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On quadratic stochastic processes and related differential equations

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Abstract. It is known that the theory of Markov process is a rapidly developing field with numerous applications to many branches of mathematics and physics, biology and so on. But there are some physical models which cannot be described by such processes. One of such models is related to population genetics. These processes are called quadratic stochastic processes (q.s.p.). In this theory it is important to construct nontrivial examples of such processes. In the present paper we are going to provide a construction of q.s.p. by means of two given processes. We should stress that such a construction allows us to produce lots of nontrivial examples of q.s.p. We also associate to given q.s.p. two kind of processes. Note that one of such processes is Markov. It is proved that such kind of processes uniquely define q.s.p. Moreover, we also derive some differential equations for q.s.p.

1. Introduction

It is known that Markov processes are well-developed field of mathematics which have various applications in physics, biology and so on. But there are some physical models which cannot be described by such processes. One of such models is a model related to population genetics. Namely, this model is described by quadratic stochastic processes (see [7, 1] for review). Let us recall the definition of the said process.

Let $E = \{1, 2, \ldots, n\}$ be a finite set. Denote

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0, \sum_{k=1}^{n} x_k = 1\}.$$

Consider a family of functions \{$P^{[s,t]}_{ij,k} : i, j, k \in E, s, t \in \mathbb{R}_+, t - s \geq 1$ with an initial state $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)}) \in S^{n-1}$. This family is said to be quadratic stochastic process (q.s.p.) if for fixed $s, t \in \mathbb{R}_+$, it satisfies the following conditions:

\begin{enumerate}
    \item $P^{[s,t]}_{ij,k} = P^{[s,t]}_{ji,k}$ for any $i, j, k \in E$;
    \item $P^{[s,t]}_{ij,k} \geq 0$ and $\sum_{k=1}^{n} P^{[s,t]}_{ij,k} = 1$ for any $i, j, k \in E$;
    \item An analogue of Kolmogorov-Chapman equation: there are two variants: for the initial point $x^{(0)}$ and $s < r < t$ such that $t - r \geq 1, r - s \geq 1$ one has
\end{enumerate}
where \( x_k^{(r)} \) is defined as follows:

\[
x_k^{(r)} = \sum_{i,j=1}^{n} P_{ij,k}^{(s,r)} x_i^{(s)} x_j^{(r)}.
\]

In what follows by \((E, P_{ij,k}^{[s,t]}, x^{(0)})\) we denote the definite process. We say that the q.s.p is of type (A) or (B) if it satisfies the fundamental equations either (iii_A) or (iii_B), respectively.

In this definition the functions \( P_{ij,k}^{[s,t]} \) denote the probability that under the interaction of the elements \( i \) and \( j \) at time \( s \) the element \( k \) comes into effect at time \( t \). Since for physical, chemical and biological phenomena, a certain time is necessary for the realization of an interaction, we shall take the greatest such time to be equal to 1 (see the Boltzmann model [4] or the biological model [7]). Thus the probability \( P_{ij,k}^{[s,t]} \) is defined for \( t - s \geq 1 \).

A set of quadratic stochastic processes one can decompose into following three classes:

(i) **homogeneous**, that is a transition functions \( P_{ij,k}^{[s,t]} \) depends only on \( t - s \) for arbitrary \( x, y \in E, s,t \in \mathbb{R}^+ \) such that \( t - s \geq 1 \);

(ii) **homogeneous per unit time**, that is transition functions satisfy \( P_{ij,k}^{[t,t+1]} = P_{ij,k}^{[0,1]} \) for all \( t \geq 1 \) and \( i,j,k \in E \), but don’t belong to the first class;

(iii) **nonhomogeneous**, if the process doesn’t belong to the second class.

It should be noted that the quadratic stochastic processes are related to quadratic transformations (see [4, 6]) in the same way as Markov processes are related to linear transformations. The equations (iii_A) and (iii_B) can be interpreted as different laws of behavior of the off spring. Some examples of q.s.p. were given in [2],[3]. We note that quadratic processes of type (A) were considered in [2],[9]. In this theory it is important to construct nontrivial examples of such processes. In the present paper we are going to provide a construction of q.s.p. by means of two given processes. We should stress that such a construction allows us to produce lots of nontrivial examples of q.s.o. We also associate to given q.s.p. two kind of processes. Note that one of such processes is Markov. It is proved that such kind of processes uniquely define q.s.p. Moreover, we also derive some differential equations for q.s.p. Note that similar kind of equations were obtained in [9] for homogeneous per unit time q.s.o.

2. Marginal Markov processes related to q.s.p.

In this section we are going to prove that every q.s.p. can be defined uniquely by two kind of Markov processes.

Recall that a matrix \((Q_{ij}), i,j \in E\) is called **stochastic** if

\[
Q_{ij} \geq 0, \quad \sum_{j=1}^{n} Q_{ij} = 1.
\]

A family of stochastic matrices \(\{(Q_i^{[s,t]}), s,t \in \mathbb{R}^+, t - s \geq 1\}\) is called **Markov process** if the following condition holds: for every \( s < r < t \) one has

\[
Q_{ij}^{[s,t]} = \sum_{k=1}^{n} Q_{ik}^{[s,r]} Q_{kj}^{[r,t]}.
\]
This equation is known as the Kolmogorov-Chapman equation. Let $(E, P^{[s,t]}_{i,j,k}, x^{(0)})$ be a q.s.p. Let us define

$$H^{[s,t]}_{ij} = \sum_{\ell=1}^{n} P^{[s,t]}_{il,j} x^{(s)}_{l}, \quad (2)$$

$$Q^{[s,t]}_{(ij)(uv)} = P^{[s,t]}_{ij,u} x^{(t)}_{v}, \quad (3)$$

where $i, j, u, v \in E$.

It is clear that $H^{[s,t]}_{ij}$ and $Q^{[s,t]}_{(ij)(uv)}$ are stochastic matrices.

**Theorem 2.1** Let $(E, P^{[s,t]}_{i,j,k}, x^{(0)})$ be a q.s.p. type (A). Then defined processes $(H^{[s,t]}_{ij})_{i,j \in E}$ and $(Q^{[s,t]}_{(ij)(uv)})_{(ij),(uv) \in E \times E}$ are Markov processes.

Proof. We have

$$\sum_{k=1}^{n} H^{[s,r]}_{ik} H^{[r,t]}_{kj} = \sum_{k=1}^{n} \left( \sum_{l=1}^{n} P^{[s,r]}_{il,k} x^{(s)}_{l} \right) \left( \sum_{m=1}^{n} P^{[r,t]}_{km,j} x^{(r)}_{m} \right)$$

$$= \sum_{l=1}^{n} \left( \sum_{k,m=1}^{n} P^{[s,r]}_{il,k} P^{[r,t]}_{km,j} x^{(s)}_{l} \right) x^{(r)}_{l}$$

$$= \sum_{l=1}^{n} \left( \sum_{k=1}^{n} P^{[s,r]}_{il,k} x^{(s)}_{l} \right) x^{(r)}_{l}$$

$$= \sum_{l=1}^{n} P^{[s,t]}_{il,j} x^{(s)}_{l} = H^{[s,t]}_{ij},$$

hence $(H^{[s,t]}_{ij})_{i,j \in E}$ is a Markov process.

Similarly, one gets

$$Q^{[s,t]}_{(ij)(uv)} = \left( \sum_{m,l=1}^{n} P^{[s,r]}_{ij,m} P^{[r,t]}_{ml,u} x^{(t)}_{l} P^{[r,t]}_{ml,v} \right) x^{(v)}_{l}$$

$$= \sum_{m,l=1}^{n} P^{[s,r]}_{ij,m} x^{(t)}_{l} P^{[r,t]}_{ml,u} x^{(v)}_{l}$$

$$= \sum_{m,l=1}^{n} Q^{[s,r]}_{(ij)(ml)} Q^{[r,t]}_{(ml)(uv)}$$

So, $Q^{[s,t]}_{(ij)(uv)}$ is also a Markov processes. This completes the proof.

**Theorem 2.2** Let $(E, P^{[s,t]}_{i,j,k}, x^{(0)})$ be a q.s.p type (B). Then the process $(H^{[s,t]}_{ij})$ defined by (2) is Markov. Moreover, the process $Q^{[s,t]}_{(ij)(uv)}$ defined by (3) satisfies the following equation

$$Q^{[s,t]}_{(ij)(uv)} = \sum_{i,j,k=1}^{n} H^{[s,r]}_{il} H^{[s,r]}_{jk} Q^{[r,t]}_{(ik)(uv)}$$
Proof. First we want to show that

\[ x_k(t) = \sum_{i,j=1}^{n} P_{i,j,k}^{[s,t]} x_i^{(s)} x_j^{(s)}. \]

Indeed, from

\[ x_k^{(s)} = \sum_{i,j=1}^{n} P_{i,j,k}^{[0,t]} x_i^{(0)} x_j^{(0)} \]

and

\[ P_{i,j,k}^{[0,t]} = \sum_{m,l,g,u=1}^{n} P_{i,m,l}^{[0,s]} P_{j,g,u}^{[0,s]} P_{l,u,k}^{[0,m]} x_m^{(0)} x_g^{(0)}, \]

one finds

\[ x_k^{(t)} = \sum_{i,j=1}^{n} \left( \sum_{m,l,g,u=1}^{n} P_{i,m,l}^{[0,s]} P_{j,g,u}^{[0,s]} P_{l,u,k}^{[0,m]} x_m^{(0)} x_g^{(0)} \right) x_i^{(0)} x_j^{(0)} \]

\[ = \sum_{l,u=1}^{n} P_{l,u,k}^{[s,t]} \left( \sum_{i,m=1}^{n} P_{i,m,l}^{[0,s]} x_i^{(s)} \right) \left( \sum_{j,g=1}^{n} P_{j,g,u}^{[0,s]} x_j^{(s)} \right) P_{l,u,k}^{[t]} \]

\[ = \sum_{u,v,j=1}^{n} H_{i,u}^{[s,r]} P_{i,u,v}^{[t]} x_v^{(v)} \]

Now using the last equality, we obtain

\[ H_{ij}^{[s,t]} = \sum_{l=1}^{n} P_{i,j,l}^{[s,t]} x_l^{(s)} \]

\[ = \sum_{l=1}^{n} \left( \sum_{m,u,g,v=1}^{n} P_{i,m,u}^{[s,r]} P_{j,g,u}^{[s,r]} P_{l,u,k}^{[0,m]} x_m^{(s)} x_g^{(s)} \right) x_l^{(s)} \]

\[ = \sum_{u,v,j=1}^{n} H_{i,u}^{[s,r]} P_{i,u,v}^{[t]} x_v^{(v)} \]

This shows \( H_{ik}^{[s,t]} \) is a Markov process.

From (iii\( \beta \)) we immediately find

\[ Q_{(ij)(uv)}^{[s,t]} = P_{i,j,a}^{[s,t]} a_{ij}^{(t)} \]

\[ = \left( \sum_{m,l,g,k=1}^{n} P_{i,m,l}^{[s,r]} P_{j,g,k}^{[s,r]} P_{l,k,u}^{[0,m]} x_m^{(s)} x_g^{(s)} \right) a_{ij}^{(t)} \]

\[ = \sum_{l,k=1}^{n} \left( \sum_{m=1}^{n} P_{i,m,l}^{[s,r]} x_m^{(s)} \right) \left( \sum_{g=1}^{n} P_{j,g,k}^{[s,r]} x_g^{(s)} \right) P_{l,k,u}^{[t]} a_{ij}^{(t)} \]

\[ = \sum_{l,k=1}^{n} H_{il}^{[s,r]} H_{jk}^{[s,r]} Q_{(lk)(uv)}^{[r,t]}, \]
Moreover, one has

Then the following assertions hold true:

Let

Theorem 2.3 Let \( E = \{1, 2, \ldots, n\} \) be a finite set, and we are given two non-homogeneous processes \((Q_{ij}^{[s,t]}, (H_{ij}^{[s,t]}), (J_{ij}^{[s,t]}))\) \( i,j,u,v \in E \). Under what conditions these two processes uniquely determine some q.s.p? To answer this question, we first fix an initial state \( x(0) = (x_i(0))_{i=1}^n \in S^{n-1} \). Let us define

\[
  z_k^{(t)} = \sum_i Q_{ik}^{[0,t]} x_i^{(0)}, \quad y_k^{(t)} = \sum_{i,j,l=1}^n Q_{ij}^{[0,t]} x_i^{(0)} x_j^{(0)}, \quad k \in E.
\]

**Theorem 2.3** Let \( E = \{1, 2, \ldots, n\} \) and \((Q_{ij}^{[s,t]}, (H_{ij}^{[s,t]}), (J_{ij}^{[s,t]}))\) be two stochastic processes on \( E \times E \) and \( E \), respectively. Assume that

(a) \( Q_{ij}^{[s,t]} = Q_{ij}^{[s,t]} \) for any \( i, j, u, v \in E \);

(b) \( \sum_{j=1}^n Q_{ij}^{[s,t]} y_j^{(s)} = J_{ij}^{[0,t]} z_v^{(t)} \) for any \( i, u, v \in E \);

(c) \( Q_{ij}^{[s,t]} = \sum_{l=1}^n Q_{ij}^{[s,t]} y_l^{(t)} \) for any \( i, j, u, v \in E \).

Let

\[
P_{ij,k}^{[s,t]} = \sum_{l=1}^n Q_{ij}^{[s,t]} (kl), \quad (4)
\]

Then the following assertions hold true:

(i) \( z_k^{(t)} = y_k^{(t)} \) for any \( k \in E \);

(ii) If \((Q_{ij}^{[s,t]}), (J_{ij}^{[s,t]})\) are Markov processes, then \((E, P_{ij,k}^{[s,t]}, x(0))\) is a q.s.p. of type (A).

(iii) If \((J_{ij}^{[s,t]})\) is a Markov process and \((Q_{ij}^{[s,t]}))\) satisfies

\[
  Q_{ij}^{[s,t]} = \sum_{m,l=1}^n \delta_{im} \delta_{jl} Q_{(ml)}^{[s,t]} \quad m,l \in \delta_{im} \delta_{jl} Q_{(ml)}^{[s,t]} \quad (5)
\]

then \((E, P_{ij,k}^{[s,t]}, x(0))\) is a q.s.p. of type (B).

Moreover, one has \( x_k^{(t)} = z_k^{(t)} \) (\( k \in E \)) and

\[
  J_{ij,k}^{[s,t]} = \sum_{j=1}^n P_{ij,k}^{[s,t]} z_j^{(s)} \quad (6)
\]
Proof. (i) Let us show that \( z^{(t)}_k = y^{(t)}_k \), \((k \in E)\). Indeed from condition (c) and the stochasticity of \( Q_{(ij)(uv)}^{[s,t]} \), one finds
\[
\sum_{m=1}^{n} Q_{(ij)(mk)}^{[s,t]} = \sum_{m,l=1}^{n} Q_{(ij)(ml)}^{[s,t]} y_k^{(t)} = y_k^{(t)},
\]
for any \( k \in E \). Hence, the last equality with (b) yields
\[
y_k^{(t)} = \sum_{m,j=1}^{n} Q_{(ij)(mk)}^{[s,t]} y_j^{(s)} = \sum_{m=1}^{n} \left( \sum_{j=1}^{n} Q_{(ij)(mk)}^{[s,t]} y_j^{(s)} \right) y_k^{(t)} = \sum_{m=1}^{n} z^{(t)}_{im} z_k^{(t)} = z_k^{(t)},
\]
which is the required assertion.

(ii) From (4) one can see that \( P_{ij,k}^{[s,t]} \) satisfies the conditions (i), (ii) of definition q.s.p. We need to check the equality \((iii_A)\). Denote
\[
x_k^{(t)} = \sum_{i,j=1}^{n} P_{ij}^{[0,t]} x_i^{(0)} x_j^{(0)},
\]
Then we find
\[
x_k^{(t)} = \sum_{i,j=1}^{n} \left( \sum_{l=1}^{n} Q_{(ij)(kl)}^{[s,t]} x_i^{(0)} x_j^{(0)} \right) y_k^{(t)} = y_k^{(t)}
\]
Hence, due to (i) one has
\[
x_k^{(t)} = y_k^{(t)} = z_k^{(t)},
\]
for any \( k \in E \). Let us directly check the fundamental equation \((iii_A)\). Indeed, for \( s, \tau, t \in \mathbb{R}_+ \) with \( \tau - s \geq 1, t - \tau \geq 1 \) and due to Markovianity of \( Q_{(ij)(uv)}^{[s,t]} \), we have
\[
P_{ij,k}^{[s,t]} = \sum_{l=1}^{n} Q_{(ij)(kl)}^{[s,t]} = \sum_{l=1}^{n} \left( \sum_{m,h=1}^{n} Q_{(ij)(mh)}^{[s,\tau]} Q_{(mh)(kl)}^{[\tau,t]} \right) = \sum_{l=1}^{n} \left( \sum_{m,h=1}^{n} Q_{(ij)(mg)}^{[s,\tau]} y_h^{(\tau)} Q_{(mh)(kl)}^{[\tau,t]} \right) = \sum_{m,h=1}^{n} P_{ij,m}^{[s,\tau]} P_{mh,k}^{[\tau,t]} y_h^{(\tau)}.
\]
(iii) From (b) one gets
\[
\sum_{j,l=1}^{n} Q_{(ij)(kl)}^{[s,t]} y_j^{(s)} = y_k^{(t)}
\]
for any \( i, k \in E \). Then using the last equality and (5) we obtain
\[
\sum_{m,l,h,g=1}^{n} P_{im,l}^{[s,\tau]} P_{jg,h}^{[\tau,t]} y_h^{(\tau)} x_l^{(0)} x_m^{(0)} =
\]
which means the q.s.p. satisfies the equation \((iii)\).

Note that from (b) we immediately find (6). This completes the proof.

From the proved Theorems 2.1.2.2 and 2.3 we conclude that any q.s.p. can be uniquely defined by two kind of Markov processes. Such processes are called marginal Markov processes associated with q.s.p.

3. Construction of q.s.p.

In this section we propose a construction of q.s.p. by means of two other such kind of processes. Using the proposed construction we provide certain concrete examples of q.s.p.

**Theorem 3.1** Let \((E, P^{[s,t]}_{ij,k}, \bar{x}^{(0)})\) and \((F, G^{[s,t]}_{ij,k}, \bar{y}^{(0)})\) be two q.s.o. of type (A). Then the process \((E \times F, (P \otimes G)^{[s,t]}_{ij,k}, \bar{x}^{(0)} \times \bar{y}^{(0)})\) defined by

\[
(P \otimes G)^{[s,t]}_{ij,k} = P^{[s,t]}_{i_1 j_1 k_1} G^{[s,t]}_{i_2 j_2 k_2}, \quad \text{where } \bar{i} = (i_1, i_2), \bar{j} = (j_1, j_2), \bar{k} = (k_1, k_2),
\]

is a q.s.o. of type (A).

**Proof.** It is clear that \((P \otimes G)^{[s,t]}_{ij,k} \geq 0\) and

\[
\sum_{\bar{k}=(k_1,k_2)} P^{[s,t]}_{k_1} G^{[s,t]}_{k_2} = \sum_{k_1=1}^{n} P^{[s,t]}_{k_1} \sum_{k_2=1}^{m} G^{[s,t]}_{k_2} = 1.
\]

From

\[
P^{[s,t]}_{i_1 j_1 k_1} G^{[s,t]}_{i_2 j_2 k_2} = P^{[s,t]}_{i_1 j_1 k_1} G^{[s,t]}_{i_2 j_2 k_2},
\]

we immediately get \((P \otimes G)^{[s,t]}_{ij,k} = (P \otimes G)^{[s,t]}_{ij,k}\). Let us denote that \(X^{(0)}_{\bar{i}} = x^{(0)}_{i_1} y^{(0)}_{i_2}\). Then, one can see that \(x^{(0)} \times y^{(0)} = (X^{(0)}_{\bar{i}})\). Let us find \(X^{(s)}_{\bar{i}}\), which is given by

\[
X^{(s)}_{\bar{i}} = \sum_{\bar{j}} (P \otimes G)^{[0,s]}_{\bar{i},\bar{j}} \bar{x}^{(0)}_{\bar{j}}
\]

\[
= \sum_{(i_1 i_2)(j_1 j_2)} G^{[0,s]}_{i_1 j_1 k_1} \times x^{(0)}_{i_1} y^{(0)}_{i_2} \times y^{(0)}_{j_1} y^{(0)}_{j_2}
\]

\[
= \left( \sum_{(i_1 i_2)} G^{[0,s]}_{i_1 j_1 k_1} x^{(0)}_{i_1} \right) \left( \sum_{(j_1 j_2)} G^{[0,s]}_{i_2 j_2 k_2} y^{(0)}_{j_2} \right)
\]

\[
= x^{(s)}_{k_1} y^{(s)}_{k_2}
\]

\[
(7)
\]
Now let us check the equation (iii). A).

Taking into account that the given processes $P^{[s,t]}_{ij,k}$ and $P^{[s,t]}_{ij,k}$ have type (A), then we have

$$\sum_{\tilde{u},\tilde{l}} (P \otimes G)^{[\tau,\tau]}_{ij,\tilde{u},\tilde{l}} (P \otimes G)^{[\tau,\tau]}_{ij,\tilde{u},\tilde{l}} X^{(s)}_{l} = \sum_{(u_1 u_2) (l_1 \bar{l})} P^{[s,t]}_{ij,11u1} G^{[s,t]}_{ij,22u2} P^{[\tau,\tau]}_{u_1,11,k1} G^{[\tau,\tau]}_{u_2,22,k2} x^{(s)}_{l_1} y^{(s)}_{l_2}$$

$$= \left( \sum_{(u_1 l_1)} P^{[s,t]}_{ij,11u1} P^{[\tau,\tau]}_{u_1,11,k1} x^{(s)}_{l_1} \right) \left( \sum_{(u_2 l_2)} G^{[\tau,\tau]}_{u_2,22,k2} y^{(s)}_{l_2} \right)$$

$$= P^{[s,t]}_{i1,j1,k1} G^{[s,t]}_{i2,j2,k2}$$

$$= (P \otimes G)^{[s,t]}_{ij,k}.$$  

This completes the proof.

**Theorem 3.2** Let $(E, P^{[s,t]}_{ij,k}, x^{(0)})$ and $(F, G^{[s,t]}_{ij,k}, y^{(0)})$ be two q.s.o. of type (B). Then the process $(E \times F, (P \otimes G)^{[s,t]}_{ij,k}, x^{(0)} \times y^{(0)})$ defined by

$$(P \otimes G)^{[s,t]}_{ij,k} = P^{[s,t]}_{i1,j1,k1} G^{[s,t]}_{i2,j2,k2},$$

where $\tilde{i} = (i_1, i_2), \tilde{j} = (j_1, j_2), \tilde{k} = (k_1, k_2),$  

$$x^{(0)} \times y^{(0)} = (x^{(0)}_{i} y^{(0)}_{j})_{i \in E, j \in F}$$

is a q.s.o. of type (B).

**Proof.** To prove the theorem it is enough to check the equation ((iii). B)). Noting that $X^{(m)}_{\bar{m}} = x^{(m)} y^{(m)}_{m_2},$ where $\bar{m} = (m_1, m_2),$ we get

$$\sum_{\bar{m}, \bar{l}} (P \otimes G)^{[s,t]}_{\bar{m},\bar{l}} (P \otimes G)^{[s,t]}_{\bar{m},\bar{l}} X^{(s)}_{\bar{l}} Y^{(s)}_{\bar{l}}$$

$$= \sum_{(m_1 m_2) (l_1 l_2) (u_1 u_2)} P^{[s,t]}_{11,m_1,l_1} G^{[s,t]}_{j2,m_2,l_2} P^{[\tau,\tau]}_{11,m_1,l_1} G^{[\tau,\tau]}_{j2,m_2,l_2}$$

$$= \left( \sum_{(m_1 l_1 l_1) (u_1) (u_1) (u_2) (u_2)} P^{[s,t]}_{11,m_1,l_1} P^{[\tau,\tau]}_{11,m_1,l_1} x^{(s)}_{l_1} y^{(s)}_{l_2} \right) \left( \sum_{(m_2 l_2 l_2) (u_2) (u_2)} G^{[\tau,\tau]}_{j2,m_2,l_2} y^{(s)}_{l_2} \right)$$

$$= P^{[s,t]}_{i1,j1,k1} G^{[s,t]}_{i2,j2,k2}$$

$$= (P \otimes G)^{[s,t]}_{ij,k}.$$  

This is the desired assertion.

From these theorems and Theorem 2.3 we immediately get the following

**Theorem 3.3** Let $(E, P^{[s,t]}_{ij,k}, x^{(0)})$ and $(F, G^{[s,t]}_{ij,k}, y^{(0)})$ be two q.s.o. with marginal processes $(Q^{[s,t]}_{P_{ij,k}(u_1)}), (H^{[s,t]}_{P_{ij,k}})$ and $(G^{[s,t]}_{G_{ij,k}(u_2)}), (H^{[s,t]}_{G_{ij,k}}),$ respectively. Then the marginal processes of $(E \times F, (P \otimes G)^{[s,t]}_{ij,k}, x^{(0)} \times y^{(0)})$ are given by

$$Q^{[s,t]}_{(i,j), (\bar{u}, \bar{v})} = Q^{[s,t]}_{P_{(i,j)k}(u_1 v_1)} Q^{[s,t]}_{G_{ij,k}(u_2 v_2)},$$

$$H^{[s,t]}_{ij} = H^{[s,t]}_{P_{i1,j1}} H^{[s,t]}_{G_{i2,j2}}.$$
where \( \vec{i} = (i_1, i_2), \vec{j} = (j_1, j_2), \vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2). \)

### 3.1. Example

In this paragraph, using the proved theorem, we are going to construct new examples of q.s.p. Let us consider the following process:

Let \( E = 1, 2 \) and \((x, 1 - x)\) be an initial distribution on \( E\), where \( x \in [0, 1]\). Consider the following system of transition probabilities:

\[
P^{[s,t]}_{11,1} = \frac{(1 - 2\epsilon)^{t-s}}{2^{t-s-1}}[(2^{t-s-1} - 1)(1 - 2\epsilon)s x + 1];
\]

\[
P^{[s,t]}_{12,1} = P^{[s,t]}_{21,1} = \frac{(1 - 2\epsilon)^{t-s}}{2^{t-s-1}}[(2^{t-s-1} - 1)(1 - 2\epsilon)x + \frac{1}{2}];
\]

\[
P^{[s,t]}_{22,1} = \frac{(1 - 2\epsilon)^{t-s}}{2^{t-s-1}}(2^{t-s-1} - 1)x;
\]

\[
P^{[s,t]}_{ij,2} = 1 - P^{[s,t]}_{ij,1}, i, j = 1, 2.
\]

It is known [9] that for \( \epsilon \in [0, \frac{1}{2}] \) these transition probabilities generate a q.s.o. which is of type (A) and type (B) simultaneously. In this case we have

\[
x_1(t) = (1 - 2\epsilon)^t x
\]

\[
x_2(t) = 1 - (1 - 2\epsilon)^t x
\]

For \( \epsilon = 0 \) this q.s.p is homogeneous; for \( \epsilon \neq 0 \) it is from class (II). Now, we multiply the same process to itself according to the theorem.

Let

\[
F = E \times E = (1, 1), (1, 2), (2, 1), (2, 2)
\]

and

\[
X^0 = (xy, x(1 - y), (1 - x)y, (1 - x)(1 - y))
\]

where \( x, y \in [0, 1]\) and define

\[
\mathcal{P}^{[s,t]}_{i_1 i_2 j_1 j_2} = \mathcal{P}^{[s,t]}_{i_1 j_1 k_1} \mathcal{P}^{[s,t]}_{i_2 j_2 k_2}
\]

According to the theorem, such a process is q.s.p type of A and B. Let us describe the process more precisely,

\[
\mathcal{P}_{11,1,1} = \frac{1}{2^{2(t-s-1)}}[x^2(2^{t-s-1} - 1)^2 + 2x(2^{t-s-1} - 1) + 1]
\]

\[
\mathcal{P}_{11,1,2} = \frac{1}{2^{2(t-s-1)}}[-x^2(2^{t-s-1} - 1)^2 + x(2^{t-s-1} - 1)(2^{t-s-1} - 2) + 2^{t-s-1} - 1]
\]

\[
\mathcal{P}_{11,2,2} = \frac{1}{2^{2(t-s-1)}}[x^2(2^{t-s-1} - 1)^2 - 2x(2^{t-s-1} - 1)^2 + 2x^2 t - s - 1 - 2^{t-s-1} + 1]
\]

\[
\mathcal{P}_{11,2,1} = \frac{1}{2^{2(t-s-1)}}[x^2(2^{t-s-1} - 1)^2 + 3x^2(2^{t-s-1} - 1) + \frac{1}{2}]
\]

\[
\mathcal{P}_{11,2,1} = \frac{1}{2^{2(t-s-1)}}[x^2(2^{t-s-1} - 1)^2 - x(2^{t-s-1} - 3/2)^2 - 3/2 + 2^{t-s-1} + 1/2]
\]

\[
\mathcal{P}_{11,2,1} = \frac{1}{2^{2(t-s-1)}}[x^2(2^{t-s-1} - 1)^2 + x(2^{t-s-1} - 3/2)^2 - 3/2 + 1/4 + 1/2^{t-s} - 1/2]
\]
\[ \mathcal{P}_{11,12,22} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{t-s-1} + 1)^2 - x(2(2^{2(t-s-1)}) - \frac{7}{2}2^{t-s-1} - \frac{3}{2} \right] \\
+ (2^{2(t-s-1)} + 2^{t-s-1} + \frac{1}{2})] \\
\mathcal{P}_{11,12,11} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{t-s-1} - 1)^2 + x(2^{t-s-1} - 1) + \frac{1}{4} \right] \\
\mathcal{P}_{11,22,12} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(1 - 2^{t-s-1})(1 + 2^{t-s-1}) + x(2(2^{t-s-1}) - 2^{t-s-1} - 1) + \frac{1}{4}2^{t-s} + \frac{1}{4} \right] \\
\mathcal{P}_{11,22,22} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{t-s-1} + 1)^2 - x(2(2^{t-s-1}) + 1)(2^{t-s-1} + 1) + (2^{t-s-1} + \frac{1}{2})^2 \right] \\
\mathcal{P}_{22,12,11} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{t-s-1} - 1)^2 + \frac{1}{4}2^{t-s}x - \frac{1}{2}x \right] \\
\mathcal{P}_{22,22,12} = \frac{1}{2^{2(t-s-1)}} \left[ -x^2(2^{t-s-1} - 2^{t-s-1} - 1) + x(2(2^{t-s-1}) - \frac{1}{2}2^{t-s-1} + \frac{1}{2}) + \frac{1}{4}2^{t-s} \right] \\
\mathcal{P}_{22,22,21} = \frac{1}{2^{2(t-s-1)}} \left[ -x^2(2^{t-s-1} - 1)^2 + x((2^{t-s-1} - 1)(2(2^{t-s-1} - 1)) \right] \\
\mathcal{P}_{22,22,22} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{t-s-1} - 1)^2 - x(4(2^{t-s-1}))(2^{t-s-1} - 1) + 2^{t-s-1}(2^{t-s-1} - \frac{1}{2}) \right] \\
\mathcal{P}_{12,12,11} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{2(t-s-1)} - 2^{t-s-1} + 1) - x \right] \\
\mathcal{P}_{12,12,12} = \frac{1}{2^{2(t-s-1)}} \left[ -x^2(2^{2(t-s-1)} - 2^{t-s-1} + 1) + x(2(2^{t-s-1}) - 2^{t-s-1} + 1) + 2^{t-s-1} \right] \\
\mathcal{P}_{12,12,21} = \frac{1}{2^{2(t-s-1)}} \left[ -x^2(2^{t-s-1} - 1)^2 - x(1 - 2^{t-s-1})(1 + 2^{t-s-1}) \right] \\
\mathcal{P}_{12,12,22} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{t-s-1} - 1)^2 - x(2(2^{t-s-1} - 1)(2^{t-s-1} - 1) + 2^{t-s-1}(2^{t-s-1} - 1)) \right] \\
\mathcal{P}_{12,22,11} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{t-s-1} - 1)^2 + \frac{1}{2}x(2^{t-s-1} - 1) \right] \\
\mathcal{P}_{12,22,12} = \frac{1}{2^{2(t-s-1)}} \left[ -x^2(2^{t-s-1} - 1)^2 + x(2(2^{t-s-1}) - \frac{1}{2}2^{t-s} + \frac{1}{2}) + \frac{1}{4}2^{t-s} \right] \\
\mathcal{P}_{12,22,21} = \frac{1}{2^{2(t-s-1)}} \left[ -x^2(2^{t-s-1} - 1)^2 + x(4(2^{t-s-1} - 2)(2^{t-s-1} - 1)) \right] \\
\mathcal{P}_{12,22,22} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{t-s-1} - 1)^2 - x(4(2^{t-s-1} - 1)(2^{t-s-1} - 1) + 2^{t-s-1}(2^{t-s-1} - \frac{1}{2}) \right] \\
\mathcal{P}_{22,22,11} = \frac{1}{2^{2(t-s-1)}} \left[ x^2 - \frac{x^2}{2^{t-s-1}} \right] \\
\mathcal{P}_{22,22,12} = \frac{x}{2^{t-s-1}} \left[ 2^{t-s-1} - 1 - x + \frac{x}{2^{t-s-1}} \right] \\
\mathcal{P}_{22,22,22} = \frac{1}{2^{2(t-s-1)}} \left[ x^2(2^{t-s-1} + 1)^2 - 2(2^{t-s-1} - 1)(2^{t-s-1} + 1) + 2^{2(t-s-1)} \right]
4. Analytical theory for quadratic stochastic processes

In this section we are going to derive differential equations related to q.s.p. In what follows, we always assume that the functions $P_{ij,k}$ are continuous with respect to variables $s$ and $t$ as well they are differentiable with respect to $s$ and $t$ with $t > s + 1$. We will consider two separate subcases with respect to types of q.s.p.

4.1. Differential equations for q.s.p. type A

In this subsection we consider quadratic stochastic processes of type (A). Assume that $(E, P_{ij,k}^{[s,t]}, x^{(0)})$ be a q.s.p. of type (A). Then using (iii) for $t > s + 1$ one gets

$$
\frac{P_{ij,k}^{[s,t+h]} - P_{ij,k}^{[s,t]}}{h} = \sum_{m,l=1}^{n} P_{ij,m}^{[s,t-1]} P_{ml,k}^{[t-1,t+h]} x_{l}^{(t-1)} - \sum_{m,l=1}^{n} P_{ij,m}^{[s,t-1]} P_{ml,k}^{[t-1,t]} x_{l}^{(t-1)}
$$

$$
= \sum_{m,l=1}^{m} P_{ij,m}^{[s,t-1]} \left( P_{ml,k}^{[t-1,t+h]} - P_{ml,k}^{[t-1,t]} \right) x_{l}^{(t-1)}.
$$

Now assuming

$$
\alpha_{ml,k}(t) = \lim_{h \to 0+} \frac{P_{ml,k}^{[t-1,t+h]} - P_{ml,k}^{[t-1,t]}}{h},
$$

provided that limit exists, and passing the limit for $h \to 0$, we have first system of differential equations

$$
\frac{\partial P_{ij,k}^{[s,t]}}{\partial t} = \sum_{m,l=1}^{m} \alpha_{ml,k}(t) x_{l}^{(t-1)} P_{ij,m}^{[s,t-1]}, \quad i, j, k = 1, \ldots, n
$$

Due to

$$
x_{k}^{(t)} = \sum_{i,j=1}^{n} P_{ij,k}^{[0,t]} x_{i}^{(0)} x_{j}^{(0)}
$$

we rewrite the equation (10) as follows

$$
\frac{\partial P_{ij,k}^{[s,t]}}{\partial t} = \sum_{m,l,r,q=1}^{n} \alpha_{ml,k}(t) x_{r}^{(0)} x_{q}^{(0)} P_{rq,l}^{[0,t-1]} P_{ij,m}^{[s,t-1]}
$$

Similarly for $t > s + 2$ we have

$$
P_{ij,k}^{[s,t]} - P_{ij,k}^{[s,t+h]} = \sum_{m,l} P_{ij,m}^{[s,s+1+h]} P_{ml,k}^{[s+1,t+h]} x_{l}^{(s+1+h)}

- \sum_{m,l} P_{ij,m}^{[s+h,s+1+h]} P_{ml,k}^{[s+1,t+h]} x_{l}^{(s+1+h)}

= \sum_{m,l} \left( P_{ij,m}^{[s,s+1+h]} - P_{ij,m}^{[s+1+h]} \right) x_{l}^{(s+1+h)} P_{ml,k}^{[s+1,t+h]}.
$$

Dividing both sides of this equality to $h$ and passing to limit for $h \to 0$, one gets the second system of differential equations

$$
\frac{\partial P_{ij,k}^{[s,t]}}{\partial s} = - \sum_{m,l=1}^{n} \alpha_{ij,m}(s+1) x_{l}^{(s+1)} P_{ml,k}^{[s,t+1]}, \quad i, j, k = 1, \ldots, n
$$
Again due to equation (11), we can rewrite (14) as follows

\[
\frac{\partial P_{ij,k}^{[s,t]}}{\partial s} = - \sum_{m,l,r,q=1}^{n} \alpha_{ij,m}(s + 1) x_r^{(0)} P_{r,q,l}^{[0,s+1]} P_{ml,k}^{[s+1,t]}
\]  

(15)

4.2. Differential equations for q.s.p. type B

Assume that \((E, P_{ij,k}^{[s,t]}, x^{(0)})\) be a q.s.p. of type (B). Let us denote

\[
\tilde{\alpha}_{ml,k}(t) = \lim_{h \to 0+} \frac{P_{ij,k}^{[t-1,t+h]} - P_{ij,k}^{[t-1,t]}}{h},
\]

(16)

and assume this limit exists.

For \(t > s + 2\), due to (iii) we have

\[
P_{ij,k}^{[s,t+h]} - P_{ij,k}^{[s,t]} = \sum_{m,l,r,q} P_{im,l}^{[s,t-1]} P_{jr,q}^{[s,t-1]} P_{lq,k}^{[t-1,t+h]} x_m^{(s)} x_r^{(s)} - \sum_{m,l,r,q} P_{im,l}^{[s,t-1]} P_{jr,q}^{[s,t-1]} x_m^{(s)} x_r^{(s)}
\]

\[
= \sum_{m,l,r,q} P_{im,l}^{[s,t-1]} x_m^{(s)} x_r^{(s)} P_{jr,q}^{[s,t-1]} P_{lq,k}^{[t-1,t+h]} - P_{lq,k}^{[t-1,t]} x_m^{(s)} x_r^{(s)}
\]

(17)

Now, dividing both sides the last equality by \(h\) and passing the limit \(h \to 0\), we derive the following system of differential equations.

\[
\frac{\partial P_{ij,k}^{[s,t]}}{\partial t} = \sum_{m,l,r,q} \tilde{\alpha}_{lq,k}(t) x_m^{(s)} x_r^{(s)} P_{im,l}^{[s,t-1]} P_{jr,q}^{[s,t-1]}
\]

\[\times\]

(18)

Now let us consider the following expressions

\[
P_{ij,k}^{[s,t]} - P_{ij,k}^{[s+h,t]} = \sum_{m,l,r,q} \left( P_{im,l}^{[s,s+1+h]} P_{jr,q}^{[s,s+1+h]} x_m^{(s)} x_r^{(s)} - P_{im,l}^{[s+h,s+1+h]} P_{jr,q}^{[s+h,s+1+h]} x_m^{(s)} x_r^{(s)} \right)
\]

\[
= \sum_{m,l,r,q} \left( P_{im,l}^{[s,s+1+h]} x_m^{(s)} x_r^{(s)} - P_{im,l}^{[s+h,s+1+h]} x_m^{(s)} x_r^{(s)} \right)
\]

\[
\times p_{jr,q}^{[s,s+1+h]} x_m^{(s)} x_r^{(s)} + P_{im,l}^{[s+h,s+1+h]} (s) x_m^{(s)} x_r^{(s)} P_{jr,q}^{[s+h,s+1+h]} x_m^{(s)} x_r^{(s)} - P_{lq,r}^{[s+1+h,t]}
\]

(19)
Hence, we get the second system of differential equations

\[
I = P_{jr,q}^{[s,s+1+h]}(s) x_m^{(s)} - P_{jr,q}^{[s,h,s+1+h]}(s+h) x_r^{(s+h)} = P_{jr,q}^{[s,s+1+h]}(s) - P_{jr,q}^{[s,s+1+h]}(s+h) + \sum_{m,l,r,q} P_{im,l}^{[s,s+1+h]}(s)\left(P_{jr,q}^{[s,s+1+h]}(s) x_m^{(s)} - P_{jr,q}^{[s,h,s+1+h]}(s+h) x_r^{(s+h)}\right) + \sum_{m,l,r,q} P_{im,l}^{[s,s+1+h]}(s)\left(P_{jr,q}^{[s,s+1+h]}(s) - P_{jr,q}^{[s,h,s+1+h]}(s+h)\right) + \sum_{m,l,r,q} P_{im,l}^{[s,s+1+h]}(s)\left(P_{jr,q}^{[s,s+1+h]}(s) - P_{jr,q}^{[s,h,s+1+h]}(s+h)\right)
\]

Similarly, one finds

\[
II = P_{im,l}^{[s,s+1+h]}(s) x_r^{(s)} - P_{im,l}^{[s,h,s+1+h]}(s+h) = P_{im,l}^{[s,s+1+h]}(s) - P_{im,l}^{[s,s+1+h]}(s+h) + \sum_{m,l,r,q} P_{jr,q}^{[s,s+1+h]}(s)\left(P_{im,l}^{[s,s+1+h]}(s) x_r^{(s)} - P_{im,l}^{[s,h,s+1+h]}(s+h) x_r^{(s+h)}\right) + \sum_{m,l,r,q} P_{jr,q}^{[s,s+1+h]}(s)\left(P_{im,l}^{[s,s+1+h]}(s) - P_{im,l}^{[s,h,s+1+h]}(s+h)\right) + \sum_{m,l,r,q} P_{jr,q}^{[s,s+1+h]}(s)\left(P_{im,l}^{[s,s+1+h]}(s) - P_{im,l}^{[s,h,s+1+h]}(s+h)\right)
\]

Let us denote,

\[
\beta_{ml,q}(s) = \lim_{\Delta \to 0^+} \frac{P_{ml,q}^{[s,s+1+\Delta]} - P_{ml,q}^{[s+h,s+1+\Delta]}}{\Delta},
\]

Then dividing (19) by \( h \) and passing to the limit (keeping in mind (20) and (21)) we obtain

\[
\frac{\partial P_{ij,k}^{[s,t]}}{\partial t} = \sum_{m,l,r,q} P_{im,l}^{[s,s+1]}(s)\left(P_{jr,q}^{[s,s+1]}(s) x_m^{(s)}\beta_{jr,q}(s) - P_{jr,q}^{[s+h]}(s+h) x_r^{(s+h)}\right) + \sum_{m,l,r,q} P_{jr,q}^{[s,s+1]}(s)\left(P_{im,l}^{[s,s+1]}(s) - P_{im,l}^{[s+h,s+1+h]}(s+h)\right) + \sum_{m,l,r,q} P_{jr,q}^{[s,s+1]}(s)\left(P_{im,l}^{[s,s+1]}(s) - P_{im,l}^{[s+h,s+1+h]}(s+h)\right)
\]

Hence, we get the second system of differential equations

\[
\frac{\partial P_{ij,k}^{[s,t]}}{\partial s} = \sum_{m,l,r,q} \left(-\gamma_{im,l}^{[s,s+1]} x_m^{(s)}\beta_{jr,q}(s) + \gamma_{im,l}^{[s,h]} x_r^{(s+h)}\right) + \sum_{m,l,r,q} \left(P_{jr,q}^{[s,s+1]}(s) - P_{jr,q}^{[s+h,s+1+h]}(s+h)\right) + \sum_{m,l,r,q} \left(P_{jr,q}^{[s,s+1]}(s) - P_{jr,q}^{[s+h,s+1+h]}(s+h)\right)
\]

for \( i, j, k = 1, \ldots, n \).

Note that similar kind of differential equations have been obtained in [9] for homogeneous per unit time q.s.p.
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