LINEAR ORBITS OF ALTERNATING FORMS ON REAL VECTOR SPACES

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Abstract. In this note we complete the calculation of the number of $GL(\mathbb{R}^n)$-orbits on $\Lambda^k(\mathbb{R}^n)^*$, by treating the cases $(n, k) = (7, 4)$ and $(8, 5)$ not covered in the literature. We also calculate the number of of non-degenerate and stable orbits, as they are of special interest to multisymplectic and special geometry.

INTRODUCTION

The last decades have seen a growing interest in multisymplectic geometry (cf., e.g., \cite{3,4,6,10} for research in this area and \cite{1,7} for the rôle of non-degenerate three-forms in special geometries). A manifold $M$ together with a differential form $\alpha$ of degree $k$ is called multisymplectic if $\alpha$ is closed and non-degenerate. The latter condition means that the map

$$T_pM \to \Lambda^{k-1}(T_pM)^*, \ v \mapsto \iota_v(\alpha_p)$$

is injective for all $p \in M$, where $\iota_v(\alpha_p)$ denotes the contraction of a vector $v$ into the $k$-form $\alpha_p$ on $T_pM$.

In contrast to the symplectic case (i.e., the case $k = 2$) already the linear theory is not trivial for general $k$. In fact, there are typically many different equivalence classes even for non-degenerate alternating $k$-forms on a real vector space $V$.

Since the knowledge of these linear equivalence classes (or linear types) are fundamental for multisymplectic geometry, we give here a complete answer to the following question:

Does the natural $GL(V)$-action on $\Lambda^k(V^*)$ have a finite or infinite number of orbits and what is this number in the former case?

Furthermore, we answer these questions also for non-degenerate and stable orbits (i.e. orbits of non-degenerate and stable forms), as well. We analyse the cases $(k, n) = (4, 7)$ and $(5, 8)$ that were hitherto not covered by the literature and resume all cases in Theorem 2.1 with proofs for the two cases mentioned above (Proposition 4.1 and Proposition 4.2) and references to proofs for all other cases.

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1. Preliminaries

Throughout this note, let $V$ be a finite-dimensional real vector space of dimension $n > 1$ and $k \in \{0, ..., n\}$ a natural number.

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Remark 1.4. The choice of a linear isomorphism from $V$ to $V^*$ induces an isomorphism $\Lambda^k(V) \to \Lambda^k(V^*)$, which maps orbits of $\theta_k$ to orbits of $\theta^k$. Thus the classification of orbits of $\theta_k$ is equivalent to the classification of orbits of $\theta^k$. In terms of $\Lambda^k(V)$ degeneracy of a multivector $\xi$ is equivalent to the existence of a subspace $W \subseteq V$ such that $\xi \in \Lambda^k(W) \subseteq \Lambda^k(V)$. We choose the perspective of orbits in $\Lambda^k(V^*)$, as it has a more immediate relation to multisymplectic geometry.

Definition 1.1. We define the natural left action of $GL(V)$ on $\Lambda^k(V)$ (resp. the natural right action of $GL(V)$ on $\Lambda^k(V^*)$) by

$$\theta_k : GL(V) \times \Lambda^k(V) \to \Lambda^k(V), \quad \theta_k, g \left( \sum_i v_i^1 \wedge ... \wedge v_i^k \right) = \sum_i (gv_i^1)^\wedge ... \wedge (gv_i^k)^\wedge$$

$$\theta^k : GL(V^*) \times \Lambda^k(V^*) \to \Lambda^k(V^*), \quad \theta^k_g(\alpha) = \alpha \circ \theta_k, g.$$  

For $g \in GL(V), \xi \in \Lambda^k(V)$ and $\alpha \in \Lambda^k(V^*)$, we will write $g \cdot \alpha$ for $\theta_k, g(\xi)$ and $g \cdot \alpha$ for $\theta^k_g(\alpha)$.

We will be especially interested in orbits of non-degenerate forms and stable orbits in $\Lambda^k(V^*)$, as they play an important role in multisymplectic geometry.

Definition 1.2.

1. We define the contraction map $V \times \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ by $(v, \alpha) \mapsto \iota_v \alpha$, where $\iota_v \alpha(v_1, ... , v_{k-1}) = \alpha(v, v_1, ... , v_{k-1})$.
2. We uniquely extend the contraction to a linear map $\Lambda^i(V) \times \Lambda^k(V^*) \to \Lambda^{i-1}(V^*)$ for any $t$, such that $\iota_{\xi \wedge \zeta} \alpha = \iota_\xi (\iota_\zeta \alpha)$ holds for any $\xi \in \Lambda^i(V)$ and $\zeta \in \Lambda^j(V)$.
3. An element $\alpha \in \Lambda^k(V^*)$ is called non-degenerate if the map $V \to \Lambda^{k-1}(V^*), v \mapsto \iota_v \alpha$ is injective and degenerate otherwise.
4. A multivector $\xi \in \Lambda^k(V)$ resp. a form $\alpha \in \Lambda^k(V^*)$ is called stable if the orbit $GL(V) \cdot \xi \subset \Lambda^k(V)$ resp. $GL(V) \cdot \alpha \subset \Lambda^k(V^*)$ is open and unstable otherwise.

Remark 1.3. Whenever at least one non-degenerate element exists in $\Lambda^k(V^*)$, all degenerate elements are automatically unstable.

As stability and non-degeneracy are preserved by the $GL(V)$-action, we will call an orbit $GL(V) \cdot \alpha \subset \Lambda^k(V^*)$ non-degenerate (resp. stable) if $\alpha$ is.

Remark 1.4. The choice of a linear isomorphism from $V$ to $V^*$ induces an isomorphism $\Lambda^k(V) \to \Lambda^k(V^*)$, which maps orbits of $\theta_k$ to orbits of $\theta^k$. Thus the classification of orbits of $\theta_k$ is equivalent to the classification of orbits of $\theta^k$. In terms of $\Lambda^k(V)$ degeneracy of a multivector $\xi$ is equivalent to the existence of a subspace $W \subseteq V$ such that $\xi \in \Lambda^k(W) \subset \Lambda^k(V)$. We choose the perspective of orbits in $\Lambda^k(V^*)$, as it has a more immediate relation to multisymplectic geometry.

2. Statement of the theorem

Theorem 2.1. Let $V$ be a real vector space of dimension $n > 1$ and $k \in \mathbb{N}$. The natural action of $GL(V)$ on $\Lambda^k(V^*)$ has the following number of orbits (resp. non-degenerate orbits and stable orbits) depending on $n$ and $k$. 

2 LEONID RYVKIN
| $k = 1$ | $2$ | $0$ | $1$ |
| $k = 2$, $n$ even | $\frac{n}{2} + 1$ | $1$ | $1$ |
| $k = 2$, $n$ odd | $\frac{n+1}{2}$ | $0$ | $1$ |
| $k = 3$, $n = 6$ | $6$ | $3$ | $2$ |
| $k = 3$, $n = 7$ | $14$ | $8$ | $2$ |
| $k = 3$, $n = 8$ | $35$ | $21$ | $3$ |
| $k = 4$, $n = 7$ | $20$ | $15$ | $4$ |
| $k = 4$, $n = 8$ | $\infty$ | $\infty$ | $0$ |
| $k = 5$, $n = 8$ | $35$ | $31$ | $3$ |
| $3 \leq k \leq n - 3$, $n \geq 9$ | $\infty$ | $\infty$ | $0$ |
| $k = n - 2$, $n \geq 6$, $n = 2 \mod 4$ | $\frac{n}{2} + 2$ | $\frac{n}{2}$ | $2$ |
| $k = n - 2$, $n \geq 5$, $n \neq 2 \mod 4$ | $\left\lceil \frac{n}{2} \right\rceil + 1$ | $\left\lceil \frac{n}{2} \right\rceil - 1$ | $1$ |
| $k = n - 1$ | $2$ | $0$ | $1$ |
| $k = n$ | $2$ | $1$ | $1$ |

**Proof.** The cases $k \in \{0, 1, 2, n-2, n-1, n\}$ and the cases with infinite number of orbits have been solved in [9]. The remaining cases for $k = 3$ have been completed in [5] together with the stability considerations in [8]. We settle the remaining cases $(n, k) = (7, 4)$ in Proposition 4.2 and $(n, k) = (8, 5)$ in Proposition 4.1. \(\square\)

**Remark 2.2.** If two $k$-forms on $V$ lie in the same $GL(V)$-orbit, one says that they are “equivalent” or they have the same “linear type”.

### 3. Preparations

In the sequel we will use the following technique which was used in [9] to identify the orbits of $(n-2)$-forms in an $n$-dimensional vector space.

**Lemma 3.1.** Let $\Omega \in \Lambda^n(V^*)$ be a volume form and $k \in \{0, ..., n\}$. Then the linear isomorphism $c: \Lambda^k(V) \to \Lambda^{n-k}(V^*)$, $\xi \mapsto c(\xi)\Omega$ has the following properties:

1. The isomorphism $c$ preserves stability.
2. For all $g \in GL(V)$ we have $g \cdot c(\xi) = det(g)c(g^{-1}\xi)$.
3. If $\xi$ lies in the orbit of $\zeta$, then $c(\xi)$ lies in the orbit of $c(\zeta)$ or in the orbit of $-c(\zeta)$. When $n-k$ is odd those two orbits coincide.
4. If $\xi$ and $\zeta$ lie in the same orbit, then $\xi$ lies in the orbit of $\zeta$ or in the orbit of $-\zeta$. When $k$ is odd those two orbits coincide.

**Proof.**

1. This follows directly from the continuity of linear maps between finite-dimensional vector spaces.
2. This follows from the formula $g \cdot (\iota_\xi \Omega) = \iota_{g^{-1}\xi}(g\Omega)$ and the fact that $g \cdot \Omega = \theta^p_\xi(g\Omega) = det(g)\Omega$ by the definition of the determinant.
3. If $\xi = g \cdot \zeta$, then $c(\xi) = \frac{1}{det(g)}g \cdot c(\xi)$, so $c(\xi)$ lies in the orbit of $det(g)c(\zeta)$.

By applying the endomorphism $-\sqrt{det(g^{-1})} \cdot id_V$ we see that $c(\xi)$ lies in the orbit of sign(det(g))c(\zeta). The second part of the statement follows from the identity $(\lambda \cdot id_V) \cdot \alpha = \lambda^{n-k} \alpha$ for $\alpha \in \Lambda^{n-k}(V^*)$ and $\lambda \in \mathbb{R}^*.$
4. The proof of this part is analogous to that of part (3). \(\square\)

The following lemma will help us deduce the number of non-degenerate orbits, once we know the total number of orbits.

**Lemma 3.2.** The degenerate orbits of $k$-forms on a $n$-dimensional vector space are in one-to-one correspondence to all orbits of $k$-forms on a $n-1$-dimensional vector space.
Proof. Let \( W \) be a \((n-1)\)-dimensional subspace of \( V \) and \( a \in V \setminus W \). Then \( V = W \oplus \mathbb{R} \cdot a \). Let \( \operatorname{pr}_W : V \to W \) be the projection. We show that the \( GL(W) \)-orbits of \( \Lambda^k(W^*) \) are in one-to-one correspondence to \( GL(V) \)-orbits of \( \Lambda^k_{\text{deg}}(V^*) \), the degenerate \( k \)-forms on \( V \), via the map

\[
\tilde{\phi} : \frac{\Lambda^k(W^*)}{GL(W)} \to \frac{\Lambda^k_{\text{deg}}(V^*)}{GL(V)}, \quad [a] = GL(W) \cdot a \mapsto GL(V) \cdot \phi(a) = [\phi(a)],
\]
induced by

\[
\phi : \Lambda^k(W^*) \to \Lambda^k(V^*) \oplus (\Lambda^{k-1}(W^*) \otimes (R \cdot a)^*) \cong \Lambda^k(V^*), \ a \mapsto (\alpha \oplus 0).
\]

Surjectivity of \( \tilde{\phi} \). Assume \( \alpha \in \Lambda^k(V^*) \) is degenerate, i.e. there exists a \( v \) in \( V \setminus \{0\} \) such that \( \iota_v \alpha = 0 \). Choosing a \( g \in GL(V) \) such that \( g^{-1}v = a \), we get \( \iota_g(g \cdot \alpha) = 0 \), i.e. \( (g \cdot \alpha)|W \in \Lambda^k(W^*) \) satisfies \( \phi((g \cdot \alpha)|W) = g \cdot \alpha \). Consequently \( \tilde{\phi}([\{g \cdot \alpha\}|W]) = [a] \) holds and the map \( \tilde{\phi} \) is surjective.

Injectivity of \( \tilde{\phi} \). Let \( \alpha, \beta \in \Lambda^k(W^*) \) and \( \phi(\alpha) \) be equivalent to \( \phi(\beta) \), i.e. there exists a map \( g \in GL(V) \) such that \( g \cdot \phi(\alpha) = \phi(\beta) \). Especially \( ga \) and \( a \) are both elements of the vector subspace \( \text{ann}(\phi(\alpha)) = \{v \in V | \iota_v \phi(\alpha) = 0\} \). We pick an element \( h \in GL(V) \) such that \( h(ga) = a \) and \( h \cdot \phi(\alpha) = \phi(\alpha) \). Then we have \( (hg) \cdot \phi(\alpha) = g \cdot (h \cdot \phi(\alpha)) = g \cdot \phi(\alpha) = \phi(\beta) \), where \( hg(a) = a \). Consequently \( \tilde{\phi} \circ (hg)|W \) yields a well-defined automorphism of \( W \) satisfying \( \phi((g \cdot \alpha)|W) \cdot \alpha = \beta \). Hence, \([a] = [\beta]\) and \( \tilde{\phi} \) is injective.

\( \square \)

4. PROOFS OF THE REMAINING CASES

**Proposition 4.1.** Let \( V \) be 8-dimensional. The natural action of \( GL(V) \) on \( \Lambda^5(V^*) \) has 35 orbits, 31 of those orbits are non-degenerate and three are stable.

**Proof.** We fix a volume form \( \Omega \) on \( V \). In the case at hand, \((n, k) = (8, 3)\), both \( k \) and \( n-k \) are odd. Parts (3) and (4) of Lemma 3.5 imply that the map \( c \) induces a bijection between the orbits of \( \Lambda^5(V^*) \) and the orbits of \( \Lambda^3(V) \). This bijection preserves stable orbits. Thus the total number of orbits and the number of stable orbits correspond to the numbers known from the case \((n, k) = (8, 3)\). To get the number of non-degenerate orbits one observes that in the case \((n, k) = (7, 5)\) there are 4 orbits and applies Lemma 3.2. \( \square \)

**Proposition 4.2.** Let \( V \) be 7-dimensional. The natural action of \( GL(V) \) on \( \Lambda^4(V^*) \) has 20 orbits, 15 of those orbits are non-degenerate and 4 are stable.

**Proof.** We fix a volume form \( \Omega \) on \( V \). Let \( \xi_1, ..., \xi_{14} \) be representatives of the orbits in \( \Lambda^3(V) \), such that \( \xi_1, ..., \xi_6 \) are degenerate and \( \xi_7 \) and \( \xi_{14} \) are stable. The degree \( k = 3 \) of the multivectors \( \xi \) is odd, so by Lemma 3.4 part (4), we know that the orbits of \( c(\xi_i) \) and \( c(\xi_j) \) are different for \( i \neq j \). Identifying the orbits in \( \Lambda^4(V^*) \) thus amounts to finding out for which \( i \) the orbits of \( c(\xi_i) \) and \( -c(\xi_i) \) coincide. Lemma 3.4 part (2) implies that the orbits coincide if and only if the stabilizer \( \text{Stab}(\xi_i) \subset GL(V) \) contains elements of negative determinant.

Fix \( i \in \{1, ..., 6\} \), i.e. \( \xi_i \) is degenerate. Then there exists an \( a \in V \setminus \{0\} \) such that \( \iota_a \xi_i = 0 \) and \( W \subset V \), a complement of \( \mathbb{R} \cdot a \). The element \( g \in GL(V) \) with \( g(a) = -a \) and \( g|_W = id_W \) has negative determinant and stabilizes \( \xi_i \). In the remaining cases \( i \in \{7, ..., 14\} \) the explicit list of stabilizers given in [2] yields the following:

- \( i \in \{7, ..., 12\} \): Two of the stabilizers \( \text{Stab}(\xi_i) \) of non-degenerate non-stable orbits contain elements of negative determinant
- \( i \in \{13, 14\} \): The stabilizers \( \text{Stab}(\xi_i) \) of stable orbits do not contain elements of negative determinant.
Consequently there are $8 + 2 \cdot 6 = 20$ types of orbits, four of which are stable. We get 15 as the number of non-degenerate orbits by Lemma 3.2.

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