QUALITATIVE GRAPH LIMIT THEORY.
CANTOR DYNAMICAL SYSTEMS AND CONSTANT-TIME DISTRIBUTED ALGORITHMS

GÁBOR ELEK

Abstract. The goal of the paper is to lay the foundation for the qualitative analogue of the classical, quantitative sparse graph limit theory. In the first part of the paper we introduce the qualitative analogues of the Benjamini-Schramm and local-global graph limit theories for sparse graphs. The natural limit objects are continuous actions of finitely generated groups on totally disconnected compact metric spaces. We prove that the space of weak equivalent classes of free Cantor actions is compact and contains a smallest element, as in the measurable case. We will introduce and study various notions of almost finiteness, the qualitative analogue of hyperfiniteness, for classes of bounded degree graphs. We prove the almost finiteness of a new class of étale groupoids associated to Cantor actions and construct an example of a nonamenable, almost finite totally disconnected étale groupoid, answering a query of Suzuki. Motivated by the notions and results on qualitative graph limits, in the second part of our paper we give a precise definition of constant-time distributed algorithms on sparse graphs. We construct such constant-time algorithms for various approximation problems for hyperfinite and almost finite graph classes. We also prove the Hausdorff convergence of the spectra of convergent graph sequences in the strongly almost finite category.

Keywords. qualitative graph limits, weak equivalence of Cantor actions, almost finiteness, constant-time distributed algorithms, spectral convergence

2010 Mathematics Subject Classification. 37B05, 68W15.

The author was partially supported by the ERC Consolidator Grant ”Asymptotic invariants of discrete groups, sparse graphs and locally symmetric spaces” No. 648017.
## Contents

1. Introduction and basic definitions  
   1.1. Qualitative graph limits  
   1.2. The space of free Cantor actions  
   1.3. Almost finiteness  
   1.4. Constant-time distributed algorithms  
2. Graph convergence in the naive sense  
   2.1. The space of rooted connected graphs  
   2.2. The space of countable graphs  
   2.3. Orbit invariant subspaces  
   2.4. Schreier graphs  
   2.5. Infinite graphs that cannot be approximated by finite graphs  
3. Qualitative weak equivalence of free Cantor actions  
   3.1. Weak containment and weak equivalence  
   3.2. Cantor subshifts  
   3.3. The smallest element  
   3.4. The space of the weak equivalence classes is compact  
4. Qualitative convergence and limits  
   4.1. Convergence of generalized Schreier graphs  
   4.2. The basic example  
   4.3. Stable actions as qualitative limits  
   4.4. The limits of cyclic Schreier graphs  
   4.5. The limits of countable graphs  
5. Almost finiteness of graphs and stable actions  
   5.1. The geometric groupoid of a stable action  
   5.2. Almost finiteness and convergence  
   5.3. The fractal construction  
   5.4. A non-amenable almost finite groupoid  
   5.5. Fractionally almost finite graphs  
6. The spectra of graphs  
   6.1. Uncountably many isospectral connected graphs  
   6.2. Spectral convergence for strongly almost finite graphs
7. Constant-time distributed algorithms 42
    7.1. Algorithms and oracles 42
    7.2. Distributed graph partitioning and almost finiteness 44
    7.3. Approximating maximum independent subsets 47
    7.4. Approximated maximum matchings 48
    7.5. The unrestricted weighted independent subset problem 51
    7.6. Distributed parameter testing 53
8. Doubling and almost finiteness 54
    8.1. Doubling graphs 54
    8.2. The Basic Algorithm 55
    8.3. The class of D-doubling graphs is almost finite 60
    8.4. The class of D-doubling graphs is strongly almost finite 61
    8.5. Distributed strong almost finiteness 63
References 63
Sparse graph limit theory deals with very large finite graphs of small vertex degrees (see [31] for a recent survey). One can study these large graphs by their subgraph statistics. Informally speaking, two large graphs are close to each other in the statistical sense if the frequencies of their small subgraphs do not differ too much. Benjamini and Schramm [7] defined limit objects for growing sequences of graphs that are closer and closer to each other in the statistical sense. Another kind of limit objects were defined by Aldous and Lyons [3], and soon turned out that sparse graph limits are intimately related to finitely generated infinite groups and their measure preserving actions. Goldreich and Ron [19] introduced the notion of property testing and parameter estimation of bounded degree graphs. A graph parameter can be estimated using finite samplings if and only if for two graphs that are close in the statistical sense the parameters are close to each other, as well. In other words, sparse graph limit theory is about the information one can extract from large graphs via sampling. In his seminal monograph: Metric Structures for Riemannian and Non-Riemannian Spaces [21], Gromov studied sampling (quantitative) and observable (qualitative) convergence of metric measure spaces. In our paper we develop a study of bounded degree graphs via observables: qualitative graph limit theory. The table below is intended to briefly summarize the analogies between various notions of the quantitative and the qualitative approaches.

| Quantitative graph limit theory | Qualitative graph limit theory |
|---------------------------------|--------------------------------|
| Benjamini-Schramm convergence   | Naive convergence              |
| Invariant Random Subgroups      | Uniformly Recurrent Subgroups  |
| Measure preserving actions      | Stable actions                 |
| Hyperfiniteness                 | Almost finiteness              |
| Sofic groups                    | LEF-groups                     |
| Local-global convergence        | Qualitative convergence        |
| Constant-time randomized algorithms | Constant-time distributed algorithms |
| Property testing                | Distributed property testing   |
| Weak convergence of the spectra | Hausdorff convergence of the spectra |
| von Neumann algebras            | $C^*$-algebras                 |

1.1. Qualitative graph limits. Let $d > 0$ be a natural number and denote by $Gr_d$ the set of all countable graphs (up to isomorphisms) of vertex degree bound $d$. For $G \in Gr_d$ and $r > 0$, let $B_r(G)$ be the set of all rooted balls of radius $r$ contained in $G$.

**Definition 1.1** (Naive convergence). A sequence of countable graphs $\{G_n\}_{n=1}^{\infty} \subset Gr_d$ is **convergent in the naive sense** if for any $r > 0$ and rooted ball $B$ of radius $r$, there exists $N_B > 0$ such that

- either $B \in B_r(G_n)$ for any $n \geq N_B$, 
One can immediately observe that naive convergence is the qualitative analogue of the Benjamini-Schramm graph convergence (defined for finite graphs). It is not hard to see that for any graph sequence \( \{G_n\}_{n=1}^\infty \) that converges in the naive sense, there exists a graph \( G \in Gr_d \) such that \( G_n \xrightarrow{\mathcal{G}} G \), that is, \( B \in \mathcal{B}_r(G) \) if and only if the rooted ball \( B \) is contained in all but finitely many of the graphs \( G_n \). A much finer notion of convergence, qualitative graph convergence can be defined as the analogue of the local-global convergence notion introduced by Hatami, Lovász and Szegedy [23]. Let \( H \in Gr_d \) and let \( Q \) be a finite set. Let \( \varphi : V(H) \to Q \) be a labeling function. We can define the \( r \)-configuration set of \( \varphi \) in the following way. Let \( U_{d, Q}^r \) denote the finite set of all rooted, \( Q \)-labeled balls of radius \( r \) and vertex degree bound \( d \) (up to rooted, labeled isomorphisms). Then \( \text{Conf}_{r, H}(\varphi) \subseteq U_{d, Q}^r \) is the set of all rooted, \( Q \)-labeled balls that occur in the labeled graph \( (H, \varphi) \).

**Definition 1.2** (Qualitative convergence). A sequence of countable graphs \( \{G_n\}_{n=1}^\infty \subset Gr_d \) is qualitatively convergent if for any finite set \( Q \), integer \( r > 1 \) and subset \( S \subseteq U_{d, Q}^r \),
- either there exists \( \varphi^n : V(G_n) \to Q \), such that \( \text{Conf}_{r, G_n}(\varphi^n) = S \) for any \( n \geq N_S \),
- or there exists no \( \varphi^n : V(G_n) \to Q \), such that \( \text{Conf}_{r, G_n}(\varphi^n) = S \) for any \( n \geq N_S \).

Clearly, if \( \{G_n\}_{n=1}^\infty \) qualitatively converges then it converges in the naive sense, as well. On the other hand, let \( \{C_n\}_{n=1}^\infty \) be a sequence of cyclic graphs, where \( |V(C_n)| = n \). Then, \( \{C_n\}_{n=1}^\infty \) converges in the naive sense and it does not converge qualitatively (see Proposition 4.1). In Section 4, we will define qualitative convergence for \( \Gamma \)-Schreier graphs as well, where \( \Gamma \) is a finitely generated group. It turns out that the natural limit objects for qualitative convergence of \( \Gamma \)-Schreier graphs are stable actions of the group \( \Gamma \) on a totally disconnected compact metric space.

### 1.2. The space of free Cantor actions
Qualitative convergence leads to the notion of weak equivalence of free Cantor actions. For minimal actions of the integers, such notion has already been introduced by Lin and Matui [28]. We will prove (Theorem 3) that the space of weak equivalence classes for free \( \Gamma \)-actions is compact, similarly to the space of weak equivalence class of measurable actions [1],[38]. We also prove the qualitative analogue of the Abert-Weiss Theorem [2] showing that the space above contains a smallest element, the continuous free analogue of the standard Bernoulli shifts. We will define the qualitative graph limits for sequences of simple graphs as well and study the associated étale groupoids.

### 1.3. Almost finiteness
Recall [12] that a class of finite graphs \( \mathcal{G} \subset Gr_d \) is hyperfinite if for any \( \varepsilon > 0 \) there exists \( L_\varepsilon > 0 \) such that
• for any $G \in \mathcal{G}$, we can remove $\varepsilon|V(G)|$ edges such that the resulting graph $G'$ consists of components of size at most $L_\varepsilon$.

The key notion of our paper is almost finiteness, the qualitative analogue of hyperfiniteness. Almost finiteness was originally introduced by Matui [34] for totally disconnected étale groupoids, thus, in particular, for free and even stable Cantor actions. Before getting further, let us recall some definitions from graph theory. Let $G \in Gr_d$ be a countable graph and $H \subset V(G)$ be a finite subset. Then,

- $\text{diam}_G(H) = \max_{x,y \in H} d_H(x, y)$,
- the boundary of $H$, $\partial(H)$, is the set of vertices $x \in H$ such that there exists a vertex $y \in V(G) \setminus H$ adjacent to $x$,
- the isoperimetric constant of $H$, $i_G(H) := \frac{|\partial(H)|}{|H|}$.

**Definition 1.3.** A class of graphs $\mathcal{G} \subset Gr_d$ is **almost finite** if for any $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that for each $G \in \mathcal{G}$ we have a partition $V(G) = \{H_1, H_2, \ldots\}$ satisfying the following properties.

1. For any $j \geq 1$, $\text{diam}_G(H_j) \leq K_\varepsilon$.
2. For any $j \geq 1$, $i_G(H) \leq \varepsilon$.

We call such a partition an $(\varepsilon, K_\varepsilon)$-partition. An infinite graph $G \in Gr_d$ is called an almost finite graph if the class consisting of the single element $G$ is almost finite.

Clearly, any almost finite graph class is hyperfinite. On the other hand, the class of finite trees is hyperfinite, but not almost finite. We say that a class $\mathcal{G} \subset Gr_d$ is **distributed almost finite** if for any $\varepsilon > 0$ there exists a constant-time distributed algorithm that computes an $(\varepsilon, K_\varepsilon)$-partition for any $G \in \mathcal{G}$ (see Section 7 for further details).

**Conjecture 1.1.** A class $\mathcal{G}$ is almost finite if and only if it is distributed almost finite.

**Remark 1.** Let $\Gamma$ be a finitely generated amenable group and let $C_\Gamma$ be a Cayley graph of $\Gamma$. Using the language of our paper, the breakthrough Tiling Theorem of Downarowicz, Huczek and Zhang [10] states that the graph $C_\Gamma$ is almost finite. On the other hand, the distributed almost finiteness of $C_\Gamma$ would imply that all free $\Gamma$-actions on the Cantor set are almost finite in the sense of Matui. Such result would have very important consequences for the reduced $C^*$-algebras of the actions [26], [37].

The following two definitions are motivated by the notion of fractional hyperfiniteness introduced by Lovász [32].

**Definition 1.4.** A class $\mathcal{G} \subset Gr_d$ is called **strongly almost finite** if for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ and $K_\varepsilon > 0$ such that for any $G \in \mathcal{G}$ we have $T_\varepsilon$
Definition 1.5. A countable graph $G$ is fractionally almost finite if for any $\varepsilon > 0$, there exists $T_{\varepsilon} > 0$ and $K_{\varepsilon} > 0$ such that we have $T_{\varepsilon}$ amount of partitions of $V(G)$, $\{H_{i}^{1}, H_{i}^{2}, \ldots \}_{i=1}^{T_{\varepsilon}}$ into subsets of diameter at most $K_{\varepsilon}$, so that for any $x \in V(G)$

\[
\frac{\{|i \mid x \in \partial(H_{j}^{i}) \text{ for some } 1 \leq j\}|}{T_{\varepsilon}} < \varepsilon.
\]

We will see that regular trees (that are clearly not almost finite graphs) are actually fractionally almost finite. The main technical result of our paper is that $D$-doubling graphs are, in fact, strongly almost finite and even distributed strongly almost finite, see Section 7 (Theorem 10). This theorem implies that stable actions with doubling graph structure are almost finite in the sense of Matui. Many minimal stable actions with the doubling property was constructed in [15], hence by the Main Result of [37] they all amount to new examples of simple $C^*$-algebras of stable rank one. We will also construct an example of an almost finite minimal étale groupoid (of a minimal stable action) which is not topologically amenable, answering a query of Suzuki [37].

1.4. Constant-time distributed algorithms. The second part of our paper is an application in computer science, and it is strongly related to the notions and results of the first part. Let us recall the classical $\text{LOCAL}$-model of distributed graph algorithms (see e.g. [6]). Let us fix a constant $d > 0$ for the rest of the paper. Let $G = (V, E)$ be a simple graph of vertex degree bound $d$, such that the vertices have a unique ID from the set $\{1, 2, \ldots, |V|\}$. The vertices are identified with processors and the edges between adjacent vertices are identified with communication ports. In each round each vertex $x \in V$

- sends some message to each of its adjacent vertices,
- receives some message from each of its adjacent vertices,
- performs some calculation based on all the received messages.

After a certain amount of rounds (the running time) the procedure halts and each of the vertices produces some output, e.g. an element of a given finite set $F$. Note that the local calculation performed by the vertices can be unbounded and are not taken into consideration in the calculation of the running time. Also, we do not bound the length of the individual messages. The process starts with the communication of the ID’s. The simplest and most basic distributed graph algorithm problem is the vertex colouring problem that are used for breaking the symmetries of the graphs. The goal is to produce a legal colouring of the vertices of the graph $G$ by $(d + 1)$-colours (that is, adjacent vertices must have different colours). Linial [29] proved that one
needs $O(\log^*(n))$ rounds for such vertex colouring. On the other hand, such $(d + 1)$-colouring can be computed within $(O(d^2) + \log^*(n))$-time [18]. Note that $\log^*(n)$ denoted the iterated logarithm, where

- $\log^*(n) := 0$ if $n \leq 1$.
- $\log^*(n) := 1 + \log^*(\log_2(n))$, otherwise.

Our goal is to study distributed algorithms that can be performed in constant-time provided that a symmetry breaking vertex colouring is already given. Let $r > 0$ be an integer, $F$ be a finite set and $\mathcal{G} \subseteq \mathcal{G}_{r,d}$ be a class of graphs. Let us assume that the vertices of the graphs $G \in \mathcal{G}$ are labeled by the finite set $Q$ in such a way that if $0 < d_G(x, y) \leq r$, then the labels of $x$ and $y$ are different. A constant-time distributed algorithm starts with such $(r, Q)$-labelings and computes a labeling of the vertices of the graphs $G$ by the finite set $F$ in at most $r$ rounds. Note that for graphs of size $n$ the required precolouring can be computed within $O(d^{2r}) + \log^*(n)$-time.

Our definition of a contant-time distributed algorithm is motivated by Theorem 2. We use a local symmetry breaking mechanism that gives away the minimum amount of information about the graphs on which our algorithm will be performed. So, our approach can be viewed as the qualitative analogue of the randomized local framework introduced by Goldreich and Ron [19], which uses a uniformly random labeling of the vertices to break most of the symmetries with high probability. We will show that for hyperfinite graph classes (see Section 7), for any $\varepsilon > 0$ we have a constant-time distributed algorithm that produces an $(1 - \varepsilon)$-approximation of the maximum independent set problem or the minimum vertex cover problem (Proposition 7.2). For a smaller class of graphs we have a constant-time distributed algorithm even for the weighted unrestricted independent set problem in the deterministic-randomness framework (Proposition 7.4). For arbitrary graphs, we can show the existence of a constant-time distributed algorithm that produces an $(1 + \varepsilon)$-approximation of the maximum matching problem (Theorem 9). This is a typical infinite-to-finite proof that uses the notion of qualitative graph limits. Finally, we consider distributed parameter testing, that is strongly related to naive graph convergence. We prove a general spectral convergence result (Theorem 7) and show that for classes of $D$-doubling graphs the spectrum of the graph can be tested in a distributed fashion.

2. Graph convergence in the naive sense

The goal of this section is to introduce the notion of naive convergence and to define the relevant compact spaces of countable graphs.

2.1. The space of rooted connected graphs. Until the end of this paper we fix an integer $d > 0$. First of all, let us recall the notion of rooted graph convergence (see e.g. [14]). Let $\mathcal{RG}_d$ be the set of all connected graphs $G$ with vertex degree bound $d$ and a distinguished vertex (root) $x \in V(G)$. We can
define a metric $d_{RG}$ on the set $RG_d$ in the following way. Let $(G, x), (H, y) \in RG_d$. Then

$$d_{RG}((G, x), (H, y)) = 2^{-n},$$

where $n$ is the largest integer for which the rooted balls $B_n(G, x)$ and $B_n(H, y)$ are rooted-isomorphic. Then, $RG_d$ is a compact metric space with respect to $d_{RG}$. We will also consider the space of Cantor-labeled rooted graphs $C RG_d$. An element of $C RG_d$ is a rooted, connected graph $(G, x)$ equipped with a vertex labeling $\varphi : V(G) \to \{0, 1\}^{|s|}$ by the Cantor set. For $s > 0$ let $\varphi_s : V(G) \to \{0, 1\}^{|s|}$ denote the projection of $\varphi$ onto the first $s$ coordinates, where $[s] = \{1, 2, \ldots, s\}$. Hence, if $(G, x, \varphi)$ is a $\{0, 1\}^{|s|}$-labeled rooted graph then $(G, x, \varphi_s)$ is a $\{0, 1\}^{|s|}$-labeled rooted graph. Again, we can define a metric on $C RG_d$ by

$$d_{C RG_d}((G, x, \varphi), (H, y, \psi)) = 2^{-n},$$

where $n$ is the largest integer for which the $\{0, 1\}^{|s|}$-labeled rooted balls $B_n(G, x, \varphi_s)$ and $B_n(H, y, \psi_s)$ are rooted-labeled isomorphic. Again, $C RG_d$ is a compact metric space with respect to $d_{C RG_d}$. Similarly, we can define a compact metric structure on the space $RG_d^G$ of rooted countable graphs of vertices labeled by the finite set $Q$.

2.2. The space of countable graphs. Now let $Gr_d$ be the set of all (not necessarily connected) countable graphs of vertex degree bound $d$. For $G \in Gr_d$ denote by $B(G)$ the set of rooted balls $B$ for which there is an $x \in V(G)$ and $k \geq 1$ such that $B_k(G, x)$ is rooted isomorphic to $B$. Let $G, H \in Gr_d$. We say that $G$ and $H$ is equivalent, if $B(G) = B(H)$. We will denote by $Gr_d$ the set of equivalence classes of $Gr_d$. We can define a metric on $Gr_d$ as follows. Let $(G, H) \in Gr_d$ representing the elements $[G], [H] \in Gr_d$. Then $d_{Gr}([G], [H]) = 2^{-n}$ if

- For any $1 \leq i \leq n$ and ball $B \in U_d^i$ (the set of all rooted balls of radius $i$ and vertex degree bound $d$), either $B \in B(G)$ and $B \in B(H)$, or $B \notin B(G)$ and $B \notin B(H)$.
- There exists $B \in U_d^{n+1}$ such that $B$ is a rooted ball in exactly one of the two graphs.

Again, we consider the Cantor labeled graphs. An element of the set $CGr_d$ is a countable graph $G \in Gr_d$ equipped with a vertex labeling $\varphi : V(G) \to \{0, 1\}^{|s|}$. For $k \geq 1$ we denote by $CU_d^k$ the set of rooted balls $B$ of radius $k$ equipped with a vertex labeling $\rho : V(B) \to \{0, 1\}^{|k|}$. So, if $(G, \varphi) \in CGr_d$ and $k \geq 1$, then for any $x \in V(G)$ we assign an element of $CU_d^k$. Now we can proceed exactly the same way as in the unlabeled case. For $G \in CGr_d$ and $B \in CU_d^k$, $B \in CB(G)$ if and only if there exists $x \in V(G)$ such that the ball $B_k(G, x, \varphi_k)$ is rooted-labeled isomorphic to $B$. We say that $(G, \varphi)$ and $(H, \psi)$ are equivalent if $CB(G) = CB(H)$. The set of equivalence classes will be denoted by $CGr_d$. The metric on $C Gr_d$ is defined as follows. Let $(G, \varphi), (H, \psi) \in CGr_d$ representing the classes $[(G, \varphi)]$ and $[(H, \psi)]$. Then,

$$d_{CGr}((G, \varphi), (H, \psi)) = 2^{-n},$$
if

- For any $1 \leq i \leq n$ and $B \in \mathcal{C}U^i_d$, either $B \in \mathcal{C}B(G)$ and $B \in \mathcal{C}B(H)$, or $B \notin \mathcal{C}B(G)$ and $B \notin \mathcal{C}B(H)$.
- There exists $B \in \mathcal{C}U^{n+1}_d$ such that $B$ is rooted-labeled isomorphic to a $\{0,1\}^{[n+1]}$-labeled ball of exactly one of the two graphs.

Similarly, we can define a metric on the equivalence classes of countable graphs $\overline{G}_d$ with vertices labeled by a finite set $Q$.

2.3. Orbit invariant subspaces. Now, let $G \in \mathcal{G}_d$. Then we can consider the set $O(G) \subset \mathcal{R}_d$, the orbit of $G$. The elements of $O(G)$ are all the rooted graphs $(G^x, x)$, where $x \in V(G)$ and $G^x$ is the component of $G$ containing $x$. The orbit closure of $G$, $\overline{O(G)}$, is the closure of $O(G)$ in the compact metric space $\mathcal{R}_d$. We say that a closed set $M \subseteq \mathcal{R}_d$ is orbit invariant if for any $(G, x) \in M$ and $y \in V(G)$, $(G, y) \in M$ as well. We denote the set of all orbit invariant closed sets by $\text{Inv}(\mathcal{R}_d)$.

**Proposition 2.1.** The metric space $\overline{G}_d$ is compact and $\overline{O} : \overline{G}_d \to \text{Inv}(\mathcal{R}_d)$ is a homeomorphism, where the topology on $\text{Inv}(\mathcal{R}_d)$ is given by the Hausdorff metric and $\overline{O}$ assigns to the graph $G$ its orbit closure.

**Proof.** Let $\{G_n\}_{n=1}^{\infty}$ be a Cauchy-sequence in $\mathcal{G}_d$. Let $\mathcal{B}$ be the set of rooted balls that are eventually contained in the graphs $\{G_n\}_{n=1}^{\infty}$. In order to prove compactness, it is enough to show that there exists $G \in \mathcal{G}_d$ such that $\mathcal{B}(G) = \mathcal{B}$. If $B \in \mathcal{B} \cap U_i$, then there exists $n_B > 0$ and for any $n \geq n_B$ a vertex $x_n^B \in V(G_n)$ such that the rooted ball of radius $i$ around $x_n^B$ is rooted-isomorphic to $B$. Let $(G_{n_k}^B, x_{n_k}^B)$ be a convergent sequence in $\mathcal{R}_d$ and let $(G_B, x_B)$ be its limit. Then, $\mathcal{B}(G_B) \subset \mathcal{B}$ and $B \in \mathcal{G}_d$. Let $G = \bigcup B \mathcal{G}_B$ be the graph that consists of the disjoint copies of the graphs $G_B$. Clearly, $\mathcal{B}(G) = \mathcal{B}$. Hence, $\mathcal{G}_d$ is compact.

Suppose that $\overline{O(G)} = \overline{O(H)}$ for some $G, H \in \mathcal{G}_d$. Then $\bigcup_{J \in \overline{O(G)}} \mathcal{B}(J) = \mathcal{B}(G)$ and also, $\bigcup_{J \in \overline{O(H)}} \mathcal{B}(J) = \mathcal{B}(H)$, therefore $G$ and $H$ are equivalent. Thus, the map $\overline{O} : \overline{G}_d \to \text{Inv}(\mathcal{R}_d)$ is injective. Also, if $M \in \text{Inv}(\mathcal{R}_d)$ then it is easy to see that $M = \overline{O(G)}$, where $G$ is the disjoint union of graphs $\{G_n\}_{n=1}^{\infty}$, where $(G_n, x_n)$ is a dense set in $M$. Hence, the map $\overline{O(G)}$ is surjective as well.

Finally, we prove that if $G_n \to G$ in the metric space $\mathcal{G}_d$, then $\overline{O(G_n)} \to \overline{O(G)}$ in the Hausdorff metric of $\text{Inv}(\mathcal{R}_d)$. Let $(H, x) \in \overline{O(G)}$. We need to show that there exists $y_k \in V(G_k)$ so that $(G_k, y_k) \to (H, x)$ in the metric space $\mathcal{R}_d$. For $k \geq 1$, let $s_k$ be the largest integer such that the following two conditions are satisfied.

1. $s_k \leq k$.
2. There exists $y_k \in G_k$ so that $B_{s_k}(G_k, y_k)$ and $B_{s_k}(H, x)$ are rooted-isomorphic.
Since \( \{G_n\}_{n=1}^{\infty} \) converges to \( G \) and \((H, x) \in \overline{Gr}_d \) we have that \( s_k \to \infty \) as \( k \to \infty \). Thus, \( \{ (G_k, y_k) \}_{k=1}^{\infty} \) tends to \((H, x) \) in the metric space \( \overline{RG}_d \). In order to finish our proof we need to show that if \((H, x) \notin O(G)\), then there is no subsequence \((H_{n_k}, x_{n_k}) \in O(G_{n_k})\) such that \((H_{n_k}, x_{n_k}) \to (H, x)\). That is, there exists \( k \geq 1 \) such that if \( n \) is large enough, then \( B_k(H, x) \notin B(H_{n_k}) \) if \((H_n, z) \in O(G_n)\). Since \((H, x) \notin O(G)\), there exists \( k \geq 1 \) so that \( B_k(H, x) \notin B(G_n) \) if \( n \) is large enough. Consequently, \( B_k(H, x) \notin B(H_n) \) if \( n \) is large enough, then \( B_k(H, x) \notin B(H_n) \) provided that \((H_n, z) \in O(G_n)\). 

Similarly, one can prove the following proposition.

**Proposition 2.2.** The metric spaces \( \overline{CG}_d \) and \( \overline{Gr}_d^Q \) are compact. Also, \( \overline{O} : \overline{CG}_d \to \text{Inv}(\overline{CRG}_d) \) and \( \overline{O} : \overline{Gr}_d^Q \to \text{Inv}(\overline{RG}_d^Q) \) are homeomorphisms.

**Remark 2.** Let \( \{G_n\}_{n=1}^{\infty} \subset Gr_d \) be a sequence of graphs that is convergent in the naive sense (see Introduction). By Proposition 2.1, there exists \( G \in Gr_d \) such that \( G_n \xrightarrow{\sim} G \).

### 2.4. Schreier graphs

In this subsection we define naive convergence for Schreier graphs. Let \( \Gamma \) be a finitely generated group and \( \Sigma \) be a finite, symmetric generating set for \( \Gamma \). Let \( H \subset \Gamma \) be a subgroup of \( \Gamma \). Recall that the Schreier graph \( S(\Gamma, H) \) is defined in the following way.

- \( V(S(\Gamma, H)) = \Gamma/H \), the set of right cosets of \( H \),
- there is a directed edge from \( Ha \) to \( Hb \) labeled by \( \sigma \in \Sigma \) if \( Ha\sigma = Hb \).

The root of \( S(\Gamma, H) \) is the subgroup \( H \) itself. Note that by definition any Schreier graph is a rooted, connected graph. The set of all Schreier graphs are denoted by \( \Gamma_{\Sigma}G \). Again, we can define a metric \( d_{\Gamma_{\Sigma}G} \) of the set \( \Gamma_{\Sigma}G \). Let \( S(\Gamma, H), S(\Gamma, K) \in \Gamma_{\Sigma}G \). Then,

\[
d_{\Gamma_{\Sigma}G}(S(\Gamma, H), S(\Gamma, K)) = 2^{-n},
\]

where \( n \) is the largest integer for which the balls \( B_n(S(\Gamma, H), H) \) and \( B_n(S(\Gamma, K), K) \) are rooted \( \Sigma \)-edge labeled isomorphic. Again, \( \Gamma_{\Sigma}G \) is a compact metric space with respect to the metric \( d_{\Gamma_{\Sigma}G} \). Note that the map \( H \to S(\Gamma, H) \) provides a homeomorphism \( \tau : \text{Sub}(\Gamma) \to \Gamma_{\Sigma}G \) ([15]). One can also define the compact space of Cantor-vertex labeled, rooted Schreier graphs \( C\Gamma_{\Sigma}G \) and the compact space of \( Q \)-labeled rooted Schreier graphs in a similar fashion. Observe that we have a natural \( \Gamma \)-action \( \alpha : \Gamma \racts \Gamma_{\Sigma}G \), where \( \alpha(\gamma)(S(\Gamma, H)) = S(\Gamma, gHg^{-1}) \). The action extends to the spaces \( C\Gamma_{\Sigma}G \) and \( \Gamma_{\Sigma}G^Q \) as well ([16]). If \( \varphi : \Gamma/H \to \mathcal{C} \) is a map, then

\[
\alpha(g)(S(\Gamma, H), \varphi) = (S(\Gamma, gHg^{-1}), \psi),
\]

where \( \psi(gHg^{-1}a) = \varphi(Ha\gamma) \).

A **generalized Schreier graph** \( S \) is the countable union of some Schreier graphs. Generalized Schreier graphs are associated to not necessarily transitive actions of \( \Gamma \) on countable sets. We denote the space of all generalized
Schreier graphs of $\Gamma$ with respect to $\Sigma$ by $G_\Sigma \Gamma \mathcal{G}$. Let $U_{r, \Sigma} \Gamma \mathcal{G}$ be the set of rooted, $\Sigma$-edge labeled balls of radius $r$ (up to isomorphisms) that occur in some Schreier graph of $\Gamma$. For $S \in G_\Sigma \Gamma \mathcal{G}$, we denote by $B_{r, \Sigma}(S)$ the set of all rooted, $\Sigma$-edge labeled balls $B$ for which there exists $x \in V(S)$ and $k \geq 1$ such that $B_k(S, x)$ is rooted-labeled isomorphic to $B$. Let $S, T \in G_\Sigma \Gamma \mathcal{G}$. We say that $S$ and $T$ are equivalent if $B_{r, \Sigma}(S) = B_{r, \Sigma}(T)$. Again, we denote by $G_\Sigma \Gamma \mathcal{G}$ the set of equivalence classes of generalized Schreier graphs. We can define a metric on $G_\Sigma \Gamma \mathcal{G}$ in the same way as we did for $G_{r,d}$. We can also consider the set of Cantor labeled generalized Schreier graphs $CG_\Sigma \Gamma \mathcal{G}$ and the set $G_\Sigma \Gamma \mathcal{Q}$ as well, together with the metric spaces $CG_\Sigma \Gamma \mathcal{G}$ and $G_\Sigma \Gamma \mathcal{Q}$.

Then, we have the following straightforward generalization of Proposition 2.1 and Proposition 2.2.

**Proposition 2.3.** The metric spaces $G_\Sigma \Gamma \mathcal{G}$, $CG_\Sigma \Gamma \mathcal{G}$ and $G_\Sigma \Gamma \mathcal{Q}$ are compact. Also, $O : G_\Sigma \Gamma \mathcal{G} \to \text{Inv}(G_\Sigma \Gamma \mathcal{G})$, $O : CG_\Sigma \Gamma \mathcal{G} \to \text{Inv}(CG_\Sigma \Gamma \mathcal{G})$ and $O : G_\Sigma \Gamma \mathcal{Q} \to \text{Inv}(G_\Sigma \Gamma \mathcal{Q})$ are homeomorphisms.

### 2.5. Infinite graphs that cannot be approximated by finite graphs.

It is known that all graphons are limits of a convergent sequence of dense finite graphs [30]. On the other hand, it is not known whether every measured graphing is the limit of sparse graphs. For naive convergence, we have the following proposition.

**Proposition 2.4.** There exists a countable, connected infinite graph of bounded vertex degree, which is not the naive limit of a sequence of finite graphs.

**Proof.** Let us recall the notion of a LEF-group [20]. Let $\Gamma$ be a finitely generated group with a symmetric generating set $\Sigma$. Let Cay($\Gamma$, $\Sigma$) be the corresponding right Cayley graph. That is,

- $V(\text{Cay}(\Gamma, \Sigma)) = \Gamma$,
- $(\gamma, \delta)$ is a directed edge of Cay$(\Gamma, \Sigma)$ labeled by $\sigma \in \Sigma$, if $\sigma \gamma = \delta$.

A $\Sigma$-graph is a finite graph $G$ such that

- for each vertex $x \in V(G)$, $\deg(x) = |\Sigma|$,
- and each edge $(x, y)$ is labeled uniquely by some element of $\Sigma$, where
- the label of $(x, y)$ is the inverse of the label of $(y, x)$.

We say that a $\Sigma$-graph $G$ is an $n$-approximant of Cay$(\Gamma, \Sigma)$ if all the $n$-balls in $G$ are edge-labeled-isomorphic to the $n$-ball of the Cayley graph Cay$(\Gamma, \Sigma)$. The group $\Gamma$ is a LEF-group if for any $n \geq 1$ Cay$(\Gamma, \Sigma)$ possesses an $n$-approximant (it is not hard to see that being LEF is independent of the choice of the generating system). Gordon and Vershik [20] showed that not all groups $\Gamma$ are LEF. Now, let $\Gamma$ be a finitely generated group and let $\Sigma = \{\sigma_i\}_{i=1}^k$ be a symmetric generating system of $\Gamma$. We encode Cay$(\Gamma, \Sigma)$ by an undirected, unlabeled graph in the following way. For any $1 \leq i \leq k$, connect $x \in \Gamma$ with
\[\sigma_ix\text{ by a path of length } 3k.\] So, from any \(x \in \Gamma\) there are \(k\) ongoing path. If the path corresponding to the element \(\sigma_i\), then glue a hanging edge to the \(i\)-th vertex of the path. Notice that the path between \(x\) and \(\sigma_ix\) will receive two hanging edges, one from the direction of \(x\) and one from the direction of \(\sigma_ix\). Let \(G\) be the resulting infinite graph. Clearly, there exists a sequence of finite graphs \(\{G_n\}_{n=1}^{\infty} \subset Gr_d\), \(G_n \rightarrow G\) if and only if \(\Gamma\) is a LEF-group. Hence our proposition follows from the existence of a non-LEF group. \(\square\)

3. Qualitative weak equivalence of free Cantor actions

The goal of this section is to study the qualitative analogue of the weak equivalence of essentially free probability measure preserving actions (see [8] for a survey). The section also serves as a preparation for our qualitative graph limit theory.

3.1. Weak containment and weak equivalence. Let \(\Gamma\) be a finitely generated group and \(\Sigma\) be a finite, symmetric generating set for \(\Gamma\). Let \(r > 0\) be an integer and \(\varphi : C \rightarrow Q\) be a continuous map, where \(Q\) is a finite set and \(\alpha : \Gamma \acts C\) is a free Cantor-action. For \(x \in C\) let the map \(\tau_{x,\alpha,\varphi}^r : B_r(\Gamma, \Sigma, e_\Gamma) \rightarrow Q\) defined by setting

\[\tau_{x,\alpha,\varphi}^r(\gamma) := \varphi(\alpha(\gamma)(x)).\]

Also,

1. Let \(\text{Conf}_{r,\alpha}(\varphi) = \bigcup_{x \in C} \tau_{x,\alpha,\varphi}^r\).
2. If \(Q = \{0, 1\}^k\) we set \(\text{Conf}_{r,k}(\alpha) = \bigcup_{\varphi : C \rightarrow \{0, 1\}^k} \text{Conf}_{r,\alpha}(\varphi)\).
3. Let \(\text{Conf}(\alpha) = \bigcup_{r,k} \text{Conf}_{r,k}(\alpha)\).

**Definition 3.1.** Let \(\alpha : \Gamma \acts C\) and \(\beta : \Gamma \acts C\) be free Cantor-actions. We say that \(\alpha\) qualitatively weakly contains \(\beta\), \(\alpha \succeq \beta\), if for any finite set \(Q\), continuous map \(\psi : C \rightarrow Q\) and \(r > 0\), there exists a continuous map \(\varphi : C \rightarrow Q\) such that 

\[\text{Conf}_{r,\alpha}(\varphi) = \text{Conf}_{r,\beta}(\psi)\].

We say that \(\alpha\) is qualitatively weakly equivalent to \(\beta\), \(\alpha \simeq \beta\), if \(\alpha \succeq \beta\) and \(\beta \succeq \alpha\).

Clearly, qualitative weak containment does not depend on the choice of the generating system \(\Sigma\). The set of all quantitative weak equivalent classes will be denoted by \(\text{Free}(\Gamma)\).

**Remark 3.** In case of free \(\mathbb{Z}\)-actions, our notion of qualitative weak equivalence coincide with the notion of weak approximate conjugacy introduced by Lin and Matui [28].

Recall that the cost and strong ergodicity are invariants with respect to weak equivalence in the measurable framework [8]. The next theorem shows that certain important properties of Cantor actions are, in fact, invariants of the qualitative weak equivalence class.
Theorem 1. Let $\alpha : \Gamma \curvearrowright \mathcal{C}$ and $\beta : \Gamma \curvearrowright \mathcal{C}$ be Cantor actions.

1. If $\alpha$ admits an invariant probability measure, then $\beta$ admits an invariant probability measure as well.
2. If $\beta$ is an amenable action, then $\alpha$ is an amenable action as well.

Hence, both amenability and admitting an invariant probability measure are qualitative weak invariants.

Proof. Let $\{\{0,1\}^k\}_{r, \Gamma, \Sigma, k} = B_r(\Gamma, \Sigma, k)$ be the set of all $\{0,1\}^k$-valued functions on the ball $B_r(\Gamma, \Sigma, e\Gamma)$ of radius $r$ centered around the unit element of $\Gamma$ in the Cayley graph of $\Gamma$ with respect to $\Sigma$. If $A \in B_r(\Gamma, \Sigma, k)$ then we can define the clopen set

$$U_A := \{x \in \mathcal{C} \mid (\beta(\gamma))(x)|_k = A(\gamma) \text{ for any } \gamma \in B_r(\Gamma, \Sigma, e\Gamma)\},$$

where $(x)|_k$ is the first $k$ coordinates of $x$. Clearly, $\cup_{A \in B_r(\Gamma, \Sigma, k)} U_A$ is a clopen partition of $\mathcal{C}$. For $\sigma \in \Sigma$ and $A, B \in B_r(\Gamma, \Sigma, k)$, we define two further clopen sets

$$U_A^{\sigma \to B} := \{x \in \mathcal{C} \mid x \in U_A, \beta(\sigma)(x) \in U_B\}.$$ 

and

$$U_B^{\sigma \to A} := \{y \in \mathcal{C} \mid y \in U_B, \beta(\sigma^{-1})(y) \in U_A\}.$$ 

Let $r, k \geq 1$ and $\varphi : \mathcal{C} \to \{0,1\}^h$ be a continuous map, where $h \geq k$. For $A \in B_r(\Gamma, \Sigma, k)$ we set

$$U_A^\varphi = \{x \in \mathcal{C} \mid \varphi_k(\alpha(\gamma))(x) = A(\gamma) \text{ for any } \gamma \in B_r(\Gamma, \Sigma, e\Gamma)\}.$$ 

(recall that $\varphi_k(x) = (\varphi(x)|_k)$). Also, if $A \in B_r(\Gamma, \Sigma, k)$, where $1 \leq l \leq k$, let $A|_l \in B_r(\Gamma, \Sigma, l)$ is the projection of the values of $A$ onto the first $l$ coordinates. If $1 \leq s \leq r$, let $A|_s \in B_s(\Gamma, \Sigma, k)$ is the restriction of $A$ onto the ball $B_s(\Gamma, \Sigma, e\Gamma)$. Finally, we use the notation $A \subseteq B$ if $B = (A|_s)|_l$. Note that $A \subseteq B$ implies that $U_A \subseteq U_B$. So, if $B \in B_s(\Gamma, \Sigma, l)$, then

$$V_B^\varphi = \cup_{A \in B_s(\Gamma, \Sigma, k), A \subseteq B} U_A^\varphi.$$ 

Since $\alpha \succeq \beta$, for all $r \geq 1$ there exists a continuous map $\varphi^r : \mathcal{C} \to \{0,1\}^r$ such that for all $A \in B_r(\Gamma, \Sigma, r)$, $V_A^{\varphi^r}$ is nonempty if and only if $U_A$ is nonempty.

Let $\{r_n\}_{n=1}^\infty$ be an increasing sequence of integers such that for all $s, l \geq 1$ and $B \in B_s(\Gamma, \Sigma, l)$,

$$\lim_{n \to \infty} \mu(V_B^{\varphi^r}) = l(B)$$

exists, where $\mu$ is a probability measure on $\mathcal{C}$ invariant under the action $\alpha$. Set $\nu(U_B) = l(B)$.

Lemma 3.1. The function $\nu$ extends to a Borel probability measure on $\mathcal{C}$ in a unique way.

Proof. If $B \in B_s(\Gamma, \Sigma, l)$, $s \leq r$, $l \leq k$ and $n$ is a large enough integer, then $\mu(V_B^{\varphi^r})$ is well-defined and equals to $\sum_{A \in B_s(\Gamma, \Sigma, k), A \subseteq B} V_A^{\varphi^r}$ and also

$$\sum_{B \in B_s(\Gamma, \Sigma, l)} \mu(V_B^{\varphi^r}) = 1.$$
Also, if for some \( B \in \mathcal{B}_s(\Gamma, \Sigma, l) \), the set \( U_B \) is empty, then \( \mu(V^\varphi_n_B) = 0 \). Hence, if \( 1 \leq l \leq k \) and \( 1 \leq s \leq r \), then
\[
\nu(U_B) = \sum_{A \in \mathcal{B}_r(\Gamma, \Sigma, k), A \subset B} \nu(U_A)
\]
and
\[
\sum_{B \in \mathcal{B}_s(\Gamma, \Sigma, l)} \nu(U_B) = 1.
\]
Also, \( \nu(U_B) \) is defined being zero, if \( U_B \) is the empty set. Hence, \( \nu \) is a premeasure on the basic sets of \( \mathcal{C} \), therefore, \( \nu \) extends to a Borel probability measure in a unique way.

**Proposition 3.1.** The measure \( \nu \) is invariant under the action \( \beta \).

**Proof.** It is enough to show that for any \( r \geq 1 \), \( \sigma \in \Sigma \) and pair \( A, B \in \mathcal{B}_r(\Gamma, \Sigma, r) \), we have that
\[
\nu(U^\alpha_{A \rightarrow B}) = \nu(U^\alpha_{B \leftarrow A}).
\]
For large enough \( n \), we can define the clopen sets \( V^\varphi_n_{A \rightarrow B} \) and \( V^\varphi_n_{B \leftarrow A} \) by
\[
V^\varphi_n_{A \rightarrow B} := \{ x \in V^\varphi_n_A \mid \alpha(\sigma)(x) \in V^\varphi_n_B \},
\]
\[
V^\varphi_n_{B \leftarrow A} := \{ z \in V^\varphi_n_B \mid \alpha(\sigma^{-1})(z) \in V^\varphi_n_A \}.
\]

**Lemma 3.2.**
\[
\nu(U^\alpha_{A \rightarrow B}) = \lim_{n \to \infty} \mu(V^\varphi_n_{A \rightarrow B}),
\]
\[
\nu(U^\alpha_{B \leftarrow A}) = \lim_{n \to \infty} \mu(V^\varphi_n_{B \leftarrow A}).
\]

**Proof.** For \( D \in \mathcal{B}_{r+1}(\Gamma, \Sigma, r) \), let us use the notation \( D \sqsubset A \rightarrow B \) if \( U_D \subset U^\alpha_{A \rightarrow B} \). Also, for \( E \in \mathcal{B}_{r+1}(\Gamma, \Sigma, r) \), let \( E \sqsupset B \leftarrow A \) if \( U_E \subset U^\alpha_{B \leftarrow A} \). Then we have that
\[
\bullet \quad U^\alpha_{A \rightarrow B} = \bigcup_{D \in \mathcal{B}_{r+1}(\Gamma, \Sigma, r), D \subset A \rightarrow B} U_D,
\]
\[
\bullet \quad U^\alpha_{B \leftarrow A} = \bigcup_{E \in \mathcal{B}_{r+1}(\Gamma, \Sigma, r), E \sqsupset B \leftarrow A} U_E.
\]
\[
\bullet \quad \text{for large enough } n,
\]
\[
V^\varphi_n_{A \rightarrow B} = \bigcup_{D \in \mathcal{B}_{r+1}(\Gamma, \Sigma, r), D \subset A \rightarrow B} V^\varphi_n_D,
\]
\[
\bullet \quad \text{for large enough } n,
\]
\[
V^\varphi_n_{B \leftarrow A} = \bigcup_{E \in \mathcal{B}_{r+1}(\Gamma, \Sigma, r), E \sqsupset B \leftarrow A} V^\varphi_n_E.
\]
Therefore our lemma follows.

Since \( \mu \) is invariant under the action \( \alpha \), for all \( n \geq 1 \), we have that
\[
\mu(V^\varphi_n_{A \rightarrow B}) = \mu(V^\varphi_n_{B \leftarrow A}).
\]
Therefore, \( \nu(U^\alpha_{A \rightarrow B}) = \nu(U^\alpha_{B \leftarrow A}) \). Thus our proposition follows.

Now, the first part of our theorem follows from Lemma 3.1 and Proposition 3.1. Let us turn to proof of the second part of our theorem. First, let us
recall the notion of an amenable action from [25]. The free action $\beta : \Gamma \curvearrowright \mathcal{C}$ is an amenable action if there exists a sequence of weak*-continuous maps $b_n : X \to \text{Prob}(\Gamma)$ such that for every $\gamma \in \Gamma$,
\begin{equation}
\limsup_{n \to \infty} \sup_{x \in \mathcal{C}} \|S(\gamma)(b_n(x)) - b_n(\beta(\gamma)(x))\|_1 = 0,
\end{equation}
where $S$ is the natural action of the group $\Gamma$ on $\text{Prob}(\Gamma)$. Since $\Gamma$ is finitely generated, it is enough to assume (2) for the generators $\sigma \in \Sigma$.

**Lemma 3.3.** Let $b : \mathcal{C} \to \text{Prob}(\Gamma)$ be a weak*-continuous function. Then for any $\varepsilon > 0$, there exists $R > 0$ and a weak*-continuous function $b' : \mathcal{C} \to \text{Prob}(\Gamma)$ such that for all $x \in \mathcal{C}$, $\text{Supp}(b'(x)) \subseteq B_R(\Gamma, \Sigma, e_\Gamma)$ and $\|b(x) - b'(x)\|_1 < \varepsilon$.

**Proof.** Suppose that there exists a sequence $\{x_n\}_{n=1}^\infty$ such that for all $n \geq 1$
\begin{equation}
b(x_n)(B_n(\Gamma, \Sigma, e_\Gamma)) \leq 1 - \frac{\varepsilon}{3}.
\end{equation}
Then, for any limit point $x$ of the sequence, $b(x)(\Gamma) \leq 1 - \frac{\varepsilon}{3}$ leading to a contradiction. That is, there must exist some $R > 0$, such that for all $x \in \mathcal{C}$,
\begin{equation}
b(x)(B_R(\Gamma, \Sigma, e_\Gamma)) > 1 - \frac{\varepsilon}{3}.
\end{equation}
We define the function $b'$ in the following way.
- Let $b'(x)(\gamma) = b(x)(\gamma)$, if $e_\Gamma \neq \gamma \in B_R(\Gamma, \Sigma, e_\Gamma)$.
- Let $b'(x)(\gamma) = 0$, if $\gamma \notin B_R(\Gamma, \Sigma, e_\Gamma)$.
- Let $b'(x)(e_\Gamma) = b(x)(e_\Gamma) + b(x)(\Gamma \setminus B_R(\Gamma, \Sigma, e_\Gamma))$.

Clearly, $b' : \mathcal{C} \to \text{Prob}(\Gamma)$ is weak*-continuous, for any $x \in \mathcal{C}$ we have that $\text{Supp}(b'(x)) \subseteq B_R(\Gamma, \Sigma, e_\Gamma)$ and $\|b(x) - b'(x)\|_1 < \varepsilon$. \hfill \Box

So, from now on we can assume that for any $n \geq 1$, there exists some $R_n > 0$ such that for all $x \in \mathcal{C}$,
\begin{equation}
\text{Supp}(b_n(x)) \subseteq B_{R_n}(\Gamma, \Sigma, e_\Gamma).
\end{equation}
The following lemma is trivial.

**Lemma 3.4.** Let $f \in \text{Prob}(\Gamma)$ such that $\text{Supp}(f) \subseteq B_R(\Gamma, \Sigma, e_\Gamma)$ for some $R > 0$. Let $k \geq 1$ be an integer. Then, there exists $g \in \text{Prob}(\Gamma)$, $\text{Supp}(g) \subseteq B_R(\Gamma, \Sigma, e_\Gamma)$, such that
- for any $\gamma \in B_R(\Gamma, \Sigma, e_\Gamma)$, $g(\gamma) = \frac{i}{k}$ for some integer $i \geq 0$,
- $\|f - g\|_1 = \sum_{\gamma \in B_R(\Gamma, \Sigma, e_\Gamma)} |f(\gamma) - g(\gamma)| \leq \frac{|B_R(\Gamma, \Sigma, e_\Gamma)|}{k}$.

**Lemma 3.5.** Let $\{b_n\}_{n=1}^\infty$ as in (3). Then, for any $n \geq 1$ and $\varepsilon > 0$, there exists $k > 1$ and continuous function $c_n : \mathcal{C} \to \text{Prob}(\Gamma)$ such that for all $\mathcal{C}$,
- $\text{Supp}(c_n(x)) \subseteq B_{R_n}(\Gamma, \Sigma, e_\Gamma)$,
- for all $\gamma \in B_{R_n}(\Gamma, \Sigma, e_\Gamma)$, $c_n(x)(\gamma) = \frac{i}{k}$ for some integer $i \geq 0$,
- $\|b_n(x) - c_n(x)\|_1 \leq \varepsilon$. 

Proof. Let \( \{U_\alpha\}_{\alpha \in I} \) be a finite clopen partition of the Cantor set \( C \) such that if \( x, y \in U_\alpha \) for some \( \alpha \), then \( \|b_n(x) - b_n(y)\|_1 \leq \epsilon/3 \). Let \( k \geq 1 \) be an integer such that

\[
\frac{|B_{R_n}(\Gamma, \Sigma, \epsilon)}{k} \leq \frac{\epsilon}{3}.
\]

For each \( \alpha \) pick an element \( x_\alpha \in U_\alpha \). Then for each \( n \geq 1 \) choose \( c_n^\alpha \in \text{Prob}(\Gamma) \) in such a way that

- \( \|c_n^\alpha - b_n(x_\alpha)\| \leq \frac{\epsilon}{k} \),
- \( \text{Supp}(c_n^\alpha) \subseteq B_{R_n}(\Gamma, \Sigma, \epsilon) \),
- for any \( \gamma \in B_{R_n}(\Gamma, \Sigma, \epsilon) \), \( c_n^\alpha(\gamma) = \frac{1}{k} \), where \( i \geq 0 \) is an integer.

Finally, define \( c_n : C \rightarrow \text{Prob}(\Gamma) \) by setting \( c_n(y) := c_n^\alpha \) if \( y \in U_\alpha \). Then the functions \( \{c_n\}_{n=1}^\infty \) satisfy the conditions of our lemma. \( \square \)

Now, let \( \alpha, \beta : \Gamma \curvearrowright \mathcal{C} \) be free Cantor actions such that \( \alpha \geq \beta \) and \( \beta \) is amenable. Let \( b : C \rightarrow \text{Prob}(\Gamma) \) be a weak*-continuous map, \( k \geq 1 \) and \( R \geq 1 \) be integers, \( \epsilon > 0 \) such that for all \( x \in C \) and \( \sigma \in \Sigma \),

- \( \text{Supp}(b(x)) \subseteq B_R(\Gamma, \Sigma, \epsilon) \),
- for any \( \gamma \in B_R(\Gamma, \Sigma, \epsilon) \), \( b(x)(\gamma) = \frac{i}{k} \), where \( i \geq 0 \) is an integer,
- \( \|S(\sigma)(b(x)) - b(\beta(\sigma))(x)\|_1 < \epsilon \).

Let \( Q \) be the finite set of elements \( c \) in \( \text{Prob}(\Gamma) \), such that \( \text{Supp}(c) \subseteq B_R(\Gamma, \Sigma, \epsilon) \) and \( c(\gamma) = \frac{i}{k} \), where \( i \geq 0 \) is an integer. Define the continuous map \( \pi : \mathcal{C} \rightarrow Q \) by setting \( \pi(x) := b(x) \in Q \). Let \( \hat{\pi} : \mathcal{C} \rightarrow Q \) be a continuous map such that

\[
\text{Conf}_{1,\alpha}(\hat{\pi}) = \text{Conf}_{1,\beta}(\pi).
\]

Now, let \( \hat{b} : \mathcal{C} \rightarrow \text{Prob}(\Gamma) \) be defined by \( \hat{b}(y) = \hat{\pi}(y) \). Then, for any \( y \in C \) and \( \sigma \in \Sigma \),

\[
\|S(\sigma)(\hat{b}(y)) - \hat{b}(\alpha(\sigma))(y)\|_1 < \epsilon.
\]

Hence, \( \alpha \) is amenable as well. This finishes the proof of our theorem. \( \square \)

3.2. Cantor subshifts.

**Definition 3.2.** Let \( Z \subset C^\Gamma \) be a closed subset invariant under the right shift action, \((R(\delta)(z))(\gamma) = z(\gamma \delta)\). We call \( Z \) a **Cantor subshift** if \( Z \) is homeomorphic to \( C \) and the action on \( Z \) is free.

Let \( \alpha : \Gamma \curvearrowright \mathcal{C} \) be a free Cantor action. Let the equivariant map \( \kappa_\alpha : C \rightarrow C^\Gamma \) be defined by setting \( \kappa_\alpha(x)(\gamma) := \alpha(\gamma)(x) \). We call \( \kappa_\alpha(C) \) the Cantor subshift of \( \alpha \). Clearly, every action is conjugate isomorphic (hence, qualitatively weakly equivalent) to its own Cantor subshift. Note that not all Cantor subshift \( Z \subset C^\Gamma \) are in the form of \( \kappa_\alpha(C) \) for some action \( \alpha \). Let \( Y \subset C^\Gamma \) be a Cantor subshift and \( r \geq 1 \). Then \( \text{Conf}_{r,Y}([t]) \subseteq \{0,1\}^t B_r(\Gamma, \Sigma, \epsilon) \) is defined in the following way. The map \( \varphi : B_r(\Gamma, \Sigma, \epsilon) \rightarrow \{0,1\}^t \) is an element of \( \text{Conf}_{r,Y}([t]) \).
if there exists \( y \in Y \) such that for all \( \gamma \in B_r(\Gamma, \Sigma, e_\Gamma) \), \( \varphi(\gamma) = (y(\gamma))[t] \). The following two lemmas will be used in our proofs.

**Lemma 3.6.** Let \( Y \subset \mathbb{C}^\Gamma \) be a Cantor subshift and \( \alpha : \Gamma \curvearrowright \mathbb{C} \) be a free Cantor action. Then \( \alpha \succeq Y \) if and only for any \( r, t \geq 1 \), there exists a continuous map \( \varphi : \mathbb{C} \to \{0,1\}^t \) such that \( \text{Conf}_{r,\alpha}(\varphi) = \text{Conf}_{r,Y}([t]) \).

**Proof.** Clearly, the condition of the lemma is necessary for \( Y \) begin qualitatively weakly contained by \( \alpha \). Let us show that the condition is sufficient as well. Let \( Q \) be a finite set, \( r \geq 1 \) and \( \psi : Y \to Q \) be a continuous map. We need to show that there exists \( \hat{\psi} : \mathbb{C} \to Q \) such that \( \text{Conf}_{r,\alpha}(\hat{\psi}) = \text{Conf}_{r,Y}(\psi) \).

For \( A \in \text{Conf}_{n,Y}[t] \) set
\[
W_A := \{ y \in Y \mid y(\gamma)[t] = A(\gamma), \text{ for any } \gamma \in B_n(\Gamma, \Sigma, e_\Gamma) \}.
\]
Since \( \psi \) is continuous, there exists \( n, t \geq 1 \) such that \( \psi \) is constant on the clopen sets \( W_A \), for all \( A \in \text{Conf}_{n,Y}[t] \). Let \( \pi : \mathbb{C} \to \{0,1\}^t \) such that
\[
\text{Conf}_{n+1,\alpha}(\pi) = \text{Conf}_{n+1,Y}([t]) \).
\]
Again, if \( A \in \text{Conf}_{n,\alpha}(\pi) = \text{Conf}_{n,Y}([t]) \), let
\[
V_A := \{ x \in \mathbb{C} \mid \pi(\alpha(\gamma))(x)) = A(\gamma), \text{ for any } \gamma \in B_n(\Gamma, \Sigma, e_\Gamma) \}.
\]
Now, we define \( \hat{\psi} : \mathbb{C} \to Q \) in the following way. If \( x \in V_A \), then \( \hat{\psi}(x) = q \) where \( q \) is the value of \( \psi \) on the clopen set \( W_A \). Then, it is easy to see that
\[
\text{Conf}_{r,\alpha}(\hat{\psi}) = \text{Conf}_{r,Y}(\psi) \),
\]
hence the lemma follows. \( \square \)

For \( z \in \{0,1\}^t \) and \( x \in \mathbb{C} = \{0,1\}^\mathbb{N} \), let \( x \downarrow z \in \mathbb{C} \) be defined in the following way.

- If \( 1 \leq i \leq t \), \( x \downarrow z(i) = z(i) \),
- if \( t + 1 \leq i \), \( x \downarrow z(i) = x(i-t) \).

Now, let \( \alpha : \Gamma \curvearrowright \mathbb{C} \) be a free Cantor action and \( \pi : \mathbb{C} \to \{0,1\}^t \) be a continuous map. Then we define an equivariant homeomorphism \( \kappa_\alpha^\pi : \mathbb{C} \to \mathbb{C}^\Gamma \) by setting
\[
\kappa_\alpha^\pi(x) = \alpha(\gamma)(x) \downarrow \pi(\alpha(\gamma)(x)) \).
\]
The following lemma is trivial.

**Lemma 3.7.** The space \( \kappa_\alpha^\pi(\mathbb{C}) \) is a Cantor subshift.

### 3.3. The smallest element

Our next goal is to prove a qualitative analogue of a result of Abért and Weiss [2]. Their theorem states that all essentially free p.m.p. action of a countable group weakly contains the Bernoulli actions.

**Theorem 2.** Let \( \Gamma \) be a finitely generated group. Then, there exists a free Cantor action \( \beta : \Gamma \curvearrowright \mathbb{C} \) such that any free Cantor action \( \alpha : \Gamma \curvearrowright \mathbb{C} \) qualitatively weakly contains \( \beta \).
Proof. First, we explicitly construct the action $\beta$. For $n \geq 1$, let $r_n$ be a positive integer such that $2^{r_n} \geq |B_n(\Gamma, \Sigma, e_\Gamma)|$. So, one can label the elements of $\Gamma$ by the set $\{0,1\}^{r_n}$ in such a way that if $0 < d_{Cay(\Gamma, \Sigma)}(x, y) \leq n$, then the labels of $x$ and $y$ are different.

**Definition 3.3.** An $n$-block is a labeling

$$\varphi : \prod_{j=1}^{n} \varphi^j : B_n(\Gamma, \Sigma, e_\Gamma) \to \prod_{j=1}^{n} \{0,1\}^{r_j},$$

where $\varphi^j : B_n(\Gamma, \Sigma, e_\Gamma) \to \{0,1\}^{r_j}$, such that if $1 \leq j \leq n$ and $0 < d_{Cay(\Gamma, \Sigma)}(x, y) \leq n$, then $\varphi^j(x) \neq \varphi^j(y)$.

Now, let $y = \prod_{j=1}^{\infty} y_j : \Gamma \to \prod_{j=1}^{\infty} \{0,1\}^{r_j}$ be a labeling such that

- for any $j \geq 1$, if $0 < d_{Cay(\Gamma, \Sigma)}(x, y) \leq j$, then $y_j(x) \neq y_j(y)$,
- for any $n \geq 1$ and $n$-block $\varphi : B_n(\Gamma, \Sigma, e_\Gamma) \to \prod_{j=1}^{n} \{0,1\}^{r_j}$, there exists $\delta \in \Gamma$ such that for any $1 \leq j \leq n$ and $\gamma \in B_n(\Gamma, \Sigma, e_\Gamma)$ we have $\varphi^j(\gamma) = y_j(\delta \gamma)$.

We call such a labeling $y$ a full labeling. Clearly, full labelings exist.

**Lemma 3.8.** Let $y$ be a full labeling and $Y \subset C^\Gamma$ be the orbit closure of $y$ in $C^\Gamma$, where $C = \{0,1\}^N$ is identified with $\prod_{j=1}^{\infty} \{0,1\}^{r_j}$. Then the restricted shift action $\beta : \Gamma \curvearrowright Y$ is free.

**Proof.** Observe that for any $r \geq 1$, there exists $s_r \geq 1$ such that if $0 < d_{Cay(\Gamma, \Sigma)}(\gamma, \delta) \leq r$ then $(y(\gamma))[s] \neq (y(\delta))[s]$. Hence, if $\hat{y} \in Y$ and $0 < d_{Cay(\Gamma, \Sigma)}(\gamma, \delta) \leq r$, then $(\hat{y}(\gamma))[s] \neq (\hat{y}(\delta))[s]$. That is, $\beta(\gamma)(\hat{y}) \neq \hat{y}$ provided that $\gamma \neq e_\Gamma$. Hence the action $\beta$ is free on $Y$. \hfill $\Box$

**Lemma 3.9.** The space $Y$ is homeomorphic to the Cantor set.

**Proof.** It is enough to show that $Y$ does not have an isolated point. Let $\hat{y} \in Y$. Observe that if we restrict $\hat{y} = \prod_{j=1}^{\infty} \hat{y}_j$ onto the ball $B_n(\Gamma, \Sigma, e_\Gamma)$ we obtain an $n$-block $\varphi$. On the other hand, there are more than one $n+1$-blocks $\psi$ such that if we restrict $\psi$ onto $B_n(\Gamma, \Sigma, e_\Gamma)$ taking only the first $\sum_{j=1}^{n} r_j$ coordinates, the resulting $n$-block coincides with $\varphi$. Hence, for any $\varepsilon > 0$, there exists $z \in Y$ such that $0 < d_Y(y, \hat{y}) < \varepsilon$. Therefore, the space $Y$ is homeomorphic to $C$. \hfill $\Box$

In order to finish the proof of our theorem. It is enough to prove the following proposition.

**Proposition 3.2.** Let $\alpha : \Gamma \curvearrowright C$ be a free Cantor action. Then, $\alpha \succeq \beta$.

**Proof.** Let $\{r_j\}_{j=1}^{\infty}$ be as above and set $k_n := \sum_{j=1}^{n} r_j$. Our goal is to construct a continuous map $\pi_n : C \to \{0,1\}^{k_n}$ such that the set $Conf_{n,\alpha}(\pi_n)$ equals to the set of all $n$-blocks. Then, we have that $Conf_{n,k_n}(\kappa_n) = Conf_{n,k_n}(\beta)$. 

Then by Lemma 3.6, it follows that \( \alpha \geq \beta \). First, we make sure that \( \text{Conf}_{n,\alpha}(\pi_n) \) contains all the \( n \)-blocks. We use an insertion process to define \( \pi_n \) partially. Let \( M_n \) be a positive integer, which is so large that

\[
\left\{ 0, 1 \right\}^{k_n} \cup B_{M_n}(\Gamma, \Sigma, e_\Gamma) \text{ contains an element } \psi^n \text{ such that } \pi_n \\text{ blocks. Now let } \psi^n \text{ for some } \eta
\]

We define the map \( \psi^n = \prod_{j=1}^n \psi^n_j : B_{M_n}(\Gamma, \Sigma, e_\Gamma) \to \left\{ 0, 1 \right\}^{k_n} \) such that

- for any \( 1 \leq j \leq n \), \( \psi^n_j(\gamma_1) \neq \psi^n_j(\gamma_2) \), whenever \( 0 < d_{\text{Cay}(\Gamma, \Sigma)}(\gamma_1, \gamma_2) \leq j \),
- for any \( n \)-block \( \varphi : B_n(\Gamma, \Sigma, e_\Gamma) \to \left\{ 0, 1 \right\}^{k_n} \), there exists \( \zeta \in B_{M_n}(\Gamma, \Sigma, e_\Gamma) \) such that \( B_n(\Gamma, \Sigma, \zeta) \) is contained in the ball \( B_{M_n}(\Gamma, \Sigma, e_\Gamma) \) and \( \psi^n(\zeta, \gamma) = \varphi(\gamma) \) for any \( \gamma \in B_n(\Gamma, \Sigma, \zeta) \).

Informally speaking, the labeling \( \psi^n \) contains an inserted copy of all the \( n \)-blocks. Now let \( U_n \) be a nonempty clopen subset of \( C \) such that if \( x \neq y \in U_n \) then

\[
\bigcup_{\gamma \in B_{2M_n}(\Gamma, \Sigma, e_\Gamma)} \alpha(\gamma)(x) \cap \bigcup_{\gamma \in B_{2M_n}(\Gamma, \Sigma, e_\Gamma)} \alpha(\gamma)(y) = \emptyset.
\]

We define the map \( \pi_n \) on the clopen set \( \bigcup_{x \in U_n} \bigcup_{B_{M_n}(\Gamma, \Sigma, e_\Gamma)} \alpha(\gamma)(x) \) in the following way. If \( y = \alpha(\gamma)(x) \), where \( x \in U_n \) and \( \gamma \in B_{M_n}(\Gamma, \Sigma, e_\Gamma) \), then let \( \pi_n(y) = \psi^n(\gamma) \). It follows from our construction that \( \text{Conf}_{n,\alpha}(\pi_n) \) will contain all the \( n \)-blocks. We only need to make sure that \( \text{Conf}_{n,\alpha}(\pi_n) \) contains only \( n \)-blocks. Thus, our proposition follows from the lemma below.

**Lemma 3.10.** Let \( r \geq 1 \) be a positive integer and \( Q \) be a finite set such that \( |Q| \geq |B_r(\Gamma, \Sigma, e_\Gamma)| \). Also, let \( W \subset C \) be a clopen set and \( \eta : W \to Q \) be a continuous function. Suppose that for all pairs \( x, y \in W \) such that \( y = \alpha(\gamma)(x) \) for some \( e_\Gamma, \gamma \in B_r(\Gamma, \Sigma, e_\Gamma) \), we have that \( \eta(x) \neq \eta(y) \). Then there exists \( \hat{\eta} : C \to Q \) such that \( \hat{\eta}|_W = \eta \) and for all pairs \( x, y \in C \) such that \( y = \alpha(\gamma)(x) \) for some \( e_\Gamma, \gamma \in B_r(\Gamma, \Sigma, e_\Gamma) \), we have that \( \hat{\eta}(x) \neq \hat{\eta}(y) \).

**Proof.** Let \( \bigcup_{i=1}^s V_i = C \) be a clopen partition such that for any \( 1 \leq i \leq s \) and \( x \neq y \in V_i \),

\[
\bigcup_{\gamma \in B_{2r}(\Gamma, \Sigma, e_\Gamma)} \alpha(\gamma)(x) \cap \bigcup_{\gamma \in B_{2r}(\Gamma, \Sigma, e_\Gamma)} \alpha(\gamma)(y) = \emptyset.
\]

By induction, it is enough to show that there exists a continuous map \( \eta' : V_1 \cup W \to Q \) such that \( \eta'|_W = \eta \) and for all pairs \( x, y \in V_1 \cup W \) such that \( y = \alpha(\gamma)(x) \) for some \( e_\Gamma, \gamma \in B_r(\Gamma, \Sigma, e_\Gamma) \), we have that \( \eta'(x) \neq \eta'(y) \). So, let us construct the map \( \eta' \). Let \( z \in V_1 \setminus W \) and \( Q = \left\{ 1, 2, \ldots, k \right\} \). Let \( \tau(z) \) be the smallest positive integer such that there exists no \( \gamma \in B_r(\Gamma, \Sigma, e_\Gamma) \) satisfying both of the two conditions below.

- \( \alpha(\gamma)(z) \in W \).
- \( \eta(\alpha(\gamma)(z)) = i \).
Observe that $\tau^{-1}(i)$ is a clopen set for each $1 \leq i \leq k$ and

$$V \cup W = W \cup \bigcup_{i=1}^{k} \tau^{-1}(i).$$

Define $\eta': V \cup W \to Q$ by setting

- $\eta'(z) = \eta(z)$ for $z \in W$,
- $\eta'(z) = i$, for $z \in \tau^{-1}(i)$.

It is easy to see that the map $\eta'$ satisfies the condition of our lemma. \hfill \Box

Since the previous lemma implies Proposition 3.2, our theorem follows. \hfill \Box

### 3.4. The space of the weak equivalence classes is compact

We can introduce a natural metric on the set $\text{Free}(\Gamma)$. Let $\text{CF}(n,k)$ be the set of all subsets of $\{\{0,1\}^k\} B_n(\Gamma, \Sigma, \epsilon_T)$. Let $\alpha, \beta : \Gamma \rhd C$ be free Cantor actions. The weak distance of the classes $[\alpha]$ and $[\beta]$ is defined in the following way.

$$d_w([\alpha],[\beta]) = 2^{-n}$$

if

- there exists $A \in \text{CF}(n,n)$ such that either $A \in \text{Conf}_{n,0}(\varphi)$ for some continuous map $\varphi : C \to \{0,1\}^n$ and $A \notin \text{Conf}_{n,\beta}(\psi)$ for any continuous map $\psi : C \to \{0,1\}^n$, or $A \in \text{Conf}_{n,0}(\varphi)$ for some continuous map $\varphi : C \to \{0,1\}^n$ and $A \notin \text{Conf}_{n,\beta}(\psi)$ for any continuous map $\psi : C \to \{0,1\}^n$,
- for any $1 \leq i \leq n - 1$ and $B \in \text{CF}(i,i)$, $B \in \text{Conf}_{i,0}(\varphi)$ for some continuous map $\varphi : C \to \{0,1\}^i$ if and only if $B \in \text{Conf}_{i,\beta}(\psi)$ for some continuous map $\psi : C \to \{0,1\}^i$.

The following theorem is the qualitative analogue of the main result of [1] (see also [38]) on the compactness of the weak equivalent classes of p.m.p. actions.

**Theorem 3.** The space $\text{Free}(\Gamma)$ is compact with respect to the metric $d_w$.

**Proof.** First, we introduce some operations on configuration sets. For $1 \leq n \leq m$ and $k \geq 1$, let

$$\rho^{m,n} : \{\{0,1\}^k\} B_m(\Gamma, \Sigma, \epsilon_T) \to \{\{0,1\}^k\} B_n(\Gamma, \Sigma, \epsilon_T),$$

be the restriction map. So, if $A \in \text{CF}(m,k)$, then $\rho^{m,n}(A) \in \text{CF}(n,k)$. Also, if $1 \leq a \leq b \leq k$ and $l = b - a + 1$, let the map $\pi^{[a,b]} : \{0,1\}^k \to \{0,1\}^l$ be defined by setting for $1 \leq i \leq k$,

$$\pi^{[a,b]}(c)(i) = c(a - i + 1).$$

That is, if $A \in \text{CF}(n,k)$, then $\pi^{[a,b]} \circ A \in \text{CF}(n,l)$. Now, let $\{\alpha_j : \Gamma \rhd C\}_{j=1}^{\infty}$ be a sequence of free actions such that the classes $\{[\alpha_j]\}_{j=1}^{\infty}$ form a Cauchy sequence in the $d_w$-metric. We need to show that there exists $\alpha : \Gamma \rhd C$ such that $\lim_{j \to \infty} [\alpha_j] = [\alpha]$. For $n \geq 1$, let us call $A \in \text{CF}(n,k)$ a **surviving configuration** if for all but finitely many $j$’s, there exists a continuous map
\( \varphi^j : \mathcal{C} \to \{0,1\}^k \) such that \( \text{Conf}_{m,\alpha_j}(\varphi^j) = \mathcal{A} \), or in shorthand notation, \( \mathcal{A} \in \text{Conf}(\alpha_j) \). Our goal is to construct a free action \( \alpha : \Gamma \curvearrowright \mathcal{C} \) such that \( \mathcal{A} \in \text{Conf}(\alpha) \) if and only if \( \mathcal{A} \) is a surviving configuration. Let \( \{\mathcal{A}_i\}_{i=1}^\infty \) be an enumeration of the surviving configurations, where \( \mathcal{A}_i \in \text{Conf}(\alpha_i) \). Let \( m_1 \leq m_2 \leq \ldots \) be a sequence of integers such that if \( 1 \leq j \leq i \), then \( m_i \geq n_j \). For all pairs \( 1 \leq j \leq i \), let \( \hat{A}_j \in \text{CF}(m_i, l_j) \) be a surviving configuration such that \( \rho^{m_i,m_j}(\hat{A}_j) = A_j \). For \( i \geq 1 \), let \( k_i = \sum_{j=1}^i l_j \). Let \( \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots \) be an infinite sequence of integers such that for any \( i \geq 1 \), \( \alpha_i + 1 = \beta_i \) and \( \beta_i - \alpha_i + 1 = l_i \). Let \( CO(i) \subset \text{CF}(m_i, k_i) \) be the set of all surviving configurations \( \mathcal{B} \) such that for any \( 1 \leq j \leq i \), \( \pi_{[a_j,b_j]} \circ \mathcal{B} = \hat{A}_j \). Note that if \( \mathcal{B} \in CO(i) \), then
\[
\pi_{[1,k_i-1]} \circ (\rho^{m_i,m_i-1}(\mathcal{B})) \in CO(i-1).
\]
Therefore by Konig’s Lemma, we have a sequence of surviving configurations \( \{\mathcal{B}_i \in CO(i)\}_{i=1}^\infty \) such that for any \( i \geq 1 \),
\[
\pi_{[1,k_i-1]} \circ (\rho^{m_i,m_i-1}(\mathcal{B}_i)) = \mathcal{B}_{i-1}.
\]
For \( n \geq 1 \), let \( 1 \leq i_n \leq n \) be the largest integer such that \( \mathcal{B}_{i_n} \in \text{Conf}(\alpha_{i_n}) \) (we can assume that \( \mathcal{B}_{i_n} \in \text{Conf}(\alpha_n) \), for every \( n \geq 1 \)). Clearly \( i_n \to \infty \) as \( n \to \infty \). Let \( \varphi_n : \mathcal{C} \to \{0,1\}^{k_{i_n}} \) be a continuous map such that \( \text{Conf}_{m_{i_n},\alpha_{i_n}}(\varphi_n) = \mathcal{B}_{i_n} \) and let \( \kappa_{\varphi_n}(\mathcal{C}) = Y_n \subset \mathcal{C}^\Gamma \) be the corresponding Cantor subshift. Note that by Lemma 3.7, \( Y_n \) is conjugate isomorphic to \( \alpha_{i_n} \). Let \( Y \subset \mathcal{C}^\Gamma \) be defined in the following way. Let \( y \in Y \), if there exists a sequence \( \{y_{nk} \in Y_{nk}\}_{k=1}^\infty \) such that \( \lim_{k \to \infty} y_{nk} = y \). Our theorem immediately follows from the following proposition.

**Proposition 3.3.** The space \( Y \) is a Cantor subshift and \( [Y_n] \to [Y] \) in \( \text{Free}(\Gamma) \).

**Proof.** Clearly, \( Y \) is closed and \( \Gamma \)-invariant.

**Lemma 3.11.** The shift action of \( \Gamma \) on \( Y \) is free.

**Proof.** Let \( \beta : \Gamma \curvearrowright \mathcal{C} \) be the smallest element in \( \text{Free}(\Gamma) \) as in Theorem 2. So, any \( \mathcal{A} \in \text{Conf}(\beta) \) is a surviving configuration. Let \( m \geq 1 \) and let \( \mathcal{A} \in \text{Conf}_{m,r_m}(\beta) \) such a configuration that \( e(\gamma_1) \neq e(\gamma_2) \), provided that \( e : B_m(\Gamma, \Sigma, e_\Gamma) \to \{0,1\}^{r_m} \in \mathcal{A} \) and \( \gamma_1 \neq \gamma_2 \in B_m(\Gamma, \Sigma, e_\Gamma) \). Therefore, there exists some constant \( s_m > 0 \) such that for large enough \( i \),
\[
(\gamma_i(\gamma_2))_{[s_m]} \neq (\gamma_i(\gamma_2))_{[s_m]},
\]
provided that \( d_{\text{Clay}(\Gamma, \Sigma)}(\gamma_1, \gamma_2) \leq m \). Therefore, for such \( \gamma_1 \) and \( \gamma_2 \),
\[
(\gamma_i(\gamma_1))_{[s_m]} \neq (\gamma_i(\gamma_2))_{[s_m]}.
\]
Consequently, if \( e_\Gamma \neq \delta \in B_m(\Gamma, \Sigma, e_\Gamma) \), then
\[
R(\delta)(y) \neq y,
\]
where \( L \) is translation action on \( \mathcal{C}^\Gamma \). Since (4) holds for all \( m \geq 1 \), the action of \( \Gamma \) on \( Y \) is free. \( \square \)
We need to show that \( \text{Conf}(Y) \) equals to the set of all surviving configurations. Let \( \pi_i : Y \rightarrow \{0, 1\}^{k_i} \) be the projection onto the first \( k_i \) coordinates and \( n \geq 1 \).

**Lemma 3.12.** The configuration \( \text{Conf}_{n,Y}(\pi_i) \) is a surviving configuration.

**Proof.** By our construction, there exists \( A \in CF(n, k_i) \) such that for all large enough \( k \geq 1 \), \( A = \text{Conf}_{n,Y}(\pi_i) \). Clearly, \( \text{Conf}_{n,Y}(\pi_i) \subseteq A \). Now, let \( c : B_n(\Gamma, \Sigma, e_\Gamma) \rightarrow \{0, 1\}^{k_i} \in A \). Then, we have a sequence \( \{y_k \in Y_k\}_{k=1}^{\infty} \) so that for any \( \gamma \in B_n(\Gamma, \Sigma, e_\Gamma) \), \( \pi_i(y_k(\gamma)) = c(\gamma) \). So, \( \pi_i(y(\gamma)) = c(\gamma) \), whenever \( y \) is a limit point of the sequence \( \{y_k\}_{k=1}^{\infty} \).

Therefore, \( c \in \text{Conf}(Y) \). □

Let \( A \in CF(n, k) \) be a surviving configuration. Then, there exists \( i \geq 1 \) and some integers \( a, b \) such that

\[ \pi^{[a,b]} \circ (\rho^{i,n}(B_i)) = A. \]

Since by the previous lemma, \( B_j \in \text{Conf}(Y) \), we can immediately see that \( A \in \text{Conf}(Y) \) as well. Now, let \( A \in \text{Conf}_{n,k}(Y) \) and let \( \varphi : Y \rightarrow \{0, 1\}^k \) be a continuous map such that \( \text{Conf}_{n,Y}(\varphi) = A \). By continuity, there exists \( i \geq 1 \), such that for any Cantor subshift \( Z \subset C^\Gamma \) for which \( \text{Conf}_{i,Z}(\pi_i) = \text{Conf}_{i,Y}(\pi_i) \), there exists some continuous map \( \varphi_Z \) so that

\[ \text{Conf}_{i,Z}(\varphi_Z) = \text{Conf}_{n,Y}(\varphi) = A. \]

Since for large enough \( k \geq 1 \),

\[ \text{Conf}_{i,Y_k}(\pi_i) = \text{Conf}_{i,Y}(\pi_i), \]

we can see that \( A \in \text{Conf}(Y_k) \) for large enough \( k \). Therefore, \( A \) is a surviving configuration and so, our proposition follows. □

This concludes the proof of our theorem.

**Remark 4.** In [16], for any free Cantor action \( \alpha : \Gamma \curvearrowright C \) we constructed a specific minimal Cantor action \( m_\alpha : \Gamma \curvearrowright C \). By the construction it is clear that \( \alpha \preceq m_\alpha \) holds for any free action \( \alpha \). Therefore, we can conclude that the smallest element of \( \text{Free}(\Gamma) \) can be represented by a minimal action.

**Remark 5.** Let \( \{[\beta_i]\}_{i=1}^{\infty} \) be a dense subset of \( \text{Free}(\Gamma) \). Then, we can consider the Cantor action \( \beta = \prod_{i=1}^{\infty} \beta_i \curvearrowright \prod_{i=1}^{\infty} C \). Clearly, for any \( i \geq 1 \), \( \beta \preceq \beta_i \). Hence, \( \beta \geq \alpha \) for any free action \( \alpha : \Gamma \curvearrowright C \). That is, \( [\beta] \) is the largest element of \( \text{Free}(\Gamma) \).

### 4. Qualitative convergence and limits

The goal of this section is to introduce and study the main notion of our paper: qualitative convergence of countable graphs.
4.1. **Convergence of generalized Schreier graphs.** First, we extend Definition 1.2 for Schreier graphs. Let $G\Gamma_\Sigma G$ be the set of all generalized Schreier graphs as in Section 2. Let $Q$ be a finite set and $U_{r,Q}^{\Gamma,\Sigma}$ be the set of all rooted, $\Sigma$-edge labeled, $Q$-vertex labeled balls of radius $r$ that occurs in some $Q$-labeled Schreier graphs. Now, let $S \in G\Gamma_\Sigma G$ and $\varphi : V(S) \to Q$ be a continuous map. Then, let $\text{Conf}_{r,S}(\varphi) \subseteq U_{r,Q}^{\Gamma,\Sigma}$, the configuration of $\varphi$, be the set of all rooted, $\Sigma$-edge labeled, $Q$-vertex labeled balls that occur in in the labeled graph $(S, \varphi)$. Also, we denote by $\text{Conf}(S)$ the set of all configurations of $S$.

**Definition 4.1.** A sequence of generalized Schreier graphs $\{S_n\}_{n=1}^\infty \subseteq G\Gamma_\Sigma G$ is qualitatively convergent, if for any finite set $Q$, integer $r \geq 1$ and $A \subseteq U_{r,Q}^{\Gamma,\Sigma}$, there exists $N_A > 0$ such that

- either for all $n \geq N_A$, there exists $\varphi_n : V(S_n) \to Q$ such that $\text{Conf}_{r,S_n}(\varphi) = A$,
- or for all $n \geq N_A$, there exists no $\varphi_n : V(S_n) \to Q$ such that $\text{Conf}_{r,S_n}(\varphi) = A$.

Again, qualitative convergence does not depend on the choice of $\Sigma$.

4.2. **The basic example.** Let $\Gamma = \mathbb{Z}$, $\Sigma = \{1, -1\}$ and $C_\Gamma$ be the cyclic Schreier graph on the set $\mathbb{Z}/n\mathbb{Z}$. That is, $a \to b$ is a 1-labeled directed edge if $b = (a + 1)(\text{mod } n)$ and $a \to b$ is a $-1$-labeled directed edge if $b = (a - 1)(\text{mod } n)$. Our goal is to explicitly describe the qualitatively convergent subsequences of $\{C_\Gamma\}_{n=1}^\infty$. A subset $W \subseteq \mathbb{N}$ is called division-closed if $n \in W$ and $m | n$ always implies that $m \in W$ as well. Division-closed subsets can be classified in the following way. Let $\{p_s\}_{s=1}^\infty$ be the sequence of primes $2, 3, \ldots$. The family of division-closed subsets are in a one-to-one correspondence $\kappa \to W_\kappa$ with the sequences $\kappa = \{\tau_s\}_{s=1}^\infty$, where $\tau_s \in \{\mathbb{N} \cup \{\infty\}\}$, and $n \in W_\kappa$ if $n = p_1^{q_1}p_2^{q_2} \ldots$ with $q_s < \tau_s$ for all $s \geq 1$.

**Example 1.** If $\kappa = \{1, 1, 1, \ldots\}$, then $W_\kappa = \{1\}$. If $\kappa = \{\infty, \infty, \ldots\}$, then $W_\kappa = \mathbb{N}$. If $\kappa = \{\infty, 1, 1, 1, \ldots\}$, then $W_\kappa = \{1, 2, 4, 8, \ldots\}$.

Let $\rho = \{n_k\}_{k=1}^\infty$ be a strictly increasing sequence of integers. We say that $\rho$ is arithmetically convergent, if for any $m > 1$

1. either $m | n_k$ for all but finitely many $k$'s,
2. or $m \nmid n_k$ for all but finitely many $k$'s.

The set of integers $m$ satisfying condition (1) form the arithmetic spectrum of $\rho$, $\text{Ar}(\rho)$. Clearly, $\text{Ar}(\rho)$ is a division-closed set for any arithmetically convergent sequence $\rho$. Also, for any division-closed set $W$, there exists an arithmetically convergent sequence $\rho$ such that $\text{Ar}(\rho) = W$.

**Proposition 4.1.** Let $\{n_k\}_{k=1}^\infty$ be a strictly increasing sequence of integers Then, $\{C_{n_k}\}$ is qualitatively convergent, if and only if $\{n_k\}_{k=1}^\infty$ is arithmetically convergent.
Proof. Let $I_r^Q$ be the set of all $Q$-labelings of the directed path $I_r = [-r, -r + 1, \ldots, r]$. We say that a labeling $b \in I_r^Q$ is a continuation of $a \in I_r^Q$, if there exists a labeling $c$ of the path $[-r, -r + 1, \ldots, r + 1]$ such that the restriction of $c$ onto the segment $[-r, -r + 1, \ldots, r]$ equals to $a$ and the restriction of $b$ onto the segment $[r + 1, -r + 2, \ldots, r + 1]$ equals to $b$. For an example, let $Q = \{1, 2, 3\}$. The element $(1, 2, 3) \in I_r^Q$ is the continuation of $(1, 1, 2), (2, 1, 2)$ and $(3, 1, 2)$. Also, the element $(1, 1, 1) \in I_1^Q$ is the continuation of itself. So, let $A \subset U_{\Sigma}^r = I_r^Q$ be as above, a set of $Q$-labeled rooted $r$-balls in cyclic Schreier graphs. We construct a directed graph $G_A$ using the structure of $A$ in the following way.

- The vertex set of $G_A$ consists of the elements of $A$,  
- $a \rightarrow b$ is a directed edge of $G_A$ if $b$ is a continuation of $a$.

Observe, that there exists a map $\varphi_n : V(C_n) \rightarrow Q$ such that $Conf_{r,C_n}(\varphi_n) = A$, if there exists a closed directed walk of length $n$ in the graph $G_A$ such that the walk passes through all the elements of $A$ at least once. We call such a walk an $A$-walk. Our proposition immediately follows from the lemma below.

**Lemma 4.1.** Let $A \subseteq I_r^Q$ be as above. Suppose that $A$-walks exist in the graph $G_A$. Then, there exists a positive integer $n_A$ with the following property. For any large enough integer $m$, there exists an $A$-walk of length $m$, if and only if $m$ is divisible by $n_A$.

**Proof.** Observe that if $w_1$ is an $A$-walk of length $m_1$ and $w_2$ is an $A$-walk of length $m_2$ then the concatenation of $w_1$ and $w_2$ is an $A$-walk of length $m_1 + m_2$. Let $n_A$ be the greatest common divisor of the $A$-walk lengths. Then, $n_A = \sum_{i=1}^{k} p_i m_i$, where $\{p_i\}_{i=1}^{k}$ are integers and $\{m_i\}_{i=1}^{k}$ are lengths of $A$-walks. So, if $n$ is a large enough integer divisible by $n_A$, then $n = \sum_{i=1}^{k} q_i m_i$, where $\{q_i\}_{i=1}^{k}$ are non-negative integers. Consequently, there exists an $A$-walk of length $n$. \[\square \]

4.3. **Stable actions as qualitative limits.** Let $X$ be a totally disconnected compact metric space of continuum size. Then $X = A \cup C$, where $A$ is a countable set. Let $\alpha : \Gamma \curvearrowright X$ be a continuous action. Following Glasner and Weiss [22], we call $\alpha$ a **stable action** if for any $\gamma \in \Gamma$ and $x \in X$ such that $\alpha(\gamma)(x) = x$, there exists a neighbourhood $x \in U$ such that $\alpha(\gamma)(y) = y$ for all $y \in U$. Clearly, any free action is stable. Let $Sub(\Gamma)$ be the compact metric space of all the subgroups of $\Gamma$. Recall that we regard $Sub(\Gamma)$ as the closed subset of the space of subsets $\{0, 1\}^\Gamma$, with the natural conjugation action of $\Gamma$. If $\beta : \Gamma \curvearrowright X$ is a continuous action, we have a natural map $\text{Stab}_\beta : X \rightarrow Sub(\Gamma)$ so that for $x \in X$, $\text{Stab}_\beta(x)$ is the stabilizer subgroup of $x$. The action $\beta$ is stable if and only if $\text{Stab}_\beta$ is continuous (see [22]). In this case, $\text{Stab}_\beta(X)$ is a $\Gamma$-invariant closed subset of $Sub(\Gamma)$. Now, let $Q$ be a finite set and $\varphi : X \rightarrow Q$ be a continuous map. Let $\text{Conf}_{r,\beta}(\varphi) \subseteq U_{\Sigma}^r$ be the set of all rooted $Q$-labeled balls $\eta$ of radius $r$ such that there exists $x \in X$ so that the $Q$-labeled ball around $x$ is isomorphic to $\eta$. 
Definition 4.2. Let $\{S_n\}_{n=1}^{\infty} \subset \Gamma T_\Sigma G$ be a qualitatively convergent sequence of generalized Schreier graphs. We say that $\{S_n\}_{n=1}^{\infty}$ is qualitatively converges to the stable action $\beta : \Gamma \curvearrowright X$, $S_n \xrightarrow{q} \beta$, if for any finite set $Q$ and $A \subseteq U_{\Gamma, \Sigma}^{r,q}$, there exists a continuous map $\psi : X \rightarrow Q$ such that $\text{Conf}_{r, \beta}(\psi) = A$ if and only if for all large enough $n \geq 1$, there exists $\varphi_n : V(S_n) \rightarrow Q$ such that $\text{Conf}_{r, S_n}(\varphi_n) = A$.

Remark 6. Let $\beta : \Gamma \curvearrowright C$ be a free Cantor action such that for some sequence of generalized Schreier graphs $S_n \xrightarrow{q} \beta$. Suppose that the free Cantor action $\alpha : \Gamma \curvearrowright C$ is qualitatively weakly equivalent to $\beta$. Then $S_n \xrightarrow{q} \alpha$, as well.

The main result of this section is the following qualitative analogue of the theorem of Hatami, Lovász and Szegedy [23] on the existence of local-global limits.

Theorem 4. For any qualitatively convergent sequence $\{S_n\}_{n=1}^{\infty} \subset \Gamma T_\Sigma G$, there exists a totally disconnected continuum $X$ and a stable action $\beta : \Gamma \curvearrowright X$ such that $S_n \xrightarrow{q} \beta$.

Proof. The proof of our theorem will be very similar to the one of Theorem 3. Again, it is enough to check convergence for finite sets $Q$, where $Q = \{0, 1\}^k$ for some $k \geq 1$. As in the previous section, we call $A \subseteq U_{\Gamma, \Sigma}^{r,(0,1)^k}$ a surviving configuration, if for large enough $n \geq 1$, $A \in \text{Conf}(S_n)$. Let $\{A_i\}_{i=1}^{\infty}$ be an enumeration of the surviving configurations, where $A_i \subseteq U_{\Gamma, \Sigma}^{r,(0,1)^{k_i}}$. Now, we construct a sequence of Cantor labelings $\{\psi^n : V(S_n) \rightarrow C\}_{n=1}^{\infty}$. Let $\prod_{j=1}^{\infty} \psi^n_j : V(S_n) \rightarrow \prod_{j=1}^{\infty} \{0, 1\}^{k_j}$ be defined in the following way.

- If $A_j \in \text{Conf}(S_n)$, then let $\psi^n_j : V(S_n) \rightarrow \{0, 1\}^{k_j}$ be a function such that $\text{Conf}_{r, S_n}(\psi^n_j) = A_j$,
- if $A_j \notin \text{Conf}(S_n)$, then let $\psi^n_j(v) = \{0, 0, \ldots, 0\}$ for all $v \in V(S_n)$.

Consider the totally disconnected compact space $\text{CGT}_\Sigma G$ of rooted $C$-labeled Schreier graphs. Let $(S, \varphi) \in \text{CGT}_\Sigma G$. Then its orbit closure, $\overline{O}((S, \varphi))$ is defined as in Section 2. Now, let us define $X \subset \text{CGT}_\Sigma G$ as follows. Let $y \in X$, if $y = \lim_{n \rightarrow \infty} y_{n_k}$ for some convergent sequence $\{y_{n_k}\}_{k=1}^{\infty} \subset \text{CGT}_\Sigma G$, where $y_{n_k} \in \overline{O}((S_{n_k}, \varphi_{n,k}))$. Clearly, $X$ is closed and invariant under $\Gamma$.

Lemma 4.2. The action of $\Gamma$ on the space $X$ is stable.

Proof. Before starting the proof let us make a remark on stability. Let $z = (T, H, \rho) \in \text{CGT}_\Sigma G$, where $T$ is a Schreier graph, $H \in \text{Sub}(\Gamma)$ and $\rho : V(T) \rightarrow C$. Then, we have a natural rooted Schreier graph structure $z' = (T', H', \rho')$ on the orbit set of $z$ in $\text{CGT}_\Sigma G$. One can observe that $z$ is not always equal to $z'$. For example, if $T = \text{Cay}(\Gamma, \Sigma)$, $H = e_\Gamma$ and $\rho : \Gamma \rightarrow C$ is a constant-valued function, then $V(T')$ consists of a singleton. Now suppose that for any $r \geq 1$, there exists some $s_r \geq 1$ such that $$(\rho(x))_{[s_r]} \neq (\rho(y))_{[s_r]}$$.
provided that $0 < d_T(x, y) \leq r$. Then, $(T, H, \rho) = (T', H', \rho')$ and also, the $\Gamma$-action on the orbit closure $\overline{O}(z)$ is stable (see e.g. the proof of Corollary 3.1. in [16]). If $Y \subset \mathcal{C}_{\Sigma}G$ is a closed $\Gamma$-invariant subset, $r \geq 1$, $Q$ is a finite set and $\varphi : Y \to Q$ is a continuous map, then one can consider the set

$$\text{Conf}_{r,Y}^\Gamma(\varphi) = \bigcup_{y \in Y} \text{Conf}_{r,T_y}(\varphi),$$

where $y = (T_y, H_y, \rho_y)$. In general, it is possible that $\text{Conf}_{r,Y}^\Gamma(\varphi) \neq \text{Conf}_{r,Y}^1(\varphi)$. However, it is clear from the discussion above that

$$(5) \quad \text{Conf}_{r,Y}^\Gamma(\varphi) = \text{Conf}_{r,Y}^1(\varphi)$$

holds, whenever the action of $\Gamma$ on $Y$ is stable. Now, let us turn back to the proof of our lemma. Observe that for any $r \geq 1$, there exists $i \geq 1$ and a surviving configuration $A_i \subseteq U_{\Gamma,\Sigma}^{r_i,\{0,1\}^{k_i}}$ such that $r_i > r$ and $A_i$ is $r$-separating. That is, for any $c \in A_i$, $c : B \to \{0,1\}^{k_i}$, where $B \in U_{\Gamma,\Sigma}^{r_i,\{0,1\}^{k_i}}$, we have that $c(u) \neq c(v)$ provided that $0 < d_B(u, v) \leq r$. Hence, for large enough $n \geq 1$,

$$(6) \quad (\hat{g}(a))[s_r] \neq (\hat{g}(b))[s_r],$$

for any $\hat{g} = (T_{\hat{g}}, H_{\hat{g}}, \rho_{\hat{g}}) \in \overline{O}(S_n, \varphi_n)$, where $0 < d_{\hat{g}}(a, b) \leq r$ and $s_r = \sum_{j=1}^i k_j$. Therefore by definition, (6) also holds if $\hat{g} \in Y$. Hence, the action of $\Gamma$ on $Y$ is stable.

Again, we need to prove that $\text{Conf}(Y)$ equals to the set of surviving configurations. We can repeat the proof of Theorem 3, to show that if $A$ is a surviving configuration, then $A \in \text{Conf}(Y)$. It is also clear from our construction, that $\text{Conf}_{r,Y}(\pi_i)$ is a surviving configuration for any $r \geq 1$ and $i \geq 1$. So, the continuity argument after Lemma 3.12 can be applied to immediately show that all elements of $\text{Conf}(Y)$ are, in fact, surviving configurations. \hfill \Box

Remark 7. Let us suppose that for a qualitatively convergent sequence of generalized Schreier graphs $\{S_n\}_{n=1}^\infty$, there exists an integer $q \geq 1$ such that each graph $S_n$ contains exactly one vertex of degree $q$. Then if $S_n \xrightarrow{\beta} \alpha$, where $\alpha : \Gamma \curvearrowright X$ is a stable action, $X$ always contains an isolated point.

Remark 8. Let $S \in \mathcal{C}_{\Gamma}G$ be a generalized Schreier graph. Then, the vertices of $S$ form a conjugacy invariant subset $A(S)$ in $\text{Sub}(\Gamma)$. Let $\overline{A(S)}$ be the closure of $A(G)$. If $S_n \xrightarrow{\beta} S$, then it is not hard to to see that $\overline{A(S)}$ is the Hausdorff limit of the sequence of compact subsets $\{\overline{A(S_n)}\}_{n=1}^\infty$. Moreover, if $S_n \xrightarrow{\beta} \beta$, where $\beta : \Gamma \curvearrowright X$ is a stable action, then the closed subset $\text{Stab}(X)$ coincides with $\overline{A(S)}$.

4.4. The limits of cyclic Schreier graphs. Let $\alpha : \mathbb{Z} \curvearrowright \mathcal{C}$ be a free minimal Cantor action of the integers. Following Lin and Matui [28], we say that the integer $n \in \mathbb{N}$ is an element of the periodic spectrum of $\alpha$, $PS(\alpha)$, if there exists a clopen set $U \subset \mathcal{C}$ such that $\bigcup_{i=0}^{n-1} \alpha(i)(U)$ is a clopen partition of $\mathcal{C}$. Clearly, $PS(\alpha)$ is a division-closed set. In [36] (Lemma 3.14), Shimomura constructed a free minimal Cantor action $\alpha_W : \Gamma \curvearrowright \mathcal{C}$ for every division-closed
set \( W \) such that \( PS(\alpha_W) = W \). Now, we characterize the qualitative limits of cyclic Schreier graphs using the periodic spectrum.

**Theorem 5.** Let \( W \) be a division-closed set and \( \{C_{n_k}\}_{k=1}^\infty \) be a qualitatively convergent sequence of cyclic Schreier graphs such that \( Ar(\{n_k\}_{k=1}^\infty) = W \). Then, \( \alpha_W \) is a qualitative limit for \( \{C_{n_k}\}_{k=1}^\infty \).

**Proof.** Let \( A \subset U^Q \) be a set as in Lemma 4.1, for which \( A \)-walks exist and let \( n_A \) be the smallest positive integer for which \( A \)-walks of length \( m \) exist for large enough integers \( m \). First, suppose that \( n_A \in W \). That is, \( A \) is a surviving configuration for \( \{C_{n_k}\}_{k=1}^\infty \).

**Lemma 4.3.** \( A \in Conf(\alpha_W) \).

**Proof.** Since \( n_A \in PS(\alpha_W) \), we have a clopen set \( U \subset C \) such that \( \bigcup_{i=0}^{n_A-1} \alpha_W(i)(U) \) is a clopen partition of \( C \). Let \( l \geq 1 \) be an integer such that if \( l \leq m \) and \( n_A \mid m \), then there exists a \( A \)-walk of length \( m \). As in Lemma 3.10, we can easily prove that there exists a clopen set \( V \subset U \) such that if \( x \in V \), then \( \alpha_W(i)(x) \notin V \) for any \( 1 \leq i \leq l \). By minimality, there exists some \( l \leq k \) such that for all \( x \in V \), \( \alpha_W(i)(x) \in V \) for some \( l \leq i \leq k \). Pick

\[
c = (\hat{q}_{-r}, \hat{q}_{-r+1}, \ldots, \hat{q}_0, \hat{q}_1, \ldots, \hat{q}_l) \in A.
\]

For any integer \( l \leq m \leq k \) such that \( n_A \mid m \), pick an \( A \)-walk \( w_k(m) \) of length \( m \), that starts and ends with \( c \). That is, for each such \( m \) we have a sequence of elements from \( A \),

\[
(w^m, w^m, \ldots, w^m),
\]

such that \( w^m_0 = w^m_{m-1} = c \), and a corresponding sequence of elements of \( Q \),

\[
(q^m_0, q^m_1, \ldots, q^m_{m-1}),
\]

where \( q^m_i = w^m_i(0) \). Now we define the continuous maps

\[
\varphi_1 : C \setminus V \to \{1, 2, \ldots, k\} \quad \text{and} \quad \varphi_2 : C \setminus V \to \{1, 2, \ldots, k\}
\]

in the following way. For \( y \in C \setminus V \), there exist unique positive integers \( i, m, i \leq m \leq k, n_A \mid m \) and \( x_1 \in V, x_2 \in V \) such that

- \( \alpha_W(i)(x_1) = y, \alpha_W(m)(x_1) = x_2 \),
- \( \alpha_W(j)(x_1) \notin V \) for \( 1 \leq j \leq m-1 \).

Then, let \( \varphi_1(y) = m, \varphi_2(y) = i \). Now, we can define the continuous map \( \psi : C \to Q \) by

- \( \psi(x) = \hat{q}_0 \) for \( x \in V \),
- \( \psi(y) = q_{\varphi_1(y)}^{\varphi_2(y)} \) for \( y \in C \setminus V \).

It is not hard to see that \( \text{Conf}_{r, \psi}(\alpha_W) = A \). Hence, the lemma follows. \( \square \)

**Lemma 4.4.** Let \( \beta \) be a minimal action such that \( A \in Conf(\beta) \). Then, \( n_A \in PS(\beta) \).
Proof. Let \( \varphi : C \to Q \) be a continuous map such that \( \text{Conf}_{r,\varphi}(\beta) = A \). Then, we have a corresponding continuous map \( \Phi : C \to A \) such that for each \( x \in C \)

\[
\Phi(x) = (\varphi(\beta(-r)(x)), \varphi(\beta(-r+1)(x)), \ldots, \varphi(\beta(r)(x)))
\]

Let \( c \in A \). By minimality, there exists some \( t \geq 1 \) such that if \( \Phi(x) = c \) \( \Phi(\beta(i)(x)) = c \) and \( i \geq t \), then \( n_A \mid i \) and the sequence

\[
(\Phi(x), \Phi(\beta(1)(x)), \ldots, \Phi(\beta(i)(x)))
\]

defines an \( A \)-walk starting and ending at \( c \). Again, let \( V \subset \Phi^{-1}(c) \) be a clopen set such that if \( x \in V \) and \( j \leq t \), then \( \beta(i)(x) \notin V \). So, we can define the continuous map \( \lambda : C \to \{0, 1, \ldots, n_A - 1\} \) by setting

- \( \lambda(x) = 0 \) if \( x \in V \),
- \( \lambda(y) = i \), if \( j \) is the smallest integer such that \( \beta(j)(x) = y \), for some \( x \in V \) and \( j \equiv i \mod(n_A) \).

Then, \( \bigcup_{i=0}^{n-1} \beta(i)(U) \) is a clopen partition of \( C \), where \( U = \lambda^{-1}(0) \). Hence, the lemma follows. □

By the previous lemmas, \( A \in \text{Conf}(\alpha_W) \) if and only if \( n_A \in W \), so our theorem follows. □

4.5. The limits of countable graphs. Let \( \alpha : \Gamma \curlyeqprec C \) be a stable action of a finitely generated group \( \Gamma \), with symmetric generating set \( S \). Then, for any \( x \in C \), we can consider the simple, connected orbit graph \( G_x \), where

- \( V(G_x) = \cup_{\gamma \in \Gamma} \alpha(\gamma)(x) \),
- the vertices \( y \neq z \in V(G_x) \) are adjacent if there exists \( \sigma \in \Sigma \) such that \( \alpha(\sigma)(y) = z \).

Let \( \varphi : C \to Q \) be a continuous map and let \( d = \max_{x \in C} \text{deg}_{G_x}(x) \). Then \( \text{Conf}_{r,\varphi}(\varphi) \subset U_d^{rQ} \) is the set of all rooted \( Q \)-labeled balls of radius \( r \), that occur in the \( Q \)-labeled graph \( (G_x, \varphi) \) for some \( x \in C \). Again, let \( \text{Conf}(\alpha) = \cup_{r,\varphi} \text{Conf}_{r,\varphi}(\varphi) \).

**Definition 4.3.** The action \( \alpha : \Gamma \curlyeqprec C \) is the limit of the qualitatively convergent graph sequence \( \{G_n\}_{n=1}^{\infty} \subset Gr_{d} \) if \( A \in \text{Conf}(\alpha) \) if and only if \( A \in \text{Conf}(G_n) \) for large enough \( n \geq 1 \).

We have the following “simple graph” version of Theorem 4.

**Proposition 4.2.** For any qualitatively convergent graph sequence \( \{G_n\}_{n=1}^{\infty} \subset Gr_d \), there exists some finitely generated group \( \Gamma \) and a stable action \( \alpha : \Gamma \curlyeqprec C \) such that \( \alpha \) is the limit of \( \{G_n\}_{n=1}^{\infty} \).

**Proof.** Before proving the proposition, let us make a short comment. Let \( Q \) be a finite set and \( G \in Gr_d \) be a simple graph. Suppose that there exists a labeling \( \tau : E(G) \to Q \) such that adjacent edges have different labels. Then, \( \tau \) defines an action of the \( Q \)-fold free power \( F_Q \) of the cyclic group of two
elements with generating system \( Q \), where \( q^2 = e_{F_Q} \) for each \( q \in Q \). The underlying graph of the Schreier graph of the action \( \tau \) is just \( G \). Suppose that \( \{G_n\}_{n=1}^{\infty} \subset Gr_d \) is a convergent graph sequence, \( \tau_n : E(G_n) \to Q \) is a sequence of maps as above and \( \{S_n\}_{n=1}^{\infty} \) is the associated sequence of Schreier graphs. If \( S_n \xrightarrow{\cong} \beta \), where \( \beta \) is a stable action of \( F_Q \), then \( \beta \) is the qualitative limit of our sequence \( \{G_n\}_{n=1}^{\infty} \). It is not hard to show that one has maps \( \tau_n : E(G_n) \to Q \) such that the associated Schreier graphs converge naively, but we do not know whether one can achieve qualitative convergence. Therefore, we need to pursue a somewhat different path towards the proof. Let \( \Gamma \) be a finitely generated group and \( \Sigma \) be a symmetric generating set, \( |\Sigma| = d \). Then, we have a natural “forgetting” map \( \mathcal{F} : C\Gamma_2 \mathcal{G} \to CR\mathcal{G}_d \), mapping each rooted Schreier graph into the underlying rooted simple graph. Clearly, \( \mathcal{F} \) is continuous and maps invariant sets into invariant sets.

Let \( Y \subset CR\mathcal{G}_d \) be a closed invariant subset. We say that \( Y \) is proper if for any \( r > 0 \), there exists \( s_r > 0 \) such that if \( (G, \psi, x) \in Y \), \( x, y \in V(G) \) and \( 0 < d_G(x, y) \leq r \), then \( (\psi(x))[s_r] \neq (\psi(y))[s_r] \). Now, let \( \bigcup_{i=1}^{q} W_i \) be a clopen partition of \( Y \) such that if \( (G, \psi, x) \in W_i \), \( 0 < d_G(x, y) \leq 2 \), then \( (G, \psi, y) \neq W_i \). Clearly, such partition exists by properness. Let \( \psi : Y \to \{1, 2, \ldots, q\} \) be the continuous function such that \( \psi((G, \psi, x)) = i \), if \( (G, \psi, x) \in W_i \). Let \( Q = \{1, 2, \ldots, q\} \) and \( \hat{Q} \) be the set of \( 2 \)-element subsets of \( Q \). Let \( F_{\hat{Q}} \) be the \( |\hat{Q}| \)-fold free power of the cyclic group of two elements with generating system \( \hat{Q} \), where \( a^2 = e_{F_{\hat{Q}}} \), if \( a \in \hat{Q} \). For \( (G, \psi, x) \in Y \) and \( y, z \in V(G) \), \( d_G(y, z) = 1 \), label the edge \( (y, z) \in E(G) \) by \( \{\psi((G, \psi, y)), \psi((G, \psi, z))\} \in \hat{Q} \). Observe that for each \( (G, \psi, x) \in Y \) we obtain a rooted \( F_{\hat{Q}} \)-Schreier graph \( \overline{(G, \psi, x)} \).

The following lemma is easy to prove.

**Lemma 4.5.** The set \( Y = \bigcup_{(G, \psi, x) \in Y} \overline{(G, \psi, x)} \) is a closed invariant subset of \( C\Gamma_2 \mathcal{G} \), where \( \Gamma = F_{\hat{Q}} \) and \( \Sigma = \hat{Q} \). The action of \( F_{\hat{Q}} \) on \( Y \) is stable and the forgetting map \( \mathcal{F} : Y \to Y \) is a homeomorphism.

Let \( Y \subset CR\mathcal{G}_d \) be a proper subset. We can define the configuration spaces as in the proof of Theorem 4. Let \( \varphi : Y \to Q \) be a continuous map, where \( Q \) is a finite set. Then, \( \text{Conf}_{r,Y}(\varphi) \subset U^r_d \) is the set of all rooted \( Q \)-labeled balls that occur in some labeled graph \( (G, \psi) \), where \( (G, \psi, x) \in Y \). Note that we do not use the function \( \psi \) only the graph structure. Again, \( \text{Conf}(Y) \) is the set of all configurations. We say that \( Y \) is a limit of a qualitatively convergent sequence \( \{G_n\}_{n=1}^{\infty} \subset Gr_d \) if \( \text{Conf}(Y) \) equals to the set of all surviving configurations. By Lemma 4.5, our proposition follows from the following lemma.

**Lemma 4.6.** For each qualitatively convergent sequence \( \{G_n\}_{n=1}^{\infty} \subset Gr_d \) there exists a proper subset \( Y \subset CR\mathcal{G}_d \) such that \( G_n \xrightarrow{\mathcal{F}} Y \).

**Proof.** The proof of the lemma is almost identical to the one of Theorem 4. We glance through the proof for completeness. It is enough to check convergence for finite sets \( Q \), where \( Q = \{0, 1\}^k \). Let \( \{A_i\}_{i=1}^{\infty} \) be an enumeration of
the surviving configurations, where $A_i \subseteq U_i^* \{0,1\}^{k_i}$. Again, we construct a sequence of Cantor labelings $\{\psi^n : V(G_n) \to C\}_{n=1}^\infty$. Let $\prod_{j=1}^\infty \psi^n_j : V(G_n) \to \prod_{j=1}^\infty \{0,1\}^{k_j}$ be defined in the following way.

- If $A_j \in \text{Conf}(G_n)$, then let $\psi^n_j : V(G_n) \to \{0,1\}^{k_j}$ be a function such that $\text{Conf}_r(G_n, \psi^n_j) = A_j$.
- If $A_j \notin \text{Conf}(G_n)$, then let $\psi^n_j(v) = \{0,0,\ldots,0\}$ for all $v \in V(G_n)$.

Now, let us define $Y \subset C\Sigma G$ as follows. Let $y \in Y$, if $y = \lim_{k \to \infty} y_{nk}$ for some convergent sequence $\{y_{nk}\}_{k=1}^\infty \subset CR_{d}$, where $y_{nk} \in \mathcal{O}(G_{nk}, \varphi_{n,k})$. The following lemma can be proven in the same way as Lemma 4.2.

**Lemma 4.7.** The space $Y \subset C\Sigma G$ is proper.

Now, we need to show that $\text{Conf}(Y)$ is equal to the set of surviving configurations. This can be done exactly the same way as in the end of the proof of Theorem 4. □

## 5. Almost finiteness of graphs and stable actions

The goal of this section is to introduce and study various notions of almost finiteness for classes of bounded degree graphs. Our definition of almost finiteness is based on the notion of almost finiteness of étale groupoids introduced by Matui [34].

### 5.1. The geometric groupoid of a stable action

Let $X$ be a compact, metrizable, totally disconnected continuum. Let $\Gamma$ be a countable group and let $\alpha : \Gamma \curvearrowright X$ be a stable action. Now using the action $\alpha$, we define a locally compact, totally disconnected, Hausdorff, principal, étale groupoid $G_{\alpha}$, the geometric groupoid of $\alpha$. One should note that $G_{\alpha}$ is isomorphic to the transformation groupoid of the action $\alpha$ only if the action is free.

1. The elements of the groupoid $G_{\alpha}$ are the pairs $(x,y)$, where $y = \alpha(\gamma)(x)$ for some $\gamma \in \Gamma$. As usual, we have the range and source maps $s(x,y) = x$, $r(x,y) = y$, and a groupoid multiplication $(x,y)(a,b) = (x,b)$, where $a = z$.

2. The basis for the topology on $G_{\alpha}$ is given in the following way. Let $\alpha(\gamma)(x) = y$. Then by stability,
   - there exist clopen sets $x \in U$ and $y \in V$ such that $\alpha(\gamma) : U \to V$ is a homeomorphism,
   - and if also $\alpha(\delta)(x) = y$, then there exists a clopen set $x \in W$ such that $\alpha(\gamma)|_W = \alpha(\delta)|_W$.

The base neighbourhoods of the element $(x,y)$ are in the form of $(U, x, y)$, where $\alpha(\gamma) : U \to V$ as above, and $(a, b) \in (U, x, y)$ if $a \in U$ and $\alpha(\gamma)(a) = b$.

One can immediately see that
(1) The base set $G^0_\alpha$ is homeomorphic to $X$. The multiplication, source and range maps are continuous.

(2) The range map is a local homeomorphism, so our groupoid is étale.

(3) The isotropy groups of all $x \in X$ are trivial.

Hence, $G_\alpha$ is a locally compact, totally disconnected, Hausdorff, principal, étale groupoid. The groupoid $G_\alpha$ is minimal if and only if the action $\alpha$ is minimal. Now, let us suppose that $\Gamma$ is finitely generated and $\Sigma$ is a symmetric generating system of $\Gamma$. Then, for any $x \in X$, the orbit set of $x$

$$\{(x,y) \in G_\alpha\}$$

is equipped with a bounded degree graph structure $G_\alpha(\Sigma)$. The element $(x,a)$ is adjacent to $(x,b)$ if $\alpha(\sigma)(a) = b$ for some generator $\sigma \in \Sigma$. Let $H_{\alpha,\Sigma}$ be the components of $G_\alpha(\Sigma)$. Now, we define the almost finiteness of $\alpha$.

**Definition 5.1.** The action $\alpha : \Gamma \curvearrowright X$ is almost finite if for any $\varepsilon > 0$ there exist $K_\varepsilon > 0$, a finite set $Q$ and a continuous map $\varphi_\varepsilon : X \to Q$ satisfying the following conditions.

1. If $x, y \in X, \alpha(\gamma)(x) = y$ for some $\gamma \in \Gamma$, and $\varphi_\varepsilon(x) = \varphi_\varepsilon(y)$, then either $d_G(x,y) \leq K_\varepsilon$ or $d_G(x,y) \geq 3K_\varepsilon$, where $G \in H_{\alpha,\Sigma}$ and $x \in V(G)$.

2. If $x$ and $G$ are as above, $i_G(H_x) \leq \varepsilon$, where

$$H_x = \{z \in V(G) \mid d_G(x,z) \leq K_\varepsilon \text{ and } \varphi_\varepsilon(x) = \varphi_\varepsilon(z)\}.$$ 

Clearly, almost finiteness of $\alpha$ does not depend on the choice of the generating system.

Note that $\varphi_\varepsilon$ defines a continuous field of almost finite partitions on the orbit graphs in $H_{\alpha,\Sigma}$. One can easily check that the partitions above satisfies the conditions given in [37] Definition 3.6. Hence, we have the following definition-proposition.

**Proposition 5.1.** The action $\alpha : \Gamma \curvearrowright X$ is almost finite in the sense of Definition 5.1 if and only if the principal étale groupoid $G_\alpha$ is almost finite in the sense of [37] Definition 3.6.

Using Definition 7.1 in Section 7 and Theorem 10, we have the following proposition.

**Proposition 5.2.** The action $\alpha$ is almost finite if the set of graphs $H_{\alpha,\Sigma}$ is a distributed almost finite class. In particular, the action $\alpha$ is almost finite if $H_{\alpha,\Sigma}$ is a $D$-doubling family for some $D > 0$.

In [15], we proved (see the remark after Proposition 5.2.) that for any real number $t \geq 1$, there exists a minimal, stable action $\alpha_t$ of some finitely generated group and some constant $C_t$ such that for any orbit graph $H$, $r \geq 1$ and $x \in V(H)$,

$$\frac{1}{C_t} r^t \leq |B_r(H,x)| \leq C_t r^t.$$
So, there exists some \( D > 0 \) such that all the orbit graphs \( H \) are \( D \)-doubling (see Section 8). Hence by Proposition 5.2, the action \( \alpha_t \) is a minimal, almost finite Cantor action. Thus, by the Main Theorem of [37], we have the following corollary.

**Corollary 5.1.** For all \( \alpha \geq 1 \), the groupoid \( C^* \)-algebra of \( \alpha_t \) is a simple \( C^* \)-algebra of stable rank one.

### 5.2. Almost finiteness and convergence.

**Proposition 5.3.** Let \( \{ G_n \}_{n=1}^{\infty} \subset Gr_d \) be an almost finite resp. a strongly almost finite family and suppose that \( G_n \xrightarrow{\delta} G \). Then, \( G \) is an almost finite graph resp. a strongly almost finite graph.

**Proof.** First, we need a lemma.

**Lemma 5.1.** Let \( G \in Gr_d \) be a countable infinite graph such that for some \( \varepsilon \) and \( K \), there exists an \( (\varepsilon, K) \)-partition of \( V(G) \). Let \( H \in Gr_d \) be equivalent to \( G \). Then, there exists an \( (\varepsilon, K) \)-partition of \( V(H) \).

**Proof.** Let \( S = \{ S_i \}_{i=1}^{\infty} \) be an \( (\varepsilon, K) \)-partition of \( V(G) \). We can define a graph structure \( G_S \) on \( S \). Let \( S_i \neq S_j \) be adjacent, if there exist vertices \( x \in S_i \) and \( y \in S_j \) so that \( d_G(x,y) \leq 3K \). Observe that for any \( i \geq 1 \), the degree of \( S_i \) in the graph \( G_S \) is less than \( d^{\delta K+1} \). Indeed, let \( x \in S_i \). Then any tile \( S_j \) adjacent to \( S_i \) is contained in the ball \( B_{3K}(G,x) \). Also, \( |B_{3K}(G,x)| < d^{\delta K+1} \).

So if \( |Q| = d^{\delta K+1} \), we can label \( S \) by elements of the set \( Q \) in such a way that adjacent elements have different labels. Let us lift the labeling above to a labeling \( \varphi : V(G) \to Q \) in such a way that \( \varphi(x) \) is the label of the tile containing \( x \). Hence, we obtain an element \( (G,\varphi) \in Gr_d Q \).

**Lemma 5.2.** There exists a labeling \( \psi : V(H) \to Q \) such that for any labeled ball \( B_R(H,z,\psi) \) there exists a labeled ball \( B_R(G,y,\varphi) \) that is rooted-labeled isomorphic to \( B_R(H,z,\psi) \).

**Proof.** Fix a vertex \( w \in V(H) \). For any \( n \geq 1 \), we define \( B_n(w,\psi_n) \) to be rooted-isomorphic to some ball \( B_n(y,\varphi) \). Then we consider a convergent subsequence,

\[
\{ B_{n_k}(w,\psi_{n_k}) \}_{k=1}^{\infty} \xrightarrow{\text{reg}_d} (H',\psi'),
\]

where \( H' \) is the component of \( H \) containing \( w \). Clearly, for any \( z \in V(H') \) and \( R > 0 \), there exists a labeled ball \( B_R(G,y,\varphi) \) that is rooted-labeled isomorphic to \( B_R(H',z,\psi') \). We can finish the proof of the lemma, by defining \( \varphi \) for all components of \( H \).

Now we can finish the proof of Lemma 5.1. We can define a partition of \( V(H) \) using the labeling \( \psi : V(H) \to Q \). Let \( x \equiv_{\psi} y \) if \( \psi(x) = \psi(y) \) and \( d_H(x,y) \leq K \). It is easy to check that \( \equiv_{\psi} \) is in fact an equivalence relation. Also, by Lemma 5.2, the partition defined by \( \equiv_H \) is an \( (\varepsilon, K) \)-partition of \( V(H) \).
Now we can conclude the proof of our proposition. Since \( \{G_n\}_{n=1}^{\infty} \) form an almost finite class, the graph \( G \) that consists of disjoint copies of the graphs \( \{G_n\}_{n=1}^{\infty} \) is itself almost finite. Let \( \varphi : V(G) \rightarrow Q \) encode an \((\varepsilon,K)\)-partition of \( V(G) \) as above. Thus we have a sequence \( \{(G_n,\varphi_n)\}_{n=1}^{\infty} \subset Gr_d^Q \) and we can consider a convergent subsequence \( (G_{n_k},\varphi_{n_k}) \rightarrow (H,\varphi) \) by Proposition 2.1, where \( H \in Gr_d \). By the argument at the end of Lemma 5.1, \( V(H) \) has an \((\varepsilon,K)\)-partition. Since \( \{G_{n_k}\}_{k=1}^{\infty} \nrightarrow G \) and \( (G_{n_k},\varphi_{n_k}) \rightarrow (H,\varphi) \), the graphs \( G \) and \( H \) are equivalent. Thus, by Lemma 5.1, \( V(G) \) has an \((\varepsilon,K)\)-partition as well. Therefore, \( G \) is almost finite. The strong almost finite version can be proven in exactly the same way. \( \square \)

For qualitative convergence, we have the following proposition.

**Proposition 5.4.** Let \( \{G_n\}_{n=1}^{\infty} \subset Gr_d \) be a set of finite graphs. Suppose that \( G_n \xrightarrow{\alpha} G \), where \( \alpha : \Gamma \curvearrowright X \) is some action of a finitely generated group \( \Gamma \). Then, the family \( \{G_n\}_{n=1}^{\infty} \) is almost finite if and only if \( \alpha \) is almost finite.

**Proof.** First, suppose that \( \alpha \) is almost finite and fix \( \varepsilon > 0 \). Let \( \psi_\varepsilon : X \rightarrow Q \) be the mapping of Definition 5.1 and consider the configuration \( \mathcal{A} = \text{Conf}_{5K,\alpha}(\psi_\varepsilon) \). Since \( \mathcal{A} \) is a surviving configuration, we have \( N > 0 \) such that for any \( n \geq N \) there exists \( \varphi_n : V(G_n) \rightarrow Q \) such that

\[
\text{Conf}_{5K,\varphi_n}(\mathcal{A}) = \mathcal{A}.
\]

Therefore, the family \( \{G_n\}_{n \geq N} \) is \((\varepsilon, K_\varepsilon)\)-almost finite. Thus, \( \{G_n\}_{n=1}^{\infty} \) is an \((\varepsilon, L_\varepsilon)\)-almost finite family, where

\[
L_\varepsilon = \max(K_\varepsilon, \max_{1 \leq i \leq N} |V(G_i)|).
\]

Now, suppose that \( \{G_n\}_{n=1}^{\infty} \) is an almost finite family. Let \( \varepsilon > 0 \). Let \( \{(G_n,\varphi_n)\}_{n=1}^{\infty} \subset Gr_d^Q \) be as in the proof of Proposition 5.3. For \( n \geq 1 \), let \( \mathcal{A}_n = \text{Conf}_{5K,\varphi_n}(\mathcal{A}) \). Then, there exists \( \mathcal{A} \) such that \( \mathcal{A} = \mathcal{A}_n \) for infinitely many \( n 's \). That is, \( \mathcal{A} \) is a surviving configuration. Hence, there exists \( \psi : X \rightarrow Q \) such that \( \mathcal{A} = \text{Conf}_{5K,\alpha}(\psi) \). So, by Definition 5.1, the action \( \alpha \) is almost finite. \( \square \)

### 5.3. The fractal construction

Now, as a preparation for Theorem 6, we construct an almost finite, connected infinite graph \( G \) of bounded vertex degrees, which is minimal in the suitable sense and contains an expander sequence of finite induced subgraphs. A somewhat similar construction has been used in [15]. Let \( K \) be a finite graph. We call a subset \( A_i \subset V(K) \) an \( i \)-subset if

- for any \( x \neq y \in A_i \), \( d(x,y) > 3^i \),
- \( \bigcup_{p \in A_i} B_{10^i}(K,p) = V(K) \).

The following lemma is easy to prove.
Lemma 5.3. For any finite 3-regular graph $L$ such that $10^k \leq \text{diam}(L)$, we have disjoint subsets $A_1, A_2, \ldots, A_k \subset V(L)$ such that $A_i$ is an $i$-subset and $\bigcup_{i=1}^{k} A_i \neq V(L)$.

Now let $\{K_j\}_{j=1}^{\infty}$ be an expander sequence of finite 3-regular graphs such that

- for any $j \geq 1$, $\text{diam}(K_j) \geq 10^j$ and
- $|V(K_j)| \geq j3^{(10^j+1)}$.

Using Lemma 5.3, we pick disjoint subsets $A^j_1, A^j_2, \ldots, A^j_l \subset V(K_j)$ and a vertex $p_j$ such that $A^j_i$ is an $i$-subset and $p_j \notin \bigcup_{i=1}^{l} A^j_i$. We call the vertex $p_j$ the connecting vertex of $K_j$.

Step 1. Let $H_1 = K_1$. The graph $H_2$ is constructed in the following way. First, we consider our graph $K_2$. For each $q \in A^2_1 \subset V(K_2)$, we choose a copy $H^2_1$ of the graph $H_1$. Also, we identify one of the $|A^2_1|$ copies with the graph $H_1$. Then, we connect the graph $H^2_1$ with the vertex $q$ by adding an edge $e$ between $q$ and the connecting vertex of $H^2_1$. The resulting graph will be denoted by $H_2$. So, $|V(H_2)| = |V(K_2)| + |A^2_1||V(H_1)|$. From now on, $p_2 \in V(K_2)$ will also be called the connecting vertex of the graph $H_2$.

Step 2. Consider the graph $K_3$. Again, for each vertex $r \in A^3_1$, we pick a copy $H^3_1$ of the graph $H_1$. Then, we connect $H^3_1$ with the vertex $r$ as above. Also, for each $s \in A^3_2$, we pick a copy $H^3_2$ of the graph $H_2$ and connect $H^3_2$ to the vertex $s$ via the connecting vertex. Again, we identify one of the copies $H^3_2$ with the graph $H_2$. So, the resulting graph $H_3$ contains $H_2$ as an induced subgraph. From now on, $p_3 \in K_3$ will also be called the connecting vertex of $H_3$.

Step $n$. Suppose that the graphs $H_1 \subset H_2 \subset \cdots \subset H_n$ have already been constructed and each graph $H_i$ contains a connecting point of degree 3. Now, for any $1 \leq i \leq n$ and for each $w \in A_i^{n+1}$, we pick a copy $H^w_i$ of the graph $H_i$. Then we connect $H^w_i$ to the vertex $w$ by an edge between $w$ and the connecting vertex of $H^w_i$. Finally, we identify for some $z \in A_n^{n+1}$ the graph $H^z_n$ with the graph $H_n$. Finally, the connecting vertex of $K_{n+1}$ will be called the connecting vertex of $H_{n+1}$.

Now, let $H = \bigcup_{i=1}^{\infty} H_i$. Observe that $H$ is a connected, infinite graph with vertex degree bound 4.

For each vertex $x \in V(H)$, there is a unique integer $j_1(x)$ such that $x$ is a vertex of a certain copy $K_{j_1(x)}(x)$ of the graph $K_{j_1(x)}$. Then, we have an integer $j_2(x)$ and a copy $K_{j_2(x)}(x)$ of the graph $K_{j_2(x)}$ such that the subgraphs $K_{j_1(x)}(x)$ and $K_{j_2(x)}(x)$ are connected by an edge in the above process of building the graph $H$. Inductively, we have an infinite sequence of integers $j_1(x) < j_2(x) < \ldots$ and disjoint induced subgraphs $K_{j_1(x)}(x), K_{j_2(x)}(x), \ldots$ so that the graphs $K_{j_n(x)}(x)$ and $K_{j_{n+1}(x)}(x)$ are connected to each other by an edge. We call the integer $j_1(x)$ the type of $x$. 
Lemma 5.4. For all $x \in V(H)$ and $l \geq 1$, there exists $y \in V(H)$ such that

- The type of $y$ equals to $l$,
- $d_H(x, y) \leq \text{diam}(H_l) + 10^l + 2$.

Proof. Suppose that $j_1(x) = m > l$. Then, $x \in K_m(x)$ and there exists an element $z$ of the distinguished $l$-subset of the copy $K_m(x)$ such that $d_H(x, z) \leq 10^l$. By our construction, $z$ is adjacent of a vertex $y$ of type $l$. Now, assume that $j_1(x) \leq l$. Let $k \geq 1$ be the integer such that $j_k(x) \leq l$, $j_{k+1}(x) > l$. By our construction, there exists a vertex $w$ of type $j_{k+1}(x)$ such that $d_H(x, w) \leq \text{diam}(H_l) + 1$. Hence, by our previous observation, the lemma follows. \hfill \Box

Proposition 5.5. The graph $H$ is almost finite.

Proof. Let $l \geq 1$ and let $S_l$ be the set of all vertices $x \in V(H)$ such that $j_1(x) > l$. Then, $S_l$ is the disjoint union of subsets $P$, where the induced subgraph $\overline{P}$ on $P$ is isomorphic to $K_i$ for some integer $i > l$. For each set $P$, let $\alpha_j^P = \{\alpha_1^P, \alpha_2^P, \ldots, \alpha_{k_l^P}^P\}$ be the distinguished $l$-subset in $\overline{P}$. So, we can partition $P$ into subsets $\bigcup_{j=1}^{k_l^P} U_{P,j}$, where for any $1 \leq j \leq k_l^P$,

$$\alpha_j^P \in U_{P,j} \subset B_{10^l}(\overline{P}, \alpha_j^P).$$

Now we define the subset $V_{P,j} \subset V(H)$ in the following way. Let $x \in V_{P,j}$ if either $x \in U_{P,j}$ or $x$ is a vertex of a copy of $H_i$, $1 \leq i \leq l$ attached to a vertex in the set $U_{P,j} \cap A_i^P$ in the construction. Observe that $\bigcup_{P} \bigcup_{1 \leq j \leq k_l^P} V_{P,j}$ is a partition of $V(H)$. Also,

$$3^{10^l+1} \leq |K_i| \leq |H_l| \leq |V_{P,j}|.$$  \hfill (7)

and

$$|\partial(V_{P,j})| \leq |U_{P,j}| \leq 3^{10^l+1}.$$  \hfill (8)

Therefore, for all $P$ and integer $1 \leq j \leq k_l^P$ we have that

$$\frac{|\partial(V_{P,j})|}{|V_{P,j}|} \leq \frac{1}{l}. $$ \hfill (9)

Thus, by (7) and (9), we can immediately see that the graph $H$ is almost finite. \hfill \Box

5.4. A non-amenable almost finite groupoid. Now we prove the main result of this section. It answers a query of Suzuki (Remark 3.7 [37]).

Theorem 6. There exists a stable minimal action $\alpha : \Gamma \curvearrowright C$ of finitely generated group such that the associated minimal geometric groupoid $G_\alpha$ is almost finite but non-amenable.

Proof. Consider the connected graph $H$ constructed in the previous subsection. Let $Q$ be a finite set and $\varphi : E(H) \to Q$ be a labeling such that adjacent edges have different labels. Again, we consider the $|Q|$-fold free power $F_Q$ of the cyclic group of two elements with symmetric generator set $Q$. The
labeling induces a transitive action of the group \(F_Q\) on the set \(V(H)\) such that \(H\) is the underlying graph of the associated Schreier graph \(S_H\). For each \(n \geq 1\), consider an \((1/n, K_n)\)-almost finite partition of \(H\). Pick a large enough integer \(k_n\) and a labeling \(\tau_n : V(H) \to \{0, 1\}^{k_n}\) that encodes the partition. That is, if \(x\) and \(y\) belong to the same part then \(\tau_n(x) = \tau_n(y)\), and if \(\tau_n(x) = \tau_n(y)\) for some \(x, y\) which are not belonging to the same part, then \(d_H(x, y) > 3K_n\). Also, for each \(n \geq 1\), pick a large enough integer \(l_n\) and a map \(\kappa_n : V(H) \to \{0, 1\}^{l_n}\) such that if \(0 < d_H(x, y) \leq n\), then \(\kappa_n(x) \neq \kappa_n(y)\). Finally, for \(n \geq 1\), let \(\mu_{2n} = \kappa_n\) and \(\mu_{2n-1} = \tau_n\). Consider \(h \in H\) and the \(C\)-labeling \(\mu = \prod_{n=1}^\infty \mu_n : V(H) \to \{0, 1\}^\mathbb{N}\) and the element \(y = (H, h, \mu) \in \Gamma_\Sigma G\), where \(\Gamma = F_Q\) and \(\Sigma = Q\). Let \(Z\) be a closed, minimal, invariant set in the orbit closure \(\overline{O(y)}\). Our theorem immediately follows from the proposition below.

**Proposition 5.6.** The restricted action \(\alpha : \Gamma \curvearrowright Z\) is stable, almost finite and the associated geometric groupoid \(G_\alpha\) is minimal.

**Proof.** By the choice of the mappings \(\kappa_n\), there exist continuous maps \(\rho_n : Z \to \{0, 1\}^{l_n}\) such that if \(x\) and \(y\) are vertices of the same graph \(G\) in \(Z\) and \(0 < d_G(x, y) \leq n\), then \(\rho_n(x) \neq \rho_n(y)\). Therefore, the action of \(F_Q\) on \(Z\) is stable. Also, by the choice of the mappings \(\tau_n\), there exist continuous maps \(\lambda_n : Z \to \{0, 1\}^{l_n}\) that define \((1/n, K_n)\)-almost finite partitions of the elements of \(Z\) as required in Definition 5.1. We only need to show that the geometric groupoid \(G_\alpha\) is non-amenable. By the definition of an amenable groupoid [4], if \(G_\alpha\) is amenable then all the orbit graphs of \(G_\alpha\) possess Yu’s Property A. It is well-known that if an infinite graph \(G\) contains a growing sequence of expanders as induced subgraphs, then \(G\) does not have Property A. Our graph \(H\) was constructed in such a way that it contains a sequence of expanders as induced subgraphs. By Lemma 5.4, for any \(n \geq 1\), there exists \(t_n > 0\) such that if \(x \in V(H)\), then the ball \(B_{t_n}(H, x)\) contains a copy of \(K_n\) as induced subgraph. Hence, if \(G\) is in the orbit closure of \(H\) and \(y \in V(G)\), then the ball \(B_{t_n}(G, y)\) contains an induced copy of \(K_n\). Therefore, the orbit graphs in \(G_\alpha\) do not have Property A. Hence our proposition (and so the theorem) follows.

### 5.5. Fractionally almost finite graphs.

The notion of fractional almost finiteness (see Definition 1.5) is closely related to the notion of fractional hyperfiniteness introduced by Lovász [32]. Observe that if \(G \in Gr_d\) is a fractionally almost finite graph and \(H \subset G\) is a subgraph, then \(H\) is fractionally almost finite as well. On the other hand, almost finiteness of \(G\) does not necessarily imply the almost finiteness of \(H\), since the Cayley graph of some amenable groups contain copies of the 3-regular tree as a subgraph. Lovász proved that fractional hyperfiniteness implies hyperfinitess for graphing. Nevertheless, we have the following proposition.

**Proposition 5.7.** The \(k\)-regular tree \(T_k\) is fractionally almost finite (and, of course, \(T_k\) is not almost finite).
Proof. Fix an infinite ray \{p_1, p_2, \ldots\} in \( T_k \). Let \( l \geq 1 \) be an integer. Now, for each integer \( 1 \leq i \leq l \) we construct a partition of \( V(T_k) \). For \( x \in V(T_k) \), let \( P(x) \) be the shortest path from \( x \) to \( p_i \). Let \( x \equiv_{l,i} y \) if there exists some \( z \in V(T_k) \) such that

- both \( P(x) \) and \( P(y) \) contains \( z \),
- \( d_{T_k}(x, z) < l \), \( d_{T_k}(y, z) < l \) and
- \( d_{T_k}(z, p_i) \) is divisible by \( l \).

Then,

1. \( \equiv_{l,i} \) is an equivalence relation.
2. The equivalence classes have bounded diameter.
3. The vertex \( x \) is on the boundary of its class if and only if

\[ d_{T_k}(x, p_i) \equiv 0 \text{ or } l - 1 \mod l. \]  

Observe that for any \( x \in V(G) \), there exists at most 2 elements \( i \) of the set \{1, 2, \ldots, l\} such that (10) holds. Hence, if \( l > \frac{2}{\varepsilon} \), then (1) is satisfied. \( \square \)

Remark 9. Using Proposition 2.10 of [9], it is not hard to see that all fractionally almost finite graphs have Property A. It would be interesting to construct a Property A graph \( G \) that is not fractionally almost finite.

6. THE SPECTRA OF GRAPHS

The main goal of this section is to prove a spectral convergence result for strongly almost finite graph classes.

6.1. Uncountably many isospectral connected graphs. Let \( G \in Gr_d \) be an infinite graph and \( \mathcal{L}_G : l^2(V(G)) \to l^2(V(G)) \) be the Laplacian operator on \( G \). That is, \( \mathcal{L}_G(f)(x) = \deg(x)f(x) - \sum_{x \sim y} f(y) \).

It is well-known that \( \mathcal{L} \) is a positive, self-adjoint operator and \( \text{Spec}(\mathcal{L}_G) \subset [0, 2d] \).

Proposition 6.1. If \( G \) and \( H \) are equivalent (see Section 2), then \( \text{Spec}(\mathcal{L}_G) = \text{Spec}(\mathcal{L}_H) \).

Proof. First, we need a lemma.

Lemma 6.1. Let \( P \) be real polynomial, then \( \|P(\mathcal{L}_G)\| = \|P(\mathcal{L}_H)\| \).

Proof. Fix some \( \varepsilon > 0 \). Let \( f \in l^2(V(G)) \) such that \( \|f\| = 1 \) and \( \|P(\mathcal{L}_G)(f)\| \geq (1 - \varepsilon)\|P(\mathcal{L}_G)\| \). We can assume that \( f \) is supported on a ball \( B_s(G, x) \) for some \( s > 0 \) and \( x \in V(G) \). Let \( t \) be the degree of \( P \). Then, \( P(\mathcal{L}_G)(f) \) is supported in the ball \( B_{s+t}(G, x) \). Since \( G \) and \( H \) are equivalent, there exists \( y \in V(H) \) such that the ball \( B_{s+t}(G, x) \) is rooted-isomorphic to the ball \( B_{s+t}(H, y) \) under the rooted-isomorphism \( j \). Then, \( \|j_*(f)\| = 1 \) and \( \|P(\mathcal{L}_G)(f)\| = \|P(\mathcal{L}_H)(j_*(f))\| = \|P(\mathcal{L}_H)(f)\| \).
\[ \|P(L_H)(j_*(f))\|, \text{ where } j_*(f)(z) = f(j^{-1}(z)). \] Therefore, \( \|P(L_H)\| \geq (1 - \epsilon)\|P(L_G)\| \) holds for any \( \epsilon > 0 \). Consequently, \( \|P(L_H)\| \geq \|P(L_G)\| \). Similarly, \( \|P(L_G)\| \geq \|P(L_H)\| \), thus our lemma follows. \( \square \)

By functional calculus, we have that
\[ \|f(L_G)\| = \|f(L_H)\| \]
holds for any real continuous function. Observe that \( \lambda \in \text{Spec}(L_G) \) if and only if for any \( n \geq 1 \) \( \|f_n^\lambda(L_G)\| \neq 0 \), where \( f_n^\lambda \) is a piecewise linear, continuous, non-negative function such that
- \( f_n^\lambda(x) = 1 \) if \( \lambda - \frac{1}{n} \leq x \leq \lambda + \frac{1}{n} \).
- \( f_n^\lambda(x) = 0 \) if \( x \geq \lambda + \frac{2}{n} \) or \( x \leq \lambda - \frac{2}{n} \).

Therefore, by (11) our proposition follows. \( \square \)

It is well-known that many isospectral finite graphs exist. It is less-known (but certainly known for experts) that many isospectral infinite connected graphs exist, so the following proposition might be interesting on its own.

**Proposition 6.2.** For any \( d > 3 \), there exist uncountably many pairwise non-isomorphic trees in \( Gr_d \) which possess the same spectra.

**Proof.** Let \( K_d \) be the set of all finite trees in \( Gr_d \). Let \( T \) be an infinite tree of vertex degree bound \( d - 1 \). We construct a tree \( \hat{T} \) in the following way. Let \( \{ t_i \}_{i=1}^\infty \) be an enumeration of the vertices of \( T \) and \( \{ T_i \}_{i=1}^\infty \) be an enumeration of the elements of \( K_d \). For each \( i \geq 1 \) let us connect the tree \( T_i \) with \( T \) by adding an edges between \( t_i \) and a vertex \( p_i \in T_i \) such that \( \text{deg}(p) < d \).

**Lemma 6.2.** The set of connected, finite induced subgraphs of \( \hat{T} \) (up to isomorphisms) is exactly \( K_d \).

**Proof.** By our construction, if \( G \in K_d \), then \( G \) is an induced subgraph of \( \hat{T} \). Conversely, let \( H \) be a connected, finite induced subgraph of \( \hat{T} \). Clearly, \( H \) is a tree of vertex degree bound \( d \).

By our lemma, for any \( T, S \) with vertex degree bound \( d - 1 \), we have that \( \hat{T} \) is equivalent to \( \hat{S} \). It is easy to check that if \( T \) and \( S \) are infinite non-isomorphic trees without leaves, then \( \hat{T} \) and \( \hat{S} \) are not isomorphic. Since there are uncountably many such trees, our proposition follows. \( \square \)

**Remark 10.** Note that the end space of the tree \( T \) and \( \hat{T} \) are the same, so one can actually has uncountably many trees with the same spectrum and pairwise different end spaces.

6.2. **Spectral convergence for strongly almost finite graphs.** The main goal of this section is to prove the following theorem.
Theorem 7. Let \( \{G_n\}_{n=1}^{\infty} \subset Gr_d \) be a strongly almost finite family of graphs such that \( G_n \xrightarrow{\nu} G \). Then \( \text{Spec} G_n \rightarrow \text{Spec}(G) \) in the Hausdorff topology.

Proof. Let \( G \) be a strongly almost finite infinite graph. Then, it is not hard to see that for any \( \delta > 0 \) and integer \( l > 1 \) there exist constants \( M_{\delta,l} > 0, L_{\delta,l} > 0, \) and finite partitions \( \{H^i_1, H^i_2, \ldots\}_{i=1}^{M_{\delta,l}} \) such that

- the diameters of the classes \( \{H^i_j\} \) are bounded by \( L_{\delta,l} \),
- for each class \( H^i_j, \frac{|\delta_l(H^i_j)|}{|H^i_j|} < \delta, \)
- for each \( x \in V(G) \),
  \[
  \{i \mid x \in \delta_l(H^i_j) \text{ for some } j \geq 1\} \leq \delta \cdot M_{\delta,l}.
  \]

We call such a partition family \( \mathcal{E} \) a \( (\delta, l, M_{\delta,l}, L_{\delta,l}) \)-almost finite partition family.

Now, let \( f : V(G) \rightarrow \mathbb{R} \) be a function such that \( \sum_{x \in V(G)} f^2(x) = 1 \). Let \( \mathcal{E} \) be a \( (\delta, l, M_{\delta,l}, L_{\delta,l}) \)-almost finite partition family and for \( 1 \leq i \leq M_{\delta,l} \) let \( f_i : V(G) \rightarrow \mathbb{R} \) be defined by setting

- \( f_i(x) = f(x) \) if \( x \in H^i_j \setminus \delta_l(H^i_j) \) for some \( j \geq 1 \).
- \( f_i(x) = 0 \), otherwise.

Lemma 6.3. We have the following inequality.

\[
\{i \mid \|f_i\|^2 < (1 - \sqrt{\delta})\} < \sqrt{\delta} M_{\delta,l}.
\]

Proof. By our condition,

\[
\sum_{x \in V(G)} \sum_{1 \leq i \leq M_{\delta,l}} f^2_i(x) \geq \sum_{x \in V(G)} (1 - \delta) M_{\delta,l} f^2(x).
\]

That is,

\[
\sum_{1 \leq i \leq M_{\delta,l}} \|f_i\|^2 \geq (1 - \delta) M_{\delta,l}.
\]

Let \( A \) be the set of integers \( 1 \leq i \leq M_{\delta,l} \) for which \( \|f_i\|^2 < (1 - \sqrt{\delta}) \). Then,

\[
|A|(1 - \sqrt{\delta}) + (M_{\delta,l} - |A|) \geq (1 - \delta) M_{\delta,l}.
\]

That is,

\[
\frac{|A|}{M_{\delta,l}} (1 - \sqrt{\delta}) + (1 - \frac{|A|}{M_{\delta,l}}) \geq (1 - \delta),
\]

Hence, \( \frac{|A|}{M_{\delta,l}} < \sqrt{\delta} \). \( \square \)

Now, let \( \mathcal{E} \) as above and \( P \) be a polynomial of degree less than \( l \). Define

\[
\|P(\Delta)\|_{\mathcal{E}} := \sup_g \frac{\|P(\Delta)g\|}{\|g\|},
\]
where the supremum is taken for all nonzero functions $g$ which are supported on $H_j^i \setminus \partial(H_j^i)$ for some class $H_j^i$. Clearly, $\|P(\Delta)\|_{\mathcal{E}} \leq \|P(\Delta)\|$. Also, let

$$\|P(\Delta)\|_{\mathcal{E}} := \sup_{g} \frac{\|P(\Delta)g\|}{\|g\|},$$

where the supremum is taken for all $L^2$-functions $g$ such that there exists $i$ for which $g(x) = 0$ if $x \in \partial(H_j^i)$ for all classes $H_j^i$. By the degree condition on $P$, $\|P(\Delta)\|_{\mathcal{E}} = \|P(\Delta)\|_{\mathcal{E}}$.

**Lemma 6.4.** Suppose that $\max_{0 \leq t \leq 2d} |P(t)| \leq 2$. Then,

$$\|P(\Delta)\| - \|P(\Delta)\|_{\mathcal{E}} < 3 \left(1 - \sqrt{1 - \sqrt{\delta}}\right).$$

**Proof.** Let $f : V(G) \to \mathbb{R}$ such that $\|f\| = 1$ and $\|P(\Delta)f\| \geq \|P(\Delta)\| - (1 - \sqrt{1 - \sqrt{\delta}})$. Let $f_i$ be as above such that $\|f_i\| > (1 - \sqrt{\delta})$. Then,

$$\|P(\Delta)f_i\| \geq \|P(\Delta)f\| - 2(1 - \|f_i\|) \geq \|P(\Delta)\| - 3 \left(1 - \sqrt{1 - \sqrt{\delta}}\right).$$

Since $\|f_i\| \leq 1$, we have that

$$\|P(\Delta)\|_{\mathcal{E}} \geq \|P(\Delta)\| - 3 \left(1 - \sqrt{1 - \sqrt{\delta}}\right).$$

Thus, our lemma follows. \qed

For any $m \geq 1$, we have polynomials, $\{P_i^m\}_{i=1}^{m2d}$ such that

- for any $0 \leq t \leq 2d$, $0 < P_i^m(t) \leq 1 + \frac{1}{m}$,
- $0 < P_i^m(t) < \frac{1}{m}$ if $t \leq \frac{1}{m2d} - \frac{1}{m}$ or $t \geq \frac{1}{m2d} + \frac{1}{m}$,
- $1 - \frac{1}{m} < P_i^m(t) < 1 + \frac{1}{m}$ if $\frac{1}{m2d} - \frac{1}{2m} < t < \frac{1}{m2d} + \frac{1}{2m}$.

Now, let $G_n \xrightarrow{n} G$, where $\{G_n\}_{n=1}^{\infty}$ is a strongly almost finite class. Thus, by Proposition 5.3 $\{G_n\}_{n=1}^{\infty}$ is a strongly almost finite class as well. By the previous lemma, for any $\delta > 0$ and $l > 0$, we have $M_{\delta,l} > 0, L_{\delta,l} > 0$ and $(\delta, l, M_{\delta,l}, L_{\delta,l})$-almost finite partition families $\mathcal{E}_n$ on $G_n$ and a $(\delta, l, M_{\delta,l}, L_{\delta,l})$-almost finite partition family $\mathcal{E}$ on $G$ such that for all $n \geq 1$ and $1 \leq i \leq m2d$,

$$\|P_i^m(\Delta_{G_n})\| - \|P(\Delta_{G_n})\|_{\mathcal{E}_n} \leq 3 \left(1 - \sqrt{1 - \sqrt{\delta}}\right),$$

$$\|P_i^m(\Delta_{G})\| - \|P(\Delta_{G})\|_{\mathcal{E}} \leq 3 \left(1 - \sqrt{1 - \sqrt{\delta}}\right).$$

Since $G_n \xrightarrow{n} G$, for large enough $n$ the set of all induced subgraphs of diameter at most $L_{\delta,l}$ in $G_n$ equals to the set of all induced subgraphs of diameter at most $L_{\delta,l}$ in $G$. Therefore, for large enough $n$, we have that for all $1 \leq i \leq m2d$,

$$\|P_i^m(\Delta_{G_n})\| - \|P_i^m(\Delta_{G})\| \leq 6 \left(1 - \sqrt{1 - \sqrt{\delta}}\right).$$

(12)
Suppose that \( \lambda \in \text{Spec}(\Delta_G) \). Then by (12), for any \( \zeta > 0 \) there exists \( N_\zeta > 0 \) such that if \( n > N_\zeta \), then we have some \( \lambda_n \in \text{Spec}(G_n) \), \( |\lambda_n - \lambda| < \zeta \). Also, there exists \( M_\zeta > 0 \) such that if \( n > M_\zeta \) and \( \mu_n \in \text{Spec}(G_n) \), then there exists \( \mu \in \text{Spec}(G) \) such that \( |\mu_n - \mu| < \zeta \). Hence, our theorem follows. \( \square \)

Remark 11. Let \( G_n \xrightarrow{\mu} G \), where \( \{G_n\}_{n=1}^\infty \) is a large girth sequence of 3-regular graphs and \( G \) is the 3-regular tree. Then for all \( n \geq 1 \), \( 0 \in \text{Spec} G_n \) and \( 0 \notin \text{Spec} G \).

7. Constant-time distributed algorithms

The theory of distributed graph algorithms is a vast subject developed in the last thirty years (see the monograph [33]). Using the ideas of our paper we introduce the notion of constant-time distributed algorithms, the qualitative analogue of constant-time randomized local algorithms.

7.1. Algorithms and oracles. First, let us give a formal definition for an \( r \)-round distributed algorithm on a graph class \( \mathcal{G} \subseteq \mathcal{G}_d \). For a finite set \( Q \) and a graph \( G \), we will denote by \( L_Q(G) \) the set of all functions \( \varphi : V(G) \rightarrow Q \).

An \( r \)-round distributed algorithm can be represented by an operator

\[
O : L_{Q_1}(G) \rightarrow L_{Q_2}(G)
\]

for some finite sets \( Q_1, Q_2 \). However, the value \( O(\varphi)(x) \) depends only on the values of \( \varphi \) on the ball \( B_r(G, x) \). Therefore, the operator \( O \) can be described by a map \( \Theta : U_{d}^{r_1, Q_1} \rightarrow Q_2 \), where

\[
O(\varphi)(x) = \Theta(B_1(G, x, \varphi)).
\]

We call \( \Theta \) an oracle function and we use the notation \( O_{\Theta} \) for the associated algorithm operator. The following examples formalize how to combine simple algorithms into more complicated ones.

Example 2. Let \( O_{\Theta_1} : U_{d}^{r_1, Q_1} \rightarrow Q_2 \) and \( O_{\Theta_2} : U_{d}^{r_2, Q_2} \rightarrow Q_3 \) be oracles. Then, there exists a unique composition oracle \( O_{\Theta_3} : U_{d}^{r_3, Q_1} \rightarrow Q_3 \) such that for any \( \varphi \in L_{Q_1}(G) \),

\[
O_{\Theta_3}(\varphi) = O_{\Theta_2}(O_{\Theta_1}(\varphi)).
\]

Example 3. Let \( \{\Theta_i : U_{d}^{r_i, M_i} \rightarrow Q_i\}_{i=1}^k \) be oracles. Also, for the functions \( \{\varphi_i : L_{M_i}(G)\}_{i=1}^k \) let \( \oplus_{i=1}^k \varphi_i \in L_{\oplus_{i=1}^k M_i}(G) \) be the function such that

\[
(\oplus_{i=1}^k \varphi_i)(x) = \oplus_{i=1}^k (\varphi_i(x)).
\]

Let \( \Theta : U_{d}^{r, \oplus_{i=1}^k M_i} \rightarrow N \) be an oracle and \( r = \max_{1 \leq i \leq k} r_i \). Then, there exists a unique oracle \( \Theta' : U_{d}^{r+k, \oplus_{i=1}^k M_i} \rightarrow N \), such that

\[
\Theta'(\oplus_{i=1}^k \varphi_i) = \Theta(\oplus_{i=1}^k \Theta_i(\varphi_i)).
\]

Now, let \( k \geq 1 \) be an integer and \( Q \) be a finite set such that \( |Q| \geq |V(B)| \) (we say that \( Q \) is \((r, d)\text{-large}) for all \( B \in U_{d}^k \). Let \( L_{Q}^{k}(G) \) denote the set of all \( k \)-separating functions \( \varphi : V(G) \rightarrow Q \). That is, functions \( \varphi \) for which
φ(x) ≠ φ(y) if 0 < d_G(x, y) ≤ k. As in the Introduction, a constant-time distributed algorithm (CTDA) on a graph class $G \subset Gr_d$, takes a $k$-separating $Q_1$-labeling on $G \in \mathcal{G}$ and produces some $Q_2$-labeling. So, the algorithm operator is given by an oracle,

$$\Theta : U^r_{d,k,Q_1} \rightarrow Q_2,$$

where $U^r_{d,k,Q_1}$ denote the set of all $k$-separating $Q_1$-labelings of the balls in $U^r_d$.

Before getting further, let us consider a simple, but important example.

**Proposition 7.1.** We have a CTDA, which produces a maximal independent subset for any graph $G \in Gr_d$.

**Proof.** First, let us make the statement of our proposition completely precise. Let $M \geq d + 1$. Then, there exists an oracle $\Theta : U^{|M|,1,1}_{d} \rightarrow \{a, b\}$ such that for any graph $G \in Gr_d$ and $\varphi \in L^{|1|}_M(G)$, the set $(O_\Theta(\varphi))^{-1}(a)$ is a maximal independent subset in $G$. We will give a combinatorial description of $\Theta$ as the composition of $|M|$ 1-round oracles. Let $\varphi \in L^{|1|}_M(G)$. In the first round, we relabel the vertices $x \in V(G)$ for which $\varphi(x) = 1$ by $a$ to obtain the labeling $\varphi_1 : V(G) \rightarrow M \cup a$. Note that $\varphi_1(y) = \varphi(y)$ if $\varphi(y) ≠ 1$. In the second round, we relabel the vertices $x$ for which $\varphi_1(x) = 2$. Let $\varphi_2(x) = a$ if $\varphi_1(y) ≠ a$ for all vertices $y$ adjacent to $x$. Otherwise, let $\varphi_2(x) = b$. So, we obtain the labeling $\varphi_2 : V(G) \rightarrow M \cup \{a, b\}$. Inductively, in the $k$-th round we start with a labeling $\varphi_{k-1} : V(G) \rightarrow M \cup \{a, b\}$. Then, we relabel the vertices $x$ for which $\varphi_{k-1}(x) = k$. Again, let $\varphi_k(x) = a$ if $\varphi_{k-1}(y) ≠ a$ for all vertices $y$ adjacent to $x$. Otherwise, let $\varphi_k(x) = b$. So, after the $|M|$-th round we obtain the labeling $\varphi_{|M|}$ which maps all vertices of $G$ into the set $\{a, b\}$ and $\{(\varphi_{|M|})^{-1}(a)\}$ is a maximal independent set. Indeed, if $\varphi_{|M|}(x) = b$, then $\varphi_{|M|}(y) = a$ for at least one vertex $y$ adjacent to $x$. □

Certain algorithms, e.g. the one described in Example 3, requires elements of the set $L^{|r|}_M(G)$ as input. However, the input functions given are from the much larger set $L^{|r|}_N(G)$, where $|N| > |M|$. The following lemma shows that we can convert $r$-separating $N$-valued functions into $r$-separating $M$-valued functions using a simple oracle.

**Lemma 7.1.** Let $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$ be finite sets such that $n > m$ and $M$ is $(r, d)$-large. Then, we have an oracle

$$\Theta : U^{|r|}_{d,n,m} \rightarrow M,$$

such that $O_\Theta$ maps $L^{|r|}_N(G)$ into $L^{|r|}_M(G)$.

**Proof.** Let $\varphi \in L^{|r|}_N(G)$. In the first round we relabel the elements $x \in V(G)$, for which $\varphi(x) = m+1$. Let $\varphi_1(x) = i$, where $1 \leq i \leq m$ is the smallest integer such that $\varphi(y) ≠ i$, provided that $0 \leq d_G(x, y) ≤ r$. By inductive relabelings, we construct the sequence $\varphi_1, \varphi_2, \ldots, \varphi_{n-m}$ in $n-m$ rounds. Then, $\varphi_{n-m}$ will be an $r$-separating $M$-labeling. □
7.2. Distributed graph partitioning and almost finiteness. We start with the notion of a distributed graph partitioning oracle, the qualitative analogue of the randomized partitioning oracles introduced by Hassidim et. al. [24]. Let $\mathcal{G} \subseteq Gr_d$ be a class of graphs, $1 \leq s \leq n$ and $\Theta : U_{d}^{n,s,M} \rightarrow Q$ be an oracle, where $M$ is some $(s,d)$-large set. Let $\varepsilon > 0$ be a real number and $K_{\varepsilon} > 0$ be an integer.

Definition 7.1. $\Theta$ is a **distributed $(\varepsilon, K_{\varepsilon})$-partitioning oracle** for the class $\mathcal{G}$, if for all $G \in \mathcal{G}$ and $\varphi \in L_{M}^{(s)}(G)$ the following two conditions are satisfied.

1. For any $x, y \in V(G)$, if $O_{\Theta}(\varphi)(x) = O_{\Theta}(\varphi)(y)$, we have that either $d_{G}(x, y) \leq K_{\varepsilon}$ or $d_{G}(x, y) \geq 3K_{\varepsilon}$.
2. For any $x \in V(G)$, $i_{G}(H_{x}) \leq \varepsilon$, where

$$H_{x} = \{z \in V(G) \mid d_{G}(x, z) \leq K_{\varepsilon} \text{ and } O_{\Theta}(\varphi)(x) = O_{\Theta}(\varphi)(z)\}.$$ 

Let $x \equiv_{(\Theta, \varphi)} y$ if $y \in H_{x}$. It is easy to see that the relation $\equiv_{(\Theta, \varphi)}$ is, in fact, an equivalence relation. So, $\Theta$ computes an $(\varepsilon, K_{\varepsilon})$-partition, indeed.

Definition 7.2. A family $\mathcal{G} \subseteq Gr_d$ is distributed almost finite if for any $\varepsilon > 0$, there exist integers $1 \leq s_{\varepsilon} \leq n_{\varepsilon}$, and $K_{\varepsilon} \geq 1$, finite sets $M_{\varepsilon}, Q_{\varepsilon}$ such that $M_{\varepsilon}$ is $(s_{\varepsilon},d)$-large and we also have an oracle function $\Theta_{\varepsilon} : U_{d}^{n_{\varepsilon},s_{\varepsilon},M_{\varepsilon}} \rightarrow Q_{\varepsilon}$ such that $\Theta_{\varepsilon}$ is a $(\varepsilon, K_{\varepsilon})$-partitioning oracle for $\mathcal{G}$.

Let $\mathcal{G} \subseteq Gr_d$ be a class of finite graphs, $1 \leq s \leq n$ be integers and $M$ be an $(s,d)$-large set. Again, let $Q$ be another finite set and $*$ be an extra symbol. Let $\Theta : U_{d}^{n,s,M} \rightarrow \{Q, *\}$ be an oracle function. We say that $\Theta$ is an **$(\varepsilon, K_{\varepsilon})$-hyperfinite partitioning oracle** for the class $\mathcal{G}$ if the following three conditions are satisfied.

1. For any $x, y \in V(G)$ and $\varphi \in L_{M}^{(s)}(G)$ if $* \neq O_{\Theta}(\varphi)(x) = O_{\Theta}(\varphi)(y)$, we have that either $d_{G}(x, y) \leq K_{\varepsilon}$ or $d_{G}(x, y) \geq 3K_{\varepsilon}$.
2. For any $x \in V(G)$, such that $O_{\Theta}(\varphi)(x) \neq *, i_{G}(H_{x}) \leq \varepsilon$, where

$$H_{x} = \{z \in V(G) \mid d_{G}(x, z) \leq K_{\varepsilon} \text{ and } O_{\Theta}(\varphi)(x) = O_{\Theta}(\varphi)(z)\}.$$ 

3. $|\{z \mid O_{\Theta}(\varphi)(z) = *\}| \leq \varepsilon|V(G)|$.

Definition 7.3. A family of finite graphs $\mathcal{G} \subseteq Gr_d$ is distributed hyperfinite if for any $\varepsilon > 0$, there exist integers $1 \leq s_{\varepsilon} \leq n_{\varepsilon}$ and $K_{\varepsilon} \geq 1$, finite sets $M_{\varepsilon}, Q_{\varepsilon}$ such that $M_{\varepsilon}$ is $(s_{\varepsilon},d)$-large and an oracle function $\Theta_{\varepsilon} : U_{d}^{n_{\varepsilon},s_{\varepsilon},M_{\varepsilon}} \rightarrow \{Q_{\varepsilon}, *\}$ such that $\Theta_{\varepsilon}$ is a $(\varepsilon, K_{\varepsilon})$-hyperfinite partitioning oracle for $\mathcal{G}$.

Finally, we define distributed strong almost finiteness.

Definition 7.4. A family $\mathcal{G} \subseteq Gr_d$ is distributed strongly almost finite if for any $\varepsilon > 0$, there exist integers $1 \leq s_{\varepsilon} \leq n_{\varepsilon}$, and $N_{\varepsilon}, K_{\varepsilon} \geq 1$, finite sets $M_{\varepsilon}, Q_{\varepsilon}$ such that $M_{\varepsilon}$ is $(s_{\varepsilon},d)$-large, and oracle functions $\{\Theta_{i} : U_{d}^{n_{\varepsilon},s_{\varepsilon},M_{\varepsilon}} \rightarrow Q_{\varepsilon}\}_{i=1}^{N_{\varepsilon}}$ such that

1. $\{\Theta_{i}\}_{i=1}^{N_{\varepsilon}}$ are $(\varepsilon, K_{\varepsilon})$- partitioning oracles for $\mathcal{G}$
(2) for all \( G \in \mathcal{G}, \varphi \in L^*_M(G) \) and \( x \in V(G) \),
\[
\frac{|\{i \mid x \in \partial_G(H^i_x)\}|}{|N_\varepsilon|} \leq \varepsilon,
\]
where \( H^i_x \) is the class defined by the labeling \( O_{\Theta_i}(\varphi) \).

The following theorem is the qualitative analogue of the main result of [24].

**Theorem 8.** Let \( \mathcal{G} \in Gr_d \) be a hyperfinite family of finite graphs. Then, \( \mathcal{G} \) is distributed hyperfinite as well.

**Proof.** First, we need two lemmas.

**Lemma 7.2.** Let \( G \in Gr_d \) be a hyperfinite family of finite graphs. Assume that for any \( 0 < \varepsilon < 1 \) and \( G \in \mathcal{G}, G \) is \((\varepsilon, K_\varepsilon)\)-hyperfinite. Then, provided that \( G \in \mathcal{G} \) and \( H \subset G \) is an induced subgraph such that \( |V(H)| \geq \varepsilon |V(G)| \), one can remove \((d + 1)\varepsilon |V(H)| \) vertices from \( H \) (with all the adjacent edges) so that for all the remaining components \( T \), \(|T| \leq K_\varepsilon 3 \) and \( i(T) \leq \varepsilon \).

**Proof.** Observe, that \( H \) is \((\varepsilon^2, K_\varepsilon)\)-hyperfinite. So, let us remove \( \varepsilon^2 |V(H)| \) vertices from \( H \) in such a way that all the remaining components have size at most \( K_\varepsilon \). Let \( A \) be the subset of the remaining components \( M \) such that \( i_H(M) > \varepsilon \). Then, we have that
\[
\varepsilon \sum_{M \in A} |M| \leq d \varepsilon^2 |V(H)|.
\]
Hence, \( \sum_{M \in A} |M| \leq d \varepsilon |V(H)| \). Thus, we can remove \((d\varepsilon + \varepsilon^2)|V(H)| \) vertices from \( V(H) \) in such a way that all the remaining components have size at most \( K_\varepsilon \) and have isoperimetric constant at most \( \varepsilon \).

**Lemma 7.3.** Let \( (1 - (d + 1)\varepsilon) > 1/2 \). Let \( H \subset G \) be graphs as above. Suppose that \( \mathcal{M} \) is a maximal system of connected induced subgraphs in \( H \) such that if \( C \neq D \in \mathcal{M} \), then \( d_H(V(C), V(D)) \geq 2 \) and if \( C \in \mathcal{M} \), then \( |V(C)| \leq K_\varepsilon \) and \( i(V(C)) \leq \varepsilon \). Then,
\[
\sum_{C \in \mathcal{M}} |V(C)| \geq \frac{\varepsilon}{4d^2 K_\varepsilon^2} |V(G)|.
\]

**Proof.** By the previous lemma, we have a system of induced subgraphs \( \mathcal{N} \) such that
- \( \sum_{A \in \mathcal{N}} |V(A)| \geq (1 - (d + 1)\varepsilon)|V(H)| \),
- if \( A \neq B \in \mathcal{N} \), then \( d_H(V(A), V(B)) \geq 2 \),
- if \( A \in \mathcal{N} \), then \(|V(A)| \leq K_\varepsilon \) and \( i(V(A)) \leq \varepsilon \).

Notice that if \( A \in \mathcal{N} \), then there exists an element \( C \in \mathcal{M} \) such that \( V(A) \cap B_2(H, V(C)) \neq \emptyset \), that is, the 2-neighbourhood of the set \( V(C) \) in \( H \) intersects...
Indeed, if such subgraph $C$ did not exist, then $\mathcal{M}$ could not be a maximal system. Since $|B_2(H, V(C))| \leq 2d^2 K_\varepsilon^3$, we have that

$$|\mathcal{M}| \geq \frac{1}{2d^2 K_\varepsilon^3} |\mathcal{N}|.$$ 

Also,

$$\sum_{A \in \mathcal{N}} |V(A)| \geq (1 - (d + 1)\varepsilon)|V(H)| \geq \frac{1}{2} |V(H)|.$$ 

Therefore $|\mathcal{N}| \geq \frac{1}{2d^2 K_\varepsilon^3} |V(G)|$. Thus,

$$\sum_{C \in \mathcal{M}} |V(C)| \geq |\mathcal{M}| \geq \frac{\varepsilon}{4d^2 K_\varepsilon^3} |V(G)|. \quad \square$$ 

Now, we can prove our theorem. Repeating the argument of Proposition 7.1, we can devise an oracle $\Theta : U_{d}^{n,M} \times \{a,b\} \rightarrow \{c,d\}$ such that for any pair $\varphi \in L_{d}^{(n)}(G)$ and $\psi \in L_{d}^{(a,b)}(G)$, $O_{\Theta}(\varphi \oplus \psi) = \rho \in L_{d}^{(c,d)}(G)$ and $\mathcal{M}$ is a maximal system in the induced subgraph $H$ satisfying the following two conditions.

1. If $C \neq D \in \mathcal{M}$, then $d_H(V(C), V(D)) \geq 2$.
2. For any $C \in \mathcal{M}$, $|V(C)| \leq K_\varepsilon^3$ and $i(V(C)) \leq \varepsilon$,

where $H$ is the induced subgraph on the set $\psi^{-1}(a)$ and $\mathcal{M}$ is the induced subgraph on the set $\rho^{-1}(c)$. Now, we start with the graph $G$ and set $H_0 = G$. We apply the distributed algorithm $O_{\Theta}$ to compute the maximal system $\mathcal{M}_0$. Then, we remove the vertices covered by the elements of $\mathcal{M}_0$ and all the neighbouring vertices to obtain the induced subgraph $H_1$. Now, we remove the vertices of the maximal system $\mathcal{M}_1$ in $H_1$ together with the neighbouring vertices to obtain $H_2$ and so on. Hence, we compute a sequence of induced subgraphs $H_0 \supset H_1 \supset \cdots \supset H_j$, where $j$ is an integer larger than $\frac{4d^2 K_\varepsilon^3}{\varepsilon}$. By Lemma 7.3, we have that $|V(H_j)| \leq \varepsilon |V(G)|$. Now, let $X$ be the union of all vertices computed in the process as vertices covered by some maximal system. Let $Y$ be the neighbours of the elements of $X$. Then,

$$V(G) = X \cup Y \cup V(H_j).$$ 

Also, $|Y| \leq d\varepsilon |X|$, since the elements of the maximal systems computed in the process are graphs $C$ such that $i(V(C)) \leq \varepsilon$. Therefore,

$$|Y \cup V(H_j)| \leq (d + 1)\varepsilon |V(G)|.$$ 

Hence, we have a CTDA that computes a set $Z$ of size at most $(d + 1)\varepsilon |V(G)|$ in the graph $G$ such that if we remove $Z$ from $V(G)$, all the remaining components have size at most $K_\varepsilon^3$. Thus, $\mathcal{G}$ is a distributed hyperfinite family of finite graphs. $\square$
7.3. Approximating maximum independent subsets.

**Definition 7.5.** Let \( \mathcal{G} \subset Gr_d \) be a class of finite graphs. We say that there exist CTDA’s for the approximated maximum independent subset problem in \( \mathcal{G} \), if for any \( \varepsilon > 0 \), there exist integers \( 1 \leq s_\varepsilon \leq n_\varepsilon \), a finite \((s_\varepsilon, d)\)-large subset \( M_\varepsilon \) and an oracle \( \Theta : U^{n_\varepsilon, s_\varepsilon, M}_d \to \{a, b\} \) satisfying the following properties.

1. For any \( G \in \mathcal{G} \) and \( \varphi \in L^{(s)}_M(G) \), \((O_\Theta(\varphi))^{-1}(a)\) is an independent subset in \( G \).
2. \( \frac{|(O_\Theta(\varphi))^{-1}(a)|}{|V(G)|} \geq \varepsilon \), where \( I_G \) is a maximum size independent subset in \( G \).

**Proposition 7.2.** Let \( \mathcal{G} \subset Gr_d \) be a hyperfinite class of finite graphs. Then, there exist CTDA’s for approximated maximum independent sets in \( \mathcal{G} \).

**Proof.** Let \( G \in Gr_d, Q \) be a finite set and \( K \geq 1 \) be an integer. Then, we call a function \( \rho : V(G) \to \{Q, *\} \) a \( K \)-subset function if the following two conditions are satisfied.

1. If \( \rho(x) = \rho(y) \), then either \( d_G(x, y) \leq K \) or \( d_G(x, y) \geq 3K \).
2. If \( \rho(x) \neq * \) and \( \rho(z) \neq * \) and \( \rho(x) \neq \rho(y) \), then \( d_G(x, y) \geq 2 \).

For such a \( K \)-subset function \( \rho \) and \( x \in V(G) \) and \( \rho(x) \neq * \), we will denote by \( T^\rho_x \) the induced subgraph on the set

\[ \{z \mid d_G(x, y) \leq K \text{ and } \rho(z) = \rho(x)\}. \]

**Lemma 7.4.** Let \( K > 0 \) and \( Q \) be a finite set. Then, there exists a \((K, d)\)-large set \( M = \{1, 2, \ldots, m\} \) and an oracle \( \Theta : U^{K,M,Q,*}_d \to \{a, b\} \), such that for all \( G \in Gr_d, \varphi \in L^{(K)}_M(G) \) and \( K \)-subset function \( \rho \in L^{(Q,*)}_M(G) \), the function \( f = O_\Theta(\varphi \oplus \rho) \) has the following properties.

- \( f^{-1}(a) \subset \rho^{-1}(Q) \).
- For any \( x \in V(G) \), such that \( \rho(x) \neq * \) the set \( f^{-1}(a) \cap V(T^\rho_x) \) is a maximal independent subset of the graph \( T^\rho_x \).

**Proof.** Let \( \Lambda \) be the finite set of not necessarily connected graphs (up to isomorphism) that occur as induced subgraph in some ball \( B \in U^K_d \). Let \( \hat{\Lambda} \) be the set of all graphs with linearly ordered vertices such that the underlying graph is an element of \( \Lambda \). Finally, for any \( \hat{J} \in \hat{\Lambda} \) fix a maximal independent subset \( I_J \subset V(\hat{J}) \). We describe \( \Theta \) by the corresponding algorithm operator \( O_\Theta \) in the following way. If \( \rho(x) \neq * \), let \( O_\Theta(\varphi \oplus \rho)(x) = a \) if in the ordered graph \( (T^\rho_x, \varphi) \), \( x \in I(Tr^\rho_x, \varphi) \). Note that the relation \( x \in I(Tr^\rho_x, \varphi) \) is well-defined. It is not hard to see that \( \Theta \) satisfies the conditions of our lemma.

Now let \( \mathcal{G} \) be a hyperfinite class of finite graphs. Combining Theorem 8 and Lemma 7.4, we can immediately see that for any \( \varepsilon > 0 \), there exists a CTDA which for any \( G \in \mathcal{G} \) produces an induced subgraph \( H \) and a maximal independent subset \( I_H \subset V(H) \) such that \( (1 - \varepsilon)|V(G)| \leq |V(H)| \). Let \( J \)
denote the restriction of $I_G$ onto $I_H$. Then, $|J| \leq |I_H|$ and $|I_G| \leq |J| + \varepsilon |V(G)|$ hold. Consequently, 

$$\frac{|I_H|}{|V(G)|} \geq \frac{|I_G|}{|V(G)|} - \varepsilon,$$

hence our proposition follows. $\square$

**Remark 12.** The proof of Proposition 7.2 illustrates the basic subroutines we use to build CTDA’s.

A: The subroutine finds a maximal $r$-separating system in the graphs $G \in \mathcal{G}$.

B1: For some $s \geq 1, 2 < s < r$, the subroutine takes a symmetry breaking function $\varphi \in L^2_{M}(G)$ and a maximal $r$-separating system $T$ as inputs. Then, independently label the balls $B_r(G, x), x \in T$ by some set $Q$ using local computations in the balls.

B2: A slight modification of the previous one. The subroutine uses the function $\varphi \in L^2_{M}(G)$, a maximal $r$-separating system $T$ and a previously constructed labeling $\psi : V(G) \to P$ to label the balls $B_r(G, x), x \in T$.

**Remark 13.** One can apply algorithm oracles to Cantor subshifts $Z \in \mathcal{C}_{\Sigma}G$ as well. It is important to note that $O_{\Theta}(Z)$ is always qualitatively weakly contained in $Z$.

### 7.4. Approximated maximum matchings

The goal of this section is to prove the following qualitative analogue of the main results of [13] and [35]. A similar result using a different concept of local algorithms was proved by Even, Medina and Ron [17] (see also, [5]).

**Theorem 9.** There exist CTDA’s for the approximated maximum matching for finite graphs in $Gr_d$.

**Proof.** Again, before getting into details, let us explain the precise meaning of the theorem. For a $(5, d)$-large set $Q$, we call the function $\rho : V(G) \to Q$ an matching function on $G$ if the following conditions hold.

1. For any $a \in V(G)$, there exists at most one $b \in V(G)$ such that $0 < d_G(a, b) \leq 5$ and $\rho(a) = \rho(b)$.
2. If $0 < d_G(a, b) \leq 5$ and $\rho(a) = \rho(b)$, then $a$ and $b$ are adjacent vertices.

Clearly, the set of adjacent pairs $(a, b)$ for which $\rho(a) = \rho(b)$ form a matching $M_\rho$ of $G$. The existence of CTDA’s for the approximated maximum matching in finite graphs means that for any $\varepsilon > 0$, there exist integers $1 \leq n \leq l$, an $(n, d)$-large set $N$, an $(5, d)$-large set $Q$ and an oracle $\Theta : U^{l, n, N}_d \to Q$ such that for any finite graph $G \in Gr_d$ and $\varphi \in L^{(n)}_N(G)$,

- $\rho = O_{\Theta}(\varphi)$ is an matching function, and
- $\frac{|M_\rho|}{|V(G)|} \geq \frac{|M_G|}{|V(G)|} - \varepsilon$, where $M_G$ is the maximum sized matching in $G$. 
We will closely follow the proof of the main result in [13]. Informally speaking, the distributed algorithm work as follows. First, we build a “local improvement” algorithm (see also [35]) which takes a matching \( M \) as an input (together with the usual symmetry breaking auxilliary function) and produces a new matching \( M' \) such that \( |M'| > |M| \) provided that \( M \) has an augmenting path of length shorter than \( T \). By Lemma 2.1 of [13], if a matching \( M \) has at most \( \varepsilon |V(G)| \) vertices from which an augmenting path shorter than \( T \) starts, then

\[
\frac{|M_G|}{|V(G)|} \leq \frac{|M|}{|V(G)|} \frac{T + 1}{T} + \frac{\varepsilon}{2} \leq \frac{|M|}{|V(G)|} + \frac{1}{T} + \frac{\varepsilon}{2}.
\]

Thus, if \( T > \frac{2}{\varepsilon} \), the repeated applications of such a local improvement algorithm lead to the required \((1 + \varepsilon)\)-approximation of our theorem. The construction of such algorithm is not very hard, however, we need to show that the number of repetitions needed is bounded for the class of finite graphs in \( Gr_d \). So, let us write down the algorithm as a crude pseudo-code, where each step requires a simple basic subroutine algorithm as described in Remark 12. Let \( T > \frac{2}{\varepsilon} \) be an integer.

10 For each finite graph \( G \in Gr_d \) we set up a starting matching function \( \rho : V(G) \to Q \).

20 We construct a finite family \( J_1, J_2, \ldots, J_t \) of maximal \( 10T \)-separating systems in \( V(G) \) such that \( \bigcup_{i=1}^t J_i = V(G) \). It is easy to see that for large enough \( t \) such algorithm exists for graphs with vertex degree bound \( d \).

30 LET \( j = 1 \).

40 LET \( i = 1 \).

50 IF \( j = \text{"BOUND"} \) THEN GO TO 90.

60 IF \( i = t + 1 \), LET \( j = j + 1 \) and GO TO 40.

70 Consider the vertices \( x \in J_i \), if there exists an augmenting path starting from \( x \) the subroutine makes the improvement inside the ball \( B_{4T}(G,x) \) to obtain a new matching function \( \rho : V(G) \to Q \) representing more edges. If there is no such augmenting path the algorithm does not change the matching inside the ball.

80 LET \( i = i + 1 \) and GO TO 50.

90 STOP.

Our theorem follows from the proposition below.
Proposition 7.3. If “BOUND” is a large enough integer, then for any finite graph $G \in Gr_d$, when the algorithm stops we end up with a matching satisfying (13).

Proof. We apply an infinite-to-finite argument motivated by a similar proof in [13]. Let $s_1 < s_2 < \ldots$, $m_1 < m_2 < \ldots$ and $k_1 < k_2 < \ldots$ be positive integers such that $M_n = \{0,1\}^{k_n}$ is an $(s_n,d)$-large set and $\Theta_n : U_d^{s_n, m_n, M_n} \to Q$ is an oracle that takes an element of $L^{(s_n)}(G)$, $G \in Gr_d$ as input, and construct a matching as in the pseudo-code above until the variable $j$ reaches $n$. Suppose that the statement of our proposition does not hold. Then, there exists a matching as in the pseudo-code above until the variable $x$ is greater than $\varepsilon |V(G_n)|$. Thus, we have $\varphi_n \in L^{(s_n)}(G_n)$ such that for the matching function $O_{\Theta}(\varphi_n)$ we have that

$$\frac{l_n}{|V(G_n)|} > \frac{\varepsilon}{2} |V(G_n)|.$$  

Note that for any $n \geq 1$ we can regard the labeled graph $(G_n, \varphi_n)$ as an element of $CGr_d$ by extending the labeling from $\{0,1\}^{k_n}$ to $\{0,1\}^{\mathbb{N}}$ as zero for all the coordinates larger than $k_n$. Now, we recall the $\mathcal{C}$-labeled version of the Benjamini-Schramm convergence (Section 3.2. [14]). Let $\{H_n, \psi_n\}_{n=1}^\infty \subset CGr_d$ be a sequence of finite graphs. For any $B \in U_d^{r,\{0,1\}}$, let $T_{G,\varphi}(B) \subset V(H, \psi)$ be the set of vertices $x$ in $(H, \psi) \in CGr_d$ such that the rooted-labeled ball of radius $r$ around $x$ is rooted-labeled isomorphic to $B$. We say that $\{H_n, \psi_n\}_{n=1}^\infty \subset CGr_d$ is convergent in the sense of Benjamini and Schramm if for any $r \geq 1$ and $B \in U_d^{r,\{0,1\}}$

$$\lim_{n \to \infty} \frac{|T_{H_n,\psi_n}(B)|}{|V(H_n)|}$$

exists. Clearly, one can pick a convergent graph sequence from any sequence of finite graphs in $Gr_d$, so we can suppose that our counterexample sequence $\{G_n, \varphi_n\}_{n=1}^\infty$ is convergent in the sense of Benjamini and Schramm. We can also suppose that $\{G_n, \varphi_n\}_{n=1}^\infty$ is convergent in $CGr_d$. Now, let us consider the infinite version of our proposition. Let $Z \subset CGr_d$ be a proper subset such that for any $n \geq 1$, if $(H, x, \psi) \in Z$ and $y, w \in V(H)$ for which $0 \leq d_H(y, w) \leq s_n$, then we have that $(\psi(y))_{\{k_n\}} \neq (\psi(w))_{\{k_n\}}$, where $\{k_n\}_{n=1}^\infty$, $\{s_n\}_{n=1}^\infty$ are the sequences as above. Note that $Z$ can be regarded as a Borel graphing (see [13]). Observe that the operators $O_{\Theta_n}(x)$ defines a Borel matching $M_n$ in $Z$.

Lemma 7.5. If $\mu$ is an invariant probability measure on the Borel graphing $Z$, then there exists an integer $m_Z > 1$ such that the $\mu$-measure of elements $z \in Z$, for which an augmenting path (with respect to the matching $M_{m_Z}$) shorter than $T$ starts at $z$ is less than $\frac{\varepsilon}{4}$.

Proof. The lemma follows from Proposition 1.1 [13], nevertheless we give a short proof for completeness. Let $e = (y, w)$ be an edge of $(H, \psi, x) \in Z$. 

In the matchings \( \{ M_n \}_{n=1}^{\infty} \) for some \( n \)'s \( e \) belongs to matching \( M_n \) for some \( n \)'s \( e \) does not belong to the matching \( M_n \). However, the membership of the edge \( e \) stabilizes. Indeed, any change of the membership of the edge \( e \) increases the number of matched vertices in the ball \( B_{ST}(H, y) \). Hence, we have a well-defined limit matching \( M \) for which there is no element \( z \in Z \) with augmenting path shorter than \( T \) starting at \( z \). Let \( S_n \subset Z \) be the set of vertices \( z \in Z \) (do not forget that \( z \) is a rooted-labeled infinite graph \( (H, x, \phi) \)), for which all edges adjacent to \( z \) in the graphing structure are stabilized after the \( n \)-th step. Since \( \mu(S_n) \to 1 \), our lemma follows. \( \square \)

Now let \( (G, \phi) \in \mathcal{CGr}_d \) be a limit of the sequence \( \{ G_n, \phi_n \}_{n=1}^{\infty} \subset \mathcal{CGr}_d \). Let \( Z \subset \mathcal{CGr}_d \) be the orbit closure of \( (G, \phi) \) in \( \mathcal{CGr}_d \) as in Section 2. Then, we have an invariant probability measure \( \mu \) on \( Z \), for which

\[
\lim_{n \to \infty} \frac{|T_{G_n, \phi_n}(B)|}{|V(G_n)|} = \mu(T_Z(B))
\]

holds for all \( r \geq 1 \) and \( B \in U^{r, (0,1)^r}_d \). Here, \( T_Z(B) \) is the clopen set of elements \( (H, x, \psi) \in Z \) such that the rooted-labeled ball \( B_r(H, x, \psi_{|r}) \) is rooted-labeled isomorphic to \( B \). That is, the measured graphing \( (Z, \mu) \) is the Benjamini-Schramm limit of the sequence \( \{ (G_n, \varphi_n) \}_{n=1}^{\infty} \). Also, for any ball \( B \) the set \( T_Z(B) \) is nonempty if and only if the sets \( \{ T_{G_n, \phi_n}(B) \}_{n=1}^{\infty} \) are nonempty for all but finitely many values of \( n \). Now let \( m_Z \) be the constant in Lemma 7.5. Notice that the sequence of \( Q \)-labeled graphs \( \{ O_{\Theta_{m_Z}}(\varphi_n) \}_{n=1}^{\infty} \) is convergent in the compact space \( \mathcal{Gr}_d^Q \). Also, it is not hard to see that for all \( r \geq 1 \) and \( B \in U^{r, Q}_d \), we have that

\[
\lim_{n \to \infty} \frac{|T_{G_n, \phi_n}(B)|}{|V(G_n)|} = \mu(T_Z(B)).
\]

Therefore,

\[
\lim_{n \to \infty} \frac{|A_{G_n, \phi_n}(B)|}{|V(G_n)|} = \mu(A_Z(B)),
\]

where \( A_{G_n, \phi_n}(B) \) is the set of vertices \( x \in G_n \) for which an augmenting path shorter than \( T \) starts at \( x \) in the matching defined by the matching function \( O_{\Theta_{m_Z}}(\phi_n) \). Similarly, \( A_Z(B) \) is the clopen set of vertices \( z \in Z \) for which an augmenting path of \( M_{m_Z} \) shorter than \( T \) starts at \( z \). Since by Lemma 7.5, \( \mu(A) \leq \frac{\varepsilon}{3} \) and

\[
\limsup_{n \to \infty} \frac{|A_{G_n, \phi_n}(B)|}{|V(G_n)|} \geq \frac{\varepsilon}{2},
\]

we obtain a contradiction. Hence, our proposition follows and so does our theorem. \( \square \)

7.5. The unrestricted weighted independent subset problem. Let \( G \in \mathcal{Gr}_d \) be a finite graph. Let \( w \in V(G) \to \mathbb{N} \cup 0 \) be an arbitrary function. The maximum \( w \)-weighted independent subset in \( G \) is an independent set \( J \subset V(G) \) such that \( \sum_{x \in J} w(x) \) is maximal among all independent subsets of
The $(1 + \varepsilon)$-approximated $w$-weighted independent problem is to find an independent set $J \subset V(G)$ such that

$$
\sum_{x \in J} w(x) \geq \sum_{x \in V(G)} w(x) - \varepsilon,
$$

where $I_{G,w}$ is a maximum $w$-weight independent subset in $G$. A local distributing algorithm algorithm for the approximated weighted independent subset problem must deal with arbitrarily large integers, hence it must use the full power of the $\text{LOCAL}$-model. The messages between the processors as well as the local computations are supposed to be unbounded. The deterministic-random local distributed algorithm for the approximated weighted independent subset problem for any $\varepsilon > 0$ takes a symmetry breaking function $\varphi \in L^{(r \varepsilon)}(G)$ as an input and for any $w : V(G) \to \mathbb{N} \cup 0$ in $r \varepsilon$-rounds it produces independent subsets $J_1, J_2, \ldots, J_{T\varepsilon}$ in $G$ in such a way that if we randomly pick one of the $J_i$’s then the probability of picking an independent subset satisfying (14) is larger than $(1 - \varepsilon)$.

Proposition 7.4. Let $G \subset GRd$ be a distributed strongly almost finite graph class (e.g. $D$-doubling graphs by Theorem 10) then there exist deterministic-random local distributed algorithms for the approximated weighted independent subset problem.

Proof. Let $G \in \mathcal{G}$ and $E = \cup_{j=1}^{T\varepsilon} H_j$ be a partition of $V(G)$ such that

$$
\sum_{x \in E} w(x) \geq (1 - \varepsilon) \sum_{x \in V(G)} w(x)
$$

where $E = \cup_{j=1}^{T\varepsilon} (H_j \setminus \partial(H_j))$. For each $j \geq 1$, pick a maximum $w$-weighted independent set $I_j^E$ in the graph induced on $H_j \setminus \partial(H_j)$ and let $J^E = \cup_{j=1}^{T\varepsilon} I_j^E$. Then, we have that

$$
\sum_{x \in J^E} w(x) + \varepsilon \sum_{x \in V(G)} w(x) \geq \sum_{x \in I_{G,w}} w(x).
$$

That is,

$$
\frac{\sum_{x \in J^E} w(x)}{\sum_{x \in V(G)} w(x)} \geq \frac{\sum_{y \in I_{G,w}} w(x)}{\sum_{x \in V(G)} w(x)} - \varepsilon.
$$

Repeating the argument of Lemma 6.3, we can immediately see that we have a CTDA for the class $\mathcal{G}$ that produces, for all $G \in \mathcal{G}$, partitions

$$
\left\{ E_i \right\}_{i=1}^{T \varepsilon} = \left\{ H_i^1, H_i^2, \ldots \right\}_{i=1}^{T \varepsilon}
$$

such that the diameter of each class $H_i^j$ is bounded by some constant $L_{\varepsilon}$, independently of $G$, and the number of $i$’s for which (15) holds is larger than $(1 - \varepsilon)T\varepsilon$. Now our proposition follows, since simple local algorithms can find maximum size $w$-independent sets in the bounded diameter parts, by checking all independent subsets (using the full power of the $\text{LOCAL}$-model).
7.6. Distributed parameter testing. Our notion of distributed parameter testing can be viewed as the qualitative version of the randomized parameter testing due to Goldreich and Ron [19]. First, recall the randomized parameter testing model for bounded degree graphs. Let \( p : \mathcal{G} \to [a, b] \) or, in general, \( p : \mathcal{G} \to K \) be a function such that \( K \) is a compact metrizable space. We say that \( p \) is testable or estimable in the class of finite graphs \( \mathcal{G} \subseteq \mathcal{G}_{r,d} \) if we have the following algorithm.

1. First, we pick \( r \) vertices of the graph \( G \in \mathcal{G} \) uniformly randomly and explore the \( s \)-neighbourhood of the picked vertices.
2. Then, the algorithm makes a “guess” \( \hat{p}(G) \) in such a way that
   \[
   \text{PROB}(d_K(\hat{p}(G), p(G)) > \varepsilon) < \varepsilon.
   \]

Now we define a distributed parameter testing algorithm for the class of finite graphs \( \mathcal{G} \subseteq \mathcal{G}_{r,d} \).

1. First, the algorithm learn \( \mathcal{B}_{s,e}(G) \), the set of all rooted balls of radius \( s \) that occur in \( G \). Note that the algorithm will not know the probability distribution on \( \mathcal{B}_{s,e}(G) \), as opposed to the case of randomized testability, where we have a very good estimate on the distribution with high probability by the Law of Large Numbers.
2. Based on the knowledge of \( \mathcal{B}_{s,e}(G) \), the algorithm makes a guess \( \hat{p}(G) \) in such a way that the inequality \( d_K(\hat{p}(G), p(G)) \leq \varepsilon \) always holds.

It is well-known that for a class \( \mathcal{G} \subseteq \mathcal{G}_{r,d} \) the parameter is testable in the randomized setting if and only if for all Benjamini-Schramm convergent sequences \( \{G_n\}_{n=1}^\infty \subseteq \mathcal{G} \) the limit \( \lim_{n \to \infty} p(G_n) \) exists. We have the following proposition for the distributed setting.

**Proposition 7.5.** The parameter \( p : \mathcal{G} \to K \) is testable in the distributed sense if and only if for all naively convergent sequence \( \{G_n\}_{n=1}^\infty \subseteq \mathcal{G} \) the limit \( \lim_{n \to \infty} p(G_n) \) exists.

**Proof.** Suppose that for all naively convergent sequence \( \{G_n\}_{n=1}^\infty \subseteq \mathcal{G} \) the limit \( \lim_{n \to \infty} p(G_n) \) exists. Then, for any \( \varepsilon > 0 \) there exists \( s > 0 \) such that if \( d_K(p(G), p(H)) < \varepsilon \), then \( \mathcal{B}_s(G) = \mathcal{B}_s(H) \). Then, we have the following testing algorithm. For all family \( \mathcal{B} \subseteq \mathcal{U}_d \) which equals to \( \mathcal{B}_s(G) \) for some graph \( G \in \mathcal{G}_{r,d} \), we pick a representative \( G_B \in \mathcal{G}_{r,d} \). The algorithm will make the guess \( \hat{p}(G) = p(G_B) \). Now, suppose that for some naively convergent sequence \( \{G_n\}_{n=1}^\infty \subseteq \mathcal{G} \) the limit \( \lim_{n \to \infty} p(G_n) \) does not exist. Then, there exists some \( \varepsilon > 0 \) such that for some sequences \( \{H_n\}_{n=1}^\infty \subseteq \mathcal{G} \) and \( \{J_n\}_{n=1}^\infty \subseteq \mathcal{G} \), \( \mathcal{B}_s(H_n) = \mathcal{B}_s(J_n) \) and \( d_K(p(H_n), p(J_n)) > \varepsilon \). Therefore, it is impossible to guess the value \( p(G) \) based on the knowledge of \( \mathcal{B}_s(G) \) for any \( k \geq 1 \). □

By Theorem 7, we have the following corollary.

**Corollary 7.1.** Let \( \mathcal{G} \subseteq \mathcal{G}_{r,d} \) be a strongly almost finite class of finite graphs. Then the parameter \( \text{p}_\text{Norm} : \mathcal{G} \to [0, 2d] \), \( \text{p}_\text{Norm}(G) = \|\Delta_G\| \) or the parameter \( \text{p}_\text{Spec} : \mathcal{G} \to \text{Cl}([0, 2d]) \), \( \text{p}_\text{Spec}(G) = \text{Spec}(\Delta_G) \) are testable parameters, where
$\text{Cl}([0,2d])$ is the compact set of all closed subsets of the interval $[0,2d]$ with the Hausdorff metric.

8. Doubling and almost finiteness

The goal of this section is to prove the following theorem.

Theorem 10. For any $D > 1$, the class of $D$-doubling graphs is distributed strongly almost finite.

8.1. Doubling graphs. Let $D$ be a positive integer. A graph $G$ of bounded vertex degree is called $D$-doubling if for any $x \in V(G)$ and integer $s \geq 1$, $|B_{2s}(G,x)| \leq D|B_s(G,x)|$. Any doubling graph $G$ has polynomial growth of order $\log_2(D)$, that is, there exists $C > 0$ such that $|B_r(G,x)| \leq Cr\log_2(D)$ holds for all $x \in V(G)$ and $r \geq 1$. In fact, the constant $C$ depends only on $D$ and the vertex degree bound of $G$. Although it is not true that all the graphs of polynomial growth are doubling, graphs of strict polynomial growth are always doubling. Recall that a graph $G$ is of strict polynomial growth if there exists $\alpha \geq 1$, $0 < C_1 < C_2$ such that $C_1r^\alpha \leq B_r(G,x) \leq C_2r^\alpha$ holds for all $x \in V(G)$ and $r \geq 1$. The following lemma is well-known and we prove it only for completeness.

Lemma 8.1. If $G$ is $D$-doubling then any ball $B_{2s}(G,x)$ can be covered by at most $D^4$-balls of radius $s$.

Proof. Let $z \in V(G)$ and $s > 0$. We need to show that the ball $B_{2s}(G,z)$ can be covered by $D^4$ balls of radius $s$. Let $X \subseteq B_{2s}(G,z)$ be a maximal set of vertices such that if $p \neq q \in X$, then $d_G(p,q) > s$. Hence, $B_{2s}(G,x) \subseteq \bigcup_{p \in X} B_s(G,p)$. So, $B_{2s}(G,x)$ can be covered by at most $|X|$ balls of radius $s$. Also, if $p \in X$ then

$$B_{4s}(G,z) \subseteq B_{8s}(G,p).$$

Furthermore, the balls $\{B_{s/2}(G,p)\}_{p \in X}$ are disjoint and contained in $B_{4s}(G,z)$. By the $D$-doubling property, for any $p \in X$ $|B_{8s}(G,p)| \leq D^4|B_{s/2}(G,p)|$. Hence by (16) if $p \in X$, then $|B_{4s}(G,z)| \leq D^4|B_{s/2}(G,p)|$. Therefore, $|X| \leq D^4$.

The following lemma is due to Lang and Schlichenmaier [27].

Lemma 8.2. Let $G$ be a $D$-doubling graph. Then for all integers $s,n \geq 1$ one has a system $\mathcal{B} = \bigcup_{i=1}^{D^4(n+3)^4} \mathcal{B}_i$ such that
(1) $\mathcal{B}_t$ is the union of balls of radius $s$ and if $B_s(G, z)$ and $B_s(G, z')$ are two elements of $\mathcal{B}_t$ so that $z \neq z'$, then $B_{2s}(G, z) \cap B_{2s}(G, z') = \emptyset$.

(2) The elements of $\mathcal{B}$ covers all the vertices of $G$.

Proof. Let $Z \subset V(G)$ be a maximal set of vertices such that $d_G(z, z') > s$ if $z, z' \in Z$ and $z \neq z'$. Then the family $\mathcal{B} := \{B_s(G, z)\}_{z \in Z}$ covers $V(G)$. By the previous lemma, for each $z \in Z$ the ball $B_{2s}(G, z)$ can be covered by $D((n+3)^4)$ balls of diameter less or equal $s$. Each of these balls contains at most one element of $Z$. Therefore, any ball $B_{2s}(G, z), z \in Z$ contains at most $D((n+3)^4)$ elements of $Z$. Hence there exists a colouring $\chi : Z \to \{1, 2, \ldots, D((n+3)^4)\}$ such that $\chi(z) \neq \chi(z')$, whenever $z, z' \in Z$ and $d_G(z, z') \leq 2n^2s$. That is, the family of balls $\mathcal{B}_t = \{B_s(G, z), \chi(z) = i\}$ satisfies the conditions of our lemma.

Let $H$ be a finite subset of the graph $G$. Then $\partial_K(H)$ denotes the set of vertices $x$ of $H$ for which there exists a vertex $y \not\in H$ such that $d_G(x, y) \leq K$. Also, $B_K(H)$ denotes the set of vertices $y$ of $G$ for which there exists a vertex $x \in H$ such that $d_G(x, y) \leq K$. The following lemma is a straightforward consequence of the definitions.

**Lemma 8.3.** Let $G$ be a $D$-doubling graph of vertex degree bound $d$, $K > 0$, $0 < p < 1$ and $\delta > 0$. Then, there exists an integer $M = M_{D, d, K, p, \delta}$ such that for any $x \in V(G)$ and $s \geq M$,

$$|\{t \mid s \leq t < 2s, \frac{|B_{t+K}(G, x)|}{|B_{t-K}(G, x)|} < 1 + \delta\}| > ps.$$ 

That is, if $s$ is large enough, then most of the balls in the form $B_t(G, x)$, $s \leq t < 2s$ has small boundary.

### 8.2. The Basic Algorithm

Let $0 < \varepsilon < \frac{1}{2}$ be a real constant and $D$ be a positive integer. We call an $N$-tuple of positive integers $S_1 > S_2 > S_3 > \cdots > S_N$ $(D, \varepsilon)$-good if for any $D$-doubling graph $G$, any integer $1 \leq i \leq N$ and $q \in V(G)$, there exists an integer $S_i \leq r_i(q) < 2S_i$ such that

$$|\frac{B_{r_i(q)+16NS_{i+1}}(q)}{|B_{r_i(q)}(q)|} < 1 + \frac{\varepsilon}{10D^3}, \text{if } 1 \leq i \leq N - 1.$$  

$$i_G(B_{r_i(q)}(q)) < \varepsilon, \text{if } 1 \leq i \leq N.$$  

For any $1 \leq i \leq N - 1$, $S_i > 4S_{i+1} + 4S_{i+2} + \cdots + 4S_N$.

$$(1 - \frac{1}{4D^3})^N < \varepsilon.$$ 

The existence of such $(D, \varepsilon)$-good tuples easily follows from Lemma 8.3. Note that since all the balls in this section are in our graph $G$, we will use the notation $B_r(x)$ instead of $B_r(G, x)$. 


The preliminary round. Let $S_1 > S_2 > S_3 > \cdots > S_N$ be a fixed $(D, \varepsilon)$-good $N$-tuple and $G$ be a $D$-doubling graph. For each integer $1 \leq i \leq N$ pick a maximal system of vertices $\{q^i_\alpha\}_{\alpha \in J_i}$ in $G$ in such a way that if $\alpha \neq \beta$ then $d_G(q^i_\alpha, q^i_\beta) > 8 S_i$. Using our definition of $(D, \varepsilon)$-goodness, for each chosen vertex $q^i_\alpha$ we pick an integer $S_i \leq r_i(q^i_\alpha) < 2 S_i$ such that
\begin{equation}
\frac{|B_{r_i(q^i_\alpha) + 16 NS_i}(q^i_\alpha)|}{|B_{r_i(q^i_\alpha)}(q^i_\alpha)|} < 1 + \frac{\varepsilon}{10D^3}, \text{ if } 1 \leq i \leq N - 1.
\end{equation}
and
\begin{equation}
i_G(B_{r_i(q^i_\alpha)}(q^i_\alpha)) < \varepsilon, \text{ if } 1 \leq i \leq N.
\end{equation}
For $1 \leq i \leq N$ we call the chosen balls $B_{r_i(q^i_\alpha)}(q^i_\alpha)$ balls of type-$i$.

The construction round. First, we discard all balls of type-2 that are intersecting a chosen ball of type-1. Inductively, we discard all chosen balls of type-$i$ that are intersecting a chosen ball $B$ of type-$j$, $j < i$ such that $B$ has not been previously discarded. Finishing the process we obtain a disjoint system of balls $B^1, B^2, \ldots$, which we call nice balls. Our main technical proposition goes as follows.

**Proposition 8.1.** Let $A \subset G$ be a finite subset such that
\begin{equation}
|\partial_{16NS_i}(A)| \leq \varepsilon 10D^3,
\end{equation}
then we have that
\begin{equation}
|\bigcup_{B \subset A} B^i| \geq (1 - \varepsilon)|A|.
\end{equation}

**Proof.** First we need a lemma.

**Lemma 8.4.** Let $1 \leq i \leq N$ and $Q \subset G$ be a finite subset such that
\begin{equation}
\frac{|\partial_{16S_i}(Q)|}{|Q|} < \frac{1}{2},
\end{equation}
then
\begin{equation}
\sum_{q^i_\alpha, q^i_\beta \in Q \setminus \partial_{8S_i}(Q)} |B_{r_i(q^i_\alpha)}(q^i_\alpha)| > \frac{1}{2D^3}|Q|.
\end{equation}

**Proof.** Observe that
\begin{equation}
\bigcup_{q^i_\alpha, q^i_\beta \in Q \setminus \partial_{8S_i}(Q)} B_{8S_i}(q^i_\beta) \supset Q \setminus \partial_{16S_i}(Q).
\end{equation}
Indeed, let $x$ be a vertex in $Q \setminus \partial_{16S_i}(Q)$. Then, by the maximality of the system $\{q^i_\alpha\}_{\alpha \in J_i}$, there exists $\beta \in J_i$ such that $x \in B_{8S_i}(q^i_\beta)$. Clearly, $q^i_\beta \in Q \setminus \partial_{8S_i}(Q)$. By (24),
\begin{equation}
D^3 \sum_{q^i_\alpha, q^i_\beta \in Q \setminus \partial_{8S_i}(Q)} |B_{r_i(q^i_\beta)}(q^i_\beta)| \geq |Q \setminus \partial_{16S_i}(Q)|,
\end{equation}
hence, our lemma follows. \qed
Now we turn to the proof of our proposition. Let \( A \subset G \) be a finite subset satisfying the inequality (23). We define a process during which we inductively pick nice balls inside \( A \) in such a way that eventually the picked balls will cover at least \((1 - \varepsilon)|A|\) vertices of \( A \). Before starting our construction we need two technical lemmas.

**Lemma 8.5.** Let \( A \subset G \) be a finite subset such that
\[
\frac{|\partial_{16S_{i+1}}(A)|}{|A|} < \frac{\varepsilon}{10D^3}.
\]
For \( 1 \leq j \leq i \), let \( C_j \subset A \) be the union of some nice balls of type-\( j \). Suppose that \( |A \cup \bigcup_{j=1}^{i} C_j| \geq \varepsilon|A| \). Then
\[
(25) \quad \frac{|\partial_{16S_{i+1}}(A \cup \bigcup_{j=1}^{i} C_j)|}{|A \cup \bigcup_{j=1}^{i} C_j|} < \frac{1}{2}.
\]

**Proof.** Let \( x \in \partial_{16S_{i+1}}(A \cup \bigcup_{j=1}^{i} C_j) \). Then at least one of the two conditions below are satisfied.

- \( x \in \partial_{16S_{i+1}}(A) \).
- For some \( 1 \leq j \leq i \) there exists \( y \in C_j \), such that \( d_G(y, C_j) \leq 16S_{i+1} \). That is \( x \in B_{16S_{i+1}}(C_j) \backslash C_j \).

Therefore by (21),
\[
|\partial_{16S_{i+1}}(A \cup \bigcup_{j=1}^{i} C_j)| \leq \frac{\varepsilon}{10}|A| + \frac{\varepsilon}{10}(\sum_{j=1}^{i} |C_j|) \leq \frac{\varepsilon}{5}|A|.
\]
Hence, if \( |A \cup \bigcup_{j=1}^{i} C_j| \geq \varepsilon|A| \), then (25) holds. \( \square \)

**Lemma 8.6.** Let \( A \) and \( \{C_j\}_{j=1}^{i} \) be as in Lemma 8.5. Let
\[
B = \partial_{16S_{i+1}}(A) \cup \bigcup_{j=1}^{i} (B_{16S_{i+1}}(C_j) \backslash C_j) .
\]
Then
\[
(26) \quad |B| \leq \frac{\varepsilon}{4D^3}|A| .
\]

**Proof.** Observe that by (23),
\[
|\partial_{16S_{i+1}}(A)| \leq \frac{\varepsilon}{10D^3}|A| .
\]
Also, if \( L \) is a ball of type-\( j \) \( j \leq i \), then by (17)
\[
|B_{16S_{i+1}}(L) \backslash L| \leq \frac{\varepsilon}{10D^3}|L| .
\]
Therefore
\[
|\bigcup_{j=1}^{i} B_{16S_{i+1}}(C_j) \backslash C_j| \leq \frac{\varepsilon}{10D^3}|A| ,
\]
hence (26) follows. \( \square \)
Now we define our covering process. First, we cover \( A \) with balls of type-1, then we inductively cover the rest of \( A \) by smaller and smaller chosen balls. The trick is that we use only smaller balls which are further and further away from the balls we used previously. In this way, we can assure that all the balls we use in the covering process are nice.

So, let \( Q_0 = A \).

**Step 1.** We pick all the chosen balls of type-1 in \( A \) that does not intersect \( \partial_4 S_1(A) \). These balls will be called \( A \)-covering balls.

- The union of \( A \)-covering ball of type-1 will be denoted by \( C_1 \).
- We set \( Q_1 := Q_0 \setminus C_1 \).

**Step 2.** We continue the covering process by picking all the chosen balls \( L \) of type-2 inside the set \( A \) that does not intersect the set \( \partial_8 S_1(A) \) and for any \( A \)-covering ball \( D \) of type-1, \( L \) does not intersect \( B_{4S_2}(D) \) either.

- Again, we will call the balls picked above \( A \)-covering balls and denote their union by \( C_2 \).
- We set \( Q_2 := Q_1 \setminus C_2 \).

We will see in a moment that all the \( A \)-covering balls of type-2 are nice.

**Step \((i+1)\).** In the first \( i \)-steps we have already defined disjoint sets \( C_1, C_2, \ldots, C_i \) inside the set \( A \), where for any \( 1 \leq j \leq i \), the set \( C_j \) is the union of nice balls picked at the step \( j \). The \( A \)-covering balls have the following properties:

- If \( L \) is an \( A \)-covering ball of type-\( j \) and \( M \) is an \( A \)-covering ball of type-\( k \), \( k < j \) then \( B_{4(j-k)S_1}(M) \cap L = \emptyset \).
- If \( L \) is an \( A \)-covering ball of type-\( j \), then \( \partial_4 S_1(A) \cap L = \emptyset \).
- \( Q_j = Q_{j-1} \setminus C_j \).

Now we continue our covering process. We pick all the chosen balls \( P \) of type-\((i + 1)\) contained in \( A \) for which both conditions below are satisfied:

1. \( \partial_{4(i+1)S_1}(A) \cap P = \emptyset \).
2. For any \( 1 \leq j \leq i \) and \( A \)-covering ball \( M \) of type-\( j \), \( B_{4(i+1-j)S_1}(M) \cap P = \emptyset \).

Finally, we set \( Q_{i+1} := Q_i \setminus C_{i+1} \). Our crucial observation is formulated in the following lemma.

**Lemma 8.7.** All the \( A \)-covering balls of type-\((i + 1)\) are nice.

**Proof.** Let \( L \) be an \( A \)-covering ball of type-\((i + 1)\). Suppose that \( L \) is not nice. Then there exists a nice ball \( M \) of type-\( j \), \( j \leq i \) such that \( M \cap L \neq \emptyset \). In our construction, \( A \)-covering balls of type-\( j \) cannot intersect an \( A \)-covering ball of type-\((i + 1)\). That is, \( M \) cannot be an \( A \)-covering ball. Since \( L \subset A \) and \( \partial_{4(i+1)S_1}(A) \cap L = \emptyset \), the ball \( M \) is a subset of \( A \) as well. Hence, the reason that \( M \) has not picked at step \( j \) was
• either that $M \cap \partial_{4j} S_{i}(A) \neq \emptyset$
• or that $M \cap B_{4(j-k)} S_{k+1}(D) \neq \emptyset$ for some $A$-covering ball $D$ of type-$k$.

**Case 1.**

1. There exists $x \in M$, $x \in \partial_{4j} S_{i}(A)$ and
2. there exists $y \in L$ such that $y \notin \partial_{4(i+1)} S_{i}(A)$ and $y \in M$.

That is, $d_{G}(x, y) \geq 4(i + 1 - j) S_{i}$ and $d_{G}(x, y) \leq \text{diam}(M) \leq 4S_{i}$ leading to a contradiction.

**Case 2.**

1. There exists an $A$-covering ball $D$ of type-$k$, $k < j$ and $x \in M$ such that $x \in B_{4(j-k)} S_{k+1}(D)$
2. there exists $y \in L \cap M$, $y \notin \partial_{4(i+1)} S_{i}(A)$.

Then $d_{G}(x, y) > 4(i+1 - j) S_{k+1}$ and $d_{G}(x, y) \leq \text{diam}(M) \leq 4S_{j}$ leading again to a contradiction.

**Lemma 8.8.** If $|Q_{i}| > \varepsilon|A|$ then $|Q_{i+1}| < (1 - \frac{1}{4D^{3}})|Q_{i}|$.

**Proof.** Observe that $Q_{i} = (A \cup \bigcup_{j=1}^{i} C_{j})$. By Lemma 8.5,

$$\frac{|\partial_{16S_{i}} Q_{i}|}{|Q_{i}|} < \frac{1}{2}.$$ 

Hence by Lemma 8.4 we have that

$$\sum_{q_{\alpha}^{i+1}, q_{\alpha}^{i+1} \in Q_{i}\backslash \partial_{8S_{i+1}}(Q_{i})} |B_{r(i+1)}(q_{\alpha}^{i+1})| > \frac{1}{2D^{3}}|Q_{i}|.$$ 

Let $q_{\alpha}^{i+1} \in Q_{i}\backslash \partial_{8S_{i+1}}(Q_{i})$ such that the ball $L = B_{r(i+1)}(q_{\alpha}^{i+1})$ does not intersect the set

$$B' = \partial_{4(i+1)} S_{i}(A) \cup \bigcup_{j=1}^{i}(B_{4(j-i)} S_{i}(C_{j}) \backslash C_{j}).$$

Then, $L$ is an $A$-covering ball of type-$(i + 1)$. On the other hand, if $L$ does intersect $B'$, then

$$L \subset \partial_{16NS_{i}}(A) \cup \bigcup_{j=1}^{i}(B_{16NS_{i+1}}(C_{j}) \backslash C_{j}).$$

Therefore by Lemma 8.6,

$$|C_{i+1}| > \frac{1}{2D^{3}}|Q_{i}| - \varepsilon|A| > \frac{1}{4D^{3}}|Q_{i}|.$$ 

That is,

$$|Q_{i+1}| = |Q_{i}\backslash C_{i+1}| < (1 - \frac{1}{4D^{3}})|Q_{i}| \quad \Box$$

By Lemma 8.8 and (20), Proposition (8.1) immediately follows. \Box
8.3. The class of D-doubling graphs is almost finite. The goal of this subsection is to prove the following proposition.

**Proposition 8.2.** The class of D-doubling graphs is almost finite.

*Proof.* Let $G$ be a $D$-doubling countable graph with vertex degree bound $d$. Fix $\varepsilon > 0$ and let $\varepsilon' = \frac{\varepsilon}{4D^{16}}$. Let $S_1 > S_2 > \cdots > S_N$ be a $(D, \varepsilon')$-good $N$-tuple of integers. Let $\mathcal{N}_i$ be the set of nice balls of type-$i$ obtained in the construction round of our Basic Algorithm of Subsection 8.2. Let $U \subset V(G)$ be the set of vertices $x$ such that $x$ is not contained in any nice ball $L \in \mathcal{N}_i$, $1 \leq i \leq N$. By (18), $i_G(L) < \varepsilon'$ if $L$ is a nice ball. Following [10], we will construct an injective map $\Psi : U \to V(G) \setminus U$ in such a way that for any nice ball $L$,

$$i_G(L \cup \Psi^{-1}(L)) < \varepsilon.$$ 

Also, we will assure that

$$\sup_{x \in V(G)} d_G(\Psi(x), x) < \infty.$$ 

Thus, we have a covering $V(G) = \bigcup_{1 \leq i \leq N} \bigcup_{L \in \mathcal{N}_i} (L \cup \Psi^{-1}(L))$ of the vertices of $G$ with disjoint sets of bounded diameter with isoperimetric constant less or equal than $\varepsilon$. This will show that $G$ is almost finite.

Now, let us construct $\Psi$. First of all, by Lemma 8.2 and Lemma 8.3, we have an integer $s > 0$ and a system $\mathcal{B} = \bigcup_{i=1}^{D^{16}} \mathcal{B}_i$ so that

- The elements of $\mathcal{B}_i$ are disjoint balls $B$ such that $s \leq \text{radius of } B \leq 2s$ and $|\partial_{i6NS_i}(B)| < \frac{s'}{10D^2}|B|$. Hence by Proposition 8.1, for any $B \in \mathcal{B}_i$, we have that $|U \cap B| < \varepsilon'|B|$.
- $\mathcal{B}$ covers $V(G)$.

For a nice ball $L$ and $1 \leq i \leq D^{16}$, let $L_1, L_2, \ldots, L_{D^{16}}$ be disjoint subsets of $L$ such that

$$2\varepsilon'|L| < |L_i| < 3\varepsilon'|L|.$$ 

Thus for any $1 \leq i \leq D^{16}$ and $B \in \mathcal{B}_i$

$$|U \cap B| < |\cup_{L \cap B} L_i|.$$ 

Let $\varphi_i : U \cap (\bigcup_{B \in \mathcal{B}_i} B) \to \bigcup_{B \in \mathcal{B}_i} \bigcup_{L \in \mathcal{B}_i} L_i$ be an arbitrary injective map such that if $x \in B_i$ then $\varphi_i(x) \in B_i$. Our injective map $\Psi : U \to V(G) \setminus U$ is defined as follows. If $x \in U$, let $\Psi(x) = \varphi_i(x)$ if $i$ is the smallest integer such that $x \in U \cap B$ for some ball $B \in \mathcal{B}_i$. Hence, if $L$ is a nice ball then

$$\partial_G(L \cup \Psi^{-1}(L)) \leq |\partial_G(L)| + |\Psi^{-1}(L)| \leq$$

$$\leq |\partial_G(L)| + 4\varepsilon'D^{16}|L| \leq \varepsilon|L|.$$ 

Therefore our proposition follows. \hfill \Box

**Remark 14.** Recently, Downarowicz and Zhang [11] proved that Cayley graphs of groups of subexponential growth are distributed almost finite. Nevertheless, they proof used the transitivity of the graphs in a significant way.
8.4. The class of D-doubling graphs is strongly almost finite. In this subsection we go one step further and strengthen Proposition 8.2.

**Proposition 8.3.** For any $D \geq 1$, the class of D-doubling graphs is even strongly almost finite.

**Proof.** We need to modify the Basic Algorithm of Subsection 8.2 to fit our purposes. Let $0 < \delta, \varepsilon < 1$ be real constants and $D$ be a positive integer. Also, let $N > 0$ be so large that $(1 - \frac{1}{4D^3})^N < \varepsilon$, $(1 - \frac{1}{4D^3})^N < \frac{\delta}{2}$. We call an $N$-tuple of positive integers $S_1 > S_2 > \cdots > S_N$ $(D, \varepsilon, \delta)$-good if there exist positive integers $\{R_i\}_{i=1}^N$ such that for any D-doubling graph $G$, for any integer $1 \leq i \leq N$ and $q \in V(G)$, there exists an integer $S_i \leq r_i(q) < 2S_i$, such that $r_i(q) + R_i < 2S_i$ and

\[
\frac{|B_{r_i(q) + j + 20NS_{i+1}}(q)|}{|B_{r_i(q)}(q)|} < 1 + \frac{\varepsilon}{10D^3}, \quad \text{if } 1 \leq i \leq N - 1, 1 \leq j \leq R_i.
\]

(27)

\[
i_G(B_{r_i(q)}(q)) < \varepsilon, \quad \text{if } 1 \leq i \leq N, 1 \leq j \leq R_i.
\]

(28)

\[
\text{For any } 1 \leq i \leq N - 1, S_i > 4S_{i+1} + 4S_{i+2} + \cdots + 4S_N.
\]

(29)

\[
\text{For any } 1 \leq i \leq N - 1, \quad \frac{10S_{i+1}}{R_i} < \frac{\delta}{2N}.
\]

(30)

Again, the existence of such $(D, \varepsilon, \delta)$-good tuples follows from Lemma 8.3. Now, we fix a $(D, \varepsilon, \delta)$-good tuple $S_1 > S_2 > \cdots > S_N$ and a system of integers $\{R_i\}_{i=1}^N$ satisfying the conditions above. Using Lemma 8.2, for each $1 \leq i \leq N$ we pick $D^{20}$ maximal systems of vertices $\{(q^{l,t}_\alpha)_{\alpha \in J, t}\}_{t=1}^{D^{20}}$ such that

- for any $1 \leq t \leq D^{20}$ and $\alpha \neq \beta$, $d_G(q^{l,t}_\alpha, q^{l,t}_\beta) > 8S_t$, 
- the balls $\{(B_{S_t}(q^{l,t}_\alpha))_{\alpha \in J, t}\}_{t=1}^{D^{20}}$ cover $V(G)$.

Then, using the definition of $(D, \varepsilon, \delta)$-goodness, for each $1 \leq i \leq N$, $1 \leq t \leq D^{20}$ and vertex $q^{l,t}_\alpha$, we pick an integer $S_i \leq r_i(q^{l,t}_\alpha) < 2S_i$ such that $r_i(q^{l,t}_\alpha) + R_i < 2S_i$ and

\[
\frac{|B_{r_i(q^{l,t}_\alpha) + j + 20NS_{i+1}}(q)|}{|B_{r_i(q^{l,t}_\alpha) + j}(q)|} < 1 + \frac{\varepsilon}{10D^3}, \quad \text{if } 1 \leq i \leq N - 1, 1 \leq j \leq R_i.
\]

(31)

\[
i_G(B_{r_i(q^{l,t}_\alpha) + j}(q)) < \varepsilon, \quad \text{if } 1 \leq i \leq N, 1 \leq j \leq R_i.
\]

(32)

We call an $N$-tuple $\{(t_i, j_i)\}_{i=1}^N$ a **code** if for any $1 \leq i \leq N$ we have that $1 \leq t_i \leq D^{20}$ and $1 \leq j_i \leq R_i$. Thus, the set of codes $M$ has $D^{20N} \prod_{i=1}^N R_i$ elements. For each code $\{(t_i, j_i)\}_{i=1}^N$ and $1 \leq i \leq N$, we have a system of chosen balls with centers $\{q^{l,t}_\alpha\}_{\alpha \in J, t_i}$ and radii $\{r_i(q^{l,t}_\alpha) + j_i\}_{\alpha \in J, t_i}$ and an associated $(\varepsilon, K_\varepsilon)$-partition given by the Basic Algorithm for a certain integer $K_\varepsilon > 0$ that does not depend on $G$. By the definition of strong almost finiteness, Proposition 8.3 easily follows from the proposition below.
Proposition 8.4. Let \( x \in V(G) \) and \( C_x \in M \) be the set of codes \( \{(t_i, j_i)\}_{i=1}^N \) for which there exists a nice ball \( B \) given by the Basic Algorithm, such that \( x \in B \) and \( x \notin \partial G(B) \). Then, \(|C_x| \geq (1-\delta)|M|\).

Proof. Let \( D_x \) be the set of codes \( \{(t_i, j_i)\}_{i=1}^N \) for which the chosen balls given by the Basic Algorithm are not covering \( x \). Also, let \( E_x \) be the set of codes \( \{(t_i, j_i)\}_{i=1}^N \) for which there exists \( 1 \leq i \leq N-1 \) and a chosen \( i \)-ball \( B = B_{r_i(q_i^t)+j_i(q_i^t)} \) so that

- \( B \cap B_{5S_{i+1}}(x) \neq \emptyset \) and
- \( B \) does not contain the ball \( B_{5S_{i+1}}(x) \).

Lemma 8.9. If \( \{(t_i, j_i)\}_{i=1}^N \notin D_x \cup E_x \), then \( \{(t_i, j_i)\}_{i=1}^N \in C_x \).

Proof. Suppose that \( \{(t_i, j_i)\}_{i=1}^N \notin D_x \cup E_x \). Let \( 1 \leq i \leq N \) be the smallest integer such that there exists a chosen \( i \)-ball \( B \) such that \( x \notin B \). Since our code is not in \( E_x \), \( x \notin \partial G(B) \). Also, there is no chosen \( j \)-ball, \( j < i \) which intersects \( B \). Hence, \( B \) is a nice ball and therefore \( \{(t_i, j_i)\}_{i=1}^N \in C_x \).

Lemma 8.10. \(|D_x| < \delta/2|M|.|\)

Proof. For any \( 1 \leq i \leq N \), there exists at least one \( 1 \leq t \leq D^{20} \) such that \( x \in \cup_{\alpha \in J_{i,t}} B_S(q_i^t) \). Hence, \(|D_x| \leq \prod_{i=1}^N (D^{20} - 1) R_i \). Thus,

\[
\frac{|D_x|}{|M|} < (1 - \frac{1}{D^{20}})^N < \frac{\delta}{2}.
\]

Lemma 8.11. \(|E_x| < \delta/2|M|.|\)

Proof. Fix an \( N \)-tuple \( \{t_i\}_{i=1}^N \), \( 1 \leq t_i \leq D^{20} \). Then, for any \( 1 \leq i \leq N-1 \), there exists at most one \( \alpha \in J_{i,t} \) such that both

\[
B_{r_i(q_{\alpha}^t)+j_i(q_{\alpha}^t)} \cap B_{5S_{i+1}}(x) \neq \emptyset
\]

(33)

\[
B_{r_i(q_{\alpha}^t)+j_i(q_{\alpha}^t)} \cap B_{5S_{i+1}}(x) \neq \emptyset
\]

(34)

hold for a certain element \( 1 \leq j \leq R_i \). Clearly, the number of \( j \)'s for which both (33) and (34) hold is not greater than \( 10S_{i+1} \). Thus,

\[
|E_x| \leq D^{20N} \left( \sum_{j=1}^{N-1} \frac{10S_{j+1}}{R_j} \right)^N \prod_{i=1}^N R_i
\]

Hence our lemma follows.

Now, Lemma 8.9, Lemma 8.10 and Lemma 8.11 immediately imply our proposition.
8.5. Distributed strong almost finiteness. Now, we finish the proof of Theorem 10. It is not hard to see that all the constructions in Proposition 8.2 and Proposition 8.3 can be done locally. Nevertheless, we show step by step how to build the partition families witnessing strong almost finiteness in a distributed fashion, using the simple subroutines described in Remark 12.

1. We pick the \( D^{20} \) maximal systems of vertices \( \{q^{(A)}_{\alpha, t}\}_{\alpha \in J_{t}} \) using an (A)-type local algorithm.
2. Using (B)-type local algorithms in the balls around the picked vertices we construct the chosen balls.
3. Again, using (B)-type local algorithms we construct the system of nice balls.
4. Now, we construct the system of balls \( \bigcup_{i=1}^{\mathcal{B}} \mathcal{B}_i \) as in the proof of Proposition 8.2 using (A)- and (B)-type local algorithms.
5. We construct the injective maps \( \Psi \) as in the proof of Proposition 8.2 for each nice ball system using a (B)-type local algorithm.

This finishes the proof of our theorem. \( \square \)

References

[1] M. Abért and G. Elek, The space of actions, partition metric, and combinatorial rigidity. preprint, arXiv:1108.2147v1.
[2] M. Abért and B. Weiss, Bernoulli actions are weakly contained in any free action. Ergodic Theory Dynam. Systems 33 (2013), no. 2, 323–333.
[3] D. Aldous and R. Lyons, Processes on unimodular random networks. Electron. J. Probab. 12 (2007), no.54, 1454–1508.
[4] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids. Monographies de L’Enseignement Mathématique 36, L’Enseignement Mathématique, Geneva, (2000)
[5] M. Astrand, V. Polischuk, J. Rybicki, J. Suomela and J. Uitto, Local algorithms in (weakly) coloured graphs. preprint arXiv:1002.0125
[6] L. Barenboim and M. Elkin, Distributed graph coloring. Fundamentals and recent developments. Synthesis Lectures on Distributed Computing Theory, 11. Morgan & Claypool Publishers, Williston, VT, (2013).
[7] I. Benjamini and O. Schramm, Recurrence of distributional limits of finite planar graphs. Electron. J. Probab. 6, (2001), no. 23, 1–13.
[8] P. Burton and A. S. Kechris, Weak containment of measure preserving group actions. preprint arXiv:1611.07921
[9] M. Dadarlat and E. Guentner, Uniform embeddability of relatively hyperbolic groups. J. Reine Angew. Math. 612 (2007), 1–15.
[10] T. Downarowicz, D. Huczek and G. Zhang, Tilings of amenable groups J. Reine Angew. Math. (to appear)
[11] T. Downarowicz and G. Zhang, The comparison property of amenable groups. preprint arXiv:1712.05129
[12] G. Elek, The combinatorial cost. Enseign. Math. (2) 53 (2007), no. 3-4, 225–235.
[13] G. Elek and G. Lippner, Borel oracles. An analytical approach to constant-time algorithms. Proc. Amer. Math. Soc. 138 (2010), no. 8, 2939–2947.
[14] G. Elek, Finite graphs and amenability. J. Funct. Anal. 263 (2012), no. 9, 2593–2614.
[15] G. Elek, Uniformly recurrent subgroups and simple \( C^* \)-algebras. J. Funct. Anal. 274 (2018), no. 6, 1657–1689.
[16] G. Elek, Free minimal actions of countable groups with invariant probability measures preprint arXiv:1805.11149
[17] G. Even, M. Medina and D. Ron, Distributed Maximum Matching in Bounded Degree Graphs Proceedings of the 2015 International Conference on Distributed Computing and Networking, ICDCN 2015, Goa, India, January 4-7, 2015 1–10.
[18] A. Goldberg, S. Plotkin and G. Shannon, Parallel symmetry-breaking in sparse graphs. SIAM Journal on Discrete Mathematics 1 (1988), no. 4, 434–446.
[19] O. Goldreich and D. Ron, Property testing in bounded degree graphs. Algorithmica 32 (2002), no. 2, 302–343.
[20] E. I. Gordon and A. M. Vershik, Groups that are locally embeddable in the class of finite groups. St. Petersburg Math. J. 9 (1998), no. 1, 49–67
[21] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces. With appendices by M. Katz, P. Pansu and S. Semmes. Progress in Math. 152 Birkhauser, Boston MA (2001)
[22] E. Glasner and B. Weiss, Uniformly recurrent subgroups. Recent trends in ergodic theory and dynamical systems, 63–75, Contemp. Math., 631, Amer. Math. Soc., Providence, RI, (2015).
[23] H. Hatami, L. Lovász and B. Szegedy, Limits of locally-globally convergent graph sequences. Geom. Funct. Anal. 24 no. 1, (2014) 269–296.
[24] A. Hassidim, J. Kelner, H. N. Nguyen and K. Onak Local graph partitions for approximation and testing. (2009) 50th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2009, 22–31, IEEE Computer Soc., Los Alamitos, CA
[25] N. Higson and J. Roe, Amenable group actions and the Novikov conjecture. J. Reine. Angew. Math. 519 (2000) 143–153.
[26] D. Kerr, Dimension, comparison, and almost finiteness. preprint, arXiv:1710.00393
[27] U. Lang and T. Schlichenmaier, Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions. Int. Math. Res. Not. (2005), no. 5, 3625–3655.
[28] H. Lin and H. Matui, Minimal dynamical systems and approximate conjugacy. Math. Ann. 332 (2005), no. 4, 795–822
[29] N. Linial, Locality in distributed graph algorithms. SIAM J. Comput. 21 (1992), no. 1, 193–201.
[30] L. Lovász and B. Szegedy, Limits of dense graph sequences. J. Combin. Theory Ser. B 96 (2006), no. 6, 933–957.
[31] L. Lovász, Large networks and graph limits. AMS. Coll. Publ. 60 Amer. Math. Soc. Providence, RI, (2012)
[32] L. Lovász, Hyperfinite graphings and combinatorial optimization. preprint arXiv:1709.03179
[33] N. A. Lynch, Distributed algorithms. The Morgan Kaufmann Series in Data Management Systems. Morgan Kaufmann, San Francisco, CA, (1996).
[34] H. Matui, Homology and topological full groups of étale groupoids on totally disconnected spaces. Proc. Lond. Math. Soc. (3) 104 (2012), no. 1, 27–56.
[35] H. N. Nguyen and K. Onak, Constant-Time Approximation Algorithms via Local Improvements, 49th Annual IEEE Symposium on Foundations of Computer Science, FOCS 327–336. (2008), IEEE, 327–336.
[36] T. Shimomura, Special homeomorphisms and approximation for Cantor systems. Topology Appl. 161 (2014), 178–195.
[37] Y. Suzuki, Almost finiteness for general étale groupoids and its applications to stable rank of crossed products. Int. Math. Res. Not. (to appear)
[38] R. D. Tucker-Drob, Weak equivalence and non-classifiability of measure preserving actions. Ergodic Theory Dynam. Systems 35 (2015), no. 1, 293–336.
DEPARTMENT OF MATHEMATICS AND STATISTICS, FYLDE COLLEGE, LANCASTER UNIVERSITY, LANCAS TER, LA1 4YF, UNITED KINGDOM

E-mail address: g.elek@lancaster.ac.uk