We propose a new numerical algorithm for computing the tensor rank decomposition or canonical polyadic decomposition of higher-order tensors subject to a rank and genericity constraint. Reformulating this computational problem as a system of polynomial equations allows us to leverage recent numerical linear algebra tools from computational algebraic geometry. We characterize the complexity of our algorithm in terms of an algebraic property of this polynomial system—the multigraded regularity. We prove effective bounds for many tensor formats and ranks, which are of independent interest for overconstrained polynomial system solving. Moreover, we conjecture a general formula for the multigraded regularity, yielding a (parameterized) polynomial time complexity for the tensor rank decomposition problem in the considered setting. Our numerical experiments show that our algorithm can outperform state-of-the-art numerical algorithms by an order of magnitude in terms of accuracy, computation time, and memory consumption.

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1 INTRODUCTION

We introduce an original direct numerical algorithm for tensor rank decomposition or canonical polyadic decomposition (CPD) in the low-rank regime. By “direct,” we mean an algorithm that does not rely on numerical optimization or other iteratively refined approximations with a data-dependent number of iterations.

Consider the vector space \( \mathbb{C}^{(n_1+1) \times \cdots \times (n_d+1)} \) whose elements represent order-\( d \) tensors in coordinates relative to a basis. We say that a tensor in such a space is of rank 1 or elementary if it is of
the following form:
\[ \alpha^1 \otimes \alpha^2 \otimes \cdots \otimes \alpha^d := \left( \alpha_{j_1}^{a_1} \alpha_{j_2}^{a_2} \cdots \alpha_{j_d}^{a_d} \right)_{0 \leq j_k \leq n_k}, \quad \text{where} \quad \alpha^k = \left( \alpha_{j_k}^k \right)_{0 \leq j_k \leq n_k} \in \mathbb{C}^{n_k+1}. \]

Every tensor \( \mathcal{A} \) can be expressed as a linear combination of rank-1 tensors:
\[ \mathcal{A} = \sum_{i=1}^{r} \alpha_i^1 \otimes \cdots \otimes \alpha_i^d, \quad \text{with} \quad \alpha_i^k = \left( \alpha_{j_k}^{k_i} \right)_{0 \leq j_k \leq n_k} \in \mathbb{C}^{n_k+1}. \quad \text{(CPD)} \]

If \( r \) is minimal among all such expressions of \( \mathcal{A} \), then \( r \) is called the rank of the tensor according to Hitchcock [1927], and Equation \( \text{(CPD)} \) is called a CPD. The problem of computing a CPD of \( \mathcal{A} \), i.e., determining a set of rank-1 tensors summing to \( \mathcal{A} \), has many applications in science and engineering, see for instance Kolda and Bader [2009] and Sidiropoulos et al. [2017].

The strategy we propose for computing (Equation \( \text{(CPD)} \)) relies on the fact that the problem is equivalent to solving a certain system of polynomial equations. Under suitable assumptions, these equations can be obtained from the nullspace of a flattening \( \mathcal{A}_{(1)} \in \mathbb{C}^{(n_1+1) \times \prod_{k=2}^{d}(n_k+1)} \) of the tensor \( \mathcal{A} \), as in Kuo and Lee [2018]. Once we have obtained these polynomial equations, whose solutions correspond to the rank-1 terms in Equation \( \text{(CPD)} \), we use recent numerical normal form techniques from Bender and Telen [2020] and Telen [2020a, b] to solve them. The following example, which is used as a running example throughout the article, illustrates how this works.

**Example 1.1 (Motivating Example).** Consider the \( 4 \times 3 \times 3 \) tensor \( \mathcal{A} \) with flattening
\[
\mathcal{A}_{(1)} = \begin{bmatrix}
\partial_{00} & \partial_{01} & \partial_{02} & \partial_{10} & \partial_{11} & \partial_{12} & \partial_{20} & \partial_{21} & \partial_{22} \\
1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & \\
1 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & \\
1 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 2 \\
\end{bmatrix}.
\]

The column of this matrix indexed by \( \partial_{k\ell} \) contains the entries \( \mathcal{A}_{jk\ell} \), \( j = 0, \ldots, 3 \). The reason for this indexing will become clear in Section 2. The kernel of \( \mathcal{A}_{(1)} \) is the transpose of
\[ R_{(1,1)}^\top = \begin{bmatrix}
x_0y_0 & x_0y_1 & x_0y_2 & x_1y_0 & x_1y_1 & x_1y_2 & x_2y_0 & x_2y_1 & x_2y_2 \\
f_1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
f_2 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\
f_3 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
f_4 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
f_5 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}. \quad (1.1)
\]

The column of \( R_{(1,1)}^\top \) corresponding to column \( \partial_{k\ell} \) of \( \mathcal{A}_{(1)} \) is now indexed by \( x_ky_\ell \): We interpret the rows as polynomials

\[
\begin{align*}
f_1 &= -x_1y_0 + x_1y_1, & f_2 &= -x_0y_0 - x_1y_0 + x_1y_2, & f_3 &= -2x_0y_0 + x_2y_0, \\
f_4 &= -x_0y_1 + x_2y_1, & f_5 &= -2x_0y_2 + x_2y_2.
\end{align*}
\]

These are bilinear forms in \( S = \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2] \). As explained in Section 2, the common zeros of \( f_1, \ldots, f_5 \) form a subvariety of \( \mathbb{P}^2 \times \mathbb{P}^2 \) consisting of the four points
\[
\begin{align*}
\zeta_1 &= ((1 : 0 : 2), (1 : 0 : 0)), & \zeta_2 &= ((1 : 0 : 1), (0 : 1 : 0)), \\
\zeta_3 &= ((1 : 1 : 2), (0 : 0 : 1)), & \zeta_4 &= ((0 : 1 : 0), (1 : 1 : 1)).
\end{align*}
\]
These points correspond to the last two factors of the rank-1 terms in

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \otimes \begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} \otimes \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} \otimes \begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} \otimes \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\]

this is the decomposition (CPD) of \( \mathcal{A} \).

**Contributions**

We formulate our results for the case \( d = 3 \), as the general case can be handled using the standard reshaping trick (Section 2.2). Our main contribution is a new numerical algorithm for computing the rank-\( r \) CPD of a third-order tensor \( \mathcal{A} \) based on linear algebra, under the assumption that \( \mathcal{A} \) is \( r \)-identifiable and a flattening of \( \mathcal{A} \) has rank \( r \), see Assumption 2.1. We call this algorithm \( \text{cpd}\_\text{hnf} \). It is based on a well-known reformulation of the problem as a system of polynomial equations, which we solve using state-of-the-art methods from computational algebraic geometry. This results in Algorithm 3.1. We show that this algorithm generalizes pencil-based algorithms [Beltrán et al. 2019; De Lathauwer 2006; Leurgans et al. 1993; Lorber 1985; Sanchez and Kowalski 1990] from very low ranks to much higher ranks in the unbalanced case; see Theorems 3.1 and 3.2 and Section 3.3.

We give a new, explicit description of the complexity of the tensor rank decomposition problem in terms of an algebraic property of aforementioned polynomial system: the multigraded regularity of a non-saturated ideal in the homogeneous coordinate ring of \( \mathbb{P}^m \times \mathbb{P}^n \) (Proposition 3.1). We characterize the regularity in terms of the rank of a structured matrix obtained directly from the rank-1 terms in Equation (CPD), see Example 4.1. These new insights allow us to formulate a conjecture regarding the regularity (Conjecture 1). We prove this conjecture for many formats, see Theorem 4.2. These results are of independent interest for the field of polynomial system solving. They have the following consequence related to the complexity of our algorithm.

**Theorem 1.1.** Consider the tensor space \( \mathcal{C}^{\ell+1} \otimes \mathcal{C}^{m+1} \otimes \mathcal{C}^{n+1} \) of dimension \( M = (\ell+1)(m+1)(n+1) \) with \( \ell \geq m \geq n \). If Conjecture 1 holds and \( \mathcal{A} \) is a generic tensor of rank \( r \leq \phi mn \) with \( \phi \in [0, 1) \) a fixed constant, then \( \text{cpd}\_\text{hnf} \) runs in polynomial time \( O(M^\frac{1}{2} r^\frac{1}{2} n) \).

Our numerical experiments in Section 6 show that the proposed algorithm is highly efficient and gives accurate results (see Figure 2). For instance, we can compute the decomposition of a \( 7 \times 7 \times 7 \times 7 \times 6 \times 5 \times 5 \) tensor of rank 1,000 in double-precision arithmetic with an accuracy of order \( 10^{-15} \) in 441 s—a feat we believe has not been matched by other tensor decomposition algorithms. Moreover, \( \text{cpd}\_\text{hnf} \) seems to behave well in the presence of noise.

A Julia implementation of \( \text{cpd}\_\text{hnf} \), including driver routines to reproduce our numerical experiments, is provided at https://gitlab.kuleuven.be/u0072863/homogeneous-normal-form-cpd.

**Related Work**

The idea of computing tensor decompositions via polynomial root finding is central in apolarity-based approaches such as Bernardi et al. [2011, 2013]; Bernardi and Taufer [2020], and Brachat et al. [2010]; Nie [2017] for the symmetric case. The Hankel operators play the role of normal forms in this context. These operators can be obtained partially from the tensor \( \mathcal{A} \). In most cases, an additional polynomial system needs to be solved to complete the Hankel operators [Mourrain 2018, Section 4].

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1A property on a variety \( \mathcal{V} \) is "generic" if the locus where the property does not hold is contained in a Zariski closed subset.
For instance, this is step (2) in Algorithm 5.1 of Brachat et al. [2010]. Although our method works only under certain assumptions on the rank of $A$, in contrast to apolarity-based methods it requires only linear algebra computations, and it operates in polynomial rings with fewer variables. Moreover, the method from Brachat et al. [2010] uses connected-to-one bases for its normal form computations. The choice of such a basis is discussed at length in Sections 4.3 and 7.2 of Bernardi and Taufer [2020]. In this article, we exploit the flexibility of truncated normal forms [Telen et al. 2018] to achieve better numerical results. For a comparison, see Section 4.3.3 of Telen [2020b].

Kuo and Lee [2018] obtained an affine system of polynomial equations as in Example 1.1, but it is solved using homotopy continuation methods. This approach is infeasible for some formats that are handled without any problems by our algorithm. For instance, for the aforementioned eighth-order tensor of rank 1,000, the method of Kuo and Lee [2018] would need to track over $40 \times 10^9$ homotopy paths. We argue that the eigenvalue-based methods proposed in this article are more natural to use in an overdetermined setting.

The state-of-the-art algorithms for tensor rank decomposition using only linear algebra computations were proposed by Domanov and De Lathauwer [2014, 2017]. Although these methods work under slightly milder conditions, our numerical experiments suggest that Algorithm 1 of Domanov and De Lathauwer [2017] often requires the construction of larger matrices than those in our algorithm. There is no explicit connection with polynomial equations. The algorithm and its complexity depend on a parameter $l$, which is chosen incrementally by trial and error for each format. In an analogous way, the complexity of our algorithm is governed by the choice of a parameter. However, our analysis in Section 4 tells us a priori which parameter value should be used, circumventing a trial-and-error approach. In Section 6, we demonstrate that cpd_hnf improves on Domanov and De Lathauwer [2014, 2017] in terms of computational complexity and accuracy.

Outline

In Section 2, we state our assumptions and show how computing the CPD of a tensor $A$ is formulated as a system of polynomial equations. In Section 3, we make the connection with normal form methods explicit. That is, we describe how a pre-normal form can be computed directly from the tensor $A$ and how this allows us to reduce the above polynomial system to an eigenvalue problem. We explain how the approach generalizes so-called pencil-based algorithms in Section 3.3. This leads to a complete, high-level description of our algorithm cpd_hnf in Section 3.4. In Section 4, we study the regularity of the ideal associated to our polynomial system. These results are the starting point for our analysis of the complexity of cpd_hnf in Section 5. In Section 6, we demonstrate the efficiency and accuracy of cpd_hnf relative to the state of the art through several numerical experiments. The article is completed in Section 7 with some final conclusions.

2 FROM TENSOR DECOMPOSITION TO POLYNOMIAL EQUATIONS

In this section, we explain how tensor rank decomposition, under some restrictions, can be reduced to solving a polynomial system of equations whose coefficients are directly obtained from the tensor $A$. The next two subsections state the restrictions under which the proposed algorithm operates. In Section 2.3, the polynomial system is constructed. We show that it gives a nice algebraic description of a certain projection of the rank-1 tensors appearing in $A$’s decomposition. Section 2.4 provides a pseudo-algorithm summarizing the main steps of the proposed numerical algorithm.

2.1 Identifiability, Flattenings, and the Main Assumption

For the moment, let us assume $A \in \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$ is a third-order tensor of rank $r$. The cpd_hnf algorithm works under the following assumption.
ASSUMPTION 2.1. The tensor $\mathcal{A} \in \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$ with $\ell \geq m \geq n > 0$ is generic of rank $r \leq \min \{\ell + 1, mn\}$.

By generic, we mean that $\mathcal{A}$ is contained in a Zariski dense open subset of the set of all rank-$r$ tensors. We will describe this open subset more explicitly in Lemma 2.2. For now, we point out that in this open set, each tensor $\mathcal{A}$ is such that

(i) $\mathcal{A}$ is $r$-identifiable, and
(ii) the standard 1-flattening $\mathcal{A}_{(1)}$ of the tensor $\mathcal{A}$ is of the same rank as $\mathcal{A}$.

These are necessary conditions for our algorithm to work. We now briefly recall their meaning.

Condition (i) is usually very weak and only of a technical nature. Recall that the set of all rank-1 tensors forms an algebraic variety, i.e., the solution set of a system of polynomial equations, called the Segre variety $\mathcal{S}$. A rank-$r$ CPD of a tensor $\mathcal{A}$ is a set of rank-1 tensors whose sum is $\mathcal{A}$. The set of all such rank-$r$ CPDs is denoted by $\mathcal{S}^{[r]} = \{X \subset \mathcal{S} \mid |X| = r\}$. Tensor rank decomposition consists of computing an element of the fiber of $f: \mathcal{S}^{[r]} \to \mathcal{S}$, where $f$ is an analytic, bijective function at a rank-$r$ tensor $\mathcal{A}$. For brevity, we denote the image of $f$ by

\[ S_r = f(\mathcal{S}^{[r]}). \]

Tensor $\mathcal{A}$ is called $r$-identifiable if the fiber $f^{-1}(\mathcal{A})$ contains exactly one element. That is, there is a unique set in $\mathcal{S}^{[r]}$ whose sum is $\mathcal{A}$. Generally, $r$-identifiability fails only on a strict closed subvariety of (the Zariski closure of) $\mathcal{S}_r$; see Bocci et al. [2014]; Chiantini and Ottaviani [2012]; Chiantini et al. [2014, 2017]. This property is called the generic $r$-identifiability of $\mathcal{S}$. These results entail that $r$-identifiability fails only on a subset of $\mathcal{S}_r$ of Lebesgue measure zero if the rank $r$ and dimensions $(\ell + 1, m + 1, n + 1)$ satisfy some very weak conditions; see Section 1 in Chiantini et al. [2014] and Section 3 in Chiantini et al. [2017] for more details and statements for higher-order tensors as well. If $\mathcal{S}_r$ is generically $r$-identifiable, then there is a Zariski-open subset of the closure of $\mathcal{S}_r$ so that $f^{-1}$ is an analytic, bijective tensor decomposition function. We aspire to solve the tensor decomposition problem only in this well-behaved setting.

Condition (ii) is more restrictive, but allows us to tackle the tensor decomposition problem using only efficient linear algebra. Recall that the standard 1-flattening of $\mathcal{A} \in \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$ consists of interpreting $\mathcal{A}$ as the matrix $\mathcal{A}_{(1)} \in \mathbb{C}^{(\ell+1) \times (m+1)(n+1)}$. For a rank-1 tensor $\mathcal{A} = \alpha \otimes \beta \otimes \gamma$ this identification is defined by

\[ \mathcal{A}_{(1)} = \alpha(\beta \otimes \gamma)^\top, \]

and the general case follows by linearity. The tensor product\(^2\) in the foregoing expression is also called the reverse-order Kronecker product

\[ \otimes : \mathbb{C}^{m+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^{(m+1)(n+1)}, \quad (\beta, \gamma) \mapsto [\beta_i \gamma_j]_{(i,j)}, \]

where the standard bases were assumed for these Euclidean spaces and the indices $(i, j)$ are sorted by the reverse lexicographic order. Note that the 1-flattening is easy to compute when $\mathcal{A}$ is given in coordinates relative to the standard tensor product basis. In that case it suffices to reshape the coordinate array to an $(\ell + 1) \times (m + 1)(n + 1)$ array (e.g., as in Julia’s or Matlab’s reshape function).

\(^2\)The tensor product is defined uniquely by the linear space into which it maps by universality [Greub 1978], so we do not make a distinction in notation.
2.2 The Reshaping Trick

For dealing with higher-order tensors \( \mathcal{A} \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_d+1} \) with \( d > 3 \), we rely on the well-known reshaping trick. It consists of interpreting a higher-order tensor as a third-order tensor. While the approach described in the remainder of the article could be applied directly to higher-order tensors as well, the range of ranks \( r \) to which this version would apply is (much) more restrictive than the range obtained from reshaping.

Recall that reshaping \( \mathcal{A} \) to a third-order tensor \( \mathcal{A}_{(I,J,K)} \) is a linear isomorphism

\[
\mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_d+1} \cong (\otimes_{i \in I} \mathbb{C}^{n_i+1}) \otimes (\otimes_{j \in J} \mathbb{C}^{n_j+1}) \otimes (\otimes_{k \in K} \mathbb{C}^{n_k+1}) \cong \mathbb{C}^{\prod_{i \in I}(n_i+1)} \otimes \mathbb{C}^{\prod_{j \in J}(n_j+1)} \otimes \mathbb{C}^{\prod_{k \in K}(n_k+1)}
\]

that identifies \( \alpha^1 \otimes \cdots \otimes \alpha^d \) with \( (\otimes_{i \in I} \alpha^i) \otimes (\otimes_{j \in J} \alpha^j) \otimes (\otimes_{k \in K} \alpha^k) \), wherein \( I \sqcup J \sqcup K \) partitions \( \{1, \ldots, d\} \). The general case follows by linearity from the universal property [Greub 1978].

It was shown in Section 7 of Chiantini et al. [2017] that under some conditions, the unique CPD of \( \mathcal{A} \) can be recovered from the CPD of a reshaping \( \mathcal{A}_{(I,J,K)} \). Here, we exploit the slightly more general result that requires no \( r \)-identifiability of \( \mathcal{A} \).

**Lemma 2.1.** Let \( \mathcal{A} \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_d+1} \) be a rank-\( r \) tensor. If there exists a partition \( I \sqcup J \sqcup K \) of \( \{1, \ldots, d\} \) such that the third-order reshaping \( \mathcal{A}_{(I,J,K)} \in \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \) is \( r \)-identifiable, then \( \mathcal{A} \) is \( r \)-identifiable and the CPD of \( \mathcal{A}_{(I,J,K)} \) is the CPD of \( \mathcal{A} \) under the isomorphism Equation (I).

**Proof.** Let \( I \sqcup J \sqcup K \) be such a partition and assume that \( \mathcal{A}_{(I,J,K)} \)'s unique CPD is \( \mathcal{A}_{(I,J,K)} = \sum_{q=1}^{r} \alpha_q \otimes \beta_q \otimes \gamma_q \). Let \( \mathcal{A} = \sum_{q=1}^{r} \alpha_q^1 \otimes \cdots \otimes \alpha_q^d \) be an arbitrary rank-\( r \) CPD. Then,

\[
\mathcal{A}_{(I,J,K)} = \sum_{q=1}^{r} \left( \otimes_{i \in I} \alpha_q^i \right) \otimes \left( \otimes_{j \in J} \alpha_q^j \right) \otimes \left( \otimes_{k \in K} \alpha_q^k \right)
\]

because of Equation (I). Since \( \mathcal{A}_{(I,J,K)} \)'s rank-\( r \) CPD is unique by assumption, it follows there is some permutation \( \pi \) and scalars \( \rho_q, \sigma_q, \tau_q = 1 \) such that we have, for all \( q = 1, \ldots, r \),

\[
\rho_q \alpha_{\pi_q} = \otimes_{i \in I} \alpha_q^i, \quad \sigma_q \beta_{\pi_q} = \otimes_{j \in J} \alpha_q^j, \quad \tau_q \gamma_{\pi_q} = \otimes_{k \in K} \alpha_q^k.
\]

This implies that the CPD of \( \mathcal{A}_{(I,J,K)} \) under the linear isomorphism Equation (I) results in a CPD of \( \mathcal{A} \).

It remains to show that it is the unique one. Denote the set of \( r \) rank-1 summands in any two CPDs of \( \mathcal{A} \) by \( \{\tau_1, \ldots, \tau_r\} \) and \( \{\tau'_1, \ldots, \tau'_r\} \), respectively. The previous paragraph showed that any CPD of \( \mathcal{A} \) implies a CPD of \( \mathcal{A}_{(I,J,K)} \). Since \( \mathcal{A}_{(I,J,K)} \)'s CPD is unique, there exist permutations \( \pi \) and \( \pi' \) such that for all \( q = 1, \ldots, r \),

\[
\left( \otimes_{i \in I} \alpha_q^i \right) \otimes \left( \otimes_{j \in J} \beta_q^j \right) \otimes \left( \otimes_{k \in K} \gamma_q^k \right) = \left( \tau_{\pi_q} \right)_{(I,J,K)} = \left( \tau'_{\pi'_q} \right)_{(I,J,K)}.
\]

Since reshaping is injective and sends rank-1 tensors in the domain to rank-1 tensors in the codomain, we conclude that \( \{\tau_1, \ldots, \tau_r\} = \{\tau'_1, \ldots, \tau'_r\} \). That is, all of \( \mathcal{A} \)'s CPDs must consist of the same rank-1 terms, which is exactly the definition of \( r \)-identifiability. \( \square \)

For decomposing higher-order tensors \( \mathcal{A} \) of rank \( r \) via the reshaping trick, we proceed as follows:

1. **Find** \( I \sqcup J \sqcup K = \{1, \ldots, d\} \) such that \( \mathcal{A}_{(I,J,K)} \) is \( r \)-identifiable and its rank satisfies (R).
2. **Compute** the CPD \( \mathcal{A}_{(I,J,K)} = \sum_{q=1}^{r} \alpha_q \otimes \beta_q \otimes \gamma_q \), e.g., as in Algorithm 2.1.
3. **Viewing** the recovered vectors as \( \alpha_q \in \otimes_{i \in I} \mathbb{C}^{n_i+1}, \beta_q \in \otimes_{j \in J} \mathbb{C}^{n_j+1} \), and \( \gamma_q \in \otimes_{k \in K} \mathbb{C}^{n_k+1} \), a CPD of \( \mathcal{A} \) is \( \mathcal{A}_1 + \cdots + \mathcal{A}_r \) where the tensors \( \mathcal{A}_q \) are isomorphic to \( \alpha_q \otimes \beta_q \otimes \gamma_q \) under Equation (I).

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Note that the rank-1 decompositions in the third step can be computed for example with a sequentially truncated higher-order singular value decomposition [Vannieuwenhoven et al. 2012], as in our implementation, or with a cross approximation [Oseledets et al. 2008].

Because of the reshaping trick, we will henceforth describe our approach only for \( d = 3 \).

### 2.3 Polynomial Systems Defined by Flattenings

Having delineated the range (R) in which the proposed cpd_hnf algorithm will work, we continue by describing it. Our strategy to compute the tensor rank decomposition of

\[
\mathcal{A} = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i \in \mathbb{C}^{l+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}
\]

is to compute the points

\[
\zeta_i = (\beta_i, \gamma_i) \in \mathbb{C}^{m+1} \times \mathbb{C}^{n+1}, \quad i = 1, \ldots, r,
\]

first as the solution of a system of (low-degree) polynomial equations. Thereafter, we compute the \( \alpha_i \)'s with efficient linear algebra by plugging \( \beta_i \) and \( \gamma_i \) into Equation (A) and solving the resulting linear system of equations. Indeed, since

\[
\mathcal{A}_{(1)} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_r \end{bmatrix} \begin{bmatrix} \beta_1 \otimes \gamma_1 & \cdots & \beta_r \otimes \gamma_r \end{bmatrix}^\top
\]

and Assumption 2.1 guarantees that the second \( r \times (m+1)(n+1) \) matrix is of full rank \( r \) (see Lemma 2.2 below), so it has a right inverse. Applying the latter on the right to \( \mathcal{A}_{(1)} \) yields the corresponding \( \alpha_i \)'s. Note that it suffices to compute the points \( \beta_i \) and \( \gamma_i \) up to a nonzero scaling factor. Therefore, it is natural to consider our problem in complex projective space.

Recall that the \( k \)-dimensional complex projective space \( \mathbb{P}^k \) is the space of equivalence classes \([x] = \{\lambda x \mid \lambda \in \mathbb{C} \setminus \{0\}\}\) for \( x \in \mathbb{C}^{k+1} \setminus \{0\} \). The entries of the vector \( x = (x_0, \ldots, x_k) \in \mathbb{C}^{k+1} \setminus \{0\} \) are called homogeneous coordinates of \([x]\). With a slight abuse of notation, we will write \( x = (x_0 : \cdots : x_k) \in \mathbb{P}^k \) for both the equivalence class \([x]\) and a set of homogeneous coordinates \( x \in \mathbb{C}^{k+1} \setminus \{0\} \).

The proposed cpd_hnf algorithm exploits the fact that the points \( \zeta_i = (\beta_i, \gamma_i) \in \mathbb{P}^m \times \mathbb{P}^n \) are defined by algebraic relations on \( X = \mathbb{P}^m \times \mathbb{P}^n \) that can be computed directly from the tensor \( \mathcal{A} \). Such algebraic relations are homogeneous polynomials in a bi-graded polynomial ring. In this context, it is natural to think of degrees as \( 2 \)-tuples \((d, e) \in \mathbb{N}^2\), where \( d \) is the degree in the variables corresponding to \( \mathbb{P}^m \) and \( e \) is the degree in the variables corresponding to \( \mathbb{P}^n \). Note that this differs from the more familiar setting where the degree is a natural number \( d \in \mathbb{N} \). For more on multi-graded rings, see Miller and Sturmfels [2005]. The flexibility provided by this bi-graded setting reduces the complexity of our algorithm. Concretely, we work in the \( \mathbb{N}^2 \)-graded polynomial ring

\[
S = \mathbb{C}[x_0, \ldots, x_m, y_0, \ldots, y_n] = \bigoplus_{(d,e)\in\mathbb{N}^2} S_{(d,e)}, \quad \text{where} \quad S_{(d,e)} = \bigoplus_{|a|=d, |b|=e} \mathbb{C} \cdot x^a y^b.
\]

Here, we used the notation \( x^a = x_0^{a_0} \cdots x_m^{a_m} \) and \( |a| = a_0 + \cdots + a_m \) for \( a = (a_0, \ldots, a_m) \in \mathbb{N}^{m+1} \), and analogously for \( b \). The graded pieces \( S_{(d,e)} \) are vector spaces over \( \mathbb{C} \). The variables \( x \) correspond to homogeneous coordinates on the factor \( \mathbb{P}^m \) and in \( S_{(d,e)} \), \( d \) is the degree in the \( x \)-variables. Analogously, elements of \( S_{(d,e)} \) have degree \( e \) in the \( y \)-variables, which are related to \( \mathbb{P}^n \).

An element \( f \in S \) is called homogeneous if \( f \in S_{(d,e)} \) for some \((d, e) \in \mathbb{N}^2\). The key reason that the ring \( S \), with its grading (2.2), is naturally associated to \( X \) is that homogeneous elements \( f \in S_{(d,e)} \) have well-defined zero sets on \( X \). By this, we mean that for \( f \in S_{(d,e)} \) and for any \( \zeta = (x, y) \in X \), \( f(x, y) = 0 \) is independent of the choice of homogeneous coordinates. Indeed, this follows from

\[
f(\lambda x, \mu y) = \lambda^d \mu^e f(x, y) \quad \text{for} \quad \lambda, \mu \in \mathbb{C} \setminus \{0\}.
\]

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Therefore, whenever \( f \) is homogeneous it makes sense to write \( f(\zeta) = 0 \) if \( f \) vanishes on some set of homogeneous coordinates for \( \zeta \), and to define the subvariety of \( X \) defined by \( f \) as \( V_X(f) = \{ \zeta \in X \mid f(\zeta) = 0 \} \). For a homogeneous ideal \( I \subset S \) (i.e., \( I \) is generated by homogeneous elements with respect to the grading \((2.2)\)), we denote the subvariety of \( X \) corresponding to \( I \) by

\[
V_X(I) = \{ \zeta \in X \mid f(\zeta) = 0, \text{ for all homogeneous } f \in I \}.
\]

That is, \( V_X(I) \) contains the solutions of the polynomial system defined by the simultaneous vanishing of all homogeneous equations \( f \in I \).

**Example 2.1.** The polynomials \( f_1, \ldots, f_3 \) in Example 1.1 are homogeneous elements of degree \((1, 1)\) in \( S \), i.e., \( f_i \in S_{(1, 1)} \). They generate the homogeneous ideal \( I \), whose corresponding subvariety is \( V_X(I) = \{ \zeta_1, \ldots, \zeta_4 \} \subset X = \mathbb{P}^2 \times \mathbb{P}^2 \).

With this notation in place, we turn back to the tensor \( \mathcal{A} \) and show that, under suitable assumptions, \( V_X(I) = \{ \zeta_1, \ldots, \zeta_r \} \) for some homogeneous ideal \( I \) defined by \( \mathcal{A} \). This will generalize the procedure in Example 1.1. Consider the flattening \( \mathcal{A}_{(1)} \) from Equation \((2.1)\). This flattening has a natural interpretation as a \( \mathbb{C} \)-linear map

\[
\mathcal{A}_{(1)} : S_{(1, 1)} \longrightarrow \mathbb{C}^{\ell+1}.
\]

Indeed, we can identify the space \( \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \) with the graded piece \( S_{(1, 1)} \) of degree \((1, 1)\) by

\[
e_k \otimes e_l \longmapsto x_k y_l, \quad 0 \leq k \leq m, \ 0 \leq l \leq n,
\]

where \( e_k \) represents the \((k + 1)\)-st standard basis vector of \( \mathbb{C}^{m+1} \), and analogously for \( e_l \) and \( \mathbb{C}^{n+1} \). For brevity, we also write \( \mathcal{A}_{(1)} \) for a matrix representation of the map \( \mathcal{A}_{(1)} \), where the standard basis of monomials \((2.4)\) is used for \( S_{(1, 1)} \). The \( \ell + 1 \) rows of \( \mathcal{A}_{(1)} \) are elements of the dual space \( (S_{(1, 1)})^\vee \) represented in its dual basis

\[
\{ \partial_{kl} = -\frac{\partial^2}{\partial x_k \partial y_l} \mid 0 \leq k \leq m, \ 0 \leq l \leq n \}.
\]

From this vantage point, the rows of the second factor in Equation \((2.1)\) are

\[
(\beta_l \otimes y_l)^T = \sum_{0 \leq k \leq m} \sum_{0 \leq l \leq n} \beta_{l,k} y_l \partial_{kl} \in S_{(1, 1)}^\vee.
\]

It is clear that for any \( f \in S_{(1, 1)} \), we have \((v \otimes w)^T(f) = f(v, w)\).

Let \( f_1, \ldots, f_s \in S_{(1, 1)} \) be a \( \mathbb{C} \)-basis for the kernel \( \ker \mathcal{A}_{(1)} \). The \( f_i \) generate a homogeneous ideal \( I = \langle \ker \mathcal{A}_{(1)} \rangle = \langle f_1, \ldots, f_s \rangle \subset S \). The following lemma characterizes precisely what we mean by generic in Assumption 2.1.

**Lemma 2.2.** Consider the space \( \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \) with \( \ell \geq m \geq n > 0 \). Then, for all \( r \leq \min \{ \ell + 1, mn \} \) there exists a Zariski dense open subset \( U \subset S_r \) such that for all tensors \( \mathcal{A} \in U \),

(i) \( \mathcal{A} \) is \( r \)-identifiable,
(ii) the flattening \( \mathcal{A}_{(1)} \) has rank \( r \),
(iii) the ideal \( I = \langle \ker \mathcal{A}_{(1)} \rangle \subset S \) is such that \( V_X(I) \) consists of the points \( \{ \zeta_1, \ldots, \zeta_r \} \), and these points have multiplicity one.

**Proof.** Items (i) and (ii) hold on a dense open subset \( U_1 \) by Proof of Proposition 8.1 in Bocci et al. [2014]. Item (iii) holds on a dense open subset \( U_2 \) by the scheme-theoretic version of the Trisecant Lemma [Russo 2016, Proposition 1.4.3]. Our subset \( U \) is \( U_1 \cap U_2 \). \( \square \)
Point (iii) in Lemma 2.2 is a technicality we will need in Section 3 to invoke the results from Section 5.5 in Telen [2020b]. For the reader who is familiar with algebraic geometry, we included a consequence of Lemma 2.2 relating $I$ to the vanishing ideal of $V_X(I)$ in Corollary A.1.

We were not able to construct an example for which (i)+(ii) are satisfied, but (iii) is not. This raises the question whether (i)+(ii) implies (iii), and if not, how our method extends to counterexamples.

2.4 The High-level Algorithm

We conclude the section by presenting a high-level pseudo-algorithm for computing the tensor rank decomposition ($\mathcal{A}$) of $\mathcal{A}$. This is presented as Algorithm 2.1. Its steps summarize the discussion up to this point.

**Algorithm 2.1:** Compute the tensor rank decomposition of $\mathcal{A} \in \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$

**Require:** $\mathcal{A}$ satisfies Assumption 2.1.

1. $\mathcal{A}_{(1)} \leftarrow$ the $(\ell + 1) \times (m + 1)(n + 1)$ 1-flattening of $\mathcal{A}$
2. $\{f_1, \ldots, f_s\} \leftarrow$ a $\mathbb{C}$-basis for $\ker \mathcal{A}_{(1)}$
3. $I \leftarrow$ the ideal $\langle f_1, \ldots, f_s \rangle \subset S$
4. $\{(\beta_i, \gamma_i)\}_{i=1,\ldots,r} \leftarrow$ homogeneous coordinates for $\{\zeta_i\}_{i=1,\ldots,r} = V_X(I) \subset X$
5. $(\alpha_1, \ldots, \alpha_r) \leftarrow$ solve the linear system defined by Equation (2.1)
6. return $\{(\alpha_i, \beta_i, \gamma_i)\}_{i=1,\ldots,r}$

Remark 2.1. In a practical implementation of Algorithm 2.1, the necessary conditions from Lemma 2.2 would be handled as follows. Condition (i) that $\mathcal{A}$ is $r$-identifiable would be treated as a non-checked precondition. After obtaining the algorithm’s output, the user should verify that $\mathcal{A} \approx \sum_{i=1}^r \alpha_i \otimes \beta_i \otimes \gamma_i$. If this holds, then specific $r$-identifiability can be checked using a posteriori certifications such as those in Chiantini et al. [2017]; Domanov and De Lathauwer [2013, 2017]; Kruskal [1977]; Sidiropoulos and Bro [2000]. In Algorithm 2.1, condition (ii) is checked in line 2 and condition (iii) in line 4; see Section 3.4.

The main focus of this article is dealing with step 4 of Algorithm 2.1. We will employ a state-of-the-art method for solving polynomial equations on $X$, based on homogeneous normal forms. This strategy is described in the next section.

3 FROM POLYNOMIAL EQUATIONS TO EIGENVALUES

In this section, we employ tools from computational algebraic geometry for computing points in a product of projective spaces in step 4 of Algorithm 2.1. The next subsection summarizes the relevant results in the present setting. We show in Section 3.1 that the solutions can be obtained from an eigenvalue problem defined by pre-normal forms. How to compute the latter is explained in Section 3.2. Thereafter, in Section 3.3, we demonstrate that so-called pencil-based algorithms for decomposing tensors of very low rank are closely related to our cpd_hnf algorithm. The full algorithm for performing step 4 of Algorithm 2.1 is presented in Section 3.4. Finally, we conclude this section with some examples.

3.1 The Eigenvalue Theorem

Our algorithm is built on a multi-homogeneous version of the eigenvalue theorem (Theorem 3.1 below), which allows us to find solutions of systems of polynomial equations via eigenvalue computations. Behind this is the theory of homogeneous normal forms. In our context, these are $\mathbb{N}^2$-graded
versions of truncated normal forms, introduced in Telen et al. [2018], and special cases of the more general toric homogeneous normal forms used by Bender and Telen [2020]; Telen [2020a] and formally introduced in Telen [2020b]. For our purpose, it suffices to work with slightly simpler objects, called pre-normal forms, so we use homogeneous normal forms only implicitly. For full proofs and more details, we refer to Section 5.5.4 in Telen [2020b].

Consider the space \( X = \mathbb{P}^m \times \mathbb{P}^n \) and its associated ring \( S \), which is the \( \mathbb{N}^2 \)-graded polynomial ring from Equation (2.2). Let \( I \subset S \) be a homogeneous ideal. The ideal \( I \) inherits the grading on \( S \):

\[
I = \bigoplus_{(d,e) \in \mathbb{N}^2} I_{(d,e)}, \quad \text{where} \quad I_{(d,e)} = I \cap S_{(d,e)}
\]

is a \( \mathbb{C} \)-subspace of \( S_{(d,e)} \), and the quotient ring

\[
S/I = \bigoplus_{(d,e) \in \mathbb{N}^2} (S/I)_{(d,e)} = \bigoplus_{(d,e) \in \mathbb{N}^2} S_{(d,e)}/I_{(d,e)}
\]

(this are quotients of vector spaces) similarly inherits this grading. Our objective is to compute a set of homogeneous coordinates of the points in \( V_X(I) \) by using linear algebra computations. To accomplish this, it is necessary to work in graded pieces of \( S, I, \) and \( S/I \). Let \( M \) denote either of the latter. The (multi-graded) Hilbert function \( \text{HF}_M : \mathbb{N}^2 \to \mathbb{N} \) is given by

\[
\text{HF}_M(d,e) = \dim_{\mathbb{C}} M_{(d,e)}
\]

and keeps track of the dimension of the vector space \( M_{(d,e)} \). Note that for a homogeneous ideal \( I \subset S \) and \( (d,e) \in \mathbb{N}^2 \), we have \( \text{HF}_{S/I}(d,e) = \text{HF}_S(d,e) - \text{HF}_I(d,e) \).

Example 3.1. The Hilbert function of the ring \( S \) is given explicitly by

\[
\text{HF}_S(d,e) = \dim_{\mathbb{C}} S_{(d,e)} = \binom{m+d}{d} \binom{n+e}{e}, \quad \text{with} \quad (d,e) \in \mathbb{N}^2.
\]

Before stating the main result of this section, Theorem 3.1, we present three auxiliary lemmas. Lemmas 3.1 and 3.2 are well-known in commutative algebra. For lack of a precise reference and completeness, we included short proofs in the Appendix.

Lemma 3.1. Let \( I \subset S \) be a homogeneous ideal such that \( V_X(I) \) consists of \( r < \infty \) points \( \{ \zeta_1, \ldots, \zeta_r \} \). For each \((d', e') \in \mathbb{N}^2 \), there exists a homogeneous polynomial \( h_0 \in S_{(d', e')} \) such that \( V_X(h_0) \cap V_X(I) = 0 \). Equivalently, we can find \( h_0 \in S_{(d', e')} \) such that \( h_0(\zeta_i) \neq 0, \; i = 1, \ldots, r \).

Lemma 3.2. Let \( I \subset S \) be a homogeneous ideal such that \( V_X(I) \) consists of \( r < \infty \) points, each with multiplicity one. There exists \( (d,e) \in \mathbb{N}^2 \) with \( (d,e) \neq (1,1) \) and \( (d-1, e-1) \geq (0,0) \) (entry-wise) such that \( \text{HF}_{S/I}(d,e) = r \).

Assumption 3.1. Henceforth, we assume the following:

1. Let \( I = \langle \ker \mathcal{A}_{(1)} \rangle \) be as in Section 2.3, where \( f_1, \ldots, f_s \in S_{(1,1)} \) form a basis for \( \ker \mathcal{A}_{(1)} \). We assume that \( \mathcal{A} \) satisfies Assumption 2.1, so that \( V_X(I) \) consists of the \( r \) points \( \zeta_1, \ldots, \zeta_r \) with multiplicity one by Lemma 2.2.

2. The tuple \((d,e) \in \mathbb{N}^2 \) is such that \((d,e) \geq (1,1)\) and \( \text{HF}_{S/I}(d,e) = r \). Note that this is satisfied for \((d,e) = (1,1)\) by construction of \( I \).

3. We write \((d', e') = (d-1, e-1)\), and \( h_0 \in S_{(d', e')} \) is such that \( h_0(\zeta_i) \neq 0, \; i = 1, \ldots, r \). This makes sense by Lemma 3.1.

\footnote{Often, we will require \((d,e) \neq (1,1)\) and \((d-1, e-1) \geq (0,0)\). This makes sense by Lemma 3.2. The problem of how to find such a tuple \((d,e)\) will be the topic of Section 4.}
Lemma 3.3. Let \( I, (d, e) \) and \( h_0 \) be as in Assumption 3.1. If a \( \mathbb{C} \)-linear map \( N : S(d,e) \rightarrow \mathbb{C}' \) satisfies
\[
\text{rank}(N) = r \quad \text{and} \quad \ker N = I(d,e),
\]
then the induced linear map \( N_{h_0} : S(1,1) \rightarrow \mathbb{C}' \), \( f \mapsto N(h_0 f) \) has rank \( r \).

Proof. Note that \( HF_{S(I)}(1,1) = r \) by construction. By Proposition 5.5.7 in Telen [2020b], the fact that \( HF_{S(I)}(d,e) = r \) implies that \( (1,1) \) and \((d', e')\) form a so-called regularity pair. Surjectivity of the map \( N_{h_0} \) then follows directly from Lemma 5.5.3 in Telen [2020b].\( \square \)

Linear maps \( N : S(d,e) \rightarrow \mathbb{C}' \) satisfying Equation (3.1) will play an important role in translating our polynomial root finding problem into an eigenvalue problem. We will therefore give them a name.

Definition 3.1 (Pre-normal Forms). A \( \mathbb{C} \)-linear map \( N : S(d,e) \rightarrow \mathbb{C}' \) satisfying Equation (3.1) is called a pre-normal form on \( S(d,e) \).

If \( r = \ell + 1 \), then Assumption 2.1 implies that \( \mathcal{A}(1) : S(1,1) \rightarrow \mathbb{C}' \) is a pre-normal form on \( S(1,1) \).

We use the terminology pre-normal form, because, following Definition 5.5.6 in Telen [2020b], the term normal form is reserved for linear maps satisfying an extra technical condition. Normal forms are closely related to a classical result in computational algebraic geometry called the eigenvalue theorem [Cox et al. 2006, Section 2.4], which is used for computing isolated solutions to polynomial systems. The pre-normal forms introduced above will be useful for formulating a multi-homogeneous version of this theorem, namely, Theorem 3.1 below. Before stating it, we need to fix some additional notation.

In Lemma 3.3, \( N_{h_0} \) is the composition of \( N \) with the linear map that represents multiplication by \( h_0 \in S(d',e') \). More generally, for \( g \in S(d',e') \), we define \( N_g : S(1,1) \rightarrow S(d,e) \) as \( N_g(f) = N(g f) \). In Theorem 3.1, we will restrict the map \( N_{h_0} \) to an \( r \)-dimensional subspace \( B \subset S(1,1) \) such that the resulting map is invertible. We denote this restriction by \( (N_{h_0})_B \). In practice, it suffices to select \( r \) columns of the matrix of \( N_{h_0} \) so that the resulting \( r \times r \) submatrix \( (N_{h_0})_B \) is invertible.

In what follows, we will write \( w_i^{\top} = (\beta_i \otimes y_i)^{\top} \in B^\vee \) for the linear functional representing evaluation at \( \zeta_i = (\beta_i, y_i) \). This is the restriction of the functional \( (\beta_i \otimes y_i)^{\top} \in S^\vee(1,1) \) to the vector space \( B \). Concretely, we set
\[
w_i^{\top}(b) = (\beta_i \otimes y_i)^{\top}(b) = b(\beta_i, y_i), \quad \text{with} \quad b(x,y) \in B \subset S(1,1).
\]
After fixing a basis for \( B^\vee \), \( w_i^{\top} \) can be represented as a row vector.

If \( g, h \in S(d', e') \setminus \{0\} \) are two homogeneous polynomials of the same degree, then the fraction \( g/h \) is a well-defined function on \( X \setminus V_X(h) \). Indeed, the evaluation of this function does not depend on the choice of homogeneous coordinates. Therefore, we may write \( (g/h)(\zeta) \) for the evaluation of this rational function at \( \zeta \in X \setminus V_X(h) \).

Theorem 3.1 (Eigenvalue Theorem). Let \( I, (d, e) \) and \( h_0 \) be as in Assumption 3.1 and let \( N \) be a pre-normal form. Let \( B \subset S(1,1) \) be any \( r \)-dimensional subspace such that the restriction \( (N_{h_0})_B : B \rightarrow \mathbb{C}' \) is invertible. For any \( g \in S(d',e') \), we have
\[
w_i^{\top} \circ M_{g/h_0} = \frac{g}{h_0}(\zeta_i) \cdot w_i^{\top}, \quad i = 1, \ldots, r,
\]
where \( M_{g/h_0} : B \rightarrow B \) is the composition \( (N_{h_0})_B^{-1} \circ (N_g)_B \).

Proof. This follows from Theorem 5.5.3, Propositions 5.5.4 and 5.5.5 in Telen [2020b] and the fact that \( (1,1) \) and \((d', e')\) form a regularity pair (see the proof of Lemma 3.3).\( \square \)
After fixing a basis for $B$ and representing $w_i$ in the dual basis for $B^\vee$, Equation (3.3) is a standard matrix eigenproblem: $w_i^TM_{g/h_0} = \lambda_i w_i^T$. That is, $(\lambda_i, w_i)$ is a left eigenpair of the $r \times r$ matrix $M_{g/h_0}$. Note that Theorem 3.1 implies that all maps of the form $M_{g/h_0}$ share a set of eigenvectors.

We now sketch one way of using Theorem 3.1 to retrieve the coordinates of $\zeta_i = (\beta_i, y_i)$ from eigenvalues, assuming a pre-normal form $N : S_{(d,e)} \to C^r$ is given. The problem of computing a pre-normal form is addressed in the next subsection. We assume $d \geq 2$. Let $h \in S_{(d-1,e)}$ and $h_0 \in S_{(d^e,e^e)}$ be homogeneous polynomials that do not vanish at any of the points $\zeta_i$. These can be chosen generically. Set $g_j = x_j h \in S_{(d^e,e^e)}, j = 0, \ldots, m$. Choose $B \subset S_{(1,1)}$ of dimension $r$ such that $(N_{h_0})_B$ is invertible and compute the matrices $M_j = M_{g_j/h_0} = (N_{h_0})^{-1}_B \circ (N_{g_j})_B$. By Theorem 3.1, the eigenvalues of $M_j$ are given by $\lambda_{ji} = (g_j/h_0)(\zeta_i)$. Writing $\beta_{ij}$ for the $j$th coordinate of $\beta_i$, we have

$$
(\lambda_{0i} : \ldots : \lambda_{mi}) = \left( \frac{\beta_{0i}h(\zeta_i)}{h_0(\zeta_i)} : \ldots : \frac{\beta_{mi}h(\zeta_i)}{h_0(\zeta_i)} \right) = (\beta_{0i} : \ldots : \beta_{mi}) = \beta_i.
$$

Note that if $(d,e) = (2,1)$, we can take $h = 1$.

Subsequently, we compute $y_i$ by solving the linear system of equations

$$
f_1(\beta_i, y) = \cdots = f_s(\beta_i, y) = 0. \quad (3.4)
$$

The foregoing approach requires that $(d,e) \neq (1,1)$. Otherwise $S_{(d^e,e^e)} = S_{(0,0)} = \mathbb{C}$, and we can only evaluate constant functions using Theorem 3.1.

### 3.2 Computing Pre-normal Forms

As illustrated in the previous subsection, once we have computed a pre-normal form, the points $\zeta_i$ can be recovered using basic linear algebra computations. A natural next issue to address is how to compute a pre-normal form.

Our starting point is a basis $f_1, \ldots, f_s \in S_{(1,1)}$ of $\ker \mathcal{A}_{(1)}$, generating our ideal $I = \langle \ker \mathcal{A}_{(1)} \rangle = \langle f_1, \ldots, f_s \rangle$. For any tuple $(d,e) \in \mathbb{N}^2$ such that $(d,e) \geq (1,1)$, the degree-$(d,e)$ part $I_{(d,e)}$ of $I$ is the $\mathbb{C}$-vector space spanned by

$$
\{ x^{d^i} y^{e^i} f_j \mid (|a'|, |b'|) = (d^i, e^i), \ i = 1, \ldots, s \} \subset S_{(d,e)}, \quad (3.5)
$$

where $(d^i,e^i) = (d - 1, e - 1)$. If $d = 0$ or $e = 0$, then we have $I_{(d,e)} = \{0\}$. In analogy with Equation (1.1), we construct a matrix whose rows are indexed by the monomial basis elements of $S_{(d,e)}$ (i.e., the monomials $\{ x^{d^i} y^{e^i} \mid |a| = d, |b| = e \}$), and whose columns are the polynomials (3.5) expanded in this basis. We denote this matrix by

$$
R_I(d,e) \in \mathbb{C}^{\text{HF}_{S_{(d,e)}}} \times s^{\text{HF}_{S_{(d^e,e^e)}}}. \quad (3.6)
$$

Such matrices represent graded resultant maps in the terminology of Section 5.5.4 in Telen [2020b]. They are multihomogeneous versions of the classical Macaulay matrices [Macaulay 1916]. We present an explicit example below in Example 3.2.

Observe that the Hilbert function $HF_I(d,e)$ is given by the rank of $R_I(d,e)$ and $HF_{S/I}(d,e)$ by its corank. This follows immediately from the observation that the columns of $R_I(d,e)$ span $I_{(d,e)}$. A left nullspace matrix $N$ of $R_I(d,e)$ represents a map $S_{(d,e)} \to S_{(d,e)}/I_{(d,e)} \approx \mathbb{C}^{HF_{S/I}(d,e)}$. This has the following consequence.

Proposition 3.1. If $(d,e) \in \mathbb{N}^2$ is such that $HF_{S/I}(d,e) = r$, then any left nullspace matrix $N$ of $R_I(d,e)$ represents a pre-normal form.

4If $d = 1$ and $e \geq 2$, then the roles of $d$ and $e$ can be swapped so that the approach still works.
We conclude that a pre-normal form $N$ can be computed, for instance, from a full singular value decomposition (SVD) of $R_1(d,e)$, where $\text{HF} S/J(d,e) = r$. This solves the problem of computing a pre-normal form, assuming that we know a degree $(d,e) \in \mathbb{N}^2$ for which $\text{HF} S/J(d,e) = r$. The problem of finding such degrees is addressed in Section 4.

3.3 Relation to Pencil-based Algorithms

To obtain the homogeneous coordinates for $\xi_1, \ldots, \xi_r$ as eigenvalues of the matrices $M_{g/h_0}$, we usually have to work with pre-normal forms on $S(d,e)$, where $(d,e) \neq (1,1)$ and $(d',e') \geq (0,0)$.

An exception is the case where $r \leq m + 1 \leq \ell + 1$. For these tensors of very low rank, a pre-normal form $N : S_{(1,1)} \to \mathbb{C}^r$ will suffice under the mild condition that

$$[\beta_1 \cdots \beta_r] \in \mathbb{C}^{(m+1)\times r} \text{ has rank } r.$$  \hfill (3.7)

The underlying reason is that vanishing at $\{\xi_1, \ldots, \xi_r\}$ gives $r$ linearly independent conditions on $S_{(1,0)}$. The proof of the following theorem is another consequence of the theory of homogeneous normal forms and is deferred to the Appendix.

**Theorem 3.2 (Eigenvalue Theorem for Low Ranks).** Let $I = \langle \ker A \rangle$ where $A$ has rank $r \leq m + 1 \leq n + 1$ and satisfies both Assumption 2.1 and Equation (3.7). Let $h_0 \in S_{(0,1)}$ be such that $h_0(\xi_i) \neq 0$ for $i = 1, \ldots, r$ and let $N : S_{(1,1)} \to \mathbb{C}^r$ be a pre-normal form on $S_{(1,1)}$. For $g \in S_{(0,1)}$, we define

$$\overline{N}_g : S_{(1,0)} \to \mathbb{C}^r \text{ given by } \overline{N}_g(f) = N(gf).$$

We have that $\overline{N}_{h_0}$ has rank $r$. For any $r$-dimensional subspace $B \subset S_{(1,0)}$ such that the restriction $(\overline{N}_{h_0})_{|B} : B \to \mathbb{C}^r$ is invertible, the eigenvalues of $M_{g/h_0} = (\overline{N}_{h_0})_{|B}^{-1} \circ (\overline{N}_{h_0})_{|B}$ are $\{\gamma_i/h_0(\xi_i)\}_{i=1,\ldots,r}$.

This theorem is exploited to compute $\xi_i = (\beta_i, \gamma_i)$ efficiently as follows. If $A$ satisfies Assumption 2.1 and Equation (3.7), then we take a basis of the $r$-dimensional row span of $A_{(1)}$ in Theorem 3.2 as our pre-normal form. This can be obtained from a compact SVD of $A_{(1)}$. Once the $\gamma_i$ are computed from the eigenvalues of the $M_{g/h_0}$, the $\beta_i$ can be obtained as in Equation (3.4).

Alternatively, one can use the eigenvectors of these commuting matrices $M_{g/h_0}$ for $j = 1, \ldots, r$ [Telen 2020b, Theorem 5.5.3].

Theorem 3.2 is intimately related to what Beltrán et al. [2019] called pencil-based algorithms for solving Equation (CPD) when the rank satisfies $r \leq m + 1 \leq \ell + 1$, such as those by Leurgans et al. [1993]; Lorber [1985]; Sanchez and Kowalski [1990]. Recall that pencil-based algorithms assume that $A \in \mathbb{C}^{n \times r \times (n+1)}$ is a rank-$r$ tensor. In addition, they assume that the $\alpha_i$ form a linearly independent set, and likewise for the $\beta_i$’s. Then, we have that the tensor contraction of $A$ with $h_0^\top \in (\mathbb{C}^{n+1})^\vee$, i.e.,

$$h_0^\top \otimes A = \sum_{i=1}^r (\alpha_i \otimes \beta_i) \cdot (h_0^\top \gamma_i) = AD_{h_0}B^\top,$$  \hfill (3.8)

is an invertible $r \times r$ matrix insofar as $h_0^\top \gamma_i \neq 0$. Herein, $A \in \mathbb{C}^{r \times r}$ (respectively, $B \in \mathbb{C}^{r \times r}$) has the $\alpha_i$’s (respectively, $\beta_i$’s) as columns, and $D_{h_0} = \text{diag}(h_0^\top \gamma_1, \ldots, h_0^\top \gamma_r)$. Let $\overline{N}_{h_0} = h_0^\top \otimes A$ and $\overline{N}_g = g^\top \otimes A$ for $h_0, g \in (\mathbb{C}^{n+1})^\vee$. Then, we have

$$M_{g/h_0} = \overline{N}_{h_0}^{-1} \overline{N}_g = B^{-\top} D_{h_0}^{-1} D_g B^\top,$$

5The decomposition problem for a rank-$r$ tensor in $\mathbb{C}^{(r+1)\times (m+1)\times (n+1)}$ with $r \leq m + 1 \leq \ell + 1$ can always be reduced to this so-called concise case [Bürgisser et al. 1997; Landsberg 2012] by computing an orthogonal Tucker decomposition [Tucker 1966] followed by a rank-$r$ decomposition of the core tensor.
so that the points $\beta_i$ can be recovered uniquely from the matrix of eigenvectors $B^T$, provided that $h_0^T y_i \neq 0$ for all $i = 1, \ldots, r$. The $\alpha_i$'s and $y_i$'s can then be recovered from the 2-flattening; see Beltrán et al. [2019]; De Lathauwer [2006]; Leurgans et al. [1993]; Lorber [1985]; Sanchez and Kowalski [1990] for more details. With the foregoing suggestive notation, it is easy to see that the matrix of $N_{h_0} : f \mapsto N(f h_0)$ with respect to the standard bases is precisely Equation (3.8). Indeed, note that since we can take $N = \mathcal{A}(1)$, we have $f h_0 \approx f \otimes h_0$ and so $N(f h_0) = \mathcal{A}(1)(f \otimes h_0)$.

Pencil-based algorithms may thus be interpreted as a special case of the proposed $\text{cpd_hnf}$ algorithm based on homogeneous normal forms when $r \leq m + 1 \leq \ell + 1$. Note that because of the numerical instabilities analyzed by Beltrán et al. [2019] caused by extracting $\beta_i$ from the eigenvectors, we prefer to extract the $\xi_i = (\beta_i, y_i)$ in a different way. We compute $\beta_i$ from the eigenvalues of $M_{\kappa_i}/h_0$ and the corresponding $y_i$ from the linear system Equation (3.4).

### 3.4 The Algorithm

The discussion so far is distilled into Algorithm 3.1. This algorithm implements step 4 of Algorithm 2.1. Note that we dropped the tilde on top of the $N_*$'s in lines 2–7 to streamline the presentation.

The first phase of the algorithm, up to line 21, constructs the pre-normal form $N$ and chooses an $N_{h_0}$. This phase depends on whether we can invoke the more efficient Theorem 3.2 ($r \leq m + 1$) or we need the full power of Theorem 3.1. In the former case, we can take $N = \mathcal{A}(1)$, while in the latter case, we need to take $N$ equal to the left null space of $R_I(d, e)$. How we choose the degree $(d, e)$ in line 9 is explained in Section 4. The matrix $R_I(d, e) \in \mathbb{C}^{\mathbb{H}_r(e) \times \mathbb{H}_s(d, e)}$ can be constructed efficiently column-by-column without polynomial multiplication. Indeed, by Equation (3.5) it suffices to copy the coefficients of $f_i$ relative to the monomial basis of $S(1, 1)$ into the correct rows; see also Example 3.2 below. The left null space $N$ can be extracted from the last $r$ columns of the $U$-factor in the SVD $R_I(d, e) = USV^H$, where $^H$ denotes the conjugate transpose. In our implementation, the matrix $N_{h_0} \in \mathbb{C}^{r \times \mathbb{H}_r(e)}$ is chosen by sampling the coefficients of $h_0 \in S(d', e')$ identically and independently distributed from a Gaussian distribution. With probability 1, $h_0$ satisfies $h_0(y_i) \neq 0$ for all $i$; hence, this is a valid choice of $h_0$.

The next phase of the algorithm, in lines 21–22, chooses a basis $B$. Although in theory Theorem 3.1 enables us to choose any $B$ such that $(N_{h_0})^{-1}_B$ is invertible, Telen et al. [2018] showed that for reasons of numerical stability it is crucial to choose $B$ such that $(N_{h_0})_B$ is well-conditioned. In practice, such a subspace $B$ can be found using a QR decomposition with optimal column pivoting or by using the SVD [Telen 2020b, Chapter 4]. We stated the QR approach in Algorithm 3.1.

The multiplication matrices are constructed straightforwardly as the formula suggests in lines 23 to 25. Note that the upper triangular matrix $(N_{h_0})_B$ does not need to be inverted explicitly, rather the system can be solved by backsubstitution.

In line 26, the matrices $M_{(h x_k)}/h_0$ need to be simultaneously diagonalized, as we have that $M_{(h x_k)}/h_0 = V^{-1} \text{diag}(\beta_k)V$. We compute $V$ from a random linear combination of the $M_{(h x_k)}/h_0$'s, and then extract $\beta_k$ as the diagonal entries from the (approximately) diagonalized matrix $V^{-1} M_{(h x_k)}/h_0 V$.

The system in line 28 is solved efficiently by noting that the coefficients of $f_j$ can be arranged in a matrix $F_j$ of size $(m + 1) \times (n + 1)$, so that $f_j(x, y) = x^T F_j y$. Hence, Equation (3.4) boils down to computing the kernel of $AY = 0$ where the rows of $A \in \mathbb{C}^{s \times (n+1)}$ are the row vectors $\beta_i^T F_j$. Note that by Assumption 2.1, ker $A$ is spanned by $y_i$. In line 29 the obtained solution $(\beta_i, y_i)$ is refined using standard Newton iterations. Here the pseudo-inverse of the Jacobian matrix of $f_1, \ldots, f_r$ is used. This matrix has full rank if and only if $(\beta_i, y_i)$ has multiplicity one. This can be used to check condition (iii) in Lemma 2.2.
Algorithm 3.1: Compute $V_X(I)$ for $I = \langle \ker \mathcal{A}(1) \rangle = \langle f_1, \ldots, f_s \rangle$

Require: $r$ is the rank of $\mathcal{A} \in \mathbb{C}^{(\ell+1) \times (m+1) \times (n+1)}$ with $\ell \geq m \geq n$.
1: if $r \leq m + 1$ and (3.7) then
2: $N \leftarrow$ an $r \times (m + 1)(n + 1)$ matrix representing the row space of $\mathcal{A}(1)$, whose columns are indexed by the symbols $\partial_{k,l}$
3: for $k = 0, \ldots, m$ do
4: $N_k \leftarrow$ the submatrix of $N$ with columns indexed by $\{ \partial_{k,j} \mid j = 0, \ldots, n \}$
5: end for
6: $N_{h_0} \leftarrow c_0 N_0 + \cdots + c_m N_m$, a random $\mathbb{C}$-linear combination of the $N_k$
7: $h \leftarrow 1$
8: else
9: $(d,e)$ $\leftarrow$ a tuple in $\mathbb{N}^2$ such that $(d,e) \neq (1,1), (d',e') \geq (0,0)$ and $HF_S(d,e) = r$
10: Construct the resultant matrix $R_I(d,e)$
11: $N \leftarrow$ an $r \times HF_S(d,e)$ matrix representing the left nullspace of $R_I(d,e)$, whose columns are indexed by the symbols $\partial_{a,b}$ with $|a| = d, |b| = e$.
12: for $(a',b')$ such that $|a'| = d', |b'| = e'$ do
13: $N_{a',b'} \leftarrow$ submatrix of $N$ with columns indexed by $\{ \partial_{a,b} \mid a-a' \geq 0, b-b' \geq 0 \}$
14: end for
15: $N_{h_0} \leftarrow \sum[a'=d',|b'|=e'] c_{a',b'} N_{a',b'}$, a random $\mathbb{C}$-linear combination of the $N_{a',b'}$
16: $h \leftarrow \sum[a''=d'-1,|b''|=e'] \hat{c}_{a'',b'} x^{a'} y^{b'}$, a random element of $S_{(d'-1,e')}$
17: for $k = 0, \ldots, m$ do
18: $N_k \leftarrow \sum[a''=d'-1,|b''|=e'] \tilde{c}_{a'',b'} N_{a''+e_k,b'}$
19: end for
20: end if
21: $Q, R, p \leftarrow$ QR factorization of $N_{h_0}$ with optimal column pivoting
22: $(N_{h_0})_B \leftarrow R[:, 1, \ldots, r]
23: for k = 0, \ldots, m do
24: $M_{(h,x_k)/h_0} \leftarrow (N_{h_0})_B^{-1} Q^H N_k[:, p(1, \ldots, r)]$
25: end for
26: $(\beta_1, \ldots, \beta_r) \leftarrow$ simultaneous diagonalization of $M_{(h,x_k)/h_0}, \ldots, M_{(h,x_m)/h_0}$
27: for $i = 1, \ldots, r$ do
28: $y_i \leftarrow$ solve $f_i(\beta_i, y) = \cdots = f_s(\beta_i, y) = 0$ for $y$
29: $(\beta_i, y_i) \leftarrow$ refine $(\beta_i, y_i)$ using Newton iteration
30: end for
31: return $(\beta_1, y_1), \ldots, (\beta_r, y_r)$

Remark 3.1. As pointed out to us by an anonymous referee and Bernard Mourrain, it is possible to replace the left nullspace computation in line 11 by a smaller linear system of equations. This interpretation corresponds to the flat extension of the quasi-Hankel operators in Bernardi et al. [2013]. The pre-normal form $N$ can be chosen such that $N_{h_0}$ is given by $\mathcal{A}(1)$. This determines $N$ on the $HF_S(1,1)$-dimensional subspace $h_0 \cdot S_{(1,1)}$. For $(a',b')$ such that $|a'| = d'$ and $|b'| = e'$, the restriction $N_{[x^{a'} y^{b'}, S_{(1,1)}]}$ should satisfy $N_{[x^{a'} y^{b'}, S_{(1,1)}]} (x^{a'} y^{b'} \cdot f_i) = 0, i = 1, \ldots, s$. These linear conditions determine the pre-normal form $N$ on the remaining $(HF_S(d,e) - HF_S(1,1))$-dimensional vector space $HF_S(d,e)/(h_0 \cdot S_{(1,1)})$.

This observation allows to perform the main linear algebra computations on a matrix of size $(HF_S(d,e) - HF_S(1,1)) \times s HF_S(d',e')$, which is smaller than the size of $R_I(d,e)$. Note that the number of rows is reduced by a factor of $1 - HF_S(1,1)/HF_S(d,e) \approx 1 - \frac{1}{m^{d-1} n^{e-1}}$. Note that the resulting pre-normal form $N$ does not represent an orthogonal projection along $I(d,e)$, and we expect that this may impact the numerical accuracy. The implementation and further investigation of this procedure are beyond the scope of this article.
Table 1. Hilbert Functions \( HF_{S/I}(i, j) \) from Example 3.2 (Left) and Example 3.3 (Right) for Small Values of \( i, j \)

| \( i \) | 0 | 1 | 2 | 3 | \ldots |
|------|---|---|---|---|-------|
| 0    | 1 | 3 | 6 | 10 | \ldots |
| 1    | 3 | 4 | 4 | 4 | \ldots |
| 2    | 6 | 4 | 4 | 4 | \ldots |
| 3    | 10| 4 | 4 | 4 | \ldots |
| \vdots | \vdots | \vdots | \vdots | \vdots | \ddots |

| \( j \) | 0 | 1 | 2 | 3 | 4 | 5 | \ldots |
|------|---|---|---|---|---|---|-------|
| 0    | 1 | 3 | 6 | 10| 15| 21| \ldots |
| 1    | 7 | 12| 15| 16| 15| 12| \ldots |
| 2    | 28| 21| 15| 12| 12| 12| \ldots |
| 3    | 84| 12| 12| 12| 12| 12| \ldots |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \ddots |

3.5 Some Examples

We now present two illustrative examples. The first one shows how to use the techniques explained above on the tensor \( \mathcal{A} \) in Example 1.1.

Example 3.2 (Example 1.1, Continued). The Hilbert function of \( S/I \), where \( I \) is generated by the five \( f_i \)'s from Example 1.1, is shown in Table 1 for small degrees. From \( HF_{S/I}((1, 1) + (d', e')) = r = 4 \) for \((d', e') \in \mathbb{N}^2\), we see that every degree \((d, e) \geq (1, 1)\) leads to a pre-normal form. Using \((d', e') = (1, 0)\), we obtain the pre-normal form \( N \) as the cokernel of \( R_I(2, 1) \in \mathbb{R}^{18 \times 15} \), whose transpose is

\[
\begin{pmatrix}
\begin{bmatrix}
x_0 f_1 \\
x_1 f_2 \\
x_2 f_3 \\
x_0 f_2 \\
x_1 f_2 \\
x_2 f_2 \\
x_0 f_3 \\
x_1 f_3 \\
x_2 f_3 \\
x_0 f_4 \\
x_1 f_4 \\
x_2 f_4 \\
x_0 f_5 \\
x_1 f_5 \\
x_2 f_5 \\
\end{bmatrix}
\end{pmatrix}
\]

The missing entries represent zeros. The row indexed by \( x_2 f_5 \) has entry \(-2\) in the column indexed by \( x_0 x_2 y_0 \) and 1 in the column indexed by \( x_2^2 y_0 \). This comes from \( x_2 f_5 = -2x_0 x_2 y_0 + x_2^2 y_0 \). The cokernel of \( R_I(2, 1) \) can be obtained, for instance, from the full SVD. We set \( h_0 = x_0 + x_1 + x_2 \) and use the subspace \( B \) spanned by \( B = \{x_0 y_0, x_0 y_1, x_0 y_2, x_1 y_0\} \). The numerical approximations of the eigenvalues of \( M_{x_0}/h_0, M_{x_1}/h_0 \) and \( M_{x_2}/h_0 \), found using Julia, are the rows of

\[
\begin{bmatrix}
-1.03745e-16 & 0.25 & 0.333333 & 0.5 \\
1.0 & 0.25 & -2.48091e-16 & -3.16351e-16 \\
-1.64372e-16 & 0.5 & 0.666667 & 0.5 \\
\end{bmatrix}
\]

These approximate the evaluations of \( x_i/h_0 \) at \( \zeta, \zeta_5, \zeta_1, \zeta_2 \) (in that order, from left to right). Consequently, the columns in the display above are homogeneous coordinates for \( \beta_4, \beta_5, \beta_1, \beta_2 \). The \( y_i \)'s can then be obtained by solving the linear system (3.4) of five equations in three unknowns. The left eigenvectors of the matrices \( M_{x_j}/h_0 \) are the columns of

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corresponding to evaluation (up to scale) of $B$ at $\zeta_4, \zeta_3, \zeta_1, \zeta_2$.

The ideal $I$ in the previous example has the property that $HF_{S/I}(1 + d', 1 + e') = r$ for all $(d', e') \in \mathbb{N}^2$. Our next example shows that this is not the case in general.

**Example 3.3 (A Format for which $HF_{S/I}(2, 1) \neq r$).** In Example 3.2, we could take any $(d', e') \in \mathbb{N}^2$ to compute a pre-normal form. However, it may be necessary to take bigger leaps in $\mathbb{N}^2$ such that $HF_{S/I}(((1, 1) + (d', e')) = r$. As a concrete example, consider a rank-12 tensor $A \in \mathbb{C}^{12} \otimes \mathbb{C}^7 \otimes \mathbb{C}^3$ with the decomposition

$$A = \sum_{i=1}^{12} \alpha_i \otimes \beta_i \otimes \gamma_i,$$

where $\beta_1, \ldots, \beta_{12}$ are the columns of a generic $7 \times 12$ matrix, $\gamma_1, \ldots, \gamma_{12}$ are the columns of a generic $3 \times 12$ matrix and $\alpha_1, \ldots, \alpha_{12}$ are the columns of any invertible $12 \times 12$ matrix. The Hilbert function of $S/I$ where $I = (\ker A_{(1)})$ is shown, for small degrees, in Table 1. By Proposition 3.1, a possible choice for $(d', e')$ is $(2, 0)$; this is underlined in the left part of Table 1. Other examples are $(2, 1), (2, 2), (1, 2), (0, 4)$. Some noteworthy non-examples are $(1, 0), (1, 1), (0, 1)$. In Section 4, we investigate choices of $(d', e')$ of the form $(d', 0)$. Our results will explain why, in this example, $d' = 1$ does not work, but $d' = 2$ does.

## 4 REGULARITY

One key step of the proposed cpd_hnf algorithm has not been investigated. As explained in the previous section (see also line 9 of Algorithm 3.1), we should determine a correct degree $(d, e)$. This choice has a major impact on the computational complexity of the proposed algorithm. Indeed, it determines the dimensions of the graded resultant matrix $R_I(d, e)$ from Equation (3.6) whose left nullspace is required. The goal of this section is determining which degree $(d, e)$ is needed for Theorem 3.1 to apply. From this we can then deduce our algorithm’s computational complexity.

As before, let $A$ be a tensor as in Equation (A) that satisfies Assumption 2.1, and let the $\mathbb{N}^2$-graded ring from Equation (2.2) be denoted by $S = \mathbb{C}[x_0, \ldots, x_m, y_0, \ldots, y_n]$. We assume that $r > m + 1$, for otherwise Theorem 3.2 applies and no choice of $(d, e)$ is required.

To compute $\beta_i$ and $\gamma_i$ via Theorem 3.1, we need to compute a pre-normal form on $S(d,e)$ for $(d,e) \neq (1, 1)$ and $(d', e') \geq (0,0)$. Proposition 3.1 tells us that we must find such a tuple $(d,e)$ for which additionally $HF_{S/I}(d,e) = r$, where $I$ is the ideal $(\ker A_{(1)})$. Motivated by this, we make the following definition.

**Definition 4.1.** For a homogeneous ideal $J \subset S$ such that $J = \langle f_{(1, 1)} \rangle$ and $V_X(J)$ consists of $r$ points with multiplicity one, we define the regularity of $J$ to be the set

$$\text{Reg}(J) = \{(d,e) \in \mathbb{N}^2 \mid (d,e) \geq (1,1) \text{ and } HF_{S/I}(d,e) = r\}.$$ 

Hence, our task is to find a tuple $(d,e) \in \text{Reg}(I) \setminus \{(1,1)\}$. Recall that such a tuple exists by Lemma 3.2. In this section, for given $\ell, m, n$ and $r$ satisfying (R), we conjecture an explicit formula for $d$ and $e$ so that $(d,1), (1,e) \in \text{Reg}(I) \setminus \{(1,1)\}$ for generic tensors of this format and rank. We prove it in many practical cases. Because the results in this section are of independent interest for solving structured, overdetermined systems of polynomial equations, we formulate them in a slightly more general context. The first statement of this section, Proposition 4.1, is concerned with homogeneous ideals $J$ of $S$...
that are generated by elements of degree \((1, 1)\). After that, we specialize to a particular type of such \((1, 1)\)-generated ideals. More precisely, to a tuple \(Z = (\zeta_1, \ldots, \zeta_r) \in X^r\), we associate an ideal \(I(Z)\), which is generated by elements of degree \((1, 1)\), and we investigate its Hilbert function (Corollary 4.1). In our tensor setting, we will have \(I(Z) = I = (\ker A_{(1)})\). After pointing out in Lemma 4.1 that for \(r \leq mn\), most configurations \(Z = (\zeta_1, \ldots, \zeta_r) \in X^r\) lead to an ideal \(I(Z)\) such that \(V_0(I(Z)) = (\zeta_1, \ldots, \zeta_r)\), where each of the \(\zeta_i\) occurs with multiplicity one, we use Corollary 4.1 to characterize \(\text{Reg}(I(Z))\) in Theorems 4.1 and 4.2 and Conjecture 1.

For \(a = (a_0, \ldots, a_m) \in \mathbb{N}^{m+1}, b = (b_0, \ldots, b_n) \in \mathbb{N}^{n+1}\), we write \(\partial_{a,b} : S \to S\) for the differential operator

\[
\partial_{a,b} = \frac{1}{a_0! \cdots a_m! b_0! \cdots b_n!} \partial^{|a|+|b|} \partial_{x_0}^{a_0} \cdots \partial_{x_m}^{a_m} \partial_{y_0}^{b_0} \cdots \partial_{y_n}^{b_n},
\]

such that the basis of \(S_{(d,e)}^\vee\) dual to \(\{x^a y^b \mid |a| = d, |b| = e\}\) is given by \(\{\partial_{a,b} \mid |a| = d, |b| = e\}\). We write \(e_k\) for the standard basis vector \((0, \ldots, 1, \ldots, 0)\) with a 1 in the \((k+1)\)st position, such that

\[
\partial_{e_k,e_l} = \frac{\partial^2}{\partial_{x_k} \partial_{y_l}}, \quad 0 \leq k \leq m, \quad 0 \leq l \leq n.
\]

Note that this differs from the shorter notation \(\partial_{kl}\) used in previous sections to have a general notation for derivatives of arbitrary order. For \((d,e) \in \mathbb{N}^2\) and \(J \subset S\) a homogeneous ideal, the linear space \(J_{(d,e)}^J \subset S_{(d,e)}^\vee\) is defined as

\[
J_{(d,e)}^J = \{v \in S_{(d,e)}^\vee \mid v(f) = 0 \text{ for all } f \in J\}.
\]

It follows from basic linear algebra that \(J_{(d,e)}^J \cong (S/I)(J_{(d,e)})\), such that \(\text{HF}_{S/J}(d,e) = \dim \mathbb{C} J_{(d,e)}^J\). As before, we denote \((d',e') = (d-1, e-1)\). To simplify the notation in this section, we will use the abbreviations

\[
\sum_{a,b} = \sum_{|a|=d} \sum_{|b|=e}, \quad \sum_{a',b'} = \sum_{|a'|=d'} \sum_{|b'|=e'}; \quad \sum_{k,l} = \sum_{0 \leq k \leq m} \sum_{0 \leq l \leq n}.
\]

We focus on ideals \(J\) that are generated by elements of degree \((1, 1)\). In this case, a functional belongs to \(J_{(1,1)}^J\) if and only if it induces functionals in \(J_{(1,1)}^{J_{(d,e)}}\).

**Proposition 4.1.** Let \(J \subset S\) be a homogeneous ideal such that \(J = (J_{(1,1)})\). An element \(v = \sum_{a,b} c_{a,b} \partial_{a,b} \in S_{(d,e)}^\vee\) is contained in \(J_{(d,e)}^J\) if and only if

\[
\sum_{a',b',k,l} c_{a'+e_k,b'+e_l} \partial_{e_k,e_l} \in J_{(1,1)}^{J_{(d,e)}} \quad \text{for all } (a',b') \text{ such that } |a'| = d', \ |b'| = e'.
\]

**Proof.** Since \(J = (J_{(1,1)})\), an element \(v = \sum_{a,b} c_{a,b} \partial_{a,b} \in S_{(d,e)}^\vee\) is contained in \(J_{(d,e)}^J\) if and only if \(v(hf) = 0\) for all \(f \in J_{(1,1)}\) and all \(h \in S_{(d',e')}\). Using Leibniz’ rule, we find

\[
0 = v(hf) = \sum_{a,b} c_{a,b} \partial_{a,b}(hf) = \sum_{a,b} c_{a,b} \sum_{k,l} \partial_{a-e_k,b-e_l}(h) \partial_{e_k,e_l}(f),
\]

with the convention that \(\partial_{a,b} = 0\) whenever \(\min(a) < 0\) or \(\min(b) < 0\). Regrouping the terms in this expression gives

\[
0 = \sum_{a',b'} \sum_{k,l} \partial_{a'+e_k,b'+e_l} \sum_{a,b} c_{a'+e_k,b'+e_l} \partial_{e_k,e_l}(f) \quad \text{for all } h \in S_{(d',e')}, f \in J_{(1,1)}.
\]

This proves the statement. \(\Box\)
In our tensor rank decomposition setting, we are mainly interested in investigating the Hilbert function for \( (1, 1) \)-generated ideals defined by point configurations in \( X = \mathbb{P}^m \times \mathbb{P}^n \). To that end, fix \( r \) points \( Z = (\zeta_1, \ldots, \zeta_r) \in X' \) and let \( \zeta_i = (\beta_i, \gamma_i) \). We denote \( w_i = (\beta_i \otimes \gamma_i)^\top \in S_{(1, 1)}^1 \) such that \( w_i(f) = f(\beta_i, \gamma_i) \) for \( f \in S_{(1, 1)} \).\(^6\) In coordinates, the \( w_i \) are \( w_i = \sum_{k,l} \beta_{k,l} Y_l Y_i \partial_{e_k, e_l} \). To the point configuration \( Z \), we associate an ideal \( J(Z) \subset S \) by setting

\[
J(Z)_{(1,1)}^1 = \text{span}_\mathbb{C}(w_1, \ldots, w_r) \quad \text{and} \quad J(Z) = \langle J(Z)_{(1,1)} \rangle.
\]

Note that the ideal \( J = \langle \ker \mathcal{A}(Z) \rangle \) from previous sections arises in this way.\(^7\) We denote \( W \subset X' \) for the Zariski-open subset in which \( w_1, \ldots, w_r \) are \( \mathbb{C} \)-linearly independent. If \( r \leq \HF_S(1, 1) \), then \( W \subset X' \) is non-empty and therefore dense in the Euclidean topology.\(^8\) We have the following consequences of Proposition 4.1.

**Corollary 4.1.** Let \( Z = (\zeta_1, \ldots, \zeta_r) \in X' \) and let \( w_1, \ldots, w_r \) and \( J(Z) \) be as above. For the maps \( \iota: S_{(d,e)}^1 \hookrightarrow S_{(1,1)}^1 \otimes S_{(d',e')}^1 \) and \( M: (S_{(d,e')}^1)^r \rightarrow S_{(1,1)}^1 \otimes S_{(d',e')}^1 \) given by

\[
\iota \left( \sum_{a,b} c_{a,b} \partial_{a,b} \right) = \sum_{a',b'} \sum_{k,l} c_{a'+e_k,b'+e_l} \partial_{e_k,e_l} \otimes \partial_{a',b'}, \quad M(v_1, \ldots, v_r) = \sum_{i=1}^r w_i \otimes v_i,
\]

we have \( \iota(J(Z)^{1}_{(d,e)}) = \text{im} \iota \cap \text{im} M \). Moreover, if \( Z \in W \), then we have that \( \HF_{S/J}(d,e) = \dim_{\mathbb{C}}(M^{-1}(\text{im} \iota)) \), where

\[
M^{-1}(\text{im} \iota) = \left\{ (v_1, \ldots, v_r) \in (S_{(d,e')}^1)^r \mid M(v_1, \ldots, v_r) \in \text{im} \iota \right\}.
\]

**Proof.** By Proposition 4.1, \( v = \sum_{a,b} c_{a,b} \partial_{a,b} \) is an element of \( J(Z)^{1}_{(d,e)} \) if and only if for all \( a', b' \) such that \( |a'| = d', |b'| = e' \), there exist \( v_{i,a',b'} \in \mathbb{C} \) such that

\[
\sum_{k,l} c_{a'+e_k,b'+e_l} \partial_{e_k,e_l} = \sum_{i=1}^r v_{i,a',b'} w_i.
\]

Writing \( v_i = \sum_{a',b'} v_{i,a',b'} \partial_{a',b'} \in S_{(d',e')}^1 \) and writing Equation (4.1) in matrix format, we see that Equation (4.1) is equivalent to the following equality in \( S_{(1,1)}^1 \otimes S_{(d',e')}^1 \):

\[
\sum_{a',b'} \sum_{k,l} c_{a'+e_k,b'+e_l} \partial_{e_k,e_l} \otimes \partial_{a',b'} = \sum_{i=1}^r w_i \otimes v_i.
\]

Linear independence of \( w_1, \ldots, w_r \) implies that \( M \) is injective. We have that injectivity of \( \iota \) and \( M \) implies along with \( \HF_{S/J}(d,e) = \dim_{\mathbb{C}}(J(Z)^{1}_{(d,e)}) \) that

\[
\dim_{\mathbb{C}}(J(Z)^{1}_{(d,e)}) = \dim_{\mathbb{C}}(\text{im} \iota) = \dim_{\mathbb{C}}(\text{im} \iota \cap \text{im} M) = \dim_{\mathbb{C}}(M^{-1}(\text{im} \iota)).
\]

This concludes the proof. \( \square \)

**Corollary 4.2.** Let \( Z = (\zeta_1, \ldots, \zeta_r) \in W \subset X' \) and \( (d,e) \geq (1, 1) \). Then, \( \HF_{S/J}(d,e) \geq r \).

\(^6\)The notation is similar to Section 2. Here, we omit the restriction to the subspace \( B \) (or, equivalently, we take \( B = S_{(1,1)} \)). We also drop the transpose on \( w_i \), as we will think of them as column vectors instead of row vectors in this section.

\(^7\)For the reader who is familiar with algebraic geometry, we note that this is our motivation for associating the ideal \( J(Z) \) to \( Z \), instead of the usual vanishing ideal of the points in \( Z \). These are different ideals, as \( J(Z) \) is usually not saturated with respect to the irrelevant ideal of \( S \).

\(^8\)This follows from the fact that the Segre variety is not contained in a hyperplane.
PROOF. The statement follows from Lemma 5.5.7 in Telen [2020b]. Nevertheless, we give an instructive proof. Let \( w'_f \in S'_{(d', c')} \), \( w'' \in S''_{(d, e)} \) be given by \( w'_f(f) = f(\beta_1, \gamma_1) \), \( w''_g(g) = g(\beta_1, \gamma_1) \) for \( f \in S_{(d', c')} \), \( g \in S_{(d, e)} \). Then, \( \iota(w''_g) = w_i \otimes w'_f \) and thus

\[
\mathcal{M}(w'_f, 0, \ldots, 0), \mathcal{M}(0, w'_f, \ldots, 0), \ldots, \mathcal{M}(0, 0, \ldots, w'_f)
\]

are all contained in \( \text{im} \iota \). Therefore, \( \mathcal{M}^{-1}(\text{im} \iota) \) contains at least \( r \) linearly independent elements, so by Corollary 4.1, we have \( \text{HS}/J(Z)(d, e) \geq r \).

The space \( S'_{(1, 1)} \otimes S''_{(d', e')} \) is identified with the space of matrices of size \( \text{HS}(1, 1) \times \text{HS}(d', e') \), where the rows are indexed by \( \partial_{e_k, e_l} \) for \( 0 \leq k \leq m, 0 \leq l \leq n \) and columns are indexed by \( \partial_{d', e'} \) where \((d', e') \in \mathbb{N}^{m+1} \times \mathbb{N}^{n+1} \) with \(|d'| = d', |b'| = e' \). For such a matrix to be contained in \( \text{im} \iota \), a collection of partial symmetry conditions needs to be satisfied. For instance, if \((a + e_k, b + e_l) = (a'' + e_k, b'' + e_l)\), then the entry in the row indexed by \( \partial_{e_k, e_l} \) and column indexed by \( \partial_{d', e'} \) should be equal to the entry in the row indexed by \( \partial_{e_k, e_l} \) and column indexed by \( \partial_{d'', e''} \) (see Example 4.1). Matrices in \( \text{im} \iota \) are called catalcatic matrices [Iarrobino and Kanev 1999, Definition 1.3].

We can use Corollary 4.1 to compute the Hilbert function of \( S(J(Z)) \) via a rank computation of a matrix whose entries are monomials evaluated at the points \( (\beta_1, \gamma_1) \). This is important for our proof of Theorem 4.2. It is most easily explained by means of an example.

**Example 4.1.** Let \((m, n) = (3, 2), (d, e) = (2, 1) \) and \( r = 6 \). We consider the ideal \( J(Z) \) defined by the tuple \( Z = (\xi_1, \ldots, \xi_6) \in X^6 = (\mathbb{P}^3 \times \mathbb{P}^2)^6 \) in the ring \( S = \mathbb{C}[x_0, x_1, x_2, x_3, y_0, y_1, y_2] \), where \( \xi_i = (\beta_1, \gamma_i) = (\beta_{10} : \beta_{11} : \beta_{12}, (y_{10} : y_{11} : y_{12})) \). As explained above, we can identify \( S'_{(1, 1)} \otimes S''_{(1, 0)} \) with \( 12 \times 4 \) matrices. The image of \((v_1, \ldots, v_6) \in (S'_{(1, 0)})^6 \), with \( v_i = \sum_{q=0}^{3} v_{iq} \partial_{e_q, 0} \), under \( \mathcal{M} \) is

\[
\begin{bmatrix}
\beta_1 \otimes y_1 & \cdots & \beta_6 \otimes y_6
\end{bmatrix} =
\begin{bmatrix}
v_1^T & \cdots & v_i & \cdots & v_6^T
\end{bmatrix}
\]

However, the image of \( \sum_{|a|=2, |b|=1} c_{a,b} \partial_{a,b} \) under \( \iota \) is the matrix

\[
\iota \left( \sum_{|a|=2, |b|=1} c_{a,b} \partial_{a,b} \right) =
\begin{bmatrix}
\partial_{e_0, 0} & \partial_{e_1, 0} & \partial_{e_2, 0} & \partial_{e_3, 0} \\
\partial_{e_0, e_0} & \partial_{e_0, e_1} & \partial_{e_0, e_2} & \partial_{e_0, e_3} \\
\partial_{e_1, e_0} & \partial_{e_1, e_1} & \partial_{e_1, e_2} & \partial_{e_1, e_3} \\
\partial_{e_2, e_0} & \partial_{e_2, e_1} & \partial_{e_2, e_2} & \partial_{e_2, e_3} \\
\partial_{e_3, e_0} & \partial_{e_3, e_1} & \partial_{e_3, e_2} & \partial_{e_3, e_3}
\end{bmatrix}
\]

For \( \mathcal{M}(v_1, \ldots, v_6) \), i.e., Equation (4.3), to be contained in \( \text{im} \iota \), it must be such that the \( (\partial_{e_1, e_0}, \partial_{e_0, 0}) \)-entry is equal to the \( (\partial_{e_0, e_0}, \partial_{e_1, 0}) \)-entry. This gives a linear condition on the \( v_{iq} \). There are 18 such conditions. Let \( v_{i,q} = (v_{1,q}, v_{2,q}, v_{3,q}, v_{4,q}, v_{5,q}, v_{6,q})^T \) be the column of the second matrix in Equation (4.3) indexed by \( \partial_{e_{q,0}} \) and let \( \Gamma = [y_{ij}] \) be the matrix that has the homogeneous coordinates \( y_{ij} \),
Let $j = 0, \ldots, 2$, of $\gamma_i$, $i = 1, \ldots, 6$, as columns. We also let $H_q = \text{diag}(\beta_{1q}, \ldots, \beta_{6q})$. The 18 symmetry conditions are

$$
\begin{bmatrix}
\Gamma H_1 & -\Gamma H_0 \\
\Gamma H_2 & -\Gamma H_0 \\
\Gamma H_3 & -\Gamma H_0 \\
\Gamma H_2 & -\Gamma H_1 \\
\Gamma H_3 & -\Gamma H_1 \\
\Gamma H_3 & -\Gamma H_2 \\
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_1 \\
\psi_2 \\
\psi_3 \\
\end{bmatrix}
= 0,
$$

where the colors in the block rows of the coefficient matrix $A(Z)$ correspond to the entries in Equation (4.4) on which they impose relations. In other words, the kernel of $A(Z)$ is the vector space $\{(\psi_1, \ldots, \psi_6) \mid M(\psi_1, \ldots, \psi_6) \in \text{im} i\}$ from Corollary 4.1. Hence, $HF_{S/J(Z)}(2, 1)$ is the corank of $A(Z)$. It is at least 6, since

$$A(Z) \begin{bmatrix} H_0 & H_1 & H_2 & H_3 \end{bmatrix}^T = 0.$$ 

These null vectors correspond to the $w'_i$ in the proof of Corollary 4.2. For generic $Z \in X'$, the corank of $A(Z)$ is exactly 6, so $HF_{S/J(Z)}(2, 1) = 6$.

From the discussion in Example 4.1, we would like to conclude that, generically, $(2, 1) \in \text{Reg}(J(Z))$. For this to make sense, i.e., to apply Definition 4.1, we need to show that for most configurations $Z$, $J(Z)$ defines $r$ points with multiplicity one. By an argument analogous to Lemma 2.2, this happens for small enough ranks.

**Lemma 4.1.** Suppose that $r \leq mn$ (this is Equation (R) for $\ell = \infty$). There is a Zariski open, dense subset $U' \subset X'$ such that for all $Z = (\xi_1, \ldots, \xi_r) \in U$, $w_1, \ldots, w_r$ are $\mathbb{C}$-linearly independent and $V_X(J(Z)) = \{\xi_1, \ldots, \xi_r\}$ consists of $r$ points with multiplicity one.

**Proof.** As before, let $\xi_i = (\beta_i, \gamma_i)$, $i = 1, \ldots, r$ and $w_i = (\beta_i \otimes \gamma_i)^\top \in S_{[1, 1]}^r$. By Theorem 2.5 in Chiantini and Ciliberto [2006], there is an open dense subset $U' \subset X'$ such that for $Z = (\xi_1, \ldots, \xi_r) \in U'$, $\text{span}_\mathbb{C}(w_1, \ldots, w_r)$ contains no points of the form $(\beta \otimes \gamma)^\top$, other than the $w_i$. We set $U = U' \cap W$, which is open and dense in $X'$. The rest of the proof is identical to that of Lemma 2.2.

Our next goal is to show that, to prove that $(d, e) \in \text{Reg}(J(Z))$ for almost all $Z \in X'$, it suffices to find one particular instance $Z^* \in W$ for which $HF_{S/J(Z^*)}(d, e) = r$.

**Proposition 4.2.** Suppose $r \leq HF_S(1, 1)$ such that $W \neq \emptyset$. For fixed $(d, e) \geq (1, 1)$, the Hilbert function $HF_{S/J(Z)}(d, e)$, as a function of $Z$, is upper semicontinuous on $W$. That is, for any $r^* \in \mathbb{N}$,

$$V_{r^*} = \{Z \in W \mid HF_{S/J(Z)}(d, e) > r^*\}$$

is Zariski closed in $W$. Consequently, either $V_{r^*} = W$ or $V_{r^*} \subseteq W$ is a strict closed subvariety.

**Proof.** We have $HF_{S/J(Z)}(d, e) = HF_S(d, e) - HF_J(Z)(d, e)$, and $J(Z)(d, e) = S_{(d', e')} \cdot J(Z)(1, 1)$. Therefore, the condition $HF_{S/J(Z)}(d, e) > r^*$ is equivalent to $\dim_{\mathbb{C}}(S_{(d', e')} \cdot J(Z)(1, 1)) < HF_S(d, e) - r^*$, which can be written as the vanishing of the $(HF_S(d, e) - r^*)$-minors of a matrix, whose entries are monomials in the coordinates of $Z$.

**Corollary 4.3.** Suppose that $r \leq mn$ and $(d, e) \geq (1, 1)$. Let $U \subset X'$ be the dense open subset from Lemma 4.1. If for some element $Z^* \in W$ we have $HF_{S/J(Z^*)}(d, e) = r$, then there is a Zariski open, dense subset $U^o$ of $U$ such that for all $Z \in U^o$, we have $(d, e) \in \text{Reg}(J(Z))$. 

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Proof. By Corollary 4.2, we know \( V_{r-1} = W \). Since \( Z^r \subset W \setminus V_r \), Proposition 4.2 implies that \( V_r \subset W \) is a strict subvariety. We set \( U^o = U \setminus V_r \). Clearly, if \( Z = (\zeta_1, \ldots, \zeta_r) \in U^o \), then \( J(Z) \) is such that \( V_X(J(Z)) = \{ \zeta_1, \ldots, \zeta_r \} \), where these points occur with multiplicity one (this uses \( U^o \subset U \)), and \( HF_{S/J(Z)} = r \) (since \( U^o \subset V_{r-1} \setminus V_r \)). \( \square \)

This result will be particularly useful for proving Theorem 4.2 below. First, we investigate which combinations of \( m, n, d, e, r \) are possible to have \( HF_{S/J(Z)}(d, e) = r \) for generic points \( Z \).

For \((d, e) \geq (1, 1) \) and \( m, n \in \mathbb{N}_0 \), we define

\[
\mathcal{R}(m, n, (d, e)) = \frac{HF_S(1, 1)HF_S(d', e') - HF_S(d', e')}{HF_S(d', e') - 1}.
\]

If \((d, e) = (1, 1)\), then we set \( \mathcal{R}(m, n, (d, e)) = \infty \). For given \( m, n \) and \((d, e) \geq (1, 1)\), the following result shows that \( \mathcal{R}(m, n, (d, e)) \) bounds the rank \( r \), for which we could possibly have \( HF_{S/J(Z)}(d, e) = r \) for \( Z \in W \subset X^r \).

**Theorem 4.1.** Let \((d, e) \geq (1, 1) \) and \( Z \in W \subset X^r = (\mathbb{P}^m \times \mathbb{P}^n)^r \), with

\[
\mathcal{R}(m, n, (d, e)) < r \leq mn.
\] (4.5)

We have \( HF_{S/J(Z)}(d, e) > r \). In particular, for \( r \) in the range (4.5) and \( Z \in U \), where \( U \subset W \) is the open subset from Lemma 4.1, we have \((d, e) \notin \text{Reg}(J(Z))\).

**Proof.** Since \( HF_{S/J(Z)}(d, e) = HF_S(d, e) - HF_{J(Z)}(d, e) \geq HF_S(d, e) - HF_S(d', e')HF_{J(Z)}(1, 1) \) and \( HF_{J(Z)}(1, 1) = HF_S(1, 1) - r \), we have

\[
HF_{S/J(Z)}(d, e) \geq HF_S(d, e) + rHF_S(d', e') - HF_S(1, 1)HF_S(d', e').
\] (4.6)

Solving \( HF_S(d, e) + rHF_S(d', e') - HF_S(1, 1)HF_S(d', e') > r \) yields the first inequality in Equation (4.5).

Note that if \((d, e) = (1, 1)\), the range (4.5) is empty. This agrees with the fact that \((1, 1) \in \text{Reg}(J(Z))\) for all \( Z \in W \). From Corollary 4.2, we know that \( HF_{S/J(Z)}(d, e) \geq r \) for \((d, e) \geq (1, 1)\) and \( Z \in W \). Combining this with a dimension argument as in the proof of Theorem 4.1, we see that

\[
HF_{S/J(Z)}(d, e) \geq \max\{r, \text{HF}_S(d, e) + r\text{HF}_S(d', e') - \text{HF}_S(1, 1)\text{HF}_S(d', e')\}.
\] (4.7)

For \( e' = 0 \) or \( d' = 0 \), we observe experimentally that equality holds for generic configurations \( Z = (\zeta_1, \ldots, \zeta_r) \in X^r \). In this case, it suffices to check for which \( r \) the maximum equals \( r \). This leads us to the following conjecture.

**Conjecture 1.** Let \( d \geq 1 \) and let \( m, n, r \) be such that

\[
r \leq \min\{\mathcal{R}(m, n, (d, 1)), mn\}.
\] (4.8)

There is a Zariski open, dense subset \( U^o \subset U \subset X^r = (\mathbb{P}^m \times \mathbb{P}^n)^r \), were \( U \) is the open subset from Lemma 4.1, such that for all \( Z \in U^o \), \((d, 1) \in \text{Reg}(J(Z))\).

**Theorem 4.2.** Conjecture 1 holds in the following cases:

1. \( m \in \mathbb{N}_0, n \in \mathbb{N}_0 \) and \( d = 1 \),
2. \( m \in \{1, 2\}, n \in \mathbb{N}_0 \) and \( d = 2 \),
3. \( 2 \leq m + 1, n + 1 \leq 50 \) and \( d = 2 \),
4. \( 2 \leq m + 1, n + 1 \leq 9 \) and \( 3 \leq d \leq 5 \),
5. \( 2 \leq m + 1, n + 1 \leq 6 \) and \( 6 \leq d \leq 10 \).
PROOF. (1) In this case, clearly $U^\circ = U$.

(2) Following Example 4.1, we can compute $\text{HF}_{S/J(Z)}(2, 1)$ as the corank of the stacked matrix $A(Z) = [G_1(Z) \cdots G_m(Z)]^\top$, where

$$
G_k(Z) = \begin{bmatrix}
0_{(n+1)\times (k-1)r} & \Gamma H_k & -\Gamma H_{k-1} \\
0_{(n+1)\times (k-1)r} & \Gamma H_{k+1} & -\Gamma H_{k-1} \\
\vdots & \vdots & \ddots \\
0_{(n+1)\times (k-1)r} & \Gamma H_m & -\Gamma H_{k-1}
\end{bmatrix},
$$

the matrix $\Gamma$ contains homogeneous coordinates of the points $y_i$ in its columns, and $H_q$ is the diagonal matrix $\text{diag}(\beta_{1q}, \ldots, \beta_{rq})$.

By Corollary 4.3, for each $r$ satisfying Equation (4.8), we must find one $Z^s \in W$ such that $\text{HF}_{S/J(Z^s)}(2, 1) = \text{corank}(A(Z^s)) = r$. In fact, since $A(Z) (H_0 \cdots H_m)^\top = 0$, we have corank($A(Z)$) $\geq r$ for any $Z \in X^r$, and by upper semicontinuity of corank, if corank($A(Z^s)$) $= r$ for some $Z^s \in X^r$, then corank($A(Z^s)$) $= r$ for all $Z$ in a dense, Zariski open subset of $X^r$. For $m = 1$, Equation (4.8) entails $r \leq n + 1$, which means that the matrix $\Gamma$ has more rows than columns. On a dense, Zariski open subset of $X^r$, $\Gamma$ has rank $r$ and, since $(\beta_{10}, \beta_{11}) \neq (0, 0)$, the matrix $A = [TH_1 - TH_2]$ has rank at least $r$. The proof for $m = 2$ is more technical. Since we could not generalize it for higher $m$, we have deferred it to the Appendix.

(3)–(5) These cases consists of a computer-assisted proof.\footnote{The code and certificates can be obtained at https://gitlab.kuleuven.be/u0072863/homogeneous-normal-form-cpd.}

For proving case (3), we generated random $Z^s$ and confirmed that $A(Z^s)$ has corank $r$ as follows.

We generate random $\beta_i \in \mathbb{Z}^{m+1}$ and $y_i \in \mathbb{Z}^{n+1}$. Then $A(Z)$ is a matrix over $\mathbb{Z}$. We can upper bound its corank by computing the corank, via a reduction to row echelon form using Gaussian elimination, over the finite field $\mathbb{Z}_p$ for some prime number $p$. We used $p = 8191$. The corank of $A(Z^s)$ over the finite field $\mathbb{Z}_p$ is then an upper bound of the corank of $A(Z^s)$ over $\mathbb{Z}$ (and so also its algebraic closure). We implemented this approach in C++ and certified for all listed cases that there exists a $Z^*$ (with integer coefficients) such that $A(Z)$ has corank $r$.

For proving cases (4) and (5), we apply Corollary 4.3 to verify Conjecture 1 by computing the Hilbert function in Macaulay2 [Grayson and Stillman] for all listed cases as follows:

\begin{verbatim}
F = Z/8191;
HF = (x,y) -> binomial(m*x,x)*binomial(n*y,y);
degs = apply(m+1,i->(1,0)) | apply(n+1,i->(0,1));
S = F[x_0..x_m,y_0..y_n, Degrees => degs];
X = matrix apply(m+1,i->apply(r,j->random(F)));
Y = matrix apply(n+1,i->apply(r,j->random(F)));
N = transpose matrix apply(r,i->(X_i**Y_i));
I = ideal (basis({1,1},S)*(gens ker N));
rk = rank N; -- check that this is r, so Z* is in W
hf = hilbertFunction({d,1},I); -- check that this equals r
\end{verbatim}

where $m, n, d$, and $r$ are, respectively, $m, n, d$, and $r$ from the theorem. \qed

Note that the role of $m$ and $n$ in Conjecture 1 can be interchanged, implying the analogous statement that for $e \geq 1$ and generic $Z \in W \subset X^e = (\mathbb{P}^m \times \mathbb{P}^n)^e$ with $r \leq \min \{\mathcal{R}(m, n, (1, e), mn)\}$, we have $(1, e) \in \text{Reg}(J(Z))$.

Let $\mathcal{A} \in \mathbb{C}^{e+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$ be a general tensor whose rank satisfies Equation (R) and let $I = \langle \ker \mathcal{A}(1) \rangle = J(Z)$, where $Z = ((\beta_i, y_i))_{i=1,\ldots,r}$. A consequence of Conjecture 1 would be that

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Example 4.2 (Example 3.3, Continued). In Example 3.3, we had $m = 6$ and $n = 2$. We have $R(6, 2, (2, 1)) = 21/2 < r = 12$, which explains why $(2, 1) \notin \text{Reg}(l)$. Also, $R(6, 2, (3, 1)) = 112/9 \geq 12$ implies, since $\mathcal{A}$ is generic, that $(3, 1) \in \text{Reg}(l)$. Similarly, the smallest $e > 1$ for which $R(6, 2, (1, e)) \geq 12$ is $e = 5$, so that $(1, 5) \in \text{Reg}(l)$, but $(1, e) \notin \text{Reg}(l)$ for $1 < e < 5$. These computations are confirmed by Table 1.

One can check that

$$R(m, n, (d, 1)) = \frac{(m+d-1)}{d-1} \left( (m+1)(n+1) - \frac{n+1}{d} (m+d) \right). \quad (4.9)$$

The first factor is always greater than 1 and the second factor is at least $mn$ if $d \geq n + 1$. If Conjecture 1 holds, then for all tensors $\mathcal{A}$ of rank $r$ in the range (R), we have $(n+1, 1) \in \text{Reg}(l)$. In that case, $\text{cpd}_{\text{hnf}}$ can take $(d, e) = (n+1, 1)$ to treat all identifiable rank-$r$ tensors in the unbalanced regime. That is, when $\ell > mn$. For such shapes, Bocci et al. [2014] proved that generic $r$-identifiability holds up to $r \leq mn$.

Conjecture 1 gives a way of finding degrees of the form $(d, 1) \in \text{Reg}(l) \setminus \{(1, 1)\}$. There might exist other tuples of the form $(d, e) \in \text{Reg}(l) \setminus \{(1, 1)\}$, which could lead to a lower computational cost. For such degrees, Equation (4.7) may be a strict inequality, which means that other tools are needed. We leave the further exploration of effective bounds for the regularity for future research.

5 COMPUTATIONAL COMPLEXITY

One motivation for studying the regularity of the ideal $I = \langle \ker \mathcal{A}(1) \rangle$ is to understand the computational complexity of Algorithm 2.1.

Assume that we are given an $r$-identifiable tensor $\mathcal{A}$ in $\mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$, where $\ell \geq m \geq n$. We additionally assume that $\ell + 1 \leq (m+1)(n+1)$. The size of the input will be denoted by $M = (\ell+1)(m+1)(n+1)$. Because of the constraint $(m+1)(n+1) \geq \ell + 1 \geq m + 1 \geq n + 1$, we have $M^{\frac{1}{3}} \leq m + 1 \leq M^{\frac{2}{3}}$ and $2 \leq n + 1 \leq M^{\frac{1}{3}}$.

Since $\text{cpd}_{\text{hnf}}$ applies only if the rank $r$ satisfies Equation (R), we have $r = O(M^{\frac{2}{3}})$.

Without going into details, it can be verified that a crude upper bound for the computational complexity of the steps in Algorithm 2.1, excluding step 4, is $O(M^3)$. The key contribution to the time complexity originates from Algorithm 3.1. Its complexity is ultimately determined by the choice of $(d, e)$ in line 9 of the algorithm. We analyze what happens when the degree is selected as $(d, 1)$. In this case, bounding the size of the matrix $R_I(d, e)$ in Algorithm 3.1 is critical. Assuming Conjecture 1 holds, we have

$$\left( \frac{3}{2} \right)^{d+1} \leq \text{HF}_S(d, 1) = \left( \frac{m+d}{d} \right)(n+1) \leq M^{\frac{2}{3}} M^{\frac{1}{3}}. \quad (5.1)$$

The upper bound follows from $\left( \frac{m+d}{d} \right) \leq (m+d)^d \leq (m+1)^d \leq ((m+1)(n+1))^d$. For establishing the lower bound, note that $(1 + \frac{m}{d})^d \leq \left( \frac{m+d}{d} \right)$ and use the facts that $d \leq n + 1 \leq m + 1$ (see the

---

10 A tensor $\mathcal{A}$ not satisfying this constraint on $\ell$ can always be represented in new bases by a coordinate array $\mathcal{B}$ in a concise [ Bürgisser et al. 1997; Landsberg 2012] tensor space $\mathbb{C}^{\ell+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$ with $\ell' \leq \ell$, $m' \leq m$, and $n' \leq n$. After permutation of the factors, a concise tensor space $\mathbb{C}^{k_1} \otimes \mathbb{C}^{k_2} \otimes \mathbb{C}^{k_3}$ always satisfies $k_2 k_3 \geq k_1 \geq k_2 \geq k_3$; see, e.g., Carlini and Kleppe [2011]. In practice, $\mathcal{B}$ can be obtained as the core tensor of the (sequentially) truncated higher-order singular value decomposition [De Lathauwer et al. 2000; Vannieuwenhoven et al. 2012].
discussion below Equation (4.9)), so that \( d \geq 2 \) implies that \( \frac{m}{d} \geq \frac{1}{2} \). Computing a basis for the left null space of \( R_I(d, 1) \) requires at most \( O((\text{HF}_S(d, 1))^3) \) and at least \( r \cdot \text{HF}_S(d, 1) \) operations, as we know \( (d, 1) \in \text{Reg}(I) \) so that the dimension of the left null space is \( r \). Consequently, line 11 has a time complexity that is at least exponential in \( d \).

**Proposition 5.1.** Consider the concise tensor space \( \mathbb{C}^{M^\frac{1}{2}} \otimes \mathbb{C}^{M^\frac{1}{2}} \otimes \mathbb{C}^{M^\frac{1}{2}} \). If Conjecture 1 holds, then for a generic tensor of rank \( r = M^\frac{1}{2} - 2M^\frac{1}{2} - 1 \) in this space, the asymptotic time complexity of Algorithm 2.1 is at least exponential in the input size \( M = M^\frac{1}{2} M^\frac{1}{2} M^\frac{1}{2} \), if the degrees are restricted to \( (d, 1) \) or \( (1, e) \).

**Proof.** It follows from Equation (4.9) and Theorem 4.1 that for sufficiently large \( M \), the degree \( d \) should be at least \( \frac{1}{2} M^\frac{1}{2} \). Combining this with the above discussion about the size of \( R_I(d, 1) \) concludes the proof. \( \square \)

One might conclude from this result that cpd\_hnf is not an effective algorithm for tensor rank decomposition. However, Hillar and Lim [2013] proved that computing the rank of a tensor over \( \mathbb{C} \) is an NP-hard problem. Hence, a polynomial-time algorithm that applies to all inputs for this problem is not anticipated. A typical instance of an NP-hard problem can often be solved more efficiently than the worst-case instance. Theorem 1.1 is in this spirit.

**Proof of Theorem 1.1.** It follows from the upper bound in Equation (5.1) that the time complexity of Algorithm 2.1 is \( O((\text{HF}_S(d, 1))^3) = O(M_d^{2d+1}) \).

We determine the smallest degree \( d \) under Conjecture 1 such that \( (d, 1) \in \text{Reg}(I) \), where \( I = \langle \ker \mathcal{A}_{(1)} \rangle \). From Theorem 4.1, we know that a necessary condition for this is that \( R(m, n, (d, 1)) \geq r = \phi mn \). The last inequality is implied by \( (m + 1)(n + 1) - \frac{n+1}{d}(m + d) \geq \phi mn \) because of Equation (4.9). This is equivalent to

\[
(1 - \phi)mn + m + n + 1 \geq \frac{1}{d} (m + d)(n + 1) = \frac{1}{d} m(n + 1) + n + 1.
\]

Hence, \( R(m, n, (d, 1)) \geq \phi mn \) is implied by

\[
d \geq \frac{1}{1 - \phi} > (n + 1) \frac{m(n + 1)}{(1 - \phi)n + 1} = \frac{m(n + 1)}{(1 - \phi)mn + m}.
\]

In other words, provided Conjecture 1 holds, it suffices to take \( d \geq \lceil \frac{1}{1 - \phi} \rceil \) to be able to cover all ranks up \( \phi mn \). As \( \phi \neq 1 \) is a constant, this proves the result. \( \square \)

We can observe that the formula \( d \geq \frac{1}{1 - \phi} \) is asymptotically sharp in the sense that as \( \phi \to 1 \) the exponent in \( O(M^\star) \) needs to blow up to \( \infty \) and no polynomial in the input size can control the growth. Indeed, Proposition 5.1 shows that exactly when \( \phi = 1 \) there exist cases that require at least an exponential growth. Note that by the discussion below Equation (4.9), assuming Conjecture 1, we can use \( d = n + 1 \) for any \( \phi \in [0, 1] \). This gives the universal, exponential complexity bound \( O(M^{\frac{n+1}{1 - \phi}}) \).

### 6 NUMERICAL EXPERIMENTS

We present several numerical results demonstrating the effectiveness of cpd\_hnf. All experiments were performed on KU Leuven/UHasselt’s Tier-2 Genius cluster of the Vlaams Supercomputer Centrum (VSC). Specifically, the supercomputer’s scheduling software allocated standard skylake nodes containing two Xeon Gold 6140 CPUs (18 physical cores, 2.3 GHz clock speed, 24.75 MB L3 cache) with 192 GB of main memory, as well as standard cascadelake nodes, which are equipped with two Xeon Gold 6240 CPUs (18 physical cores, 2.6 GHz clock speed, 24.75 MB L3 cache) with
192 GB of main memory for our experiments. In all experiments, we allowed all algorithms to use up to 18 physical cores.

The proposed algorithm was implemented in Julia v1.4.0, relying on the non-base packages Arpack.jl, DynamicPolynomials.jl, GenericSchur.jl, and MultivariatePolynomials.jl. Our implementation follows the pseudocode in Algorithms 2.1 and 3.1 and the detailed discussion in Section 3.4 quite closely. The Julia code of cpd_hnf, including driver routines to reproduce cpd_hnf’s experimental results can be found at https://gitlab.kuleuven.be/u0072863/homogeneous-normal-form-cpd.

In the experiments below, random rank-$r$ tensors $\mathbf{A} = \sum_{i=1}^I a_i \otimes \beta_i \otimes \gamma_i$ are generated by randomly sampling the elements of $a_i \in \mathbb{R}^{d_i}$, $\beta_i \in \mathbb{R}^{m_i}$, and $\gamma_i \in \mathbb{R}^{n_i}$ identically and independently distributed (i.i.d.) from a standard normal distribution.

### 6.1 Implementation Details

The algorithm is implemented for real and complex input tensors. In the former case, all computations are performed over the reals with the exception of the computation of the left null space of $R_I(d, e)$. In the real case, the algorithm continues with the real part of the output of this step.

At the start of the algorithm, we compress the $(\ell + 1) \times (m + 1) \times (n + 1)$ input tensor, which is to be decomposed into $r$ rank-1 terms, to a $\min(\ell + 1, r) \times \min(m + 1, r) \times \min(n + 1, r)$ tensor. For this, we apply ST-HOSVD compression [Vannieuwenhoven et al. 2012] with truncation rank $(\min(\ell + 1, r), \min(m + 1, r), \min(n + 1, r))$. Most of the experiments below are chosen so that this step performs no computations.

The kernel of $\mathcal{A}_{(1)}$ in Algorithm 2.1 is computed by an SVD. The linear system at the end of Algorithm 2.1 is solved by computing the Khatri–Rao product $K = [\beta_i \otimes \gamma_i]_{i=1}^{I}$ and then solving the overdetermined linear system $KA = \mathcal{A}_{(1)}^\top$ for $A$. The rows of $A$ then correspond to the $a_i$’s.

In Algorithm 3.1, the degree $(d, e)$ is determined automatically by assuming Conjecture 1 is true and selecting a valid $(d, 1)$ or $(1, e)$ based on a heuristic that takes into account the estimated computational cost and numerical considerations.

The key computational bottleneck is the computation of the left nullspace of $R_I(d, e)$ in line 11 of Algorithm 3.1. When the number of entries of $R_I(d, e)$ is smaller than 10,000, we use the standard SVD-based kernel computation. In larger cases, for efficiency, we propose to employ Arpack.jl’s eigs function to extract the left null space from the Hermitian Gram matrix $G = R_I(d, e)(R_I(d, e))^H$, where $^H$ denotes the Hermitian transpose. The eigs function (with parameters tol = $1e^{-6}$ and maxiter = 25) extracts the $r$ eigenvalues of smallest modulus and their corresponding eigenvectors. Note that Arpack.jl employs a Bunch–Kaufman factorization [Bunch and Kaufman 1977] of the positive semidefinite input matrix $G$ to perform its Lanczos iterations. Using eigs was up to 75% faster than the SVD-based approach for large problems.

The final step of Algorithm 3.1 is implemented as discussed at the end of Section 3.4. The kernel is computed with an SVD and $\gamma_i$ is taken as the left singular vector corresponding to the smallest singular value. After this step, we find $(\beta_i, \gamma_i)$. Observe that $f_j(\beta_i, \gamma_i)$ should vanish exactly for all $j = 1, \ldots, s$. We propose to refine the approximate roots $(\beta_i, \gamma_i)$ by applying three Newton iterations on $f_i = \cdots = f_s = 0$. This adds a minor computational overhead of $r(m+1)(n+1)(m+n)^2$.

### 6.2 Impact on the Accuracy of Computational Optimizations

We compare and evaluate the impact on numerical accuracy of four variants of Algorithm 3.1. Two options are considered in each combination:

---

11 Additional packages are used to support our experimental setup, but these are not required for the main algorithm.
12 For more details, see the function minimumMultiDegree in NormalFormCPD.jl.
Fig. 1. A comparison of the $\log_{10}$ of the relative backward error of the proposed cpd_hnf in the four variants discussed in Section 6.2 on random rank-$r$ tensors in $\mathbb{R}^{\ell+1} \otimes \mathbb{R}^{m+1} \otimes \mathbb{R}^{n+1}$ with $\ell \geq m \geq n$. The largest dimension and the rank satisfy $\ell + 1 = r = \min(\mathcal{R}(m,n, (2,1)), mn)$, where the function $\mathcal{R}$ is as in Equation (4.9). The color scale is the same in all plots.

(i) refining the approximate roots $(\beta_i, \gamma_i)$ with 3 Newton steps (+newton) or not, and
(ii) computing the left nullspace of $R_I(d,e)$ with Arpack.jl’s iterative eigs method applied to the Gram matrix or using the standard svd.

We append _eigs, _eigs+newton, _svd, and _svd+newton to cpd_hnf to distinguish the variants.

The experimental setup is as follows. Random rank-$r$ tensors are generated as explained at the start of this section, sampling one rank-$r$ tensor for each combination $(\ell + 1, m + 1, n + 1, r)$ with $25 \geq m + 1 \geq n + 1 \geq 2$ and $\ell + 1 = r = \min(\mathcal{R}(m,n, (2,1)), mn)$ where $\mathcal{R}$ is as in Equation (4.9). For each variant of cpd_hnf, the relative backward error $\|A - \sum_{i=1}^{r} \alpha_i' \otimes \beta_i' \otimes \gamma_i'\|_F \|A\|_F^{-1}$, and execution time is recorded, where $(\alpha_i', \beta_i', \gamma_i')$ are the estimates obtained by the algorithm.

The experiment was performed on cascadelake nodes and the results are displayed in Figure 1. It is visually evident in the right panels that the Newton refinement improves the overall relative backward error for both eigs and svd by several orders of magnitude and brings both to seemingly the same level of about $10^{-16} - 10^{-12}$. However, there is a marked difference in accuracy between eigs and svd before Newton refinement, the svd being more accurate by about 1–3 orders of magnitude. It is interesting to compare these results with the leftmost panel of Figure 2(b), which shows the corresponding results on the same tensors for a conceptually similar state-of-the-art method from the literature. The method also requires a kernel computation that, in the displayed graph, is performed with a singular value decomposition. This means that the methods in Figure 1(c) and the left panel of Figure 2(b) can be directly compared. It can be seen that the proposed cpd_hnf_svd
Fig. 2. A comparison of the log_{10} of the relative backward error of the proposed cpd_hnf and the state-of-the-art cpd_ddl from Domanov and De Lathauwer [2014, 2017] on random rank-r tensors in $\mathbb{R}^{\ell+1} \otimes \mathbb{R}^{m+1} \otimes \mathbb{R}^{n+1}$ with $\ell \geq m \geq n$. The largest dimension and the rank satisfy $\ell + 1 = r = \min\{R(m, n, (d, 1)), mn\}$, where the function $R$ is as in Equation (4.9). The outcomes for $d = 2, 3, 4$ are shown, respectively, in the left, middle, and right plots. The color scale is the same in all plots.

method, even without Newton refinement, can be up to 3 orders of magnitude more accurate (in relative backward error). The differences in accuracy are thus not exclusively attributable to the Newton refinement.

We did not include a visualization of the timings, because they visually look very similar. By adding the Newton refinement, the total execution time over all cases only modestly increased: 3.4\% for eigs and 3.8\% for svd. The relative increase was larger for the small cases while it was close to 0–5\% for the large cases ($n + 1 \geq 20$). Given that the accuracy can be improved by orders of magnitude, we conclude from these experiments that a few steps of Newton refinement of the approximate roots $(\beta_i, \gamma_i)$ is highly recommended.

After deciding that Newton refinement is our default choice and observing in the right panels of Figure 1 there is no significant difference in accuracy between the eigs and svd, we investigate their relative computational performance. In terms of total execution time over all cases, the eigs variant is 29.1\% faster. For the largest case this increases to a 40.9\% reduction in execution time.

From the above experiments, we conclude that the variant with an iterative eigensolver and Newton refinement is an appropriate default choice. In the remainder of the experiments, cpd_hnf refers to cpd_hnf_eigs+newton, the variant using Arpack’s iterative eigenvalue method applied to the Gram matrix combined with 3 Newton steps to refine the approximate roots.

6.3 Comparison with the State of the Art
In the second experiment, we compare cpd_hnf with the current state of the art in direct numerical methods for tensor rank decomposition in terms of accuracy and computational performance. The
algorithm developed by Domanov and De Lathauwer [2014, 2017] is an advanced direct numerical method that shares several high-level characteristics with `cpd_hnf`:

(i) Both methods assume the target rank \( r \) is supplied to the decomposition algorithm. They operate in a similar regime of ranks and are able to treat the full range of generically \( r \)-identifiable tensors in unbalanced tensor spaces. The method by Domanov and De Lathauwer can even deal with some ranks \( r > \ell + 1 \geq m + 1 \geq n + 1 \), while our algorithm cannot.
(ii) Like our method, Domanov and De Lathauwer rely on a simultaneous diagonalization procedure to extract one or two of the factor matrices.
(iii) To deal with high ranks, both methods rely on the construction of an auxiliary matrix whose size is parameterized by an integer. By increasing this integer, the range of ranks that is covered by the algorithms is broadened at the cost of a substantially increased computational complexity. In both algorithms, the asymptotic computational complexity is determined by the cost of computing the kernel of this auxiliary matrix. Contrary to Domanov and De Lathauwer [2014, 2017], we are able to give a precise connection between this integer and the range of ranks we can cover (subject to Conjecture 1).
(iv) When \( \ell + 1 \geq m + 1 \geq r \), both approaches can be considered pencil-based algorithms. For this reason, our comparison will focus on the case where \( \ell + 1 \geq r \geq m + 1 \geq n + 1 \).

A Matlab implementation of Domanov and De Lathauwer’s algorithm [Domanov and De Lathauwer 2017] is available as part of the Tensorlab+ repository [Hendrikx et al.] as `cpd3_gevd` in the domanov2017laa directory. We refer to it as `cpd_ddl` henceforth. `cpd_ddl` has an accurate and a fast option for computing the kernel of the auxiliary matrix. We found in our experiments that the dimension of this kernel is often misjudged by the fast implementation. Consequently, it usually increases the size of the auxiliary matrix and causes it to exceed the 16.2 GB memory limit, we put on the size of that matrix. As a concrete statistic, in the configuration with \( d = 2 \) below, the fast version of `cpd_ddl` failed in over 75% of the cases. Therefore, we exclusively compare our implementation with the accurate version `cpd_ddl`.

We generate random rank-\( r \) tensors \( A \) as described at the start of this section. We apply both `cpd_hnf` and `cpd_ddl` to these random tensors for all of the following configurations\(^\text{13}\):

\[
\begin{align*}
40 \geq m + 1 \geq n + 1 & \geq 2 \quad \text{for} \quad d = 2, \text{ and} \\
20 \geq m + 1 \geq n + 1 & \geq 2 \quad \text{for} \quad d = 3, \text{ and} \\
15 \geq m + 1 \geq n + 1 & \geq 2 \quad \text{for} \quad d = 4,
\end{align*}
\]

and in all cases, we take \( \ell + 1 = r = \min\{\mathcal{R}(m, n, (d, 1)), mn\} \). We do not provide our algorithm with the value of \( d \). For each input tensor, `cpd_hnf` determines the degree \((d, 1)\) or \((1, e)\) automatically as explained in the previous subsection. For each algorithm, we record the relative backward error, total execution time, and the size of the auxiliary matrix whose kernel is computed. As the latter is the dominant operation in both algorithms, its time and memory complexity gives a good estimate of the overall complexity.

For the first configuration `cpd_ddl` was executed on `bigmem` nodes of the supercomputer. These are the same as the `skylake` nodes, except that they are equipped with 768 GB of main memory. The

\[^{13}\text{Both algorithms were applied to the same rank-}\ r\ \text{tensors. To deal with the different programming languages and to limit storage demands, we generated a buffer of } 10^7 \text{ reals sampled i.i.d. from a standard normal distribution. The } a_i, b_i, \text{ and } y_i \text{ of the true decomposition were then generated from this buffer. This entails there is a statistical correlation between the various tensors that are decomposed. However, we judged that this does not affect the validity of our conclusions. For each line in the set of configurations, a different random buffer was generated.}\]
Fig. 3. The $\log_{10}$ of the total execution time (seconds) of cpd_hnf in the setup from Figure 2.

Fig. 4. The $\log_{10}$ of the size factor $\mu$ of cpd_ddl relative to cpd_hnf. The value $a$ means that cpd_ddl consumes $10^a \times$ the memory cpd_hnf needs to store $R_I(d, e)$.

reason was that our Matlab driver routine consumed more than 162 GB of memory while running through all configurations.

The accuracy of cpd_hnf and cpd_ddl in this set of experiments is shown in Figure 2. The newly proposed algorithm is consistently several orders of magnitude more accurate in terms of the relative backward error than the state of the art.

It took over 146 h to obtain the (incomplete) results in the left panel of Figure 2(b). To put this in perspective, we note that the computation for the left panel of Figure 2(a) took a little over 17 h. The missing values inside of the triangles in Figure 2(b) indicate that cpd_ddl wanted to allocate an auxiliary matrix that would require more than 16.2 GB of memory. The same memory constraint was also imposed on cpd_hnf, but here only the largest case with $d = 4$ and $m = n = 14$ could not be treated. The red pixel for $d = 2, m = 35, n = 13$ in Figure 2(a) indicates that cpd_hnf gave inaccurate results. This is the only case where eigs failed to find a sufficiently accurate nullspace.

The timings of our Julia implementation are shown in Figure 3. As cpd_ddl is implemented in a different programming language, we believe a direct comparison in timings is not opportune. Nevertheless, in both algorithms computing the kernel of the auxiliary matrix has the highest asymptotic time complexity. In cpd_ddl it is an $L \times L$ square matrix, and in our algorithm, depending on the choice of degree, $R_I(d, e)$ is an almost-square $M \times N$ matrix with $M \approx N$. Therefore, we decided to plot the ratio between the number of elements in the auxiliary matrix of cpd_ddl and cpd_hnf. Figure 4 visualizes this factor $\mu = L^2/(MN)$ in a logarithmic scale. This number indicates the fraction of memory that cpd_ddl requires relative to cpd_hnf. Raising it to the power $\frac{3}{2}$ gives an estimate of the speedup factor in execution time of cpd_hnf relative to cpd_ddl. Based on this estimate and the execution times we logged, it is accurate to state that the newly proposed algorithm outperforms the state of the art by up to two orders of magnitude for larger tensors.
6.4 Robustness in the Noisy Case

The next experiment illustrates that cpd_hnf can successfully decompose tensors even in the presence of some noise. The setup is as follows. We generate a random rank-$r$ tensor $A$ of size $150 \times 25 \times 10$ by randomly sampling the $\alpha_i, \beta_i,$ and $\gamma_i$ as before. Then, we add white Gaussian noise of relative magnitude $10^e$ for $e = -1, \ldots, -15$; that is, we compute $A' = A + 10^e \frac{\|A\|_F}{\|E\|_F} E$. We provide $A'$ as input to our algorithm and request a rank-$r$ decomposition.

The relative backward error between $A'$ and the computed rank-$r$ decomposition is shown in a logarithmic scale in Figure 5. Because of our setup, the rank-$r$ CPD of $A$ has relative backward error $10^e$. A good tensor decomposition algorithm should thus return a CPD with a backward error of at most $10^e$. Remarkably, for tensors with random rank-$r$ CPDs, the proposed algorithm consistently manages to reach this benchmark when $e \leq -5$. For ranks up to about half the maximum range (from $r = 1$ to 70), it even consistently manages to reach this benchmark for white noise of magnitude at most $10^{-2}$! Based on these results, we anticipate that cpd_hnf could be employed as a rank-$r$ approximation algorithm in the high signal-to-noise regime. We believe this observation warrants further research.

6.5 An Example of Higher-order Tensors

The last experiment illustrates that the reshaping trick combined with a decomposition algorithm that works well in the unbalanced regime (such as cpd_hnf and cpd_ddl [Domanov and De Lathauwer 2014, 2017]) is a powerful technique for decomposing high-order, high-rank tensors, even with a balanced shape.

As an example, we generated an eighth-order real tensor of size $7 \times 7 \times 7 \times 7 \times 6 \times 6 \times 5 \times 5$ and rank $r = 1,000$ with factor matrices whose entries are sampled i.i.d. from a standard normal distribution. Here is what happens:

Grouping $[[4, 6, 7, 8], [1, 2, 3], [5]]$ and reshaped to $(1050, 343, 6)$ tensor in 0.810859903 s
1. Performed ST-HOSVD compression to $(1000, 343, 6)$ in 0.44888179 s
   Swapped factors 2 and 3, so the tensor has size $(1000, 6, 343)$
   Selected degree increment $d_0 = [1, 0]$
2. Constructed kernel of $A_1$ of size $(1000, 2058)$ in 22.33814068 s
3. Constructed resultant map of size $(7203, 6348)$ in 72.176802896 s
4. Constructed res res’ in 1.266772414 s
5. Computed cokernel of size $(1000, 7203)$ in 108.33285903 s
6. Constructed multiplication matrices in 2.037837294 s
7. Diagonalized multiplication matrices and extracted solution in 78.097176017 s
8. Refined factor matrices Y and Z in 151.17006114 s
9. Recovered factor matrix X in 0.186951757 s
10. Recovered the full factor matrices in 0.9396202759999999 s
Computed tensor rank decomposition in 440.457582263 s
Relative backward error = 3.873171296624731e-15
Relative forward error = 7.303893102189592e-14

To our knowledge, this computation represents the first time any tensor decomposition algorithm of any type (i.e., alternating least squares, optimization-based methods, direct algebraic methods, homotopy-based methods, or heuristic methods) successfully decomposes a high-order rank-1000 tensor that cannot be reshaped to an order-3 tensor whose CPD can be computed with a pencil-based algorithm.

7 CONCLUSIONS
The cpd_hnf algorithm proposed in this article computes the CPD of tensors satisfying Assumption 2.1 using numerical linear algebra techniques for solving systems of polynomial equations. Its complexity is governed (Proposition 5.1 and Theorem 1.1) by the regularity of a homogeneous, \( \mathbb{N}^2 \)-graded ideal obtained from a flattening. We presented a formula for degrees (\( d, 1 \)) in the regularity for generic tensors of many formats (Theorem 4.2) and proposed Conjecture 1 for the general case. Our experiments show that cpd_hnf produces backward errors that are almost always of the order of the machine precision. This improves upon the previous state of the art by several orders of magnitude (see Figure 2). In the high signal-to-noise-ratio regime, it seems the algorithm can be used to approximate noisy rank-\( r \) tensors.

Possible directions for future research include a further analysis of the regularity, a relaxation of the conditions in Assumption 2.1, generalizations for (semi-)symmetric tensors, and a theoretical analysis of cpd_hnf in the noisy setting.

APPENDIX
A PROOF OF THE TECHNICAL RESULTS
For two ideals \( I, K \subseteq S \), we write \( (I : K^\infty) = \{ f \in S \mid K^k f \subseteq I \text{ for some } k \in \mathbb{N} \} \). The next result follows from Lemma 2.2.

**Corollary A.1.** Under Assumption 2.1, we have the identity \( (I : K^\infty) = J \), where \( I = \langle \ker A_{(1)} \rangle \), \( J \) is the vanishing ideal

\[
J = \langle f \in S \mid f \text{ is homogeneous and } f(\zeta_i) = 0, i = 1, \ldots, r \rangle,
\]

and \( K = \langle x_i y_j \mid 0 \leq i \leq m, 0 \leq j \leq n \rangle \) is the irrelevant ideal of \( S \).

**Proof.** By Lemma 2.2, the modules \( S/I \) and \( S/J \) define the same subscheme of \( X \) consisting of \( r \) simple points. It follows that \( K^k(J/I) = 0 \) for large enough \( k \). See, for instance, Proposition 5.3.10 in Cox et al. [2011]. □

**Proof of Lemma 3.1.** This follows from the fact that all \( h \in S_{(d', e')} \) satisfying \( h(\zeta_i) = 0 \) lie on a hyperplane \( H_i \) through the origin in the \( \mathbb{C} \)-vector space \( S_{(d', e')} \). Any \( h_0 \) in the complement of \( H_1 \cup \cdots \cup H_r \) satisfies \( h_0(\zeta_i) \neq 0, i = 1, \ldots, r \). □

**Proof of Lemma 3.2.** Let \( J = (I : K^\infty) \) be the saturation of \( I \) with respect to the irrelevant ideal \( K \) (see Corollary A.1). First, it follows from Propositions 4.4 and 6.7 in Maclagan and Smith [2004] that there exists \( (\delta_1, \epsilon_1) \in \mathbb{N}^2 \) such that \( HF_{S/J}(\delta_1 + \delta', \epsilon_1 + \epsilon') = r \) for all \( (\delta', \epsilon') \in \mathbb{N}^2 \). Second, we show that there exists \( (\delta_2, \epsilon_2) \in \mathbb{N}^2 \) such that \( J(\delta_2 + \delta', \epsilon_2 + \epsilon') = J(\delta_2 + \delta', \epsilon_2 + \epsilon') \) for all \( (\delta', \epsilon') \in \mathbb{N}^2 \).

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To see this, note that $K^k J \subset I$ for some $k \in \mathbb{N}$ (see, e.g., Chapter 4, Section 4, Proposition 9 in Cox et al. [2013]) and $K^k = \langle S_{(k,k)} \rangle$. For a finite set of homogeneous generators $g_1, \ldots, g_m$ of $J$, choose $(\delta_2, e_2)$ such that entry-wise, $(\delta_2, e_2) \geq (k, k) + \deg(g_i), i = 1, \ldots, m$. Take $(d, e) \in (\max(\delta_1, \delta_2), \max(e_1, e_2)) + \mathbb{N}^2$ such that $(d, e) \neq (1, 1)$ and $(d - 1, e - 1) \geq (0, 0)$.

**Proof of Theorem 3.2.** Under the assumptions of the theorem, $((0, 1), (1, 0))$ is a regularity pair for the ideal $J = (I : K^{\omega})$, defining the same set of points $V_X(J) = V_X(I)$ (see Corollary A.1). The statement now follows from Theorem 5.5.3, Propositions 5.5.4 and 5.5.5 in Telen [2020b].

**Proof of Item (3) in Theorem 4.2.** We prove the case $m = 2$ in item (3) of the theorem. It suffices to show that for some configuration $Z$, the matrix

$$A' = \begin{bmatrix} -\Gamma H_0 & \Gamma H_2 \\ \Gamma H_0 & -\Gamma H_1 \end{bmatrix}$$

has full rank. We set $\beta_{i0} = 1, i = 1, \ldots, r$, such that $H_0$ is the identity matrix. Condition (4.8) ensures that $A'$ has more rows than columns, so it suffices to show that $\ker A' = \{0\}$. Suppose $A' \nu = 0$, then $\nu$ can be split into $\nu_1, \nu_2 \in C'$ such that $\Gamma \nu_1 = \Gamma \nu_2 = 0$. If $r \leq n + 1$, then it is clear that this implies $\nu_1 = \nu_2 = 0$ for generic $Z$. Therefore, we assume $r > n + 1$ and make the following choice for $\Gamma$:

$$\Gamma = \begin{bmatrix} 1 & \cdots & Y_{n+2,0} \\ & \ddots & \vdots \\ 1 & \cdots & Y_{n+2,n} \end{bmatrix} = \begin{bmatrix} \id_{n+1} & \hat{\Gamma} \end{bmatrix} \in \mathbb{C}^{(n+1) \times r},$$

where $\hat{\Gamma} \in \mathbb{C}^{(n+1) \times \kappa}$ is the submatrix of $\Gamma$ consisting of its last $\kappa = r - (n + 1)$ columns. We have that $\Gamma \nu_i = 0$ implies $\nu_i = \Gamma^\perp w_i$ for some $w_i \in C^\kappa$ and $\Gamma^\perp = \begin{bmatrix} -\hat{\Gamma} \end{bmatrix}$. Hence, $A \nu = 0$ is equivalent to $[\Gamma H_2 \Gamma^\perp - \Gamma H_1 \Gamma^\perp][w_i] = 0$. Condition (4.8) implies $2\kappa \leq n + 1$, so that the coefficient matrix in this equation has more rows than columns. Upon closer inspection, we see that

$$\Gamma H_2 \Gamma^\perp = \begin{bmatrix} (\beta_{n+2,q} - \beta_{1,q})Y_{n+2,0} & \cdots & (\beta_{r,q} - \beta_{1,q})Y_{r,0} \\ (\beta_{n+2,q} - \beta_{2,q})Y_{n+2,1} & \cdots & (\beta_{r,q} - \beta_{2,q})Y_{r,1} \\ \vdots & \ddots & \vdots \\ (\beta_{n+2,q} - \beta_{n+1,q})Y_{n+2,n} & \cdots & (\beta_{r,q} - \beta_{n+1,q})Y_{r,n} \end{bmatrix} \begin{bmatrix} (\beta_{j,q} - \beta_{i+1,q})Y_{ji} \end{bmatrix}_{0 \leq i \leq n, 0 \leq j \leq r}.$$  

To make rows $\kappa + 1, \ldots, 2\kappa$ equal to zero in $\Gamma H_2 \Gamma^\perp$, we set $\beta_{\kappa+1,2} = \beta_{\kappa+2,2} = \cdots = \beta_{2\kappa,2} = \beta_{n+2,2} = \cdots = \beta_{n+3,2} = \cdots = \beta_{r,2} = 1$. All other $\beta$-coordinates are chosen at random, such that all entries of $[\Gamma H_2 \Gamma^\perp - \Gamma H_1 \Gamma^\perp]$, except those in the rows $\kappa + 1, \ldots, 2\kappa$ of $\Gamma H_2 \Gamma^\perp$, are of the form $\star Y_{ji}$, with $\star$ some non-zero complex number. Then,

$$[\Gamma H_2 \Gamma^\perp - \Gamma H_1 \Gamma^\perp] = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \\ D_1 & D_2 \end{bmatrix},$$

where $C_{ij} \in \mathbb{C}^{\kappa \times \kappa}$ and $D_i \in \mathbb{C}^{(n+1-2\kappa) \times \kappa}$. The minor corresponding to the first $2\kappa$ rows is seen to be $\det(C_{11}) \det(C_{22})$. This is a product of two nonzero polynomials in the parameters $Y_{ji}, i = 0, \ldots, n, j = n + 2, \ldots, r$. For generic choices of the parameters, this minor is non-zero, hence $w_1 = w_2 = 0$, and thus $\nu = 0$ and $\ker A' = \{0\}$. 

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