Time-Dependent Automorphism Inducing Diffeomorphisms, Open Algebras and the Generality of the Kantowski-Sachs Vacuum Geometry

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Abstract

Following the spirit of a previous work of ours, we investigate the group of those General Coordinate Transformations (GCTs) which preserve manifest spatial homogeneity. In contrast to the case of Bianchi Type Models we, here, permit an isometry group of motions $G_4 = SO(3) \otimes T_r$, where $T_r$ is the translations group, along the radial direction, while $SO(3)$ acts multiply transitively on each hypersurface of simultaneity $\Sigma_t$. The basis 1-forms, cannot be invariant under the action of the entire isometry group and hence produce an Open Lie Algebra. In order for these GCTs to exist and have a non-trivial, well-defined action, certain integrability conditions have to be satisfied; their solutions, exhibiting the maximum expected “gauge” freedom, can be used to simplify the generic, spatially homogeneous, line element. In this way an alternative proof of the generality of the Kantowski-Sachs (KS) vacuum is given, while its most general, manifestly homogeneous, form is explicitly presented.
1 Introduction

In a previous work [1] we have found and studied the group of those General Coordinate Transformations (GCTs) which leave the line element of a generic Bianchi Type Geometry, quasi-form invariant; i.e. preserve manifest spatial homogeneity. It was found that these GCTs, which mix time and space variables in the new space variables, induce special time-dependent automorphic changes on the spatial scale factor matrix $\gamma_{\alpha\beta}(t)$, along with the corresponding changes on the lapse function $N(t)$ and the shift vector $N^\alpha(t)$. These transformations—called Time-Dependent Automorphism Inducing Diffeomorphisms (AIDs)—contain the maximum expected “gauge” freedom i.e. 4 arbitrary functions of time for each and every Bianchi Type. Using this freedom one can significantly simplify the generic line element and thus the resultant Einstein’s Field Equations (EFEs). Also, the number of the expected essential constants calculated in each case, is in agreement with the suggested, corresponding, number in the established literature (e.g. [2, 3] and the references therein).

A basic point of this analysis, is that Bianchi Type Geometries, are characterized by the existence of a 3-dimensional isometry group of motions $G_3$, which acts simply transitively on each hypersurface of simultaneity $\Sigma_t$. This means that there exists an invariant basis of 1-forms $\sigma_i^\alpha(x)$, satisfying:

$$\sigma_{i,j}^\alpha(x) - \sigma_{j,i}^\alpha(x) = 2 C_{\mu\nu}^\alpha \sigma_j^\mu(x) \sigma_i^\nu(x) \quad (1.1)$$

where $C_{\mu\nu}^\alpha$ are the –space independent– structure constants of the corresponding isometry group $G_3$.

In the present letter we wish to extent this analysis in the case of open algebras; i.e. when the $C_{\mu\nu}^\alpha$s, are space dependent and thus become structure functions. This usually happens when the isometry group of motions $G_r$, acts multiply transitively on each hypersurface of simultaneity $\Sigma_t$—see [3, 4] for more details on the subject. In 3 dimensions the KS Type spaces are defined [5, 6] as those admitting an isometry group $G_4$ which acts on spacelike hypersurfaces, with no subgroup $G_3$ acting simply transitively on the hypersurfaces—but instead, an Abelian subgroup $G_2$, acting on these. Because each four dimensional Lie algebra contains a three dimensional subalgebra [6], there exists a three dimensional isometry group if a four dimensional one exists. Thus also for the KS model, there is a $G_3$ subgroup which however acts simply transitively only on two dimensional spacelike surfaces which consequently are of constant curvature (maximally symmetric two dimensional spaces). Kantowski [6] showed that two dimensional surfaces of zero and negative curvature give rise to four dimensional invariance groups, which have simply transitive three dimensional subgroups. Thus the only possibility left is that of the two dimensional surfaces of positive constant curvature, i.e. two dimensional spheres.

The Killing Vector Fields (KVF)s, in $r, \theta, \phi$ coordinates, are:

$$K_1 = -\cos(\phi) \frac{\partial}{\partial \theta} + \cot(\theta) \sin(\phi) \frac{\partial}{\partial \phi} \quad K_2 = \sin(\phi) \frac{\partial}{\partial \theta} + \cot(\theta) \cos(\phi) \frac{\partial}{\partial \phi}$$

$$K_3 = \frac{\partial}{\partial \phi} \quad K_4 = \frac{\partial}{\partial r} \quad (1.2)$$
The corresponding Lie algebra, is:

\[ \{ K_i, K_j \} = \varepsilon_{ij}^k K_k, \quad i, j, k, \in \{ 1, 2, 3 \} \]

\[ \{ K_4, K_j \} = 0, \quad j \in \{ 1, 2, 3 \} \]

where \( \varepsilon_{ij}^k \) is the Levi-Civita symbol. One may see that, \( K_1, K_2, K_3 \) correspond to the \( G_3 \) which acts on two dimensional spacelike surfaces of positive constant curvature (two spheres), while \( K_3, K_4 \) correspond to the Abelian \( G_2 \) which acts on the three dimensional spacelike hypersurfaces. If one demands that the generic 3-metric \( g_{ij}(x) \) be invariant under all KVFs i.e. \( L_{K}g_{ij} = 0 \), one obtains the form \( g_{ij}(x) = A\delta_{\alpha\beta}\sigma_\alpha^\mu(x)\sigma_\beta^\nu(x) \) where \( A \) is a positive number, \( \delta_{\alpha\beta} \) is the identity 3-matrix and \( \sigma_\alpha^\mu(x) \) a basis of 1-forms, which is invariant under the action of the \( G_2 \) only:

\[ g_{ij}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin(\theta) \end{pmatrix} \]

They satisfy:

\[ d\sigma_\alpha^\mu = C_{\mu\nu}^\alpha \sigma_\mu^\nu \Leftrightarrow \sigma_\alpha^\mu(x) - \sigma_\alpha^\mu(x) = 2 C_{\mu\nu}^\alpha(x)\sigma_\mu^\nu(x) \]

where \( C_{33}^3(x) = \text{Cot}(\theta)/2 \) is the only non vanishing, structure function.

If one considers transformations of the form \( x^i = f^i(\tilde{x}^j) \) and demands the manifest splitting to be preserved, then when an invariant basis exists, one arrives at a 3-metric \( \tilde{g}_{ij}(\tilde{x}) = \tilde{\gamma}_{\mu\nu}(\tilde{x})\sigma_\mu^\nu(\tilde{x}) \)–see \[7\] for details– where \( \tilde{\gamma}_{\mu\nu} = \Lambda_\mu^\alpha \Lambda_\nu^\beta A\delta_{\alpha\beta} \) and \( \Lambda_\mu^\alpha \) must belong to the automorphism group of the Lie Algebra i.e. must satisfy \( C_{\alpha\beta}^\gamma \Lambda_\mu^\alpha \Lambda_\nu^\beta = C_{\mu\nu}^\gamma \). For the case of the basis \((1.4)\), the elements of which form an open algebra, the notion of automorphisms can be extended; they must be the matrices satisfying \((2.12)\)with zero left-hand side. Thus the following admissible matrices \( \Lambda_\mu^\alpha \)s are obtained:

\[ \Lambda_\mu^\alpha = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_5 & 0 \\ 0 & \lambda_8 & \lambda_9 \end{pmatrix} \]

with the condition \( x^2 = \text{ArcCot}(\text{Cot}(\tilde{x}^2)/\lambda_5) \). Notice that the subgroup of the above transformations designated by \( \lambda_5 = 1 \), is exactly what one would call automorphisms from the cotangent space point of view i.e. elements of \( GL(3, \mathbb{R}) \) which leave the structure function tensor invariant in form and value. The most general 3-metric is thus given by the basis \((1.4)\) and the matrix:

\[ \tilde{\gamma}_{\mu\nu} = \Lambda_\mu^\alpha \Lambda_\nu^\beta A\delta_{\alpha\beta} = \begin{pmatrix} \tilde{\gamma}_{11} & \tilde{\gamma}_{12} & 0 \\ \tilde{\gamma}_{12} & \tilde{\gamma}_{22} & \tilde{\gamma}_{23} \\ 0 & \tilde{\gamma}_{23} & \tilde{\gamma}_{33} \end{pmatrix} \]
It is noteworthy that the \((1, 3)\) component of the matrix \((1.7)\), can not acquire a non zero value through the diffeomorphisms considered. One might think that other, more general coordinate transformations might fill this gap but this is not true: if one calculates the Ricci scalar of a 3-metric with such a non zero component of \(\tilde{\gamma}_{\alpha\beta}\) –in the given basis one forms–, one would find it to depend on \(x^2\) and therefore, would conclude that this metric is not spatially homogeneous.

2 AIDs and the generality of the KS vacuum model

In the 3+1 decomposition –see e.g. [8]– one makes use of the lapse function \(N\) and the shift vector \(N^i\) and sees the 4-dimensional line element parameterized as:

\[
ds^2 = (N^i N_i - N^2) dt^2 + 2 N_i dx^i dt + g_{ij} dx^i dx^j
\]

(EFEs corresponding to (2.1), expressed in terms of the extrinsic curvature:

\[
K_{ij} = \frac{1}{2N} (N_{ij} + N_{ij} - \frac{\partial g_{ij}}{\partial t})
\]

are:

\[
H_0 = K^i_j K^j_i - K^2 + R = 0
\]

(2.2a)

\[
H_i = K^j_{ij} - K^j_i = 0
\]

(2.2b)

\[
\partial_t K^i_j - N K^i_j + N R^i_j + g^{il} N_{ijl} - (K^i_{jl} + K^i_{lj} N^j_{il} - K^j_{lj} N^i_{il}) = 0
\]

(2.2c)

If one restricts attention to spatially homogeneous spacetimes, and uses a set of basis 1-forms to decompose the spatial part of the metric and the shift vector, one will find that the line element (2.1) assumes the form:

\[
ds^2 = (N^\alpha(t) N_\alpha(t) - N^2(t)) dt^2 + 2 N_\alpha(t) \sigma^\alpha_i(x) dt dx^i + \gamma_{\alpha\beta}(t) \sigma^\alpha_i(x) \sigma^\beta_j(x) dx^i dx^j
\]

(Latin indices are spatial, with domain of definition \(\{1, 2, 3\}\), while Greek indices number the different basis 1-forms, with the same domain of definition, and are lowered and raised by \(\gamma_{\alpha\beta}(t)\) and \(\gamma^{\alpha\beta}(t)\), respectively.

Insertion of relations (1.5), (2.3), into (2.2) results in the following set of Ordinary Differential Equations (ODEs):

\[
E_0 \doteq K^\alpha_\beta K^\beta_\alpha - K^2 + R = 0
\]

(2.4a)

\[
E_i \doteq (2K^\xi_\rho C^\rho_{\xi\epsilon} - 2K^\xi_\rho C^\rho_{\xi\epsilon} + K^\alpha_\beta \sigma^\alpha_\beta) \sigma^\beta_i(x) - K_i = 0
\]

(2.4b)
\[ E^\alpha_\beta = \dot{K}^\alpha_\beta - N KK^\alpha_\beta + NR^\alpha_\beta + 2N^\xi (C^\alpha_\xi K^\xi_\beta - C^\rho_\xi K^{\alpha}_\rho) - K^{\alpha}_{\beta, m} \sigma^m_\xi(x) N^\xi \] (2.4c)

and:

\[ K_{ij} = -\frac{1}{2N} \left( \dot{\gamma}_{i\beta} + 2 \gamma_{i\nu} C^\nu_\mu N^\mu + 2 \gamma_{\beta\nu} C^\nu_\mu N^\mu \right) \sigma^i_i \sigma^j_j \equiv K_{\alpha\beta} \sigma^i_i \sigma^j_j \] (2.5)

\[ R_{ij} = (C^\alpha_\sigma C^\sigma_\mu \gamma_{\alpha\mu} \gamma^\nu \gamma^\tau \mu + 2 C^\lambda_\alpha \gamma^\lambda_\mu \gamma^\nu \gamma^\tau \mu + 2 C^\nu_\alpha \gamma^\nu \gamma^\mu \gamma^\lambda \gamma^\tau \nu + 2 C^\lambda_\mu \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\tau \nu \nonumber + 2 C^\lambda_\mu \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\tau \nu \sigma^i_i \sigma^j_j + 2 C^\mu_\nu \sigma^\mu_{\mu\nu} - C^\mu_{\nu\delta} \sigma^\mu_{\delta} + \sigma^i_i \gamma^\epsilon_{\epsilon\delta} C^\epsilon_{\epsilon\delta\tau} \left( \sigma^i_i \sigma^j_j + \sigma^i_i \sigma^j_j \right) \equiv R_{\alpha\beta} \sigma^i_i \sigma^j_j \] (2.6)

as well as:

\[ K^i_j = g^i\omega K_{\omega j} = \sigma^i_\alpha \gamma^\alpha_{\omega\nu} \sigma^\omega_{\nu j} = \sigma^i_i \sigma^j_j K_{\alpha\beta} \]

\[ K = g^{ij} K_{ij} = \gamma^{\alpha\beta} K_{\alpha\beta} \]

\[ R = g^{ij} R_{ij} = C^\alpha_\mu C^\beta_\nu \gamma_{\alpha\mu \nu} \gamma_{\gamma\lambda} + 2 C^\mu_\alpha \gamma_{\alpha \mu \nu} \gamma_{\nu \lambda} + 4 C^\alpha_\mu \gamma_{\mu \nu} - 4 C^\alpha_{\alpha, m} \sigma^\beta_n g^{mn} \]

It is noteworthy that the quantities with Greek indices or no indices, are time dependent only—although not manifestly. Equation set (2.4), forms what is known as a—complete—perfect ideal; that is, there are no integrability conditions obtained from this system. So, with the help of the third of (2.4d), (2.3), (2.7), it can explicitly be shown, that the time derivatives of (2.4a) and (2.4b) vanish identically. The calculation is straightforward—although somewhat lengthy; the Jacobi identity \( C^\alpha_\xi \sigma^\xi_\omega \) = 0 (of course, because of the structure functions the appropriate Jacobi identity would, in general, contain a term \( C^\alpha_\xi \sigma^\xi_\omega \) which however vanishes for structure functions here considered). The vanishing of the time derivatives of the 4 constraint equations: \( E_0 = 0, E_i = 0 \), implies that these equations, are first integrals of (2.4c)—moreover, with vanishing integration constants. Based on the intuition gained from the full theory, one expects that the freedom which corresponds to these integrals, is a reflection of the only known covariance of the theory; i.e. of the freedom to make arbitrary changes of the time and space coordinates.

Let us first consider the time reparameterization invariance; if a transformation:

\[ t \rightarrow \tilde{t} = g(t) \Leftrightarrow t = f(\tilde{t}) \] (2.7a)

is inserted in the line element (2.3), it is easily inferred that:

\[ \gamma_{\alpha\beta}(t) \rightarrow \gamma_{\alpha\beta}(f(\tilde{t})) \equiv \tilde{\gamma}_{\alpha\beta}(\tilde{t}) \] (2.7b)

\[ N(t) \rightarrow \pm N(f(\tilde{t})) \frac{df(\tilde{t})}{d\tilde{t}} \equiv \tilde{N}(\tilde{t}) \]

\[ N^\alpha(t) \rightarrow N^\alpha(f(\tilde{t})) \frac{df(\tilde{t})}{d\tilde{t}} \equiv \tilde{N}^\alpha(\tilde{t}) \] (2.7c)
Accordingly, $K^\alpha_\beta$ transforms under (2.7a) as a scalar and thus (2.4a) and (2.4b) are also scalar equations, while (2.4c), gets multiplied by a factor $df(t)/dt$. Thus, given a particular solution to equations (2.4), one can always obtain an equivalent solution, by arbitrarily redefining time. Hence, the existence of one arbitrary function of time in the general solution to Einstein’s equations (2.4), is understood.

In order to understand the presence of the other arbitrary functions of time, it is natural to turn our attention to the transformations of the 3 spatial coordinates $x^i$. To begin with, consider the transformations:

$$
t = t \iff \tilde{t} = \tilde{t}
$$

$$
x^i = g^i(x^j, t) \iff \tilde{x}^i = f^i(\tilde{x}^j, \tilde{t}) = f^i(\tilde{x}^j, t)
$$

Under these transformations, the line element (2.3) becomes:

$$
ds^2 = \left( N^\alpha(t)N_\alpha(t) - N(t)^2 \right) dt^2 + 2\sigma^\alpha_i(\tilde{x}) \frac{\partial f^i}{\partial \tilde{t}} N_\alpha(t) + \frac{\partial f^i}{\partial \tilde{t}} \frac{\partial f^j}{\partial \tilde{t}} \sigma^\alpha_i(\tilde{x})\sigma^\beta_j(\tilde{x})\gamma_{\alpha\beta}(t) \right) dt^2
$$

$$+ 2\sigma^\alpha_i(\tilde{x}) \frac{\partial f^i}{\partial \tilde{x}^m} \left( N_\alpha(t) + \sigma^\beta_j(\tilde{x}) \frac{\partial f^j}{\partial \tilde{t}} \gamma_{\alpha\beta}(t) \right) d\tilde{x}^m dt
$$

$$+ \sigma^\alpha_i(\tilde{x})\sigma^\beta_j(\tilde{x})\gamma_{\alpha\beta}(t) \frac{\partial f^i}{\partial \tilde{x}^m} \frac{\partial f^j}{\partial \tilde{x}^n} d\tilde{x}^m d\tilde{x}^n
$$

(2.9)

Since our aim, is to retain spatial homogeneity of the line element (2.3), we have to refer the form of the line element given in (2.4a) to the old basis $\sigma^\alpha_i(x)$ at the new spatial point $\tilde{x}^i$. Since $\sigma^\alpha_i$ –both at $x^i$ and $\tilde{x}^i$–, as well as, $\partial f^i/\partial \tilde{x}^j$, are invertible matrices, there always exists a non-singular matrix $\Lambda^\alpha_\mu(\tilde{x}, t)$ and a triplet $P^\alpha(\tilde{x}, t)$, such that:

$$
\sigma^\alpha_i(x) \frac{\partial f^i}{\partial \tilde{x}^m} = \Lambda^\alpha_\mu(\tilde{x}, t)\sigma^\mu_m(\tilde{x}) \Rightarrow \frac{\partial f^i}{\partial \tilde{x}^m} = \Lambda^\alpha_\beta(\tilde{x}, t)\sigma^\beta_m(\tilde{x})\sigma^\alpha_i(x)
$$

$$
\sigma^\alpha_i(x) \frac{\partial f^i}{\partial \tilde{t}} = P^\alpha(\tilde{x}, t) \Rightarrow \frac{\partial f^i}{\partial \tilde{t}} = P^\alpha(\tilde{x}, t)\sigma^\alpha_i(x)
$$

(2.10)

The above relations, must be regarded as definitions, for the matrix $\Lambda^\alpha_\mu$ and the triplet $P^\alpha$. With these identifications, the line element (2.9) assumes the form:

$$
ds^2 = \left( (N_\alpha(t) + P_\alpha(\tilde{x}, t))(N^\alpha(t) + P^\alpha(\tilde{x}, t)) - N(t)^2 \right) dt^2
$$

$$+ 2(N_\alpha(t) + P_\alpha(\tilde{x}, t)) \Lambda^\beta_\mu(\tilde{x}, t)\sigma^\beta_m(\tilde{x})dt d\tilde{x}^m
$$

$$+ \gamma_{\alpha\beta}(t)\Lambda^\alpha_\mu(\tilde{x}, t)\Lambda^\beta_\mu(\tilde{x}, t)\sigma^\alpha_m(\tilde{x})\sigma^\mu_n(\tilde{x})d\tilde{x}^m d\tilde{x}^n
$$

(2.11)

Consistency requirements, such as the non trivial action of the transformation under discussion, change the character of (2.10), from definitions to a set of first-order highly non-linear Partial Differential Equations (PDEs) for the unknown functions $f^i$. The existence of local solutions to these equations is guaranteed by Frobenius theorem [1] as long as the necessary and sufficient conditions:

$$
\frac{\partial}{\partial \tilde{x}^j} \left( \frac{\partial f^i}{\partial \tilde{x}^m} \right) - \frac{\partial}{\partial \tilde{x}^m} \left( \frac{\partial f^i}{\partial \tilde{x}^j} \right) = 0
$$

6
\[
\frac{\partial}{\partial t} \left( \frac{\partial f^i}{\partial \tilde{x}^m} \right) - \frac{\partial}{\partial \tilde{x}^m} \left( \frac{\partial f^i}{\partial t} \right) = 0
\]

hold. These integrating conditions, with repeated use of (1.5), result in:

\[
\Lambda_{\beta,m}(\tilde{x}, t)\sigma^\beta_n(\tilde{x}) - \Lambda_{\beta,n}(\tilde{x}, t)\sigma^\beta_m(\tilde{x}) = 2(C_{\alpha\beta}(x)\Lambda^\alpha_\mu(\tilde{x}, t)\Lambda^\beta_\nu(\tilde{x}, t) - C_{\mu\nu}(\tilde{x})\Lambda^\xi_\omega(\tilde{x}, t))\sigma^\mu_m(\tilde{x})\sigma^\nu_n(\tilde{x})
\]  

(2.12)

\[
\frac{1}{2} P^\alpha_m(\tilde{x}, t) = \left( \frac{1}{2} \Lambda^\alpha_\beta(\tilde{x}, t) - C^\alpha_{\mu\nu}(x)\Lambda^\mu_\nu(\tilde{x}, t)\Lambda^\beta_\gamma(\tilde{x}, t) \right)\sigma^\beta_m(\tilde{x})
\]

(2.13)

The line element (2.9) can be written, more concisely:

\[
\mathrm{d}s^2 \equiv (\tilde{N}^\alpha \tilde{N}_\alpha - \tilde{N}^2)dt^2 + 2\tilde{N}_\alpha \sigma^\alpha_i(\tilde{x})d\tilde{x}^i dt + \tilde{\gamma}_{\alpha\beta}\sigma^\alpha_i(\tilde{x})\sigma^\beta_j(\tilde{x})d\tilde{x}^i d\tilde{x}^j
\]

(2.14)

with the allocations:

\[
\tilde{\gamma}_{\alpha\beta} = \Lambda^\mu_\alpha(\tilde{x}, t)\Lambda^\nu_\beta(\tilde{x}, t)\gamma_{\mu\nu}(t)
\]

(2.15a)

\[
\tilde{N}_\alpha = \Lambda^\alpha_\beta(\tilde{x}, t)(N^\beta(t) + P^\beta(\tilde{x}, t)\gamma_{\rho\beta}(t)) \text{ and thus } \tilde{N}^\alpha = S^\alpha_\beta(\tilde{x}, t)(N^\beta(t) + P^\beta(\tilde{x}, t))
\]

(2.15b)

\[
\tilde{N}(t) = N(t)
\]

(2.15c)

(where \(S = \Lambda^{-1}\))

In order for the transformation (2.8) to preserve the splitting of the 3-metric, into spatial and temporal parts, i.e. \(\tilde{\gamma}_{\alpha\beta} = \tilde{\gamma}_{\alpha\beta}(t)\) in (2.15a), one must restrict to the case \(\Lambda^\alpha_\beta(\tilde{x}, t) = \Lambda^\alpha_\beta(t)\). Indeed, such a partial –because of space independence– solution to the system (2.12), exists:

\[
\Lambda^\alpha_\beta(t) = \begin{pmatrix} \lambda_1(t) & \lambda_2(t) & 0 \\ 0 & \lambda_5(t) & 0 \\ 0 & \lambda_8(t) & \lambda_9(t) \end{pmatrix}
\]

(2.16)

under the condition:

\[
x^2 = \text{ArcCot} \left( \frac{\text{Cot}(\tilde{x}^2)}{\lambda_5(t)} \right)
\]

(2.17)

i.e.: \(x^2 = f^2(\tilde{x}^2, t)\), as far as the transformation (2.8) is concerned. This is the automorphism group for the KS Type models. Then, equations (2.13) –in view of (2.17)– result in:

\[
\Lambda^\alpha_\beta(t) = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_5 & 0 \\ 0 & \lambda_8(t) & \lambda_9 \end{pmatrix}
\]

(2.18a)
\[ P^\alpha(\tilde{x}, t) = \left( \lambda_2'(t)\tilde{x}^2 + h_1(t), 0, \sin(\tilde{x}^2) \left( h_2(t) + \lambda_8(t)\ln\left(\tan\left(\frac{\tilde{x}^2}{2}\right)\right)\right) \right) \]  
(2.18b)

where the prime, denotes temporal differentiation and \( \lambda_2(t), \lambda_8(t), h_1(t), h_2(t), \) are arbitrary functions of time.

At this point, it must be observed that spatial homogeneity implies that the Ricci scalar is space independent, on a given –albeit arbitrary– hypersurface of simultaneity \( \Sigma_t. \) Through computing facility, one may see that this requirement, restricts the possible admissible forms of the scale factor matrix \( \gamma_{\alpha\beta}(t), \) to the set:

\[
\gamma_{\alpha\beta}(t) = \begin{pmatrix}
\gamma_{11}(t) & \gamma_{12}(t) & 0 \\
\gamma_{12}(t) & \gamma_{22}(t) & \gamma_{23}(t) \\
0 & \gamma_{23}(t) & \gamma_{33}(t)
\end{pmatrix}
\]  
(2.19)

We now conclude this section, by describing the –somehow lengthy but mostly algebraic– algorithm of the application of these results to the most general admissible 3-metric for the KS vacuum. To this end, consider the most general metric of type (2.19), with a lapse function \( N(t) \) and a full shift vector \( N^\alpha(t) \) present, in a system of local coordinates \( (t, r, \theta, \phi) \), and a transformation of the type (2.3) –under the condition (2.17)– to a new set of local coordinates \( (t, \tilde{r}, \tilde{\theta}, \tilde{\phi}) \). In this new “frame”, the lapse function, the shift vector and the 3-metric, can be found using the transformation laws (2.15a), (2.15b), (2.15c), where \( \Lambda^\alpha_\beta(t) \) and \( P^\alpha(\tilde{x}, t) \) are to be calculated from (2.18).

Using the freedom provided by the temporal functions \( \lambda_2(t) \) and \( \lambda_8(t) \), the initial scale factor matrix is brought to a diagonal form, called “final scale factor matrix”, –which depends on some combinations of the elements of the initial scale factor matrix. The corresponding alterations to the lapse function and the shift vector, can also be found; the lapse function remains unchanged and the “final shift vector”, depends on some combinations of the initial shift vector, and the elements of the initial scale factor matrix. Inserting these “final quantities” into the linear constraints (2.41), it is found that the shift vector must be zero –except for its first component, in which the EFEs are “transparent” and thus at one’s disposal– and some correlations on the combinations of the elements of the initial scale factor matrix appearing, which bring the diagonal “final scale factor matrix”, to the form diag(\( \gamma_{11}(t), \gamma_{22}(t), \gamma_{22}(t) \)). Our purpose, is thus, fulfilled. Starting from the most general, admissible form of the 3-metric and a shift vector, through a particular class of GCTs, we end up to the KS metric, with no shift. This fact, is an alternative proof of not only the generality, but also of the irreducibility of the KS vacuum solution.

Our method makes it possible to give the KS vacuum metric in the most general form containing all the relevant “gauge” freedom; one has only to solve the reduced system and invert the transformation which enables the reduction. The solution of the reduced system is (in the time gauge \( \gamma_{22}(t) = t^2 \)):

\[
ds^2 = -\left(\frac{C}{t} - 1\right)^{-1}dt^2 + \left(\frac{C}{t} - 1\right)dr^2 + t^2d\theta^2 + t^2\sin^2(\theta)d\phi^2
\]  
(2.20)
where $C$, is a constant. Therefore, using (2.18) one can write the most general metric—in matrix notation—of the KS type:

$$\gamma_{\text{Most General}}(t) = \Lambda^T(t)\gamma_{KS}(f(t))\Lambda(t)$$

(2.21)

where $\Lambda_{\alpha}^\beta$ is to be given by (2.18a) and:

$$N(t) = f'(t)(\frac{C}{f(t)} - 1)^{-1/2}$$

(2.22)

$$N^\alpha(t) = S^\alpha_\beta(t)P^\beta(f(t))$$

(2.23)

where $P^\beta$ is to be given by (2.18b) and $S = \Lambda^{-1}$.

### 3 Discussion

We have considered the set of GCTs which preserve manifest spatial homogeneity for the case of symmetry groups acting multiply transitively on the hypersurfaces of simultaneity $\Sigma_t$ (1.2), (1.3). This implies that the associated basis 1-forms satisfy an open algebra (1.5). The transformations found (2.8), (2.10) mix space and time in the new space variables, their existence is guaranteed by the Frobenious Theorem (2.12), (2.13) and their effect on the general spatially homogeneous line element (2.3) and/or (2.14), (2.15) can be used to simplify the reduced EFEs, (2.4). In this way an alternative proof is given, of the fact that the well known KS vacuum metric (2.20) is the irreducible form of the most general geometry admitting $G_4$ invariant homogeneous hypersurfaces. The closed form of this metric (2.21), (2.22), (2.23), exhibiting all the relevant “gauge” freedom in arbitrary functions of time, is also given.

The merits of this analysis can be most easily brought forward by comparing it to the original derivation: the 3-dimensional KVFs (1.2) are first trivially prolonged to spacetime vectors by adding to each of them a vanishing time component; then one demands for these vectors to be symmetries of the spacetime metric sought for. This results in the well known form of the KS vacuum metric (modulo a $g_{01}$ component). Implicit in the above derivation is the following important assumption: the space variables in which the original KVFs are given, are to be identified to the spatial coordinates of the Gauss normal system of the requested metric (an assumption which justifies the choice $g_{01} = 0$, a posteriori).

On the contrary, the analysis here presented makes no such assumption. The coordinates in which the original KVFs—and thus the basis 1-forms—are given can be any of the set obtained in (2.8). Consequently the scale factor matrix can be filled with non vanishing elements—along with the existence of non vanishing shift. Then the linear constraint EFEs ensure that when, through the effect of a particular AID (2.18a), the scale factor matrix is cast into a diagonal form, the shift vanishes.

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Lastly we would like to point out that our analysis is susceptible to generalization for the cases of lesser symmetry, say when there exist only two KVF:s: one must first find a basis 1-forms –invariant under the action of these KVF:s–, assume a splitting of the spatial metric of the form $g_{ij}(t, x) = \gamma_{\alpha\beta}\sigma_i^\alpha(x)\sigma_j^\beta(x)$, where the “scale factor” matrix will be supposed to depend not only on $t$, but also on the unique combination of the $x^i$ that has zero Lie derivative with respect to the KVF:s. Then the integrability conditions will give spatial coordinate transformations which hopefully will simplify the EFEs.

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