LIMITS OF BAUMSLAG-SOLITAR GROUPS AND DIMENSION ESTIMATES
IN THE SPACE OF MARKED GROUPS

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Abstract. We prove that the limits of Baumslag-Solitar groups studied in [GS08] are non-linear hopfian C*-simple groups with infinitely many twisted conjugacy classes. We exhibit infinite presentations for these groups, classify them up to group isomorphism, describe their automorphisms and discuss the word and conjugacy problems. Finally, we prove that the set of these groups has non-zero Hausdorff dimension in the space of marked groups on two generators.

Introduction

Baumslag-Solitar groups are ubiquitous in group theory and topology [BS62, Mol69, Gl79, FM98, Why01, JS79] and offer a remarkable test bed for group-theoretic properties. Considering their limits, we obtain in the present paper a Cantor set of pairwise non-isomorphic two-generated groups with a number of combinatorial and geometrical non-closed properties. By a closed property, we mean a property that defines a closed subset of the space of marked groups. We also give the first estimates of non-vanishing Hausdorff dimension in the space of marked groups.

Let \( m \in \mathbb{Z} \setminus \{0\} \). For every sequence \((\xi_n)\) of integers in \( \mathbb{Z} \) such that \(|\xi_n|\) tends to infinity and \( \xi_n \) tends to some \( m \)-adic integer \( \xi \), define \( BS(m, \xi) = \lim_{n \to \infty} BS(m, \xi_n) \) where \( BS(p, q) = \langle a, b \mid abpa^{-1} = b^q \rangle \) denotes the Baumslag-Solitar group with \( p, q \) in \( \mathbb{Z} \setminus \{0\} \); the limit is taken with respect to the topology of the space of marked groups [Sta06b, Th. 6]. As a consequence of its definition, \( BS(m, \xi) \) enjoys the following closed properties: it is torsion-free, centerless, non-Kazhdan, contains a non-abelian free group generated by \( b \) and \( bab^{-1} \) if \(|m| > 1\), and satisfies the relation \( [abma^{-1}, b] = 1 \). We present here results of a different nature, relying on the existence of an HNN decomposition.

The limits \( BS(m, \xi) \) are first studied for their own right in [GS08], where it is shown that \( BS(m, \xi) \) acts transitively on a tree and maps homomorphically onto the special limit \( BS(1, 0) = \mathbb{Z} \wr \mathbb{Z} \). These two results are extensively used in this article.

Results. We first prove that \( BS(m, \xi) \) is a non-degenerate HNN extension of a free abelian group of infinite countable rank. More precisely, consider the free abelian groups

\[
E = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots
\]

\[
E_{m, \xi} = \mathbb{Z}me_0 \oplus \mathbb{Z}(e_1 - r_1(\xi)e_0) \oplus \mathbb{Z}(e_2 - r_2(\xi)e_0) \cdots
\]

\[
E_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots
\]

where \((r_i(\xi))\) is an integer sequence defined in Section 2.1. Let \( \tilde{BS}(m, \xi) \) be the HNN extension of basis \( E \) with conjugated subgroups \( E_{m, \xi}, E_1 \) and stable letter \( a \), where conjugacy from \( E_{m, \xi} \) to \( E_1 \) is defined by \( a(me_0)a^{-1} = e_1 \) and \( a(e_i - r_i(\xi)e_0)a^{-1} = e_{i+1} \).

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1The property (T) of Kazhdan is open by [Sha00, Th. 6.7].

2This is actually the shortest relation with respect to \( a \) and \( b \) by [Sta06a, Pr. 2].
Theorem A (Cor. 2.10 and Th. 3.1). Let \((a, b)\) be the canonical generating pair of \(\overline{BS}(m, \xi)\). Then the map defined by \(f(a) = a\) and \(f(b) = e_0\) induces an isomorphism from \(BS(m, \xi)\) to \(\overline{BS}(m, \xi)\). Moreover, the group \(\overline{BS}(m, \xi)\) admits the following infinite presentation:

\[
\langle a, b \mid [b, b_i] = 1, i \geq 1 \rangle
\]

with \(b_1 = a b^m a^{-1}\) and \(b_i = a b_{i-1} b^{-r_i-1} a^{-1}\) for every \(i \geq 2\).

In particular, we have \(BS(m, 0) = \langle a, b \mid [a^i b^m a^{-1}, b] = 1, i \geq 1 \rangle\). In addition, the latter presentation is minimal.

It follows from [GS08 Th. 4.1] that no group \(\overline{BS}(m, \xi)\) can be finitely presented. Note also that \(\overline{BS}(m, \xi)\) enjoys any property shared by all non-degenerate HNN extensions, for instance it is primitive [GG08] and has uniform exponential growth [HB00]. Using the latter HNN decomposition, we show the following:

Theorem B. Assume that \(|m| > 1\). Then we have:

- \(\overline{BS}(m, \xi)\) is hopfian but not co-hopfian (Th. 5.8). If \(m\) is prime and \(\xi\) is algebraic over \(\mathbb{Q}\), then \(BS(m, \xi)\) is not residually finite (Pr. 5.9).
- \(\overline{BS}(m, \xi)\) is \(C^*\)-simple and inner-amenable (Pr. 4.1 and Pr. 4.6).
- \(\overline{BS}(m, \xi)\) has infinitely many twisted conjugacy classes (Pr. 5.4).
- \(\overline{BS}(m, \xi)\) is not equationally noetherian and hence not linear (Pr. 5.10).
- The automorphism group of \(\overline{BS}(m, \xi)\) is a split extension of \(BS(m, \xi)\) by an infinite dihedral group (Pr. 5.7).

From our study of homomorphisms carried out in Section 5 we deduce the following classification result.

Theorem C (Th. 5.10 and Pr. 2.9). Let \(m, m' \in \mathbb{Z} \setminus \{0\}\) and let \(\xi, \xi' \in \mathbb{Z}_{m'}\). Then \(BS(m, \xi)\) is abstractly isomorphic to \(BS(m', \xi')\) if and only if there is \(e \in \{\pm 1\}\) and \(d \in \mathbb{N}\) such that \(m = cm', d = \gcd(m, \xi) = \gcd(m', \xi')\) and the \(m\)-adic numbers \(\xi/d, e\xi'/d\) project onto the same element of \(\mathbb{Z}_{m/d}\) via the canonical map \(\mathbb{Z}_m \to \mathbb{Z}_{m/d}\).

Thus two given limits are isomorphic if and only if they are isomorphic as marked groups [GS08 Th. 2.1], i.e. if and only if they represent the same point in the space of marked groups on two generators.

The first-named author has shown that the box-counting dimension of \(G_2\), the space of marked groups on two generators, is infinite [Guy07]. Estimating the Hausdorff dimension of \(\mathbb{Z}^m_{\xi}\), the set of marked groups \(BS(m, \xi)\) such that \(\xi\) is invertible in \(\mathbb{Z}_m\) (Th. 6.5), we deduce the following:

Theorem D (Cor. 6.6). The Hausdorff dimension of \(G_2\) satisfies \(\dim_H(G_2) \geq \log(2)/6\). In particular, this dimension does not vanish.

Our last result pertains to the algorithmic complexity of the word and conjugacy problem (Prop. 7.3) and their degrees of undecidability. For every problem that can be suitably represented by a language and every \(m\)-adic number there is a well-defined degree of undecidability, called the Turing degree (Sec. 7).

Theorem E (Cor. 7.7). Assume that \(\xi\) is invertible in \(\mathbb{Z}_m\). Then the following Turing degrees coincide:

- the Turing degree of the word problem for \(\overline{BS}(m, \xi)\);
- the Turing degree of the conjugacy problem for \(\overline{BS}(m, \xi)\);
- the Turing degree of \(\xi\).

In particular, the word problem is solvable for \(\overline{BS}(m, \xi)\) if and only if \(\xi\) is a computable number.

The resolution degree \(r_1(n)\) of the word problem for a finitely generated group \(\Gamma\) [Gri85 Def. 1] is a quantitative measure of the undecidability of the word problem based on Kolgomorov’s ideas. It is, intuitively, the minimal amount of information necessary to decide if \(w = 1\) in \(\Gamma\) for every
word \( w \) with \(|w| \leq n \), where \(|w|\) is the length of \( w \) with respect to a chosen generating set. Let \( \omega \) be an infinite sequence of symbols from some finite alphabet. We denote by \( \omega^{(n)} \) be the word made of the first \( n \) symbols of \( \omega \). For a word \( w \), we denote by \( KR(w) \) the Kolmogorov complexity resolution of \( w \) which is, intuitively, the minimal amount of information necessary to obtain, for every natural number \( i \leq |w| \), the \( i \)-th symbol of the word \( w \). Two functions \( f \) and \( g \) are said to be equivalent if there is some \( C \geq 0 \) such that \( f(n) \leq g(Cn) + C \) and \( g(n) \leq f(Cn) + C \) for every \( n \).

**Theorem F** (Pr. 2.6 and Th. 6.1). Let \( \Gamma = BS(m, \xi) \) and let \( \omega = (r_i(\xi)) \). Then \( r_1(n) \) is equivalent to \( KR(\omega^{(n)}) \).

For comparison, the resolution degree of the Grigorchuk group \( \Gamma = G_\omega \) is equivalent to \( KR(\omega^{([\log n])}) \) [Gri85, Th. 3]. Thus, for a typical \( \omega \) in the measure-theoretic sense, \( r_1(n) \) is linear for the corresponding \( BS(m, \xi) \) and logarithmic for \( G_\omega \) [ZL70].

## 1. Backgrounds

### 1.1. The space of marked groups.

The commonly used definition of the space of marked groups \(^3\) is due to Grigorchuk who proved by a topological argument that his intermediate growth groups cannot be finitely presented [Gri84]. The space of marked groups has then been used to prove both existence and abundance of groups with exotic properties [Ste96, Cha00] and has turned to be a remarkably suited framework for the study of Sela’s limit groups [CG05]. Isolated points were investigated in [CGP07] and an isolated group is used by de Cornulier to answer a question of Gromov concerning the existence of sofic groups which do not arise as limits of amenable groups [Cor09]. Very little is known on its topological type [Cor09a]. The box-counting dimension of the space of marked groups on \( k \) generators is infinite if \( k \geq 2 \) [Guy07]. We show that its Hausdorff dimension is non-zero in Section 6.

The free group on \( k \) generators will be denoted by \( F_k \), or \( F(S) \) with \( S = (s_1, \ldots, s_k) \), if we want to precise the names of canonical generating elements. A marked group on \( k \) generators is a pair \((G, S)\) where \( G \) is a group and \( S = (s_1, \ldots, s_k) \in G^k \) is an ordered generating set of \( G \). An isomorphism of marked groups is an isomorphism which respects the markings. A marked group \((G, S)\) is endowed with a canonical epimorphism \( \phi : F_S \to G \), which induces an isomorphism of marked groups between \( F_S/\ker \phi \) and \( G \). Hence a class of marked groups for the relation of marked isomorphism is represented by a unique quotient of \( F_S \).

Let \( w = x_1^{e_1} \cdots x_n^{e_n} \) be a reduced word in \( F_S \) (with \( x_i \in S \) and \( e_i \in \{\pm 1\} \)). The integer \( n \) is called the length of \( w \) and denoted \(|w|\). If \((G, S)\) is a marked group on \( k \) generators and \( g \in G \), the length of \( g \) is

\[
|g|_G := \min \{ n : g = s_1 \cdots s_n \text{ with } s_i \in S \cup S^{-1} \} \leq \min \{|w| : w \in F_S, \phi(w) = g\}.
\]

Let \( G_k \) be the class of marked groups on \( k \) generators up to marked isomorphism. Let us recall that the topology on \( G_k \) comes from the following ultrametric distance: for \( (G_1, S_1) \neq (G_2, S_2) \in G_k \) we set \( d((G_1, S_1), (G_2, S_2)) := e^{-\lambda} \) where \( \lambda \) is the length of a shortest element of \( F_k \) which vanishes in one group and not in the other one. One may also keep in mind the following characterization of convergent sequences [Sta06a, Pr. 1].

**Lemma 1.1.** Let \((G_n)\) be a sequence of marked groups in \( G_k \). The sequence \((G_n)\) converges if and only if for any \( w \in F_k \), we have either \( w = 1 \) in \( G_n \) for \( n \) large enough, or \( w \neq 1 \) in \( G_n \) for \( n \) large enough.

\(^3\)A very similar topology was first considered in [Gro81, Final Remarks]. The general idea of topologizing sets of groups goes back to Mahlers and Chabauty [Mah46, Cha50]. The interested reader should consult [Hav08] for a thorough account.
The free group \( F_2 = \mathbb{F}(a, b) \), the Baumslag-Solitar groups \( BS(p, q) \) and their limits \( BS(m, \xi) \) are marked by their canonical generating pair \((a, b)\).

1.2. **The ring of \( m \)-adic integers.** Let \( m \in \mathbb{Z} \setminus \{0\} \). Recall that the ring of \( m \)-adic integers \( \mathbb{Z}_m \) is the projective limit in the category of topological rings of the system
\[
\cdots \rightarrow \mathbb{Z}/m^h \mathbb{Z} \rightarrow \mathbb{Z}/m^{h-1} \mathbb{Z} \rightarrow \cdots \rightarrow \mathbb{Z}/m^2 \mathbb{Z} \rightarrow \mathbb{Z}/m \mathbb{Z}
\]
where the arrows are the canonical surjective homomorphisms. This shows that \( \mathbb{Z}_m \) is compact. This topology is compatible with the ultrametric distance given, for \( \xi \neq \eta \), by
\[
|\xi - \eta|_m = |m|^{-\max\{k \in \mathbb{N} : \xi - \eta \in m^k \mathbb{Z}_m\}}.
\]
and \( \mathbb{Z} \) embeds isomorphically and densely in \( \mathbb{Z}_m \). To avoid ambiguity, we call elements of \( \mathbb{Z} \) rational integers. We only need the following easy facts about \( m \)-adic integers. Detailed proofs can be found in the second-named author’s Ph.D. thesis [Sta05, Appendix C].

- There is a topological ring isomorphism from \( \mathbb{Z}_m \) to \( \bigoplus_{p \mid m} \mathbb{Z}_p \), where \( p \) ranges in the set of the prime divisors of \( m \).
- Any non-zero ideal can be uniquely written under the form \( d\mathbb{Z}_m \) where \( d \) is a positive rational integer whose prime divisors divide \( m \). Moreover we have \( \mathbb{Z} \cap d\mathbb{Z}_m = d\mathbb{Z} \). For any \( n \in \mathbb{Z} \) and any \( \xi \in \mathbb{Z}_m \), we can define the greatest common divisor \( \gcd(n, \xi) \) of \( n \) and \( \xi \) as the unique positive rational integer \( d \) such that \( n \) and \( \xi \) generate \( d\mathbb{Z}_m \).
- An \( m \)-adic integer \( \xi \) is invertible if and only if it does not belong to any of the ideals \( p\mathbb{Z}_m \) where \( p \) is a prime divisor of \( m \) in \( \mathbb{Z} \); equivalently \( \gcd(m, \xi) = 1 \).

Note finally that non-zero rational integers are never zero divisors in \( \mathbb{Z}_m \). Hence, we can consider the ring \( \mathbb{Z}^{-1} \mathbb{Z}_m \) whose elements are fractions of the form \( \frac{a}{b} \) with \( a \in \mathbb{Z}_m \) and \( b \in \mathbb{Z} \setminus \{0\} \) and whose laws are the classical ones for fractions.

1.3. **HNN extensions and tree actions.** In this section we fix our notations for HNN extensions and collect several facts concerning their standard tree actions. Let \( G \) be a group, let \( H, K \) be subgroups of \( G \) and let \( \tau : H \rightarrow K \) be an isomorphism. The HNN extension of base \( G \) whose stable letter \( t \) conjugates \( H \) to \( K \) via \( \tau \), is the group
\[
\text{HNN}(G, H, K, \tau) = \langle G, t \mid t \tau^{-1} = \tau(h) \text{ for every } h \in H \rangle.
\]
Let \( \Gamma = \text{HNN}(G, H, K, \tau) \). We say that \( \Gamma \) is an ascending HNN extension if either \( G = H \) or \( G = K \). We say that \( \Gamma \) is a degenerate HNN extension if \( G = H = K \), i.e. \( \Gamma = G \rtimes \mathbb{Z} \) where \( \langle t \rangle \) identifies with \( \mathbb{Z} \). Note that a given group (e.g. \( \mathbb{Z} \rtimes \mathbb{Z} \)) may have two different HNN decompositions, one being degenerate whereas the other is not.

The Normal Form Theorem. A sequence \( g_0, t^{e_1}, g_1, \ldots, t^{e_n}, g_n \), with \( n \geq 0 \), is said to be a reduced sequence if there is no consecutive subsequence \( t, g_i, t^{-1} \) with \( g_i \in H \) or \( t^{-1}, g_i, t \) with \( g_i \in K \). Britton’s lemma [LS77] page 181] asserts that the word \( g_0t^{e_1} \cdots t^{e_n}g_n \) has a non-trivial image \( \gamma \) in \( \Gamma \) if \( g_0, t^{e_1}, \ldots, t^{e_n}, g_n \) is reduced and \( n \geq 1 \). Such a word is called a reduced form for \( \gamma \). Although \( \gamma \) may have different reduced forms, the sequence \( t^{e_1}, \ldots, t^{e_n} \) is uniquely determined by \( \gamma \) and we call its length \( n = |\gamma|_t \) the \( t \)-length of \( \gamma \). Fixing a set \( T_H \) of representatives of right cosets of \( H \) in \( G \) and a set \( T_K \) of representatives of right cosets of \( K \) in \( G \) such that \( 1 \in T_H \cap T_K \), the Normal Form Theorem [LS77] Th. IV.2.1] asserts that there is a unique sequence \( g_0, t^{e_1}, \ldots, t^{e_n}, g_n \) representing \( \gamma \in \Gamma \) with the following properties:

- \( g_0 \) is an arbitrary element of \( G \),
- If \( e_i = 1 \), then \( g_i \in T_H \),
- If \( e_i = -1 \), then \( g_i \in T_K \),
- there is no consecutive subsequence \( t^*, 1, t^{-*} \).

This sequence is the normal sequence of \( \gamma \) with respect to \( T_H \) and \( T_K \) and we call the word \( g_0 t^{e_1} \cdots t^{e_n} g_n \) the normal form of \( \gamma \).

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4Note that \( Z_{-m} = Z_m \) and that \( Z_m \) is the zero ring if \( |m| = 1 \).
Action on the standard Bass-Serre tree and its boundary. The standard Bass-Serre tree of $\Gamma$ is the oriented tree $T$ with vertex set $V(T) = \Gamma/G$, with set of oriented edges $E_+(T) = \Gamma/H$ subject to the incidence relations $o(\gamma H) = \gamma G$ and $t(\gamma H) = \gamma t^{-1}G$.

Given a tree $T$, we define the boundary $\partial T$ as the set of cofinal rays. The set $\partial T$ has a natural topology defined as follows. We call a shadow the boundary of any connected component of $T$ deprived of one of its edges. The family of shadows generates a topology on $\partial T$ which is Hausdorff and totally discontinuous. If $T$ is a countable non-linear tree, e.g. the standard Bass-Serre tree of a non-degenerate HNN extension, then $\partial T$ is a perfect Baire space [HP09, Pr. 11 and 24.ii]. Note that every automorphism of $T$ induces an homeomorphism of $\partial T$.

As $\Gamma$ acts without inversion on its Bass-Serre tree $T$, every element $\gamma \in \Gamma$, viewed as a tree automorphism, is either elliptic, i.e. $\gamma$ fixes some vertex of $T$, or hyperbolic, i.e. $\gamma$ fixes no vertex of $T$ but exactly two ends of $T$ but exactly two ends of $\partial T$ [Ser77]. The action of $\Gamma$ on $\partial T$ has no fixed end if $\Gamma$ is non-ascending, exactly one fixed end if $\Gamma$ is non-degenerate and ascending, and exactly two fixed ends if $\Gamma$ is degenerate [HP09, Pr. 24].

2. HNN decomposition of the limits

We fix $m \in \mathbb{Z} \setminus \{0\}$, $\xi \in \mathbb{Z}_m$ and $(\xi_n)$ a sequence of rational integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$. A natural HNN decomposition arises from the transitive action of $\overline{BS}(m,\xi)$ on a tree constructed in [GS08]. We recall this construction.

We denote by $H_n$ (respectively $H^+_n$) the subgroup of $BS(m,\xi_n)$ generated by $b$ (respectively $b^m$) and by $T_n$ the Bass-Serre tree of $BS(m,\xi_n)$. We set

$$Y = \left( \prod_{n \in \mathbb{N}} V(T_n) \right) / \sim = \left( \prod_{n \in \mathbb{N}} BS(m,\xi_n)/H_n \right) / \sim$$

$$Y^m = \left( \prod_{n \in \mathbb{N}} E_+(T_n) \right) / \sim = \left( \prod_{n \in \mathbb{N}} BS(m,\xi_n)/H^+_n \right) / \sim$$

where $\sim$ is defined by $(x_n) \sim (y_n) \iff \exists n_0 \forall n \geq n_0 : x_n = y_n$ in both cases. We now define an oriented graph $X = X_{m,\xi}$ by

$$V(X) = \{ x \in Y : \exists w \in \mathbb{F}_2 \text{ such that } (x_n) \sim (wH_n) \}$$

$$E_+(X) = \{ y \in Y^m : \exists w \in \mathbb{F}_2 \text{ such that } (y_n) \sim (wH^+_n) \}$$

$$o((wH^+_n)) = (wH_n) = (o(wH^+_n))$$

$$t((wH^+_n)) = (wa^{-1}H_n) = (t(wH^+_n))$$

The graph $X_{m,\xi}$ is a tree and that the obvious action of $\mathbb{F}_2$ on $X_{m,\xi}$ factorizes through the canonical projection $\mathbb{F}_2 \to \overline{BS}(m,\xi)$ [GS08, Sec.3]. Let $v_0 = (H_n)$ and $e_0 = (H^+_n)$. We denote by $B$ (respectively $B_m$) the stabilizer of $v_0$ (respectively $e_0$) in $\overline{BS}(m,\xi)$. We set $B_{\xi} = aBma^{-1}$. As the action of $\overline{BS}(m,\xi)$ is clearly transitive on vertices and geometrical edges, the group $\overline{BS}(m,\xi)$ has a HNN decomposition:

**Proposition 2.1.** The group $\overline{BS}(m,\xi)$ is the HNN extension of base group $B$, stable letter $a$ and conjugated subgroups $B_m$ and $B_{\xi} = aBma^{-1}$.

We define the inner automorphism $\tau_a : \gamma \mapsto a\gamma a^{-1}$ of $\overline{BS}(m,\xi)$ and we denote by $\text{HNN}(B,B_m,B_{\xi},\tau)$ the previous HNN decomposition where $\tau$ is the isomorphism from $B_m$ to $B_{\xi}$ induced by $\tau_a$.

**Proof.** This follows from results in [Ser77]. See in particular Section I.5 and Remark 1 after Theorem 7 in Section I.4. \hfill $\Box$

The following lemma is an immediate consequence of the definition of the action:

**Lemma 2.2.** Let $w \in \mathbb{F}_2$ and let $\gamma$ be its image in $\overline{BS}(m,\xi)$. Then $\gamma \in B$ (respectively $\gamma \in B_m$) if and only if $w = b^{\lambda_n}$ (respectively $b^{m\lambda_n}$) in $BS(m,\xi_n)$ for all $n$ large enough.
As a result, $B$ is abelian. Let $\mathbb{Z}[X]$ be the ring of polynomials in the variable $X$ with integer coefficients. A thorough study of the sequence $(\lambda_n)$ of Lemma 2.2 enables us to embed $B$ isomorphically into the additive group of $\mathbb{Z}[X]$: 

**Proposition 2.3.**

- Let $w \in \mathbb{F}_2$ with image $\gamma \in B$. Then there is a unique polynomial $P_\gamma(X) \in \mathbb{Z}[X]$ such that $w = b P_{\gamma}(\xi_n/m)$ in $BS(m, \xi_n)$ for all $n$ large enough.
- The map $q : \gamma \mapsto P_\gamma(X)$ defines an injective homomorphism from $B$ into $\mathbb{Z}[X]$.

The construction of the homomorphism $q$ relies on a sequence of polynomials built up from $m$ and $\xi$ in a non obvious way. Its definition is the subject of the next section. The proof of Proposition 2.3 is therefore postponed to Section 2.2 (see Proposition 2.9).

2.1. The functions $r_i$ and $s_i$. In this section we define functions $r_i$ and $s_i$ on $\mathbb{Z}$ which describe how the $b$ exponents of a given word $w \in \mathbb{F}_2$ behave when we reduce it in $BS(m, n)$ for $n$ ranging in a given class modulo $m^h$. In [Sta06b, GS08], these functions were decisive to describe converging sequences of Baumslag-Solitar groups. It turns out that they extend continously to $\mathbb{Z}_m$ and that $\xi$, and hence $BS(m, n)$, is uniquely determined by $m$ and the sequence $(r_i(\xi))$.

**Definition 2.4.** We define the functions $r_0, r_1, \ldots, s_0, s_1, \ldots$ on $\mathbb{Z}$ depending on a parameter $m \in \mathbb{Z} \setminus \{0\}$ by the inductive formulas:

1. $r_0(n) = 0, s_0(n) = 1$;
2. $ns_{i-1}(n) = ms_i(n) + r_i(n)$ with $r_i(n) \in \{0, \ldots, |m| - 1\}$ for every $i \geq 1$.

**Proposition 2.5.** Let $n, n' \in \mathbb{Z}$ such that $\gcd(n, m) = \gcd(n', m) = d$. Let $h \in \mathbb{N} \setminus \{0\}$ and $\tilde{m} = m/d$.

1. Assume that $n \equiv n' \pmod{\tilde{m}^h \mathbb{Z}}$. Then the two following hold:
   - $(i)$ $r_i(n) = r_i(n')$ for $i = 1, \ldots, h$;
   - $(ii)$ $s_i(n) \equiv s_i(n') \pmod{\tilde{m}^{h-i} \mathbb{Z}}$ for $i = 1, \ldots, h$.
2. If $r_i(n) = r_i(n')$ for $i = 1, \ldots, h$ then $n \equiv n' \pmod{\tilde{m}^h \mathbb{Z}}$.

**Proof of Proposition 2.5.** Proof of (1). We show by induction on $i$ that 

$r_i(n) = r_i(n'), s_i(n) \equiv s_i(n') \pmod{\tilde{m}^{h-i} \mathbb{Z}}$ and $s_i(n) \cdot n \equiv s_i(n') \cdot n' \pmod{\tilde{m}^{h-i} \mathbb{Z}}$

for $i = 0, 1, \ldots, h$. The case $i = 0$ is obvious. Assume that $r_{i-1}(n) = r_{i-1}(n'), s_{i-1}(n) \equiv s_{i-1}(n') \pmod{\tilde{m}^{h-i} \mathbb{Z}}$ and $s_{i-1}(n) \cdot n \equiv s_{i-1}(n') \cdot n' \pmod{\tilde{m}^{h-i+1} \mathbb{Z}}$ for some $1 \leq i \leq h$. By construction

\[
\begin{align*}
  s_{i-1}(n) \cdot n &= s_i(n) \cdot m + r_i(n) \\
  s_{i-1}(n') \cdot n' &= s_i(n') \cdot m + r_i(n')
\end{align*}
\]

We deduce from the induction hypothesis that $r_i(n) \equiv r_i(n') \pmod{m \mathbb{Z}}$ and hence $r_i(n) = r_i(n')$. As a result, 

$s_i(n) - s_i(n') = \frac{(s_{i-1}(n) \cdot n - s_{i-1}(n') \cdot n')}{m}$

from which we deduce that

(3) $s_i(n) \equiv s_i(n') \pmod{\tilde{m}^{h-i} \mathbb{Z}}$

still by using the induction hypothesis. Look at the right member of the relation 

$s_i(n) \cdot n - s_i(n') \cdot n' = (s_i(n) - s_i(n')) \cdot n + s_i(n') \cdot (n - n') .$

The first term is a multiple of $\tilde{m}^{h-i} d$ because of equation (3) and the fact that $d$ divides $n$. The second one is a multiple of $\tilde{m}^{h-i} d$ because of hypothesis (1). Therefore, we get $s_i(n) \cdot n \equiv s_i(n') \cdot n' \pmod{\tilde{m}^{h-i} d}$ as desired.
Proof of (2). By construction, we have:

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\forall i \in \{1, \ldots, h\}, \quad s_{i-1}(n) \cdot n = s_i(n) \cdot m + r_i(n) \\
\text{and} \\
s_{i-1}(n') \cdot n' = s_i(n') \cdot m + r_i(n') 
\end{array}
\right.
\]

for all \( i \in \{1, \ldots, h\} \). By hypothesis (2), we get \( s_i(n) - s_i(n') = \frac{(s_{i-1}(n) - s_{i-1}(n')) \cdot \hat{n}}{\hat{m}} + \frac{s_{i-1}(n') \cdot \hat{n} - n'}{\hat{m}} \), which we write into the following form:

\[
(4) \quad s_{i}(n) - s_{i}(n') = \frac{(s_{i-1}(n) - s_{i-1}(n')) \cdot \hat{n}}{\hat{m}} + \frac{s_{i-1}(n') \cdot \hat{n} - n'}{\hat{m}}.
\]

where \( \hat{n} = n/d \) and \( \hat{n}' = n'/d \). We show by induction on \( k \) that \( \hat{n} \equiv \hat{n}' \mod \hat{m}^{k}Z \) for \( k = 1, \ldots, h \).

The case \( k = 1 \) follows directly from equation (4) for \( i = 1 \) (recall that \( s_0(n) = 1 = s_0(n') \)). For the inductive step, we assume that \( 2 \leq k \leq h \) and \( \hat{n} \equiv \hat{n}' \mod \hat{m}^{k-1}Z \), which implies that the second term in the right member of equation (4) is a multiple of \( \hat{m}^{k-2} \) (it is an integer in particular). We then proceed in \( k \) steps using equation (4) and the fact that \( \hat{m} \) and \( \hat{n} \) are coprime:

- For \( i = k \) we get \( s_{k-1}(n) \equiv s_{k-1}(n') \mod \hat{m}Z \);
- For \( i = k - 1 \) (if \( k \geq 3 \)), we get \( s_{k-2}(n) \equiv s_{k-2}(n') \mod \hat{m}^2Z \);
- So on forth with \( i = k - 2, \ldots, 2 \), we get the sequence of congruences \( s_{k-3}(n) \equiv s_{k-3}(n') \mod \hat{m}^3Z, \ldots, s_1(n) \equiv s_1(n') \mod \hat{m}^{k-1}Z \);
- For \( i = 1 \), we get \( \hat{n} \equiv \hat{n}' \mod \hat{m}^{k}Z \).

Finally, we obtain \( n \equiv n' \mod \hat{m}^{k}dZ \) and the proof is then complete.

\[ \square \]

**Proposition 2.6.**

(i) The functions \( r_i \) and \( s_i \) admit unique continuous extensions

\[
\begin{align*}
\forall i \in \{1, \ldots, h\}, \quad r_i : Z_m \to \{0, \ldots, |m| - 1\}, \\
s_i : Z_m \to Z_m
\end{align*}
\]

such that

\[
(1) r_0(\xi) = 0, \ s_0(\xi) = 1;
\]

\[
(2) \xi s_{i-1}(\xi) = ms_i(\xi) + r_i(\xi) \quad \text{with} \quad r_i(\xi) \in \{0, \ldots, |m| - 1\} \quad \text{for every} \ i \geq 1.
\]

Let \( \xi, \xi' \in Z_m \) such that \( \gcd(\xi, m) = \gcd(\xi', m) = d \) and let \( h \in \mathbb{N} \setminus \{0\} \). Setting \( \hat{m} = m/d \), the two following are equivalent:

(i) \( \xi, \xi' \in Z_m \) holds;

(ii) \( r_i(\xi) = r_i(\xi') \) for \( i = 1, \ldots, h \).

Let \( \xi, \xi' \in Z_m \). The two following are equivalent:

(i) \( \xi \in \mathbb{N} \) such that \( \gcd(\xi, m) = \gcd(\xi', m) = d \) and \( \pi(\xi/d) = \pi(\xi'/d) \)

where \( \pi : Z_m \to Z_{m/d} \) is the canonical ring homomorphism,

(ii) \( r_i(\xi) = r_i(\xi') \) for every \( i \).

(iii) The map \( \xi \mapsto (r_i(\xi))_{i \geq 1} \) defines a homeomorphism from \( Z_m^\times \) onto \( (Z/mZ)^\times \times (Z/mZ)^\mathbb{N} \)

endowed with its product topology. (Here we identify \( Z/mZ \) with the set \( \{0, \ldots, |m| - 1\} \) coprime with \( m \).)

**Proof.** (i) Proposition 2.4 shows that the functions \( r_i, s_i \) are uniformly continuous with respect to the \( m \)-adic topology on \( Z \) and that \( r_i \) is moreover constant on \( m \)-adic balls of radius \(|m|^{-j} \). The existence and uniqueness of the extensions to \( Z_m \) follow. The inductive formulas of Definition 2.4 extend continuously on \( Z_m \), which gives (1) and (2). Let us now pick \( n, n' \in Z \) such that \( \xi \equiv n \) and \( \xi' \equiv n' \mod \hat{m}Z_m \). Then, one has \( n \equiv n' \mod \hat{m}dZ_m \) if and only if \( \xi \equiv \xi' \mod \hat{m}dZ_m \). As \( r_i \) is constant on \( m \)-adic balls of radius \(|m|^{-j} \), we have \( r_i(n) = r_i(\xi) \) and \( r_i(n') = r_i(\xi') \) for \( i = 1, \ldots, h \). The equivalence between (1) and (2) now follows readily from Proposition 2.6.

(ii) As \( \gcd(m, \xi) = \gcd(m, r_i(\xi)) \) for every \( \xi \in Z_m \), by (2), the result immediately follows from (i).

(iii) Let \( \xi \in Z_m^\times \). By (2), we have \( r_i(\xi) \in (Z/mZ)^\times \). For any \( \xi, \xi' \in Z_m^\times \), we have \( d = \gcd(\xi, m) = \gcd(\xi', m) = 1 \). We deduce from (2) that the map \( R : \xi \mapsto (r_i(\xi))_{i \geq 1} \) defines a
continuous embedding from \( \mathbb{Z}_m^x \) into \( P := (\mathbb{Z}/m\mathbb{Z})^x \times (\mathbb{Z}/m\mathbb{Z})^h \). As \( \mathbb{Z}_m^x \) and \( P \) are compact, it is sufficient to show that \( R \) has a dense image. Let \( E_h \) be the set of integers \( k \in \{0, \ldots, |m|^h - 1\} \) coprime with \( m \). Consider the map \( R_h : \xi \mapsto (r_i(\xi))_{1 \leq i \leq h} \). This is an injective map from \( E_h \) into \( P_h := (\mathbb{Z}/m\mathbb{Z})^x \times (\mathbb{Z}/m\mathbb{Z})^{h-1} \) by the equivalence of (1) and (2). As \( E_h \) and \( P_h \) both have cardinality \( \varphi(m)|m|^{h-1} \), where \( \varphi \) denotes the Euler function, the map \( R_h \) is a bijection. Hence \( R \) maps the set of integers coprime with \( |m| \) onto a dense subset of \( P \). \( \square \)

The following proposition shows that the restriction of \( s_i \) to a suitable congruence class is a polynomial in \( n \) with coefficients in \( \mathbb{Q} \).

**Proposition 2.7.** Let \( \xi \in \mathbb{Z}_m \). We define recursively \( P_h(X) = P_h,\xi(X) \) by

\[
P_0(X) = m \text{ and } P_h(X) = XP_{h-1}(X) - r_h(\xi) \text{ for } h \geq 1.
\]

Then, we have:

(i) \( P_h(X) = mX^h - r_1(\xi)X^{h-1} - \cdots - r_h(\xi) \in \mathbb{Z}[X] \).

(ii) \( s_h(n) = \frac{1}{m} P_h(\frac{n}{m}) \) for all \( h \geq 0 \) and all \( n \in \mathbb{Z} \) such that \( n \equiv \xi \pmod{m^h} \).

**Proof.** (i) The proof is an obvious induction on \( h \).

(ii) First, observe that \( r_i(n) = r_i(\xi) \) for all \( i \leq h \) and all \( n \in \mathbb{Z} \) such that \( n \equiv \xi \pmod{m^h} \) by Proposition 2.6. An easy induction on \( h \) using the definitions of \( s_i(n) \) and \( P_i(X) \) gives the conclusion. \( \square \)

2.2. **Stabilizers.** This section is devoted to the study of the stabilizers \( B, B_m \) and \( B_\xi \).

Let \( b_0 = b, b_1 = ab^m a^{-1}, b_i = ab_{i-1}b_0^{-r_i(\xi)} a^{-1} \) for \( i \geq 2 \). The following lemma shows that \( b_i \) defines an element of \( B \) for every \( i \).

**Lemma 2.8.** Let \( i \geq 1 \). We have \( b_i = b^\xi \xi_{s_i}^{-1}(\xi) \) in \( BS(m, \xi_n) \) for all \( n \) large enough.

**Proof.** We show by induction on \( i \) that:

\[
(5) \quad \text{for every } i \geq 1 \text{ we have } b_i = b^{\xi_{s_{i-1}(\xi)}} \text{ in } BS(m, \xi_n) \text{ for all } n \text{ large enough.}
\]

Since \( b_1 = ab^m a^{-1} = b^{\xi_n} \) in \( BS(m, \xi_n) \) and \( s_0(\xi_n) = 1 \) for all \( n \), (5) holds if \( i = 1 \). Assume (5) holds for some \( i \geq 1 \). By the induction hypothesis we have \( b_{i+1} = ab_i b^{-r_i(\xi)} a^{-1} = ab^{\xi_{s_i}^{-1}(\xi_n)} b_i^{-r_i(\xi)} a^{-1} \) for all \( n \) large enough. Recall that \( \xi_n \) tends to \( \xi \) in \( \mathbb{Z}_m \) and hence \( r_i(\xi_n) = r_i(\xi) \) for all \( n \) large enough by Proposition 2.6. By Definition 2.4, we obtain \( b_{i+1} = b^{\xi_{s_i}^{-1}(\xi_n)} \) in \( BS(m, \xi_n) \) for all \( n \) large enough. Since \( \xi_n s_i(\xi_n) = \frac{n}{m} P_i(\frac{n}{m}) \) for all \( n \) large enough by Proposition 2.7, the proof is then complete. \( \square \)

We can generalize the previous lemma by assigning to every \( \gamma \in B \) a polynomial with integer coefficients.

**Proposition 2.9.**

(i) Let \( w \in \mathbb{F}_2 \) with image \( \gamma \in B \). Then there is a unique polynomial \( P_{\gamma}(X) \in \mathbb{Z}[X] \) independent of \( w \) such that \( w = b^{P_{\gamma}(\xi_n/m)} \) in \( BS(m, \xi_n) \) for all \( n \) large enough.

(ii) The map \( q : \gamma \mapsto P_{\gamma}(X) \) defines an injective homomorphism from \( B \) into \( \mathbb{Z}[X] \). The abelian group \( B = q(B) \) is freely generated by \( \{1, XP_0(X), XP_1(X), XP_2(X), \ldots\} \). Hence \( B \) is freely generated by the set \( \{b_0, b_1, b_2, b_3, \ldots\} \).

(iii) Let \( w \in \mathbb{F}_2 \) with image \( \gamma \in B \). Then \( \gamma \) belongs to \( B_m \) if and only if

\[
P_{\gamma}(X) = k_0 + k_1 mX + k_2 XP_1(X) + \cdots + k_t XP_{t-1}(X)
\]

with \( k_0 + k_1 r_1(\xi) + k_2 r_2(\xi) + \cdots + k_t r_t(\xi) \equiv 0 \pmod{m} \). Moreover the abelian group \( B_m = q(B_m) \) is freely generated by \( P_0(X) = m, P_1(X) = XP_0(X) - r_1(\xi), P_2(X) = XP_1(X) - r_2(\xi), \ldots \). Hence \( B_m \) is freely generated by \( \{b_0^m, b_1 b_0^{r_1(\xi)}, b_2 b_0^{r_1(\xi)} r_2(\xi), \ldots\} \).

(iv) The abelian group \( B_\xi = q(B_\xi) \) is freely generated by \( \{XP_0(X), XP_1(X), XP_2(X), \ldots\} \).

The abelian group \( B_\xi \) is freely generated by \( \{b_1, b_2, b_3, \ldots\} \).
(v) For any \( \gamma \in B_m \), we have \( q(a^{\gamma}a^{-1}) = XP_{\gamma}(X) \). Moreover the map \( a \mapsto (0,1), b \mapsto (1,0) \) induces a surjective homomorphism \( q_m, \xi : BS(m, \xi) \rightarrow \mathbb{Z}/\mathbb{Z} \).  

By Proposition 2.9.  

Q.E.D.

(iii) Let \( \xi \in \mathbb{Z}_m \) and \( \xi \in \mathbb{Z}/\mathbb{Z} \) be the canonical map. The maps \( \chi : \gamma \mapsto P_{\gamma}(\xi/m) \) and \( \tilde{\chi} = \pi \circ \chi \) define homomorphisms from \( B \) to \( B_m \) and \( \mathbb{Z}/\mathbb{Z} \) respectively. Their kernels satisfy: \( \ker \chi = \bigcap_{i \geq 0} a^{-i}Ba^{i} \subset \ker \tilde{\chi} \). Moreover, if \( \bigcap_{i \geq 0} a^{-i}Ba^{i} \) is non-trivial, then it contains some \( \gamma \in \mathbb{B}_m \setminus B_\xi \).

(vii) Let \( \mathcal{C} \) be the image of \( \mathbb{B}_\xi \) by the map \( \nu : P(X) \rightarrow \mathbb{X}/P(X) \). Then \( \mathcal{C} \) is a subgroup of \( \mathbb{B} \) and \( \mathbb{B}/\mathcal{C} \) is an infinite cyclic group generated by the image of 1.

Proof of Proposition $\mathcal{E}$

(i) The uniqueness of \( P_{\gamma}(X) \) follows from the fact that a non-zero polynomial with coefficients in \( \mathbb{Q} \) has only finitely many zeros in \( \mathbb{Q} \). Using the fact that \( BS(m, \xi_n) \) converges to \( BS(m, \xi) \), we also deduce that \( P_{\gamma}(X) \) is independent of \( w \). To show the existence of \( P_{\gamma}(X) \) we write \( w = b^{\sigma_n}a^\xi b^{-n}a^{-1}n \) with \( \varepsilon_j = \pm 1 \) for \( j = 1, \ldots, h \). By Lemma 2.2 \( w \) reduces to a power of \( b \) in \( BS(m, \xi_n) \) for all \( n \) large enough. Consequently we have \( \sigma_n(w) = 0 \) and hence \( h = 2t \) for some \( t \in \mathbb{N} \). By [2008] Lemma 2.6 with \( d = 1 \) there exist \( k_0, \ldots, k_t \in \mathbb{Z} \) depending only on \( w \) and \( \xi \) such that \( w = b^{\sigma(n)} \) in \( BS(m, \xi_n) \) for all \( n \) such that \( \xi_n \equiv \xi \mod m^i \mathbb{Z}_m \) with \( |\xi_n| \) large enough and

\[
\alpha(n) = k_0 + k_1 \xi_n + k_2 s_1(\xi_n) \xi_n + \cdots + k_t s_{t-1}(\xi_n) \xi_n,
\]

the latter equation being Formula (*) in the proof of [2008] Lemma 2.6. Since \( \xi_n \) tends to \( \xi \in \mathbb{Z}_m \) and \( |\xi_n| \) tends to infinity as \( n \) goes to infinity, the equality (6) holds for all \( n \) large enough. Using Proposition 2.7 we can write \( \alpha(n) = P_{\gamma}(\xi_n) \) with

\[
P_{\gamma}(X) = k_0 + k_1 XP_0(X) + k_2 XP_1(X) + \cdots + k_t XP_{t-1}(X).
\]

(ii) The map \( q : \gamma \mapsto P_{\gamma}(X) \) trivially induces a homomorphism from \( B \) to \( \mathbb{Z}/\mathbb{Z} \) such that \( q(b) = 0 \). Let \( w \in \mathbb{F}_2 \) with image \( \gamma \in B \). If \( P_{\gamma}(X) \) is the zero polynomial then \( w \) is trivial in \( BS(m, \xi_n) \) for all \( n \) large enough and hence it is trivial in \( BS(m, \xi) \). Thus \( q \) is injective. It is immediate from (7) that \( q(B) \) is a subgroup of the free abelian group with basis \( \{ 1, XP_0(X), XP_1(X), \ldots \} \). It follows from Lemma 2.2 that \( q(b_i) = XP_{i-1}(X) \) for all \( i \geq 1 \), so that \( q(B) \) coincides with the latter group.

(iii) Let \( w \in \mathbb{F}_2 \) with image \( \gamma \in B \). By (i) we can write \( w = b^{P_{\gamma}(\xi_n/m)} \) in \( BS(m, \xi_n) \) for all \( n \) large enough. Since \( \xi_n s_{i-1}(\xi_n) \equiv r_i(\xi) \mod m \mathbb{Z} \) for every \( i \geq 1 \) and for all \( n \) large enough, we deduce from (7) that \( P_{\gamma}(\xi_n/m) \equiv k_0 + k_1 r_1(\xi) + k_2 r_2(\xi) + \cdots + k_t r_t(\xi) \mod m \mathbb{Z} \) for all \( n \) large enough. By Lemma 2.2 we have: \( \gamma \in \mathbb{B}_m \) if and only if \( \alpha(n) \equiv 0 \mod m \mathbb{Z} \). This proves the first claim.

We easily deduce that \( \{ m, XP_0(X) - r_1(\xi), XP_1(X) - r_2(\xi), \ldots \} \) freely generates \( \mathbb{B}_m \). Since \( q(b_0^n) = m \) and \( q(b_i b_0^{-r_i(\xi)}) = XP_{i-1}(X) - r_i(\xi) \) the set \( \{ b_0^n, b_1 b_0^{-r_1(\xi)}, b_2 b_0^{-r_2(\xi)}, \ldots \} \) freely generates \( B_m \).

(iv) Since \( B_\xi = a B_m a^{-1} \), we deduce from (iii) and the definition of \( b_i \) that \( \{ b_1, b_2, \ldots \} \) generates \( B_\xi \). As the elements \( q(b_i) = XP_{i-1}(X) \) \( (i \geq 1) \) freely generate \( \mathbb{B}_\xi \), we deduce that \( \{ b_1, b_2, \ldots \} \) freely generates \( B_\xi \).

(v) Let \( w \in \mathbb{F}_2 \) with image \( \gamma \in B_m \). We can write \( w = b^{P_{\gamma}(\xi_n/m)} \) in \( BS(m, \xi_n) \) for all \( n \) large enough with \( P_{\gamma}(\xi_n/m) \equiv 0 \mod m \mathbb{Z} \). Thus \( a^{\gamma}a^{-1} = b^{(\xi_n/m)P_{\gamma}(\xi_n/m)} \) in \( BS(m, \xi_n) \) for all \( n \) large enough. Hence \( q(a^{\gamma}a^{-1}) = XP_{\gamma}(X) \) by definition of \( q \).

The map \( q_m, \xi \) is well-defined homomorphism by [2008] Th. 3.12. Using the first part of (v), we easily deduce by induction that \( q(b_i) = q_m, \xi (b_i) \) for every \( i \geq 0 \), which completes the proof.

(vi) The map \( \gamma \mapsto P_{\gamma}(\xi/m) \) is a well-defined homomorphism from \( B \) to \( \mathbb{Z}/\mathbb{Z} \). By Proposition 2.6 we have \( s_i(\xi) \in \mathbb{Z}_m \) for every \( i \). By Proposition 2.7 we have \( P_{\gamma}(\xi/m) = s_i(\xi) \in \mathbb{Z}_m \) for every \( n \) large enough and hence \( P_{\gamma}(\xi/m) = s_i(\xi) \in \mathbb{Z}_m \) by Proposition 2.6. Now it follows from (ii) that \( P_{\gamma}(\xi/m) \) belongs to \( \mathbb{Z}_m \) for every \( \gamma \in B \).

Let \( \gamma \in \ker \chi \) and let \( i \geq 0 \). For all \( n \) large enough \( m^i \) divides \( P_{\gamma}(\xi_n/m) \). We deduce from (i) and Lemma 2.2 that \( a^{\gamma}a^{-1} \in B \). Thus \( \gamma \in \bigcap_{i \geq 0} a^{-i}Ba^{i} \).
Let $\eta = \pi(\xi/d)$. By (v) we have $\hat{\chi}(a\gamma^{-1}a^{-1}) = \frac{1}{m}\hat{\chi}(\gamma) \in \mathbb{Z}_m$ for every $\gamma \in B \cap a^{-1}Ba$. Consider now $\gamma \in B$ such that $a\gamma^{-1}a^{-1} \in B$ for every $i$. As $\gcd(m, \eta) = 1$, $m_i$ divides $\hat{\chi}(\gamma)$ for every $i$. Therefore we have $\gamma \in \ker \hat{\chi}$.

Assume that $B' = \bigcap_{i \geq 0} a^{-i}Ba^i$ is non-trivial. Clearly $B' \subset B_m$ and any $\gamma \in B'$ such that $P_\gamma(X)$ has minimal degree does not belong to $B_\xi$ by (v).

(vii) It follows from the definition of $P_i(X)$ that multiplication by $\frac{X_i}{X_i^2}$ maps $XP_0(X)$ to $-m + XP_0(X)$ and $XP_i(X)$ to $r_i(\xi) - XP_{i-1}(X) + XP_i(X)$ for every $i \geq 1$. Therefore $\mathcal{C}$ is a subgroup of $\mathcal{B}$. A straightforward induction on $i$ shows that the image of $XP_i(X)$ in $\mathcal{B}/\mathcal{C}$ lies inside the cyclic subgroup generated by the image of 1. As $\mathcal{B}_\xi$ does not contain any constant polynomial, neither does $\mathcal{C}$. Therefore $\mathcal{B}/\mathcal{C}$ is infinite. □

Now, we can give a simple description of the HNN structure of $\overline{BS}(m, \xi)$. Let $\overline{BS}(m, \xi)$ be the HNN extension of basis $E$ with conjugated subgroups $E_{m, \xi}$ and $E_1$ stable letter $a$, where $E, E_m, E_\xi$ are free abelian groups of countable rank, namely

$$E = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots$$

$$E_{m, \xi} = \mathbb{Z}me_0 \oplus \mathbb{Z}(e_1 - r_1(\xi)e_0) \oplus \mathbb{Z}(e_2 - r_2(\xi)e_0) \cdots$$

$$E_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots$$

and conjugacy from $E_{m, \xi}$ to $E_1$ is defined by $a(me_0)a^{-1} = e_1$ and $a(e_i - r_i(\xi)e_0)a^{-1} = e_{i+1}$. We denote by $\phi$ the isomorphism from $E_{m, \xi}$ to $E_1$ induced by $\tau_a$.

Corollary 2.10. The map defined by $f(a) = a$ and $f(b) = e_0$ induces an isomorphism from $\overline{BS}(m, \xi)$ to $BS(m, \xi)$. The inverse map is determined by $f^{-1}(a) = a$ and $f^{-1}(e_i) = b_i$ for every $i \geq 0$.

Proof. It is a straightforward corollary of Propositions 2.1 and 2.0 □

We have then three different notations for the same group, namely

- $\overline{BS}(m, \xi) = \text{HNN}(B, B_m, B_\xi, \tau)$ where $B$ is generated by $b = b_0, b_1, b_2, \ldots$,
- $BS(m, \xi) = \text{HNN}(E, E_{m, \xi}, E_1, \phi)$ where $E$ is generated by $e_0, e_1, e_2, \ldots$,
- $BS(m, \xi) = \text{HNN}(\mathcal{B}, \mathcal{B}_m, \mathcal{B}_\xi, \theta)$ where $\mathcal{B}$ is generated by 1, $XP_0(X)$, $XP_1(X)$, \ldots and $\theta$ is defined in the obvious way.

Our favoured HNN extension is $\overline{BS}(m, \xi)$ as it considerably eases off notations when different parameters $m$ and $\xi$ are considered (Sec. 5 and Sec. 6). Nevertheless $BS(m, \xi)$ is useful when we still need to see this group as limit in the space of marked groups (Sec. 5) and the HNN extension $BS(m, \xi)$ involving polynomials is advantageous to study relations in the group (Sec. 5).

3. An infinite presentation built up from $\xi$

In this section, we use of the HNN decomposition of $\overline{BS}(m, \xi)$ to give an infinite group presentation which depends explicitly on $m$ and the sequence $(r_i(\xi))$. A group presentation $\langle X \mid R \rangle$ is said to be minimal if the kernel of the natural homomorphism $\langle X \rangle \to \langle X \mid R' \rangle \to \langle X \mid R \rangle$ is non-trivial for all $R' \subsetneq R$.

Theorem 3.1. The group $\overline{BS}(m, \xi)$ admits the following infinite presentation:

$$\langle a, b \mid [b, b_i] = 1, i \geq 1 \rangle$$

with $b_1 = ab^ma^{-1}$ and $b_i = ab_{i-1}b^{-r_{i-1}(\xi)}a^{-1}$ for every $i \geq 2$.

In particular, we have $\overline{BS}(m, 0) = \langle a, b \mid [b, a^{b^ma^{-1}}] = 1, i \geq 1 \rangle$. The latter presentation is moreover minimal.

In the case $\xi \neq 0$, we previously showed [GS08, Th. 4.1] that $\overline{BS}(m, \xi)$ cannot be finitely presented. The following consequence is therefore immediate.

Corollary 3.2. No group $\overline{BS}(m, \xi)$ is finitely presented.
Proof of Theorem 3.1. We divide the proof into two parts:

1. We show that $\overline{BS}(m, \xi) = \langle a, b \mid [b, b_j] = 1, i \geq 1 \rangle$ using Propositions 2.1 and 2.2.
2. We show that this presentation is minimal in the case $\xi = 0$ using basic facts on graph groups.

Proof of (1). By Proposition 2.9, we can define an isomorphism $\phi : B_m \rightarrow B_{\xi}$ by $\phi(b_{0}^{i}) = b_{1}$, $\phi(b_{0}^{-r_{i}(\xi)}) = b_{i+1}$ for $i \geq 1$. Using the HNN decomposition of Proposition 2.1, we deduce that $\overline{BS}(m, \xi)$ admits the presentation $\langle a, B \mid a^{\gamma}a^{-1} = \phi(\gamma), \gamma \in B_m \rangle$. Since $a$ and $b$ generate $BS(m, \xi)$ and $B$ is a free abelian group (Proposition 2.2), we deduce that

$$\overline{BS}(m, \xi) = \langle a, b \mid [b, b_j] = 1, 0 \leq i < j \rangle$$

(Note that the relations $abma^{-1} = \phi(b_{0}^{m})$ and $abma^{-1} = \phi(b_{0}^{-r_{i}(\xi)})$ are satisfied by definition of $b_i$.) We show by induction on $i \geq 1$ the following claim: any relation $[b, b_j] = 1$ with $0 \leq i < j$ is a consequence of relations $[b, b_k] = 1$ with $1 \leq k \leq j$. The case $i = 0$ is trivial. Assume now that $i \geq 1$. Using definitions, we can write $[b, b_j] = a_{[b, b_{i}]}b_{0}^{-r_{i}(\xi)}$, $b_{j-1}b_{0}^{-r_{j-1}(\xi)}a^{-1}$. By induction hypothesis the relation $[b_{i}, b_{j-1}] = 1$ is a consequence of relations $[b_{0}, b_{k}] = 1$ with $1 \leq k \leq j-1$. Hence $[b, b_j] = 1$ is a consequence of relations $[b, b_k] = 1$ with $1 \leq k \leq j$. The induction is then complete. Thus all relations $[b, b_j] = 1$ with $1 \leq i < j$ can be deleted from presentation $\overline{BS}(m, \xi)$.

Proof of (2). It remains to show that $\overline{BS}(m, \xi) = \langle a, b \mid [b, b_j] = 1, i \geq 1 \rangle$ is a minimal presentation when $\xi = \pi(\xi) = 0$. In this case we have $r_{i}(\xi) = 0$, and hence $b_i = a^{bma^{-1}}$, for all $i \geq 1$.

We fix some $k \geq 1$ and we show that $[b, b_k] = 1$ does not hold in the group $G_k := \langle a, b \mid [b, b_0] = 1, k \neq i \geq 1 \rangle$.

To prove this we consider the group $\overline{B} = \langle g_0, g_1, \ldots \mid [g, g_j] = 1, |i - j| \neq k \rangle$ and its subgroups $\overline{B}_m = \langle g_0, g_1, g_2, \ldots \rangle$ and $\overline{B}_0 = \langle g_0, g_1, g_2, \ldots \rangle$. We will use basic facts on graph groups, i.e. groups defined by a presentation whose relators are commutators of some pairs of the generators. (Such presentations are often encoded by a graph, whose vertices correspond to the generators and whose edges tell which ones commute; this explains the terminology.) Let $G = \langle X \mid R \rangle$ be a graph group. Any element of $G$ can be written as a word $c_1c_2\ldots c_l$ where each syllable $c_i$ belongs to some cyclic group generated by some element of $X$. We consider three types of moves that we can perform on such words:

1. Remove a syllable $c_i = 1$.
2. Replace consecutive syllables $c_i$ and $c_{i+1}$ in the same cyclic subgroup by the single syllable $(c_ic_{i+1})$.
3. For consecutive syllables $c_i \in \langle x \rangle$ and $c_{i+1} \in \langle x' \rangle$ with $x, x' \in X$, $[x, x'] \in R$, exchange $c_i$ and $c_{i+1}$.

If $g \in G$ is represented by a word $w = c_1\ldots c_l$ which cannot be changed to a shorter word using any sequence of the above moves, then $w$ is said to be a normal form for $g$. We will use the following results:

**Theorem 3.3.** (Bau81)

(Normal Form Theorem) A normal form in a graph group represents the trivial element if and only if it is the trivial word.

(Abelian Subgroups) Any abelian subgroup of a graph group is a free abelian group.

We define the partial map $\psi : \overline{B}_{\xi} \rightarrow \overline{B}_m$ by $\psi(g_i) = g_{0}^{i}$ and $\psi(g_i) = g_{i-1}$ for every $i \geq 2$. By using the Normal Form Theorem, we can readily show that $\overline{B}_{\xi} = \langle g_1, g_2, \ldots \mid [g, g_j] = 1, |i - j| \neq k \rangle$. We clearly have $[\psi(g), \psi(g_j)] = 1$ in $\overline{B}_m$ for any $i, j \geq 1$ such that $|i - j| \neq k$. Hence $\psi$ induces a surjective homomorphism from $\overline{B}_{\xi}$ onto $\overline{B}_m$.

Let $g \in \overline{B}_{\xi}$. Replacing every $g_i$ by $\psi(g_i)$ in any non-trivial normal form for $g$ in $\overline{B}$ clearly leads to a non-trivial normal form for $\psi(g)$ in $\overline{B}$. Hence $\psi$ is injective, which proves that $\psi$ is an isomorphism. Let $\bar{\phi}$ be its inverse homomorphism. We set $\overline{G}_k := \langle \bar{B}, a \mid aga^{-1} = \phi(\gamma), g \in \overline{B}_m \rangle$.

We trivially check that the map $a \mapsto a$, $b \mapsto g_0$ induces a surjective homomorphism from $G_k$ onto
\( \tilde{G}_k \) that maps \([b_0, b_k]\) to \([g_0, g_k]\). Since the graph group \( \tilde{B} \) embeds isomorphically into the HNN extension \( \tilde{G}_k \), we have \([g_0, g_k] \neq 1\) and hence \([b_0, b_k] \neq 1\).

4. C*-Simplicity

We first recall some definitions. Let \( \Gamma = BS(m, \xi) \) and let \( \sigma : BS(m, \xi) \to Z \) be the surjective homomorphism defined by \( \sigma(a) = 1 \) and \( \sigma(b) = 0 \). Let \( Z \wr Z = Z[X^{\pm 1}] \times XZ \) where the action of \( Z \) on the additive group of \( Z[X^{\pm 1}] \) is the multiplication by \( X \). This group is generated by \( \{(1, 0), (0, 1)\} \) and the map \( a \mapsto (0, 1), b \mapsto (1, 0) \) induces a surjective homomorphism \( q_{m, \xi} \) from \( \Gamma \) onto \( Z \wr Z \) \cite{GS08} Th. 3.12. By \( (v) \) and \( (vi) \) of Proposition 2.2, the restriction of \( q_{m, \xi} \) to \( B \) (which identifies with a subgroup of \( Z[X] \)) is the identity and the map \( \gamma \mapsto P_\gamma(\xi/m) \) is a well-defined homomorphism from \( B \) to the additive group of \( Z_m \).

Given a group \( \Gamma \), recall that its reduced C*-algebra \( C^*_r(\Gamma) \) is the closure for the operator norm of the reduced C*-algebra \( C(\Gamma) \) acting by the left-regular representation on the Hilbert space \( \ell^2(\Gamma) \). For an introduction to group C*-algebras, see for example \cite{Dav96} Ch.VII. A group is C*-simple if it is infinite and if its reduced C*-algebra is a simple topological algebra.

Non-abelian free groups are C*-simple. The first proof of this fact, due to Powers \cite{Pow75}, relies on a combinatorial property of free groups shared by many other groups, called for this reason Powers groups. Thus Powers groups are C*-simple. The Baumslag-Solitar group \( BS(m, n) \) is C*-simple if and only if \( |m| \neq |n| \) \cite{Iva01} Th. 4.9. In this case, it is actually a strongly Powers group \cite{HP09} Pr. 5. A group \( G \) is said to be strongly Powers if any of its subnormal subgroups is a Powers group.

**Theorem 4.1.** Let \( |m| > 1, \xi \in Z_m \). Then \( BS(m, \xi) \) is a strongly Powers group.

Amenable groups are not C*-simple \cite{Har07} Pr. 3. This is the reason why we have to exclude the groups \( BS(m, \xi) \) with \( |m| = 1 \). Actually, there is only one such marked group \cite{GS08} Th. 2.1 and it is isomorphic to the solvable group \( Z \wr Z \) \cite{Sta06} Th. 2.

We consider the action of \( \Gamma = BS(m, \xi) \) on its Bass-Serre tree \( T = X_{m, \xi} \) and use the criterion of de la Harpe and Préaux \cite{HP09} Pr. 16 on tree action to show Theorem 4.1. The latter criterion needs the action of \( \Gamma \) to be faithful in a strong sense: it has to be slender. Two other conditions are required, but both follow immediately from \cite{HP09} Pr. 24, as \( \Gamma \) is a non-ascending HNN extension. Let us define a slender action. A tree automorphism \( \gamma \) of \( T \) is slender if its fixed point set \( \partial T \gamma \) has empty interior in \( \partial T \) with respect to the shadow topology (Sec. 1.3). The action of \( \Gamma \) on \( T \) is slender if for every \( \gamma \in \Gamma \setminus \{1\} \) the automorphism of \( T \) induced by \( \gamma \), also denoted by \( \gamma \), is slender. A slender action is faithful, it is even strongly faithful in the sense of \cite{HP09} Sec. 1.

Since hyperbolic elements are obviously slender, we focus now on the fixed point set of elliptic elements. We still need more definitions to describe the fixed point set \( T^\gamma \) for any \( \gamma \in B \).

Let \( l, u \in Z \cup \{\pm \infty\} \). We denote by \( \{l \leq \sigma \leq u\} \) the subgraph of \( T \) whose vertices \( \gamma B \) satisfy \( l \leq \sigma(\gamma) \leq u \). We denote by \( T[l, u] \) the connected component of \( 1B \) in \( \{l \leq \sigma \leq u\} \).

**Lemma 4.2.** Let \( l, u \in Z \cup \{\pm \infty\} \). Assume either that \( l > -\infty \) and \( |m| > 1 \), or that \( u < +\infty \). Then, the set \( \partial[l \leq \sigma \leq u] \subset \partial T \) has empty interior in \( \partial T \) with respect to the shadow topology.

**Proof.** Assume that \( l > -\infty \) and \( |m| > 1 \). Since any vertex \( \gamma B \) of \( T \) has \( |m| \) neighbours \( \gamma'B \) such that \( \sigma(\gamma') = \sigma(\gamma) - 1 \), any shadow contains (the class of) a geodesic ray \( (\gamma_1 B, \gamma_2 B, \ldots, \gamma_n B, \ldots) \) such that \( \sigma(\gamma_n) \) tends to \( -\infty \). Such a ray does not lie in \( \partial[l \leq \sigma \leq u] \). The proof of the second case is analogous (any vertex \( \gamma B \) of \( T \) has countably many neighbours \( \gamma'B \) such that \( \sigma(\gamma') = \sigma(\gamma) + 1 \)).

Let \( \nu \) be the natural valuation on \( Z[X] \), i.e. the one defined by \( \nu(X^i) = i \) for every \( i \). Let \( \nu \) be the “valuation” defined on \( B \) by \( \mu(\gamma) = \sup \{i \geq 1 : P_\gamma(\xi/m) \in d\nu(X^iZ_m)\} \) where \( \sup = 0 \). Let \( \gamma \in B \). Observe that

- \( \gamma \in B_m \) if and only if \( \mu(\gamma) \geq 1 \),
- \( \gamma \in B_\xi \) if and only if \( \nu(P_\gamma) \geq 1 \).
Proposition 4.3. Let $\gamma \in B$. We have $T^\gamma = T[-\mu(\gamma), \nu(P_\gamma)]$.

Corollary 4.4. Let $\Gamma = \overline{BS}(m, \xi)$.

- The action of $\Gamma$ on $T$ is slender.
- The centralizer of $b$ in $\Gamma$ coincides with $B$ if $|m| > 1$.
- The centralizer of $a^k$ in $\Gamma$ coincides with $(a)$ for every $k \neq 0$.

Proof of Corollary 4.4. Let us show that the action of $\Gamma$ is slender. Remind that any hyperbolic element of $\Gamma$ is slender. Since any elliptic element is conjugated to some $\gamma \in B$ by a homeomorphism of $\partial T$, it suffices to prove that every $\gamma \in B \setminus \{1\}$ is slender. Since $\nu(P_\gamma) < \infty$ for every $\gamma \in B \setminus \{1\}$, the result follows from Proposition 4.3.

Assume that $|m| > 1$ and let $\gamma$ be an element of the centralizer of $b$ in $\Gamma$. By Proposition 4.3, we have $T^b = 1B$. Since $\gamma$ commutes with $b$, we have then $\gamma \cdot 1B = 1B$, i.e. $\gamma \in B$.

Let $k \in \mathbb{Z} \setminus \{0\}$ and let $\gamma$ be an element of the centralizer of $a^k$ in $\Gamma$. Let $a_+$ (resp. $a_-$) be the class of rays which are cofinal with $(a^kB)_{n \geq 0}$ (resp. $(a^nB)_{n \leq 0}$). Then $\gamma$ preserves the fixed point set $\{a_+, a_-\}$ of $a^k$. Since $\sigma(\gamma a^n) = \sigma(\gamma) + n$ for every $n \in \mathbb{Z}$, $\gamma$ cannot exchange $a_+$ and $a_-$ and hence fixes them both. Consequently, there is some $n \in \mathbb{Z}$ such that $\gamma \cdot B = a^nB$, i.e. there is $g \in B$ such that $\gamma = a^n g$. We deduce that $g$ centralizes $a^k$ and hence $g$ fixes $a_+$ and $a_-$. As $g$ is elliptic, Proposition 4.3 gives $\nu(P_\gamma) = +\infty$, hence $g = 1$. Thus $\gamma = a^n$. □

Proposition 4.3 is a straightforward consequence of the following lemma.

Lemma 4.5. Let $g \in \Gamma$ and let $c_0 a^{+\epsilon_1} c_1 \cdots a^{+\epsilon_k} c_k$ with $\epsilon_i \in \{\pm 1\}$, $c_i \in B$ be a reduced form of $g$. Let $\sigma_i(g) = \sum_{j=1}^i \epsilon_j$. Let $\gamma \in B$. The following are equivalent:

1. $gB \in T^\gamma$.
2. $g^{-1}\gamma g \in B$.
3. $-\mu(\gamma) \leq \sigma_i(g) \leq \nu(P_\gamma)$ for every $1 \leq i \leq k$.

If the previous conditions hold then we have: $g^{-1}\gamma g = a^{-\sigma(g)} \gamma a^{\sigma(g)}$.

Proof. (i) $\iff$ (ii) is trivial. We show the equivalence (ii) $\iff$ (iii) and the last statement of the lemma by induction on $k$. If $k = 0$, both are trivial. Assume that $k > 0$ and write $w = c_0 a^{+\epsilon_1} g'$.

(ii) $\implies$ (iii): we have $g'^{-1}\gamma' g' \in B$ with $\gamma' = a^{-\epsilon_1} \gamma a^{\epsilon_1}$. By Britton’s lemma, we have: $g'^{-1}\gamma' g' \in B$ and, either $\gamma \in B_m$ and $\epsilon_1 = -1$, or $\gamma \in B_k$ and $\epsilon_1 = 1$. We deduce that $\gamma' \in B$, $\mu(\gamma') = \mu(\gamma) + \epsilon_1$, $\nu(P_\gamma') = \nu(P_\gamma) - \epsilon_1$, $\gamma' \in B$ and $-\mu(\gamma) \leq \epsilon_1 \leq \nu(P_\gamma)$. The result then follows from the induction hypothesis.

(iii) $\implies$ (ii): As $-\mu(\gamma) \leq \epsilon_1 \leq \nu(P_\gamma)$, we have either $\epsilon_1 = -1$ and hence $\gamma \in B_m$, or $\epsilon_1 = 1$ and hence $\gamma \in B_k$. We deduce that $\gamma' \in B$. The result follows from the induction hypothesis.

Last statement: by induction hypothesis, we have

$$g'^{-1}\gamma' g' = a^{-\sigma(g') \gamma a^{\sigma(g')}} = a^{-\epsilon_1 - \sigma(g') \gamma a^{\epsilon_1} + \sigma(g')} = a^{-\sigma(g)} \gamma a^{\sigma(g)}.$$

□

Proof of Theorem 4.7. Since $\Gamma = \overline{BS}(m, \xi)$ is a non-ascending HNN extension, the action of $\Gamma$ is strongly hyperbolic on $T$ and minimal on $\partial T$ [HP09, Pr. 22]. The action of $\Gamma$ on $T$ is slender by Corollary 4.3. By [HP09, Pr. 16], $\Gamma$ is a strongly Powers group. □

Inner amenability. A countable group $G$ is said inner amenable if it admits a mean (i.e. a non-zero, finite and finitely additive measure) on the set of all the subsets of $G \setminus \{1\}$ which is invariant under inner automorphisms. We say that $G$ has the icc property if the conjugacy class of any of its non-trivial element is infinite. These two notions are motivated by the study of the von Neumann algebra $W^*(G)$ of $G$ (see [Eff75, BH86]). Amenable groups or groups that do not have icc are clearly inner amenable. The second-named author has proved that the Baumslag-Solitar group $BS(n, m)$ has icc, is inner amenable but not amenable whenever $|m| > |n| > 1$ [Sta06b, Ex. 2.4 and 3.2]. Note that for every $|m| > 1$, the group $\overline{BS}(m, \xi)$ also has icc since every Powers group does [Har85, Pr. 1].
Proposition 4.6. Let $|m| > 1, \xi \in \mathbb{Z}_m$. The group $\overline{BS}(m, \xi)$ is inner amenable and non-amenable.

The proof relies on:

Theorem 4.7. [Sta06, Pr. A.0.2] Let $\Gamma = HNN(\Lambda, H, K, \phi)$ with $H \neq \Lambda$ or $K \neq \Lambda$. Let $Z(\Lambda)$ be the center of $\Lambda$. If for every $n \geq 1$ there exist some non-trivial elements $h_i^{(n)} \in Z(\Lambda) \cap H \cap K$ such that $h_i^{(n)} = \phi(h_{i-1}^{(n)})$ for $i = 1, \ldots, n$, then $\Gamma$ is inner amenable.

Proof of Proposition 4.6 Let $n \geq 1$ and set $h_i^{(n)} := a^ib^{m-n+i}$ for $i = 0, \ldots, n$. The hypotheses of Theorem 4.7 ii are trivially satisfied by these elements, which proves that $\overline{BS}(m, \xi)$ is inner amenable. Using Britton’s Lemma, we can readily show that the subgroup generated by $a$ and $bab^{-1}$ is a non-abelian free subgroup of $\overline{BS}(m, \xi)$. Hence $\overline{BS}(m, \xi)$ is not amenable by a classical result of von Neumann. □

Actually, we can prove a stronger statement than the existence of the elements $h_i^{(n)}$ as in Theorem 4.7. Indeed, it follows easily from Lemma 4.5 that $\bigcap_{1 \leq i \leq n} \gamma_i^{-1}B \gamma_i$ is a free abelian group of infinite countable rank for every $n$ and every $\gamma_1, \ldots, \gamma_n \in \overline{BS}(m, \xi)$. By [Kro90, Lem. 1.1], this yields the following vanishing cohomological property: for every free $\mathbb{Z}[\overline{BS}(m, \xi)]$-module $F$, we have $H^i(\overline{BS}(m, \xi), F) = 0$ for all $i$.

5. HOMOMORPHISMS

This section is devoted to the study of group homomorphisms from a given limit group to another one. We classify the limits of Baumslag-Solitar groups $\overline{BS}(m, \xi)$ up to abstract group isomorphism (Theorem 5.10), we compute the automorphism group of every limit (Proposition 5.12) and we prove that every limit is hopfian (Theorem 5.8). Finally, we show that every limit has infinite twisted conjugacy classes (Proposition 5.3).

As the map $a \mapsto a, b \mapsto b$ induces an isomorphism from $BS(m, n)$ to $BS(-m, -n)$, it also induces an isomorphism from $\overline{BS}(m, \xi)$ to $\overline{BS}(-m, -\xi)$ for any $\xi \in \mathbb{Z}_m$. For these reasons, we will assume that $m > 0$.

The following lemma can be readily deduced from the definition of $\overline{BS}(m, \xi)$.

Lemma 5.1. Let $d = \gcd(m, \xi)$ and let $\eta = \pi(\xi/d) \in \pi : \mathbb{Z}_m \rightarrow \mathbb{Z}_{m/d}$ be the canonical map. The map $a \mapsto a, b \mapsto b^d$ induces an injective homomorphism from $\overline{BS}(m, \eta)$ into $\overline{BS}(m, \xi)$.

From now on, we will consider $\overline{BS}(m, \xi)$ rather than $\overline{BS}(m, \xi)$ because it will ease off notations. We fix $m, m' \in \mathbb{N} \setminus \{0\}$ and $\xi, \xi' \in \mathbb{Z}_m, \mathbb{Z}_{m'}$ and we set $\Gamma = BS(m, \xi)$ and $\Gamma' = BS(m', \xi')$.

The following proposition shows that every surjective homomorphism from $\Gamma$ onto $\Gamma'$ is conjugated to a simple one by an element of $\Gamma'$.

Proposition 5.2. Let $p : \Gamma \rightarrow \Gamma'$ be a surjective homomorphism. Then $m'$ divides $m, \sigma(p(a)) = \pm 1$ and $p(e_0)$ is conjugated to $\pm e_0$. Moreover $\sigma(p(a)) = 1$ if $m' > 1$.

Before proving Proposition 5.2, we quote the following observation for further reference.

Remark 5.3. For any $Q \in \mathbb{Z}[X^{\pm 1}]$, there exists $\psi_Q \in \operatorname{Aut}(\mathbb{Z}[X^{\pm 1}] \rtimes \mathbb{Z})$ such that $\psi_Q(1, 0) = (1, 0)$ and $\psi_Q(0, 1) = (Q, 1)$. This automorphism satisfies $\psi_Q(P, 0) = (P, 0)$ for all $P \in \mathbb{Z}[X^{\pm 1}]$.

Indeed, this can be readily checked with the well-known presentation

$$\mathbb{Z}[X^{\pm 1}] \rtimes \mathbb{Z} \cong \langle a, b \mid [a^ib^{-i}, b] = 1 \text{ for all } i \geq 1 \rangle,$$

where $a$ corresponds to $(0, 1)$ and $b$ corresponds to $(1, 0)$.

Proof of Proposition 5.2 We set $\alpha = p(a), \beta = p(e_0)$ and $\gamma = \alpha \beta a^{-1}$. First we show that $\beta$ is an elliptic element of $\Gamma'$. Assume by contradiction that $\beta$ is hyperbolic. Then $\gamma^m$ is hyperbolic with axis $\alpha(D)$, where $D$ is the axis of $\beta$. Since $e_1 = a(me_0)a^{-1}$ commutes with $e_0$ in $\Gamma$, $\beta$ commutes with $\gamma^m$. As a result, $\gamma^m$ has the same axis as $\beta$, namely $D$. Thus $D$ is invariant under $\alpha$. There are two cases: either $\alpha$ is hyperbolic (case 1) or $\alpha$ is elliptic (case 2).
Lemma 5.5. Let $\phi$. Then the following are equivalent:

1. $\phi$ is an automorphism of $\Gamma'$.
2. The group $\Gamma'$ has infinitely many twisted conjugacy classes.

Proof. By Corollary 5.4, the group $\Gamma'$ has infinitely many twisted conjugacy classes. We may assume that $\phi$ is an automorphism of $\Gamma'$. By Lemma 5.2, we can write $\phi$ as a product of elements of the form $a_i = a_i(m,e_0)a^{-1}$ for all $i$. Thus, up to conjugacy, we can assume that $\phi = e_\epsilon$.

Assume now that $m' > 1$. Let $\alpha = c_0a_1c_2a_2 \cdots c_{t-1}a_{t-1} = za_1c_{t+1}$ (with $c_\epsilon = \pm 1$ and $c_1 \in E$) be a reduced form in $\Gamma'$. As $\gamma^m = \alpha(m,e_0)a^{-1}$ commutes with $\beta = e_\epsilon$, we deduce from Corollary 4.2 that $\alpha(m,e_0)a^{-1} \in E$. It follows that $e_\epsilon = 1$ and $m'$ divides $m$ by Britton's lemma. As $a_\sigma(\alpha(m,e_0)a^{-1}) \in E$ by Lemma 4.3, it follows from Britton's lemma that $\sigma(\sigma) \geq 0$. Since $\sigma(\sigma) \in \{\pm 1\}$, we deduce that $\sigma(\sigma) = 1$. \qed

Let $G$ be a group and let $\phi$ be an automorphism of $G$. Two elements $\gamma, \gamma' \in G$ are said to be $\phi$-twisted conjugate if there is $g \in G$ such that $\gamma' = g\gamma\phi(g^{-1})$. We say that $G$ has infinitely many twisted conjugacy classes if $G$ has infinitely many $\phi$-twisted conjugacy classes for every automorphism $\phi$. The study of this property is mainly motivated by topological fixed point theory and by the problem of finding a twisted analogue of the classical Burnside-Frobenius theorem (see [FH94] for an introduction to these topics). Baumslag-Solitar groups $BS(m,n)$ with $(m,n) \neq \pm(1,1)$ [FG06] have the wreath product $\mathbb{Z} \rtimes \mathbb{Z}$ [GW06]. Cor. 4.3 have infinitely many twisted conjugacy classes (the reader may consult [Rom] for an up-to-date list of known examples).

Corollary 5.4. The group $\Gamma$ has infinitely many twisted conjugacy classes.

Proof. If $m = 1$, one has $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$. As mentioned above, it has infinitely many twisted conjugacy classes.

Suppose now that $m > 1$ and let $\phi$ be an automorphism of $\Gamma$. It follows from Proposition 5.2 that $\sigma \circ \phi = \sigma$. Thus $\sigma$ is constant on each $\phi$-twisted conjugacy class. As $\sigma$ takes infinitely many values, $\Gamma$ has infinitely many $\phi$-twisted conjugacy classes. \qed

For every $i \geq 1$, we set $w_i = a_i(m,e_0)a^{-1}(-r_i(\xi)e_0)a^{-1}(-r_{i-1}(\xi)e_0)a^{-1}$. Then $e_i = w_i$ holds in $\Gamma$ for every $i \geq 1$. Recall that

(9) $\Gamma = \left\{ a, e_0, e_1, \ldots \mid [e_i, e_j] = 1 \text{ for all } i, j \geq 0, \right.$

$\left. a(m,e_0)a^{-1} = e_1, a(e_i - r_i(\xi)e_0)a^{-1} = e_{i+1} \text{ for all } i \geq 1 \right\}$

(10) $= \langle a, e_0 \mid [e_0, w_i] = 1 \text{ for all } i \geq 1 \rangle$

by Corollary 2.10 and Theorem 5.1.

Let $J$ be the map defined by $J(a) = a$ and $J(e_0) = -e_0$. It follows from presentation (9) that $J$ induces an automorphism of $\Gamma$ such that $J(e_i) = -e_i$ for every $i$.

Lemma 5.5. Let $t_1, \ldots, t_n \in \{0, \ldots, m - 1\}$ and let

$w(m, t_1, \ldots, t_n) = a^{n+1}(me_0)a^{-1}(-t_1e_0)a^{-1}(-t_2e_0)a^{-1} \cdots (-t_ne_0)a^{-1}$.

Then the following are equivalent:

(i) The image of $w(m, t_1, \ldots, t_n)$ in $\Gamma$ belongs to $E$.
(ii) The image of $w(m, t_1, \ldots, t_n)e_0w(-m, -t_1, \ldots, -t_n)(-e_0)$ in $\Gamma$ is trivial.
(iii) $t_i = r_i(\xi)$ for every $1 \leq i \leq n$.

Proof. (i) $\Rightarrow$ (ii): As $J(w(m, t_1, \ldots, t_n)) = w(-m, -t_1, \ldots, -t_n)$ and $J$ maps any element of $E$ to its inverse, we deduce that $w(m, t_1, \ldots, t_n)e_0w(-m, -t_1, \ldots, -t_n)(-e_0) = 1$ in $\Gamma$.

(ii) $\Rightarrow$ (iii): Assume that $w = w(m, t_1, \ldots, t_n)e_0w(-m, -t_1, \ldots, -t_n)(-e_0)$ has a trivial image in $\Gamma$. We prove the following claim by induction: for every $1 \leq i \leq n$, we have $w = v_ie_0J(v_i)(-e_0)$ with $v_i = a^{n+1-i}(e_i, t_i, e_0)(-1)(-t_{i+1}e_0)a^{-1} \cdots (-t_ne_0)a^{-1}$ and $t_j = r_j(\xi)$ for every $1 \leq j \leq i$. For $i = 1$, the claim follows from the relation $a(me_0)a^{-1} = e_1$ in $\Gamma$. Assume that the claim holds for some $i \geq 1$. We have then $w = v_i e_0 J(v_i)(-e_0) = 1$ in $\Gamma$. By Britton’s lemma, the sequence $(a, e_i - t_i, e_0, a^{-1})$ is not reduced and hence $e_i - t_i, e_0 \in E_{m, t}$. Therefore $t_i = r_i(\xi)$ and $a(e_i - t_i, e_0)a^{-1} = e_{i+1}$. We deduce that $w = v_{i+1} e_0 J(v_{i+1})$ with $v_{i+1} = a^{n-i}(e_{i+1} - t_{i+1}e_0)a^{-1} \cdots (-t_ne_0)a^{-1}$, which completes the induction.

(iii) $\Rightarrow$ (i): it follows from the fact that $w(m, r_1(\xi), \ldots, r_i(\xi)) = w_i$ and $w_i = e_i$ in $\Gamma$ for $i \geq 1$. \hfill $\square$

Lemma 5.6. Assume that $m = m' > 1$. Let $\theta$ be a map such that $\sigma \circ \theta(a) = 1$ and $\theta(e_0) = e_0$. If $\theta$ induces an homomorphism from $\Gamma$ to $\Gamma'$ then $r_i(\xi) = r_i(\xi')$ for every $i \geq 1$. In this case, the restriction of $\theta$ to $E$ is the identity and we have $|\theta(\gamma)|_a = |\gamma|_a |\theta(a)|_a$ for every $\gamma \in \Gamma$. In particular $\theta$ is injective.

Proof. We can write $\theta(a) = za e E$ and $z \in E'$ such that $\sigma(z) = 0$. Assume that $\theta$ induces an homomorphism from $\Gamma$ to $\Gamma'$. We show by induction that $\theta(e_i) = w_i$ for every $i \geq 1$. First, observe that $\theta(e_i)$ commutes with $\theta(e_0) = e_0$ for every $i \geq 1$. Hence $\theta(e_i) \in E$ for every $i \geq 1$ by Corollary 4.3. In particular, $\theta(e_1) = \theta(w_1) = za(me_0)a^{-1}z^{-1} \in E$. As $z(\xi) = 0$, we deduce from Lemma 4.3 that $\theta(e_1) = w_1$. Assume now that $\theta(e_i) = w_i$ for some $i \geq 1$. As $e_{i+1} = a(e_i - r_i(\xi)e_0)a^{-1}$ in $\Gamma$, we have $\theta(e_{i+1}) = za(w_i - r_i(\xi)e_0)a^{-1}z^{-1} \in E$. We obtain $\theta(e_{i+1}) = w_{i+1}$ by Lemma 4.3, which completes the induction. As $w_i = \theta(e_i) \in E$ for every $i \geq 1$, we deduce from Lemma 4.3 that $r_i(\xi) = r_i(\xi')$ for every $i \geq 1$.

Assume that the latter conditions holds and let $\theta(a) = c_1 a^e c_2 a^{e^2} \cdots c_{n+1} a^{e_{n+1}}$ (with $e_j = \pm 1$ and $e_j \in E$ for every $j$) be a reduced form in $\Gamma'$. Since $\theta(a)(me_0)\theta(a)^{-1} = e_1$ and $\theta(a)^{-1}e_1\theta(a) = me_0$, we have $e_1 = e_i = 1$ by Britton’s lemma. It readily follows that $|\theta(\gamma)|_a = |\gamma|_a |\theta(a)|_a$ for every $\gamma \in \Gamma$. \hfill $\square$

Hopf property and residual finiteness. An important reason for considering Baumslag-Solitar groups was, at the origin, that they gave the first examples of non-hopfian one-relator groups [BS62]. Let us recall that a group $G$ is hopfian if every surjective endomorphism from $G$ is an isomorphism. It is known that the Hopf property is neither open [ABL+05] Stad06a nor closed [CGP07] Pr. 5.10 in the space of marked groups.

A group $G$ is said to be residually finite if for every non-trivial element $g$ of $G$ there is a finite quotient $F$ of $G$ such that the image of $g$ in $F$ is non-trivial. For $m = 1$, the only limit is $BS(1, 0) = \mathbb{Z}/\mathbb{Z}$. This group is residually finite, hence hopfian by a well-known theorem of Malcev; see e.g. [LS77] Theorem IV.4.10.

Definition 5.7. A group $G$ is co-hopfian if every injective homomorphism from $G$ to itself is an isomorphism.

Theorem 5.8. The group $\Gamma$ is hopfian but not co-hopfian.

We denote by $J$ the automorphism of $\Gamma$ defined by $J(a) = a$ and $J(e_0) = -e_0$.

Proof. We can assume that $m > 1$. Let $p$ be a surjective endomorphism of $\Gamma$. By Proposition 5.2, there is $\epsilon \in \{0, 1\}$ and an inner automorphism $\tau$ of $\Gamma$ such that $p' = J' \circ \tau \circ p$ satisfies $\sigma \circ \theta(a) = 1$ and $\theta'(e_0) = e_0$ with. By Lemma 5.6 $p'$ is injective and hence so is $p$. Therefore $p$ is an isomorphism.

Let $k \in \mathbb{N} \setminus \{1\}$ be coprime with $m$ and let $\theta$ be the map defined by $\theta(a) = a$, $\theta(e_0) = ke_0$. Considering the group presentation (9), we deduce that $\theta$ induces an endomorphism of $\Gamma$. We can
readily check that \( \theta(e) = ke \) for every \( e \in E \) and that \( |\theta(\gamma)|_a = |\gamma|_a \) for every \( \gamma \in \Gamma \). It follows that \( \theta \) is injective. We also deduce that \( \theta(\Gamma) \cap E = \theta(E) = ke \neq E \). Hence \( \theta \) is not surjective. 

We say that \( \xi \in \mathbb{Z}_m = \bigoplus_{p \mid m} \mathbb{Z}_p \) is \textit{algebraic} if the \( p \)-component \( \xi_p \in \mathbb{Z}_p \) of \( \xi \) is algebraic over \( \mathbb{Q} \) for every prime \( p \) dividing \( m \). It only means that \( \xi \) is a root in \( \mathbb{Z}_m \) of some polynomial with coefficients in \( \mathbb{Z} \). Let \( d = \gcd(m, \xi) \) and \( \eta = \pi(\xi/d) \) where \( \pi : \mathbb{Z}_m \rightarrow \mathbb{Z}_m/\mathbb{Z}_m \) is the canonical map. It readily follows from the definition of Proposition 2.9 that \( \xi \) (resp. \( \eta \)) is algebraic if and only if \( \ker \chi \neq 1 \) (resp. \( \ker \hat{\chi} \neq 1 \)). If \( \xi \) is invertible in \( \mathbb{Z}_m \), i.e. \( \gcd(m, \xi) = 1 \), then the two kernels coincide with \( \bigcap_{i \geq 0} a^{-i}Ea^i \).

**Proposition 5.9.** If \( \eta \) is algebraic then \( \Gamma \) is not residually finite.

**Proof.** By Lemma 5.1, \( BS(m, \eta) \) embeds isomorphically into \( BS(m, \xi) \). As a subgroup of a residually finite group is residually finite, we could assume that \( \gcd(m, \xi) = 1 \). By Proposition 2.9, we can pick \( e \in \bigcap_{i \geq 0} a^{-i}Ea^i \setminus E_1 \). We set \( \gamma = [e, ae_0a^{-1}] = (-e)e(ae_0a^{-1}e) \). By Britton’s Lemma, \( \gamma \) is not trivial in \( \Gamma \). We show that \( \gamma \) has trivial image in any finite quotient of \( \Gamma \), which proves that \( \Gamma \) is not residually finite. Let \( F \) be a finite quotient of \( \Gamma \) with cardinal \( n \). There is \( e' \in E \), such that \( e = a^{-n+1}e'a^{n-1} \) in \( \Gamma \). Since \( a^n = 1 \) in \( F \), we have \( \gamma = a^{-n+1}e'a^{n-1}ae_0a^{-1} = a'[e', e_0]a^{-1} = 1 \) in \( F \). The proof is then complete. \( \square \)

Classification of limits up to group isomorphism.

**Theorem 5.10.** The group \( \Gamma \) is isomorphic to \( \Gamma' \) if and only if \( m = m' \) and \( r_i(\xi) = r_i(\xi') \) for every \( i \geq 1 \).

**Proof.** Assume that \( \Gamma \) is isomorphic to \( \Gamma' \). If \( m' = 1 \), then \( \Gamma' \) is isomorphic to \( \mathbb{Z} \) and so is \( \Gamma \). This forces \( m = 1 \) for \( \Gamma \) would contain a non-abelian free subgroup otherwise. It follows that \( r_i(\xi) = r_i(\xi') = 0 \) for every \( i \geq 1 \). Therefore we can assume that \( m' > 1 \). By Proposition 5.2, \( m' \) divides \( m \) and there is an isomorphism \( \theta : \Gamma \rightarrow \Gamma' \) such that \( \sigma \circ \theta(a) = 1 \) and \( \theta(e_0) = e_0 \). Considering \( \theta^{-1} \), we also deduce that \( m \) divides \( m' \) and hence \( m = m' \). By Lemma 5.6 we have \( r_i(\xi) = r_i(\xi') \) for every \( i \geq 1 \).

The converse follows immediately from the group presentation (9).

Automorphism group. Let \( e \in E \) and let \( \phi_e \) be the map \( \Gamma \) defined by \( \phi_e(a) = ae \) and \( \phi(e_0) = e_0 \). We deduce from the group presentation (10) that \( \phi_e \) induces an isomorphism of \( \Gamma \) with inverse map \( \phi_{-e} \). Moreover, we have \( J \circ \phi_e \circ J = \phi_{-e} \). The following lemma is then immediate.

**Lemma 5.11.** The automorphisms \( \phi_{e_0} \) and \( J \) generate a group isomorphic to an infinite dihedral group, namely the semi-direct product \( \mathbb{Z}e_0 \ltimes \mathbb{Z}/2\mathbb{Z} \) where the action of \( \mathbb{Z}/2\mathbb{Z} \) on \( E \) is multiplication by \( -1 \).

Hence we can consider the semi-direct product \( \Gamma \ltimes (\mathbb{Z}e_0 \ltimes \mathbb{Z}/2\mathbb{Z}) \) where the action of \( \mathbb{Z}e_0 \ltimes \mathbb{Z}/2\mathbb{Z} \) on \( \Gamma \) is the obvious one. We denote by \( \text{Inn}(\Gamma) \) the group of inner automorphisms of \( \Gamma \) and by \( \text{Out}(\Gamma) = \text{Aut}(\Gamma)/\text{Inn}(\Gamma) \) the group of outer automorphisms.

**Proposition 5.12.** Assume that \( m > 1 \).

- \( \text{Out}(\Gamma) \) is isomorphic to \( \mathbb{Z}e_0 \ltimes \mathbb{Z}/2\mathbb{Z} \).
- \( \text{Aut}(\Gamma) \) is isomorphic to \( \Gamma \ltimes (\mathbb{Z}e_0 \ltimes \mathbb{Z}/2\mathbb{Z}) \).

We denote by \( C \) the subgroup of \( E \) generated by \( e_1 - me_0 \) and the elements \( e_i - e_{i-1} + r_{i-1}e_0 \) with \( i \geq 1 \). Proposition 5.12 will follow from:

**Lemma 5.13.** Assume that \( m > 1 \) and let \( e \in E \).

- The image of \( \phi_e \circ J \) in \( \text{Out}(\Gamma) \) is non-trivial.
- The image of \( \phi_e \) is trivial in \( \text{Out}(\Gamma) \) if and only if \( e \in C \).
- For every automorphism \( \phi \) of \( \Gamma \), there is \( e \in E \), \( e \in \{0, 1\} \) such that \( \phi = \phi_e \circ J^e \) holds in \( \text{Out}(\Gamma) \).
Proof. As $\Gamma$ is centerless (see e.g. Corollary 1.4) and torsion-free, $\text{Inn}(\Gamma)$ is torsion-free. The first assertion follows from the fact that $\phi_e \circ J$ has order 2.

Let $e \in E$ and assume that there is $z \in \Gamma$ such that $\phi_e = \tau_z$. As $z$ centralizes $e_0$, we deduce from Corollary 4.4 that $z \in E$. We deduce from the equality $\phi_e(a) = za^{-1}$ that $a(e + z)a^{-1} = z$.

By Britton’s lemma, we have $e + z \in E_{m, \xi}$ and hence $z \in E_1$. Identifying $E$ with $B$, we deduce from Proposition 2.9 vii, that $X(P_e(X) + P_z(X)) = P_z(X)$. Therefore $P_e(X) = \frac{X - 1}{X} P_z(X) = e(-P_z(X)) \in \mathfrak{c}$ where $\mathfrak{c}$ and $\mathfrak{z}$ are defined in Proposition 2.9 vii. Since $\mathfrak{c} = q(C)$ and $q$ is injective, we have $e \in C$. Conversely, if $e \in C$, we can readily check that $\phi_e = \tau_z$ where $z \in E_1$ is given by the formula $P_e(X) = \frac{X - 1}{X} P_z(X)$.

Consider now an arbitrary automorphism $\phi$ and let us show that $\phi = \phi_e \circ J^e$ holds in $\text{Out}(\Gamma)$ for some $e \in E$ and some $\ell \in \{0, 1\}$. By Proposition 5.2 we can assume that $\phi(e_0) = \pm e_0$ and $\phi(a) = za^\ell$ with $a \in \Gamma$ such that $\sigma(a) = 0$ and $a^\ell \in E$. Composing possibly by $J$, we can assume that $\phi(e_0) = e_0$ hence that $\phi$ and $\phi^{-1}$ both satisfy the conditions of Lemma 5.6. We deduce from Lemma 5.6 that $1 = |a|_a = |\phi(a)|_a|\phi^{-1}(a)|_a$. Therefore $|a|_a = 0$, i.e. $a \in E$ and hence $\phi = \phi_e \circ J^e$ holds in $\text{Out}(\Gamma)$.

Proof of Proposition 5.12. By Lemma 5.13 $\text{Out}(\Gamma)$ is generated by the images of $J, \phi_e$ with $e \in E$. By Proposition 2.9 vii, the quotient $E/C$ is infinite cyclic and generated by the image of $e_0$. Hence, by Lemma 5.13 the natural map $\text{Aut}(\Gamma) \to \text{Out}(\Gamma)$ induces an isomorphism from the subgroup generated by $J$ and $\phi_{e_0}$ onto $\text{Out}(\Gamma)$. It follows then from Lemma 5.11 that $\text{Out}(\Gamma)$ is isomorphic to $\mathbb{Z} e_0 \rtimes \mathbb{Z}/2\mathbb{Z}$. As a result, the exact sequence $1 \to \text{Inn}(\Gamma) \to \text{Aut}(\Gamma) \to \text{Out}(\Gamma) \to 1$ splits. Since $\Gamma$ is centerless it naturally identifies with $\text{Inn}(\Gamma)$. Thus $\text{Aut}(\Gamma)$ is isomorphic to $\Gamma \rtimes (\mathbb{Z} e_0 \rtimes \mathbb{Z}/2\mathbb{Z})$.

An immediate consequence of Proposition 5.12 is that every automorphism of $\Gamma$ is induced by an automorphism of $F(a, e_0)$. Group presentations with such property are called almost quasi-free presentations [LS77, Ch. II.2].

Recall that the map $a \mapsto (0, 1), e_0 \mapsto (1, 0)$ induces a surjective homomorphism $q_{m, \xi}$ from $\Gamma$ onto $\mathbb{Z}/\mathbb{Z} = \overline{BS}(1, 0)$. Another consequence of Proposition 5.12 is:

Corollary 5.14. If $|m| > 1$, then the kernel of $q_{m, \xi}$ is a characteristic free subgroup of $\Gamma$ of infinite rank.

Proof. The normal subgroup $N = \ker q_{m, \xi}$ is a free group by [GS08 Th. 3.11]. As $N \nsubseteq E_{m, \xi}$ and $N E_{m, \xi}$ has infinite index in $\Gamma$, $N$ is not finitely generated by [KST71 Th. 9].

To conclude, thanks to Proposition 5.12 it suffices to show that $N$ is invariant under the automorphisms $J$ and $\phi_{e_0}$. It is invariant under $J$ since the diagram

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{J} & \Gamma \\
\downarrow q_{m, \xi} & & \downarrow q_{m, \xi} \\
\overline{BS}(1, 0) & \xrightarrow{J} & \overline{BS}(1, 0)
\end{array}
$$

commutes. A similar argument works for the automorphism $\phi_{e_0}$.

Equationally noetherian groups. In this section, we determine which groups $\overline{BS}(m, \xi)$ are equationally noetherian. Equationally noetherian groups play an important role in algebraic geometry over groups [BMR99], the state-of-the-art approach to equations over groups. An equationally noetherian group enjoys the following strong form of the Hopf property: any sequence of surjective endomorphisms is stationary (see [MR00 Th. D1.2] or [OH07 Cor. 2.8]). Let us recall the definition. Given $w = w(x_1, \ldots, x_n) \in G * F(x_1, \ldots, x_n)$ and an $n$-tuple $(g_1, \ldots, g_n) \in G^n$, we denote by $w(g_1, \ldots, g_n)$ the element of $G$ obtained by replacing $x_i$ by $g_i$. For any subset $W \subseteq G * F(x_1, \ldots, x_n)$, we consider the roots

$$
\text{Root}(W) = \{(g_1, \ldots, g_n) \in G^n : w(g_1, \ldots, g_n) = 1 \text{ for all } w \in W\}.
$$
Definition 5.15. A group $G$ is equationally noetherian if, for all $n \geq 1$ and for all $W \subseteq G \ast \mathbb{F}(x_1, \ldots, x_n)$, there exists a finite subset $W_0 \subseteq W$ such that $\text{Root}(W) = \text{Root}(W_0)$.

Linear groups over a commutative, noetherian, unitary ring (e.g. a field), are equationally noetherian \[BMR97\] while any wreath product of a non-abelian group by an infinite one is not equationally noetherian \[BMR99\].

Proposition 5.16. Let $m \in \mathbb{Z} \setminus \{0\}$ and $\xi \in \mathbb{Z}_m$. The group $\text{BS}(m, \xi)$ is equationally noetherian if and only if $|m| = 1$.

We need the following result on Baumslag-Solitar groups.

Proposition 5.17. \[BMR99\] Pr. 5] Let $m, n \in \mathbb{Z} \setminus \{0\}$.

1. If either $|m| = 1$ or $|n| = 1$ or $|m| = |n|$, then the group $\text{BS}(m, n)$ is linear over $\mathbb{Q}$ and hence equationally noetherian;
2. else, the group $\text{BS}(m, n)$ is not equationally noetherian.

As we need some excerpts of the proof of Proposition 5.17, we provide it in full. Let $R$ be a commutative ring with unity. We will use the following elementary fact without further mention. If a group $G$ has a finite index subgroup which is linear over $R$, then so is $G$ \[Wei73\] Lem. 2.3].

Proof of Proposition 5.17(1). Suppose first that $|m| = 1$ or $|n| = 1$. It is well known, and easy to show, that the map $a \mapsto (x \mapsto \frac{m}{n} x), b \mapsto (x \mapsto x + 1)$ yields an injective group homomorphism from $\text{BS}(m, n)$ into the affine group over $\mathbb{Q}$. The group $\text{BS}(m, n)$ is then linear over $\mathbb{Q}$. As $\text{BS}(m, n)$ is soluble in this case, we observe that it is linear over $\mathbb{Z}$ if and only if it is polycyclic\[6\], i.e. $|n| = |m| = 1$.

If $m = n$, it is easy to check that the normal subgroup $\langle \langle a, b^m \rangle \rangle$ is isomorphic to $\mathbb{F}[m] \times \mathbb{Z}$ and hence linear over $\mathbb{Z}$. Clearly, it has index $|m|$ in $\text{BS}(m, m)$. Thus $\text{BS}(m, n)$ is linear over $\mathbb{Z}$.

Suppose finally $n = -m$. Let $\text{BS}_2(m, n) = \langle \langle a^2, b \rangle \rangle \subseteq \text{BS}(m, n)$. The subgroups $\text{BS}_2(m, m)$ and $\text{BS}_2(m, -m)$ are clearly isomorphic and have index two in $\text{BS}(m, m)$ and $\text{BS}(m, -m)$ respectively. We have shown that $\text{BS}(m, m)$ is linear over $\mathbb{Z}$. We deduce that $\text{BS}_2(m, m)$ is linear over $\mathbb{Z}$ and hence so is $\text{BS}(m, -m)$.

It follows from the above proof that $\text{BS}(m, n)$ is linear over $\mathbb{Z}$ if and only if $|m| = |n|$.

Proof of Proposition 5.17(2). As the groups $\text{BS}(m, n)$ and $\text{BS}(n, m)$ are isomorphic, we may assume that $|m| < |n|$. Then, there exists $\nu > 0$ and a prime number $p$ such that $p^\nu$ divides $n$ but not $m$. Let us consider the set

$$W = \{w_i := [x^{-i} y x^i, z] : i \in \mathbb{N} \setminus \{0\}\} \subseteq \mathbb{F}(x, y, z)$$

and the triples $(x_k = a, y_k = b^m, z_k = b)$. If $n$ divides an integer $\beta$, then we have $a^{-1} b^{n^\beta} a = b^{\beta'}$ and the factorization of $\beta'$ contains (strictly) less factors $p$ than the one of $\beta$. Consequently, for all $k$, there exists $N(k) \in \mathbb{N}$ and $\alpha(k) \in \mathbb{Z}$ such that

$$a^{-N(k)} b^k a^{N(k)} = b^{\alpha(k)}$$

and $n$ does not divide $\alpha(k)$.

Remark 5.18. (1) As $a^{-k} b^k a = b^{m_k}$, we have $N(k) \geq k$;
2. Set $\mu = \mu(m)$ to be the maximal exponent arising in the factorization of $m$. Then we obtain $N(k) \leq (\mu + 2)k$. Indeed, we have $a^{-k} b^k a = b^{m_k}$ and the exponent of $p$ in the factorization of $m_k$ is at most $(\mu + 1)k$.

The triple $(x_k = a, y_k = b^m, z_k = b)$ is a root of $w_i$ if and only if $i \leq N(k)$. Indeed:
- if $i \leq N(k)$, then $a^{-i} b^k a^i$ is a power of $b$, so that $[a^{-i} b^k a^i, b] = 1$;

\[6\] By theorems of Mal’cev and Auslander \[Seg83\] Ch. 2 and Ch. 3], a soluble group is linear over $\mathbb{Z}$ if and only if it is polycyclic.

\[6\] It is possible to use only one variable: replace $W$ by $W' = \{[a^{-i} y a^i,b] : i \in \mathbb{N} \setminus \{0\}\} \subseteq \text{BS}(m, n) \ast \mathbb{F}(y)$. 

19
• If \( i > N(k) \), then
\[
[a^{-i}a^k, b] = a^{-N(k)}[b, a^{-i-1}a^k, b] = a^{-N(k)} \cdot b^{-1}.
\]
This is reduced in \( BS(m, n) \), since \( |m| > 1 \) and \( n \) does not divide \( \alpha(k) \).

If we now consider a finite subset \( W_f = \{ w_i, \ldots, w_n \} \subset W \), then, choosing \( k \) large enough, we have \( N(k) \geq i_1, \ldots, i_k \). Consequently, the triple \( (x_k, y_k, z_k) \) is in \( \text{Root}(W_f) \setminus \text{Root}(W) \). This proves that \( BS(m, n) \) is not equationally noetherian.

**Proof of Proposition 5.16.** If \( |m| = 1 \), one has \( BS(m, \xi) = \mathbb{Z} \mathbb{Z} \) which is equationally noetherian (e.g. it is linear over the field \( \mathbb{Q}(X) \)).

Let us now assume that \( |m| > 1 \). Consider a sequence \( (\xi_n) \) of rational integers such that
\[
|\xi_n| \to \infty \text{ and } \xi_n \to \xi \text{ in } \mathbb{Z}_m \text{ for } n \to \infty.
\]
We may assume that \( |m| < |\xi_n| \) for all \( n \). Set \( W = \{ w_i := [x^{-i}y x^i, z] : i \in \mathbb{N} \setminus \{ 0 \} \} \) and \( (x_k = a, y_k = b^k, z_k = b) \), as in the proof of Proposition 5.17(2). We have proved the existence of natural numbers \( N_0(k) \) such that \( (x_k, y_k, z_k) \) is a root of \( w_i \) in \( BS(m, \xi_n) \) if and only if \( i \leq N_0(k) \). Moreover, Remark 5.18 gives the estimates
\[
k \leq N_0(k) \leq (\mu(m) + 2)k
\]
for all \( n \). Therefore, if we take a finite subset \( W_f = \{ w_i, \ldots, w_n \} \subset W \), then, choosing \( k \) large enough, we have \( N_0(k) \geq i_1, \ldots, i_k \) for all \( n \). Therefore, for all \( j \), we have \( w_j (x_k, y_k, z_k) = 1 \) in all groups \( BS(m, \xi_n) \), and, passing to the limit, we see that \( (x_k, y_k, z_k) \) is a root of \( W_f \) in the group \( BS(m, \xi) \).

On the other hand, by considering \( w_i \) with \( i > (\mu(m) + 2)k \), we see that \( w(x_k, y_k, z_k) \neq 1 \) in all \( BS(m, \xi) \). Hence, in \( BS(m, \xi) \), the triple \( (x_k, y_k, z_k) \) is not a root of \( W \). This proves that \( BS(m, \xi) \) is not equationally noetherian.

**6. Dimensions**

In this section we give the first non-trivial Hausdorff dimension estimates of a subspace of the space of marked groups on two generators. Let us recall that the map \( BS_m : \mathbb{Z}_m \to \mathcal{G}_2 : \xi \mapsto BS(m, \xi) \) is injective on \( \mathbb{Z}_m^\times \) [GSOS Th. 1]. In order to estimate Hausdorff dimensions of the subspaces
\[
Z_m^\times = BS_m(\mathbb{Z}_m^\times) = \{ BS(m, \xi) : \xi \text{ is invertible in } \mathbb{Z}_m \},
\]
we will prove that the maps between \( Z_m^\times \) and \( Z_n^\times \) satisfy Hölder conditions and then apply classical results about Hausdorff dimension. In this section, we always assume that \( m \) is a rational integer satisfying \( |m| \geq 2 \).

**6.1. Distances between limits.** The first step towards Hausdorff dimension estimates is to estimate the distance between groups \( BS(m, \xi) \) and \( BS(m, \xi') \) in terms of the \( m \)-adic distance between \( \xi \) and \( \xi' \).

**Theorem 6.1.** Let \( h \in \mathbb{N} \setminus \{ 0 \} \) and \( \xi, \xi' \in \mathbb{Z}_m \) satisfying \( d := \gcd(m, \xi) = \gcd(m, \xi') \). Setting \( \hat{m} = m/d \), we have:

1. If \( BS(m, \xi) \) and \( BS(m, \xi') \) have the same relations up to length \( 2(|m| + 1)h + 2|m| + 6 \), then \( \xi \equiv \xi' \pmod{\hat{m}h} \).

2. If \( \xi \equiv \xi' \pmod{\hat{m}h} \), then \( BS(m, \xi) \) and \( BS(m, \xi') \) have the same relations up to length \( 2h \).

**Proof of Theorem 6.1.** Thanks to Corollary 2.10, we may work in \( BS(m, \xi) \) and \( BS(m, \xi') \) instead of \( BS(m, \xi) \) and \( BS(m, \xi') \). Recall that, given the free abelian groups of countable rank
\[
E = \mathbb{Z} e_0 + \mathbb{Z} e_1 + \mathbb{Z} e_2 + \cdots
\]
\[
M = \mathbb{Z} me_0 + \mathbb{Z} (e_1 - r_1(\xi)e_0) + \mathbb{Z} (e_2 - r_2(\xi)e_0) + \cdots \leq E
\]
\[
M' = \mathbb{Z} me_0 + \mathbb{Z} (e_1 - r_1(\xi')e_0) + \mathbb{Z} (e_2 - r_2(\xi')e_0) + \cdots \leq E
\]
\[
E_1 = \mathbb{Z} e_1 + \mathbb{Z} e_2 + \cdots \leq E,
\]

20
We have
\[ \widetilde{BS}(m, \xi) = \langle a, E \mid a\psi(x)a^{-1} = x \; \forall x \in E_1 \rangle \]
\[ BS(m, \xi') = \langle a, E \mid a\psi'(x)a^{-1} = x \; \forall x \in E_1 \rangle , \]
where the isomorphism \( \psi : E_1 \to M \) is defined by \( \psi(e_1) = me_0 \) and \( \psi(e_{i+1}) = e_i - r_i(\xi)e_0 \) for \( i > 0 \), and the isomorphism \( \psi' : E_1 \to M' \) is defined similarly. Recall also that the element \( b \in BS(m, \xi) \) corresponds to \( e_0 \in BS(m, \xi) \). By Proposition 2.6 the condition \( \xi \equiv \xi' (mod \, \hat{m}h\mathbb{Z}_m) \) is equivalent to \( r_i(\xi) = r_i(\xi') \) for \( i = 1, \ldots, h \). We will consider the latter condition.

(1) Let \( w = w(m, r_1(\xi), \ldots, r_h(\xi))e_0w(-m, -r_1(\xi), \ldots, -r_h(\xi))(-e_0) \) be defined as in Lemma 5.5. As \( |w| \leq 2(|m| + 1)h + 2|m| + 6 \) and \( w = 1 \) in \( BS(m, \xi) \), we also have \( w = 1 \) in \( BS(m, \xi') \).

We deduce from Lemma 5.5 that \( r_i(\xi) = r_i(\xi') \) for \( i = 1, \ldots, h \).

(2) Let \( w \) be a (freely reduced) word on the alphabet \( \{a^\pm 1, b^\pm 1\} \) satisfying \( |w| \leq 2h \). By substituting occurrences of \( b^k \) by \( \alpha e_0 \), we obtain a sequence \( s = (x_0, a^{\varepsilon_1}x_1, \ldots, a^{\varepsilon_k}x_k) \), with \( k \geq 0 \), of length at most \( 2h \), where \( \varepsilon_i = \pm 1 \) and \( x_i \) is an element of the subgroup \( \mathbb{Z}e_0 \leq E \) for all \( i \). What we have to show is that the product of the sequence \( s \) vanishes in \( BS(m, \xi) \) if and only if it vanishes in \( BS(m, \xi') \).

We reduce the sequence \( s \) in the HNN-extension \( BS(m, \xi) \), that is we perform, as long as possible, substitutions of:

- a subsequence \( (a, x, a^{-1}) \), with \( x \in M \), by the element \( \psi^{-1}(x) \in E_1 \);
- a subsequence \( (a^{-1}, x, a) \), with \( x \in E_1 \), by the element \( \psi(x) \in M \).

We then obtain a sequence \( t = (y_0, a^{\varepsilon_1}y_1, \ldots, a^{\varepsilon_l}y_l) \), with \( l \geq 0 \), which is reduced in \( \widetilde{BS}(m, \xi) \), and whose product in the latter group is equal to the product of \( s \). The number of substitutions from \( s \) to \( t \) is trivial at most \( h \). Therefore, it is easy to see that \( t \) and the intermediate sequences contain only \( a^\pm 1 \) letters and elements of the subgroup \( \mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_h \).

Now, we use the hypothesis \( r_i(\xi) = r_i(\xi') \) for \( i = 1, \ldots, h \). Therefore, the relation
\[ M \cap (\mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_h) = M' \cap (\mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_h) \]
holds and \( \psi \) and \( \psi' \) are equal in restriction to \( \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{h+1} \). It is thus possible to reduce the sequence \( s \) in \( \widetilde{BS}(m, \xi') \) by performing the same substitutions as in \( \widetilde{BS}(m, \xi) \). Hence, the sequences \( s \) and \( t \) have the same product in \( BS(m, \xi') \). Moreover, the sequence \( t \) is also reduced in \( BS(m, \xi') \) — if not, an argument similar to the above one would show that \( t \) is not reduced in \( BS(m, \xi) \).

Finally, by structure theorems on HNN-extensions, the product of \( t \) vanishes in \( \widetilde{BS}(m, \xi) \) (resp. \( \widetilde{BS}(m, \xi') \)) if and only if \( l = 0 \) and \( y_0 = 0 \) in \( E \). This concludes the proof of part (2).

We now turn to the case \( d = 1 \), that is, to the case of invertible \( m \)-adic integers. Recall that the metric on \( G_2 \) is given by \( d(N_1, N_2) = e^{-\nu(N_1, N_2)} \) if \( N_1 \neq N_2 \), where \( \nu(N_1, N_2) = \inf \{|w| : w \in N_1 \Delta N_2\} \).

**Corollary 6.2.** Let \( h \in \mathbb{N} \setminus \{0\} \) and \( \xi, \xi' \in \mathbb{Z}_m^\times \) such that \( |\xi - \xi'|_m = |m|^{-h} \). Setting \( x = \widetilde{BS}(m, \xi) \) and \( x' = \widetilde{BS}(m, \xi') \), we have
\[ e^{-2(|m| + 1)(h + 1) - 2|m| - 6} \leq d(x, x') \leq e^{-2h - 1}. \]

**Proof.** If \( d(x, x') < e^{-2(|m| + 1)(h + 1) - 2|m| - 6} \), then \( \widetilde{BS}(m, \xi) \) and \( \widetilde{BS}(m, \xi') \) have the same relations up to length \( 2(|m| + 1)(h + 1) + 2|m| + 6 \). Theorem 6.1(1) gives then \( |\xi - \xi'|_m \leq |m|^{-h} \).

On the other hand, the relation \( |\xi - \xi'|_m = |m|^{-h} \) implies \( \xi \equiv \xi' (mod \, \hat{m}h\mathbb{Z}_m) \). Theorem 6.1(2) gives then \( d(x, x') \leq e^{-2h - 1}. \)
6.2. Hausdorff dimension estimates. We set $f$ to be the inverse of the (bijective) map $\overline{BS}_m : \mathbb{Z}_m^\times \to \mathbb{Z}_m^\times$. We now show that $f$ and $f^{-1} = \overline{BS}_m$ both satisfy a Hölder condition.

**Proposition 6.3.** For all $x, x' \in \mathbb{Z}_m^\times$, we have

$$|f(x) - f(x')|_m \leq Cd(x, x')^\alpha$$

where $\alpha = (2(|m| + 1))^{-1} \log |m|$ and $C$ is some positive constant.

*Proof.* Set $\xi = f(x)$ and $\xi' = f(x')$, so that $x = \overline{BS}(m, \xi)$ and $x' = \overline{BS}(m, \xi')$, and write $|\xi - \xi'|_m = |m|^{h}$ with $h \in \mathbb{N}$. Let us treat the case $h \in \mathbb{N} \setminus \{0\}$ first. Using Corollary 6.2 (at the second line), we get:

$$d(x, x') \geq e^{-2(|m| + 1)h - 4|m| - 8} = C_1 e^{-2(|m| + 1)h} = C_1 \left( e^{-h \log |m|} \right)^{2(|m| + 1)}$$

with $C_1 > 0$. Consequently, we have $d(x, x') \geq C_1 |f(x) - f(x')|_m^{-\alpha}$, whence $|f(x) - f(x')|_m \leq C_2 d(x, x')^\alpha$ for some $C_2 > 0$.

Finally, in case $h = 0$, that is $\xi \neq \xi' \pmod{m}$, there is a word

$$a^2 b m^{-1} a^{-1} b \cdot a^2 b^{-m} a^{-1} b^{-1}$$

which is trivial in one of the marked groups $x = \overline{BS}(m, \xi), x' = \overline{BS}(m, \xi')$ but not in the other one. This gives a constant $D > 0$ such that $d(x, x') \geq D$, hence a constant $C_3 > 0$ such that $|f(x) - f(x')|_m = 1 \leq C_3 d(x, x')^\alpha$. □

**Proposition 6.4.** For all $\xi, \xi' \in \mathbb{Z}_m^\times$, we have

$$d(f^{-1}(\xi), f^{-1}(\xi')) \leq |\xi - \xi'|_m^\beta$$

where $\beta = 2(\log |m|)^{-1}$.

*Proof.* Let us write $|\xi - \xi'|_m = |m|^{-h}$ with $h \in \mathbb{N}$. By corollary 6.2, we have $d(f^{-1}(\xi), f^{-1}(\xi')) \leq e^{-2h - 1}$ (note that for $h = 0$ this is trivially true, since $\text{diam}(G_2) = e^{-1}$). Hence, we get

$$d(f^{-1}(\xi), f^{-1}(\xi')) \leq e^{-2h} = (e^{-h \log |m|})^2 \log |m|^{-1} = |\xi - \xi'|^\beta\_m,$$

which concludes the proof. □

**Theorem 6.5.** The Hausdorff dimension of $\mathbb{Z}_m^\times$ satisfies:

$$\frac{\log |m|}{2(|m| + 1)} \leq \dim_H(\mathbb{Z}_m^\times) \leq \frac{\log |m|}{2}$$

(for all $m$ such that $|m| \geq 2$).

*Proof.* It is well-known, and easy to show, that $\dim_H(\mathbb{Z}_m^\times) = 1$ with respect to the metric chosen in Section 1.2. Set $\alpha = (2(|m| + 1))^{-1} \log |m|$ and $\beta = 2(\log |m|)^{-1}$, as in Propositions 6.3 and 6.4. Classical theory of Hausdorff dimension (see e.g. [Fal93, Pr. 2.3] or [Rog70, Th. 29]) and these propositions give $1 \leq \alpha^{-1} \dim_H(\mathbb{Z}_m^\times)$ and $\dim_H(\mathbb{Z}_m^\times) \leq \beta^{-1}$, hence the result. □

**Corollary 6.6.** The Hausdorff dimension of $G_2$ satisfies $\dim_H(G_2) \geq \log(2)/6$. In particular, this dimension does not vanish.

We have estimated the Hausdorff dimension of the subspaces $\mathbb{Z}_m^\times$, which are homeomorphic to the Cantor set (provided that $|m| \geq 2$). But many interesting subspaces of $G_2$, or $G_n$, appeared in the litterature, e.g:

- the Cantor set of Grigorchuk groups [Gri84]: many such groups have intermediate growth;
- the closure $H_n \subseteq G_n$ of non-elementary hyperbolic groups considered by Champetier [Cha00];
- the minimal Cantor subset of $G_3$ constructed by Nekrashevych [Nek07].
The first-named author has proved that the box-counting dimension (and hence the Hausdorff dimension) of the set of Grigorchuk groups vanishes \cite{Guy07}. It also holds for the set of Nekraševych groups as these groups share similar contracting properties with the latter. In the case of hyperbolic groups, we do not know whether the Hausdorff dimension vanishes or not.

7. Complexity of the word and conjugacy problems

In this section, we study isomorphism invariants of groups originating from language theory, namely the space complexity and the Turing degree of the word and conjugacy problems. Our results are inspired by the works for Grigorchuk \cite{Gri84} and Garzon and Zalcstein \cite{GZ91} on the word problem of Grigorchuk groups. First, we show that the space complexity of the word problem for $\overline{BS}(m, \xi)$ is tightly related to the space complexity of the rational integer sequence $(r_i(\xi))$ (Proposition 7.3). Second, we show that the conjugacy problem for $\overline{BS}(m, \xi)$ is Turing reducible to the word problem for $\overline{BS}(m, \xi)$ (Corollary 6.4). For the sake of simplicity, our emphasis is on the space complexity of the word problem. Analogs of Proposition 7.3 for time complexity and the conjugacy problem could be proved if one is prepared to more technicalities.

Space complexity. Let $A$ be a set. We denote by $A^*$ the set all strings (or words) on $A$. Let $s \in A^*$. We denote by $|s|_A$ the string length of $s$, that is the number of symbols of $A$ in $s$. We may simply write $|s|$ when the underlying set is clearly given by the context. A set $L$ is a language if it is a subset of $A^*$ for some finite set $A$ called alphabet.

Let $G$ be a group and let $X$ be a finite generating set of $G$. We denote by $WP(G,X)$ the set of strings $s \in (X \cup X^{-1})^*$ such that $s = 1$ in $G$, i.e. $s$ reduces to the trivial element of $G$. The decision problem of membership in $WP(G,X)$ is called the word problem with respect to $X$. The Turing time and space complexity of the language $WP(G,X)$ are group-theoretic properties independant of $X$ \cite{MOS85}; so $X$ will be omitted.

Nota Bene 7.1. A Turing machine $M$ is an off-line Turing machine if it has a read-only input tape with endmarkers and finitely many semi-infinite storage tapes. All Turing machines considered in this section are off-line Turing machines that halts on every input. We adress the reader to \cite{HU79} for the complete definitions of terms used in this section.

Let $M$ be an off-line Turing machine and let $f : \mathbb{R}_+ \rightarrow \mathbb{N}$ be a function. If for every input word of length $n$, the machine $M$ scans at most $f(n)$ cells on any storage tape, then $M$ is said to be an $f(n)$ space-bounded Turing machine. We denote by $\text{DSPACE}(f)$ (resp. $\text{NSPACE}(f)$) the class of languages which are accepted by a deterministic (resp. non-deterministic) $f(n)$ space-bounded Turing machine. A language $L$ is recursive if it is accepted by a Turing machine. A function $g : \mathbb{N}^k \rightarrow \mathbb{N}^l$ is a recursive function if it can be computed by a Turing machine (the $k$ arguments $i_1, \ldots, i_k$ of $g$ are initially placed on the input tape separated by 1’s, as $0^{i_1}10^{i_2}1\cdots10^{i_k}$, the $l$ arguments are placed similarly in some output tape). A function $g : \mathbb{N} \rightarrow \mathbb{N}$ is said to belong to $\text{DSPACE}(f)$ (resp. $\text{NSPACE}(f)$) if there exists a deterministic (resp. non-deterministic) Turing machine taking as input the binary expansion of $j$ and computing $g(j)$ in space bounded above by $f(n)$ where $n$ is the number of binary digits of $j$. A language $L$ (resp. a function $g$) is said to separate the inclusion of two space complexity classes

$$\text{DSPACE}(f) \subset \text{NSPACE}(f)$$

if $L$ (resp. $g$) belongs to $\text{NSPACE}(f)$ but not to $\text{DSPACE}(f)$. Proofs below use of the Tape Compression Theorem \cite{HU79} Th. 12.1] without mentioning it: the equality of language classes

$$\text{DSPACE}(f) = \text{DSPACE}(c f)$$

holds for any $c > 0$, with an analogue statement in the non-deterministic case.

Time complexity is analogously defined by counting the number of state transitions of a Turing machine with a read-and-write input tape. Every input word of length $n$ requires at least $n$ state transitions to be entirely read, hence $\text{DTIME}(n)$ is the smallest time complexity class. For every
function $f$, we have $\text{DTIME}(f) \subset \text{DSPACE}(f)$. We collect few facts on the word problem of finitely generated groups.

- The language $W P(G)$ is regular if and only if $G$ is a finite group [Anı71]. If $G$ is infinite then $W P(G)$ does not belong to $\text{DSPACE}(\log \log)$ [HS85].
- The language $W P(G)$ is context-free if and only if $G$ is virtually free [MS83, Dun85].
- The language $W P(G)$ belongs to $\text{DSPACE}(\log)$ if $G$ is a linear group over a field of characteristic zero [LZ77]. There exists a finitely presented non-linear group $G$ such that $W P(G) \in \text{DSPACE}(\log)$ [Waa81].
- There is no known example of a “simple” group presentation for which the word problem does not belong to $\text{DSPACE}(\log)$.
- If $G$ contains a copy of $\mathbb{Z}$ then $W P(G)$ does not belong to $\text{DSPACE}(g)$ for any $g$ such that $g(n)/\log(n)$ tends to 0 [AGM92, Th. 2]. In particular, log is a sharp bound for the space complexity of the word problem of any infinite finitely generated linear group.
- The word problem of a word hyperbolic group $G$ is solvable in real time [Hol00]. In particular $W P(G) \in \text{DTIME}(n)$.

Let $p, q \in \mathbb{Z} \setminus \{0\}$ and let $W P(p, q)$ (resp. in $W P(m, \xi)$) be the set of strings $s \in \{a^{\pm 1}, b^{\pm 1}\}^*$ such that $s = 1$ in $BS(p, q)$ (resp. $\overline{BS}(m, \xi)$). Given $\overline{BS}(m, \xi)$, we define the function $r$ on $\mathbb{N}$ by $r(0) = |m|$ and $r(n) := \epsilon_m \epsilon_n$ where $\epsilon_m$ is the sign of $m$. This definition is motivated by the fact that $W P(m, \xi) = W P(|m|, \epsilon_m \xi)$ since $\overline{BS}(m, \xi)$ and $\overline{BS}(-m, -\xi)$ are isomorphic as marked groups. The following proposition can be proved by using arguments similar to those of Lemma 1.4.

**Proposition 7.2.** $W P(p, q) \in \text{DSPACE}(n) \cap \text{DTIME}(n^2)$.

As $BS(p, q)$ is not virtually free, we observe that the language $W P(p, q)$ is not a context-free language. The complement of $W P(p, q)$ is not a context-free language either, except if $|p| = |q|$ [HRR+05]. Solvable Baumslag-Solitar groups (i.e., groups $BS(p, q)$ with $|p| = 1$ or $|q| = 1$) have a tidy real-time word problem [HR03, Th. 2.1]. We still ignore whether $W P(p, q)$ belongs to $\text{DSPACE}(\log)$ in the case $BS(p, q)$ is not linear. (Recall that $BS(p, q)$ is linear if and only if either $|p| = |q|$ or $|p| = 1$ or $|q| = 1$ by Proposition 5.1.) The reader interested in geodesic languages of Baumslag-Solitar groups should consult [Eld05, DL].

Provided $r$ belongs to $\text{DSPACE}(n)$, Proposition 7.2 holds for $W P(m, \xi)$ and it corresponds to the lowest complexity bound we obtain. Our next result relate the space complexity of $W P(m, \xi)$ to the space complexity of $r$. Let us stress on the fact that functions $r$ are “numerous” because of Proposition 5.5ii: for any $m \in \mathbb{Z} \setminus \{0\}$, $d \in \mathbb{N} \setminus \{0\}$ and for any $g : \mathbb{N} \to \{0, \ldots, |m| - 1\}$ there is some $\xi \in \mathbb{Z}_m$ such that $g(n) = r_n(\epsilon_m \xi)$ for all $n \geq 2$. Hence the following proposition can be seen as a result of density in the space hierarchy.

**Proposition 7.3.** Let $f$ be a non-decreasing function such that $f(n) \geq n$ and $\text{DSPACE}(f) \neq \text{NSPACE}(f)$. Let $\overline{BS}(m, \xi)$ be such that $r$ separates the inclusion $\text{DSPACE}(f) \subset \text{NSPACE}(f)$. Then $W P(m, \xi)$ separates the inclusion

$$\text{DSPACE}(f(n/6|m|)) \subset \text{NSPACE}(f(n)).$$

This result is an immediate consequence of the following two lemmas.

**Lemma 7.4.** Assume $r \in \text{DSPACE}(f(n))$ for some non-decreasing function $f$. Then $W P(m, \xi) \in \text{DSPACE}(n + f(n))$. Likewise for $\text{NSPACE}$. 

**Lemma 7.5.** Assume $W P(m, \xi) \in \text{DSPACE}(f(n/6|m|))$ for some non-decreasing function $f$ such that $f(n) \geq n$. Then $r \in \text{DSPACE}(f(n))$. Likewise for $\text{NSPACE}$.

---

It is incorrectly claimed in the proof of [HRR+05, Th. 13] that $|p| = |q|$ if and only if $BS(p, q)$ is virtually abelian. The condition $|p| = |q|$ is less restrictive for it means that $BS(p, q)$ contains a copy of the direct product $F_{|p|} \times \mathbb{Z}$ as a finite index subgroup, or equally that $BS(p, q)$ is linear over $\mathbb{Z}$.
Let us summarize the idea of the proof of Lemma 7.4. Applying to a given word \( w \in \{a^{\pm 1}, b^{\pm 1}\}^* \) the natural algorithm originating from Britton’s lemma, we obtain a reduced sequence for \( w \). This reduction is carried out within at most \( |w| \) steps and at each step we consider a word whose length is at most \( |m| \) times the length of the previous one. As we encode the exponents of \( a \) and \( b \) by means of their binary expansions, this stretching factor becomes an additive constant which explains the linear part of the space complexity bound. The other part of the bound comes from the fact that we need to compute \( r(n) \) to reduce words \( w \) such that \( |w|_a = n \).

As for the proof of Lemma 7.3, we notice that a Turing machine which can solve the word problem for \( \overline{BS}(m, \xi) \), can decide which of the words defined in Lemma 5.3 are trivial. Hence it can be used to compute \( r_i(\xi) \) for every \( i \).

We will work with our favoured HNN extension \( \overline{BS}(m, \xi) \) instead of \( \overline{BS}(m, \xi) \). In order to make a careful enough counting of the numbers of scanned cells, we will use following notations. We fix \( \mathcal{A} := \{a^{\pm 1}, \pm e_0, \pm e_1, \ldots \} \). Let \( m \in \mathbb{Z} \setminus \{0\} \) and \( \xi \in \mathbb{Z}_m \). Given \( w \in \mathcal{A}^* \), we can rewrite \( w \) in \((a) \ast E \) under the form

\[
(11) \quad w(0) = a^{\alpha_1} \cdot c_1 a^{\alpha_2} \cdot c_2 \cdots a^{\alpha_h} \cdot c_h
\]

with \( c_j = (\beta_{0j} e_0)(\beta_{1j} e_1) \cdots (\beta_{kj} e_k) \), \( \alpha_j, \beta_{ij} \in \mathbb{Z} \) for all \( l, j \). We denote by \( \varepsilon_j \) the sign of \( \alpha_j \). We suppose that the following holds: there is some \( j \) such that

\[
(*) \quad \varepsilon_j = -\varepsilon_{j+1} = -1 \text{ and } c_j \in E_m, \xi \text{ or } \varepsilon_j = -\varepsilon_{j+1} = 1 \text{ and } c_j \in E_1.
\]

We denote by \( \ell = \ell(w) \) the smallest \( j \) such that \((11)\) holds. Let \( w' \) be the word we get from \( w(0) \) by replacing \( a^{\alpha_i} c_i a^{\alpha_{i+1}} \) by \( a^{\alpha_i} \varepsilon_i \phi_i(c_i) a^{\alpha_{i+1} + \varepsilon_i} \) in \( w \) and reducing this new word as in \((11)\). We write \( w' = a^{\alpha_1'} c_1 a^{\alpha_2'} c_2 \cdots a^{\alpha_h'} c_h' \). Notice that a given exponent \( \alpha \) of \( a \) in \( w' \) either remains unchanged in \( w' \), vanishes or is replaced by some \( a' \) such that \( |\alpha'| = |\alpha| - 1 \). The subwords \( c_j \) remain unchanged in \( w' \) or vanish, except one which is replaced by some subword \( c' \) with \( |c'| \leq (2 + |m|)|n \) where \( n = |w| \). As long as \((*)\) holds for \( w(i) = a^{\alpha_1(i)} c_1(i) a^{\alpha_2(i)} c_2 \cdots a^{\alpha_h(i)} c_h(i) \) with \( i \geq 0 \), we can define \( w(i+1) = (w(i))' \).

By Britton’s Lemma, for any \( w \in \mathcal{A}^* \), there is some \( i = \ell(w) \geq 0 \) such that \( w(i) = 1 \) is a reduced form for \( w \). We call the previous algorithm the Britton’s algorithm.

Lemma 7.4 By hypothesis, there is an \( f(n) \) space-bounded Turing machine \( M_r \) computing \( r(n) \). We denote by \( R_1 \) its input tape and by \( R_k \) \( (2 \leq k \leq p) \) its storage tapes. We design an off-line Turing machine \( M \) that halts on every input \( w \in \{a^{\pm 1}, \pm e_0\}^* \): if a non-trivial reduced form for \( w \) has been found, it halts without accepting, else \( w \) is reduced to 1 and \( M \) halts in an accepting state. Tape I is the read-only input tape where \( w \) is displayed without accounting for any space. At the beginning, \( M \) writes the string \( s^{(0)} \) on Tape 0 that encodes \( w^{(0)} \):

\[
s^{(0)} := \varepsilon_1 \overline{\alpha}_1 \overline{\tau}_1 \varepsilon_2 \overline{\alpha}_2 \overline{\tau}_2 \cdots \varepsilon_h \overline{\alpha}_h \overline{\tau}_h.
\]

The strings \( \overline{\tau}_i \in \{0, 1\}^* \) are the binary expansions of \( |\alpha_i| \); each string \( \overline{\tau}_j \in \{0, 1, \pm \}^* \) is the concatenation of the binary expansions of the numbers \( |\beta_{ij}| \) separated by sign symbols. If \( \alpha_1 = 0 \) (respectively \( c_h = 0 \)) then \( \varepsilon_1 \overline{\alpha}_1 \) (respectively \( \overline{\tau}_h \)) is replaced by the empty string.

We now describe how \( M \) works on its storage tapes \( R_k \), \( (1 \leq k \leq p) \), \( T_0 \), \( T_1 \) and \( D \). First, the machine read the input: while the head of tape I scans the first \( j \) symbols of \( w \), \( M \) stores the number \( j \) using a counter situated in tape \( R_1 \) and then \( M \) computes and stores \( r_j(\varepsilon_m, \xi) \) in some of the tapes \( R_k \) by simulating \( M_r \). Meanwhile, the head of tape \( T_0 \) writes \( s^{(0)} \), following an obvious linearly space-bounded algorithm. Once the input is read, \( M \) goes ahead by running Britton’s algorithm. During the \( i \)-th step of this algorithm, with \( i \) even, the head of Tape \( T_0 \) writes the string \( s^{(i)} \) encoding \( w^{(i)} \) over \( s^{(i-2)} \) if condition \((*)\) holds for \( w^{(i-1)} \). The latter word is encoded by a string \( s^{(i-1)} \) stored in Tape \( T_1 \). In the next step, the head of Tape \( T_1 \) writes the string \( s^{(i+1)} \) encoding \( w^{(i+1)} \) over \( s^{(i-1)} \) if condition \((*)\) holds for \( w^{(i-1)} \). Tape \( D \) is a draft tape used to carry out two kind of arithmetical computations on binary expansions: the tests for condition \((*)\) and the computations of \( c^{(i)} \). The content of Tape \( D \) is erased after each step. The machine \( M \) halts.
in a state of acceptation if \( s^{(i)} \) is the trivial string. It halts without accepting in case condition \( \square \) does not hold for \( w^{(i)} \).

**Space bound.** The machine \( M \) scans at most \( f(n) \) cells on the storage tapes \( R_k \) while computing \( r_1(\varepsilon_m \xi), \ldots, r_n(\varepsilon_m \xi) \). It also scans at most \( C_0 n \) cells while storing each number \( j \) and all numbers \( r_j(\varepsilon_m \xi) \) for \( j \leq n \), where \( C_0 > 0 \) is independent of \( n \).

Since \( |s^{(i+1)}| \leq |s^{(i)}| + \log_2(|m|) \) and \( |s^{(0)}| \leq 2n + 2 \), we deduce that \( M \) scans at most \( C_1 n \) cells on the storage tapes \( T_0 \) and \( T_1 \), where \( C_1 > 0 \) is independent of \( n \). In order to decide if \( c^{(i)} \) belongs to \( E_{m, \xi} \) or \( E_1 \), \( M \) uses the formula of Proposition \( \text{viii} \): according to the signs of \( \alpha_j^{(i)} \) and \( \alpha_{j+1}^{(i)} \), \( M \) carries out the division of \( \gamma_m := \beta_0^{(i)} + \beta_1^{(i)} r_1(\varepsilon_m \xi) + \beta_2^{(i)} r_2(\varepsilon_m \xi) + \cdots + \beta_k^{(i)} r_k(\varepsilon_m \xi) \) by \( |m| \) or divides \( \gamma_\xi := \beta_1^{(i)} + \beta_2^{(i)} r_1(\varepsilon_m \xi) + \beta_3^{(i)} r_2(\varepsilon_m \xi) + \cdots + \beta_k^{(i)} r_k(\varepsilon_m \xi) \) by \( |m| \) if moreover \( \beta_0^{(i)} = 0 \). As \( \log_2(1 + |\gamma|) \leq |s^{(i+1)}| \), for \( \gamma = \gamma_m, \gamma_\xi \), this requires to scan at most \( C_2 n \) cells on Tape D, where \( C_2 > 0 \) is independent of \( n \). In order to compute \( c^{(i)} \), no more than \( |s^{(i+1)}| \) cells need to be scanned on Tape D. Hence the number of cells scanned by \( M \) on Tape D is linearly bounded. All in all, we get \( WP(m, \xi) \in \text{DSPACE}(n + f(n)) \).

The first part of the proof of Lemma 7.5 is based on the following facts.

**Lemma 7.6.** Let \( m, n \in \mathbb{Z} \setminus \{0\} \) with \( |m| > 1 \) and let \( \xi \in \mathbb{Z}_m \). Let \( v_k = [ab^k a^{-1}, b] \) for \( k \in \mathbb{Z} \).

Then we have: \( v_k = 1 \) in \( BS(m, \xi) \) if and only if \( k \equiv 0 \pmod{m|Z|} \).

**Proof.** Let \( |m| > 1 \). We deduce from Britton’s lemma the following claim: for every \( k \in \mathbb{Z} \), we have \( v_k = 1 \) in \( BS(m, n) \) if and only if \( k \equiv 0 \pmod{m|Z|} \). As \( BS(m, \xi n) \) tends to \( BS(m, \xi) \) as \( n \) goes to infinity, \( v_k \) is trivial in \( BS(m, \xi) \) if and only if it is trivial in \( BS(m, \xi n) \) for all \( n \) large enough, which completes the proof.

For \( h \geq 1, t_1, \ldots, t_h \in \{0, \ldots, |m| - 1\} \), we set

\[
v(|m|, t_1, \ldots, t_h) := w(|m|, t_1, \ldots, t_h)bw(-|m|, -t_1, \ldots, -t_h)b^{-1}\]

where \( w(|m|, t_1, \ldots, t_h) \) is defined as in Lemma 5.5.

**Lemma 7.6.** By hypothesis, there is a deterministic \( f(n/|m|) \) space-bounded Turing machine \( M \) that solves the word problem for \( BS(m, \xi) \). We design a Turing machine \( M' \) computing \( r(n) \) as follows. The storage tapes of \( M' \) consists of the tapes of \( M \) and two other tapes \( W \), and \( O \) (output tape). The tape \( W \) identifies with the input tape of \( M \) and \( M' \) simulates \( M \) on every tape of \( M \).

**Computation of \(|m|\).** By Lemma 7.6, we have \( |m| = \min \{k \geq 1 : v_k = 1 \text{ in } BS(m, \xi)\} \). The machine \( M' \) first writes \( v_k \) on tape \( W \) for \( k = 1 \) and runs \( M \). While \( v_k \) is not accepted by \( M \), the machine \( M' \) writes \( v_{k+1} \) over \( v_k \), adds one to a counter storing \( k \) in tape \( O \) and clears the storage tapes of \( M \). If \( v_k \) is accepted, which means \( k = |m| \), then \( M' \) clears tape \( W \).

**Computation of \( r_n(\varepsilon_m \xi) \).** Using two counters that store \( i \leq n \) and \( t \in \{0, \ldots, |m| - 1\} \) in tape \( O \), the machine \( M' \) lists recursively the words \( w_i(t) = v(|m|, r_1(\varepsilon_m \xi), \ldots, r_{i-1}(\varepsilon_m \xi), t) \) on tape \( W \). Once a word \( w_i(t) \) is written on tape \( W \), the machine \( M' \) runs \( M \). If \( w_i(t) \) is not accepted by \( M \), then \( M' \) writes \( w_i(t + 1) \) over \( w_i(t) \), clears the storage tapes of \( M \) and runs \( M \) again. If the word \( w_i(t) \) written on \( W \) is accepted by \( M \), which means \( t_i = r_i(\varepsilon_m \xi) \) by Lemma 5.5, then \( M' \) stores \( r_i(\xi) \) in tape \( O \), increment \( i \) and restarts with \( w_{i+1}(0) \) or halts if \( i = n \).

**Space bound.** Obviously, the number of cells scanned by \( M' \) to compute \(|m|\) is bounded by some constant \( C(m) \) independent of \( n \). The number of cells scanned by \( M' \) while writing words \( v(|m|, t_1, \ldots, t_n) \) on tape \( W \) is bounded by \( |v(|m|, t_1, \ldots, t_n)| \leq 6n|m| \), the number of cells used to store \( r_1(\xi), \ldots, r_n(\xi) \) is bounded by \( n \log_2(|m| + 1) \) and the number of cells scanned by \( M' \) while simulating \( M \) over its storage tapes is bounded by \( f(n) \). Hence \( WP(m, \xi) \in \text{DSPACE}(f) \).
Turing degree. Let $E, F$ be languages. The language $E$ is said to be Turing reducible to $F$ if there is Turing machine $M$ with oracle $F$ whose accepted language is $E$. The language $E$ is said Turing equivalent to $F$ if $E$ is Turing reducible to $F$ and $F$ is Turing reducible to $E$. The Turing degree of $E$ (also called the degree of unsolvability of $E$) is the class all languages that are Turing equivalent to $E$. Let $f : E \rightarrow F$ be a function. We define the Turing degree of $f$ as the Turing degree of the graph of $f$. We denote by $CP(m, \xi)$ the set of pairs $(w, w') \in \{(a^{\pm 1}, b^{\pm 1})^* \times \{a^{\pm 1}, b^{\pm 1}\}^* \}$ such that $w$ is conjugated to $w'$ in $BS(m, \xi)$. We call the Turing degrees of $WP(m, \xi), CP(m, \xi)$ the Turing degrees of the word problem and the conjugacy problem for $BS(m, \xi)$. These Turing degrees does not depend on the choice of a generating set for $BS(m, \xi)$.

Corollary 7.7. The following Turing degrees coincide:

- the Turing degree of the word problem for $BS(m, \xi)$;
- the Turing degree of the conjugacy problem for $BS(m, \xi)$;
- the Turing degree of $r$.

In particular, the word problem is solvable for $BS(m, \xi)$, i.e $WP(m, \xi)$ is a recursive language, if and only if $r$ is a recursive function.

In contrast, Britton has proved that the conjugacy problem for any HNN extension with base a finitely generated abelian group is solvable, i.e. both Turing degrees are 0 [Bri79]. It is also worth noting this optimal result of Miller: for every pair of recursively enumerable Turing degrees $a, b$ where $a$ is Turing reducible to $b$, there is a finitely presented group whose word problem has Turing degree $a$ and whose conjugacy problem has Turing degree $b$ [Mil71].

Observe that one can define recursive $m$-adic numbers in the very same way one defines recursive (equivalently computable) real numbers (see [Wei00] for a definition of computable real numbers). The Turing degree of an $m$-adic number is then defined by means of its Hensel expansion. If $\xi \in \mathbb{Z}_m$, Corollary 7.7 then says that the word problem is solvable in $BS(m, \xi)$ if and only if $\xi$ is a recursive number and that the Turing degree of the word problem coincides with the Turing degree of $\xi$.

Proof. The last claim directly follows from Lemmas 7.4 and 7.5. From the proofs of these lemmas, we can easily deduce that the Turing degree of $WP(m, \xi)$ coincides with the Turing degree of $r$.

To complete the proof we design quite informally a Turing machine with oracle $r$ that solves the conjugacy problem in $BS(m, \xi)$. We fix the set of representatives $T_{m, \xi} = \{0, e_0, \ldots, (m - 1) e_0\}$ of the cosets of $E_1$ in $E$ and the set of representatives $T_1 = \{0, e_0, 2 e_0, \ldots\}$ of the cosets of $E_{m, \xi}$ in $E$. If $r$ can be computed by means of a Turing machine, Britton’s algorithm (see the proof of Lemma 7.4) yields a reduced form in $BS(m, \xi)$ of any $w \in \{a^{\pm 1}, \pm e_0\}^*$. The process of working from the right with the relations $a(m e_0) a^{-1} = e_1$ and $a(e_i - r_i(\xi)e_0) a^{-1} = e_{i+1}$ yields a normal form for $w$ with respect to the sets of representatives $T_{m, \xi}$ and $T_1$. Thus we can design a Turing machine with oracle $r$ that computes normal forms $\tilde{v}, \tilde{w}$ of cyclically reduced conjugates of $v$ and $w$ for any $v, w \in \{a^{\pm 1}, \pm e_0\}^*$. If $|\tilde{v}|_a \neq |\tilde{w}|_a$, we deduce from Collin’s lemma [LS77] Th. 2.5 that $v$ is not a conjugate of $w$ in $BS(m, \xi)$. The machine can be designed in such a way that it halts in this case in a non-accepting state. Hence we can assume that $|\tilde{v}|_a = |\tilde{w}|_a$. Comparing the normal form $\tilde{v}$ to the normal form of each cyclic permutation of $\tilde{w}$, the machine can decide wether or not there exist, $e \in E$ and some cyclic permutation $(\tilde{w})^* \tilde{v}$ such that $\tilde{v} = e(\tilde{w})^*(-e)$. By Collin’s lemma, it is enough to decide wether $v$ is a conjugate of $w$, provided either $|\tilde{v}|_a$ or $|\tilde{w}|_a$ is not zero. Hence we can assume that $v, w$ have their images in $E$. We deduce from Lemma 7.4 that $v$ is a conjugate of $w$ in $BS(m, \xi)$ if and only if there is some $n \in \mathbb{Z}$ such that $v = a^n w a^{-n}$ in $BS(m, \xi)$. Identifying $E$ with $B$ in Proposition 2.9 i, we can consider $n(v, w) = \deg P_v(X) - \deg P_w(X)$. By means of a Turing machine with oracle $r$, we can compare the Laurent polynomials $P_v(X)$ and $X^{n(v, w)} P_w(X)$ and hence decide wether or not $v$ is a conjugate of $w$. 

\qed
Remark 7.8. We can construct a family of public-key cryptosystems based on the word problems of limits of Baumslag-Solitar groups by adapting the construction in [GZ91] based on Grigorchuk groups. The attack conceived in [GHM+04] does not threaten these new cryptosystems since such an attack would require in our case at least $|m|^N$ numbers of computations when the length of the public-key is $N$, if we follow the cryptanalysis of the authors. However, another attack conceived in [GHM+04], namely the reaction attack against the Magyarik-Wagner protocol, can be proved to be successful.

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