Friedel-Type Oscillations in the Problem of Skin Effect in Degenerate Collisionless Plasma

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It is shown that a Friedel type oscillations accompany skin effect in degenerate plasma of a metal. It was learnt earlier that Friedel oscillations take place under charge screening in quantum plasma. However the nature of Friedel oscillations is not in the quantum character of the plasma, but in the features of degenerate Fermi distribution, namely, in its sharp transformation into zero directly just the other side of the Fermi surface. This circumstance leads to the Friedel-type oscillations under anomalous skin effect.

Key words: degenerate collisionless plasma, dielectric permeability, Friedel oscillations, Kohn singularities.

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1. Introduction

It is considered [1], that under penetration into degenerate plasma the transversal electric field in the problem of skin effect in the infrared area changes according to exponential law

\[ E = E_0 e^{-x/\delta}, \quad \delta = \frac{c}{\omega_p}, \] (1)

where \(c\) is the light speed, and \(\omega_p\) is plasma (Langmuir) frequency.

It is well known also (see, for instance, [2]), that dimensionless electric field in the problem of skin effect has the following form

\[ \frac{E(x)}{E'(0)} = \frac{a l}{\pi} \int_{-\infty}^{\infty} \frac{e^{i k_1 x} d k_1}{\varepsilon_{tr}(k_1) - a k_1^2}. \] (2)

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Here $x_1$ is the dimensionless coordinate, $x_1 = \frac{x}{l}$, $k_1 = k l$ is the dimensionless wave number, $k$ is the dimensional wave number, $l = v_F \tau$ is the mean free path of electrons,

$$a = \left( \frac{\varepsilon \nu}{v_F \Omega} \right)^2, \quad \Omega = \frac{\omega}{\omega_p}, \quad \varepsilon = \frac{\nu}{\omega_p},$$

$\varepsilon_{tr}$ is the transversal dielectric permeability,

$$\varepsilon_{tr} = 1 - \frac{3}{4 \Omega (k_1 \varepsilon)^3} \left[ 2(\Omega + i\varepsilon)(k_1 \varepsilon) + \left[ (\Omega + i\varepsilon)^2 - (k_1 \varepsilon)^2 \right] \ln \frac{\Omega + i\varepsilon - k_1 \varepsilon}{\Omega + i\varepsilon + k_1 \varepsilon} \right].$$

In the present work it is shown that the dielectric permeability has Kohn singularities (see [3] – [6]), which lead to electric field Friedel kind oscillations [7] – [14].

2. Problem Solution

The quantity $\varepsilon_{tr}$ is regular under all values of the frequency of the oscillations of the electric field and the wave number. However, under the collision frequencies tending to zero, i.e. under $\varepsilon \to 0$ the wave number derivative $\varepsilon_{tr}$ has singularities. These singularities are analogous to so called Kohn singularities, which take place at quantum longitudinal dielectric permittivity. It is known that Kohn singularities result in change of the asymptotic form of the electric field of the electric charge. Instead of the Debay screening the slowly receding Friedel oscillations take place. On the Fig. 1 the graph of the wave number derivative of the dielectric permittivity is presented. We can easily see the features indicated above on the graph. In accordance with this fact the change of the asymptotic form of the electric field takes place under skin effect.

On the Fig. 1 Kohn kind singularities for the case of sodium are presented. We put the derivative $d\varepsilon_{tr}/dq$ on the vertical axis. We consider two cases of the frequencies $\Omega = 0.08$ and $\Omega = 0.1$. Singularities of the derivative $d\varepsilon_{tr}/dq$ in the points $q = 0.08$ and $q = 0.1$ is seen on the graph.

Let us carry out the change of variable of integration $k_1 = q/\varepsilon$ and note that

$$k_1 x_1 = k_1 \frac{x_1}{\varepsilon} = \frac{x}{l \varepsilon} = q \frac{x \omega_p}{v_F \tau \nu} = q \frac{\omega_p}{v_F} x.$$
Then

\[ \frac{E(x)}{E'(0)} = \frac{al}{\varepsilon \pi} \int_{-\infty}^{\infty} \frac{e^{iq\omega_p x/v_F} dq}{\varepsilon_{\text{tr}}(q) - bq^2}, \]

where

\[ b = \frac{a}{\varepsilon^2} = \left( \frac{c}{v_F \Omega} \right)^2. \]

Our aim is to analyze the asymptotic behaviour of the electric field under \( x \to \infty \). At the same time we would consider the contribution into the integral of region near of the derivative singularity \( \varepsilon_{\text{tr}} \). Our consideration would be similar to the one stated into \cite{12}. 

Figure 1: Kohn kind singularities: derivative of the dielectric permittivity, curves 1 and 2 correspond to the values \( \Omega = 0.1 \) and \( \Omega = 0.08 \).
Integrating twice by parts we receive

\[
\frac{E(x)}{E'(0)} = \frac{alv_F^2}{\varepsilon \pi \omega_p^2 x^2} \int_{-\infty}^{\infty} \frac{\varepsilon''_{tr}(q)e^{iq\omega_p x/v_F} dq}{[\varepsilon_{tr}(q) - bq^2]^2}. \tag{3}
\]

In the expression (3) we write down only terms which display the most anomalous behaviour near the Kohn singularity. In this approximation \(\varepsilon''_{tr}(q)\) we can write in the following form

\[
\varepsilon''_{tr}(q) = -\frac{3}{4\Omega q^3} \left[ \frac{\Omega + i\varepsilon + q}{\Omega + i\varepsilon - q} - \frac{\Omega + i\varepsilon - q}{\Omega + i\varepsilon + q} \right].
\]

Now instead of (3) for the electric field we obtain the following expression

\[
\frac{E(x)}{E'(0)} = \frac{3c^2 v_F}{4\pi \omega_p x^2} \int_{-\infty}^{\infty} \left[ \frac{q + \Omega + i\varepsilon}{q - \Omega - i\varepsilon} - \frac{q - \Omega - i\varepsilon}{q + \Omega + i\varepsilon} \right] \frac{e^{iq\omega_p x/v_F} dq}{q^3[\varepsilon_{tr}(q) - bq^2]^2}. \tag{4}
\]

Let us consider the case of collisionless plasma, i.e. the case when \(\varepsilon \to 0\). Then the integral (4) can be simplified significantly:

\[
\frac{E(x)}{E'(0)} = \frac{3c^2 v_F}{4\pi \omega_p x^2} \int_{-\infty}^{\infty} \left[ \frac{q + \Omega}{q - \Omega} - \frac{q - \Omega}{q + \Omega} \right] \frac{e^{iq\omega_p x/v_F} dq}{q^3[\varepsilon_{tr}(q) - bq^2]^2}. \tag{5}
\]

Here

\[
\varepsilon_{tr}(q) - bq^2 = 1 - \frac{3}{2q^2} - \frac{3}{4\Omega q^3}(\Omega^2 - q^2) \ln \frac{\Omega - q}{\Omega + q} - \left( \frac{cq}{v_F \Omega} \right)^2.
\]

For the calculation of the integral from (5)

\[
J = \int_{-\infty}^{\infty} \frac{q + \Omega}{q - \Omega} - \frac{q - \Omega}{q + \Omega} \frac{e^{iq\omega_p x/v_F} dq}{q^3[\varepsilon_{tr}(q) - bq^2]^2}
\]

we use the method stated in the monograph [12]. For this purpose we would present the integral \(J\) in the form of the difference \(J = J_1 - J_2\), where

\[
J_1 = \int_{-\infty}^{\infty} \frac{q + \Omega}{q - \Omega} \cdot \frac{e^{iq\omega_p x/v_F} dq}{q^3[\varepsilon_{tr}(q) - bq^2]^2};
\]
\[ J_2 = \int_{-\infty}^{\infty} \frac{q - \Omega}{q + \Omega} \cdot \frac{e^{i q \omega_p x / v_F}}{q^3 \left[ \varepsilon_{tr}(q) - b q^2 \right]^2} dq, \]

After the evident change of variable of integration for the second integral \( J_2 \) we get the expression:

\[ J_2 = - \int_{-\infty}^{\infty} \frac{q + \Omega}{q - \Omega} \cdot \frac{e^{-i q \omega_p x / v_F}}{q^3 \left[ \varepsilon_{tr}(q) - b q^2 \right]^2} dq. \]

Consequently, the integral \( J \) equals to:

\[ J = 2 \int_{-\infty}^{\infty} \frac{q + \Omega}{q - \Omega} \cdot \frac{\cos \left( \frac{q \omega_p x}{v_F} \right)}{q^3 \left[ \varepsilon_{tr}(q) - b q^2 \right]^2} dq. \]

Considering the singularity of the kernel \( 1/(q - \Omega) \) of this integral the most large contribution to the value of this integral is made by the values of the subintegral function near the point of singularity \( q = \Omega \). The function \( f(q) = (q + \Omega)q^{-3}[\varepsilon_{tr}(q) - b q^2]^{-2} \) change slowly in the neighbourhood of the point \( q = \Omega \). Therefore further we assume \( f(q) = f(\Omega) \) under the calculation of the integral \( J \) near the singular point. We obtain

\[ J = 2 f(\Omega) \int_{-\infty}^{\infty} \frac{\cos \left( \frac{q \omega_p x}{v_F} \right)}{q - \Omega} dq. \]

After the integration variable change \( q - \Omega = u \), noticing that

\[ \cos \left( \frac{q \omega_p x}{v_F} \right) = \cos \left( \frac{(u + \Omega) \omega_p x}{v_F} \right) = \cos \frac{\omega_p u}{v_F} x \cos x - \sin \frac{\omega x}{v_F} \sin \frac{\omega_p u}{v_F} x, \]

and using the known relation

\[ \int_{-\infty}^{\infty} \frac{\sin A x}{x} dx = \pi, \]

we receive:

\[ J = -2\pi f(\Omega) \sin \frac{\omega}{v_F} x, \]
where

\[ f(\Omega) = \frac{2}{\Omega^2 \left[ \frac{3}{2\Omega^2} + \left( \frac{c}{v_F} \right)^2 - 1 \right]^2} = \frac{2\omega_p^2}{\left[ \frac{3}{2} \omega_p^2 + \left( \frac{c}{v_F} \right)^2 \omega - \omega^2 \right]^2}, \]

or

\[ f(\Omega) = \frac{2\Omega^2}{\left[ \frac{3}{2} + \left( \frac{c}{v_F} \Omega \right)^2 - \Omega^2 \right]^2}. \]

Thereby we have found that the electric field far away from the surface \( x = 0 \) decreases according to the law:

\[ \frac{E(x)}{E'(0)} = -\frac{A}{x^2} \sin \frac{\omega}{v_F} x, \]

where

\[ A = \frac{3c^2v_F}{x^2\omega_p^3} \cdot \frac{1}{\left[ \frac{3}{2} + \left( \frac{c}{v_F} \right)^2 - \Omega^2 \right]^2} = \frac{3c^2v_F\omega_p}{\left[ \frac{3}{2} + \frac{c^2}{v_F^2} - \omega^2 \right]^2}. \]

Let us rewrite the formula (6) with the help of the dimensionless parameters in the following form:

\[ \frac{E(x)}{E'(0)} = -\frac{A}{x^2} \sin \frac{\Omega\omega_p}{v_F} x, \]

where

\[ A = \frac{3c^2v_F}{\omega_p^3 \left[ \Omega^2 \left( \frac{c^2}{v_F^2} - 1 \right) + \frac{3}{2} \right]^2}. \]

Noting that in the considered here nonrelativistic case \( v_F \ll c \), we can simplify the formula (8):

\[ A = \frac{3c^2v_F}{\omega_p^3 \left[ \frac{3}{2} + \left( \frac{c}{v_F} \Omega \right)^2 \right]^2}. \]

In the low frequencies case when \( \Omega \ll \frac{v_F}{c} \), the formula (9) can be simplified and reduced to the following form:

\[ A = \frac{4c^2v_F}{3\omega_p^3}. \]
and in the case when $\Omega \gg \frac{v_F}{c}$, the formula (9) can be reduced to the form

$$A = \frac{3v_F^5}{c^2\omega_p^3\Omega^4}.$$  

Let us consider the case of infra-red frequencies. The formula (1) is applicable near the surface. In accordance with this fact we obtain

$$\frac{E(0)}{E'(0)} = -\frac{c}{\omega_p}. \quad (10)$$

Dividing the equality (7) by (10), we have:

$$E(x) = \frac{\omega_p A(\Omega, v_F) E(0)}{x^2} \sin \left(\frac{\omega_p \Omega v_F}{v_F} x\right). \quad (11)$$

The formula (11) can be presented in the form

$$E(x) = \frac{B}{x^2} \sin \left(\frac{\omega_p \Omega}{v_F} x\right) E(0), \quad B = \frac{3cv_F E(0)}{\omega_p^2 \left[\frac{3}{2} + \left(\frac{c}{v_F} \Omega\right)^2 - \Omega^2\right]^2}.$$  

Let us carry out the graphic research of the expressions obtained. The behaviour of the variable $y = |E(x)/E(0)|$ is shown on the Figs. 2–4, where the relation $E(x)/E(0)$ is determined according to the equality (11). The distance on the horizontal axis is measured in centimetres.

We present the behaviour of the curves $y_1(x) = B/x^2$ ($E(0) = 1$) (curve 1) and $y_2(x) = e^{-\omega_p x/c}$ (curve 2) on the Figs. 5 and 6. At the same time the logarithmic scale is used on the vertical axis, and the distance on the horizontal axis is measured in microns.

### 3. Conclusion

Friedel was the first [7] – [11] to discover that asymptotic (on the large distances) decreasing of the screened potential of the point charge under quantum consideration of the degenerate collisionless plasma has not only monotonously decreasing but also oscillating character as well. The reason of such oscillations is the sharp falling (to the zero) beyond the Fermi surface of the Fermi distribution for the electrons $f_F(v)$,

$$f_F(v) = \Theta(v_F - v),$$
\( \Theta(x) \) is the Heaviside function,

\[
\Theta(x) = \begin{cases} 
1, & x > 0, \\
0, & x < 0. 
\end{cases}
\]

This peculiarity of the Fermi distribution results in so called Kohn singularities (see [3] – [14]). Kohn singularities are the consequences of the logarithmic singularities of the longitudinal dielectric permittivity of the degenerate plasma. Just the Kohn singularities result in the Friedel oscillations.

In the work [14] the dependance of the Friedel oscillations on the temperature in the collisionless plasma is found out. It was shown that under finite temperature the amplitude of the Friedel oscillation decreases exponentially with the distance. Similar dependance should be apparent under Friedel oscillations in skin effect as well.

![Figure 2: Oscillations of kind of Friedel in the case \( \Omega = 10^{-4}, 2 \cdot 10^{-5} < x < 2 \cdot 10^{-3} \). Curves 1, 2, 3 correspond to sodium, gold and aluminium.](image-url)
Figure 3: Oscillations of kind of Friedel in the case $\Omega = 10^{-3}$, $9 \cdot 10^{-5} < x < 3 \cdot 10^{-3}$. Curves 1, 2, 3 correspond to sodium, gold and aluminium.

Figure 4: Oscillations of kind of Friedel in the case of aluminium, $1.5 \cdot 10^{-3} < x < 1.8 \cdot 10^{-3}$. Curves 1, 2, 3 correspond to the values $\Omega = 10^{-4}, 10^{-3}, 10^{-2}$. 
Figure 5: Intersection of the two curves $y_1(x) = \frac{B}{x^2}$ (curve 1) and $y_2(x) = e^{-\omega_p x/c}$ (curve 2) in the point $x_* = 0.716$ micrometers under $\Omega = 10^{-2}$ (logarithmic scale on the vertical axis).

Figure 6: Intersection of the two curves $y_1(x) = \frac{B}{x^2}$ (curve 1) and $y_2(x) = e^{-\omega_p x/c}$ (curve 2) in the point $x_* = 1.176$ micrometers under $\Omega = 10^{-1}$ (logarithmic scale on the vertical axis).
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