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To cite this article: Florian J Curchod et al 2019 New J. Phys. 21 023016

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A versatile construction of Bell inequalities for the multipartite scenario

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Keywords: entanglement, Bell non-locality, multipartite, Bell inequalities

Abstract

Local measurements acting on entangled quantum states give rise to a rich correlation structure in the multipartite scenario. We introduce a versatile technique to build families of Bell inequalities witnessing different notions of multipartite nonlocality for any number of parties. The idea behind our method is simple: a known Bell inequality satisfying certain constraints, for example the Clauser–Horne–Shimony–Holt inequality, serves as the seed to build new families of inequalities for more parties. The constructed inequalities have a clear operational meaning, capturing an essential feature of multipartite correlations: their violation implies that numerous subgroups of parties violate the inequality chosen as seed. The more multipartite nonlocal the correlations, the more subgroups can violate the seed. We illustrate our construction using different seeds and designing Bell inequalities to detect k-way nonlocal multipartite correlations, in particular, genuine multipartite nonlocal correlations—the strongest notion of multipartite nonlocality. For one of our inequalities we prove analytically that a large class of pure states that are genuine multipartite entangled (GME) exhibit genuine multipartite nonlocality for any number of parties, even for some states that are almost product. We also provide numerical evidence that this family is violated by all GME pure states of three and four qubits. Our results make us conjecture that this family of Bell inequalities can be used to prove the equivalence between genuine multipartite pure-state entanglement and nonlocality for any number of parties.

1. Introduction

Quantum theory is rich in features that defy classical intuition. Systems of several particles are particularly interesting in that sense, with quantum systems exhibiting more intricate correlations than those possible within classical ones. For instance, some composite quantum systems cannot be specified by the state of their parts alone, but require a global description—a phenomenon known as quantum entanglement. When the parts of such entangled systems are separated at a distance, and local (independent) measurements are made on them, the distribution of outcomes can exhibit nonlocal correlations, in the sense that they cannot be explained by the existence of a (possibly hidden) classical common cause [1, 2]. Apart from their fundamental interest, quantum entanglement and quantum nonlocality have been identified as key resources for quantum information science.

Nonlocal correlations have been extensively studied in the simplest scenario of bipartite systems, which is sufficient to obtain powerful resources for information tasks with no classical equivalent: randomness expansion [3–5] and amplification [6], distribution of secret keys in a provably secure way [7–9] or testing the functioning of devices with minimal assumptions on their internal machinery [10], for example.

Multipartite scenarios—consisting of set-ups with at least three parties—have received far less attention due to their greater complexity. They offer, however, a much richer source of correlations than the bipartite set-up, and have already been proven useful for several tasks. Either for a better use of the potential provided by multipartite systems—which might be particularly interesting for tasks on quantum networks—or simply to
explore scenarios that go beyond the standard bipartite set-up, the study of multipartite scenarios is nowadays a central problem [11–19].

A detailed study of correlations in the multipartite scenario is an increasingly demanding task, as the complexity of the possible states of the systems and sets of correlations grows exponentially with the number of parties. A common approach to characterise multipartite correlations consists of testing whether they can be reproduced by models in which the parties share different physical resources (classical, quantum or post-quantum correlations) [11, 13, 14, 20]. These models range from completely classical—where all parties can only share classical correlations, to genuine multipartite (GM)—where all parties are required to be non-classically correlated. Intermediate models include, for instance, hybrid models where non-classical correlations are allowed inside groups of the parties, but the different groups can only be correlated classically between each other [14]. Although families of Bell inequalities that provide insight on the rich structure of multipartite scenario have been built [11, 14, 20], we are far from a complete characterisation of multipartite correlations.

Understanding the precise relation between entanglement and nonlocality in the multipartite scenario is of particular interest. While quantum entanglement is necessary for the display of quantum nonlocality, it is not sufficient. Indeed, there exist entangled mixed states for which single local measurements never generate nonlocal correlations [21, 22]. Remarkably, bipartite systems in a pure quantum state display a straightforward relation between entanglement and nonlocality: all pure entangled bipartite states are nonlocal, a result known as Gisin’s theorem [23]. This result has been extended to the multipartite scenario [24, 25], with the caveat that the used definitions of entanglement and nonlocality do not capture any truly multipartite features (with these definitions, a multipartite system is said to be non-classically correlated if at least two parties share non-classical correlations). Partial results have been obtained for the GM notions of entanglement and nonlocality: all three-qubit systems in a GM entangled (GME) pure state are GM nonlocal (GMNL), as well as any n-qubit systems in a fully-symmetric GME pure state [26, 27]. However, these results rely on the use of GM Hardy-type paradoxes [28], which have the drawback of not allowing for direct experimental tests, contrary to nonlocal correlations detected by the violation of a Bell inequality. Note that GM Hardy-type paradoxes can be used to devise indirect experimental tests of GMNL, but currently this is only possible up to four-parties, which is a strong limitation of this approach.

In this work we introduce a new technique to build Bell inequalities for the detection of truly multipartite nonlocal correlations, in a no-signalling (NS) framework [14]. These inequalities have a very clear operational meaning and capture essential features of multipartite nonlocality. Our construction takes a ‘seed’—a Bell inequality that fulfills certain constraints—to generate Bell inequalities for an arbitrary number of parties. The inequalities can be designed to detect k-way nonlocality, for any $2 \leq k \leq n$, including the extreme case of GMNL ($k = 2$). We illustrate the potential of the method by constructing several families of multipartite Bell inequalities from different seeds and for different notions of multipartite nonlocality.

Our technique is particularly fit for the detection of multipartite nonlocal correlations of pure states. Indeed, using the CHSH inequality [30] as the seed, we design two families of Bell inequalities, $I_{\text{sym}}$ and $I_{\text{dis}}$ that detect GMNL in large classes of GME pure states. Note that the CHSH inequality has already been used to prove the equivalence between pure state entanglement and nonlocality for bipartite systems [23]. Moreover, for three parties, $I_{\text{sym}}$ coincides with a Bell inequality obtained in [14], for which the authors found numerical evidence that the equivalence holds for all three-qubit states. Here we show analytically that, for any number of parties, all pure GHZ–like states that are GME contain GMNL correlations detected by $I_{\text{sym}}$, even almost separable states. We supplement these analytical results by providing numerical evidence that all four-qubit systems in a GME pure state violate $I_{\text{sym}}$. In the tripartite scenario, using $I_{\text{dis}}$, we also show analytically that all pure states symmetrical under the permutation of two parties are GMNL. The partial results obtained added to the operational meaning of our construction lead us to conjecture that the family of Bell inequalities $I_{\text{sym}}$ can be used to generalise Gisin’s theorem, proving that all GME pure states are GMNL.

2. The tripartite scenario

We start by introducing the main concepts used in this work and our results for the tripartite scenario, which is the simplest multipartite scenario for the observation of nonlocal correlations. This scenario counts with three distant observers $A_i, i \in \{1, 2, 3\}$ making rounds of measurements on multipartite quantum systems. At each

Although not directly testable, GM Hardy-type paradoxes allow for indirect experimental tests of GMNL. The idea is to use the ideal Hardy correlations as targets for experiments. The nonlocal character of the realised distributions can then be tested through linear programming and the knowledge of the extreme points of the polytope of biseparable correlations. The limitation of this approach is that computing the vertices of the biseparable polytopes with $m + 1$ parties will crucially require knowing all the extremal points of the $m$-partite no-signaling polytope. These have so far only been computed up to three parties [29], making this indirect approach impossible for more than four-parties with current knowledge.
round, the choice of local measurement performed by each party is labelled \( x_i \) and the obtained outcome \( a_i \). The generated joint conditional probability distribution \( P(a_1,a_2,a_3|x_1,x_2,x_3) \) is then said to be local if it factorises, given the additional knowledge of a (possibly hidden) common classical cause \( \lambda \):
\[
P_L(a_1,a_2,a_3|x_1,x_2,x_3) = \sum_{\lambda} q(\lambda) P_A(a_1|x_1, \lambda) P_A(a_2|x_2, \lambda) P_A(a_3|x_3, \lambda).
\]
(1)

The common cause \( \lambda \) is a discrete random variable with distribution \( q(\lambda) \geq 0, \sum_{\lambda} q(\lambda) = 1 \) and \( P_A(a_i|x_i, \lambda) \) is a probability distribution for party \( A_i \). A distribution \( P(a_1,a_2,a_3|x_1,x_2,x_3) \) that does not allow for a decomposition (1) is said to be nonlocal. Note that this definition of locality for three parties is a straightforward generalisation of the bipartite scenario, where the only difference is the addition of a third party. Because of the measurement arrangements it is assumed that the NS principle [31] holds, i.e. party \( A_1 \) cannot signal to the other parties by performing a choice of measurement
\[
P(a_1,a_2,a_3|x_1,x_2,x_3) \equiv P(a_1,a_2,a_3|x_1,x_2,x_3) = \sum_{a_i} P(a_1,a_2,a_3|x_1,x_2,x_3), \ \forall x_i
\]
(2)
and similarly for parties \( A_2 \) and \( A_3 \).

The notion of separability for a tripartite pure state \( |\psi_{123}\rangle \) is also a direct extension of the bipartite case, \( |\psi_{123}\rangle = \sum_{\lambda_0} q(\lambda_0) |\phi_0\rangle |\phi_1\rangle |\phi_2\rangle \), where \( |\phi_i\rangle \) is the state of party \( A_i \). The state \( |\psi_{123}\rangle \) is then entangled whenever it does not admit for the previous decomposition. In that case, it is already known to be nonlocal, as there always exist local measurements on it that lead to a nonlocal joint distribution [24]. This equivalence between pure state entanglement and nonlocality is however essentially the same as for bipartite systems [23], since it only requires two parties to be entangled.

Here we are interested in genuinely multipartite definitions of entanglement and nonlocality. As first noticed by Svetlichny [11], distributions generated in a tripartite scenario lead to stronger notions of nonlocality. Consider for instance a relaxation of the locality assumption, where pairs of parties are now allowed to group together and share nonlocal resources. This type of hybrid local/nonlocal models leads to joint conditional probability distributions
\[
P_{1/1}(a_1,a_2,a_3|x_1,x_2,x_3)
= \sum_{\lambda_i} q_i(\lambda_i) P_{A_i,A_i}(a_1,a_2,a_3|x_1,x_2,x_3, \lambda_i) P_{A_i}(a_1|x_1, \lambda_i)
+ \sum_{\lambda_2} q_2(\lambda_2) P_{A_2,A_2}(a_1,a_2,a_3|x_1,x_2,x_3, \lambda_2) P_{A_2}(a_2|x_2, \lambda_2)
+ \sum_{\lambda_3} q_3(\lambda_3) P_{A_3,A_3}(a_1,a_2,a_3|x_1,x_2,x_3, \lambda_3) P_{A_3}(a_3|x_3, \lambda_3)
\]
(3)
with \( q_i(\lambda_i) \geq 0 \) and \( \sum_{\lambda_i} q_i(\lambda_i) = 1 \). Distributions \( P(a_1,a_2,a_3|x_1,x_2,x_3) \) that cannot be decomposed in the form (3) are named genuine tripartite nonlocal. As shown in [13], the original notion of multipartite nonlocality by Svetlichny [11] faces operational problems. To avoid these, one assumes that the NS principle [32] also holds at the level of distributions \( P_{A,A}(a_1,a_2|x_j,x_j, \lambda) \), which implies that the marginals \( P(a_i|x_i, \lambda) = P(a_i|x_i, \lambda) = \sum_{a_j} P(a_1,a_2|x_i,x_j, \lambda) \), \( \forall x_j \), are well defined for all \( \lambda \).

In analogy, a tripartite system is said to be in a genuine tripartite entangled pure state if it cannot be decomposed as \( |\psi_{123}\rangle = \sum_{ijk} \langle \phi_{ij}\rangle |\phi_k\rangle \), where \( ijk \) is any combination of the parties. One can easily verify that local measurements on biseparable states always lead to a hybrid joint distribution (3).

Before introducing our inequalities witnessing genuine tripartite nonlocal correlations, recall that a Bell inequality is described by a bounded linear combination of probability terms
\[
\sum_{\bar{a},\bar{x}} c_{\bar{a},\bar{x}} P(\bar{a}|\bar{x}) \leq B
\]
(4)
in a experiment with \( n \) parties, where the number of observables \( \bar{x} = (x_1, x_2, \ldots, x_n) \) and respective outcomes \( \bar{a} = (a_1, a_2, \ldots, a_n) \) is fixed. The coefficients \( c_{\bar{a},\bar{x}} \) are real numbers and \( B \) is the maximum attained by local or hybrid distributions, according to the problem. In the bipartite scenario, where each party has two choices of two-outcome measurements, i.e. \( x_i, a_i \in \{0, 1\} \) for \( i = 1, 2 \), the violation of the CHSH inequality is both necessary and sufficient for \( P(a_1,a_2|x_1,x_2) \) to be nonlocal [30, 33, 34]. This is also the inequality used to show that all pure bipartite entangled states are nonlocal [23]. Here we use a variant of the CHSH inequality
\[
I_{A_2A_3} = P(00|00) - P(01|01) - P(10|10) + P(00|11) \leq 0
\]
(5)
that, for NS distributions, is equivalent to the standard expression
\[
\text{CHSH} = \langle A_2B_0 \rangle + \langle A_1B_0 \rangle + \langle A_2B_1 \rangle - \langle A_1B_1 \rangle \leq 2,
\]
(6)
where \( \langle A_2B_0 \rangle = \sum_{a,b} P(a = b|xy) = P(a = b|xy) \).
3. Bell inequalities for genuine tripartite nonlocality

We start by exemplifying our method through the construction of two Bell inequalities witnessing genuine tripartite nonlocal correlations. In both cases, we use the CHSH inequality (3) as the seed. The main idea is to make enough pairs of parties to play the nonlocal game defined by the seed, such that the inequality can only be violated by genuine tripartite correlations. The first inequality is symmetrical under permutation of the three parties

\[ I_{\text{sym}}^{A_1 A_2 A_3} = I_{0|0}^{A_1 A_2} + I_{0|0}^{A_1 A_3} + I_{0|0}^{A_2 A_3} - P(000|000) \leq 0 \]  

(7)

and the second inequality is symmetrical under the permutation of parties \( A_2 \) and \( A_3 \),

\[ I_{0|0}^{A_1 A_2 A_3} = I_{0|0}^{A_1 A_2} + I_{0|0}^{A_1 A_3} - P(000|000) \leq 0, \]  

(8)

where the term

\[ I_{0|0}^{A_1} = P(000|000) - P(010|010) - P(100|100) - P(000|110) \]  

(9)

represents the lifting\(^{[35]}\) of the seed \( I_{0|0}^{A_1 A_2} \) to the tripartite scenario, by setting observer \( A_3 \) to measurement \( x_3 = 0 \) and outcome \( a_3 = 0 \) (and similarly for the terms \( I_{0|0}^{A_1 A_2} \) and \( I_{0|0}^{A_1 A_3} \)). Intuitively, in \( I_{0|0}^{A_1 A_2 A_3} \) every pair of parties plays a (lifted) CHSH game while in \( I_{0|0}^{A_1 A_2 A_3} \) party \( A_1 \) acquires a central role by playing a CHSH game with every remaining party. Note that the local bound of the lifted inequalities remains the same as the local bound for the seed, \( I_{0|0}^{A_1} \leq 0 \). See appendix A for the proof of this property and more details on lifted Bell inequalities.

**Theorem 1.** The Bell inequalities \( I_{\text{sym}}^{A_1 A_2 A_3} \) and \( I_{0|0}^{A_1 A_2 A_3} \) witness genuine tripartite nonlocality.

**Proof.** We want to show that any hybrid distribution satisfies \( I_{\text{sym}}^{A_1 A_2 A_3} \leq 0 \) and \( I_{0|0}^{A_1 A_2 A_3} \leq 0 \). First, observe that due to the convexity of hybrid distributions (3), it is sufficient to perform the proof for the extremal distributions \( P_{A_1 A_2 (a_i a_j x_i x_j)} P_{A_1 (a_i x_j)} \).

The second basic element of our proof is that

\[ I_{0|0}^{A_1 A_2} (P_{A_1 A_2 (a_i a_j x_i x_j)} P_{A_1 (a_i x_j)}) \leq 0 \]  

(10)

for any triplet \( i, j, k \in \{1, 2, 3\} \) with \( i = j \neq k = i \). This comes from the fact that the (lifted) inequality \( I_{0|0}^{A_1 A_2} \) can only be violated if parties \( A_j \) and \( A_k \) are non-classically correlated, which is not the case when the correlations allow for a decomposition of the form \( P_{A_1 A_2 (a_i a_j x_i x_j)} P_{A_1 (a_i x_j)} \). (A proof of this property can be found in appendix A.)

After this observation, we know that for every extremal hybrid distribution, the only potentially positive term in both our inequalities is \( I_{0|0}^{A_1 A_2} (P_{A_1 A_2 (a_i a_j x_i x_j)} P_{A_1 (a_i x_j)}) \), therefore:

\[ I_{\text{sym}}^{A_1 A_2 A_3} \leq I_{0|0}^{A_1 A_2} - P(000|000) \]  

\[ I_{0|0}^{A_1 A_2 A_3} \leq I_{0|0}^{A_1 A_2} - P(000|000). \]  

(11)

The last element of our proof is then the fact that

\[ I_{0|0}^{A_1 A_2} \equiv I_{0|0}^{A_1} - P(000|000) \]

\[ = -P(010|010) - P(100|100) - P(000|110) \leq 0 \]  

(12)

for any probability distribution, which will cancel the potentially positive term.

Notice that the idea behind our construction can be extended to build a three-parameter \( \mu_{12}, \mu_{13}, \mu_{23} \) family of inequalities that also witness genuine tripartite nonlocality

\[ I_{\mu}^{A_1 A_2 A_3} = \mu_{12} I_{0|0}^{A_1 A_2} + \mu_{13} I_{0|0}^{A_1 A_3} + \mu_{23} I_{0|0}^{A_2 A_3} - P(000|000) \leq 0 \]  

(13)

for \( \mu_{ij} \in [0, 1] \) and \( \mu_{12} + \mu_{13} + \mu_{23} > 1 \) (if this last condition is not met, the inequality is trivial: \( I_{0|0}^{A_1 A_2 A_3} \leq 0 \) for any probability distribution). Indeed, following the arguments of the proof of theorem 1, one can verify that (13) holds for any hybrid distribution (3):

\[ I_{\mu}^{A_1 A_2 A_3} (P_{A_1 A_2 (a_i a_j x_i x_j)} P_{A_1 (a_i x_j)}) \leq \mu_{ij} I_{0|0}^{A_1} - P(000|000) \leq I_{0|0}^{A_1 A_2 A_3} \leq 0. \]  

(14)

It is interesting to observe that the local strategy where every party always obtains outcome \( a_i = 1 \) for any measurement \( x_i \) saturates both inequalities \( I_{0|0}^{A_1 A_2 A_3} = 0 \) and \( I_{\text{sym}}^{A_1 A_2 A_3} = 0 \), thus as well \( I_{\mu}^{A_1 A_2 A_3} = 0 \). This implies that the local and hybrid bounds of our inequalities coincide.
4. Detection of genuine tripartite nonlocality in pure states

Bell inequality \( I_{A_1 A_2 A_3}^{A_1 A_2 A_3} \) seems particularly fit for the detection of genuine tripartite nonlocality of pure states. Indeed, it belongs to class 6 of [14], where strong numerical evidence was provided indicating that all three-qubit systems in a GME pure state could generate correlations violating it. This result hints that \( I_{A_1 A_2 A_3}^{A_1 A_2 A_3} \) is a good candidate for an analytical proof of equivalence between GME and GMNL for tripartite pure states. Later, we will generalise this inequality for \( n \) parties, prove analytically that it detects GMNL in a large class of GME pure states and provide numerical evidence that all GME pure states of four qubits are GMNL.

We now focus on inequality \( I_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} \leq 0 \) and show that it is useful for the detection of genuine tripartite nonlocality of pure states. In [36], it was shown that all systems of three qubits in a pure state could be written as

\[
|\psi\rangle = h_0|000\rangle + h_1 e^{i\phi}|100\rangle + h_2|101\rangle + h_3|110\rangle + h_4|111\rangle,
\]

where \( h_i \in \mathbb{R}, \sum_i h_i^2 = 1 \) and \( \phi \in [0, \pi] \).

Theorem 2. For all tripartite pure states (15) that are GME and symmetrical under the permutation of any two parties, say \( A_2 \) and \( A_3 \), \( h_0 > 0 \) and \( h_3 = h_1 \), one can find local measurements on them such that the generated correlations violate inequality \( I_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} \leq 0 \) (8), hence generating GMNL correlations.

Proof. A complete proof of theorem 2 can be found in appendix B. The main line of it goes as follows. We will start by choosing parties \( A_2 \) and \( A_3 \) to perform the same (projective) measurements \( \langle m_{a_1} | a_2 = a_3 \rangle \). This, together with the \( A_2 \leftrightarrow A_3 \) invariance of the state, implies that the observed correlations \( P(a_1 a_2 a_3 | x_1 x_2 x_3) \) are also symmetrical with respect to the permutation of \( A_1 \) and \( A_3 \). Consequently, for these correlations, we have \( I_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} = 1_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} \) and \( I_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} \) can be simplified to

\[
I_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} = 2P_{0100} - P(000000) = P(000000) - 2P(010010) - 2P(100010) - 2P(000110) \leq 0.
\]

We now show that we can always find appropriate measurements such that we obtain a particular violation of the previous inequality

\[
\begin{align*}
P(000000) & > 0 \\
P(010010) & = P(100010) = P(000110) = 0.
\end{align*}
\]

These conditions correspond to an Hardy paradox [28] on parties \( A_1 \) and \( A_2 \). Consider the post-measurement state \( |\psi_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} \rangle \), which is the state prepared by party \( A_0 \) after making the measurement \( \langle m_{a_1} = 0 | x_1 = 0 \rangle \) on \( |\psi\rangle \):

\[
|\psi_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} \rangle \propto I_{A_0} \otimes I_{A_1} \otimes \langle m_{a_1} = 0 | x_1 = 0 \rangle |\psi\rangle.
\]

Since \(|\psi\rangle \) is GME by assumption, we can tune the measurement \( \langle m_{a_1} = 0 | x_1 = 0 \rangle \) such that the prepared state \( |\psi_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} \rangle \) is non-maximally entangled [24]. After Hardy’s construction [28], we know that for a pure non-maximally entangled state \( |\psi_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} \rangle \), we can always find a one-parameter family of measurements on \( |m_{a_1} = 0 \rangle \) leading to an Hardy paradox (17). This means that we can choose freely the first measurement, say \( \langle m_{a_2} = 0 | x_2 = 0 \rangle \), and always find three other measurements \( \langle m_{a_2} = 0 | x_2 = 1 \rangle, \langle m_{a_2} = 0 | x_2 = 0 \rangle, \langle m_{a_2} = 0 | x_2 = 1 \rangle \) that satisfy (17). Therefore, we are able to choose \( \langle m_{a_2} = 0 | x_2 = 0 \rangle \) in order to be compatible with the condition of preparing a state \( |\psi_{A_0 A_1 A_2 A_3}^{A_0 A_1 A_2 A_3} \rangle \) of non-maximally entangled. More details can be found in appendices B and C.

\[\Box\]

5. The general multipartite scenario

We proceed now to exposing our results in the general multipartite scenario. We consider any number \( n > 2 \) of observers making dichotomic choices of local measurements \( x_i \in \{0, 1\} \) on their share of a joint quantum system and obtaining outcomes \( a_i \in \{0, 1\} \), generating a distribution \( P(a_0 a_1 ... a_n | x_1 x_2 ... x_n) \equiv P(\vec{a} | \vec{x}) \).

The definition of GM nonlocality for any number of parties is more intricate than the tripartite case, but follows basically the same idea as (3). A distribution \( P(\vec{a} | \vec{x}) \) is said to be biseparable if

\[
P_{2-sep}(\vec{a} | \vec{x}) = \sum_{\vec{g}} q_{\vec{g}} (\vec{\lambda}) P(\vec{a} | \vec{x}) P(\vec{g} | \vec{x}) P(\vec{\lambda}),
\]

where \( \sum_{\vec{g}} q_{\vec{g}} (\vec{\lambda}) = 1, q_{\vec{g}} (\vec{\lambda}) > 0 \) and \( \vec{g} \) is a group consisting of a particular subset of the \( n \) observers and \( \vec{\lambda} \) its complement. We label the string of measurement choices (resp. outcomes) of the observers belonging to the group \( g \) as \( \vec{x}_g \) (\( \vec{a}_g \)). For example, for \( n = 3 \) there are only three possible inequivalent ways of making two groups: \( (g_0 = A_1 A_2, g_1 = A_3), (g_2 = A_1, g_3 = A_2) \) and \( (g_0 = A_2 A_3, g_1 = A_1) \), leading to a decomposition of the form (3). Distributions that cannot be written according to the decomposition (19) are genuine multipartite nonlocal.
We will see later that we can define intermediate types of multipartite nonlocal correlations, where one allows for more than two groups of parties.

Again, local measurements on pure biseparable states \( |\psi_1...\psi_n\rangle = |\phi_g\rangle |\overline{\phi}_g\rangle \) for some splitting \( g/\overline{g} \) of the particles, always lead to biseparable joint distributions (19). GM entanglement is necessary to generate GM nonlocal correlations.

6. Bell inequalities for GMNL

The generalisation of inequalities \( I^{A_k A_l}_{\text{sym}} \) (7) and \( I^{A_k A_l}_{\otimes} \) (8) to any number \( n \) of parties gives two distinct families of Bell inequalities that can be written in a simple form:

\[
I^{A_1...A_n}_{\text{sym}} = \sum_{i=1}^{n-1} \sum_{j>i}^{n} I^{A_i A_j}_{\text{sym}} \leq \left( \frac{n-1}{2} \right) P(\overline{0}|\overline{0}) \leq 0, \\
I^{A_1...A_n}_{\otimes} = \sum_{j>1}^{n} I^{A_i A_j}_{\otimes} - (n-2) P(\overline{0}|\overline{0}) \leq 0,
\]

where \( \left( \frac{n-1}{2} \right) = \frac{(n-1)(n-2)}{2} \) and we take the freedom of writing \( \overline{0} \equiv (0, 0, ..., 0) \), the size of the string should be obvious in the context. Similarly to (9), \( I^{A_i A_j}_{\otimes} \) is a lifting of inequality \( I^{A_i A_j} \) (5) to \( n \) observers by setting the remaining \( n-2 \) observers to have their measurement and outcome set to 0:

\[
I^{A_i A_j}_{\otimes}|_{\overline{0}} = P(0,0,\overline{0}|0,0,\overline{0}) - P(1,0,\overline{0}|1,0,\overline{0}) - P(0,1,\overline{0}|0,1,\overline{0}) - P(0,0,\overline{0}|1,1,\overline{0}).
\]

The operational meaning of these inequalities is the following. In both cases, we use as ‘seed’ the CHSH inequality (5), which defines a nonlocal game between parties \( A_1 \) and \( A_1 \) represented by the lifted inequalities \( I^{A_i A_j}_{\otimes} \). For the symmetrical family (20) every pair of parties is required to play a CHSH game while for the inequalities ‘centered’ on \( A_1 \) (21), party \( A_1 \) is required to play a CHSH game with every other party.

**Theorem 3.** The Bell inequalities \( I^{A_1...A_n}_{\text{sym}} \leq 0 \) (20) and \( I^{A_1...A_n}_{\otimes} \leq 0 \) (21) are witnesses of GM nonlocality for all \( n \geq 3 \).

**Proof.** Here we only provide an outline, the full proof can be found in appendix D.1. The idea is similar to the one for three parties. We want to show that all biseparable distributions (19) for \( n \) parties satisfy \( I^{A_1...A_n}_{\text{sym}} \leq 0 \) and \( I^{A_1...A_n}_{\otimes} \leq 0 \). Again, by convexity, it is enough to verify it for extremal biseparable distributions \( P(\overline{d}_k|\overline{x}_k)P(\overline{d}_k|\overline{x}_k) \) from (19).

If parties \( A_i \) and \( A_j \) belong to different groups, they are only classically correlated and therefore \( I^{A_i A_j}_{\otimes} \leq 0 \). Then, the only terms that can give a positive contribution \( I^{A_i A_j}_{\otimes} > 0 \) are terms where parties \( A_i \) and \( A_j \) belong to the same group. Now the trick is to kill these positive contributions by subtracting enough \( P(\overline{0}|\overline{0}) \) terms since, similarly to \( n = 3 \) (12),

\[
I^{A_i A_j}_{\otimes}|_{\overline{0}} \equiv I^{A_i A_j}_{\otimes} - P(\overline{0}|\overline{0}) \leq 0
\]

for any probability distributions.

**Symmetric family** \( I^{\text{sym}} \)—In general, if the first group \( g \) consists of \( m \) parties and \( \overline{g} \) of \( n - m \) for some \( 1 \leq m \leq n - 1 \), a total number of \( \begin{pmatrix} m \cr 2 \end{pmatrix} + \begin{pmatrix} n - m \cr 2 \end{pmatrix} \) inequalities \( I^{A_i A_j}_{\otimes} \) can in principle be positive. Since \( \left( \frac{n-1}{2} \right) > \left( \frac{m}{2} \right) + \left( \frac{n-m}{2} \right) \), \( \forall m \geq 2 \), the largest number of pairs is obtained by putting \( n-1 \) parties in one group, which means \( \left( \frac{n-1}{2} \right) \) potentially positive terms \( I^{A_i A_j}_{\otimes} \). Then

\[
I^{A_1...A_n}_{\text{sym}}(P(\overline{d}_k|\overline{x}_k)P(\overline{d}_k|\overline{x}_k)) \leq \sum_{i=1}^{n-2} \sum_{j<i}^{n-1} I^{A_i A_j}_{\otimes} - \left( \frac{n-1}{2} \right) P(\overline{0}|\overline{0}) = \sum_{i=1}^{n-2} \sum_{j>i}^{n-1} I^{A_i A_j}_{\otimes} \leq 0,
\]

where we used the fact that \( I^{A_1...A_n}_{\text{sym}} \) is invariant under permutations of parties to consider the specific partition \( g = \{1, ..., n-1\} \) and \( \overline{g} = \{n\} \).

**Centered family** \( I^{\otimes} \)—The proof follows the same idea as before. Using (23), any biseparable distribution (19) with \( m \) parties in the first group \( g \) containing party \( A_i \) and \( n - m \) in the other group \( \overline{g} \) gives

6
Two parties \( i \) and \( j \) inside the same group can potentially violate a lifted inequality \( I_{00}^{A_i A_j} \) (as represented by a dashed line between them). (i) Two groups of parties \( |g| = 2; |g| = 3 \), giving a distribution of the form \( P(a_i; x_1) = P(a_i; x_2) \) \( P(a_i; x_3; x_4; x_5) \) and a maximum number of \( \left( \frac{2}{2} \right) \times \left( \frac{3}{2} \right) = 4 \) violated inequalities \( I_{00}^{A_i A_j} > 0 \). (ii) Two groups of parties \( |g| = 1; |g| = 4 \), for \( \left( \frac{4}{2} \right) = 6 \) potentially violated inequalities \( I_{00}^{A_i A_j} > 0 \). (iii) GMNL: all parties are in the same group and thus \( \left( \frac{5}{2} \right) = 10 \) inequalities can be violated. Only (iii) can violate \( I_{\text{sym}}^{A_1 \ldots A_n} = \sum_{i=1}^{n} \sum_{j=1}^{n} I_{00}^{A_i A_j} - 6 P(0|0) \) since \( I_{00}^{A_i A_j} - P(0|0) \leq 0 \).

\[
I_{00}^{A_1 \ldots A_n} = \sum_{j \in g} \sum_{i \not\in g} I_{00}^{A_i A_j} - \left( n - 2 \right) P(0|0) \leq \sum_{j \in g} I_{00}^{A_i A_j} \leq 0
\]

since there are at most \( n - 2 \) parties together with party \( A_1 \) in the first group \( g \).

One can understand a violation of the families of inequalities (20) and (21) in the following way: (a) any (extremal) distribution that violates \( I_{\text{sym}}^{A_1 \ldots A_n} \) (20) needs to be capable of violating more than \( \left( \frac{n - 1}{2} \right) \) lifted CHSH inequalities \( I_{00}^{A_i A_j} \) between different observers \( A_i \) and \( A_j \); and (b) any distribution that violates \( I_{\text{sym}}^{A_1 \ldots A_n} \) (21) violates more than \( n - 2 \) lifted CHSH inequalities \( I_{00}^{A_i A_j} \) between \( A_1 \) and different observers \( A_j \). Only GMNL correlations, where all pairs of parties are nonlocally correlated, are able to do this. We also provide an illustration of how our inequalities work in figure 1.

More insight on the rich structure of the symmetrical family of inequalities (20) is given by noticing that they can also be written in a recursive form for \( n \geq 3 \):

\[
I_{\text{sym}}^{A_1 \ldots A_n} = \sum_{i=1}^{n} I_{00}^{A_i \backslash A_i} - \left( n - 2 \right) P(0|0) \leq 0,
\]

where \( I_{00}^{A_i \backslash A_i} \) is the symmetrical inequality for \( n - 1 \) observers lifted to \( n \) of them with observer \( A_i \)’s input and outcome set to 0. If \( n = 3 \) for example, \( I_{00}^{A_1 \backslash A_1} \) corresponds to the CHSH inequality lifted to 3 parties (9). The proof of the equivalence between the direct expression (20) and the recursive one (26) can be found in appendix D.2.

In other words, operationally a violation of the symmetrical family \( I_{\text{sym}}^{A_1 \ldots A_n} \) can also be understood as a violation of more than \( n - 2 \) inequalities \( I_{00}^{A_i A_j} \) between \( n - 1 \) parties lifted to \( n \) parties—instead of \( \left( \frac{n - 1}{2} \right) \) bipartite ones \( I_{00}^{A_i A_j} \) lifted to \( n \) parties. Since this argument can be used recursively, one concludes that GMNL correlations violating our inequalities violate numerous inequalities between subset of \( n \) parties lifted to \( n \) parties, for all \( n \).

Observe that, similar to the tripartite case, the generalised families \( I_{\text{sym}}^{A_1 \ldots A_n} \) (20) and \( I_{00}^{A_1 \ldots A_n} \) (21) are also saturated by local distributions. The local strategy is the same as for \( n = 3 \): every party \( A_i \) outputs \( a_i = 1 \) for all measurements \( x_i \). It follows that the local and biseparable bounds of our families of inequalities coincide for all \( n \).
7. Detection of GMNL in pure states

Let us now analyse how the symmetric family of Bell inequalities $I_{\text{sym}}^{A_1\ldots A_n}$ sheds light on the relation between GME and GMNL of pure states. The goal is to understand whether it is possible to find local measurements on any pure GME state that generate GMNL correlations. Our Bell inequalities seem fit to prove this result since for any pure GME state there exist local projections on any $n - 2$ parties that leave the remaining two in a pure entangled state [24], which can in turn be used to violate the CHSH inequality [23]. The main difficulty in proving the result is the need to find local measurements that simultaneously perform the desired projections but are also fit to violate the CHSH terms. For $n = 2$ our two families of inequalities coincide with the CHSH inequality, which was used to prove the equivalence of local and nonlocality and pure state entanglement [23]. For $n = 3$, there is numerical evidence that this holds for GME three-qubits pure states [14] using the symmetrical family $I_{\text{sym}}^{A_i A_j A_k}$ (8).

We consider the generalisation of these results to the scenario with $n = 4$ parties, where we obtained numerical evidence that all four-qubit systems in a pure GME state generate distributions violating the Bell inequality $I_{\text{sym}}^{A_i A_j A_k}$ (20). For this, we have randomly drawn four qubit states and numerically searched for local measurements leading to a violation of our inequality. Note that the set of separable states is of volume zero in the state space.

We now proceed to show analytically that a large class of pure GME states of the GHZ family [37] can generate GMNL correlations, for all number of parties $n \geq 3$, as detected by the symmetrical family of inequalities $I_{\text{sym}}^{A_i A_j A_k}$ (20).

**Theorem 4.** All pure GME states of the form

$$|\text{GHZ}^\theta\rangle = \cos \theta |0\rangle^{\otimes n} - \sin \theta |1\rangle^{\otimes n}$$

with $\theta \in [0, \frac{\pi}{4}]$ violate the Bell inequality $I_{\text{sym}}^{A_i A_j A_k}$ (20) for all $n \geq 3$. All parties $A_i$ make the same projective measurements, $\langle m_{a|x} \rangle = \langle m_{a|x} \rangle_{i}$, defined by

$$\langle m_{0|x} \rangle = \cos \alpha_x \langle 0 \rangle + \sin \alpha_x \langle 1 \rangle,$$

$$\langle m_{1|x} \rangle = \sin \alpha_x \langle 0 \rangle - \cos \alpha_x \langle 1 \rangle,$$

where

$$\alpha_0 = \arctan(\tan^{-1} \frac{3}{\pi - 2\theta}(\theta)),$$

$$\alpha_1 = -\arctan(\tan^{-1} \frac{1}{\pi - 4\theta}(\theta)).$$

In other words, all states of the form (27) that are GME are GMNL.

**Proof.** A detailed and constructive proof of this theorem can be found in appendix E. The key point is to impose the local measurements to be the same for every party, which makes the joint outcome distribution invariant under permutations of the parties (since the state $|\text{GHZ}^\theta\rangle$ (27) also has this invariance). This symmetry simplifies the problem and allowed us to find an analytical solution.

Interestingly, the only GME pure state of this family for which our construction fails is the maximally entangled state ($\theta = \pi/4$), which is already known to generate GMNL for any number of observers [20]. We have however found, numerically, several sets of measurements on this state that lead to distributions violating our inequality, but the amount of symmetries is reduced. Interestingly, theorem 4 implies that even states that are almost separable ($\theta \rightarrow 0$) can be used to generate GMNL correlations for any number of observers.

It is important to observe that we already knew from [27] that all $n$-qubit systems in a GME symmetric pure state are GMNL, which is a more general result than theorem 4. In particular, for three-qubit systems, the problem was completely solved: all three-qubit systems in a pure GME state can exhibit GMNL [26]. These results rely however on the violation of two families of Hardy-like paradoxes witnessing GMNL, making it directly untestable in an experiment (where even the smallest imperfections lead to values $P(abxy) = \epsilon > 0$). As mentioned already (see footnote 3), note that up to four parties it is possible to derive indirect experimental tests of GMNL from GM Hardy paradoxes, but this approach is currently unavailable for more parties. Although clearly not as general, our results are testable for any number of parties and might lead the way to a complete generalisation of Gisin’s theorem.
8. Constructing GM Bell inequalities from different seeds

The construction was so far done using as ‘seed’ the CHSH inequality \(I_{A:B}^{H:A}(5)\) to build new families of inequalities. We now show the versatility of our technique by using different inequalities as seed. In general, any inequality \(S_{A_1,\ldots,A_n}^{A_1,\ldots,A_n}\) for \(n < n\) parties that can be written as

\[
S_{A_1,\ldots,A_n}^{A_1,\ldots,A_n}(P(\vec{a}|\vec{x})) = P(\vec{0}|\vec{0}) - \sum_{\vec{a},\vec{x}=0,\overline{0}} \beta_{\vec{a},\vec{x}} P(\vec{a}|\vec{x}) \leq B_{2-\text{sep}} = 0
\]  

(30)

such that \(\beta_{\vec{a},\vec{x}} \geq 0\), \(\forall \vec{a}, \vec{x} \neq \vec{0}, \overline{0}\), and with biseparable bound \(B_{2-\text{sep}} = 0\), is a valid seed to build a Bell inequality for \(n\) parties. To see that, note that the key ingredient in our proofs is, again, that

\[
S_{A_1,\ldots,A_n}^{A_1,\ldots,A_n} \equiv S_{A_1,\ldots,A_n}^{A_1,\ldots,A_n} - P(\vec{0}|\vec{0}) = - \sum_{\vec{a},\vec{x}=0,\overline{0}} \beta_{\vec{a},\vec{x}} P(\vec{a}|\vec{x}) \leq 0
\]  

(31)

for any probability distribution. (This implies that the lifting of (31) to more parties is never positive \(S_{A_1,\ldots,A_n}^{A_1,\ldots,A_n} \leq 0\), which we used frequently in our proofs). Although the condition for a Bell inequality to be used as a seed is fairly simple, we do not know of a systematic way to find out which inequalities can be written in the form (30). We will now illustrate our construction with two different seeds.

8.1. The tilted CHSH inequality

As a first example, we use the ’tilted CHSH’ inequality [38] as the new seed. This inequality is a variation of the CHSH inequality with two free parameters, used for randomness certification [38] and self-testing of partial entangled states [39]. By setting one of the parameters to 1, the tilted CHSH inequality can be written in the form (30)

\[
I_{A:B}^{H:A} = P(00|00) - P(01|01) - P(10|10) - P(00|11) - \frac{3}{2} P_A(1|0) \leq 0,
\]  

(32)

where \(\beta \geq 0\) and \(P(a_i|x_i) = \sum_{a_i} P(a_i a_j x_i x_j), \forall x_j\), is the marginal distribution of party \(A_1\). Clearly, the inequality satisfies condition (31). Starting from the new seed \(I_{A:B}^{H:A}\) (32), we construct two new families of GMNL Bell inequalities.

Theorem 5. The families of inequalities

\[
I_{A_1,\ldots,A_n}^{H:A_1,\ldots,A_n} = \sum_{i=1}^{n-1} \sum_{j>i} I_{A_i,\overline{a},\overline{0}|\overline{0}} = \left( \frac{n-1}{2} \right) P(\overline{0}|\overline{0}) \leq 0,
\]  

(33)

\[
I_{A_1,\ldots,A_n}^{H:A_1,\ldots,A_n} = \sum_{j=1}^{n} I_{A_j,\overline{a},\overline{0}|\overline{0}} = (n-2) P(\overline{0}|\overline{0}) \leq 0
\]  

(34)

witness GMNL for any number \(n \geq 3\) of parties.

Proof. The proof that these two families of inequalities indeed witness GMNL for all \(n\) is exactly the same as for the families (20) and (21), but now using the seed \(I_{A:B}^{H:A}\) and property (31).

Although we cannot yet show that all the inequalities introduced in the previous theorem are useful, we can argue that at least some of them admit a quantum violation. Note that the CHSH inequality is a special case of the tilted family, a relation that also holds for the families of inequalities built with these seeds. Our argument follows now by continuity. Tilted inequalities ‘close’ enough to the CHSH give rise of inequalities of the families \(I_{A_1,\ldots,A_n}^{H:A_1,\ldots,A_n}\) and \(I_{A_1,\ldots,A_n}^{H:A_1,\ldots,A_n}\) that are also violated by some of the nonlocal quantum distributions that are detected by the CHSH-based family. We leave a more complete proof of usefulness of these inequalities for future work.

8.2. A tripartite inequality as seed

As a second example, we illustrate how to use a multipartite inequality as a seed. We chose a Bell inequality for three parties—that witnesses GMNL in tripartite correlations—that belongs to class 5 of [14] and that can be written as

\[
I_{A_1,\ldots,A_3}^{H:A_1,\ldots,A_3} = P(000|000) - P(010|111)
- P(000|011) - P(001|001) - P(100|110)
- P(010|010) - P(100|100) \leq 0
\]  

(35)

and hence satisfying condition (31). This allows us to construct, again, two new families of Bell inequalities witnessing GMNL for any \(n \geq 4\).
Theorem 6. The families of inequalities

\[ I^{A_i\ldots A_n}_{\text{tri,sym}} = \sum_{i=1}^{n-2} \sum_{j>i}^{n-1} \sum_{k>j}^{n} I^{A_iA_jA_k}_{\emptyset\emptyset} - \left( \frac{n-1}{3} \right) P(\emptyset\emptyset) \leq 0, \]  
\[ I^{A_i\ldots A_n}_{\text{tri,00}} = \sum_{j>2}^{n} I^{A_jA_k}_{\emptyset\emptyset} - (n-3)P(\emptyset\emptyset) \leq 0 \]

witness GMNL for \( n \geq 4 \) parties.

Proof. Although this proof is again analogous to the previous families, a tripartite inequality as the seed changes the weights associated to the term \( P(\emptyset\emptyset) \). For the symmetrical family (36), consider that a biseparable probability distribution (19) can violate at most \( \left( \frac{n-1}{3} \right) \) inequalities \( I^{A_iA_jA_k}_{\emptyset\emptyset} \) between three different parties. This comes from the fact that the best for a biseparable distribution is a grouping \( g = \{1, 2, \ldots, n-1\} \), \( \tilde{g} = \{n\} \) of the parties, allowing a maximum of \( (n-1)/3 \) to potentially violate the inequality \( I^{A_iA_jA_k}_{\emptyset\emptyset} \). For the centered family (37), consider that there are at most \( n-3 \) inequalities \( I^{A_iA_jA_k}_{\emptyset\emptyset} \) that can potentially be violated for a grouping \( g = \{1, 2, \ldots, n-1\} \), \( \tilde{g} = \{n\} \) of the parties. \( \square \)

To explore whether the constructed families of inequalities (36) and (37) are useful, we have numerically searched for quantum correlations leading to their violation in the four-party set-up. Using states of the form \( |\psi\rangle = \cos \theta |0000\rangle + \sin \theta |1111\rangle \) we have found measurements leading to the violation of both families of inequalities. For the symmetrical family \( I^{A_iA_jA_k}_{\text{tri,sym}} \), we have obtained violations for all states \(|\psi\rangle\) with \( \theta = \frac{\pi}{2} \) for \( h = 2, 3, \ldots, 9 \). For the family \( I^{A_iA_jA_k}_{\text{tri,00}} \) we managed to find a violation for states \(|\psi\rangle\) with \( \theta = \frac{\pi}{2} \) for \( h = 2, 3, \ldots, 13 \). These numerical results provide a first evidence that this family of inequalities is useful.

It would obviously be interesting to explore further up to which extent the inequalities that can be built as the ones in (33), (34), (36) and (37)—are useful to witness GMNL arising from quantum states. We leave this direction of research open for further work. Finally, it would also be insightful to consider seeds allowing for more measurement choices and/or outcomes.

9. Constructing \( k \)-way nonlocality Bell inequalities

Now we show how our construction also allows one to build families of Bell inequalities that witness intermediate types of multipartite nonlocality. Indeed, in the multipartite scenario it is possible to define a hierarchy of multipartite correlations taking into account the extent to which these are multipartite nonlocal. This can be measured, for example, by notions such as hierarchy of multipartite correlations taking into account the extent to which these are multipartite nonlocal. Indeed, in the multipartite scenario it is possible to define intermediate types of multipartite nonlocality. Now we show how our construction also allows one to build families of Bell inequalities that witness \( k^{-}\)-way nonlocality Bell inequalities. For the sake of simplicity, we use the seed inequality \( I^{A_iA_j}_{\emptyset\emptyset} \) to generalise our two families of Bell inequalities, symmetric and centered, for the detection of \( k^{-}\)-way nonlocality.

Theorem 7. The families of inequalities for \( n \) parties

\[ I^{A_i\ldots A_n}_{k-\text{sep,sym}} = \sum_{i=1}^{n-1} \sum_{j<i}^{n} I^{A_iA_jA_k}_{\emptyset\emptyset} - \left( \frac{n+1-k}{2} \right) P(\emptyset\emptyset) \leq 0, \]  
\[ I^{A_i\ldots A_n}_{k-\text{sep,00}} = \sum_{j>1}^{n} I^{A_jA_k}_{\emptyset\emptyset} - (n-k)P(\emptyset\emptyset) \leq 0 \]

witness \( k^{-}\)-way nonlocality (or non \( k^{-}\)- separability) for all \( n \geq 3 \), \( k < n \).
Proof. The proof follows the same line as the proofs for the other families of inequalities we have already constructed. By making $k$ groups instead of 2, one needs to count the maximum number of pairs of parties $A_i,A_j$ that can be made inside all the $k$ groups for the family (39). Indeed, only pairs $A_i,A_j$ inside the same group can potentially violate a lifted inequality $I_{0\bar{0}}^{A_i A_j}$. The best way to group $n$ parties into $k$ groups, in order to maximise the number of such pairs of parties, is to put $n - k + 1$ parties into one group and the remaining $k - 1$ ones into one group each. In this way, a maximum amount of $\binom{n + 1 - k}{2}$ inequalities $I_{0\bar{0}}^{A_i A_j}$ can potentially be violated, but these can be cancelled by the $\binom{n + 1 - k}{2}P(\bar{0}\bar{0})$ terms in (39) since $I_{0\bar{0}}^{A_i A_j} - P(\bar{0}\bar{0}) \leq 0$.

For the family (40), one needs to count the maximum number of pairs $A_i,A_j$ that can be made inside the group containing party $A_1$. By putting the maximal number of $n - k$ parties, plus party $A_1$, in one group, one gets that a maximum number of $n - k$ pairs $A_i,A_j$ can be formed. This implies that a maximum amount of $n - k$ inequalities $I_{0\bar{0}}^{A_i A_j}$ can potentially be violated, but these are cancelled by the $(n - k)P(\bar{0}\bar{0})$ term in (40) since $I_{0\bar{0}}^{A_i A_j} - P(\bar{0}\bar{0}) \leq 0$.

Observe that for both families of inequalities and any fixed number of parties $n$, correlations violating one of the inequalities detecting $k$-way nonlocality will also violate the inequalities in the same family for $k'$-way nonlocality for all $k' > k$. This property is easily derived observing that

$$I_{k-\text{sep},\text{sym}}^{A_1\ldots A_n} = I_{k-\text{sep},\text{sym}}^{A_1\ldots A_{k'}} = \alpha(k, k') P(\bar{0}\bar{0}) \leq 0,$$

$$I_{k-\text{sep},\text{sym}}^{A_1\ldots A_n} = I_{k-\text{sep},\text{sym}}^{A_1\ldots A_{k'}} = \beta(k, k') P(\bar{0}\bar{0}) \leq 0,$$

where

$$\alpha(k, k') = \binom{n + 1 - k}{2} - \binom{n + 1 - k'}{2}$$

$$\beta(k, k') = k' - k$$

and $n \geq k'$ is chosen to be fixed. The functions $\alpha(k, k')$ and $\beta(k, k')$ are always positive when $k' > k$. A distribution violating $I_{k-\text{sep},\text{sym}}^{A_1\ldots A_n}$ necessarily has $P(\bar{0}\bar{0}) > 0$ (since it is the only positive term of the inequality). It is now easy to conclude that the same correlation will give $I_{k-\text{sep},\text{sym}}^{A_1\ldots A_{k'}} > 0$, that is, the correlation will be detected by inequalities for lower level of nonlocality. The same holds for the centered family $I_{k-\text{sep},\text{sym}}^{A_1\ldots A_n}$.

As a consequence of this observation, we conclude that the symmetric family $I_{k-\text{sep},\text{sym}}^{A_1\ldots A_n}$ admits quantum violations for every level of nonlocality $k$. Indeed, theorem 4 can be generalised to all these inequalities: one can use the local measurements found in the theorem on any state of the form $|\text{GHZ}_n\rangle_\theta = \cos \theta |0\rangle^n - \sin \theta |1\rangle^n$ (27) with $\theta \in [0, \frac{\pi}{4}]$ to obtain a violation of $I_{k-\text{sep},\text{sym}}^{A_1\ldots A_n}$ for all $k$ and $n$.

10. Conclusion

In this work, we have introduced a versatile technique to build Bell inequalities for the $n$-partite scenario. It consists of taking a ‘seed’—a Bell inequality for $m < n$ parties obeying a simple constraint—to generate new families of Bell inequalities. Intuitively, the seed defines the nonlocal game that will be played by numerous groups of $m$ parties in the $n$-partite system. The specification of the sets of parties that are required to play the nonlocal game defines the level of multipartite nonlocality to be detected. Indeed, our construction can be used to witness $k$-nonlocal multipartite correlations, including the stronger notion of GM nonlocality (where all the parties of the system are nonclassically correlated).

To illustrate the power of our construction, we have first used the CHSH inequality as the seed to build two new families of Bell inequalities, $I_{\text{sym}}$ and $I_{\bar{0}}$, for the detection of GM nonlocality in systems with any number $n \geq 3$ of parties. We showed that these families are particularly useful for the detection of GMNL in genuine multipartite entangled (GME) pure states. Indeed, for $n = 3$ we proved that $I_{\bar{0}}^{A_1 A_2 A_3}$ is able to detect genuine tripartite nonlocality in all genuine entangled three-qubit pure states invariant under the permutation of two parties. We also showed that the family $I_{\text{sym}}$ witnesses GMNL in all pure GME GHZ-like states $\cos \theta |0\rangle^n - \sin \theta |1\rangle^n$, including those which are almost product. Note that for three parties, $I_{\text{sym}}^{A_1 A_2 A_3}$ coincides with a Bell inequality found in [14], where numerical evidence was given that it detects GMNL in all GME three-qubit pure states. We extended this numerical evidence to the four-partite case, using $I_{\text{sym}}^{A_1 A_2 A_3}$ to detect GMNL in all pure GME four-qubit states. Taking into account these partial results and the operational meaning of the
family $I_{\text{sim}}$, we conjecture that this single family of Bell inequalities can be used to show that all GME pure states display GMNL, establishing a direct relation between GM notions of pure state entanglement and nonlocality.

Apart from a proof in full generality, which seems not straightforward, it would be interesting to extend these results to more families of GM pure states. A possibility is to study the multipartite $W$-state, $|W_n\rangle = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} |i\rangle \otimes |0\rangle$, for which we already managed to obtain, numerically, violations up to $n = 5$. It would also be interesting to use further the characterisation of all three-qubit systems in a pure state [40] to obtain an analytical proof that our inequality for $n = 3$ detects GMNL when these states are GME by detecting also the states which have no symmetries. Further numerical exploration, for more observers or systems of larger dimensions, is another possibility.

We have further used our technique to build families of Bell inequalities taking two different seeds: the bipartite ‘tilted CHSH inequality’ and a tripartite inequality witnessing GMNL from [14]. We have also shown how to design families of Bell inequalities that detect $k$-way multipartite nonlocality. In all cases, the construction is quite straightforward, which shows the potential and versatility of the method.

For future work, it would be interesting to further explore the applicability of the Bell inequalities built through our method. For instance, can the family of Bell inequalities with the tilted CHSH inequality as a seed (32) be used to self-test certain classes of multipartite entangled pure states? Also generalising our seed to more settings and/or outcomes has the potential for generating Bell inequalities fit to detect multipartite correlations in a whole range of new different scenarios.

### Acknowledgements

This work is supported by the ERC CoG QITBOX, the AXA Chair in Quantum Information Science, the Templeton Foundation, the Spanish MINECO (QIBEQI FIS2016-80773-P and Severo Ochoa SEV-2015-0522), Fundacio Cellex, Generalitat de Catalunya (CERCA Program and SGR1381).

### Appendix A. Lifting Bell inequalities to more observers

The technique of lifting a Bell inequality consists in taking an inequality designed for a specific Bell set-up—with a fixed number of observers, measurements and outcomes—and extending it to a set-up with an increased number of any of these variables. Here we are interested in lifting a Bell inequality to more observers. We will briefly review its definition and prove one property of these inequalities, which is used in theorems 1 and 3.

Consider a Bell inequality for two observers (4) that, without loss of generality, can be written as

$$I = \sum_{a_1,a_2,x_1,x_2} c_{a_1,a_2}^{x_1,x_2} P(a_1,a_2|x_1,x_2) \leq 0,$$

where observer $A_1$ performs a measurement $x_1$ and obtains an outcome $a_1$. The coefficients $c_{a_1,a_2}^{x_1,x_2}$ are real numbers and $P(a_1,a_2|x_1,x_2)$ represents the observed outcome distribution, for each measurement pair. A lifting of this Bell inequality to $n$ observers consists in extending the expression (A1) by choosing a fixed measurement and outcome for observers $A_3, \ldots, A_n$:

$$I_{A_1A_2}^{\tilde{0}\tilde{0}} = \sum_{a_3,a_4,x_3,x_4} c_{a_3,a_4}^{x_3,x_4} P(a_1,a_2,\tilde{0}|x_3,x_4,\tilde{0}) \leq 0,$$

where, without loss of generality, the fixed $n-2$ measurements and outcomes are set to $\tilde{0} = \{0, \ldots, 0\}$. Notice that $P(a_1,a_2,\tilde{0}|x_3,x_4,\tilde{0}) = P(a_1,a_2|x_3,\tilde{0},\tilde{0}) P(\tilde{0}|\tilde{0})$, where $P(\tilde{0}|\tilde{0})$ is independent of measurements $x_3$ and $x_4$ according to the NS principle. This means that if the conditional distribution $P(\tilde{0}|\tilde{0},a_1,a_2|x_3,x_4) \equiv P(a_1,a_2|x_3,\tilde{0},\tilde{0})$ violates the bipartite inequality (A1), it implies that the full distribution $P(\tilde{0}|\tilde{0})$ violates the lifted inequality (A2). Therefore, the nonlocality of the conditional distribution is a sufficient condition for the nonlocality of the full distribution.

We now want to show that any biseparable distribution (19) where parties $A_1$ and $A_2$ belong to different groups of parties, $A_1 \in g$ and $A_2 \in \tilde{g}$, does not violate a lifted Bell inequality (A2):

$$I_{\tilde{0}\tilde{0}}^{\tilde{A}_1\tilde{A}_2}(P_{\text{bisp}}) \leq 0.$$

Since a Bell inequality is a linear function and $P_{\text{bisp}}$ is convex, it is enough to show that the previous inequality holds for any pure biseparable distribution $P(\tilde{a}_1|\tilde{x}_1) P(\tilde{a}_2|\tilde{x}_2)$. We have then
\begin{equation}
T_{\theta|0}^{A_1 A_2}(P(\tilde{a}_1|x_1)P(\tilde{a}_2|x_2)) = \sum_{a_1, a_2, x_1, x_2} \tilde{c}_{a_1, a_2, x_1, x_2}^{A_1 A_2} P_{A_1}(a_1|x_1, \theta, 0) P_{A_2}(a_2|x_2, \theta, 0) P_{\tilde{A}_1}(0|\theta) P_{\tilde{A}_2}(0|\theta) \leq 0,
\end{equation}

where the size of the vector $\tilde{0}$ should be clear by the context. Notice that we have again used the fact that the distributions are NS and that $P_{\tilde{A}_1}(a_1|x_1, \theta, 0)$ are well-defined local distributions.

**Appendix B. Proof of theorem 2**

We here provide a formal proof of theorem 2. Let us start with an observation that will be used in the upcoming proof.

**B.1. Observation**

On any pure, non-maximally entangled, two-qubit state

\begin{equation}
|\phi_0\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle
\end{equation}

i.e. for $\theta \in ]0, \frac{\pi}{4}[$, the measurements

\begin{align*}
M_{00} &= \cos(\alpha)|0\rangle + e^{i\phi}\sin(\alpha)|1\rangle \\
M_{11} &= \cos^2(\theta)\cos(\alpha)|0\rangle + e^{i\phi}\sin^2(\theta)\sin(\alpha)|1\rangle \\
N_{00} &= \sin^2(\theta)e^{i\phi}\sin(\alpha)|0\rangle - \cos^2(\theta)\cos(\alpha)|1\rangle \\
N_{11} &= \sin(\theta)\sin(\alpha)e^{i\phi}|0\rangle - \cos(\theta)\cos(\alpha)|1\rangle
\end{align*}

lead to correlations

\begin{equation}
P_{\theta}(ab|xy) = \langle \phi_0|(M_{a|x} \otimes N_{b|y})|\phi_0\rangle
\end{equation}

that violate inequality (5) $I_{A_1 A_2}(P_{\theta}(ab|xy)) > 0$ with the free parameters $\alpha$ and $\phi$ such that $\alpha = 0, \pi/2$ for any $\theta = 0, \pi/4$. More precisely, they lead to the particular violation of the inequality (5)

\begin{equation}
I_{A_1 A_2}(P_{\theta}(ab|xy)) = P_y(00|00) > 0
\end{equation}

and thus $P_y(01|01) = P_y(10|10) = P_y(00|11) = 0$, i.e. a realisation of the bipartite Hardy paradox [28]. A proof of this observations can be found further in the appendix C.

Since we are interested in a violation up to any extent of our inequality

\begin{equation}
I_{A_1 A_2}(P_{\theta}(ab|xy)) > 0
\end{equation}

whose bound is zero, we have taken the freedom not to normalise some of the measurements in (B.2). In other words, the observation (B.1) implies that for any non-maximally entangled pure two-qubit state $|\phi_0\rangle$ (B1), one can chose one of the measurement of one of the parties for free (as expressed by the free parameters $\alpha$ and $\phi$ such that $\alpha = 0, \pi/2$) and still find three other measurements such that the generated correlations violated the inequality.

Now we want to show that a large class of three qubit GME states violate inequality $I_{A_1 A_2}(8)$. In [36], it was shown that all three qubits in a pure state could be written as

\begin{equation}
|\Psi_i\rangle = h_0|000\rangle + h_1 e^{i\phi}|100\rangle + h_2|011\rangle + h_3|110\rangle + h_4|111\rangle,
\end{equation}

where $h_i \geq 0, \sum_i h_i^2 = 1$ and $\phi \in ]0, \pi[$. On these states, we impose the additional constrain that $h_2 = h_3$, i.e. we consider only the states (B5) which are symmetrical with respect to the permutations of the parties $A_2 \leftrightarrow A_3$. By relabelling the parties’ index, however, any state which is symmetrical with respect to the permutation of two out of the three parties can be transformed to one where the symmetry is between parties $A_2$ and $A_3$, which we chose without loss of generality. Now, party $A_2$ and $A_3$ both make the same projective measurement $\langle m_{a|x} \rangle$ for their input choice $x_2 = x_3 = 0$

\begin{equation}
\langle m_{a|x} \rangle = \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle
\end{equation}

for some (yet) free angle $\alpha$\textsuperscript{4}. The state that is prepared between parties $A_1 A_3$ (resp. $A_1 A_2$) from party $A_2$ ($A_3$) by performing measurement $\langle m_{a|x} \rangle$ (B6) on the state $|\Psi_i\rangle$ (B5) conditioned on obtaining the outcome $a_2 = 0$ ($a_3 = 0$) is

\begin{equation}
|\psi^{A_1 A_3}_{0|0}\rangle = |\psi^{A_1 A_3}_{0|0}\rangle \propto \cos(\alpha)h_0|00\rangle + (\cos(\alpha)h_1 + \sin(\alpha)h_2)|10\rangle + (\cos(\alpha)h_1 + \sin(\alpha)h_4)|11\rangle
\end{equation}

since $h_2 = h_3$ and that both the state $|\Psi_i\rangle$ (B5) and measurements $\langle m_{a|x} \rangle$ are symmetrical with respect to permutation $A_2 \leftrightarrow A_3$. Using the concurrence, the state $|\psi^{A_1 A_3}_{0|0}\rangle$ (B7) is entangled if and only if

\textsuperscript{4} Remark additionally that we do not make use of a potential second degree of freedom (the phase).
\[
\begin{pmatrix}
\cos(\alpha)h_0 & 0 \\
\cos(\alpha)h_1 + \sin(\alpha)h_2 & \cos(\alpha)h_2 + \sin(\alpha)h_4
\end{pmatrix} \neq 0 \\
\Leftrightarrow \cos(\alpha)h_0(\cos(\alpha)h_2 + \sin(\alpha)h_4) \neq 0
\]

(B8)

leading to the four conditions

\[\alpha \approx \frac{\pi}{2},\]

(B9)

\[\tan(\alpha) \approx -\frac{h_1}{h_4},\]

(B10)

\[h_0 \neq 0,\]

(B11)

\[h_2 \approx 0 \approx h_4.\]

(B12)

First, remark that both conditions (B11) and (B12) only mean that the state \(|\psi_3\rangle\) (B5) needs to be GME (as well as symmetrical \(h_2 = h_3\)). Now, since the parameter \(\alpha\) is free, we choose to avoid the two values \(\alpha = \frac{\pi}{2}\) and 

\[\alpha = -\arctan\left(\frac{h_1}{h_4}\right) \approx 0.\]

In the end, one can tune continuously the parameter \(\alpha\) (up to the forbidden values (B9) and (B10)) so that the prepared states \(|\psi_{0,1,2,3}\rangle = |\psi_{0,1,2,3}\rangle\) are not maximally entangled. One can then use observation B.1, as well as the symmetries (A) that was imposed on both state and measurements, to obtain

\[
I_{03}^{\mu} A_3 = I_{03}^{0,0} + I_{03}^{1,1} = P(000|000) = 2P(000|000) - P(000|000) \\
= P(000|000) - 2P(010|010) - 2P(010|010) - 2P(000|110) > 0
\]

(B13)

by choosing \(A_1\)’s measurements as in (B2) for the prepared (non maximally entangled) state \(|\psi_{0,1,2,3}\rangle\) (B7), i.e. realising

\[
P(000|000) > 0 \\
P(010|010) = 0 \\
P(100|100) = 0 \\
P(000|110) = 0.
\]  

(B14)

**Appendix C. Hardy’s measurements for \(n = 2\)**

From the realisation (B3), we have four conditions

\[
P(00|00) > 0 \\
P(01|01) = 0 \\
P(10|10) = 0 \\
P(00|11) = 0
\]

(C1)

to be satisfied by the measurement \(M_{0|x}\) and \(N_{0|y}\) made on the state \(|\phi_0\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle\) written in it’s Schmidt basis by A and B respectively. We start by choosing \(M_{0|0} = \cos \alpha \langle 0 | + \sin \alpha e^{i\delta} \langle 1 |\) freely and then try to satisfy these four conditions. From \(P(01|01) = 0\) we get that

\[
\begin{align*}
\cos(\alpha) \langle 0 | + \sin \alpha e^{i\delta} \langle 1 | & \otimes N_{0|11} \cdot (\cos(\theta) |00\rangle + \sin(\theta) |11\rangle) = 0 \\
\Leftrightarrow N_{0|11} \langle \cos \alpha \cos \theta |0 \rangle + e^{i\delta} \sin \alpha \sin \theta |1 \rangle) & = 0 \\
\Leftrightarrow N_{0|11} \alpha \sin \sin \sin \theta |0 \rangle & - \cos \alpha \cos \theta |1 \rangle,
\end{align*}
\]

(C2)

where we use non normalized measurements, which, again, does not make a difference when interested in conditions of the form \(P(a_0,a_2|x_1,x_2) = 0\) or \(P(a_0,a_2|x_1,x_2) > 0\). Considering projective two-outcome measurements:

\[
N_{1|11} \propto \cos \alpha \cos \theta |0 \rangle + e^{-i\delta} \sin \alpha \sin \theta |1 \rangle.
\]  

(C3)

Then, with condition \(P(00|11) = 0\)

\[
\begin{align*}
M_{1|1} \otimes N_{1|11} \langle \cos(\theta) |00\rangle + \sin(\theta) |11\rangle) & = 0 \\
\Leftrightarrow M_{1|1} \langle \cos \alpha \cos^2 \theta |0 \rangle + e^{-i\delta} \sin \alpha \sin^2 \theta |1 \rangle) & = 0 \\
\Rightarrow M_{1|1} \propto e^{-i\delta} \sin \alpha \sin^2 \theta |0 \rangle & - \cos \alpha \cos^2 \theta |1 \rangle, \\
\Rightarrow M_{0|1} \propto \cos \alpha \cos^2 \theta |0 \rangle + e^{i\delta} \sin \alpha \sin^2 \theta |1 \rangle.
\end{align*}
\]

(C4)
Finally, from condition $P(10|10) = 0$

\[ M_{01} \otimes N_{010}(\cos(\theta)|00\rangle + \sin(\theta)|11\rangle) = 0 \]

\[ \Rightarrow N_{010} \propto e^{i\theta} \sin \alpha \sin^3 \theta |0\rangle - \cos \alpha \cos^3 \theta |1\rangle, \quad (C6) \]

\[ \Rightarrow N_{01|1} \propto \cos \alpha \cos^3 \theta |0\rangle + e^{-i\theta} \sin \alpha \sin^3 \theta |1\rangle. \quad (C7) \]

Now one can check that with these measurements on the state $\cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ gives:

\[ M_{01} \otimes N_{010}(\cos(\theta)|00\rangle + \sin(\theta)|11\rangle) \propto \ldots = -\frac{e^{i\theta}}{8} \sin 2\alpha \sin 4\theta. \quad (C8) \]

That is equal to zero—i.e. $P(00|00) = 0$—if and only if $\alpha = 0, \pi/2$. In the end, the conditions (C1) are satisfied for these measurements for all non-maximally entangled states with any set of measurements of the form

\[ M_{01} = \cos(\alpha) |0\rangle + e^{i\theta} \sin(\alpha) |1\rangle \]

\[ M_{011} \propto \cos^2(\theta) \cos(\alpha) |0\rangle + e^{i\theta} \sin^2(\theta) \sin(\alpha) |1\rangle \]

\[ N_{010} \propto \sin^2(\theta) \sin(\alpha) |0\rangle - \cos^2(\theta) \cos(\alpha) |1\rangle \]

\[ N_{01|1} \propto \sin(\theta) \sin(\alpha) |0\rangle - \cos(\theta) \cos(\alpha) |1\rangle \quad (C9) \]

except for the forbidden values of $\alpha = 0, \pi/2$.

Appendix D. Properties of our families of Bell inequalities

D.1. The family of Bell inequalities $I_{\text{sym}}^{A_1A_2\ldots A_n}(20)$ witnesses GMNL

In this section we want to give a more detailed proof of theorem 3, which states that for any number $n \geq 3$ of observers, all biseparable distributions (19) satisfy our family of inequalities (20)

\[ I_{\text{sym}}^{A_1A_2\ldots A_n} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} I_{ij|0}^{A_iA_j} \leq 0, \quad (D1) \]

where \( \left( \frac{n-1}{2} \right) = \frac{(n-1)(n-2)}{2} \) and thus $I_{\text{sym}}^{A_1A_2\ldots A_n}$ witnesses GMNL in the distributions. The proof for the family of inequalities $I_{ij}^{A_iA_j}(21)$ follows exactly the same lines.

**Proof.** Our Bell inequalities $I_{ij}^{A_iA_j}$ are invariant under permutations of the observers. Since a Bell inequality is a linear function of the probability terms $P(\bar{a}_i|x_i)$, and by the convexity of biseparable distributions (19), we can restrict the proof—without loss of generality—to pure biseparable distributions of the form

\[ P_{m/(n-m)} \equiv P(a_1x_1 \ldots a_n|x_1x_2 \ldots x_m)P(a_{m+1}x_{m+2} \ldots a_n|x_{m+1}x_{m+2} \ldots x_n), \quad (D2) \]

where the first term includes the variables of the $m$ first observers and the second the remaining $n - m$. Let us recall that, inside each group, observers are allowed to share any NS nonlocal resources. Our proof consists in counting how many lifted inequalities $I_{ij}^{A_iA_j}(22)$ can be violated by a pure biseparable distribution (D2). We will see that this happens to at most \( \left( \frac{n-1}{2} \right) \) lifted inequalities. Indeed, a term $I_{ij}^{A_iA_j}$ can only be positive if observers $A_i$ and $A_j$ belong to the same group ($i,j \leq m$ or $i,j > m$), since otherwise there are only classically correlated (see appendix A). Thus

\[ I_{ij}^{A_iA_j}(P_{m/(n-m)}) \leq \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} I_{ij|0}^{A_iA_j} + \sum_{k=m+1}^{n} \sum_{l=k}^{n} I_{ij|0}^{A_iA_j} - \left( \frac{n-1}{2} \right) P(\bar{0}|\bar{0}) \]

\[ = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} I_{ij|0}^{A_iA_j} - \left( \frac{m}{2} \right) P(\bar{0}|\bar{0}) + \sum_{k=m+1}^{n} \sum_{l=k}^{n} I_{ij|0}^{A_iA_j} - \left( \frac{n-m}{2} \right) P(\bar{0}|\bar{0}) \]

\[ - (m-1)(n-m-1)P(\bar{0}|\bar{0}) \]

\[ = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} I_{ij|0}^{A_iA_j} + \sum_{k=m+1}^{n} \sum_{l=k}^{n} I_{ij|0}^{A_iA_j} - (m-1)(n-m-1)P(\bar{0}|\bar{0}) \]

\[ \leq - (m-1)(n-m-1)P(\bar{0}|\bar{0}) \quad (D3) \]

in which we have used the fact that $\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} I_{ij|0}^{A_iA_j}$ contains \( \left( \frac{m}{2} \right) \) lifted terms and $\sum_{k=m+1}^{n} \sum_{l=k}^{n} I_{ij|0}^{A_iA_j}$ contains \( \left( \frac{n-m}{2} \right) \) of them. We have further used $I_{ij|0}^{A_iA_j} \equiv I_{ij|0}^{A_iA_j} - P(\bar{0}|\bar{0}) \leq 0$, for any $i,j$ (see equation (23)). Notice that
the situation where most lifted terms $I^A_{0|0}^{A_j}$ could be positive occurs for bipartitions of one versus $n - 1$
observers, hence the $\left(\frac{n-1}{2}\right)$ factor in our Bell inequalities (D1).

\[\square\]

D.2. A recursive formula for our inequalities

Our family of Bell inequalities $I^A_{sym} \to A_k$ can also be written in a recursive form, which shows its rich multipartite
structure and operational meaning:

\[I^A_{sym} \to A_k = \frac{1}{n-2} \sum_{i=1}^{n} I^\text{all\ \!\!}_{0|0}^{A_i} - P(\bar{0}|\bar{0}) \leq 0 \quad \text{(D4)}\]

for $n \geq 3$, where $I^\text{all\ \!\!}_{0|0}^{A_i}$ is the Bell inequality testing genuine nonlocality between $n - 1$ parties lifted to $n$ parties,
with party $A_i$'s input and outcome set to 0

\[I^\text{all\ \!\!}_{0|0}^{A_i} = \frac{1}{n-3} \sum_{j=i}^{n} I^\text{all\ \!\!}_{0|0}^{A_i} - P(\bar{0}|\bar{0}). \quad \text{(D5)}\]

The seed of this recursive expression is the variant of the CHSH inequality (5).

**Proof.** We prove that the recursive expression (D4) is equivalent to the direct expression (D1) for $I^A_{sym} \to A_k$
through mathematical induction. First, we check that for $n = 3$ the equivalence holds, which can easily be done
by developing both expressions. Then, we show that if the equivalence is true for $n$, it implies that it is true also
for $n + 1$.

Suppose the equivalence holds for $n$:

\[\frac{1}{n-2} \sum_{i=1}^{n} I^\text{all\ \!\!}_{0|0}^{A_i} - P(\bar{0}|\bar{0}) = \sum_{i=1}^{n-1} \sum_{j>i}^{n} I^A_{0|0}^{A_i} - \left(\frac{n-1}{2}\right) P(\bar{0}|\bar{0}). \quad \text{(D6)}\]

For $n + 1$, we develop the recursive expression in (D4), where $I^\text{all\ \!\!}_{0|0}^{A_i}$ is now an $n$ observer inequality for
which the recurrence hypothesis (D6) can be used:

\[\frac{1}{n-1} \sum_{i=1}^{n+1} I^\text{all\ \!\!}_{0|0}^{A_i} - P(\bar{0}|\bar{0}) = (D6) \quad \frac{1}{n-1} \sum_{i=1}^{n+1} \left( \sum_{j=1}^{n+1} \sum_{k>j}^{n+1} I^A_{0|0}^{A_j} - \left(\frac{n+1}{2}\right) P(\bar{0}|\bar{0}) - P(\bar{0}|\bar{0}) \right)
\]

\[= \frac{1}{n-1} \sum_{i=1}^{n+1} \left( \sum_{j=1}^{n+1} \sum_{k>j}^{n+1} I^A_{0|0}^{A_j} - \left(\frac{n+1}{2}\right) - \left(\frac{n-1}{2}\right) + 1 \right) P(\bar{0}|\bar{0}). \quad \text{(D7)}\]

Note that the last expression can be simplified taking into account that the terms $I^{A_{j}A_{k}}_{0|0}$ are being counted
multiple times. Since the inequalities are invariant under permutations of observers, we can restrict our attention
to counting how many times the particular term $I^{A_{j}A_{k}}_{0|0}$ appears in (D7). One can check that $\sum_{j=1}^{n} \sum_{k>j}^{n+1} I^{A_{j}A_{k}}_{0|0}$ gives
one term $I^{A_{j}A_{k}}_{0|0}$ if $i \neq 1, 2$. Summing over $i$, we get a total of $n - 1$ terms, from which we obtain

\[\frac{1}{n-1} \sum_{i=1}^{n+1} \left( \sum_{j=1}^{n+1} \sum_{k>j}^{n+1} I^A_{0|0}^{A_j} - \left(\frac{n+1}{2}\right) - \left(\frac{n-1}{2}\right) + 1 \right) P(\bar{0}|\bar{0}) = \sum_{j=1}^{n+1} \sum_{k>j}^{n+1} I^A_{0|0}^{A_j} - \left(\frac{n}{2}\right) P(\bar{0}|\bar{0}), \quad \text{(D8)}\]

where we used $\frac{n+1}{n-1} \left(\frac{n-1}{2}\right) + 1 = \frac{n}{2}$. Since the last expression coincides with the direct expression (D1)
for $n + 1$ observers, we finish our proof. \[\square\]

D.3. Fully local strategies that saturate the inequalities

Interestingly, one can check that the (fully) local strategy

\[P_L(a_1, a_2, \ldots, a_n | x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } a_i = 1 \quad \forall i \text{ and } \forall x_i \\ 0 & \text{else} \end{cases} \quad \text{(D9)}\]
saturates our families of inequalities (20) and (21) since there is no term in the inequalities where all outcomes have value 1. Nonlocal resources shared between a subset of the observers are thus useless to reach better bounds on our family, only nonlocal resources shared between all observers are relevant. Remark that these observation generalise to all the families of inequalities that have the CHSH inequality $I^{A_i - A_j}$ (5) as seed.

D.4. Post-quantum NS resources that violate the inequalities

Consider a GM generalisation of the (no-signalling) PR-box [32]:

$$P_{NS}(d|\vec{x}) = \begin{cases} \frac{1}{2^{n-1}} & \text{if } \bigoplus_{i=1}^{n} a_i = \bigoplus_{i=1}^{n} \bigoplus_{j>i}^{n} x_i x_j, \\ 0 & \text{else} \end{cases}$$

(D10)

where the marginal distributions are completely random, i.e. $P_{NS}(a_i|x_i) = \frac{1}{2}$, $\forall i$. It is interesting to see that this post-quantum NS distribution violates our Bell inequalities $I^{A_i A_j - A_k}_{sym}$, for all $n \geq 2$

$$I^{A_i A_j - A_k}_{sym}(P_{NS}) = \frac{n-1}{2^{n-1}} > 0.$$  

(D11)

**Proof.** The proof follows from direct evaluation of our inequalities (D1) with the NS box (D10). First, we get that $I^{A_i A_j}_{0|0}(P_{NS}) = \frac{(2^n-1)}{2} P_{NS}(0|0) \quad \forall i, j$ because $P_{NS}(0|0)$ is the only non-vanishing term. Then

$$I^{A_i A_j - A_k}_{sym}(P_{NS}) = \sum_{i=1}^{n} \sum_{j>i}^{n} P_{NS}(0|0) - \left( \frac{n-1}{2} \right) P_{NS}(0|0)$$

$$= \frac{1}{2^{n-1}} \left[ \binom{n}{2} - \left( \frac{n-1}{2} \right) \right] = \frac{n-1}{2^{n-1}} > 0, \quad \forall n \geq 2$$

(D12)

which finishes our proof.

Appendix E. All pure GME states of the family $|GHZ^n|_0 = \cos \theta |0\rangle^{\otimes n} - \sin \theta |1\rangle^{\otimes n}$ generate GMNL correlations

Here we prove theorem 4 in detail.

**Proof.** Our proof is constructive as we will provide, for all states

$$|GHZ^n|_\theta = \cos \theta |0\rangle^{\otimes n} - \sin \theta |1\rangle^{\otimes n}$$

(E1)

with $\theta \in [0, \frac{\pi}{2}]$, local measurements that lead to explicit distributions $P_{GHZ^n}(d|\vec{x})$ violating our family of inequalities $I^{A_i - A_j}_{0|0}$ (20).

In order to provide symmetry to the problem, and significantly reduce the degrees of freedom, all the observers use the same projective measurements $m_{a_i|x_i} = m_{a_j|x_j}$:

$$m_{0|x} = \cos \alpha_x |0\rangle + \sin \alpha_x |1\rangle$$

$$m_{1|x} = \sin \alpha_x |0\rangle - \cos \alpha_x |1\rangle$$

(E2)

Since both the state (E1) and measurements (E2) are invariant under permutations of the observers, the generated distribution $P_{GHZ^n}(d|\vec{x})$ also has this symmetry. For three observers for example, we get that $P(100|100) = P(010|010) = P(001|001)$ or $P(000|011) = P(000|101) = P(000|110)$ or that the lifted inequalities are all equal $I^{A_i A_j}_{0|0} = I^{A_i A_j}_{0|0} = I^{A_i A_j}_{0|0}$. This implies that inequalities $I^{A_i - A_j}_{0|0}$ (21), when evaluated on the generated distributions, simplify to

$$I^{A_i A_j - A_k}_{sym}(P_{GHZ^n}) = \left( \frac{n}{2} \right) I^{A_i A_j}_{0|0} - \left( \frac{n-1}{2} \right) P(0|0)$$

$$= (n-1) P(000|000) - 2 \left( \frac{n}{2} \right) P(100|100) - \left( \frac{n}{2} \right) P(000|110),$$

(E3)

where we have used that $\binom{n}{2} - \left( \frac{n-1}{2} \right) = n - 1$. Using measurements (E2) on the state (E1) we obtain all the terms of (E3)
\[ P(00\bar{0}|00\bar{0}) = (\cos^4(\alpha_0)\cos(\theta) - \sin^4(\alpha_0)\sin(\theta))^2 \]
\[ P(10\bar{0}|10\bar{0}) = (\cos^{-1}(\alpha_0)\cos(\theta) + \sin^{-1}(\alpha_0)\cos(\theta))^2 \]
\[ P(00\bar{1}|11\bar{0}) = (\cos^{-2}(\alpha_0)\cos(\theta) - \sin^{-2}(\alpha_0)\sin(\theta))^2. \] (E4)

We want now to find angles \( \alpha_x \) of the local measurements (E2) such that the quantity (E3) is always positive.

A particular solution is
\[ \begin{cases}
    P(00\bar{0}|00\bar{0}) > 0 \\
    P(10\bar{0}|10\bar{0}) = P(00\bar{0}|11\bar{0}) = 0
\end{cases} \] (E5)

which holds true for angles
\[ \alpha_0 = \arctan(\tan^{-3}\theta) \]
\[ \alpha_1 = -\arctan(\tan^{-3}\theta) \] (E6)

when \( \theta \in [0, \frac{\pi}{4}] \). The value of the inequalities at these angles is
\[ I_{\text{sym}}^{A_1 \rightarrow A_4}(\mathcal{P}_{\text{GZT}}(d|\mathcal{X})) = (n - 1) P(00\bar{0}|00\bar{0}) \]
\[ = (n - 1)(\cos^4(\arctan(\tan^{-3}\theta))\cos(\theta) - \sin^4(\arctan(\tan^{-3}\theta))\sin(\theta))^2 \] (E7)

which is positive for \( \theta \in [0, \pi/4] \), as promised.

For the maximally entangled state \( \theta = \pi/4 \) we have \( P(0|0) = 0 \), which means that our construction breaks. However, this state is already known to be GM nonlocal for all number of observers [20], and moreover we numerically found several sets of measurements on it that lead to a violation of our inequalities.

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