Enhancement of pairwise entanglement via $\mathbb{Z}_2$ symmetry breaking

Andreas Osterloh and Guillaume Palacios
Institut für Theorietische Physik, Leibniz Universität Hannover, Appelstr. 2, 30167 Hannover, Germany

Simone Montangero
NEST-CNR-INFM & Scuola Normale Superiore, Piazza Cavalieri 7, I-56126 Pisa, Italy and
Institut für Theoretische Festkörperphysik, Universität Karlsruhe, 76128 Germany

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We study the effect of symmetry breaking in a quantum phase transition on pairwise entanglement in spin-1/2 models. We give a set of conditions on correlation functions a model has to meet in order to keep the pairwise entanglement unchanged by a parity symmetry breaking. It turns out that all mean-field solvable models do meet this requirement, whereas the presence of strong correlations leads to a violation of this condition. This results in an order-induced enhancement of entanglement, and we report on two examples where this takes place.

Entanglement is a non-locality inherent to quantum mechanics and an important resource for quantum optics and quantum information processing. It also has attained a lot of interest in the last decade from condensed matter physicists, especially since a connection between quantum non-locality and quantum phase transitions was proposed [1, 2]. This initiated a vast analysis of quantum critical models with respect to their entanglement and quantum information processing. It also has manifested the invariance of the concurrence under symmetry breaking. The models under consideration there fulfill a parity symmetry or global phase flip symmetry, meaning that the eigenstates can be chosen as superpositions from states all having the same parity of flipped spins (i.e.: odd or even). It is reflected by $\prod_i \sigma_z^i, H = 0$. The general 2 sites density matrix of such a system is given by

$$\rho_{ij} = \begin{pmatrix} A & a & F \\ a & B & C \\ F & b & D \end{pmatrix},$$

(1)

where indices for the entries of the density matrix have been omitted. The entries of $\rho$ are related to spin correlators as follows: $a = (\langle S^x \rangle + 2 \langle S^z S^z \rangle)/2$, $b = (\langle S^x \rangle - 2 \langle S^z S^z \rangle)/2$, $A = \langle S^x S^x \rangle + \langle S^z \rangle + 1/4$, $B = 1/4 - \langle S^x S^z \rangle$, $C = \langle S^x S^x \rangle + \langle S^y S^y \rangle$, $D = \langle S^x S^x \rangle - \langle S^z \rangle + 1/4$ and $F = \langle S^x S^z \rangle - \langle S^y S^y \rangle$. The symmetry-breaking manifests itself in $a, b \neq 0$. For $a = b = 0$, the square root of the eigenvalues of $\rho \rho^\dagger$ are $B + \epsilon$ and $\sqrt{AD} \pm \epsilon$. The concurrence of a 2-site reduced density matrix $\rho$ is computed from the positive semidefinite matrix $R := \rho(\sigma_y \otimes \sigma_y)\rho^\dagger(\sigma_y \otimes \sigma_y)$ as $C = \max \{0, 2\lambda_{\text{max}} - tr \sqrt{R}\}$, where $\lambda_{\text{max}}$ is the maximum eigenvalue of $\sqrt{R}$. Thus the concurrence is $C = 2\max \{0, |C| - \sqrt{AD}, |F| - B\} = 2\max \{0, C_{af}, C_f\}$. We refer to the concurrence as belonging to the anti-ferro- or ferromagnetic sector whenever the largest eigenvalue of $\sqrt{R}$ is from this sector and denote it by $C_{af}$ and $C_f$, respectively. Throughout all the letter will we adopt the notation $\langle S^\alpha S^\beta \rangle := \langle \hat{S}_i^\alpha \hat{S}_j^\beta \rangle$ and $\langle S^z \rangle := \langle \hat{S}_i^z \rangle$. For the models considered here, the Hamiltonian is real and so is the density matrix.

In order to study the effect of symmetry breaking, accompanied by non-zero $\langle S^x S^z \rangle$ and order parameter $\langle S^z \rangle$, one has to extract the dependence of the eigenvalues of $R$ on $a$ and $b$. This task simplifies since the eigenvector $(0, 1, -1, 0)$ (the singlet Bell state) and its
The LMG model \[27\] in transverse field and the concurrence will in general be affected by the broken symmetry. Nonetheless, to the best of our knowledge no such case has been reported in the literature, which motivates the search for further conditions that make the concurrence robust against symmetry breaking or for examples where the concurrence is affected. We will show in this letter that both cases occur.

Having taken the square of Eq.\((5)\) and inserted \(\kappa \to B + C - 2\sqrt{AD}\), the resulting expression is conveniently expressed in the new variables \(\lambda := b/a\) and \(a\), leading to

\[
32a^2(\lambda - \lambda_0)^2 \left[ 1 + a^2(\lambda - \lambda_1)(\lambda - \lambda_2) \right] = 0 ,
\]

where \(\lambda_0 = \sqrt{D/A}\), \(\lambda_1 + \lambda_2 = 2\kappa/A\) and \((\lambda_1 - \lambda_2)^2 = 4\kappa(B + C)/A^2\).

Both \(\lambda_1\) and \(\lambda_2\) are real for \(\kappa \geq 0\), which is mandatory for a non-vanishing concurrence. The real solutions to Eq.\((6)\) are constraints on the correlation functions of the model which insure the concurrence to be unaffected by symmetry breaking. The solution \(\lambda = \lambda_0\) can then be recast into a simple condition:

\[
\frac{(1 + 4 \langle S^x S^z \rangle^2 - 16 \langle S^z \rangle^2)}{1 + 4 \langle S^x \rangle^2 + 4 \langle S^z \rangle^2} = \frac{\langle S^x \rangle^2 - 2 \langle S^z \rangle^2}{\langle S^x \rangle^2 + 2 \langle S^z \rangle^2} .\tag{7}
\]

Note that this condition is automatically satisfied for mean-field solvable models as a direct consequence of the factorization of the two-point functions in this case.

Another real solution to Eq.\((6)\) exists if \(a \leq -1\) and

\[
\langle S^x \rangle \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2 + 2} \right)^2 = \pm \frac{1}{\pi} .\tag{8}
\]

to insure the reality of both \(a\) and \(b\). This leads to the following transcendental equation

\[
\frac{B + C}{(\kappa + A)^2} \ln \left( \frac{\kappa}{A[a] - 1} \right) = \frac{A(\kappa + A)}{A(\kappa + A) + 2\kappa^2 / \pi[a]} .\tag{9}
\]

Eq.\((9)\) has a unique real solution for arbitrary \(A, B\) and \(\kappa\). The rewriting of this condition in terms of correlation function is lengthy but straightforward. There is numerical evidence that \(a < -1\) only occurs for a two-site reduced density matrix of rank 3 or less. Interestingly, rank 3 and 2 are assumed for the (2-magnon) W state \(\sum_i w_i \sigma_i^x \sigma_i^y \sigma_i^z \mathbb{I} \) and GHZ (cat) state \(|\psi\rangle = |\psi\rangle + |\psi\rangle\rangle\) of the whole chain, where \(w_i \in \mathbb{C}\) and \(|\psi\rangle\rangle\) are in opposite direction fully polarized states. If Eqs.\((7)\) and \((8)\) are violated, the concurrence will be affected by symmetry-breaking, as condition \((5)\) is then violated.

LMG model The LMG model \[27\] in transverse field is described by the Hamiltonian

\[
H_{LMG} = -\frac{1}{N} \sum_{i < j} (\sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y) - h \sum_i \sigma_i^z \tag{10}
\]

where the \(\sigma_i^x\) are the Pauli matrices for the \(i\)-th lattice site and \(N\) is the chain length. Any two spins are interacting with the same coupling strength. The prefactor \(1/N\) leads to a finite free energy per spin in the
thermodynamic limit. \( H \) commutes with the total spin and the spin-flip operator \( \prod_i \sigma_i^z \) for any anisotropy parameter \( \gamma \). This discrete \( \mathbb{Z}_2 \) symmetry is broken at the quantum critical point, \( h_c = 1 \). The factorizing field of this model is \( h_f = \sqrt{\gamma} \). Due to its infinite-range coupling, the mean-field approximation for the LMG model becomes exact in the thermodynamic limit [28]. In this limit, condition (7) is satisfied for arbitrary transverse field. The non-trivial behavior of the entanglement of the LMG lies in the finite-size corrections to the mean-field solution [3].

\[ H_{XY} = -\frac{1}{4} \sum_i (\sigma_i^x \sigma_{i+1}^x + \gamma \sigma_i^y \sigma_{i+1}^y) - h \sum_i \sigma_i^z, \quad (12) \]

where \( h \) is the transverse field strength and \( \gamma \) the anisotropy parameter. Below the quantum critical field \( h_c = (1 + \gamma)/4 \) the parity symmetry is broken; the factorizing field is at \( h_f = \sqrt{\gamma}/2 \). We numerically computed the ground state nearest neighbor (n.n.) concurrence by means of the Density Matrix Renormalization Group [29]. This powerful numerical technique finds an optimal truncated bases of size \( m \sim 100 \) to describe the spin chain wave function keeping the desired precision [30]. In Fig. 2 we show the results of the numerical simulations for the concurrence of the central spins (in order to minimize boundary effects) of a chain of length \( N = 199 \) and different anisotropy values.

The concurrence for the even sector \( C_{\text{Sym}} \) and broken symmetry ground state \( C_{\text{Broken}} \) are then compared for different anisotropy values. The concurrence vanishes at the factorizing field and its derivative with respect to the transverse field at the critical point diverges as expected [1]. The even and odd sector concurrences are found to coincide for the parameter range we considered (data not shown). We used an additional field \( h_B \) along \( \sigma_x \) to break the symmetry; our results are stable with respect to changes in the field strength \( h_B \in [10^{-4} : 10^{-8}] \) and the field direction in the \( XY \) plane (data not shown). Below the factorizing field the concurrence of the broken symmetry and even ground state are clearly different while they are equal between the factorizing and the critical field as there the condition (5) holds [25]. In

![FIG. 1: Rescaled concurrence for the LMG model: The even sector (full line) and the symmetry broken (dashed line) concurrence is shown for \( N = 100 \) and \( \gamma = 0.5 \); the inset shows the difference between the rescaled concurrence in the \( \mathbb{Z}_2 \)-symmetric case and in the \( \mathbb{Z}_2 \)-broken case.](image1)

![FIG. 2: N.n. concurrence for the anisotropic XY model for \( N = 199 \).](image2)
\( (h/h_f < 1) \) the difference is not negligible. The inset of Fig. 3 reports the finite size scaling of \( |C_{\text{Sym}} - C_{\text{Broken}}| \). Differently from the LMG Model, the broken symmetry effect on the concurrence does not vanish in the thermodynamic limit. It approaches a non-zero value with oscillating behavior. This means that for the XY model conditions \([28,29]\) and thus Eq. (5) are violated below \( h_f \).

**Conclusions.** We have found conditions on spin correlation functions, which ensure that the concurrence is invariant respect to breaking of a parity symmetry. They are necessary and sufficient in a regime complementary to where the relations from Ref. [21] do apply. For mean field exact models, one of the conditions (Eq. 7) is satisfied and hence will the rescaled concurrence be unaffected by parity symmetry breaking. The further conditions (Eqs. 8,9) only emerged for the reduced two-site density matrix having rank three or less. Interestingly, this occurs for the whole system being in a W-type state with two running flipped spins in the ferromagnetically polarized state and a W or GHZ-state, respectively. It would be interesting to verify, whether such states may satisfy the corresponding condition on correlation functions: W-states would lead to a long range concurrence, which has been observed in [13] close to the factorizing point. Numerical studies reveal that the concurrence is affected by symmetry breaking for the LMG and the XY model. As a consequence, the conditions [7,9] on the spin correlations are violated up to perhaps a single value for the magnetic field in the LMG model. We gave certain conditions that might admit scenarii different from a robust concurrence between \( h_c \) and \( h_f \). Their investigation could provide important insight in the interplay of entanglement and quantum phase transitions.

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[31] an analytical proof of this assumption is left to a future publication.