Generalized optical theorem for Rayleigh scattering approximation

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March 2, 2022

Abstract

A general expression for the optical theorem for probe sources given in terms of propagation invariant beams is derived. This expression is obtained using the far field approximation for Rayleigh regime. In order to illustrate this results is revisited the classical and standard scattering elastic problem of a dielectric sphere for which the incident field can be any propagation invariant beam.

1 Introduction

In electromagnetic theory the Optical Theorem (OT) has a very long an interesting history (See R. Newton [1] and references therein). This physical concept of the OT is a useful result in scattering theory, relating the extinction cross section of a structure to the scattering amplitude in the forward direction [2, 3]. In applied sciences to understand this mechanics and how the absorption and the scattering affect wave propagation throughout a medium can help to obtain a meaningful features of the object such as the characterization and its physical properties [4].

Recently, some interesting reviews have been published [5, 6] concerning the development of generalized Lorenz-Mie theories, the Extended Optical theorem (EOT), using structured beam shape methods are presented, and the description of several electromagnetic effects by experiments, some applications using T-matrix methods for structured beam illumination.

The optical theorem has generated important and new applications, such as calculating the complex scattering amplitude on spherical and cylindrical objects, application in acoustic backscattering [7], to understand correctly the physical effects of propagation in radiation force and torque [8], without input nonphysical effect or based on numerical calculations (See [9] and references).

A generalized optical theorem for acoustic waves has been published by Marston and Zhang [10, 11], recently extended for arbitrary beams [12]. For electromagnetic fields [13, 14, 15, 16] quantum mechanics [17], in time domain, transmission lines, propagation in acoustic and electromagnetic waves in anisotropic medium [18, 19, 20], seismologic waves [21], the manipulation of the scattering pattern using non-Hermitian particles [22].

In the following section, a general derivation EOT applicable to any “nondiffracting beams” using the Huygens principle in the far field approximation for Rayleigh regime. A generalized scattering amplitude function is presented. These results are illustrated using a Bessel beam.

2 Theoretical background analysis

Let us consider a scattering particle of arbitrary form and size, with volume $V$ a boundary surface $\partial \Omega$ and, a complex permittivity $\varepsilon(\vec{r}) = \varepsilon_0 [\varepsilon'_r(\vec{r}) + i \varepsilon''_r(\vec{r})]$, where $\varepsilon_0$ is the vacuum permittivity. For simplicity the medium surrounding the scattering particle is lossless and its permittivity is $\varepsilon$ and not magnetic. Let us consider the incident arbitrary beams as $\vec{E}_i(\vec{r}), \vec{H}_i(\vec{r})$ that strikes the scattering particle, $\vec{E}_s(\vec{r}), \vec{H}_s(\vec{r})$ the scattered electromagnetic field and $\vec{E}(\vec{r}), \vec{H}(\vec{r})$ as the total fields [8, 2, 23, 24].

$$\vec{E} = \vec{E}_i + \vec{H}_s,$$  (1)
\( \vec{H} = \vec{H}_i + \vec{H}_s. \)  
(2)

The time-averaged power absorbed by the scattering object is given by

\[
P_a = -\frac{1}{2} \int_{\partial \Omega} \vec{n} \cdot \left[ \left( \vec{E}_i + \vec{E}_s \right) \times \left( \vec{H}_i^* + \vec{H}_s^* \right) \right] dS,
\]
(3)

this expression can be written in term of the Poynting vector as

\[
P_a = -\frac{1}{2} \int_{\partial \Omega} \vec{n} \cdot \left( \vec{S}_i + \vec{S}_s + \vec{S}' \right) dS,
\]
(4)

Where the first term of Eq. (4) is the incident Poynting vector

\[
\vec{S}_i = \frac{1}{2} \text{Re} \left[ \vec{E}_i \times \vec{H}_i^* \right],
\]
(5)

followed by the scattered vector

\[
\vec{S}_s = \frac{1}{2} \text{Re} \left[ \vec{E}_s \times \vec{H}_s^* \right],
\]
(6)

and the cross term interaction vector, which depend in term of the initial and scatter vectors

\[
\vec{S}' = \frac{1}{2} \text{Re} \left[ \vec{E}_i \times \vec{H}_s^* + \vec{E}_s \times \vec{H}_i^* \right].
\]
(7)

Using these equations and the right boundary condition which depend of the specific geometry. It is possible to obtain analytical and numerical solution for general scattering problem (See [23, 24] where the authors has reported a pedagogical tutorials).

In this letter, we follow and adapt the approach presented in (See. Refs [25, 26]) for an arbitrary structured beams, where in general the electromagnetic fields can be expressed as \( \vec{E}(r, t) = \text{Re}[E(r)e^{-i\omega t}] \) and \( \vec{H}(r, t) = \text{Re}[H(r)e^{-i\omega t}] \), where \( \omega \) is the harmonic frequency of the wave (single frequency).

Let us define the following incident fields as

\[
\vec{E}_i = \hat{e}_x E_0 \varphi(x, y) e^{ik_z z} e^{-i\omega t},
\]
(8)

\[
\vec{H}_i = \frac{1}{i\omega \mu_0} \nabla \times \vec{E}_i,
\]
(9)

where \( \varphi(x, y) \) physically represent a structured beam (scalar function) which satisfies the Maxwell’s equations and the transversal Helmholtz equation \( \nabla^2 \varphi + k_t^2 \varphi = 0 \), in Cartesian, circular, parabolic cylindrical, and elliptical coordinates [30, 27, 28, 29] and \( k_t \) is the transversal wave vector. For simplicity the factor \( e^{-i\omega t} \) is omitted.

In the following is considered a polarized electric field, with a complex amplitude \( E_0 \).

\[
\vec{E}_i = \hat{e}_x E_0 \varphi(x, y) e^{ik_z z},
\]
(10)

\[
\vec{H}_i \approx \hat{e}_y \frac{E_0}{\omega \mu_0} k_z \varphi(x, y) e^{ik_z z},
\]
(11)

The Poynting vector for these incident fields lies the plane XY [3].

In this approximation has been assumed that any impinging electric field can be written as invariant structured beam using the plane wave spectrum representation [30, 27, 28, 29].

\[
\varphi(x, y) = \int_0^{2\pi} A(\phi) e^{ik_z (x \cos \phi + y \sin \phi)} d\phi,
\]
(12)

where \( A(\phi) \) is an angular function. Note that for \( A(\phi) = \delta(\phi - \phi_0) \), the plane wave case is recovered, \( k_t \) is the transversal wave vector. This equation has the following interpretation. A structured or non-diffracting beam is given by a superposition of multiple plane waves having transversal wave vectors \( k_t \) on a circle. Only for particular functions \( A(\phi) \) on the Whittaker’s integral can be expressed analytically. For example higher order Bessel beams are defined by \( A(\phi) = e^{im\phi} \), where \( m \) is the azimuthal order of the beam. Mathieu beams are defined by \( A(\phi) = C(m, q, \phi) + iS(m, q, \phi) \) are the
Mathieu cosine and sine functions. Weber beam are defined by \( A(a, \phi) = \frac{1}{2(\pi|\sin \phi|+\tau^2)} e^{ia \ln |\tan \phi/2|} \) in turn is divided into even and odd case [27].

After substituting Eqs. (10) and (11) into Eq. (4) is written as

\[
P_a + P_s = -\frac{1}{2} \int_{\partial \Omega} \text{Re} \left[ \hat{E}_s \times \hat{H}_s^* + \hat{E}_s \times \hat{H}_s^* \right] \cdot \hat{n} dS,
\]

(13)

where \( \eta = \sqrt{\mu/\varepsilon} \) is denoted for the impedance. This is a general expression that represent from the physical point of view the interaction between the incident and scattered fields in term of the Green function, also related with the Huygens principle [25, 26]. Therefore, the scattered electric field in the far field can be expressed as

\[
\hat{E}_s = \int_{\partial \Omega} i \omega \mu G(\hat{r}, \hat{r}') \cdot \left[ \hat{n} \times \hat{H}_s \right] + \nabla \times G(\hat{r}, \hat{r}') \cdot \left[ \hat{n} \times \hat{E}_s \right] dS'.
\]

(15)

Alternatively, this field in the scattered vector in the far approximation can also be expressed as

\[
\hat{E}_s(\hat{r}) = e^{i k r} F(\hat{k}_i, \hat{k}_s) \cdot \hat{e}_i,
\]

(16)

where the function \( F(\hat{k}_i, \hat{k}_s) \) is called the scattering amplitude function, \( \hat{e}_i \) is an unitary vector. A similar expression is related with the pressure in acoustic [10, 11, 12].

This function can provide itself very interesting physics in scattering phenomena. Indeed, in Refs. [31, 32] the authors proposed a method how to obtain analytical expressions for the scattering amplitude function in order to explore an acoustic Bessel beam, later extended for Mathieu [33] and Weber [34] waves. For instance, in order to obtain an OT expression for arbitrary beam, we use Eq. (16) in the forward direction i.e \( \hat{k}_s = \hat{k}_i \), taking the product with \( \hat{e}_i \varphi^*(x', y') E_0^s \), and making the integration over the scattered object in the far field approximation for the electromagnetic case [35, 36, 37, 15, 16, 27]. Also a clear derivation for the ETO theorem for acoustic waves using Jones lemma can be reviewed [10, 11, 12].

\[
\hat{e}_i \varphi^*(x', y') \cdot \tilde{E}_s = \int_{\partial \Omega} \varphi^*(x', y') e^{-ik \cdot \hat{r}} \left[ \hat{e}_i \cdot (\hat{n} \times \hat{H}_s) + (\hat{k}_i \times \hat{e}_i) \cdot (\hat{n} \times \hat{E}_s) \right] dS',
\]

(17)

Therefore, without loss generality applying the same physical condition as [26, 25, 35, 36, 38] using Eq. (16) and (17), the optical theorem can be written as

\[
\sigma_{\text{ext}} = \frac{4\pi}{k} \text{Im} \left[ \hat{e}_i \cdot \varphi^*(x', y') F(\hat{k}_i, \hat{k}_s) \cdot \hat{e}_i \right].
\]

(18)
This expression recover the OT for the plane wave case,
\[
\sigma_{\text{ext}} = \frac{4\pi}{k} \text{Im} \left[ \hat{e}_i \cdot F(\hat{k}_i, \hat{k}_i) \cdot \hat{e}_i \right] \tag{19}
\]

3 Generalized scattering amplitude function

In this section, we show an application of the scattering amplitude function. Let us consider an electric field \( \vec{E} = 3 \hat{e}_i + \varepsilon \vec{E}_i \).

In order to avoid redundancy, we put forward the scattering amplitude function the Rayleigh regime reported \cite{38}, but here written in term of Whittaker integral as
\[
F(\hat{i}, \hat{o}) = k^2 a^3 \left( \frac{n^2 - 1}{n^2 + 2} \right) [\hat{e}_i - (\hat{o} \cdot \hat{e}_i) \hat{o}] e^{ikz} \int_{0}^{2\pi} A(\phi) e^{ik(x \cos \phi + y \sin \phi)} d\phi, \tag{21}
\]
where \( n \) is the complex refraction index, \( k \) is the wave vector and \( a \) is the sphere radius.

The square of this equation an integration over the solid angle \( d\Omega \) gives the differential scattering cross section as
\[
\frac{d\sigma_d}{d\Omega} = |F(\hat{i}, \hat{o})|^2 = k^4 a^6 \left( \frac{n^2 - 1}{n^2 + 2} \right)^2 I_{\text{beam}}, \tag{22}
\]
where \( I_{\text{beam}} = \left[ 1 - (\hat{o} \cdot \hat{e}_i)^2 \right] |\varphi(x, y)|^2 \), is the intensity expressed in term of a scalar potential, it is related with the incident angle and observation point, if \( \hat{e}_i \) is perpendicular to the scattering plane, and \( \hat{o} \cdot \hat{e}_i = \sin \theta \), if \( \hat{e}_i \) lies in the plane, \( |\varphi(x, y)|^2 \) is related with the intensity. Using this expression several probe fields can be used to measure the scattering cross section.

In the following, it is denoted \( \sigma_d \equiv d\sigma_d/d\Omega \).

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Figure 2: Normalized differential scattering cross section and the polarization scattered function for a Bessel beam with \( m = 0 \), using the equations \( \text{(23)} \) and \( \text{(24)} \) with \( \beta = 35^\circ \).

3.1 The Rayleigh scattering approximation using a Bessel beam

In this section, we consider an incident electric field expressed using Eq. \( \text{(20)} \) taking the angular spectrum as \( A(\phi) = e^{im\phi} \), which represent a linear polarized Bessel beam of \( m \) order is expressed as \( \vec{E}_i(\vec{r}) = \hat{e}_i J_m(k_i r) e^{ikz} \). Using this result it is possible to obtain the polarization function of the scattered radiation as
\[
\sigma_d(\theta) = \frac{k^4 a^6}{2} \left( \frac{n^2 - 1}{n^2 + 2} \right)^2 J_m^2(k_i r)(1 + \cos^2 \theta), \tag{23}
\]
In this problem, we have used the standard scattering geometry \[41, 42, 40\] to relate the scattering angle \(\theta = \arccos (\hat{e}_i \cdot \hat{o})\) to the unit vectors \(i\) and \(o\) in a particular point \(P\), where the scattered radiation is observed. In the other hand, if the incident field is unpolarized, the differential scattering function is the average over parallel and perpendicular incidents fields, then \(\sigma_d(\theta) = \frac{1}{2} \left[ \sigma_d^\parallel(\theta) + \sigma_d^\perp(\theta) \right]\), and \(\hat{o} \cdot \hat{e}_i = 0\) if \(\hat{e}_i\) is perpendicular to the scattering plane and \(\hat{o} \cdot \hat{e}_1 = \sin \theta\) if \(\hat{e}_i\) lies in the plane. Using these information we can calculate the polarization scattered function as

\[
\Pi(\theta) = \frac{\sigma_d^\perp(\theta) - \sigma_d^\parallel(\theta)}{\sigma_d^\perp(\theta) + \sigma_d^\parallel(\theta)} = \frac{\sin^2 \theta}{1 + \cos^2 \theta} J_m^2 (k_t r) .
\] (24)

Figure 3: Normalized differential scattering cross section and the polarization scattered function for a Bessel beam with \(m = 1\) and with \(\beta = 60^\circ\).

It is important to note, that integrating the equation Eq. (23) over \(d\Omega\) is straightforward to obtain \(\sigma_s\) \[38\], the scattering transversal section, while the absorbed transversal section \(\sigma_a\) have to be obtained with the Poynting vector \[2, 3, 25, 26\], since in other way, this would result in \(\sigma_{\text{ext}} = 0\). Note that taking \(r \to 0\) into (23) and (24) is recovered the plane wave case \[2\].

Finally, in order to validate our results, we show in Fig. 1 some numerical results, in which we compare the scattered radiation as function of the dispersion angle for both the differential cross section and the polarization scattered function, considering a plane wave and a Bessel beam with \(m = 0\). For simplicity, we make the analysis in one dimension and vary the angle from 0 to \(\pi\). For the Bessel beam case, \(k_t = k \sin \beta\) is the transversal wave vector, and \(\beta\) is the value of the half-cone angle.

In addition, this model reproduce the expected behavior for a plane wave \[2\]. For a plane wave case, the differential cross section present a minimum at \(\pi/2\) and the polarization scattered function a maximum at \(\pi/2\) See Figure 3, these angles the scattered radiation is linear polarized, this effect was proposed by Rayleigh to explain the blue sky \[2, 42, 41\]. At these same values for a zero Bessel beam, the differential cross section shows a reduction respect to the plane wave case, with a maximum around \(\pi/3\), while the scattered polarization function shows a clear decreasing of the scattered radiation.

In Figure 2, we show the three dimensional behavior of the differential cross section \((\sigma_s)_{BB}\) and the polarization scatter function \(\Pi(\theta)_{BB}\) with \(m = 0\) and \(\beta = 35^\circ\) for a Bessel beam. In addition, to illustrate how the scattering cross section and the polarization scattered function change with the variation of the incident field, we show in Figure 3 the behavior of these quantities for \(m = 1\) and \(\beta = 60^\circ\). Bi-dimensional scattered pattern were reported using only a zero order Bessel EM scattering in a dielectric sphere \[38\] where the author compare the incident Bessel beam as a function of the impinging angle \(\beta\).

4 Conclusions

A general optical theorem for any arbitrary beam was derived, using the amplitude scattering function and the Huygens principle in the far field approximation. the presented ordinary form of the optical theorem renders the particular case
for free space derived in previous works \cite{9,10,11,12}. A general representation for the extinction in Rayleigh scattering regime was studied and the effect for a linearly polarized Bessel beam of $m$ order as a function of the incident impinging angle $\beta$. From the physical point of view this method Eq. \eqref{22} can be extended using another spectral beam wave representation \cite{39}, it allows to study waves such as X waves, Airy, Frozen waves, among others. It would be interesting to explore as was stated \cite{12} the physics of differential cross section of scattering $d\sigma/d\Omega$ and for the extinction $d\sigma_{\text{ext}}/d\Omega$, the geometrical features and analogies between acoustic and electromagnetism. Several application such as the Rayleigh scattering of nanoparticles and optical forces \cite{43} using focused femtosecond laser pulses are in development. Applications measuring the optical extinction has been recently experimentally explored using a radial polarized cylindrical beams \cite{44}.

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