Nonlocal interactions versus viscosity in turbulence

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It is shown that nonlocal interactions determine energy spectrum in isotropic turbulence at small Reynolds numbers. It is also shown that for moderate Reynolds numbers the bottleneck effect is determined by the same nonlocal interactions. Role of the large and small scales covariance at the nonlocal interactions and in energy balance has been investigated. A possible hydrodynamic mechanism of the nonlocal solution instability at large scales has been briefly discussed. A quantitative relationship between effective strain of the nonlocal interactions and viscosity has been found. All results are supported by comparison with the data of experiments and numerical simulations.

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I. INTRODUCTION

Classic wind-tunnel experiments [1] showed that there is no scaling behavior in a presumably isotropic turbulence at low Reynolds numbers. Moreover, recent high-resolution numerical simulations [2] performed for low-Reynolds-number ($R_L \approx 10 - 60$) show no hint of scaling-like behavior of the velocity increments even when ESS [3] is applied. For moderate Reynolds numbers numerical simulations show ‘excess’ power just before the dissipation (Kolmogorov’s) wavenumber $k_d$ (a hump in the compensated energy spectra), see for instance [4]. This non-scaling effect (bottleneck effect [5]) is usually related with reducing efficiency of the energy cascade toward $k_d$ [3,6,7]. For the small Reynolds numbers applicability of the energy cascade idea is problematic for entire range of scales. It is shown in recent paper [8] that nonlocal interactions become dominating in comparison with the local ones just in the near-dissipation range of scales (cf. also [8]). In this range the viscous effects cannot be neglected and scaling asymptote corresponding to the nonlocal regime cannot be observed [8] (in the inertial range the local interactions are presumably dominating ones and the nonlocal scaling asymptote also cannot be observed). However, the nonlocal scaling asymptote can be used as a zeroth term in a perturbation approach taking into account the viscosity effects. There are many ways to develop such perturbation approach. For instance, in the paper [8] a perturbation approach giving logarithmic corrections to the scaling was developed, provided by significant role of the kink instabilities of the vortex filaments at moderate and large values of Reynolds number [10]. This ‘logarithmic’-perturbation approach is shown to be effective in a vicinity of the crossover scale $r_c$, where exchange of stability between local and nonlocal regimes takes place [8]. This vicinity is rather wide at moderate values of Reynolds number, when an overlap between these regimes is a strong phenomenon [8]. For small Reynolds numbers, however, the kink instabilities of the small vortex tubes is suppressed by the strong viscosity (see, for instance [11] and references therein). Therefore, for the small Reynolds numbers another, adequate just for this case, perturbation approach should be developed. An approach of such type is suggested in present paper. Starting from this approach and using comparison with results of numerical simulations [12] the laboratory experiments [1] it is shown that energy spectrum for small Reynolds numbers is determined by the nonlocal interactions even in isotropic turbulence. The same nonlocal interactions provide a hydrodynamic mechanism for the so-called bottleneck effect for moderate Reynolds numbers. It is also shown that large and small scales covariance at the nonlocal interactions plays a significant role in these phenomena. A quantitative relationship between effective strain of the nonlocal interactions and viscosity has been found using dynamical equations.

II. PERTURBATIONS TO SCALING

Let us following to the paper [8] consider a dimensional function $E(k)$ of a dimensional argument $k$. And let us construct a dimensionless function of the same argument

$$\alpha(k) = \frac{E^{-1}dE}{k^{-1}dk}. \quad (1)$$

If for $k_d \gg k$ we have no relevant fixed scale (scaling situation), then for these values of $k$ the function $\alpha(k)$ must be independent on $k$, i.e. $\alpha(k) \simeq \text{const}$ for $k_d \gg k$. For turbulence $k$ could be a wavenumber and $k_d$ could be a dissipation wavenumber ($k_d = 1/\eta$, where $\eta = (\nu^3/\varepsilon)^{1/4}$ is so-called viscous scale [24]). Solution of equation (1) with constant $\alpha$ can be readily found as

$$E(k) \simeq ck^\alpha \quad (2)$$
where \( c \) is a dimension constant. This is the well-known power law corresponding to the scaling situations.

Let us now consider an analytic approach, which allows us to find corrections of all orders to the approximate power law, related to the fixed scale \( k_d \). In the non-scaling situation let us denote

\[
f \equiv \ln \left( \frac{E}{A} \right), \quad x \equiv \ln \left( \frac{k}{k_d} \right)
\]

where \( A \) and \( k_d \) are dimensional constants used for normalization.

In these variables, equation (1) can be rewritten as

\[
\frac{df}{dx} = \alpha(x)
\]

In the non-scaling situation \( x \) is a dimensionless variable, hence the dimensionless function \( \alpha(x) \) can be non-constant. Since the 'pure' scaling corresponds to \( k/k_d \ll 1 \) we will use an analytic expansion in power series

\[
\alpha(x) = \alpha_0 + \alpha_1 (k/k_d) + \cdots + \frac{1}{n!} \alpha_n (k/k_d)^n + \cdots
\]

where \( \alpha_n \) are dimensionless constants. Choice of the small parameter for analytic perturbation approach is determined by physical situation, which one intends to consider. For instance, for moderate Reynolds numbers it would be generally preferable to consider the \( x^{-1} \) as a small parameter [8]. In this paper, however, we intend to consider energy spectra in isotropic turbulence at small Reynolds numbers and corresponding phenomena in a relatively close vicinity of the dissipation scale at moderate Reynolds numbers. The kink instabilities of the vortex filaments, which are a significant factor near \( r_c \) (see above and [8]) are presumably not significant in situations with strong viscous effects [11]. Therefore
the choice of $x^{-1}$ does not seem to be relevant here. On the other hand, parameter $k/k_d$ seems to be less related to the specific hydrodynamic structures dominating processes in turbulence and more relevant to a sheer taking into account corrections to the scaling providing by viscosity.

After substitution of the analytic expansion (5) into Eq. (4) the zeroth order approximation gives the power law (2) with $\alpha \equiv \alpha_0$. First order analytic approximation, when one takes only the two first terms in the analytic expansion (5), gives

$$E(k) \cong c k^{\alpha_0 \epsilon_{\alpha_1 / k_d}}.$$  

(cf, for instance, [13]-[16]). Corrections of the higher orders can be readily found in this perturbation approach.

III. NONLOCAL INTERACTIONS

Let us recall that in isotropic turbulence a complete separation of local and non-local interactions is possible in principle. It was shown by Kadomtsev [17] that this separation plays a crucial role for the local Kolmogorov’s cascade regime with scaling energy spectrum

$$E(k) \cong K \langle \varepsilon \rangle^{2/3} k^{-5/3}$$  

(7)

where $\langle \varepsilon \rangle$ is the average of the energy dissipation rate, $\varepsilon$, $k = 1/r$ is the wave-number, and $K$ is the so-called Kolmogorov constant. This separation should be effective for the both ends. That is, if there exists a solution with the local scaling (7) as an asymptote, then there should also exist a solution with the non-local scaling asymptote.
FIG. 3: As in Fig. 2 but for \( R_\lambda = 460 \). The upper part of the figure shows the same energy spectrum as the lower one but in the compensate (according to the Kolmogorov’s scaling) form. The solid curves are the best fit corresponding to the nonlocal approximation (10) with the same \( \alpha_1 = 6.0 \pm 0.1 \) as in Figs 2.

Of course, the two solutions with these asymptotes should be alternatively stable (unstable) in different regions of scales. It is expected, that the local (Kolmogorov’s) solution is stable (i.e. statistically dominating) in inertial range (that means instability of the non-local solution in this range of scales).

Roughly speaking, in non-local solution for small scales \( r \) only non-local interactions with large scales \( L \) \((1 \gg r/L)\) are dynamically significant (the interaction among the small scales is negligible compared with interaction via large scales) and the non-local interactions is determined by large scale strain/shear. This means that one should add to the energy flux \( \langle \varepsilon \rangle \)-parameter (which is a governing parameter for the both solutions) an additional parameter such as the strain \( s \) for the non-local solution. As far as we know it was noted for the first time by Nazarenko and Laval [18] that dimensional considerations applied to the non-local asymptote result in the power-law energy spectrum

\[
E(k) \simeq c \langle \varepsilon \rangle \frac{s}{k} k^{-1}
\]  

both for two- and three-dimensional cases. Linear dependence of the spectrum (8) on \( \langle \varepsilon \rangle \) is determined by the linear nature of equations corresponding to the non-local asymptote that together with the dimensional considerations results in (8) [18]. Interesting numerical simulations were performed in [19]. In these simulations local and non-local interactions have been alternatively removed. For the first case a tendency toward a spectrum flatter than -5/3 is observed near and beyond the separating scale (beyond which local interactions are ignored), that supports Eq. (8).

Following to the perturbation approach suggested above both local and non-local regimes can be corrected. The first order correction is

\[
E(k) \simeq C \langle \varepsilon \rangle^{2/3} k^{-5/3} e^{-\beta(k/k_d)}
\]  

and

\[
E(k) \simeq c \langle \varepsilon \rangle \frac{s}{k} k_{-1} e^{-\alpha_1(k/k_d)}
\]  

for the local and non-local regimes respectively \( (K, c, \alpha_1 \text{ and } \beta \text{ are dimensionless constants}) \).
IV. STRAIN AND VISCOSITY

It is shown in Ref. 8 that for sufficiently large Reynolds numbers, providing a visible inertial interval, there is an overlapping between the two regimes: non-local and local (Kolmogorov). This overlapping is based on the very nature of the stability exchange between the two statistical regimes. However, in analogy with the viscous scale \( \eta = (v^3/\langle \varepsilon \rangle)^{1/4} \) there should be a scale \( \eta_s \) such that for scales \( r < \eta_s \) contribution of the local interactions will be drastically decreased in comparison with the nonlocal ones. In the analogy with \( \eta \) one can calculate \( \eta_s \) using the dimensional considerations as

\[
\eta_s = \left( \frac{\langle \varepsilon \rangle}{8s^4} \right)^{1/2}
\]  

cf for shear flows 23).

The dynamical equations provide us with a relationship 24

\[
\langle \varepsilon \rangle = 2\nu \int_0^{k_d} k^2 E(k)dk
\]

For \( E(k) \) given by Eq. (10) the dissipation function \( k^2 E(k) \) has its maximum at \( k = k_d/6 \). If \( k_d \gg k_s = 1/\eta_s \) (see below) the maximum of the dissipation function \( k^2 E(k) \) is located just between \( k_s \) and \( k_d \). Therefore, we can estimate
Substituting (10) (with $\alpha_1 = 6$) into (13) we obtain relationship
\[
s \simeq \frac{c}{18} \langle \varepsilon \rangle^{1/2} \nu^{-1/2}
\] (14)

Using the relationship (14) one can estimate the prefactor in the nonlocal approximation to the energy spectra (10) as
\[
c \langle \varepsilon \rangle / s \simeq 18 \langle (\varepsilon) \nu \rangle^{1/2}
\] (15)

Now using the data of the DNS \[12\] (cf Figs. 2,3) let us calculate the prefactor $c \langle \varepsilon \rangle / s$. Results of these calculations are shown in figure 4 as circles ($Re_\lambda = 38, 54, 70, 125, 284, 380, 460$). This figure shows the prefactor $c \langle \varepsilon \rangle / s$ against $((\varepsilon) \nu)^{1/2}$. The straight line with the slope equals to 18 indicates agreement with Eq. (15).

It should be noted that in the DNS \[12\], $\langle \varepsilon \rangle \simeq \text{const}$ for $R_\lambda \geq 70$, in agreement with the well known Kolmogov’s hypothesis \[24\]. Therefore, for $R_\lambda \geq 70$ (when $\langle \varepsilon \rangle \simeq \text{const} \[12\]$) it follows from Eq. (14) that $s \sim \nu^{-1/2}$. This relationship provides us also with dependence of the strain $s$ on $R_\lambda$.

Due to the nonlinear character of the Navier-Stokes equations the so-called triadic type of interactions is dominating mechanism of the dynamical interactions in turbulence (see, for instance, \[9\]). A triad corresponding to the nonlocal interactions involves two short-wave-number modes and one long-wave-number mode (see a sketch in figure 5). Accordingly, at the nonlocal interactions two characteristic space scales are actively involved: large-scale characteristic scale $\eta_s$ (11) and small-scale (viscous or Kolmogorov) characteristic space scale $\eta = (\nu^3 / \langle \varepsilon \rangle)^{1/4}$. It is naturally that the large scales should be normalized by $\eta_s$ while the small scales should be normalized by $\eta$. This could cause an obvious problem at the nonlocal interactions. However, the large and small scales covariance relationship at the nonlocal interactions follows directly from the relationship (14)
\[
\frac{\eta_s}{\eta} = \left( \frac{\langle \varepsilon \rangle}{\nu s^2} \right)^{3/4} \simeq \left( \frac{18}{c} \right)^{3/2} \simeq \text{const}
\] (16)
(i.e. the relation $\eta_s/\eta$ is independent on $R_\lambda$). And vice versa, the large and small scales covariance (15) results in the relationship of the type (14). The covariance at the nonlocal interactions supports right balance between the energy flux to the small scales and their dissipative capacity (cf, for instance, \[26,27\]).
Actually, the scale \( \eta_s = 1/k_s \) should provide an edge of applicability of the approximation (10) to the real energy spectra. If one compares the \( \eta_s \) determined by this way from the Figs. 2,3,6 one can see that indeed \( \eta_s/\eta \simeq \text{const} \) in agreement with Eq. (16) (we have also checked this with the data for \( R_\lambda = 284,380 \) [12] and the data reported in [29]). Moreover, taking into account a continuity condition of the energy spectrum at the point \( k = k_s \) (and using Eqs. (7) and (10)):

\[
K \langle \varepsilon \rangle^{2/3} k_s^{5/3} \simeq c \langle \varepsilon \rangle k_s^{-1} e^{-\alpha_1 (k_s/k_d)}
\]

we obtain equation

\[
K \simeq c \exp[-(2c)^{3/2}/\alpha_1^2]
\]

Substituting the Kolmogorov constant \( K \simeq 1.6 \) (see [12,28]) and \( \alpha_1 \simeq 6 \) into this equation we obtain \( c \simeq 2 \). Then, substituting this value of \( c \) into Eq. (16) we obtain \( k_s/k_d \simeq 0.037 \). The last value is in agreement with the available data (see Figs. 2,3,6 and [29]).

Together with the universality of \( \alpha_1 \simeq 6.0 \) (which also is a consequence of the scale covariance) Eq. (16) determines the well known from DNSs [12,29] universality (independence on \( R_\lambda \)) of the position of the ‘hump’ in the axes \( k\eta \) for the bottleneck effect.

V. PASSIVE SCALAR

For passive scalar \( \theta \) in the isotropic turbulence the equations (9),(10) should be replaced [8] by the equations

\[
E^\theta(k) \propto \langle \varepsilon \rangle^{-1/3} \langle \varepsilon_\theta \rangle k^{-5/3} e^{-\gamma(k/k_d)}
\]

and

\[
E^\theta(k) \propto \langle \varepsilon_\theta \rangle \frac{1}{s} k^{-\delta} e^{-\delta(k/k_d)}
\]

for the local and non-local regimes respectively (\( \gamma \) and \( \delta \) are dimensionless constants, and \( \langle \varepsilon_\theta \rangle \) is the average value of dissipation rate of scalar variance).

Figure 6 shows three-dimensional passive scalar spectrum from a DNS performed in [30] for Péclet number \( P_\lambda = 427 \) (the Schmidt number is unity, i.e. \( P_\lambda = R_\lambda \)). The solid curves in this figure correspond to the best fit by equation (20) (nonlocal interactions). In the upper part of this figure we show the spectrum in the compensate (according to the Corrsin-Obukhov scaling [29]) form. One can clear see the hump corresponding to the bottleneck effect. The arrows show position of the \( \eta_s \) scale. Comparing with Figs. 2,3 one can see that for the passive scalar \( \eta_s/\eta \) takes the same universal value as for the velocity field (the large and small scales covariance (15)).
FIG. 6: The passive scalar DNS data (circles) for homogeneous isotropic turbulence described in [30], Péclet number $P_\lambda = 427$ and the Schmidt number is unity (i.e. $P_\lambda = R_\lambda$). The solid curves in this figure correspond to the best fit by equation (20) (nonlocal interactions).

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