Deformations of bihamiltonian structures of hydrodynamic type

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Abstract
In this paper we study the deformations of bihamiltonian PDEs of hydrodynamic type with one dependent variable. The reason we study such deformations is that the deformed systems maintain an infinite number of commuting integrals of motion up to a certain order in the deformation parameter. This fact suggests that these systems could have, at least for small times, multi-solitons solutions. Our numerical experiments confirm this hypothesis.
1 Introduction

The main purpose of this paper is to study the effects of a “deformation” of an infinite dimensional completely integrable system.

The first point is to define what a deformation is. In this paper we deal with bihamiltonian systems of hydrodynamic type for which it is quite natural to define deformations in terms of the Jacobi identity: the deformed bihamiltonian structure satisfies the Jacobi identity only up to a certain order in the deformation parameter.

The interesting deformations are the deformations that cannot be obtained from the original bihamiltonian structure by just a change of coordinates. Therefore the first problem is to select the non trivial deformations.

In order to solve this problem it is convenient formulate it in terms of Poisson cohomology. Recently Degiovanni, Magri and Sciacca (see [4]) proved that the first two Poisson cohomology groups of a Poisson manifold \((M, P)\) are trivial when \(M\) is the loop space \(\{S^1 \to \mathbb{R}^n\}\) and \(P\) is a Poisson bracket of hydrodynamic type. Getzler independently (see [11]) proved that all groups \(H^i(P, M)\) for \(i\) positive are trivial for such \((M, P)\).

This result, as we will see, simplifies remarkably our problem and allows us to solve it (in this paper we classify the deformations up to fourth order).

For the second order deformations we show explicitly how to obtain an infinite “hierarchy” of hamiltonian equations. We will see that the corresponding flows commute up to the order of the deformation.

One typical class of solutions of infinite dimensional completely integrable systems is the class of multi-soliton solutions. A natural question arises: do the equations of the deformed hierarchy have multi-solitons solutions (at least for small times)?

The numerical experiments we have performed for an equation of the deformed hierarchy show the existence of solutions analogous to two-solitons solutions.

Finally we observe that the deformations of the bihamiltonian structures of hydrodynamic type appear in the framework of Frobenius manifolds where, with some additional constraints, they play a crucial role in the problem of reconstruction of a 2D TFT from a given Frobenius manifold studied by Dubrovin and Zhang (see [8]). One of these constraints, called quasi-triviality, is analysed in the last part of this paper.

The paper is organized as follows.

The first part (section 2) is a brief introduction to Poisson cohomology. We focus our attention, in particular, on the infinite-dimensional version of Poisson cohomology. To do this we use the formalism of formal calculus of variations (see for example [10]). The main purpose of section 2 is to explain how to get the formulae for the Schouten brackets that will appear in the calculations.

In section 3 we give the classification of deformations up to fourth order in the deformation parameter.

In section 4 we construct the deformed hierarchy and we find one soliton solutions of one equation of the hierarchy.

Section 5 gives the results of the numerical experiments and section 6 gives the proof of classification theorem.
In the last section (section 7) we introduce the notion of quasi-triviality and we prove that all deformations are quasi-trivial.

2 Poisson geometry

2.1 Poisson bracket

Definition 1 Let $M$ be a smooth manifold, $P^{ij}(x)$ a bivector (i.e. a skew-symmetric contravariant tensor field of type $(2,0)$), and $f$ and $g$ two smooth functions. The expression

$$\{f, g\} := P^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

is a Poisson bracket on $M$ if it defines a structure of Lie algebra on the ring of smooth functions on $M$.

Jacoby identity is the only property of a Lie algebra that is not a consequence of the skew-symmetry of $P^{ij}$ and of the definition of Poisson bracket. It is well known that the Jacobi identity holds if and only if the tensor

$$J^{ijk} = \{\{x^i, x^j\}, x^k\} + \text{cyclic} = \partial_i P^{sk} \partial^s P^{ij} + \partial_i P^{jk} \partial^s P^{si} + \partial_i P^{ki} \partial^s P^{sj}$$

is indentically equal to 0. In fact

$$\{\{f, g\}, h\} + \text{cyclic} = J^{ijk} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \frac{\partial h}{\partial x^k}$$

for any $f$, $g$ and $h \in C^\infty(M)$

Definition 2 A vector field $\nabla H$ associated to a smooth function $H$ by the formula

$$(\nabla H)^i = P^{ij} \frac{\partial H}{\partial x^j} = \{x^i, H\}$$

is called Hamiltonian vector field.

Definition 3 A smooth function $f$ is called Casimir if $\{f, .\} = 0$.

2.2 Schouten bracket and Poisson cohomology

Let $\Gamma^i(M)$ be the space of $i$-vectors. It is well-known (see [16]) that there is an unique well defined $\mathbb{R}$-bilinear extension of the Lie-derivative to an operator

$$[., .] : \Gamma^p(M) \times \Gamma^q(M) \to \Gamma^{p+q}(M)$$

such that

$$[X_1 \wedge ... \wedge X_p, Q] = \sum_{i=1}^p (-1)^{i+1} X_1 \wedge ... \wedge \hat{X}_i \wedge ... \wedge X_p \wedge [X_i, Q]$$

(2.5)
where \( X_k \in \Gamma^1(M) \) for \( k = 1, \ldots, p \), \( Q \in \Gamma^q(M) \) and \([X_i, Q] = L_{X_i}Q\).

This bilinear map is called **Schouten bracket**. It has the following properties:

\[
[P, Q] = (-1)^{pq} [Q, P] \tag{2.6}
\]

\[
[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{pq+q} Q \wedge [P, R] \tag{2.7}
\]

\[
(-1)^{p(q-1)} [P, [Q, R]] + (-1)^{(p-1)q} [Q, [R, P]] + (-1)^{(q-1)r} [R, [P, Q]] = 0 \tag{2.8}
\]

where \( P \in \Gamma^p(M) \), \( Q \in \Gamma^q(M) \) and \( R \in \Gamma^r(M) \).

The last property is called graded Jacobi identity.

It can be proved by using (2.5) and (2.6).

In coordinates the Schouten bracket of a \( p \)-vector \( P \) and a \( q \)-vector \( Q \) is given by the formula (see [16])

\[
[P, Q]^{k_1 \ldots k_{p+q-1}} = \frac{(-1)^p}{p!(q-1)!} \delta^{k_1 \ldots k_{p+q-1}}_{i_1 \ldots i_p j_1 \ldots j_q} \frac{\partial P^{i_1 \ldots i_p}}{\partial x^{i_1}} Q^{j_1 \ldots j_q} + \frac{1}{(p-1)!q!} \delta^{k_1 \ldots k_{p+q-1}}_{i_1 \ldots i_p j_1 \ldots j_q} P^{i_1 \ldots i_p} \frac{\partial Q^{j_1 \ldots j_q}}{\partial x^{i_1}},
\]

where \( \delta^{\ldots}_{\ldots} \) is the Kronecker multi-index.

**Example 1** If \( P, Q \in \Gamma^2(M) \) then

\[
[P, Q]^{i j k} = \frac{\partial Q^{i j}}{\partial x^s} P^{s k} + \frac{\partial P^{i j}}{\partial x^s} Q^{s k} + \text{cyclic} \tag{2.9}
\]

When \( Q = P \) we obtain

\[
\frac{1}{2} [P, P] = J^{i j k} \tag{2.10}
\]

and then we have the following well-known

**Theorem 1** The Jacobi identity holds if and only if \([P, P] = 0\).

The last step before introducing the Poisson cohomology is the following

**Theorem 2** Let \( P \) be a Poisson bivector on \( M \), then the operator \( d_P : \Gamma^q(M) \to \Gamma^{q+1}(M) \) defined by the formula

\[
d_P Q := [P, Q] \tag{2.11}
\]

is a cohomology operator, i.e. \( d_P^2 = 0 \)

**Proof**

The graded Jacobi identity and the property (2.6) imply

\[
0 = [[P, Q], P] + [[P, P], Q] + [[Q, P], P] = 2 [[P, Q], P] = -2d_P^2 Q \tag{2.12}
\]

The last theorem allows us, following [13], to introduce the complex

\[
0 \to \Gamma^0(M) \to \Gamma^1(M) \to \Gamma^2(M) \to \Gamma^3(M) \to \ldots
\]

and to define the Poisson cohomology as \( HP^*(M, P) = \ker(d_P)/\text{Im}(d_P) \).
Example 2 $HP^0$
If $f$ is a smooth function then $d_P f = P^{is} \frac{\partial f}{\partial x^s}$, that is $HP^0 = \text{Casimirs}$.

Example 3 $HP^1$
The cocycles are the infinitesimal symmetries ($d_P X = L_X P$), the coboundaries are hamiltonian vector fields. Then $HP^1 = \text{Infinitesimal symmetries/Hamiltonian vector fields}$.

Example 4 $HP^2$
Let $Q$ be a bivector. $Q$ is a cocycle if and only if $[P, Q] = 0$. Therefore $P + \epsilon Q$ satisfies Jacoby identity mod$(O(\epsilon^2))$. This means that the cocycles are infinitesimal deformations of the Poisson bracket. The coboundaries are infinitesimal deformations obtained by a change of coordinates. In fact $Q$ is a coboundary if and only if $Q = L_X P$ where $X$ is a vector field. 

Summarizing:
$HP^2 = \text{Infinitesimal deformations of } P/\text{Deformations obtained by a change of coordinates}$

2.3 Bihamiltonian structure on $M$

Definition 4 A bihamiltonian structure on $M$ is a pair $(P_1, P_2)$ of Poisson bivectors such that for any $\lambda_1, \lambda_2 \in \mathbb{R}$, the linear combination

$$\lambda_1 P_1 + \lambda_2 P_2$$

is again a Poisson bivector

In terms of Schouten bracket the bivectors $(P_1, P_2)$ are a bihamiltonian structure if and only if

$$[P_1, P_1] = [P_2, P_2] = [P_1, P_2] = 0 \quad (2.13)$$

2.4 Poisson brackets on formal loop space

Now we want to extend the previous definitions to the loop space $\mathcal{L} = \{ u : S^1 \to \mathbb{R} \}$. Let $\mathcal{A}$ be the space of differential polynomials in $u$, that is

$$f \in \mathcal{A} \iff f = \sum f_{s_1...s_m} (u) u^{(s_1)} ... u^{(s_m)} \quad (2.14)$$

where $u^{(s_i)} := \frac{d^{s_i} u}{dx^{s_i}}$.

We observe that $f$ is not necessarily a polynomial in $u$.

*The role of the functions on $\mathcal{L}$ is played by the local functionals

$$I = \int_{S^1} f(u(x), u_x, u_{xx}, ... ) dx$$

where $f \in \mathcal{A}$.

\[ \text{In general } f \text{ could depend on } x. \text{ In this paper we will deal only with differential polynomials where } x \text{ doesn’t appear explicitly.} \]
Definition 5 A (non local) multivector $X$ is a formal infinite sum of the type

$$X = X^{s_1,...,s_k}(x_1,...,x_k; u(x_1),...,u(x_k), u_x(x_1),...) \frac{\partial}{\partial u^{s_1}(x_1)} \wedge ... \wedge \frac{\partial}{\partial u^{s_k}(x_k)}$$

where the coefficients satisfy the skew-symmetry condition with respect to simultaneous permutations $s_p, x_p \leftrightarrow s_q, x_q$

The wedge product of a $k$-vector $X$ by a $l$-vector $Y$ is defined as

$$(X \wedge Y)^{s_1,...,s_k+l}(x_1,...,x_{k+l}; u(x_1),...,u(x_{k+l}),...) =$$

$$\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^{sgn \sigma} X^{s_1,...,s_k}(x_{\sigma(1)},...,x_{\sigma(k)},...) Y^{s_{k+1},...,s_{k+l}}(x_{\sigma(k+1)},...,x_{\sigma(k+l)},...).$$

Definition 6 A $k$-vector is called translation invariant if

$$X^{s_1,...,s_k}(x_1,...,x_k; u(x_1),...,u(x_k),...) = \partial_{x_1}^{s_1}...\partial_{x_k}^{s_k} X(x_1,...,x_k; u(x_1),...,u(x_k),...)$$

where $X(...)$ means $X^{0,...0}(...)$ and for any $t$

$$X(x_1+t,...,x_k+t; u(x_1),...,u(x_k),...) = X(x_1,...,x_k; u(x_1),...,u(x_k),...)$$

It follows from this definition that a translation invariant multi-vector field is completely characterized by the “components”

$$X^{x_1,...,x_k} := X(x_1,...,x_k; u(x_1),...,u(x_k),...)$$

Definition 7 A $k$ form $\omega$ is a finite sum

$$\omega = \frac{1}{k!} \omega_{s_1,...,s_k} \delta u^{s_1} \wedge ... \wedge \delta u^{s_k}$$

where $\omega_{s_1,...,s_k} \in \mathcal{A}$

In order to define a Poisson structure we need to introduce a criterion for the Jacobi identity. We have seen that in the finite-dimensional case the Jacobi identity can be written in terms of the Schouten bracket. Therefore if we will be able to define an infinite-dimensional version of the Schouten bracket we will be also able to define an infinite-dimensional version of the Poisson bracket.
2.4.1 Schouten bracket of translation-invariant multivectors

In the case of translation invariant multivectors one obtains the formula for the Schouten bracket simply by translating the formula (2.9) in the new context. Heuristically this can be done by substituting sums for integrals, partial derivatives for variational derivatives, etc. The result is:

**Definition 8** Schouten bracket of a translation invariant p-vector $P^{x_1 \ldots x_p}$ and a translation invariant q-vector $Q^{x_1 \ldots x_q}$.

\[
[P, Q]^{x_1 \ldots x_{p+q-1}} = \sum_{\sigma \in S_{p+q-1}} (-1)^{sgn(\sigma)} \sum_{s=0}^{p} \left( \frac{(-1)^p}{p!(q-1)!} \left( \frac{\partial^s Q^{x_{\sigma(1)} \ldots x_{\sigma(p+1)} x_{\sigma(p+q-1)}}}{\partial u^s(x_{\sigma(i)})} \right) \right) + \sum_{i=0}^{q-1} \frac{1}{(p-1)! q!} \left( \frac{\partial^s Q^{x_{\sigma(p+i)} x_{\sigma(1)} \ldots x_{\sigma(p-1)}}}{\partial u^s(x_{\sigma(p+i)})} \right) \]

(2.16)

In the case $p=2$:

\[
[P, Q]^{x_1 \ldots x_{q+1}} = \sum_{\sigma \in S_{q+1}} (-1)^{sgn(\sigma)} \sum_{s=0}^{p} \left( \frac{1}{2!(q-1)!} \left( \frac{\partial^s Q^{x_{\sigma(3)} \ldots x_{\sigma(q+1)}}}{\partial u^s(x_{\sigma(i)})} \right) \right) + \sum_{i=0}^{q-1} \frac{1}{q!} \left( \frac{\partial^s P^{x_{\sigma(2+i)} x_{\sigma(1)}}}{\partial u^s(x_{\sigma(2+i)})} \right) \]

(2.17)

**Example 5** Schouten bracket of a bivector $P$ with the components $P^{xy}$ and a local functional $I = \int_{S^1} f(u(x), u_x, u_{xx}, \ldots) dx$

\[
[P, I]^x = \int_{S^1} \frac{\delta f}{\delta u^s(y)} P_{yx} dy
\]

(2.18)

**Example 6** Lie derivative of a translation invariant bivector $P^{xy}$ along a translation invariant vector field $Q$ with the components $Q^x$

\[
[P, Q]^{xy} = \sum_s \left( \left( \frac{\partial^s Q^x}{\partial u^s(x)} \right) \frac{\partial P^{xy}}{\partial u^s(y)} + \left( \frac{\partial^s Q^y}{\partial u^s(y)} \right) \frac{\partial P^{xy}}{\partial u^s(x)} \right) - \left( \frac{\partial^s P^{xy}}{\partial u^s(y)} \right) \frac{\partial Q^x}{\partial u^s(x)}\]

(2.19)
Example 7 Schouten bracket of two translation invariant bivectors $P$ and $Q$

$$[P, Q]^{xyz} = \frac{1}{2} \sum_s \left( \frac{\partial P^{xy}}{\partial u^s(x)} \partial_x^s Q^{xz} + \frac{\partial Q^{xy}}{\partial u^s(x)} \partial_x^s P^{xz} + \frac{\partial P^{xy}}{\partial u^s(y)} \partial_y^s Q^{yz} + \frac{\partial Q^{xy}}{\partial u^s(y)} \partial_y^s P^{yz} + \frac{\partial P^{zx}}{\partial u^s(z)} \partial_z^s Q^{zy} + \frac{\partial Q^{zx}}{\partial u^s(z)} \partial_z^s P^{zy} + \frac{\partial Q^{yz}}{\partial u^s(y)} \partial_y^s P^{zx} + \frac{\partial P^{yz}}{\partial u^s(y)} \partial_y^s P^{zx} \right)$$

(2.20)

Remark 1
The operator $\frac{\partial}{\partial u(x)}$ is the usual partial derivative when it acts on functions depending on $x$ while it is, by definition, equal to 0 when it acts on functions not depending on $x$. In other words

$$\frac{\partial u(x)}{\partial u(x)} = 1$$
$$\frac{\partial u(y)}{\partial u(x)} = 0$$

Remark 2
The formula (2.19) can be obtained by the formula (2.17) for $q = 1$ by using skew-symmetry of $P$.

Definition 9 A translation invariant Poisson bivector $P$ is a translation invariant bivector such that

$$[P, P]_{\text{Schouten}} = 0$$

As in finite dimensional case a translation-invariant Poisson bivector $P^{xy}$ defines a Poisson structure on the loop space $\mathcal{L}$. The Poisson bracket of two local functionals $I_1, I_2$ is given by the formula

$$\{I_1, I_2\} := \int_{S^1} \int_{S^1} \frac{\delta I_1}{\delta u(x)} P^{xy} \frac{\delta I_2}{\delta u(y)} dxdy$$

(2.21)

To define the Poisson cohomology on the space of translation invariant multivectors we have to prove that the operator $d_P := [P, \cdot]$ associated to a Poisson bivector $P$ satisfies the condition $d_P^2 = 0$. We have seen that this condition is satisfied (for a Poisson bivector) if the graded Jacobi identity holds and moreover that the graded Jacobi identity follows from (2.5) and (2.6).

The property (2.6) is obvious. As far as it concerns (2.5) it will be sufficient to analyse a particular case to understand how the proof works in general.

We want to check the "Leibniz rule" (2.5) when $p = 2$ and $Q$ is a bivector. In this case we have to prove:

$$[X \wedge Y, Q] = Y \wedge [X, Q] - X \wedge [Y, Q]$$

(2.22)
Left hand side
By using formula for the Schouten bracket of two bivector, we obtain

\[
[X \wedge Y, Q]_{xyz} = \\
\left( \frac{\partial X}{\partial u^s(x)} Y(y) - X(y) \frac{\partial Y}{\partial u^s(x)} \right) (\partial_x^s P_{zy}) + \left( \frac{\partial P_{zy}}{\partial u^s(x)} \right) \left( (\partial_x^s X)(y) - X(y) (\partial_x^s X) \right) \\
\left( \frac{\partial Y}{\partial u^s(x)} X(y) - Y(y) \frac{\partial X}{\partial u^s(x)} \right) (\partial_y^s P_{xz}) + \left( \frac{\partial P_{xz}}{\partial u^s(x)} \right) \left( (\partial_y^s X)(y) - X(y) (\partial_y^s X) \right) \\
\left( \frac{\partial X}{\partial u^s(z)} X(z) - X(z) \frac{\partial Y}{\partial u^s(z)} \right) (\partial_z^s P_{xy}) + \left( \frac{\partial P_{xy}}{\partial u^s(z)} \right) \left( (\partial_z^s X)(y) - X(y) (\partial_z^s X) \right)
\]

This formula can be written collecting terms in \(Y(x), Y(y), \) etc.: 

\[
= Y(x) \left( -\frac{\partial X}{\partial u^s(y)} (\partial_y^s P_{xz}) + \frac{\partial X}{\partial u^s(z)} (\partial_z^s P_{xy}) + \frac{\partial P_{xz}}{\partial u^s(y)} (\partial_y^s X(y)) + \frac{\partial P_{xy}}{\partial u^s(z)} (\partial_z^s X(z)) \right) + \ldots \quad (2.23)
\]

Right hand side

\[
Y \wedge [X, Q] - X \wedge [Y, Q] = \\
= \frac{1}{2} \sum_{s \in S_3} (-1)^{sgn(s)} Y(x_{s(1)})[X, P]_{x_{s(2)}x_{s(3)}} - \frac{1}{2} \sum_{s \in S_3} (-1)^{sgn(s)} X(x_{s(1)})[Y, P]_{x_{s(2)}x_{s(3)}} = \\
= \frac{1}{2} \left( Y(x)[X, P]_{yz} - Y(x)[X, P]_{zy} - Y(y)[X, P]_{xz} + Y(y)[X, P]_{zx} + Y(y)[X, P]_{xy} + Y(z)[X, P]_{zy} - Y(z)[X, P]_{yz} \right)
\]

Let us consider the first two terms:

\[
\frac{1}{2} \left( Y(x)[X, P]_{yz} - Y(x)[X, P]_{zy} \right)
\]

They can be written as

\[
Y(x) \left( \frac{\partial X}{\partial u^s(z)} (\partial_z^s P_{xy}) - \frac{\partial X}{\partial u^s(y)} (\partial_y^s P_{xz}) + \frac{1}{2} \frac{\partial P_{xz}}{\partial u^s(y)} (\partial_y^s X(y)) - \frac{1}{2} \frac{\partial P_{yz}}{\partial u^s(z)} (\partial_x^s Y(y)) \right) + \\
\frac{1}{2} \frac{\partial P_{yz}}{\partial u^s(z)} (\partial_x^s X(z)) - \frac{1}{2} \frac{\partial P_{zy}}{\partial u^s(z)} (\partial_x^s X(z))
\]
By comparing this expression with (2.23) we realize that they coincide because of the skew-symmetry:

\[
\frac{\partial P_{xy}}{\partial u^s(y)} = - \frac{\partial P_{yx}}{\partial u^s(y)} \quad (2.24)
\]

\[
\frac{\partial P_{xy}}{\partial u^s(z)} = - \frac{\partial P_{yx}}{\partial u^s(z)} \quad (2.25)
\]

Analogously for the other terms of the left and right hand side.

### 2.4.2 Local multivectors and their Poisson cohomology

**Definition 10** A local k-vector is a translation invariant k-vector such that its dependence on \(x_1, \ldots, x_k\) is given by a finite order distribution with the support on the diagonal \(x_1 = \ldots = x_k\).

In coordinates a multivector \(X\) has the form

\[
X = \sum_{p_2, p_3, \ldots, p_n \geq 0} X(u(x_1), u_2(x_1), \ldots) \delta^{(p_2)}(x_1 - x_2) \delta^{(p_3)}(x_1 - x_3) \cdots \delta^{(p_n)}(x_1 - x_n) \quad (2.26)
\]

It is easy to check that \(P_{xy} = u \delta'(y - x) + \frac{1}{2} u_x \delta(x - y)\) is a local bivector. In fact

\[
P_{yx} = u(y) \delta'(y - x) + \frac{1}{2} u_y \delta(y - x) = -u_x \delta'(x - y) - u_x \delta(x - y) + \frac{1}{2} u_x \delta(x - y) = -P_{xy}
\]

But \(\frac{\partial P_{xy}}{\partial u(x)}\) is not equal to \(- \frac{\partial P_{yx}}{\partial u(x)} = 0\!\)!

This problem can be solved by writing \(P_{xy}\) in a suitable form. If we write \(P_{xy} = P'_{xy} = \frac{1}{2} (P_{xy} - P_{yx})\) then

\[
\frac{\partial P'_{xy}}{\partial u(x)} = - \frac{\partial P'_{yx}}{\partial u(x)} \quad (2.27)
\]

This is true in general: if we want to use the same formulae valid for non-local multivectors we have to write the local multivector in a form “compatible” with the operators \(\frac{\partial}{\partial u^s(x)}\). The practical rule is to write the multivector \(T^{x_1, \ldots, x_n}\) in the form

\[
\frac{1}{n!} \sum_{\sigma} (-1)^{\text{sgn}(\sigma)} T^{x_{\sigma(1)}, \ldots, x_{\sigma(n)}}
\]

Analogously to the non-local case the formula (2.17) allows us to define a cohomology operator \(d_P\) starting from a local Poisson bivector \(P\):

\[
0 \rightarrow \Gamma^0 \xrightarrow{d_P} \Gamma^1 \xrightarrow{d_P} \Gamma^2_{\text{local}} \xrightarrow{d_P} \Gamma^3_{\text{local}} \rightarrow \cdots \xrightarrow{d_P} \Gamma^k_{\text{local}} \cdots
\]

where \(\Gamma^0\) is the space of local functionals, \(\Gamma^1\) is the space of non local translation-invariant vector fields and \(\Gamma^k_{\text{local}}\) is the space of local translation-invariant \(k\)-vector fields. It is easy to check that the cohomology groups we have introduced in this way have the same meaning that the cohomology groups of finite-dimensional Poisson manifolds (see the examples in the section devoted to Poisson cohomology of finite dimensional Poisson manifolds).
3 Deformations of bihamiltonian systems of hydrodynamic type

In this paper we deal with Poisson bivectors of the form

\[ P_{xy} = \phi(u)\delta^{(1)}(x - y) + \frac{1}{2}(\partial_x \phi)\delta(x - y) + \sum_k \epsilon^k P^{[k]}_{xy} \]  

where

\[ P^{[k]}_{xy} = \sum_{s=0}^{k+1} A_{k,s}(u, u_x, \ldots)\delta^{(k+1-s)}(x - y) \]  

where \( A_{k,s} \) are differential polynomials. Introducing the following gradation in the space \( A \) of differential polynomials:

\[ \text{deg}(f(u)) = 0 \]
\[ \text{deg}(u^{(k)}) = k \]

we require also that

\[ \text{deg}(A_{k,s}) = s \]  

The part of order 0 is called local Poisson bracket of hydrodynamic type (see [6]). Analogously to the finite dimensional case we have the following

**Definition 11** A bihamiltonian structure on the loop space \( L \) is a pair \((P_1, P_2)\) of local Poisson bivectors such that for any \( \lambda_1, \lambda_2 \in \mathbb{R} \), the linear combination

\[ \lambda_1 P_1 + \lambda_2 P_2 \]

is again a Poisson bivector.

**Definition 12** An evolutionary equation

\[ u_t = F(u(x), u_x, u_{xx}, \ldots) \]

is called Hamiltonian if it can be written in the form

\[ u_t = \{u, H\} \]

where \( H \) is some local functional.

**Definition 13** An evolutionary equation is called bihamiltonian if and only if it is hamiltonian with respect both Poisson bivectors of a bihamiltonian structure.
Example 8  Rescaling $x \rightarrow \epsilon x$ the KdV equation $u_t = uu_x + u_{xxx}$ one obtains

$$u_t = \epsilon (uu_x + \epsilon^2 u_{xxx})$$

One usually introduces slow time variable $t \rightarrow \epsilon t$ to rewrite the last equation into the form

$$u_t = uu_x + \epsilon^2 u_{xxx}$$

This is small dispersion expansion of the KdV equation. It is a bihamiltonian equation. The first Poisson bivector (Gardner-Zakharov-Faddeev bivector)

$$P^{xy} = \delta'(x - y)$$

and the second Poisson bivector (Magri bivector)

$$\{u(x), u(y)\}_2 = u\delta'(x - y) + \frac{1}{2} u_x \delta(x - y) - \epsilon^2 \delta^3(x - y)$$

belong to the class of bivectors introduced before.

It is easy to check that the pair $(P_1, P_2)$ of the Gardner-Zakharov-Faddeev bracket and of the Magri bracket is a bihamiltonian structure. It is called **Magri bihamiltonian structure**.

**Definition 14** The group of transformation

$$u \rightarrow \bar{u} = \sum_k \epsilon^k F_k(u, u_x, ...)$$

where $F_k \in A$, $\text{deg}(F_k) = k$ and $\frac{\partial F_k}{\partial u} \neq 0$ is called **Miura group**.

Degiovanni, Magri and Sciacca (see [4]) and Getzler (see [11]) solved independently the problem of studying the action of the Miura group on the bracket (3.2).

More precisely they study the following problem: does it exits an element of the Miura group that transforms the bracket (3.2) into the bracket (3.4)?

This problem can be reduced to a cohomological problem. In fact we have seen that it corresponds to the study of the second group of Poisson cohomology associated to the bracket (3.4) $^2$.

$HP^2(P, \Pi) = $ **Infinitesimal deformations of $P$/Deformations that can be obtained by infinitesimal change of coordinates of the form (3.6)**.

This cohomology group is trivial (see [4] and [11]).

In this paper we deal with deformations of bihamiltonian structure of hydrodynamic type.

**Definition 15** A deformation of order $k$ of a bihamiltonian structure of hydrodynamic type is a pair $(P_1, P_2)$ of the form (3.2) such that $P_1 - \lambda P_2$ satisfies Jacoby identity for every $\lambda$ up to the order $k$:

$$[P_1, P_1] = [P_2, P_2] = [P_1, P_2] = o(\epsilon^k)$$

From the triviality of second Poisson cohomology group it follows that we can always assume one of the brackets of the form (3.4).

In the next section we explain how to classify these deformations.

$^2$ In our case the role of infinitesimal change of coordinates is played by the Miura group.
3.1 Classification of deformations of bihamiltonian structure of hydrodynamic type

We are interested in the following question: which deformations of a bihamiltonian structure of hydrodynamic type are trivial? In other words which deformations can be obtained from a bihamiltonian structure of hydrodynamic type by the action of Miura group?

We start considering first order deformations, that is

\[ P_1 = P_1^{(0)} = \delta^1(x - y) \quad \text{and} \quad P_2 = P_2^{(0)} + \epsilon P_2^{(1)} \]

By definition we have

\[ [P_1, P_2] = o(\epsilon) \]  \hspace{1cm} (3.8)
\[ [P_2, P_2] = o(\epsilon) \]  \hspace{1cm} (3.9)

The equation \([P_1, P_2] = o(\epsilon)\) implies \([P_1^{(0)}, P_2^{(1)}] = d_1 P_2^{(1)} = 0\) and from the triviality of the second Poisson cohomology group it follows that there exists a vector \(X_2^{(1)}\) such that

\[ P_2^{(1)} = d_1 X_2^{(1)} \]  \hspace{1cm} (3.10)

The equation \([P_2, P_2] = o(\epsilon)\) implies that

\[ [P_2^{(0)}, P_2^{(1)}] = d_2 P_2^{(1)} = d_2 d_1 X_2^{(1)} = -d_1 d_2 X_2^{(1)} = 0 \]  \hspace{1cm} (3.11)

where we have used the fact that \((d_1 + d_2)^2 = 0\).

Among all deformations that satisfy the equation \(d_1 d_2 X_2^{(1)} = 0\) we have to select trivial deformations, that is the deformations that can be obtained by infinitesimal change of coordinates. In our case this means that there exists a vector field \(\tilde{X}\) such that

\[ \text{Lie}_{\tilde{X}} P_2^{(0)} = P_2^{(1)} \]  \hspace{1cm} (3.12)
\[ \text{Lie}_{\tilde{X}} P_1^{(0)} = 0 \]  \hspace{1cm} (3.13)

**Theorem 3** A deformation \(P_\lambda = P_1^{(0)} - \lambda(P_2^{(0)} + \epsilon d_1 X_2^{(1)})\) is trivial \(\iff X_2^{(1)} = d_1 a + d_2 b\) where \(a, b\) are local functionals.

**Proof**

\(\leftarrow\)

\(X_2^{(1)} = d_1 a + d_2 b \Rightarrow P_2^{(1)} = d_1 d_2 b = -d_2 d_1 b\)

This means \(P_2^{(1)} = \text{Lie}_{\tilde{X}} P_2^{(0)}\) with \(\tilde{X} = -d_1 b\).

Moreover \(\text{Lie}_{\tilde{X}} P_1^{(0)} = 0\).

\(\Rightarrow\)

\(\text{Lie}_{\tilde{X}} P_1^{(0)} = 0 \Rightarrow \tilde{X} = d_1 b\) (the first Poisson cohomology group is trivial).

\[-d_1 d_2 b = d_2 d_1 b = \text{Lie}_{\tilde{X}} P_2^{(0)} = P_2^{(2)} = d_1 X_2^{(2)} \Rightarrow X_2^{(2)} = -d_2 b + d_1 a\]

---

\(^3\)It is well-known that \([P_1, P_1] = 0\)
From the last theorem it follows that the elements of the group $\text{Ker}(d_1d_2)/(\text{Im}(d_1) + \text{Im}(d_2))$ are the non trivial first order deformations.

**Remark**
For higher order deformations we can repeat the same arguments and obtain the same result. Consequently, in general:
1) A $k$ order deformation $P_2^{(k)}$ can be represented in the form

$$P_2^{(0)} - \lambda P_1^{(0)} + \sum_{i=1}^{k} \epsilon^i P_2^{(i)} = P_2^{(0)} - \lambda P_1^{(0)} + \sum_{i=1}^{k} \epsilon^i d_1 X_2^{(i)} \quad (3.14)$$

where due to our definition of a gradation in the space $\mathcal{A}$ of differential polynomials (see (3.3)) we have necessarily $\text{deg}(X_2^{(i)}) = i$ (see the form of the formula (2.19) when $P_1 = \delta'(x - y)$).

2) $P_2^{(k)}$ is trivial if and only if $X_2^{(k)} = d_1 A + d_2 B$.

**Remark 1** Also in this case “trivial” means that $P_2^{(k)}$ can be eliminated by the action of Miura group. Obviously the triviality of $P_2^{(k)}$ doesn’t imply that the $k$-order deformation is trivial.

**Remark 2**
From the (2.18) it follows immediately that $d_1$ and $d_2$ increases the degree of a local functional by one. Then the degree of $A$ and $B$ must be equal to $k - 1$.

In the next section we summarize the results of the classification of deformations up to fourth order.

### 3.2 Classification of deformations: results

**Theorem 4** Up to the fourth order all deformations of a bihamiltonian structure of hydrodynamic type (see definition 15) can be reduced, by the action of Miura group to the following form:

$$u\delta^{(1)}(x - y) + \frac{1}{2} u_x \delta(x - y) - \lambda \delta^{(1)}(x - y) +$$

$$+ \epsilon^2 \left(-2s\delta^3(x - y) - 3\partial_x s\delta^2(x - y) - \partial_x^2 s\delta^1(x - y)\right) +$$

$$+ \epsilon^4 \left(-2s\delta^5(x - y) - 5(\partial_x s) \delta^4(x - y) - 10(\partial_x^2 s) \delta^3(x - y) - 10(\partial_x^3 s) \delta^2(x - y) +$$

$$- 3(\partial_x^4 s) \delta^1(x - y) + 2w\delta^3(x - y) + 3(\partial_x w) \delta^2(x - y) + (\partial_x^2 w) \delta^1(x - y)\right) + O(\epsilon^5) \quad (3.15)$$

with

$$w = 2 \frac{\partial \bar{s}}{\partial u} u_{xx} \quad (3.16)$$

where $s$ is an arbitrary function of $u$ and $\bar{s} = -2s \frac{\partial w}{\partial u}$.

From this theorem it follows immediately:

\[\text{we will identify the degree of a local functional } \int f dx \text{ with the degree of its density } f\]
Corollary 1 Up to the fourth order every deformation of Magri bihamiltonian structure is trivial

Proof
Indeed, in this case $s = constant \neq 0$ and then $\tilde{s} = 0, \ w = 0$.

4 The deformed hierarchy

4.1 Integrals of motions
As we have already been said bihamiltonian structures give rise to an infinite number of "almost-
constants" of motions. From these constants one can construct an hierarchy of hamiltonian equa-
tions.
In this section we show explicitely how to produce this hierarchy and we check that the corre-
sponding flows commute up to the order of the deformation.
The technique is well known for the KdV equation (see [9]): one looks for a solution of the equation
\[ P_\lambda v = 0 \] (4.1)
in terms of a formal series: the coefficients of 1-form $v$ are variational derivatives of the integrals
of motions.  
We apply the same technique to the second order deformations (see classification theorem).
We start with the equation $P_\lambda v = 0$. This equation implies $vP_\lambda v = 0$. From skew-symmetry of
$P_\lambda$ it follows that $0 = vP_\lambda v = \partial_x(...)$. That is $(...) = f(\lambda)$. More precisely
\[ v(-\lambda \partial_x + u \partial_x + \frac{1}{2}u_x + \epsilon^2(2s(u)\partial_x^3 + 3(\partial_x s)\partial_x^2 + (\partial_x^2 s)\partial_x)) v = 0 \] (4.2)
From (4.2) it follows:
\[ \partial_x\left(\frac{1}{2}v^2(u - \lambda) + \epsilon^2(2svv_{xx} + s_xvv_x - sv_x^2)\right) = 0 \] (4.3)
that is
\[ \frac{1}{2}v^2(u - \lambda) + \epsilon^2(2svv_{xx} + s_xvv_x - sv_x^2) = f(\lambda) \] (4.4)
By choosing $f(\lambda) = -\frac{\lambda}{2}$ we look for a solution of the form
\[ v = \sum_{i=0}^{\infty} \frac{P_i}{\lambda^i} \] (4.5)
where obviously $p_0 = 1$.
By straightforward calculation we get
\[ \frac{1}{2} \sum_{i+j=k} \frac{p_i p_j}{\lambda^{k-1}} = \sum_{i+j=k-1} \left( \frac{u}{2} p_i p_j + \epsilon^2(2sp_i p_{jxx} + s_x p_i p_{jx} - sp_{ix} p_{jx}) \right) \frac{1}{\lambda^{k-1}} \] (4.6)
\(^5\)this is equivalent to the exactness of the 1-form $v$ with respect to the vertical differential of the variational
bicomplex (see [3]).
for $k = 1$ (4.6) implies
\[ \frac{1}{2}(p_0p_1 + p_1p_0) = \frac{1}{2}up_0^2 \] (4.7)
that is
\[ p_1 = \frac{u}{2} \] (4.8)
for $k=2$:
\[ p_2 = \frac{3}{8}u^2 + \frac{\epsilon^2}{2}(2su_{xx} + s_xu_x) \] (4.9)
and so on. We observe that the coefficients will have always the form
\[ p_i = f(u) + O(\epsilon^2) + O(\epsilon^4) + ... \] (4.10)
where $f(u)$ is a polynomial in $u$.

Now we want to prove that the coefficients $p_i$ are, up to the second order variational derivatives. We consider a curve $u(t)$ on the loop space $L$. By differentiating the equation
\[ \frac{1}{2}v^2(u - \lambda) + \epsilon^2(2svv_{xx} + s_xvv_x - sv_x^2) + \frac{\lambda}{2} = 0 \] (4.11)
along the vector field $\dot{u}$, tangent to this curve, we get
\[ \dot{uv} = -2\dot{v}(u - \lambda) - \epsilon^2 \left( 4\dot{sv}_{xx} + 4s\frac{\dot{v}}{v}\dot{v}_{xx} + 4sv_{xx} + 2s_xv_x + 2\left( \dot{s}\frac{\dot{v}}{v} + 2\frac{v_x}{v}(\dot{s}v_x - s_x\dot{v}) \right) \right) \]
By straightforward calculation one can write the last equation in the form
\[ \dot{uv} = \partial_t \left( -2\frac{\lambda}{v} \right) + \epsilon^2 \left( \partial_t (-4sv_{xx} - 2s_xv_x) + \partial_x \left( 4\frac{\dot{v}}{v}v_x \right) + 2\frac{v_x}{v}(\dot{s}v_x - s_x\dot{v}) \right) \] (4.12)
This equation implies
\[ \int_{S^1} \dot{uv}dx = \frac{d}{dt} \left( \int_{S^1} f dx \right) + O(\epsilon^4) \] (4.13)
for some $f = f(u, u_x, u_{xx}, ...)$.

In fact from (4.10) it follows that $v$ has the form
\[ v = v^0 + O(\epsilon^2) \] (4.14)
where $v^0$ is a function of $u$ and
\[ \dot{s}v_x^0 - s_x\dot{v}^0 = 0 \]
Now we can formulate the
Theorem 5 \( p_i = \frac{\delta I_i}{\delta u} + O(\epsilon^4) \) for some functionals \( I_i \). Moreover
\[
\{I_i, I_j\}_1 = O(\epsilon^4)
\] (4.15)

Proof

\[
\int_{S_1} \dot{uv} \, dx = \frac{d}{dt} \left( \int_{S_1} f \, dx \right) + O(\epsilon^4) = \int_{S_1} \left( \frac{\partial f}{\partial u} \dot{u} + \frac{\partial f}{\partial u_x} \dot{u}_x + \frac{\partial f}{\partial u_{xx}} \dot{u}_{xx} + \ldots \right) \, dx + O(\epsilon^4) = \ldots
\]

By integrating by parts, we get
\[
... = \int_{S_1} \dot{u} \left( \frac{\partial f}{\partial u} - \partial_x \left( \frac{\partial f}{\partial u_x} \right) + \partial_x^2 \left( \frac{\partial f}{\partial u_{xx}} \right) + \ldots \right) \, dx + O(\epsilon^4) = \int_{S_1} \dot{u} \frac{\delta f}{\delta u} \, dx + O(\epsilon^4)
\]

Then
\[
v = \frac{\delta f}{\delta u} + O(\epsilon^4)
\] (4.16)

that is
\[
p_i = \frac{\delta I_i}{\delta u} + O(\epsilon^4)
\] (4.17)

for some functionals \( I_i \).

Moreover, by definition, the coefficients \( p_i \) of 1-form \( v \) satisfy the equation
\[
(P_2 - \lambda P_1) \left( \sum_{i=0}^{\infty} \frac{p_i}{\lambda^i} \right) = 0
\] (4.18)

In terms of \( I_i \) this condition can be written as
\[
\sum_{i=0}^{\infty} \left( P_2 \frac{\delta I_i}{\delta u} \right) \frac{1}{\lambda^i} = \sum_{i=-1}^{\infty} \left( P_1 \frac{\delta I_{i+1}}{\delta u} \right) \frac{1}{\lambda^i} + O(\epsilon^4)
\] (4.19)

From this equation it follow immediately:
\[
P_1 \frac{\delta I_0}{\delta u} = O(\epsilon^4)
\]
\[
... = P_2 \frac{\delta I_i}{\delta u} = P_1 \frac{\delta I_{i+1}}{\delta u} + O(\epsilon^4)
\]

By using these Lenard recursion relations it is easy to prove the therem (see for details [9])

4.2 Soliton solutions

Now we study one of the equation of the hierarchy. More precisely we concentrate on the equation
\[
u_t = \partial_x^2 \frac{\delta I_2}{\delta u} = \frac{3}{4} uu_x + \epsilon^2 \left( s(u) u_{xxx} + 2 \frac{\partial s}{\partial u} u_x u_{xx} + \frac{1}{2} \frac{\partial^2 s}{\partial u^2} u_x^3 \right)
\] (4.20)

When \( s = \text{constant} \) this is KdV equation.
4.2.1 One soliton solutions

We look for solutions of (4.20) of the form \(u(x, t) = u(x + ct) = u(z)\).

By substituting we get

\[
cu_z = \frac{3}{4} uu_z + \epsilon^2 \left( s(u)u_{zzz} + 2 \frac{\partial s}{\partial u} u_z u_{zz} + \frac{1}{2} \frac{\partial^2 s}{\partial u^2} u_z^3 \right)
\]  

(4.21)

This equation can be written as

\[
\partial_z \left( cu - \frac{3}{8} u^2 - \epsilon^2 \left( \partial_z (su_z) - \frac{1}{2} (\partial_z s) u_z \right) \right) = 0
\]

(4.22)

that is

\[
cu - \frac{3}{8} u^2 - \epsilon^2 \left( \partial_z (su_z) - \frac{1}{2} (\partial_z s) u_z \right) = c_1
\]

(4.23)

By multiplying for \(u_z\) we get again a total derivative

\[
\partial_z \left( \frac{1}{2} cu^2 - \frac{1}{8} u^3 - \epsilon^2 \left( \frac{1}{2} s u_z^2 \right) - c_1 u \right) = 0
\]

(4.24)

that is

\[
\frac{1}{2} cu^2 - \frac{1}{8} u^3 - \epsilon^2 \left( \frac{1}{2} s u_z^2 \right) - c_1 u = c_2
\]

(4.25)

This equation can be written as

\[
\left( \frac{du}{dz} \right)^2 = F(u)
\]

(4.26)

where

\[
F(u) = \frac{2}{\epsilon^2 s(u)} \left( -\frac{1}{8} u^3 + \frac{1}{2} cu^2 - c_1 u - c_2 \right)
\]

(4.27)

To obtain the solution one has to invert the following integral

\[
z - z_0 = \pm \int_{\text{ } u}^{u_0} \frac{1}{F(u)^{1/2}} du
\]

(4.28)

4.2.2 Case \(s(u) = u\)

In this case \(F(u) = \frac{P(u)}{u}\) where \(P(u)\) is a polynomial of degree 3. It is well known (see [5]) that one soliton solutions occur when \(F(u)\) has one simple zero and one double zero.
By using the formula (see [15])
\[
\int_{x}^{a_0} \left( \frac{x - a_3}{(a_0 - x)(x - a_1)(x - a_2)} \right)^{\frac{1}{2}} dx = \frac{2(a_0 - a_3)}{((a_0 - a_2)(a_1 - a_3))^{\frac{1}{2}}} \Pi \left( \phi, \frac{a_1 - a_0}{a_1 - a_3}, k \right)
\]
(4.29)

where \( a_0 > u > a_1 > a_2 > a_3 \) and

\[
\phi = \arcsin \left( \frac{(a_1 - a_3)(a_0 - x)}{(a_0 - a_1)(x - a_3)} \right)
\]
(4.30)

\[
k = \left( \frac{(a_0 - a_1)(a_2 - a_3)}{(a_0 - a_2)(a_1 - a_3)} \right)^{\frac{1}{2}}
\]
(4.31)

\[
\Pi(\phi, \nu, k) = \int_{0}^{\phi} \frac{d\phi}{(1 - \nu \sin^2 \phi)(1 - k^2 \sin^2 \phi)^{\frac{1}{2}}}
\]
(4.32)

we obtain, when \( a_1 = a_2 \) and \( a_3 = 0 \) (see [1])

\[
z - z_0 = \pm \frac{2a_1}{(a_1 (a_0 - a_1))^{\frac{1}{2}}} \left[ -2\ln \left( \frac{(a_1(a_0-u))^{\frac{1}{2}}}{u(a_0-a_1)} \right) + 1 \right] - \left( \frac{a_0}{a_1 - 1} \right)^{\frac{1}{2}} \arctg \left( \frac{2u (a_0 - 1)^{\frac{1}{2}}}{2u - a_0} \right)
\]
(4.34)

The speed \( c \) of the wave and the constants of integration \( c_1 \) and \( c_2 \) can be easily written in terms of the coefficients \( a_0, a_1, a_2, a_3 \):

\[
c = \frac{1}{4}(a_0 + 2a_1)
\]
(4.35)

\[
c_1 = \frac{1}{8}(2a_0a_1 + a_1^2)
\]
(4.36)

\[
c_2 = -\frac{1}{8}a_0a_1^2
\]
(4.37)

The expression (4.34) can be inverted numerically. To test the existence of two solitons solutions we have used two such solutions with different speed as initial condition. The results of numerical experiments are described in the next section.
5 Numerical experiments

In this section we analyse the equation (4.20) for $s(u) = u$, i.e.

$$u_t = \partial_x \left( \frac{\delta I_2}{\delta u} \right) = \partial_x \left( \frac{3}{8} u^2 + \frac{\epsilon^2}{2} (2uu_{xx} + u_x^2) \right)$$

We write this equation in the form

$$u_t + F_x = 0$$

where $F = -\frac{\delta I_2}{\delta u}$. To make numerical experiments we have used a two-steps Lax-Wendroff scheme (see [14]). This scheme is characterized by an auxiliary step of calculation

$$u_{j+\frac{1}{2}}^{n+1} = \frac{1}{2} (u_j^n + u_{j+1}^n) - \frac{\Delta t}{2\Delta x} (F_{j+1}^n - F_j^n),$$

where $F_{j+1}^n = F(u_{j+1}^n)$.

The main step is

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} (F_{j+\frac{1}{2}}^{n+\frac{1}{2}} - F_{j-\frac{1}{2}}^{n+\frac{1}{2}}).$$

The first figure illustrates two solitons of different height and speed we have used in numerical experiments.

We have choosen periodic boundary conditions.

The results of our experiments is showed in figure 2. The waves move from the right to the left. The three pictures of the second figure show the solitons before, during and after the collision. Like the soliton collision in integrable systems they reemerge with the same shape.
Remark
Due to numerical instability for big amplitude we performed numerical experiments with small
and slow waves.
We don’t know if the origin of this numerical instability is the algorithm we have used.

6 Classification of deformations: proof

6.1 First order deformations

We have seen that in this case the non trivial deformations are the elements of the group
$\text{Ker}(d_1d_2)/(\text{Im}(d_1) + \text{Im}(d_2))$.
We start looking for the solutions of the equation $d_1d_2X = 0$.
First of all we calculate $d_2X$

\[
d_2X = [P_2^{(0)}, X]_{xy} = \frac{1}{2}(\partial_x^s X) \frac{\partial}{\partial u^s(x)} (u\delta'(x - y) + \frac{1}{2}u_x \delta(x - y))
\]

\[
-\frac{1}{2}(\partial_y^s X) \frac{\partial}{\partial u^s(y)} (u\partial_y \delta(y - x) + \frac{1}{2}u_y \delta(y - x)) + \partial_x^s (u\partial_y \delta(y - x) + \frac{1}{2}u_y \delta(y - x)) \frac{\partial X}{\partial u^s(y)}
\]

\[
\quad -\partial_x^s (u\delta'(x - y) + \frac{1}{2}u_x \delta(x - y)) \frac{\partial X}{\partial u^s(x)}
\]

where, by definition, $\delta'(x - y) = \delta^{(1)}(x - y) = \partial_x \delta(x - y)$.
Moreover by using the formula

\[
f(y)\delta^{(s)}(x - y) = \sum_{q=0}^{s} \binom{s}{q} f^{(q)}(x)\delta^{(s-q)}(x - y)
\]  
   (6.1)
we get
\[
\frac{1}{2} \left( \frac{\partial}{\partial u^s(x)} \left( u \delta'(x - y) + \frac{1}{2} u_x \delta(x - y) \right) (\partial_x^s X) - \frac{\partial}{\partial u^s(y)} \left( u \delta'(y - x) + \frac{1}{2} u_y \delta(y - x) \right) (\partial_x^s X) \right) =
\]
\[
= \frac{1}{2} \left( X \delta'(x - y) + \frac{1}{2} \partial_x X \delta(x - y) - X \partial_y \delta(x - y) - \frac{1}{2} \partial_y X \delta(y - x) \right) =
\]
\[
= X \delta'(x - y) + \frac{1}{2} \partial_x X \delta(x - y)
\]
and
\[
\partial_y^s (u \partial_y \delta(x - y) + \frac{1}{2} u_y \delta(x - y)) \frac{\partial X}{\partial u^s(y)} - \partial_x^s (u \delta^{(1)}(x - y) + \frac{1}{2} u_x \delta(x - y)) \frac{\partial X}{\partial u^s(x)} =
\]
\[
= \partial_x^s (-u(x) \delta^{(1)}(x - y) - \frac{1}{2} u_x \delta(x - y)) \frac{\partial X}{\partial u^s(y)} +
\]
\[
- \partial_x^s (-u(y) \partial_y \delta(x - y) - \frac{1}{2} u_y \delta(y - x)) \frac{\partial X}{\partial u^s(x)} =
\]
\[
(-1)^{s+1} (u(x) \delta^{(s+1)}(x - y) + \frac{1}{2} u_x \delta^{(s)}(x - y)) \frac{\partial X}{\partial u^s(y)} +
\]
\[
- u(y) \delta^{(s+1)}(x - y) + \frac{1}{2} u_y \delta^{(s)}(x - y)) \frac{\partial X}{\partial u^s(x)}
\]

Summarizing we obtain
\[
(d_2 X)_{xy} = X \delta^{(1)}(x - y) + \frac{1}{2} \partial_x X \delta(x - y) + \sum_{s=0}^{s+1} \sum_{q=0} c_{qs} \delta^{(s+1-q)}(x - y) \tag{6.2}
\]

with
\[
c_{qs} = \left( \begin{array}{c} s + 1 \\ q \end{array} \right) \left( -1 \right)^{s+1} u \left( \partial_x^q \left( \frac{\partial X}{\partial u^s(x)} \right) \right) - \frac{\partial X}{\partial u^s(x)} u^{(q)}
\]
\[
+ \frac{1}{2} \left( \begin{array}{c} s \\ q - 1 \end{array} \right) \left( -1 \right)^{s+1} u_x \left( \partial_x^{q-1} \left( \frac{\partial X}{\partial u^s(x)} \right) \right) + \frac{\partial X}{\partial u^s(x)} u^{(q)} \tag{6.3}
\]

In this case \( \deg(X) = 1 \) i.e. \( X = s(u) u_x \). This implies that we can write the sum (6.2) as
\[
(d_2 X)_{xy} =
\]
\[
\left( \frac{1}{2} \partial_x X + c_{21} + c_{10} \right) \delta(x - y) + (X + c_{11} + c_{00}) \delta^{(1)}(x - y) + c_{01} \delta^{(2)}(x - y) \tag{6.4}
\]
with
\[ c_{21} = \left( u \frac{\partial^2 s}{\partial u^2} + \frac{1}{2} \frac{\partial s}{\partial u} \right) u_x^2 + \left( u \frac{\partial s}{\partial u} - \frac{1}{2} s \right) u_{xx} \]
\[ c_{10} = \left( -u \frac{\partial^2 s}{\partial u^2} - \frac{\partial s}{\partial u} \right) u_x^2 - u \frac{\partial s}{\partial u} u_{xx} \]
\[ c_{11} = 2u \frac{\partial s}{\partial u} u_x - su_x \]
\[ c_{00} = -2u \frac{\partial s}{\partial u} u_x \]
\[ c_{01} = 0 \]

Substituting the last equations in (6.4) we obtain \((d_2 X)_{xy} = 0\). This means that all fields \(X\) of degree 1 belong to \(\text{ker}(d_1 d_2)\).

Now we have to calculate the trivial field i.e. the fields \(X = d_1 A + d_2 B\).

where
\[ A = \int_{S^1} A(u) dx \]
\[ B = \int_{S^1} B(u) dx \]

Using the formulae
\[ d_1 A = -\partial_x \frac{\delta A}{\delta u} \] (6.5)
and
\[ d_2 B = -\partial_x \left( u \frac{\delta B}{\delta u} \right) + \frac{1}{2} \frac{\delta B}{\delta u} u_x \] (6.6)

we get
\[ d_1 A + d_2 B = \left( -\frac{\partial^2 A(u)}{\partial u^2} - \frac{1}{2} \frac{\partial B(u)}{\partial u} - u \frac{\partial^2 B(u)}{\partial u^2} \right) u_x \] (6.7)

This shows immediately that all deformations are trivial.

### 6.1.1 Explicit form of the deformations

We have to calculate \(d_1 X\) for an arbitrary field of degree 1.
\(d_2 X = 0\) implies \(d_1 d_2 X = -d_2 d_1 X = 0\). Then there exists a vector field \(\tilde{X}\) of degree 1 such that \(d_1 X = d_2 \tilde{X} = 0\).

This argument can be also used as an alternative proof of triviality of first order deformations.
6.2 Second order

Now we consider the deformation \( P_2 = P_2^{(0)} - \lambda P_1^{(0)} + \epsilon^2 P_2^{(2)} \).

Also in this case the non trivial deformations are elements of the group \( Ker(d_1 d_2) / (Im(d_1) + Im(d_2)) \).

We start again considering the solutions of the equation \( d_1 d_2 X = 0 \).

In this case the sum (6.2) becomes

\[
(d_2 X)_{xy} = \left( \frac{1}{2} \partial_x X + c_{10} + c_{21} + c_{32} \right) \delta(x - y) + (X + c_{00} + c_{11} + c_{22}) \delta(1)(x - y) + \\
(c_{01} + c_{12}) \delta(2)(x - y) + c_{02} \delta(3)(x - y)
\]

where \( X = s_0 u_{xx} + s_1 u_x^2 \) and

\[
\begin{align*}
  c_{10} &= -\partial_x \left( u \frac{\partial X}{\partial u} \right) \\
  c_{21} &= u \partial_x^2 \left( \frac{\partial X}{\partial u_x} \right) - \frac{1}{2} \partial_X u_{xx} + \frac{1}{2} \partial_x \left( \frac{\partial X}{\partial u_x} \right) u_x \\
  c_{32} &= -u \partial_x^3 \left( \frac{\partial X}{\partial u_{xx}} \right) - \frac{1}{2} \partial_X u_{xxx} - \frac{1}{2} \partial_x^2 \left( \frac{\partial X}{\partial u_{xx}} \right) u_x \\
  c_{00} &= -2u \frac{\partial X}{\partial u} \\
  c_{11} &= 2u \partial_x \left( \frac{\partial X}{\partial u_x} \right) - \partial_x \frac{\partial X}{u_x} u_x \\
  c_{22} &= -3u \partial_x^2 \left( \frac{\partial X}{\partial u_{xx}} \right) - 2 \frac{\partial_X u_{xx}}{\partial u_{xx}} - \partial_x \left( \frac{\partial X}{\partial u_{xx}} \right) u_x \\
  c_{01} &= 0, c_{12} = -3u \partial_x \left( \frac{\partial X}{\partial u_{xx}} \right) - 3 \frac{\partial_X u_{xx}}{\partial u_{xx}} u_x, c_{02} = -2u \frac{\partial X}{\partial u_{xx}}
\end{align*}
\]

Hence we can write

\[
(d_2 X)_{xy} = \sum_{k=0}^{3} b_k \delta^{(k)}(x - y) \tag{6.8}
\]

with \( \text{deg}(b_k) = 3 - k \). By straightforward calculation we get

\footnote{The triviality of first order deformations means that we can always kill the term of first order in \( \epsilon \) with a change of coordinates}
By using the formula

\[ b_0 = au_{xx} + bu_x u_{xx} + cu_x^3 = \left(-2u \frac{\partial s_0}{\partial u} + 2us_1 \right) u_{xxx} + \]

\[ \left( s_1 - 4u \frac{\partial^2 s_0}{\partial u^2} + 4u \frac{\partial s_1}{\partial u} - \frac{\partial s_0}{\partial u} \right) u_x u_{xx} + \left( \frac{u \partial^2 s_1}{\partial u^2} + \frac{1}{2} \frac{\partial s_1}{\partial u} - u \frac{\partial^3 s_0}{\partial u^3} - \frac{1}{2} \frac{\partial s_0}{\partial u^2} \right) u_x^3 \]

\[ b_1 = du_x + gu_x^2 = \left(-5u \frac{\partial s_0}{\partial u} + 4us_1 - s_0 \right) u_{xx} + \left( 2u \frac{\partial s_1}{\partial u} - 3u \frac{\partial^2 s_0}{\partial u^2} - \frac{\partial s_0}{\partial u} - s_1 \right) u_x^2 \]

\[ b_2 = hu_x = -3 \left( \frac{\partial s_0}{\partial u} + s_0 \right) u_x \]

\[ b_3 = l = -2us_0 \]

Now we can calculate explicitly the equations \( d_1d_2X = 0 \) in terms of the coefficients \( b_0, b_1, b_2, b_3 \).

By using the formula

\[ (d_1d_2)_{xyz} = \frac{1}{2} \sum_{t=0}^{3} \frac{\partial (d_2X)_{xy}}{\partial u^t(x)} \delta^{(t+1)}(x - z) - \frac{1}{2} \sum_{t=0}^{3} \frac{\partial (d_2X)_{xz}}{\partial u^t(y)} \delta^{(t+1)}(y - z) + \]

\[ \frac{1}{2} \sum_{t=0}^{3} \frac{\partial (d_2X)_{xx}}{\partial u^t(z)} \delta^{(t+1)}(z - y) - \frac{1}{2} \sum_{t=0}^{3} \frac{\partial (d_2X)_{xz}}{\partial u^t(y)} \delta^{(t+1)}(x - y) + \]

\[ \frac{1}{2} \sum_{t=0}^{3} \frac{\partial (d_2X)_{yy}}{\partial u^t(y)} \delta^{(t+1)}(y - x) - \frac{1}{2} \sum_{t=0}^{3} \frac{\partial (d_2X)_{yz}}{\partial u^t(z)} \delta^{(t+1)}(z - x) \]

that is

\[ (d_1d_2)_{xyz} = \frac{1}{2} \sum_{t=0}^{3} \left( \frac{\partial b_0}{\partial u^t(x)} \right) \delta(x - y) \delta^{(t+1)}(x - z) + \frac{1}{2} \sum_{t=0}^{2} \left( \frac{\partial b_1}{\partial u^t(x)} \right) \delta'(x - y) \delta^{(t+1)}(x - z) + \]

\[ + \frac{1}{2} \sum_{t=0}^{1} \left( \frac{\partial b_2}{\partial u^t(x)} \right) \delta(x - y) \delta^{(t+1)}(x - z) + \frac{1}{2} \sum_{t=0}^{1} \left( \frac{\partial b_3}{\partial u^t(x)} \right) \delta^{(3)}(x - y) \delta'(x - z) + \]

\[ - \frac{1}{2} \sum_{t=0}^{2} \left( \frac{\partial b_0}{\partial u^t(y)} \right) \delta(y - x) \delta^{(t+1)}(y - z) - \frac{1}{2} \sum_{t=0}^{2} \left( \frac{\partial b_1}{\partial u^t(y)} \right) \delta'(y - x) \delta^{(t+1)}(y - z) - \]

\[ \frac{1}{2} \sum_{t=0}^{1} \left( \frac{\partial b_2}{\partial u^t(y)} \right) \delta(y - x) \delta^{(t+1)}(y - z) - \frac{1}{2} \sum_{t=0}^{2} \left( \frac{\partial b_3}{\partial u^t(y)} \right) \delta^{(3)}(y - x) \delta'(y - z) + ... \]

In order to obtain a sum where appear only terms with \( \delta^{(i)}(x - y)\delta^{(j)}(x - z) \) and where the coefficients depend only on \( x \) we use the identity

\[ \delta(x - y)\delta(x - z) = \delta(y - x)\delta(y - z) = \delta(z - x)\delta(z - y) \]

(6.9)
and formula (6.1). For example we can write
\[ -\frac{1}{2} \sum_{t=0}^{3} \left( \frac{\partial b_0}{\partial u^t(y)} \right) \delta(y - x)\delta^{(t+1)}(y - z) = \]
\[ = (-1)^{t+1} \frac{1}{2} \sum_{t=0}^{3} \left( \frac{\partial b_0}{\partial u^t(y)} \right) \delta^{t+1}_x (\delta(y - x)\delta(y - z)) = \]
\[ = (-1)^{t+1} \frac{1}{2} \sum_{t=0}^{3} \left( \frac{\partial b_0}{\partial u^t(y)} \right) \delta^{t+1}_x (\delta(x - y)\delta(x - z)) = \]
\[ = \frac{1}{2} \sum_{t=0}^{3} \left( \frac{\partial b_0}{\partial u^t(x)} \right) \delta(x - y)\delta^{(t+1)}(x - z) = \]
\[ = \frac{1}{2} \sum_{t=0}^{3} \left( \frac{\partial b_0}{\partial u^t(x)} \right) \delta(x - y)\delta^{(t+1)}(x - z) \]

In this way, after a long but elementary calculation we obtain
\[ d_1 d_2 X = 2f \delta \wedge \delta^3 + 3f \delta^1 \wedge \delta^3 + 3f_{x} \delta \wedge \delta^2 + f_{xx} \delta \wedge \delta^1 \]  \hspace{1cm} (6.10)
where \( \delta^i \wedge \delta^j = \frac{1}{2} \left( \delta^{(i)}(x - y)\delta^{(j)}(x - z) - \delta^{(j)}(x - y)\delta^{(i)}(x - z) \right) \) and
\[ f = \frac{\partial b_1}{\partial u_x} - \partial_x \left( \frac{\partial b_1}{\partial u_{xx}} \right) - \frac{\partial b_2}{\partial u} + \partial_x \left( \frac{\partial b_3}{\partial u} \right) \]  \hspace{1cm} (6.11)

Therefore the equation \( d_1 d_2 X = 0 \) is equivalent to the equation \( f = 0 \). By substituting \( b_1, b_2, b_3 \) in this equation we get the condition
\[ s_1 = \frac{\partial s_0}{\partial u} \]  \hspace{1cm} (6.12)

### 6.2.1 Trivial deformations

The differentials \( d_1 \) and \( d_2 \) increase the degree by one. Because the degree of \( X \) is 2 the trivial deformations can be written as \( X = d_1 A + d_2 B = 0 \) where the degree of the densities of \( A \) and \( B \) is 1. But the variational derivative of local functionals with densities of degree 1 vanishes:
\[ \frac{\delta \int_{S_1} (f(u))u_x \, dx}{\delta u} = \sum_{i=0}^{1} (-1)^i \delta^i_x \left( \frac{\partial f}{\partial u^{(i)}} \right) = \frac{\partial f}{\partial u} u_x - \partial_x (f(u)) = 0 \]

Then all the deformations \( d_1 X_2^2 \) (with \( X_2^2 = s_0 u_{xx} + \frac{\partial s_0}{\partial u} u_x^2 \)) are not trivial.
6.2.2 Deformations: explicit form

By using the formula (2.19), we get

\[ d_1X = \sum_s \left( (\partial_{yy}^s \delta'(y - x)) \frac{\partial}{\partial u(y)}(su_{yy} + \frac{\partial s}{\partial u} u^2_y) - (\partial_{xx}^s \delta'(x - y)) \frac{\partial}{\partial u(x)}(su_{xx} + \frac{\partial s}{\partial u} u^2_x) \right) = \]

\[ \left( \frac{\partial s}{\partial u} u_{yy} + \frac{\partial^2 s}{\partial u^2} u^2_y \right) \delta^{(1)}(y - x) - \left( \frac{\partial s}{\partial u} u_{xx} + \frac{\partial^2 s}{\partial u^2} u^2_x \right) \delta^{(1)}(x - y) + \]

\[ \left( \frac{2 \partial s}{\partial u} u_y \right) \delta^{(2)}(y - x) - \left( \frac{2 \partial s}{\partial u} u_x \right) \delta^{(2)}(x - y) + s(y)\delta^{(3)}(y - x) - s(x)\delta^{(3)}(x - y) \]

By observing that

\[ \frac{\partial s}{\partial u} u_{xx} + \frac{\partial^2 s}{\partial u^2} u^2_x = \partial_x^2 s \]

and by using the identity (6.1) it is easy to get the formula

\[ P^{(2)}_2 = d_1 X^{(2)}_2 = -2s\delta^3(x - y) - 3\partial_x s\delta^2(x - y) - \partial_x^2 s\delta^1(x - y) \]  

(6.13)

6.3 Third order

The condition of compatibility \([P_1, P_2] = 0\) implies that \(P_2^3 = d_1 X^{(3)}_2\) and the Jacoby identity \([P_2, P_2] = 0\) implies that \(d_2 P^{(3)}_2 = -d_1 d_2 X^{(3)}_2 = 0\).

Remark

There are no conditions containing the second order deformations.

We start again calculating \(d_2 X\). From (2.19) it follows:

\[ (d_2 X)_{xy} = \]

\[ \left( \frac{1}{2} \partial_x X + c_{10} + c_{21} + c_{32} + c_{43} \right) \delta(x - y) + (X + c_{00} + c_{11} + c_{22} + c_{33}) \delta^{(1)}(x - y) + \]

\[ + (c_{01} + c_{12} + c_{23}) \delta^{(2)}(x - y) + (c_{02} + c_{13}) \delta^{(3)}(x - y) + c_{03} \delta^{(4)}(x - y) \]
where $X = s_0 u_{xxx} + s_1 u_x u_{xx} + s_2 u_x^3$ and

$$
c_{10} = -\partial_x \left( u \frac{\partial X}{\partial u} \right)
$$

$$
c_{21} = u \partial_x^2 \left( \frac{\partial X}{\partial u_x} \right) - \frac{1}{2} \frac{\partial X}{\partial u_x} u_{xx} + \frac{1}{2} u_x \partial_x \left( \frac{\partial X}{\partial u_x} \right)
$$

$$
c_{32} = -u \partial_x^3 \left( \frac{\partial X}{\partial u_{xx}} \right) - \frac{1}{2} \frac{\partial X}{\partial u_{xx}} u_{xxx} - \frac{1}{2} u_x \partial_x^2 \left( \frac{\partial X}{\partial u_{xx}} \right)
$$

$$
c_{43} = u \partial_x^4 \left( \frac{\partial X}{\partial u_{xxx}} \right) - \frac{1}{2} \frac{\partial X}{\partial u_{xxx}} u_{xxxx} + \frac{1}{2} u_x \partial_x^3 \left( \frac{\partial X}{\partial u_{xxx}} \right)
$$

$$
c_{00} = -2u \partial X \partial u_x
$$

$$
c_{11} = 2u \partial_x \left( \frac{\partial X}{\partial u_x} \right) - \frac{\partial X}{\partial u_x} u_x
$$

$$
c_{22} = -3u \partial_x^2 \left( \frac{\partial X}{\partial u_{xx}} \right) - 2 \frac{\partial X}{\partial u_{xx}} u_{xx} - u_x \partial_x \left( \frac{\partial X}{\partial u_{xx}} \right)
$$

$$
c_{33} = 4u \partial_x^3 \left( \frac{\partial X}{\partial u_{xxx}} \right) - 5 \frac{\partial X}{\partial u_{xxx}} u_{xxx} + \frac{3}{2} u_x \partial_x^2 \left( \frac{\partial X}{\partial u_{xxx}} \right)
$$

$$
c_{01} = 0
$$

$$
c_{12} = 3u \partial_x \left( \frac{\partial X}{\partial u_{xx}} \right) - 3 \frac{\partial X}{\partial u_{xx}} u_x
$$

$$
c_{23} = 6u \partial_x^2 \left( \frac{\partial X}{\partial u_{xxx}} \right) - 9 \frac{\partial X}{\partial u_{xxx}} u_{xx} + \frac{3}{2} u_x \partial_x \left( \frac{\partial X}{\partial u_{xxx}} \right)
$$

$$
c_{13} = 4u \partial_x \left( \frac{\partial X}{\partial u_{xxx}} \right) - 3 \frac{\partial X}{\partial u_{xxx}} u_x
$$

$$
c_{03} = 0
$$

Then we can write

$$
(d_2 X)_{xy} = \sum_{k=0}^{3} b_k \delta^{(k)}(x - y)
$$

(6.14)
with $\text{deg}(b_k) = 4 - k$ and

$$b_0 = au_{xxxx} + bu_xu_{xxx} + c(u_{xx})^2 + du_{xx}u_x + gu_x^4 =$$

$$= 0 + \left(-3u \frac{\partial s_1}{\partial u} + 3u \frac{\partial^2 s_0}{\partial u^2} + 6us_2\right) \left(u_xu_{xxx} + (u_{xx})^2\right) +$$

$$\left(-6u \frac{\partial^2 s_1}{\partial u^2} + 12u \frac{\partial s_2}{\partial u} - 3 \frac{\partial s_1}{\partial u} + 3s_2 + 6u \frac{\partial^3 s_0}{\partial u^3} + \frac{3}{2} \frac{\partial^2 s_0}{\partial u^2}\right) u_x^2u_{xx} +$$

$$+ \left(2u \frac{\partial^2 s_2}{\partial u^2} + \frac{\partial s_2}{\partial u} - \frac{1}{2} \frac{\partial^2 s_1}{\partial u^2} + u \frac{\partial^4 s_0}{\partial u^4} + \frac{1}{2} \frac{\partial^3 s_0}{\partial u^3} - u \frac{\partial^3 s_1}{\partial u^3}\right) u_x^4$$

$$b_1 = hu_{xxx} + lu_xu_{xx} + mu_x^3 =$$

$$= \left(2u \frac{\partial s_0}{\partial u} - us_1 - \frac{3}{2} s_0\right) u_{xxx} + \left(12us_2 - 3s_1 - 9u \frac{\partial s_1}{\partial u} + 12u \frac{\partial^2 s_0}{\partial u^2} + \frac{3}{2} \frac{\partial s_0}{\partial u}\right) u_xu_{xx} +$$

$$+ \left(4u \frac{\partial s_2}{\partial u} - 2s_2 - 3u \frac{\partial^2 s_1}{\partial u^2} - \frac{\partial s_1}{\partial u} + 4u \frac{\partial^3 s_0}{\partial u^3} + \frac{3}{2} \frac{\partial s_0}{\partial u^2}\right) u_x^3$$

$$b_2 = pu_{xx} + qu_x^2 =$$

$$= \left(-3us_1 + 6u \frac{\partial s_0}{\partial u} - \frac{9}{2} s_0\right) u_{xx} + \left(-3u \frac{\partial s_1}{\partial u} - 3s_1 + 6u \frac{\partial^2 s_0}{\partial u^2} + \frac{3}{2} \frac{\partial s_0}{\partial u}\right) u_x^2$$

$$b_3 = ku_x = \left(-2us_1 + 4u \frac{\partial s_0}{\partial u} - 3s_0\right) u_x$$
6.3.1 Deformations

In terms of coefficients $b_k$ the equation $(d_1d_2X)_{xy} = 0$ can be written simply by adding to the equation obtained in the previous case the terms

\[
\frac{1}{2} \left( \frac{\partial b_0}{\partial u_{xxxx}} \right) \delta(x-y)\delta^{(5)}(x-z) + \frac{1}{2} \left( \frac{\partial b_1}{\partial u_{xxxx}} \right) \delta^{(1)}(x-y)\delta^{(4)}(x-z)
\]

\[
+ \frac{1}{2} \left( \frac{\partial b_2}{\partial u_{xx}} \right) \delta^{(2)}(x-y)\delta^{(3)}(x-y) + \frac{1}{2} \left( \frac{\partial b_3}{\partial u_{xx}} \right) \delta^{(3)}(x-y)\delta^{(2)}(x-z)
\]

\[
- \frac{1}{2} \left( \frac{\partial b_0}{\partial u_{yyyy}} \right) \delta(y-x)\delta^{(5)}(y-z) - \frac{1}{2} \left( \frac{\partial b_1}{\partial u_{yyyy}} \right) \delta^{(1)}(y-x)\delta^{(4)}(y-z)
\]

\[
- \frac{1}{2} \left( \frac{\partial b_2}{\partial u_{yy}} \right) \delta^{(2)}(y-x)\delta^{(3)}(y-z) - \frac{1}{2} \left( \frac{\partial b_3}{\partial u_{yy}} \right) \delta^{(3)}(y-x)\delta^{(2)}(y-z)
\]

\[
+ \frac{1}{2} \left( \frac{\partial b_0}{\partial u_{zzzz}} \right) \delta(z-x)\delta^{(5)}(z-y) + \frac{1}{2} \left( \frac{\partial b_1}{\partial u_{zzzz}} \right) \delta^{(1)}(z-x)\delta^{(4)}(z-y)
\]

\[
+ \frac{1}{2} \left( \frac{\partial b_2}{\partial u_{zz}} \right) \delta^{(2)}(z-x)\delta^{(3)}(z-y) + \frac{1}{2} \left( \frac{\partial b_3}{\partial u_{zz}} \right) \delta^{(3)}(z-x)\delta^{(2)}(z-y)
\]

\[
- \frac{1}{2} \left( \frac{\partial b_0}{\partial u_{xxxx}} \right) \delta(x-z)\delta^{(5)}(x-y) - \frac{1}{2} \left( \frac{\partial b_1}{\partial u_{xxxx}} \right) \delta^{(1)}(x-z)\delta^{(4)}(x-y)
\]

\[
- \frac{1}{2} \left( \frac{\partial b_2}{\partial u_{xx}} \right) \delta^{(2)}(x-z)\delta^{(3)}(x-y) - \frac{1}{2} \left( \frac{\partial b_3}{\partial u_{xx}} \right) \delta^{(3)}(x-z)\delta^{(2)}(x-y)
\]

\[
+ \frac{1}{2} \left( \frac{\partial b_0}{\partial u_{yyyy}} \right) \delta(y-z)\delta^{(5)}(y-x) + \frac{1}{2} \left( \frac{\partial b_1}{\partial u_{yyyy}} \right) \delta^{(1)}(y-z)\delta^{(4)}(y-x)
\]

\[
+ \frac{1}{2} \left( \frac{\partial b_2}{\partial u_{yy}} \right) \delta^{(2)}(y-z)\delta^{(3)}(y-x) + \frac{1}{2} \left( \frac{\partial b_3}{\partial u_{yy}} \right) \delta^{(3)}(y-z)\delta^{(2)}(y-x)
\]

\[
- \frac{1}{2} \left( \frac{\partial b_0}{\partial u_{zzzz}} \right) \delta(z-y)\delta^{(5)}(z-x) - \frac{1}{2} \left( \frac{\partial b_1}{\partial u_{zzzz}} \right) \delta^{(1)}(z-y)\delta^{(4)}(z-x)
\]

\[
- \frac{1}{2} \left( \frac{\partial b_2}{\partial u_{zz}} \right) \delta^{(2)}(z-y)\delta^{(3)}(z-x) - \frac{1}{2} \left( \frac{\partial b_3}{\partial u_{zz}} \right) \delta^{(3)}(z-y)\delta^{(2)}(z-x)
\]

By introducing the function $f$ and $g$ defined by the formulae

\[
f = \frac{\partial b_1}{\partial u_x} - \partial_x \left( \frac{\partial b_1}{\partial u_{xx}} \right) + \partial_x^2 \left( \frac{\partial b_1}{\partial u_{xxx}} \right) - \partial_x \left( \frac{\partial b_1}{\partial u} \right) + \partial_x \left( \frac{\partial b_3}{\partial u} \right)
\]

\[
g = \frac{\partial b_1}{\partial u_{xxx}} - \frac{\partial b_2}{\partial u_{xx}} + \frac{\partial b_3}{\partial u_x}
\]

we can write the result in the form

\[
d_1d_2X = \sum f_{ij} \delta^i \wedge \delta^j \quad (6.15)
\]
where

\[
\begin{align*}
    f_{05} &= f_{23} = 2g \\
    f_{14} &= 5g \\
    f_{04} &= 5gx \\
    f_{13} &= 8gx \\
    f_{03} &= 2f + 4g_{xx} \\
    f_{12} &= 3f + 3g_{xx} \\
    f_{02} &= 3fx + g_{xxx} \\
    f_{01} &= fx \\
\end{align*}
\]

Therefore the equation \( d_1 d_2 X = 0 \) is equivalent to the equations \( f = 0 \) and \( g = 0 \).

By substituting the coefficients \( b_1, b_2 \) and \( b_3 \) in these equations we obtain that the last equation is identically satisfied and the first is equivalent to the condition

\[
2s_2 - \frac{\partial s_1}{\partial u} + \frac{\partial^2 s_0}{\partial u^2} = 0
\]

(6.16)

6.3.2 Trivial deformations

In this case trivial deformations are \( d_1A + d_2B \) with

\[
\begin{align*}
A &= \int_{S^1} (A_0(u)u_{xx} + A_1(u)u_x^2) \, dx \\
B &= \int_{S^1} (B_0(u)u_{xx} + B_1(u)u_x^2) \, dx
\end{align*}
\]

By using formula (2.18) we get

\[
d_1A + d_2B = \left( -4 \left( \frac{\partial^2 A_0}{\partial u^2} - \frac{\partial A_1}{\partial u} \right) - 4u \left( \frac{\partial^2 B_0}{\partial u^2} - \frac{\partial B_1}{\partial u} \right) - \frac{1}{2} \left( \frac{\partial^2 B_0}{\partial u^2} - \frac{\partial B_1}{\partial u} \right) \right) u_{xxx} + \\
&\quad + \left( - \left( \frac{\partial^3 A_0}{\partial u^3} - \frac{\partial^2 A_1}{\partial u^2} \right) - u \left( \frac{\partial^3 B_0}{\partial u^3} - \frac{\partial^2 B_1}{\partial u^2} \right) - \frac{1}{2} \left( \frac{\partial^2 B_0}{\partial u^2} - \frac{\partial B_1}{\partial u} \right) \right) u_x u_{xx}
\]

If we call \( \ddot{A} := A_1 - \frac{\partial A_0}{\partial u} \) and \( \ddot{B} := B_1 - \frac{\partial B_0}{\partial u} \), we can write

\[
d_1A + d_1B = \\
\left( 2\ddot{A} + 2u\ddot{B} \right) u_{xxx} + \left( 4 \frac{\partial \ddot{A}}{\partial u} + 4u \frac{\partial \ddot{B}}{\partial u} + \ddot{B} \right) u_x u_{xx} + \left( \frac{\partial^2 \ddot{A}}{\partial u^2} + 4u \frac{\partial^2 \ddot{B}}{\partial u^2} + \frac{\partial \ddot{B}}{\partial u} \right) u_x^3
\]

Now we can prove that all deformations \( P_2^{(3)} \) are trivial.
Proof
The trivial deformations satisfy equation (6.16). In fact
\[ 2 \left( \frac{\partial^2 \tilde{A}}{\partial u^2} + 4u \frac{\partial^2 \tilde{B}}{\partial u^2} + \frac{\partial \tilde{B}}{\partial u} \right) - \frac{\partial}{\partial u} \left( 4 \frac{\partial \tilde{A}}{\partial u} + 4u \frac{\partial \tilde{B}}{\partial u} + \tilde{B} \right) + \frac{\partial^2}{\partial u^2} \left( 2\tilde{A} + 2u\tilde{B} \right) = 0 \]

Every field \( X = s_0u_{xxx} + s_1u_xu_{xx} + s_2u_x^3 \) with coefficients \( s_0, s_1, s_2 \) satisfying equation (6.16) can be written as \( d_1A + d_2B \) by choosing
\[
\tilde{A} = \frac{s}{2} - \frac{u}{3} \left( 2 \frac{\partial s_0}{\partial u} - s_1 \right) \\
\tilde{B} = \frac{1}{3} \left( 2 \frac{\partial s_0}{\partial u} - s_1 \right)
\]
and this choice is always possible.

6.3.3 Deformations: explicit form

By using the formula (2.19)
\[
d_1X = \sum_{s \geq 0} \left( (\partial_y^s \delta'(y-x)) \frac{\partial}{\partial u^{(s)}(y)} (s_0u_{yyy} + s_1u_{yy}u_y + s_2u_y^3) + \right.
\]
\[
- (\partial_x^s \delta'(x-y)) \frac{\partial}{\partial u^{(s)}(x)} (s_0u_{xxx} + s_1u_xu_{xx} + s_2u_x^3) \right)
\]
and condition (6.16) it is easy to get the formula
\[
P_2^{(3)} = d_1X_2^{(3)} = -2t\delta^{(3)}(x-y) - 3\partial_x t\delta^{(2)}(x-y) - \partial_x^2 t\delta^{(1)}(x-y)
\]
where \( t \) is an arbitrary differential polynomial of degree 1.

6.4 Fourth order

In this section we consider the case: \( P_2 = P_2^{(0)} + \epsilon^2 P_2^{(2)} + \epsilon^4 P_2^{(4)} \).\footnote{The third order term can be always killed by changing coordinates and the second order term is \( P_2^{(2)} = d_1X_2^{(2)} \) with \( X_2^{(2)} = su_{xx} + \frac{\partial}{\partial u} u_x^2 \).}

The compatibility condition implies \( P_2^{(4)} = d_1X_2^{(4)} \) and the Jacoby identity \([P_2, P_2] = o(\epsilon^4)\) implies
\[
d_1d_2X_2^{(4)} - \frac{1}{2} [d_1X_2^{(2)}, d_1X_2^{(2)}] = 0
\]

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6.4.1 Deformation

First of all we consider the term \[d_1X_2^{(2)} \Delta_1 X_2^{(2)}\]. By using the formula (2.20) we get

\[
\begin{align*}
\left[d_1X_2^{(2)} \Delta_1 X_2^{(2)}\right] = & \quad \partial \left(d_1X_2^{(2)}\right)_{xy} \delta_x^s \left(d_1X_2^{(2)}\right)_{xz} - \partial \left(d_1X_2^{(2)}\right)_{yx} \delta_y^s \left(d_1X_2^{(2)}\right)_{yz} + \\
& + \partial \left(d_1X_2^{(2)}\right)_{xz} \delta_x^s \left(d_1X_2^{(2)}\right)_{zy} - \partial \left(d_1X_2^{(2)}\right)_{yz} \delta_y^s \left(d_1X_2^{(2)}\right)_{zy} + \\
& + \partial \left(d_1X_2^{(2)}\right)_{yx} \delta_y^s \left(d_1X_2^{(2)}\right)_{yz} - \partial \left(d_1X_2^{(2)}\right)_{zy} \delta_z^s \left(d_1X_2^{(2)}\right)_{zy} 
\end{align*}
\]

Let us focus our attention on the first term. By straightforward calculation we get

\[
\partial \left(d_1X_2^{(2)}\right)_{xy} \delta_x^s \left(d_1X_2^{(2)}\right)_{xz} = \sum_{i=1,\ldots,3; j=1,\ldots,6-i} b_{ij}(i)(x-y)(j)(x-z) \tag{6.18}
\]
where

\[ b_{11} = \left( 6 \frac{\partial^2 s}{\partial u^3} + 2 \frac{\partial s}{\partial u} \frac{\partial^3 s}{\partial u^4} \right) u_x^4 + \left( 14 \frac{(\partial^2 s)^2}{\partial u^2} + 14 \frac{\partial^3 s}{\partial u^3} \right) u_x^2 u_{xx} + \left( 8 \frac{\partial^2 s}{\partial u^2} \right) u_{xx}^2 + \left( 12 \frac{\partial s}{\partial u} \frac{\partial^2 s}{\partial u^2} \right) u_x u_{xxx} + 2 \left( \frac{\partial s}{\partial u} \right)^2 u_{xxxx} \]

\[ b_{12} = \left( 16 \frac{\partial^3 s}{\partial u^3} + 16 \left( \frac{\partial^2 s}{\partial u^2} \right)^2 \right) u_x^3 + \left( 52 \frac{\partial^2 s}{\partial u^2} \right) u_x u_{xx} + 10 \left( \frac{\partial s}{\partial u} \right)^2 u_{xxx} \]

\[ b_{21} = \left( 6 \frac{\partial^3 s}{\partial u^3} + 6 \left( \frac{\partial^2 s}{\partial u^2} \right)^2 \right) u_x^3 + \left( 24 \frac{\partial^2 s}{\partial u^2} \right) u_x u_{xx} + 6 \left( \frac{\partial s}{\partial u} \right)^2 u_{xxx} \]

\[ b_{13} = \left( 4 \frac{\partial^3 s}{\partial u^3} + 38 \frac{\partial^2 s}{\partial u^2} \right) u_x^3 + \left( 4 \frac{\partial^2 s}{\partial u^2} + 18 \left( \frac{\partial s}{\partial u} \right)^2 \right) u_{xx} \]

\[ b_{31} = \left( \frac{4}{u^2} \frac{\partial s}{\partial u} \right) u_x^2 + 4 \left( \frac{\partial s}{\partial u} \right)^2 u_{xx} \]

\[ b_{22} = \left( 42 \frac{\partial^2 s}{\partial u^2} \right) u_x^2 + 24 \left( \frac{\partial s}{\partial u} \right)^2 u_{xx} \]

\[ b_{23} = \left( 12 \frac{\partial^2 s}{\partial u^2} \right) u_x + 30 \left( \frac{\partial s}{\partial u} \right)^2 \]

\[ b_{32} = 12 \left( \frac{\partial s}{\partial u} \right)^2 u_x \]

\[ b_{14} = \left( 8 \frac{\partial^2 s}{\partial u^2} \right) u_x + 14 \left( \frac{\partial s}{\partial u} \right)^2 \]

\[ b_{24} = 12 \frac{\partial s}{\partial u} \]

\[ b_{33} = 8 \frac{\partial s}{\partial u} \]

\[ b_{15} = 4 \frac{\partial s}{\partial u} \]

The other terms in (6.17) have the same form. The only difference is that the variables \( x, y \) and \( z \) play a different role. Therefore we can apply the usual tricks and write an expression containing only terms with \( \delta^{(i)}(x - y)\delta^{(j)}(x - z) \). For example we can write
\[- \frac{\partial (d_1 X_2^{(2)})}{\partial u^s(y)} \partial_y^s \left( d_1 X_2^{(2)} \right)_{yz} = - \sum_{ij} b_{ij}(y) \delta^{(i)}(x - y) \delta^{(j)}(x - z) =
\]
\[- (-1)^{i+j} \sum_{ij} b_{ij}(y) \partial_x^i \partial_z^j \delta(y - x) \delta(y - z) =
\]
\[= - (-1)^{i+j} \sum_{ij} b_{ij}(y) \partial_x^i \partial_z^j \delta(x - y) \delta(x - z) =
\]
\[(1)^{i+1} \sum_{ij} b_{ij}(y) \partial_x^i \delta(x - y) \delta^{(j)}(x - z) =
\]
\[= - (-1)^{i+1} \sum_{ij} b_{ij}(y) \sum_{k=0}^{i} \left( \begin{array}{c} i \\ k \end{array} \right) \delta^{(k)}(x - y) \delta^{(j+i-k)}(x - z) =
\]
\[(1)^{i+1} \sum_{ij} \sum_{k=0}^{i} \sum_{l=0}^{k} \left( \begin{array}{c} i \\ k \\ l \end{array} \right) b_{ij}(y) \partial_x^{(l)} \delta^{(k-l)}(x - y) \delta^{(j+i-k)}(x - z)
\]

The final result is

\[
\left[ d_1 X_2^{(2)}, d_1 X_2^{(2)} \right] = 16 \left( s \frac{\partial s}{\partial u} \right) \delta^1 \wedge \delta^5 + 40 \left( s \frac{\partial s}{\partial u} \right) \delta^2 \wedge \delta^4 + 40 \partial_x \left( s \frac{\partial s}{\partial u} \right) \delta^1 \wedge \delta^4 + 
\]
\[+ 48 \partial_x \left( s \frac{\partial s}{\partial u} \right) \delta^2 \wedge \delta^3 + 32 \partial_x^2 \left( s \frac{\partial s}{\partial u} \right) \delta^1 \wedge \delta^3 + 8 \partial_x^3 \left( s \frac{\partial s}{\partial u} \right) \delta^1 \wedge \delta^2
\]

The calculation of the term $d_2 X_2^{(4)}$ can be done as above. In this case we have

\[
\left( d_2 X_2^{(4)} \right)_{xy} = \left( \frac{1}{2} \partial_x X + c_{10} + c_{21} + c_{32} + c_{43} + c_{54} \right) \delta(x - y)
\]
\[
(X + c_{00} + c_{11} + c_{22} + c_{33} + c_{44}) \delta^{(1)}(x - y) + (c_{01} + c_{12} + c_{23} + c_{34}) \delta^{(2)}(x - y)
\]
\[+ (c_{02} + c_{13} + c_{24}) \delta^{(3)}(x - y) + (c_{03} + c_{14}) \delta^{(4)}(x - y) + c_{04} \delta^{(5)}(x - y)
\]

where $X_2^{(4)} = X = s_0 u_{xxxx} + s_1 u_x u_{xxx} + s_2 (u_{xx})^2 + s_3 u_x^2 u_{xx} + s_4 u_x^4$ and the coefficients $c_{ij}$ have the
same expression in terms of $X_2^{(4)}$ that in the previous case, except for the new coefficients

\[
c_{54} = -u \frac{\partial^5 b}{\partial u_{xxxx}} - \frac{1}{2} \frac{\partial X}{\partial u_{xxxx}} u_{xxxxx} - \frac{1}{2} u_x \frac{\partial^4 b}{\partial u_{xxx}} \frac{\partial X}{\partial u_{xxxx}}
\]

\[
c_{44} = -5u \frac{\partial^4 b}{\partial u_{xxx}} - \frac{3}{2} \frac{\partial X}{\partial u_{xxxx}} u_{xxx} - 2u_x \frac{\partial^3 b}{\partial u_{xx}} \frac{\partial X}{\partial u_{xxxx}}
\]

\[
c_{34} = -10u \frac{\partial^3 b}{\partial u_{xx}} - \frac{7}{2} \frac{\partial X}{\partial u_{xxxx}} u_{xx} - 3u_x \frac{\partial^2 b}{\partial u_x} \frac{\partial X}{\partial u_{xxxx}}
\]

\[
c_{24} = -10u \frac{\partial^2 b}{\partial u_x} - \frac{8}{2} \frac{\partial X}{\partial u_{xxxx}} u_x - 2u_x \frac{\partial b}{\partial u} \frac{\partial X}{\partial u_{xxxx}}
\]

\[
c_{14} = -5u \frac{\partial b}{\partial u} - \frac{5}{2} \frac{\partial X}{\partial u_{xxxx}} u
\]

\[
c_{04} = -2u \frac{\partial X}{\partial u_{xxxx}}
\]

Consequently we can write

\[
\left( d_2 X_2^{(4)} \right)_{xy} = \sum_{k=0}^{5} b_k \delta^{(k)}(x - y)
\]

with \( \text{deg}(b_k) = 5 - k \) and

\[
b_0 = a_0 u_{xxxx} + a_1 u_x u_{xxxx} + a_2 u_{xx} u_{xxx} + a_3 u_x^2 u_{xxx} + a_4 u_x (u_{xx})^2 + a_5 u_x^3 u_{xx} + a_6 u_x^5 =
\]

\[
\left( 2u s_1 - 2us_2 - 2u \frac{\partial s_0}{\partial u} \right) u_{xxxx} + \left( s_1 - \frac{\partial s_0}{\partial u} - s_2 - 6u \frac{\partial s_2}{\partial u} + 6u \frac{\partial s_1}{\partial u} - 6u \frac{\partial^2 s_0}{\partial u^2} \right) u_x u_{xxx} + \left( -10u \frac{\partial s_2}{\partial u} + 10u \frac{\partial s_1}{\partial u} - 10u \frac{\partial^2 s_0}{\partial u^2} \right) u_{xxx} u_{xx} + \left( 2 \frac{\partial s_1}{\partial u} - 2 \frac{\partial s_2}{\partial u} - 2 \frac{\partial^2 s_0}{\partial u^2} + 12 u s_4 - 4u \frac{\partial s_3}{\partial u} + \right.
\]

\[
-6u \frac{\partial^2 s_2}{\partial u^2} + 10u \frac{\partial^2 s_1}{\partial u^2} - 10u \frac{\partial^3 s_0}{\partial u^3} \right) u_x^2 u_{xxx} + \left( -3 \frac{\partial s_2}{\partial u} + 3 \frac{\partial s_1}{\partial u} - 3 \frac{\partial^2 s_0}{\partial u^2} + 24 u s_4 - 8u \frac{\partial s_3}{\partial u} + \right.
\]

\[
-7u \frac{\partial^2 s_2}{\partial u^2} + 15u \frac{\partial^2 s_1}{\partial u^2} - 15u \frac{\partial^3 s_0}{\partial u^3} \right) u_x (u_{xx})^2 + \left( 6s_4 - \frac{3}{2} \frac{\partial s_3}{\partial u} - \frac{\partial^2 s_2}{\partial u^2} + \frac{3}{2} \frac{\partial s_1}{\partial u} - \frac{3}{2} \frac{\partial^3 s_0}{\partial u^3} + \right.
\]

\[
+24u \frac{\partial s_1}{\partial u} - 8u \frac{\partial^2 s_3}{\partial u^2} - 2u \frac{\partial^3 s_2}{\partial u^3} + 10u \frac{\partial^3 s_1}{\partial u^3} - 10u \frac{\partial^4 s_0}{\partial u^4} \right) u_x^3 u_{xxx} + \left( \frac{3}{2} \frac{\partial s_4}{\partial u} - \frac{1}{2} \frac{\partial^2 s_3}{\partial u^2} + \frac{1}{2} \frac{\partial^3 s_1}{\partial u^3} + \right.
\]

\[
-\frac{1}{2} \frac{\partial^4 s_0}{\partial u^4} + 3u \frac{\partial^4 s_4}{\partial u^4} - u \frac{\partial^3 s_3}{\partial u^3} + u \frac{\partial^4 s_1}{\partial u^4} - \frac{\partial^5 s_0}{\partial u^5} \right) u_x^5
\]
In terms of the formula we have found before:

\[
 b_1 = c_0 u_{xxxx} + c_1 u_x u_{xxx} + c_2 (u_{xx})^2 + c_3 u^2_{xx} u_x + c_4 u^4_x = \\
\left(-2s_0 - 7u \frac{\partial s_0}{\partial u} + 6u_1 - 6u_2\right) u_{xxxx} + \left(-2s_2 - s_1 - 2u \frac{\partial s_0}{\partial u} - 2u s_3 - 12u \frac{\partial s_2}{\partial u} + 16u \frac{\partial s_1}{\partial u} +
\right. \\
\left.-20u \frac{\partial^2 s_0}{\partial u^2}\right) u_x u_{xxx} + \left(-3s_2 - 2u s_3 - 8u \frac{\partial s_2}{\partial u} + 12u \frac{\partial s_1}{\partial u} - 15u \frac{\partial^2 s_0}{\partial u^2}\right) (u_{xx})^2 + \left(-5s_3 +
\right. \\
\left.-2u \frac{\partial s_2}{\partial u} + \frac{9}{2} \frac{\partial s_1}{\partial u} - 6 \frac{\partial^2 s_0}{\partial u^2} + 24u s_4 - 13u \frac{\partial s_3}{\partial u} - 6u \frac{\partial^2 s_2}{\partial u^2} + 24u \frac{\partial^2 s_1}{\partial u^3} - 30u \frac{\partial^3 s_0}{\partial u^3}\right) u^2_x u_{xx} + \\
\left.+ \left(-3s_4 - \frac{3}{2} \frac{\partial s_3}{\partial u} - \frac{3}{2} \frac{\partial^2 s_2}{\partial u^2} + 6u \frac{\partial s_4}{\partial u} - 3u \frac{\partial^2 s_3}{\partial u^2} + 4u \frac{\partial^3 s_1}{\partial u^3} - 5u \frac{\partial^4 s_0}{\partial u^4}\right) u^4_x \right)
\]

\[
 b_2 = d_0 u_{xxx} + d_1 u_x u_{xx} + d_2 u^3_x = \left(-6u s_2 + 6u s_1 - 10u \frac{\partial s_0}{\partial u} - 7s_0\right) u_{xxx} + \\
\left(-6u_2 - 3s_1 - 3 \frac{\partial s_0}{\partial u} - 6u s_3 - 6u \frac{\partial s_2}{\partial u} + 18u \frac{\partial s_1}{\partial u} - 30u \frac{\partial^2 s_0}{\partial u^2}\right) u_x u_{xxx} + \\
\left(-3s_3 - 3 \frac{\partial s_1}{\partial u} - 6 \frac{\partial^2 s_0}{\partial u^2} - 2u \frac{\partial s_3}{\partial u} + 6u \frac{\partial^2 s_1}{\partial u^2} - 10u \frac{\partial^3 s_0}{\partial u^3}\right) u^3_x \right)
\]

\[
 b_3 = p_0 u_x + p_1 u^2_x = \left(-8s_0 - 4u s_2 + 4u s_1 - 10u \frac{\partial s_0}{\partial u}\right) u_{xx} + \\
\left(-3s_1 - 2 \frac{\partial s_0}{\partial u} - 2u s_3 + 4u \frac{\partial s_1}{\partial u} - 10u \frac{\partial^2 s_0}{\partial u^2}\right) u^2_x \right)
\]

\[
 b_4 = q u_x = \left(-5s_0 - 5u \frac{\partial s_0}{\partial u}\right) u_x \right)
\]

\[
 b_5 = k = -2u s_0
\]

We are almost able to write the equation

\[
 d_1 d_2 X_2^{(4)} - \frac{1}{2} \left[ d_1 X_2^{(2)}, d_1 X_2^{(2)} \right] = 0
\]

(6.19)

in terms of \( b_k \). In fact to obtain the term \( d_1 d_2 X_2^{(4)} \) it is sufficient to add the following terms to the formula we have found before:

\[
\frac{1}{2} \frac{\partial b_4}{\partial u} \delta^{(4)}(x - y) \delta^{(1)}(x - z) + \frac{1}{2} \frac{\partial b_5}{\partial u} \delta^{(5)}(x - y) \delta^{(1)}(x - z) + \\
\frac{1}{2} \frac{\partial b_4}{\partial u} \delta^{(4)}(x - y) \delta^{(2)}(x - z) + \frac{1}{2} \frac{\partial b_3}{\partial u} \delta^{(3)}(x - y) \delta^{(3)}(x - z) + \\
\frac{1}{2} \frac{\partial b_2}{\partial u} \delta^{(2)}(x - y) \delta^{(4)}(x - z) + \frac{1}{2} \frac{\partial b_1}{\partial u} \delta^{(1)}(x - y) \delta^{(5)}(x - z) + \\
\frac{1}{2} \frac{\partial b_0}{\partial u} \delta(x - y) \delta^{(6)}(x - z) + ...
\]

and to rewrite the terms in the usual way.
Defining the functions \( f, g, h \) in the following way:

\[
\begin{align*}
  f &= \frac{\partial b_5}{\partial u} - \frac{\partial b_1}{\partial u_x} + \frac{\partial b_2}{\partial u_{xxx}} - \frac{\partial b_1}{\partial u_{xxxx}} \\
  g &= \frac{\partial b_1}{\partial u_{xxx}} - \frac{\partial b_2}{\partial u_{xx}} + \frac{\partial b_3}{\partial u_x} - \frac{\partial b_1}{\partial u} - 2\partial_x \left( \frac{\partial b_1}{\partial u_{xxx}} \right) + \partial_x \left( \frac{\partial b_2}{\partial u_{xx}} \right) - \partial_x \left( \frac{\partial b_1}{\partial u_x} \right) + 2\partial_x \left( \frac{\partial b_5}{\partial u} \right) \\
  h &= \frac{\partial b_1}{\partial u_x} - \partial_x \left( \frac{\partial b_1}{\partial u_{xx}} \right) + \partial_x^2 \left( \frac{\partial b_1}{\partial u_{xxx}} \right) - \partial_x^2 \left( \frac{\partial b_1}{\partial u_{xxxx}} \right) - \frac{\partial b_2}{\partial u_x} + \partial_x \left( \frac{\partial b_3}{\partial u} \right) - \partial_x^2 \left( \frac{\partial b_4}{\partial u} \right) + \partial_x^3 \left( \frac{\partial b_5}{\partial u} \right)
\end{align*}
\]

we have

\[
d_1 d_2 X = 2f \delta^1 \wedge \delta^5 + 5f \delta^2 \wedge \delta^4 + 2g \delta \wedge \delta^5 + (5f_x + 5g) \delta^1 \wedge \delta^4 + (6f_x + 2g) \delta^2 \wedge \delta^3 + 5g_x \delta \wedge \delta^4 + (4f_{xxx} + 8g_x) \delta^1 \wedge \delta^3 + (2h + 4g_{xx}) \delta \wedge \delta^3 + (f_{xxx} + 3h + 3g_{xx}) \delta^1 \wedge \delta^2 + (3h_x + g_{xxx}) \delta \wedge \delta^2 + h_{xx} \delta \wedge \delta^1
\]

Then the equation \( d_1 d_2 X_2^{(4)} - \frac{1}{2} \left[ d_1 X_2^{(2)}, d_1 X_2^{(2)} \right] = 0 \) is equivalent to the conditions

\[
\begin{align*}
  f - 4s \frac{\partial s}{\partial u} &= 0 \\
  g &= 0 \\
  h &= 0
\end{align*}
\]

The first equation connects the second order deformation with the fourth order deformation:

\[
s_0 = -2s \frac{\partial s}{\partial u} \tag{6.20}
\]

The second and the third equation give the conditions

\[
\begin{align*}
  s_1 - s_2 &= \frac{\partial s_0}{\partial u} \\
  3s_4 - \frac{\partial s_3}{\partial u} + \frac{\partial^2 s_2}{\partial u^2} &= 0 \tag{6.22}
\end{align*}
\]

### 6.4.2 Trivial deformations

In this case trivial deformations can be written as \( d_1 A + d_2 B \) with

\[
\begin{align*}
  A &= \int_{S^1} (A_0(u)u_{xxx} + A_1(u)u_xu_{xx} + A_2(u)u_x^3)dx \\
  B &= \int_{S^1} (B_0(u)u_{xxx} + B_1(u)u_xu_{xx} + B_2(u)u_x^3)dx
\end{align*}
\]

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A straightforward calculation shows that, for trivial deformations, the coefficients $s_i$ are

$$s_0 = 0$$  
$$s_1 = s_2 = 3\tilde{A} + 3u\tilde{B}$$  
$$s_3 = 6\frac{\partial \tilde{A}}{\partial u} + 6u\frac{\partial \tilde{B}}{\partial u} + \frac{3}{2}\tilde{B}$$  
$$s_4 = \frac{\partial^2 \tilde{A}}{\partial u^2} + 6u\frac{\partial^2 \tilde{B}}{\partial u^2} + \frac{1}{2}\frac{\partial \tilde{B}}{\partial u}$$

with

$$\tilde{A} = 2A_2 - \frac{\partial A_1}{\partial u} + \frac{\partial^2 A_0}{\partial u^2}$$  
$$\tilde{B} = 2B_2 - \frac{\partial B_1}{\partial u} + \frac{\partial^2 B_0}{\partial u^2}$$

Now we are able to prove that $P_2^{(4)}$ is trivial if and only if $s_0 = 0$.

**Proof**

Trivial deformations satisfy equations

$$s_0 = 0$$  
$$s_1 = s_2$$  
$$3s_3 - \frac{\partial s_3}{\partial u} + \frac{\partial^2 s_2}{\partial u^2} = 0$$

In fact

$$3\frac{\partial^2}{\partial u^2} \left( \tilde{A} + \tilde{B} \right) - \frac{3}{2} \frac{\partial \tilde{B}}{\partial u} - \frac{\partial}{\partial u} \left( 6\frac{\partial}{\partial u} \left( \tilde{A} + \tilde{B} \right) - \frac{9}{2}\tilde{B} \right) + \frac{\partial^2}{\partial u^2} \left( 3 \left( \tilde{A} + \tilde{B} \right) \right) = 0$$

Conversely, we can always write a deformation $X$ (with $s = 0$) as $X = d_1 A + d_2 B$. It is sufficient to choose $A$ and $B$ in such way that the following equations hold:

$$\tilde{B} = \frac{4}{9} \frac{\partial s_3}{\partial u} - \frac{2}{9} s_3$$  
$$\tilde{A} = \frac{1}{3} s_2 - \tilde{B}$$

### 6.4.3 Deformations: explicit form

By using the formula (2.19)

$$d_1 X = \sum_s \left( \frac{\partial^s \delta'(y - x)}{\partial u^{(s)}(y)} \left( s_0 u_{yyyy} + s_1 u_y u_{yyyy} + s_2 u_{yy}^2 + s_3 u_y^4 \right) + \frac{\partial^s \delta'(x - y)}{\partial u^{(s)}(x)} \left( s_0 u_{xxxx} + s_1 u_x u_{xxxx} + s_2 u_{xx}^2 + s_3 u_x^4 \right) \right)$$

To avoid any confusion with the other functions $s_0$ defined in this paper, in the theorem 4 we have substituted $s_0$ for $\tilde{s}$. 

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and conditions (6.20), (6.21) and (6.22) it is easy to get the formula

\[ d_1 X^{(4)}_2 = -2s_0 \delta^{(5)}(x - y) - 5 (\partial_x s_0) \delta^{(4)}(x - y) - 10 (\partial_x^2 s_0) \delta^{(3)}(x - y) + \]
\[ -10 (\partial_x^3 s_0) \delta^{(2)}(x - y) - 3 (\partial_x^4 s_0) \delta^{(1)}(x - y) + 2w \delta^{(3)}(x - y) + \]
\[ + (\partial_x w) \delta^{(1)}(x - y) \]

whith

\[ w = w_0 u_{xx} + w_1 u_x^2 = 2 \frac{\partial s_0}{\partial u} u_{xx} + w_1 u_x^2 \]

where \( w_1 \) is an arbitrary function of \( u \) and \( s_0 \) is related to the function \( s \) appearing in second order deformation by the equation

\[ s_0 = -2s \frac{\partial s}{\partial u} \]

7 Quasi-triviality

Definition 16 The group of transformations

\[ u \rightarrow \bar{u} = \sum_k \epsilon^k \frac{F_k(u, u_x, u_{xx}, \ldots)}{G_k(u, u_x, u_{xx}, \ldots)} \]

where \( F_k, G_k \in A, \ deg(F_k) - deg(G_k) = k \) and \( \frac{\partial F_k}{\partial u} \neq 0 \) is called quasi-Miura group.

A deformation is called quasi-trivial if it can be eliminated by the action of the quasi-Miura group. We have seen that, in general, the second and fourth order deformations are not trivial. In this section we show that they are quasi-trivial.

7.1 Second order

Theorem 6 If \( X^{(2)}_2 = d_1 A + d_2 B \), where \( A \) and \( B \) are local functionals whose densities are ratios of differential polynomials, then the second order deformation \( P_2 = P_1^{(0)} - \lambda(P_2^{(0)} + \epsilon^2 d_1 X^{(2)}_2) \) is trivial.

Proof

\[ X^{(2)}_2 = d_1 A + d_2 B \Rightarrow P_2^{(2)} = -d_2 d_1 B \]

This implies

\[ P_2^{(2)} = \text{Lie}_X P_2^{(0)} \]
\[ \text{Lie}_X P_1^{(0)} = 0 \]

with

\[ X = -d_1 B \]
**Theorem 7** All second order deformations are quasi-trivial.

**Proof**

If we choose

\[ A = \int_{S^1} (a(u) \frac{u_{xx}}{u_x}) dx \]  
(7.3)

\[ B = \int_{S^1} (b(u) \frac{u_{xx}}{u_x}) dx \]  
(7.4)

By using formula (2.18) we get

\[ d_1 A + d_2 B = \]
\[ = \left( 2 \frac{\partial^2 a}{\partial u^2} + 2u \frac{\partial^2 b}{\partial u^2} + \frac{1}{2} \frac{\partial b}{\partial u} \right) u_{xx} - \left( \frac{\partial a}{\partial u} + u \frac{\partial b}{\partial u} \right) \frac{u_{xx}^2}{u_x^2} + \]
\[ \left( \frac{\partial a}{\partial u} + u \frac{\partial b}{\partial u} \right) \frac{u_{xxx}}{u_x} + \left( \frac{\partial^3 a}{\partial u^3} + u \frac{\partial^3 b}{\partial u^3} + \frac{1}{2} \frac{\partial^2 b}{\partial u^2} \right) u_x^2 \]

If we put

\[ a(u) = -ub(u) + \int_{u_0}^{u} b(u) du \]  
(7.5)

Then

\[ \frac{\partial a}{\partial u} = -u \frac{\partial b}{\partial u} \]
\[ \frac{\partial^2 a}{\partial u^2} = - \frac{\partial b}{\partial u} - u \frac{\partial^2 b}{\partial u^2} \]
\[ \frac{\partial^3 a}{\partial u^3} = -2 \frac{\partial^2 b}{\partial u^2} - u \frac{\partial^3 b}{\partial u^3} \]

and these equations imply

\[ d_1 A + d_2 B = -\frac{3}{2} \left( \frac{\partial b}{\partial u} u_{xx} + \frac{\partial^2 b}{\partial u^2} u_x^2 \right) \]  
(7.6)

that is equal to \( X_2^{(2)} = s(u)u_{xx} + \frac{\partial s}{\partial u} u_x^2 \) if we choose

\[ b(u) = -\frac{2}{3} \int_{u_0}^{u} s(u) du \]  
(7.7)

### 7.2 Fourth order

**Lemma 1** A fourth order deformation is quasi-trivial if and only if there exists a vector field \( Y \) such that

\[ \text{Lie}_Y P_1^{(0)} = 0 \]  
(7.8)

\[ P_2^{(4)} - \frac{1}{2} \text{Lie}_X P_2^{(0)} = \text{Lie}_Y P_2^{(0)} \]  
(7.9)

where \( X \) is the vector field (7.2).
Proof

The reduction of the fourth order deformation to the form \( P^{(0)}_1 - \lambda P^{(0)}_2 \) can be achieved in two steps.

In the first step one kills the second order part of the deformation \( P^{(0)}_2 \) with the quasi-Miura transformation generated by the vector field (7.2):

\[
P^{(0)}_1 - \lambda \left( P^{(0)}_2 + \epsilon^2 P^{(2)}_2 + \epsilon^4 P^{(4)}_2 + O(\epsilon^5) \right) \rightarrow P^{(0)}_1 - \lambda \left( P^{(0)}_2 + \epsilon^4 \left( P^{(4)}_2 - \frac{1}{2} \text{Lie}_X P^{(0)}_2 + O(\epsilon^5) \right) \right).
\]

In the second step one kills the fourth order part of the deformation \( P^{(4)}_2 - \frac{1}{2} \text{Lie}_X P^{(0)}_2 \) with the quasi-Miura transformation generated by the vector field \( Y \).

**Theorem 8** All fourth order deformations are quasi-trivial.

To prove the theorem we need the following

**Lemma 2** If \( X^{(4)}_2 = d_1 \tilde{A} + d_2 \tilde{B} - \frac{1}{2} [d_1 B, d_2 B] \) where

\[
B = \int_{s^1} b(u) \frac{u_{xx}}{u_x} \ dx
\]

and \( b(u) \) is given by the expression (7.7), then the fourth order deformation

\[
P^{(0)}_1 - \lambda \left( P^{(0)}_2 + \epsilon^4 \left( P^{(4)}_2 - \frac{1}{2} \text{Lie}_X P^{(0)}_2 \right) \right)
\]

is trivial.

Proof

\( P^{(4)}_2 = d_1 X^{(4)}_2 = d_1 d_2 \tilde{B} - d_1 \frac{1}{2} [d_1 B, d_2 B] \)

By using the graded Jacobi identity we get

\[
P^{(4)}_2 - \frac{1}{2} \text{Lie}_X P^{(0)}_2 = \text{Lie}_Y P^{(0)}_2 \tag{7.10}
\]

where \( X = -d_1 B \) and \( Y = -d_1 \tilde{B} \). Moreover

\[
\text{Lie}_Y P^{(0)}_1 = 0
\]

Then to prove the theorem it is sufficient to show that the vector field \( X^{(4)}_2 = s_0 u_{xxxx} + s_1 u_x u_{xxx} + s_2 u_x^2 u_{xx} + s_3 u_x^2 u_{xx} + s_4 u_x^4 \) \(^9\) can be written in the form

\[
X^{(4)}_2 = d_1 \tilde{A} + d_2 \tilde{B} - \frac{1}{2} [d_1 B, d_2 B]
\]

\(^9\)We recall that the coefficients of the vector field \( X^{(4)}_2 \) satisfy the condition (6.20), (6.21) and (6.22)
We start calculating the term $-\frac{1}{2}[d_1 B, d_2 B]$. In this case the Schouten bracket coincides with the commutator of the vector fields

$$d_1 B = 2 \frac{\partial^2 b}{\partial u^2} u_{xx} - \frac{\partial b}{\partial u} u_x^2 + \frac{\partial b}{\partial u} u_{xxx} + \frac{\partial^3 b}{\partial u^3} u_x^2$$

and

$$d_2 B = ud_1 B + \frac{1}{2} \left( \frac{\partial b}{\partial u} u_{xx} + \frac{\partial^2 b}{\partial u^2} u_x^2 \right)$$

The result is

$$-\frac{1}{2}[d_1 B, d_2 B] = -\frac{1}{2} \sum_s \left( \frac{\partial_x (d_1 B)}{\partial u^{(s)}(x)} - \frac{\partial_x (d_2 B)}{\partial u^{(s)}(x)} \right) =$$

$$u_{xxxx} \left( \frac{33}{4} \frac{\partial^4 b}{\partial u^4} \right) + u_x u_{xxx} \left( \frac{27}{4} \left( \frac{\partial^2 b}{\partial u^2} \right)^2 + \frac{45}{4} \frac{\partial b}{\partial u} \frac{\partial^3 b}{\partial u^3} \right) +$$

$$+ u_{xx}^2 \left( -\frac{3}{4} \left( \frac{\partial^2 b}{\partial u^2} \right)^2 + 15 \frac{\partial b}{\partial u} \frac{\partial^4 b}{\partial u^4} \right) + u_x^2 u_{xxx} \left( \frac{45}{2} \frac{\partial^2 b}{\partial u^2} \frac{\partial^3 b}{\partial u^3} + \frac{21}{2} \frac{\partial b}{\partial u} \frac{\partial^4 b}{\partial u^4} \right) +$$

$$+ u_x^4 \left( \frac{9}{4} \frac{\partial^2 b}{\partial u^2} \frac{\partial^4 b}{\partial u^4} + \frac{3}{2} \frac{\partial^3 b}{\partial u^3} \frac{\partial b}{\partial u} + \frac{3}{4} \left( \frac{\partial^3 b}{\partial u^3} \right)^2 \right) + \frac{9}{2} \frac{u_{xxx}^2}{u_x^2} \left( \frac{\partial b}{\partial u} \right)^2 +$$

$$- \frac{9}{u_x^2} \frac{u_{xx}^2}{u_x} \left( \frac{\partial b}{\partial u} \right)^2 + 2 \frac{u_{xxxx} u_{xxx}}{u_x} \left( \frac{\partial b}{\partial u} \right)^2 + \frac{27}{2} \frac{\partial b}{\partial u} \frac{\partial^2 b}{\partial u^2} +$$

$$+ \frac{u_x^3}{u_x^2} \left( \frac{33}{4} \frac{\partial b}{\partial u} \frac{\partial^2 b}{\partial u^2} \right) + 18 \frac{u_{xx}^2 u_{xxx}}{u_x^2} \left( \frac{\partial b}{\partial u} \right)^2 - 6 \frac{u_{xx} u_{xxx}}{u_x^2} \left( \frac{\partial b}{\partial u} \right)^2$$

The problem now is to guess the form of the local functionals $\tilde{A}$ and $\tilde{B}$. We know that the degree of $\tilde{A}$ and $\tilde{B}$ is equal to 3 but all ratios of differential polynomials of the form

$$\frac{P(u, u_x, u_{xx}, \ldots)}{Q(u, u_x, u_{xx}, \ldots)}$$

where $\text{deg}(P) - \text{deg}(Q) = 3$, are allowed.

The form of the coefficients in (7.11) suggests that $\tilde{A}$ and $\tilde{B}$ contain only terms of the form

$$\frac{P(u, u_x, u_{xx}, \ldots)}{u_x^i}$$

for some integer $i$. 

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Let us try with the local functionals

\[ \tilde{a} = \int_{s^1} \left( h_1(u) \frac{u_{xx}^2}{u_x} + h_2(u) \frac{u_x^3}{u_x} + h_3(u) \frac{u_{xxx}u_{xxxx}}{u_x^2} + h_4(u) \frac{u_{xxxx}}{u_x^3} + h_5(u) \frac{u_{xx}^2u_{xxxx}}{u_x^4} + h_6(u) \frac{u_{xx}^4}{u_x^5} + \right. \\
+ h_7(u) \frac{u_{xxx}u_{xxxx}}{u_x^3} + h_8(u) \frac{u_{xxxx}^2}{u_x^2} + h_9(u) \frac{u_{xxxx}u_{xxxx}}{u_x^4} + h_{10}(u) \frac{u_{xxxx}u_{xxxx}}{u_x^5} + h_{11}(u) \frac{u_{xxxx}^2}{u_x^6} \left. \right) \, dx \]

\[ \tilde{b} = \int_{s^1} \left( k_1(u) \frac{u_{xx}^2}{u_x} + k_2(u) \frac{u_x^3}{u_x} + k_3(u) \frac{u_{xxx}u_{xxxx}}{u_x^2} + k_4(u) \frac{u_{xxxx}}{u_x^3} + k_5(u) \frac{u_{xx}^2u_{xxxx}}{u_x^4} + k_6(u) \frac{u_{xx}^4}{u_x^5} + \right. \\
+ k_7(u) \frac{u_{xxx}u_{xxxx}}{u_x^3} + k_8(u) \frac{u_{xxxx}^2}{u_x^2} + k_9(u) \frac{u_{xxxx}u_{xxxx}}{u_x^4} + k_{10}(u) \frac{u_{xxxx}u_{xxxx}}{u_x^5} + k_{11}(u) \frac{u_{xxxx}^2}{u_x^6} \left. \right) \, dx \]

In order to determine the exact form of the coefficients \( h_i \) and \( k_i \) one has just to compare \( d_1 \tilde{a} + d_2 \tilde{b} \) with \( \frac{1}{2}[d_1 B, d_2 B] \).

The first step is to kill the coefficients of rational terms in \( d_1 \tilde{a} + d_2 \tilde{b} \) that don’t appear in \( -\frac{1}{2}[d_1 B, d_2 B] \) and the second step is to eliminate the remaining rational terms.

The calculations are very long, we give here only the final result: \(^{10}\)

\[
\begin{align*}
    h_i(u) &= -uk_i(u), \quad i = 1, \ldots, 11 \\
    k_1(u) &= -\frac{1}{28}k_8(u) - \frac{15}{56}k_2(u) \\
    k_3(u) &= 4k_1(u) \\
    k_4(u) &= 0 \\
    k_5(u) &= -6k_8(u) \\
    k_6(u) &= 4k_8(u) \\
    k_7(u) &= k_8(u) \\
    k_9(u) &= k_{10}(u) = k_{11}(u) = 0 \\
    k_{2}(u) + \frac{\partial k_8}{\partial u} &= \frac{7}{5} \left( \frac{\partial b}{\partial u} \right)^2
\end{align*}
\]

With this choice and taking into account (7.7), by using formula (2.18) we get

\[
-\frac{1}{2}[d_1 B, d_2 B] + d_1 \tilde{a} + d_2 \tilde{b} = \\
-2s \frac{\partial s}{\partial u} u_{xxxx} + \left( -\frac{7}{3} \frac{\partial^2 s}{\partial u^2} - \frac{13}{3} \left( \frac{\partial s}{\partial u} \right)^2 \right) u_x u_{xxx} + \left( -\frac{1}{3} s \frac{\partial^2 s}{\partial u^2} - \frac{7}{3} \left( \frac{\partial s}{\partial u} \right)^2 \right) u_{xx}^2 + \\
+ \left( \frac{5}{3} s \frac{\partial^3 s}{\partial u^3} - 4 \frac{\partial s}{\partial u} \frac{\partial^2 s}{\partial u^2} \right) u_{xx}^2 + \left( \frac{1}{3} \left( \frac{\partial^2 s}{\partial u^2} \right)^2 + \frac{\partial s}{\partial u} \frac{\partial^3 s}{\partial u^3} + \frac{2}{3} \frac{\partial^4 s}{\partial u^4} \right) u_x^4 = \\
= s'_{0} u_{xxxx} + s'_{1} u_x u_{xxx} + s'_{2} u_{xx}^2 + s'_{3} u_{xx}^2 + s'_{4} u_x^4
\]

\(^{10}\)In fact there is a certain freedom in the choice of coefficients \( h_i \) and \( k_i \); for example it is not necessary to put \( k_{4}(u) \) equal to 0.
First of all we observe that \( s'_0 \) is equal to \( s_0 \). Moreover the coefficients \( s'_i \) satisfy equations (6.21) and (6.22):

\[
s'_1 - s'_2 = \frac{\partial}{\partial u} \left( -2s \frac{\partial s}{\partial u} \right)
\]
\[
3s'_4 - \frac{\partial s'_4}{\partial u} + \frac{\partial^2 s'_2}{\partial u^2} = \left( \frac{\partial^2 s}{\partial u^2} \right)^2 + 3 \frac{\partial s}{\partial u} \frac{\partial^2 s}{\partial u^2} + 2s \frac{\partial^4 s}{\partial u^4} - \frac{5}{3} \frac{\partial^4 s}{\partial u^4} + \frac{7}{3} \frac{\partial s}{\partial u} \frac{\partial^3 s}{\partial u^3} + 4 \left( \frac{\partial^2 s}{\partial u^2} \right)^2 - \frac{16}{3} \frac{\partial s}{\partial u} \frac{\partial^3 s}{\partial u^3} - 5 \left( \frac{\partial^2 s}{\partial u^2} \right)^2 - \frac{1}{3} \frac{\partial^4 s}{\partial u^4} = 0
\]

Then the vector field

\[
X - \left( -\frac{1}{2} [d_1 B, d_2 B] + d_1 \tilde{a} + d_2 \tilde{b} \right) = \\
= (s_0 - s'_0)u_{xxxx} + (s_1 - s'_1)u_x u_{xxx} + (s_2 - s'_2)u_{xx}^2 + (s_3 - s'_3)u_x^2 u_{xx} + (s_4 - s'_4)u_x^4
\]

is trivial. Indeed:

\[
s_0 - s'_0 = 0
\]

Moreover the equations

\[
s_1 - s_2 = \frac{\partial s_0}{\partial u}
\]
\[
s_1 - s_2 = \frac{\partial s_0}{\partial u}
\]

imply

\[
s_1 - s'_1 = s_2 - s'_2
\]

Finally

\[
3(s_4 - s'_4) - \frac{\partial}{\partial u} (s_3 - s'_3) + \frac{\partial^2}{\partial u^2} (s_2 - s'_2) = \\
3s_4 - \frac{\partial}{\partial u} s_3 + \frac{\partial^2}{\partial u^2} s_2 - \left( 3s'_4 - \frac{\partial}{\partial u} s'_3 + \frac{\partial^2}{\partial u^2} s'_2 \right) = 0
\]

This means that

\[
X - \left( -\frac{1}{2} [d_1 B, d_2 B] + d_1 \tilde{a} + d_2 \tilde{b} \right) = d_1 a + d_2 b
\]

that is

\[
X = -\frac{1}{2} [d_1 B, d_2 B] + d_1 (\tilde{a} + a) + d_2 (\tilde{b} + b)
\]
8 Conclusions

In this paper we have studied the problem of classification of deformations of bihamiltonian structure of hydrodynamic type.

The main result of the paper is that, up to the fourth order, any deformation can be reduced to the form (3.15) depending only on a functional parameter $s(u)$.

These deformations give rise to an infinite hierarchy of “almost-commuting” hamiltonian equations.

In this paper we started studying numerically one of the equations of the deformed hierarchy corresponding to second order deformations.

The results obtained indicate the existence of the analogue of 2-soliton solutions at least for small times and for small amplitudes, but a deeper analysis is still necessary.

We have seen that one of the consequence of classification theorem is that deformations of the Magri bihamiltonian structure are trivial up to fourth order. This fact suggests that in our classification problem the Magri bihamiltonian structure is a stable object according to the definition of Arnold. \footnote{A classification problem can be thought as a decomposition of a space of objects in equivalence classes (see [2]); in our case the objects are deformations; two deformations are equivalent if and only if they can be reduced to the same form by the action of Miura group. An object is called stable if a sufficiently small “neighbourhood” of this object contains only objects of the same class.}

It would be interesting to investigate if this situation is more general, that is, if completely integrable systems of bihamiltonian type are the stable objects of a corresponding classification problem. For example it is possible to apply the same techniques used in this paper to the “symplectic” case where the leading term in the parameter $\epsilon$ has the form

$$P_{1,2} = h^{ij}_{1,2} \delta(x - y)$$  \hspace{1cm} (8.1)

with

$$h^{ij}_{1} = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & -1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & -1 & 0
\end{pmatrix}$$
\[
h^{ij}_2 = \begin{pmatrix}
0 & u_1 & 0 & 0 & ... & 0 & 0 \\
-u_1 & 0 & 0 & 0 & ... & 0 & 0 \\
0 & 0 & 0 & u_2 & ... & 0 & 0 \\
0 & 0 & -u_2 & 0 & ... & 0 & 0 \\
& & & & & & \\
& & & & & & \\
0 & 0 & 0 & 0 & ... & 0 & u_n \\
0 & 0 & 0 & 0 & ... & -u_n & 0
\end{pmatrix}
\]

Also in this case the the cohomology groups \( H^1 \) and \( H^2 \) associated to the differentials \( d_{P_1} \) and \( d_{P_2} \) are trivial (see [7]) and then the non trivial deformations are classified by the group

\[
\frac{\ker (d_1d_2)}{(\text{Im}(d_1) + \text{Im}(d_2))}
\]

Finally we observe that the result about quasi-triviality suggests that this property is a consequence of the definition of deformation and it is not an additional constraint.

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