A UNIFYING APPROACH FOR ROLLING SYMMETRIC SPACES

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Abstract. The main goal of this paper is to present a unifying theory to describe the pure rolling motions of Riemannian symmetric spaces, which are submanifolds of Euclidean or pseudo-Euclidean spaces. Rolling motions provide interesting examples of nonholonomic systems and symmetric spaces appear associated to important applications. We make a connection between the structure of the kinematic equations of rolling and the natural decomposition of the Lie algebra associated to the symmetric space. This emphasises the relevance of Lie theory in the geometry of rolling manifolds and explains why many particular examples scattered through the existing literature always show a common pattern.

1. Introduction. Riemannian symmetric spaces are an important class of Riemannian manifolds that arise in a wide variety of applications, in particular in engineering problems running from path-planning of mobile robots to information geometry and emerging from areas, such as image processing, machine learning and data analysis.

The theory of Riemannian symmetric spaces was initiated and vigorously developed by É. Cartan in the late 1920s, as mentioned in the Preface of one of the most standard references concerning this theory, Helgason’s book [20]. The literature about theoretical and applied developments of this interesting area of research has...
increased significantly along the years. Here, without being exhaustive, we refer to [3, 4, 5, 6] for important contributions of A. Bloch and collaborators. These also inspired our present work.

Examples of symmetric spaces, that became popular in the already mentioned applied areas and related literature are, for instance: the Graßmann manifold, each point of which is a subspace, associated to a set of images after applying principal component analysis (PCA) [18, 22, 48, 51]; the Essential manifold, which parameterises the epipolar constraint encoding the relation between correspondences across two images of the same scene taken from two different locations [19, 46]; the manifold of special orthogonal matrices which plays an important role in robotics [9, 10, 39] and in biomedical and biomechanical applications, ranging from image enhancement and image restoration techniques to feature and object detection tasks and representations of human skeleton [34, 42, 49].

In the context of these applications, interpolation problems on Riemannian symmetric spaces become quite natural. For instance, image interpolation, one of the most elementary image processing tasks, can be formulated as an interpolation problem on the Graßmann manifold. This happens in many active vision applications, such as in time-varying dynamic models for human activities [47, 48]. Also, when human skeleton is represented as point in a Lie group which is the product of several copies of the 3-dimensional rotation group, to perform human action recognition from several skeletal data corresponds to solve an interpolation problem on that Lie group, as described in [49].

Several interpolation techniques have been proposed in the literature when the data is represented on some manifold. One which has been shown to be effective is based on rolling motions of the manifold over another manifold of the same dimension, subject to nonholonomic constraints of no-slip and no-twist. The main idea behind this approach is to use rolling motions to project the data from the manifold to a flat space where classical interpolation methods can be applied, and thereafter rolling back the solution in order to solve the initial problem on the manifold with a more complicated geometry.

The mathematical formulation of rolling motions of sub-Euclidean manifolds first appeared in [40] and [44] and led to a kinematic interpretation of Levi-Civita and normal connections on submanifolds. These were followed by the work of several authors who studied rolling motions of particular manifolds embedded in Euclidean or pseudo-Euclidean spaces. Without being exhaustive, we refer the work about rolling bodies in [1], the work of Anthony Bloch and collaborators about the rolling sphere and its connection with quantum spins in [2], [7] and [43], the optimality properties of rolling motions of spaces of constant curvature studied in [26] and [52], the kinematics of rolling orthogonal groups and Graßmann manifolds in [24]. More recently, generalisations of rolling manifolds embedded in a general Riemannian manifold were presented in [23]. Extension of rolling notions to certain pseudo-Euclidean manifolds have also appeared in [29] for the Lorentzian sphere, and in [13] and [38] for pseudo-orthogonal groups and symplectic groups. In recent years, an intrinsic viewpoint has been successfully applied to study rolling motions. In this new approach, the geometric description of the rolling does not depend on any embedding. Examples of works that follow this viewpoint are, for instance, [8, 11, 12, 17], and [37]. While the intrinsic viewpoint might be considered theoretically stronger, from the perspective of applications the extrinsic approach is more effective.
In this paper, we focus on rolling motions of symmetric homogeneous spaces over their affine tangent space at a point, both embedded in a Euclidean or pseudo-Euclidean space. The results presented here extend the results contained in [31]. Our main goal is to find a unifying theory that incorporates all the existing scattered results and explains the common pattern that is observed in the kinematic equations of many rolling motions.

The structure of the paper is the following. Section 2 introduces the necessary background, including the definition of rolling subject to the nonholonomic constraints of no-slip and no-twist, and the fundamentals of homogeneous symmetric spaces. The main contributions are given in Section 3. The most relevant result is stated in Theorem 3.2, where a strong relationship between rolling maps and the structure of the Lie algebra associated to the symmetric space is revealed. Theorem 3.3 shows how to generate left-invariant parallel vector fields on symmetric spaces from rolling maps. These results are illustrated by several examples. The first three examples concern manifolds embedded in Euclidean spaces. Example 1 shows how the Lie algebra forces the structure of the kinematic equations for the rolling sphere and this is also illustrated with the Graßmann manifold in Example 2 and the Essential manifold in Example 3. The next two examples deal with pseudo-Euclidean manifolds: the Lorentzian sphere, in Example 4, and quadratic Lie groups in Example 5. We conclude the paper with a few remarks.

2. Preliminaries. We begin this section with the notion of rolling one manifold upon another manifold of the same dimension, both embedded in some Riemannian or pseudo-Riemannian manifold $M$. Certain restrictions on these manifolds will be assumed whenever necessary. The second part of this section contains essential facts concerning homogeneous spaces that are needed for the main results presented in Section 3.

2.1. Rolling maps. In this subsection we give the definition of a rolling map for the rolling motion of $M$ upon $M_0$. This definition, which is a generalisation of Sharpe’s definition [44, Appendix B] for Euclidean submanifolds, extends naturally to a more general situation, where the embedding space $\tilde{M}$ is an orientable Riemannian or pseudo-Riemannian manifold, cf. [23].

In what follows, $\tilde{M}$ denotes an ambient space, $\mathcal{J}(\tilde{M})$ denotes the Lie group of isometries on $\tilde{M}$ and $I \subset \mathbb{R}$ is a closed interval. Derivatives with respect to the parameter $t$ will be denoted by the ‘dot’, e.g., $\dot{\chi}$ denotes partial derivative of $\chi$ with respect to $t$. “$\ast$” refers to the partial derivative with respect to the space coordinates and $T^\perp$ to the normal vector bundle. Figure 1 illustrates the rolling motion of the 2-sphere $S^2$ upon a two-dimensional surface along development curve $\sigma_0$.

Definition 2.1. Let $M$ and $M_0$ be two $n$-submanifolds of $\tilde{M}$, both orientable and isometrically embedded in $\tilde{M}$. A rolling map of $M$ on $M_0$ without slipping or twisting is a map $\chi: I \to \mathcal{J}(\tilde{M})$ satisfying the following conditions.

Rolling: There is a piecewise smooth curve $\sigma: I \to \tilde{M}$, called the rolling curve on $M$, such that, for all $t \in I$:

- $\chi(t)(\sigma(t)) \in M_0$, and
- $T_{\chi(t)(\sigma(t))}(\chi(t)(\tilde{M})) = T_{\chi(t)(\sigma(t))}M_0$.

The curve $\sigma_0: I \to M_0$ defined by $\sigma_0(t) := \chi(t)(\sigma(t))$ is called the development curve of $\sigma$.

No-slip: For all but a finite number of $t \in I$:
\[ \dot{\chi}(t) \chi(t)^{-1} (\sigma_0(t)) = 0. \]

**No-twist**: The two complementary conditions, for all but a finite number of \( t \in I \):
- **tangential**: \( (\dot{\chi}(t) \chi(t)^{-1})_* (T_{\sigma_0(t)} M_0) \subset T_{\sigma_0(t)}^\perp M_0 \), and
- **normal**: \( (\dot{\chi}(t) \chi(t)^{-1})_* (T_{\sigma_0(t)}^\perp M_0) \subset T_{\sigma_0(t)} M_0. \)

Figure 1 illustrates the rolling motion of the 2-sphere \( S^2 \) upon a two-dimensional surface along development curve \( \sigma_0 \).

**Unraveling the abstract definition above**, the *rolling* condition imply that, at each point of contact, both manifolds, \( M_0 \) and \( \chi(t)(M) \), share the same tangent space. This is identified as a subspace of the tangent space of \( \tilde{M} \) at the considered point. The *no-slip* condition expresses the fact that the rolling and development curves have the same velocity at the point of contact, i.e., \( \dot{\sigma}_0(t) = \chi(t)_*(\dot{\sigma}(t)) \).

The two complementary *no-twist* conditions assert that the infinitesimal action \( (\dot{\chi}(t) \chi(t)^{-1})_* \) has no tangential component, when acting on the tangent space \( T_{\sigma_0(t)} M_0 \), and no normal component, when acting on the normal space \( T_{\sigma_0(t)}^\perp M_0 \), at the point of contact (Figure 2). Thus, as Sharpe in his book [44, page 379] has pointed out, in suitable coordinates in a neighbourhood of \( p \in M_0 \), Figure 2. Let \( q = \chi(t_0)(p) \) and suppose that \( W \in T_q \tilde{M} \) is a tangent vector at \( q \in \tilde{M} \). Then \( \tilde{W} = (\chi^{-1}(t_0))_*(W) \) is a tangent vector at \( p \). Thus, \( \tilde{W}(t) = (\chi(t))_*(\tilde{W}) \) is a vector field along \( \gamma: I \rightarrow \tilde{M} \), given by

\[
(\dot{\chi}(t) \chi(t)^{-1})_* = \begin{bmatrix}
0 & X_{n \times r} \\
-X_{n \times r}^T & 0
\end{bmatrix}
\begin{bmatrix}
T_p M_0 \\
T_p^\perp M_0
\end{bmatrix}
\]

In essence, our main result, Theorem 3.2 captures the structure of \( (\dot{\chi} \chi^{-1})_* \), expressed by the matrix in (1), that is carried from the Lie algebra of the symmetry group acting transitively on \( M \).

**Remark 1.** In the context of Definition 2.1, the composition \( (\dot{\chi} \chi^{-1})_* \) makes sense even for an arbitrary Riemannian or pseudo-Riemannian ambient space. For any fixed \( t_0 \in I \), choose \( p = \sigma(t_0) \in \tilde{M} \). Figure 2. Let \( q = \chi(t_0)(p) \) and suppose that \( W \in T_q \tilde{M} \) is a tangent vector at \( q \in \tilde{M} \). Then \( \tilde{W} = (\chi^{-1}(t_0))_*(W) \) is a tangent vector at \( p \). Thus, \( \tilde{W}(t) = (\chi(t))_*(\tilde{W}) \) is a vector field along \( \gamma: I \rightarrow \tilde{M} \), given by
Figure 2. A vector attached to a rolling manifold generates a vector field along the path \( \chi(t)(p) \) in the ambient space; thus a tangent (normal) vector in \( T_pM \) is isometrically carried to a tangent (normal) vector in \( T_q(\chi(t_0)(M)) = T_qM_0 \).

\[ \gamma(t) = \chi(t)(p). \] Now, \( \gamma(t_0) = q \) and by the no-slip condition \( \dot{\gamma}(t_0) = 0 \). Therefore \( (\dot{\chi}\chi^{-1})_*(\dot{W}) \), at \( t = t_0 \), is equal to \( \dot{W}(t_0) \), which in turn coincides with the covariant derivative \( \nabla_t \dot{W} \) in the ambient space, cf. [23] for a proof.

In this paper we are only concerned with the Euclidean \( \mathbb{R}^{n+k} \) or pseudo-Euclidean \( \mathbb{R}^{n,k} \) ambient space. Thus in this paper the group of isometries \( \mathcal{F}(M) \) is either the Euclidean group \( \mathbf{E}(n+k) \) or the pseudo-Euclidean group \( \mathbf{E}(k,n) \), respectively. The rolling manifold \( M \subset \tilde{M} \) is a (homogeneous) symmetric space and \( M_0 \) is the affine tangent space \( T_{p_0}M \), to \( M \) at some point \( p_0 \in M \).

We strongly believe that the geometric foundations developed here are applicable to more general ambient spaces. One particular situation where this happens is illustrated in Example 6, where \( \tilde{M} = \mathbb{R}^{n+1} \) is equipped with a left-invariant metric.

2.2. Symmetric Riemannian homogeneous spaces. This section gives a brief introduction to symmetric Riemannian homogeneous spaces. There is a vast literature on this subject, including [14, 20, 27, 30, 33, 41]. We refer to these books for more details.

Suppose that a connected Lie group \( \mathfrak{G} \) acts transitively on a Riemannian manifold \( M \), i.e.,

1. there exists a smooth map \( \tau: \mathfrak{G} \times M \rightarrow M \), called a translation, given by \( (a,p) \mapsto \tau_a(p) = a(p) \), such that, for any \( p \in M \), \( \tau_a(\tau_b(p)) = \tau_{ab}(p) \), for all \( a, b \in \mathfrak{G} \);
2. \( \tau_e(p) = p \), for any \( p \in M \), where \( e \) is the identity element of \( \mathfrak{G} \);
3. for any \( p, q \in M \), there exist \( a \in \mathfrak{G} \) such that \( q = \tau_a(p) \).

For an arbitrary fixed point \( p_0 \in M \) the closed subgroup

\[ H := \{ a \in \mathfrak{G} \mid \tau_a(p_0) = p_0 \} \]

is the isotropy group of \( \mathfrak{G} \) at \( p_0 \). Then, the smooth map \( \pi: \mathfrak{G} \rightarrow M \) given by \( \pi(a) = \tau_a(p_0) \) is a submersion. It can be easily seen that \( \tau_a \circ \pi = \pi \circ L_a \), where \( L_a: \mathfrak{G} \rightarrow \mathfrak{G} \) denotes left translation. This is illustrated by the following commuting diagram.
m = g

denote the Lie algebra of left invariant vector fields on

X into

(see, for instance, [33]). Hence, any vector field

space

Riemannian geometric point of view:

2.2.1. manifold, we first describe geometric implications of this submersion.

Ignoring for a moment the group structure and treating

G on

decomposition such that

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Then, the corresponding left invariant distribution

\[ \mathcal{H}_a = (L_a)_*(\mathfrak{m}) \]
on \( \mathfrak{g} \) is also right invariant under \( \text{Ad}_H \). Moreover, for any \( h \in H \)

\[ \pi(i_h(g)) = \tau_h(\pi(g)), \quad \text{for all } g \in \mathfrak{g}, \]
where \( i_a \) denotes the inner automorphism by \( a \in \mathfrak{g} \), i.e.,
\[
i_a := L_a \circ R_{a^{-1}} : \mathfrak{g} \to \mathfrak{g}
\]
\[
g \mapsto aga^{-1},
\]
whose differential, denoted by \( \text{Ad}_a := i_a^* \), is a group homomorphism called the adjoint representation of \( \mathfrak{g} \). By (9), for any \( X \in \mathfrak{m} \) and \( h \in H \),
\[
\pi_{e_h}(\text{Ad}_h X) = \tau_h(\pi_{e_h}(X))
\]
\[
\pi_{e_h} \circ \text{Ad}_h = \tau_h \circ \pi_{e_h}.
\]

The following commutative diagram illustrates the relations above.
\[
\begin{array}{ccc}
\mathfrak{m} & \xrightarrow{\text{Ad}_a} & \mathfrak{m} \\
\pi_* & & \pi_* \\
\mathcal{T}_{\gamma} \mathcal{M} & \xrightarrow{\tau_{a_*}} & \mathcal{T}_{\gamma} \mathcal{M}
\end{array}
\]

Since the metric on \( \mathcal{M} \) is invariant under \( \mathfrak{g} \), i.e., for any \( a \in \mathfrak{g} \) the push forward \( \tau_a^- \) is an isometry, the above relationship induces an \( H \)-invariant inner product on \( \mathfrak{m} \). But consequences of (11) are even more profound.

The tangent bundle \( \mathcal{T} \mathcal{M} \) is isomorphic to the bundle \( E \) associated to the \( H \)-principal bundle \( \pi : \mathfrak{g} \to \mathcal{M} \) via the adjoint representation of \( H \) on \( \mathfrak{m} \). The isomorphism \( (\mathfrak{g} \times \mathfrak{m})/H \cong \mathcal{T} \mathcal{M} \), where \( H \) acts on \( \mathfrak{g} \times \mathfrak{m} \) by \( (a, X) \mapsto (ah^{-1}, \text{Ad}_h X) \), is given by the map
\[
\Phi : E \to \mathcal{T} \mathcal{M}
\]
\[
[(a, X)] \mapsto \tau_{a_*}(\pi_{e_h}(X)).
\]

Thus the isomorphism \( \pi_* : \mathfrak{m} \to \mathcal{T}_{\gamma} \mathcal{M} \) identifies \( \mathfrak{m} \cong \mathcal{T}_{\gamma} \mathcal{M} \) and hence provides a one-to-one correspondence between \( \mathfrak{g} \)-invariant tensor fields on \( \mathcal{M} \) and \( \text{Ad}_H \)-invariant tensors on \( \mathfrak{m} \). In particular, it identifies a \( \mathfrak{g} \)-invariant metric on \( \mathcal{M} \) with an \( \text{Ad}_H \)-invariant inner product on \( \mathfrak{m} \).

Thus, we have seen that the natural projection \( \pi : \mathfrak{g} \to \mathcal{M} \cong \mathfrak{g}/H \) induces the linear surjection \( \pi_* : \mathcal{T}_e \mathfrak{g} \to \mathcal{T}_{\gamma} \mathcal{M} \) with the following isomorphisms
\[
\mathcal{T}_{\gamma} \mathcal{M} \cong \mathcal{T}_e \mathfrak{g}/\ker \pi_* \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}.
\]

If the action of \( \mathfrak{g} \) on \( \mathcal{M} \) is reductive, i.e., it satisfies (8), we say that \( \mathcal{M} \) is a reductive Riemannian space.

**Definition 2.2.** The reductive Riemannian space \( \mathcal{M} \cong \mathfrak{g}/H \), with \( \mathfrak{g} \)-invariant metric uniquely determined by a positive definite quadratic form on \( \mathfrak{m} \), is called a symmetric Riemannian homogeneous space if the vector subspace \( \mathfrak{m} \) satisfies \( [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \).

Thus, for symmetric Riemannian homogeneous spaces the Lie algebra decomposition satisfies the following relations
\[
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}, \quad [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m} \quad \text{and} \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}.
\]

This decomposition is known as a Cartan type decomposition.

We conclude this section with the following property of symmetric homogeneous spaces. Suppose that \( \gamma : I \to \mathfrak{g} \) is an injective smooth curve. For any \( V_0 \in \mathcal{T}_{\gamma(t)} \mathcal{M} \), \( V(t) := \tau_{\gamma(t)}(V_0) \) is a vector field along \( \pi(\gamma(t)) \). Then, it can be shown that \( V(t) \) is the parallel translation of \( V_0 \) along \( \pi(\gamma(t)) \) if and only if \( \gamma \) is horizontal, i.e.,
\( \dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)} \). Thus, the horizontal distribution \( \mathcal{H} \) determines parallel translation and therefore a covariant derivative, because covariant derivative is determined by its notion of parallelity and vice versa. For this reason, \( \mathcal{H} \) is also called a connection on the principal bundle \( \pi: \mathfrak{G} \to \mathcal{M} \).

3. **Rolling symmetric homogeneous spaces.** This section deals with rolling motions of symmetric homogeneous spaces. We keep the assumptions made in Section 2.1 for the embedding space, that is \( \mathcal{M} = \mathbb{R}^m \). In what follows, \( \mathfrak{G} \) denotes the identity component of the group of isometries \( \mathcal{J}(\mathcal{M}) \) that preserve orientation. For instance, if the ambient manifold is the Euclidean space \( \mathbb{R}^m \), then \( \mathfrak{G} \) is the special Euclidean group \( \text{SE}(m) = \text{SO}(m) \ltimes \mathbb{R}^m \), the semi-direct product of the special orthogonal group \( \text{SO}(m) \) and \( \mathbb{R}^m \). In this section, we impose the following further assumptions on the rolling manifolds and on the group action:

1. \( \mathcal{M} \subset \overline{\mathcal{M}} \) is a symmetric homogeneous space (symmetric space for short),
2. \( \mathcal{M}_0 \subset \overline{\mathcal{M}} \) is the affine tangent space to \( \mathcal{M} \) at some point \( p \in \mathcal{M} \), and
3. \( \mathfrak{G} \subset \mathfrak{G} \) is a subgroup that acts transitively on \( \mathcal{M} \).

We would like to stress that all groups and manifolds considered here are real.

3.1. **Main results.** One of the main objectives of this section is to show that, under appropriate assumptions, the no-slip and no-twist nonholonomic constraints of Definition 2.1 restrict \( (\chi^{-1})_* \) to the \( m \) component of \( \mathfrak{g} \). Theorem 3.2 states sufficient conditions for this statement to hold.

In order to prove this main result, we need several preliminary results. We first assign elements \( X \) of the Lie algebra \( \mathfrak{g} \) of \( \mathfrak{G} \) to the vector space of linear maps \( \Psi_X \) from \( T_{p_0} \overline{\mathcal{M}} \) to itself, in the following way.

Choose a suitable \( \epsilon > 0 \), and let \( X \in \mathfrak{g} \) and \( \gamma_X: (-\epsilon, \epsilon) \to \mathcal{M} \) be a curve defined by \( \gamma_X(t) := \tau_{\exp(tX)}(p_0) \), on an open neighbourhood of \( p_0 \in \mathcal{M} \). Then \( \gamma_X \) is a curve in \( \mathcal{M} \) generated by the subgroup \( \exp(tX) \). Given \( V_0 \in T_{p_0} \overline{\mathcal{M}} \), define a vector field \( V_X \) in \( \overline{\mathcal{M}} \) along \( \gamma_X \) by \( V_X(t) = \tau_{\exp(tX)}(V_0) \) and the map \( \Psi: \mathfrak{g} \times T_{p_0} \overline{\mathcal{M}} \to T_{p_0} \overline{\mathcal{M}} \) by

\[
\Psi_X(V_0) := \overline{D}_tV_X(0),
\]

where \( \overline{D}_t \) denotes the covariant derivative in the ambient space. Expression (15) defining \( \Psi \) does not depend on a particular curve through \( p_0 \) because the affine connection \( \nabla_XY|_{p_0} \) depends only on \( X(p) \) and the value of \( Y \) along any curve \( \gamma: (-\epsilon, \epsilon) \to \mathcal{M} \) satisfying \( \gamma(0) = p \) and \( \dot{\gamma}(0) = X(p) \), cf. [14, p. 50]. By linearity of the Riemannian connection, \( \Psi_X \) is linear over \( \mathbb{R} \). The following proposition shows that \( \Psi_X \) is also linear over \( \mathbb{R} \) in \( X \).

**Proposition 1.** Let \( X \) and \( Y \) be two elements of the Lie algebra \( \mathfrak{g} \). Then

\[
\Psi_{aX+bY}(V_0) = a\Psi_X(V_0) + b\Psi_Y(V_0),
\]

for any tangent vector \( V_0 \in T_{p_0} \overline{\mathcal{M}} \) and \( a, b \in \mathbb{R} \).

**Proof.** Keeping the notation from the previous section

\[
\dot{\gamma}_{X+Y}(0) = \frac{d}{dt} \bigg|_{t=0} \tau_{\exp(t(X+Y))}(p_0) = \frac{d}{dt} \bigg|_{t=0} (\pi(\exp(t(X+Y)))) \\
= \pi_* (X+Y) = \pi_*(X) + \pi_*(Y) = \dot{\gamma}_X(0) + \dot{\gamma}_Y(0).
\]

Whenever vector fields \( V \) generated by curves \( \gamma_X, \gamma_Y \) and \( \gamma_{X+Y} \) are extendible, then, by the properties of the covariant derivative (cf. [33, page 57]), one has \( \overline{D}_tV = \)}
\(\nabla_\gamma \tilde{V}\), for any extension \(\tilde{V}\) of \(V\). Thus, (16) follows immediately from the linearity of the Levi-Civita connection.

Otherwise, we argue locally. In any local coordinates in a neighbourhood of \(p_0 \in M \subset \tilde{M}\), the covariant derivative of \(V\) along \(\gamma\) takes the form

\[
\overline{\nabla}_t V(0) = \left(\dot{V}^k(0) + V^j(0)\dot{\gamma}^i(0)\Gamma^k_{ij}(p_0)\right)\partial_k.
\]

(18)

We want to show that \(\dot{V}^k_{X+Y}(0) = \dot{V}^k_X(0) + \dot{V}^k_Y(0)\), for any \(X, Y \in \mathfrak{g}\). Let \(\sigma : (-\delta, \delta) \rightarrow \tilde{M}\) be a smooth curve in \(\tilde{M}\) such that \(\sigma(0) = p_0\) and \(\dot{\sigma}(0) = V_0\). Define a family of smooth curves \(\varpi_Z : (-\varepsilon, \varepsilon) \times (-\delta, \delta) \rightarrow \tilde{M}\) by

\[
\varpi_Z(t, s) := \exp(tZ)(\sigma(s)), \quad \text{for any } \quad Z \in \mathfrak{g}.
\]

Since, for any \(p \in \tilde{M}\) one has, for instance, differentiating the Zassenhaus formula,

\[
\frac{d}{dt}\bigg|_{t=0} \tau_{\exp(t(X+Y))}(p) = \frac{d}{dt}\bigg|_{t=0} \tau_{\exp(tX)}(\tau_{\exp(tY)}(p)) + \frac{d}{dt}\bigg|_{t=0} \tau_{\exp(tY)}(\tau_{\exp(tX)}(p)),
\]

then \(\dot{\varpi}_{X+Y}(0, s) = \dot{\varpi}_X(0, s) + \dot{\varpi}_Y(0, s)\). Thus, in the local coordinates

\[
\dot{V}^k_{X+Y}(0) = \frac{\partial^2 \varpi^k_{X+Y}(t, s)}{\partial t \partial s} \bigg|_{t=0 s=0} = \frac{\partial^2 \varpi^k_X(t, s)}{\partial t \partial s} \bigg|_{t=0 s=0} + \frac{\partial^2 \varpi^k_Y(t, s)}{\partial t \partial s} \bigg|_{t=0 s=0} = \frac{\partial}{\partial s} \varpi^k_X(0, s) + \frac{\partial}{\partial s} \varpi^k_Y(0, s).
\]

(19)

This implies that \(\dot{V}^k_{X+Y}(0) = \dot{V}^k_X(0) + \dot{V}^k_Y(0)\), for any \(X, Y \in \mathfrak{g}\). Substituting this expression and (17) into (18) proves (16).

Similarly, one shows that \(\overline{\nabla}_t V_a X(0) = a \overline{\nabla}_t V_X(0)\), for any \(a \in \mathbb{R}\), thus proving that \(\Psi_X\) is linear over \(\mathbb{R}\) in \(X\).

Other properties of \(\Psi_X\), defined in (15), are provided by Proposition 2 through Proposition 4.

Proposition 2. Let \(\mathfrak{h}\) be the Lie algebra of the isotropy group \(H\) of \(p_0 \in M\). Then, \(\Psi_{\mathfrak{h}}(T_{p_0} M) \subset T_{p_0} M\) and \(\Psi_{\mathfrak{h}}(T_{p_0}^\perp M) \subset T_{p_0}^\perp M\).

Proof. For any vector \(X \in \mathfrak{h}\), let \(h : (-\varepsilon, \varepsilon) \rightarrow \mathfrak{g}\) be given by \(h(t) := \exp(tX)\).

Suppose that \(V \in T_{p_0} M\) and that \(\tilde{V}\) is its horizontal lift. Because \(p_0 = \pi(e)\), then by (9) the curve \(\tau_{h(t)}(p_0) = \tau_{h(t)}(\pi(e)) = \pi(h(t)(e)) = \pi(e) = p_0\) is constant.

Hence, \(\tau_{h(t)}\) maps \(T_{p_0} M\) to itself and therefore \(\tau_{h(t)}(\pi_{\mathfrak{r}}(V)) = \pi_{\mathfrak{r}}(Ad_{h(t)} \tilde{V})\) is a vector field in the tangent space \(T_{p_0} M\). Taking derivative with respect to \(t\) of both sides of this equality, with \(h(0) = e\), yields

\[
\Psi_X(V) = \pi_{\mathfrak{r}}([X, \tilde{V}]),
\]

where we used (18). This proves the first assertion.

To prove the second assertion, suppose that \(\Lambda \in T_{p_0}^\perp M\). Since \(H\) is an isometry on \(\tilde{M}\), then \(\langle \tau_{h(t)}(V), \tau_{h(t)}(\Lambda) \rangle = \langle V, \Lambda \rangle = 0\), for any \(V \in T_{p_0} M\). So,

\[
0 = \left. \frac{d}{dt} \langle \tau_{h(t)}(V), \tau_{h(t)}(\Lambda) \rangle \right|_{t=0} = \langle \overline{\nabla}_t V(0), \tau_{h(0)}(\Lambda) \rangle + \langle \tau_{h(0)}(V), \overline{\nabla}_t \Lambda(0) \rangle = \langle \Psi_X(V), \Lambda \rangle + \langle V, \Psi_X(\Lambda) \rangle = \langle V, \Psi_X(\Lambda) \rangle.
\]

(20)

Therefore, \(\Psi_X(\Lambda)\) lies in the normal space \(T_{p_0}^\perp M\), what was to show. \(\square\)
Proposition 3. Let \( \mathfrak{m} = \mathfrak{g}/\mathfrak{h} \), where \( \mathfrak{h} \) is the Lie algebra of the isotropy group \( H \) of \( p_0 \in \mathbb{M} \). Then \( \Psi_m(T_{p_0}\mathbb{M}) \subset T_{p_0}^{\perp}\mathbb{M} \).

Proof. For any vector \( \mathfrak{m} \in \mathfrak{m} \), let \( g: (-\varepsilon, \varepsilon) \rightarrow \mathfrak{g} \) be given by \( g(t) = \exp(tX) \). It is known that \( \gamma := \pi \circ g \) is a geodesic in \( \mathbb{M} \). Because \( g \) is a horizontal curve in \( \pi: \mathfrak{g} \rightarrow \mathbb{M} \), then, by (7) and the discussion at the end of Section 2.2, the vector field \( \Pi \rightarrow \mathbb{M} \)

\[
\Psi_X(V_0) = D_t V(0) = \Pi(\gamma(0), V_0) = \Pi(\pi_*(X), V_0)
\]

(21)

belongs to the normal space \( T_{p_0}^{\perp}\mathbb{M} \).

Up to this point, properties of the map \( \Psi \) acting on tangent and normal spaces did not depend on the embedding of \( \mathbb{M} \) in \( \overline{\mathbb{M}} \). However, to determine how \( \Psi_m \) maps the normal space \( T_{p_0}^{\perp}\mathbb{M} \) it is necessary to impose additional conditions on the embedding. In the proof of Proposition 4 we need that the isometric immersion \( \mathbb{M} \rightarrow \overline{\mathbb{M}} \) has parallel second fundamental form. Let us recall the definition of this concept.

Definition 3.1. Suppose \( \iota: \mathbb{M} \rightarrow \overline{\mathbb{M}} \) is an isometric immersion of one Riemannian manifold into another. Then, the second fundamental form \( \Pi \) is called parallel if

\[
(\overline{\nabla}_Z \Pi)(X,Y) = \nabla^\perp_Z \Pi(X,Y) - \Pi(\nabla_Z X, Y) - \Pi(X, \nabla_Z Y) = 0,
\]

for any smooth vector fields \( X,Y,Z \in \mathfrak{X}(\mathbb{M}) \). The submanifold \( \mathbb{M} \) is called parallel (or extrinsic symmetric) when \( \nabla \Pi = 0 \).

Remark 2. Such immersions were extensively studied by a number of authors, cf. [15, 16, 36, 50]. Dirk Ferus classified submanifolds of Euclidean space with parallel second fundamental forms, cf. [16]. It turns out that all such submanifolds are essentially the so called symmetric \( R \)-spaces introduced by Shoshichi Kobayashi and Masaru Takeuchi, cf. [45]. Examples of symmetric \( R \)-spaces are: the sphere, the projective space (over \( \mathbb{R} \), \( \mathbb{C} \) or quaternions) and the Cayley projective plane. Martin Magid, in his paper [36], studied submanifolds of \( \mathbb{R}^m \) equipped with indefinite metrics. In particular, Magid has classified parallel isometric immersions in pseudo-Euclidean space \( \mathbb{R}^m \) equipped with metrics having signature (1, \( m-1 \)) and signature (2, \( m-2 \)).

Proposition 4. Let \( \mathfrak{m} = \mathfrak{g}/\mathfrak{h} \), where \( \mathfrak{h} \) is the Lie algebra of the isotropy group \( H \) of \( p_0 \in \mathbb{M} \). If \( \mathbb{M} \rightarrow \overline{\mathbb{M}} \) satisfies one of the following conditions:

1. \( \mathbb{M} \) is simply connected and co-dimension one (i.e., \( \dim \mathbb{M} + 1 = \dim \overline{\mathbb{M}} \)); or
2. \( \mathbb{M} \) is a parallel submanifold and \( \Pi(T_{p_0}\mathbb{M}, T_{p_0}\mathbb{M}) = T_{p_0}^{\perp}\mathbb{M} \),

then, \( \Psi_m(T_{p_0}^{\perp}\mathbb{M}) \subset T_{p_0}\mathbb{M} \).

Proof. To prove the above statement, keeping the same notation as before, we show first that given any unit normal vector \( \Lambda_0 \in T_{p_0}^{\perp}\mathbb{M} \), the action \( \tau_{g(t),\ast}(\Lambda_0) \) induces a parallel normal vector field \( \Lambda(t) \in \mathfrak{X}(\mathbb{M}) \). This claim is trivially true in co-dimension one. This is because the holonomy group of normal vectors is \( \mathbb{Z}_2 = \{-1, 1\} \), when \( \mathbb{M} \) is simply connected. In this case, any differentiable field of unit normal vectors along paths in \( \mathbb{M} \subset \overline{\mathbb{M}} \) gives a field of vectors that does not depend on the choice of curves, cf. [28, page 5].
Suppose now that the second fundamental form is parallel, i.e., $\nabla^2 \Pi = 0$. Then, in particular,
\[
\nabla_{\nabla}^2 \Pi(Y, Z) = \Pi(\nabla_X Y, Z) + \Pi(Y, \nabla_X Z), \quad \text{for any } X, Y, Z \in \mathfrak{X}(M).
\] (22)
By the hypothesis, for any normal vector $\Lambda_0 \in T_{p_0}^\perp M$ there exist two tangent vectors $U_0, V_0 \in T_{p_0} M$ such that $\Lambda_0 = \Pi(U_0, V_0)$. Then, $\Lambda(t) := \tau_g(t)_* \Lambda_0$ is clearly a normal vector field, i.e., a vector field in the normal bundle $T^\perp M$. This follows from
\[
\tau_g(t)_* \Lambda_0 = \Pi(\tau_g(t)_* U_0, \tau_g(t)_* V_0) = \Pi(U(t), V(t)),
\]
where we use the fact that $g(t)$ is an isometry. From parallelism of the second fundamental form (22)
\[
\nabla_{\nabla}^2 \Lambda = \nabla_{\nabla}^2 (\Pi(U, V)) = \Pi(\nabla_X U, V) + \Pi(U, \nabla_X V).
\]
Set $X = \dot{g}$. Then
\[
D^+_\tau \Lambda = \Pi(D_t U, V) + \Pi(U, D_t V) = 0
\]
because $U$ and $V$ are parallel by Proposition 3. Hence $\Lambda$ is normal parallel.

Since $\mathfrak{G}$ is an isometry on $\mathbf{M}$ then $\langle \tau_{g(t)}(V), \tau_{g(t)}(\Lambda) \rangle = \langle V, \Lambda \rangle = 0$, for any $V \in T_{p_0} M$. Then
\[
\frac{d}{dt} \langle \tau_{g(t)}(V), \tau_{g(t)}(\Lambda) \rangle = \langle D_t V(0), \tau_{g(0)}(\Lambda) \rangle + \langle \tau_{g(0)}(V), D_t (\Lambda(0)) \rangle
\]
\[
= \langle \Psi_X(V), \Lambda \rangle + \langle V, \Psi_X(\Lambda) \rangle = 0. \quad (23)
\]
By (21) and by the Weingarten equation [33, page 136] it follows now that
\[
\langle \psi_X(\Lambda) \rangle = -\langle \Psi_X(V), \Lambda \rangle = -\langle \Pi(\pi_*(X)), V \rangle, \langle \Xi(\Lambda(X), V) \rangle,
\]
where $\Xi(Z, \Lambda) = \Xi_\Lambda(Z) = (\nabla_X A)^\top$ is a self-adjoint endomorphism of the tangent bundle $TM$ called the Weingarten map. Since the above equality holds for any tangent vector $V \in T_{p_0} M$, then
\[
\Psi_X(\Lambda) = \Xi(\pi_*(X), \Lambda). \quad (24)
\]
This concludes the proof. \hfill \Box

In the remainder of this paper we assume that the manifold $M$, that is isometrically embedded in the ambient space $\mathbf{M} = \mathbb{R}^{m+k}$, is rolling upon its affine tangent space $T_{p_0}^\aff M$, with rolling curve $\sigma$ and development curve $\sigma_0$. We further suppose that rolling maps belong to a subgroup of the identity component $\mathfrak{G}$ of the group of isometries $\mathcal{G}(\mathbf{M})$ that is the semi-direct product $\mathfrak{G} \ltimes V$ of $\mathfrak{G}$ and a vector space $V$. These induce the following chain of inclusions $\mathfrak{G} \times V \subseteq \mathfrak{G} \subset \mathcal{G}(\mathbf{M})$. We shall denote elements of $\mathfrak{G} \ltimes V$ by pairs $(\sigma, s)$, where $\sigma \in \mathfrak{G}$ and $s \in V$. In a local frame $\tau_{(g,s)}(x) = g(x) + s$, cf. [2, pages 101–102]. Thus the “space derivative” (the differential) satisfies $\tau_{(g,s)} = \tau_g$. Since $\mathfrak{G}$ acts transitively on $M$ then, without loss of generality, the rolling curve can be written as
\[
\sigma(t) = \tau_{g^{-1}(t)}(p_0) = \pi(g^{-1}(t)) \quad \text{and} \quad \sigma(0) = p_0,
\]
where $g : I \to \mathfrak{G}$ may not be unique. For any $t_0 \in I$, let $q = \sigma_0(t_0)$ and $p = \sigma(t_0)$. Given a tangent vector $U \in T_q(T_{p_0}^\aff M)$ the push-forward $\tilde{U} = \tau_{\chi^{-1}(t_0)}(U)$ is a tangent vector in $T_p M$. Furthermore, vector $\tilde{U}_0 = \tau_{g(t_0)}(\tilde{U})$ is a tangent vector in
Theorem 3.3. Let \( \chi \) be a rolling map of a symmetric space \( M \cong \mathfrak{g}/H \) isometrically embedded in \( \mathbb{M} = \mathbb{R}^{m,k} \), whose embedding \( \mathfrak{M} \hookrightarrow \mathbb{M} \) is as in Proposition 4. Let \( \mathfrak{m} = \mathfrak{g}/\mathfrak{h} \), where \( \mathfrak{h} \) is the Lie algebra of the isotropy group \( H \) of \( p_0 \in M \). Then, \( (\dot{\chi}(t) \chi^{-1}(t))_* \) belongs to \( \mathfrak{m} \), for all \( t \in I \).

Proof. For any fixed \( t_0 \in I \), denote \( (\dot{\chi}(t_0) \chi^{-1}(t_0))_* \) by \( X \in \mathfrak{g} \). Let \( X = X_h + X_m \) be the decomposition of \( X \) into components in \( \mathfrak{h} \) and \( \mathfrak{m} \), respectively. For any vector \( V \in T_{p_0}M \), the action of \( \Psi_X \) splits according to

\[
\Psi_X(V) = \Psi_{X_h + X_m}(V) = \Psi_{X_h}(V) + \Psi_{X_m}(V),
\]

where \( \Psi_{X_h}(V) \in T_{p_0}M \) and \( \Psi_{X_m}(V) \in T_{p_0}^\perp M \), by Propositions 2 and 3, respectively. From the tangential part of the no-twist conditions, \( \Psi_X(V) \in T_{p_0}^\perp M \) thus \( \Psi_{X_h}(V) \) is zero, for all \( V \in T_{p_0}M \). By a similar reasoning with the normal part of the no-twist conditions, by Propositions 2 and 4, one shows that \( \Psi_{X_h}(\Lambda) = 0 \), for all \( \Lambda \in T_{p_0}^\perp M \). By linearity and properties of the connection \( \Psi_{X_h} \equiv 0 \) implies that \( X_h \) is zero. Hence \( X = X_m \). This completes the proof. \( \square \)

Thus we have confirmed what geometric intuition suggests. For connected, orientable, and symmetric manifolds, with the transitive Lie group \( \mathfrak{g} \), if \( \chi \) is a rolling map then \( \chi_* \) is a horizontal curve in \( \mathfrak{g} \). We make this statement more precise in Corollary 5.

Theorem 3.3. Let \( \chi = (g,s) \) be a rolling map of a symmetric space \( M \cong \mathfrak{g}/H \) and \( \sigma(t) = \tau_{g^{-1}(t)}(p_0) \) be the corresponding rolling curve. For any \( V_0 \in T_{p_0}M \) define a vector field along \( \sigma \) by

\[
V(t) := \tau_{g^{-1}(t)}(V_0).
\]

Then \( V \) is a parallel vector field along \( \sigma \) satisfying \( (L_g)_*(V) = V_0 \).}

Proof. Clearly \( V(t) \in T_{\sigma(t)}M \). Let \( L_a \) denote the left translation by \( a \in \mathfrak{g} \). Then, \( V = (L_{g^{-1}})_*(V_0) \) and

\[
V(f \circ L_g) = ((L_{g^{-1}})_*(V_0))(f \circ L_g) = V_0(f \circ L_g \circ L_{g^{-1}}) = V_0(f),
\]

for any differentiable \( f \) on \( M \). Thus \( (L_g)_*(V) = V_0 = V(0) \).
The rolling map $\chi$ generates a vector field $\tilde{V}$ along the development curve $\sigma_0(t) = \chi(t)(\sigma(t))$, and since rolling maps preserve covariant differentiation, cf. [23], then $\tilde{D}_t \tilde{V} = \tilde{D}_t \tilde{V}$, where $\tilde{D}_t$ is the covariant derivative on the affine tangent space. Because $\tilde{V}(t) = \chi(t)\ast (V(t)) = (\tau_{g(t)} \ast \tau_{g^{-1}(t)})(V_0) = V_0$ is constant, it follows that $\tilde{D}_t \tilde{V} = 0$, what was to show.

3.2. The rolling distribution. Given the group $\mathcal{G}$ acting transitively on the symmetric space $M$, fix a point $p_0 \in M$. Take $(g, q) \in \mathcal{G} \times T^{\text{aff}}_{p_0}M$ and let $q$ be the point of contact in $T^{\text{aff}}_{p_0}M$. Then, $p = \tau^{-1}_g(p_0) = \pi(g^{-1})$ is the point of contact in $M$. The element $g \in \mathcal{G}$ describes the orientation of $M$ relative to $T^{\text{aff}}_{p_0}M$, since we can suppose that $T^{\text{aff}}_{p_0}M$ is stationary. Thus $Q = \mathcal{G} \times T^{\text{aff}}_{p_0}M$ is the configuration space for the rolling problem. Since $\dim M = \dim T^{\text{aff}}_{p_0}M$, the dimension of the configuration space is

$$\dim Q = \dim \mathcal{G} + \dim M = \dim g + \dim g - \dim h = 2 \dim g - \dim h.$$  

We shall identify the differential of $\chi = (g, s) \in \mathcal{G} \times V \subset \mathcal{F}(\tilde{M})$ through $\chi_\ast = \tau_{g_s}$. By the uniqueness of isometries, there exists an injective mapping $\psi: Q \to \mathcal{F}(\tilde{M})$. For example, if $\tilde{M} = \mathbb{R}^{n+1}$ is the Euclidean space then $\mathcal{F}(\tilde{M}) = \mathcal{S}O(n+1) \ltimes \mathbb{R}^{n+1}$ and $\psi((g, q))(x) = \tau_g(x) + q - p_0$.

We are now able to identify the distribution of rolling motions of a homogeneous symmetric space $M$, without slip or twist, upon its affine tangent space $T^{\text{aff}}_{p_0}M$ at a point $p_0 \in M$, under the assumption that $M$ is also a parallel submanifold.

We suppose that on $T^{\text{aff}}_{p_0}M$ the second fundamental form and the Weingarten operator are zero, that is, $\Pi^0 \equiv 0$ and $\Xi^0 \equiv 0$. As a starting point to describe the rolling distribution we take the following system of equations, first derived by Sharpe [44, page 380] and then generalised in [23].

For any $(g, q) \in Q$, the rank $n$ rolling distribution $\mathcal{D} \subset T_{(g, q)}Q$ is given by the following set of equations

$$\begin{cases}
(\tau_g \circ \tau_{g^{-1}})(q) = 0, \\
(\tau_g \circ \tau_{g^{-1}})_\ast(V) = -\tau_{g_s}(\Pi(\tau_{g^{-1}} \ast (q), \tau_{g^{-1}}(V))), & \text{for } V \in T_qM_0 \\
(\tau_g \circ \tau_{g^{-1}})_\ast(\Lambda) = -\tau_{g_s}(\Xi(\tau_{g^{-1}} \ast (q), \tau_{g^{-1}}(\Lambda))), & \text{for } \Lambda \in T^\perp_qM_0.
\end{cases}$$

(26)

One can easily recognise the no-slip condition in the first equation and the no-twist conditions, tangential and normal, in the last two equations.

Notice first that the covariant derivatives on $M$ and on $\tilde{M}$ are $\mathcal{G}$ equivariant, hence so are the second fundamental form and the Weingarten operator. The above system of equations can therefore be simplified in the following way. For any fixed $t_0 \in I$, let $\sigma(t_0) = q$ and $\sigma(t_0) = p$. Given a tangent vector $V \in T_qM_0$, the vector $\tilde{V} = \tau_{g^{-1}}(V)$ is tangent to $M$ at $p$. Thus, with the first equality in (26) one has

$$\tau_{g_s}(\Pi_p(\tau_{g^{-1}} \ast (\sigma_0), \tau_{g^{-1}}(V))) = \tau_{g_s}(\Pi_p(\sigma, \tilde{V})) = \Pi_{\tau_g(p)}(\tau_{g_s}(\sigma), \tau_{g_s}(\tilde{V})).$$

Since the rolling curve is given by $\sigma(t) = \tau_{g^{-1}}(t)(p_0) = \pi(g^{-1}(t))$, then $\dot{\sigma} = \pi_s(\dot{g}^{-1})$ and

$$\tau_{g_s}(\dot{\sigma}) = \tau_{g_s}(\pi_s(\dot{g}^{-1})) = \pi_s(g \dot{g}^{-1}) = -\pi_s(g \dot{g}^{-1}).$$

Finally, denoting $X = g \dot{g}^{-1}$ and $\tilde{V}_0 = \tau_{g_s}(\tilde{V})$, where $\tilde{V}_0$ is tangent to $M$ at $p_0$, the second equation in (26) becomes

$$\Psi_X(\tilde{V}_0) = \Pi_{p_0}(\pi_s(X), \tilde{V}_0).$$
One now applies the same argument to the last equation in (26) with a normal vector \( \Lambda \in T_{p_0}^\perp M \) to derive the following system of equations for \( X : I \to g \):

\[
\begin{align*}
\Psi_X V &= \Pi(\pi_*(X), V), \quad \text{for } V \in T_{p_0} M \\
\Psi_X \Lambda &= \Xi(\pi_*(X), \Lambda), \quad \text{for } \Lambda \in T_{p_0}^\perp M.
\end{align*}
\]

(27)

Thus, we have the following result.

**Proposition 5.** Let \( \chi : I \to \mathcal{G} \ltimes V \) be a smooth curve passing through the identity at \( t = 0 \). Then \( \chi = (g, s) \) is a rolling map of a symmetric manifold \( \pi : \mathcal{G} \to M \) rolling upon its affine tangent space \( T_{p_0}^{\text{aff}} M \) along \( \sigma = \pi(g^{-1}) \) if and only if \( g : I \to \mathcal{G} \) is a horizontal curve.

**Proof.** Let \( X(t) = \dot{g}(t) g^{-1}(t) \), so that \( X : I \to g \) is a curve in the Lie algebra of \( \mathcal{G} \). The necessity follows from Theorem 3.2 and sufficiency from (21) and (24) applied to (27).

The above development can be illustrated with the following, well studied case of the unit sphere, cf. [24, 25, 52] and references therein.

With the standard embedding of the unit sphere \( S^n \hookrightarrow \mathbb{R}^{n+1} \), the second fundamental form is simply

\[
\Pi(U, V) = -\langle U, V \rangle p, \quad \text{for any } U, V \in T_p S^n.
\]

Let \( X = p_0 u^T - w_{p_0}^T \in \mathfrak{so}(n + 1)/\mathfrak{so}(n) \). Then, the left and right hand side of the first equation in (27) become, respectively

\[
\begin{align*}
\Psi_X(V) &= XV = (p_0 u^T - w_{p_0}^T)V = \langle u, V \rangle p_0 \\
\Pi(\pi_*(X), V) &= \Pi(-u, V) = \langle u, V \rangle p_0,
\end{align*}
\]

for any tangent vector \( V \in T_{p_0} S^n \). The second equation in (27) is obsolete in co-dimension one.

### 3.3. Examples.

Here we give a few examples of rolling symmetric spaces on their respective affine tangent spaces. These examples illustrate the main ideas behind the structure of the rolling maps and decomposition of a Lie algebra. In all the cases considered here, the manifolds are **normal homogeneous** with respect to an \( \text{Ad}(H) \)-invariant inner product on \( \mathfrak{g} \), i.e., \( \mathfrak{m} = \mathfrak{h}^\perp \).

For classical groups, as it happens for all examples below, the map \( \Psi_X \), defined by (15), has the following matrix representation, that acts on tangent vectors according to

\[
\Psi_X V = \left. \frac{d}{dt} \right|_{t=0} ((\exp tX)V) = XV,
\]

where \( X = (\chi \chi^{-1})_* \in \mathfrak{g} \) and \( V \in T_{p_0} M \).

**Example 1** (the sphere). Consider the well studied problem of rolling the sphere \( S^n \) on its affine tangent space at a point. Since \( S^n = \text{SO}(n + 1)/\text{SO}(n) \) is a homogeneous space, take any \( p_0 \in S^n \), then \( H = \text{SO}(n) \) is the isotropy group leaving \( p_0 \) fixed.

To be more precise, choose \( p_0 = (0, \ldots, 0, -1) \) to be the “south pole” of \( S^n \). The Lie algebra \( \mathfrak{g} = \mathfrak{so}(n+1) \) splits into the direct sum \( \mathfrak{m} \oplus \mathfrak{h} \), where

\[
\mathfrak{h} = \left\{ x \in \mathfrak{so}(n+1) \middle| \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A \in \mathfrak{so}(n) \right\}
\]
A sphere $S^2$ is rolling upon $M_0$ along the development curve $\sigma_0$ without slipping or twisting; the infinitesimal action $(\dot{\chi} \chi^{-1})_*$ is orthogonal, with respect to an $\text{Ad}(H)$-invariant inner product on $\mathfrak{so}(3)$, to the Lie algebra of the isotropy group $m = h_{\perp}$, with respect to the inner product $\langle X, Y \rangle = \frac{1}{2} \text{trace}(XY^T)$, is given by

$$m = \{ x \in \mathfrak{so}(n+1) \mid x = \begin{bmatrix} 0 & m \\ -m^T & 0 \end{bmatrix} \text{ and } m \in \mathbb{R}^{n \times 1} \} \cong T_{p_0}S^n.$$

It is easy to see that $m p_0 = T_{p_0}S^n$ and $h p_0 = 0$. Note that span$(p_0) = T_{p_0}S^n$. Let $\chi$ be the rolling map and $X = (\dot{\chi} \chi^{-1})_*$. Then, $X \in \mathfrak{g}$ and $\langle X(m p_0), p_0 \rangle = -\langle m p_0, X p_0 \rangle$. Together with the tangential part of the no-twist condition, it follows that $X$ must have a non-zero component in $m$. On the other hand, by the property $[m, h] \subset m$ in (14), $X$ cannot have a component in $h$. This proves that $X \in m$. Figure 3 illustrates this situation for the two-sphere.

**Example 2** (the Graßmann manifold). We now look at the Graßmann manifold rolling on its affine tangent space, cf. [24]. The Graßmann manifold $G_{k,n}$ is defined by $G_{k,n} := \{ P \in \mathfrak{s}(n) \mid P^2 = P \text{ and } \text{rank}(P) = k \}$ and considered embedded in $\mathfrak{s}(n)$, where $\mathfrak{s}(n)$ is the set of $n \times n$ symmetric matrices. Group $\mathfrak{G} = \text{SO}(n)$ acts transitively on $G_{k,n}$ by $(X, P) \mapsto XPX^T$. This action induces the Lie algebra action $(a, V) \mapsto aV + Va^T$. Take

$$P_0 = \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix}$$

and let $H \subset \mathfrak{G}$ be the isotropy group leaving $P_0$ fixed. Then

$$H = \left\{ \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \mid H_1 \in \text{SO}(k) \text{ and } H_2 \in \text{SO}(n-k) \right\}.$$ 

Then Lie algebra $\mathfrak{h}$ of the group $H$ is

$$\mathfrak{h} = \left\{ \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \mid h_1 \in \mathfrak{s}(k) \text{ and } h_2 \in \mathfrak{s}(n-k) \right\}.$$ 

The orthogonal complement $m = h^\perp$ is therefore

$$m = \left\{ \begin{bmatrix} 0 & m \\ -m^T & 0 \end{bmatrix} \mid m \in \mathbb{R}^{k \times (n-k)} \right\}.$$
The tangent and normal spaces at $P_0$ are given by
\[
T_{P_0}G_{k,n} = \begin{cases} 
0 & Z \\
Z^T & 0 \\
\end{cases} \quad Z \in \mathbb{R}^{k \times (n-k)} 
\]
and
\[
T^\perp_{P_0}G_{k,n} = \begin{cases} 
S_1 & 0 \\
0 & S_2 \\
\end{cases} \quad S_1 \in \mathfrak{sk}(k), \quad S_2 \in \mathfrak{s}(n-k).
\]

According to the developments in [24], if $\chi = (R, s)$ is a rolling map of $G_{k,n}$ upon its affine tangent space at $P_0$ and $A \in \mathfrak{s}(n)$, we have
\[
(\dot{\chi} \chi^{-1})_* A = \dot{R}^T R A - A \dot{R}^T \dot{R} = [\Omega, A],
\]
where $\Omega = \dot{\theta}^\perp$. Partitioning $\Omega$ as
\[
\Omega = \begin{bmatrix} m_1 & m_2 \\ -m_2^T & m_3 \end{bmatrix}, \quad \text{where} \quad m_1 = -m_1^T \quad \text{and} \quad m_3 = -m_3^T,
\]
and taking $A = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \in T^\perp_{P_0}G_{k,n}$, the normal part of the no-twist conditions implies that $[\Omega, A] \in T_{P_0}G_{k,n}$. That is
\[
\begin{bmatrix} m_1 & m_2 \\ -m_2^T & m_3 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} + \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} m_1^T & -m_2 \\ -m_2^T & m_3 \end{bmatrix} = \begin{bmatrix} m_1, S_1 \\ S_2, S_2 \end{bmatrix} m_2 S_2 - S_1 m_2
\]
yields $[m_1, S_1] = 0$ and $[m_3, S_2] = 0$, for any symmetric $S_1$ and $S_2$. This is only possible when $m_1 = 0$ and $m_3 = 0$, hence $(\dot{\chi} \chi^{-1})_* \in \mathfrak{m}$, as expected.

**Example 3** (the Essential manifold). The Essential manifold is defined as $\mathcal{E} = \mathcal{G}_{2,3} \times \text{SO}(3)$. We consider this manifold embedded in $\mathfrak{s}(3) \times \mathbb{R}^{3 \times 3}$ equipped with the Euclidean (Frobenius) norm. Points in $\mathcal{E}$ are represented by pairs $(U E_0 U^T, R)$, where $U, R \in \text{SO}(3)$ and $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. At the point $P_0 = (E_0, \mathbb{1})$ the tangent and normal space to $\mathcal{E}$ at $P_0$ are given respectively by
\[
T_{P_0} \mathcal{E} = \begin{cases} 
\left( \begin{bmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{bmatrix}, C \right) \quad \Lambda \in \mathbb{R}^{2 \times 1} \quad \text{and} \quad C \in \mathfrak{s}(3) \\
\end{cases};
\]
\[
T^\perp_{P_0} \mathcal{E} = \begin{cases} 
\left( \begin{bmatrix} B \\ 0 \end{bmatrix}, S \right) \quad B \in \mathfrak{s}(2), \quad b \in \mathbb{R} \quad \text{and} \quad S \in \mathfrak{s}(3) \\
\end{cases}.
\]
Rolling maps for the essential manifold have been studied in [35]. We refer to this paper for details.

The action of the Lie group $\mathfrak{G} = \text{SO}(3) \times \text{SO}(3) \times \text{SO}(3)$ on $\mathcal{E}$, defined by $(U, V, W)(P, R) := (U P U^T, V R W^T)$, is transitive. The isotropy subgroup of $\mathfrak{G}$ that leaves $P_0 = (E_0, \mathbb{1})$ invariant is the set
\[
H = \left\{ (U, V, V) \quad V \in \text{SO}(3) \quad \text{and} \quad U \in \begin{bmatrix} \text{SO}(2) & 0 \\ 0 & 0 \end{bmatrix} \right\}.
\]
The Lie algebra of $\mathfrak{G}$, $\mathfrak{g} = \mathfrak{s}(3) \oplus \mathfrak{s}(3) \oplus \mathfrak{s}(3)$, decomposes as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where
\[
\mathfrak{h} := \left\{ (\vartheta, \zeta, \zeta) \quad \vartheta = \begin{bmatrix} \beta \\ 0 \end{bmatrix}, \quad \beta \in \mathfrak{s}(2), \quad \zeta \in \mathfrak{s}(3) \right\}
\]
and
\[
\mathfrak{m} := \left\{ (\vartheta, \zeta, -\zeta) \quad \vartheta = \begin{bmatrix} 0 & 0 & m_1 \\ 0 & 0 & m_2 \\ -m_1 & -m_2 & 0 \end{bmatrix}, \quad \zeta \in \mathfrak{s}(3) \right\} = \mathfrak{h}^\perp.
\]
It is easy to check that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Thus $\mathcal{E}$ is a symmetric Riemannian homogeneous space, cf. [21].
Let $\chi$ be the rolling map of $\mathcal{E}$ on its affine tangent space at $P_0$ and $(U, V, W) \in \mathcal{E}$ be the first component of $\chi$. Then, according to the developments in [35], if $(A, C)$ is a vector in the embedding space, we have

$$(\hat{\chi}^{-1})_{\ast}(A, C) = \left(\hat{U}^TUA + AU^T\hat{U}, \hat{V}^TVC + CW^T\hat{W}\right),$$

or, defining skew-symmetric matrices $\Omega_U := \hat{U}^TU$, $\Omega_V := \hat{V}^TV$ and $\Omega_W := \hat{W}^TW$,

$$(\hat{\chi}^{-1})_{\ast}(A, C) = \left([\Omega_U, A], \Omega_VC - C\Omega_W\right).$$

Since the tangential no-twist condition requires that

$$(\hat{\chi}^{-1})_{\ast}(A, C) \in T_{P_0}\mathcal{E}, \quad \text{for} \quad (A, C) \in T_{P_0}\mathcal{E},$$

we can do some computations to conclude that $u = (\hat{\chi}^{-1})_{\ast} = (\Omega_U, \Omega_V, -\Omega_W)$, where $\Omega_U$ is of the form $\left[\begin{array}{cc} 0 & 0 \\ -m_1 & -m_2 \\ 0 & 0 \end{array}\right]$ and $\Omega_V = \Omega_W$. So,

$$(\hat{\chi}^{-1})_{\ast} = (\Omega_U, \Omega_V, -\Omega_V) \in m.$$  

**Example 4** (the Lorentzian sphere). We now look at the pseudo-Riemannian case, cf. [29]. The embedding space is $\mathbb{R}^{n+1}$ endowed with the Minkowski metric with signature $(n, 1)$, denoted by $J$. Let $S^{n-1}$ denote the Lorentzian sphere defined by

$$S^{n-1} := \left\{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle_J = 1 \right\}.$$  

The hypersurface $S^{n-1}$ is also known as the hyperboloid of one sheet or De Sitter space. The symmetry group acting transitively on $S^{n-1}$ is the pseudo group $SO(n, 1)$ defined as

$$SO(n, 1) := \left\{ X \in \mathbb{R}^{(n+1)\times(n+1)} \mid X^TJX = J \quad \text{and} \quad \det X = 1 \right\},$$

with Lie algebra

$$so(n, 1) := \left\{ \Omega \in \mathbb{R}^{(n+1)\times(n+1)} \mid \Omega^TJ = -J\Omega \right\}.$$  

It is known that $S^{n-1} = SO(n, 1)/SO(n - 1, 1)$ is a symmetric space. Choose $p_0 = (1, 0, \ldots, 0)$ and $n > 1$ then the isotropy group becomes

$$H = \left\{ X \in SO(n, 1) \mid X = \begin{bmatrix} 1 & 0 \\ 0 & SO(n - 1, 1) \end{bmatrix} \right\}.$$  

Its Lie algebra is therefore

$$\mathfrak{h} = \left\{ x \in so(n, 1) \mid x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} so(n - 1, 1) \right\}$$

and its orthogonal complement is

$$\mathfrak{m} = \left\{ x \in so(n, 1) \mid x = J \begin{bmatrix} 0 & -m^T \\ m & 0 \end{bmatrix} \quad \text{and} \quad m \in \mathbb{R}^{n\times 1} \right\}.$$  

This is consistent with the results in [29].

**Example 5** (Quadratic Lie groups). Rolling motions of quadratic Lie groups have been studied in [13] and [38] and we refer to these papers for more details. Here, we look at these motions in the context of the theory developed for symmetric spaces.

Let $J \in \mathbb{R}^{n\times n}$ be any matrix satisfying $J^2 = \alpha I$ and $J^T = \alpha J$, where $\alpha = \pm 1$. $\mathbb{R}^{n\times n}$ may be endowed with the indefinite inner product

$$\langle U, V \rangle_J := \text{trace}(J^T U^T J V).$$

(28)
To each such \( J \) one can associate a Lie group, consisting of the set of matrices \( X \in \mathbb{R}^{n \times n} \) that satisfy the constraint \( X^T J X = J \). These groups, called quadratic Lie groups, are pseudo-Euclidean submanifolds of \((\mathbb{R}^{n \times n}, J)\). In general, such groups are not connected. For instance, when \( J = 1 \) the corresponding group is the orthogonal group which consists of two connected components. Since we are interested in rolling connected manifolds, in this example the manifold \( M \) is the connected component containing the identity of the group

\[
G_J := \{ X \in \mathbb{R}^{n \times n} \mid X^T J X = J \},
\]

(29)

whose Lie algebra is

\[
\mathfrak{L}_J := \{ \Omega \in \mathbb{R}^{n \times n} \mid \Omega^T J = -J\Omega \}.
\]

For simplicity of notation, \( G_J \) will be used as the connected component containing the identity of the group defined in (29). The following vector space associated to \( J \) will also be useful.

\[
\mathcal{S}_J := \{ \Omega \in \mathbb{R}^{n \times n} \mid \Omega^T J = J\Omega \}.
\]

The tangent and normal spaces to \( G_J \) at a point \( P_0 \) are given by

\[
T_{P_0} G_J = \{ P_0\Omega \mid \Omega \in \mathcal{L}_J \} \quad \text{and} \quad T_{P_0}^\perp G_J = \{ P_0\Omega \mid \Omega \in \mathcal{S}_J \}.
\]

If we define \( \mathfrak{G} = G_J \times G_J \), then \( \mathfrak{G} \) acts transitively on \( G_J \) by

\[
((X, Y), R) \mapsto XRY^{-1}.
\]

Take any \( P_0 \in G_J \) and let \( H \subset \mathfrak{G} \) be the isotropy group leaving \( P_0 \) fixed. Then

\[
H = \{ (X, Y) \in \mathfrak{G} \mid XP_0Y^{-1} = P_0 \}.
\]

For the point \( P_0 = 1 \) the Lie algebra of \( H \) is

\[
\mathfrak{h} = \{ (x, x) \in \mathcal{L}_J \times \mathcal{L}_J \}
\]

and its orthogonal complement \( \mathfrak{m} = \mathfrak{h}^\perp \) with respect to the product metric induced by (28) is

\[
\mathfrak{m} = \{ (x, -x) \in \mathcal{L}_J \times \mathcal{L}_J \}.
\]

By the normal part of the no-twist conditions, for \( u = (m_1, m_2) \in \mathcal{L}_J \times \mathcal{L}_J \), we must have

\[
u\Omega = (m_1, m_2)\Omega = m_1\Omega - \Omega m_2 \in \mathcal{L}_J,
\]

for any \( \Omega \in \mathcal{S}_J \), hence,

\[
(m_1\Omega - \Omega m_2)^T J = -J(m_1\Omega - \Omega m_2).
\]

(30)

Using the definition of \( \mathcal{L}_J \) and of \( \mathcal{S}_J \), the left hand side of this equality is equal to

\[
\Omega^T m_1^T J - m_2^T \Omega^T J = -\Omega^T J m_1 - m_2^T J\Omega = -J\Omega m_1 - Jm_2\Omega.
\]

Therefore, the above condition (30) now reads

\[
-J(\Omega m_1 - m_2\Omega) = -J(m_1\Omega - \Omega m_2),
\]

which is equivalent to

\[
[m_1, \Omega] + [m_2, \Omega] = [m_1 + m_2, \Omega] = 0, \quad \text{for all} \quad \Omega \in \mathcal{S}_J.
\]

So, \( m_1 + m_2 = 0 \), or equivalently, \( u \in \mathfrak{m} \) as desired.
When \( J = \mathbb{1} \), \( G_J \) reduces to the special orthogonal group \( \text{SO}(n) \), when \( J = \begin{bmatrix} 1_k & 0 \\ 0 & -1_{n-k} \end{bmatrix} \), \( G_J \) is the connected component of the pseudo-orthogonal group \( \text{SO}(k, n - k) \), and when \( n = 2m \) and \( J = \begin{bmatrix} 0 & 2m \\ -1_m & 0 \end{bmatrix} \), \( G_J \) is the real symplectic group. All other quadratic Lie groups are isomorphic to the last two classes.

**Remark 3.** Although our study was reduced to the situation when the rolling manifold is a symmetric space embedded in an Euclidean or pseudo-Euclidean space, we strongly believe that the results presented here can be extended to more general situations. To emphasise this viewpoint, we finish this section with the example of a symmetric space embedded in a Riemannian manifold that is not the Euclidean space neither the pseudo-Euclidean space, but for which the results obtained here also apply.

**Example 6** (the ellipsoid). Consider the rolling ellipsoid \( \mathcal{E}^n \) isometrically embedded in the Riemannian structure \( \overline{M} = (\mathbb{R}^{n+1}, D^{-2}) \) induced by a positive definite matrix \( D = \text{diag}(d_1, d_2, \ldots, d_{n+1}) > 0 \), cf. [32]. Then

\[
\mathcal{E}^n := \left\{ p \in \overline{M} \mid \|p\|_{D^{-2}} = 1 \right\}.
\]

Here, the group acting on \( \mathcal{E}^n \) is \( \mathcal{G} = D \text{SO}(n+1) D^{-1} \). Since \( R \mapsto DRD^{-1} \) defines the group isomorphism \( \text{SO}(n+1) \cong \mathcal{G} \), this example is similar to the rolling sphere covered in Example 1. However, the metric considered here is left-invariant.

Let \( p_0 = (0, \ldots, 0, -d_{n+1}) = -De_{n+1} \) be the “south pole” of \( \mathcal{E}^n \). The subgroup \( H = D \text{SO}(n) D^{-1} \) is an isotropy group leaving \( p_0 \) fixed. The Lie algebra \( \mathfrak{g} = D \mathfrak{so}(n+1) D^{-1} \) splits into the direct sum \( \mathfrak{m} \oplus \mathfrak{h} \), where

\[
\mathfrak{h} = \left\{ x \in \mathfrak{g} \mid x = D \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} D^{-1} \quad \text{and} \quad A \in \mathfrak{so}(n) \right\}
\]

and \( \mathfrak{m} = \mathfrak{h}^\perp \) is given by

\[
\mathfrak{m} = \left\{ x \in \mathfrak{g} \mid x = D \begin{pmatrix} 0 & m \\ -m^T & 0 \end{pmatrix} D^{-1} \quad \text{and} \quad m \in \mathbb{R}^{n \times 1} \right\}.
\]

Clearly \( T_{p_0} \mathcal{E}^n \cong \mathfrak{m} \) because

\[
\mathfrak{m} p_0 = -D \begin{pmatrix} m \\ 0 \end{pmatrix} = T_{p_0} \mathcal{E}^n \quad \text{and} \quad \mathfrak{h} p_0 = 0.
\]

Let \( \chi \) be the rolling map and let \( u = (\dot{\chi} \chi^{-1})_* \). The no-twist conditions become

\[
u(T_{p_0} \mathcal{E}^n) \subseteq T_{p_0}^\perp \mathcal{E}^n \quad \text{and} \quad u(T_{p_0}^\perp \mathcal{E}^n) \subseteq T_{p_0} \mathcal{E}^n.
\]

By the same reasoning as in the spherical case, we reach the conclusion that \( u \in \mathfrak{m} \), which is in agreement with results in [32].

4. **Final Remarks.** This paper considers the rolling motion of symmetric spaces isometrically embedded in an Euclidean or pseudo-Euclidean space, having codimension one or whose embedding has covariantly constant (parallel) second fundamental form. We have proven that the natural decomposition of the Lie algebra of such spaces provides the structure for the kinematic equations that describe their rolling motion upon the affine tangent space at a point.

Several examples have been provided to illustrate the main results. Based on the analysis of other cases, in particular on the ellipsoid that, as a symmetric space, doesn’t fit in the scope of Euclidean or pseudo-Euclidean submanifolds, we strongly
believe that the theory developed here is more general. We intend to further investigate this possible generalisation in the near future, as well as address the controllability properties of the kinematic equations of rolling, based on the geometric structure of the kinematic equations.

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