COMPOSITIONS OF
BIRATIONAL ENDOMORPHISMS OF THE AFFINE PLANE

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Abstract. Besides contributing several new results in the general theory of birational endomorphisms of $\mathbb{A}^2$, this article describes certain classes of birational endomorphisms $f : \mathbb{A}^2 \to \mathbb{A}^2$ defined by requiring that the missing curves or contracting curves of $f$ are lines. The last part of the article is concerned with the monoid structure of the set of birational endomorphisms of $\mathbb{A}^2$.

Let $k$ be an algebraically closed field and let $\mathbb{A}^2 = \mathbb{A}^2_k$ be the affine plane over $k$. A birational endomorphism of $\mathbb{A}^2$ is a morphism of algebraic varieties, $f : \mathbb{A}^2 \to \mathbb{A}^2$, which restricts to an isomorphism $U \to V$ for some nonempty Zariski-open subsets $U$ and $V$ of $\mathbb{A}^2$. The set $\text{Bir}(\mathbb{A}^2)$ of birational endomorphisms of $\mathbb{A}^2$ is a monoid under composition of morphisms, and the group of invertible elements of this monoid is the automorphism group $\text{Aut}(\mathbb{A}^2)$. An element $f$ of $\text{Bir}(\mathbb{A}^2)$ is irreducible if it is not invertible and if, for every factorization $f = h \circ g$ with $g, h \in \text{Bir}(\mathbb{A}^2)$, one of $g, h$ is invertible. Elements $f, g \in \text{Bir}(\mathbb{A}^2)$ are equivalent (denoted $f \sim g$) if there exist $u, v \in \text{Aut}(\mathbb{A}^2)$ satisfying $u \circ f \circ v = g$. The elements of $\text{Bir}(\mathbb{A}^2)$ which are equivalent to the birational morphism $c : \mathbb{A}^2 \to \mathbb{A}^2, c(x, y) = (x, xy)$, are called simple affine contractions (SAC), and are the simplest examples of non invertible elements of $\text{Bir}(\mathbb{A}^2)$. It was at one time an open question whether $\text{Aut}(\mathbb{A}^2) \cup \{c\}$ generated $\text{Bir}(\mathbb{A}^2)$ as a monoid (the question arose in Abhyankar’s seminar at Purdue in the early 70s); P. Russell showed that the answer was negative by giving an example (which appeared later in [8, 4.7]) of an irreducible element of $\text{Bir}(\mathbb{A}^2)$ which is not a SAC. This was the first indication that $\text{Bir}(\mathbb{A}^2)$ could be a complicated object.

Papers [8] and [9] seem to be the first publications that study birational endomorphisms of $\mathbb{A}^2$ in a systematic way (these are our main references). We know of two more contributions to the subject: a certain family of elements of $\text{Bir}(\mathbb{A}^2)$ is described explicitly in [6], and [26] gives an algorithm for deciding whether a given element of $\text{Bir}(\mathbb{A}^2)$ is in the submonoid generated by $\text{Aut}(\mathbb{A}^2) \cup \{c\}$.

The list of references is much longer if we include another aspect of the problem. Indeed, there is a long history of studying polynomials $F \in k[X, Y]$ which appear as components of birational endomorphisms of $\mathbb{A}^2$. To explain this, we recall some definitions. A polynomial $F \in k[X, Y]$ is called a field generator if there exists $G \in k(X, Y)$ satisfying $k(F, G) = k(X, Y)$; in the special case where $G$ can be chosen in

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one says that $F$ is a good field generator. So a good field generator is just the same thing as a component of a birational endomorphism of $\mathbb{A}^2$. By a generally rational polynomial\(^1\) we mean an $F \in \mathbb{k}[X,Y]$ such that, for almost all $\lambda \in \mathbb{k}$, $F - \lambda$ is an irreducible polynomial whose zero-set in $\mathbb{A}^2$ is a rational curve (where “almost all” means “all but possibly finitely many”). If $\text{char} \, \mathbb{k} = 0$ then field generators and generally rational polynomials are one and the same thing (this is noted in [16]; see [7] for the positive characteristic case). The study of these polynomials is a classical subject, as is clear from considering the following (incomplete) list of references: [18], [19], [20], [23], [24], [13], [21], [22], [16], [15], [2], [17], [3], [25], [7], [5].

This paper is a contribution to the theory of birational endomorphisms of $\mathbb{A}^2$. Our methods are those of [8] and [9], and we place ourselves at the same level of generality as in those papers: the base field $\mathbb{k}$ is algebraically closed but otherwise arbitrary.

In [8], [9] and [6], there is a tendency to restrict one’s attention to irreducible elements of $\text{Bir}(\mathbb{A}^2)$. Going in an orthogonal direction, the present paper is mainly concerned with compositions of birational endomorphisms. This choice is motivated by many reasons. First, the examples given in [8], [9] and [6] show that $\text{Bir}(\mathbb{A}^2)$ contains a great diversity of irreducible elements of arbitrarily high complexity; since the task of finding all irreducible elements is probably hopeless, it seems to us that finding more examples of them might be less relevant than, say, trying to understand the monoid structure of $\text{Bir}(\mathbb{A}^2)$. Also, a significant portion of this paper is geared towards proving Theorem 3.15, which we need in the forthcoming [4] for proving the following fact: 

Let $\mathbb{k}$ be an arbitrary field and $A_0 \supset A_1 \supset \cdots$ an infinite, strictly descending sequence of rings such that (i) each $A_i$ is a polynomial ring in 2 variables over $\mathbb{k}$; (ii) all $A_i$ have the same field of fractions; and (iii) the ring $R = \bigcap_i A_i$ is not equal to $\mathbb{k}$; then $R = \mathbb{k}[F]$ for some $F$ is a good field generator of $A_0$ and a variable of $A_i$ for $i \gg 0$.

Moreover, if one is interested in field generators and generally rational polynomials, one should not restrict one’s attention to irreducible endomorphisms. In this respect we point out that the components of the morphisms described by Theorem 3.15 are precisely the “rational polynomials of simple type” listed in [17].

The paper is organized as follows.

Section 1 contains some preliminary observations on “admissible” and “weakly admissible” configurations of curves in $\mathbb{A}^2$.

Section 2 gives several new results in the general theory of birational endomorphisms of $\mathbb{A}^2$ (in particular 2.9, 2.13, 2.14 and 2.17, but also several useful lemmas).

Given $f \in \text{Bir}(\mathbb{A}^2)$, let $\text{Miss}(f)$ (resp. $\text{Cont}(f)$) be the set of missing curves (resp. contracting curves) of $f$; see 2.2 for definitions. Section 3 studies those $f \in \text{Bir}(\mathbb{A}^2)$ satisfying the condition that $\text{Miss}(f)$ is a weakly admissible configuration, or the stronger condition that $\text{Miss}(f)$ is an admissible configuration, or the even stronger condition that both $\text{Miss}(f)$ and $\text{Cont}(f)$ are admissible configurations. The main result of Section 3, Theorem 3.15, gives a complete description of these three subsets of $\text{Bir}(\mathbb{A}^2)$.

While Sections 2 and 3 are mainly concerned with individual endomorphisms, Section 4 considers the monoid structure of $\text{Bir}(\mathbb{A}^2)$. The first part of that section shows,

\(^1\)Generally rational polynomials are sometimes called “rational polynomials” or “generically rational polynomials”.\(^2\)
in particular, that if $S$ is a subset of $\text{Bir}(\mathbb{A}^2)$ such that $\text{Aut}(\mathbb{A}^2) \cup S$ generates $\text{Bir}(\mathbb{A}^2)$ as a monoid, then $\{ \deg f \mid f \in S \}$ is not bounded and $|S| = |k|$ (giving a very strong negative answer to the already mentioned question of Abhyankar). The second part shows that the submonoid $\mathcal{A}$ of $\text{Bir}(\mathbb{A}^2)$ generated by SACs and automorphisms is “factorially closed” in $\text{Bir}(\mathbb{A}^2)$, i.e., if $f, g \in \text{Bir}(\mathbb{A}^2)$ are such that $g \circ f \in \mathcal{A}$ then $f, g \in \mathcal{A}$.

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Conventions. All algebraic varieties (in particular all curves and surfaces) are irreducible and reduced. All varieties and morphisms are over an algebraically closed field $k$ of arbitrary characteristic ($k$ is assumed to be algebraically closed from 1.3 until the end of the paper). The word “point” means “closed point”, unless otherwise specified. All rings are commutative and have a unity. The symbol $A^*$ denotes the set of units of a ring $A$. If $A$ is a subring of a ring $B$ and $n \in \mathbb{N}$, the notation $B = A[n]$ means that $B$ is isomorphic (as an $A$-algebra) to the polynomial ring in $n$ variables over $A$.

1. Admissible configurations of curves in $\mathbb{A}^2$

Recall the following terminology. Let $k$ be a field, $A = k^2$, and $\mathbb{A}^2_k = \text{Spec} A$. We abbreviate $\mathbb{A}^2_k$ to $\mathbb{A}^2$. By a coordinate system of $A$, we mean an ordered pair $(F,G)$ of elements of $A$ satisfying $A = k[F,G]$. A variable of $A$ is an element $F \in A$ for which there exists $G$ satisfying $k[F,G] = A$.

Let $F \in A$ be an irreducible element and let $C \subset \mathbb{A}^2$ be the zero-set of $F$ (i.e., the set of prime ideals $p \in \text{Spec} A = \mathbb{A}^2$ satisfying $F \in p$); we call $C$ a line if $A/FA = k^1$, and a coordinate line if $F$ is a variable of $A$. Note that $C$ is a line iff $C \cong \mathbb{A}^1$; given a coordinate system $(X,Y)$ of $A$, $C$ is a coordinate line iff some automorphism of $\mathbb{A}^2$ maps $C$ onto the zero-set of $X$. It is clear that all coordinate lines are lines, and the Epimorphism Theorem ([1], [27]) states that the converse is true if $\text{char} \ k = 0$. It is known that not all lines are coordinate lines if $\text{char} \ k \neq 0$ (on the subject of lines which are not coordinate lines, see e.g. [12] for a recent survey). Coordinate lines are sometimes called rectifiable lines.

By a coordinate system of $\mathbb{A}^2 = \text{Spec} A$, we mean a coordinate system of $A$. That is, a coordinate system of $\mathbb{A}^2$ is a pair $(X,Y) \in A \times A$ satisfying $A = k[X,Y]$.

We adopt the viewpoint that $A$ (or $\mathbb{A}^2$) does not come equipped with a preferred coordinate system, i.e., no coordinate system is better than the others. This may be confusing to some readers, especially those who like to identify $\mathbb{A}^2$ with $k^2$, because any such identification inevitably depends on the choice of a coordinate system. So perhaps the following trivial remarks (1.1) deserve to be made.
1.1. Let $\mathcal{C}$ denote the set of coordinate systems of $\mathbb{A}^2$ (or $A$).

(a) Given $c = (X, Y) \in \mathcal{C}$ and an element $F \in A$, one can consider the map $k^2 \to k$, $(x, y) \mapsto F(x, y)$, defined by first writing $F = \sum_{i,j} a_{ij} X^i Y^j$ with $a_{ij} \in k$ (recall that $A = k[X, Y]$) and then setting $F(x, y) = \sum_{i,j} a_{ij} x^i y^j$ for $(x, y) \in k^2$. One can then consider the zero-set $Z(F) \subseteq k^2$ of that map $F$. We stress that the map $(x, y) \mapsto F(x, y)$ and the set $Z(F)$ depend on both $F$ and $c$; we should write $F_c(x, y)$ and $Z_c(F)$, but we omit the $c$.

(b) Let $P, Q \in k[T_1, T_2], \ P = \sum_{i,j} a_{ij} T_1^i T_2^j, \ Q = \sum_{i,j} b_{ij} T_1^i T_2^j, \ a_{ij}, b_{ij} \in k$.

(i) The pair $(P, Q)$ determines the map $k^2 \to k^2$, $(x, y) \mapsto (P(x, y), Q(x, y))$, where we define $P(x, y) = \sum_{i,j} a_{ij} x^i y^j$ and $Q(x, y) = \sum_{i,j} b_{ij} x^i y^j$.

(ii) Choose $c = (X, Y) \in \mathcal{C}$. Then $(P, Q, c)$ determines the morphism of schemes $f : \mathbb{A}^2 \to \mathbb{A}^2$ defined by stipulating that $f$ corresponds to the $k$-homomorphism $A \to A$ which maps $X$ to $P(X, Y) = \sum_{i,j} a_{ij} X^i Y^j$ and $Y$ to $Q(X, Y) = \sum_{i,j} b_{ij} X^i Y^j$ $(P(X, Y), Q(X, Y) \in A = k[X, Y])$.

(c) Suppose that $c = (X, Y) \in \mathcal{C}$ has been chosen. Then it is convenient to define morphisms of schemes $\mathbb{A}^2 \to \mathbb{A}^2$ simply by giving the corresponding polynomial maps $k^2 \to k^2$ (this will be done repeatedly in Section 3). To do so, we abuse language as follows: given $P, Q \in k[T_1, T_2]$, the sentence

“$f : \mathbb{A}^2 \to \mathbb{A}^2$ is defined by $f(x, y) = (P(x, y), Q(x, y))$”

means that $f : \mathbb{A}^2 \to \mathbb{A}^2$ is the morphism of schemes determined by $(P, Q, c)$ as in remark (b). For instance one can define $f : \mathbb{A}^2 \to \mathbb{A}^2$ by $f(x, y) = (x, xy)$, always keeping in mind that this $f$ depends on $c$.

1.2. Lemma. Let $F, G \in A = k^{[2]}$, where $k$ is any field, and suppose that each of $F, G$ is a variable of $A$. Consider the ideal $(F, G)$ of $A$ generated by $F$ and $G$.

(a) If $(F, G) = A$ then $G = aF + b$ for some $a, b \in k^*$.  

(b) If $A/(F, G) = k$ then $A = k[F, G]$.

Proof. Choose $Y$ such that $A = k[F, Y]$ and define $X = F$. Then $A = k[X, Y]$ and we may write $G$ as a polynomial in $X$ and $Y$, say $G = P(X, Y)$.

Suppose that $(F, G) = A$. Then $1 \in (F, G) = (X, P(X, Y)) = (X, P(0, Y))$ implies $P(0, Y) \in k^*$. Writing $P(0, Y) = b \in k^*$, we obtain that $G - b = P(X, Y) - P(0, Y)$ is divisible by $X$; as $G - b$ is irreducible, $G - b = aX = aF$ with $a \in k^*$, and (a) is proved.

To prove (b), we first observe that the fact that $G = P(X, Y)$ is a variable of $k[X, Y]$ and $P(X, Y) \not\in k[X]$ implies that $P$ is “almost monic” in $Y$, i.e.,

\begin{equation}
P(X, Y) = aY^d + p_1(X)Y^{d-1} + \cdots + p_d(X)
\end{equation}

with $d \geq 1$, $a \in k^*$ and $p_i(X) \subseteq k[X]$ for $i = 1, \ldots, d$. Now

$k = A/(F, G) = k[X, Y]/(X, P(X, Y)) = k[X, Y]/(X, P(0, Y))$

implies that $\deg P(0, Y) = 1$, so $d = 1$ in (1). Then $G = aY + p_1(X)$ and $k[F, G] = k[X, aY + p_1(X)] = k[X, Y] = A$. □
From now-on, and until the end of this paper, we assume that \( k \) is an algebraically closed field of arbitrary characteristic. Consider \( \mathbb{A}^2 = \mathbb{A}^2_k \).

1.3. Definition. Let \( C_1, \ldots, C_n \ (n \geq 0) \) be distinct curves in \( \mathbb{A}^2 \), and consider the set \( S = \{C_1, \ldots, C_n\} \). We say that \( S \) is a weakly admissible configuration if

(a) each \( C_i \) is a coordinate line;
(b) for every choice of \( i \neq j \) such that \( C_i \cap C_j \neq \emptyset \), \( C_i \cap C_j \) is one point and the local intersection number of \( C_i \) and \( C_j \) at that point is equal to 1.

1.4. Lemma. Given distinct curves \( C_1, \ldots, C_n \ (n \geq 0) \) in \( \mathbb{A}^2 \), the following are equivalent:

(a) \( \{C_1, \ldots, C_n\} \) is a weakly admissible configuration;
(b) there exists a coordinate system of \( \mathbb{A}^2 \) with respect to which all \( C_i \) have degree 1.

Proof. We show that (a) implies (b), the converse being trivial. Suppose that (a) holds. Write \( \mathbb{A}^2 = \text{Spec } A \) where \( A = k[t] \). We may assume that \( n \geq 2 \), otherwise the assertion is trivial. Let \( F_1, \ldots, F_n \in A \) be variables of \( A \) whose zero-sets are \( C_1, \ldots, C_n \) respectively. Condition (a) implies that, whenever \( i \neq j \), we have either \( (F_i, F_j) = A \) or \( A/(F_i, F_j) = k \). Consider the graph \( G \) whose vertex-set is \( \{F_1, \ldots, F_n\} \) and in which distinct vertices \( F_i, F_j \) are joined by an edge if and only if \( A/(F_i, F_j) = k \).

In the case where \( G \) is discrete, 1.2 implies that \( F_i = a_iF_1 + b_i \), \( a_i, b_i \in k^* \), for \( i = 2, \ldots, n \). Let \( X = F_1 \) and let \( Y \) be such that \( A = k[X, Y] \). Then all \( F_i \) have degree 1 with respect to the coordinate system \( (X, Y) \).

From now-on, assume that \( G \) is not discrete. Then we may assume that \( F_1, F_2 \) are joined by an edge. By 1.2, \( k[F_1, F_2] = A \). Let \( X_1 = F_1 \) and \( X_2 = F_2 \), then \( A = k[X_1, X_2] \) and for each \( i \in \{3, \ldots, n\} \) we have:

- if \( F_i, F_1 \) are not joined by an edge then 1.2 implies that \( F_i = a_iX_1 + b_i \) for some \( a_i, b_i \in k^* \), so \( F_i \) has degree 1 with respect to \( (X_1, X_2) \);
- if \( F_i, F_2 \) are not joined by an edge then \( F_i = a_iX_2 + b_i \) for some \( a_i, b_i \in k^* \), so \( F_i \) has degree 1 with respect to \( (X_1, X_2) \);
- if \( F_i \) is linked to each of \( F_1, F_2 \) by edges, then \( k[F_i, F_1] = A = k[F_1, F_2] \), so \( F_2 = aF_1 + \beta(F_i) \) for some \( a \in k^* \) and \( \beta(T) \in k[T] \); then \( \beta(F_i) = X_2 - aX_1 \) has degree 1 with respect to \( (X_1, X_2) \) and consequently \( F_i \) has degree 1.

So all \( F_i \) have degree 1 with respect to the coordinate system \( (X_1, X_2) \). \( \square \)

1.5. Let \( C_1, \ldots, C_n \ (n \geq 0) \) be distinct curves in a nonsingular surface \( W \). We say that \( \sum_{i=1}^n C_i \) is an SNC-divisor of \( W \) if:

- each \( C_i \) is a nonsingular curve;
- for every choice of \( i \neq j \) such that \( C_i \cap C_j \neq \emptyset \), \( C_i \cap C_j \) is one point and the local intersection number of \( C_i \) and \( C_j \) at that point is equal to 1;
- if \( i, j, k \) are distinct then \( C_i \cap C_j \cap C_k = \emptyset \).

If \( D = \sum_{i=1}^n C_i \) is an SNC-divisor of \( W \) we write \( \mathcal{G}(D, W) \) for the graph whose vertex set is \( \{C_1, \ldots, C_n\} \) and in which distinct vertices \( C_i, C_j \) are joined by an edge if and only if \( C_i \cap C_j \neq \emptyset \).
1.6. **Definition.** Let \( C_1, \ldots, C_n \ (n \geq 0) \) be distinct curves in \( \mathbb{A}^2 \). We say that the set \( \{C_1, \ldots, C_n\} \) is an admissible configuration if

(a) each \( C_i \) is a coordinate line;
(b) \( D = \sum_{i=1}^n C_i \) is an SNC-divisor of \( \mathbb{A}^2 \);
(c) the graph \( G(D, \mathbb{A}^2) \) defined in 1.5 is a forest.

1.7. **Proposition.** Given distinct curves \( C_1, \ldots, C_n \ (n \geq 0) \) in \( \mathbb{A}^2 \), the following are equivalent:

(a) \( \{C_1, \ldots, C_n\} \) is an admissible configuration;
(b) there exists a coordinate system \( (X, Y) \) of \( \mathbb{A}^2 \) such that \( \bigcup_{i=1}^n C_i \) is the zero-set of \( \varphi(X)Y^j \) for some \( j \in \{0, 1\} \) and some \( \varphi(X) \in k[X] \setminus \{0\} \).

**Proof.** It’s enough to show that (a) implies (b), as the converse is trivial. Assume that (a) holds. By 1.4, we may choose a coordinate system which respect to which all \( C_i \) have degree 1. As \( D = \sum_{i=1}^n C_i \) is an SNC-divisor and \( G(D, \mathbb{A}^2) \) is a forest, \( \bigcup_{i=1}^n C_i \) must be one of the following:

- a union of \( n \) parallel lines;
- a union of \( n - 1 \) parallel lines with another line.

Indeed, any other configuration of lines would either contain three concurrent lines or produce a circuit in the graph. Now it is clear that (b) is satisfied. \( \square \)

2. **Birational morphisms** \( f : X \to Y \) **of nonsingular surfaces**

**with special emphasis on the case** \( X = Y = \mathbb{A}^2 \)

Throughout this section, \( k \) is an algebraically closed field of arbitrary characteristic and we abbreviate \( \mathbb{A}_k^n \) to \( \mathbb{A}^n \). We consider birational morphisms \( f : X \to Y \), where \( X \) and \( Y \) are nonsingular algebraic surfaces over \( k \) (a morphism \( f : X \to Y \) is birational if there exist Zarisky-open subsets \( \emptyset \neq U \subseteq X \) and \( \emptyset \neq V \subseteq Y \) such that \( f \) restricts to an isomorphism \( U \to V \)). We are particularly interested in the case \( X = \mathbb{A}^2 = Y \).

Essentially all the material given in 2.1–2.8 can be found in [8]. From 2.9 to the end of the section, the material appears to be new (except 2.13(a)).

2.1. Let \( f : X \to Y \) and \( f' : X' \to Y' \) be birational morphisms of nonsingular surfaces. We say that \( f, f' \) are equivalent \( (f \sim f') \) if there exist isomorphisms \( x : X \to X' \) and \( y : Y \to Y' \) such that \( y \circ f = f' \circ x \).

2.2 ([8, 1.2]). Let \( f : X \to Y \) be a birational morphism of nonsingular surfaces. A **missing curve** of \( f \) is a curve \( C \subset Y \) whose intersection with the image of \( f \) is a finite set of points. We write \( \text{Miss}(f) \) for the set of missing curves of \( f \), \( q(f) \) for the cardinality of \( \text{Miss}(f) \) and \( q_0(f) \) for the number of missing curves of \( f \) which are disjoint from \( f(X) \). A **contracting curve** of \( f \) is a curve \( C \subset X \) such that \( f(C) \) is a point. The set of contracting curves of \( f \) is denoted \( \text{Cont}(f) \), and \( c(f) \) denotes the cardinality of \( \text{Cont}(f) \). The natural numbers \( q(f), q_0(f) \) and \( c(f) \) are invariant with respect to equivalence (2.1) of birational morphisms, i.e., \( f \sim f' \Rightarrow c(f) = c(f') \) and similarly for \( q \) and \( q_0 \). Call a point of \( Y \) a **fundamental point** of \( f \) if it is \( f(C) \) for some contracting
curve $C$ of $f$. The set of fundamental points of $f$ is also called the center of $f$, denoted $\text{cent}(f)$. The exceptional locus of $f$ is defined to be $\text{exc}(f) = f^{-1}(\text{cent}(f))$, and is equal to the union of the contracting curves of $f$.

2.3 ([8, 1.1 and 1.2]). Let $f : X \to Y$ be a birational morphism of nonsingular surfaces. There exists a commutative diagram

$$
\begin{array}{ccc}
Y_n & \xrightarrow{\pi_n} & Y_0 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

where "\hookrightarrow" denotes an open immersion and, for each $i$, $\pi_i : Y_i \to Y_{i-1}$ is the blowing-up of the nonsingular surface $Y_{i-1}$ at a point $P_i \in Y_{i-1}$.

Define $n(f)$ to be the least natural number $n$ for which there exists a diagram (2). Note that $n(f)$ is invariant with respect to equivalence of birational morphisms.

For each $i \in \{1, \ldots, n\}$, consider the exceptional curve $E_i = \pi_i^{-1}(P_i) \subset Y_i$, and let the same symbol $E_i$ also denote the strict transform of $E_i$ in $Y_n$. It is clear that the union of the contracting curves of $f$ is the intersection of $E_1 \cup \cdots \cup E_n \subset Y_n$ with the open subset $X$ of $Y_n$; thus:

$$
c(f) \leq n(f),
$$

(4) each contracting curve is nonsingular and rational, $D = \sum_{C \in \text{Cont}(f)} C$ is an SNC-divisor of $X$ and the graph $\mathcal{G}(D, X)$ is a forest.\(^2\)

Given $i \in \{1, \ldots, n\}$, consider the complete curve $E_i \subset Y_n$. Note that if $S$ is a projective nonsingular surface and $\mu : Y_n \hookrightarrow S$ is an open immersion, the self-intersection number of $\mu(E_i)$ in $S$ is independent of the choice of $(S, \mu)$; we denote this number by $(E^2_i)_{Y_n}$. Then the following holds (cf. [8, 1.2(c)]):

(5) Diagram (2) satisfies $n = n(f)$ if and only if $(E^2_i)_{Y_n} \leq -2$ holds for all $i \in \{1, \ldots, n\}$ such that $E_i \subset Y_n \setminus X$.

In particular, if Diagram (2) satisfies $n = n(f)$ then:

$$
\text{cent}(f) = \{ \pi_1 \circ \cdots \circ \pi_{i-1}(P_i) \mid 1 \leq i \leq n \}.
$$

The following remarks are trivial consequences of 2.3, but are very useful:

2.4. Remarks. Let $f : X \to Y$ be a birational morphism of nonsingular surfaces.

(a) For each $C \in \text{Miss}(f)$, we have $C \cap f(X) \subseteq \text{cent}(f)$. In particular, the condition $q_0(f) = 0$ is equivalent to “every missing curve contains a fundamental point”.

(b) Let $C \subset Y$ be a curve. Then there exists at most one curve $C' \subset X$ such that $f(C')$ is a dense subset of $C$. Moreover, $C'$ exists if and only if $C \notin \text{Miss}(f)$.

\(^2\)Note that contracting curves are not necessarily isomorphic to $\mathbb{A}^1$. So, in the case $X = \mathbb{A}^2 = Y$, $\text{Cont}(f)$ is not necessarily an admissible configuration in the sense of 1.6.
2.5. **Lemma.** If $\mathbb{A}^2 \xrightarrow{f} \mathbb{A}^2 \xrightarrow{g} \mathbb{A}^2$ are birational morphisms then $n(g \circ f) = n(g) + n(f)$.

*Proof.* Follows from [8, 2.12].

2.6. **Lemma.** Let $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be a birational morphism.

(a) $q(f) = c(f) \leq n(f)$
(b) $f$ is an isomorphism $\iff n(f) = 0 \iff c(f) = 0 \iff q(f) = 0$.
(c) Each missing curve of $f$ is rational with one place at infinity.
(d) Each fundamental point belongs to some missing curve; each missing curve contains some fundamental point.
(e) If a point $P$ is a singular point of some missing curve of $f$, or a common point of two missing curves, then $P$ is a fundamental point of $f$.

*Proof.* Equality $q(f) = c(f)$ in (a) follows from [8, 4.3(a)], and $c(f) \leq n(f)$ was noted in (3). Assertion (b) follows from the observation that if $n(f) = 0$ or $c(f) = 0$ then $f$ is an open immersion $\mathbb{A}^2 \hookrightarrow \mathbb{A}^2$ and hence an automorphism. Assertion (c) follows from result 4.3(c) of [8]. The first (resp. the second) assertion of (d) follows from [8, 2.1] (resp. from the claim that $q_0(f) = 0$, in [8, 4.3(a)]). Refer to [10, 1.6] for a proof of assertion (e). □

Several of the above facts are stated in greater generality in [8]. For instance, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are birational morphisms of nonsingular surfaces and $q_0(f) = 0$, then (by [8, 1.3]) $n(g \circ f) = n(g) + n(f)$. Also, if $X, Y$ are nonsingular surfaces satisfying $\Gamma(X, \mathcal{O}_X)^* = k^*$ and $\text{Cl}(Y) = 0$ then (by [8, 2.11]) every birational morphism $f : X \rightarrow Y$ satisfies $q_0(f) = 0$. The following generalization of 2.6(a) also deserves to be noted:

2.7. **Lemma.** Let $f : X \rightarrow X$ be a birational morphism, where $X$ is a nonsingular surface. Then $c(f) = q(f)$.

*Proof.* Follows from Remark 2.13 of [8]. □

2.8. **Lemma.** Let $f : X \rightarrow Y$ be a birational morphism of nonsingular surfaces and $\Gamma_f$ the union of the missing curves of $f$. If $X$ is affine then the following hold:

(a) $\text{cent}(f) \subseteq \Gamma_f$;
(b) $Y \setminus \Gamma_f$ is the interior of $f(X)$ and $f^{-1}(Y \setminus \Gamma_f) = X \setminus \text{exc}(f)$;
(c) the surfaces $X \setminus \text{exc}(f)$ and $Y \setminus \Gamma_f$ are affine, and $f$ restricts to an isomorphism $X \setminus \text{exc}(f) \rightarrow Y \setminus \Gamma_f$.

*Proof.* Follows from [8, Prop. 2.1] and its proof. □

2.9. **Proposition.** Let $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be a birational morphism. If $P$ is a singular point of a missing curve of $f$, then $P$ belongs to at least two missing curves of $f$.

*Proof.* By 2.6(e), $P$ is a fundamental point of $f$; so it suffices to show that if a fundamental point $P$ belongs to only one missing curve $C$, then the multiplicity $\mu(P, C)$ of $C$ at $P$ is equal to 1. So assume that $P$ is a fundamental point which belongs to only one missing curve $C$. Choose a diagram (2) satisfying $n = n(f)$, and let the notation
Let us label the missing curves as \( C_1, \ldots, C_q \), where
\[
P_1 \in C_j \iff j = 1.\]

The diagram (2) together with the ordering \( (C_1, \ldots, C_q) \) of the set of missing curves constitutes a “minimal decomposition” of \( f \), in the terminology of [8, 1.2(h)]. This minimal decomposition \( D \) determines matrices \( \mu_D, \varepsilon_D, \varepsilon'_D \), defined in [8, 2.8].

These are matrices with entries in \( \mathbb{N} \), and result [8, 4.3(b)] asserts that the product \( \varepsilon'_D \mu_D \) is a square matrix of determinant \( \pm 1 \). We shall now argue that the condition
\[
\det(\varepsilon'_D \mu_D) = \pm 1
\]
implies that \( \mu(P_1, C_1) = 1 \) (this will complete the proof of the proposition).

By (7), the \( n \times n \) matrix \( \varepsilon_D \) has the following shape:
\[
\varepsilon_D = (\varepsilon_{ij}) = \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_1 \end{pmatrix}
\]
(with \( \varepsilon_{ij} \in \mathbb{N} \) for all \( i, j \))

where \( \varepsilon_0 \) is an \( s \times s \) lower-triangular matrix with zero diagonal, and where

the first row is the only zero row of \( \varepsilon_0 \).

Consider the \( n \times n \) matrix \( \varepsilon_D \), determined by \( \varepsilon_D \) as explained in [8, 2.7]. The already mentioned properties of \( \varepsilon_D \) imply that \( \varepsilon_D \) is as follows:
\[
\varepsilon_D = (\varepsilon_{ij}) = \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_1 \end{pmatrix}
\]
(with \( \varepsilon_{ij} \in \mathbb{N} \) for all \( i, j \))

where \( \varepsilon_0 \) is an \( s \times s \) lower-triangular matrix with diagonal entries equal to 1, and where

all entries in the first column of \( \varepsilon_0 \) are positive.

Next, \( \varepsilon'_D \) is the submatrix of \( \varepsilon_D \) obtained by deleting the \( i \)-th row for each \( i \in J \), where
\[
J = \{ i \mid 1 \leq i \leq n, \ E_i \cap X = \emptyset \ \text{in} \ Y_n \}
\]
in the notation of 2.3 (\( J \) is defined in [8, 1.2(h)]). Let \( j_0 = |J \cap \{1, \ldots, s\}| \); then the \( (n - |J|) \times n \) matrix \( \varepsilon'_D \) has the form
\[
\varepsilon'_D = \begin{pmatrix} \varepsilon'_0 & 0 \\ 0 & \varepsilon'_1 \end{pmatrix},
\]
where \( \varepsilon'_0 \) is an \( (s - j_0) \times s \) matrix with entries in \( \mathbb{N} \) and
\[
\text{all entries in the first column of } \varepsilon'_0 \text{ are positive.}
\]

Finally, consider the \( n \times q \) matrix \( \mu_D \); by (8),
\[
\mu_D = \begin{pmatrix} F & 0 \\ G & H \end{pmatrix}
\]
where \( F = \begin{pmatrix} \mu(P_1, C_1) \\ \vdots \\ \mu(P_s, C_1) \end{pmatrix} \) is \( s \times 1 \).

We have
\[
\varepsilon'_D \mu_D = \begin{pmatrix} \varepsilon'_0 & 0 \\ \varepsilon'_1 \end{pmatrix} \begin{pmatrix} F & 0 \\ G & H \end{pmatrix} = \begin{pmatrix} \varepsilon'_0 F \\ \varepsilon'_1 G \end{pmatrix} \begin{pmatrix} 0 \\ \varepsilon'_1 H \end{pmatrix},
\]
where the block $\varepsilon'_0F$ has size $(s - j_0) \times 1$. By (7), $(E'_s)_Y = -1$; so $E_s \not\subseteq Y \setminus X$ by (5) and hence $s \notin J$ by definition of $J$. It follows that $s - j_0 \geq 1$. In view of (10), the fact that $\det(\varepsilon_{2j}) = \pm 1$ implies that $s - j_0 = 1$ and that the unique entry of $\varepsilon'_0F$ is $\pm 1$. We have $\{1, \ldots, s\} \setminus J = \{s\}$, so $\varepsilon'_0 = (\varepsilon_{s1} \ldots \varepsilon_{ss})$ and $\sum_{j=1}^{s} \varepsilon_{sj} \mu(P_j, C_1) = \pm 1$. Since $\varepsilon_{sj} \in \mathbb{N}$ for all $j$ and (by (9)) $\varepsilon_{s1} \geq 1$, we get $1 \leq \mu(P_1, C_1) \leq \sum_{j=1}^{s} \varepsilon_{sj} \mu(P_j, C_1) = \pm 1$, so $\mu(P_1, C_1) = 1$. This completes the proof. \[\square\]

2.10. Remark. Let $\varphi : X \to Y$ be a dominant morphism of nonsingular surfaces. By a deficient curve of $\varphi$, we mean a curve $C \subset Y$ satisfying:

$$\text{for almost all points } P \in C, |f^{-1}(P)| < [k(X) : k(Y)]_s$$

where “almost all” means “all except possibly finitely many,” “$|$” denotes cardinality, $k(X)$ and $k(Y)$ are the function fields of $X$ and $Y$ and $[k(X) : k(Y)]_s$ is the separable degree of the field extension $k(X)/k(Y)$. Note that $\varphi$ has finitely many deficient curves, and that if $\varphi$ is birational then the deficient curves are precisely the missing curves.

Then it is interesting to note that Proposition 2.9 does not generalize to all dominant morphisms $\mathbb{A}^2 \to \mathbb{A}^2$, i.e., it is not the case that each singular point of a deficient curve is a common point of at least two deficient curves. This is shown by the following example, in which we assume that $\text{char } k = 0$.

Choose a coordinate system of $\mathbb{A}^2$ and define morphisms $\mathbb{A}^2 \xrightarrow{f} \mathbb{A}^2 \to \mathbb{A}^2 \xrightarrow{h} \mathbb{A}^2$ by:

$$f(x, y) = (x, xy), \quad g(x, y) = (x + y(y^2 - 1), y), \quad h(x, y) = (x, y^2).$$

Note that $f$ is a SAC and $g \in \text{Aut}(\mathbb{A}^2)$. Define $\varphi = h \circ g \circ f : \mathbb{A}^2 \to \mathbb{A}^2$. Then $\varphi$ has two deficient curves $C_1$ and $C_2$, where:

- $C_1$ is “$y = 0$” (the deficient curve of $h$);
- $C_2$ is “$x^2 - y(y - 1)^2 = 0$” (the image by $h \circ g$ of the missing curve of $f$).

Moreover, $(0, 1)$ is a singular point of $C_2$ which is not on $C_1$.

2.11. Lemma. Let $\mathbb{A}^2 \xrightarrow{f} \mathbb{A}^2 \to \mathbb{A}^2$ be birational morphisms. Then

$$\text{cent}(g \circ f) = \text{cent}(g) \cup g(\text{cent}(f)).$$

In particular, every fundamental point of $g$ is a fundamental point of $g \circ f$.

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be birational morphisms of nonsingular surfaces. The reader may easily verify that $\text{cent}(g \circ f) \subseteq \text{cent}(g) \cup g(\text{cent}(f))$ and $g(\text{cent}(f)) \subseteq \text{cent}(g \circ f)$. In order to obtain the desired equality, there remains to show that

$$\text{cent}(g) \subseteq \text{cent}(g \circ f).$$

We claim that (11) is true whenever $q_0(f) = 0$. Indeed, consider $P \in \text{cent}(g)$. Then there exists a curve $C \subset Y$ such that $g(C) = \{P\}$. If $C \notin \text{Miss}(f)$ then (2.4(b)) there exists a curve $C' \subset X$ such that $f(C')$ is a dense subset of $C$; in particular, $(g \circ f)(C') = \{P\}$ and hence $P \in \text{cent}(g \circ f)$. If $C \in \text{Miss}(f)$ then, since $q_0(f) = 0$, 2.4(a) implies that some fundamental point $Q$ of $f$ lies on $C$; then there exists a curve $C' \subset X$ such that $f(C') = \{Q\}$; then $(g \circ f)(C') = \{P\}$ and hence $P \in \text{cent}(g \circ f)$.
By [8, 2.11], the condition $q_0(f) = 0$ is satisfied whenever $\Gamma(X, \mathcal{O}_X)^* = \mathbb{k}^*$ and $\text{Cl}(Y) = 0$. In particular, if $X = \mathbb{A}^2 = Y$ then $q_0(f) = 0$, so (11) holds and consequently $\text{cent}(g \circ f) = \text{cent}(g) \cup g\left(\text{cent}(f)\right)$. \hfill \Box

2.12. Lemma. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be birational morphisms of nonsingular surfaces and let $\Gamma_f$ (resp. $\Gamma_g$, $\Gamma_{gof}$) be the union of all missing curves of $f$ (resp. of $g$, $g \circ f$).

(a) $\Gamma_{gof}$ is equal to the union of all 1-dimensional components of $\Gamma_g \cup g(\overline{\Gamma_f})$, where $g(\overline{\Gamma_f})$ denotes the closure of $g(\Gamma_f)$ in $Z$.

(b) If $Y$ is affine then $\Gamma_{gof} = \Gamma_g \cup g(\overline{\Gamma_f})$; in particular, each missing curve of $f$ is included in $g^{-1}(\Gamma_{gof})$.

Proof. To prove (a), it’s enough to show that a curve in $Z$ is not included in $\Gamma_{gof}$ if and only if it is not included in $\Gamma_g \cup g(\overline{\Gamma_f})$. Let $C \subset Z$ be a curve such that $C \not\subset \Gamma_{gof}$. Then there exists a curve $C_0 \subset X$ such that $g(f(C_0))$ is a dense subset of $C$; consequently, the set $C_1 = \overline{f(C_0)}$ is a curve in $Y$ and $g(C_1)$ is dense in $C$, so $C$ is not a missing curve of $g$ and hence $C \not\subset \Gamma_g$. If $C \subset g(\Gamma_f)$ then there exists a missing curve $C'_1$ of $f$ such that $g(C'_1) = C$; however, $C_1$ is the only curve in $Y$ whose image by $g$ is a dense subset of $C$, and $C_1$ is not a missing curve of $f$; so $C \not\subset g(\Gamma_f)$ and hence $C \not\subset \Gamma_g \cup g(\overline{\Gamma_f})$.

Conversely, let $C \subset Z$ be a curve such that $C \not\subset \Gamma_g \cup g(\overline{\Gamma_f})$. Then $C \not\subset \Gamma_g$, so there exists a curve $C_1 \subset Y$ such that $g(C_1)$ is a dense subset of $C$. Note that $C_1$ is not a missing curve of $f$, because $C \not\subset g(\Gamma_f)$; so there exists a curve $C_0 \subset X$ such that $f(C_0)$ is a dense subset of $C_1$. Then $(g \circ f)(C_0)$ is a dense subset of $C$ and consequently $C \not\subset \Gamma_{gof}$. This proves (a).

(b) Suppose that $Y$ is affine. If a point $P \in Z$ is an irreducible component of $\Gamma_g \cup g(\overline{\Gamma_f})$ then $\{P\} = g(C)$ where $C$ is a missing curve of $f$, so $P$ is a fundamental point of $g$; since $Y$ is affine, 2.8 implies that $\text{cent}(g) \subset \Gamma_g$, so $P \in \Gamma_g$, which contradicts the hypothesis that $P$ is an irreducible component of $\Gamma_g \cup g(\overline{\Gamma_f})$. This shows that $\Gamma_g \cup g(\overline{\Gamma_f})$ is a union of curves, so $\Gamma_{gof} = \Gamma_g \cup g(\overline{\Gamma_f})$ follows from (a). \hfill \Box

Results 2.13 and 2.14 are valid in all characteristics, but are particularly interesting when $\text{char} \mathbb{k} > 0$ (recall that not all lines are coordinate lines when $\text{char} \mathbb{k} > 0$).

2.13. Proposition. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a birational morphism.

(a) If a missing curve of $f$ is nonsingular then it is a coordinate line.

(b) If a contracting curve of $f$ has one place at infinity, then it is a coordinate line.

Proof. Assertion (a) follows from result 4.6 of [8]. To prove (b), consider a contracting curve $C$ of $f$ such that $C$ has one place at infinity. We noted in (4) that $C$ is a nonsingular rational curve, so $C \cong \mathbb{A}^1$ is clear.

Let us embed $\text{dom}(f) = \mathbb{A}^2$ in $X \cong \mathbb{P}^2$, let $L$ be the function field of $X$ and $V$ the prime divisor of $L/\mathbb{k}$ whose center on $X$ is the closure of $C$ in $X$ (i.e., $V$ is the DVR $\mathcal{O}_{X,\xi}$ where $\xi \in X$ is the generic point of $C$). Also embed $\text{codom}(f) = \mathbb{A}^2$ in $Y \cong \mathbb{P}^2$, and note that the center of $V$ on $Y$ is zero-dimensional, since $C \in \text{Cont}(f)$. 

Consider the Kodaira dimension $\kappa(V)$ as defined in the introduction of Section 2 of [11]. Then $\kappa(V) < 0$ by [11, 2.1] and the fact that the center of $V$ on $Y$ is zero-dimensional; so $C$ is a coordinate line by [11, 2.4].

2.14. Corollary. Let $C$, $C'$ be curves in $\mathbb{A}^2$ such that $C \cong \mathbb{A}^1 \cong C'$, and suppose that there exists a birational morphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ such that $f(C)$ is a dense subset of $C'$. Then $f(C) = C'$. Moreover, if one of $C, C'$ is a coordinate line then both are coordinate lines.

Proof. It is a simple fact that every dominant morphism $\mathbb{A}^1 \to \mathbb{A}^1$ is finite, hence surjective; so $f(C) = C'$.

If $C$ is a coordinate line then there exists a birational morphism $g : \mathbb{A}^2 \to \mathbb{A}^2$ such that $C \in \text{Miss}(g)$ (choose a coordinate system $(X,Y)$ such that $C = \mathbb{Z}(X)$, and take $g(x,y) = (x,xy)$); then $C' \in \text{Cont}(f \circ g)$ by 2.12, so 2.13(a) implies that $C'$ is a coordinate line.

If $C'$ is a coordinate line then there exists a birational morphism $g : \mathbb{A}^2 \to \mathbb{A}^2$ such that $C' \in \text{Cont}(g)$ (choose $(X,Y)$ such that $C' = \mathbb{Z}(X)$ and take $g(x,y) = (x,xy)$); then $C \in \text{Cont}(g \circ f)$, so 2.13(b) implies that $C$ is a coordinate line.

2.15. Lemma. Let $\mathbb{A}^2 \xrightarrow{f} \mathbb{A}^2 \xrightarrow{g} \mathbb{A}^2$ be birational morphisms.

(a) If $\text{Miss}(f) \subseteq \text{Cont}(g)$ then $\text{Miss}(f)$ is admissible.

(b) If $\text{Cont}(g) \subseteq \text{Miss}(f)$ then $\text{Cont}(g)$ is admissible.

Proof. Applying statement (4) in 2.3 to the morphism $g$ gives:

\[ D' = \sum_{C \in \text{Cont}(g)} C \text{ is an SNC-divisor of } \mathbb{A}^2 \text{ and the graph } \mathcal{G}(D', \mathbb{A}^2) \text{ is a forest.} \]

If $\text{Cont}(g) \subseteq \text{Miss}(f)$ then each element of $\text{Cont}(g)$ is a nonsingular missing curve of $f$, and hence is a coordinate line by 2.13(a); then (12) implies that $\text{Cont}(g)$ is admissible, so (b) is proved.

If $\text{Miss}(f) \subseteq \text{Cont}(g)$ then, by (12), $D = \sum_{C \in \text{Miss}(f)} C$ is an SNC-divisor of $\mathbb{A}^2$ and the graph $\mathcal{G}(D, \mathbb{A}^2)$ is a forest; in particular each missing curve of $f$ is nonsingular and hence is a coordinate line by 2.13(a); so $\text{Miss}(f)$ is admissible and (a) is proved.

2.16. Lemma. Let $\mathbb{A}^2 \xrightarrow{f} \mathbb{A}^2 \xrightarrow{g} \mathbb{A}^2$ be birational morphisms.

(a) If $\text{Miss}(f) \nsubseteq \text{Cont}(g)$ then $q(g \circ f) > q(g)$.

(b) If $\text{Cont}(g) \nsubseteq \text{Miss}(f)$ then $c(g \circ f) > c(f)$.

Proof. (a) It is clear that $\text{Miss}(g) \subseteq \text{Miss}(g \circ f)$. If $C$ is a missing curve of $f$ which is not contracted by $g$ then the closure $\overline{g(C)}$ of $g(C)$ is a missing curve of $g \circ f$ but not a missing curve of $g$, so $\text{Miss}(g) \subset \text{Miss}(g \circ f)$ and hence $q(g) < q(g \circ f)$.

(b) We have $\text{Cont}(f) \subseteq \text{Cont}(g \circ f)$. If $C$ is a contracting curve of $g$ which is not a missing curve of $f$ then there exists a curve $C' \subset \mathbb{A}^2$ such that $f(C')$ is a dense subset of $C$. Then $C'$ is a contracting curve of $g \circ f$ but not one of $f$, so $\text{Cont}(f) \subset \text{Cont}(g \circ f)$ and hence $c(f) < c(g \circ f)$.
In 2.17 and 2.18, we write \( \# \Gamma \) for the number of irreducible components of a closed set \( \Gamma \), and \( \Gamma_{\tau} = \bigcup_{C \in \text{Miss}(\tau)} C \) for any birational morphism \( \tau \) of nonsingular surfaces.

2.17. **Lemma.** Let \( f : \mathbb{A}^2 \to \mathbb{A}^2 \) be a birational morphism and \( \Gamma \) a union of missing curves of \( f \) such that
\[
(13) \quad \text{each missing curve of } f \text{ is either included in } \Gamma \text{ or disjoint from } \Gamma.
\]
Then \( \# f^{-1}(\Gamma) = \# \Gamma \) and \( f \) can be factored as \( \mathbb{A}^2 \xrightarrow{g} \mathbb{A}^2 \xrightarrow{h} \mathbb{A}^2 \), where \( g, h \) are birational morphisms, \( \Gamma_h = \Gamma \) and \( \Gamma_g \cap \text{exc}(h) = \emptyset \).

Result 2.17 is an immediate consequence of:

2.18. **Proposition.** Let \( f : X \to Y \) be a birational morphism where \( X, Y \) are nonsingular affine surfaces and let \( \Gamma \subset Y \) be a union of missing curves of \( f \) satisfying (13). Then the following hold.

\( (a) \) \( f \) can be factored as \( X \xrightarrow{g} W \xrightarrow{h} Y \) where \( W \) is a nonsingular affine surface, \( g, h \) are birational morphisms, \( \Gamma_h = \Gamma \), \( c(h) = \# f^{-1}(\Gamma) \) and \( \Gamma_g \cap \text{exc}(h) = \emptyset \).

\( (b) \) If \( X, Y \) are factorial with trivial units then \( \# f^{-1}(\Gamma) = \# \Gamma \) and, in part (a), \( W \) can be chosen to be factorial with trivial units.

\( (c) \) If \( X = \mathbb{A}^2 \) and \( Y \) is factorial, then \( Y = \mathbb{A}^2 \) and we can choose \( W = \mathbb{A}^2 \) in part (a).

**Proof.** (a) We may choose a commutative diagram (2) satisfying \( n = n(f) \) and in which the blowings-up \( \pi_1, \ldots, \pi_n \) are ordered in such a way that the points over \( \Gamma \) are blown-up first, i.e., there exists \( m \in \{0, \ldots, n\} \) such that
\[
\{ i \in \{1, \ldots, n\} \mid P_i \in \Gamma \text{ or } P_i \text{ is infinitely near a point of } \Gamma \} = \{1, \ldots, m\}.
\]
Refer to 2.3 for the notations. If \( 0 \leq j \leq k \leq n \) and \( D \subset Y_j \) is a curve or a union of curves, we write \( \tilde{D}^{Y_k} \) for the strict transform of \( D \) on \( Y_k \). Let \( J \) be the set of \( j \in \{1, \ldots, m\} \) such that \( \tilde{E}_j^{Y_n} \cap X = \emptyset \) (recall that \( X \) is an open subset of \( Y_n \)) and define
\[
(14) \quad W = Y_m \setminus \left( \tilde{\Gamma}^{Y_m} \cup \bigcup_{j \in J} \tilde{E}_j^{Y_m} \right).
\]
Then \( W \) is a nonsingular surface and \( f \) factors as \( X \xrightarrow{g} W \xrightarrow{h} Y \) where \( g, h \) are birational morphisms, \( \Gamma_h = \Gamma \) and \( \text{Cont}(h) = \{ \tilde{E}_i^{Y_m} \cap W \mid i \in \{1, \ldots, m\} \setminus J \} \); thus
\[
(15) \quad q(h) = \# \Gamma \quad \text{and} \quad c(h) = \# f^{-1}(\Gamma).
\]
Let \( \Gamma' = C_1 \cup \cdots \cup C_s \) where \( C_1, \ldots, C_s \subset Y \) are the missing curves of \( f \) not included in \( \Gamma \); then \( \Gamma_f = \Gamma \cup \Gamma' \) and \( \Gamma \cap \Gamma' = \emptyset \). Moreover,
\[
(16) \quad \text{Miss}(g) = \{ \tilde{C}_i^{Y_m} \cap W \mid i = 1, \ldots, s \}.
\]
Indeed, consider \( C \in \text{Miss}(g) \). If \( h(C) \) is a point then \( C = \tilde{E}_j^{Y_m} \cap W \) for some \( j \in \{1, \ldots, m\} \), and in fact \( \tilde{E}_j^{Y_m} \cap X = \emptyset \) (so \( j \in J \)) otherwise \( C \) would not be a missing
curve of $g$; then (14) implies that $\widetilde{E}_j^m \cap W = \emptyset$, which is absurd. So $h(C)$ is a dense subset of a curve $C_s \subset Y$. Now $C_s \in \text{Miss}(f)$ by 2.12, and (14) implies that $C \notin \tilde{G}^m$, hence $C_s \notin \Gamma$; so $C_s \subseteq \Gamma'$ and consequently $C = \tilde{C}_i^m \cap W$ for some $i \in \{1, \ldots, s\}$. This proves “$\subseteq$” in (16), and “$\supseteq$” easily follows from 2.12.

From (16), we deduce that $\Gamma_g \cap \text{exc}(h) \subseteq h^{-1}(\Gamma') \cap h^{-1}(\Gamma)$, so

$$\Gamma_g \cap \text{exc}(h) = \emptyset.$$  

To prove (a), there only remains to show that $W$ is affine. Since $X$ is affine, 2.8 implies that $W \setminus \Gamma_g$ is affine; as (by (17)) $\text{exc}(h) \subset W \setminus \Gamma_g$,

$$\text{no contracting curve of } h \text{ is a complete curve.}$$

Embed $Y_0$ in a nonsingular projective surface $\overline{Y}_0$ and enlarge diagram (2) as follows:

$$\begin{array}{cccccc}
\overline{Y}_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_{m+1}} \overline{Y}_m & \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_1} & \overline{Y}_0 \\
\downarrow & & & & \downarrow \\
\overline{Y}_n & \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_{m+1}} & \overline{Y}_m & \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_1} & \overline{Y}_0 \\
\downarrow & & & & \downarrow \\
X & \xrightarrow{g} & \overline{W} & \xrightarrow{h} & Y
\end{array}$$

Let $D_i = \overline{Y}_i \setminus Y_i \ (0 \leq i \leq n)$. Since $Y = Y_0$ is affine, $D_0$ is a nonempty connected union of curves and each irreducible component of $\Gamma$ meets $D_0$ (where $\Gamma$ denotes the closure of $\Gamma$ in $\overline{Y}_0$). It follows that $D_m$ is a nonempty connected union of curves and that each irreducible component of $\tilde{C}_i^m$ meets $D_m$. Let us argue that

$$W \text{ is connected at infinity.}$$

Suppose that (19) is false; then $\overline{Y}_m \setminus W$ is not connected, so some connected component $\mathcal{C}$ of $\overline{Y}_m \setminus W$ is disjoint from $D_m$. Then $\mathcal{C}$ does not contain any irreducible component of $\tilde{C}_i^m$. By (14), it follows that $\mathcal{C} \subseteq \bigcup_{j \in J} \overline{E}_j^m$.

We have $\overline{Y}_m \setminus (W \setminus \Gamma_g) = \overline{C}_i^m \cup \cdots \cup \overline{C}_s^m \cup (\overline{Y}_m \setminus W)$ by (16); since $W \setminus \Gamma_g$ is affine,

$$\overline{C}_i^m \cup \cdots \cup \overline{C}_s^m \cup (\overline{Y}_m \setminus W) \text{ is connected.}$$

As $\overline{Y}_m \setminus W$ is not connected and $\mathcal{C}$ is a connected component of it, some $\overline{C}_i^m$ must meet $\mathcal{C}$. So there exist $i \in \{1, \ldots, s\}$ and $j \in J$ such that $\overline{C}_i^m \cap \overline{E}_j^m \neq \emptyset$. As $C_i \subseteq \Gamma'$, this implies that $P_j \in \Gamma'$ or $P_j$ is i.n. a point of $\Gamma'$; since $j \leq m$, we also have $P_j \in \Gamma$ or $P_j$ is i.n. a point of $\Gamma$; so $\Gamma \cap \Gamma' \neq \emptyset$, a contradiction. So (19) is true.

In view of (18), (19) and the fact that $Y$ is affine, applying [8, 2.2] to $h : W \to Y$ shows that $W$ is affine and concludes the proof of (a).

(b) Assume that $X, Y$ are factorial and have trivial units; then [8, 3.4] gives $q(h) \leq c(h)$, so $\# \Gamma \leq \#f^{-1}(\Gamma')$ by (15). Since $\Gamma'$ also satisfies (13), it follows that $\# \Gamma' \leq \#f^{-1}(\Gamma')$.

By 2.8 we have $\text{cent}(f) \subseteq \Gamma_f = \Gamma \cup \Gamma'$, so $f^{-1}(\Gamma) \cup f^{-1}(\Gamma')$ is exactly the union of all contracting curves of $f$; as $f^{-1}(\Gamma) \cap f^{-1}(\Gamma') = \emptyset$, we get $\#f^{-1}(\Gamma) + \#f^{-1}(\Gamma') = c(f)$. 
We have \( c(f) = q(f) \) by [8, 2.9], and it is clear that \( q(f) = \# \Gamma + \# \Gamma' \), so
\[
\# \Gamma \leq \# f^{-1}(\Gamma), \quad \# \Gamma' \leq \# f^{-1}(\Gamma') \quad \text{and} \quad \# \Gamma + \# \Gamma' = \# f^{-1}(\Gamma) + \# f^{-1}(\Gamma');
\]
consequently,
\[
(20) \quad \# \Gamma = \# f^{-1}(\Gamma)
\]
\( (21) \quad q(h) = c(h) \)
where (21) follows from (20) and (15). By (21), (19) and [8, 3.4], we get that \( W \)
\[
\text{is factorial and has trivial units, which proves (b).}
\]
If \( X = \mathbb{A}^2 \) and \( Y \) is factorial then, by (b), \( W \) may be chosen to be factorial; then [8, 4.2] implies that \( W \) and \( Y \) are isomorphic to \( \mathbb{A}^2 \), which proves (c) and completes the proof of the Proposition. \( \square \)

2.19. **Definition.** Let \( f : X \to Y \) be a birational morphism of nonsingular surfaces. Consider a diagram (2) satisfying \( n = n(f) \) and with notation as in 2.3 (for each \( i \in \{1, \ldots, n\} \), \( \pi_i : Y_i \to Y_{i-1} \) be the blowing-up of \( Y_{i-1} \) at the point \( P_i \in Y_{i-1} \)).

(a) Let \( C \) be a missing curve of \( f \). For each \( i = 0, 1, \ldots, n \), let \( C^Y_i \subset Y_i \) denote the strict transform of \( C \) on \( Y_i \) \((C^Y_0 = C)\). Then we define the natural number
\[
n(f, C) = \text{cardinality of the set } \{ i \mid 1 \leq i \leq n, \ P_i \in C^Y_{i-1} \}
\]
and note that \( n(f, C) \) depends only on \((f, C)\), i.e., is independent of the choice of diagram (2). To indicate that \( n(f, C) = k \), we say that \( "C \) is blown-up \( k \) times".

(b) For each \( i = 1, \ldots, n \), let \( \bar{P}_i \in Y_0 \) be the image of \( P_i \) by \( \pi_1 \circ \cdots \circ \pi_{i-1} : Y_{i-1} \to Y_0 \). For each \( P \in Y \), define the natural number
\[
n(f, P) = \text{cardinality of the set } \{ i \mid 1 \leq i \leq n, \ \bar{P}_i = P \}
\]
and note that \( n(f, P) \) depends only on \((f, P)\), i.e., is independent of the choice of the diagram (2) used for defining it.

2.20. **Remarks.** Let \( f : X \to Y \) be a birational morphism of nonsingular surfaces.

(a) Let \( C \in \text{Miss}(f) \). Then \( n(f, C) = 0 \iff C \cap f(X) = \emptyset \), and \( n(f, C) = 1 \) implies that there exists exactly one fundamental point of \( f \) lying on \( C \). Note that if \( X = \mathbb{A}^2 = Y \) then each missing curve contains at least one fundamental point \((2.6(d))\), so each missing curve is blown-up at least once.

(b) Let \( P \in Y \). Then \( n(f, P) > 0 \iff P \in \text{cent}(f) \), where \( \iff \) is obvious and \( \Rightarrow \) follows from (6). It is also clear that \( n(f) = \sum_{P \in Y} n(f, P) \).

2.21. **Lemma.** Let \( X \overset{f}{\to} Y \overset{g}{\to} Z \) be birational morphisms of nonsingular surfaces, and assume that \( n(g \circ f) = n(g) + n(f) \) or \( X = Y = Z = \mathbb{A}^2 \).

(a) Let \( D \in \text{Miss}(g) \); then \( D \in \text{Miss}(g \circ f) \) and \( n(g \circ f, D) = n(g, D) \).

(b) Let \( C \in \text{Miss}(f) \setminus \text{Cont}(g) \) and let \( D \) be the closure of \( g(C) \) in \( Z \). Then:

\( \bullet \) \( D \in \text{Miss}(g \circ f) \) and \( n(f, C) \leq n(g \circ f, D) \).
• \( n(f, C) = n(g \circ f, D) \implies C \cap \text{exc}(g) = \emptyset; \)
• if \( g(C) = D \) or \( C \cong \mathbb{A}^1 \) then

\[
n(f, C) = n(g \circ f, D) \iff C \cap \text{exc}(g) = \emptyset.
\]

(c) For each \( P \in Z \), \( n(g \circ f, P) = n(g, P) + \sum_{Q \in g^{-1}(P)} n(f, Q) \).

Proof. If \( X = Y = Z = \mathbb{A}^2 \) then \( n(g \circ f) = n(g) + n(f) \) by 2.5; so \( n(g \circ f) = n(g) + n(f) \) holds in all cases. Let \( m = n(f) \) and \( n = n(g) \). Choose commutative diagrams (I) and (II) as follows:

\[
\begin{array}{ccc}
Y_m \xrightarrow{\pi_m} \ldots \xrightarrow{\pi_1} Y_0 & & Z_n \xrightarrow{\rho_n} \ldots \xrightarrow{\rho_1} Z_0 \\
\uparrow f & | & \uparrow g \\
X & | & Y \\
\end{array}
\]

and use them to build the commutative diagram

\[
\begin{array}{ccc}
Z_{n+m} \xrightarrow{\rho_{n+m}} \ldots \xrightarrow{\rho_{n+2}} Z_{n+1} & & \xrightarrow{\rho_{n+1}} Z_n \xrightarrow{\rho_n} \ldots \xrightarrow{\rho_1} Z_0 \\
\uparrow Y_m \xrightarrow{\pi_m} \ldots \xrightarrow{\pi_2} Y_1 & & \xrightarrow{\pi_1} Y_0 \\
\uparrow f & & \uparrow g \\
X & | & Y \\
\end{array}
\]

In the three diagrams, “\( \hookrightarrow \)” are open immersions, \( Y_i, Z_i \) are nonsingular surfaces, \( Y_i \xrightarrow{\pi_i} Y_{i-1} \) is the blowing-up of \( Y_{i-1} \) at a point \( P_i \in Y_{i-1} \) and \( Z_i \xrightarrow{\rho_i} Z_{i-1} \) is the blowing-up of \( Z_{i-1} \) at a point \( Q_i \in Z_{i-1} \). Moreover, \( Y_{i-1} \hookrightarrow Z_{n+i-1} \) maps \( P_i \) on \( Q_{n+i} \) (let us simply write \( P_i = Q_{n+i} \)). Diagrams (I) and (II) are minimal in the sense that \( n(f) = m \) and \( n(g) = n \); since \( n(g \circ f) = n(f) + n(g) = m + n \), it follows that (III) is also minimal.

(a) Let \( D \in \text{Miss}(g) \); then \( D \in \text{Miss}(g \circ f) \) by 2.12(a). Let \( D^{Z_i} \subset Z_i \) be the strict transform of \( D \subset Z_0 \) on \( Z_i \). As \( D^{Z_n} \subset Z_n \setminus Y_0 \) and \( \text{cent}(\rho_{n+1} \circ \cdots \circ \rho_{n+m}) = \text{cent}(\pi_1 \circ \cdots \circ \pi_m) \subset Y_0 \), we see that

\[
\{ i \mid 1 \leq i \leq n + m, \ Q_i \in D^{Z_i-1} \} = \{ i \mid 1 \leq i \leq n, \ Q_i \in D^{Z_i-1} \}.
\]

Since \( n(g \circ f, D) \) (resp. \( n(g, D) \)) is by definition the cardinality of the set in the lhs (resp. rhs) of (22), we have \( n(g \circ f, D) = n(g, D) \).

(b) Let \( C \in \text{Miss}(f) \setminus \text{Cont}(g) \) and let \( D \) be the closure of \( g(C) \) in \( Z \). Then \( D \in \text{Miss}(g \circ f) \) by 2.12(a). Define \( D^{Z_i} \subset Z_i \) as before, then

\[
\{ i \mid n + 1 \leq i \leq n + m, \ Q_i \in D^{Z_i-1} \} \subseteq \{ i \mid 1 \leq i \leq n + m, \ Q_i \in D^{Z_i-1} \}.
\]

Since \( n(f, C) \) (resp. \( n(g \circ f, D) \)) is the cardinality of the set in the lhs (resp. rhs) of (23), we have \( n(f, C) \leq n(g \circ f, D) \), and moreover

\[
n(f, C) \neq n(g \circ f, D)
\]

is equivalent to

\[
\{ i \mid 1 \leq i \leq n, \ Q_i \in D^{Z_i-1} \} \neq \emptyset.
\]
By minimality of diagram (II) together with (6), (25) is equivalent to
\[(26)\quad D \cap \text{cent}(g) \neq \emptyset.\]
Now
\[(27)\quad C \cap \text{exc}(g) \neq \emptyset\]
implies (26) and, if \(g(C) = D\), the converse is true. So we have shown that
\[(28)\quad n(f, C) = n(g \circ f, D) \implies C \cap \text{exc}(g) = \emptyset,\]
and that the converse holds whenever \(g(C) = D\). Finally, we observe that if \(C \sim A^1\) then the dominant morphism \(C \xrightarrow{g} D\) is necessarily finite, hence surjective, so the converse of (28) is true whenever \(C \sim A^1\). This proves (b).

To prove (c), define \(\bar{Q}_i = (\rho_1 \circ \cdots \circ \rho_{i-1})(Q_i) \in Z_0\) for all \(i = 1, \ldots, m + n\) and observe that the trivial equality (for any \(P \in Z\))
\[\left| \left\{ i \mid \bar{Q}_i = P \right\} \right| = \left| \left\{ i \mid i \leq n \text{ and } \bar{Q}_i = P \right\} \right| + \left| \left\{ i \mid i > n \text{ and } \bar{Q}_i = P \right\} \right|\]
is the desired equality. \(\square\)

3. Compositions of simple affine contractions

Let \(k\) be an algebraically closed field of arbitrary characteristic, and let \(A^2 = A^2_k\). As in the introduction, we write \(\text{Bir}(A^2)\) for the monoid of birational endomorphisms \(f : A^2 \to A^2\), and we declare that \(f, g \in \text{Bir}(A^2)\) are equivalent \((f \sim g)\) if \(u \circ f \circ v = g\) for some \(u, v \in \text{Aut}(A^2)\). The equivalence class of \(f \in \text{Bir}(A^2)\) is denoted \([f]\). Note that the conditions \(f \sim f'\) and \(g \sim g'\) do NOT imply that \(f \circ g \sim f' \circ g'\).

The aim of this section is to describe the subsets \(S_w \supset S_a \supset S_{aa}\) of \(\text{Bir}(A^2)\) defined by:

\[
S_w = \left\{ f \in \text{Bir}(A^2) \mid \text{Miss}(f) \text{ is weakly admissible} \right\},
\]

\[
S_a = \left\{ f \in \text{Bir}(A^2) \mid \text{Miss}(f) \text{ is admissible} \right\},
\]

\[
S_{aa} = \left\{ f \in \text{Bir}(A^2) \mid \text{both Miss}(f) \text{ and Cont}(f) \text{ are admissible} \right\}
\]
(refer to 1.3, 1.4, 1.6 and 1.7); note that these sets are not closed under composition of morphisms. We learn at a relatively early stage (see 3.6(c)) that each element of \(S_w\) is a composition of simple affine contractions (SACs are defined in the introduction and again in 3.2). However, an arbitrary composition of SACs does not necessarily belong to \(S_w\) (resp. \(S_a, S_{aa}\)), so in each of the three cases one has to determine which compositions of SACs give the desired type of endomorphism. The answer is given in Theorem 3.15, which is the main result of this section.

The material of 3.1–3.5(a) can be found in [8] and [9]; everything else appears to be new.

As before, we have \(A^2 = \text{Spec } A\) where \(A = k^2\) is fixed throughout, and by a coordinate system of \(A^2\) we mean a pair \((X, Y) \in A \times A\) satisfying \(A = k[X, Y]\) (see the introduction of Section 1).
3.1. Let $\mathcal{C}$ temporarily denote the set of coordinate systems of $\mathbb{A}^2$. Given an element $c = (X, Y)$ of $\mathcal{C}$, consider the $k$-homomorphism $A \to A$ given by $X \mapsto X$ and $Y \mapsto XY$; this homomorphism determines a morphism $\mathbb{A}^2 \to \mathbb{A}^2$ which we denote $\alpha_c$; clearly, $\alpha_c \in \operatorname{Bir}(\mathbb{A}^2)$. Note that if $c_1, c_2 \in \mathcal{C}$ then $\alpha_{c_1}^m \sim \alpha_{c_2}^m$ for every $m \geq 1$. So, for each $m \geq 1$, the equivalence class $[\alpha_c^m]$ of $\alpha_c^m$ is independent of the choice of $c \in \mathcal{C}$.

3.2. **Definition.** A birational morphism $\mathbb{A}^2 \to \mathbb{A}^2$ is called a *simple affine contraction* (SAC) if it is equivalent to $\alpha_c$ for some (hence for every) coordinate system $c$ of $\mathbb{A}^2$.

Note that if $f$ is a SAC and $c \in \mathcal{C}$ then $f \sim \alpha_c$, but $f^2$ need not be equivalent to $\alpha_c^2$.

For readers who like to identify $\mathbb{A}^2$ with $k^2$, we note that $\alpha_c$ corresponds to the map $k^2 \to k^2$, $(x, y) \mapsto (x, xy)$, and that the SACs are obtained by composing this map both sides with automorphisms. See 1.1.

3.3. **Lemma.**

(a) A birational morphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ is a SAC if and only if $n(f) = 1$.

(b) If $f : \mathbb{A}^2 \to \mathbb{A}^2$ is a SAC then $f$ has one missing curve $L$ and one fundamental point $P$; moreover, $L$ is a coordinate line and $P \in L$.

(c) Let $L \subset \mathbb{A}^2$ be a coordinate line and $P \in L$ a point. Let $X \overset{\pi}{\to} \mathbb{A}^2$ be the blowing-up of $\mathbb{A}^2$ at $P$ and $U \subset X$ the complement of the strict transform of $L$. Then $U \cong \mathbb{A}^2$ and the composition $\mathbb{A}^2 \overset{\pi}{\to} U \hookrightarrow X \overset{\pi}{\to} \mathbb{A}^2$ is a SAC with missing curve $L$ and fundamental point $P$.

(d) If $f, g : \mathbb{A}^2 \to \mathbb{A}^2$ are two SAC with the same missing curve and the same fundamental point then there exists an automorphism $\theta : \mathbb{A}^2 \to \mathbb{A}^2$ such that $g = f \circ \theta$:

\[
\begin{array}{ccc}
\mathbb{A}^2 & \overset{\cong}{\overset{\theta}{\longrightarrow}} & \mathbb{A}^2 \\
g \downarrow & & \downarrow f \\
\mathbb{A}^2
\end{array}
\]

(e) Let $c$ be a coordinate system of $\mathbb{A}^2$ and $\alpha_c \in \operatorname{Bir}(\mathbb{A}^2)$ as in 3.1. For $f \in \operatorname{Bir}(\mathbb{A}^2)$,

\[q(f) = 1 \iff c(f) = 1 \iff f \sim \alpha^\nu_c(f)\]

**Proof.** Assertion (a) is [8, 4.10] or [9, 4.1]; assertions (b)–(d) are trivial; (e) is [8, 4.11] together with the fact 2.6(a) that $q(f) = c(f)$.

**Remark.** Assertion 3.3(e) can be phrased as follows: given an integer $m \geq 1$ and a coordinate system $c$ of $\mathbb{A}^2$, the equivalence class $[\alpha_c^m]$ of $\alpha_c^m$ is:

\[\begin{align*}
[\alpha_c^m] &= \{ f \in \operatorname{Bir}(\mathbb{A}^2) \mid q(f) = 1 \text{ and } n(f) = m \} \\
&= \{ f \in \operatorname{Bir}(\mathbb{A}^2) \mid c(f) = 1 \text{ and } n(f) = m \}.
\end{align*}\]

3.4. **Corollary.** If $f \in \operatorname{Bir}(\mathbb{A}^2)$ has a unique missing curve $C$, then $C$ is a coordinate line.

**Proof.** This follows from 3.3(e). One can also deduce it from 2.13(a) and 2.9.
See 2.19 for the definition of the phrase “$L$ is blown-up only once”.

3.5. **Lemma.** Let $f \in \text{Bir}(\mathbb{A}^2)$.

(a) Suppose that some missing curve $L$ of $f$ is blown-up only once. Then $L$ is a coordinate line. Moreover, if $P \in L$ is the unique fundamental point of $f$ which is on $L$ and $\gamma \in \text{Bir}(\mathbb{A}^2)$ is a SAC with missing curve $L$ and fundamental point $P$, then $f$ factors as $\mathbb{A}^2 \xrightarrow{f'} \mathbb{A}^2 \xrightarrow{\gamma} \mathbb{A}^2$ with $f' \in \text{Bir}(\mathbb{A}^2)$.

(b) Suppose that $f$ factors as $\mathbb{A}^2 \xrightarrow{f'} \mathbb{A}^2 \xrightarrow{\gamma} \mathbb{A}^2$ with $f', \gamma \in \text{Bir}(\mathbb{A}^2)$, where $\gamma$ is a SAC. Let $L$ be the missing curve of $\gamma$. Then $L$ is a missing curve of $f$ which is blown-up only once.

**Proof.** (a) is an improvement of [9, 4.6]. The proof of [9, 4.6] shows that $P$ is a nonsingular point of $L$; then [8, 4.6] implies that $L$ is a coordinate line. Choose a diagram (2) for $f$ such that $n = n(f)$ and $P_1 = P$. Let $L_{\pi_1} \subset Y_1$ denote the strict transform of $L$ on $Y_1$ and define $W = Y_1 \setminus L_{\pi_1} \subset Y_1$. As $L$ is a missing curve of $f$ and is blown-up only once, the image of $\mathbb{A}^2 \xrightarrow{f} Y_1 \xrightarrow{\pi_2 \circ \cdots \circ \pi_n} Y_1$ is included in $W$; so $f$ factors as $\mathbb{A}^2 \xrightarrow{g'} W \xrightarrow{h'} \mathbb{A}^2$ where $g', h'$ are birational morphisms and $h'$ is the composition $W \subset Y_1 \xrightarrow{\pi_1} Y_0 = \mathbb{A}^2$. By 3.3(c), $W \cong \mathbb{A}^2$ and the composition $\mathbb{A}^2 \cong W \xrightarrow{h'} \mathbb{A}^2$ is a SAC with missing curve $L$ and fundamental point $P$; so $f$ factors as $\mathbb{A}^2 \xrightarrow{\gamma} \mathbb{A}^2 \xrightarrow{h'} \mathbb{A}^2$ where $g, h \in \text{Bir}(\mathbb{A}^2)$ and $h$ is a SAC with missing curve $L$ and fundamental point $P$.

By 3.3(d), $f = \gamma \circ \theta$ for some $\theta \in \text{Aut}(\mathbb{A}^2)$. Then $f = \gamma \circ \theta \circ g$ and we are done.

(b) By 2.21(a), $L \in \text{Miss}(f)$ and $n(f, L) = n(\gamma \circ f', L) = n(\gamma, L) = 1$. \hfill $\Box$

3.6. **Proposition.** Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a birational morphism such that:

(i) $f$ is not an isomorphism;

(ii) there exists a coordinate system of $\mathbb{A}^2$ with respect to which all missing curves of $f$ have degree 1.

Then there exists a missing curve of $f$ which is blown-up only once. Moreover, if $L$ is such a curve and $P \in L$ is the unique fundamental point of $f$ which is on $L$, then the following hold.

(a) There exists a coordinate system $(X, Y)$ of $\mathbb{A}^2$ such that $L = Z(X)$ and $P = (0, 0)$, and such that the union of the missing curves of $f$ is equal to the zero-set of one of the following polynomials in $k[X, Y]$:

(i) $XY^m \prod_{i=1}^n (X - a_i)$, for some $m \in \{0, 1\}$, $n \geq 0$ and $a_1, \ldots, a_n \in k$;

(ii) $X(Y-1)^m \prod_{i=1}^n (Y - b_i X)$, for some $m \in \{0, 1\}$, $n \geq 0$ and $b_1, \ldots, b_n \in k$.

(b) If $\gamma : \mathbb{A}^2 \to \mathbb{A}^2$ is a SAC with missing curve $L$ and fundamental point $P$, then $f$ factors as $\mathbb{A}^2 \xrightarrow{f'} \mathbb{A}^2 \xrightarrow{\gamma} \mathbb{A}^2$ where $f'$ is a birational morphism such that $\text{Miss}(f')$ is admissible.

(c) $f$ is a composition of SACs.

**Proof.** By [9, 4.7], $f = h \circ g$ where $g, h \in \text{Bir}(\mathbb{A}^2)$ and $h$ is a SAC; then 3.5(b) implies that some missing curve of $f$ is blown-up only once.
Let $L$ be a missing curve of $f$ which is blown-up only once, and let $P \in L$ be the unique fundamental point of $f$ which is on $L$. Choose a coordinate system $(X, Y)$ of $\mathbb{A}^2$ such that $L = \mathbb{Z}(X)$ and $P = (0, 0)$, and with respect to which all missing curves of $f$ have degree 1. Define $\gamma_0 : \mathbb{A}^2 \to \mathbb{A}^2$ by $\gamma_0(x, y) = (x, xy)$. As $\gamma_0$ is a SAC with missing curve $L$ and fundamental point $P$, 3.5(a) implies that $f$ factors as $\mathbb{A}^2 \xrightarrow{f'} \mathbb{A}^2 \xrightarrow{\gamma_0} \mathbb{A}^2$, for some $f' \in \text{Bir}(\mathbb{A}^2)$. Let $\Gamma$ (resp. $\Gamma'$) be the union of the missing curves of $f$ (resp. of $f'$). Then

$$\Gamma = \mathbb{Z}(X) \cup \overline{\gamma_0(\Gamma')}$$

by 2.12. In particular, if $C$ is a missing curve of $f'$ then $\gamma_0(C)$ is included into a line of degree 1; from $\gamma_0(x, y) = (x, xy)$, it easily follows that $C$ is either a vertical line $\mathbb{Z}(X - a)$ (for some $a \in \mathbb{k}$), a horizontal line $\mathbb{Z}(Y - b)$ (for some $b \in \mathbb{k}$), or a hyperbola $\mathbb{Z}(X(\alpha + \beta Y) - 1)$ (for some $\alpha, \beta \in \mathbb{k}, \beta \neq 0$), where in fact the last case cannot occur because $C$ has one place at infinity by 2.6(c). So

$$\text{(30)} \quad \text{each missing curve of } f' \text{ is either a vertical line or a horizontal line.}$$

In particular, all missing curves of $f'$ have degree 1. It follows from the first part of the proof that

$$\text{(31)} \quad \text{if } f' \text{ is not an isomorphism, some missing curve of } f' \text{ is blown-up only once.}$$

Let $h$ (resp. $v$) be the number of missing curves of $f'$ which are horizontal (resp. vertical) lines. Then $\min(h, v) \leq 1$, otherwise (by 2.6(e)) every missing curve of $f'$ would contain at least two fundamental points of $f'$, and hence would be blown-up at least twice, contradicting (31). Statement (30) together with $\min(h, v) \leq 1$ imply that $\text{Miss}(f')$ is admissible, which proves the special case “$\gamma = \gamma_0$” of assertion (b); in view of 3.3(d), it follows that (b) is true.

If $h \leq 1$ then $\Gamma'$ is the zero-set of $(Y - b)^h \prod_{i=1}^{v}(X - a_i)$, for some $b \in \mathbb{k}$ and $a_1, \ldots, a_v \in \mathbb{k}$. Then, by (29), $\Gamma$ is the zero-set of $X(Y - bX)^h \prod_{i=1}^{v}(X - a_i)$. Replacing the coordinate system $(X, Y)$ by $(X, Y - bX)$, we see that (a-i) is satisfied.

If $v \leq 1$ then $\Gamma'$ is the zero-set of $(X - a)^v \prod_{i=1}^{h}(Y - b_i)$, for some $a \in \mathbb{k}$ and $b_1, \ldots, b_h \in \mathbb{k}$. Then $\Gamma$ is the zero-set of $X(X - a)^v \prod_{i=1}^{h}(Y - b_iX)$. If $a = 0$ or $v = 0$ then $\Gamma$ is the zero-set of $X(X - 1)^0 \prod_{i=1}^{h}(Y - b_iX)$, so (a-ii) holds; if $a \neq 0$ and $v \neq 0$ then $\Gamma$ is the zero-set of $X(X - a) \prod_{i=1}^{h}(Y - b_iX)$, so (a-ii) holds after replacing $(X, Y)$ by $(a^{-1}X, Y)$.

So assertion (a) is true. To prove assertion (c), consider the factorization $f = \gamma \circ f'$ given by (b). Since $n(\gamma) = 1$ by 3.3(a), we have $n(f') = n(f) - 1$ by 2.5. Moreover, the fact that $\text{Miss}(f')$ is admissible implies, by 1.7, that there exists a coordinate system of $\mathbb{A}^2$ with respect to which all missing curves of $f'$ have degree 1. It is clear that (c) follows by induction on $n(f)$. \hfill \Box

3.7. Remark. We stress that assumption (ii) of 3.6 is strictly stronger than “all missing curves are coordinate lines”. Indeed, there exists an irreducible element $f \in \text{Bir}(\mathbb{A}^2)$
with three missing curves, these being the lines $\mathbf{Z}(X + Y)$ and $\mathbf{Z}(X - Y)$ and the parabola $\mathbf{Z}(Y - X^2)$:

$$
\mathbb{A}^2 \xrightarrow{f} \mathbb{A}^2 = \begin{array}{c}
\end{array}
$$

(an example of such an $f$, due to Russell, appeared in [8, 4.7]). Here, each missing curve is a coordinate line, and hence has degree 1 with respect to a suitable choice of coordinate system. However, these three lines are not simultaneously rectifiable, so $f$ does not satisfy assumption (ii) of 3.6 (it does not satisfy the conclusion either: since $f$ is not a SAC and is irreducible, it is not a composition of SACs).

Also note that, by 1.4, assumption (ii) of 3.6 is equivalent to “Miss($f$) is weakly admissible”.

3.8. From now-on, and until the end of section 3,

we fix a coordinate system $\mathfrak{c} = (X, Y)$ of $\mathbb{A}^2$.

This allows us to identify $\mathbb{A}^2$ with $k^2$. See 1.1 for the notation $\mathbf{Z}(F)$ and for our convention regarding the definition of morphisms using coordinates.

3.9. In 3.9.1–3.9.3 below, we define three submonoids of Bir($\mathbb{A}^2$), denoted $\mathcal{H}_c$, $\mathcal{G}_c$ and $\mathcal{V}_c$, respectively. The subscript $c$ reminds us that these sets depend on the choice of $c$ made in 3.8. Since $c$ is fixed until the end of this section, there is no harm in omitting it and writing simply $\mathcal{H}$, $\mathcal{G}$, and $\mathcal{V}$. It is clear from the definitions below that these three monoids are included in the submonoid of Bir($\mathbb{A}^2$) generated by SACs and automorphisms.

3.9.1. Given $m \in \mathbb{N}$ and $p \in k[Y]$ such that\(^3\) $\deg p < m$, define $h_{m,p} \in \text{Bir}(\mathbb{A}^2)$ by $h_{m,p}(x, y) = (xy^m + p(y), y)$. Observe that $h_{m,p}$ is equivalent to $\gamma^m$, where $\gamma$ is the SAC given by $(x, y) \mapsto (xy, y)$; consequently, $n(h_{m,p}) = n(\gamma^m) = mn(\gamma)$, i.e.,

$$n(h_{m,p}) = m.$$

Define $\mathcal{H} = \mathcal{H}_c = \{ h_{m,p} \mid m \in \mathbb{N}, p \in k[Y] \text{ and } \deg p < m \}$. It is easily verified that $\mathcal{H}$ is a submonoid of Bir($\mathbb{A}^2$).

3.9.2. Let $\mathbf{M}$ be the multiplicative monoid whose elements are the $2 \times 2$ matrices $M = \begin{pmatrix} i & j \\ k & \ell \end{pmatrix}$ with $i, j, k, \ell \in \mathbb{N}$ and $i\ell - jk = \pm 1$. It is easily verified that $\mathbf{M}$ is generated by $\{(0,1), (1,0)\}$. Given $M = \begin{pmatrix} i & j \\ k & \ell \end{pmatrix} \in \mathbf{M}$, define the morphism $\gamma_M : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ by $(x, y) \mapsto (x^iy^j, x^ky^\ell)$. Note that $\gamma_{M_1} \circ \gamma_{M_2} = \gamma_{M_1M_2}$ for all $M_1, M_2 \in \mathbf{M}$, so the set

$$\mathcal{G} = \mathcal{G}_c = \{ \gamma_M \mid M \in \mathbf{M} \}$$

We adopt the convention that the zero polynomial has degree $-\infty$; consequently, the condition $\deg p < 0$ is equivalent to $p$ being the zero polynomial (so $h_{0,p} = h_{0,0}$ is the identity map).
is a monoid (under composition) generated by \( \{ \gamma(0, 1), \gamma(0, 1) \} \). As \( \gamma(0, 1) \) is a SAC and \( \gamma(0, 1) \) is an automorphism, it follows that \( \mathcal{G} \) is a submonoid of \( \text{Bir}(\mathbb{A}^2) \).

### 3.9.3. Given a polynomial \( \varphi \in k[X] \setminus \{0\} \), define \( v_{\varphi} \in \text{Bir}(\mathbb{A}^2) \) by \( v_{\varphi}(x, y) = (x, \varphi(x)y) \).

Then let

\[
\mathcal{V} = \mathcal{V}_c = \left\{ v_{\varphi} \mid \varphi \in k[X] \setminus \{0\} \right\}.
\]

Note that \( v_{\varphi} \circ v_{\psi} = v_{\varphi \psi} = v_{\psi} \circ v_{\varphi} \) for any \( \varphi, \psi \in k[X] \setminus \{0\} \), so \( \mathcal{V} \) is a submonoid of \( \text{Bir}(\mathbb{A}^2) \).

### 3.10. Lemma. For a birational morphism \( f : \mathbb{A}^2 \to \mathbb{A}^2 \), the following are equivalent:

(a) The union of the missing curves of \( f \) is included in \( \mathcal{Z}(Y) \);

(b) there exists \( (h, \theta) \in \mathcal{H} \times \text{Aut}(\mathbb{A}^2) \) such that \( f = h \circ \theta \):

\[
\begin{array}{ccc}
\mathbb{A}^2 & \overset{f}{\longrightarrow} & \mathbb{A}^2 \\
\theta & \downarrow & \downarrow h \\
\mathbb{A}^2 & \overset{\gamma}{\longrightarrow} & \mathbb{A}^2
\end{array}
\]

Moreover, if these conditions are satisfied then the pair \( (h, \theta) \) in (b) is unique.

**Proof.** We leave it to the reader to verify that (b) implies (a) and that \( (h, \theta) \) is unique, in (b). By induction on \( n(f) \), we show that (a) implies (b).

If \( n(f) = 0 \) then (b) holds with \( \theta = f \) and \( h = h_{0,0} \).

If \( n(f) > 0 \) then \( f \) is not an isomorphism, and hence has at least one missing curve; so \( \mathcal{Z}(Y) \) is the unique missing curve of \( f \); by 3.6, this missing curve is blown-up only once. This missing curve must contain a fundamental point \( (c, 0) \) of \( f \); as \( h_{1,c} \in \mathcal{H} \) is a SAC with missing curve \( \mathcal{Z}(Y) \) and fundamental point \( (c, 0) \), 3.6 implies that \( f = h_{1,c} \circ f' \) for some birational morphism \( f' : \mathbb{A}^2 \to \mathbb{A}^2 \). It is immediate that \( h_{1,c}(\Gamma) = \Gamma \), where \( \Gamma = \mathcal{Z}(Y) \) is the missing curve of \( f \); so 2.12 implies that the union of the missing curves of \( f' \) is included in \( \mathcal{Z}(Y) \). As \( n(f') = n(f) - 1 \), we may assume by induction that \( f' = h' \circ \theta \) for some \( h' \in \mathcal{H} \) and \( \theta \in \text{Aut}(\mathbb{A}^2) \). Then \( f = h_{1,c} \circ h' \circ \theta \) is the desired factorization, where we note that \( h_{1,c} \circ h' \in \mathcal{H} \).

### 3.11. Lemma. For a birational morphism \( f : \mathbb{A}^2 \to \mathbb{A}^2 \), the following are equivalent:

(a) The union of the missing curves of \( f \) is included in \( \mathcal{Z}(XY) \);

(b) there exists \( (M, h, \theta) \in \mathcal{M} \times \mathcal{H} \times \text{Aut}(\mathbb{A}^2) \) such that \( f = \gamma_M \circ h \circ \theta \):

\[
\begin{array}{ccc}
\mathbb{A}^2 & \overset{f}{\longrightarrow} & \mathbb{A}^2 \\
\theta & \downarrow & \downarrow h \\
\mathbb{A}^2 & \overset{\gamma_M}{\longrightarrow} & \mathbb{A}^2
\end{array}
\]

**Proof.** It is easily verified that (b) implies (a). We prove that (a) implies (b) by induction on \( n(f) \). Assume that \( f \) satisfies (a).

If \( n(f) = 0 \) then \( f \) is an isomorphism, so (b) holds with \( \theta = f \), \( h = h_{0,0} \) and \( M = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \).

Let \( n > 0 \) and assume that the result is true whenever \( n(f) < n \). Now consider \( f \) satisfying (a) and such that \( n(f) = n \).
If \( q(f) = 1 \) then the missing curve \( \Gamma \) of \( f \) is \( \mathbb{Z}(X) \) or \( \mathbb{Z}(Y) \). Define \( M = (1, 0) \) (resp. \( M = (0, 1) \)) if \( \Gamma = \mathbb{Z}(Y) \) (resp. \( \Gamma = \mathbb{Z}(X) \)). Then \( \gamma_M \circ f \) has a unique missing curve, and this curve is \( \mathbb{Z}(Y) \). Applying 3.10 to \( \gamma_M \circ f \) gives \( \gamma_M \circ f = h \circ \theta \) for some \( \theta \in \text{Aut}(\mathbb{A}^2) \) and \( h \in \mathcal{H} \). Noting that \( \gamma_M \circ \gamma_M \) is the identity, we get \( f = \gamma_M \circ h \circ \theta \).

From now-on, assume that \( q(f) = 2 \). Let \( \Gamma \) be the union of the missing curves of \( f \), i.e., \( \Gamma = \mathbb{Z}(XY) \). By 3.6, some element \( L \) of \( \text{Miss}(f) = \{ \mathbb{Z}(X), \mathbb{Z}(Y) \} \) is blown-up only once. As \( (0, 0) \) is a common point of the two missing curves, it must be a fundamental point of \( f \). For a suitable choice of \( M_1 \in \{(1, 1), (1, 0)\} \), \( \gamma_{M_1} \) is a SAC with missing curve \( L \) and fundamental point \( (0, 0) \). Then 3.6 implies that \( f = \gamma_{M_1} \circ f' \) for some birational morphism \( f' : \mathbb{A}^2 \to \mathbb{A}^2 \). By 2.12, the union of the missing curves of \( f' \) is included in \( \gamma_{M_1}^{-1}(\Gamma) = \Gamma \), so \( f' \) satisfies (a). As \( n(f') = n(f) - 1 \), the inductive hypothesis implies that \( f' = \gamma_{M_2} \circ h \circ \theta \) for some \( \theta \in \text{Aut}(\mathbb{A}^2) \), \( h \in \mathcal{H} \) and \( M_2 \in \mathcal{M} \). So \( f = \gamma_{M_1} \circ \gamma_{M_2} \circ h \circ \theta \), and since \( \gamma_{M_1} \circ \gamma_{M_2} = \gamma_{M_1 M_2} \), we are done. \( \square \)

3.12. Let \( \Delta = \Delta_1 \) be the subgroup of \( \text{Aut}(\mathbb{A}^2) \) whose elements are of the form \( \delta(x, y) = (x, y + q(x)) \), with \( q \in \mathbb{k}[X] \).

3.13. Lemma. Let \( \Gamma = \mathbb{Z} \left( \prod_{i=1}^{s} (X - c_i) \right) \) where \( c_1, \ldots, c_s \) (\( s \geq 0 \)) are distinct elements of \( \mathbb{k} \). For a birational morphism \( f : \mathbb{A}^2 \to \mathbb{A}^2 \), the following are equivalent:

(a) The union of the missing curves of \( f \) is included in \( \Gamma \);
(b) there exists a commutative diagram

\[
\begin{array}{ccc}
\mathbb{A}^2 & \xrightarrow{f} & \mathbb{A}^2 \\
\downarrow{\theta} & & \downarrow{\delta} \\
\mathbb{A}^2 & \xrightarrow{\varphi} & \mathbb{A}^2
\end{array}
\]

where \( \delta \in \Delta, \theta \in \text{Aut}(\mathbb{A}^2), \varphi \in \mathbb{k}[X] \setminus \{0\} \) and where the set of roots of \( \varphi \) is included in \( \{c_1, \ldots, c_s\} \).

Proof. That (b) implies (a) is left to the reader. Suppose that \( f \) satisfies (a). We prove (b) by induction on \( n(f) \).

If \( n(f) = 0 \) then \( f \) is an isomorphism, so (b) holds with \( \theta = f, \varphi = 1 \) and \( \delta = \text{id} \).

Let \( n > 0 \) be such that the result is true whenever \( n(f) < n \). Consider \( f \) satisfying (a) and such that \( n(f) = n \). Then \( f \) is not an isomorphism, and hence has at least one missing curve (so \( s > 0 \)). By 3.6, one of the missing curves (say \( L = \mathbb{Z}(X - c_j) \)) of \( f \) is blown-up only once. We know that \( L \) contains a fundamental point \( (c_j, d) \) of \( f \); let \( \delta_1 \in \Delta \) be defined by \( \delta_1(x, y) = (x, y - d) \) and let \( f_1 = \delta_1 \circ f \). Since \( L \) is a missing curve of \( f \) which is blown-up only once and \( (c_j, d) \in L \) is a fundamental point of \( f \), it follows that \( \delta_1(L) = L \) is a missing curve of \( f_1 \) which is blown-up only once and that \( \delta_1(c_j, d) = (c_j, 0) \in L \) is a fundamental point of \( f_1 \). As \( \nu_{(X-c_j)} \) is a SAC with missing curve \( L \) and fundamental point \( (c_j, 0) \), 3.6 implies that \( f_1 \) factors through \( \nu_{(X-c_j)} \). Thus
\[ \delta_1 \circ f = v_{(X-c_j)} \circ f' \text{ for some birational morphism } f' : \mathbb{A}^2 \to \mathbb{A}^2 : \]

\[
\begin{array}{ccc}
\mathbb{A}^2 & \xrightarrow{f} & \mathbb{A}^2 \\
\downarrow{f'} & & \downarrow{\delta_1} \\
\mathbb{A}^2 & \xrightarrow{v_{(X-c_j)}} & \mathbb{A}^2 \\
\end{array}
\]

Since \( \delta_1 \) maps each vertical line onto itself, the union of all missing curves of \( f_1 \) is \( \Gamma \); so, by 2.12, the union of the missing curves of \( f' \) is included in \( v_{(X-c_j)}^{-1}(\Gamma) = \Gamma \), so \( f' \) satisfies (a). As \( n(f') = n(f) - 1 \), the inductive hypothesis implies that there exists a commutative diagram (ignore the dotted arrows for now)

\[
(32)
\]

with \( \varphi' \in k[X] \setminus \{0\} \) (and all roots of \( \varphi' \) are in \( \{c_1, \ldots, c_s\} \)), \( \theta \in \text{Aut}(\mathbb{A}^2) \), and \( \delta_2 \in \Delta \) defined by \( \delta_2(x,y) = (x,y + q(x)) \) (some \( q \in k[X] \)). Now if we define \( \delta_3 \in \Delta \) by \( \delta_3(x,y) = (x,y + (x-c_j)q(x)) \), then

\[
\delta_3 \circ v_{(X-c_j)} = v_{(X-c_j)} \circ \delta_2.
\]

So diagram (32), including the dotted arrows, is commutative. Let \( \delta = \delta_3 \circ \delta_1 \in \Delta \) and \( \varphi = (X-c_j)\varphi'(X) \) (so \( v_{(X-c_j)} \circ \varphi' = \varphi \)); then \( \delta, \theta, v_{\varphi} \) give the commutative diagram displayed in the statement of assertion (b).

3.14. Lemma. Let \( \Gamma = Z(Y \prod_{i=1}^{s} (X-c_i)) \), where \( s \geq 1 \) and \( c_1, \ldots, c_s \) are distinct elements of \( k \). Let \( f : \mathbb{A}^2 \to \mathbb{A}^2 \) be a birational morphism such that:

\[
(33) \text{ The union of the missing curves of } f \text{ is equal to } \Gamma.
\]

Then there exists a commutative diagram

\[
\begin{array}{ccc}
\mathbb{A}^2 & \xrightarrow{f} & \mathbb{A}^2 \\
\downarrow{\theta} & & \downarrow{T} \\
\mathbb{A}^2 & \xrightarrow{\varphi} & \mathbb{A}^2 \\
\end{array}
\]

where \( T \in \text{Aut}(\mathbb{A}^2) \) is of the form \( T(x,y) = (x-c,y) \) with \( c \in k \), \( \theta \) is an arbitrary element of \( \text{Aut}(\mathbb{A}^2) \), and \( (\varphi, M, h) \in (k[X] \setminus \{0\}) \times M \times \mathcal{H} \).

Proof. We first settle the case \( s = 1 \). Define \( T \in \text{Aut}(\mathbb{A}^2) \) by \( T(x,y) = (x-c_1,y) \). Then the union of the missing curves of \( T \circ f \) is \( Z(xy) \), so 3.11 implies that there
exists $(M, h, \theta) \in \mathbf{M} \times \mathbf{H} \times \text{Aut}(\mathbb{A}^2)$ such that $T \circ f = \gamma_M \circ h \circ \theta = v_{\varphi} \circ \gamma_M \circ h \circ \theta$ with $\varphi = 1$, so the result is true when $s = 1$.

We proceed by induction on $n(f)$. For $f$ satisfying (33) we have $q(f) = s + 1 \geq 2$, so the least possible value for $n(f)$ is 2. If $n(f) = 2$ then $q(f) \leq n(f) = 2$, so $s = 1$ and the result is true in this case.

Let $n > 2$ be such that the result is true whenever $n(f) < n$. Consider $f$ satisfying (33) and such that $n(f) = n$.

By the first paragraph, we may assume that $s > 1$. By 3.6, one of the missing curves (say $L$) of $f$ is blown-up only once; we choose such an $L$. By 2.6(e), the points $(c_i, 0)$, $1 \leq i \leq s$, are fundamental points of $f$; so $Z(Y)$ is blown-up at least $s \geq 2$ times and hence $L = Z(X - c_j)$ for some $j \in \{1, \ldots, s\}$. As $v_{(X-c_j)}$ is a SAC with missing curve $L$ and fundamental point $(c_j, 0)$, 3.6 implies that $f = v_{(X-c_j)} \circ f'$ for some birational morphism $f': \mathbb{A}^2 \to \mathbb{A}^2$. Let $\Gamma' = \bigcup_{C \in \text{Miss} (f')} C$. By 2.12, $\Gamma' \subseteq v^{-1}_{(X-c_j)}(\Gamma) = \Gamma$; in fact it is easy to see (again by 2.12) that

$$Z(Y \prod_{i \in I} (X - c_i)) \subseteq \Gamma' \subseteq Z(Y \prod_{i = 1}^s (X - c_i)),$$

where $I = \{1, \ldots, s\} \setminus \{j\}$, so $f'$ satisfies the hypothesis of the lemma. As $n(f') = n(f) - 1$, the inductive hypothesis implies that $T \circ f' = v_{\psi} \circ \gamma_M \circ h \circ \theta$ for some $(\psi, M, h) \in (k[X] \setminus \{0\}) \times \mathbf{M} \times \mathbf{H}$ and $\theta, T \in \text{Aut}(\mathbb{A}^2)$, where $T$ is of the form $T(x, y) = (x - c, y)$ for some $c \in k$. Noting that $T \circ v_{(X-c_j)} = v_{(X+c-c_j)} \circ T$, we get

$$T \circ f = T \circ v_{(X-c_j)} \circ f' = v_{(X+c-c_j)} \circ T \circ f' = v_{(X+c-c_j)} \circ v_{\psi} \circ \gamma_M \circ h \circ \theta$$

$$= v_{\varphi} \circ \gamma_M \circ h \circ \theta,$$

where $\varphi(X) = (X + c - c_j)\psi(X) \in k[X] \setminus \{0\}$, as desired. \hfill \square

Before stating the main theorem of this section, let us recall the assumptions under which it is valid. Our base field $k$ is an algebraically closed field of arbitrary characteristic, and $\mathbb{A}^2$ is the affine plane over $k$. We fix a coordinate system $c = (X, Y)$ of $\mathbb{A}^2$; this allows us to use coordinates for defining morphisms $\mathbb{A}^2 \to \mathbb{A}^2$ (cf. Section 1). The choice of $c$ also determines the submonoids $\mathbf{V} = \mathbf{V}_c$, $\mathbf{G} = \mathbf{G}_c$ and $\mathbf{H} = \mathbf{H}_c$ of Bir($\mathbb{A}^2$) (cf. 3.9). Then we have the following result:

3.15. Theorem. Let $f: \mathbb{A}^2 \to \mathbb{A}^2$ be a birational morphism.

(a) The following conditions are equivalent:

(i) Miss($f$) is weakly admissible;

(ii) $f$ is equivalent to one of the following elements of Bir($\mathbb{A}^2$):

- $\alpha^m \circ \nu \circ \gamma \circ h$, for some $(\nu, \gamma, h) \in \mathbf{V} \times \mathbf{G} \times \mathbf{H}$, $m \in \{0, 1\}$ and $i \in \{1, 2\}$, where $\alpha_1, \alpha_2 \in \text{Bir}(\mathbb{A}^2)$ are the SACs defined by $\alpha_1(x, y) = (xy, y)$ and $\alpha_2(x, y) = (x(1 - y), 1 - y)$;

- the morphism $(x, y) \mapsto (x(p(x)y + q(x)), p(x)y + q(x))$, for some $p, q \in k[X]$ with $p \neq 0$.

(b) The following conditions are equivalent:
(i) Miss(f) is admissible;
(ii) f is equivalent to \( v \circ \gamma \circ h \) for some \( (v, \gamma, h) \in V \times G \times H \).

(c) The following conditions are equivalent:
(i) Both Miss(f) and Cont(f) are admissible;
(ii) f is equivalent to an element of \( V \cup G \).

Proof. For each of (a), (b) and (c), we show that (i) implies (ii) and leave the converse to the reader. We begin with (b).

Suppose that f satisfies (b-i). Let \( \Gamma = \bigcup_{C \in \text{Miss}(f)} C \). By 1.7, there exists \( \omega \in \text{Aut}(A^2) \) such that \( \omega(\Gamma) = Z(Y^d \prod_{i=1}^{s}(X - c_i)) \), where:
\[
d \in \{0, 1\}, \; s \geq 0 \text{ and } c_1, \ldots, c_s \text{ are distinct elements of } k.
\]

Note that the union of the missing curves of \( f_1 = \omega \circ f \) is equal to \( Z(Y^d \prod_{i=1}^{s}(X - c_i)) \); as \( f_1 \sim f \), it is enough to prove that \( f_1 \) is equivalent to \( v \circ \gamma \circ h \) for some \( (v, \gamma, h) \in V \times G \times H \). So we may as well replace f by \( f_1 \) throughout; so from now-on we assume that
\[
\Gamma = \bigcup_{C \in \text{Miss}(f)} C = Z(Y^d \prod_{i=1}^{s}(X - c_i)).
\]

If \( d = 0 \), (resp. \( s = 0 \)), then the desired conclusion follows from 3.13 (resp. from 3.10). So we may assume that \( d = 1 \) and \( s \geq 1 \). Then 3.14 gives the desired conclusion, i.e., we showed that (b-i) implies (b-ii).

Suppose that (a-i) holds. Let \( \Gamma = \bigcup_{C \in \text{Miss}(f)} C \). By 1.4, f satisfies the hypothesis of 3.6. To prove (a-ii), we may assume that Miss(f) is not admissible (otherwise (a-ii) follows from (b)). Then 3.6 implies that there exists \( \omega \in \text{Aut}(A^2) \) such that \( \omega(\Gamma) = Z(F) \) where

\[
F = Y \prod_{i=1}^{s}(X - c_i Y) \quad \text{or} \quad F = Y (Y - 1) \prod_{i=1}^{s}(X - c_i Y),
\]

where \( s \geq 2 \) and \( c_1, \ldots, c_s \in k \) are distinct. We know, also by 3.6, that some missing curve of f (say \( C_0 \in \text{Miss}(f) \)) is blown-up only once. In the second case of (34), \( \omega(C_0) \) is necessarily equal to \( Z(Y) \); in the first case, we may choose \( \omega \) in such a way that \( \omega(C_0) = Z(Y) \).

It is clear that we may replace f by \( \omega \circ f \) throughout. Then we have \( \Gamma = Z(F) \), \( Z(Y) \) is a missing curve of f which is blown-up only once, and \( (0, 0) \) is the unique fundamental point of f which lies on \( Z(Y) \). If \( F \) is as in the first (resp. the second) case of (34), let \( \alpha = \alpha_1 \) (resp. \( \alpha = \alpha_2 \)), where \( \alpha_1, \alpha_2 \in \text{Bir}(A^2) \) are defined in the statement; then \( \alpha \) is a SAC with missing curve \( Z(Y) \) and fundamental point \( (0, 0) \). By 3.6, it follows that \( f = \alpha \circ f' \) for some \( f' \in \text{Bir}(A^2) \). Let \( \Gamma' \) be the union of the missing curves of \( f' \). Using 2.12, we find

in the first case of (34), \( Z(\prod_{i=1}^{s}(X - c_i)) \subseteq \Gamma' \subseteq Z(Y \prod_{i=1}^{s}(X - c_i)) \);
in the second case of (34), \( Z(Y \prod_{i=1}^{s}(X - c_i)) \subseteq \Gamma' \subseteq Z(Y (Y - 1) \prod_{i=1}^{s}(X - c_i)) \).
In particular, $f'$ satisfies the hypothesis of 3.6; by that result, some missing curve of $f'$ is blown-up only once; so $\Gamma'$ cannot be equal to $Z(Y(Y - 1) \prod_{i=1}^{s} (X - c_i))$. It follows that $\Gamma' = Z(G)$ where

$$G = \prod_{i=1}^{s} (X - c_i) \quad \text{or} \quad G = Y \prod_{i=1}^{s} (X - c_i).$$

First consider the case $G = \prod_{i=1}^{s} (X - c_i)$; then $\alpha = \alpha_1$, because the first case of (35) can only happen in the first case of (34). By 3.13, there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{A}^2 & \xrightarrow{f'} & \mathbb{A}^2 \\
\theta \downarrow & & \downarrow \delta \\
\mathbb{A}^2 & \xrightarrow{v} & \mathbb{A}^2
\end{array}$$

where $v \in V$, $\delta, \theta \in Aut(\mathbb{A}^2)$ and $\delta$ is of the form form $\delta(x, y) = (x, y - q(x))$ for some $q \in k[X]$. Then $f = \alpha_1 \circ f' = \alpha_1 \circ \delta^{-1} \circ v \circ \theta \sim \alpha_1 \circ \delta^{-1} \circ v$. Let $p \in k[X] \setminus \{0\}$ be such that $v(x, y) = (x, p(x)y)$, then

$$(\alpha_1 \circ \delta^{-1} \circ v)(x, y) = (x(p(x)y + q(x)), p(x)y + q(x)),$$

which shows that (a-ii) holds in this case.

Consider the second case, $G = Y \prod_{i=1}^{s} (X - c_i)$. Here, $\alpha$ may be either one of $\alpha_1, \alpha_2$. By 3.14, there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{A}^2 & \xrightarrow{f'} & \mathbb{A}^2 \\
\theta \downarrow & & \downarrow T \\
\mathbb{A}^2 & \xrightarrow{v \circ \gamma \circ h} & \mathbb{A}^2
\end{array}$$

where $(v, \gamma, h) \in V \times G \times \mathfrak{H}$, $\theta \in Aut(\mathbb{A}^2)$, and $T \in Aut(\mathbb{A}^2)$ is of the form $T(x, y) = (x - c, y)$, with $c \in k$. Now $\alpha \circ T^{-1} = \nu \circ \alpha$ where $\nu \in Aut(\mathbb{A}^2)$ is given by $\nu(x, y) = (x + cy, y)$. Thus

$$f = \alpha \circ f' = \alpha \circ T^{-1} \circ v \circ \gamma \circ h \circ \theta = \nu \circ \alpha \circ v \circ \gamma \circ h \circ \theta \sim \alpha \circ v \circ \gamma \circ h,$$

showing that (a-ii) holds in this case as well. So (a-i) implies (a-ii).

Let $\varphi \in k[X] \setminus \{0\}$, $M = \left( \begin{smallmatrix} i & j \\ k & l \end{smallmatrix} \right) \in M$ and $h_{m,p}(x, y) = (xy^m + p(y), y) \in \mathfrak{H}$, where $m \in \mathbb{N}$ and $p(Y) \in k[Y]$ are such that $\deg p < m$. As a preparation for the proof that (c-i) implies (c-ii), we first show:

(36) If $\text{Cont}(\gamma_M \circ h_{m,p})$ is admissible, then $\gamma_M \circ h_{m,p} \sim \gamma$ for some $\gamma \in G$.

(37) If $\text{Cont}(\nu_\varphi \circ h_{m,p})$ is admissible, then $\nu_\varphi \circ h_{m,p}$ is equivalent to an element of $V \cup G$.

Observe:

(38) $(\gamma_M \circ h_{m,p})(x, y) = ((xy^m + p(y))^iy^j, (xy^m + p(y))^ky^j)$.

To prove (36), first consider the case $ik \neq 0$; then (38) implies that $Z(XY^m + p(Y))$ is a contracting curve (or a union of contracting curves) of $\gamma_M \circ h_{m,p}$. So, by the hypothesis
of (36), each irreducible component of \( \mathbb{Z}(XY^m + p(Y)) \) has one place at infinity. The only way to achieve this is to have \( p = 0 \), in which case we have \( h_{m,p} = \gamma_{M'} \) with \( M' = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \). Then \( \gamma_M \circ h_{m,p} = \gamma_N \in \mathcal{S} \) with \( N = MM' \), so (36) is true in this case.

Consider next the case where \( ik = 0 \). Then \( M \in \left\{ \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right\} \) for some \( j, \ell \in \mathbb{N} \). If \( M = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \), then \( (\gamma_M \circ h_{m,p})(x, y) = (y, (xy^m + p(y))y^\ell) = (y, xy^{m+\ell} + y^\ell p(y)) \), which is equivalent to the birational morphism \( (x, y) \mapsto (y, xy^{m+\ell}) \), i.e., \( \gamma_M \circ h_{m,p} \sim \gamma_N \in \mathcal{S} \) with \( N = (\begin{smallmatrix} 0 & 1 \\ 1 & m+\ell \end{smallmatrix}) \). Similarly, if \( M = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \) then \( \gamma_M \circ h_{m,p} \sim \gamma_N \in \mathcal{S} \) with \( N = (\begin{smallmatrix} 1 & m+1 \\ 0 & 1 \end{smallmatrix}) \). This completes the proof of (36).

To prove (37), we first note that if \( m = 0 \) then \( v_\varphi \circ h_{m,p} = v_\varphi \circ \text{id} = v_\varphi \in \mathcal{V} \). Likewise, if \( \varphi \in k^* \) then \( v_\varphi \) is an isomorphism, so \( v_\varphi \circ h_{m,p} \sim h_{m,p} \sim \gamma_{(\frac{1}{0} \quad m)} \in \mathcal{S} \). So we may assume from now-on that \( m > 0 \) and that \( \varphi \) has at least one root.

If \( c \in k \) is a root of \( \varphi \) then \( \mathbb{Z}(XY^m + p(Y) - c) \) is a union of contracting curves of \( v_\varphi \circ h_{m,p} \), so, by the hypothesis of (37), each irreducible component of \( \mathbb{Z}(XY^m + p(Y) - c) \) has one place at infinity. As \( m > 0 \), this implies that \( p(Y) - c \) is the zero polynomial, and this is true for each root \( c \) of \( \varphi \). So \( \varphi = a(X - c)^n \) for some \( a \in k^* \) and \( n \geq 1 \), and \( h_{m,p}(x, y) = (xy^m + c, y) \). Then \( (v_\varphi \circ h_{m,p})(x, y) = (xy^m + c, a(xy^m)^n y) \), which is equivalent to \( (x, y) \mapsto (xy^m, (xy^m)^n y) \), i.e., \( v_\varphi \circ h_{m,p} \sim \gamma_N \in \mathcal{S} \) with \( N = (\begin{smallmatrix} 1 & m \\ 0 & mn+1 \end{smallmatrix}) \). This proves (37).

To prove that (c-i) implies (c-ii), we consider \( f = v_\varphi \circ \gamma_M \circ \text{id} \) for some \( (\varphi, M, h) \in (k[X] \setminus \{0\}) \times M \times \mathcal{H} \), we assume that \( \text{Cont}(f) \) is admissible, and we have to prove (c-ii). We use the notation \( M = (\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}) \in M \) and \( h(x, y) = (xy^m + p(y), y) \) where \( m \in \mathbb{N} \) and \( p(Y) \in k[Y] \) are such that \( \deg p < m \).

The assumption that \( \text{Cont}(f) \) is admissible implies in particular:

\[
(39) \quad \text{Each contracting curve of } v_\varphi \circ \gamma_M \text{ has one place at infinity.}
\]

Indeed, suppose that \( C \in \text{Cont}(v_\varphi \circ \gamma_M) \) has more than one place at infinity; then, by 2.6(c), \( C \) is not a missing curve of \( h \) and consequently there exists a curve \( C' \subset A^2 \) such that \( h(C') \) is a dense subset of \( C \). Then \( C' \) is a contracting curve of \( f = v_\varphi \circ \gamma_M \circ \text{id} \) but has more than one place at infinity (because it dominates a curve with more than one place at infinity). This contradicts the assumption that \( \text{Cont}(f) \) is admissible, so (39) is proved.

We claim:

\[
(40) \quad ij = 0 \text{ or } \varphi(X) = aX^n, \text{ for some } a \in k^* \text{ and } n \in \mathbb{N}.
\]

Indeed, suppose that \( \varphi \) is not of the form \( aX^n \) with \( a \in k^* \) and \( n \in \mathbb{N} \); then there exists \( c \in k^* \) such that \( \varphi(c) = 0 \). Then \( \mathbb{Z}(x^iy^j - c) \) is a contracting curve of \( v_\varphi \circ \gamma_M \) and, if \( ij \neq 0 \), this curve has more than one place at infinity, contradicting (39). So (40) is proved.

Consider the case where \( \varphi(X) = aX^n \). Then \( v_\varphi = \theta \circ \gamma_{M_1} \) where \( \theta \in \text{Aut}(A^2) \) and \( M_1 = (\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}) \in M \). Then \( f = v_\varphi \circ \gamma_M \circ h = \theta \circ \gamma_{M_1} \circ \gamma_M \circ h \sim \gamma_{M_1M} \circ h \), so (36) implies that \( f \sim \gamma \) for some \( \gamma \in \mathcal{S} \), so we are done in this case.

There remains the case \( ij = 0 \); here we have \( M \in \left\{ \left( \begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix} \right), \left( \begin{smallmatrix} k & 0 \\ 0 & 1 \end{smallmatrix} \right) \right\} \) for some \( k, \ell \in \mathbb{N} \).
If $M = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ then $\gamma_M = v_{(X^k)}$ so $f = v_\varphi \circ v_{(X^k)} \circ h_{m,p} = v_{\varphi_1} \circ h_{m,p}$ where $\varphi_1 = X^k \varphi(X)$, so (37) implies that $f$ is equivalent to an element of $\mathcal{V} \cup \mathcal{G}$ (so we are done).

If $M = \begin{pmatrix} 0 & 1 \\ 1 & \ell \end{pmatrix}$ then $M = M_1 M_2$ where $M_1 = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Now $\gamma_{M_2} = \tau$, where $\tau \in \text{Aut}(\mathbb{A}^2)$ is defined by $\tau(x,y) = (y,x)$, and $\gamma_{M_1} = v_{(X^\ell)}$. So we have $f \sim f \circ \tau = v_\varphi \circ \gamma_{M_1} \circ \gamma_{M_2} \circ h_{m,p} \circ \tau = (v_\varphi \circ v_{(X^\ell)}) \circ (\tau \circ h_{m,p} \circ \tau) = v_{\varphi_1} \circ (\tau \circ h_{m,p} \circ \tau)$, where $\varphi_1 = X^\ell \varphi(X)$. We have $(\tau \circ h_{m,p} \circ \tau)(x,y) = (x, yx^m + p(x))$, so

$$(v_{\varphi_1} \circ (\tau \circ h_{m,p} \circ \tau))(x,y) = (x, \varphi_1(x)(yx^m + p(x))) = (x, x^m \varphi_1(x)y + \varphi_1(x)p(x)),$$

which is equivalent to the birational morphism $(x,y) \mapsto (x, x^m \varphi_1(x)y)$ with $\psi = X^m \varphi_1$. So $f \sim v_\psi \in \mathcal{V}$ and we have shown that (c-i) implies (c-ii). □

3.16. **Corollary.** Let $f \in \text{Bir} \mathbb{A}^2$. Suppose that all missing curves of $f$ are lines, and that these are simultaneously rectifiable. Then there exists a coordinate system of $\mathbb{A}^2$ with respect to which the configuration of missing curves is one of the following:

(a) Parallel lines $L_1, \ldots, L_s$ ($s \geq 0$).

(b) Parallel lines $L_1, \ldots, L_s$ ($s \geq 1$), plus one line $L_0$ not parallel to $L_1, \ldots, L_s$.

(c) Concurrent lines $L_1, \ldots, L_s$ ($s \geq 3$).

(d) Concurrent lines $L_1, \ldots, L_s$ ($s \geq 3$) plus one line $L_0$, where $L_0$ is parallel to one of the concurrent lines.

Conversely, each of the above configurations of lines occurs as the configuration of missing curves of some $f \in \text{Bir}(\mathbb{A}^2)$.
The proof below gives, in each of the cases (a)–(d), an example of an \( f \in \text{Bir}(\mathbb{A}^2) \) having the desired configuration of missing curves.

**Proof of 3.16.** The hypothesis on \( f \) is that Miss\((f)\) is weakly admissible, so \( f \) is described by part (a-ii) of Theorem 3.15; it follows that Miss\((f)\) must be one of the configurations (a)–(d). Note that Miss\((f)\) is admissible in cases (a) and (b). In cases (c) and (d), Miss\((f)\) is weakly admissible but not admissible.

Conversely, consider the configurations of lines (a)–(d). In each of the four cases we may choose a coordinate system \( c = (X,Y) \) of \( \mathbb{A}^2 \) with respect to which the configuration of lines is \( Z(F) \), where:

\[
F = \begin{cases} 
\prod_{i=1}^s (X - c_i) & \text{in case (a)}, \\
Y \prod_{i=1}^s (X - c_i) & \text{in case (b)}, \\
Y \prod_{i=1}^{s-1} (X - c_i Y) & \text{in case (c)}, \\
Y(Y - 1) \prod_{i=1}^{s-1} (X - c_i Y) & \text{in case (d)},
\end{cases}
\]

where \( c_1, \ldots, c_s \) (resp. \( c_1, \ldots, c_{s-1} \)) are distinct elements of \( k \) in cases (a) and (b) (resp. in cases (c) and (d)). Let us exhibit, in each case, an \( f \in \text{Bir}(\mathbb{A}^2) \) such that the union of all missing curves of \( f \) is \( Z(F) \). In cases (a) and (b), choose a univariate polynomial \( \varphi \in k[t] \) whose roots are exactly \( c_1, \ldots, c_s \), and define \( f \in \text{Bir}(\mathbb{A}^2) \) by

\[
f(x,y) = \begin{cases} 
(x, \varphi(x) y), & \text{in case (a)}, \\
(xy, \varphi(xy) y), & \text{in case (b)}.
\end{cases}
\]

Then the union of the missing curves of \( f \) is \( Z(F) \), as desired. In cases (c) and (d), first choose \( g \in \text{Bir}(\mathbb{A}^2) \) such that the union of the missing curves of \( g \) is \( Z(G) \), where

\[
G = \begin{cases} 
\prod_{i=1}^{s-1} (X - c_i) & \text{in case (c)}, \\
Y \prod_{i=1}^{s-1} (X - c_i) & \text{in case (d)}
\end{cases}
\]

(we know that \( g \) exists by cases (a) and (b)). Then define

\[
f = \begin{cases} 
\alpha_1 \circ g & \text{in case (c)}, \\
\alpha_2 \circ g & \text{in case (d)},
\end{cases}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are defined in the statement of Theorem 3.15. It follows from 2.12(b) that the union of the missing curves of \( f \) is \( Z(F) \). \( \Box \)

4. SOME ASPECTS OF THE MONOID \text{Bir}(\mathbb{A}^2)

Let \( k \) be an algebraically closed field and \( \mathbb{A}^2 = \mathbb{A}^2_k \), and consider the non commutative monoid \( \text{Bir}(\mathbb{A}^2) \) defined in the introduction. Note that this is a cancellative monoid, since it is included in the group of birational automorphisms of \( \mathbb{P}^2 \).

In view of 2.5 and 2.6(b), it is clear that each non invertible element of \( \text{Bir}(\mathbb{A}^2) \) is a composition of finitely many irreducible elements. In other words,

the monoid \( \text{Bir}(\mathbb{A}^2) \) has factorizations into irreducibles.
Essentially nothing is known regarding uniqueness of factorizations. It is natural to ask whether one can find all irreducible elements of Bir\((\mathbb{A}^2)\) up to equivalence. However, considering the examples given in [8], [9] and [6] and certain facts such as [8, 4.12], one gets the impression that the irreducible endomorphisms might be too numerous and too diverse to be listed. The first part of the present section gives some simple observations (4.1–4.5) that strengthen that impression.

Given \(f, g \in \text{Bir}(\mathbb{A}^2)\), let us write \(f \mid g\) if there exist \(u, v \in \text{Bir}(\mathbb{A}^2)\) such that \(u \circ f \circ v = g\). By a prime element of Bir\((\mathbb{A}^2)\), we mean a non invertible element \(p\) satisfying

\[
\text{for all } f, g \in \text{Bir}(\mathbb{A}^2), \quad p \mid (g \circ f) \Rightarrow p \mid f \text{ or } p \mid g.
\]

It follows from 2.5 and 2.6(b) that every prime element of Bir\((\mathbb{A}^2)\) is irreducible. It is natural to ask whether the converse is true, and in particular whether SACs are prime (SACs are certainly irreducible). These questions are open; we don’t even know if there exists a prime element in Bir\((\mathbb{A}^2)\).

We say that a submonoid \(M\) of Bir\((\mathbb{A}^2)\) is factorially closed in Bir\((\mathbb{A}^2)\) if the conditions \(f, g \in \text{Bir}(\mathbb{A}^2)\) and \(g \circ f \in M\) imply \(f, g \in M\). It is natural to ask whether \(A\) is factorially closed in Bir\((\mathbb{A}^2)\), where \(A\) is the submonoid of Bir\((\mathbb{A}^2)\) generated by SACs and automorphisms. The main result of this section, Theorem 4.8, states that \(A\) is indeed factorially closed in Bir\((\mathbb{A}^2)\).

**Remark.** It is obvious that the only irreducible elements of \(A\) are the SACs, that each non invertible element of \(A\) is a composition of irreducible elements, and that \(A\) has the following “unique factorization” property: if \(x_1, \ldots, x_m, y_1, \ldots, y_n\) are irreducible elements of \(A\) such that \(x_1 \circ \cdots \circ x_m = y_1 \circ \cdots \circ y_n\), then \(m = n\) and for each \(i = 1, \ldots, n\) we have \(x_i = u_i \circ y_i \circ v_i\) for some invertible elements \(u_i, v_i \in A\). (However, it is easy to see that \(A\) is not a unique factorization monoid in the sense defined in [14].)

**Irreducible elements and generating sets**

We write \([f]\) for the equivalence class of an element \(f\) of Bir\((\mathbb{A}^2)\).

**4.1. Lemma.** \(|\{ [f] \mid f \text{ is an irreducible element of Bir}(\mathbb{A}^2) \}| = |k|\).

**Proof.** Fix a coordinate system \((X, Y)\) of \(\mathbb{A}^2\). For each \(a \in k^*\), let \(C_a \subset \mathbb{A}^2\) be the zero-set of \(aY^2(Y - 1) + X \in k[X, Y]\).

Define \(U = \{(a_1, a_2, a_3) \in k^3 \mid a_1, a_2, a_3 \text{ are distinct and nonzero}\}\). Define an equivalence relation \(\approx\) on the set \(U\) by declaring that \((a_1, a_2, a_3) \approx (b_1, b_2, b_3)\) iff there

---

\(^4\)We do know that Bir\((\mathbb{A}^2)\) is not a “unique factorization monoid” in the sense of [14], but this by no means settles the question of uniqueness of factorizations in Bir\((\mathbb{A}^2)\). Indeed, there are several non equivalent definitions of what one might mean by “uniqueness of factorization” in non commutative monoids, and the one used in [14] seems to be particularly inadequate in the case of Bir\((\mathbb{A}^2)\).

\(^5\)The question is natural in view of the question whether SACs are prime and in view of the following trivial fact: let \(P\) be a set of prime elements in a commutative and cancellative monoid \(N\), and let \(\mathcal{P}\) be the submonoid of \(N\) generated by \(P\) and all invertible elements of \(N\); then \(\mathcal{P}\) is factorially closed in \(N\).
exists \( \theta \in \text{Aut}(A^2) \) satisfying \( \theta(C_{a_1} \cup C_{a_2} \cup C_{a_3}) = C_{b_1} \cup C_{b_2} \cup C_{b_3} \). The reader may check\(^6\) that the set \( U/\sim \) of equivalence classes has cardinality \(|k|\).

Given \( q \geq 2 \) and distinct elements \( a_1, \ldots, a_q \in k^* \), there exists an irreducible element \( f \in \text{Bir}(A^2) \) such that Miss\((f) = \{C_{a_1}, \ldots, C_{a_q}\}\) and \( n(f) = q + 2 \) (to see this, set \( m = 3 \) and \( \delta_1 = \cdots = \delta_{q-1} = 0 \) in \([8, 4.13])\).\(^7\) In particular, for each \( a = (a_1, a_2, a_3) \in U \) there exists an irreducible \( f_a \in \text{Bir}(A^2) \) such that Miss\((f_a) = \{C_{a_1}, C_{a_2}, C_{a_3}\}\). If \( a, b \in U \) are such that \( f_a \sim f_b \) then there exist \( \theta, \theta' \in \text{Aut}(A^2) \) satisfying \( \theta \circ f_a = f_b \circ \theta' \); then \( \theta(C_{a_1} \cup C_{a_2} \cup C_{a_3}) = C_{b_1} \cup C_{b_2} \cup C_{b_3} \), so \( a \approx b \). By the preceding paragraph we get \(|\{ [f_a] \mid a \in U \}| = |k|\), from which the desired conclusion follows. \( \square \)

4.2. Lemma. For any subset \( S \) of \( \text{Bir}(A^2) \), the following are equivalent:

\begin{enumerate}
\item \( \text{Aut}(A^2) \cup S \) is a generating set for the monoid \( \text{Bir}(A^2) \);
\item for each irreducible \( f \in \text{Bir}(A^2) \), \([f] \cap S \neq \emptyset\).
\end{enumerate}

Proof. Suppose that \( S \) satisfies (i) and consider an irreducible \( f \in \text{Bir}(A^2) \). By (i),
\[ f = g_1 \circ \cdots \circ g_n \]
for some finite subset \( \{g_1, \ldots, g_n\} \) of \( \text{Aut}(A^2) \cup S \).

By irreducibility of \( f \), exactly one element \( g_i \) of \( \{g_1, \ldots, g_n\} \) is not in \( \text{Aut}(A^2) \) (consequently, \( g_i \in S \)). So \( f \sim g_i \in S \), which proves that \( S \) satisfies (ii).

Conversely, suppose that (ii) holds and consider \( h \in \text{Bir}(A^2) \); we claim that
\[ h = g_1 \circ \cdots \circ g_N \]
for some finite subset \( \{g_1, \ldots, g_N\} \) of \( \text{Aut}(A^2) \cup S \).

This is clear if \( h \in \text{Aut}(A^2) \), so assume that \( h \notin \text{Aut}(A^2) \). Then \( h = f_1 \circ \cdots \circ f_n \)
for some finite collection \( \{f_1, \ldots, f_n\} \) of irreducible elements of \( \text{Bir}(A^2) \) (existence of a factorization into irreducibles is a consequence of 2.5). For each \( i \in \{1, \ldots, n\} \), we have \([f_i] \cap S \neq \emptyset\), so \( f_i = u_i \circ s_i \circ v_i \) for some \( s_i \in S \) and \( u_i, v_i \in \text{Aut}(A^2) \). Then
\[ h = (u_1 \circ s_1 \circ v_1) \circ \cdots \circ (u_n \circ s_n \circ v_n) = g_1 \circ \cdots \circ g_N \]
where \( \{g_1, \ldots, g_N\} \subset \text{Aut}(A^2) \cup S \). This proves (i). \( \square \)

4.3. Corollary. Let \( S \) be a subset of \( \text{Bir}(A^2) \) such that \( \text{Aut}(A^2) \cup S \) is a generating set for the monoid \( \text{Bir}(A^2) \). Then \(|S| = |k|\).

Proof. Follows from 4.1 and 4.2. \( \square \)

4.4. Remark. Let \( f \in \text{Bir}(A^2) \) and let \( \gamma = (X, Y) \) be a coordinate system of \( A^2 \). Then \( f : A^2 \to A^2 \) is given by \( f(x, y) = (u(x, y), v(x, y)) \) for some polynomials \( u, v \in k[X, Y] \). We define \( \deg_{\gamma} f = \max(\deg_{\gamma} u, \deg_{\gamma} v) \). We may also define \( \deg f \) to be the minimum of \( \deg_{\gamma} f \) for \( \gamma \) ranging over the set of coordinate systems of \( A^2 \). Then
\[ \deg f \geq \frac{c(f) + 2}{2}. \]
\(^6\)This is a tedious exercise. We leave it to the reader because it is completely elementary and has nothing to do with the subject matter of this paper.
\(^7\)Note that in Example 4.13 of [8] one has Miss\((f) = \{C_1, \ldots, C_q\}\). This doesn’t seem to be stated explicitly, but it is clear if one reads the construction.
Indeed, if $F_1, \ldots, F_c \in k[X,Y]$ are irreducible polynomials whose zero-sets are the contracting curves of $f$ (so $c(f) = c$) then the jacobian determinant of $(u,v)$ with respect to $(X,Y)$ is divisible by $\prod_{i=1}^c F_i$. This implies that $\deg_f f \geq (c+2)/2$, where the right hand side is independent from $\gamma$. Statement (41) follows.

4.5. **Corollary.** Let $S$ be a subset of $\text{Bir}(\mathbb{A}^2)$ such that $\text{Aut}(\mathbb{A}^2) \cup S$ is a generating set for the monoid $\text{Bir}(\mathbb{A}^2)$. Then $\{ \deg f \mid f \in S \}$ is not bounded.

**Proof.** Let $n \in \mathbb{N}$. By [8, 4.13], there exists an irreducible element $g \in \text{Bir}(\mathbb{A}^2)$ satisfying $c(g) \geq 2n$. By 4.2, there exists $f \in S$ satisfying $f \sim g$; then $c(f) = c(g) \geq 2n$, so $\deg f > n$ by (41).

**FACTORIAL CLOSEDNESS OF $\mathcal{A}$ IN $\text{Bir}(\mathbb{A}^2)$**

Let $\mathcal{A}$ be the submonoid of $\text{Bir}(\mathbb{A}^2)$ generated by SACs and automorphisms.

See 2.19 for the definition of $n(f, C)$, where $f \in \text{Bir}(\mathbb{A}^2)$ and $C \in \text{Miss}(f)$.

4.6. **Lemma.** Consider $\mathbb{A}^2 \xrightarrow{\alpha} \mathbb{A}^2 \xrightarrow{f} \mathbb{A}^2$ where $\alpha, f \in \text{Bir}(\mathbb{A}^2)$ and $\alpha$ is a SAC. Assume that the missing curve $C$ of $\alpha$ is disjoint from $\text{exc}(f)$ and let $D$ be the closure of $f(C)$ in $\mathbb{A}^2$. Then there exist a SAC $\alpha'$ and some $f' \in \text{Bir}(\mathbb{A}^2)$ satisfying $f \circ \alpha = \alpha' \circ f'$ and $\text{Miss}(\alpha') = \{D\}$. Moreover, if $f$ is a SAC then so is $f'$.

**Proof.** By 2.21(b), we have $D \in \text{Miss}(f \circ \alpha)$ and $n(f \circ \alpha, D) = n(\alpha, C) = 1$ (because $C \cap \text{exc}(f) = \emptyset$ and $C \cong \mathbb{A}^1$). Let $P$ be the unique fundamental point of $f \circ \alpha$ which lies on $D$ and let $\alpha'$ be a SAC with missing curve $D$ and fundamental point $P$. Then 3.5(a) implies that $f \circ \alpha = \alpha' \circ f'$ for some $f' \in \text{Bir}(\mathbb{A}^2)$. Then $n(f') = n(f)$, so if $f$ is a SAC then so is $f'$.

4.7. **Definition.** Let $h \in \text{Bir}(\mathbb{A}^2)$ be such that $h \notin \text{Aut}(\mathbb{A}^2)$. Let $C \in \text{Miss}(h)$.

(a) A factorization of $h$ is a tuple $\mathcal{f} = (h_1, \ldots, h_n)$ of elements of $\text{Bir}(\mathbb{A}^2)$ satisfying $h = h_1 \circ \cdots \circ h_n$ (where $n \geq 1$). If $h_1, \ldots, h_n$ are SACs, we say that $\mathcal{f}$ is a factorization of $h$ into SACs.

(b) Given a factorisation $\mathcal{f} = (h_1, \ldots, h_n)$ of $h$, we define $\text{depth}_i(h, C)$ to be the unique $i \in \{1, \ldots, n\}$ satisfying

there exists a missing curve of $h_i$ whose image by $h_1 \circ \cdots \circ h_{i-1}$ is a dense subset of $C$.

Observe that $\text{depth}_i(h, C) \geq 1$ and that $\text{depth}_i(h, C) = 1 \iff C \in \text{Miss}(h_1)$.

(c) If $h \in \mathcal{A}$ then we define

$\text{depth}(h, C) = \min \{ \text{depth}_i(h, C) \mid \mathcal{f} \text{ is a factorization of } h \text{ into SACs} \}$.

Note that $\text{depth}(h, C) \geq 1$, and that $\text{depth}(h, C) = 1$ is equivalent to the existence of SACs $\alpha_1, \ldots, \alpha_n$ satisfying $h = \alpha_1 \circ \cdots \circ \alpha_n$ and $\text{Miss}(\alpha_1) = \{C\}$.

---

\textsuperscript{8}This result answers a question posed by Patrick Popescu-Pampu.
4.8. **Theorem.** If \( f, g \in \text{Bir}(\mathbb{A}^2) \) satisfy \( g \circ f \in \mathcal{A} \), then \( f, g \in \mathcal{A} \).

**Proof.** We proceed by induction on \( n(g \circ f) \), the result being trivial for \( n(g \circ f) \leq 2 \). Let \( n \geq 3 \) be such that

\[
\forall f, g \in \text{Bir}(\mathbb{A}^2), \quad g \circ f \in \mathcal{A} \text{ and } n(g \circ f) < n \implies f, g \in \mathcal{A}.
\]

Consider \( f, g \in \text{Bir}(\mathbb{A}^2) \) such that \( g \circ f \in \mathcal{A} \) and \( n(g \circ f) = n \); we have to show that \( f, g \in \mathcal{A} \). Since \( g \circ f \in \mathcal{A} \), the number \( \text{depth}(g \circ f, C) \) is defined for every \( C \in \text{Miss}(g \circ f) \). Observe that

\[
\text{(42)} \quad \text{there exists } C \in \text{Miss}(g \circ f) \text{ satisfying } \text{depth}(g \circ f, C) = 1, \text{ and any such } C \text{ satisfies } n(g \circ f, C) = 1.
\]

Indeed, for any factorization \( g \circ f = \alpha_1 \circ \cdots \circ \alpha_n \) of \( g \circ f \) into SACs, the missing curve \( C \) of \( \alpha_1 \) satisfies \( C \in \text{Miss}(g \circ f) \) and \( \text{depth}(g \circ f, C) = 1 \), so \( C \) exists. Given any \( C \in \text{Miss}(g \circ f) \) satisfying \( \text{depth}(g \circ f, C) = 1 \), there exists a factorization \( g \circ f = \alpha_1 \circ \cdots \circ \alpha_n \) of \( g \circ f \) into SACs satisfying \( \text{Miss}(\alpha_1) = \{C\} \); then \( n(g \circ f, C) = n(\alpha_1 \circ \cdots \circ \alpha_n, C) = n(\alpha_1, C) = 1 \), where the second equality follows from 2.21(a). This proves (42).

We now proceed to prove that \( f, g \in \mathcal{A} \). We first do so in two special cases (numbered 1 and 2) and then in the general case.

**Case 1:** there exists \( C \in \text{Miss}(g) \) such that \( \text{depth}(g \circ f, C) = 1 \).

Then there exist SACs \( \alpha_1, \ldots, \alpha_n \) satisfying \( g \circ f = \alpha_1 \circ \cdots \circ \alpha_n \) and \( \text{Miss}(\alpha_1) = \{C\} \). We note that \( n(g, C) = n(g \circ f, C) = 1 \), where the first equality follows from 2.21(a) and the second from (42), and where the assumption \( C \in \text{Miss}(g) \) is needed for the first equality. As \( n(g, C) = 1 \), there is a unique fundamental point \( P \) of \( g \) lying on \( C \). Consider the fundamental point \( P_1 \) of \( \alpha_1 \); then 2.11 implies that \( P \) and \( P_1 \) are fundamental points of \( g \circ f \) (lying on \( C \)); as \( n(g \circ f, C) = 1 \), \( P = P_1 \); so \( \alpha_1 \) is a SAC with missing curve \( C \) and fundamental point \( P \). By 3.5(a), there exists \( g' \in \text{Bir}(\mathbb{A}^2) \) such that \( g = \alpha_1 \circ g' \). Then \( \alpha_1 \circ g' \circ f = g \circ f = \alpha_1 \circ \cdots \circ \alpha_n \); cancelling \( \alpha_1 \) yields \( g' \circ f = \alpha_2 \circ \cdots \circ \alpha_n \in \mathcal{A} \). As \( n(g' \circ f) = n - 1 \), we get \( g', f \in \mathcal{A} \) by (\(*\)). Then \( g = \alpha_1 \circ g' \in \mathcal{A} \) as well, so we are done in Case 1.

**Case 2:** \( n(f) = 1 \).

Note that \( f \) is a SAC; let \( C \) be its missing curve. By (42), we may consider \( D_1 \in \text{Miss}(g \circ f) \) satisfying \( \text{depth}(g \circ f, D_1) = 1 \) and \( n(g \circ f, D_1) = 1 \).

By Case 1, we may assume that \( D_1 \notin \text{Miss}(g) \). Then (by 2.12) \( D_1 \) is the closure of \( g(C) \); since \( C \cong \mathbb{A}^1 \), we have in fact \( g(C) = D_1 \) (every dominant morphism \( \mathbb{A}^1 \to \mathcal{C} \) is surjective). Let \( P \) be the unique fundamental point of \( g \circ f \) on \( D_1 \) and let \( Q \in C \) be the fundamental point of \( f \); then \( g(Q) \in D_1 \) is a fundamental point of \( g \circ f \) by 2.11, so \( g(Q) = P \).

Since \( n(g \circ f, D_1) = 1 = n(f, C) \), 2.21(b) implies that \( C \cap \text{exc}(g) = \emptyset \). By 2.8, \( g \) restricts to an isomorphism \( \mathbb{A}^2 \setminus \text{exc}(g) \to \mathbb{A}^2 \setminus \Gamma_g \), where \( \Gamma_g \) is the union of all missing curves of \( g \). Since \( C \subset \mathbb{A}^2 \setminus \text{exc}(g) \) and \( D_1 = g(C) \), it follows that \( D_1 \subset \mathbb{A}^2 \setminus \Gamma_g \). Since \( P \in D_1 \subset \mathbb{A}^2 \setminus \Gamma_g \) and \( \text{cent}(g) \subseteq \Gamma_g \), we have \( P \notin \text{cent}(g) \) and hence \( n(g, P) = 0 \); so
2.21 gives \( n(g \circ f, P) = n(g, P) + \sum_{P' \in \{Q\}} n(f, P') = 1 \) and we have shown

\[
D_1 \cap \Gamma_g = \emptyset \text{ and } n(g \circ f, P) = 1.
\]

Since \( \text{depth}(g \circ f, D_1) = 1 \), we may choose a factorization \( g \circ f = \alpha_1 \circ \cdots \alpha_n \) of \( g \circ f \) into SACs satisfying Miss(\( \alpha_1 \)) = \{D_1\}. We have cent(\( \alpha_1 \)) = \{P\} because the fundamental point of \( \alpha_1 \) is a fundamental point of \( g \circ f \) lying on \( D_1 \). Write Cont(\( \alpha_1 \)) = \{E_1\}, then by 2.21,

\[
n(\alpha_1 \circ (\alpha_2 \circ \cdots \circ \alpha_n), P) = n(\alpha_1, P) + \sum_{P' \in E_1} n(\alpha_2 \circ \cdots \circ \alpha_n, P')
\]

where the left hand side is equal to \( n(g \circ f, P) = 1 \) by (43). As \( n(\alpha_1, P) = 1 \), we have \( n(\alpha_2 \circ \cdots \circ \alpha_n, P') = 0 \) for all \( P' \in E_1 \), so cent(\( \alpha_2 \circ \cdots \circ \alpha_n \)) \( \cap E_1 = \emptyset \) and in particular cent(\( \alpha_2 \)) \( \cap E_1 = \emptyset \). It follows that the missing curve \( C_2 \) of \( \alpha_2 \) is not equal to \( E_1 \) (because cent(\( \alpha_2 \)) \( \subset C_2 \)). So the closure of \( \alpha_1(C_2) \) in \( \mathbb{A}^2 \) is a curve \( D_2 \) such that

\[
D_2 \in \text{Miss}(g \circ f) \setminus \{D_1\} = \text{Miss}(g).
\]

Then \( D_2 \subseteq \Gamma_g \), so \( D_2 \cap D_1 = \emptyset \) by (43). If \( C_2 \cap E_1 \neq \emptyset \) then \( \alpha_1(C_2) \cap \alpha_1(E_1) \neq \emptyset \), so \( P \in D_2 \), contradicting \( D_2 \cap D_1 = \emptyset \); thus

\[
C_2 \cap E_1 = \emptyset.
\]

This allows us to use 4.6. By that result, there exist SACs \( \alpha'_1, \alpha'_2 \) such that \( \alpha_1 \circ \alpha_2 = \alpha'_1 \circ \alpha'_2 \) and Miss(\( \alpha'_1 \)) = \{D_2\}. Since \( g \circ f = \alpha'_1 \circ \alpha'_2 \circ \alpha_3 \circ \cdots \circ \alpha_n \) is a factorization of \( g \circ f \) into SACs satisfying Miss(\( \alpha'_1 \)) = \{D_2\}, we have depth(\( g \circ f, D_2 \)) = 1. Since \( D_2 \in \text{Miss}(g) \), Case 1 implies that \( f, g \in \mathcal{A} \).

**General case.** The result is trivial if \( n(f) = 0 \), and follows from Case 2 if \( n(f) = 1 \). So we may assume that \( n(f) \geq 2 \). Consequently, \( n(g) \leq n - 2 \).

By (42), we may pick \( D \in \text{Miss}(g \circ f) \) satisfying \( \text{depth}(g \circ f, D) = 1 \) and \( n(g \circ f, D) = 1 \). By Case 1, we may assume that \( D \notin \text{Miss}(g) \). Then \( D \) is the closure of \( g(C) \) for some \( C \in \text{Miss}(f) \). We have \( 1 \leq n(f, C) \leq n(g \circ f, D) = 1 \), so \( n(f, C) = 1 \). Then 3.5(a) implies that there exist an SAC \( \alpha \) and some \( f' \in \text{Bir}(\mathbb{A}^2) \) such that \( f = \alpha \circ f' \) and Miss(\( \alpha \)) = \{C\}. On the other hand, the fact that \( \text{depth}(g \circ f, D) = 1 \) allows us to choose a factorization \( g \circ f = \alpha_1 \circ \cdots \circ \alpha_n \) of \( g \circ f \) into SACs satisfying Miss(\( \alpha_1 \)) = \{D\}. We have \( D \in \text{Miss}(g \circ \alpha) \) and

\[
n(g \circ \alpha, D)^{2.21(a)} = n(g \circ \alpha \circ f', D) = n(g \circ f, D) = 1.
\]

Let \( P \) be the unique fundamental point of \( g \circ \alpha \) lying on \( D \); then \( P \) is a fundamental point of \( g \circ f \) and hence is the unique fundamental point of \( g \circ f \) lying on \( D \). As the fundamental point of \( \alpha_1 \) is a fundamental point of \( g \circ f \) lying on \( D \), it follows that \( \alpha_1 \) is a SAC with missing curve \( D \) and fundamental point \( P \). Then 3.5(a) implies that
there exists $g' \in \text{Bir}(\mathbb{A}^2)$ satisfying $g \circ \alpha = \alpha_1 \circ g'$.

$$
\begin{array}{c}
\mathbb{A}^2 \xrightarrow{f} \mathbb{A}^2 \xrightarrow{g} \mathbb{A}^2 \\
\downarrow \quad \alpha \quad \downarrow \alpha_1 \\
\mathbb{A}^2 \xrightarrow{f'} \mathbb{A}^2 \xrightarrow{g'} \mathbb{A}^2
\end{array}
$$

Since $\alpha_1 \circ g' \circ f' = g \circ f = \alpha_1 \circ \cdots \circ \alpha_n$, cancelling $\alpha_1$ gives $g' \circ f' = \alpha_2 \circ \cdots \circ \alpha_n \in \mathcal{A}$.

By $\ast$, we obtain $f', g' \in \mathcal{A}$.

Since $g' \in \mathcal{A}$, it follows that $f = \alpha \circ f' \in \mathcal{A}$.

Since $g' \in \mathcal{A}$ we get $g \circ \alpha = \alpha_1 \circ g' \in \mathcal{A}$; we also have $n(g \circ \alpha) < n$, because $n(g) \leq n - 2$; so $g \in \mathcal{A}$ by $\ast$.

So $f, g \in \mathcal{A}$. \qed

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