EXTENDED SCHUR’S $Q$-FUNCTIONS AND THE FULL KOSTANT–TODA HIERARCHY ON THE LIE ALGEBRA OF TYPE $D$

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Dedicated to the memory of Hermann Flaschka

Abstract. The full Kostant–Toda hierarchy on a semisimple Lie algebra is a system of Lax equations, in which the flows are determined by the gradients of the Chevalley invariants. This paper is concerned with the full Kostant–Toda hierarchy on the even orthogonal Lie algebra. By using a Pfaffian of the Lax matrix as one of the Chevalley invariants, we construct an explicit form of the flow associated to this invariant. As a main result, we introduce an extension of the Schur’s $Q$-functions in the time variables, and use them to give explicit formulas for the polynomial $\tau$-functions of the hierarchy.

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1. Introduction

We start with fixing some notations. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank $n$. We fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{\Pi} \oplus \mathfrak{h} \oplus \mathfrak{n},$$

where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}$ (resp. $\mathfrak{\Pi}$) is the nilradical of a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ (resp. $\mathfrak{\Pi} = \mathfrak{h} \oplus \mathfrak{\Pi}$). Let $\Sigma \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and $\Sigma^+$ and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ the positive system and the simple system associated with $\mathfrak{n}$ respectively. Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be an invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. If $\mathfrak{g}$ is simple, then $\kappa$ is proportional to the Killing form of $\mathfrak{g}$. Let $\{H_i : 1 \leq i \leq n\} \cup \{X_\alpha : \alpha \in \Sigma\}$ be a Chevalley basis satisfying

$$[H_i, H_j] = 0 \quad (1 \leq i, j \leq n),$$

$$[H_i, X_\alpha] = \alpha(H_i)X_\alpha \quad (1 \leq i \leq n, \alpha \in \Sigma),$$

$$[X_\alpha, X_{-\alpha}] = H_\alpha \quad (\alpha \in \Sigma),$$

$$[X_\alpha, X_\beta] = N_{\alpha, \beta}X_{\alpha+\beta} \quad (\alpha, \beta \in \Sigma),$$

where $H_\alpha \in \mathfrak{h}$ and we use the convention $X_{\alpha+\beta} = 0$ if $\alpha + \beta \notin \Sigma$. Let $\mathcal{G}$ be a connected complex semisimple Lie group with Lie algebra $\text{Lie} \mathcal{G} = \mathfrak{g}$.

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The full Kostant–Toda hierarchy (f-KT hierarchy for short) on the Lie algebra \( g \) is defined as follows. Let \( t = (t_1, t_2, \ldots, t_n) \) be time variables and consider an element \( L(t) \in g \) of the form
\[
L(t) = \sum_{i=1}^{n} a_i(t) H_i + \sum_{i=1}^{n} X_{\alpha_i} + \sum_{\alpha \in \Sigma^+} b_\alpha(t) X_{-\alpha},
\]
which is called the Lax matrix. Here \( a_i(t) \) and \( b_\alpha(t) \) are functions of the multi-time variables \( t \).

It is known as Chevalley’s theorem that the ring \( \mathbb{C}[g]^G \) of \( G \)-invariant polynomial functions on \( g \) is generated by \( n \) algebraically independent homogeneous polynomials \( I_1, \ldots, I_n \), which are referred to as the fundamental invariants or the Chevalley invariants. Then the f-KT hierarchy is defined by
\[
\frac{\partial L}{\partial t_k} = [P_k^{\geq 0}, L] \quad \text{with} \quad P_k = (\nabla I_k)(L) \quad (1 \leq k \leq n),
\]
where \( \nabla f : g \to g \) denotes the gradient of \( f \in \mathbb{C}[g] \) given by
\[
\kappa(\nabla f(X), Z) = \left. \frac{d}{dt} f(X + tZ) \right|_{t=0} \quad (Z \in g),
\]
and \( P_k^{\geq 0} \) is the image of \( P_k \) under the projection onto \( b \) with respect to the direct sum decomposition \( g = b \oplus \pi \).

The full Kostant–Toda lattice, the first member of the hierarchy with \( I_1 = \frac{1}{2} \text{tr}(L^2) \) (i.e. \( P_1 = L \)) on \( g = \mathfrak{sl}_{n+1}(\mathbb{C}) \), was first introduced by Ercolani, Flaschka and Singer in [8], where the main purpose of their paper is to show its complete integrability by constructing a sufficient number of the first integrals, called chop integrals. The integrability of the lattice for other types of semisimple Lie algebras was shown by Gekhtman and Shapiro [6], in which they gave a Lie theoretic meaning of the chop integrals and showed that the generic orbits of the f-KT lattices are completely integrable in a noncommutative sense.

In [18, 7], the f-KT lattice with tridiagonal (or Jacobi) element \( L \) of a semi-simple Lie algebra \( g \), called simply the Kostant–Toda (KT) lattice, is studied by virtue of the representation theory. The main result of Kostant [18] is to show that the KT lattice is a completely integrable Hamiltonian system and the integration of the KT lattice is completely determined by the weight structure of the fundamental representations of the corresponding group \( G \). More precisely, the integration turns out to be an Iwasawa-type factorization problem. Then Goodman and Wallach [7] found formulas of the solutions in terms of the \( \tau \)-functions which are given by
\[
\tau_i(t) = \langle v_{\varpi_i}, g(t) \cdot v_{\varpi_i} \rangle \quad (1 \leq i \leq n),
\]
where \( \langle \cdot, \cdot \rangle \) is a non-degenerate bilinear form on the irreducible highest weight representation corresponding to the fundamental weight \( \varpi_i \) with highest weight vector \( v_{\varpi_i} \). Here the group element \( g(t) \in G \) is defined by \( g(t) = \exp(tL(0)) \) with the initial matrix \( L(0) \). In this paper, we extend this formula to the \( \tau \)-function for the f-KT hierarchy on \( g = \mathfrak{so}_{2n}(\mathbb{C}) \), and give all the polynomial \( \tau \)-functions.

There have been considerable interest on the singular solutions of the f-KT hierarchy. The singular structures are determined by the zeros of the \( \tau \)-functions, called the Painlevé divisors in [4] (see also [2]). Around a point on the Painlevé divisor, the \( \tau \)-function admits a power series expansion in \( t = (t_1, \ldots, t_n) \), whose leading order term is given by a Schur-type polynomial. In [4], Flaschka and Heine determine the obstructions of the Gauss decomposition and link the result to the singularity of the KT lattice.

The Schur-type functions also appear in the \( \tau \)-functions of the KP-type hierarchies [21, 23, 13, 9, 19]. In particular, You [24] gives the polynomial \( \tau \)-functions of the BKP hierarchy in terms of Schur’s \( Q \) functions, which were first introduced by Schur in the study of projective representations of symmetric group. In this paper, we will introduce an extension of Schur’s \( Q \)-functions, and show that these extensions give the polynomial \( \tau \)-functions to the f-KT hierarchy on \( g = \mathfrak{so}_{2n}(\mathbb{C}) \).
You [21, 25] also gives identities relating 2-reduced Schur functions and Schur’s $Q$-functions (see also [12]). In general, any 2-reduced Schur function can be expressed as a sum of products of pairs of Schur’s $Q$-functions (see [22, 8]). In the course of the proof of our main theorem, we see that some of these identities are interpreted as relations among matrix coefficients involving the half-spin representations of the spin group $\text{Spin}_{2n}(\mathbb{C})$.

Our main goal in this paper is to give a complete list of the rational solutions (i.e., the polynomial $\tau$-functions) to the f-KT hierarchy on the Lie algebra $\mathfrak{so}_{2n}(\mathbb{C})$. Sections 2 and 3 discuss general features of the f-KT hierarchy. In Section 2, we prove that the flows in (1.2) mutually commute and that any $G$-invariant polynomial is a first integral of the f-KT hierarchy (Proposition 2.1). In Section 4 we show that the solutions of the f-KT hierarchy is completely determined and expressed by the $\tau$-functions, which are defined as the matrix coefficients on the irreducible highest weight representations of $G$ (Definition 3.4 and Proposition 3.7). We also give the formula of polynomial $\tau$-functions (Proposition 3.10). Sections 4 and 5 focus on the f-KT hierarchy on the Lie algebra $\mathfrak{so}_{2n}(\mathbb{C})$. In Section 4, by choosing the specific Chevalley invariants including a Pfaffian invariant, we give an explicit formula of the f-KT hierarchy on $\mathfrak{so}_{2n}(\mathbb{C})$ (Proposition 4.4). In particular, one of the Chevalley invariants is given by a Pfaffian of the Lax matrix, and the flow associated with the Pfaffian invariant has not been found explicitly in the previous works. In Section 5 we introduce an extension of Schur’s $Q$-functions in the time variables $t = (t_1, t_3, \ldots, t_{2n-3}, s)$, where the variable $s$ is the flow parameter corresponding to the Chevalley invariant given by Pfaffian. Our main result (Theorem 5.1) provides explicit formulas for the polynomial $\tau$-functions in terms of extended Schur’s $Q$-functions.

2. The full Kostant–Toda hierarchy

In this section we prove that the flows generated by the Chevalley invariants in the f-KT hierarchy (1.2) mutually commute and that any $G$-invariant polynomial is a first integral of the f-KT hierarchy.

Let $\{Z_1, \ldots, Z_n\}$ be a basis of $\mathfrak{g}$ and $\{\zeta_1, \ldots, \zeta_N\}$ be the dual basis of $\mathfrak{g}^*$. We can identify a polynomial function $f \in \mathbb{C}[\mathfrak{g}]$ with a polynomial in $\zeta_1, \ldots, \zeta_N$. Then we have

$$\nabla f(X) = \sum_{i=1}^{N} \frac{\partial f}{\partial \zeta_i}(X)Z_i,$$

where $\{Z^1, \ldots, Z^N\}$ is the dual basis of $\mathfrak{g}$ with respect to the bilinear form $\kappa$.

In what follows, we choose the quadratic Casimir element as the first member $I_1$ of the Chevalley invariants, i.e., we put

$$I_1 = \frac{1}{2} \sum_{i=1}^{N} \zeta_i \zeta^i,$$

where $\{\zeta^1, \ldots, \zeta^N\}$ is the basis of $\mathfrak{g}^*$ dual to $\{Z^1, \ldots, Z^N\}$. Since $(\nabla I_1)(X) = X$, the first member of the hierarchy (1.2), called the f-KT lattice, is given by

$$\frac{\partial L}{\partial t_1} = [L_{\geq 0}, L].$$

Now we have the following proposition.

**Proposition 2.1.** Let $L$ be a solution to the f-KT hierarchy (1.2).

1. We have

$$\frac{\partial^2 L}{\partial t_k \partial t_1} = \frac{\partial^2 L}{\partial t_l \partial t_k}$$

for $1 \leq k, l \leq n$.

Hence the fKT flows commute with each other.
(2) If \( f \in \mathbb{C}[g]^G \), then we have
\[
\frac{\partial}{\partial t_k} f(L) = 0 \quad (1 \leq k \leq n).
\]

Hence \( f(L) \) is invariant under the \( f \)-KT flows.

In the proof of this proposition, we use the following properties of gradients.

**Lemma 2.2.**  
(1) For \( f \in \mathbb{C}[g]^G \), \( g \in G \) and \( X \in g \), we have
\[
\text{Ad}(g)(\nabla f)(X) = (\nabla f)(\text{Ad}(g)X).
\]
(2) For \( f \in \mathbb{C}[g]^G \) and \( X, Y \in g \), we have
\[
[(\nabla f)(X), Y] = \sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial \zeta_i \partial \zeta_j}(X)\zeta_j([X,Y])Z^i.
\]
(3) For \( f \in \mathbb{C}[g]^G \) and \( X \in g \), we have
\[
[(\nabla f)(X), (\nabla f)(X)] = 0.
\]
(4) For \( f, g \in \mathbb{C}[g]^G \) and \( X \in g \), we have
\[
[(\nabla f)(X), (\nabla g)(X)] = 0.
\]

**Proof.**  
(1) Since \( \kappa : g \times g \rightarrow \mathbb{C} \) is an invariant bilinear form and \( f \) is Ad(\( G \))-invariant, we have
\[
\kappa \left( (\text{Ad}(g^{-1})(\nabla f)(\text{Ad}(g)X)), Z \right) = \kappa \left( (\nabla f)(\text{Ad}(g)X), \text{Ad}(g)Z \right) = \frac{d}{dt} f(\text{Ad}(g)(X + tZ))\bigg|_{t=0} = \frac{d}{dt} f(X + tZ)\bigg|_{t=0} = \kappa \left( (\nabla f)(X), Z \right)
\]
for any \( Z \in g \). Hence we have \( \text{Ad}(g^{-1})(\nabla f)(\text{Ad}(g)X) = (\nabla f)(X) \), i.e. \( (\nabla f)(\text{Ad}(g)X) = \text{Ad}(g)(\nabla f)(X) \).

(2) We apply (1) to \( g = \exp(t \text{ad}Y) \in G \). Then by using (2.1) we have
\[
\sum_{i=1}^{N} \frac{\partial f}{\partial \zeta_i} \left( \exp(t \text{ad}Y)X \right) Z^i = \exp(t \text{ad}Y) (\nabla f(X)).
\]

By differentiating the both sides with respect to \( t \) and putting \( t = 0 \), we obtain
\[
\sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial \zeta_i \partial \zeta_j}(X)\zeta_j([X,Y])Z^i = [Y, (\nabla f)(X)].
\]

(3) is obtained from (2) by putting \( Y = X \).

(4) follows from (2) by specializing \( Y = (\nabla g)(X) \) and using (3). \( \square \)

We can use this lemma to prove Proposition 2.1.

**Proof of Proposition 2.1.**  
(1) Recall \( P_k = (\nabla I_k)(L) \) and \( P_k^{\geq 0} \) is the image of \( P_k \) under the projection from \( g \) to \( b \). By using (1.2) and the Jacobi identity, we have
\[
\frac{\partial^2 L}{\partial t_i \partial t_k} - \frac{\partial^2 L}{\partial t_k \partial t_i} = \left[ \frac{\partial P_k^{\geq 0}}{\partial t_i}, L \right] + \left[ P_k^{\geq 0}, \left[ P_k^{\geq 0}, L \right] \right] - \left[ \frac{\partial P_k^{\geq 0}}{\partial t_k}, L \right] - \left[ P_{k_i}^{\geq 0}, \left[ P_{k_i}^{\geq 0}, L \right] \right].
\]

We can use this lemma to prove Proposition 2.1.
So it is enough to prove

\((2.7)\)
\[
\frac{\partial P_k^0}{\partial t_l} - \frac{\partial P_l^0}{\partial t_k} + [P_k^0, P_l^0] = 0.
\]

First we show that

\((2.8)\)
\[
\frac{\partial P_k}{\partial t_l} = [P_l^0, P_k], \quad \frac{\partial P_l}{\partial t_k} = [P_k^0, P_l].
\]

By using (2.1), (1.2) and (2.4), we obtain
\[
\frac{\partial P_k}{\partial t_l} = [P_l^0, P_k], \quad \frac{\partial P_l}{\partial t_k} = [P_k^0, P_l].
\]

By using (2.1), (1.2) and (2.4), we obtain
\[
\frac{\partial P_k}{\partial t_l} = [P_l^0, P_k], \quad \frac{\partial P_l}{\partial t_k} = [P_k^0, P_l].
\]

We denote by \(P_k^0\) and \(P_l^0\) the images of \(P_k\) and \(P_l\) under the projection onto \(\pi\) respectively. Then \(P_k = P_k^0 + P_k^{<0}\), \(P_l = P_l^0 + P_l^{<0}\), and we can rewrite (2.8) as
\[
\frac{\partial P_k^0}{\partial t_l} + \frac{\partial P_l^0}{\partial t_k} = [P_l^0, P_k^0] + [P_l^0, P_k^0],
\]
\[
\frac{\partial P_l^0}{\partial t_k} + \frac{\partial P_l^0}{\partial t_k} = [P_k^0, P_l^0] + [P_k^0, P_l^0].
\]

Hence we have
\[
\frac{\partial P_k^0}{\partial t_l} - \frac{\partial P_l^0}{\partial t_k} + [P_k^0, P_l^0] = \frac{\partial P_l^0}{\partial t_k} - \frac{\partial P_l^0}{\partial t_k} - [P_k^0, P_l^0] - [P_k^0, P_l^0].
\]

Since \([P_k, P_l] = 0\) by (2.0), we have
\[
\frac{\partial P_k^0}{\partial t_l} - \frac{\partial P_l^0}{\partial t_k} + [P_k^0, P_l^0] = \frac{\partial P_l^0}{\partial t_k} - \frac{\partial P_l^0}{\partial t_k} + [P_k^0, P_l^0].
\]

Since the left hand side is an element of \(\mathfrak{b}\) and the right hand side is an element of \(\pi\), we obtain the desired identity (2.9) and complete the proof of (1).

(2) Since \(\frac{\partial L}{\partial t_k} = [P_k^0, L]\), we see
\[
\frac{\partial f(L)}{\partial t_k} = \sum_{i=1}^{N} \frac{\partial f(L)}{\partial \zeta_i} \frac{\partial \zeta_i(L)}{\partial t_k} = \sum_{i=1}^{N} \frac{\partial f(L)}{\partial \zeta_i} \zeta_i([P_k^0, L]).
\]

By using (2.1) and \([P_k^0, L] = \sum_{i=1}^{N} \zeta_i([P_k^0, L])Z_i\), we have
\[
\frac{\partial f(L)}{\partial t_k} = \kappa \left(\nabla f(L), [P_k^0, L] \right) = -\kappa \left(\nabla f(L), [P_k^0, L] \right).
\]

By (2.3), we conclude \(\frac{\partial f(L)}{\partial t_k} = 0\). \(\square\)

**Remark 2.3.** The Kostant–Toda (KT) lattice discussed in [18] is a special case of (1.2) with \(b_\alpha(t) = 0\) for all \(\alpha \in \Sigma^+ \setminus \Pi\), that is, the case where the Lax matrix \(L\) is a Jacobi element of \(\mathfrak{g}\). The hierarchy of this type is also referred to as the tri-diagonal f-KT hierarchy, and the KT lattice (the first member of the hierarchy) is given by
\[
\frac{\partial a_i}{\partial t_1} = b_\alpha, \quad \frac{\partial b_\alpha}{\partial t_1} = -\sum_{j=1}^{n} (C_i, a_j)b_\alpha, \quad (1 \leq i \leq n),
\]
where \( C = (C_{ij}) = (\alpha_i(H_j)) \) is the Cartan matrix of \( \mathfrak{g} \). The complete integrability of the lattice is shown by using the existence of the Chevalley invariants.

3. Solution method and the \( \tau \)-functions

In this section, we introduce the \( \tau \)-functions and prove that the solutions of the f-KT hierarchy can be described in terms of them. We fix some notations. Recall \( G \) is a connected complex semisimple Lie group with Lie algebra \( \text{Lie}(G) = \mathfrak{g} \), and denote by \( \mathcal{H}, \mathcal{N}, \mathcal{N}, \mathcal{B} \) and \( \bar{\mathcal{B}} \) the connected subgroups of \( G \) with Lie algebras \( \mathfrak{h}, \mathfrak{n}, \mathfrak{n}, \mathfrak{b} \) and \( \bar{\mathfrak{b}} \) respectively. Let \( W = N_G(\mathcal{H})/\mathcal{H} \) be the Weyl group of \( \mathfrak{g} \), where \( N_G(\mathcal{H}) \) is the normalizer of \( \mathcal{H} \) in \( G \), and fix a complete set of coset representatives \( \{ w : w \in W \} \).

3.1. Matrix coefficients. Let \( \theta : G \to G \) be the Chevalley involution satisfying
\[
\theta(h) = h^{-1} \quad (h \in \mathcal{H}),
\]
\[
\theta(\mathcal{N}) = \mathcal{N}, \quad \theta(\bar{\mathcal{N}}) = \bar{\mathcal{N}}.
\]

We denote by \( (\rho(\lambda), V(\lambda)) \) the irreducible highest weight representation of the group \( G \) with highest weight \( \lambda \) and highest weight vector \( v_\lambda \). Then the contragredient representation \( V(\lambda)^* \) is equivalent to the representation \( (\rho^\theta(\lambda), V(\lambda)^\theta) \) of \( G \) obtained from \( V(\lambda) \) by twisting by \( \theta \), i.e., \( V(\lambda)^\theta = V(\lambda) \) and \( \rho^\theta_\lambda = \rho_\lambda \circ \theta. \)

**Lemma 3.1.** For each irreducible highest weight representation \( V(\lambda) \) of \( G \), there exists a unique bilinear form \( \langle \cdot, \cdot \rangle_\lambda : V(\lambda) \times V(\lambda) \to \mathbb{C} \) satisfying
\[
\langle \theta(g)v, gw \rangle_\lambda = \langle v, w \rangle_\lambda \quad (g \in G, v, w \in V),
\]
\[
\langle v_\lambda, v_\lambda \rangle_\lambda = 1.
\]
Moreover, there exists an orthonormal basis of \( V(\lambda) \) consisting of weight vectors.

**Proof.** Since \( V(\lambda)^* \cong V(\lambda)^\theta \), the canonical pairing \( V(\lambda)^* \times V(\lambda) \to \mathbb{C} \) induces a desired bilinear form, and the uniqueness follows from Schur’s Lemma. The latter claim can be proved by showing that weight vectors of different weights are orthogonal with respect to \( \langle \cdot, \cdot \rangle_\lambda \). \( \Box \)

**Definition 3.2.** Given a dominant weight \( \lambda \) of \( G \), we define a function \( c_\lambda : G \to \mathbb{C} \) by putting
\[
c_\lambda(g) = \langle v_\lambda, g \cdot v_\lambda \rangle_\lambda \quad (g \in G),
\]
where \( \langle \cdot, \cdot \rangle_\lambda \) is the bilinear form on \( V(\lambda) \) given in Lemma [3.1]. That is, \( c_\lambda(g) \) is a matrix coefficient for the irreducible highest weight representation of \( G \).

The following lemma is useful.

**Lemma 3.3.**
1. If \( \{ u_1, \ldots, u_r \} \) is an orthonormal basis of \( V(\lambda) \), then we have for \( v, w \in V(\lambda) \),
\[
\langle v, gh \cdot w \rangle_\lambda = \sum_{i=1}^r \langle v, g \cdot u_i \rangle_\lambda \langle u_i, h \cdot w \rangle_\lambda.
\]
2. If \( g \in G \) is written as \( g = \pi hn \) with \( \pi \in \mathcal{N} \), \( h \in \mathcal{H} \) and \( n \in \mathcal{N} \), then we have
\[
c_\lambda(\pi hn) = \chi^\lambda(h),
\]
where \( \chi^\lambda : \mathcal{H} \to \mathbb{C}^\times \) is the character corresponding to \( \lambda \).
3. For dominant weights \( \lambda \) and \( \mu \), we have
\[
c_\lambda(g) \cdot c_\mu(g) = c_{\lambda+\mu}(g) \quad (g \in G).
\]
Proof. (1) Using the expansion $h \cdot w = \sum_{i=1}^{r} (u_i, h \cdot w)_{\lambda} u_i$, we have
\[
\langle v, gh \cdot w \rangle_{\lambda} = \left\langle v, g \cdot (h \cdot w) \right\rangle_{\lambda} = \left\langle v, g \cdot \left( \sum_{i=1}^{r} (u_i, h \cdot w)_{\lambda} u_i \right) \right\rangle_{\lambda} = \sum_{i=1}^{r} \langle u_i, h \cdot w \rangle_{\lambda} \left\langle v, g \cdot u_i \right\rangle_{\lambda},
\]
which is the desired formula.

(2) Since $\theta(\pi)^{-1} \cdot v_{\lambda} = v_{\lambda}$, $n \cdot v_{\lambda} = v_{\lambda}$ and $h \cdot v_{\lambda} = \lambda(h)v_{\lambda}$, we have
\[
c_{\lambda}(\pi h n) = \langle v_{\lambda}, n h n \cdot v_{\lambda} \rangle_{\lambda} = \lambda(h)n h n \cdot v_{\lambda} = \lambda(h)(v_{\lambda})_{\lambda} = \lambda(h).
\]

(3) Since $V(\lambda + \mu)$ appears in the tensor product $V(\lambda) \otimes V(\mu)$ with multiplicity one, we obtain the embedding $\iota : V(\lambda + \mu) \to V(\lambda) \times V(\mu)$ such that $\epsilon(v_{\lambda+\mu}) = v_{\lambda} \otimes v_{\mu}$. Then the restriction of the tensor product bilinear form $\langle \cdot, \cdot \rangle_{\lambda} \otimes \langle \cdot, \cdot \rangle_{\mu}$ to $V(\lambda + \mu)$ coincides with $\langle \cdot, \cdot \rangle_{\lambda+\mu}$ by the uniqueness in Lemma 3.1. Hence we have
\[
c^{\lambda+\mu}(g) = \langle v^{\lambda+\mu}, g \cdot v^{\lambda+\mu} \rangle_{\lambda+\mu} = \langle v^{\lambda} \otimes v^{\mu}, g \cdot (v^{\lambda} \otimes v^{\mu}) \rangle = 
\]
\[
= \langle v^{\lambda}, g \cdot v^{\lambda} \rangle_{\lambda} \langle v^{\mu}, g \cdot v^{\mu} \rangle_{\mu} = c^\lambda(g)c^\mu(g).
\]

\[
\boxdot
\]

3.2. $\tau$-functions and matrix factorization. To find a solution of the f-KT hierarchy \[(1.2)\], let us define (see also \[(1.1)\])
\[
\Theta(X; t) = \sum_{k=1}^{n} t_k(\nabla I_k)(X)
\]
for $X \in g$, and consider the exponential $\exp \Theta(X; t) \in G$. Then the $\tau$-functions are introduced as follows. Let $\varpi_1, \ldots, \varpi_n$ be the fundamental weights of $g$, i.e. $\varpi_i(H_j) = \delta_{i,j}$ for $1 \leq i, j \leq n$.

**Definition 3.4.** Suppose that $G$ is a connected, simply-connected complex semisimple Lie group with Lie $G = g$. Then we define $\tau_k(X; t) = \tau_k(X; t_1, \ldots, t_n)$ by the matrix coefficient $c_{\varpi_k}(g)$ with $g = \exp \Theta(X; t)$, i.e.
\[
\tau_k(X; t) = c_{\varpi_k}(\exp \Theta(X; t)) = \langle v_{\varpi_k}, \exp \Theta(X; t) \cdot v_{\varpi_k} \rangle_{\varpi_k}.
\]

More generally, for a vector $u \in V(\varpi_k)$, we define
\[
\tau_k(X; t; u) = \langle v_{\varpi_k}, \exp \Theta(X; t) \cdot u \rangle_{\varpi_k}.
\]

We shall prove that the solutions of \[(1.2)\] are described in terms of the $\tau$-functions. First we show that the coefficients $b_{\alpha_i}$ of a solution to \[(1.2)\] are uniquely determined by $a_1, \ldots, a_n$.

**Proposition 3.5.** If $L = L(t)$ is a solution of the form \[(1.1)\] to the full Kostant–Toda hierarchy \[(1.2)\], then $b_{\alpha_i}$’s ($\alpha \in \Sigma^+$) are expressed as polynomials in $a_i$’s and their derivatives. In particular, we have
\[
b_{\alpha_i} = \frac{\partial a_i}{\partial t_1} \quad (1 \leq i \leq n).
\]

**Proof.** Recall that the height $ht(\alpha)$ of a root $\alpha = \sum_{i=1}^{n} c_i \alpha_i \in \Sigma$ is defined by $ht(\alpha) = \sum_{i=1}^{n} c_i$. For a nonzero integer $k$, let $g_k$ be the span of root vectors $X_\alpha$ with $ht(\alpha) = k$. And we put $g_0 = h$. Then we have
\[
g = \bigoplus_{k \in \mathbb{Z}} g_k, \quad [g_k, g_l] \subset g_{k+l}.
\]

If we put
\[
L_1 = \sum_{i=1}^{n} X_{\alpha_i}, \quad L_0 = \sum_{i=1}^{n} a_i(t)H_i, \quad L_{-k} = \sum_{ht(\alpha)=k} h_{\alpha}(t)X_{-\alpha} \quad (k > 0),
\]

then...
then we have
\[ L = L_1 + L_0 + L_{-1} + L_{-2} + \cdots, \quad L_{\geq 0} = L_1 + L_0, \quad L_k \in \mathfrak{g}_k. \]

By comparing the graded components of (3.2), we obtain
\[ \frac{\partial L_0}{\partial t_1} = [L_1, L_{-1}], \]
\[ \frac{\partial L_{-k}}{\partial t_1} = [L_1, L_{-k+1}] + [L_0, L_{-k}] \quad (k > 0). \]

Since \([L_1, L_{-1}] = \sum_{i=1}^{n} b_{\alpha_i} H_i\), we obtain (3.8) from (3.3) by comparing the coefficients of \(H_i\).

We prove by induction on \(\text{ht}(\beta)\) that \(b_{\beta}\) can be uniquely expressed as a linear combination of \(a_i b_{\alpha}\) and \(\frac{\partial h}{\partial t_1}\) with \(1 \leq i \leq n\) and \(\text{ht}(\alpha) = \text{ht}(\beta) - 1\). Suppose \(k > 0\). Equating the coefficients of \(X_{-\alpha}\) with \(\text{ht}(\alpha) = k\) in (3.10), we obtain
\[ \frac{\partial h_{\alpha}}{\partial t_1} = -\left(\sum_{i=1}^{n} \alpha(H_i) a_i\right) b_{\alpha} + \sum_{\text{ht}(\beta) = k+1} N_{\alpha-\beta, \beta} b_{\beta} \quad (\text{ht}(\alpha) = k), \]
where \(N_{\alpha-\beta, \beta} = 0\) unless \(\alpha - \beta\) is a simple root. We regard (3.11) as a system of linear equations in the unknown variables \(b_{\beta}\) (\(\text{ht}(\beta) = k + 1\)) and consider the coefficient matrix
\[ M = (N_{\alpha-\beta, \beta})_{\text{ht}(\alpha)=k, \text{ht}(\beta)=k+1}. \]

On the other hand, it can be shown that \(M\) is the representation matrix of the linear map \(\text{ad} L_1 : \mathfrak{g}_{-(k+1)} \to \mathfrak{g}_{-k}\). Since \([L_1, X_{\alpha_i}] = \sum_{i=1}^{n} \alpha_i(H_i) X_{\alpha_i}\), we can find a principal \(\mathfrak{sl}_2\)-triple \(\{h, e, f\}\) with \(e = L_1\) such that \([h, e] = 2e, [h, f] = -2f\) and \([e, f] = h\), and we have \(\mathfrak{g}_k = \{X \in \mathfrak{g} : [h, X] = 2kX\}\) (see [16] Section 5).

By appealing to the representation theory of \(\mathfrak{sl}_2\), we see that \(\text{ad} L_1 : \mathfrak{g}_{-(k+1)} \to \dim \mathfrak{g}_{-k}\) is injective. Hence the matrix \(M\) has a full rank and the system (3.11) of linear equations has a unique solution in \((b_{\beta})_{\text{ht}(\beta)=k+1}\).

\[ \square \]

**Remark 3.6.** Noting (3.11), one can impose the following constraints (or reduction, see [15] Section 5):

\[ b_{\alpha} = 0 \quad \text{with} \quad \text{ht}(\alpha) \geq k + 1. \]

The resulting hierarchy may be called the \(k\)-banded \(f\)-KT hierarchy. For example, the KT hierarchy for a tridiagonal Lax matrix is the 1-banded \(f\)-KT hierarchy. Then the constraints lead to additional equations for the \(\tau\)-functions (see Remark 3.8).

We can find a solution of (1.2) by considering the Gauss decomposition of \(\exp \Theta(L(0); t)\) (this is a standard method to find the solution, see e.g. [11] [14]). In general we have a Bruhat decomposition
\[ \exp \Theta(L(0); t) = \overline{\pi}(t) \hat{w} b(t) \]
with \(\overline{\pi}(t) \in \overline{\mathcal{N}}, w \in W\) and \(b(t) \in \mathcal{B}\) for a particular \(t\), where \(\hat{w}\) is a fixed representative of the coset \(w \in W = N_G(H)/H\). It can be shown that \(w = e\) if and only if \(\tau_k(t) \neq 0\) for all \(1 \leq k \leq n\) (see e.g. [14]).

**Proposition 3.7.** For a generic \(t\) (i.e. \(\tau_k(t) \neq 0\) for \(1 \leq k \leq n\)), we have the Gauss decomposition
\[ \exp \Theta(L(0); t) = \overline{\pi}(t) b(t), \]
with \(\overline{\pi}(t) \in \overline{\mathcal{N}}\) and \(b(t) \in \mathcal{B}\), and the solution of the \(f\)-KT hierarchy (1.2) is given by
\[ L(t) = \text{Ad} (\overline{\pi}(t)^{-1}) L(0) = \text{Ad} (b(t)) L(0). \]

Moreover, we have
\[ a_i(t) = \frac{\partial}{\partial t_i} \ln \tau_i(L(0); t) = \frac{\tau'_i(L(0); t)}{\tau_i(L(0); t)} \quad (1 \leq i \leq n), \]
where \( \tau'_i = \frac{\partial \tau_i}{\partial t_1} \) and \( b_\alpha(t) \) are expressed as rational functions of \( \tau_i(L(0); t) \) and their derivatives.

**Remark 3.8.** In the case of the KT lattice, one can find the following formulas of the solutions in terms of \( \tau \)-functions (see also [7, 4]):

\[
a_i(t) = \frac{\partial}{\partial t_1} \ln \tau_i(t), \quad b_\alpha(t) = b_\alpha(0) \prod_{j=1}^{n} \tau_j(t)^{-C_{i,j}} \quad (1 \leq i \leq n).
\]

Note that the \( \tau \)-functions for the KT hierarchy satisfy

\[
\frac{\partial^2}{\partial t_1^2} \ln \tau_i(t) = b_\alpha(0) \prod_{j=1}^{n} \tau_j(t)^{-C_{i,j}} \quad (1 \leq i \leq n).
\]

**Proof of Proposition 3.7.** We may assume that \( \mathcal{G} \) is a matrix group.

By differentiating the both sides of (3.12) with respect to \( t_k \), we obtain

\[
P_k(0) \cdot \exp \Theta(L(0); t) = \exp \Theta(L(0); t) \cdot P_k(0) = \frac{\partial}{\partial t_k} b + \frac{\partial b}{\partial t_k},
\]

where \( P_k(0) = (\nabla I_k)(L(0)) \). Multiplying this identity on the left by \( \pi^{-1} \) and on the right by \( b^{-1} \) yields

\[
\pi^{-1} P_k(0) \pi = b P_k(0) b^{-1} = \pi^{-1} \frac{\partial}{\partial t_k} b + \frac{\partial b}{\partial t_k} b^{-1}.
\]

Here we recall that \( P_1(t) = (\nabla I_1)(L(t)) = L(t) \).

We put

\[
\tilde{P}_k(t) = \pi^{-1} P_k(0) \pi(t) = b(t) P_k(0) b(t)^{-1}, \quad \tilde{L}(t) = \tilde{P}_1(t),
\]

and prove \( \tilde{L}(t) = L(t) \). Since \( [\pi, L(0)] \in \mathfrak{h} \), we see that \( \tilde{L}(t) \) has the form (1.1).

First we show that \( \tilde{L}(t) \) solves the f-KT hierarchy (1.2). By using (2.3), we obtain

\[
\tilde{P}_k(t) = \text{Ad}(b(t)) P_k(0) = \text{Ad}(b(t))(\nabla I_k)(L(0)) = (\nabla I_k)(\text{Ad}(b(t)) L(0)) = (\nabla I_k)(\tilde{L}(t)).
\]

Also it follows from (3.15) that

\[
\tilde{P}_k^{\geq 0} = \frac{\partial}{\partial t_k} b^{-1}.
\]

Now we prove

\[
\frac{\partial \tilde{L}}{\partial t_k} = [\tilde{P}_k^{\geq 0}, \tilde{L}].
\]

Differentiating \( \tilde{L}(t) = b(t)L(0)b(t)^{-1} \) with respect to \( t_k \), we have

\[
\frac{\partial \tilde{L}}{\partial t_k} = \frac{\partial b}{\partial t_k} L(0)b^{-1} + b L(0) \cdot \left( -b^{-1} \frac{\partial b}{\partial t_k} b^{-1} \right) = \frac{\partial b}{\partial t_k} b^{-1} \cdot b L(0)b^{-1} - b L(0)b^{-1} \cdot \frac{\partial b}{\partial t_k} b^{-1} = [\tilde{P}_k^{\geq 0}, \tilde{L}].
\]

Therefore, since \( \tilde{L}(0) = L(0) \), we have \( \tilde{L}(t) = L(t) \) near the origin \( t = 0 \) by the uniqueness theorem for differential equations.

Next we show that \( L(t) \) is described in terms of the \( \tau \)-functions. We decompose \( b(t) = h(t)n(t) \) with \( h(t) \in \mathcal{H} \) and \( n(t) \in \mathcal{N} \). Then we have

\[
L^{\geq 0} = \frac{\partial b}{\partial t_1} b^{-1} = \frac{\partial h}{\partial t_1} h^{-1} + h \frac{\partial n}{\partial t_1} n^{-1} h^{-1}.
\]

Since \( \frac{\partial h}{\partial t_1} h^{-1} \in \mathfrak{h} \) and \( h \frac{\partial n}{\partial t_1} n^{-1} h^{-1} \in \mathfrak{n} \), we obtain

\[
\sum_{i=1}^{n} a_i H_i = \frac{\partial h}{\partial t_1} h^{-1}.
\]
If \( h(t) = \exp \left( \sum_{j=1}^{n} c_j(t) H_j \right) \), then we can see
\[
a_i(t) = \frac{\partial c_i(t)}{\partial t_1}.
\]

On the other hand, by using (3.3) and \( \varpi_i \left( \sum_{j=1}^{n} c_j(t) H_j \right) = c_i(t) \), we have
\[
\tau_i(t) = c_{\varpi_i}(\exp \Theta(L(0); t)) = c_{\varpi_i}(\varpi(t) h(t)n(t)) = \chi^{\varpi_i}(h(t)) = e^{c_i(t)},
\]
thus we obtain \( c_i(t) = \ln \tau_i(t) \). Hence, by using Proposition 3.5, we see that \( b_{\alpha}(t)\)s are determined by the \( \tau\)-functions.

Combining the above argument, we conclude \( \widetilde{L}(t) = L(t) \) for a generic \( t \), which completes the proof. \( \square \)

It is not always the case that \( \exp \Theta(L(0); t) \) has the decomposition (3.12). For the case \( \tau_k(L(0); t) = 0 \) at some \( t = t_* \) for some \( k \), we have the following proposition.

**Proposition 3.9.** Suppose that at \( t = t_* \) we have a decomposition
(3.16) \[
\exp \Theta(L(0); t_*) = \pi_* w_* h_* n_*
\]
with \( \pi_* \in \mathcal{N} \), \( w_* \in W \), \( h_* \in \mathcal{H} \) and \( n_* \in \mathcal{N} \). Then we have
(3.17) \[
\tau_k(L(0); t + t_*) = \chi^{\varpi_k}(h_*) \sum_{i=1}^{r} \langle u_i, \pi_* w_* \cdot v_{\varpi_k} \rangle \tau_k(L(0); t; u_i),
\]
where \( \{u_1, \ldots, u_r\} \) is an orthonormal weight basis of \( V(\varpi_k) \).

We remark that \( \langle u_i, \pi_* w_* \cdot v_{\varpi_k} \rangle = 0 \) unless the weight \( w_\mu \) of \( u_i \) satisfies \( w_\mu \leq w_{\varpi_k} \), where we write \( \lambda \leq \mu \) if \( \mu - \lambda \) is a linear combination of simple roots with nonnegative integer coefficients.

**Proof.** Since \( n_* \cdot v_{\varpi_k} = v_{\varpi_k} \) and \( h_* \cdot v_{\varpi_k} = \chi^{\varpi_k}(h_*) v_{\varpi_k} \), we have
\[
\tau_k(L(0); t + t_*) = \langle v_{\varpi_k}, \exp \Theta(L(0); t) \exp \Theta(L(0); t_*) \cdot v_{\varpi_k} \rangle
= \chi^{\varpi_k}(h_*) \langle v_{\varpi_k}, \exp \Theta(L(0); t) \pi_* w_* \cdot v_{\varpi_k} \rangle.
\]
Now the desired identity is obtained by using (3.2). \( \square \)

In this article we are interested in rational solutions of the f-KT hierarchy (1.2). By Proposition 3.7, it is enough to consider the case where \( \tau_k(L(0); t) \) are polynomials in \( t \), i.e., \( L(0) \) is nilpotent. Let
\[
\Lambda = \sum_{i=1}^{n} X_{\alpha_i}
\]
be the standard regular nilpotent element. Then we have the following proposition for the formula of polynomial \( \tau\)-functions.

**Proposition 3.10.** If \( L(0) \) is nilpotent, then there exists \( \pi \in \mathcal{N} \) such that \( L(0) = \text{Ad}(\pi) \Lambda \), and we have
\[
\tau_k(L(0); t) = \sum_{i=1}^{r} \langle u_i, \pi^{-1} \cdot v_{\varpi_k} \rangle \tau_k(\Lambda; t; u_i) \quad (1 \leq k \leq n),
\]
where \( \{u_1, \ldots, u_r\} \) is an orthonormal weight basis of \( V(\varpi_k) \).

**Proof.** We note that \( L(0) \in \Lambda + \mathfrak{b} \). It is known (see [14, Proposition 16]) that \( X \in \mathfrak{g} \) is nilpotent if and only if \( I_1(X) = \cdots = I_n(X) = 0 \). By [18, Proposition 2.3.2], the following are equivalent for \( X, Y \in \Lambda + \mathfrak{b} \):
(i) there is an element \( \pi \in \mathcal{N} \) such that \( \text{Ad}(\pi)X = Y \);
Hence we conclude that there exists an element \( \eta \in \mathcal{N} \) such that \( L(0) = \text{Ad}(\eta)\Lambda \).

By applying (2.3), we have \( \Theta(L(0); t) = \text{Ad}(\eta)\Theta(\Lambda; t) \), and \( \exp \Theta(L(0); t) = (\exp \Theta(\Lambda; t)) \eta \). Hence, by using \( \theta(\eta)^{-1} \cdot v_{\omega_k} = \eta \cdot v_{\omega_k} \) and (3.3), we have

\[
\tau_k(L(0); t) = \langle v_{\omega_k}, (\exp \Theta(\Lambda; t)) \eta \cdot v_{\omega_k} \rangle = \langle v_{\omega_k}, (\exp \Theta(\Lambda; t)) \eta \cdot v_{\omega_k} \rangle = \sum_{i=1}^r \langle u_i, \eta^{-1} \cdot v_{\omega_k} \rangle \tau_k(\Lambda; u_i).
\]

Proposition 3.10 implies that, by considering all elements \( \eta \in \mathcal{N} \) and an orthonormal weight basis \( \{u_1, \ldots, u_r\} \), one can obtain all the polynomial \( \tau \)-functions for the f-KT hierarchy on the Lie algebra \( \mathfrak{g} \).

4. Matrix realization of the f-KT hierarchy for type \( D \)

In this section we give an explicit matrix realization of the f-KT hierarchy (1.2) of type \( D_n \).

4.1. Lie algebra of type \( D \). We recall several basic facts on even orthogonal Lie algebras.

We use the following realization of the orthogonal Lie algebra \( \mathfrak{so}_{2n}(\mathbb{C}) \). Let \( S \) be the \( 2n \times 2n \) antidiagonal symmetric matrix given by

\[
S = \sum_{i=1}^n (-1)^{n-i}(E_{i,2n+1-i} + E_{2n+1-i,i}),
\]

where \( E_{i,j} \) denote the matrix unit with 1 at the \((i,j)\) entry and 0 at all other entries. Then we define the orthogonal Lie algebra \( \mathfrak{so}_{2n}(\mathbb{C}) \), the simple Lie algebra of type \( D_n \), by putting

\[
\mathfrak{so}_{2n}(\mathbb{C}) = \{X \in \mathfrak{gl}_{2n}(\mathbb{C}) : XS + SX = 0\}.
\]

Then the corresponding orthogonal and special orthogonal groups are given by

\[
\mathbf{O}_{2n}(\mathbb{C}) = \{g \in \mathbf{GL}_{2n}(\mathbb{C}) : gSg = S\}, \quad \mathbf{SO}_{2n}(\mathbb{C}) = \{g \in \mathbf{SL}_{2n}(\mathbb{C}) : gSg = S\}
\]

respectively. We denote by \( \mathbf{Spin}_{2n}(\mathbb{C}) \) the spin group, which is the simply connected Lie group of type \( D_n \).

In the remaining of this paper we write \( \mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C}) \). Let \( \mathfrak{h} \) (resp. \( \mathfrak{n}, \mathfrak{p} \)) be a subalgebra of \( \mathfrak{g} \) consisting of diagonal matrices (resp. strictly upper triangular matrices, strictly lower triangular matrices) in \( \mathfrak{g} \).

Then

\[
\mathfrak{h} = \{\text{diag}(h_1, \ldots, h_n, -h_n, \ldots, -h_1) : h_i \in \mathbb{C}\}.
\]

is a Cartan subalgebra of \( \mathfrak{g} \) and \( \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h} \oplus \mathfrak{n} \) is a triangular decomposition. Let \( \varepsilon_i : \mathfrak{h} \to \mathbb{C} \) be the linear map given by \( \varepsilon_i(\text{diag}(h_1, \ldots, h_n, -h_n, \ldots, -h_1)) = h_i \). Then \( \{\varepsilon_i : 1 \leq i \leq n\} \) forms a basis of \( \mathfrak{h}^* \) and the root system of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) is given by

\[
\Sigma = \{\pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_j) : 1 \leq i < j \leq n\}.
\]

We put

\[
H_i = E_{i,i} - E_{i+1,i+1} + E_{2n-i,2n-i} - E_{2n-i+1,2n-i+1} \quad (1 \leq i \leq n-1),
\]

\[
H_n = E_{n-1,n-1} - E_{n,n+1} + E_{n,n} - E_{n+2,n+2}.
\]

\[\text{The reason we choose this matrix } S \text{ is that the standard regular nilpotent element } \Lambda \text{ can be taken as a } (0,1)-\text{matrix. See (3.1).}\]
and

\[ X_{\varepsilon_i - \varepsilon_j} = E_{i,j} - (-1)^{j-i} E_{2n+1-j, 2n+1-i}, \]
\[ X_{\varepsilon_i + \varepsilon_j} = E_{i,2n+1-j} - (-1)^{j-i} E_{j,2n+1-i}, \]
\[ X_{-(\varepsilon_i - \varepsilon_j)} = E_{j,i} - (-1)^{j-i} E_{2n+1-i, 2n+1-j}, \]
\[ X_{-(\varepsilon_i + \varepsilon_j)} = E_{2n+1-j, i} - (-1)^{j-i} E_{2n+1-i, j}, \]
\begin{align*}
(1 \leq i < j \leq n).
\end{align*}

These elements \( \{H_i : 1 \leq i \leq n\} \cup \{X_\alpha : \alpha \in \Sigma\} \) form a Chevalley basis of \( \mathfrak{g} \). The roots \( (4.1) \)

\[ \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \quad \alpha_n = \varepsilon_n - \varepsilon_{n-1} \]

form the simple system corresponding to \( n \). Moreover the bilinear form \( \kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \) given by \( (4.2) \)

\[ \kappa(X, Y) = \text{tr}(XY) \quad (X, Y \in \mathfrak{g}) \]

is symmetric, invariant and nondegenerate.

In this setting, the Lax matrix for \( \mathfrak{so}_n(\mathbb{C}) \) \((n = 4)\) given by \( (4.3) \) has the following form:

\[
L = \begin{pmatrix}
  a_1 & 1 & b_{e_1 - e_2} & 0 \\
  b_{e_1 - e_2} & a_2 - a_1 & a_3 - a_2 + a_4 & b_{e_2 - e_3} \\
  b_{e_1 - e_3} & b_{e_2 - e_3} & a_4 - a_3 & 0 \\
  b_{e_1 + e_4} & b_{e_2 + e_4} & b_{e_3 + e_4} & 0 & a_3 - a_4 \\
  b_{e_2 - e_4} & b_{e_3 - e_4} & a_2 - a_3 - a_4 & b_{e_2 + e_3} \\
  b_{e_2 - e_3} & b_{e_3 - e_2} & a_1 - a_2 & b_{e_2 + e_4} \\
  b_{e_1 - e_2} & 0 & b_{e_2 - e_4} & b_{e_3 - e_4} & b_{e_2 + e_3} \\
  0 & b_{e_1 + e_2} & -b_{e_1 + e_3} & b_{e_1 + e_4} & -b_{e_1 - e_2} - a_1
\end{pmatrix},
\]

where all unfilled entries in the upper triangular part are zero.

Let \( W \) be the Weyl group of \( \mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C}) \). Then \( W \) acts on the Cartan subalgebra \( \mathfrak{h} \) as permutations together with an even number of sign changes in the coordinates \( h_1, \ldots, h_n \). Hence \( W \) is isomorphic to the subgroup of the symmetric group \( \mathfrak{S}_{2n} \) given by \( (4.4) \)

\[
\mathfrak{D}_n = \left\{ \omega \in \mathfrak{S}_{2n} : \begin{array}{l}
  \#\{i : 1 \leq i \leq n, \omega(i) \geq n + 1\} \text{ is even} \\
  w(i) + w(2n + 1 - i) = 2n + 1 \text{ for } 1 \leq i \leq 2n
\end{array} \right\},
\]

and the isomorphism is given by

\[
s_i \mapsto (i, i + 1)(2n - i, 2n + 1 - i) \quad (1 \leq i \leq n - 1),
\]
\[
s_n \mapsto (n - 1, n + 1)(n, n + 2),
\]

where \( s_i \) is the simple reflection corresponding to \( \alpha_i \) and \((p, q) \in \mathfrak{S}_{2n} \) denotes the transposition of \( p \) and \( q \).

\subsection*{4.2. Explicit form of the f-KT hierarchy on \( \mathfrak{so}_{2n}(\mathbb{C}) \).}

Let \( \mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C}) \) and \( \mathcal{G} = \text{SO}_{2n}(\mathbb{C}) \) or \( \text{Spin}_{2n}(\mathbb{C}) \). First we give an explicit system of algebraically independent generators of the ring \( \mathbb{C}[\mathfrak{g}]^\mathcal{G} \) of invariant polynomial functions on \( \mathfrak{g} \). We denote by \( \text{Pf}(A) \) the Pfaffian of a skew-symmetric matrix \( A \) of even order. For the invariants of the ring of polynomials \( \mathbb{C}[\mathfrak{g}] \), we have the following proposition.

\begin{proposition}
We define \( I_k \in \mathbb{C}[\mathfrak{g}] \) \((1 \leq k \leq n)\) by putting

\[ I_k(X) = \frac{1}{2k} \text{tr}(X^{2k}) \quad (1 \leq k \leq n - 1), \quad I_n(X) = 2c_n \text{Pf}(SX), \]

where \( c_n = (-1)^{n(n+1)/2} \) is a normalization constant (see \( (5.13) \)). Then \( \{I_1, \ldots, I_{n-1}, I_n\} \) forms a system of algebraically independent homogeneous generators of \( \mathbb{C}[\mathfrak{g}]^\mathcal{G} \).
**Proof.** Let \( R : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}] \) be the restriction map. Then the Chevalley restriction theorem asserts that \( R \) induces an isomorphism \( \mathbb{C}[\mathfrak{g}]^W \to \mathbb{C}[\mathfrak{h}]^W \). If we identify \( \mathbb{C}[\mathfrak{h}] \) with the polynomial algebra in \((x_1, \ldots, x_n)\), where \( x_i \) corresponds to \( \varepsilon_i \in \mathfrak{h}^* \), then we have

\[
R(I_k) = \frac{1}{k} (x_1^{2k} + \cdots + x_n^{2k}) \quad (1 \leq k \leq n - 1), \quad R(I_n) = -2x_1x_2 \cdots x_n.
\]

Also it is known (see [10, p.68]) that \( \mathbb{C}[\mathfrak{h}]^W \) is generated by \( n \) algebraically independent homogeneous polynomials

\[
f_k = x_1^{2k} + \cdots + x_n^{2k} \quad (1 \leq k \leq n - 1), \quad f_n = x_1x_2 \cdots x_n.
\]

The proposition follows from these claims. \( \square \)

Given a skew-symmetric matrix \( A \) of even order, let \( \widehat{A} \) is the skew-symmetric matrix, called the coPfaffian matrix of \( A \), whose \((i,j)\) entry, \( i < j \), is given by

\[(4.5) \quad \widehat{A}_{i,j} = (-1)^{i+j+1} \text{Pf}(A^{i,j}),
\]

where \( A^{i,j} \) denotes the submatrix of \( A \) obtained from \( A \) by removing the \( i \)th and \( j \)th rows/columns. Then we have

\[A \cdot \widehat{A} = \widehat{A} \cdot A = \text{Pf}(A) \cdot I,
\]

where \( I \) is the identity matrix. (We refer the readers to [11] for an exposition on Pfaffians.) The gradients of \( I_k \) \((1 \leq k \leq n)\) are given as follows.

**Proposition 4.2.** We have

\[(4.6) \quad (\nabla I_k) \,(X) = X^{2k-1} \quad (1 \leq k \leq n - 1), \quad (\nabla I_n) \,(X) = c_n \cdot \widehat{X} \cdot S.
\]

In the proof of the second identity, we need the following formula for the derivative of a Pfaffian.

**Lemma 4.3.** Let \( n \) be an even integer. If \( A(t) = (a_{i,j}(t))_{1 \leq i,j \leq n} \) is a skew-symmetric matrix with entries functions of \( t \), then we have

\[(4.7) \quad \frac{d}{dt} \text{Pf}(A(t)) = \frac{1}{2} \text{tr} \left( \widehat{A}(t) \cdot A'(t) \right),
\]

where \( A'(t) = (\frac{d}{dt}a_{i,j}(t))_{1 \leq i,j \leq n} \).

**Proof.** If we regard the Pfaffian \( \text{Pf}(A) \) as a function in its entries \( a_{i,j} \) \((i < j)\), then it follows from the expansion of \( \text{Pf}(A) \) along the \( i \)th row that

\[
\frac{\partial}{\partial a_{i,j}} \text{Pf}(A) = (-1)^{i+j+1} \text{Pf}(A^{i,j}) = \widehat{A}_{i,j}.
\]

Hence we see that

\[
\frac{d}{dt} \text{Pf}(A(t)) = \sum_{1 \leq i < j \leq n} \frac{\partial}{\partial a_{i,j}} \text{Pf}(A) \cdot \frac{d}{dt}a_{i,j} = \sum_{1 \leq i < j \leq n} \widehat{A}_{i,j} \frac{d}{dt}a_{i,j}.
\]

Since \( \widehat{A}_{i,i} = 0, \, \widehat{A}_{i,j} = -\widehat{A}_{j,i}, \) and \( A_{i,j}' = -A_{j,i}' \), we have

\[
\frac{1}{2} \text{tr} \left( \widehat{A}(t) \cdot A'(t) \right) = \frac{1}{2} \sum_{i,j=1}^n \widehat{A}_{i,j} A_{i,j}' = \sum_{1 \leq i < j \leq n} \widehat{A}_{i,j} A_{i,j}'.
\]

This completes the proof. \( \square \)
Proof of Proposition 4.2. The proof for the first identity is easy. We prove the second identity. If we put \( Z(t) = S(X + tY) \), then we have
\[
\kappa((\nabla I_n)(X), Y) = \frac{d}{dt} I_n(X + tY)\big|_{t=0} = 2c_n \frac{d}{dt} \text{Pf}(Z(t))\big|_{t=0}.
\]
By using Lemma 4.3, we have
\[
\kappa((\nabla I_n)(X), Y) = c_n \text{tr}\left( \left( \frac{\hat{Z}}{t} \cdot Z'(t) \right) \right) = c_n \text{tr}\left( \left( \frac{\hat{S}X}{s} \cdot SY \right) \right).
\]
Since \( \kappa \) is nondegenerate on \( \mathfrak{g} \), we obtain the desired identity. \( \square \)

We rename the time variables from \((t_1, \ldots, t_n)\) to \((t_1, t_3, \ldots, t_{2n-3}, s)\), where \( t_{2k-1} \) corresponds to the trace invariant \( I_k(X) \) and \( s \) corresponds to the Pfaffian invariant \( I_n(X) \). In summary, we obtain the following proposition.

**Proposition 4.4.** The matrix representation of the f-KT hierarchy on the Lie algebra \( \mathfrak{so}_{2n}(\mathbb{C}) \) is given as follows:
\[
\left\{
\begin{array}{l}
\frac{\partial L}{\partial t_{2k-1}} = \left[ \left( L^{2k-1} \right)^{\geq 0}, L \right] (1 \leq k \leq n-1), \\
\frac{\partial L}{\partial s} = c_n \left[ \left( \frac{\hat{S}L}{s} \right)^{\geq 0}, L \right],
\end{array}
\right.
\]
where \( c_n = (-1)^{n(n+1)/2} \).

5. Polynomial solutions of the \( \tau \)-functions for type \( D \)

In this section, we give explicit formulas for the polynomial \( \tau \)-functions for type \( D \). By Propositions 3.9 and 3.10, it is enough to consider the \( \tau \)-functions \( \tau_r(\Lambda; t; u) \) associated to the standard regular nilpotent element \( \Lambda \).

5.1. Explicit formulas for \( \tau_n \) and \( \tau_{n-1} \). Recall the definition of the \( \tau \)-functions in the case of the orthogonal Lie algebra \( \mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C}) \) of type \( D_n \). Let \( t = (t_1, \ldots, t_{2n-3}, s) \) be the time variables of the f-KT hierarchy (1.8) for \( \mathfrak{g} \). Let
\[
\Lambda = \sum_{i=1}^{n} X_{\alpha_i} = \sum_{i=1}^{n-1} (E_{i, i+1} + E_{2n-i, 2n+1-i}) + E_{n-1, n+1} + E_{n, n+2}
\]
be the standard regular nilpotent element of \( \mathfrak{g} \), and put
\[
\Theta(\Lambda; t) = \sum_{k=1}^{n-1} t_{2k-1} \Lambda^{2k-1} + sc_n \left( \frac{\hat{S}A}{s} \right) \cdot S
\]
where \( \hat{A} \) denotes the coPfaffian matrix of a skew-symmetric matrix \( A \) (see (4.5)). The fundamental weights of \( \mathfrak{g} \) corresponding to the simple roots defined by (4.1) are given by
\[
\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (1 \leq i \leq n-2),
\]
\[
\varpi_{n-1} = \frac{1}{2} (\varepsilon_1 + \cdots + \varepsilon_{n-2} + \varepsilon_{n-1} - \varepsilon_n),
\]
\[
\varpi_n = \frac{1}{2} (\varepsilon_1 + \cdots + \varepsilon_{n-2} + \varepsilon_{n-1} + \varepsilon_n).
\]
Then, from Definition 3.3, the \( \tau \)-functions are defined as the matrix coefficients on the irreducible highest weight representations \( V(\varpi_r) \) with highest weight \( \varpi_r \) and highest weight vector \( v_{\varpi_r} \):
\[
\tau_r(\Lambda; t; u) = \langle v_{\varpi_r}, \exp \Theta(\Lambda; t) \cdot u \rangle_{\varpi_r},
\]
where \( u \in V(\varpi_r) \) and \( \langle \cdot, \cdot \rangle_{\varpi_r} \) is the bilinear form given in Lemma 3.1.
To state our main results, we introduce $2$-reduced Schur functions and Schur’s $Q$-functions in the so-called Sato-variables $t' = (t_1, t_3, \ldots, t_{2n-3})$ (see [20, III.8] and [21] for Schur’s $Q$-functions). We define $q_m(t')$ by the generating function

\begin{equation}
\sum_{m \geq 0} q_m(t') z^m = \exp \left( \sum_{k=1}^{n-1} t_{2k-1} z^{2k-1} \right).
\end{equation}

For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ ($\lambda_1 \geq \cdots \geq \lambda_n \geq 0$) of length at most $n$, we define the 2-reduced Schur function $S_\lambda(t')$ by

\begin{equation}
S_\lambda(t') = \det (q_{\lambda_i-i+j}(t'))_{1 \leq i,j \leq n}.
\end{equation}

For a strict partition $\alpha = (\alpha_1, \ldots, \alpha_l)$ ($\alpha_1 > \cdots > \alpha_l > 0$) of length $l$, the corresponding Schur’s $Q$-function $Q_\alpha(t')$ is defined inductively on $l$:

1. If $l = 0$, i.e. $\lambda = \emptyset$, then $Q_\emptyset(t') = 1$.
2. If $l = 1$, i.e. $\lambda = (i)$ with $i \geq 0$, then $Q_{(i)}(t') = q_i(t')$.
3. If $l = 2$, i.e. $\lambda = (i, j)$ with $i > j > 0$, then

\begin{equation}
Q_{(i,j)}(t') = q_i(t')q_j(t') + 2 \sum_{k=1}^{j} (-1)^k q_{i+k}(t')q_{j-k}(t').
\end{equation}

4. If $l \geq 3$, then we define

\begin{equation}
Q_{\alpha}(t') = \begin{cases} 
\text{Pf } (Q_{(\alpha_i, \alpha_j)}(t'))_{1 \leq i,j \leq l} & \text{if } l \text{ is even}, \\
\text{Pf } (Q_{(\alpha_i, \alpha_j)}(t'))_{1 \leq i,j \leq l+1} & \text{if } l \text{ is odd},
\end{cases}
\end{equation}

where $\alpha_{l+1} = 0$ and $Q_{(\alpha, 0)}(t') = q_{\alpha}(t')$ when $l$ is odd.

For a strict partition $\alpha$ such that $\alpha_1 \leq n - 1$, we define an extended Schur’s $Q$-function by

\begin{equation}
\hat{Q}_\alpha(t', s) = \begin{cases} 
Q_{\alpha}(t') + (-1)^n s Q_{\alpha \setminus (n-1)}(t') & \text{if } n-1 \text{ is a part of } \alpha, \\
Q_{\alpha}(t') & \text{otherwise}\end{cases}
\end{equation}

where $\alpha \setminus (n-1)$ is the strict partition obtained from $\alpha$ by removing the part $n-1$, i.e. the shifted diagram of $\alpha \setminus (n-1)$ is the same as the skew diagram $\alpha/(n-1)$.

Let $w \in W$ be a Weyl group element and regard it as a permutation in $\mathfrak{D}_n$ (see [4,8]). Then we associate a strict partition $\alpha = \alpha(w) = (\alpha_1, \ldots, \alpha_p)$ determined by the relation

\begin{equation}
\{ \alpha_1 + n + 1, \ldots, \alpha_p + n + 1 \} = \{ w(i) : 1 \leq i \leq n, w(i) \geq n + 2 \}.
\end{equation}

One of the main results in this paper is the following explicit formula of the $\tau$-functions $\tau_n$ and $\tau_{n-1}$.

(See Proposition [5,10] for the explicit formulas for other $\tau$-functions $\tau_r$ with $1 \leq r \leq n-2$.) Then the polynomial $\tau$-functions are given by the formulas in Proposition [5,10].

**Theorem 5.1.** For $w \in W \cong \mathfrak{D}_n$, we have

\begin{equation}
\tau_n(\Lambda; t; w \cdot v_{\infty_n}) = c_{n,w} \hat{Q}_{\alpha(w)}(t', s),
\end{equation}

\begin{equation}
\tau_{n-1}(\Lambda; t; w \cdot v_{\infty_{n-1}}) = c_{n-1,w} \hat{Q}_{\alpha(w')} (t', -s),
\end{equation}

where $c_{n,w}$ and $c_{n-1,w}$ are nonzero constants, and $\mathfrak{D}_n \ni w \mapsto w^l \in \mathfrak{D}_n$ is the automorphism given by $w^l = (n, n+1)w(n, n+1)$.

Before proving this theorem in the next subsection, we give a Pfaffian expression of $\hat{Q}_{\alpha}(t', s)$. 

Lemma 5.3. If we put
\[ t \exp \Theta(\Lambda; t) = \prod_{1 \leq i < j \leq t} \left( q_{i,j} - q_{j,i} \right) \]
then we have
\[ \Lambda + \frac{t^2}{2} \Theta(\Lambda; t) \]
where \( \Lambda_{t+1} = 0 \) and \( \tilde{Q}_{(\alpha,0)} = Q_{(\alpha)} \). Moreover we have for \( n \geq 1 \geq i > j > 0 \)
\[ \tilde{Q}_{(i,j)}(t,s) = \tilde{Q}_{(i)}(t,s) \tilde{Q}_{(j)}(t,s) + 2 \sum_{k=1}^{j} (-1)^k \tilde{Q}_{(i+k)}(t',s) \tilde{Q}_{(j-k)}(t',s), \]
where \( \tilde{Q}_{(0)} = 1 \).

Proof. Equation (5.12) is an immediate consequence of (5.5) and the definition (5.7). For the proof of (5.11), it is enough to consider the case where \( \alpha_1 = n - 1 \).

If \( l \) is even, then we use the multilinearity of Pfaffians to obtain
\[ \text{Pf} \left( \tilde{Q}_{(\alpha)}(t',s) \right)_{1 \leq i < j \leq l} = \text{Pf} \left( Q_{(\alpha)}(t') \right)_{1 \leq i < j \leq l} + (-1)^n s \text{Pf} \left( 0 \begin{array}{c c} (Q_{(\alpha)}(t'))_{2 \leq i \leq l} \end{array} \right), \]
The first Pfaffian is equal to \( Q_{\alpha}(t') \). By moving the first row/column to the last and then by multiplying the last row/column by \( -1 \), we see that the second Pfaffian equals \( (-1)^{l-1} \cdot (-1)Q_{(\alpha_{2,\ldots,\alpha_{l}})}(t') = \tilde{Q}_{(\alpha_{2,\ldots,\alpha_{l}})}(t') \).

If \( l \) is odd, then we have
\[ \text{Pf} \left( \tilde{Q}_{(\alpha)}(t',s) \right)_{1 \leq i < j \leq l+1} = \text{Pf} \left( Q_{(\alpha)}(t') \right)_{1 \leq i < j \leq l+1} + (-1)^n s \text{Pf} \left( 0 \begin{array}{c c} (Q_{(\alpha)}(t'))_{2 \leq i \leq l+1} \end{array} \right), \]
The first Pfaffian is equal to \( Q_{\alpha}(t') \). By adding the last row/column to the first row/column and then by expanding the resulting Pfaffian along the first row/column, we see that the second Pfaffian equals \( \tilde{Q}_{(\alpha_{2,\ldots,\alpha_{l}})}(t') \). \( \square \)

5.2. Proof of Theorem 5.1 As the first step toward the proof, we compute the explicit form of
\[ \exp \Theta(\Lambda; t) \in \text{SO}_{2n}(\mathbb{C}). \]

Lemma 5.3. If we put
\[ \Lambda' = X_{\varepsilon_1 + \varepsilon_n} - X_{\varepsilon_1 - \varepsilon_n} = -E_{1,n} + E_{1,n+1} + (-1)^n E_{n,2n} - (-1)^n E_{n+1,2n} \]
\[ \Lambda'' = E_{1,2n} \]
then we have
\[ \exp \Theta(\Lambda; t) = \sum_{m=0}^{2n-2} q_m(t') \Lambda^m + s \Lambda' - (-1)^n s^2 \Lambda''. \]
More explicitly, this matrix looks like

\[
\begin{pmatrix}
1 & q_1 & q_2 & \cdots & q_{n-2} & q_{n-1} - s & q_{n-1} + s & 2q_n & \cdots & 2q_{2n-3} & 2q_{2n-2} - (-1)^n s^2 \\
1 & q_1 & \cdots & q_{n-3} & q_{n-2} & q_{n-2} & 2q_{n-1} & \cdots & 2q_{2n-4} & 2q_{2n-3} \\
& \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & 1 & q_1 & q_2 & \cdots & 2q_{n-1} & 2q_n & & & \\
& & & 1 & 0 & q_1 & \cdots & q_{n-3} & q_{n-2} & q_{n-1} + (-1)^n s & q_{n-1} - (-1)^n s \\
& & & & 1 & q_1 & \cdots & q_{n-3} & q_{n-2} & & & \\
& & & & & \ddots & \vdots & \vdots & \vdots & \vdots & & \\
& & & & & & 1 & q_1 & & & 1 \\
\end{pmatrix},
\]

where blank entries are all 0.

**Proof.** First we prove

\[(5.13) \quad \langle \hat{S} \Lambda \cdot S = c_n \Lambda' \quad \text{with} \quad c_n = -(-1)^{n(n+1)/2}.\]

Since the \(n\)th and \((n+1)\)st rows/columns of \(S \Lambda\) are the same and the 1st row/column is the zero vector, we see that the \((i, j)\) entry, \(i < j\), of \(S \Lambda\) vanishes unless \((i, j) = (1, n)\) or \((1, n + 1)\). The entries \(\hat{S} \Lambda_{1,n}\) and \(\hat{S} \Lambda_{1,n+1}\) can be obtained by a direct computation.

Since \(\Lambda^{2n-1} = 0\), \(\Lambda \Lambda' = \Lambda' \Lambda = 0\) and \((\Lambda')^2 = -2(-1)^n \Lambda'\), \((\Lambda')^3 = 0\), we have

\[
\exp \Theta(\Lambda; t) = \exp \left( \sum_{k=1}^{n-1} t_{2k-1} \Lambda^{2k-1} \right) \cdot \exp (s \Lambda') = \left( \sum_{m=0}^{2n-2} q_m (t') \Lambda^m \right) \cdot (I + s \Lambda' - (-1)^n s^2 \Lambda'^3).
\]

Now the explicit computation of \(\Lambda'\) completes the proof. \(\square\)

The next step is to relate the \(\tau\)-functions with minors of \(\exp \Theta(\Lambda; t)\). Let \(C^{2n}\) be the defining representation of \(SO_{2n}(C)\), and regard it as a representation of \(Spin_{2n}(C)\) via the covering map \(\pi : Spin_{2n}(C) \to SO_{2n}(C)\). Let \(e_1, \ldots, e_{2n}\) be the standard basis of \(C^{2n}\). If \(1 \leq r \leq n - 1\), then the exterior powers \(\Lambda^r C^{2n}\) gives the irreducible highest weight representation of \(SO_{2n}(C)\) or \(Spin_{2n}(C)\) with highest weight \(\varepsilon_1 + \cdots + \varepsilon_r\), and highest weight vector \(e_1 \wedge \cdots \wedge e_r\),

\[
\Lambda^r C^{2n} \cong V(\varpi_r) \quad (1 \leq r \leq n - 2), \quad \Lambda^{n-1} C^{2n} \cong V(\varpi_{n-1} + \varpi_n).
\]

On the other hand, \(\Lambda^n C^{2n}\) is not irreducible and decomposes into two irreducible components:

\[
\Lambda^n C^{2n} \cong V(2\varpi_n) \oplus V(2\varpi_{n-1}),
\]

and the highest weight vector of \(V(2\varpi_n)\) (resp. \(V(2\varpi_{n-1})\)) is given by \(e_1 \wedge \cdots \wedge e_{n-1} \wedge e_n\) (resp. \(e_1 \wedge \cdots \wedge e_{n-1} \wedge e_{n+1}\)). For a sequence \(I = (i_1, \ldots, i_r)\) of integers \(i_k \in [2n] = \{1, 2, \ldots, 2n\}\), we put \(e_I = e_{i_1} \wedge \cdots \wedge e_{i_r}\).

If \(\binom{2n}{r}\) denotes the set of strictly increasing sequences \(I = (i_1, \ldots, i_r)\) of length \(r\) with \(i_1, \ldots, i_r \in [2n]\), then \(\{e_I : I \in \binom{2n}{r}\}\) forms a basis of \(\Lambda^r C^{2n}\).

Let \((,\cdot,\cdot)\) be the standard bilinear form on \(C^{2n}\) given by \(\langle v, w \rangle = \langle vw \rangle\). We define a bilinear form on \(\Lambda^r C^{2n}\) by

\[
\langle v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r \rangle = \det (\langle v_i, w_j \rangle)_{1 \leq i, j \leq r}.
\]
Then \( \{e_I : I \in \binom{[2n]}{r} \} \) forms an orthonormal basis of \( \wedge^r \mathbb{C}^{2n} \). If we denote by \( \Delta^I_j(M) \) the minor of a \( 2n \times 2n \) matrix \( M \) with rows indexed by \( I = (i_1, \ldots, i_r) \) and columns indexed by \( J = (j_1, \ldots, j_r) \), then we have

\[
(e_I, M \cdot e_J) = \Delta^I_j(M) = \det \left( m_{i_p,j_q} \right)_{1 \leq p, q \leq r}.
\]

Note that this bilinear form satisfies the conditions in Lemma 3.1.

The half-spin representations \( V(\varpi_{n-1}) \) and \( V(\varpi_n) \) are minuscule ones (see [11, VII, §7, n°3] for properties of minuscule representations). Let \( \varpi = \varpi_{n-1} \) or \( \varpi_n \). If \( S = \{ w \in W : w \varpi = \varpi \} \) is the stabilizer of \( \varpi \) and \( \{ w_1, \ldots, w_N \} \) is a complete set of coset representatives for \( W/S \), then \( \{ w, v_{\varpi} : 1 \leq i \leq N \} \) forms a basis of \( V(\varpi) \). Moreover, by using the explicit action of \( \mathfrak{g} \) on \( V(\varpi) \) given in [5], we can see that this is an orthonormal basis with respect to the bilinear form given in Lemma 5.1.

In what follows we use the notation

\[
U(t', s) = \exp \Theta(\Lambda; t', s) = \exp \left( \sum_{i=1}^{n-1} t_{2i-1} \Lambda^{2i-1} + c_n s(\hat{S}\Lambda) \cdot S \right) \in SO_{2n}(\mathbb{C}).
\]

The determination of the \( \tau \)-functions are reduced to the computation of the minors of \( U(t', s) \).

**Lemma 5.4.**

1. If \( 1 \leq r \leq n - 2 \), then we have

\[
\tau_r(\Lambda; t', s; e_J) = \Delta_{j_1, j_2, \ldots, j_r}^1(\Lambda; t', s; e_J)
\]

for any \( J = (j_1, j_2, \ldots, j_r) \).

2. We have

\[
\left( \tau_n(\Lambda; t', s; \hat{w} \cdot v_{\varpi_n}) \right)^2 = \Delta_{\varpi(1), \varpi(n-1), \varpi(n)}(U(t', s)),
\]

\[
\left( \tau_{n-1}(\Lambda; t', s; \hat{w} \cdot v_{\varpi_{n-1}}) \right)^2 = \Delta_{\varpi(1), \varpi(n-1), \varpi(n+1)}(U(t', s))
\]

for any \( w \in W \).

**Proof.** We put \( \bar{U}(t', s) = \exp \Theta(\Lambda; t', s) \in Spin_{2n}(\mathbb{C}) \). Then the covering map \( \pi : Spin_{2n}(\mathbb{C}) \to SO_{2n}(\mathbb{C}) \) sends \( \bar{U}(t', s) \) to \( U(t', s) \).

(1) Since the representation \( V(\varpi_r) \) factors through \( SO_{2n}(\mathbb{C}) \), we have

\[
\tau_r(\Lambda; t', s; e_J) = (e_I, \bar{U}(t', s) \cdot e_J)_{\varpi_r} = (e_I, U(t', s) \cdot e_J)_{\varpi_r} = \Delta^I_j(U(t', s)),
\]

where \( I = (1, 2, \ldots, r) \).

(2) We denote by \( c^\text{Spin}_{2n} \) and \( c^\text{SO}_{2n} \) the matrix coefficients on the groups \( Spin_{2n}(\mathbb{C}) \) and \( SO_{2n}(\mathbb{C}) \) respectively. Let \( r = n \) or \( n - 1 \). Then, by using (3.1), we have

\[
\left( \tau_n(\Lambda; t', s; \hat{w} \cdot v_{\varpi_n}) \right)^2 = \left( c^\text{Spin}_{2n}(\bar{U}(t', s) \hat{w}) \right)^2 = c_{2\varpi_{r}}^\text{Spin}_{2n}(\bar{U}(t', s) \hat{w}) = c_{2\varpi_{r}}^\text{SO}_{2n}(U(t', s) \hat{w})
\]

\[
= \langle v_{2\varpi_{r}}, U(t', s) \hat{w} \cdot v_{2\varpi_{r}} \rangle_{2\varpi_{r}}.
\]

Since \( v_{2\varpi_{n}} = e_{(1, \ldots, n-1, n)} \) and \( v_{2\varpi_{n-1}} = e_{(1, \ldots, n-1, n+1)} \), we obtain the desired formula. \( \square \)

The key step of the proof of Theorem 5.1 is to express certain minors of \( U(t', s) \) in terms of 2-reduced Schur function. Let \( w \in W \), which is regarded as a permutation of \( \{1, \ldots, 2n\} \), and \( J = J(w) = (j_1, \ldots, j_n) \) be the sequence obtained by rearranging \( w(1), \ldots, w(n) \) in increasing order. We associate to \( w \in W \) the partition \( \lambda = \lambda(w) = (\lambda_1, \ldots, \lambda_n) \) given by

\[
\lambda_k = \begin{cases} j_{n+1-k} - (n + 1 - k) & \text{if } j_{n+1-k} \leq n, \\ j_{n+1-k} - (n + 1 - k) - 1 & \text{if } j_{n+1-k} \geq n + 1. \end{cases}
\]

And we define \( p = p(w) \) by

\[
p(w) = \# \{ k : j_k \geq n + 2 \} = \# \{ k : \lambda_k \geq k \},
\]
that is, we have
\[ j_1 < \cdots < j_{n-p} < n + 2 \leq j_{n-p+1} < \cdots < j_n. \]

Then \( j_{n-p} = n \) or \( n + 1 \), and \( j_{n-p} = n \) (resp. \( n + 1 \)) exactly when \( p \) is even (resp. odd).

**Lemma 5.5.** Let \( w \in W \), \( J = J(w) \) and \( \lambda = \lambda(w) \) be as above. We put \( I = (1, 2, \ldots, n) \).

1. If \( \lambda_1 \leq n - 2 \), then we have
   \[ \Delta_j^l(U(t', s)) = 2^{p-1}S_{(\lambda_1, \ldots, \lambda_n)}(t') + 2^{p-1}S_{(\lambda_1+1, \ldots, \lambda_p+1, \lambda_{p+2}, \ldots, \lambda_n)}(t'). \]

2. If \( \lambda_1 = n - 1 \), then
   \[ \Delta_j^l(U(t', s)) = 2^{p-1}S_{(\lambda_1, \ldots, \lambda_n)}(t') + 2^{p-1}S_{(\lambda_1+1, \ldots, \lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n)}(t') \]
   \[ + (-1)^n s \left( 2^{p-1}S_{(\lambda_1, \ldots, \lambda_p, \lambda_{p+2}, \ldots, \lambda_n-1)}(t') + 2^{p-1}S_{(\lambda_2, \ldots, \lambda_n)}(t') \right) \]
   \[ + s^2 \left( 2^{p-2}S_{(\lambda_1, \ldots, \lambda_n-1)}(t') + 2^{p-2}S_{(\lambda_2, \ldots, \lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n-1)}(t') \right). \]

**Proof.** Let \( V(t') \) be the \( 2n \times 2n \) matrix given by
\[ V(t') = U(t', 0) = \sum_{m=0}^{2n-2} q_m(t') \Lambda^m. \]

First we expand the minor \( \Delta_j^l(U(t', s)) \) in terms of minors of \( V(t') \). Here we note that \( i \) appears in \( J \) if and only if \( 2n + 1 - i \) does not (see (4.3)). If \( j_1 = 1 \), then by expanding \( \Delta_j^l(U(t', s)) \) along the first row, we have
\[ \Delta_{j_1, \ldots, j_n}^l(U(t', s)) = \Delta_{j_1, \ldots, j_n}^{1, \ldots, n}(V(t')) + (-1)^n s \cdot \Delta_{j_1, \ldots, j_{n-p}, \ldots, j_n}^{1, 2, \ldots, n}(V(t')) \]

where the symbol \( \hat{i} \) means that we remove \( i \) from the sequence. Since the first column of the determinant \( \Delta_{j_1, \ldots, j_{n-p}, \ldots, j_n}^{1, 2, \ldots, n}(V(t')) \) is a zero vector, we have
\[ \Delta_{j_1, \ldots, j_n}^{1, \ldots, n}(U(t', s)) = \Delta_{j_1, \ldots, j_n}^{1, \ldots, n}(V(t')). \]

If \( j_n = 2n \), then by expanding \( \Delta_j^l(U(t', s)) \) along the first row and the last column, we have
\[ \Delta_{j_1, \ldots, j_n}^l(U(t', s)) = \Delta_{j_1, \ldots, j_n}^{1, \ldots, n}(V(t')) + (-1)^n s \cdot \Delta_{j_1, \ldots, j_{n-p}, \ldots, j_n}^{1, 2, \ldots, n}(V(t')) + (-1)^n s \cdot \Delta_{j_1, \ldots, j_{n-1}}^{1, \ldots, n-1, \hat{i}}(V(t')) \]
\[ + s^2 \cdot \Delta_{j_1, \ldots, j_{n-1}}^{1, 2, \ldots, n-1, \hat{i}}(V(t')) + s^2 \cdot \Delta_{j_1, \ldots, j_{n-1}}^{1, 2, \ldots, n-1, \hat{i}}(V(t')). \]

Next we compute the above minors of \( V(t') \) and express them in terms of 2-reduced Schur functions. Since the proofs are similar, we explain the computation of \( \Delta_{j_1, \ldots, j_n}^{1, \ldots, n}(V(t')) \). In fact, we prove
\[ \Delta_{j_1, \ldots, j_n}^{1, \ldots, n}(V(t')) = 2^{p-1}S_{(\lambda_1, \ldots, \lambda_n)}(t') + 2^{p-1}S_{(\lambda_1+1, \ldots, \lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n)}(t'). \]
Consider the case \( j_{n-p} = n \). In this case, by multiplying the last column by \( 1/2 \) and the last row by 2, we see that
\[
\Delta_{j_1,j_2,\ldots,j_n}^{1,2,\ldots,n}(V(t')) = \det \begin{pmatrix}
q_{\lambda_n} & \cdots & q_{\lambda_{p+2}+n-p-2} & q_{n-1} & 2q_{\lambda_p+n-p} & \cdots & 2q_{\lambda_2+n-2} & q_{\lambda_1+n-1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
q_{\lambda_{p+2}} & q_{\lambda_{p+1}+1} & \cdots & q_{\lambda_p} & 2q_{\lambda_{p+1}} & \cdots & 2q_{\lambda_2} & q_{\lambda_1+1} \\
0 & \cdots & 0 & 2 & 2q_{\lambda_{p-1}} & \cdots & 2q_{\lambda_2} & q_{\lambda_1} 
\end{pmatrix}
\]
By splitting the last row as the sum of two vectors
\[
a = ( 0 \cdots 0 1 2q_{\lambda_{p-1}} \cdots 2q_{\lambda_2} q_{\lambda_1} ), \quad \text{and} \quad b = ( 0 \cdots 0 1 0 \cdots 0 0 ),
\]
we obtain
\[
\Delta_{j_1,j_2,\ldots,j_n}^{1,2,\ldots,n}(V(t')) = \det A + \det B,
\]
where \( A \) and \( B \) are the matrices obtained by replacing the last row with \( a \) and \( b \) respectively. Comparison with the definition \( \Delta_{j_1,j_2,\ldots,j_n}^{1,2,\ldots,n}(V(t')) = 2^{p-1}S_{(\lambda_1+1,\ldots,\lambda_p+1,\lambda_{p+2},\ldots,\lambda_n)}(t') \).

If \( j_{n-p} = n+1 \), then we have
\[
\Delta_{j_1,j_2,\ldots,j_n}^{1,2,\ldots,n}(V(t')) = \det \begin{pmatrix}
q_{\lambda_n} & \cdots & q_{\lambda_{p+2}+n-p-2} & q_{n-1} & 2q_{\lambda_p+n-p} & \cdots & 2q_{\lambda_2+n-2} & q_{\lambda_1+n-1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
q_{\lambda_{p+2}} & q_{\lambda_{p+1}+1} & \cdots & q_{\lambda_p} & 2q_{\lambda_{p+1}} & \cdots & 2q_{\lambda_2} & q_{\lambda_1+1} \\
0 & \cdots & 0 & 2 & 2q_{\lambda_{p-1}} & \cdots & 2q_{\lambda_2} & q_{\lambda_1} 
\end{pmatrix}
\]
By splitting the last row as the difference of two vectors
\[
a = ( 0 \cdots 0 1 2q_{\lambda_{p-1}} \cdots 2q_{\lambda_2} q_{\lambda_1} ), \quad \text{and} \quad b = ( 0 \cdots 0 1 0 \cdots 0 0 ),
\]
we obtain \( \Delta_{j_1,j_2,\ldots,j_n}^{1,2,\ldots,n}(V(t')) = 2^{p-1}S_{(\lambda_1+1,\ldots,\lambda_p+1,\lambda_{p+2},\ldots,\lambda_n)}(t') \).

Similarly we can show
\[
\Delta_{j_1,j_2,\ldots,j_n}^{1,2,\ldots,n}(V(t')) = 2^{p-1}S_{(\lambda_1,\ldots,\lambda_p,\lambda_{p+2}-1,\ldots,\lambda_n-1)}(t'),
\]
\[
\Delta_{j_1,j_2,\ldots,j_n}^{1,2,\ldots,n}(V(t')) = 2^{p-1}S_{(\lambda_2,\ldots,\lambda_n)}(t'),
\]
\[
\Delta_{j_1,j_2,\ldots,j_n}^{1,2,\ldots,n}(V(t')) = 2^{p-2}S_{(\lambda_2-1,\ldots,\lambda_n-1)}(t') - 2^{p-2}S_{(\lambda_2,\ldots,\lambda_p,\lambda_{p+2}-1,\ldots,\lambda_n-1)}(t'),
\]
\[
\Delta_{j_1,j_2,\ldots,j_n}^{1,2,\ldots,n}(V(t')) = 2^{p-1}S_{(\lambda_2,\ldots,\lambda_p,\lambda_{p+2}-1,\ldots,\lambda_n-1)}(t').
\]
Combining these formulas for the minors of $V(t')$, we complete the proof. □

The following lemma enables us to relate the partitions appearing in Lemma 5.5 with the strict partition $\alpha(w)$ given by (5.8). Given a partition $\mu$, we define

$$l = \# \{i : \mu_i \geq i \}, \quad \beta_i = \mu_i - i, \quad \gamma_i = \beta_i - i \quad (1 \leq i \leq l),$$

where $\beta_i$ is the conjugate partition of $\mu$, and write $\mu = (\beta_1, \ldots, \beta_i | \gamma_1, \ldots, \gamma_i)$. This representation is called the Frobenius notation of $\mu$.

**Lemma 5.6.** Let $w \in W$ and $\lambda = \lambda(w)$ the partition defined by (5.14). Then we have

$$\lambda_k = \lambda_k + 1 \quad (1 \leq k \leq p), \quad \lambda_{p+1} = p, \quad \lambda_{p+1} = \lambda_{p+1} \quad (p + 1 \leq k \leq n - 1).$$

If $\alpha = \alpha(w)$ is the strict partition given by (5.8), then we have $\alpha_k = \lambda_k - k + 1$ for $1 \leq k \leq p$ and

$$(\lambda_1, \ldots, \lambda_n) = (\alpha_1 - 1, \ldots, \alpha_p - 1 | \alpha_1, \ldots, \alpha_p),$$

$$(\lambda_1 + 1, \ldots, \lambda_p + 1, \lambda_{p+2}, \ldots, \lambda_n) = (\alpha_1, \ldots, \alpha_p | \alpha_1 - 1, \ldots, \alpha_p - 1),$$

$$(\lambda_1, \ldots, \lambda_p, \lambda_{p+2} - 1, \ldots, \lambda_n - 1) = (\alpha_1 - 1, \ldots, \alpha_p - 1 | \alpha_2, \ldots, \alpha_p, 0),$$

$$(\lambda_2, \ldots, \lambda_n) = (\alpha_2, \ldots, \alpha_p | \alpha_1 - 1, \ldots, \alpha_p - 1),$$

$$(\lambda_2 - 1, \ldots, \lambda_n - 1) = (\alpha_2 - 1, \ldots, \alpha_p - 1 | \alpha_2, \ldots, \alpha_p),$$

$$(\lambda_2, \ldots, \lambda_p, \lambda_{p+2} - 1, \ldots, \lambda_n - 1) = (\alpha_2, \ldots, \alpha_p | \alpha_2 - 1, \ldots, \alpha_p - 1).$$

**Proof.** Since $j_n - p = n$ or $n + 1$, we can see that $\lambda_{p+1} = p$.

We prove $\lambda_k = \lambda_k + 1$ for $1 \leq k \leq p$. The largest $p$ elements of $J$ are

$$j_{n+1-k} = \lambda_k + n - k + 2 \quad (1 \leq k \leq p).$$

Since $i \in J$ if and only if $2n + 1 - i \not\in J$, we see that the smallest $p$ elements of $J^c = \{1, \ldots, 2n\} \setminus J$ are

$$(2n + 1) - (\lambda_k + n - k + 2) = n + k - 1 - \lambda_k \quad (1 \leq k \leq p).$$

On the other hand, it follows from [20, 1.(1.7)] that

$$\{\lambda_i - i : 1 \leq i \leq p\} \cup \{-1 + j - \lambda_j : p + 1 \leq j \leq n - 1\} = \{0, 1, \ldots, n - 2\},$$

and that the smallest $p$ elements of $J^c$ are

$$n + k - \lambda_k \quad (1 \leq k \leq p).$$

Hence we obtain $\lambda_k = \lambda_k + 1$ for $1 \leq k \leq p$.

Similarly, by considering the largest $(n - 1 - p)$ elements of $J^c$, we can prove $\lambda_k = \lambda_k + 1$ for $p + 1 \leq k \leq n - 1$.

The latter half of the lemma follows from the definition of the Frobenius notation. □

The last ingredient of the proof of Theorem 5.1 is the following relations between 2-reduced Schur functions and Schur’s $Q$-functions.

**Lemma 5.7.**

1. (see [20] III.8 Example 10 (a))) For any partition $\lambda$, we have

$$S_{\lambda}(t') = S_{\lambda}(t').$$

2. ([24] (15)], [12] Theorem 2]) If $\lambda = (\alpha_1, \ldots, \alpha_p | \alpha_1, \ldots, \alpha_p, 0)$, then we have

$$S_{(\alpha_1, \ldots, \alpha_p, \alpha_1, \ldots, \alpha_p)}(t') = 2^{-p} (Q_{(\alpha_1, \ldots, \alpha_p)}(t'))^2.$$

3. ([25] Theorem 1.3]) If $\lambda = (\alpha_1, \ldots, \alpha_p | \alpha_1, \ldots, \alpha_p, 0)$, then we have

$$S_{(\alpha_1, \ldots, \alpha_p, \alpha_1, \ldots, \alpha_p)}(t') = 2 \cdot 2^{-p} Q_{(\alpha_1, \ldots, \alpha_p)}(t') Q_{(\alpha_1, \ldots, \alpha_p)}(t').$$

Now we are in position to finish the proof of Theorem 5.1.
Proof of Theorem 5.1. First we prove that \( \tau_n(\Lambda; \mathbf{t}; \hat{w} \cdot v_{\pi_n}) = c_{n,w} \hat{Q}_{\alpha(w)}(\mathbf{t}', s) \) (for some constant \( c_{n,w} \)) (Equation (5.9)). By Lemma 5.4 (2), we have

\[
(\tau_n(\Lambda; \mathbf{t}; \hat{w} \cdot v_{\pi_n}))^2 = \pm \Delta^I_J(U(\mathbf{t}', s)),
\]

where \( I = (1, 2, \ldots, n) \) and \( J \) is the rearrangement of \( \{w(1), w(2), \ldots, w(n)\} \) in increasing order.

We consider the case where \( n - 1 \) is a part of \( \alpha(w) \), i.e. \( \lambda_1 = \alpha_1 = n - 1 \). Then, combining Lemmas 5.5, 5.6 and 5.7, we obtain

\[
\Delta^I_J(U(\mathbf{t}', s)) = 2^p S_{(\alpha_1-1, \ldots, \alpha_p-1; \alpha_1, \ldots, \alpha_p)}(\mathbf{t}') + \hat{Q}_{\alpha_1}(\mathbf{t}')^2 + 2(\pm 1)^n s Q_{(\alpha_1, \ldots, \alpha_p)}(\mathbf{t}') Q_{(\alpha_2, \ldots, \alpha_p)}(\mathbf{t}') + s^2 (Q_{(\alpha_2, \ldots, \alpha_p)}(\mathbf{t}'))^2
\]

Similarly we can show the case where \( n - 1 \) is not a part of \( \alpha(w) \).

Equation (5.10) can be derived from (5.9) by using the Dynkin diagram automorphism of \( \mathfrak{so}_{2n}(\mathbb{C}) \). Let \( \sigma \in \mathfrak{O}_{2n}(\mathbb{C}) \) be the matrix given by

\[
\sigma = \begin{pmatrix} I_{n-1} & 0 & 1 \\ 1 & 0 & I_{n-1} \end{pmatrix}.
\]

Then \( \sigma^2 = 1 \) and the conjugation by \( \sigma \) induces the Dynkin diagram automorphism switching the simple roots \( \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n \) and \( \alpha_n = \varepsilon_{n-1} + \varepsilon_n \). And the induced automorphism on \( W \cong \mathfrak{O}_n \) is given by \( \mathfrak{O}_n \ni w \mapsto w^f = (n, n+1)w(n, n+1) \in \mathfrak{O}_n \). Since we have

\[
\sigma \cdot v_{\pi_n} = \sigma \cdot e_1 \wedge \cdots \wedge e_{n-1} \wedge e_n = e_1 \wedge \cdots \wedge e_{n-1} \wedge e_{n+1} = v_{\pi_n-1},
\]

it follows from Lemma 5.4 that

\[
(\tau_{n-1}(\Lambda; \mathbf{t}; \hat{w} \cdot v_{\pi_n}))^2 = \langle v_{\pi_n-1}, U(\mathbf{t}', s)\hat{w} \cdot v_{\pi_n-1} \rangle = \langle \sigma \cdot v_{\pi_n}, U(\mathbf{t}', s)\hat{w} \cdot v_{\pi_n} \rangle = \langle e_1, \sigma^{-1} U(\mathbf{t}', s)\sigma \cdot \sigma^{-1} \hat{w} \cdot e_1 \rangle.
\]

Here we note that \( \sigma^{-1} U(\mathbf{t}', s)\sigma \) is the matrix obtained from \( U(\mathbf{t}', s) \) by swapping the \( n \)th and \((n+1)\)st rows and the \( n \)th and \((n+1)\)st columns. Then we have

\[
\sigma^{-1} U(\mathbf{t}', s)\sigma = U(\mathbf{t}', -s),
\]

and (5.10) is obtained from (5.9). This completes the proof of Theorem 5.1.

Remark 5.8. The argument in the proof of Theorem 5.1 shows that the relations for the matrix coefficients (see Lemma 5.3 (3))

\[
c_{\pi_n}(U(\mathbf{t}', s)\hat{w}) = \langle e_1 \wedge \cdots \wedge e_{n-1} \wedge e_n, U(\mathbf{t}', s)\hat{w} \cdot e_1 \wedge \cdots \wedge e_{n-1} \wedge e_n \rangle = (c_{\pi_n}(U(\mathbf{t}', s)\hat{w}))^2,
\]

\[
c_{\pi_n-1}(U(\mathbf{t}', s)\hat{w}) = \langle e_1 \wedge \cdots \wedge e_{n-1} \wedge e_{n+1}, U(\mathbf{t}', s)\hat{w} \cdot e_1 \wedge \cdots \wedge e_{n-1} \wedge e_{n+1} \rangle = (c_{\pi_n-1}(U(\mathbf{t}', s)\hat{w}))^2
\]

are translated into bilinear expansion formulas of 2-reduced Schur functions in Schur’s \( Q \)-functions, which are the identities in Lemma 5.7 (2) and (3). Similarly, since \( n \varepsilon_n - n \varepsilon_{n-1} = \varepsilon_1 + \cdots + \varepsilon_{n-1} \), it follows from Lemma 5.3 (3) that

\[
c_{\pi_{n-1}+\pi_n}(U(\mathbf{t}', s)\hat{w}) = \langle e_1 \wedge \cdots \wedge e_{n-1}, U(\mathbf{t}', s)\hat{w} \cdot e_1 \wedge \cdots \wedge e_{n-1} \rangle = c_{\pi_{n-1}}(U(\mathbf{t}', s)\hat{w}) \cdot c_{\pi_n}(U(\mathbf{t}', s)\hat{w}).
\]
By using Theorem 5.10 and a similar argument to the proof of Lemma 5.5, we can derive some bilinear expansion formulas. For example, if \( n + 2 \leq w(n) \leq 2n - 1 \), equating the coefficients of \( s \) gives us
\[
2^{p-1}s_{(a_1,\ldots,a_p|a_2-1,\ldots,a_p-1)}(t') = -Q_{(a_1,\ldots,a_p)}(t')Q_{(a_2-1,\ldots,a_p)}(t') + Q_{(a_1,\ldots,a_p|a_2-1,\ldots,a_p-1)}(t'),
\]
which is a special case of [22, Corollary 2]. See [12, 22, 25, 8] for bilinear expansion formulas of 2-reduced Schur functions in Schur’s Q-functions.

We also have the following remark on Theorem 5.1.

**Remark 5.9.** Consider the fixed-point subgroup of the diagram automorphism \( \hat{\tau} : W \to W \):
\[
W' = \{ w \in W : w = w^\dagger \},
\]
which is isomorphic to the Weyl group of type \( B_{n-1} \). Then, for \( w \in W' \) and \( s = 0 \), we have
\[
\tau_n(\Lambda; t', 0; \hat{w} \cdot \nu_{w_n}) = \tau_{n-1}(\Lambda; t', 0; \hat{w} \cdot \nu_{w_{n-1}}) = Q_{\alpha(w)}(t')
\]
up to constant multiples. This is the polynomial \( \tau \)-function of the \( \tau \)-KT hierarchy on \( so_{2n-1}(C) \) of type \( B_{n-1} \) corresponding to the spin node [26]. Also note that both \( \tau_{n-1}(\Lambda; t', 0; \hat{w} \cdot \nu_{w_{n-1}}) \) and \( \tau_n(\Lambda; t', 0; \hat{w} \cdot \nu_{w_n}) \) satisfy the BKP hierarchy, and we expect that all the polynomial \( \tau \)-functions of the BKP hierarchy are given by these \( \tau \)-functions with \( w \in W' \) [24, 13, 0, 19].

### 5.3. Explicit formulas for \( \tau_r \) with \( 1 \leq r \leq n-2 \)

In this final subsection, we give explicit formulas for the \( \tau_r \)-functions with \( 1 \leq r \leq n-2 \) in terms of 2-reduced Schur functions.

**Proposition 5.10.** Let \( J = (j_1,\ldots,j_r) \in (\mathbb{Z}^n)_r \) and regard it as a subset of \( [2n] = \{1,\ldots,2n\} \). We define a sequence \( \lambda = (\lambda_1,\ldots,\lambda_r) \) by
\[
\lambda_k = \begin{cases} j_{r-k+1} - (r - k + 1) - 1 & \text{if } j_{r-k+1} \geq n + 1, \\ j_{r-k+1} - (r - k + 1) & \text{if } j_{r-k+1} \leq n \end{cases}
\]
and \( p = \# \{ k : j_k \geq n + 2 \} \). Then the \( \tau_r \)-functions for \( 1 \leq r \leq n-2 \) are given as follows (up to nonzero constant multiples):

1. If \( J \cap \{ n, n + 1, 2n \} = \emptyset \), then we have
   \[
   \tau_r(\Lambda; t; e_J) = 2^pS_\lambda(t').
   \]
2. If \( J \cap \{ n, n + 1, 2n \} = \{ 2n \} \), then we have
   \[
   \tau_r(\Lambda; t; e_J) = 2^pS_\lambda(t') + (-1)^{n+r+1}2^{p-1}s^2S_{(\lambda_2-1,\ldots,\lambda_r-1)}(t').
   \]
3. If \( J \cap \{ n, n + 1, 2n \} = \{ n + 1 \} \), then we have
   \[
   \tau_r(\Lambda; t; e_J) = 2^pS_\lambda(t') + (-1)^{r+1}2^pS_{(\lambda_1,\ldots,\lambda_r+1)}(t').
   \]
4. If \( J \cap \{ n, n + 1, 2n \} = \{ n \} \), then we have
   \[
   \tau_r(\Lambda; t; e_J) = 2^pS_\lambda(t') + (-1)^{-r-p+1}2^pS_{(\lambda_1,\ldots,\lambda_r+1)}(t').
   \]
5. If \( J \cap \{ n, n + 1, 2n \} = \{ n + 1, 2n \} \), then we have
   \[
   \tau_r(\Lambda; t; e_J) = 2^pS_\lambda(t') + (-1)^{r-p}2^pS_{(\lambda_1,\ldots,\lambda_r+1)}(t').
   \]
6. If \( J \cap \{ n, n + 1, 2n \} = \{ n, 2n \} \), then we have
   \[
   \tau_r(\Lambda; t; e_J) = 2^pS_\lambda(t') + (-1)^{r+1}2^pS_{(\lambda_1,\lambda_r+2,1,\ldots,\lambda_r+2)}(t') + (-1)^{n+r+1}2^{p-1}s^2S_{(\lambda_2-1,\ldots,\lambda_r-1)}(t').
   \]
7. If \( J \cap \{ n, n + 1, 2n \} = \{ n, n + 1 \} \), then we have
   \[
   \tau_r(\Lambda; t; e_J) = (1)^{r-p+1}2^pS_{(\lambda_1,\lambda_r+2,1,\ldots,\lambda_r+2)}(t').
   \]
(8) If \( J \cap \{n, n + 1, 2n\} = \{n, n + 1, 2n\} \), then we have
\[
\tau_r(\Lambda; t; e_J) = (-1)^{r-p+1}2^{p+1}S_{(\lambda_1, \ldots, \lambda_p, \lambda_{p+2}, \ldots, \lambda_r, -1)}(t').
\]

**Proof.** We can prove this proposition in a similar manner to the proof of Lemma 5.5. So we omit it. □

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