Large-scale instability in hydrodynamics with stable temperature stratification driven by small-scale helical force

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Abstract

In this work we consider the effect of a small-scale helical driving force on fluid with a stable temperature gradient with Reynolds number $Re \ll 1$. At first glance, this system does not appear to have any instability. However, we show that large-scale vortex instability appears in fluid despite its stable stratification. In the non-linear mode, this instability gets saturated and gives a large number of stationary spiral vortex structures. Among these structures there is a stationary helical soliton and a kink of a new type. The theory is built on the rigorous asymptotical method of multi-scale development.

1 Introduction

The importance of the generation processes of large-scale coherent vortex structures in hydrodynamics is well known. When these coherent structures appear in small-scale turbulence they play a key role in transport processes (see for instance [1]). Numerical and laboratory experiments [2]-[5] confirm the existence of coherent vortex structures, especially for two-dimensional or quasi two-dimensional turbulence [6]-[10]. Notably, they are well observed in geophysical hydrodynamics like various cyclones in the planet’s atmospheres [11], [12]. The theory of 3D large-scale instabilities is of major interest. An example of this theory is the generation of large-scale magnetic field by helical small-scale turbulence ($\vec{v} \times \vec{v} \neq 0$) in the MHD (dynamo effect or $\alpha-$effect) (see for instance [13], [14]). Many works dealt with the dynamo theory generalization for usual hydrodynamics, and as a result it was understood that a small-scale turbulence able to generate large-scale perturbations cannot be simply homogeneous, isotropic and helical [15], but must have additional special properties. So in works [16], [17] it was shown that parity breakdown in small-scale turbulence (the external small-scale driving forces) lead to large-scale instability, the so-called Anisotropic Kinetic Alpha effect (AKA-effect). The injection of the helical external force into hydrodynamics systems was considered in many works [18]-[21]. In some cases, the existence of large-scale instability was shown (vortex dynamo or hydrodynamic $\alpha$-effect). So, in particular, in work [19] it is shown that the large-scale instability exists in convective systems with small-scale helical turbulence. Large-scale instability was interpreted as the result of a positive feedback loop between the poloidal and toroidal perturbations of the velocity field, which is carried out through the helicity coefficient. These works, as well as the results of numerical modelling, are in details
described in review [22], which is focused essentially on the possible application of these results to the issue of tropical cyclone origination.

In this work, we formulate the problem in a different way. Let us suppose that there is a stable temperature stratification in fluid. Let us apply to this fluid with the Reynolds number \( Re \ll 1 \) a small-scale, helical, external force. This force will maintain in the fluid small-scale helical fluctuations of velocity field \((\vec{v}_{\text{rot}} \vec{v} \neq 0)\). We consider the fluid as boundless. At first glance, there are no instabilities at all in this system. However, we show in this work that despite stable stratification, a large-scale vortex instability appears in the fluid which leads to the generation of large-scale vortex structures. The theory of this instability is built rigorously using the method of asymptotical multi-scale development similar to what was done in work of Frisch, She and Sulem for the theory of AKA-effect [16]. In addition to the linear theory, we also develop and study in detail the non-linear theory of this instability saturation. We devote special attention to stationary, non-linear, periodical vortex structures which appear as a result of the saturation of the found instability. Among these structures there are a spiral vortex soliton and kink of the new type.

Our work is arranged as follows: in Section 2, we set forth the formulation of the problem and equations for stable stratification in the Boussinesq approximation; in Section 3, we examine the principal scheme of multi-scale development and we give secular equations. An overall algebraic scheme of multi-scale development over the Reynolds number up to the fifth order is described in Appendix A. In Section 4, we find the velocity field of zero approximation and we describe external force properties. Calculation of terms of \( R^2 \) order is given in Section 5. The main bulky calculations of Reynolds stress i.e. closure of secular equations are presented in Appendix B. In Appendix C we deal with some minor questions concerning the closure of temperature equations. In Section 6 we consider the overall system of secular equations, and we obtain and investigate equations for the large-scale instability. In Section 7, we examine the multi-scale development for non-linear cases. The overall algebraic scheme of this development is given in Appendix D. In Section 8, we calculate the velocity field of zero approximation for the non-linear case. The Reynolds stress calculations for non-linear case are given in Appendix E. In Section 9, we discuss the non-linear stage of the instability and its saturation. We study the equations of non-linear instability and its stationary solutions. It is shown that due to the hamiltonian nature of these equations a large number of stationary vortex structures of the spiral type appear. We demonstrate also that there are solutions in the form of the spiral soliton and a kink of new type. The obtained results are discussed in conclusions in Section 10.

2 Main equations and formulation of the problem

Let us consider the equations of motion of non-compressible fluid with a constant temperature gradient in the Boussinesq approximation:

\[
\frac{\partial \vec{V}}{\partial t} + (\vec{V} \nabla) \vec{V} = -\frac{1}{\rho_0} \nabla P + \nu \Delta \vec{V} + g \beta T \vec{e}_z + \vec{f}_0; \quad (1)
\]

\[
\frac{\partial T}{\partial t} + (\vec{V} \nabla) T = \chi \Delta T - V_z A. \quad (2)
\]

\( \nabla \vec{V} = 0, \vec{e}_z = (0,0,1) \)-is the single vector in the direction of z axis, \( \beta \)-is the thermal expansion coefficient, \( A = \frac{dT_0}{dz} \)-constant equilibrium gradient of temperature, \( A = \text{Const}, A > 0, \rho_0 = \text{const.} \ \nabla T_0 = A \vec{e}_z \)-the buoyancy force and the external force \( \vec{f}_0, \text{div} \vec{f}_0 = 0 \) are taken into account in Euler equation (1). Let us write down the
force \( \vec{f}_0 \) in the form: 
\[
\vec{f}_0 = f_0 \vec{F}_0 \left( \frac{x_0}{\lambda_0}, \frac{t}{t_0} \right),
\]
where \( \lambda_0 \)- characteristic scale, \( t_0 \)- characteristic time, \( f_0 \)- characteristic amplitude of external force. We designate the characteristic velocity, which is engendered by external force as \( v_0 = v_0 \left( \frac{x_0}{\lambda_0}, \frac{t}{t_0} \right) \). When multiplying the first equation by the parameter \( \frac{\lambda_0^2}{\nu} \), we choose dimensionless variables:
\[
\vec{x} \rightarrow \frac{\vec{x}}{\lambda_0}, t \rightarrow \frac{t}{t_0}, \vec{V} \rightarrow \frac{\vec{V}}{v_0}, \vec{F}_0 \rightarrow \frac{\vec{F}_0}{f_0}, P \rightarrow \frac{P}{\rho_0 f_0 ^2}.
\]
where
\[
t_0 = \frac{\lambda_0^2}{\nu}, P_0 = \frac{v_0 \nu}{\lambda_0}, f_0 = \frac{v_0 \nu}{\lambda_0^2}, v_0 = \frac{f_0 \lambda_0^2}{\nu}.
\]
In the dimensionless variables \((t, \vec{x}, \vec{V})\), motion equations take the form:
\[
\frac{\partial \vec{V}}{\partial t} + R(\vec{V} \nabla) \vec{V} = -\nabla P + \Delta \vec{V} + \left( \frac{\lambda_0^2}{v_0 \nu} \right) g \beta T \vec{e}_z + \vec{F}_0
\]
\[
\frac{\partial T}{\partial t} + R(\vec{V} \nabla) T = \frac{1}{Pr} \Delta T - R V_z (A \lambda_0),
\]
where \( R = \frac{\lambda_0^4 \nu \beta}{v_0} \)- Reynolds number on the scale \( \lambda_0 \), \( Pr = \frac{\nu}{\chi} \)- is Prandtl number. We introduce the dimensionless temperature \( T \rightarrow \frac{T}{T_0} \chi \lambda_0 \) and obtain the equations system:
\[
\frac{\partial \vec{V}}{\partial t} + R(\vec{V} \nabla) \vec{V} - \Delta \vec{V} = -\nabla P + \frac{Ra}{R Pr} T \vec{e}_z + \vec{F}_0,
\]
\[
\frac{1}{R} \left( \frac{\partial T}{\partial t} - \frac{1}{Pr} \Delta T \right) = -V_z - (\vec{V} \nabla) T.
\]
Here \( Ra = \frac{\lambda_0^4 \nu \beta}{v_0} \)- is Rayleigh number on the scale \( \lambda_0 \). Further for the purpose of simplification we will consider the case \( Pr = 1 \). We pass to the new temperature \( T \rightarrow \frac{T}{T_0} \), and obtain finally:
\[
\frac{\partial \vec{V}}{\partial t} + R(\vec{V} \nabla) \vec{V} - \Delta \vec{V} = -\nabla P + Ra T \vec{e}_z + \vec{F}_0, \quad (3)
\]
\[
\left( \frac{\partial T}{\partial t} - \Delta T \right) = -V_z - R(\vec{V} \nabla) T. \quad (4)
\]
\[
\text{div} \vec{V} = 0.
\]
We will consider as small parameter of asymptotical development the Reynolds number \( R = \frac{\lambda_0^4 \nu \beta}{v_0} \ll 1 \) on the scale \( \lambda_0 \). The parameter \( Ra \) will be considered neither big nor small, without any impact on development scheme ( i.e. outside of the scheme parameters).

Let us examine the following formulation of the problem. We consider the external force as being small and of high frequency. This force drives the small scale velocity and temperature fluctuations on equilibrium state background. After averaging, these quickly-oscillating fluctuations equal zero. Nevertheless, the non zero terms can occur after averaging due to the fact that small non-linear interactions appear in some orders of perturbations theory. This means that they are not oscillatory, that is to say of large scale. From a formal point of view, these terms are secular, i.e. conditions for the solvability of the large scale asymptotic development. So, finding and studying the solvability equations i.e. the equations for large scales perturbations, is actually the purpose of this work. Let us designate further small scale variables as \( x_0 = (x_0, t_0) \), and large scale ones as \( X = (X, T) \). The derivative \( \frac{\partial}{\partial x_0} \) is designated \( \partial_0 \), the derivative \( \frac{\partial}{\partial x_0} \) is designated \( \partial_t \), and derivatives of large scale variables are \( \frac{\partial}{\partial X} \equiv \nabla \) i \( \frac{\partial}{\partial T} \equiv \partial_T \) respectively. (No misunderstanding occurs
between the temperature $T$ and the large scale time $T$ since here time is argument and temperature is function).

3 The multi-scale asymptotical development

For constructing multi-scale asymptotic development we follow the method which is proposed in work [16]. First of all, we develop space and time derivatives in equations (3), (4) into asymptotical series of the form:

$$\frac{\partial}{\partial x^i} = \partial_i + R^2 \nabla + \cdots.$$  \hspace{1cm} (5)

$$\frac{\partial}{\partial t} = \partial_t + R^4 \partial_T + \cdots$$  \hspace{1cm} (6)

We develop the variables $\vec{V}, T, P$ like so:

$$\vec{V}(\vec{x}, t) = \vec{v}_0(x_0) + R^2 \vec{\nabla} + R^3 \vec{V}_3 + R^4 \vec{V}_4 + R^5 \vec{V}_5 + \cdots$$  \hspace{1cm} (7)

$$T(\vec{x}, t) = T_0(x_0) + R^2 T_2 + R^3 T_3 + R^4 T_4 + R^5 T_5 + \cdots$$  \hspace{1cm} (8)

Here $\vec{W}_1(X)$ - is the velocity which depends on large scale variables only.

$$P(\vec{x}, t) = \frac{1}{R} P_{-1}(X) + P_0(x_0) + R P_1 + R^2 P_2 + R^3 (P_3 + P_3) + R^4 P_4 + R^5 P_5 + \cdots$$  \hspace{1cm} (9)

As we will see later, in development of pressure (9) it is necessary to have two terms which are dependent only on large scales variables $P_{-1}(X)$ and $P_3(X)$. Let us put now the developments (5)-(9) in the equations system (3), (4) and write down the obtained equations up to order $R^5$ inclusive. The obtained equations have a rather bulky form and are given in Appendix A. In order to simplify the writing of equations we give the algebraic structure of development only (vector indices are not written down explicitly, but can be easily restored in the necessary places). The conditions of asymptotical development solvability (5)-(9) of the equation system (3), (4) lead to the equations for the secular terms (114), (115), (109), (105) and (104). Let us write down the full system of secular equations:

$$\partial_T W_1^k - \Delta W_1^k + \nabla_p(v_0^p v_2^k + v_0^k v_2^p) + \nabla_p(W_1^p W_1^k) = -\nabla_p \bar{P}_3(X);$$  \hspace{1cm} (10)

$$\partial_T \Theta_1 - \Delta \Theta_1 = -\nabla_p(v_2^p T_0 + v_0^p T_2) - \nabla_p(W_1^p \Theta_1).$$  \hspace{1cm} (11)

$$\nabla_p W_1^p = 0$$  \hspace{1cm} (12)

$$W_1^z = 0$$  \hspace{1cm} (13)

$$\nabla P_{-1}(X) = Ra \Theta_1(X) \vec{l}_z$$  \hspace{1cm} (14)

The equations (11)-(13) form the main equation system. The equation (14) is secondary and is used to find the field $P_{-1}(X)$, which does not enter in the main system of equations.
identical and the equations (15), (16) take the form:

\[ \partial_t W_1^x - \Delta W_1^x + \nabla_p (W_1^p W_1^x) + \beta_{2px} \nabla_p W_1^x + \beta_{2py} \nabla_p W_1^y = -\nabla_x \overline{P}_3(X); \]  

(15)

\[ \partial_t W_1^y - \Delta W_1^y + \nabla_p (W_1^p W_1^y) + \beta_{gpx} \nabla_p W_1^x + \beta_{gpy} \nabla_p W_1^y = -\nabla_y \overline{P}_3(X); \]  

(16)

\[ \beta_{2px} \nabla_p W_1^x + \beta_{2py} \nabla_p W_1^y = -\nabla_x \overline{P}_3(X). \]  

(17)

The equation \( \nabla_x W_1^x + \nabla_y W_1^y = 0 \), allows us to find from the equations (15), (16) the pressure \( \overline{P}_3(X) \). But the substitution of this pressure into equation (17) leads to a contradiction because three equations appear for two variables \( W_1^x, W_1^y \). There is just one possibility to avoid this contradiction, which is to consider the variables \( W_1^x, W_1^y \) as functions of variable \( Z \) only, i.e. \( W_1^x = W_1^x(Z), W_1^y = W_1^y(Z) \). In this case, the non-linear terms in equations (15), (16) identically vanish, the equation \( \text{div} \overline{W}_1 = 0 \), is satisfied identically and the equations (15), (16) take the form:

\[ \partial_t W_1^x - \Delta_z W_1^x + \beta_{xxz} \nabla_z W_1^x + \beta_{xzy} \nabla_z W_1^y = 0 \]  

(18)

\[ \partial_t W_1^y - \Delta_z W_1^y + \beta_{yzz} \nabla_z W_1^x + \beta_{yyz} \nabla_z W_1^y = 0 \]  

(19)

\[ \beta_{zzx} \nabla_z W_1^x + \beta_{zzy} \nabla_z W_1^y = -\nabla_z \overline{P}_3(X). \]  

(20)

Then the velocities are determined by the equations (18), (19), and the pressure is found from the equation (20). Taking into account the equation (11), the temperature \( \Theta_1 \) must also be considered as a function of the variable \( Z \) only.

4 Calculations of the zero approximation fields (linear theory)

Let us designate the operator \( \partial_t - \partial^2 \equiv D_0 \). Then, applying this operator to the first equation (102), we obtain the equation only for the velocity \( v_0^i \):

\[ D_0^2v_0^i = -D_0 \partial^i P_0 - Ra(v_0^k v_0^l)l^i + D_0 F_0^i \]  

(21)

Here \( l^i \) - vector \( l^i = (0, 0, 1) \). With help of the equation \( \partial_i v_0^i = 0 \), we find the pressure \( P_0 \):

\[ P_0 = -\frac{\partial_{pp} Ra}{\partial^2 D_0} l^{i} l^{k} v_0^k. \]  

(22)

Eliminating the pressure (22) from the equation (21), we obtain the equation for \( v_0^i \):

\[ D_0^2v_0^i = -\widehat{P}^{il}(Ra(l^k v_0^k)) + D_0 F_0^i, \]  

(23)
Here $\hat{P}^{ip}$ is the projection operator:

$$\hat{P}^{ip} = \delta^{ip} - \frac{\partial^{i}\partial^{p}}{\partial^{2}}$$  \hspace{1cm} (24)

As a result, the equation (23) can be written down in the form:

$$(D_{0}^{2}\delta^{ik} + Ra\hat{P}^{ip}p^{l}l^{k})v_{0}^{k} = D_{0}F_{i}^{0}. $$  \hspace{1cm} (25)

Let us divide this equation by $D_{0}^{2}$. Then we obtain:

$$(\delta^{ik} + Ra\hat{P}^{ip}p^{l}l^{k})v_{0}^{k} = \frac{F_{i}^{0}}{D_{0}}. $$  \hspace{1cm} (26)

Designate operator $L_{ik}$:

$$L_{ik} \equiv \delta_{ik} + Ra\hat{P}^{ip}p^{l}l^{k}. $$  \hspace{1cm} (27)

Then the equation (26) takes the form:

$$L_{ik}v_{0}^{k} = \frac{F_{i}^{0}}{D_{0}}, $$  \hspace{1cm} (28)

And the velocity $v_{0}^{k}$ is found using the inverse operator $L_{ik}^{-1}$:

$$v_{0}^{k} = L_{kj}^{-1} \frac{F_{j}^{0}}{D_{0}}, $$  \hspace{1cm} (29)

$$L_{ik}L_{kj}^{-1} = \delta_{ij}. $$  \hspace{1cm} (30)

It is easy to make sure by the direct check that the inverse operator $L_{kj}^{-1}$ has the form:

$$L_{kj}^{-1} = \delta_{kj} - \frac{Ra\hat{P}^{km}l_{m}l_{j}}{D_{0}^{2} + Ra\hat{P}_{pq}p^{l}l^{q}}. $$  \hspace{1cm} (31)

Consequently the expression for the velocity $v_{0}^{k}$ takes the form:

$$v_{0}^{k} = \left[ \delta_{kj} - \frac{Ra\hat{P}^{km}l_{m}l_{j}}{D_{0}^{2} + Ra\hat{P}_{pq}p^{l}l^{q}} \right] \frac{F_{j}^{0}}{D_{0}}. $$  \hspace{1cm} (32)

From the equation

$$D_{0}T_{0} = -l^{k}v_{0}^{k}, $$  \hspace{1cm} (33)

we can find at once the field $T_{0}$:

$$T_{0} = -\left[ 1 - \frac{Ra\hat{P}^{nm}l_{m}l_{n}}{D_{0}^{2} + Ra\hat{P}_{pq}p^{l}l^{q}} \right] \frac{(l^{j}F_{0}^{j})}{D_{0}^{2}}. $$  \hspace{1cm} (34)

In order to use these formulae we have to specify in explicit form the helical external force $F_{0}^{j}$. The simplest and most natural way is to specify the external force as deterministic. (Certainly, it is possible to specify the external force in a statistical way with specifying random field correlator, but this leads to more bulky calculations). As it is well known, helicity means that $\vec{f}_{0} \text{rot} \vec{f}_{0} \neq 0$. Let us specify the force $\vec{f}_{0}$ like so:

$$\vec{f}_{0} = f_{0} \left[ \vec{i} \cos \varphi_{2} + \vec{j} \sin \varphi_{1} + \vec{k} (\cos \varphi_{1} + \sin \varphi_{2}) \right], $$  \hspace{1cm} (35)

where
\[ \varphi_1 = k_0 x - \omega_0 t, \varphi_2 = k_0 y - \omega_0 t, \quad (36) \]

or
\[ \varphi_1 = \vec{k}_1 \vec{x} - \omega_0 t, \quad \varphi_2 = \vec{k}_2 \vec{x} - \omega_0 t, \quad (37) \]

It is evident that \( \text{rot} \vec{f}_0 = k_0 \vec{z} \vec{f}_0 \), where \( \varepsilon \) is the single pseudo scalar, i.e. helicity is equal to:
\[
\vec{f}_0 \text{rot} \vec{f}_0 = k_0 \varepsilon \vec{f}_0^2 \neq 0. \quad (38)
\]

The formulae \((35), (37)\) permit us to easily make intermediate calculations, but in the final formulae we obviously shall take \( f_0, k_0, \omega_0 \) as equal to one, since external force is dimensionless and depends only on dimensionless space and time arguments. The force \((35)\) is physically simple and can be realized in laboratory experiments and in numerical simulation.

It is easy to write down the force \((35)\) in the complex form. It is evident that:
\[
\vec{f}_0 = \vec{A} \exp(i\varphi_1) + \vec{A}^* \exp(-i\varphi_1) + \vec{B} \exp(i\varphi_2) + \vec{B}^* \exp(-i\varphi_2), \quad (39)
\]

where vectors \( \vec{A} \) and \( \vec{B} \) has the form:
\[
\vec{A} = \frac{f_0}{2} (\vec{k} - i\vec{j}), \quad \vec{B} = \frac{f_0}{2} (i\vec{i} - i\vec{k}), \quad (40)
\]

and \( \varphi_1, \varphi_2 \) are given by formulae \((37)\).

### 5 Calculations of \( R^2 \) order terms

Second approximation equations are the form \((107)\):
\[
D_0 v_2^i = -\partial^j P_2 + RaT_2 l^i - \partial_k (W_k^1 v_0^1 + v_0^k W_1^i) - \partial_k (v_1^k v_0^i + v_0^k v_1^i), \quad (41)
\]
\[
D_0 T_2 = -v_2^{k,k} - \partial_k (W_k^1 T_0 + v_0^k \Theta_1) - \partial_k (v_1^k T_0 + v_0^k T_1) \quad (42)
\]

Let us, as earlier, apply to the equation \((11)\) the operator \( D_0 \) and exclude the field \( T_2 \). For \( v_2^i \) obtain:
\[
D_0^2 v_2^i = -D_0 \partial^j P_2 - Ra(v_2^{k,k}) l^i -
-
Ra l^i \left[ \partial_k (W_k^1 T_0 + v_0^k \Theta_1) - \partial_k (v_1^k T_0 + v_0^k T_1) \right] -
-D_0 \partial_k (W_k^1 v_0^i + v_0^k W_1^i) - D_0 \partial_k (v_1^k v_0^i + v_0^k v_1^i).
\]

By excluding the pressure \( P_2 \) with the help of the equation \( \partial_i v_2^i = 0 \), we obtain as usual the equation with the projection operator \((24)\):
\[
D_0^2 v_2^i + Ra \hat{P}^p q l^k v_2^k =
-
\hat{P}^p \left\{ Ra \partial^p \partial_k (W_k^1 T_0 + v_0^k \Theta_1) + Ra \partial^p \partial_k (v_1^k T_0 + v_0^k T_1) \right\} -
-
\hat{P}^p \left\{ D_0 \partial_k (W_k^1 v_0^p + v_0^k W_1^p) + D_0 \partial_k (v_1^k v_0^p + v_0^k v_1^p) \right\}.
\]
We divide this equation by $D_0^2$, and see that it can be written down in the form:

$$L_{ik}v_k^i = -\frac{\hat{\rho}_{ip}}{D_0^2} \{ Ra\rho \hat{\rho}_k(W_{1k}^i T_0 + v_0^i \Theta_1) + Ra\rho \hat{\rho}_k(v_k^i T_0 + v_0^i T_1) +$$

$$+ D_0 \hat{\rho}_k(W_{1k}^i v_0^i + v_0^i W_{1i}^p) + D_0 \hat{\rho}_k(v_k^i v_0^i + v_0^i v_0^i W_{1i}^p) \}.$$  \hspace{1cm} (43)

Where $L_{ik}$ is the former operator \((27)\). Taking into account the expression \((31)\) for the inverse operator $L^{-1}_{kj}$, we obtain:

$$v_k^i = -\left[ \delta_{kj} - \frac{Ra\hat{P}_{knl} l_j P_{ij}}{D_0^2 + RaP_{\lambda l} l_j l_q} \right] \frac{\hat{P}_{ip}}{D_0^2} (Ra\rho W_{1m}^m \partial_m T_0 + D_0 W_{1m}^m \partial_m v_0^p).$$  \hspace{1cm} (44)

Let us write down the expression \((15)\) in the form:

$$v_k^i = -W_{1}^{m} T_{(1)}^{mk} - W_{1}^{m} T_{(2)}^{mk},$$  \hspace{1cm} (46)

Where the tensors $T_{(1)}^{mk}, T_{(2)}^{mk}$ have the form:

$$T_{(1)}^{mk} = -\left[ \delta_{kj} - \frac{Ra\hat{P}_{knl} l_j P_{ij}}{D_0^2 + RaP_{\lambda l} l_j l_q} \right] \frac{\hat{P}_{ip}}{D_0^2} Ra\rho \hat{\rho}_k \partial_m T_0,$$  \hspace{1cm} (47)

$$T_{(2)}^{mk} = \left[ \delta_{kj} - \frac{Ra\hat{P}_{knl} l_j P_{ij}}{D_0^2 + RaP_{\lambda l} l_j l_q} \right] \frac{\hat{P}_{ip}}{D_0^2} \partial_m v_0^p.$$  \hspace{1cm} (48)

The calculations of Reynolds stresses are rather bulky, which is why we perform them in Appendix B.

6 Large scale instability

In Appendix B, we calculate Reynolds stresses for the equation \((10)\); in other words, we obtain closed secular equations for large scale perturbations. We write down these equations in the explicit form:

$$\partial_T W_x + \alpha \nabla_Z W_y = \nabla_Z W_x,$$  \hspace{1cm} (49)

$$\partial_T W_y - \alpha \nabla_Z W_x = \nabla_Z W_y,$$  \hspace{1cm} (50)

$$\alpha = -\varepsilon Ra \frac{4 - 2 Ra}{(4 + Ra^2)^2}.$$  \hspace{1cm} (51)

Here $\varepsilon$ designate a single pseudo scalar because expressions $\nabla_Z W_y, - \nabla_Z W_x$ are components of $\text{rot} \hat{W}$. The equations \((19), (50)\) differ from equation of AKA -effect \((16)\) by the coefficient $\alpha$ only. They obviously contain an instability which generates large scale vortex structures. Choosing the velocities $W_x, W_y$ in the form:

$$W_x = A \exp(\gamma T) \sin kz,$$  \hspace{1cm} (52)
\[ W_y = B \exp(\gamma T) \cos kz, \] (53)

we obtain the instability increment 
\[ \gamma = \pm \alpha k z - k^2 z, \]
i.e. \( \max \gamma = \frac{\alpha^2}{2} \), with the \( k = \frac{\alpha}{2} \).

The formulae (52), (53) describe spiral vortex structure (circularly polarized plane wave) with the amplitude which increases exponentially with time. These waves are sometimes called Beltrami runaways since for them there is no usual hydrodynamical interaction \( \vec{W} \nabla \vec{W} \equiv 0 \).

If the external force has zero helicity, then the \( \alpha \)-term vanishes in accordance with the general theorem of the Reynolds stress tensor [15]. Helicity is taken into account in the external force structure itself. If the temperature gradient vanishes, then it is evident that the \( \alpha \)-term also vanishes. Besides the equations (49), (50) there is an equation to find the pressure:

\[ C (W_x + W_y) = -P_3(X) + \text{Const}, \] (54)

where \( C \)
\[ C = \frac{8 - 12Ra - 2Ra^2 - 3Ra^3 - Ra^4}{(4 + Ra^2)^3}, \] (55)

and also the equation for the large scale temperature perturbation found in Appendix C:

\[ \partial_T \Theta_1 - \Delta \Theta_1 = -2 \frac{Ra - 2}{(4 + Ra^2)^2} \nabla Z (W_x + W_y) \] (56)

and the equation to find the pressure \( P_{-1}(X) : \)

\[ \nabla_z P_{-1}(X) = Ra \Theta_1(X) \] (57)

It is easy to solve all these equations with the known fields \( W_x, W_y \).

It should be noted that the field \( W_z \) is equal to zero in the main approximation only. \( W_z \neq 0 \) in the approximation of higher orders. In compliance with this in the approximation of higher orders appear also derivatives \( \frac{\partial}{\partial X}, \frac{\partial}{\partial Y} \). This means, that in fact one has to consider that

\[ \frac{\partial}{\partial Z} \gg \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}. \] (58)

The conditions (58) means that the horizontal scales of the instable vortices \( L_x, L_y \gg L_z \) (vertical scale). \( L_z \gg \lambda_0 \).

Since the increasing fields \( W_x, W_y \) belong to Beltrami type, the general non linearity \( (W\nabla)W \) cannot limit their increase, because, as it has been already stated, it is identically equal to zero. In order to describe the process of this instability saturation it is necessary to develop a non linear theory of \( \alpha \)-effect. This theory is described in the following sections.

### 7 Multi scale development for the non linear case

In the non linear case the large scale field \( W(X) \) is no small any longer that is why the asymptotical development (7)-(9) has to be modified. Let us search the solution for the equations (3), (4) in the following form:

\[ \vec{V}(\vec{x}, t) = \frac{1}{R} \vec{W}_{-1}(X) + \vec{v}_0(x_0) + R\vec{V}_1 + R^2\vec{V}_2 + R^3\vec{V}_3 + \cdots \] (59)

\[ T(\vec{x}, t) = \frac{1}{R^2} T_{-1}(X) + T_0(x_0) + RT_1 + R^2T_2 + R^3T_3 + \cdots \] (60)
\( P(\vec{x}, t) = \frac{1}{R^3} P_{-3}(X) + \frac{1}{R^2} P_{-2}(X) + \frac{1}{R} P_{-1}(X) + P_0(x_0) + R(P_1 + \mathbf{P}_1(X)) + R^2 P_2 + R^3 P_3 + \cdots \)  

(61)

For the derivatives we use the previous development (5), (6). Substituting these expressions into the initial equations (3), (4) and gathering together the terms of the same order \( R \), up to degree \( R^3 \) inclusive, we obtain the equations of multi scale asymptotical development.

The algebraic scheme of this development is given in Appendix D. The result of this scheme is the separation of secular terms from fast oscillating ones. Let us give secular terms in the explicit form:

\[
\partial_t W^i - \Delta W^i + \nabla_k \left( v_0^k v_0^i \right) = -\nabla_i P_1; \tag{62}
\]

\[
\partial_t T - \Delta T + \nabla_k (v_0^k T_0) = 0. \tag{63}
\]

In these equations we do not write the law index \((-1)\). Besides this, there are secular equations:

\[
\nabla_i W^i = 0, \quad W^z = 0, \tag{64}
\]

\[
\nabla_k (W^k W^i) = -\nabla_i P_{-1}, \tag{65}
\]

\[
\nabla_k (W^k T) = 0. \tag{66}
\]

The equations (64)-(66) are satisfied in the previous geometry:

\[
W = (W^x(z), W^y(z), 0), \quad \text{and}, \quad P_{-1} = \text{Const}. \tag{67}
\]

There is also an equation to find the pressure \( P_{-3} \):

\[
\nabla_z P_{-3} = Ra T l. \tag{68}
\]

It is clear that the essential equations for finding the non linear alpha-effect is the equation (62). In order to obtain these equations in the closed form we need to calculate the Reynolds stresses \( \nabla_k \left( v_0^k v_0^i \right) \). Below, we will deal with the solution of this problem.

First, we have to calculate the \( v_0^k \) fields of zero approximation. From the asymptotical development in zero order we have the equations:

\[
\partial_t v_0^i - \partial^2 v_0^i + W^k \partial_k v_0^i = -\partial_i P_0 + Ra T_0 l^i + P_0^i, \tag{69}
\]

\[
\partial_t T_0 - \partial^2 T_0 + W^k \partial_k T_0 = -v_0^k l^k. \tag{70}
\]

8 Calculation of zero approximation fields in the non linear case

Let us introduce the operator \( \tilde{D}_0 \):

\[
\tilde{D}_0 = \partial_t - \partial^2 + W^k \partial_k. \tag{71}
\]

Using the operator \( \tilde{D}_0 \), we write down the equations (69) and (70) in the form:
\[ \hat{D}_0 v^i_0 = -\partial_i P_0 + Ra T_0 l^i + F^i_0, \] (72)

\[ \hat{D}_0 T_0 = -v^k_0 t^k \] (73)

Eliminating the temperature from the equation (72) we obtain:

\[ \hat{D}_0^2 v^i_0 = -\hat{D}_0 \partial_i P_0 - Ra (v^k_0 t^k) l^i + \hat{D}_0 F^i_0. \] (74)

\[ \partial_i v^i_0 = 0. \] (75)

Eliminating the pressure from the equation (74) we obtain:

\[ \hat{D}_0^2 v^i_0 = -\hat{P}_{sp} (Ra v^k_0 l^p) + \hat{D}_0 F^i_0 \] (76)

or:

\[ (\hat{D}_0^2 \delta_{ik} + \hat{P}_{sp} Ra l^p) v^k_0 = \hat{D}_0 F^i_0. \] (77)

Dividing this equation by \( \hat{D}_0^2 \), we can write it in the form:

\[ L_{ik} v^k_0 = \frac{F^i_0}{\hat{D}_0}, \] (78)

Where \( L_{ik} \) is the prior operator (27) in which \( D_0 \) is replaced by \( \hat{D}_0 \). In much the same way the inverse operator also coincides with (31) when substituting \( D_0 \) by \( \hat{D}_0 \). As a result it is easy to find the fields \( v^k_0 \) and \( T_0 \):

\[ v^k_0 = \left[ \delta_{kj} - \frac{Ra \hat{P}_{km} l^m l^j}{\hat{D}_0^2 + Ra \hat{P}_{pq} l^p l^q} \right] \frac{F^j_0}{\hat{D}_0}, \] (79)

\[ T_0 = - \left[ 1 - \frac{Ra \hat{P}_{mn} l^m l^n}{\hat{D}_0^2 + Ra \hat{P}_{pq} l^p l^q} \right] \left( \frac{1}{2} \hat{D}_0^2 \right). \] (80)

The external force \( F_0 \) has the prior form (39). The effect of the operator \( \hat{D}_0 \) on proper function \( \exp(\omega t + i \vec{k} \vec{x}) \) has obviously the form:

\[ \hat{D}_0 \exp(\omega t + i \vec{k} \vec{x}) = \hat{D}_0(\omega, \vec{k}) \exp(\omega t + i \vec{k} \vec{x}), \] where \( \hat{D}_0(\omega, \vec{k}) \) is:

\[ \hat{D}_0(\omega, \vec{k}) = i(\omega + \vec{k} \vec{W}) + k^2. \] (81)

From this it is evident that:

\[ \hat{D}_0(\omega, -\vec{k}_1) = i(\omega - \vec{k}_1 \vec{W}) + k_1^2, \] (82)

\[ \hat{D}_0(\omega, -\vec{k}_1) = \hat{D}_0(-\omega, \vec{k}_1), \] (83)

\[ \hat{D}_0(\omega, -\vec{k}_2) = i(\omega - \vec{k}_2 \vec{W}) + k_2^2, \] (84)

\[ \hat{D}_0(\omega, -\vec{k}_2) = \hat{D}_0(-\omega, \vec{k}_2). \] (85)

From the formulae (78) and (79) follows that the field \( v^k_0 \) is composed of four terms: \( v^k_0 = v^k_{01} + v^k_{02} + v^k_{03} + v^k_{04} \) where

\[ v^k_{02} = (v^k_{01})^*, v^k_{04} = (v^k_{03})^*, \]

\[ v^k_{01} = \exp(i \varphi) \left[ \delta_{kj} - \frac{Ra \hat{P}_{km} l^m l^j}{\hat{D}_0^2(-\omega_0, \vec{k}_1) + Ra \hat{P}_{pq} l^p l^q} \right] \frac{A^j}{\hat{D}_0(-\omega_0, \vec{k}_1)} \] (86)
\[ v_{03}^k = e^{i\varphi_2} \left[ \delta_{kj} - \frac{Ra\tilde{P}_{km}l_{mj}}{D_0(\omega_0, \vec{k}_2) + Ra\tilde{P}l_2} \right] \frac{B^j}{D_0(\omega_0, \vec{k}_2)}. \]  

As it was said earlier, in scalar operators \( \tilde{D}_0 \) one can take \( \omega_0 = 1, \vec{k}_1 = (1, 0, 0), \vec{k}_2 = (0, 1, 0) \). Then taking into account that \( \tilde{P}_{ll} = 1 \), we obtain:

\[ \tilde{D}_0(\omega_0, -\vec{k}_1) = 1 + i(1 - W_1) \equiv D_1, \]  
\[ \tilde{D}_0(-\omega_0, \vec{k}_1) = D_1^*, \]  
\[ \tilde{D}_0(\omega_0, -\vec{k}_2) = 1 + i(1 - W_2) \equiv D_2, \]  
\[ \tilde{D}_0(-\omega_0, \vec{k}_2) = D_2^*. \]

Taking into consideration these formulae we can write down the velocities \( v_0^k \) in the form:

\[ v_{01}^k = e^{i\varphi_1} \left[ \delta_{kj} - \frac{Ra\tilde{P}_{km}l_{mj}}{D_1^{*2} + Ra} \right] \frac{A^j}{D_1^{*2}}, \]  
\[ v_{03}^k = e^{i\varphi_2} \left[ \delta_{kj} - \frac{Ra\tilde{P}_{km}l_{mj}}{D_2^{*2} + Ra} \right] \frac{B^j}{D_2^{*2}}. \]

### 9 Non linear instability and large scale vortex structures

The calculations of the Reynolds stresses for the non linear case are performed in the Appendix E. Let us write down in the explicit form the equations for the non linear instability:

\[ \partial_T W_1 - \nabla_z^2 W_1 = -\nabla_z T_{(2)}^{31} = \]  
\[ = \nabla_z \frac{Ra(1 - W_2)}{[1 + (1 - W_2)^2][(W_2(2 - W_2) + Ra)^2 + 4(1 - W_2)^2]}, \]  
\[ \partial_T W_2 - \nabla_z^2 W_2 = -\nabla_z T_{(1)}^{32} = \]  
\[ = -\nabla_z \frac{Ra(1 - W_1)}{[1 + (1 - W_1)^2][(W_1(2 - W_1) + Ra)^2 + 4(1 - W_1)^2]}. \]

Here we introduced the following notations: \( W_1 \equiv W_x, W_2 \equiv W_y \). It is easy to verify that with small values of the variables \( W_1, W_2 \) the equations (92), (93) are reduced to the equations (49)-(51) and describe the linear stage of instability. It is clear that with the increasing of \( W_1, W_2 \) the non linear terms decrease and the instability gets saturated. As a result of the development and stabilization of instability the non linear vortex helical structures appear. The study of the form of these stationary structures is of interest. For that purpose we take \( \partial_T W_1 = \partial_T W_2 = 0 \) in the equations (92), (93). Integrating these equation over \( z \), we obtain:

\[ \frac{\partial X}{\partial z} = \frac{RaP}{(1 + P^2)[4P^2 + (1 - P^2 + Ra)^2]} + C_1, \]  
\[ \frac{\partial P}{\partial z} = -\frac{RaX}{(1 + X^2)[4X^2 + (1 - X^2 + Ra)^2]} - C_2. \]
Figure 1: Phase picture of the dynamical system with $Ra = 2$, $C_1 = C_2 = 0$. The bold line shows the phase trajectory which comes out of the point $(1, 1)$ and after the “time” $Z = L$ comes back to the same point. This trajectory presents the stationary solution of the boundary problem with the rigid boundaries in the layer whose thickness is $L = z$.

Here new variables are introduced $X = 1 - W_1$, $P = 1 - W_2$, and $C_1, C_2$- are integration constants. The system of equations (94) and (95) can be write down in the hamiltonian form:

$$\frac{\partial X}{\partial z} = \frac{\partial H}{\partial P}, \quad \frac{\partial P}{\partial z} = -\frac{\partial H}{\partial X}. \quad (96)$$

Here the variable $z$ plays the role of time and the hamiltonian $H$ has the form:

$$H = U(P) + U(X) + C_1 P + C_2 X + C_3. \quad (97)$$

Where $U(x)$ has the form:

$$U(x) = \frac{1}{4(4 + Ra)} \ln \frac{(1 + x^2)^2}{4Ra + (x^2 + 1 - Ra)^2} + \sqrt{Ra} \frac{1 + x^2 - Ra}{4(4 + Ra)} \frac{\arctan \left(\frac{1 + x^2 - Ra}{2\sqrt{Ra}}\right)}{2\sqrt{Ra}}. \quad (98)$$

The function $H$ (97), (98) is obviously the first integral of the equations system (94), (95) and can be found by the direct integration of this system. With $C_1 = 0, C_2 = 0$ the function $U(x)$ is limited above and below as well. That is why the hamiltonian section by the constant $H = H_0$, gives closed periodical trajectories on the phase plane $(X, P)$, which correspond to the helical vortex structures in the real space. Examples of phase pictures for $Ra = 2$ and $Ra = 3$ are represented in fig.1 and fig.2. With $C_1 = 0, C_2 = 0$ on the phase plane there is only one elliptical point. Closed trajectories correspond to the periodical non linear vortex structures. Thick closed lines correspond to the non linear structures which are also the solutions of the boundary problem with the rigid boundary:

$$W_1 = 0, W_2 = 0, z = 0, z = L,$$

where $L$ is the period over $z$ of phase trajectory, which gets out with $z = 0$, of $W_1 = 0, W_2 = 0$ and gets back to the same point with $z = L$. The space structures of periodical
Figure 2: Phase picture of the dynamical system with \( Ra = 3 \), \( C_1 = 0 \), \( C_2 = 0 \). The bold line shows the trajectory which corresponds to the stationary solution of the boundary problem with rigid boundaries with \( z = 0 \) and \( z = L \).

solutions is presented in fig.3-fig.5. If one of the constants, for instance \( C_1 \neq 0 \), then one hyperbolic point appears on phase picture. For instance, phase pictures with \( C_1 = 0.1 \) are presented in fig.6. The example of periodical vortex structure which corresponds to the closed trajectory on pase plane with \( Ra = 2 \) is given in fig.7. The solution which corresponds to the separatrix in fig.6 is of particular interest. This solution describes the solitary spiral turn of the velocity field around the axis \( z \) (soliton) see fig 8. Moving away from soliton the velocity field becomes constant. This kind of solitons were not known earlier. The interesting particularity of this soliton is the fact that it is also the solution of the boundary problem with free boundaries. For this boundary problem [23]:

\[
\frac{\partial W_1}{\partial z} = \frac{\partial W_2}{\partial z} = 0
\]

on the fluid boundary. In addition to that, boundaries must be on a big distance from the soliton, much bigger than the soliton’s characteristic dimensions. In cases when there are two constants \( C_1 \neq 0, C_2 \neq 0 \) two hyperbolic and two elliptical points appear on phase picture. The example of this phase picture with \( C_1 = 0.1, C_2 = 0.1 \) is shown in fig.9. As earlier, the periodic vortex structures correspond to closed trajectories around elliptical points. Localized solutions (solitons) correspond to a separatrix in fig.9. Since the separatrix connects two different hyperbolic points, the soliton now has two different limiting values, with \( z \to \pm \infty \), fig.10. This soliton is called a kink. Thereby, spiral kinks correspond to the separatrix in fig.9. These kinks are also solutions of the boundary problem with free boundaries. In conclusion, it should be remembered that the system of the equations (92),(93) is closed. The velocity field \( W_1, W_2 \) determines the pressure \( P_1 \), according to the formulae:

\[
-P_1 = T_{(1)}^{33} + T_{(2)}^{33},
\]

where \( T_{(1)}^{33}, T_{(2)}^{33} \) are given by the formulæ (235),(236). Besides, the velocity field \( W_1, W_2 \) gives the contribution to the equation for temperature (63). Closure of this equation is made in much the same way as the closure for velocity. Nevertheless, this equation is
secondary and here we do not give the result of this closure.

10 Conclusions and discussion of results

In this work, it is shown that in fluid with stable stratification, a large scale instability appears under the action of small scale helical force. The result of instability is the generation of vortex structures of the Beltrami type. The vortices have the characteristic vertical dimension $L_z \gg \lambda_0$ and the horizontal dimension is much bigger than the vertical one. Since the vertical component of the velocity $W_z$ is equal to zero in the main approximation and the stratification is stable, then the found instability has no relation with convection. The structure of the equation which describes the instability in linear approximation is the same as the equation of $\alpha$-effect or more precisely as the equation of AKA-effect. As a result, instability generates plane spiral waves with circular polarisation (Beltrami runaway). With an increase in amplitude, the instability and its stabilization are described by a non linear theory. Stationary equations appear to be hamiltonian, which is why they are a rich set of periodical spiral vortex structures. Notwithstanding the fact that attention in this work was essentially paid to a boundary free problem, it should be noted that some periodical solutions turn out to be solutions of the boundary problem with rigid boundaries. We would like to pay special attention to stationary soliton and kink, which correspond to the separatrix on the phase plane. This is the solitons of the new type. In real space it describes one spiral turn of the velocity vector field around the axis $z$. Soliton and kink are also the solutions of a boundary problem with free boundaries.

Let us return to the formulation of the problem. The external helical force $\vec{f}_0$ is given in the explicit form in order to make calculations more transparent. Strictly speaking, its explicit form is not very important for the existence itself of $\alpha$-effect. It is necessary that $\text{rot} \vec{f}_0 \approx \vec{f}_0$ only. The external force could be chosen statistically by specifying the correlator:

$$\overline{f_i f_m} = A(\tau, r) \delta_{im} + B(\tau, r) r_i r_m + G(\tau, r) \epsilon_{imn} r_n.$$  \hspace{1cm} (99)

It is fundamental that the last term $G(\tau, r)$ (helicity) in this correlator is not equal to zero, otherwise $\alpha$-effect is absent. Nevertheless the statistical method is more bulky since it requires the specification of the functions $A, B, G$ and calculations of rather complicated integrals. If we specify the external force dynamically then the averaging over fast
oscillations is performed easily.

The question is of interest about the origin of instability on qualitative level. For this, we revert to the expression for the Reynolds stresses:

$$\nabla_z (v_0^3 v_k^k + v_0^k v_0^n).$$  \hspace{1cm} (100)

As far as the direction $z$ is particular, the averages $(v_0^3 v_k^k + v_0^k v_0^n)$ which are non equal to zero must be proportional to $l_z$. From the property of the external force $A_1 A_3^* = 0$, follows, that with $k = 1$, the velocity $W_x$ is fully left of Reynolds stress. Another velocity $W_y$ enters into Reynolds stress with the coefficient $i W_y l_z B_3 B_1^*$. Since from the properties of external force it follows that $B_3 B_1^* = -\frac{i}{4}$, we obtain the factor $\left(\frac{1}{4} W_y l_z\right)$. In a similar way, with $k = 2$, because of property of the external force $B_2 B_3^* = 0$, the velocity $W_y$ is fully missed out in the second Reynolds equation. The velocity $W_x$ enters on the second Reynolds equation with the coefficient: $i W_x l_z A_2 A_3^*$. Due to the property of the external force $A_2^* A_3 = \frac{i}{4}$, we obtain the factor $\left(-\frac{1}{4} W_x l_z\right)$. (We may not write down the common factor). It is clear that as a result we obtain the components of vector product $[\vec{W} \times \vec{l}] = i W_y l_z - j W_x l_z$. Or if we take into account $\nabla_z$, we obtain the components $\text{rot} \vec{W}$. These components $\text{rot} \vec{W}$ provide a positive feedback loop between the velocity components like in the usual $\alpha$-effect which leads to instability.

**Appendix A. Asymptotical development scheme**

In the $R^{-1}$order there is only one term:

$$\partial P_{-1}(X) = 0. \hspace{1cm} (101)$$

With this the equation (101) is satisfied automatically. The equations of the order $R^0$ has the form:

$$\begin{align*}
\partial_t v_0 - \partial^2 v_0 &= -\partial P_0 + Ra T_0 \vec{l}_z + \vec{F}_0, \\
\partial_t T_0 - \partial^2 T_0 &= -v_0^2, \\
\partial t v_0 &= 0.
\end{align*} \hspace{1cm} (102)$$

From the equation (102) it follows immediately that the functions $v_0, T_0, P_0$ are oscillating due to the oscillating character of the external force $\vec{F}_0$. Approximation equations $R^1$ have the form:
The equations (103) contain already oscillating terms as well as the non-oscillating ones. Oscillating terms after averaging give zero and only the non-oscillating ones remain. That is why the solvability condition of this system is the independent vanishing of the non-oscillating (secular) terms as well as the oscillating ones. For the system (103) the condition of solvability in the approximation $R_1$ gives equations:

$$\nabla P_{-1}(X) = Ra\Theta_1(X)\vec{l}_z$$  \hspace{1cm} (104)

$$W_z^{1} = 0$$  \hspace{1cm} (105)

The oscillating part in the approximation $R^1$ is described by the equations system:

$$\partial_t v_1 - \partial^2 v_1 + \partial v_0 v_0 = -\partial P_1 - \nabla P_{-1}(X) + Ra(\Theta_1(X) + T_1)\vec{l}_z;$$
$$\partial_t T_1 - \partial^2 T_1 = -W_z^1 - v_1^2 - \partial v_0 T_0;$$
$$\partial v_1 = 0$$  \hspace{1cm} (106)

i.e. $v_1 = v_1(x_0), T_1 = T_1(x_0), P_1 = P_1(x_0)$. In the approximation $R^2$ we have the following equations:

$$\partial_t v_2 - \partial^2 v_2 = -\partial P_2 - \partial(W_1 v_0 + v_0 W_1) - \partial(v_1 v_0 + v_0 v_1) + RaT_2\vec{l}_z;$$
$$\partial_t T_2 - \partial^2 T_2 = -v_2^2 - \partial(W_1 T_0 + v_0 \Theta_1) - \partial(v_1 T_0 + v_0 T_1);$$
$$\partial v_2 = 0.$$  \hspace{1cm} (107)

It is easy to evidence that after averaging, all the terms in equations (107) give zero. Thereby secular terms do not appear in the order $R^2$ and the fields $v_2, T_2, P_2$ remain oscillating. However, they now depend on large scale variables $X$, i.e. $v_2 = v_2(x_0, X), T_2 = T_2(x_0, X), P_2 = P_2(x_0, X)$. The approximation $R^3$ gives equations:

$$\partial_t v_3 - \partial^2 v_3 = -\partial P_3 - \partial(W_1 + v_1)(W_1 + v_1) - \partial(v_2 v_0 + v_0 v_2) + RaT_3\vec{l}_z;$$
$$\partial_t T_3 - \partial^2 T_3 = -v_3^2 - \partial(W_1 + v_1)(\Theta_1 + T_1) - \partial(v_2 T_0 + v_0 T_2);$$
$$\partial v_3 + \nabla W_1 = 0.$$  \hspace{1cm} (108)
Figure 6: Phase picture of a dynamical system with $Ra = 2, C_1 = 0.1, C_2 = 0$.

From the equation (108) one can see that an averaging of the first two equations gives only zero terms, i.e. does not give any secular terms. But the third equation averaging gives a new secular term:

$$\nabla W_1 = 0 \quad (109)$$

Thereby the fields $v_3, T_3, P_3$ remain oscillating, but depend on $X$, i.e. $v_3 = v_3(x_0, X), T_3 = T_3(x_0, X), P_3 = P_3(x_0, X)$. Now we pass to the equations of the $R^4$ approximation. They have the form:

$$\partial_t v_4 - 2\partial \nabla v_2 - \partial^2 v_4 + \partial[v_3 v_0 + v_0 v_3 + v_2(W_1 + v_1) + (W_1 + v_1)v_2] + \quad (110)$$

$$\partial_t T_4 - 2\partial \nabla T_2 - \partial^2 T_4 = -v_4^2 - \partial [v_3 T_0 + v_0 T_3 + v_2(\Theta_1 + T_1)] +$$

$$+(W_1 + v_1)T_2] - \nabla[(W_1 + v_1)T_0 + v_0(\Theta_1 + T_1)];$$

$$\partial v_4 + \nabla v_2 = 0. \quad (111)$$

It is easy to see that these equations have no secular terms at all.

Finally, let us consider equations of the approximation$R^5$. Their form is rather bulky:

$$\partial_t v_5 + \partial_T W_1 - \partial^2 v_5 - 2\partial \nabla v_3 - \Delta W_1 + \partial[v_4 v_0 + v_0 v_4 + v_3(W_1 + v_1) +$$

$$(W_1 + v_1)v_3 + v_2v_2)] + \nabla[v_2 v_0 + v_0 v_2 + (W_1 + v_1)(W_1 + v_1)] = -\partial P_5 - \nabla P_3(X) - \nabla P_3 + RaT_3 \vec{\ell}_z.$$

$$\partial_t T_5 + \partial_T \Theta_1 - \partial^2 T_5 - 2\partial \nabla T_3 - \Delta \Theta_1 = -v_5^2 - \partial[v_4 T_0 + v_0 T_3 + v_3(\Theta_1 + T_1)] +$$

$$(W_1 + v_1)T_3 + v_2T_2] - \nabla[v_2 T_0 + v_0 T_2) - \nabla[(W_1 + v_1)(\Theta_1 + T_1)].$$
\[ \partial v_5 + \nabla v_3 = 0 \]

The equations of the fifth order (112), (113) give the main system of secular equations as the solvability conditions of the fifth approximation:

\[ \partial T W_1 - \Delta W_1 + \nabla (v_2 v_0 + v_0 v_2) + \nabla (W_1 W_1) = -\nabla P_3(X); \quad (114) \]

\[ \partial T \Theta_1 - \Delta \Theta_1 = -\nabla (v_2 T_0 + v_0 T_2) - \nabla (W_1 \Theta_1). \quad (115) \]

**Appendix B. Calculations of the Reynolds stresses.**

Since the external force is composed of four terms (39), then the expression for \( v_0^P \) is composed of four terms as well: \( 1v_0^P, 2v_0^P, 3v_0^P, 4v_0^P \). With this:

\[ 2v_0^P = (1v_0^P)^* \quad 4v_0^P = (3v_0^P)^* \quad (116) \]

\[ 1v_0^P = \exp i \varphi_1 \left[ \frac{\delta_{\mu \nu} - \frac{Ra \hat{P}_{p\lambda}(k_1) \lambda l_{\mu}}{D_0^2(-\omega_0, k_1) + Ra \hat{P}_{m\lambda}(k_1) \lambda l_{\nu}}}{A_\mu} \right] \frac{A_\mu}{D_0(-\omega_0, k_1)}, \quad (117) \]

\[ 3v_0^P = \exp i \varphi_2 \left[ \frac{\delta_{\mu \nu} - \frac{Ra \hat{P}_{p\lambda}(k_2) \lambda l_{\mu}}{D_0^2(-\omega_0, k_2) + Ra \hat{P}_{m\lambda}(k_2) \lambda l_{\nu}}}{B_\mu} \right] \frac{B_\mu}{D_0(-\omega_0, k_2)}. \quad (118) \]

In order to simplify equation writing, we will write down the convolution \( \hat{P}_{m\lambda l_{\mu} l_{\nu}} \) in the form \( \hat{P}(k) l \). In the similar way, for the temperature field \( T_0 \) there are four terms: \( 1T_0, 2T_0, 3T_0, 4T_0 \). At the same time:

\[ 2T_0 = (1T_0)^* \quad 4T_0 = (3T_0)^* \quad (119) \]

\[ 1T_0 = -\exp i \varphi_1 \left[ 1 - \frac{Ra \hat{P}(k_1) l \mu}{D_0^2(-\omega_0, k_1) + Ra \hat{P}(k_1) l \nu} \right] \frac{(l_{\mu} A_\mu)}{D_0^2(-\omega_0, k_1)}, \quad (120) \]
Figure 8: Helical soliton which corresponds to the separatrix in fig.6 with \( Ra = 2, C_1 = 0.1, \) \( C_2 = 0.1. \)

\[
3\hat{T}_0 = -\exp i\varphi_2 \left[ 1 - \frac{Ra\hat{P}(k_2)ll}{D_0^2(-\omega_0, k_2) + Ra\hat{P}(k_2)ll} \right] \frac{(l_\mu B_\mu)}{D_0^2(-\omega_0, k_2)} \tag{121}
\]

Let us consider the tensor \( T_{m^k}^{(2)} \). It is also composed of four terms:

\[
1\hat{T}_{m^k}^{(2)} = \left( 3\hat{T}_{m^k}^{(2)} \right)^* \hat{T}_{m^k}^{(2)} = (3\hat{T}_{m^k}^{(2)})^*. \tag{122}
\]

The equation for the tensor \( 1\hat{T}_{m^k}^{(2)} \) follows directly from the formula (48):

\[
1\hat{T}_{m^k}^{(2)} = \exp i\varphi_2 \left[ \delta_{kj} - \frac{Ra\hat{P}_{kn}(k_1)l_nl_j}{D_0^2(-\omega_0, k_1) + Ra\hat{P}(k_1)ll} \frac{\hat{P}_{pm}(k_1)l_m}{D_0^2(-\omega_0, k_1) + Ra\hat{P}(k_1)ll} \right] \times \left[ \delta_{\mu\nu} - \frac{Ra\hat{P}_{n\lambda}(k_1)l_nl_\lambda}{D_0^2(-\omega_0, k_1) + Ra\hat{P}(k_1)ll} \right] A_\mu. \tag{123}
\]

First of all we transform the expression, which contains projection operators in the formula (123):

\[
\left[ \delta_{kj} - \frac{Ra\hat{P}_{kn}(k_1)l_nl_j}{D_0^2(-\omega_0, k_1) + Ra\hat{P}(k_1)ll} \right] \hat{P}_{jp} \left[ \delta_{\mu\nu} - \frac{Ra\hat{P}_{p\lambda}(k_1)l_\mu l_\lambda}{D_0^2(-\omega_0, k_1) + Ra\hat{P}(k_1)ll} \right] =
\]

\[
\left[ \delta_{kj} - \frac{Ra\hat{P}_{kn}(k)l_nl_j}{D_0^2(\omega_0, k) - Ra\hat{P}(k)ll} \right] \hat{P}_{j\lambda} \left[ \delta_{\mu\nu} - \frac{Ral_\mu l_\lambda}{D_0^2(\omega_0, k) - Ra\hat{P}(k)ll} \right] =
\]

\[
= \hat{P}_{kn} \left[ \delta_{n\lambda} - \frac{Ra\hat{P}_{kj}(k)l_nl_j}{D_0^2(\omega_0, k) + Ra\hat{P}(k)ll} \right] \left[ \delta_{\mu\nu} - \frac{Ral_\mu l_\lambda}{D_0^2(\omega_0, k) + Ra\hat{P}(k)ll} \right]. \tag{124}
\]

Here we use the property of the projection operator.

\[
\hat{P}_{jp}\hat{P}_{p\lambda} = \hat{P}_{j\lambda}
\]

With help of the formulae (124) it is possible to write down:
Figure 9: Phase picture of the dynamical system with \( Ra = 2, C_1 = 0.1, C_2 = 0.1 \). One can see the appearance of two hyperbolical and two elliptical points.

\[
1 T_{(2)}^{mk} = \exp i\varphi_1 \frac{i k_1^m \hat{P}_{kn}(k_1)}{D_0^2(-\omega_0, k_1)} \left[ \delta_{n\lambda} - \frac{Ra \hat{P}_{\lambda j}(k_1) n l_j}{D_0^2(-\omega_0, k_1) + Ra \hat{P}(k_1) ll} \right] \times (125) \\
\times \left[ \delta_{\lambda\mu} - \frac{Ra l_\lambda l_\mu}{D_0^2(-\omega_0, k_1) + Ra \hat{P}(k_1) ll} \right] A_\mu,
\]

\[
3 T_{(2)}^{mk} = \exp i\varphi_2 \frac{i k_2^m \hat{P}_{kn}(k_2)}{D_0^2(-\omega_0, k_2)} \left[ \delta_{n\lambda} - \frac{Ra \hat{P}_{\lambda j}(k_2) n l_j}{D_0^2(-\omega_0, k_2) + Ra \hat{P}(k_2) ll} \right] \times (126) \\
\times \left[ \delta_{\lambda\mu} - \frac{Ra l_\lambda l_\mu}{D_0^2(-\omega_0, k_2) + Ra \hat{P}(k_2) ll} \right] B_\mu.
\]

From the definition follows expressions for four terms of the tensor: \( T_{(1)}^{mk} \): 

\[
1 T_{(1)}^{mk} = \exp i\varphi_1 \frac{i k_1^m \hat{P}_{kn}(k_1)}{D_0^2(-\omega_0, k_1)} \times (127) \\
\]

\[
2 T_{(1)}^{mk} = (1 T_{(1)}^{mk})^* 4 T_{(1)}^{mk} = (3 T_{(1)}^{mk})^*.
\]

\[
3 T_{(1)}^{mk} = - \exp i\varphi_1 \frac{i k_1^m Ra l_\mu}{D_0^2(-\omega_0, k_1) + Ra \hat{P}(k_1) ll} \hat{P}_{kn}(k_1) \times (128) \\
\times \left[ 1 - \frac{Ra \hat{P}(k_1) ll}{D_0^2(-\omega_0, k_1) + Ra \hat{P}(k_1) ll} \right] A_\mu,
\]

\[
3 T_{(1)}^{mk} = - \exp i\varphi_2 \frac{i k_2^m Ra l_\mu}{D_0^2(-\omega_0, k_2) + Ra \hat{P}(k_2) ll} \hat{P}_{kn}(k_1) \times (129) \\
\times \left[ 1 - \frac{Ra \hat{P}(k_2) ll}{D_0^2(-\omega_0, k_2) + Ra \hat{P}(k_2) ll} \right] B_\mu.
\]

As a matter of fact, all these expressions are considerably simplified. Actually, the operator \( D_0 = \partial_t - \partial^2 \) acts on its own function \( \exp(i\omega_0 t + i\vec{k} \vec{x}) \) and gives \( D_0 \exp(i\omega_0 t + i\vec{k} \vec{x}) = D_0(\omega_0, k_0) \exp(i\omega_0 t + i\vec{k} \vec{x}) \), where \( D_0(\omega_0, k_0) = i\omega_0 + k_0^2 \). In dimensionless variables this means that:
Figure 10: Helical kink which corresponds to the separatrix in fig.9.

\[ D_0(\omega_0, k_0) = 1 + i; D_0(-\omega_0, k_0) = 1 - i = D_0^*(\omega_0, k_0). \]  
(130)

The expression \( \tilde{P}_{kn} l_k l_n \equiv \tilde{P}ll = l^2 - \frac{(k_1)^2}{c^2} = 1 \), since \( \bar{k}_1, \bar{k}_2 \perp \bar{l} \). As a result:

\[ D_0^2(\omega_0, k_0) + Ra \tilde{P}ll = 2i + Ra; D_0^2(-\omega_0, k_0) + Ra \tilde{P}ll = (2i + Ra)^* \].  
(131)

And all tensors are simplified considerably:

\[ 1v_0^P = \exp i\varphi_1 \left[ \delta_{pm} - \frac{Ra\tilde{P}_{pm}(k_1)l_p l_m}{(2i + Ra)^*} \right] \frac{A_m}{(1 - i)} \]  
(132)

\[ 3v_0^P = \exp i\varphi_2 \left[ \delta_{pm} - \frac{Ra\tilde{P}_{pm}(k_2)l_p l_m}{(2i + Ra)^*} \right] \frac{B_m}{(1 - i)} \]  
(133)

\[ 1T_{m}^{(2)} = \exp i\varphi_1 \frac{ik_1^m \tilde{P}_{kn}(k_1)}{(1 - i)^2} \left[ \delta_{n \lambda} - \frac{Ra\tilde{P}_{\lambda j}(k_1)l_n l_j}{(2i + Ra)^*} \right] \times \]  
\[ \times \left[ \delta_{\lambda \mu} - \frac{Ral_{\lambda \mu}}{(2i + Ra)^*} \right] A_{\mu}, \]  
(134)

\[ 3T_{m}^{(2)} = \exp i\varphi_2 \frac{ik_2^m \tilde{P}_{kn}(k_2)}{(1 - i)^2} \left[ \delta_{n \lambda} - \frac{Ra\tilde{P}_{\lambda j}(k_2)l_n l_j}{(2i + Ra)^*} \right] \times \]  
\[ \times \left[ \delta_{\lambda \mu} - \frac{Ral_{\lambda \mu}}{(2i + Ra)^*} \right] B_{\mu}. \]  
(135)

\[ 1T_{m}^{(1)} = -\exp i\varphi_1 \frac{ik_1^m Ra l^m l^{\mu}}{(1 - i)^4} \tilde{P}_{kn}(k_1) \times \]  
\[ \times \left[ 1 - \frac{Ra}{(2i + Ra)^*} \right]^2 A_{\mu}, \]  
(136)

\[ 3T_{m}^{(1)} = -\exp i\varphi_2 \frac{ik_2^m Ra l^m l^{\mu}}{(1 - i)^4} \tilde{P}_{kn}(k_1) \times \]  
(137)
\[
\left[ 1 - \frac{Ra}{(2i + Ra)^2} \right] B_\mu.
\]

The reason of the further simplification of these expressions is:

\[
\hat{P}_{\mu\nu}(k) l_\nu = l_\mu.
\] (138)

Taking into account (138), we obtain:

\[
1v_0^P = \exp i\varphi_1 \left[ \delta_{\rho m} - \frac{Ra l_m}{(2i + Ra)} \right] \frac{A_m}{(1 - i)};
\] (139)

\[
3v_0^P = \exp i\varphi_2 \left[ \delta_{\rho m} - \frac{Ra l_m}{(2i + Ra)} \right] \frac{B_m}{(1 - i)};
\] (140)

\[
1T_{mk}^{(2)} = \exp i\varphi_1 \frac{i k_1^m \hat{P}_k(k_1)}{(1 - i)^2} \left[ \delta_{n\mu} - \frac{Ra l_n}{(2i + Ra)} \right] \times \left[ \delta_{\lambda\mu} - \frac{Ra l_\lambda}{(2i + Ra)2} \right] A_\mu;
\] (141)

\[
3T_{mk}^{(2)} = \exp i\varphi_2 \frac{i k_2^m \hat{P}_k(k_2)}{(1 - i)^2} \left[ \delta_{n\mu} - \frac{Ra l_n}{(2i + Ra)} \right] \times B_\mu.
\] (142)

\[
1T_{mk}^{(1)} = - \exp i\varphi_1 \frac{i k_1^m Ra l^{k\mu}}{(1 - i)^4} \left[ 1 - \frac{Ra}{(2i + Ra)2} \right]^2 A_\mu;
\] (143)

\[
3T_{mk}^{(1)} = - \exp i\varphi_2 \frac{i k_2^m Ra l^{k\mu}}{(1 - i)^4} \left[ 1 - \frac{Ra}{(2i + Ra)2} \right]^2 B_\mu.
\] (144)

We will do calculations of Reynolds stresses in several stages. To begin with, we consider the term \( v_0^P T_{mk}^{(2)} \). This average value is composed of four terms in which the oscillation phase is cancelled:

\[
v_0^P T_{mk}^{(2)} = [v_0^P T^{(2)}_{mk} + 2v_0^P T^{(2)}_{mk} + 3v_0^P T^{(2)}_{mk} + 4v_0^P T^{(2)}_{mk}] (145)
\]

The second term in the (145) is conjugated with the first one, and the fourth with the third one. Now we substitute in the (145) correspondent expressions for tensors and we obtain:

\[
\overline{v_0^P T^{(2)}_{mk}} = \text{Re} \left( \frac{-i k_1^m}{(1 + i)} \right) \left[ A_\mu - \frac{Ra l^{z\mu}}{(2i + Ra)^2} \right] \times \left[ A_k^* - k_1^k A_z^* - \frac{2Ra l_b A_z^*}{(2i + Ra)^2} \right] +
\]

\[
+ \text{Re} \left[ \frac{(1 + i)}{k_1^m} [B_\mu - \frac{Ra l^{z\mu}}{(2i + Ra)^2} |B_k^* - k_2^k B_z^* - \frac{2Ra l_b B_z^*}{(2i + Ra)^2}] \right].
\] (146)

Let us find now the components of Reynolds stresses \( v_0^P T_{k2}^{(2)} = -v_0^P W_{k2} T_{mk}^{(2)} \). Taking in consideration the (146), we obtain:

\[
\overline{v_0^P T_{k2}^{(2)}} = \text{Re} \left( \frac{i W_k}{(1 + i)} \right) \left[ A_\mu - \frac{Ra l^{z\mu}}{(2i + Ra)^2} \right] \times
\] (147)
Taking into account the Let us put the full contribution in the tensor of the Reynolds stresses \(v_0^p v_2^p + v_0^k v_2^k\) from the tensor \(T_{(2)}^{mk}\) is obtained using the symmetrization of this equation over the indices \(p, k\). As a result we obtain:

\[
+Re \left[ \frac{iW_y}{1 + i} [B^p - \frac{Ra_l B^z}{(2i + Ra)^2}] [B^*_k - k^*_{l} B^*_y - \frac{2Ra_l B^*_x}{(2i + Ra)^2} + \frac{Ra^2 l_k B^*_z}{(2i + Ra)^2}] \right] + \frac{1}{4} \left( \frac{Ra}{(2i + Ra)^2} + \frac{2Ra}{(2i + Ra)} - \frac{Ra^2}{(2i + Ra)^2} \right) + \frac{1}{4} \left( \frac{Ra}{(2i + Ra)^2} + \frac{2Ra}{(2i + Ra)} - \frac{Ra^2}{(2i + Ra)^2} \right) \right].
\]

Let us put \(p = z, k = 1\), in the equation (148), i.e. we find the component \(x\) in the Reynolds stress. It is easy to see that:

\[
\overline{\frac{v_0^x v_2^p + v_0^k v_2^p}{(1 + i)}} = Re \nabla_z \left\{ \frac{iW_y}{1 + i} \left[ 1 - \frac{Ra}{(2i + Ra)^2} B^z B^*_1 \right] \right\} + \frac{1}{4} \left( \frac{Ra}{(2i + Ra)^2} + \frac{2Ra}{(2i + Ra)} - \frac{Ra^2}{(2i + Ra)^2} \right) + \frac{1}{4} \left( \frac{Ra}{(2i + Ra)^2} + \frac{2Ra}{(2i + Ra)} - \frac{Ra^2}{(2i + Ra)^2} \right) \right].
\]

Taking into account the \(B^*_2 B_1 = \frac{i}{4}\), we obtain:

\[
\overline{\frac{v_0^x v_2^p + v_0^k v_2^p}{(1 + i)}} = Re \nabla_z W_y \left\{ \frac{1}{4(1 + i)} \left[ - \frac{Ra}{(2i + Ra)^2} + \frac{2Ra}{(2i + Ra)} - \frac{Ra^2}{(2i + Ra)^2} \right] \right\}.
\]

After calculating the real part in the (150) we obtain:

\[
\overline{\frac{v_0^x v_2^p + v_0^k v_2^p}{(1 + i)}} = \alpha_1 \nabla_z W_y,
\]

Where:

\[
\alpha_1 = -Ra \frac{(-4Ra + Ra^2 + 12)}{4(Ra^2 + 4)^2}.
\]

Putting \(p = z, k = 2\), in the (148), we find the corresponding component in the Reynolds stress. It is easy to see that:

\[
\overline{\frac{v_0^z v_2^p + v_0^k v_2^p}{(1 + i)}} = Re \nabla_z \left\{ \frac{iW_x}{1 + i} \times \right\} \times \left[ \frac{Ra}{(2i + Ra)^2} A_3 A_2^* - \frac{2Ra}{(2i + Ra)^2} A_2 A_3^* + \frac{Ra^2}{(2i + Ra)^2} A_2^2 A_3^* \right].
\]
\[ + \text{Re} \left\{ \nabla_z iW_y \left[ 1 - \frac{2Ra}{(2i + Ra)} + \frac{Ra^2}{(2i + Ra)^2} \right] B_2B_3^* \right\}. \]

As far as \( B_2B_3^* = 0, A_3A_3^* = \frac{1}{4}, \) we obtain from the (153):

\[ \frac{v_0^p v_z^2}{v_0^p v_z^2} = -\alpha_1 \text{\nabla}_z W_x. \] (154)

Now we need to calculate the contribution in the Reynolds stresses from the tensor \( T^{mk}_{(1)} \). As it was done previously

\[ \overline{v_0^P T^{mk}_{(1)}} = [1v_0^P [2 T^{mk}_{(1)}] + [2v_0^P] [1 T^{mk}_{(1)}] + [3v_0^P][4 T^{mk}_{(1)}] + [4v_0^P][3 T^{mk}_{(1)}] \] (155)

With this the second term is conjugated with the first one and the fourth is conjugated with the third one. The simple calculation of the first term gives:

\[ \overline{[1v_0^P [2 T^{mk}_{(1)}]} = - \frac{Ra}{2(1+i)^3} \left[ 1 - \frac{Ra}{(2i + Ra)} \right]^{2} (-ik_1^m)^k \times \] (156)

\[ \times \left[ A_p - \frac{Ral_p A_3}{(2i + Ra)^*} \right] A_3^*. \]

We calculate similarly the third term:

\[ \overline{[3v_0^P][4 T^{mk}_{(1)}]} = - \frac{Ra}{2(1+i)^3} \left[ 1 - \frac{Ra}{(2i + Ra)} \right]^{2} (-ik_2^m)^k \times \] (157)

\[ \times \left[ B_p - \frac{Ral_p B_3}{(2i + Ra)^*} \right] B_3^*. \]

Now it is easy to find the contribution in the \( v_0^p v_z^2 \):

\[ \frac{v_0^p v_z^2}{v_0^p v_z^2} = \] (158)

\[ = \text{Re} \left\{ - \frac{Ra}{(1+i)^3} \left[ 1 - \frac{Ra}{(2i + Ra)} \right]^{2} (iW_x)^l \right\} \left[ A_p - \frac{Ral_p A_3}{(2i + Ra)^*} \right] A_3^* + \]

\[ + \text{Re} \left\{ - \frac{Ra}{(1+i)^3} \left[ 1 - \frac{Ra}{(2i + Ra)} \right]^{2} (iW_y)^l \right\} \left[ B_p - \frac{Ral_p B_3}{(2i + Ra)^*} \right] B_3^* \} \]

After symmetrizing this tensor over the indices \( p, k \), we obtain:

\[ \frac{v_0^p v_z^2}{v_0^p v_z^2} = \] (159)

\[ = \text{Re} \left\{ - \frac{Ra}{(1+i)^3} \left[ 1 - \frac{Ra}{(2i + Ra)} \right]^{2} (iW_x)^l \right\} \left[ A_p - \frac{Ral_p A_3}{(2i + Ra)^*} \right] A_3^* + \]

\[ + \text{Re} \left\{ - \frac{Ra}{(1+i)^3} \left[ 1 - \frac{Ra}{(2i + Ra)} \right]^{2} (iW_y)^l \right\} \left[ B_p - \frac{Ral_p B_3}{(2i + Ra)^*} \right] B_3^* \} + \]

\[ + \text{Re} \left\{ - \frac{Ra}{(1+i)^3} \left[ 1 - \frac{Ra}{(2i + Ra)} \right]^{2} (iW_x)^l \right\} \left[ A_k - \frac{Ral_k A_3}{(2i + Ra)^*} \right] A_3^* + \]

\[ + \text{Re} \left\{ - \frac{Ra}{(1+i)^3} \left[ 1 - \frac{Ra}{(2i + Ra)} \right]^{2} (iW_y)^l \right\} \left[ B_k - \frac{Ral_k B_3}{(2i + Ra)^*} \right] B_3^* \} + \]
Putting \( p = z, k = 1 \), in the formula (159), we obtain the tensor \( x \)-component of the Reynolds stress:

\[
\overline{v_0^2 v_1^2} + \overline{v_0 v_2} = \text{Re} \left\{ -\frac{i Ra}{(1+i)^3} \left( 1 - \frac{Ra}{(2i+Ra)} \right)^2 \nabla_z [W_x A_1 A_3^* + W_y B_1 B_3^*] \right\}.
\] (160)

Since \( A_1 A_3^* = 0, B_1 B_3^* = \frac{i}{4} \), we get:

\[
\overline{v_0^2 v_1^2} + \overline{v_0 v_2} = \text{Re} \left\{ \frac{Ra}{4(1+i)^3} \left( 1 - \frac{Ra}{(2i+Ra)} \right)^2 \nabla_z W_y \right\}.
\] (161)

After calculating the real part, we obtain:

\[
\overline{v_0^2 v_1^2} + \overline{v_0 v_2} = \alpha_2 \nabla_z W_y
\] (162)

Where

\[
\alpha_2 = -\frac{Ra}{4(4+Ra^2)^2} (4 - Ra^2 - 4Ra).
\] (163)

Putting in the formula (159) \( p = z, k = 2 \) we easily get:

\[
\overline{v_0^2 v_2^2} + \overline{v_0 v_3} = \text{Re} \left\{ -\frac{i Ra}{(1+i)^3} \left( 1 - \frac{Ra}{(2i+Ra)} \right)^2 \nabla_z [W_x A_2 A_3^* + W_y B_2 B_3^*] \right\}.
\] (164)

Since \( A_2 A_3^* = -\frac{i}{2}, B_2 B_3^* = 0 \), then

\[
\overline{v_0^2 v_2^2} + \overline{v_0 v_3} = -\alpha_2 \nabla_z W_x
\] (165)

and designating:

\[
\alpha = \alpha_1 + \alpha_2 = -Ra \frac{4 - 2Ra}{(4+Ra^2)^2},
\] (166)

we obtain the final expressions for the tensor components of the Reynolds stresses:

\[
\overline{v_0^2 v_1^2} + \overline{v_0 v_2} = \alpha \nabla_z W_y
\] (167)

\[
\overline{v_0^2 v_2^2} + \overline{v_0 v_3} = -\alpha \nabla_z W_x
\] (168)

The terms (167) and (168) are fundamental. Nevertheless, strictly speaking one must calculate other, less important, terms. Let us consider the component \( p = z, k = 3 \) of the Reynolds stress. Putting in (148) \( p = z, k = 3 \), we get the contribution of the tensor \( T_{mk}^{(2)} \):

\[
\overline{v_0^2 v_3^2} + \overline{v_0 v_4} = \text{Re} \left\{ \frac{2i W_x}{1+i} \left[ 1 - \frac{Ra}{(2i+Ra)^*} \right] \left[ 1 - \frac{2Ra}{(2i+Ra)} \right] A_3 A_3^* \right\} + \text{Re} \left\{ \frac{2i W_y}{1+i} \left[ 1 - \frac{Ra}{(2i+Ra)^*} \right] \left[ 1 - \frac{2Ra}{(2i+Ra)} \right] B_3 B_3^* \right\}.
\] (169)
As far as \( A_3 A_3^* = B_3 B_3^* = \frac{1}{4} \), we obtain:

\[
\overline{v_0^2 v_2^3} + \overline{v_0^3 v_2} = C_2 \nabla_z (W_x + W_y), \tag{170}
\]

where

\[
C_2 = \frac{-Ra^3 + 2Ra^2 - 4Ra + 8}{(4 + Ra^2)^3}. \tag{171}
\]

Now we can find the contribution in \( v_0^2 v_2^3 + v_0^3 v_2 \) of the tensor \( T_{mn}^{(1)} \). Putting in the formula \( p = z, k = 3 \), we obtain:

\[
\overline{v_0^2 v_2^3} + \overline{v_0^3 v_2} = C_1 \nabla_z (W_x + W_y), \tag{172}
\]

where

\[
C_1 = -Ra \frac{2 + Ra}{(4 + Ra^2)^2}. \tag{173}
\]

Designating the common coefficient \( C = C_1 + C_2 \),

\[
C = \frac{8 - 12Ra - 2Ra^2 - 3Ra^3 - Ra^4}{(4 + Ra^2)^3}, \tag{174}
\]

we obtain:

\[
\overline{v_0^2 v_2^3} + \overline{v_0^3 v_2} = C \nabla_z (W_x + W_y), \tag{175}
\]

11 Appendix C. Closure of the temperature equation.

In order to close the temperature equation we have to calculate the term:

\[
\nabla_z \left( \overline{v_{(1)}^z T^{(2)}} + \overline{v_{(0)}^z T^{(2)}} \right) \tag{176}
\]

It follows from the formula \( p = z, k = 3 \), that indeed there is the contribution only of the following terms in \( T^{(2)} \):

\[
T^{(2)} = -\frac{1}{D_0} v_{(0)}^z - \frac{1}{D_0} W_p \partial_p T^{(0)}. \tag{177}
\]

From the formula \( \text{[159]} \) it is easy to find the component \( 1 T^{(0)} \), \( 2 T^{(0)} \), \( 3 T^{(0)} \), \( 4 T^{(0)} \), \( 1 T^{*} = 2 T^{(0)} \), \( 3 T^{*} = 4 T^{(0)} \) :

\[
2 T^{(0)} = -\exp(-i \varphi_1) \frac{1}{(1 + i)^2} \left[ 1 - \frac{Ra}{(2i + Ra)} \right] A_3^*, \tag{178}
\]

\[
4 T^{(0)} = -\exp(-i \varphi_2) \frac{1}{(1 + i)^2} \left[ 1 - \frac{Ra}{(2i + Ra)} \right] B_3^*. \tag{179}
\]

\[
T^{(2)} = W_{1} T^{p_{xz}} \tag{180}
\]

\[
T^{(2)} = W_{1} T^{p_{xz}} + \frac{1}{D_0} W_p \partial_p T^{(0)}. \tag{181}
\]

First of all we find the component \( \nabla_z \overline{v_{(0)}^z T^{(0)}} \) :
Calculations similar to the previous ones give:

\[ \nabla z v_z^{(2)} T(0) = 2\text{Re} \nabla z \left\{ \left[ 1 v_z^{(2)} \right] [2 T(0)] + \left[ 3 v_z^{(2)} \right] [4 T(0)] \right\}. \tag{182} \]

Since the \( v_z^{(2)} \) is composed of two terms \( 46 \), then at the beginning we find the contribution 
\(-W_q[T_{(1)}]_z [2 T(0)]\). As it was done in previous calculations we obtain:

\[ -W_q[T_{(1)}]_z [2 T(0)] = \]

\[ \frac{(-i) Ra W_x}{(1 - i)^3(1 + i)^2} \left[ 1 - \frac{Ra}{(2i + Ra)^*} \right] \left[ 1 - \frac{Ra}{(2i + Ra)} \right] A_z A_z^*. \tag{183} \]

Further we find the contribution:

\[ -W_q[T_{(2)}]_z [2 T(0)] = \frac{i W_x}{4} \left[ 1 - \frac{Ra}{(2i + Ra)^*} \right] \left[ 1 - \frac{Ra}{(2i + Ra)} \right] A_z A_z^*. \tag{184} \]

As a result we get the term:

\[ 2\text{Re} \nabla z \left[ 1 v_z^{(2)} \right] [2 T(0)] = \]

\[ = 2\text{Re} \nabla z W_x \left( -\frac{i}{4} \right) \left[ \frac{Ra}{(1 - i)^2} - 1 \right] \left[ 1 - \frac{Ra}{(2i + Ra)^*} \right] \left[ 1 - \frac{Ra}{(2i + Ra)} \right] A_z A_z^*. \tag{185} \]

Let us come now to the calculations of the term \( 2\text{Re} \nabla z \left\{ \left[ 3 v_z^{(2)} \right] [4 T(0)] \right\} :\)

\[ 2\text{Re} \nabla z \left\{ \left[ 3 v_z^{(2)} \right] [4 T(0)] \right\} = \]

\[ = 2\text{Re} \nabla z \left\{ -W_q[T_{(1)}]_z [4 T(0)] - W_q[T_{(2)}]_z [4 T(0)] \right\}. \tag{186} \]

Calculations similar to the previous ones give:

\[ -W_q[T_{(1)}]_z [4 T(0)] = \]

\[ = \frac{(-i) Ra W_y}{(1 - i)^3(1 + i)^2} \left[ 1 - \frac{Ra}{(2i + Ra)^*} \right] \left[ 1 - \frac{Ra}{(2i + Ra)} \right] B_z B_z^*, \tag{187} \]

\[ -W_q[T_{(2)}]_z [4 T(0)] = \frac{i W_y}{4} \left[ 1 - \frac{Ra}{(2i + Ra)^*} \right] \left[ 1 - \frac{Ra}{(2i + Ra)} \right] B_z B_z^*. \tag{188} \]

As a result we obtain the term \( \nabla z v_z^{(2)} T(0) \):

\[ \nabla z v_z^{(2)} T(0) = \]

\[ = 2\text{Re} \nabla z W_x \left( -\frac{i}{4} \right) \left[ \frac{Ra}{(1 - i)^2} - 1 \right] \left[ 1 - \frac{Ra}{(2i + Ra)^*} \right] \left[ 1 - \frac{Ra}{(2i + Ra)} \right] |A_z|^2 + \]

\[ +2\text{Re} \nabla z W_y \left( -\frac{i}{4} \right) \left[ \frac{Ra}{(1 - i)^2} - 1 \right] \left[ 1 - \frac{Ra}{(2i + Ra)^*} \right] \left[ 1 - \frac{Ra}{(2i + Ra)} \right] |B_z|^2. \tag{189} \]

Let us come now to the calculations of the term \( v_z^{(0)} T(2) \).
Let us write down the auxiliary expressions:

\[
[2T(2)] = \frac{1}{1+i} \left( W_q[2T_{(1)}^{qz}] + W_q[2T_{(2)}^{qz}] + iW_x[2T_{(0)}] \right);
\]

\[
[4T(2)] = \frac{1}{1+i} \left( W_q[4T_{(1)}^{qz}] + W_q[4T_{(2)}^{qz}] + iW_y[4T_{(0)}] \right).
\]

Taking into account these formulae and expressions for \([2T_{(1)}^{qz}], [2T_{(2)}^{qz}], [4T_{(1)}^{qz}], [4T_{(2)}^{qz}]\), it is not difficult to get an expression for the \(\nabla z T_{(2)}^{qz}\):

\[
\nabla z T_{(2)}^{qz} = 2 \text{Re} \left\{ \left[ 1 T_{(0)}^{qz} \right] [2T(2)] + \left[ 3 T_{(0)}^{qz} \right] [4T(2)] \right\}.
\]

(190)

12 Appendix D. Scheme of asymptotical development for the non linear case

Let us present the algebraical structure of the asymptotical development of the equations (3), (4) for the non linear theory (we will not write indices because they can be restored evidently at any moment). In the order \(R^{-3}\) there is only the equation:

\[
\partial P_{-3} = 0, \Rightarrow P_{-3} = P_{-3}(X).
\]

(195)

In the order \(R^{-2}\) we have the equation:

\[
\partial P_{-2} = 0, \Rightarrow P_{-2} = P_{-2}(X).
\]

(196)

In the order \(R^{-1}\) we get a system of equations:

\[
\partial_t W_{-1} - \partial^2 W_{-1} = -(\partial P_{-1} + \nabla P_{-3}) + RaT_{-1}l_{-2} - \partial W_{-1}W_{-1},
\]

(197)
\[ \partial_t T_{-1} - \partial^2 T_{-1} = -\partial W_{-1} T_{-1} - W_{-1}^z, \]  
(198)

\[ \partial W_{-1} = 0. \]

The system of equations \([197], [198]\) gives secular terms:

\[ - \nabla P_{-3} + Ra T_{-1} l_z = 0. \]  
(199)

\[ W_{-1}^z = 0. \]  
(200)

In zero order \(R^0\) we have the following system of equations:

\[ \partial_t v_0 - \partial^2 v_0 + \partial(W_{-1} v_0 + v_0 W_{-1}) = \]  
(201)

\[ = -(\partial P_0 + \nabla P_{-2}) + Ra T_0 l_z + F, \]

\[ \partial_t T_0 - \partial^2 T_0 + \partial(W_{-1} T_0 + v_0 T_{-1}) = -v_{0}^z. \]  
(202)

\[ \partial v_0 = 0. \]

These equations give one secular equation:

\[ \nabla P_{-2} = 0, \Rightarrow P_{-2} = \text{Const.} \]  
(203)

Consider the equations of the first approximation \(R\):

\[ \partial_t v_1 - \partial^2 v_1 + \partial(W_{-1} v_1 + v_1 W_{-1} + v_0 v_0) = \]  
(204)

\[ = -\nabla(W_{-1} W_{-1}) - (\partial P_1 + \nabla P_{-1}) + Ra T_1 l_z. \]

\[ \partial_t T_1 - \partial^2 T_1 + \partial(W_{-1} T_1 + v_1 T_{-1} + v_0 T_0) + \nabla(W_{-1} T_{-1}) = -v_{1}^z. \]  
(205)

\[ \partial V_1 + \nabla W_{-1} = 0. \]  
(206)

From this system of equations follow the secular equations:

\[ \nabla W_{-1} = 0, \]  
(207)

\[ \nabla(W_{-1} W_{-1}) = -\nabla P_{-1}, \]  
(208)

\[ \nabla(W_{-1} T_{-1}) = 0. \]  
(209)

The secular equations \([207]-[209]\), are obviously satisfied in the previous velocity field geometry:

\[ W = (W_x(Z), W_y(Z), 0); T_{-1} = T_{-1}(Z); \nabla P_{-1} = 0, \Rightarrow P_{-1} = \text{Const.} \]  
(210)

In the second order \(R^2\), we obtain equations:

\[ \partial_t v_2 - \partial^2 v_2 - 2\partial \nabla v_0 + \partial(W_{-1} v_2 + v_2 W_{-1} + v_0 v_1 + v_1 v_0) = \]  
(211)

\[ = -\nabla(W_{-1} v_0 + v_0 W_{-1}) - (\partial P_2 + \nabla P_0) + Ra T_2 l_z, \]
\[ \partial_t T_2 - \partial^2 T_2 - 2\partial \nabla T_0 + \partial (W_{-1} T_2 + v_2 T_{-1} + v_0 T_1 + v_1 T_0) = \]
\[ = -\nabla (W_{-1} T_0 + v_0 T_{-1}) - v_2. \]

\[ \partial v_2 + \nabla v_0 = 0. \tag{213} \]

It is easy to see that in the order \( R^2 \) there is no secular terms.

Let us come now to the most important order \( R^3 \). In this order we obtain equations:

\[ \partial_t v_3 + \partial_r W_{-1} - (\partial^2 v_3 + 2\partial \nabla v_1 + \Delta W_{-1}) + \nabla (W_{-1} v_1 + v_1 W_{-1} + v_0 v_0) + \]
\[ + \partial (W_{-1} v_3 + v_3 W_{-1} + v_0 v_2 + v_2 v_0 + v_1 v_1) = -(\partial P_3 + \nabla P_1) + Ra T_3 l_z. \tag{214} \]

\[ \partial_t T_3 + \partial_r T_{-1} - (\partial^2 T_3 + 2\partial \nabla T_1 + \Delta T_{-1}) + \nabla (W_{-1} T_1 + v_1 T_{-1} + v_0 T_0) + \]
\[ + \partial (W_{-1} T_3 + v_3 T_{-1} + v_0 T_2 + v_2 T_0 + v_1 T_1) = -v_3^2. \tag{215} \]

\[ \partial v_3 + \nabla v_1 = 0. \]

From this we get the main secular equation:

\[ \partial_r W_{-1} - \Delta W_{-1} + \nabla (v_0 v_0) = -\nabla P_1, \tag{216} \]

\[ \partial_r T_{-1} - \Delta T_{-1} + \nabla (v_0 T_0) = 0. \tag{217} \]

13 Appendix E. Reynolds stress in non linear case

In order to calculate the Reynolds stresses we have first of all to calculate the expression:

\[ \overline{v_0^k v_0^l} = 2Re \left( \frac{v_0^k v_0^l}{k_0^v} + \frac{v_0^k v_0^l}{k_0^v} \right). \tag{218} \]

Taking into account the formula \( \text{(90)} \), we obtain:

\[ \frac{v_0^k v_0^l}{k_0^v} + \frac{v_0^k v_0^l}{k_0^v} = T_{(1)}^{2k} = \frac{1}{|D_1|^2} (A_k A_i^* + A_k^* A_i) + \]
\[ - \frac{Ra A_k^*}{|D_1|^2} \left( \frac{l_k A_i + l_i A_k}{D_1^2 + Ra} \right) - \frac{Ra A_k^*}{|D_1|^2} \left( \frac{l_k A_i^* + l_i A_k^*}{D_1^2 + Ra} \right) + \]
\[ + \frac{2}{|D_1|^2} \frac{Ra^2 l_k l_i |A_k|^2}{|D_1^2 + Ra|^2}. \tag{219} \]

Similarly taking into account the formula \( \text{(91)} \), we obtain:

\[ \frac{v_0^k v_0^l}{k_0^v} + \frac{v_0^k v_0^l}{k_0^v} = T_{(2)}^{2k} = \frac{1}{|D_2|^2} (B_k B_i^* + B_k^* B_i) - \]
\[ - \frac{Ra B_k^*}{|D_2|^2} \left( \frac{l_k B_i + l_i B_k}{D_2^2 + Ra} \right) - \frac{Ra B_k^*}{|D_2|^2} \left( \frac{l_k B_i^* + l_i B_k^*}{D_2^2 + Ra} \right) + \]
\[ + \frac{2}{|D_2|^2} \frac{Ra^2 l_k l_i |B_k|^2}{|D_2^2 + Ra|^2}. \tag{220} \]
\[
+ \frac{2}{|D_2|^2} \frac{Ra_i l_i |B_2|^2}{|D_2^2 + Ra|^2}.
\]

It is clear that the components \(T_{31}^{(1)}\) and \(T_{32}^{(2)}\) are of interest. To begin with we consider the components of the tensor \(T_{31}^{(1)}\),

\[
T_{31}^{(1)} = \frac{1}{|D_1|^2} (A_3 A_1^* + A_3^* A_1) - \frac{Ra}{|D_1|^2} \left( \frac{A_3 A_1^*}{D_1^2 + Ra} + \frac{A_3^* A_1}{D_1^2 + Ra} \right) = 0,
\]

Since \(A_3 A_1^* = A_3^* A_1 = 0\).

\[
T_{32}^{(1)} = \frac{1}{|D_1|^2} (A_3 A_2^* + A_3^* A_2) - \frac{Ra}{|D_1|^2} \left( \frac{A_3 A_2^*}{D_1^2 + Ra} + \frac{A_3^* A_2}{D_1^2 + Ra} \right).
\]

The first bracket in the (222) is equal to zero, which is why:

\[
T_{32}^{(1)} = \frac{i}{4|D_1|^2} \left( \frac{D_2^2 - D_1^{*2}}{|D_1^2 + Ra|^2} \right).
\]

Now consider the component \(T_{32}^{(2)}\):

\[
T_{32}^{(2)} = \frac{1}{|D_2|^2} (B_3 B_2^* + B_3^* B_2) - \frac{Ra}{|D_2|^2} \left( \frac{B_3 B_2^*}{D_2^2 + Ra} + \frac{B_3^* B_2}{D_2^2 + Ra} \right) = 0.
\]

As far as \(B_3^* B_2 = B_3 B_2^* = 0\) we consider the component \(T_{31}^{(2)}\):

\[
T_{31}^{(2)} = \frac{1}{|D_2|^2} (B_3 B_1^* + B_3^* B_1) - \frac{Ra}{|D_2|^2} \left( \frac{B_3 B_1^*}{D_2^2 + Ra} + \frac{B_3^* B_1}{D_2^2 + Ra} \right).
\]

The first bracket in the formula (225) is equal to zero, then:

\[
T_{31}^{(2)} = -\frac{i}{4|D_2|^2} \left( \frac{D_2^{*2} - D_2^2}{|D_2^2 + Ra|^2} \right).
\]

Taking into account:

\[
(D_1^2 - D_1^{*2}) = 4i(1 - W_1), (D_2^{*2} - D_2^2) = -4i(1 - W_2),
\]

\[
|D_1|^2 = 1 + (1 - W_1); |D_2|^2 = 1 + (1 - W_2),
\]

\[
|D_1^2 + Ra|^2 = (W_1(2 - W_1) + Ra)^2 + 4(1 - W_1)^2,
\]

\[
|D_2^{*2} + Ra|^2 = (W_2(2 - W_2) + Ra)^2 + 4(1 - W_2)^2.
\]

The components \(T_{32}^{(1)}\), \(T_{31}^{(2)}\) take the form:
\[ T_{32}^{(1)} = \frac{Ra(1 - W_1)}{[1 + (1 - W_1)^2][(W_1(2 - W_1) + Ra)^2 + 4(1 - W_1)^2]}, \] (231)

\[ T_{31}^{(2)} = -\frac{Ra(1 - W_2)}{[1 + (1 - W_2)^2][(W_2(2 - W_2) + Ra)^2 + 4(1 - W_2)^2]}, \] (232)

Now calculate the components \( T_{33}^{(1)} \) and \( T_{33}^{(2)} \). It is easy to see that:

\[ T_{33}^{(1)} = \frac{2|A_3|^2}{|D_1|^2} \left[ 1 - Ra \frac{(D_1^2 + D_1^2)}{|D_1^2 + Ra|^2} - \frac{Ra^2}{|D_1^2 + Ra|^2} \right], \] (233)

\[ T_{33}^{(2)} = \frac{2|B_3|^2}{|D_2|^2} \left[ 1 - Ra \frac{(D_2^2 + D_2^2)}{|D_2^2 + Ra|^2} - \frac{Ra^2}{|D_2^2 + Ra|^2} \right]. \] (234)

Or in the explicit form:

\[ T_{33}^{(1)} = \frac{1}{2[1 + (1 - W_1)^2]} \times \]
\[ \times \left[ 1 - Ra \frac{2[1 - (1 - W_1)^2] + Ra}{[W_1(2 - W_1) + Ra]^2 + 4(1 - W_1)^2} \right], \] (235)

\[ T_{33}^{(2)} = \frac{1}{2[1 + (1 - W_2)^2]} \times \]
\[ \times \left[ 1 - Ra \frac{2[1 - (1 - W_2)^2] + Ra}{[W_2(2 - W_2) + Ra]^2 + 4(1 - W_2)^2} \right]. \] (236)

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