The Chordal Loewner Equation and Monotone Probability Theory

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Abstract

In [5], O. Bauer interpreted the chordal Loewner equation in terms of non-commutative probability theory. We follow this perspective and identify the chordal Loewner equations as the non-autonomous versions of evolution equations for semigroups in monotone and anti-monotone probability theory. We also look at the corresponding equation for free probability theory.

Keywords: chordal Loewner equation, evolution families, non-commutative probability, free probability, monotone probability, anti-monotone probability, quantum processes

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1 Introduction

Denote by \( \mathbb{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \) the upper half-plane. Let \((\nu_t)_{t \geq 0}\) be a family of probability measures on \( \mathbb{R} \). The chordal (ordinary) Loewner equations are given by

\[
\frac{\partial}{\partial t} g_t = \int_{\mathbb{R}} \frac{1}{g_t - u} \nu_t(du) \quad \text{for almost every } t \in [0, \infty), \quad g_0(z) = z \in \mathbb{H},
\]

\[
\frac{\partial}{\partial t} \varphi_t = \int_{\mathbb{R}} \frac{1}{u - \varphi_t} \nu_t(du) \quad \text{for almost every } t \in [0, \infty), \quad \varphi_0(z) = z \in \mathbb{H}.
\]

In the first case, the mappings \( z \mapsto g_t(z) \) are conformal mappings from \( \mathbb{H} \setminus K_t \) onto \( \mathbb{H} \), where \((K_t)_{t \geq 0}\) is a family of growing hulls, i.e. \( K_t \subset \mathbb{H} \setminus K_t \) is simply connected and \( K_s \subset K_t \) whenever \( s \leq t \). The initial condition implies \( K_0 = \emptyset \). The second equation is interpreted in a similar way.

In this note, we would like to show how these equations can be interpreted in terms of monotone probability theory (equation (1.2)) and anti-monotone probability theory (equation (1.1)). These relations are in fact quite simple. In case of the second equation (1.2), we have that \(1/\varphi_t\) is the Cauchy transform of a probability measure \(\mu_t\). The process \((\mu_t)_{t \geq 0}\), in turn, naturally corresponds to a “quantum process” \((X_t)_{t \geq 0}\), which can be seen as a collection of self-adjoint bounded

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linear operators on a Hilbert space, with monotonically independent increments such that the distribution of \( X_t \) is given by \((\mu_t)_{t \geq 0}\).

In what follows, we explain this connection in more detail. We also take a look at the corresponding differential equation in free probability theory.

## 2 Non-commutative probability

Non-commutative probability theory provides an abstract description of random variables, motivated by the role that observables play in quantum mechanics. In the following we recall some of the basic notions of free probability theory and monotone probability theory. Both are non-commutative probability theories in which the classical notion of independent random variables is replaced by freely independent/monotonically independent random variables. We refer to [3] for an introduction.

A non-commutative probability space \((\mathcal{A}, \varphi)\) consists of a unital algebra \(\mathcal{A}\) and a linear functional \(\varphi : \mathcal{A} \to \mathbb{C}\) with \(\varphi(1) = 1\). The elements of \(\mathcal{A}\) are called random variables and \(\varphi\) should be thought of as an expectation. The distribution of a random variable \(a\) is simply defined as the collection of all moments \(\varphi(a^k)\), \(k \in \mathbb{N}\).

Furthermore, \((\mathcal{A}, \varphi)\) is called \(C^*\)-probability space if \(\mathcal{A}\) is a \(C^*\)-algebra and \(\varphi\) is a state, i.e. a positive linear functional of norm 1.

**Example 2.1.** Let \(\mathcal{A}\) be the space of all \(N \times N\)-matrices with the spectral norm and let \(\varphi(a) = \frac{1}{N} \text{Tr}(a)\). Then \((\mathcal{A}, \varphi)\) is a \(C^*\)-probability space.

**Example 2.2.** Let \(H\) be a Hilbert space and let \(\mathcal{A}\) be the space \(\mathcal{B}(H)\) of all bounded linear operators on \(H\). Furthermore, let \(\Omega \in H\) be a unit vector and define \(\varphi\) by \(\varphi(a) = \langle a\Omega, \Omega \rangle\). Then \((\mathcal{A}, \varphi)\) is a \(C^*\)-probability space.

In the following, we assume that \((\mathcal{A}, \varphi)\) is a \(C^*\)-probability space. If \(a \in \mathcal{A}\) is self-adjoint, then the distribution of \(a\) as defined above can be identified with a probability measure on \(\mathbb{R}\) by using the spectral theorem: There exists a probability measure on \(\mathbb{R}\) (supported on the spectrum \(\sigma(a)\)) such that for every polynomial \(p\), the value \(\varphi(p(a))\) can be represented by

\[
\varphi(p(a)) = \int_{\sigma(a)} p(z) \mu(dz).
\]

In the following, “probability measure” always means a probability measure on \(\mathbb{R}\).

### 2.1 Free probability theory

Free probability theory has been introduced by D. Voiculescu in [38]. It is based on a non-commutative notion of independence of random variables, the free independence.

A collection \(a_1, a_2, ..., a_N \in \mathcal{A}\) of random variables is called freely independent if

\[
\varphi(p_1(a_{j(1)})...p_k(a_{j(k)})) = 0
\]

for all polynomials \(p_1, ..., p_k\) such that \(j(1) \neq j(2) \neq ... \neq j(k)\) and \(\varphi(p_i(a_{j(i)})) = 0\) for all \(i = 1, ..., k\).

To simplify the notation later on, we will call an \(N\)-tuple \((a_1, a_2, ..., a_N) \in \mathcal{A}^N\) freely independent if \(a_1, ..., a_N\) are freely independent.

The usefulness of the above definition is due to the following fact: Let \(a, b \in \mathcal{A}\) be two freely independent random variables. Then the moments of \(a + b\) can be calculated by using the moments of \(a\) and \(b\) only ([38 Proposition 4.3]). This leads to the free convolution:

Assume that \(a, b\) are freely independent and self-adjoint random variables with distributions \(\mu\) and \(\nu\). The distribution of \(a + b\), denoted by \(\mu \boxplus \nu\), is called the free convolution of \(\mu\) and \(\nu\).

**Remark 2.3.** Recall the following fact from classical probability theory: For a probability measure \(\mu\) on \(\mathbb{R}\), the characteristic function (or Fourier transform) \(F_\mu\) is given by \(F_\mu(t) = \int_{\mathbb{R}} e^{itx} \mu(dx)\). Take two classically independent random variables \(X\) and \(Y\) with distributions...
$\mu$ and $\nu$. Then the distribution $\alpha$ of $X + Y$, the classical convolution of $\mu$ and $\nu$, can be expressed in a simple way by using the characteristic functions:

$$\log F_\alpha = \log F_\mu + \log F_\nu.$$ 

The free convolution of probability measures is closely related to their Cauchy transforms: First, the Cauchy transform (or Stieltjes transform) is given by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x} \mu(dx),$$ 

where we let $z \in \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$. (Note that the measure $\mu$ can be recovered from $G_\mu$ by the Stieltjes-Perron inversion formula, see [33] Theorem F.2.) Let $V_\mu$ be its right inverse, i.e. the solution of $G_\mu(V_\mu(z)) = 1, V_\mu(z) \sim \frac{1}{z}$ for $z \sim \infty$. (In general, $V_\mu$ exists as a holomorphic function defined on a Stolz angle near $\infty$.) Finally, the $R$-transform $R_\mu(z)$ of $\mu$ is defined by $R_\mu(z) = V_\mu(z) - \frac{1}{z}$.

For probability measures $\mu$ and $\nu$ on $\mathbb{R}$, the sum $\mu \boxplus \nu$ can be calculated by the formula

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z).$$

**Example 2.4.** The delta distribution $\delta_0$ has the $R$-transform $R_{\delta_0}(z) = 0$, which is thus the neutral element w.r.t $\boxplus$.

**Example 2.5.** Let $\mu = \nu = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1$. Then $G_\mu(z) = \frac{z}{z^2 - 2}, V_\mu$ is given by $\frac{\sqrt{1+4z^2}+1}{2z}$ and

$$R_\mu(z) = \frac{\sqrt{1+4z^2} - 1}{z}, \quad R_{\mu \boxplus \mu}(z) = \frac{\sqrt{1+4z^2} - 1}{z}.$$ 

Thus, one obtains that $G_{\mu \boxplus \mu}(z) = \frac{1}{\sqrt{1+4z^2}},$ which is the Cauchy transform of the arcsine distribution $\frac{1}{\pi \sqrt{1+4x^2}} dx, x \in [-2,2]$.

**Application 2.6.** Consider the $2N \times 2N$ matrix

$$D_{2N} = \begin{pmatrix} 1 & -1 & \cdots & 1 \\ -1 & 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & \cdots & 1 \end{pmatrix}.$$ 

Its spectral measure (put the mass $1/(2N)$ in each eigenvalue) is given by $\mu = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1$. Now we introduce a random Hermitian matrix as follows: Let $U_{2N}$ be a $2N \times 2N$ CUE random matrix (circular unitary ensemble) [7] and define $E_{2N} = U_{2N}^* D_{2N} U_{2N}$. The spectral distribution $\nu$ of $E_{2N}$ is also given by $\nu = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1 = \mu$.

Now, $A_{2N} : = D_{2N} + E_{2N}$ is a random Hermitian matrix with an eigenvalue distribution $\alpha_{2N}$ which seems to be rather complicated to determine. However: $D_{2N}$ and $E_{2N}$ are “asymptotically free” and a theorem from free probability theory tells us that $\alpha_{2N}$ converges weakly to $\mu \boxplus \nu$ as $N \to \infty$, which is the arcsine distribution from Example 2.5; see [36] p.64).

**Example 2.7.** Free probability theory possesses a non-commutative analogue of the central limit theorem; see [35]. The free analogue of the normal distribution (with mean zero and variance $\sigma^2$) is given by Wigner’s semicircle distribution $\mu_{W,\sigma^2}$ given by

$$\frac{\sqrt{4\sigma^2 - x^2}}{\pi 2\sigma^2} dx, \quad x \in [-2\sigma,2\sigma].$$

Here, we have $G_{\mu_{W,\sigma^2}}(z) = \frac{2}{z + \sqrt{4\sigma^2 - x^2}},$ and $R_{\mu_{W,\sigma^2}}(z) = \sigma z$, and consequently, the semicircle distribution is freely stable:

$$\mu_{W,\sigma^2} \boxplus \mu_{W,\sigma^2} = \mu_{W,\sigma^2 + \tau^2}.$$ 

1The distribution of such a matrix is given by the Haar measure on the unitary group.
Example 2.8. Let \((\mu_t)_{t \geq 0}\) be a semigroup with respect to free convolution, i.e. \(\mu_{t+s} = \mu_t \boxplus \mu_s\). Let \(\alpha\) be an arbitrary probability measure and define \(G_t := G_{\mu_t, \boxplus \alpha}\). Then \(G_t\) satisfies the PDE
\[
\frac{\partial}{\partial t} G_t(z) = -\frac{\partial}{\partial z} G_t(z) \cdot R(G_t(z)), \quad G_0(z) = G_\alpha,
\]
where \(R\) is the R-transform \(R = R_{\mu_t}\); see [122, p.74]. In this case, \(R\) has an analytic extension to \(\mathbb{H}\) and \(R : \mathbb{H} \to \mathbb{H}\).

For example, take \(\mu_t = \nu_{W,t}\), which leads to
\[
\frac{\partial}{\partial t} G_t(z) = -\frac{\partial}{\partial z} G_t(z) \cdot G_t(z), \quad G_0(z) = G_\alpha,
\]
i.e. the “free analogue of the heat equation is the complex inviscid Burgers equation” ([120], p.44), because a realization of the process \(\nu_{W,t}\) is called a free Brownian motion, see Section 4.

2.2 Monotone independence

For a probability measure \(\mu\) we define the \(F\)-transform of \(\mu\) simply as \(F_\mu := 1/G_\mu\).

Remark 2.9. Let \(\mu, \nu\) be probability measures. Then there exist holomorphic mappings \(\omega_1, \omega_2 : \mathbb{H} \to \mathbb{H}\) such that
\[
F_{\mu}^{-1}\nu(\omega_1(z)) = F_{\mu}(\omega_2(z)) \quad \text{for all } z \in \mathbb{H}.
\]
Furthermore, also \(\omega_1, \omega_2\) have the form \(F_{\sigma_1}\) and \(F_{\sigma_2}\) for probability measures \(\sigma_1, \sigma_2\); see [125, Introduction].

Now one defines the monotone convolution \(\mu \bowtie \nu\) by
\[
F_{\mu \bowtie \nu} = F_{\mu} \circ F_{\nu}.
\]
This convolution is related to another notion of independence of random variables, the monotone independence, which was introduced by N. Muraki ([25]) and independently by De. Giosa, Lu ([9, 23]).

Let \(a_1, ..., a_N \in \mathcal{A}\). The tuple \((a_1, a_2, ..., a_N)\), as an ordered collection of random variables, is called \textit{monotonically independent} if the following two conditions hold:

(a) \(\varphi(a_{ij}^pa_{ik}) = \varphi(a_{ij}^pa_{ik})\) whenever \(i < j\) and \(j > k\).

(b) \(\varphi(a_{i_1}^{p_1}...a_{i_j}^{p_j}a_{i_1}^{q_1}...a_{i_n}^{q_n}) = \varphi(a_{i_1}^{p_1})...\varphi(a_{i_j}^{p_j})\varphi(a_{i_1}^{q_1})...\varphi(a_{i_n}^{q_n})\)

whenever \(i_1 > ... > i_2 > i_1 > i < j_1 < j_2 < ... < j_n\).

Here, all exponents are natural numbers and the cases \(m = 0\) and \(n = 0\) are included.

Remark 2.10. We note that condition (a) is sometimes replaced by the stronger condition
\[(a') a_{ij}^pa_{ik}^q = \varphi(a_{ij}^pa_{ik}^q) a_{ij}^r\]

see [22, 23]. Furthermore, there are also several other notions of independence and convolutions. Some interesting relations and decompositions for convolutions are studied in [125].

Finally, we note that N. Muraki showed in [25] that there are only five “nice”, so called natural independences: the tensor, free, Boolean, monotone and anti-monotone independence; see also [4, p.198].

Example 2.11. Let \(a \in \mathcal{A}\). Then \((a, 1)\) is monotonically independent. However, the pair \((1, a)\) is not monotonically independent in general: Let \((\mathcal{A}, \varphi)\) be defined as in Example 2.1. Let \(a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). Condition (b) is not satisfied for the pair \((1, a)\) because \(\frac{1}{2} \text{Tr}(a1a) = \frac{1}{2} \text{Tr}(a^2) = \frac{1}{2} \text{Tr}(1) = 1\), while \(\frac{1}{2} \text{Tr}(a)\frac{1}{2} \text{Tr}(1)\frac{1}{2} \text{Tr}(a) = 0\).

Let \(a, b \in \mathcal{A}\) be self-adjoint random variables such that \((a, b)\) is monotonically independent. Denote by \(\mu\) and \(\nu\) the probability measure of \(a\) and \(b\) respectively. Then we have: the distribution of \(a + b\) is exactly the measure \(\mu \bowtie \nu\), see [23, Theorem 4].

Again, the delta distribution \(\delta_0\) is neutral, i.e. \(\mu \bowtie \delta_0 = \delta_0 \bowtie \mu = \mu\).

Example 2.12. Let \(\mu_{A, \sigma^2}\) be the arcsine distribution with mean \(0\) and variance \(\sigma^2\), i.e. \(\frac{1}{\pi \sqrt{2\sigma^2 - x^2}}dx\), \(x \in [-\sqrt{2\sigma}, \sqrt{2\sigma}]\). Then \(F_{\mu_{A, \sigma^2}}(z) = \sqrt{z^2 - 2\sigma^2}\) and we have
\[
\mu_{A, \sigma^2} \bowtie \mu_{A, \tau^2} = \mu_{A, \sigma^2 + \tau^2}.
\]

The arcsine distribution plays the role of the Wigner semicircle distribution in free probability; see [28, Theorem 2] for a central limit theorem in monotone probability theory.
3 Non-autonomous evolution equations

3.1 The chordal Loewner equation

In [22], C. Loewner introduced a differential equation for conformal mappings to attack the so called Bieberbach conjecture: Let $D \subset \mathbb{C}$ be the unit disc and assume that $f : D \to \mathbb{C}$ is univalent (holomorphic and injective) with $f(0) = 0$ and $f'(0) = 1$. Let $a_n$ be the coefficients of the power series expansion $f(z) = z + \sum_{n \geq 2} a_n z^n$. Then

$$|a_n| \leq n.$$ 

Loewner could prove this inequality for $n = 3$ and the conjecture has been proven completely in 1985 by L. de Branges. Since its introduction, Loewner’s approach has been extended and the Loewner differential equations are now an important tool in the theory of conformal mappings. In the following, we describe a special differential equation that goes back to P. Kufarev. We refer to [1] for an historical overview of Loewner theory.

The so called chordal Loewner equation can be described as follows:

Take a family $\{\nu_t\}_{t \geq 0}$ of probability measures and let $M_t(z) = G_{\nu_t}(z) = \int \frac{1}{z-x} \nu_t(dx)$ be the Cauchy transform of $\nu_t$. Assume that $t \mapsto M_t(z)$ is measurable for every $z \in \mathbb{H}$. (3.1)

The chordal Loewner equation is given by the Carathéodory ODE (“a.e.” stands for “almost every”)

$$\frac{\partial}{\partial t} g_t = M_t(g_t) \quad \text{for a.e. } t \in [0, \infty), \quad g_0(z) = z \in \mathbb{H},$$

and has a unique solution ([14, Theorem 4]).

For fixed $z \in \mathbb{H}$, the solution $t \mapsto g_t(z)$ may have a finite lifetime $T(z) > 0$ in the sense that $g_t(z) \in \mathbb{H}$ for all $t < T(z)$, but $\lim_{t \to T(z)} \text{Im}(g_t(z)) = 0$.

If we fix a time $t > 0$ and let $K_t = \{z \in \mathbb{H} | T(z) \leq t\}$, then $g_t(z)$ is interpreted as the conformal mapping

$$g_t : \mathbb{H} \setminus K_t \to \mathbb{H} \quad \text{with the normalization} \quad g_t(z) = z + \frac{t}{z} + o(1/z)$$

as $z \to \infty$ non-tangentially in $\mathbb{H}$. The sets $K_t \subset \mathbb{H}$ are growing hulls, which means $\mathbb{H} \setminus K_t$ is simply connected and $K_s \subset K_t$ whenever $s \leq t$. As we start with the identity mapping, we have $K_0 = \emptyset$.

![Figure 1: The mappings $g_t$ and $f_t$.](image)

The inverse mappings $f_t = g_t^{-1}$ satisfy the Loewner PDE

$$\frac{\partial}{\partial t} f_t = -\frac{\partial}{\partial z} f_t \cdot M_t(z), \quad f_0(z) = z \in \mathbb{H}.$$ (3.3)

Now, as noted in [5], one can now consider the mapping $G_t := 1/f_t$, which is the Cauchy transform of a measure $\mu_t$, and so the Loewner equation can be interpreted as a mapping

$$\mathcal{L} : \{\nu_t\}_{t \geq 0} \mapsto \{\mu_t\}_{t \geq 0}.$$  

Example 3.1. Let $\nu_t = \delta_0$ for all $t \geq 0$. The solution $g_t$ of the Loewner equation

$$\frac{\partial}{\partial t} g_t = \frac{1}{g_t}, \quad g_0(z) = z \in \mathbb{H},$$

(3.4)
is given by \( g_t(z) = \sqrt{z^2 + 2t} \). Hence, the hull \( K_t \) is a straight line segment connecting 0 to \( \sqrt{2t} \). We have \( G_t = \frac{1}{\sqrt{z^2 - 2t}} \) and thus the measure \( \mu_t \) is the arcsine distribution with variance \( t \); see Example 3.2.

The family \( G_t \) can be characterized by the differential equation

\[
\frac{\partial}{\partial t} G_t(z) = - \frac{\partial}{\partial z} G_t(z) \cdot M_t(z), \quad G_0(z) = \frac{1}{z}.
\]

\[(3.5)\]

**Example 3.2.** As noted in [5], the only fixed point of \( L \), i.e., \( M_t = G_t \) for all \( t \), is given by \( \mu_t = \mu_{W,t} \), which follows by taking \( \alpha = \delta_0 \) in equation (2.2) and comparing it with equation (3.5).

**Example 3.3.** If \( M_t \) does not depend on \( t \), i.e.,

\[
\frac{\partial}{\partial t} f_t(z) = - \frac{\partial}{\partial z} f_t(z) \cdot M_0(z), \quad f_0(z) = z,
\]

\[(3.6)\]

then the mappings \( f_t = 1/G_t \) form a semigroup with respect to composition: \( f_{t+s} = f_t \circ f_s = f_s \circ f_t \).

From this last example we see that both (2.1) and (3.6) describe semigroups with respect to different convolutions. In (2.1) we have

\[ \mu_{t+s} = \mu_t \boxplus \mu_s, \]

while \( \mu_t \) in (3.6) satisfies

\[ \mu_{t+s} = \mu_t \lozenge \mu_s, \]

because \( f_t = F_{\mu_t} \).

Next we look at the non-autonomous versions of these equations from the perspective of monotone, anti-monotone and free probability theory.

### 3.2 Monotone evolution families

A (one-real-parameter) monotone semigroup \((\mu_t)_{t \geq 0}\) is a family of probability measures having the property \( \mu_{t+s} = \mu_t \uplozenge \mu_s \), \( \mu_0 = \delta_0 \). Now we generalize monotone semigroups to monotone evolution families.

**Definition 3.4.** We call a collection \((\sigma_{s,t})_{0 \leq s \leq t}\) of probability measures a monotone evolution family if it satisfies the conditions

(a) \( \sigma_{s,t} = \delta_0 \),

(b) \( \sigma_{s,t} = \sigma_{u,t} \lozenge \sigma_{s,u} \) whenever \( 0 \leq s \leq u \leq t \),

(c) \( \sigma_{s,u} \) converges weakly to \( \sigma_{s,t} \) as \( u \to t \).

In addition, it is called normal if the first and second moments exist and

(d) \( \int_R x^2 \sigma_{s,t}(dx) = t - s \) for all \( 0 \leq s \leq t \).

Let \((\nu_t)_{t > 0}\) be a family of probability measures such that the Cauchy transforms \( M_t = \int_R \frac{1}{z-u} \nu_t(du) \) satisfy (3.1), and consider the “time reversed” version of (3.2):

\[
\frac{\partial}{\partial t} \varphi_{s,t} = -M_t(\varphi_{s,t}) \quad \text{for a.e. \( t \in [s, \infty) \)}, \quad \varphi_{s,t}(z) = z \in \mathbb{H}.
\]

\[(3.7)\]

**Remark 3.5.** Fix some \( T > 0 \). Let \( h_t \) be the solution to \( \frac{\partial}{\partial t} h_t = M_{T-t}(h_t) \). Then \( h_T \) is the inverse of \( \varphi_{0,T} \).

According to [14] Theorem 4, (3.7) has a unique solution \( \varphi_{s,t} : \mathbb{H} \to \mathbb{H} \), which is an evolution family of holomorphic mappings in the sense that

\[ \varphi_{t,t}(z) = z, \quad \varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u} \quad \text{whenever} \ 0 \leq s \leq u \leq t. \]

These solutions are exactly the \( F \)-transforms of normal monotone evolution families.
Proposition 3.6. Let \((\nu_t)_{t \geq 0}\) and \(\varphi_{s,t}\) be defined as above. For each \(0 \leq s \leq t\), the mapping \(\varphi_{s,t}\) is the \(F\)-transform of a measure \(\sigma_{s,t}\), and \((\sigma_{s,t})_{0 \leq s \leq t}\) is a normal monotone evolution family.

Conversely, let \((\sigma_{s,t})_{0 \leq s \leq t}\) be a normal monotone evolution family and let \(\varphi_{s,t}\) be the \(F\)-transform of \(\sigma_{s,t}\). Then there exists a family \((\nu_t)_{t \geq 0}\) of probability measures such that (3.1) holds and \(\varphi_{s,t}\) satisfies the Loewner equation (3.7).

Proof. We begin with the first part of the statement:
Each \(\varphi_{s,t}\) is a univalent mapping from \(\mathbb{H}\) into itself and can be represented as
\[
\varphi_{s,t}(z) = z + \int_{\mathbb{R}} \frac{1}{u - z} \beta_{s,t}(du),
\]
where \(\beta_{s,t}\) is a finite Borel measure with \(\beta_{s,t}(\mathbb{R}) = t - s\), see [11], Theorem 4 and the definition of the class \(\mathcal{P}\) on p.1210. From [6] Proposition 2.1 it follows that \(\varphi_{s,t}\) is the \(F\)-transform of a probability measure \(\sigma_{s,t}\) which has mean 0 and variance \(\beta_{s,t}(\mathbb{R}) = t - s\).

Because of (3.8), the conditions (a) and (b) in Def. 3.4 are satisfied. Furthermore, we have
\[
|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \frac{|t - u|}{\text{Im}(z)},
\]
see [13] p.1214. By Proposition 3.3 in [36], we conclude that \(\sigma_{s,u}\) converges weakly to \(\sigma_{s,t}\) as \(u \to t\), i.e. condition (c) holds as well.

Next, let \(\sigma_{s,t}\) be a normal monotone evolution family and let \(\varphi_{s,t}\) be the \(F\)-transform of \(\sigma_{s,t}\). As \(\sigma_{s,t}\) has mean 0 and finite variance, the mapping \(\varphi_{s,t}\) and can be represented as (\*) with \(\beta_{s,t}(\mathbb{R}) = t - s\).

Condition (a) and (b) imply that \(\varphi_{s,t}\) satisfies (3.8). Furthermore, the weak convergence, condition (c), implies that \(\varphi_{s,u}\) converges locally uniformly to \(\varphi_{s,t}\) as \(u \to t\). From [13] Theorem 3, it follows that \(\varphi_{s,t}\) satisfies (3.7).

Remark 3.7. The measures \((\sigma_t)_{t \geq 0} := (\sigma_{0,t})_{t \geq 0}\) form a monotone semigroup, i.e. \(\sigma_{t+s} = \sigma_t \triangleright \sigma_s\), if and only if \(\nu_t\) does not depend on \(t\). In this case, \(\sigma_{s,t} = \sigma_{0,t-s}\), and so the whole evolution family is reduced to a semigroup.

Remark 3.8. If we consider a monotone evolution family, not necessarily normal, then the \(F\)-transforms \(\varphi_{s,t}\) correspond to the differential equation
\[
\frac{\partial}{\partial t} \varphi_{s,t} = -P_t(\varphi_{s,t}) \quad \text{for a.e. } t \in [s, \infty), \quad \varphi_{s,s}(z) = z \in \mathbb{H},
\]
where \(P_t\) has the form
\[
P_t = \gamma_t + \int_{\mathbb{R}} \frac{1 + xz}{z - x} \tau_t(dx),
\]
and \(\gamma_t \in \mathbb{R}\) and \(\tau_t\) is a positive finite measure for a.e. \(t \geq 0\). This can be proven similarly by using the monotone Lévy-Khintchine formula [16, Theorem 1.1].

To summarize, for normal monotone evolution families we have three equivalent objects:
\[
(\sigma_{s,t})_{0 \leq s \leq t} \quad \overset{\text{F–transform}}{\longleftrightarrow} \quad (\varphi_{s,t})_{0 \leq s \leq t} \quad \overset{\text{Equation (3.7)}}{\longleftrightarrow} \quad (\nu_t)_{0 \leq t}.
\]

Remark 3.9. There exists also a multiplicative monotone convolution, see [12] and the references therein, which naturally corresponds to the so-called radial Loewner equation. This differential equation has been considered originally by C. Loewner in [22].

3.3 Anti-monotone evolution families

Finally, one defines quite similarly anti-monotone independence and anti-monotone convolution, see [11]. For probability measures \(\mu, \nu\), the anti-monotone convolution \(\mu \prec \nu\) is defined by \(F_{\mu \prec \nu} = F_{\nu} \circ F_{\mu}\).

Definition 3.10. We call a collection \((\sigma_{s,t})_{0 \leq s \leq t}\) of probability measures \(\sigma\) (normal) anti-monotone evolution family if it satisfies the conditions of Definition 3.4 with (b) replaced by
\[
(b') \quad \sigma_{s,t} = \sigma_{u,t} \triangleleft \sigma_{s,u} \quad \text{whenever } 0 \leq s \leq u \leq t.
\]
The family \((\varphi_{s,t})_{0 \leq s \leq t} := (F_{s,t})_{0 \leq s \leq t}\) is now what is called a reverse evolution family in Loewner theory, see [8, Definition 1.9], and one obtains analogously to Proposition 3.6.

**Proposition 3.11.** Let \((\nu_t)_{t \geq 0}\) be a family of probability measures satisfying (3.1) and let \(M_t\) be the Cauchy transform of \(\nu_t\). Denote by \(\varphi_{s,t}\) the solution to

\[
\frac{\partial}{\partial s} \varphi_{s,t} = M_s(\varphi_{s,t}) \quad \text{for a.e. } s \in [0, t], \quad \varphi_{t,t}(z) = z \in \mathbb{H}. \tag{3.10}
\]

Then \(\varphi_{s,t}\) is the \(F\)-transform of a measure \(\sigma_{s,t}\), and \((\sigma_{s,t})_{0 \leq s \leq t}\) is a normal anti-monotone evolution family.

Conversely, let \((\sigma_{s,t})_{0 \leq s \leq t}\) be a normal anti-monotone evolution family and let \(\varphi_{s,t}\) be the \(F\)-transform of \(\sigma_{s,t}\). Then there exists a family \((\nu_t)_{t \geq 0}\) of probability measures such that (3.1) holds and \(\varphi_{s,t}\) satisfies the (reverse) Loewner equation (3.10).

**Proof.** The proof is analogous to the one of Proposition 3.6 and, instead of [14, Thm. 3, 4], we can use [8, Theorem 4.2] in this situation. \(\square\)

Equivalently (see [8, Theorem 4.2]), one can describe \(\varphi_{s,t}\) as the solution to

\[
\frac{\partial}{\partial t} \varphi_{s,t} = -\frac{\partial}{\partial z} \varphi_{s,t} \cdot M_t(\varphi_{s,t}) \quad \text{for a.e. } t \in [s, \infty), \quad \varphi_{s,s}(z) = z \in \mathbb{H}. \tag{3.11}
\]

Note that (3.11) is nothing but (3.3).

**3.4 The slit equation**

The most prominent Loewner equation is the so-called slit equation, which simply corresponds to \(\nu_t = \delta_{\gamma(t)}\), where \(U : [0, \infty) \to \mathbb{R}\) is a continuous function. Both equations, (3.7) and (3.10), are called slit equations in this case.

Let us stay now in the setting of monotone probability theory. Equation (3.7) is given by

\[
\frac{\partial}{\partial t} \varphi_{s,t} = \frac{1}{U(t)-\varphi_{s,t}} \quad \text{for a.e. } t \in [s, \infty), \quad \varphi_{s,s}(z) = z \in \mathbb{H}. \tag{3.12}
\]

If the so-called driving function \(U\) is smooth enough, then the solutions \(\varphi_{s,t}\) are conformal mappings of the form \(\varphi_{s,t} : \mathbb{H} \to \mathbb{H} \setminus \gamma_{s,t}\), where \(\gamma_{s,t}\) is a simple curve \(\gamma_{s,t} : [s, t] \to \mathbb{H}\) with \(\gamma(s) = U(s) \in \mathbb{R}\) and \(\gamma_{s,t}(s, t) \subset \mathbb{H}\). Such a curve is also called a slit of the upper half-plane. For the smoothness conditions, we refer to [24, 21, 20].

Conversely, for every slit \(\gamma\) there exists \(T > 0\) and \(U : [0, T] \to \mathbb{R}\) such that the solution of (3.12) satisfies \(\varphi_{0,T}(\mathbb{H}) = \mathbb{H} \setminus \gamma\); see [15] and the references therein. If \(U(t) \equiv u \in \mathbb{R}\), then the solution \(\varphi_{s,t} = \varphi_{0,t-s}\) to (3.12) is given by

\[
\varphi_{0,t}(z) = u + \sqrt{(z-u)^2 - 2t},
\]

which maps \(\mathbb{H}\) onto \(\mathbb{H}\) minus a straight line segment from \(u\) to \(u + i\sqrt{2t}\). The corresponding probability measure is given by \(\sigma_{0,t} = \delta_{-u} \triangleright \mu_{A,t} \triangleright \delta_{u}\).

If \(U(t)\) is not constant, then one can approximate \(\varphi_{0,t}\) by the solution of a piecewise constant driving function.

Choose \(N \in \mathbb{N}\) and let \(\Delta t = \frac{1}{N}\) be a time interval. Assume we are interested in \(\varphi_{0,K\Delta t}, K \in \mathbb{N}\).

Approximately, it can be obtained as follows: Let \(\Delta_0, \Delta_1, \ldots\) be defined by \(\Delta_k = U((k+1)\Delta t) - U(k\Delta t)\). We have \(\sigma_{k\Delta t,(k+1)\Delta t} \approx \delta_{-\Delta_k} \triangleright \mu_{A,\Delta t} \triangleright \delta_{\Delta_k}\), and consequently

\[
\sigma_{0,K\Delta t} = \sigma_{(K-1)\Delta t,K\Delta t} \circ \sigma_{0,(k-1)\Delta t} = \ldots \approx \bigtriangledown_{k=0}^{K-1} (\delta_{-\Delta_k} \triangleright \mu_{A,\Delta t} \triangleright \delta_{\Delta_k}) =: \sigma_{0,K\Delta t}. \tag{3.13}
\]

We note that for the computation of the conformal mappings, a slightly different approximation is more practical for use, see [17].

**Question 3.12.** Let \(\mu\) be a probability measure such that its \(F\)-transform \(F_\mu\) is injective and \(F_\mu(\mathbb{H}) = \mathbb{H} \setminus \gamma\) for a slit \(\gamma\). How can those probability measures \(\mu\) be characterized?

A basic property of those probability measures is the symmetry with respect to a point \(u \in \mathbb{R}\), which is the preimage of the tip of the slit \(\gamma\) with respect to the mapping \(F_\mu\).
Proposition 3.13. Let \( \mu \) be a probability measure such that \( F_\mu \) maps \( \mathbb{H} \) conformally onto \( \mathbb{H} \setminus \gamma \), where \( \gamma \) is a slit. Then \( \text{supp} \, \mu \) is a compact interval \([a, b]\), \( \mu \) has a density \( d(x) \) on \((a, b)\) and there exists \( u \in (a, b) \) and a homeomorphism \( h : (a, b) \to (a, b) \) with \( h(u) = u, h(a, u] = [u, b) \) such that \( d(h(x)) = d(x) \) for all \( x \in (a, b) \).

Proof. As the domain \( \mathbb{H} \setminus \gamma \) has a locally connected boundary, the mapping \( F_\mu \) can be extended continuously to \( \overline{\mathbb{H}} \); see [34, Theorem F.6].

There exists an interval \([a, b]\) such that \( f([a, b]) = \gamma \) and there is a unique \( u \in (a, b) \) such that \( F_\mu(u) \) is the tip of the slit. All points \([a, u] \) correspond to the left side, all points \([u, b] \) to the right side of \( \gamma \). (This orientation follows from the behaviour of \( F_\mu(x) \) as \( x \to \pm \infty \).) Hence, there exists a unique homeomorphism \( h : (a, b) \to (a, b) \) with \( h(u) = u, h(a, u] = [u, b) \) such that \( F_\mu(h(x)) = F_\mu(x) \) for all \( x \in (a, b) \).

It follows from [34, Theorem F.6] that \( \mu \) is absolutely continuous on \((a, b)\) and the density \( d(x) \) satisfies \( d(x) = \frac{1}{2} \text{Im}(1/F_\mu)(x) \). Hence, \( d(h(x)) = d(x) \) for all \( x \in (a, b) \). (If 0 is not the starting point of the slit, \( \mu \) is absolutely continuous on \([a, b]\) as \( G_\mu(a), G_\mu(b) \neq \infty \) in this case.) Finally, as \( \text{Im}(1/F_\mu)(x) = 0 \) for all \( x \in \mathbb{R} \setminus [a, b] \), we conclude that \( \text{supp} \mu = [a, b] \), again by using [34, Theorem F.6]. \( \square \)

Remark 3.14. A slit \( \gamma \) is called quasislit if \( \gamma \) approaches \( \mathbb{R} \) nontangentially and \( \gamma \) is the image of a line segment under a quasiconformal mapping. The latter property can also be stated as the “bounded turning property”

\[
\sup_{x,y \in \gamma} \frac{\text{diam}(x,y)}{|x-y|} < \infty,
\]

where \( \text{diam}(x,y) \) is the diameter of the subcurve of \( \gamma \) joining \( x \) and \( y \).

The theory of conformal welding implies: \( \gamma \) is a quasislit if and only if \( h \) is quasisymmetric; see [21, Lemma 6] and [24, Lemma 2.2].

Example 3.15 (Schramm-Loewner Evolution). Let \( B_t : [0, \infty) \to \mathbb{R} \) be a standard Brownian motion and \( \kappa \in (0, \infty) \). Let \( \nu_t = \delta_{\sqrt{\kappa/2} B_t} \). The solution \( g_t \) to the stochastic differential equation (3.2), i.e.

\[
\frac{\partial}{\partial t} g_t = \frac{1}{g_t - \sqrt{\kappa} B_t}, \quad g_0(z) = z,
\]

describes the growth of a random simple curve in \( \mathbb{H} \) from 0 to \( \infty \), which is called Schramm-Loewner evolution (SLE). SLE and its generalizations have important applications in statistical mechanics and probability theory. We refer to [13] for an introduction.

The solution to (3.12) with \( U(t) = \sqrt{\kappa/2} B_t \) is called backward SLE (see [22]). It corresponds to a (classically) random normal monotone evolution family \((\sigma_{s,t})_{0 \leq s \leq t}\). Now we can approximate \( \sigma_{0, K \Delta t} \) as follows: Let \( \Delta_0, \Delta_1, \ldots \) be a sequence of (classically) independent normally distributed random variables with mean 0 and variance \( \frac{\kappa}{2} \Delta t \). Then

\[
\sigma_{0, K \Delta t} \approx \bigcap_{k=0}^{K-1} (\delta_{-\Delta_k} \triangleright \mu_{A, \Delta t} \triangleright \delta_{\Delta_k}) \approx \sigma_{0, K \Delta t}^N.
\]

We have \( \sigma_{0, K \Delta t} = \lim_{N \to \infty} \sigma_{0, K \Delta t}^N \) in the sense of convergence in distribution with respect to the topology induced by weak convergence; see [22] for an even stronger statement.

3.5 Free evolution families

Let \( (\mu_t)_{t \geq 0} \) be a free semigroup, i.e. \( \mu_{t+s} = \mu_t \boxplus \mu_s \). In this case, \( R_\mu \) has an analytic extension to \( \mathbb{H} \) and

\[
R_\mu = \alpha + \int_{\mathbb{R}} \frac{z + x}{1 - x z} \nu(dx) (3.14)
\]

with \( \alpha \in \mathbb{R} \) and \( \nu \) is a finite positive measure. Moreover, \( \mu \) has mean 0 and finite variance \( \sigma^2 \) if and only if

\[
R_\mu = \int_{\mathbb{R}} \frac{z}{1 - x z} \nu(dx) = \int_{\mathbb{R}} \frac{1}{1/z - x} \nu(dx) = G_\nu(1/z),
\]

where \( \nu \) is a measure with \( \nu(\mathbb{R}) = \sigma^2 \); see [2] Section 4.1.

By generalizing equation (2.1), we obtain evolution families with respect to the free convolution.
Thus, for all Proposition 3.17.

We assumed that the limit Definition 3.16.

see [33], Section 2.4, we have in fact locally uniform convergence and thus D countable dense subset z for form R probability measures such that ≤ t.

It can easily be verified that the closure of the set of all Cauchy-transforms of probability measures is continuous with respect to locally uniform convergence, we obtain from [7, Proposition 2.3] that of a probability measure as H is Lipschitz continuous: as s,t →→ R s,t(z) = Gνt(1/z) for a.e. t ∈ [s,∞), R s,s(z) = 0. (3.15)

Proposition 3.17. Under the above assumptions, (3.15) has a unique solution R s,t. For all 0 ≤ s ≤ t, R s,t is the R-transform of a probability measure σ s,t and the collection (σ s,t)0≤s≤t is a normal free evolution family. Conversely, let (σ s,t)0≤s≤t be a normal free evolution family. Then there exists a family (νt)≥0 of probability measures such that Gνt satisfy (3.1) and the R-transform R s,t of σ s,t satisfies equation (3.15).

Proof. Obviously, the solution R s,t of (3.15) is simply given by

R s,t(z) = \int_s^t \int_\mathbb{R} \frac{1}{1/z-x} \nu_t(dx)dt.

As R s,t is a holomorphic mapping with R s,t(H) ⊂ H, it is easy to see that this function also has the form R s,t(z) = Gσ s,t(1/z) for a positive measure σ s,t; see [14] Lemma 1. The behaviour of R s,t for z near 0 yields that σ s,t(\mathbb{R}) = \int_0^s \nu_t(\mathbb{R}) dt = t-s. This implies that R s,t(z) is the R-transform of a probability measure σ s,t with mean 0 and variance t-s. Clearly, R s,t + R u,t = R s,u whenever 0 ≤ s ≤ u ≤ t and R t,t = 0, which implies σ s,t = σ u,t ⊗ σ s,u, σ s,t = δ0. Furthermore, as t → R s,t is continuous with respect to locally uniform convergence, we obtain from [7] Proposition 2.3 that σ s,u converges weakly to σ s,t as u → t.

Conversely, let R s,t be the R-transform of σ s,t. Fix some s ≥ 0 and let t ≥ s. Then, R s,t(z) = Gσ s,t(1/z) with σ s,t(\mathbb{R}) = t-s. This implies that, for z ∈ H, the map t → R s,t(z) is Lipschitz continuous:

|R s,t(z) − R s,u(z)| = |R u,t(z)| ≤ \int_\mathbb{R} \frac{1}{1/z-u} |Gσ s,t(du)| ≤ \frac{t-u}{\text{Im}(1/z)},

for all s ≤ u ≤ t.

Thus, t → R s,t(z) is differentiable for every t ∈ [s,∞) except a zero set N(z). By considering a countable dense subset D ⊂ H, we conclude that there exists a zero set N ⊂ [s,∞) such that t → R s,t(z) is differentiable for every z ∈ D and every t ∈ [s,∞) \ N. Now assume t → R s,t is differentiable at t0 for all z ∈ D and let h > 0. Then \( h^{-1}(R s,t_0+h - R s,t_0) = h^{-1}R t_0,t_0+h(z) \) = Gβ t_0,0(1/z) for a positive measure β t_0,0 with β t_0,0(\mathbb{R}) = h^{-1}(t+h-t) = 1.

It can easily be verified that the closure of the set of all Cauchy-transforms of probability measures is the set of all Cauchy-transforms of non-negative measures with mass ≤ 1. This family is locally bounded as every such G s satisfies |G s(z)| ≤ \frac{1}{\text{Im}(1/z)}.

We assumed that the limit \lim_{h\to0} Gβ t_0,h exists for all z ∈ D. By the Vitali-Porter theorem, see [33], Section 2.4, we have in fact locally uniform convergence and thus

\lim_{h\to0} Gβ t_0,h(z) = Gν t_0(z)

for a non-negative measure ν t_0 with ν t_0(\mathbb{R}) ≤ 1. In particular, t → R s,t(z) is differentiable for all t ∈ [s,∞) \ N and all z ∈ H.

By the proof of the first part, we have that \int_0^s \nu_t(\mathbb{R}) dt is equal to t-s; hence \nu_t(\mathbb{R}) = 1 for a.e. \tau ≥ s. Clearly, we can choose (νt) t≥0 such that νt is a probability measure for every t ≥ 0 and that z → Gνt(z) is measurable for every z ∈ H.

□
Thus, a normal free evolution family can be described as:

\[
\begin{align*}
(s,t)_{0 \leq s \leq t} & \quad \text{R-transform} \quad (R_{s,t})_{0 \leq s \leq t} & \quad \text{Equation (3.15)} \quad \left(\nu_t\right)_{0 \leq t}.
\end{align*}
\]

**Remark 3.18.** If we consider a free evolution family, not necessarily normal, then the R-transforms \(R_{s,t}\) correspond to the differential equation

\[\frac{\partial}{\partial t} R_{s,t} = P_t(R_{s,t}) \quad \text{for a.e. } t \in [s, \infty), \quad R_{s,s}(z) = z \in \mathbb{H},\]

where \(P_t\) has the form \[3.14\] with some \(\alpha_t \in \mathbb{R}\) and a positive finite measure \(\nu'_t\) for a.e. \(t \geq 0\).

**Example 3.19** ("free slit equation"). One can look at the analogue of the slit equation in the free setting in two different ways. First, let \(U_t : [0, \infty) \to \mathbb{R}\) be a continuous function and consider \(\nu_t = \delta_{U_t}\), i.e. \(G_{\nu_t}(z) = \frac{1}{z - U(t)}\). Then \(R_{s,t}(z) = \int_s^t \frac{1}{z - U(t)} \, dt\).

Secondly, we look at the analogue of \[3.13\] in the free setting, i.e. we replace the arcsine distribution by the semicircle distribution. However, as \(\delta_{-\Delta_k} \boxplus \mu_{W,\Delta_t} \boxplus \delta_{\Delta_k} = \mu_{W,\Delta_t}\) due to commutativity of \(\boxplus\), we simply obtain that

\[\sigma_{0,K,t,\Delta_t} = \sigma_{0,K,\Delta_t} = \mu_{W,K,t,\Delta_t}\]

i.e. we obtain a free Brownian motion \(\sigma_{s,t} = \mu_{W,t-s}\).

### 4 Realizations

Let \((\sigma_{s,t})_{0 \leq s \leq t}\) be a (free/monotone/anti-monotone) evolution family. Of course, one is interested in realizations of such a family of distributions as a process on a \(C^*\)-algebra.

**Definition 4.1.** A realization of \((\sigma_{s,t})_{0 \leq s \leq t}\) is a \(C^*\)-algebra \((A, \varphi)\) with a collection \((X_t)_{0 \leq s \leq t} \subset A\) of self-adjoint random variables such that

(a) \(X_0 = 0\),

(b) the distribution of \(X_t - X_s\) is given by \(\sigma_{s,t}\) for all \(0 \leq s \leq t\),

(c) the increments \((X_{t_n} - X_{t_{n-1}}, X_{t_{n-1}} - X_{t_{n-2}}, \ldots, X_{t_0} - X_{t_1})\) are independent for all \(0 \leq t_1 \leq t_2 \leq \ldots \leq t_n\),

(d) \(t \mapsto X_t\) is continuous.

Of course, instead of continuity, one can require other regularity conditions.

If \((\sigma_{s,t})_{t \geq 0}\) is a semigroup, then we say that \(X_t\) is a realization of \((\sigma_{s,t})_{t \geq 0}\) if \(X_t\) is a realization of the evolution family \((\sigma_{s,t})_{0 \leq s \leq t} := (\sigma_{t-s})_{0 \leq s \leq t}\).

**Remark 4.2.** A realization of an evolution family is also called a (free/monotone/anti-monotone) additive process, while the realization of a semigroup is called (free/monotone/anti-monotone) Lévy process; see [4] p. 112).

**Example 4.3.** A realization of the free semigroup \((\sigma_{s,t})_{t \geq 0} = \mu_{W,t}\) is called a free Brownian motion. Similarly, a realization of the monotone semigroup \((\sigma_{s,t})_{t \geq 0} = \mu_{A,t}\) is called a monotone Brownian motion, which corresponds to Example 3.1.

In general, one can switch between non-commutative and classical evolution families of probability measures by using the Lévy–Khintchine representation formulas; see [4] Remark 5.17, Theorem 4.14] and [13 Theorem 2.2].

Also, we note that the free evolution family \((\sigma_{s,t})_{0 \leq s \leq t} = (\sigma_{W,t-s})_{0 \leq s \leq t}\) is also an anti-monotone evolution family that is easy to describe. On the one hand, the R-transforms of the family satisfy \[3.15\] with \(\nu_t = \mu_{W,1}\) (see equation \[2.2\]). On the other hand, its F-transforms satisfy \[3.11\] with \(\nu_t = \mu_{W,t}\), which has been mentioned already in Example 3.2.

A similar anti-monotone evolution family appears in the context of multiple SLE processes in [10]. Here, the F-transforms satisfy \[5.11\] where \(G_{\nu_t} = G_t\) is given by

\[\frac{\partial}{\partial t} G_t(z) = -2 \frac{\partial}{\partial z} G_t(z) \cdot G_t(z), \quad G_0(z) = G_0.\]

A free Brownian motion can be realized on a Fock space as follows (see [35]):
Take the Hilbert spaces $H = L^2(\mathbb{R})$ and let $F(H)$ be the free Fock space

$$F(H) = \Omega \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} H^\otimes n,$$

where $\Omega \in H$ has norm 1. Note that $H^\otimes n$ can be identified with $L^2(\mathbb{R}^n)$.

Now we set $\mathcal{A} = \mathcal{B}(F(H))$, which is the space of all bounded linear operators on $F(H)$, and define $\varphi : \mathcal{A} \to \mathbb{C}$ by $\varphi(a) = \langle a\Omega, \Omega \rangle$. Then $(\mathcal{A}, \varphi)$ is a $C^*$-probability space.

Next, for $h \in H$, define the creation operator $a^*(h)$ by

$$a^*(h)f_1 \otimes f_2 \otimes \ldots \otimes f_n = h \otimes f_1 \otimes f_2 \otimes \ldots \otimes f_n$$

for all $n \in \mathbb{N}$ and $a^*(h)\Omega = h$. The annihilation operator $a(h)$ is the adjoint of $a^*(h)$ and acts as follows:

$$a(h)f_1 \otimes f_2 \otimes \ldots \otimes f_n = \langle h, f_1 \rangle f_2 \otimes \ldots \otimes f_n$$

for all $n \geq 2$, $a(h)f_1 = \langle h, f_1 \rangle \Omega$, $a(h)\Omega = 0$. Both $a(h)$ and $a^*(h)$ are elements of $\mathcal{A}$.

For every $t \geq 0$, define $B_t = a(1_{[0,t]}) + a^*(1_{[0,t]})$. Then $B_t$ is a self-adjoint random variable in $\mathcal{A}$ and $(B_t)_{t \geq 0}$ is a realization of a free Brownian motion.

In a similar way, one can realize a monotone Brownian motion; see [26]. Here, one can use the monotone Fock space

$$F_m(L^2(\mathbb{R}_+)) = \Omega \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathbb{L}^2(\mathbb{R}^n_+),$$

where $\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}_+^n \mid x_1 > x_2 > \ldots > x_n\}$.

Realizations can also be described by “quantum stochastic differential equations”, see (4) Theorem 4.1 for the monotone and (6) Section 5.4 and 6 for the free case, and are usually constructed on Fock spaces, see (6) p.246 for the semigroup case. Also, we refer to (4) Chapter 4 for a description of concrete applications of quantum Markov processes in quantum mechanics.

**Question 4.4. Is it possible to realize the (classically) random monotone/anti-monotone evolution family of SLE (i.e. $\nu_t = \delta_{\sqrt{\kappa/2}B_t}^\nu$)?**

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