Article

Exponential Synchronization of Hyperbolic Complex Spatio-Temporal Networks with Multi-Weights

Hongkun Ma 1 and Chengdong Yang 2,*

1 School of Economics, Shandong Normal University, Jinan 250358, China; mahongkun@sdnu.edu.cn
2 School of Information Science and Technology, Linyi University, Linyi 276005, China
* Correspondence: yangchengdong@lyu.edu.cn

Abstract: This paper deals with the leader-following synchronization of first-order, semi-linear, complex spatio-temporal networks. Firstly, two sorts of complex spatio-temporal networks based on hyperbolic partial differential equations (CSTNHPDEs) are built: one with a single weight and the other with multi-weights. Then, a new distributed controller is designed to address CSTNHPDE with a single weight. Sufficient conditions for the synchronization and exponential synchronization of CSTNHPDE are presented by showing the gain ranges. Thirdly, the proposed distributed controller addresses of CSTNHPDE with multi-weights, and gain ranges are obtained for synchronization and exponential synchronization, respectively. Finally, two examples show the effectiveness and good performance of the control methods.

Keywords: synchronization; complex networks; multi-weights; hyperbolic partial differential equations

MSC: 05C82

1. Introduction

The synchronization of complex networks, a group dynamical behavior, aims to drive nodes to perform a designated task synchronously. It has been applied to many engineering aspects, such as intelligent traffic [1,2], circuit systems [3], image processing [4–6], smart grids [7], secure communication [8,9], multi-agent systems [10], rumor propagation [11], data security [12], biological systems [13], etc.

A number of important works discuss the synchronization of complex networks [14–18]. This literature shows node dynamics depending only on time. In practice, the dynamics of all processes are spatio-temporal [19–21]. As a consequence, it is necessary to study complex spatio-temporal networks (CSTNs), which is with spatio-temporal characteristics [22]. Wu et al. studied the synchronization of CSTNs with space-independent coefficients and space-dependent coefficients, with or without spatio-temporal disturbance [23]. Huang et al. proposed a fuzzy synchronization method for nonlinear CSTNs with reaction—diffusion terms [24]. Luo et al. studied event-triggered control for the finite-time synchronization of reaction–diffusion CSTNs [25]. Yang et al. studied the boundary control of fractional-order CSTNs [26]. Zheng et al. researched synchronization analysis for fractional-order CSTNs with time delays [27]. Shen et al. studied the \( H_{\infty} \) synchronization of Markov jump CSTNs using an observer-based method [28]. Kabalan et al. studied boundary control for the synchronization of leader–follower CSTNs with in-domain coupling [29].

Most references are modeled by parabolic PDEs, while there are few methods studying hyperbolic PDEs. There are many hyperbolic PDEs systems in practice, including shallow-water systems [30], epidemic models [31], district heating networks [32], heat exchangers [33], and reactor models [34]. Therefore, it is important to study the synchronization of hyperbolic PDEs-based CSTNs (HPDECSTNs).
Chueshov presented invariant manifolds and nonlinear master–slave synchronization hyperbolic and parabolic CSTNs [35]. Li studied the synchronization, exact synchronization and approximate synchronization of HPDECSTNs [36]. Li and Lu researched exact-boundary synchronization for a kind of first-order hyperbolic system [37]. Lu proposed a local exact-boundary synchronization for a kind of first-order, quasi-linear hyperbolic system [38]. However, technical difficulties remain regarding the synchronization of a semi-linear, first-order HPDECSTNs when the convection coefficient is symmetric semi-negative definite or semi-positive definite, which motivate this paper. Multi-weights exist in many physical networks [39–43]. As a result, HPDECSTN with multi-weights is important and remains challenging.

This paper mainly studies the leader-following synchronization control of a semi-linear HPDECSTN with two sorts of boundary conditions in a one-dimensional space. This paper’s contributions are as follows: (1) Two sorts of HPDECSTN models are built, one with a single weight and the other with multi-weights. (2) A new distributed controller is designed to address CSTNHPDE with a single weight. Sufficient conditions for the synchronization and exponential synchronization of CSTNHPDE are presented by providing the gain ranges. (3) The proposed distributed controller addresses CSTNHPDE with multi-weights and gain ranges, obtained for synchronization and exponential synchronization, respectively. (4) Two examples show the effectiveness and good performance of the control methods.

Notations: Let $I_N$ denote the identity matrix with Nth order, $P > 0$ $(P < 0)$ denote symmetric positive definite (negative definite), and $\lambda_{\text{max(min)}}(\cdot)$ denote the maximum (minimum) eigenvalue.

2. Problem Formulation

This paper first studies a class of leader-following, semi-linear, hyperbolic PDE-based, complex spatio-temporal networks (HPDECSTNs) with a single weight. The following node is assumed to be

\[
\begin{aligned}
\frac{\partial z_i(\zeta, t)}{\partial t} &= \frac{\partial z_i(\zeta, t)}{\partial \zeta} + Az_i(\zeta, t) + Bf(z_i(\zeta, t)) + c \sum_{j=1}^{N} g_{ij} \Gamma z_j(\zeta, t) + u_i(\zeta, t), \\
z_i(L, t) &= 0, \\
z_i(\zeta, 0) &= z_i^0(\zeta), i \in \{1, 2, \cdots, N\},
\end{aligned}
\]

(1)

where $(\zeta, t) \in [0, L] \times [0, \infty)$ are space and time, respectively. $z_i(\zeta, t)$ and $u_i(\zeta, t) \in \mathbb{R}^n$ are the state and control input, respectively. $0 < L \in \mathbb{R}$ is a constant. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times \mathbb{R}^n}$, and $\Gamma \in \mathbb{R}^{n \times n}$ are constant matrices. $f(\cdot)$ is a nonlinear function. The coupling strength $c > 0$ is a constant. $G = (g_{ij})_{N \times N}$ satisfies $g_{ii} = -\sum_{j=1, j \neq i}^{N} g_{ij}$.

The leader node is assumed to be

\[
\begin{aligned}
\frac{\partial s(\zeta, t)}{\partial t} &= \frac{\partial s(\zeta, t)}{\partial \zeta} + As(\zeta, t) + Bf(s(\zeta, t)), \\
s(L, t) &= 0, \\
s(\zeta, 0) &= s^0(\zeta),
\end{aligned}
\]

(2)

where $s(\zeta, t) \in \mathbb{R}^n$ is the state.

This paper aims to study a distributed controller $u_i(\zeta, t)$, driving HPDECSTN (1) to the leader node (2), designed as

\[
u_i(\zeta, t) = d_i(s(\zeta, t) - z_i(\zeta, t)),
\]

(3)

where $d_i$ are the control gains to be determined.
Definition 1. HPDECSTN (1) reaches synchronization, if
\[
\lim_{t \to \infty} ||z_i(\zeta, t) - s(\zeta, t)|| = 0, i \in \{1, 2, \cdots, N\}. \tag{4}
\]

Definition 2. Given \( \rho > 0 \), HPDECSTN (1) reaches exponential synchronization, if there is a real number \( \sigma > 0 \) such that
\[
||z_i(\zeta, t) - s(\zeta, t)|| \leq \sigma \exp(-2\rho t)||z_i^0(\zeta) - s^0(\zeta)||, i \in \{1, 2, \cdots, N\}. \tag{5}
\]

Assumption 1. For any \( \zeta_1, \zeta_2 \in \mathbb{R}, \) then \( 0 < \lambda \in \mathbb{R}, \) satisfying
\[
|f(\zeta_1) - f(\zeta_2)| \leq \lambda |\zeta_1 - \zeta_2|. \tag{6}
\]

3. Synchronization of HPDECSTNs with a Single Weight

Let the synchronization error be \( e_i(\zeta, t) \overset{\Delta}{=} z_i(\zeta, t) - s(\zeta, t) \). The error system of between HPDECSTN (1) and (2) yields
\[
\begin{align*}
\frac{d e_i(\zeta, t)}{dt} &= \frac{\partial e_i(\zeta, t)}{\partial \zeta} + (I_N \otimes A)e_i(\zeta, t) + (I_N \otimes B)F(e_i(\zeta, t)) + (G \otimes \Gamma)e_i(\zeta, t) + u_i(t), \\
\end{align*}
\]
where \( e_i^0(\zeta) \overset{\Delta}{=} z_i^0(\zeta) - s^0(\zeta), \) \( u(t) \overset{\Delta}{=} [u_1^T(t), u_2^T(t), \cdots, u_N^T(t)]^T, \) \( e(\zeta, t) \overset{\Delta}{=} [e_1^T(\zeta, t), e_2^T(\zeta, t), \cdots, e_N^T(\zeta, t)]^T, \) \( F(e_i(\zeta, t)) \overset{\Delta}{=} f(z_i(\zeta, t)) - f(s(\zeta, t)), \)
and \( F(e(\zeta, t)) \overset{\Delta}{=} [F^T(e_1(\zeta, t)), F^T(e_2(\zeta, t)), \cdots, F^T(e_N(\zeta, t))]^T. \)

Theorem 1. Suppose Assumption 1 holds. HPDECSTN (1) reaches synchronization under the controller (2), if
\[
d_i > \lambda_{\max}(\Psi), \tag{8}
\]
where \( \Psi \overset{\Delta}{=} I_N \otimes \frac{A + A^T}{2} + 0.5I_N \otimes BB^T + 0.5\lambda^2 I_{N_N} + 0.5c(G \otimes \Gamma + G^T \otimes \Gamma^T). \)

Proof. Choose the Lyapunov functional candidate as follows:
\[
V(t) = 0.5 \int_0^t e^T(\zeta, t)e(\zeta, t)d\zeta. \tag{9}
\]
One has
\[
\begin{align*}
V(t) &= \int_0^L e^T(\zeta, t)\frac{d e_i(\zeta, t)}{dt}d\zeta \\
&= \int_0^L e^T(\zeta, t)\frac{\partial e_i(\zeta, t)}{\partial \zeta}d\zeta \\
&+ \int_0^L e^T(\zeta, t)(I_N \otimes A + cG \otimes \Gamma)e(\zeta, t)d\zeta \\
&+ \int_0^L e^T(\zeta, t)F(e(\zeta, t))d\zeta - \int_0^L e^T(\zeta, t)(D \otimes I_n)e(\zeta, t)d\zeta,
\end{align*}
\]
where \( D \overset{\Delta}{=} \text{diag}\{d_1, d_2, \cdots, d_N\}. \)
By integrating by parts,
\[
\int_0^L e^T(\zeta, t) \frac{\partial e(\zeta, t)}{\partial \zeta} d\zeta \\
= e^T(\zeta, t) e(\zeta, t) \big|_{\zeta=L}^{\zeta=0} \\
- \int_0^L \frac{\partial e^T(\zeta, t)}{\partial \zeta} e(\zeta, t) d\zeta \\
= -e^T(0, t) e(0, t) \\
- \int_0^L e^T(\zeta, t) \frac{\partial e(\zeta, t)}{\partial \zeta} d\zeta
\]
which implies
\[
\int_0^L e^T(\zeta, t) \frac{\partial e(\zeta, t)}{\partial \zeta} d\zeta \leq -0.5 e^T(0, t) e(0, t).
\] (12)

Under Assumption 1,
\[
\int_0^L e^T(\zeta, t) BF(e(\zeta, t)) d\zeta \\
\leq 0.5 \int_0^L e^T(\zeta, t) BB^T e(\zeta, t) d\zeta + 0.5 \int_0^L F^T(\zeta, t) F(\zeta, t) d\zeta \\
= \int_0^L e^T(\zeta, t) (0.5 I_N \otimes BB^T + 0.5 \chi^2 I_{nn}) e(\zeta, t) d\zeta.
\] (13)
The substitution of (11)–(13) into (10) yields,
\[
V(t) \leq \int_0^L e^T(\zeta, t) (\Psi - D \otimes I_n) e(\zeta, t) d\zeta,
\] (14)
where \( \Psi = I_N \otimes \frac{\Gamma + \Gamma^T}{2} + 0.5 I_N \otimes BB^T + 0.5 \chi^2 I_{nn} + c G \otimes \Gamma \) and \( D = \text{diag}\{d_1, d_2, \ldots, d_N\} \).

It is obvious that (8) implies
\[
\Psi - D \otimes I_n < 0.
\] (15)

The substitution of (15) into (14) yields, \( \dot{V}(t) \leq -\lambda_{\text{min}}(D \otimes I_n - \Psi)||e(\cdot, t)||, \) for all non-zero \( e(\zeta, t) \), implying synchronization of HPDECSTN (1). \( \square \)

**Theorem 2.** Suppose Assumption 1 holds. Given \( \rho > 0 \), HPDECSTN (1) reaches exponential synchronization under the controller (2), if
\[
d_i > \lambda_{\text{max}}(\Psi + \rho I_{nn}),
\] (16)
where \( \Psi = I_N \otimes \frac{\Gamma + \Gamma^T}{2} + 0.5 I_N \otimes BB^T + 0.5 \chi^2 I_{nn} + 0.5 c (G \otimes \Gamma + \Gamma^T \otimes \Gamma^T) \).

**Proof.**
\[
\dot{V}(t) + 2\rho V(t) \\
\leq \int_0^L e^T(\zeta, t) (\Psi + \rho I_{nn} - D \otimes I_n) e(\zeta, t) d\zeta \\
\leq 0,
\] (17)
which implies
\[
V(t) \leq V(0) \exp(-2\rho t).
\] (18)
It follows from (18) that
\[ ||e_i(\zeta, t)||^2 \leq \sigma \exp(-2\rho t), \tag{19} \]
where \( \sigma = ||e_i^0(\zeta)||^2 \). Therefore, exponential synchronization is obtained. \( \square \)

4. Synchronization of HPDECSTNs with Multi-Weights

This section studies a class of semi-linear HPDECSTNs with multi-weights, where the following node is as follows:

\[
\begin{align*}
\frac{\partial z_i(\zeta, t)}{\partial t} &= \frac{\partial z_i(\zeta, t)}{\partial \zeta} + A z_i(\zeta, t) + B f(z_i(\zeta, t)) + c_1 \sum_{j=1}^{N} g_{ij}^1 \Gamma_1 z_j(\zeta, t) \\
&\quad + c_2 \sum_{j=1}^{N} g_{ij}^2 \Gamma_2 z_j(\zeta, t) + \cdots + c_l \sum_{j=1}^{N} g_{ij}^l \Gamma_l z_j(\zeta, t) + u_i(\zeta, t), \tag{20}
\end{align*}
\]

where \( \Gamma_1 \in \mathbb{R}^{n \times n}, \Gamma_2 \in \mathbb{R}^{n \times n}, \ldots, \Gamma_l \in \mathbb{R}^{n \times n} \) are constant matrices. \( G_k = (g_{ij}^k)^N_{i,j} \) satisfies

\[
\sum_{j=1}^{N} g_{ij}^k = - \sum_{j=1}^{N} g_{ji}^k.
\]

The error system of between HPDECSTN (20) and (2) with multi-weights can be obtained as

\[
\begin{align*}
\frac{\partial e_i(\zeta, t)}{\partial t} &= \frac{\partial e_i(\zeta, t)}{\partial \zeta} + (I_N \otimes A) e(\zeta, t) + F(e(\zeta, t)) + c_1 (G_1 \otimes \Gamma_1) e(\zeta, t) \\
&\quad + c_2 (G_2 \otimes \Gamma_2) e(\zeta, t) + \cdots + c_l (G_l \otimes \Gamma_l) e(\zeta, t) + u_i(\zeta, t), \tag{21}
\end{align*}
\]

**Theorem 3.** Suppose that Assumption 1 holds. HPDECSTN (20) reaches synchronization under the controller (2), if

\[
d_i > \lambda_{\max}(\Xi), \tag{22}
\]

where \( \Xi \triangleq I_N \otimes \frac{A + A^T}{2} + 0.5I_N \otimes BB^T + 0.5\lambda^2 I_{2n} + 0.5c_1 (G_1 \otimes \Gamma_1 + G_1^T \otimes \Gamma_1^T) + 0.5c_2 (G_2 \otimes \Gamma_2 + G_2^T \otimes \Gamma_2^T) + \cdots + 0.5c_l (G_l \otimes \Gamma_l + G_l^T \otimes \Gamma_l^T). \)

**Proof.** The proof is similar to that of Theorem 1, and so it is omitted. \( \square \)

**Theorem 4.** Suppose that Assumption 1 holds. Given \( \rho > 0 \), HPDECSTN (20) reaches exponential synchronization under the controller (2), if

\[
d_i > \lambda_{\max}(\Xi + \rho I_{2n}), \tag{23}
\]

where \( \Xi \triangleq I_N \otimes \frac{A + A^T}{2} + 0.5I_N \otimes BB^T + 0.5\lambda^2 I_{2n} + 0.5c_1 (G_1 \otimes \Gamma_1 + G_1^T \otimes \Gamma_1^T) + 0.5c_2 (G_2 \otimes \Gamma_2 + G_2^T \otimes \Gamma_2^T) + \cdots + 0.5c_l (G_l \otimes \Gamma_l + G_l^T \otimes \Gamma_l^T). \)

**Proof.** The proof is similar to that of Theorem 2, and so it is omitted. \( \square \)

**Remark 1.** This paper addresses not only the synchronization of HPDECSTNs, but also the exponential synchronization. Moreover, this paper addresses HPDECSTNs not only with a single weight, but also with multi-weights.
Remark 2. Compared with the results modeled by ordinary differential equations with multi-weights [39–43], this paper addresses spatio-temporal models with multi-weights.

Remark 3. Different from the control design for synchronization of parabolic PDEs-based CSTNs [44,45], this paper deals with the synchronization of hyperbolic PDEs-based CSTNs.

Remark 4. Only a few important results discussed the synchronization, exact synchronization and approximate synchronization of HPDECSTNs [36–38]. Different from those with a single weight, this paper addresses the case with multi-weights.

5. Numerical Simulation

Example 1. Consider a single weighted HPDECSTN (1) with random initial conditions and

\[
A = \begin{bmatrix} 5.1 & 2.7 \\ -1.1 & 4.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 1.5 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad L = 1, \quad c = 0.2, \quad f(\cdot) = \tanh(\cdot).
\]

(24)

The single weight takes

\[
G = \begin{bmatrix} -5 & 1 & 2 & 2 \\ -1 & 4 & 3 & 0 \\ 1 & 1 & -3 & 1 \\ -3 & -2 & -3 & 8 \end{bmatrix}.
\]

(25)

Figure 1 shows that HPDECSTN (1) cannot reach synchronization without control. It is obvious that \( \chi = 1 \). With Theorem 1, solve \((16)\) by Matlab, the feedback gains \( d_i = 12.04 \) are obtained. Figure 2 shows that HPDECSTN (1) reaches exponential synchronization under the controller (2) with \(d_i = 12.04\). The controller (2) with the feedback gains \( d_i = 12.04 \) is shown in Figure 3.

![Figure 1: \( c(\zeta, t) \) of HPDECSTN (1) without control.](image-url)
Figure 2. $e(\zeta, t)$ of HPDECSTN (1) with control.

Figure 3. The control input of HPDECSTN (1).

**Example 2.** Consider multi-weighted HPDECSTN (20) with random initial conditions and the same parameters as those of Example 1, except:

\[ \begin{align*}
    c_1 &= 0.8, \\
    c_2 &= 0.3, \\
    c_3 &= 0.4, \\
    c_4 &= 0.5
\end{align*} \]  

(26)
The weights take

\[
G_1 = \begin{bmatrix}
-5 & 1 & 2 & 2 \\
-1 & 4 & 3 & 0 \\
1 & 1 & -3 & 1 \\
-3 & -2 & -3 & 8
\end{bmatrix},
G_2 = \begin{bmatrix}
6 & -1 & -2 & -3 \\
-2 & 4 & -3 & 1 \\
1 & 2 & -3 & 0 \\
-1 & -3 & -3 & 7
\end{bmatrix},
\]

(27)

\[
G_3 = \begin{bmatrix}
-2 & 4 & 1 & -3 \\
2 & -1 & 3 & -4 \\
-2 & -1 & -2 & 5 \\
6 & 2 & -3 & -5
\end{bmatrix},
G_4 = \begin{bmatrix}
-5 & 1 & 2 & 2 \\
1 & 3 & -2 & -2 \\
-7 & -2 & 3 & 6 \\
-3 & 1 & -3 & 5
\end{bmatrix}.
\]

(28)

Figure 4 shows that HPDECSTN (20) cannot reach synchronization without control. With Theorem 4, solving (23) using Matlab, the feedback gains \(d_i = 26.21\) are obtained. Figure 5 shows that HPDECSTN (20) reaches exponential synchronization under controller (2) with \(d_i = 26.21\). The controller (2) with the feedback gains \(d_i = 26.21\) is shown in Figure 6.

Figure 4. \(e(\zeta, t)\) of HPDECSTN (1) without control.
6. Conclusions

This paper has dealt with the leader-following synchronization control of two classes of semi-linear HPDECSTNs: one HPDECSTN with a single weight, and the other with multi-weights. To drive HPDECSTNs to synchronization, one new distributed controller was constructed. Dealing with HPDECSTNs with a single weight, sufficient conditions for synchronization and exponential synchronization of CSTNHPDE were presented by providing gain ranges. Furthermore, the proposed distributed controller was used to address
CSTNHPDE with multi-weights and gain ranges, which were obtained for synchronization and exponential synchronization, respectively. Two examples illustrated the effectiveness of the developed theoretical results. In future work, the event-triggered control and pinning control of HPDECSCTNs will be studied.

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