Pentagons and rhombuses that can form rotationally symmetric tilings

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Abstract

In this study, various rotationally symmetric tilings that can be formed using pentagons that are related to rhombus are discussed. The pentagons can be convex or concave and can be degenerated into a trapezoid. If the pentagons are convex, they belong to the Type 2 family. Because the properties of pentagons correspond to those of rhombuses, the study also explains the correspondence between pentagons and various rhombic tilings.

Keywords: pentagon, rhombus, tiling, rotationally symmetry, monohedral

1 Introduction

In \cite{8} and \cite{9}, we introduced rotationally symmetric tilings and rotationally symmetric tilings (tiling-like patterns) with an equilateral convex polygonal hole at the center formed using convex pentagonal tiles\textsuperscript{1} These tilings have different connecting methods such as edge-to-edge\textsuperscript{2} and non-edge-to-edge. The convex pentagonal tiles forming the tilings belong to the Type 1 family\textsuperscript{3}. Note that the convex pentagonal tiles in \cite{8} and \cite{9} are considered to be generated by bisecting equilateral concave octagons and equilateral convex hexagons, respectively.

Apart from the rotationally symmetric tilings with convex pentagonal tiles described above, Hirschhorn, Hunt, and Zucca demonstrated a five-fold rotationally symmetric tiling with equilateral convex pentagonal tiles belonging to the Type 2 family, as shown in Figure 1\cite{1,2,10,11,14}. In \cite{10}, we considered edge-to-edge tilings with a convex pentagon having four equal-length edges and demonstrated that the convex pentagon in Figure 1 correlates

\textsuperscript{1} A tiling (or tessellation) of the plane is a collection of sets that are called tiles, which covers a plane without gaps and overlaps, except for the boundaries of the tiles. The term “tile” refers to a topological disk, whose boundary is a simple closed curve. If all the tiles in a tiling are of the same size and shape, then the tiling is monohedral\cite{1,11}. In this study, a polygon that admits a monohedral tiling is called a polygonal tile\cite{5,7}. Note that, in monohedral tiling, it admits the use of reflected tiles.

\textsuperscript{2} A tiling by convex polygons is edge-to-edge if any two convex polygons in a tiling are either disjoint or share one vertex or an entire edge in common. Then other case is non-edge-to-edge\cite{1,5,7}.

\textsuperscript{3} To date, fifteen families of convex pentagonal tiles, each of them referred to as a “Type,” are known\cite{1,5,7,11}. For example, if the sum of three consecutive angles in a convex pentagonal tile is 360°, the pentagonal tile belongs to the Type 1 family. Convex pentagonal tiles belonging to some families also exist. Known convex pentagonal tiles can form periodic tiling. In May 2017, Michæl Rao declared that the complete list of Types of convex pentagonal tiles had been obtained (i.e., they have only the known 15 families), but it does not seem to be fixed as of March 2020\cite{11}.
to a case of a convex pentagonal tile called “C20-T2,” which has five equal-length edges (i.e., equilateral edges) and an interior angle of 72°. The results suggest that the five-fold rotationally symmetric tiling shown in Figure 1 can be formed using a convex pentagonal tile (C20-T2) with four equal-length edges, as shown in Figure 2.

As in [8] and [9], we expected that the convex pentagonal tile C20-T2 will be able to form not only five rotationally symmetric tilings, but also other rotationally symmetric tilings. We then confirm that C20-T2 is capable of forming such tilings. This study introduces the results obtained.

Figure. 1: Five-fold rotationally symmetric tiling by an equilateral convex pentagonal tile that Hirschhorn, Hunt, and Zucca demonstrated (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 2: Five-fold rotationally symmetric tiling by a convex pentagonal tile with four equal-length edges (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well. Note that the gray area in the figure is used to clearly depict the structure)
2 Conditions of pentagon that can form rotationally symmetric tilings

In this study, the vertices (interior angles) and edges of the pentagon will be referred to using the nomenclature shown in Figure 3(a). C20-T2 shown in [10] is a convex pentagon that satisfies the conditions

\[
\begin{align*}
B + D + E &= 360^\circ, \\
a &= b = c = d,
\end{align*}
\]  

(1)

and can form the representative tiling (tiling of edge-to-edge version) of Type 2 that has the relations “\(B + D + E = 360^\circ, \ 2A + 2C = 360^\circ\)” because this convex pentagon has four equal-length edges, it can be divided into an isosceles triangle \(BCD\), an isosceles triangle \(ABE\) with a base angle \(\alpha\), and a triangle \(BDE\) with \(\angle DBE = \theta\) and \(\angle BDE = \delta\), as shown in Figure 3(b). Accordingly, using the relational expression for the interior angle of each vertex of C20-T2, the conditional expressions of (1) can be rewritten as follows:

\[
\begin{align*}
A &= 180^\circ - 2\alpha, \\
B &= 90^\circ + \theta, \\
C &= 2\alpha, \\
D &= 90^\circ - \alpha + \delta, \\
E &= 180^\circ + \alpha - \theta - \delta, \\
a &= b = c = d,
\end{align*}
\]  

(2)

where

\[
\delta = \tan^{-1}\left(\frac{\sin \theta}{\tan \alpha - \cos \theta}\right)
\]

and \(0^\circ < \alpha < 90^\circ\) because \(A > 0^\circ\) and \(C > 0^\circ\) [10]. This pentagon has two degrees of freedom (\(\alpha\) and \(\theta\) parameters), besides its size. If the edge \(e\) of this pentagon exists and the pentagon is convex, then \(0^\circ < \theta < 90^\circ\). But depending on the value of \(\alpha\), even if \(\theta\) is selected in \((0^\circ, 90^\circ)\), the pentagon may not be convex or may be geometrically nonexistent. If \(a = b = c = d = 1\), then the length of edge \(e\) can be expressed as follows:

\[
e = 2\sqrt{1 - \sin(2\alpha) \cos \theta}.
\]
Let the interior angle of vertex $A$ be $\frac{360^\circ}{n}$ (i.e., $\alpha = 90^\circ - \frac{180^\circ}{n}$) so that pentagons satisfying (2) can form an $n$-fold rotationally symmetric tiling. (Remark: Due to the properties of the pentagons, the interior angle of vertex $C$, and not vertex $A$, will be $\frac{360^\circ}{n}$.) Note that $n$ is an integer greater than or equal to three, because $C > 0^\circ$. Therefore, the conditions of pentagonal tiles that can form $n$-fold rotationally symmetric tilings are expressed in (3). Note that the properties of the shape of pentagons that satisfy (3) depending on the values of $n$ and $\theta$ are summarized in Appendix.

\[
\begin{align*}
A &= \frac{360^\circ}{n}, \\
B &= 90^\circ + \theta, \\
C &= 180^\circ - \frac{360^\circ}{n}, \\
D &= \delta + \frac{180^\circ}{n}, \\
E &= 270^\circ - \theta - \delta - \frac{180^\circ}{n}, \\
a &= b = c = d.
\end{align*}
\]

(3)

3 Relationships between pentagon and rhombus

The convex pentagon shown in Figure 2 satisfies (3), where $n = 5$ and $\theta = 63^\circ$. Note that it is equivalent to the case where $\alpha = 54^\circ$ and $\theta = 63^\circ$, in (2). By using this convex pentagon of Figure 2, the method of forming tilings with pentagons satisfying the conditions of (2) or (3) is described below. In accordance with the relationship between the five interior angles of the pentagon, the vertex concentrations that can be always used in tilings are “$A + C = 180^\circ$, $B + D + E = 360^\circ$, $2A + 2C = 360^\circ$.” According to (2) and (3), the edge $e$ of the pentagon is the sole edge of different length. Therefore, the edge $e$ of one convex pentagon is always connected in an edge-to-edge manner with the edge $e$ of another convex pentagon. A pentagonal pair with their respective vertices $D$ and $E$ concentrated forms the basic unit of the tiling. This basic unit can be made of two types: a (anterior side) pentagonal pair as shown in Figure 4(a) and a reflected (posterior side) pentagonal pair as shown in Figure 4(b). Four different types of units, as shown in Figures 4(c), 4(d), 4(e), and 4(f), are obtained by combining two pentagonal pairs shown in Figures 4(a) and 4(b), so that $B + D + E = 360^\circ$ can be assembled.

As shown in Figures 4(a) and 4(b), a rhombus (red line), with an acute angle of $72^\circ$, formed by connecting the vertices $A$ and $C$ of the pentagons, is applied to each basic unit of the pentagonal pair. (Remark: In this example, because the interior angle of the vertex $A$ is $72^\circ$, the rhombus has an acute angle of $72^\circ$. That is, the interior angles of the rhombus corresponding to the pair of pentagons in Figures 4(a) and 4(b) are the same as the interior angles of vertices $A$ and $C$ in (2) and (3).) Consequently, the parts of pentagons that protrude from the rhombus match exactly with the parts that are more dented than the rhombus (refer to Figures 4(c), 4(d), 4(e), and 4(f)). In fact, tilings in which “$B + D + E = 360^\circ$, $2A + 2C = 360^\circ$” using pentagons satisfying (2) and (3) are equivalent to rhombic tilings (tilings formed by rhombuses). (Though a rhombus is a single entity, considering its internal pentagonal pattern, it will be considered as two entities.)

Rhombuses have two-fold rotational symmetry and two axes of reflection symmetry pass-
Figure 4: Relationships between pentagonal pair (basic unit) and rhombus
Figure 5: Combinations of rhombuses and pentagons
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ing through the center of the rotational symmetry (hereafter, this property is described as $D_2$ symmetry). Therefore, the rhombus and the reflected rhombus have identical outlines. Thus, the two methods of concentrating the four rhombic vertices at a point without gaps or overlaps are: Case (i) an arrangement by parallel translation as shown in Figure 5(a); Case (ii) an arrangement by rotation (or reflection) as shown in Figure 5(b). This concentration corresponds to forming a “$2A + 2C = 360°$” at the center by four pentagons. In Case (i), because the pentagonal vertices circulate as “$A \rightarrow C \rightarrow A \rightarrow C$” at the central “$2A + 2C = 360°$,” one combination (refer to Figure 5(c)) is obtained by using two units of Figure 4(c) and another combination (refer to Figure 5(d)) is obtained by using two units of Figure 4(d). In Case (ii), because the pentagonal vertices circulate as “$A \rightarrow A \rightarrow C \rightarrow C$” at the central “$2A + 2C = 360°$,” one combination (refer to Figure 5(e)) is obtained by using units of Figures 4(c) and 4(d), and another combination (refer to Figure 5(f)) is obtained by using units of Figures 4(e) and 4(f).

Only when the unit comprising eight pentagons in Figures 5(c) or 5(d) are arranged in a parallel manner, a tiling, as shown in Figure 5(g), is formed that represents a tiling of Type 2, in which “$B + D + E = 360°$, $2A + 2C = 360°$.” Because rhombuses can form rhombic belts by translation in the same direction vertically, rhombic tilings can also be formed by the belts that are freely connected horizontally by the connecting method shown in Figures 5(a) and 5(b). Further, pentagonal tilings (tilings formed by pentagons) corresponding to those rhombic tilings can be formed.

When $n$ vertices, with interior angles of $\frac{360°}{n}$, of $n$ rhombuses are concentrated at a point, an $n$-fold rotationally symmetric arrangement is formed, with adjacent rhombuses connected as shown in Figure 5(b). Therefore, an $n$-fold rotationally symmetric tiling with rhombuses can be formed by dividing each rhombus, in that arrangement, into similar shapes. By converting the rhombuses of such rhombic tiling into pentagons satisfying (3), the rotationally symmetric tilings with convex pentagons can be obtained (refer to Figure 5(h)). Therefore, when forming $n$-fold rotationally symmetric tilings from a pentagon satisfying (3), the pentagonal arrangement can be known from the corresponding $n$-fold rotationally symmetric tilings with a rhombus.

4 Rotationally symmetric tilings

Table 1 presents some of the relationships between the interior angles of convex pentagons satisfying (3) that can form the $n$-fold rotationally symmetric edge-to-edge tilings. (For $n = 3\rightarrow 10, 16$, tilings with convex pentagonal tiles are drawn. For further details, Figures 2, 6, 13.) Due to the presence of parameter $\theta$ in (3), the shapes of convex pentagons that satisfy (3) and can form an $n$-fold rotationally symmetric tiling are not fixed. Therefore, each example presented in Table 1 is a pentagon with a convex shape that can form an $n$-fold rotationally symmetric tiling. If the pentagons satisfying (3) are convex, the $n$-fold rotationally symmetric tilings with the pentagonal tiles are connected in an edge-to-edge manner and have no axis of reflection symmetry. The reason for this lack of symmetry is that the units comprising pentagons corresponding to that rhombus with $D_2$ symmetry (refer to the Schoenflies notation for symmetry in a two-dimensional point group [12, 13]). “$D_n$” represents an $n$-fold rotation axis with $n$ reflection symmetry axes. The notation for symmetry is based on that presented in [12, 13].
Table 1: Example of interior angles of convex pentagons satisfying (3) that can form the $n$-fold rotationally symmetric tilings

| $n$ | Value of interior angle (degree) | Edge length of $e$ | Figure number |
|-----|---------------------------------|-------------------|---------------|
| 3   | 120 151 60 143.96 65.04 | 1.523 | 6 |
| 4   | 90 151 90 104.5 104.5 | 1.436 | 7 |
| 5   | 72 153 108 80.01 126.99 | 1.508 | 2 |
| 6   | 60 151 120 65.04 143.96 | 1.523 | 8 |
| 7   | 51.43 146 128.57 54.37 159.63 | 1.500 | 9 |
| 8   | 45 151 135 46.89 162.11 | 1.621 | 10 |
| 9   | 40 150 140 41.07 168.93 | 1.648 | 11 |
| 10  | 36 156 144 36.88 167.12 | 1.745 | 12 |
| 11  | 32.73 156 147.27 33.31 170.69 | 1.766 |
| 12  | 30 160 150 30.49 169.51 | 1.821 |
| 13  | 27.69 160 152.31 28.04 171.96 | 1.834 |
| 14  | 25.71 164 154.29 26.03 169.97 | 1.877 |
| 15  | 24 164 156 24.25 171.75 | 1.885 |
| 16  | 22.5 170 157.5 22.72 167.28 | 1.932 | 13 |
| 17  | 21.18 170 158.82 21.36 168.64 | 1.936 |
| 18  | 20 170 160 20.16 169.84 | 1.940 |

... ... ... ... ... ...

The pentagons with $n = 3$ and $n = 6$ correspond to rhombuses with an acute angle of $60^\circ$ (i.e., they correspond to tiling of an equilateral triangle), and these pentagons are opposite to each other. (In Table 1, the interior angle of vertex $B$ is chosen to have the same value in both the cases. Note that the convex pentagonal tiles with $n = 3$ and $n = 6$ belong to both the Type 2 and Type 5 families \[1, 5–7, 11\].) According to this relationship, these tiles can form tilings with three-fold rotational symmetry that have six-fold rotational symmetry at the intersection of tilings, as shown in Figure 14. Also, in addition to $2A + 2C = 360^\circ$, “$3C = 360^\circ$, $4A + C = 360^\circ$, $6A = 360^\circ$” are valid in these tilings (refer to Figure 15). In particular, consider the unit comprising six pentagons, as shown in Figure 15(a), that has outline shape with $D_3$ symmetry. The pentagons in this unit can be reversed freely (i.e., a unit comprising six anterior pentagons can be freely exchanged with a unit comprising six posterior pentagons). Therefore, various patterns, as shown in Figures 16 and 17, can be generated by the pentagon corresponding to the rhombus with an acute angle of $60^\circ$.

In the case of $n = 4$, $A = C$ and $D = E$, and the pentagon has a line of symmetry connecting the vertex $B$ to the midpoint of the edge $e$ (refer to Figure 7), i.e., there is no distinction between its anterior and posterior sides — in the figures of this study, the posterior pentagons are marked with an asterisk mark. Accordingly, the rhombus corresponding to this case is a square. Therefore, the unit comprising eight pentagons corresponding to Figure 5(a) has $C_4$ symmetry. The convex pentagonal tiling of this case is called Cairo tiling, and the convex pentagonal tiles belong to both the Type 2 and Type 4 families \[1, 5–7, 11\].
Table 2: Trapezoids based on pentagons satisfying (3) that can form the \( n \)-fold rotationally symmetric tilings

| \( n \) | A (degree) | B (degree) | C (degree) | D (degree) | E (degree) | Edge length of \( e \) | Figure number |
|-------|-----------|-----------|-----------|-----------|-----------|----------------|--------------|
| 5     | 72        | 108       | 108       | 72        | 180       | 0.618          | 18           |
| 6     | 60        | 120       | 120       | 60        | 180       | 1              | 19           |
| 7     | 51.43     | 128.57    | 128.57    | 51.43     | 180       | 1.247          |              |
| 8     | 45        | 135       | 135       | 45        | 180       | 1.414          | 20           |
| 9     | 40        | 140       | 140       | 40        | 180       | 1.532          |              |
| 10    | 36        | 144       | 144       | 36        | 180       | 1.618          |              |
| 11    | 32.73     | 147.27    | 147.27    | 32.73     | 180       | 1.683          |              |
| 12    | 30        | 150       | 150       | 30        | 180       | 1.732          |              |
| 13    | 27.69     | 152.31    | 152.31    | 27.69     | 180       | 1.771          |              |
| 14    | 25.71     | 154.29    | 154.29    | 25.71     | 180       | 1.802          |              |
| 15    | 24        | 156       | 156       | 24        | 180       | 1.827          |              |
| 16    | 22.5      | 157.5     | 157.5     | 22.5      | 180       | 1.848          |              |
| 17    | 21.18     | 158.82    | 158.82    | 21.18     | 180       | 1.865          |              |
| 18    | 20        | 160       | 160       | 20        | 180       | 1.879          |              |
| ...   | ...       | ...       | ...       | ...       | ...       | ...            | ...          |

Equilateral pentagons that satisfy (3) exist, provided \( n = 4, 5, 6, 7 \). The pentagons that are convex and have equilateral edges are the cases with \( n = 4 \) (\( B \approx 131.41^\circ \)) and \( n = 5 \) (\( B \approx 127.95^\circ \)). Figure 1 shows the five-fold rotationally symmetric tiling with equilateral convex pentagons with \( n = 5 \). In the case of \( n = 6 \), the shape changes to a trapezoid, and the trapezoid can form three-fold or six-fold rotationally symmetric tiling (refer to Figures 19 and 21). Note that the line corresponding to edge \( e \) in the figures is shown as a blue line). In the case of \( n = 7 \), the pentagon is concave and can form a seven-fold rotationally symmetric tiling (refer to Figure 25). Here, let us introduce some \( n \)-fold rotationally symmetric tilings with \( C_n \) symmetry formed of trapezoids based on pentagons that satisfies (3), similar to the equilateral pentagonal case with \( n = 6 \). If the pentagons satisfying (3) with \( n \geq 5 \) have \( \theta = 90^\circ - A \), then “\( A = D \), \( B = C \), \( E = 180^\circ \)” Therefore, they are trapezoids with a line of symmetry. Table 2 presents some of these trapezoids. (For \( n = 5, 6, 8 \), tilings with trapezoidal tiles are drawn. For further details, Figures 18–20. Note that the line corresponding to edge \( e \) in the figures is shown as a blue line.) Because the pentagons with \( n = 3 \) and \( n = 6 \) are opposite to each other, the trapezoid for the case of \( n = 6 \) can form three-fold or six-fold rotational symmetry tilings and mixed tilings, as shown in Figure 21. The trapezoid for the case of \( n = 6 \) corresponds to a rhombus with an acute angle of \( 60^\circ \), and the shape shown in Figure 15(a) corresponds to Figure 22(a). Therefore, similar to the case of a convex pentagon, various patterns can be generated by this trapezoid. At the same time, focusing on the equilateral triangles that appear in the tiling, as shown in Figure 22(b), various patterns can be generated by replacing the trapezoids without using the shape shown in Figure 22(a).

Next, let us introduce some \( n \)-fold rotationally symmetric tilings with \( C_n \) symmetry
Table 3: Example of interior angles of concave pentagons satisfying (3) that can form the 
$n$-fold rotationally symmetric tilings

| $n$ | Value of interior angle (degree) | Edge length of $e$ | Figure number |
|-----|----------------------------------|-------------------|---------------|
| 5   | $72$ | $98$ | $108$ | $55.82$ | $206.18$ | $0.482$ | 23 |
| 6   | $60$ | $98$ | $120$ | $40.63$ | $221.37$ | $0.755$ | 24 |
| 7   | $51.43$ | $106.41$ | $128.57$ | $39.90$ | $213.69$ | $1$ | 25 |
| 8   | $45$ | $112$ | $135$ | $36.64$ | $211.36$ | $1.174$ | 26 |
| 9   | $40$ | $112$ | $140$ | $31.63$ | $216.37$ | $1.271$ | 27 |
| 10  | $36$ | $112$ | $144$ | $27.88$ | $220.12$ | $1.349$ | 28 |
| 11  | $32.73$ | $112$ | $147.27$ | $24.96$ | $223.04$ | $1.412$ | 29 |
| 12  | $30$ | $135$ | $150$ | $28.16$ | $196.84$ | $1.608$ | 30 |
| 13  | $27.69$ | $112$ | $152.31$ | $20.67$ | $227.33$ | $1.509$ | 31 |
| 14  | $25.71$ | $112$ | $154.29$ | $19.05$ | $228.95$ | $1.546$ | 32 |
| 15  | $24$ | $112$ | $156$ | $17.66$ | $230.34$ | $1.578$ | 33 |
| 16  | $22.5$ | $112$ | $157.5$ | $16.47$ | $231.53$ | $1.606$ | 34 |
| 17  | $21.18$ | $112$ | $158.82$ | $15.43$ | $232.57$ | $1.631$ | 35 |
| 18  | $20$ | $112$ | $160$ | $14.51$ | $233.49$ | $1.653$ | 36 |

... | ... | ... | ... | ... | 37 |

formed of concave pentagons that satisfies (3) similar to the equilateral pentagonal case with $n = 7$. There are two cases of concave pentagons that satisfy (3), $B > 180^\circ$ and $E > 180^\circ$. A concave pentagon with $E > 180^\circ$ is geometrically nonexistent if $n < 5$. Due to the presence of parameter $\theta$ in (3), the shapes of concave pentagons that satisfy (3) and can form an $n$-fold rotationally symmetric tiling are not fixed. Each example presented in Table 3 is such a concave pentagon with $E > 180^\circ$. (For $n = 5, 8, 10, 12$, tilings with concave pentagonal tiles are drawn. For further details, Figures 23–28.) Because the concave pentagon for the case of $n = 6$ also corresponds to a rhombus with an acute angle of $60^\circ$, similar to that of a convex pentagon, various patterns can be generated by this concave pentagon (refer to Figure 29).

5 Rotationally symmetric tilings (tiling-like patterns) with an equilateral concave polygonal hole at the center

The rhombus can form various tilings, one of which is a rotationally symmetric tiling-like pattern with a regular polygonal hole at the center [3]. Note that the tiling-like patterns are not considered tilings due to the presence of a gap, but are simply called tilings in this study. According to the properties deduced from [3], pentagons satisfying (3) can form rotationally symmetric tilings with a polygonal hole at the center, as shown in [8] and [9]. Though the rhombus has $D_2$ symmetry, the units comprising pentagons satisfying (3) corresponding to the rhombus have $C_2$ symmetry. Therefore, pentagons satisfying (3) can form rotationally symmetric tilings with an equilateral polygonal hole at the center, provided $n$ in (3) is an even number. The hole formed at the center is an equilateral concave $2n$-gon with $D_2$ symmetry,
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and the tiling with hole has $C_{\frac{n}{2}}$ symmetry.

Let us introduce figures of these tilings. Figure 30 shows a rotationally symmetric tiling with $C_4$ symmetry, with an equilateral concave 16-gonal hole with $D_4$ symmetry at the center, using a convex pentagon with $n = 8$, as presented in Table 1. Figure 31 shows a rotationally symmetric tiling with $C_5$ symmetry, with an equilateral concave 20-gonal hole with $D_5$ symmetry at the center, using a convex pentagon with $n = 10$, as presented in Table 1. Figure 32 shows a rotationally symmetric tiling with $C_8$ symmetry, with an equilateral concave 32-gonal hole with $D_8$ symmetry at the center, using a convex pentagon with $n = 16$, as presented in Table 1. As shown in these figures, the two types of rhombuses generated by pentagons (with and without gray color) are reflections of each other. That is, because these tilings are formed by alternately connecting the two types of rhombuses, they have $C_{\frac{n}{2}}$ symmetry and form an equilateral concave $2n$-gonal hole with $D_{\frac{n}{2}}$ symmetry with iterating concave and convex edges. According to the above properties, if $n$ in (3) is an odd number, the polygonal holes cannot close. As described in Section 4, due to the presence of parameter $\theta$ in (3), for pentagons satisfying the condition (3), their shape is not fixed, and they need not be convex. Figure 33 shows a rotationally symmetric tiling with $C_4$ symmetry, with an equilateral concave 16-gonal hole with $D_4$ symmetry at the center, using a trapezoid with $n = 8$, as presented in Table 2. Figure 34 shows a rotationally symmetric tiling with $C_4$ symmetry, with an equilateral concave 16-gonal hole with $D_4$ symmetry at the center, using a concave pentagon with $n = 8$, as presented in Table 3. Figure 35 shows a rotationally symmetric tiling with $C_5$ symmetry, with an equilateral concave 20-gonal hole with $D_5$ symmetry at the center, using a concave pentagon with $n = 10$, as presented in Table 3. Figure 36 shows a rotationally symmetric tiling with $C_6$ symmetry, with an equilateral concave 24-gonal hole with $D_6$ symmetry at the center, using a concave pentagon with $n = 12$, as presented in Table 3.

Using pentagons satisfying (3) with $n = 4$, similar to those shown in Figures 30–36, a rotationally symmetric tiling with $C_2$ symmetry, with an equilateral concave octagonal hole with $D_2$ symmetry at the center, can be formed. Because the concave octagonal hole corresponds to the shape of the pentagonal pair of Figure 4(a), it can be filled with two pentagons.

Using pentagons satisfying (3) with $n = 6$, corresponding to the rhombus with an acute angle of 60°, similar to those shown in Figures 30–36, a rotationally symmetric tiling with $C_3$ symmetry, with an equilateral concave 12-gonal hole with $D_3$ symmetry at the center, can be formed. Because the concave 12-gonal hole corresponds to the shape shown in Figure 15(a), it can be filled with six pentagons. Furthermore, this pentagon can form a three-fold rotational symmetric tiling as shown in Figure 6. The outline of six pentagons at the center of such a tiling corresponds to an equilateral concave 12-gon shown in Figure 15(a). Therefore, if the six pentagons at the center of such a tiling are removed, it appears as a three-fold rotationally symmetric tiling, with an equilateral concave 12-gonal hole with $D_3$ symmetry at the center. In the case of $n = 6$, as explained in Section 4, because the arrangement of pentagons inside the tilings can be replaced as shown in Figures 16 and 17, it can form different patterns with three-fold or six-fold rotational symmetry, or patterns without rotational symmetry. The above patterns of tilings with an equilateral concave 12-gonal hole at the center by the pentagons with $n = 6$ are one such variation.

The above-mentioned rotationally symmetric tiling with a regular polygonal hole at the center, using rhombuses, is formed by the following method. Because one inner angle (interior angle) of a regular $m$-gon is \(180° - \frac{360°}{m}\), the outer angle of one vertex of a regular $m$-gon is
“180° + \frac{360°}{m},” and that can be achieved by a combination of the acute and obtuse angles of the rhombus. For example, in the case of a regular octagon \((m = 8)\), the interior angle of one vertex is 135°, so the value of “360° − 135° = 225°” will be shared by one obtuse and multiple acute angles. This sharing can be done in rhombuses with an acute angle of \(\frac{360°}{8k}\), where \(k\) is an integer greater than or equal to one. For a rhombus with an acute angle of 45° (when \(k = 1\)), sharing will be “\(2 \times 45° + 135° = 225°\)”; for a rhombus with an acute angle of 22.5° (when \(k = 2\)), sharing will be “\(3 \times 22.5° + 157.5° = 225°\)” and so on. In fact, a rhombus with an acute angle of \(\frac{360°}{m+k}\) can form a rotationally symmetric tiling with a regular \(m\)-gonal hole at the center \([3]\). Therefore, a pentagon satisfying (3) with \(n\) \(D\) whether convex, concave, or trapezoidal. For trapezoids, presented in Table 2, it is clear that tiling is formed by multiple different cases of trapezoids. In the case of trapezoids, presented in Table 2, it is clear that tiling is formed by multiple different cases of trapezoids. In these examples, pentagons satisfying (3) are used, but we note that pentagons that satisfy (2), in which \(\alpha\) has same value and \(\theta\) parameter \(\theta\) is selected, and it is possible provided \(n\) is an even number, as described above.

For example, if the pentagons can form rotationally symmetric tilings with \(C_4\) symmetry, with an equilateral concave 16-gonal hole with \(D_4\) symmetry at the center, they correspond to pentagons satisfying \([3]\) whose \(n\) is a multiple of eight. Figure 30 is a case of \(k = 1\) and \(n = 8\). Figure 37 is a case of \(k = 2\), i.e., a rotationally symmetric tiling with \(C_4\) symmetry, with an equilateral concave 16-gonal hole with \(D_4\) symmetry at the center, by a convex pentagon with \(n = 16\), as presented in Table 1. Similarly, the concave pentagon with \(n = 12\), as presented in Table 3 can form a rotationally symmetric tiling with \(C_3\) symmetry, with an equilateral concave 12-gonal hole with \(D_3\) symmetry at the center, as shown in Figure 38.

6 Tilings with multiple pentagons of different shapes

Rhombuses can form edge-to-edge tilings by using different shapes of rhombuses with differing interior angles when the lengths of edges are same. Tilings using two or more types of pentagons that satisfy (2), in which \(\theta\) has same value and \(\alpha\) has different values, correspond to the tilings with two or more types of rhombuses. That is, the tiling is not monohedral. Figure 39 is an example of tiling using \(n = 3, 4, 8\), as presented in Table 1 and Figure 39(b) is an example of tiling using \(n = 8, 10\), as presented in Table 3. In these examples, pentagons satisfying (3) are used, but we note that tilings can be formed by pentagons satisfying (2), whose interior angle \(A\) is not \(\frac{360°}{n}\). In the case of trapezoids, presented in Table 2 it is clear that tiling is formed by multiple different trapezoids.

Furthermore, pentagons satisfying (2) with the same value of \(\theta\) can be used in a tiling, whether convex, concave, or trapezoidal. For \(\theta = 45°\), Figure 40 shows an example of tiling by convex pentagons satisfying (2) with \(\alpha = 54°\), trapezoids (pentagons) satisfying (2) with \(\alpha = 67.5°\), and concave pentagons satisfying (2) with \(\alpha = 75°\).

The pentagonal tilings in Figures 39 and 40 satisfying “\(B + D + E = 360°\), 2\(A + 2C = 360°\)” are rhombic tilings that are formed from rhombic belts made by translation in the same direction. By adjusting the combination of rhombuses used, tilings other than the combination of above belts can be formed. (They correspond to pentagonal tilings that admit the vertex concentrations “\(B + D + E = 360°\), 2\(A + 2C = 360°\)” and also vertex concentrations other than “\(B + D + E = 360°\), 2\(A + 2C = 360°\)”). For example, because squares and rhombuses with an acute angle of 45° can form an eight-fold rotationally symmetric tiling \([3]\), a pentagonal tiling, as shown in Figure 41, corresponding to it can be formed by convex pentagons with \(n = 4, 8\), as presented in Table 1. In addition, because rhombuses with acute angles of 72° and 36° can form a five-fold rotationally symmetric tiling, a pentagonal tiling \([3]\).
as shown in Figure 42, corresponding to it can be formed by convex pentagons satisfying (3) with \( n = 5 \) and \( \theta = 45^\circ \), and concave pentagons satisfying (3) with \( n = 10 \) and \( \theta = 45^\circ \). Note that the number of pentagons satisfying (3) included in the corresponding rhombuses can be changed. (In Figure 41, one rhombus includes 32 pentagons, and in Figure 42, one rhombus includes eight pentagons. It is possible to have a case where one rhombus includes two pentagons or \( 8 \times u^2 \) pentagons where \( u = 1, 2, 3, \ldots \).) Similarly, tilings with three or more types of rhombuses can be converted into tilings with pentagons.

7 Conclusions

In [8] and [9], we introduced convex pentagonal tiles, belonging to the Type 1 family, that can generate countless rotationally symmetric tilings. In this study, we have shown that convex pentagonal tiles belonging to the Type 2 family can generate countless rotational symmetric tilings. In addition, because the pentagons have two parameters, the study discussed that the tilings can be generated by shapes other than convex.

Because the properties of pentagons dealt with in this study correspond to those of rhombuses, it also explained the correspondence between pentagons and various rhombic tilings. Not all rhombic tilings (including tilings with holes as introduced in Section 5) can be converted into pentagonal tilings by the method discussed in this study. But, various knowledge of rhombic tilings can be used to generate various pentagonal tilings.

Livio Zucca presented interesting tilings using equilateral pentagons in [14]. However, that study does not consider pentagons with four equal-length edges and the relationship between pentagons and rhombuses.

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Appendix

We organized the properties of the shape of pentagons that satisfy (3) depending on the values of $n$ and $\theta$. If the pentagons that satisfy (3) exist geometrically, then $n > 2$, because $C > 0^\circ$. It is $0^\circ < \theta < 180^\circ$ because the edge $e$ of the pentagon that satisfies (3) exists and does not intersect the edge $b$. As mentioned in Section 4, the pentagons that satisfy (3) with $n = 3$ and $n = 6$ correspond to rhombuses with an acute angle of $60^\circ$, and these pentagons are opposite to each other. The shapes of the pentagons that satisfy (3) change depending on the values of $n$ and $\theta$ as follows:

Case where the pentagons that satisfy (3) admit $n = 4$

- $0^\circ < \theta < 90^\circ$: Convex pentagons
- $\theta = 90^\circ$: Parallelograms ($B = 180^\circ$)
- $90^\circ < \theta < 180^\circ$: Concave pentagons ($B > 180^\circ$)

Cases where the pentagons that satisfy (3) admit $n \geq 5$

- $0^\circ < \theta < 90^\circ - \frac{360^\circ}{n}$: Concave pentagons ($E > 180^\circ$)
- $\theta = 90^\circ - \frac{360^\circ}{n}$: Trapezoids ($E = 180^\circ$)
- $90^\circ - \frac{360^\circ}{n} < \theta < 90^\circ$: Convex pentagons
- $\theta = 90^\circ$: Parallelograms ($B = 180^\circ$)
- $90^\circ < \theta < 180^\circ$: Concave pentagons ($B > 180^\circ$)
Cases where the pentagons that satisfy (3) become the polygons with $a = b = c = d = e$

- Case where $n = 4$ and $\theta \approx 41.41^\circ$ (Convex pentagon with $E \approx 114.30^\circ$)
- Case where $n = 5$ and $\theta \approx 37.95^\circ$ (Convex pentagon with $E \approx 149.76^\circ$)
- Case where $n = 6$ and $\theta = 30^\circ$ (Trapezoid of $E = 180^\circ$)
- Case where $n = 7$ and $\theta \approx 16.41^\circ$ (Concave pentagon with $E \approx 213.69^\circ$)

If the pentagons that satisfy (3) have $\theta = 90^\circ$ (i.e., $B = 180^\circ$), then they are parallelograms (the case of $n = 4$ is a rectangle). Such parallelograms correspond to half of the rhombus corresponding to the basic unit of Figures 4(a) or 4(b).

If the pentagons that satisfy (3) have $E = 180^\circ$, then “$A = D$, $B = C$.” Thus, they are trapezoids with a line of symmetry and $\theta = 90^\circ - \frac{360^\circ}{n}$ holds. It is when $n \geq 5$ that the pentagons that satisfy (3) become trapezoids, because $\theta > 0^\circ$.

If the pentagons that satisfy (3) have $a = b = c = d = e = 1$, then $1 = 2\sqrt{1 - \sin(2\alpha)\cos\theta}$ where $\alpha = 90^\circ - \frac{180^\circ}{n}$ (refer to Section 2). In this case, $n \leq 7$ for $\theta$ to exist.

Finally, example figures using a concave pentagon with $B > 180^\circ$ are shown. Figure 43 shows an eight-fold rotationally symmetric tiling by a concave pentagon satisfying (3) with $n = 8$ and $B = 224^\circ$. Figure 44 shows a rotationally symmetric tiling with $C_4$ symmetry, with an equilateral concave 16-gonal hole with $D_4$ symmetry at the center, using a convex pentagon satisfying (3) with $n = 8$ and $B = 224^\circ$. 
Figure 6: Three-fold rotationally symmetric tiling by a convex pentagon of $n = 3$ in Table 1. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 7: Four-fold rotationally symmetric tiling by a convex pentagon of $n = 4$ in Table 1. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 8: Six-fold rotationally symmetric tiling by a convex pentagon of \( n = 6 \) in Table [1]. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 9: Seven-fold rotationally symmetric tiling by a convex pentagon of $n = 7$ in Table 1. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 10: Eight-fold rotationally symmetric tiling by a convex pentagon of \( n = 8 \) in Table 1 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 11: Nine-fold rotationally symmetric tiling by a convex pentagon of $n = 9$ in Table I (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 12: 10-fold rotationally symmetric tiling by a convex pentagon of \( n = 10 \) in Table I. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 13: 16-fold rotationally symmetric tiling by a convex pentagon of $n = 16$ in Table 1. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 14: Examples of tilings with three-fold and six-fold rotational symmetry by a pentagon that corresponds to rhombus with an acute angle of 60°

Figure. 15: Combinations of vertices A and C of convex pentagons that correspond to rhombuses with an acute angle of 60°
Figure 16: Examples of tilings by a pentagon that corresponds to rhombus with an acute angle of 60°, Part 1
Figure 17: Examples of tilings by a pentagon that corresponds to rhombus with an acute angle of 60°, Part 2
Figure. 18: Five-fold rotationally symmetric tiling by a trapezoid of $n = 5$ in Table 2 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 19: Six-fold rotationally symmetric tiling by a trapezoid of \( n = 6 \) in Table 2 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 20: Eight-fold rotationally symmetric tiling by a trapezoid of $n = 8$ in Table 2 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 21: Examples of tilings with three-fold and six-fold rotational symmetry by a trapezoid that corresponds to rhombus with an acute angle of 60°

Figure 22: Examples of tilings by a trapezoid that corresponds to rhombus with an acute angle of 60°
Figure. 23: Five-fold rotationally symmetric tiling by a concave pentagon of $n = 5$ in Table 3 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 24: Six-fold rotationally symmetric tiling by a concave pentagon of \( n = 6 \) in Table 3. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well.)
Figure 25: Seven-fold rotationally symmetric tiling by a concave pentagon of $n = 7$ in Table 3 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 26: Eight-fold rotationally symmetric tiling by a concave pentagon of $n = 8$ in Table 3 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 27: 10-fold rotationally symmetric tiling by a concave pentagon of \( n = 10 \) in Table 3 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 28: 12-fold rotationally symmetric tiling by a concave pentagon of $n = 12$ in Table 3 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 29: Examples of tilings by a concave pentagon that corresponds to rhombus with an acute angle of $60^\circ$
Figure 30: Rotationally symmetric tiling with $C_4$ symmetry, with an equilateral concave 16-gonal hole with $D_4$ symmetry at the center, by a convex pentagon of $n = 8$ in Table 1 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 31: Rotationally symmetric tiling with $C_5$ symmetry, with an equilateral concave 20-gonal hole with $D_5$ symmetry at the center, by a convex pentagon of $n = 10$ in Table 1 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 32: Rotationally symmetric tiling with $C_8$ symmetry, with an equilateral concave 32-gonal hole with $D_8$ symmetry at the center, by a convex pentagon of $n = 16$ in Table 1. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 33: Rotationally symmetric tiling with $C_4$ symmetry, with an equilateral concave 16-gonal hole with $D_4$ symmetry at the center, by a trapezoid of $n = 8$ in Table 2 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 34: Rotationally symmetric tiling with $C_4$ symmetry, with an equilateral concave 16-gonal hole with $D_4$ symmetry at the center, by a concave pentagon of $n = 8$ in Table 3 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure. 35: Rotationally symmetric tiling with $C_5$ symmetry, with an equilateral concave 20-gonal hole with $D_5$ symmetry at the center, by a concave pentagon of $n = 10$ in Table 3. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well.)
Figure 36: Rotationally symmetric tiling with $C_6$ symmetry, with an equilateral concave 24-gonal hole with $D_6$ symmetry at the center, by a concave pentagon of $n = 12$ in Table 3. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 37: Rotationally symmetric tiling with $C_4$ symmetry, with an equilateral concave 16-gonal hole with $D_4$ symmetry at the center, by a convex pentagon of $n = 16$ in Table 1 (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 38: Rotationally symmetric tiling with $C_3$ symmetry, with an equilateral concave 12-gonal hole with $D_3$ symmetry at the center, by a concave pentagon of $n = 12$ in Table 3. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 39: Example of tiling using convex pentagons with $n = 3, 4, 8$ in Table 1 and example of tiling by concave pentagons with $n = 8, 10$ in Table 3.
Figure 40: Example of tiling by convex pentagons satisfying (2) with $\alpha = 54^\circ$ and $\theta = 45^\circ$, trapezoids satisfying (2) with $\alpha = 67.5^\circ$ and $\theta = 45^\circ$, and concave pentagons satisfying (2) with $\alpha = 75^\circ$ and $\theta = 45^\circ$. 

< Case of $n = 5$ >
A = 72°,
B = 135°,
C = 108°,
D ≈ 82.57°,
E ≈ 142.43°,
a = b = c = d.

< Case of $n = 8$ >
A = 45°,
B = 135°,
C = 135°,
D = 45°,
E = 180°,
a = b = c = d.
Figure. 41: Eight-fold rotationally symmetric tiling by convex pentagons with $n = 4, 8$ in Table I
(The figure is solely a depiction of the area around the rotationally symmetric structure, and the tiling can be spread in all directions)
Figure. 42: Five-fold rotationally symmetric tiling by convex pentagons satisfying (3) with $n = 5$ and $\theta = 45^\circ$, and concave pentagons satisfying (3) with $n = 10$ and $\theta = 45^\circ$ (The figure is solely a depiction of the area around the rotationally symmetric structure, and the tiling can be spread in all directions)
Figure. 43: Eight-fold rotationally symmetric tiling by a concave pentagon satisfying \( n = 8 \) and \( B = 224^\circ \) (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)
Figure 44: Rotationally symmetric tiling with $C_4$ symmetry, with an equilateral concave 16-gonal hole with $D_4$ symmetry at the center, by a concave pentagon satisfying (3) with $n = 8$ and $B = 224^\circ$. (The figure is solely a depiction of the area around the rotationally symmetric center, and the tiling can be spread in all directions as well)