Inverse anisotropic conductivity from internal current densities

Guillaume Bal\textsuperscript{1}, Chenxi Guo\textsuperscript{1} and François Monard\textsuperscript{2}

\textsuperscript{1} Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027, USA
\textsuperscript{2} Department of Mathematics, University of Washington, Seattle WA, 98195, USA

E-mail: gb2030@columbia.edu, cg2597@columbia.edu and fmonard@uw.edu

Received 28 March 2013, revised 18 September 2013
Accepted for publication 27 November 2013
Published 28 January 2014

Abstract
This paper concerns the reconstruction of a fully anisotropic conductivity tensor $\gamma$ from internal current densities of the form $J = \gamma \nabla u$, where $u$ solves a second-order elliptic equation $\nabla \cdot (\gamma \nabla u) = 0$ on a bounded domain $X$ with prescribed boundary conditions. A minimum number of $n + 2$ such functionals known on $Y \subset X$, where $n$ is the spatial dimension, is sufficient to guarantee a unique and explicit reconstruction of $\gamma$ locally on $Y$. Moreover, we show that $\gamma$ is reconstructed with a loss of one derivative compared to errors in the measurement of $J$ in the general case and no loss of derivatives in the special case where $\gamma$ is scalar. We also describe linear combinations of mixed partial derivatives of $\gamma$ that exhibit better stability properties and hence can be reconstructed with better resolution in practice.

Keywords: anisotropic, conductivity, current density

1. Introduction

Hybrid medical imaging modalities are currently extensively studied in the bio-engineering community. Such methods aim to combine high contrast, such as the one found in the modalities electrical impedance tomography (EIT) or optical tomography (OT), with high resolution, as is observed in magnetic resonance imaging (MRI) or ultrasound.

The high-contrast modality EIT aims to locate unhealthy tissues by reconstructing their electrical conductivity $\gamma$ from current boundary measurements. This leads to an inverse problem known as Calderon’s problem. Extensive studies have been made on uniqueness properties and reconstruction methods for this inverse problem [33]. Unfortunately, the problem is non-injective in the anisotropic case, and injective but severely ill-posed in the isotropic case, yielding images with poor resolution.
It is sometimes possible to leverage a physical coupling between a high-contrast, low-resolution modality such as EIT and a high-resolution, low-contrast modality. Such a coupling typically provides internal functionals of the unknown coefficients of interest and greatly improve its resolution while giving access to anisotropic features that were not available otherwise [1–3, 6, 7, 22, 29, 32]. Different types of internal functionals, such as current densities and power densities, corresponding to different physical couplings have been analyzed to recover the unknown conductivity. In the case of power densities, we refer the reader to, e.g., [4, 5, 9, 16, 17, 22–25].

In the present paper, we consider the current density impedance imaging problem, also called magnetic resonance electrical impedance tomography of reconstructing an anisotropic, uniformly elliptic, conductivity tensor in the second-order elliptic equation,

\[
\nabla \cdot (\gamma \nabla u) = \sum_{i,j=1}^{n} \partial_i (\gamma^{ij} \partial_j u) = 0 \quad (X), \quad u|_{\partial X} = g.
\]

from knowledge of internal current densities of the form \( H = \gamma \nabla u \), where \( u \) solves (1). To be consistent with earlier publications, where \( H \) denotes internal functionals, we use the notation \( H \) for current densities rather than the more customary notation \( J \). Here \( X \) is an open bounded domain with a \( C^{2,\alpha} \) or smoother boundary \( \partial X \).

Internal current density functionals \( H \) can be obtained by the technique of current density imaging. The idea is to use MRI to determine the magnetic field \( B \) induced by an input current \( I \). The current density is then defined by \( H = \nabla \times B \). We thus need to measure all components of \( B \) to calculate \( H \), which may create some difficulties in practice, but this is the starting point of this paper. See [11, 30] for details.

The existing results on this problem in the literature are devoted to the reconstruction of a scalar coefficient \( \gamma \). A perturbation method to reconstruct the unknown conductivity in the linearized case was presented in [12]. In dimension \( n = 2 \), a numerical reconstruction algorithm based on the construction of equipotential lines was given in [18]. Kwon et al [19] proposed a \( J \)-substitution algorithm, which is an iterative algorithm. Assuming knowledge of the magnitude of only one current density \( |H| = |\gamma \nabla u| \), the problem was studied in [26–28] (see the latter reference for a review) in the isotropic case and more recently in [10, 21] in the anisotropic case under knowledge of the anisotropic part (the reconstructed conformal factor is hence also scalar). In [14, 20], Nachman et al and Lee independently found a explicit reconstruction formula for visualizing \( \log \gamma \) at each point in a domain. In [31], assuming that the magnetic field \( B \) is measured, Seo et al gave a reconstruction for a complex-valued, isotropic conductivity. The reconstruction with more general functionals of the form \( \gamma^p \nabla u \) (with \( \gamma \) scalar and \( t \) a real exponent) is shown in [15]. For \( t = 0 \), the functionals are given by solutions of (1), in which case general complex-valued, anisotropic tensors \( \gamma \) can be reconstructed; see [8].

The present work addresses the fully anisotropic problem (with \( t = 1 \)) and shows that a finite number of current densities \( \{ H_j = \gamma^p \nabla u \}_{j=1}^{m} \) allows us to uniquely and stably reconstruct the full tensor \( \gamma \) (and hence its anisotropic features) with explicit reconstruction algorithms. The procedures described are valid provided that the measurements considered satisfy certain hypotheses of maximality (e.g. the current densities coming from \( n \) solutions have linearly independent gradients). These technical but necessary hypotheses are formulated in section 2.1. Depending on the satisfied hypotheses, we describe algorithms to explicitly reconstruct all of or part of the conductivity tensor and present their stability properties. Successive algorithms are proposed for (i) the conformal factor, (ii) the anisotropic structure (the product of both recovers \( \gamma \), see decomposition (4)) and (iii) the mixed partials \( \{ \partial_q \gamma^{pl} - \partial_p \gamma^{ql} \}_{p,q,l} \), where \( \gamma^{pq} := (\gamma^{-1})_{pq} \). This last combination of partial derivatives of \( \gamma \) is reconstructed with the.
same stability as $\gamma$ itself, thus displaying improved stability, which in practice means better resolution. The algorithms are mainly described in section 2.2 and justified in section 3. Section 4 is then devoted to understanding under which conditions on the class of conductivity tensors and on the choice of boundary conditions the hypotheses formulated in section 2.1 are satisfied.

2. Statement of the main results

For $X \subset \mathbb{R}^n$, we denote by $\Sigma(X)$ the set of conductivity tensors with bounded components satisfying the uniform ellipticity condition

$$\kappa^{-1} \|\xi\|^2 \leq \xi \cdot \gamma \xi \leq \kappa \|\xi\|^2, \quad \xi \in \mathbb{R}^n, \quad \text{for some } \kappa \geq 1. \quad (2)$$

For $k \geq 1$ an integer and $0 < \alpha < 1$, we denote

$$C^{k,\alpha}_{\Sigma}(X) := \{ \gamma \in \Sigma(X) \mid \gamma_{pq} \in C^{k,\alpha}(X), \quad 1 \leq p \leq q \leq n \}. \quad \text{(3)}$$

In what follows, by ‘solution of (1)’ we may refer to the solution itself or the boundary condition that generates it, i.e. $g = u|_{\partial X} \in H^\gamma(\partial X)$ (elliptic theory tells us that $u \in H^1(X)$ solving (1) exists and is unique as soon as $\gamma \in \Sigma(X)$). We will consider collections of measurements of the form

$$H_i : \gamma \mapsto H_i(\gamma) = \gamma \nabla u_i, \quad 1 \leq i \leq m, \quad \text{(4)}$$

where $u_i$ solves (1) with boundary condition $g_i$. We decompose $\gamma$ into the product of a scalar factor $\beta$ with an anisotropic structure $\tilde{\gamma}$

$$\gamma := \beta \tilde{\gamma}, \quad \beta = (\det \gamma)^\frac{1}{\alpha}, \quad \det \tilde{\gamma} = 1. \quad \text{(5)}$$

Since $\gamma$ satisfies the uniform ellipticity condition (2), $\beta$ is bounded away from zero. From knowledge of a sufficiently large number of current densities, the reconstruction formulas for $\beta$ and $\tilde{\gamma}$ can be established locally on any subdomain $Y \subset X$ in terms of the current densities and their first-order derivatives known on $Y$.

The rest of the section is structured as follows. We first formulate in section 2.1 the hypotheses that allow us to justify the reconstruction procedures. These hypotheses formulate rank maximality of gradients or hessians of solutions of (1) so that algebraic inversions for the conductivity are possible. All of these hypotheses are expressed as open conditions on continuous functionals of the data and are thus robust to small perturbation in the boundary conditions or the conductivity tensor. The explicit reconstruction procedures for $\gamma$ are described in section 2.2. The available data $\{H_i\}_{1 \leq i \leq m}$ depend on appropriate choices of the boundary conditions $\{g_i\}_{1 \leq i \leq m}$. Section 2.3 presents necessary conditions on the choice of $\{g_i\}_{1 \leq i \leq m}$ so that the hypotheses formulated in section 2.1 hold.

2.1. Main hypotheses

The first hypothesis aims at making the scalar factor $\beta$ in (4) locally reconstructible via a gradient equation.

**Hypothesis 2.1.** There exist two solutions $(u_1, u_2)$ of (1) and $X_0 \subset X$ convex satisfying

$$\inf_{x \in X_0} F_1(u_1, u_2) \geq c_0 > 0 \quad \text{where} \quad F_1(u_1, u_2) := |\nabla u_1|^2|\nabla u_2|^2 - (\nabla u_1 \cdot \nabla u_2)^2. \quad \text{(6)}$$

On to the hypotheses for local reconstructibility of $\tilde{\gamma}$, we first need to have, locally, a basis of gradients of solutions of (1).

**Hypothesis 2.2.** There exist $n$ solutions $(u_1, \ldots, u_n)$ of (1) and $X_0 \subset X$ satisfying

$$\inf_{x \in X_0} F_2(u_1, \ldots, u_n) \geq c_0 > 0, \quad \text{where} \quad F_2(u_1, \ldots, u_n) := \det(\nabla u_1, \ldots, \nabla u_n). \quad \text{(7)}$$
Let us now pick \( u_1, \ldots, u_n \) satisfying hypothesis 2.2 and consider additional solutions \([u_{n+1}]_{n+1}^m\). Each additional solution decomposes in the basis \((\nabla u_1, \ldots, \nabla u_n)\) as

\[
\nabla u_{n+k} = \sum_{i=1}^n \mu_k^i \nabla u_i, \quad 1 \leq k \leq m, \tag{7}
\]

where, as shown in [5] for instance, the coefficients \( \mu_k^i \) take the expression

\[
\mu_k^i = -\frac{\det(\nabla u_1, \ldots, \nabla u_{n+k}, \ldots, \nabla u_n)}{\det(\nabla u_1, \ldots, \nabla u_n)} = -\frac{\det(H_1, \ldots, H_{n+k}, \ldots, H_n)}{\det(H_1, \ldots, H_n)},
\]

in particular, these coefficients are accessible from current densities. The subsequent algorithms will make extensive use of the matrix-valued quantities

\[
Z_k = [Z_{k,1}] \cdots [Z_{k,n}], \quad \text{where } Z_{k,i} := \nabla \mu_k^i, \quad 1 \leq k \leq m. \tag{8}
\]

In particular, the next hypothesis, formulating a sufficient condition for local reconstructibility of the anisotropic part of \( y \) is that, locally, a certain number of matrices \( Z_k \) (at least two) satisfies some rank maximality condition. Here and below, \( A_n(\mathbb{R}) \) and \( S_n(\mathbb{R}) \) denote the vector spaces of skew-symmetric and symmetric \( n \times n \) matrices, respectively, and for any real-valued matrix \( A, A^\text{sym} := \frac{1}{2}(A + A^T) \) denotes the projection of \( A \) onto \( S_n(\mathbb{R}) \).

**Hypothesis 2.3.** Assume that hypothesis 2.2 holds for some \((u_1, \ldots, u_n)\) over \( X_0 \subset X \) and denote by \( H \) the matrix with columns \( H_1, \ldots, H_n \). Then there exist \( u_{n+1}, \ldots, u_{n+m} \) solutions of (1) and some \( X' \subset X_0 \) such that the \( x \)-dependent space

\[
W := \text{span}\{(Z_k H^T \Omega)^\text{sym}, \Omega \in A_n(\mathbb{R}), 1 \leq k \leq m\} \subset S_n(\mathbb{R}) \tag{9}
\]

has codimension one in \( S_n(\mathbb{R}) \) throughout \( X' \).

An alternate approach to reconstruct \( y \) is to set up a coupled system for \( u_1, \ldots, u_n \) satisfying hypothesis 2.2 globally. This system of PDEs can be derived under the following hypothesis (part A). From this system and under an additional hypothesis (part B), we can derive an elliptic system from which to reconstruct \( u_1, \ldots, u_n \).

**Hypothesis 2.4.**

A. Suppose that hypothesis 2.2 is satisfied over \( X_0 = X \) for some solutions \((u_1, \ldots, u_n)\). There exists an additional solution \( u_{n+1} \) of (1) whose matrix \( Z_1 \) defined by (8) is uniformly invertible over \( X \), i.e.

\[
\inf_{x \in \chi} \det Z_1 \geq c_0 > 0, \tag{10}
\]

for some positive constant \( c_0 \).

B. There exist \( n + 2 \) solutions \((u_1, \ldots, u_{n+2})\) such that \((u_1, \ldots, u_n, u_{n+2})\) satisfy (A), and two \( A_n(\mathbb{R}) \)-valued functions \( \Omega_1(x), \Omega_2(x) \) such that the matrix

\[
S = (Z_2 Z_1^T \Omega_1(x) + H Z_1^T \Omega_2(x))^\text{sym} \quad \text{(with } Z_2 := Z_2^{-T}) \tag{11}
\]

satisfies the ellipticity condition (2).

The first important result to note is that the hypotheses stated above remain satisfied under small perturbations (in spaces of sufficiently smooth functions) of the boundary conditions or the conductivity tensor:

**Proposition 2.5.** Assume that hypothesis 2.1–2.4 holds over some \( X_0 \subset X \) for a given number \( m \) of solutions of (1) with boundary conditions \( g_1, \ldots, g_m \). Then for any \( 0 < \alpha < 1 \), there exists a neighborhood of \((g_1, \ldots, g_m, \gamma)\) open for the \( \mathcal{C}^{2,\alpha}(\partial X)^m \times \mathcal{C}^{1,\alpha}(X) \) topology where the same hypothesis holds over \( X_0 \). In the case of 2.4.B, it still holds with the same \( A_n(\mathbb{R}) \)-valued functions \( \Omega_1 \) and \( \Omega_2 \).
2.2. Reconstruction algorithms and stability properties

We now present successive reconstruction algorithms, which taken altogether provide explicit local reconstructions of fully anisotropic tensors $\gamma$.

**Reconstruction of $\beta$ knowing $\tilde{\gamma}$.** Under knowledge of $\tilde{\gamma}$ and using two measurements $H_1, H_2$ coming from two solutions satisfying hypothesis 2.1 over some $X_0 \subset X$, we can derive the following gradient equation for $\log \beta$

$$
\nabla \log \beta = \frac{1}{|H_1|^2} (|H_1|^2 d(\tilde{\gamma}^{-1} H_1) - (H_1 \cdot H_2) d(\tilde{\gamma}^{-1} H_2)(\tilde{\gamma} H_1, \tilde{\gamma} H_2)\tilde{\gamma}^{-1} H_1
$$

where $D := |H_1|^2|H_2|^2 - (H_1 \cdot H_2)^2$ is bounded away from zero over $X_0$ thanks to hypothesis 2.1, and where the exterior calculus notations used here are recalled in the appendix.

Equation (12) allows us to reconstruct $\beta$ under the knowledge of $\beta(X_0)$ at one fixed point in $X_0$ by integrating (12) over any curve starting from some $x_0 \in X_0$. This leads to a unique and stable reconstruction with no loss of derivatives, as formulated in the following proposition. This generalizes the result in [14] to anisotropic tensors.

**Proposition 2.6** (Local uniqueness and stability for $\beta$). Consider two tensors $\gamma = \beta \tilde{\gamma}$ and $\gamma' = \beta' \tilde{\gamma}'$, where $\tilde{\gamma}, \tilde{\gamma}' \in W^{1,\infty}(X)$ are known. Suppose that hypothesis 2.1 holds over the same $X_0 \subset X$ for two pairs $(u_1, u_2)$ and $(u'_1, u'_2)$, solutions of (1) with conductivity $\gamma$ and $\gamma'$, respectively. Then the following stability estimate holds for any $p \geq 1$

$$
\| \log \beta - \log \beta' \|_{W^p(X_0)} \leq \epsilon_0 + C \left( \sum_{i=1}^{2} \| H_i - H_i' \|_{W^{p,\infty}(X)} + \| \tilde{\gamma} - \tilde{\gamma}' \|_{W^{p,\infty}(X)} \right).
$$

Where $\epsilon_0 = | \log \beta(x_0) - \log \beta'(x_0) |$ is the error committed at some fixed $x_0 \in X_0$.

**Algebraic, local reconstruction of $\tilde{\gamma}$.** For the local reconstruction of the anisotropic structure, we start from $n + m$ solutions $(u_i, \ldots, u_{n+m})$ satisfying hypotheses 2.2 and 2.3 over some $X_0 \subset X$. In particular, the linear space $\mathcal{W} \subset S_n(\mathbb{R})$ defined in (9) is of codimension one in $S_n(\mathbb{R})$. We will see that the tensor $\tilde{\gamma}$ must be orthogonal to $\mathcal{W}$ for the inner product $\langle A, B \rangle := A_{ij}B_{ij} = \text{tr}(AB^T)$. Together with the conditions that $\text{det} \tilde{\gamma} = 1$ and $\tilde{\gamma}$ is positive, the space $\mathcal{W}$, known from the measurements $H_1, \ldots, H_{n+m}$ completely determines $\tilde{\gamma}$ over $X_0$. In light of these observations, a constructive reconstruction algorithm based on a generalization of the cross-product is proposed in section 3.2. A similar approach was recently used in [23] in the context of inverse conductivity from power densities. This algorithm leads to a unique and stable reconstruction in the sense of the following proposition.

**Proposition 2.7** (Local uniqueness and stability for $\tilde{\gamma}$). Consider two uniformly elliptic tensors $\gamma$ and $\gamma'$. Suppose that Hypotheses 2.2 and 2.3 hold over the same $X_0 \subset X$ for two $n + m$-tuples $(u_i)_{i=1}^{n+m}$ and $(u'_i)_{i=1}^{n+m}$, solutions of (1) with conductivity $\gamma$ and $\gamma'$, respectively. Then the following stability estimate holds for any integer $p \geq 0$

$$
\| \tilde{\gamma} - \tilde{\gamma}' \|_{W^{1,\infty}(X)} \leq C \sum_{i=1}^{n+m} \| H_i - H_i' \|_{W^{p+1,\infty}(X)}.
$$

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Joint reconstruction of \((\hat{\gamma}, \beta)\), stability improvement for \(\nabla \times \gamma^{-1}\). Judging from the stability estimates (14) and (13), reconstructing \(\beta\) after \(\hat{\gamma}\) is less stable than when \(\hat{\gamma}\) is assumed to be known. This is because in the former case, errors on \(W^{0, \infty}\)-norm in \(\hat{\gamma}\) are controlled by errors in \(W^{p+1, \infty}\)-norm in current densities. In particular, on the \(W^{p, \infty}\) scale, stability on \(\beta\) is no better than that of \(\hat{\gamma}\), and joint reconstruction of \((\hat{\gamma}, \beta)\) using the preceding two algorithms displays the following stability, with \(\gamma = \beta \hat{\gamma}\)

\[
\|\gamma - \gamma'\|_{W^{0, \infty}(X_0)} \leq C \sum_{i=1}^{n+m} \|H_i - H'_i\|_{W^{p+1, \infty}(X)}.
\]

However, once \(\gamma\) is reconstructed, some linear combinations of first-order partials of \(\gamma^{-1}\) can be reconstructed with better stability. These are the exterior derivatives of the columns of \(\gamma^{-1}\), a collection of \(n^2(n-1)/2\) scalar functions which we denote \(\nabla \times \gamma^{-1}\) and is reconstructed via the formula

\[
\partial_q y_{pl} - \partial_p y_{ql} = H_l^i (y_{qi} \partial_p H_j^i - y_{ql} \partial_q H_j^i), \quad 1 \leq l \leq n, \quad 1 \leq p < q \leq n,
\]

derived in section 3.3 and assuming that we are working with a basis of solutions satisfying hypothesis 2.2. The stability statement (15) is thus somewhat improved into a statement of the form

\[
\|\gamma - \gamma'\|_{W^{0, \infty}(X_0)} + \|\nabla \times (\gamma^{-1} - \gamma'^{-1})\|_{W^{0, \infty}(X_0)} \leq C \sum_{i=1}^{n+m} \|H_i - H'_i\|_{W^{p+1, \infty}(X)}.
\]

where we have defined

\[
\|\nabla \times (\gamma^{-1} - \gamma'^{-1})\|_{W^{0, \infty}(X_0)} := \sum_{l=1}^{n} \sum_{1 \leq i < j \leq n} \|\partial_q y_{pl} - \partial_p y_{ql}\|_{W^{0, \infty}(X_0)}.
\]

Global reconstruction of \(\gamma\) via a coupled elliptic system. While the preceding approach required a certain number of additional solutions, we now show how one can setup an alternate reconstruction procedure with only \(m = 2\) additional solutions satisfying hypothesis 2.4. A microlocal study of linearized current densities functionals shows that this is the minimum number of functionals necessary to reconstruct all of \(\gamma\).

The present approach consists in eliminating \(\gamma\) from the equations and writing an elliptic system of equations for the solutions \(u_j\); see [5, 22, 23] for similar approaches in the setting of power density functionals. The method goes as follows. Assume that hypothesis 2.2 holds for some \((u_1, \ldots, u_n)\) over \(X_0 = X\) and denote \([\nabla U] = [\nabla u_1, \ldots, \nabla u_n]\) as well as \(H = [H_1, \ldots, H_n]\). Since \(H = [\nabla U]\), we can thus reconstruct \(\gamma\) by \(\gamma = [\nabla U]^{-1} H\) once \([\nabla U]\) is known. We now show that we may reconstruct \([\nabla U]\) by solving a second-order elliptic system of partial differential equations.

When hypothesis 2.4.A is satisfied for some \(u_{n+1}\) and considering an additional solution \(u_{n+2}\) and its corresponding current density, we first derive a system of coupled partial differential equations for \((u_1, \ldots, u_n)\), whose coefficients depend only on measured quantities.

**Proposition 2.8.** Suppose \(n + 2\) solutions \((u_1, \ldots, u_{n+2})\) satisfy hypotheses 2.2 and 2.4.A and consider their corresponding measurements \(H_j = \{H_j\}_{j=1}^{n+2}\). Then the solutions \((u_1, \ldots, u_n)\) satisfy the coupled system of PDE’s

\[
\begin{align*}
Z^1_Z (e_p \otimes e_q - e_q \otimes e_p) : \nabla^2 u_j + \tilde{v}_{ij}^p \cdot \nabla u_i &= 0, \\
HZ^1 (e_p \otimes e_q - e_q \otimes e_p) : \nabla^2 u_j + \tilde{v}_{ij}^p \cdot \nabla u_i &= 0, \\
\end{align*}
\]

for \(1 \leq j \leq n\) and \(1 \leq p < q \leq n\), and where the vector fields \(\{\tilde{v}_{ij}^p, \tilde{v}_{ij}^q\}\) only depend on the current densities \(H_i\).
If additionally, $u_{n+2}$ is such that hypothesis 2.4.B is satisfied, we can deduce a strongly coupled elliptic system for $(u_1, \ldots, u_n)$ from (18).

**Theorem 2.9.** With the hypotheses of proposition 2.8, assume further that hypothesis 2.4.B holds for some $A_n(\mathbb{R})$-valued functions

$$\Omega_i(x) = \sum_{1 \leq p < q \leq n} \omega_{pq}^i(x)(e_p \otimes e_q - e_q \otimes e_p), \quad i = 1, 2.$$  

Then $(u_1, \ldots, u_n)$ can be reconstructed via the strongly coupled elliptic system

$$-\nabla \cdot (S \nabla u_j) + W_{ij} \cdot \nabla u_i = 0, \quad u_j|_{\partial X} = g_j, \quad 1 \leq j \leq n,$$

(19)

where $S = (Z_2^T Z_1^T \Omega_1(x) + H Z_1^T \Omega_2(x))^{sym}$ as in (11) and where we have defined

$$W_{ij} := \nabla \cdot S - \sum_{1 \leq p < q \leq n} \omega_{pq}^1(x)v_{pq}^{ij} + \omega_{pq}^2(x)\tilde{v}_{pq}^{ij}, \quad 1 \leq i, j \leq n.$$  

(20)

Moreover, if system (19) with trivial boundary conditions has only the trivial solution, $u_1, \ldots, u_n$ are uniquely reconstructed. Subsequently, $\gamma$ reconstructed as $\gamma = H[\nabla U]^{-1}$ satisfies the stability estimate

$$\|\gamma - \gamma'\|_{L^2(X)} + \|\nabla \times (\gamma^{-1} - \gamma'^{-1})\|_{L^2(X)} \leq C\|H_i - H'_i\|_{H^1(X)},$$

(21)

for data sets $H_i, H'_i$ close enough in $H^1$-norm.

### 2.3. Choice of boundary conditions and verification of main hypotheses

We now conclude with a discussion on necessary conditions ensuring that hypotheses 2.1–2.4 are satisfied. These conditions are based on appropriate choices of the boundary conditions $\{g_j\}_{1 \leq j \leq m}$ and depend on the type of tensor $\gamma$ we wish to reconstruct.

$\gamma$ close to a constant or isotropic tensor.

**Proposition 2.10.** For any smooth domain $X \subset \mathbb{R}^n$ and considering a constant conductivity tensor $\gamma_0$, there exists a non-empty $C^{2, \alpha}$-open subset of $[H^1(\partial X)]^{n+2}$ of boundary conditions fulfilling hypotheses 2.1–2.4 throughout $X$.

The second test case regards isotropic smooth tensors of the form $\gamma = \beta I_n$, where we show that the scalar coefficient $\beta$ can be reconstructed globally by using the real and imaginary parts of the same complex geometrical optics (CGOs) solution. The use of CGOs for fulfilling internal conditions was previously used in [4, 8, 25].

**Proposition 2.11.** For an isotropic tensor $\gamma = \beta I_n$ with $\beta \in H^{3+\varepsilon}(X)$ for some $\varepsilon > 0$, there exists a non-empty $C^{2, \alpha}$-open subset of $[H^1(\partial X)]^n$ fulfilling hypothesis 2.1 throughout $X$.

Thanks to proposition 2.5, we can also formulate the following without proof.

**Corollary 2.12.** Suppose $\gamma$ is a tensor as in either proposition 2.10 or 2.11. Then, for any $0 < \alpha < 1$, there exists a $C^{1, \alpha}$-neighborhood of $\gamma$ for which the conclusion of the same proposition remains valid.
**Push-forwards by diffeomorphisms.** Recall that for \( \Psi : X \to \Psi(X) \) a \( W^{1,2} \)-diffeomorphism and \( \gamma \in \Sigma(X) \), we define \( \Psi_* \gamma \) the conductivity tensor push-forwarded by \( \Psi \) from \( \gamma \) defined over \( \Psi(X) \), by

\[
\Psi_* \gamma := (|J_\Psi|^{-1} D \Psi \cdot \gamma \cdot D \Psi) \circ \Psi^{-1}, \quad J_\Psi := \det D \Psi. \quad (22)
\]

We now show that, whenever a tensor is being push-forwarded from another by a diffeomorphism, then the local or global reconstructibility of one is equivalent to that of the other, in the sense of the proposition below. While the existence of \( \Psi_* \gamma \) in \( \Sigma(\Psi(X)) \) merely requires that \( \Psi \) be a \( W^{1,2} \)-diffeomorphism, our results below will require that \( \Psi \) be smoother and that it satisfies the following uniform condition over \( X \)

\[
C_\Psi^{-1} \leq |J_\Psi| \leq C_\Psi \quad \text{for some } C_\Psi \geq 1. \quad (23)
\]

**Proposition 2.13.** Assume that hypothesis 2.1–2.4 holds over some \( X_0 \subseteq X \) for a given number \( m \) of solutions of (1) with boundary conditions \( g_1, \ldots, g_m \). For \( \Psi : X \to \Psi(X) \) a smooth diffeomorphism satisfying (23), the same hypothesis holds true over \( \Psi(X_0) \) for the conductivity tensor \( \Psi_* \gamma \) with boundary conditions \( (g_1 \circ \Psi^{-1}, \ldots, g_m \circ \Psi^{-1}) \). In the case of hypothesis 2.4.B, it holds with the following \( \Lambda_{n}(\mathbb{R}) \)-valued functions defined over \( \Psi(X) \):

\[
\Psi_* \Omega_1 := [D \Psi \cdot \Omega_1 \cdot D \Psi] \circ \Psi^{-1} \quad \text{and} \quad \Psi_* \Omega_2 := [|J_\Psi|D \Psi \cdot \Omega_2 \cdot D \Psi^t] \circ \Psi^{-1}. \quad (24)
\]

In contrast to inverse conductivity problems from boundary data, where the diffeomorphisms above are a well-known obstruction to injectivity, proposition 2.13 precisely states the opposite: if a given tensor \( \gamma \) is reconstructible in some sense, then so is \( \Psi_* \gamma \), and the boundary conditions making the inversion valid are explicitly given in terms of the ones that allow to reconstruct \( \gamma \).

**Corollary 2.14.** Suppose \( \gamma \) is a tensor as in either proposition 2.10 or 2.11 and \( \Psi : X \to \Psi(X) \) is a diffeomorphism satisfying (23). Then the conclusion of the same proposition holds for the tensor \( \Psi_* \gamma \) over \( \Psi(X) \) and boundary conditions defined over \( \delta(\Psi(X)) \).

This shows that tensors that are the push-forward of a scalar (isotropic) tensor can be explicitly reconstructed on \( X \) from knowledge of \( n + 2 \) currents on \( X \). In dimension \( n = 2 \), any (symmetric) tensor can be written as the push-forward of a scalar tensor so that knowledge four currents (corresponding to appropriate choices of the boundary conditions) is sufficient.

**Arbitrary sufficiently regular tensor.** We finally state that any \( C^{1,\alpha} \) smooth tensor is reconstructible from current densities in the sense of the following proposition. This result uses the Runge approximation property, a property equivalent to the unique continuation principle, valid for Lipschitz-continuous tensors.

**Proposition 2.15.** Let \( X \subset \mathbb{R}^n \) a \( C^{2,\alpha} \) domain and \( \gamma \in C^{1,\alpha}_X(X) \). Then for any \( x_0 \in X \), there exists a neighborhood \( X_0 \subset X \) of \( x_0 \) and \( n + 2 \) solutions of (1) fulfilling hypotheses 2.2 and 2.3 over \( X_0 \).

**Outline:** The rest of the paper is structured as follows. Section 3 presents the derivations of the local reconstruction algorithms and section 4 discusses how to fulfill the hypotheses from section 2.1, justifying the settings in which the reconstruction algorithms work. Each section starts with a more detailed outline.
3. Explicit reconstructions

We now present the derivation of the reconstruction algorithms: section 3.1 covers the local reconstruction of $\beta$ and proves proposition 2.6; section 3.2 covers the local reconstruction of $\gamma$ and the proof of proposition 2.7; section 3.3 justifies equation (16); section 3.4 discusses the global reconstruction of $\gamma$ via an elliptic system, with a proof of propositions 2.8 and 2.9.

3.1. Local reconstruction of $\beta$

In this section, we assume that $\tilde{\gamma}$ is known and with $W^{1,\infty}$ components. Assuming hypothesis 2.1 is fulfilled for two solutions $u_1, u_2$ over an open set $X_0 \subset X$, we now prove equation (12).

Proof of equation (12). Rewriting (3) as $\frac{1}{H} \tilde{\gamma}^{-1} H_j = \nabla u_j$ and applying the operator $d(\cdot)$, using identities (A.1) and (A.2), we arrive at the following equation for $\log \beta$:

$$\nabla \log \beta \wedge (\tilde{\gamma}^{-1} H_j) = d(\tilde{\gamma}^{-1} H_j), \quad j = 1, 2.$$ \hfill (25)

Let us first notice the following equality of vector fields

$$\nabla \log \beta \wedge (\tilde{\gamma}^{-1} H_1) = (\nabla \log \beta \cdot \tilde{\gamma} H_1)(\tilde{\gamma}^{-1} H_1) - |H_1|^2 \nabla \log \beta,$$

so that

$$\nabla \log \beta = \frac{1}{|H_1|^2}(\nabla \log \beta \cdot \tilde{\gamma} H_1)(\tilde{\gamma}^{-1} H_1) - \frac{1}{|H_1|^2} \nabla \log \beta \wedge (\tilde{\gamma}^{-1} H_1)(\tilde{\gamma} H_1, \cdot)$$

$$= \frac{1}{|H_1|^2}(\nabla \log \beta \cdot \tilde{\gamma} H_1)(\tilde{\gamma}^{-1} H_1) - \frac{1}{|H_1|^2} d(\tilde{\gamma}^{-1} H_1)(\tilde{\gamma} H_1, \cdot).$$

It remains thus to prove that

$$(\nabla \log \beta \cdot \tilde{\gamma} H_1) = \frac{1}{D}(|H_1|^2 d(\tilde{\gamma}^{-1} H_1) - (H_1 \cdot H_2)d(\tilde{\gamma}^{-1} H_2))(\tilde{\gamma} H_1, \tilde{\gamma} H_2),$$

which may be checked directly by computing, for $j = 1, 2$

$$d(\tilde{\gamma}^{-1} H_j)(\tilde{\gamma} H_1, \tilde{\gamma} H_2) = d \log \beta \wedge (\tilde{\gamma}^{-1} H_j)(\tilde{\gamma} H_1, \tilde{\gamma} H_2)$$

$$= (\nabla \log \beta \cdot \tilde{\gamma} H_1)H_j \cdot H_2 - (\nabla \log \beta \cdot \tilde{\gamma} H_2)(H_j \cdot H_1).$$

Taking the appropriate weighted sum of the above equations allows to extract $(\nabla \log \beta \cdot \tilde{\gamma} H_1)$, and hence (12). \hfill $\square$

Reconstruction procedures for $\beta$, uniqueness and stability. Suppose equation (12) holds over some convex set $X_0 \subset X$ and fix $x_0 \in X_0$. Equation (12) is a gradient equation $\nabla \log \beta = F$ with known right-hand side $F$. For any $x \in X_0$, one may thus construct $\beta(x)$ by integrating (12) over the segment $[x_0, x]$, leading to one possible formula

$$\beta(x) = \beta(x_0) \exp \left( \int_0^1 (x - x_0) \cdot F ((1 - t)x_0 + tx) \, dt \right), \quad x \in X_0.$$ \hfill (26)

Proof of proposition 2.6. Since $\det \tilde{\gamma} = 1$, the entries of $\tilde{\gamma}^{-1}$ are polynomials of the entries of $\tilde{\gamma}$, so that the entries of the right-hand side of (12) are polynomials of the entries of $H_1, H_2, \tilde{\gamma}$ and their derivatives, with bounded coefficients. It is thus straightforward to establish that

$$\| \nabla \log \beta - \nabla \log \beta \|_{L^\infty(X_0)} \leq C(\| H - H^\prime \|_{W^{1,\infty}(X)} + \| \tilde{\gamma} - \tilde{\gamma}^\prime \|_{W^{1,\infty}(X)})$$ \hfill (27)
for some constant \( C \). Estimate (13) then follows from the fact that
\[
\| \log \beta - \log \beta' \|_{L^\infty(X_0)} \leq | \log \beta(x_0) - \log \beta'(x_0) | + \Delta(X) \| \nabla \log \beta - \nabla \log \beta' \|_{L^\infty(X_0)},
\]
where \( \Delta(X) \) denotes the diameter of \( X \).

One could use another integration curve than the segment \([x_0, x]\) to compute \( \beta(x) \). In order for this integration to not depend on the choice of curve, the right-hand side \( F \) of (12) should satisfy the integrability condition \( dF = 0 \), a condition on the measurements which characterizes partially the range of the measurement operator.

When measurements are noisy, said right-hand side may no longer satisfy this requirement, in which case the solution to (12) no longer exists. One way to remedy this issue is to solve the normal equation to (12) over \( X_0 \) (whose boundary can be made smooth) with, for instance, Neuman boundary conditions:
\[
-\Delta \log \beta = -\nabla \cdot F \quad (X_0), \quad \partial \nu \log \beta|_{\partial X_0} = F \cdot \nu,
\]
where \( \nu \) denotes the outward unit normal to \( X_0 \). This approach salvages existence while projecting the data onto the range of the measurement operator, with a stability estimate similar to (13) on the \( H^1 \) Sobolev scale instead of the \( W^{1,\infty} \) one.

3.2. Local reconstruction of \( \tilde{\gamma} \)

We now turn to the local reconstruction algorithm of \( \tilde{\gamma} \). In this case, the reconstruction is algebraic, i.e. no longer involves integration of a gradient equation. In the sequel, we work with \( n + m \) solutions of (1) denoted \( \{u_i\}_{i=1}^{n+m} \), whose current densities \( \{H_i = \gamma \nabla u_i\}_{i=1}^{n+m} \) are assumed to be measured.

**Derivation of the space of linear constraints (9).** Apply the operator \( d(\gamma^{-1} \cdot) \) to the relation of linear dependence
\[
H_{n+k} = \mu_k^i H_i, \quad \text{where} \quad \mu_k^i := -\frac{\det(H_1, \ldots, H_{n+k}, \ldots, H_n)}{\det(H_1, \ldots, H_n)}, \quad 1 \leq i \leq n.
\]

Using the fact that \( d(\gamma^{-1} H_i) = d(\nabla u_i) = 0 \), we arrive at the following relation,
\[
Z_{k,i} \wedge \tilde{\gamma}^{-1} H_i = 0, \quad \text{where} \quad Z_{k,i} := \nabla \mu_k^i, \quad k = 1, 2, \ldots
\]
Applying this vanishing two-form to the vector fields \( \tilde{\gamma} e_p, \tilde{\gamma} e_q \), \( 1 \leq p < q \leq n \), we obtain,
\[
H_{q,i} Z_{k,i} \cdot \tilde{\gamma} e_p = H_{p,i} Z_{k,i} \cdot \tilde{\gamma} e_q.
\]
Notice that the above equation means \( \langle \tilde{\gamma} Z_k \rangle_H H_q = \langle \tilde{\gamma} Z_k \rangle_{H_p} H_q \), which amounts to the fact that \( \tilde{\gamma} H_{z} H^T \) is symmetric. This means in particular that \( \tilde{\gamma} Z_k H^T \) is orthogonal to \( A_n(\mathbb{R}) \), and for any \( \Omega \in A_n(\mathbb{R}) \), we can rewrite this orthogonality condition as
\[
0 = \text{tr}(\tilde{\gamma} Z_k H^T \Omega) = \text{tr}(\tilde{\gamma} Z_k H^T \Omega) = \tilde{\gamma} : Z_k H^T \Omega = (Z_k H^T \Omega)^{\text{sym}}, \quad (28)
\]
where the last part comes from the fact that \( \tilde{\gamma} \) is itself symmetric. Each matrix \( Z_k \) thus generates a subspace of \( S_n(\mathbb{R}) \) of linear constraints for \( \tilde{\gamma} \). Considering \( m \) additional solutions, we arrive at the space of constraints defined in (9).
Algebraic inversion of \( \tilde{y} \) via cross-product. We now show how to reconstruct \( \tilde{y} \) explicitly at any point where the space \( \mathcal{V} \) defined in (9) has codimension one. We define the generalized cross product as follows. Over an \( N \)-dimensional space \( \mathcal{V} \) with a basis \( (e_1, \ldots, e_N) \), we define the alternating \( N - 1 \)-linear mapping \( \mathcal{N} : \mathcal{V}^{N-1} \rightarrow \mathcal{V} \) as the formal vector-valued determinant below, to be expanded along the last row

\[
\mathcal{N}(V_1, \ldots, V_{N-1}) := \frac{1}{\det(e_1, \ldots, e_N)} \begin{vmatrix}
\langle V_1, e_1 \rangle & \ldots & \langle V_1, e_N \rangle \\
\vdots & \ddots & \vdots \\
\langle V_{N-1}, e_1 \rangle & \ldots & \langle V_{N-1}, e_N \rangle \\
e_1 & \ldots & e_N
\end{vmatrix}
\]

(29)

\( \mathcal{N}(V_1, \ldots, V_{N-1}) \) is orthogonal to \( V_1, \ldots, V_{N-1} \). Moreover, \( \mathcal{N}(V_1, \ldots, V_{N-1}) \) vanishes if and only if \( (V_1, \ldots, V_{N-1}) \) are linearly dependent.

With this notion of cross-product in the case \( \mathcal{V} = S_n(\mathbb{R}) \), we derive the following reconstruction algorithm for \( \tilde{y} \). Adding \( m \) additional solutions, we find that \( \mathcal{W} \) can be spanned by \( \mathcal{W} := \frac{n-1}{2} \mathcal{W} \) matrices whose expressions are given in (9), picking for instance \( \{e_i \otimes e_j - e_j \otimes e_i\}_{1 \leq i < j \leq n} \) as a basis for \( A_n(\mathbb{R}) \). The condition that \( \mathcal{W} \) is of codimension one over \( X_0 \) can be formulated as:

\[
\inf_{x \in X_0} B(x) > c_1 > 0, \quad B := \sum_{I \in \sigma(n_S - 1, \mathcal{W})} |\det \mathcal{N}(I)|^{\frac{1}{2}},
\]

(30)

where \( \sigma(n_S - 1, \mathcal{W}) \) denotes the sets of increasing injections from \([1, n_S - 1]\) to \([1, \mathcal{W}]\), and where we have defined \( \mathcal{N}(I) = \mathcal{N}(M_{i_1}, \ldots, M_{n_S}) \), where \( \mathcal{N} \) is defined by (29) with \( \mathcal{V} = S_n(\mathbb{R}) \). Then under condition (30), \( \mathcal{W} \) is of rank \( n_S - 1 \) in \( S_n(\mathbb{R}) \).

Whenever \( (M_{i_1}, \ldots, M_{n_S - 1}) \) are picked in \( \mathcal{W} \), their cross-product must be proportional to \( \tilde{y} \). The constant of proportionality can be deduced, up to sign, from the condition \( \det \tilde{y} = 1 \) so we arrive at \( \pm |\det \mathcal{N}(M_{i_1}, \ldots, M_{n_S - 1})|^{\frac{1}{2}} \tilde{y} = \mathcal{N}(M_{i_1}, \ldots, M_{n_S - 1}) \). The sign ambiguity is removed by ensuring that \( \tilde{y} \) must be symmetric definite positive, in particular its first coefficient on the diagonal should be positive. As a conclusion, we obtain the relation

\[
|\det \mathcal{N}(I)|^{\frac{1}{2}} \tilde{y} = \text{sign}(\mathcal{N}_{11}(I)) \mathcal{N}(I), \quad I \in \sigma(n_S - 1, \mathcal{W}).
\]

(31)

This relation is nontrivial (and allows to reconstruct \( \tilde{y} \)) only if \( (M_{i_1}, \ldots, M_{n_S - 1}) \) are linearly independent. When \( \text{codim} \mathcal{V} = 1 \) but \( \mathcal{W} > n_S - 1 \), we do not know a priori which \( n_S - 1 \)-subfamily of \( \mathcal{W} \) has maximal rank, so we sum over all possibilities. Equation (31) then becomes

\[
\sum_{I \in \sigma(n_S - 1, \mathcal{W})} \text{sign}(\mathcal{N}_{11}(I)) \mathcal{N}(I) = B \tilde{y},
\]

(32)

with \( B \) defined in (30). Since \( B > c_1 > 0 \) over \( X_0 \), \( \tilde{y} \) can be algebraically reconstructed on \( X_0 \) by formula (32), where \( \mathcal{N} \) is defined by (29) with \( \mathcal{V} = S_n(\mathbb{R}) \).

Uniqueness and stability. Formula (32) has no ambiguity provided condition (30), hence the uniqueness. Regarding stability, we briefly justify proposition 2.7.

Proof of proposition 2.7. In formula (32), the components of the cross-products \( \mathcal{N}(I) \) are smooth (polynomial) functions of the components of the matrices \( Z_H \), which in turn are smooth functions of the components of \( [H]_{1i}^{1+m} \) and their first derivatives, and where the only term appearing as denominator is \( \det(H_1, \ldots, H_n) \), which is bounded away from zero by virtue of hypothesis 2.2. Thus (14) holds for \( p = 0 \). That it holds for any \( p \geq 1 \) is obtained by taking partial derivatives of the reconstruction formula of order \( p \) and bounding accordingly. \( \square \)
3.3. Joint reconstruction of \((\tilde{\gamma}, \beta)\) and stability improvement

In this section, we justify equation (16), which allows to justify the stability claim (17). Starting from \(n\) solutions satisfying hypothesis 2.2 over \(X_0 \subseteq X\) and denote \(H = (H_{ij})^{n}_{i,j=1} = [H_1|\ldots|H_n]\) as well as \(H^{pq} := (H^{-1})_{pq}\). Applying the operator \(d(\gamma^{-1} \cdot)\) to both sides of (3) yields \(d(\gamma^{-1} H_j) = d(\nabla u_j) = 0\) due to (A.1). Rewritten in scalar components for \(1 \leq j \leq n\) and \(1 \leq p < q \leq n\)

\[
0 = \partial_q (\gamma^{pq} H_j) - \partial_p (\gamma^{pq} H_{ij}) = (\partial_q \gamma^{pq} - \partial_p \gamma^{pq}) H_{ij} + \gamma^{pq} \partial_q H_{ij} - \gamma^{pq} \partial_p H_{ij}.
\]

Thus (16) is obtained after multiplying the last right-hand side by \(H^p\), summing over \(j\) and using the property that \(\sum_{j=1}^{n} H_{ij} H^p = \delta_{il}\).

3.4. Reconstruction of \(\gamma\) via an elliptic system

In this section, we will construct a second-order system for \((u_1, \ldots, u_n)\) with \(n + 2\) measurements, assuming hypotheses 2.2 and 2.4A hold with \(X_0 = X\). For the proof below, we shall recall the definition of the Lie bracket of two vector fields in the Euclidean setting:

\[
[X, Y] := (X \cdot \nabla)(Y - (Y \cdot \nabla)X = (X' \partial_i)Y' e_j - (Y' \partial_i)X' e_j.
\]

**Proof of proposition 2.8.** As is shown by (28), \(\gamma Z_i H^T\) is symmetric. Multiplying both sides by \(\gamma^{-1}\) and using \(\gamma^{-1} H = \nabla U\), we see that \(Z_i[\nabla U]^T\) is symmetric. More explicitly, we have

\[
Z_{k,p} \partial_q u_i = Z_{k,q} \partial_p u_i, \quad k = 1, 2,
\]

or simply \(Z_i[\nabla U]^T = [\nabla U] Z_i^T\). Assume hypothesis 2.4A holds with \(Z_2\) invertible so that \((Z_{1,1}, \ldots, Z_{1,n})\) form a basis in \(\mathbb{R}^n\). We define its dual frame such that \(Z_{2,1}^T Z_{2,1} = \delta_{ij}\). Denote \(Z_1^* = [Z_{1,1}^*, \ldots, Z_{1,n}^*]\) and \(Z_2^* = Z_2^{-T}\). Then the symmetry of \(Z_2[\nabla U]^T\) reads,

\[
Z_{2,j}^* : \nabla u_i = Z_{2,j}^T : \nabla u_j, \quad 1 \leq i \leq j \leq n.
\]

Pick \(v\) a scalar function, we have the following commutation relation:

\[
(X \cdot \nabla)(Y \cdot \nabla) u = (Y \cdot \nabla)(X \cdot \nabla) u + [X, Y] \cdot \nabla u.
\]

Rewrite \(Z_{1,p} \partial_q = Z_{1,p} e_q \cdot \nabla\) and apply \(Z_{2,j}^* \cdot \nabla\) to both sides of (33), we have the following equation by the above relations in Lie bracket,

\[
\begin{align*}
[Z_{2,j}^*, Z_{1,p} e_q] \cdot \nabla u_i &+ (Z_{1,p} e_q \cdot \nabla) (Z_{2,j}^* \cdot \nabla) u_i = [Z_{2,j}^*, Z_{1,q} e_p] \cdot \nabla u_i \\&
\quad + (Z_{1,q} e_p \cdot \nabla) (Z_{2,j}^* \cdot \nabla) u_i
\end{align*}
\]

where \(Z_{k,l} = Z_k : e_l \otimes e_l\). Plugging (34) to the above equation gives,

\[
\begin{align*}
(Z_{1,p} e_q \cdot \nabla) (Z_{2,j}^* \cdot \nabla) u_j &+ [Z_{2,j}^*, Z_{1,p} e_q] \cdot \nabla u_i = (Z_{1,q} e_p \cdot \nabla) (Z_{2,j}^* \cdot \nabla) u_j \\&
\quad + [Z_{2,j}^*, Z_{1,q} e_p] \cdot \nabla u_i
\end{align*}
\]

Looking at the principal part, the first term of the LHS reads

\[
(Z_{1,p} e_q \cdot \nabla) (Z_{2,j}^* \cdot \nabla) u_j = (Z_{1,q} Z_{2,j}^* e_p \otimes e_q) : \nabla^2 u_j + (Z_{1,p} e_q \cdot \nabla) Z_{2,j}^* \cdot \nabla u_j.
\]

Therefore, (35) amounts to the following coupled system,

\[
Z_{1,q} Z_{2,j}^* (e_p \otimes e_q - e_q \otimes e_p) : \nabla^2 u_j + v_{ij}^{pq} : \nabla u_i = 0, \quad u_j|_{\partial X} = g_j, \quad 1 \leq p < q \leq n
\]

where

\[
v_{ij}^{pq} := \delta_{ij} [(Z_{1,p} e_q - Z_{1,q} e_p) \cdot \nabla] Z_{2,j}^* + [Z_{2,j}^*, Z_{1,p} e_q - Z_{1,q} e_p].
\]

Notice that \(H = \gamma^{\nabla U}\) implies that \(H^{-T}[\nabla U]^T\) is symmetric. Compared with equation (33), we can see that the same proof holds if we replace \(Z_2\) by \(H^{-T}\). In this case, the dual
frame of $H^{-T}$ is simply $H$. So (36) and (37) hold by replacing $Z^*$ by $H$ and defining $\tilde{v}_{ij}^{pq}$ accordingly.

We now suppose that hypothesis 2.4.B is satisfied and proceed to the proof of theorem 2.9.

**Proof.** Starting from hypothesis 2.4.B with $A_n(\mathbb{R})$-valued functions of the form

$$\Omega_i(x) = \sum_{1 \leq p < q \leq n} \omega_{ij}^p(x)(e_p \otimes e_q - e_q \otimes e_p), \quad i = 1, 2,$$

we take the weighted sum of equations (18) with weights $\omega_{ij}^1, \omega_{ij}^2$. The principal part becomes $S : \nabla^2 u_i$, which upon rewriting it as $\nabla \cdot (S \nabla u_i) - (S \cdot \nabla) \nabla u_i$ yields system (19).

On to the proof of stability, pick another set of data $H_i^0 := \{H_i^0\}_{i=1}^{+\infty}$ close enough to $H_i$ in $W^{1,\infty}$ norm, and write the corresponding system for $u_i^0, \ldots, u_n^0$:

$$-\nabla \cdot \nabla u^0_j + W^0_j \cdot \nabla u^0_j = 0, \quad 1 \leq j \leq n,$$

(38)

where $S_i$ and $W^0_j$ are defined by replacing $H_i$ in (20) by $H_j^0$. Subtracting (38) from (19), we have the following coupled elliptic system for $v_j = u_j - u^0_j$:

$$-\nabla \cdot \nabla v_j + W_j \cdot \nabla v_j = \nabla \cdot (S - S_i) \nabla u^0_j + (W^0_j - W_j) \cdot \nabla u^0_j, \quad v_j|_{\partial X} = 0 \quad (39)$$

The proof is now a consequence of the Fredholm alternative (as in [5, theorem 2.9]). We recast (39) as an integral equation. Denote the operator $L_0 = -\nabla \cdot (S \nabla)$ and $L_0^{-1} : H^{-1}(X) \ni f \mapsto v \in H_0^1(X)$, where $v$ is the unique solution to the equation

$$-\nabla \cdot (S \nabla v) = f \quad (X), \quad v|_{\partial X} = 0.$$ 

By the Lax–Milgram theorem, we have $\|v\|_{H_0^1(X)} \leq C\|f\|_{H^{-1}(X)}$, where $C$ only depends on $X$ and $S$. Thus $L_0^{-1} : H^{-1}(X) \rightarrow H_0^1(X)$ is continuous, and by Rellich imbedding, $L_0^{-1} : L^2(X) \rightarrow H_0^1(X)$ is compact. Define the vector space $\mathcal{H} = (H_0^1(X))^n, v = (v_1, \ldots, v_n), h \in (L_0^{-1} f_1, \ldots, L_0^{-1} f_n)$, where $f_j = \nabla \cdot (S - S_i) \nabla u^0_j + (W^0_j - W_j) \cdot \nabla u^0_j$, and the operator $P : \mathcal{H} \rightarrow \mathcal{H}$ by,

$$P : \mathcal{H} \ni v \rightarrow P v := (L_0^{-1}(W_1 \cdot \nabla v_1), \ldots, L_0^{-1}(W_n \cdot \nabla v_n)) \in \mathcal{H}.$$

Since the $W_j$ are bounded, the differential operators $W_j \nabla : H_0^1 \rightarrow L^2$ are continuous. Together with the fact that $L_0^{-1} : L^2 \rightarrow H_0^1$ is compact, we get that $P : \mathcal{H} \rightarrow \mathcal{H}$ is compact. After applying the operator $L_0^{-1}$ to (19), the elliptic system is reduced to the following Fredholm equation:

$$(\mathbf{I} + P)v = h.$$ 

By the Fredholm alternative, if $-1$ is not an eigenvalue of $P$, then $\mathbf{I} + P$ is invertible and bounded $\|v\|_{\mathcal{H}} \leq \|(\mathbf{I} + P)^{-1}\|_{\mathcal{L}(\mathcal{H})}\|h\|_{\mathcal{H}}$. Since $L_0^{-1} : H^{-1}(X) \rightarrow H_0^1(X)$ is continuous, $h \in (H_0^1(X))^n$ is bounded by $f = (f_1, \ldots, f_n)$ in $(H^{-1}(X))^n$.

$$\|h\|_{\mathcal{H}} \leq \|L_0^{-1}\|_{\mathcal{L}(H^{-1}, H_0^1)}\|f\|_{H^{-1}(X)}.$$ 

Then we have the estimate,

$$\|v\|_{\mathcal{H}} \leq \|(\mathbf{I} + P)^{-1}\|_{\mathcal{L}(\mathcal{H})}\|L_0^{-1}\|_{\mathcal{L}(H^{-1}, H_0^1)}\|f\|_{H^{-1}(X)}.$$ 

Noting that $L_0^{-1}$ is continuous and the RHS of (39) is expressed by $H_i - H_j^0$ and their derivatives up to second order, we have the stability estimate

$$\|u - u^0\|_{H_0^1(X)} \leq C\|H_i - H_j^0\|_{H^0(X)}$$

where $C$ depends on $H_i$ but can be chosen uniform for $H_i$ and $H_j^0$ sufficiently close. Then $\gamma$ is reconstructed by $\gamma = H[\nabla U]^{-1}$ and $\nabla \times \gamma^{-1}$ by (16), with a stability of the form

$$\|\gamma - \gamma^0\|_{L^2(X)} + \|\nabla \times (\gamma^{-1} - \gamma^{-1})\|_{L^2(X)} \leq C\|H_i - H_j^0\|_{H^0(X)}.$$

\hfill $\square$
4. Fulfilling the hypotheses

Now that we have presented the reconstruction algorithms, we describe for which choices of the boundary conditions the necessary hypotheses are satisfied. This choice of boundary conditions depends on the type of tensor we wish to reconstruct. Satisfying the hypotheses requires a sufficient qualitative control on the solutions of (1). Such solutions can be constructed explicitly when $\gamma$ is close to constant (proposition 2.10 in section 4.1) and by means of complex geometric optics solutions when $\gamma$ is close to isotropic (proposition 2.11 in section 4.1). In the general case, sufficiently explicit solutions can only be constructed locally with boundary conditions on $\partial X$ that are known to exist by means of a Runge approximation property, leading to a generic local reconstructibility of $C^{1,\alpha}$ conductivity tensors in section 4.2 (see proof of proposition 2.15).

We show in section 4.3 how boundary conditions allowing successful reconstructions can be modified in the case where the conductivity is push-forwarded from a known case by a given diffeomorphism (proof of proposition 2.13). We then show in section 4.4 that the reconstruction algorithms are robust with respect to small perturbations of the boundary conditions and conductivity tensors in certain topologies (see proof of proposition 2.5).

The explicit constructions of section 4.1 are new and specific to the current density problem considered in this paper. Sections 4.2–4.4 adapt methods that have been developed in the context of other hybrid inverse problems [3, 7, 8, 23] to the specific setting of the hypotheses stated in section 2.1.

4.1. Test cases

**Constant tensors.** We first prove that hypotheses 2.1–2.4 can be fulfilled with explicit constructions in the case of constant coefficients.

**Proof of proposition 2.10.** Hypotheses 2.2 is trivially satisfied throughout $X$ by choosing the collection of solutions $u_i(x) = x_i$ for $1 \leq i \leq n$, then hypothesis 2.1 is fulfilled by picking any two distinct solutions of the above family.

**Fulfilling hypothesis 2.3.** Let us pick

$$u_i(x) := x_i, \quad 1 \leq i \leq n,$$

$$u_{n+1}(x) := \frac{1}{2} x\gamma_0^{-\frac{1}{2}} \sum_{j=1}^{n} t_j (e_j \otimes e_j)\gamma_0^{-\frac{1}{2}} x, \quad \sum_{j=1}^{n} t_j = 0, \quad t_p \neq t_q \text{ if } p \neq q,$$

$$u_{n+2}(x) := \frac{1}{2} x\gamma_0^{-\frac{1}{2}} \sum_{j=1}^{n-1} (e_j \otimes e_{j+1} + e_{j+1} \otimes e_j)\gamma_0^{-\frac{1}{2}} x.$$ (40)

In particular, $H = \gamma_0$ and $Z_i = \nabla^2 u_{n+1}$ for $i = 1, 2$, do not depend on $x$ and admit the expression

$$Z_1 = \gamma_0^{-\frac{1}{2}} \sum_{j=1}^{n} t_j (e_j \otimes e_j)\gamma_0^{-\frac{1}{2}} \quad \text{and} \quad Z_2 = \gamma_0^{-\frac{1}{2}} \sum_{j=1}^{n-1} (e_j \otimes e_{j+1} + e_{j+1} \otimes e_j)\gamma_0^{-\frac{1}{2}}.$$

We will show that the $(x$-independent) space

$$\mathcal{W} = \text{span}\{(Z_1 H^T \Omega)^{\text{sym}}, (Z_2 H^T \Omega)^{\text{sym}}, \Omega \in A_n(\mathbb{R})\}$$

has codimension one in $S_n(\mathbb{R})$ by showing that $\mathcal{W} \subset \mathbb{R}_{\gamma_0}$, the other inclusion $\supset$ being evident.
Fulfilling hypothesis 2.4 with \
un\\n\gamma_0. The symmetry of \(AZ_iHT\) implies that \(\sum_{j=1}^{n} t_i e_j \otimes e_j \gamma_0^{-\frac{1}{2}} A\gamma_0^{-\frac{1}{2}}\) is symmetric. Denote \(B = \gamma_0^{-\frac{1}{2}} A\gamma_0^{-\frac{1}{2}} \in S_n(\mathbb{R})\), we deduce that 
\[t_i B_{ij} = t_j B_{ji}, \quad \text{for } 1 \leq i, j \leq n.\]

Since \(B\) is symmetric and \(t_i \neq t_j\) if \(i \neq j\), the above equation gives that \(B_{ij} = 0\) for \(i \neq j\), thus \(B\) is a diagonal matrix, i.e. \(B = \sum_{i=1}^{n} B_i e_i \otimes e_i\). The symmetry of \(AZ_iHT\) implies that 
\[\sum_{j=1}^{n-1} (e_j \otimes e_j + e_{j+1} \otimes e_j) \gamma_0^{-\frac{1}{2}} A\gamma_0^{-\frac{1}{2}}\] is symmetric, which means that 
\[
\sum_{1 \leq i, j \leq n, 1 \leq j \leq n-1} B_{ii}(e_i \otimes e_j)(e_j \otimes e_{j+1} + e_{j+1} \otimes e_j) = \sum_{j=1}^{n-1} B_{jj}(e_j \otimes e_j) (e_j \otimes e_{j+1} + e_{j+1} \otimes e_j).
\]

Write the above equation explicitly, we get 
\[
\sum_{j=1}^{n-1} B_{j+1,j+1} e_j \otimes e_{j+1} + B_{jj} e_j \otimes e_j = \sum_{j=1}^{n-1} B_{j+1,j+1} e_{j+1} \otimes e_j + B_{j+1,j+1} e_{j+1} \otimes e_j.
\]

Which amounts to 
\[
\sum_{j=1}^{n-1} (B_{j+1,j+1} - B_{jj}) (e_{j+1} \otimes e_j - e_{j+1} \otimes e_j) = 0.
\]

Notice that \((e_{j+1} \otimes e_j - e_{j+1} \otimes e_j)\) and \((e_{j+1} \otimes e_j + e_{j+1} \otimes e_j)\) are linearly independent in \(A_n(\mathbb{R})\), so \(B_{j+1,j+1} = B_{jj}\) for \(1 \leq j \leq n-1\), i.e. \(B\) is proportional to the identity matrix. This means that \(A\) must be proportional to \(\gamma_0\) and thus \(W \subseteq \mathbb{R} \gamma_0\). Hypothesis 2.3 is fulfilled throughout X.

**Fulfilling hypothesis 2.4 with \(\gamma = I_n\)**, We split the proof according to dimension.

**Even case** \(n = 2m\). Suppose that \(n = 2m\), pick \(u_i = x_i\) for \(1 \leq i \leq n\), \(u_{n+1} = \sum_{i=1}^{m} x_{2i-1} x_{2i}\) and \(u_{n+2} = \sum_{i=1}^{m} (\frac{x_{2i-1} - x_{2i}}{2})\). Then simple calculations show that 
\[
Z_1 = \sum_{i=1}^{m} (e_{2i-1} \otimes e_{2i} + e_{2i} \otimes e_{2i-1}) \quad \text{and} \quad Z_2 = \sum_{i=1}^{m} (e_{2i-1} \otimes e_{2i} - e_{2i} \otimes e_{2i}).
\]

We have \(\det Z_1 = (-1)^m \neq 0\) so 2.4.A is fulfilled. Let us choose 
\[
\Omega_1 := \sum_{p=1}^{m} (e_{2p} \otimes e_{2p-1} - e_{2p-1} \otimes e_{2p}) \quad \text{and} \quad \Omega_2 = 0,
\]
then direct calculations show that \(S = (Z_2^T \Omega_1 + HZ_2^T \Omega_2)^{ym} = I_n\), which is clearly uniformly elliptic, hence 2.4.B is fulfilled.

**Odd case** \(n = 3\). Pick \(u'_i = x_i\) for \(1 \leq i \leq 3\), \(u'_{3+1} = x_1 x_2 + x_2 x_3\) and \(u'_{3+2} = \frac{x_1}{2} x_2^2 + \frac{x_2}{2} x_3^2 + \frac{x_3}{2} x_1^2\), where \(t_1, t_2, t_3\) are to be chosen. In this case, \(H' = I_3\), 
\[
Z_1' = 2(e_1 \otimes e_2 + e_2 \otimes e_3) \quad \text{and} \quad (Z_2')^* = \sum_{i=1}^{3} t_i e_i \otimes e_i \quad \text{(note that \(Z_2'\) fulfills 2.4.A)}. Pick \(\Omega'_1(x) = e_2 \otimes e_1 - e_1 \otimes e_2\), \(\Omega'_2(x) = e_2 \otimes e_3 - e_3 \otimes e_2\), simply calculations show that, 
\[
S' = ((Z_2')^T \Omega'_1(x) + H'(Z_2')^T \Omega'_2(x))^{ym} = \begin{bmatrix} t_1 & 0 & \frac{n+1}{2} \\ 0 & -n_2 - 1 & 0 \\ \frac{n+1}{2} & 0 & 1 \end{bmatrix}.
\]

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(t_1, t_2, t_3) must be such that \( S' \) is positive definite and \( \text{tr}(Z_1') = 0 \) (because \( u_2' \) solves (1)). This entails the conditions

\[
t_1 > 0, \quad t_1(t_1 + 1) < 0, \quad -(t_2 + 1) \left( t_1 - \left( \frac{t_1 + 1}{2} \right)^2 \right) > 0 \quad \text{and} \quad t_1 = -\frac{t_2 t_3}{t_2 + t_3}.
\]

These conditions can be jointly satisfied for instance by picking \( t_1 = 6, t_2 = -2 \) and \( t_3 = 3 \), thus hypothesis 2.4.B is fulfilled in the case \( n = 3 \).

**Odd case** \( n = 2m + 3 \). When \( n = 2m + 3 \) for \( m \geq 0 \), we build solutions based on the previous two cases. Let us pick

\[
u_i = x_i, \quad 1 \leq i \leq n,
\]

\[
u_{n+1} = \sum_{i=1}^{m} x_{2i-1} x_{2i} + x_{2m+1} x_{2m+2} + x_{2m+2} x_{2m+3},
\]

\[
u_{n+2} = \sum_{i=1}^{m} \left( \frac{x_{2i-1} - x_{2i}}{2} \right)^2 + \frac{1}{12} x_{2m+1}^2 - \frac{1}{4} x_{2m+2}^2 + \frac{1}{6} x_{2m+3}^2.
\]

Then one can simply check that \( Z_j \) is of the form

\[
\tilde{Z}_j = \left[ \begin{array}{c} Z_j \\ 0_{3 \times 2m} \end{array} \right], \quad j = 1, 2,
\]

where \( Z_j/Z'_j \) are constructed as in the case \( n = 2m/n = 3 \), respectively. Accordingly, let us construct \( \Omega_{1,2} \) by block using the previous two cases,

\[
\tilde{\Omega}_j = \left[ \begin{array}{c} \Omega_j \\ 0_{3 \times 2m} \end{array} \right],
\]

and the \( S \) matrix so obtained becomes

\[
\tilde{S} = (Z_1' Z_1^T \tilde{\Omega}_1 + H Z_1^T \tilde{\Omega}_2)_{\text{sym}} = \left[ \begin{array}{c} I_{2m} \\ 0_{3 \times 2m} \end{array} \right],
\]

where \( S' \) is the definite positive matrix constructed in the case \( n = 3 \). Again, hypothesis 2.4.B is fulfilled.

**Fulfilling hypothesis 2.4 with \( y \) constant.** Let \( \{v_i\}_{i=1}^{n+2} \) denote the harmonic polynomials constructed in any case above (i.e. \( n \) even or odd) with \( y = \|v\|_e \), and denote \( Z_i', Z_i^0, H', \Omega_i^0, \Omega_i^0 \) and \( S^0 = (Z_i'^T Z_i^0 + H' \Omega_i^0)^{\text{sym}} \) the corresponding matrices. Define here, for \( 1 \leq i \leq n \),

\[
u_i(x) := v_i(x) \quad \text{and} \quad i = n + 1, n + 2, u_i(x) = v_i(y^{-1} x), \quad \text{all solutions of (1) with constant} \ y \ \text{constant}.
\]

Then we have that \( Z_i = y^{-1} Z_i^0 y^{-1} \) for \( i = 1, 2 \) and \( H = y \). Upon defining

\[
\Omega_i := y^{-1} \Omega_i^0 y^{-1} \in \mathbb{R}^n \quad \text{for} \quad i = 1, 2,
\]

direct calculations show that

\[
S = (Z_1' Z_1^T \Omega_1 + H Z_1^T \Omega_2)_{\text{sym}} = y^{-1} S^0 y^{-1}.
\]

Whenever \( Z_1^0 \) is non-singular, so is \( Z_1 \) and whenever \( S_0 \) is symmetric definite positive, so is \( S \).

The proof is complete. \( \square \)

**Isotropic tensors.** As a second test case, we show that, based on the construction of CGOs solutions, hypothesis 2.1 can be satisfied globally for an isotropic tensor \( \gamma = \beta \mathbb{1}_n \) when \( \beta \) is smooth enough. CGO solutions find many applications in inverse conductivity/diffusion problems, and more recently in problems with internal functionals \([4, 8, 25]\). As established in \([7]\), when \( \beta \in H^{2 + \frac{3}{2} + \epsilon}(X) \), one is able to construct a complex-valued solution of (1) of the form

\[
u_{\rho} = \frac{1}{\sqrt{\beta}} e^{\rho y_1} (1 + \psi_{\rho}),
\]

\[\text{(42)}\]
where $\rho \in \mathbb{C}^n$ is a complex frequency satisfying $\rho \cdot \rho = 0$, which is equivalent to taking $\rho = \rho(k + ik)$ for some unit orthogonal vectors $k, k^\perp$ and $\rho = |\rho|/\sqrt{2} > 0$. The remainder $\psi_\rho$ satisfies an estimate of the form $\rho |\psi_\rho| = O(1)$ in $C^1(\overline{X})$. The real and imaginary parts of $\nabla u_\rho$ are almost orthogonal, modulo an error term that is small (uniformly over $X$) when $\rho$ is large. We use this property here to fulfill hypothesis 2.1.

**Proof of proposition 2.11.** Pick two unit orthogonal vectors $k$ and $k^\perp$, and consider the CGO solution $u_\rho$ as in (42) with $\rho = \rho(k + ik)$ for some $\rho > 0$ which will be chosen large enough later. Computing the gradient of $u_\rho$, we arrive at

$$\nabla u_\rho = \frac{1}{\sqrt{\beta}} e^{\rho x}(\rho + \varphi_\rho), \quad \text{with} \quad \varphi_\rho := \nabla \psi_\rho + \rho \psi_\rho - (1 + \psi_\rho) \nabla \log \sqrt{\beta},$$

with $\sup_{X} |\varphi_\rho| \leq C$ independent of $\rho$. Splitting into real and imaginary parts, each of which is a real-valued solution of (1), we obtain the expression

$$\nabla u_\rho^\Re = \frac{\rho e^{kx}}{\sqrt{\beta}} \left( (k + \rho^{-1} \varphi_\rho^\Im) \cos(\rho k^\perp \cdot x) - (k^\perp + \rho^{-1} \varphi_\rho^\Re) \sin(\rho k^\perp \cdot x) \right),$$

$$\nabla u_\rho^\Im = \frac{\rho e^{kx}}{\sqrt{\beta}} \left( (k^\perp + \rho^{-1} \varphi_\rho^\Re) \cos(\rho k^\perp \cdot x) + (k + \rho^{-1} \varphi_\rho^\Im) \sin(\rho k^\perp \cdot x) \right),$$

from which we compute directly that

$$|\nabla u_\rho^\Re| \nabla u_\rho^\Im | - (\nabla u_\rho^\Re \cdot \nabla u_\rho^\Im)^2 = \frac{\rho^2 e^{2k \cdot x}}{\beta} (1 + o(\rho^{-1})).$$

Therefore, for $\rho$ large enough, the quantity in the left-hand side above remains bounded away from zero throughout $X$, and the proof is complete. \hfill \Box

### 4.2. General reconstructibility

We now briefly explain how any $C^{1,\alpha}$-smooth conductivity tensor is locally reconstructible from current densities. The proof relies on the Runge approximation for elliptic equations, which is equivalent to the unique continuation principle, valid for conductivity tensors with Lipschitz-continuous components. The $C^{1,\alpha}$ ($\alpha > 0$) regularity is required for forward elliptic estimates.

This scheme of proof was recently used in the context of other inverse problems with internal functionals [8, 23], and the interested reader is invited to find more detailed proofs there. The proof follows three main steps:

(i) For $x_0 \in X$ fixed and denoting $\gamma_0 := \gamma(x_0)$, we first construct solutions of the constant-coefficient problem by picking the functions defined in (40) (call them $v_{i}, \ldots, v_{n+2}$) and by defining, for $1 \leq i \leq n + 2$, $u_i^0(x) := v_i(x) - v_i(x_0)$. These solutions satisfy $\nabla \cdot (\gamma_0 \nabla u_i) = 0$ everywhere and fulfill hypotheses 2.2 and 2.3 globally.

(ii) Using the maximum principle and interior regularity results, the solutions above are then $C^2$-approximated on a neighborhood of $x_0$ with conductivity solutions involving the true conductivity $\gamma$.

(iii) Using the Runge approximation, the solutions of the previous step are then $C^2$ approximated on a neighborhood of $x_0$ with solutions of (1). The crucial point is that these last solutions can be controlled from the boundary. Because these solutions are $C^2$-close enough to $[u_i]_{i=1}^{n+1}$ on a neighborhood of $x_0$, they will fulfill hypotheses 2.2 and 2.3 there, thus guaranteeing that the reconstruction algorithms work in this neighborhood. Doing this for any $x_0 \in X$ completes the proof.
Remark 4.1 (On generic global reconstructibility). Let us mention that from the local reconstructibility statement above, one can establish a global reconstructibility one. Heuristically, by compactness of \( \hat{X} \), one can cover the domain with a finite number of either neighborhoods as above or subdomains diffeomorphic to a half-ball if the point \( x_0 \) is close to \( \partial \hat{X} \), over each of which \( \gamma \) is reconstructible. One can then patch together the local reconstructions using for instance a partition of unity, and obtain a globally reconstructed \( \gamma \). The additional technicalities that this proof incurs may be found in [8].

As a conclusion, for any \( C^{1,\alpha} \)-smooth tensor \( \gamma \), there exists a finite \( N \) and non-empty open set \( \mathcal{O} \subset (C^{2,\alpha} (\partial \hat{X}))^N \) such that any \( [g_i]_{i=1}^N \in \mathcal{O} \) generates current densities that reconstruct \( \gamma \) uniquely and stably (in the sense of estimate (17)) throughout \( \hat{X} \).

### 4.3. Push-forward by diffeomorphism

Let \( \Psi : X \to \Psi(X) \) be a \( W^{1,2} \)-diffeomorphism where \( X \) has smooth boundary. Then for \( \gamma \in \Sigma(X) \), the push-forwarded tensor \( \Psi_\gamma \) defined in (22) belongs to \( \Sigma(\Psi(X)) \) and \( \Psi \) pushes forward a solution \( u \) of (1) to a function \( v = u \circ \Psi^{-1} \) satisfying the conductivity equation

\[
-\nabla_x \cdot (\Psi_\gamma \nabla_x v) = 0 \quad (\Psi(X)), \quad v|_{\partial(\Psi(X))} = g \circ \Psi^{-1},
\]

moreover \( \Psi \) and \( \Psi|_{\partial X} \) induce respective isomorphisms of \( H^1(\hat{X}) \) and \( H^1(\partial \hat{X}) \) onto \( H^1(\Psi(X)) \) and \( H^1(\partial(\Psi(X))) \).

**Proof of proposition 2.13.** The hypotheses of interest all formulate the linear independence of some functionals in some sense. We must see first how these functionals are push-forwarded via the diffeomorphism \( \Psi \). For \( 1 \leq i \leq m \), we denote \( v_i := \Psi_{\star} u_i = u_i \circ \Psi^{-1} \) as well as \( \Psi_{\star} H_i := [\Psi_{\star} \gamma] \nabla_x v_i \) where \( \gamma \) denotes the variable in \( \Psi(X) \). Direct use of the chain rule allows to establish the following properties, true for any \( x \in X \):

\[
\begin{align*}
\nabla u_i (x) &= [D\Psi]^T (x) \nabla_x v_i (\Psi(x)), \\
H_i (x) &= \gamma \nabla u_i (x) = [J_\Psi (x)] [D\Psi]^{-1} \Psi_{\star} H_i (\Psi(x)), \\
Z_i (x) &= [D\Psi]^T (x) \Psi_{\star} Z_i (\Psi(x)),
\end{align*}
\]

(43)

where we have defined \( \Psi_{\star} Z_i \) the matrix with columns

\[
[\Psi_{\star} Z_i]_{j,k} = -\nabla_v \frac{\det(\nabla v_1, \ldots, \nabla v_{j-1}, \nabla v_j, \nabla v_{j+1}, \ldots, \nabla v_{n})}{\det(\nabla v_1, \ldots, \nabla v_n)}, \quad 1 \leq j \leq n.
\]

**Hypotheses 2.1 and 2.2.** Since \( [D\Psi] \) is never singular over \( X \), relations (43) show that for any \( 1 \leq k \leq n \), the vectors fields \( (\nabla u_1, \ldots, \nabla u_k) \) are linearly dependent at \( x \) if and only if the vectors fields \( (\nabla v_1, \ldots, \nabla v_k) \) are linearly dependent at \( \Psi(x) \). The case \( k = 2 \) takes care of hypothesis 2.1 while the case \( k = n \) takes care of hypothesis 2.2.

**Hypothesis 2.3.** If we denote

\[
\Psi_{\star} \mathcal{W} (\Psi(x)) = \text{span}(\Psi_{\star} Z_1 (\Psi, H) \Omega)_{\text{sym}}, \quad \Omega \in A_\ast (\mathbb{R}), \quad 1 \leq k \leq m,
\]

direct computations show that

\[
\mathcal{W}(x) = [D\Psi(x)]^T \cdot \Psi_{\star} \mathcal{W}(\Psi(x)) \cdot [D\Psi(x)],
\]

thus since \( D\Psi(x) \) is non-singular, we have that \( \dim \mathcal{W}(x) = \dim \Psi_{\star} \mathcal{W}(\Psi(x)) \), so the statement of proposition holds for hypothesis 2.3.

**Hypothesis 2.4.** The transformation rules (43) show that \( Z_1 \) is non-singular at \( x \) iff \( \Psi_{\star} Z_1 \) is non-singular at \( \Psi(x) \), so the statement of the proposition holds for hypothesis 2.4A.
Second, for two \( A_\alpha(\mathbb{R}) \)-valued functions \( \Omega_1(x) \) and \( \Omega_2(x) \), and upon defining \( \Psi_1, \Omega_1, \Psi_2, \Omega_2 \) as in (24), as well as
\[
\Psi, S := ([\Psi, Z_2]^T[\Psi, Z_1]^T \Psi, \Omega_1 + [\Psi, H][\Psi, Z_1]^T \Psi, \Omega_2)^{gm},
\]
direct use of relations (43) yield the relation
\[
S(x) = [D\Psi(x)]^{-1} \cdot \Psi, S(\Psi(x)) \cdot [D\Psi(x)]^{-T}, \quad x \in X,
\]
and since \( D\Psi \) is uniformly non-singular, \( S \) is uniformly elliptic if and only if \( \Psi, S \) is, so the statement of the proposition holds for hypothesis 2.4.B. \( \square \)

### 4.4. Openness properties and stability with respect to small perturbations

In this section, we prove proposition 2.5 which allows us to deduce that fulfilling the hypotheses stated in 2.1 (or equivalently, successfully reconstructing a conductivity tensor via one of the above algorithms) is stable under small enough \( C^{2,\alpha} \)-perturbations of the boundary conditions, and small enough \( C^{1,\alpha} \)-perturbations of the conductivity tensor, for any \( 0 < \alpha < 1 \).

We will make use of the following result, based on Schauder estimates for elliptic equations. It is for instance stated in [13].

**Proposition 4.2.** For \( k \geq 2 \) an integer and \( 0 < \alpha < 1 \), if \( X \) is a \( C^{k+1,\alpha} \)-smooth domain, then the mapping \( (g, \gamma) \mapsto u, \) solution of (1), is continuous in the functional setting
\[
C^{k,\alpha}(\partial X) \times C^{1,\alpha}_{\Sigma}(X) \to C^{k,\alpha}(X).
\]

As a consequence, we can claim that, with the same \( k, \alpha \) as above, the current density operator \( (g, \gamma) \mapsto \gamma \nabla u \) is continuous in the functional setting
\[
C^{k,\alpha}(\partial X) \times C^{1,\alpha}_{\Sigma}(X) \to C^{k,\alpha}(X).
\]
Moreover, this fact allows us to prove proposition 2.5.

**Proof of proposition 2.5.** Fixing some domain \( X_0 \subset X \) and using proposition 4.2, it is clear that the mappings
\[
f_1 : (C^{2,\alpha}(\partial X))^2 \times C^{1,\alpha}_{\Sigma}(X) \ni (g_1, g_2, \gamma) \mapsto \inf_{X_0} F_1(u_1, u_2),
\]
\[
f_2 : (C^{2,\alpha}(\partial X))^n \times C^{1,\alpha}_{\Sigma}(X) \ni (g_1, \ldots, g_n, \gamma) \mapsto \inf_{X_0} F_2(u_1, \ldots, u_n),
\]
with \( F_1, F_2 \) defined in (5), (6), are continuous, so \( f_1^{-1}((0, \infty)) \) and \( f_2^{-1}((0, \infty)) \) are open, which takes care of hypotheses 2.1 and 2.2. Further, hypothesis 2.3 is fulfilled if and only if condition 30 holds. Again, using proposition 4.2, the mapping \( f_3 := \inf_{X_0} B \) with \( B \) defined in (30) is a continuous function of \( (g_1, \ldots, g_{n+1}, \gamma) \in (C^{2,\alpha}(\partial X))^{n+1} \times C^{1,\alpha}_{\Sigma}(X) \) so that \( f_3^{-1}((0, \infty)) \) is open.

Along the same lines, hypothesis 2.4.A is stable under such perturbations because the mapping
\[
(C^{2,\alpha}(\partial X))^{n+1} \times C^{1,\alpha}_{\Sigma}(X) \ni (g_1, \ldots, g_{n+1}, \gamma) \mapsto \inf_{X} \det Z_1,
\]
is continuous whenever \( u_1, \ldots, u_n \) satisfy (6) over \( X \). Finally, fixing two \( A_\alpha(\mathbb{R}) \)-valued functions \( \Omega_1(x) \) and \( \Omega_2(x) \), hypothesis 2.4.B is fulfilled whenever
\[
(g_1, \ldots, g_{n+2}, \gamma) \in \bigcap_{i=1}^{n} S_i^{-1}((0, \infty)),
\]

(44)
where we have defined the functionals, for $1 \leq i \leq n$

$$s_i : (\mathcal{C}^{2,\alpha}(\partial X))^{n+2} \times \mathcal{C}^{1,\alpha}(X) \ni (g_1, \ldots, g_{n+2}, \gamma) \mapsto \inf_X \det\{S_{pq}\}_{1 \leq p, q \leq i},$$

with $S = \{S_{pq}\}_{1 \leq p, q \leq n}$ defined as in (11). Such functionals are, again, continuous, in particular the set in the right-hand side of (44) is open. This concludes the proof. □

Acknowledgments

This work was supported in part by NSF grant DMS-1108608 and AFOSR grant NSSEFFFA9550-10-1-0194. FM is partially supported by NSF grant DMS-1025372.

Appendix. Exterior calculus and notations

Throughout this paper, we use the following convention regarding exterior calculus. Because we are in the Euclidean setting, we will avoid the flat operator notation by identifying vector fields with one-forms via the identification $\mathbf{e}_i \equiv \mathbf{e}_i^\ast$ where $\{\mathbf{e}_i\}_{i=1}^n$ and $\{\mathbf{e}^\ast_i\}_{i=1}^n$ denote bases of $\mathbb{R}^n$ and its dual, respectively. In this setting, if $V = V^i \mathbf{e}_i$ is a vector field, $dV$ denotes the two-vector field

$$dV = \sum_{1 \leq i < j \leq n} (\partial_i V^j - \partial_j V^i) \mathbf{e}_i \wedge \mathbf{e}_j.$$ 

A two-vector field can be paired with two other vector fields via the formula

$$A \wedge B(C, D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C),$$

which allows to make sense of expressions of the form

$$dV(A, \cdot) = \sum_{1 \leq i < j \leq n} (\partial_i V^j - \partial_j V^i)((A \cdot \mathbf{e}_i)\mathbf{e}_j - (A \cdot \mathbf{e}_j)\mathbf{e}_i).$$

Note also the following well-known identities for $f$ a smooth function and $V$ a smooth vector field, rewritten with the notation above:

$$d(\nabla f) = 0, \quad f \in \mathcal{C}^2(X),$$ 

(A.1)

$$d(fV) = \nabla f \wedge V + fdV.$$ 

(A.2)

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