$L^p$ compression of some HNN extensions

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Abstract

In [GJ03], the authors introduce a framework to prove that a large class of HNN extensions have the Haagerup property, the main motivation being Baumslag-Solitar groups. Using this framework and new tools on locally compact groups developed in [CdCMT12], we are able to obtain quantitative results on embeddings into Lebesgue spaces for a large class of HNN extensions.

1 Introduction

In [GK04], Guentner and Kaminker introduced the notion of compression exponent between metric spaces. Roughly speaking, it gives a way of quantifying how well a metric space coarsely embeds into another. We recall here the definitions. Let $X$ and $Y$ be metric spaces. A map $f : X \to Y$ is said to be large-scale Lipschitz if we can find some constants $A, B \geq 0$ so that the following inequality holds for every $x, y \in X$:

$$d(f(x), f(y)) \leq Ad(x, y) + B.$$ 

In this case, we define $R(f)$ to be the supremum over the $\alpha \in [0, 1]$ such that there exist some constants $C, D \geq 0$ so that $d(f(x), f(y)) \geq Cd(x, y)^\alpha - D$, for every $x, y \in X$. Then, the compression exponent, or simply compression of $X$ with target $Y$ is defined as $\alpha^*_Y(X) := \sup_f R(f)$, where the supremum is taken over all Large-scale Lipschitz maps $f$ from $X$ into $Y$. When $Y = L^p(\Omega)$ with $\Omega$ a standard Borel space, we set $\alpha^*_p = \alpha^*_p$. If $G$ is a compactly generated locally compact group (e.g. a finitely generated group), we view it as a metric space with the word metric. The study of groups seen as metric spaces has

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offered some striking results. Let us recall one. If $G$ is a finitely generated group and if $\alpha^*_p(G) > 0$ for some $p \in (1, +\infty)$, then $G$ satisfies the Novikov conjecture (see [KY06]).

In [GJ03] the authors introduced the notion of an $\mathcal{N}$-BS group. These are groups arising as HNN extensions satisfying properties similar to Baumslag-Solitar groups. In order to prove the Haagerup property for such groups, they developed a framework that we shall heavily rely on and that we now recall.

Let $\mathcal{N}$ be a locally compact compactly generated group and let $G$ be a closed subgroup of $\mathcal{N}$. Let $i_1, i_2 : H \to G$ be two inclusions of a group $H$ onto open subgroups of finite index, and assume $i_1$ and $i_2$ are conjugated by an automorphism $\varphi$ of $\mathcal{N}$. The $\mathcal{N}$-BS group $\Gamma$ is then the HNN extension $\text{HNN}(G, H, i_1, i_2)$ whose presentation is given by $\langle S, t | R, ti_1(h)t^{-1} = i_2(h) \forall h \in H \rangle$, where $G = \langle S | R \rangle$.

Theorem 1 Let $\mathcal{N}$ be a connected Lie group and $G$ a closed cocompact subgroup of $\mathcal{N}$ and let $\Gamma$ be an HNN extension as above. Then, for all $p > 1$, $\alpha^*_p(\Gamma) = 1$.

The strategy to prove Theorem 1 is to construct a metric space $M$ on which $\Gamma$ acts continuously, properly, cocompactly and by isometries, so that, $M$ and $\Gamma$ are quasi-isometric by Svarc Lemma: this is done in section 2, where we also give a quantitative comparison between two natural metrics on $M$. In the last section, we prove Theorem 1 and treat some concrete examples.

2 The space $M$

Let $\hat{\mathcal{N}} = \mathcal{N} \times \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathcal{N}$ by iterations of $\varphi$ and let $j_\mathcal{N} : \Gamma \to \hat{\mathcal{N}}$ be the homomorphism defined by $g \mapsto (g, 0)$ for $g \in G$ and $t \mapsto (1, 1)$. Then consider $T$, the Bass-Serre tree associated with the HNN extension $\Gamma$ and denote by $j_T : \Gamma \to \text{Aut}(T)$ the monomorphism induced by the action of $\Gamma$ on $T$. For later use, recall the following result from [GJ03].

Theorem 2 The homomorphism $j : \Gamma \to \hat{\mathcal{N}} \times \text{Aut}(T)$, $g \mapsto (j_\mathcal{N}(g), j_T(g))$ is injective and has closed image. In particular, it is proper. \(\square\)

Following Proposition 2.1 in [CdCMT12], we will define a metric space $Y$ on which $\hat{\mathcal{N}}$ acts continuously, properly, cocompactly and by isometries. Endow $\hat{\mathcal{N}}$ with a left-invariant Riemannian metric. For each coset $L_i = \mathcal{N} \times \{i\}$ of $\hat{\mathcal{N}}$ in $\hat{\mathcal{N}}$ we consider a strip $L_i \times [0, 1]$, equipped with the product Riemannian metric, and attach it to $\hat{\mathcal{N}}$ by identifying $(l, 0)$ to $l$ and $(l, 1)$
to $l \cdot (1,1)$. Denote by $Y$ the space obtained in this way. $Y$ has a natural shortest-path metric induced by the riemannian metric on each of the strips. Furthermore, $Y$ is naturally homeomorphic (but not necessarily isometric!) to $\mathcal{R} \times \mathbb{R}$. Using this obvious parametrization, $\mathcal{R}$ acts on $Y$ by $(n,k) \cdot (y,s) = (n\varphi^k(y), k+s)$, for $(n,k) \in \mathcal{R}$ and $(y,s) \in Y$. As in Proposition 2.1 in [CdCMT12], $Y$ is a locally compact, geodesic metric space on which $\mathcal{R}$ acts continuously, properly, cocompactly and by isometries. We denote by $b$ the projection map $(y,s) \mapsto s$.

Let us recall briefly the construction of the Bass-Serre tree $T$ of $\Gamma$. It is an oriented graph whose vertices are the left-cosets $\Gamma/G$ and the edges correspond to the left cosets $\Gamma/i_1(H)$. The edge $\gamma/i_1(H)$ is directed from $\gamma t^{-1} G$ to $\gamma G$. As the $i_k(H)$ are of finite index in $G$, $T$ is locally finite. Then, by construction, $\Gamma$ acts naturally on $T$ by left multiplication.

Now, let $p : \Gamma \to \mathbb{Z}$ be the homomorphism defined on the generators by $p(t) = 1$ and $p(g) = 0$, for every $g \in G$. Since the vertices of $T$ correspond to the left cosets of $G$ in $\Gamma$, we can define a map $c$ on the vertices of $T$ by $c(xG) = p(x)$ and extend it to the metric tree $T$ by affine interpolation. This allows us to define the fibre product $M$:

$$M = \{(x,y) \in T \times Y : c(x) = b(y)\}.$$ 

The subspace $M$ is $\Gamma$-invariant for the diagonal action of $\Gamma$ on $T \times Y$. Indeed, for all $x \in T$, $c(g \cdot x) = c(x)$ if $g \in G$ and $c(t \cdot x) = c(x) + 1$. In a similar fashion, for all $y \in Y$, $b(g \cdot y) = b(y)$ if $g \in G$ and $b(t \cdot y) = b(y) + 1$. Hence, if $c(x) = b(y)$, it implies that $c(\gamma \cdot x) = b(\gamma \cdot y)$ for any $\gamma \in \Gamma$. Other similar fibre products have already been considered, namely horocyclic products and millefeuille spaces. Those spaces are defined using so-called Busemann functions (see Section 7 in [CdCMT12]). It is worth noting that, in general, our functions $c$ or $b$ are not Busemann functions.

We endow $T \times Y$ with the product metric, namely, $d((x,y),(x',y')) = d_T(x,x') + d_Y(y,y')$.

**Lemma 1** $M$ is path-connected. Furthermore, denoting by $d_M$ the shortest-path metric induced by $d$ on $M$, the metrics $d$ and $d_M$ are bilipschitz equivalent.

**Proof:** First, observe that, for any point $y = (n,s) \in Y$, the path

$$\alpha_y : \mathbb{R} \to Y : u \mapsto (n,u+s)$$

is a geodesic such that $\alpha_y(0) = y$ and $b(\alpha_y(u)) = b(y) + u$, $\forall u \in \mathbb{R}$. Similarly, for any point $x \in T$ one can choose a geodesic path $\beta_x : \mathbb{R} \to T$ such that
\[ \beta_x(0) = x \] and \[ c(\beta_x(u)) = c(x) + u. \] Let \((x_0, y_0), (x_1, y_1) \in M.\) We will build a path linking those points in two steps. For the first one, let \(\sigma : [0, d_T(x_0, x_1)] \to T\) be the geodesic from \(x_0\) to \(x_1.\) Let \(\theta_1\) be the path defined by
\[ \theta_1(u) = (\sigma(u), \alpha_{y_0}(c(\sigma(u)) - b(y_0))). \]

The left component links \(x_0\) to \(x_1,\) while the right component starts from \(y_0\) and ends at a certain point \(y_2.\) Moreover, the path \(\theta_1\) is contained in \(M.)\)

Indeed, for all \(u \in [0, d_T(x_0, x_1)],\) we have:
\[ b(\alpha_{y_0}(c(\sigma(u)) - b(y_0))) = b(y_0) + c(\sigma(u)) - b(y_0) = c(\sigma(u)). \]

So, \(\theta_1\) connects \((x_0, y_0)\) to a point \((x_1, y_2) \in M\) satisfying \(b(y_2) = c(x_1) = b(y_1).\) For the second step, we will find a path in \(M\) between \((x_1, y_2)\) and \((x_1, y_1).\) In a similar way, let \(\tilde{\sigma} : [0, d_Y(y_2, y_1)] \to Y\) be a geodesic path linking \(y_2\) to \(y_1\) in \(Y.\) Then, it is easy to check that the path
\[ \theta_2 : [0, d_Y(y_2, y_1)] \to M : \theta_2(u) = (\beta_{x_1}(b(\tilde{\sigma}(u)) - c(x_1)), \tilde{\sigma}(u)) \]
does the job. This shows that \(M\) is path-connected. Now, the inequality \(d \leq d_M\) being immediate, we need to analyze the length of the path we just considered in order to finish the proof. Denoting by \(L(\theta_j)\) the length of the path \(\theta_j,\) we get the following estimates:
\[ L(\theta_1) \leq 2d_T(x_0, x_1) \]
and
\[ L(\theta_2) \leq 2d_Y(y_2, y_1) \leq 2(d_T(y_0, y_1) + d_Y(y_1, y_2)) \]
By construction, \(d_Y(y_0, y_2) \leq d_T(x_0, x_1).\) We can conclude:
\[
\begin{align*}
d_M((x_0, y_0), (x_1, y_1)) & \leq L(\theta_1) + L(\theta_2) \\
& \leq 2d_T(x_0, x_1) + 2d_Y(y_0, y_1) + 2d_Y(y_1, y_2) \\
& \leq 4d_T(x_0, x_1) + 2d_Y(y_0, y_1) \\
& \leq 4 \cdot d((x_0, x_1), (y_0, y_1)).
\end{align*}
\]

\[ \square \]

3 Proof of Theorem \[ \square \] and Applications

In order to apply Svarc Lemma, we prove that the action of \(\Gamma\) is proper and cocompact.
Lemma 2  The $\Gamma$-action on $T \times Y$ is proper. That is, for all $(x, y) \in T \times Y$, there exists $r > 0$ so that $\{ \gamma \in \Gamma : \gamma \cdot B((x, y), r) \cap B((x, y), r) \neq \emptyset \}$ is relatively compact in $\Gamma$.

In particular, as $M$ is a closed subset of $T \times Y$, we get immediately the following Corollary.

Corollary 1  The $\Gamma$-action on the fibre product $M$ is proper.

Proof of Lemma 2:  The action of $\text{Aut}(T) \times \tilde{\mathcal{N}}$ on $T \times Y$ is proper, therefore, by Theorem 2, it is also the case for the action of $\Gamma$. As $M$ is closed and $\Gamma$-invariant, we can conclude.

Lemma 3  The action of $\Gamma$ on $M$ is cocompact.

Proof:  It is enough to see that, for any sequence $(x_k, y_k)_k \subset M$, we can find a sequence $(\gamma_k)_k \subset \Gamma$ so that the sequence $(\gamma_k \cdot (x_k, y_k))_k$ converges. Since $\Gamma$ acts transitively on the edges of $T$, we can assume that the sequence $(x_k)_k$ belongs to the edge $[G, tG]$. This implies that $0 \leq c(x_k) = b(y_k) \leq 1$, for all but possibly finitely many $k$, so that the sequence $(y_k)_k$ is contained in the strip of $Y$ corresponding to the coset $\mathcal{N}$ in $\tilde{\mathcal{N}}/\mathcal{N}$. Using the fact that the action of $G$ on $\mathcal{N}$ is cocompact, we can multiply by elements of $G$ in such a way that the sequence $(y_k)_k$ converges. But since $G$ stabilizes the vertex $G$ in $T$, this process maintains the sequence $(x_k)_k$ inside the edges adjacent to $G$. Since there are only finitely many on these, the sequence $(x_k, y_k)_k$ converges up to extracting a subsequence. This concludes the proof.

We are now able to prove Theorem 1.

Proof of Theorem 1:  Firstly, we show that $\alpha_p^*\mu(\Gamma) \geq \alpha_p^*\mu(\tilde{\mathcal{N}})$. Indeed, by Svarc Lemma, $\Gamma$ is quasi-isometric to $(M, d_M)$, which is quasi-isometric to $(M, d)$ by Lemma 4. Moreover, $Y$ is quasi-isometric to $\mathcal{N}$. Hence, $\alpha_p^*(\Gamma) = \alpha_p^*(M) \geq \alpha_p^*(T \times Y)$ and $\alpha_p^*(Y) = \alpha_p^*(\tilde{\mathcal{N}})$. Then, the lower bound follows from the propositions:

- For a tree $T$, $\alpha_p^*(T) = 1$, for all $p > 1$. (See Theorem 2.6 in [BS08])
- For two metric spaces $X$ and $X'$, the compression of $X \times X'$ is the minimum of the compressions of the factors. (See [GK04])

Finally, we conclude the proof by noting that $\alpha_p^*(\tilde{\mathcal{N}}) = 1$, which follows from the following propositions:
• Any semi-direct product of a connected Lie group with \( \mathbb{Z} \) is quasi-isometric to a connected Lie group. (By an unpublished result of Y. Cornulier)

• Let \( K \) be a connected Lie group. Then, \( \alpha_p^*(K) = 1 \), for all \( p > 1 \). (See [Tes11])

We remark that, if \( \mathcal{R} \) is a soluble connected Lie group, then Cornulier’s result is a simple consequence of a lemma of Mostow. Here is a short proof that we owe to Alain Valette. In this case, \( \mathcal{R} \) is soluble and Noetherian (i.e. every closed subgroup is compactly generated). A lemma of Mostow (see Lemma 5.2 in [Mos71]) asserts that there exist a compact normal subgroup \( K \) of \( \mathcal{R} \) and a soluble almost connected Lie group \( M \) such that the quotient \( \mathcal{R}/K \) is isomorphic to \( L \), where \( L \) is a cocompact, closed subgroup of \( M \). Then, the connected component of unity \( M^0 \) is quasi-isometric to \( \mathcal{R} \). Indeed, on one hand, by compactness, \( \mathcal{R} \) is quasi-isometric to \( \mathcal{R}/K \) and by Mostow we deduce that \( \mathcal{R} \) is quasi-isometric to \( L \). On the other hand, by cocompactness, \( L \) is quasi-isometric to \( M \) and, since \( M \) has only finitely many connected components, it is quasi-isometric to \( M^0 \).

In particular, Theorem \( \square \) allows us to cover all the examples appearing in [GJ03].

**Corollary 2** The following groups have compression 1.

1. The Baumslag-Solitar groups \( BS^p_q = \langle x, t \mid x^p = tx^q t^{-1} \rangle = HNN(\mathbb{Z}, \mathbb{Z}, p, q) \), with parameters \( p, q \in \mathbb{Z}_+ \). (For a different proof, see also [CV12].)

2. Torsion free, finitely presented abelian-by-cyclic groups.

3. Let \( \mathcal{R} \) be a homogeneous nilpotent Lie group. So, it admits a dilating automorphism \( \varphi \). Suppose that \( \mathcal{R} \) contains a discrete, cocompact subgroup \( G \) which is invariant by \( \varphi \). Then, for any finite index subgroup \( H \) in \( G \), the extension \( HNN(G, H, i_1, \varphi|_H) \), where \( i_1 \) is the canonical injection, has compression 1.

**Proof:** The Baumslag-Solitar groups \( BS^p_q \) can be seen as HNN extension of \( \mathbb{Z} \) with itself, considering the inclusions \( i_1, i_2 : \mathbb{Z} \to \mathbb{Z} \) defined by \( i_1(n) = pn \) and \( i_2(n) = qn \). Then, apply Theorem \( \square \) with the automorphism of \( \mathcal{R} = \mathbb{R} \) given by \( \varphi(x) = \frac{p}{q}x \).
For the second class of examples, it is known (see for instance [FM00]) that these groups are HNN extensions of $\mathbb{Z}^n$ with itself with respect to $i_1, i_2 : \mathbb{Z}^n \to \mathbb{Z}^n$, where $i_1$ is the identity and $i_2 \in GL_n(\mathbb{Z})$. Again, apply Theorem 1 with the automorphism of $\mathfrak{N} = \mathbb{R}^n$ given by $i_2^{-1}$.

The case of the last class of examples is clear by construction.

Remark 1 It is important to note that the assumption about finite presentation is necessary to treat the second class of examples. Indeed, the wreath product $\mathbb{Z} \wr \mathbb{Z}$ is torsion free, abelian-by-cyclic and finitely generated. However, it is computed in [ANP09] that $\alpha_2(\mathbb{Z} \wr \mathbb{Z}) = \frac{2}{3}$.

Remark 2 In the case where $\mathfrak{N}$ is a generic compactly generated locally compact (not necessarily a connected Lie group), it is also possible to find a space $Y$ admitting a geometric action of $\tilde{\mathfrak{N}}$, by Proposition 2.1 in [CdCMT12]. However, it is not clear how to endow $Y$ with a natural fibration $b$ compatible with the semi-direct product structure on $\tilde{\mathfrak{N}}$ in order to generalize Theorem 1.

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