Explicit expressions for stationary states of the Lindblad equation for a finite state space

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Abstract
The Lindblad equation describes the time evolution of a density matrix of a quantum mechanical system. Stationary solutions are obtained by time-averaging the solution, which will in general depend on the initial state. We provide an analytical expression for the steady states of the Lindblad equation using the quantum jump unraveling, a version of an ergodic theorem, and the stationary probabilities of the corresponding discrete-time Markov chains. Our result is valid when the number of states appearing in the quantum trajectory is finite. The classical case of a Markov jump-process is recovered as a special case, and differences between the two are discussed.

Keywords: Lindblad equation, (quantum-)master equation, stationary solution, quantum-jump unraveling, discrete-time Markov chain, absorbing sets and steady states

(Some figures may appear in colour only in the online journal)

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1. Introduction

The Lindblad equation is a quantum master equation describing the time evolution of a density matrix of an open quantum system. It is given by

\[
\partial_t \rho(t) = L(\rho) = -i[H, \rho(t)] + \sum_{k \in I} \gamma_k \left( V_k \rho(t) V_k^\dagger - \frac{1}{2} \{ V_k^\dagger V_k, \rho(t) \} \right)
\]

\[
= -i (H_c \rho(t) - \rho(t) H_c) + \sum_{k \in I} \gamma_k \left( V_k \rho(t) V_k^\dagger \right),
\]

where \([ \cdot, \cdot ]\) and \(( \cdot, \cdot )\) denote the commutator and anticommutator, respectively, and the so-called conditional Hamiltonian \(H_c\) is defined as

\[
H_c := H - \frac{i}{2} \sum_{k \in I} \gamma_k (V_k^\dagger V_k) =: H - \frac{i}{2} \Lambda.
\]

We have set \(\hbar = 1\). An open quantum system is a quantum mechanical system in contact with an external environment. The Lindbladian \(L(\rho)\) of equation (1) is the most general form of an (infinitesimal) generator that ensures that the solution \(\rho(t)\) remains a density matrix, in particular that is positive semi-definite (with respect to the complex numbers) and has trace one. This theorem has first been proven by Gorini, Kossakowski, and Sudarshan for the finite dimensional case, with Lindblad generalizing the statement to bounded generators [BP02]. At that time, generators of the form \(L(\rho)\) have already appeared multiple times in the physical literature [BP02]. In addition to these previous theorems, which are based on quantum Markov semi-groups, there are two types of microscopic derivations of the Lindblad equation, namely the weak-coupling limit and the low-density limit. The basis for both derivations is the ad-hoc assumption that the combined system of system and bath can be written initially as a product state. From this initial state, one computes the dynamics and traces over the environmental degrees of freedom to arrive at the reduced density matrix of the system.

When conducting a numerical simulation of a specific model, one often relies on unravellings of the Lindblad equation, which are ensembles of stochastic trajectories whose average yields the solution of the Lindblad equation [BP02]. While different unravellings may yield qualitative different types of dynamics, all measurable quantities depend only on the solution \(\rho(t)\), while additional properties depending on either a certain type of unravelling or a specific quantum trajectory are not accessible to experimental observation.

The Lindblad equation contains the von Neumann equation, describing a closed quantum mechanical system, as a special case. The solution of the von Neumann equation is given by \(e^{-iHt} \rho_0 e^{iHt}\), with the self-adjoint Hamiltonian \(H' = H\). Since \(H_c\) is not self-adjoint, the (conditional) time evolution operator \(e^{-iH_c t}\) is not unitary.

The additional terms can be interpreted as modeling the influence of the environment, in particular by inducing transitions between states of the system. The so-called Lindblad operators \(\{ V_k \mid k \in I \}\) can be interpreted as causing transitions \(\psi \to \frac{V_k \psi}{\| V_k \psi \|}\), where \(0 \neq I \subset \mathbb{N}\) is a finite index set and \(\gamma_k > 0\) are positive transition rates. We will restrict ourselves to the finite-dimensional case, that is \(\rho, V_k, H \in \mathbb{C}^{N \times N}\), with \(N \in \mathbb{N}, N \geq 2\).
One important aspect in the study of quantum master equation is its long-term behaviour \( \lim_{t \to \infty} \rho(t) \) and its steady states \( \rho^\infty(\rho_0) \) (with \( \rho^\infty \) defined by \( \mathcal{L}(\rho^\infty) = 0 \)), as this determines the state where a quantum mechanical system will eventually be. When certain algebraic conditions on the Lindblad operators are satisfied, the steady state is unique and asymptotically stable. The most prominent theorem about such conditions was given by Spohn \cite{Spo77}. It requires that the set of Lindblad operators form a basis, i.e. that the environment couples to all degrees of freedom, which applies only to a limited set of physical systems and cannot easily be generalized.

Most papers studying the stationary states of the Lindblad equation require additional properties of the Lindblad operators. Buča and Prosen propose a method which assumes that the underlying symmetry of the system is already known \cite{BP12}. While being numerically cheap, finding all the symmetries of an open system can be highly non-trivial, especially when both ‘strong’ and ‘weak’ symmetry is involved (see \cite{TM21} for a summary). Another approach was made by Trushechkina, where special kinds of Lindblad operators were considered, which allow a ‘backwards’ transition for every ‘forward’ transition (see \cite{Tru18}). For the dimension \( N = 2 \) a full discussion can be found in \cite{AIIN22}.

The general nature of the Lindbladian in equation (1) makes it difficult to compute analytical expressions for all possible steady states. In this paper, we aim at statements about the stationary states, without making additional assumptions on the Lindblad operators. We will address the nature and number of these stationary states, how they can be determined, and how they are related to the long-term behaviour that results from a given initial state.

The standard way to compute the stationary state from a given initial state \( \rho_0 \) would be to determine the solution \( t \mapsto \rho(t) \) of the initial value problem of equation (1) for all positive times \( t \geq 0 \) and to evaluate the time average \( \langle \rho_{\rho_0}(t) \rangle_{t \geq 0} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \rho_{\rho_0}(t)\,dt \). We propose a different method: We express the density matrix \( \rho(t) \) in terms of the average of an ensemble of stochastic quantum trajectories. Then we make use of the fact that the limit of the time average of a single quantum trajectory exists and that it may be interchanged with the ensemble limit \cite{KM04}. We will show that both the time average and the ensemble average of a quantum trajectory can be explicitly evaluated with the theory of discrete-time Markov chains when certain conditions (in particular a finite state space) are satisfied. Possible generalizations are discussed in appendix B.

2. The concept of an unravelling of the Lindblad equation

Unravelings are ensembles of stochastic quantum trajectories that are equivalent to the Lindblad equation in the sense that the state of the system \( \rho(t) \) at a time \( t \geq 0 \) is given by the average over all possible trajectories,

\[
\rho(t) = \int_{\mathcal{U}} \Theta(t, \omega) d\mathcal{P}(\omega).
\]  (3)

The parameter \( \omega \in \mathcal{U} \) labels the quantum trajectories \( \mathbb{R}_{\geq 0} \ni t \mapsto \Theta(t, \omega) \in \{\text{density matrices}\} \) (see figure 1 for all illustration).

In particular, the expectation values of any quantum mechanical observable \( A \), given the system is in the state \( \rho \), can be computed from these unraveling, according to
Figure 1. Illustration of an ensemble of quantum trajectories, originating from the quantum jump unravelling, which is a special type of piecewise deterministic processes interrupted by quantum jumps. The average of the trajectories at a given time $t \geq 0$ yields the density matrix $\rho(t)$.

$$E_{\rho}[A] = \text{Tr}[\rho A] = \int_{\mathcal{U}} \text{Tr}[\Theta(\omega)A]d\mathcal{P}(\omega).$$ (4)

Unravellings provide different perspectives on the subject of time evolution of open quantum systems and can yield new insights since they are tackled by a different set of mathematical tools: Instead of linear differential equations one has the theory of Markov chains, piecewise deterministic processes, and stochastic differential equations at hand [BP02].

Moreover, they are often superior when it comes to numerical simulations of specific models as a Ket-state unravelling (see section 2.3) requires less memory storage (linear with the dimension of the system, instead of quadratic [Lid19]).

While unravellings are usually done in terms of ket states, we want to focus on the so called mixed-state unravellings, which assume values in the set of density matrices [BP02].

Among the different possible types of unravellings, we will use the quantum-jump unravelling. For a given Hamiltonian $H$ and a given set of Lindblad operators $\{V_k : k \in I\}$, the quantum jump unravelling is a piecewise deterministic process (see [BP02, Lid19]) where the continuous, deterministic time evolution is interrupted by discontinuous quantum jumps at times $t_n$, where a Lindblad operator $V_{\pi_n}$ is applied.

So a quantum trajectory is completely determined by the sequence of times $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ (which record when a quantum jump takes place) and the sequence $(\tau_n)_{n \in \mathbb{N}} \in \mathbb{N}$ of indices of Lindblad operators (which records which operator has been applied at time $t_n$). We define $\mathcal{U}$ to be the set of all possible quantum trajectories, $\tau_n := t_{n+1} - t_n$ to be the length of the time interval between two jumps and $\Theta_n := \Theta(t_n)$ the state of the unravelling at time $t_n$.

Note that we use with $\Theta(t, \omega)$ a different symbol for a state in the quantum trajectory $\omega \in \mathcal{U}$, to distinguish it from the solution $\rho(t)$ of the Lindblad equation (1) at time $t_n$, which is the average over all quantum trajectories, $\rho(t) = \langle \Theta(\omega, t) \rangle_{\omega \in \mathcal{U}}$. 


2.1. Definitions and preparatory considerations

2.1.1. (Pseudo-) Time-evolution operators and quantum jump operators. To avoid unnecessarily complicated notation, we denote by $U_t$ the (non-unitary) time evolution according to the conditional Hamiltonian $H_c$, and by $J_k$ the quantum jump operator, irrespective of the nature of the quantum state, be it a density matrix or a ket state. Their action on these two types of quantum states is thus given by

$$
U_t : \mathbb{C}^{N \times N} \rightarrow \mathbb{C}^{N \times N}, \quad U_t, J_k : \mathbb{C}^N \rightarrow \mathbb{C}^N
$$

$$
U_t(\Theta) = e^{-iH_c t \Theta} C \Theta, \quad U_t(\psi) = e^{-iH_c t \psi}
$$

$$
J_k(\Theta) = V_k \Theta V_k^\dag, \quad J_k(\psi) = V_k \psi.
$$

During the time interval $[t_n, t_{n+1})$ between two quantum jumps there is a deterministic time evolution $\Theta(t) = \frac{U_{t_n} \Theta(t_{n+1})}{Tr[U_{t_n} \Theta(t_{n+1})]}$, and directly after a quantum jump $\gamma$ induced by the Lindblad operator $V_k$ the state of the quantum trajectory is given by $\Theta(t_{n+1}) = \frac{J_k(\Theta)}{Tr[J_k(\Theta)]}$ (we write $\frac{A}{Tr[A]}$ instead of $\frac{A}{Tr[A]}$).

2.1.2. The distribution of the waiting time. Given that the quantum trajectory is in state $\Theta$, the waiting time $t$ for the next jump is distributed according to

$$
f(t|\Theta) := -\frac{d}{dt} \left[ \frac{U_t(\Theta)}{e^{-iH_c t \Theta} C \Theta} \right] = -\text{Tr} \left[ \left( -iH_c \right) U_t(\Theta) + U_t(\Theta) \left( iH_c^\dag \right) \right] \Theta
$$

$$
= i \text{Tr} [H U_t(\Theta)\Theta] - \frac{1}{2} \text{Tr} [\Theta U_t(\Theta) H] + \frac{1}{2} \text{Tr} [U_t(\Theta) \Theta A] + \frac{1}{2} \text{Tr} [U_t(\Theta) \Theta A]
$$

$$
= \text{Tr} [\Theta U_t(\Theta)] \sum_{k \in J} \gamma_k \Theta V_k V_k^\dag
$$

$$
= \sum_{k \in J} \gamma_k \Theta V_k e^{-iH_c t \Theta} \Theta V_k^\dag = \sum_{k \in J} f^{(k)}(t|\Theta)
$$

with

$$
f^{(k)}(t|\Theta) := \gamma_k J_k \circ U_t(\Theta) = \gamma_k V_k e^{-iH_c t \Theta} \Theta e^{iH_c t \Theta} V_k^\dag \Theta (\text{see [BP02]}).
$$

Note that $f(\cdot|\Theta)$ need not be a probability distribution since $\int_{\mathbb{R}^+} f(t|\Theta) dt < 1$ is possible. In this case, we call $\Theta$ a possible trapping state, which will be discussed below.

2.1.3. The time-weighted average state. For a finite waiting time $\tau < \infty$ we define the time-weighted average state between two jumps as

$$
I_\Theta(\tau) := \int_0^\tau \frac{U_t(\Theta)}{\text{Tr}[U_t(\Theta)]} dt = \tau \cdot \langle \Theta(t) \rangle_{t \in [0,\tau]},
$$

which is the waiting time $\tau$ times the average state between two quantum jumps.
2.2. The algorithm for an unravelling

The algorithm for obtaining an unravelling is the following (see [Lid19] or [BP02] for a detailed proof):

1. Choose an initial state $\rho_0$ and set $\Theta_0 := \rho_0 = \rho(t_0)$.
2. Given that the system at time $t_n$ is in state $\Theta_n$, we calculate the waiting time $\tau_n$ to the next jump according to the following considerations: The probability for $\tau_n$ to lie in the interval $[0, T]$ is given by

$$
\mathcal{P}(\tau_n \in [0, T] \mid \Theta_n) = \int_0^T f(t \mid \Theta_n) dt = 1 - \text{Tr}[U_T(\Theta_n)].
$$

(9)

One way of determining the waiting time is the so-called inversion method (see: [BP02]):

We draw a random number $\eta_n$ from the uniform distribution of the interval $[0, 1]$ and set

$$
\tau_n := \begin{cases} 
  f^{-1}(\eta_n \mid \Theta_n), & \text{if } \eta_n \in \text{image}(f(\cdot \mid \Theta_n)) \\
  \infty, & \text{otherwise.}
\end{cases}
$$

(10)

An illustration of this procedure is given in figure 2.

Now we have to distinguish two cases:

(i) When $\lim_{t \to \infty} \text{Tr}[U_t(\Theta_n)] = 0$, then the quantum jump will occur at a finite time.

(ii) When $\lim_{t \to \infty} \text{Tr}[U_t(\Theta_n)] \in (0, 1]$, then there is a nonzero chance that the process no longer jumps, i.e. the waiting time is infinite. In this case, the state of the quantum trajectory after the time $t_n$ is given by $\frac{U_{t_n}(\Theta_n)}{\text{Tr}[\ldots]}$, and we call $\Theta_n$ a possible trapping state.
If the waiting time is finite \((\tau_n < 1)\), choose the operator \(V_{\pi_n}\) that is applied at time \(t_n\) according to

\[
P_{\pi_n} = k \Theta_n, \quad \tau_n = f(k), \quad \Theta_n \in J_{\pi_n}(\Theta_n)
\]

Then for a fixed unravelling \(\omega \in U\), for positive times \(t_n \in \mathbb{R}_{>0}\), and (possibly infinite) times \(t_{n+1} \in \mathbb{R}_{>0} \cup \{\infty\}\), we have

\[
\begin{align*}
\Omega_{\rho_n}(\omega) &= \{ \rho_0 = \Theta_0 \} \cup \left\{ \frac{U_{t_{n+1}} \circ \cdots \circ U_{t_n} \circ \Theta_n}{\text{Tr}[\cdots]} : n \in \mathbb{N}, \omega = (t_i, \pi_i)_{i \in \mathbb{N}} \right\} \\
&= \{ \Theta_n(\omega) : n \in \mathbb{N}, \omega = (t_i, \pi_i)_{i \in \mathbb{N}} \}
\end{align*}
\]

and the sequence \(\{\Theta_n\}_{n \in \mathbb{N}} = \{\Theta_{\rho_n}(t_n, \omega)\}_{n \in \mathbb{N}}\) the discrete quantum trajectory. Figure 3 serves as an illustration.

Such a process is called renewal process [GS20, DMPS18, Bré13].

2.3. Ket-state unravellings

If some state in the quantum trajectory is a pure state \(\Theta_n = P_{\psi_n} := |\psi_n\rangle \langle \psi_n|\), the unravelling reduces to a ket-state unravelling from then on, and equations (6), (9), (11), and (12) become

\[
f(t|\psi) = -\text{Tr}[U_t(P_\psi)] = -\text{Tr}[P_{U_t(\psi)}] = -\|U_t\psi\|^2
\]

\[
P(\tau_n \in [0,T] | \psi_n) = \frac{\int_0^T dt \frac{f(t|\psi)}{-\frac{d}{dt}U_T(\psi_n)^2}}{1 - \|U_T(\psi_n)\|^2}
\]
\[ P(\tau_n = k | \psi_n, \tau = \tau_n) = \frac{\mathcal{F}(\tau_n | \psi_n)}{\sum_{j \in J} \mathcal{F}(\tau_n | \psi_n)} \quad (11') \]

The states occurring during a ket state unravelling are given by
\[ \Psi_{\psi_n}(t, \omega) = \frac{U_{t-n} \circ J_{\tau_n+1} \circ \cdots \circ J_{\tau_1} \circ U_{\tau_n}(\Psi_0)}{\cdots} \Psi_n := \psi(t_n, \omega). \quad (12') \]

The transition to a ket-state unravelling occurs for instance when a Lindblad operator of rank one \( \langle V = |\psi\rangle\langle\varphi| \rangle \) is applied at some point in the quantum trajectory,
\[ \Theta_{n+1} = \frac{|\psi\rangle \langle \varphi | U_{\tau_n}(\Theta_n) | \varphi \rangle \langle \psi |}{\text{Tr}[\cdots]} = |\psi\rangle \langle \psi | \pm \psi. \quad (14) \]

So, in some sense, Lindblad operators of rank one are ‘quasi-classical’.

3. Computing the stationary solution of the Lindblad equation based on quantum-jump unravelling

The stationary solution \( \rho^\infty(\rho_0) \) for an initial state \( \rho_0 \) is obtained by taking the long-time average of expression (3) and interchanging the time average with the ensemble average,
\[ \rho^\infty(\rho_0) = \int_\mathcal{U} \Theta^\infty_{\rho_0}(\omega) \, d\mathcal{P}_{\rho_0}(\omega), \quad (15) \]

where the time average for a single trajectory is given by
\[ \Theta^\infty_{\rho_0}(\omega) = \langle \Theta_{\rho_0}(t, \omega) \rangle_{t \geq 0} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Theta_{\rho_0}(t, \omega) \, dt. \quad (16) \]

We therefore need to calculate the time averages of quantum trajectories and their probabilities. Instead of computing the probability density function \( \mathcal{P}(\omega) \) for all \( \omega \in \mathcal{U} \) separately, we will divide the set of all possible unravellings into a finite partition \( \mathcal{U} = \bigcup_{i=1}^{n} \mathcal{U}(B_i) \) of disjoint measurable sets, such that \( \mathcal{U}(B_i) \) contains all quantum trajectories with the same long-term behaviour \( \Theta_{B_i} \),
\[ \rho^\infty(\rho_0) = \sum_{i=1}^{n} \int_{\mathcal{U}(B_i)} \Theta^\infty_{\rho_0}(\omega) \, d\mathcal{P}_{\rho_0}(\omega) = \sum_{i=1}^{n} \mathcal{P}(\mathcal{U}(B_i) | \rho_0) \Theta_{B_i}. \quad (17) \]

Calculating the different building blocks of this expression is the goal of the next subsections.

An explicit expression for the time average \( \Theta^\infty_{\rho_0}(-) \) of a single quantum trajectory will be obtained in section 3.1. To find the different possible long-time behaviors of quantum trajectories, we view an unravelling as a discrete-time Markov chain in section 3.2, where the next state \( \Theta_{n+1} \) of the discrete quantum trajectory depends only on the previous state \( \Theta_n \). In section 3.3 we explicitly construct the set of possible states in the Markov chain and the matrix of transition probabilities, and we illustrate this procedure by three examples in section 3.4.
This gives us both the stationary states of the Markov chain and the probabilities for the different long-term behaviors.

In section 3.8, we will evaluate the time averaged states between jumps. By combining all these results, we can obtain in section 3.9 the final expression for the stationary solutions of the Lindblad equation (1).

3.1. Time average of a single quantum trajectory

The time average of a single quantum trajectory is given by

$$\Theta^\infty(\omega) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Theta_{\rho_0}(t,\omega) \, dt =: \langle \Theta_{\rho_0}(t,\omega) \rangle_{t \geq 0}.$$  \hspace{1cm} (18)

according to the theorem quoted in appendix A.1.

In order to evaluate this expression further, we denote by

$$J(T) := \{ n \in \mathbb{N} : t_n \leq T \}$$  \hspace{1cm} the number of jumps before the time $T > 0$, and by

$$J_s(T) := \{ n \in \{1, \ldots, J(T)\} : \Theta_n = \Theta_s \}$$  \hspace{1cm} the number of times the state $\Theta_s$ has appeared in the quantum trajectory before the time $T$.

We break the time interval $[0, T)$ into a disjoint union of smaller time intervals $[t_n, t_{n+1})$ between jumps,

$$[0, T) = \bigcup_{n=0}^{J(T)-1} [t_n, t_{n+1}) \cup [t_{J(T)}, T),$$

and compute

$$\Theta^\infty(\omega) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Theta_{\rho_0}(t,\omega) \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \sum_{n=0}^{J(T)-1} \int_{t_n}^{t_{n+1}} U_{t-t_n}(\Theta_n) \, dt + \lim_{T \to \infty} \frac{1}{T} I_{\Theta(t,T)}(T-t_{J(T)})$$

$$= \lim_{T \to \infty} \frac{1}{T} \sum_{n=0}^{J(T)-1} \int_0^{t_{n+1}} U_{t-t_n}(\Theta_n) \, dt + \lim_{T \to \infty} \frac{1}{T} I_{\Theta(t,T)}(T-t_{J(T)})$$

$$= \lim_{T \to \infty} \left( \frac{1}{T} \sum_{n=0}^{J(T)-1} I_{\Theta_n}(\tau_n) \right) + \lim_{T \to \infty} \left( \frac{1}{T} I_{\Theta(t,T)}(T-t_{J(T)}) \right),$$

where the time-weighted average $I_{\Theta}(\tau)$ was defined in equation (8).

The number of quantum jumps can be finite or infinite, so in the following we have to distinguish these two cases. As we will see, the ‘jump part’ vanishes in the case of finitely many jumps and the time average will be determined by the ‘remainder part’, and vice versa when the number of jumps is infinite.
3.1.1. The number of quantum jumps is finite. When the number of quantum jumps is finite, we can set \( J(T) \) large enough \( J_{\text{Max}} := J_M = J_{\text{Max}} \). So the discrete quantum trajectory will stop at the state \( \Theta_{\text{trap}} := \Theta_{J_M} \). As mentioned before, this can only happen if \( \int_{t_{\text{trap}}}^{t_{\text{stop}}} f(t \mid \Theta_{\text{trap}}) \, dt < 1 \).

We claim that the time average of this particular quantum trajectory equals \( \Theta_{\rho_0} = \lim_{T \to \infty} \frac{1}{T} \int_0^T U_i(\Theta_{\text{trap}}) \, dt \).

**Proof.** (a) The ‘jump part’:

\[
\left\| \frac{1}{T} \sum_{n=0}^{J(T)-1} I_{\Theta_n}(\tau_n) \right\| \leq \max_{n \in \{0, \ldots, J(T)-1\}} \sup_{\tau_n} \| I_{\Theta_n}(\tau_n) \| \xrightarrow{T \to \infty} 0. \tag{21}
\]

(b) The ‘remainder part’:

\[
\frac{I_{\Theta_T}(T - t_{J(T)})}{T} = \frac{I_{\Theta_{J_M}}(T - t_{J_M})}{T - t_{J_M}} \left( \frac{T - t_{J_M}}{T} \right) \xrightarrow{T \to \infty} \lim_{T \to \infty} \frac{1}{T} I_{\Theta_{\text{trap}}}(T) = \lim_{T \to \infty} \frac{1}{T} \int_0^T U_i(\Theta_{\text{trap}}) \, dt = (\Theta_{\text{trap}}(t))_{t \geq t_{J(T)}}. \tag{22}
\]

3.1.2. The number of quantum jumps is infinite. When the number of quantum jumps is infinite, the discrete quantum trajectory does not stop. We show that the ‘remainder part’ vanishes and derive an expression for the ‘jump part’.

(a) The ‘remainder part’:

The ‘remainder’ term \( \lim_{T \to \infty} \frac{I_{\Theta_T}(T - t_{J(T)})}{T} \) tends to zero when \( T \) approaches infinity,

\[
\left\| \frac{I_{\Theta_T}(T - t_{J(T)})}{T} \right\| = \left\| \frac{I_{\Theta_{J_M}}(T - t_{J_M})}{T - t_{J_M}} \left( \frac{T - t_{J_M}}{T} \right) \right\| \leq \sup_{n \in \{0, \ldots, J(T)-1\}} \sup_{|\tau_n|} \| U_i(\Theta_n) \| \left( 1 - \frac{t_{J(T)}}{T} \right) \xrightarrow{T \to \infty} 0, \tag{23}
\]

where we used the fact that \( \lim_{T \to \infty} \frac{t_{J(T)}}{T} = 1 \) (see appendix A.6).

(b) The ‘jump part’:

For a fixed quantum trajectory \( \omega \in \mathcal{U} \) that does not stop we denote by \( \alpha_s(k) \in \mathbb{N} \) the position in that quantum trajectory where the corresponding state \( \Theta_s \in \Omega_{\rho_0}(\omega) \) appears for the \( k \)th time, so

\[
\alpha_s(k) := \min_{M \in \mathbb{N}} \left\{ \{ \Theta_n : n \leq M, \Theta_n = \Theta_s \} \right\} = k. \tag{24}
\]

Note that we define \( \alpha_s(k) \) only for those states that actually appear in \( \omega \in \mathcal{U} \).

In particular, we have \( \Theta_{\alpha_s(k)} = \Theta_s \) for all \( s \in \mathcal{S} \) and for all \( k \in \mathbb{N} \), and the waiting time \( \tau_{\alpha_s(k)} \) has the probability density function \( f_{\tau_{\alpha_s(k)}}(\theta) = f_\Theta(\theta | \Theta_s) \) (see equation (6) for a definition).
The sums of the form \( \frac{1}{J(T)} \sum_{n=0}^{J(T)-1} x_n \), with \( x_n \in \{ \tau_n, I_{\Theta_n}(\tau_n) \} \) in the ‘jump part’ of equation (20) can be rearranged to

\[
\frac{1}{J(T)} \sum_{n=0}^{J(T)-1} x_n = \sum_{n \in S} \frac{J_n(T)}{J(T)} \sum_{k=1}^{J_n(T)} x_{n(k)}.
\]

(25)

For all states \( s \in S \), both the waiting times \( \{ \tau_{\alpha_s(k)} : k \in \mathbb{N} \} \) and the time-weighted average states \( \{ I_{\Theta_s}(\tau_{\alpha_s(k)}) : k \in \mathbb{N} \} \) are independent and identically distributed, and by the law of large numbers we have

\[
\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \tau_{\alpha_s(k)} \xrightarrow{K \to \infty} \tau_s,
\]

\[
\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} I_{\Theta_s}(\tau_{\alpha_s(k)}) \xrightarrow{K \to \infty} \int_0^\infty I_{\Theta_s}(\tau) f(\tau | \Theta_s) d\tau =: \hat{I}_s(\tau_s),
\]

pointwise, almost surely.

The term \( \frac{J_n(T)}{J(T)} \) can be interpreted as the fraction of jumps in the quantum trajectory whose outcome yields the state \( \Theta_s \).

For long times, the trajectory will eventually be in a minimal absorbing set \( B \subseteq \Omega_{\rho_0}(\omega) \), which is a subset of states that is both absorbing (which means that there are no transitions out of this set) and minimal in the sense of set inclusions (when \( B \) is a minimal absorbing set and \( A \subseteq B \) is absorbing, we have \( A = B \)), (see section 3.2 and appendix A.3). Therefore, the fraction \( \frac{J_n(T)}{J(T)} \) of jumps landing in state \( \Theta_s \) converges to a stationary value \( q(\Theta_s | B) \) associated with this minimal absorbing set,

\[
\frac{J_n(T)}{J(T)} \rightarrow \frac{1}{J(T)} \sum_{k=0}^{J(T)-1} \delta_{\Theta_s, \Theta_s} \xrightarrow{T \to \infty} q(\Theta_s | B),
\]

(27)

with \( q(\Theta_s | B) \) being the relative frequency with which the state \( \Theta_s \in \Omega_{\rho_0}(\omega) \) appears in the discrete quantum trajectory \( (\Theta_n)_{n \in \mathbb{N}} \). Since the numbers \( \{q(\Theta_s | B)\}_{s \in S} \) are non-negative and add up to one, we can interpret \( q(\Theta_s | B) \) as the probability of state \( \Theta_s \), provided the discrete quantum trajectory is eventually in the minimal absorbing set \( B \subseteq \Omega_{\rho_0}(\omega) \).

This idea will be picked up in section 3.2.

If the probability for a state \( \Theta_s \in \Omega_{\rho_0}(\omega) \) in a quantum trajectory is strictly positive, then it must appear infinitely often in the corresponding trajectory: \( q(\Theta_s | B) = \lim_{T \to \infty} \frac{J_n(T)}{J(T)} > 0 \implies \lim_{T \to \infty} J_n(T) = \infty \).
When the number of quantum jumps is infinite, this quantum trajectory also ends in a minimal absorbing set, which is in this case given by

$$\Theta_{\infty} (\tau_s) = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{\lfloor T/\tau_s \rfloor - 1} I_{\Theta_s} (\tau_s)$$

Summarizing the two cases.

The assumption of a finite state space \(|\mathcal{S}| < \infty\) is needed in order to avoid proving the interchangeability of two limits. However, it suffices to assume, that the number of states in every quantum trajectory is finite. An example is given in section B.2 in the appendix. For further generalization, a specific structure of the state space is needed, since for arbitrary state spaces, the stationary solution of the corresponding Markov chain need not exist (compare the example in section A.8 in the appendix).

3.1.3. Summarizing the two cases. When the number of quantum jumps is infinite, this minimal absorbing set is a subset of the possible states of the discrete quantum trajectory, \(B \subseteq \bigcup_{\omega \in \mathcal{U}} \Omega_{\rho_0} (\omega)\). When on the other hand the number of quantum jumps is finite, that is when the discrete quantum trajectory ends at a possible trapping state \(\Theta_{\tau_0}\), then the discrete quantum trajectory also ends in a minimal absorbing set, which is in this case given by \(B = \left\{ \lim_{T \to \infty} \frac{1}{T} I_{\Theta_{\tau_0}} (T) \right\}\).

Taking the two expressions for the average of the quantum trajectory together, we obtain

$$\Theta_B := \Theta_{\rho_0} (\omega) = \begin{cases} \lim_{T \to \infty} \frac{1}{T} I_{\Theta_{\tau_0}} (T), & \text{for finitely many jumps} \\ \sum_{n \in \mathbb{N}} q (\Theta_s | B) I_{\tau_s} (\tau_s), & \text{when the number of jumps is infinite.} \end{cases}$$

This result shows that the long-term behaviour of a quantum trajectory does not depend on the trajectory \(\omega \in \mathcal{U}\) itself, but only on the minimal absorbing set \(B\) it is eventually captured in.

3.2. Discrete-time Markov chains with finite state space

Now that we have computed an expression for the time-averaged state \(\Theta_{\rho_0} (\omega)\) for a fixed \(\omega \in \mathcal{U}\), we need to evaluate two quantities in order to obtain the stationary states of the Lindblad equation (see figure 4 for an illustration):
Figure 4. Illustrating the significance of a Markov chain in a quantum state unraveling: In order to compute the stationary state we need both the long-term behaviour $\Theta_B$ of a single quantum trajectory contained in the minimal absorbing set $B_i$, as well as the probability $P(U(B_i))$ for such a trajectory to be captured by this minimal absorbing set $B_i$, where $i \in \{1, \ldots, n\}$.

(i) First, the probability $P(U(B) | \rho_0)$ for a quantum trajectory to land in a minimal absorbing set $B$.

(ii) Second, the expression for the stationary states $\Theta_B$, corresponding to the minimal absorbing set.

Depending on the nature of the minimal absorbing set, we need for the expression of the stationary state either

(iii) the probabilities $q(\Theta_s | B)$ for the states $\Theta_s \in B$ within the minimal absorbing set $B$ as well as their time-weighted ensemble average $I_s(\tau_s)$ or

(iv) the long-term average $\lim_{T \to \infty} \frac{1}{T} I_{\Theta_{\text{trap}}}(T)$, where $\Theta_{\text{trap}}$ is a possible trapping state.

Both the probability for reaching each minimal absorbing set $B \subseteq \Omega_\rho$, $P(U(B) | \rho_0)$, as well as the probability for each state $q(\Theta_i | B)$ within such a minimal absorbing set can be obtained by considering only the states $\Theta_s \in \Omega_\rho$ of the discrete quantum trajectory and the transition probabilities between them. The discrete quantum trajectory is a Markov chain, since the probability for the next state $\Theta_{n+1}$ depends only on the previous state $\Theta_n$,

$$P(\Theta_{n+1} | \Theta_n, \Theta_{n-1}, \ldots, \Theta_0) = P(\Theta_{n+1} | \Theta_n).$$

We will therefore in the following summarize those results for classical discrete-time Markov chains that will be relevant for obtaining the two types of probabilities. The connection to the quantum trajectories and the evaluation of the time averages between jumps will follow in the subsequent subsections.

We consider a classical, discrete-time Markov process $(X_n)_{n \in \mathbb{N}_0}$ taking values in the finite state space $\Omega$. The transition probabilities are defined via

$$Q_{ij} := q_{j \rightarrow i} := P(X_{n+1} = i | X_n = j) = P(X_{n+1} = i | X_n = j, \ldots, X_0 = i_0).$$
Figure 5. Example of a state transition network of the Markov chain with the minimal absorbing sets $B_1$, $B_2$, $B_3$ and the transition matrix.

This specifies the transition matrix $Q \in (\mathbb{R}_{\geq 0})^{\Omega \times \Omega}$, which is a column-stochastic matrix that satisfies
\[
Q_{ij} \geq 0, \text{ for all } i, j \in \Omega \text{ and } \sum_{i=1}^{\Omega} Q_{ij} = 1, \text{ for all } j \in \Omega. \tag{32}
\]

The states and the nonzero transition probabilities form a network $(\Omega, \mathcal{E})$, with the states $\Omega$ being the nodes and the nonzero transitions $\mathcal{E} \subseteq \Omega \times \Omega$ being the directed links between them. When there is no transition from state $i$ to state $j$, the associated transition probability $q_{i \rightarrow j}$ vanishes. The transition probability indicate the strength of the link.

For each finite-size Markov chain there is an associated state transition network (a directed weighted graph). An example is given in figure 5:

\[
Q = \begin{pmatrix}
0 & q_{2 \rightarrow 1} & 0 & 0 & 0 & 0 & 0 & 0 \\
q_{1 \rightarrow 2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_{1 \rightarrow 3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
q_{1 \rightarrow 4} & 0 & 0 & q_{4 \rightarrow 1} & q_{5 \rightarrow 4} & 0 & 0 & 0 \\
0 & 0 & 0 & q_{4 \rightarrow 5} & q_{5 \rightarrow 5} & 0 & 0 & 0 \\
0 & q_{2 \rightarrow 6} & 0 & 0 & 0 & q_{6 \rightarrow 1} & q_{7 \rightarrow 6} & q_{8 \rightarrow 6} \\
0 & 0 & 0 & 0 & q_{6 \rightarrow 7} & q_{7 \rightarrow 7} & q_{8 \rightarrow 7} & q_{8 \rightarrow 8} \\
0 & 0 & 0 & 0 & 0 & q_{6 \rightarrow 8} & q_{7 \rightarrow 8} & q_{8 \rightarrow 8}
\end{pmatrix}
\]

For all subsets of states $B \subseteq \Omega$ there is a corresponding subnetwork $(B, \mathcal{E}_B)$, where $\mathcal{E}_B := \{(i, j) \in \mathcal{E} : i, j \in B\}$ consists of all links in the subset $B \subseteq \Omega$.

We call a subset $B \subseteq \Omega$ absorbing if there are no edges pointing out of $B$, that is $q_{i \rightarrow j} = 0$ for all $i \in B$ and $j \in \Omega \setminus B$. An absorbing subset $B \subseteq \Omega$ is called minimal if for all absorbing subsets $C \subseteq \Omega$ with $C \subseteq B$ we have $C = B$. In particular, there can be more than one minimal absorbing subset [FD22].
We denote with $q_n(i)$ the probability of the system to be in state $i$ at the time step $n \in \mathbb{N}_0$, with $q_n(i) \geq 0$, $\sum_{i \in \Omega} q_n(i) = 1$ for all $n \in \mathbb{N}$.

Given an initial probability distribution $\mathbf{q}_0 \in [0, 1]^{|\Omega|}$, with $\|\mathbf{q}_0\|_1 = 1$, the probability distribution after $n \in \mathbb{N}$ time steps is given by

$$
\mathbf{q}_n(\mathbf{q}_0) = \mathbf{Q}^n \mathbf{q}_0 = \sum_{j=1}^{|\Omega|} q_0(j) \mathbf{Q}^j \mathbf{e}_j = \sum_{j=1}^{|\Omega|} q_0(j) \mathbf{P}(X_n = j | X_0 = j, \mathbf{Q})
$$

with $\mathbf{P}(X_n = i | X_0 = j) = (\mathbf{Q}^j)_{ij} = \sum_{\alpha_i \in \Omega} \sum_{\alpha_{n-1} \in \Omega} Q_{i,\alpha_{n-1}} \cdots Q_{\alpha_1,j}$.

The probability that a probability starting in state $j \in \Omega$ will be in state $i \in \Omega$ after $n \in \mathbb{N}$ steps is given by the sum over all possible trajectories $(j = X_0, X_1, \ldots, X_n = i) \in \Omega^n$, namely

$$
\mathcal{P}(\omega = (X_n)_{n \in \mathbb{N}_0} \text{ with } X_0 = i \text{ and } X_n = j) = \mathcal{P}(X_n = i | X_0 = j) = \sum_{\alpha_i \in \Omega} \sum_{\alpha_{n-1} \in \Omega} Q_{i,\alpha_{n-1}} \cdots Q_{\alpha_1,j}.
$$

A limiting distribution $\lim_{k \to \infty} \mathbf{Q}^k \mathbf{q}_0$ does not necessarily exist for a discrete-time Markov chain since oscillations are possible. However, the time average $\lim_{k \to \infty} \frac{1}{k} \sum_{k=0}^{k-1} \mathbf{Q}^k \mathbf{q}_0$ exists for all finite state space $\Omega$, and for all initial distributions $\mathbf{q}_0$ and converges to (one of) the stationary solutions, which are eigenvectors of the transition matrix $\mathbf{Q}$ to the eigenvalue $\lambda = 1$, see appendix A.4.

There is a one-to-one correspondence between minimal absorbing sets of the state-transition network and the stationary solutions of a Markov chain [FD22]: The number of minimal absorbing sets equals the number of (linearly independent) stationary solutions, and for every minimal absorbing set we can construct an associated eigenvector whose non-zero entries correspond to the states in the minimal absorbing network, see equation (81) in appendix A.5.

The algorithm for obtaining the stationary solution corresponding to the initial condition $\mathbf{q}_0$ is the following:

(i) Find all minimal absorbing sets $B_1, \ldots, B_n$ of the network $\Omega$, with $n \in \mathbb{N}$, $n \leq |\Omega|$.
(ii) Compute probability vectors $\mathbf{q}^\infty(\mathbf{q}_0 \in B_i)$ for all $i \in \{1, \ldots, n\}$ according to formula (79) in the appendix, which will be pairwise orthogonal.
(iii) Enlarge the orthonormal system $(\mathbf{q}^\infty(\mathbf{q}_0 \in B_i))_{i \in \{1, \ldots, n\}}$ to an ordered basis

$$
(\mathbf{q}^\infty(\mathbf{q}_0 \in B_1), \ldots, \mathbf{q}^\infty(\mathbf{q}_0 \in B_n), \mathbf{h}^{(n+1)}, \ldots, \mathbf{h}^{|\Omega|}),
$$

where the first $n$ basis vectors are $\mathbf{q}^\infty(\mathbf{q}_0 \in B_1), \ldots, \mathbf{q}^\infty(\mathbf{q}_0 \in B_n)$.
(iv) When the basis representation of the initial condition is given by

\[ q_0 = \sum_{i=1}^{n} \mu B_i q_\infty^\infty(q \in B_i) + \sum_{i=n+1}^{\Omega} \mu_i h \]

then the corresponding stationary solution is given by

\[ q_\infty(q_0) = \sum_{i=1}^{n} \mu B_i q_\infty(q_0 \in B_i) = \sum_{i=1}^{n} \mathcal{P}(B_i | q_0) q_\infty(q_0 \in B_i). \tag{35} \]

An example is given in appendix A.7.

Note that while the focus in [FD22] was on continuous-time Markov chains, the above steps for discrete-time Markov chains follow directly from there. Stationary solutions of a discrete-time Markov chain are identical to the stationary solutions of the corresponding continuous-time Markov chain since eigenvectors to the eigenvalue \( \lambda = 1 \) of transition matrix \( Q \) of the discrete-time Markov chain are eigenvalues to the eigenvalue \( \lambda = 0 \) of transition matrix \( \Gamma Q = Q - 1 \) of the associated continuous-time Markov chain: \( \Gamma Q q_\infty = 0 = (Q - 1) q_\infty \).

3.3. Constructing the Markov chain for quantum trajectories

In order to apply the theory of Markov chains, we need to determine both the states and the transition probabilities. Unlike in classical Markov chains, it can happen that an unravelling stops, as the column sum of the transition matrix need not be one. This is the case for possible trapping states. When constructing the matrix \( Q \), this has to be taken into account. In the following, we will first compute the set of possible states of the Markov chain, before we derive expressions for the transition probabilities.

3.3.1. The states of the Markov chain. We denote the set of possible states in the Markov chain by

\[ \{ \Theta : s \in S \} : = \Omega_{\rho_0} \]

\[ = \bigcup_{\omega \in \Omega} \Omega_{\rho_0}(\omega) \cup \left\{ \lim_{T \to \infty} \frac{I_{\text{trap}}(T)}{T} : \Theta_{\text{trap}} \in \bigcup_{\omega \in \Omega} \Omega_{\rho_0}(\omega) \text{ and } \lim_{T \to \infty} \text{Tr}[U_t(\Theta_{\text{trap}})] \neq 0 \right\}. \tag{36} \]

The last set is only relevant for the possible trapping states \( \Theta_{\text{trap}} \) with \( \lim_{T \to \infty} \text{Tr}[U_t(\Theta_{\text{trap}})] \in (0, 1] \). In that case, there is a non-zero chance for a transition from the state \( \Theta_{\text{trap}} \) to the minimal absorbing set \( \{ \lim_{T \to \infty} \frac{I_{\text{trap}}(T)}{T} \} \) (see figure 6 for an illustration). An explicit expression for the time average of a quantum trajectory that is captured in this minimal absorbing set is given below in section 3.8.2.

We choose the initial distribution vector \( q_{\rho_0} \) to be the first unit vector \( q_{\rho_0} = e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{|S|} \), corresponding to the initial state \( \rho_0 = \Theta_0 \in \Omega_{\rho_0} \), which is first element of the set of possible states in the quantum trajectory.

We assume that in all quantum trajectories only finitely many states appear, \( |S| < \infty \).
3.3.2. The transition probabilities of the Markov chain. The transition probability $P(\Theta_{s_1} \rightarrow \Theta_{s_2})$ is the probability that the next state in the quantum trajectory equals $\Theta_{s_2}$, provided that the last state was $\Theta_{s_1}$. It can be computed to

$$P(\Theta_{s_2} | \Theta_{s_1}) = \int_{\mathbb{R}_{\geq 0}} \frac{p(\tau, \Theta_{s_2} | \Theta_{s_1})}{f(\tau | \Theta_{s_1}) \cdot P(\Theta_{s_2} | \tau, \Theta_{s_1})} \, d\tau = \sum_{k \in I(s_1 \rightarrow s_2)} \int_{\mathbb{R}_{\geq 0}} f(k(\tau | \Theta_{s_1}) d\tau, \tag{37}$$

where the sum goes over all Lindblad operators $V_k$ that yield a transition from state $\Theta_{s_1}$ to $\Theta_{s_2}$, or more formally: $I(s_1 \rightarrow s_2) := \{ k \in I : J_k \circ U_t(\Theta_{s_1}) = \Theta_{s_2} \text{ for some } t > 0 \}$.

The following examples illustrate both the states of all possible discrete quantum trajectories as well as the transition probabilities.

3.4. Examples

In this section we present three examples that illustrate the state transition network $\Omega_{\rho_0}$ for a specific choice of Hamiltonian and Lindblad operators. We note that $\Omega_{\rho_0}$ depends on the initial state $\rho_0$ and certain transition probabilities can vanish for special initial conditions (e.g. $\rho_{11}(0) = 0$). The transition network consists of only those states, that can be reached from $\rho_0$.

3.5. Example with trivial Hamiltonian and two transition operators

We consider a system of dimension $N = 2$ with a trivial Hamiltonian and two transition operators as Lindblad operators, that is:

$$H \propto \mathbb{1}, \quad V_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}. \tag{38}$$

The set of possible states appearing in the Markov chain can be computed to:

$$\Omega_{\rho_0} = \left\{ \Theta_{s_1} = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix}, \Theta_{s_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \Theta_{s_3} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \tag{39}$$
Figure 7. State transition network for example (a): with probability one ($\rho_{11}(0) + \rho_{22}(0) = 1$) the system jump from the initial state $\Theta_{s1} = \rho_0$ to one of the states constituting the unique, minimal absorbing set of this network $B = \{ \Theta_{s2}, \Theta_{s3} \}$ and then oscillates between them, with the time average yielding the stationary state $\rho^\infty = \frac{1}{\gamma_1 + \gamma_2} \begin{pmatrix} \frac{1}{\gamma_1} & 0 \\ 0 & \frac{1}{\gamma_2} \end{pmatrix}$.

The transition matrix equals

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ \rho_{11}(0) & 0 & 1 \\ \rho_{22}(0) & 1 & 0 \end{pmatrix}, \text{ with the ordered states } (\Theta_{s1} = \rho_{0}, \Theta_{s2}, \Theta_{s3}).$$

(40)

In this example there are no possible trapping states. The state transition network is depicted in figure 7.

3.6. Example with trivial Hamiltonian and one projection matrix as Lindblad operator

We consider a system of dimension $N = 2$ with a trivial Hamiltonian and a projection operator for a Lindblad operators, that is:

$$H \propto \mathbb{1}, V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \Lambda = \begin{pmatrix} \gamma_1 & 0 \\ 0 & 0 \end{pmatrix}.$$  (41)

The set of possible states appearing in the Markov chain can be computed to:

$$\Omega_{\rho_0} = \left\{ \Theta_{s1} = \begin{pmatrix} \rho_{11}(0) \\ \rho_{21}(0) \end{pmatrix}, \Theta_{s2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \Theta_{s3} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$  (42)

The transition matrix is given by $Q = \begin{pmatrix} 0 & 0 & 0 \\ \rho_{11}(0) & 1 & 0 \\ \rho_{22}(0) & 0 & 1 \end{pmatrix}.$
Figure 8. State transition network for example (b): starting at the initial state $\Theta_{s_1} = \rho_0$, there are two different types of dynamics: Either the Lindblad operator $V_1$ is applied and the system 'jumps' to the state $\Theta_{s_2}$. Once the system is in state $\Theta_{s_2}$, the operator $V_1$ is repeatedly applied, without ever changing the state. The other dynamics is only possible, because $\Theta_{s_1} = \rho_0$ is a possible trapping state: With probability $\rho_{22}(0)$, no Lindblad operator is ever applied, while the system converges to the state $\Theta_{s_3}$, as indicated by the gray arrow with transition probability $\rho_{22}(0)$. The gray arrow with transition probability one is 'artificial' in the sense that this transition never happens in the dynamics of the unraveling, but is required in order for the state transition network to form a Markov chain.

The initial state $\rho_0 = \Theta_{s_1}$ is also a possible trapping state. While the state $\Theta_{s_2}$ appears in every quantum trajectory, the state $\Theta_{s_3}$ is only attained in the limit $t \to \infty$, when no quantum jump takes place after the initial state $\rho_0$ (compare figure 8).

3.7 Example of a state transition network with $|\Omega| = 5$ states and a non-trivial Hamiltonian

We consider the following system with dimension $N = 4$, where the Hamilton operator and the two Lindblad operators are given by:

\[
H = \begin{pmatrix}
E_1 & 0 & 0 & 0 \\
0 & E_1 & 0 & 0 \\
0 & 0 & E_2 & 0 \\
0 & 0 & 0 & E_2
\end{pmatrix},
V_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
V_2 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
H_c = \begin{pmatrix}
E_1 - \frac{i\gamma_1}{2} & 0 & 0 & 0 \\
0 & E_1 - \frac{i\gamma_1}{2} & 0 & 0 \\
0 & 0 & E_2 - \frac{i\gamma_2}{2} & 0 \\
0 & 0 & 0 & E_2 - \frac{i\gamma_2}{2}
\end{pmatrix}.
\]
The state transition network is given in figure 9. The state transition network for a system of dimension $N=4$, where the Hamiltonian and Lindblad operators are given by equation (43). Starting with the initial state $\rho_0$, the second state in the discrete quantum trajectory is either $\Theta_{s_2}$ or $\Theta_{s_3}$, which determines its future evolution: Either we have an oscillation between the states $\Theta_{s_2}$ and $\Theta_{s_3}$ in the minimal absorbing set $B_1 = \{\Theta_{s_2}, \Theta_{s_3}\}$, or an oscillation between the states $\Theta_{s_4}$ and $\Theta_{s_5}$ in the minimal absorbing set $B_2 = \{\Theta_{s_4}, \Theta_{s_5}\}$.

The set of possible states appearing in the Markov consists of the five states

$$\Omega_{\rho_0} = \{\Theta_{s_1} = \rho_0, \Theta_{s_2}, \Theta_{s_3}, \Theta_{s_4}, \Theta_{s_5}\},$$

with

$$\Theta_{s_1} = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) & \rho_{13}(0) & \rho_{14}(0) \\ \rho_{21}(0) & \rho_{22}(0) & \rho_{23}(0) & \rho_{24}(0) \\ \rho_{31}(0) & \rho_{32}(0) & \rho_{33}(0) & \rho_{34}(0) \\ \rho_{41}(0) & \rho_{42}(0) & \rho_{43}(0) & \rho_{44}(0) \end{pmatrix},$$

$$\Theta_{s_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Theta_{s_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Theta_{s_4} = \begin{pmatrix} \frac{\rho_{11}(0) + \rho_{22}(0)}{\rho_{11}(0) + \rho_{22}(0)} & \frac{\rho_{11}(0) + \rho_{23}(0)}{\rho_{11}(0) + \rho_{23}(0)} & \frac{\rho_{11}(0) + \rho_{24}(0)}{\rho_{11}(0) + \rho_{24}(0)} & 0 \\ \frac{\rho_{21}(0) + \rho_{32}(0)}{\rho_{21}(0) + \rho_{32}(0)} & \frac{\rho_{21}(0) + \rho_{33}(0)}{\rho_{21}(0) + \rho_{33}(0)} & \frac{\rho_{21}(0) + \rho_{34}(0)}{\rho_{21}(0) + \rho_{34}(0)} & 0 \\ \frac{\rho_{31}(0) + \rho_{42}(0)}{\rho_{31}(0) + \rho_{42}(0)} & \frac{\rho_{31}(0) + \rho_{43}(0)}{\rho_{31}(0) + \rho_{43}(0)} & \frac{\rho_{31}(0) + \rho_{44}(0)}{\rho_{31}(0) + \rho_{44}(0)} & 0 \\ \frac{\rho_{41}(0) + \rho_{52}(0)}{\rho_{41}(0) + \rho_{52}(0)} & \frac{\rho_{41}(0) + \rho_{53}(0)}{\rho_{41}(0) + \rho_{53}(0)} & \frac{\rho_{41}(0) + \rho_{54}(0)}{\rho_{41}(0) + \rho_{54}(0)} & 0 \end{pmatrix},$$

and the transition probabilities

$$q_{s_1 \to s_2} = \rho_{11}(0) + \rho_{22}(0),$$

$$q_{s_1 \to s_3} = \rho_{13}(0) + \rho_{44}(0).$$

The state transition network is given in figure 9.
\[ \rho_B(\rho_0) = \sum_{B \in \mathcal{B}} \mathcal{P}(B | \rho_0) \sum_{\Theta \in \Theta_B} q_\Theta(\Theta_s | B) \frac{\{ \Theta_s(t) \}_{t \in [0, \tau]} f(\Theta_s)}{\text{Tr}[\ldots]} \]

\[ \overset{(*)}{=} \sum_{q_{B_1} \to q_{B_2}} \mathcal{P}(B_1 | \rho_0) \sum_{\Theta \in \Theta_{B_1}} q_\Theta(\Theta_s | B_1) \frac{\{ \Theta_s(t) \}_{t \in [0, \tau]} f(\Theta_s)}{\text{Tr}[\ldots]} + \sum_{q_{B_2} \to q_{B_2}} \mathcal{P}(B_2 | \rho_0) \sum_{\Theta \in \Theta_{B_2}} q_\Theta(\Theta_s | B_2) \frac{\{ \Theta_s(t) \}_{t \in [0, \tau]} f(\Theta_s)}{\text{Tr}[\ldots]} \]

We evaluate the stationary states \( \Theta_B \) occurring in equation (29). When \( \Theta_s \) is not a possible trapping state, we can compute the (conditional) time evolution operator \( e^{-iH_c \tau} \) (for details see appendix A.2) and get

3.8. Evaluating the stationary state \( \Theta_B \) of the Lindbladian for the minimal absorbing set \( B \)

We now evaluate the stationary states \( \Theta_B \) occurring in equation (29).

We write the conditional Hamiltonian \( H_c \) (see equation (2)) as a direct sum of Jordan blocks

\[ H_c = \bigoplus_{m=1}^M J_m = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_M \end{pmatrix}, \quad \text{with} \]

\[ J_m = \begin{pmatrix} \delta_m & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \delta_m & 1 & 0 \end{pmatrix} \in \mathbb{C}^{s_m \times s_m} \]

being a Jordan block of size \( s_m \in \mathbb{N} \) and \( \sum_{m=1}^M s_m = N \).

We discuss separately the two cases of a finite and an infinite number of quantum jumps.

3.8.1. Infinitely many quantum jumps. When the quantum trajectory contains infinitely many quantum jumps, we need to evaluate the expression \( \overline{I}(\tau) \) defined in equation (26).

When \( \Theta_s \) is not a possible trapping state, we can compute the (conditional) time evolution operator \( e^{-iH_c \tau} \) (for details see appendix A.2) and get
\[ I_k(\tau) = \int_0^\infty f(\tau | \Theta_s) \frac{d\tau}{\frac{1}{\tau} \text{Tr}[U_\tau(\Theta_s)]} \]

\[ = \frac{\text{integration}}{\text{partial}} \frac{-\text{Tr}[U_\tau(\Theta_s)]}{\frac{1}{\tau} \text{Tr}[U_\tau(\Theta_s)]} \int_0^\tau U_\tau(\Theta_s) \frac{d\tau}{\frac{1}{\tau} \text{Tr}[U_\tau(\Theta_s)]} + \int_0^\infty \text{Tr}[U_\tau(\Theta_s)] \frac{U_\tau(\Theta_s)}{\frac{1}{\tau} \text{Tr}[U_\tau(\Theta_s)]} d\tau \]

\[ = -\lim_{\tau \to \infty} \text{Tr}[U_\tau(\Theta_s)] \int_0^\tau U_\tau(\Theta_s) \frac{d\tau}{\frac{1}{\tau} \text{Tr}[U_\tau(\Theta_s)]} + \int_0^\infty U_\tau(\Theta_s) d\tau \]

\[ = \frac{A_2}{\pi} \int_0^\infty U_\tau(\Theta_s) d\tau \int_0^\infty e^{-i H_c \tau} \Theta_s (e^{-i H_c \tau})^\dagger \tau \]

\[ = \int_0^\infty \left( \begin{array}{ccc} e^{-i H_c \tau} & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-i H_c \tau} \end{array} \right) \left( \begin{array}{ccc} \Theta^{(1,1)} & \cdots & \Theta^{(1,M)} \\ \vdots & \ddots & \vdots \\ \Theta^{(M,1)} & \cdots & \Theta^{(M,M)} \end{array} \right) \left( \begin{array}{ccc} e^{i J_f^\dagger \tau} & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{i J_f^\dagger \tau} \end{array} \right) d\tau \]

\[ = \left( \begin{array}{ccc} \int_0^\infty e^{-i J_{i0} \tau} \Theta^{(1,1)} e^{i J_f^\dagger \tau} d\tau & \cdots & \int_0^\infty e^{-i J_{i0} \tau} \Theta^{(1,M)} e^{i J_f^\dagger \tau} d\tau \\ \vdots & \ddots & \vdots \\ \int_0^\infty e^{-i J_{i0} \tau} \Theta^{(M,1)} e^{i J_f^\dagger \tau} d\tau & \cdots & \int_0^\infty e^{-i J_{i0} \tau} \Theta^{(M,M)} e^{i J_f^\dagger \tau} d\tau \end{array} \right) \]

(47)

For the component \( \sum_{i=1}^{n-1} s_i + j + \sum_{i=1}^{n-1} s_{i'} + k \), we get with \( \delta_k = R_s + i J_s \):

\[ \left( I_k(\tau) \right) \left( \sum_{i=1}^{n-1} s_i + j + \sum_{i=1}^{n-1} s_{i'} + k \right) \]

\[ = \left( \int_0^\infty e^{-i J_{i0} \tau} \Theta^{(m,n)} e^{i J_f^\dagger \tau} d\tau \right) \]

\[ \left( \begin{array}{c} \sum_{i=1}^{n_{\alpha}} \sum_{j'=k} \left( \alpha - j + \beta - k \right) \left( \Theta^{(m,n)} \right)_{ij'} \\
\sum_{i=1}^{n_{\alpha}} \sum_{j'=k} \left( \alpha - j \right) \left( \Theta^{(m,n)} \right)_{ij'} \end{array} \right) \]

(48)

In step (*) in the last line we used the fact that since \( \Theta_s \) is not a possible trapping state, a diagonal block of \( \Theta_s \) must vanish whenever the corresponding eigenvalue of \( \Lambda \) vanishes \( \lambda_m = 0 \implies \Theta^{(m,n)} = 0^{n_{\alpha \times n_{s'}}} \), see appendix A.2, and an inequality that is valid for all density matrices: \( |(\Theta_s)_{ij}| \leq (\Theta_s)_{ii} (\Theta_s)_{jj} \).

An important special case is when both Jordan blocks are of size one, that is \( s_{m} = 1 = s_{n'} \). In that case we have
When the number of jumps is finite, we have to evaluate one, the block matrices together. Our starting point was the quantum jump unravelling (section A.1). Now that we have obtained all building blocks for the stationary solution, let us put them immediately after jumps is finite average of a single trajectory exists. We assumed that the space of density matrices occurring of a single trajectory Markov chain always exists. We evaluated the time average

\[
\left( \int_0^{\infty} e^{-i\Delta t} \Theta^{(m,n)} e^{i\Delta t} dt \right)_{11} = \begin{cases} \Theta^{(m,n)}_{[t(R_s-R_a)+I_n+I_m]}, & \text{if } I_n \neq 0 \neq I_m \\ 0, & \text{else.} \end{cases}
\]  

When the conditional Hamiltonian \( H_c \) is diagonalizable, then all Jordan block are of size one, the block matrices \( \Theta^{(m,n)} \in \mathbb{C} \) are scalars, and expression (48) becomes:

\[
(I_c(\Theta))_{mn} = \frac{(\Theta^{(m,n)})_{11}}{1(\delta_0^t - \delta_0^m)} = \begin{cases} \Theta^{(m,n)}_{[t(R_s-R_a)+I_n+I_m]}, & \text{if } I_n \neq 0 \neq I_m \\ 0, & \text{else.} \end{cases}
\]  

3.8.2. Finitely many quantum jumps. When the number of jumps is finite, we have to evaluate \( \Theta_B = \left( \frac{U_t(\Theta_{\text{trap}})}{\|U_t\|} \right)_{t \geq 0} \), with \( \Theta_{\text{trap}} \) being a possible trapping state.

By definition of a possible trapping state, we know that \( \text{Tr}[U_t(\Theta_{\text{trap}})] \) converges to a positive value:

\[
\text{Tr}[U_t(\Theta_{\text{trap}})] \xrightarrow{t \to \infty} q \in (0, 1].
\]

Then we have

\[
\left( U_t(\Theta_{\text{trap}}) \right)_{\text{Tr}} \frac{\sum_{\alpha=0}^{s_n} \sum_{\beta=0}^{s_n} e^{i(R_s-R_a)\Delta t} \Theta^{(m,n)}_{\text{Tr}} \left( \sum_{i=1}^{s_n} x_i \right)_{\alpha,\beta}}{\text{Tr}[U_t(\Theta_{\text{trap}})] \xrightarrow{t \to \infty} 0}
\]  

and

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{i\Delta E t} dt = \begin{cases} 1, & \text{if } \Delta E = 0 \\ 0, & \text{else.} \end{cases}
\]

When we combine the equations (51) and (52), we get

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left( U_t(\Theta_{\text{trap}}) \right)_{\text{Tr}} \left( \sum_{\alpha=0}^{s_n} \sum_{\beta=0}^{s_n} e^{i(R_s-R_a)\Delta t} \Theta^{(m,n)}_{\text{Tr}} \left( \sum_{i=1}^{s_n} x_i \right)_{\alpha,\beta} \right) dt = \delta_{I_n,0} \delta_{I_m,0} \delta_{R_s,R_a} \sum_{\mu \in \{1, \ldots, M, I_n=0\}} (\Theta^{(m,n)}_{\text{trap}})_{\mu,\mu}.
\]  

3.9. The stationary solution of the Lindblad equation

Now that we have obtained all building blocks for the stationary solution, let us put them together. Our starting point was the quantum jump unravelling (section 2) and the fact that the time and ensemble average can be exchanged (section A.1), due to the fact that the time average of a single trajectory exists. We assumed that the space of density matrices occurring immediately after jumps is finite \(|S| < \infty\), so that a stationary solution of the corresponding Markov chain always exists. We evaluated the time average \( \Theta_B := \Theta^{(\omega,\mu)}_{\text{trap}} = \left( \Theta^{(\omega,\mu)}_{\text{trap}}(\omega, t) \right)_{t \geq 0} \) of a single trajectory \( \omega \in \mathcal{U} \) (see equation (29) in section 3.1), and we showed that it depends
only on the minimal absorbing set \( B \subseteq \Omega_{\rho_0} \), the quantum trajectory is eventually captured in (section 3.2). We therefore separated the integral over the different unravellings into a sum over the minimal absorbing sets in the network, where the integrand \( \Theta_B \) is constant over the set \( \mathcal{U}(B) \) of quantum trajectories that all land in the associated minimal absorbing set \( B \subseteq \Omega_{\rho_0} \). Mathematically, all this gives the following steps for obtaining the formula for the limiting density matrix \( \rho^\infty(\rho_0) \), depending on the initial state \( \rho_0 \):

\[
\rho^\infty(\rho_0) = \lim_{t \to \infty} \frac{1}{T} \int_0^T \rho(t) \, dt = (\rho_{\rho_0}(t))_{t \geq 0} = (\langle T \rho_{\rho_0}(t, \omega) \rangle_{\omega \in U})_{t \geq 0}
\]

\[
= \langle T \rho_{\rho_0}(t, \omega) \rangle_{\omega \in U} = \int_{\mathcal{U} = \bigcup_{i \in S} \mathcal{U}(B_i)} \Theta_B^\infty(\omega) \, d\mathcal{P}_{\rho_0}(\omega)
\]

\[
= |S| < \infty \sum_{i=1}^n \int_{\mathcal{U}(B_i)} \Theta_B^\infty(\omega) \, d\mathcal{P}_{\rho_0}(\omega) = \sum_{i=1}^n \mathcal{P}(\mathcal{U}(B_i) | \rho_0) \Theta_B.
\]

By mapping the quantum trajectories onto discrete-time Markov chains (section 3.3), we were able to compute the probability \( \mathcal{P}(\mathcal{U}(B_i) | \rho_0) \) of landing in the minimal absorbing set \( B_i \) as the coefficient \( \mu_{B_i} \) in an expansion of the initial probability distribution vector \( q_{\rho_0} \) into an ordered basis of generalized eigenvectors of the transition matrix \( Q \) (see section 3.2),

\[
q_{\rho_0} = \sum_{i=1}^{|S|} \mu_i \mathbf{h}^i =: \sum_{i=1}^n \mu_{B_i} q^\infty(q_{\rho_0} \in B_i) + \sum_{i=n+1}^{|S|} \mu_i \mathbf{h}^i.
\]

The first \( n \) generalized eigenvectors \( q^\infty(q_{\rho_0} \in B_i) \) for \( i \in \{1, \ldots, n\} \) are the actual eigenvectors of \( Q \) to the eigenvalue \( \lambda = 1 \) (see equation (79) in appendix A.5). The second term contains the remaining generalized eigenvectors \( \mathbf{h}^i \), and it declines toward zero under repeated application of the transition matrix \( Q \).

An explicit expression for \( \Theta_B \) was given in section 3.8. It is the stationary solution of the Lindblad equation that lies in the minimal absorbing set \( B \subseteq \Omega_{\rho_0} \).

3.10. The classical case

The classical case can be recovered when the Hamiltonian vanishes \( (H = 0) \) and \( \Lambda = \beta \) is proportional to the identity operator for some real number \( \beta \in \mathbb{R}_{>0} \). In this case, there are no possible trapping states, and we get for the stationary solution of a minimal absorbing set \( B \subseteq \Omega_{\rho_0} \) and for the stationary solution of equation (1)

\[
\Theta_B = \sum_{i \in S} q(\Theta_s | B) \underbrace{\mathcal{I}(\tau_i)}_{\beta} \| \Theta_s \|^\frac{1}{\beta} \| \Theta_s \|^{-\frac{1}{\beta}} \frac{1}{\beta} \sum_{i \in S} q(\Theta_s | B) \Theta_s \frac{1}{\beta} = \sum_{i \in S} q(\Theta_s | B) \Theta_s
\]

\[
\rho^\infty(\rho_0) = \sum_{i=1}^n \mathcal{P}_{\rho_0}(\mathcal{U}(B_i)) \Theta_B = \sum_{i \in S} \left( \prod_{j=1}^{|S|} \mathcal{P}_{\rho_0}(\mathcal{U}(B_i)) \right) q(\Theta_s | B) \Theta_s = \sum_{i \in S} q(\Theta_s) \Theta_s.
\]

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3.11. Computational complexity

We now want to talk briefly about the computational complexity needed to compute all classical stationary solutions \( \rho^\infty(\rho_0 \in B) \), which include the classical probabilities \( \mathcal{P}(B \mid \rho_0) \) and the classical probability vector \( q^\infty(\rho_0 \in B) \). It is possible to search numerically for strongly connected components in linear time [GM78]. When merging these strongly connected components into a single state via graph condensation [GM78] and following a single path to its end, one arrives at a minimal absorbing set. This requires \( O(|\Omega| + |E|) \) time, where \( |\Omega| \) is the number of states and \( |E| \) the number of links. Computing the stationary solution \( \rho_B \) for a minimal absorbing set \( B \in \{ \text{min. ab. sets} \} \) can be done in \( O(|E_B| \cdot N_B) \), where \( N_B \) is the total number of in-trees of the subnetwork \( S_B := \{ B, \{(i,j) \in E : i, j \in B \} \} \). This results in a total time of \( O(|\Omega| + |E| + \sum_{B \in B} |E_B| \cdot N_B) \).

The construction of the state transition network can be done in \( |I| \cdot |\Omega_{\rho_0}| \), where \( |I| \) is the number of Lindblad operators and \( |\Omega_{\rho_0}| \) the number of quantum states.

4. Discussion and conclusion

We have shown that the stationary states of the Lindblad equation (1) can be determined by using the concept of a quantum jump unravelling. The final formula is given by

\[
\rho^\infty(\rho_0) = \sum_{i=1}^n \mathcal{P}(\mathcal{U}(B_i) \mid \rho_0) \Theta_{B_i},
\]

where the sum goes over all minimal absorbing sets of the discrete-time Markov chain associated with the quantum jump process, whose states \( \{ \Theta_s : s \in \mathcal{S} \} \) and transition probabilities \( \mathcal{P}((\Theta_{s_1} \rightarrow \Theta_{s_2})) \) are given in sections 3.3.1 and 3.3.2, respectively.

The factor \( \Theta_{B_i} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \Theta(\omega) d\omega \) denotes the time average of all quantum trajectories \( \omega \in \mathcal{U} \) which are eventually captured by the minimal absorbing set \( B_i \subseteq \Omega_{\rho_0} \). Explicit expressions for \( \Theta_{B_i} \) are derived in section 3.8, depending on whether the number of quantum jumps is finite or infinite. \( \mathcal{P}(\mathcal{U}(B_i) \mid \rho_0) \) denotes the probability for a quantum trajectory being captured by the minimal absorbing set \( B_i \), provided the initial state is given by \( \rho_0 \). It can be computed from the coefficients \( \mu_{B_i} \) of the vector \( q_{\rho_0} \) (representing the initial state \( \rho_0 \in \Omega_{\rho_0} \)) when expressing it in a basis of generalized eigenvectors of the transition matrix \( Q \) of the discrete-time Markov chain, which is defined in section 3.3.2. When there is no change of the states between jumps, the classical case is recovered, see section 3.10.

In the case of infinitely many states, such a method is in general no longer applicable: Although the existence of stationary density matrices \( \rho^\infty \in \mathbb{C}^{N \times N} \) for the Lindbladian is guaranteed (see [KM04]), the existence of stationary vectors \( q^\infty \in \mathbb{R}^{\mid \mathcal{S} \mid} \) for the corresponding Markov chain is not, as shown in appendix A.8. This limitation can only be dissolved if an analogous expression to formula (81), but for special kinds of Markov chains with infinite state space is known. One possible condition could be to require the Markov chain to be positive recurrent, in which case a stationary solution exists [Pri13, DMPS18, Bré13]. An interesting question for further research would be which conditions the Lindblad operators must have for the states of the quantum trajectory to be positive recurrent or whether this is a necessary condition for the Lindblad operators.
However, it is possible to compute the stationary density matrix $\rho^\infty$ when the sequence of states in the quantum trajectory, together with the corresponding average waiting times, converges: $(\Theta_k, \tau_k) \xrightarrow{k \to \infty} (\Theta^\infty, \tau^\infty)$. In this case, we can evaluate the time average of a quantum trajectory to

$$\langle \Theta(t, \omega) \rangle_{t \geq 0} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{J(T)-1} I_{\Theta_k}(\tau_k) = \frac{1}{\tau^\infty} \int_0^{\tau^\infty} \frac{U_i(\Theta^\infty)}{\text{Tr}[]} \, \text{d}t.$$ (58)

This means that while we can no longer use the formula for $\Theta_B$ given by equation (29) when the minimal absorbing set $B$ contains infinitely many states, we can use equation (57) when the number of minimal absorbing sets is finite. We only have to substitute the expression of equation (58) for $\Theta_B$. Unfortunately, this is not quite the same as giving an explicit recipe for obtaining the steady state, as was possible in the case with finitely many states. But it might still be possible to obtain the stationary density matrices this way for many more cases that are physically relevant.

Data availability statement

No new data were created or analysed in this study.

Appendix A

A.1 The time average and the ensemble average [KM04]

For almost all $\omega \in \mathcal{U}$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \text{d}t \, \Theta_{\rho_0}(t, \omega) = \Theta^\infty_{\rho_0}(\omega).$$

Here $\Theta^\infty_{\rho_0}$ is a random variable whose expectation value is the time average of $U(t) \rho_0$,

$$\langle \langle \Theta_{\rho_0}(t, \omega) \rangle_{t \geq 0} \rangle_{\omega \in \mathcal{U}} = \langle \langle \Theta_{\rho_0}(t, \omega) \rangle_{\omega \in \mathcal{U}} \rangle_{t \geq 0}.$$

So, in this sense, the time average commutes with the ensemble average.

A.2 Auxiliary calculations for section 3.8.1

A.2.1 The form of the (conditional) time evolution operator. With respect to a suitable basis, we can write the conditional Hamiltonian in the Jordan normal form, that is

$$H_c = \bigoplus_{m=1}^M J_m = \begin{pmatrix} J_1 & \cdots & \cdots \cr \cdots & \cdots & \cdots \cr J_M \end{pmatrix}, \text{ with}$$

$$J_m = \begin{pmatrix} \delta_m & 1 & \cdots & \cdots \cr \cdots & \cdots & \cdots & \cdots \cr \cdots & \cdots & \cdots & \cdots \cr \delta_m & 1 \end{pmatrix} \in \mathbb{C}^{s_m \times s_m}, \text{ with } \sum_{m=1}^M s_m = N.$$ (59)

The eigenvalues of $H_c$ are then given by $\{\delta_1, \ldots, \delta_M\}$. 
This makes it easier to compute matrix exponentials, in particular the conditional time evolution operator

\[
    e^{-i H_\tau t} = \begin{pmatrix}
        e^{-i J_{1\tau} t} & 0 & \cdots & 0 \\
        \vdots & \ddots & \ddots & \vdots \\
        0 & \cdots & e^{-i J_{M\tau} t}
    \end{pmatrix} = \bigoplus_{m=1}^{M} e^{-i J_{m\tau} t} = \bigoplus_{m=1}^{M} e^{-i \delta_{m\tau} t} \begin{pmatrix}
        1 & \frac{i}{\tau} & \cdots & \frac{i^{(m-1)} \tau}{(s_n-1)!} \\
        \frac{i}{\tau} & \frac{i^2}{\tau^2} & \cdots & \frac{i^{(m-2)} \tau^2}{(s_n-2)!} \\
        \vdots & \vdots & \ddots & \vdots \\
        \frac{i^{M-1}}{\tau^{M-1}} & \frac{i^M}{\tau^M} & \cdots & 1
    \end{pmatrix}.
\]

(60)

When applying the conditional time evolution to a density matrix \(\Theta_s\), we get

\[
    U_\tau(\Theta_s) = \begin{pmatrix}
        e^{-i J_{1\tau} \tau} & 0 & \cdots & 0 \\
        0 & e^{-i J_{2\tau} \tau} & \cdots & 0 \\
        \vdots & \vdots & \ddots & \vdots \\
        0 & 0 & \cdots & e^{-i J_{M\tau} \tau}
    \end{pmatrix} \begin{pmatrix}
        \Theta^{(1,1)} & \cdots & \Theta^{(1,M)} \\
        \vdots & \ddots & \vdots \\
        \Theta^{(M,1)} & \cdots & \Theta^{(M,M)}
    \end{pmatrix} \begin{pmatrix}
        e^{i J_{1\tau} \tau} & 0 & \cdots & 0 \\
        \vdots & \ddots & \vdots & \vdots \\
        0 & \cdots & e^{i J_{M\tau} \tau}
    \end{pmatrix} = \begin{pmatrix}
        e^{-i J_{1\tau} \tau} \Theta^{(1,1)} e^{i J_{1\tau} \tau} & \cdots & e^{-i J_{1\tau} \tau} \Theta^{(1,M)} e^{i J_{1\tau} \tau} \\
        \vdots & \ddots & \vdots \\
        e^{-i J_{M\tau} \tau} \Theta^{(M,1)} e^{i J_{M\tau} \tau} & \cdots & e^{-i J_{M\tau} \tau} \Theta^{(M,M)} e^{i J_{M\tau} \tau}
    \end{pmatrix}.
\]

(61)

When we look at the components \(\sum_{\mu=1}^{s_\mu} \sum_{\nu=1}^{\tau \delta_{m\nu}} \Theta_{\gamma} \rho_{\mu \nu+k}\) of equation (61), we get:

\[
    \langle U_\tau(\Theta_s) \rangle \sum_{\mu=1}^{s_\mu} \sum_{\nu=1}^{\tau \delta_{m\nu}} \Theta_{\gamma} \rho_{\mu \nu+k} = \left( e^{-i J_{1\tau} \tau} \Theta^{(m,n)} e^{i J_{1\tau} \tau} \right)_{j k}
\]

\[
    = \sum_{\alpha=1}^{s_m} \sum_{\beta=1}^{s_n} \left( e^{-i J_{\alpha\beta} \tau} \right)_{j k} \left( \Theta^{(m,n)} \right)_{\alpha \beta} \left( e^{i J_{\alpha\beta} \tau} \right)_{j k}
\]

\[
    = \sum_{\alpha=1}^{s_m} \sum_{\beta=1}^{s_n} \Theta^{(m,n)} \rho_{\alpha \beta} \left( e^{i J_{\alpha\beta} \tau} \right)_{j k}
\]

(62)

A.2.2. Auxiliary calculations for equation (47). When \(\Theta_s \in \Omega_{\rho_0}\) is a state in the quantum trajectory that is not a possible trapping state, then

\[
    \lim_{\tau \to \infty} \text{Tr}[U_\tau(\Theta_s)] = \int_0^\tau U_\tau(\Theta_s) \text{d}t = 0.
\]

Proof. It suffices to show that the left hand side of equation (63) is a positive semi-definite matrix with vanishing trace. To see this, we take an arbitrary vector \(\psi \in \mathbb{C}^N\) and compute

\[
    \langle \psi | U_\tau(\Theta_s) \psi \rangle = \left( e^{i H^\tau \psi} | \Theta_s e^{i H^\tau \psi} \right) \geq 0,
\]

(63)

(64)

where we used the fact that the density matrix \(\Theta_s\) is positive semi-definite. Since multiplying by a positive scalar and integrating over the interval \([0, \tau]\) does not change positivity, we have shown the first part of our claim.
For the second part we use the Jordan normal form of $H_\epsilon$ (see equation (59)).

We recall that for non-possible trapping states $\Theta_s$ the trace $U_\epsilon(\Theta_s)$ vanishes as $t$ tends to infinity (see the algorithm for the unravelling in section 2.2, (ii)) and compute

$$
\Theta_s \text{ is no possible trapping state} \quad \lim_{t \to \infty} \text{Tr}[U_\epsilon(\Theta_s)] = \lim_{t \to \infty} \sum_{\alpha, \beta} \frac{e^{-2t\omega_s}}{\alpha! \beta!} \left( D_{\alpha, \beta} \right) \left( \Theta_s \right) \cdot \Theta_s = 0.
$$

This means that whenever the imaginary part of an eigenvalue of the conditional Hamiltonian equals zero, then the corresponding block of $\Theta_s$ must vanish

$$
\text{Im}[\delta_{\alpha, \beta}] = 0 \implies \Theta_s = 0.
$$

When we now take the trace over the left side of equation (63), we get

$$
\lim_{t \to \infty} \text{Tr}[U_\epsilon(\Theta_s)] \cdot \text{Tr} \left[ \int_0^\tau U_\epsilon(\Theta_s) \, dt \right] = 0,
$$

since multiplying equation (65) by a factor of $\tau$ does not influence the limiting behavior. \hfill \Box

### A.3. In a classical, discrete-time Markov chain with a finite state space a trajectory $(\Theta_{\Theta_s})_{s \in \mathbb{N}}$ will eventually be captured by a minimal absorbing set, that is $\mathcal{P}(\Theta_s \in \{\text{minimal absorbing sets}\}) = 1$

**Proof.** Let $B_1, \ldots, B_n$ be the minimal absorbing sets of the associated state transition network for the Markov chain. We know that for every state $s \in \Omega$ in the network there is a path to a minimal absorbing set $B \subseteq \Omega$. This means that there is a finite number of states $s = s_0, \ldots, s_L \in B$ with $q_{s_j \rightarrow s_{j+1}} > 0$ [FD22]. We now let $l_i$ be the length of the shortest (non-zero) path to the minimal absorbing set number $i \in \{1, \ldots, n\}$ and $q_i$ be the largest probability for reaching the set $B_i$ in $l_i$ time steps, namely

$$
l_i := \min_{l \in \mathbb{N}_0} \left\{ l \in \mathbb{N}_0 : (\omega_0 = s, \ldots, \omega_l \in B_i) \text{ with } q_{s_j \rightarrow s_{j+1}} > 0 \text{ for all } j \in \{0, \ldots, l\} \right\}
$$

$$
L := \max_{i \in \{1, \ldots, n\}} \{ l_i \}
$$

$$
q_i := \max_{\omega \in \Omega^{l_i+1}} \left\{ \prod_{j=0}^{l_i} q_{\omega_j \rightarrow \omega_{j+1}}, \text{ with } \omega_0 = s \text{ and } \omega_l \in B_i \right\}
$$

$$
q_0 := \min_{i \in \{1, \ldots, n\}} \{ q_i \}.
$$

Further, be let $L(q_0)$ be the largest (smallest) of the $l_i (q_i)$. For a fixed natural number $m \in \mathbb{N}$ we define the event that a trajectory $\omega \in \Omega^m$ after $L \cdot m$ time steps is not contained in a minimal absorbing set
\[ A_m := \left\{ \omega \in \Omega^N : \omega_{L_m} \notin \bigcup_{i=1}^{n} B_i \right\}. \] (68)

The corresponding probability for \( A_m \) to occur can be estimated to \( P(A_m) \leq (1 - q_0)^m \). Since the sum over all probabilities of \( A_m \) is finite \( \left( \sum_{m \in \mathbb{N}} P(A_m) < \infty \right) \), we know by the lemma of Borel–Cantelli that the probability for \( A_m \) to occur infinitely often vanishes, \( P(A_m \text{ infinitely often}) = 0 \).

\[ A.4. \text{ The time average of a classical, discrete-time Markov chain with finite state space} \]

Let \( Q \) be the transition matrix of a discrete-time Markov chain that is the entries represent the transition probabilities \( Q_{ij} = \mathcal{P}(j \rightarrow i) \) and \( q_i(q_0) = Q^k q_0 \) is the probability distribution after \( k \in \mathbb{N}_0 \) jumps. While the limit \( k \rightarrow \infty \) need not exist, since oscillation are possible (consider the aperiodic Markov chain with the transition matrix \( Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)), the time average does and equals

\[ \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} Q^k q_0 = \sum_{i=1}^{n} \mathcal{P}(B_i | q_0) q_\infty(q_0 \in B_i), \] (69)

where \( B_1, \ldots, B_n \) are the minimal absorbing sets of the associated state-transition network.

**Proof.** For every analytical function \( f \) and any Jordan block

\[ J = \begin{pmatrix} \lambda & 1 \\ \vdots & \ddots & \ddots \\ \lambda & 1 \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{C}^{D \times D} \]

of size \( D \in \mathbb{N}_{\geq 2} \), we have

\[ f(J_D) = \begin{pmatrix} f(\lambda) & f'(\lambda) & \cdots & f^{(D-1)}(\lambda) \\ \vdots & \ddots & \ddots & \vdots \\ f(\lambda) & f'(\lambda) & \cdots & f(\lambda) \end{pmatrix}. \] (70)

When we choose a specific function

\[ f(x) = \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} x^k = 0, \] (71)

for all \( x \in \{ z \in \mathbb{C} : |z| \leq 1 \} \setminus \{1\} \),
we get
\[
\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} (J_{\lambda \neq 1})^k = f(J_{\lambda \neq 1}) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.
\]

Since we can interpret the matrix \( \Gamma_Q := Q - \mathbb{I} \) as a transition matrix of a continuous-time Markov chain, and \( \text{ker}(\Gamma_Q - \lambda \mathbb{I}) = \text{ker}(Q - (\lambda + 1) \mathbb{I}) \) we conclude that for the eigenvalue \( \lambda = 1 \) of \( Q \) the algebraic multiplicity \( a_{\lambda = 1} \) must coincide with the geometric multiplicity \( d_{\lambda = 1} \)

\[
d_{\lambda = 1}(Q) = d_{\lambda = 0}(\Gamma_Q = Q - \mathbb{I}) = a_{\lambda = 1}(Q). \tag{73}
\]

This means that we can write the transition matrix as \( Q = S^{-1}(\mathbb{I} \quad 0 \quad \mathbb{J})S \), with

\[
\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{J}^k = 0.
\]

When writing the initial state as a linear combination of generalized eigenvectors \( q_0 = \sum_{i=1}^{[\Omega]} \mu_i h^{(i)}_\lambda(Q) \), we can compute the time average of the Markov chain as

\[
q^\infty(q_0) := \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} Q^k q_0 = S \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{J}^k S^{-1} q_0 = S \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{J} \end{pmatrix} S^{-1} q_0 = \sum_{i=1}^{[\Omega]} \mu_i h^{(i)}_\lambda(Q) \quad \sum_{i=1}^{[\Omega]} \mu_i h^{(i)}_\lambda(Q) \times \sum_{i=1}^{[\Omega]} P(B_i \mid q_0) q^\infty(q_0 \in B_i). \tag{74}
\]

In the last step (*) we have used the fact that the probability \( P(B_i \mid q_0) \) for a single trajectory to land in the minimal absorbing set \( B_i \), provided it started with the initial distribution \( q_0 \), is given by the coefficient \( \mu_i \) that is determined by the initial condition \( q_0 = \sum_{i=1}^{[\Omega]} \mu_i h^{(i)}_\lambda(Q) \) and the form of the stationary states \( q^\infty(q_0 \in B_i) = h^{(i)}_\lambda(Q) \), for \( i \in \{1, \ldots, n\} \):

\[
P(B_i \mid q_0) \overset{\text{Def}}{=} \sum_{j \in B_i} (q^\infty(q_0))_j = \mu_i \left( \sum_{j \in B_i} q^\infty(q_0 \in B_i) \right) = \mu_i. \tag{75}
\]

It can easily be verified that the coefficients \( \mu_i \) form indeed a probability distribution, namely they are non-negative and their sum equals one.
Alternatively, we can interpret $P(B_i | q_0)$ not only as the probability for a single trajectory being captured by the minimal absorbing set $B_i$, but also as the fraction of trajectories being captured by $B_i$, whose initial distribution is given by $q_0$. To be more concrete, let $\omega \in \Omega^I$ be a trajectory starting at some state $\omega_0 = j \in \Omega$. We know from appendix A.3 that such a trajectory will almost surely be captured by some minimal absorbing set $B \subset \Omega$, which means there exists a natural number $L_j \in \mathbb{N}_0$ such that after $L_j - 1$ time steps the trajectory will only assume values in $B$ ($\omega_k \in B$ for all $k \geq L_j$). Then we can compute the probability for $\omega$ to

$$P(\omega | \omega_0 = j) = \prod_{i=0}^{L_j-1} q_{\omega_i \rightarrow \omega_{i+1}} = \prod_{i=0}^{L_j-1} Q_{\omega_{i+1}, \omega_i}. \tag{76}$$

When we now set $U(B | \omega_0 = j)$ to be the set of all trajectories starting at some state $j \in \Omega$ that are eventually captured by the minimal absorbing set $B \subset \Omega$, $U(B | \omega_0 = j) := \{ \omega \in \Omega^I : \omega_0 = j \text{ and } \omega_k \in B \text{ eventually} \}$. The probabilities of this set of trajectories can also be computed to

$$P(U(B) | \omega_0 = j) = \sum_{\omega \in U(B | \omega_0 = j)} P(\omega | \omega_0 = j). \tag{77}$$

When we do not have a fixed starting point $j \in \Omega$, but an ensemble of starting points $q_0 = \sum_{j=1}^{[\Omega]} \mu_j \epsilon_j$, the corresponding probability equals

$$P(B | q_0) := P(U(B) | q_0) = \sum_{j=1}^{[\Omega]} \mu_j P(U(B) | \epsilon_j) = \sum_{j=1}^{[\Omega]} \mu_j \sum_{\omega \in U(B)} P(\omega | \omega_0 = j). \tag{78}$$

So the conditional probability $P(B | q_0)$ equals the sum over the probabilities over all trajectories, which land eventually in the minimal absorbing set $B \subset \Omega$ and whose initial conditions were distributed in $\Omega$ according to $q_0 \in (\mathbb{R}_{\geq 0})^{[\Omega]}$, with $\mu_j \geq 0$ being the fraction of trajectories that start at the state $j$ (note that we do not distinguish between the state $j \in \Omega$ and the corresponding unit vector $\epsilon_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T$).

### A.5. The form of the (unique) stationary solution of a continuous-time Markov chain for a minimal absorbing set

We call a network $\Omega$ with no self-loops an in-tree (also called anti-arborescence [GM78]) rooted at state $\omega_0 \in \Omega$ if for all states $\omega \in \Omega$ there is a unique directed path leading from state $\omega$ towards the root $\omega_0$. An example is given in figure 10, where the root is state number 2. The weight of an in-tree is the product of the link-strength of all edges of the in-tree: $\prod_{(i,j) \in E(\Omega)} \gamma_{i \rightarrow j}$, with $E(\Omega)$ denoting the set of edges of the network.

When $\Gamma$ is generator for the so-called Master equation $\dot{p} = \Gamma p$, with $\Gamma_{ij} = \begin{cases} \gamma_{j \rightarrow i}, & i \neq j; \\ -\sum_{k=1}^{[\Omega]} \gamma_{j \rightarrow k}, & i = j. \end{cases}$, it can be shown [CK78, MG13, FD22] that the unique stationary solution of the Master equation for a strongly connected network with no self-loops can...
Figure 10. Example of an in-tree rooted at state number 2. There is a unique directed path from each state of the tree leading to state number 2. Its weight equals $\prod_{(i,j) \in \mathcal{E}(T)} \gamma_{i \rightarrow j} = \gamma_{1 \rightarrow 2} \gamma_{1 \rightarrow 2} \gamma_{7 \rightarrow 2} \gamma_{4 \rightarrow 7} \gamma_{6 \rightarrow 7} \gamma_{3 \rightarrow 4} \gamma_{5 \rightarrow 6}$. Reproduced from [FD22].

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be computed according to equation (79), where $T_{i-\leftarrow}(\Omega)$ is the set of all in-trees of $\Omega$, which are rooted in state number $i \in \Omega$. An example is given in figure 11 below

$$
p_{\infty} = \left( \sum_{T \in T_{i-\leftarrow}(\Omega)} \prod_{(i,j) \in \mathcal{E}(T)} \gamma_{i \rightarrow j} \right)
$$

For the discrete-time Markov associated to a strongly connected, weighted network with possible self-loops, we know from appendix A.4 that the stationary solution $q_{\infty} := \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} Q^k q_0 \in \ker(Q - 1 \cdot \mathbb{1})$ exists and is independent for all initial condition $q_0 \in (\mathbb{R}_{\geq 0})^{|\Omega|}$.

Since we can interpret $\Gamma Q := Q - 1$ as the generator for a continuous time Markov chain, we can compute the compute the expression in equation (80) to

$$
q_{\infty} = \left( \sum_{T \in T_{i-\leftarrow}(\Omega)} \prod_{(i,j) \in \mathcal{E}(T)} q_{i \rightarrow j} \right)
$$

A.6. When the number of quantum jumps in a quantum trajectory is not bounded, then $t_{J(T)} \to \infty$ in probability.

Proof. Fix $\epsilon > 0$, then we have:

$$
P \left( \left| \frac{t_{J(T)}}{T} - 1 \right| > \epsilon \right) = P \left( T - t_{J(T)} > \epsilon T \right) = \text{Tr}[U_{\epsilon T} \Theta_{J(T)}] \to_{T \to \infty} 0,
$$
Figure 11. Example of a strongly connected network (a) together with the corresponding in-trees rooted in state number 1 (b) and 3 (c), and the two in-trees rooted in state number 2 ((d) and (e)). The kernel of $\Gamma$ is the span of a vector whose $i$th component is the sum over all in-trees rooted in state number $i$ of the product of the rates of all edges that constitute that particular in-tree. In this example we have:

$$\operatorname{kern} (\Gamma) = \text{span} \left\{ \begin{pmatrix} \gamma_{3 \to 2} \cdot \gamma_{2 \to 1} \\ \gamma_{1 \to 2} \cdot \gamma_{3 \to 2} + \gamma_{1 \to 3} \cdot \gamma_{3 \to 2} \\ \gamma_{2 \to 1} \cdot \gamma_{1 \to 3} \end{pmatrix} \right\}.$$

Reproduced from [FD22].

where we used the fact that $\Theta_{J(T)}$ can never be a possible trapping state, since the number of quantum jumps is by assumption not bounded, so the quantum trajectory will almost surely jump at some point.

\[\square\]

A.7. Illustrating the algorithm for obtaining the stationary probabilities of a discrete-time Markov chain

When the state transition network is given by figure 12, the initial distribution

$$q_0 = \begin{pmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \end{pmatrix} = \begin{pmatrix} q_2(0) + q_1(0) \frac{q_{1 \to 2}}{q_{1 \to 2} + q_{1 \to 3}} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} q_3(0) + q_1(0) \frac{q_{1 \to 3}}{q_{1 \to 2} + q_{1 \to 3}} \\ 0 \\ 0 \end{pmatrix}$$

(83)
A.8. The need for a finite state space

Now that we have derived a formula for the stationary solution of the Lindblad equation, which is valid when certain assumptions are satisfied, we will look at these assumptions and check whether they can be dropped.

One crucial assumption was the need for a finite state space $|S| < \infty$. When dropping this assumption, the formula does no longer hold true in general, as the following counter-example shows:

$$V_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda = \sum_{k=1}^{2} \gamma_k V_k^\dagger V_k = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 + \gamma_3 \end{pmatrix}$$

For $\gamma_1 = \gamma_2$, the set of possible states appearing in the Markov chain can be computed to

$$\Omega_{\rho_0} = \left\{ \rho_0, \Theta_m : m \in \mathbb{N}_0 \right\}$$

with

$$\Theta_{2m} = \frac{1}{\rho_{11}(0) + \rho_{22}(0) 2^m} \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) 2^m \\ \rho_{21}(0) 2^m & \rho_{22}(0) \end{pmatrix},$$

$$\Theta_{2m+1} = \frac{1}{1 + 2^m} \begin{pmatrix} 1 & 1/2^m \\ 1/2^m & 1 \end{pmatrix}.$$
The probability for applying the Lindblad operators $V_1$ and $V_2$ is given by

$$P(\pi_{n+1} = 1 | \Theta_n = \Theta_{2n+1}) = 1 - \frac{1}{2} \frac{1}{2^m + 1} =: q_{k=1}(m) = 1 - q_{k=2}(m)$$

$$P(\pi_{n+1} = 1 | \Theta_n = \Theta_{2n}) = 1 - \frac{1}{2} \frac{\rho_{22}(0)}{\rho_{11}(0) + \rho_{22}(0)} =: q_{k=0}^{\rho_0}(m) = 1 - q_{k=2}^{\rho_0}(m).$$  \hspace{1cm} (88)$$

Then, by the lemma of Borel–Cantelli, the probability for applying the operator $V_2$ infinitely often is zero (for $\rho_{11}(0) \neq 0$), since $\sum_{n=0}^{\infty} P(\pi_{n+1} = 2 | \Theta_n) < \infty$.

Hence we know almost surely that from a certain time step $k \in \mathbb{N}$ onward only the Lindblad operator $V_1$ will be applied (so the system will eventually follow the black arrow of figure 13). This means that all states are transient states and hence no stationary distribution for this Markov chain exists.

Clearly, this can not happen, for finitely many states. The fact that there is no stationary solution of the corresponding Markov chain in the previous example, does not mean, that no stationary solution for the original Lindblad equation exist. And indeed, the stationary solution is given by $\rho^\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The reason for this discrepancy is the fact that the sequence of states in the discrete quantum trajectory $(\Theta_n)_{n \in \mathbb{N}}$ converges, while the classical states are unit vectors $(e_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$, which do not converge (with $e_n = (0, \ldots, 0, 1, 0, \ldots)$).
Appendix B. Possible generalizations for a finite state space

B.1. Lindblad operators of rank one are not necessary for a finite state space

We have seen in section A.8, that the assumptions of a finite state space cannot easily be dropped.

The question arises, of when this conditions is fulfilled. A sufficient criterion is surely, when all Lindblad operators $V_k$ have rank one ($\text{rank}(V_k) = 1$) and the conditional Hamiltonian is diagonalizable, as is the case when each Lindblad operator models the transitions between discrete states, like ladder operators and lasers.

This is not a necessary condition, however, as the following example shows:

$$H = \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_2 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \gamma_2 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 \\ 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_1 \end{pmatrix}.$$

The set of possible states appearing in the Markov chain can be computed to:

$$\Omega_{\rho_0} = \left\{ \Theta_{s_1} = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) & \rho_{13}(0) & \rho_{14}(0) \\ \rho_{21}(0) & \rho_{22}(0) & \rho_{23}(0) & \rho_{24}(0) \\ \rho_{31}(0) & \rho_{32}(0) & \rho_{33}(0) & \rho_{34}(0) \\ \rho_{41}(0) & \rho_{42}(0) & \rho_{43}(0) & \rho_{44}(0) \end{pmatrix}, \Theta_{s_2} = \frac{1}{\rho_{33}(0) + \rho_{44}(0)} \begin{pmatrix} \rho_{33}(0) & \rho_{34}(0) & 0 & 0 \\ \rho_{43}(0) & \rho_{44}(0) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Theta_{s_3} = \frac{1}{\rho_{11}(0) + \rho_{22}(0)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{11}(0) & \rho_{12}(0) \\ 0 & 0 & \rho_{21}(0) & \rho_{22}(0) \end{pmatrix} \right\}. \quad (89)$$

The state transition network is (qualitatively) similar to that of example a) in section 3.5, with $\tau_{s_2} = \frac{1}{\gamma_2}$ and $\tau_{s_3} = \frac{1}{\gamma_1}$, where the stationary solution is given by

$$\rho = \begin{pmatrix} q(\Theta_{s_2}) \Theta_{s_2} \frac{1}{\tau_{s_2}} + q(\Theta_{s_3}) \Theta_{s_3} \frac{1}{\tau_{s_3}} \\ q(\Theta_{s_2}) \frac{1}{\tau_{s_2}} + q(\Theta_{s_3}) \frac{1}{\tau_{s_3}} \end{pmatrix}$$

$$= \frac{1}{\tau_{s_2} + \tau_{s_3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{33}(0) & \rho_{34}(0) \\ 0 & 0 & \rho_{43}(0) & \rho_{44}(0) \end{pmatrix}.$$
with recurrent states is finite.

B.2. Example of a state transition network with an infinite state space, where the number of recurrent states is finite

Let the Hamiltonian and the Lindblad operators be given by

\[
H = \begin{pmatrix}
E_1 & 0 & 0 & 0 \\
0 & E_2 & 0 & 0 \\
0 & 0 & E_3 & 0 \\
0 & 0 & 0 & E_3
\end{pmatrix}, \quad V_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad V_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad V_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \Lambda = \begin{pmatrix}
2\gamma_1 & 0 & 0 & 0 \\
0 & 2\gamma_1 & 0 & 0 \\
0 & 0 & \gamma_2 & 0 \\
0 & 0 & 0 & \gamma_3
\end{pmatrix}, \quad (92)
\]

\[
H_c = \begin{pmatrix}
E_1 - i\gamma_1 & 0 & 0 & 0 \\
0 & E_2 - i\gamma_1 & 0 & 0 \\
0 & 0 & E_3 - \frac{i\gamma_2}{2} & 0 \\
0 & 0 & 0 & E_3 - \frac{i\gamma_3}{2}
\end{pmatrix}
\]

The set of possible states appearing in the Markov consists of the five states

\[
\Theta_{\rho_0} = \{ \Theta_i = \rho_0, \Theta_{x_1}(\tau), \Theta_{x_2}, \Theta_{x_3}, \Theta_{x_4} \}. \quad (93)
\]

with

\[
\Theta_i = \begin{pmatrix}
\rho_{11}(0) & \rho_{12}(0) & \rho_{13}(0) & \rho_{14}(0) \\
\rho_{21}(0) & \rho_{22}(0) & \rho_{23}(0) & \rho_{24}(0) \\
\rho_{31}(0) & \rho_{32}(0) & \rho_{33}(0) & \rho_{34}(0) \\
\rho_{41}(0) & \rho_{42}(0) & \rho_{43}(0) & \rho_{44}(0)
\end{pmatrix}, \quad (94)
\]

\[
\Theta_{x_1}(\tau) = \begin{pmatrix}
\frac{1}{\gamma_1} \\
\frac{1}{\gamma_1} + \frac{1}{\gamma_2}
\end{pmatrix} \times \begin{pmatrix}
\rho_{22}(0) & \rho_{21}(0) e^{i\tau(E_1 - E_2)} & \rho_{21}(0) e^{i\tau(E_1 - E_2)} & \rho_{22}(0) \\
\rho_{12}(0) e^{-i\tau(E_1 - E_2)} & \rho_{11}(0) & \rho_{11}(0) & \rho_{12}(0) e^{-i\tau(E_1 - E_2)} \\
\rho_{12}(0) e^{-i\tau(E_1 - E_2)} & \rho_{11}(0) & \rho_{11}(0) & \rho_{12}(0) e^{-i\tau(E_1 - E_2)} \\
\rho_{22}(0) & \rho_{21}(0) e^{i\tau(E_1 - E_2)} & \rho_{21}(0) e^{i\tau(E_1 - E_2)} & \rho_{22}(0)
\end{pmatrix}.
\]
Figure 14. State transition network for a system of dimension $N = 4$, with an infinite number of states due to the set of states $\{\Theta_s(\tau) : \tau \geq 0\}$, but only finitely many states within a minimal absorbing set $B = \{\Theta_s, \Theta_{s2}\}$, between which every quantum trajectory oscillates.

$$\Theta_s = \left( \frac{1}{\rho_{11}(0) + \rho_{22}(0)} \right) \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) & \rho_{12}(0) & \rho_{11}(0) \\ \rho_{21}(0) & \rho_{22}(0) & \rho_{22}(0) & \rho_{21}(0) \\ \rho_{11}(0) & \rho_{12}(0) & \rho_{12}(0) & \rho_{11}(0) \\ \rho_{21}(0) & \rho_{22}(0) & \rho_{22}(0) & \rho_{21}(0) \end{pmatrix},$$

$$\Theta_{s2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } \Theta_{s4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the transition probabilities

- $q_{s1 \rightarrow s1} = 0$, $q_{s2 \rightarrow s2} = 0$, $q_{s3 \rightarrow s3} = 0$, $q_{s4 \rightarrow s4} = 0$,
- $q_{s1 \rightarrow s2} = \rho_{12}(0) + \rho_{22}(0)$, $q_{s2 \rightarrow s3} = \frac{1}{2}$, $q_{s3 \rightarrow s4} = \frac{1}{2} \rho_{11}(0) + \rho_{22}(0)$,
- $q_{s4 \rightarrow s1} = 0$, $q_{s1 \rightarrow s3} = 0$, $q_{s2 \rightarrow s4} = 0$, $q_{s4 \rightarrow s2} = \frac{1}{2} \rho_{11}(0) + \rho_{22}(0)$,
- $q_{s3 \rightarrow s2} = 0$, $q_{s2 \rightarrow s1} = 0$, $q_{s3 \rightarrow s1} = 0$, $q_{s4 \rightarrow s3} = 0$,
This results in the following state transition network (compare figure 14), which consists of an infinite number of states:

Since the number of states in each quantum trajectory remains finite, the stationary solution can still be computed, according to:

\[
\rho_\infty(\rho_0) = \sum_{B \in \mathcal{B}} P(B | \rho_0) \sum_{\tau \in B} q_\infty(\theta_{\tau} | B) \frac{\tau \cdot (\theta_{\tau}(t))_{t \in [0, \tau]} }{\text{Tr}[\cdots]} \frac{1}{\tau - f(\theta_{\tau})} \frac{B(B_1 =)}{\langle \langle \theta_{\sigma}, \theta_{\sigma} \rangle \rangle} \sum_{i \in \{4, 5\}} q_\infty(\theta_{\sigma_i} | B) \frac{1}{\text{Tr}[\cdots]} \theta_{\sigma_i} \\
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\gamma_a + \gamma_b} & 0 \\
0 & 0 & 0 & \frac{1}{\gamma_a + \gamma_b}
\end{pmatrix} .
\]

**B.3. Finitely many cluster points**

We have seen in section A.8 that our result need no longer hold, when simply dropping the assumption of a finite state space. In this section we want to explore possibilities about how to weaken this assumption, without completely abolishing it.

The crucial part, where the finite state space is needed, are the calculation in equation (28). We could weaken this assumption, when we only assume that the number of cluster points (also called accumulation points) of the sequence \((\Theta(t_n))_{n \in \mathbb{N}}\) is finite, that is \(|\text{CP}(\omega)| := |\text{cluster points of } \omega| < \infty\).

This means that for every cluster point \(c \in \text{CP}\) there exists a subsequence \((\Theta_{\psi_i(n)})_{n \in \mathbb{N}}\) such that \(\Theta_{\psi_i(n)} \xrightarrow{n \to \infty} \Theta_c\), where \(\psi_i : \mathbb{N} \to \mathbb{N}\) are a strictly increasing functions such that \(\text{image}(\psi_i) \cap \text{image}(\psi_j) = \emptyset\), for \(i \neq j\).

This results in a similar formula, namely

\[
\lim_{T \to \infty} \frac{\sum_{t=0}^{T-1} I_{\theta_{\psi_i(n)}}(\tau_{\psi_i(n)})}{\sum_{t=0}^{T-1} \frac{J(T)}{\lambda(T)} \sum_{l=0}^{J(T)-1} \tau_{\psi_i(n)} + \left(\frac{T-\tau_{\psi_i(n)}}{\lambda(T)}\right)}
= \sum_{c \in \text{CP}} \frac{1}{q(\theta_c)} \frac{\sum_{t=0}^{J(T)} I_{\theta_{\psi_i(n)}}(\tau_{\psi_i(n)})}{\sum_{t=0}^{J(T)} \frac{J(T)}{\lambda(T)} \sum_{l=0}^{J(T)-1} \tau_{\psi_i(n)} + \left(\frac{T-\tau_{\psi_i(n)}}{\lambda(T)}\right)}
= \frac{\sum_{c \in \text{CP}} q(\theta_c) I_{\theta_{\psi_i(n)}}(\tau_{\psi_i(n)})}{\sum_{c \in \text{CP}} q(\theta_c) \tau_{\psi_i(n)}} ,
\]

(95)
while the formal analogy to equation (28) is remarkable, there is an important drawback: We no longer have minimal absorbing sets and there is no analytical expression for the probabilities $q(\Theta_c)$, as was the case for a finite state space.

It might be possible to find exact solutions of these stationary probabilities in simple examples, but in most cases one has to rely on numerical methods.

What is more, the sequence $(\Theta(t_n))_{n\in\mathbb{N}}$ could well have an infinite number of cluster points, as a rotation matrix $V_1 = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$ shows, where $\varphi$ is not a rational multiple of $\pi$.

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