Abstract. We prove that Schrödinger operators with meromorphic potentials \((H_{\alpha,\theta} u)_n = u_{n+1} + u_{n-1} + \frac{g(\theta + n\alpha)}{f(\theta + n\alpha)} u_n\) have purely singular continuous spectrum on the set \(\{E : L(E) < \delta(\alpha, \theta)\}\), where \(\delta\) is an explicit function, and \(L\) is the Lyapunov exponent. This extends results of [15] for the Maryland model and of [4] for the almost Mathieu operator, to the general family of meromorphic potentials.

1. Introduction

We study operators of the form:

\[(H_{\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + \frac{g(\theta + n\alpha)}{f(\theta + n\alpha)} u_n\]

acting on \(l^2(\mathbb{Z})\), where \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\) is the frequency, \(T = \mathbb{R}/\mathbb{Q}\), \(\theta \in T\) is the phase, \(f\) is an analytic function and \(g\) is Lipshitz. This class contains all meromorphic potentials and therefore both the almost Mathieu (\(f = 1, g = \lambda \cos 2\pi \theta\)) and Maryland (\(f = \sin 2\pi \theta, g = \lambda \cos 2\pi \theta\)) families as particular cases.

Let \(p_n/q_n\) be the continued fraction approximants of \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\).

Assume \(g/f\) has \(m\) poles, \(m \geq 0\). We denote them by \(\theta_i, i = 1, ..., m\), including multiplicities. We now define index \(\delta\) as follows:

\[
\delta(\alpha, \theta) = \limsup_{n \to \infty} \frac{\sum_{i=1}^{m} \ln \|q_n(\theta - \theta_i)\|_{\mathbb{R}/\mathbb{Z}} + \ln q_{n+1}}{q_n}.
\]

where \(\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{l \in \mathbb{Z}}|x - l|\). Let \(L(E)\) be the Lyapunov exponent, see [21]. \(L\) depends also on \(\alpha\) but we suppress it from the notation as we keep \(\alpha\) fixed.

Our main result is:

**Theorem 1.1.** Let \(\delta(\alpha, \theta)\) be as in (2.1). Then

1. \(H_{\alpha,\theta}\) has no eigenvalues on \(\{E : L(E) < \delta(\alpha, \theta)\}\).
2. If \(L(E) > 0\) for a.e. \(E\) (in particular, if \(m > 0\)), then \(H_{\alpha,\theta}\) has purely singular continuous spectrum on \(\{E : L(E) < \delta(\alpha, \theta)\}\).

**Remark.** Since absence of absolutely continuous spectrum follows from a.e. positivity of the Lyapunov exponents and holds for all unbounded potentials [13], part (2) immediately follows from part (1), on which we therefore concentrate.

Recently, there has been an increased interest in obtaining arithmetic conditions (in contrast to a.e. statements) for various quasiperiodic spectral results. In particular, there have been remarkable advances in the theory of the almost Mathieu operator

\[(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + \lambda \cos 2\pi(\theta + n\alpha) u_n\]
(see e.g. [17] [12] for the review and background in physics). Define

\[ \beta = \beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}, \]

which describes how Liouvillian \( \alpha \) is. We say that \( \alpha \) is Diophantine if \( \beta(\alpha) = 0 \). Note that for almost every phase \( \theta \) (only depends on \( \alpha \)) we have \( \delta(\alpha, \theta) = \beta(\alpha) \).

It was conjectured in 1994 [9] that \( \lambda = e^\beta \) is the phase transition point from singular continuous spectrum to pure point spectrum for \( \alpha \)-Diophantine \( \theta \) (and that the transition is at larger \( \lambda \) for non-\( \alpha \)-Diophantine \( \theta \)). The history of partial results towards this conjecture include [10] [2]. Recently, Avila, You and Zhou proved [4]

**Theorem 1.2.** For \( \lambda > e^\beta \), the spectrum is pure point with exponentially decaying eigenfunctions for a.e. \( \theta \), and for \( 1 < \lambda < e^\beta \), the spectrum is purely singular continuous for all \( \theta \).

**Remark.** The spectrum is known to be absolutely continuous for all \( \alpha, \theta \) for \( \lambda < 1 \) (the final result in [11]).

A fully arithmetic version of the localization statement

**Theorem 1.3.** For \( \lambda > e^\gamma \), the spectrum is pure point with exponential decaying eigenfunctions for \( \alpha \)-Diophantine \( \theta \)

was established recently in [13].

Define also

\[ \gamma = \gamma(\alpha, \theta) = \limsup_{n \to \infty} -\frac{\ln \left\| 2\theta + n\alpha \right\|_{\mathbb{R}/\mathbb{Z}}}{|n|}. \]

We say that \( \theta \) is \( \alpha \)-Diophantine if \( \gamma(\alpha, \theta) = 0 \).

It was also conjectured in [9] [11] that \( \lambda = e^\gamma \) is the phase transition point from singular continuous spectrum to pure point spectrum for Diophantine \( \alpha \) (and that the transition is at larger \( \lambda \) for non-Diophantine \( \alpha \)). Partial results towards this conjecture include [16] [10]. The conjecture was recently fully established in [14]:

**Theorem 1.4.** For \( \lambda > e^{\gamma(\alpha, \theta)} \), the spectrum is pure point with exponentially decaying eigenfunctions for Diophantine \( \alpha \), and for \( \lambda < e^{\gamma(\alpha, \theta)} \), the spectrum is singular continuous for all \( \alpha \).

Therefore, for the almost Mathieu operator the precise transition from pure point to singular continuous spectrum is understood for either Diophantine \( \alpha \) and all \( \theta \) or for all \( \alpha \) and \( \alpha \)-Diophantine \( \theta \), but not yet for all parameters.

Another case with a significant recent arithmetic results is the Maryland model

\[ (H_{\lambda, \alpha, \theta} u)_n = u_{n+1} + u_{n-1} + \lambda \tan \pi(\theta + n\alpha)u_n \]

It is the prototypical operator of form (1.1). This model was proposed by Grempel, Fisherman, and Prange [6] as a linear version of the quantum kicked rotor. It is an exactly solvable example of the family of incommensurate models, thus attracting continuing interest in physics, e.g. [8]. The complete description of spectral transitions for the Maryland model (depending on arithmetic properties of all parameters) was given recently in [15].

Namely, an index \( \delta(\alpha, \theta) \in [-\infty, \infty] \) was introduced in [15]:

\[ \delta(\alpha, \theta) = \limsup_{n \to \infty} \frac{\ln \left\| q_n(\theta - \frac{1}{2}) \right\|_{\mathbb{R}/\mathbb{Z}} + \ln q_{n+1}}{q_n} \]

The main result of [15] regarding the singular continuous part is:

**Theorem 1.5.** [15] \( H_{\alpha, \theta} \) has purely singular continuous spectrum on \( \{ E : L(E) < \delta(\alpha, \theta) \} \).
Lemma 2.1. then for some absolute constant $C > 0$

Theorem 1.6. \[15\] $H_{\alpha, \theta}$ has pure point spectrum on \{ $E : L(E) > \delta(\alpha, \theta)$ \}.

Our result therefore is an extension of Theorem [15] (to which Theorem [11] specializes for $f = \cos 2\pi \theta$, $g = \lambda \sin 2\pi \theta$) to the general family of singular potentials. For $f \equiv 1, g = \lambda \cos 2\pi \theta$ we recover the singular continuous part of Theorem 1.2 (note that the proof of [4] also extends in this case to $f \equiv 1$ and a Lipshitz condition on $g$ without many changes). Theorem 1.6 shows that our result is sharp for the Maryland model. However, Theorems 1.3, 1.4 show that it is not sharp for the almost Mathieu operator other than for $\alpha$-Diophantine $\theta$. Based on this, we do not expect sharpness for general Lipshitz or even analytic potentials ($f \equiv 1$), and conjecture that sharpness (that is point spectrum in the complementary regime other than possibly on the transition line) may be a corollary of certain monotonicity.

2. Preliminaries: cocycle, Lyapunov exponent

Assume without loss of generality, $f(\theta) = (e^{2\pi i \theta} - e^{2\pi i \theta_1}) \cdots (e^{2\pi i \theta - e^{2\pi i \theta_m})}, m = 1, \cdots.$

Let $\Theta = \cup_{l=1}^N \theta_l + Z \alpha + Z$. From now on we fix $E$ in the spectrum and $\theta \in \Theta^c$ such that $L(E) < \delta(\alpha, \theta)$. We will show $H_{\lambda, \alpha, \theta}$ cannot have an eigenvalue at $E$.

A formal solution of the equation $H_{\alpha, \theta}u = Eu$ can be reconstructed via the following equation

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$$

where $A(\theta) = \begin{pmatrix} E - g(\theta) & 1 \\ 1 & 0 \end{pmatrix}$ is the so-called transfer matrix.

The pair $(\alpha, A)$ is the cocycle corresponding to the operator [14]. It can be viewed as a linear skew-product $(x, \omega) \mapsto (x + \alpha, A(x) \cdot \omega)$. Generally, one can define $M_n$ for an invertible cocycle $(\alpha, M)$ by $(\alpha, M)^n = (n\alpha, M_n)$, $n \in Z$ so that for $n \geq 0$:

$$M_n(x) = M(x + (n - 1)\alpha)M(x + (n - 2)\alpha) \cdots M(x),$$

and $M_{-n}(x) = M_n(x - n\alpha)$.

The Lyapunov exponent of a cocycle $(\alpha, M)$ is defined by

$$L(\alpha, M) = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{T}} \ln \| M_n(x) \| dx.$$ 

Let $A(x) = \frac{1}{f(x)} D(x)$ where

$$D(x) = \begin{pmatrix} E f(x) - g(x) & -f(x) \\ f(x) & 0 \end{pmatrix}$$

be the regular part of $A(x)$. Since $\int_T \ln |f(x)| dx = 0$, we have

$$L(E) := L(\alpha, A) = L(\alpha, D).$$

Lemma 2.1. \[2\] Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $\theta \in \mathbb{R}$ and $0 \leq j_0 \leq q_n - 1$ be such that

$$| \sin \pi (\theta + j_0 \alpha) | = \inf_{0 \leq j \leq q_n - 1} \sin \pi (\theta + j \alpha),$$

then for some absolute constant $C > 0$,

$$-C \ln q_n \leq \sum_{j=0, j \neq j_0}^{q_n-1} \ln | \sin \pi (\theta + j \alpha) | + (q_n - 1) \ln 2 \leq C \ln q_n$$
We will also use that the denominators of continued fraction approximants of $\alpha$ satisfy
\[ \| k\alpha \|_{\mathbb{R}\setminus\mathbb{Z}} \geq \| q_n\alpha \|_{\mathbb{R}\setminus\mathbb{Z}}, 1 \leq k < q_{n+1}, \]
and
\[ \frac{1}{2q_{n+1}} \leq \| q_n\alpha \|_{\mathbb{R}\setminus\mathbb{Z}} \leq \frac{1}{q_{n+1}}. \] (2.2)

A quick corollary of subadditivity and unique ergodicity is the following upper semicontinuity statement:

**Lemma 2.2.** (e.g. [3]) Suppose $(\alpha, A)$ is a continuous cocycle. Then for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$, such that for any $x \in \mathbb{T}$ we have
\[ \| A_n(x) \| \leq C e^{n(L(A)+\varepsilon)}. \]

**Remark 2.1.** Applying this to 1-dimensional continuous cocycles, we get that if $g$ is a continuous function such that $\ln|g| \in L^1(\mathbb{T})$, then
\[ \left| \prod_{l=a}^{b} g(x + l\alpha) \right| \leq e^{(b-a+1)(\int \ln|g|d\theta) + \varepsilon}. \]

**3. Absence of point spectrum**

Let $\varphi$ be a solution to $H_{\alpha,\theta}\varphi = E\varphi$ satisfying $\| \begin{pmatrix} \varphi_0 \\ \varphi_{-1} \end{pmatrix} \| = 1$. We have the following restatement of Gordon’s lemma. We state a precise form that will be convenient for us.

**Theorem 3.1.** If there exists a constant $c > 0$ and a subsequence $q_{n_i}$ of $q_n$ such that the following estimates holds:
\[ \| (A^2_{q_{n_i}}(\theta) - A_{2q_{n_i}}(\theta)) \begin{pmatrix} \varphi_0 \\ \varphi_{-1} \end{pmatrix} \| \leq e^{-c q_{n_i}}, \] (3.1)
and
\[ \| (A^{-1}_{q_{n_i}}(\theta) - A^{-1}_{q_{n_i}}(\theta - q_n\alpha)) \begin{pmatrix} \varphi_0 \\ \varphi_{-1} \end{pmatrix} \| \leq e^{-c q_{n_i}}, \] (3.2)

then we have
\[ \max\{\| (\varphi_{q_{n_i}}) \|, \| (\varphi^{-q_{n_i}}_{-1}) \|, \| (\varphi^{2q_{n_i}}_{-1}) \| \} \geq \frac{1}{4}. \] (3.3)

**Proof.** This is a standard argument, going back to [7]. The key idea is to use the following two equalities:
\[
\begin{align*}
A_{q_{n_i}}(\theta) - \text{Tr} A_{q_{n_i}}(\theta) \cdot Id + A_{q_{n_i}}^{-1}(\theta) = 0 \\
A^2_{q_{n_i}}(\theta) - \text{Tr} A_{q_{n_i}}(\theta) \cdot A_{q_{n_i}}(\theta) + Id = 0
\end{align*}
\]
and separate the cases $|\text{Tr} A_{q_{n_i}}(\theta)| > \frac{1}{2}$, $|\text{Tr} A_{q_{n_i}}(\theta)| < \frac{1}{2}$.

\[ \square \]
3.1. Proof of Theorem 4.1. Assume $\varphi$ is a decaying solution of $H_{\alpha, \theta} \varphi = E \varphi$, satisfying $\| (\varphi_{0} \varphi_{-1}) \| = 1$. On one hand, it must be true that for any $\eta > 0$, there exists $N$ such that $\| (\varphi_{k}) \| \leq \eta$ for $|k| > N$. On the other hand, we will prove the following lemma in section 5:

Lemma 3.2. For any $\varepsilon > 0$ there exists a subsequence $\{q_{n_{i}}\}$ of $\{q_{n}\}$ so that we have the following estimates:

\begin{align*}
\| (A_{q_{n_{i}}}^{-1}(\theta) - A_{q_{n_{i}}}^{-1}(\theta - q_{n_{i}} \alpha)) (\varphi_{0} \varphi_{-1}) \| &\leq \varepsilon^{-q_{n_{i}}(L(E) - \delta(\alpha, \theta) + 4\varepsilon)}, \\
\text{and} \\
\| (A_{q_{n_{i}}}^{2}(\theta) - A_{2q_{n_{i}}}(\theta)) (\varphi_{0} \varphi_{-1}) \| &\leq \varepsilon^{-q_{n_{i}}(L(E) - \delta(\alpha, \theta) + 4\varepsilon)}.
\end{align*}

Then combining Lemma 3.2 and Theorem 3.1 we get a contradiction, which shows the absence of point spectrum. \hfill \Box

4. KEY LEMMAS

Let $| \sin \pi(\theta - \theta_{l} + j\alpha) | = \inf_{0 \leq j \leq q_{n_{i}}} | \sin \pi(\theta - \theta_{l} + j\alpha) |$.

Lemma 4.1. If $\delta(\alpha, \theta) > 0$, then for any $\varepsilon > 0$, there exists a subsequence $q_{n_{i}}$ of $q_{n}$ such that the following estimate holds

$$\prod_{l=1}^{m} | \sin \pi(\theta - \theta_{l} + j\alpha) | \geq \frac{e^{q_{n_{i}}(\delta - \frac{\varepsilon}{4})}}{q_{n_{i}+1}}.$$  

Proof. By the definition of $\delta(\alpha, \theta)$, there exists a subsequence $q_{n_{i}}$ of $q_{n}$ such that

$$\sum_{l=1}^{m} \ln \left( \frac{q_{n_{i}}(\theta - \theta_{l})}{q_{n_{i}}} \right) + \ln q_{n_{i}+1} > \delta(\alpha, \theta) - \frac{\varepsilon}{4},$$

thus

$$\| q_{n_{i}}(\theta - \theta_{l}) \| \cdot \| q_{n_{i}}(\theta - \theta_{m}) \| > \frac{e^{q_{n_{i}}(\delta - \frac{\varepsilon}{4})}}{q_{n_{i}+1}}.$$  

In particular, $\| q_{n_{i}}(\theta - \theta_{l}) \| > \frac{e^{q_{n_{i}}(\delta - \frac{\varepsilon}{4})}}{q_{n_{i}+1}}$ for any $1 \leq l \leq m$. Since

\begin{align*}
| \sin \pi(\theta - \theta_{l} + j\alpha) | \\
\geq 2 \| (\theta - \theta_{l} + j\alpha) \| \\
\geq 2 \| q_{n_{i}}(\theta - \theta_{l} + j\alpha) \| \\
\geq \frac{2 \| q_{n_{i}}(\theta - \theta_{l}) \| - 2q_{n_{i}}}{q_{n_{i}}} \\
\geq \frac{\| q_{n_{i}}(\theta - \theta_{l}) \|}{q_{n_{i}}},
\end{align*}

We have

$$\prod_{l=1}^{m} | \sin \pi(\theta - \theta_{l} + j\alpha) | \geq \prod_{l=1}^{m} \frac{\| q_{n_{i}}(\theta - \theta_{l}) \|}{q_{n_{i}}} > \frac{e^{q_{n_{i}}(\delta - \frac{\varepsilon}{4})}}{q_{n_{i}+1}} \cdot \frac{1}{(q_{n_{i}})^{m}} > \frac{e^{q_{n_{i}}(\delta - \frac{\varepsilon}{4})}}{q_{n_{i}+1}}.$$  

\hfill \Box
Lemma 4.2. The following estimate holds
\[ \prod_{j=0}^{q_{n_i}-1} |f(\theta + j\alpha)| \geq \frac{e^{q_{n_i}(\delta - \varepsilon)}}{q_{n_i+1}}. \]

Proof.
\[ \prod_{j=0}^{q_{n_i}-1} |f(\theta + j\alpha)| = 2^{m_{q_{n_i}}} \prod_{l=0}^{m} \prod_{j=0}^{q_{n_i}-1} |\sin \pi(\theta - \theta_l + j\alpha)| \]
\[ = 2^{m_{q_{n_i}}} \left( \prod_{l=0}^{m} \prod_{j=0, j \neq j_l}^{q_{n_i}-1} |\sin \pi(\theta - \theta_l + j\alpha)| \right) \left( \prod_{l=0}^{m} |\sin \pi(\theta - \theta_l + j(\theta + j\alpha))| \right). \]
Combining Lemma 2.1 and 4.1
\[ \prod_{j=0}^{q_{n_i}-1} |f(\theta + j\alpha)| \geq 2^{m_{q_{n_i}}} e^{m(-C \ln q_{n_i} - (q_{n_i}-1) \ln 2)} \cdot \frac{e^{q_{n_i}(\delta - \varepsilon)}}{q_{n_i+1}} \geq \frac{e^{q_{n_i}(\delta - \varepsilon)}}{q_{n_i+1}}. \]

5. PROOF OF LEMMA 4.2

We give a detailed proof of (3.4). (3.5) could be proved in a similar way.

Proof. Let
\[ A^{-1}(x) = \frac{1}{f(x)} \begin{pmatrix} 0 & f(x) \\ -f(x) & Ef(x) - g(x) \end{pmatrix} \triangleq \frac{F(x)}{f(x)}. \]
Consider
\[ \Psi_{n_i} = \left( A^{-1}(\theta) - A^{-1}_{q_{n_i}}(\theta - q_{n_i}\alpha) \right) \begin{pmatrix} \varphi_0 \\ \varphi_{-1} \end{pmatrix}. \]
For simplicity let us introduce some notations: fixing \( \theta \), for any function \( z(x) \) on \( T \) denote \( z_j = z(\theta + j\alpha) \); for any matrix function \( M(x) \) denote \( M^j = M(\theta + j\alpha) \). Then, by telescoping,
\[ \Psi_{n_i} = \left( \frac{F^0}{f_0} \frac{F^1}{f_1} \cdots \frac{F_{q_{n_i}-1}}{f_{q_{n_i}-1}} - \frac{F^{q_{n_i}}}{f_{q_{n_i}}} \cdots \frac{F^{1}}{f^{1}} \right) \begin{pmatrix} \varphi_0 \\ \varphi_{-1} \end{pmatrix} \]
\[ = \sum_{j=0}^{q_{n_i}-1} \left( \frac{F^0 F^1 \cdots F^{j-1}}{f_0 f_1 \cdots f_{j-1}} \right) \left( \frac{F^j}{f_j} - \frac{F^{-q_{n_i}+j}}{f^{q_{n_i}+j}} \right) \left( \frac{F^{-q_{n_i}+j} \cdots F^{-1}}{f^{-q_{n_i}+j} \cdots f^{-1}} \right) \begin{pmatrix} \varphi_0 \\ \varphi_{-1} \end{pmatrix}, \]
where for \( j = 0 \) the first, and for \( j = q_{n_i} - 1 \) the last, multiple are set to be equal to one.

Thus
\[ \Psi_{n_i} = \sum_{j=0}^{q_{n_i}-1} \left( \prod_{l=0}^{j-1} \frac{F^l}{f_l} \right) \left( \frac{F^j}{f_j} - \frac{F^{-q_{n_i}+j}}{f^{-q_{n_i}+j}} \right) \begin{pmatrix} \varphi_{-q_{n_i}+j+1} \\ \varphi_{-q_{n_i}+j} \end{pmatrix} \]
\[ = \sum_{j=0}^{q_{n_i}-1} \left( \prod_{l=0}^{j-1} \frac{F^l}{f_l} \right) \left( \frac{F^j f^{-q_{n_i}+j} - F^{-q_{n_i}+j} f^{q_{n_i}+j} + F^{-q_{n_i}+j} f^{-q_{n_i}+j} f^{q_{n_i}+j} f_j}{f_j f^{-q_{n_i}+j}} \right) \begin{pmatrix} \varphi_{-q_{n_i}+j+1} \\ \varphi_{-q_{n_i}+j} \end{pmatrix} \]
\[ = \sum_{j=0}^{q_{n_i}-1} \left( \prod_{l=0}^{j-1} \frac{F^l}{f_l} \right) \left( \frac{F^j f^{-q_{n_i}+j}}{f_j} \begin{pmatrix} \varphi_{-q_{n_i}+j+1} \\ \varphi_{-q_{n_i}+j} \end{pmatrix} + \frac{f^{-q_{n_i}+j} - f_j}{f_j} \begin{pmatrix} \varphi_{-q_{n_i}+j} \\ \varphi_{-q_{n_i}+j-1} \end{pmatrix} \right). \]
Since $\phi$ is a decaying solution, there exists a constant $C > 0$ such that
\[
\| (\varphi_k - \varphi_{k-1}) \| \leq C.
\]
Observe that $sup_{\theta} \| F(\theta + q_n \alpha) - F(\theta) \| < \frac{C}{q_n + 1}$. Now we can get, using Lemma 2.2, Remark 2.1 and Lemma 4.2 in the second inequality
\[
\| (A_q^{-1}(\theta) - A_p^{-1}(\theta - q_n \alpha)) \left( \begin{array}{c} \varphi_0 \\ \varphi_{-1} \end{array} \right) \|
\leq C q_n e^{q_n(L(E) + \epsilon)} e^{q_n(\delta - \epsilon)}
\leq C q_n e^{q_n(L(E) - \delta + 4r)}.
\]

\[\Box\]

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