gives:

$$\Gamma^0_{\alpha\beta}(q \to 0, \omega = 0)_{T=0} = \frac{1}{\pi u} \left[\frac{1}{2} + \frac{m_\alpha m_\beta}{m^2_\alpha - m^2_\beta} \ln \left(\frac{m^2_\beta}{m^2_\alpha}\right)\right].$$

We obtain the final expressions of the static and uniform components of spin susceptibility for $T \to 0$ as:

$$\chi^{11}_{0}(0) = \chi^{22}_{0}(0) = \frac{1}{\pi u} \left\{\cos^2 \alpha \left[1 - \frac{mm_3}{(m^2_\alpha - m^2_\beta) \ln(m^2_3/m^2)}\right] + \sin^2 \alpha \left[1 - \frac{mm_4}{(m^2_\alpha - m^2_\beta) \ln(m^2_4/m^2)}\right]\right\} + O(T^{-1/2}e^{-\min(m_\alpha)/T})$$

$$\chi^{33}_{0}(0) \approx \frac{2\pi m}{u} T^{-1/2}e^{-\frac{u}{T}}. \quad (40)$$

The result shows that the in-plane static susceptibility does not vanish for $T \to 0$, whereas it decays exponentially, $T^{-1/2}e^{-\Delta'J_{z}/(2\pi T)}$ along the DM vector. A similar result holds in the case of the anisotropic spin 1 chain where the in-plane magnetic susceptibility is also finite at zero magnetic field. Restoration of the $SU(2)$ symmetry by the KSEA interaction gives exponentially decaying susceptibilities in all directions: $\chi^{11}_{0}(0) = \chi^{22}_{0}(0) = \chi^{33}_{0}(0) \approx T^{-1/2}e^{-\Delta'J_{z}/(2\pi T)}$, where $\Delta'' = \lambda^2\bar{J}_{z}/2\pi$, similarly to an XXX ladder. This is shown in Fig. 2, where a remarkable feature is that at low $T$ the spin susceptibility is lower in presence of a DM plus KSEA interaction, with a larger spectral gap opening.

C. NMR relaxation rate

In this section, we calculate the NMR longitudinal relaxation rate $T_1^{-1}$. By definition, $T_1^{-1}$ is related to the imaginary part of the dynamical susceptibility in the longitudinal direction:

$$(T_1^{-1}) \propto T \lim_{\omega \to 0} \sum_k \frac{\text{Im} \chi(k, \omega)}{\omega}. \quad (41)$$

The susceptibility $\chi(k, \omega)$ can be decomposed into two contributions, one form the uniform part of the magnetization, $\chi_0(k, \omega)$ and one from the staggered component $\chi_{\pi}(k, \omega)$. It is well known that in the case of a ladder, the staggered component $\chi_{\pi}$ of the spin density makes at low temperatures a negligible contribution to $T_1^{-1}$ compared with the uniform part $\chi_0$, so we can focus on this latter contribution which is easily obtained from the method of Ref. [30]. In the following, we will denote $\chi_0(k, \omega)$ simply by $\chi$ to simplify notations. We have:

$$\chi_a(k, i\omega_n) = \int_0^\beta e^{i\omega_n\tau} \langle \vec{J}_a(k, \tau)\vec{J}_a(0, 0) \rangle,$$  

and $\vec{J}_a$ ($a = 1, 2$) is the total spin current in chain 1 or 2. It can be derived by Eq.s (A2)-(A4)-(A5) in the Appendix A. Using the polarization function, we get the following expression for the longitudinal susceptibility:

$$\chi^a(q, i\omega_n) = [-\cos^2 \alpha(\Gamma^{23}(q, i\omega_n) + \Gamma^{14}(q, i\omega_n) + \Gamma^{31}(q, i\omega_n) + \Gamma^{24}(q, i\omega_n))$$

$$-\sin^2 \alpha(\Gamma^{12}(q, i\omega_n) + \Gamma^{13}(q, i\omega_n) + \Gamma^{14}(q, i\omega_n) + \Gamma^{23}(q, i\omega_n))$$

$$-(\Gamma^{12}(q, i\omega_n) + \Gamma^{34}(q, i\omega_n))]. \quad (43)$$
Taking into account that the fermionic branches 1, 2 have the same spectrum, we can reduce the above expression to a simpler form:

\[ \chi^a(q, i\omega_n) = -[2\Gamma^{23}(q, i\omega_n) + \Gamma^{14}(q, i\omega_n) + (\Gamma^{12}(q, i\omega_n) + \Gamma^{34}(q, i\omega_n))]. \]  

(44)

The detailed calculations are reported in the Appendix B. Here, we only quote the final results. For \( m_\alpha \neq m_\beta \), we have\[44\]

\[ T \sum_q \lim_{\omega \to 0} \frac{\text{Im}\Gamma^{\alpha\beta}(q, \omega)}{\omega} = \int_{\text{max}(m_\alpha, m_\beta)}^\infty \frac{d\epsilon}{4\pi} \left[ \frac{\epsilon^2 + m_\alpha m_\beta}{\sqrt{\epsilon^2 - m_\alpha^2} \sqrt{\epsilon^2 - m_\beta^2}} \right] \text{sech}^2 \left( \frac{\beta \epsilon}{2} \right). \]  

(45)

For \( m_\alpha = m_\beta \), the NMR relaxation rate is divergent\[43\] for \( \omega = 0 \). For \( \omega \sim 0 \), the following expression is obtained:

\[ T \sum_q \frac{\text{Im}\Gamma^{\alpha\alpha}(q, \omega)}{\omega} \sim \frac{m_\alpha}{\pi} e^{-m_\alpha/T} \ln \left( \frac{4\epsilon^T}{\omega} \right), \]  

(46)

where \( \gamma \) is Euler’s constant\[43\].

From (45) and (46), the final expression for the longitudinal relaxation rate reads:

\[
\frac{1}{T_1(\omega)} = \int_{m_3}^\infty \frac{d\epsilon}{4\pi} \frac{\epsilon^2 + mm_3}{\sqrt{\epsilon^2 - m^2} \sqrt{\epsilon^2 - m_3^2}} \text{sech}^2 \left( \frac{\beta \epsilon}{2} \right) + \int_{m_4}^\infty \frac{d\epsilon}{4\pi} \left[ \frac{\epsilon^2 + mm_4}{\sqrt{\epsilon^2 - m^2} \sqrt{\epsilon^2 - m_3^2}} + \frac{\epsilon^2 + m_3 m_4}{\sqrt{\epsilon^2 - m_3^2} \sqrt{\epsilon^2 - m_4^2}} \right] \text{sech}^2 \left( \frac{\beta \epsilon}{2} \right)
\]

\[ + \frac{m}{\pi} e^{-m/T} \ln \left( \frac{4\epsilon^T}{\omega} \right) \]  

(47)

The first two terms give the in-plane singlet-triplet contributions, while the last two terms are the contributions along the DM \( \hat{z} \)-axes. Compared with the results in Ref. \[43\] the in-plane singlet-triplet contributions are different. For a DM plus KSEA interaction these contributions become equal, leading to same result of Ref. \[50\] with \( J_\perp \to J_\perp \), where an explicit dependence on DM interaction is still present.

V. CORRELATION FUNCTIONS IN APPLIED MAGNETIC FIELD AND ESR SPECTRA

The electron-spin resonance (ESR), as well as neutron scattering (NMR), have revealed powerful experimental techniques to test magnon excitations in quasi-one-dimensional Heisenberg antiferromagnets. In applied magnetic field, ESR experiments exhibit thermally activated resonances, depending on the orientation, that could be interpreted as transitions between magnon states whose energies are split by magnetic fields and crystal-field anisotropy. The ESR power absorption is proportional to the imaginary part of the dynamical susceptibility \( I(\omega) \propto \omega \chi^{\text{imag}}(\omega) \). We calculate the ESR transition frequencies in two cases, with applied magnetic field \( h \) along the DM vector and perpendicular to it. The first case is straightforward, the field couples to the doublet \( \xi^a \xi^b \) and ESR intensity is non-zero only when \( \omega h = h \), where \( \omega \) is the microwave frequency. From the point of view of ESR spectra the resonance has zero width. The result holds both in presence and not of DM interaction, so that a measure of ESR spectra with applied field along \( \mathbf{D} \) does not permit to disregard the anisotropic from the isotropic interactions. The analysis of ESR spectra for the field perpendicular to DM vector is more involved. We will begin with the case in which Dzialoshinskii-Moriya interactions exist only along the rungs. The field dependence of the dispersion relation is no longer determined by symmetry considerations alone. The Hamiltonian is given by:

\[
H = -\frac{i\nu}{2} \sum_{a=1}^4 \int dx (\xi^a_R \partial_x \xi^a_R - \xi^a_L \partial_x \xi^a_L) - \frac{i}{\hbar} \sum_{a=1}^4 m_a \int dx \xi^a_R \xi^a_L + i\hbar \sum_{\nu=L,R} \int dx [\cos \alpha \xi^\nu_0 \xi^\nu_0 - \sin \alpha \xi^\nu_0 \xi^\nu_0].
\]  

(48)

Compared to the case without DM interaction, the field couples to both singlets \( \xi^3 \) and \( \xi^4 \) so that also singlet modes participate to ESR resonances. To diagonalize the Hamiltonian \[43\] we express it in Fourier space as:

\[
H = \sum_{k>0} \xi^\nu(-k) \mathcal{H}(k) \xi^\nu(k)
\]  

(49)

where:
Using the expressions (29) of the uniform component of the spin density, we obtain:

\[ \zeta'(k) = \left( \begin{array}{c} \xi_R^2(k) \\ \xi_L^2(k) \\ \xi_R^3(k) \\ \xi_L^3(k) \\ \xi_R^4(k) \\ \xi_L^4(k) \end{array} \right) = \frac{1}{\sqrt{L}} \int_0^L dx \zeta(x)e^{-ikx}, \]

and:

\[ \mathcal{H}(k) = \begin{pmatrix} uk & im & ih \cos \alpha & 0 & ih \sin \alpha & 0 \\ -im & -uk & 0 & ih \cos \alpha & 0 & ih \sin \alpha \\ -ih \cos \alpha & 0 & uk & im_3 & 0 & 0 \\ 0 & -ih \cos \alpha & -im_3 & -uk & 0 & 0 \\ -ih \sin \alpha & 0 & 0 & 0 & uk & im_4 \\ 0 & -ih \sin \alpha & 0 & 0 & -im_4 & -uk \end{pmatrix}. \]

The spectrum of \( \mathcal{H} \) is obtained by solving the equation \( \det(\epsilon - \mathcal{H}) = 0 \). This equation reduces to the form:

\[ y^3 + a_2y^2 + a_1y + a_0 = 0 \]

where \( y = \epsilon(k)^2 \), and whose coefficients are listed in the Appendix \ref{supplementary}. In presence of the KSEA interaction we have a slight simplification coming from the condition \( m_1 = m_2 \). This permits a reduction of the matrix to blocks simplifying the diagonalization. Some asymptotic expressions for the spectrum are obtained in the large and small \( h \) limit and they are explicitly described in the Appendix \ref{supplementary}. As discussed in Ref. \ref{note}, the ESR signal will result from transitions between the two lowest branches \( \epsilon_i(k, \alpha, h) \) (\( i = 1, 2 \)). Transitions will occur at values of field and wavevector satisfying \( |\epsilon_1(k, \alpha, h) - \epsilon_2(k, \alpha, h)| = \omega h \), for a fixed DM interaction. In the anisotropic case the difference will depend on momentum \( k \), and \( \alpha \) causing a resonance broadening. Comparing with the results of ESR analysis of the one-dimensional Heisenberg antiferromagnet, in presence of a pure DM interaction the three magnon branches mix even for a field along one of the symmetry axes due to the easy-plane effect. If DM interaction would not be present a branch would be unaffected by the external field, and only the other two would split. ESR experiments with field orientation along symmetry axes would help to confirm the picture of a DM induced mixing. As shown in Fig. \ref{fig}, under the application of the magnetic field perpendicular to the DM vector, a remarkable result is that in contrast to the case of an isotropic ladder, the magnetic field does not close the gap, thus gapped but incommensurate correlations develop. This result is in consistent agreement with neutron diffraction studies\ref{note} on CaCuO\(_3\), whose structure is similar to that of prototype two-leg spin ladder compound SrCuO\(_3\). The results show that the magnetic structure is incommensurate in the direction of the frustrated interchain interaction in presence of an applied magnetic field along the \( a \)-axes. An evaluation of the DM interaction reveals that DM coupling in this system may be as large as several meV. In presence of a KSEA interaction, the numerical results show a closure of the gap upon application of the magnetic field. The KSEA interaction makes the axis parallel to \( D \) equivalent to the one perpendicular to it, restoring the \( SU(2) \) symmetry. This result is shown in Fig. \ref{fig}. The most general orientation of the magnetic field in the plane perpendicular to \( D \) will be dealt elsewhere.

We now briefly discuss the case in which we have Dzialoshinskii-Moriya interaction both along the rung and along the chain direction. Then, we have to perform a gauge transformation \( \mathbb{I} \). As a result, the coupling to the field parallel to \( x \) direction becomes:

\[ H_{\text{field}} = -h \sum_n (\cos(\delta n)S_n^x - \sin(\delta n)S_n^y). \]

Using the expressions (29) of the uniform component of the spin density, we obtain:

\[ H_{\text{field}} = -ih \sum_{\nu} \int dx (\cos \alpha \xi_\nu^3 - \sin \alpha \xi_\nu^4) \left[ \cos \left( \frac{\delta x}{a} \right) \xi_\nu^2 + \sin \left( \frac{\delta x}{a} \right) \xi_\nu^1 \right]. \]

Thus, the magnetic field is now acting as a periodic potential on the Majorana fermions and band structure in the spectrum should be expected. The discussion of the effects of such band structure on ESR response will be discussed elsewhere.
VI. CONCLUSIONS

We have presented a Majorana fermion description of spin 1/2 ladders in presence of anisotropic superexchange spin interactions along the rungs which allows for the calculation of static and dynamic magnetic properties. We have shown that in presence of a pure DM interaction, the systems always has a spectral gap and the lower lying excitations are Majorana doublets, instead of triplets for the regular ladder, indicating that the DM interaction alone is acting as an easy-plane effect. We have computed both the asymptotic behavior and temperature dependence of the anisotropic spin susceptibility helpful for NMR experiments. Our analysis reveals that the easy-plane effect induced by a pure DM interaction gives shorter correlation lengths along the DM vector, compared to the orthogonal direction. The symmetric counterpart of the DM interaction, i.e. the KSEA interaction, restores the SU(2) symmetry and the same correlation length is found in all direction. We have demonstrated that the interplay of the DM interaction and KSEA interaction results in very interesting behaviors of the static susceptibility and ESR spectra that are of direct experimental relevance. In presence of a pure DM interaction gapped but incommensurate correlations develop while the effect of a superimposed KSEA interaction is the closure of the gap when an external magnetic field is applied along one of the symmetry axes. The overall analysis reveals that there is a net way of determining the DM interaction and the interplay of the DM and KSEA interactions in experiments. It would be interesting to extend the present analysis to the strong-coupling limit, or analyse the effect of a canted antiferromagnetism over the DM and KSEA interactions.

VII. ACKNOWLEDGMENTS

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APPENDIX A: UNIFORM COMPONENTS OF THE CORRELATION FUNCTIONS

According to bosonization, we can decompose the spin density into a uniform and a staggered component as:

\[ S_p = J_{R,p} + J_{L,p} + e^{i \pi/a} n_p \]  

We have discussed the staggered component in the previous sections. In the present section, we derive the expression of the uniform component of the spin density as a function of the Majorana fermion operators. Using the definition of the current and the transformation, we obtain the following expressions for the uniform component of the spin density:

\[
(J_{R1} + J_{R2})^1 = -i (\cos \alpha \xi_R^1 \xi_R^1 - i \sin \alpha \xi_R^1 \xi_R^1) \\
(J_{R1} + J_{R2})^2 = -i (\cos \alpha \xi_R^2 \xi_R^2 + \sin \alpha \xi_R^2 \xi_R^2) \\
(J_{R1} + J_{R2})^3 = -i \xi_R^1 \xi_R^2
\]  

(A2)

And:

\[
(J_{L1} + J_{L2})^1 = -i (\cos \alpha \xi_L^1 \xi_L^1 - \sin \alpha \xi_L^1 \xi_L^1) \\
(J_{L1} + J_{L2})^2 = -i (\cos \alpha \xi_L^2 \xi_L^2 + \sin \alpha \xi_L^2 \xi_L^2) \\
(J_{L1} + J_{L2})^3 = -i \xi_L^1 \xi_L^2
\]  

(A3)

For the differences of the currents, we obtain:

\[
(J_{R1} - J_{R2})^1 = -i (\cos \alpha \xi_R^1 \xi_R^1 + \sin \alpha \xi_R^1 \xi_R^1) \\
(J_{R1} - J_{R2})^2 = -i (\sin \alpha \xi_R^2 \xi_R^2 + \cos \alpha \xi_R^2 \xi_R^2) \\
(J_{R1} - J_{R2})^3 = i \xi_R^1 \xi_R^2
\]  

(A4)

\[
(J_{L1} - J_{L2})^1 = -i (\cos \alpha \xi_L^1 \xi_L^1 + \sin \alpha \xi_L^1 \xi_L^1) \\
(J_{L1} - J_{L2})^2 = -i (\sin \alpha \xi_L^2 \xi_L^2 + \cos \alpha \xi_L^2 \xi_L^2) \\
(J_{L1} - J_{L2})^3 = i \xi_L^1 \xi_L^2
\]  

(A5)

From the previous expressions we evaluate the slowly varying component of the Matsubara spin-spin correlation functions which are useful for neutron scattering experiments and NMR relaxation rate as reported in the main text.

APPENDIX B: DETAILS ON THE CALCULATION OF THE NMR RELAXATION RATE AND MAGNETIC SUSCEPTIBILITY

To obtain the NMR relaxation rate or the magnetic susceptibility we have to calculate the following quantity

\[ T \int \frac{dq}{2\pi} \lim_{\omega \to 0} \frac{\text{Im} \Gamma^{\alpha\beta}(q, \omega)}{\omega} \]  

(B1)

where \( \Gamma^{\alpha\beta}(q, \omega) \) is defined in Eq. (14).

We use the following decompositions:

\[ G_{RR}^\alpha(k, i\omega_n) = -\frac{1}{2} \left( 1 + \frac{uk}{\epsilon_\alpha(k)} \right) \frac{1}{i\omega_n + \epsilon_\alpha(k)} + \left( 1 - \frac{uk}{\epsilon_\alpha(k)} \right) \frac{1}{i\omega_n - \epsilon_\alpha(k)} \]  

(B2)

\[ G_{RL}^\alpha(k, i\omega_n) = -\frac{im}{2\epsilon_\alpha(k)} \left( \frac{1}{i\omega_n + \epsilon_\alpha(k)} - \frac{1}{i\omega_n - \epsilon_\alpha(k)} \right) \]  

(B3)

To obtain after performing the Matsubara sum:
\[ \Gamma_{\alpha\beta}(q, \omega_n) = \Gamma_{\alpha\beta}^{pp}(q, \omega_n) + \Gamma_{\alpha\beta}^{ph}(q, \omega_n), \]

\[ \Gamma_{\alpha\beta}^{pp}(q, \omega_n) = \frac{dk}{4\pi} \left( 1 - \frac{u^2 k (k - q) + m_\alpha m_\beta}{\epsilon_\alpha(k) \epsilon_\beta(k - q)} \right) (1 - n_F(\epsilon_\alpha(k)) - n_F(\epsilon_\beta(k - q))) \]

\[ \times \left[ \frac{1}{i \omega_n + \epsilon_\alpha(k) + \epsilon_\beta(k - q)} - \frac{1}{i \omega_n - \epsilon_\alpha(k) - \epsilon_\beta(k - q)} \right], \]

\[ \Gamma_{\alpha\beta}^{ph}(q, \omega_n) = \frac{dk}{4\pi} \left( 1 + \frac{u^2 k (k - q) + m_\alpha m_\beta}{\epsilon_\alpha(k) \epsilon_\beta(k - q)} \right) (n_F(\epsilon_\alpha(k)) - n_F(\epsilon_\beta(k - q))) \]

\[ \times \left[ \frac{1}{i \omega_n + \epsilon_\beta(k - q) - \epsilon_\alpha(k)} - \frac{1}{i \omega_n + \epsilon_\alpha(k) - \epsilon_\beta(k - q)} \right]. \]

1. NMR rate

From Equation [B4], it is clear that only \( \Gamma_{\alpha\beta}^{ph}(q, \omega_n) \) can make a non-zero contribution to \( \text{Im} \Gamma_{\alpha\beta} \) for \( \omega \to 0 \). Making the analytic continuation \( i \omega_n \to \omega + i0 \) and considering the imaginary part, we obtain:

\[ \text{Im} \Gamma_{\alpha\beta}^{ph}(q, \omega) = \frac{dk}{4} \left( 1 + \frac{u^2 k (k - q) + m_\alpha m_\beta}{\epsilon_\alpha(k) \epsilon_\beta(k - q)} \right) (n_F(\epsilon_\alpha(k)) - n_F(\epsilon_\beta(k - q))) \]

\[ \times \left[ \delta(\omega - \epsilon_\beta(k - q) + \epsilon_\alpha(k)) - \delta(\omega - \epsilon_\alpha(k) + \epsilon_\beta(k - q)) \right] \]

(B7)

Performing the integration over \( q \), and making the variable change \( k - q \to k' \), we obtain the following expression for the contribution of \( \Gamma_{\alpha\beta}^{ph} \) to the NMR rate:

\[ \frac{T}{8\pi} \int dk \int dk' \left( 1 + \frac{u^2 k k' + m_\alpha m_\beta}{\epsilon_\alpha(k) \epsilon_\beta(k')} \right) (n_F(\epsilon_\alpha(k)) - n_F(\epsilon_\beta(k'))) \frac{\delta(\omega - \epsilon_\beta(k') + \epsilon_\alpha(k)) - \delta(\omega - \epsilon_\alpha(k) + \epsilon_\beta(k'))}{\omega} \]

(B8)

The term containing \( kk' \) vanishes by symmetry. Since the rest is invariant under \( k \to -k \) or \( k' \to -k' \), we can reduce the integration on \( (k, k') \to [0, \infty] \times [0, \infty] \) and multiply by a factor of 4. It is now convenient to introduce the variable change: \( \epsilon = \epsilon_\alpha(k), \epsilon' = \epsilon_\beta(k') \) to find that the contribution to the NMR rate reads:

\[ \frac{T}{2\pi} \int_{m_\alpha}^\infty d\epsilon \int_{m_\beta}^\infty d\epsilon' \left( \epsilon' + m_\alpha m_\beta \right) (n_F(\epsilon) - n_F(\epsilon')) \frac{\delta(\omega + \epsilon - \epsilon') - \delta(\omega + \epsilon' - \epsilon)}{\omega} \]

(B9)

For \( m_\alpha > m_\beta \) and \( \omega > 0 \), this expression is rewritten as:

\[ \lim_{\omega \to 0} \omega \text{Im} \Gamma_{\alpha\beta}^{ph}(q, \omega) = \frac{1}{4\pi} \int_{m_\alpha}^\infty d\epsilon \frac{\epsilon^2 + m_\alpha m_\beta}{\sqrt{\epsilon^2 - m_\alpha^2} \sqrt{\epsilon^2 - m_\beta^2}} \cosh^{-2} \left( \frac{\epsilon}{2T} \right) \]

(B10)

For \( m_\alpha = m_\beta \), some extra care is needed as the NMR relaxation rate contains a logarithmic divergence for \( \omega \to 0 \). This time, the contribution to the NMR relaxation rate is:

\[ \frac{T}{\omega} \text{Im} \Gamma_{\alpha\beta}^{ph}(q, \omega) = \frac{1}{4\pi} \int_{m_\alpha}^\infty d\epsilon \frac{\epsilon^2 + m_\alpha^2}{\sqrt{\epsilon^2 - m_\alpha^2} \sqrt{\epsilon^2 - m_\alpha^2}} \cosh^{-2} \left( \frac{\epsilon}{2T} \right) \]

(B11)

Since the divergence comes from \( \epsilon \sim m \), to estimate the singular part we can approximate the integral by:

\[ \frac{m_\alpha}{\pi} \int_{m_\alpha}^\infty \frac{e^{-\epsilon/T} d\epsilon}{\sqrt{\epsilon - m_\alpha} \sqrt{\epsilon + \omega - m_\alpha}} \]

(B12)

Provided \( m \gg T \). With the variable change:

\[ \epsilon = m + \frac{\omega}{2} (\cosh t - 1) \]

(B13)

The integral is rewritten:

\[ \frac{m_\alpha}{\pi} e^{-(m_\alpha - \omega/2)/T} \int_0^\infty dt e^{-\frac{\omega^2}{4T} \cosh t} \sim \frac{m_\alpha}{\pi} e^{-m/T} \left[ K_0 \left( \frac{\omega}{2T} \right) + o(1) \right] \]

Expansion of the Bessel function for \( \omega \to 0 \) leads to the formula (3.9) of Ref. [3]
2. Susceptibility

The calculation of the susceptibility for \( m_\alpha = m_\beta \) is straightforward. To obtain the susceptibility for \( m_\alpha \neq m_\beta \), we must note that there are two contributions, one from \( \Gamma^{pp} \), the other from \( \Gamma^{ph} \). Interestingly, the contribution for \( \Gamma^{pp} \) does not vanish for \( T = 0 \). Below, we give a detailed calculation of \( \Gamma^{pp} \). Starting from Eq. (B3) for \( \omega_n = 0 \), we obtain:

\[
\Gamma^{pp}(q \to 0, \omega_n = 0) = \int \frac{dk}{2\pi} \left( 1 - \frac{(uk)^2 + m_\alpha m_\beta}{\epsilon_\alpha(k) \epsilon_\beta(k)} \right) \frac{1 - n_F(\epsilon_\alpha(k)) - n_F(\epsilon_\beta(k))}{\epsilon_\alpha(k) + \epsilon_\beta(k)}.
\]

(B15)

Multiplying numerator and denominator by \( \epsilon_\alpha - \epsilon_\beta \), and taking \( T = 0 \), we obtain:

\[
\Gamma^{pp}(q \to 0, \omega_n = 0) = \frac{1}{m_\alpha^2 - m_\beta^2} \int \frac{dk}{2\pi} \left[ \epsilon_\alpha(k) + \frac{(uk)^2 + m_\alpha m_\beta}{\epsilon_\alpha(k)} - \epsilon_\beta(k) - \frac{(uk)^2 + m_\alpha m_\beta}{\epsilon_\beta(k)} \right].
\]

(B16)

To calculate the integral (B16), we consider:

\[
\int_0^K \frac{dk}{2\pi} \left( \epsilon_\alpha(k) + \frac{(uk)^2 + m_\alpha m_\beta}{\epsilon_\alpha} \right),
\]

(B17)

where \( K \) is a cut-off. Using the variable change: \( uk = m_\alpha \cos \theta \), we find:

\[
\int_0^K \frac{dk}{2\pi} \left( \epsilon_\alpha(k) + \frac{(uk)^2 + m_\alpha m_\beta}{\epsilon_\alpha} \right) = \frac{1}{\pi u} \left[ uK \sqrt{uK^2 + m_\alpha^2 + m_\alpha m_\beta \ln \left( \frac{2uK}{m_\alpha} \right)} \right].
\]

(B18)

With this result, we find that:

\[
\Gamma^{pp}(q \to 0, \omega_n = 0) = \frac{1}{\pi u(m_\alpha^2 - m_\beta^2)} \lim_{K \to \infty} \left[ \frac{uK(m_\alpha^2 - m_\beta^2)}{\sqrt{(uK)^2 + m_\alpha^2 + m_\alpha m_\beta \ln \left( \frac{m_\beta}{m_\alpha} \right)}} + m_\alpha m_\beta \ln \left( \frac{m_\beta}{m_\alpha} \right) \right],
\]

(B19)

i.e.

\[
\Gamma(q = 0, \omega = 0) = \frac{1}{\pi u} \left[ \frac{1}{2} + \frac{m_\alpha m_\beta}{m_\alpha^2 - m_\beta^2} \ln \left( \frac{m_\beta}{m_\alpha} \right) \right].
\]

(B20)

APPENDIX C: ENERGY LEVELS OF THE DM LADDER IN A MAGNETIC FIELD

The secular equation that gives the eigenvalues of the Hamiltonian (48) is of the form:

\[
y^3 + a_2 y^2 + a_1 y + a_0 = 0
\]

(C1)

where \( y = \epsilon(k)^2 \). We put \( \epsilon_i(k) = k^2 + m_i^2 \). The coefficients \( a_i \) are given by

\[
a_2 = -(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + 2h^2) \\
a_1 = h^4 + 2h^2 A(m, \alpha) + \epsilon_1 \epsilon_2^2 + \epsilon_1 \epsilon_3^2 + \epsilon_2 \epsilon_3^2 \\
a_0 = -h^4 B(k, m', \alpha) + 2h^2 C(k, m, \alpha) - \epsilon_1^2 \epsilon_2^2 \epsilon_3^2.
\]

(C2)

\[
A(m, \alpha) = (m_2^2 \cos^2 \alpha - m_1 m_4 \sin^2 \alpha + m_3^2 \sin^2 \alpha - m_1 m_3 \cos^2 \alpha) \\
B(k, m', \alpha) = (m_2^2 \cos^4 \alpha + 2m_3 m_4 \sin^2 \alpha \cos^2 \alpha + m_3^4 \sin^4 \alpha + k^2) \\
C(k, m, \alpha) = (m_1 m_3 m_4^2 \cos^2 \alpha + k^2 m_4^2 \cos^2 \alpha + m_1 m_3 m_4 \sin^2 \alpha + k^2 m_3 m_4 \sin^2 \alpha + k^2 m_4 m_5 \cos^2 \alpha + k^4),
\]

(C3)

where \( m = (m_1, m_3, m_4) \) and \( m' = (m_3, m_4) \). For some specific values of \( h \) the solutions of (C1) are plotted in Figs. For small \( h \) or large \( h \) we derive the following asymptotic expressions for the solutions:
1. Behavior in strong magnetic field

For strong magnetic field, it is convenient to search for solutions of the form: $\eta = h^2 u(h)$. Doing such replacement, we obtain the equation for $u$ in the form:

$$u(u - 1)^2 - \frac{c_1^2 + c_2^2 + c_3^2}{h^2} u^2 + \left( \frac{2A}{h^2} + \frac{e_1^2 c_2^2 + e_2^2 c_3^2 + c_3^2 c_2^2}{h^4} \right) u - \left( \frac{B}{h^2} + \frac{c_1^2 c_2^2 c_3^2}{h^6} \right) = 0$$

(C4)

Thus, one can search for solutions of the form:

$$u = \frac{1}{E/h + C/2}$$

(C5)

$$u = \frac{E/h + C/2}{1}$$

(C6)

In particular, we find the solution $u = B/h^2$, which leads to $\epsilon(k) = \sqrt{B}$ which is an effectively non-magnetic mode. This non-magnetic mode can be obtained in the numerical calculations for high magnetic field ($h \geq 9$). We note that for $\alpha = 0$ we recover the singlet mode of the spin ladder. The other solution leads to an energy dispersion of the form:

$$\epsilon(k) = \pm \gamma(k/2, m, \alpha) \pm h,$$

(C7)

where $\gamma(k/2, m) = \sqrt{k^2 + (m + m_3 \cos^2 \alpha + m_4 \sin^2 \alpha)^2}/4$.

We note that for $\alpha = 0$ and $m = m_3$ we recover the magnetic modes of the spin ladder. So, in high fields, the Dzialoshinskii Moriya ladder appears to behave similarly to a regular spin ladder albeit with renormalized gaps.

2. Behavior in small magnetic field

At small $h$, we can expand equation (C1) in powers of $h^2$ and find a solution valid to first order in $h^2$. The equation expanded to first order in $h^2$ reads:

$$\prod_{i=1,2,3} (\epsilon^2 - \epsilon_i^2) - 2h^2 \epsilon^4 + 2h^2 A(m, \alpha) \epsilon^2 + 2h^2 C(m, k, \alpha) = 0.$$  

(C8)

One can search for solutions of the form: $\epsilon^2(k) = \epsilon_i^2(k) + \beta_i h^2$. We find:

$$\beta_i = - \frac{C(k, m, \alpha) + A(m, \alpha) \epsilon_i(0)^2 - \epsilon_i(0)^4}{\epsilon_i(0) \Pi_{\neq j}(\epsilon_i(0)^2 - \epsilon_j(0)^2)}.$$  

(C9)

From the knowledge of the spectrum we evaluate the total magnetization through the following relation:

$$M = \sum_{i=1}^{3} \sum_{\epsilon_i(k) < 0} \frac{\partial \epsilon_i(h)}{\partial h} = - \sum_{i=1}^{3} \sum_{\epsilon_i(k) < 0} \frac{h}{\epsilon_i(0)} \left[ -\epsilon_i^4(0) + A^2(m, \alpha) \epsilon_i^2(0) + C(k, m, \alpha) \right] \Pi_{\neq j}(\epsilon_i^2(0) - \epsilon_j^2(0)) + O(h^3).$$  

(C10)

We can perform analytically the integral and the result gives a finite magnetic susceptibility in zero magnetic field:

$$\chi(0) \approx \frac{m^2 m_3^2 \ln(m/m_3) + (m^2 + m^2 m_4^2 - m_4^2) \ln(m/m_4) + (m_3^2 + m_3^2 m_4^2 - m_4^2) \ln(m/m_4)}{(m^2 - m_4^2)(m^2 - m_4^2)(m_3^2 - m_4^2)}.$$  

(C11)

Thus, at zero temperature, there is a finite susceptibility in the direction perpendicular to the DM vector, but zero susceptibility parallel to the DM vector.

In presence of the DM+KSEA interaction the equation (C8) still holds with the prescription $\epsilon_1 = \epsilon_2$ and $m = m_3$, thus we search for solutions of the form: $\epsilon^2(k) = \epsilon_i^2(k) + \alpha_1 h + \alpha_2 h^2$ and $\epsilon^2(k) = \epsilon_i^2(k) + \alpha_3 h^2$. We find:

$$\alpha_1 = \sqrt{\frac{C(k, m, \alpha) + A(m, \alpha) \epsilon_1(0)^2 - \epsilon_1(0)^4}{\epsilon_1(0)^2 - \epsilon_3(0)^2}}; \alpha_2 = 0$$  

$$\alpha_2 = \sqrt{\frac{B(k, m', \alpha) - \epsilon_1(0)^2}{\epsilon_1(0)^2 - \epsilon_3(0)^2}}; \alpha_1 = 0$$  

$$\alpha_3 = \frac{C(k, m, \alpha) + A(m, \alpha) \epsilon_3(0)^2 - \epsilon_3(0)^4}{\epsilon_3(0)(\epsilon_3(0)^2 - \epsilon_1(0)^2)}.$$  

(C12)
The first solution gives zero contribution to the net magnetization, thus we find:

\[
M = \sum_{\epsilon_{1,3}(k) < 0} \hbar \epsilon_1(0) \sqrt{\frac{B(k, m', \alpha) - \epsilon_1(0)^2}{\epsilon_1(0)^2 - \epsilon_3(0)^2}} - \frac{\hbar}{\epsilon_3(0)} \frac{\epsilon_3^4(0) + A^2(m, \alpha)\epsilon_3^2(0) + C(k, m, \alpha)}{\epsilon_3^2(0) - \epsilon_1^2(0)} + O(h^3). \tag{C13}
\]

The integration of such expression gives the zero temperature susceptibility in presence of the DM+KSEA interaction.
FIG. 1. A plot of specific heat versus temperature for a spin ladder with and without DM interaction. A remarkable feature is that at low temperature, the specific heat of the ladder with DM interaction is lower while it is higher at high temperature.

FIG. 2. A plot of spin susceptibility along the $z$-axes versus temperature for a spin ladder with DM interaction and DM plus KSEA interaction, for $D/J_\perp = 0.8$. The KSEA interaction restores the SU(2) symmetry but a larger spectral gap appears in the low-T limit.
FIG. 3. A plot of the dispersion of the spin ladder with DM interaction for fixed magnetic field $h = 3$, $u = 1$; $m$ is the Majorana doublet mass, $m_{3,4}$ are the masses of the singlets. We notice that there is no closure of any gap upon application of a quite strong field. From the dispersion, we should expect gapped but incommensurate correlations to develop in the system.

FIG. 4. A plot of the dispersion of the spin ladder with DM+KSEA interaction for fixed magnetic field $h = 1.5$, $u = 1$. We notice a closure of the gap upon application of the magnetic field. Thus, the KSEA interaction makes the $\hat{x}$ axis equivalent to the $\hat{z}$ axis.