COMMUNICATIONS

ELEMENT ORDERS IN COVERS
OF FINITE SIMPLE GROUPS

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The set of element orders of a finite group $G$ is called the spectrum of $G$ and is denoted by $\omega(G)$. We say that a finite group $H$ is a cover of a group $G$ if $G = H/K$ for some normal subgroup $K$ of $H$; if, in addition, $K \neq 1$, then the corresponding cover is proper. A group $G$ is recognizable by spectrum among covers if $\omega(G) \neq \omega(H)$ for any proper cover $H$ of $G$. If $G$ has a nontrivial normal soluble subgroup, then it is not recognizable by spectrum among covers; moreover, there are infinitely many pairwise nonisomorphic covers $H$ of $G$ with $\omega(H) = \omega(G)$ [1, Lemma 1].

We study the property of being recognizable among covers for finite non-Abelian simple groups. Our main object is to prove a conjecture which says that this property is shared by all finite non-Abelian simple groups except for some Lie-type groups of low Lie rank.

**THEOREM 1.** Let $G$ be a finite non-Abelian simple group. Suppose that $G \neq 3D_4(2)$ and $G$ is not a classical group of dimension less than 11. Then $G$ is recognizable by spectrum among covers.

By 2011, the conclusion of Theorem 1 was proved for sporadic groups [2], alternating groups [3], Ree and Suzuki groups [4-6], groups $G_2(q)$ [7] and $E_8(q)$ [8], as well as for linear groups of dimension at least five [9] and unitary groups of dimension at least six [10]. Also we have recently become aware that the group $3D_4(2)$ is unrecognizable among covers [11]. In [10], the following intermediate result on symplectic and orthogonal groups was obtained: if $G$ is one of the groups $S_{2n}(q)$ and $O_{2n+1}(q)$ with $n \geq 3$, or $O_{2n}^{\pm}(q)$ with $n \geq 4$, and $V$ is a nonzero $G$-module over a field of characteristic coprime to $q$, then $\omega(V \ltimes G) \neq \omega(G)$. A motivation for this result is a simple

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observation that a group $G$ is recognizable be spectrum among covers iff $\omega(G) \neq \omega(H)$ for any split extension $H = V \times G$, where $V$ is a nonzero $G$-module over a field of positive characteristic. Thus, modulo the results mentioned above, Theorem 1 follows from the two propositions below.

PROPOSITION 2. Let $G$ be one of the simple groups $3D_4(q)$, $F_4(q)$, $E_6(q)$, $2E_6(q)$, and $E_7(q)$, and let $V$ be a nonzero $G$-module over a field of positive characteristic coprime to $q$. If $H$ is a natural semidirect product of $V$ and $G$, then $\omega(H) \neq \omega(G)$.

PROPOSITION 3. Let $G$ be one of the groups $3D_4(q)$ with $q > 2$, $F_4(q)$, $E_6(q)$, $2E_6(q)$, and $E_7(q)$, or one of the simple groups $S_{2n}(q)$ (where $n = 3$ and $q$ is odd or $n \geq 4$), $O_{2n+1}(q)$ (where $q$ is odd and $n \geq 3$), and $O_{2n}^\pm(q)$ (where $n = 5$ and $q$ is even, or $n = 4$, or $n \geq 6$). Suppose that $V$ is a nonzero $G$-module over a field of characteristic dividing $q$, and $H$ is a natural semidirect product of $V$ and $G$. Then $\omega(H) \neq \omega(G)$.

It is worth noting that Proposition 2 and [10, Thm. 1] give an affirmative answer to [12, Question 17.74].

Information on element orders in covers of the simple group $O_{2n+1}(q)$ has also been used to solve a problem that arose in studying quasirecognizability of symplectic groups. A simple non-Abelian group $G$ is said to be quasirecognizable by spectrum if every finite group $H$ with $\omega(H) = \omega(G)$ has exactly one non-Abelian composition factor and that factor is isomorphic to $G$. According to [13, Thm. 3], if $G = S_{2n}(q)$, where $n \geq 4$, and $H$ is a finite group with $\omega(H) = \omega(G)$, then non-Abelian composition factors of $H$, which are Lie-type groups over a field of characteristic dividing $q$, are contained in the set $\{S_{2n}(q), O_{2n+1}(q), O_{2n}^-(q)\}$. The groups $S_{2n}(q)$ and $O_{2n+1}(q)$ are not isomorphic if $q$ is odd and $n \geq 3$, but nevertheless their spectra coincide very closely and share the same subsets of odd numbers, of numbers coprime to $q$, and of numbers of the form $2m$, where $m$ is an odd prime. For that reason, eliminating the case where a non-Abelian composition factor of $H$ is isomorphic to $O_{2n+1}(q)$ is a severe challenge. By comparing the spectra of proper covers of $O_{2n+1}(q)$ with the spectrum of $S_{2n}(q)$, we reduce this case to the situation where $H$ is an almost simple group with socle $O_{2n+1}(q)$.

PROPOSITION 4. Let $G$ be a simple group $S_{2n}(q)$ and $S$ a simple group $O_{2n+1}(q)$, where $q$ is odd and $n \geq 3$. Suppose that $H$ is a finite group with $\omega(H) = \omega(G)$ and $H$ has a composition factor isomorphic to $S$. Then $S \leq H \leq \text{Aut}(S)$.

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