ON THE CLASSICAL HARDNESS OF SPOOFING LINEAR CROSS-ENTROPY BENCHMARKING

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ABSTRACT. Recently, Google announced the first demonstration of quantum computational supremacy with a programmable superconducting processor (Arute et al. 2019). Their demonstration is based on collecting samples from the output distribution of a noisy random quantum circuit, then applying a statistical test to those samples called Linear Cross-Entropy Benchmarking (Linear XEB). This raises a theoretical question: How hard is it for a classical computer to spoof the results of the Linear XEB test? In this short note, we adapt an analysis of Aaronson and Chen (2017) to prove a conditional hardness result for Linear XEB spoofing. Specifically, we show that the problem is classically hard, assuming that there is no efficient classical algorithm that, given a random n-qubit quantum circuit C, estimates the probability of C outputting a specific output string, say 0^n, with variance even slightly better than that of the trivial estimator that always estimates 1/2^n. Our result automatically encompasses the case of noisy circuits.

1. INTRODUCTION

A research team based at Google has announced a demonstration of quantum computational supremacy, by sampling the output distributions of random quantum circuits (Arute et al. 2019). To verify that their circuits were working correctly, they tested their samples using Linear Cross-Entropy Benchmarking (Linear XEB). This test simply checks that the observed samples tend to concentrate on the outputs that have higher probabilities under the ideal distribution for the given quantum circuit. More formally, given samples z_1, ..., z_k, Linear XEB entails checking that E_i[P(z_i)] is greater than some threshold, where P(z) is the probability of observing z under the ideal distribution. In the regime of 40-50 qubits, these probabilities can be calculated by a classical supercomputer with enough time.

While there is some support for the conjecture that no classical algorithm can efficiently sample from the output distribution of a random quantum circuit (Bouland et al. 2019), less is known about the hardness of directly spoofing a test like Linear XEB. Results about the hardness of sampling are not quite results about the hardness of spoofing Linear XEB; a device could score well on Linear XEB while being far from correct in total variation distance by, for example, always outputting the items with the k highest probabilities.

However, Aaronson and Chen (2017) were able to prove the hardness of a different, related verification procedure from a strong assumption they called the Quantum Threshold Assumption (QUATH). Informally, QUATH states that it is impossible for a polynomial-time classical algorithm to guess whether a specific output string like 0^n has greater-than-median probability of being observed as the output of a given n-qubit quantum circuit, with
success probability $1/2 + \Omega(1/2^n)$. They went on to investigate algorithms for breaking QUATH by estimating the output amplitudes of quantum circuits. For certain classes of circuits output amplitudes can be efficiently calculated, but in general even efficiently sampling from the output distribution is impossible unless the polynomial hierarchy collapses (Aaronson and Arkhipov [2011], Bremner et al. [2016]). Aaronson and Chen found an algorithm for calculating amplitudes of arbitrary circuits that runs in time $d^{O(n)}$, where $d$ is the circuit depth. This is now used in some state-of-the-art simulations, but is still too slow and of the wrong form to violate QUATH, as there is no way to trade the accuracy for polynomial efficiency.

Here, we formulate a slightly different assumption that we call XQUATH and show that it implies the hardness of spoofing Linear XEB. Like QUATH, the new assumption is quite strong, but makes no reference to sampling. In particular, while we don’t know a reduction, refuting XQUATH seems essentially as hard as refuting QUATH. Note that our result says nothing, one way or the other, about the possibility of improvements to algorithms for calculating amplitudes. It just says that there’s nothing particular to spoofing Linear XEB that makes it easier than nontrivially estimating amplitudes.

Indeed, since the news of the Google group’s success broke, at least two results have potentially improved on the classical simulation efficiency, beyond what Google had considered. First, [Gray 2019] was able to optimize tensor network contraction methods to obtain a faster classical amplitude estimator, though it’s not yet clear whether this will be competitive for calculating millions of amplitudes at once. Second, [Pednault et al. 2019] argued that, by using extra memory, existing classical supercomputers should be able to simulate the experiments done at Google in a few days. Our result provides some explanation for why these improvements had to target the general problem of amplitude estimation, rather than doing anything specific to the problem of spoofing Linear XEB.

2. Preliminaries

Throughout this note we will refer to random quantum circuits. Our results apply to circuits chosen from any reasonable distribution $\mathcal{D}$ over circuits on $n$ qubits. For every such distribution there is a corresponding version of XQUATH. For instance, we could consider a distribution where $d$ alternating layers of random single- and neighboring two-qubit gates are applied to a square lattice of $n$ qubits, as in Google’s experiment. Our assumption XQUATH states that no efficient classical algorithm can estimate the probability of such a random circuit $C$ outputting $0^n$, with variance even slightly lower than the trivial algorithm that always estimates $1/2^n$.

**Definition 1** (XQUATH, or Linear Cross-Entropy Quantum Threshold Assumption). There is no polynomial-time classical algorithm that takes as input a quantum circuit $C \leftarrow \mathcal{D}$ and produces an estimate $p$ of $p_0 = \Pr[C \text{ outputs } 0^n]$ such that

$$E[(p_0 - 2^{-n})^2] = E[(p_0 - p)^2] + \Omega(2^{-3n})$$

where the expectations are taken over circuits $C$ as well as the algorithm’s internal randomness.

The simplest way to attempt to refute XQUATH might be to try $k$ random Feynman paths of the circuit, all of which terminate at $0^n$, and take the empirical mean over their

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1The reason for the bound being $2^{-3n}$ will emerge from our analysis.
contributions to the amplitude. However, this approach will only yield an improvement in variance over the trivial algorithm that decays exponentially with the number of gates in the circuit, rather than the number of qubits. As mentioned above, even the best existing quantum simulation algorithms do not appear to significantly help in refuting XQUATH.

The problem XHOG is to generate outputs of a given quantum circuit that have high expected amplitude.

**Problem 1** (XHOG, or Linear Cross-Entropy Heavy Output Generation). Given a circuit $C$, generate $k$ distinct samples $z_1, \ldots, z_k$ such that $\mathbb{E}_z[|\langle z| C |0^n\rangle|^2] \geq b/2^n$.

When the depth is large enough, the output probabilities $p$ of almost all circuits are empirically observed to be accurately described by the Porter-Thomas distribution $2^n e^{-2^n p}$, although this has only been rigorously proven in some special cases (Boixo et al. [2018], Arute et al. [2019]). Under this assumption, for observed outputs $z$ from ideal circuits $C \leftarrow \mathcal{D}$ we have

$$\mathbb{E}[|\langle z| C |0^n\rangle|^2] \approx \int_0^{2^n} \frac{x}{2^n} xe^{-x} dx \approx \frac{2}{2^n}$$

So we expect an ideal circuit to solve XHOG with $b \approx 2$, and a noisy circuit to solve XHOG with $b > 1$ is hard to do classically with many samples and high probability. For completeness, we show in the Appendix that with Google’s number of samples and estimated circuit fidelity, they would be expected to solve XHOG with sufficiently high probability.

## 3. Proof of the Reduction

We now provide a reduction from the problem in XQUATH to XHOG. Since we only call the XHOG algorithm once in the reduction, solving XHOG actually requires as many computational steps as solving the problem in XQUATH, minus $O(k)$.

**Theorem 1.** Assuming XQUATH, no polynomial-time classical algorithm can solve XHOG with probability $s > \frac{1}{2} + \frac{1}{2k}$, and

$$k \geq \frac{1}{((2s - 1)b - 1)(b - 1)}.$$

**Proof.** Suppose that $A$ is such a classical algorithm solving XHOG. Given a quantum circuit $C \leftarrow \mathcal{D}$, first draw a uniformly random $z \in \{0, 1\}^n$, and apply NOT gates at the end of $C$ on qubits $i$ where $z_i = 1$ to get a circuit $C'$. It is easy to see that $C'$ is distributed exactly the same as $C$, even conditioned on a particular $z$. Also, $\langle 0^n | C |0^n \rangle = \langle z | C' |0^n \rangle$, so $\Pr[C$ outputs $0^n] = \Pr[C'$ outputs $z]$. Call this probability $p_0$.

Run $A$ on input $C'$ to get $z_1, \ldots, z_k$ with $\mathbb{E}_z[|\langle z_i| C' |0^n\rangle|^2] \geq 2^{-n}$. If $z \in \{z_i\}$, then our algorithm outputs $b2^{-n}$; otherwise it outputs $2^{-n}$.

Let $X = (p_0 - 2^{-n})^2 - (p_0 - p)^2$. Simple calculations in the Appendix show that

$$\mathbb{E}[X \mid z \in \{z_i\} \text{ and } A \text{ succeeded}] \geq 2^{-2n}(b - 1)^2$$

$$\mathbb{E}[X \mid z \in \{z_i\} \text{ and } A \text{ failed}] \geq -2^{-2n}(b^2 - 1)$$
Since $\mathbb{E}[X \mid z \not\in \{z_i\}] = 0$, and since $z$ is uniformly random even conditioned the output of $A$ and its success or failure,

$$
\mathbb{E}[X] = 2^{-n}k \cdot \mathbb{E}[X \mid z \in \{z_i\} \text{ and } A \text{ succeeded}]
+ 2^{-n}k(1 - s) \cdot \mathbb{E}[X \mid z \in \{z_i\} \text{ and } A \text{ failed}]
\geq 2^{-3n}k((2s - 1)b - 1)(b - 1)
$$

which is $\Omega(2^{-3n})$ as long as $k \geq 1/((2s - 1)b - 1)(b - 1)$. This completes the proof. □

One simple instance of the theorem is to take $s = \frac{3}{4} + \frac{1}{10}$ and $k = 2(b - 1)^{-2}$. Note that even with $s = 1$, we need $k \geq (b - 1)^{-2}$ samples for the proof to work. In fact, if the number of samples $k$ is much smaller than $(b - 1)^{-2}$, then even sampling uniformly at random would pass XHOG with non-negligible probability.

4. Open Problems

We conclude with two open problems related to our reduction.

- Can the classical hardness of spoofing Linear XEB be based on a more secure assumption? Is there a similar assumption to XQUATH that is equivalent to the classical hardness of XHOG?
- Is XQUATH true? What is the relationship of XQUATH to QUATH?

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Appendix

Probability of Failing XHOG. Let \( Y = |\langle z | C | 0^n \rangle|^2 \), where \( z \) is sampled from our XHOG device which was given \( C \leftarrow D \). Here we show that, if \( \mathbb{E}[Y] \geq (2b - 1)/2^n \) and \( k > (b - 1)^{-2} \), then the device will solve XHOG with high enough probability for Theorem 1 to apply. We choose \( s = 3/4 + 1/4b \) for simplicity.

If our device is noise-free,
\[
\operatorname{Var}(Y) = \int_0^{2^n} x e^{-x} \left( \frac{x}{2^n} - \frac{2}{2^n} \right)^2 \, dx \approx \frac{2}{2^n}
\]
and if we are sampling from the uniform distribution,
\[
\operatorname{Var}(Y) = \int_0^{2^n} e^{-x} \left( \frac{x}{2^n} - \frac{1}{2^n} \right)^2 \, dx \approx \frac{1}{2^n}
\]
By the law of total variance, for a noisy device sampling from the ideal distribution with probability \( p \) and the uniform distribution with probability \( 1 - p \),
\[
\operatorname{Var}(Y) = \frac{2p}{2^{2n}} + \frac{1 - p}{2^{2n}} + \frac{p(2 - p - 1)^2}{2^{2n}} + \frac{(1 - p)(1 - p - 1)^2}{2^{2n}} = \frac{1 + 2p - p^2}{2^{2n}} \leq \frac{2}{2^{2n}}
\]
So for a noisy device, the standard deviation is at most \( \sqrt{2}/2^n \). Let \( \bar{Y}_k \) be the average of \( Y \) over \( k \) samples. Then the standard deviation of \( \bar{Y}_k \) is at most \( \sqrt{2}/k2^n \).

Suppose \( \mathbb{E}[Y] \geq (b + \varepsilon)/2^n \). Then by Chebyshev’s inequality,
\[
\operatorname{Pr}[\bar{Y}_k \leq b/2^n] \leq \operatorname{Pr}[|\bar{Y}_k - \mathbb{E}[Y]| \geq \varepsilon/2^n] \leq \frac{2}{\varepsilon^2 k^2}
\]
So, to succeed XHOG with probability \( 3/4 + 1/4b \), we only need
\[
k > \frac{2\sqrt{2}}{\varepsilon} \sqrt{\frac{b}{b - 1}}
\]
With \( \varepsilon = b - 1 \), a device producing \( k > (b - 1)^{-2} \) samples should therefore pass XHOG with probability larger than \( 3/4 + 1/4b \). The Google team estimated that they were sampling \( \mathbb{E}[Y] \approx 1.002/2^n \) for their largest circuits, so with \( n = 53, k = 30 \times 10^6, b = 1.001, \) and \( \varepsilon = 0.001 \), they needed 1 million samples, much less than the 30 million they did take.

Algebra in the Proof of Theorem 1. These calculations were left out of the proof of Theorem 1 in the main text for brevity.

\[
\mathbb{E}[X | \ z \in \{z_i\} \text{ and } A \text{ succeeded}] = 2 \cdot 2^{-n}(b - 1) \cdot \mathbb{E}[p_0 | \ z \in \{z_i\} \text{ and } A \text{ succeeded}] + 2^{-2n}(1 - b^2) \\
\geq 2 \cdot 2^{-n}(b - 1)(b2^{-n}) + 2^{-2n}(1 - b^2) \\
= 2^{-2n}(b - 1)^2
\]

\[
\mathbb{E}[X | \ z \in \{z_i\} \text{ and } A \text{ failed}] = 2 \cdot 2^{-n}(b - 1) \cdot \mathbb{E}[p_0 | \ z \in \{z_i\} \text{ and } A \text{ failed}] + 2^{-2n}(1 - b^2) \\
\geq -2^{-2n}(b^2 - 1)
\]

\[
\mathbb{E}[X] = 2^{-n}ks \cdot \mathbb{E}[X | \ z \in \{z_i\} \text{ and } A \text{ succeeded}] + 2^{-n}k(1 - s) \cdot \mathbb{E}[X | \ z \in \{z_i\} \text{ and } A \text{ failed}] \\
\geq 2^{-3n}k((2s - 1)b - 1)(b - 1) \\
\geq 2^{-3n}
\]