On Algebraic Hyperbolicity of Log Varieties

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Abstract. Kobayashi conjecture says that every holomorphic map $f : \mathbb{C} \to \mathbb{P}^n \setminus D$ is constant for a very general hypersurface $D \subset \mathbb{P}^n$ of degree $\deg D \geq 2n + 1$. As a corollary of our main theorem, we show that $f$ is constant if $f(\mathbb{C})$ is contained in an algebraic curve.

1. Introduction and Statement of Results

1.1. Kobayashi Conjecture. The hyperbolicity in the title refers to Kobayashi hyperbolicity $K$. A complex manifold $M$ is hyperbolic in the sense of S. Kobayashi if the hyperbolic pseudo-metric defined on $M$ is a metric. One consequence of a complex manifold $X$ being hyperbolic is that there does not exist nonconstant holomorphic map from $\mathbb{C}$ to $X$; actually, this is also sufficient if $X$ is compact by R. Brody $B$.

It is usually very hard to prove a complex manifold to be hyperbolic, even for very simple ones, such as hypersurfaces and their complements in projective spaces. S. Kobayashi conjectured that the following is true.

Conjecture 1.1 (Kobayashi Conjecture). For $n \geq 2$,
1. a very general hypersurface $D \subset \mathbb{P}^{n+1}$ of degree $\deg D \geq 2n + 1$ is hyperbolic;
2. $\mathbb{P}^n \setminus D$ is hyperbolic and hyperbolically embedded in $\mathbb{P}^n$ for a very general hypersurface $D \subset \mathbb{P}^n$ of degree $\deg D \geq 2n + 1$.

There is an ample literature on the problem. Please see, e.g., $D$ and $Z1$ for surveys on the subject. There have been some major breakthroughs recently on the conjecture. J.P. Demailly and J. El Goul proved the conjecture for $n = 2$ and $\deg D \geq 21$ $DEG$. Independently, M. McQuillan also proved the conjecture for $n = 2$ and $\deg D \geq 36$ $M$. Later, for the second part of the conjecture, J. El Goul improved the bound for $\deg D$ to 15 $EG$.

The purpose of this paper is not on Kobayashi conjecture itself but rather on a closely related problem. Our main question is:

Question 1.2. Does $\mathbb{P}^n \setminus D$ contain an algebraic torus $\mathbb{C}^* \hookrightarrow \mathbb{P}^n \setminus D$, embedded algebraically, for $\deg D \geq 2n + 1$?

The answer to this question has to be positive if Kobayashi conjecture holds true; otherwise, there will be a nonconstant map $\mathbb{C} \to \mathbb{C}^* \hookrightarrow \mathbb{P}^n \setminus D$ with $\mathbb{C} \to \mathbb{C}^*$ the covering map given by $z \to e^z$. Indeed, this question would have to be addressed if there were to be a proof for the full statement of the conjecture.
Simple as this problem sounds, surprisingly there is no proof for it existing in the literature or it may have escaped the attention of the researchers in this field, especially algebraic geometers. In any case, except the result of G. Xu in $n = 2$ which we will discuss later, there is no affirmative answer to this question in general. Let us first set ourselves a rather modest goal to prove the following.

\textbf{Claim 1.3.} There is no $\mathbb{C}^* \subset \mathbb{P}^n \setminus D$ for a very general hypersurface $D$ of sufficiently large degree.

\textbf{Proof.} Since the statement is algebraic and generic in nature, we may approach it via a degeneration argument. First, let us work with some special $D$, say, $D = H_1 \cup H_2 \cup \ldots \cup H_{2n+1}$ a union of $2n + 1$ hyperplanes in general position. To show there is no $\mathbb{C}^*$ in $\mathbb{P}^n \setminus D$, it is in essence to show there is no rational curve $C \subset \mathbb{P}^n$ which meet $D$ (set-theoretically) at no more than two distinct points. To achieve the least number of intersections between a curve $C \subset \mathbb{P}^n$ and $D = H_1 \cup H_2 \cup \ldots \cup H_{2n+1}$, the best one can do is to choose $C$ such that $C$ meets $H_1, H_2, \ldots, H_n$ only at the point $p = H_1 \cap H_2 \cap \ldots \cap H_n$ and meets $H_{n+1}, H_{n+2}, \ldots, H_{2n}$ only at the point $q = H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{2n}$. But then $H_{2n+1}$ must meet $C$ at a third point since $p, q \notin H_{2n+1}$ when $H_i$ are chosen to be in general position. So every curve $C \not\subset D$ meets $D$ at no less than three distinct points. More generally, if $D$ is a union of $k$ hypersurfaces that meet properly among themselves, i.e., any $n + 1$ of them have empty intersection, then every curve $C \not\subset D$ meets $D$ at no less than $\lceil k/n \rceil$ distinct points.

But we are really interested in irreducible $D$’s. An obvious way to do this is to degenerate $D$ to a union of hypersurfaces. The basic setup is as follows.

Let $Z = \mathbb{P}^n \times \Delta$ and $W \subset Z$ be a pencil of hypersurfaces of degree $d$ whose central fiber $W_0$ is a union $D_1 \cup D_2 \cup \ldots \cup D_k$ of $k$ hypersurfaces. Throughout the paper, we always use the notation $\Delta$ to denote the disk parametrized by $t$.

Let $Y$ be a reduced irreducible flat family of rational curves over $\Delta$ with the commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\pi} & Z \\
\downarrow & & \downarrow \\
\Delta & \longrightarrow & \Delta
\end{array}
$$

(1.1)

where $\Delta \to \Delta$ is a base change and $\pi : Y \to Z$ is a proper morphism. We assume that $\pi(Y_t)$ meets $W_t$ properly in $Z$. Our goal is, of course, to show that $\pi(Y_t)$ meets $W_t$ at no less than three distinct points for
It is not 0 if \( d \gg 0 \). We do know that \( \pi(Y_0) \) meets \( W_0 \) at no less than \([k/n]\) distinct points. It seems that we are done as long as we take \( d = k \geq 2n + 1 \) since the number of intersections \( \pi(Y_t) \cap W_t \) is obviously lower semi-continuous in \( t \). However, there is a serious flaw in our argument: even if \( \pi(Y_t) \) meets \( W_t \) properly, \( \pi(Y_0) \) could very well fail to meet \( W_0 \) properly, i.e., some component of \( \pi(Y_0) \) may be contained in \( W_0 = D_1 \cup D_2 \cup \ldots \cup D_k \).

There are more sophisticated ways we will later introduce to deal with the situation that \( \pi(Y_0) \) and \( W_0 \) do not intersect properly. For the moment, let us get around the problem using a very simple argument. Since \( Y_t \) is rational, every component of \( Y_0 \) is rational. Thus we may make the degree of \( D_i \) large enough so that \( D_i \) does not contain any rational curves for \( i = 1, 2, \ldots, k \). This is possible thanks to a theorem of H. Clemens, which we will discuss in details later. It says that a very general hypersurface \( D \) of degree \( \deg D \geq 2n - 1 \) in \( \mathbb{P}^n \) does not contain any rational curves. So it is enough to take \( \deg D_i \geq 2n - 1 \). This, inevitably, will make \( d = \deg D \) really big. But at least we have achieved our modest goal 1.3. q.e.d.

Actually, we have proved:

**Proposition 1.4.** Let \( D \) be a very general hypersurface of degree \( d \) in \( \mathbb{P}^n \). Then every rational curve \( C \not\subset D \) meets \( D \) at no less than

\[
\left\lceil \frac{1}{n} \left\lfloor \frac{d}{2n - 1} \right\rfloor \right\rceil
\]

distinct points. Consequently, \( \mathbb{P}^n \setminus D \) does not contain \( C^* \) if \( d \geq 4n^2 - 1 \).

This is, however, nowhere near a satisfactory answer to our question. The bound \( 4n^2 - 1 \) in the above proposition is a far cry from the conjectured \( 2n + 1 \), especially in the light of G. Xu’s result in \( n = 2 \). In \[X2\], he proved that every irreducible curve (not only rational ones) \( C \not\subset D \) meets \( D \) at no less than \( d - 2 \) distinct points for a very general curve \( D \) of degree \( d \) in \( \mathbb{P}^2 \). In the end of his paper, he asked whether the following similar statement holds in higher dimensions.

**Question 1.5.** If \( D \subset \mathbb{P}^n \) is a very general hypersurface of degree \( d \) and \( C \subset \mathbb{P}^n \) is an irreducible curve with \( C \not\subset D \), what is the least number of distinct points that \( C \cap D \) must contain? The expected answer is \( d - 2n + 2 \).

An answer to the above question, of course, also answers Question 1.2.

Xu’s proof in \( n = 2 \) is based upon the study of the deformation of the pair \( (C, D) \) with \( |C \cap D| \) fixed, where \( |C \cap D| \) is the number of distinct
points in the (set-theoretical) intersection $C \cap D$. Xu’s method seems very hard to generalize to higher dimensions or to underlying spaces other than $\mathbb{P}^n$. An alternative approach based upon degeneration was adopted in [C1]. It can be summarized as follows.

With $Y, Z$ and $W$ defined as before, we try to give a lower bound for $|\pi(Y_t) \cap W_t|$ based upon the information on $|\pi(Y_0) \cap W_0|$. Here $Z = \mathbb{P}^2 \times \Delta$ and $W \subset Z$ is a pencil of degree $d$ curves with reducible central fiber. As we pointed out before, if $\pi(Y_0)$ fails to meet $W_0$ properly, we cannot say much about $|\pi(Y_t) \cap W_t|$ in general. However, we can work around this problem due to the fact that $\pi^* Y$ is a divisor in $Z$ here. The intersection $\pi^* Y_t \cap W_t$ can be regarded as a member in the linear series $\mathbb{P}H^0(W_t, O_{W_t}(\pi^* Y_t))$. There is a well-defined notion of taking limit of a linear series, i.e., $\lim_{t \to 0} \mathbb{P}H^0(W_t, O_{W_t}(\pi^* Y_t))$. Correspondingly, we may take the limit $\lim_{t \to 0} (\pi^* Y_t \cap W_t)$ as a member of the limit linear series, which can be described explicitly in terms of the information we have on $\pi^* Y_0, W_0$ and the base locus of the pencil $W$.

This approach seems more adaptable than Xu’s. It can be easily extended to surfaces other than $\mathbb{P}^2$ as in [C1]. However, this line of argument still cannot be carried over to higher dimensions, since in order for this to work, we need to treat $\pi^* Y_t \cap W_t$ as an element in the Chow group $A_0(W_t)$; however, with our present knowledge of Chow groups, we have no idea how to take the limit $\lim_{t \to 0} A_0(W_t)$, even only for $W_t$ surfaces.

Note that both the above and Xu’s argument did not use the information on the genus of $Y_t$, while the fact that $Y_t$ is rational is used in an essential way in [1.3]. In order to answer Question [1.2], it is suggested and carried out for surfaces in [C2] to bound the geometric genus $g(C)$ and $|C \cap D|$ at the same time. More precisely, we are trying to bound the quantity

\begin{equation}
2g(C) - 2 + |C \cap D|.
\end{equation}

Of course, we are done if we can show that (1.3) is positive for every reduced irreducible curve $C \not\subseteq D$ if $\deg D \geq 2n + 1$.

Another subtle point is that instead of using the number of the set-theoretical intersections between $C$ and $D$, we should use a more natural notion of intersection as defined in [C2].

**Definition 1.6.** Let $D$ be an effective divisor on $X$ and $C \subset X$ be a reduced irreducible curve such that $C \not\subseteq D$. Let $\nu : C^\nu \to C \subset X$ be the normalization of $C$ and then $i_X(C, D)$ is the number of distinct points in the set $\nu^{-1}(D)$. 


Roughly, $i_X(C, D)$ is the intersection between the normalization of $C$ and $D$. This is a more natural notion than $|C \cap D|$ since $i_X(C, D)$ depends only on the complement $X \setminus D$ while $|C \cap D|$ also depends on the choice of the compactification of $X \setminus D$. For example, suppose that $C$ meets $D$ at an ordinary double point, i.e., a node $p$ of $C$. Let $X'$ be the blowup of $X$ at $p$, $D'$ be the total transform of $D$ and $C'$ be the proper transform of $C$. Then we have $X \setminus D \cong X' \setminus D'$ and $i_X(C, D) = i_{X'}(C', D')$ but $|C' \cap D'| = |C \cap D| + 1$.

So we are actually trying to bound the quantity

$$2g(C) - 2 + i_X(C, D).$$

(1.4)

Our main result is the following.

**Theorem 1.7.** Let $D \subset \mathbb{P}^n$ be a very general hypersurface of degree $d$. Then

$$2g(C) - 2 + i_{\mathbb{P}^n}(C, D) \geq (d - 2n) \deg C$$

for all reduced irreducible curves $C$ with $C \not\subset D$.

Although we do not really answer Question 1.5 in the above theorem, it does give us what we need as far as Question 1.2 is concerned. Namely, we have the following corollary of Theorem 1.7.

**Corollary 1.8.** For a very general hypersurface $D \subset \mathbb{P}^n$ of degree $\deg D \geq 2n + 1$, $\mathbb{P}^n \setminus D$ does not contain any algebraic torus $\mathbb{C}^*$. Therefore, a holomorphic map $f : \mathbb{C} \to \mathbb{P}^n \setminus D$ is constant if $f(\mathbb{C})$ is contained in an algebraic curve.

In addition, (1.3) also shows that $i_{\mathbb{P}^n}(C, D)$ goes up as $\deg C$ goes up, which is not predicted by Question 1.5. Also it is sharp in the sense that for each $d \geq 2n$, there exists a line meeting $D$ at exactly $d - 2n + 2$ distinct points (see e.g. [22]) and hence the equality in (1.5) holds.

**1.2. Basic strategy.** In [22], we proved Theorem 1.7 for $n = 2$. In order to prove it for $n > 2$, we need to generalize the techniques in [22] to higher dimensions. This is mainly what this paper is about.

The proof of Theorem 1.7 for $n > 2$ is actually not very much different to that for $n = 2$ or the proof for 1.3 at least in principle. It basically consists of two parts:

1. we first prove that (1.3) holds when $D$ is a union of $d$ hyperplanes in general position;
2. we then prove (1.3) for $D$ irreducible by degenerating it to a union of hyperplanes.
However, both parts of the proof are technically harder than those for \( n = 2 \), especially the degeneration part.

Let us first recall some definitions and notations employed in \( \mathbb{C}^2 \).

We have already defined the number \( i_X(C, D) \) for a log pair \((X, D)\) and a curve \( C \subset X \). For technical reasons, we will define \( i_X(C, D, P) \) involving an extra term \( P \) as follows.

**Definition 1.9.** Let \( X \) be a scheme, \( D \subset X \) be a closed subscheme pure of codimension one, \( P \subset X \) be a closed subscheme of \( X \) and \( C \) be a reduced irreducible curve on \( X \). Let \( f : \tilde{X} \to X \) be the blowup of \( X \) along \( P \) and let \( \tilde{C} \) and \( \tilde{D} \) be the proper transforms of \( C \) and \( D \) under \( f \), respectively. If \( \tilde{C} \) and \( \tilde{D} \) exist and meet properly, then we define
\[
\begin{align*}
\qquad i_X(C, D, P) &= i_{\tilde{X}}(\tilde{C}, \tilde{D}). 
\end{align*}
\]

(1.6)

It is also convenient to extend the definition of \( i_X(C, D, P) \) to the following situations.

1. Suppose that \( C \subset D = \bigcup_{i \in I} D_i \) with \( D_i \) irreducible. We define
\[
\begin{align*}
\qquad i_X(C, D) &= i_X(C, \bigcup_{i \notin J} D_i),
\end{align*}
\]
where \( J = \{ j \in I : C \subset D_j \} \).

2. Suppose that \( \tilde{C} \) does not exist, i.e., \( C \) is contained in the exceptional locus of \( f \). Then \( i_X(C, D, P) \) is defined by
\[
\begin{align*}
\qquad i_X(C, D, P) &= \inf_{\Gamma \subset f^{-1}C} i_{\tilde{X}}(\Gamma, \tilde{D})
\end{align*}
\]
where we consider all curves \( \Gamma \subset \tilde{X} \) that map birationally onto \( C \) by \( f \).

3. If \( C \) is nonreduced or reducible, we define
\[
\begin{align*}
\qquad i_X(C, D, P) &= \sum_{\Gamma \subset C} \mu_{\Gamma} i_X(\Gamma, D, P),
\end{align*}
\]
where we sum over all irreducible components \( \Gamma \subset C \) and \( \mu_{\Gamma} \) is the multiplicity of \( \Gamma \) in \( C \).

A central theme of \( \mathbb{C}^2 \) is the algebraic hyperbolicity of a log variety \((X, D)\).

**Definition 1.10.** Let \( X \subset \mathbb{P}^N \) be a projective variety, \( D \) be an effective divisor on \( X \) and \( P \subset D \) be a closed subscheme of \( X \) of codimension at least 2. We call \((X, D, P)\) *algebraically hyperbolic* if there exists a positive number \( \epsilon \) such that
\[
\begin{align*}
\qquad 2g(C) - 2 + i_X(C, D, P) &\geq \epsilon \deg C
\end{align*}
\]
(1.10)
for all reduced irreducible curves $C \subset X$ with $C \not\subset D$. And we call $(X, D)$ algebraically hyperbolic if $\deg D \geq 2n + 1$.

So Theorem 1.7 implies that $(\mathbb{P}^n, D)$ is algebraically hyperbolic if

$$2g(C) - 2 \geq \epsilon \deg C$$

(1.11)

for all reduced irreducible curves $C \subset X$. Demailly proved that there are no nonconstant maps from an abelian variety to an algebraically hyperbolic variety [D, Theorem 2.1, p. 293]. Using his argument, one can show a similar statement for log varieties.

**Proposition 1.11.** If $(X, D)$ is algebraically hyperbolic, then there are no nonconstant maps from a semiabelian variety to $X \setminus D$.

In order to state our next theorem, we need to introduce the following terms.

**Definition 1.12.** We call a nonempty finite set $\{D_i\}_{i \in I}$ of base point free (BPF) divisors on $X$ an effective adjunction sequence if for any two disjoint subsets $I_1$ and $I_2$ of $I$ satisfying $|I_1| + |I_2| = |I| - 1$, $\sum_{i \in I_1} D_i$ is ample when restricted to $\cap_{i \in I_2} D_i$. Here by a BPF divisor, we refers to a general member of a BPF linear system. Note that if $I_1 = \emptyset$, $\sum_{i \in I_1} D_i = 0$; for it to be ample on $\cap_{i \in I_2} D_i$, it is necessary that $\dim \cap_{i \in I_2} D_i \leq 0$.

**Remark 1.13.** Let $n = \dim X$. Obviously, $n + 1$ BPF ample divisors on $X$ form an effective adjunction sequence. So $n + 1$ hyperplanes in $\mathbb{P}^n$ form an effective adjunction sequence. If $\{D_i\}_{i \in I}$ is an effective adjunction sequence on $M$ and $\{E_j\}_{j \in J}$ is an effective adjunction sequence on $N$, then $\{\pi_M^* D_i\}_{i \in I} \cup \{\pi_N^* E_j\}_{j \in J}$ is an effective adjunction sequence on $M \times N$, where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the projections from $M \times N$ to $M$ and $N$, respectively.

We will justify the name “effective adjunction sequence” in [2.1].

We use $C_1(X)$ to denote the free abelian group generated by the curves on a variety $X$ and let $N_1(X) = C_1(X)/\sim_{num}$, where $\sim_{num}$ is the numerical equivalence. We call $\varphi : N_1(X) \to \mathbb{R}$ an additive function on $N_1(X)$ if $\varphi \in \text{Hom}(N_1(X), \mathbb{R})$. Of course, if $X$ is nonsingular, $\text{Hom}(N_1(X), \mathbb{R}) \cong N^1(X) \otimes \mathbb{R}$ where $N^1(X) = \text{Div}(X)/\sim_{num}$, i.e., every additive function $\varphi$ on $N_1(X)$ is given by a $\mathbb{R}$-divisor $D$ such that
\[ \varphi(C) = D \cdot C \] for all curves \( C \subset X \). Let \( \overline{NE}(X) \subset N_1(X) \) be the cone of effective curves on \( X \). We call \( \varphi : N_1(X) \to \mathbb{R} \) a **subadditive** function on \( N_1(X) \) if
\[ \varphi(C_1) + \varphi(C_2) \geq \varphi(C_1 + C_2) \quad (1.12) \]
for any \( C_1, C_2 \in \overline{NE}(X) \).

**Definition 1.14.** Let \{\( D_i \}_{i \in I} \) be a finite set of BPF divisors on a variety \( X \), \( \varphi : C_1(X) \to \mathbb{R} \) be a function on \( C_1(X) \) and \( j_1, j_2 \in \mathbb{R} \). We say that \{\( D_i \)\}_{i \in I} satisfies the condition \( \mathcal{C}_X(\varphi, j_1, j_2) \) on \( X \) if the following holds: for any two disjoint subsets \( J_1 \) and \( J_2 \) of \( I \) satisfying \(|J_1| \leq j_1, |J_2| \leq j_2\) and
\[ \prod_{j \in J_1} D_j \neq 0, \]
there exists a partition \( I = I_1 \sqcup I_2 \sqcup J_2 \) of \( I \) with the properties that \( J_1 \subset I_1, \{D_i\}_{i \in I_1} \) is an effective adjunction sequence on \( X \) and
\[ \sum_{i \in I_2} D_i \cdot C \geq \varphi(C) \quad (1.14) \]
holds for all curves \( C \subset X \).

**Theorem 1.15.** Let \( X \) be a smooth projective variety of dimension \( n \), \{\( \mathbb{P}L_i \)\}_{i \in I} be a finite set of BPF linear systems on \( X \) and \( D_i \) be a very general member of \( \mathbb{P}L_i \) for each \( i \in I \).

Let \( \varphi : C_1(X) \to \mathbb{R} \) be a function on \( C_1(X) \) such that \{\( D_i \)\}_{i \in I} satisfies the condition \( \mathcal{C}_X(\varphi, 0, n - 1) \) on \( X \).

Let \( F \) be a fixed effective divisor of \( X \) and \( P \subset F \cap D \) be a closed subscheme of \( X \) pure of codimension two or empty, where \( D = \sum_{i \in I} D_i \).

Then
\[ 2g(C) - 2 + i_X(C, D, P) \geq \varphi(C) \quad (1.15) \]
for all reduced irreducible curves \( C \subset X \) and \( C \not\subset F \).

**Remark 1.16.** Note that \( F \) is a fixed divisor while \( D_i \) varies in \( \mathbb{P}L_i \). The conclusion of the theorem should be more precisely phrased as “for a very general choice of \( D_i \in \mathbb{P}L_i \) for each \( i \in I \), (1.15) holds for ...”. Also note that as \( D_i \) varies, \( P \) varies as well. But since \( P \) is a closed subscheme of \( X \) pure of codimension two and \( P \subset D \cap F \), \( P \) is determined by \( D_i \) up to finitely many choices. That is, if we let
\[ F = \sum_{j \in J} d_j F_j \] where \( F_j \) are the irreducible components of \( F \), then \( P \) is given by fixing a set \( \mathcal{P} \subset I \times J \) and \( d_j \leq d_j \) and setting

\[
P = \bigcup_{(i,j) \in \mathcal{P}} (D_i \cap \hat{d}_j F_j). \tag{1.16}
\]

Let \( X = \mathbb{P}^n \) and \( D = \sum D_i \) be a union of \( d \) hyperplanes. Then we may apply the above theorem by taking \( F = \emptyset \), and \( \varphi(C) = (d - 2n) \deg C \). It is easy to check that \( \{D_i\} \) satisfies the condition \( C_X(\varphi,0,n-1) \) if \( d \geq 2n \) and hence (1.5) holds for a union \( D \) of \( d \) hyperplanes in general position. A natural way to go from Theorem 1.15 to 1.7 is to degenerate an irreducible \( D \) to a union of hyperplanes. That is what we are going to do next, albeit in a general setting.

**Theorem 1.17.** Let \( X \) be a smooth projective variety, \( \{\mathbb{P} L_i\}_{i \in I} \) be a finite set of BPF linear systems on \( X \) and \( D_i \) be a very general member of \( \mathbb{P} L_i \) for every \( i \in I \).

Let

\[
I = \left( \bigcup_{\alpha \in A} I_\alpha \right) \sqcup \left( \bigcup_{\beta \in B} I_\beta \right),
\]

be a partition of \( I \), \( D_i \) be a very general member of the linear system \( \mathbb{P} L_i \), for \( \gamma \in A \cup B \), \( S = \cap_{\alpha \in A} D_{I_\alpha} \) and \( D = \sum_{\beta \in B} D_{I_\beta} \), where

\[
\mathcal{L}_J = \bigotimes_{j \in J} \mathcal{L}_j \tag{1.18}
\]

for \( J \subset I \).

Let \( \varphi : N_1(X) \to \mathbb{R} \) be a subadditive function on \( N_1(X) \) such that \( \{D_i\}_{i \in I} \) satisfies the condition \( C_X(\varphi, \dim X - 2, \dim S - 1) \). Then

\[
2g(C) - 2 + i_X(C,D) \geq \varphi(C) \tag{1.19}
\]

for all reduced curves \( C \subset S \).

**Remark 1.18.** For example, we let \( X = \mathbb{P}^n \), \( \varphi(C) = (d - 2n) \deg C \), \( A = \emptyset \), \( \{D_i\} \) consist of \( d \) hyperplanes and \( D \in |\sum D_i| \) be a very general hypersurface of degree \( d \). It is not hard to check that \( \{D_i\} \) satisfies the condition \( C_X(\varphi, n - 2, n - 1) \) if \( d \geq 2n \). Therefore, (1.5) holds for \( d \geq 2n \). It is easy to check that (1.5) holds for \( d < 2n \) as well. Therefore, Theorem 1.7 follows.

We do not require \( D \) to be irreducible in the above theorem. Therefore, Theorem 1.7 holds for \( D \) reducible as well.
Corollary 1.19. Let $D = \bigcup D_k \subset \mathbb{P}^n$, where $D_k \subset \mathbb{P}^n$ is a very general hypersurface of degree $d_k$. Then
\begin{equation}
2g(C) - 2 + i_{\mathbb{P}^n}(C, D) \geq (d - 2n) \deg C
\end{equation}
for all reduced curves $C \subset \mathbb{P}^n$ where $d = \sum d_k$.

The above corollary is useful since the hyperbolicity of $\mathbb{P}^n \setminus D$ has been studied for reducible $D$'s as well (see e.g. [G1], [G2] and [DSW]).

As another corollary of Theorem 1.17, we take $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_k}$. Let $H_j$ be the pullback of the hyperplane divisors under the projections $X \to \mathbb{P}^{n_j}$, for $j = 1, 2, \ldots, k$. We let $\mathcal{A} = \emptyset, \{D_i\}$ consist of $d_1$ divisors in $|H_1|$, $d_2$ divisors in $|H_2|$, ... and $d_k$ divisors in $|H_k|$. Let $D$ be a very general member of $|\sum D_i|$ and $\varphi(C) = \epsilon(H_1 + H_2 + \ldots + H_k)C$, where
\begin{equation}
\epsilon = \min_{1 \leq j \leq k} (d_j - n_j - n)
\end{equation}
and $n = \sum n_j$. It is not hard to check that $\{D_i\}$ satisfies the condition $C_X(\varphi, n - 2, n - 1)$ when $\epsilon \geq 0$. So we have the following corollary.

Corollary 1.20. Let $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_k}$ and $D \subset X$ be a very general hypersurface of type $(d_1, d_2, \ldots, d_k)$. Then
\begin{equation}
2g(C) - 2 + i_X(C, D) \geq \epsilon \deg C
\end{equation}
for all reduced curves $C \subset X$, where $\epsilon$ is given by (1.21) and $\deg C = (H_1 + H_2 + \ldots + H_k)C$.

So $(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_k}, D)$ is algebraically hyperbolic if $d_j > n_j + n$ for $j = 1, 2, \ldots, k$ and it is reasonable to expect $(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_k}) \setminus D$ to be hyperbolic for such $D$, which can be regarded as a generalized Kobayashi conjecture. This result is again sharp in the sense that the equality can be achieved in (1.22) when $\epsilon \geq 0$.

When $\mathcal{A} \neq \emptyset$, $C$ is contained in the complete intersection $S$. Although the introduction of the case $C \subset S$ is mainly for the purpose of induction, it is quite interesting in its own right. For example, we let $X = \mathbb{P}^n, \mathcal{B} = \emptyset, \{D_1\}$ consist of $d$ hyperplanes and $D_{1\alpha}$ be a very general hypersurface of degree $d_\alpha$ for $\alpha \in \mathcal{A}$, where $d = \sum d_\alpha$. It is easy to check that $\{D_1\}$ satisfies $C_X(\varphi, n - 2, n - a - 1)$ with $\varphi(C) = (d + a - 2n) \deg C$ for $d + a \geq 2n$, where $a = |\mathcal{A}|$. So we obtain the following corollary.

Corollary 1.21. Let $D_k \subset \mathbb{P}^n$ be a very general hypersurface of degree $d_k$ for $k = 1, 2, \ldots, a$.
\begin{equation}
2g(C) - 2 \geq \left( \sum_{k=1}^a d_k + a - 2n \right) \deg C
\end{equation}
for all reduced curves $C \subset S = \cap D_k$.

This is a well-known result due to Clemens \cite{Cl} for hypersurfaces and to Ein \cite{E1} for complete intersections. See also \cite{E2, V, CLR, C-L} and \cite{X1}.

As before, we may formulate a generalization of Corollary 1.21 in

$$X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_k}.$$  

**Corollary 1.22.** Let $D_i \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_k}$ be a very general hypersurface of type $(d_{i1}, d_{i2}, \ldots, d_{ik})$ for $i = 1, 2, \ldots, a$. Then

$$2g(C) - 2 \geq \min_{1 \leq j \leq k} \left( \sum_{i=1}^{a} d_{ij} + a - n_j - n \right) \deg C$$  

for all reduced curves $C \subset \cap D_i$, where $n = \sum n_j$ and $\deg C = (H_1 + H_2 + \ldots + H_k)C$.

Similar corollaries hold in other homogeneous spaces, e.g., Grassmannians, which we will not formulate here.

**1.3. Some questions.** It is interesting to notice that although Corollary 1.19 and 1.20 are sharp, Corollary 1.21 and 1.22 are most likely not. The latter two imply that the corresponding complete intersections are algebraically hyperbolic if the RHS’s of (1.23) and (1.24) are positive. However, there are no obvious reasons why they should not be algebraically hyperbolic if the RHS’s are zero. Actually, we expect they are and some cases have already been proved in \cite{C2}.

**Question 1.23.** Under the hypothesis of Corollary 1.22, is $\cap D_i$ algebraically hyperbolic if

$$\min_{1 \leq j \leq k} \left( \sum_{i=1}^{m} d_{ij} + m - n_j - n \right) = 0?$$  

Especially, it is interesting to know whether a very general hypersurface of degree $d = 2n - 1$ in $\mathbb{P}^n$ is algebraically hyperbolic.

There are discussions on the significance of this question in \cite{C2}.

Alternatively, one may ask what is the smallest $d$ such that a very general hypersurface $D$ of degree $d$ in $\mathbb{P}^n$ is algebraically hyperbolic. If $d = 2n - 3$, $D$ contains lines (see e.g. \cite{H}) and hence cannot be algebraically hyperbolic. So the only degrees in doubt are $d = 2n - 2$ and $2n - 1$. We believe that $D$ is algebraically hyperbolic if $d = 2n - 1$ while it is not if $d = 2n - 2$. While we have some evidences for our conjecture on $d = 2n - 1$, our assertion on $d = 2n - 2$ is pure speculation. But even for $d = 2n - 1$, we do not know the answer even for $n = 3$. 
That is, we do not even know whether a very general quintic surface in $\mathbb{P}^3$ is algebraically hyperbolic or not, although some similar statements for complete intersections have been proved in \[C^2\].

Despite the fact we have proved that some classes of quasi-projective varieties, which are conjectured to be hyperbolic, are algebraically hyperbolic, we still do not know whether “hyperbolic” implies “algebraically hyperbolic”.

**Question 1.24.** If $X\setminus D$ is hyperbolic (and hyperbolically embedded), does this imply that $(X, D)$ is algebraically hyperbolic?

This is known for $D = \emptyset$ by Demailly \[D\], Theorem 2.1, p. 293]. However Demailly’s argument for the projective case does not carry over to the quasi-projective case since the integral of the curvature of the hyperbolic metric on an affine curve is not finite. However, we do have the following.

**Proposition 1.25.** Suppose that $(X, D, T_X)$ has a logarithmic $k$-jet metric with strictly negative curvature for some $k$. Then $(X, D)$ is algebraically hyperbolic.

Please see \[D\] for the definitions of jet differentials and jet metrics and \[D-L\] for those of logarithmic jet differentials and logarithmic jet metrics.

The proof of the above statement follows the same argument of Theorem 8.1 in \[D\], where we only have to change from “jets” to “logarithmic jets” and from $T_C$ to $T_C(-\log \nu^{-1} D)$. It is conjectured by Demailly that $X$ is hyperbolic if and only if $(X, T_X)$ has strictly negative $k$-jet curvature for $k$ large enough \[D\], Conjecture 7.13, p. 324]. We may likewise form a conjecture for log varieties: $X\setminus D$ is hyperbolic if and only if $(X, D, T_X)$ has strictly negative logarithmic $k$-jet curvature.

**1.4. Conventions.**

1. Throughout the paper, we will work exclusively over $\mathbb{C}$.

2. By a variety $X$ being very general, we mean that $X$ lies on the corresponding parameter space (Hilbert scheme or moduli space) with countably many proper closed subschemes removed. So the notion of being very general relies on the fact that the base field $\mathbb{C}$ we work with is uncountable.

**2. Proof of Theorem 1.15**

**2.1. Preliminaries.** An effective adjunction sequence has the following properties.
Lemma 2.1. Let $f: Y \to X$ be a morphism between two projective varieties $X$ and $Y$. Suppose that $\{D_i\}_{i \in I}$ is an effective adjunction sequence on $X$. Let $\bar{D}_i = f^*D_i$ for $i \in I$. Then the following holds.

1. If $f$ is finite over $f(Y)$, $\{\bar{D}_i\}_{i \in I}$ is an effective adjunction sequence on $Y$.

2. If $f$ is finite over $f(Y)$, for each $J \subseteq I$, $\{\bar{D}_i : i \in I \setminus J\}$ is an effective adjunction sequence when restricted to $\bar{D}_J = \cap_{j \in J} \bar{D}_j$.

3. If $Y$ is smooth and $f$ is generically finite over $f(Y)$, $K_Y + \sum_{i \in I} \bar{D}_i$ is effective.

The last statement is the reason we use the term “effective adjunction sequence”.

Proof of Lemma 2.1. The first two statements are obvious and we will leave their proofs to the readers.

We will prove the effectiveness of $K_Y + \sum_{i \in I} \bar{D}_i$ by repeatedly cutting $f(Y)$ by $D_i$’s and applying Kawamata-Viehweg vanishing theorem.

Obviously, $\bar{D}_i$ is BPF on $Y$. By Bertini’s theorem, we may assume that $\bar{D}_i$ is smooth for each $i$. Since $\{D_i\}_{i \in I}$ is an effective adjunction sequence, $\sum_{i \neq \alpha} D_i$ is ample for any $\alpha \in I$. So at least one of $\bar{D}_i$ is nonempty. Let us fix $\alpha \in I$ such that $D_\alpha \cap f(Y) \neq \emptyset$.

Since $\sum_{i \neq \alpha} D_i$ is ample, $\sum_{i \neq \alpha} \bar{D}_i$ is big and nef on $Y$. Then

$$h^1(K_Y + \sum_{i \neq \alpha} \bar{D}_i) = 0$$

by Kawamata-Viehweg vanishing theorem. Therefore, we have the surjection

$$H^0(K_Y + \sum_{i \in I} \bar{D}_i) \to H^0(\mathcal{O}_{\bar{D}_\alpha}(K_{\bar{D}_\alpha} + \sum_{i \neq \alpha} \bar{D}_i)).$$

Therefore, to show that $K_Y + \sum \bar{D}_i$ is effective on $Y$, it suffices to show that $K_{\bar{D}_\alpha} + \sum_{i \neq \alpha} \bar{D}_i$ is effective on $\bar{D}_\alpha$. Since $\{D_i : i \neq \alpha\}$ is an effective adjunction sequence when restricted to $D_\alpha \cap f(Y)$, $\sum_{i \neq \alpha} D_i$ is ample when restricted to $D_\alpha \cap f(Y)$ and hence there is at least one $D_\beta \in \{D_i : i \neq \alpha\}$ such that $D_\alpha \cap D_\beta \cap f(Y) \neq \emptyset$. Continue cutting $\bar{D}_\alpha$ by $\bar{D}_\beta$ and we obtain

$$H^0(K_Y + \sum_{i \in I} \bar{D}_i) \to H^0(\mathcal{O}_{\bar{D}_\alpha}(K_{\bar{D}_1} + \sum_{i \neq \alpha} \bar{D}_i))$$

$$\to H^0(\mathcal{O}_{\bar{D}_{\alpha\beta}}(K_{\bar{D}_{\alpha\beta}} + \sum_{i \neq \alpha, \beta} \bar{D}_i))$$
where \( \bar{D}_{\alpha \beta} = \bar{D}_{\alpha} \cap \bar{D}_{\beta} \). We will carry on this argument and eventually end up with

\[
(2.4) \quad H^0(K_\mathcal{Y} + \sum_{i \in I} \bar{D}_i) \to H^0(O_{\bar{D}_J}(K_{\bar{D}_J} + \sum_{i \notin J} \bar{D}_i))
\]

where \( J \subset I \) is an index subset such that \( |J| = \dim \mathcal{Y} - 1 \) and

\[
(2.5) \quad f(\mathcal{Y}) \cdot \prod_{j \in J} D_j \neq 0.
\]

So it remains to justify that \( K_{\bar{D}_J} + \sum_{i \notin J} \bar{D}_i \) is effective. Since \( \bar{D}_J \) is a curve, it suffices to show that \( \bar{D}_J \cdot \sum_{i \notin J} \bar{D}_i \geq 2 \), i.e.,

\[
(2.6) \quad f(\mathcal{Y}) \cdot \prod_{j \in J} D_j \cdot \sum_{i \notin J} D_i \geq 2.
\]

If \((2.6)\) fails, then at most one of \( f(\mathcal{Y}) \cdot D_i \cdot \prod_{j \notin J} D_j \) is positive and the rest are zeroes for \( i \notin J \). Let \( D_\gamma \in \{ D_i : i \notin J \} \) be the divisor such that \( f(\mathcal{Y}) \cdot D_i \cdot \prod_{j \notin J} D_j = 0 \) for each \( i \notin J \cup \{ \gamma \} \). Then

\[
(2.7) \quad f(\mathcal{Y}) \cdot \prod_{j \in J} D_j \cdot \sum_{i \notin J \cup \{ \gamma \}} D_i = 0.
\]

But \((2.7)\) contradicts the fact that \( \sum_{i \notin J \cup \{ \gamma \}} D_i \) is ample when restricted to \( \cap_{j \in J} D_j \). This finishes the proof of \((2.6)\) and hence the effectiveness of \( K_\mathcal{Y} + \sum_{i \in I} \bar{D}_i \).

q.e.d.

Regarding the condition \( \mathcal{C}_X(\varphi, j_1, j_2) \), we have the following observations.

**Lemma 2.2.** If \( \{ D_i \}_{i \in I} \) satisfies the condition \( \mathcal{C}_X(\varphi, j_1, j_2) \) on \( X \), then the following holds.

1. Any \( \{ D_k \}_{k \in K} \supset \{ D_i \}_{i \in I} \) satisfies \( \mathcal{C}_Y(\varphi, j_1', j_2') \) on \( Y \) for any \( j_1' \leq j_1, j_2' \leq j_2 \) and closed subscheme \( Y \subset X \).

2. For any two disjoint subsets \( N_1 \) and \( N_2 \) of \( I \) satisfying \( |N_1| \leq j_1 \) and \( |N_2| \leq j_2 \), \( \{ D_i : i \in I \setminus (N_1 \cup N_2) \} \) satisfies \( \mathcal{C}_X(\varphi, j_1 - |N_1|, j_2 - |N_2|) \) when restricted to \( Y = \cap_{i \in N_1} D_i \).

3. Let \( Y \) be a smooth projective variety and \( f : Y \to X \) be a proper morphism which is generically finite over its image. Suppose that \( j_1 \geq 0 \). Then for any \( J \subset I \) satisfying \( |J| \leq j_2 \), the curves \( C \subset Y \) satisfying

\[
(2.8) \quad \left( K_Y + \sum_{i \notin J} f^* D_i \right) C < \varphi(f_* C)
\]
cannot cover $Y$.

Proof. The first statement is obvious.

For (2), let assume that $Y \neq \emptyset$; otherwise, there is nothing to prove. Let $J_1$ and $J_2$ be two disjoint subsets of $I \setminus (N_1 \cup N_2)$ satisfying $|J_1| \leq j_1 - |N_1|$ and $|J_2| \leq j_2 - |N_2|$. Since $\{D_i\}_{i \in I}$ satisfies $C_X(\varphi, j_1, j_2)$, there exists a partition $I = I_1 \sqcup I_2 \sqcup (J_2 \cup N_2)$ such that $J_1 \cup N_1 \subset I_1$, $\{D_i\}_{i \in I_1}$ is an effective adjunction sequence on $X$ and (1.14) holds. By Lemma 2.1, $\{D_i \in I \setminus N_1\}$ is an effective adjunction sequence on $Y$. Then the partition $I \setminus (N_1 \cup N_2) = (I_1 \setminus N_1) \sqcup I_2 \sqcup J_2$ has all the required properties in order for $\{D_i : i \in I \setminus (N_1 \cup N_2)\}$ to satisfy $C_Y(\varphi, j_1 - |N_1|, j_2 - |N_2|)$ on $Y$.

For (3), since $\{D_i\}_{i \in I}$ satisfies $C_X(\varphi, j_1, j_2)$ and $|J| \leq j_2$, there exists a partition $I = I_1 \sqcup I_2 \sqcup J$ of $I$ such that $\{D_i\}_{i \in I_1}$ is an effective adjunction sequence on $X$ and

$$(2.9) \quad \sum_{i \in I_2} f^*D_i \cdot C \geq \varphi(f_*C).$$

If

$$(2.10) \quad \left( K_Y + \sum_{i \in I_1} f^*D_i \right) C \geq 0$$

then

$$(2.11) \quad \left( K_Y + \sum_{i \notin J} f^*D_i \right) C = \left( K_Y + \sum_{i \in I_1} f^*D_i \right) C + \sum_{i \in I_2} f^*D_i \cdot C \geq \varphi(f_*C).$$

Therefore, if $C$ satisfies (2.8), we necessarily have

$$(2.12) \quad \left( K_Y + \sum_{i \in I_1} f^*D_i \right) C < 0.$$

So such $C$ must be contained in the base locus of the linear series $|K_Y + \sum_{i \in I_1} f^*D_i|$. By Lemma 2.1, $K_Y + \sum_{i \in I_1} f^*D_i$ is effective and hence $C$ is contained in a fixed proper closed subscheme of $Y$. Consequently, such curves cannot cover $Y$. q.e.d.

2.2. Deformation lemmas. Let us first recall how a similar statement of Theorem 1.15 was proved in [C2, Theorem 1.8]. To explain the ideas of [C2] in a nutshell, we will work out an example by showing how to prove (1.5) for $D$ a union of five lines of $\mathbb{P}^2$. 

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\[ \text{\textcopyright 2023 AI Language Lab} \]
Let $D \subset \mathbb{P}^2$ be a union of five lines in general position. We claim that
\begin{equation}
2g(C) - 2 + i_{\mathbb{P}^2}(C, D) \geq \deg C
\end{equation}
for any reduced irreducible curves $C \subset \mathbb{P}^2$ with $C \not\subset D$. This was proved in two steps in [C2].

First, we prove the following: if $D \subset \mathbb{P}^2$ is a union of four lines with no three lines passing through the same point, then there are at most countably many reduced irreducible curves $C \subset \mathbb{P}^2$ violating (2.13). This was proved in [C2] using the results on the deformation of maps with tangency conditions (see e.g. [H-M, Chap 3, Sec B]). However, we find it a little cumbersome to generalize this deformational argument to higher dimensions. So a slightly different approach will be taken upon here (see Lemma 2.4 below).

Second, after we proved (2.13) with only countably many exceptions for $D$ a union of four lines, the following argument “eliminates” these exceptions once we have an extra line. Let $D = \bigcup_{i=1}^{5} L_i$ be a union of five lines in general position and let $D' = \bigcup_{i=1}^{4} L_i$. If $C$ satisfies (2.13), we are done. Otherwise, we necessarily have
\begin{equation}
2g(C) - 2 + i_{\mathbb{P}^2}(C, D') < \deg C.
\end{equation}
Since there are only countably many curves $C$ satisfying (2.14), by Bertini, $L_5$ meets $C$ transversely and $D' \cap L_5 \cap C = \emptyset$. Actually, we may choose $D'$ to be the union of any four lines of $D$ and run the same argument as above. Eventually, we conclude that either (2.13) holds or $C$ meets $D$ transversely. If it is the latter case, $i_{\mathbb{P}^2}(C, D) = 5 \deg C$ and (2.13) is trivially satisfied.

It turns out that things become more complicated if we go up in dimensions. Let us first introduce the following definition (see also [CLR]).

**Definition 2.3.** Let $X$ be a scheme. We say that the subschemes $C \subset X$ with a certain property $P$ are at most $r$-filling if there exists a union $W \subset X$ of countably many closed subschemes of $X$ of dimension $r$ such that every $C$ with property $P$ lies in $W$.

Let $D \subset \mathbb{P}^n$ be a union of $2n + 1$ hyperplanes in general position. We want to generalize the above argument to show that
\begin{equation}
2g(C) - 2 + i_{\mathbb{P}^n}(C, D) \geq \deg C
\end{equation}
for any reduced irreducible curves $C \subset \mathbb{P}^n$ with $C \not\subset D$. This is done in three steps.
**Step 1:** We show that the curves $C$ violating (2.15) are at most $(n-1)$-filling in $\mathbb{P}^n$ if $D$ is a union of $n+2$ hyperplanes.

**Step 2:** We show that the curves $C$ violating (2.15) are at most $(n-r)$-filling if $D$ is a union of $n+r+1$ hyperplanes for $2 \leq r \leq n-1$.

**Step 3:** We show that there are no curves $C$ violating (2.15) if $D$ is a union of $2n+1$ hyperplanes.

Of course, things become more technical when we work in the general setting of Theorem 1.13. But our basic approach is still the same.

The following two lemmas conclude the first step of our proof of Theorem 1.13. These are deformation-theoretical results in essence.

Let us first recall the following definitions.

A variety $X$ is of normal crossing (has a normal-crossing singularity, etc.) at a point $p \in X$ if $X$ is locally analytically given by $\{ (z_1, z_2, ..., z_n) : z_1 z_2 ... z_m = 0 \}$ at $p$ for some $m \leq n$. A variety is of normal crossing (has normal crossing, has only normal-crossing singularities, etc.) if it is of normal crossing everywhere. And a variety $X$ has simple normal crossing if each of its irreducible components is smooth in addition that $X$ has normal crossing.

**Lemma 2.4.** Let $X$ be a smooth projective variety of dimension $n$ and $D$ be a reduced effective divisor on $X$. Then the reduced irreducible curves $C \subset X$ with the following properties are at most $(n-1)$-filling: $C \not\subset D$, $C$ meets $D$ only at points where $D$ has normal crossing and

\[ 2g(C) - 2 + i_X(C, D) < (K_X + D)C. \]  

**Proof.** Let us first prove the lemma assuming that $C$ only meets $D$ at smooth points of $D$.

Let $M$ be the variety parameterizing such curves. It does not really matter how we construct $M$. For example, we may realize $M$ as a subvariety of the moduli space of stable maps to $X$ with marked points. A point in $M$ is $(f, q_1, q_2, ..., q_l)$, where $f : C^\nu \to X$ is a stable map such that $C = f(C^\nu)$ has the required properties and $f^*D = m_1q_1 + m_2q_2 + ... + m_lq_l$ with $l = i_X(C, D)$. If we construct $M$ this way, we have got a natural compactification of $M$ and a universal family $\pi : Y \to M$ over $M$. Compactify $M$ and let us assume that $Y$ and $M$ are projective.

There is a natural map $f : Y \to X$. The statement of the lemma is equivalent to that $f$ is not dominant. Let us assume that $f$ is dominant.

Without the loss of generality, let us assume that $M$ is irreducible; otherwise, we replace $M$ by one of its irreducible component $M_0$ such that $\pi^{-1}(M_0)$ dominates $X$. We may also assume that $\dim M = \dim X - 1$; if not, we may replace $M$ by a subvariety $M_0 \subset M$ satisfying that
dim \( M_0 = \dim X - 1 \) and \( \pi^{-1}(M_0) \) dominates \( X \). So we may assume that \( f : Y \to X \) is a surjective generically finite map. Let \( Q_1, Q_2, \ldots, Q_l \) be the \( l \) sections of \( \pi : Y \to M \) which restrict to the \( l \) marked points on a general fiber of \( \pi \).

After we desingularize \( M \) and \( Y \), let us assume that \( M \) and \( Y \) are nonsingular. Since \( f \) is surjective, we have the exact sequence

\[
0 \to f^* \Omega_X \to \Omega_Y \to \Omega_{Y/X} \to 0. \tag{2.17}
\]

Hence \( c_1(\Omega_Y) = c_1(f^* \Omega_X) + c_1(\Omega_{Y/X}) \). Since \( f \) is generically finite, \( \Omega_{Y/X} \) is a torsion sheaf on \( Y \). A local computation shows that \( Q_k \subset \text{supp}(\Omega_{Y/X}) \) and \( \Omega_{Y/X} \) has length at least \( m_k - 1 \) along \( Q_k \) for \( k = 1, 2, \ldots, l \). Therefore, the divisor

\[
c_1(\Omega_Y) - c_1(f^* \Omega_X) - \sum_{k=1}^{l} (m_k - 1)Q_k \tag{2.18}
\]

is effective. And its restriction to a general fiber \( C' \) of \( \pi : Y \to M \) is effective, too. Obviously, \( K_Y|_{C'} = K_{C'} \) and \( \sum_{k=1}^{l} (m_k - 1)Q_k|_{C'} = \sum_{k=1}^{l} (m_k - 1)q_k = f^*D - \sum_{k=1}^{l} q_k \). Therefore,

\[
K_{C'} - f^*(K_X + D) + \sum_{k=1}^{l} q_k \tag{2.19}
\]

is effective, which implies \( 2g(C) - 2 + i_X(C, D) \geq (K_X + D)C \). This is a contradiction.

Now let us handle the case that \( C \) meets \( X \) at singular points of \( D \). Let \( [C] \) be a general point of \( M \). Suppose that \( C \) meets \( D \) at \( p \) where \( D \) has normal crossing. Let \( D_{\text{sing}} \) be the singular locus of \( D \) and \( P \) be the irreducible component of \( D_{\text{sing}} \) containing \( p \). We blow up \( X \) along \( P \) and let \( f : \tilde{X} \to X \) be the corresponding map. Let \( \tilde{C} \) be the proper transform of \( C \) under \( f \) and \( \tilde{D} = \text{supp}(f^*D) \) be the reduced total transform of \( D \). It is easy to check that

\[
g(\tilde{C}) = g(C), \tag{2.20}
\]

\[
i_X(\tilde{C}, \tilde{D}) = i_X(C, D), \tag{2.21}
\]

and

\[
K_{\tilde{X}} + \tilde{D} = f^*(K_X + D). \tag{2.22}
\]

If \( \tilde{C} \) meets \( \tilde{D} \) only at nonsingular points of \( \tilde{D} \) over \( p \), we are done. If not, \( \tilde{C} \) will meet \( \tilde{D} \) at a point \( p' \) on the exceptional divisor \( E \) of \( f \),
where $\tilde{D}$ again has normal crossing. Then we continue to blow up $\tilde{X}$ along the irreducible component of $\tilde{D}_{\text{sing}}$ containing $p'$. This process will eventually end and (2.21)-(2.22) always hold during the process. After a finite sequence of blowups over $p$, let $\tilde{X}$ be the resulting variety, $\tilde{C}$ be the proper transform of $C$ and $\tilde{D}$ be the reduced total transform of $D$. Then $\tilde{C}$ will eventually meet $\tilde{D}$ only at nonsingular points of $\tilde{D}$ over $p$; meanwhile, we always have

$$2g(\tilde{C}) - 2 + i_{\tilde{X}}(\tilde{C}, \tilde{D}) - (K_{\tilde{X}} + \tilde{D})\tilde{C}$$

by (2.21)-(2.22). Do this for every $p \in C \cap D$ where $D$ is singular and we reduce the lemma to the case that $C$ meets $D$ only at smooth points of $D$, which we have already proved. q.e.d.

**Remark 2.5.** It is essential that $C$ only meets $D$ at normal crossing singularities of $D$; otherwise, the above lemma may not hold if worse singularities present themselves in $C \cap D$ due to the fact that (2.22) may not hold if we blow up along $P$ where $D$ does not have normal crossing. For example, let us consider the lines $L$ meeting a cubic curve in $\mathbb{P}^2$ at only one point. We would expect that there are only finitely many such lines according to the lemma. However, this is simply false if the cubic has a triple point, i.e., it is the union of three lines meeting at a point $p$ for which $L$ could be any line passing through $p$; on the other hand, it is fine for a nodal cubic for which $L$ could only be one of the two tangent lines at the node or one of the three flex lines.

**Lemma 2.6.** Let $X$ be a smooth projective variety of dimension $n$, $\{\mathbb{P}L_i\}_{i \in I}$ be a finite set of BPF linear systems on $X$ and $D_i$ be a general member of $\mathbb{P}L_i$ for each $i \in I$. Let $F$ be a fixed effective divisor of $X$ and $P \subset F \cap D$ be a closed subscheme of $X$ pure of codimension two or empty, where $D = \sum_{i \in I} D_i$. Then the reduced irreducible curves $C \not\subset D$ satisfying

$$2g(C) - 2 + i_X(C, D, P) < (K_X + D)C$$

are at most $(n - 1)$-filling in $X$.

There is a similar statement for surfaces in $\mathbb{C}^2$, Lemma 2.1]. The proofs of the two lemmas are very close. Basically, we will try reduce the lemma to the case $P = \emptyset$ and $C$ meets $D$ only along the nonsingular locus of $D$ so that we may apply Lemma 2.4. However, there are some technical issues presenting themselves in high dimensions. In the case of surfaces, $P$ is a finite set of points and we assume $P$ to be reduced
2. PROOF OF Theorem 1.15

in [C2, Lemma 2.1]; hence the blowup \( \tilde{X} \) of \( X \) along \( P \) is smooth and it is obvious that

\begin{equation}
K_{\tilde{X}} + f_*^{-1}D = f^*(K_X + D),
\end{equation}

where \( f_*^{-1}D \) is the proper transform of \( D \) under the blowup \( f: \tilde{X} \to X \). Based on this observation, we reduced the lemma to the case \( P = \emptyset \) in [C2, Lemma 2.1]. However, things are not that easy in high dimensions: \( P \) could be singular; as a consequence, the blowup \( \tilde{X} \) of \( X \) along \( P \) could very well be singular and we no longer have (2.25). Actually, \( K_{\tilde{X}} \) will even fail to be \( \mathbb{Q} \)-Cartier if \( P \) is really “bad”. To overcome these obstacles, we need first to “remove” the bad singularities of \( P \). This is achieved by making \( F \) into an effective divisor with simple-normal-crossing support.

**Proof of Lemma 2.6.** It is a well-known fact that there exists a series of blowups \( f: \tilde{X} \to X \) of \( X \) with smooth centers such that \( f^*F \) is supported on an effective divisor of simple normal crossing. Let \( E_f \subset \tilde{X} \) be the exceptional locus of \( f \), \( \tilde{F} = f^*F, \tilde{D} = f^*D, \tilde{P} = f^{-1}(P) \) and \( \tilde{C} = f_*^{-1}C \) be the proper transform of \( C \) under \( f \), where we assume that \( C \subset X \) is a reduced irreducible curve such that \( C \not\subset f(E_f) \) (those curves contained in \( f(E_f) \) are at most \((n-2)\)-filling anyway). We claim that

\begin{align}
(2.26) & \quad g(C) = g(\tilde{C}), \\
(2.27) & \quad (K_X + D)C \leq (K_{\tilde{X}} + \tilde{D})\tilde{C}, \quad \text{and} \\
(2.28) & \quad i_X(C, D, P) \geq i_{\tilde{X}}(\tilde{C}, \tilde{D}, \tilde{P})
\end{align}

where (2.26) is obvious and (2.27) follows directly from the fact that \( \tilde{X} \) is the blowup of \( X \) with smooth centers and \( C \not\subset f(E_f) \), while (2.28) requires some explanation.

Let \( \text{Bl}_{\tilde{P}} \tilde{X} \) and \( \text{Bl}_P X \) be the blowups of \( \tilde{X} \) and \( X \) along \( \tilde{P} \) and \( P \), respectively. We have the commutative diagram

\begin{equation}
\begin{array}{ccc}
\text{Bl}_{\tilde{P}} \tilde{X} & \xrightarrow{\tilde{g}} & \text{Bl}_P X \\
\downarrow \pi & & \downarrow \pi \\
\tilde{X} & \xrightarrow{f} & X
\end{array}
\end{equation}

Let \( \tilde{\pi}_*^{-1}\tilde{D} \subset \text{Bl}_{\tilde{P}} \tilde{X} \) and \( \pi_*^{-1}D \subset \text{Bl}_P X \) be the proper transforms of \( \tilde{D} \) and \( D \) under \( \tilde{\pi} \) and \( \pi \), respectively. Obviously, we have the commutative
diagram

\[
\begin{array}{c}
\tilde{\pi}_*^{-1}\tilde{D} \xrightarrow{g} \pi_*^{-1}D \\
\pi \downarrow \quad \pi \downarrow \\
\tilde{D} \xrightarrow{f} D
\end{array}
\]

(2.30)

and hence \( g(\tilde{\pi}_*^{-1}\tilde{D}) = \pi_*^{-1}D \). Consequently,

\[
g\left(\tilde{C} \cap \tilde{\pi}_*^{-1}\tilde{D}\right) \subset C \cap \pi_*^{-1}D
\]

(2.31)

and (2.28) follows.

We conclude from (2.26)-(2.28) that for every \( C \subset X \) satisfying (2.24) and \( C \not\subset f(E_f) \),

\[
2g(\tilde{C}) - 2 + i_X(\tilde{C}, \tilde{D}, \tilde{P}) < (K_X + \tilde{D})\tilde{C}.
\]

(2.32)

So the conclusion of the lemma holds for \((\tilde{X}, \tilde{D}, \tilde{P})\) as long as it holds for \((X, D, P)\), where \( \tilde{P} \subset \tilde{D} \cap \tilde{F} \) and \( \text{supp}(\tilde{F}) \) has simple normal crossing.

Therefore, we may simply assume that \( F \) has simple-normal-crossing support at the very beginning.

Let \( X' \) be the blowup of \( X \) along \( P \) and \( \tilde{X} \) be a desingularization of \( X' \). We claim that (2.25) holds for \( f : \tilde{X} \to X \). This follows from the following explicit construction of \( \tilde{X} \).

Let \( P = \cup_{i=1}^\alpha \mu_i P_i \), where \( P_i \)'s are the irreducible components of \( P \) and \( \mu_i \) is the multiplicity of \( P_i \) in \( P \).

If \( \mu_i = 1 \) for each \( i \), i.e., \( P \) is reduced, \( \tilde{X} \) can be constructed by subsequently blowing up \( X \) along \( P_1, P_2, \ldots, P_\alpha \). That is, we first blow up \( X \) along \( P_1 \) to obtain \( X_1 \), then blow up \( X_1 \) along the proper transform of \( P_2 \) to obtain \( X_2 \), next blow up \( X_2 \) along the proper transform of \( P_3 \) to obtain \( X_3 \) and so on. We obtain a sequence of blowups:

\[
\tilde{X} = X_\alpha \xrightarrow{f_\alpha} X_{\alpha-1} \xrightarrow{f_{\alpha-1}} \ldots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = X.
\]

(2.33)

It is not hard to see that (2.33) is a sequence of blowups with smooth centers due to our assumption on \( F \) and \( D \). Therefore, \( \tilde{X} \) is smooth and it is easy to check (2.25) by observing that

\[
K_{X_i} + D^{(i)} = f_i^*(K_{X_{i-1}} + D^{(i-1)})(i)
\]

for each \( i \), where \( D^{(i)} \) is the proper transform of \( D \) under the map \( X_i \to X \).

The construction of \( \tilde{X} \) becomes more complicated if \( P \) is nonreduced. Note that at a general point \( p \in P_i \), \( P \) is locally given by \( x = y^{\mu_i} = 0 \), where \( x = 0 \) defines \( D \) and \( y^\mu = 0 \) defines \( F \) for some \( \mu \geq \mu_i \). So if \( \mu_i > 1 \), \( X' \) has a singularity \( p' \) over \( p \), which is locally
given by \( xz = y^{\mu_1} \). This suggests that \( \tilde{X} \) is locally constructed over \( p \) by blowing up \( X \) along \( P_i \mu_i \) times (see Remark 2.8 for explanations). Correspondingly, the blowup \( f_i : X_i \to X_{i-1} \) in (2.33) is replaced by a sequence of \( \mu_i \) blowups:

\[
(2.35) \quad X_i = X_{i,\mu_i} \xrightarrow{f_{i,\mu_i}} X_{i,\mu_i-1} \xrightarrow{f_{i,\mu_i-1}} \cdots \xrightarrow{f_{i,1}} X_{i,0} = X_{i-1}
\]

where \( f_{i,1} : X_{i,1} \to X_{i-1} \) is the blowup of \( X_{i-1} \) along \( \widetilde{P}_i \) and \( f_{i,j} : X_{i,j} \to X_{i,j-1} \) is the blowup of \( X_{i,j-1} \) along \( E_{i,j-1} \cap D^{(i,j-1)} \) for \( j > 1 \). Here \( \widetilde{P}_i \) is the proper transform of \( P_i \) under the map \( X_{i-1} \to X \), \( E_{i,j-1} \) is the exceptional divisor of \( f_{i,j-1} \) and \( D^{(i,j-1)} \) is the proper transform of \( D \) under the map \( X_{i,j-1} \to X \). Obviously, (2.33) is a sequence of blowups with smooth centers and it is easy to check that

\[
(2.36) \quad K_{X_{i,j}} + D^{(i,j)} = f_{i,j}^*(K_{X_{i,j-1}} + D^{(i,j-1)}))
\]

for each \( j \). And hence (2.34) and (2.33) follow.

It follows from our construction of \( f : \tilde{X} \to X \) that \( f^{-1}(P) \) is a Cartier divisor of \( \tilde{X} \). Hence \( f \) factors through \( X' \):

\[
(2.37) \quad f : \tilde{X} \xrightarrow{g} X' \to X
\]

by the universal property of blowups. Therefore, \( g : \tilde{X} \to X' \) is a desingularization of \( X' \). Let \( \tilde{D} \) and \( D' \) be the proper transforms of \( D \) under the maps \( \tilde{X} \to X \) and \( X' \to X \), respectively. Then

\[
(2.38) \quad i_{X'}(\tilde{C}, \tilde{D}) \leq i_X(C', D') = i_X(C, D, P),
\]

where \( \tilde{C} \) and \( C' \) are the proper transforms of \( C \) under the maps \( \tilde{X} \to X \) and \( X' \to X \), respectively, and we assume that \( C \subset X \) is a curve not contained in the image of the exceptional locus of \( f \). Therefore,

\[
(2.39) \quad 2g(\tilde{C}) - 2 + i_X(\tilde{C}, \tilde{D}) - (K_\tilde{X} + \tilde{D})\tilde{C} \\
\leq 2g(C) - 2 + i_X(C, D, P) - (K_X + D)C.
\]

Therefore, to prove the lemma for \( (X, D, P) \), it suffices to prove it for \( (\tilde{X}, \tilde{D}, \emptyset) \). This reduces the lemma to the case \( P = \emptyset \). Note that we can no longer assume \( D_k \) to be a general member of a BPF linear system but we do have that \( D \) is of simple normal crossing and that is all we need in order to apply Lemma 2.4.

q.e.d.

Remark 2.7. If \( P \) is reduced, \( g : \tilde{X} \to X' \) is a small morphism, i.e., the exceptional locus of \( g \) has codimension at least two in \( \tilde{X} \). Note that we may blow up \( X \) along \( P_i \)'s in an arbitrary order of \( \{P_i\} \). By choosing
a different order of \(\{P_i\}\), we usually arrive at a different desingularization of \(X\), which is a flop of \(\tilde{X}\). For example, let \(X\) be a threefold and \(P = P_1 \cup P_2\) be a curve in \(X\). Suppose that \(P_1\) and \(P_2\) meet at a point \(p\) which is a node of the curve \(P\). Then \(X'\) has a rational double point \(p'\) over \(p\). Let \(X_{12}\) be the blowup of \(X\) along \(P_1\) followed by the blowup along \(P_2\) and \(X_{21}\) be the blowup of \(X\) along \(P_2\) followed by the blowup along \(P_1\). Then the rational map \(X_{12} \rightarrow X_{21}\) induced by (2.40)

\[
X_{12} \rightarrow X' \leftarrow X_{21}
\]

is a flop.

**Remark 2.8.** To resolve the singularity \(X = \{xz = y^\mu\}\), we first blow up \(X\) along \(P = \{x = y = 0\}\) to obtain \(X'\), which has a singularity given by \(x'z = y^{\mu-1}\) on \(P' = \{x' = y = 0\}\) with \(x' = x/y\). Blow up \(X'\) along \(P'\) to obtain \(X''\), which has a singularity given by \(x''z = y^{\mu-2}\) with \(x'' = x'/y\). Continue this process and we resolve the singularity \(xz = y^\mu\) by blowing up \(X\) along \(P\) \(\mu\) times.

**2.3. Proof of Theorem 1.15.** If \(C \subset D\), say \(C \subset D_\alpha\) for some \(\alpha \in I\), then we may replace \((X,D)\) by \((X',D')\) with \(X' = D_\alpha\) and \(D' = (D - D_\alpha)|_{D_\alpha}\) while observing that \(\{D_i : i \neq \alpha\}\) satisfies \(C_{X'}(\varphi, 0, n - 2)\) on \(X'\). So let us assume that \(C\) meets \(D\) properly. It suffices to prove the following statement.

**Proposition 2.9.** Under the hypotheses of Theorem 1.15, the following holds: for each \(J \subset I\) with \(|J| = n - r\), the reduced irreducible curves \(C \not\subset D \cup F\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \varphi(C)
\]

are at most \((n - r)\)-filling for \(1 \leq r \leq n\).

**Proof.** We argue by induction on \(r\).

By Lemma 2.2, for each subset \(J \subset I\) with \(|J| = n - 1\), the curves \(C\) satisfying

\[
\left(K_X + \sum_{i \notin J} D_i\right) C < \varphi(C)
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]

are at most \((n - 1)\)-filling; and by Lemma 2.6, the reduced irreducible curves \(C\) satisfying

\[
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i, P) < \left(K_X + \sum_{i \notin J} D_i\right) C
\]
are at most \((n-1)\)-filling. Therefore, the reduced irreducible curves \(C\) satisfying (2.41) are at most \((n-1)\)-filling for \(|J| = n - 1\). So the proposition holds for \(r = 1\).

We use the notation \(\sum_{i \notin J} D_i\) to denote the closure of the locally noetherian subscheme of \(X\) swept out by the reduced irreducible curves \(C \not\subseteq D \cup F\) satisfying (2.41). The statement of the proposition is equivalent to that

\[
\dim \sum_{i \notin J} D_i \leq n - r \tag{2.44}
\]

for each \(J \subset I\) with \(|J| = n - r\).

Suppose that

\[
\dim \sum_{i \notin J} D_i \leq n - r + 1 \tag{2.45}
\]

for each \(J \subset I\) with \(|J| = n - r + 1\). We want to show that (2.44) holds. Assume the contrary:

\[
\dim \sum_{i \notin J} D_i = n - r + 1 \tag{2.46}
\]

for some \(J \subset I\) with \(|J| = n - r\). Fix \(k \notin J\) and let \(J' = J \cup \{k\}\). Then according to the inductive hypothesis,

\[
\dim \sum_{i \notin J'} D_i \leq n - r + 1. \tag{2.47}
\]

And since

\[
D_{k'} + \sum_{i \notin J'} D_i = \sum_{i \notin J} D_i \subseteq \sum_{i \notin J'} D_i, \tag{2.48}
\]

\[
\dim D_{k'} + \sum_{i \notin J'} D_i = \dim \sum_{i \notin J'} D_i. \tag{2.49}
\]

Therefore, the irreducible components of \(D_{k'} + \sum_{i \notin J'} D_i\) of dimension \(n - r + 1\) must also be the components of \(\sum_{i \notin J'} D_i\). Now if we choose \(D'_{k} \in P_L\) and \(D'_{k} \neq D_{k}\), we still have

\[
D'_{k} + \sum_{i \notin J'} D_i \subset \sum_{i \notin J'} D_i \tag{2.50}
\]

and

\[
\dim D'_{k} + \sum_{i \notin J'} D_i = \dim \sum_{i \notin J'} D_i. \tag{2.51}
\]
Therefore, as $D_k$ varies in the linear series $\mathbb{P}\mathcal{L}_k$, the $(n-r+1)$-dimensional irreducible components of $D_k + \sum_{i \notin J'} D_i$ vary as irreducible components of $\sum_{i \notin J'} D_i$. When we fix $\sum_{i \notin J'} D_i$, the scheme $\sum_{i \notin J'} D_i$ is fixed and hence so are its components. Consequently, the $(n-r+1)$-dimensional components of $D_k + \sum_{i \notin J'} D_i$ remain fixed as we vary $D_k$ and fix the rest of $D_i$'s. Therefore, the $(n-r+1)$-dimensional components of $D_k + \sum_{i \notin J'} D_i$ and $D_k' + \sum_{i \notin J'} D_i$ are the same. From now on, we will drop all the components of dimension less than $n-r+1$ from the definition of $\sum_{i \notin J} D_i$, i.e., we will pretend that $\sum_{i \notin J} D_i$ is pure of dimension $n-r+1$. Then we have

\begin{equation}
D_k + \sum_{i \notin J'} D_i = D_k' + \sum_{i \notin J'} D_i \tag{2.52}
\end{equation}

for any general choice of $D_k, D_k' \in \mathbb{P}\mathcal{L}_k$. Note that $k$ is chosen arbitrarily from $I \setminus J$. So $\text{(2.52)}$ actually implies that

\begin{equation}
\sum_{i \notin J} D_i = \sum_{i \notin J} D_i' \tag{2.53}
\end{equation}

for any general choices of $D_i, D_i' \in \mathbb{P}\mathcal{L}_i$. Or equivalently, $\sum_{i \notin J} D_i$ remains fixed even if we vary all $D_i$'s at the same time.

A word of warning is in order for a potential misunderstanding of $\text{(2.53)}$. It does not imply that a curve $C \subset X$ satisfying $\text{(2.41)}$ will also satisfy

\begin{equation}
2g(C) - 2 + i_X(C, \sum_{i \notin J} D_i', P) < \varphi(C). \tag{2.54}
\end{equation}

The curves satisfying $\text{(2.41)}$ and $\text{(2.54)}$ may very well be different but they do sweep out the same subscheme in $X$, if we assume that $\text{(2.46)}$ holds.

Let $Y$ be an irreducible component of $\sum_{i \notin J} D_i$ and $f : \tilde{Y} \to Y \subset X$ be a desingularization of $Y$. Let $\tilde{D}_i = f^*D_i, F = f^*F$ and $\tilde{P} = f^{-1}(P)$. Since $Y$ is independent of the choices of $D_i$ by $\text{(2.53)}$, $\tilde{D}_i$ is a very general member of a BPF linear series on $\tilde{Y}$.

For a reduced irreducible curve $C \subset Y$ and $C \not\subset D$, we have $g(\tilde{C}) = g(C)$ and

\begin{equation}
i_X(C, \sum_{i \notin J} \tilde{D}_i, \tilde{P}) \leq i_X(C, \sum_{i \notin J} D_i, P) \tag{2.55}
\end{equation}
by the same argument for (2.28), where $\tilde{C}$ is the proper transform of $C$ under $f$. Hence

$$2g(\tilde{C}) - 2 + i_{\tilde{Y}}(\tilde{C}, \sum_{i \not \in J} \tilde{D}_i, \tilde{P}) \leq 2g(C) - 2 + i_X(C, \sum_{i \not \in J} D_i, P).$$  

Therefore, $\tilde{Y}$ is covered by the reduced irreducible curves $\tilde{C}$ satisfying

$$2g(\tilde{C}) - 2 + i_{\tilde{Y}}(\tilde{C}, \sum_{i \not \in J} \tilde{D}_i, \tilde{P}) < \varphi(C).$$

Since such curves $\tilde{C}$ cover $\tilde{Y}$, we must have

$$(K_{\tilde{Y}} + \sum_{i \not \in J} \tilde{D}_i)\tilde{C} \geq \varphi(C)$$

by Lemma 2.2. Therefore,

$$2g(\tilde{C}) - 2 + i_{\tilde{Y}}(\tilde{C}, \sum_{i \not \in J} \tilde{D}_i, \tilde{P}) < (K_{\tilde{Y}} + \sum_{i \not \in J} \tilde{D}_i)\tilde{C}. 
$$

This contradicts with Lemma 2.6, which says the curves satisfying (2.59) are at most $(\dim \tilde{Y} - 1)$-filling in $\tilde{Y}$. q.e.d.

### 3. A Special Case of Theorem 1.17

Due to the technicality of the proof of Theorem 1.17, we feel it is better to first work out a special case. So we will delay the proof of Theorem 1.17 to the next section. Here instead we will study a special case, as the major ingredients of the proof are already present in this special case.

#### 3.1. The complement of a surface of degree 7 in $\mathbb{P}^3$. Our proof of Theorem 1.17 consists of two main steps of degeneration.

**Step 1:** First we specialize $D$ by degenerating $D_{1\beta}$ to $\sum_{i \in I_\beta} D_i$ for each $\beta \in B$. In this way, we reduce the theorem to the case $|I_\beta| = 1$ for all $\beta \in B$.

**Step 2:** Next we specialize $S$ by degenerating $D_{1\alpha}$ to $\sum_{i \in I_{\alpha}} D_i$ for each $\alpha \in A$. Eventually, we reduce the theorem to Theorem 1.15, which we have already proved.

The second step of our proof relies on a concept developed in [C2], called virtual genus of a curve lying on a reducible variety. Here we will work out an example where $A = \emptyset$. So it only involves the first step of degeneration. The case $\dim X = 2$ was handled in [C2]. Let us consider
the first nontrivial case in $X = \mathbb{P}^3$: the complement of a very general surface $D$ of degree 7. We want to prove that

$$2g(C) - 2 + i_{\mathbb{P}^3}(C, D) \geq \deg C$$

for all reduced curves $C \subset \mathbb{P}^3$. Our inductive hypothesis is that the theorem holds in $\dim X - |A| < 3$. In particular, we assume:

**Hypothese 3.1.** On a very general sextic surface $S \subset \mathbb{P}^3$

$$2g(C) - 2 \geq \deg C$$

for all reduced curves $C \subset S$.

As planned, the proof of (3.1) is carried out by degenerating $D$ to a union of seven planes in general position for which Theorem 1.15 can be applied. However, due to technical difficulties, we find it better to degenerate one plane at a time instead of degenerating $D$ directly to seven planes. Namely, we will first degenerate $D$ to a union of a sextic surface and a plane, then to a union of a quintic surface and two planes and so on. The argument for each step of degeneration is similar. We will illustrate it by carrying out the first step of degeneration. That is, we will prove (3.1) under the inductive hypothesis that

**Hypothese 3.2.** (3.1) holds for $D = S \cup G \subset \mathbb{P}^3$ a very general union of a sextic surface $S$ and a plane $G$, i.e.,

$$2g(C) - 2 + i_{\mathbb{P}^3}(C, S \cup G) \geq \deg C$$

for all reduced curves $C \subset \mathbb{P}^3$.

Thus we will degenerate $D$ to $S \cup G$. The basic set up is as follows.

Let $Z = \mathbb{P}^3 \times \Delta$ and $W \subset Z$ be a pencil of surfaces in $\mathbb{P}^3$ of degree 7 such that $W$ is irreducible and $W_0 = S \cup G$.

Let $Y$ be a reduced flat family of curves over $\Delta$ with the commutative diagram (1.1). Our goal is to prove that

$$2g(Y_t) - 2 + i_Z(\pi_*Y_t, W) \geq \deg \pi_*Y_t$$

for $t \neq 0$ by analyzing what happens on the central fiber. If $\pi_*Y_0$ is reduced and meets $W_0$ properly, then

$$2g(Y_0) - 2 + i_{\mathbb{P}^3}(\pi_*Y_0, W_0) \geq \deg \pi_*Y_0$$

by Hypothesis [3.2] and (3.4) follows due to the obvious semi-continuity of its LHS. However, $\pi_*Y_0$ could very well be nonreduced and even worse, it may fail to meet $W_0$ properly. Overcoming these difficulties is essentially what our proof of Theorem 1.17 is about.

It turns out that the biggest problem we could have is that $\pi_*Y_0$ has a component contained in $G$ while the other problems mentioned
above can be easily overcome. Fortunately, there is a very common construction we can use to resolve the problem.

3.2. Construction of a fan. Let \( f : \tilde{Z} \to Z \) be the blowup of \( Z \) along \( G \) and let \( R \) be the exceptional divisor (see Figure 1). Obviously, the central fiber \( \tilde{Z}_0 \) of \( \tilde{Z} \) is the union \( R \cup X \), where \( R \) is a \( \mathbb{P}^1 \) bundle over \( G \) and \( R \cap X = G \). Actually, \( R \) is the projectivization of the normal bundle \( N_{G/Z} \) of \( G \) in \( Z \). It is not hard to see that

\[
N_{G/Z} = \mathcal{O}_G \oplus \mathcal{O}_G(G) = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1).
\]

(3.6)

So \( R \) is the \( \mathbb{P}^1 \) bundle over \( \mathbb{P}^2 \) given by \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \).

Such construction has been extensively used by Z. Ran in his study of Severi varieties of plane curves [R] and more recently by C. Ciliberto and R. Miranda in their works on Nagata conjecture [CM1] and [CM2]. In their cases, \( Z = \mathbb{P}^2 \times \Delta \) and \( G \subset Z_0 \cong \mathbb{P}^2 \) is a line. Then blow up \( Z \) along \( G \) and we will obtain the ruled surface \( R \cong \mathbb{F}_1 \) over \( G \) on the central fiber \( \tilde{Z}_0 \) as the exceptional divisor. Such construction can be easily generalized. More generally, \( Z \) can be an arbitrary flat family of varieties over \( \Delta \) and \( G \subset Z_0 \) be a closed subscheme of the central fiber \( Z_0 \). We blow up \( Z \) along \( G \) to obtain \( \tilde{Z} \). The central fiber \( \tilde{Z}_0 \) consists of the proper transform of \( Z_0 \) and the exceptional divisor \( R \), which is a \( \mathbb{P}^n \) bundle over \( G \). We will use the terminology of Ran to call \( \tilde{Z}_0 \) a fan.

The main purpose of constructing a fan \( \tilde{Z}_0 \) is to study the infinitesimal behavior of a flat family \( Y \subset Z \) in the neighborhood of \( G \). In our case, we want to “separate” \( \pi(Y) \) and \( W \) along \( G \) via the construction of \( \tilde{Z} \).

Let \( \tilde{W} \) be the proper transform of \( W \) under the blowup \( \tilde{Z} \to Z \). The central fiber \( \tilde{W}_0 \) of \( \tilde{W} \) is the union of the proper transform of \( S \), which we still denote by \( S \), and a surface \( \tilde{G} \subset R \) (see Figure 1).

It is not hard to figure out what \( \tilde{G} \) is. First of all, we obviously have

\[
S \cap G = \tilde{G} \cap G.
\]

(3.7)

Second, it is not hard to see that \( f_* (\tilde{G}) = G \), where \( f : R \to G \) is the projection. Indeed, \( \tilde{G} \) is a member of the linear series

\[
\mathbb{P} H^0(\mathcal{O}_R(G) \otimes f^*(\mathcal{O}_G(7)))
\]

(3.8)

where \( \mathcal{O}_R(G) \) is the tautological line bundle of \( R \) and \( \mathcal{O}_G(1) \) is the hyperplane bundle of \( G \cong \mathbb{P}^2 \). In addition, for a generic choice of the pencil \( W \), the corresponding \( \tilde{G} \) is a general member of the linear series (8.8).
Figure 1. The blowup of $Z = \mathbb{P}^3 \times \Delta$ along a plane $G \subset Z_0$

Now we have a rational map $Y \to \tilde{Z}$ factoring through $Z$. After resolving the indeterminacies of this map, we obtain

\[
\begin{array}{ccc}
    \tilde{Y} & \xrightarrow{\tilde{\pi}} & \tilde{Z} \\
    \downarrow & & \downarrow f \\
    Y & \xrightarrow{\pi} & Z.
\end{array}
\]

(3.9)

Since the blowup $f$ did nothing to the general fibers, we have

\[
2g(Y_t) - 2 + i_Z(\pi_*Y_t, W) = 2g(\tilde{Y}_t) - 2 + i_{\tilde{Z}}(\tilde{\pi}_*\tilde{Y}_t, \tilde{W})
\]

for $t \neq 0$.

We are trying to bound the RHS of (3.10) with the information on $\tilde{Y}_0$, which is a curve lying on the union of two smooth varieties $X$ and $R$ meeting transversely. There is a result [C2, Theorem 1.17] dealing with this situation. We will put it in a more general form suitable for our purpose.

**Proposition 3.3.** Let $X$ be a flat family of projective varieties over $\Delta$, whose general fibers $X_t$ are irreducible and smooth and whose central fiber $X_0 = D = \bigcup_{i \in I} D_i$ is of normal crossing along $\partial D_i$ for each $i \in I$, where $D_i$ are irreducible components of $X_0$ and we write

1. $\partial D_i = D_i \cap (\bigcup_{j \neq i} D_j)$ for $i \in I$,
2. more generally, \( D_J = \cap_{j \in J} D_j \), \( \partial D_J = D_J \cap (\cup_{i \notin J} D_i) \) for every \( J \subset I \),

3. and \( \partial D = \cup_{i \in I} \partial D_i = \cup_{|J|=2} D_J \).

Let \( Q = \partial D \cap X_{\text{sing}} \) be the singular locus of \( X \) along \( \partial D \) and suppose that \( \dim(Q \cap D_J) \leq \dim D_J - 1 \) for every \( J \subset I \).

Let \( W \subset X \) be an effective divisor of \( X \) that is flat over \( \Delta \). Suppose that \( W \) meets \( D_J \) and \( D_J \cap Q \) properly in \( X \) for each \( J \subset I \).

Let \( Y \) be a reduced flat family of curves over \( \Delta \) with the commutative diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{\cdot} & \Delta
\end{array}
\]

where \( \pi : Y \to X \) is a proper map and \( \Delta \to \Delta \) is a base change.

Then

\[
2g(Y_t) - 2 + i_X(\pi_*Y_t, W) \geq \sum_{\Gamma \subset \pi_*Y_0} \mu_\Gamma \Phi_{X,W}(\Gamma)
\]

where we sum over all irreducible components \( \Gamma \subset \pi_*Y_0 \), \( \mu_\Gamma \) is the multiplicity of \( \Gamma \) in \( \pi_*Y_0 \) and

\[
\Phi_{X,W}(\Gamma) = 2g(\Gamma) - 2 + i_X(\Gamma, D \cup W, \partial D_J \cap Q)
\]

with \( J = \{ j \in I : \Gamma \subset D_j \} \).

Furthermore, suppose that there exists a morphism \( f : X \to \hat{X} \) which blows down \( D_\alpha \) for some \( \alpha \in I \) while inducing isomorphisms on \( \cup_{i \neq \alpha} D_i \) and \( X_t \) for \( t \neq 0 \). Let \( \hat{W} = f_*W \). Then

\[
2g(Y_t) - 2 + i_X(\pi_*Y_t, W) \geq \sum_{\Gamma \subset \pi_*Y_0, \Gamma \notin D_\alpha} \mu_\Gamma \Phi_{X,W}(\Gamma) + \sum_{\Gamma \notin D_\alpha} \mu_\Gamma \Phi_{\hat{X},\hat{W}}(f_*\Gamma)
\]

where we sum over the irreducible components \( \Gamma \subset \pi_*Y_0 \) not contracted by \( f \).

**Modification:** if we assume that \( Y \) is irreducible, then (3.12) and (3.14) can be slightly improved as follows. Let \( U_Y \) be the union of the irreducible components \( \Gamma \subset \pi_*Y_0 \) with the following properties:

1. \( \Phi_{X,W}(\Gamma) < 0 \);
2. \( \Gamma \cap \partial D = \emptyset \);
3. \( B \cdot \Gamma \geq 0 \) for every irreducible component \( B \) of \( W \) satisfying that \( \Gamma \subset B \) and \( \pi(Y) \notin B \).
If $U_Y \subset \text{supp}(\pi_*Y_0)$, we may exclude $\Gamma \subset U_Y$ in the summations of (3.12) and (3.14), i.e., (3.12) and (3.14) continue to hold when we sum over $\Gamma \not\subset U_Y$.

**Remark 3.4.** If we take $W = \emptyset$, (3.12) becomes

$$g(Y_t) \geq g^\text{vir}_Q(\pi_*Y_0) \tag{3.15}$$

with $g^\text{vir}_Q(\pi_*Y_0)$ the virtual genus of $\pi_*Y_0$ with respect to $Q$ defined in \cite{C2} and this is exactly what \cite[Theorem 1.17]{C2} says.

We will prove Proposition 3.3 much later in 4.3. Now let us apply it to our situation with $\tilde{Y}$ being the family of curves and $f : \tilde{Z} \to Z$ being the map blowing down $R$. We obtain

$$2g(Y_t) - 2 + i_Z(\pi_*Y_t, W) \geq \sum_{\Gamma \subset R} \mu_f \Phi_{\tilde{Z}, \tilde{W}}(\Gamma) + \sum_{\Gamma \not\subset R} \mu_f \Phi_{Z,W}(\Gamma) \tag{3.16}$$

where we sum over the irreducible components $\Gamma \subset \tilde{\pi}_*\tilde{Y}_0$ not contracted by $f$.

By the definition of $\Phi$, we have

$$\Phi_{\tilde{Z}, \tilde{W}}(\Gamma) = 2g(\Gamma) - 2 + i_R(\Gamma, G \cup \tilde{G}) \quad \text{for } \Gamma \subset R, \tag{3.17}$$
$$\Phi_{Z,W}(\Gamma) = 2g(\Gamma) - 2 \quad \text{for } \Gamma \subset S, \text{ and} \tag{3.18}$$
$$\Phi_{Z,W}(\Gamma) = 2g(\Gamma) - 2 + i_X(\Gamma, S \cup G) \quad \text{for } \Gamma \not\subset S \cup R. \tag{3.19}$$

The RHS's of (3.18) and (3.19) can be easily bounded by the inductive hypothesis, i.e.,

$$2g(\Gamma) - 2 \geq \deg \Gamma \tag{3.20}$$

for $\Gamma \subset S$ by (3.2) and

$$2g(\Gamma) - 2 + i_X(\Gamma, S \cup G) \geq \deg \Gamma \tag{3.21}$$

for $\Gamma \subset X$ by (3.3). Therefore, in order to prove (3.4), it suffices to show

$$2g(\Gamma) - 2 + i_R(\Gamma, G \cup \tilde{G}) \geq \deg f_\ast \Gamma \tag{3.22}$$

for $\Gamma \subset R$ and $f_\ast \Gamma \neq 0$, since $\deg \pi_*Y_t = \sum \mu_{\Gamma'} \deg f_{\ast \Gamma'}$. So we have eventually reduced (3.1), which is a statement on the log pair $(X,D)$, to (3.22), which is a statement on the log pair $(R,G \cup \tilde{G})$. Note that $R$ is a $\mathbb{P}^1$ bundle over $G \cong \mathbb{P}^2$. To prove (3.22), we need to further degenerate $\tilde{G}$. This will lead to the construction of a fan by blowing up $R \times \Delta$ along $F \subset R$, where $F = f^*L$ is the pullback of a line $L \subset G$. 


under the projection $f : R \to G$. The exceptional divisor of this blowup is a $\mathbb{P}^1$ bundle $R'$ over $F$ and $R'$ can also be regarded as a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over $L \cong \mathbb{P}^1$ (see Figure 2). We call $R'$ a projective tower over $L$.

3.3. Projective Tower. Let $X$ be a scheme and $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n$ be vector bundles over $X$. The projective tower of $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n$ over $X$, denoted by $\mathbb{P}_X(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$ or $\mathbb{P}(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$ if $X$ is clear from the context, is constructed inductively as follows. First, let $X_0 = X$ and $X_1 = \mathbb{P}\mathcal{E}_1$. Next, let $X_2 = \mathbb{P}(\pi_1^* \mathcal{E}_2)$ be the projectivization of the vector bundle $\pi_1^* \mathcal{E}_2$ over $X_1$, where $\pi_1$ is the projection $X_1 \to X_0$. Similarly, $X_3$ is the projectivization of the pullback of $\mathcal{E}_3$ over $X_2$ and so on. Finally, $X_n$ is the projectivization of the pullback of $\mathcal{E}_n$ over $X_{n-1}$ and we call $X_n = \mathbb{P}(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$ the projective tower of $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n$ over $X$. The order of $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n$ in the construction is not important, e.g., $\mathbb{P}(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = \mathbb{P}(\mathcal{E}_2, \mathcal{E}_1, ..., \mathcal{E}_n)$. Indeed, $\mathbb{P}(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$ is just the fiber product $\mathbb{P}\mathcal{E}_1 \times_X \mathbb{P}\mathcal{E}_2 \times_X \mathbb{P}\mathcal{E}_n$ over $X$. The Picard group
of \( \mathbb{P}(E_1, E_2, ..., E_n) \) is \( \text{Pic} X \oplus \mathbb{Z}^n \), where the \( \mathbb{Z}^n \) part is generated by \( M_1, M_2, ..., M_n \) with \( M_i \), the pullback of the tautological divisor of \( \mathbb{P}E_i \) under the projection \( \mathbb{P}(E_1, E_2, ..., E_n) \to \mathbb{P}E_i \). We call \( M_i \) the tautological divisors of \( \mathbb{P}(E_1, E_2, ..., E_n) \). Let \( D \in \text{Pic} X \) be a divisor on \( X \) and \( \pi \) be the projection \( \mathbb{P}(E_1, E_2, ..., E_n) \to X \). Then the global sections of a divisor \( a_1 M_1 + a_2 M_2 + ... + a_n M_n + \pi^* D \) on \( \mathbb{P}(E_1, E_2, ..., E_n) \) can be naturally identified with the global sections of the vector bundle

\[
(3.23) \quad \text{Sym}^{a_1} E_1^\vee \otimes \text{Sym}^{a_2} E_2^\vee \otimes \ldots \otimes \text{Sym}^{a_n} E_n^\vee \otimes O_X(D)
\]
on \( X \), where \( E_i^\vee \) is the dual of \( E_i \).

Before we proceed, we want to state a simple principle that was used throughout [2] and will be used extensively here as well. We will glorify it by calling it “Riemann-Hurwitz on curves with marked points”. Basically, it says the following.

**Proposition 3.5.** Let \( \pi_{\Gamma} : \Gamma \to \Sigma \) be a surjective map between two reduced curves \( \Gamma \) and \( \Sigma \) and let \( M \subset \Sigma \) be a finite set of points on \( \Sigma \). Let \( \pi_{\Gamma}^{-1}(M) \) be the (set-theoretical) inverse image of \( M \) and we assume that \( \pi_{\Gamma}^{-1}(M) \) is contained in the nonsingular locus of \( \Gamma \). Then

\[
(3.24) \quad 2g(\Gamma) - 2 + |\pi_{\Gamma}^{-1}(M)| \geq \gamma(2g(\Sigma) - 2 + |M|)
\]

where \( \gamma \) is the degree of \( \pi_{\Gamma} \).

This is more or less obvious by noting that the total ramification index of the points in \( \pi_{\Gamma}^{-1}(M) \) is at least \( \gamma|M| - |\pi_{\Gamma}^{-1}(M)| \).

**Proposition 3.6.** Let \( E \) be the rank two vector bundle over \( \mathbb{P}^k \) given by \( E = \mathcal{O} \oplus \mathcal{O}(1) \) and

\[
(3.25) \quad R = \mathbb{P}(E, E, ..., E) = \mathbb{P}E^{(n-k)}
\]
be the projective tower of \( n - k \) \( E \)'s over \( \mathbb{P}^k \). Let \( M_i \) be the pullback of the tautological divisor under the \( i \)-th projection \( p_i : R \to \mathbb{P}E \) and let \( F \) be the pullback of the hyperplane divisor of \( \mathbb{P}^k \) under the projection \( p : R \to \mathbb{P}^k \). Suppose that \( L \) is a very general member of the linear series \( |M + aF| \) and \( F_1, F_2, ..., F_b \) are \( b \) very general members of the linear series \( |F| \), where \( M = M_1 + M_2 + ... + M_{n-k} \) and \( a \geq n - k \geq 0 \). Then

\[
(3.26) \quad 2g(C) - 2 + i_R(C, L \cup M \cup F) \geq (a + b - 2n)(C \cdot F)
\]
for every reduced irreducible curve \( C \subset R \) satisfying \( p_s C \neq 0 \), where \( F = F_1 + F_2 + ... + F_b \). Here we assume that \( 0 < k < 3 \).
In (3.22), \( R = \mathbb{P}^2 \mathbb{E}, G \in \mathbb{P}H^0(\mathcal{O}_R(M)) \) and \( \tilde{G} \in \mathbb{P}H^0(\mathcal{O}_R(M + 7\mathcal{F})) \). Therefore, (3.22) follows directly from the above proposition by taking \( n = 3, k = 2, a = 7 \) and \( b = 0 \).

**Proof of Proposition 3.6.** First, let us prove the proposition when \( a = n - k \). Let \( \Gamma = \text{supp}(p_*C) \) be the reduced image of \( C \) under the projection \( p \) and let \( p_C : C \to \Gamma \) be the restriction of \( p \) to \( C \).

Applying Proposition 3.5 to the map \( p_C \) by noticing that \( p_{\Gamma}^{-1}(\Gamma \cap F_{\beta}) \subset C \cap F_{\beta} \), for \( \beta = 1, 2, ..., b \), we obtain (see Figure 3)

\[
2g(C) - 2 + i_{R}(C, F) \geq (\deg p_C) (2g(\Gamma) - 2 + i_{p_{\Gamma}}(\Gamma, p(F))).
\] (3.27)

Obviously, \( p(F) = p(F_1) \cup p(F_2) \cup ... \cup p(F_b) \) is the union of \( b \) hyperplanes of \( \mathbb{P}^k \) in general position. Therefore,

\[
2g(\Gamma) - 2 + i_{p_{\Gamma}}(\Gamma, p(F)) \geq (b - 2k) \deg \Gamma
\] (3.29)

by Theorem 1.15. Combining (3.28) and (3.29) yields

\[
2g(C) - 2 + i_{R}(C, F) \geq (b - 2k)(C \cdot F).
\] (3.30)

And hence (3.26) holds when \( a = n - k \).

Second, it is also easy to verify that the proposition holds when \( n = k \) (we set \( M_i = 0 \) when \( n = k \)). When \( n = k \), \( R \cong \mathbb{P}^k \), \( F \) is a union of \( b \) hyperplanes in \( \mathbb{P}^k \) in very general position and \( L \) is a very general hypersurface of degree \( a \) in \( \mathbb{P}^k \). Since \( k < 3 \) and we have the inductive hypothesis that Theorem 1.17 holds in \( \dim X < 3 \),

\[
2g(C) - 2 + i_{R}(C, F \cup L) \geq (a + b - 2k) \deg C
\] (3.31)

and (3.26) follows.

So the proposition holds for \( a = n - k \) or \( n = k \). We will also prove the proposition for \( k = 1 \). These are the starting points of our induction. Next, we will use a degeneration argument to bring down
the value of \(a\), \(k\) or \(n\) (the proof for the case \(k = 1\) is included in the following argument).

Let \(\hat{L}\) be a very general member of the linear series \(|M + (a - 1)F|\) and \(G\) be a very general member of the linear series \(|F|\). By degenerating \(L\) to \(\hat{L} \cup G\), we will lower the value of \(a\), \(k\) or \(n\). Repeating this process, we will eventually reduce the proposition to one of the cases we have already verified.

Let \(Z = R \times \Delta, W_M = g^*M, W_F = g^*F\) and \(W_L \subset Z\) be a pencil in \(|M + aF|\) whose general fibers are general members of the linear series and whose central fiber is \(\hat{L} \cup G\), where \(g\) is the projection \(Z \to R\).

Let \(Y\) be a reduced irreducible flat family of curves over \(\Delta\) with the commutative diagram (1.1). We assume that \(\pi_!Y_t \cdot F \neq 0\). Our goal is to prove that

\[
2g(Y_t) - 2 + i_Z(\pi_!Y_t, W) \geq (a + b - 2n)(\pi_!Y_t \cdot F)
\]

for \(t \neq 0\), where \(W = W_M + W_F + W_L\).

If \(k > 1\), we blow up \(Z\) along \(G\). Let \(f : \tilde{Z} \to Z\) be the blowup map and \(\tilde{W}\) be the proper transform of \(W\). As before, we have the commutative diagram (3.9) and the identity (3.10). Let \(E \subset \tilde{Z}\) be the exceptional divisor of \(f\). Then the central fiber of \(Z\) is the union \(R \cup E\) and \(R \cap E = G\).

Note that we do not blow up \(Z\) if \(k = 1\). If \(k = 1\), we let \(\tilde{Z} = Z, f\) be the identity map and \(\tilde{W} = W\) in the following argument.

After resolving the indeterminacies of the rational map \(Y \to \tilde{Z}\), we obtain the commutative diagram (3.9).

Again, we may apply Proposition 3.3 to \((\tilde{Y}, \tilde{Z}, \tilde{W})\) and obtain

\[
2g(Y_t) - 2 + i_{\tilde{Z}}(\pi_!Y_t, \tilde{W}) \geq \sum_{\Gamma \subset E, f_*\Gamma \neq 0} \mu_{\Gamma}\Phi_{\tilde{Z}, \tilde{W}}(\Gamma) + \sum_{\Gamma \notin E} \mu_{\Gamma}\Phi_{Z, W}(f_*\Gamma)
\]

(3.33)

where we sum over the irreducible components \(\Gamma \subset \pi_!\tilde{Y}_0\) satisfying that \(f_*\Gamma \neq 0\) and \(\Gamma \not\subset U_{\tilde{Y}}\). Here we need to exclude \(\Gamma \subset U_{\tilde{Y}}\) in the summations, for reasons that will be clear later.

Therefore, we just have to show

\[
\Phi_{\tilde{Z}, \tilde{W}}(\Gamma) \geq (a + b - 2n)(f_*\Gamma \cdot F)
\]

(3.34)

for all \(\Gamma \subset E\) and \(f_*\Gamma \neq 0\) and

\[
\Phi_{Z, W}(f_*\Gamma) \geq (a + b - 2n)(\Gamma \cdot F)
\]

(3.35)

for all \(\Gamma \subset R\) and \(\Gamma \not\subset U_{\tilde{Y}} \cup G\); (3.32) will follow easily.
Suppose that $\Gamma \subset E$ and $f_*\Gamma \neq 0$. Let us first figure out what $E$ and $\hat{W} \cap E$ look like.

Obviously, $E = \mathbb{P}_{G}E$ while $G \cong \mathbb{P}_{p^{k-1}}E^{(n-k)}$ is the projective tower of $n-k$ $\mathcal{E}$’s over $\mathbb{P}^{k-1}$. Therefore, $E$ is the projective tower of $n-k+1$ $\mathcal{E}$’s over $\mathbb{P}^{k-1}$, i.e., $E \cong \mathbb{P}_{p^{k-1}}E^{(n-k+1)}$.

It is not hard to see that

$$\hat{W} \cap E = \left( \bigcup_{j=1}^{n-k} M_j' \right) \cup \left( \bigcup_{\beta=1}^{b} F'_\beta \right) \cup L'$$

where

1. $M_j' \cap G = M_j \cap G$ for $1 \leq j \leq n-k$ and $M_1', M_2', ..., M_{n-k}'$ and $M'_{n-k+1} = G$ are the $n-k+1$ tautological divisors of $E \cong \mathbb{P}_{p^{k-1}}E^{(n-k+1)}$;
2. $F_\beta' \cap G = F_\beta \cap G$ and $F_\beta'$ is a very general member of the linear series $\mathbb{P}H^0(O_E(F'))$ for $\beta = 1, 2, ..., b$ with $F'$ the pullback of the hyperplane divisor under the projection $E \to \mathbb{P}^{k-1}$;
3. $L' \cap G = \hat{L} \cap G$ and $L'$ is a very general member of the linear series $\mathbb{P}H^0(O_E(M' + aF'))$ with $M' = M_1' + M_2' + ... + M_{n-k+1}'$.

Thus

$$\Phi_{\hat{Z},\hat{W}}(\Gamma) = 2g(\Gamma) - 2 + i_E(\Gamma, (\hat{W} \cap E) \cup G')$$

$$= 2g(\Gamma) - 2 + i_E(\Gamma, M' \cup F' \cup L')$$

$$\geq (a + b - 2n)(\Gamma \cdot F') = (a + b - 2n)(f_*\Gamma \cdot F)$$

by the inductive hypothesis, where $F' = F_1' + F_2' + ... + F_b'$.

Suppose that $\Gamma \subset R$, $\Gamma \not\subset U_{\hat{Y}} \cup G$ and $p_*\Gamma = 0$, i.e., $\Gamma$ is contained in a fiber of $p : R \to \mathbb{P}^{k}$. Note that $\Phi_{Z,W}(f_*\Gamma)$ could be negative in this case if we do not exclude $\Gamma \subset U_{\hat{Y}}$. However, since $\Gamma \not\subset U_{\hat{Y}}$,

$$\Phi_{Z,W}(f_*\Gamma) \geq 0$$

by the definition of $U_{\hat{Y}}$. This is the reason that we need to exclude those $\Gamma \subset U_{\hat{Y}}$.

Suppose that $\Gamma \subset R$, $p_*\Gamma \neq 0$ and $\Gamma \not\subset \hat{L}$. Then

$$\Phi_{Z,W}(f_*\Gamma) = 2g(\Gamma) - 2 + i_R(\Gamma, W_0)$$

$$= 2g(\Gamma) - 2 + i_R(\Gamma, M \cup F \cup G \cup \hat{L})$$

$$\geq ((a - 1) + (b + 1) - 2n)(\Gamma \cdot F)$$

$$= (a + b - 2n)(\Gamma \cdot F)$$

by the inductive hypothesis.
Suppose that $\Gamma \subset R$, $p_\ast \Gamma \neq 0$ and $\Gamma \subset \hat{L}$. Then
\[
\Phi_{Z,W}(f_\ast \Gamma) = 2g(\Gamma) - 2 + i_R(\Gamma, M \cup F)
\]  
(3.40)
\[= 2g(\Gamma) - 2 + i_L(\Gamma, (M \cup F) \cap \hat{L}).\]

Let $\psi : X \to M_{n-k} = N$ be the projection from $X$ to $M_{n-k}$ where $X$ is regarded as the projectivization of $E$ over $N = M_{n-k}$. It is not hard to see that $N$ is the projective tower of $n - k - 1$ $E$'s over $\mathbb{P}^k$, i.e.,
\[
N \cong \mathbb{P}_{pk} \mathcal{E}^{(n-k-1)}.
\]  
(3.41)

Let $F'_\beta = F_\beta \cap N$ for $\beta = 1, 2, ..., b$, $M'_j = M_j \cap N$ for $j = 1, 2, ..., n-k-1$ and $L' = \hat{L} \cap N$. It is not hard to see that

1. $F'_\beta$ is a very general member of the linear series $\mathbb{P} H^0(\mathcal{O}_{N}(\mathcal{F}'))$ for $\beta = 1, 2, ..., b$, where $\mathcal{F}' = \mathcal{F} \cap N$ is the pullback of the hyperplane divisor under the projection $N \to \mathbb{P}^k$;
2. $M'_1, M'_2, ..., M'_{n-k-1}$ are the $n - k - 1$ tautological divisors of $N \cong \mathbb{P}_{pk} \mathcal{E}^{(n-k-1)}$;
3. $L'$ is a very general member of the linear series
\[
\mathbb{P} H^0 \left(\mathcal{O}_N (M' + (a - 2)\mathcal{F}')\right),
\]
where $M' = M'_1 + M'_2 + ... + M'_{n-k-1}$.

Since $\psi$ induces an isomorphism between $\hat{L}$ and $N$, we have
\[
i_L(\Gamma, (M \cup F) \cap \hat{L}) = i_N(\psi_\ast \Gamma, M' \cup F' \cup L')
\]  
(3.43)
where $F' = F'_1 + F'_2 + ... + F'_b$. Apply the inductive hypothesis to $(N, M' \cup F' \cup L')$ and we obtain
\[
\Phi_{Z,W}(f_\ast \Gamma) = 2g(\Gamma) - 2 + i_N(\psi_\ast \Gamma, M' \cup F' \cup L')
\]  
(3.44)
\[\geq ((a - 2) + b - 2(n - 1)) (\psi_\ast \Gamma \cdot \mathcal{F}')
\[= (a + b - 2n)(\Gamma \cdot F).
\]

Combine (3.38), (3.39) and (3.44) and we obtain (3.35). q.e.d.

4. Proof of Theorem 1.17

We will carry out the proof of Theorem 1.17 in two steps as outlined in 3.1: we will first degenerate $D$ and then $S$.

One of our main tools is Proposition 3.3, which will be proved at the end of this section.
4. PROOF OF Theorem 1.17

4.1. Degeneration of $D$. Without the loss of generality, we assume that $D_i \neq 0$ for all $i \in I$ and $I_\gamma \neq \emptyset$ for all $\gamma \in A \cup B$. Hence $\dim S \leq \dim X - |A|$. Our inductive hypothesis is:

**HYPOTHESIS 4.1.** The theorem holds in $\dim S < n$.

Of course, to start the induction, we have to verify the following.

**CLAIM 4.2.** The theorem holds in $\dim S = 1$.

**Proof.** By Bertini, $S$ meets $D$ transversely. Therefore, $i_X(S, D) = D \cdot S$ and

$$2g(S) - 2 + i_X(S, D) = \left( K_X + \sum_{i \in I} D_i \right) S. \tag{4.1}$$

Let $C$ be an irreducible component of $S$. Since $S$ is cut by general members of BPF linear systems, the monodromy actions on the irreducible components of $S$ are transitive, which implies that any two irreducible component of $S$ are numerically equivalent. Therefore,

$$2g(C) - 2 + i_X(C, D) = \left( K_X + \sum_{i \in I} D_i \right) C. \tag{4.2}$$

Since $\{D_i\}_{i \in I}$ satisfies the condition $C_X(\varphi, \dim X - 2, 0)$, the curves $C$ satisfying

$$\left( K_X + \sum_{i \in I} D_i \right) C < \varphi(C) \tag{4.3}$$

cannot cover $X$ by Lemma 2.2. However, if $C$ is an irreducible component of $S$, the curves in the numerical class of $C$ obviously covers $X$. Consequently, we necessarily have

$$\left( K_X + \sum_{i \in I} D_i \right) C \geq \varphi(C) \tag{4.4}$$

and (1.19) follows. q.e.d.

If $|I_\beta| = 1$ for every $\beta \in B$, then we proceed directly to 4.2. Otherwise, let $|I_\gamma| > 1$ for some $\gamma \in B$. We will degenerate $D_{I_\gamma}$ to $\sum_{i \in I_\gamma} D_i$. However, as in the special case carried out in last section, instead of degenerating $D_{I_\gamma}$ directly to $\sum_{i \in I_\gamma} D_i$, we will degenerate $D_{I_\gamma}$ one component at a time.

Pick an arbitrary $\kappa \in I_\gamma$ and let $\hat{I}_\gamma = I_\gamma \setminus \{\kappa\}$ and $D_{\hat{I}_\gamma}$ be a very general member of the linear series $\mathbb{P}L_{\hat{I}_\gamma}$. We will degenerate $D_{I_\gamma}$ to the union $D_{\hat{I}_\gamma} \cup D_\kappa$ and keep doing this until $|I_\beta| = 1$ for every $\beta \in B$. 
The argument for each step of degeneration, i.e., the degeneration of $D_I$ to $D_I \cup D_\kappa$, has been amply illustrated by the special case we did in the previous section. Basically, it involves the construction of a fan over $G = D_\kappa$ and then the proof of a statement similar to Proposition 3.6.

Let $Z = X \times \Delta$ and $D_{I_\alpha} \subset Z$ be a pencil in the linear series $\mathbb{P}L_{I_\gamma}$ whose general fibers are general members of the linear series and whose central fiber is $D_{I_\gamma} \cup D_\kappa$.

Let $egin{align*}
Z = X \times \Delta \quad \text{and} \quad D_{I_\gamma} \subset Z \quad \text{be a pencil in the linear series} \quad \mathbb{P}L_{I_\gamma}
\end{align*}$

whose general fibers are general members of the linear series and whose central fiber is $D_{I_\gamma} \cup D_\kappa$.

Let $D_{I_\alpha} = p^*D_{I_\alpha}$ for $\tau \in \mathcal{A} \cup \mathcal{B}$ and $\tau \neq \gamma$, $S = \cap_{\alpha \in \mathcal{A}}D_{I_\alpha}$ and $D = \sum_{\beta \in \mathcal{B}}D_{I_\beta}$, where $p : Z \to X$ is the projection from $Z$ to $X$.

Let $Y$ be a reduced flat family of curves with the commutative diagram (1.1). We assume that $\pi(Y) \subset S$. Our goal is to prove

$$2g(Y_t) - 2 + i_Z(\pi_*Y_t, D) \geq \varphi(\pi_*Y_t)$$

for $t \neq 0$ under the inductive hypothesis:

$$2g(C) - 2 + i_X(C, D_0) \geq \varphi(C)$$

for all reduced curves $C \subset S$.

Let $f : \tilde{Z} \to Z$ be the blowup of $Z$ along $G = D_\kappa$ and let $R$ be the exceptional divisor. Then $\tilde{Z}_0 = X \cup R$ and $X \cap R = G$. Let $\tilde{D}$ and $\tilde{S}$ be the proper transforms of $D$ and $S$ under $f$, respectively.

After resolving the indeterminacies of the rational map $Y \to \tilde{Z}$, we obtain the commutative diagram (3.9).

By Proposition 3.3, we have

$$2g(Y_t) - 2 + i_Z(\pi_*Y_t, D) \geq \sum_{\Gamma \subset \tilde{R}, \Gamma \neq 0} \mu_{\Gamma} \Phi_{\tilde{Z}, \tilde{D}}(\Gamma) + \sum_{\Gamma \subset \tilde{R}} \mu_{\Gamma} \Phi_{\tilde{Z}, \tilde{D}}(f_*\Gamma)$$

where we sum over the irreducible components $\Gamma \subset \tilde{\pi}_*\tilde{Y}_0$ not contracted by $f$.

Let $P = D_{I_\kappa}$. By the definition of $\Phi$, we have

$$\Phi_{\tilde{Z}, \tilde{D}}(\Gamma) = 2g(\Gamma) - 2 + i_R(\Gamma, G \cup (\tilde{D} \cap R)) \quad \text{for} \quad \Gamma \subset R,$$

$$\Phi_{\tilde{Z}, \tilde{D}}(f_*\Gamma) = 2g(\Gamma) - 2 + i_Z(f_*\Gamma, D) \quad \text{for} \quad \Gamma \subset P, \quad \text{and}$$

$$\Phi_{\tilde{Z}, \tilde{D}}(f_*\Gamma) = 2g(\Gamma) - 2 + i_X(f_*\Gamma, D_0) \quad \text{for} \quad \Gamma \not\subset R \cup P.$$
for any \( \Gamma \subset S \cap P \) and
\[
2g(\Gamma) - 2 + i_X(\Gamma, D_0) \geq \varphi(\Gamma)
\]
for any \( \Gamma \subset S \) and \( \Gamma \not\subset P \cup G \).

It is easy to see that (4.13) follows directly from the inductive hypothesis (4.6). And (4.12) follows from Hypothesis 4.1 by the following argument. Let \( \tilde{D} = D - D_{i_\gamma} \), \( \tilde{S} = S \cap P = \cap_{\alpha \in A} D_{I_\alpha} \cap D_{i_\gamma} \). By Lemma 2.2, \( \{D_i : i \in I, i \neq \kappa\} \) satisfies \( C_X(\varphi, \dim X - 2, \dim \hat{S} - 1) \). Therefore,
\[
2g(C) - 2 + i_X(C, \tilde{D}) \geq \varphi(C)
\]
for all reduced curves \( C \subset \hat{S} \) by Hypothesis 4.1, since \( \dim \hat{S} \leq \dim S - 1 \). Then (4.12) follows by observing \( i_Z(\Gamma, D) = i_X(\Gamma, \tilde{D}) \) for \( \Gamma \subset \hat{S} \).

It remains to justify (4.11). This leads to the following proposition similar to 3.6, from which (4.11) follows.

**Proposition 4.3.** Let \( X \) be a smooth projective variety, \( \{\mathbb{P}L_i\}_{i \in I} \) be a finite set of BPF linear systems on \( X \), \( D_i \) be a very general member of \( \mathbb{P}L_i \) and \( \mathcal{E}_i = \mathcal{O} \oplus \mathcal{O}(D_i) \) for \( i \in I \).

Let \( J \subset I \) and \( R = \mathbb{P}X \mathcal{E}(J) = \mathbb{P}X(\mathcal{E}_j)_{j \in J} \) be the projective tower of \( \mathcal{E}_j \) for \( j \in J \), i.e., the fiber product
\[
\prod_{j \in J} \mathbb{P}X \mathcal{E}_j
\]
over \( X \). Let \( M_j \) be the pullback of the tautological divisor of \( \mathbb{P}X \mathcal{E}_j \) under the projection \( p_j : R \to \mathbb{P}X \mathcal{E}_j \) and \( M = \sum_{j \in J} M_j \).

Let
\[
I = \left( \bigcup_{\alpha \in A} I_\alpha \right) \sqcup \left( \bigcup_{\beta \in B} I_\beta \right) \sqcup I_\gamma \sqcup J
\]
be a partition of \( I \). Let \( D_{i_\tau} \) be a very general member of \( \mathbb{P}L_{i_\tau} \) and \( F_\tau = p^*D_{i_\tau} \) for \( \tau \in A \cup B \), where \( p : R \to X \) is the natural projection. Let \( L \) be a very general member of the linear series
\[
\mathbb{P}(\bigotimes_{j \in J}(\mathcal{C} \oplus \mathcal{L}_j)) \otimes \mathcal{L}_I \subset \mathbb{P}H^0(\mathcal{O}_R(M + \sum_{j \in J \setminus I_\gamma} p^*D_j)),
\]
\( S = \cap_{\alpha \in A} D_{I_\alpha} \), \( S = \cap_{\alpha \in A} F_\alpha \), \( D = \sum_{\beta \in B} D_{I_\beta} \) and \( F = \sum_{\beta \in B} F_\beta \). Here we use the identification between the linear systems on \( X \) and those on \( R \), as given in (3.23).

Let \( \varphi : N_1(X) \to \mathbb{R} \) be a subadditive function on \( N_1(X) \) such that \( \{D_i\}_{i \in I \setminus J} \) satisfies the condition \( C_X(\varphi, \dim X - 2, \dim S - 1) \) on \( X \).
Suppose that $0 < \dim S < n$. Then
\begin{equation}
2g(C) - 2 + i_R(C, M \cup F \cup L) \geq \varphi(p_* C)
\end{equation}
for all reduced irreducible curves $C \subset S$ satisfying $p_* C \neq 0$.

**Proof.** First, let us verify the case that $I_\gamma = \emptyset$. In this case, since $\dim S < n$,
\begin{equation}
2g(\Gamma) - 2 + i_X(\Gamma, D) \geq \varphi(\Gamma)
\end{equation}
for all reduced curves $\Gamma \subset S$ by Hypothesis 4.1. Let $\Gamma$ be the reduced image of $C$ under the projection $p$ and let $p_C : C \to \Gamma$ be the restriction of $p$ to $C$. As in the proof of Proposition 3.5, we may apply Proposition 3.6 to the map $p_C$ to obtain (see Figure 3):
\begin{equation}
2g(C) - 2 + i_R(C, F) \geq (\deg p_C)(2g(\Gamma) - 2 + i_X(\Gamma, D)).
\end{equation}
Then (4.18) follows from (4.19) and (4.20).

Second, the proposition obviously holds when $J = \emptyset$, i.e., $X = R$ by Hypothesis 4.4.

So far we have verified the proposition for $I_\gamma = \emptyset$ or $X = R$. We will also prove the proposition for $\dim S = 1$. Next we will try to reduce it to one of these cases by degenerating $L$ (the proof for $\dim S = 1$ is included in the following argument).

Pick an arbitrary $\kappa \in I_\gamma$ and let $\hat{I}_\gamma = \hat{I}_\gamma \cup \{\kappa\}$ and $\hat{L}$ be a very general member of the linear series
\begin{equation}
P(\bigotimes_{j \in J}(C \oplus L_j)) \otimes L_{\hat{I}_\gamma} \subset \mathbb{P}H^0(\mathcal{O}_R(M + \sum_{j \in \hat{I}_\gamma} p^* D_j)).
\end{equation}
By degenerating $L$ to $\hat{L} \cup D_\kappa$, we will lower the value of $|I_\gamma|$, $\dim X$ or $\dim R - \dim X$. So repeating this process, we will eventually reduce the proposition to one of the cases we have proved.

Let $Z = R \times \Delta, W_\beta = g^* F_\beta$ for $\beta \in \mathcal{B}$ and $W_\gamma \subset Z$ be a pencil in the linear series (4.17) whose general fibers are general members of the linear series and whose central fiber is $\hat{L} \cup D_\kappa$, where $g$ is the projection $Z \to R$. Let $W = W_\gamma + \sum_{\beta \in \mathcal{B}} W_\beta + \sum_{j \in \mathcal{J}} g^* M_j$.

Let $Y$ be a reduced irreducible flat family of curves with the commutative diagram (1.1). We assume that $\pi(Y) \subset S$ and $p_* \pi(Y) \neq 0$. Our goal is to prove
\begin{equation}
2g(Y_t) - 2 + i_Z(\pi_* Y_t, W) \geq \varphi(\pi_* Y_t).
\end{equation}
If $\dim S > 1$, we blow up $Z$ along $G = D_\kappa$. Let $f : \tilde{Z} \to Z$ be the blowup map and let $E$ be the exceptional divisor. Then $\tilde{Z}_0 = R \cup E$
and $R \cap E = G$. Let $\widetilde{W}_\beta, \widetilde{W}_\gamma$ and $\widetilde{W}$ be the proper transforms of $W_\beta$, $W_\gamma$ and $W$, respectively.

Note that we do not blow up $Z$ if $\dim S = 1$. If $\dim S = 1$, we let $\overline{Z} = Z$, $f$ be the identity map and $\overline{W} = W$ in the following argument.

After resolving the indeterminacies of the rational map $Y \to \overline{Z}$, we obtain the commutative diagram (3.9).

By Proposition 3.3,

$$2g(Y_t) - 2 + i_Z(\pi_s Y_t, W) \geq \sum_{\Gamma \subset E} f_* \Phi_{\overline{Z}, \overline{W}}(\chi_{E_{\overline{Z}, \overline{W}}}(\Gamma)) + \sum_{\Gamma \not\subset E} f_* \Phi_{Z, W}(f_*(\Gamma))$$

where we sum over the irreducible components $\Gamma \subset \overline{π}_s \overline{Y}_0$ satisfying that $f_* \Gamma \neq 0$ and $\Gamma \not\subset U_{\overline{Y}}$. Here we need to exclude $\Gamma \subset U_{\overline{Y}}$ due to the presence of the components $\Gamma \subset R$ with $p_*(\Gamma) = 0$.

By the definition of $Φ$, we have

$$\Phi_{Z, W}(f_*(\Gamma)) = 2g(\chi_{E_{Z, W}}(\Gamma)) - 2 + i_Z(\chi_{Z, W}(f_*(\Gamma)),$$

(4.24)  \[ \Phi_{Z, W}(f_*(\Gamma)) = 2g(\chi_{E_{Z, W}}(\Gamma)) - 2 + i_Z(\chi_{Z, W}(f_*(\Gamma)) \]

for $\Gamma \subset \hat{L}$, and

$$\Phi_{Z, W}(f_*(\Gamma)) = 2g(\chi_{E_{Z, W}}(\Gamma)) - 2 + i_R(\chi_{R, W}(\Gamma))$$

(4.26)  \[ \Phi_{Z, W}(f_*(\Gamma)) = 2g(\chi_{E_{Z, W}}(\Gamma)) - 2 + i_R(\chi_{R, W}(\Gamma)) \]

for $\Gamma \not\subset E \cup \hat{L}$.

Observe that $E$ is actually the projective tower $\mathbb{P}_G \mathcal{E}^{(j)} \times_G \mathbb{P}_G \mathcal{E}_\kappa$ over $G$ and

$$\widetilde{W} \cap E = \left( \bigcup_{j \in J} M'_j \right) \cup \left( \bigcup_{\beta \in \mathcal{B}} F'_{\beta} \right) \cup L'$$

(4.27)  \[ \widetilde{W} \cap E = \left( \bigcup_{j \in J} M'_j \right) \cup \left( \bigcup_{\beta \in \mathcal{B}} F'_{\beta} \right) \cup L' \]

where

1. $M'_j \cap G = M_j \cap G$ and $M'_j$ and $G$ are the tautological divisors of $E$ over $G$ for $j \in J$;
2. $F'_{\beta} \cap G = F_\beta \cap G$ and $F'_{\beta} = f^*(D_{I_\beta} \cap G)$;
3. $L' \cap G = \hat{L} \cap G$ and $L'$ is a very general member of the linear series

$$\mathbb{P} \left( \bigotimes_{j \in J} (\mathcal{C} \oplus L_j) \right) \otimes (\mathcal{C} \oplus \mathcal{L}_\kappa) \otimes \mathcal{L}_I,$$

(4.28)  \[ \mathbb{P} \left( \bigotimes_{j \in J} (\mathcal{C} \oplus L_j) \right) \otimes (\mathcal{C} \oplus \mathcal{L}_\kappa) \otimes \mathcal{L}_I \]

with $\mathcal{L}_I$ restricted to $G$.

Note that we have $\dim G < \dim X$ and $\{D_i : i \in I, i \neq \kappa\}$ satisfies the condition $\mathcal{C}_G(\varphi, \dim G - 2, \dim S - 1)$ when restricted to $G$. Therefore, we may apply the inductive hypothesis to $(E, M'_U \cup F' \cup L')$ and obtain

$$\Phi_{Z, W}(\Gamma) = 2g(\chi_{E_{Z, W}}(\Gamma)) - 2 + i_E(\chi_{E_{Z, W}}(\Gamma), M'_U \cup F' \cup L') \geq \varphi(p_* f_*(\Gamma))$$

(4.29)  \[ \Phi_{Z, W}(\Gamma) = 2g(\chi_{E_{Z, W}}(\Gamma)) - 2 + i_E(\chi_{E_{Z, W}}(\Gamma), M'_U \cup F' \cup L') \geq \varphi(p_* f_*(\Gamma)) \]
for any $\Gamma \subset E$, $f_*\Gamma \neq 0$ and $f(\Gamma) \subset S$, where $M' = \sum_{j \in J} M_j' + G$ and $F' = \sum_{\beta \in B} F'_\beta$.

Suppose that $\Gamma \subset R$, $\Gamma \not\subset U_{\tilde{Y}} \cup G$ and $p_*\Gamma = 0$, i.e., $\Gamma$ is contained in a fiber of $p$. Note that $\Phi_{Z,W}(f_*\Gamma)$ could be negative if we do not exclude $\Gamma \subset U_{\tilde{Y}}$. But since $\Gamma \not\subset U_{\tilde{Y}}$, it is easy to check that
\[
\Phi_{Z,W}(f_*\Gamma) \geq 0
\]
by the definition of $U_{\tilde{Y}}$. This is the reason that we need to exclude those $\Gamma \subset U_{\tilde{Y}}$.

Suppose that $\Gamma \subset R$, $p_*\Gamma \neq 0$ and $\Gamma \not\subset \tilde{L}$. Then it follows directly from the inductive hypothesis that
\[
\Phi_{Z,W}(f_*\Gamma) = 2g(\Gamma) - 2 + i_R(\Gamma, W_0) \geq \varphi(p_*\Gamma)
\]
for any $\Gamma \subset S$, $p_*\Gamma \neq 0$ and $\Gamma \not\subset \tilde{L}$.

Suppose that $\Gamma \subset R$, $p_*\Gamma \neq 0$ and $\Gamma \subset \tilde{L}$. We do exactly the same thing as we did in the proof of Proposition 3.4 by projecting $R$ to $M_r$ for some $\tau \in J$. The projection $\psi : R \to M_r = N$ induces an isomorphism between $\tilde{L}$ and $N$. It is not hard to see that $N$ is the projective tower
\[
P_X\mathcal{E}(\tilde{J}) = P_X((\mathcal{E}_j)_{j \in \tilde{J}})
\]
over $X$ with $\tilde{J} = J \setminus \{\tau\}$.

Let $F'_\beta = F_\beta \cap N$ for $\beta \in B$, $M'_j = M_j \cap N$ for $j \in \tilde{J}$ and $L' = \tilde{L} \cap N$. It is not hard to see that
1. $F'_\beta = p'_ND_{I\beta}$, where $p_N$ is the projection $N \to X$;
2. $M'_j$ are the tautological divisors of $N$ for $j \in \tilde{J}$;
3. $L'$ is a very general member of the linear series
\[
P\left( \bigotimes_{j \in \tilde{J}} (C \oplus \mathcal{L}_j) \right) \otimes \mathcal{L}_{\tilde{I}'}
\]
on $N$.

Since $\psi$ induces an isomorphism between $\tilde{L}$ and $N$, we have
\[
i_{\tilde{L}}(\Gamma, (M \cup F) \cap \tilde{L}) = i_N(\psi_*\Gamma, M' \cup F' \cup L')
\]
where $M' = \sum_{j \in \tilde{J}} M'_j$ and $F' = \sum_{\beta \in B} F'_\beta$. Let $S' = S \cap \tilde{L}$. Since $\dim N - \dim X < \dim R - \dim X$ and $\{D_i : i \in I, i \neq \kappa\}$ satisfies the condition $\mathcal{L}_X(\varphi, \dim X - 2, \dim S'-1)$, we may apply the inductive hypothesis to $(N, M' \cup F' \cup L')$ and obtain
\[
2g(\psi_*\Gamma) - 2 + i_N(\psi_*\Gamma, M' \cup F' \cup L') \geq \varphi(p_*\Gamma)
\]
for all $\Gamma \subset S'$ and $p_*\Gamma \neq 0$. Therefore,
\[
\Phi_{X,W}(f_*\Gamma) = 2g(\Gamma) - 2 + i_{\tilde{L}}(\Gamma, (M \cup F) \cap \tilde{L}) - i_N(\psi_*\Gamma, M' \cup F' \cup L') \geq \varphi(p_*\Gamma)
\]
for all $\Gamma \subset S'$ and $p_*\Gamma \neq 0$.

Combining (4.23), (4.29), (4.30), (4.31) and (4.35), we arrive at (4.22).

4.2. Degeneration of $S$. After we have reduced the theorem to the case that $|I_\beta| = 1$ for all $\beta \in B$ by degenerating $D$, we will finish the proof by degenerating $S$.

For every $\alpha \in A$, let $Z_\alpha \subset X \times \Delta$ be a pencil in the linear series $\mathbb{P}\mathcal{L}_{I_\alpha}$ whose general fibers are general members of the linear series and whose central fiber is the union $\cup_{i \in I_\alpha} D_i$. Let $Z = \cap_{\alpha \in A} Z_\alpha$ and $W = p^*D$, where $p$ is the projection $Z \to X$.

Let $Y$ be a reduced flat family of curves with the commutative diagram (1.1). Our goal is to prove that

\[(4.36) \quad 2g(Y_t) - 2 + i_Z(\pi_*Y_t, W) \geq \varphi(\pi_*Y_t).\]

The central fiber of $Z$ is a union of projective varieties of simple normal crossing with components $\cap_{\alpha \in A} D_{i_\alpha}$, where $i_\alpha \in I_\alpha$. By Proposition 3.3,

\[(4.37) \quad 2g(Y_t) - 2 + i_Z(\pi_*Y_t, W) \geq \sum_{\Gamma \subset \pi_*Y_0} \mu_{\Gamma} \Phi_{Z,W}(\Gamma).\]

Obviously, in order to show (4.36), it suffices to show

\[(4.38) \quad \Phi_{Z,W}(\Gamma) \geq \varphi(\Gamma)\]

for every irreducible component $\Gamma \subset \pi_*Y_0$.

Let $J = \{j \in I : \Gamma \subset D_j\}$, $M_\alpha = I_\alpha \setminus J$ and $D_J = \cap_{j \in J} D_j$. Then

\[(4.39) \quad \Phi_{Z,W}(\Gamma) \geq 2g(\Gamma) - 2 + i_{D_J}(\Gamma, (\cup_{i \in I \setminus J} D_i) \cap D_J, Q \cap \partial D_J)\]

where $Q$ is the singular locus of $Z$ and

\[(4.40) \quad \partial D_J = D_J \cap \left( \bigcup_{\alpha \in A} \bigcup_{i \in M_\alpha} D_i \right).\]

Let us assume that $\dim D_J \geq 2$ since (4.38) follows from Lemma 2.2 if $\dim D_J = 1$ by the argument for the claim 4.2.

The singular locus $Q \cap \partial D_J$ can be described as follows: let $F_\alpha$ be a general member of the pencil $Z_\alpha$ and then

\[(4.41) \quad Q \cap \partial D_J = D_J \cap \left( \bigcup_{\alpha \in A} \bigcup_{i \in M_\alpha} (D_i \cap F_\alpha) \right).\]
Since \( \{D_i\}_{i \in I} \) satisfies the condition \( C_\Gamma(\varphi, \dim X - 2, \dim Z_0 - 1) \), \( \{D_i\}_{i \in I \setminus J} \) satisfies the condition \( C_{D,J}(\varphi, \dim D_J - 2, \dim D_J - 1) \) when restricted to \( D_J \) by Lemma 2.2. Then it follows from Theorem 1.15 that

\[
2g(\Gamma) - 2 + i_{D,J}(\Gamma, (\cup_{i \in I \setminus J} D_i) \cap D_J, Q \cap \partial D_J) \geq \varphi(\Gamma)
\]

for all reduced irreducible curves \( \Gamma \) that are not contained in \( \cup_{\alpha \in A} F_\alpha \).

Note that \( F_\alpha \) is a general member of the linear series \( \mathbb{P}L_{I_\alpha} \). Fix \( Q \) and we consider the linear subseries \( \mathcal{F}_Q \) of \( \mathbb{P}L_{I_\alpha} \) given by

\[
\mathcal{F}_Q = \{ F_\alpha \in \mathbb{P}L_{I_\alpha} : F_\alpha \cap (\cup_{i \in M_\alpha} D_i) = Q \cap (\cup_{i \in M_\alpha} D_i) \}.
\]

Since \( M_\alpha \subset I_\alpha \), the base locus of \( \mathcal{F}_Q \) is exactly \( Q \cap (\cup_{i \in M_\alpha} D_i) \). Therefore, it is possible to choose \( F_\alpha \in \mathcal{F}_Q \) such that \( \Gamma \not\subset F_\alpha \). This is true for every \( \alpha \in A \). So it is possible to choose \( \{F_\alpha\}_{\alpha \in A} \) with \( Q \) fixed such that \( \Gamma \not\subset \cup_{\alpha \in A} F_\alpha \). Then (4.42) follows.

### 4.3. Proof of Proposition 3.3

The proof of Proposition 3.3 is almost the same as that of [C2, Theorem 1.17]. First let us quote two lemmas in [C2] (Lemma 2.3 and 4.1).

**Lemma 4.4.** Let \( X \subset \Delta^r_{x_1 x_2 \ldots x_n} \times \Delta \) be the hypersurface given by \( x_1 x_2 \ldots x_n = t \) for some \( n \leq r \), where \( \Delta^r_{x_1 x_2 \ldots x_r} \) is the \( r \)-dimensional polydisk parameterized by \( (x_1, x_2, \ldots, x_r) \) and \( \Delta \) is the disk parameterized by \( t \). Let \( X_0 = D = \cup_{k=1}^n D_k \) with \( D_k = \{ x_k = t = 0 \} \).

Let \( Y \) be a flat family of curves over \( \Delta \) with the commutative diagram (3.11). Suppose that \( \pi_Y Y_0 \not\equiv 0 \). Then for each \( D_k \) there exists a curve \( \Gamma' \subset \pi(Y_0) \) with \( \Gamma' \subset D_k \).

**Lemma 4.5.** Let \( X \subset \Delta^r_{x_1 x_2 \ldots x_n} \times \Delta \) be the hypersurface given by \( x_1 x_2 \ldots x_n = t f(t, x_1, x_2, \ldots, x_r) \), where \( n < r \), \( f(0,0,0,\ldots,0) = 0 \) and \( f(t, x_1, x_2, \ldots, x_r) \not\equiv 0 \) along \( x_1 = x_2 = \ldots = x_n = t = 0 \). Let \( X_0 = D = \cup_{k=1}^n D_k \) where \( D_k = \{ x_k = t = 0 \} \). And let \( Q \) be the singular locus of \( X \), i.e., \( Q \) is cut out on \( X \) by \( f(t, x_1, x_2, \ldots, x_r) = 0 \).

Let \( Y \) be a flat family of curves over \( \Delta \) with the commutative diagram (3.11). Suppose that there exists \( J \subset \{1,2,\ldots,n\} \) and a reduced irreducible curve \( \Gamma \subset \pi(Y_0) \) such that \( \Gamma \subset D_J \), \( \Gamma \not\subset \partial D_J \) and \( i_{D,J}(\Gamma, \partial D_J, Q \cap \partial D_J) > 0 \). Then there exists a curve \( \Gamma' \subset \pi(Y_0) \) such that \( \Gamma' \subset \cup_{k \not\in J} D_k \).

Please see [C2] for the proofs of these two lemmas.

**Definition 4.6.** Let \( X, D \) and \( P \) be given as in Definition 1.3 and let \( f : C \to X \) be a proper map from a curve \( C \) to \( X \). We define

\[
i_X(C, D, P)_f = i_X(f_*C, D, P).
\]
When there is no confusion on what \( f \) is, we just write \( i_X(C, D, P)_f \) as \( i_X(C, D, P) \). For a point \( p \in C \), we define \( i_{X, p}(C, D, P) \) by choosing an open neighborhood (analytic or étale) \( U \) of \( f(p) \) and letting
\[
i_{X, p}(C, D, P) = i_U(V, D \cap U, P \cap U)
\]
where \( V \) is the connected component of \( f^{-1}(U) \) that contains the point \( p \). Obviously, if \( C \) is smooth,
\[
i_X(C, D, P) = \sum_{p \in C} i_{X, p}(C, D, P).
\]

Finally, we define \( I_X(C, D, P) \subset C \) to be the set
\[
I_X(C, D, P) = \{ p \in C : i_{X, p}(C, D, P) \neq 0 \}.
\]

Now we are ready to prove Proposition 3.3.

Without the loss of generality, let us assume that \( Y \) is irreducible and birational over the image \( \pi(Y) \), which meets \( W \) properly. Furthermore, we may normalize \( Y \) and apply semistable reduction to the map \( Y \to X \). In the end, we may assume that \( \pi : Y \to X \) is a family of semistable maps with marked points \( Y_t \cap \pi^*(W) \) for \( t \neq 0 \). More specifically, we assume that the following holds:

1. \( Y \) is smooth and \( Y_0 \) is nodal;
2. \( Y_t \cap \pi^*(W) \) extends to disjoint sections of the fibration \( Y \to \Delta \), i.e., the flat limit \( \lim_{t \to 0} (Y_t \cap \pi^*(W)) \) consists of \( i_X(\pi_*Y_t, W) \) distinct points lying on the nonsingular locus of \( Y_0 \);
3. \( \pi : Y \to X \) is minimal with respect to these properties.

For each component \( \Gamma \subset Y_0 \), we define \( \sigma(\Gamma) \) to be the set of points on \( \Gamma \) that consists of the marked points \( \Gamma \cap \lim_{t \to 0} (Y_t \cap \pi^*(W)) \) and the intersections between \( \Gamma \) and the other components of \( Y_0 \). The basic observation is (see [C2]):
\[
2g(Y_t) - 2 + i_X(\pi_*Y_t, W) = \sum_{\Gamma \subset Y_0} (2p_a(\Gamma) - 2 + |\sigma(\Gamma)|)
\]
where \( p_a(\Gamma) \) is the arithmetic genus of \( \Gamma \) and we sum over all irreducible components \( \Gamma \subset Y_0 \). Therefore, we will be done with (3.12) as long as we can prove
\[
2p_a(\Gamma) - 2 + |\sigma(\Gamma)| \geq (\deg \pi_\Gamma) \Phi_{X, W}(F)
\]
for each irreducible component \( \Gamma \subset Y_0 \), where \( F = \text{supp}(\pi(\Gamma)) \), \( \pi_\Gamma : \Gamma \to F \) is the restriction of \( \pi \) to \( \Gamma \) and we let \( \Phi_{X, W}(F) = 0 \) if \( F = \text{pt} \), i.e., \( \Gamma \) is contractible under \( \pi \).

Obviously, if \( \Gamma \) is contractible under \( \pi \),
\[
2p_a(\Gamma) - 2 + |\sigma(\Gamma)| \geq 0
\]
due to the minimality of $\pi : Y \to X$.

Suppose that $\Gamma$ is noncontractible under $\pi$. Let $\rho(\Gamma) \subset \Gamma$ be the set
\begin{equation}
\rho(\Gamma) = I_X(\Gamma, W \cup D, Q \cap \partial D_J)
\end{equation}
where $J = \{ j \in I : F \subset D_j \}$. By Lemma 4.4 and 4.5 and using the same argument as in \cite{22}, we can show that
\begin{equation}
\rho(\Gamma) \subset \sigma(\Gamma).
\end{equation}
And by Proposition 3.5,
\begin{equation}
2g(\Gamma) - 2 + |\rho(\Gamma)| \geq (\deg \pi_\Gamma)(2g(F) - 2 + i_X(F, W \cup D, Q \cap \partial D_J)).
\end{equation}
Then (4.49) follows.

Let us consider the modified statement that excludes the components $\Gamma \subset U_Y$ from the RHS of (3.12). Let
\begin{equation}
Z = \pi^{-1}(U_Y).
\end{equation}
If $Z = Y_0$, there is nothing to prove since $U_Y = \text{supp}(\pi_*Y_0)$ in this case. Otherwise, we write the RHS of (4.48) as
\begin{equation}
\sum_{\Gamma \subset Y_0} (2p_a(\Gamma) - 2 + |\sigma(\Gamma)|) \newline = \sum_{\Gamma \not\subset Z} (2p_a(\Gamma) - 2 + |\rho(\Gamma)|) \\
+ \sum_{\Gamma \not\subset Z} (|\sigma(\Gamma)| - |\rho(\Gamma)|) + \sum_{\Gamma \subset Z} (2p_a(\Gamma) - 2 + |\sigma(\Gamma)|).
\end{equation}
We have already proved that
\begin{equation}
2g(\Gamma) - 2 + |\rho(\Gamma)| \geq (\deg \pi_\Gamma)\Phi_{X,W}(F)
\end{equation}
for $\Gamma \not\subset Z$. So it suffices to show that
\begin{equation}
\sum_{\Gamma \not\subset Z} (|\sigma(\Gamma)| - |\rho(\Gamma)|) + \sum_{\Gamma \subset Z} (2p_a(\Gamma) - 2 + |\sigma(\Gamma)|) \geq 0.
\end{equation}
Let $V \subset Z$ be a maximal connected component of $Z$ and let us assume that
\begin{equation}
\sum_{\Gamma \subset V} (2p_a(\Gamma) - 2 + |\sigma(\Gamma)|) < 0;
\end{equation}
otherwise, there is nothing to prove. This happens only if (see Figure 1)
1. every component of $V$ is smooth and rational;
2. the dual graph of $V$ is a tree;
3. $V$ meets the rest of $Y_0$ at a single point $p_V$ and let us assume that $p_V = V \cap \Gamma_V$ for some irreducible component $\Gamma_V \not\subset Z$;
4. \( \pi_*V \neq 0 \), i.e., \( V \) is noncontractible under \( \pi \);

5. \( V \cap M = \emptyset \), where \( M \) is the closure of \{\( Y_t \cap \pi^*(W) \)\}_{t \neq 0} in \( Y \);

6. the LHS of (4.57) is exactly \(-1\).

\[
p_v(V) = 0, \quad V^2 = -1 \quad \text{and} \quad V \cdot M = 0, \quad \text{where} \quad M \quad \text{are the sections given by the marked points} \quad Y_t \cap \pi^*(W).
\]

**Figure 4. Configuration of \( V \)**

Note that \( p_V \in \sigma(\Gamma_V) \). If \( p_V \not\in \rho(\Gamma_V) \), then

\[
|\sigma(\Gamma_V)| - |\rho(\Gamma_V)| + \sum_{\Gamma \subset V} (2p_\alpha(\Gamma) - 2 + |\sigma(\Gamma)|) \geq 0.
\]

If this holds for every maximal connected component \( V \subset Z \), (4.56) easily follows. Otherwise, suppose that

\[
p_V \in \rho(\Gamma_V).
\]

for some \( V \). Since \( \pi(V) \cap \partial D_f = \emptyset \), (4.59) can happen only if there exists an irreducible component \( B \) of \( W \) such that \( \pi(\Gamma_V) \not\subset B \) and \( F_V \subset B \) for some \( F_V \subset \pi_*V \). Let us consider the divisor \( \pi^*(B) \) on \( Y \). We can write it as

\[
\pi^*(B) = \hat{M} + \hat{V} + N
\]

where \( \hat{M} \subset M \), \( \text{supp}(\hat{V}) \subset V \), \( F_V \subset \pi(\hat{V}) \), \( \text{supp}(N) \subset Z_0 \) and \( \hat{V} \cap N = \emptyset \). Since \( V \cap M = \emptyset \), \( \hat{V} \cap \hat{M} = \emptyset \). This leads to the contradiction:

\[
(\hat{V})^2 = \hat{V}(\hat{M} + \hat{V} + N) = \pi_*\hat{V} \cdot B \geq 0.
\]

Therefore, (4.59) cannot hold for any maximal connected component \( V \) of \( Z \) and (4.56) follows. This justifies (3.12) with \( \Gamma \subset U_Y \) excluded in the summation.

Now let us turn to (3.14). Let \( G = f(D_\alpha) \subset D_\beta \) for some \( \beta \in I \) and \( \beta \neq \alpha \). By our assumption on \( f \), \( G \cap D_i = \emptyset \) for \( i \neq \alpha, \beta \) and \( X \) is the blowup of \( \hat{X} \) along \( G \).
Let $\hat{Y}$ be a flat family of curves that fits in the commutative diagram:

$$
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow \phi & & \downarrow f \\
\hat{Y} & \rightarrow & \hat{X}
\end{array}
$$

(4.62)

and we assume that $\hat{\pi} : \hat{Y} \rightarrow \hat{X}$ is a family of semistable maps with marked points $\hat{Y}_t \cap \hat{\pi}^*(\hat{W})$. Obviously, the map $\phi : Y \rightarrow \hat{Y}$ simply contracts some $-1$ or $-2$ smooth rational curves $\Gamma$ with $f_*\pi_*\Gamma = 0$.

For an irreducible component $\hat{\Gamma} \subset \hat{Y}_0$ with $\hat{\pi}_*\hat{\Gamma} \neq 0$, there exists a unique component $\Gamma \subset Y_0$ that dominates $\hat{\Gamma}$ by $\phi$. Actually, $\phi : \Gamma \rightarrow \hat{\Gamma}$ is a partial normalization of $\hat{\Gamma}$. Let $\sigma(\Gamma), \rho(\Gamma) \subset \Gamma$ and $\sigma(\hat{\Gamma}), \rho(\hat{\Gamma}) \subset \hat{\Gamma}$ be the sets defined as before.

In order to show (3.14), it is enough to show

$$2p_a(\hat{\Gamma}) - 2 + |\sigma(\hat{\Gamma})| \geq 2g(\Gamma) - 2 + |\rho(\Gamma)|$$

(4.63)

for every component $\Gamma \subset Y_0$ with $F = \text{supp}(\pi_*\Gamma) \subset D_\alpha$ and $f_*\pi_*\Gamma \neq 0$, since the RHS of (4.63) is at least $(\deg \pi_\Gamma)\Phi_{X,W}(F)$ by (4.55).

In order to show (4.63), it in turns suffices to show

$$\rho(\Gamma) \subset \phi^{-1}(\sigma(\hat{\Gamma})).$$

(4.64)

So it remains to verify (4.64). Let $p \in \rho(\Gamma)$.

There are two possibilities that $p \in I_{D_\alpha}(\Gamma, G)$ or $p \in I_X(\Gamma, W)$.

Suppose that $p \in I_{D_\alpha}(\Gamma, G)$. Then by Lemma 4.4, there exists a component $\Gamma' \subset Y_0$ such that $\pi(\Gamma') \subset D_\beta$, $\pi_*\Gamma' \neq 0$ and $\Gamma$ and $\Gamma'$ are joined by a chain of curves which are contracted to $\pi(p)$ by $\pi$ (see Figure 5). Obviously, $\phi_*\Gamma' \neq 0$ and hence $p \in \phi^{-1}(\sigma(\hat{\Gamma}))$.

![Figure 5. Application of Lemma 4.4](image-url)
Suppose that $p \in I_X(\Gamma, W)$ and $p \not\in \phi^{-1}(\sigma(\hat{\Gamma}))$. This can happen only if there exists a connected union of components $V \subset Y_0$ such that

1. $p = V \cap \Gamma$ and $p$ is the only intersection between $V$ and the rest of $Y_0$;
2. $\phi_* V = 0$ and $V$ is a tree of smooth rational curves;
3. $L = \text{supp}(\pi_* V)$ is a fiber of the projection $f : D_\alpha \to G$;
4. $V \cap M = \emptyset$;
5. there exists an irreducible component $B \subset W$ such that $\pi(\Gamma) \not\subset B$ and $L \subset B$.

Again we have the configuration of $V$ as in Figure 4. Next, we argue in exactly the same way as for (4.59) by observing that $B \cdot L \geq 0$. This finishes the proof of (4.64) and hence the proof of (3.14).

By an almost identical argument as before, which we will not repeat, we can prove (3.14) with $\Gamma \subset U_Y$ excluded in the summation.
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