Deconfined criticality and bosonization duality in easy-plane Chern-Simons two-dimensional antiferromagnets

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Two-dimensional quantum systems with competing orders can feature a deconfined quantum critical point, yielding a continuous phase transition that is incompatible with the Landau-Ginzburg-Wilson scenario, predicting instead a first-order phase transition. This is caused by the LGW order parameter breaking up into new elementary excitations at the critical point. Canonical candidates for deconfined quantum criticality are quantum antiferromagnets with competing magnetic orders, captured by the easy-plane CP1 model. A delicate issue however is that numerics indicates the easy-plane CP1 antiferromagnet to exhibit a first-order transition. Here we show that an additional topological Chern-Simons term in the action changes this picture completely in several ways. We find that the topological easy-plane antiferromagnet undergoes a second-order transition with quantized critical exponents. Further, a particle-vortex duality naturally maps the partition function of the Chern-Simons easy-plane antiferromagnet into one of massless Dirac fermions.

Introduction — It is well known that some quantum critical systems exhibit a phase structure evading the traditional Landau-Ginzburg-Wilson (LGW) theory of phase transitions [1–3]. Typical examples are two-dimensional quantum systems with competing orders, like for instance antiferromagnetic (AF) and valence-bond solid (VBS) orders originating from general quantum spin models with SU(2) symmetry [3, 4]. The LGW scenario predicts a first-order phase transition for such a system. However, the interplay between emergent instanton excitations (i.e., spacetime magnetic monopoles) and staggered Berry phases [4] causes the actual phase transition to become a second-order one, leading in this way to a quantum critical point separating the AF and VBS phases. For similar reasons discussed in studies of the deconfined transition in high-energy physics, this type of critical point has been dubbed a "deconfined quantum critical point" [1]. At such a critical point, order parameters on both sides of the transition fall apart into "elementary particles" called spinons and we speak of spinon deconfinement.

A well studied effective theory in this context is the quantum O(3) nonlinear sigma model (NLσM),

\[ \mathcal{L}_{\text{NL}\sigma M} = \frac{1}{2g} (\partial_{\mu} n)^2 + \ldots, \] (1)

where \( n^2 = 1 \), supplemented by instanton-suppressing terms, here symbolically represented by ellipses [1, 2, 5–8]. Physically, the model is an effective theory of antiferromagnets capturing the long-distance interactions, and the unit vector \( n \) is the direction of the magnetization. When tuning the coupling constant \( g \), the system undergoes a quantum phase transition from an AF ordered phase to a paramagnetic phase separated by a critical coupling \( g_c \). By means of the Hopf map, \( n = z_1^a \sigma_a z_2 \), where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is a Pauli matrix vector, the \( O(3) \) NLσM is shown to be equivalent to the CP1 model,

\[ \mathcal{L}_{\text{CP}1} = \frac{1}{g} \sum_{a=1,2} |(\partial_{\mu} - ia_{\mu}) z_a|^2 + \ldots, \] (2)

where the constraint \( |z_1|^2 + |z_2|^2 = 1 \) holds and the gauge field is an auxiliary field given by \( a_{\mu} = (i/2) \sum_a (z_a^a \partial_{\mu} z_a - z_a \partial_{\mu} z_a^a) \).

Although the gauge field \( a_{\mu} \) is an auxiliary field at the level of field equations, it becomes dynamical when quantum fluctuations of the spinon fields \( z_a \) are accounted for, causing a Maxwell term to be generated in the low-energy regime [9]. In this context it is also interesting to consider generalizations with \( N \) complex fields, yielding an \( O(2N) \) symmetric version, the CP\( N^{-1} \) model. It has been recently demonstrated [10] that the large \( N \) limit in a instanton-suppressed CP\( N^{-1} \) model implies a second-order phase transition. The result agrees with the standard field theory analysis of the large \( N \) limit [9, 11]. Nevertheless, lower values of \( N \) were shown numerically to exhibit a first-order phase transition, specifically for \( N = 4, 10, 15 \); though the \( N = 2 \) case remained inconclusive [10, 12]. This result contrasts with the large \( N \) limit without instanton suppression, where a first-order phase transition occurs [13, 14].

A well-studied model since the early days of DC [1, 2, 6, 7] is the easy-plane CP1 model with Lagrangian,

\[ \mathcal{L}_{\text{ep}} = \mathcal{L}_M + \mathcal{L}_{\text{CP}1} + \frac{K}{2g^2} |z_1|^2 - |z_2|^2, \] (3)

which follows directly from the NLσM by adding the easy-plane anisotropy term, \( \mathcal{L}_{\text{anis}} = Kn_\perp^2/2g^2 \), where \( K > 0 \). Instanton suppression in the above Lagrangian is achieved by means of a Maxwell term [2, 5],

\[ \mathcal{L}_M = \frac{1}{2g^2} (\epsilon_{\mu\nu\lambda}\partial_{\mu} n_{\nu} n_{\lambda})^2. \] (4)
An exact particle-vortex duality transformation of the lattice Villain model version of $\mathcal{L}_{ep}$ shows that the model is self-dual [1, 2, 5, 12]. Partly on the basis of this self-duality, it was originally argued [1, 2] that the easy-plane CP$^1$ model undergoes a second-order phase transition, featuring therefore a deconfined quantum critical point. However, it was later demonstrated numerically that the phase transition is actually a first-order one [6, 7], a result that is also corroborated by renormalization group (RG) results [8].

Here we consider the topological easy-plane CP$^1$ lagrangian including a Chern-Simons (CS) term, i.e., $\mathcal{L} = \mathcal{L}_{ep} + \mathcal{L}_{CS}$, where,

$$\mathcal{L}_{CS} = i\kappa \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda,$$

(5)
describes a CS Lagrangian in Euclidean spacetime. For arbitrary $\kappa$ the CS action is invariant under any topologically trivial gauge transformation, since the surface arbitrary real describes a CS Lagrangian in Euclidean spacetime. For RG analysis of the CP$^1$ experimentally.

as we will elaborate later, this is of direct relevance to features a CS term of the form, $\kappa b$, that the scaling behavior of the topological theory cannot be smoothly connected to the limit where $\kappa \rightarrow 0$. In the second part of the paper we show that the dual model features a CS term of the form,

$$\tilde{\mathcal{L}}_{CS} = -\frac{i}{2\kappa} \epsilon_{\mu\nu\lambda} (b_{1\mu} + b_{2\mu}) \partial_\nu (b_{1\lambda} + b_{2\lambda}),$$

(6)

two gauge fields $b_{1\mu}$ and $b_{2\mu}$. Finally, in the third part we show that for $\kappa = 1/(2\pi)$ the duality of the second part actually corresponds to a bosonization duality [18, 19] involving massless Dirac fermions [20].

Renormalization group analysis — Let us start by discussing the nature of the phase transition of the easy-plane CP$^1$ in the Lagrangian $\mathcal{L} = \mathcal{L}_{ep} + \mathcal{L}_{CS}$, and consider a soft constraint version of the model,

$$\mathcal{L} = \mathcal{L}_M + \mathcal{L}_{CS} + \sum_{a=1,2} \left[ (i\partial_\mu - ia_\mu) z_a \right]^2 + m_a^2 |z_a|^2$$

$$+ \frac{\kappa}{2} |z_1|^2 - |z_2|^2)^2 + \frac{K}{2} (|z_1|^2 - |z_2|^2)^2.$$  

(7)

details of the RG calculations are presented in Supplemental Material (SM). There we show that the original theory features two IR fixed points for the renormalized dimensionless couplings $\hat{u}$, $\hat{K}$, and $\hat{\epsilon}^2$. Importantly, $\epsilon^2$ sets a UV scale for the renormalized dimensionless gauge coupling $\epsilon^2$, in the sense that the IR stable fixed point $\epsilon^2$ is also reached when $\epsilon^2 \rightarrow \infty$ (see SM). One of the fixed points is $O(2) \times O(2)$-symmetric, while the second one corresponds to an emergent $O(4)$-symmetry. Interestingly, the Abelian Higgs CS critical exponents do not belong to the XY universality class, as they are $\kappa$-dependent.

An important outcome of the RG analysis is that the limit $\kappa \rightarrow 0$ with $\epsilon^2$ finite does not reduce to the RG equations expected for a $U(1) \times U(1)$ Abelian Higgs model [9]. This happens because the presence of the CS term causes the one-loop gauge field bubble in the scalar field vertex function to vanish at zero external momenta (see SM for details on this point).

From the RG analysis it follows that the correlation length critical exponents for the $O(2) \times O(2)$- and $O(4)$-symmetric IR fixed points are quantized and depend on the level of the CS term. In particular, for a level 1 CS term this yields $\nu \times O(2) = 49/80 \approx 0.613$. This value is nearly the same as the one-loop result $\nu = 5/8$ of the XY universality class. For the $O(4)$-symmetric criticality we obtain a larger value, $\nu \times O(4) = 2/3$, which is independent of the CS level at the one-loop order.

The anomalous dimension $\eta_N$ is defined by the critical magnetization correlation function at large distances, $\langle n(x) \cdot n(0) \rangle \sim 1/|x|^{1+\eta_N(n)}$. For a level 1 CS term we obtain, $\eta_N \times O(2) = 59/49 \approx 1.2$ and $\eta_N \times O(4) = 164/147 \approx 1.12$, for the $O(2) \times O(2)$ and $O(4)$ symmetric cases, respectively. This clearly shows that a new universality class emerges.

At this point the following remark is in order. Typically, DC implies considerably larger anomalous dimensions $\eta_N$ as compared to the case of the LGW paradigm of phase transitions. However, it is rather rare that these values exceed unity. The leading order value in the easy-plane case without a CS term is $\eta_N = 1$ (Gaussian approximation) [1]. For the $J - Q$ model the result is $\eta_N = 0.35$, but the easy-plane $J - Q$ model is reported to deliver a much larger value, $\eta_N \approx 0.91$ [21]. On the other hand, the theory considered here exhibits anomalous dimensions $\eta_N > 1$. An example where this also occurs is in a lattice boson model with an emergent $Z_2$ gauge symmetry [22], where the anomalous dimension is numerically calculated to be $\eta \approx 1.493$.

Duality analysis — We start the discussion of the duality transformation by changing to polar coordinates $z_a = \rho_a e^{i\theta_a}$ in the partition function of the easy-plane CP$^1$ model. After integrating $\rho_2$ out and assuming a strong anisotropy ($K \gg g^2$), we obtain $\rho_1^2 \approx \rho_2^2 \approx 1/2$, leading to an effective action depending only on the phase
fields coupled to the gauge field, 
\[ S_{\text{eff}} = S_{\text{CS}} + \frac{1}{2g} \sum_{a=1,2} \int d^3x (\partial \mu \partial \alpha - a\mathbf{)}^2, \]  
(8)

where the CS action \( S_{\text{CS}} \) corresponds to the Lagrangian \( (5) \). The effective action \( S_{\text{eff}} \) is equivalent to a two-component CS superconductor in the London limit where the amplitudes of the order parameter are constrained to be equal.

The traditional way to perform a duality transformation is to carry it out on the lattice [23]. Nevertheless, while it is a straightforward task to define a Maxwell term on the lattice [24], fundamental difficulties arise when one tries to define the CS term on the lattice. It is known that the CS action \( \partial \mathbf{h}^\mu \) transforms introduces auxiliary fields \( \delta \mathbf{h}^\mu \) to perform the subsequent calculations directly in the continuum.

Even though we are working directly in the continuum, in order for the theory to be well-defined at the short distances, we need to regularize it. So we include an additional Maxwell term \( \rho \). The first step of our duality transformation introduces auxiliary fields \( \mathbf{h}^\mu \), \( I = 1, 2 \), such that
\[ S_{\text{eff}} = \sum_{I=1,2} \int d^3x \left[ \frac{g}{2} h^\mu_I - i h^\mu_I (\partial \nu \partial \theta^I - a\mathbf{)} \right] \]  
(9)

To account for the periodicity of \( \theta^I \), the following decomposition in terms of longitudinal phase fluctuations and vortex gauge fields holds \( [23] \), \( \partial_\nu \partial_\alpha = \partial_\nu \varphi^I + 2\pi \nu_{\alpha}^I \), where \( \varphi^I \in \mathbf{R} \) and the vorticity,
\[ w_{\nu I} = \epsilon_{\mu \nu \lambda} \partial_\mu \nu_{\alpha}^I (x) = \sum_{c} n_{Ic} \int_{L_{Ic}} d\mathbf{y} \delta^3 (x - y^{(c)}), \]  
(10)

with quanta \( n_{Ic} \in \mathbf{Z} \) and the integral is over a path along the \( c \)-th vortex loop \( L_{Ic} \).

Integrating out both \( \varphi^I \) and \( a\mathbf{)} \) leads to the action,
\[ \tilde{S} = \sum_{I=1,2} \int d^3x \left[ \frac{g}{2} h^\mu_I + i 2\pi \nu_{\alpha}^I h^\mu_I \right] \]  
(11)

\[ + \frac{1}{2} \int d^3x \int d^3x' \partial_\mu \nu_{\alpha}^I (x - x') (h^\mu_I + h^\nu_{\alpha}^I) (h'^\mu_I + h'^\nu_{\alpha}^I), \]

where \( h^\mu_I \) denotes dependence on \( x' \), and the propagator in momentum space,
\[ D_{\mu \nu} (p) = \frac{g \kappa^2}{2 (g^2 p^2 + 4 \kappa p^2)} \left( \delta_{\mu \nu} - \frac{\epsilon_{\mu \nu \lambda} p_{\lambda}}{p^2} \right), \]  
(12)

is the Fourier transform of \( D_{\mu \nu} (x) \). Here, the longitudinal contribution is absent due to the constraint \( \partial_\nu h^\mu = 0 \) which appears after integrating out fields \( \varphi^I \). This also leads to \( h^\mu_I \) being expressed in terms of new auxiliary fields \( b^\mu_I = \epsilon_{\mu \nu \lambda} \partial_\nu b^\lambda_I \).

As we are interested in the case of easy-plane \( \mathbf{S} \) model, we can send \( \epsilon^2 \rightarrow \infty \) after performing explicitly the calculations in Eq. (11) and obtain the following dual Lagrangian,
\[ \mathcal{L}_{\text{dual}} = \sum_{I=1,2} \left[ \frac{g}{2} (\epsilon_{\mu \nu \lambda} b^\lambda_I)^2 + i 2\pi w_{\nu I} b^\mu_I \right] \]  
(13)

One notices that the presence of the CS term in the original model leads to the appearance of the mixed CS term anticipated in Eq. (6). Thus, the dual action \( (63) \) features gauge fields coupled to an ensemble of vortex loops \( w_{\nu I} \). The latter represent the worldlines of the particles of the original model \( [24, 32] \).

As mentioned earlier in the context of the original theory using a soft constraint, an IR stable fixed point for the dimensionless renormalized gauge coupling is reached as \( \epsilon^2 \rightarrow \infty \). This result remains valid in the hard constraint case. In Eq. (63) \( 1/g \) assumes the role of \( \epsilon^2 \) of the original theory. Note that \( g = \tilde{g}/\Lambda \), where \( \tilde{g} \) is dimensionless and \( \Lambda \) is a UV cutoff, so the theory with a hard constraint reaches a UV nontrivial fixed point \( \tilde{g}_s \) as \( \Lambda \rightarrow \infty \), so \( g \rightarrow 0 \). Thus, the duality establishes a mapping between the UV and IR regimes of the theory.

**Bosonization duality** — Having obtained a bosonic dual theory, we will show now that the theory of CS easy-plane antiferromagnets is actually self-dual at criticality and leads to the bosonization duality for massless Dirac fermions. We proceed to show this by first integrating out the fields \( b^\mu_I \) in Eq. (63). This yields the dual action in terms of vortex loop fields,
\[ \tilde{S} = 2\pi^2 \int d^3x \int d^3x' \tilde{D}_{\mu \nu} (x - x') (w_{\nu 1} + w_{\nu 2}) (w'_{\nu 1} + w'_{\nu 2}) \]  
\[ + \frac{\pi}{g} \int d^3x \int d^3x' (w_{\nu 1} - w_{\nu 2}) (w'_{\nu 1} - w'_{\nu 2}) \frac{1}{|x - x'|}, \]  
(14)

where as before we are using primes to denote the dependence on \( x' \) and \( \tilde{D}_{\mu \nu} (x - x') \) in momentum space reads,
\[ \tilde{D}_{\mu \nu} (p) = \frac{g \kappa^2}{2 (g^2 p^2 + 4 \kappa p^2)} \left( \delta_{\mu \nu} - \frac{\epsilon_{\mu \nu \lambda} p_{\lambda}}{p^2} \right). \]  
(15)

Now, we will show that, similarly to the standard easy-plane theory \([5]\), the model considered here is self-dual in the large distance regime \( g^2 p^2 \ll 1 \). In this case the vortices \( w_{\nu 1} \) and \( w_{\nu 2} \) balance, so we can write approximately, \( w_{\nu 1} \approx w_{\nu 2} \equiv w_{\nu} \), so that (for details, see SM),
\[ S_{\text{dual}} = \int d^3x \left( 2\pi^2 g^2 w_{\nu}^2 + i 2\pi^2 \kappa v_{\nu} w_{\nu} \right). \]  
(16)

On the other hand, letting \( g \rightarrow 0 \) in the initial Abelian Higgs CS action \( (9) \) and integrating out \( h_{2\mu} \) yields \( \alpha =
\[ \partial_{\mu} \theta_2. \] Subsequent integration of \( h_{1\mu} \) enforces \( \theta_1 = \theta_2 \equiv \theta. \] At the end, this yields,
\[ S = \int d^3x \left( \frac{2\pi^2}{e^2} \mu^2 + i2\pi^2 \kappa v_{\mu} w_{\mu} \right), \] (17)
and therefore we obtain the duality for the partition function,
\[ Z_{\text{dual}}(e^2 = \infty, g, \kappa) = Z(g' = 0, e^2 = 1/(g\kappa^2), \kappa). \] (18)

Underlying the above result is the duality relation between the couplings, \( g e^2 = 1/\kappa^2 \). For a level 1 CS term the latter reduces to \( g e^2 = (2\pi)^2 \), which is the Dirac quantization associated to particle-vortex duality. It is interesting to note that Eq. (18) constitutes a topological version of the "frozen superconductor" regime in the particle-vortex duality for the Abelian Higgs model in 2+1 dimensions derived by Peskin [24] and Dasgupta and Halpern [33].

We are now ready to explore the critical dual theory which, as was discussed above, is obtained by setting \( g \to 0 \) in the Lagrangian (16). This yields up to an overall normalization the partition function,
\[ \bar{Z}_{\text{crit}} = \sum_{\text{loops}} \exp \left[ \frac{i\pi N}{2} \sum_{a,b} n_an_b \right] \times \int_{L_a} dx^{(a)} \int_{L_b} dx^{(b)} \epsilon_{\mu\nu\lambda} \left( \frac{x^{(b)} - x^{(a)}}{|x^{(b)} - x^{(a)}|^2} \right), \] (19)
where we sum over all loops \( L_a \) and \( L_b \), not excluding \( a = b \) contributions, which will turn out to be a crucial point [34, 35]. For \( a \neq b \) the double integral above yields a contribution \( e^{i2\pi^2 N_{a\mu}\kappa} N_{ab} \in \mathbb{Z} \), in virtue of the Gauss linking number formula [36, 37]. Despite looking at first sight singular, the \( a = b \) contributions are actually finite and proportional to the so called writhe of the (vortex) loop [38–40]. The latter can be conveniently written in terms of a suitable parametrization, \( x_{\mu}(s), s \in [0, 1], \) by defining the unit vector, \( u_{\mu}(s,s') = (x_{\mu}(s) - x_{\mu}(s'))/[x(s) - x(s')], \) in which case the writhe is recast as,
\[ W_{\alpha} = \frac{1}{4\pi} \int_{L_a} ds' \epsilon_{\mu\nu\lambda} \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds'} (x_{\lambda}(s) - x_{\lambda}(s')) \]
\[ = \frac{1}{4\pi} \int_{L_a} ds' \epsilon_{\mu\nu\lambda} u_{\mu}(s) \partial_s u_{\nu}(s) \partial_s' u_{\lambda}. \] (20)
The result is reminiscent of the point-splitting regularization employed to calculate expectation values of Wilson loops [41]. This is in agreement with Ref. [31], where it is shown that the point-splitting procedure yields the topological invariant which coincides with the writhe in theories containing a Maxwell term in addition to a CS one when \( e^2 \to \infty \).

We now consider a specific case of a level 1 CS theory in the original model corresponding to \( \kappa = 1/(2\pi) \). Consequently, the dual partition function at criticality (19) takes the form,
\[ \bar{Z}_{\text{crit}} = \sum_{\text{loops}} (-1)^{N_{ab}} e^{i\pi \sum_a n_a^2 W_a}. \] (21)
The contribution from the linking number formula generates weight factors \((-1)^n\) in the dual model, where \( n \) is integer. This result is reminiscent of the lack of gauge invariance of the partition function under topologically nontrivial gauge transformations in the dual model [42, 43]. This result makes apparent that the considered duality corresponds to a form of bosonization akin to the one discussed by Polyakov for the CP^1 model with a CS term [34, 44]. This contribution is sometimes referred to as the Polyakov spin factor [34, 35, 45–48]. Equation (21) relates to the representation of the partition function of a Dirac fermion in 2+1 Euclidean dimensions in terms of loops [35, 46–48], with the difference that in our case the parity anomaly factor implies that the fermions are massless [15, 49–51].

As far as the writhe is concerned, it is worth to recall that it arises quite naturally in the partition function of Wilson fermions on an euclidean cubic spacetime lattice [35]. However, the analysis of Ref. [35] and previous ones [34, 45–48, 52] requires massive fermions.

It is remarkable that even if the analysis above does not explicitly employ fermions, still a result that can only follow from massless fermions is obtained. To elaborate this point further we recall that a topologically nontrivial gauge transformation \( \gamma, a_{\mu} \to a_{\mu}^{\gamma} \), in a continuous deformation of the gauge field, leads to the subsequent transformation of the fermion determinant \( \det(\partial + i\gamma) \to (-1)^n \det(\partial + i\gamma) \), with \( n \) being the winding number [43, 49–51]. Therefore, integrating over \( a_{\mu} \) requires to account for redundant gauge configurations and sum over all possible winding numbers corresponding to different topological sectors in the partition function.

To further substantiate our bosonization claim, we redrew this result using the flux attachment approach to duality [19], which involves a path integral formalism corresponding to a “Fourier transform” for quantized fluxes. In order for this to work in our case we have to attach fluxes to both fermions and bosons. The end result is that the dual Lagrangian (63) is the bosonized version of massless Dirac fermions with half-quantized CS flux attached. (The explicit derivation can be found in the SM). Therefore, our derivation is consistent with the flux attachment technique, but in contrast to it, does not assume any conjectures as a starting point. Thus, our analysis provides yet another check for these conjectures.

Final remarks — We have demonstrated through RG analysis that the topological easy-plane CP^1 model undergoes a second-order phase transition. Following this result, we established a dual theory, which at criticality exhibits a parity anomaly. This occurs at the particular value of a CS coupling \( \kappa \) that provides topological gauge invariance. We relate that to massless Dirac fermions, thereby establishing an explicit bosonization duality [18]. Since the theory we consider here possesses a \( U(1) \times U(1) \)
symmetry, our analysis subscribes into the so-called beyond flavor bound scenario of duality [53, 54]. Additionally, let us consider these results within an experimental context. The dual theory (63) with $\kappa = 1/(2\pi)$ and gauge fields rescaled as $b_{\mu} \rightarrow b_{\mu}/(2\pi)$ features a CS term as it occurs in the (1, 1, 1) quantum Hall (QH) state associated to a bilayer QH system [55–57]. As mentioned, the initial model corresponds to a two-component CS superconductor. Therefore, the duality picture discussed here naturally connects the observed resonant tunneling in bilayer QH ferromagnets [58] to a Josephson-like effect in a system that is not superconducting [59–61]. Our analysis shows that such an experimental setup represents the dual physical system to the actual easy-plane CS antiferromagnet. They belong to the same universality class so that the bilayer QH ferromagnet offers a controllable experimental system for a deconfined critical point. Moreover, in view of the connection to massless Dirac fermions established in this letter, bilayer QH ferromagnets would in principle offer a platform to experimentally explore the bosonization duality in 2+1 dimensions. It would be interesting to check whether experiments can reveal the critical behavior with quantized exponents as we predict here.

Another system of interest where our approach may (with appropriate modifications) be relevant is the topological field theory for magic-angle graphene [62], where a duality between superconductivity and insulating regimes occur.

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SUPPLEMENTAL MATERIAL

I. RENORMALIZATION GROUP ANALYSIS

First we consider the easy-plane CP$^3$ model with a soft constraint and without any additional gauge terms,

$$\mathcal{L} = \sum_{\alpha=1,2} |(\partial_\mu - ia_\mu)z_\alpha|^2 + \frac{K}{2}|z_1|^2|z_2|^2$$

$$+ m_0^2(|z_1|^2 + |z_2|^2) + \frac{u}{2}(|z_1|^2 + |z_2|^2)^2. \quad (22)$$

We define the couplings $U = u + K$ and $V = u - K$ and derive the renormalized couplings at one-loop order,

$$U_r = U - (5U^2 + V^2)I(m^2), \quad (23)$$

$$V_r = V - 2(V^2 + 2UV)I(m^2), \quad (24)$$

where,

$$I(m^2) = \int \frac{1}{(p^2 + m^2)^2} = \frac{m^{d-4}}{(4\pi)^{d/2}} \Gamma \left(2 - \frac{d}{2}\right), \quad (25)$$

with $m$ being the renormalized mass, and we have introduced the notation $\int d^dp = \int \frac{d^dp}{(2\pi)^d}$. Since the CS is absent, we are generalizing the calculation to $d$ dimensions. We define the dimensionless renormalized couplings by $\hat{U} = m^{d-4}U_r$ and $\hat{V} = m^{d-4}V_r$, along with the rescalings $\hat{U} \to \hat{U}/c_d$, $\hat{V} \to \hat{V}/c_d$, where $c_d = (4 - d)(4\pi)^{-d/2}\Gamma(2 - d/2)$, so that the RG $\beta$ functions, $\beta_{\hat{U}} = m\hat{d}\hat{U}/dm$ and $\beta_{\hat{V}} = m\hat{d}\hat{V}/dm$ are obtained,

$$\beta_{\hat{U}} = m\frac{d\hat{U}}{dm} = -(4 - d)\hat{U} + 5\hat{U}^2 + \hat{V}^2, \quad (26)$$

$$\beta_{\hat{V}} = m\frac{d\hat{V}}{dm} = -(4 - d)\hat{V} + 2(\hat{V}^2 + 2\hat{U}\hat{V}). \quad (27)$$

We obtain three fixed points, namely, the Gaussian fixed point, $(\hat{U}_G, \hat{V}_G) = (0, 0)$, the XY fixed point, $(\hat{U}_{XY}, \hat{V}_{XY}) = (\epsilon/5, 0)$, and the easy-plane anisotropy fixed point, $(\hat{U}_e, \hat{V}_e) = (\epsilon/6)(1, 1)$, where $\epsilon = 4 - d$. The XY fixed point is stable for $\hat{V} = 0$, but becomes unstable for any small $\hat{V}$. Note that this fixed point actually corresponds $\hat{U} = 2\hat{K}$, so this yields a fixed point $\hat{K}_c = \epsilon/10$ associated to the $O(2) \times O(2)$ symmetry.

The fixed point $(\hat{U}_e, \hat{V}_e)$ actually corresponds to vanishing anisotropy, i.e., $\hat{K}_e = 0$, since $\hat{U}_e = \hat{V}_e$, implying $\epsilon = \epsilon/6$. Hence, this fixed point governs the $O(4)$ universality class. This fixed point becomes IR unstable only in a region where $K < 0$, which would correspond to easy-axis rather than easy-plane anisotropy.

Next, we consider the coupling to the gauge field $a_\mu$. We include CS and Maxwell terms,

$$\mathcal{L}_{\text{gauge}} = \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2 + \frac{i}{2}e_{\mu\nu\lambda}a_\mu \partial_\nu a_\lambda. \quad (28)$$

The gauge field propagator in the absence of interactions is given in the Landau gauge by,

$$D_{\mu\nu}(p) = \frac{1}{p^2 + M^2} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - M\epsilon_{\mu\nu\lambda} \frac{\partial_\lambda}{p^2} \right), \quad (29)$$

where $M = e^2\kappa$, and where we used the rescaling $a_\mu \to e^0 a_\mu$.

Since the CS term is defined in three spacetime dimensions, the renormalized couplings $U_r$ and $V_r$ will be calculated for fixed dimensionality $d = 3$ [63]. This has the drawback of making $\epsilon = 4 - d$ as a control parameter in principle unavailable to us (see section II for a more thorough discussion on this point). Instead, we generalize the easy-plane model to a system featuring a global $O(N) \times O(N)$ symmetry with $N$ even and explicitly consider the large $N$ limit. This is achieved by considering $N/2$ complex fields $z_{2a}$, $a = 1, \ldots, N/2$. In this case the special case where the anisotropy is absent (i.e., $K = 0$) will correspond to an $O(2N)$ global symmetry.

The coupling to a dynamical gauge field will cause $U_r$ and $V_r$ to receive a contribution from the diagram at Fig. 1 through the square of the wave function renormalization. This diagram is the only one giving a momentum dependent contribution to the total self-energy at one-loop. Its explicit expression is given by,

$$\Sigma(p) = -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(2p_\mu - k_\mu)(2p_\nu - k_\nu)}{(p - k)^2 + m^2} D_{\mu\nu}(k)$$

$$= 4e^2 \left\{ \int \frac{d^3k}{(2\pi)^3} \frac{(k \cdot p)^2}{(p - k)^2 + m^2[(k^2 + M^2)}$$

$$- \int \frac{d^3k}{(2\pi)^3} \frac{1}{[(p - k)^2 + m^2][(k^2 + M^2) \}}. \quad (30)$$

In the small external momentum $|p|$ limit, we obtain,

$$\Sigma(p) = -\frac{2e^2}{3\pi} \frac{p^2}{m + |M|} + O(p^4). \quad (31)$$

FIG. 1. Scalar field self-energy

$$\Sigma(p) = -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(2p_\mu - k_\mu)(2p_\nu - k_\nu)}{(p - k)^2 + m^2} D_{\mu\nu}(k)$$

$$= 4e^2 \left\{ \int \frac{d^3k}{(2\pi)^3} \frac{(k \cdot p)^2}{(p - k)^2 + m^2[(k^2 + M^2)}$$

$$- \int \frac{d^3k}{(2\pi)^3} \frac{1}{[(p - k)^2 + m^2][(k^2 + M^2) \}}. \quad (30)$$

In the small external momentum $|p|$ limit, we obtain,

$$\Sigma(p) = -\frac{2e^2}{3\pi} \frac{p^2}{m + |M|} + O(p^4). \quad (31)$$
Another diagram contributing to both $U_r$ and $V_r$ is shown in Fig. 2. However, the latter vanishes at zero external momenta (see the Appendix A in Ref. [64]).

Therefore, we obtain the wave function renormalization,

$$Z = 1 + \frac{2}{3\pi m} \frac{e^2}{m + e^2|\kappa|}, \quad (32)$$

and dimensionless renormalized couplings,

$$\hat{U} = Z^2 \left[ \frac{U}{m} - \frac{(N + 8)U^2 + N\hat{V}^2}{16\pi m^2} \right],$$

$$\approx \left[ \frac{U}{m} - \frac{e^2}{2\pi m|\kappa|} + \frac{4e^2U}{3\pi m(m + e^2|\kappa|)} \right. - \left. \frac{(N + 8)U^2 + N\hat{V}^2}{8\pi m^2} \right]. \quad (33)$$

$$\hat{V} = Z^2 \left[ \frac{V}{m} - \frac{2V^2 + (N + 2)UV}{8\pi m^2} \right],$$

$$\approx \left[ \frac{V}{m} - \frac{e^2}{2\pi m|\kappa|} + \frac{4e^2V}{3\pi m(m + e^2|\kappa|)} \right. - \left. \frac{2V^2 + (N + 2)UV}{8\pi m^2} \right], \quad (34)$$

where now the dimensionality is fixed to $d = 3$. We define an additional dimensionless coupling, $e^2 = e_r^2/m$, where $e_r^2$ is the renormalized gauge coupling which is calculated at one-loop order by considering the vacuum polarization diagram of Fig. 3.

The new RG $\beta$ functions are given by,

$$\beta_U = - \left[ 1 + \frac{4\hat{e}^2}{3\pi(1 + \hat{e}^2|\kappa|)^2} \right] \hat{U} + \frac{(N + 8)\hat{U}^2 + N\hat{V}^2}{16\pi}, \quad (35)$$

$$\beta_V = - \left[ 1 + \frac{4\hat{e}^2}{3\pi(1 + \hat{e}^2|\kappa|)^2} \right] \hat{V} + \frac{2\hat{V} + (N + 2)\hat{U}}{8\pi}, \quad (36)$$

along with the $\beta$ function,

$$\beta_{\hat{e}^2} = - \hat{e}^2 + \frac{N\hat{e}^4}{24\pi}. \quad (37)$$

On the other hand, the non-renormalization of the CS term [65] implies,

$$\beta_\kappa = \left( \frac{N\hat{e}^2}{24\pi} - 1 \right) \kappa. \quad (38)$$

Therefore, at the charged fixed point an arbitrary value of $\kappa$ is allowed, leading to a critical behavior featuring continuously varying critical exponents as a function of $\kappa$. The vanishing of $\beta_{\hat{e}^2}$ at the IR stable fixed point (i.e., $\hat{e}^2 \neq 0$), automatically implies the vanishing of $\beta_\kappa$ for arbitrary $\kappa$. Thus, the fixed point structure of the $\beta$ functions (35) and (36) at the IR stable fixed point $\hat{e}^2 = 24\pi/N$ is similar to the one of the $\beta$ functions for the charge neutral system given by Eqs. (26) and (27). Plugging the fixed point $e^2$ into Eqs. (35) and (36) we find that these $\beta$ functions have two nontrivial fixed points, namely,

$$(\hat{U}_*, \hat{V}_*) = \left( \frac{16\pi[32N + (N + 12n)^2]}{(N + 8)(N + 12n^2)}, \frac{(N + 8)(N + 12n^2)}{16\pi} \right), \quad (39)$$

corresponding to the $O(N) \times O(N)$ symmetry regime, while the $O(2N)$ symmetric case,

$$(\hat{U}_*, \hat{V}_*) = \left( \frac{8\pi[32N + (N + 12n)^2]}{(N + 4)(N + 12n^2)}, \frac{(N + 4)(N + 12n^2)}{8\pi} \right), \quad (40)$$

where we have assumed a level $n$ CS term, $\kappa = n/(2\pi)$. An additional fixed point $(\hat{U}_1, \hat{V}_1)$ corresponding to a regime where $K > u$ is obtained for $N > 4$ (recall that $N$ is even), where,

$$\hat{U}_1 = \frac{8\pi[32N + (N + 12n)^2]}{(N + 8)(N + 12n^2)}, \quad (41)$$

$$\hat{V}_1 = (4 - N)\hat{U}_1. \quad (42)$$

Note that for $N = 2$ the fixed point $(\hat{U}_1, \hat{V}_1)$ coincides with the $O(4)$ symmetric one.

The flow diagram in terms of the original couplings $u$ and $K$ is shown in Fig. 4. Interestingly, we see that the $O(4)$ symmetric fixed point occurring for a vanishing anisotropy is IR stable. This implies that for the CS CP1 theory a deconfined critical point occurs in the more symmetric case. The anisotropy fixed point is stable along the line $\hat{u} = \hat{K}$, corresponding to the case of a scalar self-coupling interaction of the form, $|z_1|^4 + |z_2|^4$.

Hence, we arrive at two non-trivial fixed points that govern second-order phase transitions. Clearly, a new
Thus, for a level 1 CS term this yields $\nu^{O(2) \times O(2)} = 49/80 \approx 0.613$. This is nearly the same as the one-loop value $\nu = 5/8$ of the $XY$ universality class.

For the $O(4)$ symmetric criticality we obtain a larger correlation length critical exponent, $\nu^{O(4)} = 2/3$, which at this order is independent of the CS level. Interestingly, the same value is obtained for the limit case of a neutral system.

Finally, we would like to calculate the anomalous dimension $\eta_N$ of the critical magnetization correlation function. Using the relation,

$$\sigma_{\alpha\beta}^a \sigma_{\gamma\delta}^a = 2\delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\beta} \delta_{\gamma\delta}, \quad (46)$$

we obtain that,

$$\mathcal{S}(x) = \langle \mathbf{n}(x) \cdot \mathbf{n}(0) \rangle = 2\langle z_\alpha^*(x)x_\beta(x)z_\alpha(0)x_\beta(0) \rangle - \langle |z_\alpha(x)|^2 |z_\beta(0)|^2 \rangle. \quad (47)$$

Note that in the hard constraint case the second term in the equation above is unity.

The calculation of $\eta_N$ amounts to finding the anomalous dimension of the operator $z_\alpha^*(x) x_\beta(x)$ \[66\]. The anomalous dimension of this operator is one of the eigenvalues occurring in the matrix,

$$[\eta_2] = -N \frac{\hat{U}_\ast}{16\pi} [P] - \left[ \frac{\hat{U}_\ast}{8\pi} - \frac{2\hat{e}_\ast^2}{3\pi(1 + \hat{e}_\ast^2|\kappa|)^2} \right] [I], \quad (48)$$

where $N = 2$ corresponds to the $O(2) \times O(2)$ symmetric case and $N = 4$ to the $O(4)$ symmetric one. The matrix elements of $[P]$ and $[I]$ are given by,

$$[I]_{\alpha\beta,\gamma\delta} = \frac{1}{N} \delta_{\alpha\delta} \delta_{\beta\gamma}, \quad (49)$$

$$[P]_{\alpha\beta,\gamma\delta} = \frac{1}{N} \delta_{\alpha\beta} \delta_{\gamma\delta}, \quad (50)$$

where the indices run from 1 to 4. The trace of $[\eta_2]$ yields $\eta_2 \delta_{\alpha\beta}$, which corresponds to insertions of $|z_\alpha|^2$. In other words, we recover the formula for $1/\nu$ via the well known relation $\eta_2 = 1/\nu - 2$. The insertion of the operator $z_\alpha^*(x) x_\beta(x)$ with $\alpha \neq \beta$ corresponds to the zero eigenvalue of $[P]$, and so we obtain,

$$\bar{\eta}_2 = \frac{\hat{U}_\ast}{8\pi} + \frac{2\hat{e}_\ast^2}{3\pi(1 + \hat{e}_\ast^2|\kappa|)^2}, \quad (51)$$

which is related to $\eta_N$ by the formula $\eta_N = 1 - 2\bar{\eta}_2$. In the $O(2) \times O(2)$ symmetric case we obtain,

$$\eta_N^{O(2) \times O(2)} = \frac{1}{5} \left[ 7 - \frac{48}{(1 + 6n)^2} \right], \quad (52)$$

and similarly for $O(4)$ symmetry,

$$\eta_N^{O(4)} = \frac{4}{3} \left[ 1 - \frac{56}{5(1 + 6n)^2} \right]. \quad (53)$$

For a level 1 CS term we obtain, $\eta_N^{O(2) \times O(2)} = 59/49 \approx 1.2$ and $\eta_N^{O(4)} = 164/147 \approx 1.12$. 

II. CONTROLLING THE RG ANALYSIS

The advantage of the $\epsilon$-expansion is that it allows for a reliable expansion parameter in perturbation theory. Of course, one then is ultimately interested in the $\epsilon = 1$ case, and several mathematical techniques have been used in the past to show that the perturbation series actually converge [66]. There is also the fixed dimension approach by Parisi [63], but this typically applies to scalar field theories without the coupling to a gauge field. The difficulty can be seen in our case, where for $N = 2$, which is the case we are interested in, the RG fixed point $\hat{\epsilon}_*^2 = 12\pi$ is too large. We cannot use the $\epsilon$-expansion to obtain $\hat{\epsilon}_*^2 \sim O(\epsilon)$ in this case because the CS term imposes a fixed dimension $d = 3$ from the outset. Furthermore, thanks to the CS term the Feynman diagram of Fig. 2 vanishes. This diagram is a known obstruction towards reaching a fixed point for the scalar couplings, since it leads to a large contribution in the $\beta$ functions, even within the $\epsilon$-expansion. Only for $N$ sufficiently large the theory can become critical for a nonzero gauge coupling. Hence, fixed dimensionality $d = 3$ is a desirable feature in our case.

Introducing a larger global symmetry group at fixed dimension $d = 3$ provides a way to control the perturbation expansion, since all fixed points behave as $\sim O(1/N)$ for $N$ large. However, let us give an additional argument that even though we take $N = 2$ at the end, the results for universal quantities can be relied upon.

The main source of difficulty for $N = 2$ is the fixed point value for the gauge coupling, which is $\hat{\epsilon}_*^2 = 12\pi$ in this case. However, quite generally, the renormalization of the gauge coupling follows from the self-energy of the gauge field propagator, which is obtained from the vacuum polarization diagram of Fig. 3. This leads to the effective Maxwell Lagrangian,

$$L_M = \frac{1}{2\hat{\epsilon}^2} |1 + \Pi(0)| (\epsilon_{\mu\nu}\partial_\mu a_\nu)^2,$$

where $\Pi(0) = Ne^2/(24\pi m)$ is the vacuum polarization at $p = 0$. From this we read off the dimensionless renormalized gauge coupling,

$$\hat{\epsilon}^2 = \frac{e^2 m^{-1}}{1 + \Pi(0)} = \frac{e^2 m^{-1}}{1 + \frac{N e^2}{24\pi m}}.$$  \hfill (54)

Thus, the $\beta$ function of Eq. (37) is easily obtained by simple differentiation, $\beta_{\hat{\epsilon}^2} = m\hat{\epsilon}^2/dm$ without any need of a series expansion. Furthermore, we note the following two important facts. First, the result obtained in Eq. (55) is the same as the one obtained within an $1/N$ expansion. Second, the fixed point follows from Eq. (55) in two different ways, namely, either by directly letting $m \to 0$, or by taking the limit where the bare (dimensionful) gauge coupling $e^2 \to \infty$. The latter limit highlights the strong coupling character of the theory at $d = 3$. Furthermore, this is the regime of interest to us in the duality analysis.

As far as the couplings $\hat{U}$ and $\hat{V}$ are concerned, $\hat{e}^2$ enters only via the wavefunction renormalization, since the diagram of Fig. 2 vanishes. Since the CS mass also depends on $\hat{e}^2$, the perturbative results in Eqs. (33) and (34) are not jeopardized by the strong-coupling character of the gauge coupling.

Finally, we could, somewhat artificially, make an $\epsilon$-expansion analysis in which we compute Feynman diagrams in $d$ dimensions for the cases where $\epsilon_{\mu\nu\lambda}^{\epsilon}$ does not play any role, while still keeping $d = 3$ in the diagram of Fig. 2, since in this case $\epsilon_{\mu\nu\lambda}$ plays a crucial role. It is worth to carry out this calculation as well, in order to clearly show the need of the fixed dimension approach in this case. In fact, we will show below that while fixed points exist as before, they lead to unphysical values of the critical exponent $\nu$ in the $O(2) \times O(2)$ invariant case.

Most of what we need for this calculation is already available, since we have discussed the $d$-dimensional example in absence of the gauge coupling earlier in the previous section. It remains to discuss the changes in the diagram of Fig. 1. We have,

$$\Sigma(p) = \frac{4e^2 m d^{d-4}}{(4\pi)^{d/2}} \left(1 - \frac{1}{d}\right) \Gamma \left(1 - \frac{d}{2}\right) \times \frac{1 - m/M_0^{d-2}}{1 - M^2/m^2} p^2 + O(p^4),$$

which upon expanding around $d = 4$ yields the wavefunction renormalization at one-loop order,

$$Z = 1 + \frac{3\hat{\epsilon}^2}{8\pi^2\epsilon}.$$  \hfill (57)

Hence, the $\beta$ functions become for $N = 2$,

$$\beta_{\hat{U}} = - (\epsilon + 6\hat{\epsilon}^2)\hat{U} + 5\hat{U}^2 + \hat{V}^2,$$

$$\beta_{\hat{V}} = - (\epsilon + 6\hat{\epsilon}^2)\hat{V} + 2(\hat{V}^2 + 2\hat{U}\hat{V}),$$

$$\beta_{\hat{e}^2} = - \hat{e}^2 + \frac{\hat{e}^4}{3},$$

$$\beta_{\kappa} = \left(\frac{\hat{e}^4}{3} - \epsilon\right)\kappa,$$

where we have performed a rescaling similar to the one described above Eqs. (26) and (27).

Now, if we consider the correlation length exponent for the $O(2) \times O(2)$ case, we obtain to order $\epsilon$,

$$\nu_{O(2) \times O(2)} = \frac{1}{2} + 8\epsilon/3 \approx \frac{1}{2} + \frac{2\epsilon}{3} - \frac{\epsilon^2}{3},$$

and we see after setting $\epsilon = 1$ at the end that $\nu_{O(2) \times O(2)} < 0$ and therefore unphysical. Even if one does not completely adhere to the $\epsilon$-expansion and use Eq. (62) without making the expansion, a result smaller than $1/2$ is obtained after setting $\epsilon = 1$. This is also unphysical, since the critical exponent $\nu$ should be larger than or equal to its mean-field value for a local field theory of this type.
III. SELF-DUALITY

Let us introduce a change of the variables for the gauge fields, \( b_{\mu} = (b_{1\mu} + b_{2\mu})/2 \), \( b_{-\mu} = (b_{1\mu} - b_{2\mu})/2 \). Then, the Eq. (13) of the paper takes the form,

\[
\mathcal{L}_{\text{dual}} = g[(\varepsilon_{\mu\nu\lambda}\partial_{\rho} b_{\lambda})^2 + (\varepsilon_{\mu\nu\lambda}\partial_{\rho} b_{-\lambda})^2] + i2\pi(w_{1\mu} + w_{2\mu})b_{+\mu} + i2\pi(w_{1\mu} - w_{2\mu})b_{-\mu} - \frac{i}{2\kappa}\varepsilon_{\mu\nu\lambda}b_{\nu\lambda}. \tag{63}
\]

To integrate out the gauge fields, we need to find the propagator \( \tilde{D}_{\mu\nu} \), which is the inverse of the tensor,

\[
M_{\mu\nu} = 2g(p^2\delta_{\mu\nu} - p_{\mu}p_{\nu}) + \frac{2}{\alpha}p_{\mu}p_{\nu} + \frac{4}{\kappa}\varepsilon_{\mu\nu\lambda}p_{\lambda}, \tag{64}
\]

with a gauge fixing \( \alpha \). After a straightforward calculation one obtains the propagator \( \tilde{D}_{\mu\nu} \) in the momentum space,

\[
\tilde{D}_{\mu\nu}(p) = \frac{1}{2(2g^2\kappa^2p^2 + 4)} \left( \delta_{\mu\nu} - \frac{2\varepsilon_{\mu\nu\lambda}p_{\lambda}}{kp^2} \right), \tag{65}
\]

where the Landau gauge (\( \alpha = 0 \)) was used and we have dropped a longitudinal part \( \sim p_{\mu}p_{\nu} \), since the zero divergence constraint of the vortex loop variables causes such a term to give a vanishing contribution. Therefore, the effective action for vortex fields in the momentum space is

\[
S_{\text{dual}} = 2\pi^2g^2k^2\int_{p} D_{\mu\nu}(p)(w_{1\mu} + w_{2\mu})(p)(w_{1\mu} + w_{2\mu})(-p) + \frac{\pi^2}{g}\int_{p} (w_{1\mu} - w_{2\mu})(p)(w_{1\mu} - w_{2\mu})(-p). \tag{66}
\]

And in the coordinate space one obtains,

\[
S_{\text{dual}} = 2\pi^2g^2k^2\int_{x} D_{\mu\nu}(x - x')(w_{1\mu} + w_{2\mu})(w_{1\mu}' + w_{2\mu}') + \frac{\pi^2}{g}\int_{x} (w_{1\mu} - w_{2\mu})(w_{1\mu}' - w_{2\mu}'). \tag{67}
\]

We perform explicit calculations with the propagator Eq. (65), the first term in Eq. (66) becomes

\[
2\pi^2g^2k^2\int_{p} D_{\mu\nu}(p)(w_{1\mu} + w_{2\mu})(p)(w_{1\mu} + w_{2\mu})(-p) = 2\pi^2g^2k^2 \left( \int_{p} (w_{1\mu} + w_{2\mu})(p)(w_{1\mu} + w_{2\mu})(-p) \right) - \int_{p} \varepsilon_{\mu\nu\lambda}p_{\lambda}(w_{1\mu} + w_{2\mu})(p)(w_{1\nu} + w_{2\nu})(-p) - \frac{i}{8\pi^2\kappa}\varepsilon_{\mu\nu\lambda}p_{\lambda}(w_{1\mu} + w_{2\mu})(p)(w_{1\nu} + w_{2\nu})(-p), \tag{68}
\]

In the case of \( g^2\kappa^2 \ll 1 \), the last line of the expression simplifies. Let us take a closer look at the integral not taking coefficients into account,

\[
\int_{p} \frac{\varepsilon_{\mu\nu\lambda}p_{\lambda}(w_{1\mu} + w_{2\mu})(p)(w_{1\nu} + w_{2\nu})(-p)}{p^2} = \frac{\int_{p} \varepsilon_{\mu\nu\lambda}p_{\lambda}(w_{1\mu} + w_{2\mu})(p)(w_{1\nu} + w_{2\nu})(y)}{p^2} = \frac{i}{4\pi}\int_{p} \int_{y} e^{i\nu(x-y)}(w_{1\mu} + w_{2\mu})(x)(w_{1\nu} + w_{2\nu})(y), \tag{69}
\]

where we used a Fourier transform and an exponential representation of the \( \delta \)-function. Further calculations lead to

\[
\frac{i}{4\pi}\varepsilon_{\mu\nu\lambda}\int_{y} \int_{x} \frac{(x_{\nu} - y_{\nu})}{|x - y|^3}(w_{1\mu} + w_{2\mu})(x)(w_{1\nu} + w_{2\nu})(y) = -\frac{i}{4\pi}\int_{y} \int_{x} (x_{\nu} - y_{\nu})\partial_{\nu}(v_{1\beta} + v_{2\beta})(x)(w_{1\beta} + w_{2\beta})(y) = i\int d^4x(v_{1\beta} + v_{2\beta})(w_{1\beta} + w_{2\beta}). \tag{70}
\]

Finally, the Eq. (67) takes the form,

\[
S_{\text{dual}} = \frac{\pi}{4g}\int_{x} \int_{x'} e^{-\frac{2\pi^2g^2k^2}{\kappa^2}(w_{1\mu} + w_{2\mu})(x)(w_{1\mu} + w_{2\mu})(x')} |x - x'| + \frac{\pi^2}{g}\int_{x} \int_{x'} (w_{1\mu} - w_{2\mu})(w_{1\mu}' - w_{2\mu}) |x - x'| + \frac{1}{2}\int d^4x(v_{1\beta} + v_{2\beta})(w_{1\beta} + w_{2\beta}). \tag{71}
\]

IV. FLUX ATTACHMENT BOSONIZATION DUALITY

In this section we conider the bosonization duality that we obtain in the scope of a duality web approach [18, 19]. To do so, we first need to write down the field theory for the dual bosonic system obtained in Eq. (13) of the main body of the paper. To this end we introduce complex scalar fields \( \phi_I, I = 1,2 \) yielding a second-quantized representation for the ensemble of vortex loops [23]. This yields the Lagrangian,

\[
\mathcal{L}_{\text{dual}} = \sum_{I=1,2} \left[ |(\partial_{\mu} - ib_{I\mu})\phi_I|^2 + m^2|\phi_I|^2 + \frac{\lambda}{2}|\phi_I|^4 \right] - \frac{i}{8\pi^2\kappa}\varepsilon_{\mu\nu\lambda}(b_{I\mu} + b_{2\mu})(\partial_{\nu}(b_{I\lambda} + b_{2\lambda})), \tag{72}
\]

where we have rescaled gauge fields \( b_{I\mu} \rightarrow b_{I\mu}/(2\pi) \), thus assigning a unit charge to both \( \phi_1 \) and \( \phi_2 \). Following the technique employed in Ref. [19], we awoke the bosonization conjectures,

\[
Z_{fQED}[A]e^{\frac{i}{2}S_{CS}[A]} = Z_{bQED + flux}[A], \tag{73}
\]

\[
Z_{fQED + flux}[A] = Z_{bQED}[A]e^{-S_{CS}[A]}, \tag{74}
\]
where $S_{CS}[A]$ is the action for a level 1 CS term, $A_\mu$ is the background field. The fermionic and bosonic partition functions are, respectively,

$$
Z_{fQED}[A] = \int D\bar{\psi} D\psi e^{-S_{fQED}[A]},
$$

$$
S_{fQED}[A] = \int d^3x \bar{\psi} (\partial - iA_\mu)\psi,
$$

while the flux attachment operates as follows,

$$
Z_{bQED+\text{flux}}[A] = \int D\phi^* D\phi e^{-S_{bQED}[A]},
$$

$$
S_{bQED}[A] = \int d^3x \left[ (\partial_\mu - iA_\mu)\phi \right]^2 + m^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4,
$$

where the BF term is given by,

$$
S_{BF}[a; A] = \frac{i}{2\pi} \int d^3x A_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda.
$$

Now that we have recalled the basic flux attachment dualities (73) and (74), we can derive the duality described in the main text by multiplying both these relations together,

$$
Z_{fQED}[A] e^\frac{i}{\hbar} S_{CS}[A] Z_{fQED+\text{flux}}[A] = Z_{bQED}[A] Z_{bQED}[A] e^{-S_{CS}[A]},
$$

(80)

After we promote the background field $A_\mu$ to a dynamical field $b_\mu$, the left-hand side of the Eq. (80) takes the form,

$$
\int \prod_{I=1,2} Db_I D\bar{\psi}_I D\psi_I e^{-S},
$$

$$
S = \int d^3x \left\{ \sum_{I=1,2} \left[ \bar{\psi}_I (\partial - iA_\mu)\psi_I - \frac{i}{8\pi} b_{1\mu} \epsilon_{\mu\nu\lambda} \partial_\nu b_{1\lambda} \right] - \frac{i}{2\pi} b_{1\mu} \epsilon_{\mu\nu\lambda} \partial_\nu b_{2\lambda} \right\}.
$$

(81)

Integrating out $\psi_2$ generates a level 1/2 CS term with a minus sign. We can integrate out the dynamical field $b_{2\mu}$, which enforces $b_{1\mu} = b_{2\mu}$. Eventually, we can write down the left-hand side of the Eq. (80),

$$
\int Db_D D\bar{\psi} D\psi e^{-\int d^3x \left[ \bar{\psi} (\partial - iA_\mu)\psi + \frac{i}{8\pi} b_{1\mu} \epsilon_{\mu\nu\lambda} \partial_\nu b_{1\lambda} \right]},
$$

(82)

where we have set $\psi_1 \equiv \psi$.

Now, let us write explicitly the right-hand side of the Eq. (80),

$$
\int \prod_{I=1,2} Db_I D\phi^*_I D\phi_I e^{-S},
$$

$$
S = \int d^3x \sum_{I=1,2} \left[ (\partial_\mu - i b_{1\mu})\phi_I \right]^2 + \ldots
$$

$$
+ \frac{i}{4\pi} \epsilon_{\mu\nu\lambda} (b_{1\mu} + b_{2\mu}) \partial_\nu (b_{1\lambda} + b_{2\lambda}),
$$

(83)

where the ellipses represent scalar field self-interactions. This is precisely the time-reversal transformed version of the dual Lagrangian (72) for $\kappa = 1/(2\pi)$. Therefore, we have obtained that our derivation is consistent with the bosonization duality performed via flux attachments to fermions and bosons.