On the existence of a rainbow 1-factor in proper coloring of $K_{rn}^{(r)}$

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Abstract

El-Zanati et al proved that for any 1-factorization $\mathcal{F}$ of the complete uniform hypergraph $\mathcal{G} = K_{rn}^{(r)}$ with $r \geq 2$ and $n \geq 3$, there is a rainbow 1-factor. We generalize their result and show that in any proper coloring of the complete uniform hypergraph $\mathcal{G} = K_{rn}^{(r)}$ with $r \geq 2$ and $n \geq 3$, there is a rainbow 1-factor.

Keywords: edge-colored graph, rainbow 1-factor, rainbow matching

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1 Introduction

A hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of vertices and a set $\mathcal{E}$ of subsets of $\mathcal{V}$ called edges. An edge subset $\mathcal{E}'$ of disjoint edges of $\mathcal{E}$ is called independent. A proper coloring of $\mathcal{E}$ is a partition of $\mathcal{E}$ into independent sets with each partition set is given a color, say $1, 2, \cdots, l$. For a given coloring of $\mathcal{G}$, a subhypergraph $\mathcal{G}'$ is called rainbow if each edge of $\mathcal{G}'$ has distinct color. A 1-factor of a hypergraph $(\mathcal{V}, \mathcal{E})$ is an independent edge set which partition

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A 1-factorization of $\mathcal{G} = \mathcal{G}(V, E)$ is a partition of $E$ into 1-factors. For positive integers $r \geq 2$ and $n$, the complete $r$-uniform hypergraph on $n$ vertices is the hypergraph $K_n^{(r)}$, with a vertex set $V$ of order $n$ and an edge set $E$ consisting of all $r$-subsets of $V$. Note that $K_n^{(2)}$ is $K_n$, the simple complete graph of order $n$. In order for $K_n^{(r)}$ to contain a 1-factor, it is clearly necessary that $r$ divides $n$. In 1973, Baranyai [1] showed that $K_n^{(r)}$ has a 1-factorization. Given a 1-factorization $\mathcal{F}$ of $K_n^{(r)}$, $n \geq 3$, Woolbright in 1978 showed that there exists a 1-factor in $K_n^{(r)}$ whose edges belong to at least $n - 1$ different 1-factors of $\mathcal{F}$. In 1998, Woolbright and Fu [4] proved that for any 1-factorization of $K_{2n}$ there is a rainbow 1-factor.

In [2], El-Zanati et al proved that for any 1-factorization $\mathcal{F}$ of the complete uniform hypergraph $\mathcal{G} = K_n^{(r)}$ with $r \geq 2$ and $n \geq 3$, there is a rainbow 1-factor. It is clear that a 1-factorization is a very special case of proper colorings. In the present paper, we want to use a weaker condition, the proper coloring condition, to replace their stronger condition, the 1-factorization condition, and to generalize their result as follows: for any proper coloring of the complete uniform hypergraph $\mathcal{G} = K_n^{(r)}$ with $r \geq 2$ and $n \geq 3$, there is a rainbow 1-factor. To show the result, we divide the proof into three cases: $r = 2$ and $n \geq 3$, $r > 2$ and $n = 3$, $r > 2$ and $n > 3$. Notice that the proof of Theorem 3 for the case $r > 2$ in [2] can be used directly to show that for $r > 2, n > 3$ there is a rainbow 1-factor in any proper coloring of the complete uniform hypergraph $\mathcal{G} = K_n^{(r)}$. For the case $r > 2, n = 3$, we give a prove in Theorem 2.3. The substantial part of our proof is to show the result for the case $r = 2$ and $n \geq 3$, which will be given in Theorems 2.1 and 2.2. As a result we have

**Theorem 1.1.** For any proper coloring of the complete uniform hypergraph $\mathcal{G} = K_n^{(r)}$ with $r \geq 2$ and $n \geq 3$, there is a rainbow 1-factor.

### 2 Existence of a rainbow 1-factor in proper coloring of $K_{2n}$ and $K_{3r}^{(r)}$

Let $G = (V, E)$ be a graph and $C$ be a proper coloring of $G$. A rainbow matching of $G$ is a matching whose edges have pairwise different colors. For $e \in E$, let $C(e)$ denote the color of $e$. For $v \in V$, let $C(v) = \{C(e) | e$ is incident with $v\}$. For any subset $E'$ of $E$, let $C(E') = \{C(e) | e \in E'\}$ and $F(E') = C(E) - C(E')$. 


Lemma 2.1. For any proper coloring of \( K_{2n} \), there is a rainbow perfect matching when \( n = 3 \) or \( n = 4 \).

Proof. For \( n = 3 \), let the vertices of \( K_6 \) be \( v_1, v_2, \cdots, v_6 \) and \( C \) be a proper coloring of \( K_6 \), let \( 1, 2, \cdots, l \) be the colors used, we will show that there is a rainbow \( 3K_2 \) in \( C \). Suppose 1 is the color that appears least times in \( C \) on \( E(K_6) \). If 1 appears on three edges, then \( C \) is a 1-factorization of \( K_6 \) and by the proof in [1], there is a rainbow 1-factor in \( C \). If 1 appears on two edges, say \( C(x_1x_2) = C(x_3x_4) = 1 \), assume that \( C(x_5x_1) = 2, C(x_5x_2) = 3, C(x_5x_3) = 4, C(x_5x_4) = 5 \), \( C(x_5x_6) = 6 \). Since \( \{x_5x_1, x_6x_2, x_3x_4\} \) is independent, to avoid the existence of rainbow perfect matching, it must be that \( C(x_6x_2) = 2 \). Similarly, \( \{x_5x_2, x_6x_1, x_3x_4\} \) is independent and \( C(x_6x_1) = 3 \); \( \{x_1x_2, x_5x_3, x_6x_4\} \) is independent and \( C(x_6x_4) = 4 \); \( \{x_1x_2, x_5x_4, x_6x_3\} \) is independent and \( C(x_6x_3) = 5 \). Now both \( \{x_5x_2, x_6x_3, x_1x_4\} \) and \( \{x_5x_3, x_6x_2, x_1x_4\} \) are independent, whatever color the edge \( x_1x_4 \) receives, we will have a rainbow perfect matching. If 1 appears only once in \( C \) and \( c(x_1x_2) = 1 \), to avoid the existence of rainbow perfect matching, there is no rainbow \( 2K_2 \) in the subgraph induced by \( \{v_2, v_3, v_4, v_5\} \). The only such coloring is \( c(x_3x_4) = c(x_5x_6) = 2, c(x_3x_5) = c(x_4x_6) = 3, c(x_3x_6) = c(x_5x_4) = 4 \). Assume \( c(x_3x_1) = 5 \). Since both \( \{x_1x_3, x_5x_6, x_2x_4\} \) and \( \{x_1x_3, x_4x_6, x_2x_5\} \) are independent, we have \( C(x_2x_4) = 5 \) and \( C(x_2x_5) = 5 \), a contradiction. So in any proper coloring of \( K_6 \), there is a rainbow perfect matching.

For \( n = 4 \), let the vertices of \( K_8 \) be \( v_1, v_2, \cdots, v_8 \) and \( C \) be a proper coloring of \( K_8 \), we will show that there is a rainbow \( 4K_2 \) in \( C \). Starting with any triangle, it is possible to find in \( C \) of \( G = K_8 \) at least two rainbow \( K_4 \), say \( G[\{v_1, v_2, v_3, v_4\}] \) and \( G[\{v_1, v_2, v_3, v_5\}] \) are both rainbow. If there is at least one rainbow \( 2K_2 \) in \( G[\{v_5, v_6, v_7, v_8\}] \), since \( \{v_1v_2, v_3v_4\} \), \( \{v_1v_3, v_2v_4\} \) and \( \{v_1v_4, v_2v_3\} \) are all independent and each edge has a distinct color, we can find a rainbow \( 4K_4 \) in \( C \). Similarly, there is no rainbow \( 2K_2 \) in \( G[\{v_4, v_6, v_7, v_8\}] \). But it is impossible that both \( G[\{v_5, v_6, v_7, v_8\}] \) and \( G[\{v_4, v_6, v_7, v_8\}] \) have no rainbow \( 2K_2 \), and the proof is complete. \(\square\)

Theorem 2.2. For \( n \geq 3 \), any proper coloring of \( K_{2n} \) contains a rainbow perfect matching.
Proof. By Lemma 1, we can assume \( n \geq 5 \). Let \( C \) be any proper coloring of \( K_{2n} \) with the colors named 1, 2, \( \cdots \), \( l \), \( l \geq 2n - 1 \). Let \( \mathcal{M} \) be any maximal rainbow matching with \( |\mathcal{M}| = k \). Suppose \( k < n \), we will show that there must be a rainbow matching with \( k + 1 \) edges. Recall that \( C(\mathcal{M}) \) denotes the set of colors of \( \mathcal{M} \) and \( F(\mathcal{M}) \) denotes the complementary set of colors. Let \( s, t \) be two unmatched vertices. We may assume that \( C(\mathcal{M}) = \{1, 2, \cdots, k\} \) and \( C(st) = 1 \). Note that by maximality of \( \mathcal{M} \), any edge incident with \( s \) whose color is in \( F(\mathcal{M}) \) must be incident with an edge of \( \mathcal{M} \).

Let \( C(s) = C_1 \cup C_2 \) with \( C_1 \subseteq C(\mathcal{M}) \), \( C_2 \subseteq F(\mathcal{M}) \), \( |C_1| = p \leq k \), \( |C_2| = 2n - 1 - p \). Consider all the \((s, t)\)-paths of length three, whose first edge is colored with a color \( \alpha \) in \( F(\mathcal{M}) \), and the second edge is in \( \mathcal{M} \); we call them the candidate 3-paths relative to \( \mathcal{M} \). We can assume that each of these paths has its third edge colored with a color either in \( C(\mathcal{M}) - \{1\} \), or the color \( \alpha \) again, for otherwise we could augment \( \mathcal{M} \) to \( k + 1 \) edges simply by deleting the second edge of the path from it and adding the first and third edges. There are \( 2n - 1 - p \) of these candidate paths and only \( k - 1 \) colors in \( C(\mathcal{M}) - \{1\} \). So it follows that at least \( 2n - p - k \) of these paths have the first and third edges colored with the same color in \( F(\mathcal{M}) \); we call such paths \( \mathcal{M} \)-symmetric \((s, t)\)-paths.

Let \( C(t) = C'_1 \cup C'_2 \) with \( C'_1 \subseteq C(\mathcal{M}) \), \( C'_2 \subseteq F(\mathcal{M}) \), \( |C'_1| = q \leq k \), \( |C'_2| = 2n - 1 - q \). Consider the \( 2n - 1 - q \) edges incident with \( t \) whose colors are in \( F(\mathcal{M}) \). Each of these edges must be incident with an edge of \( \mathcal{M} \), by the maximality of \( \mathcal{M} \); at most 2 of them, say the ones colored \( k + 1 \) and \( k + 2 \), are incident with the edge of \( \mathcal{M} \) colored 1. Now let \( L = C(t) \setminus (C(\mathcal{M}) \cup \{k+1, k+2\}) \) and \( |L| = 2n - q - 3 \).

For each color \( i \in L \), we define a slight variation of the \((\mathcal{M}, st)\) pair. If the edge of color \( i \) incident with vertex \( t \) is \( e_t = \{t, z_i\} \) of \( \mathcal{M} \), we let the corresponding matching be \( \mathcal{M}_i = (\mathcal{M} - \{e_i\}) \cup \{e_t\} \); now \( t_i \) is unmatched (in \( \mathcal{M}_i \)), and we let our starting/ending vertex pair be \( s, t_i \), respectively. Note that \( F(\mathcal{M}_i) = (F(\mathcal{M}) - i) \cup \{C(e_i)\} \). Also note that \( C(e_i) \neq 1 \), because \( i \) is neither \( k + 1 \) or \( k + 2 \).

As in the previous discussion, for each such \( i \), there are \( 2n - 1 - p \) candidate 3-paths relative to \( \mathcal{M}_i \), starting at \( s \), ending at \( t_i \), whose first edge is colored with a color in \( F(\mathcal{M}_i) \), and whose second edge is in \( \mathcal{M}_i \). Again, we assume that at least \( 2n - p - k \) of these paths are symmetric. Thus, listing the symmetric
paths for \( i \in L \), we get a total of at least \((2n - q - 3)(2n - p - k)\) paths in the list of symmetric candidate paths. However, because in each of these symmetric paths either the middle edge is in \( M \), or the path has the form \( sz_i t_i \), and therefore has the same first and third edges as \( sz_i t_i \), each of the symmetric candidate paths is uniquely determined by its first edge. Moreover, the color \( \alpha \) of the starting/ending edge in these symmetric paths cannot be \( c(e_i) = c(\{z_i, t_i\}) \) (the only possible such path has vertex sequence \( sz_i t_i \), which is not symmetric because \( c(e_i) \neq 1 \)), so \( \alpha \) must be in \( C_2 \). Therefore, each of the possible starting color can only start one path in the list. It follows that \( 2n - p - 1 \geq (2n - q - 3)(2n - p - k) \). Let \( x = 2n - p \), \( y = 2n - q \), then \( x - 1 \geq (y - 3)(x - k) \), and \( x(y - 4) < k(y - 3) \). Since \( q \leq k < n \), \( y = 2n - q = n + n - q \geq n + n - k > n \), \( y > 4 \). Then we have \( \frac{x}{k} < \frac{y - 3}{y - 4} \) and \( x < k \), that is \( p + k > 2n \), which is contrary to \( p \leq k < n \).

We conclude that there must be a rainbow matching with \( k + 1 \) edges, and so the result follows. \( \square \)

For the case \( n = 3 \), it is easy to see that in any 1-factorization of \( K_{3r}^{(r)} \) there is a rainbow 1-factor, and the proof was omitted in [2]. But in a proper coloring, the proof is not straightforward, and we prefer to give the details in the following.

**Theorem 2.3.** For \( r \geq 2 \), any proper coloring of \( K_{3r}^{(r)} \) contains a rainbow perfect matching.

**Proof.** Let the vertex set of \( K_{3r}^{(r)} \) be \( V = \{x_1, x_2, \cdots, x_{3r}\} \) and \( C \) be a proper coloring of \( K_{3r}^{(r)} \). Take any two independent edges having a same color 1, say \( m_1 = \{x_1, x_2, \cdots, x_r\} \) and \( m_2 = \{x_{r+1}, x_{r+2}, \cdots, x_{2r}\} \). Then for any edge \( m^1 \subset V - m_1 \) other than \( m_2 \) and \( m^{1*} = V - (m_1 \cup m^1) \), \( m^1 \) and \( m^{1*} \) have the same color and there is no other edge in this color, otherwise \( \{m_1, m^1, m^{1*}\} \) is a rainbow 1-factor. Similarly, for any edge \( m_2 \subset V - m_2 \) other than \( m_1 \) and \( m^{2*} = V - (m_2 \cup m^2) \), \( m^2 \) and \( m^{2*} \) have the same color and there is no other edge in this color. Let \( m^1 = \{x_{r+1}, x_{r+2}, \cdots, x_{2r-1}, x_{2r+1}\} \), \( m^2 = \{x_1, x_{2r+2}, x_{2r+3}, \cdots, x_{3r}\} \), then \( \{m^1, m^2, \{x_2, x_3, \cdots, x_r, x_{2r}\}\} \) is a rainbow 1-factor. \( \square \)
References

[1] Zs. Baranyai, On the factorization of the complete uniform hypergraph, In: Infinite and finite sets I Colloq Math Soc János Bolyai 10, North-Holland, Amsterdam, 1975, pp.91-108.

[2] S.I. El-Zanati, M.J. Plantholt, P.A. Sissokho and L.E. Spence, On the existence of rainbow 1-factor in 1-factorizations of $K_{rn}^{(r)}$, J. Combin. Des. 2007.

[3] D.E. Woolbright, On the size of partial 1-factor of 1-factorization of the complete $k$-uniform hypergraph on $kn$ vertices, Ars Combin. 6(1978), 185-192.

[4] D.E. Woolbright and H.L. Fu, On the existence of rainbows in 1-factorizations of $K_{2n}$, J. Combin. Des. 6(1998), 1-20.