QUADRATIC MAPS BETWEEN GROUPS

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Dedicated to Mamuka Jibladze on the occasion of his fiftieth birthday

Abstract. The notion of quadratic maps between arbitrary groups appeared at several places in the literature on quadratic algebra. Here a unified extensive treatment of their properties is given; the relation with a relative version of Passi’s polynomial maps and groups of degree 2 is established and used to study the structure of the latter.

2000 Mathematics Subject Classification: 20F18, 16S34.
Key words and phrases: Quadratic map, polynomial group, augmentation quotient.

INTRODUCTION

Polynomial maps have been appearing in nilpotent group theory for a long time, originally in the form of rational (numerical) functions, for example in the Hall–Petrescu formula or the group law of torsion-free nilpotent groups when written with respect to a Mal’cev basis. An intrinsic notion of polynomial maps from groups to abelian groups was introduced by Passi [30], together with a universal example $G \to P_n(G)$ where the abelian group $P_n(G)$ is called a “polynomial group”. Passi’s motivation came from the study of dimension subgroups; since then, his construction turned out to provide a key tool in the study of many other problems: in the theory of group schemes [7] as well as in the theory of nilpotent groups, concerning their second (co)homology [15], [16], automorphism groups or simplicial objects [13], [17]. However, a need to study polynomial maps between arbitrary groups comes from unstable homotopy theory; after Baues’ [3] and the author’s [12] study of metastable homotopy groups and Moore spaces [4], the foundations of “quadratic algebra” were laid in [5] where the notion of quadratic maps with a non-abelian target group first appeared. Since then, in the steadily growing literature on quadratic algebra and its applications, various variants and properties were exhibited when needed, in the works of Baues, Jibladze, Muro, Pirashvili and the author (most of these articles can be found on ArXiv). So the purpose of this paper is to provide a thorough unified treatment of what is called weakly quadratic maps, because of their good functorial behaviour; for brevity we drop the word “weakly” in this paper. So several of our formulas and properties appear also elsewhere in the literature, but we include and prove most of them here for the sake of a coherent exposition. However, we here work in slightly more general framework of quadratic maps relative to a subgroup, inspired by Passi’s study.
of relative dimension subgroups, see [32], later on extended by Kuz’min [24] and in [20]. The relative viewpoint here leads to the construction of various categories of relative quadratic maps generalizing the “quadratic envelope of the category of 2-step nilpotent groups” introduced by Jibladze and Pirashvili [23]. In particular, the category $\mathbf{CP}$ of central quadratic pair maps introduced here turns out to play a fundamental role in quadratic algebra since it allows one to refound the basic notions and to define modules over square ringoids instead of only square groups as in [5]; this is work in progress and will be presented in [11] and in a forthcoming book on quadratic algebra jointly written with H. Gaudier, F. Goichot, B. Loiseau and T. Pirashvili.

In Section 1 we introduce and study relative quadratic maps and their universal examples $G \rightarrow Q(G, B)$ which constitute a nonabelian version of the relative version of Passi’s polynomial maps and groups in degree 2. The latter are studied in Section 2, first for an arbitrary degree before focussing on degree 2 where the relative Passi groups $P_2(G, B)$ turn out to be precisely the abelianization of the groups $Q(G, B)$; this fact allows us to derive their structure from the (easy) non-abelian case in a rather simple way, and to deduce several exact sequences for $P_2(G, B)$.

Finally we note that most of the theory of this paper can be generalized to the abstract setting of semi-abelian categories [21]; in particular, all algebraic theories containing a group law as part of the structure (for example, algebras over any non-unitary $k$-linear operad, like Lie algebras), admit a theory of quadratic maps. Also, the theory of quadratic maps between modules is inaugurated in [10], and the notion of polynomial maps between non-abelian groups of arbitrary degree is introduced by the author, from an inductive viewpoint generalizing the one adopted in this paper; this is work in progress. There is, however, a different approach due to Leibman which also has interesting applications [26], [27]; the precise relation between the two approaches remains to be clarified.

1. Quadratic Maps between Groups

Throughout this paper, the symbols $G$ and $H$ denote groups. The commutator of elements $a, b \in G$ is defined as $[a, b] = aba^{-1}b^{-1}$, and the conjugation is denoted by $^ab = aba^{-1}$. We write $G^{ab} = G/G'$ and $ab : G \rightarrow G^{ab}$ for the natural projection. Recall that the lower central series of $G$ is defined by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. We say that $G$ is $n$-step nilpotent if $\gamma_{n+1}(G) = \{1\}$. If $G$ is 2-step nilpotent, the commutator map $[-, -] : G \times G \rightarrow G$ is well known to be bilinear.

Let $f : G \rightarrow H$ be some function between arbitrary groups. We shall, however, write the group law of $H$ additively since in many applications $H$ is abelian, and in those in [5], [11] where $H$ is genuinely nonabelian, it is written additively anyway to match the conventions in homotopy theory which originally motivated these developments.

Define the deviation function or cross effect of $f$ to be the map

$$d_f : G \times G \rightarrow H \text{ by } d_f(a, b) = f(ab) - f(b) - f(a).$$
Furthermore, let $I_f$ resp. $D_f$ denote the subgroup of $H$ generated by $\text{Im}(f)$ resp. $\text{Im}(d_f)$.

**Definitions 1.1.** We say that $f$ as above is

(a) **linear** if $d_f = 0$, i.e., $f$ is a group homomorphism;

(b) **quadratic** if $d_f$ is **bilinear** and $D_f$ is **central** in $I_f$, or more explicitly, $\forall a, b, c \in G$, $[d_f(a, b), f(c)] = 0$.

This definition of quadratic maps first appears in [23] under the same definition of weakly quadratic maps.

Note that linear maps are quadratic. We denote by $\text{Quad}(G, H)$ the set of quadratic maps from $G$ to $H$.

We also need the following relative version of quadratic maps. Let $B$ be a subgroup of $G$. Then we say that $f$ as above is **quadratic relative to $B$** if $f$ is quadratic and $d_f(B \times G) = d_f(G \times B) = 0$. Note that $f$ is quadratic iff it is quadratic relative to $\{1\}$. We define the **radical** of a quadratic map $f$ to be the set $\text{rad}(f)$ consisting of all elements $a$ of $G$ such that for all $b \in G$, $d_f(a, b) = d_f(b, a) = 0$. Note that $\text{rad}(f)$ is a subgroup of $G$ and that $G/\text{rad}(f)$ is normal and $G/\text{rad}(f)$ is abelian. It is also clear that $\text{rad}(f)$ is the largest subgroup $B$ of $G$ such that $f$ is quadratic relative to $B$.

In the following proposition we collect the basic properties of quadratic maps which are easily deduced from the definition.

**Proposition 1.2.** Let $f : G \to H$ be a quadratic map relative to some subgroup $B$ of $G$.

(a) One has the following identities for $a, b \in G$:

\[
\begin{align*}
    f(ab) &= d_f(a, b) + f(a) + f(b) = f(a) + f(b) + d_f(a, b), \\
    d_f(a, b) &= -f(a) + f(ab) - f(b);
\end{align*}
\]

(b) $f$ is normalized, i.e., $f(1) = 0$;

(c) the restriction of $f$ to $B$ is linear, whence $f(B)$ is a subgroup of $H$;

(d) there is a canonical linear map

\[
w_f : G/BG' \otimes G/BG' \to H
\]

such that $w_f(\bar{a} \otimes \bar{b}) = d_f(a, b)$ for $a, b \in G$.

**Examples 1.3.** (0) Let $R$ be any ring and $a, b \in R$. Then the function $f : R \to R$, $f(x) = ax^2 + bx$, is a quadratic map between additive groups. More generally, quadratic forms on vector spaces are quadratic maps.

(1) For any subgroup $B$ of $G$, the 2-fold diagonal map $\delta^2 : G \to G/BG' \otimes G/BG'$, $\delta(a) = \bar{a} \otimes \bar{a}$, is quadratic relative to $B$.

(2) Let $L$ be a 2-step nilpotent Lie algebra (i.e., $[[L, L], L] = 0$) over $\mathbb{Z}[\frac{1}{2}]$. Then a multiplicative group law on $L$ is defined by the truncated Campbell-Baker-Hausdorff-formula, i.e., $x \circ y = x + y + \frac{1}{2}[x, y]$ for $x, y \in L$. Indeed, $(L, \circ)$
is a uniquely 2-divisible 2-step nilpotent group, and this construction provides a functorial equivalence between groups of this type and 2-step nilpotent Lie algebras over $\mathbb{Z}[\frac{1}{2}]$. This is a special case of a more general result of Lazard [25], providing “abelian models” for sufficiently divisible nilpotent groups. Now in the 2-step nilpotent case above, the identity map $id : (L, \circ) \to (L, +)$ is quadratic with $d_{id}(x, y) = \frac{1}{2}[x, y]$. 

We note that generalizing the above equivalence of Lazard we constructed the functorial abelian models for arbitrary 2-step nilpotent groups (actually, for central group extensions with abelian cokernel), cf. [13] and also [17]; this construction is based on the properties of relative quadratic maps with values in abelian groups, see Section 2 below.

(3) Let $k$ be some commutative ring with unit and $G$ be the subgroup $1 + Tk[[T]]$ of the group of units of the algebra $k[[T]]$ of power series over $k$. For $n \geq 0$, let $c_n : G \to k$, $c_n(\sum_{i \geq 0} a_i T^i) = a_n$. Then $c_1$ is linear and $c_2$ is quadratic since $c_2(fg) = c_2(f) + c_2(g) + c_1(f)c_1(g)$, $f, g \in G$.

(4) Let $\Sigma(n, 3n - 3)$ denote the pointed homotopy category of suspensions $\Sigma X$ of topological spaces $X$ such that $\Sigma X$ is an $(n - 1)$-connected $(3n - 3)$-dimensional CW-space. Then for $\Sigma X, \Sigma Y \in \Sigma(n, 3n - 3)$ the second James-Hopf-invariant $\gamma_2 : [\Sigma X, \Sigma Y] \to [\Sigma X, \Sigma Y \wedge Y]$ (see [37]) is a quadratic map where the group structure on $[\Sigma X, Z]$ is induced by the cogroup structure of $\Sigma X$. Moreover, if also $\Sigma Z \in \Sigma(n, 3n - 3)$ and $f : \Sigma X \to \Sigma Y$ is a continuous map then the map $[f]^* : [\Sigma Y, \Sigma Z] \to [\Sigma X, \Sigma Z]$ is a quadratic map, see [3, Appendix]. This example gave rise to the notion of quadratic categories, see [5].

More examples appear in the following proposition showing that quadratic maps are intimately related to 2-step nilpotent groups.

**Proposition 1.4.** Let $G$ be a group and $n \geq 2$. Then the following properties are equivalent.

1. $G$ is 2-step nilpotent.
2. The $(n - 1)$-fold multiplication map $\mu_{n-1} : G^n \to G$, $\mu_{n-1}(a_1, \ldots, a_n) = a_1 \cdots a_n$, is a quadratic map.
3. For all groups $K$ and all linear maps $f_1, \ldots, f_n : K \to G$, the product map $f_1 \cdots f_n : K \to G$, $x \mapsto f_1(x) \cdots f_n(x)$, is quadratic.
4. The map $2_G : G \to G$, $a \mapsto a^2$, is quadratic.

Each of these implies

5. The map $n_G : G \to G$, $a \mapsto a^n$, is quadratic.

Note that property (2) neatly generalizes the often useful fact that $G$ is abelian iff $\mu_1$ is linear.
Proof. We first note that if $G$ is 2-step nilpotent the following relations hold for $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $G^n$:

$$a_1b_1 \cdots a_nb_n = \prod_{1 \leq i < j \leq n} [b_i, a_j](a_1 \cdots a_n)(b_1 \cdots b_n), \quad (5)$$

$$d_{\mu_{n-1}}(a, b) = \prod_{1 \leq i < j \leq n} [b_i, a_j]. \quad (6)$$

In fact, if $i < j$, we have $b_ia_j = [b_i, a_j]a_jb_i$; in shuffling all the factors $b_i$ in the product $a_1b_1 \cdots a_nb_n$ to the right, one introduces all the commutators $[b_i, a_j]$ with $i < j$. But the latter are central in $G$, so we can gather them on the left, which proves the first formula. It implies the second one because $d_{\mu_{n-1}}(a, b) = (a_1b_1 \cdots a_nb_n)(b_1 \cdots b_n)^{-1} (a_1 \cdots a_n)^{-1}$.

Now we prove the desired equivalences.

(1) $\Rightarrow$ (2). We have $D_{\mu_{n-1}} \subset G' \subset Z(G)$ by equation (6) and as $G$ is 2-step nilpotent. Each commutator $[b_i, a_j]$ being bilinear identity (6) shows that $d_{\mu_{n-1}}$ is bilinear, too.

(2) $\Rightarrow$ (3). The map $K \to G^n, x \mapsto (f_1(x), \ldots, f_n(x))$, is linear. Hence by Proposition 1.8 below the composite map with the quadratic map $\mu_{n-1} : G^n \to G$ is still quadratic.

(3) $\Rightarrow$ (4). It suffices to take two of the maps $f_i$ equal to the identity of $G$ and the others equal to the trivial map.

(4) $\Rightarrow$ (1). Using (3) we get $d_{2_G}(a, b) = [-a, b]$, hence $[-a, b] \cdot [b-a, b'] = [-a, bb'] = d_{2_G}(a, bb') = d_{2_G}(a, b)d_{2_G}(a, b') = [-a, b]([-a, b']$, for $a, b, b' \in G$.

It follows that $b[-a, b'] = [-a, b']$, whence $G'$ is central in $G$ and $G$ is 2-step nilpotent.

(5) It suffices to take $K = G$ and each $f_i = id$ in assertion (3).

We now recall some relations already proved in Lemma 2 in [23].

**Proposition 1.5.** Let $f : G \to H$ be a quadratic map. Then the following relations hold for $a, b \in G$.

$$f(a^{-1}) = -f(a) + df(a, a), \quad (7)$$

$$f(ab^{-1}) = f(a) - f(b) - df(ab^{-1}, b), \quad (8)$$

$$f[a, b] = [f(a), f(b)] + df(a, b) - df(b, a), \quad (9)$$

$$f(ab) = f(a)f(b) + df(a, b) - df(b, a), \quad (10)$$

$$d_{-f}(a, b) = -df(b, a) - f[a, b]. \quad (11)$$

We point out that relation (9) has a conceptual meaning which makes it a crucial ingredient in the structure theory in Section 2; it is also the only relation
which generalizes to the setting of semi-abelian categories where it again plays a key role [21].

**Composition of quadratic maps.** The content of this section is essentially due to my former student O. Perriquet [33]. Composing two quadratic maps does not give a quadratic map in general, see Example 1.3(0). However, the following sufficient condition assuring stability under composition is often satisfied in practice, see [5], [11].

**Definition 1.6.** Let \( K \xrightarrow{g} G \xrightarrow{f} H \) be two quadratic maps relative to some subgroup \( A \) of \( K \) and \( B \) of \( G \), resp. We say that the couple \((f, g)\) is a quadratic pair if \( g(A) \subset B \) and \( df(D_g \times G) = df(G \times D_g) = 0 \).

For example, if \( g \) and \( f \) are quadratic maps such that \( D_g \subset G' \) then for \( A = K' \) and \( B = G' \), \((f, g)\) is a quadratic pair by (9).

**Proposition 1.7.** If \((f, g)\) as above is a quadratic pair, then the composite map \( f \circ g \) is quadratic relative to \( A \), with

\[
d_{f \circ g} = f_\ast d_g + (g \times g) \ast d_f . \tag{12}
\]

Moreover, if \( D_g \subset B G' \), \( g \) induces a linear map \( \bar{g} : K/AK' \rightarrow G/BG' \) and we have

\[
w_{f \circ g} = f_\ast w_g + (\bar{g} \otimes \bar{g}) \ast w_f . \tag{13}
\]

We call (12) or (13) the derivation property of a quadratic pair.

**Proof.** Writing \( G \) and \( H \) additively we have for \( a, b \in K \)

\[
d_{f \circ g}(a, b) = fg(ab) - fg(b) - fg(a)
\]

\[
= f(d_g(a, b) + g(a) + g(b)) - fg(b) - fg(a) = f(d_g(a, b)) + f(g(a) + g(b)) - fg(b) - fg(a) \quad \text{since } d_f(D_g \times G) = 0
\]

\[
= f(d_g(a, b)) + df(g(a), g(b)) + fg(a) + fg(b) - fg(b) - fg(a) = f(d_g(a, b)) + df(g(a), g(b)).
\]

Next we check that \( d_{f \circ g} \) is linear in the first variable; the argument for the second variable is similar. For \( a' \in K \), we have

\[
d_{f \circ g}(aa', b) = f(d_g(aa', b)) + df(g(aa'), g(b))
\]

\[
= f(d_g(a, b) + d_g(a', b)) + df(g(a) + g(a') + d_g(a, a'), g(b))
\]

\[
(\ast) \quad = f(d_g(a, b)) + f(d_g(a', b)) + df(g(a), g(b)) + df(g(a'), g(b))
\]

\[
(\ast\ast) \quad = f(d_g(a, b)) + df(g(a), g(b)) + f(d_g(a', b)) + df(g(a'), g(b)) = d_{f \circ g}(a, b) + df(g(a'), g(b)) .
\]

Equations \((\ast)\) and \((\ast\ast)\) hold since \( df(D_g \times G) = 0 \) and since \( D_f \subset Z(I_f) \), resp.

Moreover, if \( a \) or \( b \) is in \( A \) then \( d_{f \circ g}(a, b) = 0 \) since then \( d_g(a, b) = 0 \) and \( df(g(a), g(b)) = 0 \) as \( g(A) \subset B \). It remains to check that \( D_{f \circ g} \subset Z(I_{f \circ g}) \). Let \( a, b, c \in K \). Then \([df(g(a), g(b)), f(g(c))] = 0 \) since \( D_f \subset Z(I_f) \), and

\[
[f(d_g(a, b)), f(g(c))] = f[d_g(a, b), g(c)] - df(d_g(a, b), g(c)) + df(g(c), d_g(a, b))
\]
by (9). But the first of the latter three terms is trivial since $D_g \subset Z(I_g)$, the other two since $d_f(D_g \times G) = d_f(G \times D_g) = 0$. \hfill \Box

**Corollary 1.8.** Pre- or postcomposing a quadratic map by a linear map gives a quadratic map. More precisely, if $f : G \to H$ is quadratic and $g : K \to G$, $h : H \to L$ are linear maps of groups then $hfg$ is quadratic with $d_hfg = h_g(g \times g)^*d_f$. \hfill \Box

**Corollary 1.9.** There is a functor $\text{Quad}(G,-)$ from the category of groups to the category of sets sending a group $H$ to $\text{Quad}(G,H)$ and a homomorphism $f : H \to K$ to the map $f_* : \text{Quad}(G,H) \to \text{Quad}(G,K)$. The following relations are crucial in dealing with quadratic categories, see [5].

**Proposition 1.10.** Let $f, g : G \to H$ be two functions between groups.

1. If $(2_H, f)$ is a quadratic pair, then $D_f$ is central in $H$.
2. If $(2_H, g)$ is a quadratic pair, then the following relations hold for $a, b \in G$.

\[
d_{f+g}(a, b) = d_f(a, b) + d_g(a, b) + d_{2_H}(g(a), f(b)) \]
\[
d_f(a, b) + d_g(a, b) + [f(b), g(a)].
\]

**Proof.** If $2_H$ is quadratic then $d_{2_H}(a, b) = [-a, b]$ by (3), whence (1) follows from the equations $[H, D_f] = [-H, D_f] = d_{2_H}(H \times D_f) = 0$. To prove (2) calculate

\[
d_{f+g}(a, b) = (f + g)(a + b) - (f + g)(b) - (f + g)(a)
\]
\[
= f(a + b) + g(a + b) - g(b) - f(b) - g(a) - f(a)
\]
\[
= d_f(a, b) + f(a) + f(b) + d_g(a, b) + g(a)
+ g(b) - g(b) - f(b) - g(a) - f(a)
\]
\[
= d_f(a, b) + d_g(a, b) + [f(b), g(a)].
\]

The last equation follows from the fact that $H'$ is central in $H$ as $H$ is 2-step nilpotent since $2_H$ is quadratic, see Proposition 1.4. Equation (**) is due to the fact that $D_g$ is central in $H$ by assertion (1). Finally, as $H$ is 2-step nilpotent, $[f(b), g(a)] = -[g(a), f(b)] = [-g(a), f(b)] = d_{2_H}(g(a), f(b))$. \hfill \Box

**The category of quadratic pairs.**

**Definition 1.11.** A pair of groups $(G, B)$ consists of a group $G$ together with a subgroup $B$ of $G$. A pair map $f : (G, B) \to (H, C)$ between pairs of groups is a function $f$ from $G$ to $H$ such that $f(B) \subset C$. Moreover, a pair map $f$ is a linear pair map if the function $f$ is linear, and $f$ is a quadratic pair map if $f$ is quadratic relative to $B$ such that $C$ contains $D_f$.

Note that any quadratic map $f : G \to H$ is a quadratic pair map from $(G, G')$ to $(H, H'D_f)$ by (9) and from $(G, \text{rad}(f))$ to $(H, f(\text{rad}(f))D_f)$ by Proposition 1.2(c).
Proposition 1.12. Let \((K, A) \xrightarrow{g} (G, B) \xrightarrow{f} (H, C)\) be quadratic pair maps. Then the composite map \(fg : (K, A) \to (H, C)\) is a quadratic pair map whose deviation satisfies the derivation rules (12) and (13).

This is immediate from Proposition 1.7; it leads to the following generalization of the “quadratic envelope of the category of 2-step nilpotent groups” constructed by Jibladze and Pirashvili [23]: this is the category, denoted by \(\text{Ni}_q\), whose objects are 2-step nilpotent groups and whose morphisms from \(G\) to \(H\) are quadratic maps \(f : G \to H\) such that \(D_f \subseteq H'\).

Corollary 1.13. Pairs of groups and quadratic pair maps between them form a category denoted by \(\text{QP}\) which we call the quadratic envelope of the usual category of linear pair maps.

In fact, the category \(\text{Ni}_q\) fully embeds into \(\text{QP}\) by sending \(G\) to the pair \((G, G')\).

Denote by \(\text{Gr}\) and \(\text{Ab}\) the category of groups and abelian groups, resp. Then the following is again an immediate consequence of Proposition 1.7.

Proposition 1.14. Let \(\text{NQP}\) resp. \(\text{AQP}\) resp. \(\text{CP}\) be the full subcategory of \(\text{QP}\) consisting of those pairs of groups \((G, B)\) for which \(B\) is normal in \(G\), resp. abelian, resp. central containing \(G'\).

(a) There are functors \(\text{Gr} : \text{QP} \to \text{Ab} \times \text{Gr}\), \(\text{Gr}_N : \text{NQP} \to \text{Gr} \times \text{Gr}\), \(\text{Gr}_A : \text{AQP} \to \text{Ab} \times \text{Ab}\), \(\text{Gr}_C : \text{CP} \to \text{Ab} \times \text{Ab}\) such that \(\text{Gr}\) and \(\text{Gr}_A\) send an object \((G, B)\) to the object \((G/G'B, B)\) while \(\text{Gr}_N\) and \(\text{Gr}_C\) send \((G, B)\) to \((G/B, B)\), and all of them send \(f : (G, B) \to (H, C)\) to \((\bar{f}, f_B)\) where \(f_B : B \to C\) is the restriction of \(f\) and \(\bar{f}\) is induced by \(f\).

(b) There is a bifunctor \(D : \text{AQP}^{op} \times \text{AQP} \to \text{Ab}\) defined by
\[
D((G, B), (H, C)) = \text{Hom}((G/G'B) \otimes (G/G'B), C) \quad \text{and} \quad D(g^{op}, f) = f_*(\bar{g} \otimes \bar{g})^*.
\]

(c) Assigning to a map \(f : (G, B) \to (H, C)\) in \(\text{AQP}\) its defect \(w_f \in D((G, B), (H, C))\) defines a derivation from \(\text{AQP}\) to \(D\), see [2].

Note that \(\text{CP}\) is contained in the intersection of \(\text{NQP}\) and \(\text{AQP}\), and that for \((G, B) \in \text{CP}\) the group \(G\) is 2-step nilpotent. Thus the category \(\text{Ni}_q\) also identifies with the full subcategory of \(\text{CP}\) consisting of the objects \((G, G')\).

Moreover, \(\text{CP}\) is a right quadratic category in the sense of [5]; from several points of view, it plays the same role in quadratic algebra as the category of abelian groups plays in classical algebra, see [11]; it can therefore be considered as a different generalization of the category of abelian groups than the category of square groups constructed in [6]. The relation between these two generalizations, however, is not yet understood.

Universal relative quadratic map. We now show that the functor \(\text{Quad}(G, -)\) is representable; this result is also obtained in [23]. We thus get an endofunctor \(Q\) of the category of groups the properties of which are studied.
Theorem 1.15. Let G be a group and B a subgroup of G.

(i) There exists a universal quadratic map relative to B, \( q = q_{G,B} : G \to Q(G,B) \) to some group \( Q(G,B) \), i.e., for any quadratic map of groups \( f : G \to H \) relative to B there exists a unique linear map \( \hat{f} q = f \).

(ii) The sequence of group homomorphisms

\[
0 \longrightarrow G/BG' \otimes G/BG' \xrightarrow{w_q} Q(G,B) \xrightarrow{\hat{id}} G \longrightarrow 1
\]

is exact. Actually, it is a central group extension represented by the bilinear 2-cocycle

\[
\bar{D} = \langle \bar{a}, \bar{b} \rangle = \langle \bar{x}, \bar{y} \rangle
\]

where \( \bar{a} = q(a) = q(a) = q(a) = q(a) \) is a bijection natural in \( G \).

Proof. The trick is to define the group \( Q(G,B) \) by the cocycle \( D \), i.e., we let \( Q(G,B) = (G/BG' \otimes G/BG') \times G \) endowed with the group law \( (x,a) + (y,b) = (x + y - \bar{a} \otimes \bar{b}, ab) \). Furthermore, let \( q : G \to Q(G,B), q(a) = (0,a) \). Then for \( a,b \in G, q(a) + q(b) = (-\bar{a} \otimes \bar{b},ab) = (-\bar{a} \otimes \bar{b}, 1) + q(ab) \), whence \( d_q(a,b) = (\bar{a} \otimes \bar{b}, 1) \). This term is bilinear, central in \( Q(G,B) \), and trivial whenever \( a \) or \( b \) is in \( B \), so \( q \) is a quadratic map relative to \( B \) with \( w_q(x) = (x,1) \). In order to prove its universal property let \( f : G \to H \) be some quadratic map relative to \( B \). Define \( \hat{f} : Q(G,B) \to H \) by \( \hat{f}(x,a) = w_f(x) + f(a) \). Then \( \hat{f} \) satisfies \( \hat{f} q = f \) and for \( (x,a),(y,b) \in Q(G,B) \), we have

\[
\hat{f}((x,a) + (y,b)) = w_f(x + y - \bar{a} \otimes \bar{b}) + f(ab)
\]

\[
= w_f(x) + w_f(y) - d_f(a,b) + d_f(a,b) + f(a) + f(b)
\]

\[
= w_f(x) + f(a) + w_f(y) + f(b) \quad \text{since} \quad \text{Im}(w_f) = D_f \subset Z(I_f)
\]

\[
= \hat{f}(x,a) + \hat{f}(y,b)
\]

To prove the uniqueness of \( \hat{f} \) let \( g : Q(G,B) \to H \) be a linear map such that \( gg = f \). Then \( g(0,a) = gg(a) = f(a) \) and \( g(\bar{a} \otimes \bar{b},1) = g d_q(a,b) = d_{gq}(a,b) \) (by 1.8) = \( d_f(a,b) = w_f(\bar{a} \otimes \bar{b}) \), whence, by linearity, \( g(x,1) = w_f(x) \) for all \( x \in G/BG' \otimes G/BG' \). Thus \( g(x,a) = g((x,1) + (0,a)) = w_f(x) + f(a) = \hat{f}(x,a) \), whence \( g = \hat{f} \). Finally, we have \( \hat{id}(x,a) = w_{id}(x)a = a \) as \( \hat{id} \) is linear, so \( \hat{id} \) is the projection to the second factor which proves the exactness of the sequence in (ii) as we have already seen that \( w_q \) is the canonical injection of the first factor.

Write \( Q(G) = Q(G,\{1\}) = Q(G,G') \) and \( q = q_G = q_{G,\{1\}} \). Note that Theorem 1.15(i) says that the map

\[
q^* : \text{Hom}(Q(G), H) \to \text{Quad}(G, H)
\]

is a bijection natural in \( H \).

Corollary 1.16. Let \( X \) be a set and \( F \) a free group with basis \( X \). Then there is a (non natural) isomorphism \( \phi : Q(F) \to (F^a)^b \times F \) such that \( \phi q_F \) is
given by
\[ \phi_{qF}\left(\prod_{i=1}^{n} x_i^{\epsilon_i}\right) = \left(\sum_{i=1}^{n} \frac{1}{2} \epsilon_i \bar{x}_i \otimes \bar{x}_i + \sum_{1 \leq i < j \leq n} \epsilon_i\epsilon_j \bar{x}_i \otimes \bar{x}_j, \prod_{i=1}^{n} x_i^{\epsilon_i}\right) \] (16)
for \( x_i \in X, \epsilon_i = \pm 1 \). Consequently, for any group \( H \) there is a bijection
\[ \text{Quad}(F, H) \cong \{ (\chi, \psi) \in H^X \times H^{X \times X} \mid [\text{Im}(\chi), \text{Im}(\psi)] = [\text{Im}(\chi), \text{Im}(\psi)] = \{1\} \} \] (17)
which carries \( f \in \text{Quad}(F, H) \) to \((f_{|x}, d_{f_{|x \times x}})\).

**Proof.** The map \( q_{F_{|x}} \) induces a splitting of the central extension (14). The formula for \( \phi_{qF} \) is obtained using (2) and (7). To determine \( \text{Quad}(F, H) \) now use the bijection \( q_{F}^{\ast} \) in (15) and the fact that \( X \times X \to \mathbb{F}_{ab} \otimes \mathbb{F}_{ab}, (x, y) \mapsto x \otimes y \), is a basis of the abelian group \( \mathbb{F}_{ab} \otimes \mathbb{F}_{ab} \).

**Proposition 1.17.** There is an endofunctor \( Q \) of the category of groups sending \( G \) to \( Q(G) \) and \( f : G \to H \) to \( Q(f) = q_{Hf} : Q(G) \to Q(H) \). It has the following properties.

(a) If \( G \) is \( n \)-step nilpotent for \( n \geq 2 \), then so is \( Q(G) \).

(b) If \( G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \to 1 \) is an exact sequence of groups, then so is the sequence
\[ Q(G_1) \times G_1^{ab} \otimes G_2^{ab} \times G_2^{ab} \otimes G_1^{ab} \xrightarrow{\xi} Q(G_2) \xrightarrow{\overline{Q} \beta} Q(G_3) \to 1 \] (18)
where \( \xi(x, y, z) = Q(\alpha)(x) + w_q(\alpha^{ab} \otimes 1)(y) + w_q(1 \otimes \alpha^{ab})(z) \). One also has the identity
\[ \text{Ker}(Q(\beta)) = \text{Im}(q_{G_2} \alpha) + \text{Im}(w_{q_{G_2}}(\alpha \otimes \alpha + \alpha^{ab} \otimes 1_{G_2} + 1_{G_2} \otimes \alpha^{ab})). \] (19)

Similarly, assigning the group \( Q(G, B) \) to a pair \((G, B)\) defines a functor from the category of linear pair maps to the category of groups.

**Proof.** (a): Let \( a \in Q(G), b \in \gamma_n(Q(G)) \); we must show that \([a, b] = 1 \). Modulo the central subgroup \( \text{Im}(w_q) \) we may assume that \( a = qa', b = qb' \) for some \( a', b' \in G \). Then by (9), \([a, b] = q[a', b'] - w_q(\overline{a'} \otimes \overline{b'} - \overline{b'} \otimes \overline{a'}) = 1 \) since \([a', b'] = 1 \) as \( b' \in \gamma_n(G) \) and since \( \overline{b} = 0 \) as \( n \geq 2 \).

For (b) use the naturality of sequence (14) with respect to the maps \( \alpha \) and \( \beta \); then the exactness of sequence (18) follows by an easy diagram chase together with the right exactness of the tensor product. Identity (19) then follows using the fact that \( Q(G_1) = \langle \text{Im}(q) \rangle + \text{Im}(w_q) \).

**Corollary 1.18.** If \( G \) is generated by a subset \( X \), then \( Q(G) \) is generated by the subset \( q(X) \cup d_q(X \times X) \).

Just apply Proposition 1.17(b) to the epimorphism from the free group of basis \( X \) to \( G \) and use Corollary 1.16.
\textbf{Corollary 1.19.} Two quadratic maps \( f,g : G \to H \) coincide if and only if they coincide on a generating subset \( X \) of \( G \) and \( d_f \) coincides with \( d_g \) on \( X \times X \).

Just note that \( f = g \) iff \( \hat{f} = \hat{g} \) as \( q \) is injective (having \( \hat{id} \) as a retraction).

The above properties of the functor \( Q \) allow us to construct quadratic maps on groups defined by generators and relations, as follows. Let \( G = \langle X | R \rangle \) be a presentation of \( G \), i.e. let \( X \) be a set, \( F \) a free group with basis \( X \), \( R \) a subset of \( F \) and \( \pi : F \to G \) a homomorphism whose kernel is the normal subgroup of \( F \) generated by \( R \). For \( r \in R \), write \( r = \prod_{i=1}^{n_r} x_{ri}^{\epsilon_{ri}} \) with \( x_{ri} \in X \) and \( \epsilon_{ri} = \pm 1 \), and \( \bar{r} = rF' = \sum_{x \in X} k_{rx} x \) in \( F^{ab} \), with \( k_{rx} \in \mathbb{Z} \).

\textbf{Proposition 1.20.} For some group \( H \) let \( (\chi, \psi) \in H^X \times H^{X \times X} \). Then there exists a quadratic map \( f : G \to H \) such that

\[ f \pi(x) = \chi(x) \quad \text{and} \quad d_f(\pi(x),\pi(y)) = \psi(x,y) \quad \text{for } x,y \in X \quad (20) \]

if and only if for all \( r \in R \) and \( y \in X \) the following three conditions hold.

(i) \( [\text{Im}(\chi), \text{Im}(\chi)] = [\text{Im}(\chi), \text{Im}(\psi)] = \{1\}; \)

(ii) \( \sum_{i=1}^{n_r} (\epsilon_{ri} \chi(x_{ri}) + \frac{1-\epsilon_{ri}}{2} \psi(x_{ri}, x_{ri})) + \sum_{1 \leq i < j \leq n} \epsilon_{ri} \epsilon_{rj} \psi(x_{ri}, x_{rj}) = 0; \)

(iii) \( \sum_{x \in X} k_{rx} \psi(x,y) = \sum_{x \in X} k_{rx} \psi(y,x) = 0. \)

Moreover, any quadratic map from \( G \) to \( H \) is induced by maps \( \chi \) and \( \psi \) as above.

\textbf{Proof.} Let \( F_1 \) be a free group with basis \( F \times R \). Then the sequence \( F_1 \xrightarrow{\delta} F \xrightarrow{\pi} G \xrightarrow{\delta} 1 \) is exact with \( \delta(a,r) = a r \) for \( (a,r) \in F \times R \). Hence by (15) and Proposition 1.17(b) there exists a quadratic map \( f : G \to H \) satisfying (20) iff there is a homomorphism \( \kappa : Q(F) \to H \) factoring through \( Q(\pi) \) and satisfying the property

\[ \kappa q_F(x) = \chi(x) \quad \text{and} \quad \kappa d_{q_F}(x,y) = \psi(x,y) \quad \text{for } x,y \in X. \quad (21) \]

By Corollary 1.16 a homomorphism \( \kappa \) satisfying (21) exists iff condition (i) holds, so suppose this to be true in the sequel. By Proposition 1.17(b) \( \kappa \) factors through \( Q(\pi) \) iff \( \kappa Q(\delta) = \kappa w_{q_F}(\delta^{ab} \otimes 1 + 1 \otimes \delta) = 0 \). But \( \text{Im}(\delta^{ab}) = (\text{Im}(R \to F^{ab})) \) whence the second identity holds if for all \( (r,x) \in R \times X, \)

\[ \kappa w_{q_F}(\bar{r}, x) = \kappa w_{q_F}(\bar{x}, \bar{r}) \]

which by expanding \( \bar{r} \) is equivalent to condition (iii). Next consider \( \kappa Q(\delta) \). By Corollary 1.18 \( Q(F_1) \) is generated by \( q_{\delta} \delta(F \times R) + \text{Im}(w_{q_F}(\delta^{ab} \otimes \delta^{ak})) \). Let \( (a,r) \in F \times R \). Then \( q_{\delta}(a,r) = q_{\delta}(a)q_F(r) + w_{q_F}(a \otimes \bar{r} \otimes \bar{a}) \) by (10). Hence if \( \kappa \) satisfies conditions (iii) it annihilates \( \text{Im}(Q(\delta)) \) iff it annihilates \( q_{\delta}(R) \) which is equivalent to condition (ii) by (16).

\[ \square \]

\textbf{2. Relation with Passi’s Construction}

In this section we study relative polynomial maps in the sense of Passi by using the nonabelian theory of the first section as an essential tool. We will see that the proof of the main properties in the quadratic case becomes more natural in this way than in former approaches in the literature.
For a commutative ring $R$ with unit let $I_R(G)$ denote the augmentation ideal of the group algebra $R(G)$; for $k \geq 0$, $I_R^k(G)$ denotes its $k$-th power, with the convention $I_R^0(G) = R(G)$. If $R = \mathbb{Z}$ we also write $I(G) = I_\mathbb{Z}(G)$. For a function $f : G \to A$ to some abelian group $A$ let $\bar{f} : \mathbb{Z}(G) \to A$ denote the extension of $f$ to a $\mathbb{Z}$-linear homomorphism.

**Definition 2.1.** Let $G$ be a group and $B$ be a normal subgroup of $G$. We say that a function $f : G \to A$ as above is polynomial of degree $\leq n$ relative to $B$ if $\bar{f}$ annihilates the subset $1 + I(B)I(G) + I^{n+1}(G)$ of $\mathbb{Z}(G)$.

Moreover, we say that $f$ is (normalized) polynomial of degree $\leq n$ if it is polynomial of degree $\leq n$ relative to $\{1\}$.

**Remark 2.2.** The notion of (absolute) polynomial map from groups to abelian groups is due to Passi [30]. The relative case was only implicitly presented in most of the works in the literature based on this notion, notably when related to the dimension subgroup problem. See [32] for a thorough treatment of the subject.

Note that $f$ is polynomial of degree $\leq 0$ iff $f = 0$. Moreover, if $f$ is polynomial of degree $\leq n$, $f$ is also polynomial of degree $\leq n$ relative to $\gamma_n(G)$ as $I(\gamma_n(G)) \subset I^n(G)$; this is immediate by the inductive application of the formula

$$[a, b] - 1 = [a - 1 , b - 1]a^{-1}b^{-1}$$

(22)

for $a, b \in G$ where $[\cdot, \cdot]$ denotes the group commutator on the left and the ring commutator on the right.

The following inductive characterization of polynomial maps is useful for proving polynomiality, see [14] where a more general version is developed (with respect to an arbitrary $N$-series). For the reader’s convenience we give a direct proof of our special case here which is very short and easy anyway.

**Proposition 2.3.** Let $f : G \to A$ be any normalized function from a group $G$ to some abelian group $A$. Then we have the following properties.

1. For $a, b \in G$, $d_f(a, b) = \bar{f}((a - 1)(b - 1))$.
2. For $n \geq 1$, $f$ is polynomial of degree $\leq n$ relative to some given normal subgroup $B$ of $G$ if and only if the following two conditions hold.
   
   a. The map $d_f(a, -) : G \to A$ (or equivalently, $d_f(-, a) : G \to A$) is polynomial of degree $\leq n - 1$ for all $a \in G$.
   
   b. For all $(b, a) \in B \times G$, $f(ba) = f(b) + f(a)$ or equivalently, $d_f(b, a) = 0$.

**Proof.** (1) is immediate from expanding the right-hand term. To prove (2) let $a \in G$. Then for $b \in G$, $d_f(a, -(b - 1)) = d_f(a, b) - d_f(a, 1) = \bar{f}((a - 1)(b - 1))$ by (1). As the elements $b - 1, b \in G$, generate $I(G)$ as a $\mathbb{Z}$-module, it follows by linearity that $d_f(a, -(b - 1)) = \bar{f}((a - 1)x)$ for all $x \in I(G)$. Hence $d_f(a, -(I^n(G))) = \bar{f}((a - 1)I^n(G))$, which implies that property (a) is equivalent to $f$ being polynomial of degree $\leq n$. Moreover, (b) is equivalent to $\bar{f}((b -
1) \((a - 1)) = d_f(b, a) = 0\) for all \((b, a) \in B \times G\), which in turn means that \(f(I(B)I(G)) = 0\).

In low degrees, we obtain the following characterization of (relative) polynomial maps.

**Corollary 2.4.** Let \(f : G \to A\) and \(B\) as in 2.3.

1) \(f\) is polynomial of degree 1 relative to \(B\) iff it is linear.

2) \(f\) is polynomial of degree 2 relative to \(B\) iff \(d_f\) is bilinear and annihilates \(B\) in the first variable.

**Corollary 2.5.** Let \(G\) be a group, \(B\) be a central subgroup of \(G\) and \(f : G \to A\) as in 2.3. 2.3. Then \(f\) is polynomial of degree \(\leq 2\) relative to \(B\) if and only if \(f\) is quadratic relative to \(B\).

This is immediate from 2.4, just note that the centrality of \(B\) implies that \(d_f(a, b) = \bar{f}((a - 1)(b - 1)) = \bar{f}((b - 1)(a - 1)) = d_f(b, a)\) for \((a, b) \in G \times B\).

Before exploiting Corollary 2.5 we give some examples, the verification of which is based on 2.3 and 2.4.

**Examples 2.6.** Let \(G\) be a group.

1) If \(G\) is 3-step nilpotent, the commutator map \(G \times G \to G, (a, b) \mapsto [a, b]\), is bipolynomial of degree \(\leq 2\).

2) Recall that the non-abelian tensor square \(G \otimes G\) of \(G\) is a group closely related to the homotopy group \(\pi_3 \Sigma K(G, 1)\) and also to the second homology group \(H_2(G)\), see [8] and [9]. Now if \(G\) is 2-step nilpotent, the natural map \(G \times G \to G \otimes G, (a, b) \mapsto a \otimes b\), is bipolynomial of degree \(\leq 2\), see [16] where this fact is used to compute \(G \otimes G\) for 2-step nilpotent groups.

3) For \(G\) abelian the \(n\)-fold diagonal map \(\delta^n : G \to G^n, \delta^n(a) = a \otimes \cdots \otimes a\), is polynomial of degree \(\leq n\), see [14].

In order to introduce universal relative polynomial maps we recall the following definition and facts from [15].

**Definition 2.7.** Let \(G\) be a group, \(R\) as above, and \(B \triangleleft G\) a normal subgroup. We define the quotient \(R\)-algebra without unit

\[ P_{n,R}(G, B) = I_R(G)/\left( I_R(B)I_R(G) + I_R^{n+1}(G) \right) \]

The canonical quotient map from \(I_R(G)\) to \(P_{n,R}(G)\) or to \(P_{n,R}(G, B)\) is denoted by \(\rho\) or \(\rho_n\). If \(f : (G, B) \to (H, C)\) is a linear pair map in \(\text{NQP}\), it induces a morphism of \(R\)-algebras \(P_{n,R}(f) : P_{n,R}(G, B) \to P_{n,R}(H, C)\) defined by \(P_{n,R}(f)\rho(a - 1) = \rho(f(a) - 1)\) for \(a \in G\).

We point out that \(P_{n,R}(G) = P_{n,R}(G, \{1\})\) is the polynomial group constructed by Passi [30]; the relative version was introduced and studied modulo torsion in [15]. The aim of this section is to investigate in detail the structure of \(P_{2,\mathbb{Z}}(G, B)\) for central \(B\) which is the crucial ingredient of our abelian models for 2-step nilpotent groups in [13, 17].
Using the elementary identification
\[ I_R(G)/I_R(B)R(G) \xrightarrow{\cong} I_R(G/B), \quad \overline{a} - 1 \mapsto \overline{a} - 1 \quad \text{for} \ a \in G, \] (23)
we see that multiplication in the ring \( P_{n,R}(G, B) \) gives rise to an \( R \)-linear map
\[ \mu_n : P_{n-1,R}(G/B) \otimes_{R(G)} P_{n-1,R}(G, B) \to P_{n,R}(G, B) \] (24)
such that for \( x, y \in I_R(G) \), \( \mu_n(\rho_{n-1}(x) \otimes \rho_{n-1}(y)) = \rho_n(xy) \). This shows that via left multiplication, \( P_{n,R}(G, B) \) is a nilpotent left \( R(G/B) \)-module of class \( \leq n \); recall that a left \( R(G) \)-module \( B \) is called nilpotent of class \( \leq k \) if \( I^k_R(G) \cdot B = 0 \).

Now consider the map
\[ p_{n,R} : G \to P_{n,R}(G, B), \quad p_{n,R}(a) = \rho(a - 1). \]
In the case where \( R = \mathbb{Z} \) we omit the subscript \( R \). Recall that for a group homomorphism \( f : G \to H \) and an \( R(H) \)-module \( M \), an \( f \)-derivation from \( G \) to \( M \) is a map \( d : G \to M \) such that \( d(ab) = f(a)d(b) + d(a) \) for \( a, b \in G \).

**Proposition 2.8.** The maps \( p_{n,R} \) and \( p_n \) have the following universal properties:

(i) \( p_{n,R} : G \to P_{n,R}(G, B) \) is a universal \((G \to G/B)\)-derivation from \( G \) into nilpotent \( R(G/B) \)-modules of class \( \leq n \).

(ii) \( p_n : G \to P_n(G, B) \) is a universal polynomial map of degree \( \leq n \) relative to \( B \) from \( G \) into abelian groups.

**Proof.** (i) follows from the well known fact that the map \( G \to I_{n,R}(G), a \mapsto a - 1 \), is a universal derivation from \( G \) into arbitrary \( G \)-modules, see [22] VI.5. To prove (ii) we first show that \( p_n \) is polynomial of degree \( \leq n \) relative to \( B \). The linear extension \( \overline{p}_n : \mathbb{Z}(G) \to P_n(G, B) \) of \( p_n \) satisfies \( \overline{p}_n(a - 1) = \rho(a - 1) \) for \( a \in G \), so \( \overline{p}_n(x) = \rho(x) \) for all \( x \in I(G) \) by linearity. In particular, \( \overline{p}_n(1 + I(B))I(G) + I^{n+1}(G)) = \rho(I(B))I(G) + I^{n+1}(G)) = 0 \), whence the assertion. To prove the universal property, let \( f : G \to A \) be any polynomial map of degree \( \leq n \) relative to \( B \). Then \( \tilde{f} : \mathbb{Z}(G) \to A \) factors through a \( \mathbb{Z} \)-linear map, also denoted by \( \tilde{f} \), \( \mathbb{Z}(G)/(I(B))I(G) + I^{n+1}(G)) \to A \). Denoting the restriction of \( \tilde{f} \) to \( P_n(G, B) \), again by \( \tilde{f} \) we have \( \tilde{f}p_n = f \) as \( \tilde{f}(1) = 0 \). Furthermore, \( \tilde{f} \) is the unique \( \mathbb{Z} \)-linear map with this property since the elements \( p_n(a), a \in G \), generate \( P_n(G, B) \) as a \( \mathbb{Z} \)-module. \( \square \)

Property (i) implies a canonical isomorphism \( P_{n,R}(G, B) \cong R(G/B) \otimes_{\mathbb{Z}(G)} P_n(G) \) of left \( R(G/B) \)-modules.

Let \( f : G \to A \) be a polynomial map of degree \( \leq n \) relative to \( B \) and \( \tilde{f} : P_n(G, B) \to A \) the canonical induced \( \mathbb{Z} \)-linear map according to property (ii). Define the homomorphism
\[ w_f = \tilde{f}\mu_n : P_{n-1}(G/B) \otimes_{\mathbb{Z}(G)} P_{n-1}(G, B) \to A. \] (25)
Then by 2.3 (1) we have the following commutative diagram of factorizations induced by $f$:

$$
\begin{array}{ccc}
G & \xrightarrow{f} & A & \xrightarrow{d_f} & G \times G \\
\| & & \uparrow \phi & \searrow \psi & \| \\
G & \xrightarrow{p_n} & P_n(G,B) & \xrightarrow{\mu_n} & P_{n-1}(G/B) \otimes_{\mathbb{Z}(G)} P_{n-1}(G,B)
\end{array}
$$

(26)

Note that $w_{p_n} = \mu_n$.

Finally, suitable group homomorphisms induce ring homomorphisms on the constructions introduced above in the obvious way.

From (23) and the inclusion $I_R(\gamma_n(G)) \subset I_R^G(G)$ we deduce the natural isomorphisms of $R$-algebras

$$
P_{n,R}(G/\gamma_{n+1}(G), B\gamma_{n+1}(G)/\gamma_{n+1}(G)) \xrightarrow{\sim} P_{n,R}(G,B)
$$

$$
\xrightarrow{\sim} P_{n,R}(G,B\gamma_n(G)).
$$

(27)

Now let

$$
G: B \xrightarrow{i} G \xrightarrow{\pi} Q
$$

be a group extension with abelian kernel $B$. We will frequently identify $B$ with $i(B)$ and omit $i$ from the notation. It is an elementary fact that the sequence

$$
R \otimes_{\mathbb{Z}} B \xrightarrow{p_{n,R}i} P_{n,R}(G,B) \xrightarrow{P_{n,R}(\pi)} P_{n,R}(Q) \rightarrow 0
$$

(29)

is an exact sequence of $R(Q)$-linear homomorphisms, where the $Q$-action on $R \otimes_{\mathbb{Z}} B$ is given by the $R$-linear extension of the $Q$-action on $B$ induced by conjugation in $G$. Moreover, the map $p_{n,R}i$ here denotes the $R$-linear extension of the map $p_{n,R}i$ defined above.

Now consider the case where $B$ is central in $G$. Then $I_R(B)I_R(G) = I_R(G)I_R(B)$, whence the map $\mu_n$ in (24) factors through another $R$-linear map, also denoted by $\mu_n$,

$$
\mu_n: P_{n-1,R}(G/B) \otimes_{R(G/B)} P_{n-1,R}(G/B) \rightarrow P_{n,R}(G,B)
$$

(30)

such that $\mu_n(p_{n-1,R}(a) \otimes p_{n-1,R}(b)) = p_{n,R}(a)p_{n,R}(b)$, $a,b \in G$. Consequently, if $f: G \rightarrow A$ is polynomial map of degree $\leq n$ relative to $B$, then $w_f$ in (25) factors through another $\mathbb{Z}$-linear map, also denoted by $w_f$,

$$
w_f: P_{n-1}(G/B) \otimes_{\mathbb{Z}(G/B)} P_{n-1}(G/B) \rightarrow A
$$

(31)

such that $w_f(p_{n-1}(a) \otimes p_{n-1}(b)) = d_f(a,b)$, $a,b \in G$.

Now consider the case we are mainly interested in here, that is $n = 2$ and $B$ is central in $G$. Noting that $P_1(G/B)$ is a trivial $G/B$-module and using the canonical identifications $P_1(G/B) \cong (G/B)^{ab} \cong G/BG'$ we see that here $w_f$ is equivalent to the following linear map, also denoted by $w_f$,

$$
w_f: G/BG' \otimes G/BG' \xrightarrow{\sim} P_1(G/B) \otimes_{\mathbb{Z}(G/B)} P_1(G/B) \rightarrow A.
$$

(32)
This map satisfies \( w_f(\bar{a} \otimes \bar{b}) = d_f(a, b) \) for \( a, b \in G \), whence it coincides with the map \( w_f \) defined in Proposition 1.2(d) so that there is essentially no ambiguity in our notation.

Now we are ready for comparing the two universal constructions for quadratic and degree 2 polynomial maps, respectively.

**Proposition 2.9.** Let \( B \) be a central subgroup of a group \( G \). Then there is a natural isomorphism of abelian groups \( \alpha : Q(G, B)^{ab} \cong P_2(G, B) \) such that \( \alpha \cdot ab \cdot q = p_2 \) and \( \alpha \cdot ab \cdot w_q = w_{p_2} \).

**Proof.** By 2.5 the map \( p_2 : G \to P_2(G, B) \) is quadratic relative to \( B \) and the map \( ab \cdot q : G \to Q(G, B)^{ab} \) is polynomial of degree \( \leq 2 \). Hence \( \alpha \) and its inverse are induced by the universal properties of \( q \) and \( p_2 \), resp. So the equation \( \alpha \cdot ab \cdot q = p_2 \) holds by construction. Then the second one follows from the definition of \( w_q \) and from the identity \( w_{p_n} = \mu_n \).

Let \( G \) be a group and \( B \) a central subgroup of \( G \). Then we have the natural homomorphisms of abelian groups

\[
BG'/\gamma_3(G) \xrightarrow{c_2} G/BG' \wedge G/BG' \xrightarrow{l_2} G/BG' \otimes G/BG' \tag{33}
\]
defined by \( c_2(\bar{a} \wedge \bar{b}) = [a, b]_{\gamma_3}(G) \) and \( l_2(\bar{a} \wedge \bar{b}) = \bar{a} \otimes \bar{b} - \bar{b} \otimes \bar{a} \).

**Theorem 2.10.** Let \( G \) be a group and \( B \) a central subgroup of \( G \). Then the following natural sequences of abelian groups are exact:

\[
0 \to \text{Ker}(c_2) \xrightarrow{l_2} G/BG' \otimes G/BG' \xrightarrow{p_2} P_2(G, B) \xrightarrow{p_1} G^{ab} \to 1, \tag{34}
\]

\[
0 \to G/\text{BG'} \wedge G/\text{BG'} \xrightarrow{(c_2, l_2)^T} (G/\text{BG'} \otimes G/\text{BG'}) \xrightarrow{(p_2, p_1)} P_2(G, B) \xrightarrow{p_2} G/\text{BG'} \to 1, \tag{35}
\]

\[
0 \to B\gamma_3(G)/\gamma_3(G) \xrightarrow{p_2} P_2(G, B) \xrightarrow{P_2(\pi)} P_2(G/B) \to 0, \tag{36}
\]

where \( \rho_1 p_2(a) = [a, B'G] \) and \( \rho_2 p_2(a) = aB'G \) for \( a \in G \), \( \pi \) denotes the injection of \( B\gamma_3(G)/\gamma_3(G) \) or \( B\gamma_3(G)/\gamma_3(G) \) into \( G/\gamma_3(G) \), and \( \pi : (G, B) \to (G/B, \{1\}) \) is the natural projection.

**Remarks 2.11.** Taking \( B = \{1\} \) in (34) we rediscover the natural isomorphism \( T^2(G)/T^3(G) \cong U_2L(G) \) in [1]. Moreover, the centrality of \( B \) is a crucial hypothesis in Theorem 2.10 as in general (36) has to be replaced by the natural exact sequence

\[
\text{Tor}_1^G(\text{BG'}, \text{BG'}) \xrightarrow{[\cdot, \cdot]} B\gamma_3(G)/B'\gamma_3(G) \xrightarrow{p_2} P_2(G, B) \xrightarrow{P_2(\pi)} P_2(G/B) \to 0,
\]

where the map \([\cdot, \cdot]\) sends a typical generator \( \langle aB'G, k, bB'G \rangle \) with \( a, b \in G \), \( k \in \mathbb{Z} \) such that \( a^k, b^k \in B'G \) (see [29, V.6]) to the element \([a, b^k]B'\gamma_3(G)\) which is nontrivial in general, see Theorem 2.6 and Example 2.4 in [18].
Proof of Theorem 2.10. By the right-hand isomorphism in (27) we may assume that \( \gamma_3(G) = 1 \). Now recall the central extension

\[
0 \to G/BG' \otimes G/BG' \overset{w_q}{\to} Q(G, B) \overset{\tilde{id}}{\to} G \to 1
\]

from 1.15. Putting \( P = \tilde{id}^{-1} G' \) and writing \([-,-]\) for the respective commutator maps we have the following commutative diagram with exact rows.

\[
\begin{array}{ccc}
Q(G, B) \times Q(G, B) & \overset{\tilde{id} \times \tilde{id}}{\to} & G \times G \\
\downarrow [-,-] & \searrow c & \downarrow [-,-] \\
G/BG' \otimes G/BG' & \overset{w_q}{\to} & P \\
\| & \downarrow \text{inc} & \| \\
G/BG' \otimes G/BG' & \overset{w_q}{\to} & Q(G, B) \overset{\tilde{id}}{\to} G
\end{array}
\]

The factorization through \( c \) of the commutator map of \( Q(G, B) \) exists as \( \text{Im}(w_q) \) is central in \( Q(G, B) \); it satisfies

\[
c(a, b) = [q(a), q(b)] = q[a, b] - w_q(\bar{a} \otimes \bar{b} - \bar{b} \otimes \bar{a})
\]

by (9). As \([-, -]: G \times G \to G'\) is bilinear and \( q \) is linear on \( G' \) by Proposition 1.2(c), \( c \) is bilinear; it annihilates \( BG' \) in both variables since \( B \) is central and \( G' \) is abelian. Now \( P \) is abelian being a split central extension of the abelian group \( G' \) by construction of \( Q(G, B) \). Hence \( c \) gives rise to a homomorphism \( \bar{c}: G/BG' \wedge G/BG' \to P \) such that \( \bar{c}(\bar{a} \wedge \bar{b}) = [q(a), q(b)] \) and \( Q(G, B)' = (\text{Im}(c)) = \text{Im}(\bar{c}) \). Consider the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
G/BG' \otimes G/BG' & \overset{w_q}{\to} & P \\
\uparrow -l_2 & \uparrow \bar{c} & \| \\
\ker(c_2) & \overset{\text{inc}}{\to} & G/BG' \wedge G/BG' \overset{c_2}{\to} G'
\end{array}
\]

The right-hand square clearly commutes, and by (37), \( \bar{c} = gc_2 - w_q l_2 \), whence the left hand square commutes, too. It follows that \( Q(G, B)^{ab} = \text{coker}(\text{inc} \cdot \bar{c}) \) fits into the exact sequence

\[
0 \to (G/BG' \otimes G/BG')/l_2 \ker(c_2) \overset{ab \cdot w_q}{\to} Q(G, B)^{ab} \to G^{ab} \to 1
\]

which becomes sequence (34) under the identification \( Q(G, B)^{ab} \cong P_2(G, B) \), see 2.9. As to sequence (35), first note that it is exact in \( P_2(G, B) \) since the composite isomorphism \( \text{coker}(\mu_2) \cong I(G)/I^2(G) \cong G/G' \) takes \( p_2i(b) \) to \( bG' \) for...
b ∈ B. Furthermore, we have \((p_2i, \mu_2)(c_2, -l_2)^t = 0\) by the relation \(p_2ic_2 = \mu_2l_2\) which follows from the identity

\[
p_2([a, b]) = [p_2(a), p_2(b)] = p_2(a)p_2(b) - p_2(b)p_2(a)
\]

for \(a, b \in G\); this is immediate from equation (22). The map \((c_2, -l_2)^t\) is injective as \(l_2\) is; it remains to show that the map \((p_2i, \mu_2) : \Pi : = \text{coker}(c_2, -l_2)^t \to P_2(G, B)\) induced by \((p_2i, \mu_2)\) is injective. Consider the commutative diagram (39) below where \(\phi\) is the natural projection and \(i_1, i_2\) are the natural inclusions into \(BG' \oplus (G/BG' \otimes G/BG')\) followed by the natural projection to \(\Pi\).

\[
\begin{array}{ccccccccc}
G/BG' \wedge G/BG' & \xrightarrow{c_2} & BG' & \xrightarrow{\phi} & BG'/G' \\
\downarrow l_2 & & \downarrow i_1 & & \\
G/BG' \otimes G/BG' & \xrightarrow{i_2} & \Pi & \xrightarrow{(\phi, 0)} & BG'/G' \\
\downarrow & & \| & & \\
G/BG' \otimes G/BG' / l_2\text{Ker}(c_2) & \xrightarrow{\overline{c}_2} & \Pi & \xrightarrow{(\phi, 0)} & BG'/G' \\
\| & & \downarrow (p_2i, \mu_2) & & \uparrow \text{inc} \\
G/BG' \otimes G/BG' / l_2\text{Ker}(c_2) & \xrightarrow{p_2i} & P_2(G, B) & \xrightarrow{\rho_1} & G^{ab}
\end{array}
\]

The rows are exact: for the bottom row this follows from sequence (34), for the second and the third row from the fact that the upper left hand square is a cocartesian square of abelian groups (or by easy direct arguments). The injectivity of \((p_2i, \mu_2)\) is now immediate.

Finally, to prove the exactness of sequence (36) it suffices to check the injectivity of \(p_2i\), see (29). But in the above diagram \(i_1\) is injective as \(l_2\) is, so \(p_2i = (p_2i, \mu_2) \circ i_1\) is injective, too. □

References

1. F. Bachmann and L. Grünenfelder, Homological methods and the third dimension subgroup. *Comment. Math. Helv.* 47(1972), 526–531.
2. H. J. Baues and G. Wirsching, Cohomology of small categories. *J. Pure Appl. Algebra* 38(1985), No. 2–3, 187–211.
3. H. J. Baues, Homotopy type and homology. *Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York*, 1996.
4. H. J. Baues and M. Hartl, On the homotopy category of Moore spaces and the cohomology of the category of abelian groups. *Fund. Math.* 150(1996), No. 3, 265–289.
5. H. J. Baues, M. Hartl, and T. Pirashvili, Quadratic categories and square rings. *J. Pure Appl. Algebra* 122(1997), No. 1-2, 1–40.
6. H. J. Baues and T. Pirashvili, Quadratic endofunctors of the category of groups. *Adv. Math.* 141(1999), No. 1, 167–206.
7. L. Breen, Fonctions thêta et théorème du cube. *Lecture Notes in Mathematics*, 980. *Springer-Verlag, Berlin*, 1983.
8. R. Brown and J.-L. Loday, Van Kampen theorems for diagrams of spaces. With an appendix by M. Zisman. *Topology* **26**(1987), No. 3, 311–335.
9. G. J. Ellis, Nonabelian exterior products of groups and exact sequences in the homology of groups. *Glasgow Math. J.* **29**(1987), No. 1, 13–19.
10. H. Gaudier and M. Hartl, Quadratic maps between modules (submitted); arxiv:0809.0194.
11. F. Goichot and M. Hartl, Modules over square ringoids, in preparation.
12. M. Hartl, Quadratische Funktoren und Homotopie von Moore-Räumen. *Diplomarbeit, Rheinische Friedrich-Wilhelms-Universität Bonn*, 1985.
13. M. Hartl, Abelsche Modelle nilpotenter Gruppen. *Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn*, 1991.
14. M. Hartl, Some successive quotients of group ring filtrations induced by N-series. *Comm. Algebra* **23**(1995), No. 10, 3831–3853.
15. M. Hartl, Polynomiality properties of group extensions with a torsion-free abelian kernel. *J. Algebra* **179**(1996), No. 2, 380–415.
16. M. Hartl, The nonabelian tensor square and Schur multiplicator of nilpotent groups of class 2. *J. Algebra* **179**(1996), No. 2, 416–440.
17. M. Hartl, Structures polynomiales en théorie des groupes nilpotents. *Mémoire d’habilitation à diriger des recherches. Prépublication de l’Institut de Recherche Mathématique Avancée, Strasbourg*, 1998, 94 pp.
18. M. Hartl, The relative second Fox and third dimension subgroup. *Indian J. Pure Appl. Math.* **39**(2008) No. 5, 435–451.
19. M. Hartl, On Fox quotients of arbitrary group algebras. *Internat. J. Algebra Comput.* (to appear); arxiv:0707.0281.
20. M. Hartl, R. Mikhailov, and I. B. S. Passi, Dimension quotients. Centenary Volume *J. Indian Math. Soc.* (to appear); arXiv:0803.3290.
21. M. Hartl, Quadratic functors with values in semi-abelian categories (in preparation).
22. P. J. Hilton and U. Stammbach, A course in homological algebra. *Graduate Texts in Mathematics*, Vol. 4. *Springer-Verlag, New York–Berlin*, 1971.
23. M. Jibladze and T. Pirashvili, Quadratic envelope of the category of class two nilpotent groups. *Georgian Math. J.* **13**(2006), No. 4, 693–722.
24. Yu. V. Kuz’min, Dimension subgroups of extensions with an abelian kernel. (Russian) *Mat. Sb.* **187**(1996), No. 5, 65–70; English transl.: *Sb. Math.* **187**(1996), No. 5, 685–691.
25. M. Lazard, Sur les groupes nilpotents et les anneaux de Lie. *Ann. Sci. Ecole Norm. Sup. (3)* **71**(1954), 101–190.
26. A. Lehman, Polynomial mappings of groups. *Israel J. Math.* **129**(2002), 29–60.
27. A. Lehman, Polynomial sequences in groups. *J. Algebra* **201**(1998), No. 1, 189–206.
28. G. Losey, On the structure of $Q_2(G)$ for finitely generated groups. *Canad. J. Math.* **25**(1973), 353–359.
29. S. Mac Lane, Homology. *Die Grundlehren der mathematischen Wissenschaften*, Bd. 114. *Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin–Göttingen–Heidelberg*, 1963.
30. I. B. S. Passi, Polynomial maps on groups. *J. Algebra* **9**(1968), 121–151.
31. I. B. S. Passi, Polynomial functors. *Proc. Cambridge Philos. Soc.* **66**(1969), 505–512.
32. I. B. S. Passi, Group rings and their augmentation ideals. *Lecture Notes in Mathematics*, 715. Springer, Berlin, 1979.
33. O. Perriquet, Square ringoid. *Mémoire de DEA, Université de Lille* 1, 1999.
34. D. G. Quillen, On the associated graded ring of a group ring. *J. Algebra* 10(1968), 411–418.
35. R. Sandling, Dimension subgroups over arbitrary coefficient rings. *J. Algebra* 21(1972), 250–265.
36. R. Sandling, The dimension subgroup problem. *J. Algebra* 21(1972), 216–231.
37. G. W. Whitehead, Elements of homotopy theory. *Graduate Texts in Mathematics*, 61. Springer-Verlag, New York–Berlin, 1978.

(Received 1.07.2007)

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