PRIME DIVISORS OF SEQUENCES OF INTEGERS

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Abstract. In this paper, we develop Furstenberg’s topological proof of infinity of primes, and prove several results about prime divisors of sequences of integers, including the celebrated Schur’s theorem. In particular, we give a simple proof of a classical result which says that a non-degenerate linear recurrence sequence of integers of order $k > 1$ has infinitely many prime divisors.

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1. Introduction

Given a sequence of integers $\{a_n\}_{n=0}^{\infty}$, a prime $p$ is called a prime divisor of $\{a_n\}_{n=0}^{\infty}$ if $p|a_n$ for some $n$. This paper is mainly concerned with the question that when a sequence of integers $\{a_n\}_{n=0}^{\infty}$ has infinitely many prime divisors.

Euclid’s theorem of infinity of primes says that the sequence of natural numbers has infinitely many prime divisors.

Euclid’s proof of infinity of primes is very beautiful and simple, which can be further applied to prove Schur’s theorem [6, 12]:

Theorem 1.1. Let $f(x) \in \mathbb{Z}[x]$ be a nonconstant integral polynomial. Then the sequence $\{f(n)\}_{n=0}^{\infty}$ has infinitely many prime divisors.

In 1955, Furstenberg gave a mystery proof of Euclid’s theorem, using the language of topology as follows:

For $a, b \in \mathbb{Z}$, $b > 0$, let $\text{Con}(a,b)$ be the congruence class $\{x \in \mathbb{Z} \mid x \equiv a \pmod{b}\}$. Then we obtain a topology $\mathcal{F}$ (Furstenberg’s topology) by taking the classes $\text{Con}(a,b)$ as a basis for the open sets. We note that each $\text{Con}(a,b)$ is closed as well. If the set of primes $\mathbb{P}$ were finite, then

$$\mathbb{Z} \setminus \{1, -1\} = \bigcup_{p \in \mathbb{P}} \text{Con}(0,p)$$
would be also closed. Consequently \( \{1, -1\} \) would be an open set, but this contradicts the definition of the topology \( F \).

In [1, [1]], it was shown that the topological language can be avoided in Furstenberg’s proof. In [2, p.56], Furstenberg’s proof is treated as simply a reductio version of Euclid’s. But, as we will see in the following, Furstenberg’s non-constructive proof has special advantage in many cases; even the topological language turns out to be very convenient!

Throughout this paper, \( Z \) denotes the set of integers endowed with the Furstenberg topology \( F \), and \( \mathbb{N} \) and \( \mathbb{N}_0 \) denote the sets of positive and non-negative integers considered as topological subspaces of \( Z \), respectively.

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2. Schur’s theorem and its generalization

In this section, we give a topological proof of Schur’s theorem, and derive a generalization. We first extend Furstenberg’s proof in the following form:

**Theorem 2.1.** Let \( f : \mathbb{N} \to \mathbb{Z} \) be a continuous map, which is unbounded on any congruence class. Then the sequence \( \{f(n)\}_{n=1}^{\infty} \) has infinitely many prime divisors.

**Proof.** Choose an \( m \in \mathbb{N} \) such that \( |f(m)| = k > 0 \). Let \( U \) denote the open set \( \text{Con}(0,k) = k\mathbb{Z} \); thus \( U \) is an open neighborhood of \( k \). By the continuity of \( f \), there is an open neighborhood \( V \) of \( m \) (in \( \mathbb{N} \)) of the form \( V := \text{Con}(m,b)|_{\mathbb{N}} = \{m + nb : n \in \mathbb{N}_0\} \) such that \( f(V) \subset U \), i.e., \( f(m + nb) \) \((n = 0, 1, 2 \cdots )\) is divisible by \( k \). Now, for each \( n \in \mathbb{N}_0 \), set \( g(n) = \frac{1}{k}f(m + nb) \). Then \( g(n) \) is also a continuous map from \( \mathbb{N}_0 \) to \( Z \). It suffices to show that \( \{g(n)\}_{n=0}^{\infty} \) has infinitely many prime divisors. Assume that the set \( \mathbb{P}_g \) of prime divisors of \( \{g(n)\}_{n=0}^{\infty} \) is finite. Then

\[
M = \mathbb{N}_0 \setminus \bigcup_{p \in \mathbb{P}_g} g^{-1}(0,p)
\]

is an open subset of \( \mathbb{N}_0 \), consisting of those \( n \) on which \( g \) takes the value \( \pm 1 \). But by the definition of \( g \), we have \( |g(0)| = 1 \). Hence \( M \) is nonempty, and must contain a congruence class. This contradicts our assumption that \( f \) is unbounded on any congruence class. \( \square \)

Since both addition and multiplication are continuous with respect to Furstenberg’s topology, any integral polynomial defines a continuous map from \( \mathbb{N} \) to \( \mathbb{Z} \). Hence Schur’s theorem follows directly from Theorem 2.1.

Polynomial sequences belong to a very special kind of recurrence sequences. In fact, an integral polynomial \( f(x) \) of degree \( k \) satisfies

\[
f(n + k + 1) = \sum_{i=0}^{k} (-1)^i \binom{k+1}{i+1} f(n + k - i).
\]
In the theorem below we generalize Schur’s theorem to a class of recurrence sequences.

**Theorem 2.2.** Let \( \{a_n\}_{n=0}^\infty \) be a recurrence sequence of integers satisfying
\[
a_{n+k+1} = \pm a_n + f(a_{n+1}, \ldots, a_{n+k})
\]
where \( f \in \mathbb{Z}[x_1, \ldots x_k] \). We further assume that \( \lim_{n \to \infty} |a_n| = \infty \). Then the recurrence sequence has infinitely many prime divisors.

**Proof.** A routine argument about linear recurrence sequences [4, p.45] shows that \( \{a_n\}_{n=0}^\infty \) is periodic modulo \( m \), i.e., for any positive integer \( m \), there exists \( s \in \mathbb{N} \) such that \( a_n \equiv a_{n+s} \pmod{m} \) for all \( n \geq 0 \). Hence a function \( h : \mathbb{N}_0 \to \mathbb{Z} \) of the form \( h(n) = a_n \) is continuous. Now applying Theorem 2.1 we conclude the proof. \( \square \)

Notice however that Theorem 2.2 does not cover even the linear recurrence sequences satisfying the simple relation
\[ a_{n+2} = a_{n+1} + 2a_n. \]

In order to treat more recurrence sequences, we need to further generalize Theorem 2.1.

For a nonzero integer \( m \), we define a variant of Furstenberg’s topology \( \mathcal{F}_m \) on \( \mathbb{Z} \) by taking the classes \( \text{Con}(a,b) \) as a basis for the open sets, where \( b \) runs over all positive integers prime to \( m \); then the symbol \( \mathbb{Z}_m \) denotes the topological space \( (\mathbb{Z}, \mathcal{F}_m) \). The following result is a simple generalization of Theorem 2.1; the proof is exactly the same.

**Theorem 2.3.** Let \( m \neq 0 \) and let \( f : \mathbb{N} \to \mathbb{Z}_m \) be a continuous map, which is unbounded on each congruence class. If \( f(n) \) is prime to \( m \) for each \( n \in \mathbb{N} \), then \( \{f(n)\}_{n=0}^\infty \) has infinitely many prime divisors.

3. Linear recurrence sequences

It is well-known that a non-degenerate linear recurrence sequence of integers of order \( k > 1 \) has infinitely many prime divisors [3, 8, 9, 11, 13, 16].

In this section, using Theorem 2.3 we give a simple proof of this result.

First we give some preliminaries about linear recurrence sequences. We say that a sequence of integers \( \{a_n\}_{n=0}^\infty \) is a linear recurrence sequence of order \( k \) if the following linear recurrence relation of order \( k \) is satisfied
\[
a_{n+k} = r_1a_{n+k-1} + \cdots + r_ka_n \quad (n = 0, 1, 2, \ldots),
\]
where \( r_1, \ldots, r_k \in \mathbb{Z} \) are constant, and \( r_k \neq 0 \). We note that a linear recurrence sequence may satisfy linear recurrence relations of different orders. For example, the Fibonacci sequence
\[ a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, \ldots \]
satisfies both
\[ a_{n+2} = a_{n+1} + a_n, \]
and
\[ a_{n+3} = 3a_{n+2} - a_{n+1} - 2a_n, \]
which are of order 2 and 3, respectively. When we say about an order of a linear recurrence sequence and recurrence relation, we always mean the minimal one.

Let
\[ g(x) = 1 - \sum_{i=1}^{k} r_i x^i \]
be the characteristic polynomial of the recurrence relation (2). If \( \{a_n\}_{n=0}^{\infty} \) is a linear recurrence sequence of order \( k \), satisfying (2) one can easily check that
\[ f(x) = g(x) \left( \sum_{n=0}^{\infty} a_n x^n \right) \]
is an integral polynomial of degree less than \( k \). Hence, the generating function of \( \{a_n\}_{n=0}^{\infty} \) is a rational function
\[ \sum_{n=0}^{\infty} a_n x^n = \frac{f(x)}{g(x)}, \]
and \( f(x) \) and \( g(x) \) are co-prime (otherwise, \( \{a_n\}_{n=0}^{\infty} \) would be of order less than \( k \), see below for details). Thus, for a linear recurrence sequence of order \( k \), the generating function extends to a meromorphic function on \( \mathbb{C} \) with \( k \) poles (the poles are counted with their multiplicities). On the other hand, if the generating function of a sequence of integers \( \{a_n\}_{n=0}^{\infty} \) is of the form \( 3 \), where \( g(x) = 1 - \sum_{i=1}^{k} r_i x^i \in \mathbb{Z}[x] \), \( r_k \neq 0 \), and \( f(x) \in \mathbb{Z}[x] \) is a nonzero polynomial of degree less than \( k \), and \( f(x) \) and \( g(x) \) are co-prime, then \( \{a_n\}_{n=0}^{\infty} \) is a linear recurrence sequence of order \( k \), satisfying (3).

A recurrence sequence is called degenerate if its characteristic polynomial has two distinct roots whose ratio is a root of unity, and non-degenerate otherwise.

For
\[ g(x) = 1 - \sum_{i=1}^{k} r_i x^i = \prod_{i=1}^{k} (1 - \psi_i x) \in \mathbb{Z}[x], \]
and \( b \in \mathbb{N} \), set
\[ (\phi_b g)(x) = \prod_{i=1}^{k} (1 - \psi_i^b x) \in \mathbb{Z}[x]. \]
If the ratio of two distinct roots of \( g(x) \) is not a root of unity, then the same is true for \( \phi_b g \).

**Lemma 3.1.** Let \( \{a_n\}_{n=0}^{\infty} \) be a non-degenerate linear recurrence sequence of order \( k > 1 \), and let
\[ g(x) = 1 - \sum_{i=1}^{k} r_i x^i \]
be the associated characteristic polynomial. Then for \( 0 \leq c < b \), the subsequence \( \{a_{c+bn}\}_{n=0}^{\infty} \) is also a non-degenerate linear recurrence sequence of order \( k \), whose characteristic polynomial is \( \phi_b g \).
Proof. Let the generating function of \(\{a_n\}_{n=0}^\infty\) be of the form (3), with the polynomial \(f(x)\) prime to \(g(x)\). Then the generating function of \(\{a_{c+bn}\}_{n=0}^\infty\) is

\[
\frac{1}{b} \sum_{i=1}^b e^{-ci} x_{c/b} f(\zeta_{b} x_{c/b}^{1/b}) = \frac{h(x)}{\phi_bg(x)},
\]

where \(\zeta_b = e^{2\pi i/b}\) and \(h(x) \in \mathbb{C}[x]\) is of degree less than \(k\) (because the left hand side of (4) is a rational function vanishing at infinity). As the left hand side of (4) lies in \(\mathbb{Z}[[x]]\), \(h(x)\) is in fact an integral polynomial. By counting the poles of both sides of (4), we see that \(\phi_bg(x)\) and \(h(x)\) are co-prime; hence the lemma follows. \(\square\)

For completeness of the paper, we give a short proof of a weaker version of the celebrated Skolem-Mahler-Lech theorem \([14]\) (asserting that for every sequence \(\{a_n\}\) as in Corollary 3.2 below, \(|a_n| \to \infty\) as \(n \to \infty\)).

**Corollary 3.2.** A non-degenerate linear recurrence sequence \(\{a_n\}_{n=0}^\infty\) of order \(k > 1\) is unbounded on any congruence class.

**Proof.** By lemma 3.1 it suffices to show that \(\{a_n\}_{n=0}^\infty\) itself is unbounded. Assume that there exists a positive integer \(m\) such that \(|a_n| < m\) for all \(n\). As \(\{a_n\}_{n=0}^\infty\) is eventually periodic modulo \(2m\), there exists a positive integer \(k\) such that the numbers \(a_{nk}\) are congruent to each other modulo \(2m\), for \(n\) sufficiently large. Then \(|a_n| < m\) forces \(\{a_{nk}\}_{n=0}^\infty\) to be eventually constant; this contradicts Lemma 3.1. \(\square\)

A positive integer \(m\) will be said to be a null divisor of a sequence of integers \(\{a_n\}_{n=0}^\infty\) if \(a_n\) is divisible by \(m\) for \(n\) sufficiently large. For a prime \(p\), the largest integer \(j \in \mathbb{N}\) such that \(p^j\) is a null divisor of \(\{a_n\}_{n=0}^\infty\) will be called the index of \(p\) in \(\{a_n\}_{n=0}^\infty\). We need the following result from \([15]\) about null divisors.

**Lemma 3.3.** Let \(\{a_n\}_{n=0}^\infty\) be a non-degenerate linear recurrence sequence of order \(k > 1\), and let

\[
g(x) = 1 - \sum_{i=1}^k r_i x^i
\]

be the associated characteristic polynomial. Assume that \(\text{GCD}(r_1, \cdots, r_k) = 1\). Then for any prime \(p\), the index of \(p\) in \(\{a_n\}_{n=0}^\infty\) is finite.

Now we are in the position to give a simple proof of the following theorem \([3]\); cf. \([3, 9, 11, 13, 16]\).

**Theorem 3.4.** Let \(\{a_n\}_{n=0}^\infty\) be a non-degenerate linear recurrence sequence of order \(k > 1\), satisfying

\[
a_{n+k} = r_1 a_{n+k-1} + \cdots + r_k a_n,
\]

where \(r_k \neq 0\). Then \(\{a_n\}_{n=0}^\infty\) has infinitely many prime divisors.
Proof. Let
\[ g(x) = 1 - \sum_{i=1}^{k} r_i x^i = \prod_{i=1}^{k} (1 - \psi_i x) \in \mathbb{Z}[x] \]
be the associated characteristic polynomial. Let \( \mathbb{Q}(\psi_1, \cdots, \psi_k) \) be the splitting field of \( g(x) \) and let \( A \) be the integral closure of \( \mathbb{Z} \) in \( \mathbb{Q}(\psi_1, \cdots, \psi_k) \).

For \( \alpha_1, \cdots, \alpha_n \in A \), let \( (\alpha_1, \cdots, \alpha_n) \) denote the ideal of \( A \) generated by \( \alpha_1, \cdots, \alpha_n \). The ideal \( (\psi_1, \cdots, \psi_k) \) of \( A \) is invariant under the Galois group. Hence by the ideal theory of Dedekind rings (cf. [7, Theorem 2, p.18] and [7, Corollary 2, p.26]), there exist two positive integers \( s, t \) such that
\[
(\psi_1^s, \cdots, \psi_k^s) = (\psi_1^t, \cdots, \psi_k^t) = (t).
\]
Set
\[
h(x) = 1 - \sum_{i=1}^{k} m_i x^i = \prod_{i=1}^{k} (1 - \psi_i x) \in \mathbb{Z}[x].
\]
We see that \( m_i \in \mathbb{Z} \) belongs to \((\psi_1^s, \cdots, \psi_k^s)^i = (t^i)\), hence \( m_i \) is divisible by \( t^i \).

Claim 3.5.
\[
GCD(m_1/t^1, \cdots, m_k/t^k) = 1.
\]

Proof. First, by (5) we have \( \psi_i^s/t \in A \) for \( 1 \leq i \leq k \), and \( (\psi_1^s/t, \cdots, \psi_k^s/t) = A \).

Now we show by contradiction that \((m_1/t^1, \cdots, m_k/t^k) = A \) and this would imply immediately the claim. Assume the contrary that \((m_1/t^1, \cdots, m_k/t^k) \neq A \). Then there exists a prime ideal \( B \) of \( A \) such that \((m_1/t^1, \cdots, m_k/t^k) \in B \).

As \((\psi_1^s/t, \cdots, \psi_k^s/t) = A \), there exists a partition \( I \cup J = \{1, 2, \cdots, k\} \) such that \( \psi_i^s/t \in B \) if and only if \( i \in I \), and \( J \) is nonempty. Set \( j = |J| \), and let \( f_j(x_1, \cdots, x_k) \) be the \( j \)-th elementary symmetric polynomial. Then each product in \( f_j(\psi_1^s/t, \cdots, \psi_k^s/t) \) belongs to \( B \) except for \( \prod_{i \in J} \psi_i^s/t \), hence \( f_j(\psi_1^s/t, \cdots, \psi_k^s/t) \notin B \). On the other hand, by (4), \( f_j(\psi_1^s/t, \cdots, \psi_k^s/t) = (-1)^{j-1}m_j/t^j \in B \).

By Lemma 3.1, \( \{a_{sn}\}_{n=0}^\infty \) is still a non-degenerate linear recurrence sequence of order \( k \), with the characteristic polynomial
\[
h(x/t) = 1 - \sum_{i=1}^{k} m_i x^i.
\]
By induction, we see that \( a_{sn} \) is divisible by \( t^n \), for each \( n \geq 0 \).

Set \( b_n = a_{sn}/t^n \). Then \( \{b_n\}_{n=0}^\infty \) is a non-degenerate linear recurrence sequence of order \( k \), with the characteristic polynomial
\[
h'(x/t) = 1 - \sum_{i=1}^{k} m_i x^i/t^i.
\]
As the prime divisors of \( \{b_n\}_{n=0}^\infty \) are also prime divisors of \( \{a_n\}_{n=0}^\infty \), it suffices to prove the theorem by replacing \( \{a_n\}_{n=0}^\infty \) with \( \{b_n\}_{n=0}^\infty \). Hence by Claim \( \ref{claim:arithmetic-property} \) we can assume that \( \gcd(r_1, \cdots, r_k) = 1 \). This condition still holds for subsequence \( \{a_{n+k}\}_{n=0}^\infty \) where \( i > 0 \).

Let \( p_1, \cdots, p_m \) be all the prime divisors of \( r_k \). By Lemma \( \ref{lemma:prime-divisors} \) we can choose a positive integer \( l \) which is larger than the index of \( p_1 \) in \( \{a_n\}_{n=0}^\infty \). Then there exists a positive integer \( j \) such that \( a_n = a_{n+j} (\mod p_1^l) \) for \( n \) sufficiently large. As \( p_1^l \) is not a null divisor of \( \{a_n\}_{n=0}^\infty \), we can choose a subsequence \( \{a_{n+l}\}_{n=0}^\infty \) such that the terms \( a_{n+l} \) are not divisible by \( p_1^l \), and are congruent to each other modulo \( p_1^l \). Let \( p_1^l \) be the highest power of \( p_1 \) dividing \( a_l \). We can replace \( \{a_n\}_{n=0}^\infty \) by \( \{a_{n+l}/p_1^l\}_{n=0}^\infty \) whose terms are prime to \( p_1 \).

Continuing in this way, we can finally find a subsequence \( \{a_{cn+d}\}_{n=0}^\infty \), \( (c > 0) \) and a positive integer \( e \) such that each term \( a_{cn+d} \) is divisible by \( e \) and the quotient \( a_{cn+d}/e \) is prime to \( p_1, \cdots, p_m \). By Corollary \( \ref{corollary:existence} \) the sequence \( \{a_{cn+d}/e\}_{n=0}^\infty \) gives a continuous map from \( \mathbb{N}_0 \) to \( \mathbb{Z}_{r_k} \), satisfying all the conditions of Theorem \( \ref{theorem:existence} \). Now applying Theorem \( \ref{theorem:existence} \) we conclude the proof.

**Remark 3.6.** We note that, for each polynomial \( f(x) \in \mathbb{Z}[x] \) of degree \( k \), the sequence \( \{f(n)\}_{n=0}^\infty \) is a non-degenerate linear recurrence sequence of order \( k+1 \) (see \( \ref{remark:non-degenerate} \)). Hence Theorem \( \ref{theorem:generalization} \) is another generalization of Schur’s theorem.

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