DATA STRUCTURES FOR REAL MULTIPARAMETER
PERSISTENCE MODULES

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ABSTRACT. A theory of modules over posets is developed to define computationally feasible, topologically interpretable data structures, in terms of birth and death of homology classes, for persistent homology with multiple real parameters. To replace the noetherian hypothesis in the general setting of modules over posets, for theoretical as well as computational purposes, a finitely encoded condition is defined combinatorially and developed algebraically. The finitely encoded hypothesis captures topological tameness of persistent homology, and poset-modules satisfying it can be specified by fringe presentations that reflect birth-and-death descriptions of persistent homology.

The homological theory of modules over posets culminates in a syzygy theorem characterizing finitely encoded modules as those that admit finite presentations or resolutions by direct sums of upset modules or downset modules, which are analogues over posets of flat and injective modules over multigraded polynomial rings.

The geometric and algebraic theory of modules over posets focuses on modules over real polyhedral groups (partially ordered real vector spaces whose positive cones are polyhedral), with a parallel theory over discrete polyhedral groups (partially ordered abelian groups whose positive cones are finitely generated) that is simpler but still largely new in the generality of finitely encoded modules. Existence of primary decomposition is proved over arbitrary polyhedral partially ordered abelian groups, but the real and discrete cases carry enough geometry and, crucially in the real case, topology to induce complete theories of minimal primary and secondary decomposition, associated and attached faces, minimal generators and cogenerators, socles and tops, minimal upset covers and downset hulls, Matlis duality, and minimal fringe presentation. In particular, when the data are real semialgebraic, that property is preserved by functorial constructions. And when the modules in question are subquotients of the group itself, minimal primary and secondary decompositions are canonical.

Tops and socles play the roles of functorial birth and death spaces for multiparameter persistence modules. They yield functorial QR codes and elder morphisms for modules over real and discrete polyhedral groups that generalize and categorify the bar code and elder rule for persistent homology in one parameter. The disparate ways that QR codes and elder morphisms model bar codes coalesce, in ordinary persistence with one parameter, to a functorial bar code.

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1. Introduction

Families of topological spaces in data analysis often arise from filtrations: collections of subspaces of a single topological space. Inclusions of subspaces induce a partial order on such collections. Applying the homology functor then yields a commutative diagram $M$ of vector spaces indexed by the partially ordered set $Q$ of subspaces. This $Q$-module $M$ is called the persistent homology of the filtration, referring to how classes are born, persist for a while, and then die as the parameter moves up in the poset $Q$.

Ordinary persistent homology, in which $Q$ is totally ordered—usually the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, or a subset $\{1, \ldots, m\}$—is well studied; see [EH10], for example. Persistence with multiple parameters was introduced by Carlsson and Zomorodian.
[CZ09] for $Q = \mathbb{N}^n$, and it has been developed since then in various ways, assuming that the module $M$ is finitely generated. In contrast, the application that drives the developments here has real parameters, and it fails to be finitely generated in other fundamental ways. It is therefore the goal here to define computationally feasible, topologically interpretable data structures, in terms of birth and death of homology classes, for persistent homology with multiple real parameters.

The data structures are made possible by a finitely encoded hypothesis that captures topological tameness of persistent homology. A syzygy theorem characterizes finitely encoded modules as those that admit appropriately finite presentations and resolutions, all amenable to computation. The technical heart of the paper is a development of basic commutative algebra—especially socles and minimal primary decomposition—for modules over the poset $\mathbb{R}^n$, or equivalently, multigraded modules over rings of polynomials with real instead of integer exponents. This algebraic theory yields functorial QR codes and elder morphisms for modules over real and discrete polyhedral groups that generalize and categorify the bar code and elder rule for ordinary single-parameter persistence. The disparate ways that QR codes and elder morphisms model bar codes coalesce, in ordinary persistence with one parameter, to make ordinary bar codes functorial.

1.1. Acknowledgements. First, special acknowledgements go to Ashleigh Thomas and Justin Curry. Both have been and continue to be long-term collaborators on this project. They were listed as authors on earlier drafts, but their contributions lie more properly beyond these preliminaries, so they declined in the end to be named as authors on this installment. Early in the development of the ideas here, Thomas put her finger on the continuous rather than discrete nature of multiparameter persistence modules for fly wings. She computed the first examples explicitly, namely those in Example 1.3, and produced the biparameter persistence diagrams there. And she suggested the term “QR code” (Remark 16.4). Curry contributed, among many other things, clarity and intuition regarding the topology of endpoints—the limits defining upper and lower boundary functors—as well as regarding the elder rule. He also pointed out connections from the combinatorial viewpoint taken here, in terms of modules over posets, to higher notions in algebra and category theory, particularly those involving constructible sheaves, which are in the same vein as Curry’s proposed uses of them in persistence [Cur14]; see Remarks 2.4, 2.7, 2.12, and 5.14.

The author is indebted to David Houle, whose contribution to this project was seminal and remains ongoing; in particular, he and his lab produced the fruit fly wing images. Paul Bendich and Joshua Cruz took part in the genesis of this project, including early discussions concerning ways to tweak persistent (intersection) homology for the fly wing investigation. Banff International Research Station provided an opportunity for valuable feedback and suggestions at the workshop there on Topological Data Analysis (August, 2017) as this research was being completed; many participants, especially the organizers, Uli Bauer and Anthea Monod, as well as Michael Lesnick, shared important perspectives and insight. Thomas Kahle requested that Proposition 4.7
be an equivalence instead of merely the one implication it had stated. Hal Schenck
gave helpful comments on the Introduction. Some passages in Section 1.2 are taken
verbatim, or nearly so, from [Mil15].

1.2. Biological origins. This investigation of data structures for real multiparameter
persistence modules intends both senses of the word “real”: actual—from genuine data,
with a particular dataset in mind—and with parameters taking continuous as opposed
to discrete values. Instead of reviewing the numerous possible reasons for considering
multiparameter persistence, many already having been present from the outset [CZ09,
Section 1.1], what follows is an account of how real multiparameter persistence arises
in the biological problem that the theory here is specifically designed to serve.

The normal *Drosophila melanogaster* fruit fly wing depicted on the left differs from
the abnormal other two in topology as well as geometry. The dataset, as provided by
biologist David Houle [Hou03] as part of ongoing work with his lab on the ideas here,
presents the veins in each wing as an embedded planar graph, with a location for each
vertex and an algebraic curve for each arc. The graph in the middle has an extra edge,
and hence two extra vertices, while the graph on the right is lacking an intersection.
These topological variants, along with many others, occur in natural *D. melanogaster*
populations, but rarely. On the other hand, different species of *Drosophila* exhibit a
range of wing vein topologies. How did that come to be? Wing veins serve several
key purposes, as structural supports as well as conduits for airways, nerves, and blood
cells [Bla07]. Is it possible that some force causes aberrant vein topologies to occur
more frequently than would otherwise be expected in a natural population—frequently
enough for evolutionary processes to take over?

Waddington [Wad53] famously observed hereditary topological changes in wing vein
phenotype (loss of crossveins) after breeding flies selected for crossveinless phenotype
due to embryonic heat shock. Results generated by Weber, and later with more power
by Houle’s lab, show that selecting for continuous wing deformations results in skews
toward deformed wings with normal vein topology [Web90, Web92, Hou03]. But this
selection also unexpectedly yields higher rates of topological novelty; this finding, as yet
unpublished and not yet precisely formulated, is what needs to be tested statistically.

There are many options for statistical methods to test the hypothesis, some of them
elementary, such as a linear model taking into account a weighted sum of (say) the
number of vertices and the total edge length. Whatever the chosen method, it has to
grapple with the topological vein variation, giving appropriate weight to new or deleted
singular points in addition to varying shape. Real multiparameter persistence in its
present form was conceived to serve the biology in this way, but the problem has since turned around: fly wings supply a testing ground for the feasibility and effectiveness of multiparameter persistent homology as a statistical tool.

Example 1.1. Let $Q = \mathbb{R}_- \times \mathbb{R}_+$ with the coordinatewise partial order, so $(r, s) \in Q$ for any nonnegative real numbers $-r$ and $s$. Let $X = \mathbb{R}^2$ be the plane in which the fly wing is embedded and define $X_{rs} \subseteq X$ to be the set of points at distance at least $-r$ from every vertex and within $s$ of some edge. Thus $X_{rs}$ is obtained by removing the union of the balls of radius $r$ around the vertices from the union of $s$-neighborhoods of the edges. In the following portion of a fly wing, $-r$ is approximately twice $s$:

![Diagram of a fly wing](image)

The biparameter persistent homology module $M_{rs} = H_0(X_{rs})$ summarizes wing vein structure for our purposes. And the nature of wing vein formation from gene expression levels during embryonic development (see [Bla07] for background) gives reason to believe that this type of persistence models biological reality reasonably faithfully.

Remark 1.2. The parameters in Example 1.1 govern intersections between edges and vertices. The biparameter persistence here can therefore be viewed as a multiscale generalization of persistent intersection homology [BH11] in which interactions among strata are tuned by relations among the parameters.

1.3. Encoding modules over arbitrary posets. Biparameter persistence can only serve as an effective summary of a fly wing for statistical purposes if it can be computed from the initial spline data. In general, computation with persistent homology is only conceivable in the presence of some finiteness condition on the $Q$-module or tameness on the topology that gives rise to it. The prior standard for finiteness in multiparameter persistence has been the setting where $Q = \mathbb{N}^n$ and the $Q$-module is finitely generated. Those conditions quickly fail for fly wings.

Example 1.3. Using the setup from Example 1.1, the zeroth persistent homology for the toy-model “fly wing” at left in Figure 1 is the $\mathbb{R}^2$-module $M$ shown at center. Each point of $\mathbb{R}^2$ is colored according to the dimension of its associated vector space in $M$, namely 3, 2, or 1 proceeding up (increasing $s$) and to the right (increasing $r$). The structure homomorphisms $M_{rs} \to M_{r's'}$ are all surjective. This $\mathbb{R}^2$-module fails to be finitely presented for three fundamental reasons: first, the three generators sit infinitely far back along the $r$-axis. (Fiddling with the minus sign on $r$ does not help: the natural maps on homology proceed from infinitely large radius to 0 regardless of how the picture is drawn.) Second, the relations that specify the transition from vector spaces of dimension 3 to those of dimension 2 or 1 lie along a real algebraic curve, as
do those specifying the transition from dimension 2 to dimension 1. These curves have uncountably many points. Third, even if the relations are discretized—restrict $M$ to a lattice $\mathbb{Z}^2$ superimposed on $\mathbb{R}^2$, say—the relations march off to infinity roughly diagonally away from the origin. (The remaining image is explained in Example 1.4.)

To replace the noetherian hypothesis in the setting of modules over arbitrary posets, for theoretical as well as computational purposes, the finitely encoded condition is introduced combinatorially (Definition 2.6). It stipulates that the module $M$ should be pulled back from a $P$-module along a poset morphism $Q \to P$ in which $P$ is a finite poset and the $P$-module has finite dimension as a vector space over the field $k$.

**Example 1.4.** The right-hand image in Example 1.3 is a finite encoding of $M$ by a three-element poset $P$ and the $P$-module $H = k^3 \oplus k^2 \oplus k$ with each arrow in the image corresponding to a full-rank map between summands of $H$. Technically, this is only an encoding of $M$ as a module over $Q = \mathbb{R}_- \times \mathbb{R}_+$. The poset morphism $Q \to P$ takes all of the yellow rank 3 points to the bottom element of $P$, the olive rank 2 points to the middle element of $P$, and the blue rank 1 points to the top element of $P$. (To make this work over all of $\mathbb{R}^2$, the region with vector space dimension 0 would have to be subdivided, for instance by introducing an antidiagonal extending downward from the origin, thus yielding a morphism from $\mathbb{R}^2$ to a five-element poset.) This encoding is semialgebraic (Definition 2.9): its fibers are real semialgebraic sets.

The finitely encoded hypothesis captures topological tameness of persistent homology in situations from data analysis, making precise what it means for there to finitely many topological transitions as the parameters vary. But there is nuance: the isotypic regions (Definition 2.23), on which the homology remains constant, need not be situated in a manner that makes them the fibers of a poset morphism (Example 2.24). Nonetheless, over arbitrary posets, modules with finitely many isotypic regions always admit finite encodings (Theorem 2.33), although the isotypic regions are typically subdivided by the encoding poset morphism. In the case where the poset is a real vector space, if the isotypic regions are semialgebraic then a semialgebraic encoding is possible.

In ordinary totally ordered persistence, finitely encoded means simply that the bar code should have finitely many bars: the poset being finite precludes infinitely many non-overlapping bars (the bar code can’t be “too long”), while the vector space having
finite dimension precludes a parameter value over which lie infinitely many bars (the bar code can’t be “too thick”).

Finite encoding has its roots in combinatorial commutative algebra in the form of sector partitions [HM05] (or see [MS05, Chapter 13]). Like sector partitions, finite encoding is useful, theoretically, for its enumeration of all topologies encountered as the parameters vary. However, enumeration leads to combinatorial explosion outside of the very lowest numbers of parameters. And beyond its inherent inefficiency, poset encoding lacks many of the features that have come to be expected from persistent homology, including the most salient: a description of homology classes in terms of their persistence, meaning birth, death, and lifetime.

1.4. Discrete persistent homology by birth and death. The perspective on finitely generated \( \mathbb{Z}^n \)-modules arising from their equivalence with multiparameter persistence is relatively new to commutative algebra. Initial steps have included descriptions of the set of isomorphism classes [CZ09], presentations [CSV14] and algorithms for computing [CSZ09, CSV12] or visualizing [LW15] them, as well as interactions with homological algebra of modules, such as persistence invariants [Knu08] and certain notions of multiparameter noise [CLR+15].

Algebraically, viewing persistent homology as a module rather than (say) a diagram or a bar code, a birth is a generator. In ordinary persistence, with one parameter, a death is more or less the same as a relation. However, in multiparameter persistence the notion of death diverges from that of relation. The issue is partly one of geometric shape in the parameter poset, say \( \mathbb{Z}^n \) (the shaded regions indicate where classes die):

If death is to be dual to birth, then a nonzero homology class at some parameter should die if it moves up along any direction in the poset. Birth is not the bifurcation of a homology class into two independent ones; it is the creation of a new class from zero. Likewise, genuine death is not the joining of two classes into one; it is annihilation. And death should be stable, in the sense that wiggling the parameter and then pushing up should still kill the homology class.

In algebraic language, death is a cogenerator rather than a relation. For finitely generated \( \mathbb{N}^n \)-modules, or slightly more generally for finitely determined modules (Example 2.8 and Definition 4.1), cogenerators are irreducible components, cf. [MS05, Section 5.2]. Indeed, irreducible decomposition suffices as a dual theory of death in the finitely generated case; this is more or less the content of Theorem 4.33. The idea there is that surjection from a free module covers the module by sending basis elements to
births in the same (or better, dual) way that inclusion into an injective module envelops
the module by capturing deaths as injective summands. The geometry of this process
in the parameter poset on the injective side is as well understood as it is on the free side
[MS05, Chapter 11], and in finitely generated situations it is carried out theoretically or
algorithmically by finitely generated truncations of injective modules [Mil02, HM05].

Combining birth by free cover and death by injective hull leads naturally to flange
presentation (Definition 4.26), which composes the augmentation map $F \to M$ of a
flat resolution with the augmentation map $M \leftarrow E$ of an injective resolution to get
a homomorphism $F \to E$ whose image is $M$. The indecomposable summands of $F$
capture births and those of $E$ deaths. Flange presentation splices a flat resolution to
an injective one in the same way that Tate resolutions (see [Coa03], for example) segue
from a free resolution to an injective one over a Gorenstein local ring of dimension 0.

Why a flat cover $F \to M$ instead of a free one? There are two related reasons:
first, flat modules are dual to injective ones (Remark 4.20), so in the context of finitely
determined modules the entire theory is self-dual; and second, births can lie infinitely
far back along axes, as in the toy-model fly wing from Example 1.3.

1.5. Discrete to continuous: fringe presentation. That multiparameter persis-
tence modules can fail to be finitely generated, like Example 1.3 does, in situations
reflecting reasonably realistic data analysis was observed by Patriarca, Scolamiero,
and Vaccarino [PSV12, Section 2]. Their “monomial module” view of persistence cov-
ers births much more efficiently, for discrete parameters, by keeping track of generators
not individually but gathered together as generators of monomial ideals. Huge num-
bers of predictable syzygies among generators are swallowed: monomial ideals have
known syzygies, and there are lots of formulas for them, but nothing new is learned
from them topologically, in the persistent sense.

Translating to the setting of continuous parameters, and including the dual view
of deaths, which works just as well, suggests an uncountably more efficient way to
cover births and deaths than listing them individually. This urge to gather births or
deaths comes independently from the transition to continuous parameters from discrete
ones. To wit, any $\mathbb{R}^n$-module $M$ can be approximated by a $\mathbb{Z}^n$-module, the result of
restricting $M$ to, say, the rescaled lattice $\varepsilon\mathbb{Z}^n$. Suppose, for the sake of argument,
that $M$ is bounded, in the sense of being zero at parameters outside of a bounded
subset of $\mathbb{R}^n$; think of Example 1.3, ignoring those parts of the module there that
lie outside of the depicted square. Ever better approximations, by smaller $\varepsilon \to 0$,
yield sets of lattice points ever more closely hugging an algebraic curve. Neglecting the difficulty of computing where those lattice points lie, how is a computer to store or manipulate such a set? Listing the points individually is an option, and perhaps efficient for particularly coarse approximations, but in \( n \) parameters the dimension of this storage problem is \( n - 1 \). As the approximations improve, the most efficient way to record such sets of points is surely to describe them as the allowable ones on one side of an algebraic curve. And once the computer has the curve in memory, no approximation is required: just use the (points on the) curve itself. In this way, even in cases where the entire topological filtration setup can be approximated by finite simplicial complexes, understanding the continuous nature of the un-approximated setup is both more transparent and more efficient.

Combining flange presentation with this monomial module view of births and deaths yields fringe presentation (Definition 5.1), the analogue for modules over an arbitrary poset \( Q \) of flange presentation for finitely determined modules over \( Q = \mathbb{Z}^n \). The role of indecomposable free or flat modules is played by upset modules (Example 2.11) which have \( k \) in degrees from an upset \( U \) and 0 elsewhere. The role of indecomposable injective modules is played similarly by downset modules.

Fringe presentation is expressed by monomial matrix (Definition 5.4), an array of scalars with rows labeled by upsets and columns labeled by downsets. For example,

\[
\begin{bmatrix}
\varphi_{11}
\end{bmatrix}
\]

represents a fringe presentation of \( M = k \) as long as \( \varphi_{11} \in k \) is nonzero. The monomial matrix notation specifies a homomorphism \( k[a] \rightarrow k[A] \) whose image is \( M \), which has \( M_a = k \) over the yellow parameters \( a \) and 0 elsewhere. The blue upset specifies the births at the lower boundary of \( M \); unchecked, the classes would persist all the way up and to the right. But the red downset specifies the deaths along the upper boundary of \( M \).

When the birth upsets and death downsets are semialgebraic, or otherwise manageable algorithmically, monomial matrices render fringe presentations effective data structures for real multiparameter persistence. Fringe presentations have the added benefit of being topologically interpretable in terms of birth and death.

Although the data structure of fringe presentation is aimed at \( \mathbb{R}^n \)-modules, it is new and lends insight already for finitely generated \( \mathbb{N}^n \)-modules (even when \( n = 2 \)), where monomial matrices have their origins [Mil00, Section 3]. The context there is more or less that of finitely determined modules; see Definition 4.28, in particular, which is
really just the special case of fringe presentation in which the upsets are localizations of \(\mathbb{N}^n\) and the downsets are duals—that is, negatives—of those.

1.6. **Homological algebra of modules over posets.** Even in the case of filtrations of finite simplicial complexes by products of intervals—that is, *multifiltrations* (Example 2.5) of finite simplicial complexes—persistent homology is not naturally a module over a polynomial ring in \(n\) (or any finite number of) variables. This is for the same reason that single-parameter persistent homology is not naturally a module over a polynomial ring in one variable: though there can only be finitely many topological transitions, they can (and often do) occur at incommensurable real numbers. That said, observe that filtering a finite simplicial complex automatically induces a finite encoding. Indeed, the parameter space maps to the poset of simplicial subcomplexes of the original simplicial complex by sending a parameter to the simplicial subcomplex it represents. That is not the smallest poset, of course, but it leads to a fundamental point: one can and should do homological algebra over the finite encoding poset rather than (only) over the original parameter space.

This line of thinking culminates in a syzygy theorem (Theorem 5.16) to the effect that finitely encoded modules are characterized as those admitting, equivalently,
- finite fringe presentations,
- finite resolutions by finite direct sums of upset modules, or
- finite resolutions by finite direct sums of downset modules.

This result directly reflects the closer-to-usual syzygy theorem for finitely determined \(\mathbb{Z}^n\)-modules (Theorem 4.33), with upset and downset resolutions being the arbitrary-poset analogues of free and injective resolutions, respectively, and fringe presentation being the arbitrary-poset analogue of flange presentation.

The moral is that the finitely encoded condition over arbitrary posets appears to be the right notion to stand in lieu of the noetherian hypothesis over \(\mathbb{Z}^n\): the finitely encoded condition is robust, has separate combinatorial, algebraic, and homological characterizations, and makes algorithmic computation possible, at least in principle.

The proof of the syzygy theorem works by reducing to the finitely determined case over \(\mathbb{Z}^n\). The main point is that given a finite encoding of a module over an arbitrary poset \(Q\), the encoding poset can be embedded in \(\mathbb{Z}^n\). The proof is completed by pushing the data forward to \(\mathbb{Z}^n\), applying the syzygy theorem there, and pulling back to \(Q\).

1.7. **Geometric algebra over partially ordered abelian groups.** One of the most prominent features of multiparameter persistence that differs from the single-parameter theory is that elements in modules over \(\mathbb{R}^n\) or \(\mathbb{Z}^n\) can die in many ways. A hint of this phenomenon occurs already over one parameter: an element can die after persisting finitely—that is, its bar can have a right endpoint—or it can persist indefinitely. Over two parameters, a class could persist infinitely along the \(x\)-axis but die upon moving up sufficiently along the \(y\)-axis, or vice versa, or it could persist infinitely in both directions, or die in both directions.
Example 1.5. Let $D$ be the downset in $\mathbb{R}^2$ consisting of all points beneath the upper branch of the hyperbola $xy = 1$. Then $D$ canonically decomposes as the union

\[
\begin{array}{c}
\text{leftmost subset on the right} \cup \\
\text{middle subset} \cup \\
\text{rightmost subset}
\end{array}
\]

of its subsets that die pure deaths of some type (Theorem 3.19): every red point in the

- leftmost subset on the right dies when pushed over to the right or up far enough;
- middle subset dies in the localization of $D$ along the $x$-axis (Definition 3.12 or Definition 3.23) when pushed up far enough; and
- rightmost subset dies locally along the $y$-axis when pushed over far enough.

Generally, in multiparameter situations, elements can persist forever along any face of the positive cone and die upon exiting far enough from that face. Worse, single elements can do combinations of these things; take, for example, any nonzero element of degree $0$ in the $\mathbb{Z}^2$-module $\mathbb{k}[x, y]/\langle xy \rangle$, which persists along the $x$-axis but dies upon leaving it, while it also persists along the $y$-axis but dies upon leaving it.

These phenomena have already been discussed implicitly in prior sections: types of death in, say, the category of finitely determined $\mathbb{Z}^n$-modules correspond to isomorphism classes of indecomposable injective $\mathbb{Z}^n$-modules, which come in $2^n$ flavors, up to translation along $\mathbb{Z}^n$ (also known as $\mathbb{Z}^n$-graded shift). The injective half of a finite flange presentation places a module $M$ inside of a finite direct sum of such modules, thereby decomposing every element of $M$ as a sum of components each of which dies a pure death of one of these $2^n$ types. Gathering injective summands of the same type, every element of $M$ is a sum of elements each of which dies a pure death of some distinct type. In algebraic language, this is a primary decomposition of $M$ (Definition 3.28).

The natural setting in which to carry out primary decomposition is, in the best tradition of classical mathematics [Bir42, Cli40, Rie40], over partially ordered abelian groups (Definition 3.1). Those provide an optimally general context in which posets have some notion of “face” along which one can localize without altering the ambient poset. That is, a partially ordered group $Q$ has an origin—namely its identity $0$—and hence a positive cone $Q_+$ of elements it precedes. A face of $Q$ is a submonoid of the positive cone that is also a downset therein (Definition 3.6). And as everything takes place inside of the ambient group $Q$, every localization of a $Q$-module along a face (Definition 3.23) remains a $Q$-module; that much is needed to functorially isolate all module elements that die in only one way, which is accomplished by local support functors, as in ordinary commutative algebra and algebraic geometry (Definition 3.28). Preoccupation with the potential for algorithmic computation draws the focus to the case where $Q$ is polyhedral, meaning that it has only finitely many faces (Definition 3.6).
This notion is apparently new. Its role here is to guarantee finiteness of primary decomposition of finitely encoded modules (Theorem 3.32).

It bears mentioning that primary decomposition of downset modules, or equivalently, expressions of downsets as unions of coprimary downsets (Definition 3.18), is canonical (Theorem 3.19 and Corollary 3.20), generalizing the canonical primary decomposition of monomial ideals in ordinary polynomial rings. Topologically speaking, coprimary downsets are those cogenerated by poset elements that die pure deaths, so this canonical expression as a union tells the fortune of every downset element.

Notably lacking from primary decomposition theory over arbitrary polyhedral partially ordered abelian groups is a notion of minimality—alas, a lack that is intrinsic.

**Example 1.6.** The union in Example 1.5 results in a canonical primary decomposition of the downset module $k[D]$ over $\mathbb{R}^2$ (Corollary 3.20). Although all three of the pure death types are required in the union decomposing $D$ (Example 1.5), the final two summands in the decomposition of $k[D]$ are redundant. One can, of course, simply omit the redundant summands, but for arbitrary polyhedral partially ordered groups no criterion is known for detecting a priori which summands should be omitted.

The failure of minimality here stems from geometry that can only occur in partially ordered groups more general than finitely generated free ones. More specifically, $D$ contains elements that die pure deaths of type “$x$-axis” but the boundary of $D$ fails to contain an actual translate of the face of $\mathbb{R}^2_+$ that is the positive $x$-axis. This can be seen as a certain failure of localization to commute with taking homomorphisms into $k[D]$; this is the content of the crucial Remark 6.50, which highlights the difference between real-graded algebra and integer-graded algebra. It is the source of much of the subtlety in the theory developed in this paper, particularly Sections 6–10, which is a development of substantial portions of basic commutative algebra of finitely determined $\mathbb{R}^n$-modules with one hand (the noetherian one) tied behind the back.

The purpose of that theory is partly to rectify, for real multiparameter persistence, the failure of minimality in Example 1.6 by pinpointing what it means for $\mathbb{R}^n$-modules to have minimal cogenerators (Section 6), and what it means for a primary decomposition to be minimal (Section 10). But more importantly, thinking of cogenerators topologically as deaths, the minimal cogenerator theory and its dual for minimal generators and births (Section 12) provide exactly the ingredients needed for multiparameter functorial generalizations of bar codes and elder rules (Sections 16, 17, and 19).
1.8. Minimal generators and cogenerators. Even in ordinary, single-parameter persistence, the intervals in a bar code need not be closed: they are usually half-open, being typically closed on the left (at birth) and open on the right (at death). This subtlety becomes more delicate with multiple parameters. Sections 6–13 build theory to handle the notions of generator and cogenerator for modules over real polyhedral groups (Definition 3.8), which is $\mathbb{R}^n$ with an arbitrary polyhedral positive cone.

Intuitively, a generator of a module (or upset) is an element that is not present when approached from below but present when approached from above. Dually, a cogenerator of a module (or downset) is an element that is present when approached from below but not present when approached from above—think of the right endpoint of a bar in ordinary persistence, be it a closed endpoint or an open one. With multiple parameters, (co)generators can also have positive dimension, being parallel to any face of the positive cone. For death, this is what it means to persist along the face; for birth, the dual concept is less familiar but occurs already in the toy-model fly wing (Example 1.3). The single-parameter case, namely bars with infinite length, have in practice been handled in an ad hoc manner, but that is not an option in multiparameter persistence, where infinite bars come in distinct polyhedral flavors.

Phrased more geometrically, upsets and downsets in a real polyhedral group $Q$ need not be closed subsets of $Q$. Points along the frontier—in the topological closure but outside of the original set—feel like they are minimal or maximal, respectively, but in reality they only are so in a limiting sense. For this reason, a $Q$-module need not be minimally generated or cogenerated, even if it is $Q$-finite (Definition 2.32), has finite isotypic subdivision (Definition 2.23), and is bounded, in the sense of having nonzero homogeneous elements only in degrees from a set that is bounded in $Q \cong \mathbb{R}^n$. The indicator module for the interior of the unit cube in $\mathbb{R}^n_+$ provides a specific example.

The main idea is to express frontier elements using limits and colimits. To get a feel for the theory, it is worth a leisurely tour through the single-parameter case, where $Q$ is the real polyhedral group $\mathbb{R}$ with positive cone $\mathbb{R}_+$. To that end, fix a module $M$ over $\mathbb{R}$.

- **Closed right endpoint at** $a \in \mathbb{R}$. Let $k_a$ be the $\mathbb{R}$-module that is zero outside of degree $a$ and has a copy of $k$ in degree $a$. Each nonzero element $k_a$ dies whenever it is pushed to the right by any positive distance along $\mathbb{R}$. A closed right endpoint at $a$ is a submodule of $M$ isomorphic to $k_a$. Equivalently a closed right endpoint at $a$ is a nonzero homomorphism $k_a \to M$, so $1 \in k_a$ lands on an element of degree $a$ in $M$ that dies when pushed any distance to the right along $\mathbb{R}$. Thus closed right endpoints of $M$ at $a$ are detected functorially by $\text{Hom}_\mathbb{R}(k_a, M)$.

- **Open right endpoint at** $a \in \mathbb{R}$. Consider $\lim_{a' < a} M_a$. This vector space sits at $a$ but records only what happens strictly to the left of $a$. This direct limit being nonzero means only that $a$ is not a leftmost endpoint of $M$. More precisely, this direct limit detects exactly those bars that extend strictly to the left from $a$. Now compare the direct limit with the vector space $M_a$ itself via the natural homomorphism $\varphi_a : \lim_{a' < a} M_a \to M_a$ induced by the universal property of
direct limit. Elements outside of the image of \( \varphi_a \) are left endpoints at \( a \). (Hence, without meaning to at this stage, we have stumbled upon what it means to be a closed generator at \( a \); see Section 18.) Elements in the image of \( \varphi_a \) persist from “just before \( a \)” to \( a \) itself and hence could be closed right endpoints at \( a \) but not open ones. Elements in the kernel of \( \varphi_a \), on the other hand, persist until “just before \( a \)” and no further; these are the open right endpoints sought.

- Infinite right endpoint. A class persists indefinitely if its bar contains a translate of the ray \( \mathbb{R}_+ \). Functorially, the goal is to stick the upset module \( k[a + \mathbb{R}_+] \) into \( M \). An element of Hom\((k[a + \mathbb{R}_+], M)\) is nothing more or less than simply an element of \( M_a \), since \( k[a + \mathbb{R}_+] \) is a free \( k[\mathbb{R}_+] \)-module of rank 1 generated in degree \( a \). How is the homomorphism ensured to be injective? By localizing along the face \( \mathbb{R}_+ \), which annihilates torsion and hence non-injective homomorphisms. Note that any other choice of \( a' \) on the ray \( a + \mathbb{R}_+ \) should morally select the very same bar, so these injective homomorphisms should be taken modulo translation along the face \( \tau \) in question, in this case \( \tau = \mathbb{R}_+ \) itself. Thus, functorially, infinite right endpoints are detected by Hom\(_\mathbb{R}_+(k[\mathbb{R}_+], M)_{\mathbb{R}_+}/\mathbb{R}_+\), where the underline on Hom means to try sticking all translates (graded shifts) of \( k[\mathbb{R}_+] \) into \( M \), the subscript \( \mathbb{R}_+ \) means localization, and the quotient means modulo the translation action of \( \mathbb{R}_+ \) on the localization.

To unify the closed and open right endpoints, and hence to indicate the generalization to multiple parameters, consider the upper boundary module \( \delta M = \delta^{(0)} M \oplus \delta^{\mathbb{R}_+} M \), where \( \delta^{(0)} M = M \) is viewed in each degree \( a \in \mathbb{R} \) as the trivial direct limit of \( M_{a'} \) over \( a' \in a - \sigma^\circ \) for the relative interior \( \sigma^\circ \) of the face \( \sigma = \{0\} \) of the positive cone \( \mathbb{R}_+ \), and \( \delta^{\mathbb{R}_+} M \) in degree \( a \) is the direct limit over \( a - \sigma^\circ \) for \( \sigma = \mathbb{R}_+ \) itself (Definition 6.15). The universal property of direct limits induces a homomorphism from the \( \mathbb{R}_+ \)-summand to the \{0\}-summand. Therefore \( \delta M \) carries an action of \( \mathbb{R} \times \mathcal{F} \), where \( \mathbb{R} \) acts on each summand and the face poset \( \mathcal{F} \) of the cone \( \mathbb{R}_+ \) takes the \( \sigma \)-summand to the \( \sigma' \)-summand for any \( \sigma' \subseteq \sigma \), which in this case means only the \( \mathbb{R}_+ \)-summand to the \{0\}-summand. Functorially, a right endpoint—be it closed or open, without specification—is detected at \( a \) by the cogenerator functor Hom\(_{\mathbb{R} \times \mathcal{F}}^\circ \left( k[\mathbb{R}_+], \delta M \right) \), which computes the degree \( a \) piece of the socle of \( M \) (Definition 6.27).

The general real multiparameter case (Definition 6.59) is the main contribution of Section 6. It takes the join of the finite and infinite endpoint cases, combining the upper boundary functor with homomorphisms from \( k[\tau] \), localization, and quotient modulo the span of the face \( \tau \), being careful to do these operations in the correct order, because some of them fail to commute (Remark 6.50). The definition comes with a small galaxy of useful foundations; see the opening of Section 6 for an overview of those.

The functorial treatment of left endpoints is dual to that of right endpoints presented here, with inverse instead of direct limits, and so on. Armed with geometric intuition from this detailed single-parameter discussion of right endpoints as a guide to Section 6, reflecting the picture left-to-right should suffice as intuition for its dual, Section 12.
There are subtleties stemming from the asymmetry between the exactness properties of direct and inverse limits—see the opening of Section 12 for an overview of those—but they affect only the outer confines of the theoretical development and should cause no concern in practice, when all of the vector spaces in sight have finite dimension.

The exposition remains concerned solely with cogenerator theory and its consequences from Section 6 through Section 10, as opposed to the dual theory of tops in Section 12 and its consequences in Section 13. Generators are left until Section 18 because they require more, namely elder morphisms (Section 17).

1.9. **Infinitesimal geometry of real polyhedral modules.** Although primary decomposition and local support work over arbitrary polyhedral partially ordered groups (Section 3), the generality must be restricted to the setting of real polyhedral groups for much of the new theory of socles and cogenerators in Sections 6–10. (The simpler parallel theory over *discrete polyhedral groups*—see Definition 3.10—holds and has value, but it is barely new, being based on more elementary foundations; it is recorded for posterity in Section 11.) The specific technical reason for the restriction is detailed in Remark 6.6. It is related to the omnipresent phenomenon that drives the novelty, namely that boundaries of upsets and downsets need not be closed or open, but can a priori be anything in between.

The miracle, however, is that “anything in between” is hardly so: it turns out to be rigidly constrained. The distinct approaches to a boundary point of a downset in a real polyhedral group $Q$ are indexed by the faces of $Q$ (Proposition 6.10). This rigidity renders the cogenerator theory finite. In particular, viewing upper boundary modules as gathering all ways of taking direct limits of vector spaces $M_a$ beneath a fixed degree $a$, this rigidity is what confines upper boundary modules to only finitely many summands.

It might rightly be observed that in most current uses of ordinary persistent homology with real parameters, the left endpoints are closed, so there is no need to develop notions of closed and open generators. However, for the same reason the right endpoints are usually open, so the notion of open socle is critical. In addition, allowing only closed births and open deaths would obscure the natural self-duality of the theory via Matlis duality for finitely encoded modules (Section 4.4).

Infinitesimal geometry of real polyhedral groups prevents minimal generating sets in the usual sense—or dually, minimal irreducible decomposition (see Remark 8.12)—from necessarily existing in this setting. That remains true even if one is willing to accept uncountably many summands. Most of the this trouble is alleviated by the concept of *shape* (Proposition 6.10), which codifies how closed is the principal upset of a generator or coprincipal downset of a cogenerator.

**Example 1.7.** Consider the following upsets and downsets in $\mathbb{R}^2$. (The choice of which of these to make an upset and which a downset was arbitrary: the picture dualizes by
1. The upset $U_1 = \mathbb{R}^2_+ \setminus y$-axis is a “half-open” positive quadrant. Writing it as a union of (translated) closed positive quadrants in $U_1$ requires infinitely many. As long as the $x$-axis contains enough of their lower corners so that 0 is an accumulation point, that suffices. However, the theory of generators makes $U_1$ principal: its sole open generator lies infinitesimally near the origin on the $x$-axis.

2. The upset $U_2 = \mathbb{R}^2_+ \setminus \{0\}$ is a positive quadrant that is closed away from its missing origin. It is a union of closed positive quadrants as $U_1$ is, except the lower corners must have 0 as an accumulation point in each of the axes. Generator theory detects two open generators, one being $U_1$ and the other being its reflection across the diagonal. Note the similarity with the discrete case: the maximal ideal in $\mathbb{N}^2$ has two generators, one as close to the origin as possible on each axis. That description is made precise in $\mathbb{R}^2_+$ by Theorem 8.10.

3. Plucking out a single point from the hyperbolic boundary of the downset in Example 1.5 has an odd effect. At the frontier point, cogenerator theory (Definition 6.59) detects two open cogenerators, dual to the generators of of $U_2$, but they are redundant: $D_3$ equals the union of the closed negative orthants cogenerated by the points along the rest of the hyperbola.

4. The ability to omit (co)generators is even more striking upon deleting an interval from the hyperbola, instead of a single point, to get the downset $D_4$. Along the deleted curve, cogenerator theory detects cogenerators of the same shape as those for $D_3$. Hence $D_4$ is the union of coprincipal downsets of these shapes along the deleted curve together with closed coprincipal downsets along the rest of the hyperbola. However, any finite number of the coprincipal downsets along the deleted curve can be omitted, as can be checked directly. In fact, any subset of them that is dense in the deleted curve can be omitted (Theorem 8.10). Note that a closed negative orthant is required at the lower endpoint of the deleted curve, because the endpoint has not been deleted, whereas the cogenerator at the upper endpoint of the deleted curve can always be omitted because of open negative orthants hanging from points along the hyperbola just below it.

1.10. **Topologies on generators and cogenerators.** What, in the end, does it take to express a downset in a real polyhedral group as a union of coprincipal downsets? This question stems from practical considerations about birth and death in multiparameter
persistence: any generalization of bar code, functorial or otherwise, requires knowing at least what are the analogues of the set of left endpoints and the set of right endpoints. The general answer comes down to density in what is more or less a topology on the set of cogenerators (Theorem 8.10) that arises by viewing certain cones related to faces of the positive cone $Q_+$ as being open in $Q$. These ideas and their consequences are akin to those in [KS17], but the main results lie along different lines.

The same concept of density governs criteria for injectivity of homomorphisms between finitely encoded modules. In noetherian commutative algebra, socles are essential submodules, meaning that any nonzero submodule—the kernel of a homomorphism at hand, say—must intersect the socle nontrivially. Here, socles of $M$ are no longer submodules, having been constructed from auxiliary modules derived from $M$, namely upper boundary modules $\delta M$. Nonetheless, socles are still functorial, so the question of whether they detect injectivity remains valid, and the answer is positive (Theorem 7.7). Furthermore, in noetherian commutative algebra, no proper submodule of the socle detects injectivity. But over real polyhedral groups, the same geometric considerations just discussed for cogeneration of downsets have the consequence that a subfunctor of $\text{soc}$ detects injectivity precisely when it is dense in the same sense (Theorem 8.15).

The fact that socles are not submodules raises another relevant point: in the closed cogenerator case there is a submodule containing the socle element, but for open cogenerators there must still be a submodule to witness the injectivity, because injectivity comes from there being no actual submodule that goes to 0. The cogenerator merely indicates the presence of such a submodule, rather than being an element of it. Reconstructing an honest submodule in Section 9 requires much of the theory in earlier sections. It has the consequence that a submodule is an essential submodule of an ambient module precisely when the socle of the submodule is dense in that of the ambient one (Theorem 9.5), a result pivotal in the characterization of coprimary modules via socles (Theorem 10.2), the first step in minimal primary decomposition.

1.11. Primary decomposition over real polyhedral groups. Detecting injectivity of homomorphisms is vital to primary decomposition because such decompositions can be expressed as inclusions into direct sums of coprimary modules (Definition 3.28), as already seen in Example 1.6. What makes such a decomposition minimal? Given that injectivity of homomorphisms is characterized by arbitrary inclusion of socles, primary decompositions ought to be considered minimal if their socle inclusions are dense. However, even though it is a priori harder to achieve isomorphism on socles than density of inclusion, the stronger conclusion nonetheless holds: minimal primary decompositions exist for finitely determined modules over real polyhedral groups with socle isomorphism as the minimality criterion (Definition 10.14 and Theorem 10.16). Moreover, these minimal decompositions are canonical for downset modules or, more generally, subquotients of $\mathbb{k}[Q]$ (Theorem 10.6 or Corollary 10.17, respectively), just as they are for monomial ideals in ordinary polynomial rings.
Example 1.8. When $\mathbb{R}^2$ is considered as a real polyhedral group, the two redundant components in Example 1.6 are not part of the minimal primary decomposition in Theorem 10.6 because the downset $D$ has no cogenerators, in the functorial sense of Definition 6.59, along the $x$-axis or $y$-axis. The illustrations in Example 1.5 show that $k[D]$ has elements supported on the face of $\mathbb{R}^2$ that is the (positive) $x$-axis, and it has elements supported on the face of $\mathbb{R}^2$ that is the (positive) $y$-axis, but the socle of $k[D]$ along each axis is 0, as the entire boundary curve of $k[D]$ is supported on the origin.

In finitely generated situations, socle-minimality as in Definition 10.14 is equivalent, when applied to an injective hull or irreducible decomposition, to there being a minimal number of summands. In contrast, minimal primary decompositions in noetherian commutative algebra are not required to be socle-minimal in any sense: they stipulate only minimal numbers of summands, with no conditions on socles. This has unfortunate consequences for uniqueness: components for embedded (i.e., nonminimal) primes are far from unique. Requiring socle-minimality as in Definition 10.14 recovers a modicum of uniqueness over arbitrary noetherian rings, as socles are functorial even if primary components themselves need not be. This tack is more commonly taken in combinatorial commutative algebra, typically involving objects such as monomial or binomial ideals. In particular, the “witnessed” forms of minimality for mesoprimary decomposition [KM14, Definition 13.1 and Theorem 13.2] and irreducible decomposition of binomial ideals [KMO16] serve as models for the type of minimality in primary decompositions considered here.

That said, in ordinary noetherian commutative algebra a socle-minimal primary decomposition is automatically produced by the usual existence proof, which leverages the noetherian hypothesis to create an irreducible decomposition. Indeed, a primary decomposition is socle-minimal if and only if each primary component is obtained by gathering some of the components in a minimal irreducible decomposition. In lieu of truly minimal irreducible decompositions over real polyhedral groups, whose impossibility stems from infinitesimal geometry, one is forced to settle for the density theorems reviewed in Section 1.10. The existence of primary decompositions at all, let alone ones with a mite of canonical minimality, should therefore come as a relief.

The Matlis duals of (minimal) primary decomposition and associated primes are (minimal) secondary decomposition and attached primes, covered briefly over real and discrete polyhedral groups in Section 12. Prior theory surrounding those is lesser known, even to algebraists, but has existed for decades [Kir73, Mac73, Nor72] (see [Sha76, Section 1] for a brief summary of the main concepts). Not enough of secondary decomposition is required in later sections to warrant going into much detail about it, especially as Matlis duality provides such a clear dual picture that is equivalent for finitely encoded modules over real or discrete polyhedral groups, but some is needed. The flying situation in Example 1.3 highlights the naturality of allowing births to extend backward indefinitely in one or more directions, for instance. The unfamiliarity of secondary decomposition and its related functors is another reason why the bulk
of the technical development over real polyhedral groups is carried out in terms of cogenerators and socles instead of generators and tops.

It would seem that birth and generators are in adamantine antisymmetry with death and cogenerators, but when it comes to interactions between the two, the symmetry of the theory is broken by the partial order on $Q$: elements in $Q$-modules move from birth inexorably toward death. It is possible to treat elements of modules over partially ordered groups functorially, of course, since they are homomorphisms from the monoid algebra of the positive cone, but the dual of an element is not an element. (It is instead a homomorphism to the injective hull of the residue field.) That makes generators more complicated to deal with than cogenerators, cementing the choice to develop the theory in terms of cogenerators.

1.12. **Minimal homological algebra over real polyhedral groups.** Minimal primary decomposition provides an opportunity to revisit the syzygy theorem for modules over posets (see Section 1.6) in the context of real polyhedral groups (Theorem 14.2): fringe presentations, upset presentations, and downset presentations of modules from data analysis applications can be chosen minimal, and semialgebraic if the modules start off that way. This line of thought raises issues concerning minimal upset and downset resolutions, especially termination of minimal resolutions and length bounds on resolutions that are not necessarily minimal, discussed more in Sections 14 and 20.

1.13. **Functorial invariants of multiparameter persistence.** Fringe presentation is a convenient, compact, computable representation of multiparameter persistent homology, but it is not canonical—not unique up to isomorphism, not necessarily minimal, not overtly exhibiting invariants of the module. For these things, in ordinary persistence with one parameter, one turns to bar codes.

The bar code in single-parameter persistence has two interpretations that diverge in the presence of multiple parameters. One is that it records birth and genuine death of persistent homology classes, meaning parameters where classes go to 0 [ZC05, Cra13]. The other is that it records birth and elder-death, meaning parameters where homology classes join to persisting classes born earlier. These two interpretations yield the same result up to isomorphism for modules over the poset $\mathbb{R}$ or $\mathbb{Z}$. The first interpretation generalizes functorially to QR codes (Section 16), while the second generalizes functorially to elder morphisms (Section 17). Both have rightful claims to be considered “the” functorial multiparameter analogue of bar code, but neither exactly captures the essence of bar code, which in single-parameter situations should functorially be

- a “top” vector space spanned by open, closed, and infinite left endpoints;
- a “socle” vector space spanned by open, closed, and infinite right endpoints; and
- a linear map from the top vector space to the socle vector space.

QR codes get the latter two items right but fail on the first, being forced by functoriality to use birth spaces that are entire graded pieces of $M$ (technically: $M^{\rho,\xi}/\rho$ from Definition 12.26) that merely surject onto the top spaces of $M$. Elder morphisms get the
first one right but fail on the second, being forced by the filtered nature of birth spaces to use death spaces that come not from $M$ itself but rather from the elder quotients of $M$. This divergence of QR code and elder morphism traces back to the divergent notions of birth and death as opposed to generator and relation in Section 1.4.

What is a QR code? Bar codes for ordinary persistence specify births, deaths, and matchings between them. QR codes in multiparameter situations categorify these notions as a partially ordered direct sum of birth vector spaces (Definitions 15.1 and 15.10 over discrete and real polyhedral groups, respectively), a partially ordered direct product of death vector spaces (Definitions 15.3 and 15.13 over discrete and real polyhedral groups, respectively), with a functorial linear map to relate them (Theorem 16.1 and Definition 16.2). The death vector spaces in the product are simply socles. It would be desirable for the birth direct summands to be tops, but alas there are no functorial homomorphisms from tops to socles, so more or less entire graded pieces of $M$ are used instead; see Remark 16.6 for why this is both necessary and not so bad. The QR code is functorial and faithful enough that $M$ is easily recovered canonically from its QR code if $M$ is finitely encoded (Theorem 16.5; see also Remarks 16.7 and 16.9).

The claim that there are no functorial homomorphisms from tops to socles merits clarification: it means no functorial homomorphisms from tops to socles of the same module $M$. The reason is that given a generator of $M$ in birth degree $\beta$, it determines an element of $\text{top}(M)$ in degree $\beta$ that is only well defined modulo elements born at parameters earlier than $\beta$. But elements born earlier than $\beta$ can die earlier, at the same parameter, or after the given generator dies, so modifying the given generator by adding an elder element can alter the generator’s map to the death module. This analysis demonstrates why the birth summands of QR codes are bigger than top spaces of $M$, but it also points the way to functorial elder morphisms.

The partial order on the birth poset is utilized to define, in the previous paragraph, what it means for an element to be “elder” than a degree $\beta$ generator (Definition 17.3). With that precision in hand, the previous paragraph shows why there is a well defined linear map from the top of $M$ in degree $\beta$ to the socle of the quotient $M/M_{<\beta}$ of $M$ modulo its elder submodule at $\beta$ (Theorem 17.5; see also Remark 17.9). That is the elder morphism (Definition 17.7). It functorializes the well known the elder rule from single-parameter persistence, which says that a class dies when it “joins with an older class”. The elder objects in Section 17 make this joining precise, particularly what happens the instant before an extant class joins an elder one: the dying ember of an extant class is a socle element of the elder quotient $M/M_{<\beta}$. In multiparameter persistence, a single extant class can, when pushed up along the parameter poset, give rise to dying embers over many death parameters of the elder quotient $M/M_{<\beta}$.

The failures of QR codes and elder morphisms to satisfy all three desiderata are solely multiparameter phenomena; in one parameter, functorial QR codes and elder morphisms can be combined to succeed on all three items simultaneously. The result is the functorial bar code in Theorem 19.3, whose minor cost is that the socle in
death degree $\alpha$ must—by a combination of the two forces already encountered, namely functoriality and the filtered nature of birth spaces—be replaced by an associated graded vector space. In other words, the vector space spanned by all of the closed right endpoints over $a \in \mathbb{R}$ is functorial, as is the the vector space spanned by all of the open right endpoints over $a$, but their subspaces spanned by right endpoints of intervals born over $b$ is only functorial modulo the right endpoints of intervals born strictly earlier than $b$.

The idea of the proof is enlightening. An extant element suffers elder-death if pushing it infinitesimally forward along the real line lands the extant element in the elder submodule. Subtracting the image elder element ("elder projection") from the extant one yields an extant element that suffers genuine death and not merely elder-death. This proof works over $\mathbb{Z}$ but fails in the presence of multiple parameters for a telling reason: when the extant element suffers elder-death in independent directions, it can land on different elder elements when pushed up in different directions, so there is no way to modify the extant element to die genuinely in all directions simultaneously.

**Example 1.9.** Let $M$ be the $\mathbb{Z}^2$-module generated by $\{e_x, e_y, e_{xy}\}$, where $\deg e_m = \deg m \in \mathbb{N}^2$, with relations $\langle xye_x - xe_{xy}, yxe_y - ye_{xy} \rangle$. If $\beta = [1, 1]^t$ then $M_{\prec \beta} = \langle e_x, e_y \rangle$, so $M/M_{\prec \beta} = M/\langle e_x, e_y \rangle = \langle e_{xy} \rangle/\langle xe_{xy}, ye_{xy} \rangle \cong \mathbb{k}_{xy}$ is the indicator subquotient for the point $[1, 1] \in \mathbb{Z}^2$. Therefore $M/M_{\prec \beta}$ is its own socle, and it is supported on the face $\{0\}$. But $M$ itself has no socle elements along $\{0\}$ of degree $[1, 1]$. (In fact, $M$ has no elements supported on $\{0\}$, regardless of degree.) Indeed, every element of degree $[1, 1]$ is expressible as $\alpha ye_x + \beta xe_y + \gamma e_{xy}$. If this element lies in $\soc \{0\} M$ then multiplying it by $xy$ forces $\alpha + \beta = \gamma$, but multiplying it by $x$ forces $\beta = 0$, while multiplying it by $y$ forces $\alpha = 0$. The relations of $M$ are specifically designed so that any generator of $M$ with degree $[1, 1]$ maps to different elements of $M_{\prec \beta} = \langle e_x, e_y \rangle$ upon multiplication by $x$ and by $y$, because $xe_{xy} = xye_x$ whereas $ye_{xy} = xye_y$.

This example brings up one final point: in the presence of multiple parameters, generically it is impossible to peel off summands using an elder rule. This fuels an intuition that the Krull–Schmidt theorem on existence of decompositions as direct sums of indecomposables carries little power for persistence over posets not of finite representation type. Heuristically, if the filtered topological space is connected or has otherwise trivial topology, then although lots of interesting things may happen along the way to triviality, the infinitely persisting class must lie in some summand. That doesn’t a priori mean anything—it is true in one real parameter, after all—but not being able to peel off summands by elder rule obstructs nontrivial direct sum decomposition.

**Advice for the reader.** The heart of this paper is pictorial algebra. Nearly every statement and proof was written with visions of upsets and downsets and cones and faces at the forefront. It is likely best understood with that kind of geometry in mind.
2. Encoding poset modules

2.1. Multiparameter persistence.

Definition 2.1. Let \( Q \) be a partially ordered set \((\text{poset})\) and \( \leq \) its partial order. A module over \( Q \) (or a \( Q \)-module) is

- a \( Q \)-graded vector space \( M = \bigoplus_{q \in Q} M_q \) with
- a homomorphism \( M_q \to M_{q'} \) whenever \( q \leq q' \) in \( Q \) such that
- \( M_q \to M_{q''} \) equals the composite \( M_q \to M_{q'} \to M_{q''} \) whenever \( q \leq q' \leq q'' \).

A homomorphism \( M \to N \) of \( Q \)-modules is a degree-preserving linear map, or equivalently a collection of vector space homomorphisms \( M_q \to N_{q'} \), that commute with the structure homomorphisms \( M_q \to M_{q'} \) and \( N_q \to N_{q'} \).

The last bulleted item is commutativity: it reflects that inclusions of subspaces induce functorial homology morphisms in the motivating examples of \( Q \)-modules.

Definition 2.2. Let \( X \) be a topological space and \( Q \) a poset.

1. A filtration of \( X \) indexed by \( Q \) is a choice of subspace \( X_q \subseteq X \) for each \( q \in Q \) such that \( X_q \subseteq X_{q'} \) whenever \( q \leq q' \).
2. The \( i \)th persistent homology of the filtered space \( X \) is the associated homology module, meaning the \( Q \)-module \( \bigoplus_{q \in Q} H_i X_q \).

Convention 2.3. The homology here could be taken over an arbitrary ring, but for simplicity it is assumed throughout that homology is taken with coefficients in a field \( \mathbb{k} \).

Remark 2.4. There are a number of abstract, equivalent ways to phrase Definition 2.2. For example, a filtration is a functor from \( P \) to the category \( \mathcal{S} \) of subspaces of \( X \), or a natural transformation from the category \( P \) to \( \mathcal{S} \), or an \( \mathcal{S} \)-valued sheaf on the topological space \( P \), where a base for the topology is the set of principal dual order ideals. For background on and applications of many of these perspectives, see Curry’s dissertation [Cur14], particularly §4.2 there.

Example 2.5. A real multifiltration of \( X \) is a filtration indexed by \( \mathbb{R}^n \), with its partial order by coordinatewise comparison. Example 1.1 is a real multifiltration of \( X = \mathbb{R}^2 \) with \( n = 2 \). The monoid \( \mathbb{R}_+^n \subset \mathbb{R}^n \) of nonnegative real vectors under addition has monoid algebra \( \mathbb{k}[\mathbb{R}_+^n] \) over the field \( \mathbb{k} \), a “polynomial” ring whose elements are (finite) linear combinations of monomials \( x^a \) with real, nonnegative exponent vectors \( a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n \). It contains the usual polynomial ring \( \mathbb{k}[\mathbb{N}^n] \) as a \( \mathbb{k} \)-subalgebra. The persistent homology of a real \( n \)-filtered space \( X \) is an \( \mathbb{R}_+^n \)-graded module over \( \mathbb{k}[\mathbb{R}_+^n] \), which is the same thing as an \( \mathbb{R}_+^n \)-module.
2.2. Finite encoding.
For practical purposes, it is important to encode finiteness properties of poset-graded vector spaces that at least provide hope for effective computation.

**Definition 2.6.** Fix a poset $Q$. An *encoding* of a $Q$-module $M$ by a poset $P$ is a poset morphism $\pi : Q \to P$ together with a $P$-module $H$ such that $M \cong \pi^* H = \bigoplus_{q \in Q} H_{\pi(q)}$, the *pullback of $H$ along $\pi$*, which is naturally a $Q$-module. The encoding is *finite* if
1. the poset $P$ is finite, and
2. the vector space $H_p$ has finite dimension for all $p \in P$.

**Remark 2.7.** Encoding of a $Q$-module $M$ by a poset morphism to $P$ is equivalent to viewing $M$ as a sheaf on $P$ that is constructible in the Alexandrov topology, in the sense defined independently by Lurie [Lur17, Definitions A.5.1 and A.5.2]. The focus here is on the characterizations and consequences of the finitely encoded condition rather than on arbitrary poset encodings.

**Example 2.8.** Take $Q = \mathbb{Z}^n$ and $P = \mathbb{N}^n$. The *convex projection* $\mathbb{Z}^n \to \mathbb{N}^n$ sets to 0 every negative coordinate. The pullback under convex projection is the Čech hull [Mil00, Definition 2.7]. More generally, suppose $a \preceq b$ in $\mathbb{Z}^n$. The interval $[a, b] \subseteq \mathbb{Z}^n$ is a box (rectangular parallelepiped) with lower corner at $a$ and upper corner at $b$. The *convex projection* $\pi : \mathbb{Z}^n \to [a, b]$ takes every point in $\mathbb{Z}^n$ to its closest point in the box. A $\mathbb{Z}^n$-module is *finitely determined* if it is finitely encoded by $\pi$.

Effectively computing a finite encoding of a real multifiltered space requires keeping track of the fibers of the morphism $\mathbb{R}^n \to P$ in addition to the data on $P$. The fact that applications of persistent homology often arise from metric considerations, which are semialgebraic in nature, suggests the following condition for algorithmic developments.

**Definition 2.9.** Fix a partially ordered real vector space $Q$ of finite dimension (see Definition 3.1, or just take $Q = \mathbb{R}^n$ for now). A finite encoding $\pi : Q \to P$ is *semialgebraic* if its fibers are real semialgebraic varieties. A module over $Q$ is *semialgebraic* if it has a semialgebraic encoding.

**Lemma 2.10.** For any poset $Q$, the category of finitely encoded $Q$-modules is abelian. If $Q$ is a partially ordered real vector space of finite dimension, then the category of semialgebraic modules is abelian.

*Proof.* The category in question is a subcategory of the category of $Q$-modules. It suffices to prove that it is a full subcategory, meaning that the kernel and cokernel of any homomorphism $M_1 \to M_2$ of finitely encoded $Q$-modules is finitely encoded, as is any finite direct sum of finitely encoded $Q$-modules. For all of these it suffices to show that any two finitely encoded $Q$-modules can be finitely encoded by a single poset morphism. To that end, suppose that $M_i$ is finitely encoded by $\pi_i : Q \to P_i$ for $i = 1, 2$. Then $M_1$ and $M_2$ are both finitely encoded by $\pi_1 \times \pi_2 : Q \to P_1 \times P_2$. The semialgebraic case has the same proof. \(\square\)
2.3. Upsets, downsets, and indicator modules.

Example 2.11. Given a poset $Q$, the vector space $\mathbb{k}[Q] = \bigoplus_{q \in Q} \mathbb{k}$ that assigns $\mathbb{k}$ to every point of $Q$ is a $Q$-module. It is encoded by the morphism from $Q$ to the trivial poset $P$ with one point and vector space $H = \mathbb{k}$. This leads to less trivial examples in a way that works for an arbitrary poset $Q$, although our main example is $Q = \mathbb{R}^n$.

1. An upset (also called a dual order ideal) $U \subseteq Q$, meaning a subset closed under going upward in $Q$ (so $U + \mathbb{R}^n_+ = U$, when $Q = \mathbb{R}^n$) determines an indicator submodule or upset module $\mathbb{k}[U] \subseteq \mathbb{k}[Q]$.

2. Dually, a downset (also called an order ideal) $D \subseteq Q$, meaning a subset closed under going downward in $Q$ (so $D - \mathbb{R}^n_+ = D$, when $Q = \mathbb{R}^n$) determines an indicator quotient module or downset module $\mathbb{k}[Q] \to \mathbb{k}[D]$.

When $Q = \mathbb{R}^n$, an indicator module of either sort is semialgebraic if the corresponding upset or downset is a semialgebraic subset of $\mathbb{R}^n$.

Remark 2.12. Indicator submodules $\mathbb{k}[U]$ and quotient modules $\mathbb{k}[D]$ are $Q$-modules, not merely $U$-modules or $D$-modules, by setting the graded components indexed by elements outside of the ideals to 0. It is only by viewing indicator modules as $Q$-modules that they are forced to be submodules or quotients, respectively. For relations between these notions and those in Remark 2.4, again see Curry’s thesis [Cur14]. For example, upsets form the open sets in the topology from Remark 2.4.

Example 2.13. Ising crystals at zero temperature, with polygonal boundary conditions and fixed mesh size, are semialgebraic upsets in $\mathbb{R}^n$. That much is by definition: fixing a mesh size means that the crystals in question are (staircase surfaces of finitely generated) monomial ideals in $n$ variables. Remarkably, such crystals remain semialgebraic in the limit of zero mesh size; see [Oko16] for an exposition and references.

Example 2.14. Monomial ideals in polynomial rings with real exponents, which correspond to upsets in $\mathbb{R}^n_+$, are considered in [ASW15], including aspects of primality, irreducible decomposition, and Krull dimension. Upsets in $\mathbb{R}^n$ are also considered in [MMc15], where the combinatorics of their lower boundaries, and topology of related simplicial complexes, are investigated in cases with locally finite generating sets.

For future reference, here are some basic facts about upset and downset modules.

Definition 2.15. A poset $Q$ is

1. connected if for every pair of elements $q, q' \in P$ there is a sequence $q = q_0 \preceq q_1 \preceq q_1' \preceq \cdots \preceq q_k = q'$ in $Q$;

2. upper-connected if every pair of elements in $Q$ has an upper bound in $Q$;

3. lower-connected if every pair of elements in $Q$ has a lower bound in $Q$; and

4. strongly connected if $Q$ is upper-connected and lower-connected.

Example 2.16. $\mathbb{R}^n$ is strongly connected. The same is true of any partially ordered abelian group (see Section 3.1 for basic theory of those posets).
Example 2.17. A poset $Q$ is upper-connected if (but not only if) it has a maximum element—one that is preceded by every element of $Q$. Similarly, $Q$ is lower-connected if it has a minimum element—one that precedes every element of $Q$.

Definition 2.18. For an upset $U$ and a downset $D$ in a poset $Q$, write $U \preceq D$ if $U$ and $D$ have nonempty intersection: $U \cap D \neq \emptyset$.

Lemma 2.19. Fix a poset $Q$.

1. For an upset $U$ and a downset $D$, a nonzero homomorphism $\mathbb{k}[U] \to \mathbb{k}[D]$ of $Q$-modules exists if and only if $U \preceq D$.
2. $\text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D]) = \mathbb{k}$ if $U \preceq D$ and either $U$ is lower-connected as a subposet of $Q$ or $D$ is upper-connected as a subposet of $Q$.
3. If $U$ and $U'$ are upsets and $Q$ is upper-connected, then $\text{Hom}_Q(\mathbb{k}[U'], \mathbb{k}[U]) = \mathbb{k}$ if $U' \subseteq U$ and $0$ otherwise.
4. If $D$ and $D'$ are upsets and $Q$ is lower-connected, then $\text{Hom}_Q(\mathbb{k}[D], \mathbb{k}[D']) = \mathbb{k}$ if $D \supseteq D'$ and $0$ otherwise.

Proof. The first claim is immediate from the definitions. For the second, compare the action of $\varphi : \mathbb{k}[U] \to \mathbb{k}[D]$ on the copy of $\mathbb{k}$ in any two degrees $q$ and $q'$ with the action of $\varphi$ on the copy of $\mathbb{k}$ in any degree that is a lower bound for $q$ and $q'$. The proof of the last claim is the same, after replacing $D$ with $D'$ and $U$ with $D$. The proof of the remaining claim is dual to that of the last claim.

Example 2.20. Consider the poset $\mathbb{N}^2$, the upset $U = \mathbb{N}^2 \smallsetminus \{0\}$, and the downset $D$ consisting of the origin and the two standard basis vectors. Then $\mathbb{k}[U] = m = \langle x, y \rangle$ is the graded maximal ideal of $\mathbb{k}[\mathbb{N}^2] = \mathbb{k}[x, y]$ and $\mathbb{k}[D] = \mathbb{k}[\mathbb{N}^2]/m^2$. Now calculate

$$\text{Hom}_{\mathbb{N}^2}(\mathbb{k}[U], \mathbb{k}[D]) = \text{Hom}_{\mathbb{N}^2}(m, \mathbb{k}[\mathbb{N}^2]/m^2) = \mathbb{k}^2,$$

a vector space of dimension 2: one basis vector preserves the monomial $x$ while killing the monomial $y$, and the other basis vector preserves $y$ while killing $x$. In general, $\text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D])$ is the vector space spanned by the set $\pi_0(U \cap D)$ of connected components (Definition 2.15) of $U \cap D$ as a subposet of $Q$. This proliferation of homomorphisms is undesirable for both our computational and theoretical purposes. Indeed, for an extreme example, consider the case over $\mathbb{R}^2$ in which $U$ is the closed half-plane above the antidiagonal line $y = -x$ and where $D = -U$, so that $U \cap D$ is totally disconnected: $\pi_0(U \cap D) = U \cap D$. This example motivates the following concept.

Definition 2.21. For an upset $U$ and a downset $D$ in a poset $Q$ with $U \preceq D$, a homomorphism $\varphi : \mathbb{k}[U] \to \mathbb{k}[D]$ is connected if there is a scalar $\lambda \in \mathbb{k}$ such that $\varphi$ acts as multiplication by $\lambda$ on the copy of $\mathbb{k}$ in degree $q$ for all $q \in U \cap D$.

Remark 2.22. Equivalently, $\mathbb{k}[U] \to \mathbb{k}[D]$ is connected if it factors through $\mathbb{k}[Q]$. 
2.4. Isotypic subdivision.
The main result of Section 2, namely Theorem 2.33, says that finite encodings always exist for $Q$-modules with finitely many domains of constancy, defined as follows.

**Definition 2.23.** Fix a $Q$-module $M$. The *isotypic subdivision* of $Q$ induced by $M$ is the equivalence relation $Q/M$ generated by the relation that sets $a \sim b$ whenever $a \preceq b$ in $Q$ and the induced homomorphism $M_a \to M_b$ is an isomorphism. The equivalence classes of the isotypic subdivision are called *regions*, or *$M$-regions* if the context does not make it clear.

**Example 2.24.** The quotient map $Q \to Q/M$ of sets need not be a morphism of posets. Indeed, there is no natural way to impose a poset structure on the set of isotypic regions. Take, for example, $Q = \mathbb{R}^2$ and $M = k_0 \oplus k[\mathbb{R}^2]$, where $k_0$ is the $\mathbb{R}^2$-module whose only nonzero component is at the origin, where it is a vector space of dimension 1. This module $M$ induces only two isotypic regions, namely the origin and its complement. Neither of the two isotypic regions has a stronger claim to precede the other, but at the same time it would be difficult to justify forcing the isotypic regions to be incomparable.

**Example 2.25.** Example 1.3 shows an isotypic subdivision of $\mathbb{R}^2$ which happens to form a poset and therefore produces an encoding.

2.5. Uptight posets.
Constructing finite encodings relies on combinatorics that works for all posets.

**Definition 2.26.** Fix a poset $Q$ and a set $\Upsilon$ of upsets. For each poset element $a \in Q$, let $\Upsilon_a \subseteq \Upsilon$ be the set of upsets from $\Upsilon$ that contain $a$. Two poset elements $a, b \in Q$ lie in the same *uptight region* if $\Upsilon_a = \Upsilon_b$.

**Remark 2.27.** Every uptight region is the intersection of a single upset (not necessarily one of the ones in $\Upsilon$) with a single downset. Indeed, the intersection of any family of upsets is an upset, the complement of an upset is a downset, and the intersection of any family of downsets is a downset. Hence the uptight region containing $a$ equals $(\bigcap_{U \in \Upsilon_a} U) \cap (\bigcap_{U \notin \Upsilon_a} \overline{U})$, with the first intersection being an upset and the second being a downset.

**Proposition 2.28.** In the situation of Definition 2.26, the uptight regions form a poset in which $A \preceq B$ whenever $a \preceq b$ for some $a \in A$ and $b \in B$.

**Proof.** The stipulated relation on the set of uptight regions is

- reflexive because $a \preceq a$ for any element $a$ in any uptight region $A$;
- transitive because the relation on $Q$ is transitive; and
- antisymmetric for the following reason. Suppose uptight regions $A$ and $B$ satisfy $A \preceq B$ and $B \preceq A$, so $a \preceq b$ and $b \preceq a'$ for some $a, a' \in A$ and $b, b' \in B$. Then $\Upsilon_a \subseteq \Upsilon_b = \Upsilon_{b'} \subseteq \Upsilon_{a'} = \Upsilon_a$, with the containments following from $a \preceq b$ and $b' \preceq a'$. Therefore $\Upsilon_a = \Upsilon_b$, so $A = B$. \qed
Definition 2.29. Fix a $Q$-module $M$. An isotypic upset of $Q$ induced by $M$ is either
1. an upset $U_I$ generated by an isotypic region $I$ of $M$, or
2. the complement of a downset $D_I$ cogenerated by an isotypic region $I$ of $M$.
Write $P_M$ for the poset of uptight regions determined by the set $\Upsilon_M$ of isotypic upsets.

Proposition 2.30. Each uptight region determined by the set $\Upsilon_M$ of isotypic upsets
is contained a single $M$-isotypic region.

Proof. Suppose that $A$ is an uptight region that contains points from isotypic regions $I$ and $J$. Any point in $I \cap A$ witnesses the containments $A \subseteq D_I$ and $A \subseteq U_I$ of $A$ inside the isotypic upset and downset generated and cogenerated by $I$. Any point $j \in J \cap A$ is therefore sandwiched between elements $i, i' \in I$, so $i \preceq j \preceq i'$, because $j \in U_I$ (for $i$) and $j \in D_I$ (for $i'$). By symmetry, switching $I$ and $J$, there exists $j' \in J$ with $i' \preceq j'$. The sequence $M_i \to M_j \to M_i'$ of $Q$-module structure homomorphisms induces isomorphisms $M_i \to M_i'$ and $M_j \to M_j'$ by definition of isotypic region. Elementary homological algebra implies that $M_i \to M_j$ is an isomorphism, so $I = J$, as desired. \qed

Example 2.31. Proposition 2.30 does not claim that $I = U_I \cap D_I$, and in fact that claim is often not true. Consider $Q = \mathbb{R}^2$ and $M = k \oplus k[\mathbb{R}^2]$, as in Example 2.24, and take $I = \mathbb{R}^2 \setminus \{0\}$. Then $U_I = D_I = \mathbb{R}^2$, so $U_I \cap D_I$ contains the other isotypic region $J = \{0\}$. The uptight poset $P_M$ has precisely four elements:
1. the origin $\{0\} = U_J \cap D_J$;
2. the complement $U_J \setminus \{0\}$ of the origin in $U_J$;
3. the complement $D_J \setminus \{0\}$ of the origin in $D_J$; and
4. the points $\mathbb{R}^2 \setminus (U_J \cup D_J)$ lying only in $I$ and in neither $U_J$ nor $D_J$.
Oddly, uptight region 4 has two connected components, the second and fourth quadrants $A$ and $B$, that are incomparable: any chain of relations from Definition 2.23 that realizes the equivalence $a \sim b$ for $a \in A$ and $b \in B$ must pass through the positive quadrant or the negative quadrant, each of which accidentally becomes comparable to the other isotypic region $J$ and hence lies in a different uptight region.

Definition 2.32. For any poset $Q$, a module $M$ is $Q$-finite if the vector spaces $M_a$ have finite dimension for all $a \in Q$.

$Q$-finiteness is one of two requirements for the existence of a finite encoding. The other stipulates that $Q$ be decomposable in a well behaved manner with respect to $M$, and this is implied by having only finitely many isotypic regions.

Theorem 2.33. A $Q$-finite module $M$ over a poset $Q$ admits a finite encoding if the isotypic subdivision of $Q$ induced by $M$ is finite. Moreover, in that case $M$ has a finite encoding by its uptight poset $P_M$. This finite encoding is semialgebraic if $Q = \mathbb{R}^n$ and every isotypic region is semialgebraic.
Proof. The first sentence follows from the second. Assume that the $M$-isotypic subdivision is finite. $P_M$ is a finite set because the number of uptight regions is bounded above by $2^{|Q/M|}$; every element of $Q$ lies inside or outside of each isotypic upset and isotypic downset. The set $P_M$ is a poset by Proposition 2.28, and hence the set map $Q \to P_M$ is a poset morphism by definition of the partial order on $P_M$. The module $M$ is constant on the fibers of this quotient morphism, in the sense that the vector spaces $M_a$ are naturally isomorphic for all $a$ in any single uptight region $A \in P_M$, by Proposition 2.30. For each such region $A$, choose a representative $a(A) \in A$, and define $H_A = M_{a(A)}$. The vector spaces $H_A$ for $A \in P_M$ naturally comprise a $P_M$-module by virtue of the structure maps for $M$, and the pullback of $H_A$ to $Q$ is isomorphic to $M$ by construction.

For the final claim, the Minkowski sums $I + \mathbb{R}^n_+$ and $I - \mathbb{R}^n_+$ are semialgebraic whenever $I$ is a semialgebraic isotypic region, and finite intersections of these and their complements are semialgebraic. □

By Theorem 2.33, the notion of finite encoding extends the notion of finiteness encapsulated by finite isotypic subdivision. The next example demonstrates that this is a proper extension.

**Example 2.34.** Let $M$ be the $\mathbb{R}^2$-module that has $M_a = 0$ for all $a \in \mathbb{R}^2$ except for those on the antidiagonal line spanned by $[1\ -1] \in \mathbb{R}^2$, where $M_a = k$. There is only one such $\mathbb{R}^2$-module because all of the degrees of nonzero graded pieces of $M$ are incomparable, so all of the structure homomorphisms $M_a \to M_b$ are zero. Thus every point on the line is a singleton isotypic region. However, these uncountably many isotypic regions can all be gathered together: $M$ has a poset encoding by the chain with three elements, where the fiber over the middle element is the antidiagonal line, and the fibers over the top and bottom elements are the open half-spaces above and below the line, respectively. (This is the uptight poset for the two upsets that are the closed and open half-spaces bounded below by the antidiagonal.) In contrast, using the diagonal line spanned by $[1\ 1] \in \mathbb{R}^2$ instead of the antidiagonal line yields a module with no finite encoding; see Example 2.35.

The direction of the line in Example 2.34 is important: antidiagonal lines, whose points form an antichain in $\mathbb{R}^2$, behave radically differently than diagonal lines.

**Example 2.35.** Let $M$ be an $\mathbb{R}^2$-module with $M_a = k$ whenever $a$ lies in the closed diagonal strip between the lines of slope 1 passing through any pair of points. The structure homomorphisms $M_a \to M_b$ could all be zero, for instance, or some of them could be nonzero. But the length $|a - b|$ of any nonzero such homomorphism must in any case be bounded above by the Manhattan (i.e., $\ell^\infty$) distance between the two points, since every longer structure homomorphism factors through a sequence that exits and re-enters the strip. In particular, the structure homomorphism between any pair of points on the upper boundary line of the strip is zero because it factors through
a homomorphism that points upward first; therefore such pairs of points lie in distinct isotypic regions. The same conclusion holds for pairs of points on the lower boundary line of the strip. Consequently, in any encoding of $M$, the poset must be uncountable.

3. Primary decomposition over partially ordered groups

3.1. Polyhedral partially ordered groups.
The next definition, along with elementary foundations surrounding it, can be found in Goodearl’s book [Goo86, Chapter 1].

Definition 3.1. An abelian group $Q$ is partially ordered if it is generated by a submonoid $Q_+$, called the positive cone, that has trivial unit group. The partial order is: $q \leq q' \iff q' - q \in Q_+$.

Example 3.2. The finitely generated free abelian group $Q = \mathbb{Z}^n$ can be partially ordered with any positive cone $Q_+$, polyhedral or otherwise, although the free commutative monoid $Q_+ = \mathbb{N}^n$ of integer vectors with nonnegative coordinates is most common and serves as a well behaved, well known foundational case to which substantial parts of the general theory reduce.

Example 3.3. The group $Q = \mathbb{R}^n$ can be partially ordered with any positive cone $Q_+$, polyhedral or otherwise, although the orthant $Q_+ = \mathbb{R}^n_+$ of vectors with nonnegative coordinates is most useful for multiparameter persistence.

The following allows the free use of the language of either $Q$-modules or $Q$-graded $k[Q_+]$-modules, as appropriate to the context.

Lemma 3.4. A module over a partially ordered abelian group $Q$ is the same thing as a $Q$-graded module over $k[Q_+]$. □

Example 3.5. When $Q = \mathbb{Z}^n$ and $Q_+ = \mathbb{N}^n$, the relevant monoid algebra is the polynomial ring $k[\mathbb{N}^n] = k[\mathbf{x}]$, where $\mathbf{x} = x_1, \ldots, x_n$ is a sequence of $n$ commuting variables.

Primary decomposition of $Q$-modules depends on certain finiteness conditions. In ordinary commutative algebra, where $Q = \mathbb{Z}^n$, the finiteness comes from $Q_+$, which is assumed to be finitely generated (so it is an affine semigroup). This condition implies that finitely generated $Q$-modules are noetherian: every increasing chain of submodules stabilizes. Primary decomposition is then usually derived as a special case of the theory for finitely generated modules over noetherian rings. But the noetherian condition is stronger than necessary: in the presence of a finitely encoded hypothesis, it suffices for the positive cone to have finitely many faces, in the following sense. To our knowledge, the notion of polyhedral partially ordered group is new, as there is no existing literature on primary decomposition in this setting.

Definition 3.6. A face of the positive cone $Q_+$ of a partially ordered abelian group $Q$ is a submonoid $\sigma \subseteq Q_+$ such that $Q_+ \setminus \sigma$ is an ideal of the monoid $Q_+$. Sometimes it is simpler to say that $\sigma$ is a face of $Q$. Call $Q$ polyhedral if it has only finitely many faces.
Example 3.7. The partially ordered groups in Example 3.3 are polyhedral so long as $Q_+$ has only finitely many extreme rays. This case arises so often in the sequel—it is crucial in Section 6—that it is isolated in Definition 3.8. A non-polyhedral partial order is induced on $Q = \mathbb{R}^3$ by taking $Q_+$ to be a cone over a circle.

Definition 3.8. A real polyhedral group is a group $Q \cong \mathbb{R}^n$, for some $n$, partially ordered so that its positive cone $Q_+$ is an intersection of finitely many closed half-spaces. For notational clarity, $\mathbb{R}^n_+$ always means the nonnegative orthant in $\mathbb{R}^n$, so $Q_+ = \mathbb{R}^n_+$ means the standard componentwise partial order on $\mathbb{R}^n$.

Remark 3.9. The positive cone $Q_+$ in a real polyhedral group $Q = \mathbb{R}^n$ is the set of nonnegative real linear combinations of finitely many vectors that all lie in a single open half-space in $Q$. The cone need not be rational; that is, the vectors that generate it—or the linear functions defining the closed half-spaces—need not have rational entries.

Definition 3.10. A discrete polyhedral group is a finitely generated free abelian group partially ordered so that its positive cone is a finitely generated submonoid.

Remark 3.11. A finitely generated free abelian group can be partially ordered by a positive cone that is not a finitely generated submonoid, such as $Q = \mathbb{Z}^2$ with $Q_+ = C \cap \mathbb{Z}^2$ for the cone $C \subseteq \mathbb{R}^2$ generated by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \pi \end{bmatrix}$. Such objects are ruled out as discrete polyhedral groups because the faces of the positive cone in $Q \otimes \mathbb{R}$ need not be in bijection with the faces of $Q_+$ itself, and the image of $Q$ need not be discrete in the quotient of $Q \otimes \mathbb{R}$ modulo the subgroup spanned by one of those real faces.

3.2. Primary decomposition of downsets.

Definition 3.12. Fix a face $\tau$ of the positive cone $Q_+$ in a polyhedral partially ordered group $Q$. Write $\mathbb{Z}\tau$ for the subgroup of $Q$ generated by $\tau$. Let $D \subseteq Q$ be a downset.

1. The localization of $D$ along $\tau$ is the subset $D_\tau = \{ q \in D \mid q + \tau \subseteq D \}$.
2. An element $q \in D$ is globally supported on $\tau$ if $q \notin D_{\tau'}$ whenever $\tau' \not\subseteq \tau$.
3. The part of $D$ globally supported on $\tau$ is $\Gamma_\tau D = \{ q \in D \mid q \text{ is supported on } \tau \}$.
4. An element $q \in D$ is locally supported on $\tau$ if $q$ is globally supported on $\tau$ in $D_\tau$.
5. The local $\tau$-support of $D$ is the subset $\Gamma_\tau(D_\tau) \subseteq D$ consisting of elements globally supported on $\tau$ in the localization $D_\tau$.
6. The $\tau$-primary component of $D$ is the downset $P_\tau(D) = \Gamma_\tau(D_\tau) - Q_+$ cogenerated by the local $\tau$-support of $D$. 
Example 3.13. The coprincipal downset \( a + \tau - Q_+ \) inside of \( Q = \mathbb{Z}^n \) cogenerated by \( a \) along \( \tau \) is globally supported along \( \tau \). It also equals its own localization along \( \tau \), so it equals its local \( \tau \)-support and is its own \( \tau \)-primary component. Note that when \( Q_+ = \mathbb{N}^n \), faces of \( Q_+ \) correspond to subsets of \([n] = \{1, \ldots, n\} \), the correspondence being \( \tau \leftrightarrow \chi(\tau) \), where \( \chi(\tau) = \{i \in [n] \mid e_i \in \tau\} \) is the characteristic subset of \( \tau \) in \([n]\). (The vector \( e_i \) is the standard basis vector whose only nonzero entry is 1 in slot \( i \).)

Remark 3.14. The localization of \( D \) along \( \tau \) is acted on freely by \( \tau \). Indeed, \( D_\tau \) is the union of those cosets of \( \mathbb{Z}\tau \) each of which is already contained in \( D \). The minor point being made here is that the coset \( q + \mathbb{Z}\tau \) is entirely contained in \( D \) as soon as \( q + \tau \subseteq D \) because \( D \) is a downset: \( q + \mathbb{Z}\tau = q + \tau - \tau \subseteq q + \tau - Q_+ \subseteq D \) if \( q + \tau \subseteq D \).

Remark 3.15. The localization of \( D \) is defined to reflect localization at the level of \( Q \)-modules: enforcing invertibility of structure homomorphisms \( k[D]_q \to k[D]_{q+f} \) for \( f \in \tau \) results in a localized indicator module \( k[D][\mathbb{Z}\tau] = k[D_\tau] \).

Example 3.16. In Definition 3.12, assume that \( Q \) is a real polyhedral group. Then an element \( q \in D \) is globally supported on \( \tau \) if and only if it lands outside of \( D \) when pushed far enough in any direction outside of \( \tau \)—that is, every element \( f \in Q_+ \setminus \tau \) has a nonnegative integer multiple \( \lambda f \) with \( \lambda f + q \notin D \).

One implication is easy: if every \( f \in Q_+ \setminus \tau \) has \( \lambda f + q \notin D \) for some \( \lambda \in \mathbb{N} \), then any element \( f' \in \tau' \setminus \tau \) has a multiple \( \lambda f' \in \tau' \) such that \( \lambda f' + q \notin D \), so \( q \notin D_{\tau'} \). For the other direction, use that \( Q_+ \setminus \tau \) is generated as an upset (i.e., an ideal) of \( Q_+ \) by the nonzero vectors along the rays of \( Q_+ \) that are not contained in \( \tau \). By hypothesis, \( q \in \Gamma_\rho D \Rightarrow q \notin D_\rho \) for all such rays \( \rho \), so along each \( \rho \) there is a vector \( v_\rho \) with \( v_\rho + q \notin D \). Given \( f \in Q_+ \setminus \tau \), choose \( \lambda \in \mathbb{N} \) big enough so that \( \lambda f \geq v_\rho \) for some \( \rho \).

Example 3.17. The \( \tau \)-primary component of \( D \) in Definition 3.12 need not be supported on \( \tau \). Take \( D \) to be the "under-hyperbola" downset in Example 1.5. Then \( D = P_0(D) \), where \( 0 \) is the face of \( Q_+ \) consisting of only the origin. Points outside of \( Q_+ \) are not supported at the origin, being instead supported at either the \( x \)-axis (if the point is below the \( x \)-axis) or the \( y \)-axis (if the point is behind the \( y \)-axis). The subsets depicted in Example 1.5 are \( D \) itself; the global support on \( 0 \), which equals the local support on \( 0 \); the local support on the \( x \)-axis; and the local support on the \( y \)-axis. In contrast, the global support on (say) the \( y \)-axis consists of the part of the local support that sits strictly above the \( x \)-axis.

Definition 3.18. Fix a downset \( D \) in a polyhedral partially ordered group \( Q \).

1. The downset \( D \) is coprimary if \( D = P_\tau(D) \) for some face \( \tau \) of the positive cone \( Q_+ \). If \( \tau \) needs to specified then \( D \) is called \( \tau \)-coprimary.

2. A primary decomposition of \( D \) is an expression \( D = \bigcup_{i=1}^r D_i \) of coprimary downsets \( D_i \), called components of the decomposition.
Theorem 3.19. Every downset $D$ in a polyhedral partially ordered group $Q$ is the union $\bigcup_\tau \Gamma_\tau(D_\tau)$ of its local $\tau$-supports for all faces $\tau$ of the positive cone.

Proof. Given an element $q \in D$, finiteness of the number of faces implies the existence of a face $\tau$ that is maximal among those such that $q \in D_\tau$; note that $q \in D = D_0$ for the trivial face $0$ consisting of only the identity of $Q$. It follows immediately that $q$ is supported on $\tau$ in $D_\tau$. \hfill $\square$

Corollary 3.20. Every downset $D$ in a polyhedral partially ordered group $Q$ has a canonical primary decomposition $D = \bigcup_\tau P_\tau(D)$, the union being over all faces $\tau$ of the positive cone with nonempty support $\Gamma_\tau(D_\tau)$.

Remark 3.21. The union in Theorem 3.19 is not necessarily disjoint. Nor, consequently, is the union in Corollary 3.20. There is a related union, however, that is disjoint: the sets $(\Gamma_\tau D) \cap D_\tau$ do not overlap. Their union need not be all of $D$, however; try Example 3.17, where the negative quadrant intersects none of the sets $(\Gamma_\tau D) \cap D_\tau$.

Algebraically, $(\Gamma_\tau D) \cap D_\tau$ should be interpreted as taking the elements of $D$ globally supported on $\tau$ and then taking their images in the localization along $\tau$, which deletes the elements that aren’t locally supported on $\tau$. That is, $(\Gamma_\tau D) \cap D_\tau$ is the set of degrees where the image of $\Gamma_\tau \mathbb{k}[D] \rightarrow \mathbb{k}[D]_\tau$ is nonzero.

Example 3.22. The decomposition in Theorem 3.19—and hence Corollary 3.20—is not necessarily minimal: it might be that some of the canonically defined components can be omitted. This occurs, for instance, in Example 1.6. The general phenomenon, as in this hyperbola example, stems from geometry of the elements in $D_\tau$ supported on $\tau$, which need not be bounded in any sense, even in the quotient $Q/\mathbb{Z}_\tau$. In contrast, for (say) quotients by monomial ideals in the polynomial ring $\mathbb{k}[\mathbb{N}^n]$, only finitely many elements have support at the origin, and the downset they cogenerate is consequently artinian.

3.3. Primary decomposition of finitely encoded modules.

This section leverages the finitely encoded condition to produce primary decompositions. First, the support $\Gamma_\tau$ on a face $\tau$ needs to be defined as a functor on modules.

Definition 3.23. Fix a face $\tau$ of a polyhedral partially ordered group $Q$. The localization of a $Q$-module $M$ along $\tau$ is the tensor product

$$M_\tau = M \otimes_{\mathbb{k}[Q_+]} \mathbb{k}[Q_+ + \mathbb{Z}_\tau],$$

viewing $M$ as a $Q$-graded $\mathbb{k}[Q_+]$-module. The submodule of $M$ globally supported on $\tau$ is

$$\Gamma_\tau M = \bigcap_{\tau' \subset \tau} \ker(M \rightarrow M_{\tau'}) = \ker(M \rightarrow \bigoplus_{\tau' \subset \tau} M_{\tau'}).$$

Example 3.24. Definition 3.12.2 says that $1_q \in \mathbb{k}[D]_q = \mathbb{k}$ lies in $\Gamma_\tau \mathbb{k}[D]$ if and only if $q \in \Gamma_\tau D$, because $q \notin D_{\tau'}$ if and only if $1_q \mapsto 0$ under localization of $\mathbb{k}[D]$ along $\tau'$. 
Lemma 3.25. The kernel of any natural transformation between two exact covariant functors is left-exact. In more detail, if \( \alpha \) and \( \beta \) are two exact covariant functors \( \mathcal{A} \to \mathcal{B} \) for abelian categories \( \mathcal{A} \) and \( \mathcal{B} \), and \( \gamma_X : \alpha(X) \to \beta(X) \) naturally for all objects \( X \) of \( \mathcal{A} \), then the association \( X \mapsto \ker \gamma_X \) is a left-exact covariant functor \( \mathcal{A} \to \mathcal{B} \).

Proof. This can be checked by diagram chase or spectral sequence. \( \square \)

Proposition 3.26. The global support functor \( \Gamma_\tau \) is left-exact.

Proof. Use Lemma 3.25: global support is the kernel of the natural transformation from the identity to a direct sum of localizations. \( \square \)

Proposition 3.27. For modules over a polyhedral partially ordered group, localization commutes with taking support: \( (\Gamma_\tau \cdot M)_\tau = \Gamma_\tau \cdot (M_\tau) \), and both sides are 0 unless \( \tau' \supseteq \tau \).

Proof. Localization along \( \tau \) is exact, so

\[
\ker (M \to M_{\tau''})_\tau = \ker (M_\tau \to (M_{\tau''})_\tau) = \ker (M_\tau \to (M_{\tau'})_\tau).
\]

Since localization along \( \tau \) commutes with intersections of submodules, \( (\Gamma_\tau \cdot M)_\tau \) is the intersection of the leftmost of these modules over \( \tau'' \not\subseteq \tau' \). But \( \Gamma_{\tau'}(M_\tau) \) equals the same intersection of the rightmost of these modules by definition. And if \( \tau'' \not\supseteq \tau \) then one of these \( \tau'' \) equals \( \tau \), so \( M_\tau \to (M_{\tau'})_\tau = M_\tau \) is the identity map, whose kernel is 0. \( \square \)

Definition 3.28. Fix a \( Q \)-module \( M \) for a polyhedral partially ordered group \( Q \).

1. The \textit{local \( \tau \)-support} of \( M \) is the module \( \Gamma_\tau \cdot M_\tau \) of elements globally supported on \( \tau \) in the localization \( M_\tau \), or equivalently (by Proposition 3.27) the localization along \( \tau \) of the submodule of \( M \) globally supported on \( \tau \).

2. The module \( M \) is \textit{coprimary} if for some face \( \tau \), the localization map \( M \hookrightarrow M_\tau \) is injective and \( \Gamma_\tau M_\tau \) is an essential submodule of \( M_\tau \). If \( \tau \) needs to be specified then \( M \) is called \( \tau \)-coprimary.

3. A \textit{primary decomposition} of \( M \) is an injection \( M \hookrightarrow \bigoplus_{i=1}^r M/M_i \) into a direct sum of coprimary quotients \( M/M_i \), called \textit{components} of the decomposition.

Remark 3.29. Primary decomposition is usually phrased in terms of primary submodules \( M_i \subseteq M \), which by definition have coprimary quotients \( M/M_i \), satisfying \( \bigcap_{i=1}^r M_i = 0 \) in \( M \). This is equivalent to Definition 3.28.3.

Example 3.30. A primary decomposition \( D = \bigcup_{i=1}^r D_i \) of a downset \( D \) yields a primary decomposition of the corresponding indicator quotient, namely the injection \( k[D] \hookrightarrow \bigoplus_{i=1}^r k[D_i] \) induced by the surjections \( k[D] \twoheadrightarrow k[D_i] \).

The existence of primary decomposition in Theorem 3.32 is intended for finitely encoded modules, but because it deals with essential submodules and not generators, it only requires the downset half of a fringe presentation.
Definition 3.31. A downset hull of a module $M$ over an arbitrary poset is an injection $M \hookrightarrow \bigoplus_{j \in J} E_j$ with each $E_j$ being a downset module. The hull is finite if $J$ is finite. The module $M$ is downset-finite if it admits a finite downset hull.

Theorem 3.32. Every downset-finite module over a polyhedral partially ordered group admits a primary decomposition.

Proof. If $M \hookrightarrow \bigoplus_{j=1}^k E_j$ is a downset hull of the module $M$, and $E_j \hookrightarrow \bigoplus_{i=1}^\ell E_{ij}$ is a primary decomposition for each $j$ afforded by Corollary 3.20 and Example 3.30, then let $E^\tau$ be the direct sum of the downset modules $E_{ij}$ that are $\tau$-coprimary. Set $M^\tau = \ker(M \to E^\tau)$. Then $M/M^\tau$ is coprimary, being a submodule of a coprimary module. Moreover, $M \to \bigoplus_\tau M/M^\tau$ is injective because its kernel is the same as the kernel of $M \to \bigoplus_{ij} E_{ij}$, which is a composite of two injections and hence injective by construction. Therefore $M \to \bigoplus_\tau M/M^\tau$ is a primary decomposition. □

4. Finitely determined $\mathbb{Z}^n$-modules

Unless otherwise stated, this section is presented over the discrete polyhedral group $Q = \mathbb{Z}^n$ with $Q_+ = \mathbb{N}^n$. It begins by reviewing the structure of finitely determined $\mathbb{Z}^n$-modules, including (minimal) injective and flat resolutions. These serve as models for the concepts of socle, cogenerator, and downset hull over real polyhedral groups (Sections 6–10)—as well as their dual notions of top, generator, and upset covers (Section 12)—and is the foundation underlying the syzygy theorem for finitely encoded modules (Section 5.3), including the existence of fringe presentations (Section 5.1).

Our main references for $\mathbb{Z}^n$-modules are [Mil00, MS05]. The development of homological theory for injective and flat resolutions in the context of finitely determined modules is functorially equivalent to the development for finitely generated modules, by [Mil00, Theorem 2.11], but it is convenient to have on hand the statements in the finitely determined case directly. The characterization of finitely determined modules in Proposition 4.7 and (hence) Theorem 4.33 is apparently new.

4.1. Definitions.

The essence of the finiteness here is that all of the relevant information about the relevant modules should be recoverable from what happens in a bounded box in $\mathbb{Z}^n$.

Definition 4.1. A $\mathbb{Z}^n$-finite module $N$ is finitely determined if for each $i = 1, \ldots, n$ the multiplication map $\cdot x_i : N_b \to N_{b+e_i}$ (see Example 3.5 for notation) is an isomorphism whenever $b_i$ lies outside of some bounded interval.

Remark 4.2. This notion of finitely determined is the same notion as in Example 2.8. A module is finitely determined if and only if, after perhaps translating its $\mathbb{Z}^n$-grading, it is $a$-determined for some $a \in \mathbb{N}^n$, as defined in [Mil00, Definition 2.1].
Remark 4.3. For $\mathbb{Z}^n$-modules, the finitely determined condition is weaker—that is, more inclusive—than finitely generated, but it is much stronger than finitely encoded. The reason is essentially Example 2.8, where the encoding has a very special nature. For a generic sort of example, the restriction to $\mathbb{Z}^n$ of any $\mathbb{R}^n$-finite $\mathbb{R}^n$-module with finitely many isotypic regions of sufficient width is a finitely encoded $\mathbb{Z}^n$-module, and there is simply no reason why the isotypic regions should be commensurable with the coordinate directions in $\mathbb{Z}^n$. Already the toy-model fly wing modules in Examples 1.3 and 1.4 yield infinitely generated but finitely encoded $\mathbb{Z}^n$-modules, and this remains true when the discretization $\mathbb{Z}^n$ of $\mathbb{R}^n$ is rescaled by any factor.

Example 4.4. The local cohomology of an affine semigroup rings is finitely encoded but usually not finitely determined; see [HM05] and [MS05, Chapter 13], particularly Theorem 13.20, Example 13.17, and Example 13.4 there.

4.2. Injective hulls and resolutions.

Remark 4.5. Every $\mathbb{Z}^n$-finite module that is injective in the category of $\mathbb{Z}^n$-modules is a direct sum of downset modules $k[D]$ for downsets $D$ cogenerated by vectors along faces (Example 3.13). This statement holds over any discrete polyhedral group (Definition 3.10) by [MS05, Theorem 11.30].

Minimal injective resolutions work for finitely determined modules just as they do for finitely generated modules. The standard definitions are as follows.

Definition 4.6. Fix a $\mathbb{Z}^n$-module $N$.

1. An injective hull of $N$ is an injective homomorphism $N \to E$ in which $E$ is an injective $\mathbb{Z}^n$-module (see Remark 4.5). This injective hull is
   - finite if $E$ has finitely many indecomposable summands and
   - minimal if the number of such summands is minimal.
2. An injective resolution of $N$ is a complex $E^\ast$ of injective $\mathbb{Z}^n$-modules whose differential $E^i \to E^{i+1}$ for $i \geq 0$ has only one nonzero homology $H^0(E^\ast) \cong N$ (so $N \hookrightarrow E^0$ and $\text{coker}(E^{i-1} \to E^i) \hookrightarrow E^{i+1}$ are injective hulls for all $i \geq 1$). $E^\ast$
   - has length $\ell$ if $E^i = 0$ for $i > \ell$ and $E^\ell \neq 0$;
   - is finite if $E^\ast = \bigoplus_i E^i$ has finitely many indecomposable summands; and
   - is minimal if $N \hookrightarrow E^0$ and $\text{coker}(E^{i-1} \to E^i) \hookrightarrow E^{i+1}$ are minimal injective hulls for all $i \geq 1$.

Proposition 4.7. The following are equivalent for a $\mathbb{Z}^n$-module $N$.

1. $N$ is finitely determined.
2. $N$ admits a finite injective resolution.
3. $N$ admits a finite minimal injective resolution.

Any finite minimal resolution is unique up to isomorphism and has length at most $n$. 
Proof. The proof is based on existence of finite minimal injective hulls and resolutions for finitely generated \( \mathbb{Z}^n \)-modules, along with uniqueness and length \( n \) given minimality, as proved by Goto and Watanabe [GW78].

First assume \( N \) is finitely determined. Translating the \( \mathbb{Z}^n \)-grading affects nothing about existence of a finite injective resolution. Therefore, using Remark 4.2, assume that \( N \) is a-determined. Truncate by taking the \( N_{\geq 0} \)-graded part of \( N \) to get a positively a-determined—and hence finitely generated—module \( N_{\geq 0} \); see [Mil00, Definition 2.1]. Take any minimal injective resolution \( N_{\geq 0} \to E^* \). Extend backward using the Čech hull [Mil00, Definition 2.7], which is exact [Mil00, Lemma 2.9], to get a finite minimal injective resolution \( \check{C}(N_{\geq 0} \to E^*) = (N \to \check{C}E^*) \), noting that \( \check{C} \) fixes indecomposable injective modules whose \( \mathbb{Z}^n \)-graded parts are nonzero and is zero on all other indecomposable injective modules [Mil00, Lemma 4.25]. This proves \( 1 \Rightarrow 3 \).

That \( 3 \Rightarrow 2 \) is trivial. The remaining implication, \( 2 \Rightarrow 1 \), follows because every indecomposable injective is finitely determined and the category of finitely determined modules is abelian. (The category of \( \mathbb{Z}^n \)-modules each of which is nonzero only in a bounded set of degrees is abelian, and constructions such as kernels, cokernels, or direct sums in the category of finitely determined modules are pulled back from there.) \( \square \)

4.3. Socles of finitely determined modules.
The nature of minimality in Definition 4.6 plays a decisive role in the theory developed in later sections. Its homological manifestation via socles is particularly essential. Expressing that manifestation requires interactions with localization and restriction.

Definition 4.8. Fix a face \( \tau \) of \( \mathbb{N}^n \) and a \( \mathbb{Z}^n \)-module \( N \).

1. The restriction of \( N \) to the subgroup \( \mathbb{Z} \tau \subseteq \mathbb{Z}^n \) is \( N|_{\tau} = \bigoplus_{a \in \mathbb{Z} \tau} N_a \).

2. If \( N = M_\tau \) is the localization of a \( \mathbb{Z}^n \)-module along \( \tau \), then \( N|_{\tau} \) is the quotient-restriction \( M/\tau \) of \( M \) along \( \tau \), where \( \tau \) is the complement \( [n] \setminus \tau \).

Remark 4.9. The restriction that defines \( M/\tau \) can equivalently be viewed as a quotient: \( \mathbb{Z} \tau \) acts freely on \( M_\tau \), and \( M/\tau \) is the quotient by this action; hence the nomenclature. Another way of viewing \( M/\tau \) as a quotient is via \( M/\tau = M/I_\tau M \), where \( I_\tau = \langle x_i - 1 \mid i \in \chi(\tau) \rangle \). Note that it is not necessary to localize along \( \tau \) before restricting, if one is willing to accept restriction along a translate of \( \mathbb{Z} \tau \). More precisely, for any \( b \) with sufficiently large \( \tau \)-coordinates, \( M/\tau = M|_{b+\mathbb{Z} \tau} = \bigoplus_{a \in \mathbb{Z} \tau} M_{b+a} \). This statement depends on the finitely determined condition; it is false in general.

\( \mathbb{Z}^n \)-finite injective modules have minimal cogenerators whose degrees are canonical when taken in \( \mathbb{Z} \tau = \mathbb{Z}^n/\mathbb{Z} \tau \). This generalizes the dual notion of minimal generators for \( \mathbb{Z}^n \)-graded free modules (or, more generally, minimal generators for finitely generated modules over graded or local rings). The notion of cogenerator extends to arbitrary finitely determined \( \mathbb{Z}^n \)-modules by functoriality, as follows.
Definition 4.10. Fix a $\mathbb{Z}^n$-module $M$ and a face $\tau$ of $\mathbb{N}^n$. A **cogenerator along** $\tau$ of degree $a \in \mathbb{Z}^n$ is a nonzero element $y \in M_a$ that generates a submodule isomorphic to $k[a + \tau] = x^a k[\tau]$. The *cogenerator functor along* $\tau$ takes $M$ to its *socle along* $\tau$: 

$$\overline{\text{soc}}_\tau M = \bigoplus_{a \in \mathbb{Z}^n} \text{Hom}_\mathbb{Z}(k_a, M/\tau)$$

where the *skyscraper* module $k_a$ is the indicator subquotient for the singleton $\{a\} \subseteq \mathbb{Z}^n$.

The bar over soc is explained after Definition 6.22 and in Remark 6.28.

Proposition 4.11. An injective hull $M \to E$ is minimal as in Definition 4.6 if and only if $\overline{\text{soc}}_\tau M \to \overline{\text{soc}}_\tau E$ is an isomorphism for all faces $\tau$ of $\mathbb{N}^n$.

Proof. See [MS05, Section 11.5]. □

4.4. Matlis duality.

Finitely determined flat modules and their properties are best described and deduced using a duality that turns $\mathbb{Z}^n$ upside down and takes vector space duals. However, since this duality is needed in greater generality later, the next definitions are more general.

Definition 4.12. A poset $Q$ is **self-dual** if it is given a poset automorphism $q \mapsto -q$.

Example 4.13. Inversion makes partially ordered abelian groups self-dual as posets.

Definition 4.14. Fix a self-dual poset $Q$. The *Matlis dual* of a $Q$-module $M$ is the $Q$-module $M^\vee$ defined by

$$(M^\vee)_q = \text{Hom}_q(M_{-q}, k),$$

so the homomorphism $(M^\vee)_q \to (M^\vee)_{q'}$ is transpose to $M_{-q'} \to M_{-q}$.

Example 4.15. The Matlis dual over a partially ordered group $Q$ is equivalently

$$M^\vee = \text{Hom}_Q(M, k[Q^+]^\vee),$$

where $\text{Hom}_Q(M, N) = \bigoplus_{q \in Q} \text{Hom}(M, N(q))$ is the direct sum of all degree-preserving homomorphisms from $M$ to $Q$-graded translates of $N$. This is proved using the adjunction between Hom and $\otimes$; see [MS05, Lemma 11.16], noting that the nature of the grading group is immaterial. And as in [MS05, Lemma 11.16],

$$\text{Hom}_Q(M, N^\vee) = (M \otimes_Q N)^\vee.$$

Example 4.16. It is instructive to compute the Matlis dual of localization along a face $\tau$ over a partially ordered abelian group: the Matlis dual of $M_\tau$ is

$$(M_\tau)^\vee = \text{Hom}(k[Q^+]_\tau \otimes M, k) = \text{Hom}(k[Q^+]_\tau, \text{Hom}(M, k)) = \text{Hom}(k[Q^+]_\tau, M^\vee),$$

the module of homomorphisms from a localization of $k[Q^+]$ into $M$. The unfamiliarity of this functor is one of the reasons for developing most of the theory in this paper in terms of socles and cogenerators instead of tops and generators.
Lemma 4.17. $(M')^\vee$ is canonically isomorphic to $M$ if $\dim_k M_q < \infty$ for all $q \in Q$. □

Remark 4.18. The Matlis dual of Remark 4.5 says that every $Q$-finite flat module over a discrete polyhedral group $Q$ is isomorphic to a direct sum of upset modules $\mathbb{k}[U]$ for upsets of the form $U = b + \mathbb{Z}\tau + Q_+$. These upset modules are the graded translates of localizations of $\mathbb{k}[Q_+]$ along faces.

Lemma 4.19. $\text{Hom}(\mathbb{k}[Q_+], -)$ is exact for all faces $\tau$ of any partially ordered group $Q$.

Proof. Apply the final line of Example 4.15, using that $\mathbb{k}[Q_+]$ is flat (it is a localization of $\mathbb{k}[Q_+]$) and $(-)^\vee$ is exact and faithful. □

Remark 4.20. What really drives the lemma is the observation that while the opposite notion to injective is projective (reverse all of the arrows in the definition), the adjoint notion to injective is flat. That is, a module is flat if and only if its Matlis dual is injective. This is an instance of a rather general phenomenon that can be phrased in terms of a monoidal abelian category $C$ possessing a Matlis object $E$ for a Matlis dual pair of subcategories $\mathcal{A}$ and $\mathcal{B}$ such that $\text{Hom}(-, E)$ restricts to exact contravariant functors $\mathcal{A} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{A}$ that are inverse to one another. The idea is to set $M' = \text{Hom}(M, E)$, the Matlis dual of any object $M$ of $C$, and define an object of $C$ to be $\mathcal{B}$-flat if $F \otimes -$ is an exact functor on $\mathcal{B}$. Then an object $F$ of $\mathcal{A}$ is $\mathcal{B}$-flat if and only if $\text{Hom}(F, -)$ is an exact functor on $\mathcal{A}$. Examples of this situation include artinian and noetherian modules over a complete local ring; modules of finite length over any local ring (in both cases $E = E(R/m)$ is the injective hull of the residue field); and of course $Q$-finite modules over a partially ordered abelian group $Q$. The latter two examples feature a single Matlis self-dual subcategory.

4.5 Flat covers, flat resolutions, and generators.

Minimal flat resolutions are not commonplace, but the notion is Matlis dual to that of minimal injective resolution. In the context of finitely determined modules, flat resolutions work as well as injective resolutions. The definitions are as follows.

Definition 4.21. Fix a $\mathbb{Z}^n$-module $N$.

1. A flat cover of $N$ is a surjective homomorphism $F \to N$ in which $F$ is a flat $\mathbb{Z}^n$-module (see Remark 4.18). This flat cover is
   - finite if $F$ has finitely many indecomposable summands and
   - minimal if the number of such summands is minimal.

2. A flat resolution of $N$ is a complex $F$ of flat $\mathbb{Z}^n$-modules whose differential $F_{i+1} \to F_i$ for $i \geq 0$ has only one nonzero homology $H_0(F_i) \cong N$ (so $F_0 \to N$ and $F_{i+1} \to \ker(F_i \to F_{i-1})$ are flat covers for all $i \geq 1$). The flat resolution $F$.
   - has length $\ell$ if $F_i = 0$ for $i > \ell$ and $F_\ell \neq 0$;
   - is finite if $F_\ell = \bigoplus F_i$ has finitely many indecomposable summands; and
   - is minimal if $F_0 \to N$ and $F_{i+1} \to \ker(F_i \to F_{i-1})$ are minimal flat covers for all $i \geq 1$. 


Definition 4.22. Fix a $\mathbb{Z}^n$-module $M$ and a face $\tau$ of $\mathbb{N}^n$. A generator of $M$ along $\tau$ of degree $a \in \mathbb{Z}^n$ is a nonzero element $y \in M_a$ that maps to a socle element in a quotient of $M$ isomorphic to $k[a - \tau] = k[-a + \tau]$. The generator functor along $\tau$ takes $M$ to its top along $\tau$: 

$$\text{top}_\tau M = \mathbb{k} \otimes_{\mathbb{Z}^n} (\text{Hom}_{\mathbb{Z}^n}(k[\mathbb{N}^n + \mathbb{Z} \tau], M)/\tau),$$

where $\mathbb{k} = k_0$ is the skyscraper $\mathbb{Z} \tau$-module in degree 0.

Remark 4.23. The definition of generator may feel unfamiliar. Usually a minimal generator of a module $M$ over (say) a complete local ring $R$ with maximal ideal $m$ is an element $y \in M$ whose image is part of a basis for $M/mM$. Equivalently, $y$ is nonzero in a quotient of $M$ that is isomorphic to $R/m$. What has been relaxed here is that to consider generators along a prime ideal $p$ of positive dimension, the definition requires $y$ to map to the socle of a quotient of $M$ isomorphic to the Matlis dual of $R/p$; the generator $y$ then “extends backward in $M$ along $R/p$”. The generator functor $\text{top}_\tau$ encapsulates this by first attempting to insert the localization $k[\mathbb{N}^n]_\tau$ into $M$—which only works if there is an element of $M$ that can be “extended backward along $\tau$”—and then taking the ordinary generators of the quotient-restriction along $\tau$.

The functorial relationship between top and socle is the main mode of proof for tops.

Proposition 4.24. The generator functor over $\mathbb{Z}^n$ along a face $\tau$ of $\mathbb{N}^n$ is Matlis dual to the cogenerator functor over $\mathbb{Z}^n$ along $\tau$ on the Matlis dual:

$$\overline{\text{top}}_\tau M = (\overline{\text{soc}}_\tau(M^\vee))^\vee.$$

Proof. This is an exercise in the adjointness of $\text{Hom}$ and $\otimes$, using Example 4.16 (applied to $M^\vee$). To simplify notation in the argument, set $N = \text{Hom}(k[\mathbb{N}^n]_\tau, M)$. Then

$$(\overline{\text{soc}}_\tau(M^\vee))^\vee = \text{Hom}_{\mathbb{Z}^n}(k, M^\vee/\tau)^\vee$$

$$= \text{Hom}_{\mathbb{Z}^n}(k[\mathbb{Z} \tau], (M^\vee)_\tau)^\vee/\tau$$

$$= \text{Hom}_{\mathbb{Z}^n}(k[\mathbb{Z} \tau], \text{Hom}(k[\mathbb{N}^n]_\tau, M))^{\vee}/\tau$$

$$= \text{Hom}_{\mathbb{Z}^n}(k[\mathbb{Z} \tau], N^{\vee})^{\vee}/\tau$$

$$= \text{Hom}_{\mathbb{Z}^n}(k[\mathbb{Z} \tau], \text{Hom}(N, k))^{\vee}/\tau$$

$$= \text{Hom}_{\mathbb{Z}^n}(k[\mathbb{Z} \tau] \otimes_{\mathbb{Z}^n} N, k)^{\vee}/\tau$$

$$= (k[\mathbb{Z} \tau] \otimes_{\mathbb{Z}^n} N)^{\vee}/\tau$$

$$= (k[\mathbb{Z} \tau] \otimes_{\mathbb{Z}^n} N)/\tau$$

$$= k \otimes_{\mathbb{Z} \tau} (N/\tau).$$

Here is a prototypical example of how to use this dual-to-socle characterization.

Corollary 4.25. A flat cover $F \to M$ is minimal as in Definition 4.21 if and only if $\text{top}_\tau F \to \text{top}_\tau M$ is an isomorphism for all faces $\tau$ of $\mathbb{N}^n$.

Proof. This is Matlis dual to Proposition 4.11, by Proposition 4.24.
4.6. Flange presentations.

Definition 4.26. Fix a $\mathbb{Z}^n$-module $N$.

1. A flange presentation of $N$ is a $\mathbb{Z}^n$-module morphism $\varphi : F \to E$, with image isomorphic to $N$, where $F$ is flat and $E$ is injective in the category of $\mathbb{Z}^n$-modules.
2. If $F$ and $E$ are expressed as direct sums of indecomposables, then $\varphi$ is based.
3. If $F$ and $E$ are finite direct sums of indecomposables, then $\varphi$ is finite.
4. If the number of indecomposable summands of $F$ and $E$ are simultaneously minimized then $\varphi$ is minimal.

Remark 4.27. The term flange is a portmanteau of flat and injective (i.e., “flainj”) because a flange presentation is the composite of a flat cover and an injective hull.

Flange presentations are effective data structures via the following notational trick. Topologically, it highlights that births occur along generators of the flat summands and deaths occur along cogenerators of the injective summands, with a linear map, over the ground field, to relate them.

Definition 4.28. Fix a based finite flange presentation $\varphi : \bigoplus_p F_p = F \to E = \bigoplus_q E_q$. A monomial matrix for $\varphi$ is an array of scalar entries $\varphi_{pq}$ whose columns are labeled by the indecomposable flat summands $F_p$ and whose rows are labeled by the indecomposable injective summands $E_q$:

$$
\begin{bmatrix}
E_1 & \cdots & E_\ell \\
F_1 & [\varphi_{11} & \cdots & \varphi_{1\ell} ] \\
\vdots & \vdots & \ddots & \vdots \\
F_k & [\varphi_{k1} & \cdots & \varphi_{k\ell} ]
\end{bmatrix}
$$

$F_1 \oplus \cdots \oplus F_k = F \xrightarrow{\varphi} E = E_1 \oplus \cdots \oplus E_\ell.$

The entries of the matrix $\varphi_{..}$ correspond to homomorphisms $F_p \to E_q$.

Lemma 4.29. If $F = \mathbb{k}[a + \mathbb{Z} \tau' + \mathbb{N}^n]$ is an indecomposable flat $\mathbb{Z}^n$-module and $E = \mathbb{k}[b + \mathbb{Z} \tau - \mathbb{N}^n]$ is an indecomposable injective $\mathbb{Z}^n$-module, then $\text{Hom}_{\mathbb{Z}^n}(F, E) = 0$ unless $(a + \mathbb{Z} \tau' + \mathbb{N}^n) \cap (b + \mathbb{Z} \tau - \mathbb{N}^n) \neq \emptyset$, in which case $\text{Hom}_{\mathbb{Z}^n}(F, E) = \mathbb{k}$.

Proof. This is a special case of Lemma 2.19. □

Definition 4.30. In the situation of Lemma 4.29, write $F \preceq E$ if their degree sets have nonempty intersection: $(a + \mathbb{Z} \tau' + \mathbb{N}^n) \cap (b + \mathbb{Z} \tau - \mathbb{N}^n) \neq \emptyset$.

Proposition 4.31. With notation as in Definition 4.28, $\varphi_{pq} = 0$ unless $F_p \preceq E_q$. Conversely, if an array of scalars $\varphi_{pq} \in \mathbb{k}$ with rows labeled by indecomposable flat modules and columns labeled by indecomposable injectives has $\varphi_{pq} = 0$ unless $F_q \preceq E_q$, then it represents a flange presentation.

Proof. Lemma 4.29 and Definition 4.30. □
The unnatural hypothesis that a persistence module be finitely generated results in data types and structure theory that are asymmetric regarding births as opposed to deaths. In contrast, the notion of flange presentation is self-dual: their duality interchanges the roles of births \((F)\) and deaths \((E)\).

**Proposition 4.32.** A \(\mathbb{Z}^n\)-module \(N\) has a finite flange presentation \(F \rightarrow E\) if and only if the Matlis dual \(E^\vee \rightarrow F^\vee\) is a finite flange presentation of the Matlis dual \(N^\vee\).

**Proof.** Matlis duality is an exact, contravariant functor on \(\mathbb{Z}^n\)-modules that takes the subcategory of finitely determined \(\mathbb{Z}^n\)-modules to itself (these properties are immediate from the definitions), interchanges flat and injective objects therein, and has the property that the natural homomorphism \((N^\vee)^\vee \rightarrow N\) is an isomorphism for finitely determined \(N\); see [Mil00, Section 1.2] for a discussion of these properties. \(\square\)

**4.7. Syzygy theorem for \(\mathbb{Z}^n\)-modules.**

**Theorem 4.33.** A \(\mathbb{Z}^n\)-module is finitely determined if and only if it admits one, and hence all, of the following:

1. a finite flange presentation; or
2. a finite flat presentation; or
3. a finite injective copresentation; or
4. a finite flat resolution; or
5. a finite injective resolution; or
6. a minimal one of any of the above.

Any minimal one of these objects is unique up to noncanonical isomorphism, and the resolutions have length at most \(n\).

**Proof.** The hard work is done by Proposition 4.7. It implies that \(N\) is finitely determined \(\Leftrightarrow N^\vee\) has a minimal injective resolution \(\Leftrightarrow N\) has a minimal flat resolution of length at most \(n\), since the Matlis dual of any finitely determined module \(N\) is finitely determined. Having both a minimal injective resolution and a minimal flat resolution is stronger than having any of items 1–3, minimal or otherwise, so it suffices to show that \(N\) is finitely determined if \(N\) has any of items 1–3. This follows, using that the category of finitely determined modules is abelian as in the proof of Proposition 4.7, from the fact that every indecomposable injective or flat \(\mathbb{Z}^n\)-module is finitely determined. \(\square\)

**Remark 4.34.** Conditions 1–6 in Theorem 4.33 remain equivalent for \(\mathbb{R}^n\)-modules, with the standard positive cone \(\mathbb{R}^n_+\), assuming that the finite flat and injective modules in question are finite direct sums of localizations of \(\mathbb{R}^n\) along faces and their Matlis duals. (The equivalence, including minimality, is a consequence of the generator and cogenerator theory over real polyhedral groups, particularly Theorem 14.2.) The equivalent conditions do not characterize \(\mathbb{R}^n\)-modules that are pulled back under convex projection from arbitrary modules over an interval in \(\mathbb{R}^n\), though, because all sorts of infinite things can happen inside of a box, such as having generators along a curve.
5. Fringe presentations and syzygy theorem for poset modules

For modules over arbitrary posets, the conditions of flatness and injectivity still make sense, but it is too restrictive to ask for resolutions or presentations that are finite direct sums of indecomposables in such generality, as demonstrated by the formidable infinitude of such objects in the case of $\mathbb{R}^n$-modules like fly wing modules. The idea here, both for theoretical and computational purposes, is to allow arbitrary upset and downset modules instead of only flat and injective ones. The data structures in the title of this paper refer, in practical terms, to monomial matrices constructed from fringe presentations and indicator resolutions.

5.1. Fringe presentations.

Definition 5.1. Fix any poset $Q$. A fringe presentation of a $Q$-module $M$ is

- a direct sum $F$ of upset modules $\mathbb{k}[U]$, 
- a direct sum $E$ of downset modules $\mathbb{k}[D]$, and 
- a homomorphism $F \to E$ of $Q$-modules with
  - image isomorphic to $M$ and
  - components $\mathbb{k}[U] \to \mathbb{k}[D]$ that are connected (Definition 2.21).

A fringe presentation is

1. finite if the direct sums are finite;
2. subordinate to an encoding of $M$ (Definition 2.6) if each summand $\mathbb{k}[U]$ of $F$ and $\mathbb{k}[D]$ of $E$ is constant on every fiber of the encoding poset morphism; and
3. semialgebraic if $Q$ is a partially ordered real vector space of finite dimension and the fringe presentation is subordinate to a semialgebraic encoding (Definition 2.9).

Lemma 5.2. An indicator module is constant on every fiber of a poset morphism $\pi : Q \to P$ if and only if it is the pullback along $\pi$ of an indicator $P$-module.

Proof. The “if” direction is by definition. For the “only if” direction, observe that if $U \subseteq Q$ is an upset that is a union of fibers of $P$, then the image $\pi(U) \subseteq P$ is an upset whose preimage equals $U$. The same argument works for downsets. □

Example 5.3 (Pullbacks of flat and injective modules). An indecomposable flat $\mathbb{Z}^n$-module $\mathbb{k}[b + \mathbb{Z}\tau + \mathbb{N}^n]$ is a downset module for the poset $\mathbb{Z}^n$. Pulling back to any poset under a poset map to $\mathbb{Z}^n$ therefore yields a downset module for the given poset. The dual statement holds for any indecomposable injective module $\mathbb{k}[b + \mathbb{Z}\tau - \mathbb{N}^n]$: its pullback is a downset module.

Definition 5.4. Fix a finite fringe presentation $\varphi : \bigoplus_p \mathbb{k}[U_p] = F \to E = \bigoplus_q \mathbb{k}[D_q]$. A monomial matrix for $\varphi$ is an array of scalar entries $\varphi_{pq}$ whose columns are labeled
by the birth upsets $U_p$ and whose rows are labeled by the death downsets $D_q$:

\[
\begin{bmatrix}
D_1 & \cdots & D_\ell \\
U_1 \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1\ell} \\
\vdots & \ddots & \vdots \\
U_k \begin{bmatrix} \varphi_{k1} & \cdots & \varphi_{k\ell} \\
\end{bmatrix}
\end{bmatrix} 
\end{bmatrix}
\]

\[
\mathbb{k}[U_1] \oplus \cdots \oplus \mathbb{k}[U_k] \xrightarrow{F} E = \mathbb{k}[D_1] \oplus \cdots \oplus \mathbb{k}[D_\ell].
\]

**Proposition 5.5.** With notation as in Definition 5.4, $\varphi_{pq} = 0$ unless $U_p \preceq D_q$. Conversely, if an array of scalars $\varphi_{pq} \in \mathbb{k}$ with rows labeled by upsets and columns labeled by downsets has $\varphi_{pq} = 0$ unless $U_p \preceq D_q$, then it represents a fringe presentation.

**Proof.** Lemma 2.19.1 and Definition 2.21.

Pullbacks have particularly transparent monomial matrix interpretations.

**Proposition 5.6.** Fix a poset $Q$ and an encoding of a $Q$-module $M$ via a poset morphism $\pi : Q \to P$ and $P$-module $H$. Any monomial matrix for a fringe presentation of $H$ pull back to a monomial matrix for a fringe presentation subordinate to the encoding by replacing the row labels $U_1, \ldots, U_k$ and column labels $D_1, \ldots, D_\ell$ with their preimages, namely $\pi^{-1}(U_1), \ldots, \pi^{-1}(U_k)$ and $\pi^{-1}(D_1), \ldots, \pi^{-1}(D_\ell)$.

**Remark 5.7.** The term “fringe” is a portmanteau of “free” and “injective” (i.e., “frinj”), the point being that it combines aspects of free and injective resolutions while also conveying that the data structure captures trailing topological features at both the birth and death ends.

### 5.2. Indicator resolutions

**Definition 5.8.** Fix any poset $Q$ and a $Q$-module $M$.

1. An **upset resolution** of $M$ is a complex $F$ of $Q$-modules, each a direct sum of upset submodules of $\mathbb{k}[Q]$, whose differential $F_i \to F_{i-1}$ decreases homological degrees and has only one nonzero homology $H_0(F) \cong M$.

2. A **downset resolution** of $M$ is a complex $E^*$ of $Q$-modules, each a direct sum of downset quotient modules of $\mathbb{k}[Q]$, whose differential $E^i \to E^{i+1}$ increases cohomological degrees and has only one nonzero homology $H^0(E^*) \cong M$.

An upset or downset resolution is called an **indicator resolution** if the up- or down- nature is unspecified. The **length** of an indicator resolution is the largest (co)homological degree in which the complex is nonzero. An indicator resolution is

3. **finite** if the number of indicator module summands is finite,

4. **subordinate** to an encoding of $M$ (Definition 2.6) if each indicator summand is constant on every fiber of the encoding poset morphism, and

5. **semialgebraic** if $Q$ is a partially ordered real vector space of finite dimension and the resolution is subordinate to a semialgebraic encoding (Definition 2.9).
Definition 5.9. Monomial matrices for indicator resolutions are defined similarly to how they are for fringe presentations in Definition 5.4, except that for the cohomological case the row and column labels are source and target downsets, respectively, while in the homological case the row and column labels are target and source upsets, respectively:

\[
\begin{pmatrix}
\cdots & D_q & \cdots \\
E^i & \varphi_{pq} & E^{i+1} \\
\vdots & \vdots & \vdots \\
D_p & \cdots & \cdots \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\cdots & U_{i+1} & \cdots \\
F_i & \varphi_{pq} & F_{i+1} \\
\vdots & \vdots & \vdots \\
U_p & \cdots & \cdots \\
\end{pmatrix}
\]

(Note the switch of source and target from cohomological to homological, so the map goes from right to left in the homological case, with decreasing homological indices.)

As in Proposition 5.6, pullbacks have transparent of monomial matrix interpretation.

Proposition 5.10. Fix a poset \(Q\) and an encoding of a \(Q\)-module \(M\) via a poset morphism \(\pi : Q \to P\) and \(P\)-module \(H\). Monomial matrices for any indicator resolution of \(H\) pull back to monomial matrices for an indicator resolution of \(M\) subordinate to the encoding by replacing the row and column labels with their preimages under \(\pi\).

Definition 5.11. Fix any poset \(Q\) and a \(Q\)-module \(M\).

1. An upset presentation of \(M\) is an expression of \(M\) as the cokernel of a homomorphism \(F_1 \to F_0\) each of which is a direct sum of upset modules.
2. A downset copresentation of \(M\) is an expression of \(M\) as the kernel of a homomorphism \(E^0 \to E^1\) each of which is a direct sum of downset modules.

These indicator presentations are finite, or subordinate to an encoding of \(M\), or semi-algebraic as in Definition 5.8.

Remark 5.12. It is tempting to think that a fringe presentation is nothing more than the concatenation of the augmentation map of an upset resolution (that is, the surjection at the end) with the augmentation map of a downset resolution (that is, the injection at the beginning), but there is no guarantee that the components \(F_i \to E_j\) of the homomorphism thus produced are connected (Definition 2.21).

5.3. Syzygy theorem for modules over posets.

Proposition 5.13. For any inclusion \(\iota : P \to Z\) of posets and \(P\)-module \(H\) there is a \(Z\)-module \(\iota_* H\), the pushforward to \(Z\), whose restriction to \(\iota(P)\) is \(H\) and is universally repelling: \(\iota_* H\) has a canonical map to every \(Z\)-module whose restriction to \(\iota(P)\) is \(H\).

Proof. At \(z \in Z\) the pushforward \(\iota_* H\) places the colimit \(\lim H\leq z\) of the diagram of vector spaces indexed by the elements of \(P\) whose images precede \(z\). The universal property of colimits implies that \(\iota_* H\) is a \(Z\)-module with the desired universal property. □
Remark 5.14. With certain perspectives as in Remark 2.4, the pushforward is a left Kan extension [Cur14, Remark 4.2.9].

Definition 5.15. An encoding of a $Q$-module dominates a fringe presentation, indicator resolution, or indicator presentation if any of these is subordinate to the encoding.

Theorem 5.16. A module $M$ over a poset $Q$ is finitely encoded if and only if it admits one, and hence all, of the following:

1. a finite fringe presentation; or
2. a finite upset presentation; or
3. a finite downset copresentation; or
4. a finite upset resolution; or
5. a finite downset resolution; or
6. any of the above subordinate to any given finite encoding; or
7. a finite encoding dominating any given one of items 1–5.

The equivalences remain true over any partially ordered real vector space of finite dimension if “finitely encoded” and all occurrences of “finite” are replaced by “semialgebraic”.

Proof. The plan is first to show that a finitely encoded $Q$-module $M$ has finite upset and downset resolutions and (co)presentations, as well as a finite fringe presentation. Next a dominating finite encoding is constructed from a given presentation or resolution. The semialgebraic case uses the same arguments, mutatis mutandis, and will not be mentioned further.

Let $M$ be a $Q$-module finitely encoded by a poset morphism $\pi : Q \to P$ and $P$-module $H$. The finite poset $P$ has order dimension $n$ for some positive integer $n$; as such $P$ has an embedding $i : P \hookrightarrow \mathbb{Z}^n$. The pushforward $i_*H$ is finitely determined because it is pulled from any box containing $i(P)$. The desired presentation or resolution is pulled back to $Q$ (via $i \circ \pi : Q \to \mathbb{Z}^n$) from the corresponding flange, flat, or injective presentation or resolution of $i_*H$ afforded by Theorem 4.33; these pullbacks to $Q$ are finite indicator resolutions of $M$ subordinate to $\pi$ by Example 5.3 and Lemma 5.2. In the fringe case, the component homomorphisms are connected because, by Lemma 2.19, the components of flange presentations are automatically connected.

To produce a finite encoding $\pi$ given a fringe presentation $\varphi$, let $P$ be the poset of uptight regions (Proposition 2.28) for the set $\Upsilon(\varphi)$ of upsets comprising the birth upsets and the complements of the death downsets of $\varphi$. The proof is similar if indicator presentations or resolutions are given: each upset in the set $\Upsilon$ is either the degree set of one of the summands (in the case of upset presentation or resolution) or is the complement of the degree set of one of the summands (in the case of downset copresentation or resolution).

Example 5.17. The persistent homology of a fly wing (Example 1.1) admits a finite isotypic subdivision of $\mathbb{R}^2$ into semialgebraic sets. Such a module therefore has a finite fringe presentation subordinate to a semialgebraic encoding by Theorems 2.33 and 5.16.
Remark 5.18. Comparing Theorems 5.16 and 4.33, what happened to minimality? It is not clear in what generality minimality can be characterized. Much of this paper can be seen as a case study for posets arising from abelian groups that are either finitely generated and free (the discrete polyhedral case) or real vector spaces of finite dimension (the real polyhedral case); see in particular Theorem 14.2. The answer is rather more nuanced in the real case, obscuring how minimality might generalize beyond discrete or real polyhedral groups. For discrete polyhedral groups, the result is not far beyond what is in the literature, although it highlights that minimality ought to be phrased in terms of isomorphisms on tops and socles rather than by counting summands. (That point that is especially pertinent for primary decomposition; see Section 1.11.)

Remark 5.19. In the situation of the proof of Theorem 5.16, composing two applications of Proposition 5.6—one for the encoding $\pi : Q \rightarrow P$ and one for the embedding $\iota : P \hookrightarrow \mathbb{Z}^n$—yields a monomial matrix for a fringe presentation of $M$ directly from a monomial matrix for a flange presentation.

Remark 5.20. Lesnick and Wright consider $\mathbb{R}^n$-modules [LW15, Section 2] in finitely presented cases. As they indicate, homological algebra of such $\mathbb{R}^n$-modules is no different than finitely generated $\mathbb{Z}^n$-modules. This can be seen by finite encoding: any finite poset in $\mathbb{R}^n$ is embeddable in $\mathbb{Z}^n$, because a product of finite chains is all that is needed.

6. Socles and cogenerators

The main result in this section is Definition 6.59, which introduces the notions of cogenerator functor and socle along a face with a given nadir. Its concomitant foundations include ways to decompose the cogenerator functors into continuous and discrete parts (Proposition 6.65), interactions with localization (Proposition 6.67), left-exactness (Proposition 6.71), and preservation of finitely encoded or semialgebraic conditions (Theorem 6.72), along with the crucial calculation of socles in the simplest case, namely the indicator function of a single face (Example 6.70). The theory is built step by step, starting with local geometry of downsets near their boundaries (Section 6.1) and the functorial view of the geometry (Section 6.2), and then proceeding to cogenerator functors and socles with increasing levels of complexity (Sections 6.3–6.6). Each stage includes preliminary versions of the foundations whose take-away versions are those in Section 6.6, although the foundations in earlier stages are developed in increasing generality, over arbitrary posets instead of real polyhedral groups, for example.

6.1. Tangent cones of downsets.

Definition 6.1. The tangent cone $T_aD$ of a downset $D$ in a real polyhedral group $Q$ (Definition 3.8) at a point $a \in Q$ is the set of vectors $v \in -Q_+$ such that $a + \varepsilon v \in D$ for all sufficiently small (hence all) $\varepsilon > 0$.

Remark 6.2. Since the real number $\varepsilon$ in Definition 6.1 is strictly positive, the vector $v = 0$ lies in $T_aD$ if and only if $a$ itself lies in $D$, and in that case $T_aD = -Q_+$. 
Example 6.3. The tangent cone defined here is not the tangent cone of $D$ as a stratified space, because the cone here only considers vectors in $-Q_+$. A specific simple example to see the difference is the closed half-plane $D$ beneath the line $y = -x$ in $\mathbb{R}^2$, where the usual tangent cone at any point along the boundary line is the half-plane, whereas $T_aD = -\mathbb{R}_+^2$. Furthermore, $T_aD$ can be nonempty for a point $a$ in the boundary of $D$ even if $a$ does not lie in $D$ itself. For an example of that, take $D^o$ to be the interior of $D$; then $T_aD^o = -\mathbb{R}_+^2 \setminus \{0\}$ for any $a$ on the boundary line.

The most important conclusion concerning tangent cones at points of downsets, Proposition 6.10, says that such cones are certain unions of relative interiors of faces. Some definitions and preliminary results are required.

Definition 6.4. Fix a real polyhedral group $Q$.

1. For any face $\sigma$ of the positive cone $Q_+$, write $\sigma^o$ for the relative interior of $\sigma$.
2. For any set $\nabla$ of faces of $Q_+$, write $Q_\nabla = \bigcup_{\sigma \in \nabla} \sigma^o$, the cone of shape $\nabla$.
3. A cocomplex in $Q_+$ is an upset in the face poset $\mathcal{F}_Q$ of $Q_+$, where $\sigma \preceq \tau$ if $\sigma \subseteq \tau$.

Example 6.5. The cocomplex $\nabla \sigma = \{\text{faces } \sigma' \text{ of } Q_+ \mid \sigma' \supseteq \sigma\}$ is the open star of the face $\sigma$. It determines the cone $Q_{\nabla \sigma}$ of shape $\nabla \sigma$, which plays an important role.

Remark 6.6. The next proposition is the reason for specializing this section to real polyhedral groups instead of arbitrary polyhedral partially ordered groups, where limits might not be meaningful. For example, although limits make formal sense in the integer lattice $\mathbb{Z}^n$ with the usual discrete topology, it is impossible for a sequence of points in the relative interior of a face to converge to the origin of the face. This quantum separation has genuine finiteness consequences for the algebra of $\mathbb{Z}^n$-modules that usually do not hold for $\mathbb{R}^n$-modules.

Proposition 6.7. If $\{a_k\}_{k \in \mathbb{N}}$ is any sequence converging to $a$, then $\bigcup_{k=0}^{\infty} (a_k - Q_+) \ni a - Q_+^o$. If the sequence is contained in $a - \sigma^o$, then the union equals $a - Q_{\nabla \sigma}$.

Proof. For each point $b \in a - Q_+^o$, every linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ that is nonnegative on $Q_+$ eventually takes values on the sequence that are bigger than $\ell(b)$; thus $b$ lies in the union as claimed. When the sequence is contained in $a - \sigma^o$, the union is contained in $a - \sigma^o - Q_+$ by hypothesis, but the union contains $a - \sigma^o$ by the first claim applied with $\sigma$ in place of $Q_+$. The union therefore equals $a - \sigma^o - Q_+$ because it is a downset. The next lemma completes the proof. \qed

Lemma 6.8. If $\sigma$ is any face of the positive cone $Q_+$ then $\sigma^o + Q_+ = Q_{\nabla \sigma}$.

Proof. Fix $f + b \in \sigma^o + Q_+$. If $\ell(f + b) = 0$ for some linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ that is nonnegative on $Q_+$, then $\ell(f) = 0$, too. Therefore the support face of $f + b$ (the smallest face in which it lies) contains $\sigma$. On the other hand, suppose $b$ lies interior to some face of $Q_+$ that contains $\sigma$. Then pick any $f \in \sigma^o$. If $\ell(b) = 0$ then also $\ell(f) = 0$, because the support face of $b$ contains $\sigma$. But if $\ell(b) > 0$, then $\ell(b) > \ell(c_f)$ for
any sufficiently small positive $\varepsilon$. As $Q_+$ is an intersection of only finitely many closed half-spaces, a single $\varepsilon$ works for all relevant $\ell$, and then $b = \varepsilon f + (b - \varepsilon f) \in \sigma^c + Q_+$. \hfill $\blacksquare$

For $Q_+ = \mathbb{R}_+^n$ the following is essentially [MMc15, Lemma 5.1].

**Corollary 6.9.** If $D \subseteq Q$ is a downset in a real polyhedral group, then $a$ lies in the closure $D$ if and only if $D$ contains the interior $a - Q_+^c$ of the negative cone with origin $a$.

**Proposition 6.10.** If $a \in \overline{D}$ for a downset $D$, then $T_a D = -Q_\nabla$ is the negative cone of shape $\nabla$ for some cocomplex $\nabla$ in $Q_+$. In this case $\nabla = \nabla^a_\nabla$ is the shape of $D$ at $a$.

**Proof.** The result is true when $n = 1$ because there are only three possibilities for $a \in \mathbb{R}$: either $a \in D$, in which case $T_a D = -\mathbb{R}_+ = Q_\nabla$ for $\nabla = \mathcal{F}_Q$ (Remark 6.2); or $a \not\in D$ but $a$ lies in the closure of $D$, in which case $T_a D = Q_\nabla$ for $\nabla = \{Q_+^c\} \subseteq \mathcal{F}_Q$; or $a$ is separated from $D$ by a nonzero distance, in which case $T_a D = Q_\{\}$ is empty.

Write $D_\sigma$ for the intersection of $D$ with the $a$-translate of the linear span of $\sigma$:

$$D_\sigma = D \cap (a + \mathbb{R}\sigma).$$

If $\sigma \not\subseteq Q_+$, then $T_a D_\sigma = \sigma_\nabla$ for some upset $\nabla \subseteq \mathcal{F}_\sigma$ by induction on the dimension of $\sigma$. In actuality, only the case $\dim \sigma = n - 1$ is needed, as the face posets $\mathcal{F}_\sigma$ for $\dim \sigma = n - 1$ almost cover $\mathcal{F}_Q$: only the open maximal face $Q_\sigma^c$ itself lies outside of their union, and that case is dealt with by Corollary 6.9. \hfill $\blacksquare$

6.2. **Upper boundary functors.**

**Definition 6.11.** For a module $M$ over a real polyhedral group $Q$, a face $\sigma$ of $Q_+$, and a degree $a \in Q$, the **upper boundary atop $\sigma$ at $a$** in $M$ is the vector space

$$(\delta^\sigma M)_a = M_{a - \sigma} = \lim_{\rightarrow a'=a-\sigma^c} M_{a'}.$$

**Lemma 6.12.** The functor $M \mapsto \delta^\sigma M = \bigoplus_{a \in Q} (\delta^\sigma M)_a$ is exact.

**Proof.** Direct limits are exact in categories of vector spaces (or modules over rings). \hfill $\blacksquare$

**Lemma 6.13.** The structure homomorphisms of $M$ as a $Q$-module induce natural homomorphisms $M_{a - \sigma} \to M_{b - \tau}$ for $a \preceq b$ in $Q$ and faces $\sigma \supseteq \tau$ of $Q_+$.

**Proof.** The natural homomorphisms come from the universal property of colimits. First a natural homomorphism $M_{a - \sigma} \to M_{b - \sigma}$ is induced by the composite homomorphisms $M_c \to M_{c + b - a} \to M_{b - \sigma}$ for $c \in a - \sigma^c$ because adding $b - a$ takes $a - \sigma^c$ to $b - \sigma^c$. For $M_{b - \sigma} \to M_{b - \tau}$ the argument is similar, except that existence of natural homomorphisms $M_c \to M_{b - \tau}$ for $c \in b - \sigma^c$ requires Proposition 6.7 and Lemma 6.8. \hfill $\blacksquare$
Remark 6.14. The face poset $\mathcal{F}_Q$ of the positive cone $Q_+$ can be made into a commutative monoid in which faces $\sigma$ and $\tau$ of $Q_+$ have sum

$$\sigma + \tau = \sigma \cap \tau.$$  

Indeed, these monoid axioms use only that $(\mathcal{F}_Q, \cap)$ is a bounded meet semilattice, the monoid unit element being the maximal semilattice element—in this case, $Q_+$ itself. When $\mathcal{F}_Q$ is considered as a monoid in this way, the partial order on it has $\sigma \preceq \tau$ if $\sigma \supseteq \tau$, which is the opposite of the default partial order on the faces of a polyhedral cone. For utmost clarity, $\mathcal{F}_Q^{\text{op}}$ is written when this monoid partial order is intended.

Definition 6.15. Fix a module $M$ over a real polyhedral group $Q$ and a degree $\mathbf{a} \in Q$. The upper boundary functor takes $M$ to the $Q \times \mathcal{F}_Q^{\text{op}}$-module $\delta M$ whose fiber over $\mathbf{a} \in Q$ is the $\mathcal{F}_Q^{\text{op}}$-module

$$(\delta M)_\mathbf{a} = \bigoplus_{\sigma \in \mathcal{F}_Q} M_{\mathbf{a} - \sigma} = \bigoplus_{\sigma \in \mathcal{F}_Q} (\delta^\sigma M)_\mathbf{a}.$$  

The fiber of $\delta M$ over $\sigma \in \mathcal{F}_Q^{\text{op}}$ is the upper boundary $\delta^\sigma M$ of $M$ atop $\sigma$.

Remark 6.16. The face of $Q_+$ that contains only the origin $0$ is an absorbing element: it acts like infinity, in the sense that $\sigma + \{0\} = \{0\}$ in the monoid $\mathcal{F}_Q^{\text{op}}$ for all faces $\sigma$. Adding the absorbing element $0$ in the $\mathcal{F}_Q^{\text{op}}$ component therefore induces a natural $\mathbb{R}^n \times \mathcal{F}_Q^{\text{op}}$-module projection from the upper boundary $\delta M$ to $M$. At a degree $\mathbf{a} \in \mathbb{R}^n$, this projection is $M_{\mathbf{a} - \sigma} \to M_{\mathbf{a} - 0} = M_\mathbf{a}$. Interestingly, the frontier of a downset $D$—those points in the topological closure but outside of $D$—is the set of nonzero degrees of a functor, namely $\ker(\delta^\sigma M \to M)$ for $\sigma = Q^\circ_+$. The proof is by Corollary 6.9.

There is no natural map $M \to \delta^\sigma M$ when $\sigma \neq \{0\}$ has positive dimension, because an element of degree $\mathbf{a}$ in $M$ comes from elements of $\delta^\sigma M$ in degrees less than $\mathbf{a}$. However, that leaves a way for Lemma 6.13 to afford a notion of divisibility of upper boundary elements by elements of $M$.

Definition 6.17. An element $y \in M_\mathbf{b}$ divides $x \in (\delta^\sigma M)_\mathbf{a}$ if $\mathbf{b} \in \mathbf{a} - Q_{\nabla\sigma} = \mathbf{a} - \sigma^\circ - Q_+$ (Lemma 6.8) and $y \mapsto x$ under the natural map $M_\mathbf{b} \to M_{\mathbf{a} - \sigma}$ (Lemma 6.13). The element $y$ is said to $\sigma$-divide $x$ if, more restrictively, $\mathbf{b} \in \mathbf{a} - \sigma^\circ$.

Lemma 6.18. If $\sigma \in \mathcal{F}_Q$ and $D$ is a downset in $Q$ then $\delta^\sigma \mathbb{R}[D] = \mathbb{R}[\delta^\sigma D]$, where

$$\delta^\sigma D = \bigcup_{x \in Q} D \cap (x + \mathbb{R}\sigma) = D \cup \bigcup_{x \in \partial D} D \cap (x + \mathbb{R}\sigma).$$

It suffices to take the middle union over $x$ in any subspace complement to $\mathbb{R}\sigma$.

Proof. For the second displayed equality, observe that the middle union contains the right-hand union because the middle one contains $D$. For the other containment, if $x + \mathbb{R}\sigma$ contains no boundary point of $D$, then $D \cap (x + \mathbb{R}\sigma) = \overline{D} \cap (x + \mathbb{R}\sigma)$ is already closed, so the contribution of $D \cap (x + \mathbb{R}\sigma)$ to the middle union is contained in $D$. 

For the other equality, \( \delta^\sigma k[D] \) is nonzero in degree \( a \) if and only if \( a - \sigma^\circ \subseteq D \) and that condition is equivalent to \( a - \sigma^\circ \subseteq D \cap (a + R\sigma) \) because \( a - \sigma^\circ \subseteq a + R\sigma \). Translating \( D \cap (a + R\sigma) \) back by \( a \) yields a downset in the real polyhedral group \( R\sigma \), with \( (R\sigma)_+ = \sigma \), thereby reducing to Corollary 6.9.

**Proposition 6.19.** If \( \sigma \in \mathcal{F}_Q \) and \( D \) is a downset in a real polyhedral group then

\[
\delta^\sigma k[D] = k[\delta^\sigma D]
\]

is the indicator quotient for a downset \( \delta^\sigma D \) satisfying

1. \( D \subseteq \delta^\sigma D \subseteq \overline{D} \);
2. \( \delta^\sigma D = \{ a \in \overline{D} \mid \sigma \in \nabla^a \} \); and
3. if \( D \) is semialgebraic then so is \( \delta^\sigma k[D] \).

**Proof.** Item 1 follows from item 2. What remains to show is that \( \delta^\sigma D \) is a downset in \( \overline{D} \) characterized by item 2 and that it is semialgebraic if \( D \) is.

First, \( \sigma \in \nabla^a \) means that \( a - \sigma^\circ \subseteq D \), which immediately implies that \( a \in \delta^\sigma D \) by Lemma 6.18. Conversely, suppose \( a \in \delta^\sigma D \). Lemma 6.18 and Corollary 6.9, the latter applied to the downset \( -a + D \cap (a + R\sigma) \) in \( R\sigma \), imply that \( a - \sigma^\circ \subseteq D \), and hence \( \sigma \in \nabla^a \) by definition, proving item 2. Given that \( a - \sigma^\circ \subseteq D \), Proposition 6.7 yields \( D \cap (a - Q_+) \supseteq a - Q_{\nabla^a} \). Consequently, if \( b \in a - Q_+ \) then \( D \cap (b - \sigma) \supseteq b - \sigma^\circ \), whence \( b \in \delta^\sigma D \). Thus \( \delta^\sigma D \) is a downset.

The semialgebraic claim follows from a general result, Lemma 6.20, the case here being \( Y = Q/R\sigma \) and \( X = D \) by Lemma 6.18.

**Lemma 6.20.** If \( X \subseteq \mathbb{R}^n \) and \( X \to Y \) is a morphism of semialgebraic varieties, then the family \( X|_Y \) obtained by taking the closure in \( \mathbb{R}^n \) of every fiber of \( X \) is semialgebraic.

**Proof.** This is a consequence of Hardt’s theorem [Har80, Theorem 4] (see also [Shi97, Remark II.3.13]), which says that over a subset of \( Y \) whose complement in \( Y \) has dimension less than \( \dim Y \), the family \( X \to Y \) is trivial.

**Proposition 6.21.** If \( M \) is finitely encoded over a real polyhedral group then so is its upper boundary \( \delta^\sigma M \). If \( M \) is semialgebraic then so is \( \delta M \). The same are true with \( \delta^\sigma \) in place of \( \delta \) for a fixed face \( \sigma \).

**Proof.** Since the relevant categories are abelian by Lemma 2.10, it suffices to treat the \( \delta^\sigma \) case. As the direct sums that form a \( Q \)-module from its graded pieces are exact, as is the formation of \( \delta M \) from \( M \) (Lemma 6.12) the claim reduces to Proposition 6.19 by Theorem 5.16.3.

### 6.3. Closed socles and closed cogenerator functors.

In commutative algebra, the socle of a module over a local ring is the set of elements annihilated by the maximal ideal. These elements form a vector space over the residue field \( k \) that can alternately be characterized by taking homomorphisms from \( k \) into the module. Either characterization works for modules over partially ordered groups, but only the latter generalizes readily to modules over arbitrary posets.
Definition 6.22. Fix an arbitrary poset $P$. The skyscraper $P$-module $\mathbb{k}_p$ at $p \in P$ has $\mathbb{k}$ in degree $p$ and 0 in all other degrees. The closed cogenerator functor $\text{Hom}_P(\mathbb{k}, -)$ takes each $P$-module $M$ to its closed socle: the $P$-submodule

$$\overline{\text{soc}} M = \text{Hom}_P(\mathbb{k}, M) = \bigoplus_{p \in P} \text{Hom}_P(\mathbb{k}_p, M).$$

When it is important to specify the poset, the notation $P\overline{-}\text{soc}$ is used instead of $\overline{\text{soc}}$. A closed cogenerator of degree $p \in P$ is a nonzero element in $(\overline{\text{soc}} M)_p$.

The bar over “soc” is meant to evoke the notion of closure or “closed”. The bar under Hom is the usual one in multigraded commutative algebra for the direct sum of homogeneous homomorphisms of all degrees (see [GW78, Section I.2] or [MS05, Definition 11.14], for example).

Example 6.23. The closed socle of $M$ consists of the elements that are annihilated by moving up in any direction, or that have maximal degree. In particular, the indicator quotient module $\mathbb{k}[D]$ for any downset $D \subseteq P$ has closed socle

$$\overline{\text{soc}} \mathbb{k}[D] = \mathbb{k}[\text{max } D],$$

the indicator subquotient supported on the set of elements of $D$ that are maximal in $D$.

Lemma 6.24. The closed cogenerator functor over any poset is left-exact.

Proof. A $P$-module is the same thing as a module over the path algebra of (the Hasse diagram of) $P$ with relations to impose commutativity, namely equality of the morphism induced by $p < p''$ and the composite morphism induced by $p < p' < p''$ (see [Yuz81], for example). The purpose of viewing things this way is merely to demonstrate that the category of $P$-modules is a category of modules (graded by $P$) over a ring, where Hom from a fixed source is automatically left-exact. □

Lemma 6.25. If $D$ is a semialgebraic downset in a real polyhedral group then $\text{max } D$ is semialgebraic, as well.

Proof. The proof uses standard operations on semialgebraic subsets that preserve the semialgebraic property; see [Shi97, Chapter II], for instance.

Inside of $\mathbb{R}^n \times \mathbb{R}^n$, consider the subset $X$ whose fiber over each point $a \in D$ is $a + m$, where $m = Q \setminus \{0\}$ is the maximal monoid ideal of $Q_+$. Note that $m$ is semialgebraic because it is defined by linear inequalities and a single linear inequation. The subset $X \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is semialgebraic because it is the image of the algebraic morphism $D \times m \to D \times \mathbb{R}^n$ sending $(a, q) \mapsto (a, a + q)$. The intersection of $X$ with the semialgebraic subset $D \times D$ remains semialgebraic, as does the projection of this intersection to $D$. The image of the projection is $D \setminus \text{max } D$ because $(a + m) \cap D = \emptyset$ precisely when $a \in \text{max } D$. Therefore $\text{max } D = D \setminus (D \setminus \text{max } D)$ is semialgebraic. □

Proposition 6.26. If a module $M$ over any poset is finitely encoded then so is its closed socle $\overline{\text{soc}} M$. If $M$ is semialgebraic over a real polyhedral group then so is $\overline{\text{soc}} M$. 

Proof. If $M$ is finitely encoded then $M = \ker(E^0 \to E^1)$ with $E^i$ a finite direct sum of indicator quotients by Theorem 5.16.3. Thus $\overline{\soc} M = \ker(\overline{\soc} E^0 \to \overline{\soc} E^1)$ by Lemma 6.24. For the finitely encoded claim it therefore suffices, by Lemma 2.10, to prove that $\overline{\soc} k[D]$ is finitely encoded if $k[D]$ is. But Example 6.23 implies that $\overline{\soc} k[D] = k[\max D] = \ker(k[D] \to k[D \setminus \max D])$ is a downset copresentation, where $D \setminus \max D$ is a downset by definition.

The same argument shows the semialgebraic claim by Lemma 6.25. \qed

6.4. Socles and cogenerator functors.

**Definition 6.27.** The **cogenerator functor** takes a module over a real polyhedral group to its socle:

$$\soc M = \overline{\soc} \delta M,$$

the closed socle, computed over the poset $Q \times \mathcal{F}^{op}_Q$, of the upper boundary module of $M$.

**Remark 6.28.** Notationally, the lack of a bar over “soc” serves as a visual cue that the functor is over a real polyhedral group, as the upper boundary $\delta$ is not defined in more generality. This visual cue persists throughout the more general notions of socle.

**Lemma 6.29.** The cogenerator functor $M \mapsto \soc M$ is left-exact.

**Proof.** Use exactness of upper boundaries atop $\sigma$ (Lemma 6.12), exactness of the direct sums forming $\delta M$ from $\delta^* M$, and left-exactness of closed socles (Lemma 6.24). \qed

Sometimes it is useful to apply the closed socle functor to $\delta M$ over $Q \times \mathcal{F}^{op}_Q$ in two steps, first over one poset and then over the other. These yield the same result.

**Lemma 6.30.** The functors $Q^{\soc}$ and $\mathcal{F}^{op}_Q^{\soc}$ commute. In particular,

$$\mathcal{F}^{op}_Q^{\soc} (Q^{\soc} \delta M) \cong \soc M \cong Q^{\soc} (\mathcal{F}^{op}_Q^{\soc} \delta M).$$

**Proof.** By taking direct sums over $a$ and $\sigma$, this follows from the natural isomorphisms $\Hom_{\mathcal{F}^{op}_Q} (k_\sigma, \Hom_Q (k_a, -)) \cong \Hom_{Q \times \mathcal{F}^{op}_Q} (k_{a, \sigma}, -) \cong \Hom_Q (k_a, \Hom_{\mathcal{F}^{op}_Q} (k_\sigma, -)).$ \qed

The fundamental examples—indicator quotients for downsets—require a notation.

**Definition 6.31.** In the situation of Definition 6.4, write $\partial \nabla$ for the antichain of faces of $Q_+$ that are minimal under inclusion in $\nabla$. 


Remark 6.32. The reason for writing $\partial \nabla$ instead of max or min is that it would be ambiguous either way, since both $\mathcal{F}_Q$ and $\mathcal{F}_Q^{\text{op}}$ are natural here. Taking the “op” perspective, the $\mathcal{F}_Q^{\text{op}}$-module $k[\partial \nabla]$ with basis $\partial \nabla$ resulting from Definition 6.31 is really just a $\mathcal{F}_Q^{\text{op}}$-graded vector space: the antichain condition ensures that every non-identity element of $\mathcal{F}_Q^{\text{op}}$ acts by 0, unless $\partial \nabla = \{0\}$, in which case all of $\mathcal{F}_Q^{\text{op}}$ acts by 1.

The case of most interest here is $\nabla = \nabla^a_D$, the shape of $D$ at $a$ (Proposition 6.10).

Example 6.33. For a downset $D$ in a real polyhedral group, $\mathcal{F}_Q^{\text{op}}$-$\text{soc} \delta k[D]$ has $k[\partial \nabla^a_D]$ in each degree $a$, because $\delta k[D]$ itself has $k[\nabla^a_D]$ in each degree $a$ by Definition 6.15 and Proposition 6.19. What $Q$-$\text{soc}$ then does is find the degrees $a \in Q$ maximal among those where $\sigma \in \partial \nabla^a_D$, by Proposition 6.19.2 and Example 6.23.

Taking socles in the other order, first $Q$-$\text{soc} \delta k[D]$ asks whether $\sigma \in \nabla^a_D$ but $\sigma \not\in \nabla^b_D$ for any $b \succ a$ in $Q$. That can happen even if $\sigma$ contains a smaller face where it still happens. What $\mathcal{F}_Q^{\text{op}}$-$\text{soc}$ then does is return the smallest faces at $a$ where it happens.

Corollary 6.34. The socle of the indicator quotient $k[D]$ for any downset $D$ in a real polyhedral group $Q$ is nonzero only in degrees lying in the topological boundary $\partial D$.

Proof. By Proposition 6.19.1, $\delta k[D]$ is a direct sum of indicator quotients. Example 6.23 and Proposition 6.10 show that the socle of an indicator quotient over a real polyhedral group lies along the boundary of the downset in question. □

Proposition 6.35. If a module $M$ over a real polyhedral group is finitely encoded then so is its socle $\text{soc} M$. If $M$ is semialgebraic then so is $\text{soc} M$.

Proof. By Lemma 6.30, $\text{soc} M$ is the closed socle of the $Q$-module $\mathcal{F}_Q^{\text{op}}$-$\text{soc} \delta M$. The $\mathcal{F}_Q^{\text{op}}$-graded component of this $Q$-module in degree $\sigma$ is the intersection of the kernels of the $Q$-module homomorphisms $\delta^\sigma M \to \delta^{\sigma'} M$ for $\sigma \supseteq \sigma'$. Now apply Proposition 6.21 and Lemma 2.10. □

6.5. Closed socles along faces of positive dimension.

Definition 6.36. For a partially ordered abelian group $Q$ and a face $\tau$ of $Q_+$, write $Q/\mathbb{Z} \tau$ for the quotient of $Q$ modulo the subgroup generated by $\tau$. If $Q$ is a real polyhedral group then write $Q/\mathbb{R} \tau = Q/\mathbb{Z} \tau$.

Remark 6.37. The image $Q_+/\mathbb{Z} \tau$ of $Q_+$ in $Q/\mathbb{Z} \tau$ is a submonoid that generates $Q/\mathbb{Z} \tau$, but $Q_+/\mathbb{Z} \tau$ can have units, so $Q/\mathbb{Z} \tau$ need not be partially ordered in a natural way. However, if $Q$ is a real polyhedral group then the group of units (lineality space) of the cone $Q_+/\mathbb{R} \tau$ is just $\mathbb{R} \tau$ itself, because $Q_+$ is pointed, so $Q/\mathbb{R} \tau$ is a real polyhedral group whose positive cone $(Q/\mathbb{R} \tau)_+ = Q_+/\mathbb{R} \tau$ is the image of $Q_+$. Similar reasoning applies to the intersection of the real polyhedral situation with any subgroup of $Q$; this includes the case of normal affine semigroups, where the subgroup of $Q$ is discrete.
Lemma 6.38. The subgroup $\mathbb{Z}\tau \subseteq Q$ of a partially ordered group $Q$ acts freely on the localization $M_\tau$ of any $Q$-module $M$ along a face $\tau$. Consequently, if $I_\tau \subseteq k[Q_+]$ is the augmentation ideal $\langle m - 1 \mid m \in k[\tau] \text{ is a monomial} \rangle$, then the $Q/\mathbb{Z}\tau$-graded module $M/\tau = M/I_\tau M$ over the monoid algebra $k[Q/\mathbb{Z}\tau]$ satisfies

$$M_\tau \cong \bigoplus_{a \to \tilde{a}} (M/\tau)_a.$$

Proof. The monomials of $k[Q_+]$ corresponding to elements of $\tau$ are units on $M_\tau$ acting as translations along $\tau$. Since the augmentation ideal sets every monomial equal to 1, the quotient $M \to M/\tau$ factors through $M_\tau$. □

Definition 6.39. The $k[Q/\mathbb{Z}\tau]$-module $M_\tau$ in Lemma 6.38 is the quotient-restriction $M/\tau$ of $M$ along $\tau$.

Remark 6.40. Over (any subgroup of) a real polyhedral group $Q$, the functor $M_\tau \mapsto M/\tau$ has a “section” $M/\tau \to M|_{\tau^+}$, where $N|_{\tau^+} = \bigoplus_{a \in \tau^+} N_a$ is the restriction of $N$ to any linear subspace $\tau^+$ complementary to $\mathbb{R}\tau$. (When $Q_+ = \mathbb{R}_+$, a complement is canonically spanned by the face orthogonal to $\tau$, or equivalently, the unique maximal face of $\mathbb{R}_n$ intersecting $\tau$ trivially.) The restriction is a module over the real polyhedral group $\tau^+$ with positive cone $(Q_+ + \mathbb{R}\tau) \cap \mathbb{R}\tau^+$, which projects isomorphically to the positive cone of $Q/\mathbb{R}\tau$. Thus the quotient-restriction is both a quotient and a restriction of $M_\tau$. While a section can exist over polyhedral partially ordered groups that are not real, it need not. For example, when $Q = \mathbb{Z}^2$ and the columns of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ generate $Q_+$, taking $\tau = \langle [2] \rangle$ to be the face along the $x$-axis yields a quotient monoid $Q_+/\mathbb{Z}\tau \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{N}$ with torsion, preventing $k[Q_+]_\tau \mapsto k[Q_+] / \tau$ from having a section to any category of modules over a subgroup of $Q$.

Lemma 6.41. The quotient-restriction functors $M \mapsto M/\tau$ are exact.

Proof. Localizing along $\tau$ (Definition 3.23) is exact because the localization $k[Q_+ + \mathbb{Z}\tau]$ of $k[Q_+]$ is flat as a $k[Q_+]$-module. The exactness of the functor that takes each $k[Q_+ / \mathbb{Z}\tau]$-module $M_\tau$ to $M/\tau$ can be checked on each $Q/\mathbb{Z}\tau$-degree individually. □

The definition of closed socle and closed cogenerator are expressed in terms of Hom functors analogous to those in Definition 6.22. They are more general in that they occur along faces of $Q$, but more restrictive in that $Q$ needs to be a partially ordered group instead of an arbitrary poset for the notion of face to make sense.

Definition 6.42. Fix a face $\tau$ of a partially ordered group $Q$. The skyscraper $Q$-module at $a \in Q$ along $\tau$ is $k[a + \tau]$, the subquotient $k[a + Q_+] / k[a + m_\tau]$ of $k[Q]$, where $m_\tau = Q_+ \setminus \tau$. Set

$$\text{Hom}_Q(k[\tau], -) = \bigoplus_{a \in Q} \text{Hom}_Q(k[a + \tau], -).$$
Definition 6.43. Fix a partially ordered group $Q$, a face $\tau$, and a $Q$-module $M$.

1. The (global) closed cogenerator functor along $\tau$ takes $M$ to its (global) closed socle along $\tau$: the $k[Q_+/Z\tau]$-module
   
   $\text{soc}_\tau M = \text{Hom}_Q(k[\tau], M)/\tau$.

2. If $Q/Z\tau$ is partially ordered, then the local closed cogenerator functor along $\tau$ takes $M$ to its local closed socle along $\tau$: the $Q/Z\tau$-module
   
   $\text{soc}(M/\tau) = \text{Hom}_{Q/Z\tau}(k, M/\tau)$.

   Elements of $\text{soc}(M/\tau)$ are identified with elements of $M/\tau$ via $\varphi \mapsto \varphi(1)$.

3. Regard the $Q$-module $\text{Hom}_Q(k[\tau], M)$ naturally as contained in $M$ via $\varphi \mapsto \varphi(1)$. A homogeneous element in this $Q$-submodule that maps to a nonzero element of $\text{soc}_\tau M$ is a (global) closed cogenerator of $M$ along $\tau$. If $D \subseteq Q$ is a downset, then a closed cogenerator of $D$ is the degree in $Q$ of a closed cogenerator of $k[D]$.

4. Regard $\text{soc}(M/\tau)$ naturally as contained in $M/\tau$ via $\varphi \mapsto \varphi(1)$. A nonzero homogeneous element in $\text{soc}(M/\tau)$ is a local closed cogenerator of $M$ along $\tau$.

Remark 6.44. Notationally, a subscript on “soc” serves as a visual cue that the functor is over a partially ordered group, as faces of posets are not defined in more generality. This visual cue persists throughout the more general notions of socle.

Remark 6.45. The closed cogenerator functor over a partially ordered group is the global closed cogenerator functor along the trivial face: $\text{soc} = \text{soc}_{\{0\}}$ and it equals the local cogenerator functor along $\{0\}$.

Remark 6.46. In looser language, a closed cogenerator of $M$ along $\tau$ is an element

- annihilated by moving up in any direction outside of $\tau$ but that
- remains nonzero when pushed up arbitrarily along $\tau$.

Equivalently, a closed cogenerator along $\tau$ is an element whose annihilator under the action of $Q_+$ on $M$ equals the prime ideal $m_\tau = Q_+ \setminus \tau$ of the positive cone $Q_+$.

Example 6.47. The closed socle along a face $\tau$ of the indicator quotient $k[D]$ for any downset $D$ in a partially ordered group $Q$ with partially ordered quotient $Q/Z\tau$ is

$\text{soc}_\tau k[D] = k[\max_\tau D]$, where $\max_\tau D$ is the image in $Q/Z\tau$ of the set of closed cogenerators of $D$ along $\tau$:

$\max_\tau D = \{ a \in D \mid (a + Q_+) \cap D = a + \tau \}/Z\tau$.

The set of closed cogenerators of $D$ along $\tau$ can also be characterized as the elements of $D$ that become maximal in the localization $D_\tau$ of $D$ (Definition 3.12.1).

Every global closed cogenerator yields a local one.
Proposition 6.48. Fix a partially ordered group \( Q \). There is a natural injection
\[
\text{soc}_\tau M \hookrightarrow \text{soc}(M/\tau)
\]
for any \( Q \)-module \( M \) if \( \tau \) is a face with partially ordered quotient \( Q/\mathbb{Z}_\tau \).

Proof. Localizing any homomorphism \( k[a + \tau] \to M \) along \( \tau \) yields a homomorphism \( k[a + \mathbb{Z}_\tau] \to M_\tau \), so \( \text{Hom}_Q(k[\tau], M)_\tau \) is naturally a submodule of \( \text{Hom}_Q(k[\mathbb{Z}_\tau], M_\tau) \). The claim now follows from Lemma 6.41 and the next result. \( \square \)

Lemma 6.49. If \( Q \) and \( Q/\mathbb{Z}_\tau \) are partially ordered, there is a canonical isomorphism
\[
\text{Hom}_Q(k[\mathbb{Z}_\tau], M_\tau)/\tau \cong \text{Hom}_{Q/\mathbb{Z}_\tau}(k, M/\tau).
\]

Proof. Follows from the definitions, using that \( k[\mathbb{Z}_\tau]/\tau = k \) in \((Q/\mathbb{Z}_\tau)\)-degree 0. \( \square \)

The following crucial remark highlights the difference between real-graded algebra and integer-graded algebra. It is the source of much of the subtlety in the theory developed in this paper, particularly Sections 6–10.

Remark 6.50. In contrast with taking support on a face (Proposition 3.27) and also with socles in commutative algebra over noetherian local or graded rings (e.g., Definition 4.10), localization need not commute with taking closed socles along faces of positive dimension in real polyhedral groups. In other words, the injection in Proposition 6.48 need not be surjective: there can be local closed cogenerators that do not lift to global ones. The problem comes down to the homogeneous prime ideals of the monoid algebra \( k[Q_+] \) not being finitely generated, so the quotient \( k[\tau] \) fails to be finitely presented; it means that \( \text{Hom}_{k[Q_+]}(k[\tau], -) \) need not commute with \( A \otimes_{k[Q_+]} - \), even when \( A \) is a flat \( k[Q_+] \)-algebra such as a localization of \( k[Q_+] \). The context of \( \mathbb{R}^n \)-modules complicates the relation between support on \( \tau \) and closed cogenerators along \( \tau \) because the “thickness” of the support can approach 0 without ever quantum jumping all the way there and, importantly, remaining there along an entire translate of \( \tau \), as it would be forced to for a discrete group like \( \mathbb{Z}^n \). See Examples 3.17 and 3.22, for instance, where the support on the \( x \)-axis has no closed socle along the \( x \)-axis. This issue is independent of the density phenomenon explored in Section 8; indeed, the downset in Example 3.17 is closed, so its socle equals its closed socle and is closed.

Proposition 6.51. The global closed cogenerator functor \( \text{soc}_\tau \) along any face \( \tau \) of a partially ordered group is left-exact, as is the local version if \( Q/\mathbb{Z}_\tau \) is partially ordered.

Proof. For the global case, \( \text{Hom}_Q(k[\tau], -) \) is exact because it occurs in the category of graded modules over the monoid algebra \( k[Q_+] \), and quotient-restriction is exact by Lemma 6.41. For the local case, use exactness of \( M \mapsto M/\tau \) again (Lemma 6.41) and left-exactness of closed socles (Lemma 6.24), the latter applied over \( Q/\mathbb{Z}_\tau \). \( \square \)

Lemma 6.52. If \( D \) is a semialgebraic downset in a real polyhedral group \( Q \) then \( \max_\tau D \) is semialgebraic in \( Q/\mathbb{R}_\tau \) for any face \( \tau \) of \( Q_+ \).
Proof. The projection of a semialgebraic set is semialgebraic, so by Example 6.47 it suffices to prove that the set of degrees of closed cogenerators of \( k[D] \) along \( \tau \) is semi-algebraic. The argument comes in two halves, both following the framework of the proof of Lemma 6.25. For the first half, simply replace \( m \) by \( m = Q_+ \setminus \tau \) to find that \( \{ a \in D \mid (a + Q_+) \cap D \subseteq a + \tau \} \) is semialgebraic. The second half uses \( \tau \) instead of \( m \), and this time it intersects the subset \( X \) with \( D \times (Q \setminus D) \) to find that \( \{ a \in D \mid (a + Q_+) \cap D \supseteq a + \tau \} \) is semialgebraic. The desired set of degrees is the intersection of these two semialgebraic sets. \( \square \)

Proposition 6.53. If a module \( M \) over a partially ordered group is finitely encoded then so is its closed socle \( \text{soc}_\tau M \) along any face \( \tau \). If \( M \) is semialgebraic over a real polyhedral group then so is \( \text{soc}_\tau M \).

Proof. If \( M \) is finitely encoded then \( M = \ker(E^0 \to E^1) \) with \( E^i \) a finite direct sum of indicator quotients. Thus \( \text{soc}_\tau M = \ker(\text{soc}_\tau E^0 \to \text{soc}_\tau E^1) \) by Proposition 6.51. For the finitely encoded claim it therefore suffices, by Lemma 2.10, to prove that \( \text{soc}_\tau k[D] \) is finitely encoded if \( k[D] \) is. Writing \( D/\mathbb{Z}_\tau \) for the image of \( D \) in \( \mathbb{Q}/\mathbb{Z}_\tau \), Example 6.47 implies that \( \text{soc}_\tau k[D] = \ker(k[D/\mathbb{Z}_\tau] \to k[D/\mathbb{Z}_\tau \setminus \text{max}_\tau D]) \) is an indicator copresentation, where the set \( D/\mathbb{Z}_\tau \setminus \text{max}_\tau D \) is downset because \( \text{max}_\tau D \) is contained in the set \( \text{max}(D/\mathbb{Z}_\tau) \) of maximal elements of \( D/\mathbb{Z}_\tau \).

The same argument shows the semialgebraic claim by Lemma 6.52. \( \square \)

Remark 6.54. Closed socles, without reference to faces, work over arbitrary posets and are actually used that way in this work (over \( \mathcal{F}_Q^{\text{op}} \), for instance, in Section 6.4). That explains why this separate section on closed socles along faces of positive dimension is required, instead of simply doing Section 6.3 in this specificity in the first place.

6.6. Socles along faces of positive dimension.

Lemma 6.55. If \( \tau \) is a face of a real polyhedral group \( Q \) then the face poset of the quotient real polyhedral group \( Q/\mathbb{R}_\tau \) is isomorphic to the open star \( \nabla_\tau \) from Example 6.5 by the map \( \nabla_\tau \to (Q/\mathbb{R}_\tau)_+ \) sending \( \sigma \in \nabla_\tau \) to its image \( \sigma/\tau \) in \( Q/\mathbb{R}_\tau \).

Proof. See Remark 6.37. \( \square \)

Definition 6.56. In the situation of Lemma 6.55, endow \( \nabla_\tau \) with the monoid and poset structures from Remark 6.14, so \( \sigma \preceq \sigma' \) in \( \nabla_\tau \) if \( \sigma \supseteq \sigma' \). The upper boundary functor along \( \tau \) takes \( M \) to the \( Q \times \nabla_\tau \)-module \( \delta_\tau M = \bigoplus_{\sigma \in \nabla_\tau} \delta_\sigma M = \delta M/\bigoplus_{\sigma \notin \nabla_\tau} \delta_\sigma M \).

The notation is such that \( \delta_\sigma \neq 0 \Leftrightarrow \sigma \supseteq \tau \).

Definition 6.57. Fix a partially ordered group \( Q \), a face \( \tau \), and an arbitrary commutative monoid \( P \). The skyscraper \( (Q \times P) \)-module at \( (a, \sigma) \in Q \times P \) along \( \tau \) is \( k_\sigma[a + \tau] = k[a + \tau] \otimes_k k_\sigma \),
the right-hand side being a module over the ring $k[Q_+] \otimes_k k[P] = k[Q_+ \times P]$ with tensor factors as in Definitions 6.22 and 6.42. Set

$$\text{Hom}_{Q \times P}(k[\tau], -) = \bigoplus_{(a, \sigma) \in Q \times P} \text{Hom}_{Q \times P}(k_\sigma[a + \tau], -).$$

**Remark 6.58.** When $P$ is trivial, this notation agrees with Definition 6.42, because $Q \times \{0\} \cong Q$ canonically, so $\text{Hom}_{Q \times \{0\}}(k[\tau], -) = \text{Hom}_Q(k[\tau], -)$.

**Definition 6.59.** Fix a real polyhedral group $Q$, a face $\tau$, and a $Q$-module $M$.

1. The *(global)* cogenerator functor along $\tau$ takes $M$ to its *(global)* socle along $\tau$:

   $$\text{soc}_\tau M = \underline{\text{Hom}}_{Q \times \nabla\tau}(k[\tau], \delta_\tau M)/\tau.$$

   The $\nabla\tau$-graded components of $\text{soc}_\tau M$ are denoted by $\text{soc}_\sigma^\tau M$ for $\sigma \in \nabla\tau$.

2. The *local* cogenerator functor along $\tau$ takes $M$ to its local socle along $\tau$:

   $$\text{soc}(M/\tau) = \text{soc}(\delta(M/\tau) = \underline{\text{Hom}}_{Q/\nabla\tau \times \nabla\tau}(k, \delta(M/\tau)),$$

   where the upper boundary is over $Q/\nabla\tau$ and the closed socle is over $Q/\nabla\tau \times \nabla\tau$. Elements of $\text{soc}(M/\tau)$ are identified with elements of $\delta(M/\tau)$ via $\varphi \mapsto \varphi(1)$. A homogeneous element $s$ in this submodule that maps to a nonzero element of $\text{soc}_\tau M$ is a *(global)* cogenerator of $M$ along $\tau$, and if $s \in \delta^\tau_\tau M$ then it has nadir $\sigma$. If $D \subseteq Q$ is a downset, then a cogenerator of $D$ along $\tau$ with nadir $\sigma$ is the degree in $Q$ of a cogenerator of $k[D]$ with nadir $\sigma$ along $\tau$.

3. Regard $\underline{\text{Hom}}_{Q \times \nabla\tau}(k[\tau], \delta_\tau M)$ as a $(Q \times \nabla\tau)$-submodule of $\delta_\tau M$ via $\varphi \mapsto \varphi(1)$. A homogeneous element $s$ in $\text{soc}(M/\tau)$ is a local *cogenerator* of $M$ along $\tau$.

**Remark 6.60.** The reason to quotient by $\tau$ in Definition 6.59.1 is to lump together all cogenerators with nadir $\sigma$ along the same translate of $\nabla\tau$. This lumping makes it possible for a socle basis to produce a downset hull that is (i) as minimal as possible and (ii) finite. The lumping also creates a difference between the notion of socle element and that of cogenerator: a socle element is a class of cogenerators, these classes being indexed by elements in the quotient-restriction. In contrast, a local cogenerator is a cogenerator of the quotient-restriction itself, so a local cogenerator is already an element in the socle of the quotient-restriction. This difference between socle element and cogenerator already arises for closed socles along faces (Definition 6.43) but disappears in the context of socles not along faces (see Remark 6.45), be they over real polyhedral groups (Definition 6.27) or closed over posets (Definition 6.22).

**Remark 6.61.** If localization commuted with cogenerator functors, then the restriction from $\mathcal{F}_Q^{\text{op}}$ to $\nabla\tau$ in Definition 6.59.1 would happen automatically, because localizing $M$ along $\tau$ would yield a module over $Q_+ + \nabla\tau$, whose face poset is naturally $\nabla\tau$. But in this real polyhedral setting, the restriction from $\mathcal{F}_Q^{\text{op}}$ to $\nabla\tau$ must be imposed manually because the Hom must be taken before localizing (Remark 6.50), when the default face poset is still $\mathcal{F}_Q^{\text{op}}$. 
Remark 6.62. If \( a \) is a cogenerator of \( D \) along \( \tau \), then the topology of \( D \) at \( a \) is induced by downsets of the form \( a' - \sigma^0 \) for faces \( \sigma \in \nabla \tau \) and elements \( a' \in a + \tau^0 \).

This subtle issue regarding shapes of cogenerators along \( \tau \) is a vital reason for using \( \nabla \tau \) instead of \( \mathcal{F}_Q^{op} \). It is tempting to expect that if a face \( \sigma \) is minimal in the shape \( \nabla_D^a \), then any expression of \( D \) as an intersection of downsets must induce the topology of \( D \) at \( a \) by explicitly taking \( a - \sigma^0 \) into account in one of the intersectands. One way to accomplish that would be for an intersectand to be a union of downsets of the form \( b - Q\nabla_b \) (see Definition 6.4) in which one of the elements \( b \) is \( a \). But if \( \sigma \in \nabla_D^a \) for all \( a' \in a + \tau \), or even merely for a single element \( a' \in a + \tau^0 \), then

\[
    a - \sigma^0 = a' - (a' - a - \sigma^0) \in a' - (\tau^0 + \sigma^0) \subseteq a' - (\tau \vee \sigma)^0.
\]

As the purpose of cogenerators is to construct downset decompositions as minimally as possible, it is counterproductive to think of \( \sigma \) as being a valid \( \mathcal{F}_Q^{op} \)-socle degree unless \( \sigma \in \nabla \tau \), because otherwise it fails to give rise to an essential cogenerator. See Theorem 8.10 for the most general possible view of considerations in this Remark.

Remark 6.63. Although \( \text{soc}_\tau M \) is a module over \( Q/\mathbb{R} \tau \times \nabla \tau \) by construction, the actions of \( Q/\mathbb{R} \tau \) and \( \nabla \tau \) on it are trivial, in the sense that attempting to move a nonzero homogeneous element up in one of the posets either takes the element to 0 or leaves it unchanged. (The latter only happens if the degree is unchanged, which occurs only when acting by the identity \( 0 \in Q/\mathbb{R} \tau \) or when acting by \( \sigma \in \nabla \tau \) on an element of \( \nabla \tau \)-degree \( \sigma' \subseteq \sigma \).) That is what it means to be a direct sum of skyscraper modules. It implies that any direct sum decomposition of \( \text{soc}_\tau M \) as a vector space graded by \( Q/\mathbb{R} \tau \times \nabla \tau \) is also a decomposition of \( \text{soc}_\tau M \) as a \( Q/\mathbb{R} \tau \)-module or as a \( \nabla \tau \)-module.

Lemma 6.64. If \( \tau \) is a face of \( a \) is a real polyhedral group \( Q \) and \( N = \bigoplus_{\sigma \in \nabla \tau} N_\sigma \) is a module over \( Q \times \nabla \tau \), then \( \text{Hom}_{\nabla \tau}(k_\sigma, N)/\tau \cong \text{Hom}_{\nabla \tau}(k_\sigma, N/\tau) \), and hence

\[
    (\nabla_{\tau, \text{soc}} N)/\tau \cong \nabla_{\tau, \text{soc}} (N/\tau).
\]

Proof. \( \text{Hom}_{\nabla \tau}(k_\sigma, N) \) is the intersection of the kernels of the \( Q \)-module homomorphisms \( N_\sigma \to N_{\sigma'} \) for faces \( \sigma \supset \sigma' \), so the isomorphism of Hom modules follows from Lemma 6.41. The socle isomorphism follows by taking the direct sum over \( \sigma \in \nabla \tau \).

Proposition 6.65. The functors \( \text{soc}_\tau \) and \( \nabla_{\tau, \text{soc}} \) commute. In particular,

\[
    \nabla_{\tau, \text{soc}} (\text{soc}_\tau, \delta_\tau M) \cong \text{soc}_\tau M \cong \text{soc}_\tau (\nabla_{\tau, \text{soc}} \delta_\tau M).
\]

Proof. By taking direct sums over \( a \) and \( \sigma \), this is mostly the natural isomorphisms

\[
    \text{Hom}_{\nabla \tau}(k_\sigma, \text{Hom}_Q(k[a + \tau], -)) \cong \text{Hom}_{Q \times \nabla \tau}(k_\sigma[a + \tau], -)
    \cong \text{Hom}_Q(k[a + \tau], \text{Hom}_{\nabla \tau}(k_\sigma, -))
\]

that result from the adjunction between \( \text{Hom} \) and \( \otimes \). Taking the quotient-restriction along \( \tau \) (Definition 6.39) almost yields the desired result; the only issue is that the left-hand side requires Lemma 6.64.
Example 6.66. If \( a \) is a cogenerator of a downset \( D \subseteq Q \) along \( \tau \) with nadir \( \sigma \), then reasoning as in Example 6.33 and using Definition 6.31, computing \( \nabla_{\tau^{-}\soc} \) first in Proposition 6.65 shows that \( \sigma \in \partial(\nabla_{D} \cap \nabla_{\tau}) \). What \( \soc_{\tau} \) then does is verify that the image \( \tilde{a} \) of \( a \) in \( Q/\mathbb{R}\tau \) is maximal with this property, by Example 6.47.

Proposition 6.67. There is a natural injection

\[
\soc_{\tau} M \hookrightarrow \soc(M/\tau)
\]

for any module \( M \) over a real polyhedral group \( Q \) and any face \( \tau \) of \( Q \).

Proof. By Proposition 6.48 \( \soc_{\tau} N \hookrightarrow \soc(N/\tau) \) for \( N = \nabla_{\tau^{-}\soc} \delta_{\tau} M \) viewed as a \( Q/\mathbb{R}\tau \)-module. Proposition 6.65 yields \( \soc_{\tau} N = \soc_{\tau} M \). It remains to show that \( (Q/\mathbb{R}\tau^{-}\soc)(N/\tau) = \soc(M/\tau) \). To that end, first note that

\[
(\nabla_{\tau^{-}\soc} \delta_{\tau} M/\tau) \cong \nabla_{\tau^{-}\soc} \left( (\delta_{\tau} M/\tau) \right) \cong \nabla_{\tau^{-}\soc} \delta(M/\tau),
\]

the first isomorphism by Lemma 6.64 and the second by Lemma 6.68, which shows that the modules acted on by \( \nabla_{\tau^{-}\soc} \) are isomorphic. Now apply the last isomorphism in Lemma 6.30, with \( Q \) replaced by \( Q/\mathbb{R}\tau \) so that automatically \( \mathcal{F}_{Q}^{\text{op}} \) must be replaced by \( \nabla_{\tau} \) via Lemma 6.55. \( \square \)

Lemma 6.68. If \( \sigma \supseteq \tau \) then \( (\delta^{\sigma} M/\tau) \cong \delta^{\sigma/\tau}(M/\tau) \).

Proof. Explicit calculations from the definitions show that in degree \( a/\tau \) both sides equal

\[
\lim_{\substack{a' \in a - \sigma^{\circ} \\
v \in \tau}} \mathcal{M}_{a' + v},
\]

although they take the colimits in different orders: \( v \) first or \( a' \) first. The hypothesis that \( \sigma \supseteq \tau \) enters to show that any direct limit over \( \{a' \in Q \mid a'/\tau \in a/\tau - (\sigma/\tau)^{\circ}\} \) can equivalently be expressed as a direct limit over \( a' \in a - \sigma^{\circ} \). \( \square \)

Corollary 6.69. An indicator quotient for a downset in a real polyhedral group has at most one linearly independent socle element along each face with given nadir and degree. In fact, the degrees of independent socle elements along \( \tau \) with fixed nadir are incomparable in \( Q/\mathbb{R}\tau \), and nadirs of socle elements with fixed degree are incomparable in \( \nabla_{\tau} \).

Proof. A socle element of an indicator quotient \( E \) along a face \( \tau \) of \( Q \) is a local socle element of \( E \) along \( \tau \) by Proposition 6.67. Local socle elements along \( \tau \) are socle elements (along the minimal face \( \{0\} \)) of the quotient-restriction along \( \tau \) by Definition 6.59.2. But \( E/\tau \) is an indicator quotient of \( \mathbb{K}[Q/\mathbb{R}\tau] \), so its socle degrees with fixed nadir \( \sigma \) are incomparable, as are its nadirs with fixed socle degrees, by Example 6.33. \( \square \)

Example 6.70. Propositions 6.65 and 6.67 ease some socle computations. To see how, consider the indicator \( Q \)-module \( \mathbb{K}[\rho] \) for a face \( \rho \) of \( Q \). Proposition 6.67 immediately implies that \( \soc_{\tau} \mathbb{K}[\rho] = 0 \) unless \( \rho \supseteq \tau \), because localizing along \( \tau \) yields \( \mathbb{K}[\rho]_{\tau} = 0 \) unless \( \rho \supseteq \tau \).
Next compute $\delta^\sigma \mathbb{k}[\rho]$. When either $a \not\in \rho$ or $\sigma \not\subseteq \rho$, the direct limit in Definition 6.11 is over a set $a - \sigma^\circ$ of degrees in which $\mathbb{k}[\rho] = 0$ in a neighborhood of $a$. Hence the only faces that can appear in $\delta^\sigma \mathbb{k}[\rho]$ lie in the interval between $\tau$ and $\rho$, so assume $\tau \subseteq \sigma \subseteq \rho$. If $(\delta^\sigma \mathbb{k}[\rho])_a \neq 0$ then it equals $\mathbb{k}$ because $\mathbb{k}[\rho]$ is an indicator module for a subset of $Q$. Moreover, if $(\delta^\sigma \mathbb{k}[\rho])_a = \mathbb{k}$ then the same is true in any degree $b \in a + \rho$ because $(b - a) + (a - \sigma^\circ) \cap \rho \subseteq (b - \sigma^\circ) \cap \rho$. Thus $\delta^\sigma \mathbb{k}[\rho]$ is torsion-free as a $\mathbb{k}[\rho]$-module.

The soc, on the left side of Proposition 6.65, which by Definition 6.43 is a quotient-restriction of a module $\text{Hom}_Q(\mathbb{k}[\tau], \delta^\tau \mathbb{k}[\rho])$, can only be nonzero if $\tau = \rho$, as any nonzero image of $\mathbb{k}[\tau]$ is a torsion $\mathbb{k}[\rho]$-module. Hence the socle of $\mathbb{k}[\rho]$ along $\tau$ equals the closed socle along $\tau = \rho$, which is computed directly from Definition 6.59.1 and Definition 6.39 to be $\text{Hom}_Q(\mathbb{k}[\tau], \mathbb{k}[\tau]) / \tau = \mathbb{k}[\tau] / \tau$. In summary,

$$\text{soc}_{\tau} \mathbb{k}[\rho] = \begin{cases} \mathbb{k}_0 & \text{for } 0 \in Q / \mathbb{R} \tau \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.71. The global cogenerator functor $\text{soc}_{\tau}$ along any face $\tau$ of a real polyhedral group is left-exact, as is the local cogenerator functor along $\tau$.

Proof. Proposition 6.51 and Lemma 6.12.

Here is the final version of the statement that a module in the category of finitely encoded modules or semialgebraic algebraic modules remains there upon taking socles. Previous versions are used in the proof, but for modules over real polyhedral group this is the only statement worth remembering, as all of the others are special cases. That said, it is also worth noting Proposition 6.26, which treats the finitely encoded category over an arbitrary poset.

Theorem 6.72. If a module $M$ over a real polyhedral group is finitely encoded then so is its socle $\text{soc}_{\tau} M$ along any face $\tau$. If $M$ is semialgebraic then so is $\text{soc}_{\tau} M$. For any face $\sigma \supseteq \tau$, these statements remain true for the socle along $\tau$ with nadir $\sigma$.

Proof. By Proposition 6.65, $\text{soc}_{\tau} M$ is a composite of the functors $\delta$, $\text{soc}_{\tau}$, and $\nabla_{\tau} \text{soc}$ in some order. For $\delta$ use Proposition 6.21. For $\text{soc}_{\tau}$ use Proposition 6.53. For $\nabla_{\tau} \text{soc}$ use Lemma 2.10: the $\nabla_{\tau}$-graded component of $\nabla_{\tau} \text{soc} N$ is the intersection of the kernels of the homomorphisms $N_{\sigma} \to N_{\sigma'}$ for $\sigma \supset \sigma'$. This argument for $\nabla_{\tau} \text{soc}$ already proves the claim concerning a fixed nadir $\sigma$ (see also Remark 6.63). 

7. Essential property of socles

In this section, $Q$ is a real polyhedral group unless otherwise stated.

The culmination of the foundations developed in Section 6 says that socles and cogenerators detect injectivity of homomorphisms between finitely encoded modules over real polyhedral groups (Theorem 7.7), as they do for noetherian rings in ordinary commutative algebra. The theory is complicated by there being no actual submodule
containing a given non-closed socle element; that is why socles are functors that yield submodules of localizations of auxiliary modules rather than submodules of localizations of the given module itself. Nonetheless, it comes down to the fact that, when \( D \subseteq Q \) is a downset, every element can be pushed up to a cogenerator. Theorem 7.5 contains a precise statement that suffices for the purpose of Theorem 7.7, although the definitive version of Theorem 7.5 occurs in Section 8, namely Theorem 8.10.

The proof of Theorem 7.5 requires a definition—essentially the notion dual to that of shape (Proposition 6.10). Informally, it is the set of faces \( \sigma \) such that a neighborhood of \( a \) in \( a + \sigma^o \) is contained in the downset \( D \). The formal definition reduces by negation to the discussion surrounding tangent cones of downsets (Section 6.1), noting that the negative of an upset is a downset.

**Definition 7.1.** The upshape of a downset \( D \) in a real polyhedral group \( Q \) at \( a \) is

\[
\Delta^D_a = \mathcal{F}_Q \setminus \nabla^{-a}_U,
\]

where \( U = Q \setminus D \) is the upset complementary to \( D \).

**Lemma 7.2.** The upshape \( \Delta^D_a \) is a polyhedral complex (a downset) in \( \mathcal{F}_Q \). As a function of \( a \), for fixed \( D \) the upshape \( \Delta^D_a \) is decreasing, meaning \( a \preceq b \Rightarrow \Delta^D_a \supseteq \Delta^D_b \).

**Proof.** These claims are immediate from the discussion in Section 6.1. \( \Box \)

**Remark 7.3.** The upshape \( \Delta^D_a \) is a rather tight analogue of the Stanley–Reisner complex of a simplicial complex, or more generally the lower Koszul simplicial complex [MS05, Definition 5.9] of a monomial ideal in a degree from \( \mathbb{Z}^n \). (The complex \( K_b(I) \) would need to be indexed by \( b - \text{supp}(b) \) to make the analogy even tighter.) Similarly, the shape of a downset at an element of \( Q \) is analogous to the upper Koszul simplicial complex of a monomial ideal [MS05, Definition 1.33].

The general statement about pushing up to cogenerators relies on the special case of closed cogenerators for closed downsets.

**Lemma 7.4.** If \( D \subseteq Q \) is a downset and the part of \( D \) above \( b \in D \) is closed, so \( (b + Q_+) \cap D = (b + Q_+) \cap \overline{D} \), then \( b \preceq a \) for some closed cogenerator \( a \) of \( D \).

**Proof.** It is possible that \( b + Q_+ \subseteq D \), in which case \( D = Q_+ \) and \( b \) is by definition a closed cogenerator along \( \tau = Q_+ \). Barring that case, the intersection \( (b + Q_+) \cap \partial D \) of the principal upset at \( b \) with the boundary of \( D \) is nonempty. Among the points in this intersection, choose \( a \) with minimal upshape. Observe that \( \{0\} \in \Delta^D_a \) because \( a \in D \), so \( \Delta^D_a \) is nonempty.

Let \( \tau \in \Delta^D_a \) be a facet. The goal is to conclude that \( \Delta^D_a = \mathcal{F}_\tau \) has no facet other than \( \tau \), for then \( \Delta^D_{a'} = \mathcal{F}_\tau \) for all \( a' \succeq a \) in \( D \) by upshape minimality and Lemma 7.2, and hence \( a \) is a cogenerator of \( D \) along \( \tau \) by Definition 6.43 (see also Remark 6.46).

Suppose that \( \rho \in \mathcal{F}_Q \) is a ray that lies outside of \( \tau \). If \( \rho \in \Delta^D_a \) then upshape minimality implies \( \rho \in \Delta^D_{a'} \) for any \( a' \in (a + \tau^o) \cap D \), and such an \( a' \) exists by definition.
of upshape. Consequently, some face containing both $\rho$ and $\tau$ lies in $\Delta_a^D$; if $v$ is any sufficiently small vector along $\rho$, then $a' + v = a + (a' - a) + v \in D$, and the smallest face containing $(a' - a) + v$ contains both the interior of $\tau$ (because it contains $a' - a$) and $\rho$ (because it contains $v$). But this is impossible, so in fact $\Delta_a^D = F_\rho$.

**Theorem 7.5.** If $D$ is a downset in a real polyhedral group $Q$, then there are faces $\tau \subseteq \sigma$ of $Q_+$ and a cogenerator $a$ of $D$ along $\tau$ with nadir $\sigma$ such that $b \preceq a$.

**Proof.** It is possible that $b + Q_+ \subseteq D$, in which case $D = Q$ and $b$ is by definition a closed cogenerator along $\tau = Q_+$, which is the same as a cogenerator along $Q_+$ with nadir $Q_+$. Barring that case, the intersection $(b + Q_+) \cap \partial D$ of the principal upset at $b$ with the boundary of $D$ is nonempty. Among the points in this intersection, there is one with minimal shape, and it suffices to treat the case where this point is $b$ itself.

Minimality of $\nabla_D^b$ implies that the shape does not change upon going up from $b$ while staying in the closure $\overline{D}$. Consequently, given any face $\sigma \in \nabla_D^b$, the shape of $D$ at every point in $b + Q_+$ that lies in $\overline{D}$ also contains $\sigma$. Equivalently by Proposition 6.19.2, $(b + Q_+) \cap \delta^\sigma D = (b + Q_+) \cap D$. Lemma 7.4 produces a closed cogenerator $a$ of $\delta^\sigma D$, along some face $\tau$, satisfying $b \preceq a$. Since $\nabla_D^b$ is a nonempty cocomplex, its intersection with $\nabla\tau$ is nonempty, so assume $\sigma \in \nabla_D^b \cap \nabla\tau$. The closed cogenerator $a$ of $\delta^\sigma D$ need not be a cogenerator of $D$, but if $\sigma$ is minimal under inclusion in $\nabla_D^b \cap \nabla\tau$, then $a$ is indeed a cogenerator of $D$ along $\tau$ with nadir $\sigma$ by Proposition 6.65—specifically the first displayed isomorphism—applied to Example 6.23. □

**Remark 7.6.** The arguments in the preceding two proofs are essential to the whole theory of socles, which hinges upon them. The structure of the arguments dictate the forms of all of the notions of socle, particularly those involving cogenerators along faces.

Theorem 7.7 is intended for finitely encoded modules, but because it has no cause to deal with generators, in actuality it only requires half of a fringe presentation (or a little less; see Remark 5.12). The statement uses divisibility (Definition 6.17), which works verbatim for $\delta, M$, by Definition 6.56, because it refers only to upper boundaries atop a single face $\sigma$.

**Theorem 7.7** (Essentiality of socles). Fix a homomorphism $\varphi : M \to N$ of modules over a real polyhedral group $Q$.

1. If $\varphi$ is injective then $\soc_{\tau} \varphi : \soc_{\tau} M \to \soc_{\tau} N$ is injective for all faces $\tau$ of $Q_+$.

2. If $\soc_{\tau} \varphi : \soc_{\tau} M \to \soc_{\tau} N$ is injective for all faces $\tau$ of $Q_+$ and $M$ is downset-finite, then $\varphi$ is injective.

If $M$ is downset-finite then each homogeneous element of $M$ divides a cogenerator of $M$.

**Proof.** Item 1 is a special case of Proposition 6.71. Item 2 follows from the divisibility claim, for if $y$ divides a cogenerator $s$ along $\tau$ then $\varphi(y) \neq 0$ whenever $\soc_{\tau} \varphi(\tilde{s}) \neq 0$, where $\tilde{s}$ is the image of $s$ in $\soc_{\tau} M$. 
For the divisibility claim, fix a downset hull $M \hookrightarrow \bigoplus_{j=1}^{k} E_j$ and a nonzero $y \in M_b$. For some $j$ the projection $y_j \in E_j$ of $y$ divides a cogenerator of $E_j$ along some face $\tau$ with some nadir $\sigma$ by Theorem 7.5. Choose one such cogenerator $s_j$, and suppose it has degree $a \in Q$. There can be other indices $i$ such that $(\text{soc}_\tau E_i)\bar{a} \neq 0$, where $\bar{a}$ is the image of $a$ in $Q/\mathbb{R} \tau$. For any such index $i$, as long as $y_i \neq 0$ it divides a unique cogenerator in $s_i \in \delta^\tau E_i$ by Corollary 6.69. Therefore the image of $y$ in $E = \bigoplus_{j=1}^{k} E_j$ divides the sum of these cogenerators $s_j$. But that sum is itself another cogenerator of $E$ along $\tau$ with nadir $\sigma$ in degree $a$, and the fact that $y$ divides it places the sum in the image of the injection (Lemma 6.12) $\delta^\sigma M \hookrightarrow \delta^\sigma E$. □

Corollary 7.8. Fix a downset-finite module $M$ over a real polyhedral group.

1. $M = 0$ if and only if $\text{soc}_\tau M = 0$ for all faces $\tau$.
2. $\text{soc}_\tau M' \cap \text{soc}_\tau M'' = \text{soc}_\tau(M' \cap M'')$ in $\text{soc}_\tau M$ for submodules $M'$ and $M''$ of $M$.

Proof. That $M = 0 \Rightarrow M = 0$ is trivial. On the other hand, if $\text{soc}_\tau M = 0$ for all $\tau$ then $M$ is a submodule of 0 by Theorem 7.7.2.

The second item is basically left-exactness (Proposition 6.71):

$$\text{soc}_\tau(M' \cap M'') = \text{soc}_\tau \ker(M' \to M/M'')$$

$$= \ker(\text{soc}_\tau M' \to \text{soc}_\tau(M/M''))$$

$$= \ker(\text{soc}_\tau M' \to \text{soc}_\tau M/\text{soc}_\tau M'')$$

$$= \text{soc}_\tau M' \cap \text{soc}_\tau M'' ,$$

where the penultimate equality is because $\text{soc}_\tau M''$ is the kernel of the homomorphism $\text{soc}_\tau M \to \text{soc}_\tau(M/M'')$, so that $\text{soc}_\tau M/\text{soc}_\tau M'' \hookrightarrow \text{soc}_\tau(M/M'')$. □

There is a much stronger statement connecting socles to essential submodules (Theorem 9.5), but it requires language to speak of density in socles as well as tools to produce submodules from socle elements, which are the main themes of Section 8.

8. Minimality of socle functors

Socles capture the entirety of a downset by maximal elements in closures along faces; that is in some sense the main content of socle essentiality (Theorem 7.7), or more precisely Theorem 7.5. But since closures are involved, it is reasonable to ask if anything smaller still captures the entirety of every downset. Algebraically, for arbitrary modules, this asks for subfunctors of cogenerator functors. The particular subfunctors here concern the graded degrees of socle elements, for which notation is needed.

Definition 8.1. The degree set of any module $N$ over a poset $P$ is

$$\text{deg} N = \{a \in P \mid N_a \neq 0\} .$$

Write $\text{deg}_P = \text{deg}$ if more than one poset could be intended.
8.1. Neighborhoods of group elements.

The topological condition characterizing when enough cogenerators are present is a sort of density in the set of all cogenerators. Lemma 6.18 has a related closure notion.

**Definition 8.2.** Fix faces $\sigma \supseteq \tau$ of a real polyhedral group $Q$.

1. A $\sigma$-neighborhood of a point $\tilde{a} \in Q/\mathbb{R}\tau$ in a subset $X \subseteq Q/\mathbb{R}\tau$ is the intersection of $X$ with a subset of the form $(a - v + Q_+)/\mathbb{R}\tau$ with $v \in \sigma^\circ$ and $\tilde{a} = a + \mathbb{R}\tau$.
2. A $\sigma$-limit point of a subset $X \subseteq Q/\mathbb{R}\tau$ is a point $\tilde{a} \in Q/\mathbb{R}\tau$ that is a limit (in the usual topology) of points in $X$ each of which lies in a $\sigma$-neighborhood of $\tilde{a}$.
3. The $\sigma$-closure of $X \subseteq Q/\mathbb{R}\tau$ is the set of points $\tilde{a} \in Q/\mathbb{R}\tau$ such that $X$ has at least one point in every $\sigma$-neighborhood of $\tilde{a}$.

**Remark 8.3.** The sets $X$ to which Definition 8.2 is applied are typically decomposed as finite unions of antichains (but see Proposition 8.6 for an instance where this is not the case). Such sets “cut across” subsets of the form $(a - v + Q_+)/\mathbb{R}\tau$, rather than being swallowed by them, so $\sigma$-neighborhoods have a fighting chance of reflecting some concept of closeness in antichains. If $\sigma = Q_+$ and $\tau = \{0\}$, for example, and $X$ is an antichain in $Q$, then a $\sigma$-neighborhood of a point $a \in X$ is the same thing as a usual open neighborhood of $a$ in $X$, so $\sigma$-closure is the usual topological closure. If, at the other extreme, $\sigma = \tau$, then every antichain in $Q/\mathbb{R}\tau$ is $\tau$-closed.

**Example 8.4.** Let $Q = \mathbb{R}^2$ and $\tau = \{0\}$. Take for $X \subset \mathbb{R}^2$ the convex hull of $0, e_1, e_2$ but with the first standard basis vector $e_1$ removed. If $\sigma$ is the $x$-axis of $\mathbb{R}^2_+$, then in

constitute $\sigma$-neighborhoods of $e_1$: in $X$ take all of the blue points and in the half-open hypotenuse take only the bold blue segment. The point $e_1$ is a $\sigma$-limit point of the half-open hypotenuse.

The next result is applied in the proof of Theorem 8.15. It is close to—but not quite—what is needed for the $\sigma$-neighborhoods to form a base for a topology on $X$; to be a base, the intersections would have to allow $\sigma$-neighborhoods of distinct points.

**Lemma 8.5.** Any finite intersection of $\sigma$-neighborhoods of a point $\tilde{a} \in Q/\mathbb{R}\tau$ in a subset $X \subseteq Q/\mathbb{R}\tau$ contains a $\sigma$-neighborhood of $\tilde{a} \in Q/\mathbb{R}\tau$ in $X$.

**Proof.** If $v \in \sigma^\circ$ then $a - v + \sigma$ contains an open neighborhood of $a$ in $a - \sigma$. Therefore so does a finite intersection $U$ of sets of the form $a - v + \sigma$. Any point $v' \in U \cap (a - \sigma^\circ)$ yields the desired $\sigma$-neighborhood $a - v' + Q_+$. That proves the case $\tau = \{0\}$. Reducing modulo $\mathbb{R}\tau$ proves the general case. \qed
The concept of \( \sigma \)-neighborhood provides a means to connect socles (Definition 6.59) with support (Definition 3.23) and primary decomposition (Definition 3.28).

**Proposition 8.6.** In a real polyhedral group \( Q \), every cogenerator of a downset \( D \) along \( \tau \) with nadir \( \sigma \) has a \( \sigma \)-neighborhood \( O \) in \( D \subseteq Q \) (so \( \sigma \supseteq \{0\} \) are the faces in Definition 8.2) such that \( k[O] \subseteq k[D] \) is \( \tau \)-coprimary and globally supported on \( \tau \).

**Proof.** Let \( a \) be such a cogenerator of \( D \). Suppose \( \{a_k\}_{k \in \mathbb{N}} \subseteq a - \sigma \subseteq D \) is any sequence converging to \( a \). If \( a_k \) is supported on a face \( \tau' \), then \( \tau' \supseteq \tau \) because \( a \succeq a_k \) and \( a \) remains a cogenerator of the localization of \( D \) along \( \tau \) by Proposition 6.67. The same argument shows that the \( \sigma \)-neighborhood \( O = a_k + Q_+ \) yields a submodule \( k[O] \subseteq k[D] \) such that \( k[O] \twoheadrightarrow k[D] \) is \( \tau \)-coprimary and globally supported on \( \tau \). The goal is therefore to show that some \( a_k \) is supported on \( \tau \), for then all of \( O \) is supported on \( \tau \), as support can only decrease upon going up in \( Q \).

If each \( a_k \) is supported on a face properly containing \( \tau \), then, restricting to a subsequence if necessary, assume that it is the same face \( \tau' \) for all \( k \). (This uses the finiteness of the number of faces.) But then \( a + \tau' = \lim_k (a_k + \tau') \) is contained in \( \delta \sigma D \), contradicting the fact that \( a \) is supported on \( \tau \) in \( \delta \sigma D \). \( \square \)

**Example 8.7.** All three of the downsets

\[
\begin{array}{ccc}
D_1 & D_2 & D_3 \\
\hline
\end{array}
\]

in \( \mathbb{R}^2 \) have a cogenerator at the open corner \( a \) along the face \( \tau = \{0\} \), but their behaviors near \( a \) differ in character. Write \( \sigma_x \) and \( \sigma_y \) for the faces of \( \mathbb{R}^2_+ \) that are its horizontal and vertical axes, respectively.

1. Here \( a \) has two nadirs: it is a cogenerator along \( \tau = \{0\} \) for both \( \sigma_x \) and \( \sigma_y \) by Proposition 6.19 and Example 6.23. The blue set in Example 8.4 constitutes a \( \tau \)-coprimary \( \sigma_x \)-neighborhood of \( a \) globally supported on \( \tau \), as in Proposition 8.6, if the open point there is also \( a \).

2. Here \( a \) has only the nadir \( \sigma_x \), because the downset has no points in \( a + \mathbb{R} \sigma_y \) to take the closure of in Lemma 6.18. Again, the blue set in Example 8.4 constitutes the desired \( \sigma_x \)-neighborhood of \( a \).

3. Here \( a \) has only the nadir \( \sigma_y \). It is possible to compute this directly, but it is more apropos to note that Proposition 8.6 rules out \( \sigma_x \) as a nadir. Indeed, every \( \sigma_x \)-neighborhood of \( a \) in \( D_3 \) is an infinite vertical strip. None of these \( \sigma_x \)-neighborhoods are supported on \( \tau = \{0\} \), since elements therein persist forever along \( \sigma_y \). In contrast, every choice of \( v \in \sigma_y^\circ \) yields a \( \sigma_y \)-neighborhood \( -v + \mathbb{R}^2_+ \cap D_3 \) supported on \( \{0\} \); that is, the entire negative \( y \)-axis is supported on \( \{0\} \).
Compare the following with Example 6.70; it is the decisive more or less explicit calculation that justifies the general theory of socles and provides its foundation. Note that being $\tau$-coprimary only requires an essential submodule to be globally supported on $\tau$; see the under-hyperbola in Example 3.17. Dually, being globally supported on $\tau$ allows for elements with support strictly contained in $\tau$.

**Corollary 8.8.** Fix a face $\tau$ of a real polyhedral group $Q$ and a subquotient $M$ of $k[Q]$ that is $\tau$-coprimary and globally supported on $\tau$. Then $\soc_{\tau'} M = 0$ unless $\tau' = \tau$.

**Proof.** Proposition 6.67 implies that $\soc_{\tau'} M = 0$ unless $\tau' \supseteq \tau$ by definition of global support: localizing along $\tau'$ yields $M_{\tau'} = 0$ unless $\tau' \subseteq \tau$. On the other hand, $M$ being a subquotient of $k[Q]$ means that $M \subseteq k[D]$ for some downset $D$. By left-exactness of socles (Proposition 6.71), every cogenerator of $M$ is a cogenerator of $k[D]$. Applying Proposition 8.6 to any such cogenerator along $\tau'$ implies that $\tau = \tau'$, because no $\tau$-coprimary module has a submodule supported on a face strictly contained in $\tau$. □

Definition 8.2.2 stipulates no condition the generators of the relevant $\sigma$-neighborhoods—the vectors $a - v$ in Definition 8.2.1. The difference between being a $\sigma$-limit point and lying in the $\sigma$-closure is hence that for $\sigma$-closure, the convergence is stipulated on the generators of the $\sigma$-neighborhoods rather than on the points of $X$. That said, the a priori weaker (that is, more inclusive) notion of $\sigma$-limit point is equivalent: the generators can be forced to converge.

**Proposition 8.9.** If a sequence $\{a'_k\}_{k \in \mathbb{N}}$ in a real polyhedral group $Q$ has $a'_k \to a$ and $a'_k \in a_k + Q_+$ for some $a_k \in a - \sigma^\circ$, where $\sigma$ is a fixed face, then it is possible to choose the elements $a_k$ so that $a_k \to a$. Consequently, if $\sigma \supseteq \tau$ then the $\sigma$-closure of any set $X \subseteq Q/\mathbb{R}\tau$ equals the set of its $\sigma$-limit points.

**Proof.** Writing $a'_k = a - v_k + z_k$ with $v_k \in \mathbb{R}\sigma$ and $z_k \in \sigma^\perp$, the only relevant thing the hypothesis $a'_k \in a_k + Q_+$ does is force $z_k$ to land in $Q_+/\mathbb{R}\sigma$ when projected to $Q/\mathbb{R}\sigma$. This leads us to consider the set $Z \subseteq \sigma^\perp$ of vectors in $\sigma^\perp$ whose images in $Q_+/\mathbb{R}\sigma$ lie in $Q_+/\mathbb{R}\sigma$ and have magnitude $\leq 1$. Let $V \subseteq \mathbb{R}\sigma$ be the ball of radius 1. Find $s \in \sigma^\circ$ so that $s + V + Z \subseteq Q_+$. To see that such an $s$ exists, first find $s$ so that $s + Z \subseteq Q_+$, which exists by rescaling any element $s' \in \sigma^\circ$ because the projection of $(s' + \sigma^\perp) \cap Q_+$ to $Q/\mathbb{R}\sigma$ contains a neighborhood of $0$ in $Q_+/\mathbb{R}\sigma$. Then observe that the condition $s + Z \subseteq Q_+$ remains true after adding any element of $\sigma$ to $s$. In particular, add the center of any ball in $\sigma^\circ$ of radius 1, which exists because $\sigma^\circ$ is nonempty, open in $\mathbb{R}\sigma$, and closed under positive scaling.

Having fixed $s$ with $s + V + Z \subseteq Q_+$, note that $\varepsilon s + \varepsilon V + \varepsilon Z \subseteq Q_+$. On the other hand, the magnitudes of $v_k$ and $z_k$ are bounded above by $\varepsilon_k = |a'_k - a|$. Therefore, $a'_k \in a_k + Q_+$, where $a_k = a - \varepsilon_k s \to a$, because $a'_k = a - v_k + z_k \in a + \varepsilon V + \varepsilon Z$, and $a + \varepsilon V + \varepsilon Z = (a - \varepsilon s) + (\varepsilon s + \varepsilon V + \varepsilon Z) \subseteq a - \varepsilon s + Q_+$.  

The claim involving $\tau$ follows, when $\tau = \{0\}$, from Proposition 6.7: it implies that each element of $a - \sigma^o$ precedes some $a_k$, and hence the $\sigma$-neighborhood it generates contains $a'_k$. The case of arbitrary $\tau$ reduces to $\tau = \{0\}$ by working modulo $\mathbb{R}\tau$. \qed

8.2. Dense cogeneration of downsets.

The subfunctor version of density in socles for modules requires first a geometric version for downsets. For geometric intuition, it is useful to recall Lemma 6.8, which says that $Q_{\nabla\sigma} = \sigma^o + Q_+$. Thus $a - Q_{\nabla\sigma}$ is the "coprincipal" downset with apex $a$ and shape $\nabla\sigma$. Adding $\tau$ to get $a + \tau - Q_{\nabla\sigma}$ takes the union of these downsets along $a + \tau$.

Theorem 8.10. Let $A_{\sigma}^\tau \subseteq Q$ be a set of cogenerators of a downset $D$ in a real polyhedral group $Q$ along a face $\tau$ with nadir $\sigma$ for each $\sigma \in \nabla\tau$. If every cogenerator of $D$ along $\tau$ with nadir $\sigma$ maps to a $\sigma$-limit point of the image of $A_{\tau} = \bigcup_{\sigma \subseteq \tau} A_{\sigma}^\tau$ in $Q/\mathbb{R}\tau$, then

$$D = \bigcup_{\text{faces } \sigma, \tau \text{ with } \sigma \supseteq \tau} \bigcup_{a \in A_{\sigma}^\tau} a + \tau - Q_{\nabla\sigma}.$$ 

Proof. Theorem 7.5 is equivalent to the desired result in the case that every $A_{\sigma}^\tau$ is the set of all cogenerators of $D$ along $\tau$ with nadir $\sigma$, by Example 6.66 and Remark 6.62. Hence it suffices to show that

$$\bigcup_{\sigma' \supseteq \tau} \bigcup_{a' \in A_{\sigma'}^\tau} a' + \tau - Q_{\nabla\sigma'} \supseteq a + \tau - Q_{\nabla\sigma}$$

for any fixed cogenerator $a$ of $D$ along $\tau$ with nadir $\sigma$. In fact, by definition of $\sigma$-limit point, it is enough to show that

$$\bigcup_{k=1}^{\infty} a'_k + \tau - Q_{\nabla\sigma_k} \supseteq a + \tau - Q_{\nabla\sigma},$$

where $\{a'_k\}_{k \in \mathbb{N}}$ is a sequence of elements of $A_{\tau}$ such that

- $a_k$ lands in a $\sigma$-neighborhood of the image $\tilde{a}$ of $a$ when projected to $Q/\mathbb{R}\tau$, and
- these images $\tilde{a}'_k$ converge to $\tilde{a}$ in $Q/\mathbb{R}\tau$

and $\sigma_k$ is a nadir of the cogenerator $a'_k$ along $\tau$.

Note that there is something to prove even when $\sigma = \tau$ (see the end of Remark 8.3) because $A_{\sigma}^\tau$ only needs to have at least one closed cogenerator in $Q$ for each closed socle degree in $Q/\mathbb{R}\tau$, whereas the set of all closed cogenerators along $\tau$ mapping to a given socle degree might not be a single translate of $\sigma$. On the other hand, $\tau - Q_{\nabla\tau} = \tau - \tau^o - Q_+$ by Lemma 6.8, and this is just $\mathbb{R}\tau - Q_+$. Therefore $a + \tau - Q_{\nabla\tau}$ contains the translate of the negative cone $-Q_+$ at every point mapping to $\tilde{a}$, cogenerator or otherwise, completing the case $\sigma = \tau$.
For general \( \sigma \supseteq \tau \), again Lemma 6.8 says that \( \mathbf{a} + \tau - Q_{\nabla \sigma} = \mathbf{a} + \tau - \sigma^\circ - Q_+ \), and as before this set is preserved under translation by \( \mathbb{R}\tau \) because \( -Q_+ \) contains \( -\tau \). Therefore the question reduces to the quotient \( Q/\mathbb{R}\tau \), where it becomes

\[
\bigcup_{k=1}^{\infty} \mathbf{a}_k' - Q_{\nabla \sigma_k}/\tau \supseteq \mathbf{a} - Q_{\nabla \sigma}/\tau.
\]

But as \( Q_{\nabla \sigma_k}/\tau = \sigma_k^\circ/\tau + (Q/\mathbb{R}\tau)_+ \) by Lemma 6.8, it does no harm (and helps the notation) to assume that \( \tau = \{0\} \). The desired statement is now

\[
\bigcup_{k=1}^{\infty} \mathbf{a}_k' - Q_{\nabla \sigma_k} \supseteq \mathbf{a} - Q_{\nabla \sigma},
\]

the hypotheses being those of Proposition 8.9. The proof is completed by applying Proposition 6.7 to the sequence \( \{a_k\}_{k \in \mathbb{N}} \) produced by Proposition 8.9, noting that \( a_k - Q_+ \subseteq a_k' - Q_{\nabla \sigma_k} \) as soon as \( a_k \in a_k' - Q_{\nabla \sigma_k} \), because \( a_k' - Q_{\nabla \sigma_k} \) is a downset. \( \square \)

**Example 8.11.** Consider the downsets in Example 8.7. The question is whether \( \mathbf{a} \) is forced to appear in the union from Theorem 8.10 or not.

1. The point \( \mathbf{a} \) is a \( \sigma_x \)-limit point of \( D_1 \) by Example 8.4, which shares its geometry with \( D_1 \) on the relevant set, namely the \( x \)-axis and above. Theorem 8.10 therefore wants to force \( \mathbf{a} \) to appear. However, every \( \sigma_y \)-neighborhood of \( \mathbf{a} \) in \( D_1 \) contains exactly one cogenerator, namely \( \mathbf{a} \). Therefore \( \mathbf{a} \) is indeed forced to appear.

2. In contrast, \( \mathbf{a} \) is not needed for \( D_2 \). Abstractly, this is because \( D_2 \) is missing precisely the negative \( y \)-axis that caused \( \mathbf{a} \) to be forced in \( D_1 \). But geometrically it is evident that \( D_2 \) equals the union of the closed negative quadrants hanging from the open diagonal ray.

3. Here \( \mathbf{a} \) is the sole cogenerator of \( D_3 \) along \( \tau = \{0\} \), so it is forced to appear.

**Remark 8.12.** Theorem 8.10 is the analogue for real polyhedral groups of the fact that monomial ideals in affine semigroup rings admit unique irredundant irreducible decompositions [MS05, Corollary 11.5]. To see the analogy, note that expressing a downset as a union is the same as expressing its complementary upset as an intersection.

In Theorem 8.10 the union is neither unique nor irredundant, but only in the sense that a topological space can have many dense subsets, each of which can usually be made smaller by omitting some points. The union in which \( \deg_{Q/\mathbb{R}\tau} \soc_\tau k[D] \) modulo \( \mathbb{R}\tau \) is still canonical, though redundant in a predictable manner.

**Corollary 8.13.** Fix a cogenerator \( \mathbf{a} \) of a downset \( D \) along a face \( \tau \) with nadir \( \sigma \) in a real polyhedral group. If \( \mathbf{b} \in D \) and \( \mathbf{b} \preceq \mathbf{a} \), then the image \( \tilde{\mathbf{a}} \) of \( \mathbf{a} \) in \( Q/\mathbb{R}\tau \) has a \( \sigma \)-neighborhood \( \mathcal{O} \) in \( \deg_{Q/\mathbb{R}\tau} \soc_\tau k[D] \) such that \( \tilde{\mathbf{b}} \preceq \tilde{\mathbf{a}}' \) for all \( \tilde{\mathbf{a}}' \in \mathcal{O} \).

**Proof.** Assume \( \mathbf{b} \in D \) and \( \mathbf{b} \preceq \mathbf{a} \). Theorem 8.10 implies that \( \mathbf{b} \in \mathbf{a} + \tau - Q_{\nabla \sigma} = \mathbf{a} + \mathbb{R}\tau - \sigma^\circ - Q_+ \). Therefore \( \mathbf{a} + \mathbb{R}\tau = \mathbf{b} + \mathbb{R}\tau + s + q \) for some \( s \in \sigma^\circ \) and \( q \in Q_+ \). The \( \sigma \)-neighborhood in question is \( (\tilde{\mathbf{b}} + \tilde{q} + Q_+) \cap \deg_{Q/\mathbb{R}\tau} \soc_\tau k[D] \). \( \square \)
8.3. Dense subfunctors of socles.

In general, a subfunctor \( \Phi : A \to B \) of a covariant functor \( \Psi : A \to B \) is a natural transformation \( \Phi \to \Psi \) such that \( \Phi(A) \subseteq \Psi(A) \) for all objects \( A \in A \) [EM45, Chapter III]; denote this by \( \Phi \subseteq \Psi \). (This notation assumes that the objects of \( B \) are sets, which they are here; in general, \( \Phi(A) \to \Psi(A) \) should be monic.)

**Definition 8.14.** A subfunctor \( S_\tau = \bigoplus_{\sigma \in \Sigma_\tau} S_\sigma^\tau \subseteq \text{soc}_\tau \) from modules over \( Q \) to modules over \( Q/\mathbb{R}\tau \times \nabla \tau \) is dense if the \( \sigma \)-closure of \( \text{deg}_{Q/\mathbb{R}\tau} S_\tau k[D] \) contains \( \text{deg} \text{soc}_\sigma^\tau k[D] \) for all faces \( \sigma \supseteq \tau \) and downsets \( D \subseteq Q \). An \( S \)-cogenerator of a \( Q \)-module \( M \) is a cogenerator of \( M \) along some face \( \tau \) whose image in \( \text{soc}_\tau M \) lies in \( S_\tau M \).

**Theorem 8.15.** Fix subfunctors \( S_\tau \subseteq \text{soc}_\tau \) for all faces \( \tau \) of a real polyhedral group. Theorem 7.7 holds with \( S \) in place of \( \text{soc} \) if and only if \( S_\tau \) is dense in \( \text{soc}_\tau \) for all \( \tau \).

*Proof.* Every subfunctor of any left-exact functor takes injections to injections; therefore Theorem 7.7.1 holds for any subfunctor of \( \text{soc}_\tau \) by Proposition 6.7.1. The content is that Theorem 7.7.2 is equivalent to density of \( S_\tau \) in \( \text{soc}_\tau \) for all \( \tau \).

First suppose that \( S_\tau \) is dense in \( \text{soc}_\tau \) for all \( \tau \). It suffices to show that each homogeneous element \( y \in M \) divides some \( S \)-cogenerator \( s \), for then \( \varphi(y) \neq 0 \) whenever \( S_\tau \varphi(\tilde{s}) \neq 0 \), where \( \tilde{s} \) is the image of \( s \) in \( S_\tau M \subseteq \text{soc}_\tau M \). There is no harm in assuming that \( M \) is a submodule of its downset hull: \( M \subseteq E = \bigoplus_{j=1}^{k} E_j \). Theorem 7.7 produces a cogenerator \( x \) of \( E \) that is divisible by \( y \), and \( x \) is automatically a cogenerator of \( M \)—say \( x \in \delta^\tau_M \subseteq \delta^\tau_E \)—because \( y \) divides \( x \). Write \( x = \sum_{j=1}^{k} x_j \in \delta^\tau_E = \bigoplus_{j=1}^{k} \delta^\tau E_j \). For any index \( j \) such that \( x_j \neq 0 \), Corollary 8.13 and the density hypothesis yield a \( \sigma \)-neighborhood of \( \tilde{a} \) containing a socle element \( \tilde{s}_j \) mapped by an \( S \)-cogenerator \( s_j \) that is divisible by \( y_j \). An \( S \)-cogenerator \( s \) of \( M \) divisible by \( y \) is constructed from \( s_j \) just as an ordinary cogenerator is constructed from \( s_j \) in the second paragraph of the proof of Theorem 7.7.

Now suppose that \( S_\tau \) is not dense in \( \text{soc}_\tau \) for some face \( \tau \), so some downset \( D \subseteq Q \) has a cogenerator \( a \in Q \) whose image \( \tilde{a} \in \text{deg} \text{soc}_\sigma^\tau k[D] \subseteq Q/\mathbb{R}\tau \) has a \( \sigma \)-neighborhood \( \text{deg}_{Q/\mathbb{R}\tau} \text{soc}_\tau k[D] \cap (a - v + Q_+)/\mathbb{R}\tau \) devoid of images of \( S \)-cogenerators along \( \tau \). Appealing to Lemma 8.5, the intersection of \( a - v + Q_+ \) with a \( \sigma \)-neighborhood \( O \) of \( a \) in \( D \) from Proposition 8.6 contains another \( \sigma \)-neighborhood \( O' \) of \( a \) that still satisfies the conclusion of Proposition 8.6 because every submodule of any \( \tau \)-coprimary module globally supported on \( \tau \) is also \( \tau \)-coprimary and globally supported on \( \tau \). The injection \( k[O'] \hookrightarrow k[D] \) yields an injection \( S_\tau k[O'] \hookrightarrow S_\tau k[D] \), but by construction \( S_\tau k[D] \) vanishes in all degrees from \( \text{deg}_{Q/\mathbb{R}\tau} \text{soc}_\tau k[O'] \), so \( S_\tau k[O'] = 0 \). On the other hand, \( \text{soc}_\tau k[O'] = 0 \) for \( \tau' \neq \tau \) by Corollary 8.8, so the subfunctor \( S_{\tau'} \) vanishes on \( k[O'] \) for all faces \( \tau' \). Consequently, applying \( S_{\tau'} \) to the homomorphism \( \varphi : k[O'] \to 0 \) yields an injection \( 0 \hookrightarrow 0 \) for all faces \( \tau' \) even though \( \varphi \) is not injective. \( \square \)
9. Essential submodules via density in socles

The $\sigma$-neighborhoods in Proposition 8.6 transfer cogenerators back into honest submodules; they are, in that sense, the reverse of Definition 6.11. In fact this transference of cogenerators into submodules works not merely for indicator quotients but for arbitrary modules with finite downset hulls, as in Theorem 9.5. The key is the generalization of $\sigma$-neighborhoods to arbitrary downset-finite modules.

**Definition 9.1.** Fix a module $M$ over a real polyhedral group $Q$ and a face $\tau$.

1. A $\sigma$-divisor (Definition 6.17) $y \in M$ of a cogenerator of $M$ along $\tau$ with nadir $\sigma$ (Definition 6.59) is nearby if $y$ is globally supported on $\tau$ (Definition 3.23).
2. A $\sigma$-neighborhood in $M$ of a cogenerator $s \in \delta_\sigma M$ is a submodule of $M$ generated by a nearby $\sigma$-divisor of $s$.
3. A neighborhood in $\text{soc}_\tau M$ of a homogeneous socle element $\tilde{x} \in \text{soc}_\sigma M$ is $\text{soc}_\tau N$ for a $\sigma$-neighborhood $N$ in $M$ of a cogenerator in $\delta_\sigma M$ that maps to $\tilde{x}$.
4. A submodule $S_\tau \subseteq \text{soc}_\tau M$ is dense if for all $\sigma \supseteq \tau$, every neighborhood of every homogeneous element of $\text{soc}_\sigma M$ contains a nonzero element of $S_\tau$.

**Lemma 9.2.** Every neighborhood in $M$ of every homogeneous element in $\text{soc}_\sigma M$ is a $\tau$-coprimary submodule of $M$ globally supported on $\tau$.

**Proof.** Let $y$ be a nearby $\sigma$-divisor of a cogenerator $s \in \delta_\sigma M$. Let $x$ be a homogeneous multiple of $y$. That $x$ is supported on $\tau$ is automatic from the hypothesis that $y$ is supported on $\tau$. To say that $\langle y \rangle$ is $\tau$-coprimary means, given that it is supported on $\tau$, that $\langle y \rangle$ is a submodule of its localization along $\tau$. But $s$ remains a cogenerator after localizing along $\tau$ by Proposition 6.67, so $x$ must remain nonzero because it still divides $s$ after localizing. □

**Proposition 9.3.** Fix a downset-finite module $M$ over a real polyhedral group with faces $\sigma \supseteq \tau$. Every cogenerator in $\delta_\sigma M$ has a $\sigma$-neighborhood in $M$.

**Proof.** Let $s \in \delta_\sigma M$ be the cogenerator, and let its degree be $\text{deg}_Q(s) = a \in Q$. Choose a downset hull $M \hookrightarrow E = \bigoplus_{j=1}^k E_j$, so $E_j = \mathbb{k}[D_j]$ for a downset $D_j$. Express $s = s_1 + \cdots + s_k \in \delta^\sigma E = \bigoplus_{j=1}^k \delta^\sigma E_j$. Proposition 8.6 produces a $\sigma$-neighborhood $\mathcal{O}_j$ of $a$ in $Q$, for each index $j$, such that $\mathbb{k}[\mathcal{O}_j \cap D_j]$ is a $\sigma$-neighborhood in $E_j$ of the image $\tilde{s}_j \in \text{soc}_\sigma E_j$. Lemma 8.5 then yields a single $\sigma$-neighborhood $\mathcal{O} = a - v + Q_+$ of $a$ in $Q$ that lies in the intersection $\bigcap_{j=1}^k \mathcal{O}_j$. The cogenerator $s \in \delta^\sigma$ is a direct limit over $a - \sigma^\circ$; since $\mathcal{O}$ contains a neighborhood (in the usual topology) of $a$ in $\sigma^\circ$, some element $y \in M$ with degree in $\mathcal{O}$ is a $\sigma$-divisor of $s$. This element $y$ is nearby $s$ by construction. □

The following generalization of Corollary 8.8 to modules with finite downset hulls is again the decisive computation.

**Corollary 9.4.** Fix a downset-finite $\tau$-coprimary $Q$-module $M$ globally supported on a face $\tau$ of a real polyhedral group $Q$. Then $\text{soc}_{\tau'} M = 0$ unless $\tau' = \tau$. 

Proof. Proposition 6.67 implies that \( \text{soc}_\tau M = 0 \) unless \( \tau' \supseteq \tau \) by definition of global support: localizing along \( \tau' \) yields \( M_{\tau'} = 0 \) unless \( \tau' \subseteq \tau \). On the other hand, applying Proposition 9.3 to any cogenerator of \( M \) along a face \( \tau' \) implies that \( \tau = \tau' \), because no \( \tau \)-coprimary module has a submodule supported on a face strictly contained in \( \tau \). □

**Theorem 9.5.** In a downset-finite module \( M \) over a real polyhedral group, \( M' \) is an essential submodule if and only if \( \text{soc}_\tau M' \) is dense in \( \text{soc}_\tau M \) for all faces \( \tau \).

**Proof.** First assume that \( M' \) is not an essential submodule, so \( N \cap M' = 0 \) for some nonzero submodule \( N \subseteq M \). Let \( s \in \delta^\sigma N \) be a cogenerator. Any \( \sigma \)-neighborhood of \( s \) in \( N \), afforded by Proposition 9.3, has a socle along \( \tau \) that is a neighborhood of \( \tilde{s} \) in \( \text{soc}_\tau M \) whose intersection with \( \text{soc}_\tau M' \) is 0. Therefore \( \text{soc}_\tau M' \) is not dense in \( \text{soc}_\tau M \).

Now assume that \( \text{soc}_\tau M' \) is not dense in \( \text{soc}_\tau M \) for some \( \tau \). That means \( \text{soc}_\tau M \) for some nadir \( \sigma \) has an element \( \tilde{s} \) with a neighborhood \( \text{soc}_\tau N \) that intersects \( \text{soc}_\tau M' \) in 0. But \( \text{soc}_\tau N \cap \text{soc}_\tau M' = \text{soc}_\tau(N \cap M') \) by Corollary 7.8.2. The vanishing of this socle along \( \tau \) means that \( \text{soc}_\tau(N \cap M') = 0 \) for all faces \( \tau' \) by Corollary 9.4, and thus \( N \cap M' = 0 \) by Corollary 7.8.1. Therefore \( M' \) is not an essential submodule of \( M \). □

**Example 9.6.** The convex hull of \( 0, e_1, e_2 \) in \( \mathbb{R}^2 \) but with the first standard basis vector \( e_1 \) removed defines a subquotient \( M \) of \( k[\mathbb{R}^2] \). It has submodule \( M' \) that is the indicator function for the same triangle but with the entire \( x \)-axis removed. All of the cogenerators of both modules occur along the face \( \tau = \{0\} \) because both modules are globally supported on \( \{0\} \). However the ambient module—but not the submodule—has a cogenerator \( y \in \delta^\sigma M \) with nadir \( \sigma = x \)-axis of degree \( e_1 \):

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{example.png}
\end{array}
\]

Every such neighborhood contains socle elements in \( \text{soc}_\tau M' \), so \( M' \subseteq M \) is an essential submodule by Theorem 9.5. Trying to mimic this example in a finitely generated context is instructive: pixelated rastering of the horizontal lines either isolates the socle element at the right-hand endpoint of the bottom edge or prevents it from existing in the first place by aligning with the right-hand end of the line above it.

10. **Primary decomposition over real polyhedral groups**

This section takes the join of Section 3, which develops primary decomposition as far as possible over arbitrary polyhedral partially ordered groups, and Section 6, which develops socles over real polyhedral groups. That is, it investigates how socles interact with primary decomposition in real polyhedral groups.
10.1. Associated faces.
What makes the theory for real polyhedral groups stronger than for arbitrary polyhedral partially ordered groups is the following notion familiar from commutative algebra, except that (as noted in Section 9) socle elements do not lie in the original module.

**Definition 10.1.** A face $\tau$ of a real polyhedral group $Q$ is associated to a downset-finite $Q$-module $M$ if $\text{soc}_\tau M \neq 0$. If $M = \mathbb{k}[D]$ for a downset $D$ then $\tau$ is associated to $D$. The set of associated faces of $M$ or $D$ is denoted by $\text{Ass} M$ or $\text{Ass} D$.

**Theorem 10.2.** A downset-finite module $M$ over a real polyhedral group is $\tau$-coprimary if and only if $\text{soc}_\tau M = 0$ whenever $\tau' \neq \tau$, or equivalently, $\text{Ass}(M) = \{\tau\}$.

**Proof.** If $M$ is not $\tau$-coprimary then either $M \to M_\tau$ has nonzero kernel $N$, or $M \to M_\tau$ is injective while $M_\tau$ has a submodule $N_\tau$ supported on a face strictly containing $\tau$. In the latter case, moving up by an element of $\tau$ shows that $N = N_\tau \cap M$ is nonzero. In either case, any cogenerator of $N$ lies along a face $\tau' \neq \tau$, so $0 \neq \text{soc}_{\tau'} N \subseteq \text{soc}_{\tau'} M$.

On the other hand, if $M$ is $\tau$-coprimary then $\Gamma_\tau M$ is an essential submodule of $M$ because every nonzero submodule of $M \subseteq M_\tau$ has nonzero intersection with $\Gamma_\tau M_\tau$, and hence with $M \cap \Gamma_\tau M_\tau = \Gamma_\tau M$, inside of the ambient module $M_\tau$ by Definition 3.28.2. Theorem 9.5 says that $\text{soc}_{\tau'} \Gamma_\tau M$ is dense in $\text{soc}_{\tau'} M$ for all $\tau'$. But $\text{soc}_{\tau'} \Gamma_\tau M = 0$ for $\tau' \neq \tau$ by Corollary 9.4, so density implies $\text{soc}_{\tau'} M = 0$ for $\tau' \neq \tau$. \hfill $\square$

10.2. Canonical primary decompositions of downsets.

**Lemma 10.3.** A downset $D$ in a real polyhedral group is $\tau$-coprimary if and only if

$$D = \bigcup_{\text{faces } \sigma \text{ with } \sigma \nsubseteq \tau} \bigcup_{a \in A_\sigma^\tau} a + \tau - Q_\sigma$$

for sets $A_\sigma^\tau \subseteq Q$ such that the image in $Q/\mathbb{R}_\tau \times \nabla_\tau$ of $\bigcup_{a \in A_\tau^\sigma} A_\sigma^\tau \times \{\sigma\}$ is an antichain, and in that case $A_\tau^\sigma$ projects to a subset of $\text{deg}_{Q/\mathbb{R}_\tau} \text{soc}^\sigma_{\tau'} \mathbb{k}[D] \subseteq Q/\mathbb{R}_\tau$ for each $\sigma$.

**Proof.** If $D$ is $\tau$-coprimary, then it is such a union by Theorem 10.2 and Theorem 8.10, keeping in mind the antichain consequences of Example 6.33.

On the other hand, if $D$ is such a union, then first of all it is stable under translation by $\mathbb{R}_\tau$ because every member of the union is. Working in $Q/\mathbb{R}_\tau$, therefore, assume that $\tau = \{0\}$. Example 6.33 implies that every element of $A_\tau^\sigma$ is a cogenerator of $D$ with nadir $\sigma$. Proposition 8.6 produces a $\sigma$-neighborhood $O_\alpha^\sigma$ of $a$ in $D$ that is globally supported on $\{0\}$ (and hence $\{0\}$-coprimary). But every element $b \in D$ that precedes $a$ also precedes some element in $O_\alpha^\sigma$: that is, $b \preceq a \Rightarrow (b + Q_\sigma) \cap O_\alpha^\sigma \neq \emptyset$. The union of the $\sigma$-neighborhoods $O_\alpha^\sigma$ over all faces $\sigma$ and elements $a \in A_\tau^\sigma$ therefore cogenerates $D$, so $D$ is coprimary by Definition 3.18. \hfill $\square$
Remark 10.4. The antichain condition in Lemma 10.3 is necessary: \( Q \) itself is the union of all translates of \(-Q\), but \( Q \) is \( Q_+\)-coprimary, whereas \(-Q\) is \( \{0\}\)-coprimary. Moreover, the \( \nabla \tau \) component of the antichain condition is important; that is, the nadirs also come into play. For a specific example, take \( D \subseteq \mathbb{R}^2 \) to be the union of the open negative quadrant cogenerated by \( 0 \) and the closed negative quadrant cogenerated by any point on the strictly negative \( x \)-axis. The \( Q \)-components of the two cogenerators are comparable in \( Q \), but the nadirs are comparable the other way (it is crucial to remember that the ordering on the nadirs is by \( F_{Q}^{op} \), not \( F_{Q} \), so smaller faces are higher in the poset). Of course, no claim can be made that deg\( Q/R_{\tau} \) soc\( \sigma \tau \) soc\( \tau \) \([D]\) equals the image in \( Q/R_{\tau} \) of \( A_{\tau}^{\sigma} \); only the density claim in Theorem 8.10 can be made.

Definition 10.5. A primary decomposition (Definition 3.18) \( D = \bigcup_{j=1}^{k} D_j \) of a downset in a real polyhedral group is minimal if

1. each face associated to \( D \) is associated to precisely one of the downsets \( D_j \), and
2. the natural map \( \text{soc}_{\tau} \) \( k[D] \rightarrow \text{soc}_{\tau} \bigoplus_{j=1}^{k} k[D_j] \) is an isomorphism for all faces \( \tau \).

Theorem 10.6. Every downset \( D \) in a real polyhedral group has a canonical minimal primary decomposition

\[
D = \bigcup_{\tau \in \text{Ass} \ D} \bigcup_{\sigma \supseteq \tau} \bigoplus_{a \in A_{\tau}^{\sigma}} a + \tau - Q_{\nabla \sigma},
\]

where \( A_{\tau}^{\sigma} \subseteq Q \) is the set of cogenerators of \( D \) along \( \tau \) with nadir \( \sigma \).

Proof. That \( D \) equals the union is a special case of Theorem 8.10. The new content is that for fixed \( \tau \), the inner union is \( \tau \)-coprimary (which follows from Lemma 10.3) and the socle maps are isomorphisms (which follows from Theorem 10.2 and Lemma 10.3).

Remark 10.7. The \( \tau \)-primary component in Theorem 10.6 is canonical even though the set of cogenerators used to express the inner union need not be; see Theorem 8.10.

Example 10.8. The canonical \( \tau \)-primary component in Theorem 10.6 can differ from the \( \tau \)-primary component \( P_{\tau}(D) \) in Definition 3.12.6 and especially Corollary 3.20, although it takes dimension at least 3 to do it. For a specific case, let \( \tau \) be the \( z \)-axis in \( \mathbb{R}^3 \), and let \( D_1 \) be the \( \{0\}\)-coprimary (Lemma 10.3) downset in \( \mathbb{R}^3 \) cogenerated by the nonnegative points on the surface \( z = 1/(x^2 + y^2) \). Then every point on the positive \( z \)-axis is supported on \( \tau \) in \( D_1 \). That would suffice, for the present purpose, but for the fact that \( \tau \) fails to be associated to \( D_1 \). The remedy is to force \( \tau \) to be associated by taking the union of \( D_1 \) with any downset \( D_2 = a + \tau - \mathbb{R}_+^3 \) with \( a = (x, y, z) \) satisfying \( xy < 0 \), the point being for \( D_2 \nsubseteq D_1 \) to be \( \tau \)-coprimary but not contain the \( z \)-axis itself. The canonical \( \tau \)-primary component of \( D = D_1 \cup D_2 \) is just \( D_2 \) itself, but by construction \( \Gamma_{\tau} \) \( D \) also contains the positive \( z \)-axis. (Note: \( D = D_1 \cup D_2 \) is not the canonical primary decomposition of \( D \) because \( D_2 \) swallows an open set of cogenerators of \( D_1 \), so these cogenerators must be omitted from the \( \{0\}\)-primary
component to induce an isomorphism on socles.) The reason why three dimensions are needed is that \( \tau \) must have positive dimension, because elements supported on \( \tau \) must be cloaked by those supported on a smaller face; but \( \tau \) must have codimension more than 1, because there must be enough room modulo \( \mathbb{R} \tau \) to have incomparable elements.

10.3. Minimal downset hulls of modules.

**Definition 10.9.** A downset hull \( M \to E = \bigoplus_{j=1}^{k} E_{j} \) (Definition 3.31) of a module over a real polyhedral group is

1. **coprimary** if \( E_{j} = k[D_{j}] \) is coprimary for all \( j \), so \( D_{j} \) is a coprimary downset, and
2. **minimal** if the induced map \( \text{soc}_{\tau} M \to \text{soc}_{\tau} E \) is an isomorphism for all faces \( \tau \).

**Theorem 10.10.** Every downset-finite module \( M \) over a real polyhedral group admits a minimal coprimary downset hull.

**Proof.** Suppose that \( M \to \bigoplus_{j=1}^{k} E_{j} \) is any finite downset hull. Replacing each \( E_{j} \) by a primary decomposition of \( E_{j} \), using Theorem 10.6, assume that this downset hull is coprimary. Let \( E' \) be the direct sum of the \( \tau \)-coprimary summands of \( E \). Then \( \text{soc}_{\tau} E = \text{soc}_{\tau} E' \) by Theorem 10.2. Replacing \( M \) with its image in \( E' \), it therefore suffices to treat the case where \( M \) is \( \tau \)-coprimary and \( E = E' \).

The proof is by induction on the number \( k \) of summands of \( E \). If \( k = 1 \) then \( M \subseteq k[D] \) is a submodule of a \( \tau \)-coprimary downset module. Let \( D' \) be the union of the coprincipal downsets \( a + \tau - Q_{\sigma} \) over all \( a \in Q \) and faces \( \sigma \) such that the projection of \( a \) to \( Q/\mathbb{R}\tau \) is the degree of an element in the image of the natural map \( \text{soc}_{\tau} M \to \text{soc}_{\tau} k[D] \). Since that natural map is injective for all \( \sigma \) by Theorem 7.7.1, it is a consequence of Theorem 8.10 that there is a surjection \( k[D] \to k[D'] \). But Lemma 10.3 implies that \( \text{soc}_{\tau} k[D'] \subseteq \text{soc}_{\tau} M \), so \( \text{soc}_{\tau} M \to \text{soc}_{\tau} k[D'] \) is an isomorphism.

When \( k > 1 \), let \( M' = \ker(M \to E_{k}) \). Then \( M' \leftarrow \bigoplus_{j=1}^{k-1} E_{j} \), so it has a minimal coprimary hull \( M' \to E' \) by induction. The \( k = 1 \) case proves that \( M'' = M/M' \) has a minimal coprimary hull \( M'' \to E'' \). The exact sequence \( 0 \to M' \to M \to M'' \to 0 \) yields an exact sequence

\[
0 \to \text{soc}_{\tau} M' \to \text{soc}_{\tau} M \to \text{soc}_{\tau} M''
\]

which, if exact, automatically splits by Remark 6.63. Hence it suffices to prove that \( \text{soc}_{\tau} M \to \text{soc}_{\tau} M'' \) is surjective. For that, note that the image of \( \text{soc}_{\tau} M \) in \( \text{soc}_{\tau} E \) surjects onto its projection to \( \text{soc}_{\tau} E_{k} \), but the image of \( \text{soc}_{\tau} M \to \text{soc}_{\tau} E_{k} \) is the image of the injection \( \text{soc}_{\tau} M'' \to \text{soc}_{\tau} E_{j} \) by construction. \( \square \)

**Remark 10.11.** The proof of the theorem shows more than the statement: any coprimary downset hull \( M \leftarrow E = E_{1} \oplus \cdots \oplus E_{k} \) of a coprimary module \( M \) induces a filtration \( 0 = M_{0} \subset M_{1} \subset \cdots \subset M_{k} = M \) such that \( \text{soc}_{\tau} M = \bigoplus_{j=1}^{k} \text{soc}_{\tau}(M_{j}/M_{j-1}) \), and furthermore \( M \to E \) can be “minimalized”, in the sense that a minimal hull \( E' \) can be constructed from \( E \) so that \( \text{soc}_{\tau} M \cong \text{soc}_{\tau} E' \) decomposes as direct sum of factors \( \text{soc}_{\tau}(M_{j}/M_{j-1}) \cong \text{soc}_{\tau} E'_{j} \). Reordering the summands \( E_{j} \) yields another filtration
of $M$ with the same property. That $\text{soc}_\tau M$ breaks up as a direct sum in so many ways should not be shocking, in view of Remark 6.63. The main content is that all of the socle elements of $M/M_{k-1}$ are inherited from $M$, essentially because $M_{k-1}$ is the kernel of a homomorphism to a direct sum of downset modules, so $M_{k-1}$ has no generators that are not inherited from $M$.

**Remark 10.12.** Theorem 10.10 is the analogue of existence of minimal injective hulls for finitely generated modules over noetherian rings [BH98, Section 3.2] (see also Proposition 4.7). The difference here is that a direct sum—as opposed to direct product—can only be attained by gathering cogenerators into finitely many bunches.

**Example 10.13.** The indicator module for the disjoint union of the strictly negative axes in the plane injects in an appropriate way into one downset module (the punctured negative quadrant) or a direct sum of two (negative quadrants missing one boundary axis each). Thus the “required number” of downsets for a downset hull of a given module is not necessarily obvious and might not be a functorial invariant. This may sound bad, but it should not be unexpected: the quotient by an artinian monomial ideal in an ordinary polynomial ring can have socle of arbitrary finite dimension, so the number of coprincipal downsets required is well defined, but if downsets that are not necessarily coprincipal are desired, then any number between 1 and the socle dimension would suffice. This phenomenon is related to Remark 6.63: breaking the socle of a downset into two reasonable pieces expresses the original downset as a union of the two downsets cogenerated by the pieces.

10.4. Minimal primary decomposition of modules.

**Definition 10.14.** A primary decomposition $M \hookrightarrow \bigoplus_{i=1}^r M/M_i$ (Definition 3.28.3) of a module over a real polyhedral group is minimal if $\text{soc}_\tau M \hookrightarrow \text{soc}_\tau \bigoplus_{i=1}^r M/M_i$ is an isomorphism for all faces $\tau$.

**Definition 10.15.** Given a coprimary downset hull $M \hookrightarrow E$ of an arbitrary downset-finite module $M$ over a real polyhedral group, write $E^\tau$ for the direct sum of all summands of $E$ that are $\tau$-coprimary. The kernel $M^\tau$ of the composite homomorphism $M \to E \to E^\tau$ is the $\tau$-primary component of 0 for this particular downset hull of $M$.

**Theorem 10.16.** Every downset-finite module $M$ over a real polyhedral group admits a minimal primary decomposition. In fact, if $M \hookrightarrow E$ is a coprimary downset hull then $M \hookrightarrow \bigoplus_\tau M/M^\tau$ is a primary decomposition that is minimal if $M \hookrightarrow E$ is minimal.

**Proof.** Fix a coprimary downset hull $M \hookrightarrow E$. The quotient $M/M^\tau$ is $\tau$-coprinary since it is a submodule of the coprimary module $E^\tau$ by construction, and $M \to \bigoplus_\tau M/M^\tau$ is injective because the injection $M \hookrightarrow \bigoplus_\tau E^\tau = E$ factors through $\bigoplus_\tau M/M^\tau \subseteq E$. 


Theorem 10.2 implies that soc\(_\tau(M/M^\tau) = 0\) unless \(\tau = \tau'\), regardless of whether \(M \hookrightarrow E\) is minimal. And if the hull is minimal, then soc\(_\tau M \rightarrow soc\_\tau E^\tau\) is an isomorphism (by hypothesis) that factors through the injection soc\(_\tau(M/M^\tau) \rightarrow soc\_\tau E^\tau\) (by construction), forcing soc\(_\tau M \cong soc\_\tau(M/M^\tau)\) to be an isomorphism for all \(\tau\).

\[\square\]

**Corollary 10.17.** Every indicator subquotient \(k[S]\) of \(k[Q]\) over a real polyhedral group \(Q\) has a canonical minimal primary decomposition. (That \(k[S]\) is a subquotient means that \(S \subseteq Q\) is the intersection of an upset and a downset in \(Q\).)

**Proof.** Let \(D\) be the downset cogenerated by \(S\). Let \(k[D] \hookrightarrow E\) be the minimal copri-
mary downset hull of \(k[D]\) resulting from the canonical minimal primary decomposi-
tion of \(D\) in Theorem 10.6. Then the composite map \(k[S] \hookrightarrow k[D] \hookrightarrow E\) is a minimal
downset hull of \(k[S]\) to which Theorem 10.16 applies. \(\square\)

11. **Socles and essentiality over discrete polyhedral groups**

The theory developed for real polyhedral groups in Sections 6–12 applies as well
to discrete polyhedral groups (Definition 3.10). The theory is easier in the discrete
case, in the sense that only closed cogenerator functors are needed, and none of the
density considerations in Sections 8–9 are relevant. The deduction of the discrete case
is elementary, but it is worthwhile to record the results, both because they are useful
and for comparison with the real polyhedral case.

For the analogue of Theorem 7.7, the notion of divisibility in Definition 6.17 makes
sense, when \(\sigma = \{0\}\), verbatim in the discrete polyhedral setting: an element \(y \in M_b\)
divides \(x \in M_a\) if \(b \in a - Q_+\) and \(y \mapsto x\) under the natural map \(M_b \rightarrow M_a\).

**Theorem 11.1** (Discrete essentiality of socles). Fix a homomorphism \(\varphi : M \rightarrow N\) of
modules over a discrete polyhedral group \(Q\).

1. If \(\varphi\) is injective then \(soc\_\tau \varphi : soc\_\tau M \rightarrow soc\_\tau N\) is injective for all faces \(\tau\) of \(Q_+\).
2. If \(soc\_\tau \varphi : soc\_\tau M \rightarrow soc\_\tau N\) is injective for all faces \(\tau\) of \(Q_+\) and \(M\) is downset-
finite, then \(\varphi\) is injective.

In fact, each homogeneous element of \(M\) divides some closed cogenerator of \(M\).

**Proof.** Item 1 is a special case of Proposition 6.51. Item 2 follows from the divisibility claim, for if \(y\) divides a closed cogenerator \(s\) along \(\tau\) then \(\varphi(y) \neq 0\) whenever \(soc\_\tau \varphi(\tilde{s}) \neq 0\), where \(\tilde{s}\) is the image of \(s\) in \(soc\_\tau M\). The divisibility claim follows from the case where \(M\) is generated by \(y \in M_b\). But \(<y>\) is a noetherian \(k[Q_+]-\)module and
hence has an associated prime. This prime equals the annihilator of some homogeneous
element of \(<y>\), and the quotient of \(k[Q_+]\) modulo this prime is \(k[\tau]\) for some face \(\tau\)
[MS05, Section 7.2]. That means, by definition, that the homogeneous element is a
closed cogenerator along \(\tau\) divisible by \(y\). \(\square\)
The discrete analogue of Theorem 8.15 is simpler in both statement and proof.

**Theorem 11.2.** Fix subfunctors \( S_\tau \subseteq \text{soc}_\tau \) for all faces \( \tau \) of a discrete polyhedral group. Theorem 11.1 holds with \( S \) in place of \( \text{soc} \) if and only if \( S_\tau = \text{soc}_\tau \) for all \( \tau \).

**Proof.** Every subfunctor of any left-exact functor takes injections to injections; therefore Theorem 11.1 holds for any subfunctor of \( \text{soc}_\tau \) by Proposition 6.51. The content is that Theorem 11.1 fails as soon as \( S_\tau M \not\subseteq \text{soc}_\tau M \) for some module \( M \) and some face \( \tau \). To prove that failure, suppose \( \tilde{s} \in \text{soc}_\tau M \setminus S_\tau M \) for some closed cogenerator \( s \) of \( M \) along \( \tau \). Then \( \langle s \rangle \subseteq M \) induces an injection \( S_\tau \langle s \rangle \to \text{soc}_\tau M \), but by construction the image of this homomorphism is 0, so \( \text{soc}_\tau \langle s \rangle = 0 \) for all \( \tau' \neq \tau \) because \( \langle s \rangle \) is abstractly isomorphic to \( k[\tau] \), which has no associated primes other than the kernel of \( k[Q_+] \to k[\tau] \). Consequently, applying \( S_\tau \) to the homomorphism \( \varphi : \langle s \rangle \to 0 \) yields an injection \( 0 \to 0 \) for all faces \( \tau' \) even though \( \varphi \) is not injective. □

The analogue of Theorem 9.5 is similarly simpler.

**Theorem 11.3.** In any module \( M \) over a discrete polyhedral group, \( M' \) is an essential submodule if and only if \( \text{soc}_\tau M' = \text{soc}_\tau M \) for all faces \( \tau \).

**Proof.** First assume that \( M' \) is not an essential submodule, so \( N \cap M' = 0 \) for some nonzero submodule \( N \subseteq M \). Any closed cogenerator \( s \) of \( N \) along any face \( \tau \) maps to a nonzero element of \( \text{soc}_\tau M \) that lies outside of \( \text{soc}_\tau M' \). Conversely, if \( \text{soc}_\tau M' \neq \text{soc}_\tau M \), then any closed cogenerator of \( M \) that maps to an element \( \text{soc}_\tau M \setminus \text{soc}_\tau M' \) generates a nonzero submodule of \( M \) whose intersection with \( M' \) is 0.

The analogue of Theorem 10.6 uses slightly modified definitions but its proof is easier.

**Definition 11.4.** A primary decomposition (Definition 3.18) \( D = \bigcup_{j=1}^{k} D_j \) of a downset in a discrete polyhedral group is **minimal** if

1. the downsets \( D_j \) are coprimary for distinct associated faces of \( D \), and

2. the natural map \( \text{soc}_\tau k[D] \to \text{soc}_\tau \bigoplus_{j=1}^{k} k[D_j] \) is an isomorphism for all faces \( \tau \),

where \( \tau \) is **associated** if some element generates an upset in \( D \) that is a translate of \( \tau \).

**Theorem 11.5.** Every downset \( D \) in a discrete polyhedral group has a canonical minimal primary decomposition \( D = \bigcup_{\tau} D^\tau \) as a union of coprimary downsets

\[
D^\tau = \bigcup_{a_\tau \in \deg \text{soc}_\tau k[D]} a_\tau - Q_+,
\]

where \( a_\tau \) is viewed as an element in \( Q/\mathbb{Z}_\tau \) to write \( a_\tau \in \deg \text{soc}_\tau k[D] \) but \( a_\tau \subseteq Q \) is viewed as a coset of \( \mathbb{Z}_\tau \) to write \( a_\tau - Q_+ \).
Proof. The downset $D$ is contained in the union by the final line of Theorem 11.1, but the union is contained in $D$ because every closed cogenerator of $D$ is an element of $D$. It remains to show that $D^\tau$ is coprimary and that the socle maps are isomorphisms.

Each nonzero homogeneous element $y \in k[D^\tau]$ divides an element $s_y$ whose degree lies in some coset $a_\tau$ by construction. As $D^\tau \subseteq D$, each such element $s_y$ is a closed cogenerator of $D^\tau$ along $\tau$. Therefore $k[D^\tau]$ is coprimary, inasmuch as no prime other than the one associated to $k[\tau]$ can be associated to $D^\tau$. The same argument shows that these elements $s_y$ generate an essential submodule of $D^\tau$, and then Theorem 11.3 yields the isomorphism on socles.

Remark 11.6. Each of the individual coprincipal downsets in Theorem 11.5 is an irreducible component of $D$. Thus the theorem can also be interpreted as saying that every downset in a discrete polyhedral group has a unique irredundant irreducible decomposition; the irredundant condition stems from the socle isomorphisms.

Definition 11.7. A downset hull $M \rightarrow E = \bigoplus_{j=1}^k E_j$ (Definition 3.31) of a module over a discrete polyhedral group is

1. coprimary if $E_j = k[D_j]$ is coprimary for all $j$, so $D_j$ is a coprimary downset, and
2. minimal if the induced map $\text{soc}_\tau M \rightarrow \text{soc}_\tau E$ is an isomorphism for all faces $\tau$.

The discrete analogue of Theorem 10.10 appears to be new.

Theorem 11.8. Every downset-finite module $M$ over a discrete polyhedral group admits a minimal coprimary downset hull.

Proof. The argument follows that of Theorem 10.10. In the course of the proof, note that the discrete analogue of Theorem 10.2 is the definition of associated prime and that the analogue of Remark 6.63 holds (more easily) in the discrete polyhedral setting.

Remark 11.9. Remark 10.11 holds verbatim over discrete polyhedral groups.

Finally, here is the discrete version of minimal primary decomposition.

Definition 11.10. A primary decomposition $M \hookrightarrow \bigoplus_{i=1}^r M/M_i$ (Definition 3.28) of a module over a discrete polyhedral group is minimal if $\text{soc}_\tau M \rightarrow \text{soc}_\tau \bigoplus_{i=1}^r M/M_i$ is an isomorphism for all faces $\tau$.

Definition 11.11. Given a coprimary downset hull $M \hookrightarrow E$ of an arbitrary downset-finite module $M$ over a discrete polyhedral group, write $E^\tau$ for the direct sum of all summands of $E$ that are $\tau$-coprimary. The kernel $M^\tau$ of the composite homomorphism $M \rightarrow E \rightarrow E^\tau$ is the $\tau$-primary component of $0$ for this particular downset hull of $M$.

Theorem 11.12. Every downset-finite module $M$ over a discrete polyhedral group admits a minimal primary decomposition. In fact, if $M \hookrightarrow E$ is a coprimary downset hull then $M \hookrightarrow \bigoplus_\tau M/M^\tau$ is a primary decomposition that is minimal if $M \hookrightarrow E$ is minimal.

Proof. Follow the proof of Theorem 10.16. \qed

Note: Theorem 11.10, which appears to be new, states that every downset-finite module over a discrete polyhedral group admits a minimal coprimary downset hull. The proof follows the argument of Theorem 10.10, noting that the discrete analogue of Theorem 10.2 is the definition of associated prime and that Remark 6.63 holds more easily in the discrete polyhedral setting.

Remark 11.9 highlights the verbatim nature of Remark 10.11 over discrete polyhedral groups.

Finally, the discrete version of minimal primary decomposition is introduced, with Definition 11.10 defining a minimal primary decomposition and Definition 11.11 defining the $\tau$-primary component of $0$ for a particular downset hull of $M$.

Theorem 11.12 asserts that every downset-finite module over a discrete polyhedral group admits a minimal primary decomposition. In fact, if $M \hookrightarrow E$ is a coprimary downset hull, then $M \hookrightarrow \bigoplus_\tau M/M^\tau$ is a primary decomposition that is minimal if $M \hookrightarrow E$ is minimal. The proof follows the proof of Theorem 10.16. \qed
Corollary 11.13. Every indicator subquotient \( \mathbb{k}[S] \) of \( \mathbb{k}[Q] \) over a discrete polyhedral group \( Q \) has a canonical minimal primary decomposition.

Proof. The proof of Corollary 10.17 works verbatim, citing the discrete analogues of Theorems 10.6 and 10.16, namely Theorems 11.5 and 11.12.

12. Generator functors and tops

The theory of generators is Matlis dual (Section 4.4) to the theory of cogenerators. Every result for socles, downsets, and cogenerators therefore has a dual. All of these dual statements can be formulated so as to be straightforward, but sometimes they are less natural (see Remarks 12.14 and 12.15, for example), sometimes they are weaker (see Remark 12.5), and sometimes there are natural formulations that must be proved equivalent to the straightforward dual (see Definition 12.23 and Theorem 12.25, for example). This section presents those Matlis dual notions that are used in later sections, along with a few notions or results for which no direct dual is present in earlier sections, such as Proposition 12.16, Proposition 12.29, and Lemma 12.27.

12.1. Lower boundary functors.

The following are Matlis dual to Definition 6.11, Lemma 6.13, and Definition 6.15.

Definition 12.1. For a module \( M \) over a real polyhedral group \( Q \), a face \( \xi \) of \( Q_+ \), and a degree \( b \in Q \), the lower boundary beneath \( \xi \) at \( b \) in \( M \) is the vector space

\[
(\partial^\xi M)_b = M_{b+\xi} = \lim_{b' \in b+\xi^o} M_{b'}.
\]

Lemma 12.2. The structure homomorphisms of \( M \) as a \( Q \)-module induce natural homomorphisms \( M_{b+\xi} \to M_{c+\eta} \) for \( b \preceq c \) in \( Q \) and faces \( \xi \subseteq \eta \) of \( Q_+ \).

Remark 12.3. In contrast with Remark 6.14, the relevant monoid structure here on the face poset \( \mathcal{F}_Q \) of the positive cone \( Q_+ \) is dual to the monoid denoted \( \mathcal{F}_Q^\text{op} \). In this case the monoid axioms use that \( \mathcal{F}_Q \) is a bounded join semilattice, the monoid unit being \( \{0\} \). The induced partial order on \( \mathcal{F}_Q \) is the usual one, with \( \xi \preceq \eta \) if \( \xi \subseteq \eta \).

Definition 12.4. Fix a module \( M \) over a real polyhedral group \( Q \) and a degree \( b \in Q \). The lower boundary functor takes \( M \) to the \( Q \times \mathcal{F}_Q \)-module \( \partial M \) whose fiber over \( b \in Q \) is the \( \mathcal{F}_Q \)-module

\[
(\partial M)_b = \bigoplus_{\xi \in \mathcal{F}_Q} M_{b+\xi} = \bigoplus_{\xi \in \mathcal{F}_Q} (\partial^\xi M)_b.
\]

The fiber of \( \partial M \) over \( \xi \in \mathcal{F}_Q \) is the lower boundary \( \partial^\xi M \) of \( M \) beneath \( \xi \).
Remark 12.5. Direct and inverse limits play differently with vector space duality. Consequently, although the notion of lower boundary functor is categorically dual to the notion of upper boundary functor, the duality only coincides unfettered with vector space duality in one direction, and some results involving tops are necessarily weaker than the corresponding results for socles; compare Theorem 7.7 with 13.3 and Example 13.5, for instance. To make precise statements throughout this section on generator functors, starting with Lemma 12.7, it is necessary to impose a finiteness condition that is somewhat stronger than $Q$-finiteness (Definition 2.32).

Definition 12.6. A module $M$ over a real polyhedral group $Q$ is infinitesimally $Q$-finite if its lower boundary module $\partial M$ is $Q$-finite.

Lemma 12.7. If $\xi$ is a face of a real polyhedral group $Q$, then
1. $\partial^\xi (M^\vee) = (\delta^\xi M)^\vee$ for all $Q$-modules $M$, and
2. $(\partial^\xi M)^\vee = \delta^\xi (M^\vee)$ if $M$ is infinitesimally $Q$-finite.

Proof. Degree by degree $b \in Q$, the first of these is because the vector space dual of a direct limit is the inverse limit of the vector space duals. Swapping “direct” and “inverse” only works with additional hypotheses, and one way to ensure these is to assume infinitesimal $Q$-finiteness of $M$. Indeed, then $M = \partial^{(b)} M$ is $Q$-finite, so replacing $M$ with $M^\vee$ in the first item yields $\partial^\xi M = (\delta^\xi (M^\vee))^\vee$ by Lemma 4.17. Thus $(\partial^\xi M)^\vee = \delta^\xi (M^\vee)$, as $\partial^\xi M$—and hence $(\delta^\xi (M^\vee))^\vee$ and $\delta^\xi (M^\vee)$—is also $Q$-finite.

Example 12.8. Any module $M$ that is a quotient of a finite direct sum of upset modules (“upset-finite” in Definition 13.2) over a real polyhedral group is infinitesimally $Q$-finiteness of $M$. Indeed, the Matlis dual of such a quotient is a downset hull demonstrating that $M^\vee$ is downset-finite and hence $Q$-finite. Proposition 6.19 and exactness of upper boundary functors (Lemma 6.12) implies that $\delta (M^\vee)$ remains downset-finite and hence $Q$-finite. Applying Lemma 12.7.1 to $M^\vee$ and using that $(M^\vee)^\vee = M$ (Lemma 4.17) on the left-hand side yields that $\partial M$ is $Q$-finite. This example includes all finitely encoded modules by Theorem 5.16.2.

Proposition 12.9. The category of infinitesimally $Q$-finite modules over a real polyhedral group $Q$ is a full abelian subcategory of the category of $Q$-modules. Moreover, the lower boundary functor is exact on this subcategory.

Proof. Use Matlis duality, in the form of Lemma 12.7, along with Lemma 6.12.

12.2. Closed generator functors.
Here is the Matlis dual of Definition 6.22. Recall the skyscraper $P$-module $\mathbb{k}_p$ there.

Definition 12.10. Fix an arbitrary poset $P$. The closed generator functor $\mathbb{k} \otimes_P -$ takes each $P$-module $N$ to its closed top: the quotient $P$-module
\[ \overline{\text{top}} N = \mathbb{k} \otimes_P N = \bigoplus_{p \in P} \mathbb{k}_p \otimes_P N. \]
When it is important to specify the poset, the notation $P\text{-}\text{top}$ is used instead of $\text{top}$. A closed generator of degree $p \in P$ is a nonzero element in $(\text{top}N)_p$.

**Remark 12.11.** $P\text{-}\text{soc} N \hookrightarrow N$ is the universal $P$-module monomorphism that is 0 when composed with all nonidentity maps induced by going up in $P$. The Matlis dual notion is $N \twoheadrightarrow P\text{-}\text{top} N$, the universal $P$-module epimorphism that is 0 when composed with all nonidentity maps induced by going up in $P$.

### 12.3. Closed generator functors along faces.

Generators along faces of partially ordered groups make sense just as cogenerators along faces do; however, they are detected not by localization but by the Matlis dual operation in Example 4.16, which is likely unfamiliar (and is surely less familiar than localization). An element in the following can be thought of as an inverse limit of elements of $M$ taken along the negative of the face $\rho$. This is Matlis dual to the construction of the localization $M_{\rho}$ as a direct limit.

**Definition 12.12.** Fix a face $\rho$ of a partially ordered group $Q$ and a $Q$-module $M$. Set

$$M_{\rho} = \text{Hom}_Q(k[Q_+]_\rho, M).$$

The following is Matlis dual to Definition 6.43.1; see Theorem 12.18. Duals for the rest of Definition 6.43 are omitted for reasons detailed in Remarks 12.14 and 12.15.

**Definition 12.13.** Fix a partially ordered group $Q$, a face $\rho$, and a $Q$-module $M$. The closed generator functor along $\rho$ takes $M$ to its closed top along $\rho$:

$$\text{top}_{\rho} M = (k[\rho] \otimes_Q M)^{\rho}/\rho.$$ 

**Remark 12.14.** The notion of (global) closed cogenerator has a Matlis dual, but since the dual of an element—equivalently, a homomorphism from $k[Q_+]$—is not an element, the notion of closed generator is not Matlis dual to a standard notion related to socles. See Remark 18.4, which defines a generator along a face $\rho$ as an element of $M_{\rho}$ that is not a multiple of any generator of lesser degree. Making this precise requires care regarding what “lesser” means.

**Remark 12.15.** The notion of local socle has a Matlis dual, but it is not in any sense a local top, because localization does not Matlis duality to localization (Example 4.16). Instead, Matlis dualizing the local socle yields a functor $k \otimes_{Q/\mathbb{Z}^\rho} M_{\rho}$ that surjects onto $\text{top}_{\rho} M$ by the Matlis dual of Proposition 6.67. Local socles found uses in proofs here and there, such as Corollary 6.69, Proposition 8.6, Corollary 8.8, Lemma 9.2, and Corollary 9.4, via Proposition 6.67. But since Matlis duals of statements hold regardless of their proofs, given appropriate finiteness conditions (Definition 12.6), local socles and their Matlis duals have no further use in this paper.

That said, a related but simpler functor that surjects onto $\text{top}_{\rho}$ plays a crucial role in the construction of birth modules: the next result captures the general concept that the “elements of $M$ of degree $b_{\rho} \in Q/\mathbb{Z}^\rho$” surject onto the top of $M$ along $\rho$ in degree $b_{\rho}$.
Proposition 12.16. Over any partially ordered group $Q$ there is a natural surjection

$$M^\rho / \rho \twoheadrightarrow \text{top}_\rho M$$

of modules over $Q / \mathbb{Z}_\rho$ for any face $\rho$ of $Q$.

Proof. Tensor the surjection $k[Q_+] \twoheadrightarrow k[\rho]$ with $M$ to get a surjection $M \twoheadrightarrow k[\rho] \otimes_Q M$. Then apply Lemma 4.19 and Lemma 6.41.

In the next lemma, a prerequisite to the duality of closed socles and tops, note that localization of $N$ along $\rho$ on the left-hand side is hiding in the quotient-restriction.

Lemma 12.17. For any module $N$ over a partially ordered group and any face $\rho$,

$$(N/\rho)^\vee = (N^\vee)^\rho / \rho \quad \text{and} \quad (N^\vee)/\rho = (N^\rho / \rho)^\vee.$$ 

Proof. This is Example 4.16 plus the observation that quotient-restriction along $\rho$ commutes with Matlis duality on modules that are already localized along $\rho$, namely

$$(N/\rho)^\vee = (N^\rho / \rho)^\vee = (N^\rho / \rho)^\vee / \rho,$$

as can be seen directly from Definitions 4.14 and 6.39.

Theorem 12.18. If $Q$ is a partially ordered group and $M$ is any $Q$-module, then

$$(\text{top}_\rho M)^\vee = \overline{\text{soc}}_\rho (M^\vee) \quad \text{and} \quad \overline{\text{top}}_\rho (M^\vee) = (\overline{\text{soc}}_\rho M)^\vee.$$

Proof. The two are similar, but for the record both are written out:

$$(\overline{\text{top}}_\rho M)^\vee = (k[\rho] \otimes_Q M)^\rho / \rho)^\vee$$

$$= (k[\rho] \otimes_Q M)^\rho / \rho \text{ by Lemma 12.17}$$

$$= \text{Hom}_Q(k[\rho], M^\vee)^\rho / \rho \text{ by Example 4.15}$$

$$= \overline{\text{soc}}_\rho (M^\vee),$$

and

$$(\overline{\text{soc}}_\rho M)^\vee = (\text{Hom}_Q(k[\rho], M)^\rho / \rho)^\vee$$

$$= (\text{Hom}_Q(k[\rho], M)^\rho / \rho \text{ by Lemma 12.17}$$

$$= (k[\rho] \otimes_Q M^\vee)^\rho / \rho \text{ by Example 4.15}$$

$$= \overline{\text{top}}_\rho (M^\vee).$$

12.4. Generator functors over real polyhedral groups.

Here is the Matlis dual to Definition 6.56, using Definition 12.4.

Definition 12.19. For a face $\rho$ of real polyhedral group, set $\Delta \rho = (\nabla \rho)^{\text{op}}$ the open star of $\rho$ (Example 6.5) with the partial order opposite to Definition 6.56, so

$$\xi \preceq \eta \text{ in } \Delta \rho \text{ if } \xi \subseteq \eta.$$ 

The lower boundary functor along $\rho$ takes $M$ to the $Q \times \Delta \rho$-module $\partial_\rho M = \bigoplus_{\xi \in \Delta \rho} \partial^\rho M$. 

**Proposition 12.20.** Fix a face $\rho$ of a real polyhedral group $Q$. The Matlis dual $N^\vee$ over $Q$ of any module $N$ over $Q \times \Delta\rho$ is naturally a module over $Q \times \nabla\rho$, and the same holds with $\nabla\rho$ and $\Delta\rho$ swapped. Moreover,

$$(\Delta\rho\text{-top } N)^\vee = \nabla\rho\text{-soc}(N^\vee) \quad \text{and} \quad \Delta\rho\text{-top}(N^\vee) = (\nabla\rho\text{-soc } N)^\vee,$$

where the Matlis duals are taken over $Q$.

**Proof.** Matlis duality over $Q$ already reverses the arrows in the $\Delta\rho$-module structure on $N$, making $N^\vee$ into a module over $Q \times \nabla\rho$. An adjointness calculation then yields

$$(\Delta\rho\text{-top } N)^\vee = (k \otimes_{\Delta\rho} N)^\vee = \text{Hom}_{\nabla\rho}(k, N^\vee) = \nabla\rho\text{-soc}(N^\vee),$$

and the other adjointness calculation is similar. \hfill \Box

**Corollary 12.21.** For a face $\rho$ over a real polyhedral group $Q$, the lower boundary is Matlis dual over $Q$ to the upper boundary: as modules over $Q \times \Delta\rho$,

1. $\partial\rho(M^\vee) = (\delta\rho M)^\vee$ for all $Q$-modules $M$, and
2. $(\partial\rho M)^\vee = \delta\rho (M^\vee)$ if $M$ is infinitesimally $Q$-finite.

**Proof.** Lemma 12.7 plus the first part of Proposition 12.20. \hfill \Box

Next is the Matlis dual of Definition 6.57, using the skyscraper modules $k_\xi[b + \rho]$ there, followed by the Matlis dual of Definition 6.59.

**Definition 12.22.** Fix a partially ordered group $Q$, a face $\rho$, and an arbitrary commutative monoid $P$. Define a functor $k[\rho] \otimes_{Q \times P}$ on modules $N$ over $Q \times P$ by

$$k[\rho] \otimes_{Q \times P} N = \bigoplus_{(b, \xi) \in Q \times P} k_\xi[b + \rho] \otimes_{Q \times P} N.$$

**Definition 12.23.** Fix a real polyhedral group $Q$, a face $\rho$, and a $Q$-module $M$. The generator functor along $\rho$ takes $M$ to its top along $\rho$: the $(Q/\mathbb{R}\tau \times \Delta\rho)$-module

$$\text{top}_\rho M = (k[\rho] \otimes_{Q \times \Delta\rho} \partial\rho M)^\rho/\rho.$$

The $\Delta\rho$-graded components of $\text{top}_\rho M$ are denoted by $\text{top}_\rho^\xi M$ for $\xi \in \Delta\rho$.

**Proposition 12.24.** The functors $\text{top}_\rho$ and $\Delta\rho\text{-top}$ commute. In particular,

$$\Delta\rho\text{-top}(\text{top}_\rho \partial\rho M) \cong \text{top}_\rho M \cong \text{top}_\rho(\Delta\rho\text{-top} \partial\rho M).$$
Proof. This is Matlis dual to Proposition 6.65, but it holds without any infinitesimal $Q$-finiteness restriction by proving it directly:

$$
\Delta \rho - \text{top} \left( \text{top}_\rho N \right) = k \otimes \Delta \rho \left( k[\rho] \otimes_Q N \right)^{\sigma} \rho \text{ by Definitions 12.10 and 12.23}
= k \otimes \Delta \rho \left( \text{Hom} \left( k[Q_+], k[\rho] \otimes_Q N \right) / \rho \right) \text{ by Definition 12.12}
= \left( k \otimes \Delta \rho \text{Hom} \left( k[Q_+], k[\rho] \otimes_Q N \right) \right) / \rho \text{ by Lemma 6.41}
= \left( k[\rho] \otimes_Q N \otimes \Delta \rho k \right)^{\sigma} \rho \text{ by Lemma 4.19}
= \text{top}_\rho (\Delta \rho - \text{top} N),
$$

whose penultimate line is equal to $\left( k[\rho] \otimes_Q \Delta \rho N \right)^{\sigma} \rho$. Now $N = \partial_\rho M$ yields $\text{top}_\rho M$. \(\square\)

**Theorem 12.25.** Over a real polyhedral group, the generator functor along a face $\rho$ is Matlis dual to the cogenerator functor along $\rho$:

1. $\text{top}_\rho (M^\vee) = (\text{soc}_\rho M)^\vee$ for all $Q$-modules $M$, and
2. $(\text{top}_\rho M)^\vee = \text{soc}_\rho (M^\vee)$ if $M$ is infinitesimally $Q$-finite.

Proof. The two calculations are similar, as in Theorem 12.18, so only one is included—the one that requires $M$ to be infinitesimally $Q$-finite, to see exactly where that enters:

$$(\text{top}_\rho M)^\vee = (\Delta \rho - \text{top} (\text{top}_\rho \partial_\rho M))^{\vee} \text{ by Proposition 12.24}
= \nabla \rho - \text{soc} ((\text{top}_\rho \partial_\rho M)^\vee) \text{ by Proposition 12.20}
= \nabla \rho - \text{soc} (\text{soc}_\rho ((\partial_\rho M)^\vee)) \text{ by Theorem 12.18}
= \nabla \rho - \text{soc} (\text{soc}_\rho \delta_\rho (M^\vee)) \text{ by Corollary 21.2.12}
= \text{soc}_\rho (M^\vee) \text{ by Proposition 6.5.}\)

\(\square\)

**Definition 12.26.** For any faces $\rho \subseteq \xi$ of a real polyhedral group, set

$$M^{\rho, \xi} = \text{Hom}_Q \left( k[Q_{\xi}], M \right),$$

where $Q_{\xi} = \xi^0 + Q_+$ as in Lemma 6.8 and localization along $\rho$ is as in Definition 3.23.

**Lemma 12.27.** Any faces $\rho \subseteq \xi$ of a real polyhedral group yield a natural isomorphism

$$(\partial^\xi M)^\rho \cong M^{\rho, \xi}.$$ 

Proof. A homomorphism $k[b_\rho + Q_+] \to \partial^\xi M$ for $b_\rho = b + \mathbb{R}_\rho$ is a family of elements $x_{b+r} \in (\partial^\xi M)_{b+r}$ for $r \in \mathbb{R}_\rho$. Each of these is an inverse limit of elements in $M_{b'}$ for $b' \in b + r + \xi$. The union of these sets $b + r + \xi$ is $b_\rho + Q_{\xi}$. The universal property of inverse limits produces a family of elements $x_{b'} \in M_{b'}$ for $b' \in b_\rho + Q_{\xi}$. These elements are the images of $1 \in k[b_\rho + Q_{\xi}]_{b'}$ under a homomorphism $k[b_\rho + Q_{\xi}] \to M$. And any such homomorphism yields a coherent family of elements indexed by $b_\rho + Q_{\xi}$ whose inverse limits specify a homomorphism $k[b_\rho + Q_+] \to \partial^\xi M$. \(\square\)

**Proposition 12.28.** For any faces $\rho \subseteq \xi$ of a real polyhedral group $Q$, the functor $M \mapsto M^{\rho, \xi}$ is exact on the category of infinitesimally $Q$-finite modules.
Finally, here is real analogue of the surjection in Proposition 12.16.

Proposition 12.29. Over any real polyhedral group $Q$ there is a natural surjection
\[ M^{\rho,\xi}/\rho \to \text{top}_{\rho}^{\xi} M \]
of modules over $Q/\mathbb{R}\rho$ for any faces $\rho$ and $\xi$ with $\xi \supseteq \rho$.

Proof. Regard $k[\rho] \otimes_{Q \times \partial_{\rho} M}$ as a quotient $(Q \times \Delta_{\rho})$-module of $\partial_{\rho} M$. Then $\left( \partial_{\rho} M \right)^{\rho} \to \left( k[\rho] \otimes_{Q \times \partial_{\rho} M} \right)^{\rho}$ by Lemma 4.19. The result follows by applying Lemma 12.27 to the component of the surjection in $\Delta_{\rho}$-degree $\xi$ and then taking the quotient-restriction along $\rho$. □

Example 12.30. Fix a face $\rho$ and suppose faces $\eta$ and $\xi$ both contain $\rho$. Then the top of $k[b_{\rho} + Q\vee]_{\eta}$ along any face other than $\rho$ vanishes, and $\text{top}_{\rho}^{\eta}[k[b_{\rho} + Q\vee]] = k[b_{\rho}]$ if $\eta = \xi$ and vanishes otherwise. The vanishing along faces other than $\rho$ is because $k[b_{\rho} + Q\vee]$ is a secondary module by the Matlis dual to Corollary 8.8. The rest of the vanishing is because the tensor product
\[ k[\rho] \otimes_{Q \times \Delta_{\rho}} \partial_{\rho} M = k[\rho] \otimes_{Q} \partial_{\rho} M \otimes_{\Delta_{\rho}} k \]
is almost always 0: the $k[\rho] \otimes_{Q}$ takes care of $Q/\mathbb{R}\rho$-degrees other than $b_{\rho}$, while the $\otimes_{\Delta_{\rho}} k$ takes care $\Delta_{\tau}$-degrees other than $\xi$. Finally, the calculation in degree $(b_{\rho}, \xi)$ reduces by Proposition 12.29 and Corollary 6.69$^{\vee}$ to $\text{Hom}_{Q}(k[b_{\rho} + Q\vee], k[b_{\rho} + Q\vee]) = k$.

13. Essential properties of tops

Begin with the dual to Definition 10.1 and Theorem 10.2.

Definition 13.1. Fix a face $\rho$ and a module $M$ over a real or discrete polyhedral group.
1. The face $\rho$ is attached to $M$ if $\text{top}_{\rho} M \neq 0$.
2. If $M = k[U]$ for an upset $U$ then $\rho$ is attached to $U$.
3. The set of attached faces of $M$ or $U$ is denoted by $\text{Att}(M)$ or $\text{Att}(U)$.
4. The module $M$ is $\rho$-secondary if $\text{Att}(M) = \{ \rho \}$.

Definition 13.2. An upset cover of a module $M$ over an arbitrary poset is a surjection $\bigoplus_{j \in J} F^j \to M$ with each $F^j$ an upset module. The cover is
1. finite if $J$ is finite.

The module $M$ is upset-finite if it admits a finite upset cover. If the poset is a real or discrete polyhedral group, then the cover is
2. secondary if $F^j = k[U_j]$ is secondary for all $j$, so $U_j$ is a secondary upset, and
3. minimal if the induced map $\text{top}_{\rho} E \to \text{top}_{\rho} M$ is an isomorphism for all faces $\rho$.

That was Matlis dual to Definition 10.9. Next are the duals to Theorems 7.7 and 11.1.
Theorem 13.3 (Essentiality of real tops). Fix a homomorphism $\varphi : N \to M$ of modules over a real polyhedral group $Q$.

1. If $\varphi$ is surjective with $M$ and $N$ both being infinitesimally $Q$-finite modules, then $\text{top}_{\rho} \varphi : \text{top}_{\rho} N \to \text{top}_{\rho} M$ is surjective for all faces $\rho$ of $Q_+$.

2. If $\text{top}_{\rho} \varphi : \text{top}_{\rho} N \to \text{top}_{\rho} M$ is surjective for all faces $\rho$ of $Q_+$ and $M$ is upset-finite, then $\varphi$ is surjective. □

Theorem 13.4 (Essentiality of discrete tops). Fix a homomorphism $\varphi : N \to M$ of modules over a real or discrete polyhedral group $Q$.

1. If $\varphi$ is surjective then $\text{top}_{\rho} \varphi : \text{top}_{\rho} N \to \text{top}_{\rho} M$ is surjective for all faces $\rho$ of $Q_+$.

2. If $\text{top}_{\rho} \varphi : \text{top}_{\rho} N \to \text{top}_{\rho} M$ is surjective for all faces $\rho$ of $Q_+$ and $M$ is upset-finite, then $\varphi$ is surjective. □

Example 13.5. Some hypothesis is needed in Theorem 13.3.1, in contrast to Theorem 7.7.1 or indeed Theorem 13.4.1. Let $M = k[U]$ for the open half-plane $U \subset \mathbb{R}^2$ above the antidiagonal line $y = -x$. Then $M$ is $\{0\}$-secondary, with $\text{top}_{\{0\}} M$ surjects onto $M$, but the map on tops fails to hit any element in $\mathbb{R}^2$-degree $0$. This kind of behavior might lead one to wonder: why is the Matlis dual not a counterexample to Theorem 7.7.1? Because $M^\vee$ does not possess a well defined map to a direct sum indexed by $a \neq 0$ along the antidiagonal line, only to a direct product. Any sequence of points $v_k \in -U$ converging to $0$ yields a sequence of elements $z_k \in M^\vee$. The image of the sequence $\{z_k\}_{k=1}^\infty$ in any particular one (or finite direct sum) of the downset modules of the form $k[a - Q_{\gamma_{v_k}}]$ with $a \neq 0$ is eventually $0$, but in the direct product the sequence $\{z_k\}_{k=1}^\infty$ survives forever. The direct limit of the image sequence witnesses the nonzero socle of the direct product at the missing point $0$.

Theorem 13.6. Every upset-finite module $M$ over a real or discrete polyhedral group admits a minimal secondary upset hull.

Proof. This is the Matlis dual of Theorems 10.10 and 11.8, using Example 12.8 to allow the results of Section 12 to be applied at will. □

The straightforward dualization of primary decomposition in Sections 10.4 and 11 to secondary decomposition is omitted.

14. Minimal presentations over discrete or real polyhedral groups

Definition 14.1. Fix a module $M$ over a real or discrete polyhedral group.

1. An upset presentation $F_1 \to F_0$ of $M$ is minimal if $F_1 \to \ker(F_0 \to M)$ and $F_0 \to M$ are minimal upset covers (Definition 13.2).

2. An upset resolution $F_\bullet$ of $M$ is minimal if the upset presentation $F_1 \to F_0$ and upset covers $F_{i+1} \to \ker(F_i \to F_{i-1})$ for all $i \geq 1$ are minimal.
3. A downset copresentation $E^0 \rightarrow E^1$ of $M$ is minimal if $\text{coker}(M \rightarrow E^0) \hookrightarrow E^1$ and $M \hookrightarrow E^0$ are minimal downset hulls (Definition 10.9).

4. A downset resolution $E^\bullet$ of $M$ is minimal if the downset copresentation $E^0 \rightarrow E^1$ and downset hulls $\text{coker}(E^{i-1} \rightarrow E^i) \hookrightarrow E^{i+1}$ for all $i \geq 1$ are minimal.

5. A fringe presentation $F \rightarrow E$ of $M$ is minimal if it is the composite of a minimal upset cover of $M$ and a minimal downset hull of $M$.

**Theorem 14.2.** A module $M$ over a real or discrete polyhedral group $Q$ is finitely encoded if and only if $M$ admits

1. a minimal finite fringe presentation; or
2. a minimal finite upset presentation; or
3. a minimal finite downset copresentation.

When $M$ is semialgebraic over a real polyhedral group $Q$, these minimal presentations are all semialgebraic.

**Proof.** In both the real and discrete cases, any one of these minimal presentations is, in particular, finite, so the existence of any of them implies that $M$ is finitely encoded by Theorem 5.16. It is the other direction that requires the theory in Sections 6–12.

In the real polyhedral case, any finite downset hull can be minimalized by Theorem 10.10 and Remark 10.11. The Matlis dual of this statement says that any finite upset cover can be minimalized, as well. Composing these from a given finite fringe presentations yields a minimal finite fringe presentation. In addition, the cokernel of any downset hull (minimal or otherwise) of a finitely encoded module is finitely encoded by Lemma 2.10, so the cokernel has a minimal finite downset hull by Theorem 10.10 again. That yields a minimal finite downset copresentation. The Matlis dual of a minimal finite downset copresentation of the Matlis dual $M^\vee$ is a minimal upset presentation of $M$ by Theorem 12.25 (which applies unfettered to finitely encoded modules by Example 12.8).

The discrete polyhedral case follows the parallel proof, using Theorem 11.8 and Remark 11.9 instead of Theorem 10.10 and Remark 10.11.

If $M$ is semialgebraic, then the minimalization procedure in Theorem 10.10 and Remark 10.11 is semialgebraic by induction on the number $k$ of summands there, the base case being the canonical primary decomposition of a semialgebraic downset in Theorem 10.6, which is semialgebraic by Theorem 6.72. □

**Remark 14.3.** Comparing Theorem 14.2 to Theorem 5.16, various items are missing.

1. Theorem 14.2 makes no claim concerning whether the presentations can minimalized if an encoding $\pi : Q \rightarrow P$ has been specified beforehand. It is a priori possible that deleting nonminimal generators of upsets and cogenerators of downsets could prevent an indicator summand from being constant on fibers of $\pi$.

2. Theorem 14.2 makes no claim concerning finite encodings dominating any one of the three presentations there, but as each of these presentations is finite, existence is already implied by Theorem 5.16, including semialgebraic considerations.
3. Theorem 14.2 makes no claim concerning minimal finite resolutions. Minimal resolutions of finitely encoded modules over real or discrete polyhedral groups can be constructed from scratch by Theorem 10.10, Theorem 11.8, and their Matlis duals (in the real case that is Theorem 13.6), but there is no guarantee that a minimal indicator resolution must terminate after finitely many steps.

**Remark 14.4.** Remark 14.3.3 raises an intriguing point about indicator resolutions: the bound on the length in Theorem 5.16 comes from the order dimension of the encoding poset, which is more or less unrelated to the dimension of the real or discrete polyhedral group. It seems plausible that the geometry of the polyhedral group asserts control to prevent the lengths from going too high, just as it does to prevent the cohomological dimension of an affine semigroup ring from going too high via Ishida complexes to compute local cohomology [MS05, Section 13.3.1]. This points to potential value of developing a derived functor side of the top-socle / birth-death / generator-cogenerator story for indicator resolutions to solve Conjecture 14.6, which would be a true indicator analogue of the Hilbert Syzygy Theorem.

**Definition 14.5.** Fix a poset $Q$ and a $Q$-module $M$.
1. The downset-dimension of $M$ is the smallest length of a downset resolution of $M$.
2. The upset-dimension of $M$ is the smallest length of an upset resolution of $M$.
3. The indicator-dimension of $M$ is maximum of its downset- and upset-dimensions.
4. The indicator-dimension of $Q$ is the maximum of the indicator-dimensions of its finitely encoded modules.

**Conjecture 14.6.** The indicator-dimension of any real or discrete polyhedral group $Q$ equals the rank of $Q$ as a free module (over the field $\mathbb{R}$ or group $\mathbb{Z}$, respectively).

**Remark 14.7.** It is already open to find a module of indicator-dimension as high as 2 over $\mathbb{R}^2$. It would not be shocking if the rank of $Q$ were an upper bound instead of an equality for the indicator-dimension in the conjecture: the use of upset modules instead of free modules could prevent the final syzygies that, in finitely generated situations, come from elements supported at the origin by local duality.

# 15. Birth and death posets and modules

## 15.1. Discrete polyhedral case.
This subsection works in the generality of discrete polyhedral groups (Definition 3.10). Notation and concepts from Sections 6 and 12, particularly those surrounding quotient-restriction and closed cogenerators along faces of polyhedral groups (Section 6.5) as well as closed generator functors along faces (Section 12.3) are used freely here, without further cross-reference. For example, see Definition 12.12 for the meaning of $M^p$. Basic poset concepts from Section 2 are also required, such as pullbacks in Definition 2.6.

**Definition 15.1.** Fix a module $M$ over a discrete polyhedral group $Q$. 
1. The birth poset of $Q$ is the disjoint union $B_Q = \bigsqcup \rho Q/\mathbb{Z}\rho = \bigsqcup \rho B^\rho_Q$ with 

$$(b_\rho \in Q/\mathbb{Z}\rho) \preceq (b'_\rho \in Q/\mathbb{Z}\rho') \iff b_\rho + Q_+ \supseteq b'_\rho + Q_+,$$

where $b_\rho \subseteq Q$ is viewed as a coset of $\mathbb{Z}\rho$ in $Q$ for the purpose of writing $b_\rho + Q_+$.

2. The death poset of $M$ is the subposet $B_M = \bigsqcup \rho \deg \top_\rho M \subseteq B_Q$. Write 

$$\iota_\rho : \deg \top_\rho M \to Q/\mathbb{Z}\rho$$

for the $\rho$-component $B^\rho_M \hookrightarrow B^\rho_Q$ of the inclusion $B_M \hookrightarrow B_Q$.

3. The birth module of $M$ is the $B_M$-graded vector space 

$$\text{Birth}_M = \bigoplus \rho \iota_\rho^*(M^\rho/\rho),$$

so the component in degree $b_\rho \in B^\rho_M$ is $\text{Birth}_{b_\rho} M = (M^\rho/\rho)_{b_\rho}$.

**Lemma 15.2.** The birth module $\text{Birth}_M$ is naturally a $B_M$-module.

**Proof.** If $b_\rho \in Q/\mathbb{Z}\rho$ and $b'_\rho \in Q/\mathbb{Z}\rho'$, then $b_\rho + Q_+ \supseteq b'_\rho + Q_+$ implies that any homomorphism $k[b_\rho + Q_+] \to M$ restricts to a homomorphism $k[b'_\rho + Q_+] \to M$. Note that $b_\rho + Q_+$ equals its translate by an element of $\mathbb{Z}\rho$, so these all yield the same restriction to $b'_\rho + Q_+$; hence the restriction descends to $M^\rho/\rho = \text{Hom}_Q(k[Q_+\rho], M)/\rho$. $\square$

**Definition 15.3.** Fix a module $M$ over a discrete polyhedral group $Q$.

1. The death poset of $Q$ is the disjoint union $D_Q = \bigsqcup \tau Q/\mathbb{Z}\tau = \bigsqcup \tau D^\tau_Q$ with 

$$(a_\tau \in Q/\mathbb{Z}\tau) \preceq (a'_\tau \in Q/\mathbb{Z}\tau') \iff a_\tau - Q_+ \subseteq a'_\tau - Q_+,$$

where $a_\tau \subseteq Q$ is viewed as a coset of $\mathbb{Z}\tau$ in $Q$ for the purpose of writing $a_\tau + Q_+$.

2. The death poset of $M$ is the subposet $D_M = \bigsqcup \tau D^\tau_M = \bigsqcup \tau \deg \soc_\tau M \subseteq D_Q$.

3. The death module of $M$ is 

$$\text{Death}_M = \prod_{a_\tau \in D_M} (\soc_\tau M)_{a_\tau}$$

whose factor in degree $a_\tau \in D^\tau_Q$ is also denoted $\text{Death}_{a_\tau} M = (\soc_\tau M)_{a_\tau}$.

**Remark 15.4.** The socles and tops are closed because the polyhedral group is discrete.

**Lemma 15.5.** The death module $\text{Death}_M$ is naturally a $D_M$-module and a $D_Q$-module.

**Proof.** This follows from the (simpler) discrete polyhedral analogue of Remark 6.63. $\square$

**Remark 15.6.** $D_Q = B^\emptyset_Q$, in the sense that $a_\tau \succeq a'_\tau$ in $D_Q \iff -a_\tau \preceq -a'_\tau$ in $B_Q$.

**Remark 15.7.** Given that the poset-module structure on $\text{Death}_M$ is trivial, why bother with it? The reason is to be able to compare degrees of elements in $\text{Death}_M$ with degrees of elements in $\text{Birth}_M$. The partial orders on $B_Q$ and $D_Q$ are defined so that if a localization $F$ of $k[Q_+]$ has a nonzero map to an indecomposable injective $k[a_\tau - Q_+]$, then $F$ has a nonzero map to all of the indecomposable injectives indexed by elements above $a_\tau$ in $D_Q$. This relation is formalized in Proposition 15.8.
Proposition 15.8. The disjoint union $B_Q \sqcup D_Q$ is partially ordered by

$$b_\rho \preceq a_\tau \iff (b_\rho + Q) \cap (a_\tau - Q) \neq \emptyset$$

along with the given partial orders on $B_Q$ and $D_Q$, if $Q$ is a discrete polyhedral group. □

Remark 15.9. The birth and death posets of $Q$ can be thought of as partial compactifications of $Q$, in essence the integer points on a toric variety. In the case $Q = \mathbb{Z}^n$, for example, with its standard positive cone $\mathbb{N}^n$, the birth poset can be thought of as the set of “lattice points” in a space homeomorphic to the union of the faces of the unit hypercube $[0,1]^n$ containing the origin in $\mathbb{R}^n$. In any neighborhood of the origin in $\mathbb{R}^n$ this conception is realized by exponentiating elements of $\mathbb{Z}^n$ and its quotients modulo coordinate subspaces, setting $e^{-\infty} = 0$. The death poset of $\mathbb{Z}^n$ is similarly described, but the relevant set of faces consists of those containing the point $(1,\ldots,1)$ instead of the origin. This geometric view of birth and death posets makes the disjoint union in Proposition 15.8 particularly clear, although it points out that really the union should not be disjoint but rather should identify the copies of $Q$ sitting inside of $B_Q$ and $D_Q$.

15.2. Real polyhedral case.

This subsection works in the generality of real polyhedral groups (Definition 3.8). Notation and concepts from Sections 6 and 12, particularly those surrounding face posets and cogenerators along faces of real polyhedral groups (Sections 6.1 and 6.6) as well as generator functors along faces (Section 12.4) and quotient-restriction (Section 6.5) are used freely here, without further cross-reference. For example, see Definition 12.26, Lemma 12.27, and Proposition 12.29 for the meaning of $M^\rho,\xi$. Basic poset concepts from Section 2 are also required, such as pullbacks in Definition 2.6.

Definition 15.10. Fix a module $M$ over a real polyhedral group $Q$.

1. The birth poset of $Q$ is the disjoint union $B_Q = \bigsqcup_\rho Q/\mathbb{R}\rho \times \Delta \rho$ with

$$b_\rho, \xi \preceq (b_\rho', \xi') \iff b_\rho + Q\xi \supseteq b_\rho' + Q\xi',$$

where $b_\rho \subseteq Q$ is viewed as a coset of $\mathbb{R}\rho$ in $Q$ for the purpose of writing $b_\rho + Q\xi$. Set $B_Q^\rho = Q/\mathbb{R}\rho \times \Delta \rho$ and $B_Q^\rho,\xi = Q/\mathbb{R}\rho$.

2. The birth poset of $M$ is the subposet $B_M = \bigcup_\rho \deg \top_\rho M \subseteq B_Q$. Write

$$\iota_\rho : \deg \top_\rho M \to Q/\mathbb{R}\rho \times \Delta \rho \quad \text{and} \quad \iota_\rho^\xi : \deg \top_\rho^\xi M \to Q/\mathbb{R}\rho$$

for the components $B_M^\rho \hookrightarrow B_Q^\rho$ and $B_M^\rho,\xi \hookrightarrow B_Q^\rho,\xi$ of the inclusion $B_M \hookrightarrow B_Q$.

3. The birth module of $M$ is the $B_M$-graded vector space

$$\text{Birth} M = \bigoplus_\rho \bigoplus_{\xi \in \Delta \rho} (\iota_\rho^\xi)^*(M^\rho,\xi/\rho),$$

so the component in degree $(b_\rho, \xi) \in B_M^\rho$ is $\text{Birth}_{b_\rho}^\xi M = (M^\rho,\xi/\rho)_{b_\rho}$. 

Definition 15.13. Fix a module $\mathcal{B}_R$ of the real polyhedral group $R$ is totally ordered. Heuristically, it feels like $R$ but with each point doubled and a negative infinity thrown in. More precisely, let $-\infty = (R/R, R_+)$ be the unique coset of $R = RR_+$ in $R$. For each real number $b$, let $b_\bullet$ be the pair $(b, 0)$ and $b_\circ$ the pair $(b, R_+)$. Then $\mathcal{B}_R$ consists of the point $-\infty$ along with $b_\bullet$ and $b_\circ$ for all $b \in R$, with the total order that has $-\infty < b_\bullet < b_\circ < b'_\bullet$ if $b < b'$. More geometrically, $\mathcal{B}_R$ is the set of positive-pointing rays in $R$ totally ordered by inclusion. The rays come in three flavors, namely $(-\infty, \infty) < (b, \infty) < (b, \infty)$, and the latter is $< [b', \infty)$ if $b < b'$.

Lemma 15.12. The birth module $\text{Birth} M$ is naturally a $\mathcal{B}_M$-module.

Proof. If $(b_\rho, \xi) \preceq (b'_\rho, \xi')$ then any homomorphism $k[b_\rho + Q\nabla\xi] \rightarrow M$ restricts to a homomorphism $k[b'_\rho + Q\nabla\xi'] \rightarrow M$. Note that $b_\rho + Q\nabla\xi$ equals its translate by an element of $R\rho$, so any of these translates yield the same restriction to $b'_\rho + Q\nabla\xi'$; hence the restriction descends to $M^{\rho, \xi}/\rho = \text{Hom}_Q(k[Q\nabla\xi], M)/\rho.$

Definition 15.13. Fix a module $M$ over a real polyhedral group $Q$.

1. The death poset of $Q$ is the disjoint union $\mathcal{D}_Q = \bigcup_r Q/\mathbb{R} \times \nabla r$ with
   $$(a_r, \sigma) \preceq (a'_r, \sigma') \iff a_r - Q_{\nabla \sigma} \subseteq a'_r - Q_{\nabla \sigma},$$
   where $a_r \subseteq Q$ is viewed as a coset of $\mathbb{R} \times \nabla r$ in $Q$ for the purpose of writing $a_r + Q_{\nabla \sigma}$. Set $\mathcal{D}_Q^r = Q/\mathbb{R} \times \nabla r$ and $\mathcal{D}_Q^{r, \sigma} = Q/\mathbb{R} \times \nabla r$.

2. The death poset of $M$ is the subposet $\mathcal{D}_M = \bigcup_r \mathcal{D}_M^r = \bigcup_r \text{deg soc}_r M \subseteq \mathcal{D}_Q$.

3. The death module of $M$ is
   $$\text{Death} M = \prod_{a_r \in \mathcal{D}_M^r} \text{soc}_r M_{a_r} = \prod_{(a_r, \sigma) \in \mathcal{D}_M^{r, \sigma}} (\text{soc}_r^\sigma M)_{a_r}$$
   whose factor in degree $a_r \in \mathcal{D}_Q^r$ is also denoted $\text{Death}_{a_r} M = (\text{soc}_r^\sigma M)_{a_r}$ and whose factor in degree $(a_r, \sigma) \in \mathcal{D}_Q^{r, \sigma}$ is also denoted $\text{Death}_{a_r}^\sigma M = (\text{soc}_r^\sigma M)_{a_r}$.

Example 15.14. The death poset of the real polyhedral group $R$ is the negative (or, if you like, Matlis dual) of the birth poset in Example 15.11, so $\mathcal{D}_\mathbb{R}$ has $a'_\rho < a_\circ < a_\bullet < \infty$ for all $a' < a \in \mathbb{R}$. Geometrically, $\mathcal{D}_\mathbb{R}$ is the set of negative-pointing rays in $\mathbb{R}$ totally ordered by inclusion, meaning $(-\infty, a'] < (-\infty, a) < (-\infty, a] < (-\infty, \infty)$ when $a' < a$.

Lemma 15.15. The death module $\text{Death} M$ is naturally a module over $\mathcal{D}_M$ and $\mathcal{D}_Q$.

Proof. This follows from Remark 6.63.

Proposition 15.16. The disjoint union $\mathcal{B}_Q \cup \mathcal{D}_Q$ is partially ordered by
   $$(b_\rho, \xi) \preceq (a_r, \sigma) \iff (b_\rho + Q\nabla\xi) \cap (a_r - Q_{\nabla \sigma}) \neq \emptyset$$
   along with the given partial orders on $\mathcal{B}_Q$ and $\mathcal{D}_Q$, if $Q$ is a real polyhedral group. □
Remark 15.17. The geometric picture in Remark 15.9 remains roughly valid, but now the face-poset factors $\Delta \rho$ and $\nabla \tau$ come into consideration. Whereas passing from $(b_\rho, \xi)$ to $(b_\rho', \xi)$ for a fixed face $\xi$ should be thought of as a macroscopic motion, passing from $(b_\rho, \xi)$ to $(b_\rho, \xi')$ should be thought of as an infinitesimal nudge. Thus every point in the partial compactification becomes “arithmetically thickened” in a manner reminiscent of the way an inertia group on an orbifold keeps track of automorphisms along a stratum, noting that indeed the nature of the thickening is precisely dependent on the stratum because along $\rho$ only infinitesimal motions to faces $\xi \supseteq \rho$ are allowed.

16. Death functors and QR codes

16.1. Death functors.

Theorem 16.1. Fix a module $M$ over a polyhedral partially ordered group $Q$.

1. If $Q$ is discrete and $a_\tau \in D_M$ is a death degree, then there is a death functor

$$\partial_{a_\tau} : M_{b_\rho}^\rho \to \text{Death}_{a_\tau} M,$$

natural in $b_\rho \in B_Q$, where $M_{b_\rho}^\rho = \text{Hom}_Q(k[b_\rho + Q_\xi], M)$ as in Definition 12.12.

2. If $Q$ is real and $(a_\tau, \sigma) \in D_M$ is a death degree, then there is a death functor

$$\partial_{a_\tau} : M_{b_\rho}^{\rho, \xi} \to \text{Death}_{a_\tau}^\sigma M,$$

natural in $(b_\rho, \xi) \in B_Q$, where $M_{b_\rho}^{\rho, \xi} = \text{Hom}_Q(k[b_\rho + Q_\tau \xi], M)$ as in Definition 12.26.

Proof. The proof for the discrete case is strictly easier, since it is little else than the real case with $\xi = \rho$ and $\sigma = \tau$. The following real proof is written to work verbatim after changing every $Q_\tau \xi$ to $Q_+\xi$, placing a bar over every soc, and erasing every $\xi$ and $\sigma$.

The image of any given homogeneous homomorphism $\varphi : k[b_\rho + Q_\tau \xi] \to M$ is a quotient $k[S]$ of $k[b_\rho + Q_\tau \xi]$, where $S \subseteq b_\rho + Q_\tau \xi$ is a downset of $b_\rho + Q_\tau \xi$. Therefore $\varphi$ canonically induces an inclusion $\varphi_* : k[S] \hookrightarrow M$. If $a_\tau \in D_M^\tau$ then either $a_\tau \notin D_{k[S]}^\tau$ or $(b_\rho, \xi) \not\preceq (a_\tau, \sigma)$ and, for some $b \in b_\rho + Q_\tau \xi$, the element $1 \in k[S]_b$ divides a cogenerator $s_{a_\tau}$ of $k[S]$ along $\tau$ with nadir $\sigma$ in some degree $a$ in the coset $a_\tau$. All choices of $a$, regardless of which $b \in b_\rho + Q_\tau \xi$ is used, result in cogenerators $s_{a_\tau}^\sigma$ with the same image $s_{a_\tau}^\sigma \in \text{soc}_\tau^\omega k[S]$ because of the quotient-restriction modulo $\tau$ in Definition 6.59.1. The socle element $s_{a_\tau}^\sigma$ is taken functorially to $\text{soc}_\tau^\rho M$ by the map $\text{soc}_\tau^\rho \varphi_* : \text{soc}_\tau^\rho k[S] \hookrightarrow \text{soc}_\tau^\rho M$ in Theorem 11.1. The desired functor takes $\varphi$ to

$$\partial_{a_\tau} \varphi = \begin{cases} \text{soc}_\tau^\rho \varphi_* (s_{a_\tau}^\sigma) & \text{if } (b_\rho, \xi) \preceq (a_\tau, \sigma) \text{ and } a_\tau \in D_M^\tau, \\ 0 & \text{otherwise.} \end{cases}$$

The naturality of the death functor is because $(b_\rho, \xi) \preceq (b_\rho', \xi)$ implies that the element $1 \in k[S]_{b'}$ for some $b' \in b_\rho + Q_\tau \xi'$ also divides one of the aforementioned closed cogenerators $s_{a_\tau}^\sigma$ of $k[S]$ along $\tau$ with nadir $\sigma$ in some degree $a \in a_\tau$. \qed
16.2. **QR codes.**

**Definition 16.2.** The *(functorial) QR code* of a $Q$-module $M$ is the homomorphism

$$\partial = \prod_{a_r \in D_M} \partial_{a_r} : \text{Birth} M \to \text{Death} M$$

induced by Theorem 16.1.1 if $Q$ is a discrete polyhedral group and

$$\partial = \prod_{(a_r, \sigma) \in D_M} \partial^\sigma_{a_r} : \text{Birth} M \to \text{Death} M$$

induced by Theorem 16.1.2 if $Q$ is a real polyhedral group.

**Remark 16.3.** Functoriality of the death map means that $\partial$ is a morphism of modules over posets, in the sense that if $(b_{\rho}, \xi) \preceq (b'_{\rho'}, \xi')$ then the composite homomorphism

$$\text{Birth}_{b_{\rho}} M \to \text{Birth}_{b'_{\rho'}} M \to \prod_{(b'_{\rho'}, \xi') \leq (a_r, \sigma)} \text{Death}^\sigma_{a_r} M$$

is also expressible as the composite homomorphism

$$\text{Birth}_{b_{\rho}} M \to \prod_{(b_{\rho}, \xi) \leq (a_r, \sigma)} \text{Death}^\sigma_{a_r} M \to \prod_{(b'_{\rho'}, \xi') \leq (a_r, \sigma)} \text{Death}^\sigma_{a_r} M.$$
via the homomorphism $\mathbb{k}[b_\rho + Q_{\nabla\xi}] \to \mathbb{k}[a_\tau - Q_{\nabla\sigma}]$ defined by $1 \in \mathbb{k}$ if $(b_\rho, \xi) \preceq (a_\tau, \sigma)$ and $0$ otherwise. Using the inclusion $Q = Q/\{0\} \times \{Q_+\} \subseteq B_Q$ to make sense of the relevant direct limits, $\text{Birth} M$ pushes forward to a $Q$-module

$$\text{Birth}_Q^Q M = \bigoplus_{a \in Q} \lim_{\to} \text{Birth}_{b_\rho}^Q M$$

universal among $Q$-modules with a map from $\text{Birth} M$. The map $\hat{\partial}$ factors through $\text{Birth}_Q^Q M$ by functoriality in Theorem 16.1; write $\hat{\partial}_Q$ for the induced map on $\text{Birth}_Q^Q M$.

The direct sum in Eq. (16.2) maps surjectively onto $M$ by Theorem 13.3.2 (or Theorem 13.4.2 in the discrete case). Universality of the pushforward forces this surjection to factor through $\text{Birth}_Q^Q M$. On the other hand, $\hat{\partial}_Q$ factors through this surjection $\text{Birth}_Q^Q M \to M$ by construction of the death functor in Theorem 16.1: every element of $\text{Birth} M$ that maps to $0$ in $M$ certainly maps to $0$ in $\text{Death} M$. But the induced map from $M$ to the product in Eq. (16.2) is injective by Theorem 7.7 (or Theorem 11.1 in the discrete case) because it induces an injection on socles by construction. (Note: the socle of the product in Eq. (16.2) along a face could be much bigger than that of $M$, but composing the map from $M$ with the projection to any factor of the product shows that the induced map on socles must at least be injective.)

**Remark 16.6.** Keeping track of entire graded components of $M$ (technically: $M_{\rho,\xi}/\rho$) feels wasteful compared to keeping track merely of vector spaces $\text{top}_\rho^\xi M$, whose graded components are best viewed as vector spaces that count minimal generators. This waste feels unpleasant, but it is required to make the QR morphism functorial. And perhaps it shouldn’t feel so bad: writing down $\text{top}_\rho^\xi M$ as the quotient that it naturally is requires knowing $\text{Birth}_{b_\rho} M$ as well as the kernel of its surjection to $\text{top}_\rho^\xi M$ (see Theorem 17.5 for more about that), so it is not horrible to request that the filtered object serve as the data structure instead of the associated graded object.

In finitely generated cases—or better, finitely determined ones from Section 4—the QR code is a functorial homomorphism between vector spaces of finite dimension that encapsulates all of a given module. As such, it is similar to flange presentation (Definition 4.26). The latter, however, sacrifices functoriality for efficiency by choosing bases.

**Remark 16.7.** It is worth unwinding the definitions that lead to the reconstruction homomorphism $\hat{\partial}$ featured in Theorem 16.5. Over a discrete polyhedral group, $\text{Birth}_{b_\rho} M = (M^{\rho,\xi}/\rho)_{b_\rho}$ from Definition 15.1.3 and $M^\rho = \underline{\text{Hom}}_Q(\mathbb{k}[Q_+], M)$ from Definition 12.12, so the direct summands in Eq. (16.1) are

$$\mathbb{k}[b_\rho + Q_+] \otimes_\mathbb{k} \underline{\text{Hom}}_Q(\mathbb{k}[b_\rho + Q_+], M).$$

Similarly, over a real polyhedral group, $\text{Birth}_{b_\rho} M = (M^{\rho,\xi}/\rho)_{b_\rho}$ from Definition 15.10.3 and $M^{\rho,\xi} = \underline{\text{Hom}}_Q(\mathbb{k}[Q_{\nabla\xi}], M)$ from Definition 12.26, so the summands in Eq. (16.2) are

$$\mathbb{k}[b_\rho + Q_{\nabla\xi}] \otimes_\mathbb{k} \underline{\text{Hom}}_Q(\mathbb{k}[b_\rho + Q_{\nabla\xi}], M).$$
Note the lack of underlines on the Hom functors: these are single graded pieces of modules, not entire graded modules.

**Definition 16.8.** The life module of $M$ over a discrete or real polyhedral group $Q$ is

$$\text{Life } M = \bigoplus_{b_\rho \in B_M} k[b_\rho + Q_+] \otimes_k \text{Birth}_{b_\rho} M \quad \text{or} \quad \text{Life } M = \bigoplus_{(b_\rho, \xi) \in B_M} k[b_\rho + Q_\xi] \otimes_k \text{Birth}_{b_\rho}^\xi M,$$

according to whether $Q$ is discrete or real, respectively.

**Remark 16.9.** The image of the reconstruction homomorphism $\hat{\partial}$ in Theorem 16.5 is naturally isomorphic to $M$, so using the notion of life module, $\hat{\partial}$ induces a natural surjection $\text{Life } M \twoheadrightarrow M$, also denoted $\hat{\partial}$ by abuse of notation, that is easy to describe explicitly: the displayed expressions in Remark 16.7 map to $M$ by $y \otimes \varphi \mapsto \varphi(y)$.

### 17. Elder Morphisms

**Definition 17.1.** If $\beta \in B_Q$ is any element of the birth poset of $Q$, then define $\text{Life}_{\prec \beta} M$ by taking the direct sum in Definition 16.8 over birth poset elements strictly preceding $\beta$, and similarly define $\text{Life}_{\preceq \beta} M$ by taking the direct sum over birth poset elements weakly preceding $\beta$.

**Convention 17.2.** For any birth degree $\beta \in B_Q$ over a real or polyhedral group $Q$, set

$$\text{top}_\beta M = \begin{cases} (\text{top}_\rho M)_{b_\rho} & \text{if } Q \text{ is discrete and } \beta = b_\rho \\ (\text{top}_\xi^\rho M)_{b_\rho} & \text{if } Q \text{ is real and } \beta = (b_\rho, \xi) \end{cases}$$

$$k[\beta + Q_\beta] = \begin{cases} k[b_\rho + Q_+] \otimes_k \text{Birth}_{b_\rho} M & \text{if } Q \text{ is discrete and } \beta = b_\rho \\ k[b_\rho + Q_\xi] \otimes_k \text{Birth}_{b_\rho}^\xi M & \text{if } Q \text{ is real and } \beta = (b_\rho, \xi). \end{cases}$$

Parallel notations work for $\text{Birth}_\beta M$ and $\text{Life}_\beta M = k[\beta + Q_\beta] \otimes_k \text{Birth}_\beta M$, as well as dually for $\text{soc}_\alpha M$ and $\text{Death}_\alpha M$ when $\alpha \in D_Q$.

**Definition 17.3.** Fix a module $M$ over a real or discrete polyhedral group $Q$ and a birth degree $\beta \in B_Q$. (Thus $\beta$ has the form $b_\rho$ if $Q$ is discrete and $(b_\rho, \xi)$ if $Q$ is real.)

1. The **elder submodule** of $M$ at $\beta$ is the submodule

   $$M_{\prec \beta} = \widehat{\partial}(\text{Life}_{\prec \beta} M) \subseteq M.$$

2. The **extant submodule** of $M$ at $\beta$ is the submodule

   $$M_{\preceq \beta} = \widehat{\partial}(\text{Life}_{\preceq \beta} M) \subseteq M.$$

3. The **elder quotient** of $M$ at $\beta$ is $M_{\preceq \beta}/M_{\prec \beta}$. 
Example 17.4. When $Q = \mathbb{R}$ and $M$ is expressed as a direct sum of bars—indicator modules for intervals that on each end may be open, closed, or infinite—the elder submodule $M_{\preceq \beta}$ is the submodule of $M$ that is the direct sum of those bars whose left endpoints occur strictly earlier than $\beta$. (See Example 15.11 for the explicit description of $B_\mathbb{R}$ as a totally ordered set.) The extant submodule of $M$ is the direct sum of those bars whose left endpoints occur at $\beta$ or earlier. The elder quotient is the direct sum of those bars whose left endpoints occur exactly at $\beta$. This description has been phrased in terms of a direct sum decomposition of $M$ specified by the bar code of $M$, but one of the main points of the theory is that the result is functorial: the direct sum of bars with fixed left endpoint is canonically a subquotient of $M$.

Theorem 17.5. If $M$ is upset-finite over a discrete or real polyhedral group $Q$ then
\[ \text{Birth}(M_{\preceq \beta}/M_{\leq \beta}) = \text{top}_\beta(M_{\preceq \beta}/M_{\leq \beta}) = \text{top}_\beta M \]
for any birth degree $\beta \in B_Q$, and $\text{top}_\gamma(M_{\preceq \beta}/M_{\leq \beta}) = 0$ if $\gamma \neq \beta$.

Proof. The surjection $\widehat{\partial} : \text{Life} M \twoheadrightarrow M$ induces a map $\text{top}_\gamma \widehat{\partial} : \text{top}_\gamma \text{Life} M \to \text{top}_\gamma M$ that is an isomorphism for all $\gamma \in B_Q$ by construction, using Example 12.30 to calculate $\text{top}_\gamma \text{Life} M$ (at least in the real case; the easier discrete analogue of Example 12.30 is true and applied without further comment). As the map $\text{Life}_\beta \to M_{\preceq \beta}/M_{\leq \beta}$ is a finite upset cover, it induces a surjection $\text{top}_\rho \text{Life}_\beta M \to \text{top}_\rho(M_{\preceq \beta}/M_{\leq \beta})$ for all faces $\rho$ by Theorems 13.3 and 13.4. Example 12.30 therefore implies all of the claimed vanishing as well as $\text{top}_\beta M = \text{top}_\beta \text{Life}_\beta M \to \text{top}_\beta(M_{\preceq \beta}/M_{\leq \beta})$. The only thing that could a priori ruin the claimed right-hand equality is if the composite $\text{Life}_{\prec \beta} \to \text{Life}_{\preceq \beta} \to M_{\preceq \beta}/M_{\leq \beta}$ managed to affect $\text{top}_\beta$, but $\text{top}_\beta$ vanishes on $\text{Life}_{\preceq \beta} M$ by Example 12.30 again. Using Proposition 12.28 to produce a surjection $\text{Birth}_\beta \text{Life}_\beta M \twoheadrightarrow \text{Birth}_\beta(M_{\preceq \beta}/M_{\leq \beta})$, the remaining claimed equality follows by Proposition 12.29 from $\text{Birth}_\beta \text{Life}_\beta M = \text{top}_\beta \text{Life}_\beta M$, given that $\text{top}_\beta \text{Life}_\beta M \cong \text{top}_\beta(M_{\preceq \beta}/M_{\leq \beta})$ is an isomorphism. \hfill \Box

Example 17.6. In the situation of Example 17.4, the theorem says what is expected: the entire degree $\beta$ piece of the elder quotient $M_{\preceq \beta}/M_{\leq \beta}$ equals the vector space spanned by the left endpoints of bars that sit at $\beta$.

Definition 17.7. The elder morphism of an upset-finite module over a real or discrete polyhedral group is the functorial QR code
\[ \partial^\beta : \text{top}_\beta M \to \text{Death}(M_{\leq \beta}/M_{\preceq \beta}) \]
of $M_{\leq \beta}/M_{\preceq \beta}$ afforded by Theorem 17.5 and Definition 16.2.

Remark 17.8. Defining an elder morphism only requires that the module $M$ be upset-finite, but no claims are made about whether the definition is reasonable unless $M$ is finitely encoded, so that information about $M$, as well as its elder and extant submodules, can be recovered from their QR codes.
Remark 17.9. The elder morphism could have been defined instead to have the target \( \text{Death}(M/M_{\leq \beta}) \), because of the natural inclusion \( \text{Death}(M_{\leq \beta}/M_{< \beta}) \subseteq \text{Death}(M/M_{< \beta}) \) from Theorems 7.7 and 11.1. And linguistically it is useful to speak of classes or elements dying in a quotient of \( M \) rather than in a quotient of \( M_{\leq \beta} \), even though the class in question of course persists in the submodule \( M_{\leq \beta}/M_{< \beta} \) of the module \( M/M_{< \beta} \) if it was born at \( \beta \).

Remark 17.10. Crawley-Boevey constructed a functor from \( \mathbb{R} \)-indexed persistence modules to the category of vector spaces which counts number of intervals of a given type [Cra13]. Cochoy and Oudot extended this functorial construction to special 2-parameter persistence modules satisfying an “exact diamond condition”, which guarantees thin summands [CO16]. Elder morphisms can be thought of as multiparameter generalizations: given a birth parameter \( \beta \in \mathcal{B}_Q \), the elder morphism functorially isolates all possible elder-death parameters for classes born at \( \beta \) in the form of the death poset \( \mathcal{D}_{M/M_{\leq \beta}} \) of the elder quotient at \( \beta \). The number of “bars” born at \( \beta \) that die an elder-death by joining the elder submodule at \( \alpha \in \mathcal{D}_{M/M_{\leq \beta}} \) is categorified by functorial vector spaces, namely tops and socles. The elder morphism then associates a functorial linear birth-to-death map (a QR code) between those vector spaces.

18. Generators

Remark 18.1. Generators of graded modules are homogeneous elements that do not lie in the submodule generated in any lower degree. That is, roughly speaking, how generators are defined here, as well, but with two key enhancements in the context of finitely encoded modules over polyhedral groups \( Q \). First, generators are only elements of modules in the discrete case, although even then they are more properly thought of as homomorphisms from localizations \( k[Q_+]_\rho \) along faces. In the real case, generators are homomorphisms from “open localizations” \( k[Q_{\Delta \rho}]_\rho \) along faces. These should be thought of as rather slight generalizations of the usual notion of element, which is a homomorphism from the free module \( k[Q_+] \) itself. The second enhancement makes precise the notion of submodule generated in lower degree, given that generators need not be elements—and even in the discrete case should be thought of as equivalence classes of elements under translation along a face. The elder submodule fills that role.

Definition 18.2. Fix a module \( M \) over a polyhedral partially ordered group \( Q \).

1. If \( Q \) is discrete then a generator of \( M \) of degree \( b \in Q \) along a face \( \rho \) is an element in \( M^\rho_b \) that lies outside of the elder submodule \( M_{< b_\rho} \), where \( b_\rho = b + \mathbb{Z}_\rho \in Q/\mathbb{Z}_\rho \).

2. If \( Q \) is real then a generator of \( M \) of degree \( b \in Q \) along a face \( \rho \) with nadir \( \xi \in \Delta \rho \) is an element in \( M^\rho_{b,\xi} \) that lies outside of the elder submodule \( M_{< (b_\rho, \xi)} \).

Remark 18.3. Some recall of the objects, including their notation and relations between them, is in order. In the discrete case, \( M^\rho = \text{Hom}_Q(k[Q_+]_\rho, M) \) from Definition 12.12, and the condition of lying outside of the elder submodule means that the
image of a degree $b$ generator along $\rho$ maps to a nonzero element in $\top_\rho M$, necessarily of degree $b$, under the surjection in Proposition 12.16.

Similarly, in the real case, $M^{\rho, \xi} = \text{Hom}_Q(\mathbb{k}[Q_\|], M) = \text{Hom}_Q(\mathbb{k}[Q_+], \partial^\xi M)$ as in Definition 12.26 and Lemma 12.27. The condition of lying outside of the elder submodule means that the generator in question maps to a nonzero element of $\top_\rho M$, again necessarily of degree $b$, by Proposition 12.29.

This Remark is intended to lay bare a duality between generators and cogenerators; compare Definition 6.43.3 for the discrete case and Definition 6.59.3 for the real case.

**Remark 18.4.** A **closed generator** could be defined as an honest element of $M$, namely the image of $1 \in \mathbb{k}[Q_\|]$, assuming either the discrete case or that the nadir of the generator in question is $\rho$ in the real case. This is analogy with closed cogenerators in Definition 6.43.3, which are similarly honest elements of modules.

19. **Functorial bar codes**

The notation from Convention 17.2 is in effect throughout this section.

**Lemma 19.1.** If $M$ is a module over a real or discrete polyhedral group $Q$ and $\alpha \in D_Q$, then $\text{soc}_\alpha M$ has a canonical filtration by the poset $B_Q$:

$$(\text{soc}_\alpha M)_{\leq \beta} = \text{soc}_\alpha (M_{\leq \beta}).$$

**Proof.** If $\beta \leq \beta'$ then $M_{\leq \beta} \subseteq M_{\leq \beta'}$, so $\text{soc}_\alpha (M_{\leq \beta}) \subseteq \text{soc}_\alpha (M_{\leq \beta'})$ by Theorem 7.7. □

**Remark 19.2.** The equality in the lemma allows the parentheses to be removed: both sides are $\text{soc}_\alpha M_{\leq \beta}$. It is perhaps simpler to think of filtering $\text{soc}_\alpha M$ by the subposet $B_M \subseteq B_Q$ instead, as this is a finite filtration (in every case of computational interest) containing all of the interesting information about the filtration by $B_Q$.

**Theorem 19.3.** Fix a module $M$ over the partially ordered group $Q = \mathbb{R}$ or $\mathbb{Z}$. For all birth parameters $\beta \in B_Q$ and death parameters $\alpha \in D_Q$ there is a canonical isomorphism

$$\text{soc}_\alpha (M_{\leq \beta}/M_{< \beta}) \cong \text{soc}_\alpha M_{\leq \beta}/\text{soc}_\alpha M_{< \beta} = \text{gr}_\beta \text{soc}_\alpha M.$$

**Proof.** When $Q = \mathbb{R}$ there are two cases: in the notation of Example 15.14, either $\alpha = \infty$ or not. First assume not, so $\alpha = (a, \sigma)$ with $\sigma = \{0\}$ if $\alpha = a$ and $\sigma = \mathbb{R}_+$ if $\alpha = a$. Suppose $s \in \text{soc}_\alpha (M_{\leq \beta}/M_{< \beta})$. Choose an arbitrary lift $\tilde{s} \in (M_{\leq \beta})_{a-\sigma}$ of $s$, with notation as in Definition 6.11. The homomorphisms in Lemmas 6.13 and 12.2 induce a natural map $\varphi_\alpha : (M_{\leq \beta})_{a-\sigma} \to (M_{\leq \beta})_{a+\xi}$, where $\xi = \mathbb{R}_+$ if $\sigma = \{0\}$ (i.e., if $\alpha = a$) and $\xi = \{0\}$ if $\sigma = \mathbb{R}_+$ (i.e., if $\alpha = a$). The hypothesis on $s$ means that $\varphi_\alpha(\tilde{s})$ lies not merely in $(M_{\leq \beta})_{a+\xi}$ but in $(M_{\leq \beta})_{a+\xi}$. Let $\psi_\alpha$ be the restriction of $\varphi_\alpha$ to $(M_{\leq \beta})_{a-\sigma}$. The preimage of $\varphi_\alpha(\tilde{s})$ in $(M_{\leq \beta})_{a-\sigma}$ under $\psi_\alpha$ is well defined modulo $\text{soc}_\alpha M_{< \beta}$ by definition of socle. The desired isomorphism is

$$\omega_\alpha : s \mapsto \tilde{s} - \psi_\alpha^{-1} \circ \varphi_\alpha(\tilde{s}).$$
It is well defined as an element of \((M_{\leq \beta})_{a-\sigma}/\soc M_{\leq \beta}\) because if \(y \in (M_{\leq \beta})_{a-\sigma}\) then 
\(\psi_a^{-1} \circ \varphi_a(\tilde{s}) = y\) modulo \(\soc M_{\leq \beta}\). And although \(\omega_a\) a priori takes \(\tilde{s}\) to an arbitrary element of \((M_{\leq \beta})_{a-\sigma}/\soc M_{\leq \beta}\), in fact this element is annihilated by \(\varphi_a\) by construction and hence lies in \(\soc M_{\leq \beta}/\soc M_{\leq \beta}\). That \(\omega_a\) is surjective follows because an element of \(\soc M_{\leq \beta}\) automatically provides an element of \(\soc M_{\leq \beta}/\soc M_{\leq \beta}\) and remains unmodified by \(\omega_a\). That \(\omega_a\) is injective follows because if \(s\) is nonzero then \(\tilde{s}\) lies outside of \((M_{\leq \beta})_{a-\sigma}\), whereas \(\omega_a\) modifies \(\tilde{s}\) by an element of \((M_{\leq \beta})_{a-\sigma}\).

The above argument for \(Q = \mathbb{Z}\) is more elementary: every \(\alpha = (a, \sigma)\) is simply \(\alpha = a\), every subscript \(a - \sigma\) should simply be \(a\), and every subscript \(a + \xi\) should be \(a + 1\). The natural map \(\varphi_a : (M_{\leq \beta})_a \to (M_{\leq \beta})_{a+1}\) requires no lemmas to justify its existence.

The case \(\alpha = \infty\) is much easier, by virtue of being completely general: over any partially ordered group \(Q\), the socle along \(Q_+\) is \(\text{Hom}_Q(k[Q_+], M)/Q_+ = M/Q_+\), so the isomorphism in question is by exactness of quotient-restriction (Lemma 6.41).

**Remark 19.4.** The isomorphism in the theorem can, if justified properly, be seen as an instance of the second isomorphism theorem: if \(N\) has submodules \(K\) and \(N'\), and \(K' = K \cap N'\), then \(\ker(N/N' \to N/(N'+K)) = (N'+K)/N' \cong K/K'\). The relevant ambient module is \(N = (M_{\leq \beta})_{a-\sigma}\), with submodules \(K = \soc \sigma M_{\leq \beta}\) and \(N' = (M_{\leq \beta})_{a-\sigma}\).

**Definition 19.5.** Fix a module \(M\) over the real polyhedral group \(\mathbb{R}\).

1. The total top of \(M\) is \(\text{Top } M = \bigoplus_{\beta \in \mathcal{B}_M} \text{top } \beta M\).
2. The graded total socle of \(M\) is \(\text{gr Death } M = \prod_{\alpha \in \mathcal{D}_M} \text{gr soc } \alpha M\).
3. The functorial bar code of \(M\) is the linear map
   \[\text{Top } M \to \text{gr Death } M\]
   obtained by composing the elder morphism of \(M\) in Definition 17.7 with the elder projection in Theorem 19.3.

**Theorem 19.6.** Fix a module \(M\) over the real polyhedral group \(\mathbb{R}\). In any decomposition of \(M\) as a direct sum of indicator subquotients of \(k[\mathbb{R}]\)—that is, indicator modules on intervals—the left endpoints of the intervals form a basis for the source vector space \(\text{Top } M\) and the right endpoints form a basis for the target vector space \(\text{gr Death } M\).

**Proof.** Functoriality of bar codes (see Definitions 17.7 and 19.5) reduces to the case of a single indicator subquotient. That case falls under Examples 15.11 and 15.14.

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20. Future directions

20.1. **Fly wing implementation.**

The plan for statistical analysis of the fly wing dataset is to summarize wings using biparameter persistence and then to statistically analyze the rank function \(\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{N}\) defined by \((a \leq b) \mapsto \text{rank}(H_a \to H_b)\) [CZ09, CDF10]. In more detail, the plan is:
fly wing $\rightsquigarrow$ 2D persistence module $\rightsquigarrow$ encoding by monomial matrix $\rightsquigarrow$ rank function. There is potential loss at steps 1 and 3; in contrast, step 2 is lossless by Theorem 5.16.

The question of loss in step 1 is one of statistical sufficiency. It is plausible that such loss does not occur locally—deforming a fly wing should yield a nontrivial deformation of its biparameter persistence module (Example 1.1)—although perhaps globally it might. Proofs of such statements would likely rely on QR codes, since the different persistence modules in question ought to have different birth or death posets.

The question of loss in step 3 amounts to a question about moduli: is the variation in fly-wing modules solely from the placements of birth upsets and death downsets in a fringe presentation (Definition 5.4), or can the scalar entries vary continuously? There is reason to believe that continuous variation of scalar entries does not occur—because the homology is dimension 0 or codimension 1 in $\mathbb{R}^2$, both of which come with canonical bases—but formulating such a statement precisely requires theoretical development.

Doing statistics with rank functions requires effective data structures for them. This is a general problem, not limited to the fly wing case although well exemplified by it. Rank functions from an $n$-parameter filtration are nonnegative integer-valued functions on $\mathbb{R}^{2n}$ whose domains of constancy are—in the fly wing case, as in many data science applications—semialgebraic. Fringe presentations and surrounding objects are suited for such functions. Once rank functions are effectively represented, statistical analysis can proceed largely as in the ordinary single-parameter case [RT16].

20.2. Computation.
The plan for analysis of the fly wing dataset, in the persistence setup from Example 1.1, requires various aspects of the theory to be made more explicitly algorithmic. These and other tasks are relevant in general, not merely for fly wings or $n = 2$ parameters.

1. Compute a fringe presentation from given spline data. This should involve real semialgebraic geometry along with careful bookkeeping of planar graph topology.

2. Compute rank functions from any of the data structures for multiparameter persistence in Theorem 5.16. Naive algorithms and data structures are easy enough to write down when a finite encoding is given as in Definition 2.6, but these—the data structures as well as the algorithms—are likely suboptimal.

3. Define and devise algorithms to compute data structures for QR codes. This is almost analogous to computing flange presentations of finitely determined modules, in that QR codes record the degrees of generators, the degrees of co-generators, and a linear map, just as flange presentations do. The differences are that the source vector spaces in QR codes are (morally) entire graded pieces of the module and the linear maps are filtered through these.

4. Any data structure for QR codes is automatically a data structure for a single elder morphism, which after all is the QR code of a subquotient, but it could be important for statistical purposes to specify all elder morphisms at once. The problem is that the source and especially target vector spaces vary depending
on the birth parameter. Thus the *total elder morphism* is a family of elder morphisms fibered over the set of parameters. It suffices to fiber over the birth poset.

5. Algorithmically compute the equivalences in the syzygy theorems (Theorems 5.16 and 14.2) over real and discrete polyhedral groups. Compute QR codes and elder morphisms from fringe presentations or any of the other hallmarks of finitely encoded modules.

6. Evaluate cogenerator functors on downset modules in such a way that the cohomology can be effectively calculated. Dually, evaluate generator functors on upset modules. These operations are linchpins of the computational theory. For example, the socle of an $\mathbb{R}^n$-module is extracted from a downset copresentation by applying cogenerator functors to it, which yields a categorified semialgebraic subset of $\mathbb{R}^n$ mapping to another categorified semialgebraic subset of $\mathbb{R}^n$. The kernel is the desired socle. Part of the problem is to make these categorifications precise, in the language of (say) constructible sheaves.

Minimal flange presentations as well as injective or flat resolutions of finitely determined $\mathbb{Z}^n$-modules are computable algorithmically starting from generators and relations in the finitely generated case [HM05], and consequently for finitely determined modules using the categorical equivalences between finitely generated and finitely determined modules [Mil00, Definition 2.7, Table 1, and Theorem 2.11]. Note that some software exists for multiparameter persistence of finitely generated $\mathbb{Z}^n$-modules; see [LW15] and [Gäf17], the latter being a library for multiparameter persistent homology.

20.3. Homological algebra.

20.3.1. Stability. With discrete parameters, “algebraic invariants of multiparameter persistence modules such as minimal number of generators, Betti tables, Hilbert polynomials etc. tend to change drastically when the initial data is altered even slightly. That is why the classical commutative algebra invariants are not useful for data analysis. For data analysis we need stable invariants” [GC17]. Real multiparameter persistence provides a counterpoint: even though functorial QR codes and elder morphisms are algebraic in nature, continuity of the poset $\mathbb{R}^n$ allows them to move only slightly when the input data are perturbed. For example, wiggling a fly wing should nudge only the upsets and downsets in a fringe presentation; it should not, in some generic sense, alter the validity of a given finite encoding poset, as long as one is willing to perturb the encoding poset morphism so as to nudge the fibers. That means the homological algebra of the perturbed module $M'$ is pulled back from the same homological algebra over the encoding poset that governs the homological algebra of $M$. This needs to be made precise, particularly the word “generic” and the concept of “perturbation”—of data and modules as well as poset morphisms—but the salient point is that when the scalar matrix in a fringe presentation is fixed and only the upsets and downsets move, the Hilbert function and rank function vary in predictable geometric ways. This is therefore a call for stability theorems in real multiparameter persistence.
The motion of Betti numbers in the discrete case corresponds in the real case, at least in homological degree 0, to motion of the birth poset—or, more accurately, motion of the constructible function on the birth poset whose values are dimensions of generator functors (Section 12). These are more subtle than Hilbert or rank functions but should still enjoy some kind of upper semicontinuity, in the finitely encoded context, along the lines of upper semicontinuity in flat families in the discrete, infinitely generated context.

20.3.2. **Indicator resolutions.** Regarding Conjecture 14.6, although minimal upset and downset resolutions exist, it is unclear whether they must terminate after finitely many steps. And although finite upset and downset resolutions exist—this much is part of the largely unrestricted syzygy theorem over an arbitrary poset $Q$—no uniform bound on their length is known when $Q$ has quotients with unbounded order dimension.

20.3.3. **Block decomposition of exact modules.** The purpose of the papers by Crawley-Boevey [Cra13] and by Cochoy and Oudot [CO16], discussed in Remark 17.10, is barcode decomposition of $\mathbb{R}$-finite modules over $\mathbb{R}$ and so-called “exact” $\mathbb{R}^2$-finite modules over $\mathbb{R}^2$, respectively. The exactness condition specifies that if $y \in M_a$ is the image of an element $y_i \in M_{a-\varepsilon_i e_i}$ for every choice of $i$, then some class $y' \in M_{a'}$, for $a' = a - \sum_i \varepsilon_i e_i$, satisfies $y' \mapsto y$. Thinking in the context of arbitrary $n$, this condition ought to enable the proof of Theorem 19.3 to be generalized to arbitrary $n$; see Example 1.9 and the paragraph preceding it.

20.4. **Geometry of birth and death.** Birth and death posets of modules over real polyhedral groups should inherit topologies from Definitions 8.2 and 9.1. The death module should be a sheaf over the death poset, and the QR code should take the birth module $\text{Birth } M$ to sections of the death module $\text{Death } M$ in a well behaved—probably constructible—manner. Functorial QR codes are defined without this extra topological nuance, but these extra structures should be imposable on it. Moreover, although these topological structures are not required for recovery of a module from its QR code, they should play a central role in the characterization of maps $\text{Birth } M \to \text{Death } M$ that arise as QR codes of finitely encoded modules. Even in the discrete case this characterization is not easy: any filtered map from $\text{Birth } M$ to $\text{Death } M$ ought to define a module with specified tops and socles, but it is not a priori clear what conditions on the filtered map guarantee that the module thus defined is finitely encoded.

20.5. **Multiscale intersection homology.** The view of the particular form of multiparameter persistence in Example 1.1 as a manifestation of persistent intersection homology (Remark 1.2) has yet to be formalized. Once it is, properties such as stability and suitability for data analysis in general remain to be investigated.


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