Dynamics of coherence-induced state ordering under Markovian channels

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We study the dynamics of coherence-induced state ordering under incoherent channels, particularly four specific Markovian channels: amplitude damping channel, phase damping channel, depolarizing channel and bit flit channel for single-qubit states. We show that the amplitude damping channel, phase damping channel, and depolarizing channel do not change the coherence-induced state ordering by \(l_1\) norm of coherence, relative entropy of coherence, geometric measure of coherence, and Tsallis relative \(\alpha\)-entropies, while the bit flit channel does change for some special cases.

**Keywords:** \(l_1\)-norm of coherence, relative entropy of coherence, geometric measure of coherence, Tsallis relative \(\alpha\)-entropies of coherence, ordering state.

I. INTRODUCTION

Quantum coherence is a fundamental feature of quantum mechanics, which distinguishes the quantum world from the classical physics realm. It is an important aspect in many research fields such as low-temperature thermodynamics [1, 2], quantum biology [3, 4], and nanoscale physics [5, 6, 7]. Quantifying the coherence of quantum states [8] has become a topic of interest for researchers. Baumgratz et al. [1–5], quantum biology [6–11], and nanoscale physics [12, 13]. Quantifying the coherence of quantum states [14] has been recently proposed a strict framework to quantify quantum coherence [15]. Various coherence measures have been defined based on this framework, such as \(l_1\)-norm of coherence, relative entropy of coherence [16], geometric measure of coherence [17] and Tsallis relative \(\alpha\)-entropies of coherence measure [17]. Here, the Tsallis relative \(\alpha\)-entropy of coherence measure violates the condition of a coherence measure that does not increase under mixing of states, while it satisfies a generalized monotonicity of average coherence under subselection based on measurement.

Different coherence measures employed based on different physical contexts thus give rise to different values of coherence. Questions about ordering states with various coherence measures have also been discussed [17–19]. Whether or not the quantum operators change the coherence-induced state ordering proposed by Zhang et al. [18], forms another interesting problem.

Focused on single-qubit states, in this study, we investigate such ordering problems under incoherent channels. In particular, we consider four Markovian channels: amplitude damping channel, phase damping channel, depolarizing channel, and bit flit channel. Note that for some special cases, Zhang et al have already studied the problem for single-qubit states by using the amplitude damping channel and phase damping channel [18]. Here, we also consider the geometric measure of coherence for more general situations. We extend the results of Ref. [18] to general cases. Furthermore, we show that the depolarizing channel does not change the coherence-induced state ordering while the bit flit channel changes it when \(p = \frac{1}{2}\).

II. PRELIMINARIES

In this section, we first recapitulate some concepts related to quantum coherence. Let \(\mathcal{H}\) be a \(d\)-dimensional Hilbert space and \(\{\ket{i}\}_{i=0}^{d-1}\) be an orthonormal basis of \(\mathcal{H}\). An incoherent state is defined as \(\rho = \sum_{i=0}^{d-1} p_i|\psi_i\rangle\langle\psi_i|\), where \(p_i \geq 0\), \(\sum p_i = 1\). Let \(\mathcal{I}\) denote the set of incoherent states. An incoherent operation is defined as \(\Lambda(\rho) = \sum_{n} K_{n} \rho K_{n}^\dagger\), where \(\Sigma_{n} K_{n} K_{n}^\dagger = I\) and \(K_{n} K_{n}^\dagger \subset \mathcal{I}\). Baumgratz et al. proposed a framework to quantify quantum coherence, that is, a function \(C\) can be taken as a coherence measure if it satisfies the following postulates [15]:

- (C1) \(C(\rho) \geq 0\), \(C(\rho) = 0\) if and only if \(\rho = \mathbb{I}\;
- (C2) \(C(\Lambda(\rho)) \leq C(\rho)\) for any incoherent operation \(\Lambda\);
- (C3) \(\sum_{n} p_{n} C(\rho_{n}) \leq C(\rho)\), where \(p_{n} = \text{Tr}(K_{n} \rho K_{n}^\dagger)\) and \(p_{n} = K_{n} \rho K_{n}^\dagger / p_{n}\), \(\{K_{n}\}\) is a set of incoherent Kraus operators;
- (C4) \(\sum_{n} p_{n} C(\rho_{n}) \leq \sum_{n} p_{n} C(\rho_{n})\) for any set of quantum states \(\{\rho_{n}\}\) and any probability distribution \(\{p_{n}\}\).

Several coherence measures have been put forward based on this framework. Here, we give the definitions of the following four coherence measures for further use.

Let \(\rho\) be a state defined on \(\mathcal{H}\), then

\[
C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}| \tag{1}
\]
is the $l_1$ norm of coherence, where $\rho_{ij}$ are the entries of $\rho$. The relative entropy of coherence is defined by

$$C_r(\rho) = \min_{\sigma \in \mathcal{I}} S(\rho \| \sigma) = S(\rho_{\text{diag}}) - S(\rho),$$

where $S(\rho \| \sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma)$ is the quantum relative entropy, $S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy, and $\rho_{\text{diag}} = \sum_i \rho_{ii} |i\rangle \langle i|$ is the diagonal part of $\rho$. The geometric measure of coherence is defined by

$$C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho, \sigma),$$

where $F(\rho, \sigma) = \left( \text{Tr} \sqrt{\rho \sqrt{\sigma} \sqrt{\rho}} \right)^2$ is the fidelity of two density operators $\rho$ and $\sigma$. The Tsallis relative $\alpha$-entropy of coherence is defined by

$$C_\alpha(\rho) = \min_{\delta \in \mathcal{I}} D_\alpha(\rho \| \delta) = \frac{r^\alpha - 1}{\alpha - 1},$$

where $r = \sum_i \langle i | \rho^\alpha | i \rangle^{\frac{1}{\alpha}}$ and $\alpha \in (0, 1) \cup (1, 2]$.

Any single-qubit state can be expressed as

$$\rho = \frac{1}{2} (I + \vec{k} \vec{\sigma}) = \frac{1}{2} (I + t \vec{n} \vec{\sigma}),$$

where $\vec{k} = (k_x, k_y, k_z)$ is a real vector satisfying $\| \vec{k} \| \leq 1$, $t = \| \vec{k} \|$, $\vec{n} = (n_x, n_y, n_z)$ is a unit vector, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. Here, we note that $n_x$, $n_y$, $n_z$ represent the length of vector $\vec{k}$ along the direction $\sigma_x$, $\sigma_y$, $\sigma_z$, respectively.

A non-coherence-generating channel (NC) $\tilde{\Phi}$ is a completely positive and trace preserving (CPTP) map from an incoherent state to another incoherent state: $\tilde{\Phi}(\mathcal{I}) \subseteq \mathcal{I}$, where $\mathcal{I}$ denotes the set of incoherent states [20]. Any quantum channel $\Phi$ is called an incoherent channel if there exists a Kraus decomposition $\Phi \cdot = \sum_n K_n \cdot K_n^\dagger$ such that $\rho_n = \frac{K_n(\rho) K_n^\dagger}{\text{Tr}(K_n(\rho) K_n^\dagger)}$ is incoherent for any incoherent state $\rho$.

A rank-$2$ qubit channel is an NC if and only if it has the Kraus decomposition either as [20]

$$\Phi^{(1)}(\cdot) = E_1^{(1)}(\cdot) E_1^{(1)\dagger} + E_2^{(1)}(\cdot) E_2^{(1)\dagger}$$

with

$$E_1^{(1)} = \begin{pmatrix} e^{i\eta} \cos \theta \cos \phi & 0 \\ -\sin \theta \sin \phi & e^{i\xi} \cos \phi \end{pmatrix}, \quad E_2^{(1)} = \begin{pmatrix} \sin \theta \cos \phi & e^{i\xi} \sin \phi \\ e^{-i\eta} \cos \theta \sin \phi & 0 \end{pmatrix},$$

or as

$$\Phi^{(2)}(\cdot) = E_1^{(2)}(\cdot) E_1^{(2)\dagger} + E_2^{(2)}(\cdot) E_2^{(2)\dagger}$$

with

$$E_1^{(2)} = \begin{pmatrix} \cos \theta & 0 \\ 0 & e^{i\xi} \cos \phi \end{pmatrix}, \quad E_2^{(2)} = \begin{pmatrix} 0 & \sin \phi \\ e^{i\xi} \sin \theta & 0 \end{pmatrix},$$

where $\theta, \phi, \xi, \eta$ are all real numbers. Here $\Phi^{(1)}$ is not an incoherent channel unless $\sin \theta \cos \theta \sin \phi \cos \phi = 0$ and $\Phi^{(2)}$ is an incoherent channel.

### III. MAIN RESULTS

In this section, we first study the coherence-induced ordering problem under arbitrary incoherent channels for single-qubit states via the coherence measures $C_1$, $C_r$, $C_\alpha$, and $C_g$. Then we study the dynamics of coherence-induced state ordering under specific Markovian channels for single-qubit states by four Markovian channels namely, amplitude damping, phase damping channel, depolarizing channel, and bit flip channel.

Suppose that an incoherent channel is defined as in Eq. (10). Let $a = \frac{1 - b}{2}$ and $b = \frac{t(n_x - in_y)}{2}$ with $b = | b | e^{i\beta}$. Then $\Phi(\rho) = \begin{pmatrix} A & B \\ B^* & 1 - A \end{pmatrix}$ with $A = a \cos^2 \phi + (b^* e^{i\xi} + be^{-i\xi}) \sin \theta \sin \phi \cos \phi + (1 - a) \sin^2 \phi$, $B = b e^{i\eta - i\xi} \cos \theta \cos^2 \phi +$
The coherence measure $\mathcal{C}_l(\Phi(\rho)) = 2 | b e^{i\xi+i\eta} \cos^2 \phi + b^* e^{i\xi+i\eta} \cos \phi |$. Thus, if $\sin \phi = 0$, then $\Phi$ is an incoherent operation and $\mathcal{C}_l(\Phi(\rho)) = 2 | b | \sqrt{\frac{\cos \phi}{\cos^2 \phi}} + \frac{e^{i\xi} e^{i\eta} \sin \phi}{\cos \phi}$. We find that the value of $\mathcal{C}_l(\rho)$ depends on both $b$ and the channel itself. In other words, there may exist incoherent channels such that $\mathcal{C}_l(\Phi(\rho_1)) < \mathcal{C}_l(\Phi(\rho_2))$ though $\mathcal{C}_l(\rho_1) > \mathcal{C}_l(\rho_2)$.

Suppose that an incoherent channel is defined as in Eq. (3). Then $\Phi(\rho) = \left( \frac{C}{D} \right)$ with $C = a \cos^2 \theta + (1-a) \sin^2 \phi$ and $D = e^{i\xi} (b \cos \theta \cos \phi + b^* \sin \theta \sin \phi)$. Thus, $\mathcal{C}_l(\Phi(\rho)) = 2 | b | \frac{\cos \phi}{\cos^2 \phi} (\theta - \phi) + \sin^2 \beta \cos^2 (\theta + \phi)$. Also, we know that the value of $\mathcal{C}_l(\rho)$ depends on both $b$ and the channel itself. In other words, there may exist incoherent channels such that $\mathcal{C}_l(\Phi(\rho_1)) < \mathcal{C}_l(\Phi(\rho_2))$ though $\mathcal{C}_l(\rho_1) > \mathcal{C}_l(\rho_2)$.

According to the above discussion, we can conclude that there exist incoherent channels changing the coherence-induced state ordering under the coherence measure $\mathcal{C}_l$. This is true also for the coherence measure $\mathcal{C}_g$, since $\mathcal{C}_l$ and $\mathcal{C}_g$ give the same ordering for single-qubit states [21].

A. Amplitude damping channel

The amplitude damping channel is characterized by the Kraus’ operators: $K_0 = |0\rangle \langle 0 | + \sqrt{1-p} |1\rangle \langle 1 |$, $K_1 = \sqrt{1-p} |0\rangle\langle 0 |$, where $p \in [0,1]$. It can be directly verified that [18],

$$\varepsilon(\rho) = \left( \frac{1+tn_x}{2} - \frac{1+tn_y}{2} \right) = \varepsilon(\rho_1) = (1-p) t \sqrt{1-n_z^2},$$

$$\mathcal{C}_l(\varepsilon(\rho)) = h \left( \frac{1+t'n'_z}{2} \right) - h \left( \frac{1+t'}{2} \right),$$

where $h(x) = -x \log x - (1-x) \log(1-x)$, $r = \left[ \left( \frac{1+t'}{2} \right)^{\alpha+1} + \left( \frac{1+t'}{2} \right)^{\alpha-1} + \left( \frac{1+n'_z}{2} \right)^{\alpha+1} + \left( \frac{1+n'_z}{2} \right)^{\alpha-1} \right]^\frac{1}{\alpha}$, $t' = \sqrt{(1-p)t^2(1-n_z^2)} + (p + (1-p)n_z t^2)$, $n'_x = \sqrt{\frac{1-pn_z t}{p}}$, $n'_y = \sqrt{\frac{1-pn_z t}{p}}$, and $n'_z = \frac{p + (1-p)n_z t}{p}$.

It is clear that for the case $p = 1$, the amplitude damping channel transforms any single-qubit state to an incoherent state. For $p = 0$, any single-qubit state is unchanged under amplitude damping channel.

It has been proved that the amplitude damping channel does not change the coherence-induced state ordering under the coherence measure $\mathcal{C}_l$ [18]. In the following we study the case $p \in (0,1)$ for the coherence measures $\mathcal{C}_r$ and $\mathcal{C}_\alpha$ for $\alpha \in (0,1) \cup (1,2)$. Numerical calculation shows that for any $p \in (0,1)$, the amplitude damping channel does not change the coherence-induced state ordering by $\mathcal{C}_r$ with fixed $n_z$ or fixed $t$, since $\mathcal{C}_r$ is an increasing function with respect to $t$ for every fixed $n_z$ while a decreasing function with respect to $n_z$ for every fixed $t$; see figures [1] and [2] for the cases of $p = \frac{1}{4}$ and $\frac{3}{4}$.

For the coherence measure $\mathcal{C}_\alpha$, Zhang et al have proved that for $p = \frac{1}{2}$, the amplitude damping channel maintains the coherence-induced state ordering with fixed $n_z$ or fixed $t$. In fact, we find that it holds for any $p \in (0,1)$ and $\alpha \in (0,1) \cup (1,2)$. In Fig. [3] we show the variation of $C_2$ for $p = \frac{1}{4}, \frac{3}{4}, \frac{3}{2}, \frac{5}{2}$ and $p = \frac{7}{4}$. In Fig. [4] we show the variation of $C_\alpha$ for fixed $p = \frac{1}{2}$ and $\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$.

B. Phase damping channel

Now we study the dynamics of coherence-induced state ordering under phase damping channel, which can be characterized by the Kraus' operators $K_0 = \sqrt{1-p} |0\rangle \langle 0 |$, $K_1 = \sqrt{1-p} |1\rangle \langle 1 |$, where $0 \leq p \leq 1$. By applying the phase damping channel to the state represented by Eq. (5), we get

$$\varepsilon(\rho) = \left( \frac{1+t'n_x}{2} - \frac{1+t'n_y}{2} \right).$$
FIG. 1: The variation of $C_r(\varepsilon(\rho))$ with respect to $t$ and $n_z$ under amplitude damping channel.

FIG. 2: For $p = \frac{1}{4}$, $p = \frac{1}{2}$ and $p = \frac{3}{4}$, $C_r(\varepsilon(\rho))$ is an increasing function with respect to $t$ for the cases $n_z = 0.3$ (red line), $n_z = 0.6$ (green line) and $n_z = 0.9$ (blue line).

For $p = 0$, the phase damping channel transforms a state into an incoherent one. In the following, we study the case $p \neq 0$. For simplicity, we define $A = 1 + (p^2 - 1)(1 - n_z)^2$, $B = \frac{1 + t\sqrt{A}}{2}$, $C = (\sqrt{A} + n_z)^2$, and $D = p^2(1 - n_z^2)$. Substituting $\varepsilon(\rho)$ into eq. (1), (2), and (4), we have

$$C_{l_1}(\varepsilon(\rho)) = pt\sqrt{1 - n_z} = pC_{l_1}(\rho),$$

$$C_{r}(\varepsilon(\rho)) = h(\frac{1 + tn_z}{2}) - h(B),$$

$$C_{\alpha}(\varepsilon(\rho)) = \frac{r^{\alpha} - 1}{\alpha - 1},$$

where $r = (B^\alpha C_{l_1} + (1 - B)^\alpha \frac{C_{r}}{C_{l_1}})^\frac{1}{\alpha} + ((1 - B)^\alpha C_{l_1} + B^\alpha \frac{C_{r}}{C_{l_1}})^\frac{1}{\alpha}$. According to Eq. (16) the phase damping channel does not change the coherence-induced state ordering by $C_{l_1}$ for single-qubit states.

Next we consider the coherence measure $C_r$. On differentiating Eq. (16) with respect to $t$, we get

$$\frac{\partial C_r(\varepsilon(\rho))}{\partial t} = \frac{n_z}{2} \log \frac{1 - tn_z}{1 + tn_z} + \frac{\sqrt{A}}{2} \log \frac{1 + t\sqrt{A}}{1 - t\sqrt{A}} \geq \frac{n_z}{2} \log \frac{1 - tn_z}{1 + tn_z} + \frac{n_z}{2} \log \frac{1 + tn_z}{1 - tn_z} = 0,$$

FIG. 3: For $p = \frac{1}{4}$, $p = \frac{1}{2}$, and $p = \frac{3}{4}$, $C_r(\varepsilon(\rho))$ is a decreasing function with respect to $n_z$ for the cases $t = 0.3$ (red line), $t = 0.6$ (green line), and $t = 0.9$ (blue line).
FIG. 4: The variation of $C_2$ with respect to $t$ and $n_z$ under amplitude damping channel for $p = \frac{1}{8}$, $p = \frac{3}{8}$, $p = \frac{5}{8}$, and $p = \frac{7}{8}$.

FIG. 5: For fixed $p = \frac{1}{2}$, the variation of $C_\alpha$ with respect to $t$ and $n_z$ under amplitude damping channel for $\alpha = \frac{1}{4}$, $\alpha = \frac{3}{4}$, $\alpha = \frac{5}{4}$, and $\alpha = \frac{7}{4}$.

Since $\sqrt{A} \log \frac{1 + t \sqrt{A}}{1 - t \sqrt{A}}$ is an increasing function with respect to $p \in (0, 1]$. Moreover, since $\frac{\partial C_r(\rho)}{\partial t} \geq 0$, the phase damping channel does not change the coherence-induced state ordering by $C_\alpha$ for single-qubit states with fixed $n_z$.

On differentiating Eq. (15) with respect to $n_z$, we obtain

$$
\frac{\partial C_r(\rho(\epsilon))}{\partial n_z} = \left( \frac{t}{2} \ln \frac{1 - t n_z}{1 + t n_z} - \frac{(p^2 - 1)n_z t}{2 \sqrt{A}} \ln \frac{1 + t \sqrt{A}}{1 - t \sqrt{A}} \right) / \ln 2.
$$

Set $f(p) = \frac{(p^2 - 1)}{\sqrt{A}} \ln \frac{1 + t \sqrt{A}}{1 - t \sqrt{A}}$. Then

$$
f'(p) = \frac{p(p^2 - 1)}{\sqrt{A}} \ln \frac{1 + t \sqrt{A}}{1 - t \sqrt{A}} + \frac{2t(p^2 - 1)(p^2 - 1)}{\sqrt{A} \ln \frac{1 + t \sqrt{A}}{1 - t \sqrt{A}}} \geq \frac{p(p^2 - 1)}{\sqrt{A}} \ln \frac{1 + t \sqrt{A}}{1 - t \sqrt{A}} + \frac{2t(p^2 - 1)(p^2 - 1)}{\sqrt{A} \ln \frac{1 + t \sqrt{A}}{1 - t \sqrt{A}}} = \frac{p(1 + 1) \ln \frac{1 + t \sqrt{A}}{1 - t \sqrt{A}} - 2t}{\sqrt{A}} \geq 0,
$$
with fixed $p$ induced state ordering by $C$.

Substituting $\varepsilon$ ordering by $C$.

FIG. 6: The variation of $C_\alpha$ with respect to $t$ and $n_s$ under phase damping channel for fixed $p = \frac{1}{2}$ and $\alpha = \frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{4}$, and $\alpha = \frac{7}{4}$.

since $\frac{A^{t+1}}{t} \ln \frac{1+t\sqrt{A}}{1-t\sqrt{A}} - 2t$ is an increasing function with respect to $t \geq 0$. Thus,

$$\frac{\partial C_\alpha(\varepsilon(\rho))}{\partial n_s} \leq \left(\frac{t}{2} \ln \frac{1 - tn_s}{1 + tn_s} - \frac{n_s t}{2} f(0)\right) / \ln 2 = 0.$$ 

Therefore, the phase damping channel keeps the coherence-induced state ordering by $C_\alpha$ for single-qubit states with fixed $t$ as $\frac{\partial C_\alpha(\varepsilon(\rho))}{\partial n_s} \leq 0$.

According to Eq. (10), for the coherence measure $C_\alpha$, $\alpha \in (0, 1) \cup (1, 2)$, we have $\frac{\partial C_\alpha(\varepsilon(\rho))}{\partial n_s} = \frac{\alpha}{\alpha} \alpha^{-1} \frac{\partial \tau}{\partial t}$, where

$$\frac{\partial \tau}{\partial t} = \frac{\sqrt{A}}{\varepsilon(\rho)} \left\{ \left[ B^\alpha \frac{C}{C + D} + (1 - B)^\alpha \frac{D}{C + D} \right]^{\frac{1}{2} - 1} \left[ B^\alpha - (1 - B)^\alpha \frac{D}{C + D} \right] \right\}^{\frac{1}{2} - 1} \left[ B^\alpha - (1 - B)^\alpha \frac{D}{C + D} \right].$$

If $\alpha \in (0, 1)$, $\frac{\partial \tau}{\partial t} \leq \left[ B^\alpha \frac{C}{C + D} + (1 - B)^\alpha \frac{D}{C + D} \right]^{\frac{1}{2} - 1} \left[ B^\alpha \alpha^{-1} - (1 - B)^\alpha \right] \leq 0.

If $\alpha \in (1, 2)$, $\frac{\partial \tau}{\partial t} \geq \left[ B^\alpha \frac{C}{C + D} + (1 - B)^\alpha \frac{D}{C + D} \right]^{\frac{1}{2} - 1} \left[ B^\alpha \alpha^{-1} - (1 - B)^\alpha \right] \geq 0.

Then $\frac{\partial C_\alpha(\varepsilon(\rho))}{\partial n_s} \geq 0$. Since $\frac{\partial C_\alpha(\varepsilon(\rho))}{\partial n_s} \geq 0$, the phase damping channel does not change the coherence-induced state ordering by $C_\alpha$ for single-qubit states with fixed $n_s$. In fact, the phase damping channel does not change the coherence-induced state ordering by $C_\alpha$ for single-qubit states with fixed $t$. In general, it is very difficult to discuss the monotony of $C_\alpha$ for all parameters $\alpha \in (0, 1) \cup (1, 2)$ and $p \in (0, 1]$ with respect to $n_s$. In Fig. 6 we present the variation of $C_\alpha$ with fixed $p = \frac{1}{2}$ for $\alpha = \frac{1}{2}, \frac{3}{4}, \frac{5}{4}$, and $\alpha = \frac{7}{4}$.

C. Depolarizing channel

Now we study the dynamics of coherence-induced state ordering under depolarizing channel. The state of the quantum system after depolarizing channel is given by $\varepsilon(\rho) = \frac{p}{2} I + \left(1 - p\right) \rho$,

$$\varepsilon(\rho) = \left(\frac{1 + tn_s(1 - p)}{(1 - p)(n_s^2 + n_s)}\right)^{\frac{1}{4}} \left(\frac{1 - p}{1 - tn_s(1 - p)}\right)^{\frac{1}{4}}.$$ (19)

Substituting $\varepsilon(\rho)$ into eq. (1), (2), and (4), we have

$$C_\alpha(\varepsilon(\rho)) = \left(1 - p\right) t \sqrt{1 - n_s} = \left(1 - p\right) C_\alpha(\rho),$$ (20)
\[ C_r(\varepsilon(\rho)) = h\left(\frac{1 + t n_z(1-p)}{2}\right) - h\left(\frac{1+t(1-p)}{2}\right), \]  
(21)

\[ C_\alpha(\varepsilon(\rho)) = \frac{r^\alpha - 1}{\alpha - 1}, \]
(22)

where \( r = \left[ E^\alpha F + (1 - E)\alpha(1 - F) \right]^{\frac{1}{\alpha}} + [E^\alpha(1 - F) + (1 - E)\alpha F]^{\frac{1}{\alpha}}, \) and \( E = \frac{1+t(1-p)}{2}, \ F = \frac{1+n_z}{2}. \)

According to Eq. (20), the depolarizing channel keeps the coherence-induced state ordering under \( C_t \) for single-qubit states.

Next, we consider the coherence measure \( C_r \). \( C_r(\varepsilon(\rho)) \) is clearly a decreasing function with respect to \( n_z \), since

\[ \frac{\partial C_r(\varepsilon(\rho))}{\partial n_z} = \frac{(1-p)n_z}{2} \log \frac{1-t n_z(1-p)}{1+n_z(1-p)} = \frac{t(1-p)^2 n_z}{1-t^2 n_z^2 (1-p)^2} \leq 0. \]

Therefore,

\[ \frac{\partial C_r(\varepsilon(\rho))}{\partial t} = \frac{(1-p)n_z}{2} \log \frac{1-t n_z(1-p)}{1+n_z(1-p)} + \frac{1-p}{2} \log \frac{1+t(1-p)}{1-t(1-p)} \geq 0. \]

In addition, \( C_r(\rho) \) is also an increasing function with respect to \( t \). Thus, the depolarizing channel does not change the coherence-induced state ordering by \( C_r \) for single-qubit states with fixed \( t \).

In the end, we consider the coherence measure \( C_\alpha \), where \( \alpha \in (0, 1) \cup (1, 2) \). First of all, we show \( \frac{\partial r}{\partial t} \geq 0 \) if \( \alpha \in (1, 2) \) and \( \frac{\partial r}{\partial n_z} \leq 0 \) if \( \alpha \in (0, 1) \). We have

\[ \frac{\partial r}{\partial t} = \frac{1-p}{2} \{ \left[ E^\alpha F + (1 - E)\alpha(1 - F) \right]^{\frac{1}{\alpha}-1} \left[ E^\alpha-1 F - (1 - E)\alpha-1 (1 - F) \right] + \left[ E^\alpha(1 - F) + (1 - E)\alpha F \right]^{\frac{1}{\alpha}-1} \left[ E^{\alpha-1} (1 - F) - (1 - E)^{\alpha-1} F \right] \}. \]

Since \( x^\alpha (1 - x)^\alpha (1 - y) \geq x^\alpha (1 - y) + (1 - x)^\alpha y \), where \( \alpha > 0, \frac{1}{\alpha} \leq x, y \leq 1 \), we have

\[ \frac{\partial r}{\partial t} \geq \frac{1-p}{2} \left[ E^\alpha F + (1 - E)\alpha(1 - F) \right]^{\frac{1}{\alpha}-1} \left[ E^{\alpha-1} - (1 - E)^{\alpha-1} \right] \geq 0 \]
(24)

if \( \alpha \in (1, 2) \). If \( \alpha \in (0, 1) \), we have

\[ \frac{\partial r}{\partial t} \leq \frac{1-p}{2} \left[ E^\alpha F + (1 - E)\alpha(1 - F) \right]^{\frac{1}{\alpha}-1} \left[ E^{\alpha-1} - (1 - E)^{\alpha-1} \right] \leq 0. \]
(25)

Thus, \( \frac{\partial C_r(\varepsilon(\rho))}{\partial t} = \frac{\alpha r^\alpha - 1}{\alpha - 1} \frac{\partial r}{\partial t} \geq 0 \). Since \( \frac{\partial C_r(\rho)}{\partial n_z} \geq 0 \), we arrive at the conclusion that the depolarizing channel does not change the coherence-induced state ordering by \( C_\alpha \) for single-qubit states with fixed \( n_z \).

On the other hand, as

\[ \frac{\partial r}{\partial n_z} = \frac{1}{2\alpha} \left[ E^\alpha - (1 - E)^\alpha \right] \left[ \left( E^\alpha -1 F - (1 - E)^{\alpha-1} (1 - F) \right)^{\frac{1}{\alpha}-1} - \left( E^\alpha (1 - F) + (1 - E)^{\alpha} F \right)^{\frac{1}{\alpha}-1} \right] \]

and \( x^\alpha (1 - x)^\alpha (1 - y) \geq x^\alpha (1 - y) - (1 - x)^\alpha y \) for \( \alpha \geq 0 \) and \( \frac{1}{\alpha} \leq x, y \leq 1 \), one has \( \frac{\partial C_r(\varepsilon(\rho))}{\partial n_z} \leq 0 \). Also, \( \frac{\partial C_r(\rho)}{\partial n_z} \leq 0 \), therefore, we conclude that the depolarizing channel keeps the coherence-induced state ordering by \( C_\alpha \) for single-qubit states with fixed \( t \).

D. Bit flit channel

Now we study the dynamics of coherence-induced state ordering under bit flit channel, which can be characterized by the Kraus’ operators \( K_0 = \sqrt{\sigma_I}, \ K_1 = \sqrt{1 - \rho \sigma_z} \), where \( 0 \leq p \leq 1 \). Applying the bit flit channel to the state represented by Eq. (5), we get

\[ \varepsilon(\rho) = \begin{pmatrix} \frac{1 + t n_z(2p-1)}{2} & \frac{t n_z - i n_z(2p-1)}{2} \\ \frac{tn_z + i n_z(2p-1)}{2} & \frac{1 - t n_z(2p-1)}{2} \end{pmatrix}. \]
(26)
Substituting this $\varepsilon(\rho)$ into Eqs. (1), (2), and (4), we have

$$C_{l_1}(\varepsilon(\rho)) = \sqrt{t^2 n_s^2 + (2p - 1)^2 t^2 n_y^2} = \sqrt{(2p - 1)^2 C_{l_1}^2(\rho) + 4(p - p^2)t^2 n_s^2},$$

(27)

$$C_r(\varepsilon(\rho)) = h\left(\frac{1 + t n_z (2p - 1)}{2}\right) - h(H),$$

(28)

$$C_\alpha(\varepsilon(\rho)) = \frac{r^\alpha - 1}{\alpha - 1},$$

(29)

where $r = [H^\alpha \frac{M}{M+N} + (1 - H)^\alpha \frac{N}{M+N}]^{\frac{1}{2}} + [(1 - H)^\alpha \frac{M}{M+N} + H^\alpha \frac{N}{M+N}]^{\frac{1}{2}}$, and $G = \sqrt{1 + 4(p^2 - p)(1 - n_s^2)}$, $H = \frac{1 + \sqrt{G}}{2}$, $M = n_s^2 + (2p - 1)^2 n_y^2$ and $N = (\sqrt{G} - (2p - 1)n_s^2)^2$.

Let us consider the special case $p = \frac{1}{2}$. Thus,

$$C_{l_1}(\varepsilon(\rho)) = t n_x,$$

(30)

$$C_r(\varepsilon(\rho)) = 1 - h\left(\frac{1 + t n_x}{2}\right),$$

(31)

$$C_\alpha(\varepsilon(\rho)) = \frac{r^\alpha - 1}{\alpha - 1},$$

(32)

where $r = 2\left[\frac{1}{2}(1 + t n_x)^{\frac{1}{2}} + \frac{1}{2}(1 - t n_x)^{\frac{1}{2}}\right]$. Hence, $\frac{\partial C_{l_1}(\varepsilon(\rho))}{\partial n_x} = t > 0$, $\frac{\partial C_r(\varepsilon(\rho))}{\partial n_x} = \frac{1}{2} \log \frac{1 + t n_x}{1 - t n_x} \geq 0$ and

$$\frac{\partial C_\alpha(\varepsilon(\rho))}{\partial n_x} = \frac{\alpha t}{2(\alpha - 1)} r^{\alpha - 1}\left(\frac{1 + t n_x}{2}\right)^{\frac{1}{2} - 1} - \frac{1}{2} - \frac{1}{2} \left(\frac{1 - t n_x}{2}\right)^{\frac{1}{2} - 1} \geq 0.$$

Let $\rho_1 = \frac{1}{2}(I + t_1 n_1 \vec{\sigma})$ and $\rho_2 = \frac{1}{2}(I + t_2 n_2 \vec{\sigma})$, where $n_1 = (n_{1x}, n_{1y}, n_{1z})$, $n_2 = (n_{2x}, n_{2y}, n_{2z})$. Assume $t_1 = t_2$, $n_{1x} < n_{2x}$ and $n_{1z} < n_{2z}$. Then we find that $C_{l_1}(\rho_1) > C_{l_1}(\rho_2), C_r(\rho_1) < C_r(\rho_2), C_\alpha(\rho_1) > C_\alpha(\rho_2)$, and $C_\alpha(\varepsilon(\rho_1)) < C_\alpha(\varepsilon(\rho_2))$. Thus, the bit flit channel changes the coherence-induced state ordering by the coherence measures $C_{l_1}$, $C_r$, and $C_\alpha$ for single-qubit states with fixed $t$, where $\alpha \in (0, 1) \cup (1, 2)$.

Now assume $t_1 > t_2$, $n_{1x} < n_{2x}$ and $n_{1z} = n_{2z}$, such that $t_1 n_{1x} < t_2 n_{2x}$. Then we find that $C_{l_1}(\rho_1) > C_{l_1}(\rho_2), C_r(\varepsilon(\rho_1)) < C_r(\varepsilon(\rho_2)), C_\alpha(\rho_1) > C_\alpha(\rho_2)$ and $C_\alpha(\varepsilon(\rho_1)) < C_\alpha(\varepsilon(\rho_2))$, since the coherence measures $C_{l_1}$, $C_r$ and $C_\alpha$ are all increasing functions with respect to $t n_x$. Thus, bit flit channel changes the coherence-induced state ordering by the coherence measures $C_{l_1}$, $C_r$ and $C_\alpha$ for single-qubit states with fixed $n_z$, where $\alpha \in (0, 1) \cup (1, 2)$.

IV. CONCLUSION

We have discussed whether or not a quantum channel changes the coherence-induced state ordering, for four specific Markovian channels: – amplitude damping channel, phase flit channel, depolarizing channel, and bit flit channel. We have shown that the depolarizing channel does not change the coherence-induced state ordering by $C_{l_1}$, $C_r$, $C_\alpha$, and $C_{l_2}$. For the bit flit channel, we have shown that it does change the coherence-induced state ordering under these four coherence measures for the case of $p = \frac{1}{2}$. Our results enrich the understanding of coherence-induced state ordering under quantum channels.

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