GEOMETRY OF QUADRILATERAL NETS:
SECOND HAMILTONIAN FORM.

SERGEY M. SERGEEV

Abstract. Discrete Darboux-Manakov-Zakharov systems possess two distinct Hamiltonian forms. In the framework of discrete-differential geometry one Hamiltonian form appears in a geometry of circular net. In this paper a geometry of second form is identified.

The circular net [1] – a special type of three-dimensional quadrilateral net [2] – is an example of geometrically integrable (see [3] and references therein) system endowed by a discrete space-time Hamiltonian structure [4] what brings together geometrically integrable and completely integrable Hamiltonian systems. A class of analytical equations describing the three-dimensional quadrilateral nets is usually refereed to as discrete Darboux-Manakov-Zakharov systems [5-7]. In this paper we discuss another special type of quadrilateral net whose geometry is described by the second Hamiltonian form of DMZ systems [8].

Following [2], the 3D quadrilateral net is a $\mathbb{Z}^3$ lattice imbedded into a multidimensional linear space,

$$(n_1, n_2, n_3) \in \mathbb{Z}^3 \rightarrow \mathbf{x}(n_1, n_2, n_3) \in \mathbb{R}^M, \quad M \geq 3,$$

such that each quadrilateral, e.g.

$$(x, x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$$

is the planar one. A local cell (hexahedron) of quadrilateral net is shown in Fig. 1. Geometric integrability is based on the axiomatic statement [2]: given the points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_{12}, \mathbf{x}_{13}, \mathbf{x}_{23}$ of the hexahedron, its corners $\mathbf{x}$ and $\mathbf{x}_{123}$ can be obtained uniquely by a two-dimensional ruler (ruler which draws a plane via three non-collinear points).

The circular net is the quadrilateral net such that each its hexahedron can be inscribed into a sphere. In this paper, instead of the circular condition, suppose firstly that the target space is four-dimensional Euclidean space,

$$(Q, \text{net}): \mathbb{Z}^3 \rightarrow \mathbb{E}_4.$$

Each hexahedron is an element of a three-dimensional hyperplane. In the framework of discrete-differential geometry [3], the quadrilateral net can be viewed as a planar mesh of three-dimensional manifold embedded into four-dimensional space. The hyperplanes are more general objects then quadrilaterals since such net is not necessarily quadrilateral.

1991 Mathematics Subject Classification. 37K15.

Key words and phrases. Quadrilateral net, discrete-differential geometry, discrete Hamiltonian structure.
Let $e_1, e_2, e_3$, 

$$e_1 = \frac{x_{123} - x_{23}}{|x_{123} - x_{23}|}, \quad \text{etc.,}$$

be unit vectors defining the orientation of hexahedron in Fig. 1 and let 

$$n = \frac{\varepsilon(e_1 \wedge e_2 \wedge e_3)}{V(e_1, e_2, e_3)}, \quad \text{in indices: } (n)_\alpha = \frac{\varepsilon_{\alpha\beta\gamma\delta}(e_1)^{\beta}(e_2)^{\gamma}(e_3)^{\delta}}{V(e_1, e_2, e_3)},$$

be the unit normal vector to the hyperplane $(e_1, e_2, e_3)$. Here 

$$V(e_1, e_2, e_3) = \text{volume of parallelepiped with the edges } (e_1, e_2, e_3).$$

Consider now node $x_{123}$ of the net: the junction of eight hyperplanes shown on the left of Fig. 2. This junction is the subject of an extra “orthogonality” condition:
\[ (7) \quad \frac{V(e_1, e_2, e_3)V(e_1', e_2, e_3')V(e_1', e_2', e_3)V(e_1', e_2', e_3')}{V(e_1', e_2', e_3')V(e_1', e_2, e_3)V(e_1', e_2, e_3')V(e_1, e_2, e_3)} = 1. \]

In some sense this condition is analogous to extra condition for the circular net.

It is convenient to label the “octants” on the left of Fig. 2 by corners of dual cube, see right part of Fig. 2. For instance,

\[ (8) \quad n_h \sim *(e_1 \wedge e_2 \wedge e_3), \quad n_d \sim *(e_1 \wedge e_2 \wedge e_3'), \quad n_e \sim *(e_1 \wedge e_2' \wedge e_3'), \quad \text{etc.} \]

Orientation of each hyperplane is \( e_1' \wedge e_2' \wedge e_3'. \)

Consider now four hyperplanes \( n_c, n_e, n_h \) and \( n_d \) surrounding the edge \( e_1 \). Evidently, four hyperplanes in four-dimensional linear space have a common edge if their normal vectors are linearly dependent,

\[ (9) \quad n_c - u_1 \cdot n_c - w_1 \cdot n_h + \chi_1 u_1 w_1 \cdot n_d = 0. \]

Numerical coefficients \( u_1, w_1, \chi_1 \) in (9) are associated with the edge \( e_1 \) which is orthogonal to all \( n_c, n_e, n_h \) and \( n_d \). Analogous relations for edges \( e_2 \) and \( e_3 \) are respectively

\[ (10) \quad n_h - u_2 \cdot n_d - w_2 \cdot n_b + \chi_2 u_2 w_2 \cdot n_f = 0, \]

\[ n_c - u_3 \cdot n_h - w_3 \cdot n_g + \chi_3 u_3 w_3 \cdot n_b = 0, \]

and such equations for outgoing edges \( e_1' \) are

\[ (11) \quad n_g - u_1' \cdot n_a - w_1' \cdot n_h + \chi_1' u_1' w_1' \cdot n_f = 0, \]

\[ n_c - u_2' \cdot n_e - w_2' \cdot n_g + \chi_2' u_2' w_2' \cdot n_a = 0, \]

\[ n_e - u_3' \cdot n_d - w_3' \cdot n_a + \chi_3' u_3' w_3' \cdot n_f = 0. \]

All numerical coefficients \( u_i', w_i' \) and \( \chi_i' \) can be expressed in terms of angular data as follows. Let \( \theta_{ce} \) be an angle between \( n_c \) and \( n_e \),

\[ (12) \quad (n_c, n_e) = \cos \theta_{ce}. \]

Let further \( \varphi_{1,e} \) be a dihedral angle of hyperplane \( n_e \) for the edge \( e_1 \). In terms of unit vectors of Fig. 2, \( \varphi_{1,e} \) is the dihedral angle between planes \((e_1, e_2')\) and \((e_1, e_3')\). We extend straightforwardly these self-explanatory notations to whole dual graph of the junction, Fig. 2. Then the coefficients in relation (9) are given by

\[ (13) \quad u_1 = \frac{\sin \varphi_{1,h} \sin \theta_{eh}}{\sin \varphi_{1,d} \sin \theta_{ed}}, \quad w_1 = \frac{\sin \varphi_{1,e} \sin \theta_{ce}}{\sin \varphi_{1,d} \sin \theta_{dh}}, \quad \chi_1 = \frac{\sin \varphi_{1,e} \sin \varphi_{1,d}}{\sin \varphi_{1,h} \sin \varphi_{1,e} \sin \varphi_{1,h}}, \]

and similarly for all other relations and their coefficients. The geometry of junction without condition (7) provides

\[ (14) \quad \chi_1 \chi_2 = \chi_1 \chi_2, \quad \chi_2 \chi_3 = \chi_2 \chi_3. \]

Since there are at most four linearly independent vectors among eight \( n_a, \ldots, n_h \), the consistency of equations (911) relates the fields \( u_i', w_i' \) on outgoing edges and fields \( u_i, w_i \) on
incoming edges of Fig. 2 as follows (see e.g. [9]):

\( u'_1 = \Lambda_2^{-1} w_3^{-1}, \quad u'_2 = \Lambda_1^{-1} u_3, \quad u'_3 = \Lambda_1 u_2, \)

\( w'_1 = \Lambda_3 w_2, \quad w'_2 = \Lambda_3^{-1} w_1, \quad w'_3 = \Lambda_2^{-1} u_1^{-1}, \)

where

\[ \begin{align*}
\Lambda_1 &= u_1^{-1} u_3 - u_1^{-1} w_1 + \kappa_1 u_1 w_2^{-1}, \\
\Lambda_2 &= \frac{\kappa_1}{\kappa_2} u_2^{-1} w_3^{-1} + \frac{\kappa_2}{\kappa_3} u_1^{-1} w_2^{-1} - \frac{\kappa_1 \kappa_2}{\kappa_3} u_2^{-1} w_2^{-1}, \\
\Lambda_3 &= w_1 w_3^{-1} - u_3 w_3^{-1} + \kappa_3 w_2^{-1} u_3.
\end{align*} \]

The “orthogonality” condition (17) provides

\[ \kappa_i = \kappa'_i, \quad i = 1, 2, 3, \]

so that \( \kappa_i \) become invariants. Map (15) is the Hamiltonian one, it preserves the local symplectic form

\[ \begin{align*}
\sum_{i=1}^{3} \frac{d u_i \wedge d w_i}{u_i w_i} &= \sum_{i=1}^{3} \frac{d u'_i \wedge d w'_i}{u'_i w'_i},
\end{align*} \]

and with the orthogonality condition (17) it satisfies the functional tetrahedron equation [10]. In what follows, condition (17) is implied.

Thus, due to (18), there exists a generating function,

\[ dG(u; u') = \sum_{i=1}^{3} (\log w'_i \, d \log u'_i - \log w_i \, d \log u_i)_{u_i u_3 = u'_i u'_3}, \]

where \( u_i^\# \) and \( w_i^\# \) are related by (15, 16). In the definition of generating function \( u_i, u'_i \) are chosen as independent variables bounded by condition \( u_2 u_3 = u'_2 u'_3 \) following from (15).

Let \( L(z) \) be Roger’s dilogarithm,

\[ L(z) = \int_0^z \log(1 - x) d \log x, \]

with the branch cut \( z \geq 1 \). Then the generating function is given by

\[ G(u; u') = \log \frac{u'_3}{\kappa_1} \log \frac{u'_1}{u_1} + \log \kappa_3 \log \frac{u'_1}{u_2} + L\left( \frac{\kappa_2 u_2}{u_1} \right) + L\left( \frac{u'_2}{u_1} \right) - L\left( \frac{u_2}{u_1} \right) - L\left( \frac{1}{\kappa_1} \right)
\]

\[ = \log u_3 \log \frac{u'_1}{u_1} + \log \kappa_3 \log \frac{u'_1}{u_2} + \log \kappa_2 \log \frac{u_2}{u_2} - L\left( \frac{\kappa_1 u'_1}{u_2} \right) - L\left( \frac{u_1}{u_2} \right) + L\left( \frac{u'_1}{u_2} \right) + L\left( \frac{1}{\kappa_2} \right). \]

Positiveness of \( w_i^\# \) guarantees that arguments of all dilogarithms for one of the lines of (21) are out of the branch cut and therefore the generation function is real.

Quantization of local symplectic structure (18) \( \{ u, w \} = uw \) produces the local Weyl algebra \( uw = q^2 wu \). Quantum counterpart of Hamiltonian form of (15) is an intertwiner \( R_{123} \) in the
tensor cube of proper representations of local Weyl algebras such that

\[ u'_i = R_{123} u_i R_{123}^{-1}, \quad w'_i = R_{123} w_i R_{123}^{-1}, \quad i = 1, 2, 3. \]

For instance, the modular representation \([11]\) of the local Weyl algebra is given by

\[ u = e^{2\pi b x}, \quad w = -e^{2\pi b p}, \quad \kappa = -e^{2\pi b \lambda}, \]

where \(x, p\) is the self-conjugated Heisenberg pair

\[ [x, p] = \frac{i}{2\pi} \Rightarrow q = e^{i\pi b^2}, \]

and “physical” regime for \(b\) is

\[ \eta \overset{\text{def}}{=} \frac{b + b^{-1}}{2} > 0. \]

Modular partner to \((23)\) is

\[ \tilde{u} = e^{2\pi b^{-1} x}, \quad \tilde{w} = -e^{2\pi b^{-1} p}, \quad \tilde{\kappa} = -e^{2\pi b^{-1} \lambda}. \]

Form of the map \((22)\) for \(\tilde{u}, \tilde{w}, \tilde{\kappa}\) coincides with that for \(u, w, \kappa\); in the strong coupling regime \(0 < \eta < 1\) partner equations are Hermitian conjugated.

Kernel of the intertwiner \((22)\) in the coordinate representation of Heisenberg pairs \((24)\) is

\[ \langle x_1 x_2 x_3 | R | x'_1 x'_2 x'_3 \rangle = \delta(x_2 + x_3 = x'_2 + x'_3) e^{2\pi i ((x'_2 - x_2 - \lambda_1)(x_1 - x'_1) + (\lambda_3 - i\eta)(x_2 - x'_1))} \]

\[ \frac{\varphi(x_2 - x_1 - \lambda_1) \varphi(x'_2 - x'_1 + \lambda_2)}{\varphi(x'_2 - x_1 - i\eta + i\epsilon) \varphi(x_2 - x'_1 + \lambda_2 - \lambda_1 - i\eta + i\epsilon)}, \]

where function \(\varphi\) is the non-compact quantum dilogarithm \([11]\)

\[ \varphi(z) \overset{\text{def}}{=} \exp \left( \frac{1}{4} \int_{R+i0} \frac{e^{-2\pi zw}}{\sinh(wb) \sinh(wb/b)} \frac{dw}{w} \right). \]

Symbols \(\epsilon\) in denominator of \((27)\) define circumventions of poles. Operator \((27)\) satisfies the quantum tetrahedron equation with free \(\lambda_i\).

The choice of negative signs near \(w\) and \(\kappa\) in \((23)\) provides the unitarity of operator \((27)\) for real \(\lambda_i, R_{123}^{-1} = R_{123}^\dagger\). Positive “geometric” signs can be obtained by the analytical continuation \(\lambda_i \rightarrow \lambda_i + i\eta\) and non-unitary gauge transformation \(w \rightarrow e^{-2\pi \eta x} w e^{2\pi \eta x} = -q^{-1} w\). In that case the kernel of \(R\)-matrix \((27)\) has the semi-classical \((b \rightarrow 0\) and \(e^{2\pi b x} \rightarrow u)\) asymptotic

\[ \log \left( e^{-2\pi \eta x} (x | R | x') e^{2\pi \eta x'} \right) \xrightarrow{b \rightarrow 0} -\frac{G(u; u')}{2\pi i b^2}, \]

where the generating function is given by \((21)\).

It worth mentioning the cyclic representations of Weyl algebra with \(q^{2N} = 1\). The cyclic representation is a \(Z_N\) fiber over the base of centers \(\tilde{u} = u^N, \tilde{w} = w^N\) [12]. Equations of motion for \(\mathbb{C}\)-valued centers follow from quantum map \([15]\), they just coincide with classical
equations of motion. It is natural then to identify the evolving centers \( \tilde{u}_i, \tilde{w}_i \) directly with the geometric data \(^{[13]}\) and pose quantum problems in Hilbert space

\[ \mathcal{H} = \mathbb{Z}_N^\otimes \text{(size of net's section)} \]

in the presence of external classical geometry. The structure of \( \mathbb{Z}_N^\otimes^3 \) intertwiners and modified tetrahedron equations are discussed in more details in e.g. \(^{[13, 14]}\). Homogeneous point \( \tilde{u}'_i = \tilde{u}_i, \tilde{w}'_i = \tilde{w}_i \) of the Zamolodchikov-Bazhanov-Baxter model \(^{[15]}\) is complex one, it is not a geometrical regime.

**Acknowledgements.** I am grateful to V. Bazhanov and V. Mangazeev for valuable discussions and fruitful collaboration. Also I would like to thank M. Hewett, P. Vassiliou and J. Ascione for an encouragement.

**References**

[1] Konopelchenko, B. G. and Schief, W. K. *Three-dimensional integrable lattices in Euclidean spaces: conjugacy and orthogonality.* R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **454** (1998) 3075–3104.

[2] Doliwa, A. and Santini, P. M. *Multidimensional quadrilateral lattices are integrable.* Phys. Lett. A **233** (1997) 365–372.

[3] Bobenko, A. and Suris, Y. *Discrete differential geometry. Consistency as integrability.* Monograph pre-published at [http://www.arxiv.org/math/0504358](http://www.arxiv.org/math/0504358), 2005.

[4] Bazhanov, V. V., Mangazeev, V. V., and Sergeev, S. M. *Quantum geometry of 3-dimensional lattices.* J. Stat. Mech. (2008) P07006. [arXiv:0801.0129](http://arxiv.org/abs/0801.0129)

[5] Darboux, G. *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes.* Gauthier-Villars, Paris, 1910.

[6] Zakharov, V. E. and Manakov, S. V. *Construction of multidimensional nonlinear integrable systems and their solutions.* Funktsional. Anal. i Prilozhen. **19** (1985) 11–25, 96.

[7] Bogdanov, L. V. and Konopelchenko, B. G. *Lattice and q-difference Darboux-Zakharov-Manakov systems via \( T \)-dressing method.* J. Phys. A **28** (1995) L173–L178.

[8] Sergeev, S. M. *Quantization of three-wave equations.* J. Phys. A **40** (2007) 12709–12724.

[9] Sergeev, S. M. *Quantum 2 + 1 evolution model.* J. Phys. A: Math. Gen. **32** (1999) 5693–5714.

[10] Kashaev, R. M., Korepanov, I. G., and Sergeev, S. M. *The functional tetrahedron equation.* Teoret. Mat. Fiz. **117** (1998) 370–384.

[11] Faddeev, L. D. *Discrete Heisenberg-Weyl group and modular group.* Lett. Math. Phys. **34** (1995) 249–254.

[12] Bazhanov, V. V. and Reshetikhin, N. Y. *Remarks on the quantum dilogarithm.* J. Phys. A **28** (1995) 2217–2226.

[13] Sergeev, S. *Complex of three-dimensional solvable models.* J. Phys. A **34** (2001) 10493–10503.

[14] von Gehlen, G., Pakuliak, S., and Sergeev, S. *Explicit free parametrization of the modified tetrahedron equation.* J. Phys. A **36** (2003) 975–998.

[15] Sergeev, S. M., Mangazeev, V. V., and Stroganov, Y. G. *The vertex formulation of the Bazhanov-Baxter model.* J. Stat. Phys. **82** (1996) 31–50.