ANOSOV REPRESENTATIONS, STRONGLY CONVEX COCOMPACT GROUPS AND WEAK EIGENVALUE GAPS

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Abstract. We provide characterizations of Anosov representations of word hyperbolic groups into real semisimple Lie groups in terms of the existence of equivariant limit maps on the Gromov boundary, the Cartan property and the uniform gap summation property introduced by Guichard-Guéritaud-Kassel-Wienhard in [22]. We also study representations of finitely generated groups satisfying weak uniform gaps in eigenvalues and establish conditions to be Anosov. As an application, we also obtain a characterization of strongly convex cocompact subgroups of the projective linear group $\mathrm{PGL}_d(\mathbb{R})$.

1. Introduction

Anosov representations of fundamental groups of closed negatively curved Riemannian manifolds were introduced by Labourie [33] in his study of the Hitchin component. Labourie’s definition was later extended by Guichard–Wienhard in [23] for general word hyperbolic groups. Anosov representations have been extensively studied during the last decade by Guichard–Wienhard [23], Kapovich–Leeb–Porti [27, 28, 29], Guéritaud–Guichard–Kassel–Wienhard [22], Bochi–Potrie–Sambarino [7], Danciger–Guéritaud-Kassel [17], Zimmer [39] and others, and are now recognized as a higher rank analogue of convex cocompact representations of word hyperbolic groups into simple Lie groups of real rank 1. Moreover, recently, there have been introduced certain generalizations of classical Anosov representations for relatively hyperbolic groups, we refer to the work of Kapovich–Leeb [26], Zhu [38] and Weisman [40] for more details.

Based on the existing characterizations established in [23, 22, 27, 28, 29, 7, 31], one may define Anosov representations of a hyperbolic group into a semisimple Lie group in terms of the existence of a pair of well-behaved limit maps from the Gromov boundary of the domain group to the corresponding flag spaces, or entirely in terms of uniform gaps in the Cartan or Lyapunov projection of the image of the representation. The purpose of the present paper is to provide new characterizations and strengthen some of the existing ones. Our characterizations are in terms of the existence of limit maps, the Cartan property (see subsection 1.1) and the uniform gap summation property introduced in [22]. As an application of our main results we also obtain characterizations of strongly convex cocompact subgroups of the projective linear group $\mathrm{PGL}_d(\mathbb{R})$ (see subsection 1.3). More generally, we study linear representations of finitely generated groups satisfying weak uniform gaps in eigenvalues and we establish sufficient conditions for the domain group to be word hyperbolic and the representation to be Anosov (see sub-section 1.2). In order to provide such conditions, we study the relation between strong property (U), introduced by Kassel–Potrie in [31], and the uniform gap summation property. More precisely, we prove that a finitely generated non-virtually nilpotent group $\Gamma$ which admits a linear representation with the uniform gap summation property (see Definition 4.7), then $\Gamma$ satisfies strong property (U) which is a condition relating the word length and the stable translation length of certain group elements (see Theorem 1.7).

1.1. Characterizations in terms of limit maps and the Cartan property. Let $\Gamma$ be an infinite word hyperbolic group, $G$ be a linear, non-compact semisimple Lie group with finitely many connected
components and fix $K$ a maximal compact subgroup of $G$. We also fix a Cartan subspace $\mathfrak{a}$ of $\mathfrak{g}$, $\mathfrak{T}^+$ a closed Weyl chamber of $\mathfrak{a}$, a Cartan decomposition $G = K \exp(\mathfrak{T}^+) K$ and consider the Cartan projection $\mu : G \to \mathfrak{T}^+$. For a linear form $\varphi \in \mathfrak{a}^*$ and $H \in \mathfrak{a}$, define $\langle \varphi, H \rangle := \varphi(H)$.

Every subset $\theta \subset \Delta$ of simple restricted roots of $G$ defines a pair of opposite parabolic subgroups $P_\theta^+$ and $P_\theta^-$, well defined up to conjugation. Labourie’s dynamical definition of a $P_\theta$-Anosov representation $\rho : \Gamma \to G$ requires the existence of a pair of continuous $\rho$-equivariant maps from the Gromov boundary $\partial_{\infty} \Gamma$ to the flag spaces $G/P_\theta^+$ and $G/P_\theta^-$ called the Anosov limit maps of $\rho$ (see Definition 2.2). Our first characterization of Anosov representations is based on the existence of a pair of transverse continuous and equivariant limit maps on the Gromov boundary of the groups one of which satisfies the Cartan property:

\textbf{Theorem 1.1.} Let $\Gamma$ be a word hyperbolic group, $G$ a real semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of $G$ and $\rho : \Gamma \to G$ a representation. Then $\rho$ is $P_\theta$-Anosov if and only if the following conditions are simultaneously satisfied:

(i) $\rho$ is $P_\theta$-divergent.
(ii) There exists a pair of continuous, $\rho$-equivariant transverse maps

$$\xi^+ : \partial_{\infty} \Gamma \to G/P_\theta^+ \quad \text{and} \quad \xi^- : \partial_{\infty} \Gamma \to G/P_\theta^-$$

and the map $\xi^+$ satisfies the Cartan property.

Let us now shortly explain the assumptions of Theorem 1.1. We say that two $\rho$-equivariant maps $\xi^+ : \partial_{\infty} \Gamma \to G/P_\theta^+$ and $\xi^- : \partial_{\infty} \Gamma \to G/P_\theta^-$ are transverse, if for any two distinct points $x^+, x^- \in \partial_{\infty} \Gamma$ there is $g \in G$ such that $\xi^+(x^+) = gP_\theta^+$ and $\xi^-(x^-) = gP_\theta^-$. The representation $\rho : \Gamma \to G$ is $P_\theta$-divergent if for every infinite sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of $\Gamma$ and $\alpha \in \theta$, the sequence $((\alpha, \mu(\rho(\gamma_n))))_{n \in \mathbb{N}}$ goes to infinity. The map $\xi^+ : \partial_{\infty} \Gamma \to G/P_\theta^+$ satisfies the Cartan property if for every sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of $\Gamma$ converging to a point $x \in \partial_{\infty} \Gamma$ in the Gromov boundary, then $\xi^+(x) = \lim_n k_n P_\theta^+$, where $\rho(\gamma_n) = k_n \exp(\mu(\rho(\gamma_n)))k'_n$, $k_n, k'_n \in K$, is written in the Cartan decomposition of $G$. Examples of maps with this property are the limit maps of an Anosov representation (see [7] and [22, Thm. 1.3 (4) & 5.3 (4)]). We discuss the Cartan property in more detail in §4, where we prove (see Corollary 4.6) that for a Zariski dense representation $\rho : \Gamma \to G$ a (necessarily unique if it exists) continuous $\rho$-equivariant map $\xi : \partial_{\infty} \Gamma \to G/P_\theta^+$ has to satisfy the Cartan property.

In Theorem 1.1 the assumption that the map $\xi^+$ satisfies the Cartan is necessary and cannot be dropped (see Example 10.2). Moreover, we do not assume that the image $\rho(\Gamma)$ contains a $P_\theta$-proximal element in $G/P_\theta^+$ or that the pair of maps $(\xi^+, \xi^-)$ is compatible at some point $x \in \partial_{\infty} \Gamma$, i.e. the intersection $\text{Stab}_G(\xi^+(x)) \cap \text{Stab}_G(\xi^-(x))$ is a parabolic subgroup of $G$. Under the assumption that both maps $(\xi^+, \xi^-)$ satisfy the Cartan property, Theorem 1.1 also follows from [29, Thm. 1.7]. We explain how Theorem 1.1 is related to [29, Thm. 1.7], [27, Thm. 5.47] and [22, Thm. 1.3] at the end of this section.

Let $\Gamma$ be a finitely generated group. We fix a left invariant word metric $d_\Gamma$ on $\Gamma$ induced by a finite generating subset of $\Gamma$ and let $| \cdot | : \Gamma \to \mathbb{N}$ be the word length function defined by $|\gamma|_\Gamma = d_\Gamma(\gamma, e), \gamma \in \Gamma$. As an application of Theorem 1.1, we deduce the following characterization of Anosov representations entirely in terms of the growth of the Cartan projection of the image of a representation.

\textbf{Corollary 1.2.} Let $\Gamma$ be an infinite word hyperbolic group, $G$ a real semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of $G$, $\{\omega_\alpha\}_{\alpha \in \theta}$ the associated set of fundamental weights. Fix $| \cdot | : \Gamma \to \mathbb{N}$ a word length function on $\Gamma$. A representation $\rho : \Gamma \to G$ is $P_\theta$-Anosov if and only if the following conditions are simultaneously satisfied:

(i) There exist $C, c > 1$ such that for every $\gamma \in \Gamma$ and $\alpha \in \theta$,

$$\langle \alpha, \mu(\rho(\gamma)) \rangle \geq c \log |\gamma|_\Gamma - C.$$
(ii) There exist $B, b > 0$ such that for every $\gamma \in \Gamma$,
\[
\max_{x \in \Theta} \langle \omega_0, 2\mu(\rho(x)) - \mu(\rho(x^2)) \rangle \leq B(2|\gamma|_\Gamma - |\gamma^2|_\Gamma) + b.
\]

Now let $\rho : \Gamma \to G$ be a Zariski dense representation which admits a pair of $\rho$-equivariant limit maps $\xi^+ : \partial_\infty \Gamma \to G/P^+_\theta$ and $\xi^- : \partial_\infty \Gamma \to G/P^-_\theta$. In [23, Thm. 5.11], Guichard–Wienhard proved that $\rho$ is $P_\theta$-Anosov if and only if $\xi^+$ and $\xi^-$ are compatible and transverse. By using Theorem 1.1 and Corollary 4.6, we obtain the following slightly improved version of their theorem. For a quasi-convex subgroup $H$ of $\Gamma$ we denote by $i_H : \partial_\infty H \hookrightarrow \partial_\infty \Gamma$ the Cannon–Thurston map extending the inclusion $H \hookrightarrow \Gamma$.

**Theorem 1.3.** Let $\Gamma$ be a word hyperbolic group, $H$ a quasiconvex subgroup of $\Gamma$, $G$ a semisimple Lie group, $\theta < \Delta$ a subset of simple restricted roots of $G$ and $\rho : \Gamma \to G$ a Zariski dense representation. Suppose that $\rho$ admits continuous $\rho$-equivariant maps $\xi^+ : \partial_\infty \Gamma \to G/P^+_\theta$ and $\xi^- : \partial_\infty \Gamma \to G/P^-_\theta$. Then the restriction $\rho|_H : H \to G$ is $P_\theta$-Anosov if and only if the maps $\xi^+ \circ i_H : \partial_\infty H \to G/P^+_\theta$ and $\xi^- \circ i_H : \partial_\infty H \to G/P^-_\theta$ are transverse.

For a matrix $g \in \text{GL}_d(\mathbb{R})$ we denote by $\ell_1(g) \geq \ldots \geq \ell_d(g)$ and $\sigma_1(g) \geq \ldots \geq \sigma_d(g)$ the eigenvalues and the singular values of $g$ in non-increasing order. Let $\rho_i : \Gamma \to \text{SL}_{m_i}(\mathbb{R})$, $i \in \{1, 2\}$, be two representations such that $\rho_2$ is $P_1$-Anosov. We recall that the stretch factors associated with the representations $\rho_1$ and $\rho_2$ are:
\[
\text{dil}_-(\rho_1, \rho_2) := \inf_{\gamma \in \Gamma} \frac{\log \ell_1(\rho_1(\gamma))}{\log \ell_1(\rho_2(\gamma))}, \quad \text{dil}_+(\rho_1, \rho_2) := \sup_{\gamma \in \Gamma} \frac{\log \ell_1(\rho_1(\gamma))}{\log \ell_1(\rho_2(\gamma))},
\]
where $\Gamma_\infty$ denotes the set of infinite order elements of $\Gamma$. Observe that since $\rho_2$ is a quasi-isometric embedding (see Theorem 2.3(i)), the stretch factors $\text{dil}_\pm(\rho_1, \rho_2)$ are well-defined. As a corollary of Theorem 1.1 we obtain the following approximation result for particular pairs of representations $(\rho_1, \rho_2)$, which refines a consequence of the density result of Benoist obtained in [4] in this case.

**Corollary 1.4.** Let $\Gamma$ be a word hyperbolic group and fix $| \cdot | : \Gamma \to \mathbb{N}$ a word length function on $\Gamma$. Suppose that $\rho_1 : \Gamma \to \text{SL}_{m_1}(\mathbb{R})$ and $\rho_2 : \Gamma \to \text{SL}_{m_2}(\mathbb{R})$ are two representations such that $\rho_2$ is $P_1$-Anosov and $\rho_1$ satisfies one of the following conditions:
(i) $\rho_1$ is $P_1$-Anosov.
(ii) $\rho_1(\Gamma)$ is contained in a semisimple $P_1$-proximal Lie subgroup of $\text{SL}_{m_1}(\mathbb{R})$ of real rank 1.
Then for every $\delta > 0$ and $p, q \in \mathbb{N}$ with $\text{dil}_-(\rho_1, \rho_2) \leq \frac{\delta}{q} \leq \text{dil}_+(\rho_1, \rho_2)$, there exists an infinite sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of $\Gamma$ such that for every $n \in \mathbb{N}$:
\[
\left| \frac{p}{q} - \frac{\log \sigma_1(\rho_1(\gamma_n))}{\log \sigma_1(\rho_2(\gamma_n))} \right| \leq \frac{\delta}{q} \frac{|\gamma_n|_\Gamma}{|\gamma_n|_\Gamma}.
\]

1.2. Weak uniform gaps in eigenvalues and strong property (U). Kassel–Potrie introduced the following definition in [31]:

**Definition 1.5.** Let $\Gamma$ be a finitely generated group, $\rho : \Gamma \to \text{GL}_d(\mathbb{R})$ a representation and fix $1 \leq i \leq d-1$. The representation $\rho$ has a weak uniform $i$-gap in eigenvalues if there exists $\varepsilon > 0$ such that for every $\gamma \in \Gamma$ we have
\[
\log \frac{\ell_i(\rho(\gamma))}{\ell_{i+1}(\rho(\gamma))} \geq \varepsilon|\gamma|_\infty,
\]
where $|\gamma|_\infty = \lim_n |\gamma^n|_\infty / n$ denotes the stable translation length of $\gamma$.

The existence of a uniform $i$-gap in eigenvalues for $\rho$ is not a sufficient condition to guarantee that the representation is Anosov, and it is a natural question to determine additional conditions guaranteeing that this happens. Guéritaud–Guichard–Kassel–Wienhard proved that if $\Gamma$ is word hyperbolic, $\rho$ has a weak uniform $i$-gap in eigenvalues and admits a pair of continuous, $\rho$-equivariant, dynamics preserving and transverse maps $\xi^+ : \partial_\infty \Gamma \to \text{Gr}_i(\mathbb{R}^d)$ and
\( \xi^- : \partial \gamma \to \text{Gr}_{d-1}(\mathbb{R}^d) \), then \( \rho \) is \( P_t \)-Anosov (see [22, Thm. 1.7 (c)]). Kassel–Potrie proved [31, Prop. 4.12] that if \( \Gamma \) satisfies weak property (U) (see Definition 5.1) and \( \rho \) has a weak uniform \( i \)-gap in eigenvalues, then \( \rho \) has a strong \( i \)-gap in singular values: there exist \( C, c > 0 \) such that for every \( \gamma \in \Gamma \),

\[
\log \frac{\sigma_i(\rho(\gamma))}{\sigma_{i+1}(\rho(\gamma))} \geq c |\gamma|_{\Gamma} - C,
\]

hence \( \Gamma \) is hyperbolic and \( \rho \) is \( P_t \)-Anosov by the work of Kapovich–Leeb–Porti [28] and Bochi–Potrie–Sambarino [7]. The following theorem, motivated by [31, Question 4.9], provides further conditions under which a linear representation \( \rho : \Gamma \to \text{GL}_d(\mathbb{R}) \) of a finitely generated group \( \Gamma \) with a weak uniform \( i \)-gap in eigenvalues is \( P_t \)-Anosov.

**Theorem 1.6.** Let \( \Gamma \) be a finitely generated infinite group which is not virtually cyclic and fix \( |.| : \Gamma \to \mathbb{N} \) a word length function on \( \Gamma \). Suppose that \( \rho : \Gamma \to \text{GL}_d(\mathbb{R}) \) is a representation which has a weak uniform \( i \)-gap in eigenvalues for some \( 1 \leq i \leq d - 1 \). Then the following conditions for \( \Gamma \) and \( \rho \) are equivalent:

(i) \( \Gamma \) is word hyperbolic and \( \rho \) is \( P_t \)-Anosov.
(ii) There exists a Floyd function \( f \) such that the Floyd boundary \( \partial_f \Gamma \) of \( \Gamma \) is uncountable.
(iii) \( \Gamma \) admits a representation \( \rho_1 : \Gamma \to \text{GL}_m(\mathbb{R}) \) satisfying the uniform gap summation property.
(iv) \( \Gamma \) admits a semisimple representation \( \rho_2 : \Gamma \to \text{GL}_r(\mathbb{R}) \) with the property

\[
\lim_{|\gamma| \to \infty} \frac{1}{\log |\gamma|_{\Gamma}} \log \frac{\sigma_1(\rho(\gamma))}{\sigma_c(\rho(\gamma))} = +\infty.
\]

We prove that each one of the conditions (ii), (iii) and (iv) implies that \( \Gamma \) has strong property (U) (see Definition 5.1), so (i) will follow by the eigenvalue gap characterization from [31, Prop. 1.2]. The uniform gap summation property is a summability condition for gaps between singular values, see [22, Def. 5.2] and Definition 4.7 for the precise definitions. For example, condition (iii) of the previous theorem is satisfied when there exist \( 1 \leq j \leq m - 1 \) and \( C, c > 1 \) such that for every \( \gamma \in \Gamma \)

\[
\log \frac{\sigma_j(\rho(\gamma))}{\sigma_{j+1}(\rho(\gamma))} \geq c \log |\gamma|_{\Gamma} - C.
\]

For the proof of implication (ii) \( \Rightarrow \) (i) in Theorem 1.6 we establish that a torsion free finitely generated group whose Floyd boundary is uncountable, satisfies strong property (U).

**Theorem 1.7.** Let \( \Gamma \) be a finitely generated group and fix \( |.| : \Gamma \to \mathbb{N} \) a word length function on \( \Gamma \). Suppose that there exists a Floyd function \( f : \mathbb{N} \to (0, \infty) \) such that the Floyd boundary \( \partial_f \Gamma \) of \( \Gamma \) is non-trivial. Let \( H \) be a torsion free subgroup of \( \Gamma \) whose limit set \( \Lambda(H) \) in \( \partial_f \Gamma \) contains at least three points. Then there exists a finite subset \( F \) of \( H \) and \( C > 0 \), depending only on the group \( H \), with the property: for every \( \gamma \in H \) there exists \( g \in F \) such that

\[
|g\gamma|_{\Gamma} - |g\gamma|_X \leq C.
\]

In particular, if \( \Gamma \) is virtually torsion free then it satisfies strong property (U).

As a corollary of the previous theorem we deduce that a non-virtually nilpotent group which admits a representation with the uniform gap summation property admits a non-trivial Floyd boundary.

**Corollary 1.8.** Let \( \Gamma \) be a finitely generated group which is not virtually nilpotent, \( G \) a semisimple Lie group and \( \theta \subset \Delta \) a subset of simple restricted roots of \( G \). Suppose that there exists a representation \( \rho : \Gamma \to G \) which satisfies the uniform gap summation property with respect to \( \theta \) and a Floyd function \( f : \mathbb{N} \to (0, \infty) \). Then the Floyd boundary \( \partial_f \Gamma \) of \( \Gamma \) with respect to \( f \) is non-trivial. In particular, \( \Gamma \) satisfies strong property (U).
1.3. Characterizations of strongly convex cocompact groups. Anosov representations of hyperbolic groups are closely related with real projective geometry and geometric structures. Fix an integer $d \geq 3$. A subset $\Omega$ of the projective space $\mathbb{P}(\mathbb{R}^d)$ is called \textit{properly convex} if it is contained in an affine chart on which $\Omega$ is bounded and convex. The domain $\Omega$ is called \textit{strictly convex} if it is properly convex and $\partial \Omega$ does not contain projective line segments.

Let $\Gamma$ be a discrete subgroup of $\text{PGL}_d(\mathbb{R})$ which preserves a properly convex domain $\Omega$ of $\mathbb{P}(\mathbb{R}^d)$. The full orbital limit set $\Lambda_{\Omega}(\Gamma)$ of $\Gamma$ in $\Omega$ is the set of accumulation points of all $\Gamma$-orbits in $\partial \Omega$ (see [17, Def. 1.10]). The group $\Gamma$ acts \textit{convex cocompactly} on $\Omega$ if the convex hull of $\Lambda_{\Omega}(\Gamma)$ in $\Omega$ is non-empty and has compact quotient by $\Gamma$ (see [17, Def. 1.11]). The group $\Gamma$ is called \textit{strongly convex cocompact} in $\mathbb{P}(\mathbb{R}^d)$ if it acts convex cocompactly on some properly convex domain $\Omega$ with strictly convex and $C^1$-boundary. The work of Danciger–Guéritaud–Kassel [17] and independently of Zimmer [39], shows that Anosov representations can be essentially (up to composition with a Lie group homomorphism) viewed as convex cocompact actions on properly convex domains in some real projective space. We refer the reader to [17, Thm. 1.4 & 1.15] and [39, Thm. 1.22 & 1.25]. There are also related results in the setting of naively convex cocompact groups, see [24, Thm. 1.13].

For the definition of a $P_\theta$-Anosov representation $\rho : \Gamma \to G$, where $G$ is either $\text{PGL}_d(\mathbb{R})$ or $\text{GL}_d(\mathbb{R})$, we refer to Definition 2.2. The following result from [17] offers a connection between Anosov representations and strongly convex cocompact actions on properly convex domains.

**Theorem 1.9.** ([17, Thm. 1.4]) Let $\Gamma$ be an infinite discrete subgroup of $\text{PGL}_d(\mathbb{R})$ which preserves a properly convex domain of $\mathbb{P}(\mathbb{R}^d)$. Then $\Gamma$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^d)$ if and only if $\Gamma$ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow \text{PGL}_d(\mathbb{R})$ is $P_\theta$-Anosov.

For a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ let $d_\Omega$ be the Hilbert metric defined on $\Omega$. As an application of Theorem 1.1, we obtain the following geometric characterization of strongly convex cocompact subgroups of $\text{PGL}_d(\mathbb{R})$ which are semisimple, i.e. their Zariski closure in $\text{PGL}_d(\mathbb{R})$ is a reductive Lie group.

**Theorem 1.10.** Let $\Gamma$ be a finitely generated subgroup of $\text{PGL}_d(\mathbb{R})$. Suppose that $\Gamma$ preserves a strictly convex domain of $\mathbb{P}(\mathbb{R}^d)$ with $C^1$-boundary and the natural inclusion $\Gamma \hookrightarrow \text{PGL}_d(\mathbb{R})$ is semisimple. Then the following conditions are equivalent:

(i) $\Gamma$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^d)$.

(ii) The inclusion $\Gamma \hookrightarrow \text{PGL}_d(\mathbb{R})$ is a quasi-isometric embedding, $\Gamma$ preserves a properly convex domain $\Omega$ of $\mathbb{P}(\mathbb{R}^d)$ and there exists a $\Gamma$-invariant closed convex subset $C$ of $\Omega$ such that $(C, d_\Omega)$ is Gromov hyperbolic.

The previous theorem generalizes the well known fact that a discrete subgroup $\Gamma$ of $\text{PO}(d,1)$, $d \geq 2$, is convex cocompact if and only if the inclusion $\Gamma \hookrightarrow \text{PO}(d,1)$ is a quasi-isometric embedding.

1.4. Gromov product. We also introduce a definition of a Gromov product on $G \times G$ which we use for the proof of Theorem 1.10 (see Lemma 8.1). Let us remark that there are similar definitions of Gromov products in [6, §3] and [7, §8], however these are defined on the flag spaces associated to $G$. The Gromov product from [7] is also vector valued into a Cartan subspace of the Lie algebra of $G$.

**Definition 1.11.** Let $G$ be a real semisimple Lie group. For every linear form $\varphi \in \mathfrak{a}^*$, define the Gromov product relative to $\varphi$ to be the map $(\cdot)_\varphi : G \times G \to \mathbb{R}$ defined as follows: for $g, h \in G$,

$$
(g \cdot h)_\varphi := \frac{1}{4} \langle \varphi, \mu(g) + \mu(g^{-1}) + \mu(h) + \mu(h^{-1}) - \mu(g^{-1}h) - \mu(h^{-1}g) \rangle.
$$

We prove that for every $P_\theta$-Anosov representation $\rho : \Gamma \to G$, the restriction of the Gromov product on $\rho(\Gamma) \times \rho(\Gamma)$, with respect to a fundamental weight $\omega_\alpha$, $\alpha \in \theta$, grows coarsely as the Gromov product on $\Gamma \times \Gamma$ with respect to a world length function on $\Gamma$. 

Proposition 1.12. Let $G$ be a real semisimple Lie group, fix $\theta \subset \Delta$ a subset of simple restricted roots of $G$ and let $\{\omega_\alpha\}_{\alpha \in \theta}$ be the associated set of fundamental weights. Suppose that $\Gamma$ is a word hyperbolic group and $\rho : \Gamma \to G$ is a $P_\theta$-Anosov representation. There exist $C, c > 1$ with the property that for every $\alpha \in \theta$ and $\gamma, \delta \in \Gamma$ we have

$$C^{-1}(\gamma ; \delta)_{c} - c \leq (\rho(\gamma) \cdot \rho(\delta))_{\omega_\alpha} \leq C(\gamma ; \delta)_{c} + c.$$ 

We remark that in the case where $\omega_\alpha = \varepsilon_1$, where $\varepsilon_1(x_1, \ldots, x_m) = x_1$ is the projection in the first coordinate, the double inequality in the previous proposition is not enough to guarantee that $\rho$ is $P_1$-Anosov (see Example 10.3). However, if $\rho : \Gamma \to \mathrm{PGL}_d(\mathbb{R})$ preserves a properly convex domain $\Omega$ of $\mathbb{P}(\mathbb{R}^d)$ with strictly convex and $C^1$-boundary and the Gromov product on the Cartan projection of $\rho(\Gamma)$ with respect to $\varepsilon_1 \in \mathfrak{a}^*$ grows coarsely as the Gromov product on $\Gamma$, then $\rho$ is $P_1$-Anosov (see Proposition 8.1).

We prove Proposition 1.12 as follows: by [22, Prop. 1.8] any semisimplification $\rho^{ss}$ of $\rho$ is $P_\theta$-Anosov and hence, by using Lemma 2.11, we may replace $\rho$ with $\rho^{ss}$. Then we compare the Gromov product relative to the fundamental weight $\{\omega_\alpha\}_{\alpha \in \theta}$ with the Gromov product with respect to the Hilbert metric $d_\Omega$ for some properly convex domain and then use Theorem 1.9.

Comparison to previous characterizations and related results. We first explain how Theorem 1.1 is related to the equivalence $(3) \iff (5)$ in [29, Thm. 1.7], see also [27, Thm. 5.47]. A subgroup $\Gamma$ of a real reductive Lie group $G$ is called $\tau_{mod}$-asymptotically embedded [29, Def. 6.12], if it is $\tau_{mod}$-regular, $\tau_{mod}$-antipodal, word hyperbolic and there exists a $\Gamma$-equivariant homeomorphism $\nu : \partial_x \Gamma \to \Lambda_{\tau_{mod}}(\Gamma)$. Here $\tau_{mod}$ corresponds to the choice of a subset of simple restricted roots $\eta \subset \Delta$ of $G$, $\tau_{mod}$-antipodal means that the map $\nu$ is transverse to itself i.e. for $x \neq y$ the pair $(\nu(x), \nu(y))$ is transverse and $\tau_{mod}$-regular corresponds to $P_\theta$-divergence.

Theorem 1.1 follows from a theorem of Kapovich–Leeb–Porti [29, Thm. 1.7] in the case where both limit maps $\xi^+ : \partial_x \Gamma \to G/P_{\theta}^+$ and $\xi^- : \partial_x \Gamma \to G/P_{\theta}^-$ satisfy the Cartan property (see Definition 4.1). Under this assumption, there exists $\rho$-equivariant embedding $\xi : \partial_x \Gamma \to G/P$ with $P = P_{\theta}^+ \cap P_{\theta}^-$, where $\ast : \Delta \to \Delta$ denotes the opposition involution and $\theta^* = \{\alpha^* : \alpha \in \theta\}$. Note that the pair of maps $(\xi^+, \xi^-)$ is compatible and transverse, hence $\xi$ is injective. The map $\xi$ satisfies the Cartan property, maps onto the $\tau_{mod}$-limit set $\Lambda_{\tau_{mod}}(\rho(\Gamma))$ hence $\rho(\Gamma)$ is $\tau_{mod}$-asymptotically embedded and the assumptions of [29, Thm. 1.7] are satisfied.

We also remark that Guichard-Guéritaud-Kassel-Wienhard proved in [22, Thm. 1.3, (1)$\iff$(2)] that a representation $\rho : \Gamma \to G$ is $P_\theta$-Anosov if and only if $\rho$ is $P_\theta$-divergent and admits a pair of continuous, $\rho$-equivariant, dynamics preserving and transverse maps $\xi^\pm : \partial_x \Gamma \to G/P_{\theta}^\pm$.

Theorem 1.1 follows by [22, Thm. 1.3, (1)$\iff$(2)] under the additional assumption that both limit maps are dynamics preserving.

Organization of the paper. In §2 we provide the necessary background from Lie theory, hyperbolic groups and the Floyd boundary and recallLabourie’s dynamical definition of Anosov representations. In §3 we prove some preliminary results which we use for the proof of Theorem 1.1. In §4 we define the Cartan property for an equivariant map $\xi : \partial_x \Gamma \to G/P_{\theta}^\pm$ and discuss the uniform gap summation property of [22] in the more general setting of finitely generated groups. In §5 we discuss (strong) property (U) and prove Theorem 1.6 and Corollary 1.8. In §6 we define a Gromov product for a representation $\rho$ and prove that is comparable with the usual Gromov product on the domain group when $\rho$ is Anosov. Next, in §7 we prove Theorem 1.1 and in §8 we give the proof of Theorem 1.10. In §9 we provide conditions for the direct product of two representations to be Anosov. Finally, in §10 we provide examples of discrete and faithful representations of surface groups showing that the assumptions of our main results are necessary.

Acknowledgements. I would like to thank Richard Canary and Fanny Kassel for their support during the course of this work and for their comments in previous versions of this paper. I would also like to thank Michael Kapovich, Giuseppe Martone, Andrés Sambarino and Feng Zhu.
for helpful discussions. The author was partially supported by grants DMS-1564362 and DMS-1906441 from the National Science Foundation as well as from the European Research Council (ERC) under the European’s Union Horizon 2020 research and innovation programme (ERC starting grant DiGGeS, grant agreement No 715982).

2. Background

In this section, we recall definitions from Lie theory, review several facts for hyperbolic groups, the Floyd boundary, provideLabourie’s dynamical definition of Anosov representations and also discuss several facts for semisimple representations. We mainly follow the notation from [22, §2].

Conventions. Throughout this paper Γ is a finitely generated group equipped with a finite generating subset S, inducing a left invariant word metric dΓ on the Cayley graph CΓ of Γ. For γ ∈ Γ we set |γ|Γ := dΓ(γ, e). A linear representation ρ : Γ → GLd(ℝ), d ≥ 2, is called irreducible if ρ(Γ) does not preserve any proper subspace of ℝd. The representation ρ is called strongly irreducible if for every finite-index subgroup H of Γ the restriction ρ|H is irreducible.

2.1. Lie theory. We will always consider G to be a semisimple Lie subgroup of SLm(ℝ), m ≥ 2, of non-compact type with finitely many connected components. The Zariski topology on G is the subspace topology induced from real algebraic subsets of SLm(ℝ).

We fix a maximal compact subgroup K of G, unique up to conjugation, a Cartan decomposition g = t ⊕ p where t = Lie(K), p is the orthogonal complement of t with respect to the Killing form on g, and the Cartan subspace a which is a maximal abelian subalgebra of g contained in p. The real rank of G is the dimension of a as a real vector space.

For a linear form β ∈ a∗ we shall use the notation ρ(β) = β(H) for H ∈ a. There is a decomposition of g into the common eigenspaces of the transformations X → [H, X], H ∈ a, called the restricted root decomposition

\[ g = g_0 \oplus \bigoplus_{\alpha \in \Sigma} g_\alpha \]

where \( g_\alpha = \{ X \in g : [H, X] = \langle \alpha, H \rangle X \ \forall H \in a \} \) and \( \Sigma = \{ \alpha \in a^* : g_\alpha \neq 0 \} \) is the set of restricted roots of G. We fix \( H_0 \in a \) with \( \langle \alpha, H_0 \rangle \neq 0 \) for every \( \alpha \in \Sigma \). Denote by \( \Sigma^+ = \{ \alpha \in \Sigma : \langle \alpha, H_0 \rangle > 0 \} \) the set of positive roots and fix \( \Delta \subset \Sigma^+ \) the simple positive roots. For any simple restricted root \( \alpha \in \Delta \), denote by \( \omega_\alpha \) the fundamental weight with respect to \( \alpha \in \Delta \), e.g. see [22, §3.1].

For every \( \theta \in \Delta \), \( \Sigma_{\theta} \) denotes the set of all roots in \( \Sigma \) which are linear combinations of elements of \( \theta \). We consider the parabolic Lie algebras

\[ p^\pm_\theta = g_0 \oplus \bigoplus_{\alpha \in \Sigma^+ \cup \Sigma_{\Delta \setminus \theta}} g_\alpha \]

and denote by \( P^\pm_\theta = N_G(p^\pm_\theta) \). A subgroup \( P \) of \( G \) is parabolic if it normalizes some parabolic subalgebra. Two parabolic subgroups \( P^+ \) and \( P^- \) of \( G \) are called opposite if there \( \theta \in \Delta \) and \( g \in G \) such that \( P^\pm = gP^\pm g^{-1} \).

Let \( \overline{\alpha}^+ := \{ H \in a : \langle \alpha, H \rangle \geq 0, \ \forall \alpha \in \Delta \} \). There exists a decomposition

\[ G = K \text{ exp}(\overline{\pi}^+)K \]

called the Cartan decomposition where each element \( g \in G \) is written as

\[ g = k_g \exp(\mu(g))k'_g, \ k_g, k'_g \in K, \]

and \( \mu(g) \) denotes the Cartan projection of \( g \). The map \( \mu : G \to \overline{\pi}^+ \) is continuous and proper and is called the Cartan projection. The Lyapunov projection \( \lambda : G \to \overline{\pi}^+ \) is the map defined as follows for \( g \in G \),

\[ \lambda(g) = \lim_{n \to \infty} \frac{1}{n} \mu(g^n). \]
An element \( g \in G \) is called \( P_\theta \)-proximal if \( \min_{x \in \Theta} \langle \alpha, \lambda(\rho(\gamma)) \rangle > 0 \). Equivalently, \( g \) has two fixed points \( x_g^+ \in G/P_\theta^+ \) and \( V_g^- \in G/P_\theta^- \) such that the pair \( (x_g^+, V_g^-) \) is transverse and for every \( x \in G/P_\theta^+ \) transverse to \( V_g^- \), we have \( \lim_n g^nx = x_g^+ \). The element \( g \) is called \( P_\theta \)-biproximal if \( g \) and \( g^{-1} \) are both \( P_\theta \)-proximal and we denote by \( x_g^- \) the attracting fixed point of \( g^{-1} \) in \( G/P_\theta^- \).

**Example 2.1.** The case of \( G = SL_d(\mathbb{R}) \). We denote by \( (e_1, \ldots, e_d) \) the canonical basis of \( \mathbb{R}^d \) and set \( e^\perp_j := \oplus_{j \neq i} \mathbb{R} e_j \). The group \( \text{SO}(d) = \{ g \in SL_d(\mathbb{R}) : gg^\dagger = I_d \} \) is the unique, up to conjugation, maximal compact subgroup of \( SL_d(\mathbb{R}) \). A Cartan subspace for \( g \) is the subspace \( a = diag_0(d) \) of all diagonal matrices with zero trace. Let \( \varepsilon_i \in a^* \) be the projection to the \( (i, i) \)-entry. The closed dominant Weyl chamber of \( a \) is \( \mathbb{R}^+ := \{ \text{diag}(a_1, \ldots, a_d) : a_1 \geq \ldots \geq a_d, \sum_{i=1}^d a_i = 0 \} \) and we have the Cartan decomposition \( SL_d(\mathbb{R}) = \text{SO}(d) \exp(\mathbb{R}^+) \text{SO}(d) \). The restricted root decomposition is \( SL_d(\mathbb{R}) = a \oplus \bigoplus_{i \neq j} \mathbb{R} E_{ij} \), where \( E_{ij} \) denotes the \( d \times d \) elementary matrix with 1 at the \((i, j)\) entry and 0 everywhere else. The set of restricted roots is \( \{ \varepsilon_i - \varepsilon_j : i \neq j \} \) and of simple positive roots \( \{ \varepsilon_i - \varepsilon_{i+1} : i = 1, \ldots, d-1 \} \). For each \( i = 1, \ldots, d-1 \), the associated fundamental weight is \( \omega_{i, \varepsilon_i - \varepsilon_{i+1}} = \sum_{k=1}^i \varepsilon_k \). For an element \( g \in SL_d(\mathbb{R}) \) we denote by \( \sigma_i(g) \) and \( \ell_i(g) \) the \( i \)-th singular value and eigenvalue of \( g \). Recall the connection between eigenvalues and singular values \( \sigma_i(g) = \sqrt{\ell_i(g g^\dagger)} \). The Cartan and Lyapunov projections of \( g \in SL_d(\mathbb{R}) \) are

\[
\mu(g) = \text{diag}(\log \sigma_1(g), \ldots, \log \sigma_d(g)) \\
\lambda(g) = \text{diag}(\log \ell_1(g), \ldots, \log \ell_d(g)).
\]

For each \( 1 \leq i \leq \frac{d}{2} \) we denote by \( P_i^+ \) (resp. \( P_i^- \)) the stabilizer of the plane \( \langle e_1, \ldots, e_i \rangle \) (resp. \( \langle e_{i+1}, \ldots, e_d \rangle \)). The pair of parabolic subgroups \( (P_i^+, P_i^-) \) is opposite. An element \( g \in GL_d(\mathbb{R}) \) is \( P_i \)-proximal if and only if \( \ell_i(g) > \ell_{i+1}(g) \). In this case \( g \) admits a unique attracting fixed point in the space \( G/P_i^+ = GR_i(\mathbb{R}^d) \).

### 2.2. Gromov hyperbolic spaces.

Let \((X, d)\) be a proper geodesic metric and \( x_0 \in X \) a fixed basepoint. For an isometry \( \gamma : X \to X \) define \( \gamma|X = d(\gamma x_0, x_0) \). The translation length and the stable translation length of the isometry \( \gamma \) respectively are:

\[
\ell_X(\gamma) = \inf_{x \in X} d(\gamma x, x), \quad |\gamma|_X = \lim_{n \to \infty} \frac{|\gamma^n|_X}{n}
\]

The **Gromov product** with respect to \( x_0 \) is the map \( X \times X \to [0, \infty) \) defined as follows

\[
(x \cdot y)_{x_0} := \frac{1}{2} \left( d(x, x_0) + d(y, x_0) - d(x, y) \right).
\]

A proper geodesic metric space space \((X, d)\) is called **Gromov hyperbolic** if there exists \( \delta \geq 0 \) with the following property: for every \( x, y, z \in X \)

\[
(x \cdot y)_{x_0} \geq \min \{(x \cdot z)_{x_0}, (z \cdot y)_{x_0}\} - \delta
\]

and we denote by \( \partial_X X \) the **Gromov boundary** of \( X \).

A finitely generated group \( \Gamma \) is called **word hyperbolic** (or **Gromov hyperbolic**) if the Cayley graph of \( \Gamma \) equipped with the word metric \( d_\Gamma \) is a Gromov hyperbolic space. In this case, every infinite order element \( \gamma \in \Gamma \) has exactly two fixed points \( \gamma^+, \gamma^- \in \partial_\Gamma \Gamma \) called the attracting and repelling fixed points of \( \gamma \) respectively. For more details on Gromov hyperbolic spaces and their boundaries we refer the reader to [8, Ch. III.H & III.I.G] and [16].

### 2.3. The Floyd boundary.

A function \( f : \mathbb{N} \to (0, \infty) \) is called a **Floyd function** if it satisfies the following two conditions:

(i) \( \sum_{n=1}^\infty f(n) < +\infty \).

(ii) there exists \( \delta \in (0, 1) \) such that \( \delta f(n + 1) \leq f(n) \leq f(n + 1) \) for every \( n \in \mathbb{N} \).

Let \( \Gamma \) be a finitely generated group. Given a Floyd function \( f \) there exists a metric \( d_f \) on the Cayley graph of \( \Gamma \) with respect to \( S \) defined as follows (see [19]): for two adjacent vertices \( g, h \in \Gamma \) the distance is defined as \( d_f(g, h) = f(\max\{|g|, |h|\}) \). The length of a finite path \( p \) defined
by the sequence of adjacent vertices \( p = \{x_0, x_1, \ldots, x_k\} \) is \( L_f(p) = \sum_{i=0}^{k-1} d_f(x_i, x_{i+1}) \). For two arbitrary vertices \( g, h \in \Gamma \) their distance is \( d_f(g, h) = \inf \{ L_f(p) : p \text{ is a path from } g \text{ to } h \} \). It is easy to verify that \( d_f \) defines a metric on \( \Gamma \) and let \( \bar{\Gamma} \) be the the metric completion of \( \Gamma \) with respect to \( d_f \). Every two points \( x, y \in \bar{\Gamma} \) are represented by Cauchy sequences \( (\gamma_n)_{n \in \mathbb{N}}, (\delta_n)_{n \in \mathbb{N}} \) with respect to \( d_f \) and their distance is \( d_f(x, y) = \lim_n d_f(\gamma_n, \delta_n) \). The Floyd boundary of \( \Gamma \) with respect to \( f \) is defined to be the complement \( \partial_f \Gamma := \bar{\Gamma} \setminus \Gamma \) equipped with the metric \( d_f \). The Floyd boundary \( \partial_f \Gamma \) is called non-trivial if it contains at least three points. For every infinite order element \( \gamma \in \Gamma \) the limit \( \lim_{n \to \infty} \gamma^n \) exists (see for example [30, Prop. 4]) and is denoted by \( \gamma^\pm \).

If \( \Gamma \) is a word hyperbolic group, there exists \( \epsilon > 0 \) such that the Floyd boundary of \( \Gamma \) with respect to \( f(x) = e^{-\epsilon x} \) is the Gromov boundary of \( \Gamma \) equipped with a visual metric (see [20]). For more details and properties of the Floyd boundary we refer the reader to [19, 20, 30].

2.4. Flow spaces for hyperbolic groups. Flow spaces for hyperbolic groups were introduced by Gromov in [20] and further developed by Champetier [14] and Mineyev [34]. For any word hyperbolic group \( \Gamma \) there exists a metric space \( (\hat{\Gamma}, \varphi_t) \) equipped with an \( \mathbb{R} \)-action \( \{ \varphi_t \}_{t \in \mathbb{R}} \) called the geodesic flow with the following properties:

(a) The action of \( \Gamma \) commutes with the action of the geodesic flow.
(b) The group \( \Gamma \) acts properly discontinuously and cocompactly with isometries on the flow space \( \hat{\Gamma} \).
(c) There exist \( C, c > 0 \) such that for every \( m \in \hat{\Gamma} \), the map \( t \mapsto \varphi_t(m) \) is a \( (C, c) \)-quasi-isometric embedding \( (\mathbb{R}, d_\mathbb{R}) \to (\hat{\Gamma}, d_\hat{\Gamma}) \).

The last property guarantees that the map \( (\tau^+, \tau^-) : \hat{\Gamma} \to \hat{\partial}_x \Gamma \times \hat{\partial}_x \Gamma \setminus \{(x, x) \mid x \in \hat{\partial}_x \Gamma\} \)
\[
(\tau^+(m), \tau^-(m)) = \left( \lim_{t \to \infty} \varphi_t(m), \lim_{t \to -\infty} \varphi_{-t}(m) \right)
\]
is well defined, continuous and equivariant with respect to the action of \( \Gamma \). For example, if \( (M, g) \) is a closed negatively curved Riemannian manifold, a flow space for \( \pi_1(M) \) satisfying the previous conditions is the unit tangent bundle \( T^1\hat{\mathcal{M}} \) equipped with the standard geodesic flow.

Benoist proved that a torsion free, discrete subgroup \( \Gamma \subset \text{PGL}_d(\mathbb{R}) \) acting geometrically on a strictly convex domain \( \Omega \subset \mathbb{P}(\mathbb{R}^d) \) is word hyperbolic (see [5, Thm. 1]). A choice of a flow space for \( \Gamma \) is the manifold \( T^1\hat{\Omega} \) equipped with the Hilbert geodesic flow.

2.5. Anosov representations. Let \( \rho : \Gamma \to G \) be a representation and fix \( \theta \in \Delta \) a subset of simple restricted roots of \( G \). We denote by \( L_\theta = P^+_{\rho} \cap P^-_{\rho} \) the common Levi subgroup. There exists a \( G \)-equivariant map \( G/L_\theta \to G/P^+_{\rho} \times G/P^-_{\rho} \) mapping the coset \( gL_\theta \) to the pair \( (gP^+_{\rho}, gP^-_{\rho}) \).

The tangent space of \( G/L_\theta \) at \( (gP^+_{\rho}, gP^-_{\rho}) \) splits as the direct sum \( \mathcal{T}_{gP^+_{\rho}}G/P^+_{\rho} \oplus \mathcal{T}_{gP^-_{\rho}}G/P^-_{\rho} \) and induces a \( G \)-equivariant splitting of the tangent bundle \( \mathcal{T}(G/L_\theta) = \mathcal{E} \oplus \mathcal{E}^- \). We consider the quotient spaces:
\[
\mathcal{X}_{\rho}^\pm = \Gamma \backslash (\hat{\Gamma} \times G/L_\theta), \quad \mathcal{E}_{\rho}^\pm = \Gamma \backslash (\hat{\Gamma} \times \mathcal{E})^\pm
\]
where the action of \( \gamma \in \Gamma \) on \( \mathcal{T}(G/L_\theta) \) is given by the differential of the left translation by \( \rho(\gamma) \) denoted \( L_{\rho(\gamma)} : G/L_\theta \to G/L_\theta \). Let \( \pi : \mathcal{X}_{\rho} \to \Gamma \hat{\Gamma} \) and \( \pi_{\pm} : \mathcal{E}_{\rho}^\pm \to \mathcal{X}_{\rho} \) be the natural projections. The projections \( \pi_{\pm} \) define vector bundles over the space \( \mathcal{X}_{\rho} \) where the fiber over the point \( [\hat{m}, (gP^+_{\rho}, gP^-_{\rho})] \) is identified with the vector space \( \mathcal{T}_{gP^+_{\rho}}G/P^+_{\rho} \).

The geodesic flow \( \{ \varphi_t \}_{t \in \mathbb{R}} \) commutes with the action of \( \Gamma \) and there exists a lift of the geodesic flow on the quotients \( \mathcal{X}_{\rho} \) and \( \mathcal{E}_{\rho}^\pm \) which we continue to denote by \( \{ \varphi_t \}_{t \in \mathbb{R}} \).

Definition 2.2. ([23, 33]) Let \( \Gamma \) be a word hyperbolic group and fix \( \theta \in \Delta \) a subset of restricted roots of \( G \). A representation \( \rho : \Gamma \to G \) is called \( P_\theta \)-Anosov if:

(i) There exists a section \( \sigma : \Gamma \hat{\Gamma} \to \mathcal{X}_{\rho} \) flat along the flow lines.
(ii) The lift of the geodesic flow \( \{ \varphi_t \}_{t \in \mathbb{R}} \) on the pullback bundle \( \sigma_* \mathcal{E}^+ \) (resp. \( \sigma_* \mathcal{E}^- \)) is dilating (resp. contracting).
Two maps \( \xi^+ : \partial \xi \Gamma \rightarrow G/P^\pm_\theta \) and \( \xi^- : \partial \xi \Gamma \rightarrow G/P^-_\theta \) are called transverse if for any pair of distinct points \((x, y) \in \partial \xi \Gamma \) there exists \( h \in G \) such that \( (\xi^-(x), \xi^-(y)) = (hP^+_\theta, hP^-_\theta) \). The previous definition is equivalent to the existence of a pair of continuous \( \rho \)-equivariant transverse maps \( \xi^+ : \partial \xi \Gamma \rightarrow G/P^+_\theta \) and \( \xi^- : \partial \xi \Gamma \rightarrow G/P^-_\theta \) defining the flat section \( \sigma : \Gamma \backslash \Gamma \rightarrow X_\rho \)

\[
\sigma([\hat m]) = [\hat m, (\xi^+(\hat m)), \xi^-((\hat m))] \Gamma,
\]

and a continuous equivariant family of norms \( (\| \cdot \|_x)_{x \in \Gamma \backslash \Gamma} \) with the property that there exist \( C, a > 0 \) such that for every \( x = [\hat m] \Gamma \), \( t \geq 0 \), and \( v \in T_{\xi^+(\hat m)}G/P^+_\theta \) (resp. \( v \in T_{\xi^-((\hat m))}G/P^-_\theta \)):

\[
\| \varphi_t(X^+_v) \|_{\varphi_t((x))} \leq C e^{-at} \| X^+_v \|_x \quad \text{(resp. } \| \varphi_t(X^-_v) \|_{\varphi_t((x))} \leq C e^{-at} \| X^-_v \|_x \text{)}
\]

where \( X^+_v \) (resp. \( X^-_v \)) denotes the copy of the vector \( v \in \pi^+_1(x) \) (resp. \( v \in \pi^-_1(x) \)).

We recall now some of the key properties of Anosov representations. For more background and for the main properties of Anosov representations see [11, 22, 23, 27, 28, 29, 33]. For a coset \( gP^\pm_\theta \), the stabilizer \( \text{Stab}_G(gP^\pm_\theta) \) is the parabolic subgroup \( gP^\pm_\theta g^{-1} \) of \( G \). A pair of maps \( \xi^+ : \partial \xi \Gamma \rightarrow G/P^+_\theta \) and \( \xi^- : \partial \xi \Gamma \rightarrow G/P^-_\theta \) are called compatible if for any \( x \in \partial \xi \Gamma \) the intersection \( \text{Stab}_G(\xi^+((x))) \cap \text{Stab}_G(\xi^-(x)) \) is a parabolic subgroup of \( G \). We also say that \( \xi^+ \) (resp. \( \xi^- \)) is dynamics preserving if for every infinite order element \( \gamma \in \Gamma \), \( \rho(\gamma) \) is proximal in \( G/P^+_\theta \) (resp. \( G/P^-_\theta \)) and \( \xi^+((\gamma \cdot x)) \) (resp. \( \xi^-((\gamma \cdot x)) \)) is the attracting fixed point of \( \rho(\gamma) \) in \( G/P^+_\theta \) (resp. \( G/P^-_\theta \)).

We fix an Euclidean norm \( \| \cdot \|_x \) on the Cartan subspace \( a \subset \mathfrak{g} \) and recall that \( \mu : G \rightarrow \mathfrak{a}^+ \) denotes the Cartan projection.

**Theorem 2.3.** ([23, 33, 28]) Let \( \Gamma \) be a word hyperbolic group and \( \theta \subset \Delta \) a subset of simple restricted roots of \( G \). Suppose that \( \rho : \Gamma \rightarrow G \) is a \( P_0 \)-Anosov representation.

(i) There exist \( C, c > 1 \) such that for every \( \gamma \in \Gamma \),

\[
\min_{\alpha \in \Delta} \langle \alpha, \mu(\rho(\gamma)) \rangle \geq c^{-1} \| \mu(\rho(\gamma)) \| \geq C^{-1} |\gamma|_\Gamma - C.
\]

In particular, \( \rho \) is a quasi-isometric embedding, \( \ker(\rho) \) is finite and \( \rho(\Gamma) \) is discrete in \( G \).

(ii) \( \rho \) admits a pair of compatible, continuous, \( \rho \)-equivariant maps, dynamics preserving and transverse maps \( \xi^+ : \partial \xi \Gamma \rightarrow G/P^+_\theta \) and \( \xi^- : \partial \xi \Gamma \rightarrow G/P^-_\theta \).

(iii) The set of \( P_0 \)-Anosov representations of \( \Gamma \) in \( G \) is open in \( \text{Hom}(\Gamma, G) \) and the map assigning a \( P_0 \)-Anosov representation to its Anosov limit maps is continuous.

Let \( G \) be a semisimple linear Lie group. A representation \( \tau : G \rightarrow \text{GL}_d(\mathbb{R}) \) is called proximal if \( \tau(G) \) contains a \( P_1 \)-proximal element. For an irreducible and proximal representation \( \tau \) we denote by \( \chi_\tau \) the highest weight of \( \tau \). The functional \( \chi_\tau \in \mathfrak{a}^* \) is of the form \( \chi_\tau = \sum_{\alpha \in \Delta} n_\alpha \omega_\alpha \) and the representation \( \tau \) is called \( \theta \)-compatible if \( \theta = \{ \alpha \in \Delta : n_\alpha > 0 \} \).

The following result is the content of [23, Prop. 4.3] and [22, Lem. 3.7] and is used to reduce statements for \( P_0 \)-Anosov representations to statements for \( P_1 \)-Anosov representations.

**Proposition 2.4.** ([22, 23]) Let \( G \) a real semisimple Lie group, \( \theta \subset \Delta \) a subset of simple restricted roots of \( G \). There exists an irreducible \( \theta \)-compatible representation \( \tau : G \rightarrow \text{GL}_d(\mathbb{R}) \), \( d = d(G, \theta) \), such that \( \tau(P^+_\theta) \) and \( \tau(P^-_\theta) \) stabilize the line \( e_1 \) and the hyperplane \( e_1^\perp = \langle e_1, \ldots, e_{d-1} \rangle \) respectively, so that there exist continuous and \( \tau \)-equivariant embeddings

\[
\iota^+ : G/P^+_\theta \hookrightarrow \mathbb{P}(\mathbb{R}^d), \quad \iota^- : G/P^-_\theta \hookrightarrow \text{Gr}_{d-1}(\mathbb{R}^d)
\]

induced by \( \tau \). Moreover, a representation \( \rho : \Gamma \rightarrow G \) is \( P_0 \)-Anosov if and only if \( \tau \circ \rho : \Gamma \rightarrow \text{GL}_d(\mathbb{R}) \) is \( P_1 \)-Anosov. In this case, the pair of Anosov limit maps of \( \tau \circ \rho \) is \( (\iota^+ \circ \xi^+, \iota^- \circ \xi^-) \), where \( (\xi^+, \xi^-) \) is the pair of the limit maps of \( \rho \).
2.6. **Semisimple representations.** Let $G$ be a semisimple Lie subgroup of $\text{SL}_d(\mathbb{R})$ and $\rho : \Gamma \to G$ a representation. The representation $\rho$ is called *semisimple* if $\rho$ is a direct sum of irreducible representations. In this case the Zariski closure of $\rho(\Gamma)$ in $G$ is a reductive algebraic Lie group.

The following result was proved by Benoist using a result of Abels–Margulis–Soifer [1] and allows one to control the Cartan projection of a semisimple representation in terms of its Lyapunov projection. We refer the reader to [22, Thm. 4.12] for a proof.

**Theorem 2.5.** ([1] & [3]) Let $G$ be a real reductive Lie group, $\Gamma$ be a discrete group and $\rho_i : \Gamma \to G$, $1 \leq i \leq s$, semisimple representations. Then there exists $C > 0$ and a finite subset $F$ of $G$ such that for every $\gamma \in \Gamma$ there exists $f \in F$ with the property:

$$\max_{1 \leq i \leq s} \left| \mu(\rho_i(\gamma)) - \lambda(\rho_i(\gamma)f) \right| \leq C$$

Guéritaud–Guichard–Kassel–Wienhard in [22] observe that from $\rho$ one may define the *semisimplification* $\rho^s$ which is a semisimple representation and a limit of conjugates of $\rho$. We shall use the following result for the semisimplification of a representation.

**Proposition 2.6.** ([22, Prop. 1.8]) Let $\Gamma$ be a finitely generated group, $G$ a real semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of $G$ and $\rho : \Gamma \to G$ be a representation with semisimplification $\rho^s : \Gamma \to G$. Then, $\lambda(\rho(\gamma)) = \lambda(\rho^s(\gamma))$ for every $\gamma \in \Gamma$ and $\rho$ is $P_0$-Anosov if and only if $\rho^s$ is $P_0$-Anosov.

2.7. **Convex cocompact groups.** A subset $\Omega$ of the projective space $\mathbb{P}(\mathbb{R}^d)$ is called *properly convex* if it is contained in an affine chart in which $\Omega$ is bounded and convex. The domain $\Omega$ is called *strictly convex* if it is properly convex and $\partial \Omega$ does not contain projective line segments. Suppose that $\Omega$ is bounded and convex in some affine chart $A$. We fix an Euclidean metric $d_E$ on $A$. We denote by $d_\Omega$ the Hilbert metric on $\Omega$ defined as follows

$$d_\Omega(x, y) = \frac{1}{2} \log \frac{d_E(x, a)d_E(x, b)}{d_E(a, x)d_E(y, b)},$$

where $a, b$ are the intersection points of the projective line $[x, y]$ with $\partial \Omega$, $x$ is between $a$ and $y$, and $y$ is between $x$ and $b$. The group $\text{Aut}(\Omega) = \{ g \in \text{PGL}_d(\mathbb{R}) : g\Omega = \Omega \}$ is a Lie subgroup of $\text{PGL}_d(\mathbb{R})$ and acts by isometries for the Hilbert metric $d_\Omega$. Any discrete subgroup of $\text{Aut}(\Omega)$ acts properly discontinuously on $\Omega$.

We shall use the following estimate obtained by Danciger–Guérataud–Kassel in [17] showing that the inclusion of a convex cocompact subgroup in $\text{PGL}_d(\mathbb{R})$ is a quasi-isometric embedding.

**Proposition 2.7.** ([17, Prop. 10.1]) Let $\Omega$ be a properly convex domain of $\mathbb{P}(\mathbb{R}^d)$. For any $x_0 \in \Omega$, there exists $\kappa > 0$ such that for any $g \in \text{Aut}(\Omega)$,

$$\log \frac{\sigma_1(g)}{\sigma_d(g)} \geq 2d_\Omega(gx_0, x_0) - \kappa.$$

Let $\Gamma$ be a subgroup of $\text{PGL}_d(\mathbb{R})$ preserving a properly convex domain $\Omega$. By using the previous proposition we can control the Gromov product with respect to $\varepsilon_1 \in \frak{a}^*$ as follows:

**Lemma 2.8.** Let $\Gamma$ be a subgroup of $\text{PGL}_d(\mathbb{R})$ which preserves a properly convex domain $\Omega$ of $\mathbb{P}(\mathbb{R}^d)$. Suppose that the natural inclusion of $\Gamma \hookrightarrow \text{PGL}_d(\mathbb{R})$ is semisimple. Then for every $x_0 \in \Omega$ there exists $C > 0$ such that for every $\gamma, \delta \in \Gamma$,

$$\left| \frac{1}{2} \log \frac{\sigma_1(f)}{\sigma_d(f)} - d_\Omega(\gamma x_0, x_0) \right| \leq C, \left| (\gamma \cdot \delta)_{\varepsilon_1} - (\gamma x_0 \cdot \delta x_0)_{x_0} \right| \leq C.$$

**Proof.** By Theorem 2.5 there exists a finite subset $F$ of $\Gamma$ and $M > 0$ such that for every $\gamma \in \Gamma$ there exists $f \in F$ such that $\log \frac{\sigma_1(f)}{\sigma_d(f)} \geq \log \frac{\sigma_1(\gamma)}{\sigma_d(\gamma)} - M$. The translation length of an isometry
Let \( g \in \text{Aut}(\Omega) \) is exactly \( \frac{1}{2} \log \frac{\ell_1(q)}{\ell_2(q)}, \) see [15, Prop. 2.1]. In particular, if \( \gamma \in \Gamma \) and \( f \in F \) are as previously, we have that

\[
2d_\Omega(\gamma x_0, x_0) \geq 2d_\Omega(\gamma f x_0, x_0) - 2d_\Omega(f x_0, x_0) \\
\geq \log \frac{\ell_1(\gamma f)}{\ell_2(\gamma f)} - 2d_\Omega(f x_0, x_0) \\
\geq \log \frac{\sigma_1(\gamma)}{\sigma_2(\gamma)} - M - 2d_\Omega(f x_0, x_0).
\]

(1)

Then, by Proposition 2.7 and (1), we obtain a uniform constant \( L > 0 \) such that

\[
\left| \log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} - 2d_\Omega(\gamma x_0, x_0) \right| \leq L
\]

for every \( \gamma \in \Gamma \). The conclusion follows. \( \square \)

**Definitions 2.9.** ([17]) Let \( \Gamma \) be an infinite discrete subgroup of \( \text{PGL}_d(\mathbb{R}) \) preserving a properly convex domain \( \Omega \) of \( \mathbb{P}(\mathbb{R}^d) \) and \( \Lambda_\Omega(\Gamma) \subset \Omega \) be the set of accumulation points of all \( \Gamma \)-orbits. The group \( \Gamma \) acts convex cocompactly on \( \Omega \), if the convex hull of \( \Lambda_\Omega(\Gamma) \) in \( \Omega \) is non-empty and acted on cocompactly by \( \Gamma \). The group \( \Gamma \) is called strongly convex cocompact in \( \mathbb{P}(\mathbb{R}^d) \) if \( \Gamma \) acts convex cocompactly on some strictly convex domain \( \Omega \) with \( C^1 \)-boundary.

The following lemma follows immediately from [17, Thm. 1.4] and [39, Thm. 1.27] and is used to pass from a \( P_1 \)-Anosov representation to a convex cocompact action in some projective space.

**Lemma 2.10.** Let \( V_d \) be the vector space of \( d \times d \)-symmetric matrices and \( S_d : \text{GL}_d(\mathbb{R}) \to \text{GL}(V_d) \) be the representation defined as follows \( S_d(g)X = gXg^t \) for \( X \in V_d \). For every \( P_1 \)-Anosov representation \( \rho : \Gamma \to \text{GL}_d(\mathbb{R}) \), the representation \( S_d \circ \rho \) is \( P_1 \)-Anosov and \( S_d(\rho(\Gamma)) \) is a strongly convex cocompact subgroup of \( \text{GL}(V_d) \).

Given two representations \( \rho_1 : \Gamma \to \text{GL}_m(\mathbb{R}) \) and \( \rho_2 : \Gamma \to \text{GL}_d(\mathbb{R}) \), we say that \( \rho_1 \) uniformly dominates \( \rho_2 \) if there is \( \delta \in (0, 1) \) with the property that for every \( \gamma \in \Gamma \),

\[
(1 - \delta) \log \ell_1(\rho_1(\gamma)) \geq \log \ell_1(\rho_2(\gamma)).
\]

We will also need the lemma for the proof of Proposition 1.12, which allows us to control the Cartan projection of an Anosov representation \( \rho \) in terms of the Cartan projection of a semisimplification \( \rho^{ss} \) of \( \rho \). We expect that this fact follows by the techniques of Guichard–Wienhard in [23, §5] showing that Anosov representations have strong proximality properties.

**Lemma 2.11.** Let \( \Gamma \) be a word hyperbolic group, \( G \) a real semisimple Lie group, \( \theta \subset \Delta \) a subset of simple restricted roots of \( G \). Suppose \( \psi : \Gamma \to G \) is a \( P_0 \)-Anosov representation with semisimplification \( \psi^{ss} : \Gamma \to G \). There is a constant \( C_\psi > 0 \), depending only on \( \psi \), such that for every \( \gamma \in \Gamma \),

\[
\max_{\alpha \in \theta} \left| \langle \omega_\alpha, \mu(\psi(\gamma)) - \mu(\psi^{ss}(\gamma)) \rangle \right| \leq C_\psi.
\]

**Proof.** Let us first observe that for any linear representation \( \phi : \Gamma \to \text{GL}_m(\mathbb{R}) \) and any semisimplification \( \phi^{ss} \) of \( \phi \), by Theorem 2.5, there exists a constant \( Q > 0 \), depending on \( \phi \), such that

\[
\log \sigma_1(\phi(\gamma)) \geq \log \sigma_1(\phi^{ss}(\gamma)) - Q
\]

for every \( \gamma \in \Gamma \). Moreover, for every \( \alpha \in \theta \), there exists \( N_\alpha > 0 \) such that \( N_\alpha \omega_\alpha \) is the highest weight of an irreducible proximal representation \( \tau_\alpha : G \to \text{GL}_m(\mathbb{R}) \). In particular, there exists \( Q' > 0 \), depending only on \( \tau_\alpha \), such that \( |\log \sigma_1(\tau_\alpha(h)) - N_\alpha \langle \omega_\alpha, \mu(h) \rangle| \leq Q' \) for every \( h \in G \).

The previous two facts show that there exists \( C_\rho > 0 \), depending only on \( \rho \) and \( G \), such that for every \( \gamma \in \Gamma \),

\[
\langle \omega_\alpha, \mu(\psi(\gamma)) - \mu(\psi^{ss}(\gamma)) \rangle \geq -C_\rho.
\]
Now we prove that by enlarging $C_\rho > 0$, for every $\gamma \in \Gamma$ we also have,
\[
\langle \omega_\alpha, \mu(\psi(\gamma)) - \mu(\psi^{ss}(\gamma)) \rangle \leq C_\rho.
\] (2)

By Proposition 2.4, we may compose $\psi$ with an irreducible representation $\tau : G \to \text{GL}_n(\mathbb{R})$ such that $\rho := \tau \circ \psi$ and its semisimplification $\rho^{ss} = \tau \circ \psi^{ss}$ are $P_1$-Anosov. Up to composing $\rho$ with the representation $S_\alpha$ from Lemma 2.10, we may further assume that $\rho(\Gamma)$ and the dual $\rho^*(\Gamma)$ preserve (possibly different) properly convex domain in $\mathbb{P}(\mathbb{R}^n)$. Up to conjugating $\rho$, and possibly considering the dual representation of this conjugate, we may assume $\rho(\Gamma)$ preserves a properly convex domain $\Omega_0 \subset \mathbb{P}(\mathbb{R}^n)$ and there is a decomposition $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_{\ell}$ such that
\[
\rho = \begin{pmatrix} \rho_1 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \rho_{\ell} \end{pmatrix}, \quad \rho^{ss} = \begin{pmatrix} \rho_1 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \rho_{\ell} \end{pmatrix}
\]
where $\rho_i : \Gamma \to \text{GL}(V_i)$ are irreducible representations, $\rho_1$ is $P_1$-Anosov and uniformly dominates $\rho_i$ for $2 \leq i \leq \ell$. In particular, $\rho_1$ is the restriction of $\rho^{ss}$ on the vector subspace $\langle \xi_\rho, \cdot \rangle(\hat{\mathcal{C}} \Gamma)$.

By using induction, it is enough to consider the case when $\ell = 2$ and
\[
\rho(\gamma) = \begin{pmatrix} \rho_1(\gamma) & u(\gamma) \\ \rho_2(\gamma) \end{pmatrix}, \quad \gamma \in \Gamma,
\]
where $u : \Gamma \to \text{Hom}(V_2, V_1)$ is a matrix valued function. The group $\rho_1(\Gamma)$ preserves the properly convex domain $\Omega_0 \cap \mathbb{P}(V_1)$ of $\mathbb{P}(V_1)$. By [17, 39], there exists a closed $\rho_1(\Gamma)$-invariant properly convex domain $\Omega_1 \subset \mathbb{P}(V_1)$ and a $\rho_1(\Gamma)$-invariant closed convex subset $\mathcal{C} \subset \Omega_1$ such that $\rho_1(\Gamma) \mathcal{C}$ is compact. We fix a basepoint $x_0 \in \mathcal{C}$ such that every point of $\mathcal{C}$ is within $d_{\Omega_1}$-distance $M > 0$ from the orbit $\rho_1(\Gamma) \cdot x_0$. Let $g \in \Gamma$ and consider $x_0, x_1, \ldots, x_k \in [x_0, g x_0]$ with $\frac{1}{2} \leq d_{\Omega_1}(x_i, x_{i+1}) \leq 1$. For every $0 \leq i < k$, choose $g_i \in \Gamma$ such that $d_{\Omega_1}(\rho_1(g_i) x_0, x_0) \leq M$, where $g_0 = e$ and $g_k = g$.

Now we define $\{h_i\}_{i=0}^{k+1}$ as follows: $h_0 = e$, $h_i = g_{i-1}^{-1} g_i$, $1 \leq i < k$ and $h_{k+1} = e$. Observe that $g = h_1 \cdots h_k$ and a straightforward computation shows that
\[
u(u) = u(h_1 \cdots h_k) = \sum_{i=0}^{k-1} \rho_1(h_0 \cdots h_i) u(h_{i+1}) \rho_2(h_{i+2} \cdots h_{k+1}).
\] (3)

By using Theorem 2.5 and the fact that $\rho_1$ is semisimple, $P_1$-Anosov and uniformly dominates $\rho_2$, we can find constants $A, E, a, b, \varepsilon > 0$ such that for every $\gamma \in \Gamma$:
\[
b \frac{\sigma_1(\rho_1(\gamma))}{\sigma_1(\rho_2(\gamma))} \geq 1, \quad \sigma_1(\rho_1(\gamma)) \geq A e^{a d_{\Omega_1}(\rho_1(\gamma)x_0, x_0)}, \quad \left| \log \frac{\sigma_1(\rho_1(\gamma))}{\sigma_1(\rho_1(\gamma))} - 2 d_{\Omega_1}(\gamma x_0, x_0) \right| \leq E.
\] (4)

To simplify notation, for $i = 1, \ldots, k$, set $w_i := h_1 \cdots h_i$ and the triangle inequality shows
\[
d_{\Omega_1}(\rho_1(w_i) x_0, x) \leq M, \quad d_{\Omega_1}(\rho_1(w_i) x_0, g x_0) \leq M.
\] (5)

Note that there exists $R > 0$, independent of $g \in \Gamma$, such that $h_i \in \Gamma$ lie in a metric ball of radius $R > 0$ of $\Gamma$. Therefore, by (3), (4) and (5), there exists $C_R > 0$ independent of $g \in \Gamma$ such that:
\[
\left\| \nu(u) \right\| \leq C_R \sum_{i=1}^{k-1} \sigma_1(\rho_1(h_1 \cdots h_i)) \cdot \sigma_1(\rho_2(h_{i+1} \cdots h_k)) \leq b C_R \sum_{i=0}^{k-1} \frac{\sigma_1(\rho_1(w_i^{-1} g)) \sigma_1(\rho_1(w_i))}{\sigma_1(\rho_1(w_i^{-1} g))^2}
\]
\[
= b C_R \sum_{i=0}^{k-1} \frac{1}{\sigma_1(\rho_1(g^{-1} w_i))} \cdot \frac{\sigma_1(\rho_1(w_i^{-1} g))}{\sigma_1(\rho_1(w_i^{-1} g))} \cdot \frac{\sigma_1(\rho_1(w_i))}{\sigma_1(\rho_1(w_i))} \cdot \frac{1}{\sigma_1(\rho_1(w_i^{-1} g))^2}
\]
\[
\leq b C_R \sum_{i=0}^{k-1} e^{2 E d_{\Omega_1}(\rho_1(w_i^{-1} g) x_0, x_0)} \cdot e^{2 E d_{\Omega_1}(\rho_1(w_i) x_0, x_0)} \cdot (A^{-\epsilon} e^{-a \varepsilon |w_i^{-1} g|})
\]
\[
= b C_R e^{2 E} \sum_{i=0}^{k-1} e^{2 E} \cdot e^{2 E d_{\Omega_1}(\rho_1(w_i) x_0, x_0)} \cdot e^{-a \varepsilon d_{\Omega_1}(\rho_1(w_i^{-1} g) x_0, x_0)}
\]
We conclude that there exists $L > 0$, depending only on $\rho$, such that for every $g \in \Gamma$,
\[ \sigma_1(\rho s^g) \leq \sigma_1(\rho) \leq L \sigma_1(\rho s^g), \]
hence (2) holds true for some $C_\rho > 0$, depending only on $\rho$. The highest weight of the $\theta$-compatible representation $\pi$ is of the form $\chi_\pi = \sum_{\alpha \in \theta} n_\alpha \omega_\alpha$, $n_\alpha > 0$, and since there is $D > 0$ such that $\langle \omega_\alpha, \mu(\rho(g)) - \mu(\rho s^g(g)) \rangle \geq -D$ for every $g \in \Gamma$. Thus, we have
\[ \langle \chi_\pi, \mu(\rho(g)) - \mu(\rho s^g(g)) \rangle \geq n_\alpha \langle \omega_\alpha, \mu(\rho(g)) - \mu(\rho s^g(g)) \rangle - C \sum_{\beta \in \theta \setminus \{\alpha\}} n_\beta, \]
and in particular, for every $g \in \Gamma$ we conclude that
\[ \langle \omega_\alpha, \mu(\rho(g)) - \mu(\rho s^g(g)) \rangle \leq \frac{1}{n_\alpha} \left( C_\rho + C \sum_{\beta \in \theta \setminus \{\alpha\}} n_\beta \right). \]

\[ \square \]

3. The contraction property

Let $\Gamma$ be a word hyperbolic group, $(\hat{\Gamma}, \varphi_\Gamma)$ a flow space on which $\Gamma$ acts properly discontinuously and cocompactly and $\mathcal{F} \subset \hat{\Gamma}$ be a compact subset of $\hat{\Gamma}$ whose $\Gamma$-translates cover $\hat{\Gamma}$. Let $\rho : \Gamma \to \text{GL}_d(\mathbb{R})$ be a representation admitting a pair of transverse $\rho$-equivariant maps $\xi^+ : \partial_+ \Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^- : \partial_- \Gamma \to \text{Gr}_{d-1}(\mathbb{R}^d)$ defining the flat section $\sigma : \Gamma \setminus \hat{\Gamma} \to \mathcal{X}_\rho$ of the fiber bundle $\pi : \mathcal{X}_\rho \to \Gamma \setminus \hat{\Gamma}$. We fix an equivariant family of norms $(|| \cdot ||_x)_{x \in \Gamma \setminus \hat{\Gamma}}$ on the fibers of the bundle $\pi^\pm : \mathcal{E}^\pm_\rho \to \Gamma \setminus \hat{\Gamma}$. For a given point $\hat{m} \in \hat{\Gamma}$, fix an element $h \in G$ so that $\xi^+(\tau^+(\hat{m})) = hP_1^+$ and $\xi^-(\tau^-(\hat{m})) = hP_1^-$ and denote by $L_h : G \to G$ the left translation by $h \in G$. Then we consider the tangent spaces
\[ T_{hP_1^+} \mathbb{P}(\mathbb{R}^d) = \left\{ dL_h d\pi^+(X) : X \in \bigoplus_{i=2}^d \mathbb{R} E_{1i} \right\}, \]
\[ T_{hP_1^-} \text{Gr}_{d-1}(\mathbb{R}^d) = \left\{ dL_h d\pi^-(X) : X \in \bigoplus_{i=2}^d \mathbb{R} E_{1i} \right\}. \]
For $u \in \{0\} \times \mathbb{R}^{d-1}$ we denote by $X_u^+ \in T_{hP_1^+} \mathbb{P}(\mathbb{R}^d)$ and $X_u^- \in T_{hP_1^-} \text{Gr}_{d-1}(\mathbb{R}^d)$ the tangent vectors
\[ X_u^+ = \left[ \hat{m}, (\xi^+(\tau^+(\hat{m})), \xi^-(\tau^-(\hat{m}))), dL_h d\pi^+ \left( \begin{array}{c} 0 \\ u \\ 0 \end{array} \right) \right]_\Gamma, \]
\[ X_u^- = \left[ \hat{m}, (\xi^+(\tau^+(\hat{m})), \xi^-(\tau^-(\hat{m}))), dL_h d\pi^- \left( \begin{array}{c} 0 \\ u \\ 0 \end{array} \right) \right]_\Gamma, \]
in the fibers of the bundles $\sigma_\rho \mathcal{E}^\pm \to \Gamma \setminus \hat{\Gamma}$ over $x = [\hat{m}]_\Gamma$ and $\pi^+, \pi^-$ are the projections from $\text{SL}_d(\mathbb{R})$ to $\mathbb{P}(\mathbb{R}^d)$ and $\text{Gr}_{d-1}(\mathbb{R}^d)$ respectively.

The following lemma shows that when the geodesic flow on $\sigma_\rho \mathcal{E}^-$ is weakly contracting, then the geodesic flow on $\sigma_\rho \mathcal{E}^+$ is weakly dilating.

**Lemma 3.1.** Let $\rho : \Gamma \to \text{GL}_d(\mathbb{R})$ be a representation. Suppose there exists a pair of continuous, $\rho$-equivariant transverse maps $\xi^+ : \partial_+ \Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^- : \partial_- \Gamma \to \text{Gr}_{d-1}(\mathbb{R}^d)$. Then for any $x = [\hat{m}]_\Gamma \in \hat{\Gamma}$ and $u \in \{0\} \times \mathbb{R}^{d-1}$ we have:
The continuity of the family of norms \( \| \cdot \|_x \) in \( \Gamma \) and since \( k_1, k_2 \in K \) lie in a compact group, imply

\[
\| \varphi_{t_n}(X^+_x) \|_{\varphi_{t_n}(x)} \geq \frac{\| A_n u \|}{|\lambda_n|} \quad \text{and} \quad \| \varphi_{t_n}(X^-_x) \|_{\varphi_{t_n}(x)} \geq |s_n| \cdot \| B_n^{-1} u \|,
\]

where \( \| \cdot \| \) denotes the usual Euclidean norm on \( \mathbb{R}^{d-1} \). Up to passing to a subsequence, we may assume that \( \lim_n \gamma_n \varphi_{t_n}(\tilde{m}) = \tilde{m}' \). Since the maps \( \tau^\pm \) are continuous we conclude, up to subsequence, that \( (\gamma_n \tau^+(\tilde{m})) \) and \( (\gamma_n \tau^-(\tilde{m})) \) converge to \( \tau^+(\tilde{m}') \) in \( \tilde{\phi}_N \) and \( \tau^-(\tilde{m}') \) in \( \tilde{\phi}_L \) respectively. We have \( \xi^+ (\tau^+(\gamma_n \tilde{m})) = k_1 \mathcal{P}^+_1 \) and \( \xi^- (\tau^-(\gamma_n \tilde{m})) = k_2 \mathcal{P}^-_1 \) and by transversality, there exist \( p_n \in \mathcal{P}^+_1, q_n \in \mathcal{P}^-_1 \) and \( g \in G \) such that \( \lim_n k_1 p_n = \lim_n k_2 q_n = g \). Then there exist \( z_n, z'_n \in \mathbb{R} \) so that \( z_n k_1 e_1 = g e_1 \) and \( z'_n k_2 e_1 = g^-1 e_1 \) and we observe that \( |z_n|, |z'_n| \) converge respectively to \( |g e_1| \) and \( |g^-1 e_1| \). Notice that \( \lim_n z_n z'_n \langle k_1 e_1, k_2 e_1 \rangle = \langle g^-1 e_1, g e_1 \rangle = 1 \) and so \( \lim_n \langle k_1 e_1, k_2 e_1 \rangle = \frac{1}{|g e_1| |g^-1 e_1|} \). Recall that \( k_2^{-1} k_1 \left( \begin{array}{cc} \lambda_n & 0 \\ 0 & A_n \end{array} \right) = \left( \frac{s_n}{\lambda_n} \right) \), hence by looking at the \((1,1)\) entry of both sides, we obtain \( \frac{s_n}{\lambda_n} = \langle k_1 e_1, k_2 e_1 \rangle \) and so \( L := \inf_{n \in \mathbb{N}} \left| \frac{s_n}{\lambda_n} \right| > 0 \). Furthermore, we observe that

\[
(\begin{array}{cc} \lambda_n & 0 \\ 0 & A_n \end{array})^{-1} \left( \begin{array}{c} l \\ 0 \end{array} \right) \right) = \left( \begin{array}{c} 0 \\ B_n^{-1} \end{array} \right) \left( \begin{array}{c} k_1 \left( \begin{array}{cc} \lambda_n & 0 \\ 0 & A_n \end{array} \right) \\ 0 \\ B_n \end{array} \right) \]
Proposition 3.2. Let $\rho : \Gamma \to \text{GL}_d(\mathbb{R})$ be a representation which admits a pair of continuous $\rho$-equivariant transverse maps $\xi^+ : \partial^+ \Gamma \to \mathbb{R}[\mathbb{R}^d]$ and $\xi^- : \partial^- \Gamma \to \mathbb{R}^d \Gamma$. Fix $x = [\hat{m}]_\Gamma$, $u \in \{0\} \times \mathbb{R}^d$ and suppose $\xi^+(\tau^+(\hat{m})) = hP^+_1$ and $\xi^-(\tau^-(\hat{m})) = hP^-_1$. Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\tilde{\Gamma}$ such that $(\gamma_n \phi_{n^+}(\hat{m}))_{n \in \mathbb{N}}$ lies in a compact subset of $\tilde{\Gamma}$.

(i) $\lim_{n \to \infty} ||\phi_{n^+}(X_u^+)||_{\phi_{n^+}(x)} = +\infty$ if and only if

$$\lim_{n \to \infty} \frac{||\rho(\gamma_n)h u||}{||\rho(\gamma_n)h e_1||} = +\infty.$$ 

(ii) $\lim_{n \to \infty} ||\phi_{n^+}(X_u^-)||_{\phi_{n^+}(x)} = 0$ if and only if

$$\lim_{n \to \infty} \frac{||\rho^*(\gamma_n)h^{-t} u||}{||\rho^*(\gamma_n)h^{-t} e_1||} = 0.$$

Proof. Suppose that $\rho(\gamma_n)h = k_{1n} \left( \begin{array}{cc} \lambda_n & 0 \\ 0 & A_n \end{array} \right) = k_{2n} \left( \begin{array}{cc} s_n & 0 \\ 0 & B_n \end{array} \right)$. Let $(\gamma_{n^+})_{n \in \mathbb{N}}$ be a subsequence of $(\gamma_n)_{n \in \mathbb{N}}$. A straightforward calculation shows that

$$\frac{||Ar_n u||}{|\lambda_{n^+}|} = \frac{||\rho(\gamma_n)h u||}{||\rho(\gamma_n)h e_1||} \sin \xi \left( \rho(\gamma_n)h e_1, \rho(\gamma_n)h u \right)$$

where $\xi^+(x) = hP^+_1$ and $hu \in \xi^-(y)$. Up to passing to subsequence, we may assume that $\lim_n \gamma_{n^+}(\phi_{n^+}(\hat{m}))$ exists and so $\lim_n \gamma_n^{-1} \phi_{n^+}(\hat{m}) \neq \lim_n \gamma_n^{-1} \phi_{n^+}(\hat{m})$. The maps $\xi^+$ and $\xi^-$ are transverse, hence there exists $g \in G$ and $p_n \in P^+_1$, $q_n \in P^-_1$ such that $\lim_n \rho(\gamma_n)h p_n = \lim_n \rho(\gamma_n)h q_n = g$. Let $v_x \in e^+_{\Gamma}$ be a limit point of the sequence $(\frac{q_n^{-1} u}{||q_n^{-1} u||})_{n \in \mathbb{N}}$. Then, $\lim_n \frac{1}{||q_n^{-1} u||} \rho(\gamma_n)h u = gv_x$ and hence $\lim_n \sin \xi \left( \rho(\gamma_n)h e_1, \rho(\gamma_n)h u \right) = \sin \xi \left( gv_x, ge_1 \right) > 0$. Since we started with an arbitrary subsequence, there exists $\varepsilon > 0$ with $|\sin \xi \left( \rho(\gamma_n)h e_1, \rho(\gamma_n)h u \right)| \geq \varepsilon$ for every $n \in \mathbb{N}$. Therefore, $\frac{||Ar_n u||}{|\lambda_{n^+}|} = \frac{||\rho(\gamma_n)h u||}{||\rho(\gamma_n)h e_1||}$. By Proposition 3.1 we have that

$$||\phi_{n^+}(X_u^+)||_{\phi_{n^+}(x)} = \frac{||A_n u||}{|\lambda_n|}, \quad n \to \infty,$$

and so part (i) follows. The argument for part (ii) is similar. \qed

4. The Cartan property and the uniform gap summation property

Let $G$ be a real semisimple Lie group of non-compact type, $K$ a maximal compact subgroup of $G$ and $\mu : G \to \mathfrak{a}^*$ be the Cartan projection. The restricted Weyl group of $\mathfrak{a}$ in $\mathfrak{g}$ is the group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, where $N_K(\mathfrak{a})$ (resp. $Z_K(\mathfrak{a})$) is the normalizer (resp. centralizer) of $\mathfrak{a}$ in $K$. The group $W$ is finite, acts simply transitively on the set of Weyl chambers of $\mathfrak{a}$ and contains a unique order two element $wZ_K(\mathfrak{a}) \in W$ such that $\text{Ad}(w)\mathfrak{a}^+ = -\mathfrak{a}^+$. The element $w \in K$ defines an involution $^* : \Delta \to \Delta$ on the set of simple restricted roots $\Delta$ as follows: if $\alpha \in \Delta$ then $\alpha^* \in \Delta$ is the unique root with $\alpha^*(H) = -\alpha(\text{Ad}(w)H)$ for every $H \in \mathfrak{a}$.

Let $\Gamma$ be an infinite, finitely generated group. A representation $\rho : \Gamma \to G$ is $P_0$-divergent if

$$\lim_{|\gamma|| \to \infty} \langle \alpha, \mu(\rho(\gamma)) \rangle = +\infty$$

for every $\alpha \in \theta$. Notice that the representation $\rho$ is $P_0$-divergent if and only if $\rho$ is $P_0'$-divergent. For an element $g = k_{\rho} \exp(\mu(g))k_{\rho}'$ written in the Cartan decomposition of $G$, define

$$\Xi_0^+(g) := k_{\rho} P^+_g \quad \text{and} \quad \Xi_0^-(g) := k_{\rho} w P^-_g.$$

For a $\rho$-equivariant map $\xi^+ : \partial^+ \Gamma \to G/P^+_0$, the map $\xi^* : \partial^+ \Gamma \to G/P^*_0$ is defined as follows

$$\xi^*(x) = k_x w P^+_0,$$

where $\xi^-(x) = k_x P^-_0$ and $k_x \in K$. 
Definition 4.1. Let $G$ be a real semisimple Lie group, $\Gamma$ a word hyperbolic group and $\rho: \Gamma \to G$ a $P_0$-divergent representation.

1. Suppose that $\rho$ admits a continuous $\rho$-equivariant map $\xi^+: \partial_x \Gamma \to G/P_0^+$. The map $\xi^+$ satisfies the Cartan property if for any $x \in \partial_x \Gamma$ and every infinite sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of $\Gamma$ with $\lim_n \gamma_n = x$,

$$\xi^+(x) = \lim_{n \to \infty} \Xi^+_\theta (\rho(\gamma_n))$$

2. Suppose that $\rho$ admits a continuous $\rho$-equivariant map $\xi^-: \partial_x \Gamma \to G/P_0^-$. The map $\xi^-$ satisfies the Cartan property if the map $\xi^*: \partial_x \Gamma \to G/P_0^+$ satisfies the Cartan property. In other words, for every $x \in \partial_x \Gamma$ and every infinite sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of $\Gamma$ with $\lim_n \gamma_n = x$,

$$\xi^-(x) = \lim_{n \to \infty} \Xi^-_\theta (\rho(\gamma_n))$$

Remark 4.2. Let $\rho: \Gamma \to G$ be a $P_0$-divergent representation. The Cartan property for a continuous $\rho$-equivariant map $\xi^+: \partial_x \Gamma \to G/P_0^+$ (resp. $\xi^-$) is independent of the choice of the Cartan decomposition of $G$. This follows by the fact that all Cartan subspaces of $G$ are conjugate under the adjoint action of $G$ and the second part of [22, Cor. 5.9].

The following fact is immediate from the definition of the Cartan property:

Fact 4.3. Suppose that $\rho, \Gamma, G$ and $\theta$ are defined as in Definition 4.1 and let $\xi: \partial_x \Gamma \to G/P_0^+$ be a continuous $\rho$-equivariant map. Suppose that $\tau: G \to \text{GL}_d(\mathbb{R})$ is an irreducible $\theta$-compatible proximal representation as in Proposition 2.4 so that $\tau(P_0^+)$ stabilizes a line in $\mathbb{R}^d$ and induces a $\tau$-equivariant embedding $\iota^+: G/P_0^+ \to \mathbb{P}(\mathbb{R}^d)$. The map $\xi$ satisfies the Cartan property if and only if $\iota^+ \circ \xi$ satisfies the Cartan property.

We need the following estimates which help us verify, in several cases, the Cartan property of limit maps into the homogeneous spaces $G/P_0^+$ and $G/P_0^-$. The second part of the following proposition has been established in [7, Lem. A4] and [22, Lem. 5.8 (i)] but for completeness we give a short proof.

Proposition 4.4. Let $G$ be a real semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of $G$ and $\tau: G \to \text{GL}_d(\mathbb{R})$ an irreducible, $\theta$-proximal representation such that $\tau(P_{0\theta}^+)$ stabilizes the line $[e_1]$ in $\mathbb{R}^d$. Let $\chi_\tau \in \mathfrak{a}^*$ be the highest weight of $\tau$ and $g, r \in G$.

(i) If $g$ is $P_{0\theta}$-proximal in $G/P_{0\theta}^+$ with attracting fixed point $x_\theta^+ \in G/P_{0\theta}^+$, then

$$d_{G/P_{0\theta}^+}(x_\theta^+, \Xi_{\theta}^+(g)) \leq \exp \left( -\min_{\alpha \in \theta} \langle \alpha, \mu(g) \rangle + \langle \chi_\tau, \mu(g) - \lambda(g) \rangle \right)$$

(ii) If $\min_{\alpha \in \theta} \langle \alpha, \mu(g) \rangle > 0$ and $\min_{\alpha \in \theta} \langle \alpha, \mu(g) \rangle > 0$, then

$$d_{G/P_{0\theta}^+}(\Xi_{\theta}^+(gr), \Xi_{\theta}^+(g)) \leq C_{d,r} \exp \left( -\min_{\alpha \in \theta} \langle \alpha, \mu(g) \rangle \right)$$

where $C_{d,r} = \sigma_1(\tau(r)) \sigma_1(\tau(r^{-1})) \sqrt{d-1}$.

Proof. By the definition of the metric $d_{G/P_{0\theta}^+}$ and Proposition 2.4 we may assume that $G = \text{SL}_d(\mathbb{R}), \theta = \{e_1 - e_2\}$ and $G/P_{0\theta}^+ = \mathbb{P}(\mathbb{R}^d)$.

(i) Since $g$ is proximal there exist $h \in \text{GL}_d(\mathbb{R}), A_g \in \text{GL}_{d-1}(\mathbb{R})$ and $k_g, k_g' \in O(d)$ such that

$$g = h \left( \ell_1' (g) \atop A_g \right) h^{-1} = k_g \exp(\mu(g)) k_g', \quad |\ell_1'(g)| = \ell_1(g).$$

We can write $\Xi_{\theta}^+(g) = k_g P_{1\theta}^+$ and $x_\theta^+ = h P_{1\theta}^+ = w_1 P_{1\theta}^+$ for some $w_1 \in O(d)$. Note that

$$h \left( \ell_1'(g) \atop A_g \right) h^{-1} = w_1 \left( \ell_1'(g) \atop w_1^{-1} \right)$$
hence \( k^{-1} w_1 \left( L^1 \right) = \exp(\mu(g)) k'_g w_1 \) and \( \langle k^{-1} w_1 e_1, e_i \rangle = \frac{\sigma_i(g)}{\ell_1(g)} \langle k'_g w_1 e_1, e_i \rangle \) for \( i > 1 \).

Therefore,

\[
\ell_2^2 \left( x^+_g, \Xi^+_1(g) \right)^2 = \sum_{i=2}^d \langle k^{-1} w_1 e_1, e_i \rangle^2 = \sum_{i=2}^d \frac{\sigma_i(g)}{\ell_1(g)^2} \langle k'_g w_1 e_1, e_i \rangle^2 \leq \frac{\sigma_2(g)^2 \ell_1(g)^2}{\ell_1(g)^2}.
\]

Since \( \min_{\alpha \theta} \langle \alpha, \mu(\gamma) \rangle = \log \frac{\sigma_1(\tau(g))}{\sigma_1(\gamma g)} \langle \chi, \mu(g) \rangle = \log \sigma_1(\tau(g)) \) and \( \langle \chi, \lambda(g) \rangle = \log \ell_1(\tau(g)) \) for \( g \in G \), part (i) follows.

(ii) We have \( k_{gr} \exp(\mu(g')) = k_g \exp(\mu(g)) k'_g r \), \( k_{gr}, k'_g \in K \), and in particular

\[
\langle k^{-1} k_{gr} e_1, e_i \rangle = \frac{\sigma_i(g)}{\sigma_1(g')} \langle k'_g r(k'_g)^{-1} e_1, e_i \rangle
\]

for every \( 2 \leq i \leq d \). Note that since \( \sigma_1(\gamma g) \geq \frac{\sigma_1(g)}{\sigma_1(\gamma)}, \) and \( \langle k_{gr} r(k'_g)^{-1} e_1, e_i \rangle \leq \sigma_1(\gamma) \), we have

\[
\langle k^{-1} k_{gr} e_1, e_i \rangle \leq \frac{\sigma_i(g)}{\sigma_1(\gamma)} \sigma_1(\gamma)^{-1}.
\]

Finally, we obtain

\[
d^p(\Xi^+(gr), \Xi^+(g))^2 = \sum_{i=2}^d \langle k^{-1} k_{gr} e_1, e_i \rangle^2 = \sum_{i=2}^d \frac{\sigma_i(g)}{\sigma_1(\gamma)} \langle k'_g r(k'_g)^{-1} e_1, e_i \rangle^2 \leq \frac{C_d^2 \sigma_2(g)^2}{\sigma_1(\gamma)^2}.
\]

This finishes the proof of the lemma.

Let \( \mathcal{M} \) be a compact metrizable space and \( \Gamma \) a group acting on \( \mathcal{M} \) by homeomorphisms. The action is called a \textit{convergence group action} if for any infinite sequence \((\gamma_n)_{n \in \mathbb{N}}\) of elements of \( \Gamma \) there exists a subsequence \((\gamma_{k_n})_{n \in \mathbb{N}}\) and \( a, b \in \mathcal{M} \) such that for every compact subset \( C \subseteq \mathcal{M} \setminus \{a\} \), \( \gamma_{k_n} \mid C \) converges uniformly to the constant map \( b \). For an infinite order element \( \gamma \in \Gamma \), we denote by \( \gamma^\pm \) the local uniform limit of the sequence \((\gamma_{k_n}^\pm)_{n \in \mathbb{N}}\). Examples of convergence group actions include the action of a non-elementary word hyperbolic group on its Gromov boundary (see [20]) and the action of a finitely generated group \( \Gamma \) on its Floyd boundary \( \partial \Gamma \) (see [21] and [30, Thm. 2]).

We prove a version of [12, Lem. 9.2] which shows, in many cases, that a representation \( \rho \) is \( P_d \)-divergent when it admits a continuous \( \rho \)-equivariant limit map.

**Lemma 4.5.** Let \( \mathcal{M} \) be a compact metrizable perfect space, \( \Gamma \) a torsion free group acting on \( \mathcal{M} \) by homeomorphisms and \( \rho : \Gamma \to \text{GL}_d(\mathbb{R}) \) a representation. Suppose that \( \Gamma \) acts on \( \mathcal{M} \) as a convergence group and there exists a continuous \( \rho \)-equivariant non-constant map \( \xi : \mathcal{M} \to \mathbb{P}(\mathbb{R}^d) \). Then for every infinite sequence \((\gamma_n)_{n \in \mathbb{N}}\) of elements of \( \Gamma \) we have

\[
\lim_{n \to \infty} \frac{\sigma_1(\rho(\gamma_n))}{\sigma_{d-p+2}(\rho(\gamma_n))} = +\infty, \quad \text{where} \quad p = \text{dim}_{\mathbb{R}} \langle \xi(\mathcal{M}) \rangle.
\]

**Proof.** We first prove the statement when \( p = d \). If the result does not hold, then there exists \( \varepsilon > 0 \) and a subsequence, which we continue to denote by \((\gamma_n)_{n \in \mathbb{N}}\), such that \( \frac{\sigma_1(\rho(\gamma_n))}{\sigma_{d-p+2}(\rho(\gamma_n))} < \varepsilon \). We may write \( \rho(\gamma_n) = k_n \exp(\mu(\rho(\gamma_n))) k'_n \), where \( k_n, k'_n \in O(d) \). Up to passing to a subsequence, there exist \( \eta, \eta' \in \mathcal{M} \) such that \( x \neq \eta' \) then \( \lim_{n} \gamma_n x = \eta \) and we may also assume that the sequences \((k_n)_{n \in \mathbb{N}}, (k'_n)_{n \in \mathbb{N}}\) converge to \( k, k' \in O(d) \) respectively and \( \lim_{n} \sigma_1(\rho(\gamma_n)) = C > 0 \). Let \( x \neq \eta \) and write \( \xi(x) = k_{\varepsilon} P^1_+ \) for some \( k_\varepsilon \in O(d) \). Since \( \lim_{n} \rho(\gamma_n) \xi(x) = \xi(\eta) \), up to passing to a further subsequence, we may assume that

\[
\lim_{n \to \infty} \frac{\exp(\mu(\rho(\gamma_n))) k'_n k_\varepsilon e_1}{\exp(\mu(\rho(\gamma_n))) k'_n k_\varepsilon e_1} = \varepsilon k^{-1} k_0 e_1
\]
where $\xi(\eta) = k_\eta P_1^+$, $\epsilon \in \{-1, 1\}$. This shows that for every $x \in M$, there exists $\lambda_x \in \mathbb{R}$ such that:
$$
\langle k'k_x e_1, e_1 \rangle = \lambda_x \langle k^{-1} k_\eta e_1, e_1 \rangle, \quad \langle k'k_x e_1, e_2 \rangle = \lambda_x C^{-1} \langle k^{-1} k_\eta e_1, e_2 \rangle.
$$
Since $\langle \xi(M \setminus \{\eta'\}) \rangle = \mathbb{R}^d$ and $M$ is perfect, there exists $x_0 \neq \eta'$ such that $\lambda_{x_0} \neq 0$. Then for every $x \neq \eta'$ we observe that
$$
\langle k'k_x e_1, e_1 \rangle = \frac{\lambda_x}{\lambda_{x_0}} \langle kk_{x_0} e_1, e_1 \rangle, \quad \langle k'k_x e_1, e_2 \rangle = \frac{\lambda_x}{\lambda_{x_0}} \langle kk_{x_0} e_1, e_2 \rangle.
$$
Therefore, for every $x \neq \eta'$, $k\xi(x)$ lies in the subspace $V = \langle kk_{x_0} e_1 \rangle + e_1^+ \cap e_2^+$, a contradiction since $\dim(V) \leq d - 1$.

In the case where $p < d$, consider the subspace $V = \langle \xi(M) \rangle$ and the restriction $\hat{\rho} : \Gamma \to \text{GL}(V)$ of $\rho$. The map $\xi$ is $\hat{\rho}$-equivariant and a spanning map for $\hat{\rho}$. The conclusion follows by observing that for any $\gamma \in \Gamma$ we have $\sigma_1(\rho(\gamma)) / \sigma_2(\rho(\gamma)) \leq \sigma_1(\hat{\rho}(\gamma)) / \sigma_{d-p+2}(\hat{\rho}(\gamma))$.

**Corollary 4.6.** Let $\Gamma$ be a word hyperbolic group, $G$ a real semisimple Lie group and $\theta \subset \Delta$ a subset of simple restricted roots of $G$.

(i) Suppose that $\rho : \Gamma \to \text{SL}_d(\mathbb{R})$ is an irreducible representation admitting a continuous $\rho$-equivariant map $\xi : \hat{\rho}(\Gamma) \to \mathbb{P}(\mathbb{R}^d)$. Then $\rho$ is $P_1$-divergent and $\xi$ satisfies the Cartan property.

(ii) Suppose that $\rho' : \Gamma \to G$ is a Zariski dense representation admitting a continuous $\rho'$-equivariant map $\xi' : \hat{\rho'}(\Gamma) \to G/P_1^+$. Then $\rho'$ is $P_0$-divergent and $\xi'$ satisfies the Cartan property.

**Proof.** (i) We first claim that if $\rho(\gamma)$ is $P_1$-proximal, then $\xi(\gamma^+)$ is the attracting fixed point in $\mathbb{P}(\mathbb{R}^d)$. Indeed, since $\rho$ is irreducible we have $\langle \xi(\hat{\rho}(\Gamma)) \rangle = \mathbb{R}^d$. If $\rho(\gamma)$ is $P_1$-proximal, we can find $x \in \hat{\rho}(\Gamma) \setminus \{\gamma^-\}$ such that $\xi(x)$ is not in the repelling hyperplane $V^-$. Since $\lim_n \gamma^+ x = \gamma^+$, we have $\lim_n \gamma^+ x = \gamma^+$. Since $\rho$ is irreducible it follows by Lemma 4.5 that $\rho$ is $P_1$-divergent. Let $(\gamma_n)_{n \in \mathbb{N}}$ be an infinite sequence of elements of $\Gamma$ such that $\lim_n \gamma_n = x$. By the sub-additivity of the Cartan projection $\mu$ (see [22, Fact 2.18]) and Theorem 2.5, there exists a finite subset $F$ and $C > 0$ such that for every $\gamma \in \Gamma$, there exists $f \in F$ with $||\lambda(\rho(\gamma f)) - \mu(\rho(\gamma f))|| \leq C$. Then for large $n \in \mathbb{N}$, there exist $f_n \in F$ such that $\rho(\gamma_n f_n)$ is $P_1$-proximal and
$$
\log \xi(\rho(\gamma_n f_n)) - \log \sigma_1(\rho(\gamma_n f_n)) \geq -C.
$$
Notice also $\lim_n \gamma_n = \lim_n \gamma_n f_n = \lim_n (\gamma_n f_n)^+ = x$ in the compactification $\Gamma \cup \hat{\rho}(\Gamma)$ and so $\lim_n (x_{\rho(\gamma_n f_n)}) = \lim_n (\xi(\gamma_n f_n)^+) = \xi(x)$. Then, by using Proposition 4.4, for every $n \in \mathbb{N}$ we obtain the estimate:
$$
d_\rho(x_{\rho(\gamma_n f_n)}, \Xi_1^+(\rho(\eta))) \leq d_\rho(x_{\rho(\gamma_n f_n)}, \Xi_1^+(\rho(\gamma_n f_n))) + d_\rho(\Xi_1^+(\rho(\gamma_n f_n)), \Xi_1^+(\rho(\gamma_n)))
\leq \left( e^C + \sup_{f \in F} C_{d,f} \right) \frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}
$$
where $C_{d,f} > 0$ is defined as in Proposition 4.4 (ii). This shows $\xi(x) = \lim_n \Xi_1^+(\rho(\gamma_n))$ and finally that $\xi$ satisfies the Cartan property.

(ii) Let $\tau$ be as in Proposition 2.4. Since $\rho'$ is Zariski dense, the representation $\tau \circ \rho'$ is irreducible. By Lemma 4.5 the representation $\tau \circ \rho'$ is $P_1$-divergent and hence $\rho'$ is $P_0$-divergent. By part (i), the $\tau \circ \rho'$-equivariant map $\tau^+ \circ \xi'$ satisfies the Cartan property. It follows by Fact 4.3 that $\xi'$ satisfies the Cartan property.

We are now aiming to generalize the uniform gap summation property [22, Def. 5.2] for representations of arbitrary finitely generated groups.

**Definition 4.7.** Let $\Gamma$ be a finitely generated group, $\rho : \Gamma \to G$ be a representation and $\theta \subset \Delta$ a finite subset of restricted roots of $G$. We say that $\rho$ satisfies the uniform gap summation property with respect to $\theta$ and the Floyd function $f : \mathbb{N} \to (0, \infty)$, if there exists $C > 0$ such that
$$
\langle \alpha, \mu(\rho(\gamma)) \rangle \geq - \log f(|\gamma|) - C
$$
for every \( \gamma \in \Gamma \) and \( \alpha \in \theta \). We say that the representation \( \rho \) satisfies the uniform gap summation property if there exists a Floyd function \( f \), a subset of simple roots \( \theta \subset \Delta \) and \( C > 0 \) with the previous properties.

Let \( \rho : \Gamma \to G \) be a representation. If \( \Gamma \) is word hyperbolic group and \( \rho \) satisfies the uniform gap summation property, then it admits a pair of \( \rho \)-equivariant, continuous limit maps which satisfy the Cartan property (see [22, Thm. 5.3 (3)]). If \( \Gamma \) is not word hyperbolic, we may similarly construct a pair of \( \rho \)-equivariant continuous maps from a Floyd boundary of \( \Gamma \), \( \partial_f \Gamma \), into the flag spaces \( G/P_\rho^+ \) and \( G/P_\rho^- \). Note that when \( \partial_f \Gamma \) is non-trivial, the action of \( \Gamma \) on \( \partial_f \Gamma \) is a convergence group action (see [30, Thm. 2]) so we obtain additional information for the action of \( \rho(\Gamma) \) on its limit set in \( G/P_\rho^\pm \). We prove the following lemma that we use in the following section for the proof of Theorem 1.6.

**Lemma 4.8.** Let \( \Gamma \) be a finitely generated group, \( G \) a real semisimple Lie group, \( \theta \subset \Delta \) a subset of simple restricted roots of \( G \) and \( \rho : \Gamma \to G \) a representation. Suppose that \( \rho \) satisfies the uniform gap summation property with respect to \( \theta \) and the Floyd function \( f : \mathbb{N} \to (0, \infty) \). There exists a constant \( C > 0 \), depending only on \( \rho \), such that

\[
d_{G/P_\rho^+} (\Xi^+ \rho(g)), \Xi^+ \rho(h)) \leq C d_f (g, h)
\]

for all but finitely many \( g, h \in \Gamma \). In particular, there exists a pair of continuous \( \rho \)-equivariant maps

\[
\xi^+_{\rho} : \partial_f \Gamma \to G/P_\rho^+ \quad \text{and} \quad \xi^-_{\rho} : \partial_f \Gamma \to G/P_\rho^-.
\]

Moreover, if \( \rho(\Gamma) \) contains a \( P_0 \)-proximal element, then \( \xi^+_{\rho} (\partial_f \Gamma) \) maps onto the proximal limit set of \( \rho(\Gamma) \) in \( G/P_\rho^+ \).

**Proof.** As in the proof of Proposition 2.4, we may assume that \( \theta = \{ \varepsilon_1 - \varepsilon_2 \} \) and \( G = SL_d(\mathbb{R}) \) and \( G/P_\rho^+ = \mathbb{P}(\mathbb{R}^d) \). By definition, there exists a constant \( C > 0 \) such that for every \( \gamma \in \Gamma \),

\[
\frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \leq C f(|\gamma|_\Gamma).
\]

Let \( p \) be a path in the Cayley graph of \( \Gamma \) defined by the sequence of adjacent vertices \( g_0 = g, \ldots, h = g_n \) with \( L_f(p) = d_f(g, h) \). Since \( g_i^{-1} g_{i+1} \) lie in a fixed generating subset of \( \Gamma \), by using Proposition 4.4, we may find \( C' > 0 \), depending only on \( \rho \), such that:

\[
d_\rho (\Xi^+ \rho(g)), \Xi^+ \rho(h)) \leq \sum_{i=0}^{n-1} d_\rho (\Xi^+ \rho(g_i)), \Xi^+ \rho(g_{i+1})) \leq C' \sum_{i=0}^{n-1} \frac{\sigma_2(\rho(g_i))}{\sigma_1(\rho(g_i))}
\]

\[
\leq C' C \sum_{i=0}^{n-1} f(|g_i|_\Gamma) = C' K d_f (g, h).
\] (6)

Now define the map \( \xi^+_{\rho} : \partial_f \Gamma \to \mathbb{P}(\mathbb{R}^d) \) as follows: for a point \( x \in \partial_f \Gamma \) represented by a Cauchy sequence \( (\gamma_n)_{n \in \mathbb{N}} \) with respect to the metric \( d_f \), then

\[
\xi^+_{\rho} (x) = \lim_{n \to \infty} \Xi^+ \rho(\gamma_n) \]

The previous inequality shows that the limit \( \lim_{n \to \infty} \Xi^+ \rho(\gamma_n) \) is independent of the choice of the sequence \( (\gamma_n)_{n \in \mathbb{N}} \) representing \( x \), since for any other sequence \( (\gamma'_n)_{n \in \mathbb{N}} \) with \( x = \lim_{n \to \infty} \gamma'_n \), we have \( \lim_{n \to \infty} d_f(\gamma_n, \gamma'_n) = 0 \). Finally, \( \xi^+_{\rho} \) is well defined and Lipshitz by (6), and hence continuous. By identifying \( G/P_\rho^- \) with \( G/P_\rho^+ \), \( \xi^-_{\rho} \) similarly obtain the limit map \( \xi^-_{\rho} \).

Suppose that \( \rho(\Gamma) \) is \( P_1 \)-proximal. By the definition of the map \( \xi^+_{\rho} \), if \( \rho(\gamma_0) \) is \( P_1 \)-proximal, then \( \xi^+_{\rho}(\gamma^+_0) \) is the unique attracting fixed point of \( \rho(\gamma_0) \) in \( \mathbb{P}(\mathbb{R}^d) \). Since \( \Gamma \) acts minimally on \( \partial_f \Gamma \) we have \( \Delta_\Gamma(\xi^+_{\rho}(\partial_f \Gamma)) = \Delta_\Gamma(\xi^-_{\rho}(\partial_f \Gamma)) \).

\( \Box \)
5. Property (U), weak eigenvalue gaps and the uniform gap summation property

In this section, we prove Theorem 1.6 providing conditions under which a representation with a weak uniform gap in eigenvalues is Anosov and we also discuss (strong) property (U), and its relation with the uniform gap summation property.

Property (U) and strong property (U) were introduced by Delzant–Guichard–Labourie–Mozes [18] and Kassel–Potrie [31] respectively and are related to the growth of the translation length and stable translation length of group elements in terms of their word length.

Definition 5.1. Let $\Gamma$ be a finitely generated group and fix $|\cdot| : \Gamma \to \mathbb{N}$ a word length function on $\Gamma$. The group $\Gamma$ satisfies property (U) (resp. strong property (U)) if there exists a finite subset $F$ of $\Gamma$ and $C, c > 0$ with the following property: for every $\gamma \in \Gamma$ there exists $f \in F$ such that

$$\ell_f(f\gamma) \geq c|\gamma| - C \quad (\text{resp. } |f\gamma| \geq c|\gamma| - C).$$

Note that (strong) property (U) is independent of the choice of the left invariant word metric on $\Gamma$ since any two such metrics on $\Gamma$ are quasi-isometric.

Delzant–Guichard–Labourie–Mozes proved in [18] that every word hyperbolic group and every finitely generated group which admits a semisimple quasi-isometric embedding into a reductive Lie group satisfy (strong) property (U). We now prove Theorem 1.7 which implies that virtually torsion free finitely generated groups with non-trivial Floyd boundary satisfy property (U) (and hence property (U)).

Let us recall that the Floyd boundary $\partial_f \Gamma$ of $\Gamma$ with respect to a Floyd function $f$ is called non-trivial if $|\partial_f \Gamma| \geq 3$. For a subgroup $H$ of $\Gamma$, its limit set $\Lambda(H) \subset \partial_f \Gamma$ is the set of accumulation points of infinite sequences of elements of $H$ in $\partial_f \Gamma$.

Proof of Theorem 1.7. Let $G : [1, \infty) \to (0, \infty)$ be the function $G(x) := 10 \sum_{k=1}^{\infty} f(k)$. Note that $G$ is decreasing and $\lim_{x \to \infty} G(x) = 0$. By Karlsson’s estimate, see [30, Lem. 1], we have

$$d_f(g, h) \leq G((g \cdot h)_{e}), \quad d_f(g, g^\pm) \leq G\left(\frac{1}{2}|g|_{r}\right)$$

for every $g, h \in \Gamma$, where $g$ has finite order. Since $\Lambda(H)$ contains at least 3 points, by [30, Prop. 5], we may find $f_1, f_2 \in H$ non-trivial such that the sets $\{f_1^\pm, f_1^-\}$ and $\{f_2^\pm, f_2^-\}$ are disjoint. Let us set

$$\varepsilon := \frac{1}{100} \min \{d_f(f_1^+, f_2^+), d_f(f_1^+, f_2^-), d_f(f_1^-, f_2^+), d_f(f_1^-, f_2^-)\}$$

and make the following three choices of constants $M, R, N > 0$ as follows:

(i) $M > 0$ is chosen such that $G(x) \geq \frac{5}{100}$ if and only if $x \leq M$,
(ii) $R > 0$ is chosen such that $G(x) \leq \frac{x}{100}$ for every $x \geq R$,
(iii) $N > 0$ is chosen such that $\min \{|f_1^N|_{r}, |f_2^N|_{r}\} > 10(M + R)$.

Now we prove the following claim:

Claim. 1 Let $F := \{f_1^N, f_2^N, \varepsilon\}$. Now we claim that for every non-trivial $\gamma \in H$, there exists $g \in F$ such that $d_f(g\gamma^+, \gamma^-) \geq \varepsilon$.

If $d_f(\gamma^+, \gamma^-) \geq \varepsilon$ we choose $g = e$. So we may assume that $d_f(\gamma^+, \gamma^-) \leq \varepsilon$. We can choose $n_0 \in \mathbb{N}$ such that $G\left(\frac{1}{2}|\gamma|^n_{r}\right) < \varepsilon$ for $n \geq n_0$. Notice that we can find $i \in \{1, 2\}$ such that $d_f(\gamma^+, f_{i}^\pm) \geq 5\varepsilon$ and $d_f(\gamma^+, f_i^-) \geq 5\varepsilon$. Indeed, if we assume that $\text{dist}(\gamma^+, \{f_{i}^\pm, f_i^-\}) < 5\varepsilon$ then $d_f(\gamma^+, f_{i}^\pm) \geq \text{dist}(f_{i}^\pm, f_{i}^-) - 5\varepsilon \geq 5\varepsilon$. Without loss of generality we may assume $d_f(\gamma^+, f_{1}^\pm) \geq 5\varepsilon$ and $d_f(\gamma^+, f_{1}^-) \geq 5\varepsilon$. By our choices of $N, n_0 > 0$ we have

$$d_f(\gamma^+, f_{1}^-) \geq d_f(\gamma^+, f_{1}^-) - d_f(f_{1}^-, f_{1}^-) - d_f(\gamma^+, \gamma^-) \geq 5\varepsilon - G\left(\frac{1}{2}|f_{1}^N|_{r}\right) - G\left(\frac{1}{2}|\gamma|^N_{r}\right) \geq 48\varepsilon,$$
hence $G((\gamma^n \cdot f_1^{-N})_e) \geq \varepsilon$ for $n \geq n_0$. By the choice of $M > 0$ we have that $(\gamma^n \cdot f_1^{-N})_e \leq M$ for $n \geq n_0$. Then, we choose a sequence $(k_n)_{n \in \mathbb{N}}$ such that $|f_1^{k_n-N}|_\Gamma < |f_1^{k_n}|_\Gamma$ for every $n \in \mathbb{N}$. For $n \geq n_0$ we have

$$2(f_1^{N\gamma^n} \cdot f_1^{k_n})_e = |f_1^{N\gamma^n}|_\Gamma + |f_1^{k_n}|_\Gamma - |f_1^{N-k_n\cdot \gamma^n}|_\Gamma = |\gamma^n|_\Gamma + |f_1^{k_n}|_\Gamma - 2(\gamma^n \cdot f_1^{-N})_e + |f_1^{k_n}|_\Gamma - |f_1^{N-k_n\cdot \gamma^n}|_\Gamma \geq -2M + |f_1^{N}|_\Gamma + |f_1^{k_n}|_\Gamma - |f_1^{N-k_n}|_\Gamma \geq |f_1^{N}|_\Gamma - 2M \geq \frac{|f_1^{N}|_\Gamma}{2} \geq 2R.$$

Thus, by the choice of $R > 0$ we have $G((f_1^{N\gamma^n} \cdot f_1^{k_n})_e) \leq \varepsilon, n \geq n_0$. It follows that $d_f(f_1^{N\gamma^n}, f_1^{k_n}) \leq \varepsilon$ so $d_f(f_1^{N\gamma^n}, \gamma^n) \geq d_f(f_1^{N\gamma^n}, f_1^{k_n}) - d_f(f_1^{N\gamma^n}, f_1^{k_n}) - d_f(\gamma^n, \gamma^n) \geq 48\varepsilon$ and Claim 1 follows.

Now, let $L := 10 \max_{g \in F} |g|_\Gamma + 2R$. If $\gamma \in H$ and $|\gamma|_\Gamma < L$, then we choose $g = \varepsilon$ and obviously $|\gamma|_\Gamma - |\gamma|_\infty \leq L$. Suppose that $\gamma \in H$ and $|\gamma|_\Gamma \geq L$. We may choose $g \in F$ such that $d_f((g\gamma g^{-1})^+, \gamma^-) \geq \varepsilon, \gamma^+ \in \partial_{+1}\Gamma$. We observe that

$$d_f((g\gamma g^{-1})^+, (g\gamma)^+)_e \leq d_f((g\gamma g^{-1})^+, g\gamma g^{-1}) + d_f(g\gamma g^{-1}, g\gamma) + d_f((g\gamma)^+, g\gamma) \leq G\left(\frac{1}{2}|g\gamma g^{-1}|_\Gamma\right) + G((g\gamma g^{-1}, g\gamma),_e) + G\left(\frac{1}{2}|g\gamma|_\Gamma\right) \leq 3G\left(\frac{1}{2}|\gamma|_\Gamma - 2|g|_\Gamma\right) \leq \frac{3\varepsilon}{100},$$

$$d_f(\gamma^-, \gamma^{-1}) \leq d_f(\gamma^-, \gamma^{-1}) + d_f(\gamma^{-1}, \gamma^{-1} g^{-1}) \leq G\left(\frac{1}{2}|\gamma|_\Gamma\right) + G((\gamma^{-1}, \gamma^{-1} g^{-1}),_e) \leq 2G\left(\frac{1}{2}|\gamma|_\Gamma - 2|g|_\Gamma\right) \leq \frac{\varepsilon}{50},$$

since $|\gamma|_\Gamma - 2|g|_\Gamma \geq 2R$. Therefore, by the previous bounds we have

$$d_f((g\gamma)^+, \gamma^{-1} g^{-1}) \geq d_f((g\gamma)^+, \gamma^-) - d_f((g\gamma)^+, (g\gamma)^+)_e - d_f(\gamma^{-1}, \gamma^{-1} g^{-1}) \geq \frac{\varepsilon}{2}.$$ 

This shows that there is $n_1 > 0, G((g\gamma)^n \cdot (g\gamma)^{-1})_e \geq \frac{\varepsilon}{2}$ and $(g\gamma)^n \cdot (g\gamma)^{-1} \leq M$ for $n \geq n_1$. We can find a sequence $(m_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \left(\left|(g\gamma)^{m_n+1}\right|_\Gamma - \left|(g\gamma)^{m_n}\right|_\Gamma\right) \leq |g\gamma|_\infty$$

so

$$\lim_{n \to \infty} 2\left|(g\gamma)^{m_n} \cdot (g\gamma)^{-1}\right)_e \geq |g\gamma|_\Gamma - |g\gamma|_\infty.$$ 

Finally, since $R > M$, we conclude that

$$|g\gamma|_\Gamma - |g\gamma|_\infty \leq 2M \leq L.$$ 

This completes the proof of the theorem. \qed

5.1. Weak uniform gaps in eigenvalues. Recall that a linear representation $\rho : \Gamma \to \text{GL}_d(\mathbb{R})$ has a weak uniform $i$-gap in eigenvalues if there exists $\varepsilon > 0$ such that for every $\gamma \in \Gamma$,

$$\log \frac{\ell_i(\rho(\gamma))}{\ell_{i+1}(\rho(\gamma))} \geq \varepsilon |\gamma|_\infty.$$

For a group $\Gamma$ the lower central series

$$\ldots \leq g_3(\Gamma) \leq g_2(\Gamma) \leq g_1(\Gamma) \leq g_0(\Gamma) := \Gamma$$

is inductively defined as $g_{k+1}(\Gamma) = [\Gamma, g_k(\Gamma)]$ for $k \geq 1$. For every $k$, $g_k(\Gamma)$ is a characteristic subgroup of $\Gamma$ and the quotient $g_k(\Gamma)/g_{k+1}(\Gamma)$ is a central subgroup of $\Gamma/g_{k+1}(\Gamma)$. The group $\Gamma$ is nilpotent if there exists $m \geq 0$ with $g_m(\Gamma) = 1$. 

First, we prove the following technical lemma showing that a nilpotent group $\Gamma$ which admits a representation with weak uniform eigenvalue $i$-gap has to be virtually cyclic.

**Lemma 5.2.** Let $\Gamma$ be a finitely generated nilpotent group. Suppose that $\rho : \Gamma \to \text{GL}_d(\mathbb{R})$ has a weak uniform $i$-gap in eigenvalues for some $1 \leq i \leq d - 1$. Then $\Gamma$ is virtually cyclic.

**Proof.** We need the following elementary observation: for a group $G_1$ and a central subgroup $G_2 \subset Z(G_1)$ of $G_1$, if the quotient $G_1/G_2$ is virtually cyclic, then $G_1$ is virtually abelian.

Let $G$ be the Zariski closure of $\rho(\Gamma)$ in $\text{GL}_d(\mathbb{R})$. We consider the Levi decomposition $G = L \times U$, where $U$ is a connected normal unipotent subgroup of $G$ and $L$ is a reductive Lie group. The projection $\pi \circ \rho : \Gamma \to L$ is Zariski dense and $\lambda(\pi(\rho(\gamma))) = \lambda(\rho(\gamma))$ for every $\gamma \in \Gamma$. The Lie group $L$ is reductive and $\pi(\rho(\Gamma))$ is solvable, so $L$ has to be virtually abelian since it has finitely many connected components. We may find a finite-index subgroup $H$ of $\Gamma$ such that $g_1(H) = [H, H]$ is a subgroup of $\ker(\pi \circ \rho)$. Therefore, for $k \geq 1$ we obtain a well defined representation $\rho_k : H/g_k(H) \to \text{GL}_d(\mathbb{R})$ such that $\rho_k \circ \pi_k = \pi \circ \rho$, where $\pi_k : H \to H/g_k(H)$ is the quotient map. Note that for every $k \geq 1$ there exists $c_k \geq 1$ such that for every $h \in H$, $|\pi_k(h)|_{H/g_k(H), \infty} \leq c_k|h|_{H, \infty}$.

Since $\lambda(\rho_k(h)) = \lambda(\rho(h))$ for every $h \in H$, $\rho_k$ has a weak uniform $i$-gap in eigenvalues for every $k \geq 1$. We may use induction on $k \in \mathbb{N}$ to see that $H/g_k(H)$ is virtually cyclic. The group $H/g_1(H)$ is abelian and satisfies strong property $(U)$, so $\rho_1 = P_{r_1}$-Anosov by [31, Prop. 4.12] and $H/g_1(H)$ has to be virtually cyclic. Now suppose that $H/g_k(H)$ is virtually cyclic. Note that $g_k(H)/g_{k+1}(H)$ is a central subgroup of $H/g_{k+1}(H)$ with virtually cyclic quotient $H/g_k(H)$. It follows that $H/g_{k+1}(H)$ is virtually abelian. In particular, $H/g_{k+1}(H)$ satisfies strong property $(U)$, so $\rho_{k+1} = P_{r_1}$-Anosov and $H/g_{k+1}(H)$ is virtually cyclic. Therefore, $H/g_k(H)$ has to be virtually cyclic for every $k \geq 1$ and $H$ is virtually cyclic since $g_m(H) = 1$ for some $m \geq 1$. □

As a corollary of Theorem 1.7, we obtain Corollary 1.8 which shows that a non-virtually nilpotent group $\Gamma$ which admits a representation with the uniform gap summation property satisfies strong Property $U$.

**Proof of Corollary 1.8.** By Proposition 2.4 we may assume that $G = \text{SL}_d(\mathbb{R})$ and $\theta = \{\varepsilon_1 - \varepsilon_2\}$. Since $\rho$ satisfies the uniform gap summation property $\ker(\rho)$ is finite. It suffices to prove that a finite-index subgroup of $\Gamma' = \Gamma/\ker(\rho)$ satisfies strong property $(U)$. By Selberg’s lemma [36] $\Gamma'$ is virtually torsion free, so we may assume that $\Gamma$ is torsion free and $\rho$ is faithful. By Lemma 4.8 there exists a continuous $\rho$-equivariant map $\xi_f : \partial_f \Gamma \to \mathbb{P}(\mathbb{R}^d)$ for some Floyd function $f$. We first prove that $\partial_f \Gamma$ is not a singleton.

Suppose that $|\partial_f \Gamma| = 1$. By the definition of the map $\xi_f$, the image $\xi_f(\partial_f \Gamma)$ identifies with the $\tau_{\text{mod}}$-limit set of $\Gamma$ in $\mathbb{P}(\mathbb{R}^d)$. Since $\Gamma$ is not virtually nilpotent, we may use [26, Cor. 5.10] to check that $\partial_f \Gamma$ contains at least two points. We provide here the following different argument. Since $\partial_f \Gamma$ is assumed to be a singleton, up to conjugation, we may assume that $\xi_f(\partial_f \Gamma) = \{e_1\}$ and find a group homomorphism $a : \Gamma \to \mathbb{R}^*$ such that for every $\gamma \in \Gamma$,

$$\rho(\gamma)e_1 = a(\gamma)e_1.$$  

We consider the representation $\hat{\rho}(\gamma) = \frac{1}{a(\gamma)}\rho(\gamma)$. Note that $\hat{\rho}$ satisfies the uniform gap summation property (since $\rho$ does), $\xi_f$ is $\hat{\rho}$-equivariant and we can write

$$\hat{\rho}(\gamma) = \begin{pmatrix} 1 & u(\gamma) \\ 0 & \rho_0(\gamma) \end{pmatrix}$$

for some group homomorphism $\rho_0 : \Gamma \to \text{GL}_{d-1}(\mathbb{R})$. Let $g \in \Gamma \setminus \{e\}$. Since $\xi_f$ is constant we have $\lim_n \Xi_1^+(\hat{\rho}(g^n)) = \lim_n \Xi_1^+(\hat{\rho}(g^{-n})) = [e_1]$. Let us write $\hat{\rho}(g^n) = k_n \exp\left(\mu(\hat{\rho}(g^n))\right)k_n^{-1}$ in the

---

\footnote{This is not true when $\Gamma$ is assumed to be solvable. The Baumslag–Solitar group $BS(1, 2)$ admits a faithful representation into $\text{GL}_2(\mathbb{R})$ with a weak uniform 1-gap (see [31, Ex. 4.8]).}
Cartan decomposition of $G$, and up to passing to a subsequence, we may assume $\lim_n k_n = k_G$ and $\lim_n k'_n = k'_G$. Then $k'_n P^+_1 = w P^+_1$, $(k'_n e_1, e_1) = 0$ and $|\langle k'_n e_1, e_1 \rangle| = 1$, so
\[
\lim_{n \to \infty} \hat{\rho}(g^n) = k_G E_{11} k'_G \in \oplus_{i=2}^d \mathbb{R} E_i.
\]
If $\ell_1(\hat{\rho}(g)) > 1$, then $\ell_1(\rho_0(g)) = \ell_1(\hat{\rho}(g))$. Let $p_1$ and $p_2$ be the largest possible size of a Jordan block for an eigenvalue of maximum modulus of $\hat{\rho}(g)$ and $\rho_0(g)$ respectively. We have
\[
\sigma_1(\hat{\rho}(g^n)) = np_1^{-1} \ell_1(\hat{\rho}(g^n)), \quad \sigma_1(\rho_0(g^n)) = np_2^{-1} \ell_1(\hat{\rho}(g^n))
\]
and $p_1 > p_2$ since $\lim_n \frac{\rho_0(g^n)}{\sigma_1(\rho_0(g^n))} = 0$. In particular, there exists $C > 0$ such that
\[
\left\| u(g^n) \right\| = \left\| \sum_{i=0}^n \rho_0(g^i) u(g) \right\| \leq \left\| u(g) \right\| \sum_{i=0}^n i^{p_2-1} \ell_1(\hat{\rho}(g))^i \leq C n^{p_2-1} \ell_1(\hat{\rho}(g))^n
\]
for every $n \in \mathbb{N}$. Since $p_1 > p_2$ and $\ell_1(\rho_0(g)) > 1$ we have
\[
\lim_{n \to \infty} \frac{1}{n^{p_1-1} \ell_1(\hat{\rho}(g))^n} \sum_{i=0}^n i^{p_2-1} \ell_1(\hat{\rho}(g))^i = 0.
\]
Therefore, $\lim_n \frac{\|u(g^n)\|}{\sigma_1(\rho_0(g^n))} = 0$ which is impossible since $\lim_n \frac{\hat{\rho}(g^n)}{\sigma_1(\rho_0(g^n))}$ has at least one of its $(1, 2), \ldots, (1, d)$ entries non-zero. It follows that $\ell_1(\hat{\rho}(g)) \leq 1$ and $\ell_1(\rho_0(g)) \leq |a(g)|$. Similarly, we obtain $\ell_d(\rho_0(g))^{-1} = \ell_1(\rho_0(g^{-1})) \leq |a(g^{-1})|$. It follows that all the eigenvalues of $\rho(g)$ have modulus equal to 1. Therefore, by Theorem 2.5, any semisimplification of $\rho$ has compact Zariski closure. Then, by using [2, Thm. 3] and [26, Thm. 10.1], we conclude that $\rho(\Gamma)$ (and hence $\Gamma$) is virtually nilpotent. We have reached a contradiction, therefore, $\xi_f$ is non-constant and $\partial_f \Gamma$ contains at least two points.

Now we conclude that $\Gamma$ has strong property (U) by showing that $|\partial_f \Gamma| \geq 3$.

If $|\partial_f \Gamma| = 2$, consider the restriction $\rho_V : \Gamma \to \text{GL}(V)$ where $V = \langle \xi_f(\partial_f \Gamma) \rangle$ and $\dim(V) = 2$. We show that all elements of $\rho(\text{ker}(\rho_V))$ have all of their eigenvalues of modulus 1. For this, since $\xi_f(\partial_f \Gamma)$ contains two points, up to passing to a finite-index subgroup of $\Gamma$ and conjugating by $\rho_V$ by an element of $\text{GL}(V)$, we may assume that $\rho_V(\Gamma)$ lies in the diagonal subgroup $\text{GL}(V)$. Let $g \in \text{ker}(\rho_V)$. We may write $\rho(g^n) = w_n \exp(\mu(g^n)) w'_n$ and assume, up to conjugating $\rho$, that
\[
\lim_n w_n = w_x, \lim_n w'_n = w'_x, \text{ where } w_x P^+_1 = P^+_1.
\]
We see that $\lim_n \frac{\rho(g^n)}{\|\rho(g^n)\|} = w_x E_{11} w'_x \in \oplus_{i=1}^d \mathbb{R} E_i$ and we may write for $n \in \mathbb{N}$,
\[
\rho(g^n) = \left( I_2 \left( \sum_{i=0}^n A_i \right)^t B \right)
\]
such that $\lim_n \frac{1}{\|\rho(g^n)\|} A^n$ is the zero matrix. If $A$ has an eigenvalue of modulus greater than 1, then $\ell_1(A) = \ell_1(\rho(g))$. By working similarly as in the previous case, we have $\lim_n \frac{1}{\|\rho(g^n)\|} \sum_{i=0}^n \|A^n\| = 0$ and $\lim_n \frac{1}{\|\rho(g^n)\|} \rho(g^n)$ has all of its $(1, i)$ entries equal to zero, which is absurd. This shows that $\rho(g^{\pm 1})$ has all of its eigenvalues of modulus at most 1 for $g \in \text{ker}(\rho_V)$.

We deduce that $\rho(\text{ker}(\rho_V))$ (and hence $\ker(\rho_V)$) is virtually nilpotent and finitely generated. The quotient $\Gamma/\ker(\rho_V)$ is abelian, so $\Gamma$ has to be virtually polycyclic. Since $|\partial_f \Gamma| > 1$, a theorem of Floyd [19, p. 211] implies that $\Gamma$ has two ends, so $\Gamma$ is virtually cyclic. Since $\Gamma$ is assumed not to be virtually nilpotent, this is again a contradiction, hence $\partial_f \Gamma$ cannot contain two points.

Finally, it follows that $|\partial_f \Gamma| \geq 3$. Therefore, Theorem 1.7 shows that $\Gamma$ satisfies strong property (U).

\[\square\]

\textbf{Proof of Theorem 1.6.} Suppose that (i) holds, i.e $\rho$ is $P_1$-Anosov. Then (ii) holds since the Floyd boundary identifies with the Gromov boundary of $\Gamma$. Moreover, by Theorem 2.3 and Proposition 2.6, (iii) and (iv) hold true for any semisimplification $\rho^{ss}$ of the $P_1$-Anosov representation $\rho$. Now let us prove the other implications.
We assume that there exists $\varepsilon > 0$ such that for every $\gamma \in \Gamma$, 
\[
\log \frac{\ell_i(\rho(\gamma))}{\ell_{i+1}(\rho(\gamma))} \geq \varepsilon |\gamma|_\infty.
\]
By [31, Prop. 4.12] it is enough to prove that $\Gamma$ satisfies strong property (U).

(ii) $\Rightarrow$ (i). We first observe that for every element $g \in \ker(\rho)$ we have $|g|_\infty = 0$. We next show that $N := \ker \rho$ is finite. If not, $N$ is an infinite normal subgroup of $\Gamma$ and $\Lambda(N) = \partial_f \Gamma$ since $\Gamma$ acts minimally on $\partial_f \Gamma$. By [30, Thm. 1] there exists a non cyclic free subgroup $H$ of $N$ with $|\Lambda(H)| \geq 3$. By Theorem 1.7 we can find $\gamma \in H$ such that $|\gamma|_\infty > 0$. This is a contradiction since $\gamma \in N$. It follows that $N$ is finite.

The Floyd boundary of $\Gamma' = \Gamma/N$ is non-trivial since $\Gamma'$ is quasi-isometric to $\Gamma$. Note that the representation $\rho$ induces a faithful representation $\rho' : \Gamma' \to \mathrm{GL}_d(\mathbb{R})$ which also has a weak uniform $i$-gap in eigenvalues. Selberg’s lemma [36] implies that $\Gamma'$ is virtually torsion free so, by Theorem 1.7, $\Gamma'$ satisfies strong property (U). We conclude that $\Gamma'$ and $\Gamma$ are word hyperbolic and $\rho$ is $P_1$-Anosov.

(iii) $\Rightarrow$ (i). If $\Gamma$ is virtually nilpotent, Lemma 5.2 implies that $\Gamma$ is virtually cyclic, contradicting our assumption. If $\Gamma$ is not virtually nilpotent, since $\rho_1$ satisfies the uniform gap summation property, $\Gamma$ has to satisfy strong property (U) by Corollary 1.8. Therefore, (i) holds.

(iv) $\Rightarrow$ (i). Let $\rho^{ss}$ be a semisimplification of $\rho$. By Proposition 2.6, $\lambda(\rho(\gamma)) = \lambda(\rho^{ss}(\gamma))$ for every $\gamma \in \Gamma$ so there exists $c_2 > 0$, depending on $\rho_2$, such that
\[
\log \frac{\ell_i(\rho^{ss}(\gamma))}{\ell_{i+1}(\rho^{ss}(\gamma))} \geq \varepsilon |\gamma|_\infty \geq \varepsilon c_2 \|\lambda(\rho_2(\gamma))\|
\]
for every $\gamma \in \Gamma$. By Theorem 2.5 there exists a finite subset $F$ of $\Gamma$ and $C > 0$ such that for every $\gamma \in \Gamma$ there exists $f \in F$ with $\|\mu(\rho^{ss}(\gamma)) - \lambda(\rho(\gamma)f)\| \leq C$ and $\|\mu(\rho_2(\gamma)) - \lambda(\rho_2(\gamma)f)\| \leq C$. In particular, we may choose $R > 0$ such that
\[
\log \frac{\sigma_i(\rho^{ss}(\gamma))}{\sigma_{i+1}(\rho^{ss}(\gamma))} \geq \varepsilon c_2 \|\mu(\rho_2(\gamma))\| - R
\]
for every $g \in \Gamma$. By assumption, for all but finitely many $g \in \Gamma$ we have $\|\mu(\rho_2(\gamma))\| \geq 2/(\varepsilon c_2) \log |g|_\Gamma$, so there exists $R' > 0$ such that
\[
\log \frac{\sigma_i(\rho^{ss}(\gamma))}{\sigma_{i+1}(\rho^{ss}(\gamma))} \geq 2 \log |g|_\Gamma - R'
\]
for all $g \in \Gamma$. In particular, the semisimplification $\rho^{ss}$ of $\rho$ satisfies the uniform gap summation property. Therefore, since $\rho^{ss}$ has a weak uniform $i$-gap in eigenvalues, by implication (iii) $\Rightarrow$ (i), $\rho^{ss}$ is $P_1$-Anosov and $\Gamma$ is word hyperbolic. In particular, $\rho$ is $P_1$-Anosov. \qed

6. Gromov products

In this section, we recall the definition of the Gromov product (see Definition 1.11) associated to an Anosov representation and prove Proposition 1.12, and we show that it is comparable with the Gromov product on the domain hyperbolic group with respect to a fix word metric.

**Definition 6.1.** Let $G$ be a real semisimple Lie group. For every linear form $\varphi \in \mathfrak{a}^*$, define the Gromov product relative to $\varphi$ to be the map $(\cdot)_\varphi : G \times G \to \mathbb{R}$ defined as follows: for $g, h \in G$,
\[
(g \cdot h)_\varphi := \frac{1}{4} \langle \varphi, \mu(g) + \mu(g^{-1}) + \mu(h) + \mu(h^{-1}) - \mu(g^{-1}h) - \mu(h^{-1}g) \rangle.
\]

For a line $\ell \in \mathcal{P}(\mathbb{R}^d)$ and a hyperplane $V \in \mathrm{Gr}_{d-1}(\mathbb{R}^d)$, the distance $\mathrm{dist}(\ell, V)$ is computed by the formula
\[
\mathrm{dist}(\ell, V) = |\langle k_\ell e_1, k_V e_d \rangle|,
\]
where $\ell = [k_\ell e_1]$, $V = [k_V e_d]$ and $k_\ell, k_V \in O(d)$. The following proposition relates the Gromov product with the limit maps of a representation $\rho$ and will be used in the following sections.
Proposition 6.2. Let $\Gamma$ be a word hyperbolic group and $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ a representation. Suppose $\rho$ is $P_1$-divergent and there are continuous $\rho$-equivariant maps $\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^- : \partial_\infty \Gamma \to \text{Gr}_{d-1}(\mathbb{R}^d)$ satisfying the Cartan property. Then for $x, y \in \partial_\infty \Gamma$ and two sequences $(\gamma_n)_{n \in \mathbb{N}}, (\delta_n)_{n \in \mathbb{N}}$ of elements of $\Gamma$ with $\lim_n \gamma_n = x$ and $\lim_n \delta_n = y$ we have
\[
\lim_{n \to \infty} \exp \left( -4 \left( \rho(\gamma_n) \cdot \rho(\delta_n) \right)_{\varepsilon_1} \right) = \text{dist} \left( (\xi(x), \xi^-(y)) \cdot \text{dist} \left( (\xi(y), \xi^-(x)) \right. \right).
\]

Proof. We may write $\rho(\gamma_n) = w_n \exp(\mu(\rho(\gamma_n))) w_n^t$, and $\rho(\delta_n) = k_n \exp(\mu(\rho(\delta_n))) k_n^t$, where $w_n, w_n^t, k_n, k_n^t \in \text{PO}(d)$. Since $\rho$ is $P_1$-divergent, $\lim_n \sigma_d(\rho(\gamma_n)) = \lim_n \sigma_d(\rho(\delta_n)) = 0$ for $1 \leq j \leq d - 1$. Recall that $E_{ij}$ denotes the $d \times d$ elementary matrix with 1 on the $(i, j)$-entry. Then we notice that
\[
\lim_{n \to \infty} \exp \left( -4 \left( \rho(\gamma_n) \cdot \rho(\delta_n) \right)_{\varepsilon_1} \right) = \lim_{n \to \infty} \frac{\sigma_1(\rho(\gamma_n)^{-1} \rho(\delta_n)) \sigma_1(\rho(\delta_n)^{-1} \rho(\gamma_n))}{\sigma_1(\rho(\gamma_n)) \sigma_1(\rho(\delta_n)) \sigma_1(\rho(\delta_n)) \sigma_1(\rho(\gamma_n))} \cdot \left| E_{1d} w_n^{-1} k_n E_{11} \right| \cdot \left| E_{1d} k_n^{-1} w_n E_{11} \right|
\]
\[
= \lim_{n \to \infty} \left| E_{1d} w_n^{-1} k_n E_{11} \right| \cdot \left| E_{1d} k_n^{-1} w_n E_{11} \right|
\]
\[
= \lim_{n \to \infty} \left| w_n^{-1} k_n e_1, e_d \right| \cdot \left| k_n^{-1} w_n e_1, e_d \right|
\]
\[
= \lim_{n \to \infty} \text{dist} \left( (\Xi^+_1(\rho(\gamma_n)), \Xi_1(\rho(\delta_n))) \right) \cdot \text{dist} \left( (\Xi^+_1(\rho(\delta_n)), \Xi_1(\rho(\gamma_n))) \right)
\]
\[
= \text{dist} \left( (\xi(x), \xi^-(y)) \cdot \text{dist} \left( (\xi(y), \xi^-(x)) \right. \right)
\]
since $\xi$ and $\xi^-$ satisfy the Cartan property. This finishes the proof of the proposition. \hfill \square

Proof of Proposition 1.12. Fix $\alpha \in \theta$. By [22, Lem. 3.2], there exists $N_0 > 0$ and an irreducible $\theta$-proximal representation $\tau_\alpha : G \to \text{GL}_d(\mathbb{R})$ whose highest weight is $N_0 \omega_\alpha$, $N_0 \in \mathbb{N}$. Since $\rho$ is $P_{(\alpha)}$-Anosov, the representation $\tau_\alpha \circ \rho$ is $P_1$-Anosov. Note that there exists $C_1 > 0$, depending only on $\tau$, such that
\[
\left| \log \sigma_1(\tau_\alpha(g)) - N_0 \langle \omega_\alpha, \mu(g) \rangle \right| \leq C_1
\]
for every $g \in G$. In particular, there exists $C_2 > 0$, depending only on $\tau$, such that
\[
\left| N_0 (\rho(\gamma) \cdot \rho(\delta))_{\omega_\alpha} - (\tau_\alpha(\rho(\gamma)) \cdot \tau_\alpha(\rho(\delta)))_{\varepsilon_1} \right| \leq C_2
\]
for every $\gamma, \delta \in \Gamma$. Since $\rho$ is $P_{(\alpha)}$-Anosov, by Lemma 2.11, we may replace $\rho$ with a semisimplification $\rho^{ss}$ such that there exists $C_3 > 0$ with
\[
\left| (\rho(\gamma)) \cdot (\rho(\delta))_{\omega_\alpha} - (\rho^{ss}(\gamma)) \cdot (\rho^{ss}(\delta))_{\omega_\alpha} \right| \leq C_3
\]
for every $\gamma, \delta \in \Gamma$. Therefore, we may continue by assuming that $\rho$ is semisimple. By using Lemma 2.10, we may further assume that $\tau_\alpha(\rho(\Gamma))$ has reductive Zariski closure in $\text{GL}_d(\mathbb{R})$ and preserves a properly convex open domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$. Let us fix $x_0 \in \Omega$. By Lemma 2.8 we can find $C_4 > 0$ such that
\[
\left| (\tau_\alpha(\rho(\gamma)) \cdot \tau_\alpha(\rho(\delta)))_{\varepsilon_1} - (\tau_\alpha(\rho(\gamma)) x_0 \cdot \tau_\alpha(\rho(\delta)) x_0)_{x_0} \right| \leq C_4
\]
for every $\gamma, \delta \in \Gamma$. By [17] and [39], since $\tau_\alpha \circ \rho$ is $P_1$-Anosov, $\tau_\alpha(\rho(\Gamma))$ acts cocompactly on a closed convex subset $\mathcal{C} \subset \Omega$. Fix $x_0 \in \mathcal{C}$. The Svarc–Milnor lemma implies that the orbit map $\gamma \mapsto \tau_\alpha(\rho(\gamma)) x_0$ is a quasi-isometry between the Gromov hyperbolic spaces $(\Gamma, d_\Gamma)$ and $(\mathcal{C}, d_0)$. In particular, there exist $C_5, c_5 > 0$ such that for every $\gamma, \delta \in \Gamma$,
\[
C_{\gamma}^{-1}(\gamma \cdot \delta)_{x} - c_5 \leq (\tau_\alpha(\rho(\gamma)) x_0 \cdot \tau_\alpha(\rho(\delta)) x_0)_{x_0} \leq C_5(\gamma \cdot \delta)_{x} + c_5.
\]

Therefore, by (7), (8) and (9) we obtain the conclusion.

7. Characterizations of Anosov representations

This section is devoted to the proof of Theorems 1.1 and 1.3 and Corollary 1.2. Note that in Theorem 1.1 we do not assume that the group \( \rho(\Gamma) \) contains a \( P_\theta \)-proximal element, the pair of limit maps \((\xi^+, \xi^-)\) is compatible or the map \( \xi^- \) satisfies the Cartan property.

**Proof of Theorem 1.1.** If \( \rho \) is a \( P_\theta \)-Anosov, the Anosov limit maps of \( \rho \) are transverse and dynamics preserving and \( \rho \) is \( P_\theta \)-divergent (see Theorem 2.3). Also, the fact that the Anosov limit maps satisfy the Cartan property is contained in [22, Thm. 1.3 (4) & 5.3 (4)].

Now we assume that \( \rho \) satisfies (i) and (ii). We first reduce to the case where \( \Gamma \) is torsion free. Since \( \rho \) is \( P_\theta \)-divergent, every element of the kernel \( \ker(\rho) \) has finite order, hence \( \ker(\rho) \) is finite. The quotient group \( H = \Gamma/\ker(\rho) \) is quasi-isometric to \( \Gamma \) and by Selberg’s lemma [36] \( H \) contains a torsion free and finite-index subgroup \( H_1 \). It is enough to prove that the induced representation \( \hat{\rho} : H_1 \to G \) is \( P_\theta \)-Anosov. Notice that \( \hat{\rho} \) satisfies the same assumptions as \( \rho \) and the source group is torsion free.

Thanks to Proposition 2.4, we may assume that \( G = SL_d(\mathbb{R}), \theta = \{e_1 - e_2\}, P_\theta^+ = \text{Stab}_G(\mathbb{R}e_1) \) and \( P_\theta^- = \text{Stab}_G(e_1^+). \) Recall the definition of the bundle \( \mathcal{X}_\rho \) over the flow space \( \Gamma\hat{\Gamma} \) as in subsection 2.5. The pair of transverse maps \((\xi^+, \xi^-)\) defines the section \( \sigma : \Gamma\hat{\Gamma} \to \mathcal{X}_\rho \),

\[
\sigma([\hat{\mu}]_\Gamma) = [\hat{\mu}, (\xi^+(\hat{\tau}(\hat{\mu})), \xi^-(\hat{\tau}(\hat{\mu})))].
\]

which induces the splitting \( \sigma_* \mathcal{E} = \sigma_* \mathcal{E}^+ \oplus \sigma_* \mathcal{E}^- \), where \( \mathcal{E}^\pm \subset \mathcal{T}(G/L_\theta) \) are sub-bundles defined in subsection 2.5. Then we fix \( x = [\hat{\mu}]_\Gamma \) and choose an element \( h \in G \) so that \( \xi^+(\hat{\tau}(\hat{\mu})) = hP_1^+ \) and \( \xi^-(\hat{\tau}(\hat{\mu})) = hP_1^- \). Let \( (\gamma_n)_{n \in \mathbb{N}} \) be an increasing unbounded sequence and consider a sequence \((\gamma_n \tau_n(\hat{\mu}))_{n \in \mathbb{N}} \) lies in a compact subset of \( \Gamma \). We observe that \( \lim_{n} \gamma_n^{-1} = \hat{\tau}(\hat{\mu}) \) in the bordification \( \Gamma \cup \hat{\partial}_d \Gamma \). Moreover, observe that we can write \( \rho(\gamma_n^{-1}) = (\gamma_n^{-1})^{-1} w \exp(\mu(\rho(\gamma_n^{-1})) \omega k_n^{-1}) \), where \( w = \sum_{i=1}^{d} E_{i(\hat{d}+1-i)} \in O(d) \). Since \( \xi^+ \) is assumed to satisfy the Cartan property and \((\gamma_n)_{n \in \mathbb{N}} \) is \( P_\theta \)-divergent, up to subsequence, we may assume that \( \lim_{n} \gamma_n^{-1} = (\gamma_n^{-1})^{-1} \) is \( P_\theta^+ \). Equivalently, if \( k' = \lim_{n} k_n' \) then \( k' h = (s \ 0 \ B) \) for some \( B \in GL_{d-1}(\mathbb{R}) \). Fix \( u \in \{0\} \times \mathbb{R}^{d-1} \). Then, since \( k' h^t = I_d \), we observe that

\[
k' h^t u = w_{d-1} B^{-t} u + 0e_d, \quad k' h^t e_1 = \frac{1}{s} e_d + \sum_{i=1}^{d-1} \xi_i e_i
\]

for some \( s \neq 0, \xi_1, \ldots, \xi_{d-1} \in \mathbb{R} \) and \( w_{d-1} \in O(d-1) \) is a permutation matrix with \( w_{d-1} e_1 = e_{d-1} \) and \( w_{d-1} e_{d-1} = e_1 \). Equivalently, we write:

\[
k' h^t u = \sum_{i=1}^{d} \chi_{i,n} e_i, \quad k' h^t e_1 = \sum_{i=1}^{d} \xi_i e_i
\]

and we have that \( \lim_n \chi_{d,n} = 0 \), \( \lim_n \xi_{d,n} = \frac{1}{s} \). A computation shows that

\[
\frac{\|\rho^*(\gamma_n h^t u)\|^2}{\|\rho^*(\gamma_n h^t e_1)\|^2} = \sum_{i=1}^{d} \chi_{i,n} \sigma_i(\rho(\gamma_n))^{-2} = \sum_{i=1}^{d-1} \chi_{i,n} \sigma_i(\rho(\gamma_n))^{-2} + \chi_{d,n}^2
\]

and

\[
\frac{\|\rho^*(\gamma_n h^t e_1)\|^2}{\|\rho^*(\gamma_n h^t e_1)\|^2} = \sum_{i=1}^{d} \xi_i^2 \sigma_i(\rho(\gamma_n))^{-2} + \xi_{d,n}^2
\]

We deduce that \( \lim_n \frac{\|\rho^*(\gamma_n h^t u)\|^2}{\|\rho^*(\gamma_n h^t e_1)\|^2} = 0 \) and hence by Proposition 3.2 (ii) we conclude that

\[
\lim_{n \to \infty} \|\varphi_{\nu_n}(X_u^t)\|_{\varphi_{\nu_n}(x)} = 0.
\]

The sequence we started with was arbitrary, therefore the (lift of the) geodesic flow (see Def. 2.2) on \( \sigma_\theta \mathcal{E}^- \) is weakly contracting. By Lemma 3.1 we conclude that the flow on \( \sigma_\theta \mathcal{E}^+ \) is weakly dilating. The compactness of \( \Gamma\hat{\Gamma} \) implies that the geodesic flow on \( \sigma_\theta \mathcal{E}^+ \) (resp. \( \sigma_\theta \mathcal{E}^- \)) is uniformly

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dilating (resp. contracting). Finally, we conclude that $\rho$ is $P_0$-Anosov with Anosov limit maps $\xi^+$ and $\xi^-$. \qed

**Proof of Corollary 1.2.** Assume that conditions (i) and (ii) hold. Let $\tau : G \to \text{GL}_d(\mathbb{R})$ be an irreducible and $\theta$-proximal representation as in Proposition 2.4. It is enough to prove that $\rho' = \tau \circ \rho$ is $P_1$-Anosov. By using [22, Thm. 5.3 (1)] (see also Lemma 4.8), there exists a pair of continuous, $\rho'$-equivariant maps $\xi^+ : \partial_x \Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^- : \partial_x \Gamma \to \text{Gr}_{d-1}(\mathbb{R}^d)$ satisfying the Cartan property. Let $x, y \in \partial_x \Gamma$ be two distinct points and $(\gamma_n)_{n \in \mathbb{N}}$ a sequence of elements of $\Gamma$ with $x = \lim_n \gamma_n$ and $y = \lim_n \gamma_{n}^{-1}$. The second condition, shows that

$$\sup_{n \in \mathbb{N}} \left( 2 \log \sigma_1(\rho'(\gamma_n)) - \log \sigma_1(\rho'(\gamma_{n}^{-1})) \right) < +\infty.$$ 

By Proposition 6.2 we have that $\text{dist}(\xi^+(x), \xi^-(y)) \cdot \text{dist}(\xi^+(y), \xi^-(y)) > 0$ so the pair $(\xi^+(x), \xi^-(y))$ is transverse. The maps $\xi^+$ and $\xi^-$ are transverse, $\rho'$ is $P_1$-divergent by (i), hence, it follows by Theorem 1.1 that $\rho'$ is $P_1$-Anosov.

Conversely, part (i) follows immediately by Theorem 2.3 (i). Note that there is $N \geq 1$ such that $N \omega_{\alpha}$ is the highest weight $\chi_{\tau_{\alpha}}$ of an irreducible proximal representation $\tau : G \to \text{GL}_d(\mathbb{R})$ such that $N(\omega_{\alpha}, \mu(h)) = \log \sigma_1(\tau_{\alpha}(h))$ for every $h \in H$ (see [22, Lem. 3.2]). By Proposition 1.12 (i) we can find $B, b > 0$ such that for every $\alpha \in \theta$ and $\gamma \in \Gamma$ we have

$$\langle \omega_{\alpha}, 2\mu(\rho(\gamma)) - \mu(\rho(\gamma^2)) \rangle \leq \langle \omega_{\alpha}, 2\mu(\rho(\gamma)) + 2\mu(\rho(\gamma^{-1})) - \mu(\rho(\gamma^2)) - \mu(\rho(\gamma^{-2})) \rangle \leq B(\gamma \cdot \gamma^{-1}) + b.$$

Therefore, $\langle \omega_{\alpha}, 2\mu(\rho(\gamma)) - \mu(\rho(\gamma^2)) \rangle \geq 0$ for every $\gamma \in G$. This concludes the proof of the corollary. \qed

Let $\Gamma$ be a word hyperbolic group and $H$ be a subgroup of $\Gamma$. The group $H$ is quasiconvex in $\Gamma$ if and only if $\Gamma$ is finitely generated and quasi-isometrically embedded in $\Gamma$. In this case, there exists a continuous injective $H$-equivariant map $\iota_H : \partial_x H \to \partial_x \Gamma$ called the Cannon-Thurston map extending the inclusion $H \to \Gamma$.

**Proof of Theorem 1.3.** Corollary 4.6 shows that the representation $\rho$ is $P_0$-divergent and $\xi^+$ satisfies the Cartan property. Since $\iota_H$ is an $H$-equivariant embedding, the map $\xi^+ \circ \iota_H$ also satisfies the Cartan property. Theorem 1.1 shows that the representation $\rho|_H$ is $P_0$-Anosov. \qed

Example 10.4 provides a Zariski dense surface group representation $\rho_1 : \pi_1(S_g) \to \text{PSL}_4(\mathbb{R})$ which is not $P_1$-Anosov and admits a pair of continuous $\rho_1$-equivariant maps $(\xi^+, \xi^-)$. The representation $\rho_1$ is $P_1$-divergent and $\rho_1(\gamma)$ is $P_1$-proximal for every $\gamma \in \pi_1(S_g)$ non-trivial. However, for every finitely generated free subgroup $F$ of $\pi_1(S_g)$, the maps $\xi^+ \circ \iota_F$ and $\xi^- \circ \iota_F$ are transverse and $\rho_1|_F$ is $P_1$-Anosov.

### 8. Strongly convex cocompact subgroups of $\text{PGL}_d(\mathbb{R})$

In this section, we prove Theorem 1.10. For our proof we need the following proposition characterizing $P_1$-Anosov representations in terms of the Gromov product under the assumption that the group preserves a properly convex domain with strictly convex and $C^1$-boundary.

**Proposition 8.1.** Let $\Gamma$ be a word hyperbolic subgroup of $\text{PGL}_d(\mathbb{R})$ which preserves a strictly convex domain $\Omega$ of $\mathbb{P}(\mathbb{R}^d)$ with $C^1$-boundary. Then the following are equivalent.

(i) The natural inclusion $\Gamma \to \text{PGL}_d(\mathbb{R})$ is $P_1$-Anosov.

(ii) There exist constants $J, k > 0$ such that for every $\gamma, \delta \in \Gamma$,

$$J^{-1}(\gamma \cdot \delta)_\epsilon - k \leq (\gamma \cdot \delta)_\epsilon \leq J(\gamma \cdot \delta)_\epsilon + k.$$ 

**Proof.** (ii) $\Rightarrow$ (i). We observe that $\Gamma$ is a discrete subgroup of $\text{PGL}_d(\mathbb{R})$. Let $(\gamma_n)_{n \in \mathbb{N}}$ be an infinite sequence of elements of $\Gamma$ and $x_0 \in \Omega$. We may pass to a subsequence such that $\lim_n \gamma_n x_0 \in \Omega$ exists. Since $\Omega$ is strictly convex we conclude that $\lim_n \gamma_n x_0$ is independent of the basepoint $x_0$. Therefore, as in [17, Lem. 7.5] or Lemma 4.5, we conclude that $\lim_n \frac{d}{d_n}(\gamma_n) = 0$ and $\Gamma$ has to be $P_1$-divergent.
Now let \((\gamma_n)_{n \in \mathbb{N}}, (\delta_n)_{n \in \mathbb{N}}\) be two sequences of elements of \(\Gamma\) which converge to the same point \(x \in \partial_\infty \Gamma\). We claim that the limits \(\lim_n \gamma_n x_0, \lim_n \delta_n x_0\) exist and are equal. Note that the limits will be independent of the choice of \(x_0\). We may write
\[
\gamma_n = w_{\gamma_n} \exp(\mu(\gamma_n)) w'_{\gamma_n} \quad \text{and} \quad \delta_n = w_{\delta_n} \exp(\mu(\delta_n)) w'_{\delta_n}
\]
where \(w_{\gamma_n}, w'_{\gamma_n}, w_{\delta_n}, w'_{\delta_n} \in \text{PO}(d)\). Since \(\Gamma\) is \(P_1\)-divergent, there exist subsequences \((\gamma_{k_n})_{n \in \mathbb{N}}, (\delta_{s_n})_{n \in \mathbb{N}}\) such that \(a_1 = \lim_n \gamma_{k_n} x_0 = \lim_n \Xi_1^+ (\gamma_{k_n}), a_2 = \lim_n \delta_{s_n} x_0 = \lim_n \Xi_1^- (\delta_{s_n}), \lim_n \Xi_1^- (\gamma_{k_n}) = a_1^- \) and \(\lim_n \Xi_1^- (\delta_{s_n}) = a_2^-\), where \(\Xi_1^+ (\gamma_{k_n}) = [w_{\gamma_{k_n}} e_1]\) and \(\Xi_1^- (\gamma_{k_n}) = [w_{\gamma_{k_n}} e_d^+]\). Proposition 6.2 and the fact that \(\Xi_1^- (\gamma_{k_n})\) is \(\Gamma\)-equivariant continuous map \(\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)\) defined by the formula \(\xi(\lim_n \gamma_n) = \lim_n \gamma_n x_0\). Let \(x = \lim_n \delta_n\) and suppose \(\lim_n x_n = x \in \partial_\infty \Gamma\). We may write \(x_n = \lim_n \gamma_n x_0\). For every \(n \in \mathbb{N}\) there are \(k_n, m_n \in \mathbb{N}\), such that \(\langle \gamma_{k_n}, \delta_{m_n} \rangle > \langle \gamma_{k_n}, \delta_{m_n} \rangle > n\) and \(d_\partial (\gamma_{k_n}, \delta_{m_n}, x(x_n)) \leq \frac{1}{n}\). Then, \(\lim_n \gamma_{k_n} x_0, \lim_n \delta_{m_n} x_0\) equal and is equal to \(\xi(x) = \lim_n x_n = x\). So the map \(\xi\) is continuous. By definition \(\xi\) has the Cartan property.

The dual convex set \(\Omega^\ast\) has strictly convex boundary since the boundary of \(\Omega\) is of class \(C^1\). By considering the standard identification of \(\mathbb{F}(\mathbb{R}^d)^\ast\) with \(\mathbb{P}(\mathbb{R}^d)\), we obtain a properly convex domain \(\Omega^\ast\) of \(\mathbb{F}(\mathbb{R}^d)\) which is \(\Gamma\)-invariant and has strictly convex boundary. Since \(\langle \gamma^{-t}, \delta^{-t} \rangle = \gamma \cdot \delta\), we obtain a continuous \(\Gamma^\ast\)-equivariant limit map \(\xi^\ast : \partial_\infty \Gamma \to \mathbb{F}(\mathbb{R}^d)\) satisfying the Cartan property. From \(\xi^\ast\) we obtain a \(\Gamma^\ast\)-equivariant continuous map \(\xi^\ast : \partial_\infty \Gamma \to \mathbb{F}(\mathbb{R}^d)\) as follows: if \(\xi^\ast (x) = [k_x e_1]\) then \(\xi^\ast (x) = [k_x e_1]\).

For two distinct boundary points \(x, y \in \partial_\infty \Gamma\) denote by \(\langle x, y \rangle\) their Gromov product. By definition, we may choose sequences \((\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}}\) in \(\Gamma\) with \(x = \lim_n \alpha_n, y = \lim_n \beta_n\) and \(\langle x, y \rangle = \lim_n \langle \alpha_n, \beta_n \rangle\). By assumption we have that \(\lim_n \langle \rho(\alpha_n), \rho(\beta_n) \rangle \geq J^{-1} (\langle x, y \rangle) - k\) and hence by Proposition 6.2 we obtain the lower bound
\[
\text{dist}(\xi(x), \xi^\ast (y)) \cdot \text{dist}(\xi(y), \xi^\ast (x)) \geq e^{-4J(x,y)} - 4k > 0.
\]
Therefore, the pair of maps \((\xi, \xi^\ast)\) is transverse. Finally, the inclusion \(\Gamma \hookrightarrow \text{PGL}_d(\mathbb{R})\) is \(P_1\)-dilative, admits a pair \((\xi, \xi^\ast)\) of \(\Gamma\)-equivariant, continuous transverse maps with the Cartan property, so Theorem 1.1 shows that the inclusion \(\Gamma \hookrightarrow \text{PGL}_d(\mathbb{R})\) is \(P_1\)-Anosov.

The converse is a direct consequence of Proposition 1.12.

\[\square\]

**Proof of Theorem 1.10.** The implication (i) \(\Rightarrow\) (ii) follows immediately by the Svarc–Milnor lemma. Now assume that (ii) holds. By [17, Thm. 1.4] it is enough to prove that \(\Gamma \hookrightarrow \text{PGL}_d(\mathbb{R})\) is \(P_1\)-Anosov. Let \(x_0 \in \mathcal{C}\). Lemma 2.8 shows that the orbit map \(x_0 \mapsto \gamma x_0\) is a quasi-isometric embedding of \(\Gamma\) into \((\mathcal{C}, d_\Omega)\), hence \(\Gamma\) is word hyperbolic. By using Lemma 2.8 we deduce that there exist constants \(J, k > 0\) such that for every \(\gamma, \delta \in \Gamma\),
\[
J^{-1} (\gamma \cdot \delta) - k \leq \langle \rho(\gamma), \rho(\delta) \rangle \leq J(\gamma \cdot \delta) + k.
\]

Proposition 8.1 then finishes the proof.
9. Distribution of singular values

For $q \in \mathbb{N}$ consider the $q$-symmetric power $\text{Sym}^q : \text{GL}_d(\mathbb{R}) \to \text{GL}(\text{Sym}^q(\mathbb{R}^d))$. Note also that with respect to the standard Cartan decomposition we have $\sigma_1(\text{sym}^q g) = (\sigma_1(g))^q$.

By using Theorem 1.1 we exhibit conditions guaranteeing that the product of two linear representations of a hyperbolic group is $P_1$-Anosov.

**Theorem 9.1.** Let $\Gamma$ be a word hyperbolic group and $\rho_L : \Gamma \to \text{SL}_m(\mathbb{R})$, $\rho_R : \Gamma \to \text{SL}_d(\mathbb{R})$ two representations. Suppose there is an infinite order element $\gamma_0 \in \Gamma$ with $\ell_1(\rho_L(\gamma_0)) > \ell_1(\rho_R(\gamma_0))$.

Furthermore, suppose that $\rho_L$ is $P_1$-Anosov and $\rho_R$ satisfies one of the following conditions:

(i) $\rho_R$ is $P_1$-Anosov.

(ii) $\rho_R(\Gamma)$ is contained in a semisimple proximal Lie subgroup of $\text{SL}_d(\mathbb{R})$ of real rank 1.

Then, the following conditions are equivalent:

1. The representation $\rho_L \times \rho_R : \Gamma \to \text{SL}_{m+d}(\mathbb{R})$ is $P_1$-Anosov.
2. $\lim_{|\gamma| \to \infty} \frac{\ell_1(\rho_L(\gamma))}{\sigma_1(\rho_R(\gamma))} = +\infty$.
3. $\lim_{|\gamma| \to \infty} \frac{\ell_1(\rho_L(\gamma))}{\ell_1(\rho_R(\gamma))} = +\infty$.
4. There exist $C, c > 0$ such that $|\log \sigma_1(\rho_L(\gamma)) - \log \sigma_1(\rho_R(\gamma))| \geq c \log |\gamma| - C$, $\forall \gamma \in \Gamma$.
5. There exist $C, c > 0$ such that $|\log \ell_1(\rho_L(\gamma)) - \log \ell_1(\rho_R(\gamma))| \geq c \log |\gamma| - C$, $\forall \gamma \in \Gamma$ of infinite order.

**Proof.** Let $G$ be a $P_1$-proximal Lie subgroup of $\text{SL}_d(\mathbb{R})$ of real rank 1 with Cartan projection $\mu_G : G \to \mathbb{R}$. Up to conjugation, we may write $G = K_G \exp (\mathbb{R}T_0)K_G$, $K_G \subset h\text{SO}(d)h^{-1}$ for some $h \in \text{SL}_d(\mathbb{R})$ and $\exp (\mathbb{R}T_0) = \text{diag}(e^{a_1}, \ldots, e^{a_d})$ with $a_1 > a_2 \geq \ldots \geq a_{d-1} > a_d$. The sub-additivity of the Cartan projection shows that there exists $M > 0$ such that

$$
|\log \sigma_1(g) - a_i \mu_G(g)| \leq M
$$

for every $g \in G$ and $1 \leq i \leq d$. In particular, there exists $M' > 0$ such that

$$
\log \frac{\sigma_1(g)}{\sigma_2(g)} \geq \frac{a_1 - a_2}{a_1} \log \sigma_1(g) - M'
$$

for every $g \in G$. Since we assume that $\rho_R$ is $P_1$-Anosov or $\mu(\rho_R(\Gamma))$ is contained in a proximal, rank 1 Lie subgroup $G$ of $\text{SL}_d(\mathbb{R})$, by the previous remarks we can find $A, a > 0$ such that

$$
\log \frac{\ell_1(\rho_R(\gamma))}{\ell_2(\rho_R(\gamma))} \geq a \log \ell_1(\rho_R(\gamma)), \quad \log \frac{\sigma_1(\rho_R(\gamma))}{\sigma_2(\rho_R(\gamma))} \geq a \log \sigma_1(\rho_R(\gamma)) - A
$$

for every $\gamma \in \Gamma$.

Let $\rho := \rho_L \times \rho_R$. We obtain continuous, $\rho$-equivariant and transverse maps $\xi^+_{LR} : \partial_+ \Gamma \to \mathbb{P}(\mathbb{R}^{m+d})$ and $\xi^-_{LR} : \partial_- \Gamma \to \mathbb{P}(\mathbb{R}^{m+d})$ defined as follows:

$$
\xi^+_{LR}(x) = \xi^+_{L}(x), \quad \xi^-_{LR}(x) = \xi^-_{L}(x) \oplus \mathbb{R}^d
$$

where $\xi^+_{L}$ and $\xi^-_{L}$ are the Anosov limit maps of $\rho_L$. For every element $\gamma \in \Gamma$ we observe that the following estimates hold:

$$
\left| \log \frac{\sigma_1(\rho_L(\gamma))}{\sigma_2(\rho_L(\gamma))} \right| \geq \log \frac{\sigma_1(\rho_R(\gamma))}{\sigma_2(\rho_R(\gamma))},
$$

$$
\log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} \geq \min \left( \log \frac{\sigma_1(\rho_L(\gamma))}{\sigma_2(\rho_L(\gamma))}, \log \frac{\sigma_1(\rho_L(\gamma))}{\sigma_2(\rho_L(\gamma))} \right),
$$

$$
\log \frac{\ell_1(\rho(\gamma))}{\ell_2(\rho(\gamma))} \geq \min \left( \log \frac{\ell_1(\rho_L(\gamma))}{\ell_2(\rho_L(\gamma))}, \log \frac{\ell_1(\rho_L(\gamma))}{\ell_2(\rho_L(\gamma))} \right).
$$
(2) \Rightarrow (1). We observe that condition (2) and estimate (11) together show that \( \rho \) is \( P_1 \)-divergent. Since \( \xi^+ \) satisfies the Cartan property and \( \sigma_1(\rho_L(\gamma)) > \sigma_1(\rho_R(\gamma)) \) as \( |\gamma| \to \infty \), the map \( \xi_{LR}^+ \) has the Cartan property. The maps \( \xi^+_{LR} \) and \( \xi_{LR}^- \) are transverse, hence Theorem 1.1 shows that \( \rho_L \times \rho_R \) is \( P_1 \)-Anosov. 

\[ (3) \Rightarrow (1). \] We are proving that \( (3) \Rightarrow (2) \). Let \( \rho_L^{ss}, \rho_R^{ss} \) be semisimplifications of \( \rho_L, \rho_R \) respectively. By Proposition 2.6, it is enough to show that \( \rho_L^{ss} \times \rho_R^{ss} \) is \( P_1 \)-Anosov. By Theorem 2.5 there exists \( C > 0 \) and a finite subset \( F \) of \( \Gamma \) such that for every \( \gamma \in \Gamma \), there exists \( f \in F \) such that \[ \log \ell_1(\rho_L(\gamma f)) - \log \sigma_1(\rho_L^{ss}(\gamma)) \leq C \] and \[ \log \ell_1(\rho_R(\gamma f)) - \log \sigma_1(\rho_R^{ss}(\gamma)) \leq C. \] Let \( (\gamma_n)_{n \in \mathbb{N}} \) be an infinite sequence of elements of \( \Gamma \). For every \( n \) we choose \( f_n \in F \) satisfying the previous bounds. The triangle inequality shows \( ||\lambda(\rho_L(\gamma_n f_n))|| \geq ||\mu(\rho_L(\gamma_n))|| - C \), hence \( \lim_n |\gamma_n f_n| = +\infty \). Therefore, \( \lim_n (\log \ell_1(\rho_L^{ss}(\gamma_n)) - \log \sigma_1(\rho_L^{ss}(\gamma_n))) = +\infty \) so \( \lim_n (\log \sigma_1(\rho_L^{ss}(\gamma_n)) - \log \sigma_1(\rho_R^{ss}(\gamma_n))) = +\infty \). The claim now follows by \( (2) \Rightarrow (1) \). 

\[ (4) \Rightarrow (1). \] We first assume that \( c > 1 \). By estimate (11), there exists a constant \( C_1 > 0 \) such that \[ \log \sigma_1(\rho(\gamma)) \geq c \log |\gamma| - C_1 \] for every \( \gamma \in \Gamma \). Therefore, by [22, Thm. 5.3], we obtain a \( \rho \)-equivariant map \( \xi : \hat{\partial}\Gamma \to \mathbb{P}(\mathbb{R}^{m+d}) \) which satisfies the Cartan property. Then, since \( \rho(\gamma_0) \) is \( P_1 \)-proximal, we have \( \xi(\gamma^+_0) = \xi_{LR}^+ \). The minimality of the action of \( \Gamma \) on \( \hat{\partial}\Gamma \) shows that \( \xi = \xi_{LR}^+ \). Then \( \xi_{LR}^+ \) satisfies the Cartan property, \( \xi_{LR}^- \) and \( \xi_{LR}^+ \) are transverse and \( \rho \) is \( P_1 \)-divergent. Theorem 1.1 shows that \( \rho \) is \( P_1 \)-Anosov.

Now suppose that \( c \leq 1 \). We choose \( n \in \mathbb{N} \) large enough and consider the symmetric powers \( \text{sym}^n \rho_L, \text{sym}^n \rho_R \) of \( \rho_L, \rho_R \) respectively. Then \( \text{sym}^n \rho_L \) is \( P_1 \)-Anosov and \( \text{sym}^n \rho_R \) satisfies condition (i) or (ii). Since \( \log \sigma_1(\text{sym}^n \rho_R(\gamma)) = n \log \log \sigma_1(\rho_R(\gamma)) \) for \( \gamma \in \Gamma \), the representation \( \text{sym}^n \rho_L \times \text{sym}^n \rho_R \) satisfies condition (3) for \( c > 1 \). Therefore, the previous argument implies that the representation \( \text{sym}^n \rho_L \times \text{sym}^n \rho_R \) is \( P_1 \)-Anosov. Therefore, by estimate (10), we obtain uniform constants \( R, k > 0 \) such that \[ |\log \sigma_1(\rho_L(\gamma)) - \log \sigma_1(\rho_R(\gamma))| \geq k|\gamma| - R \] for every \( \gamma \in \Gamma \). Again we verify that \( \rho \) is \( P_1 \)-Anosov.

\[ (5) \Rightarrow (1). \] It is enough to prove that the semisimplification \( \rho_L^{ss} \times \rho_R^{ss} \) of \( \rho \) is \( P_1 \)-Anosov. Note that the representation \( \rho_L^{ss} \) is \( P_1 \)-Anosov and \( \rho_R^{ss} \) satisfies either (i) or (ii). By Theorem 2.5 there exists \( L > 0 \) and a finite subset \( F \) of \( \Gamma \) such that for every \( \gamma \in \Gamma \) there exists \( f \in F \) with \[ ||\lambda(\rho_L(\gamma f)) - \mu(\rho_R^{ss}(\gamma))|| \leq L \] and \[ ||\lambda(\rho_R(\gamma f)) - \mu(\rho_R^{ss}(\gamma))|| \leq L. \] Since \( \rho_L \) is a quasi-isometric embedding, by using the previous inequality, we may find \( M > 0 \) such that \( |\gamma f| \geq \frac{1}{M}|\gamma| - M \), where \( \gamma \in \Gamma \) and \( f \in F \) are as previously. Finally, we obtain \( L', c > 0 \) such that for every \( \gamma \in \Gamma \) we have

\[ |\log \sigma_1(\rho_L^{ss}(\gamma)) - \log \sigma_1(\rho_R^{ss}(\gamma))| \geq c \log |\gamma| - L'. \]

Therefore, \( \rho_L^{ss} \times \rho_R^{ss} \) is \( P_1 \)-Anosov from \( (3) \Rightarrow (1) \).

\[ (1) \Rightarrow (2), (3), (4), (5). \] Since \( \ell_1(\rho_L(\gamma_0)) > \ell_1(\rho_R(\gamma_0)) \), \( \xi_{LR}^+(\gamma_0) \) is the attracting fixed point of \( \rho(\gamma_0) \) in \( \mathbb{P}(\mathbb{R}^{m+d}) \). The action of \( \Gamma \) on \( \hat{\partial}\Gamma \) is minimal, hence \( \xi_{LR}^+ \) has to be the Anosov limit map of \( \rho \) in \( \mathbb{P}(\mathbb{R}^{m+d}) \). In particular, \( \xi_{LR}^+ \) satisfies the Cartan property. This shows that for any sequence \( (\gamma_n)_{n \in \mathbb{N}} \) of elements of \( \Gamma \) we have \( \lim_n (\log \sigma_1(\rho_L(\gamma_n)) - \log \sigma_1(\rho_R(\gamma_n))) = +\infty \). In particular, there exists \( \varepsilon > 0 \) such that \( (1 - \varepsilon) \log \ell_1(\rho_L(\gamma)) \geq \log \ell_1(\rho_R(\gamma)) \) for every \( \gamma \in \Gamma \). By estimates (10), (11) and Theorem 2.3 (ii) we deduce that \( (3), (4), (5) \) hold.

**Proof of Corollary 1.4.** We consider the representation \( \rho = \text{sym}^p \rho_1 \times \text{sym}^q \rho_2 \). The representation \( \text{sym}^p \rho_2 \) is \( P_1 \)-Anosov and \( \text{sym}^q \rho_1 \) satisfies either condition (i) or (ii) of Theorem 9.1. The choice of \( p, q \in \mathbb{N} \) shows that the representation \( \text{sym}^p \rho_2 \) cannot uniformly dominate \( \text{sym}^q \rho_1 \), so \( \rho \) cannot be \( P_1 \)-Anosov. Then. Theorem 9.1 (3) shows that for every \( n \in \mathbb{N} \) we can find an element \( \gamma_n \in \Gamma \) with \( |\gamma_n| > n \) and \( |q \mu_1(\rho_1(\gamma_n)) - p \mu_1(\rho_2(\gamma_n))| \leq \delta \log(\mu_1(\rho_1(\gamma_n))) \). The conclusion follows.
Remarks 9.2. (i) In Theorem 9.1, in the particular case where both $\rho_L(\Gamma)$ and $\rho_R(\Gamma)$ are contained in a proximal real rank 1 Lie subgroup of $\mathcal{S}L_m(\mathbb{R})$ and $\mathcal{S}L_d(\mathbb{R})$ respectively, the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are contained in [22, Thm. 1.14]. In the case where $\rho_L$ and $\rho_R$ take values in $\text{Aut}_b(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) for some bilinear form $b$ (see [22, §7] for background), the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (4) of Theorem 9.1 are contained in [22, Prop. 7.13 & Lem. 7.11 & Thm. 1.3].

(ii) By Theorem 2.5 and Corollary 1.4 we deduce that the closure of the set of ratios

$$\left\{ \frac{\log \ell_1(\rho_1(\gamma))}{\log \ell_1(\rho_2(\gamma))} : \gamma \in \Gamma_\mathcal{X} \right\}$$

is the closed interval $[\text{dil}_-(\rho_1, \rho_2), \text{dil}_+(\rho_1, \rho_2)]$. We may replace both $\rho_1$ and $\rho_2$ with their semisimplifications, and this fact also follows by the limit cone theorem of Benoist in [3, 4]. In the case where $\rho_1$ and $\rho_2$ are convex cocompact into a rank 1 Lie group, the previous fact also follows by [9, Thm. 2].

10. Examples and Counterexamples

In this section, we discuss examples of representations of surface groups enjoying some of the properties of Anosov representations which are not $P_1$-Anosov. The examples show that the assumptions of the main results of this paper are necessary. Throughout this section $S_g$ denotes a closed orientable surface of genus $g \geq 2$.

Example 10.1. There exists a strongly irreducible representation $\rho : \pi_1(S_g) \to \mathcal{S}L_{12}(\mathbb{R})$ which satisfies the following properties:

- $\rho$ is a quasi-isometric embedding, $P_1$-divergent and preserves a properly convex domain $\Omega$ of $\mathbb{P}(\mathbb{R}^{12})$.
- $\rho$ admits continuous, injective, $\rho$-equivariant maps $(\xi_1, \xi_{11}) : \partial \pi_1(S) \to \mathbb{P}(\mathbb{R}^{12}) \times \text{Gr}_{11}(\mathbb{R}^{12})$ satisfying the Cartan property. The proximal limit set of $\rho(\pi_1(S_g))$ in $\mathbb{P}(\mathbb{R}^{12})$ is $\xi_1(\partial \pi_1(S))$ and does not contain projective line segments.
- $\rho$ admits continuous, $\rho$-equivariant transverse maps $(\xi_4, \xi_8) : \partial \pi_1(S_g) \to \text{Gr}_4(\mathbb{R}^{12}) \times \text{Gr}_8(\mathbb{R}^{12})$.
- $\rho$ is not $P_k$-Anosov for any $k = 1, \ldots, 11$.

The previous example shows that the assumption of transversality in Theorem 1.1 is necessary. Moreover, the maps $\xi_4$ and $\xi_8$ are transverse although $\rho$ is not $P_4$-Anosov, therefore Zariski density is also necessary in Theorem 1.3.

Proof. Let $S_g$ be a closed orientable surface of genus at least 2 and $\phi : S_g \to S_k$ a pseudo-Anosov homeomorphism. The mapping torus $M_\phi$ of $S_g$ with respect to $\phi$ is a closed 3-manifold whose fundamental group is isomorphic to the HNN extension

$$\pi_1(M_\phi) = \left\langle \pi_1(S_g), t \mid tat^{-1} = \phi_a(t), a \in \pi_1(S_g) \right\rangle$$

where $\phi_a$ is the automorphism of $\pi_1(S_g)$ induced by $\phi$. Thurston in [37] (see also Otal [35]) proved that there exists a convex cocompact representation $\rho_0 : \pi_1(M_\phi) \to \mathcal{P}O(3,1)$. The representation $\rho_0$ lifts to a $P_1$-Anosov representation in $\mathcal{S}L_4(\mathbb{R})$ which we continue to denote by $\rho_0$ and let $\rho_{\text{Fiber}} := \rho_0|_{\pi_1(S_g)}$. By a result of Cannon-Thurston [13], there exists a continuous equivariant surjection $\theta : \partial \pi_1(S_g) \to \partial \pi_1(M_\phi)$. By precomposing $\theta$ with the Anosov limit map of $\rho_0$ in $\mathbb{P}(\mathbb{R}^4)$, we obtain a $\rho_{\text{Fiber}}$-equivariant continuous map $\xi_{\text{Fiber}} : \partial \pi_1(S_g) \to \mathbb{P}(\mathbb{R}^{12})$. Let $\gamma \in \pi_1(S_g)$ be an element representing a separating simple closed curve on $S_g$. We may choose a Zariski dense, Hitchin representation $\rho_H : \pi_1(S_g) \to \mathcal{S}L_3(\mathbb{R})$ with $2 \log \ell_1(\rho_{\text{Fiber}}(\gamma)) = \log \ell_1(\rho_H(\gamma))$. We claim that $\rho = \rho_{\text{Fiber}} \otimes \rho_H : \pi_1(S) \to \mathcal{S}L_{12}(\mathbb{R})$ satisfies the required properties.

Let $\otimes : \mathcal{S}O(3,1) \times \mathcal{S}L_3(\mathbb{R}) \to \mathcal{S}L_{12}(\mathbb{R})$ be the irreducible tensor product representation $(g_1, g_2) \mapsto g_1 \otimes g_2$. Let $G$ be the Zariski closure of $\rho_{\text{Fiber}} \times \rho_H$ into $\mathcal{S}O(3,1) \times \mathcal{S}L_3(\mathbb{R})$. Note that the projection of the identity component $G^0$ into $\mathcal{S}O(3,1)$ (resp. $\mathcal{S}L_3(\mathbb{R})$) is normalized.
by $\rho_{\text{Fiber}}(\pi_1(S))$ (resp. $\rho_H(\pi_1(S))$), so it has to be surjective. Since the Zariski closures of $\rho_{\text{Fiber}}$ and $\rho_H$ are simple and not locally isomorphic, it follows by Goursat’s lemma that $G = \text{SO}(3, 1) \times \text{SL}_3(\mathbb{R})$. We conclude that $\rho$ is strongly irreducible.

We obtain a properly convex domain $\Omega$ of $\mathbb{P}(\mathbb{R}^{12})$ preserved by $\rho(\pi_1(S_g))$ as follows. Let $\Omega_1$ and $\Omega_2$ be properly convex domains of $\mathbb{P}(\mathbb{R}^4)$ and $\mathbb{P}(\mathbb{R}^3)$ preserved by $\rho_{\text{Fiber}}(\pi_1(S))$ and $\rho_H(\pi_1(S_g))$ respectively, and $\Omega_i'$ a properly convex cone lifting $\Omega_i$ for $i = 1, 2$. The compact set

$$C = \{[u_1 \otimes u_2] : u_i \in \overline{\Omega_i}, u_i \in \overline{\Omega_i}'\}$$

is connected, spans $\mathbb{R}^{12}$ and is contained in an affine chart $\mathcal{A} \subset \mathbb{P}(\mathbb{R}^{12})$. We finally take $\Omega$ to be the interior of the convex hull of $C$ in $\mathcal{A}$.

The representations $\rho_{\text{Fiber}}$ and $\rho_H$ are $P_1$-divergent hence $\rho$ is also $P_1$-divergent. Note that

$$\sigma_1(\rho(\gamma)) = \sigma_1(\rho_{\text{Fiber}}(\gamma))\sigma_1(\rho_H(\gamma)) \quad \forall \gamma \in \Gamma,$$

hence $\rho$ is a quasi-isometric embedding. Let $\xi_H^+ : \partial_+ \pi_1(S_g) \to \mathbb{P}(\mathbb{R}^3)$ and $\xi_H^- : \partial_- \pi_1(S_g) \to \text{Gr}_2(\mathbb{R}^3)$ be the Anosov limit maps of $\rho_H$. The map $\xi_1 : \partial_\pi \pi_1(S_g) \to \mathbb{P}(\mathbb{R}^{12})$ defined as

$$\xi_1(x) = [k_x e_1 \otimes k_y e_1],$$

where $\xi_{\text{Fiber}}(x) = [k_x e_1]$ and $\xi_H(x) = [k_y e_1]$, is continuous and $\rho$-equivariant. Since $\rho$ is strongly irreducible, the proof of Corollary 4.6 shows that the map $\xi_1$ satisfies the Cartan property. The image of $\xi_1$ is the $P_1$-proximal limit set of $\rho(\pi_1(S_g))$ in $\mathbb{P}(\mathbb{R}^{12})$. Similarly, the dual representation $\rho^* = \rho_{\text{Fiber}}^* \otimes \rho_H^*$ admits a $\rho^*$-equivariant map $\xi_1^* : \partial_\pi \pi_1(S_g) \to \mathbb{P}(\mathbb{R}^{12})$, so we obtain the $\rho$-equivariant map $\xi_1$.

The maps $\xi_4 : \partial_- \pi_1(S_g) \to \text{Gr}_2(\mathbb{R}^{12})$ and $\xi_8 : \partial_+ \pi_1(S_g) \to \text{Gr}_2(\mathbb{R}^{12})$ defined as

$$\xi_4(x) = \mathbb{R}^4 \otimes \mathbb{R} \xi_H(x), \quad \xi_8(x) = \mathbb{R}^4 \otimes \mathbb{R} \xi_H^-(x) \quad x \in \partial_\pi \pi_1(S_g),$$

are, by their definition, $\rho$-equivariant, continuous and transverse. Also for every $x \in \partial_\pi \pi_1(S_g)$ we have $\xi_1(x) = \xi_4(x)$, and hence $\xi_1$ is injective. It follows that $\xi_1(\partial_{\pi} \pi_1(S_g)) = \text{Ann}_p(\pi_1(S_g)) \cong S^1$. For $x \neq y$ the projective line segment $[\xi_H(x), \xi_H(y)]$ intersects $\text{Ann}_p(\Gamma)$ at the set $\{\xi_H(x), \xi_H(y)\}$, hence $[\xi_1(x), \xi_1(y)] \cap \text{Ann}_p(\Gamma) = \{\xi_1(x), \xi_1(y)\}$.

The choice of the element $\gamma \in \pi_1(S_g)$ shows that $\rho(\gamma)$ cannot be $P_k$-proximal for $k = 2, 4, 6$, so $\rho$ is not $P_k$-Anosov for these values of $k$. Let $g \in \pi_1(S_g)$ be a non-trivial element. The infinite sequence of elements $(\phi^{(n)}_g(y))_{n \in \mathbb{N}}$ has the property that $(\phi^{(n)}_g(y))_{n \in \mathbb{N}}$ is unbounded and there exists $M > 0$ such that

$$\frac{\|\xi_1(\rho(\phi^{(n)}_g(y)))\|}{\|\xi_1(\rho(\phi^{(n)}_g(y)))\|} \leq M$$

for every $n \in \mathbb{N}$. Then, it is easy to check that the ratio

$$\frac{\|\xi_1(\rho(\phi^{(n)}_g(y)))\|}{\|\xi_1(\rho(\phi^{(n)}_g(y)))\|}$$

are uniformly bounded for $i = 1, 3, 5$, so $\rho$ is not $P_k$-Anosov for $k = 1, 3, 5$. \qed

**Example 10.2. Necessity of the Cartan property.** The representation $\rho \times \rho_H : \pi_1(S_g) \to \text{SL}_2(\mathbb{R})$ (where $\rho$ and $\rho_H$ are from Example 10.1) is $P_1$-divergent and admits a pair of continuous, equivariant, compatible and transverse maps $\xi^+ : \partial_\pi \pi_1(S_g) \to \mathbb{P}(\mathbb{R}^{15})$ and $\xi^- : \partial_\pi \pi_1(S_g) \to \text{Gr}_2(\mathbb{R}^{15})$ induced from the Anosov limit maps of $\rho_H$. However, $\rho \times \rho_H$ is not $P_1$-Anosov since $\rho$ cannot uniformly dominate $\rho_H$. This shows that the assumption of the Cartan property for the map $\xi^+$ in Theorem 1.1 is necessary.

**Example 10.3. Necessity of regularity of $\partial \mathcal{O}$ in Proposition 8.1.** Let $n \geq 2$ and $\Gamma$ be a convex cocompact subgroup of $\text{SU}(n, 1) \subset \text{SL}_{n+1}(\mathbb{C})$. Let $\tau_2 : \text{SL}_{n+1}(\mathbb{C}) \to \text{SL}_{2n+2}(\mathbb{R})$ be the standard inclusion defined as

$$\tau_2(g) = \begin{pmatrix} \text{Re}(g) & -\text{Im}(g) \\ \text{Im}(g) & \text{Re}(g) \end{pmatrix}, \quad g \in \text{SL}_{n+1}(\mathbb{C}).$$

The group $\text{sym}^2(\tau_2(\Gamma)) \subset \text{SL}_{2n+2}(\mathbb{R})$ is a $P_2$-Anosov subgroup and there exist $J, k > 0$ such that

$$J^{-1}(\gamma \cdot \delta) e - k \leq (\text{sym}^2(\tau_2(\gamma)) \cdot \text{sym}^2(\tau_2(\delta)))_{\xi_1} \leq J(\gamma \cdot \delta) e + k$$

for every $\gamma, \delta \in \Gamma$. Moreover, $\text{sym}^2(\tau_2(\Gamma))$ preserves a properly convex domain in $\mathbb{P}(\text{Sym}^2(\mathbb{R}^{2n+2})$ but it cannot preserve a strictly convex domain since it is not $P_1$-divergent. Similar counterexamples are given by convex cocompact subgroups of the rank 1 Lie group $\text{Sp}(n, 1) \subset \text{GL}_{n+1}(\mathbb{H})$. 

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Example 10.4. Necessity of transversality in Theorem 1.1 in the Zariski dense case. There exists a Zariski dense representation $\rho_1 : \pi_1(S_g) \to \text{PSL}_4(\mathbb{R})$ which admits a pair of continuous $\rho_1$-equivariant maps $\xi^+: \partial_\infty \pi_1(S_g) \to \mathbb{R}^4$ and $\xi^- : \partial_\infty \pi_1(S_g) \to \text{Gr}_3(\mathbb{R}^4)$ but is not $P_1$-Anosov.

Let $M^3$ be a closed hyperbolic 3-manifold fibering over the circle (with fiber $S_g$) which also contains a totally geodesic surface. By Johnson–Millson [25] the natural inclusion $j : \pi_1(M^3) \hookrightarrow \text{PO}(3, 1)$ admits a non-trivial Zariski dense deformation $j' : \pi_1(M^3) \to \text{PSL}_4(\mathbb{R})$ which by can be chosen to be $P_1$-Anosov thanks to the openness of Anosov representations (see [33, 23]). Let $\xi^+_1$ and $\xi^-_1$ be the Anosov limit maps of $j'$ into $\mathbb{R}^4$ and $\text{Gr}_3(\mathbb{R}^4)$ respectively. By the theorem of Cannon–Thurston [13] there exists a continuous, $\pi_1(S)$-equivariant map $\theta : \partial_\infty \pi_1(S_g) \to \partial_\infty \pi_1(M^3)$. The restriction $\rho_1 := j'|_{\pi_1(S_g)}$ is Zariski dense, not a quasi-isometric embedding and $\xi^+_1 \circ \theta$ and $\xi^-_1 \circ \theta$ are continuous, non-transverse and $\rho_1$-equivariant maps. In addition, by [10], every finitely generated free subgroup $F$ of $\pi_1(S_g)$ is a quasiconvex subgroup of $\pi_1(M^3)$. Hence, $j'|_F$ is $P_1$-Anosov and $\xi^+_1 \circ \iota_F$ and $\xi^-_1 \circ \iota_F$ are transverse.

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