‘Muhammad Ali effect’ and incoherent destruction of Wannier-Stark localization in a stochastic field

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We calculate an exact expression for the probability propagator for a noisy electric field driven tight-binding lattice. The noise considered is a two level jump process or a telegraph process (TP) which jumps randomly between two values ±µ. In the absence of a static field and in the limit of zero jump rate of the noisy field we find that the dynamics yields Bloch oscillations with frequency µ, while with an additional static field ϵ we find oscillatory motion with a superposition of frequencies (ϵ ± µ). On the other hand, when the jump rate is rapid, and in the absence of a static field, the stochastic field averages to zero if the two states of the TP are equally probable a-priori. In that case we see a delocalization effect. The intimate relationship between the rapid relaxation case and the zero field case is a manifestation of what we call the Muhammad Ali effect. It is interesting to note that even for zero static field and rapid relaxation, Bloch oscillations ensue if there is a bias δp in the probabilities of the two levels. Remarkably, the Wannier-Stark localization caused by an additional static field is destroyed if the latter is tuned to be exactly equal and opposite to the average stochastic field µδp. This is an example of incoherent destruction of Wannier-Stark localization.

Anecdotes recount how when the boxer Muhammad Ali was asked about the speed of his punches, he responded that they were so fast that it was as if the opponent felt no punches at all. In a variety of physical phenomena where a rapidly changing external field is involved, this intuition of equivalence with the corresponding zero (time-dependent) field phenomenon is indeed borne out1–6. We term it the ‘Muhammad Ali effect’. In the specific context of an ac electric field applied on to a one-dimensional tight-binding lattice, the large frequency limit can be shown to be mathematically equivalent to simply renormalizing the hopping parameter6,7, thus corresponding to the zero-field case. However, for certain delicate choices of the ratio of the amplitude and frequency, a dynamical localization7–10 may be engineered via a band collapse mechanism. On the other hand, the zero frequency limit when the electric field is time-independent is characterized by the familiar Bloch oscillations11–14. Other phenomena such as coherent destruction of Wannier Stark (WS) localization15,16 and super Bloch oscillations17–20 arise when an additional static field is added on an existing sinusoidal field. The former occurs when the static field is resonantly tuned with the frequency of the sinusoidal field while the latter for a slight detuning from the resonance condition.

Random disorder, in the zero electric field case is known to localize the particle via the famous phenomenon of Anderson localization21. Since the work of Anderson, transport in the presence of a fluctuating environment has also been studied22–26 both analytically and numerically. The aim here has been to understand the diffusion of a quantum particle in the presence of dynamic disorder. This dynamic disorder originates from the lattice vibrations where the modes of phonons are randomly excited and the process is modeled by a stochastic process27. In the presence of an electric field, disorder dephases the Bloch oscillations depending on the strength of disorder28–30. However for a slowly varying disordered potential the Bloch oscillations are known to survive31,32. An increased diffusion has been found to be the effect of scattering on the motion of a charged particle with a time-dependent field33.

Here, we consider the effect of a stochastic noise on top of an electric field on the motion of the particle and focus on how the Bloch oscillations are influenced by this type of dynamic disorder. The particular form of the stochastic noise is the “telegraphic noise”27,34–36, where the noise consists of jumps randomly between two levels ±µ. When such a telegraphic noise term appears as fluctuations in the site energies without any linear variation (the limit when the electric field is zero), exact analytical results for the diffusion coefficient have been obtained26. Here, we consider the case where the noise term acts as fluctuations to an electric field. This noise can also be thought of as an aperiodic form of a square wave driving (periodic square wave driving with proper tuning can yield dynamic localization37,38).

The central findings (Table I) of our Letter are as follows. For a stochastic electric field characterized by telegraph noise, we find the exact expression for the probability propagator \( P_m(t) \), defined as the probability of a particle to remain at site \( m \) at any time \( t \) given that it was at the origin at \( t = 0 \). The limit of the rapidly changing stochastic field is given particular emphasis. Denoting the bias in the probabilities of the two levels of the field to be \( \delta p \), we show that this is equivalent to an effective dc field of \( \mu \delta p \), yielding Bloch oscillations with frequency \( \mu \delta p \) (although these oscillations are exponentially damped in the infinite time limit). If an additional static field is present, we recover Bloch oscillations with a renormalized frequency in the rapid relaxation limit - this is another instance of the Muhammad Ali effect. Remarkably, by choosing the additional static field to have a precise magnitude, a destruction of WS localization15,16.

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| Deterministic $F = c_0 F_{dc} + c_1 F_{ac}$  
(High frequency regime) | Stochastic $F = c_0 F_{dc} + c_1 F_{TP}$  
(Rapid relaxation regime) |
|---|---|
| DC ($c_1 = 0$) | • Bloch oscillations (WS localization).  
| | • Equivalent to no-field case (Muhammad Ali effect).  
| | • Dynamical localization with proper tuning. |
| AC/TP ($c_0 = 0$) | • Bloch oscillations (WS localization).  
| | • Equivalent to no-field case (Muhammad Ali effect). |
| DC + AC/TP ($c_0, c_1 \neq 0$) | • Coherent destruction of WS localization at resonance.  
| | • Super Bloch oscillations at off-resonance.  
| | • Incoherent destruction of WS localization with proper tuning of bias.  
| | • Bloch oscillations with bias-dependent re-normalized frequency (another case of Muhammad Ali effect). |

Table I. The table contrasts the various phenomena that arise due to ac drive and telegraph noise.

may be engineered. Since no frequency is involved in the present context, and rather the noise may be a result of connection to a bath, this may be termed an incoherent destruction of WS localization. When the two levels of the stochastic field are equiprobable ($\delta p = 0$), we recover the well-known scenario that the rapid relaxation limit is equivalent to the zero-field limit: the ‘Muhammad Ali effect’. A complementary numerical approach is used to independently verify our findings.

The Hamiltonian for a 1D tight binding model with a time dependent electric field is

$$ H = -\frac{\Delta}{4} \sum_{n=-\infty}^{\infty} c_n^\dagger c_{n+1}^\dagger + c_{n+1} c_n + F(t) \sum_{n=-\infty}^{\infty} n c_n^\dagger c_n, \quad (1) $$

where $F(t)$ is the electric field. Natural units ($\hbar = e = 1$) are adopted for all the calculations. For a constant electric field, the dynamics gives the well known Bloch oscillations, while a periodic driving can give rise to dynamical localization when the amplitude and frequency are tuned appropriately. Here, we consider the case where the time dependent electric field is described by a two state jump process or a telegraph process.

In Wannier space the probability propagator is given by

$$ P_m(t) = \langle m | \rho(t) | m \rangle, \quad (2) $$

where $\rho(t)$ is the density matrix of the system. As shown in the supplementary section, the probability propagator can be written as a double-integral:

$$ P_m(t) = \left( \frac{1}{2\pi} \right)^2 \int_{-\pi}^{\pi} dk \int_{-\pi}^{\pi} dk' e^{-i(k-k')m} \times e^{-i \int_0^t dt' \left( V^+_k(t') - V^+_k(t') \right) }, \quad (3) $$

where $V^+_k(t) = -\frac{\Delta}{4} \left[ e^{i (k+n) t} + e^{-i (k+n) t} \right]$, with $n(t) = \int_0^t F(t') dt'$. The mean squared width of the wave-packet is then expressed in terms of the probability propagator as $\sigma^2(t) = \langle m^2 \rangle = \sum_m m^2 P_m(t)$.

The particular form of the field is taken as a telegraph noise where electric field is time-dependent and randomly fluctuates between two levels $\pm \mu$. Let $\sigma$ and $\tau$ be the rate of switching from level $+\mu$ to $-\mu$ and $-\mu$ to $+\mu$ respectively. The probability of being at any time in state $+\mu$ is given by $p_+ = \tau/(\tau + \sigma)$, whereas the probability of being in state $-\mu$ is $p_- = \sigma/(\tau + \sigma)$. It is useful to define $\lambda = \sigma + \tau$.

A clever way to make progress is to evaluate $i\eta(t) = i \int_0^t F(t') dt'$ to a $2 \times 2$ matrix:

$$ i\eta(t) = i e I + i t \mu \sigma_z + \lambda W, \quad (4) $$

in which $I$ is the identity matrix, $\sigma_z$ is the Pauli $z$ matrix, and the relaxation matrix $W$ is defined as

$$ W = \begin{bmatrix} -p_+ & p_- \\ p_+ & -p_- \end{bmatrix} = \lambda \begin{bmatrix} -\frac{\sigma}{\tau + \sigma} & \frac{\sigma}{\tau + \sigma} \\ \frac{\tau}{\tau + \sigma} & \frac{\tau}{\tau + \sigma} \end{bmatrix}. \quad (5) $$

As a consequence of this operation, the probability propagator (Eqn. 2) is also now a $2 \times 2$ matrix. The first term added in Eqn. 4 is to account for the static electric field $\epsilon$ and the two stochastic states are $|+\rangle = \left( \frac{1}{0} \right)$, $|-\rangle = \left( 0 \frac{1}{1} \right)$ corresponding to the fields $+\mu$ and $-\mu$ respectively.

Eqn. 4 can be decomposed in terms of Pauli matrices as

$$ i\eta(t) = -t(\gamma - i\epsilon) I + i \sigma_z (\gamma \delta p + \mu) + \gamma t (\sigma_x + i \delta p \sigma_y), \quad (6) $$

where $\gamma = \frac{1}{2}$ and $\delta p = (p_+ - p_-)$. The exponential of Eqn. 6 can be written in a compact form: $e^{i \eta(t)} = e^{-i (\gamma - i \epsilon) t} e^{i (h, \sigma)}$, where $h_x = \gamma, h_y = i \gamma \delta p$ and $h_z = (\gamma \delta p + \mu)$ and $|h| = \sqrt{\gamma^2 - \mu^2 + 2\gamma \mu \delta p} = \nu$. Using the Pauli spin identity: $e^{i (a, \sigma)} = I \cos |a| + i (\dot{a}, \sigma) \sin |a|$, the above expression can be written as

$$ e^{i \eta(t)} = \frac{1}{2} e^{-t(\gamma - i \epsilon)} \left[ e^{\nu t} (1 + \dot{h}, \sigma) + e^{-\nu t} (1 + \dot{h}, \sigma) \right]. \quad (7) $$

Similarly, an equation for the complex conjugation can be written with $h_x' = \gamma, h_y' = -i \gamma \delta p, h_z' = \gamma \delta p - i \mu$. After
some calculations, the exponential part of Eqn. 3 can be written as
\[ e^{-i \int_0^t dt'(V_k(t') - V_k'(t'))} = e^{ig_0(t)} e^{i(H, \sigma)}, \]
where the complicated expressions for \( g_0(t), \alpha(t) \) and \( \beta(t) \) are relegated to the supplementary section. Finally, we have the compact form
\[ e^{-i \int_0^t dt'(V_k(t') - V_k'(t'))} = \mathcal{P}(H, \sigma), \]

where \( H_x = \alpha(t), H_y = i \delta \rho \alpha(t) \) and \( H_z = \delta \rho \alpha(t) + \beta(t) \) and \( \mathcal{P}(H, \sigma) = \sqrt{\alpha^2(t) + \beta^2(t) + 2 \delta \rho \alpha(t) \beta(t)} \). Again using the Pauli spin identity, we get
\[ e^{i(H, \sigma)} = \left[ \mathcal{I}, \cos |H| + i(\hat{H}, \mathcal{I}) \sin |H| \right]. \tag{10} \]

For the final expression of probability, we need to calculate the restricted average \( \mathcal{P}(H, \sigma) = \sum_{ab} \rho(ab) \mathcal{P}(H, \sigma) \alpha \beta \]

The averages of \( \sigma_y \) and \( \sigma_z \) give \( \delta p \) and \( \delta \rho \) respectively, whereas \( \sigma_x \) averages to unity. While \( \mathcal{P}(H, \sigma) \) is a matrix, the average \( \mathcal{P}(H, \sigma) \) is just a number. The final expression for the average probability can be written as
\[ \mathcal{P}(H, \sigma) = \left( \frac{1}{2\pi} \right)^2 \int \frac{d\kappa \times e^{-i(k-k')im} e^{ig_0(t)}}{1} \right] \cos |H| + i \sin |H| \alpha(t) \frac{\mathcal{P}(H, \sigma)}{|H|} + i \delta \rho \sin |H| \beta(t) \frac{\mathcal{P}(H, \sigma)}{|H|}, \tag{11} \]

which is one of our key results. It is straightforward to verify the well-known results for the zero field \( \epsilon = 0 \) and static field \( \epsilon \neq 0 \) case where \( \mu \) vanishes, hence \( \beta(t) = 0 \) and \( \nu \to \gamma \). In the former case the probability propagator decays in time and the mean squared width becomes unbounded in time. Hence, an initially localized particle will delocalize. In the latter case of static field, both the probability and the mean squared width are bounded and exhibit the familiar Bloch oscillations with frequency \( \omega_B = \epsilon \). Further, considering the effect of telegraph noise the zero relaxation limit where \( \gamma = 0, \nu = i \mu \) (and hence \( \alpha(t) = 0 \)) is straightforward. A simplification of the probability propagator in this limit yields a superposition of probabilities for the two `static fields` \( \epsilon \pm \mu \).

The rapid relaxation condition \( \gamma >> \mu, \epsilon \) is the core emphasis of our Letter, and will be imposed in the rest of the discussion ahead. In this limit, \( \alpha^2(t) >> \beta^2(t) \) and an expansion of \( |H|, \alpha(t) \frac{\mathcal{P}(H, \sigma)}{|H|} \) and \( \beta(t) \frac{\mathcal{P}(H, \sigma)}{|H|} \) simplifies the integrand of the probability propagator (Eqn. 11) as
\[ e^{i g_0(t)} \left[ \cos |H| + i \sin |H| \alpha(t) \frac{\mathcal{P}(H, \sigma)}{|H|} + i \delta \rho \sin |H| \beta(t) \frac{\mathcal{P}(H, \sigma)}{|H|} \right] \approx e^{i g_0(t) + i \alpha(t) + i \delta \rho \beta(t)} e^{i \frac{\delta \rho \alpha(t)}{2 \Delta} \cos k}, \tag{12} \]

We consider separately the cases where both the levels of the stochastic field are equally probable \( \delta p = 0 \) and where one level is more probable than the other \( \delta p \neq 0 \).

Figure 1. The return probability of an initially localized wave-packet \( (\delta_m, 0) \) from the exact calculation and exact numerics. Here we present data for the case of zero bias \( \delta p = 0 \) in the rapid relaxation regime \( \sigma = \tau = 100 \) with \( \Delta = 2.0 \). (a) Muhammad Ali effect in the zero static field limit \( \epsilon = 0.0 \). The inset shows the unbounded growth of mean squared width, analogous to the zero-field scenario. (b) Bloch oscillations in the finite static field limit \( \epsilon = 0.4 \). In both the figures the numerics are performed for a system of size \( L = 400 \) with averaging carried out over 100 realizations of the disorder.

With \( \delta p = 0 \), and \( \gamma >> \mu \), the expression for \( \nu \) can be expanded up to \( \mathcal{O}(\frac{\mu^2}{\gamma}) \) as \( \nu = \sqrt{\gamma^2 - \mu^2} \approx \gamma - \frac{\mu^2}{2\gamma} \). For the zero static field case \( \epsilon = 0 \), the expressions for \( g_0(t), \alpha(t) \) and \( \beta(t) \) can be written as (up to \( \mathcal{O}(\mu/\gamma) \))
\[ g_0(t) \approx \eta_+ (\cos k - \cos k'), \quad \alpha(t) \approx \eta_+ (\cos k - \cos k') \quad \beta(t) \approx -\frac{\mu}{\gamma} \eta_+ (\sin k - \sin k'), \tag{13} \]

where \( \eta_\pm(t) = \frac{\Delta}{4 \gamma^2} [2\gamma t \pm (1 - e^{-2\gamma t})] \). Substituting the values of \( g_0(t), \alpha(t) \) and \( \beta(t) \) and taking the long time limit, we get
\[ e^{ig_0(t) + i\alpha(t)} e^{\frac{\delta \rho \alpha(t)}{2 \Delta} \cos k} \approx e^{\frac{\Delta \eta_+}{2} (\cos k - \cos k')}, \tag{14} \]

where \( \Delta_{\text{eff}} = \Delta \left[ 1 + \frac{1}{8} \left( \frac{\mu}{\gamma} \right)^2 \right] \frac{\sin k - \sin k'}{\cos k - \cos k'} \).

Hence in this limit, the effect is identical to the case of no field, a phenomenon we have called `Muhammad Ali effect’. This is the case where the electric field is so rapidly fluctuating between \( \pm \mu \), that for all practical purposes the system feels no effect at all. This effect is shown in Fig. 1, where the probability propagator and the mean squared width of the wave-packet are plotted with time. The return
probability decays in time and the wave-packet width becomes unbounded signifying the delocalization of an initially localized wave-packet. In the presence of the static field ($\epsilon \neq 0$), we have the approximation

$$g_0(t) + \alpha(t) \approx \frac{\Delta}{2\epsilon} \left[ e^{-\frac{\nu^2}{4\Delta^2} t} \sin(k + \epsilon t) - \sin k \right] - \frac{\Delta}{2\epsilon} \left[ e^{-\frac{\nu^2}{4\Delta^2} t} \sin(k' + \epsilon t) - \sin k' \right].$$

In the limit $\gamma \gg \mu$, the ratio $\frac{\delta_p(t)}{\delta_p(t)}$ becomes very small and can be neglected. Also the term $e^{-\frac{\nu^2}{4\Delta^2} t}$ becomes unity, unless $t$ is very large. So in this limit, one obtains Bloch oscillations with frequency $\epsilon$ for small times (Fig. 1); however the rapidly fluctuating noise causes in the long time limit for these oscillations to damp out exponentially.

Another interesting case of rapid relaxation arises when the two levels are not equiprobable ($\delta p \neq 0$). Here $\nu = \sqrt{\gamma^2 - \mu^2 + 2i\gamma\mu\delta p}$. We can expand $\nu$ upto $O\left(\frac{\delta p^2}{\gamma^2}\right)$ as $\nu = \sqrt{\gamma^2 - \mu^2} + i\mu\delta p$, and $\gamma - \nu = \frac{\delta p^2}{\gamma^2} - i\mu\delta p$. With these approximations and defining $\xi = \epsilon + \mu\delta p$, the exponent of the first part of Eqn. 12 can be simplified to

$$g_0(t) + \alpha(t) + \delta p \beta(t) \approx \frac{\Delta}{4\xi} \left[ e^{-\frac{\nu^2}{4\Delta^2} t} \sin(k + \xi t) - \sin k \right] - \frac{\Delta}{4\xi} \left[ e^{-\frac{\nu^2}{4\Delta^2} t} \sin(k' + \xi t) - \sin k' \right].$$

The above expression is similar to Eqn. 15 with $\epsilon$ replaced by $\xi$. Hence, Bloch oscillations with the average field and frequency $\xi$ appear, which in the long time limit damp out exponentially. Also, unlike the case of $\delta p = 0$, Bloch oscillations with frequency $\mu\delta p$ arise even in the zero static field case. Tuning the bias: $\delta p = -\frac{\xi}{\mu}$ in order to precisely cancel the effect of the static field, causes the average electric field to become zero, as a consequence of which Bloch oscillations are destroyed. This can be termed as incoherent destruction of WS localization as no frequency is involved in this scenario. This is to be contrasted with coherent destruction of WS localization\(^{15,16}\), where a resonant tuning of the drive provides the mechanism in a system that is subjected to a combined dc and time periodic ac field. The incoherent destruction of localization here is to be seen as a contrast with the ‘Muhammad Ali effect’. All these effects are plotted in Fig. 2, where the return probability and the mean squared width of the wave-packet are given as a function of time. The details of the numerical generation of the telegraph noise are given in the supplementary section.

To summarize, we studied the effect of an electric field subjected to random telegraphic noise on a nearest-neighbor tight-binding chain. Our first result is the derivation of an exact general expression for the probability propagator, which is then employed to illuminate several special cases. As expected, in the zero relaxation case, the probability shows oscillatory behavior, with a superposition of the frequencies ‘$\epsilon \pm \mu$’. The rapid relaxation scenario forms the core emphasis of our work, and may be subdivided into two cases: one where the rates for the two levels are the same and the other where one level has greater lifetime than the other. In the former case, a delocalization effect is obtained in zero static field and Bloch oscillations in the presence of a static field. We identify this limit as a manifestation of what we call the ‘Muhammad Ali effect’. In the latter case, a finite difference in the probabilities of the two levels renormalizes the Bloch frequency to $\omega_B = \epsilon + \mu\delta p$. A precise tuning of the bias $\delta p$ leads to incoherent destruction of WS localization. The exact results are also verified by an independent numerical approach as well.

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Supplementary material for “‘Muhammad Ali effect’ and incoherent destruction of Wannier-Stark localization in a stochastic field”

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We first provide a detailed description of a transformation that converts the original translation symmetry breaking Hamiltonian into a basis where momentum $k$ is a good quantum number. This basis also facilitates the calculation of the probability propagator for a telegraph noise as described in the following section. The last section carries information of the numerical implementation of telegraph noise.

HAMILTONIAN TRANSFORMATION

To write the Hamiltonian in a compact form where the time dependence appears as a phase term we start by introducing the unitary operators $\hat{K}, \hat{K}^\dagger$ and $\hat{N}$ [1] defined as

$$\hat{K} = \exp (-i \hat{\kappa} d) = \sum_{n=-\infty}^{\infty} |n\rangle \langle n+1|, \quad \hat{K}|n\rangle = |n-1\rangle$$

$$\hat{K}^\dagger = \exp (i \hat{\kappa} d) = \sum_{n=-\infty}^{\infty} |n+1\rangle \langle n|, \quad \hat{K}^\dagger |n\rangle = |n+1\rangle$$

$$\hat{N} = \sum_{n=-\infty}^{\infty} n |n\rangle \langle n|.$$  \hfill (1)

These operators follow the commutation rules

$$[\hat{K}, \hat{N}] = \hat{K}, \quad [\hat{K}^\dagger, \hat{N}] = -\hat{K}^\dagger, \quad [\hat{K}, \hat{K}^\dagger] = 0.$$  \hfill (2)

The eigenvectors of $\hat{K}$ are the Bloch states $|\kappa\rangle$ with eigenvalues $e^{i \kappa d}$. The connection between the Wannier basis and the Bloch basis is given by $|k\rangle = \sqrt{\frac{d}{2\pi}} \sum_n e^{-i\kappa d} |n\rangle$ and $|n\rangle = \frac{1}{\sqrt{d}} \int_{-\kappa d/2}^{\kappa d/2} dk \ e^{i\kappa d} |k\rangle$.

In terms of these new operators, the tight-binding Hamiltonian can be written as

$$\hat{H}(t) = V^+ + H_0(t),$$  \hfill (3)

where $V^\pm = -\frac{\Delta}{4} \left( \hat{K} \pm \hat{K}^\dagger \right)$ and $H_0(t) = d\mathcal{F}(t)\hat{N}$.

Following Refs 1 and 2, the Hamiltonian can be recast in terms of the unitary operators: $\hat{K}, \hat{K}^\dagger$ and $\hat{N}$ as

$$H(t) = V^+ + H_0(t),$$  \hfill (4)

where $V^\pm = -\frac{\Delta}{4} \left( \hat{K} \pm \hat{K}^\dagger \right)$ and $H_0(t) = a\mathcal{F}(t)\hat{N}$. The time evolution of the density matrix $\rho$ in Heisenberg picture is given by $\frac{\partial \rho}{\partial t} = -i [H(t), \rho(t)]$. By considering the transformation $\hat{\rho}(t) = e^{i \int_0^t H_0(t') dt'} \rho(t) e^{-i \int_0^t H_0(t') dt'}$, the equation of motion for $\hat{\rho}(t)$ can be written as

$$\frac{\partial \hat{\rho}}{\partial t} = -i \left[ \hat{V}^+(t), \hat{\rho}(t) \right],$$  \hfill (5)

where $\hat{V}^+(t) = e^{i \int_0^t H_0(t') dt'} V^+ e^{-i \int_0^t H_0(t') dt'}$. The time evolution of $\hat{\rho}$ can now be solved as

$$\hat{\rho}(t) = e^{-i \int_0^t \hat{V}^+(t') dt'} \rho(0) e^{i \int_0^t \hat{V}^+(t') dt'},$$  \hfill (6)

where $\rho(0) = |0\rangle \langle 0|$. It turns out that $[\hat{V}^+(t), \hat{V}^+(t')] = 0$ even for $t \neq t'$, and therefore no complicated time-ordering is essential.
Using the Baker-Campbell-Hausdorff (BCH) formula
\[ e^{XY}e^{-X} = Y + [X,Y] + \frac{1}{2!} [X,[X,Y]] + \ldots, \]
and the commutation relations between the unitary operators \([1]\), we can simplify the effective Hamiltonian which governs the dynamics of the density matrix \(\tilde{\rho}(t)\) as

\[ \tilde{V}^+(t) = -\frac{\Delta}{4} \left( \tilde{K} te^{i\eta(t)} + \tilde{K} e^{-i\eta(t)} \right), \]

where \(\eta(t) = \int_0^t \mathcal{F}(t') \, dt'\). The effective Hamiltonian (whose time dependence appears in the phase term) can be diagonalized in the momentum basis:

\[ \langle k|\tilde{V}^+(t)|k'\rangle = -\frac{\Delta}{4} \delta(k - k') \left[ e^{ik+i\eta(t)} + e^{-ik-i\eta(t)} \right]. \]

Furthermore, the transformed density matrix \(\tilde{\rho}(t)\) can also be written in \(k\) basis as

\[ \langle k|\tilde{\rho}(t)|k'\rangle = e^{-i \int_0^t dt' \mathcal{V}^+_k(t')} \langle k|0\rangle \langle 0|k'\rangle e^{i \int_0^t dt' \mathcal{V}^+_k(t')}, \]

where \(\mathcal{V}^+_k(t) = -\frac{\Delta}{4} \left[ e^{i(k+i\eta(t))} + e^{-i(k+i\eta(t))} \right].\)

In Wannier space the probability propagator is given by

\[ \mathcal{P}_m(t) = \langle m|\rho(t)|m\rangle = \langle m|\tilde{\rho}(t)|m\rangle, \]

where we have used the fact that \(H_0\) is diagonal in Wannier basis. The expression for the probability can be simplified to

\[ \mathcal{P}_m(t) = \int dk \int dk' \langle m|k\rangle \langle \tilde{\rho}(t)|k'\rangle \langle k'|m\rangle = \left( \frac{1}{2\pi} \right)^2 \int_{-\pi}^{\pi} dk \int_{-\pi}^{\pi} dk' e^{-i(k-k')m} e^{-i \int_0^t dt' (\mathcal{V}^+_k(t') - \mathcal{V}^+_k(t))}, \]

where \(\mathcal{V}^+_k(t) = -\frac{\Delta}{4} \left[ e^{i(k+i\eta(t))} + e^{-i(k+i\eta(t))} \right],\) with \(\eta(t) = \int_0^t \mathcal{F}(t') \, dt'.\)

**PROBABILITY CALCULATION**

For a telegraph noise we have (with \(\eta(t) = \int_0^t \mathcal{F}(t') dt'\))

\[ i\eta(t) = -t(\gamma - i\epsilon) \mathcal{I} + t\sigma_z (\gamma \delta p + i\mu) + \gamma t(\sigma_x + i\delta p \sigma_y), \]

where \(\gamma = \frac{\lambda z}{2}\) and \(\delta p = (p_+ - p_-)\). Using a Pauli spin identity: \(e^{i(a,\sigma)} = \mathcal{I} \cos |a| + i \langle \hat{n} \sigma \rangle \sin |a|\), The exponential of Eqn. 12 can be written as

\[ e^{i\eta(t)} = \frac{1}{2} e^{-t(\gamma - i\epsilon)} \left[ e^{it} (1 + \hat{h} \cdot \hat{\sigma}) + e^{-it} (1 + \hat{h} \cdot \hat{\sigma}) \right]. \]

Also, the conjugate equation is \((h'_x = \gamma, h'_y = -i\gamma \delta p, h'_z = \gamma \delta p - i\mu)\)

\[ e^{-i\eta(t)} = \frac{1}{2} e^{-t(\gamma + i\epsilon)} \left[ e^{it} (1 + \hat{h}' \cdot \hat{\sigma}) + e^{-it} (1 + \hat{h}' \cdot \hat{\sigma}) \right]. \]

Introducing \(z = e^{it}\) and \(z' = e^{it'}\), the expression for \((\mathcal{V}^+_k(t) - \mathcal{V}^+_k(t))\) can be solved to

\[ \mathcal{V}^+_k(t) - \mathcal{V}^+_k(t) = -\frac{\Delta}{8} e^{-it} \left\{ (z - z') e^{it} \left[ e^{it} (1 + \hat{h} \cdot \hat{\sigma}) + e^{-it} (1 - \hat{h} \cdot \hat{\sigma}) \right] + (z^* - z'^*) e^{-it} \left[ e^{it} (1 + \hat{h}' \cdot \hat{\sigma}) + e^{-it} (1 - \hat{h}' \cdot \hat{\sigma}) \right] \right\}. \]

Finally, we need to solve the integration

\[ -i \int_0^t dt' \left[ \mathcal{V}^+_k(t') - \mathcal{V}^+_k(t') \right] = \frac{i\Delta}{8} \left\{ (z - z') \left[ 1 - e^{-(\gamma - \nu)t + i\epsilon t} \right][\gamma - \nu] - i\epsilon \right\} + \left( z - z' \right) \left( \hat{h} \right) \left[ 1 - e^{-(\gamma + \nu)t + i\epsilon t} \right][\gamma + \nu] - i\epsilon \right\} + c.c. \]}
Using the relations

\[ \hat{h}.\vec{\sigma} = \frac{\gamma}{\nu} \sigma_x + \frac{i\gamma\delta p}{\nu} \sigma_y + \frac{\nu}{\nu} \sigma_z, \quad \hat{h}'.\vec{\sigma} = \frac{\nu}{\nu} \sigma_x + \frac{i\gamma\delta p}{\nu} \sigma_y + \frac{\nu}{\nu} \sigma_z \]

the exponential of the above equation can be written as

\[ e^{-i \int dt' (V^+_{\alpha}(t') - V^+_{\beta}(t'))} = i [g_0(t)\mathcal{I} + g_1(t)\sigma_x + g_2(t)\sigma_y + g_3(t)\sigma_z], \]

where

\[
\begin{align*}
g_0(t) &= \frac{\Delta}{8} \left\{ (z - z') \left[ \frac{1 - e^{-\gamma i\epsilon t + \nu t}}{\gamma - i\epsilon - \nu} + \frac{1 - e^{-\gamma i\epsilon t - \nu t}}{\gamma - i\epsilon + \nu} \right] + (z^* - z'^*) \left[ \frac{1 - e^{-(\gamma + i\epsilon)t + \nu^* t}}{\gamma + i\epsilon - \nu^*} + \frac{1 - e^{-(\gamma + i\epsilon)t - \nu^* t}}{\gamma + i\epsilon + \nu^*} \right] \right\}, \\
g_1(t) &= \frac{\Delta\gamma}{8} \left\{ (z - z') \left[ \frac{1 - e^{-\gamma i\epsilon t + \nu t}}{\gamma - i\epsilon - \nu} - \frac{1 - e^{-\gamma i\epsilon t - \nu t}}{\gamma - i\epsilon + \nu} \right] + (z^* - z'^*) \left[ \frac{1 - e^{-(\gamma + i\epsilon)t + \nu^* t}}{\gamma + i\epsilon - \nu^*} - \frac{1 - e^{-(\gamma + i\epsilon)t - \nu^* t}}{\gamma + i\epsilon + \nu^*} \right] \right\}, \\
g_2(t) &= \frac{i\Delta\gamma\delta p}{8} \left\{ (z - z') \left[ \frac{1 - e^{-\gamma i\epsilon t + \nu t}}{\gamma - i\epsilon - \nu} - \frac{1 - e^{-\gamma i\epsilon t - \nu t}}{\gamma - i\epsilon + \nu} \right] + (z^* - z'^*) \left[ \frac{1 - e^{-(\gamma + i\epsilon)t + \nu^* t}}{\gamma + i\epsilon - \nu^*} - \frac{1 - e^{-(\gamma + i\epsilon)t - \nu^* t}}{\gamma + i\epsilon + \nu^*} \right] \right\}, \\
g_3(t) &= \frac{i\Delta\mu}{8} \left\{ (z - z') \left[ \frac{1 - e^{-\gamma i\epsilon t + \nu t}}{\gamma - i\epsilon - \nu} - \frac{1 - e^{-\gamma i\epsilon t - \nu t}}{\gamma - i\epsilon + \nu} \right] + (z^* - z'^*) \left[ \frac{1 - e^{-(\gamma + i\epsilon)t + \nu^* t}}{\gamma + i\epsilon - \nu^*} - \frac{1 - e^{-(\gamma + i\epsilon)t - \nu^* t}}{\gamma + i\epsilon + \nu^*} \right] \right\}, \\
\end{align*}
\]

Also the expressions for \( g_2(t) \) and \( g_3(t) \) can be related to \( g_1(t) = \alpha(t) \) as

\[ g_2(t) = i\beta\alpha(t), \quad g_3(t) = \delta\alpha(t) + \beta(t), \]

where \( \beta(t) \) is the second part of \( g_3(t) \). The expression for \( |\mathbf{H}| \) can be solved to

\[ |\mathbf{H}| = \sqrt{\alpha^2(t) + \beta^2(t) + 2\delta\alpha(t)\beta(t)}. \]

Finally, substituting these into the expression for the probability propagator and taking the restricted averages [3], a simplified expression for the probability propagator for the case of telegraph noise can be obtained.

**NUMERICAL IMPLEMENTATION OF TELEGRAPHIC NOISE**

The different cases considered above for the telegraphic noise can be verified independently from an exact numerical approach. The numerical approach involves the implementation of telegraphic noise followed by the diagonalization of the Hamiltonian at each instant of time. The probability propagator can then be calculated by looking at the dynamics of an initial state.

For the numerical generation of the telegraph noise we follow Refs. 4–8. The method works as follows: Let \( \sigma \) and \( \tau \) be the rate of switching from level \( a \) to \( b \) and \( b \) to \( a \) respectively. The probability of being at any time in state \( a \) is given by \( \tau/(\tau + \sigma) \), whereas the probability of being in state \( b \) is \( \sigma/(\tau + \sigma) \). Furthermore, let \( w_{ij} = (i|W|j) \) with \( i, j = \{a, b\} \) be the matrix elements of the relaxation matrix which gives the transition rate to jump from a state \( j \) to \( i \). The condition of detailed balance implies

\[ p_b(a|W|b) = p_a(b|W|a), \]

where \( p_a \) and \( p_b \) are the probability to remain in state \( a \) and \( b \) respectively. Invoking conservation of probability along with Eqn. 22, the matrix element of the relaxation matrix can be expressed as

\[ w_{ab} = \lambda p_a, \quad w_{ba} = \lambda p_b, \]

where \( \lambda = w_{ab} + w_{ba} \).

The relaxation matrix can thus be written as

\[ W = \lambda \begin{pmatrix} -p_b & p_a \\ -p_a & -p_b \end{pmatrix}. \]
By substituting the values of $p_a$ and $p_b$, the relaxation matrix $W$ can be expressed as
\[
W = \lambda \begin{bmatrix}
\frac{\tau}{\tau + \sigma} & \frac{\tau}{\tau + \sigma} \\
\frac{\sigma}{\tau + \sigma} & \frac{\tau}{\tau + \sigma}
\end{bmatrix},
\]
(25)
where, $\lambda = \tau + \sigma$. The difference of the probabilities between the two levels can be extracted as: $\delta p = \frac{\tau - \sigma}{\tau + \sigma}$.

Also the various conditional probabilities can be expressed in terms of the elements of the relaxation matrix as follows [5, 6]:
\[
\begin{align*}
P_{aa} &= P(a, t_{n+1} | a, t_n) = \frac{\sigma}{\tau + \sigma} + \frac{\tau}{\tau + \sigma} \exp\left(-\left(\tau + \sigma\right)dt\right) \\
P_{ba} &= P(a, t_{n+1} | b, t_n) = \frac{\sigma}{\tau + \sigma} - \frac{\tau}{\tau + \sigma} \exp\left(-\left(\tau + \sigma\right)dt\right) \\
P_{bb} &= P(b, t_{n+1} | b, t_n) = \frac{\tau}{\tau + \sigma} + \frac{\sigma}{\tau + \sigma} \exp\left(-\left(\tau + \sigma\right)dt\right) \\
P_{ab} &= P(b, t_{n+1} | a, t_n) = \frac{\tau}{\tau + \sigma} - \frac{\tau}{\tau + \sigma} \exp\left(-\left(\tau + \sigma\right)dt\right)
\end{align*}
\]
(26)

Finally, the numerical simulation is done as follows. Let the starting state be $a$. A random number between 0 and 1 is generated from the computer, and is compared against the conditional probability $P_{aa}$. If the conditional probability is greater than the random number, the next state will remain $a$, otherwise the next state will be changed to $b$. If the state changes to $b$, then for the next time, a random number is again generated and contrasted against the conditional probability $P_{ba}$. If this conditional probability is greater than the random number, the next state is taken as $a$ else it will remain $b$. If the starting state is $b$, the random number is compared against the conditional probability $P_{bb}$. Again if this conditional probability is greater than the random number, the next state will remain $b$, otherwise it will be changed to $a$. If a flip happens to $a$, then a random number is generated and compared against the conditional probability $P_{ab}$. If this conditional probability is greater than the random number, the next state will flip to $b$, else it will remain $a$. This process is repeated in time units of length $dt$ until the final time is reached. The different cases of the telegraphic noise can then be generated by setting the values $\sigma$ and $\tau$.

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