Topological Properties of Phase Singularities in Wave Fields

Yi-Shi Duan, Ji-Rong Ren, and Tao Zhu

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, P. R. China

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Phase singularities as topological objects of wave fields appear in a variety of physical, chemical, and biological scenarios. In this paper, by making use of the ϕ-mapping topological current theory, we study the topological properties of the phase singularities in two and three dimensional space in details. The topological inner structure of the phase singularities are obtained, and the topological charge of the phase singularities are expressed by the topological numbers: Hopf indices and Brouwer degrees. Furthermore, the topological invariant of the closed and knotted phase singularities in three dimensional space are also discussed in details.

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I. INTRODUCTION

In physics, the solutions of wave equations in two and three dimensional space often possess phase singularities[1, 2]. At the point of phase singularity, the phase of the wave is undefined and wave intensity vanishes. This phase singularity is a common property to all waves, it exists in many area of physical, chemical, and biological scenarios, such as the quantized vortices in superfluid or superconductor systems[3, 4], the vortices phase singularities in Bose-Einstein condensates[5], the streamlines or singularities in quantum mechanical wavefunctions[6, 7], the optical vortices in optical wave systems[8, 9, 10, 11, 12, 13], the vortex filaments in chemical reaction and molecular diffusion[13, 14, 15, 16, 17], and the phase singularities in biological systems[17, 18]. In recent years, the phase singularities have drawn great interest because it is of importance for understanding fundamental wave physics and have many important applications, and a great deal of works on the phase singularities in wave fields have been done by many physicists[1, 2, 12, 13].

Phase singularities are generically points in two dimensional space and form a network of lines in three dimensional space. Around the phase singularity points in two dimensions, this topological current theory[23], we study the topological current of the phase singularity objects (points in two dimensions and lines in three dimensions), and the topological inner structure of these topological objects are obtained. For the case that the phase singularity lines in three dimensional space are closed and knotted curves, we discussed the topological invariant of these knotted family in details. This paper is arranged as follows. In Sec.II, we construct a topological current density of the phase singularity points in two dimensions, this topological current density don’t vanish only when the phase singularity points exist, the topological charge of phase singularity points are expressed by the topological quantum numbers, the Hopf indices and Brouwer degrees of the ϕ-mapping. In Sec.III is the topology of the phase singularity lines in three dimensions. In Sec.IV, we introduce a important topological invariant to describe the phase singularity lines when they are linked and knotted, it is just the sum of all the self-linking and all the linking numbers of the knot family. In Sec.V is our concluding remarks.

II. TOPOLOGICAL STRUCTURE OF PHASE SINGULARITIES IN TWO DIMENSIONS

Let us study a complex scalar field ψ(⃗x, t), the wave fields function ψ(⃗x, t) is maps from space (of either two or three dimensions) to the complex numbers, so ψ : R², R³ → C. The complex scalar field ψ(⃗x, t) is time dependent as well as space dependent, and it can be written as

\[ ψ(⃗x, t) = ||ψ|| e^{iθ} = ϕ^1(⃗x, t) + iϕ^2(⃗x, t), \]  

where ϕ^1(⃗x, t) and ϕ^2(⃗x, t) can be regarded as complex representation of a two-dimensional vector field ⃗φ = (ϕ^1, ϕ^2) over two or three dimensional space, ||ψ|| = \sqrt{ψ*ψ} is the modulus of ψ, and θ is the phase factor. In this section, we will restrict our attention to the phase

*Corresponding author. Email : zhut05@lzu.cn
singularities in two dimensional space, and labeled points \( \vec{x} = (x^1, x^2) \) in cartesian coordinates. The phase singularities in three dimensional space will be discussed in Sec.III and Sec.IV.

It is known that the current density \( \vec{J} \) associated with \( \psi(\vec{x}, t) \) is defined as

\[
\vec{J} = \text{Im}(\psi^* \nabla \psi) = \|\psi\|^2 \vec{v},
\]

(2)

the \( \vec{v} \) is the velocity field. From the expressions in Eq.(1) and Eq.(2), the velocity field \( \vec{v} \) can be rewritten as

\[
\vec{v} = \frac{1}{2\pi} \frac{1}{\|\psi\|^2} (\psi^* \nabla \psi - \nabla \psi^*) = \nabla \theta,
\]

(3)

it becomes the gradient of the phase factor \( \theta \). The vorticity field \( \vec{\omega} \) of the velocity field \( \vec{v} \) is defined as

\[
\vec{\omega} = \frac{1}{2\pi} \nabla \times \vec{v}.
\]

(4)

Obviously, in two dimensional space the \( \vec{\omega} \) only has \( z \)-component, i.e., \( \vec{\omega} = \omega \cdot \vec{e}_z \). Form Eq.(4), we directly obtain a trivial curl-free result: \( \vec{\omega} = \frac{1}{2\pi} \nabla \times \nabla \theta = 0 \). But in topology, because of the existence of phase singularities in the wave fields \( \psi \), the vorticity \( \vec{\omega} \) does not vanish.4 So in the following discussions, we will study that what the exact expression for \( \vec{\omega} \) is in topology.

Introducing the unit vector \( n^a = \phi^a/\|\phi\| (a = 1, 2; n^a n^a = 1) \), and defining a topological current in \((2+1)\) dimensions space-time,

\[
j^i = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b, \quad i = 0, 1, 2.
\]

(5)

Obviously, the topological current is identically conserved,

\[
\partial_i j^i = 0.
\]

(6)

By making use of the definition of the unit vector \( n^a \), one can reexpress the velocity field \( \vec{v} \) as

\[
\vec{v} = \epsilon_{ab} n^a \nabla n^b,
\]

(7)

and the vorticity \( \vec{\omega} \) is

\[
\omega = j^0 = \frac{1}{2\pi} \epsilon^{jk} \epsilon_{ab} \partial_j n^a \partial_k n^b \quad j, k = 1, 2,
\]

(8)

it is just the time component of the topological current \( j^i \). According to the \( \phi \)-mapping topological current theory23, one can prove that

\[
j^i = \delta^2(\vec{\phi}) D^i(\vec{\phi}/x),
\]

(9)

where

\[
D^i(\vec{\phi}/x) = \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} \partial_j \phi^a \partial_k \phi^b
\]

(10)

is the Jacobian vector, in which the time component \( D^0(\phi/x) \) is the usual two dimensional Jacobian \( D(\phi/x) \).

The expression of the topological current in Eq.(9) shows that the topological current \( j^i \) does not vanish only at the zero points of \( \vec{\phi} \), i.e., the phase singularities of the wave fields \( \psi \). According to the Eq.(8) and Eq.(9), the vorticity can be expressed

\[
\omega = j^0 = \delta^2(\vec{\phi}) D(\vec{\phi}/x),
\]

(11)

obviously, \( \omega \) also does not vanish only when the phase singularities of the wave fields \( \psi \) exist. The topological current \( j^i \) and vorticity \( \omega \) play an important role in the topological property of the phase singularities, so it is necessary to study the zero points of \( \phi \) to determine the non-zero solutions of \( j^i \).

According to the implicit function theory24, while the regular condition

\[
D^i(\phi/x) \neq 0
\]

is satisfied, the general solutions of

\[
\phi^1(x^1, x^2, t) = 0, \quad \phi^2(x^1, x^2, t) = 0,
\]

(12)

can be expressed as

\[
x^1 = x^1_k(s, t), \quad x^2 = x^2_k(s, t), \quad k = 1, 2, \ldots, N,
\]

(13)

which represent \( N \) zero points \( \vec{z}_k \) (\( k=1, 2, \ldots, N \)) where \( \psi(\vec{r}, t) = 0 \) in two-dimensional space. These zero points solutions are just the so-called phase singularity points of the wave fields \( \psi \), they represent the zeroes of the real and imaginary part of the complex scalar wave fields:

\[
\Re(\psi) = \phi^1 = 0 \quad \text{and} \quad \Im(\psi) = \phi^2 = 0.
\]

In the theory of \( \delta \) function of \( \vec{\phi}(\vec{r}) \), one can prove that in 2-dimensional space25

\[
\delta^2(\vec{\phi}) = \sum_{k=1}^{N} \frac{\delta_k}{|D(\phi/x)\vec{z}_k|} \delta^2(\vec{x} - \vec{z}_k),
\]

(14)

where the positive integer \( \beta_k \) is called the Hopf index of map \( x \rightarrow \phi \).12,13 The meaning of \( \beta_k \) is that when the point \( \vec{z}_k \) covers the neighborhood \( \Sigma_k \) of zero \( \vec{z}_k \) once, the vector field \( \vec{\phi} \) covers the corresponding region \( \beta_k \) times.

By substituting Eq.(13) into Eq.(9), one can obtain that

\[
j^i = \sum_{k=1}^{N} W_k \delta^2(\vec{x} - \vec{z}_k) \frac{dx^i}{dt},
\]

(15)

where \( W_k = \beta_k \eta_k \) is the winding number of \( \vec{\phi} \) around zero point \( \vec{z}_k \), and \( \eta_k = \text{sgn}(D(\phi/x)) = \pm 1 \) is the Brouwer degrees of \( \phi \)-mapping. The sign of Brouwer degrees are very important, for the case of vortices phase singularities, the \( \eta_k = +1 \) corresponds to the vortex, and \( \eta_k = -1 \) corresponds to the antivortex.

The topological charge (the strength of phase singularities) of the phase singularities point \( \vec{z}_k \) is defined by the Gauss map \( \vec{n} : \partial \Sigma_k \rightarrow S^1 \):26

\[
W_k = \frac{1}{2\pi} \int_{\partial \Sigma_k} n^a (\epsilon_{ab} n^b \delta a/n^b)
\]

(16)
where \( n^* \) is the pullback of Gauss map \( n \), \( \partial \Sigma_k \) is the boundary of a neighborhood \( \Sigma_k \) of point \( z_k \) and \( \Sigma_k \cap \Sigma_m = \emptyset \) for \( \Sigma_m \) is the neighborhood of another arbitrary zero point \( z_m \). In topology it means that, when the point \( z_k \) covers \( \partial \Sigma_k \) once, the unit vector \( \vec{n} \) will cover \( S^1 \), or \( \phi \) covers the corresponding region \( W_k \) times, which is a topological invariant. By using the Stokes’ theorem and in term of Eq. (8), one can obtain that

\[
W_k = \frac{1}{2\pi} \int_{\partial \Sigma_k} \epsilon_{ab} \epsilon_{ij} \partial_j n^a \partial_k n^b \partial^2 x
\]

This expression tells us that the topological charges of the phase singularity points of which the zero point \( \Sigma_m \) labeled the above section in two dimensions space are also useful for \( \Sigma_k \) in topology. It is clear that the topological charges densities of phase singularity points is just the non-vanishing vorticity \( \omega \) of the velocity field \( \vec{v} \). Using the notation \( j = (\omega, \vec{j}) \), Eq. (6) can be rewrite as

\[
\frac{\partial \omega}{\partial t} + \nabla \cdot \vec{j} = 0, \tag{18}
\]

In this section, we will study the phase singularities in three dimensions space, some results which discussed in two dimensions space are also useful here. Differently with in two dimensions, the point is labeled \( \vec{x} = (x^1, x^2, x^3) \) and the vorticity \( \vec{\omega} \) have its all three components. In three dimensional space, the \( \vec{\omega} \) can be reexpressed as

\[
\omega^i = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b, \tag{19}
\]

it is clearly that the vorticity \( \vec{\omega} \) is just the topological current of phase singularities in three dimensions space. Using the \( \phi \)-mapping theory [23], the topological current \( \vec{\omega} \) is rewritten as

\[
\omega^i = \delta^2(\vec{\phi}) D^i(\vec{\phi}/x), \tag{20}
\]

we can see from this expression that the vorticity \( \vec{\omega} \) is non-vanishing only if \( \vec{\phi} = 0 \), i.e., the existence of the phase singularities, so it is necessary to study these zero solutions of \( \vec{\phi} \). In three dimensions space, these solutions are some isolet zero lines, which are the so-called phase singularity lines in three dimensions space.

Under the regular condition

\[
D^i(\phi/x) \neq 0,
\]

the general solutions of

\[
\phi^1(x^1, x^2, x^3, t) = 0, \quad \phi^2(x^1, x^2, x^3, t) = 0 \tag{21}
\]

can be expressed as

\[
x^1 = x^1_k(s, t), \quad x^2 = x^2_k(s, t), \quad x^3 = x^3_k(s, t), \tag{22}
\]

which represent the world surfaces of \( N \) moving isolated singular strings \( L_k \) with string parameter \( s (k = 1, 2, \cdots, N) \). These singular strings solutions are just the phase singularities solutions in three dimensions space.

In \( \delta \)-function theory [23], one can obtain in three dimensions space

\[
\delta^2(\vec{\phi}) = \sum_{k=1}^N \beta_k \int_{L_k} \frac{\delta^3(\vec{x} - \vec{x}_k)}{|D(\vec{\phi}/x)|} ds, \tag{23}
\]

where

\[
D(\vec{\phi}/x)|_{\Sigma_k} = \frac{1}{2} \epsilon^{ijk} \epsilon_{mn} \frac{\partial \phi^m}{\partial u^j} \frac{\partial \phi^n}{\partial u^k},
\]

and \( \Sigma_k \) is the \( k \)th planar element transverse to \( L_k \) with local coordinates \( (u^1, u^2) \). The \( \beta_k \) is the Hopf index of \( \phi \), which means that when \( \vec{x} \) covers the neighborhood of the zero point \( \vec{x}_k(s) \) once, the vector field \( \vec{\phi} \) covers the corresponding region in \( \phi \) space \( \beta_k \) times. Meanwhile the direction vector of \( L_k \) is given by

\[
\frac{dx^i}{ds}|_{x_k} = \frac{D^i(\phi/x)}{D(\phi/x)|_{x_k}}.
\tag{24}
\]

Then from Eq. (20) and Eq. (21) one can obtain the inner structure of \( \omega^i \):

\[
\omega^i = \sum_{k=1}^N W_k \int_{L_k} \frac{dx^i}{ds} \delta^3(\vec{x} - \vec{x}_k(s)) ds, \tag{25}
\]

where \( W_k = \beta_k \eta_k \) is the winding number of \( \vec{\phi} \) around \( L_k \), with \( \eta_k = \text{sgn} D(\phi/u)|_{x_k} = \pm 1 \) being the Brouwer degree of \( \phi \) mapping. The sign of Brouwer degrees are very important, for the case of vortices phase singularities, the \( \eta_k = +1 \) corresponds to the vortex, and \( \eta_k = -1 \) corresponds to the antivortex. Hence the topological charge of the phase singularity line \( L_k \) is [21]

\[
Q_k = \int_{\Sigma_k} \omega^i d\sigma_i = W_k. \tag{26}
\]

The results in this section show us the topological inner structure of the topological current density \( \omega^i \). The topological charge of the \( k \)th phase singularity line in three dimensions can be expressed by the topological numbers: \( Q_k = W_k = \beta_k \eta_k \).
IV. TOPOLOGICAL ASPECT ON KNOTTED PHASE SINGULARITY LINES

An important case is that the phase singularity lines in three dimensions space from closed and knotted curves. Topology has a very important role in understanding these knot configurations, so it is necessary to study the topology in the knotted phase singularity lines. In order to do that, we define a the helicity integral \[ H = \frac{1}{4\pi^2} \int \vec{v} \cdot \nabla \times \vec{v} d^3x, \] (27)

this is an important topological knot invariant and it measures the linking of the phase singularity lines. From the Eq. (4), the helicity integral can be changed as

\[ H = \frac{1}{2\pi} \int \vec{v} \cdot \mathcal{A} d^3x. \] (28)

Substituting Eq. (24) into Eq. (28), one can obtain

\[ H = \frac{1}{2\pi} \sum_{k=1}^{N} W_k \int_{\xi_k} \vec{v} \cdot d\vec{x}, \] (29)

where these phase singularities are closed and knotted lines, i.e., a family of knots \( \xi_k (k = 1, 2, \ldots, N) \), Eq. (29) becomes

\[ H = \frac{1}{2\pi} \sum_{k=1}^{N} W_k \oint_{\xi_k} \vec{v} \cdot d\vec{x}. \] (30)

It is well known that many important topological numbers are related to a knot family such as the self-linking number and Gauss linking number. In order to discuss these topological numbers of knotted phase singularity lines, we define a Gauss mapping:

\[ \vec{m} : S^1 \times S^1 \rightarrow S^2, \] (31)

where \( \vec{m} \) is a unit vector

\[ \vec{m}(\vec{x}, \vec{y}) = \frac{\vec{y} - \vec{x}}{|\vec{y} - \vec{x}|}, \] (32)

where \( \vec{x} \) and \( \vec{y} \) are two points, respectively, on the knots \( \xi_k \) and \( \xi_l \) (in particular, when \( \vec{x} \) and \( \vec{y} \) are the same point on the same knot \( \xi \), \( \vec{m} \) is just the unit tangent vector \( T \) of \( \xi \) at \( \vec{x} \)). Therefore, when \( \vec{x} \) and \( \vec{y} \), respectively, cover the closed curves \( \xi_k \) and \( \xi_l \) once, \( \vec{m} \) becomes the section of sphere bundle \( S^2 \). So, on this \( S^2 \) we can define the two-dimensional unit vector \( \vec{e} = \vec{e}(\vec{x}, \vec{y}) \). \( \vec{e}, \vec{m} \) are normal to each other, i.e.,

\[ \vec{e}_1 \cdot \vec{e}_2 = \vec{e}_1 \cdot \vec{m} = \vec{e}_2 \cdot \vec{m} = 0, \]
\[ \vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = \vec{m} \cdot \vec{m} = 1. \] (33)

In fact, the velocity field \( \vec{v} \) can be decomposed in terms of this two-dimensional unit vector \( \vec{e} \): \( v_i = \epsilon_{ab} \epsilon^a \partial_i e^b \), where \( \theta \) is a phase factor. Since one can see from the expression \( \mathcal{A} = \frac{1}{2} \nabla \times \vec{v} \) that the \( \partial \theta \) term does not contribute to the integral \( H \), \( v_i \) can in fact be expressed as

\[ v_i = \epsilon_{ab} \epsilon^a \partial_i e^b. \] (34)

Substituting it into Eq.(14), one can obtain

\[ H = \frac{1}{2\pi} \sum_{k=1}^{N} W_k \oint_{\xi_k} \epsilon_{ab} \epsilon^a \partial_i e^b dx_i \wedge dy_j. \] (35)

Noticing the symmetry between the points \( \vec{x} \) and \( \vec{y} \) in Eq. (32), Eq. (34) should be reexpressed as

\[ H = \frac{1}{2\pi} \sum_{k,l=1}^{N} W_k W_l \oint_{\xi_k} \oint_{\xi_l} \epsilon_{ab} \partial_i e^a \partial_j e^b dx_i \wedge dy_j. \] (36)

In this expression there are three cases: (1) \( \xi_k \) and \( \xi_l \) are two different phase singularities (\( \xi_k \neq \xi_l \)), and \( \vec{x} \) and \( \vec{y} \) are therefore two different points (\( \vec{x} \neq \vec{y} \)); (2) \( \xi_k \) and \( \xi_l \) are the same phase singularities (\( \xi_k = \xi_l \)), but \( \vec{x} \) and \( \vec{y} \) are two different points (\( \vec{x} \neq \vec{y} \)); (3) \( \xi_k \) and \( \xi_l \) are the same phase singularities (\( \xi_k = \xi_l \)), and \( \vec{x} \) and \( \vec{y} \) are the same points (\( \vec{x} = \vec{y} \)). Thus, Eq. (36) can be written as three terms:

\[ H = \sum_{k=1}^{N} \sum_{k,l=1}^{N} \frac{1}{2\pi} W_k^2 \oint_{\xi_k} \oint_{\xi_l} \epsilon_{ab} \partial_i e^a \partial_j e^b dx_i \wedge dy_j \]
\[ + \frac{1}{2\pi} \sum_{k=1}^{N} W_k \oint_{\xi_k} \oint_{\xi_l} \epsilon_{ab} \partial_i e^a \partial_j e^b dx_i \wedge dy_j, \] (37)

By making use of the relation \( \epsilon_{ab} \partial_i e^a \partial_j e^b = \frac{1}{2} \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m}) \), the Eq. (37) is just

\[ H = \sum_{k=1}^{N} \frac{1}{4\pi} W_k^2 \oint_{\xi_k} \oint_{\xi_l} \vec{m}^* (dS) \]
\[ + \frac{1}{2\pi} \sum_{k=1}^{N} W_k \oint_{\xi_k} \oint_{\xi_l} \epsilon_{ab} \partial_i e^a \partial_j e^b dx_i \]
\[ + \sum_{k,l=1}^{N} \frac{1}{4\pi} W_k W_l \oint_{\xi_k} \oint_{\xi_l} \vec{m}^*(dS), \] (38)

where \( \vec{m}^* (dS) = \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m}) dx_i \wedge dy_j (\vec{x} \neq \vec{y}) \) denotes the pullback of the \( S^2 \) surface element.

In the following we will investigate the three terms in the Eq. (38) in detail. Firstly, the first term of Eq. (38) is just related to the writhing number \( \chi \) of \( \xi_k \)

\[ Wr(\xi_k) = \frac{1}{4\pi} \oint_{\xi_k} \oint_{\xi_l} \vec{m}^* (dS). \] (39)
For the second term, one can prove that it is related to the twisting number $T_w(\xi_k)$ of $\xi_k$

$$\frac{1}{2\pi} \oint_{\xi_k} \epsilon a e^\theta \partial e^b d\xi = \frac{1}{2\pi} \oint_{\xi_k} (\vec{T} \times \vec{V}) \cdot d\vec{V} = T_w(\xi_k), \quad (40)$$

where $\vec{T}$ is the unit tangent vector of knot $\xi_k$ at $\vec{x}$ ($\vec{m} = \vec{T}$ when $\vec{x} = \vec{y}$) and $\vec{V}$ is defined as $e^\theta = e^{ab}V^b(\vec{V} \perp \vec{T}, \vec{e} = \vec{T} \times \vec{V})$. In terms of the White formula\[30\]

$$SL(\xi_k) = Wr(\xi_k) + T_w(\xi_k), \quad (41)$$

we see that the first and the second terms of Eq.\[35\] just compose the self-linking numbers of knots.

Secondly, for the third term, one can prove that

$$\frac{1}{4\pi} \oint_{\xi_k} \oint_{\xi_l} \vec{m}^* (dS) = \frac{1}{4\pi} e^{ijk} \oint_{\xi_k} dx^i \oint_{\xi_l} dy^j \frac{(x^k - y^k)}{||\vec{x} - \vec{y}||^3} = Lk(\xi_k, \xi_l) \quad (k \neq l), \quad (42)$$

where $Lk(\xi_k, \xi_l)$ is the Gauss linking number between $\xi_k$ and $\xi_l$\[29\]. Therefore, from Eqs.\[39\], \[40\], \[41\] and \[42\], we obtain the important result:

$$H = \sum_{k=1}^{N} W_k^2 SL(\xi_k) + \sum_{k,l=1}^{N} W_k W_l Lk(\xi_k, \xi_l). \quad (43)$$

This precise expression just reveals the relationship between $H$ and the self-linking and the linking numbers of the phase singularity knots family\[29\]. Since the self-linking and the linking numbers are both the invariant characteristic numbers of the phase singularity knots family in topology, $H$ is an important topological invariant required to describe the linked phase singularities in wave fields.

V. CONCLUSION

In the present study, by making use of the $\phi$-mapping topological current theory, the topology of the phase singularity in wave fields is studied. Firstly, we obtain the inner structure of the phase singularities in two and three dimensional space. The phase singularity objects have been found at the every zero point of the wave function $\psi$ under the condition that the Jacobian determinate $D^i(\phi/x) \neq 0$. One shows that the topological charge (strength of the phase singularity) of the phase singularity objects are determined by the topological numbers: Hopf indices and Brouwer degrees. Secondly, we have studied the topological invariant of the knotted phase singularity lines in three dimensional space in details. This topological invariant can be expressed as the sum of all the self-linking and all the linking numbers of the knotted phase singularity lines.

Finally, it should be pointed out that in the present paper, when we discussed the topological properties of the phase singularities, the condition $D^i(\phi/x) \neq 0$ must be satisfied. Now the question is coming, when this condition fails, what will happen about the phase singularity objects? The answer is related to the evolution of the phase singularity objects\[31\]. The topological object will generate, annihilate, split, or merge when the condition fails, and these dynamics properties of the phase singularity objects will be discussed in further work.

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[1] L. M. Pismen, Vortices in Nonlinear fields, (Clarendon Press, Oxford, 1999).
[2] J. Adachi, G. Ishikawa, arxiv: math/0608721.
[3] R. J. Donnelly, Quantized Vortices in Helium II (Cambridge University Press, Cambridge, England, 1991).
[4] Y. S. Duan, X. Liu and P. M. Zhang, J. Phys: Condens. Matter 14, 7941 (2002).
[5] S. Inouye, S. Gupta, T. Rosenband, A. P. Chikkatur, A. Gürliat, T. L. Gustavson, A. E. Leanhardt, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 87, 080402 (2001).
[6] J. O. Hirschfelder, A. C. Christoph, and W. E. Palke, J. Chem. Phys. 61, 5456 (1974).
[7] J. Riess, Phys. Rev. D 2, 647 (1970); J. Phys. A: Math. Gen. 20, 5179 (1987).
[8] A. S. Desyatnikov, Y. S. Kivshar, and L. Torner, Optical Vortices and Vortex Solitons, Progress in Optics Vol. 47 (North-Holland, Amsterdam, 2000).
[9] J. F. Nye, Natural focusing and fine structure of light (Institute of Physics Publishing, 1999).
[10] M. V. Berry and M. R. Dennis, Proc. R. Soc. A 456, 2059 (2000).
[11] M. V. Berry and J. F. Nye, Proc. R. Soc. A 336, 165 (1974).
[12] Y. S. Duan, G. Jia and H. Zhang, J. Phys. A: Math. Gen. 32, 4943 (1999).
[13] Y. S. Duan, H. Zhang and G. Jia, Phys. Letts. A 253, 57 (1999).
[14] E. Babaev, L. D. Faddeev, and A. J. Niemi, Phys. Rev. B 65, 100512 (2002); E. Babaev, Phys. Rev. Lett. 88, 177002 (2002).
[15] A. T. Winfree and S. H. Strogatz, Physica D 8, 35 (1983);
Physica D 9, 65 (1983); Physica D 9, 333 (1983); Physica D 13, 221 (1984); Physica D 17, 109 (1985).
[16] A. T. Winfree, Physica D 84, 126 (1995).
[17] A. T. Winfree, Nature 371, 233 (1994); Science 266, 1003 (1994).
[18] A. T. Winfree, The Geometry of Biological Time (Springer, 1980).
[19] M. V. Berry and M. R. Dennis, Proc. R. Soc. A 457, 2251 (2001).
[20] J. Leach, M. R. Dennis, J. Courtial and M. J. Padgett, Nature 432, 165 (2004), New J. Phys. 7, 55 (2005).
[21] Y. S. Duan, X. Liu, and L. B. Fu, Phys. Rev. D 67, 085022 (2003).
[22] F. Faddeev and A. J. Niemi, Nature 387, 58 (1997).
[23] Y. S. Duan, SLAC-PUB-3301, 1984; Y. S. Duan, H. Zhang, G. Jia, Phys. Letts. A 253, 57 (1999).
[24] E. Goursat, A Course in Mathematical Analysis, translated by E. R. Hedrick (Dover, New York, 1904), Vol.I.
[25] J. A. Schouten, Tensor Analysis for Physicists (Clarendon Press, Oxford, 1951).
[26] Y. S. Duan, S. Li and G. H. Yang, Nucl. Phys. B 514, 705 (1998).
[27] H. Aref and I. Zawadzki, Nature 354, 50 (1991).
[28] N. D. Mermin, T. L. Ho, Phys. Rev. Lett. 36,594 (1976).
[29] E. Witten, Commun. Math. Phys. 121, 351 (1989); A. M. Polyakov, Mod. Phys. Lett. A 3, 325 (1988).
[30] D. Rolfsen, Knots and Links (Publish or Perish, Berkeley, CA, 1976).
[31] L. B. Fu, Y. S. Duan, and H. Zhang, Phys. Rev. D 61, 045004 (2000).