Multifractal Analysis of Some Inhomogeneous Multinomial Measures with Distinct Analytic Olsen’s $b$ and $B$ Functions

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Abstract We construct exact dimensional measures whose support is the whole interval $[0, 1]$ and whose Olsen’s multifractal functions $b$ and $B$ are real analytic and agree at two points only. These measures $\nu$ satisfy an extended multifractal formalism in the sense that, for $\alpha$ in some interval, the Hausdorff dimension of the level sets $X(\alpha)$ of the local Hölder exponent of $\nu$ is the Legendre transform of $b$ whereas their packing dimension is the Legendre transform of $B$. We first construct such measures $\mu$ on a symbolic space. Then we obtain the measures $\nu$ by projecting $\mu$ on $[0, 1]$ after composition with a Gray code.

Keywords Multifractal analysis · Hausdorff dimension · Packing dimension · Inhomogeneous multinomial measures · Olsen’s $b$ and $B$ functions · Gray code

Mathematics Subject Classification 28A80 · 28A78

1 Introduction

The multifractal analysis borrows its methods from the thermodynamic formalism. Given a measure $\mu$ one associates with it a “partition function” of which one takes the thermodynamic limit. One so gets a quantity which can be called “free energy” in analogy with the usual thermodynamic theory. One says that $\mu$ satisfies the multifractal formalism if the Legendre transform of this free energy yields the Hausdorff dimension of the level set of the local Hölder exponent of $\mu$ (sometimes called its singularity strength). This is the essence of the pioneering works [6,8] on this topic.

For the measures we consider in this article, the thermodynamic limit does not exist: the free energy splits into two functions given by the upper and lower limits, and the Legendre transforms of both of these functions have an interpretation in terms of dimensions of the sets of iso-singularities. While in the standard formalism discontinuities of the free energy or...
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one of its derivatives correspond to phase transitions, here we are facing a new phenomenon. It would be of interest to know whether physical systems exhibiting such a behavior exist.

We construct a family of probability measures on $[0, 1]$ such that any of its element $\mu$ has the following properties:

- it has full support and is exact dimensional,
- its Olsen’s multifractal dimensional functions $b$ and $B$ are real analytic,
- $b(q) < B(q)$ if $q \notin \{0, 1\}$,
- the graphs of $b$ and $B$ are tangent for $q = 0$ or $q = 1$,
- it obeys the refined multifractal formalism. This means that the Hausdorff and packing dimensions of the level sets of local Hölder exponents of $\mu$ are the Legendre transforms of the Olsen’s $b$ and $B$ functions respectively.

The motivations of this work come from several sources.

Ben Nasr et al. [4] constructed measures whose functions $b$ and $B$ coincide at one or two points only. Thus such measures can fulfill the multifractal formalism at one or two points only. They give two constructions. The first one provides $b$ and $B$ functions with Lipschitz regularity. The second one provides real analytic functions, but in this case the support of the measure is a Cantor set of Hausdorff dimension less than 1. They use inhomogeneous Bernoulli measures. This is the idea that we refine to get our results.

Ben Nasr and Peyrière [5], revisited the first example in [4]. Indeed in [4] no interpretation of Olsen’s function was given in terms of dimensions. Then it was proven in [5] that, for a certain range of $\alpha$, the Hausdorff dimension of the set $X(\alpha)$, where the local Hölder exponent assumes the value $\alpha$, is given by the value of the Legendre transform of $b$ at $\alpha$ whereas its packing dimension is the value of the Legendre transform of $B$ at $\alpha$. So, according to the terminology later introduced by Barral [1], these measures satisfy the refined multifractal formalism.

It is well known that the Bernoulli measures obey the multifractal formalism with Olsen’s functions $b = B$. Therefore, to obtain the inequality $B > b$, Ben Nasr et al. [4] step a little out of this family of measures and introduce some inhomogeneous properties when constructing the measures. Let us explain this on the symbolic space $\{0, 1\}^N$ instead of the interval $[0, 1]$. Two different numbers $a, a' \in (0, 1)$ are given as well as an increasing sequence $T_k$ of positive integers such that $\lim_{k \to \infty} T_{k+1}/T_k = \infty$. Then consider the measure $\mu$ such that for any $w = w_1 \cdots w_n \in \{0, 1\}^n$,

$$\mu([w]) = \prod_{j=1}^{n} \left( p_j^{1-w_j} (1 - p_j)^w_j \right),$$  

where $[w]$ stands for the cylinder defined by $w$, and, $p_j = a$ if $T_{2k - 1} \leq j < T_{2k}$, and $p_j = a'$ if $T_{2k} \leq j < T_{2k+1}$ for some $k$. Then, if we set

$$\theta_p(q) = \log_2 (p^q + (1 - p)^q),$$

it is shown in [4] that

$$b(q) = \min \{ \theta_a(q), \theta_{a'}(q) \} \quad \text{and} \quad B(q) = \max \{ \theta_a(q), \theta_{a'}(q) \}.$$

Batakis and Testud [3] also consider measures of Eq. (1). In contrast with [4], they show that if $p_n$ has a limit, then the measure $\mu$ obeys the usual multifractal formalism. Also, the authors in [3] showed the existence of a probability measure presenting a dense set of phase transitions on $(1, +\infty)$. 

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During the completion of the present article, we just happen to know the work by Barral [1] in which he proves that, given two functions fulfilling fairly general conditions, there exists a compactly supported, positive, and finite Borel measure $\mu$ on $\mathbb{R}^d$ whose $b$ and $B$ functions are just the two given functions. Moreover, this measure is exact dimensional and obeys the refined multifractal formalism. Nevertheless, in contrast to our results, this measure is not fully supported (it is always supported on a Cantor set). Also, our construction, although less general, is much simpler.

In this article, we consider a generalization of measures of Eq. (1). We use alphabets consisting of more than 2 letters. And then the two groups of parameters $\{a, 1 - a\}$ and $\{a', 1 - a'\}$ are replaced by two different groups of non negative numbers $(a_j)_{1 \leq j \leq n_a}$ and $(b_j)_{1 \leq j \leq n_b}$ such that $\sum a_j = \sum b_j = 1$ (the integers $n_a$ and $n_b$ need not be different). We keep the same time sequence $\{T_k\}$ as in [4]. The corresponding measure (see Eq. (3)) lives in what we call a “mixed symbolic space”. So in our case, since we choose to take two different groups of parameters, the condition in [3] ensuring the validity of the multifractal formalism is not satisfied. In fact, the sequence $(\tau_n)$ (which will be defined later) has a upper limit $B$ and a lower limit $b$, which differ except at two points, so the sequence does not converge for $q \notin \{0, 1\}$.

Set $\theta_a(q) = \log_{n_a} \sum a_j^q$ and define $\theta_b(q)$ similarly. Then

$$b(q) = \min\{\theta_a(q), \theta_b(q)\} \text{ and } B(q) = \max\{\theta_a(q), \theta_b(q)\}.$$

Since we wish that both functions $b$ and $B$ be analytic, we have to carefully investigate the way the graphs of $\theta_a$ and $\theta_b$ intersect. Indeed we show, with the help of the generalized Descartes’ rule [9], that we can choose the parameters so that these two graphs have two common points only, where they are tangent with intersections of even orders. With this choice, the corresponding inhomogeneous multinomial measure $\mu$ is shown to fulfill the refined multifractal formalism. With the help of a generalized Gray code, we obtain, by projecting $\mu$ on $[0, 1]$, a doubling measure which has the desired properties.

This article is organized as follows. In Sect. 2, we introduce the basic notations, definitions and some known consequences. In Sect. 3, we define the inhomogeneous multinomial measures on symbolic spaces and compute Olsen’s $b$ and $B$ functions. Then in Sect. 4 we present the measure with analytic Olsen’s functions and in Sect. 5 we prove that this measure obeys the refined multifractal formalism. In the final Section, we consider the projection, after composition with a suitable Gray code, of the measure onto the real line and obtain the same results as on the symbolic space.

2 Notations and Definitions

2.1 The Olsen Measures

We deal with a metric space $X$ possessing the Besicovitch property:

There exists a constant $C_B \in \mathbb{N}$ such that, given any bounded subset $\{x_i\}_{i \in I} \subseteq X$ and any collection $\{B(x_i, r_i)\}_{i \in I}$ of balls in $X$, one can extract from it $C_B$ countable families $\{\{B(x_{j,k}, r_{j,k})\}_{k \geq 1}\}_{1 \leq j \leq C_B}$ so that

- $\bigcup_{j,k} B(x_{j,k}, r_{j,k}) \supseteq \{x_i\}_{i \in I}$,
- for any $j$ and $k \neq k'$, $B(x_{j,k}, r_{j,k}) \cap B(x_{j,k'}, r_{j,k'}) = \emptyset$.

This definition comes from a theorem by Besicovitch asserting that Euclidean spaces have this property.
Let $\mu$ be a Borel probability measure on $\mathbb{X}$. If $E$ is a nonempty subset of $\mathbb{X}$ and if $q, t \in \mathbb{R}$ and $\delta > 0$, we introduce the quantities:

$$\mathcal{H}_{\mu, \delta}^{q,t}(E) = \inf \left\{ \sum_{i}^{n} r_i^t \mu(B(x_i, r_i)) : (B(x_i, r_i))_i \text{ centered } \delta - \text{ covering of } E \right\},$$

$$\mathcal{H}_{\mu}^{q,t}(E) = \sup_{\delta > 0} \mathcal{H}_{\mu, \delta}^{q,t}(E),$$

$$\mathcal{P}_{\mu, \delta}^{q,t}(E) = \sup_{F \subseteq E} \mathcal{P}_{\mu, \delta}^{q,t}(F);$$

and

$$\mathcal{P}_{\mu}^{q,t}(E) = \inf_{\delta > 0} \mathcal{P}_{\mu, \delta}^{q,t}(E),$$

$$\mathcal{P}_{\mu}^{q,t}(E) = \inf_{E \subseteq \bigcup E_i} \sum_{i} \mathcal{P}_{\mu}^{q,t}(E_i).$$

The star means that we omit in the summation the terms which are obviously infinite (i.e. zero raised to a negative power). $B(x_i, r_i)$ stands for the open ball centered at point $x_i \in X$ with radius $r_i$, and we denote by $B_i$ for short in the context. Let $(B(x_i, r_i))_i$ be a countable family of balls of $\mathbb{X}$. When we say $(B(x_i, r_i))_i$ is a $\delta$-covering of $E$, we mean that $\bigcup B(x_i, r_i) \supseteq E$ and for any $i, r_i < \delta$. When we say $(B(x_i, r_i))_i$ is a centered $\delta$-covering of $E$, we mean it is not only a $\delta$-covering of $E$, but also for any $i, x_i \in E$. At last, when we say $(B(x_i, r_i))_i$ is a $\delta$-packing of $E$, we mean that for any $i, x_i \in E, r_i < \delta$ and for any $j \neq k, B(x_j, r_j) \cap B(x_k, r_k) = \emptyset$.

We can see that respectively, the functions $\mathcal{H}_{\mu}^{q,t}$, $\mathcal{P}_{\mu}^{q,t}$ and $\mathcal{P}_{\mu}^{q,t}$ are multifractal extensions of the centered Hausdorff measure $\mathcal{H}$, the packing measure $\mathcal{P}$, and the packing premeasure $\mathcal{P}$.

### 2.2 The Olsen’s Functions

The functions $\mathcal{H}_{\mu}^{q,t}$, $\mathcal{P}_{\mu}^{q,t}$ and $\mathcal{P}_{\mu}^{q,t}$ induce dimensions to each subset $E$ of $\mathbb{X}$. They are defined by

$$b_{\mu,E}(q) = \sup \left\{ s : \mathcal{H}_{\mu}^{q,s}(E) = \infty \right\} = \inf \left\{ s : \mathcal{H}_{\mu}^{q,s}(E) = 0 \right\},$$

$$B_{\mu,E}(q) = \sup \left\{ s : \mathcal{P}_{\mu}^{q,s}(E) = \infty \right\} = \inf \left\{ s : \mathcal{P}_{\mu}^{q,s}(E) = 0 \right\},$$

$$\tau_{\mu,E}(q) = \sup \left\{ s : \mathcal{P}_{\mu}^{q,s}(E) = \infty \right\} = \inf \left\{ s : \mathcal{P}_{\mu}^{q,s}(E) = 0 \right\}.$$

They are multifractal extensions of the Hausdorff dimension $\dim E$, the packing dimension $\dim E$, and the packing predimension $\Delta E$.

Denote by $S_{\mu}$ the support of $\mu$. For simplicity, we will write $b_{\mu} = b_{\mu, S_{\mu}}$, $B_{\mu} = B_{\mu, S_{\mu}}$, $\tau_{\mu} = \tau_{\mu, S_{\mu}}$. And if the measure $\mu$ is clear in the context, we will omit $\mu$ and write $b$, $B$, $\tau$ respectively. They satisfy the following properties (see [10]):

- $b(0) = \dim S_{\mu}, B(0) = \dim S_{\mu}, \tau(0) = \Delta S_{\mu},$
- $b(1) = B(1) = \tau(1) = 0.0$
- $b \leq B \leq \tau$
- $b$ is decreasing and $B$ and $\tau$ are convex and decreasing.
There is another way to describe $\tau_{\mu,E}$. Fix $\lambda < 1$ and define:

$$\tilde{\tau}_{\mu,E}(q) = \limsup_{\delta \to 0} -\frac{1}{\log \delta} \log \sup \left\{ \sum_i r_i^q \mu(B_i) : (B_i)_i \text{ packing of } E \text{ with } \lambda \delta < r_i \leq \delta \right\},$$

$$\tilde{\tau}_{\mu,E}(q) = \inf \left\{ s : \tilde{\tau}_{\mu,E}(s) = \infty \right\}.$$

**Lemma 1** (see [5]) $\tilde{\tau}_{\mu,E} = \tau_{\mu,E}$. So for any $\lambda < 1$, one has

$$\tau_{\mu,E}(q) = \limsup_{\delta \to 0} -\frac{1}{\log \delta} \log \sup \left\{ \sum_i r_i^q \mu(B_i) : (B_i)_i \text{ packing of } E \text{ with } \lambda \delta < r_i \leq \delta \right\}.$$

### 2.3 The Multifractal Formalism

#### 2.3.1 Level Sets of Local Hölder Exponents

Let $\mu$ be a Borel measure on $\mathbb{X}$. For $x \in \mathbb{X}$, we define the local dimensions or local Hölder exponents of the measure $\mu$ at point $x$ by

$$\alpha_{\mu}(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \alpha_{\mu}(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

**Definition 1** We say that the measure $\mu$ is exact dimensional if there exists a constant $\delta$ such that for $\mu-$ almost every $x \in \mathbb{X}$, $\alpha_{\mu}(x) = \alpha_{\mu}(x) = \delta$.

Then for $\alpha, \beta \in \mathbb{R}$, we introduce the sets:

$$X_{\mu}(\alpha) = \left\{ x \in S_{\mu} : \alpha_{\mu}(x) \leq \alpha \right\},$$

$$X_{\mu}(\alpha) = \left\{ x \in S_{\mu} : \alpha_{\mu}(x) \geq \alpha \right\},$$

$$X_{\mu}(\alpha) = X_{\mu}(\alpha) \cap X_{\mu}(\alpha).$$

As previously, we will omit $\mu$ if the measure is clear.

#### 2.3.2 Some Consequences

As is known, the multifractal formalism aims at giving expressions of the dimension of the level sets of local Hölder exponents of measure $\mu$ in terms of the Legendre transform of some free energy function (see [6, 8]).

Let $f^*(x) = \inf_y (xy + f(y))$ denote the Legendre transform of the function $f$. Olsen proves the following general estimation, showing that the Legendre transforms of the dimension functions $b$ and $B$ are upper bounds of the dimensions of level sets.

**Theorem 2** (see [10]) Let $\mu$ be a probability measure on $\mathbb{X}$. Define $\underline{a} = \sup_{q > 0} -\frac{b(q)}{q}$ and $\overline{a} = \inf_{q < 0} -\frac{b(q)}{q}$. For all $\alpha \in (\underline{a}, \overline{a})$, we have

$$\dim X(\alpha) \leq b^*(\alpha),$$

$$\dim X(\alpha) \leq B^*(\alpha).$$
**Definition 2** Suppose that $B'(q)$ exists. We say that the measure $\mu$ obeys the multifractal formalism at point $q$, if

$$\dim X(-B'(q)) = \text{Dim} X(-B'(q)) = b^*(B'(q)) = B^*(-B'(q)).$$

And we say that the measure $\mu$ obeys the refined multifractal formalism at point $q$, if

$$\dim X(-B'(q)) = b^*(B'(q)),$$
$$\text{Dim} X(-B'(q)) = B^*(B'(q)).$$

It always needs some extra conditions to obtain a lower bound for the dimensions of level sets.

**Lemma 3** (see [5]) Let $\mu, \nu$ be two probability measures on $X$. Fix $\lambda < 1$. Set

$$\varphi(x) = \limsup_{\delta \to 0} -\frac{1}{\log \delta} \log \sup \left\{ \sum_{i} \mu(B_i^x) \nu(B_i^y) : (B_i^x) \text{ packing of } S_\mu \text{ with } \lambda \delta < r_i \leq \delta \right\}.$$ 

Assume that $\varphi(0) = 0$, $\nu(S_\mu) > 0$, and that $\varphi'(0)$ exists. Let $E = X_\mu(-\varphi'(0))$, then one has

$$\dim E \geq \text{ess sup} \text{ lim inf}_{x \in E, \nu} \text{ lim sup}_{r \to 0} \frac{\log \nu(B(x, r))}{\log r},$$
$$\text{Dim} E \geq \text{ess sup} \text{ lim sup}_{x \in E, \nu} \text{ lim sup}_{r \to 0} \frac{\log \nu(B(x, r))}{\log r},$$

where

$$\text{ess sup} f(x) = \inf \{ t \in \mathbb{R} : f(x) \leq t, \text{ for } \nu - \text{ almost every } x \in E \}.$$ 

Ben Nasr et al. give a sufficient condition and a necessary condition for a valid multifractal formalism, showing that the knowledge of the so-called Gibbs measure is quite unnecessary. That is

**Theorem 4** (see [4, 5]) Let $\mu$ be a probability measure on $X$ and $q \in \mathbb{R}$. Suppose that $B'(q)$ exists.

(i) If $\mathcal{H}_\mu^q, B(q)(S_\mu) > 0$, then

$$\dim X(-B'(q)) = \text{Dim} X(-B'(q)) = b^*(B'(q)) = B^*(-B'(q)).$$

(ii) Conversely, if $\dim X(-B'(q)) \geq b^*(B'(q))$, then $b(q) = B(q)$.

From the second part, when $B'(q)$ exists, $b(q) = B(q)$, known as the Taylor regularity condition, is the necessary condition for a valid multifractal formalism.

**Remark 1** In [11], Peyrière developed a formalism in order to perform the multifractal analysis of vector valued functions of balls in a metric space. In particular, he gave a generalization of Theorem 4.
3 Some Measures on Symbolic Spaces

3.1 The Symbolic Spaces

Let $c \geq 2$, $\mathcal{A} = \{0, 1, \ldots, c - 1\}$. We consider $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$, the set of all finite words on the $c$-letter alphabet $\mathcal{A}$.

If $w = \varepsilon_1 \cdots \varepsilon_n$ and $v = \varepsilon_{n+1} \cdots \varepsilon_{n+p}$, denote by $w \cdot v$ (or simply by $wv$ if it is not ambiguous) the word $\varepsilon_1 \cdots \varepsilon_{n+p}$. With this operation, $\mathcal{A}^*$ is a monoid whose identity element is the empty word $\varepsilon$. If a word $v$ is a prefix of the word $w$, we write $v \prec w$. This defines an order on $\mathcal{A}^*$ and endowed with this order, $\mathcal{A}^*$ becomes a tree whose root is $\varepsilon$. At last, the length of a word $w$ is denoted by $|w|$. If $w$ and $v$ are two words, $w \wedge v$ stands for their largest common prefix. It is well known that the function $d(w, v) = c - |w \wedge v|$ defines an ultrametric distance on $\mathcal{A}^*$.

The completion of $(\mathcal{A}^*, d)$ is a compact space which is the disjoint union of $\mathcal{A}^*$ and $\partial \mathcal{A}^*$, whose elements can be viewed as infinite words. Each finite word $w \in \mathcal{A}^*$ defines a cylinder $\left[ w \right] = \{ x \in \partial \mathcal{A}^* | w \prec x \}$, which can also be viewed as a ball. It is clear that the symbolic space $\partial \mathcal{A}^*$ has the Besicovitch property. For a Borel measure $\mu$ on $\partial \mathcal{A}^*$, we simply write $\mu(\left[ w \right]) = \mu(w)$. Thus we identify the Borel measure $\mu$ on $\partial \mathcal{A}^*$ with a mapping from $\mathcal{A}^*$ to $[0, +\infty]$ so that for any $w \in \mathcal{A}^*$,

$$\mu(w) = \sum_{j \in \mathcal{A}} \mu(wj).$$

Since the diameters of balls in $\partial \mathcal{A}^*$ are $c^{-n}$, one can compute the function $\tau$ in the following way according to Lemma 1. That is

$$\sum_{w \in \mathcal{A}^n} \mu(w)^q = c^n \tau_n(q),$$

(2)

$$\tau(q) = \limsup_{n \to \infty} \tau_n(q).$$

Also, we denote

$$\tau(q) = \liminf_{n \to \infty} \tau_n(q).$$

Now we define the mixed symbolic spaces. Let $c_1, c_2 \geq 2$, $\mathcal{A}_1 = \{0, 1, \ldots, c_1 - 1\}$, $\mathcal{A}_2 = \{0, 1, \ldots, c_2 - 1\}$ be two alphabets. Let $(T_k)$ be a sequence of integers such that

$$T_1 = 1, T_k < T_{k+1} \quad \text{and} \quad \lim_{k \to \infty} T_{k+1}/T_k = +\infty.$$ 

Consider the set of infinite words

$$\partial \mathcal{A}_{1,2}^* = \mathcal{A}_1^{T_2 - T_1} \mathcal{A}_2^{T_3 - T_2} \cdots = \prod_{j} X_j,$$

where

- if $T_{2k-1} \leq j < T_{2k}$ for some $k$, $X_j = \mathcal{A}_1$,
- if $T_{2k} \leq j < T_{2k+1}$ for some $k$, $X_j = \mathcal{A}_2$.

We call $\partial \mathcal{A}_{1,2}^*$ the mixed symbolic space with respect to $(\mathcal{A}_1, \mathcal{A}_2, (T_k))$. Let $N_n$ be the number of integers $j \leq n$ such that $X_j = \mathcal{A}_1$. We can immediately get that

$$\liminf_{n \to \infty} \frac{N_n}{n} = 0.$$
and
\[
\limsup_{n \to \infty} \frac{N_n}{n} = 1.
\]

For any two different elements \(w, v \in \partial \mathcal{A}^*_{1,2}\) with \(|w \wedge v| = n\), we define that \(d(w, v) = c_1^{-N_n} c_2^{-(n-N_n)}\). As previously, this defines an ultrametric distance and one checks that \(\partial \mathcal{A}^*_{1,2}\) also has the Besicovitch property.

One sees that when \(c_1 = c_2 = c\), the mixed symbolic space becomes ordinary symbolic space.

### 3.2 Inhomogeneous Multinomial Measures

In [4], the authors presented a measure which has an analytic function \(B\) and a linear function \(b\), satisfying that the graph of \(B\) is tangent to the graph of \(b\) at point \((1, 0)\), and that the support of the measure is a Cantor set with Hausdorff dimension less than 1. Here, we wish to show that there exists a measure \(\mu\) with full support, such that the Olsen’s functions \(B\) and \(b\) are both analytic and their graphs differ except at two points where they are tangent; what is more, the measure \(\mu\) obeys the refined multifractal formalism. We remark here that it is natural to ask for two intersections of these two graphs of \(B\) and \(b\), since \((1, 0)\) is already one, and \((0, 1)\) is going to be the other one once noticing that the measure is fully supported.

We first work on the symbolic spaces and begin with the following result, giving birth to the so-called inhomogeneous multinomial measures.

**Theorem 5** Let \(\mathcal{A}_1 = \{0, 1, \ldots, c_1 - 1\}, \mathcal{A}_2 = \{0, 1, \ldots, c_2 - 1\}\) and let \((T_k)\) be a sequence of integers such that

\[
T_1 = 1, T_k < T_{k+1} \quad \text{and} \quad \lim_{k \to \infty} T_{k+1}/T_k = +\infty.
\]

Let \(a_i, b_j \in (0, 1)\) and \(a_1 + \cdots + a_{c_1} = b_1 + \cdots + b_{c_2} = 1\). There exists a probability measure \(\mu\) on \(\partial \mathcal{A}^*_{1,2}\) such that for every \(q \in \mathbb{R}\),

\[
B(q) = \sup \{\log_{\mathcal{A}_1}(a_1^q + \cdots + a_{c_1}^q), \log_{\mathcal{A}_2}(b_1^q + \cdots + b_{c_2}^q)\},
\]

\[
b(q) = \inf \{\log_{\mathcal{A}_1}(a_1^q + \cdots + a_{c_1}^q), \log_{\mathcal{A}_2}(b_1^q + \cdots + b_{c_2}^q)\}.
\]

To avoid tedious notations, we write the proof with \(c_1 = c_2 = 3\). The reader will realize that the general case can be handled with minor modifications.

Now \(\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A} = \{0, 1, 2\}\). We define the measure \(\mu\) on \(\partial \mathcal{A}^*\) such that for every cylinder \([\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n]\), one has

\[
\mu(\varepsilon_1 \cdots \varepsilon_n) = \prod_{j=1}^{n} p_j,
\]

where

- if \(T_{2k-1} \leq j < T_{2k}\) for some \(k\), \(p_j = a_{\varepsilon_{j+1}}\),
- if \(T_{2k} \leq j < T_{2k+1}\) for some \(k\), \(p_j = b_{\varepsilon_{j+1}}\).

To prove the theorem, we first compute the \(\tau\) function with respect to the measure \(\mu\). Notice that \(N_n\) is the number of integers \(j \leq n\) such that \(p_j \in \{a_1, a_2, a_3\}\), we can easily deduce from Eq. (2) that

\[
\tau_n(q) = \frac{N_n}{n} \log_3(a_1^q + a_2^q + a_3^q) + \left(1 - \frac{N_n}{n}\right) \log_3(b_1^q + b_2^q + b_3^q).
\]
By using that \( \liminf_{n \to \infty} \frac{N_n}{n} = 0 \) and \( \limsup_{n \to \infty} \frac{N_n}{n} = 1 \), we can conclude that
\[
\tau(q) = \sup \{ \log_3(a_1^q + a_2^q + a_3^q), \log_3(b_1^q + b_2^q + b_3^q) \},
\]
\[
\overline{\tau}(q) = \inf \{ \log_3(a_1^q + a_2^q + a_3^q), \log_3(b_1^q + b_2^q + b_3^q) \}.
\]

In [3], the authors consider the multifractal analysis of inhomogeneous Bernoulli products, which are also known as coin tossing measures. They prove that the multifractal formalism for such measures holds if the sequence \( (\tau_n) \) converges. However, we see from above that in our case, the sequence \( (\tau_n) \) does not converge for \( q \notin \{0, 1\} \).

**Lemma 6** Let \( \mathcal{A} = \{0, 1, 2\}, q \in \mathbb{R} \). We can construct a probability measure \( \nu \) on \( \partial \mathcal{A}^n \) and a subsequence of integers \( (n_k)_{k \geq 1} \), such that
\[
\nu(w) \leq \mu(w)^q 3^{-n_k(q)}, \text{ if } w \in \mathcal{A}^n,
\]
and for every \( \varepsilon > 0 \),
\[
\nu(w) \leq \mu(w)^q 3^{-n_k(\tau(q) - \varepsilon)}, \text{ if } w \in \mathcal{A}^{n_k} \text{ with } k \text{ large}.
\]

**Proof** We define a mapping \( \nu \) from \( \mathcal{A}^n \) to \( [0, +\infty] \) such that for any \( w \in \mathcal{A}^n \),
\[
\nu(w) = \frac{\mu(w)^q}{\sum_{z \in \mathcal{A}^n} \mu(z)^q} \frac{\mu(w)^q}{3^{n_k(\tau(q) - \varepsilon)}}.
\]

Then it is easy to see that \( \nu \) is a probability measure on \( \partial \mathcal{A}^n \). Let \( (n_k)_{k \geq 1} \) be a subsequence such that \( \tau(q) = \lim_{k \to +\infty} \tau_{n_k}(q) \).

On one hand, observing that \( \overline{\tau}(q) \leq \tau_n(q) \), we obtain that
\[
\forall n \geq 1, \forall w \in \mathcal{A}^n, \nu(w) \leq \mu(w)^q 3^{-n_k(\tau(q) - \varepsilon)}.
\]

On the other hand, if \( \varepsilon > 0 \) and if \( k \) is sufficiently large, we have \( \tau(q) - \varepsilon \leq \tau_{n_k}(q) \). Thus for any \( w \in \mathcal{A}^{n_k} \), we conclude that
\[
\nu(w) \leq \mu(w)^q 3^{-n_k(\tau(q) - \varepsilon)}.
\]

So we finish the proof. \( \square \)

**Lemma 7** For any fixed \( q \), one has
\[
\mathcal{K}^{q, \tau(q) - \varepsilon}(S_\mu) > 0, \text{ for every } \varepsilon > 0,
\]
\[
\mathcal{K}^{q, \overline{\tau}(q) + \varepsilon}(S_\mu) < +\infty, \text{ for every } \varepsilon > 0,
\]
\[
\mathcal{K}^{q, \tau(q)}(S_\mu) > 0.
\]

**Proof** We mainly apply Lemma 6. To prove the first inequality, one takes any family of \( \{E_i\} \) such that \( S_\mu = \cup E_i \) and for each \( i \) one computes \( \mathcal{K}^{q, \tau(q) - \varepsilon}(E_i) \).

For any \( \delta > 0 \), for any \( x \in S_\mu \), there exists an integer \( n_x \) and a word \( w_x \in \mathcal{A}^{n_x} \) such that \( w_x < x, 3^{-n_x} < \delta \) and \( \nu(w_x) \leq \mu(w_x)^q 3^{-n_x(\tau(q) - \varepsilon)} \). When we identify a finite word with a cylinder as well as a ball, by Besicovitch property, we can extract from \( \{w_x\}_{x \in E_i} \) \( C_B \) countable families \( \{w_{j,k}\}_{1 \leq j \leq C_B, k \geq 1} \) such that \( \cup_{j,k} w_{j,k} \supseteq E_i \) and for any \( j \), \( \{w_{j,k}\}_{k \geq 1} \) is a \( \delta \)-packing of \( E_i \).

Then one gets
\[
v^*(E_i) \leq \sum_{j,k} v^*(w_{j,k}) \leq \sum_{j,k} v(w_{j,k}) \leq \sum_{j,k} \mu(w_{j,k})^q 3^{-|w_{j,k}|(\tau(q) - \varepsilon)},
\]
\( \square \) Springer
where $v^*$ stands for the outer measure associated to $v$, meaning that for any subset $A \subseteq S_\mu$,

$$v^*(A) = \inf \left\{ \sum_j v(A_j) : A \subseteq \bigcup_j A_j, \ A_j \subseteq S_\mu \text{ is a Borel set} \right\}.$$ 

So there exists $j$ such that \(\sum_k \mu(w_{j,k})q^{3-|w_{j,k}|(\tau(q) - \varepsilon)} \geq \frac{1}{C_B} v^*(E_i)\). Thus

$$\sum_i \mathcal{H}^{q,\tau(q) - \varepsilon}_\mu(E_i) \geq \frac{1}{C_B} \sum_i v^*(E_i) \geq \frac{1}{C_B} v^*(S_\mu),$$

from which it follows

$$\mathcal{H}^{q,\tau(q) - \varepsilon}_\mu(S_\mu) \geq \frac{1}{C_B} v^*(S_\mu) > 0.$$ 

To prove the second inequality, one notices that for $\varepsilon > 0$, there exists a subsequence $\{n_k\}$ such that $\tau_{n_k}(q) < \tau(q) + \varepsilon$, for every $k \geq 1$. Take any subset $F \subseteq S_\mu$, and we choose the natural centered $3^{-n_k}$-covering of $F$, which is a set of elements belonging in $\mathcal{A}^{n_k}$. Now

$$\mathcal{H}^{q,\tau(q) + \varepsilon}_\mu,F \leq \sum_{w \in \mathcal{A}^{n_k}} \mu(w)q^{3-n_k\tau_{n_k}(q) + \varepsilon} \leq \sum_{w \in \mathcal{A}^{n_k}} \mu(w)q^{3-n_k\tau_{n_k}(q)} = 1,$$

which means

$$\mathcal{H}^{q,\tau(q) + \varepsilon}_\mu,F \leq 1,$$

and

$$\mathcal{H}^{q,\tau(q) + \varepsilon}_\mu(S_\mu) \leq 1.$$ 

To prove the last inequality, it is sufficient to show $\mathcal{H}^{q,\tau(q)}_\mu(S_\mu) > 0$. For any $\delta$-covering $\{w_i\}_i$ of $S_\mu$, we have

$$\sum_i \mu(w_i)q^{3-|w_i|\tau(q)} \geq \sum_i v(w_i) \geq v(S_\mu).$$

This implies

$$\mathcal{H}^{q,\tau(q)}_\mu(S_\mu) \geq v(S_\mu),$$

which yields

$$\mathcal{H}^{q,\tau(q)}_\mu(S_\mu) > 0.$$ 

So, the proof is finished.

Remark 2 In fact, here one easily checks that in the symbolic space, $\mathcal{H}$ and $\mathcal{H}^{\delta}$ are the same. So in the proof of the second inequality, one does not need to introduce the subset $F$.

Now we come back to compute the functions $B$ and $b$. To obtain the equalities $B(q) = \tau(q)$, $b(q) = \tau(q)$, it is now sufficient to prove that $\tau(q) \leq B(q)$, $\tau(q) \leq b(q)$ and $\tau(q) \geq b(q)$, which are just consequences of Lemma 7.
So Theorem 5 has been done for \( c_1 = c_2 = 3 \). In general case, we can compute by Eq. (2)

\[
\tau_n(q) = \frac{\frac{N_n}{n} \log \left( a_1^q + \cdots + a_{c_1}^q \right)}{\frac{N_n}{n} \log c_1 + \left( 1 - \frac{N_n}{n} \right) \log c_2} + \frac{\left( 1 - \frac{N_n}{n} \right) \log \left( b_1^q + \cdots + b_{c_2}^q \right)}{\frac{N_n}{n} \log c_1 + \left( 1 - \frac{N_n}{n} \right) \log c_2},
\]

which implies

\[
\tau(q) = \sup \left\{ \log_{c_1} \left( a_1^q + \cdots + a_{c_1}^q \right), \log_{c_2} \left( b_1^q + \cdots + b_{c_2}^q \right) \right\},
\]

\[
\tau(q) = \inf \left\{ \log_{c_1} \left( a_1^q + \cdots + a_{c_1}^q \right), \log_{c_2} \left( b_1^q + \cdots + b_{c_2}^q \right) \right\}.
\]

Then we use the very same method as above, but replacing the metric \( 3^{-n} \) with \( c_1^{-N_n} c_2^{-(n-N_n)} \), and one checks that Theorem 5 is valid.

### 4 Measures with Analytic Olsen’s Functions

The purpose of this section is to find a measure whose Olsen’s functions \( B \) and \( b \) both are analytic and their graphs are tangent to each other at two special points, which means, the graphs of the two functions have four intersections, counted with their orders. So we recall some results about the number of zeros of generalized Dirichlet polynomial by G.J.O. Jameson.

**Definition 3** Let \( f : \mathbb{R} \to \mathbb{R} \) be an analytic function. \( x_0 \) is called a zero of \( f \) of order \( k \) \((k \geq 0)\) if

\[
f(x_0) = f'(x_0) = \cdots = f^{(k-1)}(x_0) = 0, \text{ and } f^{(k)}(x_0) \neq 0.
\]

Let \( f_1, f_2 : \mathbb{R} \to \mathbb{R} \) be two analytic functions. \((x_0, y_0)\) is called an intersection of the graphs of \( f_1 \) and \( f_2 \), of order \( k \) \((k \geq 0)\), if \( x_0 \) is the zero of function \((f_1 - f_2)\) of order \( k \), and \( y_0 = f_1(x_0) \).

**Definition 4** A (generalized) Dirichlet polynomial is a function of the form

\[
F(x) = \sum_{j=1}^{n} a_j e^{p_j x}, \quad x \in \mathbb{R},
\]

where the \( p_j \) can be any real numbers (listed in descending order).

The length of a Dirichlet polynomial is the number of non-zero terms in its defining expression.

Among Dirichlet polynomials, a special case is when each \( a_j \) is either 1 or -1, with equally many of each occurring. We call this type bipartite.

With the same notations as above, Jameson proves the next theorem:

**Theorem 8** (see [9]) If \((a_j)\) is bipartite of length \( 2n \), then the number of zeros of \( F \) (counted with their orders) is not greater than \( n \).

**Corollary 9** The Olsen’s functions \( B \) and \( b \) of the measure \( \mu \) in Theorem 5, when \( c_1 = c_2 = 3 \), cannot be analytic.
Proof Denote by

\[ \theta_1(q) = \log_3(a_1^q + a_2^q + a_3^q), \]
\[ \theta_2(q) = \log_3(b_1^q + b_2^q + b_3^q). \]

The number of intersections of the functions \( \theta_1 \) and \( \theta_2 \) is the same as the number of zeros of the Dirichlet polynomial

\[ F(q) = a_1^q + a_2^q + a_3^q - (b_1^q + b_2^q + b_3^q). \]

But this is a bipartite of length 6, so it has at most 3 zeros, counted with their orders. Since 0 and 1 are already two zeros of \( F \), the graphs of \( \theta_1 \) and \( \theta_2 \) cannot be tangent to each other. However, when \( c_1 = c_2 = 4 \), for Olsen’s functions \( B \) and \( b \) of the measure \( \mu \), the corresponding Dirichlet polynomial

\[ F(q) = a_1^q + a_2^q + a_3^q + a_4^q - (b_1^q + b_2^q + b_3^q + b_4^q) \]

will be a bipartite of length 8, which means, \( F \) has at most 4 zeros according to Jameson. So it is not impossible that 0 and 1 are both zeros of order 2. And this is exactly the condition we are looking for. The object of next theorem is to find two such proper groups of parameters \((a_j)\) and \((b_j)\).

Denote

\[ \theta(x_1, x_2, x_3, x_4; q) = \log(x_1^q + x_2^q + x_3^q + x_4^q), \]

then one easily computes

\[ \theta'(x_1, x_2, x_3, x_4; q) = \frac{x_1^q \log x_1 + x_2^q \log x_2 + x_3^q \log x_3 + x_4^q \log x_4}{x_1^q + x_2^q + x_3^q + x_4^q}, \]

where \( \theta' \) stands for the derivative of the function \( \theta \) with respect to \( q \).

**Theorem 10** There exist two different groups \((a_j)\) and \((b_j)\) \((a_j, b_j \in (0, 1), j = 1, 2, 3, 4)\) such that \( \sum_{j=1}^{4} a_j = \sum_{j=1}^{4} b_j = 1 \) and

\[ \theta(a_1, a_2, a_3, a_4; 0) = \theta(b_1, b_2, b_3, b_4; 0), \]
\[ \theta(a_1, a_2, a_3, a_4; 1) = \theta(b_1, b_2, b_3, b_4; 1), \]
\[ \theta'(a_1, a_2, a_3, a_4; 0) = \theta'(b_1, b_2, b_3, b_4; 0), \]
\[ \theta'(a_1, a_2, a_3, a_4; 1) = \theta'(b_1, b_2, b_3, b_4; 1). \]

**Proof** Only the last two equalities are to be proved. Take

\[ (a_1, a_2, a_3, a_4) = (a + t, b, c, d - t), \]
\[ (b_1, b_2, b_3, b_4) = (a + u, b + v, c + w, d - (u + v + w)), \]

where \( a, b, c, d \) are positive constants to be fixed later on subject to the condition \( a + b + c + d = 1 \), and \( t, u, v, w \) are real numbers. Define
\begin{align*}
\varphi(t, u, v, w) &= (a + t) \log(a + t) + b \log b + c \log c + (d - t) \log(d - t) \\
&\quad - (a + u) \log(a + u) - (b + v) \log(b + v) - (c + w) \log(c + w) \\
&\quad - (d - (u + v + w)) \log(d - (u + v + w)), \\
\psi(t, u, v, w) &= \log(a + t) + \log b + \log c + \log(d - t) - \log(a + u) \\
&\quad - (b + v) - \log(c + w) - \log(d - (u + v + w)).
\end{align*}

Notice that they are just the differences of derivatives of function \( \theta \) (with respect to \((a_j)\) and \((b_j)\)) at points 1 and 0. We wish to find suitable small non-zero numbers \( t, u, v, w \) such that both of the functions \( \varphi \) and \( \psi \) vanish.

It is easy to get
\[
\begin{align*}
\frac{\partial \varphi}{\partial t} &= \log \frac{a + t}{d - t} \\
\frac{\partial \varphi}{\partial u} &= \log \frac{d - (u + v + w)}{a + u} \\
\frac{\partial \varphi}{\partial v} &= \log \frac{d - (u + v + w)}{b + v} \\
\frac{\partial \varphi}{\partial w} &= \log \frac{d - (u + v + w)}{c + w}
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial \psi}{\partial t} &= \frac{1}{a + t} - \frac{1}{d - t} \\
\frac{\partial \psi}{\partial u} &= \frac{1}{d - (u + v + w)} - \frac{1}{a + u} \\
\frac{\partial \psi}{\partial v} &= \frac{1}{d - (u + v + w)} - \frac{1}{b + v} \\
\frac{\partial \psi}{\partial w} &= \frac{1}{d - (u + v + w)} - \frac{1}{c + w}
\end{align*}
\]

The Jacobian matrix at point \((0,0,0,0)\) is
\[
\left. \frac{\partial (\varphi, \psi)}{\partial (t, u, v, w)} \right|_{(0,0,0,0)} = \begin{pmatrix}
\log \frac{a}{d} & \log \frac{d}{a} & \log \frac{d}{b} & \log \frac{d}{c} \\
\frac{1}{a} - \frac{1}{d} & \frac{1}{a} - \frac{1}{d} & \frac{1}{a} - \frac{1}{d} & \frac{1}{b} - \frac{1}{d} - \frac{1}{c}
\end{pmatrix}
\]
(4)

Take out the middle two columns and one can expect the determinant of this submatrix nonzero. In fact, just let
\[
(a, b, c, d) = \begin{pmatrix}
1 & 2 & 3 & 4 \\
10 & 10 & 10 & 10
\end{pmatrix}
\]
then
\[
\left. \frac{\partial (\varphi, \psi)}{\partial (u, v)} \right|_{(0,0)} = \begin{pmatrix}
\log 4 & \log 2 \\
-\frac{15}{2} & -\frac{5}{2}
\end{pmatrix}
\]
which is a nondegenerate matrix.

At last, noticing that \( \varphi(0, 0, 0, 0) = \psi(0, 0, 0, 0) = 0 \) and recalling the implicit function theorem, we obtain a map \( F \) satisfying that for any small \((t, w)\), we have \((u, v) = F(t, w)\) such that
\[
\varphi(t, u, v, w) = \psi(t, u, v, w) = 0,
\]
from which the conclusion follows. \( \square \)
Remark 3 One sees from the matrix (4) that the first two columns should not be taken out, since it is obviously a degenerate matrix. This is because, when $v$ and $w$ are given, $t$ and $u$ can be never found to vanish the functions $\varphi$ and $\psi$.

For example, take $v = 0$, $w \neq 0$ but very small. Now the two groups of parameters are $(a + t, b, c, d - t)$ and $(a + u, b, c + w, d - (u + w))$ and we cannot find suitable $t$ and $u$. Otherwise, it is equivalent to obtain a bipartite of length 6 but with 4 zeros, which just causes a contradiction.

Besides, any other two columns in the matrix (4) are available.

Remark 4 If we take

$$(a, b, c, d) = \left(\frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{4}{9}\right),$$

then we can still obtain a map $F$ satisfying that for any small $(t, w)$, we have $(u, v) = F(t, w)$ such that

$$\varphi(t, u, v, w) = \psi(t, u, v, w) = 0.$$ 

Let $t = 0$, $w \neq 0$ but very small. Then $(a_1, a_2, a_3, a_4) = \left(\frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{4}{9}\right)$. If we denote $(e_1, e_2) = \left(\frac{1}{4}, \frac{2}{3}\right)$, then we have

$$\log_4(a_1^q + a_2^q + a_3^q + a_4^q) = \log_2(e_1^q + e_2^q).$$

**Proposition 11** Let $\mathcal{A} = \{0, 1, 2, 3\}$. There exists a probability measure $\mu$ on $\partial \mathcal{A}^*$ with full support such that its Olsen’s functions $B$ and $b$ are analytic and their graphs differ except at two points where they are tangent, with $B(0) = b(0)$, $B(1) = b(1)$, and $B(q) > b(q)$ for all $q \neq 0, 1$. Moreover $B$ and $b$ are convex and $B'(\mathbb{R})$ and $b'(\mathbb{R})$ both are intervals of positive length.

At the same time, if we let $\mathcal{A}_1 = \{0, 1\}$, $\mathcal{A}_2 = \{0, 1, 2, 3\}$ and $(T_k)$ be a suitable sequence of integers, then the statements above are true for the mixed symbolic space $\partial \mathcal{A}_{1,2}^*$.

**Proof** Choose $(a_j)$ and $(b_j)$ in Theorem 10. By Theorem 5 (set $c_1 = c_2 = 4$), there exists a probability measure $\mu$ on $\partial \mathcal{A}^*$ such that

$$B(q) = \sup\left\{\log_4(a_1^q + a_2^q + a_3^q + a_4^q), \log_4(b_1^q + b_2^q + b_3^q + b_4^q)\right\},$$

$$b(q) = \inf\left\{\log_4(a_1^q + a_2^q + a_3^q + a_4^q), \log_4(b_1^q + b_2^q + b_3^q + b_4^q)\right\}.$$

However, all intersections of $\theta(a_1, a_2, a_3, a_4; q)$ and $\theta(b_1, b_2, b_3, b_4; q)$ are clear: $(0, 1)$ and $(1, 0)$ are all of the four intersections, both of order 2. So these two curves are tangent to each other and one curve is always on top of the other (the order of the intersection is even). This follows the first part of the conclusion.

For the second part, just by using Remark 4, we can find $(a_j)$, $(b_j)$ and $(e_j)$ such that

$$\log_4(a_1^q + a_2^q + a_3^q + a_4^q) = \log_2(e_1^q + e_2^q).$$

So the same conclusion follows. \[\square\]

**Remark 5** We know from Theorem 4 that the measures above cannot satisfy the classical multifractal formalism at any point $q \neq 0, 1$.

**Corollary 12** The measure $\mu$ in Proposition 11 is exact dimensional.
The strong law of large numbers shows that
\[
\alpha_\mu(x) = \liminf_{n \to \infty} \frac{\log_4 \mu(B(x, 4^{-n}))}{-n} = \min\{h(a), h(b)\},
\]
\[
\overline{\alpha}_\mu(x) = \limsup_{n \to \infty} \frac{\log_4 \mu(B(x, 4^{-n}))}{-n} = \max\{h(a), h(b)\},
\]
for \(\mu\)-almost every \(x\), where
\[
h(a) = -\sum_{j=1}^{4} a_j \log_4 a_j \quad \text{and} \quad h(b) = -\sum_{j=1}^{4} b_j \log_4 b_j.
\]

But it is easy to see that \(h(a) = h(b) = -B'(1)\) according to the construction of the measure \(\mu\), from which the conclusion follows.

\[
\boxdot
\]

5 Interpretation of Legendre Transforms of \(b\) and \(B\)

In this section, we are to compute the dimensions of the level sets of local Hölder exponents of the measure \(\mu\) on \(\partial \mathcal{W}^*\) with analytic dimension functions (see Proposition 11).

We can see from the construction of measure \(\mu\) that \(0 < a_1 < a_2 < a_3 < a_4 < 1\) and \(0 < b_1 < b_2 < b_3 < b_4 < 1\). Fix such two groups of parameters and let us present an analytic result first.

**Lemma 13** For any \(\alpha \in (-\log_4 a_4, -\log_4 a_1)\), there exist four positive real numbers \(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3\) and \(\tilde{\alpha}_4\) such that \(\sum_{i=1}^{4} \tilde{\alpha}_i = 1\), and that
\[
-\sum_{i=1}^{4} \tilde{\alpha}_i \log_4 a_i = \alpha,
\]
\[
\log \frac{\tilde{\alpha}_2}{\tilde{\alpha}_1} = \log \frac{\tilde{\alpha}_3}{\tilde{\alpha}_1} = \log \frac{\tilde{\alpha}_4}{\tilde{\alpha}_1}.
\]

**Proof** Assume that \(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4 \in (0, 1)\) satisfying \(\sum_{i=1}^{4} \tilde{\alpha}_i = 1\).

For a given \(q\), to insure that
\[
\log \frac{\tilde{\alpha}_2}{\tilde{\alpha}_1} = \log \frac{\tilde{\alpha}_3}{\tilde{\alpha}_1} = \log \frac{\tilde{\alpha}_4}{\tilde{\alpha}_1} = q,
\]
onceqn
\[
\tilde{\alpha}_i = \tilde{\alpha}_1 \left( \frac{a_i}{a_1} \right)^q, \quad i = 2, 3, 4.
\]

So
\[
\sum_{i=1}^{4} \tilde{\alpha}_i = \tilde{\alpha}_1 \left( 1 + \sum_{i=2}^{4} \left( \frac{a_i}{a_1} \right)^q \right) = 1,
\]
which implies
\[
\tilde{\alpha}_i = \frac{a_i^q}{\sum_{j=1}^{4} a_j^q}, \quad i = 1, 2, 3, 4.
\]
One sees that $q$ can take any value in $(-\infty, +\infty)$. Now, consider the decreasing and convex function

$$\theta(q) = \log \sum_{j=1}^{4} a_j^q,$$

then

$$-\sum_{i=1}^{4} \tilde{a}_i \log_4 a_i = -\sum_{i=1}^{4} \frac{a_i^q}{\sum_{j=1}^{4} a_j^q} \log_4 a_i = -\sum_{i=1}^{4} \frac{a_i^q \log_4 a_i}{\sum_{j=1}^{4} a_j^q} = -\theta'(q),$$

which reaches any value in $(-\log_4 a_4, -\log_4 a_1)$.

Without loss of generality, we may assume that $a_1 < b_1$. Then by construction of the measure $\mu$, the graph of $\log_4(a_1^q + a_2^q + a_3^q + a_4^q)$ is always on top of the graph of $\log_4(b_1^q + b_2^q + b_3^q + b_4^q)$. So we have $a_4 > b_4$ and

$$\begin{align*}
B(q) &= \log_4(a_1^q + a_2^q + a_3^q + a_4^q), \\
b(q) &= \log_4(b_1^q + b_2^q + b_3^q + b_4^q).
\end{align*}$$

**Theorem 14** For any $\alpha \in (-\log_4 b_4, -\log_4 b_1)$, we have

$$\dim X(\alpha) = b^+(\alpha),$$

$$\operatorname{Dim} X(\alpha) = B^+(\alpha).$$

**Proof** Let $\alpha \in (-\log_4 b_4, -\log_4 b_1)$. By Lemma 13, we can construct a new probability measure $\nu$ on the tree just as $\mu$, but replacing $(a_i, b_i)$ with $(\tilde{a}_i, \tilde{b}_i)$ such that

$$\begin{align*}
\sum_{i=1}^{4} \tilde{a}_i &= \sum_{i=1}^{4} \tilde{b}_i = 1, \\
\alpha &= -\sum_{i=1}^{4} \tilde{a}_i \log_4 a_i = -\sum_{i=1}^{4} \tilde{b}_i \log_4 b_i, \\
\frac{\log \tilde{a}_2}{\tilde{a}_1} &= \frac{\log \tilde{a}_3}{\tilde{a}_1} = \frac{\log \tilde{a}_4}{\tilde{a}_1}, \\
\frac{\log \tilde{b}_2}{\tilde{b}_1} &= \frac{\log \tilde{b}_3}{\tilde{b}_1} = \frac{\log \tilde{b}_4}{\tilde{b}_1}.
\end{align*}$$

Fix $\lambda < 1$ and define

$$\varphi(x) = \limsup_{\delta \to 0} \frac{-1}{\log \delta} \log \sup \left\{ \sum_i \mu(B_i)^x \nu(B_i) : \right.$$

$$(B_i)_i \text{ packing of } S_\mu \text{ with } \lambda \delta < r_i \leq \delta \left. \right\}.$$

Then it is easy to compute

$$\varphi(x) = \log_4 \max \left\{ \sum_{i=1}^{4} a_i^x \tilde{a}_i, \sum_{i=1}^{4} b_i^x \tilde{b}_i \right\}.$$
It is clear that \( \varphi(0) = 0 \). And the method of choosing \( \{\tilde{a}_i, \tilde{b}_i\} \) insures that \( \varphi'(0) \) exists and is equal to \( -\alpha \).

Now we estimate the bounds of the dimensions of the level sets. The strong law of large numbers shows that

\[
\liminf_{n \to \infty} \frac{\log_4 \nu(B(x, 4^{-n}))}{-n} = \min\{h(\tilde{a}), h(\tilde{b})\},
\]

\[
\limsup_{n \to \infty} \frac{\log_4 \nu(B(x, 4^{-n}))}{-n} = \max\{h(\tilde{a}), h(\tilde{b})\},
\]

for \( \nu \)-almost every \( x \), where

\[
h(\tilde{a}) = -\sum_{i=1}^{4} \tilde{a}_i \log_4 \tilde{a}_i,
\]

\[
h(\tilde{b}) = -\sum_{i=1}^{4} \tilde{b}_i \log_4 \tilde{b}_i.
\]

So it deduces from Lemma 3 that

\[
\dim X(\alpha) \geq \min\{h(\tilde{a}), h(\tilde{b})\},
\]

\[
\text{Dim } X(\alpha) \geq \max\{h(\tilde{a}), h(\tilde{b})\}.
\]

To compute \( h(\tilde{a}) \) and \( h(\tilde{b}) \), set

\[
q_a = \frac{\log \frac{\tilde{a}_2}{\tilde{a}_1}}{\log \frac{\tilde{a}_2}{\tilde{a}_1}} = \frac{\log \frac{\tilde{a}_3}{\tilde{a}_1}}{\log \frac{\tilde{a}_3}{\tilde{a}_1}} = \frac{\log \frac{\tilde{a}_4}{\tilde{a}_1}}{\log \frac{\tilde{a}_4}{\tilde{a}_1}},
\]

then

\[
B'(q_a) = \frac{a_1^{q_a} \log_4 a_1 + a_2^{q_a} \log_4 a_2 + a_3^{q_a} \log_4 a_3 + a_4^{q_a} \log_4 a_4}{a_1^{q_a} + a_2^{q_a} + a_3^{q_a} + a_4^{q_a}}
\]

\[
= \frac{\log_4 a_1 + a_2^{q_a} \log_4 a_2 + a_3^{q_a} \log_4 a_3 + a_4^{q_a} \log_4 a_4}{1 + a_1^{q_a} + a_2^{q_a} + a_3^{q_a}}
\]

\[
= \frac{\log_4 a_1 + \tilde{a}_2 \log_4 a_2 + \tilde{a}_3 \log_4 a_3 + \tilde{a}_4 \log_4 a_4}{1 + \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + \tilde{a}_4}
\]

\[
= \varphi'(0) = -\alpha.
\]

Moreover,

\[
B(q_a) + q_a \alpha = \log_4(a_1^{q_a} + a_2^{q_a} + a_3^{q_a} + a_4^{q_a}) + q_a \alpha
\]

\[
= \log_4 a_1^{q_a} \left( 1 + \frac{a_2^{q_a}}{a_1^{q_a}} + \frac{a_3^{q_a}}{a_1^{q_a}} + \frac{a_4^{q_a}}{a_1^{q_a}} \right) + q_a \alpha
\]

\[
= q_a \log_4 a_1 + \log_4 \left( 1 + \tilde{a}_2 \tilde{a}_1 + \tilde{a}_3 \tilde{a}_1 + \tilde{a}_4 \tilde{a}_1 \right) + q_a \alpha
\]

\[
= q_a \log_4 a_1 - \log_4 \tilde{a}_1 - q_a (\tilde{a}_1 \log_4 a_1 + \cdots + \tilde{a}_4 \log_4 a_4)
\]
\begin{align*}
&= -\log_4 \tilde{a}_1 + \tilde{a}_2 \log_4 \frac{\tilde{a}_1}{\tilde{a}_2} + \tilde{a}_3 \log_4 \frac{\tilde{a}_1}{\tilde{a}_3} + \tilde{a}_4 \log_4 \frac{\tilde{a}_1}{\tilde{a}_4} \\
&= -\log_4 \tilde{a}_1 + \tilde{a}_2 \log_4 \frac{\tilde{a}_1}{\tilde{a}_2} + \tilde{a}_3 \log_4 \frac{\tilde{a}_1}{\tilde{a}_3} + \tilde{a}_4 \log_4 \frac{\tilde{a}_1}{\tilde{a}_4} \\
&= -\tilde{a}_1 \log_4 \frac{\tilde{a}_1}{\tilde{a}_1} - \tilde{a}_2 \log_4 \frac{\tilde{a}_1}{\tilde{a}_2} - \tilde{a}_3 \log_4 \frac{\tilde{a}_1}{\tilde{a}_3} - \tilde{a}_4 \log_4 \frac{\tilde{a}_1}{\tilde{a}_4} \\
&= h(\tilde{a}).
\end{align*}

And set

\begin{align*}
q_b &= \log \frac{\tilde{b}_2}{\tilde{b}_1} = \log \frac{b_2}{b_1} = \log \frac{\tilde{b}_4}{\tilde{b}_1},
\end{align*}

with the very same arguments, we have

\begin{align*}
b'(q_b) &= -\alpha, \\
b(q_b) + q_b \alpha &= h(\tilde{b}).
\end{align*}

Thus

\begin{align*}
h(\tilde{a}) &= B(q_a) + q_a \alpha = B(q_a) - q_a B'(q_a) = B^*(-B'(q_a)), \\
h(\tilde{b}) &= b(q_b) + q_b \alpha = b(q_b) - q_b b'(q_b) = b^*(-b'(q_b)),
\end{align*}

which give the lower bounds of the dimensions of the level sets:

\begin{align*}
\dim X(\alpha) &\geq b^*(-b'(q_b)) = b^*(\alpha), \\
\text{Dim } X(\alpha) &\geq B^*(-B'(q_a)) = B^*(\alpha).
\end{align*}

But we already have the opposite inequalities according to Theorem 2, by noticing that \((-\log_4 b_4, -\log_4 b_1) \subseteq (a, \overline{a})\), so finally we get

\begin{align*}
\dim X(\alpha) &= b^*(\alpha), \\
\text{Dim } X(\alpha) &= B^*(\alpha).
\end{align*}

\[\square\]

6 Measures Onto the Real Line

6.1 Generalized Gray Codes

Let \(\mathcal{A} = \{0, 1, \ldots, c - 1\}\). There is a natural way to enumerate the \(n\)-cylinders of \(\partial \mathcal{A}^n\): for \(w = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n\), we set

\[\iota(w) = \sum_{j=0}^{n-1} \varepsilon_{n-j} c^j,\]

but we need other orderings of the cylinders.

We define a transformation \(g\) on \(\mathcal{A}^n\), such that \(g(j) = j\) for all \(j \in \mathcal{A}\) and that, for any \(w = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n \in \mathcal{A}^n\) \((n \geq 2)\),

\[g(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n) = g(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{n-1}) \cdot k,\]
where \( k \in \mathcal{A} \) and \( k \equiv (\varepsilon_n - \varepsilon_{n-1}) \pmod{c} \).

We can see from the definition that if \( w, v \in \mathcal{A}^n \) are two contiguous words under natural enumeration, then \( g(w) \) and \( g(v) \) differ by one digit exactly. This property extends the one of the classical Gray code [7], which is the one we obtain when \( c = 2 \).

Also, it is obvious that \( v \prec w \) implies \( g(v) < g(w) \). So \( g \) induces a transformation on \( \partial \mathcal{A}^* \), still denoted by \( g \): for any \( x = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n \cdots \), \( g(x) \) is the unique element of \( \bigcap_{n \geq 1} [g(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n)] \). It is easy to see that \( g \) is an isometry.

**Remark 6** When \( c \geq 3 \), the way to define a Gray code is not unique. In fact, let \( \tilde{g}(j) = j \) for \( j \in \mathcal{A} \), and, for \( w \neq \epsilon \) and \( j \in \mathcal{A} \),

\[
\tilde{g}(w \cdot j) = \tilde{g}(w) \cdot k,
\]

where \( k = c - 1 - j \) when \( \iota(w) \) is odd, and \( k = j \) when \( \iota(w) \) is even. Then \( \tilde{g} \) has the same properties as \( g \).

### 6.2 Measures on \([0, 1]\)

In this final section, we work on \([0, 1]\) and try to get the same conclusion as Proposition 11, i.e. to find a probability measure on \([0, 1]\) with full support such that its Olsen’s functions \( B \) and \( b \) are analytic and their graphs differ except at two points where they are tangent, and that it obeys the refined multifractal formalism.

There is a natural map \( \gamma \) from \( \partial \mathcal{A}^* \) onto \([0, 1]\):

\[
\text{for } x = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n \cdots \in \partial \mathcal{A}^*, \quad \gamma(x) = \sum_{n \geq 1} \varepsilon_n c^{-n}.
\]

This map sends cylinders onto \( c \)-adic intervals; more precisely, if \( w \in \mathcal{A}^n \), the image of \( [w] \) under \( \gamma \) is the interval \([\iota(w)c^{-n}, (\iota(w) + 1)c^{-n}]\). In [2], the authors prove that for a continuous quasi-Bernoulli measure \( \mu \) on the symbolic space, \( \mu \) and its projection \( \gamma_\mathcal{A}(\mu) \) obey the multifractal formalism and all their Olsen’s functions are equal. However, it has no obvious reasons for which the two measures possess the same multifractal properties since it is easily checked that in our case, the measure \( \mu \) is no longer quasi-Bernoulli. So this is why we apply the Gray code before we project the measures onto the real line.

Now if \( \mu \) is a measure considered in Theorem 5, with the restriction \( c_1 = c_2 = c \), the measure \( \nu \) image of \( \mu \) under \( \gamma \circ g^{-1} \) (i.e., \( \nu(E) = \mu(g \circ \gamma^{-1}(E)) \) for any Borel set \( E \subseteq [0, 1] \)) is doubling, because of the properties of the Gray code \( g \) (see [4]). And obviously, \( S_\nu = [0, 1] \).

Since \( \nu \) is doubling, it is well known that to perform its multifractal analysis one can use coverings and packings with \( c \)-adic intervals as well as coverings and packings with general intervals. So, since \( c \)-adic intervals correspond to cylinders, \( \mu \) and \( \nu \) share the same multifractal analysis.

In particular, if we are given the measure \( \mu \) in Theorem 14, we can conclude that the \( b \) and \( B \) functions with respect to the measure \( \nu \) are analytic, that their graphs have two intersections of order two at points \((1, 0) \) and \((0, 1) \), and that, for \( q \neq \{0, 1\} \), \( b(q) < B(q) \). Moreover the measure \( \nu \) is exact dimensional, has full support, and obeys the refined multifractal formalism for a certain range of \( \alpha \).

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