Sequential Batch Learning in Finite-Action Linear Contextual Bandits

Yanjun Han, Zhengqing Zhou, Zhengyuan Zhou, Jose Blanchet, Peter W. Glynn and Yinyu Ye

April 15, 2020

Abstract

We study the sequential batch learning problem in linear contextual bandits with finite action sets, where the decision maker is constrained to split incoming individuals into (at most) a fixed number of batches and can only observe outcomes for the individuals within a batch at the batch’s end. Compared to both standard online contextual bandits learning or offline policy learning in contextual bandits, this sequential batch learning problem provides a finer-grained formulation of many personalized sequential decision making problems in practical applications, including medical treatment in clinical trials, product recommendation in e-commerce and adaptive experiment design in crowdsourcing.

We study two settings of the problem: one where the contexts are arbitrarily generated and the other where the contexts are iid drawn from some distribution. In each setting, we establish a regret lower bound and provide an algorithm, whose regret upper bound nearly matches the lower bound. As an important insight revealed therefrom, in the former setting, we show that the number of batches required to achieve the fully online performance is polynomial in the time horizon, while for the latter setting, a pure-exploitation algorithm with a judicious batch partition scheme achieves the fully online performance even when the number of batches is less than logarithmic in the time horizon. Together, our results provide a near-complete characterization of sequential decision making in linear contextual bandits when batch constraints are present.

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*Yanjun Han is with the Department of Electrical Engineering, Stanford University, email: yjhan@stanford.edu. Zhengqing Zhou is with the Department of Mathematics, Stanford University, email: zqzhou@stanford.edu. Zhengyuan Zhou is with the Stern School of Business, New York University, email: zzhou@stern.nyu.edu. Jose Blanchet, Peter W. Glynn and Yinyu Ye are with the Department of Management Science and Engineering, Stanford University, email: {jose.blanchet,glynn,yinyu-ye}@stanford.edu.
1 Introduction

With the rapid advances of digitization of the economy, massive amounts of user-specific data have become increasingly available. Among its varied implications, one that holds center-stage importance is the advent of the new era of data-driven personalized decision making: equipped with such user-specific data, decision makers across a wide variety of domains are now able to personalize the service decisions based on individuals’ characteristics, thereby improving the outcomes. Fundamentally, such improved outcomes are achieved by intelligently exploring and exploiting the heterogeneity in a given population, which manifests itself as different individuals responding differently to the same treatments/recommendations/actions. Such heterogeneity is ubiquitous across a variety of applications, including medical treatment selection in clinical trials, product recommendation in marketing, order provisioning in inventory management, hospital staffing in operation rooms, ads selection in online advertising [BM07, LCLS10, KHW+11, HDMZ12, Cha14, COPSL15, BB15, SBF17a, FSLW18, BR19].

Situated in this broader context and rising to materialize the value from personalization, contextual bandits have emerged to be the predominant mathematical framework that is at once rich and elegant. Its three modelling cores, contexts, actions, and rewards (representing individual features, recommendations and outcomes respectively), capture the salient aspects of personalized decision making and provide fertile ground for developing algorithms that contribute to making quality decisions at the fine-grained individual level.

As such, in the past decade, dedicated efforts have been devoted to this area, yielding a flourishing line of literature. Broadly speaking, the existing contextual bandits literature falls into two main categories: offline contextual bandits and online contextual bandits. In offline contextual bandits [DLL11, ZTD+12, ZZRK12, ZZL+14, SJ15, RS16, AW17, KT18, KZ18, ZAW18, JSR18], the decision maker is given a full batch dataset that has been collected from historical observations.
The decision maker aims to learn from this dataset an effective policy (i.e., a function that maps contexts to decisions) that will be deployed in the future and that (hopefully) yields good performance. As such, in offline contextual bandits, the decision maker is not allowed to perform active learning: selecting a policy in this setting is one-shot and the decision maker is solely concerned with identifying the best policy using available data.

On the other hand, in online contextual bandits, the decision maker actively interacts with the data-collection process: as data arrive sequentially, the decision maker can adapt his decisions based on what has been observed in the past, thereby deciding what data is collected. Typically, in such settings, a decision is made on the current individual based on all the past feedback, yielding an outcome that is immediately observed and incorporated to make the next decision. A rich literature has studied this setting—[LCLS10, RT10, FCGS10, RZ10, CLRS11, GZ13, AG13a, AG13b, RVR14, RVR16, JBNW17, LLZ17a, AL⁺17, DZA18, LLZ17b], for a highly incomplete list—and has developed online contextual bandits algorithms (most notably UCB-based algorithms and Thompson sampling based algorithms) that effectively balance exploration with exploitation trade-off, a key challenge therein. See further [BCB⁺12, LS18, S⁺19] for three articulate surveys that more systematically describe the field.

However, in practice, it is often the case that neither setting provides a close approximation of the underlying reality. Specifically, that the decision maker can only perform one-shot policy learning—the key setting in offline contextual bandits—is simply too restrictive and pessimistic for almost all applications. On the other hand, assuming that the decision maker can constantly observe and incorporate feedback at a per-individual scale—the key setting online contextual bandits—is also an over-simplification and too optimistic for many applications. In fact, reality often stands somewhere in between: decision makers across many applications are typically able to perform active learning and incorporate feedback from the past to adapt their decisions in the future; however, due to the physical and cost constraints, such adaptation is often limited to a fixed number of rounds of interaction. For instance, in clinical trials [Rob52, Tho33, KHW⁺11], each trial involves applying medical treatments to a group of patients, where the medical outcomes are observed and collected for the entire group at the end of the trial. The data collected from previous trials are then analyzed to design the medical treatment selection schemes for the next trial. Note that as the medical information from previous trials is incorporated to inform treatment selection in future trials, medical decision makers do have the flexibility in adaptive learning. However, such flexibility is limited since in practice, only a limited number of trials (e.g., 3) can be conducted, far less than the number of patients in the trials, hence rendering the standard online learning models inapplicable here. Another example where adaptive learning is possible but with batch constraints is product recommendation in marketing [BM07, SBF17b]. In this case, the marketer sends out product offers to a batch of customers at once when running a promotions campaign. Customers’ feedback will then be batch collected at the end and analyzed to design the next round of promotions in the targeted population. Other examples include crowdsourcing [KCS08] and simulations [CG09], where both cases exhibit the characteristic that a single-run experiment consists of a batch of individuals.

1.1 Related Work

Motivated by these considerations, we study the problem of sequential batch learning in linear contextual bandits in this paper, where a decision maker can adaptively learn to make decisions subject to the batch constraints that, as described above, commonly occur in practice. This batch constrained bandits problem have recently been studied in the simple multi-armed bandits (MAB) case. In particular, [PRC⁺16] studied sequential batch learning in the 2-armed MAB problem,
where they have given a successive elimination algorithm during each batch and established that $O(\log \log T)$ batches are needed in order to achieve the same regret bound as in the standard online learning setting. Very recently, [GHRZ19] generalized the results to the $K$-armed bandit setting (despite the seeming simplicity, the generalization is not easy) and obtained a tight $\Theta(\log \log T)$ result therein even when the batch sizes can be chosen adaptively.

However, these initial efforts on MAB settings, despite providing interesting insights, cannot capture individuals’ characteristics: in MABs, decisions can only be made at a population level (i.e. the decision maker aims to select an action that is the best for the entire population), rather than personalized at an individual level, which severely limits its practical applicability. In this paper, we fill in this gap by providing a comprehensive inquiry into sequential batch learning in linear contextual bandits with a finite number of actions, a canonical setting where decisions are provisioned based on the individual features. Our goal is then to delineate, in this more general setting, how the batch constraints impact the performance of adaptive decision making and characterize in depth the fundamental limits therein, thereby shedding light on how practical adaptive decision making can be most efficiently done in practice.

Our work is also related to but distinct from learning on bandits (either contextual or MABs) with delayed feedback [NAGS10, DHK+11, JGS13, QK15, GST16, BZC+19, ZXB19]: since the decision maker is not able to observe rewards in the interim of a batch (as they only come at the batch’s end), feedback are delayed from the decision maker’s perspective. However, a key difference exists between learning in bandits and our sequential batch learning model: the former setting works with exogenous delays–drawn either from some stochastic process or from some arbitrary sequence–that, completely contrary to the latter setting, is neither influenced nor known by the decision maker. The literature on learning in bandits with delayed feedback then develops adapted algorithms in the presence of delays and study how regret bounds scale as a function of the underlying delays. On the other hand, feedback delays in sequential batch learning are endogenous and arise as a consequence of the batch constraints; in particular, the decision maker in this setting is at full discretion of choosing the batch sizes which in turn determines how the feedback for each individual is delayed. Consequently, the frameworks provided by the learning-in-bandits-with-delays literature–both the algorithms and analyses–are not applicable here. Of course, it should also be pointed out that when viewed through the lens of learning with delayed feedback, the delays in our setting exhibit a particular structure: if a batch has size $B$, then the reward for the first item in the batch is delayed by $M - 1$ time units, the reward for the second item in the batch is delayed by $M - 2$ time units, and so on, and the reward for the last item in the batch has no delays. Consequently our results here also do not directly imply regret bounds in that literature.

1.2 Our Contributions

We study the sequential batch learning problem in linear contextual bandits when the number of actions is finite (and not too large, to be made precise later). Our main contributions are twofold. First, we consider the adversarial contexts setting, where the contexts can be generated arbitrarily or even adversarially. In this setting, we provide a UCB-style algorithm that is adapted to the sequential batch setting and achieves $\text{polylog}(T) \cdot (\sqrt{dT} + dT/M)$ expected regret, where $d$ is the dimension of the context, $M$ is the number of allowed batches and $T$ is the learning horizon. This regret bound highlights that in the adversarial context setting, $\Theta(\sqrt{dT})$ batches–rather than $T$ batches–are sufficient to achieve the $\tilde{\Theta}(\sqrt{dT})$ regret bound that is minimax optimal in the standard online learning setting. Further, we characterize the fundamental limits of learning in this setting by establishing a $c \cdot (\sqrt{dT} + \min\{T\sqrt{d}/M, T/\sqrt{M}\})$ lower bound for the expected regret, where $c$ is some universal constant. This regret lower bound highlights that at least $O(\sqrt{T})$ batches are
occurs); but for simplicity, let’s assume they are. Then the regret incurred on the first batch–since
critically on how contexts were chosen in the previous times (otherwise, there is no learning that
∥
the third batch contains the rest. Now, for the moment, let’s assume that the contexts selected
observing all the
O
that, as before, comes from the normalization factor on the covariance matrix. Finally, after
achieves a certain level of accuracy. How accurate is our estimate now? From standard theory
and running a least squares on that, we now have an estimate of the underlying parameters that
but simple analysis of instantaneous regret). Now after observing the results from the first batch,
second batch is the number of units in that batch times each individual regret, yielding ˜
O
improve to
θ
imagine
= 1 in mean square). In this case, our bound indicates that the optimal
performance is ˜
O
T/d
4
7
/7
/7
)(log log(T/d)) batches
consider the special case where
M
= 3 (not uncommon in a typical clinical trial) and the context distribution follows
N(0, I/d)
(dividing the identity matrix by
d
simply ensures that the norm of each context is bounded by 1 in mean square). In this case, our bound indicates that the optimal
performance is ˜
O
(d
5
/14
T
4
7
/7
), which is already quite close to ˜
O
(d
√dT )); further, pure exploitation–selecting the best action given our current estimate–on each batch is able to achieve this regret bound.
Why? To get a rough sense, let’s allocate the
T
units into three batches in the following way: the first batch contains ˜
O
(d
6
/7
T
4
7
/7
/7
) units, the second batch contains ˜
O
(d
2
/7
T
6
7
/7
/7
) units and the third batch contains the rest. Now, for the moment, let’s assume that the contexts selected
over time are iid: of course they are not iid, because how one context is chosen at time
T
depends critically on how contexts were chosen in the previous times (otherwise, there is no learning that occurs); but for simplicity, let’s assume they are. Then the regret incurred on the first batch–since
we haven’t observed anything and hence know nothing–is ˜
O
(d
6
/7
T
4
7
/7
/7
/7
) = ˜
O
(d
5
/14
T
4
7
/7
), since each unit incurs ˜
O
(1/√d)
regret (this is a consequence of the normalization as well as a separate but simple analysis of instantaneous regret). Now after observing the results from the first batch, and running a least squares on that, we now have an estimate of the underlying parameters that
achieves a certain level of accuracy. How accurate is our estimate now? From standard theory
on linear regression (which says that if each covariate sample is iid drawn from a standard multi-
variate Gaussian and there are
n
samples, then with high probability
∥θ − θ∗∥2 = ˜
O
(√d/n), we can deduce that after observing ˜
O
(d
6
/7
T
4
7
/7
/7
) samples in the first batch, we would be able to achieve the following estimation accuracy: ˜
∥θ − θ∗∥2 = ˜
O
p(√d/√(d
6
/7
T
4
7
/7
)) = ˜
O
p(d
5
/14
T
3
7
/7
), where the extra √d factor is again a result of our normalization on the covariance matrix. Now, a more accurate such estimate on
θ∗
would yield smaller regret for each individual unit, and if each individual’s regret is inversely proportional to ∥θ − θ∗∥2 as it turns out to be–then the total regret for the second batch is the number of units in that batch times each individual regret, yielding ˜
O
(1/√d) × ˜
O
(d
2
/7
T
6
7
/7
/7
/7
)) = ˜
O
(d
5
/14
T
4
7
/7
), where the ˜
O
(1/√d)
factor is the proportionality constant that, as before, comes from the normalization factor on the covariance matrix. Finally, after observing all the ˜
O
(d
2
/7
T
6
7
/7
/7
/7
) units in the second batch, our estimation accuracy would further improve to ˜
∥θ − θ∗∥2 = ˜
O
p(√d/√(d
2
/7
T
6
7
/7
/7
)) = ˜
O
p(d
5
/14
T
3
7
/7
). Consequently, since there are ˜
O
(T
) units in the third batch, the total regret in this batch would be–by applying the same line of reasoning as in the second batch–O((1/√d) × ˜
O
(T/(d
6
/7
T
3
7
/7
)) = ˜
O
(d
5
/14
T
4
7
/7
). As such, aggregating over all three batches, we obtain ˜
O
(d
5
/14
T
4
7
/7
) total regret. It then remains to complete the reasoning by showing that the selected contexts form a well-conditioned matrix–at least well-
conditioned enough to ensure the same estimation rate in standard linear regression applies—despite being selected in a non-iid way to maximize the rewards, as it turns out to be true.

Additionally, we also give gap-dependent regret bounds—both upper and lower bounds—in the stochastic contexts setting (note that there is no notion of gap when the contexts can be arbitrary). These bounds are typically sharper compared to their gap-independent counterparts (to which the above-mentioned regret bounds belong) when the gap is large; in particular, the dependence on $T$ would be logarithmic rather than $\sqrt{T}$. Section 5 provides a more detailed discussion on this, including how the pure-exploitation algorithm should be (slightly) modified in this case to achieve the minimax optimal gap-dependent regret bound. Finally, we mention that our analyses easily yield high-probability regret bounds as well for all settings, although for simplicity, we have chosen only to present bounds on expected regret.

2 Problem Formulation

We introduce the problem of sequential batch learning on finite-action linear contextual bandits.

2.1 Notation

We start by fixing some notation that will be used throughout the paper. For a positive integer $n$, let $[n] \triangleq \{1, \ldots, n\}$. For real numbers $a, b$, let $a \wedge b \triangleq \min\{a, b\}$. For a vector $v$, let $v^\top$ and $\|v\|_2$ be the transpose and $\ell_2$ norm of $v$, respectively. For square matrices $A, B$, let $\text{Tr}(A)$ be the trace of $A$, and let $A \preceq B$ denote that the difference $B - A$ is symmetric and positive semi-definite. We adopt the standard asymptotic notations: for two non-negative sequences $\{a_n\}$ and $\{b_n\}$, let $a_n = O(b_n)$ iff $\limsup_{n \to \infty} a_n / b_n < \infty$, $a_n = \Omega(b_n)$ iff $b_n = O(a_n)$, and $a_n = \Theta(b_n)$ iff $a_n = O(b_n)$ and $b_n = O(a_n)$. We also write $\tilde{O}(\cdot), \tilde{\Omega}(\cdot)$ and $\tilde{\Theta}(\cdot)$ to denote the respective meanings within multiplicative logarithmic factors in $n$. For probability measures $P$ and $Q$, let $P \otimes Q$ be the product measure with marginals $P$ and $Q$. If measures $P$ and $Q$ are defined on the same probability space, we denote by $\text{TV}(P, Q) = \frac{1}{2} \int |dP - dQ|$ and $D_{\text{KL}}(P||Q) = \int dP \log \frac{dP}{dQ}$ the total variation distance and Kullback–Leibler (KL) divergences between $P$ and $Q$, respectively.

2.2 Decision Procedure and Reward Structures

Let $T$ be the time horizon of the problem. At the beginning of each time $t \in [T]$, the decision maker observes a set of $K$ $d$-dimensional feature vectors (i.e. contexts) $\{x_{t,a} \mid a \in [K]\} \subseteq \mathbb{R}^d$ corresponding to the $t$-th unit. If the decision maker selects action $a \in [K]$, then a reward $r_{t,a} \in \mathbb{R}$ corresponding to time $t$ is incurred (although not necessarily immediately observed). We assume the mean reward is linear: that is, there exists an underlying (but unknown) parameter $\theta^*$ such that

$$r_{t,a} = x_{t,a}^\top \theta^* + \xi_t,$$

where $\{\xi_t\}_{t=0}^\infty$ is a sequence of zero-mean independent sub-Gaussian random variables with a uniform upper bound on the sub-Gaussian constants. Without loss of generality and for notational simplicity, we assume each $\xi_t$ is 1-sub-Gaussian: $\mathbb{E}[e^{\lambda \xi_t}] \leq e^{\lambda^2/2}$, $\forall \lambda \in \mathbb{R}$. Further, without loss of generality (via normalization), we assume $\|\theta^*\|_2 \leq 1$. We denote by $a_t$ and $r_{t,a_t}$ the action chosen and the reward obtained at time $t$, respectively. Note that both are random variables; in particular, $a_t$ is random either because the action is randomly selected based on the contexts $\{x_{t,a} \mid a \in [K]\}$ or because the contexts $\{x_{t,a} \mid a \in [K]\}$ are random, or both.

As there are different (but equivalent) formulations of contextual bandits, we briefly discuss the meaning of the above abstract quantities and how they arise in practice. In general, at each round $t$,
an individual characterized by \( v_t \) (a list of characteristics associated with that individual) becomes available. When the decision maker decides to apply action \( a_t \) to this individual, a reward \( y_t(v_t, a_t) \), which depends (stochastically) on both \( v_t \) and \( a_t \), is obtained. In practice, for both modelling and computational reasons, one often first featureizes the individual characteristics and the actions. In particular, with sufficient generality, one assumes \( \mathbb{E}[y_t(v_t, a_t) | v_t, a_t] = g_\theta(\phi(v_t, a_t)) \), where \( g_\theta(\cdot) \) is the parametrized mean reward function and \( \phi(v_t, a_t) \) extracts the features from the given raw individual characteristics \( v_t \) and action \( a_t \). In the above formulation, as is standard in the literature, we assume the feature map \( \phi(\cdot) \) is known and given and \( x_{t,a} = \phi(v_t, a) \). Consequently, we directly assume access to contexts \( \{x_{t,a} | a \in [K]\} \). Note that the linear contextual bandits setting then corresponds to \( g_\theta(\cdot) \) is linear.

### 2.3 Sequential Batch Learning

In the standard online learning setting, the decision maker immediately observes the reward \( r_{t,a} \) after selecting action \( a_t \) at time \( t \). Consequently, in selecting \( a_t \), the decision maker can base his decision on all the past contexts \( \{x_{\tau,a} | a \in [K], \tau \leq t\} \) and all the past rewards \( \{r_{\tau,a} | \tau \leq t - 1\} \).

In contrast, we consider a sequential batch learning setting, where the decision maker is only allowed to partition the \( T \) units into (at most) \( M \) batches, and the reward corresponding to each unit in a batch can only be observed at the end of the batch. More specifically, given a maximum batch size \( M \), the decision maker needs to choose a sequential batch learning algorithm \( \text{Alg} \) that has the following two components:

1. A grid \( \mathcal{T} = \{t_1, t_2, \ldots, t_M\} \), with \( 0 = t_0 < t_1 < t_2 < \cdots < t_M = T \). Intuitively, this grid partitions the \( T \) units into \( M \) batches: the \( k \)-th batch contains units \( t_{k-1} + 1 \) to \( t_k \). Note that without loss of generality, the decision maker will always choose a grid of \( M \) batches (despite being allowed to choose less than \( M \) batches) since choosing less than \( M \) batches will only decrease the amount of information available and hence yields worse performance. More formally, for any sequential batch learning algorithm that uses a grid of size less than \( M \), there exists another sequential batch learning algorithm that uses a grid of size \( M \) whose performance is no worse.

2. A sequential batch policy \( \pi = (\pi_1, \pi_2, \ldots, \pi_T) \) such that each \( \pi_t \) can only use reward information from all the prior batches (as well as all the contexts that can be observed up to \( t \)). More specifically, for any \( t \in [T] \), define the batched history \( \mathcal{H}^t = \{x_{\tau,a} | a \in [K]\}_{\tau=1}^{t-1} \cup \{r_{\tau,a} \}_{\tau=1}^{j(t)-1} \), where \( j(t) \) is the unique integer satisfying \( t_{j(t)-1} < t \leq t_{j(t)} \). Intuitively, there are \( j(t) - 1 \) batches prior to unit \( t \): only the rewards of those batches have been observed. With this definition, a sequential batch policy is any policy \( \pi \) such that each \( \pi_t \) is adapted to \( \mathcal{H}_t \) for each \( t \in [T] \).

**Remark 1.** \( M = T \) yields the standard online learning setting, where the decision maker need not select a grid. Consequently, the sequential batch learning setting has a more complex decision space—one that entails selecting both the grid and the policy.

### 2.4 Performance Metric: Regret

To assess the performance of a given sequential batch learning algorithm \( \text{Alg} \), we compare the cumulative reward obtained by \( \text{Alg} \) to the cumulative reward obtained by an optimal policy, if the decision maker had the prescient knowledge of the optimal action for each unit (i.e., an oracle that knows \( \theta^* \)). This is formalized by the following definition of regret.
Definition 1. Let \( \text{Alg} = (T, \pi) \) be a sequential batch learning algorithm. The regret of \( \text{Alg} \) is:

\[
R_T(\text{Alg}) \triangleq \sum_{t=1}^{T} \left( \max_{a \in [K]} x_{t,a}^\top \theta^* - x_{t,a_t}^\top \theta^* \right),
\]

where \( a_1, a_2, \ldots, a_T \) are actions generated by \( \text{Alg} \) in the online decision making process.

Remark 2. Although the form of regret defined here is the same as that in standard online learning, the goal here is much more ambitious, because batches induce delays in obtaining reward feedback, and hence the decision maker cannot immediately incorporate the feedback into his subsequent decision making process. Nevertheless, we still make the performance comparison to the oracle that is used in the standard online learning setting. Further, note that \( R_T(\text{Alg}) \) is a random variable since as mentioned earlier, \( a_t \) is random. For simplicity, we will mostly focus on bounding the expected regret, where the expectation is taken with respect to all sources of randomness (to be made precise later in various settings). However, high-probability regret bounds can also be obtained in our setting, and we will discuss them in relevant places throughout the paper.

2.5 Adversarial Contexts v.s. Stochastic Contexts

The regret defined in Definition 1 is also a function of all the contexts that arrive over a horizon of \( T \). In the bandits literature, depending on how these contexts are generated, there are two main categories. The first category is adversarial contexts, where at each \( t \), an adversary can choose the contexts \( \{x_{t,a} \mid a \in [K]\} \) that will be revealed to the decision maker. The second category is stochastic contexts, where at each \( t \), \( \{x_{t,a} \mid a \in [K]\} \) is iid drawn from some fixed underlying distribution.

In the standard online learning case (i.e., \( M = T \)), the optimal regret bounds under adversarial or stochastic contexts are both \( \Theta(\sqrt{dT}) \) (see [CLRS11] and Theorem 2 below). However, it turns out that in the sequential batch setting, there is a sharp difference between the adversarial contexts case and the stochastic contexts case. In particular, it turns out that the power of the adversary in choosing the contexts has a profound impact on what regret bounds can be obtained. Consequently, we divide our presentation into these two distinct regimes: we study the adversarial contexts case in Section 3 and the stochastic contexts case in Section 4. In each of these two settings, we give upper and lower bounds for regret. Finally, throughout the paper, we consider the standard low-dimensional contextual bandits regime where the action set is not too large, as made precise by the following assumption.

Assumption 1. \( K = O(\text{poly}(d)) \) and \( T \geq d^2 \).

That is, we are assuming that the number of actions cannot be too large (at most polynomial in the dimension of the context), for there is a phase transition on the regret when \( K \) can be exponential in \( d \) (see [ABL03]). Further, the time horizon must be large enough, otherwise one cannot even estimate the true parameter \( \theta^* \) within a constant \( \ell_2 \) risk at the end of the time horizon \( T \). Equivalently, we are in the low-dimensional contextual bandits setting where the number of covariates is small compared to the number of samples we receive over the entire horizon.

3 Learning with Adversarial Contexts

In this section, we focus on the adversarial contexts case, where at time \( t \in [T] \), the contexts \( x_{t,1}, \ldots, x_{t,K} \) can be arbitrarily chosen by an adversary who observes all past contexts and rewards,
with \( \| x_{t,a} \|_2 \leq 1 \) for any \( a \in [K] \). We first state the main results in Section 3.1, which characterize the upper and lower bounds of the regret. We then give a UCB-based algorithm in the sequential batch setting in Section 3.2 and describe several important aspects of the algorithm, including a variant that is used for theoretical bound purposes. Next, in Section 3.3, we show that the proposed sequential batching algorithm achieves the regret upper bound in Theorem 1. Finally, we prove the regret lower bound in Section 3.4 and therefore establish that the previous upper bound is close to be tight.

3.1 Main Results

**Theorem 1.** Let \( T, M \) and \( d \) be the learning horizon, number of batches and each context’s dimension, respectively. Denote by \( \text{polylog}(T) \) all the poly-logarithmic factors in \( T \).

1. Under Assumption 1, there exists a sequential batch learning algorithm \( \text{Alg} = (\mathcal{T}, \pi) \), where \( \mathcal{T} \) is a uniform grid defined by \( t_m = \lfloor \frac{mT}{M} \rfloor \) and \( \pi \) is explicitly defined in Section 3.2, such that:

\[
\sup_{\theta^*: \| \theta^* \|_2 \leq 1} \mathbb{E}_{\theta^*}[R_T(\text{Alg})] \leq \text{polylog}(T) \cdot \left( \sqrt{dT} + \frac{dT}{M} \right).
\]

2. Conversely, for \( K = 2 \) and any sequential batch learning algorithm, we have:

\[
\sup_{\theta^*: \| \theta^* \|_2 \leq 1} \mathbb{E}_{\theta^*}[R_T(\text{Alg})] \geq c \cdot \left( \sqrt{dT} + \left( \frac{T \sqrt{d}}{M} \wedge \frac{T}{\sqrt{M}} \right) \right),
\]

where \( c > 0 \) is a universal constant independent of \( (T, M, d) \).

Our subsequent analysis easily gives high-probability regret upper bounds. However, for simplicity and to highlight more clearly the matching between the upper and lower bounds, we stick with presenting results on expected regret. Theorem 1 shows a polynomial dependence of the regret on the number of batches \( M \) under adversarial contexts, and the following corollary is immediate.

**Corollary 1.** Under adversarial contexts, \( \Theta(\sqrt{dT}) \) batches achieve the fully online regret \( \tilde{\Theta}(\sqrt{dT}) \).

According to Corollary 1, \( T \) batches are not necessary to achieve the fully online performance under adversarial contexts: \( \Theta(\sqrt{Td}) \) batches suffice. Since we are not in the high-dimensional regime (per Assumption 1, \( d \leq \sqrt{T} \)), the number of batches needed without any performance suffering is at most \( O(T^{0.75}) \), a sizable reduction from \( O(T) \). Further, in the low-dimensional regime (i.e. when \( d \) is a constant), only \( O(\sqrt{T}) \) batches are needed to achieve fully online performance. Nevertheless, \( O(\sqrt{dT}) \) can still be a fairly large number. In particular, if only a constant number of batches are available, then the regret is linear. The lower bound indicates that not much better can be done in the adversarial contexts. This is because the power of the adversary under adversarial contexts is too strong when the learner only has a few batches: the adversary may simply pick any batch and choose all contexts anterior to this batch to be orthogonal with the contexts within this batch, such that the learner can learn nothing about the rewards in any given batch.

3.2 A Sequential Batch UCB Algorithm

The overall idea of the algorithm is that, at the end of every batch, the learner computes an estimate \( \hat{\theta} \) of the unknown parameter \( \theta^* \) via ridge regression as well as a confidence set that contains \( \theta^* \) with high probability. Then, whenever the learner enters a new batch, at each time \( t \) he simply picks the action with the largest upper confidence bound. Finally, we choose the uniform grid, i.e., \( t_m = \lfloor \frac{mT}{M} \rfloor \) for each \( m \in [M] \). The algorithm is formally illustrated in Algorithm 1.
Algorithm 1: Sequential Batch UCB (SBUCB)

Input: time horizon $T$; context dimension $d$; number of batches $M$; tuning parameter $\gamma > 0$.

Grid choice: $T = \{t_1, \cdots, t_M\}$ with $t_m = \lfloor mT/M \rfloor$.

Initialization: $A_0 = I_d \in \mathbb{R}^{d \times d}$, $\hat{\theta}_0 = 0 \in \mathbb{R}^d$, $t_0 = 0$.

for $m \leftarrow 1$ to $M$ do

  for $t \leftarrow t_{m-1} + 1$ to $t_m$ do

    Choose $a_t = \arg\max_{a \in [K]} x_{t,a}^\top \hat{\theta}_{m-1} + \gamma \sqrt{x_{t,a}^\top A_{m-1}^{-1} x_{t,a}}$ (break ties arbitrarily).

    Receive rewards in the $m$-th batch: $\{r_{t,a}\}_{t_m+1 \leq t \leq t_m}$.

    $A_m = A_{m-1} + \sum_{t = t_{m-1} + 1}^{t_m} x_{t,a} x_{t,a}^\top$.

    $\hat{\theta}_m = A_m^{-1} \sum_{t = t_{m-1} + 1}^{t_m} r_{t,a} x_{t,a}$.

end

Remark 3. Note that when $M = T$ (i.e. the fully online setting), Algorithm 1 degenerates to the standard LinUCB algorithm in [CLRS11].

To analyze the sequential batch UCB algorithm, we need to first show that the constructed confidence bound is feasible. By applying [CLRS11, Lemma 1] to our setting, we immediately obtain the following concentration result that the estimated $\hat{\theta}_{m-1}$ is close to the true $\theta^*$:

Lemma 1. Fix any $\delta > 0$. For each $m \in [M]$, if for a fixed sequence of selected contexts $\{x_{t,a}\}_{t \in [t_m]}$ up to time $t_m$, the (random) rewards $\{r_{t,a}\}_{t \in [t_m]}$ are independent, then for each $t \in [t_{m-1} + 1, t_m]$, with probability at least $1 - \frac{\delta}{T}$, the following holds for all $a \in [K]$:

$$|x_{t,a}^\top (\hat{\theta}_{m-1} - \theta^*)| \leq \left(1 + \frac{1}{2} \log \left(\frac{2KT}{\delta}\right)\right) \sqrt{x_{t,a}^\top A_{m-1}^{-1} x_{t,a}}.$$  

Remark 4. Lemma 1 rests on an important conditional independence assumption of the rewards $\{r_{t,a}\}_{t \in [t_m]}$. However, this assumption does not hold in the vanilla version of the algorithm as given in Algorithm 1. This is because a future selected action $a_t$ and hence the chosen context $x_{t,a}$ depends on the previous rewards. Consequently, by conditioning on $x_{t,a}$, previous rewards, say $r_{t_1}, r_{t_2}$ ($t_1, t_2 < t$), can become dependent. Note the somewhat subtle issue here on the dependence of the rewards: when conditioning on $x_{t,a}$, the corresponding reward $r_t$ becomes independent of all the past rewards $\{r_r\}_{r < t}$. Despite this, when a future $x_{t',a'}$ is revealed ($t' > t$), these rewards (i.e. $r_t$ and all the rewards prior to $r_t$) become coupled again: what was known about $r_t$ now reveals information about the previous rewards $\{r_r\}_{r < t}$, because $r_t$ itself would not determine the selection of $x_{t',a'}$: all those rewards have influence over $x_{t',a'}$. Consequently, a complicated dependence structure is thus created when conditioning on $\{x_{t,a}\}_{t \in [t_m]}$.

This lack of independence issue will be handled with a master algorithm variant of Algorithm 1 discussed in the next subsection. Using the master algorithm to decouple dependencies is a standard technique in contextual bandits that was first developed in [Aue02]. Subsequently, it has been used for the same purpose in [CLRS11, LLZ17b], among others. We will describe how to adapt the master algorithm in our current sequential batch learning setting next. We end this subsection by pointing out that, strictly speaking, our regret upper bound is achieved only by this master algorithm, rather than Algorithm 1. However, we take the conventional view that the master algorithm is purely used as a theoretical construct (to resolve the dependence issue) rather than a practical algorithm that
should actually be deployed in practice. In practice, Algorithm 1 should be used instead. For that reason, we discuss the master algorithm only in the proof.

3.3 Regret Analysis for Upper bound

We start with a simple fact from linear algebra that will be useful later.

**Lemma 2.** ([Aue02, Lemma 11]) Let $A$ be a symmetric matrix such that $I_d \preceq A$, and $x \in \mathbb{R}^d$ be a vector satisfying $\|x\|_2 \leq 1$. Then the eigenvalues $\lambda_1, \ldots, \lambda_d$ of $A$ and the eigenvalues $\nu_1, \ldots, \nu_d$ of $A + xx^\top$ can be rearranged in a way such that $\lambda_i \leq \nu_i$ for all $i \in [d]$, and

$$\text{Tr}(A^{-1}xx^\top) \leq 10 \sum_{j=1}^d \frac{\nu_j - \lambda_j}{\lambda_j}.$$  

We next establish a key technical lemma that will be used in establishing our regret upper bound.

**Lemma 3.** Define $X_m = \sum_{t=t_{m-1}+1}^{t_m} x_{t,a_t}x_{t,a_t}^\top$. We have:

$$\sum_{m=1}^M \sqrt{\text{Tr}(A_{m-1}^{-1}X_m)} \leq \sqrt{10 \log(T+1)} \cdot \left(\sqrt{Md + d \sqrt{\frac{T}{M}}} \right).$$

**Proof.** We start by noting that with the above notation, we have $A_m = A_{m-1} + X_m$ for any $m \in [M]$ with $A_0 = I_d$. Applying Lemma 2 repeatedly, we may rearrange the eigenvalues $\lambda_{m,1}, \ldots, \lambda_{m,d}$ of $A_m$ in such a way that $\lambda_{m-1,j} \leq \lambda_{m,j}$ for all $m \in [M], j \in [d]$, and

$$\sum_{m=1}^M \sqrt{\text{Tr}(A_{m-1}^{-1}X_m)} \leq \sqrt{10} \sum_{m=1}^M \sqrt{d \sum_{j=1}^d \frac{\lambda_{m,j} - \lambda_{m-1,j}}{\lambda_{m-1,j}}}.$$  

(2)

Note that $\lambda_{0,j} = 1$ for all $j \in [d]$. Note further that $\lambda_{M,j} \leq 1 + T, \forall j \in [d]$, which follows from the fact that $z^\top(A_M)z = z^\top(I_d + \sum_{t=1}^T x_{t,a_t}x_{t,a_t}^\top)z = \|z\|_2^2 + \sum_{t=1}^T \|z^\top x_{t,a_t}\|_2^2 \leq (T+1)\|z\|_2^2$, since $\|x_{t,a_t}\|_2 \leq 1$. Consequently, every eigenvalue of $A_M$ must be bounded by $T+1$.

Utilizing the above two pieces of information on $\lambda_{0,j}$ and $\lambda_{M,j}$, we then have the following:

$$\sum_{m=1}^M \sqrt{\frac{d}{j=1} \lambda_{m,j} - \lambda_{m-1,j}} \leq \sqrt{M \sum_{m=1}^{M-1} \sum_{j=1}^d \frac{\lambda_{m,j} - \lambda_{m-1,j}}{\lambda_{m,j}}} = \sqrt{M \sum_{j=1}^d \sum_{m=0}^{M-1} \frac{\lambda_{m+1,j} - \lambda_{m,j}}{\lambda_{m+1,j}}} \leq \sqrt{M \sum_{j=1}^d \frac{\lambda_{M,j}}{x_{\lambda_{0,j}}} \frac{dx}{x}} = \sqrt{M \sum_{j=1}^d \log \lambda_{M,j} \leq \sqrt{Md \log(T+1)},}$$

(3)

where the first inequality follows from $(\sum_{i=1}^n x_i)^2 \leq n \sum_{i=1}^n x_i^2$, for any real numbers $x_1, \ldots, x_n$. 

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where step (a) follows from the basic inequality $\sqrt{a} - \sqrt{b} \leq (a - b)/\sqrt{a}$ for $a, b \geq 0$, and step (b) is due to Cauchy–Schwarz.

Note further that $\lambda_{m,j} - \lambda_{m-1,j} \leq \text{Tr}(X_m) = \sum_{t=m-1+1}^{t_m} \|x_{t,a}\|^2_2 \leq t_m - t_{m-1} = T/M$, we therefore have:

$$
\sum_{m=1}^{M} \left( \sqrt{\sum_{j=1}^{d} \frac{\lambda_{m,j} - \lambda_{m-1,j}}{\lambda_{m,j}}} - \sqrt{\sum_{j=1}^{d} \frac{\lambda_{m,j} - \lambda_{m-1,j}}{\lambda_{m,j}}} \right) \leq \sum_{m=1}^{M} \sqrt{\sum_{j=1}^{d} \frac{(\lambda_{m,j} - \lambda_{m-1,j})^3}{\lambda_{m,j}^2 \lambda_{m,j}}} \leq \sqrt{\frac{T}{M}} \sum_{m=1}^{M} \sum_{j=1}^{d} \frac{\lambda_{m,j} - \lambda_{m-1,j}}{\lambda_{m,j}} \cdot \sqrt{\frac{T}{M}}
$$

where the second inequality follows from the fact that $\lambda_{m-1,j} \geq \lambda_{0,j} = 1$ for any $m \in [M]$. Now combining (2), (3) and (4) completes the proof.

We are now ready to prove the regret upper bound stated in Theorem 1.

**Proof of Statement 1 in Theorem 1.**

1. **Regret bound under conditional independence assumption.**

For a given $\delta > 0$, set the hyper-parameter $\gamma$ in Algorithm 1 to be $1 + \sqrt{\frac{1}{2} \log(\frac{2K^2T}{\delta})}$ for the entire proof. Under the conditional independence assumption in Lemma 1, by a simple union bound over all $t \in [T]$, we have with probability at least $1 - \delta$, the following event holds:

$$
\forall m \in [M], \forall t \in [t_{m-1} + 1, t_m], \forall a \in [K], \ |x_{t,a}^T (\hat{\theta}_{m-1} - \theta^*)| \leq \gamma \sqrt{x_{t,a}^T A_{m-1}^{-1} x_{t,a}}.
$$
On this high probability event (with probability $1 - \delta$), we can bound the regret as follows:

$$R_T(\text{Alg}) = \sum_{t=1}^{T} \left( \max_{a \in [K]} x_{t,a}^\top \theta^* - x_{t,a}^\top \theta^* \right) = \sum_{m=1}^{M} \sum_{t=m-1+1}^{t_m} \left( \max_{a \in [K]} x_{t,a}^\top \theta^* - x_{t,a}^\top \theta^* \right)$$

$$\leq \sum_{m=1}^{M} \sum_{t=m-1+1}^{t_m} \left( \max_{a \in [K]} \left( x_{t,a}^\top \hat{\theta}_{m-1} + \gamma \sqrt{x_{t,a}^\top A_{m-1}^{-1} x_{t,a}} - x_{t,a}^\top \theta^* \right) \right)$$

$$= \sum_{m=1}^{M} \sum_{t=m-1+1}^{t_m} \left( x_{t,a}^\top \hat{\theta}_{m-1} + \gamma \sqrt{x_{t,a}^\top A_{m-1}^{-1} x_{t,a}} - x_{t,a}^\top \theta^* \right)$$

$$\leq \sum_{m=1}^{M} \sum_{t=m-1+1}^{t_m} 2\gamma \sqrt{x_{t,a}^\top A_{m-1}^{-1} x_{t,a}} = 2\gamma \cdot \sum_{m=1}^{M} \sum_{t=m-1+1}^{t_m} 1 \cdot \sqrt{x_{t,a}^\top A_{m-1}^{-1} x_{t,a}}$$

$$\leq 2\gamma \sqrt{\frac{T}{M}} \cdot \sum_{m=1}^{M} \left( \sum_{t=m-1+1}^{t_m} x_{t,a}^\top A_{m-1}^{-1} x_{t,a} \right) = 2\gamma \sqrt{\frac{T}{M}} \cdot \sum_{m=1}^{M} \sqrt{\text{Tr}(A_{m-1}^{-1} X_m)}, \quad (5)$$

where the inequality in (5) follows from Cauchy–Schwartz and the choice of a uniform grid (without loss of generality we assume that $T/M$ is an integer).

Next, setting $\delta = \frac{1}{T}$ (and hence resulting in $\gamma = 1 + \sqrt{\frac{1}{2} \log (2KT^2)}$) and applying Lemma 3 to the upper bound in (5), we immediately obtain that again on this high-probability event:

$$R_T(\text{Alg}) \leq 2\sqrt{10} \left( \sqrt{\frac{1}{2} \log (2KT^2)} + 1 \right) \log(T + 1) \sqrt{\frac{T}{M}} \left( \sqrt{Md} + d \sqrt{\frac{T}{M}} \right)$$

$$= \text{polylog}(T) \cdot (\sqrt{dT} + \frac{dT}{M}). \quad (6)$$

Consequently, taking the expectation of $R_T(\text{Alg})$ yields the same bound as in Equation (6), since with probability at most $\frac{1}{T}$, the total regret over the entire horizon is at most $T$ (each time accumulates at most a regret of 1 by the normalization assumption). Since the regret bound is independent of $\theta^*$, it immediately follows that $\sup_{\theta^* \in \Theta^*} E_\theta[R_T(\pi)] = \text{polylog}(T) \cdot (\sqrt{dT} + \frac{dT}{M})$.

2. **Building a Master algorithm that satisfies conditional independence**

To complete the proof, we need to validate the conditional independence assumption in Lemma 1. Since the length of the confidence intervals does not depend on the random rewards, this task can be done by using a master algorithm SupSBUCB (Algorithm 2), which runs in $O(\log T)$ stages at each time step $t$ similar to [Aue02], which is subsequently adopted in the linear contextual bandits setting [CLRS11] and then in the generalized linear contextual bandits setting [LLZ17b] for the same purpose of meeting the conditional independence assumption. Note that SupSBUCB is responsible for selecting the actions $a_t$ and it does so by calling BaseSBUCB (Algorithm 3), which merely performs regression. This master-base algorithm pair has by now become a standard trick to get around the conditional dependency in the vanilla UCB algorithm for a variety of contextual bandits problems (by sacrificing at most $O(\log T)$ regret).
Algorithm 2: SupSBUCB

**Inputs:** $T, M \in \mathbb{Z}_{++}$, Grid $\mathcal{T} = \{t_1, t_2, \cdots, t_M\}$.

$S \leftarrow \log(T), \Psi^s_1 \leftarrow \emptyset$ for all $s \in [S]$.

for $m = 1, 2, \cdots, M$ do

Initial $\Psi^s_{m+1} \leftarrow \Psi^s_m$ for all $s' \in [S]$.

for $t = t_{m-1} + 1, \ldots, t_m$ do

$s \leftarrow 1$ and $\hat{A}_1 \leftarrow [K]$.

**Repeat:**

Use BaseSBUCB with $\Psi^s_m$ to compute $\theta^s_m$ and $A^s_m$.

For all $a \in \hat{A}_s$, compute $w^s_{t,a} = \gamma \sqrt{x^T_{t,a} (A^{-1}_m) x_{t,a}}, \hat{r}^s_{t,a} = \langle \theta^s_m, x_{t,a} \rangle$.

(a) If $w^s_{t,a} \leq 1/\sqrt{T}$ for all $a \in \hat{A}_s$, choose $a_t = \arg \max_{a \in \hat{A}_s} (\hat{r}^s_{t,a} + w^s_{t,a})$.

(b) Else if $w^s_{t,a} \leq 2^{-s}$ for all $a \in \hat{A}_s$,

$\hat{A}_{s+1} \leftarrow \{a \in \hat{A}_s | \hat{r}^s_{t,a} + w^s_{t,a} \geq \max_{a' \in \hat{A}_s} (\hat{r}^s_{t,a'} + w^s_{t,a'}) - 2^{1-s}\}$,

$s \leftarrow s + 1$.

(c) Else choose any $a_t \in \hat{A}_s$ such that $w^s_{t,a_t} > 2^{-s}$, Update $\Phi^s_{m+1} \leftarrow \Phi^s_{m+1} \cup \{t\}$.

Until an action $a_t$ is found.

end

end

Algorithm 3: BaseSBUCB

**Input:** $\Psi_m$.

$A_m = I_d + \sum_{\tau \in \Psi_m} x_{t,a} x_{t,a}'$

$c_m = \sum_{\tau \in \Psi_m} r_{t,a} x_{t,a}$

$\theta_m = A^{-1}_m c_m$

**Return** $(\theta_m, A_m)$.

More specifically, the master algorithm developed by [ACBF02] has the following structure: each time step is divided into at most $\log T$ stages. At the beginning of each stage $s$, the learner computes the confidence interval using only the previous contexts designated as belonging to that stage and selects any action whose confidence interval has a large length (exceeding some threshold). If all actions has a small confidence interval, then we end this stage, observe the rewards of the given contexts and move on to the next stage with a smaller threshold on the length of the confidence interval. In other words, conditional independence is obtained by successive manual masking and revealing of certain information. One can intuitively think of each stage $s$ as a color, and each time step $t$ is colored using one of the $\log T$ colors (if colored at all). When computing confidence intervals and performing regression, only previous contexts that have the same color are used, instead of all previous contexts.

Adapting this algorithm to the sequential batch setting is not difficult: we merely keep track of the sets $\Psi^s_m$ ($s \in [\log T]$) per each batch $m$ (rather than per each time step $t$ as in the fully online learning case). Note that we are still coloring each time step $t$, the difference here lies in the frequency at which we are running BaseSBUCB to compute the confidence bounds and rewards. Due to great similarity we omit the details here and refer to [Aue02, Section 4.3]. In particular, by establishing similar results to [Aue02, Lemma 15, Lemma 16], it is straightforward to show that the regret of the master algorithm SupSBUCB here is enlarged
at most by a multiplicative factor of \(O(\log T)\), which leads to the upper bound in Theorem 1. \(\square\)

### 3.4 Regret Analysis for Lower bound

In this section, we establish the regret lower bound and show that for any fixed grid \(T = \{t_1, \ldots, t_M\}\) and any learner’s policy on this grid, there exists an adversary who can make the learner’s regret at least \(\Omega(\sqrt{Td})\) even if \(K = 2\). Since the lower bound \(\Omega(\sqrt{Td})\) has been proved in [CLRS11] even in the fully online case, it remains to show the lower bound \(\Omega(T \sqrt{d/M} \wedge \sqrt{M})\). Note that in the fully online case, the lower bound \(\Omega(\sqrt{Td})\) given in [CLRS11] is obtained under the same assumption \(d^2 \leq T\) as in Assumption 1.

**Proof of Statement 2 in Theorem 1.** First we consider the case where \(M \geq d/2\), and without loss of generality we may assume that \(d' = d/2\) is an integer (if \(d\) is odd, then we can take \(d' = \frac{d-1}{2}\) and modify the subsequent procedure only slightly). By an averaging argument, there must be \(d'\) batches \(\{i_1, i_2, \ldots, i_{d'}\} \subset [M]\) such that

\[
\sum_{k=1}^{d'} (t_{i_k} - t_{i_{k-1}}) \geq \frac{d'T}{M}.
\]

Now \(\theta^*\) is chosen as follows: Flip \(d'\) independent fair coins to obtain \(U_1, \ldots, U_{d'} \in \{1, 2\}\), and set \(\theta^* = (\theta_1, \ldots, \theta_{d'})\) with \(\theta_{2k-1} = \frac{1}{\sqrt{d'}} \mathbb{1}(U_k = 1), \theta_{2k} = \frac{1}{\sqrt{d'}} \mathbb{1}(U_k = 2), \forall k \in [d']\). (If \(d\) is odd, then the last component \(\theta_{d'}\) is set to 0.)

Note that \(\theta^*\) is a random variable and clearly \(\|\theta^*\|_2 = 1\) (surely). Next the contexts are generated in the following manner: for \(t \in (t_{m-1}, t_m]\), if \(m = i_k\) for some \(k \in [d']\), set \(x_{t,1} = e_{2k-1}, x_{t,2} = e_{2k}\), where \(e_j\) is the \(j\)-th basis vector in \(\mathbb{R}^d\); otherwise, set \(x_{t,1} = x_{t,2} = 0\).

Now we analyze the regret of the learner under this environment. Clearly, for any \(k \in [d']\), the learner has no information about whether \((\theta_{2k-1}, \theta_{2k}) = (1/\sqrt{d'}, 0)\) or \((0, 1/\sqrt{d'})\) before entering the \(i_k\)-th batch, while an incorrect action incurs an instantaneous regret \(1/\sqrt{d'}\). Consequently, averaged over all possible coin flips \((U_1, \ldots, U_{d'}) \in \{1, 2\}^{d'}\), the expected regret is at least:

\[
\frac{1}{2} \sum_{k=1}^{d'} \frac{t_{i_k} - t_{i_{k-1}}}{\sqrt{d'}} \geq \frac{1}{2\sqrt{2}} \frac{T \sqrt{d'}}{M}
\]

due to (7), establishing the lower bound \(\Omega(T \sqrt{d'/M})\) when \(M \geq d/2\).

Next, in the case where \(M < d/2\), choose \(d' = M\). Here, we obviously have \(\sum_{k=1}^{d'} (t_{i_k} - t_{i_{k-1}}) = T\). In this case, again flip \(d'\) independent fair coins to obtain \(U_1, \ldots, U_{d'} \in \{1, 2\}\), and set \(\theta^* = (\theta_1, \ldots, \theta_{d'})\) with \(\theta_{2k-1} = \frac{1}{\sqrt{d'}} \mathbb{1}(U_k = 1), \theta_{2k} = \frac{1}{\sqrt{d'}} \mathbb{1}(U_k = 2), \forall k \in [d']\). Set all remaining components of \(\theta\) to 0. The contexts are generated as follows: for \(t \in (t_{m-1}, t_m]\), if \(m \leq M\), set \(x_{t,1} = e_{2m-1}, x_{t,2} = e_{2m}\). In this case, we again average over all possible coin flips \((U_1, \ldots, U_{d'}) \in \{1, 2\}^{d'}\), and the expected regret is at least:

\[
\frac{1}{2} \sum_{m=1}^{M} \frac{t_m - t_{m-1}}{\sqrt{d'}} = \frac{1}{2} \frac{T}{\sqrt{M}}
\]

Combining the above two cases yields a lower bound of \(\Omega(T \sqrt{d'/M} \wedge \frac{T}{\sqrt{M}})\). \(\square\)
4 Learning with Stochastic Contexts

In this section, we focus on the stochastic contexts case, where at each time $t \in [T]$, each context $x_{t,a}$ is drawn from $\mathcal{N}(0, \Sigma)$, with a possibly unknown covariance matrix $\Sigma$. Note that for each $t$, $x_{t,a}$’s can be arbitrarily correlated across different $a$’s. This is a simple setting that presents an interesting case for study: at a population level, each one of the $K$ actions is equally good; in particular, if the decision maker is not allowed to personalize the action based on the context and hence restricted to choose a single-action policy (i.e. always choose action 1 or action 2 no matter what the contexts are), then all the actions perform equally well. However, as we shall see, being able to select different actions based on the realized contexts allows the decision maker to do much more. We start by making an assumption on the covariance matrix.

**Assumption 2.** The covariance matrix $\Sigma$ satisfies $\frac{\kappa}{d} \leq \lambda_{\min}(\Sigma) \leq \lambda_{\max}(\Sigma) \leq \frac{1}{d}$ for some numerical constant $\kappa > 0$, where $\lambda_{\min}(\Sigma), \lambda_{\max}(\Sigma)$ denote the smallest and the largest eigenvalues of $\Sigma$, respectively.

The upper bound $\lambda_{\max}(\Sigma) \leq 1/d$ in Assumption 2 ensures that $\mathbb{E}\|x_{t,a}\|_2^2 \leq 1$, and therefore the stochastic contexts share the similar constraint with the previous adversarial contexts. The lower bound $\lambda_{\min}(\Sigma) \geq \kappa/d$ ensures that each stochastic context is approximately distributed as an isotropic Gaussian random vector, with a bounded condition number no less than $\kappa^{-1}$. We assume that $\kappa > 0$ is a fixed constant (say 0.1) and will not optimize the dependence on $\kappa$.

The next theorem presents tight regret bounds for the stochastic contexts case.

**Theorem 2.** Let $T, M = O(\log \log T)$ and $d$ be the learning horizon, number of batches and each context’s dimension, respectively. Denote by $\text{polylog}(T)$ all the poly-logarithmic factors in $T$.

1. Under Assumptions 1 and 2, there exists a sequential batch learning algorithm $\text{Alg} = (\mathcal{T}, \pi)$ (explicitly defined in Section 4.1) such that:

$\sup_{\theta^* : \|\theta^*\|_2 \leq 1} \mathbb{E}_{\theta^*}[R_T(\pi)] \leq \text{polylog}(T) \cdot \sqrt{\frac{dT}{\kappa}} \left( \frac{T}{d^2} \right)^{\frac{1}{2(2^M - 1)}}$.

2. Conversely, even when $K = 2$ and contexts $x_{t,a} \sim \mathcal{N}(0, I_d/d)$ are independent over all $a \in [K], t \in [T]$, for any $M \leq T$ and any sequential batch learning algorithm, we have:

$\sup_{\theta^* : \|\theta^*\|_2 \leq 1} \mathbb{E}_{\theta^*}[R_T(\pi)] \geq c \cdot \sqrt{dT} \left( \frac{T}{d^2} \right)^{\frac{1}{2(2^M - 1)}}$,

where $c > 0$ is a numerical constant independent of $(T, M, d)$.

Theorem 2 completely characterizes the minimax regret for the sequential batch learning problem in linear contextual bandits with stochastic contexts, and shows a doubly exponential dependence of the optimal regret on the number of batches $M$. The following corollary is immediate.

**Corollary 2.** Under stochastic contexts, it is necessary and sufficient to have $\Theta(\log \log(T/d^2))$ batches to achieve the fully online regret $\tilde{\Theta}(\sqrt{dT})$.

In contrast to Corollary 1, the above corollary shows that a much smaller number of batches are capable of achieving the fully online performance, which suits better for many practical scenarios. Note that for smaller number of batches, Theorem 2 also gives the tight regrets within logarithmic factors, e.g., the optimal regret is $\tilde{\Theta}(Td^{-1/2})$ when $M = 1$, is $\tilde{\Theta}(T^{2/3}d^{1/6})$ when $M = 2$, is $\tilde{\Theta}(T^{4/7}d^{5/14})$ when $M = 3$, and so on.
4.1 A Sequential Batch Pure-Exploitation Algorithm

In contrast to the adversarial contexts, under stochastic contexts the decision maker enjoys the advantage that he can choose to learn the unknown parameter $\theta^*$ from any desired direction. In other words, the exploration of the learner is no longer subject to the adversary’s restrictions, and strikingly, making decisions based on the best possible inference of $\theta^*$ is already sufficient.

**Algorithm 4:** Sequential Batch Pure-exploitation

**Input:** Time horizon $T$; context dimension $d$; number of batches $M$.

Set $a = \Theta \left( \sqrt{T \cdot \left( \frac{d}{\pi^{d/2}} \right)^{2^d-1}} \right)$

**Grid choice:** $T = \{t_1, \cdots, t_M\}$, with $t_1 = ad$, $t_m = [a\sqrt{t_{m-1}}]$, $m = 2, 3, \cdots, M$.

**Initialization:** $A = 0 \in \mathbb{R}^{d \times d}$, $\hat{\theta} = 0 \in \mathbb{R}^d$.

for $m \leftarrow 1$ to $M$ do

for $t \leftarrow t_{m-1} + 1$ to $t_m$ do

choose $a_t = \arg \max_{a \in [K]} x_{t,a}^\top \hat{\theta}$ (break ties arbitrarily).

receive reward $r_{t,a}$.

end

end

$A \leftarrow A + \sum_{t=t_{m-1}+1}^{t_m} x_{t,a_t} x_{t,a_t}^\top$.

$\hat{\theta} \leftarrow A^{-1} \sum_{t=t_{m-1}+1}^{t_m} r_{t,a_t} x_{t,a_t}$.

The algorithm we use in this setting is quite simple (see Algorithm 4). Specifically, under a particularly chosen grid $T = \{t_1, \cdots, t_M\}$, the learner, at the beginning of each batch, uses the least squares estimate $\hat{\theta}$ of $\theta^*$ based on the data in the previous batches, and then simply selects the action $a \in [K]$ which maximizes the estimated reward $x_{t,a}^\top \hat{\theta}$ for any time $t$ in this batch. Then at the end of each batch, the learner updates his estimate $\hat{\theta}$ of $\theta^*$ based on the new observations from the current batch.

How do we select the grid $T$? Intuitively, in order to minimize overall regret, we must ensure that the regret incurred on each batch is not too large, because the overall regret is dominated by the batch that has the largest regret. Guided by this observation, we can see intuitively an optimal way of selecting the grid must ensure that each batch’s regret is the same (at least orderwise in terms of the dependence of $T$ and $d$): for otherwise, there is a way of reducing the regret order in one batch and increasing the regret order in the other and the sum of the two will still have smaller regret order than before (which is dominated by the batch that has larger regret order). As we shall see later, the following grid choice satisfies this equal-regret-across-batches requirement:

$t_1 = ad$, $t_m = [a\sqrt{t_{m-1}}]$, $m = 2, 3, \cdots, M$, \hspace{1cm} (8)

where the parameter $a = \Theta \left( \sqrt{T \cdot \left( \frac{d}{\pi^{d/2}} \right)^{2^d-1}} \right)$ is chosen so that $t_M = T$.

4.2 Regret Analysis for Upper bound

We now turn to establishing the upper bound in Theorem 2. We again execute a two-step program. First, we prove that Algorithm 4 with the grid $T = \{t_1, \cdots, t_M\}$ in (8) attains the regret upper bound in Theorem 2, assuming the conditional independence assumption (cf. Lemma 5) holds. Second, similar to the master algorithm in the previous section, we then modify Algorithm 4
slightly to validate this condition. One thing to note here is that, unlike in the adversarial contexts case, here the modification is much simpler, as we shall see later.

We start by establishing that the least squares estimator \( \hat{\theta} \) is close to the true parameter \( \theta^* \) at the beginning of every batch with high probability. By the theory of least squares, this would be obvious if the chosen contexts \( x_{t,a_t} \) were i.i.d. Gaussian. However, since the action \( a_t \) depends on all contexts \( (x_{t,a})_{a \in |K|} \) available at time \( t \), the probability distribution of \( x_{t,a_t} \) may be far from isotropic. Consequently, a priori, there might be one or more directions in the context space that were never chosen, hence yielding inaccurate estimation of \( \theta^* \) along that (or those) direction(s). However, as we shall see next, this is not a concern: we establish that the matrix formed by the selected contexts are reasonably well-conditioned, despite being selected in a greedy fashion.

**Lemma 4.** For each \( m \in [M] \), with probability at least \( 1 - O(T^{-4}) \) we have

\[
\lambda_{\min} \left( \sum_{t=t_m-1}^{t_m} x_{t,a_t} x_{t,a_t}^\top \right) \geq c \cdot \frac{\kappa(t_m - t_{m-1})}{d},
\]

where \( c > 0 \) is a numerical constant independent of \( (K, T, d, m, \kappa) \).

The proof of the above lemma is a bit long and hence deferred to the appendix. Based on Lemma 4, we are ready to show that the least squares estimator \( \hat{\theta} \) is close to the true parameter \( \theta^* \) with high probability. For \( m \in [M] \), let \( \hat{\theta}_m \) be the estimate at the end of \( m \)-th batch, and \( A_m = \sum_{t=1}^{t_m} x_{t,a_t} x_{t,a_t}^\top \) be the regression matrix.

**Lemma 5.** For each \( m \in [M] \), if the rewards \( \{r_{t,a_t}\}_{t \in [t_m]} \) up to time \( t_m \) are mutually independent given the selected contexts \( \{x_{t,a_t}\}_{t \in [t_m]} \), then with probability at least \( 1 - O(T^{-3}) \),

\[
\| \hat{\theta}_m - \theta^* \|_2 \leq C d \cdot \sqrt{\frac{\log T}{\kappa t_m}}
\]

for a numerical constant \( C > 0 \) independent of \( (K, T, d, m, \kappa) \).

**Proof.** Proof. By the standard algebra of linear regression, we have:

\[
\hat{\theta}_m - \theta^* = A_m^{-1} \sum_{t=1}^{t_m} x_{t,a_t} (r_{t,a_t} - x_{t,a_t}^\top \theta^*).
\]

Hence, conditioned on the contexts \( \{x_{t,a_t}\}_{t \in [t_m]} \), the noise terms \( r_{t,a_t} - x_{t,a_t}^\top \theta^* \) are independent by the assumption, and each noise term \( r_{t,a_t} - x_{t,a_t}^\top \theta^* \) is 1-sub-Gaussian.

Next, we show that the random vector \( \hat{\theta}_m - \theta^* \) is \( \sigma^2 \)-sub-Gaussian conditioned on the contexts with \( \sigma^2 = \lambda_{\min}(A_m)^{-1} \). To see this, we start by recalling that a centered (i.e. zero-mean) random vector \( V \) is \( v \)-sub-Gaussian if the scalar random variable \( \langle V, u \rangle \) is \( v \)-sub-Gaussian for any unit vector \( u \). Consequently, take any unit vector \( u \in \mathbb{R}^d \), we have:

\[
\langle \hat{\theta}_m - \theta^*, u \rangle = \langle A_m^{-1} \sum_{t=1}^{t_m} x_{t,a_t} (r_{t,a_t} - x_{t,a_t}^\top \theta^*), u \rangle = \sum_{t=1}^{t_m} u^T A_m^{-1} x_{t,a_t} (r_{t,a_t} - x_{t,a_t}^\top \theta^*).
\]

Since each term in the summand is \( (u^T A_m^{-1} x_{t,a_t})^2 \)-sub-Gaussian, and since all of them are independent (after being conditioned on \( \{x_{t,a_t}\}_{t \in [t_m]} \)), their sum is also sub-Gaussian with the sub-Gaussian
constant equal to the sum of the sub-Gaussian constants:

\[
\sum_{t=1}^{t_m} (u^T A_m^{-1} x_{t,a_t})^2 = \sum_{t=1}^{t_m} u^T A_m^{-1} x_{t,a_t} x_{t,a_t}^T A_m^{-1} u = u^T A_m^{-1} \left( \sum_{t=1}^{t_m} x_{t,a_t} x_{t,a_t}^T \right) A_m^{-1} u \\
= u^T A_m^{-1} A_m A_m^{-1} u = u^T A_m^{-1} u \leq \lambda_{\max}(A_m^{-1}) = \lambda_{\min}(A_m)^{-1}.
\]

Since the above inequality holds for any unit vector \( u \), choosing \( \sigma^2 = \lambda_{\min}(A_m)^{-1} \) establishes the claim.

Proceeding further, by Lemma 4, we have for each \( m \in [M] \), with probability at least \( 1 - O(T^{-4}) \)
\[
\lambda_{\min}\left(\sum_{t=t_m-1+1}^{t_m} x_{t,a_t} x_{t,a_t}^T\right) \geq c \frac{\kappa(t_m-t_{m-1})}{d}.
\]
Consequently, by a union bound over all \( M \) (which is at most \( T \)), we have with probability at least \( 1 - O(T^{-3}) \),
\[
\lambda_{\min}\left(\sum_{t=t_m-1+1}^{t_m} x_{t,a_t} x_{t,a_t}^T\right) \geq c \cdot \frac{\kappa(t_m-t_{m-1})}{d}
\]
for all \( m \in [M] \). Since \( \lambda_{\min}(X + Y) \geq \lambda_{\min}(X) + \lambda_{\min}(Y) \) for any symmetric matrices \( X, Y \), it then follows that with probability at least \( 1 - O(T^{-3}) \):
\[
\lambda_{\min}(A_m) = \lambda_{\min}\left(\sum_{t=1}^{m} \sum_{t=t_{l-1}+1}^{t_l} x_{t,a_t} x_{t,a_t}^T\right) \geq \sum_{l=1}^{m} \lambda_{\min}\left(\sum_{t=t_{l-1}+1}^{t_l} x_{t,a_t} x_{t,a_t}^T\right) \geq \frac{c \kappa t_m}{d}.
\]

Finally, since \( \hat{\theta}_m - \theta^* \) is a \( \frac{\sigma}{\sqrt{d} c t_m} \)-sub-Gaussian random vector, \( \|\hat{\theta}_m - \theta^*\|_2^2 \) is a sub-exponential random variable. Therefore, conditioned on the above event for the stochastic contexts, the sub-exponential concentration gives the claimed upper bound on \( \|\hat{\theta}_m - \theta^*\|_2 \) with a further probability at least \( 1 - O(T^{-3}) \) over the random noises. Finally, taking a union bound to complete the proof. \( \square \)

Lemma 5 shows that given the conditional independence assumption, the estimator \( \hat{\theta} \) given by pure exploitation essentially achieves the rate-optimal estimation of \( \theta^* \) even if one purely explores. This now positions us well to prove the upper bound of Theorem 2. Of course, bear in mind that when using Algorithm 4, the conditional independence assumption does not hold, for the choice of future contexts depends on the rewards in the previous batches. Therefore, we will use sample splitting to build another master algorithm to gain independence at the cost of the sample size reduction by a multiplicative factor of \( M \) (recall that \( M = O(\log \log T) \)). The following proof implements these two steps; note that in this setting, the master algorithm is entirely different from and much simpler than the one given in the adversarial case.

**Proof of Statement 1 in Theorem 2.**

1. **Regret bound under conditional independence assumption.**

Consider the \( m \)-th batch with any \( m \geq 2 \), and any time point \( t \) inside this batch. By the definition of \( a_t \), we have \( x_{t,a_t}^\top \hat{\theta}_{m-1} \geq x_{t,a}^\top \hat{\theta}_{m-1} \) for any \( a \in [K] \). Consequently,
\[
\max_{a \in [K]} (x_{t,a} - x_{t,a_t})^\top \theta^* \leq \max_{a \in [K]} (x_{t,a} - x_{t,a_t})^\top (\theta^* - \hat{\theta}_{m-1}) \\
\leq \max_{a,a' \in [K]} (x_{t,a} - x_{t,a'})^\top (\theta^* - \hat{\theta}_{m-1}) \\
\leq 2 \max_{a \in [K]} |x_{t,a}^\top (\theta^* - \hat{\theta}_{m-1})|.
\]

For fixed \( a \in [K] \), marginally we have \( x_{t,a} \sim \mathcal{N}(0, \Sigma) \) independent of \( \hat{\theta}_{m-1} \). Therefore, conditioning on the previous contexts and rewards, we have \( x_{t,a}^\top (\theta^* - \hat{\theta}_{m-1}) \sim \mathcal{N}(0, \sigma^2) \) with
\[
\sigma^2 = (\theta^* - \hat{\theta}_{m-1})^\top \Sigma (\theta^* - \hat{\theta}_{m-1}) \leq \frac{\|\theta^* - \hat{\theta}_{m-1}\|_2^2}{d}.
\]
by Assumption 2. By a union bound over \( a \in [K] \), with probability at least \( 1 - O(T^{-3}) \) over the randomness in the current batch we have

\[
\max_{a \in [K]} (x_{t,a} - x_{t,a_t})^\top \theta^* \leq 2 \max_{a \in [K]} |x_{t,a}^\top (\theta^* - \hat{\theta}_{m-1})| = O \left( \|\theta^* - \hat{\theta}_{m-1}\|_2 \sqrt{\frac{\log(KT)}{d}} \right).
\]

Applying Lemma 5 and another union bound, there exists some numerical constant \( C' > 0 \) such that with probability at least \( 1 - O(T^{-3}) \), the instantaneous regret at time \( t \) is at most

\[
\max_{a \in [K]} (x_{t,a} - x_{t,a_t})^\top \theta^* \leq C' \sqrt{\log(KT) \log T} \cdot \sqrt{\frac{d}{\kappa t \ln m}}.
\]

Now taking the union bound over \( t \in [T] \), the total regret incurred after the first batch is at most

\[
\sum_{m=2}^M C' \sqrt{\log(KT) \log T} \cdot \frac{d}{\kappa t \ln m} \leq C' \sqrt{\log(KT) \log T} \cdot M \cdot a \sqrt{d} \tag{9}
\]

with probability at least \( 1 - O(T^{-2}) \), where the inequality is due to the choice of the grid in (8).

As for the first batch, the instantaneous regret at any time point \( t \) is at most the maximum of \( K \) Gaussian random variables \( \mathcal{N}(0, (\theta^*)^\top \Sigma \theta^*) \). Since \( \|\theta^*\|_2 \leq 1 \) and \( \lambda_{\max}(\Sigma) \leq 1/d \), we conclude that the instantaneous regret is at most \( C'' \sqrt{\log(KT)/d} \) for some constant \( C'' > 0 \) with probability at least \( 1 - O(T^{-3}) \). Now by a union bound over \( t \in [t_1] \), with probability at least \( 1 - O(T^{-2}) \) the total regret in the first batch is at most

\[
C'' \sqrt{\log(KT)/d} \cdot t_1 = C'' \sqrt{\log(KT)} \cdot a \sqrt{d}. \tag{10}
\]

Now combining (9), (10) and the choice of \( a \) in Algorithm 4 gives the desired regret bound in Theorem 2 with high probability (note that \( M = O(\log \log T) \)), and consequently in expectation.

2. Building a Master algorithm that satisfies conditional independence

We start by proposing a sample splitting based master algorithm (see Algorithm 5) that ensures that when restricting to the subset of observations used for constructing \( \hat{\theta} \), the rewards are conditionally independent given the contexts. The key modification in Algorithm 5 lies in the computation of the estimator \( \hat{\theta}_m \) after the first \( m \) batches. Specifically, instead of using all past contexts and rewards before \( t_m \), we only use the past observations inside the time frame \( T^{(m)} \subseteq [t_m] \) to construct the estimator. The key property of the time frames is the disjointness, i.e., \( T^{(1)}, \ldots, T^{(M)} \) are pairwise disjoint. Then the following lemma shows that the conditional independence condition holds within each time frame \( T^{(m)} \).

**Lemma 6.** For each \( m \in [M] \), the rewards \( \{r_{t,a_t}\}_{t \in T^{(m)}} \) are mutually independent conditioning on the selected contexts \( \{x_{t,a_t}\}_{t \in T^{(m)}} \).

**Proof.** For \( t \in T^{(m)} \), the action \( a_t \) only depends on the contexts \( \{x_{t,a}\}_{a \in [K]} \) at time \( t \) and the past estimators \( \hat{\theta}_1, \ldots, \hat{\theta}_{m-1} \). However, for any \( m' \in [m - 1] \), the estimator \( \hat{\theta}_{m'} \) only depends on the contexts \( x_{\tau,a_{\tau}} \) and rewards \( r_{\tau,a_{\tau}} \) with \( \tau \in T^{(m')} \). Repeating the same arguments for the action \( a_{\tau} \) with \( \tau \in T^{(m')} \), we conclude that \( a_t \) only depends on the contexts \( \{x_{\tau,a}\}_{a \in [K], \tau \in \cup_{m' \leq m-1} T^{(m')}} \) and rewards \( \{r_{\tau,a_{\tau}}\}_{\tau \in \cup_{m' \leq m-1} T^{(m')}} \). Consequently, by the disjointness of \( T^{(m)} \) and \( \cup_{m' \leq m-1} T^{(m')} \), the desired conditional independence holds. \( \square \)
Algorithm 5: Batched Pure-exploitation (with sample splitting)

**Input:** Time horizon $T$; context dimension $d$; number of batches $M$; grid $\mathcal{T} = \{t_1, \cdots, t_M\}$ same as in Algorithm 4.

**Initialization:** Partition each batch into $M$ intervals evenly, i.e., $(t_m, t_{m+1}] = \cup_{j=1}^{M} \mathcal{T}_{m}^{(j)}$.

for $m \leftarrow 1$ to $M$ do
  if $m = 1$ then
    choose $a_t = 1$ and receives reward $r_{t,a_t}$ for any $t \in [1, t_1]$.
  end
  else
    for $t \leftarrow t_{m-1} + 1$ to $t_m$ do
      choose $a_t = \arg \max_{a \in [K]} x_{t,a}^\top \hat{\theta}_{m-1}$ (break ties arbitrarily).
      receive reward $r_{t,a_t}$.
    end
    $T^{(m)} \leftarrow \bigcup_{m'=1}^{m} T^{(m')}$.
    $A_m \leftarrow \sum_{t \in T^{(m)}} x_{t,a_t} x_{t,a_t}^\top$. 
    $\hat{\theta}_m \leftarrow A_m^{-1} \sum_{t \in T^{(m)}} r_{t,a_t} x_{t,a_t}$.
  end

Output: resulting policy $\pi = (a_1, \cdots, a_T)$.

By Lemma 6, the conditional independence condition of Lemma 5 holds for Algorithm 5. Moreover, the sample splitting in Algorithm 5 reduces the sample size by a multiplicative factor at most $M$ at each round, and $M = O(\log \log T)$, therefore all proofs in Section 4.1 continue to hold with a multiplicative penalty at most doubly logarithmic in $T$. As a result, Algorithm 5 achieves the regret upper bound in Theorem 2.

4.3 Lower bound

In this section we prove the minimax lower bound of the regret under stochastic contexts for $K = 2$. The lower bound argument for the stochastic context case is quite involved and we start by establishing the following key lemma.

**Lemma 7.** For any fixed grid $0 = t_0 < t_1 < \cdots < t_M = T$ and any $\Delta \in [0, 1]$, the following minimax lower bound holds for any policy $\pi$ under this grid:

$$
\sup_{\theta^* : \|\theta^*\|_2 \leq 1} \mathbb{E}[R_T(\pi)] \geq \Delta \cdot \sum_{m=1}^{M} \frac{t_m - t_{m-1}}{10\sqrt{d}} \exp \left( -\frac{16t_{m-1}\Delta^2}{d^2} \right).
$$

Proof. Let $\theta^* = \theta \sim \text{Unif}(\Delta S^{d-1})$ be uniformly distributed on the $d$-dimensional sphere centered at the origin with radius $\Delta$. Clearly $\|\theta^*\|_2 \leq 1$ surely since $\Delta \leq 1$. Hence,

$$
\sup_{\theta^* : \|\theta^*\|_2 \leq 1} \mathbb{E}[R_T(\pi)] \geq \mathbb{E}_\theta[\mathbb{E}[R_T(\pi)] = \sum_{t=1}^{T} \mathbb{E}_\theta \left( \mathbb{E} \left[ \max_{i \in \{1,2\}} (x_{t,i} - x_{t,a})^\top \theta \right] \right).
$$

We will lower bound each term in the RHS of (11) separately. Note that there are multiple sources of randomness involved in the expectation: the randomness in the parameter $\theta$, in the contexts...
Lemma 8. Let \( x_{t,i} \), and in all the past rewards which determine the random action \( a_t \). Throughout the proof, \( \mathbb{E}_\theta \) denotes taking expectation with respect to \( \theta \), \( \mathbb{E}_x \) denotes taking expectation with respect to all (past and current) random contexts, and \( P_{\theta,x}^t \) denotes the distribution of all random rewards observable before time \( t \) conditioned on the parameter \( \theta \) and contexts \( x \), with \( \mathbb{E}_{P_{\theta,x}^t} \) being the corresponding expectation.

Note that for each \( t \in [T] \), we have \( \max_{i\in\{1,2\}} \langle x_{t,i} - x_{t,a_t} \rangle^\top \theta = \langle x_{t,1} - x_{t,2}, \theta \rangle_+ + \mathbbm{1}(a_t = 2) + \langle x_{t,1} - x_{t,2}, \theta \rangle_+ \cdot \mathbbm{1}(a_t = 1) \), where we define \( \langle u, v \rangle_+ = \max\{0, u^\top v\} \) and \( \langle u, v \rangle_- = \max\{0, -u^\top v\} \).

Taking expectations on both sides gives

\[
\mathbb{E}_\theta \mathbb{E}_{P_{\theta,x}^t} \left[ \max_{i\in\{1,2\}} \langle x_{t,i} - x_{t,a_t} \rangle^\top \theta \right] = \mathbb{E}_\theta \left[ \langle x_{t,1} - x_{t,2}, \theta \rangle_+ + \mathbbm{1}(a_t = 2) + \langle x_{t,1} - x_{t,2}, \theta \rangle_+ \cdot \mathbbm{1}(a_t = 1) \right]
\]

\[
= Z_0 \cdot \left( \mathbb{E}_{Q_1} P_{\theta,x}^t (a_t = 2) + \mathbb{E}_{Q_2} P_{\theta,x}^t (a_t = 1) \right),
\]

where in the last identity (12) we define two new probability distributions of \( \theta \) via

\[
\frac{dQ_1}{dQ_0}(\theta) = \frac{\langle x_{t,1} - x_{t,2}, \theta \rangle_+}{Z_0}, \quad \frac{dQ_2}{dQ_0}(\theta) = \frac{\langle x_{t,1} - x_{t,2}, \theta \rangle_-}{Z_0},
\]

where \( Q_0 = \text{Unif}(\Delta \mathbb{S}^{d-1}) \) is the original probability measure of \( \theta \), \( Z_0 \) is the common normalization factor, and \( \mathbb{E}_{Q_i} P_{\theta,x}^t \) denotes the mixture distribution of \( z \sim P_{\theta,x}^t \) where \( \theta \sim Q_i \), for \( i \in \{1, 2\} \). The following lemma investigates some properties of \( Q_1 \) and \( Q_2 \).

**Lemma 8.** Let \( x_{t,1} - x_{t,2} = r_t u_t \) with \( r_t \geq 0, \|u_t\|_2 = 1 \). Then \( \theta \sim Q_1 \) if and only if \( \theta - 2(u_t^\top \theta)u_t \sim Q_2 \).

Moreover, we have

\[
Z_0 = r_t \Delta \cdot \begin{cases} \frac{2^d}{\pi d/2} (d/2)^{-1}, & \text{if } d \text{ is even} \\ \frac{1}{2^d} (d-1)/2, & \text{if } d \text{ is odd} \end{cases}, \quad \mathbb{E}_{Q_1}(u_t^\top \theta)^2 = \mathbb{E}_{Q_2}(u_t^\top \theta)^2 = \frac{2\Delta^2}{d+1}.
\]

The proof of Lemma 8 is postponed to the appendix. Continuing from (12), we have

\[
\mathbb{E}_{Q_1} P_{\theta,x}^t (a_t = 2) + \mathbb{E}_{Q_2} P_{\theta,x}^t (a_t = 1) \overset{(a)}{=} 1 - \text{TV}(\mathbb{E}_{Q_1} P_{\theta,x}^t, \mathbb{E}_{Q_2} P_{\theta,x}^t)
\]

\[
\geq \frac{1}{2} \exp \left( -D_{\text{KL}}(\mathbb{E}_{Q_1} P_{\theta,x}^t \| \mathbb{E}_{Q_2} P_{\theta,x}^t) \right)
\]

\[
\overset{(c)}{=} \frac{1}{2} \exp \left( -D_{\text{KL}}(\mathbb{E}_{Q_1} P_{\theta,x}^t \| \mathbb{E}_{Q_2} P_{\theta-2(u_t^\top \theta)u_t,x}^t) \right)
\]

\[
\overset{(d)}{=} \frac{1}{2} \exp \left( -D_{\text{KL}}(P_{\theta,x}^t \| P_{\theta-2(u_t^\top \theta)u_t,x}^t) \right),
\]

where step (a) follows from Le Cam’s first lemma (cf. e.g., [Tsy08]), step (b) is due to Lemma 9 in Appendix A, step (c) follows from Lemma 8, and step (d) is due to the joint convexity of the KL divergence. For \( t \in (t_{m-1}, t_m] \), the learner can only observe rewards up to time \( t_{m-1} \) at time \( t \), and
Combining (12) to (16), we arrive at

\[
\mathbb{E}_\theta \mathbb{E}_{P_{\theta,x}^{t}} \left[ \max_{i \in \{1,2\}} (x_{t,i} - x_{t,a_t})^\top \theta \right] \geq \frac{r_t \Delta}{10 \sqrt{d}} \exp \left( - \frac{4 \Delta^2}{d+1} u_t^\top \left( \sum_{\tau=1}^{t_{m-1}} x_{t,a_{\tau}} x_{t,a_{\tau}}^\top \right) u_t \right) \geq \frac{r_t \Delta}{10 \sqrt{d}} \exp \left( - \frac{4 \Delta^2}{d+1} u_t^\top \left( \sum_{\tau=1}^{t_{m-1}} (x_{t,1} x_{t,1}^\top + x_{t,2} x_{t,2}^\top) \right) u_t \right).
\]

Finally, we take the expectation with respect to the contexts \( x \). Using the independence of \( \{x_{t,i}\}_{\tau<t} \) and \( (r_t, u_t) \), the convexity of \( x \mapsto \exp(-x) \) and \( \mathbb{E}[x_{t,i} x_{t,\tau}^\top] = I_d/d \), we arrive at

\[
\mathbb{E}_x \mathbb{E}_\theta \mathbb{E}_{P_{\theta,x}^{t}} \left[ \max_{i \in \{1,2\}} (x_{t,i} - x_{t,a_t})^\top \theta \right] \geq \frac{\mathbb{E}[r_t] \Delta}{10 \sqrt{d}} \exp \left( - \frac{8 \Delta^2 t_{m-1}}{d(d+1)} \right) \geq \frac{\Delta}{10 \sqrt{d}} \exp \left( - \frac{16 \Delta^2 t_{m-1}}{d^2} \right),
\]

where in the last inequality we have used

\[
\mathbb{E}[r_t] = \mathbb{E}\|x_{t,1} - x_{t,2}\|_2 \geq \frac{\mathbb{E}\|x_{t,1} - x_{t,2}\|_1}{\sqrt{d}} = \frac{2}{\sqrt{\pi}} > 1.
\]

Combining (11) and (17) completes the proof of Lemma 7. \( \square \)

We are now ready to put everything together and complete the proof of the lower bound.

**Proof of Statement 2 in Theorem 2.** For any fixed grid \( T = \{t_1, \cdots, t_M\} \), define \( s = \min\{m \in [M] : t_m \geq d^2\} \), which always exists due to our assumption that \( T \geq d^2 \). Now choosing some candidates of \( \Delta \in \{1, \frac{d}{\sqrt{t_s}}, \frac{d}{\sqrt{t_{s+1}}}, \cdots, \frac{d}{\sqrt{T}}\} \subset [0, 1] \) in Lemma 7 gives

\[
\sup_{\theta^* : \|\theta^*\|_2 \leq 1} \mathbb{E}[R_T(\pi)] \geq c \cdot \max \left\{ \frac{t_s}{\sqrt{d}}, \frac{t_{s+1}}{\sqrt{d}}, \frac{t_{s+2}}{\sqrt{d}}, \cdots, \frac{T}{\sqrt{t_{M-1}}} \right\}
\]

for some numerical constant \( c > 0 \). After some algebra, the right-hand side of (18) may be further lower bounded by

\[
\sup_{\theta^* : \|\theta^*\|_2 \leq 1} \mathbb{E}[R_T(\pi)] \geq c \sqrt{dT} \cdot \left( \frac{T}{d^2} \right)^{\frac{1}{2(2^{M-s}+1)-1}} \geq c \sqrt{dT} \cdot \left( \frac{T}{d^2} \right)^{\frac{1}{2(2^M-1)}}.
\]

\( \square \)
5 Problem-Dependent Regret Bounds

The regret bounds given in the previous two sections are problem-independent regret bounds (also known as gap-independent regret bounds in the bandits literature): they do not depend on the underlying parameters of the probability distribution. When the contexts are stochastic, under certain “margin” conditions, we can also consider problem-dependent regret bounds that can result in sharper bounds than those problem-independent ones. When the number of contexts is small (e.g., \( K = 2 \)), there could be a large margin between the performance of the optimal context and any sub-optimal contexts if \( \| \theta^* \|_2 \) is bounded away from zero, raising the possibility that a problem-dependent regret bound sometimes better than the worst-case regret \( \Theta(\sqrt{dT}) \) could be obtained in sequential batch learning. The next theorem characterizes this.

**Theorem 3.** Assume \( K = 2 \), and let \( T, M = O(\log T) \), \( d \) be the learning horizon, number of batches and the dimension of each context respectively. Denote by \( \text{polylog}(T) \) all the poly-logarithmic factors in \( T \). Assume without loss of generality \( \| \theta^* \|_2 > 0 \).

1. Under Assumptions 1 and 2, there exists a sequential batch learning algorithm \( \text{Alg} = (T, \pi) \) (explicitly defined below) that achieves the following regret:

\[
\mathbb{E}_{\theta^*}[R_T(\pi)] \leq \text{polylog}(T) \cdot \frac{(d/\kappa)^{3/2}}{\| \theta^* \|_2} \left( \frac{T}{d^2} \right)^{\frac{1}{M}}.
\]

2. Conversely, when the contexts \( x_{t,a} \sim \mathcal{N}(0, I_d/d) \) are independent over all \( a \in [K], t \in [T] \), for any \( M \leq T \) and any sequential batch learning algorithm, we have:

\[
sup_{\theta^*: \| \theta^* \|_2 \leq 1} \| \theta^* \|_2 \cdot \mathbb{E}_{\theta^*}[R_T(\pi)] \geq c \cdot d^{3/2} \left( \frac{T}{d^2} \right)^{\frac{1}{M}},
\]

where \( c > 0 \) is a numerical constant independent of \( (T, M, d) \).

**Corollary 3.** In this setting, it is necessary and sufficient to have \( \Theta(\log(T/d^2)) \) batches to achieve the optimal problem-dependent regret \( \tilde{\Theta}(d^{3/2}/\| \theta^* \|_2) \). Here we are not aiming to get the tightest dependence on \( \log T \) (note that \( \tilde{\Theta}(\cdot) \) hides polylog factors).

Note that the dependence on \( T \) is significantly better than \( \sqrt{T} \) in the problem-dependent bound, showing that a large \( \| \theta^* \|_2 \) makes learning simpler. We remark that although the problem-dependent regret in Theorem 3 only holds for \( K = 2 \), the generalization to a generic \( K \) is straightforward. Moreover, the margin between the optimal context and the sub-optimal context shrinks quickly as \( K \) gets larger, and therefore the margin-based problem-dependent bound is not that useful compared with the worst-case regret bound in Theorem 2 for large \( K \).

5.1 Proof of the Upper Bound in Theorem 3

The sequential batch learning algorithm which achieves the claimed upper bound is exactly the batched pure-exploitation algorithm with sample splitting shown in Algorithm 5, with a different choice of the grid: we consider a geometric grid \( T' = \{t'_1, t'_2, \ldots, t'_M\} \) with

\[
t'_1 = bd^2, \quad t'_m = [bt'_{m-1}], \quad m = 2, 3, \ldots, M,
\]

where \( b = \Theta((T/d^2)^{1/M}) \) so that \( t'_M = T \). Next we show that with the above choice of the grid, Algorithm 5 attains the regret upper bound in Theorem 3.
Consider the $m$-th batch with any $m \geq 2$, and any time point $t$ inside this batch. Define $v_t = x_{t,1} - x_{t,2}$, then our algorithm chooses the wrong arm if and only if $v_t^\top \theta^*$ and $v_t^\top \hat{\theta}_{m-1}$ have different signs. Hence, the instantaneous regret at time $t$ is
\[ v_t^\top \theta^* \cdot 1(v_t^\top \theta^* \geq 0, v_t^\top \hat{\theta}_{m-1} \leq 0) - v_t^\top \theta^* \cdot 1(v_t^\top \theta^* \leq 0, v_t^\top \hat{\theta}_{m-1} \geq 0), \]
and by the symmetry of $v_t \sim \mathcal{N}(0, 2\Sigma)$, it holds that
\[ \mathbb{E} \left[ \max_{a \in \{1, 2\}} (x_{t,a} - x_{t,1})^\top \theta^* \right] = 2 \mathbb{E} \left[ v_t^\top \theta^* \cdot 1(v_t^\top \theta^* \geq 0, v_t^\top \hat{\theta}_{m-1} \leq 0) \right]. \]
Set $\delta = \sqrt{d \log T / (\kappa t'_{m-1})}$, and partition the non-negative axis $\mathbb{R}_+$ into $\bigcup_{i=0}^{\infty} (i\delta, (i+1)\delta)$. Using this partition gives
\[ \mathbb{E} \left[ v_t^\top \theta^* \cdot 1(v_t^\top \theta^* \geq 0, v_t^\top \hat{\theta}_{m-1} \leq 0) \cdot 1(\|v_t\|_2 \leq \sqrt{10 \log T}) \right] 
= \sum_{i=0}^{\infty} \mathbb{E} \left[ v_t^\top \theta^* \cdot 1(v_t^\top \theta^* \in [i\delta, (i+1)\delta), v_t^\top \hat{\theta}_{m-1} \leq 0) \cdot 1(\|v_t\|_2 \leq \sqrt{10 \log T}) \right] 
\leq \sum_{i=0}^{\infty} (i+1)\delta \cdot \mathbb{P} \left( v_t^\top \theta^* \in [i\delta, (i+1)\delta), v_t^\top \hat{\theta}_{m-1} \leq 0, \|v_t\|_2 \leq \sqrt{10 \log T} \right) 
\leq \sum_{i=0}^{\infty} (i+1)\delta \cdot \mathbb{P} \left( v_t^\top \theta^* \in [i\delta, (i+1)\delta), v_t^\top (\theta^* - \hat{\theta}_{m-1}) \geq i\delta, \|v_t\|_2 \leq \sqrt{10 \log T} \right) 
\leq \sum_{i=0}^{\infty} (i+1)\delta \cdot \mathbb{P} \left( v_t^\top \theta^* \in [i\delta, (i+1)\delta) \right) \cdot \mathbb{P} \left( v_t^\top (\theta^* - \hat{\theta}_{m-1}) \geq i\delta \|v_t\|_2 \leq \sqrt{10 \log T} \right). \tag{19} \]

We deal with each term in (19) separately. For $\mathbb{P} \left( v_t^\top \theta^* \in [i\delta, (i+1)\delta) \right)$, note that $v_t^\top \theta^*$ is a normal random variable with variance $(\theta^*)^\top \Sigma \theta^* \geq \lambda_{\min}(\Sigma)\|\theta^*\|_2^2 \geq \kappa \|\theta^*\|_2^2 / d$, thus the probability density of this random variable is upper bounded by $\sqrt{d / (2\pi \kappa \|\theta^*\|_2^2)}$ everywhere. Therefore,
\[ \mathbb{P} \left( v_t^\top \theta^* \in [i\delta, (i+1)\delta) \right) \leq \delta \cdot \frac{\sqrt{d}}{2\pi \kappa \|\theta^*\|_2}. \tag{20} \]
For the second term of (19), the proof of Lemma 5 shows that the random vector $\theta^* - \hat{\theta}_{m-1} \in \mathbb{R}^d$ is $d/(\kappa t'_{m-1})$-subGaussian for some absolute constant $c > 0$, and is also independent of $v_t$. Hence, conditioning on $\|v_t\|_2 \leq \sqrt{10 \log T}$, the random variable $v_t^\top (\theta^* - \hat{\theta}_{m-1})$ is also subGaussian with parameter $\|v_t\|_2^2 d / (\kappa t'_{m-1}) \leq 10d \log T / (\kappa t'_{m-1})$. Consequently, subGaussian concentration gives
\[ \mathbb{P} \left( v_t^\top (\theta^* - \hat{\theta}_{m-1}) \geq i\delta \|v_t\|_2 \leq \sqrt{10 \log T} \right) \leq \exp \left( -\frac{c \kappa^2 \delta^2 t'_{m-1}}{20d \log T} \right). \tag{21} \]
Combining (19), (20), (21) and the choice of $\delta$, we conclude that
\[ \mathbb{E} \left[ v_t^\top \theta^* \cdot 1(v_t^\top \theta^* \geq 0, v_t^\top \hat{\theta}_{m-1} \leq 0) \cdot 1(\|v_t\|_2 \leq \sqrt{10 \log T}) \right] \leq \frac{d^{3/2} \log T}{\sqrt{2\pi \kappa \|\theta^*\|_2}} \sum_{i=0}^{\infty} (i+1)e^{-ci^2/20} \leq C \cdot \frac{d^{3/2} \log T}{\kappa^{3/2} t'_{m-1} \|\theta^*\|_2}. \]
Moreover, since \( v_t^\top \theta^* \leq 2 \) almost surely and \( \mathbb{P}(\|v_t\|_2 \geq \sqrt{10 \log T}) \leq T^{-5} \), we also have

\[
\mathbb{E} \left[ v_t^\top \theta^* \cdot \mathbb{1}(v_t^\top \theta^* \geq 0, v_t^\top \theta_{m-1} \leq 0) \cdot \mathbb{1}(\|v_t\|_2 > \sqrt{10 \log T}) \right] \leq 2T^{-5}.
\]

Therefore, by the choice of the grid, the expected total regret in the \( m \)-th batch is at most

\[
\left( C \cdot \frac{d^{3/2} \log T}{\kappa^{3/2} \|\theta^\star\|_2} + 2T^{-5} \right) \cdot t_m' = O \left( \frac{d^{3/2} \log T}{\kappa^{3/2} \|\theta^\star\|_2} \cdot \left( \frac{T}{d^2} \right)^{1/M} \right).
\]

The first batch is handled in the same way as the upper bound proof of Theorem 2. Specifically, the expected total regret in the first batch is

\[
O \left( t_1' \cdot \sqrt{\frac{\log T}{d}} \right) = O \left( \sqrt{d^3 \log T \left( \frac{T}{d^2} \right)^{1/3}} \right) = O \left( \sqrt{\frac{d^3 \log T}{\kappa^{3/2} \|\theta^\star\|_2^2} \left( \frac{T}{d^2} \right)^{1/3}} \right).
\]

Finally summing up all batches \( m = 1, 2, \ldots, M \) completes the proof.

### 5.2 Proof of the lower bound in Theorem 3

The proof is entirely analogous to the lower bound proof of Theorem 2. First we observe that by Lemma 7, for any \( \Delta \in [0, 1] \) and fixed grid \( T = \{t_1, t_2, \ldots, t_M\} \) we have

\[
\inf_{\pi} \sup_{\|\theta^\star\|_2 \leq 1} \|\theta^\star\|_2 \cdot \mathbb{E}_{\theta^\star}[R_T(\pi)] \geq \Delta \cdot \inf_{\pi} \sup_{\theta^\star: \|\theta^\star\|_2 \leq 1} \mathbb{E}_{\theta^\star}[R_T(\pi)] \geq \Delta^2 \cdot \sum_{m=1}^{M} \frac{t_m - t_{m-1}}{10 \sqrt{d}} \exp \left( - \frac{16t_{m-1} \Delta^2}{d^2} \right).
\]

Now define \( s = \min\{m \in [M] : t_m \geq d^2\} \), which always exists due to the assumption \( T \geq d^2 \). Choosing \( \Delta \in \{1, d^2/t_s, d^2/t_{s+1}, \ldots, d^2/t_M\} \subseteq [0, 1] \) in the above inequality gives

\[
\inf_{\pi} \sup_{\|\theta^\star\|_2 \leq 1} \|\theta^\star\|_2 \cdot \mathbb{E}_{\theta^\star}[R_T(\pi)] \geq c \cdot \max \left\{ \frac{t_s}{\sqrt{d}}, \frac{d^{3/2} t_{s+1}}{t_s}, \frac{d^{3/2} t_{s+2}}{t_{s+1}}, \ldots, \frac{d^{3/2} t_M}{t_{M-1}} \right\}
\]

for some absolute constant \( c > 0 \). Finally, applying \( \max\{a_1, \ldots, a_n\} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \) gives

\[
\inf_{\pi} \sup_{\|\theta^\star\|_2 \leq 1} \|\theta^\star\|_2 \cdot \mathbb{E}_{\theta^\star}[R_T(\pi)] \geq c \cdot d^{3/2} \left( \frac{T}{d^2} \right)^{1/3} \geq c \cdot d^{3/2} \left( \frac{T}{d^2} \right)^{1/3},
\]

as claimed.

### 6 Conclusion

As we have shown in this paper, sequential batch learning provides an interesting and nontrivial departure from the traditional online learning setting where feedback is immediately observed and incorporated into making the next decision. We studied sequential batch learning in the linear contextual bandits setting and provided an in-depth inquiry into the algorithms and theoretical performance. An important insight here is that the nature of the contexts-adversarial or stochastic—has a significant impact on the optimal achievable performance, as well as the algorithms that would achieve the minimax optimal regret bounds.
Several questions immediately suggest themselves. First, in the stochastic context setting, our current regret upper bound depends heavily on the Gaussian assumption of the contexts. It would be interesting to see how far we can move beyond the Gaussian family. It would be unlikely that the same result holds for any distribution and hence, characterizing a (hopefully large) class of distributions under which the same tight bounds are achievable would be interesting. Another direction would be to look at more complex reward structures that go beyond linear bandits and see to what extent can the current set of results be generalized. We leave them for future work.

A Definitions and Auxiliary Results

Definition 2. Let \((\mathcal{X}, \mathcal{F})\) be a measurable space and \(P, Q\) be two probability measures on \((\mathcal{X}, \mathcal{F})\).

1. The total-variation distance between \(P\) and \(Q\) is defined as:
   \[
   \text{TV}(P, Q) = \sup_{A \in \mathcal{A}} |P(A) - Q(A)|.
   \]

2. The KL-divergence between \(P\) and \(Q\) is:
   \[
   D_{\text{KL}}(P\|Q) = \begin{cases} 
   \int \log \frac{dP}{dQ} dP & \text{if } P << Q \\
   +\infty & \text{otherwise}
   \end{cases}
   \]

Lemma 9. \([Tsy08, \text{Lemma 2.6}]\) Let \(P\) and \(Q\) be any two probability measures on the same measurable space. Then
   \[
   1 - \text{TV}(P, Q) \geq \frac{1}{2} \exp \left( -D_{\text{KL}}(P\|Q) \right).
   \]

Lemma 10. \([Wai19, \text{Theorem 6.1}]\) Let \(x_1, x_2, \cdots, x_n \sim \mathcal{N}(0, I_d)\) be i.i.d. random vectors. Then for any \(\delta > 0,
   \[
   \mathbb{P} \left( \sigma_{\text{max}} \left( \frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top \right) \geq 1 + \sqrt{\frac{d}{n} + \delta} \right) \leq \exp \left( -\frac{n\delta^2}{2} \right),
   \]
   where \(\sigma_{\text{max}}(A)\) denotes the largest singular value of \(A\).

B Proof of Main Lemmas

B.1 Proof of Lemma 4

Let \(y_{t,a} = \Sigma^{-1/2} x_{t,a}\), then each \(y_{t,a}\) is marginally distributed as \(\mathcal{N}(0, I_d)\). Define
   \[
   B \triangleq \frac{1}{t_m - t_m - 1} \sum_{t=t_m+1}^{t_m} y_{t,a} y_{t,a}^\top.
   \]

Recall that \(a_t = \arg \max_{a \in [K]} x_{t,a}^\top \hat{\theta} = \arg \max_{a \in [K]} y_{t,a}^\top (\Sigma^{1/2} \hat{\theta})\) for any \(t \in [t_{m-1} + 1, t_m]\), and \(\hat{\theta}\) is an estimate of \(\theta^*\) that is independent of all contexts in the current batch \([t_{m-1} + 1, t_m]\). By rotational invariance of \(\mathcal{N}(0, I_d)\), we can without loss of generality assume \(\Sigma^{1/2} \hat{\theta} = ce_d\) for some \(c > 0\). Consequently, each \(y_{t,a_t}\) follows the distribution \(\mu_t = \mathcal{N}(0, 1) \otimes \cdots \otimes \mathcal{N}(0, 1) \otimes \nu_t\), where
\( \nu_t \) is the probability distribution of \( \max_{a \in [K]} Z_{t,a} \), where each \( Z_{t,a} \) is a standard Gaussian and the \( Z_{t,a} \)'s can be correlated across different \( a \)'s.

Now for \( y = (y_1, y_2, \cdots, y_d) \sim \mu_t \) and any unit vector \( u \in \mathbb{R}^d \), we show that there exist numerical constants \( c_1, c_2 > 0 \) independent of \((d, K)\) such that

\[
\mathbb{P}\left( |y^\top u| \geq c_1 \right) \geq c_2. \tag{22}
\]

To establish (22), we distinguish into two cases. If \( |u_d| < \frac{1}{2} \), using the fact that \( \mathbb{P}(\max(0, 1) + t \geq c) \) is minimized at \( t = 0 \) for any fixed \( c > 0 \), we conclude that

\[
\mathbb{P}\left( |y^\top u| \geq c_1 \right) \geq \mathbb{P}\left( \left| \sum_{i=1}^{d-1} y_i u_i \right| \geq c_1 \right) = \mathbb{P}(\max(0, 1 - u_d^2) \geq c_1) \geq \mathbb{P}\left( \mathcal{N}(0, \frac{3}{4}) \geq c_1 \right)
\]

is lower bounded by some positive constant. If \( |u_d| \geq \frac{1}{2} \), we have

\[
\mathbb{P}\left( |y^\top u| \geq c_1 \right) \geq \frac{1}{2} \mathbb{P}(\max(0, 1) \geq c_1) \geq \frac{1}{2} \mathbb{P}(|y_d| \geq 2c_1) \geq \frac{1}{2} \mathbb{P}(Z_{t,1} \geq 2c_1) = \frac{1}{2} \mathbb{P}(\mathcal{N}(0, 1) \geq 2c_1),
\]

which is again lower bounded by a numerical constant. Hence the proof of (22) is completed.

Based on (22) and the deterministic inequality

\[
u^\top \left( \frac{1}{t_m - t_{m-1}} \sum_{t=t_{m-1}+1}^{t_m} y_{t,a_1} y_{t,a_1}^\top \right) \cdot u \geq \frac{c_1^2}{t_m - t_{m-1}} \sum_{t=t_{m-1}+1}^{t_m} 1 \left( |y_{t,a_1}^\top u| \geq c_1 \right),
\]

the Chernoff inequality yields that for any unit vector \( u \in \mathbb{R}^d \), we have

\[
\mathbb{P}\left( u^\top Bu \geq \frac{c_1^2 c_2}{2} \right) \geq 1 - e^{-c_3(t_m - t_{m-1})}, \tag{23}
\]

where \( c_3 > 0 \) is some numerical constant.

Next we prove an upper bound of \( \lambda_{\max}(B) \), i.e., the largest eigenvalue of \( B \). Since \((a + b)(a + b)^\top \leq 2(aa^\top + bb^\top)\) for any vectors \( a, b \in \mathbb{R}^d \), for \( y_t \sim \mu_t \) we have

\[
y_t y_t^\top \leq 2(v_t v_t^\top + w_t w_t^\top),
\]

where \( v_t = (v_{t,1}, \cdots, v_{t,d-1},0) \) with \( v_{t,i} \sim \mathcal{N}(0,1) \), and \( w_t = (0, \cdots, 0, w_{t,d}) \) with \( w_{t,d} \sim \nu_t \). By concentration of Wishart matrices (cf. Lemma 10), with probability at least \( 1 - e^{-\Omega(t_m - t_{m-1})} \),

\[
\lambda_{\max} \left( \frac{1}{t_m - t_{m-1}} \sum_{t=t_{m-1}+1}^{t_m} v_t v_t^\top \right) \leq c_4
\]

holds for some numerical constant \( c_4 > 0 \). For the second term, since \( w_{t,d} \sim \nu_t \) is the maximum of \( K \) arbitrary \( \mathcal{N}(0, 1) \) random variables, the Gaussian tail and the union bound imply that \( |w_{t,d}| \leq \sqrt{c_5 \log(KT)} \) with probability at least \( 1 - O(T^{-5}) \). Hence, with probability at least \( 1 - O(T^{-4}) \), we have

\[
\lambda_{\max} \left( \frac{1}{t_m - t_{m-1}} \sum_{t=t_{m-1}+1}^{t_m} w_t w_t^\top \right) = \frac{1}{t_m - t_{m-1}} \sum_{t=t_{m-1}+1}^{t_m} w_{t,d}^2 \leq c_5 \log(KT).
\]
Combining all the previous results, and using $\lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B)$ for symmetric matrices $A, B$, we conclude that with probability at least $1 - e^{-\Omega(t_m - t_{m-1}) - O(T^{-4})}$, we have
\[
\lambda_{\max}(B) \leq c_6 \log(KT)
\]
holds for some numerical constant $c_6 > 0$.

Finally, we are ready to prove a lower bound on $\lambda_{\min}(B)$ via an $\varepsilon$-net argument. Let $\mathcal{N}_d(\varepsilon)$ be an $\varepsilon$-net of the unit ball in $\mathbb{R}^d$ (both in $\ell_2$ norm) with cardinality at most $(1 + \frac{2}{\varepsilon})^d$. Standard $\varepsilon$-net techniques (cf. [Tao12, Section 2.3.1]) give
\[
\min_{u:\|u\|_2=1} u^\top Bu \geq \min_{u \in \mathcal{N}_d(\varepsilon)} u^\top Bu - 2\varepsilon \lambda_{\max}(B).
\]
Hence, choosing $\varepsilon = \frac{c_2}{8c_6 \log(KT)}$ and combining (23), (24) and the union bound over $\mathcal{N}_d(\varepsilon)$ gives
\[
P\left( \lambda_{\min}(B) \geq \frac{c_2^2}{4} \right) \geq 1 - e^{O(d \log \log KT) - \Omega(t_m - t_{m-1}) - O(T^{-4})}.
\]

By noting that $t_m - t_{m-1} = \Omega(dvT)$ due to the choice of the grid in (8), the parameter $a$ in (??), and the assumption $M = O(\log \log T)$, we conclude that $\lambda_{\min}(B) \geq c_7$ for some numerical constant $c_7 > 0$ with probability at least $1 - O(T^{-4})$. The proof is completed by noting that
\[
\frac{1}{t_m - t_{m-1}} \sum_{t=t_{m-1}+1}^{t_m} x_{t,a_t}^\top x_{t,a_t}^\top = \Sigma_1^{1/2} B \Sigma_1^{1/2} \geq \Sigma_1^{1/2} (c_7 I_d) \Sigma_1^{1/2} = c_7 \Sigma
\]
whenever $\lambda_{\min}(B) \geq c_7$ and the assumption $\lambda_{\min}(\Sigma) \geq \kappa/d$. \qed

### B.2 Proof of Lemma 8

Let $v_1, \cdots, v_d$ be an orthonormal basis of $\mathbb{R}^d$ with $v_1 = u_t$. By rotational invariance of the uniform distribution on spheres, we have $(v_1^\top \theta, v_2^\top \theta, \cdots, v_d^\top \theta) \sim \text{Unif}(\Delta S^{d-1})$ under $Q_0$. Now recall that
\[
\frac{dQ_1}{dQ_0}(\theta) = \frac{r_t(v_1^\top \theta)_+}{Z_0}, \quad \frac{dQ_2}{dQ_0}(\theta) = \frac{r_t(v_1^\top \theta)_-}{Z_0},
\]
we conclude that if $\theta' = \theta - 2(v_1^\top \theta)v_1$, we have
\[
\frac{dQ_1}{dQ_0}(\theta) = \frac{dQ_2}{dQ_0}(\theta').
\]
As a result, it is equivalent to have $\theta \sim Q_1$ or $\theta' = \theta - 2(v_1^\top \theta)v_1 \sim Q_2$.

For the identity (13), recall that the density of $\theta = (\theta_1, \cdots, \theta_d) \sim \text{Unif}(\Delta S^{d-1})$ is
\[
f(\theta) = f(\theta_2, \cdots, \theta_d) = \left( \frac{d\pi^{d/2} \Delta^{d-1}}{\Gamma\left(\frac{d}{2} + 1\right)} \right)^{-1} \frac{2\Delta}{\Delta^2 - \theta_2^2 - \cdots - \theta_d^2} \cdot 1 \left( \sum_{i=2}^{d} \theta_i^2 \leq \Delta^2 \right),
\]
where $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$ is the Gamma function. Hence, by rotational invariance, we have
\[
Z_0 = \frac{r_t}{2} \mathbb{E}_{Q_0}[\|\theta_1\|] = r_t \Delta \cdot \int_{\sum_{i=2}^{d} \theta_i^2 \leq \Delta^2} \left( \frac{d\pi^{d/2} \Delta^{d-1}}{\Gamma\left(\frac{d}{2} + 1\right)} \right)^{-1} d\theta_2 \cdots d\theta_d
\]
\[
= r_t \Delta \left( \frac{d\pi^{d/2} \Delta^{d-1}}{\Gamma\left(\frac{d}{2} + 1\right)} \right)^{-1} \frac{\Delta^{d-1} \pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d}{2} + 1\right)} = r_t \Delta \left\{ \begin{array}{ll}
\frac{2\Delta}{\pi d/2} \frac{1}{\Gamma\left(\frac{d}{2} + 1\right)}, & \text{if } d \text{ is even} \\
\frac{1}{\pi^{d-1} / 2} \frac{1}{\Gamma\left(\frac{d}{2} - 1\right)}, & \text{if } d \text{ is odd}
\end{array} \right.. \]
Using Stirling’s approximation $\sqrt{2\pi n (\frac{n}{e})^n} \leq n! \leq e \sqrt{n (\frac{n}{e})^n}$ for any $n \geq 1$, we have

$$\frac{2}{e^2} \sqrt{\frac{\pi}{n}} \leq \frac{1}{2^{2n}} \left( \frac{2n}{n} \right) \leq \frac{e}{\pi \sqrt{2n}} \quad (26)$$

for all $n \geq 1$, and the rest of (13) follows from (26).

As for the second moment in (14), we use the spherical coordinates

$$\begin{align*}
\theta_2 &= r \cos \varphi_1, \\
\theta_3 &= r \sin \varphi_1 \cos \varphi_2, \\
\vdots \\
\theta_{d-1} &= r \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{d-3} \cos \varphi_{d-2}, \\
\theta_d &= r \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{d-3} \sin \varphi_{d-2}.
\end{align*}$$

to obtain

$$\mathbb{E}_{Q_1}[(v_i^\top \theta)^2] = \frac{r_i \Delta}{Z_0} \cdot \mathbb{E}_{Q_0}[|\theta_1|^3]$$

$$\begin{align*}
&= \frac{r_i \Delta}{Z_0} \cdot \int_{0}^{\Delta} \int_{0}^{\pi} \cdots \int_{0}^{\pi} 2\pi \left( \frac{d\pi^{d/2} \Delta^{d-1}}{\Gamma(\frac{d}{2} + 1)} \right)^{-1} (\Delta^2 - \theta_2^2 - \cdots - \theta_d^2) d\theta_2 \cdots d\theta_d \\
&= \frac{r_i \Delta}{Z_0} \cdot \int_{0}^{\Delta} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \cdots \int_{0}^{\frac{\pi}{2}} (\Delta^2 - \theta_2^2 - \cdots - \theta_d^2) d\theta_2 \cdots d\theta_d \\
&= \frac{r_i \Delta}{Z_0} \left( \frac{d\pi^{d/2} \Delta^{d-1}}{\Gamma(\frac{d}{2} + 1)} \right)^{-1} \cdot 2\Delta^d + 1 \cdot \frac{\Gamma(\frac{d}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{d-1}{2})} \cdot \frac{\Gamma(\frac{d-3}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{d-2}{2})} \cdots \frac{\Gamma(1) \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \cdot 2\pi \\
&= \frac{r_i \Delta}{Z_0} \left( \frac{d\pi^{d/2} \Delta^{d-1}}{\Gamma(\frac{d}{2} + 1)} \right)^{-1} \cdot 2\Delta^d + 1 \cdot \frac{2\pi^{d+1}}{d^2 - 1} \cdot \frac{2\pi^{d+1}}{d^2 - 1} \\
&= \frac{2\Delta^2}{d+1}.
\end{align*}$$

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