Mean square cordial labelling related to some acyclic graphs and its rough approximations

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Abstract. In this paper we investigate that the path $P_n$, comb graph $P_n \bigodot K_1$, centipede graph $(n,2)$ and star $S_n$ admits mean square cordial labeling. Also we proved that the induced sub graph obtained by the upper approximation of any sub graph $H$ of the above acyclic graphs admits mean square cordial labeling.

1. Introduction
Graph labeling [1] is one of the most important area in graph theory and it has lot of applications in many fields like circuit designing, communication networks, database management system, astronomy etc. Here we consider a simple, finite, connected and undirected graph $G = (V,E)$. For all other terminology and notations in graph theory, we follow Harary [2]. Rough set theory [3] was proposed by Z. Pawlak in 1982 and it is defined through upper and lower approximations.

The concept of cordial labeling was introduced by Cahit in the year 1987 in [4]. Mean cordial labeling of a graph defined by Ponraj et al [5]. A. Nellaimurugan et al introduced the mean square cordial labeling and they have studied it for some special graphs [6]. Also they have studied mean square cordial labeling of sometree and cycle related graphs [7], [8]. Dhanalakshmi et al have discussed graceful and even graceful labeling of rough approximations [9], [10]. Dhanalakshmi et al have studied prime labeling of rough approximations for some graphs [11].

In this paper we investigate that the graph path $P_n$, star graph $S_n$, comb graph $P_n \bigodot K_1$, centipede graph $(n,2)$ and star $S_n$ admits mean square cordial labeling. Also we have proved that the induced sub graph obtained by the upper approximation of any sub graph $H$ of the above acyclic graphs admits mean square cordial labeling.

2. Preliminaries
Definition 2.1 Comb is a graph obtained by joining a single pendant edge to each vertex of a path.

Definition 2.2 $n$-Centipede graph is a graph of $2n$ vertices obtained by joining a single pendant edge to each vertex of a path.

Definition 2.3 Centipede graph $(n,2)$ is a graph of $3n$ vertices obtained by joining a two pendant edges which are adjacent in each vertex of a path.

Definition 2.4 Star graph $S_n$ is a graph obtained by adding $n$ leaves in one internal node (apex vertex).

Definition 2.5 Let $G = (V,E)$ be a graph with $p$ vertices and $q$ edges. A Mean Square Cordial labeling of a Graph $G$ with vertex set $V$ is a bi bijection from $V$ to $\{0,1\}$ such that each edge $uv$ is assigned the
label \( \left\lfloor \frac{(f(v))^2 + f(v)^3}{2} \right\rfloor \) where \( \left\lceil x \right\rceil \) is the least integer greater than or equal to \( x \) with the condition that the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. The graph that admits a Mean Square Cordial Labeling is with 1 differ by at most 1. The graph that admits a mean square cordial labeling is Mean square cordial graph

**Definition 2.6** Let \( H \) be a sub graph of any graph \( G \) then the neighborhood of \( v \) is \( N(v) = \{v\} \cup \{u \in V(G); vu \in E(G)\} \)

**Definition 2.7** For any sub graph \( H \) of graph \( G \) then we define

(i) the lower approximation operation as \( V(H) = \{v \in V(G) / N(v) \subseteq V(H)\} \)

(ii) the upper approximation operation as \( \overline{V(H)} = \overline{\{N(v)/v \in V(H)\}} \)

The induced sub graph obtained by the lower approximation operation of any sub graph \( H \) of \( G \) has mean square cordial labeling then we can say the subgraph is lower approximation \( H \)- mean square cordial labeling on \( G \).

**Definition 2.8** **Lower approximation \( H \)- mean square cordial labeling on \( G \)** The induced sub graph obtained by the lower approximation operation of any sub graph \( H \) of \( G \) has mean square cordial labeling then we can say the subgraph is lower approximation \( H \)- mean square cordial labeling on \( G \).

**Definition 2.9** **Upper approximation \( H \)- mean square cordial labeling on \( G \)** The induced sub graph obtained by the upper approximation operation of any sub graph \( H \) of \( G \) has mean square cordial labeling then we can say the sub graph \( H \) is upper approximation \( H \)- mean square cordial labeling on \( G \).

3. Main results

**Theorem: 3.1** Path graph \( P_n, n \geq 2 \) admits mean square cordial labeling.

**Proof:** Let \( G = (V,E) \) be a path graph \( P_n \) Let \( v_1,v_2, \ldots, v_n \) be the vertices of a path graph \( P_n \).

Consider \( V(G) = \{v_i; 1 \leq i \leq n\} \) and \( E(G) = \{(v_i,v_{i+1}); 1 \leq i \leq n-1\} \)

Define \( f: V(G) \rightarrow \{0,1\} \)

**Case(i)** : If \( n \) is odd

\[
f(v_i) = \begin{cases} 
0,1 \leq i \leq \frac{n+1}{2} \\
1, \frac{n+3}{2} \leq i \leq n 
\end{cases}
\]

The induced edge labeling are as follows

\[
f(v_iv_{i+1}) = \begin{cases} 
0,1 \leq i < \frac{n+1}{2} \\
1, \frac{n+1}{2} \leq i \leq n-1 
\end{cases}
\]

**Case(ii)** : If \( n \) is even

\[
f(v_i) = \begin{cases} 
0,1 \leq i \leq \frac{n}{2} \\
1, \frac{n}{2} + 1 \leq i \leq n 
\end{cases}
\]
The induced edge labeling are as follows 

\[
f(v_i v_{i+1}) = \begin{cases} 
0, & \frac{n}{2} \leq i < \frac{n}{2} + 1 \\
1, & \frac{n}{2} + 1 \leq i \leq n - 1
\end{cases}
\]

In the view of abovelabeling pattern, we have \[|v(f(0) - v(f)(1)| \leq 1 \] and \[|e(f(0) - e(f)(1)| \leq 1\].

Hence path graph \(P_n\) admits mean square cordial labeling.

Illustration 3.1: Mean square cordial labeling of a path graph \(P_6\) of even and \(P_5\) of odd vertices shown in the Figure 1 and Figure 2

![Figure 1](image1)

![Figure 2](image2)

**Theorem: 3.2** Comb graph \(P_n \square K_1\), \(n \geq 2\) admits mean square cordial labeling.

**Proof:** Let \(G = (V,E)\) be a comb graph \(P_n \square K_1\).

Consider \(V(G) = \{v_i, u_i : 1 \leq i \leq n\}\) and \(E(G) = \{(u_i u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_i v_{i+1}) : 1 \leq i \leq n-1\}\)

Define \(f : V(G) \rightarrow \{0, 1\}\)

- \(f(u_i) = 0, 1 \leq i \leq n\)
- \(f(v_i) = 1, 1 \leq i \leq n\)

The induced edge labeling are as follows

\[
f(u_i u_{i+1}) = 0, 1 \leq i \leq n-1
\]

\[
f(u_i v_i) = 1, 1 \leq i \leq n
\]

In the view of abovelabeling pattern, we have \[|v(f(0) - v(f)(1)| \leq 1 \] and \[|e(f(0) - e(f)(1)| \leq 1\].

Hence comb graph \(P_n \square K_1\) admits mean square cordial labeling.

![Figure 3](image3)

Illustration 3.2: Mean square cordial labeling of a comb graph \(P_6 \square K_1\) shown in the Figure 3
Remark 3.2.1 n-Centipede graph also admits mean square cordial labeling by the above labeling pattern.

Illustration 3.2.2: Mean square cordial labeling of a 5-centipede graph shown in the Figure 4

![Figure 4](image-url)

Theorem 3.3 Centipede graph (n,2), n\geq 2 admits mean square cordial labeling.

Proof: Let G = (V,E) be a Centipede graph (n,2), n\geq 2.

Consider \( V(G) = \{v_i,u_i,w_i | 1 \leq i \leq n\} \) and \( E(G) = [(u_i,v_i) : 1 \leq i \leq n-1] \cup [(v_i,v_{i+1}) : 1 \leq i \leq n-1] \cup [(v_i,w_i) : 1 \leq i \leq n]\)

Define \( f: V(G) \rightarrow \{0,1\} \) by

Case(i): If \( n \) is odd

\[
\begin{align*}
  f(u_i) &= \begin{cases} 
    1, & 1 \leq i \leq \frac{n+1}{2} \\
    0, & \frac{n+3}{2} \leq i \leq n, i = 1, 2, \ldots, n 
  \end{cases} \\
  f(v_i) &= f(w_i) = \begin{cases} 
    1, & 1 \leq i \leq \frac{n-1}{2} \\
    0, & \frac{n+1}{2} \leq i \leq n, i = 1, 2, \ldots, n 
  \end{cases}
\end{align*}
\]

The induced edge labeling are as follows

\[
\begin{align*}
  f(u_i,v_i) &= \begin{cases} 
    1, & 1 \leq i \leq \frac{n+1}{2} \\
    0, & \frac{n+3}{2} \leq i \leq n, i = 1, 2, \ldots, n 
  \end{cases} \\
  f(v_i,v_{i+1}) &= \begin{cases} 
    1, & 1 \leq i < \frac{n+1}{2} \\
    0, & \frac{n+1}{2} \leq i \leq n-1, i = 1, 2, \ldots, n 
  \end{cases} \\
  f(v_i,w_i) &= \begin{cases} 
    1, & 1 \leq i \leq \frac{n-1}{2} \\
    0, & \frac{n+1}{2} \leq i \leq n, i = 1, 2, \ldots, n 
  \end{cases}
\end{align*}
\]
Case (ii) : If n is even
\[ f(v_i) = f(v_i) = f(w_i) = \begin{cases} 
0, & 1 \leq i \leq \frac{n}{2} \\
1, & \frac{n}{2} + 1 \leq i \leq n 
\end{cases} \]

The induced edge labeling are as follows

\[ f(u_i v_i) = f(v_i w_i) = \begin{cases} 
1, & 1 \leq i \leq \frac{n}{2} \\
0, & \frac{n}{2} + 1 \leq i \leq n, i = 1, 2, \ldots, n 
\end{cases} \]

\[ f(v_i v_{i+1}) = \begin{cases} 
1, & 1 \leq i < \frac{n}{2} + 1 \\
0, & \frac{n}{2} + 1 \leq i \leq n - 1, i = 1, 2, \ldots, n 
\end{cases} \]

\[ f(v_i w_i) = \begin{cases} 
1, & 1 \leq i \leq \frac{n-1}{2} \\
0, & \frac{n+1}{2} \leq i \leq n, i = 1, 2, \ldots, n 
\end{cases} \]

In the view of above labeling pattern, we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1..\)
Hence Centipede graph (n,2), n≥2 admits mean square cordial labeling.

Illustration 3.4: Mean square cordial labeling of acenipede graph (6,2) of even and acenipede graph (7,2) odd vertices shown in the Figure 5 and Figure 6
Theorem: 3.4 Star graph $S_n, n \geq 2$ admits mean square cordial labeling. Proof: Let $G = (V,E)$ be a star graph. Let $v$ be the apex vertex, $v_1, v_2, \ldots, v_n$ be the pendent vertices.

Consider $V(G) = \{v, v_1, v_2, \ldots, v_n\}$ and $E(G) = \{(vv_i): 1 \leq i \leq n\}$.

Define $f: V(G) \rightarrow \{0,1\}$ by

- $f(v) = 0, \ 1 \leq i \leq n$
- $f(v_i) = \begin{cases} 1, & i \equiv 1 \mod 2 \\ 0, & i \equiv 0 \mod 2 \end{cases}$

The induced edge labeling are as follows

- $f(uv_i) = \begin{cases} 1, & i \equiv 1 \mod 2 \\ 0, & i \equiv 0 \mod 2 \end{cases}$

In the view of abovelabeling pattern, we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence star graph $S_n$ admits mean square cordial labeling.

Illustration 3.4 Mean square cordial labeling of a star graph $S_5$ shown in the Figure 7

4. Mean square cordial labeling of a rough approximation for the above acyclic graphs:

Theorem 4.1: Any sub graph of a path graph $P_n, n \geq 2$ has upper approximation $H$ – mean square cordial labeling on $G$. 

Figure 6

![Figure 6]

Figure 7

![Figure 7]
Proof: Let the path graph be G ie., \( V(G) = \{v_1, v_2, v_3, \ldots, v_n\} \) and \( H \) be any sub graph of G ie., \( \{u_1, u_2, u_3, \ldots, u_k\} \in V(H) \).

Then the upper approximation of \( V(H) \) is \( V(H) = \{v_1, v_2, v_3, \ldots, v_k, v_{k+1}\} \) is also a path graph and by theorem 3.1 , the upper approximation \( V(H) \) is a path graph admits mean square cordial labeling. Hence the upper approximation of any sub graph \( H \) of a path graph admits mean square cordial labeling on \( G \).

Theorem 4.2: Any sub graph of a comb graph \( P_n \setminus K_1, n \geq 2 \) has upper approximation H – mean square cordial labeling on \( G \).

Proof: Let the comb graph be G ie., \( V(G) = \{u_1, u_2, u_3, \ldots, u_n, v_2, v_3, \ldots, v_n\} \) and \( H \) be any subgraph of \( G \). Based on the sub graph of \( G \) we have some cases of upper approximation and its labeling pattern which is given below

Case (i): Consider the sub graph \( H \) as the pendent vertices of \( G \), ie., \( \{v_1, v_2, v_3, \ldots, v_k\} \in V(H) \).

Then the upper approximation of \( V(H) \) is \( \overline{V(H)} = \{u_1, u_2, u_3, \ldots, u_k, v_1, v_2, v_3, \ldots, v_k, v_{k+1}\} \) is a comb graph and by theorem 3.2 , the upper approximation \( V(H) \) admits mean square cordial labeling. The comb graph be \( G \) ie., \( V(G) = \{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, \ldots, v_n\} \) is a comb graph and by remark of theorem 3.2, the upper approximation \( V(H) \) admits mean square cordial labeling. Hence the upper approximation of any sub graph \( H \) of a comb graph admits mean square cordial labeling.

Theorem 4.3: Any sub graph of a centipede graph \( (n, 2), n \geq 2 \) has upper approximation \( H \) – mean square cordial labelling on \( G \).

Proof: Let the centipede graph \( (n, 2) \) be G ie., \( V(G) = \{u_1, u_2, u_3, \ldots, u_n, v_2, v_3, \ldots, v_n, w_1, w_2, \ldots, w_n\} \) and \( H \) be any subgraph of \( G \). Based on the sub graph of \( G \) we have some cases of upper approximation and its labeling pattern which is given below

Case (i): Consider the sub graph \( H \) as the pendent vertices of \( G \), ie., \( \{v_1, v_2, v_3, \ldots, v_k\} \in V(H) \). Then the upper approximation of \( V(H) \) is \( \overline{V(H)} = \{u_1, u_2, u_3, \ldots, u_k, v_1, v_2, v_3, v_{k+1}, v_{k+2}, \ldots, v_{k+1}, w_1, w_2, \ldots, w_n\} \) is a path graph and by theorem 3.4 , the upper approximation \( V(H) \) admits mean square cordial labeling. Case (ii): Consider the sub graph \( H \) as the pendent vertices of \( G \), ie., \( \{u_1, u_2, u_3, \ldots, u_k\} \in V(H) \) . Then the upper approximation of \( V(H) \) is \( \overline{V(H)} = \{u_1, u_2, u_3, \ldots, u_k, v_1, v_2, v_3, v_{k+1}, \ldots, v_{k+1}, w_1, w_2, \ldots, w_n\} \) is a comb graph admits mean square cordial labeling. Case (iii): Consider the sub graph \( H \) as the pendent vertices of \( G \), ie., \( \{u_1, u_2, u_3, \ldots, u_k, v_{k+1}, v_1, v_2, v_3, \ldots, v_{k+1}, w_1, w_2, \ldots, w_n\} \) is a comb graph admits mean square cordial labeling. Case (iv): Consider the sub graph \( H \) as the pendent vertices of \( G \), ie., \( \{u_1, u_2, u_3, \ldots, u_k, v_1, v_2, v_3, \ldots, v_k\} \in V(H) \). Then the upper approximation of \( V(H) \) is \( \overline{V(H)} = \{u_1, u_2, u_3, \ldots, u_k, v_1, v_2, v_3, v_{k+1}, v_{k+2}, \ldots, v_{k+1}, w_1, w_2, \ldots, w_n\} \) and by the labeling pattern of case (i), we can say that upper approximation admits mean square cordial labeling. Case (v): Consider the sub graph \( H \) as the
vertices \{v_1,v_2,\ldots,v_k,w_1,\ldots,w_k\} \in V(H)\text{Then the upper approximation of } V(H) \text{ is } \overline{V(H)} = \{ u_1,u_2,u_3,\ldots,u_k,u_{k+1},v_1,v_2,\ldots,v_k+1, w_1,w_2,\ldots,w_{k+1}\} \text{ and by the labeling pattern of case (i), we can say that upper approximation admits mean square cordial labeling. Hence the upper approximation of any sub graph } H \text{ of a centipede graph } (n,2) \text{ admits } H – \text{ mean square cordial labeling on } G.

Theorem 4.4: Any sub graph of a star graph } S_n, n \geq 2 \text{ has upper approximation } H – \text{ mean square cordial labeling on } G.

Proof: Let the star graph be } G \text{ ie., } V(G) = \{v,v_1,v_2,\ldots,v_k\} \text{ and } H \text{ be any sub graph of } G. \text{ Based on the sub graph of } G \text{ we have some cases of upper approximation and its labeling pattern which is given below. Case (i): } \text{Let the sub graph be the set of all pendent vertices ,ie., } \{v_1,v_2,v_3,\ldots,v_k\} \in V(H)\text{Then the upper approximation of } V(H) \text{ is } \overline{V(H)} = \{v,v_1,v_2,v_3,\ldots,v_k\} \text{ is also a star graph and by theorem 3.4 the upper approximation } \overline{V(H)} \text{ is a star graph admits mean square cordial labeling. Case (ii): } \text{Let the sub graph be the apex vertex ,ie., } \{v\} \in V(H)\text{Then the upper approximation of } V(H) \text{ is } \overline{V(H)} = \{v,v_1,v_2,v_3,\ldots,v_k\} \text{ is also a star graph and by theorem 3.4, the upper approximation } \overline{V(H)} \text{ is a star graph admits mean square cordial labeling on } G. \text{ Hence the upper approximation of any sub graph } H \text{ of a star graph admits } H – \text{ mean square cordial labeling on } G.

5. Concluding Remarks
The investigation of labelled graph is very important due to its various applications in diverse fields. It is very interesting to study the various types of graphs which admit mean square cordial labeling. The above proved results of mean square cordial labeling and also its rough approximations of some acyclic graphs are demonstrated by means of illustrations which is helpful for your better understanding. It is an open area of research to discuss some more similar results for various graphs.

References
[1] Gallian 2011, A Dynamic Survey of Graph Labeling, Electronic Journal of Combinatorics, Vol. 18, pp. 1-219
[2] Harary Graph Theory Narosa Publishing House, New Delhi.
[3] Pawlak Rough sets International journal of computer and information science.11(1982),341-356
[4] Cahit Cordial Graphs(1987) A Weaker Version of Graceful and Harmonious Graphs ArsCombinatoria, Vol. 23, No. 3, pp. 201-207.
[5] NellaiMurugan, Heerajohn(2012), Mean cordial labeling of graphs,Open journal of Discrete Mathematics, 2(4), 145-14
[6] NellaiMurugan, Heerajohn(2012) Special Class of Mean Square Cordial Graphs,International Journal of Applied Research ; 1(11): 128-131
[7] NellaiMurugan, S.HeerajohnTree Related Mean Square Cordial Graphs ,outreach IX 2016 126-131 A multidisciplinary refereed journal
[8] NellaiMurugan, S.HeerajohnTree related of Mean Square Cordial Graphs, International Journal of research and development organization.
[9] S.Dhanalakshmi and N.Parvathi 2017, Even Graceful Labeling of RoughApproximations For Pn and Star Related Graphs International Journal of Pure and Applied Mathematics, 114
[10] Dhanalakshmi and N.Parvathi(2017) Prime labeling of Rough approximations for some special graphs, Indian journal of science and technology, 10
[11] Dhanalakshmi and N.Parvathi 2016, Lower and upper approximation H-graceful for some classes of graphs, *Global journal of pure and applied mathematics*, ISSN 0973-1768 12