Gross-Witten-Wadia transition in a matrix model of deconfinement

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We study the deconfining phase transition at nonzero temperature in a \( SU(N) \) gauge theory, using a matrix model which was analyzed previously at small \( N \). We show that the matrix model is soluble at infinite \( N \), and exhibits a Gross-Witten-Wadia transition. In some ways, the deconfining phase transition is of first order: at a temperature \( T_d \), the Polyakov loop jumps discontinuously from 0 to \( \frac{1}{2} \), and there is a nonzero latent heat \( \sim N^2 \). In other ways, the transition is of second order: e.g., the specific heat diverges as \( C \sim 1/(T - T_d)^{3/5} \) when \( T \rightarrow T_d^+ \). Other critical exponents satisfy the usual scaling relations of a second order phase transition. In the presence of a nonzero background field \( h \) for the Polyakov loop, there is a phase transition at the temperature \( T_h \) where the value of the loop = \( \frac{1}{2} \), with \( T_h < T_d \). Since \( \partial C/\partial T \sim 1/(T - T_h)^{1/2} \) as \( T \rightarrow T_h^+ \), this transition is of third order.

The properties of the deconfining phase transition for a \( SU(N) \) gauge theory at nonzero temperature are of fundamental interest. At small \( N \), this transition can only be understood through numerical simulations on the lattice [1]. Large \( N \) can be studied through numerical simulations [2] and in reduced models [3]. In the pure glue theory, this transition can be modeled through an effective model, such as a matrix model [4–10]. One limit in which the theory can be solved analytically is by putting it on a sphere of femto-scale dimensions [11, 12]. An effective theory is constructed directly by integrating out all modes with nonzero momentum, and gives a matrix model which is soluble at large \( N \) [13–16]. As a function of temperature, it exhibits a Gross-Witten-Wadia transition [17]. That is, it exhibits aspects of both first order and second order phase transitions; thus it can be termed “critical first order” [12]. Since the theory has finite spatial volume, however, there is only a true phase transition at infinite \( N \). Thus on a femtosphere, the Gross-Witten-Wadia transition appears to be mere curiosity.

Matrix models have been developed as an effective theory for deconfinement in four spacetime dimensions (and infinite volume). These models, which involve zero [6], one [7], and two parameters [8, 9], are soluble analytically for two and three colors, and numerically for four or more colors. In this paper we show that these models are also soluble analytically for infinite volume. These models, which involve zero [6], one [7], and two parameters [8, 9], are soluble analytically for infinite volume. These models, which involve zero [6], one [7], and two parameters [8, 9], are soluble analytically for infinite volume.

We expand about a constant background field for the vector potential,

\[
A_0^{ij} = \frac{2\pi T}{q_i} \delta^{ij}, \tag{1}
\]

where \( i, j = 1 \ldots N \). This \( A_0 \) field is a diagonal \( SU(N) \) matrix, and so \( \sum_{i=1}^{N} q_i = 0 \). The thermal Wilson line is the matrix \( L = \exp(2\pi i q) \); its trace is the Polyakov loop in the fundamental representation, \( \ell_1 = \text{tr}L/N \). At any \( N \), this represents a possible ansatz for the region where the expectation value of the Polyakov loop is less than unity. This region has been termed the “semi” quark gluon plasma (QGP) [5]. At infinite \( N \), this ansatz is the simplest possible for the master field in the semi-QGP.

The potential we take is a sum of two terms,

\[
\tilde{V}_{\text{eff}}(q) = -d_1(T) \tilde{V}_1(q) + d_2(T) \tilde{V}_2(q), \tag{2}
\]

where

\[
\tilde{V}_n(q) = \sum_{i,j=1}^{N} |q_i - q_j|^n (1 - |q_i - q_j|)^n. \tag{3}
\]

The term \( \sim \tilde{V}_2(q) \) is generated perturbatively at one loop order; that \( \sim \tilde{V}_1(q) \) is added to drive the transition to the confined phase. Previously, the functions \( d_1 \) and \( d_2 \) were chosen as \( d_1(T) = (2\pi^2/15) c_1 T^2 T_d^2 \) and \( d_2(T) = 2\pi^2/3 (T^4 - c_2 T^2 T_d^2) \), where \( T_d \) is the temperature for deconfinement [7–9]. These matrix models also included terms independent of the \( q \)’s, \( \sim c_3 T^2 T_d^2 \) and \( \sim BT_d^2 \).
The values of these parameters were chosen to agree with results from numerical simulations on the lattice [7–9]. As we show, however, when $N$ is infinite, at the transition temperature the nature of the solution is independent not only of the values of these parameters, but even of the choice of the functions $d_1(T)$ and $d_2(T)$ (modulo modest assumptions, given later).

The matrix model in Eqs. (2) and (3) is rather different from that on a femtosphere [11, 12]. On a femtosphere the dominant term driving confinement is the Vandermonde determinant, $\sim \Pi_{i,j} \log(\exp(2\pi i q_i) - \exp(2\pi i q_j))$: in the present model it is the terms $\sim \tilde{V}_n(q)$. The logarithmic singularities of the Vandermonde determinant are stronger than those of the from the absolute values $\sim |q_i - q_j|^n$ in the $\sim \tilde{V}_n(q)$.

To treat infinite $N$, we introduce the variable $x = i/N$, so that $q_i \to q(x)$, and the potential is an integral over $x$. It is useful to introduce the eigenvalue density, $\rho(q) = dx/dq$ [13]. The integrals over $x$ then become integrals over $q$, weighted by $\rho(q)$. The eigenvalue density must be positive, and by definition is normalized to

$$\int_{-\pi}^{\pi} dq \rho(q) = 1 .$$

(4)

Polyakov loops are traces of powers of the thermal Wilson line,

$$\ell_j = \frac{1}{N} \text{tr} L^j = \int_{-\pi}^{\pi} dq \rho(q) \cos(2\pi j q) .$$

(5)

As noted before, the first Polyakov loop, $\ell_1$, is that in the fundamental representation. For $j \geq 2$, the relationship of the $\ell_j$ to Polyakov loops in irreducible representations is more involved [5], but all $\ell_j$ are gauge invariant, and so physical quantities.

By a global $O(2)$ rotation we can assume that the expectation value of $\ell_1$ is real. Consequently, we take $\rho(q)$ to be even in $q$, $\rho(q) = \rho(-q)$. Anticipating the results, we also assume that the integral over $q$ does not run the full range from $-\frac{1}{2}$ to $\frac{1}{2}$, but only over a limited range, from $-q_0$ to $+q_0$.

Going to integrals over $q$, we can take out overall factors of $N^2$ from the potentials, with $\tilde{V}_n(q) = N^2 V_n(q)$, where

$$V_n(q) = \int dq \int dq' \rho(q) \rho(q') |q - q'|^n (1 - |q - q'|)^n .$$

(6)

In this expression and henceforth, all integrals over $q$ run from $-q_0$ to $+q_0$, as in Eqs. (4) and (5).

We then define $\tilde{V}_{\text{eff}}(q) = N^2 V_{\text{eff}}(q)$, where $V_{\text{eff}} = -d_1 V_1 + d_2 V_2$. Solving the model at infinite $N$, then, is just a matter of finding the (minimal) stationary point of $V_{\text{eff}}(q)$ with respect to the $q_i$’s.

The equations of motion follow by differentiating the potential in Eq. (2) with respect to $q_i$, and then taking the large $N$ limit. Doing so, we find

$$0 = [d_1 + d_2] q - \frac{d_1}{2} \int dq' \rho(q') \text{sign}(q - q') + d_2 \int dq' \rho(q') \left[-3(q - q')|q - q'| + 2(q - q')^3\right] ,$$

(7)

where sign$(x) = \pm 1$ for $x \geq 0$. For simplicity we write $d_1(T)$ and $d_2(T)$ just as $d_1$ and $d_2$.

To solve the equation of motion in Eq. (7), we follow Jurkiewicz and Zalewski [16] and use the following trick. What is difficult is that Eq. (7) is an integral equation for $\rho(q)$. To reduce this to a differential equation, take $\partial/\partial q$ of Eq. (7),

$$0 = d_1 + d_2 - d_1 \rho(q)$$

(8)

$$+ 6 d_2 \int dq' \rho(q') \left[-(q - q') \text{sign}(q - q') + (q - q')^2\right] .$$

Notice that this does not give us the second variation of the potential with respect to an arbitrary variation of $q$, which is related to the mass squared. Instead, we take the derivative of the equation of motion, with respect to a solution of the same.

We then continue until we eliminate any integral over $q'$. Taking $\partial/\partial q$ of Eq. (8) gives

$$d_1 \frac{d\rho(q)}{dq} = 6 d_2 \int dq' \rho(q') \left[-\text{sign}(q - q') + 2(q - q')\right] .$$

(9)

Lastly, by taking one final derivative, we obtain

$$\frac{d^2}{dq^2} \rho(q) + d_2 \left[\rho(q) - 1\right] = 0 .$$

(10)

In this expression we introduce the ratio

$$d_2(T) = \frac{12}{d_1(T)} .$$

(11)

We assume that like the solution at small $N$ [7–9], that $d(T)$ increases with $T$, and $d(T) \to \infty$ as $T \to \infty$. We note that the only detailed property of $d(T)$ which we require is that its expansion about $T_d$ is linear in $T - T_d$. This is a minimal assumption which is standard in mean field theory.

We thus need to solve Eqs. (7) - (10), subject to the condition of Eq. (4). The solution of Eq. (10) is trivial,

$$\rho(q) = 1 + b \cos(d q) , \quad q : -q_0 \to q_0 ,$$

(12)

where $b$ is a constant to be determined. We assume that $\rho(q) = 0$ for $|q| > q_0$. We have checked numerically that a multi gap solution [16], where $\rho(q) \neq 0$ over a set of gaps in $q$, does not minimize the potential; see the discussion at the end of Sec. (III).

When $q_0 < \frac{1}{2}$, $\rho(q_0) \neq 0$, and the solution drops discontinuously to zero at the endpoints. This stepwise discontinuity is characteristic of the model, and presumably reflects the singularities from the absolute values in the potential.
The eigenvalue density in Eq. (12) is simpler than that in the Gross-Witten model [11, 12, 14–16], where
\[
\rho_{GW}(q) = \frac{1}{2} \cos(\pi q) \left[ 1 - \frac{\sin^2(\pi q)}{\sin^2(\pi q_0)} \right]^{1/2} .
\] (13)
For any \( q_0 \), this vanishes at the endpoints, \( \rho_{GW}(\pm q_0) = 0 \), while at the transition, \( q_0 = \frac{1}{2} \). Due to the Vandermonde determinant in the potential, the density \( \rho_{GW}(q) \) has a nontrivial analytic structure in the complex \( q \)-plane, while \( \rho(q) \) does not. Since the Vandermonde potential is so different from \( V_{GW} \), though, it is natural to find that \( \rho_{GW}(q) \) is unlike \( \rho(q) \) in Eq. (12).

Eq. (12) solves Eq. (9) without further constraint. To solve the remaining equations, remember that all integrals run from \(-q_0 \) to \( q_0 \). The normalization condition of Eq. (4) gives \( b \sin(d q_0) = d \left( \frac{1}{2} - q_0 \right) \). After some algebra, one can show that Eqs. (7) and (8) are equivalent, with the solution
\[
\cot(d q_0) = \frac{d}{3} \left( \frac{1}{2} - q_0 \right) - \frac{1}{d (1/2 - q_0)} , \quad (14)
\]
and
\[
b^2 = \frac{d^4}{9} \left( \frac{1}{2} - q_0 \right)^4 + \frac{d^2}{3} \left( \frac{1}{2} - q_0 \right)^2 + 1 . \quad (15)
\]
Thus in the end, we only have to solve two coupled algebraic equations, Eqs. (14) and (15), for \( q_0 \) and \( b \) as functions of \( d = d(T) \).

At low temperature, \( d \) is small, and the theory is in the confined phase, where \( b = 0 \) and \( q_0 = \frac{1}{2} \). The eigenvalue density is constant, \( \rho(q) = 1 \), and all Polyakov loops vanish, \( \ell_j = 0 \). Thus the confined phase is characterized by the maximal repulsion of eigenvalues. The Gross-Witten model also has a constant eigenvalue density in the confined phase, which is expected, as only a constant eigenvalue density gives \( \ell_j = 0 \) for all loops.

In the limit of high temperature \( d \to \infty \). The solution is \( q_0 = 6/d^2 \) and \( b = d^2/12 \). The eigenvalue density is \( \rho \approx d^2/12 \), which becomes a delta-function \( \delta q \) for infinite \( d \). That is, at high temperatures all eigenvalues coalesce into the origin, and all Polyakov loops equal one, \( \ell_j = 1 \).

As the temperature and so \( d(T) \) is lowered, the transition occurs when \( q_0 = \frac{1}{2} \), for which \( d(T_d) = 2 \pi \). At the transition point, the eigenvalue density is
\[
\rho(q) = 1 + \cos(2 \pi q) \quad ; \quad T = T_d .
\] (16)
From Eq. (5),
\[
\ell_1(T_d^+ ) = \frac{1}{2} , \quad \ell_j(T_d) = 0 , j \geq 2 . \quad (17)
\]
Thus at the transition, only the Polyakov loop in the fundamental representation is nonzero, equal to \( \frac{1}{2} \).

What is unforeseen is that at \( T_d^+ \), the eigenvalue density in the present model, Eq. (16), coincides identically with that in the Gross-Witten model, Eq. (13). Consequently, properties exactly at \( T_d^+ \), such as the expectation values of the \( \ell_j \), are the same in the two models. Since they differ away from \( T_d \), other properties are similar, but not necessarily identical.

Consider the behavior in the deconfined phase just above the transition point, taking \( d = 2 \pi (1 + \delta d) \). The solution is \( q_0^0 = \frac{1}{2} (1 - \delta d) \), where
\[
\delta q = \frac{45}{\pi^4} \delta d^{1/5} + \frac{1}{7} \left( \frac{375}{\pi^2} \right)^{1/5} \delta d^{3/5} + \frac{25}{49} \delta d + \ldots . \quad (18)
\]

Using this, one finds that
\[
\ell_1 = \frac{1}{2} + \frac{1}{4} \left( \frac{25 \pi^2}{3} \right)^{1/5} \delta d^{2/5} + \ldots , \quad (20)
\]
while all \( \ell_j \sim \delta d \) for \( j \geq 2 \).

That is, near the transition \( \ell_1(T) \) exhibits a power like behavior which is characteristic of a second order phase transition — although \( \ell_1(T_d^+ ) \neq 0 \). For arbitrary \( d \), after some algebra one finds that at \( q_0^0 \), the solution of Eqs. (14) and (15), the potential equals
\[
V_{eff}(q_0^0) - V_{eff}^{conf} = -d^2 \left( \frac{16}{15} \left( \frac{5}{2} - q_0 \right) \right)^5 . \quad (22)
\]
The potential in the confined phase is \( V_{eff}^{conf} = V_{eff}(\frac{5}{2}) = -d_1/6 + d_2/30 \). In these matrix models, the pressure is
\[
p(T) = -V_{eff}(q_0^0) + V_{eff}^{conf} . \quad (23)
\]
This subtraction ensures that the pressure, and the associated energy density, are suppressed by \( \sim 1/N^2 \) in the confined phase. In the models of Refs. [7–9], the leading term in Eq. (24) \( \sim \delta d \) shows that the first derivative of the pressure with respect to temperature, which is related to the energy density \( e(T) \), is nonzero at \( T_d^+ \). Since the pressure and the energy density are suppressed by \( \sim 1/N^2 \) in the confined phase, the latent heat is nonzero and \( \sim N^2 \sim e(T_d) \).
Using the explicit forms for $d_1(T)$ and $d_2(T)$, we find that the latent heat is $e(T_d^+)/\left(N^2 T_d^2\right) = 1/\pi^2 \sim 10\ldots$. This is about four times smaller than the lattice results of Ref. [2] who find $\sim 0.39$ for the same quantity. The lattice results can be accomodated by adding a term like a MIT bag constant to the model [8]. Such a term is $\sim T_d^2$ but independent of the $q$'s, and so only changes the latent heat, but does not affect any other result.

The second term in Eq. (24) shows that the second derivative of the pressure with respect to temperature diverges as $T \to T_d^+$,

$$\frac{\partial^2}{\partial T^2} p(T) \sim \frac{1}{(T - T_d)^\alpha}, \quad \alpha = \frac{3}{5}. \quad (25)$$

This is the usual divergence of the specific heat for a second order phase transition.

**II. NONZERO BACKGROUND FIELD, $T = T_d$**

Background fields can be added for each loop $\ell_j$. In this paper we just consider a background field for the simplest loop, $\ell_1$, since only that is nonzero at $T_d$, Eq. (17). We add

$$V_h(q) = -\frac{d_1}{(2\pi)^2} h \ell_1 \quad (26)$$

to the potential $V_{\text{eff}}(q)$, and find the solution as before. After taking three derivatives of the equation of motion, with respect to a solution, we obtain the analogy of Eq. (10),

$$\frac{d^2}{dq^2} \rho(q) + d^2 [\rho(q) - 1] + (2\pi)^2 h \cos(2\pi q) = 0. \quad (27)$$

This equation is valid for any $d$. It is necessary to treat the case of $T_d$, where $d = 2\pi$, seperately from $T \neq T_d$.

In this section we consider the point of phase transition, where $d = 2\pi$. The solution of Eq. (27) is

$$\rho(q) = 1 + b \cos(2\pi q) - \pi q \sin(2\pi q), \quad (28)$$

where $q : -q_0 \to q_0$. Notice that the $h$-dependent term $q \sin(2\pi q)$ arises because when $T = T_d$, Eq. (27) represents a driven oscillator at the resonance frequency. The value of the constants $b$ and $q_0$ now depend upon both $d(T)$ and the background field, $h$.

The analogy of Eq. (9) is solved by Eq. (28). The normalization condition, Eq. (4), plus the analogy of Eq. (8), gives two equations for $b$ and $q_0$; as before, Eq. (7) does not give a new condition.

When $b \neq 0$, the explicit form of the analogy of Eq. (4) is elementary, but that of Eq. (8) is rather ungainly. We thus present the results of the solution in the limit of small background field, $h \ll 1$. We find that $q_0^d = \frac{1}{2}(1 - \delta q)$, where

$$\delta q = \left(\frac{45}{2 \pi^4}\right)^{1/5} h^{1/5} + \frac{3}{14} \left(\frac{3}{200 \pi^2}\right)^{1/5} h^{3/5} + \ldots \quad (29)$$

and

$$b = 1 + \frac{1}{2} \left(\frac{25}{12}\right)^{1/5} h^{2/5} + \frac{39}{56} \left(\frac{27}{80}\right)^{1/5} h^{4/5} + \ldots \quad (30)$$

For this solution, at the minimum the $h$-dependence of the potential is

$$V_{\text{eff}}(q_0^d, h) = -\frac{d_1}{8\pi^2} h + \frac{d_1}{112\pi} \left(\frac{25}{12}\right)^{1/5} h^{7/5} + \ldots. \quad (31)$$

The expectation value of the loop $\ell_1$ is

$$\ell_1 = \frac{1}{2} \left(\frac{25}{12}\right)^{1/5} h^{2/5} + \frac{39}{112} \left(\frac{27}{80}\right)^{1/5} h^{4/5} + \ldots. \quad (32)$$

Hence $\ell_1 - \frac{1}{2} \sim h^{1/2}$, where $\delta = 5/2$. This shows that the critical exponents of this model satisfy the usual Griffiths scaling relation,

$$2 - \alpha = \beta(1 + \delta). \quad (33)$$

The effective potential, as a function of $\ell_1$, is computed by taking the Legendre transform,

$$\Gamma(\ell_1) = V_{\text{eff}}(h) + \frac{d_1}{(2\pi)^2} h_1 \ell_1. \quad (34)$$

Expanding the potential in $\delta \ell_1 = \ell_1 - \frac{1}{2}$ at $T_d^+$,

$$\Gamma(\ell_1) = \frac{128}{35 \pi^3} \delta \ell_1^{7/2} + \frac{32 d_1}{5} \delta \ell_1 + \ldots \quad (35)$$

This is a very flat potential, starting only as $(\ell_1 - \frac{1}{2})^{7/2}$. This is in contrast to the femtosphere, where the potential behaves as $\sim (\ell_1 - \frac{1}{2})^3$ about the similar point [11, 12].

Expanding at $T_d^-$ gives the expansion of the potential about $\ell_1 = 0$. One can show, and we verify in the next section, that this potential vanishes. This implies that the potential has an unusual form: it is zero from $\ell_1 : 0 \to \frac{1}{2}$, and then turns on as in Eq. (35). Graphically, this potential is like that on the femtosphere; see, e.g., Fig. (1) of Ref. [12].

**III. NONZERO BACKGROUND FIELD, $T \neq T_d$**

Consider now the theory in a nonzero background field for $\ell_1$, Eq. (26), away from the transition, so $d \neq 2\pi$. The eigenvalue density again solves Eq. (27). The solution is simpler when $d \neq 2\pi$, and is just the sum of the solution for $h = 0$ and an $h$-dependent term,

$$\rho(q) = 1 + b \cos(d q) + \frac{1}{1 - (d/2\pi)^2} h \cos(2\pi q). \quad (36)$$

The solution follows as previously, and we simply summarize the results.
We first consider the confined phase, defined to be the solution for which \( q_0 = \frac{1}{2} \) and \( b = 0 \). The expectation value of the loop \( \ell_1 \) is
\[
\ell_1 = \frac{1}{1 - (d/2\pi)^2} \frac{h}{2}.
\]
(37)

For this solution the potential equals
\[
V_{\text{eff}}^\text{conf}(h) - V_{\text{eff}}^\text{conf} = + \frac{1}{1 - (d/2\pi)^2} \frac{h^2}{8\pi^2}.
\]
(38)

Performing the Legendre transformation, we find
\[
\Gamma(\ell_1) = \left(1 - \frac{d^2}{4\pi^2}\right) \frac{1}{\pi^2} \ell_1^2.
\]
(39)

This shows that in the confined phase, when \( d < 2\pi \) the mass squared of the \( \ell_1 \) loop is positive, as expected. It also shows that this mass vanishes at \( T_d \) when \( h = 0 \); this justifies the statements about the potential at the end of the previous section.

Consider a special value of \( d \), \( d_h^2 = 4\pi^2(1 - h) \); the corresponding temperature is defined to be \( T_h \), \( d(T_h) = d_h \). At this temperature, the eigenvalue density of Eq. (36) coincides exactly with that at the transition in zero background field, Eq. (16). Notably, the values of the loop at \( h \neq 0 \) and \( T = T_h \) are the same as for \( h = 0 \) and \( T = T_d \): \( \ell_1(T_h) = \ell_1(T_d) = \frac{1}{2} \), with \( \ell_j = 0 \) for \( j \geq 2 \), Eq. (17). Thus we may suspect that something special happens at \( T = T_h \). For example, the confined phase is only an acceptable solution when \( T < T_h \), as only then is the eigenvalue density positive definite.

This suggests that a phase transition occurs at \( d_h \). To show this, we compute for about this value of \( d \), taking \( d^2 = d_h^2 + 4\pi^2 h \delta d \). Solving the model as before, in the deconfined phase the solution is \( q_0^2 = \frac{1}{2}(1 - \delta q) \), where
\[
\delta q = \frac{1}{\pi} \left(\frac{3}{2}\right)^{1/2} \delta d^{1/2} + \frac{\sqrt{6}}{40\pi}(8h - 5) \delta d^{3/2} + \ldots
\]
(40)

\[
b = -\frac{4}{5} \sqrt{6} (1 - h)^{3/2} \csc(\sqrt{1 - h} \pi) \delta d^{5/2} + \ldots
\]
(41)

With this results we compute the potential in the deconfined phase, to find
\[
V_{\text{eff}}(h) - V_{\text{eff}}^\text{conf}(h) = \frac{3\sqrt{6}}{5\pi^3} \delta d^{3/2} + \ldots
\]
(42)

Taking \( \delta d \sim T_h - T \), we find that the third derivative of the pressure, with respect to temperature, diverges at \( T_h \),
\[
\frac{\partial^3}{\partial T^3} p(T) \sim \frac{1}{(T - T_h)^{1/2}}, \quad T \to T_h^+.
\]
(43)

In zero background field, then, there is a critical first order transition at a temperature \( T_d \). Turning on a background field \( \sim h \ell_1 \), the first order transition is immediately wiped out for any \( h \neq 0 \). Even so, there remains a third order phase transition, at a temperature \( T_h < T_d \), where the expectation value of the loop \( \ell_1 = \frac{1}{2} \). This behavior is the same as on a femtosphere [11, 12].

In principle one can also add a background field for any loop, \( \ell_j \) for \( j \geq 2 \). It is direct to derive the equations of motion and obtain a solution for the eigenvalue density. Obtaining the minimum of the potential is not elementary, though. The original model of Gross and Witten [14] involves the Vandermonde determinant plus a term \( \sim |\text{tr} L|^2 \). The solution for the eigenvalue density is a function which is nonzero on one interval, between \(-q_0 \) and \( q_0 \). Jurkiewicz and Zalewski [16] showed that when terms such as \( |\text{tr} L|^2 \) are added to the Gross-Witten model, that in general it involves functions which are nonzero on more than one interval. We have checked numerically that when only \( h_1 \neq 0 \), that such multi-gap solutions do not minimize the potential. We do find, however, that multi-gap solutions do minimize the potential in the presence of background fields for \( \ell_j \) when \( j \geq 2 \). Since only \( \ell_1 \neq 0 \) at \( T_d \) and \( T_h \), we defer the problem of background \( \ell_j \) for \( j \geq 2 \).

### IV. FINITE N

The model can be solved numerically at finite \( N \). This confirms, as expected on general grounds [8], that the deconfining transition is of first order for any \( N \geq 3 \). It also shows that the critical behavior found at infinite \( N \) is smoothed out at large but finite \( N \).

Using the numerical solution of the model, in the Figure we show the behavior of the specific heat, divided by \( N^2 - 1 \), for different values of \( N \). To see the putative divergence of the specific heat at infinite \( N \), rather large values of \( N \) are necessary, \( N \geq 40 \).

This Figure also shows that the increase in the specific heat only manifests itself very close to the transition, within \( \sim 0.2\% \) of \( T_d \). At present, direct numerical simulations on the lattice treat moderate values of \( N \sim 4 - 10 \) [2]. For most quantities there seems to be a weak variation with \( N \).

The present matrix model suggests that very near \( T_d \), a novel phase transition may occur at large \( N \). The values of \( N \) at which critical first order behavior arise can presumably be studied only in reduced models [3].

This begs the question of whether or not the Gross-Witten-Wadia transition does in fact occur at infinite \( N \) [18]. On the femtosphere, one can easily solve the model in the presence of additional couplings, such as \( |\text{tr} L|^2 \). Such additional couplings turn the Gross-Witten-Wadia transition into an ordinary first order transition [12]. We have not been able to solve the present model in the presence of additional couplings.

The most likely possibility is that as on a femtosphere, the presence of additional couplings washes out the Gross-Witten-Wadia transition. Nevertheless, gauge theories are remarkable things. Certainly it is worth studying \( SU(N) \) gauge theories, at very large values of
FIG. 1. Plot of the specific heat, divided by $(N^2 - 1)T^3$, for different values of $N$.

To see if there is a Gross-Witten-Wadia transition at infinite $N$.

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The original analysis of Gross and Witten [14, 15] considered a lattice gauge theory in 1 + 1 dimensions. For the Wilson action with lattice coupling $\beta$, there is a third order phase transition with respect to $\beta$. Different lattice actions give a variety of phase transitions, but again with respect to the lattice couplings [16]. This is in contrast to the femtosphere (at infinite $N$) or the present model (at any $N$), which are true thermodynamic phase transitions with respect to temperature.

It is also possible to develop a matrix model to study deconfinement in three spacetime dimensions [19]. This involves different functions of $q$ than those in four dimensions, Eq. (3). It is not clear if this model is soluble analytically at infinite $N$, or if it exhibits a Gross-Witten-Wadia transition in that limit.

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