A fractal method for chaos in conservative closed systems of several dimensions

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Abstract

A fractal method to detect, locate and quantify chaos in multi-dimensional-conservative-closed systems, based on the creation of artificial exits, is presented. The method is invariant under space-time changes of coordinates and can be used to analyse both classical and relativistic Hamiltonian systems of more than two degrees of freedom. As an application of the method we study a couple of two standard maps associated to a periodically kicked rotor of $2\frac{1}{2}$ degrees of freedom.

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1 Introduction

In the last century the mathematical basis of deterministic chaos in compact phase space (closed) systems has become clear [1], and a number of methods have allowed the study of a vast class of particular cases. Important examples of these methods are the Melnikov method, the Poincaré section method, and the Lyapunov exponents method [1, 2]. Alas, the first and second methods are hard to be used in high dimensional systems and the Lyapunov exponents are coordinate dependent [3]. Accordingly, the majority of the coordinate invariant results obtained so far refer to systems of $1\frac{1}{2}$ or two degrees of freedom.

Fractal methods, on the other hand, are coordinate independent and can be used in any dimension. These methods have been largely applied in the characterisation of the chaotic dynamics of dissipative systems, especially in the study of attractors and basin boundaries [2]. Fractal techniques have also been employed to analyse scattering processes and chaotic transients in conservative systems with exits (open systems) [4, 5].

The aim of this Letter is to propose an invariant method to detect, locate and quantify chaos in closed systems of several dimensions. More specifically, we are interested in bounded (recurrent) motions taking place in the absence of attractors and natural exits. The establishment of such a method is pertinent since the standard fractal methods cannot be applied to these cases. Roughly speaking, our method consists of the definition of adequate artificial exits in the original phase space, and the application of fractal type techniques to analyse the exit orbits. This fractal method for closed systems (FMCS) provides a graphic means to locate the chaotic and regular regions on phase space slices. It also leads to an algorithm to determine the chaotic and regular fractions of the phase space volume. Different from the above mentioned methods, the FMCS can be used to study classical as well as relativistic Hamiltonian systems of more than two degrees of freedom. In some sense the FMCS may be seen as a ‘higher-dimensional generalisation of the Poincaré section method’.

First in this Letter we establish a conjecture that will support the FMCS. In Section 3 we present the method divided in three parts: (i) to determine whether the system is chaotic or not; (ii) to locate the chaotic and regular regions on a two-dimensional surface of section; (iii) to compute the chaotic and regular fractions of the phase space volume. In Section 4 we consider
the application of the FMCS to a couple of two standard maps. Finally, we present our conclusions in the last Section.

2 Chaos and fractals

One could say that a system is chaotic if it presents both sensitive dependence on initial conditions and mixing in a nonzero volume of the phase space. This definition, however, is not adequate since it depends on the time parametrization of the system. A time independent definition that solves this problem was given by Churchill [6]: The system is chaotic if it is topologically transitive and the set of (points of) compact orbits is dense, in some positive volume of the phase space. In less technical terms, we define the system as chaotic if it presents, besides transitivity [7], a somewhere dense set of infinitely many unstable periodic orbits. Chaotic systems may exhibit not only chaotic but also nonchaotic (regular) regions. This concept of chaos is consistent with the general principle that chaos prevents integrability in the chaotic regions.

We shall consider each $m$-dimensional manifestly invariant part of the phase space (e.g., each energy surface) of autonomous closed systems, for which we define: (1) An exit $E$ is an $m$-dimensional region of the phase space, so that orbits are considered out of the system when they arrive at $E$. (2) The attraction basin $B_E$ is the closure of the set of initial conditions whose orbits reach $E$. (3) The invariant set $I_E$ is the set of interior points of $B_E$ whose orbits do not reach $E$.

The possible chaotic behaviour of the system is determined by the nature of its invariant sets. For a small exit defined in a chaotic region, the Hausdorff dimension [2] of the invariant set is fractional and tends to the maximum value $m$ (the dimension of the ambient space) when the size of the exit is arbitrarily reduced. (In the case of nonhyperbolic dynamics, the maximum value $m$ can be obtained for exits of finite size [3].) On the other hand, invariant sets associated to exits defined in regular regions do not present any fractal structure, and their dimensions jump (discontinuously) avoiding fractional values when the exits are removed. We conjecture that this behaviour

\footnote{The exit has to be sufficiently small in order to avoid the complete outcome of the invariant set.}
is typical for dynamical systems in general.\footnote{There are pathological examples of nonchaotic systems that exhibit fractal properties when exits are created (e.g., systems with degenerate resonances on a Cantor set \cite{9}). It happens when the invariant set of the exit system is fractal but nonchaotic. This behaviour is, however, atypical.}

This conjecture states that chaos in closed systems and fractals in exit systems are both determined by unstable periodic orbits. Fractal invariant sets are typically chaotic in exit systems and chaos in closed systems is associated to the existence of fractal invariant sets. Therefore, the introduction and removal of exits leads from one situation to the other. Physically, an exit system with fractal invariant set evolves chaotically for a period of time before being scattered. When the exits are removed the system evolves chaotically forever. Further details about this conjecture can be found in Ref. \cite{10}.

3 Fractal method

The above conjecture can be used to obtain a method to study chaos in conservative closed systems of several dimensions (2 or more degrees of freedom in the Hamiltonian case). In what follows we present one possible implementation of such a method, the FMCS.

In order to have an insight about where to look for chaos, we consider the volume of the phase space accessed by a sample of orbits. The idea is that in a chaotic region almost all orbits access approximately the same volume (the volume of that chaotic component). Regular orbits are expected to access lower dimensional surfaces that are arranged in families (of torus in Hamiltonian systems) with increasing area.

First we divide the phase space with a grid and then we evolve a sample of random initial conditions for a large period of time, counting the number of cells of the grid visited by each orbit. In a histogram of the number of visited cells, peaks suggest the possible existence of chaotic regions. We refer to these regions as chaotic candidates. With a search program, a subregion (a ball, for example) can be located inside each chaotic candidate. We define an exit in the subregion of the chaotic candidate that we want to study, and we compute (numerically) the dimension of the corresponding invariant set. Then we use our conjecture to conclude whether the region is chaotic or not. That is the first part of the FMCS.
Given an exit $E$ defined in a chaotic region, the chaotic region itself corresponds to the attraction basin of $E$. To locate the chaotic and regular regions on a two-dimensional surface $S$ (section) of the phase space, we define one exit in each chaotic component of the system. Then we take initial conditions on a grid in $S$ and we evolve these points until the rate of orbits arriving at the exits becomes negligible. The regular regions, corresponding to points whose orbits do not reach the exits, can be appropriately plotted in a two-dimensional graph. In this graph, the chaotic regions are represented by the blank area. That is the second part of the FMCS.

A straightforward extension of the preceding paragraph’s procedure allows us to compute the chaotic and regular fractions of the phase space volume. That is the third part of the FMCS, where the chaotic regions are identified from the evolution of points randomly chosen everywhere in the phase space. The chaotic fraction ($c_f$) is then given by the quotient between the number of initial points in the chaotic regions and the total number of initial conditions. Naturally, the regular fraction ($r_f$) is determined by the condition $r_f + c_f = 1$.

Finally, consistency tests can be made in order to check the results and the hypotheses involved in the FMCS. The possibility that two or more disconnected chaotic components are associated to the same peak of the histogram, for instance, can be verified by comparing the fraction of the volume of each chaotic component with the fraction of points in the corresponding peak of the histogram. The volume of each chaotic region is limited by the corresponding fraction of visited cells. In addition, the stability of the results should be tested by taking different grid sizes, periods of evolution, exits, etc.

The last remark of this section concerns the computation of the Hausdorff dimension of invariant sets. A technique that demands a small computational effort consists of computing the uncertainty dimension of the set of singularities of the escape time function defined on a smooth (one-dimensional) curve $C$: the escape time $t = t(\lambda)$ is the time required by the orbit of an initial point $x(\lambda)$ in $C$ to reach $E$, where $\lambda$ denotes the curve parameter; the singularities of $t$ correspond to points whose escape time is infinite (points of the invariant set). We measure the uncertainty dimension by applying a statistical method presented in [8]. A parameter value $\lambda_0$ is called $\varepsilon$-uncertain if $|t(\lambda_0 + \varepsilon) - t(\lambda_0 - \varepsilon)| \geq \Delta$, where $\Delta$ is a positive number. We compute the fraction $f(\varepsilon)$ of $\varepsilon$-uncertain points for a large number of random values of
the parameter $\lambda$, and for different values of $\varepsilon$ going to zero. The asymptotic behaviour of this function is expected to be of the form $f(\varepsilon) \approx \varepsilon^{(1-d)}$ for any $\Delta$, where $d$ is the uncertainty dimension. The nonfractal case corresponds to $d = 0$ and the fractal one to $0 < d \leq 1$. In the fractal case, the Hausdorff dimension of the (total) invariant set is supposed to be typically equal to $D = m - 1 + d$.

### 4 Example

The KAM surfaces cannot isolate the chaotic layers in nonintegrable Hamiltonian systems of three or more degrees of freedom [11]: All the chaotic volume is joined together into a single global structure or, at most, into a finite number of disjoint components. In such components, chaotic orbits are expected to get arbitrarily close to any point energetically accessible. The hypotheses of our method are therefore satisfied for each value of the energy, and the FMCS is supposed to work in Hamiltonian systems of more than two degrees of freedom. We follow with an example of a Hamiltonian-like system that has an analytical expression for its discrete form.

Section maps can be explicitly obtained for periodically kicked rotors of $2\frac{1}{2}$ degrees of freedom. A simple example is given by the time dependent Hamiltonian

$$H = \frac{I_1^2}{2m_1} + \frac{I_2^2}{2m_2} + \frac{f(\theta_1, \theta_2)}{(2\pi)^2} \sum_n \delta(t - n\tau),$$

(1)

where $\tau/m_1 = \tau/m_2 = 1$ and $f(\theta_1, \theta_2) = K \cos(2\pi \theta_1) \cos(4\pi \theta_2)$, which is equivalent to an autonomous Hamiltonian of three degrees of freedom for a fixed value of the energy. Integrating the delta function at $t = n\tau$ we obtain a four-dimensional map on $[0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$: 

$$I_1^{(n+1)} = I_1^{(n)} + \frac{K}{2\pi} \sin(2\pi \theta_1^{(n)}) \cos(4\pi \theta_2^{(n)}) \mod 1,$$

(2)

$$I_2^{(n+1)} = I_2^{(n)} + \frac{K}{\pi} \cos(2\pi \theta_1^{(n)}) \sin(4\pi \theta_2^{(n)}) \mod 1,$$

(3)

$$\theta_1^{(n+1)} = \theta_1^{(n)} + I_1^{(n+1)} \mod 1,$$

(4)

$$\theta_2^{(n+1)} = \theta_2^{(n)} + I_2^{(n+1)} \mod 1.$$

(5)
These expressions are coupled standard maps, and they also represent a perturbed twist mapping [11].

Let us consider system (2-5) for $K = 0.5$. We divide the phase space with a square grid of $16 \times 16 \times 16 \times 16$ (65536 cells). Then we compute the number $N$ of cells visited by the orbits of 150000 random initial points in 400000 iterations. The result is plotted in the histograms of Fig. 1. In Fig. 1a it can be seen that the histogram presents only two main peaks. In Figs. 1b and 1c we show refinements of each main peak. These peaks do not present smaller isolated peaks. The peak near $N = 0$ (peak I) is consistent with the hierarchical distribution expected for regular regions. The peak near $N = 64000$ (peak II), however, may be associated to chaotic regions. We define as region I and region II the sets of initial conditions associated to peak I and peak II, respectively. Considering these initial conditions we find that all the points in the ball of radius $r = 0.23$ and centre $(I_1, \theta_1, I_2, \theta_2) = (0.278, 0.510, 0.450, 0.761)$ are in region II. We denote this ball by $E_0$, and we use it as an exit to analyse region II.

In Fig. 2a we show the escape time corresponding to points randomly taken on the curve $C := \{I_1 = \theta_1 = \lambda, I_2 = \theta_2 = 0.5 \mid 0.1 < \lambda < 0.2\}$. The complicated structure of this graph is present in arbitrarily large magnifications of the interval. In Fig. 2b we show a 50 times magnification that exhibits this property. That is a graphic characteristic of fractal dimension. In fact, a numerical computation of the uncertainty dimension in the interval of Fig. 2a results in $d = 0.95 \pm 0.01$ for $10^{-13} < \varepsilon < 10^{-9}$ (±0.01 represents the statistical error). In the interval of Fig. 2b the dimension results in $d = 0.97 \pm 0.01$ for $10^{-13} < \varepsilon < 10^{-9}$. By numerically estimating the dimension in smaller and smaller intervals, we approach asymptotically to $d = 1$. This result is consistent with the nonhyperbolic character of the system, since in nonhyperbolic systems the invariant set dimension is supposed to be maximal [8]. Independently of the exact value of $d$, the fact that $0 < d \leq 1$ is enough to conclude that region II is chaotic.

Now, let us consider the section $S := \{(I_1, \theta_1, I_2, \theta_2) \mid I_2 = \theta_2 = 0.5\}$. In order to determine the regular and chaotic regions of $S$, we compute the intersection with $S$ of the attraction basin associated to the exit $E_0$. For initial conditions taken on a grid of $400 \times 400$, with a good approximation, the chaotic points correspond to orbits that outcome in less than 50000 iterations. Both chaotic and regular regions are plotted in Fig. 3, from which some physical results can be directly obtained. For example, since the chaotic
region contains parts of the lines $I_1 = 0$ and $I_1 = 1$, the periodicity in $I_1$ implies that the chaotic region actually runs from $I_1 = -\infty$ to $I_1 = +\infty$. Thus, the evolution of an initial condition in the chaotic region goes to arbitrarily large energies.

In addition, we compute the chaotic fraction of the phase space volume. The attraction basin of $E_0$ is estimated by evolving 150000 random initial points over 50000 iterations. For $K = 0.5$ the chaotic fraction results in $c_f = 0.975 \pm 0.005$. The confidence on this result remains in the fact that no significant changes are observed for 100000 and 400000 iterations. As a function of the parameter $K$, the chaotic fraction goes from zero for $K = 0$ (nonchaotic) to one for $K \approx 1.1$ (completely chaotic).

5 Final remarks

In this communication we presented a fractal method to study chaos in closed systems of several dimension, the FMCS. The method applies to conservative systems in general, not necessarily Hamiltonian. The FMCS is simple from the conceptual viewpoint and is of easy numerical implementation, since it consists of the adequate definition of exits and the subsequent analysis of the corresponding attraction basins and invariant sets.

The importance of this method is double: (1) In classical systems, that are provided by an absolute time parameter, it appears as a systematic method to study chaos in phase spaces of several dimensions; (2) In relativistic systems, that are invariant under space-time diffeomeophisms, this invariant method presents the nice property of avoiding coordinate effects, i.e. the results are independent of the space-time parameters used.

Finally, we consider the limitations of our method. The first one concerns the difficulty of the method in locating small chaotic regions. The method is more efficient in systems with large chaotic regions. In addition, the definition of exits completely inside the chaotic regions may be, sometimes, impossible. The difficulty is due to the presence of small regular regions embedded in the chaotic regions. The effect of these small regions is expected to be negligible, however. The last problem refers to points near the boundaries between chaotic and regular regions. For these points, it is difficult to determine whether they are chaotic or regular. All these problems are intrinsic of numerical methods and are also present, for instance, in the Poincaré section.
method.

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Figure 1: Histograms of the number of cells visited in a grid of $16 \times 16 \times 16 \times 16$ for $K = 0.5$. The orbits of 150000 random initial conditions were computed over 400000 iterations. (a) The histogram of all orbits. (b) Refinement of peak I. (c) Refinement of peak II.

Figure 2: Escape time relative to the exit $E_0$ for $K = 0.5$. (a) 10000 points randomly taken on $C := \{I_1 = \theta_1 = \lambda, I_2 = \theta_2 = 0.5 \mid 0.1 < \lambda < 0.2\}$. (b) A portion of (a) magnified 50 times.

Figure 3: Portrait of the regular (in black) and chaotic (in blank) regions on $S := \{(I_1, \theta_1, I_2, \theta_2) \mid I_2 = \theta_2 = 0.5\}$ for $K = 0.5$. 
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