Q-functions as models of physical reality

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We show that one can interpret physical reality using the Q-function, the expectation of a coherent state projection operator, giving a probabilistic interpretation of quantum dynamical evolution. This implies that a physical universe exists in space-time without observers, with a probability equal to its Q-function. By including the meter dynamics, we show that phase-space trajectories have measurement properties without explicit wave-function collapse. We treat continuous and discrete variable measurements, and prove that Bell inequality violations can occur for correlated spins.

The quantum divide between macroscopic and microscopic worlds has never been more important. Experimentalists now probe systems of ever larger size, yet more deeply quantum properties. It is commonplace for theoretical cosmologists to adopt the view of Coleman \cite{6,12} and others \cite{3}, that the entire universe is quantum mechanical. Such progress leaves the paradoxes of the Copenhagen interpretation of quantum mechanics and the Schrödinger cat \cite{1} in a central position in physics. Yet, to use the standard postulates of measurement theory \cite{5}, one must make a fundamental division between the microscopic and macroscopic. This is not accepted by those who see the universe as undivided \cite{6}, and is inconsistent with quantum cosmology.

Here we show that a well-defined ontological model of physical reality is obtainable using quantum phase-space. The probability of existence of a realization is given by the Q-function of quantum field theory. Every sample classical field in the stochastic evolution of the distribution is a possible outcome. The theory includes vacuum fluctuations and is frame invariant. This approach resolves paradoxes of quantum measurement \cite{1,7,8}. No separate observer and system is needed. The introduction of a model for the meter generates measurement predictions identical to the standard ones.

The Q-function probability is known to correspond physically to amplified measurements \cite{9}. We interpret the underlying fields directly as an ontological model of macroscopic reality \cite{10,11}. This can be applied to the physical universe as a whole. No explicit projection is needed to obtain sharply defined eigenvalues, although one must include the physics of measurement. This eliminates the observer-dependent wave-function collapse criticized by Wigner \cite{13} as being unsatisfactory. To demonstrate this, we will study measurement models of discrete and continuous measurements. We also prove the existence of Bell violations \cite{14} in this model of reality.

In our approach, we accept the fundamental principle that measurement is central to interpreting quantum theory. To measure a spin projection, one must include the orientation of the polarizer. Until one does this, the spin wave-function on its own is an incomplete description. Therefore, we take as fundamental that an interpretation of reality should take account of measurement. One can describe a system with or without a meter. However, the resulting behavior may change, as Bohr emphasized \cite{15}, since the meter is part of the universe.

Foundational problems in quantum mechanics have been the subject of much study, with recent reviews \cite{16,17}, and theorems claiming that the wave-function is ontological, i.e., a real object \cite{18,20}. In these works, reality is defined as what is prepared in a laboratory at one time, with causality proceeding from past to future. In view of the relativistic requirement of frame independence, we regard physical reality as existing at all space-time events, so that the most viable ontological candidates are therefore space-time fields. In our approach, it is the probabilistic trajectories of classical fields in space-time that are real objects. This permits the wave-function to have a statistical interpretation \cite{20,21}. Our wave-function interpretation is therefore not ontological in the sense of Pusey et al \cite{18,20}. These recent ontology theorems do not apply to our model \cite{20} because of backwards in time causation \cite{22,27} which occurs in Q-function dynamics through negative diffusion terms \cite{28}.

Previous models include de Broglie-Bohm and related theories with both wave-function and particle coordinates \cite{0,29}, discrete models used in quantum information \cite{30,31}, and a non-relativistic phase-space model with an epistemic restriction \cite{32}. By comparison, our model applies to quantum fields, is compatible with relativity, and includes particle statistics. This approach fulfills Einstein’s requirements \cite{33} that it is complete, formulated in terms of space/time fields, and objective, without needing observers. Other methods in the literature are less explicit: for example, the relative state or “many-worlds” theory \cite{34} depends on observers.

Related theories include consistent \cite{35} or decoherent histories \cite{36,38}, and quantum Darwinism \cite{39}. Instead of these methods, we generate a definite probability for obtaining one universe. This gives a well-defined procedure for early universe quantum models. The only requirement to understand measurement is the inclusion of a meter. Our use of a meter to understand measurement follows both Bohr and Bell \cite{15,40}, although without wave-function collapse or hidden variables. No ad-
ditional decoherence mechanism [41, 43] or nonlinearity [44] is necessary to achieve this.

Macroscopic physical reality is the marks in a notebook, or the state of a memory [40]. Since this involves measurement, it corresponds to real experiments. To obtain measured results, the results of microscopic experiments are amplified to macroscopic levels, which must be taken into account physically. Because of this, we will show that our proposal is not ruled out by no-go theorems [45, 46] for phase-space models, that state that such models cannot be realistic theories. However, these theorems ignore the fact that a meter must be included in a realistic theory of measurement. Omitting this leads to large vacuum fluctuations if one assumes a standard measurement hypothesis. We will demonstrate that the inclusion of a meter eliminates such problems, and in this sense our model of reality is contextual [47].

For bosons, a Q-function [48] \( Q(\phi) \) is the simultaneous probability of measurement of two complementary field quadratures. We propose that the complex vector field \( \phi \) is an element of reality. Conceptually, this is operationally measurable either using a beam splitter to divide a field into two halves [49, 50], or via amplification to a macroscopic level [51]. One can measure momentum and position at the same time, albeit with quantum fluctuations. The representation is for anti-normally ordered operators, and it includes vacuum fluctuations, which are small for macroscopic observations, as we will demonstrate. Since the physical universe comprises fermions and bosons, one must include a fermionic projector as well, which gives a complete resolution of unity [52, 53].

Hence, we define the probability:

\[
Q(\lambda) = Tr \left[ \hat{\rho} \hat{\Lambda}(\lambda) \right].
\]

Here \( \lambda = [\phi, \xi] \) where \( \phi = [\phi_1, \phi_2, \ldots] \) are distinct classical fields representing bosons, \( \xi = [\xi_1, \xi_2, \ldots] \) are real antisymmetric matrices representing fermions, and \( \hat{\Lambda}(\lambda) = \prod_b A_b(\phi_b) \Lambda_f(\xi_f) \) is a Gaussian [54] operator for bosonic \( (b) \) and fermionic \( (f) \) fields.

The projector \( \hat{\Lambda} \) is normalized such that \( \int \hat{\Lambda}(\lambda) d\lambda = 1 \), which means that \( \int Q(\lambda) d\lambda = 1 \). We show elsewhere that quantum dynamics of such Q-functions corresponds to an action principle that has both past and future boundary conditions. The corresponding trajectory dynamics is equivalent to quantum mechanics, so it is consistent with known physical observations. Because the distribution is positive and normalized, it also satisfies classical probability axioms at all times, which any statistical theory of macroscopic reality must satisfy.

The Q-function is the probability at one time, but the fields have relativistically invariant trajectories \( \lambda(t) \) defined at all times in all frames. Such Q function distributions include vacuum fluctuations. We must regard these as real events since the fields are ontological. From the Heisenberg uncertainty principle, the Q-function is not infinitely sharp, leading to an epistemic restriction on \( Q(\lambda) \). Yet measurements of sharp eigenvalues exist, and are fundamental to quantum measurement theory.

To understand this, we will treat the theory of measurement. For simplicity, we expand the quantum fields in mode operators. We first consider a model for the measurement of a continuous variable, the \( \hat{X} \) quadrature of the radiation field, which also can be used for particle position measurements [55]. We let \( \hat{X} = \hat{a} + \hat{a}^\dagger \) and \( \hat{Y} = (\hat{a} - \hat{a}^\dagger) / i \) be the field quadratures of mode \( \hat{a} \), with corresponding Q function amplitudes \( \alpha = (X + iY) / 2 \).

The Hamiltonian of our meter is a parametric amplifier:

\[
\hat{H} = \frac{i\hbar g}{2} [\hat{a}^{12} - \hat{a}^{22}] .
\]

We can either solve the the Q-function Fokker-Planck equations, or equivalently use the Heisenberg equations, which have operator solutions with \( \hat{X}(t) = \hat{X}(0) e^{gt} \) and \( \hat{Y}(t) = \hat{Y}(0) e^{gt} \). Given an initial vacuum state, in which \( \langle \Delta \hat{X}^2(0) \rangle = \langle \Delta \hat{Y}^2(0) \rangle = 1 \), the \( \hat{Y} \) quadrature is squeezed, with a variance below the vacuum level, and the \( \hat{X} \) quadrature develops a large variance.

Suppose the quantum system is prepared with a superposition of eigenstates, \( X_0 \), of the \( \hat{X} \) quadrature, with variance \( \langle \Delta \hat{X}^2(0) \rangle \). The parametric amplifier then amplifies the quadrature to a macroscopic level. After measurement, if the gain is \( G = e^{gt} \), the resulting variances in phase-space are:

\[
\langle \Delta X^2 \rangle = 1 + G^2 \langle \Delta \hat{X}^2(0) \rangle, \quad \langle \Delta Y^2 \rangle = 1 + \langle \Delta \hat{Y}^2(0) \rangle / G^2.
\]

This shows that the initial vacuum noise contribution to the value of the ontological variable \( X(t) \) is relatively large. Yet experimentalists can identify a reproducible
eigenvalue $X_0$ after the measurement. This is possible because the final, measured variance in $X$ - regarded as a real event - is dominated by $\langle \Delta \hat{X}^2 (0) \rangle$. After measurement, the phase-space variable $X$ is the result of measuring the $\hat{X}$ quadrature, including an amplification factor due to gain, together with a small vacuum noise term. The Q-function equation for $X$ has a negative diffusion with time-reversed causation, which is why vacuum fluctuations are relatively small after measurement.

In greater detail, the resulting Q-function coordinate is $X = GX_0 + \epsilon$, where $\epsilon$ is the vacuum noise with $\langle \epsilon^2 \rangle = 1$. From the amplified macroscopic value $X$, the experimentalist infers an eigenvalue of $\hat{X} = X_0 + \epsilon/G$, with a probability distribution of $P(\hat{X}) = \left(G/\sqrt{2\pi}\right) \exp \left(-G^2 \left(\hat{X} - X_0 \right)/2 \right)$, as shown in Fig 1. A sharp eigenvalue $X_0$ is recovered from the measured data in the limit of an ideal, infinite gain meter, as $g \to \infty$, with no further assumptions.

We next consider measurement of a qubit. Suppose the qubit is in a superposition state, $|\psi\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. This is equivalent to a spin 1/2 state, with a Pauli spin operator $\hat{\sigma}_z = |\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|$. The outcome of measuring $\hat{\sigma}_z$ is either 1 or -1. To describe this using a Q-function in phase-space, we consider the measurement for $\hat{\sigma}_z$ with a commonly used experimental Hamiltonian $[56–58]$:

$$H_M = \hbar g \hat{\sigma} \hat{n}.$$ (4)

The measurement is performed by coupling the qubit to an optical field. The field is a single mode with boson operator $a$ and number operator $\hat{\hat{n}} = a^\dagger a$. The optical “meter” field is prepared in a coherent state $|G\rangle_c$ and coupled for a time $\tau$. Here $G$ is the gain of the meter. This approach is even applicable in the limit of $G \to 0$, where it yields the large fluctuations found in weak measurement theory $[59]$. With a input superposition incident on the measurement device, the final state after a measurement time of $\tau = \pi/2g$ is:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|\uparrow\rangle |iG\rangle_c + |\downarrow\rangle |-iG\rangle_c\right].$$ (5)

This describes the final state of the two-mode system and the final state of the meter field. A measurement is then made of the quadrature $\hat{Y}$ of the meter field, regarded as an element of reality. For $G$ large, the two different values $\pm 1$ for $\hat{\sigma}_z$ are measurable by the different sign of the outcomes for $\hat{Y}$.

We therefore define an inferred spin

$$\hat{\sigma} = \frac{1}{2G} \hat{Y}.$$ (6)

The resulting probability distributions, as they develop in time, are shown in Fig 2 for different gains $G$. This plots $P(\hat{\sigma})$, the probability of an inferred value of $\hat{\sigma}$ obtained from the Q-function for $\hat{Y}$, after integrating over the transverse coordinate $X$:

$$P(\hat{\sigma}) = \frac{G}{2\sqrt{\pi}} \left[ e^{-G|\hat{\sigma} - 1|^2} + e^{-G|\hat{\sigma} + 1|^2} \right].$$ (7)

We have shown that a Q-function phase-space coordinate has a distribution that becomes relatively sharper after a high-gain measurement. This allows an observer to determine which set of trajectories includes the objectively real one, and hence to calculate with a reduced phase-space ensemble. This corresponds to an “epistemic”, or information based, projection of $\hat{\rho}$.

The strongest objections to realistic interpretations are through correlated measurements that violate Bell inequalities. We now consider correlated, spatially separated spins in the Bell state $[14]$. $|\psi\rangle = (|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B)/\sqrt{2}$. In the experiment, four different types of correlated measurements are made for spins $\sigma^\alpha_A$ and $\sigma^\beta_B$ in the $\theta$ and $\phi$ directions, and the results compared. The Clauser-Horne-Shimony-Holt Bell inequality is that, for a hidden variable model of quantum mechanics, given four different correlations $E(\theta_i, \phi_j) = \langle \sigma^A_i \sigma^B_j \rangle$ of spin, one must have $[60, 62]$:

$$B = E(\theta_1, \phi_1) - E(\theta_1, \phi_2) + E(\theta_2, \phi_2) + E(\theta_2, \phi_1) \leq 2.$$ (8)

With the view that the phase space coordinates $\alpha$ are elements of reality, we can use the Q-function for a calculation of these four correlations. Each corresponds to a different measurement Hamiltonian, namely:

$$H \left( \theta, \phi \right) = \hbar g \left[ \sigma^\alpha \hat{n}_c + \sigma^\beta \hat{n}_d \right].$$ (9)

Given this choice, we can calculate the four correlations from the inferred spins through a process of binning the phase space coordinate. We find that the Bell violation is a function of the gain, and is simply given by $B =$
The inferred Bell violation is obtained for gains \( G \gtrsim 1 \).

2\(\sqrt{2}\eta^2 (G) \), where \( \eta (G) = \frac{1}{2} (erf(G) + 1 - erf(-G)) \) is the binning efficiency which is reduced at low gain. The resulting Bell violation is plotted in Fig.3. It is clear that a Bell violation is obtained for high enough efficiency, and indeed \( G > 1 \) is already enough to observe this.

To conclude, we summarize possible objections to this ontological interpretation of quantum mechanics, and their resolution, as follows:

**Quantum fluctuations:** Since Q-functions are statistical, including vacuum fluctuations, how can they represent eigenstates of measurements? In a full representation of physical reality, the measuring device should be included. Three examples are treated above, and in all cases the vacuum fluctuations are suppressed relative to the measurement outcome, by a factor equal to the overall gain.

**The Einstein-Podolsky-Rosen paradox:** Given the EPR argument [12, 63], surely “elements of reality” must be a type of hidden variable? This can be treated through an analysis of the parametric interaction [64] used to demonstrate the EPR argument. The dynamics of the Q-function requires diffusive propagation in a backwards time direction. Correlations permitted in backward time propagation are inconsistent with the definitions of EPR’s “elements of reality”, but they are allowed in Q-function dynamics, thus providing a specific implementation of Bohr’s response to EPR [15].

**Schrödinger’s cat paradox:** How does this resolve Schrödinger’s question [4], that a state such as (5) after the measurement suggests a macroscopic object “in two places \( X_1 \) and \( X_2 \) at once, like a cat simultaneously dead and alive”? The Q function evolves dynamically as the system interacts with the measurement device. The projection along \( X \) at any given time is plotted in Fig 2. Once \( X_1 \) and \( X_2 \) are macroscopically separated beyond the level of vacuum fluctuations, this gives a definite result compatible with quantum predictions, and consistent with the pointer being either at \( X_1 \) or \( X_2 \), as discussed in [12, 65]. This does not exclude interference fringes, but that would require a different type of measurement.

**Bell’s theorem and causality:** Surely Bell’s theorem [14] proves that if one attempts to complete quantum mechanics with local hidden variables, there will be a contradiction with quantum experiments, and with even stronger multipartite correlations [66, 71]? We have shown that Bell violations are obtained using phase-space variables as elements of reality, if one includes the gain of the measurement. It is already known from electrodynamical absorber theory [23, 24, 25], that future time boundary conditions can cause violation of Bell inequalities. A similar backward time propagation occurs in Q-function dynamics [72]. One can have relativistic locality, with no instantaneous information transfer [73], while violating the Bell inequality. In the multipartite case, up to 60 simultaneous spatially separated measurements with genuine entanglement have been simulated using Q-function methods [74]. Due to vacuum fluctuations, a model of the measurement is needed for “all-or-nothing” effects, as explained above.

**Particle statistics:** Husimi Q-functions are defined for bosons, yet doesn’t the physical world include fermions? Research on fermionic Gaussian operators [75] shows that fermionic Q-functions exist [53, 76]. These have a real and positive distribution. This complete phase space is based on the bounded homogeneous spaces of group theory [77, 78]. Any fermionic operator expectation value and its dynamics [79] can be computed, in a similar way to the Husimi Q-function.

**Uniqueness:** Why should coherent state projectors represent macroscopic realism, as opposed to other measurement operators? Ontological models should not depend on how measurements are implemented. Coherent state projectors provide a minimal, unbiased implementation of the group symmetry of the field commutators in the standard model. Their role was recognized by Schrödinger [80] and others [51] who proved that they have classical behavior at the macroscopic level.

The reason why this model is successful is that coherent states are complete and provide a unique set “addresses” in the quantum world, with positive probability. There is also an exact mapping from unitary quantum field dynamics to Q-function dynamics, which is readily obtained from the operator identities. This uses the fact that quantum field Hamiltonians are at most quartic in the quantum fields. Quantum dynamics can therefore be re-expressed as differential equations that have a time-symmetric action principle and path integral.

In greater detail, the Q-function dynamical equations for unitary evolution have a traceless diffusion, with equal positive and negative diffusion terms. This leads to both causal and retro-causal effects from boundary conditions in the future. Related phenomena in electrodynamics were studied by Tetrode, Wheeler and Feynman [22, 24]. Negative diffusion terms occur in parametric amplifiers.
These are directly responsible for the reduction in vacuum noise and sharp measurement results given above.

As a result, Bell’s arguments about hidden variable theories do not apply to Q-functions. This model has a different type of time-evolution to any hidden variable theory. As we have shown, it is quite possible to have discrete spins and Bell violations. Naturally, one can study more complicated effects, and include decoherence if necessary. However, this is not essential to the development of measurement results. The only required ingredient is the meter itself, as indeed one might expect physically.

We have shown that a realistic ontological model for quantum mechanics is obtainable by utilizing a phase-space of classical fields. The Q-function gives the probability of a given field configuration as a generalized phase-space coordinate. Vacuum fluctuations are present as well. We show that these fluctuations are suppressed in a measurement with gain. After the gain is included, sharp eigenvalues are obtained. Thus, the fields can represent an objective physical reality, without an explicit collapse on measurement. In summary, this Letter demonstrates the existence of an objective, relativistically invariant ontological model for quantum mechanics.

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