Density limit in a first principles model of a magnetized plasma in the Debye–Hückel approximation

A. Carati* M. Zuin† A. Maiocchi‡ M. Marino§
E. Martines¶ L. Galgani∥

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Abstract

A crucial problem concerning a large variety of fusion devices is that the confinement due to an external magnetic field is lost above a critical density, while a widely accepted first principles explanation of such a fact is apparently lacking. In the present paper, making use of standard methods of statistical mechanics in the Debye–Hückel approximation, we give indications that for a plasma there exists a density threshold corresponding to a transition from order to chaos, the ordered motions being those in which the confining Lorentz force on a single electron prevails over the diffusive effect of the Coulomb forces. The density limit, which is proportional to the square of the magnetic field, turns out to fit not too badly the empirical data for plasma collapses in a large set of fusion devices.

1 Introduction

It is well known that in most fusion devices a density limit for plasma confinement exists, while “there is no widely accepted first principles model.”

* Dipartimento di Matematica, Università degli Studi di Milano, Milano, Italy
† Consorzio RFX, Associazione EURATOM-ENEA sulla Fusione, Padova, Italy
‡ Dipartimento di Matematica, Università degli Studi di Milano, Milano, Italy
§ Dipartimento di Matematica, Università degli Studi di Milano, Milano, Italy
¶ Consorzio RFX, Associazione EURATOM-ENEA sulla Fusione, Padova, Italy
∥ Dipartimento di Matematica, Università degli Studi di Milano, Milano, Italy
for it (see the review [1]). In the present paper, using the methods of statistical mechanics in the Debye–Hückel approximation, we show that for a magnetized plasma there exists a transition from order to chaos at a critical density. By ordered motions we just mean those for which the confining magnetic Lorentz force acting on a single electron prevails against the sum of the diffusive Coulomb forces of all the other particles, so that gyration prevails over diffusion.

The transition occurs beyond a density limit which is comparable to that observed for plasma collapses in a large set of fusion devices. Let us recall that, according to Alfvén [2] (see also page 382 of the first scientific paper [3] of Bohr), a plasma can present a diamagnetic behavior (and thus a confining pressure) only when it is in an out–of–equilibrium state, which thus can persist only in the presence of sufficiently ordered motions, corresponding to the existence of a suitable adiabatic invariant. Instead, in the presence of a strong chaoticity, diamagnetism is quickly lost (see [4]). This is the reason why the transition from order to chaos discussed here may be related to the collapses observed in fusion devices.

So the problem is that of comparing the size of the confining Lorentz force acting on a single electron to that of the diffusive force due to the Coulomb interactions with all the other particles. The estimate is trivial for the confining Lorentz force, for it is sufficient to estimate the velocity of the electrons, and this is immediately obtained in terms of temperature. For what concerns the size of the diffusive force due to all the Coulomb interactions, we estimate it using the standard methods of statistical mechanics, which compel one to take into account the $n$–body collisions for all $n$, so that the Debye length $\lambda_D$ then naturally enters into play (a comment on the possibility of using the methods of kinetic theory instead of the general ones of statistical mechanics will be given later).

Now, rigorous estimates of the sum of the Coulomb forces on a single particle can be found in the literature only for systems of pure electrons. Indeed, the Gibbs distribution can be consistently dealt with in such a case, because the results do not depend too strongly on the cutoffs that have to be introduced to manage the long range character of the forces. Instead, in the presence of the neutralizing ions one meets with the problem that the potential energy is not bounded from below. This requires introducing short range cutoffs, and the results now strongly depend on them, i.e., are model dependent (for a general introduction to the problem see [5], [6], [7], [8] and the huge bibliography in [9]). On the other hand, one expects that the force due to the ions makes the system more chaotic, so that, in order to give a model independent estimate of the threshold, in the present paper we limit ourselves to considering the contribution of the electrons, which should
correspond to giving an upper bound to the threshold.

We will show that this leads to a theoretical electron density limit \( n_e \) given by the law

\[
n_e = 3 \frac{B^2}{\mu_0 mc^2},
\]

where \( B \) is the magnetic field, \( \mu_0 \) the vacuum permeability, \( c \) the speed of light and \( m \) the electron mass. Using \( n_e = 1/a^3 \) where \( a \) is the mean interelectron distance, relation (1) can also be put in the particularly expressive form

\[
\frac{1}{2\mu_0} B^2 a^3 = \frac{1}{6} mc^2,
\]

according to which the transition from order to chaos occurs when the magnetic energy inside a cell of volume \( a^3 = 1/n_e \) just equals (apart from a factor \( 1/6 \) the electron rest energy \( mc^2 \).

We will show below that the theoretical formula (1) fits not too badly the phenomenological density limit for plasma collapses in a large set of fusion devices, and this suggests that the contributions of the ions, which we have here neglected, should be of the same order of magnitude as that of the electrons. This conjecture is supported by a further result, which will be published elsewhere (see [10] for the case of a smeared out background). Namely, one can discuss the stability properties of the equilibrium solutions of a neutral system of electrons and ions in a constant magnetic field, and one finds a bifurcation to an unstable solution, at a critical density which is the same as (1) apart from a factor of order one.

2 The theoretical density limit

In order to apply the methods of statistical mechanics, dealing with pure Coulomb interactions, we neglect any boundary effect, and just consider an infinite system of point electrons in a constant magnetic field, disregarding the role of the ions. If \( \mathbf{x}_j \) and \( \mathbf{v}_j \) denote the position vector and the velocity of the \( j \)th electron, \( e \) its charge, \( \mathbf{B} = Be_z \) a constant magnetic field directed along the \( z \) axis, and \( \epsilon_0 \) the vacuum dielectric constant, the system of equations of motion in the nonrelativistic approximation is then

\[
m\ddot{\mathbf{x}}_j = e\mathbf{v}_j \wedge \mathbf{B} + \frac{e^2}{4\pi\epsilon_0} \sum_{k\neq j} \frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|^3}.
\]

It goes without saying, that we are considering here a model describing an autonomous system, i.e., an isolated one, on which no power is injected from
outside. In terms of fusion devices, this model is presumably better suited for high field devices, which are, in general, characterized by large energy confinement times, and hence need a lower sustainment.

Obviously, the two forces at the right hand side of (3) play opposite roles, the magnetic Lorentz force producing ordered motions with confinement, while the Coulomb repulsions of the other electrons produce a diffusive effect, which we consider as a perturbation depending parametrically on density. Thus a critical situation should occur when the Lorentz force and the transverse component \( F_\perp \) of the total Coulomb force somehow balance. The size of the Lorentz force, to which only the transverse component \( v_\perp \) of the velocity contributes, is immediately estimated by

\[
v_\perp \simeq \sqrt{2k_BT/m}
\]

where \( T \) is absolute temperature and \( k_B \) the Boltzmann constant.

Much more delicate is the estimate of the vector sum of the Coulomb forces due to all the other electrons, or rather of the modulus \( F_\perp \) of its transverse component. This problem can be tackled by statistical methods, considering \( F_\perp \) as a random variable. As \( F_\perp \) obviously has zero mean, according to Chebishev’s theorem its typical value is given by its standard deviation \( \sigma_\perp \). The computations, with respect to the Gibbs ensemble at a given inverse temperature \( \beta = 1/k_BT \), will be performed in the Debye–Hückel approximation, i.e., at the lowest order in the density.

To compute the standard deviation \( \sigma_\perp \) of \( F_\perp \) (the modulus of the transverse component of the total Coulomb force acting on a single electron, the \( j \)th one), one uses the fact that the correlations between \( x_j - x_k \) and \( x_j - x_l \) can be neglected in the Debye–Hückel approximation (see [11]), so that one has

\[
\sigma_\perp^2 = \frac{2}{3} \left( \frac{e^2}{4\pi \epsilon_0} \right)^2 \sum_{k \neq j} \frac{1}{|x_j - x_k|^4} \left< \frac{1}{|x_j - x_k|^4} \right>
\]

where \( < \ldots > \) denotes canonical average. On the other hand, the probability of the relative distance \( r \) between two particles is well known to be distributed according to the Gibbs ensemble relative to the effective Debye–Hückel potential

\[
V_{\text{eff}}(r) = \frac{e^2}{4\pi \epsilon_0 r} \exp \left( -r/\lambda_D \right),
\]

where

\[
\lambda_D = \sqrt{\epsilon_0 k_B T/(n_e e^2)}
\]

is the Debye length. So eventually at first order one finds

\[
\sigma_\perp^2 = \frac{2}{3} \frac{e^4}{4\pi \epsilon_0^2 a^2} \int_0^{+\infty} \frac{1}{r^2} \exp \left[ -\beta V_{\text{eff}}(r) \right] dr.
\]
Bringing the integrand into dimensionless form one sees that, if \( a/\lambda_D \) is small, significant values of the integrand are assumed only in a region about the Bjerrum length \( b \). This is defined (see [6]) by
\[
b = \frac{e^2}{4\pi\epsilon_0 k_B T},
\]
so that one also has \( 4\pi b \lambda_D^2 = a^3 \). So the integral in (4) is well approximated by the integral
\[
\int_0^{+\infty} \frac{1}{r^2} \exp \left( -\frac{b}{r} \right) \, dr = \frac{1}{b} = \frac{4\pi \lambda_D^2}{a^3},
\]
and finally one gets
\[
F_\perp \simeq \sigma_\perp \simeq \sqrt{\frac{2}{3}} \frac{e^2}{\epsilon_0} \frac{\lambda_D}{a^3}.
\]
Thus, in the Debye–Hückel approximation a balance between confining Lorentz force and diffusive long range Coulomb forces on each electron occurs when
\[
B \sqrt{\frac{2k_B T}{m}} \simeq \sqrt{\frac{2}{3}} \frac{e}{\epsilon_0} \frac{\lambda_D}{a^3}.
\]
(5)
So temperature altogether disappears, and the transition from order to chaos occurs when density and magnetic field are related by
\[
n_e = 3 \epsilon_0 \frac{B^2}{m},
\]
i.e., by (1), if \( \epsilon_0 \) is expressed in terms of the vacuum permeability \( \mu_0 = 1/(\epsilon_0 c^2) \).

### 3 Comparison with the empirical density limit in fusion devices

We now check whether the transition from order to chaos discussed here has anything to do with the empirical data for collapses in fusion machines. A proportionality of the density limit to the square of the magnetic field in tokamaks was suggested by Granetz [12] on the basis of empirical data, but apparently was not confirmed by successive observations [29, 11]. It is well known that, while at first a proportionality to the magnetic field (through \( B/R \), where \( R \) is the major radius of the torus) had been proposed on an empirical basis for tokamaks by Murakami [13], in the plasma physics community the common opinion is rather that the density limit for tokamaks should be proportional to the Greenwald parameter \( I_p/r_a^2 \), where \( I_p \) is the plasma current and \( r_a \) the minor radius of the torus (see [11]).

We do not enter here a discussion of this point, and only content ourselves with plotting in figure 1 a collection of available data of the critical density
Figure 1: Density limit values vs $B$ for various devices: conventional tokamaks, for which recent data are shown (see references from [14] to [29]) along with the original ones of Murakami (see [13]), stellarator devices (from [30] to [33]), and spherical tokamaks (from [34] to [36]). Dotted line is the theoretical density limit (1).
for several fusion devices versus their operating magnetic field $B$ in log–log scale, comparing the data to the theoretical formula (1). One sees that the theoretical law appears to correspond not so badly to the data for the high field devices (tokamak and stellarators), whereas a sensible discrepancy is met for the low field devices (spherical tokamaks), for which the experimental data are larger by even an order of magnitude.

One should not forget however that we are discussing here a model describing an isolated, non sustained, system (i.e., with no input heating power), whereas one should expect (see the empirical Sudo limit for stellarators [33]) that larger densities are accessible as the input power is increased (although this is not so clear for tokamaks [1]). This is illustrated, in the figure, by the three points reported for the same device (the stellarator WS-A7 [30]) at essentially the same applied field, which however correspond to three different (increasing) input heatings. Now, the low field devices for which a sensible discrepancy is shown in the figure, are just the ones characterized, in general, by lower confinement time and thus by larger sustainment, so that a discrepancy corresponding to larger experimental values might be expected. It would thus be of interest to extend our model by including some forcing describing the operations of sustained devices.

4 Comments

In view of the lack of any first principles rationale for the existence of a density limit in fusion devices, the partial agreement of the theoretical formula (1) with the experimental data seems encouraging. In our opinion, a key feature characterizing the present approach is that, at variance with the treatments involving the continuum approximation, such as magnetohydrodynamics, we are dealing with the plasma as a discrete system of particles. Indeed in our treatment a key role is played by the fluctuations of the force acting on a single particle, and so the instability found here would be lost in the continuum approximation, or in any other approximation involving high–frequency cutoffs. For an analogous role of discreteness of matter in cosmology, see [37] and [38].

On the other hand, even in plasma physics theory there exists a literature in which the discrete nature of matter is taken into account. We refer to the works (see for example [39], [40] and [41]) in which the approach of kinetic theory is followed, along the lines of the classical works of Balescu, Lenard, Lifshitz and Pitaevskii. Now, both the kinetic theoretical approach and the statistical mechanical one followed here, are originating from a common source, namely, the previously mentioned work of Bogolyubov [5], so that
their results should agree. Thus the result found here should in principle be obtained also within the kinetic theory approach. We leave this interesting problem for future possible work.

Finally, it is worth mentioning that the proportionality of the density limit to the square of the magnetic field predicted by the theoretical law (1), if confirmed, might have relevant implications for future tokamaks. However, for what concerns the analytical side of the problem, in order to produce more exact fits with the experimental data one should push the theory much more forward. First one should settle the problem of the contribution of the ions, and include sustainment in the model. Then one should consider the boundary effects, and the inhomogeneities of the macroscopic quantities characterizing the plasma, such as temperature, magnetic field and density. More in general, one should consider relativistic effects, i.e., the retardations of the fields, and thus emission of radiation. Furthermore, one should consider higher order perturbation effects, going beyond the Debye–Hückel approximation. Useful information on all these problems should also be obtained through numerical studies, along the lines for example of the recent work [42] on strongly coupled plasmas.

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