Convex ordering for stochastic Volterra equations and their Euler schemes

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Abstract

In this paper, we are interested in comparing solutions to stochastic Volterra equations for the convex order on the space of continuous $\mathbb{R}^d$-valued paths and for the monotonic convex order when $d = 1$. Even if in general these solutions are neither semi-martingales nor Markov processes, we are able to exhibit conditions on their coefficients enabling the comparison. Our approach consists in first comparing their Euler schemes and then taking the limit as the time step vanishes. We consider two types of Euler schemes depending on the way the Volterra kernels are discretized. The conditions ensuring the comparison are slightly weaker for the first scheme than for the second one and this is the other way round for convergence. Moreover, we extend the integrability needed on the starting values in the existence and convergence results in the literature to be able to only assume finite first order moments, which is the natural framework for convex ordering.

Keywords: Stochastic Volterra equations, quadratic rough Heston model, convex order, Euler schemes

AMS Subject Classification (2020): 60E15, 60F15, 60G22, 91G30.

Introduction

The stochastic version of Volterra equations

$$X_t = X_0 + \int_0^t K(t, s)b(s, X_s)ds + \int_0^t K(t, s)\sigma(s, X_s)dW_s \quad (0.1)$$

goes back to the 1980’s in settings corresponding to smooth kernels (see [24]). Because of the dependence of the kernel $K(t, s)$ on its first argument, the solution is in general neither a semi-martingale nor a Markov process. Rather than addressing singular kernels, more attention was paid to other extensions like anticipative integrands using Skorokhod integration in connection with Malliavin calculus (see [23]). The more demanding case of singular kernels exploding on the diagonal (i.e. $K(t, s) \to +\infty$ as $s \to t$) appears in the literature in the early 2000’s (see [11]) motivated, among other applications, by the revival of interest for the fractional Brownian motion

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as a way to model long memory in financial mathematics and econometrics (see e.g. [10]). More recently, these stochastic Volterra equations with singular kernels have been brought back to light in a long series of papers devoted to the modeling of stochastic volatility in Finance which started with the empirical observation in [14] that volatility paths have low $H$-Hölder regularity exponent ($H \simeq 0.1$). Volterra equations with kernel $K(t, s) = (t-s)^{H-\frac{1}{2}}$ are a convenient model to reproduce this empirical feature since the Hölder exponent of their paths is $H$. Moreover, such Volterra equations with singular kernels appear as the limiting dynamics when modeling the order book by “nearly unstable” Hawkes processes (see [16]).

The resulting Volterra equation can be seen as the “rough” counterpart of the classical Cox-Ingersoll-Ross (CIR) process modeling the volatility in the classical/regular Heston stochastic volatility model. The rough Heston model obtained by substituting the Volterra process to the original CIR process can be used to price and hedge equity derivative products (see e.g. [6], [12]).

In the case when, more generally, $K(t, s)$ is a function of $t-s$, the stochastic Volterra equation is a stochastic convolution equation which can be studied using tools originally developed for the study of deterministic convolution equations, in particular resolvents (see [2, 1]).

On the other hand, convex ordering of random vectors has been developed for long as a refined way to quantify or compare risk in Econometrics and in Quantitative Finance. It can be defined between two distributions $\mu$ and $\nu$ on $\mathbb{R}^d$ (having a finite first moment) by

$$\mu \leq_{\text{cvx}} \nu \quad \text{if} \quad \forall \varphi : \mathbb{R}^d \to \mathbb{R} \text{ convex}, \quad \int \varphi(x) \mu(dx) \leq \int \varphi(y) \nu(dy).$$

This implies that both distributions have the same expectation. Two $\mathbb{R}^d$-valued random vectors $X$ and $Y$ are ordered for the convex order if their distributions are. This approach had more recently applications to determine the sign and to compare the sensitivities of derivatives products in an extensive (in fact functional) sense (see [7], [21], [18], [19]). It is also an important tool to define and identify the tracking error in the hedging process of derivative products when the volatility dynamics is misspecified. The aim of this paper is to extend some functional convex ordering results established for Brownian, jumpy or McKean-Vlasov diffusion processes (see [18], [21]) in the above cited papers (and called functional convex order or ordering) to the solution of Volterra equations. A typical result, consequence of Theorems 1.11 and 1.13 is the following: consider two solutions to the following equations (defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$):

$$X_t = X_0 + \int_0^t (t-s)^{H-\frac{1}{2}} \sigma(X_s) dW_s \quad \text{and} \quad Y_t = Y_0 + \int_0^t (t-s)^{H-\frac{1}{2}} \tilde{\sigma}(Y_s) dW_s, \; t \in [0, T]$$

($H \in (0, 1]$) where $W$ is standard Brownian motion and $\sigma, \tilde{\sigma} : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous. If $X_0, Y_0 \in L^1(\mathbb{P})$ with $X_0 \leq_{\text{cvx}} Y_0$ and

$$|\sigma| \quad \text{or} \quad |\tilde{\sigma}| \quad \text{is convex} \quad \text{and} \quad |\sigma| \leq |\tilde{\sigma}|,$$

then for every lower semi-continuous (l.s.c.) convex functional $F : (C([0, T], \mathbb{R}), \| \cdot \|_{\text{sup}}) \to \mathbb{R}$,

$$\mathbb{E} F((X_t)_{t \in [0, T]}) \leq \mathbb{E} F((Y_t)_{t \in [0, T]})$$

(where both expectations have a sense, see Section 1.2). Moreover if $|\sigma|$ is convex, then $x \mapsto \mathbb{E} F((X^x_t)_{t \in [0, T]})$ is convex (where $X^x_0 = x$). The extension of these results to the multidimensional
setting where $\sigma, \tilde{\sigma} : \mathbb{R}^d \to \mathbb{R}^{d \times q}$, $W$ is a $q$-dimensional Brownian motion and $X_0, Y_0$ are $\mathbb{R}^d$-valued random vectors holds under appropriate generalizations of the condition $|\sigma| \leq |\tilde{\sigma}|$ and the convexity of either $\sigma$ or $\tilde{\sigma}$.

The main strategy adopted in this paper to establish convex ordering properties on Volterra processes is two fold: first we establish these properties on a discrete time Euler scheme and then we transfer them to the continuous time Volterra process using a convergence theorem — here pathwise in $L^p$ — of the time discretization scheme to the process when the time step goes to 0. This roadmap has already been used in a Markovian framework by various authors (see [7, 8, 9, 22] for diffusions and [19] for McKean-Vlasov diffusions). Such an approach is important for applications: an approximation scheme which can be easily simulated that preserves (and transfers) convexity and convex ordering is a key asset since it avoids to create artificial arbitrage opportunities in hedging portfolios of derivative products computed by discretization and more generally permits a better monitoring of market risks.

Compared to the aforementioned diffusions models, several difficulties arise to implement this strategy and establish such a kind of result under natural $L^1(\mathbb{P})$ integrability assumptions on the starting values when dealing with Volterra equations, especially with singular kernels like in the above example. First, of course, since the solution is not a semi-martingale nor a Markov process, powerful tools like stochastic calculus or Markov properties cannot be used/called upon to establish convex ordering. The other more hidden difficulty follows from existing results about integrability and pathwise regularity of the solutions to the Volterra equation. Thus following [30], the existence and uniqueness of a pathwise continuous and integrable solution is established under the stringent assumption that $X_0$ and $Y_0$ lie in all $L^p(\mathbb{P})$-spaces i.e. $X_0, Y_0 \in \cap_{p>0}L^p(\mathbb{P})$ (under an additional appropriate integrability and regularity condition on the kernels which are satisfied by $K(t, s) = (t - s)^{H-\frac{1}{2}}$). We show in Theorem 1.1 that this result can be extended to the natural case where $X_0$ and $Y_0$ lie in some $L^p(\mathbb{P})$ for some $p \in (0, +\infty)$ by proving a representation formula of solutions as functionals of the Brownian path and the starting value (see Appendix D).

According to [29] or [26], the same restrictions in terms of integrability come out for the convergence of the $K$-discrete Euler scheme (1.9) of these equations where the second variable in the kernels is also chosen in the discretization grid. Moreover, the assumptions on the kernels $K(t, s)$ which ensure the $L^p(\mathbb{P})$-convergence (in sup-norm) of the continuous time extension of this Euler scheme toward the solution to the Volterra equation (see Theorem 1.4) are more stringent than those ensuring the existence of a strong solution to the equation itself (see Theorem 1.1). To overcome this problem we will also consider and analyze the convergence of the $K$-integrated Euler scheme (1.12) which can also be simulated in many cases of interest (depending on the form of the kernel). We will show in Theorem 1.2 that, if $X_0 \in L^p(\mathbb{P})$ for some $p > 0$, then it converges in $L^p(\mathbb{P})$ for the sup-norm under the same assumptions (up to time regularity of the coefficients of the equation) as those ensuring strong existence-uniqueness for the Volterra process. Propagating convexity and convex ordering through such a scheme appears as the adequate tool to establish quite general results on convex ordering for the Volterra equation, especially as concerns the assumptions on the kernels. The comparison between the two schemes remains balanced in the sense that, if relying on the $K$-discrete time scheme requires slightly more demanding assumptions on the kernels to get convergence, this scheme is sometimes easier and more stable to simulate for very small steps. The integrability restriction on $X_0$ needed in [26] is also relaxed for this scheme in Theorem 1.4. Of course both schemes have a higher complexity in terms of simulation than their
counterparts for regular diffusions due to the non-markovianity of the Volterra process (we refer to [4] for alternative approaches devised to improve the simulation of such processes).

The first section is devoted to the statement of the main results: existence and uniqueness for the stochastic Volterra process, convergence to this process of the discretization schemes as the time step goes to 0, convex ordering between two Volterra processes and monotonic convex ordering in dimension \( d = 1 \). It also gives an application to VIX options in the quadratic rough Heston model introduced in [15]. In Section 2, we state and prove the convex ordering for the \( K \)-discrete and \( K \)-integrated Euler schemes. Section 3 addresses the monotonic convex ordering for the discretization schemes in dimension \( d = 1 \). In Section 4, we deduce the ordering for the Volterra processes by implementing the second step of our strategy.

**Notation**
- We denote \( \mathbb{R}^{d \times q} \) the set of matrices with \( d \) rows and \( q \) columns.
- We denote \( A^* \) the transpose matrix of \( A \) (idem for vectors).
- We denote by \( S^+(d) = \{ S \in \mathbb{R}^{d \times d}, \text{ symmetric, positive semi-definite} \} \) and \( \mathcal{O}(d) = \{ P \in \mathbb{R}^{d \times d} : PP^* = I_d \} \).
- For \( S, T \in S^+(d), S \leq T \) if \( T - S \in S^+(d) \).
- For \( S \in S^+(d) \), we denote by \( \sqrt{S} \) the unique element of \( S^+(d) \) such that \( \sqrt{S} \sqrt{S} = S \).
- For \( A \in \mathbb{R}^{d_1 \times d_1} \) and \( B \in \mathbb{R}^{d_2 \times d_2} \), we denote by \( A \otimes B \in \mathbb{R}^{(d_1 \times d_2) \times (d_1 \times d_2)} \) the Kronecker product of \( A \) and \( B \) defined by

\[
A \otimes B = \begin{pmatrix}
A_{11}B & A_{12}B & \cdots & A_{1d_1}B \\
A_{21}B & A_{22}B & \cdots & A_{2d_1}B \\
\vdots & \vdots & & \vdots \\
A_{d_11}B & A_{d_12}B & \cdots & A_{d_1d_1}B
\end{pmatrix}
\]

- For a function \( f : E \to \mathbb{R}, \|f\|_{\sup} = \sup_{x \in E} |f(x)|. \)
- \( \lambda_d \) denotes the Lebesgue measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R}^d)) \), often denoted \( \lambda \) when \( d = 1 \).
- \( C([0,T], \mathbb{R}^d) \) (resp. \( C_0([0,T], \mathbb{R}^d) \)) denotes the set of continuous functions (resp. null at 0) from \([0,T]\) to \( \mathbb{R}^d \). \( \mathcal{B}(C_d) \) (resp. \( \mathcal{B}(C_{d,0}) \)) denotes its Borel \( \sigma \)-field induced by the sup-norm topology.
- \( \perp \) stands for independence of random variables, vectors or processes.
- \( L^0_{\mathbb{R}^d}(\mathbb{P}) \) or simply \( L^0(\mathbb{P}) \) denote the set of \( \mathbb{R}^d \)-valued random vectors defined on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\). For \( p \in (0, +\infty) \), \( L^p_{\mathbb{R}^d}(\mathbb{P}) \) or simply \( L^p(\mathbb{P}) \) denote the set of \( \mathbb{R}^d \)-valued random vectors \( X \) defined on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) such that \( \|X\|_p := E^{1/p}|X|^p < +\infty \).

1 Convex ordering of Volterra processes: main result

1.1 Volterra processes and their Euler schemes

We are interested in the stochastic Volterra equation

\[
X_t = X_0 + \int_0^t K_1(t, s)b(s, X_s)ds + \int_0^t K_2(t, s)\sigma(s, X_s)dW_s, \quad t \in [0, T],
\] (1.2)
where \( b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \), \( \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times q} \), \( K_i : \{ (t, s) : 0 \leq s < t \leq T \} \to \mathbb{R}_+ \), \( i \in \{1, 2\} \) are measurable and \( (W_t)_{t \in [0, T]} \) is a standard \( q \)-dimensional Brownian motion independent from the \( \mathbb{R}^d \)-valued random vector \( X_0 \). Let \( (\mathcal{F}_t)_{t \in [0, T]} \) be a filtration such that \( X_0 \in \mathcal{F}_0 \) and \( W \) is an \( \mathcal{F}_t \)-Brownian motion. The following existence and uniqueness result for \([1.2]\) is an improved version of \([30] \) Theorem 3.1 and Theorem 3.3, at least for Volterra equations having the above form.

**Theorem 1.1** Assume that the kernels \( K_i, i = 1, 2 \), satisfy the integrability assumption

\[
(K_{\beta}^{\text{int}}) \quad \sup_{t \in [0, T]} \int_0^t \left( K_1(t, s)^{2\beta} + K_2(t, s)^{2\beta} \right) ds < +\infty \tag{1.3}
\]

for some \( \beta > 1 \) and the continuity assumption

\[
(K_{\theta}^{\text{cont}}) \quad \exists \kappa < +\infty, \forall \delta \in (0, T),
\eta(\delta) := \max_{i=1,2} \sup_{t \in [0, T]} \left[ \int_0^t |K_i((t + \delta) \wedge T, s) - K_i(t, s)|^i ds \right]^{\frac{1}{i}} \leq \kappa \delta^\theta \tag{1.4}
\]

for some \( \theta \in (0, 1] \). Assume that the functions \( \mathbb{R}^d \ni x \mapsto b(t, x) \) and \( \mathbb{R}^d \ni x \mapsto \sigma(t, x) \) are Lipschitz with Lipschitz coefficient uniform in \( t \in [0, T] \), i.e.

\[ \exists C_r = C_{b, \sigma, T} \text{ such that } \forall t \in [0, T], \forall x, y \in \mathbb{R}^d, \ |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C_r |x - y| \]

and \( \sup_{t \in [0, T]} (|b(t, 0)| + |\sigma(t, 0)|) < +\infty \). Finally assume that \( X_0 \in L^p(\mathbb{P}) \) for some \( p \in (0, +\infty) \).

Then the Volterra equation \([1.2]\) admits, up to a \( \mathbb{P} \)-indistinguishability, a unique \((\mathcal{F}_t)\)-adapted solution \( X = (X_t)_{t \in [0, T]} \), pathwise continuous, in the sense that, \( \mathbb{P} \)-a.s.,

\[
\forall t \in [0, T], \quad X_t = X_0 + \int_0^t K_1(t, s)b(s, X_s)ds + \int_0^t K_2(t, s)\sigma(s, X_s)dW_s.
\]

This solution satisfies

\[
\forall s, t \in [0, T], \quad \|X_t - X_s\|_p \leq C_{p, T}(1 + \|X_0\|_p)|t - s|^\theta \frac{\beta - 1}{2\beta} \tag{1.5}
\]

Moreover,

\[
\forall a \in (0, \theta + \frac{\beta - 1}{2\beta}), \quad \left\| \sup_{s \neq t \in [0, T]} \frac{|X_t - X_s|}{|t - s|^a} \right\|_p < C_{a, p, T}(1 + \|X_0\|_p) \tag{1.6}
\]

for some positive real constant \( C_{a, p, T} = C_{a, b, \sigma, T} \). In particular

\[
\left\| \sup_{t \in [0, T]} |X_t| \right\|_p \leq C'_{a, p, T}(1 + \|X_0\|_p). \tag{1.7}
\]

Finally, if the condition

\[
(K_{\theta}^{\text{cont}}) \quad \exists \hat{\kappa} < +\infty, \forall \delta \in (0, T), \quad \hat{\eta}(\delta) := \max_{i=1,2} \sup_{t \in [0, T]} \left[ \int_{(t-\delta)^+}^t K_i(t, u)^i du \right]^{1/i} \leq \hat{\kappa} \delta^{\hat{\theta}} \tag{1.8}
\]

is satisfied for some \( \hat{\theta} \in (0, 1) \), then one can replace \( \frac{\beta - 1}{2\beta} \) by \( \hat{\theta} \) in \([1.5]\) and \([1.6]\).
Comments & Remarks. • In [30] Theorem 3.3, the starting value $X_0$ of the equation is assumed to lie in $\cap_{r>0} L^r(\mathbb{P})$. In fact a careful reading shows that the proof establishes a slightly more general result than announced: if $X_0 \in L^p(\mathbb{P})$ for $p > p_{eu} := \frac{1}{2} \vee \frac{2\beta}{3+\theta}$, then existence and uniqueness (up to $\mathbb{P}$-indistinguishability) of an $(\mathcal{F}_t)$-adapted pathwise $\alpha$-Hölder continuous solution for some small enough $\alpha > 0$ holds true. No bound on $\alpha$ is provided. The upper-bounds (1.5), (1.6) and (1.7) are established for $p \geq 2$ and their dependence in $\|X_0\|_p$ is not (always) specified.

• Hence the above theorem appears – in our setting – as an extension of [30] Theorem 3.3 in terms of existence and uniqueness and in terms of moment control since we simply assume $X_0 \in L^p(\mathbb{P})$ for a fixed $p \in (0, +\infty)$. The extended existence and uniqueness is proved in Appendix B. Moreover, the moment control in (1.5), (1.6) and (1.7) with respect to the starting value $X_0$ are mostly new. These controls combined with the representation of the solution as a functional $F(X_0, W)$ of the starting value $X_0$ and the Brownian motion $W$ (like for diffusions) is the key to our extension achieved through Lemma 14.1. This extension is proved in Appendix B. Its main interest is of course to include the case $p = 1$ in view of our convex ordering and convexity results.

• Assumptions $(K^\text{int}_\beta)$ and $(\hat{K}^\text{cont}_\delta)$ are very close in the singular case, e.g. when $K_i(t, s) = \varphi_i(t-s), \ 0 \leq s < t \leq T$ with $\varphi_i$ decreasing from $+\infty$ at 0.

If $(K^\text{int}_\beta)$ is in force, then $(\hat{K}^\text{cont}_\delta)$ holds with \( \hat{\theta} = \frac{\beta-1}{2\beta} \) since Hölder’s inequality implies, for $\delta \in [0, T]$, \[
\int_{(t-\delta)^+}^t K_1(t, u)du \leq \left( \int_0^t K_1(t, u) \frac{2\beta}{3+\theta} du \right)^\frac{2\theta}{2\beta} (\delta \wedge t)^\frac{\beta-1}{2\beta} \leq C_{1,\beta,T} \delta^\frac{\beta-1}{2\beta}
\]
and \[
\left( \int_{(t-\delta)^+}^t K_2(t, u)^2 du \right)^{1/2} \leq \left( \int_0^t K_2(t, u)^{2\beta} du \right)^{\frac{1}{2\beta}} (\delta \wedge t)^{\frac{\beta-1}{2\beta}} \leq C_{2,\beta,T} \delta^\frac{\beta-1}{2\beta}.
\]
Consequently, in that case, we may always assume that $\hat{\theta} \geq \frac{\beta-1}{2\beta}$.

There is no converse in general. However if, for every $t \in [0, T]$, \[
[0, t) \ni s \mapsto K_i(t, s) \text{ is non-decreasing, } i = 1, 2,
\]
(which includes usual singular kernels) then $(\hat{K}^\text{cont}_\delta)$ implies $(K^\text{int}_\beta)$ for every $\beta$ such that $\frac{\beta-1}{2\beta} < \hat{\theta}$.

In that case $\hat{\theta}$ appears as the supremum (not attained) of $\frac{\beta-1}{2\beta}$ over all $\beta$ is such that $(K^\text{int}_\beta)$ holds. This easily follows from the fact that, taking advantage of the monotonicity of the kernels $K_i$, one has under $(\hat{K}^\text{cont}_\delta)$, for $0 \leq s < t \leq T$, \[
(t-s)^{1/i} K_i(t, s) \leq \left( \int_s^t K_i(t, u)^i du \right)^{1/i} \leq \tilde{\kappa}(t-s)^{\hat{\theta}} \text{ so that } K_i(t, s) \leq \tilde{\kappa}(t-s)^\hat{\theta}^{-1/i}
\]
and $(K^\text{int}_\beta)$ holds as soon as $\frac{2\beta}{\beta+1}(\hat{\theta} - 1) > -1$ and $2\beta(\hat{\theta} - \frac{1}{2}) > -1$ which reads for both kernels $\frac{\beta-1}{2\beta} < \hat{\theta}$.

A significant difference between regular diffusion processes and Volterra processes from a technical viewpoint comes from the presence of the kernels which introduces some memory in the dynamics of the process, depriving us of the Markov property and usual tools of stochastic calculus.

Moreover, on the more practical level of simulation, it naturally suggests two ways to devise a discrete time Euler scheme associated to a regular mesh $(t^n_k = \frac{kT}{n})_{0 \leq k \leq n}$ of $[0, T]$ with step $h = \frac{T}{n}$.
\( \triangleright \text{K-discrete Euler scheme:} \) this Euler scheme is defined inductively by

\[
\bar{X}_{t_k^n} = X_0 + \sum_{\ell=1}^{k} \left( K_1(t_{k}^n, t_{k-1}^n)b(t_{k-1}^n, X_{t_{k-1}^n}) + K_2(t_{k}^n, t_{k-1}^n)\sigma(t_{k-1}^n, X_{t_{k-1}^n})(W_{t_k^n} - W_{t_{k-1}^n}) \right), \quad k = 0, \ldots, n.
\]

The notation is slightly ambiguous, but we choose not to alleviate this ambiguity by denoting \( \bar{X}^n \) in place of \( \bar{X} \), since, on one hand, we only consider the Euler scheme with \( n \) steps throughout the paper and, on the other hand, for \( x_0 \in \mathbb{R}^d \), we will at some places use the notation \( \bar{X}^x_0 \) to emphasize the dependence of the Euler scheme on the initial condition \( X_0 = x_0 \).

Note that simulation of this scheme only requires to compute pointwise values of the kernels \( K_i \) and simulate standard Brownian increments.

In view of producing a priori error bounds/convergence rate for the approximation of the stochastic Volterra process by its Euler scheme (1.10), we naturally extend it into its continuous time (or “genuine” version) by setting

\[
\bar{X}_t = X_0 + \int_0^t K_1(t, \underline{s}, \bar{X}_{\underline{s}})ds + \int_0^t K_2(t, \underline{s}, \bar{X}_{\underline{s}})dW_s, \ t \in [0, T],
\]

where \( \underline{s} = t_k^n := \frac{kT}{n} \) when \( s \in [t_k^n, t_{k+1}^n) \). It reads “in extension” when \( t \in [t_k^n, t_{k+1}^n) \),

\[
\bar{X}_t = X_0 + K_1(t, t_k^n) b(t_k^n, \bar{X}_{t_k^n})(t - t_k^n) + \sigma(t_k^n, \bar{X}_{t_k^n})K_2(t, t_k^n)(W_t - W_{t_k^n}) + \sum_{\ell=1}^{k} \left( K_1(t, t_{\ell-1}^n) b(t_{\ell-1}^n, \bar{X}_{t_{\ell-1}^n}) + K_2(t, t_{\ell-1}^n)\sigma(t_{\ell-1}^n, \bar{X}_{t_{\ell-1}^n})(W_{t_{\ell}^n} - W_{t_{\ell-1}^n}) \right), \quad k = 0, \ldots, n.
\]

\( \triangleright \text{K-integrated discrete time Euler scheme:} \) This scheme reads as follows

\[
\bar{X}_{t_k^n} = X_0 + \sum_{\ell=1}^{k} \left( \int_{t_{\ell-1}^n}^{t_{\ell}^n} K_1(t_{k}^n, s)b(t_{\ell-1}^n, X_{t_{\ell-1}^n}) + \sigma(t_{\ell-1}^n, X_{t_{\ell-1}^n}) \int_{t_{\ell-1}^n}^{t_{\ell}^n} K_2(t_{k}^n, s)dW_s \right),
\]

\( k = 0, \ldots, n. \)

When a closed or a semi-closed form is available for the vectors \( \left[ \int_{t_{\ell-1}^n}^{t_{\ell}^n} K_1(t_{k}^n, s)ds \right]_{1 \leq \ell \leq k}, \ k = 1, \ldots, n \) and the covariance matrices \( \left[ \int_{t_{\ell-1}^n}^{t_{\ell}^n} K_2(t_{k}^n, s)K_2(t_{k'}^n, s)ds \right]_{1 \leq k, k' \leq n}, \ \ell = 1, \ldots, n, \) then the above K-integrated Euler below becomes simulable (see practitioner’s corner further on). Especially when he kernels \( K_i \) are singular (i.e. \( \lim_{t \to t^-} K_i(t, s) = +\infty \)), the main asset of this second scheme is that the terms corresponding to \( \ell = k \) are computed/simulated much more accurately than their above counterparts in the K-discrete scheme (see the practitioner’s corner at the end of the current section in the case \( K_i(t, s) = (t - s)^{\alpha_i} \)). This may have a positive significant impact in practical simulations when the step \( \frac{T}{n} \) is a bit coarse.

This variant also admits a continuous time version given with the same notations by

\[
\bar{X}_t = X_0 + \int_0^t K_1(t, s)b(\underline{s}, \bar{X}_{\underline{s}})ds + \int_0^t K_2(t, s)\sigma(\underline{s}, \bar{X}_{\underline{s}})dW_s, \ t \in [0, T],
\]
which also reads “in extension” when \( t \in [t^*_k, t^*_{k+1}] \),

\[
\bar{X}_t = X_0 + \int_{t_k}^{t} K_1(t, s) ds b(t_k, \bar{X}_s) + \sigma(t_k, \bar{X}_s) \int_{t_k}^{t} K_2(t, s) dW_s \\
+ \sum_{\ell=1}^{k} \left( \int_{t_{\ell-1}}^{t_k} K_1(t, s) ds b(t_{\ell-1}, \bar{X}_{t_{\ell-1}}) + \sigma(t_{\ell-1}, \bar{X}_{t_{\ell-1}}) \int_{t_{\ell-1}}^{t_k} K_2(t, s) dW_s \right), \quad k = 0, \ldots, n. \quad (1.14)
\]

▷ Lack of Markovianity. Because of the lack of Markovianity, for both variants, \( \bar{X}_{t_k} \) is in general not a function of \((\bar{X}_{t_{k-1}}, W_{t_k} - W_{t_{k-1}})\). Nevertheless, \( \bar{X}_{t_k} \) can be computed in a unique way from the values of \((\bar{X}_0, \ldots, \bar{X}_{t_{k-1}})\) and either the vector of Brownian increments \((W_{t_k} - W_{t_0}, \ldots, W_{t_k} - W_{t_{k-1}})\) for (1.9) or the Gaussian vector \((\int_{t_{\ell-1}}^{t_k} K_2(t_k, s) dW_s)_{\ell=1,\ldots,k}\) for (1.12) so that these Euler schemes are well defined by induction. Note that the complexity for the computation of \((\bar{X}_{t_k})_{k=0,\ldots,n}\) is \(O(n^2)\) and that for the \(K\)-integrated scheme, one should generate the independent Gaussian vectors \(Y^{(a)} = (\int_{t_{\ell-1}}^{t_k} K_2(t_k, s) dW_s)_{k=t,\ldots,n}\), \(\ell = 1, \ldots, n\), (see practitioner’s corner further on).

We start by a theorem of convergence for the \(K\)-integrated Euler scheme since less assumptions are needed than for the convergence of the \(K\)-discrete Euler scheme.

**Theorem 1.2 (\(K\)-integrated Euler scheme)** Let \( T > 0 \) and let \( p \in (0, +\infty) \) being fixed. Assume \( b \) and \( \sigma \) satisfy the following time-space Hölder-Lipschitz continuity assumption

\[
(L_H) \quad \exists C_{b,\sigma} < +\infty, \forall s, t \in [0, T], \forall x, y \in \mathbb{R}^d, \\
|b(t, y) - b(s, x)| + ||\sigma(t, y) - \sigma(s, x)|| \leq C_{b,\sigma}((1 + |x| + |y|)|t - s|^\gamma + |x - y|) \quad (1.15)
\]

for some \( \gamma \in (0, 1) \). Assume the kernels \( K_i, i = 1, 2, \) satisfy the integrability condition \((K^int_\beta)\) (see (1.3)) for some \( \beta > 1 \) and the continuity condition \((K^cont_\theta)\) (see (1.4)) for some \( \theta \in (0, 1) \). If \( \tilde{X} \) denotes the \(K\)-integrated Euler scheme (1.13) with time step \( \frac{T}{n} \), then \( \tilde{X} \) has a pathwise continuous modification.

Assume furthermore that \((K^cont_\theta)\) holds for some \( \hat{\theta} \in (0, 1) \). Then there exists a real constant \( C = C_{K_1, K_2, \beta, p, b, \sigma, T} \in (0, +\infty) \) (not depending on \( n \)) such that, for every \( n \geq 1 \),

\[
\forall s, t \in [0, T], \quad \|
\bar{X}_t - \bar{X}_s \|_p \leq C(1 + \|X_0\|_p)|t - s|^{\theta \wedge \hat{\theta}} \quad (1.16)
\]

and

\[
\max_{k=0,\ldots,n} \| X_{t_k} - \bar{X}_{t_k} \|_p \leq \sup_{t \in [0, T]} \| X_t - \bar{X}_t \|_p \leq C(1 + \|X_0\|_p)(\frac{T}{n})^{\gamma \wedge \theta \wedge \hat{\theta}}. \quad (1.17)
\]

Moreover, there exists for every \( \varepsilon \in (0, 1) \) a real constant \( C_\varepsilon = C_{\varepsilon, K_1, K_2, \beta, p, b, \sigma, T} \in (0, +\infty) \) such that

\[
\max_{k=0,\ldots,n} \| X_{t_k} - \bar{X}_{t_k} \|_p \leq \sup_{t \in [0, T]} \| X_t - \bar{X}_t \|_p \leq C_\varepsilon(1 + \|X_0\|_p)(\frac{T}{n})^{(\gamma \wedge \theta \wedge \hat{\theta})(1 - \varepsilon)} \quad (1.18)
\]

The proof of this theorem is postponed to Appendices C and F.
**Remark 1.3** If \((\tilde{K}^{\text{cont}}_{\theta})\) is not satisfied, the above theorem remains valid by replacing mutatis mutandis \(\hat{\theta}\) by \(\frac{\hat{\theta} - 1}{2\theta}\) in all three inequalities.

**Theorem 1.4** (K-discrete Euler scheme) Let \(T > 0\) and let \(p \in (0, +\infty)\) being fixed. Assume \(\|X_0\|_p < +\infty, b\) and \(\sigma\) satisfy \((L\mathcal{H}_r)\). Assume that the kernels \(K_i, i = 1, 2\), satisfy \((K^\text{int}_\beta)\) for some \(\beta > 0\). This gap is filled in Appendix F using a general representation formula of the solutions and tandis \(\hat{\beta}\) for some \(\hat{\beta} > 0\). Hence one may always assume that \(\beta \geq \hat{\beta}\) and \(\hat{\beta} \geq \hat{\beta}\) in all three inequalities.

Let \(X\) denote the K-discrete time Euler scheme (1.10) with step \(\frac{T}{n}\).

(a) Under the above assumptions, \(X\) has continuous paths and there exists a real constant \(C = C_{K_1, K_2, \beta, p, b, \sigma, T} \in (0, +\infty)\) such that, for every \(n \geq 1\),

\[
\forall s, t \in [0, T], \quad \|\bar{X}_t - \bar{X}_s\|_p \leq C(1 + \|X_0\|_p)|t - s|^{\hat{\theta} \hat{\beta}}.
\]

(b) Assume moreover that

\[
(\tilde{K}^{\text{cont}}_{\hat{\theta}}) \quad \exists \hat{\beta} < +\infty, \quad \forall n \geq 1, \quad \max_{i=1,2} \sup_{t \in [0, T]} \left( \int_0^T |K_i(t, u) - K_i(t, \bar{u})|^i du \right)^{1/i} \leq \hat{\beta} \left( \frac{T}{n} \right)^{\hat{\theta}}.
\]

for some \(\hat{\theta} > 0\). Then there exists a finite constant \(C\) such that

\[
\max_{k=0, \ldots, n} \|X_{t_k} - \bar{X}_{t_k}\|_p \leq \sup_{t \in [0, T]} \|X_t - \bar{X}_t\|_p \leq C(1 + \|X_0\|_p)(\frac{T}{n})^{(\gamma \wedge \hat{\beta} \wedge \hat{\theta})}.\]

Moreover, for every \(\varepsilon \in (0, 1)\), there exists a real constant \(C_\varepsilon = C_{\varepsilon, K_1, K_2, \beta, \gamma, \gamma' , p, b, \sigma, T} \in (0, +\infty)\) such that

\[
\max_{k=0, \ldots, n} \|X_{t_k} - \bar{X}_{t_k}\|_p \leq \sup_{t \in [0, T]} \|X_t - \bar{X}_t\|_p \leq C_\varepsilon(1 + \|X_0\|_p)(\frac{T}{n})^{(\gamma \wedge \hat{\beta} \wedge \hat{\theta})(1 - \varepsilon)}.
\]

The main difference with the original reference [26] is that we claim that the rate holds for any \(p > 0\). This gap is filled in Appendix F using a general representation formula of the solutions and a “splitting” lemma (Lemma D.1) both established in Appendix D.

**Remark 1.5** Note that under \((\tilde{K}^{\text{cont}}_{\hat{\theta}})\) for some \(\hat{\theta} > 0\), we have \((\tilde{K}^{\text{cont}}_{\hat{\theta}}) \Rightarrow (K^{\text{cont}}_{\hat{\theta}}) \Rightarrow (K^{\text{cont}}_{\hat{\theta}})\). Hence one may always assume that \(\theta \geq \hat{\theta}\) and \(\hat{\theta} \geq \hat{\theta}\) even if in practice these two inequalities may hold as equalities (see Practitioner’s corner below when \(K_i(t, s) = (t - s)^{\alpha_i}, \alpha_i > -1/i\)).
Practitioner’s corner. (a) If $K_i(t, s) = (t - s)^{\alpha_i}$ with $\alpha_1 > -1$ and $\alpha_2 > -\frac{1}{2}$, then one checks (see e.g. Example 2.1 in [26]) that the assumptions of the above theorems are satisfied with $\beta \in (1, \frac{1}{2(2\alpha_1 + 1)} \wedge \frac{1}{2\alpha_2})$, $\theta = \hat{\theta} = \theta = \hat{\theta} = \hat{\theta} = \min(\alpha_1 + 1, \alpha_2 + \frac{1}{2}, 1)$.

Hence if $\alpha_1 = \alpha_2 = \alpha \in (-\frac{1}{2}, 0)$, one has $\frac{\beta - 1}{\beta} = \frac{1}{\alpha + 1} < \alpha + \frac{1}{2} = \hat{\theta} = \theta = \hat{\theta} = \hat{\theta}$. As a consequence, the final rate of convergence is $O((\frac{\tau}{n})^{\gamma(\alpha + \frac{1}{2})})$ at a fixed time $t$ and $O((\frac{\tau}{n})^{\gamma(\alpha + \frac{1}{2})(1 - \varepsilon)})$ for every $\varepsilon \in (0, 1)$ uniformly over the interval $[0, T]$.

In particular, when considering a Volterra equation driven by a pseudo-fractional Brownian motion with Hurst constant $H \in (0, 1]$ with $b$ and $\sigma$ Lipschitz in time ($\gamma = 1$) which is the standard framework of rough volatility models (see [14]) when $H \in (0, \frac{1}{2})$ or of long memory volatility models when $H \in (\frac{1}{2}, 1)$ (see [10]), then $\alpha = H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$. Note that in the long memory setting, rates $O((\frac{\tau}{n})^{\alpha + \frac{1}{2}})$ or $O((\frac{\tau}{n})^{(\alpha + \frac{1}{2})(1 - \varepsilon)})$ (for small enough $\varepsilon > 0$) are faster than $O((\frac{\tau}{n})^{\beta})$.

(b) Simulation of the $K$-integrated scheme [12]. The exact simulation of the Euler scheme [12] boils down on the one hand to computing the weight matrix

$$\left( \int_{t_{\ell-1}}^{t_{\ell}} K_1(t^m_k, s)ds \right)_{1 \leq \ell \leq k \leq n}$$

and, the other hand, to simulating in an exact way the independent random vectors

$$Y^{(\ell)} = \left( \int_{t_{\ell-1}}^{t_{\ell}} K_2(t^m_k, s)dW_s \right)_{\ell \leq k \leq n}, \quad \ell = 1, \ldots, n.$$  

The $(n - \ell + 1) \times (n - \ell + 1)$ (symmetric) covariance matrix of $Y^{(\ell)}$ is given by

$$\Sigma^{(\ell)} = \left[ \int_0^{T/n} K_2(t^m_k, t^m_{\ell-1} + u)K_2(t^m_j, t^m_{\ell-1} + u)du \right]_{\ell \leq i, j \leq n}.$$  

If furthermore, for every $\ell = 1, \ldots, n$, the functions $K_2(t^m_k, t^m_{\ell-1} + \cdot)$, $k = \ell, \ldots, n$, are linearly independent (for Lebesgue a.e. equality on $[0, T/n]$), then the covariance matrices $\Sigma^{(\ell)}$ are all positive definite and admit a Cholesky decomposition

$$\Sigma^{(\ell)} = T^{(\ell)}T^{(\ell)*} \quad \text{with} \quad T^{(\ell)} \text{ lower triangular}$$

so that

$$(Y^{(\ell)})_{\ell=1,\ldots,n} \overset{d}{=} (T^{(\ell)}Z^{(\ell)})_{\ell=1,\ldots,n}, \quad \text{where} \quad Z^{(\ell)}, \quad \ell = 1, \ldots, n, \quad \text{are independent and} \quad Z^{(\ell)} \sim \mathcal{N}(0, I_{n-\ell+1}).$$

In case the functions $K_2(t^m_k, t^m_{\ell-1} + \cdot)$, $k = \ell, \ldots, n$ are not linearly independent, one considers the (unique) positive symmetric square root of the covariance matrix $\Sigma^{(\ell)}$ as an alternative.

(c) The case of the kernels $K_i(t, s) = (t - s)^{\alpha_i}$, $\alpha_1 > -1$ and $\alpha_2 > -\frac{1}{2}$. For the “drift kernel” $K_1$ with $\alpha_1 > -1$, closed forms are straightforward since, for every $\ell = 1, \ldots, n$ and every $i \in \{0, \ldots, n - \ell\}$

$$\int_{t_{\ell-1}}^{t_{\ell}} K_1(t^m_{\ell+i}, s)ds = \frac{1}{\alpha_1 + 1} \left( \frac{T}{n} \right)^{\alpha_1 + 1} \left( (i + 1)^{\alpha_1 + 1} - i^{\alpha_1 + 1} \right).$$
As for the “pseudo-diffusion kernel” $K_2$ with $\alpha_2 > -1/2$, elementary computations show that
\[
\Sigma^{(\ell)}_{i,i} = \left(\frac{T}{n}\right)^{2\alpha_2+1} \int_0^1 ((i + u)(j + u))^{\alpha_2} du, \quad 0 \leq i \leq j \leq n - \ell. \tag{1.24}
\]
When $i = j$, \[
\Sigma^{(\ell)}_{i,i} = \frac{1}{2\alpha_2 + 1} \left(\frac{T}{n}\right)^{2\alpha_2+1} \left( (i + 1)^{2\alpha_2+1} - i^{2\alpha_2+1} \right).
\]
When $i \neq j$, $\Sigma^{(\ell)}_{i,i,j}$ can be computed offline. In the particular case $i = 0$, one may use for this purpose that
\[
\Sigma^{(\ell)}_{0,\ell} = \frac{1}{\alpha_2 + 1} \left(\frac{T}{n}\right)^{2\alpha_2+1} \int_0^1 (j + v^{1/(\alpha_2+1)})^{\alpha_2} dv.
\]

1.2 Convex ordering: from random vectors to continuous processes

**Definition 1.6** Let $U, V : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^d, \text{Bor}(\mathbb{R}^d))$ be two integrable random vectors.

(a) Convex ordering. We say that $U$ is dominated by $V$ for the convex ordering – denoted by $U \preceq_{\text{cvx}} V$ – if, for every convex function $f : \mathbb{R}^d \to \mathbb{R}$,
\[
\mathbb{E} f(U) \leq \mathbb{E} f(V). \tag{1.25}
\]
(b) Increasing convex ordering ($d = 1$). We say that $U$ is dominated by $V$ for the increasing convex ordering – denoted by $U \preceq_{\text{icv}} V$ – if (1.25) holds for every non-decreasing convex function $f : \mathbb{R} \to \mathbb{R}$.
(c) Decreasing convex ordering ($d = 1$). We say that $U$ is dominated by $V$ for the decreasing convex ordering – denoted by $U \preceq_{\text{dcv}} V$ – if (1.25) holds for every non-increasing convex function $f : \mathbb{R} \to \mathbb{R}$.

Note that any convex function $f : \mathbb{R}^d \to \mathbb{R}$ is bounded from below by an affine function so that for $U$ integrable, $\mathbb{E} \max(-f(U), 0) < +\infty$ and $\mathbb{E} f(U)$ always makes sense in $\mathbb{R} \cup \{+\infty\}$.

According to Lemma A.1 [3] for the regular convex ordering (resp. Lemma 1.1 [19] for the increasing and decreasing convex orderings), the restriction of (1.25) to convex (resp. convex non-decreasing, resp. convex non-increasing) functions $f$ with at most affine growth is enough to characterize the order between $U$ and $V$. Reasoning like in the proof of Lemma 1.9 below, one easily checks that the test functions $f$ may even be supposed to be Lipschitz continuous.

**Lemma 1.7** For all integrable $\mathbb{R}^d$-valued (resp. $\mathbb{R}$-valued) random vectors $U$ and $V$, we have $U \preceq_{\text{cvx}} V$ (resp. $U \preceq_{\text{icv}} V$, resp. $U \preceq_{\text{dcv}} V$) if and only if, for every Lipschitz convex (resp. convex and non-decreasing, resp. convex non-increasing) function $f : \mathbb{R}^d \to \mathbb{R}$ (resp. $f : \mathbb{R} \to \mathbb{R}$, resp. $f : \mathbb{R} \to \mathbb{R}$), $\mathbb{E} f(U) \leq \mathbb{E} f(V)$.

Let us extend the definition to continuous processes.

**Definition 1.8** Let $X,Y : (\Omega, \mathcal{F}, \mathbb{P}) \to C([0,T], \mathbb{R}^d)$ be two integrable continuous processes such that $\mathbb{E} [\|X\|_{\sup} + \|Y\|_{\sup}] < +\infty$.

(a) Convex ordering. We say that $X$ is dominated by $Y$ for the convex ordering – denoted by $X \preceq_{\text{cvx}} Y$ – if, for every l.s.c. (for the uniform convergence topology) convex functional $F : C([0,T], \mathbb{R}^d) \to \mathbb{R}$,
\[
\mathbb{E} F(X) \leq \mathbb{E} F(Y). \tag{1.26}
\]
(b) Increasing convex ordering \((d = 1)\). We say that \(X\) is dominated by \(Y\) for the increasing convex ordering - denoted by \(X \preceq_{icv} Y\) - if \((1.20)\) holds for every l.s.c. convex functional \(F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}\) non-decreasing for the pointwise partial order on continuous functions.

(c) Decreasing convex ordering \((d = 1)\). We say that \(X\) is dominated by \(Y\) for the decreasing convex ordering - denoted by \(X \preceq_{dcv} Y\) - if \((1.20)\) holds for every l.s.c. convex functional \(F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}\) non-increasing for the pointwise partial order on continuous functions.

Note that any l.s.c. convex functional \(F : \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}\) is bounded from below by a continuous affine functional according to Lemma 7.5 [5] (which applies since \(\mathcal{C}([0, T], \mathbb{R}^d)\) endowed with the supremum norm is, like any normed vector space, a locally convex Hausdorff space) so that for \(X\) integrable, \(\mathbb{E}\max(-F(X), 0) < +\infty\) and \(\mathbb{E} F(X)\) always makes sense in \(\mathbb{R} \cup \{+\infty\}\).

**Lemma 1.9** For all integrable \(\mathcal{C}([0, T], \mathbb{R}^d)\)-valued (resp. \(\mathcal{C}([0, T], \mathbb{R})\)-valued) processes \(X\) and \(Y\), we have \(X \preceq_{cvx} Y\) (resp. \(X \preceq_{icv} Y\), resp. \(X \preceq_{dcv} Y\)) if and only if \((1.26)\) holds for every Lipschitz convex (resp. Lipschitz convex and non-decreasing for the pointwise partial order on continuous functions, resp. Lipschitz convex and non-increasing for the pointwise partial order on continuous functions) functional \(F : \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}\) (resp. \(F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}\), resp. \(F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}\)).

**Proof.** Let \(F : \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}\) be l.s.c. and convex (resp. l.s.c., convex and non-decreasing for the pointwise partial order on continuous functions, resp. l.s.c., convex and non-increasing for the pointwise partial order on continuous functions). We are going to establish the existence of a non-decreasing and bounded from below by a continuous affine functional sequence of Lipschitz convex (resp. Lipschitz convex and non-decreasing for the pointwise order on continuous functions, resp. Lipschitz convex and non-increasing for the pointwise order on continuous functions) functionals having \(F\) as the monotone convergence theorem then ensures that \(\mathbb{E} F(X) \leq \mathbb{E} F(Y)\). Since \(F\) is arbitrary, \(X \preceq_{cvx} Y\) (resp. \(X \preceq_{icv} Y\), resp. \(X \preceq_{dcv} Y\)).

Let \(F : \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}\) be l.s.c. and convex. For \(n \in \mathbb{N}\), we define \(F_n\) as the inf-convolution of \(F\) with \(n\)-times the norm:

\[
F_n(x) = \inf_{z \in \mathcal{C}([0, T], \mathbb{R}^d)} (F(z) + n\|z - x\|_{\sup}).
\]

By Lemma 7.5 [5], there exists a continuous affine functional \(G : \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}\) such that \(F \geq G\). Hence, denoting by \(\|G\|\) the norm of the linear part of \(G\), we have for each \(x \in \mathcal{C}([0, T], \mathbb{R}^d)\),

\[
\forall z \in \mathcal{C}([0, T], \mathbb{R}^d), \quad F(z) + n\|z - x\|_{\sup} \geq G(x) + G(z) - G(x) + n\|z - x\|_{\sup} \geq G(x) + (n\|G\|\|z - x\|_{\sup}),
\]

so that \(F_n(x) \geq G(x) > -\infty\) as soon as \(n \geq \|G\|\), which we now suppose from now on. Let \(x, y \in \mathcal{C}([0, T], \mathbb{R}^d)\) be such that \(F_n(x) \geq F_n(y)\) and \((z_k)_{k \in \mathbb{N}}\) be a \(\mathcal{C}([0, T], \mathbb{R}^d)\)-valued sequence such that \(F_n(y) = \lim_{k \to \infty} (F(z_k) + n\|z_k - y\|_{\sup})\). Then, by the definition of \(F_n(x)\) and the triangle inequality,

\[
F_n(x) - F_n(y) \leq \liminf_{k \to \infty} (F(z_k) + n\|z_k - x\|_{\sup} - F(z_k) - n\|z_k - x\|_{\sup}) \leq n\|x - y\|_{\sup},
\]

so that \(F_n\) is Lipschitz with constant \(n\). Since, by convexity of \(F\) and \(\| \cdot \|_{\sup}\), for \(x, y, z, w \in \mathcal{C}([0, T], \mathbb{R}^d)\) and \(\alpha \in [0, 1]\),

\[
F(\alpha z + (1 - \alpha)w) + n\|\alpha z + (1 - \alpha)w - \alpha x - (1 - \alpha)y\|_{\sup} \leq \alpha (F(z) + n\|z - x\|_{\sup}) + (1 - \alpha) (F(w) + n\|w - y\|_{\sup}),
\]

12
taking the infimum over \(z\) and \(w\), we obtain that the function \(F_n\) is convex. It is also bounded from above by \(F\) (choice \(z = x\) in the definition of \(F_n(x)\)). Moreover, still by Lemma 7.5 \([5]\), for every \(x \in \mathcal{C}([0,T], \mathbb{R}^d)\) and \(\alpha < F(x)\) there exists a continuous affine functional \(G_{x,\alpha}\) on \(\mathcal{C}([0,T], \mathbb{R}^d)\) such that \(G_{x,\alpha}(x) = \alpha\) and \(\forall z \in \mathcal{C}([0,T], \mathbb{R}^d), \ F(z) > G_{x,\alpha}(z)\). Replacing \(G\) by \(G_{x,\alpha}\) as described in (1.27), we get that for \(n \geq \|G_{x,\alpha}\|,\ F_n(x) \geq \alpha\). Hence \(\forall x \in \mathcal{C}([0,T], \mathbb{R}^d),\ \lim_{n \to \infty} F_n(x) = F(x)\). Moreover this convergence is monotone since, clearly, \(F_n \leq F_{n+1}\).

If \(d = 1\) and \(F\) is non-decreasing (resp. non-increasing) for the pointwise partial order on continuous functions, then let \(x, y \in \mathcal{C}([0,T], \mathbb{R})\) with \(x \leq y\) (resp. \(y \leq x\)) for this partial order and let \((z_k)_{k \in \mathbb{N}}\) be a \(\mathcal{C}([0,T], \mathbb{R})\)-valued sequence such that \(F_n(y) = \lim_{k \to \infty} (F(z_k) + n\|z_k - y\|_{\text{sup}})\). For each \(k \in \mathbb{N}\), we have \(z_k + x - y \leq z_k\) (resp. \(z_k \leq z_k + x - y\)) for the partial order so that \(F(z_k + x - y) \leq F(z_k)\) and, by definition of \(F_n(x)\),

\[
F_n(x) \leq \lim_{k \to +\infty} (F(z_k + x - y) + n\|z_k + x - y\|_{\text{sup}}) \leq \lim_{k \to +\infty} (F(z_k) + n\|z_k - y\|_{\text{sup}}) = F_n(y).
\]

Hence the monotonicity for the pointwise partial order is transferred from \(F\) to \(F_n\). \(\square\)

### 1.3 Main result

We introduce a second Volterra process \((Y_t)_{t \in [0,T]}\), similar to the first one, solving the following stochastic Volterra equation

\[
Y_t = Y_0 + \int_0^t \tilde{K}_1(t,s)b(s,Y_s)ds + \int_0^t \tilde{K}_2(t,s)\tilde{\sigma}(s,Y_s)dW_s, \quad t \in [0,T],
\]

where \(\tilde{K}_1, \tilde{K}_2 : \{(t,s) : 0 \leq s < t \leq T\} \to \mathbb{R}_+\), \(\tilde{b} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d\) and \(\tilde{\sigma} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d \times q}\) and \(Y_0\) satisfy the assumptions made in Theorem 1.1 on \((K_1, K_2, b, \sigma, X_0)\) so that (1.28) admits a unique pathwise continuous solution.

\(\triangleright\) Assumptions for the \(K\)-discrete Euler scheme. We make the following comparison assumption mixing the kernels \(K_2\) and \(\tilde{K}_2\) and the diffusion coefficients \(\sigma\) and \(\tilde{\sigma}\).

\[
(C\mathcal{K}_2^{\text{disc}}\sigma) \quad \forall j \in \mathbb{N}^*, \forall x \in \mathbb{R}^d, \forall s_0, s_1, \ldots, s_j \in [0,T] \text{ with } 0 \leq s_0 < s_1 \leq \cdots \leq s_j \leq T, \quad K_2\mathcal{K}_2^*(s_j, \cdots, s_1, s_0) \otimes \sigma^*(s_0, x) \leq \tilde{K}_2\tilde{\mathcal{K}}_2^*(s_j, \cdots, s_1, s_0) \otimes \tilde{\sigma}^*(s_0, x),
\]

where

\[
K_2(s_j, \cdots, s_1, s_0) = \begin{pmatrix}
K_2(s_1, s_0) \\
K_2(s_2, s_0) \\
\vdots \\
K_2(s_j, s_0)
\end{pmatrix}
\quad \text{and} \quad \tilde{K}_2(s_j, \cdots, s_1, s_0) = \begin{pmatrix}
\tilde{K}_2(s_1, s_0) \\
\tilde{K}_2(s_2, s_0) \\
\vdots \\
\tilde{K}_2(s_j, s_0)
\end{pmatrix}.
\]

**Remark 1.10** • If \(d = 1\) the condition (1.29) reads

\[
(C\mathcal{K}_2^{\text{disc}}\sigma_{1d}) \quad \forall (s,x) \in [0,T] \times \mathbb{R}, \exists \lambda(s,x) \in [0,1], \forall t \in (s,T),
\]

\[
K_2(t,s)|\sigma(s,x)| = \lambda(s,x)\tilde{K}_2(t,s)|\tilde{\sigma}(s,x)|
\]

\[
(1.30)
\]

13
since one checks that, for \( y, z \in \mathbb{R}^1 \), \( yy^* \leq zz^* \) if only if there exists \( \lambda \in [-1, 1] \) such that \( y = \lambda z \).

- If \( \widetilde{K}_2 = K_2 \) then the above condition \( (1.29) \) reads \( K_2 K_2^*(s_j, \cdots, s_1, s) \otimes (\sigma \sigma^*(s, x) - \sigma^*(s, x)) \in \mathcal{S}^*_+(j \times d) \). It follows from Lemma A.3 in Appendix A that this condition is satisfied as soon as

\[
(C \sigma) \quad \forall (s, x) \in [0, T] \times \mathbb{R}^d, \sigma \sigma^*(s, x) \leq \sigma \sigma^*(s, x).
\]  

(1.31)

- If \( \widetilde{\sigma} = \sigma \) and for each \( s \in [0, T] \) there exists \( x \in \mathbb{R}^d \) such that \( \sigma \sigma^*(s, x) \) is positive definite, then \( (1.29) \) holds if and only if

\[
(C K_2) \quad \exists \lambda : [0, T] \to [0, 1], \forall 0 \leq s < t \leq T, K_2(t, s) = \lambda(s) \widetilde{K}_2(t, s).
\]  

(1.32)

Note that if both kernels are normalized in the sense that \( \int_0^1 K_2(t, s)ds = \int_0^1 \widetilde{K}_2(t, s)ds \) for \( t \in [0, T] \), then \( \lambda(s) = 1 \) ds-a.e. i.e. \( K_2(t, \cdot) = \widetilde{K}_2(t, \cdot) \) a.e. for every \( t \in [0, T] \).

\( \triangleright \) Assumptions for the \( K \)-integrated Euler scheme. While Condition \( (1.29) \) is tailored to deal with the approximation of the Volterra equation \( (1.2) \) by the \( K \)-discrete Euler scheme, its counterpart for the \( K \)-integrated Euler scheme writes

\[
(C K_2^\text{int}) \forall j \in \mathbb{N}^*, \forall x \in \mathbb{R}^d, \forall s_0, s_1, \cdots, s_j \in [0, T] \) with \( 0 \leq s_0 < s_1 \leq \cdots \leq s_j \leq T,
\]

\[
\int_{s_0}^{s_1} K_2 K_2^*(s_j, \cdots, s_1, s)ds \otimes \sigma \sigma^*(s_0, x) \leq \int_{s_0}^{s_1} \widetilde{K}_2 \widetilde{K}_2^*(s_j, \cdots, s_1, s)ds \otimes \sigma \sigma^*(s_0, x).
\]  

(1.33)

If \( K_2 \) and \( \widetilde{K}_2 \) are continuous in their second variable, for \( j \geq 2 \), dividing by \( \frac{1}{s_1 - s_0} \) the inequality with \( (s_j, \cdots, s_1, s) \) replaced by \( (s_j, \cdots, s_2, s) \) (which is a consequence of the full inequality \( (1.33) \)), and letting \( s_1 \downarrow s_0 \), we recover \( (C K_2^\text{disc}) \) with \( (s_j, \cdots, s_1, s_0) \) replaced by \( (s_j, \cdots, s_2, s_0) \). Without the continuity assumption, no comparison of \( (C K_2^\text{int}) \) and \( (C K_2^\text{disc}) \) is possible.

Both conditions are implied by

\[
(C K_2) \forall j \in \mathbb{N}^*, \forall x \in \mathbb{R}^d, \forall s_0, s_1, \cdots, s_j \in [0, T] \) with \( 0 \leq s_0 \leq s < s_1 \leq \cdots \leq s_j \leq T,
\]

\[
K_2 K_2^*(s_j, \cdots, s_1, s) \otimes \sigma \sigma^*(s_0, x) \leq \widetilde{K}_2 \widetilde{K}_2^*(s_j, \cdots, s_1, s) \otimes \sigma \sigma^*(s_0, x).
\]  

(1.34)

Indeed \( (C K_2^\text{disc}) \) is deduced for the choice \( s = s_0 \) and \( (C K_2^\text{int}) \) by integration with respect to \( s \) between \( s_0 \) and \( s_1 \).

\( \triangleright \) Sufficient condition for \( (C K_2) \):

\[
(C \sigma) \text{(see (1.31))} \quad \text{and} \quad (C \sigma) \text{(see (1.31))}.
\]

**Proof.** Combining the decomposition

\[
K_2 \widetilde{K}_2^*(s_j, \cdots, s_1, s) \otimes \sigma \sigma^*(s_0, x) - K_2 K_2^*(s_j, \cdots, s_1, s) \otimes \sigma \sigma^*(s_0, x)
\]

\[
= \left( K_2 \widetilde{K}_2^*(s_j, \cdots, s_1, s) - K_2 K_2^*(s_j, \cdots, s_1, s) \right) \otimes \sigma \sigma^*(s_0, x)
\]

\[
+ K_2 K_2^*(s_j, \cdots, s_1, s) \otimes (\sigma \sigma^*(s_0, x) - \sigma \sigma^*(s_0, x))
\]

with Lemma A.3, one indeed easily checks that \( (1.32) \) and \( (1.31) \) imply \( (C K_2) \) (see \( (1.34) \)).
In terms of regularity of the kernels we need for the $K$-discrete Euler scheme the global assumption

$$(K^{\text{cont}}) \exists \tilde{\theta}, \theta, \tilde{\theta} \in (0, 1] \text{ such that } (K_1, K_2) \text{ and } (\tilde{K}_1, \tilde{K}_2) \text{ satisfy } (K^{\text{cont}}_1), (K^{\text{cont}}_2) \text{ and } (K^{\text{cont}}_\theta).$$

When dealing with the $K$-integrated Euler scheme, we only need

$$(K^{\text{cont}}) \exists \theta, \tilde{\theta} \in (0, 1] \text{ such that } (K_1, K_2) \text{ and } (\tilde{K}_1, \tilde{K}_2) \text{ satisfy } (K^{\text{cont}}_\theta) \text{ and } (K^{\text{cont}}_{\tilde{\theta}}).$$

which is, according to Remark 1.5, less stringent than the former $(K^{\text{cont}})$.

\textbf{Convexity assumption on }$\sigma$. We make the following convexity assumption on the matrix field $\sigma$:

For every $(t, x, y, \alpha) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times [0, 1]$, there exists $V = V_{t,x,y,\alpha} \in \mathcal{O}(q)$ such that

$$\tag{1.37} \text{(Conv)} \quad \sigma^*(t, ax + (1-\alpha)y) \leq (\alpha \sigma(t, x) + (1-\alpha)\sigma(t, y)V)(\alpha \sigma(t, x) + (1-\alpha)\sigma(t, y)V)^*. $$

This condition has already been introduced in a less general form in [18] (with $V = I_q$) and in the present form in [17]. It is further discussed at the end of the current section.

\textbf{Theorem 1.11 (Convex ordering)} Let $X = (X_t)_{t \in [0, T]}$ and $Y = (Y_t)_{t \in [0, T]}$ be two Volterra processes defined by (1.2) and (1.28) respectively with $(b, \sigma)$ and $(\tilde{b}, \tilde{\sigma})$ satisfying $(\mathcal{L}_\gamma)$ for some $\gamma \in (0, 1]$ and $(K_1, K_2)$ and $(\tilde{K}_1, \tilde{K}_2)$ both satisfying $(K^{\text{int}}_\beta)$ for some $\beta > 1$.

(a) Regular convex ordering. Assume common drift kernel $K_1 = \tilde{K}_1$ and common affine drift function

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad b(t, x) = \tilde{b}(t, x) = \mu(t) + \nu(t)x$$

for functions $\mu : [0, T] \rightarrow \mathbb{R}^d$, $\nu : [0, T] \rightarrow \mathbb{R}^{d \times d}$.

Assume furthermore that $\sigma$ or $\tilde{\sigma}$ satisfies (Conv) and that

either $(K^{\text{cont}})$, $(CK^{\text{int}}_2, \sigma)$ hold or $(K^{\text{cont}})$ and $(CK^{\text{disc}}_2, \sigma)$ hold.

Finally assume that $X_0, Y_0 \in L^1(\mathbb{P})$ and that $X_0 \preceq_{\text{exc}} Y_0$. Then,

$$X \preceq_{\text{exc}} Y.$$

(b) Increasing convex ordering when $d = q = 1$. Assume either $x \mapsto b(t, x)$ and $x \mapsto |\sigma(t, x)|$ are convex and non-decreasing for every $t \in [0, T]$ or $x \mapsto \tilde{b}(t, x)$ and $x \mapsto |\tilde{\sigma}(t, x)|$ are convex and non-decreasing for every $t \in [0, T]$. Also assume either $(K^{\text{cont}})$, $(CK_2, \sigma)$ and

$$\forall 0 \leq s_0 < s_1 \leq s_2 \leq T, \forall x \in \mathbb{R}, \quad b(s_0, x) \int_{s_0}^{s_1} K_1(s_2, s) ds \leq \tilde{b}(s_0, x) \int_{s_0}^{s_1} \tilde{K}_1(s_2, s) ds \tag{1.39}$$

or $(K^{\text{cont}})$, $(CK^{\text{disc}}_2, \sigma_1)$ (see (1.30)) and

$$\forall 0 \leq s < t \leq T, \forall x \in \mathbb{R}, \quad K_1(t, s) b(s, x) \leq \tilde{K}_1(t, s) \tilde{b}(s, x). \tag{1.40}$$

Finally assume that $X_0, Y_0 \in L^1(\mathbb{P})$ with $X_0 \preceq_{\text{exc}} Y_0$. Then,

$$X \preceq_{\text{exc}} Y.$$
(c) Decreasing convex ordering when $d = q = 1$. Assume either $x \mapsto -b(t, x)$ and $x \mapsto |\sigma(t, x)|$ are convex and non-increasing for every $t \in [0, T]$ or $x \mapsto -\bar{b}(t, x)$ and $x \mapsto |\bar{\sigma}(t, x)|$ are convex and non-increasing for every $t \in [0, T]$. Also assume either $(K^\text{cont})$, $(\mathcal{C}K_2 \sigma)$ and

$$\forall 0 \leq s_0 < s_1 \leq s_2 \leq T, \forall x \in \mathbb{R}, \ b(s_0, x) \int_{s_0}^{s_1} K_1(s_2, s)ds \geq \bar{b}(s_0, x) \int_{s_0}^{s_1} \bar{K}_1(s_2, s)ds \quad (1.41)$$

or $(K^\text{cont})$, $(\mathcal{C}K_2 \text{disc} \sigma_1 \delta)$ and

$$\forall 0 \leq s < t \leq T, \forall x \in \mathbb{R}, \ K_1(t, s)b(s, x) \geq \bar{K}_1(t, s)\bar{b}(s, x). \quad (1.42)$$

Finally assume that $X_0, Y_0 \in L^1(\mathbb{P})$ with $X_0 \lesssim_{\text{dcv}} Y_0$. Then

$$X \lesssim_{\text{dcv}} Y.$$

Remark 1.12 Under (1.38), for $b = \bar{b}$ to satisfy $(\mathcal{L}H_1)$ (see (1.15)), it is enough that both functions $\mu$ and $\nu$ are Hölder continuous with exponent $\gamma$.

Theorem 1.13 (Convexity) Let $(b, \sigma)$ satisfy $(\mathcal{L}H_\gamma)$ for some $\gamma \in (0, 1]$ and $(K_1, K_2)$ satisfy $(K^\text{int}_\beta)$ for some $\beta > 1$ and $(K^\text{cont}_\theta)$ and $(\bar{K}^\text{cont}_\theta)$ for some $\theta, \bar{\theta} > 0$. Let $X^x = (X^x_t)_{t \in [0, T]}$ denote the solution starting from $x \in \mathbb{R}^d$ to the Volterra stochastic differential equation (1.2).  

(a) Convexity in the initial condition. Assume

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \ b(t, x) = \mu(t) + \nu(t)x \text{ for functions } \mu : [0, T] \to \mathbb{R}^d, \ \nu : [0, T] \to \mathbb{R}^{d \times d},$$

and $\sigma$ satisfies the convexity assumption $(\text{Conv})$. Then, for every convex functional $F : \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}$ with at most polynomial growth with respect to the sup-norm, $\mathbb{R}^d \ni x \mapsto \mathbb{E} F(X^x)$ is convex.

(b) Non-decreasing convexity in the initial condition when $d = q = 1$. Assume that $x \mapsto b(t, x)$ and $x \mapsto |\sigma(t, x)|$ are convex and non-decreasing for every $t \in [0, T]$. Then, for every convex functional $F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}$ with at most polynomial growth with respect to the sup-norm and non-decreasing for the pointwise partial order on continuous functions, $\mathbb{R} \ni x \mapsto \mathbb{E} F(X^x)$ is convex and non-decreasing.

(c) Non-Increasing convexity in the initial condition when $d = q = 1$. Assume that $x \mapsto -b(t, x)$ and $x \mapsto |\sigma(t, x)|$ are convex and non-increasing for every $t \in [0, T]$. Then, for every convex functional $F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}$ with at most polynomial growth with respect to the sup-norm and non-increasing for the pointwise partial order on continuous functions, $\mathbb{R} \ni x \mapsto \mathbb{E} F(X^x)$ is convex and non-increasing.

Let us state some equivalent formulations of the convexity condition $(\text{Conv})$ that may appear more “symmetric” or more “intrinsic” and explain why, in the previous theorem, we suppose when $d = q = 1$ that $x \mapsto |\sigma(t, x)|$ is convex for every $t \in [0, T]$.

Proposition 1.14 (a) The convexity condition $(\text{Conv})$ (see (1.37)) is equivalent to

(i) For every $(t, x, y, \alpha) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times [0, 1]$, there exists $U = U_{t, x, y, \alpha}, \ V = V_{t, x, y, \alpha} \in \mathcal{O}(q)$ such that

$$\sigma \sigma^*(t, \alpha x + (1 - \alpha)y) \leq (\alpha \sigma(t, x)U + (1 - \alpha)\sigma(t, y)V)(\alpha \sigma(t, x)U + (1 - \alpha)\sigma(t, y)V)^*.$$
(ii) For every \((t, x, y, \alpha) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times [0, 1]\)
\[
\exists \zeta, \tilde{\zeta} \in \mathbb{R}^{d \times q}, \quad \begin{cases}
\zeta \zeta^* = \sigma \sigma^*(t, x), & \tilde{\zeta} \tilde{\zeta}^* = \sigma \sigma^*(t, y), \\
\sigma \sigma^*(t, \alpha x + (1 - \alpha) y) \leq (\alpha \zeta + (1 - \alpha) \tilde{\zeta}) (\alpha \zeta + (1 - \alpha) \tilde{\zeta})^*. 
\end{cases}
\] \hspace{1cm} (1.43)

(b) If \(q \leq d\), the convexity condition \((1.37)\) implies that:
for every \((t, x, y, \alpha) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times [0, 1]\), there exists \(V = V_{t,x,y,\alpha} \in \mathcal{O}(d)\) such that
\[
\sigma \sigma^*(t, \alpha x + (1 - \alpha) y) \leq \left( \alpha \sqrt{\sigma \sigma^*(t, x)} + (1 - \alpha) \sqrt{\sigma \sigma^*(t, y)} \right) \left( \alpha \sqrt{\sigma \sigma^*(t, x)} + (1 - \alpha) \sqrt{\sigma \sigma^*(t, y)} \right)^*,
\]
with equivalence when \(q = d\).

(c) When \(d = q = 1\), the convexity condition \((1.37)\) reads \(x \mapsto |\sigma(t, x)|\) is convex for each \(t \in [0, T]\).

Proof. (a) The equivalence of (i) and (ii) is an easy consequence of the fact that \(A, B \in \mathbb{R}^{d \times q}\) are such that \(AA^* = BB^*\) if and only if \(A = BO\) for some \(O \in \mathcal{O}(q)\) (see Lemma A.1 in Appendix A).

Condition (i) implies \((1.37)\) (with \(U = I_q\)). These are in fact equivalent since, when \(U \in \mathcal{O}(q)\), by introducing \(U^* U = I_q\) between the two terms of the product in the left-hand side,
\[
(\alpha \sigma(t, x) U + (1 - \alpha) \sigma(t, y) V) (\alpha \sigma(t, x) U + (1 - \alpha) \sigma(t, y) V)^*
\]
\[
= (\alpha \sigma(t, x) + (1 - \alpha) \sigma(t, y) V U^*) (\alpha \sigma(t, x) + (1 - \alpha) \sigma(t, y) V U^*)^*
\]
with \(V U^* \in \mathcal{O}(q)\) if \(V\) also belongs to \(\mathcal{O}(q)\).

(b) When \(q \leq d\), since the matrix \(\pi \in \mathbb{R}^{q \times d}\) with only non vanishing entries \(\pi_{ii} = 1\) for \(i \in \{1, \cdots, q\}\), satisfies \(\pi_i \pi^* = I_q\), we have that for \(V \in \mathcal{O}(q)\),
\[
(\alpha \sigma(t, x) + (1 - \alpha) \sigma(t, y) V) (\alpha \sigma(t, x) + (1 - \alpha) \sigma(t, y) V)^*
\]
\[
= (\alpha \sigma(t, x) \pi + (1 - \alpha) \sigma(t, y) V \pi) (\alpha \sigma(t, x) \pi + (1 - \alpha) \sigma(t, y) V \pi)^*.
\]

Since the matrices \(\zeta = \sigma(t, x) \pi \in \mathbb{R}^{d \times d}\) and \(\tilde{\zeta} = \sigma(t, y) V \pi \in \mathbb{R}^{d \times d}\) are such that \(\zeta \zeta^* = \sigma \sigma^*(t, x)\) and \(\tilde{\zeta} \tilde{\zeta}^* = \sigma \sigma^*(t, y)\), we conclude by \((a)(ii)\) applied to \(\sqrt{\sigma \sigma^*}\) in place of \(\sigma\). The reverse implication when \(q = d\) also follows from \((a)(ii)\).

(c) When \(d = q = 1\), then the right-hand side of \((1.37)\) is maximal for \(U\) equal to 1 or \(-1\) depending on whether \(\sigma(t, x)\) and \(\sigma(t, y)\) share the same sign or not. Claim (c) easily follows. \(\Box\)

Remark 1.15 One derives from claim (b) the following sufficient natural criterion for \((\text{Conv})\) to hold when \(q = d\): for every \((t, x, y, \alpha) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times [0, 1]\),
\[
\sigma \sigma^*(t, \alpha x + (1 - \alpha) y) \leq \left( \alpha \sqrt{\sigma \sigma^*(t, x)} + (1 - \alpha) \sqrt{\sigma \sigma^*(t, y)} \right) \left( \alpha \sqrt{\sigma \sigma^*(t, x)} + (1 - \alpha) \sqrt{\sigma \sigma^*(t, y)} \right)^*.
\]

Note that, when \(q \leq d\), we can always assume that \(q = d\) by replacing mutatis mutandis \(\sigma\) by \(\sigma_{\pi}(t, x) \pi\) where \(\pi \in \mathbb{R}^{q \times d}\) is the matrix with only non vanishing entries \(\pi_{ii} = 1\) for \(i \in \{1, \cdots, q\}\) and adding to the \(q\)-dimensional standard Brownian motion \((W_t)_{t \in [0, T]}\) \(d - q\) coordinates corresponding to an independent \(d - q\) dimensions standard Brownian motion.

The strategy of proof of both Theorems \((1.11)\) and \((1.13)\) will rely on two steps like in \[21]\:

– establish the result for the Euler schemes by combining propagation of convexity (resp. increasing convexity) in Section \[2.1]\ (resp. Section \[3.1]\) and comparison in Section \[2.2]\ (resp. Section \[3.2]\) for the convex ordering (resp. increasing convex ordering).

– transfer the convex ordering by making the schemes converge (see Section \[4]\).
1.4 Application to VIX options in the quadratic rough Heston model

Let us consider the auxiliary variance process in the quadratic rough Heston model (see [13]):

\[ V_t = a(Z_t - b)^2 + c \quad \text{with} \quad a > 0, \ b, \ c \geq 0 \]

and, for \( H \in (0, 1/2) \), \( \lambda \in \mathbb{R} \), \( \sigma > 0 \) and \( f : [0, T] \to \mathbb{R} \) Hölder continuous with exponent \( \gamma \in (0, 1) \),

\[ Z_t = Z_0 + \int_0^t (t - s)^{H-\frac{1}{2}} \lambda (f(s) - Z_s)ds + \sigma \int_0^t (t - s)^{H-\frac{1}{2}} \sqrt{a(Z_s - b)^2 + c} \, dW_s. \quad (1.44) \]

Note that \( z \mapsto \sqrt{a(z - b)^2 + c} \) is convex and Lipschitz. Then the above Volterra equation has a unique strong solution, temporarily denoted by \( (Z_t^{(\sigma)})_{t \geq 0} \) to emphasize the dependence on the parameter \( \sigma \). Let \( V_t^{(\sigma)} = a(Z_t^{(\sigma)} - b)^2 + c \) denote the resulting squared volatility.

According to Theorem [13], one has \( (Z_t^{(\sigma)})_{t \in [0, T]} \leq \text{ess} \ (Z_t^{(\bar{\sigma})})_{t \in [0, T]} \) for \( \sigma \in (0, \bar{\sigma}] \).

The price at time 0 of a VIX contract with maturity \( T > 0 \) in such a model is given by

\[ \mathbb{E} \left( \sqrt{\frac{1}{T} \int_0^T V_t^{(\sigma)} \, dt} \right) = \mathbb{E} \left( F((Z_t^{(\sigma)})_{t \in [0, T]}) \right) \quad \text{where} \quad F((x_t)_{t \in [0, T]}) = \sqrt{\frac{1}{T} \int_0^T (a(x_t - b)^2 + c) \, dt}. \]

For \( (x_t)_{t \in [0, T]}, (y_t)_{t \in [0, T]} \in C([0, T], \mathbb{R}) \) and \( \alpha \in [0, 1] \), by the convexity of \( z \mapsto \sqrt{a(z - b)^2 + c} \) and the monotonicity of the square function on \( \mathbb{R}_+ \), then the convexity of the \( L^2(dt) \) norm,

\[ F((\alpha x_t + (1 - \alpha) y_t)_{t \in [0, T]}) \leq \sqrt{\frac{1}{T} \int_0^T \left( \alpha \sqrt{a(x_t - b)^2 + c} + (1 - \alpha) \sqrt{a(y_t - b)^2 + c} \right)^2 \, dt} \]
\[ \leq \alpha F((x_t)_{t \in [0, T]}) + (1 - \alpha) F((y_t)_{t \in [0, T]}). \]

Moreover, the functional \( F \) has at most at most affine growth with respect to the sup-norm. Hence, by Theorem [13]

\[ \mathbb{E} \left( F((Z_t^{(\sigma)})_{t \in [0, T]}) \right) \leq \mathbb{E} \left( F((Z_t^{(\bar{\sigma})})_{t \in [0, T]}) \right) \]
which also writes

\[ \mathbb{E} \left( \sqrt{\frac{1}{T} \int_0^T V_t^{(\sigma)} \, dt} \right) \leq \mathbb{E} \left( \sqrt{\frac{1}{T} \int_0^T V_t^{(\bar{\sigma})} \, dt} \right) \]

i.e. the premium is a non-decreasing function of the parameter \( \sigma \).

Moreover, owing to Theorem [13] if we now denote by \( Z^{z_0} \) the solution to \( (1.44) \) starting from \( Z_0 = z_0 \in \mathbb{R} \) and \( V_t^{z_0} = a(Z_t^{z_0} - b)^2 + c \), then \( z_0 \mapsto \mathbb{E} F(Z^{z_0}) \) is convex for every convex functional \( F : C([0, T], \mathbb{R}) \to \mathbb{R} \) with at most polynomial growth with respect to the sup-norm. In particular, \( z_0 \mapsto \mathbb{E} f(V_t^{z_0}) \) is convex for every non-decreasing convex function \( f : \mathbb{R} \to \mathbb{R} \) with polynomial growth and every \( t \in [0, T] \).

2 Convex ordering and convexity propagation for the Euler schemes

Let us recall in a synthetic way the two Euler discretization schemes associated to the regular mesh \( (t^n_k = \frac{kT}{n})_{0 \leq k \leq n} \) with \( n \geq 1 \) steps of the Volterra stochastic differential equations [12].
Note that for the convexity assumption (Conv) (see (1.37)),
\[ b(t^n_k, x) = \tilde{b}(t^n_k, x) = \mu(t^n_k) + \nu(t^n_k)x \quad \text{with} \quad (\mu(t^n_k), \nu(t^n_k)) \in \mathbb{R}^d \times \mathbb{R}^{d \times d} \]
and either both Euler schemes are $K$-discrete and $(K, \sigma)$ and $(\tilde{K}, \tilde{\sigma})$ satisfy the comparison inequality \((CK^{\text{disc}}_2 \sigma)\) (see (1.29)) or both Euler schemes are $K$-integrated, (2.48) holds, and $(K, \sigma)$ and $(\tilde{K}, \tilde{\sigma})$ satisfy the comparison inequality \((CK^{\text{int}}_2 \sigma)\) (see (1.33)). Then, for every $n \geq 1$,
\[
X^{0:n}_n \leq_{\text{cvx}} Y^{0:n}_n \quad \text{and, in particular, } (\tilde{X}^{n}_k)_{k=0, \ldots, n} \leq_{\text{cvx}} (\tilde{Y}^{n}_k)_{k=0, \ldots, n}.
\]

\textbf{Remark 2.2} \quad Assumption (2.49) clearly implies that the part concerning $(b, \tilde{b})$ in assumption (2.47) is satisfied.

- The conclusion still holds for the $K$-discrete (resp. $K$-integrated) Euler scheme when $\sigma$ or $\tilde{\sigma}$ satisfies the convexity assumption (Conv) for $t$ restricted to \(\{t^n_k : 0 \leq k \leq n - 1\}\) and $(K, \sigma)$ and $(\tilde{K}, \tilde{\sigma})$ satisfy the comparison inequality \((CK^{\text{int}}_2 \sigma)\) (resp. $(CK^{\text{int}}_2 \sigma)$) with \((s_j, \cdots, s_1, s_0)\) of the form \((t^n_0, t^n_{a-1}, \cdots, t^n_{n+1-j}, t^n_{n-j})\) for $j \in \{0, \cdots, n\}$. Note that the comparison inequalities are only used to derive (2.50) below.

\section{2.1 Backward propagation of convexity by the Euler schemes}

Let for $k \in \{1, \ldots, n\}$, \(x^{0:k}_0 = (x_0, x_1, \cdots, x^n_k, x_{n+1}, \cdots, x^n_n)\) denote a generic element of $\mathcal{E}_k = \mathbb{R}^d \times (\mathbb{R}^d)^{n} \times (\mathbb{R}^d)^{n-1} \times \cdots \times (\mathbb{R}^d)^{n+1-k}$ and let $x^{0:0}_0 = x_0$ be a generic element of $\mathcal{E}_0 = \mathbb{R}^d$. Set, for every $k \in \{0, \ldots, n - 1\}$, \(\ell \in \{k+1, \cdots, n\}\), \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d\)
\[
B_k(t^n_\ell, x, y) = x + \left(\mu(t^n_k) + \nu(t^n_k)y\right) \int_{t^n_k}^{t^n_{k+1}} K_1(t^n_s, \hat{s}) \mathrm{d}s.
\] (2.50)

Let $\Phi_k : \mathcal{E}_n \rightarrow \mathbb{R}$ be a function with $p$-polynomial growth for some $p \geq 1$. We define the functions $\Phi_k : \mathcal{E}_k \rightarrow \mathbb{R}$, $k \in \{0, \cdots, n - 1\}$ and the auxiliary functions $\Psi_k$ by a backward induction as follows: for every $(x^{0:k}_0, u) \in \mathcal{E}_k \times \mathbb{R}^{d \times q}$,
\[
\Psi_k(x^{0:k}_0, u) = \mathbb{E} \Phi_{k+1}\left(x^{0:k}_0, B_k(t^n_{k+1}, x^{k+1}_k, x^k_k) + u \int_{t^n_k}^{t^n_{k+1}} K_2(t^n_s, \hat{s}) \mathrm{d}W_s, \cdots, B_k(t^n_n, x^{n}_k, x^k_k) + u \int_{t^n_k}^{t^n_n} K_2(t^n_s, \hat{s}) \mathrm{d}W_s\right). \quad (2.51)
\]

and
\[
\forall x^{0:k}_0 \in \mathcal{E}_k, \quad \Phi_k(x^{0:k}_0) = \Psi_k(x^{0:k}_0, \sigma(t^{n}_k, x^k_k)), \quad (2.52)
\]

where for $k = 0$, we use the convention $x^{0}_0 = x^1_0 = \cdots = x^n_0 = x_0$ (consistent with the definition of the $X^\ell$ when $\ell \geq 1$). Note that under the part in (2.47) concerning $\sigma$ and the part in (2.48) concerning $(K_1, K_2)$ (which appears in the assumptions of the proposition just below), the functions \(x \mapsto \sigma(t^n_k, x)\) have at most affine growth and
\[
\int_{t^n_k}^{t^n_{k+1}} K_2(t^n_s, \hat{s}) \mathrm{d}W_s, \cdots, \int_{t^n_k}^{t^n_n} K_2(t^n_s, \hat{s}) \mathrm{d}W_s
\]
is a Gaussian random vector and thus belongs to $L^p(\mathbb{P})$. With the affine property of the functions \((x, y) \mapsto B_k(t^n_\ell, x, y)\) for $\ell \geq k+1$, one easily deduce by backward induction that the random variable in the expectation in the right-hand side of (2.51) is integrable and the subsequently well-defined functions $\Psi_k$ and $\Phi_k$ have $p$-polynomial growth for $k \in \{0, \cdots, n - 1\}$.
Proposition 2.3 (Backward propagation of convexity) Assume (2.49),
\[
\forall k \in \{0, \ldots, n-1\}, \quad \int_0^{t_{k+1}^n} \left( K_1(t_{k+1}^n, \ddot{s}) + K_2(t_{k+1}^n, \ddot{s})^2 \right) ds + \sup_{x \in \mathbb{R}^d} \frac{\| \sigma(t_k^n, x) \|}{1 + |x|} < +\infty,
\]
for all \( k \in \{0, \ldots, n-1\} \). Let us check by backward induction that they are convex.

Proof. The \( p \)-polynomial growth of the functions \( \Phi_k \) has been established just before the proposition. Let us check by backward induction that they are convex. For \( u \in \mathbb{R}^{d \times q} \), the \( \mathbb{R}^{(n-k)d} \)-valued random vector \( \left( u \int_{t_k^n}^{t_{k+1}^n} K_2(t_{k+1}^n, \ddot{s}) dW_s, \ldots, u \int_{t_k^n}^{t_{k+1}^n} K_2(t_{n}^n, \ddot{s}) dW_s \right) \) is distributed according to the centered Gaussian distribution \( \mathcal{N}_{(n-k)d}(0, \Gamma_k \otimes uu^*) \) where
\[
\Gamma_k = \int_{t_k^n}^{t_{k+1}^n} K_2(t_{k+1}^n, \ddot{s}) K_2(t_{k+1}^n, \ddot{s})^2 ds
\]
is a symmetric semidefinite positive matrix. As a consequence,
\[
\forall (x_{0:k}^*, u, v) \in \mathcal{E}_k \times \mathbb{R}^{d \times q} \times \mathbb{R}^{d \times q} \text{ with } uu^* = vv^*, \quad \Psi_k(x_{0:k}^*, u) = \Psi_k(x_{0:k}^*, v).
\]
For \( v \in \mathbb{R}^{d \times q} \) such that \( uu^* \in \mathcal{S}^+(d) \) then \( \Gamma_k \otimes vv^* - \Gamma_k \otimes uu^* = \Gamma_k \otimes (vv^* - uu^*) \in \mathcal{S}^+(n-kd) \) owing to Lemma A.3.

Since we know that two centered Gaussian distributions satisfy \( \mathcal{N}(0, \Sigma_1) \leq_{cvx} \mathcal{N}(0, \Sigma_2) \) if and only if \( \Sigma_2 \Sigma_1^{-1} - \Sigma_1 \Sigma_1^+ \in \mathcal{S}^+(d) \) (see for instance [17, Lemma 3.2 and Remark 3.1]), we deduce that
\[
\left( u \int_{t_k^n}^{t_{k+1}^n} K_2(t_{k+1}^n, \ddot{s}) dW_s, \ldots, u \int_{t_k^n}^{t_{k+1}^n} K_2(t_{n}^n, \ddot{s}) dW_s \right) \leq_{cvx} \left( v \int_{t_k^n}^{t_{k+1}^n} K_2(t_{k+1}^n, \ddot{s}) dW_s, \ldots, v \int_{t_k^n}^{t_{k+1}^n} K_2(t_{n}^n, \ddot{s}) dW_s \right).
\]
If, for \( k \in \{0, \ldots, n-1\} \), \( \Phi_{k+1} \) is convex, then so is
\[
\mathcal{E}_k \times (\mathbb{R}^d)^{n-k} \ni (x_{0:k}^*, w_{k+1}, \ldots, w_n) \mapsto \Phi_{k+1}(x_{0:k}^*, B_k(t_{k+1}^n, x_{k+1}^k) + w_{k+1}, \ldots, B_k(t_{n}^n, x_{k}^n, x_{k+1}^k) + w_n)
\]
by the affine property of \( \mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto B_k(t_{n}^n, x, y) \) for \( \ell \in \{k+1, \ldots, n\} \) (see (2.51)). We deduce that
\[
\forall x_{0:k}^* \in \mathcal{E}_k, \forall u, v \in \mathbb{R}^{d \times q} \text{ s.t. } uu^* \leq vv^*, \quad \Psi_k(x_{0:k}^*, u) \leq \Psi_k(x_{0:k}^*, v)
\]
and \( \Psi_k \) is convex. For \( x_{0:k}^*, y_{0:k}^* \in \mathcal{E}_k \) and \( \alpha \in [0, 1] \), with the existence of \( O \in \mathcal{O}(q) \) (possibly depending on \( t_{k+1}^n, x_{k}^n, y_{k}^n, \alpha \)) such that
\[
\sigma^*(t_{k}^n, \alpha x_{k}^n + (1-\alpha) y_{k}^n \leq \left( \alpha \sigma(t_{k}^n, x_{k}^n) + (1-\alpha) \sigma(t_{k}^n, y_{k}^n) O \right) \left( \alpha \sigma(t_{k}^n, x_{k}^n) + (1-\alpha) \sigma(t_{k}^n, y_{k}^n) O \right)^*.
\]
(see (1.37)), the convexity of \( \Psi_k \) and (2.53), we conclude that

\[
\Phi_k(\alpha x_{0:k}^* + (1 - \alpha)y_{0:k}^*) = \Psi_k(\alpha x_{0:k}^* + (1 - \alpha)y_{0:k}^*, \sigma(t^n_k, \alpha x_k^* + (1 - \alpha)y_k^*))
\leq \Psi_k(\alpha x_{0:k}^* + (1 - \alpha)y_{0:k}^*, \alpha\sigma(t^n_k, x_k^*) + (1 - \alpha)\sigma(t^n_k, y_k^*))O
\leq \alpha \Phi_k(x_{0:k}^*, \sigma(t^n_k, x_k^*)) + (1 - \alpha)\Psi_k(y_{0:k}^*, \sigma(t^n_k, y_k^*))O.
\]

In particular, \( x_0 \mapsto \Phi_0(x_{0:0}^*) = \phi_0(x_0) \) is convex.

\[
\Phi_k(X_{0:k}^*) = \mathbb{E}(\Phi_k(X_{0:k}^*) \mid \mathcal{F}_n^k) \quad \text{for} \quad k \in \{0, \ldots, n\},
\]

and, in particular,

\[
\mathbb{E}\Phi_0(X_0) = \cdots = \mathbb{E}\Phi_k(X_{0:k}) = \cdots = \mathbb{E}\Phi_n(X_{0:n}).
\]

2.2 Proof of Proposition 2.1

Let \( \mathcal{F}_t = \sigma(X_0, W_s, s \in [0, t], \mathcal{N}_T) \), \( \Phi_n : \mathcal{E}_n \to \mathbb{R} \) be convex with at most affine growth. We define the sequence \( (\Phi_k)_{k=0,\ldots,n} \) by the backward induction equalities (2.52), (2.51). It is clear by backward induction that, when \( ||X_0||_1 < +\infty \), then

\[
\Phi_k(X_{0:k}^*) = \mathbb{E}(\Phi_k(X_{0:k}^*) \mid \mathcal{F}_n^k) \quad \text{for} \quad k \in \{0, \ldots, n\},
\]

and, in particular,

\[
\mathbb{E}\Phi_0(X_0) = \cdots = \mathbb{E}\Phi_k(X_{0:k}) = \cdots = \mathbb{E}\Phi_n(X_{0:n}).
\]

We also set \( \tilde{\Phi}_n = \Phi_n \) and define \( \tilde{\Phi}_k \) by backward induction: for \( k \in \{0, \ldots, n-1\} \) and \( x_{0:k}^* \in \mathcal{E}_k \) (with the convention \( x_0^* = x^*_0 = \cdots = x_0^* = x_0 \) when \( k = 0 \)),

\[
\tilde{\Phi}_k(x_{0:k}^*) = \mathbb{E}\tilde{\Phi}_{k+1}(x_{0:k}^*, B_k(t^n_{k+1}, x_{k+1}^*, x_k^*) + \tilde{\sigma}(t^n_k, x_k^*) \int_{t^n_k}^{t^n_{k+1}} \tilde{K}_2(t^n_s, \tilde{s})dW_s, \cdots,
B_k(t^n_{k+1}, x_{k+1}^*, x_k^*) + \tilde{\sigma}(t^n_k, x_k^*) \int_{t^n_k}^{t^n_{k+1}} \tilde{K}_2(t^n_s, \tilde{s})dW_s),
\]

where \( B_k \) is still given by (2.54). Note that these equalities still hold with \( \Phi \), \( \tilde{\sigma} \), and \( \tilde{K}_2 \) replaced by \( \Phi \), \( \sigma \), and \( K_2 \) and that, similarly to (2.55), \( \mathbb{E}[\tilde{\Phi}_0(Y_0)] = \mathbb{E}[\tilde{\Phi}_n(Y_{0:n})] = \mathbb{E}[\Phi_n(Y_{0:n})] \).

Let us check by backward induction that for each \( k \in \{0, \ldots, n\} \), \( \Phi_k \leq \tilde{\Phi}_k \). The induction hypothesis holds with equality for \( k = n \). Let us assume that it holds at rank \( k + 1 \). When both the Euler schemes are \( K \)-integrated, then by (1.33),

\[
\int_{t^n_k}^{t^n_{k+1}} K_2^s(t^n_s, \cdots, t^n_{k+1}, \tilde{s})d\sigma^*(t^n_k, x_k^*) \leq \int_{t^n_k}^{t^n_{k+1}} \tilde{K}_2^s(t^n_s, \cdots, t^n_{k+1}, \tilde{s})d\tilde{\sigma}^*(t^n_k, x_k^*).
\]

This inequality still holds as a consequence of (1.23) when both the Euler schemes are \( K \)-discrete since then \( \tilde{s} = t^n_{k+1}^* \) for \( s \in [t^n_k, t^n_{k+1}] \). The left-hand and right-hand sides of the above inequality are the respective covariance matrices of the centered Gaussian vectors

\[
\left( \sigma(t^n_k, x_k^*) \int_{t^n_k}^{t^n_{k+1}} K_2(t^n_{k+1}^*, \tilde{s})dW_s, \cdots, \sigma(t^n_k, x_k^*) \int_{t^n_k}^{t^n_{k+1}} K_2(t^n_s, \tilde{s})dW_s \right).
\]
and \( \left( \tilde{\sigma}(t^n_k, x_k^j) \int_{t^n_k}^{t^n_{k+1}} \tilde{K}_2(t^n_{k+1}, \tilde{s}) dW_s, \ldots, \tilde{\sigma}(t^n_k, x_k^j) \int_{t^n_k}^{t^n_{k+1}} \tilde{K}_2(t^n_{k+1}, \tilde{s}) dW_s \right) \) so that the former is smaller than the latter in the convex order. By Proposition 2.3 when \( \sigma \) (resp. \( \tilde{\sigma} \)) satisfies the convexity assumption \((1.37)\), then the function \( \Phi_{k+1} \) (resp. \( \Phi_{k+1} \)) is convex so that

\[
\mathbb{E} \Phi_{k+1} \left( x^*_{0,k}, B_k(t^n_{k+1}, x_k^{k+1}, x_k^k) + \sigma(t^n_k, x_k^j) \int_{t^n_k}^{t^n_{k+1}} K_2(t^n_{k+1}, s) dW_s, \ldots, 
\right.

\[
B_k(t^n_k, x_k^j, x_k^k) + \sigma(t^n_k, x_k^j) \int_{t^n_k}^{t^n_{k+1}} K_2(t^n_{k+1}, s) dW_s, 
\]

\[
\leq \mathbb{E} \Phi_{k+1} \left( x^*_{0,k}, B_k(t^n_{k+1}, x_k^{k+1}, x_k^k) + \tilde{\sigma}(t^n_k, x_k^j) \int_{t^n_k}^{t^n_{k+1}} \tilde{K}_2(t^n_{k+1}, \tilde{s}) dW_s, \ldots, 
\right.

\[
B_k(t^n_k, x_k^j, x_k^k) + \tilde{\sigma}(t^n_k, x_k^j) \int_{t^n_k}^{t^n_{k+1}} \tilde{K}_2(t^n_{k+1}, \tilde{s}) dW_s \right)
\]

with the left-hand side equal to \( \Phi_k(x^*_{0,k}) \) and the right-hand side not greater than \( \Phi_{k+1}(x^*_{0,k}) \) by the induction hypothesis (resp. the same inequality holds with \( \Phi_{k+1} \) replaced by \( \Phi_{k+1} \), the right-hand side being equal to \( \Phi_{k+1}(x^*_{0,k}) \) and the left-hand side not smaller than \( \Phi_k(x^*_{0,k}) \) by the induction hypothesis). Hence the induction hypothesis holds at rank \( k \).

For \( k = 0 \), with the convexity of either \( \Phi_0 \) (when \( \sigma \) satisfies \((1.37)\)) or \( \Phi_0 \) (when \( \tilde{\sigma} \) satisfies \((1.37)\)) and the inequality \( X_0 \leq_{\text{ex}} Y_0 \), it yields \( \mathbb{E} \Phi_0(X_0) \leq \mathbb{E} \Phi_0(Y_0) \) which also writes \( \mathbb{E} [\Phi_n(X^*_{0,n})] \leq \mathbb{E} [\Phi_n(Y^*_{0,n})] \). We conclude with Lemma 1.7.

### 3 Increasing convex ordering for the Euler schemes \((d = q = 1)\)

The comparison for the Euler schemes is again obtained as a consequence of the comparison for the companion processes \( X^*_{0,n} \) and \( X^*_{0,n} \) where the notation has been introduced at the beginning of Section 2. We recall that, under \((2.47)\) and \((2.48)\), when \( \|X_0\|_p + \|Y_0\|_p < +\infty \) for some \( p \geq 1 \), then \( \|X^*_{0,n}\|_p + \|Y^*_{0,n}\|_p < +\infty \).

**Proposition 3.1** (Increasing convex ordering) Assume \((2.47)\), that either both functions \( \mathbb{R} \ni x \mapsto b(t^n_k, x) \) and \( \mathbb{R} \ni x \mapsto |\sigma(t^n_k, x)| \) are convex and non-decreasing for every \( k \in \{0, \ldots, n-1\} \) or both functions \( \mathbb{R} \ni x \mapsto \tilde{b}(t^n_k, x) \) and \( \mathbb{R} \ni x \mapsto |\tilde{\sigma}(t^n_k, x)| \) are convex and non-decreasing for every \( k \in \{0, \ldots, n-1\} \). Also assume that either both the Euler schemes are \( K \)-discrete and the comparison assumptions \((1.40)\) between \((K_1, b)\) and \((\tilde{K}_1, \tilde{b})\) and \((C\mathcal{K}^{\text{disc}}_2 \sigma_{1d})\) (see \((1.30)\)) between \((K_2, \sigma)\) and \((\tilde{K}_2, \tilde{\sigma})\) hold or both the Euler schemes are \( K \)-integrated and \((2.48)\), \((1.39)\) and \((C\mathcal{K}^{\text{int}}_2 \sigma)\) hold. Last assume that \( X_0 \leq_{\text{i.c.}} Y_0 \) with \( X_0, Y_0 \in L^1(\mathbb{P}) \). Then, for every \( n \geq 1 \),

\[
X^*_{0,n} \leq_{\text{i.c.}} Y^*_{0,n} \quad \text{and, in particular,} \quad (\tilde{X}^*_{1,n})_{k=0, \ldots, n} \leq_{\text{i.c.}} (\tilde{Y}^*_{1,n})_{k=0, \ldots, n}.
\]

**Remark 3.2** For the conclusion to hold, it is enough that

- when both schemes are \( K \)-integrated, \((1.39)\) holds for \( (s_0, s_1, s_2) \in \{(t^n_k, t^n_{k+1}, t^n_n) : 0 \leq k < \ell \leq n\} \) and \((C\mathcal{K}^{\text{int}}_2 \sigma)\) holds for \( (s_j, \ldots, s_1, s_0) \) of the form \( (t^n_n, t^n_{n-1}, \ldots, t^n_{n+1-j}, t^n_{n-j}) \) for \( j \in \{0, \ldots, n\} \),
when both schemes are $K$-discrete, the comparison assumptions \([1.40]\) between \((K_1, b)\) and \((\bar{K}_1, \bar{b})\) and \((C \mathcal{K}_2^{int}\sigma_1d)\) between \((K_2, \sigma)\) and \((\bar{K}_2, \bar{\sigma})\) hold for \((s, t) \in \{(t^n_k, t^n_\ell) : 0 \leq k < \ell \leq n\}\).

### 3.1 Backward propagation of increasing convexity by the Euler scheme

Let \(\Phi_n : \mathcal{E}_n \to \mathbb{R}\) be a function with \(p\)-polynomial growth for some \(p \geq 1\). We define the functions \(\Phi_k : \mathcal{E}_k \to \mathbb{R}, k \in \{0, \ldots, n - 1\}\) using some auxiliary functions \(\hat{\Psi}_k\) by the following backward induction: for every \((x_{0:k}^\bullet, u, v) \in \mathcal{E}_k \times \mathbb{R} \times \mathbb{R},\)

\[
\hat{\Psi}_k(x_{0:k}^\bullet, u, v) = \mathbb{E} \Phi_{k+1} \left( x_{0:k}^\bullet, x_k^{k+1} + u \int_{t^n_k}^{t^n_{k+1}} K_1(t^n_{k+1}, \hat{s})d\hat{s} + v \int_{t^n_k}^{t^n_{k+1}} K_2(t^n_{k+1}, \hat{s})dW_\hat{s}, \ldots, x_k^n + u \int_{t^n_k}^{t^n_{k+1}} K_1(t^n_n, \hat{s})d\hat{s} + v \int_{t^n_k}^{t^n_{k+1}} K_2(t^n_n, \hat{s})dW_\hat{s} \right)
\]

(3.57)

and \(\forall x_{0:k}^\bullet \in \mathcal{E}_k, \Phi_k(x_{0:k}^\bullet) = \hat{\Psi}_k(x_{0:k}^\bullet, b(t^n_k, x_k), \sigma(t^n_k, x_k))\),

(3.58)

with the convention \(x_{0:0}^0 = x_{0:1}^1 = \cdots = x_{0:n}^n = x_0\) when \(k = 0\). Note that under what concerns \((K_1, K_2, b, \sigma)\) in \([2.47]\) and \([2.48]\) (which appears in the assumptions of the proposition just below), the functions \(x \mapsto b(t^n_k, x)\) and \(x \mapsto \sigma(t^n_k, x)\) have at most affine growth and

\[
\left( \int_{t^n_k}^{t^n_{k+1}} K_2(t^n_{k+1}, \hat{s})dW_\hat{s}, \ldots, \int_{t^n_k}^{t^n_{k+1}} K_2(t^n_n, \hat{s})dW_\hat{s} \right)
\]

is a Gaussian random vector and thus belongs to \(L^p(\mathbb{P})\). One easily deduces still by backward induction that the random variable in the expectation in the right-hand side of (3.57) is integrable and both functions \(\hat{\Psi}_k\) and \(\Phi_k\) have \(p\)-polynomial growth for every \(k \in \{0, \ldots, n - 1\}\).

**Proposition 3.3 (Backward propagation of increasing convexity)** Assume

\[
\forall k \in \{0, \ldots, n - 1\}, \int_0^{t^n_{k+1}} \left( K_1(t^n_{k+1}, \hat{s}) + K_2(t^n_{k+1}, \hat{s})^2 \right)ds + \sup_{x \in \mathbb{R}^d} \left| b(t^n_k, x) + ||\sigma(t^n_k, x)|| \right| < +\infty
\]

and that both functions \(\mathbb{R} \ni x \mapsto b(t^n_k, x)\) and \(\mathbb{R} \ni x \mapsto ||\sigma(t^n_k, x)||\) are convex and non-decreasing for every \(k \in \{0, \ldots, n - 1\}\). When \(\Phi_n : \mathcal{E}_n \to \mathbb{R}\) is convex with \(p\)-polynomial growth for some \(p \geq 1\) and non-decreasing in each of its variables, then so are the functions \(\mathcal{E}_k \ni x_{0:k}^\bullet \mapsto \Phi_k(x_{0:k}^\bullet), k = 0, \ldots, n,\) defined by \((3.58)\) and \((3.57)\).

In particular, if \(\Phi_n(x_{0:n}^\bullet) = F(x_0, x_1, \ldots, x_n)\) with \(F : (\mathbb{R}^d)^{n+1} \to \mathbb{R}\) convex with polynomial growth and non-decreasing in each of its variables, then convexity and monotonicity are transferred to the function \(\mathbb{R}^d \ni x_0 \mapsto \Phi_0(x_0) = \mathbb{E}(F(\Xi_{t^n_k}^{x_0})_{k=0,\ldots,n})\) where \((\Xi_{t^n_k}^{x_0})_{k=0,\ldots,n}\) denotes any of the two Euler schemes (either \(K\)-discrete or \(K\)-integrated) starting from \(X_0 = x_0\).

**Proof.** If \(\Phi_{k+1}\) is convex w.r.t. \(x_{k+1}^\bullet\), then \(\hat{\Psi}_k\) is convex w.r.t. \((x_{0:k}^\bullet, u, v)\) and, by a reasoning similar to the above derivation of \((2.53)\) and \((2.54)\), satisfies

\[
\forall x_{0:k}^\bullet \in \mathcal{E}_k, \forall u, v \in \mathbb{R}, \hat{\Psi}_k(x_{0:k}^\bullet, u, v) = \hat{\Psi}_k(x_{0:k}^\bullet, |u|, |v|),
\]

(3.59)

and \(\forall u \in \mathbb{R}\) s.t. \(|v| \leq |w|, \hat{\Psi}_k(x_{0:k}^\bullet, u, |v|) \leq \hat{\Psi}_k(x_{0:k}^\bullet, u, w).\)

(3.60)
If $\Phi_{k+1}$ is non-decreasing in each of its variables, then $\hat{\Psi}_k$ is non-decreasing in $u \in \mathbb{R}$ since $K_1 \geq 0$. Hence when $\Phi_{k+1}$ is convex and non-decreasing in each of its variables, then using that $x \mapsto b(t_k^n, x)$ and $x \mapsto |\sigma(t_k^n, x)|$ are non-decreasing, we deduce that $\Phi_k$ is non-decreasing in each of its variables. Moreover, using \eqref{3.58} and \eqref{3.59} for the equalities, the monotonicity of $\hat{\Psi}_k$ in its two last variables and the convexity of $x \mapsto |\sigma(t_k^n, x)|$ and $x \mapsto b(t_k^n, x)$ for the first inequality, then the convexity of $\hat{\Psi}_k$ for the second inequality, we obtain that for $x_{0:k}^*, y_{0:k}^* \in \mathcal{E}_k$ and $\alpha \in [0, 1]$,

\[
\Phi_k(\alpha x_{0:k}^* + (1 - \alpha)y_{0:k}^*) = \hat{\Psi}_k(\alpha x_{0:k}^* + (1 - \alpha)y_{0:k}^*, b(t_k^n, \alpha x_k^* + (1 - \alpha)y_k^*), |\sigma(t_k^n, \alpha x_k^* + (1 - \alpha)y_k^*)|)
\leq \hat{\Psi}_k(\alpha x_{0:k}^*, b(t_k^n, x_k^*), |\sigma(t_k^n, x_k^*)| + (1 - \alpha)|\sigma(t_k^n, y_k^*)|)
\leq \alpha \hat{\Psi}_k(x_{0:k}^*, b(t_k^n, x_k^*), |\sigma(t_k^n, x_k^*)|) + (1 - \alpha)\hat{\Psi}_k(y_{0:k}^*, b(t_k^n, y_k^*), |\sigma(t_k^n, y_k^*)|)
= \alpha \Phi_k(x_{0:k}^*) + (1 - \alpha)\Phi_k(y_{0:k}^*).
\]

By backward induction, we conclude that when $\Phi_n$ is convex and non-decreasing in each of its variables, then so is $\Phi_k$ for each $k \in \{0, \ldots, n\}$.

**Remark 3.4** A simpler argument for the derivation of \eqref{3.61} in this more elementary framework is that, when $f$ is convex and $Z$ is a centered random variable, then

\[(x, v) \mapsto \mathbb{E} f(x + vZ) \quad \text{is convex and non-decreasing in } v \text{ on } \mathbb{R}_+\]

as it is bounded from below by its value at 0 by Jensen’s inequality.

### 3.2 Proof of Proposition 3.1

Let $\Phi_n : \mathcal{E}_n \to \mathbb{R}$ be a convex function with at most affine growth, non-decreasing in each of its variables. We are going to check that $\mathbb{E}[\Phi_n(X_{0:n}^*)] \leq \mathbb{E}[\Phi_n(Y_{0:n}^*)]$ so that the conclusion follows from Lemma 1.7. We define the sequence $(\Psi_k, \Phi_k)_{k=0, \ldots, n-1}$ by the backward induction equalities \eqref{3.57} and \eqref{3.58}. It is clear by backward induction that

\[
\Phi_k(X_{0:k}^*) = \mathbb{E}(\Phi_n(X_{0:n}^*) \mid \mathcal{F}_k^n) \quad \text{for } k \in \{0, \ldots, n\}.
\]

In particular,

\[
\mathbb{E}(\Phi_0(X_0)) = \cdots = \mathbb{E}(\Phi_k(X_{0:k}) = \cdots = \mathbb{E}(\Phi_n(X_{0:n}^*). \quad \text{(3.61)}
\]

We also set $\bar{\Phi}_n = \Phi_n$ and define $(\bar{\Phi}_k)_{k=0, \ldots, n-1}$ by backward induction: for $k \in \{0, \ldots, n - 1\}$ and $x_{0:k}^* \in \mathcal{E}_k$ (with $x_0^* = x_0^1 = \cdots = x_n^* = x_0$),

\[
\bar{\Phi}_k(x_{0:k}^*) = \mathbb{E}(\bar{\Phi}_{k+1}(x_{0:k}^* + b(t_k^n, x_k^*) \int_{x_k^n}^{t_{k+1}^n} \tilde{K}_1(t_{k+1}^n, s)ds + \tilde{\sigma}(t_{k+1}^n, x_k^n) \int_{x_k^n}^{t_{k+1}^n} \tilde{K}_2(t_{k+1}^n, s)ds)dW_s, \cdots,
\]

\[
x_k^n + b(t_k^n, x_k^n) \int_{x_k^n}^{t_{k+1}^n} \tilde{K}_1(t_{k+1}^n, s)ds + \tilde{\sigma}(t_{k+1}^n, x_k^n) \int_{x_k^n}^{t_{k+1}^n} \tilde{K}_2(t_{k+1}^n, s)ds)dW_s).
\]

Note that these equalities still hold with $\hat{\Phi}, \bar{b}, \tilde{K}_1, \tilde{\sigma}$ and $\tilde{K}_2$ replaced by $\Phi, b, K_1, \sigma$ and $K_2$ and that, similarly to \eqref{3.61}, $
\mathbb{E}[\Phi_0(Y_0)] = \mathbb{E}[\Phi_n(Y_{0:n})] = \mathbb{E}[\Phi_n(Y_{0:n})].$

25
Let us check by backward induction that for each $k \in \{0, \cdots, n\}$, $\Phi_k \leq \Phi_k$. The induction hypothesis holds with equality for $k = n$. Let us assume that it holds at rank $k + 1$. When both Euler schemes are $K$-integrated, then by (1.33),
\[
\sigma^2(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} K_2(t^n_{k+1}, \cdot, \hat{s}) ds \leq \sigma^2(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} K_2 \bar{K}_2(t^n_{k+1}, \cdot, \hat{s}) ds.
\]
This inequality still holds as a consequence of (1.39) when both Euler schemes are $K$-discrete since then $\hat{s} = t^n_k$ for $s \in [t^n_k, t^n_{k+1})$. The left-hand side and the right-hand side are the respective covariance matrices of the centered Gaussian random vector $s$

and

\[
\left(\sigma(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} K_2(t^n_{k+1}, \cdot, \hat{s}) ds, \ldots, \sigma(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} K_2(t^n_{k+1}, \hat{s}) ds\right)
\]

so that the former is smaller than the latter in the convex order. Moreover, when both Euler schemes are $K$-integrated, by (1.39),
\[
b(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} K_1(t^n_{k+1}, \cdot, \hat{s}) ds \leq \hat{b}(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} \bar{K}_1(t^n_{k+1}, \hat{s}) ds
\]

for $\ell \in \{k + 1, \cdots, n\}$. This inequality still holds as a consequence of (1.40) when both the Euler schemes are $K$-discrete. Hence
\[
\forall \ell \in \{k + 1, \cdots, n\}, x^n_k + b(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} K_1(t^n_{k+1}, \cdot, \hat{s}) ds \leq x^n_k + \hat{b}(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} \bar{K}_1(t^n_{k+1}, \hat{s}) ds.
\]

By Proposition 3.3 when the functions $\mathbb{R} \ni x \mapsto b(t, x)$ and $\mathbb{R} \ni x \mapsto |\sigma(t, x)|$ (resp. $\mathbb{R} \ni x \mapsto \hat{b}(t, x)$ and $\mathbb{R} \ni x \mapsto |\hat{\sigma}(t, x)|$) are both convex and non-decreasing for every $t \in [0, T]$, then the function $\Phi_{k+1}$ (resp. $\Phi_{k+1}$) is convex and non-decreasing in each of its variables so that
\[
\mathbb{E} \Phi_{k+1}\bigg(x_{0:k}, x_{k+1}^{k+1} + b(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} K_1(t^n_{k+1}, \cdot, \hat{s}) ds + \sigma(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} K_2(t^n_{k+1}, \hat{s}) ds, \ldots, \bigg)\nonumber
\]
\[
\leq \mathbb{E} \Phi_{k+1}\bigg(x_{0:k}, x_{k+1}^{k+1} + \hat{b}(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} \bar{K}_1(t^n_{k+1}, \cdot, \hat{s}) ds + \hat{\sigma}(t^n_k, x^n_k) \int_{t^n_k}^{t^n+k+1} \bar{K}_2(t^n_{k+1}, \hat{s}) ds, \ldots, \bigg)
\]

with the left-hand side equal to $\Phi_k(x_{0:k})$ and the right-hand side not smaller than $\Phi_k(x_{0:k})$ by the induction hypothesis (resp. the same inequality holds with $\Phi_{k+1}$ replaced by $\Phi_{k+1}$, the right-hand side being equal to $\Phi_k(x_{0:k})$ and the left-hand side not smaller than $\Phi_k(x_{0:k})$ by the induction hypothesis). Hence the induction hypothesis holds at rank $k$.

For $k = 0$, with the convexity and monotonicity of either $\Phi_0$ (when $\mathbb{R} \ni x \mapsto b(t, x)$ and $\mathbb{R} \ni x \mapsto |\sigma(t, x)|$ are convex and non-decreasing) or $\Phi_0$ (when $\mathbb{R} \ni x \mapsto \hat{b}(t, x)$ and $\mathbb{R} \ni x \mapsto |\hat{\sigma}(t, x)|$ are convex and non-decreasing) and the inequality $X_0 \leq_{icv} Y_0$, it yields $\mathbb{E}[\Phi_0(X_0)] \leq \mathbb{E}[\Phi_0(Y_0)]$ which also writes $\mathbb{E}[\Phi_n(X_{0:n}^n)] \leq \mathbb{E}[\Phi_n(Y_{0:n}^n)]$. □
4 From discrete to continuous time

4.1 \(L^p\)-convergence of the affine interpolation of the Euler scheme

To transfer our discrete time results on convex ordering to continuous time, we combine Theorems 1.2 and 1.4 with the \(C([0, T], \mathbb{R}^d)\)-valued interpolation operator \(i_n\) associated to the mesh \((t^n_k = \frac{kT}{n})_{0 \leq k \leq n}\), already introduced in [21]:

\[
i_n : (\mathbb{R}^d)^{n+1} \ni x_{0:n} \mapsto \left( t \mapsto \sum_{k=1}^{n} 1_{[t^n_{k-1}, t^n_k)}(t) \left( \frac{t^n_k - t}{t^n_k - t^n_{k-1}} x_{k-1} + \frac{t - t^n_{k-1}}{t^n_k - t^n_{k-1}} x_k \right) + 1_{\{t=t^n_k\}} x_n \right).
\]

**Corollary 4.1** Let \(p \in [1, +\infty)\) be such that \(\|X_0\|_p < +\infty\) and suppose either that \((\bar{X}^n_k)_{k=0,...,n}\) is the \(K\)-integrated Euler scheme and the assumptions of Theorem 1.2 (b) are satisfied or \((\bar{X}^n_k)_{k=0,...,n}\) is the \(K\)-discrete Euler scheme and the assumptions of Theorem 1.4 (b) are satisfied. Then

\[
\|i_n((\bar{X}^n_k)_{k=0,...,n}) - X\|_{\text{sup}} \to 0 \quad \text{as} \quad n \to +\infty.
\]

**Proof.** For \(x_{0:n} \in (\mathbb{R}^d)^{n+1}\), the function \(i_n(x_{0:n})\) being continuous and piecewise affine with affinity breaks at times \(t^n_k\) where it is equal to \(x_k\),

\[
\|i_n(x_{0:n})\|_{\text{sup}} = \max_{k=0,...,n} |x_k|.
\]

For \(x_{0:n}, y_{0:n+1} \in (\mathbb{R}^d)^{n+1}\), since \(i_n(x_{0:n}) - i_n(y_{0:n}) = i_n(x_{0:n} - y_{0:n})\), we deduce that \(i_n\) is Lipschitz with constant 1 from \((\mathbb{R}^d)^{n+1}\) to \(C([0, T], \mathbb{R}^d)\), \(\| \cdot \|_{\text{sup}}\):

\[
\|i_n(x_{0:n}) - i_n(y_{0:n})\|_{\text{sup}} = \max_{k=0,...,n} |x_k - y_k|.
\]

As a consequence, one has

\[
\|i_n((\bar{X}^n_k)_{k=0,...,n}) - i_n((X^n_k)_{k=0,...,n})\|_{\text{sup}} = \max_{k=0,...,n} |\bar{X}^n_k - X^n_k|.
\]

Let \(p > 0\) be such that \(\|X_0\|_p + \|Y_0\|_p < +\infty\). Calling upon either Theorem 1.4 or Theorem 1.2 we deduce that

\[
\|i_n((\bar{X}^n_k)_{k=0,...,n}) - i_n((X^n_k)_{k=0,...,n})\|_p \to 0 \quad \text{as} \quad n \to +\infty.
\]

Now, the uniform continuity modulus \(w(\xi, \delta) = \sup_{0 \leq s \leq t \leq (s+\delta) \land T} |\xi(t) - \xi(s)|\) of a function \(\xi \in C([0, T], \mathbb{R}^d)\) converges to 0 as \(\delta \to 0\). One easily checks that

\[
\|X - i_n((X^n_k)_{k=0,...,n})\|_{\text{sup}} \leq w(X; \frac{T}{n}) \leq 2\|X\|_{\text{sup}}.
\]

As, according to the last statement in Theorem 1.1 \(\|X\|_{\text{sup}}\|_p < +\infty\), it follows by dominated convergence that

\[
\|X - i_n((X^n_k)_{k=0,...,n})\|_{\text{sup}} \to 0 \quad \text{as} \quad n \to +\infty.
\]

Combining these two convergences yields

\[
\|i_n((\bar{X}^n_k)_{k=0,...,n}) - X\|_p \to 0 \quad \text{as} \quad n \to +\infty.
\]

\(\square\)
4.2 Proofs of Theorem 1.11 and Theorem 1.13

Proof of Theorem 1.11. In this proof, \((\bar{X}_{t_k}^n)_{k=0:n}\) denotes the \(K\)-discrete Euler scheme when \((K_1, K_2)\) and \((\bar{K}_1, \bar{K}_2)\) satisfy \((\mathcal{K}^{\text{cont}})\) and the \(K\)-integrated Euler scheme when this condition is replaced by the weaker condition \((\mathcal{K}^{\text{cont}})\) up to a reinforcement of the comparison assumptions between \((K_2, \sigma)\) and \((\bar{K}_2, \bar{\sigma})\) (resp. \((K_1, b)\) and \((\bar{K}_1, \bar{b})\)).

(a) By Corollary 4.1, the 1-Wasserstein distance \(W_1(i_n((\bar{X}_{t_k}^n)_{k=0:n}), X)\) between the law of \(i_n((\bar{X}_{t_k}^n)_{k=0:n})\) and that of \(X\) goes to 0 as \(n \to \infty\). For each \(\text{Lip}(F)\)-Lipschitz functional \(F : C([0, T], \mathbb{R}^d) \to \mathbb{R}\), since \(|\mathbb{E} F \circ i_n((\bar{X}_{t_k}^n)_{k=0:n})) - \mathbb{E} F(X)| \leq \text{Lip}(F) W_1(i_n((\bar{X}_{t_k}^n)_{k=0:n}, (\bar{X}_{t_k}^n)_{k=0:n}))\), we have

\[
\mathbb{E} F \circ i_n((\bar{X}_{t_k}^n)_{k=0:n})) \to \mathbb{E} F(X) \text{ as } n \to \infty.
\]

In the same way, \(\mathbb{E} F \circ i_n((\bar{Y}_{t_k}^n)_{k=0:n})) \to \mathbb{E} F(Y)\).

Let now \(F : C([0, T], \mathbb{R}^d) \to \mathbb{R}\) be a Lipschitz convex functional. The function \(F_n : (\mathbb{R}^d)^{n+1} \to \mathbb{R}\) defined by \(F_n(x_{0:n}) = F(i_n(x_{0:n}))\) is \(\text{Lip}(F)\)-Lipschitz since \(i_n\) is \(1\)-Lipschitz, convex as the composition of a convex function with a linear application and such that \(\mathbb{E} F \circ i_n((\bar{X}_{t_k}^n)_{k=0:n})) = \mathbb{E} F_n((\bar{X}_{t_k}^n)_{k=0:n})\) and \(\mathbb{E} F \circ i_n((\bar{Y}_{t_k}^n)_{k=0:n})) = \mathbb{E} F_n((\bar{Y}_{t_k}^n)_{k=0:n})\). It follows from Proposition 2.1 that, under the assumptions made on the coefficients \((K_1, b, K_2, \sigma, \bar{K}_1, \bar{b}, \bar{K}_2, \bar{\sigma})\) and the initial conditions \((X_0, Y_0)\) that, for every \(n \geq 1\),

\[
\mathbb{E} F_n((\bar{X}_{t_k}^n)_{k=0:n})) \leq \mathbb{E} F_n((\bar{Y}_{t_k}^n)_{k=0:n})).
\]

Consequently,

\[
\mathbb{E} F(X) = \lim_n \mathbb{E} F_n((\bar{X}_{t_k}^n)_{k=0:n})) \leq \lim_n \mathbb{E} F_n((\bar{Y}_{t_k}^n)_{k=0:n}))) = \mathbb{E} F(Y).
\]

By Lemma 1.3, we conclude that \(X \preceq_{cvx} Y\).

(b) One easily checks that if \(d = 1\) and the convex functional \(F\) is non-decreasing for the pointwise partial order on \(C([0, T], \mathbb{R})\), then \(F_n\) is non-decreasing for the componentwise partial order on \(\mathbb{R}^{n+1}\). One concludes by the same reasoning as in the proof of (a) with Proposition 2.1 replaced by Proposition 3.1.

(c) Setting \((\bar{X}_t)_{t \in [0, T]} = (-X_t)_{t \in [0, T]}\), \((\bar{Y}_t)_{t \in [0, T]} = (-Y_t)_{t \in [0, T]}\) and \((\bar{W}_t)_{t \in [0, T]} = (-W_t)_{t \in [0, T]}\), we have that \((\bar{W}_t)_{t \in [0, T]}\) is a Brownian motion independent from \(\bar{X}_0\) and

\[
\bar{X}_t = \bar{X}_0 + \int_0^t K_1(t, s)(-b(s, -\bar{\bar{X}}_s))ds + \int_0^t K_2(t, s)\bar{\sigma}(s, -\bar{\bar{X}}_s)d\bar{W}_s, \quad t \in [0, T],
\]

\[
\bar{Y}_t = \bar{Y}_0 + \int_0^t \bar{K}_1(t, s)(-\bar{b}(s, -\bar{\bar{Y}}_s))ds + \int_0^t \bar{K}_2(t, s)\bar{\bar{\sigma}}(s, -\bar{\bar{Y}}_s)d\bar{W}_s, \quad t \in [0, T].
\]

The inequality \(X_0 \preceq_{dcv} Y_0\) implies \(\bar{X}_0 \preceq_{dcv} \bar{Y}_0\) (see the reasoning just below concerning the functionals \(F\) and \(G\)). When \(x \mapsto -b(t, x)\) and \(x \mapsto |\sigma(t, x)|\) (resp. \(x \mapsto -\bar{b}(t, x)\) and \(x \mapsto |\bar{\bar{\sigma}}(t, x)|\)) are convex and non-increasing, then \(x \mapsto -b(t, -x)\) and \(x \mapsto |\sigma(t, -x)|\) (resp. \(x \mapsto -\bar{b}(t, -x)\) and \(x \mapsto |\bar{\bar{\sigma}}(t, -x)|\)) are convex and non-decreasing. Moreover 1.12 implies that

\[
\forall 0 \leq s < t \leq T, \forall x \in \mathbb{R}, K_1(t, s)(-b(s, -x)) \leq \bar{K}_1(t, s)(-\bar{b}(s, -x))
\]
and \((1.41)\) implies that

\[
\forall 0 \leq s_0 < s_1 \leq s_2 \leq T, \forall x \in \mathbb{R}, \quad -b(s_0, -x) \int_{s_0}^{s_1} K_1(s_2, s) ds \leq -\tilde{b}(s_0, -x) \int_{s_0}^{s_1} \tilde{K}_1(s_2, s) ds.
\]

Let \(F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}\) be l.s.c., convex and non-increasing for the pointwise partial order on continuous functions. Then the functional \(G\) defined by \(G((x_t)_{t \in [0, T]}) = F(-(x_t)_{t \in [0, T]})\) is also l.s.c., convex but non-decreasing for the pointwise partial order. By \((b)\), \(\mathbb{E}G(\tilde{X}) \leq \mathbb{E}G(\tilde{Y})\), which also writes \(\mathbb{E}F(X) \leq \mathbb{E}F(Y)\). This completes the proof. \(\square\)

**Proof of Theorem 1.13.** In this proof, \((\tilde{X}_{t_k}^{x_0})_{k=0:n}\) denotes the \(K\)-integrated Euler scheme starting from \(x_0 \in \mathbb{R}^d\).

(a) Let \(p \in [1, +\infty)\) denote the polynomial growth order of the convex functional \(F : \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}\). We know from \([20]\) Lemma 2.1.1 that \(F\) is then continuous for the sup-norm. By Corollary 1.1 the \(p\)-Wasserstein distance \(W_p(i_n((\tilde{X}_{t_k}^{x_0})_{k=0:n}), X^{x_0})\) between the law of \(i_n((\tilde{X}_{t_k}^{x_0})_{k=0:n})\) and that of the solution \(X^{x_0}\) to \((1.2)\) starting from \(x_0 \in \mathbb{R}^d\) goes to 0 as \(n \to \infty\). Hence

\[
\mathbb{E}F \circ i_n((\tilde{X}_{t_k}^{x_0})_{k=0:n})) \to \mathbb{E}F(X^{x_0}) \text{ as } n \to \infty.
\]

Since \((\mathbb{R}^d)^{n+1} \ni x_{0:n} \mapsto F_n(x_{0:n}) = F(i_n(x_{0:n}))\) is convex with \(p\)-polynomial growth, by Proposition 2.3 \(\mathbb{R}^d \ni x_0 \mapsto \mathbb{E}F \circ i_n((\tilde{X}_{t_k}^{x_0})_{k=0:n}))\) is convex. We conclude that \(\mathbb{R}^d \ni x_0 \mapsto \mathbb{E}F(X^{x_0})\) is convex as the pointwise limit of convex functions.

(b) If \(d = 1\) and the convex functional \(F\) is non-decreasing for the pointwise partial order on \(\mathcal{C}([0, T], \mathbb{R})\), then \(F_n\) is non-decreasing for the componentwise partial order on \(\mathbb{R}^{n+1}\). One concludes by the same reasoning as in the proof of \((a)\) with Proposition 2.3 replaced by Proposition 3.3.

(c) The conclusion follows from \((b)\) combined with the change of sign argument used in the proof of Theorem 1.11 \((c)\). \(\square\)

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Moreover, \( Q \) equivalently when \( \text{Im} Q = \text{Im} \sqrt{AA^*} \) is such that \( QQ^* \) is the orthogonal projection on \( \text{Im} \sqrt{AA^*} \).

The proof of the above corollary is obvious since, by definition of radial symmetry, \( OZ \sim Z \) for every \( O \in O(q) \). Lemma A.1 is not needed when \( Z \sim N_q(0, I_q) \), since then \( AZ \sim N_q(0, AA^*) \).

The proof below is inspired by similar (though less general) results from [17].

**Proof of lemma A.1**: If \( A = BO \) with \( O \in O(q) \), then \( AA^* = BOO^*B^* = BB^* \).

\( \Rightarrow \): In order to prove (simultaneously) the necessary condition and the equality \( A = \sqrt{AA^*}Q \) in the three cases \( d = q, d < q \) and \( d > q \), let us characterize the matrices \( Q \in \mathbb{R}^{d \times q} \) such that \( QQ^* \) is the orthogonal projection on \( \text{Im} AA^* \).

Since clearly \( \text{Im} AA^* \subseteq \text{Im} A \) and \( \text{Ker} AA^* = \text{Ker} A^* = (\text{Im} A)^\perp \) so that

\[
\dim \text{Im} AA^* = d - \dim \text{Ker} AA^* = d - \dim (\text{Im} A)^\perp = \dim \text{Im} A \leq d \land q,
\]

\( \text{Im} A = \text{Im} AA^* = \sqrt{\text{Im} AA^*} \) where the second equality is obtained by replacing \( A \) by \( \sqrt{AA^*} \).

When \( Q \) has \( \dim \text{Im} AA^* \) columns which form an orthonormal basis of \( \text{Im} AA^* \) and its other columns vanish, it is clear that \( QQ^* \) is the orthogonal projection on \( \text{Im} \sqrt{AA^*} \). Let us conversely suppose that \( Q \in \mathbb{R}^{d \times q} \) with columns \((Q_j)_{1 \leq j \leq q}\) such that \( QQ^* \) is the orthogonal projection on \( \text{Im} \sqrt{AA^*} \). Since \( \text{Im} Q = \text{Im} QQ^* = \text{Im} \sqrt{AA^*} \), the vectors \( Q_j \) belong to \( \text{Im} \sqrt{AA^*} \) and \( Q_j = QQ^*Q_j \). Hence for \( j \in \{1, \cdots, q\} \),

\[
|Q_j|^2 = Q_j^*QQ^*Q_j = \sum_{i=1}^{q} (Q_i^*Q_j)^2 = |Q_j|^2 + \sum_{i \neq j} (Q_i^*Q_j)^2,
\]

which ensures that the vectors \( Q_j \) are orthogonal. Hence \( |Q_j|^2Q_j = QQ^*Q_j = Q_j \) so that either \( |Q_j| = 0 \) or \( |Q_j| = 1 \). Therefore \( Q \) has exactly \( \dim \text{Im} \sqrt{AA^*} \) orthonormal columns which belong to \( \text{Im} \sqrt{AA^*} \) and its other columns vanish.

**Case \( d = q \)**. By the singular value decomposition, \( A = UDV \) and \( B = \tilde{U}D\tilde{V} \) for \( U, V, \tilde{U}, \tilde{V} \in O(q) \) and \( D, \tilde{D} \in \mathbb{R}^{q \times q} \) diagonal with non-negative diagonal elements. Then \( AA^* = UD^2U^* \) and \( \sqrt{AA^*} = UD\sqrt{U^*} \) so that \( A = \sqrt{AA^*} \). Denoting by \( P \in \mathbb{R}^{q \times q} \) the matrix of the orthogonal projection on \( \text{Im} \sqrt{AA^*} \) in the canonical basis of \( \mathbb{R}^q \), one has \( \sqrt{AA^*} \) since \( (\text{Im} \sqrt{AA^*})^\perp = \text{Ker} \sqrt{AA^*} \). Hence \( A = \sqrt{AA^*}PUV \) where \( PUV(PUV)^* = P \). In the same way, \( B = \sqrt{BB^*}UV \). When \( AA^* = BB^* \), we conclude that \( A = BO \) with \( O = V^*U^*UV \in O(q) \).

---

**A. Matrices, covariances and random vectors**

**Lemma A.1** Let \( A, B \in \mathbb{R}^{d \times q} \). Then

\[
AA^* = BB^* \iff \exists O \in O(q) \text{ such that } A = BO.
\]

Moreover, \( A = \sqrt{AA^*}Q \) where \( Q \in \mathbb{R}^{d \times q} \) is such that \( QQ^* \) is the orthogonal projection on \( \text{Im} \sqrt{AA^*} \) or equivalently \( Q \) has \( \dim (\text{Im} \sqrt{AA^*}) \) columns which form an orthonormal basis of \( \text{Im} \sqrt{AA^*} \) and its other columns vanish.

**Corollary A.2** Let \( A, B \in \mathbb{R}^{d \times q} \). If \( AA^* = BB^* \), then for any radially symmetric \( \mathbb{R}^q \)-valued random vector \( Z, AZ \sim BZ \) (such is the case for \( \mathcal{N}(0, I_q) \)-distributed random vectors).

---
Case \(d < q\). We define \(\tilde{A}, \tilde{B} \in \mathbb{R}^{q \times q}\) by

\[
(\tilde{A}_{ij}, \tilde{B}_{ij}) = \begin{cases} (A_{ij}, B_{ij}) & \text{for } i = 1 : d, j = 1 : q, \\ (0, 0) & \text{for } i = d + 1 : q, j = 1 : q. \end{cases}
\]

When \(AA^* = BB^*, \tilde{A}^* \tilde{A} = \tilde{B} \tilde{B}^*\), so that, by the equality of dimensions step, there exists \(O \in \mathcal{O}(q)\) such that \(\tilde{A} = \tilde{B}O\) and, by restriction of this matrix equality to the \(d\) first lines, \(A = BO\). The equality of dimensions step also ensures that \(\tilde{A} = \sqrt{AA^*} \tilde{Q}\) with \(\tilde{Q} \in \mathbb{R}^{q \times q}\) such that \(\tilde{Q} \tilde{Q}^*\) is the orthogonal projection on \(\text{Im} \sqrt{AA^*}\).

Since \(\sqrt{AA^*} = (1_{i,j \leq d})\sqrt{AA^*}_{ij}\) \(1_{i,j \leq q}\), we deduce that \(A = \sqrt{AA^*} \tilde{Q}\) where \(\tilde{Q} = (\tilde{Q}_{ij})_{1 \leq i \leq d, 1 \leq j \leq q}\). By the above characterization, \(\tilde{Q}\) has dim \(\text{Im} \tilde{A} = \text{dim} \tilde{A}\) columns which form an orthonormal basis of \(\text{Im} \tilde{A}\) and, by definition of \(A\) have vanishing coordinates with indices larger than \(d\) so that the corresponding columns of \(\tilde{Q}\) form an orthonormal basis of \(\text{Im} \sqrt{\tilde{A} \tilde{A}^*}\). The other columns of \(\tilde{Q}\) and therefore of \(\tilde{Q}\) vanish. Hence by the characterization, \(\tilde{Q}\) is such that \(\tilde{Q} \tilde{Q}^*\) is the orthogonal projection on \(\text{Im} \sqrt{\tilde{A} \tilde{A}^*}\).

Case \(d > q\). The equality \(AA^* = BB^*\) implies that \(\text{Im} B = \text{Im} \sqrt{AA^*}\). Let \(Q \in \mathbb{R}^{d \times q}\) be such that \(QQ^*\) is the orthogonal projection on \(\text{Im} \sqrt{AA^*}\). Then \(\sqrt{AA^*} = QQ^* \sqrt{AA^*}\), \(A = QQ^* A, B = QQ^* B\) and \(Q^* AA^* Q = Q^* BB^* Q\) so that, by the equality of dimensions step, \(Q^* A = Q^* BO\) for some \(O \in \mathcal{O}(q)\). Left-multiplying by \(Q\) the last equality leads to \(A = BO\). Since \(Q^* \sqrt{AA^*} QQ^* \sqrt{AA^*} Q = Q^* AA^* Q\) and \(Q^* \sqrt{AA^*} Q\) is symmetric, \(Q^* \sqrt{AA^*} Q = Q^* \sqrt{AA^*} Q\). By the equal dimensions step this implies that \(Q^* A = Q^* \sqrt{AA^*} Q\) for some \(V \in \mathcal{O}(q)\) and, by left-multiplication by \(Q\), \(A = \sqrt{AA^*} Q\) where \(QQ^* Q = QQ^*\) is the orthogonal projection on \(\text{Im} \sqrt{AA^*}\).

\section*{Lemma A.3 (Kronecker product)}

Let \(S \in \mathcal{S}^+(d_1)\) and \(T \in \mathcal{S}^+(d_2)\), \(d_1, d_2 \in \mathbb{N}\). Then (with obvious notations) the Kronecker product \(S \otimes T := [S_{ij} T]_{1 \leq i,j \leq d_1, d_1 \times d_2}\). When \(S\) and \(T\) are both positive definite, then so is \(S \otimes T\).

\textbf{Proof.} First note that \(S \otimes T\) is clearly symmetric by construction. Let \(u \in \mathbb{R}^{d_1 \times d_2}\) be defined as column vector and for \(i = 1 : d_1\), let \(U_i = (u_{(i-1)d_1+k})_{k=1:d_2} \in \mathbb{R}^{d_2}\). Then

\[
u^* S \otimes T u = \sum_{i,j=1}^{d_1} S_{ij} U_i^* T U_j
\]

The matrix \(T\) can be diagonalized as \(T = P \Delta P^*\) where \(P \in \mathcal{O}(d_2)\) and \(\Delta = \text{Diag}(\delta_1 : \delta_{d_2})\), \(\delta_k \geq 0, k = 1, \ldots, d_2\). Consequently,

\[
u^* (S \otimes T) u = \sum_{i,j=1}^{d_1} S_{ij} (P^* U_i)^* \Delta (P^* U_j) = \sum_{k=1}^{d_2} \delta_k \sum_{i,j=1}^{d_1} S_{ij} (P^* U_i)^k (P^* U_j) k \geq 0
\]

since the \(\delta_k\) are non-negative and \(S\) is positive semi-definite. □

\section*{B First elements of proof of Theorem 1.1}

The aim of this appendix is to establish the existence and uniqueness of pathwise continuous solutions to the Volterra equation \(\ref{e:volterra}\) when \(X_0 \in L^0(\mathbb{P})\). The \(L^p\) pathwise regularity when \(X_0 \in L^p(\mathbb{P})\) for some \(p > 0\), will be established in Appendix \ref{app:volterra}

Taking advantage of the structure of our stochastic Volterra equation with distinct kernels, the assumptions ensuring existence and uniqueness of pathwise continuous (even \(a\)-Hölder continuous for some small enough \(a > 0\)) solutions in \cite[Theorem 3.3]{Kramkov2005} when \(X_0 \in \bigcap_{p \geq 0} L^p(\mathbb{P})\) boil down to check that \(K_1\) and \(K_2\) satisfy assumptions \(H1'\), \(H2\)-\(H3\)-\(H4\) with the notations of \cite{Kramkov2005}. Using again its notations, this corresponds to \(g(t) = X_0\) and \(\kappa_1 = \kappa_2 = \kappa := K_1 + K_2\) and

\[
\forall 0 \leq s < t < t', \quad \lambda(t', t, s) = \sum_{i=1,2} |K_i(t', s) - K_i(t, s)|^\gamma.
\]
Thus, condition \((K^\text{int}_\beta)\) is a criterion for \(\kappa\) to lie in the class \(\mathcal{H}_{> 1}\) as requested in assumption \(H2\) of \[30\] Theorem 3.3] (this is mentioned in \[30\]). Condition \((K^\text{cont}_\beta)\) ensures that \(H4\) is satisfied. Hence, when \(X_0 \in \cap_{r>0} L^r(\mathbb{P})\), all the assumptions of \[30\] Theorem 3.3] are satisfied (the additional Hölder condition on the random adapted function \(g\) is here empty since here \(g(t) = X_0\)). So \[30\] Theorem 3.3] directly applies.

Consequently, we may start from the fact that, under the assumptions of our theorem on the kernels and the coefficients of Equation (1.2) and if \(X_0 \in L^\infty(\mathbb{P}) \cap \cap_{r>0} L^r(\mathbb{P})\), this equation has a unique pathwise continuous solution.

▷ **Existence.** We now suppose that \(X_0 \in L^0(\Omega,F_0,\mathbb{P})\). For every \(k \in \mathbb{N}^*\), let \(A_k = \{|X_0| < k\}\) and \(X^{(k)}\) denote the \((\mathcal{F}_t)\)-adapted continuous unique solution to (1.2) starting from \(X_0^{(k)} = X_0 1_{A_k} \in L^\infty(\Omega,F_0,\mathbb{P})\). We set \(X = \sum_{k \in \mathbb{N}^*} X^{(k)} 1_{A_k \setminus A_{k-1}}\) where \(A_0 = \emptyset\). Let \(t \in [0,T]\). Since

\[
\int_0^t K_2(t,s)\|\sigma(s,X_s)\|^2 ds = \sum_{k \in \mathbb{N}^*} 1_{A_k \setminus A_{k-1}} \int_0^t K_2(t,s)\|\sigma(s,X_s^{(k)})\|^2 ds < +\infty, \mathbb{P}\text{-a.s.},
\]

the stochastic integral \(\int_0^t K_2(t,s)\sigma(s,X_s) dW_s\) makes sense. Moreover, since stochastic integrals with respect to an \((\mathcal{F}_t)\)-Brownian motion commute with \(\mathcal{F}_0\)-measurable random variables, for each \(k \in \mathbb{N}^*\),

\[
1_{A_k \setminus A_{k-1}} \int_0^t K_2(t,s)\sigma(s,X_s) dW_s = \int_0^t K_2(t,s) 1_{A_k \setminus A_{k-1}} \sigma(s,X_s) dW_s = \int_0^t K_2(t,s)\sigma(s,X_s^{(k)}) dW_s, \mathbb{P}\text{-a.s.},
\]

Hence

\[
\int_0^t K_2(t,s)\sigma(s,X_s) dW_s = \sum_{k \in \mathbb{N}^*} 1_{A_k \setminus A_{k-1}} \int_0^t K_2(t,s)\sigma(s,X_s^{(k)}) dW_s, \mathbb{P}\text{-a.s.}
\]

Since, one clearly has \(\int_0^t K_1(t,s)b(s,X_s) ds = \sum_{k \in \mathbb{N}^*} 1_{A_k \setminus A_{k-1}} \int_0^t K_1(t,s)b(s,X_s^{(k)}) ds, \mathbb{P}\text{-a.s.}\), we deduce that

\[
X_t = \sum_{k \in \mathbb{N}^*} 1_{A_k \setminus A_{k-1}} \left( X_0^{(k)} + \int_0^t K_1(t,s)b(s,X_s^{(k)}) ds + \int_0^t K_2(t,s)\sigma(s,X_s^{(k)}) dW_s \right) = X_0 + \int_0^t K_1(t,s)b(s,X_s) ds + \int_0^t K_2(t,s)\sigma(s,X_s) dW_s, \mathbb{P}\text{-a.s.}
\]

with the sum over \(k \in \mathbb{N}^*\) providing a continuous modification of the right-hand side.

▷ **Uniqueness.** Let \(X\) and \(\tilde{X}\) denote two \((\mathcal{F}_t)\)-adapted continuous solutions to (1.2) starting from the same initial condition \(X_0 \in L^0(\Omega,F_0,\mathbb{P})\) and let \(X^0\) denote the solution starting from 0. For \(k \in \mathbb{N}^*\), \(1_{A_k} X + 1_{A_k^c} X^0\) and \(1_{A_k} \tilde{X} + 1_{A_k^c} X^0\) are both solutions to (1.2) starting from \(X_0 1_{A_k} \in L^\infty(\Omega,F_0,\mathbb{P})\). By uniqueness, \(\mathbb{P}(\{X = \tilde{X}\} \cap A_k) = \mathbb{P}(A_k)\). Letting \(k \to \infty\), we conclude that \(\mathbb{P}(X = \tilde{X}) = 1\). Hence uniqueness holds starting from any random variable lying in \(L^0(\Omega,F_0,\mathbb{P})\).

**C** Properties of the \(K\)-integrated Euler scheme and \(L^p\)-convergence at fixed times

▷ **Preliminaries.** As a first preliminary, we will establish the following lemma which provides a control in \(L^p(\mathbb{P})\) of the increments of general Lebesgue or stochastic integrals involving the kernels \(K_i\). It will be used several times throughout the next appendices.
Lemma C.1 Let \((H_t)_{t \in [0,T]}\) (resp. \((\tilde{H}_t)_{t \in [0,T]}\)) be an \((\mathcal{F}_t)_{t \in [0,T]}\)-progressively measurable process having values in \(\mathbb{R}^d\) (resp. \(\mathbb{M}_{d,q}(\mathbb{R})\)) such that sup \(\|H_t\|_r + \sup_{t \in [0,T]} \|\tilde{H}_t\|_r < +\infty\) for some \(r \geq 2\). Assume that the kernels \(K_i\) satisfy \((K^\text{int}_\beta)\) for some \(\beta > 1\) and \((K^\text{cont}_\theta)\) for some \(\theta \in (0,1]\). Let \(0 \leq T_0 < T_1 \leq T\). Then both processes

\[
[T_0, T] \ni t \mapsto \int_{T_0}^{t \wedge T_1} K_1(t, s) H_s ds \quad \text{and} \quad [T_0, T] \ni t \mapsto \int_{T_0}^{t \wedge T_1} K_2(t, s) \tilde{H}_s dW_s
\]

are \(\theta \wedge \frac{1}{2\beta}\)-Hölder from \([T_0, T]\) to \(L^r(\mathbb{P})\). To be more precise, there exists a real constant \(C = C_{r,T,K_1,K_2} > 0\) such that

\[
\left\| \int_{T_0}^{t \wedge T_1} K_1(t, u) H_u du - \int_{T_0}^{s \wedge T_1} K_1(s, u) H_u du \right\|_r + \left\| \int_{T_0}^{t \wedge T_1} K_2(t, u) \tilde{H}_u dW_u - \int_{T_0}^{s \wedge T_1} K_2(s, u) \tilde{H}_u dW_u \right\|_r \\
\leq C \sup_{t \in [0,T]} \|H_t\|_r \wedge \|\tilde{H}_t\|_r |t - s|^\theta \wedge \frac{1}{2\beta}.
\]

If furthermore \((\tilde{K}^\text{cont}_\theta)\) holds true, one can replace \(\frac{1}{2\beta}\) by \(\tilde{\theta}\) mutatis mutandis in the above claims.

Proof. Let \(s, t \in [T_0, T]\) with \(s < t\). By using successively the Burkholder-Davis-Gundy, generalized Minkowski and Hölder (with exponents \((\beta, \frac{\beta}{\beta - 1})\)) inequalities, one gets since \(\frac{\beta}{\beta - 1} \geq 1\),

\[
\left\| \int_{T_0}^{t \wedge T_1} K_2(t, u) \tilde{H}_u dW_u - \int_{T_0}^{s \wedge T_1} K_2(s, u) \tilde{H}_u dW_u \right\|_r^2 \\
\leq \left( C_{r,BDG} \right)^2 \left( \left\| \int_{s \wedge T_1}^{t \wedge T_1} K_2(t, u) \tilde{H}_u^2 du \right\|_r \wedge \left\| \int_{T_0}^{s \wedge T_1} \left( K_2(t, u) - K_2(s, u) \right)^2 \tilde{H}_u^2 du \right\|_r \right) \\
\leq \left( C_{r,BDG} \right)^2 \left( \int_{s}^{t} K_2(t, u)^2 \|\tilde{H}_u\|_r^2 du + \int_{T_0}^{s} \left( K_2(t, u) - K_2(s, u) \right)^2 \|\tilde{H}_u\|_r^2 du \right) \\
\leq \left( C_{r,BDG} \right)^2 \sup_{u \in [T_0, T]} \|\tilde{H}_u\|_r^2 \left( \sup_{t \in [0,T]} \left( \int_{T_0}^{t} K_2(t, u)^2 du \right)^{\frac{\beta}{2}} (t-s)^{\frac{\beta}{2\beta - 1}} + \eta(t-s)^2 \right).
\]

Hence there exists a real constant \(\kappa = \kappa_{\eta, \beta, T}\) only depending on \(\eta, \beta\) and \(T\) such that

\[
\left\| \int_{T_0}^{t \wedge T_1} K_2(t, u) \tilde{H}_u dW_u - \int_{T_0}^{s \wedge T_1} K_2(s, u) \tilde{H}_u dW_u \right\|_r \leq \kappa C_{r,BDG} \sup_{u \in [T_0, T]} \|\tilde{H}_u\|_r (t-s)^{\theta \wedge \frac{1}{2\beta}}
\]

owing to (1.13) and (1.3). The variant under \((\tilde{K}^\text{cont}_\theta)\) is straightforward since \(\int_{s}^{t} K_2(t, u)^2 du \leq \tilde{\eta}(t-s)^2\) for \(0 \leq s \leq t\). One proceeds likewise for \(t \mapsto \int_{T_0}^{t \wedge T_1} K_1(t, u) H_u du\) (only using the generalized Minkowski inequality).

Throughout the remaining of this section, we assume that the assumptions of Theorem 1.2 are in force and that \(X_0 \in L^p(\mathbb{P})\) for the value of \(p\) under consideration.

First, we define to alleviate notations \(\varphi_a(t) = \left( \int_{0}^{t} K_1(t, s)^a ds \right)^{1/a}\) and \(\psi_a(t) = \left( \int_{0}^{t} K_2(t, s)^a ds \right)^{1/a}\), \(\varphi^*_a = \sup_{t \in [0,T]} \varphi_a(t)\) and \(\psi^*_a\) likewise for \(a > 0\). Note that \(\varphi^*_a + \psi^*_a < +\infty\) owing to Assumption \((K^\text{int}_\beta)\) and Hölder’s inequality since \(\beta > 1\).

Property 1. \(L^p\)-integrability and pathwise continuity of \(X_t\), \(p > 0\). First we prove that \(X^{1}_{t} \in L^p(\mathbb{P})\) by induction on \(k\). If \(X^{1}_{t} \in L^p(\mathbb{P})\) for \(\ell = 0, \ldots, k-1\), it follows easily from equation (1.12) that \(X^{1}_{t} \in L^p(\mathbb{P})\).
since both \( b \) and \( \sigma \) have linear growth on their space variable uniformly w.r.t. their time variable as a consequence of \([1,15]\) and \( \tilde{X}_{t_{n-1}} \) and the Gaussian Wiener integral \( \int_{t_{n-1}}^{t_n} K_2(t_k, s) dW_s \) are independent. The induction relies on the Minkowski inequality when \( p \geq 1 \) and on its subadditive counterpart for \( \| \cdot \|_p \) when \( p \in (0,1) \).

For the continuous time \( K \)-integrated Euler scheme, one derives likewise from \([1,12]\) and Lemma \( [C.1] \) that, as \( \tilde{X}_{t_k} \in L^p(\mathbb{P}) \) for every \( \ell = 0, \ldots, k \), one has \( \sup_{t \in [t_k, t_{k+1}]} \| \tilde{X}_t \|_p < +\infty \). As a consequence, \( \sup_{t \in [0, T]} \| \tilde{X}_t \|_p < +\infty \).

As for the pathwise continuity, first note that the functions \( \left( \int_{t_k}^{t_{k+1}} K_1(t, s) ds \right)_{t \in [t_k, T]} \), \( \ell = 0, \ldots, n - 1 \), are continuous owing to \((K^\text{cont})\). So are \( \left( \int_{t_k}^{t_{k+1}} K_1(t, s) ds \right)_{t \in [t_k, T]} \), \( \ell = 0, \ldots, n - 1 \), owing to \((K^\text{int})\) and \((K^\text{cont})\).

Applying the above Lemma \([C.1]\) with \( \tilde{H} = 1 \), \( T_0 = t_k^0 \), \( T_1 = t_{k+1}^0 \) with \( k \in \{0, \ldots, n - 1\} \) and \( r > \frac{1}{\theta_1 \gamma} \), yields that the processes \( \left( \int_{t_k}^{t_{k+1}} K_2(t, s) dW_s \right)_{t \in [t_k, T]} \) have a continuous modification (null at \( t_k^0 \)) owing to Kolmogorov’s criterion (see \([25]\) Theorem 2.1, p.26, 3rd edition]). Then one concludes that the \( K \)-integrated Euler scheme has a continuous modification.

**Property 2. Moment control of \( \tilde{X}_t \), \( p \geq 2 \).** Let \( \rho \in (0,1) \) and \( A > 0 \) be two real numbers to be specified later on. Under Assumption \((L\mathcal{H}_\gamma)\), \( b \) and \( \sigma \) have linear growth on \( x \) uniformly in \( t \in [0, T] \) i.e. there exists a real constant \( C(0) = C_{b,\sigma,T} > 0 \) such that, for every \( t \in [0, T] \) and every \( x \in \mathbb{R} \),

\[
|b(t,x)| + |\sigma(t,x)| \leq C(0)(1 + |x|^2)^{1/2} \leq C(0)(1 + |x|).
\]  

(C.62)

Set \( \bar{f}_p(t) = \sup_{0 \leq s \leq t} \| \tilde{X}_s \|_p \). We already know that \( \bar{f}_p(T) < \infty \) and are now going to derive an estimation not depending on the number \( n \) of time-steps. It follows from the above equation and the generalized Minkowski inequality that

\[
\left\| \int_0^t K_1(t,s)b(s,\tilde{X}_s)ds \right\|_p \leq \left( \int_0^t K_1(t,s)\|b(s,\tilde{X}_s)\|_p ds \right) \leq C(0) \left( \int_0^t K_1(t,s)(1 + \|\tilde{X}_s\|_p) ds \right) \leq C(0) \left( \phi_1(t) + \int_0^t K_1(t,s)\|\tilde{X}_s\|_p ds \right).
\]

As \( s \leq t \), \( \| \tilde{X}_s \|_p \leq \bar{f}_p(s) \) and the above inequality implies

\[
\bar{f}_p(t) \leq \| X_0 \|_p + C(0) \left( \phi_1(t) + \int_0^t K_1(t,s)\bar{f}_p(s) ds \right) + \sup_{s \leq t} G_p(s),
\]

where \( G_p(t) = \left\| \int_0^t K_2(t,s)\sigma(s,\tilde{X}_s) dW_s \right\|_p \). Set \( \rho = \frac{1}{2} \left( 1 + \frac{2}{\beta} \right) = \frac{\beta + 1}{2\beta} \in (0,1) \) since \( \beta > 1 \). As \( \bar{f}_p \) is non-decreasing, one derives

\[
\bar{f}_p(t) \leq \bar{f}_p(0) + C(0) \left( \phi_1(t) + \bar{f}_p(t) \int_0^t K_1(t,s)\bar{f}_p(s)^{1-\rho} ds \right) + \sup_{s \leq t} G_p(s)
\]

\[
\leq \bar{f}_p(0) + C(0) \phi_1(t) + C(0) \bar{f}_p(t) \frac{2A}{\beta+1} \left( \int_0^t \bar{f}_p(s) ds \right)^{1-\rho} + \sup_{s \leq t} G_p(s)
\]

\[
= \bar{f}_p(0) + C(0) \phi_1(t) + C(0) \frac{2A}{\beta+1} \left( t \frac{\bar{f}_p(t)}{A} \right)^{1-\rho} + \sup_{s \leq t} G_p(s),
\]

(C.63)
where we used Hölder’s inequality in the second line with conjugate exponents \( (2\beta \overline{\beta}, 2\beta - 1) \). Applying now Young’s inequality with the same conjugate exponents yields

\[
\tilde{f}_p(t) \leq \tilde{f}_p(0) + C(0) \varphi_1(t) + C(0) \varphi_{\frac{2\beta}{\beta+1}}(t) \left( \frac{\rho}{A^{1/\rho}} \tilde{f}_p(t) + (1 - \rho) A^{1/(1 - \rho)} \int_0^t \tilde{f}_p(s) ds \right) + \sup_{s \leq t} G_p(s). \tag{C.64}
\]

We proceed likewise with the Volterra pseudo-diffusion term, up to the use of the \( L^p \)-Burkholder-Davis-Gundy (BDG) inequality. First we get for every \( t \in [0, T] \),

\[
G_p(t) \leq C_p^{BDG} \left\| \int_0^t K_2(t, s)^2 \sigma(s, \bar{X}_s)^2 ds \right\|_{p/2}^{1/2}.
\]

As \( p \geq 2 \), it follows from the linear growth property for \( \sigma \) and both regular and generalized Minkowski inequalities that

\[
G_p^2(t) = (\tilde{C}^{(0)})^2 \left( \int_0^t K_2(t, s)^2 ds + \left\| \int_0^t K_2(t, s)^2 \bar{X}_s^2 ds \right\|_{p/2} \right)
\]

\[
\leq (\tilde{C}^{(0)})^2 \left( \int_0^t K_2(t, s)^2 ds + \int_0^t K_2(t, s)^2 \left\| X_s \right\|_p^2 ds \right)
\]

where \( \tilde{C}^{(0)} = \tilde{C}_{\beta,0,p}^{BDG} \) : \( C_p^{BDG} C^{(0)} \). Using that \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b}, a, b \geq 0 \), yields

\[
G_p(t) \leq \tilde{C}^{(0)} \left( \int_0^t K_2(t, s)^2 ds \right)^{1/2} + \left( \int_0^t K_2(t, s)^2 \left\| X_s \right\|^2_2 ds \right)^{1/2}
\]

\[
\leq \tilde{C}^{(0)} \psi_2(t) + \tilde{C}^{(0)} \left( \int_0^t K_2(t, s)^2 \tilde{f}_p(s) ds \right)^{1/2}.
\]

Then, using again that \( \tilde{f}_p \) is non-decreasing, it follows, still with \( \rho = \frac{1}{2} (1 + \frac{1}{\beta}) \) that

\[
G_p(t) \leq \tilde{C}^{(0)} \psi_2(t) + \tilde{C}^{(0)} \tilde{f}_p(t) \left( \int_0^t K_2(t, s)^2 \tilde{f}_p(s)^{2(1 - \rho)} ds \right)^{1/2}
\]

\[
\leq \tilde{C}^{(0)} \psi_2(t) + \tilde{C}^{(0)} \psi_{2\beta}(t) \tilde{f}_p(t) \left( \int_0^t \tilde{f}_p(s)^{2(1 - \rho \beta)} ds \right)^{\frac{\beta+1}{2}},
\]

where we applied Hölder’s inequality with conjugate exponents \( (\beta, \frac{\beta}{\beta + 1}) \). Noting that \( 2(1 - \rho \beta) = 1 \), applying Young’s inequality with conjugate exponents \((1/\rho, 1/(1 - \rho))\) and introducing \( A > 0 \) like for the drift term, we obtain

\[
G_p(t) \leq \tilde{C}^{(0)} \psi_2(t) + \tilde{C}^{(0)} \psi_{2\beta}(t) \left( \frac{\tilde{f}_p(t)}{A^{1/\rho}} + (1 - \rho) A^{1/(1 - \rho)} \left( \int_0^t \tilde{f}_p(s) ds \right) \right). \tag{C.65}
\]

Taking advantage of the fact that \( \tilde{f}_p \) is non-decreasing, we derive that

\[
\sup_{s \leq t} G_p(s) \leq \tilde{C}^{(0)} \sup_{0 \leq s \leq t} \psi_2(s) + \tilde{C}^{(0)} \sup_{0 \leq s \leq t} \psi_{2\beta}(s) \left( \frac{\tilde{f}_p(t)}{A^{1/\rho}} + (1 - \rho) A^{1/(1 - \rho)} \left( \int_0^t \tilde{f}_p(s) ds \right) \right).
\]

Plugging this inequality in \( \text{(C.64)} \) yields

\[
\tilde{f}_p(t) \leq \tilde{f}_p(0) + C^{(1)} + C^{(2)} \left( \frac{\tilde{f}_p(t)}{A^{1/\rho}} + (1 - \rho) A^{1/(1 - \rho)} \int_0^t \tilde{f}_p(s) ds \right).
\]

36
where
\[ C^{(1)} = C^{(1)}_{K_1,K_2,\beta,b,\sigma,T} = C^{(0)} \varphi_1^* + \bar{C}^{(0)} \psi_2^* \]
and
\[ C^{(2)} = C^{(2)}_{K_1,K_2,\beta,b,\sigma,T} = C^{(0)} \varphi_2^{*\perp} + \bar{C}^{(0)} \psi_2^*. \]
Both positive real constants \( C^{(1)} \) and \( C^{(2)} \) are finite owing to \( K^{\text{int}}_\beta \).

We choose now \( A \) large enough, namely \( A = (2 \rho C^{(2)})^p \), so that \( 1 - \rho C^{(2)} A^{-1/p} = \frac{1}{2} < 1 \). Consequently, for every \( t \in [0,T] \),
\[ \bar{f}_p(t) \leq 2 \bar{f}_p(0) + 2C^{(1)}(1 - \rho)A^{1/(1-p)} \int_0^t \bar{f}_p(s)ds. \]
Since \( \bar{f}_p(0) = \|X_0\|_p \), one concludes by Grönwall’s lemma that
\[ \bar{f}_p(t) \leq 2(\|X_0\|_p + C^{(1)})e^{C^{(3)}t} \]
with \( C^{(3)} = C^{(3)}_{K_1,K_2,\beta,b,\sigma,T} = 2C^{(2)}(1 - \rho)A^{1/(1-p)} \). As a conclusion to this step, we get
\[ \sup_{t \in [0,T]} \|\bar{X}_t\|_p = \bar{f}_p(T) \leq C^{(4)}(1 + \|X_0\|_p) \]
with \( C^{(4)} = 2 \text{max}(1, C^{(1)}) e^{C^{(3)}T} \geq 2 \). Since, according to \cite[Theorem 3.1]{31}, \( \sup_{t \in [0,T]} \|X_t\|_p < \infty \), by the same reasoning applied to the Volterra process, we also derive
\[ \sup_{t \in [0,T]} \|X_t\|_p \leq C^{(4)}(1 + \|X_0\|_p). \]

**PROPERTY 3. Control of the increments** \( \|\bar{X}_t - \bar{X}_s\|_p \) of the \( K \)-integrated Euler scheme \( I \) (for \( p \geq 2 \)). Our aim in this step is to prove that there exists a positive real constant \( C_{p,T} \) not depending on \( n \) such that
\[ \forall s, t \in [0,T], \quad \|\bar{X}_t - \bar{X}_s\|_p \leq C_{p,T}(1 + \|X_0\|_p)(t - s)^{\theta \wedge \tilde{\theta}}. \]
We recall the definition \((1.13)\)
\[ \bar{X}_t = X_0 + \int_0^t K_1(t,s)b(s,\bar{X}_s)ds + \int_0^t K_2(t,s)\sigma(s,\bar{X}_s)dW_s, \quad t \in [0,T]. \]
which strongly suggests to call upon Lemma \((C.1)\) (with \( r = p \)) with \( H_s = b(s,\bar{X}_s) \) and \( \tilde{H}_s = \sigma(s,\bar{X}_s) \). First note that
\[ \|b(s,\bar{X}_s)\|_p \leq C^{(0)}(1 + \|\bar{X}_s\|_p) \leq C^{(0)}(1 + C^{(4)}(1 + \|\bar{X}_0\|_p)) \leq 2C^{(0)}C^{(4)}(1 + \|\bar{X}_0\|_p) \]
owing to \((C.02)\) and Property 2 and the fact that \( C^{(4)} \geq 1 \). One shows likewise that, for every \( u \in [0,T], \)
\[ \|\sigma(s,\bar{X}_s)\|_p \leq 2C^{(0)}C^{(4)}(1 + \|\bar{X}_0\|_p). \]
Consequently, it follows from Lemma \((C.1)\) under \( \tilde{K}^{\text{cont}}_{\bar{\theta}} \) that
\[ \|\bar{X}_t - \bar{X}_s\|_p \leq C_{r,T,K_1,K_2} \left( \sup_{t \in [0,T]} \|b(u,\bar{X}_u)\|_p + \sup_{t \in [0,T]} \|\sigma(u,\bar{X}_u)\|_p \right)(t - s)^{\theta \wedge \tilde{\theta}} \]
\[ \leq C^{(5)}(1 + \|\bar{X}_0\|_p)(t - s)^{\theta \wedge \tilde{\theta}}, \]
37
where $C^{(5)} = 4C_{p,T,K_1,K_2}C^{(0)}C^{(4)}$ with $C_{r,T,K_1,K_2}$ coming from Lemma C.1. Otherwise, if only $(K_{\beta}^{\text{int}})$ is in force, replace $\theta$ by $\frac{\theta}{\beta - 1}$.

**PROPERTY 4.** Rate of convergence of $\|X_t - \tilde{X}_t\|_p$ at fixed time $t$ for $p > p_{cu}$. Have in mind that $p_{cu} = \frac{1}{2} \vee \frac{2\beta}{\beta - 1}$ and that for such values of $p$ we know from a careful reading of the proof of [30, Theorem 3.3] that Equation (1.2) has a unique adapted pathwise continuous solution $(X_t)_{t \in [0,T]}$. Then, for every $t \in [0,T]$,

$$X_t - \tilde{X}_t = \int_0^t K_1(t,s)(b(s,X_s) - b(s,X_s))ds + \int_0^t K_2(t,s)(\sigma(s,X_s) - \sigma(s,X_s))dW_s \quad \mathbb{P}\text{-a.s.}$$

We denote by $\tilde{A}_1^p(t)$ and $\tilde{A}_2^p(t)$ the two terms of the sum in the right-hand side of the above equation. Set

$$g_p(t) = \sup_{s \in [0,t]} \|X_s - \tilde{X}_s\|_p, \quad t \in [0,T].$$

This non-decreasing function is finite owing to Theorem 1.1 and Property 1 since $X_0 \in L^p(\mathbb{P})$. We straightforwardly deduce that

$$g_p(t) \leq \sup_{s \leq t} (\|\tilde{A}_1^p(s)\|_p + \|\tilde{A}_2^p(s)\|_p). \quad (C.66)$$

Let $[b]_{H,L}$ be the mixed Hölder-Lipschitz coefficient such that, for every $s, t \in [0,T]$ and every $x, y \in \mathbb{R}$,

$$|b(s,x) - b(t,y)| \leq [b]_{H,L}(|t - s|^{\gamma}(1 + |x| + |y|) + |x - y|).$$

The coefficient $[\sigma]_{H,L}$ is defined likewise.

As for $\tilde{A}_1^p(t)$, the generalized Minkowski inequality implies (with in mind the notation $h = \frac{2}{p}$ for the time step)

$$\|\tilde{A}_1^p(t)\|_p \leq \int_0^t \|K_1(t,s)|b(s,X_s) - b(s,X_s)|\|ds$$

$$\leq [b]_{H,L} \int_0^t K_1(t,s)(|s - \tilde{s}|^{\gamma}(1 + \|X_s\|_p + \|\tilde{X}_s\|_p) + \|X_s - \tilde{X}_s\|_p)ds$$

$$\leq [b]_{H,L}(1 + 2 \sup_{t \in [0,T]} (\|X_s\|_p \vee \|\tilde{X}_s\|_p))\varphi_1^p h^{\gamma} + \int_0^t K_1(t,s)\|X_s - \tilde{X}_s\|_p ds$$

$$\leq 3C^{(4)}\varphi_1^p[b]_{H,L}(1 + \|X_0\|_p)h^{\gamma} + [b]_{H,L} \left( \int_0^t K_1(t,s)g_p(s)ds + \int_0^t K_1(t,s)\|X_s - \tilde{X}_s\|_p ds \right)$$

where we used that $\|X_s - \tilde{X}_s\|_p \leq g_p(s) + \|\tilde{X}_s - \tilde{X}_s\|_p$.

Following the strategy developed in (C.66), we get for every $A > 0$ and $\rho = \frac{\theta}{2}(1 + \frac{1}{\theta}) \in (0,1)$,

$$\|\tilde{A}_1^p(t)\|_p \leq [b]_{H,L}\varphi_1^p (A^{1/(1 - \rho)}(1 - \rho)A^{1/(1 - \rho)} + \int_0^t g_p(s)ds)$$

$$+ [b]_{H,L} \int_0^t K_1(t,s)\|X_s - \tilde{X}_s\|_p ds + 3C^{(4)}\varphi_1^p[b]_{H,L}(1 + \|X_0\|_p)h^{\gamma}. \quad (C.67)$$

As for $\tilde{A}_2^p(t)$, combining the BDG inequality and both regular and generalized Minkowski inequalities implies

$$\|\tilde{A}_2^p(t)\|_p^2 \leq (C_{pBDG}^B)^2 \left( \int_0^t K_2(t,s)^2 \|\sigma(s,X_s) - \sigma(s,X_s)\|^2 ds \right)_{p/2}$$

$$\leq (C_{pBDG}^B)^2 \|\sigma\|_{H,L} \int_0^t K_2(t,s)^2 (|s - \tilde{s}|^{\gamma}(1 + \|X_s\|_p + \|\tilde{X}_s\|_p) + \|X_s - \tilde{X}_s\|_p)^2 ds$$

$$\leq 2(C_{pBDG}^B)^2 \|\sigma\|_{H,L} \left( (3C^{(4)})^2 \psi_2(\frac{1}{2})h^{2\gamma} + \int_0^t K_2(t,s)^2 \|X_s - \tilde{X}_s\|_p^2 ds \right).$$
Following this time the strategy developed to establish (C.65) and setting again \( \rho = \frac{1}{2}(1 + \frac{1}{2}) \in (0, 1) \), we derive
\[
\|\bar{A}_1^n(t)\|_p \leq \sqrt{2}C_p^{BDG}[^{\text{cont}}]_{H,L} \left[ \psi_{2\rho}(t) \left( \rho \frac{g_p(t)}{A^{1/\rho}} + (1 - \rho)A^{1/(1-\rho)} \int_0^t g_p(s)ds \right) \right. \\
\left. + \left( \int_0^t K_2(t,s)^2(\bar{X}_s - \bar{X}_\Delta)^2_\rho ds \right)^{1/2} + 3C^{(4)}\psi_2(t)(1 + \|X_0\|_p)h^\gamma \right]. \quad (C.68)
\]

Combining (C.67) and (C.68) yields
\[
\|\bar{A}_1^n(t)\|_p + \|\bar{A}_2^n(t)\|_p \leq C^{(6)} \left( \rho \frac{g_p(t)}{A^{1/\rho}} + (1 - \rho)A^{1/(1-\rho)} \int_0^t g_p(s)ds \right) + R(t) \quad (C.69)
\]
with \( C^{(6)} = C_{K_1,K_2,\beta,b,\sigma,p} = [b]_{H,L} \varphi_2^* + \sqrt{2}C^{BDG}_p[^{\text{cont}}]_{H,L} \psi_2^* \) and
\[
R(t) = C^{(7)} \left( 1 + \|X_0\|_p)h^\gamma + \int_0^t K_1(t,u)\|\bar{X}_u - \bar{X}_\Delta\|_p du + \left( \int_0^t K_2(t,u)^2(\bar{X}_u - \bar{X}_\Delta)^2_\rho du \right)^{1/2} \right) \quad (C.70)
\]
with
\[
C^{(7)} = C_{K_1,K_2,\beta,b,\sigma,p}^{(7)} = \max \left( 3C^{(4)}[b]_{H,L} \varphi_2^* + 3\sqrt{2}C^{BDG}_p[^{\text{cont}}]_{H,L} \psi_2^*, [b]_{H,L}, \sqrt{2}C^{BDG}_p[^{\text{cont}}]_{H,L} \right).
\]

Then setting \( A = (2\rho C^{(6)})^\rho \) yields for every \( t \in [0,T] \) (since \( g_p(0) = 0 \))
\[
\|\bar{A}_1^n(t)\|_p + \|\bar{A}_2^n(t)\|_p \leq \frac{g_p(t)}{2} + C^{(6)}(1 - \rho)A^{1/(1-\rho)} \int_0^t g_p(s)ds + R(t). \quad (C.71)
\]

Consequently, it follows from (C.66) and Grönwall’s lemma that
\[
g_p(t) \leq 2C^{(6)}(1 - \rho)A^{1/(1-\rho)} \int_0^t g_p(s)ds + 2 \sup_{s \leq t} R(s) \leq 2eC^{(8)t} \sup_{s \leq t} R(s)
\]
with \( C^{(8)} = C^{(8)}_{K_1,K_2,\beta,b,\sigma,p} = 2C^{(6)}(1 - \rho)A^{1/(1-\rho)} \). In particular
\[
\sup_{t \in [0,T]} \|X_t - \bar{X}_t\|_p = g_p(T) \leq 2eC^{(8)T} \sup_{t \in [0,T]} R(t). \quad (C.72)
\]

Now, coming back to \( R(t) \), we have that
\[
\left( \int_0^t K_2(t,s)^2(\bar{X}_s - \bar{X}_\Delta)^2_\rho ds \right)^{1/2} \leq \psi_2^* \sup_{s \in [0,t]} \|\bar{X}_s - \bar{X}_\Delta\|_p
\]
whereas obviously,
\[
\int_0^t K_1(t,s)\|\bar{X}_s - \bar{X}_\Delta\|_p ds \leq \varphi_1^* \sup_{s \in [0,t]} \|\bar{X}_s - \bar{X}_\Delta\|_p.
\]
Hence, for every \( t \in [0,T] \),
\[
\sup_{s \leq t} R(s) \leq C^{(7)}(1 + \|X_0\|_p)h^\gamma + C^{(7)}(\varphi_1^* + \psi_2^*) \sup_{s \in [0,t]} \|\bar{X}_s - \bar{X}_\Delta\|_p.
\]

By combining these bounds with Property 3, one concludes that, under \( \widehat{\mathbb{K}}^{(\text{cont})} \)
\[
\forall t \in [0,T], \quad \|X_t - \bar{X}_t\|_p \leq C^{(9)}(1 + \|X_0\|_p) \left( h^\gamma + h^\rho + \hat{\theta} \right)
\]
with \( C^{(9)} = C^{(9)}_{K_1,K_2,\beta,b,\sigma,p,T} = 2eC^{(8)T}C^{(7)} \max (1, (\varphi_1^* + \psi_2^*)C^{(5)}) \).

Note that if only \( \widehat{\mathbb{K}}^{(\text{int})} \) is in force, then the final bound holds mutatis mutandis with \( \frac{2-1}{2^3} \) instead of \( \hat{\theta} \).
D Splitting lemma and representation of the solution of the Volterra equation and its Euler schemes

Theorem 1.4 is a revisited version of [26, Theorem 2.2]. Surprisingly, in its original formulation the error bounds only hold for starting values $X_0 \in L^p(\mathbb{P})$ when $p$ is large enough depending on some integrability characteristics of the kernels. This unusual restriction can be circumvented thanks to the lemma below applied to the functional $\Phi : C([0,T],\mathbb{R}^d)^2 \rightarrow \mathbb{R}$ defined by

$$\Phi(x,y) = \sup_{t\in[0,T]} |x(t) - y(t)|.$$ 

Lemma D.1 (Splitting lemma) Assume that the functions $b$ and $\sigma$ are Lipschitz in space uniformly in time, that $\sup_{t\in[0,T]} (\|b(t,0)\| + \|\sigma(t,0)\|) < +\infty$ and that the kernels $K_i$, $i = 1, 2$, satisfy $(K^\text{int}_i)$ and $(K^\text{cont}_i)$ for some $\beta > 1$ and $\theta \in (0,1]$ respectively and, if dealing with the $K$-discrete Euler scheme, $(\hat{K}^\text{cont}_i)$ and $(\hat{K}^\text{cont}_i)$ for some $\hat{\theta}, \hat{\theta} \in (0,1]$. Let $\Phi : C([0,T],\mathbb{R}^d)^2 \rightarrow \mathbb{R}$ be a Borel functional and let $n \in \mathbb{N}$. Assume there exists $\bar{p} > 0$ and a real constant $C > 0$ possibly depending on $n$, such that, for every $x_0 \in \mathbb{R}^d$,

$$\|\Phi(X^{x_0}, \bar{X}^{x_0})\|_p \leq C(1 + \|x_0\|),$$

where $X^{x_0}$ and $\bar{X}^{x_0}$ respectively denote the solutions of the Volterra equation and any of its two (genuine) Euler schemes starting from $x_0$. Then, for every $p \in (0, \bar{p})$ and every random vector $X_0$ independent of $W$, the solution $X = (X_t)_{t\in[0,T]}$ and the Euler scheme under consideration starting from $X_0$ satisfy

$$\|\Phi(X, \bar{X})\|_p \leq 2^{(1/p-1)+}C(1 + \|X_0\|_p).$$

Proof. According to Theorem D.2 below (whose proof is an adaptation to Volterra equations of the proof of Blagoveščenki-Freidlin’s theorem [27, Theorem 13.1, Section V.12-13, p.136]), there exists a bi-measurable functional $F : \mathbb{R}^d \times C_0([0,T],\mathbb{R}^d) \rightarrow C([0,T],\mathbb{R}^d)$ such that the processes $X^{x_0}$ reads

$$X^{x_0} = F(x_0, (W_t)_{t\in[0,T]})$$

and such that, for any starting random value $X_0 \in L^p(\mathbb{P})$, $p > 0$, $F(X_0, (W_t)_{t\in[0,T]})$ solves the Volterra equation (1.2) starting from $X_0$.

The same holds true for (any of) the two Euler schemes (both denoted $\bar{X}$ here): there exists a measurable functional $\bar{F}_n : \mathbb{R}^d \times C_0([0,T],\mathbb{R}^d) \rightarrow C([0,T],\mathbb{R}^d)$ such that

$$\bar{X} = \bar{F}_n(X_0, (W_t)_{t\in[0,T]}),$$

(see Proposition D.3) further on for the $K$-integrated scheme whereas, for the $K$-discrete Euler scheme, this can be proved by induction in an elementary way, the continuity of this scheme under $\hat{K}^\text{cont}_i$ and $\hat{K}^\text{cont}_i$ for some $\hat{\theta}, \hat{\theta} > 0$, being established in Appendix F.2 below).

This entails that the distribution $\mathbb{P}_{(X, \bar{X})}$ on $C([0,T],\mathbb{R}^d)^2$ of $(X, \bar{X}) = (F(X_0, W), \bar{F}_n(X_0, W))$ satisfies

$$\mathbb{P}_{(X, \bar{X})}(dx, d\bar{x}) = \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx_0)\mathbb{P}_{(X^{x_0}, \bar{X}^{x_0})}(dx, d\bar{x}).$$

Consequently, using the monotonicity of probabilistic $L^r(\mathbb{P})$-norms and pseudo-norms (in the third line) and
the elementary inequality \((a + b)^p \leq a^p + b^p\), for \(a, b \geq 0\) and \(\rho = \frac{p}{\bar{p}} \in [0, 1]\), we derive

\[
\|\Phi(X, \bar{X})\|^p_p = \mathbb{E} |\Phi(X, \bar{X})|^p = \int_{\mathbb{R}^d} \mathbb{P}_X(dx_0)\mathbb{E} |\Phi(X^{x_0}, \bar{X}^{x_0})|^p \\
\leq \int_{\mathbb{R}^d} \mathbb{P}_X(dx_0)\left(\mathbb{E} |\Phi(X^{x_0}, \bar{X}^{x_0})|^{p/\bar{p}}\right)^{p/\bar{p}} \\
\leq \int_{\mathbb{R}^d} \mathbb{P}_X(dx_0)\left(C^{\bar{p}}(1 + |x_0|)^{\bar{p}}\right)^{p/\bar{p}} \\
\leq C^p \int_{\mathbb{R}^d} \mathbb{P}_X(dx_0)(1 + |x_0|^p) = C^p(1 + \|X_0\|_p^p) \\
\leq C^p 2^{(1-p)^+}(1 + \|X_0\|_p^p)
\]

so that, finally,

\[
\|\Phi(X, \bar{X})\|_p \leq 2^{(1/p-1)^+}C(1 + \|X_0\|_p).
\]

Our next task is to establish representations for the Volterra process and its two Euler schemes in order to be able to apply the splitting lemma.

The next theorem deals with the flow \(x \mapsto (X^x_t)_{t \in [0,T]}\) of the Volterra equation. It proves the existence of a bi-measurable functional \(F : \mathbb{R}^d \times C_0([0,T], \mathbb{R}^d) \to C([0,T], \mathbb{R}^d)\) such that the solution \(X = (X_t)_{t \in [0,T]}\) of equation (1.2) reads \(X = F(X_0, W)\) for any (finite) starting value \(X_0\).

**Theorem D.2 (Blagoveščenki-Freidlin like theorem: representation of Volterra’s flow)** Assume that the functions \(b\) and \(\sigma\) are lipschitz in space uniformly in time, that \(\sup_{t \in [0,T]}(\|b(t,0)\| + \|\sigma(t,0)\|) < +\infty\) and that the kernels \(K_i\), \(i = 1, 2\), satisfy \((K^\text{int}_\beta)\) and \((K^\text{cont}_\beta)\) for some \(\beta > 1\) and \(\theta \in (0,1]\) respectively.

(a) Let \(X^x\) denotes the solution to the Volterra equation (1.2) starting from \(x \in \mathbb{R}^d\) and let \(\lambda \in (\frac{1}{\beta}, 1)\). There exists \(p^* = p^*_\beta,\theta,\lambda, d\) (made explicit in the proof) such that for every \(p > p^*\),

\[
\forall x, y \in \mathbb{R}^d, \sup_{t \in [0,T]} \left\|X^x_t - X^y_t\right\|_p \leq C|x - y|^{\lambda},
\]

for some positive real constant \(C = C_{p,\beta,\theta,\lambda, d}\).

(b) There exists a functional \(F : \mathbb{R}^d \times C_0([0,T], \mathbb{R}^d) \ni (x, w) \mapsto F(x, w) \in C([0,T], \mathbb{R}^d)\) bi-measurable (where the spaces of continuous functions are equipped with the Borel sigma-field induced by the uniform convergence topology) and continuous in \(x\) such that, for any stochastic basis \((\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}\), any \(q\)-dimensional \((\mathcal{F}_t)\)-Brownian motion \(W\) and any \(\mathcal{F}_0\)-measurable \(\mathbb{R}^d\)-valued random vector \(X_0 \in L^q(\mathbb{P})\), the solution to the Volterra equation (1.2) is \(X = F(X_0, W)\).

**Proof.** The proofs of claims (a) and (b) are intertwined. 

**Step 1 (Continuity of the flow).** It follows from [30] Theorem 3.3 that, for every \(p > p_{cu} = \frac{1}{\lambda} \vee \frac{2\beta}{\theta - 1}\), the Volterra equation (1.2) has a unique strong solution starting from any random vector \(X_0 \in L^p(\mathbb{P})\) which can be proved to be pathwise continuous (and even \(\alpha\)-Hölder pathwise continuous for some small enough \(\alpha > 0\)) and satisfying furthermore \(\mathbb{E} \sup_{t \in [0,T]} \|X^x_t\|^p < +\infty\).

By mimicking the proof of Property 4 in Appendix [C] to control \(\|X_t - \bar{X}_t\|_p\) (see also the proof of Theorem 3.3 in [30]), one derives that there exists a positive real constant \(C_{p,T} = 2e^{C^{(\theta/2)}}\) such that,

\[
\forall p > p_{cu}, \sup_{t \in [0,T]} \|X^x_t - X^y_t\|_p \leq C_{p,T}|x - y|.
\]
We also know from Property 3 in Appendix [3] (see also [20]) that for every \( p \geq 2 \), there exists a positive real constant \( C'_{p,T} \) (not depending on \( n \)) such that the \( K \)-integrated Euler scheme with step \( \frac{T}{n} \) satisfies

\[
\forall s, t \in [0, T], \forall x \in \mathbb{R}^d, \quad \| \bar{X}_t^x - \bar{X}_s^x \|_p \leq C'_{p,T}(1 + |x|)|t - s|^{\bar{\theta}}
\]

with \( \bar{\theta} = \theta \wedge \frac{\theta - 1}{2\theta} \). On the other hand, it follows from Property 4 in Appendix [3] that, for every \( t \in [0, T] \), \( \bar{X}_t \to X_t \) as \( n \to +\infty \) in \( L^p(\mathbb{P}) \) if \( p > p_{eu} \). Hence, for every \( p > p_{eu} \),

\[
\forall s, t \in [0, T], \forall x \in \mathbb{R}^d, \quad \| \bar{X}_t^x - \bar{X}_s^x \|_p \leq C'_{p,T}(1 + |x|)|t - s|^{\bar{\theta}}.
\]

Now, let \( \lambda \in (0, 1) \) be fixed. One has

\[
\| X_t^x - X_t^y - (X_s^x - X_s^y) \|_p \leq (\| X_t^x - X_t^y \|_p + \| X_s^x - X_s^y \|_p) (\| X_t^x - X_s^x \|_p + \| X_t^y - X_s^y \|_p)^{1-\lambda} \\
\leq 2C'_{p,T}(\lambda - 1) \| x - y \|^{\lambda} (1 + |x| + |y|)^{1-\lambda} + \lambda |t - s|^{\bar{\theta}(1-\lambda)}
\]

or, equivalently,

\[
\forall s, t \in [0, T], \forall x \in \mathbb{R}^d, \quad \mathbb{E}|X_t^x - X_t^y - (X_s^x - X_s^y)|^p \leq \tilde{C}_{p,T}(1 + |x| + |y|)^{p(1-\lambda)}|x - y|^{p\lambda} + s^{\bar{\theta}(1-\lambda)}.
\]

Assume from now on that \( p > p^* = p_{eu} \wedge \frac{1}{\theta(1-\lambda)} \vee \frac{d}{\lambda} \), where the second bound from below (by \( \frac{1}{\theta(1-\lambda)} \)) will permit to apply Kolmogorov’s criterion in time and the third to apply this criterion in space. It follows from the proof of Kolmogorov’s criterion [25] Theorem 2.1 where tracking the constants ensures that the final constant has linear growth in the constant which appears in the hypothesis, that, for any \( a \in (0, (1-\lambda)\bar{\theta} - \frac{d}{p}) \), there exists a positive real constant \( C_{a,p,T} \) such that

\[
\forall x, y \in \mathbb{R}^d, \quad \mathbb{E} \sup_{s, t \in [0, T]} \left( \frac{|X_t^x - X_t^y - (X_s^x - X_s^y)|}{|t - s|^{\lambda}} \right)^p \leq C_{a,p,T}(1 + |x| + |y|)^{p(1-\lambda)}|x - y|^{p\lambda}.
\]

Setting \( s = 0 \) yields

\[
\forall x, y \in \mathbb{R}^d, \quad \mathbb{E} \sup_{t \in [0, T]} |X_t^x - X_t^y|^p \leq 2^{p-1} \left( |x - y|^p + T^n C_{a,p,T}(1 + |x| + |y|)^{p(1-\lambda)}|x - y|^{p\lambda} \right) \\
\leq C_{a,p,T}(1 + |x| + |y|)^{p(1-\lambda)}|x - y|^{p\lambda}
\]

where we used the rough inequality \( |x - y|^p \leq (1 + |x| + |y|)^{p(1-\lambda)} \).

Let \( R \in \mathbb{N} \) be fixed. For every \( x \in B(0, R) \), we consider a pathwise continuous modification \( (X_t^x)_{t \in [0, T]} \) so that the mapping \( B(0, R) \ni x \mapsto (X_t^x)_{t \in [0, T]} \) can be viewed as having values in the Polish space \( C([0, T], \mathbb{R}^d) \), equipped with the distance \( \rho_T \) derived from the sup-norm on \([0, T] \). Then for every \( x, y \in B(0, R) \)

\[
\forall x, y \in B(0, R), \quad \mathbb{E} \sup_{t \in [0, T]} |X_t^x - X_t^y|^p \leq C_{a,p,T}(1 + 2R)^{p(1-\lambda)}|x - y|^{p\lambda}.
\]

As \( p \lambda > d \), a new application of Kolmogorov’s criterion [25] Theorem 2.1 ensures the existence of a \( \mathbb{P} \)-modification \( (\bar{X}^{(R)}(t), x)_{t \in B(0, R)} \) of the \( C([0, T], \mathbb{R}^d) \)-valued process \( (X_t^x)_{x \in B(0, R)} \) such that \( \mathbb{P} \)-a.s. \( x \mapsto \bar{X}^{(R)}(x) \) is \( \rho_T \)-continuous and more precisely \( a' \)-Hölder-continuous for \( a' \in (0, \lambda - \frac{d}{p}) \).

It is clear that the restriction of \( (\bar{X}^{(R)}(t), x)_{t \in B(0, R+1)} \) to \( B(0, R) \) is also a \( \mathbb{P} \)-modification with continuous “paths” in \( x \). Hence they are \( \mathbb{P} \)-indistinguishable. Consequently, there exists a \( \mathbb{P} \)-modification \( (\bar{X}^{(R)}(t))_{x \in \mathbb{R}^d} \) such that \( \mathbb{P}(d\omega) \text{-a.s. } \ x \mapsto (\bar{X}_t^x)_{t \in [0, T]} \) is continuous from \( \mathbb{R}^d \) to \( C([0, T], \mathbb{R}^d) \) and \( (\bar{X}_t^x)_{t \in [0, T]} \) is solution to (1.2) for \( X_0 = x \).
STEP 2 (The functional $F$). Now we adapt to Volterra equations the classical proof of Blagoveščenskii-Freidlin’s theorem (see e.g. [23, Theorem 10.4]) originally written for standard Brownian diffusion processes. We only take advantage of the strong existence and uniqueness of pathwise continuous strong solutions for every $\mu$-distributed starting random vector $X_0$ having a finite $L^p$-moment to exhibit a functional $F_\mu : \mathbb{R}^d \times \mathcal{C}_0([0, T], \mathbb{R}^q) \to \mathcal{C}([0, T], \mathbb{R}^d)$ (adapted w.r.t. to the canonical filtrations of both spaces) such that, for any strong solution to (1.2) driven by a $q$-dimensional $(\mathcal{F}_t)$-Brownian motion $B = (B_t)_{t \in [0, T]}$ on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbb{P})$ and any $\mathcal{F}_0$-measurable starting random vector $X_0$ with distribution $\mu$:

$$\mathbb{P}\text{-a.s. } X = F_\mu(X_0, B).$$

Let us be more precise. We consider the canonical space $\widetilde{\Omega}_0 = \mathbb{R}^d \times \mathcal{C}_0([0, T], \mathbb{R}^q)$ equipped with the product $\sigma$-field $\text{Bor}(\mathbb{R}^d) \otimes \text{Bor}(\mathcal{C}_0, \mathbb{R})$ and the product probability measure $\mathbb{P}^0 = \mu \otimes \mathbb{Q}_w$ where $\mu$ has a finite $p$-th moment for some $p > p_eu$ and $\mathbb{Q}_w$ denotes the $q$-dimensional Wiener measure. Let $(X_0, B)$ denote the canonical process on $\widetilde{\Omega}_0$ defined by $(X_0, B)(x, w) = (x, w)$. As $p > p_eu$ the Volterra SDE (1.2) has a unique pathwise strong solution $(\xi_t)_{t \in [0, T]}$ defined on $\widetilde{\Omega}_0$ (and adapted to the $\mathbb{P}^0$-completed natural filtration $\mathcal{F}_0^t = \sigma(N_{0, t}, X_0, B_s, 0 \leq s \leq t), t \in [0, T], \text{etc}$). We set

$$\bar{F}_\mu(x, w) = \xi(x, w).$$

Then one remarks that on any filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbb{P})$ on which lives a $q$-dimensional $(\mathcal{F}_t)$-Brownian motion $W$ and $X_0$ an $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random vector, then $\bar{F}_\mu(X_0, W)$ is a strong solution to the Volterra equation related to $W$ and by strong uniqueness it is $\mathbb{P}$-indistinguishable of the “natural” strong solution $X$ starting from $X_0$.

Thus, the functional $\bar{F}_\mu$ being defined on the canonical space $\widetilde{\Omega}_0$, does not depend on the selected stochastic basis (nor the Brownian motion under consideration). In particular, for every $x \in \mathbb{R}^d$, setting

$$F(x, w) := \bar{F}_\mu(x, w), \quad (x, w) \in \mathbb{R}^d \times \mathcal{C}_0([0, T], \mathbb{R}^q),$$

one has $\bar{X}^x = F(x, W) \mathbb{P}\text{-a.s.}$ (by uniqueness of strong solutions of the Volterra equation since the starting value $x$ is deterministic).

If $\mathbb{Q}_w$ denotes the Wiener measure on $\mathcal{C}([0, T], \mathbb{R}^q)$

$$\int \rho_t(F(x, w), F(x', w))^p \mathbb{Q}_w(dw) = \mathbb{E}\rho_t(\bar{X}^x, \bar{X}^y)^p$$

so that, we show like for $\bar{X}^x$, that $x \mapsto F(x, w)$ admits an $\alpha'$-Hölder $\mathbb{Q}_w$-modification since $p > p^*$ owing to Step 1. Hence $x \mapsto F(x, W)$ and $(\bar{X}^x)_{x \in \mathbb{R}^d}$ are $\mathbb{P}$-indistinguishable. Note that since $F$ is continuous in $x$ and measurable in $w$, this functional is bi-measurable according to Lemma 4.51 [23].

STEP 3 (Representation for $X_0 \in L^p(\mathbb{P})$, $p$ large). First let us consider the case where $X_0$ is the pathwise continuous solution to the Volterra equation starting from a random vector $X_0 \in L^p(\mathbb{P})$, $p > p^*$, independent of $W$ given by [30, Theorem 3.3]. We will prove that $X = F(X_0, W)$, mimicking the proof to establish (D.73), one shows that, if $X'$ denotes the pathwise continuous solution to the Volterra equation starting from $X_0' \in L^p(\mathbb{P})$ independent of $W$,

$$\sup_{t \in [0, T]} \|X_t - X'_t\|_p \leq C_{p, T} \|X_0 - X_0'\|_p.$$  \hspace{1cm} (D.75)

Let $\varphi_k : \mathbb{R}^d \to \mathbb{Q}^d$ such that $|\varphi_k(x)| \leq |x|$ for every $x \in \mathbb{R}^d$ and $\sup_{x \in \mathbb{R}^d} |\varphi_k(x) - x| \to 0$ as $k \to +\infty$ and set $X_0^{(k)} = \varphi_k(X_0)$. The set $N = \cup_{x \in \mathbb{Q}^d} \{\omega \in \Omega : \bar{X}^x(\omega) \neq F(x, W(\omega))\} \cup N_0$ is $\mathbb{P}$-negligible by construction.

Since $X_0^{(k)}$ is a $\mathbb{Q}^d$-valued, $\mathcal{F}_0$-measurable random vector, independent of $W$, $F(X_0^{(k)}, W)$ is a pathwise continuous solution to the Volterra equation so that, by (D.75),

$$\sup_{t \in [0, T]} \|X_t - F(X_0^{(k)}, W)_t\|_p \leq C_{p, T} \|X_0 - X_0^{(k)}\|_p \to 0 \text{ as } k \to +\infty.$$
With the triangle inequality, one deduces that

\[
\limsup_{k \to \infty} \sup_{t \in [0,T]} \|X_t - F(X_0, W)_t\|_p \leq \limsup_{k \to \infty} \|\rho_T (F(X_0, W), F(X_0^{(k)}, W))\|_p.
\]

Using that \(X_0\) and \(W\) are independent and the bound \((\text{D.74})\), we get

\[
\mathbb{E} \rho_T (F(X_0, W), F(X_0^{(k)}, W))^p = \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx_0) \mathbb{E} \rho_T (F(x_0, W), F(\varphi_k(x_0), W))^p
\]

\[
= \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx_0) \mathbb{E} \rho_T (X_{x_0}, W)^p
\]

\[
\leq C'_{a,p,T} \sum_{R \in \mathbb{N}} (1 + 2R)^{p(1-\lambda)} \mathbb{E} \left[ |X_0 - \varphi_k(X_0)|^{\lambda p} 1_{\{|X_0| \leq R\}} \right]
\]

\[
\leq C'_{a,p,T} \left(3^{p(1-\lambda)} \mathbb{E} \left[ |X_0 - \varphi_k(X_0)|^{\lambda p} 1_{\{|X_0| \leq 1\}} \right] + \sum_{R \geq 2} (1 + 2R)^{p(1-\lambda)} \mathbb{E} \left[ |X_0 - \varphi_k(X_0)|^{\lambda p} 1_{\{|X_0| \leq R\}} \right] \right)
\]

\[
\leq C_{a,p,T,X_0} \left(1 + \sum_{R \geq 2} (1 + 2R)^{p(1-\lambda)} \frac{1}{(R-1)^p} \right) \mathbb{E} \left[ |X_0 - \varphi_k(X_0)|^{\lambda p} |X_0|^p \right]
\]

where we used Markov inequality in the last line. The series is converging since \(\frac{(1+2R)^{p(1-\lambda)}}{(R-1)^p} \sim \frac{1}{R^{\lambda p}}\) and \(p\lambda > d \geq 1\). Hence

\[
\mathbb{E} \rho_T (F(X_0, W), F(X_0^{(k)}, W))^p \leq C'_{a,p,T,X_0} \mathbb{E} \left[ |X_0 - \varphi_k(X_0)|^{\lambda p} |X_0|^p \right] \leq C_{a,p,T,X_0} \sup_{x \in \mathbb{R}^d} |\varphi_k(x) - x| \mathbb{E} |X_0|^p \to 0
\]

as \(k \to +\infty\). Consequently, for every \(t \in [0,T]\), \(X_t = F(X_0, W)_t\) \(\mathbb{P}\)-a.s. Both being pathwise continuous \(X = F(X_0, W)\) \(\mathbb{P}\)-a.s.

**Step 4 (Representation for \(X_0 \in L^0(\mathbb{P})\)).** Let \(X_0 \in L^0(\mathbb{P})\). We consider like in Appendix \([12]\) the sequence of truncated starting values \(X_0^{(k)} = X_0 1_{A_k}\) with \(A_k = \{|X_0| < k\}\), \(k \geq 1\) and the resulting pathwise continuous solution to \((1.2)\) still denoted \(X^{(k)}\). As \(X_0^{(k)} \in L^p(\mathbb{P})\) for \(p > p^*\), then \(\mathbb{P}\)-a.s.

\[
X^{(k)} = F(X_0^{(k)}, W), \quad k \geq 1.
\]

Hence, under the convention \(A_0 = \emptyset\),

\[
X = \sum_{k \geq 1} X^{(k)} 1_{A_k} = \sum_{k \geq 1} F(X_0^{(k)}, W) 1_{A_k} = F(X_0, W).
\]

**Proposition D.3 (Representation formula for the \(K\)-integrated Euler scheme)** Assume \((K^{int}_j)\) and \((K^{cont}_j)\) are in force. Let \(n \geq 1\). There exists a bi-measurable functional \(\tilde{F}_n : \mathbb{R}^d \times C_0([0,T], \mathbb{R}^q) \to C([0,T], \mathbb{R}^q)\) such that, for any stochastic basis \((\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}), any q-dimensional \((F_t)\)-Brownian motion \(W\) and any \(\mathcal{F}_0\)-measurable \(\mathbb{R}^d\)-valued random vector \(X_0 \in L^0(\mathbb{P})\), the \(K\)-integrated Euler scheme \(\tilde{X}\) with step \(\frac{T}{n}\) starting from \(X_0\) defined by \((1.13)\) writes

\[
\tilde{X} = \tilde{F}_n(X_0, W).
\]

**Proof.** As for the \(K\)-integrated scheme, it amounts to prove that, for every \(k \in \{0, \cdots, n - 1\}\), the process

\[
[t_k^n, T] \ni t \mapsto \int_{t_k^n}^{t \wedge t_{k+1}^n} K_2(t, u) dW(u)
\]

44
can be written as a function of $W$. First we can assume temporarily that $W$ is defined on the canonical space $\Omega_0 = \mathcal{C}_0([0, T], \mathbb{R}^d)$ — i.e. $W_t(w) = w(t)$, for every $w \in \Omega_0$ — equipped with its Borel $\sigma$-field $\mathcal{B}or(\mathcal{C}_0(0))$ induced by the sup-norm topology and the Wiener measure $\mathbb{Q}_w$. For every $t \in [t^n_k, T]$, $\int_{t^n_k}^{t \wedge t^n_{k+1}} K_2(t, u) dW_u$ is $\sigma(N_{t^n_k}, W_{u} - W_{t^n_k}, t^n_k \leq u \leq t \wedge t^n_{k+1})$-measurable hence $\sigma(N_{t^n_k}, W_{u}, 0 \leq u \leq T)$-measurable so that

$$\int_{t^n_k}^{t \wedge t^n_{k+1}} K_2(t, u) dW_u = F^k(t, W).$$  \hspace{1cm} (D.76)

(A more precise result involving adaptation to the canonical filtration can be obtained likewise but it is not useful for our purpose here.) The pathwise continuity of $\left(\int_{t^n_k}^{t \wedge t^n_{k+1}} K_2(t, u) dW_u\right)_{t \in [t^n_k, T]}$ follows from Lemma 3.4 applied with $\tilde{H} \equiv 1$ and implies that, for $\mathbb{Q}_w(dw)$-almost every $w \in \Omega_0$, $[t^n_k, T] \ni t \mapsto F^k(t, w) \in \mathcal{C}_0([0, T], \mathbb{R}^d)$. Then for any standard Brownian motion $W$ on a stochastic basis $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ the representation $\left(\int_{t^n_k}^{t \wedge t^n_{k+1}} K_2(t, u) dW_u\right)$ holds true (with the same functional) since the Gaussian law of $\left(W, \int_{t^n_k}^{t \wedge t^n_{k+1}} K_2(t, u) dW_u\right)$ does not depend on the choice of $W$.

This being done, one checks by induction on the discretization times using (1.14) that the $K$-integrated Euler scheme starting from any (finite) $\mathbb{R}^d$-valued starting random vector $X_0$ admits a representation

$$\hat{X} = F_n(X_0, W),$$

where $\hat{F}_n : (\mathbb{R}^d \times \Omega_0, \mathcal{B}or(\mathbb{R}^d) \otimes \mathcal{B}or(\mathcal{C}_0(0))) \to (\mathcal{C}([0, T], \mathbb{R}^d), \mathcal{B}or(\mathcal{C}_d))$ is measurable.

\section{End of the proof of Theorem 1.1}

\textbf{Step 1 (\textit{a}-Hölder version/modification when $X_0 \in L^p(\mathbb{P})$, $p$ large enough).} Our bound (1.16) is more general and more precise than its counterpart in [30, Theorem 3.3] since it takes the form

$$\left\| \sup_{s \neq t \in [0, T]} \frac{|X_t - X_s|}{|t - s|^a} \right\|_p \leq C_{a,p,T}(1 + \|X_0\|_p),$$  \hspace{1cm} (E.77)

where $C_{a,p,T}$ does not depend on $X_0$, $a$ is any element of $(0, \theta \wedge \frac{2\beta}{\beta + 1} - 1)$ and only $X_0 \in L^p(\mathbb{P})$ is requested. This turns out to be the key for the extension to lower exponents $p$ in what follows.

At this stage we no longer rely on [30] whose proof in Theorem 3.1 is sub-optimal. We could provide a direct proof to evaluate $\|X_t - X_s\|_p$, first for $p \geq 2$ by just mimicking that of Property 3 in Appendix C. In fact it follows from Properties 3 and 4 from Appendix C the following two results: first, for $p \geq 2$ there exists a positive real constant $C$ (not depending on $n$) such that for every $n \geq 1$ the genuine $K$-integrated Euler scheme $\hat{X}$ with step $\frac{T}{n}$ satisfies the Hölder regularity property

$$\forall s, t \in [0, T], \quad \|\hat{X}_t - \hat{X}_s\|_p \leq C(1 + \|X_0\|_p)|t - s|^\theta \wedge \hat{\theta}$$

and, for $p > p_{eu} = \frac{1}{\theta} \wedge \frac{2\theta}{\beta + 1} > 2$, the “marginal” convergence property of the $K$-integrated Euler scheme toward $X$ holds i.e. $\|X_t - \hat{X}_t\|_p \to 0$ as $n \to +\infty$ for every $t \in [0, T]$. As a consequence, letting $n \to 0$, yields that, for $p > p_{eu}$,

$$\forall s, t \in [0, T], \quad \|X_t - X_s\|_p \leq C(1 + \|X_0\|_p)|t - s|^\theta \wedge \hat{\theta}.$$  \hspace{1cm} (E.78)

At this stage, we can proceed as follows: let $a \in (0, \theta \wedge \hat{\theta})$ and let $p > \frac{1}{\theta} \wedge \frac{2\theta}{\beta + 1} \wedge p_{eu}$ so that $a < \theta \wedge \hat{\theta} - \frac{1}{p}$. Tracking once again the constants in the proof of Kolmogorov’s criterion, e.g. in [23] Chapter 2, p.26], to check that the final constant depends linearly in the initial one, one derives that (E.77) does hold true for such $p$.
Step 2 ($L^p$-pathwise regularity $p \in (0, +\infty)$). To extend (E.77) to every $p > 0$ such that $\|X_0\|_p < +\infty$, one can take advantage of the form of the control to apply the splitting Lemma (D.1) with $p < \bar{p}$, $\bar{p} > \frac{1}{\theta_\delta - \alpha} \sqrt{p_eu}$ and the functional $\Phi : \mathcal{C}([0, T], \mathbb{R}^d)^2 \to \mathbb{R}$ defined by

$$\Phi(x, y) = \sup_{s \neq t \in [0, T]} \frac{|x(t) - x(s)|}{|t - s|^{\alpha}}.$$ 

The marginal bound (E.78) can be extended likewise by considering $\Phi(x, y) = |x(t) - x(s)|$. $\Box$

F. Proofs of Theorems 1.2 and 1.4

F.1 Proof of Theorem 1.2 (convergence of the $K$-integrated Euler scheme)

In this section, we complete the proof of Theorem 1.2 based on the splitting Lemma established in Appendix D.

Step 1 (Moment control, $p \in (0, 2)$). Let us denote $\bar{X}^{x_0} = (\bar{X}_t^{x_0})_{t \in [0, T]}$ the Euler scheme (1.13) starting from $x_0$ at $t = 0$. Property 2 in Appendix C applied with $p = 2$ implies that there exists a real constant $C = C_{K_1, K_2, \beta, b, \sigma, T} > 0$ (not depending on the time step $\frac{T}{n}$ i.e. on $n$) such that, if $X_0 = x_0$,

$$\sup_{t \in [0, T]} \|\bar{X}_t^{x_0}\|_2 \leq C(1 + |x_0|).$$

Then it follows from Lemma (D.1) applied with $\bar{p} = 2$ and $\Phi(x, y) = \sup_{t \in [0, T]} |y(t)|$ that

$$\forall p \in (0, 2], \quad \sup_{t \in [0, T]} \|\bar{X}_t\|_p \leq 2^{(1/p - 1)^+} C(1 + \|X_0\|_p).$$

Step 2 ($L^p$-control of the increments of the $K$-integrated Euler scheme for $p \in (0, 2)$). If $p \in (0, 2)$, we proceed likewise replacing Property 2 in Appendix C by Property 3, with the functional $\Phi(x, y) = |y(t) - y(s)|$, $s, t \in [0, T]$ in order to prove that

$$\forall s, t \in [0, T], \quad \|\bar{X}_t - \bar{X}_s\|_p \leq 2^{(1/p - 1)^+} C_{2, T}(1 + \|X_0\|_p)|t - s|^\theta \delta^{\hat{\delta}}.$$ 

Step 3 ($L^p$-rate of convergence for the marginals at fixed time $t$ for $p > 0$). We fix $\bar{p} > p_{eu}$ and combine Property 4 in Appendix C and Lemma (D.1) applied with $\Phi(x, y) = |x(t) - y(t)|$ to $p \in (0, \bar{p})$.

Step 4 ($L^p$-rate of convergence for the sup-norm, $p$ large enough). To switch from the $L^p$-convergence of marginals, namely $\|X_t - \bar{X}_t\|_p$, to this one, we rely on Corollary 4.4 [20] which is a form close to Kolmogorov’s $C$-tightness criterion of the Garsia-Rodemich-Rumsey lemma (see [13]).

Theorem F.1 (GRR lemma) Let $(Y^n)_{n \geq 1}$ be a sequence of continuous processes where the processes $Y^n = (Y^n_t)_{t \in [0, T]}$ are defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $p \geq 1$. Assume there exists $a > 1$, a sequence $(\delta_n)_{n \geq 1}$ of positive real numbers converging to 0 and a real constant $C > 0$ such that

$$\forall n \geq 1, \forall s, t \in [0, T], \quad \mathbb{E}|Y^n_t - Y^n_s|^p \leq C|t - s|^a \delta_n^p. \quad (F.79)$$

Then there exists a real constant $C_{p, T} > 0$ such that

$$\forall n \geq 1, \quad \mathbb{E}\sup_{t \in [0, T]} |Y^n_t - Y^n_0|^p \leq C_{p, T} \delta_n^p.$$
Let $\varepsilon \in (0, 1)$. We aim at applying the above inequality to the sequence of processes $Y^n = X - \bar{X}$ to prove that the order of convergence of the Euler scheme derived at Property 4 from Appendix C is preserved for the supremum over time up to multiplication by the factor $1 - \varepsilon$.

Set $\theta^* = \theta \wedge \hat{\theta}$. We first deal with large values of $p$, namely $p > p_* = 2\varepsilon / (2\varepsilon - 1)$. Then let $\lambda := \frac{1}{p^*} + \frac{\varepsilon}{2} \in (0, \varepsilon)$. For every $s, t \in [0, T]$, one has

$$\|Y^n_t - Y^n_s\|_p \leq \|Y^n_t - Y^n_s\|^\lambda \|Y^n_t - Y^n_s\|^{1 - \lambda} \leq (\|X_t - X_s\|_p + \|X_t - \bar{X}_s\|_p)^\lambda (\|X_t - \bar{X}_s\|_p + \|X_s - \bar{X}_s\|_p)^{1 - \lambda}.$$ 

By Property 4 from Appendix C there exists a positive real constant $C_1$ such that

$$\sup_{t \in [0, T]} \|X_t - \bar{X}_t\|_p \leq C_1 (1 + \|X_0\|_p) \left(\frac{T}{n}\right)^{\theta^* \wedge \gamma}.$$ 

With Property 3, still from Appendix C we deduce the existence of a positive real constant $C_2$ such that

$$\forall s, t \in [0, T], \quad \|X_t - X_s\|_p + \|X_t - \bar{X}_s\|_p \leq C_2 (1 + \|X_0\|_p) |t - s|^\theta^*.$$ 

Hence, using that $a := \lambda \theta^* = 1 + \frac{\varepsilon}{2} \theta^* > 1$, we derive

$$E(Y^n_t - Y^n_s)^p \leq C_3^\theta (1 + \|X_0\|_p)^p |t - s|^a \left(\frac{T}{n}\right)^{p(\theta^* \wedge \gamma)(1 - \lambda)},$$

with $C_3 = C_1^\lambda / C_2^\lambda$. Hence, as $Y^n_0 = 0$ and $a > 1$, it follows from the above GRR lemma that there exists a positive real constant $\kappa_{\varepsilon,p} = \kappa_{\varepsilon,p,\beta,\gamma,\theta,b,\sigma,T}$ such that, for every $n \geq 1$,

$$\left\| \sup_{t \in [0, T]} |X_t - \bar{X}_t| \right\|_p \leq \kappa_{\varepsilon,p} (1 + \|X_0\|_p) \left(\frac{T}{n}\right)^{(\theta^* \wedge \gamma)(1 - \varepsilon)}.$$ 

As $\lambda \leq \varepsilon$, it is clear that, up to an updating of the constant $\kappa_{\varepsilon,p}$ to take into account the values of $n \in \{1, \ldots, \lfloor T \rfloor\}$, one has for every $n \geq 1$,

$$\left\| \sup_{t \in [0, T]} |X_t - \bar{X}_t| \right\|_p \leq \kappa_{\varepsilon,p} (1 + \|X_0\|_p) \left(\frac{T}{n}\right)^{\theta^* \wedge \gamma}(1 - \varepsilon).$$

Now we take advantage of Appendix D to extend this result to the low integrability setting of $X_0$, i.e. $X_0 \in L^p(\bar{P})$, $p \in (0, p_*]$.\]

**Step 5 (Lp-rate of convergence of the supremum over time, p \in (0, p_0))**. It follows from Lemma D.1 applied with $\bar{p} = p_0 + 1$ and $\Phi(x, y) = \sup_{t \in [0, T]} |x(t) - y(t)|$ that

$$\left\| \sup_{t \in [0, T]} |X_t - \bar{X}_t| \right\|_p \leq 2^{(1/p_0 - 1)} \left(\frac{T}{n}\right)^{(\gamma^* \wedge \gamma)(1 - \varepsilon)} (1 + \|X_0\|_p).$$

This completes the proof of this step and of the theorem. \hfill $\square$

### F.2 Proof of Theorem 4.4 (continuity and convergence of the K-discrete time Euler scheme)

Since $(X_{\text{cont}}^n) n$ holds for some $\underline{\theta} > 0$, then for each $k \in \{0, \cdots, n - 1\}$, $t \mapsto (K_1(t, t_k), K_2(t, t_k))$ is locally Hölder continuous with exponent $\underline{\theta}$ on $(t_k, T)$. For $k \in \{0, \cdots, n - 1\}$, since the K-discrete Euler scheme reads on every interval $(t_k, T)$

$$\forall t \in (t_k, t_{k+1}], \quad \bar{X}_t = X_0 + \sum_{\ell=0}^{k-1} \left( K_1(t, t_{\ell}) b(t_{\ell}, \bar{X}_{t_{\ell}}) \frac{T}{n} + K_2(t, t_{\ell}) \sigma(t_{\ell}, \bar{X}_{t_{\ell}})(W_{t_{\ell+1}} - W_{t_{\ell}}) \right) + \bar{K}_1(t, t_k) b(t_k, \bar{X}_{t_k})(t - t_k) + K_2(t, t_k) \sigma(t_k, \bar{X}_{t_k})(W_t - W_{t_k}),$$

(F.80)
one easily deduces that it is continuous on the interval $(t^n_k, t^n_{k+1}]$ with the sum of the two first terms in the right-hand side even continuous on the closed interval $[t^n_k, t^n_{k+1}]$. Since $(\hat{K}_{\hat{\theta}}^{cont})$ holds for some $\hat{\theta} > 0$, the choice $\delta = t - t^n_k$ in this condition ensures that $\forall t \in (t^n_k, t^n_{k+1}]$, $K_i(t, t^n_k) \leq \tilde{\delta}(t - t^n_k)^{\hat{\theta} - 1/4}$. Using the law of the iterated logarithm satisfied by the Brownian motion, we deduce that the sum of the two last terms in the right-hand side of (F.80) goes to 0 as $t \to t^n_k+$ and the $K$-discrete Euler scheme has continuous paths. Under the assumptions, it is established in Theorem 1.1 that $X$ is itself pathwise continuous. The authors in [26] claim the conclusions of the theorem are valid for $p$ large enough as far as (1.21) and (1.22) are concerned and whenever $X_0 \in \cap_{r > 0} L^r(\mathbb{P})$ for (1.23). It is straightforward to show they hold true for any $p \in (0, +\infty)$ by applying Lemma [D.1] with the functionals $\Phi(x, y) = |y(t) - y(s)|$, $\Phi(x, y) = |x(t) - y(t)|$ and $\Phi(x, y) = \sup_{t \in [0, T]} |x(t) - y(t)|$, $x, y \in C([0, T], \mathbb{R}^d)$ respectively. The proof is quite similar to that of Theorem 1.2 at this stage. $\square$