PROGRAM SYNTHESIS FOR THE OEIS

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Discovering patterns in mathematical objects

- Discovery in 1995 of a more efficient formula for generating the digits of $\pi$ by Simon Plouffe.
- In 2005, Hadi Kharaghani and Behruz Tayfeh-Rezaie published their construction of a Hadamard matrix of order 428.
- In 2012, Geoffrey Exoo has found an edge colorings of $K_{35}$ that have no complete graphs of order 4 in the first color, and no complete graphs of order 6 in the second color proving that $R(4, 6) \geq 36$.
- Discovery in 2023 of a chiral aperiodic monotile by David Smith.
Search: **seq:2,3,5,7,11**

Displaying 1-10 of 1163 results found.

| A000040 | The prime numbers. |
|----------|--------------------|
| (Formerly M0652 N0241) |

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271

(list; graph; refs; listen; history; text; internal format)

OFFSET
1,1

COMMENTS
See [A065091](#) for comments, formulas etc. concerning only odd primes. For all information concerning prime powers, see [A000961](#). For contributions concerning "almost primes" see [A002808](#).

A number p is prime if (and only if) it is greater than 1 and has no positive divisors except 1 and p.

A natural number is prime if and only if it has exactly two (positive) divisors. A prime has exactly one proper positive divisor, 1.
A synthesize and test approach

OEIS sequence

\[0, 1, 3, 6, 10, 15, \ldots, 1431\]

Synthesized program

\[f(x) = (x \times x + x) \div 2\]

Test/Filter:

\[f(0) = 0, \ f(1) = 1, \ f(2) = 3, \ f(3) = 6, \ldots, \ f(53) = 1431\]
Test: criteria for selecting programs

OEIS sequence

0, 1, 3, 6, 10, 15, …, 1431

An undesirable large program

if \( x = 0 \) then 0 else
if \( x = 1 \) then 1 else
if \( x = 2 \) then 3 else
if \( x = 3 \) then 6 else ...
if \( x \geq 53 \) then 1431

Small program (Occam’s Razor)

\[
f(x) = \sum_{i=1}^{x} i
\]

Fast program (efficiency criterion)

\[
f(x) = (x \times x + x) \div 2
\]

Possible other criteria: usefulness criterion?
Synthesize: a Turing-complete language

- Constants: 0, 1, 2
- Variables: x, y
- Arithmetical operators: +, −, ×, div, mod
- Condition: if . . . ≤ 0 then . . . else . . .
- \(\text{loop}(f, a, b) := u_a\) where \(u_0 = b\),

\[u_n = f(u_{n-1}, n)\]

- Two other loop constructs: \(\text{loop2}\), a while loop

Example:

\[2^x = \prod_{y=1}^{x} 2 = \text{loop}(2 \times x, x, 1)\]
\[x! = \prod_{y=1}^{x} y = \text{loop}(y \times x, x, 1)\]
Synthesize: tokens by tokens

OEIS sequence

\[ S = 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

(}
Synthesize: tokens by tokens

OEIS sequence

\[ S = 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

\( (x) \)
OEIS sequence

\[ S = 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

\[(x \times)\]
Synthesize: tokens by tokens

OEIS sequence

\[ S = 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

\( (x \times x) \)
Synthesize: tokens by tokens

OEIS sequence

\[ S = 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

\[ \left( x \times x + \right. \]
OEIS sequence

\[ S = 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

\[(x \times x + x)\]
OEIS sequence

\[ S = 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

\[ (x \times x + x) \]
Synthesize: tokens by tokens

OEIS sequence

\[ S = 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

\[ (x \times x + x) \div \]
Synthesize: tokens by tokens

OEIS sequence

\[ S = 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

\[ \left( x \times x + x \right) \div 2 \]
Synthesize: probabilistically

OEIS sequence

0, 1, 3, 6, 10, 15, ..., 1431

Synthesized program

\( (0.2 \ldots) \)
Synthesize: probabilistically

OEIS sequence

\[0, 1, 3, 6, 10, 15, \ldots, 1431\]

Synthesized program

\[
\begin{pmatrix}
0.2 & x_{0.3}
\end{pmatrix}
\]
Synthesize: probabilistically

OEIS sequence

0, 1, 3, 6, 10, 15, …, 1431

Synthesized program

\( (0.2 \times 0.3 \times 0.12) \)
Synthesize: probabilistically

OEIS sequence

\[ 0, 1, 3, 6, 10, 15, \ldots, 1431 \]

Synthesized program

\[
(0.2 \times 0.3 \times 0.12 \times 0.99)
\]
Synthesize: probabilistically

OEIS sequence

0, 1, 3, 6, 10, 15, …, 1431

Synthesized program

\( (0.2 \times 0.3 \times 0.12 \times 0.99 + 0.1) \)
Synthesize: probabilistically

OEIS sequence

0, 1, 3, 6, 10, 15, ... , 1431

Synthesized program

\((0.2 \times 0.3 \times 0.12 \times 0.99 + 0.1 \times 0.25)\)
Synthesize: probabilistically

OEIS sequence

0, 1, 3, 6, 10, 15, …, 1431

Synthesized program

\((0.2 \times 0.3 \times 0.12 \times 0.99 + 0.1 \times 0.25)0.48\)
Synthesize: probabilistically

OEIS sequence

0, 1, 3, 6, 10, 15, … , 1431

Synthesized program

\((0.2 \times 0.3 \times 0.12 \times 0.99 + 0.1 \times 0.25) \div 0.02\)
Synthesize: probabilistically

OEIS sequence

0, 1, 3, 6, 10, 15, ..., 1431

Synthesized program

\((0.2 \times 0.3 \times 0.99 + 0.1 \times 0.25) \times 0.48 \div 0.02 \times 2^{0.09}\)
Synthesize: probabilistically

OEIS sequence

0, 1, 3, 6, 10, 15, \ldots, 1431

Synthesized program

\[ (0.2 \times 0.3 \times 0.12 \times 0.99 + 0.1 \times 0.25) \times 0.48 \div 0.02 \times 2_{0.09} \]

The probability of generating this program is:

\[ 0.2 \times 0.3 \times 0.12 \times 0.99 \times 0.1 \times 0.25 \times 0.48 \times 0.02 \times 0.09 = 1.54\ldots \times 10^{-7} \]
Synthesize: probabilistically

OEIS sequence

\[0, 1, 3, 6, 10, 15, \ldots, 1431\]

Synthesized program

\[(0.2 \times 0.3 \times 0.12 \times 0.99 + 0.1 \times 0.25) \div 2 = 2.09\]

The probability of generating this program is:

\[0.2 \times 0.3 \times 0.12 \times 0.99 \times 0.1 \times 0.25 \times 0.48 \times 0.02 \times 0.09 = 1.54\ldots \times 10^{-7}\]

In general, we are given a probability function \(\mathcal{P}(S, P, T)\).

\[
\mathcal{P}([0, 1, 3, 6, \ldots] , \ [ ] , \ "(" , ) \ = \ 0.2
\]

\[
\mathcal{P}([0, 1, 3, 6, \ldots] , \ ["("] , \ x) \ = \ 0.3
\]

\[
\mathcal{P}([0, 1, 3, 6, \ldots] , \ ["(" , x] , \ \times) \ = \ 0.12
\]

\[
\mathcal{P}([0, 1, 3, 6, \ldots] , \ ["(" , x] , \ +) \ = \ 0.67
\]

\[
\mathcal{P}([2, 3, 5, 7, \ldots] , \ ["(" , x] , \ +) \ = \ 0.60
\]
Synthesize: updating a probabilistic function

Having synthesized the program (fastest so far) \((x \times x + x) \div 2\) generating

\[0, 1, 3, 6, 10, 15, \ldots, 1431\]

how do we update a probabilistic function \(P\)?

\[P_{desired}([0, 1, 3, 6, \ldots] \ , \ [] \ , \ "(" \ ) = 0.2 \rightarrow 1\]
\[P_{desired}([0, 1, 3, 6, \ldots] \ , \ ["("] \ , \ x ) = 0.3 \rightarrow 1\]
\[P_{desired}([0, 1, 3, 6, \ldots] \ , \ ["(" , \ x ) = 0.12 \rightarrow 1\]
\[P_{desired}([0, 1, 3, 6, \ldots] \ , \ ["(" , \ x ) = 0.67 \rightarrow 0\]

A neural network finds a smooth curve approximating \(P_{desired}\). Then \(P'\) is used to sample new programs from OEIS sequences.
Figure: Number $y$ of OEIS sequences (with at least one program) after $x$ iterations

$P_0$ is a random probability distribution function.  
$P_x$ synthesizes/samples 240 programs for each OEIS sequence.  
A program may be correct for an other OEIS sequence.  
$P_x$ is updated to $P_{x+1}$.  

Self-learning: five different runs
Let's now see some examples of synthesized programs.
A10445: squares modulo 84

OEIS sequence

0, 1, 4, 9, 16, 21, 25, 28, 36, 37, 49, 57, 60, 64, 72, 81

Synthesized program

\[ \{ x \mid (x^4 - x) \mod 84 = 0 \} \]

with 84 = 2 \times f^2(2) and \( f(x) = x \times x + x \)

Proof: Left to the listener.
Example: characteristic function of primes

OEIS sequence

0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, \ldots, 1, 0

Synthesized program

\[
(((x \times x!) \mod (1 + x)) \mod 2
\]

Proof: Left to the listener.
Example: prime numbers

OEIS sequence

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, …, 271

Synthesized program

\{ x \mid 2^x \equiv 2 \mod x \}

Proof: 341 is a counterexample
Example: A294082 by David Cerna

1, 1, 1, 1, 2, 1, 1, 4, 3, 1, 1, 14, 9, 4, 1, 1, 184, 75, 16, 5, 1, 1, 33674, 5553, 244 . . .

OEIS description: Square array read by antidiagonals:
\[ T(m, n) = T(m, n - 1)^2 - T(m, n - 2)^2 + T(m, n - 2) \text{ with } T(1, n) = 1, T(m, 0) = 1, \text{ and } T(m, 1) = m. \]

Program:

```
loop2(1 + (((x * x) - x) + y), y, 0 - (1 + loop(x - (if x <= 0 then 0 else y), x, x)),
1, loop(loop(y, x - y, x), x, x))
```

Proof: Left to the reader.
### Selection of 123 Solved Sequences

**Table: Samples of the solved sequences.**

| URL                                      | Description                                                                 |
|------------------------------------------|-----------------------------------------------------------------------------|
| https://oeis.org/A317485                 | Number of Hamiltonian paths in the $n$-Bruhat graph.                        |
| https://oeis.org/A349073                 | $a(n) = U(2*n, n)$, where $U(n, x)$ is the Chebyshev polynomial of the second kind. |
| https://oeis.org/A293339                 | Greatest integer $k$ such that $k/2^n < 1/e$.                               |
| https://oeis.org/A1848                   | Crystal ball sequence for 6-dimensional cubic lattice.                      |
| https://oeis.org/A8628                   | Molien series for $A_5$.                                                    |
| https://oeis.org/A259445                 | Multiplicative with $a(n) = n$ if $n$ is odd and $a(2^s) = 2$.              |
| https://oeis.org/A314106                 | Coordination sequence Gal.6.199.4 where G.u.t.v denotes the coordination sequence for a vertex of type v in tiling number t in the Galebach list of u-uniform tilings |
| https://oeis.org/A311889                 | Coordination sequence Gal.6.129.2 where G.u.t.v denotes the coordination sequence for a vertex of type v in tiling number t in the Galebach list of u-uniform tilings. |
| https://oeis.org/A315334                 | Coordination sequence Gal.6.623.2 where G.u.t.v denotes the coordination sequence for a vertex of type v in tiling number t in the Galebach list of u-uniform tilings. |
| https://oeis.org/A315742                 | Coordination sequence Gal.5.302.5 where G.u.t.v denotes the coordination sequence for a vertex of type v in tiling number t in the Galebach list of u-uniform tilings. |
| https://oeis.org/A004165                 | OEIS writing backward                                                        |
| https://oeis.org/A83186                  | Sum of first n primes whose indices are primes.                             |
| https://oeis.org/A88176                  | Primes such that the previous two primes are a twin prime pair.             |
| https://oeis.org/A96282                  | Sums of successive twin primes of order 2.                                  |
| https://oeis.org/A53176                  | Primes $p$ such that $2p + 1$ is composite.                                 |
| https://oeis.org/A267262                 | Total number of OFF (white) cells after $n$ iterations of the "Rule 111" elementary cellular automaton starting with a single ON (black) cell. |
Thank you for your attention!

https://github.com/Anon52MI4/oeis-alien