The Angular Fractal Dimension in Galaxy’s Distributions

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Abstract. The study of large scale structure (LSS) of the Universe has entered a precision era due to all-sky surveys and numerical simulations. The new data has provided a way to bring new methods to bear to analyze the cosmology as probed by large scale structure. We use wavelet packets to investigate fractal point-processes on galactic scales. In particular, we develop a method to calculate the angular fractal dimension of galaxy distributions as a function of cosmological comoving distance. Taking advantage of the self-similarity and localization properties of discrete wavelets, we compute the angular fractal dimension of galaxies in narrow redshift bins. The narrow bins assure that dynamical evolution in the range being studied has not occurred to a significant extent. We use both real and simulated data from the Baryon Oscillation Spectroscopic Survey (BOSS) and the Mock Galaxy Catalogs produced by the Sloan Digital Sky Survey (SDSS). Using the wavelet packet power spectrum, we find areas in the galaxy distribution which have power law like behavior indicating fractal processes are present. The exponent of the power law is the Hurst exponent $H$, which is directly related to the fractal dimension of spatial point processes. We find the fractal dimension ranges from $D = 1.1$ to $D = 1.4$ for BOSS Galaxies while it ranges from $D = 1.4$ to $D = 1.8$ for Mock Galaxy Catalogs. The results are mildly dependent on the number of galaxies present in each redshift bin and less so on the resolution at which the data is binned. We conclude that this method can be used to characterize large scale structure and its evolution as a function of redshift. There are hints that the galaxy distribution may be fractal at higher redshifts than previously reported, however more data is necessary before a firmer conclusion is reached.

1. Introduction

Fractals were introduced to the modern reader by the ground breaking work of Mandelbrot[1]. This work spurred some early work to determine if fractals might be present in spatial distribution of galaxies [2, 3]. These early studies strongly suggested that at small distances, $r < 10h^{-1}$ Mpc, the galaxy distribution was fractal. These early studies were limited in both depth and coverage of the sky, but did establish that at small scales the cosmological principle, that the universe is both isotropic and homogeneous, did not hold. With the advent of all sky surveys that probed the matter field deeper in redshift as well as across the entire celestial sphere, fractal studies could be extended to probe the scales at which these results hold.

In this work, we calculate the angular fractal dimension of the angular galaxy distributions using wavelet packet transform methods at redshifts between $0.5 < z < 1$. The novel feature of this study is the use of the fractal dimension as a function of redshift (or comoving distance) to probe the evolution of large scale structure. To do so, we calculate the angular fractal dimension
in narrow redshift bins. To find the fractal dimension we make use of the both the self-similar and compact support properties of wavelet packets. Unlike the original studies, we have access to a rich data set that covers most of the northern sky and at much deeper redshifts. Specifically, we use the Sloan Digital Sky Survey’s Baryonic Oscillation Spectroscopic Survey (SDSS BOSS) [4]and SDSS Mock Galaxy Catalogs. This survey covered 10,000 square degrees of the sky and found the redshifts of about 1.5 million luminous galaxies and gives one of the most complete galaxy surveys in existence.

2. Fractals
The exact definition of a fractal varies from study to study but all contain the following traits. First, a fractal is self-similar. That is, the distribution is copy of itself when seen at different scales and/or when translated. When seen at a single scale, the geometry appears irregular and not smooth. The associated idea of fractal dimension is a dimension \( d \), which is a description of the capacity of the data set to fill the embedding space.

Secondly, the measures used to determine the scaling behavior, themselves scale. This is the origin of the oft-used box counting dimension in which a distribution is covered by boxes of one scale, then covered again with boxes of a different scale. The behavior of number of boxes versus scale is a measure of the fractalness of the distribution. It is this property that leads us to use wavelets to determine the existence of fractal behavior in the galaxy distribution.

Generally, fractal behavior is suggested if the power spectrum follows a power law over some range of frequencies. That is, there is a strong indicator of fractal behavior if the spectrum of system has a form

\[
S(f) \approx f^{-\alpha}
\]

where \( S(f) \) is the power at frequency \( f \) and \( \alpha \) is called the power law index. It is this fact that we will use in subsequent sections to determine any fractal behavior in the galaxy distributions. In essence, determining the power spectrum of our process will let us know if there are fractals, and the index \( \alpha \) will be related to the fractal dimension. One of the important features of this work is exactly how we find the power spectrum.

The use of the fractal dimension in the study of large scale structure is not the most common measure. The power spectrum and the two point correlation function are established statistical measures to classify large scale structure since the earliest days of galaxy surveys. Following the development of fractal geometry, fractal analysis of galaxies has been added to the statistical measures of cosmology to better understand the large scale structure. In principle, all statistical measures contain the same information so long as one has complete access to all the data and multiple data sets are available. Neither of these is the case in cosmology where there is only one universe and all observational data is limited due to the constraints of the measuring instruments. This means that in practice different statistical measures extract different kinds of information. To form a complete picture of the universe therefore, multiple lines of inquiry are needed. From a cosmological perspective as applied to galaxy distribution, the fractal dimension is a measure of the degree of inhomogeneity in the distribution, i.e., how irregular the distribution is. It provides information about the dominance of voids in structure, hence, it is an important tool in studying matter clustering. In addition, the transition from the homogeneous universe to the irregular universe can be used as guide as to when nonlinear effects must be accounted for.

3. Wavelet Transform
The wavelet transform is an integral transform that has gained increasing use in all types of signal analysis. In the study of large scale structure, wavelets have been used to study the interstellar medium [5], the distribution of quasars [6, 7], galactic clustering and sub-structure [8, 9], and even to analyze the cosmic microwave background [10, 11]. The main reason for this is that
wavelets can analyze a signal and get both frequency and spatial information simultaneously. Further, the basis functions that are used to perform the transform can be built to fit a particular set of boundary conditions. This has led to a myriad of wavelet-like tools such as needlets, spherical wavelets, maximal overlap wavelet, etc. For good reviews on wavelets, see for example [12, 13], and for reviews on the statistical applications of wavelets [14, 15].

While wavelets come in various forms, they can be categorized into two broad classes, discrete and continuous. The difference between the two is in how the basis functions treat scaling. Continuous wavelets use exponential-like scaling with powers less than 2. The discrete wavelets analyze the scale information of a distribution in powers of 2. This difference determines the choice of general wavelet to use. The discrete wavelets form an orthogonal basis which makes perfect reconstruction possible and statistical analysis much more convenient. Continuous wavelets allow for finer localization and frequency resolution, but are over-redundant and there is significant overlap between wavelets at each scale and between scales. Because our interest are in analyzing the galaxy as a function scale in order to detect fractal behavior, the fact that the discrete wavelet keeps the information between wavelets independent determines our choice to use the discrete wavelet transform.

Here we will use the language of signal analysis to describe discrete wavelets in more detail. Typically discrete wavelets are constructed using two functions, the wavelet functions which act like high pass filters and the scaling functions that work like low pass filters. These two filters work on a signal, producing two new sub-signals, one which contains the low frequency information and one that contains the high frequency information. Another way to look at this is that the low pass filter generates a signal that contains the local density information, while the high pass filters generates a signal that contains the local fluctuations.

In the discrete wavelet transform, the high and low pass filters are translated in discrete steps, $l$ across the signal at a particular scale $j$. The resulting coarse grained local density that results from the action of the low pass filter is then subjected to the filtering process again. Two new sub-signals emerge. This process is repeated and forms the basis for multi-resolution analysis and can be used to define a wavelet spectrum.

One can consider signals to be points in a vector space. Here we are concerned with the wavelet basis, $\phi_s(t)$ for this vector space so that any signal, $s(t)$ can be written as

$$s(t) = \sum_{-\infty}^{\infty} \langle \phi(k)s(k) \rangle \phi(t)$$

where $\langle \phi(k)s(k) \rangle$ is the inner product. These coefficients are called the wavelet coefficients and the $\phi$ are made up of two separate filters. The filters can vary and depend on the length extent that the filter is designed to span. The simplest filters are the Haar filters and in the following we focus on these.

The expansion in Eq.(2) can be recast in terms of the low and high pass filters as

$$s(t) = \sum_{-\infty}^{\infty} \langle \phi(k)s(k) \rangle \phi(t) = \sum a_0 \phi_0(t) + \sum d_j \psi_j(t)$$

where $\phi_0(t)$ is now the low pass filter and $\psi_j(t)$ the high pass filters and the wavelet coefficients of expansion are now $a_0$ and $d_j$.

As with the Fourier transform, the coefficients contain the crucial information we seek and in the case of the Haar wavelet are particularly simple to construct. The coefficients are given
Figure 1. The Discrete Wavelet Tree. The original signal in the top row is acted on by the low pass filter $G$ and the high pass filter $H$. The result is two sub-signals as represented by the wavelet coefficients. The $a$- coefficients capture the local density, while the $d$ capture the local fluctuations.

by

$$a_j = \frac{s_j + s_{j+1}}{2}$$  \hspace{1cm} (4)

$$d_j = \frac{s_j - s_{j-1}}{2}$$  \hspace{1cm} (5)

where $s_j$ is the $j$th data point. We see immediately that the approximation coefficients, $a_j$ are the average of the signal at two successive points $j$ and $j_1$, and the detail coefficients, $d_j$ are the difference of the two points. In general, the approximation and detail coefficients are obtained via a convolution between the filter and the signal. That is,

$$a_j = f(x) \ast G$$  \hspace{1cm} (6)

$$d_j = f(x) \ast H$$  \hspace{1cm} (7)

This gives two sub-signals characterized by the coefficients, $a^l_j$, $d^l_j$, where $l$ gives the location of sub-signal, and 1 is the pass that the signal has been acted on by the wavelet. The process now continues, with wavelet operating on the sub-signal composed of the $a^l_j$ coefficients. The process is illustrated in figure 1. At each pass, the wavelet generates a coarse grained signal that reveals features, both local densities and local fluctuations, at that scale. The resolution of the signal is reduced each pass by a half.

3.1. The Wavelet Packet Transform

While the wavelet transform is useful as is, it does suffer from rather low frequency resolution as each pass reduces the resolution at which the signal is analyzed by one-half. This coarse graining if often too large to capture important features in a signal.
The wavelet packet transform is one way in which we can keep the useful features of the wavelet transform, while increasing the frequency resolution at which one can analyze a signal. In the wavelet packet transform, the filters are applied to both approximation and detail coefficients after the first level of decomposition. In other words, another branch appears underneath the initial $d_j^l$ coefficients shown in figure (1) as shown now in figure (2). Each box in figure 2 represents a set of coefficients that capture the information in a frequency band. The total frequency band covered by the boxes (called leafs or nodes) ranges from 0 to the Nyquist frequency. The bands are evenly divided in this range.

In figure 2, the coefficients in each node are labeled as being acted on by the low pass filter, $a$’s or high pass filters, $d$’s. In this example, the original signal contains 8 data points labeled $s_0 \cdots s_7$. After the first pass, we have four coefficients, $a_0 \cdots a_3$ that contain information on the lower frequencies of the signal, and four coefficients $d_0 \cdots d_3$ that contain information on the higher frequencies. After the second pass, things get more interesting. The original approximation coefficients are subjected to both filters as was the case with the wavelet transform. However, the detail coefficients are also passed through the high and low pass filters. At the second pass, $j = 2$ the first leaf shows two coefficients labeled, $(a(a_1^o, a_1^1))$ and $(a(a_1^2, a_1^3))$. This means that the low pass filter has acted on each pair of coefficients and produced 2 new coefficients. These new coefficients cover the lowest frequency range of the signal. The second node is similar but instead the high pass filter is applied to the coefficients. This is similar to the wavelet case. The third node has two coefficients labeled $(a(d_1^o, d_1^1))$ and $(a(d_1^2, a_1^3))$. This means that the low pass filter has been acted on each pair of difference coefficients. Here we see the real difference between wavelets and wavelet packets. The process continues in a similar manner as shown in the rest of the figure.

The entire frequency range analyzed by the wavelet packet transform is 0 to the Nyquist frequency. In figure 2, each node at $j = 2$ contains information on 1/4 this range, while at $j = 3$, each node contains information at 1/8 the range. Thus the more passes (higher $j$) the finer the frequency resolution. However at some $j$, the number of coefficients in each node become so few that no statistically valid information can be obtained.

3.2. Wavelet Packet Spectrum
It is well known in Fourier analysis that using Parseval’s theorem one can relate the power (or variance) of a signal as a function of frequency to the Fourier coefficients. That is, the power, $P(k)$ as function of frequency, is

$$P(k) = \sum |\delta(k)|^2$$

where $\delta(k)$ are the Fourier coefficients in a frequency band centered at $k$.

Because the wavelet transform and wavelet packet transform form an orthogonal basis, Parseval’s theorem hold and it can be shown that wavelet packet power spectrum [16]

$$P_j = \sum |\tilde{\delta}_j|^2$$

where the $\tilde{\delta}_j$ are the wavelet coefficients at node $j$.

A signal that is self-similar will have a power spectrum that scales over some frequency range. That is, power law behavior in the power spectrum is a strong indicator of fractal behavior. In terms of the wavelet packet transform, using Eq. (9) fractals are indicated when,

$$P_j = \sum |\tilde{\delta}_j|^2 \approx j^\alpha.$$  \hspace{1cm} (10)

It is this probe that in subsequent sections, we will use to look for fractal behavior.
Figure 2. The WPT tree. The vector $S$ is the original signal. After the first pass or sweep $j = 1$, the original signal vector is transformed into approximation coefficients $a_i^1$ and detail coefficients $d_i^1$. The approximation coefficients are the result of the action of the low pass filter on the signal. The detail coefficients are the result of the action of the high pass filter on the signal. At the next pass, both sets of coefficients are passed through the filtering process.

4. Data
We used the Sloan Digital Sky Survey Data Release DR 13’th Baryon Oscillation Spectroscopic Survey (BOSS) [4]. The redshift covered in these surveys is $0 < z < 0.8$. The surveys covers two continuous regions of sky covering approximately 10400 deg$^2$, with a total of 2.5 million spectra. Once the data were obtained from the target objects, the spectra were processed by the BOSS data pipeline to classify the objects types (star, galaxy, quasar, etc.).

4.1. Mock Galaxy Catalogs
In addition the BOSS survey, we used also the Mock Galaxy Catalogs produced by the SDSS [17]. Mock Catalogs are used to optimize the survey, to create the covariance matrix for the clustering measurements, and to test the accuracy of the analysis. The technique to create the mock catalogs is the N-body simulation. The catalogs are created using the quick particle mesh method with the benchmark cosmological model as outlined in [17]. The model is flat $\Lambda$ CDM model with $\Omega_m = 0.274$, $\Omega_b = 0.046$, $\Omega_\Lambda = 0.726$, $H = 70$ km s$^{-1}$ Mpc, $n = 0.95$, and $\sigma_8 = 0.8$. The assumptions made in creating the mock catalogs are: (1) Galaxies form and remain in the potential wells of the dark matter halos, (2) Halos represent over dense regions of the mass field that is (100-300) times greater than the mean density that arise form the non-linear gravitational collapse. The steps of creating the mock catalogs are:

(i) Predicting the evolution of the mass field.
(ii) Locating the dark matter halos and characterize its properties.
(iii) Populating the halos with mock galaxies.
(iv) Apply survey characteristics to the galaxies box.

An updated version of the TreePM$^2$ code was used to evolve $3000^3$ particles with mass $a$ of $(5.9 \times 10^{10}h^{-1}M_p$ in a box of side $(2750)h^{-1}$ Mpc. The number of data points we used was...
approximately 32 million. The initial conditions of this simulation were: (1) Displacing the particles from a regular grid using 2nd order Lagrangian Perturbation theory, (2) $z = 25$, (3) The RMS displacement is 1.4 Mpc, (4) Time steps are sampled using the 2nd order leap frog method (friends of friends algorithm), (5) In populating galaxies by evolving the density field, the particles are selected such that they match the one and two point statistics of the dark matter halos.

5. Estimating the Fractal Dimension $D$

We used the raw SDSS galaxies data described earlier as identified using the SDSS data processing pipeline. The data files contain the right ascension, declination and red shift. The distribution of galaxies lies on the celestial sphere. We pixelize this data into matrices that span right ascension and declination in a specific redshift range. We use these matrices as inputs to the signal matrix to perform the wavelet packet transform. We estimate the power spectrum by radially averaging the two dimensional power spectrum using Eq. (9). If fractals are present, the power spectrum should have a power law behavior like that shown by Eq. (10). Fitting a log-log plot of the power spectrum gives the power law index, $\alpha$. This exponent is related to quantity known as the Hurst exponent, $H$. Finally, we calculate the fractal dimension $D$ from the Hurst exponent.

So the procedure for calculating the fractal dimension is:

(i) Using the HealPix and SDSSPix schemes, create the 2-d matrices that represent the galaxies distribution on the celestial sphere.

(ii) Apply the WPT on $2^n \times 2^n$ squared sampled matrices, and estimate the radially averaged power spectrum

$$P_i = \text{var}[C_i] \Rightarrow \text{var}(C_i, k) = \frac{1}{2^j} \sum_{k=0}^{2^j-1} |C_i, k|^2$$

where $(C_i, k)$ are the wavelet packet coefficients at level $i$ and frequency $k$.

(iii) Plot the log-log plot of the power spectrum vs. the frequency, and fit a line to the curve over the appropriate frequencies,

(iv) Calculate the Hurst exponent $H$ from the slope of the line $\alpha$. If $\alpha = (-1, -3)$, so $H$ is given by

$$H = \frac{\alpha + 3}{2}$$

If $\alpha = (-1, 1)$, so $H$ is given by

$$H = \frac{\alpha + 1}{2}$$

The Hurst exponent is a dimensionless parameter that indicates self-similarity present in the auto-correlation of a data set. Thus, it can also be defined as a measure of the self-similarity of a time series.

(v) From the Hurst exponent $H$, calculate the fractal dimension by

$$D = n + 1 - H$$

where $n$ is known as the embedding dimension, i.e., (the minimum dimension where the object can lives in). Since we are treating galaxies as points (dimension 0) our embedding dimension is $n = 1$. Therefore, equation 14 takes the form, $D = 2 - H$.

In our analysis, we slice the galaxy distribution into redshift (or comoving) spherical shells and compute the angular fractal dimension as a function of cosmological comoving space.
The term angular fractal dimension is used because the analysis is being done on spherical shells with no depth. Thus, the roughness of the distribution as measured by the Hurst exponent and fractal dimension represent roughness across the spherical shell.

6. Results
We used five different resolutions in creating the two dimensional matrices and three particular types of wavelets, Daubechies “db”, Symlets “sym” and Coiflets “coif”, with different filters. The three wavelets are orthogonal and compactly supported but they have different vanishing moments. The Symlets are remarkably similar to the Daubechies wavelets. They are nearly symmetrical wavelets proposed by Daubechies as modifications to the “db” family, and they are also the least asymmetric Daubechies wavelets. The Coiflets were introduced by Coifman. The wavelet function has 2N moments equal to 0 and the scaling function has 2N – 1 moments equal to 0. The two functions have a support of length 6N – 1. Finally, they are the compactly supported wavelets with highest number of vanishing moments for the basis functions for a given support width. We expect our results to be similar regardless the wavelet type used as long as the wavelet is orthogonal and compactly supported.

Each resolution generated pixels spanning different solid angles. The smaller the resolution, the larger the solid angle and the more galaxies that were binned into a pixel. Clearly a small resolution would cause too many galaxies to lose their spatial resolution. On the other hand, a large resolution meant a very sparse matrix with many pixels having no galaxies. This generated a large amount of shot noise that overwhelmed the signal. We chose a range of resolutions to show that, within a reasonable range, our results were not resolution dependent.

In dealing with the issue of edge distortions, we used different ways to extend the signal at the boundaries. There was very little effect on our results as a function of the boundaries. After trying out the different options, we settled down to the simplest choice: symmetric extension. In this scheme, the signal at these boundaries is replicated by symmetric values from the original signal. The powerful localization property of wavelets kept the error in the replicated signal, if any, extended to a few coefficients.

In our analysis, we used the proper distance instead of the redshift to handle the data properly. We computed the proper distance at corresponding redshifts. In reproducing the distributions of galaxies, we used a redshift bin width of \( z = 0.04 \) which corresponds to \( (167.9885) \text{ Mpc} \) at redshift 0.5. We tested different bin widths such as \( z = 0.01 \) and \( z = 0.1 \). This bin width was chosen because it was a good compromise between the 2-D projection onto a plane not being too deep while still having a good number of galaxies to process.

In fitting the line, we used the correlation coefficient \( R^2 \) to determine the goodness of the fit. We used only data that had at least an \( R^2 > 0.8 \). After calculating the different values of \( H \) at different resolutions, we calculated the fractal dimension using the relation \( D = 2 – H \). We present our results of the fractal dimension plotted against the proper distance computed using different wavelets types and filters at three different resolutions, 16, 32 and 64.

In figures 3 and 4 we show a typical power spectrum result. Figure 4 shows the log-log plot of the power spectrum in which the power nature of the power spectrum is evident over several epochs. At large frequencies, the plot flattens out as the distribution becomes more smooth. This region is not fractal. However at the lower frequencies, the power law behaviour is evident and indicates the presence of fractals at these scales. Unfortunately, the resolutions with which we could examine the data does not allow us to go below about 8 Mpc. Nevertheless, this particular spectrum shows clear power law behavior on scales of 26 Mpc to 8 Mpc, strongly hinting at a fractal structure at these scales.

At resolution 8, the values of the Hurst exponent for the BOSS Galaxies is \( H = 0.6 \pm 0.2 \). For the Mock Catalogs Galaxies, the Hurst exponent is almost constant around \( H = 0.4 \pm 0.1 \). The fractal dimension for BOSS galaxies is \( D = 1.4 \pm 0.2 \) and it has the lowest value towards the
Figure 3. The power spectrum plotted against the physical scale obeys the power law distribution at redshift $z = 0.55$.

Figure 4. Log-log plot of the power spectrum vs. frequency.

Figure 5. $D$ at resolution 32 using “db1”.

Figure 6. $D$ at resolution 64 using “db1”.

Figure 7. $D$ at resolution 32 using “sym1”.

Figure 8. $D$ at resolution 64 using “sym1”.

highest proper distance. For Mock Catalogs Galaxies, the fractal dimension is almost constant.
around $D = 1.6$. This is the lowest resolution we present. Again the spatial extent of each pixel is quite large at this resolution and this result may suffer from too large a coarse graining due to pixelization size used. We address this issue by increasing the resolution and note differences.

At resolution 16, the Hurst exponent for BOSS Galaxies is $H = 0.7 \pm 0.1$. For the Mock Catalogs Galaxies, it is $H = 0.4 \pm 0.1$. The fractal dimension of the Mock Catalogs Galaxies is almost constant around $D = 1.6 \pm 0.1$. For BOSS Galaxies, the fractal dimension is almost constant around $D = 1.3 \pm 0.1$. Despite doubling the resolution, or decreasing the pixel size, the results are remarkably close to those generated the previous resolution.

At resolution 32, the Hurst exponent of BOSS Galaxies is $H = 0.8 \pm 0.1$. For Mock Catalogs Galaxies, it has a higher fractal dimension around $D = 1.2 \pm 0.1$. For Mock Catalogs Galaxies, it has a higher fractal dimension than those generated at the previous resolutions, although the results are consistent within error bars. At this resolution, there are many more empty pixels than at the previous resolutions and this is probably affecting our results. The results produced by the Symlets and Coiflets wavelets are similar to the results of Daubechies wavelets as expected. Using the Coiflets wavelets, the fractal dimension for BOSS Galaxies is around $D = 1.2 \pm 0.1$. For Mock Catalogs Galaxies, it has a higher fractal dimension around $D = 1.7 \pm 0.2$. At this resolution, there are differences in the fractal dimension than those generated at the previous resolutions, although the results are consistent within error bars. At this resolution, there are many more empty pixels than at the previous resolutions and this is probably affecting our results. The results produced by the Symlets and Coiflets wavelets are similar to the results of Daubechies wavelets as expected. Using the Coiflets wavelets, the fractal dimension for BOSS Galaxies is around $D = 1.2 \pm 0.1$. For Mock Catalogs Galaxies, it has a higher fractal dimension around $D = 1.7 \pm 0.2$. The results using all three wavelets are in excellent agreement with each other. Moreover, the fractal dimension computed using Symlets and Coiflets wavelets lies with in the error bars of results produced by Daubechies wavelets. Changing the filter from 1 to 3 to 6 has produced a minor difference in the value of $D$. The results of the three wavelets are consistent with each other regardless the filter number used. This coherence in results regardless the wavelet type and filter used did establish that, wavelets can detect the fractal structure as long as they are orthogonal and compactly supported. The fractal dimension does not evolve as one would expect when it reaches the maximum near $D = 2$.

At higher redshifts, we expect the distribution to be smoother and the fractality of the structure to be suppressed. For the mock data, the fractal dimension is near 2 at a proper distance of 2200 Mpc and we cannot conclude that galaxies are fractals. Before then, $D$ is not 2 but because there are fewer galaxies so it is difficult to disentangle the effects of small number of galaxies.

At low redshifts, one would expect the distribution to become rougher and irregularities are more pronounced. Our results indicate that at low redshifts, the galaxies do appear more fractal. While this fractality might be a real effect, it is difficult again to confirm what causing this because the number of galaxies decreases. We did not generate resolutions that probed galaxies at scales down to 1 Mpc. To generate such a data at co-moving distances, we would have to reproduce the galaxy distribution using very high resolutions. As mentioned earlier, the
generated matrix would be extremely sparse and the shot noise generated swamps the signal.

Our results strongly hint that at least at some redshifts, galaxies are fractals probably out to distances about $10^{-20}$ Mpc as figure 3 indicates. The lack of data at higher redshifts present challenges to our analysis. However, our method does show that it is possible to study the redshift evolution of a fractal dimension in the galaxy distribution if enough data is available.

One important observation from the results is that the chosen resolution has only a small effect - the results are consistent within error bars regardless the resolution used. If our results were dependent on the chosen resolution, it would seriously raise questions on our method.

Another consistency in all the cases is that the Hurst exponent and fractal dimension values lie within the same range. The Hurst exponents are in the range of $H = (0.5 \pm 0.3)$ for BOSS Galaxies and $H = (0.4 \pm 0.2)$ for Mock Catalogs Galaxies. For the fractal dimension, the values of $D$ are always in the range of $(1 - 2)$ for both the BOSS and Mock Catalogs Galaxies. This consistency in ranges at all resolutions is another indication of the robustness of the statistics we used.

7. Conclusion
We developed a novel way of using wavelet transform to calculate the angular fractal dimension of galaxy distributions as a function of cosmological redshift. The fractal dimension as applied to galaxies quantifies the degree of homogeneity or regularity in the structure. We used data from the BOSS survey and Mock Galaxies Catalogs of the SDSS. We estimate the power spectrum of the galaxy distribution using wavelet packets and found that the power spectrum, generally, followed a power law. We extracted the power law index, also called the Hurst exponent, and from that the fractal dimension. The Hurst exponent is $H = (0.5 \pm 0.3)$ for BOSS Galaxies and $H = (0.4 \pm 0.2)$ for Mock Galaxies Catalogs. The fractal dimension reported for BOSS Galaxies is $D = (1.3 \pm 0.2)$ and $D = (1.6 \pm 0.2)$ for Mock Catalogs. The consistency of the results at all resolutions indicates the robustness of the statistics we used. What we conclude is the distribution may be fractal but more data is needed before we can make a more firm conclusion.

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