Lie-Isoclinism of Pairs of Leibniz Algebras

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Abstract
The aim of this paper is to consider the relation between Lie-isoclinism and isomorphism of two pairs of Leibniz algebras. We show that, unlike the absolute case for finite dimensional Lie algebras, these concepts are not identical, even if the pairs of Leibniz algebras are Lie-stem. Moreover, throughout the paper, we provide some conditions under which Lie-isoclinism and isomorphism of Lie-stem Leibniz algebras are equal. In order to get this equality, the concept of factor set is studied as well.

Keywords
Pair of Leibniz algebras · Lie-isoclinic pairs · Lie-stem pair · Factor set

Mathematics Subject Classification 17A32 · 18B99

1 Introduction

The isoclinism in group theory, that is an equivalence relation on groups which generalizes isomorphism, was first introduced by Hall [9] for the purpose of classifying finite p-groups of small order. This concept was studied by several authors, including Tappe [18] and Weichsel [19]. In 1994, Moneyhun [17] extended this concept to Lie algebras that produces a partition on the class of all Lie algebras into equivalence classes. By this equivalence relation, she showed that the isoclinic family of Lie algebras contains at least one stem Lie algebra. Also, she proved that the concepts of isoclinism and isomorphism between Lie algebras of the same finite dimension are identical. The iso-
clinism of a pair of Lie algebras was studied by Moghaddam et al. [15] in 2009. They generalized the first result of Moneyhun for the pair of Lie algebras. In addition, it showed that two pairs of finite dimensional stem Lie algebras are isoclinic if and only if they are isomorphic. Furthermore, it has been investigated, in [16], that the concept of isoclinism and isomorphism between the central extensions of finite dimensional Lie algebras, under some conditions, are identical.

In the last decades, a prominent research line consists in the extension of properties from Lie algebras to Leibniz algebras, which are non-anti-commutative versions of Lie algebras [13,14]. In more detail, a vector space $q$ equipped with a bilinear map $[-, -] : q \times q \to q$ is called Leibniz algebra if satisfying the Leibniz identity:

$$[[x, y, z]] = [[[x, y], z] - [[x, z], y], x, y, z \in q.$$ 

The investigations on Leibniz algebras theory show that some results of the theory of Lie algebras can be extended to Leibniz one. It is of interest to know whether the above-mentioned works, in particular, the equivalence between isoclinism and isomorphism in the presence of finite dimension, [15,17], are still true for the Leibniz algebras. So the main goal of this paper is to answer this question, for that we focus on the relative framework, that is the context relative to the Liezation functor as we explain below.

In the papers, [1,5] was initiated a study of properties of Leibniz algebras relative to the Liezation functor $\text{Lie} : \text{Leib} \longrightarrow \text{Lie}$, which assigns to a Leibniz algebra $q$ the Lie algebra $q_{\text{Lie}} = q/\langle\{[x, x] : x \in q\}\rangle$, as opposed to the absolute ones. In fact, the absolute case corresponds to the abelianization functor $\text{ab} : \text{Leib} \longrightarrow \text{Vect}$ which assigns to a Leibniz algebra $q$ the abelian Leibniz algebra $q/[q, q]$. The origin of this point of view comes from the general theory of central extensions relative to a chosen subcategory of a base category introduced in [10] and considered in the context of semi-abelian categories relative to a Birkhoff subcategory in [11].

Continuing with this study, in the first section, we introduce the concept of Lie-isoclinism for pairs of Leibniz algebras that is an equivalence relation. Similar to the pair of Lie algebras, [15], we prove that Lie-isoclinic family of Leibniz algebras contains at least one Lie-stem Leibniz algebra which is the smallest dimension and give some results about this concept, as well.

In Sect. 3, we use a function, named factor set, which is introduced by non-abelian extension of Leibniz algebras. Note that, this function (without indicating on factor set) has given by Liu et al. [12] to classify non-abelian extensions of Leibniz algebras by the second non-abelian cohomology of Leibniz algebras.

Finally, in Sect. 4, we show that two pairs of the same finite dimensional Lie-isoclinic (Lie-stem) Leibniz algebras are not isomorphic and indicate some relevant counterexamples. Moreover, by using the concept of factor set, we present as our main result some conditions that Lie-isoclinism and isomorphism, for finite dimensional Lie-stem Leibniz algebras, are equal.

Throughout, all Leibniz algebras are considered over a fixed field $\mathbb{K}$, unless otherwise stated. Our basic assumptions are the following.
Definition 1 Let \( m \) be a two-sided ideal of the Leibniz algebra \( q \), then \((m, q)\) is said to be a pair of Leibniz algebras.

Definition 2 The Lie-commutator and Lie-center of the pair of \((m, q)\) are both two-sided ideals of \( q \) contained in \( m \)

\[
[m, q]_{\text{Lie}} = \{([m, q] + [q, m] \mid m \in m, q \in q)\}
\]

\[
Z_{\text{Lie}}(m, q) = \{m \in m \mid [m, q] + [q, m] = 0 \text{ for all } q \in q\} = Z_{\text{Lie}}(q) \cap m.
\]

Remark 1 When \( m = q \), then \( Z_{\text{Lie}}(q, q) \) coincides with the Lie-center of \( q \) given in [5].

2 Lie-Isoclinism of Pairs of Leibniz Algebras

We begin with the following definition which is the corresponding relative version of the isoclinism of Lie algebras given in [15] (absolute case for Lie algebras).

Definition 3 The pairs of Leibniz algebras \((m_i, q_i), i = 1, 2\), are said to be Lie-isoclinic if there exist isomorphisms \( \alpha : q_1 / Z_{\text{Lie}}(m_1, q_1) \rightarrow q_2 / Z_{\text{Lie}}(m_2, q_2) \) with \( \alpha(m_1 / Z_{\text{Lie}}(m_1, q_1)) = m_2 / Z_{\text{Lie}}(m_2, q_2) \) and \( \beta : [m_1, q_1]_{\text{Lie}} \rightarrow [m_2, q_2]_{\text{Lie}} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
Z_{\text{Lie}}(m_1, q_1) & \times & Z_{\text{Lie}}(m_2, q_2) \\
\alpha \times \alpha & \downarrow & \beta \\
Z_{\text{Lie}}(q_1, m_1) & \rightarrow & [m_1, q_1]_{\text{Lie}} \\
C_1 & \Rightarrow & C_2 \\
\end{array}
\]

where \( C_i (\tilde{m}_i, \tilde{q}_i) = [m_i, q_i] + [q_i, m_i] \), for all \( \tilde{m}_i \in Z_{\text{Lie}}(q_i, m_i) \) and \( \tilde{q}_i \in Z_{\text{Lie}}(m_i, q_i), i = 1, 2 \). In this case, the pair \((\alpha, \beta)\) is called a Lie-isoclinism between \((m_1, q_1)\) and \((m_2, q_2)\) and we write \((m_1, q_1) \sim (m_2, q_2)\).

The following Proposition provides an equivalent condition for Lie-isoclinism between two pairs of Leibniz algebras.

Proposition 1 Let \( \pi_i : q_i \rightarrow Z_{\text{Lie}}(m_i, q_i), i = 1, 2 \), be the canonical surjective homomorphisms and \( \alpha : q_1 / Z_{\text{Lie}}(m_1, q_1) \rightarrow q_2 / Z_{\text{Lie}}(m_2, q_2) \) with \( \alpha(m_1 / Z_{\text{Lie}}(m_1, q_1)) = m_2 / Z_{\text{Lie}}(m_2, q_2) \) and \( \beta : [m_1, q_1]_{\text{Lie}} \rightarrow [m_2, q_2]_{\text{Lie}} \) be isomorphisms. The pair \((\alpha, \beta)\) is a Lie-isoclinism between \((m_1, q_1)\) and \((m_2, q_2)\), if and only if \( \beta([m_1, q_1] + [q_1, m_1]) = [m_2, q_2] + [q_2, m_2] \), where \( m_2 \in m_2, q_2 \in q_2, \alpha(\pi_1(q_1)) = \pi_2(q_2) \) and \( \alpha(\pi_1(m_1)) = \pi_2(m_2) \).

Proof Direct checking. \(\square\)

Remark 2 When \( m_i = q_i, i = 1, 2 \), then we recover the notion of Lie-isoclinism of Leibniz algebras given in [1].
An immediate result from Definition 3 is the following

**Corollary 1** Let the pairs of Leibniz algebras \((m, q)\) and \((n, p)\) be Lie-isoclinic. Then \(m\) and \(n\) are Lie-isoclinic.

The following Lemma yields information about the Lie-isoclinism between a pair of Leibniz algebras and its quotient pair by a two-sided ideal.

**Lemma 1** Let \((m, q)\) be a pair of Leibniz algebras and \(n\) a two-sided ideal of \(q\) contained in \(m\). Then, \((\frac{m}{n}, \frac{q}{n}) \sim (\frac{m}{m \cap [m, q]_{\text{Lie}}}, \frac{q}{m \cap [m, q]_{\text{Lie}}})\). In particular, \(n \cap [m, q]_{\text{Lie}} = 0\) if and only if \((\frac{m}{n}, \frac{q}{n}) \sim (m, q)\).

**Proof** We set \(\overline{q} = \frac{q}{n}, \overline{m} = \frac{m}{n}, \tilde{q} = \frac{q}{m \cap [m, q]_{\text{Lie}}}\) and \(\tilde{m} = \frac{m}{m \cap [m, q]_{\text{Lie}}}\).

It is easy to check that the map \(\gamma : q \rightarrow \overline{q}\) given by \(\gamma(q) = \gamma(q + (n \cap [m, q]_{\text{Lie}})) = \overline{q} = q + n\), is a surjective homomorphism such that \(\gamma(Z_{\text{Lie}}(\overline{m}, \overline{q})) = Z_{\text{Lie}}(\overline{m}, \overline{q})\), then it induces a surjective homomorphism \(\alpha : Z_{\text{Lie}}(\overline{m}, \overline{q}) \rightarrow Z_{\text{Lie}}(\tilde{m}, \tilde{q})\), given by \(\alpha(\overline{q} + Z_{\text{Lie}}(\overline{m}, \overline{q})) = \overline{q} + Z_{\text{Lie}}(\tilde{m}, \tilde{q})\). Moreover, \(\alpha\) is injective, because \(\alpha(\overline{q} + Z_{\text{Lie}}(\tilde{m}, \tilde{q})) = 0\), implies that \(\overline{q} \in Z_{\text{Lie}}(\overline{m}, \overline{q})\), that is \(\overline{q} = \gamma(\overline{q})\) with \(\overline{q} \in Z_{\text{Lie}}(\overline{m}, \overline{q})\). Consequently, \(\alpha\) is an isomorphism.

On the other hand, the restriction of \(\gamma\) provides the surjective homomorphism \(\beta : [\tilde{m}, \tilde{q}]_{\text{Lie}} \rightarrow [\overline{m}, \overline{q}]_{\text{Lie}},\) given by \(\beta([\tilde{m}, \tilde{q}] + [\tilde{q}, \tilde{m}]) = [\overline{m}, \overline{q}] + [\overline{q}, \overline{m}]\). Moreover, it is easy to check that \(\beta\) is injective.

Now, the commutativity of diagram (1) is obvious.

For the second statement, if \(n \cap [m, q]_{\text{Lie}} = 0\) then \(\tilde{m} \cong m\) and \(\tilde{q} \cong q\) and so \((\frac{m}{n}, \frac{q}{n}) \sim (m, q)\). Conversely, the isomorphism \(\beta : [m, q]_{\text{Lie}} \rightarrow [\overline{m}, \overline{q}]_{\text{Lie}}\) actually is induced by the canonical projection \(\pi : q \rightarrow \frac{q}{n}\) and \(\text{Ker}(\beta) = n \cap [m, q]_{\text{Lie}}\) \(\square\)

**Definition 4** The pair of Leibniz algebras \((m, q)\) is said to be a Lie-stem pair of Leibniz algebras, when \(Z_{\text{Lie}}(m, q) \subseteq [m, q]_{\text{Lie}}\).

**Example 1** An example of Lie-stem pair is provided by the three-dimensional Leibniz algebra \(q\) with basis \(\{a_1, a_2, a_3\}\) and bracket operation given by \([a_1, a_3] = a_1, [a_2, a_3] = a_2\) (class 3 \(a\) in [4]), and the two-sided ideal \(m = \text{span}\{a_1\}\). Obviously, \(0 = Z_{\text{Lie}}(m, q) \subseteq [m, q]_{\text{Lie}} = \text{span}\{a_1\}\).

**Proposition 2** The pair of Leibniz algebras \((m, q)\) is a Lie-stem pair if and only if the unique two-sided ideal \(s\) of \(q\), such that \(s \subseteq m\) and \(s \cap [m, q]_{\text{Lie}} = 0\), is the trivial one.

**Proof** Let \(s\) be a two-sided ideal of \(q\) such that \(s \subseteq m\) and \(s \cap [m, q]_{\text{Lie}} = 0\), then \(s \subseteq Z_{\text{Lie}}(m, q)\) and so \(s = s \cap Z_{\text{Lie}}(m, q) \subseteq s \cap [m, q]_{\text{Lie}} = 0\). Conversely, assume that on the contrary \(Z_{\text{Lie}}(m, q) \nsubseteq [m, q]_{\text{Lie}}\), then there exists \(z \in Z_{\text{Lie}}(m, q) \setminus [m, q]_{\text{Lie}}\), \(z \neq 0\). Put \(s\) the two-sided ideal of \(q\) spanned by \(z\), which is contained in \(m\). So from the assumption, we have \(s \cap [m, q]_{\text{Lie}} = 0\). Hence \(s = 0\) and a contradiction follows. \(\square\)

Plainly, Lie-isoclinism between the pairs of Leibniz algebras is an equivalence relation, (see for instance [11]). By the Lie-isoclinic family, we mean equivalent classes that contains the class of all pairs of Leibniz algebras. The following Theorem ensures the existence of a Lie-stem pair in the Lie-isoclinic family of pairs of Leibniz algebras.
Theorem 1 Every Lie-isoclinic family \( C \) of pairs of Leibniz algebras contains at least one Lie-stem pair of Leibniz algebras.

Proof Let \((m, q)\) be an arbitrary pair of Leibniz algebras in \( C \) and \( A = \{s \mid s \subseteq q, s \subseteq m, s \cap [m, q]_{\text{Lie}} = 0\} \). The set \( A \) is non-empty because it contains at least the zero ideal. We define a partial ordering on \( A \) by inclusion and evidently, by Zorn’s lemma, we can find a maximal two-sided ideal \( s \) in \( A \). Since \( s \cap [m, q]_{\text{Lie}} = 0 \), it follows from Lemma 1 that \((m/s, q/s) \in C\). Now, suppose that \( h/s \) is a two-sided ideal of \( q/s \) contained in \( m/s \) such that \( h/s \cap [m/s, q/s]_{\text{Lie}} = 0 \). Note that such a \( h/s \) always exists; for instance \( h = s \). Then we have \( h \cap [m, q]_{\text{Lie}} \subseteq s \cap [m, q]_{\text{Lie}} = 0 \) and so \( h \in A \). Moreover, \( s \subseteq h \), so by the maximality of \( s \), it follows that \( h = s \) and then \( h/s = 0 \). Therefore, by virtue of Proposition 2, \((m/s, q/s)\) is a Lie-stem pair of Leibniz algebras, as required.

One of the main results in this paper is the following

Theorem 2 Let \( C \) be a Lie-isoclinic family of finite dimensional pairs of Leibniz algebras and \((n, p) \in C\). Then \((n, p)\) is a Lie-stem pair if and only if \( \dim(p) = \min(\dim(q) | (m, q) \in C) \).

Proof Let \((m, q)\) and \((n, p)\) be arbitrary pairs in \( C \) and assume that \((n, p)\) is a Lie-stem pair such that \( p \) is finite dimensional. Then we have

\[
\frac{[m, q]_{\text{Lie}}}{[m, q]_{\text{Lie}} \cap Z_{\text{Lie}}(m, q)} \cong \frac{[m, q]_{\text{Lie}} + Z_{\text{Lie}}(m, q)}{Z_{\text{Lie}}(m, q)} \\
\cong \frac{[m, q]_{\text{Lie}}}{Q_{\text{Lie}}(m, q)} + \frac{Q_{\text{Lie}}(m, q)}{Z_{\text{Lie}}(m, q)} \\
\cong \frac{[n, p]_{\text{Lie}}}{Z_{\text{Lie}}(n, p)},
\]

and \([m, q]_{\text{Lie}} \cong [n, p]_{\text{Lie}}\). Therefore, \( \dim(Z_{\text{Lie}}(n, p)) = \dim([m, q]_{\text{Lie}} \cap Z_{\text{Lie}}(m, q)) \leq \dim(Z_{\text{Lie}}(m, q)) \). On the other hand, \( \frac{p}{Z_{\text{Lie}}(n, p) \cap Z_{\text{Lie}}(m, q)} \cong \frac{q}{Z_{\text{Lie}}(m, q)} \). Therefore, \( \dim(p) \leq \dim(q) \).

Conversely, let \((n, p)\) be in the family \( C \) such that \( p \) has the minimum dimension. Owing to Theorem 1, there is a two-sided ideal \( t \) of \( p \) contained in \( Z_{\text{Lie}}(n, p) \) such that \((n, p) \sim (\frac{n}{t}, \frac{p}{t})\) and \( Z_{\text{Lie}}(n, p) = ([n, p]_{\text{Lie}} \cap Z_{\text{Lie}}(n, p)) \oplus t \). But \( p \) has minimum dimension, which implies that \( t = 0 \), therefore \( Z_{\text{Lie}}(n, p) \subseteq [n, p]_{\text{Lie}} \) and this completes the proof.

The above Theorem provides the following interesting consequence which will be used in the sequel and our main result in the last section.

Corollary 2 If \((m, q)\) and \((n, p)\) are two Lie-isoclinic Lie-stem pairs of Leibniz algebras then \( Z_{\text{Lie}}(m, q) \cong Z_{\text{Lie}}(n, p) \).
Proof Let \((m, q)\) and \((n, p)\) be two Lie-isoclinic pairs of Leibniz algebras. In view of proof of Theorem 2 and isomorphism \(\beta : [m, q]_{\text{Lie}} \rightarrow [n, p]_{\text{Lie}}\), we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & Z_{\text{Lie}}(m, q) & \longrightarrow & [m, q]_{\text{Lie}} & \longrightarrow & 0 \\
& & \downarrow{\beta_1} & & \pi_1 & & \\
0 & \longrightarrow & Z_{\text{Lie}}(n, p) & \longrightarrow & [n, p]_{\text{Lie}} & \longrightarrow & 0,
\end{array}
\]

where \(\beta(Z_{\text{Lie}}(m, q)) \subseteq Z_{\text{Lie}}(n, p)\) since for all \(x \in [m, q]_{\text{Lie}}\), \(\alpha(x + Z_{\text{Lie}}(m, q)) = \beta(x) + Z_{\text{Lie}}(n, p)\). Hence, for \(x \in Z_{\text{Lie}}(m, q)\), \(0 = \alpha(\pi_1(x)) = \beta(x) + Z_{\text{Lie}}(n, p)\), so \(\beta(x) \in Z_{\text{Lie}}(n, p)\). Now, Snake Lemma [2] yields \(\beta_1\) is a surjective homomorphism and so \(\beta(Z_{\text{Lie}}(m, q)) = Z_{\text{Lie}}(n, p)\). Moreover, the left-hand square is a pull-back diagram, then \(\beta_1\) is a monomorphism. Therefore, \(Z_{\text{Lie}}(m, q) \cong Z_{\text{Lie}}(n, p)\). \(\square\)

3 Factor Sets of a Pair of Leibniz Algebras

Chevalley and Eilenberg in [6] defined the factor sets for Lie algebras and now we recall factor sets for Leibniz algebras from [12] and we analyze the interplay with the Lieization functor for a pair of Leibniz algebras.

Let \(0 \rightarrow m \overset{\subseteq}{\rightarrow} q \overset{\pi}{\longrightarrow} q^* \rightarrow 0\) be a non-abelian extension of Leibniz algebras [12, Definition 2.5]. We choose a splitting \(\tau : q^* \rightarrow q\) that is a linear map such that \(\pi \circ \tau = \text{Id}_{q^*}\). For each \(x \in q^*\), we have two linear maps \(L_x, R_x : m \rightarrow m\) given by \(L_x(m) = [m, \tau(x)]\) and \(R_x(m) = [\tau(x), m]\). Associated with any pair of elements \(x, y \in q^*\), there is an element \(f(x, y) \in m\) such that

\[
f(x, y) = [\tau(x), \tau(y)] - \tau([x, y]).
\]

The linear map \(f\) is called the factor set corresponding to the function \(\tau\).

For any \(x, y, z \in q^*\), the following identities concerning factor sets hold:

\[
[[\tau(x), \tau(y)], \tau(z)] = [f(x, y), \tau(z)] + [\tau([x, y]), \tau(z)]
\]
\[
= L_z(f(x, y)) + f([x, y], z) + \tau([x, y], z)],
\]
\[
[[\tau(x), \tau(z)], \tau(y)] = [f(x, z), \tau(y)] + [\tau([x, z]), \tau(y)]
\]
\[
= L_y(f(x, z)) + f([x, z], y) + \tau([x, z], y)],
\]
\[
[\tau(x), [\tau(y), \tau(z)]] = [\tau(x), f(y, z)] + [\tau(x), \tau([y, z])]
\]
\[
= R_x(f(y, z)) + f(x, [y, z]) + \tau([x, [y, z]]).
\]

From the above identities, the following equation is immediately derived:

\[
f([x, y], z) - f([x, z], y) - f(x, [y, z]) + L_z(f(x, y)) - L_y(f(x, z)) - R_x(f(y, z)) = 0
\]

\(\square\) Springer
Given a splitting $\tau$, we define on the $\mathbb{K}$-vector space $m \oplus q^*$ the bracket operation

$$[(m_1, x_1), (m_2, x_2)] = [(m_1, m_2)] + R_{x_1}(m_2) + L_{x_2}(m_1) + f(x_1, x_2), [x_1, x_2]].$$

A routine computation having in mind Eq. (2) shows that $(m \oplus q^*, [-, -])$ is a Leibniz algebra, which will be denoted $m \times_f q^*$.

Note that $\tau$ is a homomorphism if and only if $f(x, y) = 0$, for all $x, y \in q^*$, that is $f$ measures the deficiency of $\tau$ to be a homomorphism. If the exact sequence $0 \rightarrow m \xrightarrow{i} q \xrightarrow{\psi} q^* \rightarrow 0$ splits by a homomorphism $\tau : q^* \rightarrow q$, then $m$ is endowed with an action from $q^*$ given by $[x, m] = [\tau(x), i(m)]q$, $[m, x] = [i(m), \tau(x)]_q$, $m \in m, x \in q^*$. Hence, the semi-direct product $m \rtimes q^*$ can be constructed, which gives rise to the split extension $0 \rightarrow m \rightarrow m \rtimes q^* \xrightarrow{pr} q^* \rightarrow 0$ [14].

**Definition 5** We say that two pairs of Leibniz algebras $(m, q)$ and $(n, p)$ are isomorphic if there exist an isomorphism $\varphi : q \rightarrow p$ such that $\varphi|m : m \cong n$.

**Lemma 2** Let $(m, q)$ be a pair of Leibniz algebras. Then, there exists the factor set $f : \frac{q}{m} \times \frac{q}{m} \rightarrow m$ such that $(\text{Ker}(\pi), m \times_f \frac{q}{m}) \cong (m, q)$, where $\pi : m \times_f \frac{q}{m} \rightarrow \frac{q}{m}$ is the canonical projection.

**Proof** Let $h$ be a vector space complement of $m$ in $q$ and $\rho : \frac{q}{m} \rightarrow q$ be a linear map given by $\rho(\bar{x}) = h$, where $x = h + m$ with $h \in h$ and $m \in m$, which is a splitting of the extension $0 \rightarrow m \rightarrow q \xrightarrow{pr} \frac{q}{m} \rightarrow 0$.

Now, we define $f : \frac{q}{m} \times \frac{q}{m} \rightarrow m$ by $f(\bar{x}, \bar{y}) = [\rho(\bar{x}), \rho(\bar{y})] - \rho([\bar{x}, \bar{y}])$ that is well-defined, since $pr(\rho(\bar{x}), \rho(\bar{y})) = [\bar{x}, \bar{y}] = pr(\rho([\bar{x}, \bar{y}]))$. Hence, $f$ is the factor set and $m \times_f \frac{q}{m}$ is isomorphic to $q$ via $(m, \bar{x}) \mapsto m + \rho(\bar{x})$. Also, it is easy to check that $f|_{\text{Ker}(\pi)} : \text{Ker}(\pi) \cong m$ and this completes the proof. 

For the pair of Leibniz algebras $(m, q)$ we consider the extension

$$0 \rightarrow Z_{\text{Lie}}(m, q) \xrightarrow{\subseteq} m \xrightarrow{\pi} m/Z_{\text{Lie}}(m, q) \rightarrow 0$$

and the factor set $f$ corresponding to the splitting $\tau : m/Z_{\text{Lie}}(m, q) \rightarrow m$ given by Lemma 2. Moreover, $Z_{\text{Lie}}(m, q) \times_f m/Z_{\text{Lie}}(m, q)$ is a Leibniz algebra isomorphic to $m$. We henceforth assume that $m_f := Z_{\text{Lie}}(m, q) \times_f m/Z_{\text{Lie}}(m, q)$ is given as just described. It is easy to see that the following map is an isomorphism:

$$\kappa : Z_{\text{Lie}}(m, q) \rightarrow Z_{\text{Lie}}^{m_f} = \{(z, 0) \mid z \in Z_{\text{Lie}}(m, q)\}, \quad z \mapsto (z, 0).$$

This notation and the previous Lemma give rise to the following

**Proposition 3** Let $(m, q)$ and $(n, p)$ be two Lie-isoclinic Lie-semi pairs of Leibniz algebras. Then, there exists a factor set $f : m/Z_{\text{Lie}}(m, q) \times m/Z_{\text{Lie}}(m, q) \rightarrow Z_{\text{Lie}}(m, q)$ such that

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\begin{align*}
    n &\cong Z_{\text{Lie}}(m, q) \times_f \frac{m}{Z_{\text{Lie}}(m, q)} = m_f \\
    \text{Proof:} \quad \text{Let the pair } (\alpha, \beta) \text{ be the Lie-isoclinism between } (m, q) \text{ and } (n, p). \text{ By Corollary 2, we have the isomorphism } \beta : Z_{\text{Lie}}(m, q) \cong Z_{\text{Lie}}(n, p) \text{ and owing to Lemma 2 there exists a factor set } g : n/Z_{\text{Lie}}(n, p) \times n/Z_{\text{Lie}}(n, p) \longrightarrow Z_{\text{Lie}}(n, p) \text{ such that } n \cong n_g. \text{ Hence, we define the map } f : m/Z_{\text{Lie}}(m, q) \times m/Z_{\text{Lie}}(m, q) \longrightarrow Z_{\text{Lie}}(m, q) \text{ given by } f(\vec{m}_1, \vec{m}_2) = \beta^{-1}(g(\alpha \times (\vec{m}_1, \vec{m}_2))), \text{ for all } \vec{m}_i \in m/Z_{\text{Lie}}(m, q), i = 1, 2, \text{ which is } \beta^{-1} \text{ of the factor set corresponding to } \rho_2 \circ \alpha, \text{ where } \rho_2 \text{ is the splitting of } \pi_2 : n \twoheadrightarrow n/Z_{\text{Lie}}(n, p). \text{ One readily see that the mapping }
    \theta : Z_{\text{Lie}}(m, q) \times_f \frac{m}{Z_{\text{Lie}}(m, q)} \longrightarrow Z_{\text{Lie}}(n, p) \times_g \frac{n}{Z_{\text{Lie}}(n, p)}
\end{align*}

defined by \(\theta(z_1, \vec{m}_1) = (\beta(z_1), \alpha(\vec{m}_1)), z_1 \in Z_{\text{Lie}}(m, q), \vec{m}_1 \in \frac{m}{Z_{\text{Lie}}(m, q)},\) is an isomorphism, as required. 

We close this section by the following Lemma and Proposition which are of interest in their own account.

**Lemma 3** \textit{Let } \(f\) \text{ and } \(g\) \text{ be two factor sets on the pair of Leibniz algebras } (m, q) \text{ and } (n, p), \text{ respectively. If } \eta : m_f \twoheadrightarrow n_g \text{ is an isomorphism such that } \eta(Z_{\text{Lie}}^m) = Z_{\text{Lie}}^n, \text{ then } \eta \text{ induces isomorphisms } \eta_1 : m/Z_{\text{Lie}}(m, q) \twoheadrightarrow n/Z_{\text{Lie}}(n, p) \text{ and } \eta_2 : Z_{\text{Lie}}(m, q) \longrightarrow Z_{\text{Lie}}(n, p).\}

\textbf{Proof:} \quad \text{We consider the following commutative diagram:}

\[
\begin{array}{ccccccc}
0 & \longrightarrow & Z_{\text{Lie}}^m & \longrightarrow & m_f & \longrightarrow & m_f/Z_{\text{Lie}}^m & \longrightarrow & 0 \\
& \downarrow{\eta} & \downarrow{\eta} & & \downarrow{\tilde{\eta}} & \downarrow{\tilde{\eta}} & \downarrow{\tilde{\eta}} & \downarrow{\tilde{\eta}} & \downarrow{\tilde{\eta}} & \downarrow{\tilde{\eta}} \\
0 & \longrightarrow & Z_{\text{Lie}}^n & \longrightarrow & n_g & \longrightarrow & n_g/Z_{\text{Lie}}^n & \longrightarrow & 0,
\end{array}
\]

where \(\tilde{\eta}((z, \vec{m}) + Z_{\text{Lie}}^m) = \eta(z, \vec{m}) + Z_{\text{Lie}}^n.\)

Now, we define \(\eta_1 : m/Z_{\text{Lie}}(m, q) \twoheadrightarrow n/Z_{\text{Lie}}(n, p) \text{ by } \eta_1((0, \vec{m}) + Z_{\text{Lie}}^m) = \eta((0, \vec{m}) + Z_{\text{Lie}}^m) = (0, \eta_1(\vec{m})), \text{ and } \eta_2 : Z_{\text{Lie}}(m, q) \longrightarrow Z_{\text{Lie}}(n, p) \text{ given by } \eta(z, 0) = (\eta_2(z), 0), \text{ for any } z \in Z_{\text{Lie}}(m, q). \text{ It is easy to check that } \eta_1 \text{ and } \eta_2 \text{ are isomorphisms.} \]

**Proposition 4** \textit{Let } \(f\) \text{ and } \(g\) \text{ be two factor sets on the pair of Leibniz algebras } (m, q) \text{ and } (n, p), \text{ respectively. Let } \eta, \eta_1 \text{ and } \eta_2 \text{ be as in Lemma 3. Then, there exists a linear map } d : n/Z_{\text{Lie}}(n, p) \longrightarrow Z_{\text{Lie}}(n, p) \text{ such that}

\[
\begin{align*}
    \eta_2([z_1, z_2] + [\mu(\vec{m}_1), z_2] + [z_1, \mu(\vec{m}_2)] + f(\vec{m}_1, \vec{m}_2)) + d(\eta_1([\vec{m}_1, \vec{m}_2])) \\
    = [\eta_2(z_1) + d(\eta_1(\vec{m}_1)), \eta_2(z_2) + d(\eta_1(\vec{m}_2))] + [\eta_2(z_1) + d(\eta_1(\vec{m}_1)), \nu(\eta_1(\vec{m}_2))] \\
    + [\nu(\eta_1(\vec{m}_1)), \eta_2(z_2) + d(\eta_1(\vec{m}_2))] + g(\eta_1(\vec{m}_1), \eta_1(\vec{m}_2)).
\end{align*}
\]
for all \((z_1, \bar{m}_1), (z_2, \bar{m}_2) \in m_f\), where \(\mu \) and \(\nu\) are the corresponding splittings of \(\pi_1 : m \to m/Z_{\text{Lie}}(m, q)\) and \(\pi_2 : n \to n/Z_{\text{Lie}}(n, p)\) associated with \(f\) and \(g\), respectively.

**Proof** By Lemma 3, for all \(z \in Z_{\text{Lie}}(m, q)\) and \(z \in Z_{\text{Lie}}(m, q)\), there exists \(z_{\eta_1(\bar{m})} \in Z_{\text{Lie}}(n, p)\) such that \(\eta(0, \bar{m}) = (0, \eta_1(\bar{m})), 0\). We define the map \(d : Z_{\text{Lie}}(n, p) \to Z_{\text{Lie}}(n, p)\) by \(d(\eta_1(\bar{m})) = z_{\eta_1(\bar{m})}\), which is a linear map. So we have \(\eta(z, \bar{m}) = \eta(z, 0) + \eta(0, \bar{m}) = (\eta_2(z, 0) + (0, \eta_1(\bar{m}))), 0\). Applying this equality, for all \((z_1, \bar{m}_1), (z_2, \bar{m}_2) \in m_f\), one gets,

\[
\begin{align*}
\eta([z_1, \bar{m}_1], (z_2, \bar{m}_2)) &= \eta([z_1, z_2] + [\mu(\bar{m}_1), z_2] + [z_1, \mu(\bar{m}_2)]) + f(\bar{m}_1, \bar{m}_2), [\bar{m}_1, \bar{m}_2]) \\
&= (\eta_2([z_1, z_2] + [\mu(\bar{m}_1), z_2] + [z_1, \mu(\bar{m}_2)]) + f(\bar{m}_1, \bar{m}_2)) \\
&\quad + d(\eta_1([\bar{m}_1, \bar{m}_2]), \eta_1([\bar{m}_1, \bar{m}_2])).
\end{align*}
\]

On the other hand,

\[
\begin{align*}
\eta([z_1, \bar{m}_1], (z_2, \bar{m}_2)) &= [\eta(z_1, \bar{m}_1), \eta(z_2, \bar{m}_2)] \\
&= ([\eta_2(z_1) + d(\eta_1(\bar{m}_1)), \eta_1(\bar{m}_1)], (\eta_2(z_2) + d(\eta_1(\bar{m}_2)), \eta_1(\bar{m}_2))] \\
&= ([\eta_2(z_1) + d(\eta_1(\bar{m}_1))], (\eta_2(z_2) + d(\eta_1(\bar{m}_2))) \\
&\quad + [\nu(\eta_1(\bar{m}_1)), \eta_2(z_2) + d(\eta_1(\bar{m}_2))]) \\
&\quad + [\nu(\eta_1(\bar{m}_1)), \eta_2(z_2) + d(\eta_1(\bar{m}_2))]) \\
&\quad + g(\eta_1(\bar{m}_1), \eta_1(\bar{m}_2)), [\eta_1(\bar{m}_1), \eta_1(\bar{m}_2))].
\end{align*}
\]

Now, the statement follows from the equality of the first component of both computations. \(\square\)

Under assumptions of above Proposition and Proposition 3, we have the following isomorphism between pairs of Liebniz algebras:

\[
(Z_{\text{Lie}}(m, q), m_f) \cong (Z_{\text{Lie}}(n, p), n_f) \cong (Z_{\text{Lie}}(n, p), n)
\]

4 Lie-Isoclinism and Isomorphism Between Pairs of Leibniz Algebras

If two pairs of Leibniz algebras are isomorphic, it is easy to check that they are Lie-
isoclinic. But in this chapter, we show that the converse is not necessarily valid for finite dimensional (Lie-stem) Leibniz algebras, whereas isoclinic and isomorphism are equal for finite dimensional (stem) Lie algebras [17], and Pair of (stem) Lie algebras [15]. Nevertheless, we provide some conditions that these concepts are equal for finite dimensional Lie-stem Leibniz algebras.

**Example 2** \(q = \text{span}\{a_1, a_2, a_3\}\), with nonzero multiplication \([a_1, a_3] = a_1\) (it belongs to the class 2 (d)), where \(Z_{\text{Lie}}(q) = \text{span}\{a_2\}\) and \([q, q]_{\text{Lie}} = \text{span}\{a_1\}\), and
\( p = \text{span}\{g_1, g_2, g_3\} \), with nonzero multiplications \([g_1, g_3] = g_1, [g_2, g_3] = g_2\) and \([g_3, g_2] = -g_2\) (it belongs to the class 2 (e) with \(\alpha = 1\)), where \(Z_{\text{Lie}}(p) = \text{span}\{g_2\}\) and \([p, p]_{\text{Lie}} = \text{span}\{g_1\}\).

Now, we define isomorphisms \(\omega : \frac{q}{Z_{\text{Lie}}(q)} \longrightarrow \frac{p}{Z_{\text{Lie}}(p)}\) given by \(\omega(a_i) = g_i, i = 1, 3,\) and \(\tau : [q, q]_{\text{Lie}} \longrightarrow [p, p]_{\text{Lie}}\) given by \(\tau(a_1) = g_1\).

One easily verifies that \(p\) and \(q\) are Lie-isoclinic.

In certain circumstances, even with additional conditions like the Leibniz algebras are Lie-stem, this result is not true, as well. In the following, we investigate two Lie-stem (pairs of) Leibniz algebras, with the same finite dimension, that they are not isomorphic since they belong to different classes in \([8, \text{Theorem 4.2.6}]\) and \([3, \text{Proposition 3.11}]\), but we check they are Lie-isoclinic.

**Example 3** (a) Consider the following five-dimensional non-isomorphic Lie-stem Leibniz algebras given in \([8, \text{Theorem 4.2.6}]\): \(A_1 = \text{span}\{a_1, a_2, a_3, a_4, a_5\}\), with nonzero multiplications \([a_1, a_1] = a_3, [a_2, a_1] = a_4\) and \([a_1, a_3] = a_5\), in which \(Z_{\text{Lie}}(A_1) = \text{span}\{a_4, a_5\}\) is included in \([A_1, A_1]_{\text{Lie}} = \text{span}\{a_3, a_4, a_5\}, \) and \(A_7 = \text{span}\{g_1, g_2, g_3, g_4, g_5\}\), with nonzero multiplications \([g_1, g_1] = g_3, [g_1, g_2] = g_4, [g_2, g_1] = g_5\) and \([g_1, g_3] = g_5\), in which \(Z_{\text{Lie}}(A_7) = \text{span}\{g_4, g_5\}\) is included in \([A_7, A_7]_{\text{Lie}} = \text{span}\{g_3, g_4, g_5\}\).

We define \(\omega : \frac{A_1}{Z_{\text{Lie}}(A_1)} \longrightarrow \frac{A_7}{Z_{\text{Lie}}(A_7)}\) by \(\omega(\hat{a}_1) = \hat{g}_1, \omega(\hat{a}_2) = \hat{g}_2, \omega(\hat{a}_3) = \hat{g}_3, \omega(\hat{a}_4) = \hat{g}_4, \omega(\hat{a}_5) = \hat{g}_5\) and \(\tau : [A_1, A_1]_{\text{Lie}} \longrightarrow [A_7, A_7]_{\text{Lie}}\) by \(\tau(a_3) = g_3, \tau(a_4) = g_4 + g_5, \tau(a_5) = g_5\). Now, it is easy to check that \(\omega\) and \(\tau\) are isomorphisms making diagram (1) commutative, hence \(A_1 \sim A_7\). Note that the definition of \(\tau\) reproduces the isomorphism given by Corollary 2, namely \(\tau : Z_{\text{Lie}}(A_1) \cong Z_{\text{Lie}}(A_7)\).

(b) Consider the following four-dimensional non-isomorphic Lie-stem Leibniz algebras given in \([3]\): \(q = \text{span}\{e_1, e_2, e_3, e_4\}\), with nonzero multiplications \([e_1, e_4] = e_1, [e_2, e_4] = e_2\) and \([e_4, e_4] = e_3\) (class \(L_26(\mu_2)\), with \(\mu_2 = 1\), in \([3, \text{Proposition 3.11}]\)). Take the two-sided ideal \(m = \text{span}\{e_1, e_2, e_3\}\) of \(q\). Then, \((m, q)\) is a Lie-stem pair since \(Z_{\text{Lie}}(m, q) = \text{span}\{e_3\}\) and \([m, q]_{\text{Lie}} = \text{span}\{e_1, e_2, e_3\}\), and \(p = \text{span}\{a_1, a_2, a_3, a_4\}\), with nonzero multiplications \([a_1, a_4] = a_2, [a_3, a_4] = a_3\) and \([a_4, a_4] = a_1\) (class \(L_40\) in \([3, \text{Proposition 3.11}]\)).

Take the two-sided ideal \(n = \text{span}\{a_1, a_2, a_3\}\) of \(p\). Then, \((n, p)\) is a Lie-stem pair since \(Z_{\text{Lie}}(n, p) = \text{span}\{a_2\}\) and \([n, p]_{\text{Lie}} = \text{span}\{a_1, a_2, a_3\}\). Now, we define the isomorphisms \(\omega : \frac{q}{Z_{\text{Lie}}(m, q)} \longrightarrow \frac{p}{Z_{\text{Lie}}(n, p)}\) by \(\omega(\hat{e}_1) = \hat{a}_1, \omega(\hat{e}_2) = \hat{a}_3, \omega(\hat{e}_4) = \hat{a}_4\) and \(\tau : [m, q]_{\text{Lie}} \longrightarrow [n, p]_{\text{Lie}}\) by \(\tau(e_1) = a_2, \tau(e_2) = a_3, \tau(e_3) = a_1\). Now, it is easy to check that \(\omega\) and \(\tau\) are isomorphism making diagram (1) commutative; hence, \((m, q)\) and \((n, p)\) are Lie-isoclinic.

The aim of the rest of the paper is to find conditions under what Lie-isoclinism between two Lie-stem Leibniz algebras implies their isomorphism.

**Lemma 4** For any Lie-stem Leibniz algebra \(m\), \(Z(m) = Z_{\text{Lie}}(m)\).

**Proof** Since for every \(y \in m\) and \(z \in Z_{\text{Lie}}(m)\), \([y, z] = 0\), it follows that \([z, y] = [y, z] = 0\) and therefore \(Z_{\text{Lie}}(m) \subseteq Z(m)\). A direct checking shows that \(Z(m) \subseteq Z_{\text{Lie}}(m)\). \(\square\)
Theorem 3  Let \((m, q)\) and \((n, p)\) be two Lie-isoclinic pairs of finite dimensional complex Leibniz algebras such that:

(a) \(m\) and \(n\) are Lie-stem Leibniz algebras.
(b) For all elements \(m_1, m_2 \in m\) there exists \(\epsilon_{12} \in \mathbb{C}\) such that \([m_1, m_2] = \epsilon_{12}[m_2, m_1]\).

Then, \(m\) and \(n\) are isomorphic.

Proof First of all, we claim that \((m, q)\) and \((n, p)\) are Lie-stem pairs. Indeed, \(Z_{\text{Lie}}(m, q) \subseteq Z_{\text{Lie}}(m) \subseteq [m, m]_{\text{Lie}} \subseteq [m, q]_{\text{Lie}}\).

Owing to Proposition 3, there exist two factor sets \(f : \frac{m}{Z_{\text{Lie}}(m, q)} \times \frac{m}{Z_{\text{Lie}}(m, q)} \rightarrow Z_{\text{Lie}}(m, q)\) and \(g : \frac{n}{Z_{\text{Lie}}(n, p)} \times \frac{n}{Z_{\text{Lie}}(n, p)} \rightarrow Z_{\text{Lie}}(n, p)\) such that \(n \cong m_f\) and \(m \cong n_g\).

Let \((\omega, \tau)\) be the Lie-isoclinism between \(m_f\) and \(n_g\) provided by Corollary 1, then the following diagram is commutative,

\[
\begin{array}{c}
m_f/Z_{\text{Lie}}(m_f) \times m_f/Z_{\text{Lie}}(m_f) \xrightarrow{C_1} [m_f, m_f]_{\text{Lie}} \\
\downarrow \omega \times \omega \quad \quad \quad \quad \quad \quad \downarrow \tau \\
n_g/Z_{\text{Lie}}(n_g) \times n_g/Z_{\text{Lie}}(n_g) \xrightarrow{C_2} [n_g, n_g]_{\text{Lie}},
\end{array}
\]

We know that:

\[
\omega((z, \tilde{m}) + Z_{\text{Lie}}(m_f)) = \omega((0, \tilde{m}) + Z_{\text{Lie}}(m_f)) = (0, \omega_1(\tilde{m})) + Z_{\text{Lie}}(n_g)
\]

where \(\omega_1 : m/Z_{\text{Lie}}(m, q) \rightarrow n/Z_{\text{Lie}}(n, p)\) is an isomorphism.

On the other hand, from the following diagram

\[
\begin{array}{c}
Z_{\text{Lie}}^{n_g} \cong Z_{\text{Lie}}(n, p) \subseteq Z_{\text{Lie}}(n) \subseteq [n, n]_{\text{Lie}} \cong [m_f, m_f]_{\text{Lie}} \\
\downarrow \tau \\
Z_{\text{Lie}}^{m_f} \cong Z_{\text{Lie}}(m, q) \subseteq Z_{\text{Lie}}(m) \subseteq [m, m]_{\text{Lie}} \cong [n_g, n_g]_{\text{Lie}}
\end{array}
\]

and keeping in mind that \(Z_{\text{Lie}}(m, q)\) is a two-sided ideal of \([m, m]_{\text{Lie}}\), then from the proof of Theorem 2, Corollary 2 and the following commutative diagram

\[
\begin{array}{c}
\xymatrix{ 0 \ar[r] & Z_{\text{Lie}}(n, p) \ar[r]^{\cong} \ar[l]_{\cong} & Z_{\text{Lie}}^{n_g} \ar[r]^{\cong} \ar[l]_{\tau_1} & [n, n]_{\text{Lie}} \ar[r]^{\cong} \ar[l]_{\tau} & Z_{\text{Lie}}(n, p) \ar[r]^{\cong} \ar[l]_{\tau} & 0 \\
0 \ar[r] & Z_{\text{Lie}}(m, q) \ar[r]^{\cong} \ar[l]_{\cong} & Z_{\text{Lie}}^{m_f} \ar[r]^{\cong} \ar[l]_{\tau_1} & [m, m]_{\text{Lie}} \ar[r]^{\cong} \ar[l]_{\tau} & Z_{\text{Lie}}(m, q) \ar[r]^{\cong} \ar[l]_{\tau} & 0,}
\end{array}
\]

we conclude that \(\tau_1 : Z_{\text{Lie}}^{n_g} \cong Z_{\text{Lie}}^{m_f}\).

Now, we define \(\tau_2 : Z_{\text{Lie}}(m, q) \rightarrow Z_{\text{Lie}}(n, p)\) by \(\tau(z, 0) = (\tau_2(z), 0)\), for all \(z \in Z_{\text{Lie}}(m, q)\).
Using Lemma 4, for all \((z_i, \vec{m}_i) \in \mathfrak{m}_f, i = 1, 2,\)
\[
\tau([z_1, \vec{m}_1], [z_2, \vec{m}_2]) = \tau([z_1, \vec{m}_1], [z_2, \vec{m}_2]) = \tau([z_2, \vec{m}_2], [z_1, \vec{m}_1]) = \tau_2(f(\vec{m}_1, \vec{m}_2) + f(\vec{m}_2, \vec{m}_1), 0) + \tau(0, [\vec{m}_1, \vec{m}_2] + [\vec{m}_2, \vec{m}_1]).
\]
where \(\mu\) is as Proposition 4. On the other hand,
\[
\tau([z_1, \vec{m}_1], [z_2, \vec{m}_2]) = C_2(\omega_1([\vec{m}_1]), \omega([z_2, \vec{m}_2] + Z_{\text{Lie}}(\mathfrak{m}_f)) + Z_{\text{Lie}}(\mathfrak{n}_g)) = C_2((0, \omega_1([\vec{m}_1])) + Z_{\text{Lie}}(\mathfrak{n}_g), (0, \omega_1([\vec{m}_2])) + Z_{\text{Lie}}(\mathfrak{n}_g)) = (g(\omega_1([\vec{m}_1]), \omega_1([\vec{m}_2])) + g(\omega_1([\vec{m}_2]), \omega_1([\vec{m}_1]))).
\]
Let \(d(\omega_1([\vec{m}_1], [\vec{m}_2] + [\vec{m}_2, \vec{m}_1]))\) be the first component of \(\tau(0, [\vec{m}_1, \vec{m}_2] + [\vec{m}_2, \vec{m}_1])\) where \(d : \frac{\mathfrak{n}}{Z_{\text{Lie}}(\mathfrak{n}, \mathfrak{p})} \to Z_{\text{Lie}}(\mathfrak{n}, \mathfrak{p})\) is a linear map. Now, the isomorphism \(\tau\) yields:
\[
\tau_2(f(\vec{m}_1, \vec{m}_2) + f(\vec{m}_2, \vec{m}_1)) = g(\omega_1([\vec{m}_1]), \omega_1([\vec{m}_2])) + g(\omega_1([\vec{m}_2]), \omega_1([\vec{m}_1])).
\]
Applying assumption \((b)\) and Lemma 4, we conclude that
\[
\tau_2(f(\vec{m}_1, \vec{m}_2) + d(\omega_1([\vec{m}_1], [\vec{m}_2]))) = g(\omega_1([\vec{m}_1]), \omega_1([\vec{m}_2])) = g(\omega_1(\vec{m}_1), \omega_1(\vec{m}_2)) \tag{3}
\]
Now, we define \(\lambda : \mathfrak{m}_f \to \mathfrak{n}_g\) by \(\lambda(z, \vec{m}) = (\tau_2(z) + d(\omega_1(\vec{m})), \omega_1(\vec{m})).\) By Lemma 4, for all \((z_i, \vec{m}_i) \in \mathfrak{m}_f, i = 1, 2,\) we get,
\[
\lambda([z_1, \vec{m}_1], [z_2, \vec{m}_2]) = \tau_2(z_1, [\vec{m}_1], [\vec{m}_2] + [z_1, \mu(\vec{m}_2)] + f(\vec{m}_1, \vec{m}_2))
\]
\[
+ d(\omega_1([\vec{m}_1], [\vec{m}_2]))), \omega_1([\vec{m}_1], [\vec{m}_2]))
\]
\[
= \tau_2(f(\vec{m}_1, \vec{m}_2) + d(\omega_1([\vec{m}_1], [\vec{m}_2])), \omega_1([\vec{m}_1], [\vec{m}_2])).
\]
On the other hand,
\[
[\lambda(z_1, \vec{m}_1), \lambda(z_2, \vec{m}_2)] = \left(\tau_2(z_1) + d(\omega_1(\vec{m}_1)), \tau_2(z_2) + d(\omega_1(\vec{m}_2))\right)
\]
\[
+ [\mu(\vec{m}_1)], \nu(\vec{m}_2)] + [\nu(\vec{m}_1)), \tau_2(z_2) + d(t_1(\vec{m}_2))]
\]
\[
+ g(\omega_1(\vec{m}_1), \omega_1(\vec{m}_2), \omega_1(\vec{m}_1), \omega_1(\vec{m}_2))
\]
\[
= (g(\omega_1(\vec{m}_1), \omega_1(\vec{m}_2)), [\omega_1(\vec{m}_1), \omega_1(\vec{m}_2)]).
\]
where \( \nu \) is as in Proposition 4. It now follows from equality 3, that \( \lambda \) is a homomorphism. The following diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & Z_{\text{Lie}}^{\pi_1} \cong Z_{\text{Lie}}(n, p) \\
& \downarrow \tau_2^{-1} & \\
0 & \longrightarrow & Z_{\text{Lie}}^{\pi_2} \cong Z_{\text{Lie}}(m, q)
\end{array}
\]

is commutative, where \( \omega_1 \) and \( \tau_2 \) are the above isomorphisms. Now by Short Five Lemma [2], it follows that \( \lambda \) is an isomorphism. \( \Box \)

The above Theorem still holds for finite dimensional stem pair of Lie algebras if we drop the assumptions (a) and (b), see [15] for more details.

**Example 4** An example of non-Lie Leibniz algebra satisfying condition (b) of Theorem 3 is the two-dimensional Leibniz algebra with basis \( \{a_1, a_2\} \) and the bracket operation given by \([a_2, a_2] = \lambda a_1, \lambda \in \mathbb{C}\setminus\mathbb{C}^2 \) [7]. Here, the parameter \( \varepsilon = 1 \).

It should be mention that, in Example 3 (a), Leibniz algebra \( A_7 \) does not satisfy in the condition (b) of Theorem 3.

Note that the parameter \( \varepsilon \) in condition (b) of Theorem 3 does not require to be unique, that is every pair of elements \( \bar{m}_i, \bar{m}_j \), has a parameter \( \varepsilon_{ij} \) such that
\[
[\bar{m}_i, \bar{m}_j] = \varepsilon_{ij}[\bar{m}_j, \bar{m}_i].
\]

An example of Leibniz algebra satisfying this condition is the four-dimensional Leibniz algebra with basis \( \{a_1, a_2, a_3, a_4\} \) with bracket operation \([a_1, a_1] = a_3, [a_2, a_4] = -[a_4, a_2] = a_2, [a_4, a_4] = -2a_2 \) (class \( L_{16} \) in [3, Proposition 3.10]). Here the parameters are \( \varepsilon_{11} = 1, \varepsilon_{24} = -1, \varepsilon_{44} = 1 \).

In Lie algebra theory, it is well-known that the isoclinism between two finite dimensional stem Lie algebras implies isomorphism. For a deeper discussion, we refer the reader to [17]. In the following, we provide some conditions to have analogous result.

**Corollary 3** Let \( q \) and \( p \) be Lie-isoclinic Lie-stem of finite dimensional complex Leibniz algebras and for all elements \( x_1, x_2 \in q \) there exist \( \varepsilon_{12} \in \mathbb{C} \) such that \([x_1, x_2] = \varepsilon_{12}[x_2, x_1] \). Then, \( q \) and \( p \) are isomorphic.

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