Chiral currents and static properties of nucleons in holographic QCD

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Abstract

We analyze static properties of nucleons in the two flavor holographic QCD model of Sakai and Sugimoto described effectively by a five-dimensional \( U(2) \) Yang-Mills theory with the Chern-Simons term on a curved background. The baryons are represented in this model as a soliton, which at a time slice is approximately the BPST instanton with a fixed size. First, we construct a chiral current in four dimensions from the Noether current of local gauge transformations which are non-vanishing on the boundaries of the extra dimension. We examine this chiral current for nucleons with quantized collective coordinates to compute their charge distribution, charge radii, magnetic moments and axial vector coupling. Most of the results are better close to the experimental values than in the Skyrme model. We discuss the problems of our chiral current; non-uniqueness of the local gauge transformation for defining the current, and its gauge-noninvariance.

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1 Introduction

Holographic QCD of Sakai and Sugimoto [1, 2] has attracted much attention recently as a phenomenologically and theoretically interesting model. This model is based on a configuration of $N_f$ D8-$\bar{D}8$-branes and $N_c$ D4-branes, and the D8-D4, $\bar{D}8$-D4 and D4-D4 open strings correspond to the left-handed quark, the right-handed quark and the gluon, respectively. Spontaneous chiral symmetry breaking is implemented in this model by the fact the D8- and $\bar{D}8$-branes are smoothly connected. Using holographic approximation to replace the D4-branes with the near-horizon geometry of the corresponding SUGRA solution, the low energy effective action of the D8-branes is expressed as a five-dimensional Yang-Mills (YM) theory with the Chern-Simons (CS) term on a curved background. (We call this five-dimensional YM+CS theory simply the SS-model hereafter.) Mode-expanding the fields with respect to the coordinate $z$ of the extra fifth dimension, we obtain a four-dimensional theory consisting of the Nambu-Goldstone boson and an infinite tower of massive vector mesons. Their coupling constants are determined by only two parameters of the original theory; the ’t Hooft coupling $\lambda$ and the mass scale $M_{KK}$ which specifies the radius of the $S^1$ compactification of the D4-brane.

In the low energy limit of discarding all the massive vector mesons in the SS-model, we get the Skyrme model [3], a low energy effective theory of Nambu-Goldstone bosons. The Skyrme model has a novel property that it has a stable soliton solution describing baryons. Upon quantization of the collective coordinate of the $SU(N_f)$ rotation of the baryon solution, the static properties of nucleons such as masses, charge radii and the magnetic moments have been computed [4]. The results agree fairly well with experiments with of course a number of exceptions.

The SS-model, namely, the five-dimensional YM+CS theory mentioned above, also has a soliton solution which describes the baryon [5, 6, 7]. This baryon solution at a time slice is approximately a BPST instanton with a fixed size in flat four-dimensional YM theory. In [5], the collective coordinate quantization of the baryon solution is carried out in the $N_f = 2$ case to give the baryon spectra containing negative-parity baryons as well as the baryons with higher spins and isospins. By taking $M_{KK}$ of about 500 MeV, the obtained baryon spectra is in qualitatively good agreement with experiment. The extension to the $N_f = 3$ case is not so straightforward: the original CS term of [1, 2] cannot give the desired first class constraint which selects baryon states with correct spins. In [8], the collective coordinate quantization in the $N_f = 3$ case was carried out by using a new CS term which reproduces the desired constraint.

The purpose of this paper is to continue and accomplish the study of static properties of nucleons in the SS-model with two flavors [9]. We will examine the charge distributions, charge radii, magnetic moments and the axial vector coupling, namely, the quantities studied in [4] in the $N_f = 2$ Skyrme model. The most nontrivial and difficult point in this study is how to define, in the SS-model which is a five-dimensional YM+CS theory, the chiral $U(N_f)_L \times U(N_f)_R$ current observed in our four dimensional spacetime. One way to define the chiral current is

\* * See [7] and [9], for a different approach to the properties of nucleons in holographic QCD.
via the bulk-boundary correspondence \cite{10,11}: we introduce the external gauge fields on the boundaries \( z = \pm \infty \) of the extra fifth dimension \( z \), and read off the current from the coupling with the external fields. The current obtained this way is defined in terms of fields on the boundary. Therefore, it is invariant under gauge transformations which does not change the boundary behavior of the fields. However, we do not adopt this current in our study of static properties of nucleons since the baryon solution is localized near the origin \( z = 0 \) of the extra fifth dimension and hence the current vanishes for the baryon configuration (see, however, sec. 4.5 for a possibility of getting non-vanishing results for static properties of nucleons).

In this paper, we take another definition of the chiral current. For introducing it, let us recall that the chiral transformation in four dimensions corresponding to \((g_L, g_R) \in U(N_f)_L \times U(N_f)_R\) is realized in the SS-model in five dimensions as the local gauge transformation taking the values \( g_L \) and \( g_R \) on the boundaries \( z = \infty \) and \(-\infty\), respectively. Therefore, we first consider in the SS-model the conserved Noether current of the infinitesimal local gauge transformation which is non-vanishing at \( z = \pm \infty \). Then, the chiral current in four dimensions is obtained by integrating the five-dimensional current over \( z \). However, the chiral current defined this way has two problems which are related each other. First, this chiral current is not determined uniquely by the gauge transformation on the boundary \( z = \pm \infty \), but it depends on the way of interpolating the local gauge transformation for finite \( z \). Second, this chiral current is not a gauge invariant quantity. It is not invariant even under the local gauge transformation that does not change the behavior of the fields at \( z = \pm \infty \).

Despite these problems, we adopt the chiral current defined from the Noether current of local gauge transformation to compute the static properties of nucleons. As the interpolating function of \( z \) of the gauge transformation, we take the zero-modes of the Laplace operator of \( z \). There are two facts that supports our choice of chiral current. One is that it reproduces the chiral current of the Skyrme model in the low energy limit of discarding all the massive \( z \)-modes. The second fact is our result of the computation of the static properties of nucleons itself. We find that our results in the SS-model are generally closer to the experimental values than in the Skyrme model \cite{4}. In particular, we get a surprising improvement for the axial vector coupling compared with the Skyrme model.

The organization of the rest of this paper is as follows. In sec. 2, we introduce our chiral current from the Noether current of local gauge transformation. We also discuss the relationship of our current to the chiral current from the bulk-boundary correspondence, and the problems of our current. In sec. 3, we consider the low energy limit of our chiral current and show that it reproduces the chiral current of the Skyrme model. Sec. 4 is the main part of this paper. We first summarize the baryon classical solution in the SS-model and its collective coordinate quantization. Then, we compute the various static properties of nucleon using our chiral current. In sec. 5, we summarize the paper and discuss remaining problems. In appendix A, we examine the vector current in terms of the baryon solution in the regular gauge to explain why we have to take the singular gauge solution in the analysis of sec. 4.
2 Four-dimensional chiral current in the SS-model

For analyzing the static properties of nucleons such as the electric charge density, magnetic moments and axial-vector coupling, we have to first of all define the conserved four-dimensional current of chiral $U(N_f)_L \times U(N_f)_R$ symmetry in the SS-model which is a YM+CS theory in five-dimensions. In this section, we propose a definition of the conserved chiral current which will be used in later sections for the various analysis. Our chiral current is given in terms of five-dimensional Noether current of local gauge symmetry transformation which is non-vanishing on the boundaries $z = \pm \infty$. We will also discuss the relationship of our chiral current with that defined via the bulk-boundary correspondence \cite{10, 11}, and the problems with our current, i.e., the uniqueness and the gauge-invariance.

2.1 Action of the SS-model

First, let us summarize the action of the SS-model, which is the effective theory of $N_f$ probe D8-branes in the background of $N_c$ D4-branes. Concretely, it is a Yang-Mills (YM) theory in five dimensions with gauge group $U(N_f)$ supplemented with the Chern-Simons (CS) term:

$$S[A] = S_{YM}[A] + S_{CS}[A],$$

$$S_{YM}[A] = -\kappa \int d^4x dz \, \text{tr} \left[ \frac{1}{2} h(z) F_{\mu\nu} F^{\mu\nu} + k(z) F_{\mu z} F^{\mu z} \right],$$

$$S_{CS}[A] = \frac{N_c}{24\pi^2} \int \omega_5(A),$$

where $\mu, \nu = 0, 1, 2, 3$ are the four-dimensional Lorentz indices and $z$ is the extra fifth dimension, and they are raised/lowered by the flat metric $\eta_{MN} = \eta^{MN} = \text{diag}(-1, 1, 1, 1, 1)$ ($M, N = 0, 1, 2, 3, z$). The one-form $A = A_\mu dx^\mu + A_z dz$ is the $U(N_f)$ gauge field, and the corresponding field strength is given by $F = dA + iA^2$. In $S_{YM}$ (2.2), $\kappa$ is a constant given in terms of the 't Hooft coupling $\lambda$ and the number of colors $N_c$ as

$$\kappa = a\lambda N_c, \quad \left( a = \frac{1}{216\pi^3} \right),$$

and the warp factors $h(z)$ and $k(z)$ are

$$h(z) = (1 + z^2)^{-1/3}, \quad k(z) = 1 + z^2.$$

In the CS term $S_{CS}$ (2.3), $\omega_5(A)$ is the CS five-form:

$$\omega_5(A) = \text{tr} \left( A^2 - \frac{i}{2} A^3 F - \frac{1}{10} A^5 \right).$$

In \cite{8}, another expression of the CS term is proposed by extending the gauge field $A$ to a six-dimensional space $M_6$ whose boundary $\partial M_6$ is the original five-dimensional space of the SS-model:

$$S^\text{new}_{CS} = \frac{N_c}{24\pi^2} \int_{M_6} \text{tr} F^3.$$
Although the two expressions (2.3) and (2.7) are naively the same due to $\text{tr} F^3 = d\omega_5(\mathcal{A})$, it is crucial to adopt (2.7) in the case of $N_f = 3$ for reproducing the baryon states with correct spins [8]. However, for the chiral current in the $N_f = 2$ case, there is essentially no difference between the two CS terms.

The equations of motion (EOM) of $\mathcal{A}_\mu$ and $\mathcal{A}_z$ obtained from the action (2.1) read respectively as follows:

\begin{align}
2\kappa \left[ D_\nu (h(z)F^{\nu\mu}) + D_z (k(z)F^{z\mu}) \right] + \frac{N_c}{32\pi^2} \epsilon^{\muNPQR} F_{NP} F_{QR} &= 0, \quad (2.8) \\
2\kappa D_\mu (k(z)F^{\mu z}) + \frac{N_c}{32\pi^2} \epsilon^{\mu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} &= 0, \quad (2.9)
\end{align}

with $\epsilon^{0123z} = \epsilon^{0123} = 1$. Here, we have used that the CS five-form changes under an infinitesimal deformation $\delta \mathcal{A}$ of the gauge field $\mathcal{A}$ as

$$\delta \omega_5(\mathcal{A}) = 3 \text{tr}(\delta \mathcal{A} F^2) + d\beta(\delta \mathcal{A}, \mathcal{A}),$$

with $\beta$ given by

$$\beta(\delta \mathcal{A}, \mathcal{A}) = \text{tr} \left[ \delta \mathcal{A} \left( \mathcal{A} + \mathcal{A}^2 - \frac{i}{2} \mathcal{A}^3 \right) \right].$$

The last $\epsilon$-tensor terms in (2.8) and (2.9) remain unchanged even if we adopt another CS term (2.7).

### 2.2 Conserved current of local gauge symmetry

In order to propose a definition of the chiral current in the SS-model, let us first recall how the $U(N_f)_L \times U(N_f)_R$ symmetry in four dimensions is realized in the model. In the SS-model in a gauge with the boundary condition,

$$\mathcal{A}_M(x,z) \to 0 \quad (z \to \pm \infty),$$

the pion field $U(x)$ in four dimensions is defined by [1, 2]

$$U(x) = P \exp \left( -i \int_{-\infty}^{\infty} dz \mathcal{A}_z(x,z) \right),$$

where the path-ordering is from the right at $z = -\infty$ to the left at $z = +\infty$. Then, the chiral transformation corresponding to $(g_L, g_R) \in U(N_f) \times U(N_f)_R$ is realized as the local gauge transformation

$$\mathcal{A} \to \mathcal{A}^g = g(\mathcal{A} - id)g^{-1},$$

with $g(x,z) \in U(N_f)$ which tends to constants as $z \to \pm \infty$,

$$g(x,z) \to \begin{cases} g_L & (z \to +\infty) \\ g_R & (z \to -\infty) \end{cases}.$$
and hence keeps the boundary condition (2.12). In fact, under this local gauge transformation, the pion field transforms as

\[ U(x) \rightarrow g_L U(x) g_R^{-1}. \] (2.16)

Motivated by the realization of the chiral symmetry transformation in the SS-model as a local gauge transformation, we propose the following construction of the conserved chiral current in four dimensions:

1. First, let us consider the conserved Noether current in five-dimensions, \( J^M_\zeta (x, z) \), corresponding to the infinitesimal local gauge symmetry transformation,

\[ \delta_\zeta A_M = \mathcal{D}_M \zeta = \partial_M \zeta + i[A_M, \zeta], \] (2.17)

with \( \zeta(x, z) \) satisfying the boundary condition

\[ \zeta(x, z) \rightarrow \begin{cases} 
\zeta_L = \zeta_L^a t_a & (z \rightarrow +\infty) \\
\zeta_R = \zeta_R^a t_a & (z \rightarrow -\infty) 
\end{cases}, \] (2.18)

where \( \zeta_L^a, \zeta_R^a \) are arbitrary constants, and \( t_a \ (a = 0, 1, 2, \cdots, N_f^2 - 1) \) is the hermitian generator of \( U(N_f) \) satisfying

\[ [t_a, t_b] = if_{abc} t_c, \quad \text{tr}(t_a t_b) = \frac{1}{2} \delta_{ab}, \quad t_0 = \frac{1}{\sqrt{2N_f}} 1_{N_f}. \] (2.19)

2. Then, we define the conserved four-dimensional current \( j_\mu^\zeta(x, z) \) by

\[ j_\mu^\zeta(x) = \int_{-\infty}^{\infty} dz J^\mu_\zeta(x, z). \] (2.20)

Since \( J^M_\zeta (x, z) \) is conserved in the five-dimensional sense,

\[ \partial_M J^M_\zeta = \partial_\mu J^\mu_\zeta + \partial_z J^z_\zeta = 0, \] (2.21)

the four-dimensional current \( j_\mu^\zeta(x) \) satisfies the conservation law

\[ \partial_\mu j_\mu^\zeta = 0, \] (2.22)

provided the \( z \)-component \( J^z_\zeta \) vanishes on the boundary:

\[ J^z_\zeta(x, z \rightarrow \pm\infty) = 0. \] (2.23)

3. It follows (using (2.27) and (2.28) given below) that the five-dimensional current \( J^M_\zeta(x, z) \) and the pion field \( U(x) \) (2.13) satisfy the Ward identity

\[ \partial_M \langle J^M_\zeta(x, z) U(x') O \rangle = -\delta^4(x - x') \langle U(x; \infty, z) \mathcal{D}_\zeta \zeta(x, z) U(x; z, -\infty) O \rangle + \cdots, \] (2.24)

\[ \text{If we take into account the gauge-fixing and the Faddeev-Popov ghost terms necessary for the quantization, the current } J^M_\zeta \text{ is no longer conserved for a generic } \zeta \text{ but its divergence is given by a BRST-exact form} \] [12, 13],

\[ \partial_M J^M_\zeta = \{ Q_B, \ast \}, \]

with \( Q_B \) being the BRST charge. However, the Ward identities (2.24) and (2.28) remain valid since the pion field is a physical operator satisfying \( [Q_B, U(x)] = 0 \) (we assume that the Faddeev-Popov ghost \( c(x, z) \) vanishes at infinity).
where $U(x; z_1, z_2)$ is defined by

$$U(x; z_1, z_2) = \mathrm{P} \exp \left( -i \int_{z_2}^{z_1} dz \mathcal{A}_z(x, z) \right), \quad (2.25)$$

and $\mathcal{O}$ and the dots on the RHS represent other operators and the terms containing their gauge transformation, respectively. Integrating \((2.24)\) over $z$ and using \((2.23)\), we obtain

$$\partial_\mu \langle j^\mu_\zeta(x') U(x') \mathcal{O} \rangle = -\delta^4(x - x') \langle (\zeta_L U(x) - U(x) \zeta_R) \mathcal{O} \rangle + \ldots . \quad (2.26)$$

This is the desired chiral Ward identity in four dimensions. The four-dimensional current $j^\mu_\zeta(x)$ with $\zeta_R/L = 0$ is the left/right-current.

The conserved five-dimensional current $J^M_\zeta(x, z)$ is obtained by making on the action the infinitesimal transformation,

$$\delta A_M(x, z) = \epsilon(x, z) D_M \zeta(x, z), \quad (2.27)$$

with $\epsilon(x, z)$ vanishing at infinity, and using the identification,

$$\delta S = \int d^4xdz J^M_\zeta(x, z) \partial_M \epsilon(x, z). \quad (2.28)$$

For the action \((2.1)\), we get

$$J^M_\zeta = J^M_{\mathrm{YM}\zeta} + J^M_{\mathrm{CS}\zeta}, \quad (2.29)$$

with $J^M_{\mathrm{YM}\zeta}$ and $J^M_{\mathrm{CS}\zeta}$ from $S_{\mathrm{YM}}$ \((2.2)\) and $S_{\mathrm{CS}}$ \((2.3)\), respectively, given by

$$J^\mu_{\mathrm{YM} \zeta}(x, z) = -2\kappa \mathrm{tr} \left( h(z) F^{\mu
u} D_\nu \zeta + k(z) F^{\mu z} D_z \zeta \right), \quad (2.30)$$

$$J^\mu_{\mathrm{YM} \zeta}(x, z) = -2\kappa k(z) \mathrm{tr} \left( F^{\mu
u} D_\nu \zeta \right), \quad (2.31)$$

$$J^M_{\mathrm{CS} \zeta}(x, z) = -\frac{N_c}{64\pi^2} \epsilon^{MNPQR} \mathrm{tr} \left( \{ F_{NP}, F_{QR} \} \zeta \right). \quad (2.32)$$

Using the EOM, \((2.8)\) and \((2.9)\), we can confirm that $J^M_\zeta$ satisfies the conservation equation \((2.21)\).

### 2.3 Relation to the current from bulk-boundary correspondence

In this subsection, let us consider the relationship of our chiral current introduced above with the chiral current $\tilde{j}_{\mu}^{L/R}(x)$ defined a la the bulk-boundary correspondence [10, 11]. The latter is obtained as follows. We solve the EOM under the boundary condition,

$$A_\mu(x, z) \rightarrow \begin{cases} A_\mu^L(x) & (z \rightarrow +\infty) \\ A_\mu^R(x) & (z \rightarrow -\infty) \end{cases} \quad (2.33)$$
insert the solution depending on $A_{\mu}^{L/R}$ into the action, and read off the currents $\tilde{j}_L^{\mu}(x)$ from the coupling
\[ \int d^4x \, \text{tr} \left( A_{\mu}^{L}(x) \tilde{j}_L^{\mu}(x) + A_{\mu}^{R}(x) \tilde{j}_R^{\mu}(x) \right). \] (2.34)

This is equivalent to making the following shift in the action,
\[ A_\mu(x, z) \to A_\mu(x, z) + A_{\mu}^{L}(x)\psi_L(z) + A_{\mu}^{R}(x)\psi_R(z), \] (2.35)

with $\psi_{L/R}(z)$ being functions satisfying the boundary condition
\[ \psi_{L/R}(z) \to \begin{cases} 1 & (z \to \pm\infty) \\ 0 & (z \to \mp\infty) \end{cases}, \] (2.36)

and keeping terms linear in $A_{\mu}^{L/R}(x)$ by using that $A_M(x, z)$ is a solution to the EOM with the boundary condition (2.12). We find that the chiral current $\tilde{j}_L^{\mu}(x)$ obtained this way is
\[ \tilde{j}_L^{\mu}(x) = \mp 2\kappa k(z)\mathcal{F}^{\mu z}(x, z) \big|_{z=\pm\infty}. \] (2.37)

The relation between the two currents, (2.20) and (2.37), is as follows. First, the five-dimensional current (2.29) is rewritten as
\[ J_\zeta^{\mu}(x, z) = -2\kappa \partial_\nu \text{tr} \left[ h(z)\mathcal{F}^{\mu\nu}(x, z)\zeta(x, z) \right] \]
\[ -2\kappa \partial_\zeta \text{tr} \left[ k(z)\mathcal{F}^{\zeta\nu}(x, z)\zeta(x, z) \right] - \text{tr} \left[ (\text{LHS of (2.8)})\zeta(x, z) \right]. \] (2.38)

Integrating this over $z$, we find that
\[ \tilde{j}_L^{\mu}(x) = \text{tr} \left( \zeta_L \tilde{j}_L^{\mu}(x) + \zeta_R \tilde{j}_R^{\mu}(x) \right) + \partial_\nu \chi^{\mu\nu}(x) + (\text{EOM-term}), \] (2.39)

where $\zeta_{L/R}$ are the Lie-algebra valued constants of (2.18), and the anti-symmetric tensor $\chi^{\mu\nu}(x)$ is defined by
\[ \chi^{\mu\nu}(x) = -\chi^{\nu\mu}(x) = -2\kappa \int_{-\infty}^{\infty} dz \, \text{tr} \left[ h(z)\mathcal{F}^{\mu\nu}(x, z)\zeta(x, z) \right]. \] (2.40)

This relation implies that the two currents, $j_\zeta^{\mu}(x)$ and $\tilde{j}_L^{\mu}(x)$, are “equivalent” in the sense that their difference is an identically conserving term $\partial_\nu \chi^{\mu\nu}$. In particular, the integrated charges $Q = \int d^3x \, j_\zeta^{0}(x)$ are the same between the two if we are allowed to discard the surface integral from the $\partial_\nu \chi^{0i}$ term. However, the local currents themselves do differ from each other, and this difference can lead to different physics. Moreover, we will see in sec. 4 that, in the quantization of baryons, $\tilde{j}_L^{\mu}(x)$ vanishes and the main contribution to $j_\zeta^{\mu}(x)$ (2.39) is from the $\partial_\nu \chi^{\mu\nu}(x)$ term.

\[ \tilde{j}_{\text{CS L/R}}^{\mu}(x) = \mp \frac{N_c}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} \left( \mathcal{F}_{\nu\rho}A_\sigma + A_\nu \mathcal{F}_{\rho\sigma} - iA_\nu A_\rho A_\sigma \right) \big|_{z=\pm\infty}, \]
from the CS term (2.3) due to the $d\beta$ term in (2.10). Here, we have ignored this $\tilde{j}_{\text{CS L/R}}^{\mu}(x)$ since it vanishes at $z = \pm\infty$, in contrast with (2.37) which is multiplied by $k(z)$ (2.5). We have no $\tilde{j}_{\text{CS L/R}}^{\mu}(x)$ at all in the case of another CS term (2.7) since we simply have $\delta \text{tr} \mathcal{F}^3 = 3 d\text{tr}(\delta A \mathcal{F}^2).$
2.4 Gauge non-invariance of $j_\mu^\zeta$ and non-uniqueness of $\zeta$

Here, we discuss the problems inherent in our four-dimensional chiral current $j_\mu^\zeta(x)$ (2.20) itself. First recall that our chiral current should be considered, by its construction, for the gauge fields $A_M$ satisfying the boundary condition (2.12). The most serious problem with our chiral current $j_\mu^\zeta(x)$ is that it is not a gauge invariant quantity. It is not invariant even under the gauge transformation which keeps the boundary condition (2.12). The integrated conserved charge $Q_\zeta = \int d^3x j_0^\zeta(x)$, which can be expressed as a spatial surface integration as seen from (2.38), is invariant under the gauge transformation which does not change the boundary behavior of $A$. However, since we have to treat the local currents $j_\mu^\zeta(x)$ itself in the analysis of static properties of nucleons, the gauge-noninvariance of our chiral current is an important problem to be resolved.

Another and related problem with our chiral current is the non-uniqueness of the function $\zeta(x,z)$ defining the current. In the construction of our chiral current given in sec. 2.2, the only condition on $\zeta(x,z)$ is the boundary condition (2.18), which of course cannot uniquely determine $\zeta(x,z)$. This problem of the non-uniqueness of $\zeta(x,z)$ is intimately related with the gauge-noninvariance of our chiral current. Note that the five-dimensional current $J_M^\zeta$ (2.29) and hence the four-dimensional one $j_\mu^\zeta(x)$ are invariant under the simultaneous gauge transformation of both the gauge field $A_M(x,z)$ and $\zeta(x,z)$:

$$A_M \rightarrow A'^g = g(A_M - i\partial_M)g^{-1},$$

$$\zeta \rightarrow \zeta'^g = g\zeta g^{-1}.$$ (2.41)

This fact implies that we cannot choose a specific function $\zeta(x,z)$ for our chiral current: For giving the same chiral current among different $A$’s related to each other via gauge transformations which keep the boundary behavior, we have to adjust the corresponding $\zeta(x,z)$ according to the formula (2.41).

On the other hand, the current $\tilde{j}_\mu^{L/R}(x)$ (2.37) of the bulk-boundary correspondence are free from both of the above two problems. In particular, since it is defined by the fields on the boundaries $z = \pm \infty$, it is inert under local gauge transformations which do not change the boundary behavior of the fields. However, the current $\tilde{j}_\mu^{L/R}(x)$ does not seem to give physically sensible results. First, $\tilde{j}_\mu^{L/R}(x)$ can reproduce only a part of the chiral current of the Skyrme model in the low energy limit (see the end of sec. 3.2). Second, $\tilde{j}_\mu^{L/R}(x)$ vanishes for the baryon configuration since the baryon is localized at the origin $z = 0$ and cannot be seen on the boundary (see the end of sec. 4.1).

We do not have definite solutions to the problems of our chiral current $j_\mu^\zeta(x)$. However, in the following sections, we continue our analysis by using the present chiral current (2.20) with $J_M^\zeta$ given by (2.29)–(2.32). For definiteness, we adopt as $\zeta(x,z)$ the following one:

$$\zeta(x,z) = \psi_\pm(z)t_a,$$ (2.42)

where $\psi_\pm(z)$ are the zero-modes of the differential operator $-k(z)^{1/3}\partial_z k(z)\partial_z$ [1, 2]:

$$\psi_\pm(z) = \frac{1}{2} \pm \frac{1}{\pi} \arctan z \rightarrow \begin{cases} 1 & (z \rightarrow \pm \infty) \\ 0 & (z \rightarrow \mp \infty). \end{cases}$$ (2.43)
This $\psi_\pm(z)$ has been adopted in \cite{1,2} as $\psi_{L/R}(z)$ of (2.35) in introducing the external fields $A^{L/R}_\mu(x)$. Since $\psi_\pm(z)$ is the zero-mode, no mass terms are developed for $A^{L/R}_\mu(x)$. However, we have no convincing reason for adopting $\psi_\pm(z)$ in the context of the Noether current of local gauge transformation. The only reasons why we adopt our chiral current (2.20) with $\zeta(x, z)$ of (2.42) are that it reproduces in the low energy limit the chiral current of the Skyrme model, and that the various static properties of nucleons are obtained as non-vanishing and finite numbers which are close to the experimental values, as we will see in secs. 3 and 4, respectively.

Explicitly, the chiral current $j^\mu_{L/R,a}(x)$ corresponding to (2.42) is given by

$$ j^\mu_{L/R,a}(x) = \int_{-\infty}^{\infty} dz \, J^\mu_{L/R,a}(x, z), $$

(4.44)

with

$$ J^\mu_{L/R,a}(x, z) = -2i\kappa \left\{ \left( \bar{h}(z) [F^{\mu\nu}, A^\nu] + k(z) [F^{\mu z}, A^z] \right) t_a \right\} \psi_\pm(z) $$

$$ -2\kappa k(z) \text{tr}(F^{\mu z} t_a) \frac{d\psi_\pm(z)}{dz} - \frac{N_c}{64\pi^2} \epsilon^{\mu NPQR} \text{tr} \left( \{ F_{NP}, F_{QR} \} t_a \right) \psi_\pm(z). $$

(4.45)

The vector and the axial-vector currents are obtained by replacing $\psi_\pm(z)$ in (4.45) with the following $\psi_V(z)$ and $\psi_A(z)$, respectively:

$$ \psi_V(z) = \psi_+(z) + \psi_-(z) = 1, \quad \psi_A(z) = \psi_+(z) - \psi_-(z) = \frac{2}{\pi} \arctan z. $$

(4.46)

In this case, the $z$-component of the five-dimensional current which we have to confirm to vanish at $z = \pm\infty$ (recall (2.23)) is

$$ J^2_{L/R,a}(x, z) = -2i\kappa k(z) \text{tr} \left( [F_{z\nu}, A^\nu] t_a \right) \psi_\pm(z) - \frac{N_c}{32\pi^2} \epsilon^{\mu\rho\sigma} \text{tr} \left( F_{\mu\nu} F_{\rho\sigma} t_a \right) \psi_\pm(z). $$

(4.47)

### 3 Chiral currents in the Skyrme approximation

In the last section, we proposed a definition of the four-dimensional chiral current $j^\mu_{\zeta}(x)$ (2.20) as the $z$-integration of the five-dimensional Noether current $J^\mu_{\zeta}(x, z)$ of the local gauge symmetry transformation with the boundary condition (2.18). In this section, as a test of the validity of this definition of the chiral current in the SS-model, we evaluate it in the Skyrme approximation (i.e., in the low energy limit). It has been known that the SS-model is reduced to the Skyrme model in the low energy limit \cite{1,2}. The Skyrme model is an ordinary four-dimensional field theory having the pion field (Skyrme field) $U(x) \in U(N_f)$

---

$\dagger$ If we take $A^{L/R}_\mu$ of the form $A^{L/R}_\mu(x) = \partial_M \varepsilon^{L/R}_a(x) t_a$, the shift (2.35) is essentially equivalent to the infinitesimal transformation (2.27) with $\zeta(x, z)$ given by (2.42) with the identification $\psi_{L/R}(z) = \psi_\pm(z)$: The shift of $F_{MN}$ under (2.35) is $\partial_M \varepsilon^L_a(x) D_M(\psi_+(z) t_a) + \partial_N \varepsilon^R_a(x) D_M(\psi_-(z) t_a)$, while we have $\delta F_{MN} = \varepsilon [F_{MN}, \zeta] + (\partial_M \varepsilon) D_N \zeta$ under (2.24).
as its dynamical variable, and the chiral currents of the symmetry transformation (2.16) is simply the corresponding Noether current. Using this Noether current of chiral symmetry, various properties of the model including the static properties of nucleons have been analyzed [4]. What we wish to test here is whether the chiral current proposed in sec. 2 is reduced to the chiral current in the Skyrme model.

### 3.1 SS-model in the low energy limit

Before considering the low energy limit of our chiral current in the SS-model, we in this subsection review the derivation of the Skyrme model as the low energy limit of the SS-model. For this purpose, it is convenient to move, from the gauge with the boundary condition (2.12), to the gauge with $A_z(x, z) = 0$. Let us take as the function $g(x, z) \in U(N_f)$ in (2.14) for realizing $A_z(x, z) = 0$ the following one:

$$g(x, z) - 1 = P \exp \left(-i \int_{-\infty}^{z} dz' A_z(x, z') \right).$$

(3.1)

In the rest of this section, the gauge fields $A$ and $A^g$ denote the one with the boundary condition (2.12) and that in the $A_z^g = 0$ gauge, respectively.

In the low energy limit of the SS-model in the $A_z^g = 0$ gauge, we mode-expand the $z$-dependence of $A^g_\mu(x, z)$ in terms of the eigenfunctions of $-k(z)^{1/3} \partial_z k(z) \partial_z$ and keep only the zero-modes $\psi_\pm(z)$ (2.43). Taking into account that

$$A_\mu^g(x, z) \rightarrow \begin{cases} R_\mu(x) & (z \rightarrow +\infty) \\ 0 & (z \rightarrow -\infty) \end{cases},$$

(3.2)

with $R_\mu(x)$ defined by

$$R_\mu(x) = -iU(x)^{-1}\partial_\mu U(x),$$

(3.3)

we have in the present approximation

$$A^g_\mu(x, z) = R_\mu(x)\psi_+(z),$$

(3.4)

where all the massive-modes have been dropped on the RHS. Then, the field strengths are given by

$$F^g_{\mu
u}(x, z) = -i [R_\mu(x), R_\nu(x)] \psi_+(z) \psi_-(z),$$

(3.5)

$$F^g_{z\nu}(x, z) = R_\nu(x) \frac{d\psi_+(z)}{dz}.$$  

(3.6)

Plugging the expressions (3.5) and (3.6) into the YM part action (2.2) with $F_{MN}$ replaced by the gauge transformed one $F^g_{MN}$ and carrying out the $z$-integration, we get the four-dimensional Skyrme lagrangian of the field $U(x)$ (2.13) in the low energy limit:

$$L_U = \kappa \text{tr} \left( -\frac{1}{\pi} R_\mu R^\mu + \frac{c_S}{2} [R_\mu, R_\nu] [R^\mu, R^\nu] \right),$$

(3.7)
where $c_S$ is a constant given by

\[
c_S = \int_{-\infty}^{\infty} dz \, h(z) \left[ \psi_+ (z) \psi_- (z) \right]^2 = 0.156 \ldots . \tag{3.8}
\]

The expression of the low energy limit of the CS term depends on whether we adopt the original one (2.3) or another one (2.7) proposed in [8] (see sec. 5.5 of [1] for the former, and appendix D of [8] for the latter). In both the cases, the CS term is reduced in the low energy limit to $N_c \Gamma_{WZW}[U]$ with the Wess-Zumino-Witten term $\Gamma_{WZW}[U]$ given by an apparently common form:

\[
\Gamma_{WZW}[U] = \frac{1}{240 \pi^2} \int \text{tr}(-i U^{-1} dU)^5. \tag{3.9}
\]

For the original CS term of (2.3), the integration is over the five-dimensional space-time of $(x^\mu, z)$, and $U$ is $g(x, z)$ of (3.1). In the case of (2.7), the integration of (3.9) is over the five-dimensional space-time, $M_6$ with the $z$ part removed, and $U$ is given by (2.13) with $A_z$ replaced by that on $M_6$. In summary, the low energy limit of the SS-model is the Skyrme model with the WZW term:

\[
S_{\text{Skyrme}}[U] = \int d^4 x \, \mathcal{L}_U + N_c \Gamma_{WZW}[U]. \tag{3.10}
\]

### 3.2 Low energy limit of the chiral currents in the SS-model

Now we wish to show that our chiral current in the SS-model, $j_{L/R, a}^\mu(x)$ (2.44) with (2.45), reduces in the low energy limit to the corresponding one in the Skyrme model (3.10). For this purpose we first look more carefully at the process of moving to the $A^g_z = 0$ gauge.

In the low energy limit of the SS-model in a gauge with the boundary condition (2.12), $A_z(x, z)$ has only the mode $\hat{\phi}_0(z)$,

\[
\hat{\phi}_0(z) = \pm \frac{d\psi_\pm(z)}{dz} = \pm \frac{1}{\pi} \frac{1}{1 + z^2}, \tag{3.11}
\]

and is given by

\[
A_z(x, z) = \varphi(x) \hat{\phi}_0(z), \tag{3.12}
\]

in terms of a Lie-algebra valued function $\varphi(x)$. Then, $g(x, z)^{-1}$ (3.11) for moving to the $A^g_z = 0$ gauge and the pion field $U(x)$ (2.13) are given in terms of $\varphi(x)$ as

\[
g(x, z)^{-1} = \exp(-i \varphi(x) \psi_+ (z)), \tag{3.13}
\]
\[
U(x) = g(x, z = \infty)^{-1} = \exp(-i \varphi(x)). \tag{3.14}
\]

For this $g(x, z)^{-1}$ we actually have

\[
A^g_z = g (A_z - i \partial_z) g^{-1} = 0. \tag{3.15}
\]
The key equations used in the rest of this subsection are the projector-like properties of the zero-modes \( \psi_\pm(z) \) in the low energy approximation of dropping the massive \( z \)-modes:

\[
\psi_\pm(z)^2 \simeq \psi_\pm(z), \quad \psi_+(z)\psi_-(z) \simeq 0, \quad \psi_+(z) + \psi_-(z) = 1.
\] (3.16)

The first equation is due to that a function \( f(z) \) with the boundary behavior \( f(z \to \pm\infty) = 1 \) and \( f(z \to \mp\infty) = 0 \) is mode-expanded as \( f(z) = \psi_\pm(z) + \text{massive-modes} \), while the second one is due to that \( f(z) \) which vanish both at \( z = \pm\infty \) contains only the massive-modes. The third equation is an exact one. Using the first equation, \( g(x,z)^{-1} \) is approximated as

\[
g(x,z)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-i\varphi(x)\psi_+^n(z))^n \simeq 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-i\varphi(x))^n\psi_+(z) = \psi_-(z) + U(x)\psi_+(z).
\] (3.17)

Likewise we have

\[
g(x,z) \simeq \psi_-(z) + U(x)^{-1}\psi_+(z),
\] (3.18)

and we can confirm \( g^{-1}g \simeq 1 \) by using (3.16). Note, however, that it is dangerous to use (3.17) and (3.18) in expressions containing the derivatives with respect to \( z \), such as (3.15). For example, differentiating \( \psi_+^n \simeq \psi_+ (n \geq 2) \) with respect to \( z \) and using again (3.16), we get a conflicting relation \( n\psi_+\phi_0 \simeq \phi_0 \).

Having finished a preparation, let us turn to considering our chiral current in the SS-model in the low energy limit. Note first that our chiral current \( j_\mu^\zeta(x) \) with \( \zeta(x,z) \) of (2.42) has a meaning for configurations satisfying the boundary condition (2.12). For using the low energy expressions (3.4), (3.5) and (3.6) in the \( A_\mu^g = 0 \) gauge, we use the fact that \( j_\mu^\zeta(x) \) is invariant under the simultaneous gauge transformations (2.41) of both \( A_M \) and \( \zeta \). For \( \zeta = \psi_\pm t_a \) corresponding to the right-current \( j_\mu^{\psi_\pm}(x) \), \( \zeta^g \) in the \( A_\mu^g = 0 \) gauge is the same as the original one in the low energy limit:

\[
\zeta^g \simeq \left( \psi_- + U^{-1}\psi_+ \right)\psi_+ t_a \left( \psi_- + U \psi_+ \right) \simeq \psi_- t_a,
\] (3.19)

where we have used the expressions (3.17) and (3.18) for \( g^{-1}g \) and \( g \), respectively, and the projector property (3.16). Therefore, \( j_\mu^{\psi_\pm}(x) \) in the low energy limit is obtained by simply replacing \( A_\mu \) and \( F_{MN} \) in (2.45) with those in the \( A_\mu^g = 0 \) gauge, (3.4), (3.5) and (3.6), and carrying out the integration over \( z \):

\[
j_\mu^{\psi_\pm}(x) \simeq -2\kappa \text{tr} \left\{ \left( \frac{1}{\pi} R^\mu + c_S \left[ [R^\mu, R^\nu], R^\rho \right] \right) t_a \right\} - \frac{iN_c}{48\pi^2} \epsilon^{\mu\nu\rho\lambda} \text{tr} (R_\nu R_\rho R_\lambda t_a).
\] (3.20)

This agrees with the Noether current of the right-transformation \( U(x) \to U(x)g_R^{-1} \) in the Skyrme model (3.10). It is obvious that the \( z \)-component of the five-dimensional current (2.47) vanishes at \( z = \pm\infty \) and hence satisfies the condition (2.23) necessary for the conservation of the four-dimensional current (3.20).

Next, let us consider the left-current \( j_\mu^{\psi_\pm}(x) \) corresponding to \( \zeta = \psi_+ t_a \). In this case, the gauge transformation to the \( A_\mu^g = 0 \) gauge effects a nontrivial change on \( \zeta \):

\[
\zeta^g \simeq \left( \psi_- + U^{-1}\psi_+ \right)\psi_+ t_a \left( \psi_- + U \psi_+ \right) \simeq \psi_+ (z)U(x)^{-1}t_a U(x).
\] (3.21)
Plugging this $\zeta^g$ together with (3.3), (3.5) and (3.6) into (2.30) and (2.32), and using, in particular, that $D^\nu g_{\nu} \zeta^g \simeq -i[R_\nu U^{-1} t_a U] \psi_+ \psi_-$, we obtain the Noether current of the left-transformation in the Skyrme model:

$$j^\mu_{L,a}(x) \simeq 2\kappa \text{tr} \left\{ \left( \frac{1}{\pi} L^\mu + cs [L^\mu, L^\nu], L_\nu \right) t_a \right\} + \frac{iN_c}{48\pi^2} \epsilon^{\mu\nu\rho\lambda} \text{tr}(L_\nu L_\rho L_\lambda t_a), \quad (3.22)$$

with

$$L_\mu(x) = U(x)R_\mu(x)U(x)^{-1} = iU(x)\partial_\mu U(x)^{-1}. \quad (3.23)$$

Another way to get the same $j^\mu_{L,a}(x)$ in the low energy limit is to repeat the arguments leading to (3.20) by replacing $g(x,z)^{-1}$ of (3.17) with $g(x,z)^{-1}U(x)^{-1} \simeq \psi_-(z) + U(x)^{-1}\psi_+(z)$ which also realizes $A^U g_{\nu} = 0$.

Finally in this section, we comment on the low energy limit of another candidate chiral current $\tilde{j}^\mu_{L/R}(x)$ (2.37) defined on the boundary $z = \pm \infty$. This current should also be considered under the boundary condition (2.12). Using the relation $F_{\mu z}(x,z) = g(x,z)^{-1}F^\mu_{g \nu}(x,z)g(x,z)$ together with $g(x,z = -\infty) = 1$ and $g(x,z = \infty) = U(x)^{-1}$, we find that the low energy limit of $\tilde{j}^\mu_{L/R}(x)$ can reproduce only the first term $\pm(2\kappa/\pi) \text{tr}((L/R)^\mu t_a)$ of the whole Noether currents (3.20) and (3.22) of the Skyrme model.

4 Static properties of nucleons

Baryons in the SS-model (2.1) are described by a soliton solution. In [5], explicit construction of the baryon solution and its collective coordinate quantization were given in the two flavor ($N_f = 2$) case and in the approximation of large 't Hooft coupling $\lambda$. The baryon solution in this approximation on a time slice is the BPST instanton solution [14] with a fixed size, which is determined by the balance between the contraction force from the warp factors and the expansive one from the self-interaction via the CS term. (See [8] for an extension to the three flavor case.)

In this section, we calculate the various static properties of nucleons in the $N_f = 2$ SS-model by using the chiral current (2.44) and the baryon solution with quantized collective coordinates given in [5]. The quantities we calculate are

- Electric charge distribution and charge radii
- Magnetic moments and magnetic charge radii
- Axial-vector coupling constant $g_A$.

The present analysis is an SS-model extension of that given in [4] for the Skyrme model. At each step, we compare our results with the corresponding ones in the Skyrme model [4] and the experimental values.
4.1 Baryon solution and its collective coordinate quantization

In this subsection, we summarize the baryon solution in the $N_f = 2$ SS-model and its collective coordinate quantization in the approximation of large $\lambda$ \[5\]. They are consistently given by assuming that the space-time coordinates $x^M = (x^\mu, z)$ and the $U(2)$ gauge field $A_M$ are of the following orders with respect to $\lambda (\gg 1)$:

\[
\begin{align*}
    x^{M=\mu, z} &= \mathcal{O} (\lambda^{-1/2}) , \\
    x^0 &= \mathcal{O} (\lambda^0) , \\
    A_{M=\mu, z} &= \mathcal{O} (\lambda^{1/2}) , \\
    A_0 &= \mathcal{O} (\lambda^0) , \\
    F_{MN} &= \mathcal{O} (\lambda^0) , \\
    F_{0M} &= \mathcal{O} (\lambda^{1/2}) , \\
    (M, N \neq 0).
\end{align*}
\]

At the leading order in this large $\lambda$ approximation, the warp factors (2.5) are set equal to 1, and the $S_{YM}$ (2.2) is reduced to the YM action on the flat space. The non-trivial effects of the warp factors as well as the CS term (2.3) are of order $\lambda^{-1}$.

In \[5\], they expressed the action and the EOM in terms of the rescaled coordinates and fields which are of order $\lambda^0$. Here, we continue using the original coordinates and fields since we need chiral currents as functions of the real space coordinate.

Decomposing the $U(2)$ gauge field $A_M$ into the $SU(2)$ part $A_M$ and the $U(1)$ part $\hat{A}_M$ as

\[
A_M = A_M + \hat{A}_M \frac{1}{2} \mathbf{1}_2 = A_M^a t_a + \hat{A}_M \frac{1}{2} \mathbf{1}_2 , \\
\left( t_a = \frac{1}{2} \tau_a \right),
\]

the static baryon solution $A^{cl}_M$ sitting at the origin $(x, z) = (0, 0)$ is given to the leading order by

\[
\begin{align*}
    A^{cl}_{M=\mu, z}(x, z) &= -i \tilde{f}(\xi) g_{\text{inst}}(x, z)^{-1} \partial_M g_{\text{inst}}(x, z) , \\
    A^{cl}_0(x, z) &= 0 , \\
    \hat{A}^{cl}_{M=\mu, z}(x, z) &= 0 , \\
    \hat{A}^{cl}_0(x, z) &= -\frac{1}{8\pi^2 a \lambda} \frac{1}{\xi^2} \left[ 1 - \frac{\rho^4}{(\xi^2 + \rho^2)^2} \right],
\end{align*}
\]

with

\[
\begin{align*}
    g_{\text{inst}}(x) &= \frac{1}{\xi} \left( z \mathbf{1}_2 + ix^t \tau_i \right) \in SU(2) , \\
    \tilde{f}(\xi) &= \frac{\rho^2}{\xi^2 + \rho^2} , \\
    \xi &= \sqrt{x^2 + z^2}.
\end{align*}
\]

The explicit expressions of the $SU(2)$ part of the gauge field and the field strengths are

\[
\begin{align*}
    A^{cl}_i(x, z) &= \frac{2}{\xi^2} \tilde{f}(\xi) \left( z t_i - \epsilon_{ija} x^j t_a \right) , \\
    A^{cl}_z(x, z) &= -\frac{2}{\xi^2} \tilde{f}(\xi) x^0 t_a ,
\end{align*}
\]

and

\[
\begin{align*}
    F^{cl}_{ij}(x, z) &= - \frac{4}{\rho^2} \tilde{f}(\xi)^2 \epsilon_{ija} g_{\text{inst}}^{-1} t_a g_{\text{inst}} , \\
    F^{cl}_{i2}(x, z) &= \frac{4}{\rho^2} \tilde{f}(\xi)^2 g_{\text{inst}}^{-1} t_i g_{\text{inst}}.
\end{align*}
\]

\(\)

Accordingly, various constants in this paper (for example, (4.10), (4.25) and (4.27)) differ from the corresponding ones in \[5\] and \[8\] by $\lambda$ or $1/\lambda$. 

14
\[ g_{\text{inst}}(x, z)^{-1} g_{\text{inst}}(x, z) = \frac{1}{\xi^2} \left[ (z^2 - x^2) t_a + 2x^a x^b t_b - 2\epsilon_{abc} x^d t_c \right]. \]  

(4.9)

The \( SU(2) \) part of the solution is nothing but the BPST instanton solution \[14\] with size \( \rho \) in the \((x, z)\) space. The size \( \rho \) is determined by minimizing the subleading part of the energy as

\[ \rho_{st}^2 = \frac{1}{8\pi^2a\lambda} \sqrt{\frac{6}{5}}. \]  

(4.10)

This implies that the baryon has a very small size of order \( \lambda^{-1/2} \). The mass of the solution with \( \rho \) of (4.10) is given by

\[ M = 8\pi^2\kappa + \sqrt{\frac{2}{15}} N_c, \]  

(4.11)

where the second subleading term is from the CS term and the \( z^2 \) terms of the warp factors.

The baryon solution we presented above is in the singular gauge, namely, the gauge where \( A_{\text{cl}}^M \) is singular at the origin \( \xi = 0 \) but is regular at the infinity \( \xi = \infty \). Besides the singular gauge solution, we have the solution in the regular gauge where \( A_{\text{cl}}^M \) is regular at the origin but is singular at the infinity. The two are connected by the gauge transformation in terms of \( g_{\text{inst}} \). The reason why we adopt the singular gauge solution here is related to the boundary condition (2.12) as we will discuss in appendix A. In fact, the large \( \xi \) behavior of the \( SU(2) \) gauge field \( A_{\text{cl}}^M \) in the singular gauge is \( A_{\text{cl}}^M \sim -i(\rho^2/\xi^2) g_{\text{inst}}^{-1} \partial_M g_{\text{inst}} = \mathcal{O}(1/\xi^3) \), while it is \( -i g_{\text{inst}} \partial_M g_{\text{inst}}^{-1} = \mathcal{O}(1/\xi) \) in the regular gauge.

The collective coordinate quantization of this baryon solution is carried out in a standard manner \[5\]. Besides the center-of-mass \( x^i \) coordinate and the \( SU(2) \) rotation which are genuine zero-modes, we take the size \( \rho \) and the center-of-mass \( z \) coordinate as approximate collective coordinates for quantization since the energies of these modes are much lighter than other kinds of massive modes for large \( \lambda \). Then, the gauge field one-form \( \mathcal{A} \) with the collective coordinates of the center-of-mass and the size, \( X^\alpha(t) = (X^M(t), \rho(t)) \), and that of the \( SU(2) \) rotation, \( W(t) \in SU(2) \), incorporated is given by

\[ \mathcal{A}(x, z, t) = W(t) \left( \mathcal{A}_{\text{cl}}(x, z; X^\alpha(t)) + \Phi(x, z, t) dt - id \right) W(t)^{-1}, \]  

(4.12)

where \( \Phi \) is determined by the Gauss law as

\[ \Phi(x, z, t) = \sum_{a=1}^{3} \chi^a(t)\Phi_a - X^M(t)A_{\text{cl}}^M(x, z; X^\alpha(t)), \]  

(4.13)

with

\[ \chi^a(t) = -2i \operatorname{tr} \left( t_a W(t)^{-1} \hat{W}(t) \right), \]  

(4.14)

\[ \Phi_a = f(\xi; X^\alpha(t)) t_a, \]  

(4.15)

\[ \text{in this paper we adopt the way of introducing the collective coordinate given in [8], which is related to that in [5] by a gauge transformation.} \]
\[ f(\xi) = 1 - \frac{\xi^2}{\xi^2 + \rho^2}. \] (4.16)

In \( A^c_l(x, z; X^\alpha(t)) \), we must replace \( x^i, z \) and \( \rho \) in the initial expression \([1,3]\) by \( x^i - X^i(t), z - Z(t) \) and \( \rho(t) \), respectively. This is the case also for \( f(\xi; X^\alpha(t)) \) in (4.15). In later subsections, we use the following expressions of the gauge field and the field strength which are derived from (4.12):

\[ A_i(x, z, t) = W(t) A^c_i(x, z; X^\alpha(t)) W(t)^{-1}, \] (4.17)

\[ F_{MN}(x, z, t) = W(t) F^c_{MN}(x, z; X^\alpha(t)) W(t)^{-1}, \quad (M, N \neq 0), \] (4.18)

\[ F_{0M}(x, z, t) = W(t) \left( \dot{X}^i F^c_{Mi} + \dot{Z} F^c_{Mz} + \dot{\rho} \frac{\partial}{\partial \rho} A^c_M - \chi^a D^c_{M} \Phi_a \right) W(t)^{-1}. \] (4.19)

Substituting the expressions of the fields in terms of the collective coordinates into the original action (2.1), we obtain the lagrangian of the collective coordinates:

\[ L = L_X + L_Z + L_\rho + L_{\rho W}, \] (4.20)

where the component lagrangians are given by

\[ L_X = -8\pi^2 \kappa + \frac{m_X}{2} \dot{X}^2, \] (4.21)

\[ L_Z = \frac{m_Z}{2} \left( \dot{Z}^2 - \omega_Z^2 Z^2 \right), \] (4.22)

\[ L_\rho = \frac{m_\rho}{2} \left( \dot{\rho}^2 - \omega_\rho^2 \rho^2 \right) - \frac{Q}{\rho^2}, \] (4.23)

\[ L_{\rho W} = \frac{1}{8} m_{\rho \rho}^2 \rho^2 \sum_{a=1}^{3} (\chi^a)^2 = \mathcal{I}(\rho) \text{tr} \left( -i W^{-1} \dot{W} \right)^2, \] (4.24)

with the various quantities defined by

\[ m_X = m_Z = \frac{m_\rho}{2} = 8\pi^2 \kappa = \frac{\lambda N_c}{27\pi}, \] (4.25)

\[ \omega_Z^2 = \frac{2}{3}; \quad \omega_\rho^2 = \frac{1}{6}, \] (4.26)

\[ Q = \frac{27\pi N_c}{5\lambda}, \] (4.27)

\[ \mathcal{I}(\rho) = \frac{1}{4} m_{\rho \rho}^2 \rho^2 = 4\pi^2 \kappa \rho^2. \] (4.28)

The collective coordinates \( X^i(t), Z(t), \rho(t) \) and \( W(t) \) are quantized by using the lagrangian (4.20). In this paper, we consider only nucleons at rest and hence omit the collective coordinate \( X^i(t) \). The quantization of other collective coordinates are summarized as follows [5]:

- Upon quantization of the \( SU(2) \) rotation \( W(t) \), baryons are classified by the spin \( J_i \) and the isospin \( I_a \), which are the Noether charges of the right and the left transformations on \( W, W \rightarrow g_J W g_J^{-1} \), respectively. Explicitly, we have

\[ J_i = 2\mathcal{I}(\rho) \text{tr} \left( -i W^{-1} \dot{W} t_i \right) = \mathcal{I}(\rho) \chi^i(t), \] (4.29)
\[ I_a = 2i \langle \rho \rangle \text{tr}(i \hat{W} W^{-1} t_a). \]  

(4.30)

Since \( J_i \) and \( I_a \) are related by \( I_a t_a = -W J_i t_i W^{-1} \), their representations must be the same.

- For the \( I = J = \ell/2 \) state (\( \ell \) is an odd integer), the wave function \( R_{\ell,n_\rho}(\rho) \) of \( \rho \) (under the measure \( \int_0^\infty d\rho \rho^3 \)) is given by
  \[ R_{\ell,n_\rho}(\rho) = \rho e^{\ell F(\tilde{\ell} + 2, m_\rho \omega \rho^2)} \exp\left(-\frac{1}{2} m_\rho \omega \rho^2\right), \quad (n_\rho = 0, 1, 2, \ldots), \]  

(4.31)

where \( F(\alpha, \gamma; z) \) is the confluent hypergeometric function, and \( \tilde{\ell} \) is related to \( \ell \) by \( \tilde{\ell} = -1 + \sqrt{(\ell + 1)^2 + 2m_\rho Q} \).

- The lagrangian \( L_Z (4.22) \) for \( Z \) is simply that of a harmonic oscillator. We denote its quantum number by \( n_Z (= 0, 1, 2, \ldots) \).

Therefore, the baryon states are specified by a set of quantum numbers \( (\ell, n_\rho, n_Z) \). The mass of the corresponding state is given in the \( M_{KK} = 1 \) unit by

\[ M_{\ell,n_\rho,n_Z} = 8\pi^2 \kappa + \sqrt{\frac{(\ell + 1)^2}{6} + \frac{2}{15} N_c^2} + \sqrt{\frac{2}{3}} (n_\rho + n_Z + 1). \]  

(4.32)

The nucleon \( N \) and \( \Delta(1232) \) correspond to \( (\ell, n_\rho, n_Z) = (1, 0, 0) \) and \( (3, 0, 0) \), respectively. The mass of the nucleon and the \( N-\Delta \) mass difference are

\[ M_N = 8\pi^2 \kappa + \sqrt{\frac{2}{3}} + \frac{2}{15} N_c^2, \]  

\[ M_\Delta - M_N = \sqrt{\frac{8}{3}} + \frac{2}{15} N_c^2 - \sqrt{\frac{2}{3}} + \frac{2}{15} N_c^2. \]  

(4.33)

(4.34)

In the calculations of physical quantities in the following subsections, we make the following treatments on \( M_{KK}, M_N, \rho \) and \( Z \):

- We determine the value of \( M_{KK} \) by equating the \( N-\Delta \) mass difference (4.34) with its experimental value, \((1232 - 939) \) MeV:
  \[ M_{KK} = 488 \text{ MeV,} \quad 1/M_{KK} = 0.404 \text{ fm}. \]  

(4.35)

- Since our analyses below on currents are at the leading order in \( 1/\lambda \), we take only the first term \( 8\pi^2 \kappa \) of the nucleon mass (4.33):
  \[ M_N = 8\pi^2 \kappa. \]  

(4.36)
Since we have $F(0, \tilde{\ell} + 2, m_\rho \omega_\rho \rho^2) \equiv 1$, the wave function (4.31) of $\rho$ is given for nucleons by

$$R_{1,0}(\rho) = \rho N \exp \left( -\frac{1}{2} m_\rho \omega_\rho \rho^2 \right), \quad \left( \tilde{\ell}_N = -1 + 2\sqrt{1 + \frac{N_c^2}{5}} \right).$$

(4.37)

In evaluating the various quantities $O(\rho)$ depending on $\rho$, we take the expectation value using the wave function (4.37) and the measure $\int_0^\infty d\rho \rho^3$:

$$\langle O(\rho) \rangle_\rho = \frac{\int_0^\infty d\rho \rho^3 O(\rho) R_{1,0}(\rho)^2}{\int_0^\infty d\rho \rho^3 R_{1,0}(\rho)^2}.$$  

(4.38)

In particular, $\langle \rho^2 \rangle_\rho$, which repeatedly appears in the following subsections, is given by

$$\langle \rho^2 \rangle_\rho = \frac{2 + \tilde{\ell}_N}{m_\rho \omega_\rho} = \frac{\sqrt{6}}{8\pi^2 \kappa} \left( \frac{1}{2} + \sqrt{1 + \frac{N_c^2}{5}} \right).$$

(4.39)

This agrees with $\rho_{st}^2$ (4.10) in the large $N_c$ limit. Using (4.36) and

$$M_N = 939 \text{ MeV} = 1.92 M_{KK},$$

(4.40)

the numerical value of (4.39) for $N_c = 3$ is

$$\sqrt{\langle \rho^2 \rangle_\rho} = 0.672 \text{ fm},$$

(4.41)

where we have used (4.35) for expressing the result in fm unit.

- Since the $Z(t)$ dependence in the five-dimensional current of the form $z - Z(t)$ disappears after the $z$-integration necessary in obtaining the four-dimensional current, we put $Z = 0$ from the start. The time-derivative terms of $Z$ appearing in our chiral current is only the $\dot{Z}$ term coming from (4.19), and no terms containing more time-derivatives arise. Since the expectation value of $\dot{Z}$ is equal to zero for energy eigenstates of the $Z$ harmonic oscillator, we drop this $\dot{Z}$ term in the current.

In the following subsections we calculate the various static properties of nucleons using the $U(N_f)_{L/R}$ currents defined by (2.44) and (2.45). We see from (4.18) and (4.19) that another chiral current $\tilde{j}_L/R(x)$ (2.37) defined by the field strength on the boundary $z = \pm \infty$ does vanish. This is the case both in the singular gauge adopted here and also in the regular gauge. Quite similarly, the $z$-component of the five-dimensional current, (2.47), vanishes on the boundary for the baryon configuration and hence satisfies the condition (2.23).

Although the local current $\tilde{j}_L/R(x)$ itself vanishes for the baryon configuration, it may happen that the space integration of a quantity containing the current is non-vanishing if we take the $z \to \pm \infty$ limit after carrying out the integration. This is the case for the isovector magnetic moment computed in sec. 4.3. We will briefly summarize in sec. 4.5 the results of computation using $\tilde{j}_L/R(x)$.  

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4.2 Charge density

As a static property of nucleons, let us first consider its charge density. For this we need the isospin density \( J_{0,V,a}(x) \) and the baryon number density. First, the isospin density before the \( z \)-integration, \( J_{0,V,a}(x,z) \) \((a=1,2,3)\), to the leading order in the large \( \lambda \) approximation of (4.1) is obtained from the non-abelian part of (2.45) by neglecting the warp factors and using that \( \psi_V(z) = 1 \) for the vector current (see (2.46)):

\[
J_{0,V,a}(x,z) = 2i\kappa \text{tr} \left\{ \left( [F_{0i}, A_i] + [F_{0z}, A_z] \right) t_a \right\}
= 2i\kappa \text{tr} \left\{ \left( \sum_{M=i,z} i \left[ \frac{\partial}{\partial \rho} A^\text{cl}_M, A^\text{cl}_M \right] \rho(t) - \sum_{M=i,z} i [D^\text{cl}_M \Phi_b, A^\text{cl}_M] \chi^b(t) \right) W(t)^{-1} t_a W(t) \right\},
\]

(4.42)

where we have dropped the \( \dot{X}^i \) and \( \dot{Z} \) terms in (4.19) as announced above. Using

\[
\sum_{M=i,z} i [D^\text{cl}_M \Phi_a, A^\text{cl}_M] = \frac{8}{\rho^2} \bar{f}(\xi)^3 t_a,
\]

(4.43)

and that \( (\partial/\partial \rho) A^\text{cl}_M, A^\text{cl}_M = 0 \), which is due to \( (\partial/\partial \rho) A^\text{cl}_M \propto A^\text{cl}_M \), we get

\[
J_{0,V,a}(x,z) = \frac{2}{\pi^2 (\xi^2 + \rho^2)^3} I_a,
\]

(4.44)

where \( I_a \) is the isospin operator (4.30), and we have used the formula \( \text{tr}(t_b W^{-1} t_a W) \chi^b = \text{tr}(-i\bar{W}W^{-1} t_a) \). The isospin density in four dimensions is obtained by integrating (4.44) over \( z \):

\[
J_{0,V,a}(t) = \int_{-\infty}^{\infty} dz J_{0,V,a}(x,z) = \frac{3}{4\pi} \frac{\rho^2}{(r^2 + \rho^2)^{5/2}} I_a.
\]

(4.45)

We can confirm that our \( J_{0,V,a} \) is indeed the isospin density:

\[
I_a = \int d^3x J_{0,V,a}(t, x).
\]

(4.46)

Next is the baryon number density in four dimensions. In the SS-model, a (topologically) conserved baryon number current in five dimensions has been defined \([1,2]\):

\[
J^M_B(x,z) = \frac{1}{32\pi^2} \epsilon^{MNPQR} \text{tr}(F_{NP}F_{QR}).
\]

(4.47)

Our baryon solution has in fact a unit baryon number; namely, we have \( \int d^3x dz J^0_B = 1 \) for the solution (4.3). It is interesting to notice that this topological current \( J^M_B \) (4.47) is obtained as the Noether current of the \( U(1)_V \) symmetry, \( J^M_\zeta \) (2.29) with \( \zeta = 1 \) (see also (2.43)), divided
by $N_c$, though all the fields in the action (2.1) are inert under the $U(1)_V$ transformation. From (4.47), the four-dimensional baryon number density for the baryon configuration (4.18) is

$$j_B^0(t, \mathbf{x}) = \int_{-\infty}^{\infty} dz J_B^0(x, z) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} dz \epsilon_{ijk} \text{tr} (F_{ij}^c F_{kz}^c) = \frac{15}{8\pi (r^2 + \rho^2)^{7/2}}. \quad (4.48)$$

Integration of (4.48) further with respect to $\mathbf{x}$ gives one as mentioned above.

Now, let us evaluate physical quantities from the isospin density (4.45) and the baryon number density (4.48). First, the charge density is given by

$$j_{em}^0(t, \mathbf{x}) = j_{V,a=3}^0(t, \mathbf{x}) + \frac{1}{2} j_B^0(t, \mathbf{x}) = \frac{3}{4\pi (r^2 + \rho^2)^{5/2}} I_3 + \frac{15}{16\pi (r^2 + \rho^2)^{7/2}}, \quad (4.49)$$

and we actually have $\int d^3x j_{em}^0(\mathbf{x}) = I_3 + (1/2)$. In fig. 1 we plot the radial charge distribution $4\pi r^2 \langle j_{em}^0(r)\rangle_\rho$ for proton and neutron. The curves in fig. 1 are close to those in the Skyrme model (see fig. 2 in [4]). However, they are in disagreement with experimental results [15], which show that the neutron is almost locally neutral in the region $r \gtrsim 0.05\text{fm}$, and that the charge distribution of proton decays exponentially for large $r$ (while our $4\pi r^2 \langle j_{em}^0(r)\rangle_\rho$ decays as $1/r^5$).

Next, the isoscalar mean square charge radius is calculated as

$$\sqrt{\langle r^2 \rangle_{I=0}} = \sqrt{\left\langle \int d^3x r^2 j_B^0(t, \mathbf{x}) \right\rangle_\rho} = \frac{3}{2} \langle \rho^2 \rangle_\rho = 0.82\text{ fm}, \quad (\text{ANW} : 0.59\text{ fm}, \ \text{Exp} : 0.80\text{ fm}), \quad (4.50)$$

** $\langle r^2 \rangle_{I=0}^{1/2}$ as a function of $\rho^2$ was obtained before by S. Sugimoto (private communication).**
where we have used (4.41), and ANW and Exp denote the result of [4] and the experimental value [16], respectively. The (numerator of) isovector mean square charge radius,
\[
\langle r^2 \rangle_{I=1} = \frac{\langle \int d^3x \, r^2 J_{V,a=3}^0(t, \mathbf{x}) \rangle}{\langle \int d^3x \, J_{V,a=3}^0(t, \mathbf{x}) \rangle},
\]
is logarithmically divergent at \( r = \infty \) as in the Skyrme model [4]. This phenomenon may be ascribed to the fact that all the quarks and hence the pions are massless in the SS-model [17].

### 4.3 Magnetic moments

In this subsection, we examine quantities related with the magnetic moment of nucleons:
\[
\mathbf{\mu} = \frac{1}{2} \int d^3x \, \mathbf{x} \times \mathbf{j}_{\text{em}}(\mathbf{x}) = \frac{1}{2} \mathbf{\mu}_{I=1} + \frac{1}{2} \mathbf{\mu}_{I=0},
\]
where \( \mathbf{j}_{\text{em}} \) is the electro-magnetic current vector,
\[
\mathbf{j}_{\text{em}} = \mathbf{j}_{V,a=3} + \frac{1}{2} \mathbf{j}_B,
\]
and the isovector and the isoscalar magnetic moments are defined respectively by\textsuperscript{††}
\[
\mathbf{\mu}_{I=1} = \int d^3x \, \mathbf{x} \times \mathbf{j}_{V,a=3}(\mathbf{x}),
\]
\[
\mathbf{\mu}_{I=0} = \frac{1}{2} \int d^3x \, \mathbf{x} \times \mathbf{j}_B(\mathbf{x}).
\]

For this purpose we need the space components of the isospin current and the baryon number current. The space component of the five dimensional \( SU(N_f)_{V/A} \) currents at the leading order in the large \( \lambda \) approximation are given from (2.45) and (4.1) by
\[
J_{V/A,a}^i(x, z) = -2\kappa \text{tr} \left\{ \left( \sum_{M=j,z} i \left[ F_{iM}^{\text{cl}}, A_M^{\text{cl}} \right] \right) \psi_{V/A}(z) + F_{iz}^{\text{cl}} \frac{d\psi_{V/A}(z)}{dz} \right\} W(t)^{-1} t_a W(t). \tag{4.56}
\]
Here, we need the vector current. Using \( \psi_V(z) = 1 \) and
\[
\sum_{M=j,z} i \left[ F_{iM}^{\text{cl}}, A_M^{\text{cl}} \right] = \frac{16\rho^4}{\xi^2(\xi^2 + \rho^2)^3} \left( z t_i - \epsilon_{ija} x^j t_a \right), \tag{4.57}
\]
the space component of the four-dimensional \( SU(N_f)_V \) current is obtained as
\[
\mathbf{j}_{V,a}^i(t, \mathbf{x}) = \int_{-\infty}^{\infty} dz \, J_{V,a}^i(x, z) = \frac{4\pi\kappa}{\rho^2} \left( \frac{8}{r} - \frac{8r^4 + 20\rho^2 r^2 + 15\rho^4}{(r^2 + \rho^2)^{5/2}} \right) \epsilon_{ijk} x^j \text{tr}(t_k W(t)^{-1} t_a W(t)). \tag{4.58}
\]
\textsuperscript{††} Our \( SU(N_f)_{V/A} \) current \( j_{V/A,a}^i \) is half of that in [4].
As for the space component of the baryon number current, we first have from (4.47), (4.18) and (4.19) that

\[ J_i^B(x, z) = \frac{1}{8\pi^2} \epsilon_{ijk} \text{tr}(F_{jk}F_{0z} + 2F_{0j}F_{kz}) \]

\[ = \frac{3}{\pi^2 (\xi^2 + \rho^2)^4} \left[ (\delta_{ia} - \epsilon_{ija}x^j) \chi^a(t) + 2x^i \frac{d}{dt} \ln \rho(t) \right], \tag{4.59} \]

and hence

\[ j_B^i(t, x) = \int_{-\infty}^{\infty} dz J_B^i(x, z) = \frac{15}{16} \frac{\rho^4}{(r^2 + \rho^2)^{7/2}} \left[ -\epsilon_{ija}x^j \chi^a(t) + 2x^i \frac{d}{dt} \ln \rho(t) \right]. \tag{4.60} \]

Then, the isovector and isoscalar magnetic moments, (4.54) and (4.55), are calculated as follows:

\[ (\mu_{I=1})_i = \epsilon_{ijk} \int d^3x x^j j_B^k(t, x) = -8\pi^2 \kappa \rho^2 \text{tr} \left( t_i W(t)^{-1} t_3 W(t) \right), \tag{4.61} \]

\[ (\mu_{I=0})_i = \frac{1}{2} \epsilon_{ijk} \int d^3x x^j j_B^k(t, x) = \frac{\rho^2}{4} \chi^i(t) = \frac{1}{16\pi^2 \kappa} J_i, \tag{4.62} \]

where \( J_i \) is the spin operator (4.29). From this we can read off the isovector \( g \)-factor \( g_{I=1} \) and the isoscalar one \( g_{I=0} \) defined for nucleon states in the Pauli matrix representation of spin and isospin, \( J_i = \sigma_i/2 \) and \( I_a = \tau_a/2 \), by

\[ \mu_{I=1} = \frac{g_{I=1}}{2M_N} \frac{\sigma}{2} \otimes \tau_3, \tag{4.63} \]

\[ \mu_{I=0} = \frac{g_{I=0}}{2M_N} \frac{\sigma}{2}. \tag{4.64} \]

For \( g_{I=1} \) we use the relation valid for nucleon states (see eq. (22) in [4]),

\[ \langle N' | \text{tr} \left( t_i W^{-1} t_a W \right) | N \rangle = -\frac{1}{6} \langle N' | \sigma_i \otimes \tau_a | N \rangle, \tag{4.65} \]

to get

\[ g_{I=1} = \frac{16\pi^2 \kappa}{3} M_N \langle \rho^2 \rangle_\rho \sqrt{\frac{2}{3} \left( 1 + 2 \sqrt{1 + \frac{N^2}{5}} \right)} M_N = 6.83, \quad (\text{ANW : 6.38, Exp : 9.41}), \tag{4.66} \]

where we have used (4.39) and (4.40). On the other hand, \( g_{I=0} \) is exactly equal to one if we adopt (4.36) for \( M_N \):

\[ g_{I=0} = \frac{M_N}{8\pi^2 \kappa} = 1, \quad (\text{ANW : 1.11, Exp : 1.76}). \tag{4.67} \]

The results (4.66) and (4.67) are restated into the nucleon magnetic moments \( \mu_{p,n} \) in units of Bohr magneton \( e\hbar/(2M_N) \) as follows:

\[ \mu_p = 1.96 \quad (\text{ANW : 1.87, Exp : 2.79}), \tag{4.68} \]
\[ \mu_n = \frac{1}{4} (g_{I=0} - g_{I=1}) = -1.46, \quad (\text{ANW} : -1.31, \ \text{Exp} : -1.91). \quad (4.69) \]

Their ratio is
\[ \left| \frac{\mu_p}{\mu_n} \right| = 1.34, \quad (\text{ANW} : 1.43, \ \text{Exp} : 1.46). \quad (4.70) \]

Finally, the isoscalar magnetic mean square radius is defined by
\[ \langle r^2 \rangle_{M,I=0} = \frac{\int_0^\infty dr r^2 \rho_{M,I=0}(r)}{\int_0^\infty dr \rho_{M,I=0}(r)}, \quad (4.71) \]

in terms of the isoscalar magnetic moment density \( \rho_{M,I=0}(r) \) giving the coefficient of \( \chi^i(t) \) in (4.62); \( \mu_{I=0} = \int_0^\infty dr \rho_{M,I=0}(r) \chi^i(t) \). In the present case, we have
\[ \rho_{M,I=0}(r) = \frac{5}{4} \rho r^4 \left( r^2 + \rho^2 \right)^{3/2}. \quad (4.72) \]

Therefore, \( \langle r^2 \rangle_{M,I=0} \) is logarithmically divergent at \( r = \infty \) in contrast with the case of the Skyrme model where it is finite [1].

### 4.4 Axial-vector coupling

Let us consider the \( SU(N_f)_A \) current to obtain the axial-vector coupling \( g_A \). First, (the space component of) the five-dimensional current \( J^i_{A,a}(x,z) \) is given from (4.56) by using (4.57), (4.8) and (4.9) for the baryon solution:
\[ J^i_{A,a}(x,z) = -\frac{8\kappa \rho^2}{\xi^2(\xi^2 + \rho^2)^2} \left\{ \frac{4\rho^2 z \psi_A(z)}{\xi^2 + \rho^2} + \left( z^2 - \frac{x^2}{3} \right) \frac{d\psi_A(z)}{dz} \right\} \text{tr}(t_i W(t)^{-1} t_a W(t)), \quad (4.73) \]

where we have kept only the terms which are even in \( z \) (\( \psi_A(z) \) is odd in \( z \)) since we eventually carry out the \( z \)-integration, and have carried out the spatial angle averaging to replace \( x^a x^b \) in (4.9) by \( \delta_{ab} r^2 / 3 \). Since we are considering the leading order of the large \( \lambda \) approximation with the orders of the respective quantities given by (4.1) and we have already approximated the warp factors by 1 in (4.73), the function \( \psi_A(z) = (2/\pi) \arctan z \) (2.46) should consistently be approximated in (4.73) by
\[ \psi_A(z) = \frac{2}{\pi} z. \quad (4.74) \]

We will comment later on what happens to the \( SU(N_f)_A \) current if we keep the original expression of the function \( \psi_A(z) \). Using (4.74), the five-dimensional \( SU(N_f)_A \) current reads
\[ j^i_{A,a}(t,x) = \int_{-\infty}^\infty dz J^i_{A,a}(x,z) = \frac{32\kappa}{3\rho^2} \left[ 8r - \frac{8r^4 + 12\rho^2 r^2 + 3\rho^4}{(r^2 + \rho^2)^{3/2}} \right] \text{tr}(t_i W(t)^{-1} t_a W(t)). \quad (4.75) \]

The axial-vector coupling constant \( g_A \) is obtained by identifying the nucleon matrix element of
\[ \int d^3 x j^i_{A,a}(x,t) = -\frac{32\pi \kappa \rho^2}{3} \text{tr}(t_i W^{-1} t_a W), \quad (4.76) \]
with the $q \to 0$ limit of the non-relativistic expression [4]:

\[
\langle N'(p') | j_{A,a}^{\dagger}(0) | N(p) \rangle = \frac{1}{2} g_A(q^2) \left( \delta^{ij} - \frac{q^i q^j}{q^2} \right) \langle N'| \sigma_j \otimes \tau_a | N \rangle, \quad (q = p' - p). \tag{4.77}
\]

Making the angle averaging of $q$ to replace $q^i q^j / q^2$ in (4.77) by $\delta^{ij} / 3$ and using (4.65), we get

\[
g_A = g_A(0) = \frac{16 \pi \kappa}{3} \langle \rho^2 \rangle_\rho = \frac{\sqrt{6}}{3\pi} \left( 1 + 2 \sqrt{1 + \frac{N_c^2}{5}} \right) = 1.13, \quad (\text{ANW}: 0.61, \text{Exp}: 1.24), \tag{4.78}
\]

where we have used (4.39) for $\langle \rho^2 \rangle_\rho$. Compared with the result in the Skyrme model, we have got a surprisingly good agreement with the experimental value. Note that $g_A$ (4.78) is independent of our choice of $M_{KK}$ (4.35). If we had simply replaced $\langle \rho^2 \rangle_\rho$ in (4.78) by the minimum of the potential $\rho^2$ given by (4.10) without treating $\rho$ as an approximate collective coordinate, we would have obtained, instead of (4.78), a less attractive result; $g_A = 2^{1/2} N_c / (\sqrt{15} \pi) = 0.70$. This is nothing but the large $N_c$ limit of (4.78).

We will make some comments on the evaluation of $g_A$, which has been done by the four-dimensional integration $\int d^3 x \int dz$ of the five-dimensional current (4.73). First, note that the four-dimensional integration of the $(z^2 - x^2/3) d \psi_A(z) / dz$ part is superficially logarithmically divergent at $\xi = \infty$ for $\psi_A(z)$ of (4.74). However, it can actually be convergent since the sum of the coefficients of $z^2$ and $(x^2)^2$ is equal to zero, $1 - (1/3) \times 3 = 0$, and it can take any value depending the way of integration. The integration vanishes if we adopt the symmetric integration using the four-dimensional polar coordinates. Our $g_A$ (4.78) has been obtained by carrying out first the $z$-integration to give $j_{A,a}^{\dagger}$ and then the $x$-integration to take the zero-momentum limit. If we reverse the order of integrations, we get

\[
\int dz \int d^3 x J_{A,a}(x, z) = 0. \tag{4.79}
\]

This is seen from the following formula valid for any $\psi_A(z)$:

\[
\int d^3 x J_{A,a}(x, z) = \frac{8 \pi^2 \kappa}{3 \rho^2} \frac{d}{dz} \left[ Q(z) \psi_A(z) \right] \text{tr} \left( t_i W^{-1} t_a W \right), \tag{4.80}
\]

with

\[
Q(z) = 8 |z|^3 - \frac{8 z^4 + 4 \rho^2 z^2 - \rho^4}{\sqrt{z^2 + \rho^2}} = \mathcal{O} \left( \frac{1}{z^3} \right), \quad (z \to \pm \infty). \tag{4.81}
\]

Eq. (4.80) implies that we have (4.79) also for the original expression $\psi_A(z) = (2/\pi) \arctan z$ (2.46). Since this four-dimensional integration is absolutely and uniformly convergent, we can freely exchange the order of integrations to conclude that $g_A = 0$ for the original $\psi_A(z)$ (2.46) and the trivial warp factors.

†† Eq. (4.78) before taking the expectation value of $\rho^2$ agrees with $g_{A,\text{mag}} = 4C/\pi$ given in eq. (5.35) of [7] using a different approach. The constant $C$ is given below eq. (5.17) of [7].
4.5 Computations using $\tilde{j}_L/R$.

In this subsection, we briefly summarize the computations using the chiral current $\tilde{j}_L/R,a(x)$ (2.37) of the bulk-boundary correspondence. As mentioned at the end of sec. 4.1, although the local current $\tilde{j}_L/R,a(x)$ itself vanishes for the baryon configuration, the space integration of a quantity containing $\tilde{j}_L/R,a(x)$ can be non-vanishing if we take the $z \to \infty$ limit after carrying out the space integration. Approximating the warp factor $k(z)$ by 1 as before and replacing the boundaries $z = \pm \infty$ by $z = \pm \Lambda$, the space component of the $SU(N_f)L/R$ current (2.37) is given by

$$
\tilde{j}_{iL/R,a}(x) = \pm \frac{2}{\kappa} \frac{\pi^{2}}{(\xi^2 + \rho^2)^2} \frac{1}{(2\Lambda^2 + r^2)^3} \epsilon_{ijk} x^j \frac{1}{2} \frac{1}{W(t) - t_3 W(t)} \left( x^i x^j x^k - 2 \epsilon_{ijk} x^j x^k \right),
$$

(4.82)

where we have used (4.8) and (4.9) for the singular gauge field strength. Therefore, the space components of the vector and the axial-vector currents are

$$
\tilde{j}_{iV,a}(x) = 32 \pi^2 \kappa \rho^2 \Lambda \left( 2 \Lambda^2 + r^2 \right)^3 \frac{1}{2} \frac{1}{W(t) - t_3 W(t)} \left( x^i x^j x^k - 2 \epsilon_{ijk} x^j x^k \right),
$$

(4.83)

$$
\tilde{j}_{iA,a}(x) = -16 \pi^2 \kappa \rho^2 \Lambda \left( 2 \Lambda^2 + r^2 \right)^3 \frac{1}{2} \frac{1}{W(t) - t_3 W(t)} \left( x^i x^j x^k - 2 \epsilon_{ijk} x^j x^k \right),
$$

(4.84)

where we have ignored $\rho^2$ in the denominators since we take the limit $\Lambda \to \infty$ in the end.

First, the isovector magnetic moment is calculated using (4.83) as

$$
\langle \tilde{\mu}_{I=1} \rangle_i = \lim_{\Lambda \to \infty} \epsilon_{ijk} \int d^3 x x^j \tilde{j}_{V,a}(t, x) = -16 \pi^2 \kappa \rho^2 \Lambda \left( 2 \Lambda^2 + r^2 \right)^3 \frac{1}{2} \frac{1}{W(t) - t_3 W(t)} \left( x^i x^j x^k - 2 \epsilon_{ijk} x^j x^k \right).
$$

(4.85)

This is twice our previous result (4.61). Therefore, the corresponding isovector $g$-factor is

$$
\tilde{g}_{I=1} = 2 g_{I=1} = 13.6,
$$

(4.86)

which should be compared with the values in (4.60). The factor of two difference between the two isovector magnetic moments, (4.61) and (4.85), can be understood from the relation (2.39) between the currents. In the present case, we have to cutoff the $z$-integrations for $j^\mu_L(x)$ (2.20) and $\chi^{\mu\nu}(x)$ (2.40) at $z = \pm \Lambda$. One can confirm first that $\mu_{I=1}$ remains the same even if we take the $\Lambda \to \infty$ limit at the end, namely, even if we exchange the order of the $x$ and $z$ integrations. Second, we can show that the contribution of the $\partial_j \chi^{ij}$ term of (2.39) to the isovector magnetic moment, $\epsilon_{ijk} \int d^3 x \chi^{ij}$, is just what is necessary to explain the factor of two difference.

Next, the axial-vector coupling $g_A$ becomes zero if we adopt the axial-vector current of (4.84):

$$
\lim_{\Lambda \to \infty} \int d^3 x \tilde{j}_{A,a}(x, t) = 0.
$$

(4.87)

This is also consistent with the relation (2.39). In this case, the space integration of $\partial_j \chi^{ij}$ vanishes, and (4.87) is equivalent to our previous (4.79).
5 Summary and discussions

In this paper, we computed static properties of nucleons in the SS-model, namely, the five-dimensional YM+CS theory realized as the low energy effective theory of mesons in the holographic QCD model of Sakai and Sugimoto [1, 2]. For this purpose, we first constructed a chiral current in four dimensions from the Noether current of local gauge transformation which effects non-trivial transformations on the boundaries of the extra fifth dimension. We confirmed that our chiral current is reduced in the low energy limit to the chiral current in the Skyrme model. The baryons in the SS-model is realized as a soliton, which at a time slice is approximately the BPST instanton with a fixed size. We considered the chiral current in the collective coordinate quantization of the baryon solution within the large $\lambda$ approximation, and calculated electric charge distributions (fig. 1), charge radius (4.50), magnetic moments [(4.66) and (4.67)], and the axial-vector coupling (4.78) of nucleons by taking the masses of the nucleon and $\Delta(1232)$ as inputs. For most of these quantities, the obtained numerical values in the SS-model are better close to the experimental values than in the Skyrme model [4]. We emphasize that in this calculation it is important to quantize the scale $\rho$ of the baryon solution as an approximate collective coordinate. If we treat $\rho$ simply as a constant determined as the minimum of the potential, agreement with the experimental values becomes worse for many of the static properties as we mentioned explicitly for the axial vector coupling in sec. 4.4.

Although the results of our computation of static properties of nucleons are phenomenologically rather satisfactory ones, we have to recall that there remain two problems in our chiral current $j_{L/R}^\mu(x)$ in the SS-model as we explained in sec. 2.4. One is the ambiguity of the interpolating function $\zeta(x, z)$ satisfying the boundary condition (2.18), and the other is the gauge-noninvariance of the current. As the function $\zeta(x, z)$, we took (2.42) using the zero-modes $\psi_{\pm}(z)$ (2.43) in this paper. However, we have no convincing reason for choosing this particular $\zeta(x, z)$. The choice of $\zeta(x, z)$ directly affects the results of our computation. In particular, in the calculation of the axial-vector coupling in sec. 4.4, the first term (4.74) of the Taylor series at $z = 0$ of $\zeta(x, z)$ for the axial-vector current was the source of our nice result (4.78). The gauge-noninvariance is related with the choice of $\zeta(x, z)$ as we explained in sec. 2.4. It is indispensable to resolve these two problems in order to make the computations carried out in this paper really meaningful.

Reconsideration of another chiral current $\tilde{j}_{L/R}^\mu(x)$ (2.37) obtained from the coupling with the external gauge fields on the boundaries or from the bulk-boundary correspondence may also be necessary. The two chiral currents are related by (2.39): They are equal to each other up to the EOM and a trivially conserving term. An advantage of the current $\tilde{j}_{L/R}^\mu(x)$ is that it is free from the two problems of $j_{L/R}^\mu(x)$ mentioned above. However, $\tilde{j}_{L/R}^\mu(x)$ does not reproduce the chiral current of the Skyrme model in the low energy limit. Moreover, since the baryon solution in the SS-model is localized near the origin of the extra fifth dimension, the local current $\tilde{j}_{L/R}^\mu(x)$ itself, which is given in terms of the field strengths on the boundaries, vanishes for the baryon configuration. We saw in sec. 4.5 that a static property given as an integration of a quantity containing the chiral current could be non-vanishing if we take the limit of going to the boundary after carrying out the integration. This is the case for the
isovector $g$-factor. It might be that, by taking into account more subtle points in the current $\tilde{j}^{\mu}_{L/R}(x)$ or by going beyond the large $\lambda$ approximation, we could get nontrivial results from $\tilde{j}^{\mu}_{L/R}(x)$ for all the static properties of nucleons including the local charge density.

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A Vector current in the regular gauge

In sec. 4, we gave an analysis of the static properties of nucleons based on the baryon solution (4.3) in the singular gauge. Fortunately, the singularity of the solution at the origin $\xi = 0$ did not cause any trouble. In this appendix, we outline the analysis of the isospin density using the baryon solution in the regular gauge to explain problems of this gauge (see sec. 4.2 for the isospin density in the singular gauge).

The baryon solution in the regular gauge is related to the singular gauge solution via the gauge transformation by $g^{\text{inst}}(x, z)$. Concretely, the $SU(2)$ part of the regular gauge solution is given by (4.3) with $g^{\text{inst}}(x, z)$ and $g^{\text{inst}}(x, z)^{-1}$ exchanged and $f(\xi)$ replaced with $f(\xi)$ (4.16). The $U(1)$ part of the solution, $\hat{A}^{\text{cl}}_{M}$, remains unchanged from (4.4). The field strengths of the solution are given by (4.8) with $g^{\text{inst}}t_ag^{\text{inst}}$ replaced with $t_a$. After the introduction of the collective coordinates, the gauge fields and the field strengths are given by (4.17)–(4.19) with $\Phi_a$ (4.13) replaced by

$$\Phi_a = g^{\text{inst}}\Phi_ag^{\text{inst}}^{-1} = f(\xi; X^a(t))g^{\text{inst}}t_ag^{\text{inst}}^{-1}.$$  

(A.1)

Now, let us consider the five-dimensional isospin density $J_{V,a}^0(x, z)$ (4.42) in the regular gauge. Instead of the formula (4.43), here we use the following one in the regular gauge:

$$\sum_{M=i,z} i \left[D_{M}^{\text{cl}} \Phi_a, A_{M}^{\text{cl}} \right] \bigg|_{\text{reg. gauge}} = -\frac{8\rho^2\xi^2}{(\xi^2 + \rho^2)^3} g^{\text{inst}}t_ag^{\text{inst}}^{-1}. \quad \text{(A.2)}$$

Since $g^{\text{inst}}t_ag^{\text{inst}}^{-1}$ in (A.2), which is given by the RHS of (4.9) with the replacement $x^i \rightarrow -x^i$, is not spherically symmetric, we carry out the spatial angle averaging; $\langle x^a x^b \rangle \Omega = \delta^{ab}\langle x^2 \rangle \Omega = 0$. Then, we get

$$J_{V,a}^0(x, z) \bigg|_{\text{reg. gauge}} = -\frac{2}{\pi^2} \frac{1}{(\xi^2 + \rho^2)^3} \left( z^2 - \frac{x^2}{3} \right) I_a,$$  

(A.3)

which should be compared with (4.44) in the singular gauge. Note that the four-dimensional integration $\int d^3x \int dz$ of (A.3) is superficially logarithmically divergent, and it can take any
value depending on the way of integration. Concretely, this causes the following problem on the isospin density. Integrating (A.3) over $z$, the isospin density in four dimensions in the regular gauge is given by

$$j^0_{V,a}(t,x)\Bigr|_{\text{reg. gauge}} = \int_{-\infty}^{\infty} dz J^0_{V,a}(x,z)\Bigr|_{\text{reg. gauge}} = -\frac{\rho^2}{4\pi} \frac{1}{(r^2 + \rho^2)^{5/2}} I_a. \quad (A.4)$$

Integrating this further over $x$, we get a conflicting result:

$$\int d^3 x j^0_{V,a}(t,x)\Bigr|_{\text{reg. gauge}} = -\frac{1}{3} I_a. \quad (A.5)$$

On the other hand, if we carry out the $x$ integration first and then the $z$ integration, we obtain the consistent result:

$$\int_{-\infty}^{\infty} dz \int d^3 x J^0_{V,a}(x,z)\Bigr|_{\text{reg. gauge}} = I_a. \quad (A.6)$$

We can show (A.6) for a generic $\psi_V(z)$ satisfying the condition $\psi_V(z \to \pm \infty) = 1$, not restricted to $\psi_V(z) = 1$ used in the above calculations. In fact, using that $D_z \Phi_a\Bigr|_{\text{reg. gauge}} = 2\rho^2 z/(\xi^2 + \rho^2)^2 t_a$ after the angle averaging, we get

$$\int d^3 x J^0_{V,a}(x,z)\Bigr|_{\text{reg. gauge}} = \frac{1}{2} \frac{d}{dz} \left( \frac{z}{\sqrt{z^2 + \rho^2}} \psi_V(z) \right) I_a, \quad (A.7)$$

and hence (A.6). Note that (4.44) in the singular gauge is of order $1/\xi^6$ for large $\xi$ and there is no problem of the way of integrations.

The origin of the above explained trouble in the regular gauge is the fact that the gauge fields do not vanish sufficiently fast as $\xi \to \infty$. The baryon solution $A^a_M\Bigr|_{\text{reg. gauge}}$ in the regular gauge tends to the pure gauge $-i g_{\text{inst}} \partial_M g_{\text{inst}}^{-1} = O(1/\xi)$ as $\xi \to \infty$ in contrast with the $O(1/\xi^3)$ behavior of the singular gauge solution (4.7). This slow falloff of the regular gauge solution would be insufficient for the condition (2.12).

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