A REMARK ON MINIMAL FANO THREEDFOLDS

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Abstract. We prove in the case of minimal Fano threefolds a conjecture stated by Dubrovin at the ICM 1998 in Berlin. The conjecture predicts that the symmetrized/alternated Euler characteristic pairing on $K_0$ of a Fano variety with an exceptional collection expressed in the basis of the classes of the exceptional objects coincides with the intersection pairing of the vanishing cycles in Dubrovin’s second connection. We show that the conjecture holds for $V_{22}$, a minimal Fano threefold of anticanonical degree 22, and for $V_5$, the minimal Fano threefold of anticanonical degree 40, by applying the modularity result for the rank 1 Fano threefolds established in [Gol07]. The truth of the conjecture for $\mathbb{P}^3$ and the three-dimensional quadric is known; we consider these cases for the sake of completeness.

1. The conjecture

1.1. Exceptional collections and autoequivalences. Let $F$ be an algebraic variety. By $D^b(F)$ denote the bounded derived category of coherent sheaves on $F$.

Any derived category $\mathcal{D}$ is triangulated. This means that given are the shift functor $[1] : \mathcal{D} \to \mathcal{D}$ which is an additive autoequivalence, and the class of distinguished triangles $X \to Y \to Z \to X[1]$, which satisfies the standard axioms. An additive functor $F : \mathcal{D} \to \mathcal{D}'$ between two triangulated categories is said to be exact if it commutes with the shift functor and sends distinguished triangles to distinguished triangles. The group of isomorphism classes of exact equivalences $\mathcal{F} : D^b(F) \to D^b(F)$ is called the group of autoequivalences of $D^b(F)$ and denoted $\text{AutEq}(D^b(F))$.

By $\text{Hom}^i(X,Y)$ denote $\text{Hom}(X,Y[i])$. An object $E \in \text{Ob}(D^b(F))$ is said to be exceptional if it satisfies

$$\text{Hom}^i(E,E) = 0 \text{ when } i > 0, \quad \text{Hom}^0(E,E) = \mathbb{C}.$$ 

An ordered set $E_0, \ldots, E_n$ of exceptional objects is said to be exceptional if for any $i$

$$\text{Hom}^i(E_j,E_k) = 0 \text{ when } j > k.$$ 

An exceptional collection is full if it generates the derived category.

One may choose to work with more specific classes of varieties:

1.2. Fano varieties: cellular, minimal, Tate. An $m$-dimensional Fano variety $F$ is said to be minimal if its cohomology is as small as it can be ($H^{2k+1}(F,\mathbb{Z}) = 0, H^{2k}(F,\mathbb{Z}) = \mathbb{Z}$). A Fano is said to be Tate if its motive has no non-Tate constituents. A Fano $F$ is said to be cellular if $F$ is a union of affine spaces: $F = \bigcup A^{i(j)}_j$, $A_{j_1} \cap A_{j_2} = \emptyset$ if $j_1 \neq j_2$. Cellular Fanos and minimal Fanos are Tate.
It has been noted that Tate Fanos ‘tend to’ possess full exceptional collections, though no precise conjecture has been made, to our knowledge. It is believed that Fanos with exceptional collections are Tate. It is also believed that a minimal Fano should have a full exceptional collection.

Minimal Fano threefolds are known to have full exceptional collections, by Beilinson, Kapranov, Orlov and Kuznetsov.

1.3. Definition. We call a basis \( v_0, \ldots, v_n \) of a linear space \( V \) endowed with a bilinear form \( \chi \) semiorthonormal if \( \chi(v_i, v_j) = 0 \) when \( i > j \), and \( \chi(v_k, v_k) = 1 \) for all \( k \).

The classes of the elements of a full exceptional collection in \( K_0(F) \otimes \mathbb{Q} \) form a semiorthonormal basis with respect to the Riemann-Roch form \( \chi([O_1],[O_2]) = \sum (-1)^i \dim \text{Hom}^i(O_1, O_2) \).

Nogin described semiorthogonal bases for minimal Fano threefolds in [Nog94].

The statements of 1.3, 1.5 though not necessary for the proof, are worth to be kept in mind.

1.4. The Coxeter element. Let \( A \) be the matrix of the bilinear form \( \chi \) in a semiorthonormal basis \( v_0, \ldots, v_n \) of a space \( V \).

\( s \) Let \( \chi_s \) be the symmetrization of \( \chi \), that is \( \chi_s(w_1, w_2) = \chi(w_1, w_2) + \chi(w_2, w_1) \), and let \( I_0, \ldots, I_n \) be the reflections with respect to the vectors \( v_0, \ldots, v_n \) in the orthogonal space \( (V, \chi_s) \), \( I_j v \mapsto v - \chi_s(v, v_j) v_j \). Then one has \( I_0 I_1 \ldots I_n = -A^{-1} A^t \).

\( a \) Let \( \chi_a \) be the alternation of \( \chi \), that is \( \chi_a(w_1, w_2) = \chi(w_1, w_2) - \chi(w_2, w_1) \), and let \( I_0, \ldots, I_n \) be the reflections with respect to the vectors \( v_0, \ldots, v_n \) in the symplectic space \( (V, \chi_a) \). Then one has \( I_0 I_1 \ldots I_n = A^{-1} A^t \).

Now note that in the case \( V = K_0(F) \otimes \mathbb{Q} \) Serre’s duality yields \( \chi([O_1],[O_2]) = \chi([O_2],[O_1 \otimes K_F[m]]) \) for any pair of objects \( O_1, O_2 \in \text{Ob} \mathcal{D}^b(F) \).

Let \( A \) be the matrix of the form \( \chi \) in a basis that consists of the classes of the elements of an exceptional collection. Then the identity \( x^t A y = y^t A (A^{-1} A^t)x \) shows that in this basis the matrix of the operator defined on \( K_0 \otimes \mathbb{Q} \) by the functor \( \otimes K_F[m] \) is \( A^{-1} A^t \). Hence one has

1.5. Proposition. In \( K_0(F) \otimes \mathbb{Q} \):

for \( F \) odd-dimensional, the product of all orthogonal reflections with respect to the classes of the elements of an exceptional collection is the multiplication by \( [K_F] \);

for \( F \) even-dimensional, the product of all symplectic reflections with respect to the classes of the elements of an exceptional collection is the multiplication by \( [K_F] \).

1.6. The conjecture. Let \( F \) be an odd-dimensional (resp. even-dimensional) variety \( F \) with a full exceptional collection \( E = \langle E_1, E_2, \ldots, E_n \rangle \). Let \( \chi^* \) be the symmetrized (resp. alternated) Riemann-Roch form. Consider the linear space \( \overline{V} = (K_0(F) \otimes \mathbb{Q})/ \ker \chi^* \). The respective non-degenerate form will also be denoted by \( \chi^* \).

Let \( x = \langle x_1, \ldots, x_n \rangle \) be an ordered set of different points on \( A^1 \). Put \( U = A^1 \setminus \{x_1, \ldots, x_n\} \) and consider the representation

\[ \varphi_{E,x}: \pi_1(U^{an}) \to O(\overline{V}, \chi^*) \quad (\text{respectively}, \quad \pi_1(U^{an}) \to Sp(\overline{V}, \chi^*)) \]
determined by requiring that the loop around the \( i \)-th point act by reflection with respect to \([E_i]\). This representation defines a local system \( \mathcal{L}_{E,x} \) on \( U \).

The conjecture that was stated by Dubrovin in [Dub98] (see also preceding discussion in [Zas96]) says, roughly, that if \( x \) is chosen to be an ordered set of the critical values of the so-called Landau–Ginsburg potential \( u \), then \( \mathcal{L}_{E,x} \) is the monodromy that arises in the local system of the middle cohomology of the level set of the potential, \( R^{\text{mid}}u_*(\mathbb{Z}) \).

As the notion of a Landau–Ginsburg model has no rigorous definition yet, one interprets this statement by relating the local system to the monodromy of Dubrovin’s second structural connection.

2. Regularized quantum \( D \)-module.

As above, let \( F \) be an \( m \)-dimensional Fano variety of index \( d \), so that \(-K_F = dH\). Consider the matrix \( M_{-K_F} \) of quantum multiplication by \(-K_F\). It has entries in \( \mathbb{Q}[q_i, q_i^{-1}] \), where \( q_i \)'s correspond, as usual, to the generators of the lattice of numerical classes of curves on \( F \). Let \( h_i \) be the anticanonical degree of the class \( q_i \). One may specialize the matrix to \( M \) in \( \text{Mat}(\mathbb{Q}[t, t^{-1}]) \) by sending \( q_i \) to \( t^{h_i} \). There is no need to do that when \( H^2(F) = \mathbb{Z} \) which we will assume henceforth. The anticanonical quantum \( D \)-module on \( \mathbb{G}_{\text{m}} \) is given by the connection \( t \frac{\partial}{\partial t} \eta = \eta M \), and the regularized quantum \( D \)-module is the Fourier transform of its [middle] extension to \( \mathbb{A}^1 \).

The following is a version of the conjecture.

2.1. We shall say that the **exceptional collection/vanishing cycles conjecture** holds for \( F \) if:

i) the regularized quantum \( D \)-module is regular singular (this is expected to hold for any \( F \)), and the finite singularities \( x_1, \ldots, x_n \) are simple;

ii) \( F \) has an exceptional collection \( \mathbf{E} \), and \( x_1, \ldots, x_n \) can be ordered to a tuple \( x \) so that the local system \( \mathcal{L}_{\mathbf{E},x} \) is isomorphic to the monodromy of the regularized quantum \( D \)-module.

2.2. Regularized quantum \( D \)-modules of the minimal Fano threefolds \( \mathbb{P}^3, Q, V_5, V_{22} \) have been identified with \( d \)-pullbacks of Pickard–Fuchs equations in twisted Kuga–Sato families over the curves \( X_0(N)/W_N \) where \( N = 2, 3, 5, 11 \) respectively (\( N = -K_F^2/2d^2 \), see details in [Gol07]). In particular, the sets \( x \) for the minimal Fano threefolds \( F \) above have cardinality 4 and are formed by the \( d \)-th roots of the elliptic points \( X_0(N)/W_N \), see below.

3. \( V_{22} \)

3.1. **Theorem.** The variety [rather, family of varieties] \( V_{22} \) satisfies the exceptional collection/vanishing cycles conjecture.

**Proof.** Kuznetsov showed in [Kuz96] that \( V_{22} \) possesses exceptional collections. An instance is \( \mathbf{E} = (\mathcal{O}, S^*, E^*, \Lambda^2 S^*) \), in Kuznetsov’s notation. The matrices of
\(\chi; \chi^\bullet = \chi^{sym}\) in this basis have the form

\[
X = \begin{pmatrix} 1 & 7 & 8 & 18 \\ 0 & 1 & 4 & 13 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X^{sym} = \begin{pmatrix} 2 & 7 & 8 & 18 \\ 7 & 2 & 4 & 13 \\ 8 & 4 & 2 & 4 \\ 18 & 13 & 4 & 2 \end{pmatrix}.
\]

On the other hand, the monodromy of the regularized quantum D–module was described in [Gol07] as follows.

### 3.2.
Let \(X_0(N)^\circ\) stand for \(X_0(N) - \{\text{cusps}\} - \{\text{elliptic points}\}\). Let \(\varphi\) be the tautological projective representation \(\varphi: \pi_1(X_0(N)^\circ) \to PSL_2(\mathbb{Z})\). The monodromy that acts on \(H^1\) of the fiber of a universal elliptic curve is given by a lift of \(\varphi\) to a linear representation

\[
\overline{\varphi}: \gamma \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad c = 0 \mod N.
\]

In a suitable basis its symmetric square is

\[
\psi: \gamma \mapsto Sym^2_N \varphi(\gamma) = \begin{pmatrix} d^2 & 2cd & -c^2/N \\ bd & bc + ad & -ac/N \\ -Nb^2 & -2Nab & a^2 \end{pmatrix}.
\]

Let \(W\) be the Atkin–Lehner involution given by the action of \(W = \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}\) on \(X_0(N)\). Delete the \(W\)-invariant points from \(X_0(N)^\circ\) and let \(X_0(N)^{W\circ}\) be the quotient of the resulting curve by \(W\). The fundamental group of \(X_0(N)^{W\circ}\) is then generated by \(\pi_1(X_0(N)^\circ)\) and a loop \(\iota\) around the point that is the image of a point \(s\) on the upper half-plane stabilized by \(W\). Extend \(\psi\) to \(\iota\), by setting \(\psi(\iota) = I\) with

\[
I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

For a Fano \(F\) with \(d = 1\), \(N = \frac{-K_F^3}{2d^2}\), the resulting representation is the monodromy representation of the regularized quantum D–module.

Having said all that, and made choices, one may describe the monodromy of the regularized quantum D-module for \(V_{22}\) as follows. Let \(\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34} \in \Gamma_0(11)\) respectively denote

\[
\begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}, \quad \begin{pmatrix} 6 & 1 \\ 11 & 2 \end{pmatrix}, \quad \begin{pmatrix} 15 & 2 \\ 22 & 3 \end{pmatrix}, \quad \begin{pmatrix} 7 & 1 \\ -22 & -3 \end{pmatrix}, \quad \begin{pmatrix} 23 & 3 \\ -77 & -10 \end{pmatrix}, \quad \begin{pmatrix} 8 & 1 \\ -33 & -4 \end{pmatrix}
\]

so that \(\gamma_{12}\gamma_{23} = \gamma_{13}, \gamma_{12}\gamma_{24} = \gamma_{14}, \gamma_{23}\gamma_{34} = \gamma_{24}\). Notice that \(Tr \gamma_{ij} = X_{ij}\).

Let \(W = \begin{pmatrix} 0 & -1 \\ 11 & 0 \end{pmatrix}\) as above. Then the images of the points on the upper half plane stabilized by \(W, W\gamma_{12}, W\gamma_{13}, W\gamma_{14}\) are the four elliptic points on \(X_0(11)^\circ\). The monodromies around these points are, respectively, the reflections \(I, I\psi(\gamma_{12}), I\psi(\gamma_{13}), I\psi(\gamma_{14})\). These monodromies are orthogonal with respect to the form \(U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -22 & 0 \\ -1 & 0 & 0 \end{pmatrix}\), that is, \(IUI = \psi(\gamma_{ij})U\psi(\gamma_{ij}) = U\). The length of 2 vectors of the reflections are, respectively, \(v_1 = (-1, 0, 1), v_2 = (-4, 1, 3), v_3 = \ldots\).
\((-6, 1, 2), v_4 = (-15, 2, 3). One notices that \((v_i, v_j) = X^{sym}_{ij}. Therefore, the exceptional collection/vanishing cycles conjecture holds for \(V_{22}\).

4. \(V_5\)

The proof in the case of the variety \(V_5\) goes along the same lines, except that the index is now 2, and the monodromy of the regularized quantum D–module is realized as the restriction of \(\psi\) to an index 2 subgroup of \(\pi_1(X_0(5)^W)\) which corresponds to its double cover ramified over the unique cusp and the elliptic point that is the image of the two elliptic points on \(X_0(5)\).

Let \(\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34} \in \Gamma_0(5)\) respectively denote
\[
\begin{pmatrix}
2 & 1 \\
5 & 3
\end{pmatrix},
\begin{pmatrix}
3 & 1 \\
5 & 2
\end{pmatrix},
\begin{pmatrix}
6 & 1 \\
5 & 1
\end{pmatrix},
\begin{pmatrix}
4 & 1 \\
-5 & -1
\end{pmatrix},
\begin{pmatrix}
13 & 2 \\
-20 & -3
\end{pmatrix},
\begin{pmatrix}
7 & 1 \\
-15 & -2
\end{pmatrix}.
\]
They generate an index 2 subgroup of \(\Gamma_0(5)\); the respective cover ramifies over the 2 cusps and the 2 elliptic points on \(X_0(5)\).

Let \(W = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}\). Then the images of the points on the upper half plane stabilized by \(W, W\gamma_{12}, W\gamma_{13}, W\gamma_{14}\) are the four elliptic points on the double cover of \(X_0(5)^W\). The monodromies around these points are, respectively, the reflections \(I, I\psi(\gamma_{12}), I\psi(\gamma_{13}), I\psi(\gamma_{14})\). These monodromies are orthogonal with respect to the form \(U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -10 & 0 \\ -1 & 0 & 0 \end{pmatrix}\). The length 2 vectors of the reflections are, respectively, \(v_1 = (-1, 0, 1), v_2 = (-2, 1, 3), v_3 = (-3, 1, 2), v_4 = (-6, 1, 1)\). Then \((v_i, v_j) = X^{sym}_{ij}\) with
\[
X = \begin{pmatrix}
1 & 5 & 5 & 7 \\
0 & 1 & 3 & 10 \\
0 & 0 & 1 & 5 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Orlov proved that \(E = (\mathcal{O}, Q, S^*, \mathcal{O}(1))\) as in [Orl91] is an exceptional collection with \(\chi([E_i], [E_j]) = X_{ij}\) as above. Therefore, the exceptional collection/vanishing cycles conjecture holds for \(V_5\).

5. THE QUADRIC

The index is now 3, and the monodromy of the regularized quantum D–module is realized as the restriction of \(\psi\) to an index 3 subgroup of \(\pi_1(X_0(3)^W)\) which corresponds to its cyclic 3–cover ramified over the unique cusp and the elliptic point that is the image of the unique elliptic point on \(X_0(3)\).

Let \(\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34} \in \Gamma_0(3)\) respectively denote
\[
\begin{pmatrix}
2 & 1 \\
3 & 2
\end{pmatrix},
\begin{pmatrix}
4 & 1 \\
3 & 1
\end{pmatrix},
\begin{pmatrix}
13 & 2 \\
6 & 1
\end{pmatrix},
\begin{pmatrix}
5 & 1 \\
-6 & -1
\end{pmatrix},
\begin{pmatrix}
20 & 3 \\
-27 & -4
\end{pmatrix},
\begin{pmatrix}
7 & 1 \\
-15 & -2
\end{pmatrix}.
\]
They generate an index 3 subgroup of \(\Gamma_0(3)\); the respective cover ramifies over the 2 cusps and the elliptic point on \(X_0(3)\).
Let \( W = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} \) as above. Then the images of the points on the upper half plane stabilized by \( W, W\gamma_{12}, W\gamma_{13}, W\gamma_{14} \) are the four elliptic points on the 3–cover of \( X_0(3)^W \). The monodromies around these points are, respectively, the reflections \( I, I\psi(\gamma_{12}), I\psi(\gamma_{13}), I\psi(\gamma_{14}). \) These monodromies are orthogonal with respect to the form \( U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \). The length 2 vectors of the reflections are, respectively, \( v_1 = (-1, 0, 1), v_2 = (-2, 1, 2), v_3 = (-4, 1, 1), v_4 = (-13, 2, 1). \) Then \( (v_i, v_j) = X^{sym}_{ij} \) with

\[
X = \begin{pmatrix}
1 & 4 & 5 & 14 \\
0 & 1 & 4 & 16 \\
0 & 0 & 1 & 5 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Let \( S \) be the spinor bundle on the quadric. The collection \( E = (\mathcal{O}, S^*, \mathcal{O}(1), \mathcal{O}(2)) \) is an exceptional collection with \( \chi([E_i], [E_j]) = X_{ij} \) as above (e.g. [Nog94]). Thus, we have shown that the exceptional collection/vanishing cycles conjecture holds for the 3–dimensional quadric.

6. \( \mathbb{P}^3 \)

The index is now 4, and the monodromy of the regularized quantum D–module is realized as the restriction of \( \psi \) to an index 4 subgroup of \( \pi_1(X_0(2)^W) \) which corresponds to its cyclic 4–cover ramified over the unique cusp and the elliptic point that is the image of the unique elliptic point on \( X_0(2). \)

Let \( \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34} \in \Gamma_0(2) \) respectively denote

\[
\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 19 & 3 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ -6 & -1 \end{pmatrix}, \begin{pmatrix} 13 & 2 \\ -20 & -3 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ -22 & -3 \end{pmatrix}.
\]

They generate an index 4 subgroup of \( \Gamma_0(2) \); the respective cover ramifies over the 2 cusps and the elliptic point on \( X_0(2). \)

Let \( W = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \). Then the images of the points on the upper half plane stabilized by \( W, W\gamma_{12}, W\gamma_{13}, W\gamma_{14} \) are the four elliptic points on the 4–cover of \( X_0(2)^W \). The monodromies around these points are, respectively, the reflections \( I, I\psi(\gamma_{12}), I\psi(\gamma_{13}), I\psi(\gamma_{14}). \) These monodromies are orthogonal with respect to the form \( U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{pmatrix} \). The length 2 vectors of the reflections are, respectively, \( v_1 = (-1, 0, 1), v_2 = (-3, 1, 1), v_3 = (-9, 2, 1), v_4 = (-19, 3, 1). \) Then
\[(v_i, v_j) = X^{sym}_{ij}\text{ with}\]
\[X = \begin{pmatrix}
1 & 4 & 10 & 20 \\
0 & 1 & 4 & 10 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 1
\end{pmatrix}\]

By the Kodaira vanishing theorem, \(E = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3))\) is an exceptional collection with \(\chi([E_i], [E_j]) = X_{ij}\) as above. This is another explanation why the exceptional collection/vanishing cycles conjecture holds for \(\mathbb{P}^3\).

### 6.1. Remarks and acknowledgements.

A generating set of Gromov–Witten invariants for \(V_{22}\) had been computed by Kuznetsov (published in [BM04]), and a set for \(V_5\), by Beauville [Bea95], thus providing the input for constructing Dubrovin’s connection. The truth of the conjecture for projective spaces and odd–dimensional quadrics is well–known and is probably due to Kontsevich. See [Guz99] for a version with the first structural connection and the Stokes matrix, or an \(\ell\)–adic version in [Gol01]. An issue we do not even mention, that of constructibility at the categorial level, is worked out by Polishchuk in [Pol07].

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