THE UNEXPECTED RESURGENCE OF WEYL GEOMETRY IN LATE 20-TH CENTURY PHYSICS

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ABSTRACT. Weyl’s original scale geometry of 1918 (“purely infinitesimal geometry”) was withdrawn by its author from physical theorizing in the early 1920s. It had a comeback in the last third of the 20th century in different contexts: scalar tensor theories of gravity, foundations of gravity, foundations of quantum mechanics, elementary particle physics, and cosmology. It seems that Weyl geometry continues to offer an open research potential for the foundations of physics even after the turn to the new millennium.

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Date: 2017 03/08.
Introduction

In the 1970s three groups of authors started, basically independent from each other, to reconsider Weyl's generalization of Riemannian geometry from 1918. Weyl had proposed the latter in the perspective of building a geometrically unified theory of gravitation and electromagnetism. By the end of the 1920s, after the successful reformulation of the underlying gauge idea in relativistic quantum physics, most physicists including Weyl himself had given up the idea of extending the geometry of spacetime by a “localized” scaling degree of freedom. It was not to be expected that half a century later researchers of the next generation would try again to give Weyl geometry a new role in the changed context of late 20th century physics. But some of them did. A group of authors, in particular F. Ehlers, F. Pirani, and A. Schild, used it as a conceptual framework for clarifying the foundations of gravity; others explored extended gravity theories in the generalized geometrical structure, and still others, like W. Drechsler and H. Tann in Munich, investigated connections between gravity and quantum physics. In several of these approaches a scalar field extending the gravitational structure played a crucial role. Although none of the attempts found an immediate broader response, many of them led to follow up papers. In the result, different research perspectives exploring questions of recent physics from a Weyl geometric viewpoint emerged, but they remained too heterogeneous for coalescing to a coherent literary tradition or even forming a common research community.

The call for papers for the Mainz conference proceedings was a splendid incentive for taking stock of the broader range of Weyl geometric investigations in physics, which took a new start in the last three decades of the 20th century. Of course the following survey cannot be complete; it rather has to be confined by specified boundaries. So this paper is restricted to the more classical parts of gravity with some, relative limited outlooks at connections to quantum theory. Not covered in this survey is the whole range of Weyl geometric methods in Kaluza-Klein theories, in supergravity, and in string theory.

In order to facilitate the reading of the following survey, the paper starts with a very short introduction to, or a reminder of, central features of Weyl geometry and gravity (section 1.1). Because a considerable amount of the following developments utilize a scale covariant scalar field coupled to the Hilbert term similar to the one in Jordan–Brans–Dicke (JBD) gravity, the second part of the first section is devoted to a short glance at JBD theory from a Weyl geometric perspective (section 1.2). The other sections give a partly historical, partly systematic survey of the attempts for using Weyl geometric methods in recent physics.

In section 2 three different, partially overlapping, approaches of the 1970s are described. The already mentioned paper of Ehlers, Pirani and Schild (EPS) on the foundations of gravity and some follow up papers are dealt with in section 2.1. A completely different retake arose from proposals put forward by a group of Japanese physicists, M. Omote, R. Utiyama et al. and independently by P.A.M. Dirac. They investigated a scalar field coupling to the Hilbert term similar to JBD gravity, but in the scale invariant
approach of Weyl geometry. Dirac’s and the Japanese physicists’ interpretations of the Weylian scale connection were not the same. They and their respective immediate successors had different research contexts in mind, gravity, astrophysics, cosmology and electromagnetism in Dirac’s case, nuclear and elementary particle physics in Utiyama’s (section 2.2). Finally, although less noticed in the wider community, a specific road to Weyl geometric structures arose in the research on gauge theories of gravity arising from the Kibble-Sciama program of deriving gravitational structures (fields) from “localizing” symmetries in Minkowski space, often considered from a wider perspective than that of the Poincaré group. In this view Weyl geometry appeared as a special case of Cartan geometry, and Weyl geometric gravity ought generically be extended by a translational connection component, viz. torsion. It is a surprising fact that these three re-starts of Weyl geometric gravity, although arising from completely different backgrounds and pursuing different goals, were undertaken and published in the short interval 1971 – 1974, exactly the time when the basics for the standard model of elementary particle physics were established (section 2.3).

Before we come to the follow up investigations which made use of these approaches in the standard model of elementary particle physics and/or in astrophysics and cosmology, we turn towards an even more surprising recourse to Weyl geometry in attempts to geometrize quantum mechanics (QM) in the wake of the Bohmian heterodoxy (section 3). In order to make this kind of geometrization accessible to readers not versed in Bohmian quantum mechanics, the basic ideas necessary to understand the geometrization proposals are shortly resumed in section 3.1. A survey of a peculiar road towards geometrizing configurations spaces of QM by Weyl geometry, developed in the 1980s by E. Santamato’s and continued after the turn to the 2010s with his colleague F. De Martini, follows (section 3.2). A more fragile idea of a Bohm-type quantization procedure in cosmology leading to a Weyl geometric framework, proposed by A. and F. Shohai and M. Golshani is the topic of section 3.3.

In section 4 we turn towards different attempts at using Weyl geometric structures (mainly scale invariance and the scale invariant affine connection) and fields (Weylian scale connection and/or an additional scale covariant scalar field) in elementary particle physics. Three interrelated questions arise naturally if one wants to bring gravity closer to the physics of the standard model (SM):

(i) Is it possible to bring conformal, or at least scale covariant generalizations of classical (Einsteinian) relativity into a coherent common frame with the standard model SM?

(ii) Is it possible to embed classical relativity in a quantized theory of gravity or, the other way round, to derive classical relativity as an effective theory arising from a more fundamental quantum gravity theory at the classical level?

1Such an attempt seemed to be supported experimentally by the phenomenon of (Bjorken) scaling in deep inelastic electron-proton scattering experiments. The latter indicated, at first glance, an active scaling symmetry of mass/energy in high energy physics; but it turned out to hold only approximatively and was of restricted range.
The fact that all the SM fields, with the only exception of the Higgs field,
have conformally invariant Lagrangians, in the context of special relativity,
i.e., Minkowski space, was considered among others by F. Englert and coau-
thors already in the mid-1970s. It cried out for investigations in a Weyl
geometric perspective which then, of course, would invite generalizing the
spacetime environment of all SM fields, at least in their pseudo-classical
form,\(^2\) to Lorentzian or Weylian manifolds. In this context the Weylian
scale connection was identified by L. Smolin at the end of the 1970s as a
new, hypothetical, field which after quantization would lead to a particle
with mass close to the Planck scale. Roughly ten years later this particle
was found again and called a “Weylon” by H. Cheng (section 4.1). Again
roughly a decade later the question of “mass generation” by breaking the
scale symmetry in a Weyl geometric approach to SM fields was studied at
Munich by W. Drechsler and H. Tann (section 4.2).

This question continued to attract the interest of researchers at least until
the empirical detection of the Higgs boson in 2012. In the last few years in
particular H. Nishino and S. Rajpoot, but not only they, have studied the
question of how the symmetry of the standard model may be enhanced by
a scale degree of freedom and may be broken by a peculiar interplay of an
initially scale covariant scalar field and the “Weylon”. All this was discussed
at the pseudo-classical level (section 4.3). In the recent years some authors
have turned towards the difficult questions of Weyl scaling at the quantum
level. A group of Italian authors, G. Codello, G. D’Orico, C Pagani, and
R. Percacci brought forward new arguments with regard to the commonly
shared view that scale symmetry is necessarily broken at the quantum level.
They have proposed quantization procedures under which scale invariance
can be preserved under quantization. H. Ohanian has recently discussed the
transition between a a scale invariant phase of fields close to the Planck scale
to a lower energy regime with broken scale symmetry and Einstein gravity
as effective field theory (section 4.4).

The rescaling allowed in Weyl geometry may change the geometrical pic-
ture underlying our usually assumed cosmological models. Scalar fields with
conformal rescaling have been in use for a long time in “early universe”
modelling (section 5.1). They invite Weyl geometric investigations and were
dominated for several decades by N. Rosen and M. Israelit, the first one an
early protagonist of the Dirac approach to Weyl geometric gravity. In the
last few decades also other authors jumped in with slightly different ideas
(section 5.2f.). A coherent tradition with a larger group of researchers in
astrophysical and cosmological studies has formed in Brazil around M. Nov-
ello. It invokes a (weak) Weyl geometric framework and has defolded its
research questions for more than two decades, more stable and with a wider
group of contributors than any others line of research considered in this
survey (section 5.4). But the question of dark matter effects, if considered

\(^2\)SM fields are here called pseudo-classical if they are considered before, or better ab-
tracting from, so-called second quantization. Mathematically they are classical fields
(spinor fields or gauge connections), but the field components do not correspond to phys-
ically measurable quantities. Observationally relevant information can be extracted only
after applying perturbative quantization methods.
from the gravitational side, has to be measured at the successes of modified Newtonian dynamics, MOND. This remained outside the scope of the Brazilian school. First steps of reconstructing MOND-like phenomenology in a Weyl geometric approach to gravity, made recently at Wuppertal, seem sufficiently striking to include it here (section 5.3).

A survey of a side-stream issue of recent research, as it is attempted in this paper, cannot claim to tell a coherent, perhaps even success, story. It rather has to collect views from necessarily heterogeneous perspectives and brings them together in one panorama. In this way it may invite for a look backward and forward, in order to reflect on the development of methods and views in recent mathematical and theoretical physics (section 6).

1. Preliminaries: Weyl geometric gravity and Jordan-Brans-Dicke theory

1.1. Weyl geometry and gravity.

1.1.1. Basics of the geometrical framework. Weyl geometry is a generalization of Riemannian geometry, arising from two insights: (i) The mathematical automorphisms of both, of Euclidean geometry and of special relativity, are the similarities (of Euclidean, or respectively of Lorentz signature) rather than the congruences. No unit of length is naturally given in Euclidean geometry, and likewise the basic structures of special relativity (inertial motion and causal structure) can be given without the use of clocks and rods. (ii) The development of field theory and general relativity demands a conceptual implementation of this insight in a consequently localized mode (physics terminology). In a more physical language (i) and (ii) can be given the form of the postulate that fundamental field theories have to be formulated covariantly under point dependent rescalings of the basic units of measurement, while the Lagrangian densities and the dynamical laws (the “natural laws”) are invariant under point dependent rescaling (see Dicke’s postulate cited in section 1.2). It remains an open question whether the resulting extension of the mathematical automorphism group of the theories may be of physical import, or whether it is purely mathematical refinement.

Based on these insights, Weyl developed what he called purely infinitesimal geometry (reine Infinitesimalgeometrie) building upon a conformal generalization of a (pseudo-) Riemannian metric $g$ with coefficient matrix $(g_{\mu\nu})$ with (point-dependent) rescaling $\tilde{g}(x) = \Omega(x)^2 \, g(x)$ ($\Omega$ a nowhere vanishing positive function), and a scale (“length”) connection given by a real valued differential form $\varphi = \varphi_{\mu} dx^\mu$ (Weyl 1918a). If one rescales the metric by $\Omega$ one has to gauge transform $\varphi$ by $\tilde{\varphi} = \varphi - d \log \Omega$. The scale connection $(\varphi_{\mu})$ expresses how to compare lengths of vectors (or other metrical quantities) at two infinitesimally close points, both measured in terms of a representative $(g_{\mu\nu})$ of the conformal class. The typical symmetry of the geometry, at the infinitesimal level is thus the scale extended Poincaré group, sometimes called the Weyl group (although the same name is used in Lie
group theory in a completely different sense). In 1918 to roughly 1921/22 it seemed clear to Weyl that this extension of Riemannian geometry can be used for unifying gravity and electromagnetism; later he gave up this hope and considered his scale geometry as purely mathematical enterprise the most important features of which were transplanted to the $U(1)$-gauge theory of electromagnetism.

In hindsight, Weyl’s generalization of Riemannian geometry may be embedded in E. Cartan’s even wider program of geometries with infinitesimal symmetries. In the case of the scale extended Poincaré group one then arrives at a Cartan-Weyl geometry with a translational Cartan connection and torsion as the typical extension of the structure. With the exception of section 2.3 this paper will be restricted to the original form of Weyl geometry without torsion: in large parts it even deals with the most simple case of an integrable scale connection. The reasons for this restriction will become apparent below.

Metrical quantities in Weyl geometry are directly comparable only if they are measured at the same point $p$ of the manifold. Quantities measured at different points $p \neq q$ of finite, i.e., non-infinitesimal, distance can be compared metrically only after an integration of the scale connection along a path from $p$ to $q$. Weyl realized that this structure is compatible with a uniquely determined affine connection $\Gamma = (\Gamma^\mu_{\nu\lambda})$, the affine connection of Weyl geometry. If the Levi-Civita connection of the Riemannian part $g$ is denoted by $g\Gamma^\mu_{\nu\lambda}$, the Weylian affine connection is given by

$$\Gamma^\mu_{\nu\lambda} = g\Gamma^\mu_{\nu\lambda} + \delta^\mu_\nu \varphi_\lambda + \delta^\mu_\lambda \varphi_\nu - g_{\nu\lambda} \varphi^\mu.$$  

In the following the covariant derivative with regard to $\Gamma$ will be denoted as $\nabla = \nabla_\Gamma$. Similarly the curvature expressions for the Riemann tensor, Ricci tensor and scalar curvature $R_{\text{Riem}}, R_{\text{Ric}}, R$ will denote the Weyl geometric ones. The corresponding scale gauge dependent Riemannian analogues derived from $g\Gamma^\mu_{\nu\lambda}$ will be written as $g\nabla, gR_{\text{Riem}}, gR_{\text{Ric}}, gR$. The Weylian scalar curvature, e.g., is

$$R = gR - (n - 1)(n - 2) \varphi_\mu \varphi^\mu - 2(n - 1) g_{\mu\nu} \varphi^\mu,$$

with $n$ the dimension of the manifold. A change of scale neither changes the connection (the left hand side of (1)) nor the covariant derivative; only the composition from the underlying Riemannian part and the corresponding scale connection (right hand side) is shifted.

As every connection defines a unique curvature tensor, curvature concepts known from “ordinary” (Riemannian) differential geometry follow. The Riemann and Ricci tensors, $R_{\text{Riem}}, R_{\text{Ric}}$, are scale invariant by construction, although their expressions contain terms in $\varphi$. On the other hand, the scalar curvature involves “lifting” of indices by the inverse metric and is thus scale covariant of weight $-2$ (see below).

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4 For more historical and philosophical details see, among others, Vizgin 1994; Goenner 2004; Ryckman 2005; Scholz 1999; Scholz et al. 2001.

5 For a modern presentation of Cartan geometry, including the Cartan-Weyl case, see, e.g., Sharpe 1997 chap. 7; for the physical aspects of the extension studied since the 1970s Blagojević/Hehl 2013 chap. 8.
For vector and tensor fields (of dimensionful quantities) the appropriate scaling behaviour under change of the metrical scale has to be taken into account. If a field, expressed by \(X\) (leaving out indices) with regard to the metrical scale \(g(x)\) transforms like \(\tilde{X} = \Omega X\) with regard to the scale choice \(\tilde{g}(x)\) as above, \(X\) is called a scale covariant field of scale weight, or Weyl weight \(w(X) := k\) (usually an integer or a fraction). It is the negative of the mass weights used in particle physics. In general the covariant derivative, \(\nabla X\), of a scale covariant quantity \(X\) is no longer scale covariant; but a scale covariance can be recovered. Adding a weight dependent term solves the problem. The scale covariant derivative \(D\) of \(X\) is defined by
\[
D X := \nabla X + w(X) \varphi \otimes X,
\]
in coordinate description
\[
D \mu \nu := \nabla \mu X \nu + w(X) \varphi \mu X \nu.
\]
For example, the derivative \(\nabla g\) is not scale covariant, but \(D g\) is – even with the result zero:
\[
Dg = \nabla g + 2 \varphi \otimes g = 0.
\]
In Weyl geometry the metric is thus no longer constant with regard to the derivative \(\nabla\) but with regard to the scale covariant derivative \(D\). From the point of view of Riemannian geometry this appears as a “non-metricity” of the connection (in the literature often called “semi-metricity”). From the Weyl geometric point of view it is nothing but the metric compatibility condition for \(\Gamma\).

Here it may suffice to have recalled these basic properties. More details on Weyl geometry can be found in Weyl’s original papers \(\cite{Weyl1918a,b}\), those of his successors (\cite{Eddington1923, Bergmann1942, Dirac1973}) and more recent literature.\(^6\)

1.1.2. Weyl geometric gravity. Weyl’s generalization of Riemannian geometry arose with the perspective of generalizing Einstein gravity, which would allow a geometrical unification of gravity and electromagnetism \(\cite{Vizgin1994, Goenner2004}\). Any meaningful Lagrangian in this framework underlies the constraint of scale symmetry. Because of \(w(\sqrt{|g|}) = 4\), while the Weyl geometric scalar curvature \(R\) is of weight \(w(R) = -2\), Weyl could not work with the Hilbert-Einstein term but considered quadratic expressions in the curvature terms for a generalization of the gravitational Lagrangian, e.g.
\[
\mathcal{L}_W = L_W \sqrt{|g|} \quad \text{with} \quad L_W = \alpha_1 R^\mu_{\nu \lambda \kappa} R^\nu_{\mu \lambda \kappa} + \alpha_2 R^2.
\]
\(^6\)Presentations of Weyl geometry can be found, among others, in \(\cite{Blagojevic2002, Israelit1999, Drechsler/Tann1999, Perlick1989\, (difficult to access)}. For selected aspects see \(\cite{Codello et al.2013, Ohanian2016\, sec. 4)}\). Integrable Weyl geometry is presented in \(\cite{Dahia et al.2008, Romero et al.2011, Almeida et al.2014a, Quiros2014}\). \(\cite{Scholz2011\, sec. 2.1)}\). Be aware of different conventions for the scale connection. Expressions for Weyl geometric derivatives and curvature quantities are derived in \(\cite{Gilkey et al.2011, Yuan/Huang2013, Miritzis2004\, App.}\). For a more mathematical perspective consult \(\cite{Folland1970, Calderbank/Pedersen1998, Gauduchon1995, Higa1993, Ornea2004, Gilkey et al.2011}\).

\(^7\)As the “most simple and natural” expression \(\alpha_2 = 0\) in \(\cite{Weyl1918a, 4th ed., 5th ed.}\) and \(\alpha_1 = 0\) as the most simple example in \(\cite{Weyl1918a}\).
Of course, he added a term in the scale curvature $f = d\varphi$ ($f_{\mu\nu} = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu$) looking like the Maxwell action:

$\mathcal{L}_f = L_f \sqrt{|g|}$ with $L_f = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu}$

Only much later – in fact, about half a century later – other gravitational Lagrangians with Weyl geometric scale symmetry started to be considered. They arose from the idea of a coupling between gravity, here the Weyl geometric scalar curvature $R$, and a scalar field with “correct” complementary weight (subsection 2.2).

In the period covered here we encounter two modes of Weyl geometric gravity. One is farther away from Einstein gravity and uses square curvature Lagrangians, sometimes called Weyl gravity (in the strong sense); the other is closer to Einstein gravity and works with a modified Hilbert term coupled to a scalar field, in the physics literature it is often called Weyl geometric scalar tensor theory (WST). The latter goes back to independent proposals by M. Omote and R. Utiyama on the one hand and P.A.M. Dirac on the other for making use of a scalar field modification of the Hilbert term, analogous to Jordan-Brans-Dicke theory (see section 2.2). Here the gravitational structure is characterized by an equivalence class of triples $(g, \varphi, \phi)$, with $g = g_{\mu\nu} dx^\mu dx^\nu$ the Riemannian component of the Weylian metric, $\varphi = \varphi_\mu dx^\mu$ its scale connection, and $\phi$ an additional scalar field. The equivalence is given by combined rescaling transformations $g \mapsto \tilde{g} = \Omega^2 g = e^{2\omega} g$, $\varphi \mapsto \tilde{\varphi} = \varphi - d\omega$, $\phi \mapsto \tilde{\phi} = e^{-\omega} \phi$. Because of scaling freedom, a Weylian metric with nowhere vanishing scalar curvature can be gauged to $R = \text{const}$. Here, and elsewhere in this paper, $\equiv$ denotes an equality which holds only in a certain gauge specified by the context. Weyl considered this as the “natural” gauge, we prefer to call it the Weyl gauge.

If the scale connection is an exact form,

$\varphi = -dw$,  

with a scalar potential $w$ scale transforming by $w \mapsto \tilde{w} = w + \omega$, we work in an integrable Weyl geometric scalar tensor theory (IWST). Then the gravitational structure reduces to the Riemannian component of the metric plus, at face value, two scalar fields $(g, \phi = e^v, w)$ with equivalence under rescaling. As $v \mapsto \tilde{v} = v - \omega$, the sum $v + w$ is a scale invariant scalar field of the gravitational structure and the only crucial one. Because of the scale gauge freedom $\phi$, respectively $v$ or $w$, can be given any chosen value, e.g. a constant. In the integrable case, two scale gauges are of particular importance in addition to Weyl gauge: The Riemann gauge in which the scale connection is “integrated away” (for $\omega = -w$), then $\tilde{\varphi} \equiv 0$. The other one is the scalar field gauge in which the scalar field is scaled to a constant, $\tilde{\phi}(x) \equiv \phi_0 = \text{const}$ (for $\omega = v$). If the value of $\phi_0$ is specified such that it

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8 In a way, this may be called a geometrical “tensor-vector scalar” theory sui generis, in which all components have geometrical meaning.

9 The dynamical consequences of this interdependence have been clarified by (Israelit, 1999b, a), see section 5.2.1.
hooks up to Einstein gravity, \( \phi_o = (8\pi G)^{-1} \) (up to a hierarchy factor if need be), it is called Einstein gauge.\(^{10}\)

For a vanishing scale invariant sum,

\[ v + w = 0, \]

the scalar field \( \phi \) is essentially the potential for the scale connection, more precisely,

\[ \varphi = dv = d\ln \phi. \]

Then and only then, Einstein gauge and Riemann gauge coincide and IWST reduces to Einstein gravity. The Palatini approach varying the metric and the affine connection of a Lagrange density \( L = \phi R \sqrt{|g|} \) independently enforces the constraint \( v + w = 0 \) in addition to the integrability of the scale connection. This implies a reduction of a Palatini-IWST to Einstein gravity. The latter is then only re-written in scale covariant form, but without any modification of the dynamics.\(^{11}\) If one considers IWST from the point of view of the metric-affine scheme, one better uses variational constraints like in (Cotsakis/Miritzis, 1999) rather than the Palatini approach. Then the condition (8) is not enforced and the scalar field, respectively the integrable scale connection (9), express an additional dynamical degree of freedom.

1.2. Jordan-Brans-Dicke (JBD) gravity.

1.2.1. Basics of JBD theory. At the turn to the 1950s Pascual Jordan (Hamburg) and, a decade later, Carl Brans and Robert Dicke (Princeton) proposed a generalization of Einstein gravity by considering a varying gravitational parameter. The motivations at Hamburg and at Princeton were different, but there was a wide overlap of the ensuing theory, here abbreviated by JBD. Jordan started from an action principle (Jordan, 1952, p. 140)

\[ \mathcal{L}_J(\chi, g) = (\chi R - \frac{\xi}{\chi} \partial^\mu \chi \partial_\mu \chi)\sqrt{|\det g|}, \]

with a parameter \( \xi \) and a real scalar field \( \chi \) functioning as a kind of space-time dependent (reciprocal) gravitational “constant”. \( R \) here of course the Riemannian scalar curvature of the metric \( g \) (Jordan, 1952, 2nd. ed., 163, (3)).\(^{12}\) A Lagrange term \( L_m \) for classical matter could be foreseen (e.g., (Brans, 1961, equ. (6))). The hypothesis of a “varying gravitational constant” had been brought up already more than a decade earlier by P.A.M. Dirac, when he speculated about “large numbers” relations in physics.\(^{13}\) Pauli reminded Jordan that his “extended gravity” allowed for a class of

\(^{10}\)Obviously the Einstein gauge exists also in the non-integrable case.

\(^{11}\)Cf. sections 5.2.2, 5.4.

\(^{12}\)Warning: One has to check carefully the sign convention used in the definition of Riemann and scalar curvature. Jordan, e.g., used sign inverted definitions of the curvature terms with respect to those used here and in much of the present literature (Jordan, 1952, 40). In Fujii/Maeda’s notation (see below) this would correspond to \( \epsilon = -1 \) and thus to a “ghost” field.

\(^{13}\)For Dirac’s role in this story see (Kragh, 2016), for a larger view at JBD theory (Brans, 1999, 2014).
conformal transformations which not only affect the metric but also the scalar field,
\begin{equation}
\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} , \quad \tilde{\chi} = \Omega^{-2} \chi .
\end{equation}
Jordan included this generalization into the second edition of his book (Jordan, 1952, 2nd ed.,169).\textsuperscript{14}

A few years later, Robert Dicke and Carl Brans restarted the study of scale covariant scalar fields (including a classical matter term \(L_m\) in (10) (Brans, 1961, 8). Their motivation was to formulate a theory of gravity which took account of Mach’s principle as understood by D.W. Sciama.\textsuperscript{15} For the two US physicists the main function of the scalar field was “the determination of the local value of the gravitational constant” (Brans, 1961, 929). More clearly than in Jordan’s work, the wave character of the dynamical equation of \(\chi\) was emphasized by them (Brans [1961], equ. (9), (13)). Moreover, they had a different view of the role of scale transformations.

Their methodological goal was a scale independent foundation of physical theories, with a “passive” interpretation of scale transformations in mind (Brans, 1961, 927), while Jordan and Pauli tended to think in terms of “active” scale transformations of material structures. Dicke started an article dedicated to transformations of units in GRT (Dicke, 1962) announcing as “evident” the following principle:

> It is evident that the particular values of the units of mass, length, and time employed are arbitrary and that the laws of physics must be invariant under a general coordinate-dependent transformation of units. (Dicke, 1962, 2163)

That was very much in the spirit of Weyl’s intentions of 1918, from which the latter had disassociated himself with the shift of his gauge idea to quantum physics Weyl discussed this new view at different occasions in the 1940s, e.g. in (Weyl, 1949/2016, p. 165).\textsuperscript{16} It seems that Dicke “reinvented” the idea of scale gauge invariance of the natural laws anew. He systematically discussed the scale transformations of physical quantities, based on the (quasi-axiomatic) principle of the invariance of the velocity of light \(c\) and the Planck constant \(\hbar\). In particular, “all three quantities, time, length, and reciprocal mass transform in the same way” (Dicke, 1962, 2164), i.e.,

\[ l' = \Omega l , \quad t' = \Omega t , \quad m' = \Omega^{-1} m . \]

In this sense, Weyl’s scale gauge transformations reappeared in the principles of Jordan-Brans-Dicke theory without being mentioned as such. It may be

\textsuperscript{14}The conformal factor \(\Omega\) was (unnecessarily) restricted by the condition \(\Omega^2 = \chi^\gamma\) for some constant \(\gamma \in \mathbb{R}\).

\textsuperscript{15}In the 1950s Sciama had proposed to consider the possibility that the gravitational “constant” was related to the mass and the “radius” of the visible universe.

\textsuperscript{16}Also in the English edition of Philosophy of Mathematics and Natural Sciences Weyl expressed this disassociation quite clearly, appealing to the constants of atomic physics which regulate the frequencies of spectral lines (Weyl, 1949, 83). But this was only one part of his perspective. In the appendix he argued that for a deeper insight it would be necessary to understand how the “adaptation” of the mass of the electron to the local field constellation is achieved (Weyl, 1949, 288f.). This was close to the intentions of his 1918 approach, although no longer a claim that the goal had been achieved. Einstein, in his later papers, agreed (Einstein, 1949 555f.); see (Lehmkuhl, 2014).
that at the time nobody but Pauli was aware of this close resemblance to Weyl’s theory. Weyl’s choice of (scale) gauge was translated by Dicke into the choice of a frame of measuring units, complementing the choice of a coordinate system.

In more recent papers the scalar field and the JBD parameter are written in slightly different form. With \( \phi = \sqrt{2\xi^{-1}}\chi \), scale weight \( w(\phi) = -1 \), and \( \xi = \frac{\epsilon}{4\omega} \) the Lagrangian (10) turns into

\[
L_{BD} = \left( \frac{1}{2} \xi \phi^2 R - \frac{1}{2} \epsilon \partial_\mu \phi \partial_\mu \phi + L_{mat} \right) \sqrt{|\det g|},
\]

where sig \( g = (3,1) \sim (-+++) \) and generally \( \epsilon = 1 \), while only in exceptional cases \( \epsilon = -1 \) or 0 [Fujii/Maeda 2003, p. 5].\(^{17}\) In the following discussion this notation will be used as a standard.

A famous exception with \( \epsilon = -1 \) is the special constellation of coefficients (13)

\[
L_{cc} = \phi^2 R + 6 \partial_\mu \phi \partial^\mu \phi .
\]

Then the Lagrangian is invariant (up to an exact differential) under conformal transformations [Penrose 1965]. This case of conformal coupling allowed to study versions of gravity theory “in which scale invariance of matter is a consistency requirement on its coupling to gravitation” (Deser 1970). Deser considered a conformally coupled scalar field as a paradigmatic example for matter and observed that the addition of a quadratic term of the form \( \frac{1}{2} \mu^2 \phi^2 \), with \( \mu \) a parameter of mass dimension, implies breaking of the conformal symmetry. In this case, the range \( \phi \) “must clearly be cosmological in order not to lead to a clash with observation ” (Deser 1970, p. 252).

But in general, the three founding authors, Jordan, Brans and Dicke, considered it as evident that the “conformal transformations” (scale transformations) do not reduce the geometrical considerations to those of a purely conformal structure. They rather considered it as clear that JBD theory possesses a covariant derivative \( \nabla \), specified by the reference metric \( g \) underlying (10) from which Jordan and Brans/Dicke started. Later this scale was called Jordan frame (although Jordan was undecided, which scale might be the “natural” one). Because the Levi Civita connection of the Jordan frame metric determines the free fall trajectories of test particles, many authors consider this one as the “physical frame”, the other frames then appear as mathematical auxiliary devices. On the other hand, the JBD-field \( \phi \) can be scaled to a constant. Then the gravitational part of the Lagrangian looks like the Hilbert term of Einstein gravity, while the remnants of the JBD scalar field appears in additional expressions of the Lagrange density.\(^{18}\) The resulting Einstein frame satisfies the Riemannian “energy conservation” condition for matter tensors. Since roughly the 1990s it has found an increasing number of supporters who now propose it as the proper frame for a “physical” interpretation of JBD gravity. But no consensus in the JBD community

\(^{17}\) \( \epsilon = 1 \) corresponds to a normal field having a positive energy, in other words, not a “ghost”. Fuji/Maeda add that \( \epsilon = -1 \) looks unacceptable because it seems to indicate negative energy, but “this need not be an immediate difficulty owing to the presence of the nonminimal coupling” (ibid.).

\(^{18}\) See, e.g., (Capozziello/Faraoni 2011, chap. 3.6).
has been achieved; the discussion has remained undecided, to say the least, (Faraoni/Nadeau, 2007; Quiros et al., 2013).

1.2.2. JBD in a Weyl geometric perspective. The perspective of (integrable) Weyl geometry may help clarifying some aspects underlying this debate. Let us denote the affine connection referred to by

\[ \nabla := g\nabla, \]

where the r.h.s. expresses the Levi-Civita connection of \( g \) in the JBD Lagrangian (10). \( \nabla \) is kept unaffected, i.e. invariant, under scale transformations in JBD theory. A structural view of Weyl geometry shows that the combination of a conformal structure \([g]\) of pseudo-Riemannian metrics \( g \) and a specification of an invariant affine connection \( \nabla \) with a compatibility condition, inbuilt here because of (14), determines a Weyl structure on a differentiable manifold \( M \). In this way JBD gravity may be embedded in the theoretical frame of Weyl geometry, independent of whether or not a single author knows. Usually this is not being done (see, however, section 5.2.2).

2. Contributions to Weyl geometric gravity in the 1970s and 1980s

2.1. Ehlers/Pirani/Schild and subsequent work.

2.1.1. An axiomatic approach to the foundations of gravity. Weyl already discussed the relation between the physical concept of a causal structure and the mathematical concept of a conformal structure on a differentiable manifold (Weyl, 1918 4th. ed., appendix I). He deemed it inadequate to think of an empirical determination of the metrical coefficients \( g_{\mu\nu} \) by “rods and clocks” and looked for another empirical specification of a Weylian metric \((g, \varphi)\). In a note added to a letter to F. Klein (a little later published in Göttlinger Nachrichten as (Weyl, 1921)) he sketched an idea how this can be achieved. Assuming his framework of the generalized “purely infinitesimal” geometry, Weyl showed that two of his generalized metrics which have identical conformal structure and the same projective geodesic path structure will coincide. This meant that, at least in the framework of Weyl geometry, conformal and projective path structures specify a Weylian metric uniquely.

Weyl’s argument on the combination of projective and conformal structure was taken up and extended by Jürgen Ehlers, Felix Pirani and Alfred Schild (EPS in the sequel), about the same time in which Dirac studied Weyl geometry in the context of scalar tensor theories (Ehlers et al., 1972). This paper was written for a Festschrift in the honour of J.L. Synge. Synge had become known for his proposal to base general relativity on the behaviour of standard clocks (chronometric approach). From the foundational point of view, clocks could appear as a problematic choice, because they are realized by complicated material systems. The question arose whether more basic

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19 See the discussion in Quiros et al. (2013) and Scholz (2017).
20 Weyl (1920)
21 Weyl’s note (Weyl, 1921) became better known by his calculation and discussion of projective and conformal curvature tensors, which followed.
signal structures of gravitational theory (light rays, particle trajectories) might do the job.

Using Hilbert’s words, EPS “laid the foundations deeper”, combining Weyl’s idea of 1920/21 and the recently developed mathematical language and symbolic technology of differentiable manifolds with Hilbert’s axiomatic method. They started from three sets, $\mathcal{M} = \{p, q, \ldots\}$, $\mathcal{L} = \{L, N, \ldots\}$, $\mathcal{P} = \{P, Q, \ldots\}$, with $\mathcal{L}, \mathcal{P} \subset \mathcal{M}$, and called the three sets respectively collections of events, light rays and particles. By postulates close to physical experimental concepts of light signal exchange between particles EPS formulated different groups of axioms in the Hilbertian style of foundations of geometry ($D_1, \ldots D_4$, $L_1, L_2$, $P_1, P_2$, $C$), which allowed them to introduce a $C^3$ differentiable structure on $\mathcal{M}$ on which $\mathcal{L}$ and $\mathcal{P}$ then described smooth curves (axiom group $D$). Moreover, a $C^2$ conformal structure was defined by $\mathcal{L}$ (axioms $L$), and a differentiable projective path structure by $\mathcal{P}$ (axioms $P$).

With a compatibility axiom $C$, basically postulating that light rays can be approximated arbitrarily well by particle trajectories, EPS could derive their main result.

**Theorem 1** (Ehlers/Pirani/Schild 1972). A light ray structure $\mathcal{L}$ and a set of particle trajectories $\mathcal{P}$ defined on an event set $\mathcal{M}$ which satisfy axioms $D, L, P, C$ endow $\mathcal{M}$ with the structure of a $(C^3)$-differentiable manifold $\mathcal{M}$ and a $(C^2)$-Weylian metric $[(g, \varphi)]$. The latter is uniquely determined by the condition that its causal and geodesic structures coincide with $\mathcal{L}$ and $\mathcal{P}$ respectively.

EPS posed the question, how a (pseudo-)Riemannian structure of classical (Einsteinian) relativity might arise from the Weylian one. A simple additional Riemannian axiom, postulating the vanishing of the scale curvature, $d\varphi = 0$, could serve the purpose. Such a postulate did not seem nonsensical, as Weyl’s interpretation of the scale connection $\varphi$ as electromagnetic (e.m.) was obsolete anyhow and EPS did not adhere to it. But the authors did not exclude the possibility that a scale connection field $\varphi$ of nonvanishing scale curvature might play the role of a “true”, although still unknown, field.

2.1.2. **Subsequent work.** The paper of Ehlers, Pirani and Schild triggered a line of investigations in the foundations of general relativity, sometimes called the causal inertial approach (Coleman/Korté), sometimes subsumed under the more general search for a constructive axiomatics of GRT (Maier/Schmidt, Audretsch, Lämmerzahl, Perlick and others). These investigations turned towards a basic conceptual analysis from the point of view of foundations of inertial geometry (Coleman/Korté 1984), some even looking for Desargues type characterization of free fall lines (Pfister 2004). How a kind of “standard clocks” can be introduced in the Weyl geometric setting without taking refuge to atomic processes, by just using the observation of light rays and inertial trajectories, was studied by (Perlick 1989, 1987, 1991). Another line of follow up works explored the extension of the foundational argument of the causal inertial approach to quantum physics, where particle trajectories might no longer appear acceptable as a foundational concept.

\[22\text{See (Trautman 2012).}\]
This debate was opened by (Audretsch, 1983). It was soon continued by the collective work again of three authors (Audretsch et al., 1984), cited in the sequel by AGS, and had follow up studies, among them (Audretsch, 1994). Audretsch argued that the “gap” between Weylian and Riemannian geometry can “be closed if quantum theory as a theory of matter is made part of the total scheme” (Audretsch, 1983, 2872). He postulated that quantum theory in the sense of Dirac or Klein-Gordon (K-G) fields on a Weylian manifold are compatible with the latter’s geometry, if and only if the WKB (Wentzel-Kramers-Brillouin) approximation of the Dirac (or K-G) field leads to streamlines which in the limit $\hbar \to 0$ agree with geodesics (Audretsch’s compatibility condition).

Working with scale covariant mass factors $m$ of Weyl weight $w(m) = -1$ Audretsch found that compatibility is possible only if the mass factor $m$ of the Dirac particle has vanishing covariant derivative, $\nabla_{\mu} m = 0$ in some gauge. He observed that this implied vanishing of Weyl’s scale curvature $d\varphi = 0$ (Audretsch, 1983, equ. (6.14)) and concluded a bit rash:

> The consequence of the requirement is therefore that the Weyl space reduces to a Riemann space and the gap [between Weylian and Riemannian geometry, ES] described in Sec. I is closed. (Audretsch, 1983, 2881, emph. in original)

Because $D_\mu m = 0$ in any gauge if $\nabla_{\mu} m$ vanishes in Riemann gauge, Audretsch had only shown that the limiting condition for streamlines of the WKB approximation of the Dirac field to classical geodesic trajectories implied integrability of the Weylian metric. The question whether this would also imply the choice of the Riemann gauge as “physical” was not posed; it rather was imputed as self-evident.

In the AGS paper this question was taken up again and stated carefully in the language of conformal fibre bundles for Dirac- and for Klein-Gordon fields. AGS showed that Audretsch’s compatibility condition implies the possibility to reduce the “conformal” group, here understood as $R^+ \times SO(1, 3)$, to the orthogonal group.

**Theorem 2** (Audretsch/Gähler/Straumann 1984). A Weylian manifold $(M, ([g, \varphi]))$ of Lorentzian signature is locally integrable, iff the WKB (Wentzel-Kramers-Brillouin) approximation of a (locally defined) Dirac or Klein-Gordon field $\psi$ on $M$ leads to streamlines which agree with geodesics in the limit $h \to 0$.

The three authors formulated their consequence more carefully than Audretsch had done in his first paper. They did not claim that their investigation had completely filled the gap to Riemannian geometry.

All in all, the three authors gave a more precise and mathematically modernized presentation of Audretsch’s insight. The gap between the Weylian and the Riemannian structure in the foundations of GRT was reduced but not completely closed. It could seem natural to choose the Riemann gauge of the Weyl metric in order to reduce the structure group to $SO(3, 1)$, but nothing compelled to do so. The classical interpretation of geodesics as trajectories of mass points was foreign to the field theoretic context anyhow. It was now substituted by postulating coherence between geodesic structure and the flow-lines associated to pseudo-classical quantum fields.
2.2. Dirac’s and Omote/Utiyama’s retake of Weyl geometry.

2.2.1. Dirac on scale covariant “varying” gravity. In the 1970s P.A.M. Dirac introduced Weyl geometry into the discourse of the rising scalar-tensor theories. He was still fascinated by the interrelation of certain constellations of large numbers in physics, the “large number hypothesis” [Dirac, 1973]. Largely following Eddington’s notation and terminology [Eddington, 1923], he introduced the readers to Weyl geometry which was no longer generally known in the younger generation of physicists. He then introduced a scalar-tensor theory of gravitation coupled in an “oldfashioned”, i.e. outdated, way to electromagnetism. Like Weyl in 1918, he identified the potential of the electromagnetic field $F_{\mu\nu}$ with the Weylian scale curvature $f = d\phi$

$$F_{\mu\nu} = f_{\mu\nu}.$$ 

In the sequel I call this the electromagnetic (em) dogma. On the other hand, he replaced Weyl’s original gravity Ansatz in the Lagrangian (using square curvature terms) by a JBD-type Lagrangian using a real scalar field $\beta$ of weight $w(\beta) = -1$. He added a biquadratic scale invariant potential term,

$$(15) \quad \mathcal{L}_{Dir} = -\beta^2 R + k D\lambda \beta D\lambda \beta + c\beta^4 + \frac{1}{4} f_{\mu\nu} f^{\mu\nu},$$

with constant $k$. For $k = 6$, the scale connection terms of the Lagrangian essentially cancel (i.e., they reduce to boundary terms and thus are variationally negligible) like in (13). So Dirac wrote the Lagrangian in the form

$$(16) \quad \mathcal{L}_{Dir} = -\beta^2 g R + 6 D\lambda \beta D\lambda \beta + c\beta^4 + \frac{1}{4} f_{\mu\nu} f^{\mu\nu},$$

known to be conformally invariant [Penrose, 1965], if $g R$ denotes the sign inverted Riemannian scalar curvature with respect to the generally accepted convention, while in (15) $R$ is the sign inverted scalar curvature of the Weylian metric.

Dirac derived dynamical equations and Noether identities for diffeomorphisms and scale transformations. He distinguished Riemann gauge, $\varphi = 0$,

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23 Dirac presented his proposal for a retake of Weyl geometry at the occasion of the symposium Honouring his 70th birthday, 1972 at Trieste. This talk remained unpublished. According to [Charap/Tait, 1974, p. 249 footnote] the talk was close to his 1973 publication. For the broader historical context of this enterprise, the background in Dirac’s reflection on large numbers in the 1920s, and a surprising link to geophysics see [Kragh, 2016].

24 The qualifications “sign inverted” and “generally accepted” refers to the sign convention which agrees with the coordinate free definition $\text{Riem}(Y,Z)X = \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla\{Y,Z\}X$. It is preferred in the mathematical literature including [Weyl, 1918, 5th ed., 131] and also used in the majority of the more recent physics books. The “sign inverted” convention in some of the physics literature goes back to Einstein, e.g. [Einstein, 1916, 801], who in turn may have followed Ricci and Levi-Civita. It was continued in much of the physics literature of the first half of the 20th century, [Eddington, 1923 § 37], [Pauli, 1921] up to the influential [Weinberg, 1972 eqn. (2.1.3)]. Weyl, on the other hand, used the above convention long before the coordinate free definition of the Riemann tensor was available. It seems to be dominant in the more recent literature on GRT, although Rindler speaks of a 50% distribution among the two conventions [Rindler, 2006, 219].
which existed, of course, only for vanishing e.m. field $f_{\mu\nu} = 0$, from “Einstein gauge” (gravitational parameter constant, $\beta = 1$) and “atomic gauge” (Weyl’s natural gauge). He warned that “all three gauges are liable to be different” (Dirac [1973, 411]).

In a discussion at the end of his article, why one should believe in the proposed “drastic revision of our ideas of space and time”, Dirac announced a part of his research agenda, which was independent of the large number hypothesis:

There is one strong reason in support of the theory. It appears as one of the fundamental principles of Nature that the equations expressing basic laws should be invariant under the widest possible group of transformations . . . The passage to Weyl’s geometry is a further step in the direction of widening the group of transformations underlying physical laws [in addition to general coordinate transformations, E.S.]. One now has to consider transformations of gauge as well as transformations of curvilinear coordinates and one has to take on’s physical laws to be invariant under all these transformations, which imposes stringent conditions on them. (Dirac, 1973, 418)

So far, Dirac’s explanations agreed with the view of C. Brans and R. Dicke. He followed a tendency of the time for probing possible extensions of the symmetries (automorphisms) of fundamental physics. In distinction to Pauli’s insistence on a preferred scale, taken over into the general discourse of JBD theory, Dirac argued that at least three different gauges, Riemann, Einstein, and “atomic” gauge, indicated by atomic clocks, had to be considered in different theoretical or observational contexts. He saw no chance of a single preferred gauge; but sometimes the “atomic” gauge was assumed to be identical with Weyl gauge.\footnote{Weyl had argued that the atomic clocks somehow adapt to the local field constellation via the Weylian scalar curvature.}

2.2.2. Some remarks on Dirac’s followers. Dirac’s proposal for reconsidering Weyl geometry in a modified theory of gravity was taken up by field theorists and a few astronomers. An immediate and often quoted paper by Vittorio Canuto and coauthors gave a broader and more detailed introduction to Dirac’s view of Weyl geometry in gravity and field theory (Canuto et al., 1977). The opening remark of the paper motivated the renewed interest in Weyl geometry with actual developments in high energy physics:

In recent years, owing to the scaling behavior exhibited in high-energy particle scattering experiments there has been considerable interest in manifestly scale-invariant theories. (Canuto et al., 1977, 1643)

With the remark on “considerable interest in manifestly scale-invariant theories” in high energy physics the authors referred to Bjorken scaling and, in particular, the seminal paper (Callan et al., 1970). But the authors were careful not to claim field theoretic reality for Dirac’s scalar function $\beta$ (Canuto et al. 1977, 1645). They rather developed model consequences
for the approach in several directions: cosmology, including “LNH (large number hypothesis as a gauge condition”, modification of the Schwarzschild solution in the Dirac framework, consequences for planetary motion, and stellar structure. At the end the authors indicated certain heuristic links to gauge fields in high energy physics of the late 1970s.

Canuto was interested in exploring Dirac’s idea that, perhaps, the gravitational units of measurement, expressed by a locally dependent parameter of gravity (in place of a constant), and a frequency change of gravitational clocks, like the period of planets revolving a star, might differ from the atomic units. This would imply a violation of the strong equivalence principle. In careful evaluations of the astrophysical data available at the beginning of the 1980s he and his coauthor Itzhak Goldman concluded that a tiny difference might still be possible (Canuto/Goldman, 1983).

For some years Dirac’s approach attracted also some interest from astronomers at the Geneva observatory, Pierre Bouvier, André Maeder and coworkers. In November 1977, only a few months after the publication of (Canuto et al., 1977), the two Geneva astronomers submitted a theoretical vindication of “Weyl’s geometry as a framework for gravitation” to the journal Astrophysics and Space Science (Bouvier/Maeder, 1977). This paper was meant as a background for a larger research program. Maeder intended to “build some new mechanics” on Dirac-Weyl geometrical gravity. He conjectured that the determination of gravitating mass in gravitationally bound large systems (clusters, super clusters) on the basis of the virial theorem was affected by the “new mechanics” that the missing mass identified observationally around the middle of the 1970s by astronomers and astrophysicists, might vanish (ibid, 341f.). First empirical investigations on the Coma cluster seemed to support Maeder’s conjecture (Maeder, 1978a, b). But during the next years the evidence in favour of his conjecture dissolved. So the first attempt to bring Dirac’s theory to bear in observational cosmology faded out at the turn to the 1980s. We come back to this issue in section 5.3.

But theory development has an open horizon. Dirac’s program continued to be pursued during the following decades on the theoretical level among others by Nathan Rosen working during this time at the Technion Haifa and the University of Beer Sheva and Mark Israelit, who immigrated to Israel in 1971 and acquired his PhD at Haifa in 1975. In their continuation of the Dirac program, Rosen and Israelit stucked as far as possible to the e.m. dogma for a non-conformally coupled scalar field, $k \neq 6$, but with a light massive (Proca-type) photon. But already in his 1982 paper Rosen discussed the possibility of interpreting $\varphi_{\mu}$ as the potential of a new, hypothetical, heavy massive boson field (see below). During the 1990s he and Israelit shifted to the last interpretation as the preferred physical view of the Weylian scale connection.

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26 For Canuto and Maeder compare (Kragh, 2006, pp. 126ff.)
27 For an illuminating historical reports on the rise of dark matter see (Sanders, 2010). From a methodological point of view Maeder’s hypothesis was not so far away from the later, more pragmatic and more successful approach of modified Newtonian dynamics, MOND, by Mordechai Milgrom.
Rosen extended Dirac’s approach in several respects. He added a scale invariant mass term $\mathcal{L}_m$ to the Lagrangian, studied the dynamical equations, the corresponding currents, the Noether relations, and revisited the question of different gauges (Rosen 1982). Although he recognized the importance of the scale covariant derivatives corresponding to our (3) for giving the Lagrange density a scale invariant form, he did not write the dynamical equations scale invariantly. The left hand side of the Einstein equation, e.g., appeared with the Einstein tensor of the Riemannian component of the Weyl metric, $G = R_{\text{Riem}} - \frac{1}{2} R g$ in the notation of our section 1.1, rather than with the respective (scale invariant) Weyl geometric tensors $G = R_{\text{Weyl}} - \frac{1}{2} R g$. Similarly the right hand side expressions for the energy-momentum of mass and the scalar field were neither scale covariant nor scale invariant. All terms of the dynamical equations were stripped down to their Riemannian cores. This deprived the Weyl geometric framework of much of its conceptual strength, even though the equalities were valid in every scale gauge (Rosen 1982, equ. (121)). This remained so in all of his and Israelit’s work. A scale covariant form for the dynamical equations was introduced only a decade later in the work of Hung Cheng and Drechsler/Tann (section 4.2).

Rosen also posed the question how Dirac’s “atomic gauge”, in the sense of Weyl gauge, might be made consistent with a non-integrable Weyl geometric structure in order to remove the old problem which Einstein had raised in 1918 as an objection against Weyl’s generalized geometry. He tried to back the “atomic gauge” by introducing what he called a “standard vector” (field). For any timelike vector field $u$ of Weyl weight $w(u) = -1$ the norm $|u| = g(u,u)^{1/2}$ is scale invariant ($w|u| = 2 - 1 - 1 = 0$). If $|u|$ is scale covariantly constant, i.e. $D|u| = 0$ ($D$ the scale covariant derivative), Rosen called it a standard vector field and considered the hypothesis that atoms carry a “standard vector” field with them (Rosen 1982, p. 220f.). But he was cautious enough not to declare this hypothesis as a definitive solution of the measurement problem in Weyl geometric gravity.

He found that the Noether relations due to the diffeomorphism invariance of the Lagrangian imply the equations of motion for matter, while the Noether relations induced by its scale invariance show that the scalar field equation is a consequence of the Einstein equation and the generalized “electrodynamical” (i.e., scale curvature) equation (Rosen 1982, p. 230). Moreover, studying Dirac’s Lagrangian (15) with general coefficient $k$, he realized that for the case of non-conformal coupling the scale curvature equation acquires the form of a generalized Proca equation

$$\nabla_\nu f^{\mu\nu} + m^2 \varphi^\mu = 0$$

with $(m |\mathfrak{h}|)^2 = \frac{1}{2}(6 - k)$. This was consistent with Smolin’s observation regarding the Weylian scale connection (cf. section 4.1.), which Rosen apparently did not know. He concluded that in the case $k \neq 6$ two physical interpretations for the scale connection were possible: $\varphi^\mu$ might represent an electromagnetic field with massive photons of very small
mass, or a "meson" field extremely weak interacting with ordinary matter. He added:

These mesons could conceivably accumulate at the center of galaxies and galaxy clusters and could provided (sic!) the "missing mass" that is needed to give a closed universe. (Rosen [1982] p. 234)

Rosen thus considered an early "dark matter" hypothesis for the Weyl field, at a time when the conditions for the present understanding of dark matter in galaxies and structure formation was just forming (Sanders [2010]). He mainly related it to the missing mass for cosmological models of positive spatial curvature and alluded at best implicitly to Zwicky’s early observations of a mass problem in galaxy clusters.

For cosmological investigations Rosen also considered a vanishing scale curvature. That led to an integrable Weyl geometry with the logarithm of the Dirac scalar field as the potential of the scale connection like in our equ. (9) (Rosen, 1982 equ. (136)). Although this implies a dynamically trivial extension of Einstein gravity, Rosen found it interesting to discuss scaling effects from a geometrical point of view. For a Robertson-Walker type metric $\gamma_{\mu\nu}$ of the form

\begin{equation}
\text{ds}^2 = dt^2 - \frac{a(t)^2}{a_o^2} dt^2
\end{equation}

he introduced

$g_{\mu\nu} = \frac{a_o^2}{a(t)^2} \gamma_{\mu\nu}$.

as the cosmic gauge. After an appropriate reparametrization of the time coordinate this led to a static Riemannian metric for the model (18),

\begin{equation}
g_{\mu\nu} : \text{ds}^2 = d\beta^2 - dt^2,
\end{equation}

and $\beta = \frac{a(t)}{a_o}$ (Rosen [1982], 234ff.). He showed that the cosmological redshift $z$ of a light signal, emitted at time $T_o$ and received at $T_1$, remains invariant under rescaling. In the "cosmic gauge" it appears no longer due to a spatial expansion of the geometry, but to the scalar field $\beta$, with $z + 1 = \frac{\beta(T_1)}{\beta(T_o)}$, or equivalently, what Rosen did not mention, due to the Weylian scale connection in the "cosmic gauge" (cf. subsection 5.2.2).

2.2.3. Omote, Utiyama and the Japanese group. Already in 1971 and unnoticed by Dirac, a Lagrangian field theory of gravity with a scale covariant scalar field coupling to the Hilbert term like in JBD theory, but now explicitly formulated in the framework of Weyl geometry had been formulated by M. Omote, Tokyo, (Omote [1971]) 28 A little later, and more or less at the time of Dirac’s retake of Weyl geometry Ryoyu Utiyama (Toyonaka/Osaka) headed toward a similar goal, although referring to A. Bregman’s paper discussed in section 2.3 and with a main interest in elementary particle

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28Rosen’s “meson” was a hypothetical massive fundamental boson, no bound state of quarks like the ones of the SM.

29This was more than a year before the Trieste symposium at which Dirac talked about his ideas. Apparently the paper remained unknown to Dirac. A second paper by Omote followed after Dirac’s publication and after Utiyama had jumped in Omote [1974].
Different to Dirac, he left the *em* dogma behind and tried to understand the (nontrivial) Weylian scale connection as a new fundamental field. In a series of papers he ventured toward its bosonic interpretation (Utiyama 1973, 1975a,b) and presented his results at the Seventh International Conference on Gravitation and Relativity (Tel Aviv, June 1974). Utiyama emphasized that a Brans-Dicke field $\phi$ of weight $-1$, imported to Weyl geometry, could serve as a kind of *measure field* (Utiyama’s terminology) with respect to which gauge invariant measurable quantities could be expressed starting from any gauge (Utiyama 1973, 1975a).

The import of the scalar field into a Weyl geometric structure would let it appear natural that $\phi$ is accompanied by a Weylian scale connection $\varphi$ with non-vanishing curvature (“Weyl’s gauge field”). So Utiyama proposed to explore the ordinary Yang-Mills Lagrangian term for a Weylian scale connection

$$
\mathcal{L}_\varphi = -\varepsilon \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \sqrt{|\det g|} \quad \text{(here with } \varepsilon = 1) 
$$

(19) (Utiyama 1975b, (2.4)) He studied conditions under which “Weyl’s gauge field” admitted plane wave solutions, and came to the conclusion that it would be “tachyonic”, i.e. a field which allowed superluminal propagation of perturbations. In Utiyama’s view the “boson” had therefore to be confined to the interior of matter particles. Nevertheless he thought that this “unusual field $\varphi_\mu$ might play some role in establishing a model of a stable elementary particle” (Utiyama, 1973, 2089).

Utiyama’s results were not generally accepted. Kenji Hayashi and Taichiro Kugo, two younger colleagues from Tokyo resp. Kyoto, reanalyzed his calculations and argued that, with slight adaptations of the other parameters, the sign $\varepsilon$ could just as well be switched. Then an ordinary, at least non-tachyonic, field would result (Hayashi/Kugo, 1979, 340f.). Even then the scale connection would still have strange physical properties. The two physicists showed, after a careful introduction of Weyl geometric spinor fields and their Lagrangians (using scale covariant derivatives), that the scale connection terms canceled. As they considered only the kinetic term of fermionic Lagrangians, no Yukawa term, in their approach neither the scalar $\phi$-field nor the scale connection $\varphi$ coupled effectively to spinor fields.

At the very moment that a Weylian scale connection $\varphi$ was interpreted as a “physical” field beyond electromagnetism, it started to puzzle its investigators and, at first, posed more riddles than it was able to solve. It seemed not to couple to matter fields at all (Hayashi/Kugo), looked either “tachyonic” (Utiyama) or, as we shall see below (Smolin, Nieh, Hung Cheng), appeared to be of Planck mass, far beyond anything observable.

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30 In his first paper of 1973 Omote was not mentioned; it was taken up, however, in the references of [Utiyama 1975a].

31 Dirac included a similar scale curvature term in his Lagrangian, but did not study its consequences.

32 Apparently Hayashi/Kugo used different signature conventions from Utiyama, which resulted in another sign flip in $\varepsilon$.

33 $\phi$ and $\varphi$ not even coupled to the electromagnetic field, as Hayashi and Shirafuri showed in another paper the same year.
2.3. Cartan-Weyl geometric approaches.

2.3.1. The Cartan geometric approach to gravity. Another input to Weyl geometric gravity came from the research tradition started by Dennis Sciama and Thomas Kibble who developed a theory of gravity by “localizing” the symmetries of Minkowski space, i.e., the Poincaré group \( \text{(Sciama, 1962; Kibble, 1961)} \). They treated the “external” symmetries of spacetime similar to the “internal” ones investigated in the gauge theories over Minkowski space in elementary particle physics (isospin, \( SU_2 \), later \( SU_3 \) and generalizations) which arose from the works of Yang/Mills and Utiyama. Without explicit recourse to Cartan they reproduced basic structures of Cartan geometry in field theoretic terms written in classical tensor calculus. The dynamical nature of the infinitesimal translations component of the “localized” Poincaré group found its expression in the asymmetry of the linear connection, viz. torsion. In the sequel different Lagrangians were investigated, and more general groups, in particular the scale extended Poincaré group or the affine group, were studied. In this way a broad field of gauge theories of gravitation arose (Blagojević/Hehl, 2013).

During the 1970s several authors introduced Cartan geometric methods into this research program, particularly prolific among them André Trautman, Warszaw, and Friedrich Hehl, Cologne. They showed that Cartan geometry offered a tailor-made geometric framework for infinitesimalizing (“localizing” in the language of physicists) the symmetries and the currents known from Minkowski space and special relativity. About the same time also the first publications studying the scale extended Poincaré group, often called the Weyl group,

\[
\mathfrak{W} = \mathbb{R}^n \times (SO(1, n - 1) \times \mathbb{R}^+),
\]

appeared. In the global view \((\mathbb{R}^+, \cdot)\) operates as the dilation group on the translations and, in case of a global view, on the underlying Minkowski space \( \mathbb{R}^{(1,3)} \approx \mathfrak{W}/(SO(1, 3) \times \mathbb{R}^+) \) in the case \( n = 4 \). Under localization, or equivalently in the corresponding Cartan space, the infinitesimal groups (Lie algebras) are related in such a way that \( so(1, 3) \oplus \mathbb{R} \) operates on the infinitesimal translations, \( \mathbb{R}^4 \). \( \mathbb{R}^4 \) is “soldered” point dependently to the tangent spaces of the underlying differentiable space \( M \) by specifying a tetrad field or more generally a frame field, i.e., a family of bases of the tangent spaces. In more recent mathematical terms this corresponds to the choice of a Cartan gauge in a Cartan space modelled after \( \mathfrak{W}/(SO(1, 3) \times \mathbb{R}^+) \), respectively the corresponding Lie algebras (Sharpe, 1997). What appears in the global view as an operation on the space itself was thus reshaped, in the infinitesimalized situation, as a mere change of a Cartan gauge. Weyl’s intentions of his 1918 geometry and the ideas of Dicke and Dirac regarding unit scaling were well expressed in this approach, and at the same time extended by introducing translational curvature, torsion in Cartan’s terminology.

2.3.2. Alexander Bregmann, … at that time working at Kyoto, inferred from (Omote, 1971) that localized rescaling could be separated from Weyl’s...
geometrical interpretation of the infinitesimal length transport. He argued that the point-dependent scale transformations could be treated “analogous to the introduction of a space-time dependence into the constant parameters of Isospin or Poincaré transformations”. The global scale dimensions \( d \) of a physical field \( X \) could then be taken over as “Weyl weight” (Bregman’s terminology) of \( X \) to the localized theory (Bregman 1973, p. 668). He first developed a Kibble-like approach to gravity built upon the Poincaré group with tetrad fields \( h^\mu_a, (a = 0, \ldots, 3 \text{ indexing the tetrads}, \mu = 0, \ldots, 3 \text{ the coordinates}) \). He introduced a covariant derivative in terms of tetrad coordinates allowing for torsion, and a spin connection expressed by coefficient systems of the form \( A^{\mu}_{\nu} \) with regard to generators \( S_{mn} \) of the Lorentz group.\(^{35}\) Then he went on “to accommodate” the Weylian scale transformations to the tetrad calculus, in particular rescaling the tetrads with weight \(-1\) (ibid. pp. 675ff.)\(^{36}\)

\[
\begin{align*}
\tilde{h}^\mu_a &= \Omega^{-1} h^\mu_a
\end{align*}
\]

This expressed an operation of the scale group on the tetrads, not on the tangent vectors which remained unaffected by rescaling.\(^{37}\) That made it necessary to extend the spin connection by a component in the Lie algebra \( R \) of the scale group, i.e., a Weylian scale connection \( \varphi = \varphi_\mu dx^\mu \)

\[
\begin{align*}
\hat{A}^{mn}_\mu &= A^{mn}_\mu - (h^m_\mu h_{\nu}^n - h^n_\mu h_{\nu}^m )\varphi_\nu \quad \text{(Bregman 1973, equ. (3.6)).}
\end{align*}
\]

Bregman remarked that the modified spin connection represented by \( \hat{A}^{mn}_\mu \) is “Weyl invariant” and used it to define an associated scale covariant derivative \( D_k = h^\mu_k \tilde{D}_\mu \) with the property \( \tilde{D}_\lambda g_{\mu\nu} = -\varphi_\lambda g_{\mu\nu} \), typical for a Weylian metric like in our equ. \(^{34}\). The corresponding linear connection \( \hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \delta^\lambda_{\mu} \varphi_\nu + \delta^\lambda_{\nu} \varphi_\mu - g_{\mu\nu} \varphi^\lambda \) generalized the Weylian affine connection but was no longer symmetric; it rather included the scale invariant torsion tensor

\[
\begin{align*}
T^\lambda_{\mu\nu} &= \hat{\Gamma}^\lambda_{\nu\mu} - \hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu}.
\end{align*}
\]

In retrospect we can see in Bregman’s paper a symbolism for working in a Cartan space modelled after the homogeneous space \( \mathbb{R}^4/(SO(1,3) \times \mathbb{R}^+) \), later called a Cartan-Weyl space (or the other way round)\(^{38}\). This terminology was not Bregman’s; he used Cartan geometric language rather parsimoniously, only with regard to the underlying Riemann-Cartan structure, modelled after \( \mathfrak{g}/SO(1,3) \) with \( \mathfrak{g} = \mathbb{R}^4 \times (SO(1,3) \text{ the Poincaré group}. \) He did not think in geometric terms about the extension of this structure by rescaling the tetrads, the physical (spinor, vector etc.) fields, the associated spin connection etc.

Bregman was more interested in showing how to form Weyl invariant Lagrangians \( L \) from matter Lagrangians \( L^M \) (his notation) of scale dimension \(-4\) with regard to “constant parameter scale transformations”. He noticed that many Lagrange densities studied in field theory are also invariant under

\begin{footnotesize}
\begin{enumerate}
\item Notations have been slightly adapted.
\item For \( g_{\mu\nu} = h^\mu_a h_{\nu}^a \), the convention \(^{21}\) boils down to \( g_{\mu\nu} \mapsto g_{\mu\nu} = \Omega^2 g_{\mu\nu} \).
\item Bregman, like many other of our authors, used a sign inverted convention for the scale connection form.
\item Cf. (Sharpe, 1997).
\end{enumerate}
\end{footnotesize}
all conformal transformations of the Minkowski space, including the special conformal ones (“in particular this is generally true of theories whose quantized versions are renormalizable”) and added:

In our case such a wider invariance of $L^M$ implies in turn that the Poincare gauge invariant lagrangian $L^P$ is already Weyl invariant with $L = L^P$ \cite{Bregman1973} p. 678).

This sharp minded remark generalized Pauli’s observation that a massless Dirac-spinor field is invariant under Weyl transformations without assuming a coupling to a Weylian scale connection \cite{Pauli1940}.

Finally Bregman gave a short discussion of an integrable scale connection with potential $\sigma, \varphi_\mu = -\partial_\mu \sigma$. He considered $\sigma$ as an “independent dynamical variable” which is “connected to the translation or spin gauge fields” only through the field equations (ibid. p. 687). This approach facilitated the building of Weyl invariant Lagrange densities. As an example he presented a Lagrangian, which was similar to Omote’s Lagrangian (and Dirac’s not yet published one) including an additional torsion term \cite{Bregman1973} equ. (5.2)) \footnote{The torsion term $\frac{2}{3} T^\mu_{\nu\rho} \partial^\rho \sigma^2$ in Bregman’s equation is not scale invariant for itself, but his whole Lagrangian density is.} All in all, this was a remarkable paper which seems to have been underestimated in the following development.

2.3.3. Charap/Tait. About a year later John Charap and W. Tait, London, presented a “gauge theory of the Weyl group” building upon the papers \cite{Yang/Mills1954, Utiyama1956, Kibble1961, Dirac1973}, while Bregman’s paper remained unnoticed by them. They introduced the Weyl group as the “simplest possible non-trivial enlargement of the Poincaré group” \cite{Charap/Tait1974} p. 250), where by “non-trivial” they apparently hinted at the semidirect product operation of $\mathbb{R}^+$ on the translations. Like Bregman before them they explored “the consequences of demanding for a theory of matter fields that it be invariant under the transformations of the Weyl group” (ibid.).

They started by studying the infinitesimal Weyl transformations on Minkowski space endowed with a globally Weyl invariant Lagrangian $L(\chi, \chi')$ depending on a couple of fields $\chi$ and their first derivatives (indicated by $\chi'$). They derived the Noether relations with regard to translations, rotations and dilations without mentioning Noether \footnote{This was characteristic for the time. Over several decades the knowledge about the invariance properties of Lagrangian field theories and the know-how of dealing with it spread with marginal or no reference at all to Noether’s seminal paper \cite{Noether1918}, cf. \cite{Kosmann-Schwarzbach2011}.}. If the Euler-Lagrange equations for all fields are satisfied (“on shell”) conservation laws for expressions corresponding to the symmetries follow (Noether’s first theorem). Most of them could easily be identified with well known physical quantities and were called \textit{canonical currents}, : the canonical energy momentum current $T^\mu$, the canonical angular momentum current $M^\mu_{\nu\lambda}$ and additionally a canonical dilation current $\Delta^\mu$. The latter evaded an immediate physical interpretation. But in analogy to the angular momentum, which can be decomposed into an internal (spin) contribution of the fields and an external,
orbital component, the dilation current could be decomposed into

\begin{equation}
\Delta \mu = J^\mu + T^\mu_{\nu} x^\nu, \quad J^\mu = \frac{\partial L}{\partial (\partial_\mu \chi)} w(\chi) \chi,
\end{equation}

with an internal component \( J^\mu \) (summation of the field components understood to be included) and an external one depending on the origin of the coordinate system in Minkowski space. The external component was, of course, due to the dilational operation of \( \mathbb{R}^+ \) on the underlying space. It will be interesting to see what became of these under “localization” of the symmetries.

In the next step Charap and Tait localized the approach following Kibble’s path; i.e., they made the group operations point dependent by introducing a tetrad field \( h^\mu_\nu \) and gauge fields (geometrically spoken connections), the first one with values in the Lorentz algebra, given by coefficients \( A_{ij}^\mu \) with respect to the algebra generators \( S_{ij} \), and another one with values in the real numbers, given by a system of \( A^\mu \). This allowed to define the scale covariant derivative

\begin{equation}
D^\mu \chi = \partial^\mu \chi + \frac{1}{2} A_{ij}^\mu S_{ij} \chi - A^\mu w(\chi) \chi \quad (\text{Charap/Tait 1974, equ. (3.7)}).
\end{equation}

Like Bregman they considered Lagrange densities \( \mathcal{L} \) constructed from globally Weyl invariant ones, \( L \), by substituting partial derivatives by scale covariant derivatives (“minimal coupling” in the later physics idiom) and using the volume element \( |h|^{-1} dx \) with \( |h| = \det(h^\mu_\nu) \). They analyzed the resulting gauge field dynamics and considered their sources, usually the “right hand side” of the equations, as “modified ‘currents’” \( ^4 \).

Among the localized canonical (Noether) currents only the one referring to energy-momentum was explicitly mentioned by them. In analogy to the global case they defined

\begin{equation}
\Xi^\lambda = \frac{\partial \mathcal{L}}{\partial (\partial_\lambda \chi)} \partial_\mu \chi - \delta^\lambda_\mu \mathcal{L}
\end{equation}

and commented upon its difference from the dynamical energy momentum, arising from variational derivation of the matter Lagrangian with respect to the gravitational field. The latter written with respect to coordinate basis as a “world tensor”, \( T^\lambda_\mu \), can be derived from the canonical energy momentum, \( \Xi^\lambda_\mu \) by adding terms in dynamical spin and dilation quantities. \( ^{22} \)

But neither the corresponding conservation theorem nor the other Noether currents were mentioned, not even the canonical dilation current which was round the corner,

\begin{equation}
\Delta^\mu = -\frac{\partial \mathcal{L}}{\partial (\partial_\mu \chi)} w(\chi) \chi.
\end{equation}

\( ^{41} \Xi^\mu_\nu = \frac{\partial \mathcal{L}}{\partial v^\mu}, \quad \delta^\mu_\nu = -2 \frac{\partial \mathcal{L}}{\partial A^\mu_\nu}, \quad D^\mu = \frac{\partial \mathcal{L}}{\partial A^\mu} \quad (\text{Charap/Tait 1974 p. 256, notation slightly changed}). \) Compare the dynamical currents by Hehl et al. below \( ^{29} \). For the dynamical matter energy current variational and partial derivatives usually coincide. Charap and Tait may have (wrongly) generalized this property to the other currents.

\( ^{42} \Xi^\mu_\nu = \Xi^\mu_\nu - \frac{1}{2} \delta^\lambda_\mu A^\lambda_\nu - \mathcal{D}^\mu A^\nu \quad (\text{Charap/Tait 1974 equ. (3.22))}. \)
In a separate passage on the “geometrical interpretation” of their theory they explained how Weyl’s scale geometry of 1918 was taken up by their approach and how it was generalized by including torsion. Cartan geometry was neither mentioned nor used.

2.3.4. Hehl et al. at the Kiel conference. In the next decade several papers on gravity in a Cartan-Weyl environment appeared; among them [Kasuya, 1975] [Obukhov, 1982] [Nieh, 1982]. Not all of these contained new insights, nor were their statements always reliable; but they indicate a slow broadening of an interest in Weyl geometric gravity with torsion. This interest acquired a wider context with the rise of Cartan geometric metric-affine studies of gravity. A general discussion of this research would go far beyond the limits of this survey. But at least one of the papers of this field has to be commented upon here [Hehl et al., 1988c]. This paper was presented by Friedrich Hehl and coauthors at the Weyl centenary conference at Kiel in 1985. It discussed a view of Cartan geometric metric-affine of gravity with particular emphasis on the Weyl group $W \subset \mathbb{R}^n \rtimes GL(n, \mathbb{R})$ and may be taken as paradigmatic for how Weyl geometric aspects were dealt with in this research program.

In modernized language, it worked in a Cartan geometry modelled after the Klein space $\mathfrak{A}/GL(n, \mathbb{R})$ with the affine group $\mathfrak{A} = \mathbb{R}^n \rtimes GL(n, \mathbb{R})$, respectively the corresponding Lie algebras. Additionally an independently given (Lorentzian) metric $g$ was assumed. The local description involved again an $n$-frame field $h^a_\mu(a, \mu = 0, \ldots, n - 1)$ and its dual system of forms $h^a_\mu$ characterizing a Cartan gauge or the translational connection. Moreover, a connection with values in the Lie algebra of the isotropy group $gl(n, \mathbb{R})$ (generalizing the rotations of Cartan-Riemann geometry) was given by the coefficient system $A^b_\mu{}^a$, and a metric of Lorentzian signature by $g_{\mu\nu} = h^a_\mu h^b_\nu g_{ab}$. Let the corresponding covariant derivative be denoted by $D_a$, respectively $D_\mu$ if transcribed to coordinate indices. Of course, in general the derivative of the metric does not vanish, $D_\lambda g_{\mu\nu} = -Q_\lambda{}^{\mu\nu}$, with $Q_\lambda{}^{\mu\nu}$ usually called the non-metricity of the derivative, which is symmetric in its last two indices. With respect to coordinate bases the linear connection is $\Gamma^\lambda_{\mu\nu} = h^a_\mu h^b_\nu A^b_\mu{}^a_{\nu}$. In the case of a symmetric $\Gamma$ it can be decomposed, $\Gamma = g + q\Gamma$, into its Riemannian (Levi-Civita) component $g\Gamma$ with respect to $g$ and a component due to the non-metricity $q\Gamma$. Also $Q_{\lambda\mu\nu}$ can be decomposed into a traceless part $Q_{\lambda\mu\nu}$, with regard to the last two indices, and its trace $q_\lambda$. Hehl and coauthors introduced what they now introduced as the “celebrated Weyl vector”, the trace part $q_\lambda$ of the non-metricity:

$$q_\lambda = \frac{1}{n} Q_{\lambda\nu}^{\nu} \quad (\text{Hehl et al.}, 1988c, p. 252)$$

In our notation $\varphi_\lambda = \frac{1}{2} q_\lambda$ is a part of the connection and can also be expressed by $\varphi_\lambda = \frac{1}{n} A^a_{a\lambda}$. In the case of vanishing $Q$ the “Weyl vector”

---

43 All of them are mentioned in [Blagojević/Hehl, 2013 chap. 8].
44 In particular F. Hehl and his varying coauthors were active in this field. [Hehl et al., 1976a, 1976b, 1988a, 1989, 1995]. More papers by other authors, while only a few of those just mentioned, appear in [Blagojević/Hehl, 2013 chap. 9].
45 $Q_{\mu\nu}^{\lambda} = \frac{1}{2}(Q_{\mu\nu}^{\lambda} - Q_{\nu\mu}^{\lambda} + Q_{\nu\mu}^{\lambda})$
satisfies the metric compatibility condition of Weyl geometry, our equ. (4). Then also the linear connection $\Gamma$ coincides with the Weylian invariant affine connection. This allowed to embed Weyl geometry in the wider framework of metric-affine geometry.

In this framework the data $(h^\mu_a, A^a_{\mu\lambda}, g_{ab})$ were considered as dynamically independent field components of an extended theory of gravity (Hehl et al., 1988c, sec. 6, notation changed). For a set of matter fields $\Psi$ and a matter Lagrangian $\mathcal{L}_m$ minimally coupled to the affine-metric structure $\mathcal{L}_m(g_{ab}, h^\mu_a, \Psi, D_\mu \Psi)$ the authors defined the dynamical matter hypermomentum current of their generalized theory by the variational derivative with regard to the full isotropy connection,

$$ (29) \quad \Delta_a^{\alpha \mu} = |h| \frac{\delta \mathcal{L}_m}{\delta A^\alpha_a_{\mu}}, \quad \text{with} \quad |h| = \text{det}(h^\mu_a), $$

and decomposed it into its rotational, dilational and shear components (Hehl et al., 1988c, p. 274f.). In the context of the talk, the dynamical dilational current

$$ (30) \quad \Delta^\mu := \Delta_a^{\alpha \mu} = |h| \frac{\delta \mathcal{L}_m}{\delta A^a_{\mu}} = |h| \frac{\delta \mathcal{L}_m}{\delta \phi^\mu}, $$

was of particular importance. The authors conceded that the shear current was “remote from physical experience”; but for the dilational current they saw “supporting evidence” in Bjorken scaling of deep inelastic scattering experiments and, on the theoretical level, in certain models of supergravity (Hehl et al., 1988c, pp. 244, 275). This was a central point for their talk. Already in their abstract they announced:

In the light of modern developments in particle physics, this coupling of the Weyl vector to the material dilation current is an unalterable part in any viable theory of a general-relativistic type, which comprises a Weylian piece (Hehl et al., 1988c, p. 241).

This thesis stood somehow in contrast to the observation of Bregman (who was cited by our authors) that for matter fields with conformally invariant Lagrangians there was no need for assuming their coupling to the scale connection (the “Weyl vector”). But the authors gave an example of a scalar field with non-vanishing dilational current (see below).

Hehl and his coauthors analyzed the dynamics of the metric affine theory on a quite general level with, at first, no particular Lagrangian specified. They described the form of the three dynamical equations corresponding to the decomposition of the general linear group into shear, rotational, and dilational components and argued that only two of them were dynamically independent because of the interdependencies due to the relations of the second Noether theorem (Hehl et al., 1988c, sec. 8). A short discussion of specific Lagrangians followed, among them some consequences of a scale covariant “primordial scalar field, the so-called dilaton field $\sigma(x)$” (Hehl et al. 1988c, p. 282, emph. in original) of Weyl weight $w(\sigma) = -\frac{2}{2}$ and the

26
usual scale invariant quadratic kinetic term

\[ L_\sigma = \frac{|h|}{2} D_\mu \sigma D^\mu \sigma. \]  

As a “primordial” field \( \sigma \) was considered to be a part of the matter sector, and because of the scale covariant derivative in \( L_\sigma \) it contributed to the matter dilational current with

\[ \Delta^\mu(\sigma) = \frac{n - 2}{2n} \sigma D^\mu \sigma. \]  

But the author team warned that one should not expect easy empirical repercussions in laboratory experiments:

Local scale invariance of fundamental interactions is expected to be valid only approximately in the high energy limit of Bjorken scaling or exactly at the onset of the big bang (ibid, p. 285). They assumed a breaking of scale symmetry down to the Poincaré group “after a very short time lag” (to the big bang) and proposed a quartic potential for \( \sigma \) with a symmetry breaking quadratic term similar to the Higgs potential. In the end, their gravitational Lagrangian boiled down to Einstein gravity with cosmological constant “plus some supplementary terms known from Poincaré gauge theory” (ibid, p. 242). By the “supplementary terms” they apparently referred to the torsion-spin coupling which arises in Einstein-Cartan gravity. It becomes a serious modification of Einstein gravity only at extremely high mass densities.

In the framework of Hehl et al., Weyl geometric modifications of gravity were to be expected only under even more extreme condition than for torsion. In their view, Weyl geometric effects seemed to be banned to a speculative realm close to the “big bang”, one of the great adventure playgrounds of late 20th century physics. In the sections 4, 5 we shall see that this need not necessarily be so, if other perspectives are taken into account. But before we turn to these researches, we have to pay attention to another road towards reviving Weyl’s scale geometry. It had different roots from those of the Omote-Dirac-Utiyama and the Cartan geometric approaches discussed in this section and arose from an attempt to geometrize the dynamics of non-relativistic quantum mechanics.

3. Weyl’s scale connection a geometrical clue to quantum mechanics?

3.1. Bohmian mechanics as a background.

3.1.1. Bohm’s “causal” approach to QM. In the early 1950s David Bohm proposed an alternative approach to non-relativistic quantum mechanics (QM) with an often discussed heterodox “causal” interpretation of the latter. His core idea was to reintroduce exact particle trajectories into the

46 According to later estimates the torsion-spin coupling of Einstein-Cartan gravity becomes important only close to \( 10^{38} \) times the density of a neutron star, which signifies energy densities at the hypothetical grand unification scale of elementary particle interactions (Trautman, 2006, p. 194). (Blagojević/Hehl, 2013, p. 108).

47 Bohm, 1952a,b
description of quantum systems, which were guided by a pilot wave evolving according to the Schrödinger equation of ordinary quantum mechanics. In this move he took up earlier ideas of Louis de Broglie on the dualism of wave and particle aspects in QM, which had been critically debated in the late 1920s. This approach was mathematically close to a hydrodynamic picture of the Schrödinger equation, considered by Ernst Madelung in 1926. Madelung noticed that his “hydrodynamical” current was subject to a non-classical term which could be interpreted as a kind of force function due to the “‘inner forces’ of the continuum” (Madelung 1926, p. 323). Bohm extended these older ideas, among others, by an analysis of the measuring process. He could thus avoid to stipulate a “collapse” of the wave function, which was usually assumed for extracting real valued measuring values from the observables given by the Hermitian operators of QM (Bacciagaluppi 2009, chap. 11). Bohm wanted to challenge the mainstream (“Copenhagen”) interpretation of QM which he accepted as consistent but as unsatisfactory from a foundational and natural philosophic point of view. His goal was to find an alternative interpretation of QM which did not affect the dynamics, at least not “in the domain of dimensions of the order of \(10^{-13}\) cm”. It ought to permit

\[
\text{...to conceive of each individual system as being in a precisely definable state, whose changes with time are determined by definite laws, analogous to (but not identical with) the classical equations of motion (Bohm, 1952a, p. 167).}
\]

Bohm started from the observation that to any Schrödinger equation for a wave function \(\psi(x) = a(x)e^{iS(x)} (x \in \mathbb{R}^3)\) governed, e.g. in the case of a single particle of mass \(m\) in an external potential \(V(x)\), by the equation

\[
(33) \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi,
\]

one can associate a pair of coupled differential equations for the phase \(S(x)\) of \(\psi(x)\) and the probability density \(\rho(x) = a(x)^2\):

\[
(34) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \frac{\nabla S}{m} \right) = 0,
\]

\[
(35) \quad \frac{\partial S}{\partial t} + H(x,t) = 0,
\]

where

\[
H(x,t) = \frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{4m} \left( \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right).
\]

(Bohm, 1952a, equs. (6), (7)). Equation (35) has the form of a Hamilton-Jacobi equation for a point particle with the principal function \(S\) and conjugate momenta \(p_k = \partial_k S\), equivalently the velocity \(v = \nabla S / m\). The total potential \(\tilde{V}(x) = V(x) + U(x)\) deviates from the classical \(V(x)\) by

\[
(36) \quad U(x) = -\frac{\hbar^2}{4m} \left( \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right) = -\frac{\hbar^2}{2m} \frac{\nabla^2 a}{a}.
\]

Bohm realized the kinship of his approach to the earlier proposals of de Broglie only after he had finished his manuscript (Bohm, 1952a, p. 167). In a footnote added in proof he also referred to Madelung’s “similar” approach of 1926, adding the remark “...but like de Broglie he did not carry this interpretation to a logical conclusion” (ibid.).
Bohm considered $U(x)$ as a kind of quantum potential added to the classical one. The trajectories of the Hamilton-Jacobi system have velocities $v = \nabla S$ normal to the level surfaces of constant values of $S$. Thus (34) acquires the form of a continuity equation for an ensemble of point particles following the family of trajectories with the density $\rho$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$  

He argued that this might be “the nucleus of an alternative interpretation for Schrödinger’s equation” (Bohm, 1952a, p. 170). A pair of equations similar to (35, 37) had already been investigated by Madelung with a hydrodynamical interpretation. But Madelung was more cautious. He did not consider his equations as more fundamental than Schrödinger’s wave mechanics, but rather as a “model representation” from which one could derive the essential features of the latter (Madelung, 1926).

In Bohm’s alternative interpretation of the equations a quantum particle could seem to be no longer subject to the Heisenberg indeterminacy, because it appeared as though it follows a specified trajectory of the system (35); but the Heisenberg uncertainty was implicitly preserved because only a probability satisfying (37) could be given for a trajectory passing specified regions in the level surfaces. Via the quantum potential (36) a wave function satisfying (33) would operate as a non-local guiding structure, a “pilot wave” in a terminology not used by Bohm, for the motion of the quantum particle. This was quite close to de Broglie’s theory of the 1920s (Bacciagaluppi, 2009).

Although Bohm extended de Broglie’s and Madelung’s view by an analysis of the measuring process, his proposal did not receive immediate positive response in the quantum physics community (Myvold, 2003). Only in the longer run different authors took it up and pursued programs along his lines, although sometimes with a different outlook on the underlying ontology and enriched by new mathematical ideas. Independent of differing views on ontology or mathematical techniques, they belong to common family of de Broglie – Madelung – Bohm (dBMB) approaches.

Important for our context was the stepwise extension of the Bohmian approach to relativistic quantum mechanics, in particular for Klein–Gordon particles given by a complex field of spin zero, $\psi(x) = a(x)e^{i\bar{\varphi}}$. It had values in the complex numbers like a Schrödinger wave function, but lived on the Minkowski space with $x = (x^0, \ldots, x^3)$. Moreover, $\psi(x)$ demanded a more intricate interpretation than Born’s probability rule. In case of an electromagnetic interaction with potential $A_\mu$, the wave field of a Klein-Gordon particle of mass $m$ and charge $e$ satisfies the dynamical equation

$$\left(\frac{\hbar}{i} \partial_\mu - \frac{e}{c} A_\mu\right) \left(\frac{\hbar}{i} \partial_\mu - \frac{e}{c} A_\mu\right) \psi = (mc)^2 \psi$$

49”Die hydrodynamischen Gleichungen sind also gleichwertig mit denen von Schrödinger und liefern alles, was jene geben, d. h. sie sind hinreichend, um die wesentlichen Momente der Quantentheorie der Atome modellmäßig darzustellen” (Madelung, 1926, p. 325).

50 I thank O. Passon for his helpful explanations of Bohmian mechanics. For relativistic generalizations, see, e.g., (Nicolic, 2005).
in the signature \((+−−−)\) of the Minkowski space \((∂_o = c^{-1}\partial_t)\). Here the Bohmian method lead to the Hamilton-Jacobi and continuity equations

\[
(39) \quad \left(∂_μ S - \frac{e}{c} A_μ\right) \left(∂^μ S - \frac{e}{c} A^μ\right) = m^2 c^2 + \hbar^2 Q,
\]

\[
(40) \quad ∂_μ(a^2 ∂^μ(S - e/c A_μ)) = 0,
\]

with a “quantum term” similar to \((36)\). In this context he considered the right hand side of \((39)\) as a kind of “variable rest mass” which had to be calculated in the “immediate vicinity of the particle” \((\text{de Broglie} 1960\text{, p. 116})\):

\[
(42) \quad M_o = \sqrt{m_o^2 + \frac{\hbar^2}{c^2} ∂ν∂^ν a}.
\]

Some authors would later call \(M_o\) the “quantum mass” of a Klein-Gordon field (see subsection \(3.3\)).

3.1.2. A geometrization idea by de Broglie. In his later years de Broglie himself joined in the renewed research program. Among others, he pondered about a connection between the Jacobi flow of the Klein-Gordon field and general relativity \((\text{de Broglie} 1960\text{, pp. 118ff.})\). He defended a hypothesis according to which quantum particles are constituted by extremely dense tiny regions of the field, governed by unknown non-linear equations, while in the exterior of these regions the known linear equations of quantum mechanics hold. He called these regions “singular” and investigated whether the motion of such “singular” regions may follow geodesics similar to the motion of singular regions in general relativity, which had been studied by Einstein and Grommer in the 1920s. In this context he considered the right hand side of \((39)\) as a kind of “variable rest mass” which had to be calculated in the “immediate vicinity of the particle” \((\text{de Broglie} 1960\text{, p. 116})\):

\[
(42) \quad M_o = \sqrt{m_o^2 + \frac{\hbar^2}{c^2} ∂ν∂^ν a}.
\]

De Broglie considered the trajectories \(x_μ(s)\) of the Hamilton-Jacobi flow of a particle “in the absence of electromagnetic and gravitational fields” \((\text{ibid. p. 119, emph. in original})\) with 4-velocity, \(u^μ = \frac{dx^μ}{ds}\), normalized to \(u_μu^μ = 1\),

\[
(43) \quad u^μ = (M_o)^{-1}∂^μ S.
\]

and found that they satisfy the geodesic equations of a metric \(g_μν\) arising from the Minkowski metric \(η_μν\) by conformal rescaling

\[
(44) \quad g_μν = \frac{M_o^2}{m_o^2} η_μν.
\]

He concluded:

Thus, even if the particle is not subjected to any gravitational or electromagnetic field, its possible trajectories (…) are the same as if space-time possessed non-Euclidean metrics defined by \([g_μν]\) \((\text{de Broglie} 1960\text{, p. 120})\).

\(^{51}\text{Cf. (Nicolic 2005, p. 554) for vanishing em potential.}\)

\(^{52}\text{According to de Broglie it was J.-P. Vigier who made him awar of a parallel between his hypothesis and the work of Einstein and Grommer} \((\text{de Broglie} 1960\text{, p. 92})\).\)
This was an interesting geometrization argument and de Broglie did not remain the only one to ponder on a connection between the dBMB quantum mechanics and general relativity. Here we are mainly interested in later authors who tried to make progress by attempting a geometrization of QM in the framework of Weyl geometry.

3.2. Santamato's proposal for geometrizing quantum mechanics.

3.2.1. Two phases of work on the program. In the 1980s Enrico Santamato, Napoli, proposed a new approach to quantum mechanics (Santamato, 1984a,b, 1985). It was based on studying weak random processes of ensembles of point particles moving in a Weyl geometrically modified configuration space. He compared his approach with that of Madelung-Bohm and the stochastic program of Feynès-Nelson. While the latter dealt with stochastic (Brownian) processes, Santamato's approach was closer to the view of Madelung and Bohm because it assumed only random initial conditions, with classical trajectories given in Hamilton-Jacobi form (this explains the attribute “weak” above). One can read Bohm’s particle trajectories as deviating from those expected in Newtonian mechanics by some “quantum force”. Santamato found this an intriguing idea but deplored the latter’s “mysterious nature” which “prevents carrying out a natural and acceptable theory along this line”. He hoped to find a rational explanation for the effects of the “quantum force” by geometry with a modified affine connection of the system’s configuration space. Then the deviation from classical mechanics would appear as the outcome of “fundamental properties of space” (Santamato, 1984a, p. 216), which has to be understood in the sense of configuration space, as we may add.

In his first paper paper Santamato started from a configuration space with coordinates \((q^1, \ldots, q^n)\) endowed with a Euclidean metric. More generally, his approach allowed for a general positive definite metric \(g_{ij}\), and later even a metric of indefinite signature, for dealing with general coordinates of \(n\)-particle systems and perhaps, in a further extension, with spin. The Lagrangian of the system, and the corresponding Hamilton-Jacobi equation, contained the metric explicitly or implicitly. This Euclidean, or more generally Riemannian, basic structure was complemented by a Weylian scale connection. Santamato’s central idea was that the modification of the Hamilton-Jacobi equation induced by a properly determined scale connection can be used to express the quantum modification of the classical Hamiltonian like in the Madelung-Bohm approach. Then the quantum aspects of the systems would be geometrized in terms of the Weyl geometry, surely a striking and even beautiful idea, if it works.

Santamato thus headed towards a new program of geometrical quantization sui generis. It had nothing to do with the better known geometric quantization program initiated more than a decade earlier by J.-N. Souriau, B. Kostant and others, which was already well under way in the 1980s (Souriau, 1966; Kostant, 1970; Simms, 1978). In the latter geometrical methods underlying the canonical quantization were studied. Starting from a symplectic

\[\text{For E. Nelson’s program to re-derive the quantum dynamics from classical stochastic processes and classical probability see (Bacciagaluppi, 2005).}\]
phase space manifold of a classical system, the observables were “pre-quantized” in a Hermitian line bundle, and finally the Hilbert space representation of quantum mechanics was constructed on this basis. Santamato’s geometrization was built upon a different structure, Weyl geometry rather than symplectic geometry, and he had rather different goals.

Like other proposals in the dBMB (de Broglie-Madelung-Bohm) family, Santamato’s program did not find immediate positive response. In the following decades he shifted the center of his research to nonlinear optics of liquid crystals and to quantum optics, even with a strong empirical component, and stopped publishing on the foundational topic. Perhaps a critical paper by Carlos Castro Perelman, a younger colleague who knew the program nearly from its beginnings, contributed to the extended period of interruption? Castro discussed “a series of technical points” which seemed important for Santamato’s program from the physical point of view (Castro, 1992, p. 872). Among the problems he mentioned were several of a more foundational than of purely technical import: (i) the problem of specifying the random initial data for the ensemble of particle paths, (ii) the Hilbert space interpretation of the theory, (iii) the relationship of Santamato’s approach to the Feynman path integral quantization. He also criticized the lack of a rigorous hypothesis in the choice of the particle’s Lagrangian, the non-definite character of the probability density in the case of a Klein-Gordon particle (which could appear if the foliation with respect to the principal function $S$ is not timelike), the un-understood dependence of the particle’s effective mass on the Weylian scalar curvature (in the configuration space), and some other more technical points.

After the turn to the new century/millennium Santamato came back to foundational questions in close cooperation with his colleague Francesco De Martini from the University of Rome. Both had cooperated in quantum optics already for many years. In the 2010s they turned to geometrical quantization in a series of joint publications and continued the program started by Santamato three decades earlier. They showed how to deal with spinor fields in this framework, in particular with the Dirac equation (Santamato/DeMartini, 2013) and discussed the famous Einstein-Podolsky-Rosen (EPR) non-locality question (De Martini/Santamato, 2014a). Moreover, they analyzed the helicity of elementary particles and showed that the spin-statistics relationship of relativistic quantum mechanics can be derived in their framework without invoking arguments from quantum field theory (De Martini/Santamato, 2015, 2014c, 2016). In this new series of papers...
Minkowski space formed the starting point for the construction of the configuration spaces which could be extended by internal degrees of freedom. Moreover, a transition from point dynamical Lagrangians as the dominant view to a dynamically equivalent description in terms of scale invariant field theoretic Lagrangians in two scalar fields enlarged the perspective (see below).

It is unnecessary to go into details of these often quite technical articles; here I concentrate on the basic question of the geometrization program. Here we want to see how Santamato’s intriguing idea of introducing a Weyl geometric structure on the configuration space, in order to model the Bohmian effects of quantum systems, works.

3.2.2. The geometrization of the configuration space. Summing up, Santamato’s idea was to consider dynamical systems with finite degrees of freedom, parametrized by a configuration space $V$ with parameters $q^1, \ldots, q^n$, endowed with a pseudo-Riemannian metric $g_{ij}$ which could be of any signature (Santamato 1984a). In the case of a non-relativistic $k$-particle system without inner degrees of freedom it could be the product of Euclidean 3-metrics, for relativistic particles in Minkowski space with metric $\eta = \text{diag}(-1, 1, 1, 1)$ it was of signature $(3k, 3)$ (Santamato 1984b).

In the case of a relativistic 1-particle system with spin the product of the Minkowski space $\mathbb{M}$ and the Lorentz group served as configuration space, $V = \mathbb{M} \times SO(3, 1)$, where the second factor parametrizes “hidden” rotational degrees of freedom of the particle (Santamato/DeMartini 2013, p. 634). By an astute choice of coordinates $(q^1, \ldots, q^n) = (x^\mu, \theta^\alpha)$ (with $\alpha = 1, \ldots, 6$) in $V$, with generalized “Euler angles” $\theta^\alpha$ for parametrizing $SO(3, 1)$, the authors introduced a metric $(g_{ij})$ by a block matrix composed of the Minkowski metric $\eta_{\mu\nu}$ and a “metric of the parameter space of the Lorentz group” $g_{\alpha\beta}$ with signature $(+++-)$. A frame given by $e^a_\mu$ can be characterized by the Lorentz transformation $\theta$ which transforms the standard basis into the given one, which may now be written as $e^a_\mu(\theta)$. The metric on the Lorentz group component was derived from the group operation on the frames, by measuring the Minkowski squared norm induced by infinitesimal rotations of the Euler angles (summation over all frame vectors):

\begin{equation}
    g_{\alpha\beta}(\theta) = -a^2 \eta_{\mu\rho} \eta_{\nu\sigma} \omega^{\mu\nu}_{\rho\sigma}(\theta) \omega^{\rho\sigma}_{\alpha\beta}(\theta)
\end{equation}

with

\begin{equation}
    \omega^{\mu\nu}_{\alpha\beta}(\theta) = g^{\rho\nu} e^a_{\rho}(\theta) \frac{\partial}{\partial \theta^\alpha} e^a_\mu(\theta)
\end{equation}

The factor $a^2$ was not mentioned at this place and only made explicit by the authors in passing elsewhere (De Martini/Santamato 2014a, p. 3313). Here $a$ expressed the gyromagnetic radius of a relativistic top $a = \sqrt{\frac{\hbar}{2m}}$ and was important, because due to it the geometry would “know” about the mass of the spinning particle. For the metric they found a constant Riemannian scalar curvature $gR = \frac{6}{a^2} = \frac{(mc)^2}{\hbar^2}$ induced from the Lorentz component (De Martini/Santamato 2014a, p. 3313).
This was a surprising Riemannian geometrization of the configuration space of the, up to here, non-quantum, relativistic top by a non-definite metric $g_{ij}$ of signature $(3 + 3, 1 + 3)$ with constant scalar curvature.

3.2.3. Santamato’s random processes in the 1980s. At first, the particle motion in the configuration space had to be analyzed. Santamato characterized it as a (weak) random process described by an ensemble of trajectories $q^i(t, \omega)$ with $\omega$ “the sample tag” and a well defined and normalized probability density $\rho(q, t)$ satisfying the continuity equation (Santamato, 1984a, p. 217)

$$\partial_t \rho + \partial_i (\rho v^i) = 0.$$ (46)

He gave a peculiar derivation for the velocity field $v^i$ of his random process associated to a given Lagrangian $L(q, \dot{q}, t)$. After shifting the Lagrangian to $L^* = L + \frac{d}{dt}S$ for some sufficiently differentiable function $S(q, t)$ he analyzed the averaged action functional

$$I(t_0, t_1) = E \left( \int_{t_0}^{t_1} L^*(q(t, \omega), \dot{q}(t, \omega), t) dt \right)$$ with $E(\ldots)$ the expectation value. He looked for the minimum of $I$ under variation of $v^i = \dot{q}^i$, with respect to all random motions obeying a flow equation and satisfying given initial data. As a necessary condition for the existence of such a minimum it turned out that $S$ has to solve the Hamilton-Jacobi equation (Santamato, 1984a, app. A)

$$\partial_t S + H(q, \nabla S, t) = 0,$$ (48)

with $H(q, p, t)$ the classical Hamiltonian corresponding to $L(q, \dot{q}, t)$. Then the minimizing velocity field of (47) is the corresponding Hamilton-Jacobi flow.

Santamato could hope that the wave equations of QM might be derivable from his random processes if a classical Lagrangian (if there is any) was modified in a convincing way. “Convincing” would mean for him a change of the Lagrangian by geometrical terms, where the geometry is influenced by the particle’s (random) motion.

Geometry is not prescribed; rather it is determined by physical reality. In turn, geometry acts as a “guidance field” for matter. (Santamato, 1984a, p. 216)

He argued that such a “feedback mechanism between geometry of space and particle motion” was “quite analogous” to general relativity and might lead to “a theory that is physically indistinguishable from traditional quantum mechanics” (ibid.).

At this point Santamato complemented the originally Euclidean, or more generally Riemannian, basic structure of the configuration space by a Weylian scale connection. He called it a “vector transplantation law” and denoted it

57That is, $\omega \in \Omega$, the sample space of a probability triple $(\Omega, \mathcal{F}, P)$, where $\mathcal{F} \subset \mathcal{P}(\Omega)$ are the random events and $P$ is a probability measure on $\Omega$.

58$L^*$ has has the same Euler-Lagrange equations as the original $L$. 34
by \( \varphi_k \), corresponding to our \(- \varphi_k \). In the case of a non-Euclidean Riemannian component of the metric \( g_{ij} \) the continuity equation for the adapted probability density \( \rho = |g|^{-\frac{1}{2}} \rho \) turns into the covariant equation:

\[
\partial_t \rho + g^i_{\nu} (\rho v^\nu) = 0
\]

A classical Lagragian \( L_c(q, \dot{q}, t) \) on the original non-relativistic configuration space was then modified on the Weylianized space according to Santamato’s 1-st postulate (Santamato, 1984a, equ. (8)):

\[
L(q, \dot{q}, t) = L_c(q, \dot{q}, t) + \gamma \frac{\hbar^2}{2m} R(q, t), \quad \text{with} \quad \gamma = \frac{n - 2}{4(n - 1)},
\]

where \( R(q, t) \) denotes the complete Weylian scalar curvature. With a sign inverted convention for the scalar curvature (2), Santamato wrote it as

\[
R = gR + (n - 1)(n - 2) \varphi_i \varphi^i - 2(n - 1) g_{ij} \varphi^j.
\]

The term in \( R \) enters \( \delta S \) like an add on to the potential. The Hamilton-Jacobi equation of the random flow \( (48) \) thus becomes

\[
\partial_t S + H_c(q, \nabla S, t) - \gamma \frac{\hbar^2}{m} R = 0.
\]

According to our author’s program, \( R \) should depend on the random process and was assumed to be time dependent. Therefore the \( \varphi_k \) cannot be arbitrarily given but ought to be determined by the probability density of the matter flow in the configuration space. Santamato applied his averaged least action principle \( (47) \) another time and evaluated it with \( (50) \) for vanishing \( L_c \), i.e., for \( R(q, t) \) alone, with the encouraging result (ibid. equ. (19))

\[
\varphi_i = - (n - 2)^{-1} \partial_i \ln \rho.
\]

Then the scalar curvature \( R = gR + \varphi R \) turned out to contain (Santamato, 1984a, equ. (20))

\[
\varphi R = \frac{1}{\gamma \sqrt{\rho}} \left( g_{ij} \partial_i \sqrt{\rho} \right).
\]

This form stood in striking accord with Bohm’s quantum potential \( (36) \). Santamato jumped without hesitation from the recognition of the formal agreement to a realistic conclusion:

\[\text{...according to Eq. (53), the geometric properties of space (...) are indeed affected by the presence of the particle itself. In turn, this alteration of geometry of space acts on the particle through the quantum force } f_i = \gamma \frac{\hbar^2}{m} \partial_i R, \text{which, according to Eq. (51), depends on the gauge vector and its first and second derivatives. (Santamato, 1984a, 219, equ. numbers adapted)}\]

This was a strong statement. It suggested a close kinship of Santamato’s modification of geometry to the one in the general theory of relativity (GR), although his modification did not refer to the spacetime manifold of GR, the “extensive medium of the world” as Weyl liked to formulate, but to the configuration space of a dynamical system.

\[\text{Cf. fn 24.}\]
In his next paper Santamato derived the Klein-Gordon equation (38) in the same way starting from a random process. He used a configuration space arising from Minkowski space $\mathbb{M}$ by superimposing a Weylian scale connection (51). Including electromagnetic terms his Lagrangian for the relativistic ensemble was

$$L(x, \dot{x}) = \left(1 + \frac{1}{\gamma \hbar^2 (mc)^2} R(x)\right)^{1/2} |x| + \frac{e}{mc^2} A_\mu \dot{x}^\mu,$$

with $R = gR$ the Weylian scalar curvature ($gR = 0$). Taking into account equ. (54) it followed that a complex function $\psi = \sqrt{\rho} e^{iS}$ constructed as usual (up to a factor $\hbar^{-1}$ in the exponent) from the flow quantities “obeys the Klein-Gordon equation” (Santamato, 1984b, p. 2479). This was no small achievement; but Santamato did not continue his research along these lines for many years.

3.2.4. A look at the second phase in cooperation with De Martini. After a long interruption Santamato, now in joint work with his colleague De Martini, gave a new derivation for the Weyl geometric approach to the foundations of quantum mechanics. Moreover, in the new series of papers we find a much clearer emphasis on the underlying scale co-invariant structure. The paper (De Martini/Santamato, 2014a) started from a field theoretic Lagrangian in a metric-affine approach (cf. pp. 25ff.). It involved two scalar fields $\rho, \sigma$ with weights $w(\rho) = -2, w(\sigma) = 0$ under conformal rescaling and a scalar curvature term $R$ defined with regard to a metric $g_{ij}$ and an independently defined affine (torsion free) connection $\Gamma^k_{ij}$. $\sigma$ now took over the role of the former Hamilton-Jacobi principal function $S$ (De Martini/Santamato, 2014a, equ. (1)).

$$\mathcal{L} = \rho (\partial_\mu \sigma \partial^{\mu} \sigma + \gamma \hbar^2 R) \sqrt{|g|}.$$  

Variation with regard to the scalar fields leads to the dynamical equations:

$$\partial_\mu \sigma \partial^{\mu} \sigma + \gamma \hbar^2 R = 0$$  

$$\partial_\mu \left( \sqrt{|g|} \rho \partial^{\mu} \sigma \right) = 0 \quad \leftrightarrow \quad g^{\mu\nu} \partial_\mu (\rho \partial_\nu \sigma) = 0$$

(57) has the same form as the Hamilton-Jacobi equation of an uncharged Klein-Gordon field (39), where the scalar curvature, up to sign, takes the place of the “quantum potential”. (58) may be read as a continuity equation for a flow with density $\rho$ and velocity given by $\partial^\mu \sigma$ (if timelike). By variation with regard to the affine connection [60] the authors concluded that the affine connection has the Weyl geometric form (1) with the scale connection as in (53), just like the one Santamato had derived in the 1980s from his average action principle [61].

The authors did not consider a variation of the metric because they had a de Broglie–Madelung–Bohm context in mind in which the Riemannian metric of the configuration space was determined by the Lagrangian of a classical system. They immediately turned to it by a mechanical interpretation of

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[60] Compare subsection 5.4.1

[61] A sign error in the formula of the Weyl geometric affine connection (De Martini/Santamato, 2014a, equ. (4)) notwithstanding.
their scalar field theory. In the relativistic case the equations \(^{(57)}, (58)\) can be derived just as well as the Hamilton-Jacobi and continuity equations of a variational problem \(\delta \int L \, d\tau = 0\) with

\[
 L_r = \sqrt{-\gamma \hbar^2 R(q)g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu}
\]

This fits well to the program of geometrizing a configuration space with Riemannian metric related to a classical process, which is amended by a Weylian scale connection standing in “backreaction” with a solution pair \((\sigma, \rho)\) of \((57), (58)\).

With \(\gamma = \frac{n-2}{4(n-1)}\) like above, and \(n = 4\), the Weylian component of the scalar curvature is in fact

\[
(60) \quad \varphi R = \frac{1}{4\gamma} (2\rho^{-1} \sqrt{|g|} \partial^i \rho - \rho^{-2} \partial_i \rho \partial^i \rho) = \gamma^{-1} \frac{\sqrt{|g|} \partial^i \sqrt{\rho}}{\sqrt{\rho}}
\]

\[
(61) \quad = \gamma^{-1} \frac{2m}{\hbar^2} U = \gamma^{-1} Q,
\]

where \(U\) and \(Q\) are the additional terms (“quantum potentials”) \((36), (41)\) on the right hand side of the Hamilton-Jacobi equations of a Schrödinger, respectively a Klein-Gordon particle. It has to be understood that \((59)\) holds only for relativistic particles (Klein-Gordon and Dirac), while for the non-relativistic case of a Schrödinger particle with \(R = \varphi R\) the Lagrangian is \((50)\).

For investigating relativistic spinning particles De Martini and Santamato considered a point dynamics with internal degrees of freedom in the configuration space \(V = \mathbb{M} \times SO(3,1)\) described in subsection 3.2.2. The Hamilton-Jacobi equation of a process governed by the Lagrangian \((59)\) plus an electromagnetic term \(L_{em}\) is given by \((\text{Santamato/DeMartini, 2013, equ. (7)})\)

\[
(62) \quad (\partial_\mu S - \frac{e}{c} A_\mu)(\partial^\nu S - \frac{e}{c} A^\nu) + \hbar^2 \gamma R = 0,
\]

where \(S\) satisfies the divergence equation

\[
(63) \quad D_\mu (\partial^\mu S - \frac{e}{c} A^\mu) = 0
\]

with the scale covariant derivative, here \(D_\mu = \nabla_\mu - 2\phi_\mu\) in our weight convention with \(w(g_{\mu\nu}) = 2\), and with \(\nabla_\mu\) the Weyl geometric covariant derivative. For a current defined by \(j^\mu = \chi^{-(n-2)} \sqrt{|g|} (\partial^\mu S - \frac{e}{c} A^\mu)\) this boils down to an ordinary continuity equation

\[
(64) \quad \partial_\mu j^\mu = 0.
\]

Obviously \(j^\mu\) is scale invariant.

The transition to a complex wave function depending on all coordinates \(q\) of the configuration space

\[
(65) \quad \psi(q) = \sqrt{\rho} e^{\frac{i}{\hbar} S}
\]

\[\text{De Martini/Santamato 2014a p. 3310}\]
transforms the equs. (62), (64) into the linear differential equation of second order
\[(\hat{p}^\mu - \frac{e}{c} A^\mu)(\hat{p}_\mu - \frac{e}{c} A_\mu) \psi + \hbar^2 \gamma_\mu \gamma_R \psi = 0,\]
where \(\hat{p}\) denotes the differential operator with \(\hat{p}_\mu = -i\hbar \partial_\mu\). It has the form of the Klein-Gordon equation (38) with a mass factor which contains only the Riemannian part of the scalar curvature, \(\hbar^2 \gamma_\mu \gamma_R = \hbar^2 \gamma_6 = m^2 c^2\). The Weylian component \(\phi^R\) is controlled via (53) by the density of the quantum flow. The authors commented

This is a striking result as it demonstrates that the Hamilton-Jacobi equation, applied to a general dynamical problem can be transformed into a linear eigenvalue equation, the foremost ingredient of the formal structure of quantum mechanics and of the Hilbert space theory. (Santamato/DeMartini, 2013, p. 636)

The first step towards a reconstruction of the Hilbert space quantization of the relativistic top was achieved.

In the next step the authors analyzed the decomposition of a solution \(\psi\) of (66) into components \(\psi_{u,v}\) lying in finite dimensional representations of \(SO(3,1)\) of type \(D^{(u,v)}\) with \(2u, 2v \in \{0, 1, 2, \ldots\}\). Then \(\psi_{u,v}(q)\) can be factorized into functions of the spatial variable \(x\) with values in the representation space of \(D^{(u,v)}\), and \(\theta\)-dependent representation matrices operating on the latter; in spinor notation similar to van der Waerden’s symbolism:

\[
\psi_{u,v}(q) = D^{(u,v)}(\Lambda(\theta))_\sigma^\sigma' \psi_{\sigma'}^\sigma(x) + D^{(v,u)}(\Lambda(\theta))_\sigma^\sigma' \psi_{\sigma'}^\sigma(x)
\]

For the particular choice \(u = v = \frac{1}{2}\) this leads to a pair of 2-component spinor field on Minkowski space, equivalent to a 4-component Dirac field \(\Psi(x) = \begin{pmatrix} \psi_{\sigma'}^\sigma(x) \\ \psi_{\sigma'}'(x) \end{pmatrix}\) in the Weyl representation. Then the equation (66) acquires a form which, after neglecting an extremely small term in the electromagnetic field strength,

\[
D_+ D_- \Psi = D_- D_+ \Psi = 0,
\]

where \(D_\pm = \gamma^\mu (p_\mu - \frac{e}{c} A_\mu) \pm m\)

with the Dirac matrices \(\gamma^\mu (\mu = 0, \ldots, 3)\) (Santamato/DeMartini, 2013, p. 639).

Solutions of (68) can be decomposed into a superposition of the linear Dirac equation with positive and with negative mass. The authors proposed

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63 For van der Waerden’s spinor symbolism see [Schneider, 2011].
64 This term, \(\frac{e^2}{2c^2}(H^2 - E^2)\), is comparable with the linear term in the field strenghts only under the condition of very large field strenghts, \(E \sim 10^{18} V m^{-1}, H \sim 10^9 T\). “To have an idea how large is this field, an electron at rest is accelerated by such field up to \(10^9 GeV\) in a linear accelerator 1 m long” (De Martini/Santamato, 2014a, p. 3315).
65 It remains unclear to me (E.S.) how the representation matrices of the “Euler angles” of configuration space are suppressed, while the change of coordinate frames in Minkowski space gets represented on the spinor fields.
that the negative mass contributions have to be “disregarded as unphysi-
cal” (Santamato/DeMartini, 2013, p. 641). Even without trying to assess
this proposal, it is clear that by this model of relativistic spinning particles
Santamato and De Martini had achieved a surprising step forward for the
geometrization program of the dBMB approach started in the 1980s.

They did not stop here, but went on by investigating the nonlocality of
EPR systems in their approach. Their considerations led to a justification of
the spin-statistic relation which usually is derived by quantum field methods
(De Martini/Santamato, 2014a, 2015). In order not to blow our survey
these derivations, although central for the content of their papers, have to
be shunted here.

3.3. An attempt at bridge building to gravity. We still have to review
attempts at connecting the dBMB approach to gravity with a specific refer-
tence to Weyl geometry. Different authors tried to do so. The main thrust
in this direction was developed independently of the two Italian authors by
Fatimah Shojai, Ali Shojai and Mehdi Golshani working at Tehran. An-
other, to my taste slightly more bizarre, step in this direction was made by
Giorgio Papini and Robert Wood at the occasion of a symposium honour-
ing J.-P. Vigier (Wood/Papini, 1997). Some years earlier they had tried to
fix a defect of Dirac’s 1972 proposal to revive Weyl’s original interpretation
the scale connection as the electromagnetic potential, resulting from the
non-integrability of the scale connection.66 Papini and Wood proposed to
solve this problem by considering “bubbles” in the environment of atoms,
in which the scale symmetry is broken, while it holds in the large, outside
the “bubble” (Wood/Papini, 1992). For the Vigier symposium they recycled
their idea by establishing a connection to a dBMB approach governing the
dynamics in the bubble, similar to de Broglie’s proposal.

At the end of the 1990s F. and A. Shojai, sometimes coauthored by Gol-
shani, started with investigations of their own, in which they hoped to be
able to use a Bohmian approach for a peculiar way of quantizing a part of the
gravitational structure (Shojai et al., 1998a, b, c; Shojai/Golshani, 1998; Sho-
jaï/Shojai, 2000). To do so they used methods from scalar-tensor theories
of gravity. A specific emphasis of conformal ideas brought their approach
close to Weyl geometry. During a sojourn at the Max Planck Institute for
Gravitational Physics at Potsdam they laid this connection open and pro-
claimed it as the correct framework of their approach (Shojai/Shojai, 2003).
We want to see what that meant.

In the 1980s Jayant Narlikar and Thanu Padmanabhan had started to
study a simplified version of quantum gravity which was invariant under
conformal changes of the metric, $g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$. They proposed to quantize
only the factor $\Omega$, viz. the scale degree of freedom of the metric. This had the
great advantage of keeping the conformal structure unaffected by the quan-
tization and circumvented the infamous obstacle of a fuzzy causal structure,
which other approaches towards quantum gravity encountered. On this basis

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66 Among others, this had led to the measurement problem by atomic clocks.
Narlikar and Padmanabhan calculated semiclassical approximations for cosmological solutions of the Einstein equation \cite{NarlikarPadmanabhan1983,Padmanabhan1989}. In one of their early joint papers F. Shojai and M. Golshani took this idea up. In contrast to Narlikar and Padmanabhan they attempted a Bohmian path towards quantizing the scale factor \cite{ShojaiGolshani1998}. This was quite daring because Bohmian quantum mechanics had been developed for systems of finite degrees of freedom only. Shojai and Golshani, however, invoked the idea of de Broglie to re-interpret the “quantum mass” $M_2 = m^2 + \frac{\hbar^2}{m^2} Q$ of a Klein-Gordon system as a conformal modification of the Minkowski metric using $\Omega^2 = \frac{\hbar^2}{m^2} - 1 + \frac{\hbar^2}{m^4 c^2} \nabla_\mu \partial_\mu \sqrt{\rho} \sqrt{a}$. They considered this rescaling factor as a representative for the quantum degrees of freedom of a globally defined Klein-Gordon field.

Another problem was that $M_2$ could become negative. The Shojais and Golshani solved it by passing over to the exponential \cite{ShojaiGolshani1998, equ. (12)}

\begin{equation}
M^2 = m^2 e^{\frac{\hbar^2}{m^4 c^2} \nabla_\mu \partial_\mu \sqrt{\rho} \sqrt{a}}.
\end{equation}

The linear approximation coincides with $M^2_0$, and the conformal factor became

\begin{equation}
\Omega^2 = \frac{M^2}{m^2} = e^{\frac{\hbar^2}{m^4 c^2} \nabla_\mu \partial_\mu \sqrt{\rho} \sqrt{a}}.
\end{equation}

They started from a Lagrangian with Einstein-Hilbert term and a matter Lagrangian in terms of a Hamilton-Jacob function $S$ and flow density $\rho$,

\begin{equation}
\mathcal{L}_m = \frac{\hbar^2}{m} \left( \frac{\rho}{\hbar^2} \partial_\mu S \partial^\mu S - \frac{m^2}{\hbar^2} \rho \right) \sqrt{|g|},
\end{equation}

characteristic for a classical Jacobi-Hamilton system. Santamato’s viewpoint (which was apparently unknown to the Tehran authors) had been that the introduction of the quantum potential turned the corresponding dynamical system into the Hamilton-Jacobi form of a Klein-Gordon system. The Shojais proceeded differently. Sustained by de Broglie’s argumentation they argued:

\ldots the de Broglie remark leads to the conclusion that the introduction of the quantum potential which contains the quantal behaviors of the particles is equivalent to the introduction of a conformal factor $\Omega^2 = \frac{\hbar^2}{m^2} \nabla_\mu \partial_\mu \sqrt{\rho} \sqrt{a}$ in the metric \cite{ShojaiGolshani1998, p. 683, emphasis E.S.}.

This was a puzzling statement. De Broglie had considered a geometrization for a single particle in the absence of electromagnetic and gravitational fields \cite{subsection 3.1.2}. It remained unclear whether the argument could be transferred to the case of gravitational fields and in which sense such an “equivalence” was to be understood.

\footnote{In the physics literature, so also in the paper by Shojai and Golshani, $\Omega$ is often talked about as the “conformal” degree of freedom of the metric, or even the “conformal structure”. The latter is clearly mistaken, the first one at least misleading. Therefore I avoid this terminology in favour of scaling degree of freedom.}
In Santamato’s geometrization of a dBMB Hamilton-Jacobi system the “prepotential” of a Weylian scale connection on the configuration space \( \rho \) was \( \ln \rho \), up to a constant factor. It leads to the Weylian curvature expression \( \pi \) equivalent to a Bohmian “quantum potential” in the dynamical equation \( \Sigma \). De Broglie and with him the Shojais used a different geometrization idea. Their “prepotential” of the Weylian scale connection was the scale factor \( \Omega \) between a classical metric and the metric describing a quantum system. In the work of our authors it was the exponential expression \( \Theta \). Following de Broglie, one had to consider geodesic flows with the implicit constraint of orthogonal initial conditions to a level surface of a related Hamilton-Jacobi principal function \( S \). The modification \( \Theta \) of the usual “quantum mass” formula implies that we cannot expect equivalence in the literal sense. Even if one wants to read the argument as a motivation for a new type of dBM-like quantization procedure, following the de Broglie paradigm, a justification for the attempted generalization from de Broglie’s case (no gravitation) to the general case considered had to be given.

But our authors did not hesitate to take this step as a starting point for investigating cosmological models in which matter fields were given in different versions of scalar tensor theories\(^{68}\). In the result a Klein-Gordon field appeared on large scales, rather than as a descriptor of the motion of a single quantum particle. At some places it played the role of a matter field (Shojai/Golshani, 1998 p. 683), (Shojai, 2000a p. 1762), at others that of a “quantum gravity” modification of the metric field (Shojai et al., 1998a p. 2728). The Shojais were convinced

...that the theory works for a particle as well as for a real ensemble of the particle under consideration and that it includes pure quantum gravity effects (Shojai/Shojai, 2000, 1763).

But it remained unclear what “quantum gravity” would mean here.

One of the papers dealt explicitly with conformal transformations in scalar-tensor theories (Shojai et al., 1998a).\(^{68}\) The three authors distinguished between a “background metric”, in which they considered the quantum effects being encoded by the varying “quantum mass” \( M \), while in a “physical metric” \( \bar{M} \) was rescaled to a constant value \( \bar{m} \). Then “some part of the curvature of space-time represent the quantum effects” (Shojai et al., 1998a p. 2726). Independent of the physical interpretation and reasonability of this and some other observations one might wonder whether a reformulation in Weyl geometry could at least help to clarify the mathematical side of such statements.

This is what A. and F. Shojai attempted in (Shojai/Shojai, 2003) and a following preprint (Shojai/Shojai, 2004). In the meantime they had adopted Dirac’s theory of 1972 (see section 2.2.1), but did not follow Dirac’s em dogma. They rather considered the scale connection as “a part of the geometry of the space-time”, implicitly constrained in their context by the

\(^{68}\)Shojai/Golshani, 1998; Shojai et al., 1998a, b; Shojai/Shojai, 2000, 2001.

\(^{69}\)The authors made a difference between “scale transformations” and “conformal transformations”. In their terminology the first operated only on the metric, while the latter rescaled all physical fields according to their weights.
integrability condition. But without much hesitation they declared that Dirac’s scalar field $\beta$ in (15) “represents the quantum mass field” in the sense of their embryonic theory outlined above (Shojai/Shojai 2003, p. 7 preprint). They did not discuss how the different Lagrangians for the Dirac field and their Klein-Gordon field could be related to each other. Only a rather opaque perturbative argument was given as to why a solution of the $\beta$-scalar field equation may be identified with an expression of the “quantum mass” type, $\beta \mapsto M$ (Shojai/Shojai 2003, p. 13f). On the other hand, this identification allowed to clarify their discussion of different frames a little. They now considered “different conformal frames” as “identical pictures of the gravitational and quantum phenomena” (ibid., p. 9).

In the light of such open spots A. and F. Shojai’s conclusion that Weyl geometry “provides a unified geometrical framework for understanding the gravitational and quantum forces” (Shojai/Shojai 2003, p. 10 preprint) was at least premature and reads like too grand a speculation. Not all readers had this impression. Their program found at least one active successor, R. Carroll (Carroll, 2004). But the critical points of justification for the “Tehran” program seem not to be clarified in this work either.

4. Scale covariance in the standard model of elementary particle physics

About the middle of the 1970s the standard model of elementary particle physics (SM) started to become widely accepted as the key to the basic structures of matter (Kragh 1999 chap. 22), (Pickering 1988). Besides the point dependent (localized) internal symmetries of the electroweak forces, $SU(2) \times U(1)$ and the chromodynamic symmetry of the strong forces $SU(3)$ the new paradigm of gauge field theories worked with non-localized (“global”) external symmetries of special relativity, the Lorentz group. Characteristic for the paradigm was a global but only nearly respected scale invariance of the field Lagrangians, broken only by the mass term of the Higgs field. The Higgs field $\Phi$, a scalar field with values in an isospin $\frac{1}{2}$ representation of the electroweak group, was the clue for making electroweak symmetry of elementary particles consistent with mass terms. The latter was understood as a “spontaneous breaking” of the electroweak symmetry and became to be known as the “Higgs mechanism” (Borrelli, 2015). In this section we look at some attempts for bridging the gap between the Higgs field and the scalar field of gravity.

4.1. Englert, Smolin and Cheng, 1970/80s.

4.1.1. A conformal approach. One of the originators of this theory (Higgs mechanism), François Englert tried to play a similar game of “spontaneous
symmetry breaking” in gravity, here with a real valued scalar field with scale symmetry in the sense of conformal rescaling.

In a common paper written with Edgar Gunzig, C Truffin and P. Windey, the authors established an explicit link to JBD gravity (Englert et al., 1975). But in contrast to (Deser, 1970), Englert and coworkers considered conformal gravity as part of the quantum field program. They assumed a “dimensionless”, i.e. scale invariant, Lagrangian for gravitation with a square curvature term of an affine connection \( \Gamma \) not bound to the metric, 
\[
\mathcal{L}_{\text{grav}} = R^2 \sqrt{|\det g|},
\]
in addition to a Lagrangian matter term (Englert et al., 1975). In consequence, the authors varied with respect to the metric \( g \) and the connection \( \Gamma \) independently.

“To make contact with General Relativity” (p. 74) the authors assumed the scalar curvature as expressed by a scalar function, \( R \sim \omega^2 \) (they used the symbol \( \varphi \) instead of \( \omega \)). The Euler-Lagrange equation of the affine connection resulted in a relation like \( 1 \) for the Weyl geometric case, with an integrable integrable scale connection \( \varphi = d \log \omega \) (Englert et al., 1975, equs.(7), (8)). By such a specialization, their approach looked as though it was touching upon a Weyl structure. But this was not the point of view of the authors; they rather proceeded as “conformal” as possible on their search for connecting paths between quantum field theory of scalar fields and general relativity.

After some tentative quantum considerations the authors came back to a “classical phenomenological description” of their theory (Englert et al., 1975, 76). For this description they introduced a scalar field \( \phi(x) = \lambda^{-1} e^{\lambda \sigma(x)} \) coupled to gravity like in our equ. \( 12 \), with the necessary specification \( \xi = \frac{1}{6} \) in order to secure conformal symmetry (Englert et al., 1975, equ. (16)). They considered \( \sigma \) to be a “dilation field” (sic!) which represented a “Nambu-Goldstone boson” coupling to the mass terms.

After some turns and twists they summed up that their original action principle

\[ \ldots \text{matches all the results of General Relativity at a classical level, provided mass originates in dynamical breakdown of symmetry. Thus, the fundamental finite component fields must be massless and of the kind currently used in gauge field theories, but without scalar mesons (Englert et al., 1975, 76).} \]

In one of the following papers Englert, now with Truffin as only coauthor, studied the perturbative behaviour of his version of conformal gravity (\( \xi = \frac{n-2}{4(n-1)} \)) coupled to massless fermions and photons in \( n \geq 4 \) dimensions. He came to the conclusion that anomalies arising in the calculations for non-conformal actions disappeared at the tree and 1-loop levels in their approach. The two authors took this as an indicator that gravitation might perhaps arise in a “natural way from spontaneous breakdown of conformal invariance” (Englert et al., 1976, 426).

4.1.2. Smolin introduces Weyl geometry. Englert’s e. a. paper was one of the early steps into the direction (i) of our introduction. Other authors

\[ \text{75 The motivation or considering } n \geq 4 \text{ was the method of dimensional regularization for the quantization of the theory.} \]
followed and extended this view, some of them explicitly in a Weyl geometric setting, others continued to use the language of conformal geometry. The first strategy was chosen by Lee Smolin in his paper (Smolin, 1979). In section 2 of the paper he gave an explicit and clear introduction to Weyl geometry. The “conformally metric gravitation”, as he called it, was built upon a matter-free Lagrangian built from Weyl geometric curvature terms $R, \text{Ric} = (R_{\mu\nu}), f = (f_{\mu\nu})$ for scale curvature and used a gravitational Lagrangian of order two. In a slight adaptation of notation using the scale covariant Weylian derivatives $D$ it was (Smolin, 1979, equ. (13)):

\begin{equation}
|\text{det} g|^{-\frac{1}{2}} \mathcal{L}_{\text{grav}} = -\frac{1}{2}c \phi^2 R + [-e_1 R^{\mu\nu} R_{\mu\nu} - e_2 R^2] + \frac{1}{2} D_{\mu} \phi D_{\mu} \phi - \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} - \lambda \phi^4 \tag{72}
\end{equation}

where $c, e_1, e_2, g, \lambda$ are coupling coefficients. For coefficients of the quadratic curvature terms (in square brackets) with $e_2 = -\frac{1}{3} e_1$, the latter is variationally equivalent (equal up to divergence) to the squared conformal curvature $C^2 = C_{\mu\nu\kappa\lambda} C^{\mu\nu\kappa\lambda}$. Smolin introduced the scalar field $\phi$ not only by formal reasons (“to write a conformally invariant Lagrangian with the required properties”), but with a physical interpretation similar to those given by Englert e.a., namely “as an order parameter to indicate the spontaneous breaking of the conformal invariance” (Smolin, 1979, 260). His Lagrangian used a modified adaptation from JBD theory, “with some additional couplings” between the scale connection $\varphi$ and the scalar field $\phi$. Smolin emphasized that “these additional couplings go against the spirit of Brans-Dicke theory” because from the Riemannian point of view they introduced a non-vanishing divergence of the non-gravitational fields.

For low energy considerations Smolin dropped the square curvature term (square brackets in (72)), added an “effective” potential term of the scalar field $V_{\text{eff}}(\phi)$ and derived the equations of motion by varying with respect to $g, \phi, \varphi$. Results were Einstein equation, scalar field equation, and Yang-Mills equation for the scale connection.

\[^{76}\text{In his bibliography he went back directly to (Weyl 1922) and (Weyl 1918a); he did not quote any of the later literature on Weyl geometry.}\]

\[^{77}\text{Signs have to be taken with caution. They may depend on conventions for defining the Riemann curvature, the Ricci contraction, and the signature. Smolin, e.g., used a different sign convention for }\text{Riem} \text{ to the one used in this survey. Signs given here are adapted to }\text{signature} g = (3, 1). \text{ The Riemann tensor and Ricci contraction are those usually adopted in the mathematical literature, see fn. 24.}\]

\[^{78}\text{This seems to have been widely known. For an explicit statement see, e.g., (Hehl et al. 1996).}\]

\[^{79}\text{(Englert et al. 1975) was not quoted by Smolin.}\]
Smolin’s reduced Lagrangian contained terms in the scale connection

\[
- \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \frac{1}{8} (1 + 6c) F^2 \phi'^2
\]

That looked like a mass term for the scale connection \( \phi \), the potential of the scale curvature field \( f_{\mu\nu} \) called “Weyl field” by Smolin. By comparison with the Lagrangian of the Proca equation in electromagnetic theory, Smolin concluded that the “Weyl field” has mass close to the Planck scale, given by

\[
M^2_\phi = \frac{1}{4} (1 + 6c) F^2 .
\]

He commented that in his Weyl geometric gravitation theory “general relativity couples to a massive vector field” \( \phi \). The scalar field \( \phi \) on the other hand, “may be absorbed into the scalar parts” of \( g_{\mu\nu} \) and \( \phi' \) by a change of variables and “remains massless” (Smolin, 1979, 263). In this way, Smolin brought Weyl geometric gravity closer to the field theoretic frame of particle physics. He did not discuss mass and interaction fields of the SM. Moreover, the huge mass of the “Weyl field” must have appeared irritating.

4.1.3. Interlude. At the time Smolin’s paper appeared, the program of so-called induced gravity, entered an active phase. Its central goal was to derive the action of conventional or modified Einstein gravity from an extended scheme of standard model type quantization. Among the authors involved in this program Stephen Adler and Anthony Zee stand out. We cannot go into this story here.\(^{82}\)

Smolin’s view that the structure of Weyl geometry might be suited to bring classical gravity into a coherent frame with standard model physics did not find much immediate response. But it was “rediscovered” at least twice (plus an independently developed conformal version). In 1987/88 Hung Cheng at the MIT, and a decade later Wolfgang Drechsler and Hanno Tann at Munich, arrived at similar insights and established an explicit extension of Weyl geometric gravity to standard model (SM) fields (Cheng, 1988; Drechsler/Tann, 1999; Drechsler, 1999). Simultaneous to Cheng, the core of the idea was once more discovered by Moshé Flato (Dijon) and Ryszard Račka (during that time at Trieste), although they formulated it in a strictly conformal framework without Weyl structure (Flato/Račka, 1988). Neither Cheng, nor Flato/Račka or Drechsler/Tann seem to have known Smolin’s proposal (at least Smolin was not cited by them), nor did they refer to the

\(^{80}\)In scalar field gauge with \( \phi = \phi_0 = F \), his reduced Lagrangian (square gravitational terms dropped) was (Smolin, 1979, equ. (3.17))

\[
\left| \text{det} g \right|^{-\frac{1}{2}} L_{\text{grav}} \doteq - \frac{1}{2} F^2 \phi'^2 - \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \frac{1}{8} (1 + 6c) F^2 \phi'^2 - V_{\text{eff}}(F) .
\]

\(^{81}\)It is possible to choose the scale gauge such that \( \phi \) becomes constant (scalar field gauge, see section 1.3)

\(^{82}\)For a survey of the status of investigations in 1981 see (Adler, 1982); but note in particular (Zee, 1982, 1983). The topic of “origin of spontaneous symmetry breaking” by radiative correction was much older (Borrelli, 2015; Karaca, 2013). A famous paper was (Coleman/Weinberg, 1973). In fact, Zee’s first publication on the subject preceded Smolin’s. (Zee, 1979) was submitted in December 1978 and published in February 1979; (Smolin, 1979) was submitted in June 1979.
papers of each other. All three approaches had their own achievements. Here we can give only give a short presentation of the main points of the work directly related to Weyl geometry.

4.1.4. Hung Cheng and his “vector meson”. Hung Cheng started out from a Weyl geometric background, apparently inherited from the papers of the Japanese group of authors around Utiyama. The latter had taken up Weyl geometry in the early 1970s in a way not too different from Smolin’s later approach (see section 2.2.3). Cheng extended Utiyama’s theory explicitly to the electroweak sector of the SM. He replaced the complex scalar field \( \phi \) by the Higgs field \( \Phi \), again of weight \(-1\) but now with values in an isospin \( \frac{1}{2} \) representation, and coupled it to the Weyl geometric scalar curvature \( R \) and postulated:

\[
L_R = \frac{1}{2} \beta \Phi^* \Phi R \sqrt{|\det g|} \frac{i^2}{2}
\]

\[
L_\Phi = \frac{1}{2} \tilde{D}_\mu \Phi^* \tilde{D}_\mu \Phi \sqrt{|\det g|} \frac{i^2}{2}
\]

The scale covariant derivatives were extended to a localized electroweak (ew) group \( SU(2) \times U(1) \). With the usual denotation of the standard model, \( W_\mu \) for the field components of the \( su(2) \) part (with respect to the Pauli matrices \( \sigma_j \) \((j = 0, 1, 2)\)) and \( B_\mu \) for \( u(1)_Y \cong \mathbb{R} \) and coupling coefficients \( g, g' \) the derivative read:\n
\[
\tilde{D}_\mu \Phi = (\partial_\mu - \varphi_\mu + \frac{1}{2} ig W_\mu^j \sigma_j + \frac{1}{2} g' B_\mu) \Phi.
\]

The sign of the kinetic term of the Higgs field \((76)\) shows that Cheng supposed \( \text{sign} g = (+ - - -) \), which agrees with his high energy context, while the sign of \((75)\) indicates that he used the sign inverted convention for curvature. He added Yang-Mills interaction Lagrangians for the ew interaction fields \( F \) and \( G \) of the potentials \( W \) (values in \( su_2 \)), respectively \( B \) (values in \( u(1)_Y \)), and added a scalar curvature term in \( f = (f_\mu) = d\varphi \)

\[
L_{YM} = -\frac{1}{4} (f_\mu f^\mu + F_\mu F^{\mu} + G_\mu G^{\mu}) \sqrt{|\det g|} \frac{i^2}{2}.
\]

Finally he introduced spin \( \frac{1}{2} \) fermion fields \( \psi \) with the weight convention \( w(\psi) = -\frac{3}{2} \), and a Lagrangian \( L_\psi \) similar to the one formulated later by Drechsler, discussed below (82) [87].

---

83Flato/Racka’s paper appeared as a preprint of the Scuola Internazionale Superiore di Studi Avanzati, Trieste, in 1987: the paper itself was submitted in December 1987 to Physics Letters B and published in July 1988. Cheng’s paper was submitted in February 1988, published in November. Only a decade later, in March 2009, Drechsler and Tann got acquainted with the other two papers. This indicates that the Weyl geometric approach in field theory had not yet acquired the coherence of a research program with a stable communication network.

84In the sequel the isospin extended scalar field will be denoted by \( \Phi \).

85Cheng added another coupling coefficient for the scale connection, which is here suppressed.

86See fn 24.

87The second term in \((82)\) is missing in Cheng’s publication. That is probably not intended, but a misprint. Moreover he did not discuss scale weights for Dirac matrices in the tetrad approach.
Thus Cheng’s general relativistic scalar field $\Phi$ resembled very much the Higgs field of the SM, which at that time was still a highly hypothetical object. He called the scale connection, respectively its curvature, Weyl’s meson field. Referring to Hayashi’s e.a. observation that the scale connection does not influence the equation of motion of the spinor fields, he concluded:

...Weyl’s vector meson does not interact with leptons or quarks. Neither does it interact with other vector mesons. The only interaction the Weyl’s meson has is that with the graviton. (Cheng [1988], 2183)

Because of the tremendously high mass of “Weyl’s vector meson” Cheng conjectured that even such a minute coupling might be of some cosmological import. More precisely, he wondered, “whether Weyl’s meson may account for at least part of the dark matter of the universe” (ibid.). Similar conjectures were stated once and again over the next decades, if theoretical entities were encountered which might represent massive particles without experimental evidence. Weyl geometric field theory was not spared this fate.

4.1.5. Can gravity do what the Higgs does? In the same year in which Cheng’s paper appeared, Moshé Flato and Ryszard Rączka sketched an approach in which they put gravity into a quantum physical perspective. In our context, this paper matters because it introduced a scale covariant Brans-Dicke like field in an isospin representation similar to Hung Cheng’s, but in a strictly conformal framework (Flato/Rączka, 1988).

Six years later, R. Rączka took up the thread again, now in cooperation with Marek Pawłowski. In the meantime Pawłowski had joined the research program by a paper in which he addressed the question whether gravity “can do what the Higgs does” (Pawłowski, 1990). In a couple of preprints and two refereed papers (Pawłowski/Rączka [1994a], 1995b) the two physicists proposed a “Higgs free model for fundamental interactions”, as they described it. This proposal was formulated in a strictly conformal setting. Although it is very interesting in itself, we cannot discuss it here in more detail.

4.2. Mass generation and Weyl geometric gravity “at Munich”, 1980/90s.

4.2.1. 1990: Drechsler and Tann. A view closer to Cheng’s establishing a connection between gravity and electroweak fields in the framework of Weyl geometry was developed a decade later by Wolfgang Drechsler and his PhD student Hanno Tann at Munich. Drechsler had been active for more than twenty years in differential geometric aspects of field theory. In cooperation with D. Hartley he developed an approach of his own to Weyl geometric gravity evolving form investigations in Kaluza-Klein theories (Drechsler/Hartley).
1994). Tann joined the activity a little later during his work on his PhD thesis (Tann 1998), coming from a background interest in geometric properties of the de Broglie-Bohm interpretation of quantum mechanics (see section 3.1). In their joint work (Drechsler/Tann 1999), as well as in their separate publications (Tann 1998, Drechsler 1999) Weyl geometric structures were used in a coherent way, clearer than in most of the other physical papers discussed up to now.

Tann studied a complex valued scalar field $\Phi$, Drechsler, and the common paper of both, investigated a scalar field with values in an isospin $\frac{1}{2}$ representation of the $\text{ew}$ group (like Cheng) with gravitational Lagrangian

$$L_{\text{grav}} = L_R + L_{R^2}$$

where $L_{R^2} = \alpha R^2 \sqrt{|\det g|}$ and $L_R = \frac{1}{2} \Phi^* \Phi R$ (Drechsler 1999). A common form of their linear gravitational Lagrangian with modified Hilbert term $L_R$ and the kinetic term of the scalar field is

$$L_{R,\Phi} = \frac{\beta}{2} \Phi^* \Phi R + \frac{1}{2} (D_\mu \Phi)^* D^\mu \Phi, \quad \beta = \frac{1}{6},$$

with $\Phi^*$ the adjoint (often written as $\Phi^\dagger$) which in the case of Tann reduces to complex conjugation (often $\Phi$), $R$ the Weyl geometric scalar curvature, signature of $g$ $(1,3) \sim (+---)$ and $D_\nu$ the scale covariant derivation, in Drechsler’s case extended to the electroweak bundle. Both authors tried to straddle the gap between the gravitational scalar field and a Higgs-like scalar field of electroweak theory.

In their common paper, Drechsler and Tann introduced fermionic Dirac fields into the analysis of Weyl geometry (Drechsler/Tann, 1999). Their gravitational Lagrangian had the form (79) For the development of a Weyl geometric framework this was an unnecessary restriction, because scale covariance holds for any $\beta$. In addition, Tann wrote the modified Hilbert term with a negative sign, because the used the sign inverted convention for the Riemann tensor, see fn. 24.

In the appendix Drechsler and Tann showed that the squared Weyl geometric conformal curvature $\mathcal{C}^2 = C_{\lambda \mu \nu \rho} C^{\lambda \mu \nu \rho}$ arises from the conformal curvature of the Riemannian component $\mathcal{C}^2$ by adding a scale curvature term: $\mathcal{C}^2 = \mathcal{C}^2 + \frac{1}{2} f_{\mu \nu} f^{\mu \nu}$ (Drechsler/Tann 1999) (A 54)). So one may wonder, why they did not replace the square term $L_{R^2}$ by the Weyl geometric conformal curvature term $L_{\text{conf}} = \alpha C^2 \sqrt{|\det g|}$. 

92 (Tann 1998, equ. (372)), (Drechsler 1999, equ. (2.29)). Both authors used coefficients like in the case of conformal coupling in Riemannian geometry, $\beta = \frac{1}{6}$. In the Weyl geometric framework this was an unnecessary restriction, because scale covariance holds for any $\beta$. In addition, Tann wrote the modified Hilbert term with a negative sign, because the used the sign inverted convention for the Riemann tensor, see fn. 24.

93 (Tann 1998, equ. (372)), (Drechsler 1999, equ. (2.46)).
geometric theory of the Dirac field, they introduced an adapted Lagrangian
\( \mathcal{L}_\psi = \frac{i}{2} (\psi^* \gamma^\mu D_\mu \psi - D_\mu^* \psi^* \gamma_\mu \psi) + \gamma |\Phi| \psi^* \psi \)
with (scale invariant) coupling constant \( \gamma \) and Dirac matrices \( \gamma^\mu \) with symmetric product \( \frac{1}{2} [\gamma^\mu, \gamma^\nu] = g^{\mu\nu}1 \) \(^{[82]}\). Here the covariant derivative had to be lifted to the spinor bundle, it included an \( U(1) \) electromagnetic potential \( A = (A_\mu) \),
\( D_\mu \psi = \left( \partial_\mu + i \tilde{\Gamma}_\mu + \frac{iq}{\hbar c} A_\mu \right) \psi , \)
\( q \) electric charge of the fermion field, \( w(\psi) = -\frac{3}{2}, \tilde{\Gamma} \) spin connection lifted from the Weylian affine connection. This amounted to a (local) construction of a spin \( \frac{1}{2} \) bundle. Assuming the underlying spacetime \( M \) to be spin, they worked in a Dirac spin bundle \( D \) over the Weylian manifold \( (M, [(g, \varphi)]) \). Its structure group was \( G = Spin(3, 1) \times R^+ \times U(1) \cong Spin(3, 1) \times C^* \), where \( C^* = C \setminus \{0\} \).
The two authors considered \( \mathcal{L}_\psi \) as Lagrangian of a “massless” theory, because the masslike factor of the spinor field \( \gamma |\Phi| \) was scale invariant \(^{[97]}\) and proposed to proceed to a theory with masses by introducing a “scale symmetry breaking” Lagrange term
\( \mathcal{L}_B \sim R^\frac{6}{6} + \left( \frac{mc^2}{\hbar} \right)^2 |\Phi|^2 \)
with fixed (non-scaling) \( m \) \(^{[84]}\). But they did not associate such a transition from a (seemingly) “massless” theory to a massive one with any kind of hypothetical “phase transition”.

At the end of the paper they even commented:

It is clear from the role the modulus of the scalar field plays in this theory (…) that the scalar field with nonlinear self-coupling is not a true matter field describing scalar particles. It is a universal field necessary to establish a scale of length in a theory and should probably not be interpreted as a field having a particle interpretation. \(^{[1999, 1050]}\)

Their interpretation of the scalar field \( \Phi \) was rather geometric than that of an ordinary quantum field; but their term \( \mathcal{L}_B \) looked ad-hoc to the uninitiated.

\(^{[82]}\) can equivalently be written with a Weylianized scale covariant derivative \( D_\mu = \left( \partial_\mu + i \tilde{\Gamma}_\mu + w(\psi) \varphi_\mu + \frac{iq}{\hbar c} A_\mu \right) \). Because \( \varphi_\mu \) is real, the scale connection terms \( w(\psi) \varphi_\mu \) in the Lagrangian cancel.

\(^{[96]}\) One could then just as well consider a complex valued connection \( z = (z_\mu) \) with values \( z_\mu = \varphi_\mu + \frac{iq}{\hbar c} A_\mu \) in \( C = \mathbb{C} \) and weight \( W(\psi) = (-\frac{3}{2}, q) \). Then \( D_\mu \psi = (\partial_\mu + \tilde{\Gamma}_\mu + W(\psi) z_\mu) \psi \), presupposing an obvious convention for applying \( W(\psi)z \).

\(^{[97]}\) This argument is possible, but not compelling \( \gamma |\Phi| \) has the correct scaling weight of mass and may be considered as such.

\(^{[98]}\) Similar already in Tann’s PhD dissertation.

\(^{[99]}\) Note that one could just as well do without \( \mathcal{L}_B \) and proceed with fully scale covariant masses – compare last footnote.
4.2.2. Drechsler on mass acquirement of electroweak bosons. Shortly after the joint article with Tann, Drechsler extended the investigation to a gravitationally coupled electroweak theory (Drechsler, 1999). Covariant derivatives were lifted as $\tilde{D}$ to the electroweak bundle. It included the additional connection components and coupling coefficients $g$ and $g'$ with regard to $SU(2)$ and $U(1)_Y$ like in Cheng’s work (77). The Weyl geometric Lagrangian could be generalized and transferred to the electroweak bundle (Drechsler, 1999, (2.29)),

\[
L = L_{\text{grav}} + L_\Phi + L_\psi + L_{YM},
\]

with contributions like in (79), (76), (82), and (78) (ew terms only). Lagrangians for the fermion fields had to be rewritten similar to electromagnetic Dirac fields (82) and were decomposed into the chiral left and right contributions.

In principle, Drechsler’s proposal coincided with Cheng’s; but he proceeded with more care and with more detailed explicit constructions. He derived the equations of motion with respect to all dynamical variables (Drechsler, 1999, equs. (2.35) – (2.41)) and calculated the energy-momentum tensors of all fields occurring in the Lagrangian.

The symmetry reduction from the electroweak group $G_{ew}$ to the electromagnetic $U(1)_{em}$ could then be expressed similar to the procedure in the standard model. $SU(2)$ gauge freedom allows to chose a (local) trivialization of the electroweak bundle such that the $\Phi$ assumes the form considered in the ordinary Higgs mechanism

\[
\hat{\Phi} = \begin{pmatrix} 0 \\ \phi_o \end{pmatrix},
\]

where $\Phi_o$ denotes a real valued field, and “$\hat{\sim}$” equality in a specific gauge. $\hat{\Phi}$ has the isotropy group $U(1)$ considered as $U(1)_{em}$ and was called the electromagnetic gauge of $\Phi$.

In two respects Drechsler went beyond what had been done before. He reconsidered the standard interpretation of symmetry breaking by the Higgs mechanism (Drechsler, 1999, 1345f.). And he calculated the consequences of nonvanishing electroweak curvature components for the energy-momentum tensor of the scalar field $\hat{\Phi}$ (Drechsler, 1999, 1353ff.). With regard to the first point, he made clear that he saw nothing compelling in the interpretation of symmetry reduction as “spontaneous symmetry breaking due to a nonvanishing vacuum expectation value of the scalar field” (Drechsler, 1999, 1345). He analyzed the situation and came to the conclusion that the transition from our $\Phi$ to $\hat{\Phi}$ is to be regarded as a “choice of coordinates” for the representation of the scalar field in the theory and has, in the first place, nothing to do with a “vacuum expectation value” of this field.

100 In other parts of the literature (e.g., the work of Ráczka and Pawlowski) it is called “unitary gauge”, cf. also Flato/Ráczka, 1988.

101 Mathematically spoken, it is a change of trivialization of the $SU(2) \times U(1)$-bundle.
He compared the stabilizer $U(1)_{em}$ of $\hat{Φ}$ with the “Wigner rotations” in the study of the representations of the Poincaré group. With regard to the second point, the energy-momentum tensor of the scalar field could be calculated roughly like in the simpler case of a complex scalar field, \cite{81}. Different to what one knew from the pseudo-Riemannian case, the covariant derivatives $D_\mu \Phi$ etc. in \cite{81} were then dependent on scale or $U(1)_{em}$ curvature.

After breaking the Weyl symmetry by a Lagrangian of form \cite{84} (ibid. sec. 3), Drechsler calculated the curvature contributions induced by the Yang-Mills potentials of the $ew$ group and its consequences for the energy-momentum tensor $T_\Phi$ of the scalar field. Typical contributions to components of $T_\Phi$ had the form of mass terms
\begin{equation}
\begin{aligned}
   m_W^2 W_\mu^+ W^- \mu, & \quad m_Z^2 Z_\mu^+ Z^- \mu, \\
   \text{with} & \quad m_W^2 = \frac{1}{4} g^2 |\Phi_o|^2, \quad m_Z^2 = \frac{1}{4} g_o^2 |\Phi_o|^2,
\end{aligned}
\end{equation}
g^2 = g^2 + g'^2, \text{for the bosonic fields } W^\pm, Z \text{ corresponding to the generators } \tau_\pm, \tau_o \text{ of the electroweak group.} \text{\cite{Drechsler 1999 1353 ff.}} \text{They are identical with the mass expressions for the } W \text{ and } Z \text{ bosons in conventional electroweak theory. According to Drechsler, the terms } \text{\cite{87} in } T_\Phi \text{ indicate that the } \gamma \text{ boson and fermion mass terms appear in the total energy-momentum tensor” through the energy tensor of the scalar field after “breaking the Weyl symmetry”}\cite{104}.

Drechsler and Tann studied their scalar fields (complex or Higgs-like) as possibilities for an extension of the gravitational structure of spacetime. In their scale covariant theory of mass acquirement they tried to understand how \textit{mass generation} is linked to the gravitational structure. Drechsler added that in his view the scalar field “...should probably not be interpreted as a field having a particle interpretation” \cite{Drechsler/Tann 1999 1050}. This was an interesting remark at a time when elementary particle physicists started to collect information on a possible scalar boson of the Higgs field. But the empirical confirmation of the existence of a Higgs-like boson was still far out of sight; it did not materialize before the LHC started to operate at a sufficient level of energy and luminosity in 2012\cite{104}. Even so, a more indirect link between the Higgs field and gravity, in contrast to the perspective of our two authors compatible with a bosonic interpretation of the scalar field, would be an interesting point. Back in the 1990s Drechsler did not expect that the search for a bosonic quantum of Higgs type might ever be confirmed by experiment.

102 $W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \mp i W_\mu^2)$, $Z_\mu = \cos \Theta W_\mu^3 - \sin \Theta B_\mu$.

103 One has to be careful, however. Things become more complicated if one considers the trace. In fact, $tr T_\Phi$ contains a mass terms of the Dirac field of form $\gamma |\Phi_o|^2 \bar{\psi}^* \psi$, with $\gamma$ coupling constant of the Yukawa term (\bar{\psi} indicating electromagnetic gauge). One of the obstacles for making quantum matter fields compatible with classical gravity is the vanishing of $tr T_\psi$, in contrast to the (nonvanishing) trace of the energy momentum tensor of classical matter. Might Drechsler’s analysis indicate a way out of this impasse? – Warning: The mass-like expressions for $W$ and $Z$ in \cite{87} cancel in $tr T_\Phi$ \cite{Drechsler 1999 equ. (3.55)} like in the energy-momentum tensor of the $W$ and $Z$ fields themselves.

104 See, e.g. \cite{Franklin 2014}.
4.3. The “Higgs” and Weyl scaling after 2000. In the years following the onset of the new millennium, but still before a Higgs-like boson would be observed at the LHC, different authors continued to explore the near to scale invariance of the standard model and attempted to bridge the gap between the SM and gravity, keeping as closely as possible to the original Higgs “mechanism” developed in the special relativistic framework. They did not adhere to a common research program; the researchers used different geometric/conceptual frameworks and worked in differing perspectives. Weyl geometric methods did not always stand in the center of the investigations; some scientists worked with global scale invariance and unimodular gravity (Shaposhnikov/Zenhäusern 2009b, a), others preferred a conformal approach without making use of Weyl geometric concepts (Meissner/Nicolai, 2009; Bars et al., 2014), and some started from conformal symmetry but were mainly interested in models with radiative breaking of scale symmetry (Foot et al., 2007a, b; Foot/Kobakhidze, 2013). A small group of authors, however, continued in the line of Weyl geometric studies (Nishino/Rajpoot, Quiros, Ohanian e.a). They often were not aware of the whole range of studies made in the 1970s to 1990s and took up just one filament of the latter. With few exceptions⁴⁶ the majority of the mentioned authors worked with two scalar fields, a Higgs-like one $\Phi$ with values in a spin $\frac{1}{2}$ representation of the electroweak group, and a real-valued one, here denoted by $\phi$, the gravitational scalar field. Keeping track with our main theme, we shall concentrate on the last group of authors who worked in the framework of Weyl geometry.

The modified Hilbert-Weyl term and kinetic terms of the scalar fields of these authors were, up to notational conventions, of the common form

\begin{equation}
L_{HW} = -\frac{\epsilon_{sigg}}{2} (\zeta_1 \phi^2 + \zeta_2 \Phi^4 R),
\end{equation}

\begin{equation}
L_{\phi} = \epsilon_{sigg} \frac{\alpha_1}{2} D_{\nu} \phi D^{\nu} \phi, \quad L_{\Phi} = \epsilon_{sigg} \frac{\alpha_1}{2} (D_{\nu} \Phi) D^{\nu} \Phi^\dagger,
\end{equation}

\begin{equation}
\mathcal{L} = L \sqrt{|g|}, \quad \epsilon_{sigg} = \begin{cases} +1 & \text{for } sigg = (+-+-) \\ -1 & \text{for } sigg = (-+++) \end{cases}
\end{equation}

(in most cases $\alpha_1 = \alpha_2 = 1$), with Weyl geometric scalar curvature $R$ and electroweak and Weyl geometric covariant derivatives $D_{\mu}$. A quadratic curvature term $L_{R^2}$ was added by some, not by all, authors. Yang-Mills terms of the electroweak connections (potentials) $W$ for the $SU_2$-component, $B$ for hypercharge $U(1)$, and $\varphi$ for the scale connection with field strength $f = d\varphi$, were added,

\begin{equation}
L_{YM} = \frac{1}{4} (tr(W_{\mu\nu}W^{\mu\nu}) + B_{\mu\nu}B^{\mu\nu} + f_{\mu\nu} f^{\mu\nu}).
\end{equation}

Similarly Dirac kinetic terms $L_{\Psi, \text{kin}}$ and Yukawa mass terms $L_{\Psi, \text{Y}}$ for the different fermions $\Psi_{\ell_i}$, with indices taking care for the various types and properties ($f = q, l$ for quark or lepton, $g = 1, 2, 3$ generation, $i = u, d$ (“up, down”) for the 3-component of weak isospin, $h = R, L$ helicity) were

⁴⁶For an exception still standing under the spell of Drechsler/Tann, although with a consistently scale covariant approach without an explicit scale symmetry breaking term, see, e.g., (Scholz, 2011a).

⁴⁷In the high energy physical context, and accordingly in our section 3, signature of $g = (+-+-)$. For sign conventions regarding curvature see fn 24.
added in a form adapted to the Weyl geometric framework. The Dirac terms could be written with or without the Weyl geometric scale connection term because, even if it is included, it finally cancels in the total expression. This had been noticed already by Hayashi and Kugo (see section 2.2).

We need not reproduce the explicit form of the fermionic terms here, but have to keep in mind that the Yukawa terms contained the matrices with relative mass coefficients (“mass matrix”) of the SM and a scale covariant Higgs field.

Breaking of scale invariance without an explicit mass terms of the Higgs field became the crucial points for our authors. Because of its scaling behaviour \( w(\phi) = -1 \) the gravitational scalar field \( \phi \) already specifies a preferred scale in which it assumes a constant value \( \phi_0 \) (scalar field gauge in the terminology of section 1.1):

\[
\phi(x) \doteq \phi_0 = \text{const}
\]

This was the reason behind Utiyama calling \( \phi \) a “measuring field” already in the 1970s.

But the question still remains how such an, at first sight only mathematical, specification may be incorporated in the material structures lying at the basis of measuring processes. In the context of the search of a connection between gravity and the electroweak sector of fundamental fields it seemed natural to search for a relation between the two scalar fields \( \phi \) and \( \Phi \). For this a biquadratic/quartic potential in the two scalar fields, and a corresponding Lagrange term, plays a crucial role. Using the abbreviation \(|\Phi|^2 = \Phi^\dagger \Phi\) it is:

\[
V(\Phi, \phi) = \frac{\lambda_1}{4} |\Phi|^4 - \frac{\mu}{2} |\Phi|^2 \phi^2 + \frac{\lambda'}{4} \phi^4
\]

\[
= \frac{\lambda_1}{4} \left( |\Phi|^2 - \frac{\mu}{\lambda_1} \phi^2 \right)^2 + \frac{\lambda'}{4} \phi^4, \quad \lambda = \lambda' - \frac{\mu^2}{\lambda_1} > 0
\]

\[
\mathcal{L}_V = -V(\Phi, \phi) \sqrt{|g|}
\]

Chromodynamics was usually not considered; our group of authors concentrated on the electroweak sector of the SM and its possible link to gravity.

4.3.1. Nishino/Rajpoot. In 2004 two theoretical high energy physicists at California State University, Hitoshi Nishino and Subhash Rajpoot posed the goal of “extending the standard model with Weyl’s scale invariance”, adding that the scale invariance is “badly broken” at the order of the Planck mass/energy. They made it clear that in the “philosophy advocated in the present work the standard model Higgs is not eliminated, and is the sought for particle” (Nishino/Rajpoot 2004, 1).

For adapting the fermionic fields to the differential geometric setting, the authors outlined the usual spinor calculus in a Weyl geometric approach with scale dependent tetrads consisting of point-dependent bases \( e_a = e_\mu^a \partial_\mu \) (\( a = 0, \ldots, 3 \)) and their dual forms, here denoted by \( \vartheta^a = \vartheta_\mu^a dx^\mu \), and the metric

\[107\] See also Blagojević 2002, p. 81.

\[108\] For an explicit form of Dirac kinetic terms and Yukawa mass terms see, e.g., Nishino/Rajpoot 2009, equ. (1.2)).
\[ g_{\mu\nu} = \tilde{g}^\mu e_a \nabla_\nu. \] With \( g(x) \mapsto \tilde{g} = e^{2\Lambda(x)} g(x) \) the tetrads have to be rescaled like

\[ (\tilde{g}^a)^\mu = e^{\Lambda(x)} g^a_\mu \quad (\tilde{e}_a)^\mu = e^{-\Lambda(x)} e_\mu^a, \]

that is \( w(\tilde{g}^\mu) = 1 \), \( w(e_a) = -1 \). The Weyl geometric affine connection, the corresponding spin connection, Weyl geometric covariant derivatives, and curvature expressions were developed by the two authors, although not always completely reliable. On this background they described a two stage process of symmetry breaking. In the first step they dealt with breaking the scale symmetry, formulated in terms of the compactified scaling group \( \tilde{U}(1) \). The breaking was expressed “by setting” the value of the gravitational scalar field to a constant \( \phi_o \)

\[ \phi(x) = \phi_o \quad \text{with} \quad \zeta_1 \phi_o^2 = (8\pi G)^{-1}, \]

in our terminology they introduced scalar field (Einstein) gauge. In the second step the \( ew \) symmetry was assumed to be broken “spontaneously” like in the special relativistic SM case \((SU_2 \times U(1)_Y) \mapsto U(1)_{em})\). For the first step they gave a physical interpretation which has some analogy with the Higgs “mechanism”:

At this stage the scalar field \( \sigma \) [here denoted \( \phi \), E.S.] becomes the Goldstone boson . . . . The vector particle associated with \( \tilde{U}(1) \) breaking, the Weylon, absorbs the Goldstone field and becomes massive with mass \( M_S \) given by 

\[ M_S = \sqrt{\frac{3f^2}{4\pi G_N}} \approx 0.5 \times f \times M_P \quad [f \quad \text{a coupling constant of the scale connection, E.S.].} \]

Then the quartic potential (93) is reduced to the Higgs potential like in the SM plus a cosmological term \( \frac{\lambda}{4} \phi_4^4 \). In the ground state of the Higgs field only the cosmological term survives and the transition to scalar field gauge endows the Higgs field with mass

\[ m_H = \sqrt{\mu \phi_o}. \]

After a short outline of how to adapt the parameters to the mass generation scheme of the SM the authors concluded

Our contention is that the present model presents a viable scheme in which gravity is unified, albeit in a semi-satisfactory way, with the other interactions. (…) When the complete theory of all interactions is found, the model in its present form, it is hoped, will serve as its low energy limit.

To conclude, we have accommodated Weyls scale invariance as a local symmetry in the standard electroweak model.

\[ \text{The expression for the scalar curvature is given in the paper (and also in the later papers by the same authors) as} \quad R = g R - 6 \nabla_\mu \phi^\mu + 6 \phi_\mu \phi^\mu, \quad \text{where a coupling constant} \quad f \quad \text{introduced by the authors is here set to} \quad f = 1 \quad \text{and transcribed into our notation,} \]

\[ \text{[Nishino/Rajpoot, 2004, equ. (14)]. The correct Weyl geometric value (2) would be} \]

\[ R = g R - 6 \nabla_\mu \phi^\mu + 6 \phi_\mu \phi^\mu; \quad \text{cf. [Weyl, 1918, p. 21], [Drechsler/Tann, 1999, equ. (A 31)] and others. Because of} \]

\[ \nabla_\mu \phi^\mu = \sqrt{g} \nabla_\mu \phi^\mu + 4 \phi_\mu \phi^\mu \quad \text{this implies} \]

\[ R = g R - 6 \nabla_\mu \phi^\mu + 18 \phi_\mu \phi^\mu \]
This inevitably leads to the introduction of general relativity.

(Nishino/Rajpoot, 2004, 8)

This paper remained in a preprint stage. Although its content seems to have been presented at different conferences it never was published in a scientific journal. The reason may have been that the authors considered it only as a first, provisional step. In the following years they extended their approach to a SU(5) grand unified theory (GUT) (Nishino/Rajpoot, 2007) and revised their presentation by taking up an idea going back to Stueckelberg (Nishino/Rajpoot, 2009, 2011).

In the late 1930s Ernst Stueckelberg had introduced a massive scalar field $B$ complementing an $U(1)$ potential $A_\mu$, which expressed a field of electromagnetic type, but with mass (i.e. similar to a Proca field). $B$ was given a peculiar gauge behaviour involving a mass parameter $m$ under $U(1)$ gauge transformations $A_\mu \mapsto \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda(x)$

\begin{equation}
B(x) \mapsto \tilde{B} = B(x) + m \Lambda(x).
\end{equation}

Stueckelberg’s context was the search for an interaction of a scalar field with nucleons. Transformations of type (98) were taken up by Pauli and others. They became to be known as Stueckelberg transformations and $B$ as Stueckelberg (compensating) field. With an appropriate $\Lambda$, the Stueckelberg field allowed to specify a peculiar gauge with $\tilde{B} = 0$, without breaking the $U(1)$ symmetry which is only given a “different realization” (in Drechsler’s terms quoted above, p. 50). This turned out to be crucial for the renormalizability of the theory and made the “Stueckelberg trick” attractive for quantizing the electromagnetic field or its relatives like Proca like fields.

The careful reader may have noted the kinship between the Stueckelberg “trick” for $U(1)$ and the Higgs “mechanism” for the electroweak group. So did Nishino/Rajpoot. Moreover, they realized that, just by taking the logarithm, the transition to the Weylian scalar field gauge can be given the form of a Stueckelberg transformation. Transliterated to our notation they introduced an exponential expression of the form

\begin{equation}
\phi(x) = \xi_1^{-\frac{1}{2}} M_P e^{\frac{1}{2} \beta(x)}.
\end{equation}

Then the scale gauge transformation $\phi \mapsto \tilde{\phi} = e^{-\Lambda} \phi$ is expressed by

\begin{equation}
\beta \mapsto \tilde{\beta} + M_P \Lambda,
\end{equation}

and the transition to scalar field gauge corresponds to $\tilde{\beta} = 0$, exactly like in the case of the the Stueckelberg “trick”.

Nishino/Rajpoot thus rewrote their basic Lagrange density equivalent to our equations \(88, 89, 91, 93\) in terms of the logarithmized scalar field \(Nishino/Rajpoot, 2009\) equ. (2.3)) and normed it to scalar field gauge

\begin{itemize}
\item \textsuperscript{110}The non-broken $U(1)$ symmetry is important for the BRST relations, the quantum analogue of the Noether relations. See Ruegg et al., 2003, 75ff.
\item \textsuperscript{111}The factor $\xi_1^{-\frac{1}{2}}$ in \(Nishino/Rajpoot, 2009\) equ. (2.1)) was set by them to $\xi_1 = 1$ while transforming the Lagrangian into their equ. (2.3). The follow up paper \(Nishino/Rajpoot, 2011\) second paragraph of section 2) shows that this reduction was intended. Of course, a different factor $\xi_1$ would heavily influence the mass calculation in \(101\).
\end{itemize}
Then the mass expression $m_\phi$ for the scale connection field ("Weyl field") could be read off. In scalar field gauge the kinetic terms (89) of $\phi$ and $\Phi$ acquire forms which makes them contribute to $m_\phi$. For $\phi$ it is

$$
\frac{1}{2} D_\nu \phi D^\nu \phi = \frac{1}{2} (fM_P)^2 \varphi_\nu \varphi^\nu,
$$

while for $\Phi$ the contribution to the mass of $\varphi$ is $f(\Phi^\dagger \Phi)$ (after *ew symmetry breaking* $f v^2$, with $v^2$ the vacuum expectation value of the operator $\Phi^\dagger \Phi$).

In any case the contribution due to $\Phi$ is much less than the one from $\phi$ and from the modified Hilbert term (88), both of which are at the order of the Planck scale. It may safely be neglected at several orders of magnitude.

In the imaginative language of the elementary particle community Nishino and Rajpoot commented that the scalar field is "now eaten up by the Weylon". A little later they added, more technically:

```
After all, the Weylon $\tilde{S}_\mu$ [our $\varphi_\mu$, E.S.] acquires the mass $f M_P$, the compensator $\varphi$ [our $\beta$, E.S.] is absorbed into the longitudinal component of $\tilde{S}_\mu$, and the potential terms are reduced to the Higgs potential in SM . . . (Nishino/Rajpoot, 2009, 3)
```

With this explanation they clad the mass derivation for the scale connection field in the mantle of a narrative which is widely spread in their community and usually accepted as scientifically explanatory.

In a follow up paper, the two California State physicists came back to the topic and presented the results of some results concerning a quantized version of their theory. They started from their Lagrangian given in terms of the logarithmized scalar field (Nishino/Rajpoot, 2009, equ. (2.3)) and with modified Hilbert term

$$
L_{HW} = -\frac{1}{2} \left( \zeta_1 M_P^2 e^{2M_P^{-1} \beta(x)} + \zeta_2 \Phi^4 \right) R
$$

(*R* Weylian scalar curvature written there as $\tilde{R}$).

At this point Nishino and Rajpoot left the track of Weyl geometry and decided to switch to the JBD paradigm. They considered the initial Lagrangian a "Jordan frame" and wanted to transform it to "Einstein frame".

---

112 Warning: Nishino/Rajpoot used the notation $\varphi$ for the Stueckelberg “compensator”, i.e. our $\beta$, and $S_\mu$ for the scale connection (the potential of the "Weyl field"), our $\varphi_\mu$. In order to avoid confusion the notation in the present paper has been homogenized for the authors discussed here.

113 Nishino/Rajpoot did not consider the contribution of the modified Hilbert term, in contrast to Smolin and Cheng (see section 4.1).

114 Compare (Stoeltzner, 2014).

115 Strictly speaking their framework does not contain any meaningful “Jordan frame”, because their Weyl structure is not integrable, and thus the purely Riemannian representation of the affine connection presupposed in ordinary Jordan frame does not exist. Einstein frame, on the other hand, is meaningful in any Weyl geometric gravity approach with a scale covariant scalar field and corresponds to scalar field gauge (92).
i.e. a “field re-definition” which did not include the corresponding transformations of the scale covariant fields and the scale connection. Referring to calculations in the framework of JBD theory they arrived at a reduction of the Hilbert term to a form which depends only on the Riemannian component $gR$ of the scalar curvature. According to their calculation, the scale connection contributions drop out of the Lagrangian (but not the Yang-Mills term for the scale curvature). In other word, a reduction to Einstein frame form of JBD with two scalar fields and an additional Yang-Mills field was achieved (Nishino/Rajpoot, 2011, equ. (2.10)).

On this basis our authors performed a series of calculations at the quantum level. They determined (Adler-Bell-Jackiw and trace) anomalies, studied the possibility for cancelling the remaining (trace-) anomalies, considered quantum corrections to the cosmological constant, and studied the perturbative renormalizability of their model and the possible new divergences. All in all, these were remarkable results; but they were arrived at in a hybrid approach which started in a setting of Weyl geometric gravity and ended in JBD gravity, after performing an artificial and methodologically unconvincing transition by a “field re-definition” type of rescaling. In spite of such shortcomings the derivations were a notable step towards connecting the electroweak sector of elementary particle physics with gravitational structures, mainly formulated in a Weyl geometric framework.

4.3.2. Hao Wei, Rong-Gen Ca, Quirios. H. Nishino and S. Rajpoot were not the only researchers who thought about the question how to establish a connection between gravity and the SM fields by exploiting Weyl geometric methods. Even though we have to be selective here, it has to be clear that the Weyl geometric approach continues to be alive in the era of the Higgs boson (or some close relative) being found in experimental observations. A talk given in July 2004 by Hung Cheng at the Institute for Theoretical Physics of the Chinese Academy of Science, Beijing seems to have initiated interest in Weyl geometric methods by Chinese theoretical physicists Hao Wei, Rong-Gen Cai and others. It was natural for them to take the “Cheng-Weyl vector field” (i.e., the Weylian scale connection with massive boson studied by Cheng in the late 1980s) and Cheng’s view as their starting point for a new look at the standard model of elementary particle physics Wu (2004); Cai/Wei (2007).

Another road was taken by Israel Quirios, at the time we are interested in here, placed at Guanajuato, Mexico. Coming from a background in Jordan-Brans-Dicke gravity and cosmology (see section 5.2.2) he developed thoughts of his own about how “scale invariance and broken electroweak symmetry may coexist together” (Quirios, 2013). In this conceptually clear paper he gave a nice introduction to the basic ideas of integrable Weyl geometry and

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116 Note that the Weyl rescaling we made is a field re-definition, but it is not a part of any local scale transformation which is defined to act not only on $g_{\mu\nu}$ but also on $\Phi$ and $\varphi$ as in (2.2) [the equation for the full gauge transformation, E.S.] (Nishino/Rajpoot, 2011, p. 4).

117 In the light of the error for the scalar curvature indicated in fn. 109 one may be inclined to doubt the correctness of such a complete cancellation.

118 (Cai/Wei, 2007, Acknowledgments)
showed that the scale covariance of the SM fields can not only be imported into a general relativistic framework, if Weyl geometric gravity is used, but can even be upheld after breaking the \( ew \) symmetry. One only need to accept, and to use, mass parameters \( m \) which scale with weight \( w(m) = -1 \).

For his presentation Quiros used a simplified version of the Lagrangian (88ff.) similar to the one of Nishino/Rajpoot, whose papers he probably did not yet know. He encoded the gravitational scalar field in terms of a point-dependent scalar exponent written by him as \( \varphi \) – in order to avoid confusion we shall transliterate it like above as \( \beta \) – of the factor in the Hilbert-Weyl term. Compared with our notation above he wrote

\[
\zeta_1 \phi(x)^2 = M_p e^{\beta(x)}
\]

(103) and considered Weylian scale connections exclusively of the form

\[
\varphi = \varphi_\mu \, dx^\mu = d\beta \quad \iff \quad \varphi_\mu = \partial_\mu \beta 
\]

(Quiros 2013a, equ. (8)). This implies the restriction

\[
\phi = \text{const} \iff \beta = 0 \Rightarrow \varphi = 0 .
\]

In our terminology (103) implies an inbuilt identification of Riemann gauge and Einstein gauge. That was probably unnoticed by the author, and is widely spread among scientists who entered into Weyl geometric methods from a JBD background. For the basically geometrical and conceptual, purposes of the paper this restriction may have been of no particular disadvantage, but the dynamical role of the scalar field was trivialized by this specialization.

4.4. Towards Weyl scaling at the quantum level.

4.4.1. Scale invariant quantization procedures. Problems on a more fundamental have been posed by a group of theoretical physicists working at Trieste. Alessandro Codello, Giulio D’Orico, Carlo Pagani and Roberto Percacci recently reconsidered the question of how scale invariance behaves under quantization if one approaches it with the method of the so-called “renormalization group” (RG) and the use of functional integral methods. In (Codello et al., 2013) they gave a report on their work and rebutted the general view that quantization necessarily leads to a breaking of (point-dependent) scale symmetry even if the classical Lagrangian is scale invariant. In a step by step argumentation they show how the functional integrals can be given a scale invariant form by using an integrable Weyl geometric background and a gravitational scalar field \( \chi \) of weight \( w(\chi) = -1 \), called a “dilaton”, as external fields which are not quantized at the first stage.

They started from the basic idea that “one can make any action Weyl-invariant by replacing all dimensionful couplings by dimensionless couplings multiplied by the powers of the dilaton” (Codello et al. 2013, p. 2). Then a dimensional coupling coefficient of scaling dimension \( k \), let aus say \( \mu \), is turned into a coupling parameter of the form \( \chi^{-k} \hat{\mu} \) with a “dimensionless”, i.e. non-scaling, constant \( \hat{\mu} \). The authors achieve scale covariance/invariance of the fields, respectively actions, by using Weyl geometric expression with regard to an integrable scale connection with coefficients

\[
b_\mu = -\chi^{-1} \partial_\mu \chi \quad \text{(Codello et al. 2013, p. 3)}
\]
Like Nishino and Rajpoot they consider this as a gravitational equivalent to the “Stückelberg trick”. Their main work then consisted in showing that the Weyl invariance which is easily achievable for the classical action is left intact, in their framework, for the functional integrals, the differential equation governing the renormalization flow equation, and the UV and IR endpoints of the flow.

Classical quantum matter fields (scalar or Dirac spinors) have a vanishing trace of the energy-momentum tensor, while the expectation value of the quantized trace no longer vanishes. This so-called trace anomaly of quantization has puzzled theoretical physicists for a long time and is usually taken as a sign that scale invariance is broken at the quantum level. Our authors came to a different conclusion. They explained that, although the “trace anomaly” is still present in their approach, it no longer signifies breaking of the local scale invariance. The reason lies in a cancellation of the trace terms of the quantized fields a by corresponding counter-terms arising from the scalar field, the “dilaton” in the language of the paper.

After some comments on the quantization of the metric field, and further discussions of the difference between strictly conformal theories and the Weyl geometrically “conformalized” ones, the authors finished with the remark:

The present work provides a general proof that with a suitable quantization procedure, the equivalence between conformal frames can also be maintained in the quantum theory (Codello et al., 2013, p. 21).

But they also stated clearly that their quantization procedure does not lead to new physical effects. In this sense their research shows a certain analogy to Kretschmann’s view of diffeomorphism invariant re-formulations of physical theories which do not per se lead to new physical insights.

Even so, the authors have achieved to show that the extension of the mathematical automorphism group of the underlying theories (SM fields, implicitly also gravity theory) can be upheld under quantization. Whether a further enrichment of the theories delivers new insights at the quantum level will be a question for the future. Probably this can only be the case, if the scalar field and/or the scale connection acquires a dynamical role beyond its purely mathematical “compensatory” character in the scale transformation.

4.4.2. Ohanian’s retake of a “spontaneous” breaking of symmetry. An attempt at giving the scale connection a dynamical role has been made by Hans Ohanian from the University of Vermont. He proposed a model which connects the standard model fields with general relativity in a Weyl geometric framework. A complex scalar field $\chi$ (“dilaton”) acts as the crucial mediator. It undergoes spontaneous breaking of local scaling symmetry which the author preferred to call conformal symmetry by a mechanism very similar to the breaking of electrodynamic $U(1)$ symmetry in a model studied by Coleman/Weinberg (1973). If gravitational effects can be neglected, Ohanian’s adaptation leads to the SM field content in flat spacetime.

Ohanian preserved the label “scale transformation” for a global usage in Minkowski space, where, in addition to the rescaling of the fields $X \mapsto \tilde{X} = \Omega^k X$, a space dilation $x \mapsto \tilde{x} = \Omega x$ is applied (Ohanian, 2016, p. 25).
If, on the other hand, gravity is taken into account, the transition from quantum to classical matter being leapfrogged, it leads to Einstein gravity as an “effective field theory”. Regarding the conformal expression of fields Ohanian used a “conformalization” procedure with additional terms in the (Riemannian) scalar curvature (in place of the more natural Weyl geometric expressions). Ohanian proposed to assimilate the result of Coleman/Weinberg by a simple substitution of coefficients and concluded:

After symmetry breaking, neither the scalar field nor the vector field reveal themselves at the macroscopic level, and we can ignore the effects of the Weyl gauge-vector on the transport of lengths . . . . (Ohanian, 2016, 10f.)

Because of the conformal coupling of the scalar fields to the Riemannian scalar curvature Ohanian found that in his approach a modification of Riemannian geometry is excluded in the long-range regime and comments:

This is in contrast to the standard Brans-Dicke theory, in which the massless scalar field makes a contribution to long-range gravitational effects, . . . (ibid.)

In the high energy, short-range, regime Weyl geometric curvature does play a role in this model, as Ohanian discussed in his section 4. Then the scale connection constitutes a “vector” field of its own, similar to the electromagnetic field, but with a mass term and with the dynamical current of the scale symmetry $\mathcal{J}^\mu = \frac{\partial L}{\partial \phi^\mu}$ as right hand side of the dynamical equation.

In his outlook Ohanian conjectured that certain problematic features in the purely conformal approaches are essentially due to the lack of a coherent metrical structure. In Weyl geometry the scale connection is the clue for making a Weylian metric consistent with conformal rescaling. Ohanian therefore finished his paper with a remark which went right to the heart of the matter:

If the analysis of Ehlers et al. is correct, the absence of a Weyl vector and its geometric paraphernalia is a fatal mistake – if no Weyl vector, then no conformally invariant theory with a geometric interpretation (Ohanian, 2016, p. 16).

In this approach Ohanian proposed a model which indicated why and how a Weyl field with curvature at the short-range, high energy level looses its curvature in the low energy regime and leads to Einstein gravity in the long-range limit.

Ohanian, like many other authors, perceived the transition between the energy regimes (high – low) exclusively in the sense of hypothetical successive temporal stages in the cosmic development. This fits in with the mainstream narrative connecting cosmology and high energy physics shortly after the big bang. Philosophically inclined reader may notice that one could interpret such kind of transition non-temporally, as a structural passage between different energy levels, present at any time and any place of the world. This

\[ \partial_v \left( \sqrt{|g|} f^\nu \right) = \mathcal{J}^\nu. \]In Ohanian’s Lagrangian \( \phi \) couples only to the “dilaton” scalar field \( \chi \). This leads to a form for the variation of the Lagrangian under scale transformations such that the dynamical current coincides with the Noether current (Ohanian, 2016, equ. (13)).
would be independent of the view regarding the reality content of the big bang picture.\footnote{Physicists may well claim that, e.g., the LHC experiments are important because they explore how the world has looked like a few “nanoseconds after the big bang”. But one need not take such stories at face value in order to appreciate the activities aiming at gaining knowledge about the respective energy levels and the transitions between them.}

5. **Weyl geometric models in astrophysics and cosmology since the 1990s**

5.1. **The broader context: scalar fields in gravity, conformal rescaling.**

In the 1970s JBD theory underwent a contradictory development: On the one hand, increasing precision of radar tracking observations in the planetary system showed that Einstein gravity is an extremely good description of gravity.\footnote{See C. Will’s contribution to this volume.} A tentative modification of the latter by a Brans-Dicke type scalar field has at least to be suppressed on this level, e.g. by an extremely high value of the coupling coefficient $\xi$ of the kinetic term in (10), or may it be not adequate at all. On the other hand, the rise of particle cosmology as a new subfield of theoretical physics opened ample space for studying models in an assumed very early phase of the universe. Here it appeared reasonable to think about modified gravity and elementary particle physics as an ensemble. A fertile environment for studying speculative models emerged, some of which were designed for combining the gravitational scalar field and a Higgs-type scalar field of elementary particle physics (Kaiser, 2006, 2007). This environment gave new motivations and incentives for studying scalar-tensor theories, completely different from those of the 1960/70s (Capozziello/Faraoni, 2011, chaps. 3, 7). One of the new roles rehearsed for the scalar field on this stage was that of an agent, called *inflaton*, which drives a hypothetical phase of very early accelerated expansion of the spacetime.

Another role arose from string theory where a new type of scalar field, a so-called *dilaton*, entered the stage. Originally it coupled to the trace of the (2-dimensional) stress tensor of the string. But in the form of a constraint for restoring conformal symmetry, after its breaking under quantization, the dilaton re-appeared as a source term in a classical Einstein-like equation. That gave rise to speculate about deriving Einstein gravity as an effective theory arising from string theory, with the dilaton scalar field and conformal symmetry as mediators (Brans, 2005, p. 14f.).

All in all, a vast field for studying scalar field theories in generalized theories of gravity arose.\footnote{For extensive surveys of this field see Fujii/Maeda, 2003, Capozziello/Faraoni, 2011.} Only few authors of this field remembered Weyl geometry and took it up for their purpose. This was the case, e.g., in string models; but they remain outside the scope of this survey. They would need a study of their own; here we look at more mundane manifestations of Weyl geometry in cosmology and astrophysics during the last two decades. Because of the close kinship between Weyl geometric rescaling and conformal invariance of field theories in a Riemannian environment I here bring only a few examples for recent conformal approaches in cosmology to the mind.
They are far from exhaustive and have been selected because they connect in specific ways to our core topic.

**Conformal approaches in cosmology.** An unusual analysis of the “dark” sectors of recent cosmology was given by Philip Mannheim and Demosthenes Kazanas. They argued that the flat rotation curves of galaxies can be explained on the basis of a conformal approach to gravity (Mannheim [1989]). In their conformal theory, a static spherically symmetric matter distribution was described by the solution of a fourth order Poisson equation

\[ \nabla^4 B(r) = f(r) \]

with a typical coefficient \( B(r) \) proportional to \(-g_{oo} = g_{rr}^{-1}\) of a metric \( ds^2 = g_{oo} dt^2 - g_{rr} dr^2 - r^2 d\Omega^2 \) (up to a conformal factor). The r.h.s. of the Poisson equation, \( f(r) \), depended on the mass distribution, e.g., in a spiral galaxy. The result of a comparison of their theory with data for 11 galaxies with different behaviour of rotation curves led to a good fit and encouraged the authors to present their approach as a possible candidate for a modified gravity explanation of dark matter phenomena (Mannheim [1989, 1994]).

During the following years the approach was extended to the question of dark energy in a peculiar perspective. In the special case of conformally flat models, like Robertson-Walker geometries, Mannheim proposed to consider the Hilbert-Einstein Lagrangian term \(-\frac{1}{12} |\phi|^2 R \sqrt{|\text{det} g|}\) of a conformally coupled scalar field \( \phi \) as part of the matter Lagrangian. Due to this sign choice, he arrived at a version of the Einstein equation with inverted sign. He interpreted this as a kind of “repulsive gravity” which supposedly operates on cosmic scales in addition to the “attractive gravity” on smaller scales, indicated by the conformally modified Schwarzschild solution. In his eyes, such a repulsive gravity might step into the place of the dark energy of the cosmological constant term of standard gravity (Mannheim [2000, 729]).

In spite of such a grave difference to Einstein gravity, Mannheim did not consider his conformal view to disagree with the standard model of cosmology and its accelerated expansion. He rather argued that his approach may lead to a more satisfying explanation of the expansion dynamics. In his view, “repulsive gravity” would take over the role of dark energy. Moreover he expected that a conformal approach with quadratic curvature terms may shed new light on the initial singularity and, perhaps, also on the black hole singularities inside galaxies.

A completely different approach using local conformal symmetry in particle physics and cosmology is due to Ishak Bars, Paul Steinhardt and Neil Turok. A silent background for their interest in this question seems to have been the idea of a cyclic, respectively oscillating, model of the universe, proposed a decade earlier by two of them as an alternative to the “inflationary” paradigm (Steinhardt/Turok [2002]). In the latter proposal the minima of the oscillation were related to some kind of speculative physics of the string and brane type. In (Bars et al. [2014]) the three authors explored the possibility that a conformal theory of gravity and the standard model

\[ \text{[125]} \]

For a historical discussion of oscillating models see (Kragh [2009]) and, in an even wider perspective, H. Kragh’s contribution to this book; C. Smeenk (this volume) nicely describes the rise of the inflationary paradigm.
fields might suffice for understanding the bridging process between two cycles without necessarily much new speculative physics. They worked with a locally scale invariant version of the standard model, combined with gravity, similar to Nishino/Rajpoot (section 4.3), but in the framework of purely conformal geometry rather than Weyl geometry. They considered a complex valued gravitational scalar field \( \phi \), called a dilaton, in addition to the Higgs field \( \Phi \), both scaling with the same weight (in our notation \( w = -1 \)). The dilaton couples only to the Higgs field by a common biquadratic potential like in (93) and to the right-handed singlet neutrinos by Yukawa terms of its own (Bars et al., 2014, p. 6). All other masses are “generated” by coupling to the Higgs field like in the standard model.

The authors investigated possible general forms for locally scale invariant gravitational Lagrangians including a kinetic term for the dilaton (equ. (10), loc. cit.). They were heading towards “a fully scale-invariant approach to all physics” (p. 5, loc. cit.) by several reasons. At first, the “dimensionless constants in a conformally invariant theory are logarithmically divergent as opposed to the quadratic divergence of a bare Higgs mass term” and the recent studies of (Codello et al., 2013) have shown that “the local scale invariance survives even though there is a trace anomaly” (Bars et al., 2014, p. 2). Moreover, so they claimed, the conformal freedom of choosing different scale gauges makes their cosmological models geodesically complete. That was a bit cavalier, but it is not the aim of this paper to evaluate such claims critically. More important, in our context, is to recognize the similarity in outlook between the Weyl geometric proposals for combining gravity with standard model fields in a consequently scale invariant approach and the concern of our three authors. Here a perspective on a putative “geodesic completeness” of cosmological models came in sight, although in a rather peculiar way, not taking into account the problem of an invariant characterization of the proper time along timelike geodesics.

This question was discussed in more detail by R. Penrose in his recent proposal for embedding the standard model of cosmology in a long cycle of iterations connected by conformal bridges between Riemannian phases of cosmic evolution (Penrose, 2006). He argues that for very high energy states in the past timelike trajectories lose their physical meaning anyhow and the whole physically relevant information can be described by the structure on the lightcone. By some not yet understood processes a similar argument is imputed for states in the asymptotic future. This idea developed a purely conformal perspective of how to extend Riemann-Einstein gravity beyond the conformally compactified past and future infinities and would probably fit well to the Bars/Steinhardt/Turok approach. Both proposals assumed that it is possible to develop a meaningful physics of the bridging process between to cycles under abstraction from all those geometrical features which distinguish Weyl geometry from a purely conformal structure.

\[\text{126} \text{The authors declared geodesic incompleteness as “an artifact of an unsuitable frame choice: geodesically incomplete solutions in Einstein frame may be completed in other frames, even though the theories are entirely equivalent away from the singularity” (Bars et al., 2014, p. 13).}\]
5.2. Diverse views of Weyl geometry in cosmology.

5.2.1. Continuation of Rosen’s work. M. Israelit investigated Weyl geometric methods in cosmology in the first half of the 1990s together with his mentor N. Rosen. After Rosen’s death in 1995 he continued publishing on his own for nearly two decades. In this work the question of dark matter was studied from different perspectives, always based on geometrical fields. 

Israelit/Rosen 1992 explored the neutral massive boson interpretation of the Weylian scale connection, hinted at by Rosen already in his 1982 paper (cf. section 2.2.2). The authors assumed a “chaotic Weylian microstructure”, constituted physically by a “Weylon gas”. On large distances the scale curvature effects were negligible and a Riemannian space structure arose in their approach. On this basis Rosen and Israelit started to study a hypothetical Bose-Einstein Weylon gas satisfying the equation of state $\rho = 3p$ and its consequences for different cosmological models in Einstein gravity (Israelit/Rosen, 1993).

In one of their next papers they turned toward the scalar field and tried to find out in which way it may contribute to dark matter “pervading all of cosmic space”, i.e., on the largest scales, not in the sense of local inhomogeneities in galaxies (like in theories of the MOND family) and in galaxy clusters. Although in their approach the Einstein gauge ($\beta = 1$) leads to “the usual formalism of general relativity” (Israelit/Rosen, 1995, p. 764) our two authors believed that different gauges with non-constant $\beta$-field might lead to new physical insight. They declared:

> Although the gauge function is arbitrary, it leads to the presence of dark matter which, in principle, can be observed. (Israelit/Rosen, 1995, pp. 777)

This was not particularly convincing, and it remained open how such observation “in principle” could be made.

In some papers of the late 1990s and several at the beginning of the new millennium Israelit continued this research line using integrable Weyl geometric gravity. In this context he realized that even under the assumption of an integrable Weylian scale connection the resulting modification of Einstein gravity can be non-trivial, if the potential of the scale connection $w$ is different from the scalar field, respectively its logarithm (Israelit, 1999, chap. 7), (Israelit, 1999a, equs. (17)f.) (compare our equ. (7), (9)). Israelit derived the dynamical equations with regard to $g_{\mu\nu}$, $\varphi_\mu$ and $\beta$, and also the Noether relations due to diffeomorphism invariance and to the scale invariance of the Lagrange density. The latter showed that on shell of the Einstein equation the dynamical equations of the scale connection $\varphi_\mu$ and of the scalar field $\beta$ are equivalent.

Israelit’s aim was to explain not only dark matter but also the accelerated expansion of standard cosmology by the gravitational scalar field which he

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127 Israelit died in 2015 at the age of 87. His last paper known to me is Israelit 2012, a slightly changed version of Israelit 2010.

128 Because of scale invariance there is, in fact, only one true scalar field degree of freedom (compare subsection 1.1).
called the “Dirac gauge function”\(^{129}\). In these papers Israelit tried to convince his colleagues that “cosmic matter was created by geometry”, viz. out of the energy of the gravitational scalar field (Israelit, 2002\(^a\), p. 295). According to him, his scalar field was able to generate dark matter and the magical substrate *quintessence* flourishing in the mainstream narratives on the early “history” of the universe. These were imaginative proposals.\(^{130}\)

In one of his latest papers Israelit came back to considering a non-integrable Weylian scale connection, now no longer as a representative of the electromagnetic potential but again as a field with massive bosons, “Weylons”, of spin \(-1\) and mass \(> 10\,\text{MeV}\). On a microlevel, so his argument, the Weyl geometric structure appears non-integrable, “chaotic”, while on larger scale there remains an effective gauge “vector field” with vanishing curvature. The author concluded with the remark:

“...the purpose of the present work was to show that on the basis of the Weyl-Dirac theory one can build up a model, where conventionally matter, DM and DE are created by geometry. This aim is achieved. (Israelit, 2010, sec. 8)"

The “creation” described by Israelit did not even claim to establish a connection between geometry and the standard model fields. His discussion appeared as a reflex from afar on the cosmological mainstream in which elementary particle physicists had been so successful in occupying the debate on the “early history” of the universe (Kaiser, 2006). Perhaps this is one of the reasons why in none of the investigations of our last section, nor in the ones discussed in the next subsection, we find much overlap with those of Rosen and Israelit.

5.2.2. *Weyl geometric extensions of gravity: trivial or provocative?* Coming from Jordan-Brans-Dicke theory, *Israel Quiros* got interested in Weyl geometry while still working at Santa Clara, Cuba, several years before his work mentioned in section 4.3.2. He was one of those in the JBD community who took Dicke’s proposal seriously, which postulated to state natural laws in a form that does not depend on (localized) choices of measurement units (Quiros et al., 2000; Quiros, 2000\(^a\), 2000\(^b\)). At first he developed the formulation of “dual” views for the interchange from Jordan to Einstein frame (Quiros, 2000\(^b\)). A decade later, after he had moved to León, Mexico, he wrote a joint paper with three other Mexican authors, *Jose E. Madriz, Ricardo García-Salcedo, Tonatiuh Matos* in which the authors explained how the different frames of JBD theory may be interpreted as “complementary geometrical descriptions of a same phenomenon” (Quiros et al., 2013\(^b\)).

From there it was only a small step to entering Weyl geometric gravity. As we have seen in the last section, Quiros looked, and still looks, for a common perspective on gravity and a scale invariant formulation of SM model fields, (Quiros, 2014\(^b\)). In a recent paper he investigated the purely conformal approach to scale invariant Lagrangian field theories and criticized them for

\(^{129}\) Israelit, 1996, 1999\(^a\), 2002\(^a\), 2002\(^b\); chapters 6 and 7 in his book (Israelit, 1999\(^b\)).

\(^{130}\) I doubt that they stand on a solid base, although I am unable to check them in detail (E.S.). In any case, they are too multifarious for being discussed in this survey.

\(^{131}\) Compare on this point (Capozziello/Faraoni, 2011, pp. 86ff.).
lacking a well-defined metrical structure with a uniquely determined affine connection. He concluded

\[ \ldots \text{that there will be problems with a theory which pretends to be Weyl-invariant only because the action – and the derived field equations – is invariant under (2) point-dependent scale transformations, E.S., but which is sustained by spacetimes whose geometrical structure does not share the gauge symmetry of the action.} \] (Quiros 2014a, p. 3)

Quiros therefore pleaded for the use of Weyl geometry as an appropriate framework for his research goal.

But alas, simplifying the Lagrangian used in (Quiros 2014b) he only foresaw a kinetic term for the Higgs (or a Higgs-like) field \( \Phi \), not for the gravitational scalar field \( \phi \) coupling to the Hilbert term. With a gravitational Lagrangian including quartic potential

\begin{equation}
(108) \quad L_{\text{grav}} = \frac{1}{12} \phi^2 R + \lambda \phi^4 \quad \text{(Quiros 2014a, equ. (20))}
\end{equation}

he found that his scalar field equation for \( \phi \) reduced to the trace of the Einstein equation like in the case of conformal coupling in Riemannian geometry. After pondering about the possibility of having “an infinity of feasible fully equivalent geometrical descriptions” and the resulting paradoxical picture of an “infinity of possible patterns of cosmological evolution” he passed over to Einstein scalar-field gauge as “simplest gauge one may choose”.

For the choice of (108) as the gravitational Lagrangian this resulted in the Hilbert action of Einstein gravity “minimally coupled to the standard model of particles with no new physics beyond the standard model at low energies” (Quiros 2014a, p. 9). His following discussion reduced to the simple observation that conformal rescaling allows to scale singularities away. All this remained without new physical insights or effects; in this sense the Weyl geometric extension of gravity considered by Quiros up to 2014 remained physically trivial. But it was characterized by a conceptually clear exposition of ideas and methods, so we may hope that in the further development of Quiros’ research program he will go beyond these limitations.

Carlos Castro, after the turn of the millennium working at the Centre for Theoretical Studies of Physical Systems in Atlanta, USA, had become acquainted with Weyl geometry already in the early 1990s (see section 3.2.1). At that time Santamato’s proposal for using Weyl’s scale connection for geometrizing the quantum potential stood at the center of his interest (Castro 1992). When he became aware of the new attempts at using Weyl geometric methods in gravity and in high energy physics, he took up the Weylian thread again. His guiding questions were now how Weyl’s scale geometry may be used for understanding dark energy and, perhaps, the Pioneer anomaly which at that time could still appear as a challenge for gravity theories (Castro 2007, 2009). Castro speculated with grand visions for his newly detected interest in Weyl geometric methods, in contrast to Quiros’ more sober perspective. An even sharper contrast comes forward with regard to

\footnote{A few years later high precision numerical modelling showed that thermal effects can completely account for the observations known as the flyby anomaly of the Pioneer spacecrafts (Rievers/Lämmerzahl 2011).}
Another unconventional view was put forward by the present author (Erhard Scholz, Wuppertal). His historical studies on the work of H. Weyl lead him to the impression that already the comparatively simple modification of Riemannian geometry by integrable Weyl geometry, combined with a non-trivial scalar field extension of Einstein gravity (in the sense of our section 1.1 with \( v + w \neq 0 \)), may shed new light on certain points of present day cosmology. He was glad to find some recent activities in Weyl geometric gravity among the Munich “group”, although it went in a different direction (section 4.2).

He found it most intriguing to see that in a Weyl geometric approach to gravity the cosmological redshift need no longer be due to an expansion of the spacelike folia of Friedmann-Robertson-Walker manifolds. This was clear because, in the transition from the Riemann gauge to Einstein gauge, the warp function may be scaled away partially or completely (Scholz, 2005b). Thus a part of the cosmological redshift \( z \) may be due to the time component \( \varphi_o \) of the scale connection, rather than to a spatial expansion of the “universe”. The reason for this observation is the scale invariance of \( z \), if scale covariant geodesics of weight \( w = -1 \) are used. With regard to cosmological observers defined by a timelike geodesic flow \( X \), the redshift is given by the quotient of energies \( E_0, E_1 \) of light signals (idealized “photons”) at the event of emission \( p_o \) and of observation \( p_1 \),

\[
(109) \quad z + 1 = \frac{E_0}{E_1} = \frac{g(\gamma'(\tau_o), X(p_o))}{g(\gamma'(\tau_1), X(p_1))},
\]

where \( \gamma(\tau) \) denotes the null-geodesic representing the trajectory of the signal. Because of the parametrization of the geodesics with weight \( w = -1 \) the quotient is scale invariant. Although for Robertson-Walker models with non-trivial (i.e., not constant) scalar field and warp function \( a(t) \) in Riemann gauge, the redshift seems to result from an expanding warp function, in Einstein scalar-field gauge it may at least partially be due to the scale connection, i.e., to a field effect of the additional component of the gravitational structure.

This effect is particular striking in certain models which appear expanding in Riemann gauge but have a static metric in Einstein gauge (Scholz, 2005a, 2009). Here the cosmological redshift in Einstein gauge turned out to be completely due to the time component of the Weylian scale connection, \( H = \varphi_o \), with \( H \) the Hubble parameter. Although Scholz initially overestimated the physical import of this example, it can probably be a fruitful epistemic provocation. As a toy model it may continue to serve as an incentive for critically rethinking the foundations of our standard picture of the universe.

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133 A similar argument was already given by Rosen (see section 2.2.2) and more recently by (Romero et al., 2012, sec. 7); compare (Perlick, 1989, chap. 5).

134 It was the topic of the author’s talk at the Mainz conference.
5.3. Attempts at dark matter, MOND-like. The success of modified Newtonian dynamics, MOND, since the 1980s for explaining the rotation curves of galaxies and the Tully-Fisher relation between the luminosity of spiral galaxies and their angular velocity led to diverse attempts for general relativistic generalizations (Sanders, 2010). Some of them introduced non-geometrical structures, like an additional vector field in the so-called tensor-vector-scalar field theory, TeVeS; but the earliest attempt at a relativistic MOND-like theory was formulated by Mordechai Milgrom and Jacob Bekenstein in the framework of JBD gravity (Bekenstein/Milgrom, 1984).

This approach worked with a non-quadratic kinetic Lagrangian for the scalar field with MOND-typical transition function; therefore its name “relativistic a-quadratic” rAQUAL theory. Bekenstein and Milgrom were torn between the Jordan frame and Einstein frame. In a later review paper by one of the authors the Jordan frame was declared to be the “physical metric”, while Einstein frame was considered as the “primitive metric” (Bekenstein, 2004, p. 6). In this framework the MOND-like free fall of particles in an extremely weak gravitational field could be derived. This approach was not free of shortcomings, as the authors themselves remarked: Lensing effects seemed unexplainable by the approach, because the conformal change between the two “dual” frames seemed not to affect light-like geodesics. Moreover, the scalar field allows perturbations which propagate with superluminal velocity. The authors relativized this problem, however, by adding that such perturbations probably “cannot induce acausal effects in the behavior of particles and electromagnetic fields”, because they only relate to the conformal factor of the metric (Bekenstein/Milgrom, 1984, p. 14). An additional critical point, not only for rAQUAL but for all theories of the original MOND family, was their inability to explain the anomalous dynamics in galaxy clusters, without assuming some additional unseen matter.

Because of the close relation between integrable Weyl geometric gravity and JBD theory, Bekenstein’s and Milgrom’s rAQUAL may be an interesting challenge for testing what happens if it is transformed into a scale invariant framework. In two recent papers the present author investigated this problem (Scholz, 2016a,b). The first paper contained a Weyl geometrical reformulation of rAQUAL, at least for the so-called “deep MOND” regime and the upper transitional regime. Different from Bekenstein/Milgrom’s

135Bekenstein considered it still in 2004 as “evident” that measurements with “clocks and rods” are expressed by the Jordan metric. Moreover the latter’s Levi-Civita connection governs the free fall of test particles. But the dynamics does not satisfy the “usual Einstein equation” in Jordan frame (because of explicit terms in the scalar field). The Einstein frame represented for him the “primitive metric” because here the gravitational action reduces to the classical form of the Hilbert term, and the dynamics is given by the Einstein equation (Bekenstein, 2004, p. 5f.). In their common paper, Milgrom and Bekenstein used the terminology of “dual descriptions” working in “gravitational units” (Einstein frame) respectively “atomic units” (Jordan frame), which sounded a bit like Dirac’s distinction (Bekenstein/Milgrom, 1984, p. 14).

136In his later review paper Bekenstein qualified this point by stating that only as long as the scalar field “...contributes comparatively little to the energy-momentum tensor, it cannot affect light deflection, which will thus be due to the visible matter alone” (Bekenstein, 2004, p. 6). One can read this observation the other way round: If the scalar field carries a considerable contribution to the energy-momentum it influences light deflection.
view, in the Weyl geometric approach observable quantities are most directly expressed in Einstein gauge. Spacelike components of the Weylian scale connection, $\varphi_j$, $j = 1, 2, 3$ express additional accelerations in comparison with those induced by the Riemannian part of the metric (corresponding to Newtonian ones in the weak gravity limit). In the extremely weak gravity regime two different components of additional accelerations, $a_\varphi$, $a_\psi$, can be distinguished. The first one is part of the Riemannian acceleration, in Einstein gauge, and due to the energy density of the scalar field $\varphi$; the second one results from the the Weylian scale connection $\varphi$ (in Einstein gauge). In extremely weak static gravitational constellations (i.e., order of magnitude of Newton acceleration $a_N$ close to the MOND acceleration $a_o \approx 1.2 \times 10^{-10} \text{ms}^{-2}$), MOND-like phenomenology is reproduced similar to rAQUAL. But here half of the additional acceleration is due to the scalar field’s energy. It thus influences the light trajectories. Whether this suffices for explaining the observed lensing effects remains to be seen.

Moreover, contributions on different length scales to local inhomogeneities of the scalar field’s energy density can add up to a common effect. This seems to have striking consequences for the dynamics of galaxy clusters. In a heuristic investigation of data from 17(+2) clusters our author found an encouraging agreement of accelerations predicted by the Weyl geometric scalar tensor theory with the corresponding empirical values. This was done on the basis of the observed baryonic masses alone, without assuming additional unseen, “dark”, matter.

Calculations with ordinary MOND, or even its relativistic generalization TeVeS, reduces the need of assuming additional hypothetical dark matter, but cannot do completely without it. R. Sanders argued that sterile neutrinos could do the job. In this context it is interesting to see how, in the Weyl geometric framework, an outlandish kinetic term of the scalar field seems to suffice for explaining the otherwise anomalous dynamics of galaxy clusters.

Of course, there remained problems: For safeguarding the dynamics on the solar system level the author invented an (ad-hoc) hypothesis postulating that scalar field inhomogeneities are suppressed in regions where the value of at least one sectional curvature of the Riemannian component exceeds a certain threshold. That saves the dynamics, but the origin of such a hypothetical suppression remained unclear. Moreover, the cosmological consequences of this approach are far from clear. In spite of such shortcomings this model may be of some value for exploring the possibilities of Weyl geometric scalar fields in the realm of dark matter phenomena.

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137 Cf. last footnote.
138 The data for 2 clusters are outliers, already from the phenomenological point of view.
139 The famous Coma cluster which led Zwicky introduce the hypothesis of dark matter is among the galaxy clusters for which the Weyl geometric model is consistent with most recent empirical data on mass distributions and accelerations.
140 Unpublished calculations indicate scenarios of a cosmic evolution in agreement with many features of standard cosmology: initial singularity, large parts of the cosmological redshift due to the expansion of spatial folia in Einstein gauge, accelerated “late time” expansion etc.
5.4. The Brazilian approach. A challenge to the standard big bang picture, drawing upon Weyl geometric methods, came from Brazil. Interest in Weyl geometrical approaches to cosmology have been present in the Brazilian theoretical physics community since the 1990s. The central person for this development, Mário Novello, acquired his doctorate in 1972 at Geneva under the supervision of J.M. Jauch. Already as a young PhD student he published a paper on Dirac spinors expressed in quaternionic calculus in a Weyl space. Back in Brazil and working at the Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, he cooperated with many international guests. In 2003 he became the founding director of the Instituto de Cosmologia Relatividade e Astrofísica (ICRA). Due to his influence Weyl geometric ideas were introduced in the Brazilian community of theoretical physicists in the course of the 1990s. They flourished and turned into an research tradition of its own, the Brazilian approach to Weyl geometric gravity as I want to call it.

5.4.1. A Palatini-type path to integrable Weyl geometry. In the early 1980s Novello and a co-author from Cologne, H. Heintzmann, reflected possible consequences for cosmology if one allows to model it in a slightly more general framework than Riemannian geometry. Like other authors before them, they used a metric-affine approach to gravity, presupposing a metric $g$ and an independent affine connection $\Gamma$. This allows to define curvature tensors like in Riemannian geometry, including the scalar curvature $R$. Starting from a gravitational Lagrangian which included a term of the form

$$L_R = -e^\omega R \sqrt{|g|}$$

with point-dependent function $\omega(x)$, metric and affine connection were varied independently according to the so-called Palatini approach. They found that the variation with regard to the connection implies

$$\nabla_\lambda g_{\mu\nu} = -\partial_\lambda \omega g_{\mu\nu}.$$  

Our authors immediately realized that this relation can be identified with the Weyl geometrical compatibility condition of our equation \[1\] for the integrable scale connection

$$\varphi = \frac{1}{2} d\omega$$

(in our notation). This approach was not without limitations: it identified the scale gauges in which the coefficient $e^\omega$ of the Hilbert term in \[1\] becomes constant and the one in which the scale connection \[2\] vanishes. In our terminology above no difference between Riemann gauge and scalar field – Einstein gauge could be conceived! This structural identification of the two gauges made the Palatini approach to Weyl geometric gravity a trivial extension of Einstein gravity, if the full scale invariance of the Lagrange density is observed. But such a comparison was not in the mind of our authors.

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141In this paper Novello still thought in terms of Weyl's first interpretation of the scale connection, the “em dogma” in the terminology above.

142In this paper $\omega(x)$ was not yet introduced as a scalar field of its own, but via the square of the electromagnetic potential $A_\mu$, i.e., $\omega = \log A_\mu A^\mu$. 

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Referring to Canuto et al. and in the wake of Dirac (cf. section 2.2), they pondered about the possibility that atomic clocks and gravitational clocks at different places might be related by a variable factor $\omega(x)$. If $\omega(x)$ is asymptotically constant, different “Riemannian domains” would arise, possibly connected by “Weyl integrable regions of space”. Moreover, the “age” of the universe might become “arbitrarily large” (Novello/Heintzmann 1983).

In the following years Novello developed broad activities in gravitation theory, elementary particle physics, and cosmology; in particular he was interested in understanding how the initial singularity of standard Riemann-Einstein cosmology can be avoided. In a joint paper with Edgar Elbaz, a colleague from France, Jose M. Salim and L.A.R. Oliveira from his group, he and his co-authors proposed an imaginative model for what they called the “creation of the universe” (a clause from the title of the paper) (Novello et al. 1992). Using some Weyl geometric features and a scalar field $\omega$, the authors were able to display a “cosmic” development from a flat vacuum state (described by Minkowski space) via a contracting phase, “bouncing” at a minimum of a scale function, to an expanding “inflationary” phase. Without going into details of this study we want to see how and why this paper became a classical point of reference for the Brazilian tradition in Weyl geometric methods.

Weyl geometry was introduced, like in (Novello/Heintzmann 1983), i.e., by the Palatini method of variation (111). This led to an integrable Weyl geometry characterized by a scalar function $\omega(x)$, the potential of the scale connection $\varphi = \frac{1}{2} d \omega$. For Novello, Elbaz et al. the above mentioned identification of Riemann gauge and Einstein did not appear detrimental, because their goal was not a modification of Einstein gravity. They rather set out modelling semi-classical quantum “perturbations of the system of measurement units” described by $\delta \omega$, such that

$$\delta(\nabla \omega) = (\delta \omega) g_{\mu \nu}.$$  

Perturbations of such a kind are inconsistent with Riemannian geometry but consistent with Weyl geometry, as the authors noted with references to (Ehlers et al. 1972; Audretsch 1983; Perlick 1991). Like many physicists in the last third of the 20th century, they thought in terms of a time-evolution of the cosmos, here even in the sense of a *temporal evolution of its geometrical structure*. They hoped to find “a definite conceptual context ... for the description of such structural transitions” during the cosmic evolution in the Weyl geometric approach (Novello et al. 1992, p. 650). For this goal they considered a process governed by the Lagrangian

$$\mathcal{L}_{\text{vac}} = (R + \xi \nabla \omega) \sqrt{|g|}$$  

(114) \hspace{0.5cm} \text{(Novello et al. 1992, equ. (4.2))},

In many papers of the Brazilian tradition (Novello et al. 1992) is quoted as a starting point (Salim/Sauté 1996; de Oliveira 1997; Fonseca-Neto 2011; Romero et al. 2012), to cite just a few. Sometimes it is even called the “first approach to scalar-tensor theory in WIST” [Weyl integrable space-time] Pucheu et al. (2016).
where $\nabla_\nu$ is (our) notation for the Weyl geometric derivative and $R$ denotes the Weyl geometric scalar curvature. From the point of view of Weyl geometry this was a hybrid approach; the Lagrange density was not scale invariant, although Weyl geometric concepts and expressions were used. The authors considered this an advantage, because a difference between “gravitational” units (expressed by a point dependent gravitational “constant”) and atomic units, originally assumed by Dirac, Canuto et al. (see section 2.2.1.), appeared unacceptable to them. They rather assumed a broken (active) scale symmetry (Novello et al. 1992, p. 653); a mere transformation of units in Dicke’s sense, i.e. a passive conception of scale covariance was not their case. Understandably, they decided for Einstein gauge as the expression for the broken symmetry state. They thus understood (114) as an “effective canonical action” of a broken underlying scale symmetric dynamics with some surviving residual Weylian terms. Guided by such kind of physical intuition the authors avoided a reduction of their approach to Einstein gravity, which would have become necessary, had they assumed full scale invariance.

The resulting dynamical equation could be expressed, without loss of content, in Riemannian terms,

$$(115) \quad gR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = \lambda^2 \omega_\mu \omega_\nu - \frac{\lambda^2}{2} \omega_\alpha \omega^\alpha g_{\mu\nu},$$

with $\lambda^2 = \frac{1}{7}(4\xi - 3)$. Then it was “equivalent to an Einstein equation in which the WIST\(^{145}\) field $\omega$ provides the source of the Riemannian curvature” (Novello et al. 1992, p. 655). As $\omega$ was the integral of the scale connection, it had a “purely geometrical origin” and appeared acceptable to them, although it had strange physical properties: negative energy density, positive pressure of the same value (“stiff” matter).

The scalar field equation derived from (114) and the Einstein equation (115) evaluated for a homogeneous, isotropic spacetime led to a model without initial singularity. In the far past it looks like a contracting Minkowski space with a non-trivial and in this sense “excited” scalar field $\omega$. After a first phase of an accelerated contraction, their warp function $a(t)$ reaches a minimum value $a_0$, after which it turns into an expansionary phase. The authors interpreted the first, contracting phase as a vacuum with a geometrical scalar field excitation. Near the minimum they sketched quantum processes of photon and baryon genesis “driven” by the scalar field. Then an expansionary phase follows, ending in a state which, so they argued, could connect to the radiation dominated phase of the standard model of cosmology. All in all the calculations were embedded in an imaginative narrative which claimed to solve several pressing problems inherent in the standard picture of the “hot big bang” (no initial singularity, causal horizon and flatness problems, matter anti-matter asymmetry).

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\(^{144}\) In a side remark the authors reminded that $-2\xi \omega_\lambda \omega^\lambda$ is a variationally equivalent kinetic term, because the difference to the kinetic term in (114) is a total divergence (Novello et al. 1992, p. 654).

\(^{145}\) WIST was (and is) the abbreviation, preferred by the Brazilian authors, for “Weyl integrable scalar tensor theory”.

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5.4.2. Cosmological models with fluid matter. Several follow up papers appeared, among them (Salim/Sautu 1996, de Oliveira 1997). In the first one Salim and S.L. Sautu added different types of “external fields” representing matter and its interaction to the vacuum Lagrangian (114). At first they dealt with an electromagnetic field and an external scalar field. Their matter terms of the Lagrangian had a scale invariant form (Salim/Sautu, 1996, equ. (12)). More important for cosmology, in the next step they adapted the Lagrangian of a perfect fluid following the trajectories of a timelike vector field to their framework. Here the hybrid form of the approach with the specified scale gauge was of great advantage, because it facilitated the adaptation of the fluid Lagrangian. The authors derived the dynamical equations and constraints in their framework, first in terms of the Weyl geometric derivative and curvature expressions, with particular taking care for the interaction with the geometrical scalar field ω (Salim/Sautu, 1996, equs. (34)–(40)). After that they rewrote the Einstein equation and the scalar field equation in Riemannian terms and derived the corresponding generalized Raychaudhuri equation for the homogeneous isotropic case (equ. (47)). Rewriting the coupling constant ξ of (114) by $\lambda = \frac{1}{2}(4\xi - 3)$ they concluded that “... depending on the sign of λ, the cosmological solution under consideration can be non-singular and inflationary” (Salim/Sautu, 1996, p. 359).

That was a considerable step forward and generalized the effect observed for the case of the special vacuum solution in (Novello et al., 1992). The authors rightly concluded:

We have shown that the Weyl integrable geometry can be used in a natural way to geometrize a long-range scalar field. Using a general principle to prescribe the interaction of the geometric scalar field with other physical systems, we can describe in WIST all the classical situations studied by EGR [Einstein gravity, E.S.]. (Salim/Sautu, 1996, p. 359)

In their next paper the two authors, now supported by Henrique P. de Oliveira, studied “non-singular inflationary cosmologies in Weyl integrable spacetime” (de Oliveira, 1997). To the gravitational Lagrangian (114) of Novello et al. they added a self-interaction potential of the scalar field $V(\omega)$ and the fluid Lagrangian of (Salim/Sautu, 1996). Referring to the same parameter λ as above, they came to the conclusion that for $\lambda > 0$ the Friedmann-like solutions had strong similarities to those of Einstein gravity, while for $\lambda < 0$ interesting “novelties appear in WIST” (p. 2835).

The three authors studied the qualitative behaviour of the modified Friedmann and scalar field equations of their model in the parameter plane $(x, y)$ with

$$x = \frac{\dot{a}}{a}, \quad y = \dot{\omega}.$$  

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146 This adds flavour to the hybrid approach mentioned above.
147 The fluid Lagrangian was taken over from Ray (1972).
For vanishing potential, $V(ω) = 0$, and for an exponential potential $V(ω) = V_0 e^{βω}$ ($V_0, β$ constants) they found that the solutions of the Friedmann equation are generically singularity free, while the solutions with initial or final singularity are stable (see our fig. 1). This was a striking result. The authors commented:

Depending on the parameter $λ$, we obtained non-singular models as a general feature. (...) The non-singular behaviour is explained by the violation of the strong energy condition provided by the geometric scalar field. (de Oliveira, 1997, p. 2842f.)

The explanation indicates that the “strong energy condition” was understood in the geometrical sense ($R_{μν}V^μV^ν > 0$ for any timelike vector field $V^μ$) \(^{148}\). Later investigations of the Brazilian school would show that the geometrical energy condition may be violated, while the physical energy condition may still be satisfied (see below).

5.4.3. A tension between the Palatini approach and scale invariance. Many more papers on Weyl geometric gravity were published by the Brazilian group. Some young researchers joined the network and started to publish with colleagues from the older generation in different constellations; among them and not yet mentioned before Tony S. Almeida, F.A.P. Alves, Adriano B. Barreto, J.B. Fonseca-Neto, F.P. Pouli and Carlos Romero (in alphabetical order). They dealt with the relationship of the Palatini variant of Weyl geometric gravity to Einstein gravity and to JBD theory, and continued to study singularity behaviour of cosmological models in their slightly extended framework.

\(^{148}\)The “physical sense” of the strong energy condition is $T_{μν} - \frac{1}{2} tr g_{μν} V^μV^ν$. Geometrical and physical conditions are equivalent in Einstein gravity; see, e.g., (Curie, 2017, p. 49).
Several of these papers (Fonseca-Neto, 2011; Romero et al., 2012) used a Lagrangian of the form:

\[
\mathcal{L} = e^{(1 - \frac{\omega}{2})\varphi} (R + 2\Lambda e^{-\omega} + \kappa e^{-\omega} L_m) \sqrt{|g|},
\]

where the scalar curvature is to be understood in the metric-affine sense \((R = R(g, \Gamma))\). After a Palatini type variation like in the transition from (110) to (111) \(R\) turns into the scalar curvature of a Weylian metric given by the pair \((g, \varphi = \frac{1}{2}(d\omega))\). The authors emphasized the importance of what appeared to them a “new kind of invariance, namely with respect to Weyl transformations” (Romero et al., 2012, 8) without, however, keeping coherently to scale invariance as a guiding principle of their investigation. Rewritten in Riemannian terms this Lagrangian acquires the form of a Brans-Dicke Lagrangian with a conformally coupled scalar field. Not very convincingly, this was presented as a “geometrization” of JBD scalar fields in general and, in addition, as an argument for a compatibility of Einstein gravity and scale invariance in the sense of Weyl geometry.

Such a generalization of Einstein gravity is clearly too weak to lead to interesting new features (see section 5.2.2); it even does not allow to recuperate the Lagrangian (114), so important for the Brazilian tradition. Other papers thus start from a metric-affine generalization of JBD-type Lagrangian with general coupling coefficient \(\varphi\), written in the form

\[
\mathcal{L}_{JBD} = e^{-\varphi}(R + \omega \partial_{\mu}\phi \partial^{\mu}\phi) \sqrt{|g|},
\]

with \(R = R(g, \Gamma)\) as above. Again the Palatini variation implied the relation (111) and, in this way, a motivation for specifying the metric and connection in the sense of integrable Weyl geometry (Almeida et al., 2014b, equ. (2)).

But then the kinetic term, taken over without change from usual, Riemannian, Brans-Dicke theory breaks the scale invariance for general \(\omega\). Accordingly the authors used a restricted Weylian scale transformation only. For the transition to the “Einstein frame” they transformed the quantities \(e^{-\varphi}\) and \(R\) only, while leaving the core expression of the kinetic term unaffected (a “field substitution” rather than a gauge transformation), with the result

\[
\mathcal{L} = (\tilde{R} + \omega \partial_{\mu}\phi \partial^{\mu}\phi) \sqrt{|g|},
\]

plus a matter action \(\mathcal{L}_m\). They were thus led back to the archetypical form (114) of the gravitational Lagrangian in the Brazilian tradition.

The tension between the Weyl geometric frame and the general methodology did not pass unnoticed by the authors. But it seems that they were prevented from resolving it, because their adherence to the Palatini method of variation and the difficulty to express the kinetic term of the scalar field in a scale invariant form without using Weyl geometric scale covariant derivatives (89). In the conclusion of one of the papers they wrote ‘that neither

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149 Fonseca-Neto, 2011 eq. (7)), (Romero et al., 2012 equ. (12))

150 An important conclusion (…) is that general relativity can perfectly ‘survive’ in a non-Riemannian environment” (Fonseca-Neto, 2011) etc.

151 (Almeida et al., 2014b equ. (3.24)), (Almeida et al., 2014d equ. (16)); compare in the light of fn. 144
the action nor the field equations of the proposed theory are invariant under Weyl transformations”, admitting that “it would perhaps be desirable, at least from the aesthetic viewpoint, that the whole theory should exhibit Weyl invariance” (Almeida et al., 2014, p. 8). In another paper three of them even gave a twisted explanation why this seemingly must be so (Almeida et al., 2014b, p. 39). This is surprising, because two years earlier two of them, in this case added by Fonseca-Neto, had already noticed that the Lagrangian (119) can be scale-transformed into the form of the general JBD Lagrangian (12). So they were quite close to bringing the Brazilian approach into a coherently scale invariant form but by some reason or other the members of the Brazilian group did not dare, or felt unable, to transcribe their approach in a scale invariant mode.

Gravitational Lagrangians of the form (119) also played a role in recent qualitative studies of “isotropic cosmologies in Weyl geometry” by John Miritzis from Athens and by authors from the Brazilian network itself (Miritzis, 2004; Pucheu et al., 2016). We find qualitative studies of cosmological models with or without (initial or final) singularities. The questions and results extended those of the 1990s. New glances at global singularities in the slightly extended framework described above were added (Lobo et al., 2015). After a detailed investigation of the Raychaudhuri equation, the authors of the last mentioned paper showed that the geometrical version of the strong energy condition can be violated in the Brazilian approach, while the physical one may be maintained due to contributions of the energy-momentum of the scalar field. This was a sharp observation and may be of wider import.

6. Discussion

6.1. A rich history aside the mainstream. Our survey over the reapperance of Weyl geometry has encountered four different entrance channels through which central concepts of Weyl’s scaling invariant but still fully metrical, geometry of 1918 were reintroduced into late 20th century physics. They differed in motivation and systematics; three of them were opened even twice by essentially independent research initiatives and slightly differing systematic ideas (these are characterized by an “and” in the following list):

1. Axiomatic foundations of gravity: Ehlers/Pirani/Schild (section 2.1)
2. Scale co/invariant scalar tensor theory of gravity: Omote/Utiyama and Dirac (section 2.2)
3. Cartan geometric approach: Bregman and Charap/Tait (section 2.3)
4. de Broglie-Bohm-Madelung (dBMB) approach: Santamato and Sho-jai/Shojai/Golshani (section 3)

The first three were initiated in the short time interval 1971–1974, and two of the openings were taken twice. Dirac’s motivations for his multiple gauge approach to gravity was quite idiosyncratic and played a minor role.

152 One only needed to put the JBD Lagrangians in a Weyl geometric framework. Alternatively, if one wants to start from the Brazilian point of view, one may read the constant coefficient of the Hilbert term in (119) as the value of a scale covariant scalar field \( \chi \) in (Einstein-) scalar field gauge, \( \chi_o = 1 \), and \( \partial \phi \partial \phi \) as the scalar field gauged expression of the scale covariant kinetic term \( D_{\mu} \chi D^\mu \chi \) with scale covariant derivative (3).
for its reception. The broader scenario indicates an intellectual environment which let it appear natural to come back to Weyl’s proposal of generalizing Riemannian geometry in a new field theoretic context. Bjorken scaling had attracted attention in the late 1960s, but was known to be only approximately valid already at that time. So it could not have been a major driving force. On the other hand, the field structures of elementary particle physics were just acquiring the form and status of a new, gauge theoretic standard model, due to to the renormalizability results of ’t Hooft and Veltman (1972) and the experimental detection of quark binding states, called “J-Ψ” (1974). Their Lagrangians were basically (globally) scale invariant in Minkowski space, with only the mass term of the hypothetical Higgs field as a scale breaking term. This context may have strongly motivated researches which explored possible connections to gravity in an enlarged scale covariant framework. In such a wider perspective a new look at Weyl geometric generalizations of Einstein gravity must have appeared a promising perspective. In this respect it was important that the Jordan-Brans-Dicke research program of scalar tensor theories had shown already a decade earlier how one could model gravity without taking recourse to a quadratic curvature term. It was natural to do so also in a renewed Weyl geometric setting. This brought it much closer to Einstein gravity than the quadratic gravity theories studied since the time of Weyl. That was important because during this time Einstein gravity lived through a vivid phase of new empirical confirmations.

The fourth opening was anchored in the completely different intellectual context of the de Broglie-Bohm program for reconsidering the foundations of quantum mechanics. Its two research lines started a decade later than the first three (Santamato), or even two decades later (Shojai/Shojai/Golshani) which in the following will be referred to as the “Tehran approach”. The two lines differed among each other more strongly than the respective double starts in items 2 and 3. In the 1990s the Bohmian approaches entered a latency phase (Santamato) or were just heading towards a new beginning, still developed in a JBD framework (Iranian approach). The authors of the latter started to use Weyl geometric concepts explicitly only after the turn to the new millennium.

All in all, the time until roughly 2000 was a first phase of exploration for all the approaches. For several years the immediate continuators of the Dirac line explored astrophysical consequences of Dirac’s distinction of an “atomic gauge” and “Einstein gauge” (Bouvier, Maeder, Canuto et al.) or refined and extended the theory (Rosen). At first they stuck to Dirac’s interpretation of the scale connection as an electromagnetic potential, the em dogma. In the 1980s such a literal allegiance of Dirac’s ideas faded out. Those who continued to appeal to Dirac’s approach, like Rosen and Israelit, enriched the perspective by considering the scale connection as a representative of a Proca-like massive gauge field, or saw it in a different context anyhow like Smolin. This boiled down to a merging of the modified Dirac

\footnote{Only for the authors of (Hehl et al., 1988) this appeared to be different, see section 2.3.}

\footnote{Cf. C. Will’s contribution to this volume.}
line with the research following Omote/Utiyama’s initiative (Hayashi, Kugo et al.), which intensified with the attempts to bring Weyl’s scale geometry in contact with the field content of the rising standard model (see below).

On the other hand, the later researches of Rosen and those of Israelit explored a vast terrain of theoretical possibilities, many of them quite speculative, of how the energy-momentum of a Weyl geometric gravitational scalar field or a hypothetical “Weylon gas” might contribute to dark matter phenomena and/or to the accelerated expansion diagnosed in the usual Riemannian approach to gravity. But these studies remained on a relatively general level and remained without closer links to astrophysical or astronomical observations (section 5.2.1).

The Cartan-Weyl geometric approach was soon relegated to a very special case in the broader Cartan geometric metric-affine theories (Hehl et al.) or was studied in relation to Kaluza-Klein theory. As the latter are not included in this survey, it disappear more or less from the range of our panorama. The foundational studies of Ehlers/Pirani/Schild, on the other hand, found a broad and continued reception and development in the philosophy of physics and remained a point of orientation for foundational studies of gravity.

For some authors (Englert, Smolin, Cheng, later Drechsler, Tann) the rise of the standard model suggested to connect Weyl geometric gravity, or at least scale covariant gravity in the case of Englert et al., with standard model fields, in particular the Higgs field. Cheng’s seminal paper of 1988 was the first relatively detailed account of the electroweak sector of the SM assimilated to a Weyl geometric context, although only on the pseudo-classical level of the theory (section 4.1). Here we also find explicit references to the papers of Dirac and the Utiyama research tradition, indicating the merging of these lines mentioned above. Nearly a decade later Drechsler and Tann found much of the electroweak structure in their own development of Weyl geometry, but with the peculiar idea of considering the Higgs field as a part of the gravitational structure.

In the new century this peculiar idea was superseded by the studies of Nishina and Rajpoot who continued the research opened up by Cheng and stayed closer to the mainstream expectations of a massive Higgs field which at that time was still hypothetical (section 4.3.1). With the empirical detection of the Higgs quantum excitation (‘‘particle’’) this line was accentuated as the most realistic among the Weyl geometric approaches to SM fields. But the question how scale symmetry is related to the quantum level remains still open. Ohanian’s attempt for convincing us that scale symmetry is ‘‘spontaneously broken’’ near the Planck scale and leads back to Einstein gravity developed a nice toy model (section 4.4.2), but the investigations

\footnote{155E.g. \cite{Drechsler/Hartley 1994}.}

\footnote{156Studies of QFT on Weylian manifolds, comparable to the corresponding researches for Lorentzian manifolds, discussed in R. Wald’s contribution to this volume, are still a desideratum.}

\footnote{157The two authors neither referred to the Dirac tradition in Weyl geometric gravity nor to Utiyama’s; their Weyl geometric starting point was “self-made” \cite{Drechsler/Hartley 1994} aside from Weyl’s original papers.}
of Codello et al. show that the last word has not yet been spoken (section 4.4.1).

With regard to astrophysics and cosmology the first exploratory phase of investigations, as it was called above, was superseded in the Brazilian research tradition of Weyl geometric gravity, initiated by the work of Novello et al. Although this research line has been confined to a geometrically “hybrid” approach which would imply a dynamically inert scalar field, if the Weyl geometric scaling symmetry would be taken seriously, this group of authors followed the physical intuition of their founding “father”, or at least of the founding paper of the tradition (Novello et al., 1992) which assumed an effective action of a broken underlying scale symmetry with some surviving residual Weylian terms (section 5.4). This allowed to investigate concrete cosmological models which give some impression of what possibilities a Weyl geometric extension of present Riemann-Einstein cosmological models might offer. From a different side a bridge to the family of MOND-like theories of dark matter has been established; it has widened the research horizon for the Weyl geometric extension of gravity theories even further (section 5.3).

All in all the scale covariant, and often explicitly Weyl geometric approaches to gravity, elementary particle fields, foundations of quantum mechanics, astrophysics and cosmology have developed a rich panorama of models since the 1970s. In many cases less known scientists contributed to this research. Once in while it attracted the attention of internationally renown physicists. Although the Weyl geometric perspective remained a side-stream in all of the mentioned fields up to now, it may well offer interesting challenges and openings for the future.

6.2. ... and an open research horizon. Our panorama has shown a variety of approaches which do yet not form a coherent research program as a whole. The Bohmian research lines, e.g., still stand separate from the other approaches, although some formal connections to the scalar fields of Dirac/Omote/Utiyama type have been established in the later phase. It is not clear, however, whether the “Tehran” perspective stands on solid grounds, and if so whether it can be integrated with the “Italian” approach into a consistent common picture. Other filaments of the whole field indicate perspectives which may reinforce each other.

Although cosmology has increased its observational basis in the past few decades so tremendously, it continues to call for alternative approaches to its many conundrums. Several contributions to this volume deal with such alternatives. The Weyl geometric approach joins this challenge, although for the time being with minor strength. The Brazilian work has made the most concrete contribution to this subject, but it is still hampered by the constraints resulting from the Palatini approach of variation (section 5.4). Moreover it has not yet started to fully explore the consequences of the rescaling freedom in Weyl geometric Robertson-Walker models and the possibility that part of the cosmological redshift may be due to the scale connection rather than to a “real” expansion (section 5.2.2). Such a turn towards a more field theoretic explanation of the cosmological redshift would

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158 The following perspectives of an “open research horizon” are necessarily subjective, but may be of help for orientation.
open new vistas for the geometry of cosmological model building. The Weyl geometric approach is clearly well suited for such investigations.

Among the most recent papers we have come across first steps towards a field quantization scheme in a Weyl geometric environment, which preserves scale symmetry at the quantum level (Codello et al., section 4.4). If this quantization procedure, or another one with the property of preserving the scale symmetry, can be extended to the complete set of standard model fields plus the Weyl geometric scale connection and the gravitational scalar field, we may arrive at a modest integration of gravity and the SM, in which only the scale degree of freedom of the metric is quantized. Bars/Steinhardt/Turok have already argued that a theory with scale symmetry at the quantum level may lead to a cancelling of the quadratically divergent terms in the radiative corrections to the Higgs mass, which constitute the hard core of the naturalness problem in present elementary particle physics.\textsuperscript{159}

This is a highly interesting observation, although still an unproven expectation. Together with the long standing speculations of the scalar field and/or the “Weylon” (scale connection) field as candidates for dark matter (sections 2.2.2 and 5.2.1) the Weyl geometric approach seems to offer chances for attacking the naturalness problem of the SM and the dark matter problem jointly, essentially by extending the underlying automorphism group of gravity and field theory. This complex of expectations has fed much of the research dynamics of the supersymmetry program; here we seem to be approaching a similar thematic complex in a more modest form. We also have seen that a classical, “effective” view of the gravitational field can lead to MOND-like phenomenology if also unusual kinematical terms for the scalar field are taken into account (section 5.3). We may thus look forward with interest and curiosity to see what the future research will lead to.

\textbf{Acknowledgements.} This paper owes its existence to David Rowe’s initiative in several respects. He encouraged me to present heterodox ideas on Weyl geometric methods in cosmology at the Mainz conference and invited me to rethink the case after a cool reception of the talk by the other participants. That gave me the chance to place my views in the wider range of the recent attempts for using Weyl geometric methods in physics. After an interruption of several years, an earlier first draft of this paper (Scholz \textit{2011b}) had be to be rewritten completely for the final version of this book. The new version overlaps nicely with the wider ambit of the investigations of the interdisciplinary group \textit{Epistemology of the LHC} with center at Wuppertal, generously supported by the DFG/FWF \textit{159}. This group offers the chance for a close communication between historians and philosophers of science and colleagues from the elementary particle community. David generously accepted the resulting oversize of the paper.

\textsuperscript{159} Bars et al. \textit{2014}, p. 2

\textsuperscript{160} Preprint number ELHC 2017-002.
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