Geometry of bracket-generating distributions of step 2 on graded manifolds

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1. Introduction

A smooth distribution $D \subset TM$ is said to be bracket-generating if all iterated brackets among its sections generate the whole tangent space to the manifold $M$. [1, 8]. $D$ is a bracket-generating distribution of step 2 if $D^2 = TM$, where $D^2 = D + [D, D]$. Bejancu showed that a distribution of rank $k < m = \dim M$ is a bracket-generating distribution of step 2, if and only if, the curvature of $D$ is of constant rank $m - k$ on $M$, [1].

In this paper, a $Z_2$-graded analogue of bracket-generating distribution is given. Let $\mathcal{D}$ be a distribution of rank $(p, q)$ on an $(m, n)$-dimensional graded manifold $\mathcal{M}$, we attach to $\mathcal{D}$ a linear map $F$ on $\mathcal{D}$ defined by the Lie bracket of graded vector fields of the sections of $\mathcal{D}$. Then $\mathcal{D}$ is a bracket-generating distribution of step 2, if and only if $F$ is of constant rank $(m - p, n - q)$ on $\mathcal{M}$.

2. Preliminaries

Let $M$ be a topological space and let $\mathcal{O}_M$ be a sheaf of super $\mathbb{R}$-algebras with unity. A graded manifold of dimension $(m, n)$ is a ringed space $\mathcal{M} = (M, \mathcal{O}_M)$ which is locally isomorphic to $\mathbb{R}^{m|n}$, (see [6]).

Let $\mathcal{M}$ and $\mathcal{N}$ be graded manifolds. Let $\phi : M \to N$ be a continuous map such that $\phi^* : \mathcal{O}_N \to \mathcal{O}_M$ takes $\mathcal{O}_N(V) = \mathcal{O}_M(\phi^{-1}(V))$ for each open set $V \subset N$, then we say that $\Phi = (\phi, \phi^*) : \mathcal{M} \to \mathcal{N}$ is a morphism between $\mathcal{M}$ and $\mathcal{N}$.

Let $A$ be a super $\mathbb{R}$-algebra, $\varphi \in \text{End}_\mathbb{R}A$ is called a derivation of $A$, if for all $a, b \in A$,

$$
\varphi(ab) = \varphi(a)b + (-1)^{|\varphi||a|}a\varphi(b),
$$

where for a homogeneous element $x$ of some graded object, $|x| \in \{0, 1\}$ denotes the parity of $x$ (see [6]).

A vector field on $\mathcal{M}$ is a derivation of the sheaf $\mathcal{O}_M$. Let $U \subset M$ be an open subset, the $\mathcal{O}_M(U)$—super module of derivations of $\mathcal{O}_M(U)$ is defined by

$$
T_\mathcal{M}(U) := \text{Der}(\mathcal{O}_M(U)).
$$

The $\mathcal{O}_M$—module $T_\mathcal{M}$ is locally free of dimension $(m, n)$ and is called the tangent sheaf of $\mathcal{M}$. A vector field is a section of $T_\mathcal{M}$.

If $\Omega^1(\mathcal{M}) := T^\ast \mathcal{M}$ is the dual of the tangent sheaf of a graded manifold $\mathcal{M}$, then it is the sheaf of super $\mathcal{O}_M$-modules and

$$
\Omega^1(\mathcal{M}) := \text{Hom}(T_\mathcal{M}, \mathcal{O}_M).
$$
It is called the cotangent sheaf of a graded manifold \( \mathcal{M} \), and the sections of \( \Omega^1(\mathcal{M}) \) are called super differential 1-forms [2, 6].

Let \( \mathcal{M} = (M, \mathcal{F}) \) be an \( (m, n) \)-dimensional graded manifold and \( \mathcal{D} \) be a distribution of rank \( (p, q) \) \( (p < m, q < n) \) on \( \mathcal{M} \). Then for each point \( x \in M \) there is an open subset \( U \) over which any set of generators \( \{D_i, D_\mu \mid 1 \leq i \leq p, 1 \leq \mu \leq q \} \) of the module \( \mathcal{D}(U) \) can be enlarged to a set

\[
\left\{ \gamma_a, D_i, D_\mu, C_a \mid \begin{array}{c}
1 \leq i \leq p \\
p + 1 \leq a \leq m \\
1 \leq \mu \leq q \\
q + 1 \leq a \leq n \\
|\gamma_a| = 0 \\
|D_i| = 0 \\
|\mu_\nu| = 1 \\
|C_a| = 1
\end{array} \right. \}
\]

of free generators of \( \text{Der}\mathcal{D}(M) \) [3].

We attach to \( \mathcal{D} \) a sequence of distributions defined by,

\[
\mathcal{D} \subset \mathcal{D}^2 \subset \ldots \subset \mathcal{D}^r \subset \ldots \subset \text{Der}\mathcal{D}(M),
\]

with

\[
\mathcal{D}^2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}], \ldots, \mathcal{D}^{r+1} = \mathcal{D}^r + [\mathcal{D}, \mathcal{D}^r],
\]

where

\[
[\mathcal{D}, \mathcal{D}^r] = \text{span}\{[X,Y] : X \in \mathcal{D}, Y \in \mathcal{D}^r\}.
\]

As in the classical case, we say that \( \mathcal{D} \) is a bracket-generating distribution, if there exists an \( r \geq 2 \) such that \( \mathcal{D}^r = \text{Der}\mathcal{D}(M) \). In this case \( r \) is called the step of the distribution \( \mathcal{D} \).

Suppose that \( X, Y \in \mathcal{D} \) and consider the linear map on \( \mathcal{D} \) as follows:

\[
F(X, Y) = -(-1)^{|X||Y|}[X, Y] \mod \mathcal{D}.
\]

With respect to the above local basis \( \{D_i, C_a, D_\mu, C_\alpha\} \) of \( \text{Der}\mathcal{D}(M) \), if

\[
\begin{align*}
[D_i, D_j] &= D_{ij} D_k + D_{ji} D_k + D_{ij}^\gamma D_\gamma + D_{ji}^\gamma D_\gamma, \\
[D_i, D_\gamma] &= D_{i\gamma} D_k + D_{\gamma i} D_k + D_{i\gamma}^\nu D_\nu + D_{\gamma i}^\nu D_\nu, \\
[D_\nu, D_\mu] &= D_{\nu\mu} D_k + D_{\mu\nu} D_k + D_{\nu\mu}^\gamma D_\gamma + D_{\mu\nu}^\gamma D_\gamma, \\
[D_\mu, D_a] &= D_{\mu a} D_k + D_{a\mu} D_k + D_{\mu a}^\nu D_\nu + D_{a\mu}^\nu D_\nu, \\
[D_\mu, C_a] &= D_{\mu a}^\gamma D_\gamma + D_{a\mu}^\gamma D_\gamma
\end{align*}
\]

then, by using (2.3), we conclude that

\[
\begin{align*}
F(D_i, D_j) &= D_{ij} D_k + D_{ji} D_k + D_{ij}^\gamma D_\gamma + D_{ji}^\gamma D_\gamma \mod \mathcal{D}, \\
F(D_i, D_\gamma) &= D_{i\gamma} D_k + D_{\gamma i} D_k + D_{i\gamma}^\nu D_\nu + D_{\gamma i}^\nu D_\nu \mod \mathcal{D}, \\
F(D_\nu, D_\mu) &= D_{\nu\mu} D_k + D_{\mu\nu} D_k + D_{\nu\mu}^\gamma D_\gamma + D_{\mu\nu}^\gamma D_\gamma \mod \mathcal{D}, \\
F(D_\mu, D_a) &= D_{\mu a} D_k + D_{a\mu} D_k + D_{\mu a}^\nu D_\nu + D_{a\mu}^\nu D_\nu \mod \mathcal{D}, \\
F(D_\mu, C_a) &= D_{\mu a}^\gamma D_\gamma + D_{a\mu}^\gamma D_\gamma
\end{align*}
\]

Each component \( D_{ik}^\alpha \) of \( F \) is a superfunction on \( U \).

Let \( \mathcal{U} \) be an open subset of \( M \) such that \( \mathcal{U} \cap \mathcal{U} \neq \emptyset \). If we change the basis of \( \text{Der}\mathcal{D}(U \cap \mathcal{U}) \) to \( \{\bar{D}_j, \bar{C}_a, \bar{D}_\mu, \bar{C}_\alpha\} \) then we have

\[
\begin{align*}
\bar{D}_j &= f_{ij} D_i + f_{ij}^\alpha D_\mu, \\
\bar{D}_\nu &= f_{i\nu} D_i + f_{i\nu}^\mu D_\mu, \\
\bar{C}_a &= f_{a\nu} D_i + g_{a\nu}^\mu C_\mu + f_{a\nu}^\mu D_\mu + g_{a\nu}^\alpha C_\alpha, \\
\bar{C}_\beta &= f_{\mu\nu} D_i + g_{\mu\nu}^\alpha C_\alpha + f_{\mu\nu}^\mu D_\mu + g_{\mu\nu}^\gamma C_\gamma,
\end{align*}
\]

where

\[
\begin{bmatrix}
  f_{ij} & f_{ij}^\alpha \\
  f_{i\nu} & f_{i\nu}^\mu \\
  f_{a\nu} & f_{a\nu}^\mu \\
  f_{\mu\nu} & f_{\mu\nu}^\mu
\end{bmatrix}
\]

are nonsingular supermatrices of smooth functions on \( \mathcal{U} \cap \mathcal{U} \). Both of these matrices are even. With respect to the basis \( \{\bar{D}_j, \bar{C}_a, \bar{D}_\mu, \bar{C}_\alpha\} \) on \( \mathcal{U} \), if \( \{\bar{D}_{kh}^i, \bar{D}_{kh}^\mu, \bar{D}_{kh}^\alpha, \bar{D}_{kh}^\gamma\} \) are the local components of \( F \), then we have

\[
\begin{bmatrix}
  \bar{D}_{kh}^i & \bar{D}_{kh}^\mu \\
  \bar{D}_{kh}^\gamma & \bar{D}_{kh}^\alpha \\
  g_{\mu_i} & g_{\mu_i}^\mu \\
  g_{\mu_i}^\alpha & g_{\mu_i}^\gamma
\end{bmatrix} = \begin{bmatrix}
  f_{ij} & 0 & f_{ij}^\alpha & 0 \\
  0 & f_{i\nu} & 0 & f_{i\nu}^\mu \\
  0 & f_{a\nu} & 0 & f_{a\nu}^\mu \\
  0 & f_{\mu\nu} & 0 & f_{\mu\nu}^\mu
\end{bmatrix} \begin{bmatrix}
  f_{ik} & 0 & f_{ik}^\alpha & 0 \\
  0 & f_{i\kappa} & 0 & f_{i\kappa}^\mu \\
  0 & f_{a\kappa} & 0 & f_{a\kappa}^\mu \\
  0 & f_{\kappa\kappa} & 0 & f_{\kappa\kappa}^\mu
\end{bmatrix} \begin{bmatrix}
  \bar{D}_{kh}^i & \bar{D}_{kh}^\mu \\
  \bar{D}_{kh}^\gamma & \bar{D}_{kh}^\alpha \\
  g_{\mu_i} & g_{\mu_i}^\mu \\
  g_{\mu_i}^\alpha & g_{\mu_i}^\gamma
\end{bmatrix} .
\]
Since \( \begin{bmatrix} f_i^j & f_i^\mu \\ f_j^i & f_j^\nu \end{bmatrix} \) is invertible at \( x \in U \cap U \), we see that \( \begin{bmatrix} f_i^j & 0 \\ 0 & f_j^\nu \end{bmatrix} \) is invertible and from (2.5) we conclude that if

\[
D(x) = \begin{bmatrix}
D_{12}^{p+q+1} & D_{13}^{p+q+1} & \cdots & D_{12}^{p+q+1} \\
D_{13}^{p+q+1} & D_{13}^{p+q+1} & \cdots & D_{13}^{p+q+1} \\
\vdots & \vdots & \ddots & \vdots \\
D_{12}^{m+n} & D_{12}^{m+n} & \cdots & D_{12}^{m+n}
\end{bmatrix}
\]

\( (x) \)

then \( \text{rank } D(x) = \text{rank } \bar{D}(x) \).

Now we can define the rank of \( F \), which is related to its coefficients matrix. Before doing this, in view of (2.4), we note that the submatrices

\[
\begin{bmatrix}
D_{\mu i}^{\alpha}(x) & D_{\mu i}^{\alpha}(x) \\
D_{\nu j}^{\alpha}(x) & D_{\nu j}^{\alpha}(x)
\end{bmatrix}
\]

are even and odd respectively. The rank of the first submatrix can be defined but for the second submatrix, since \( D_{\mu i}^{\alpha}(x) \) and \( \bar{D}_{\mu i}^{\alpha}(x) \) are even, we consider the matrix

\[
\begin{bmatrix}
D_{\mu i}^{\alpha}(x) & D_{\mu i}^{\alpha}(x) \\
D_{\mu i}^{\alpha}(x) & D_{\mu i}^{\alpha}(x)
\end{bmatrix}
\]

to define its rank. Now set

\[ r := \text{rank } \begin{bmatrix}
D_{\mu i}^{\alpha}(x) & D_{\nu j}^{\alpha}(x) \\
D_{\mu i}^{\alpha}(x) & D_{\nu j}^{\alpha}(x)
\end{bmatrix} \text{ and } s := \text{rank } \begin{bmatrix}
D_{\mu i}^{\alpha}(x) & D_{\mu i}^{\alpha}(x) \\
D_{\mu i}^{\alpha}(x) & D_{\mu i}^{\alpha}(x)
\end{bmatrix}, \]

where \( i, j = 1, \ldots, p, a = p + 1, \ldots, m \) and \( \mu, \nu = 1, \ldots, q, \alpha = q + 1, \ldots, n \). Thus we define

\[ \text{rank } F(x) = (r, s). \]

If \((\xi_\alpha, \zeta_\mu)\) are local supercoordinates on a coordinate neighborhood \( U \) of \( x \in M \), \((\tilde{a} = 1, \ldots, m, \tilde{\mu} = 1, \ldots, n)\), then \( \mathcal{D} \) is locally given by the graded 1-forms

\[
\begin{align*}
\phi_b &= \phi_b^a d\xi_a + \tilde{\phi}_b^\tilde{\mu} d\zeta_{\tilde{\mu}} = 0, \quad \tilde{b} = 1, \ldots, p \\
\phi_a &= \phi_a^b d\xi_b + \tilde{\phi}_a^\tilde{\mu} d\zeta_{\tilde{\mu}} = 0, \quad \tilde{a} = 1, \ldots, q.
\end{align*}
\]

Since \( \mathcal{D} \) is a distribution of rank \((p, q)\), we may assume that the submatrices \((\phi_b^a), 1 \leq \tilde{a}, \tilde{b} \leq p, \) and \((\tilde{\phi}_a^\tilde{\mu}), 1 \leq \tilde{a}, \tilde{\mu} \leq q\) are invertible. Let the matrix \( \psi = (\psi^\alpha_\mu) \) denotes the inverse of the matrix \( (\phi_a^b, \tilde{\phi}_a^\tilde{\mu}) \), \( 1 \leq \tilde{a}, \tilde{b} \leq p, 1 \leq \tilde{a}, \tilde{\mu} \leq q \) and suppose

\[
\tilde{\phi}_b = \psi_b^\alpha \phi_\alpha + \tilde{\phi}_b^\tilde{\mu} \phi_{\tilde{\mu}}, \quad \phi_a = \psi_a^b \phi_b + \tilde{\phi}_a^\tilde{\mu} \phi_{\tilde{\mu}}.
\]

Therefore, the new notation

\[
y_a = q_{a, i} = q_i, i = 1, \ldots, p, \quad \alpha = r + 1, \ldots, m, \\
\zeta_\mu = \zeta_{\mu, i} = \zeta_{\mu, i}, \mu = 1, \ldots, q, \quad \alpha = q + 1, \ldots, n,
\]

for the coordinates, may be performed to bring the local basis of \( \Omega^1(\mathcal{M}) \) into the form \( \{dx_i, d\eta_\mu, dy_a + r_{\alpha}^a dx_i + r_{\mu}^\alpha d\eta_\mu, d\zeta_\alpha + r_{\alpha}^\mu dx_i + r_{\mu}^\alpha d\eta_\mu \} \). It is easy to check that

\[
\begin{align*}
\delta_{\xi_i} &= \frac{\partial}{\partial \xi_i} - r_{\alpha}^a \frac{\partial}{\partial y_a} - r_{\mu}^\alpha \frac{\partial}{\partial \eta_\mu}, i = 1, \ldots, p, \\
\delta_{\eta_\mu} &= \frac{\partial}{\partial \eta_\mu} + r_{\alpha}^\mu \frac{\partial}{\partial y_a} - r_{\mu}^\alpha \frac{\partial}{\partial \eta_\mu}, \mu = 1, \ldots, q.
\end{align*}
\]

(2.6)

are (respectively even and odd) generators of \( \mathcal{D} \) on \( U \) and \((\delta_{\xi_i}, \delta_{\eta_\mu}, \partial/\partial y_a, \partial/\partial \zeta_\alpha) \) is a local basis for \( \text{Der}(\mathcal{M}(U)) \). (see also [4, 5]).

With respect to this basis, if we put

\[
F(\delta_{\xi_i} / \xi_i, \delta_{\eta_\mu} / \eta_\mu) = F_i^a \frac{\partial}{\partial y_a} + F_i^\mu \frac{\partial}{\partial \xi_i} \mod \mathcal{D},
\]

\[
F(\delta_{\eta_\mu} / \eta_\mu, \delta_{\xi_i} / \xi_i) = F_\mu^a \frac{\partial}{\partial y_a} + F_\mu^\mu \frac{\partial}{\partial \eta_\mu} \mod \mathcal{D},
\]

\[
F(\delta_{\xi_i} / \xi_i, \delta_{\eta_\mu} / \eta_\mu) = F_i^a \frac{\partial}{\partial y_a} + F_i^\mu \frac{\partial}{\partial \eta_\mu} \mod \mathcal{D},
\]

\[
F(\delta_{\eta_\mu} / \eta_\mu, \delta_{\xi_i} / \xi_i) = F_\mu^a \frac{\partial}{\partial y_a} + F_\mu^\mu \frac{\partial}{\partial \xi_i} \mod \mathcal{D},
\]

(2.7)
then by using (2.3) and (2.6), we deduce that

\[
\begin{align*}
F_i^a \frac{\partial}{\partial y_a} + F_j^a \frac{\partial}{\partial a} &= \left( \frac{\delta}{\delta x_1} - \delta \right) \frac{\partial}{\partial t} \mod \mathcal{D}, \\
F_i^a \frac{\partial}{\partial y_a} + F_j^a \frac{\partial}{\partial a} &= \left( - \frac{\delta}{\delta \eta^a} \right) \frac{\partial}{\partial \theta} \mod \mathcal{D}, \\
F_i^a \frac{\partial}{\partial y_a} + F_j^a \frac{\partial}{\partial a} &= \left( \frac{\delta}{\delta x_1} - \delta \right) \frac{\partial}{\partial \theta} \mod \mathcal{D}, \\
F_i^a \frac{\partial}{\partial y_a} + F_j^a \frac{\partial}{\partial a} &= \left( - \frac{\delta}{\delta \eta^a} \right) \frac{\partial}{\partial \theta} \mod \mathcal{D},
\end{align*}
\]

(2.8)

Now let us consider a distribution \( \mathcal{D} \) of corank one on \( \mathcal{M} \). For each \( z \in \mathcal{M} \), there are two cases.

**Case 1.** Let \( \text{rank} \mathcal{D}(z) = (m-1, n) \). Then there exist a coordinate system \( (x_i, t, \eta_{\mu}), i = 1, \ldots, m-1, \mu = 1, \ldots, n \), defined in a neighborhood \( U \) of \( z \) such that \( \mathcal{D} \) is locally given by

\[
dt + r_i dx_i + r_j d\eta_j = 0.
\]

**Case 2.** Let \( \text{rank} \mathcal{D}(z) = (m, n-1) \). Then there exist a coordinate system \( (x_j, \eta_{\nu}, \theta), j = 1, \ldots, m, \nu = 1, \ldots, n-1 \) defined in a neighborhood \( U \) of \( z \) such that \( \mathcal{D} \) is locally given by

\[
d\theta + r_j dx_j + r_i d\eta_i = 0.
\]

Note that in the first case, (2.8) becomes

\[
\begin{align*}
F_i^a \frac{\partial}{\partial y_a} &= \left( \delta \right) \frac{\partial}{\partial t} \mod \mathcal{D}, \\
F_i^a \frac{\partial}{\partial y_a} &= \left( - \delta \right) \frac{\partial}{\partial \theta} \mod \mathcal{D}, \\
F_i^a \frac{\partial}{\partial y_a} &= \left( \delta \right) \frac{\partial}{\partial \theta} \mod \mathcal{D}, \\
F_i^a \frac{\partial}{\partial y_a} &= \left( - \delta \right) \frac{\partial}{\partial \theta} \mod \mathcal{D},
\end{align*}
\]

(2.9)

where \( F_i, F_j, F_{\mu} \) and \( F_{\nu} \) are the local components of \( F \) with respect to the local basis \( \{ \delta / \delta x_i, \delta / \delta y_a, \partial / \partial t \} \).

3. **Bracket-generating distribution of step 2**

In this section, we want to find the conditions under which a distribution \( \mathcal{D} \) is bracket-generating of step 2. As mentioned in the previous section, we attach to \( \mathcal{D} \) a linear map \( F \) on \( \mathcal{D} \) defined by the Lie bracket of graded vector fields of the sections of \( \mathcal{D} \). We will have several types of possibilities for the rank of \( F \). Using this, we find conditions to describe the problem.

**Theorem 3.1.** Let \( \mathcal{D} \) be a distribution of rank \( (p, q) \) \( (p < m, q < n) \) on an \( (m, n) \)-dimensional manifold \( \mathcal{M} \) such that

\[
m - p \leq \frac{p(p-1)}{2} + \frac{q(q-1)}{2}, n - q \leq \frac{q(q-1)}{2},
\]

Then \( \mathcal{D} \) is a bracket-generating distribution of step 2, if and only if the linear map \( F \) associated to \( \mathcal{D} \) is of constant rank \( (m - p, n - q) \) on \( \mathcal{M} \).

**Proof.** Let \( x \in \mathcal{M} \). Suppose \( \mathcal{D} \) is a bracket-generating distribution of step 2 and let \( \{ \delta / \delta x_i, \delta / \delta \eta_{\mu}, \partial / \partial y_a, \partial / \partial \eta_{\nu} \} \) be a basis of \( \text{Der}_{\mathcal{M}}(U) \) in a coordinate neighborhood \( U \) of \( x \). Then \( \text{rank} \mathcal{D}(x) = (m - p, n - q) \). This means that the number of linearly independent graded vector fields of the set \( \{ \delta / \delta x_j, \delta / \delta \eta_{\nu}, \partial / \partial y_a \} \), \( 1 \leq i, j \leq p, 1 \leq \mu, q \leq q \), (respectively \( \{ \delta / \delta x_j, \delta / \delta \eta_{\nu}, \partial / \partial \eta_{\nu} \} \), \( 1 \leq i, j \leq p, 1 \leq \mu, q \leq q \)) is \( m - p \) (respectively \( n - q \)). Therefore the coefficient matrix, the matrix consisting of the coefficients of the Lie brackets of graded vector fields \( \{ \delta / \delta x_j, \delta / \delta \eta_{\nu}, \partial / \partial y_a \} \) at the point \( x \), denoted by

\[
\begin{bmatrix}
D_{ij}^a(x) \\
D_{ij}^a(x)
\end{bmatrix}, \quad a = 1, \ldots, m - p, i, j = 1, \ldots, n - q, \mod \mathcal{D},
\]

having the rank \( m - p \), is invertible. Similarly, the coefficient matrix

\[
\begin{bmatrix}
D_{ij}^a(x) \\
D_{ij}^a(x)
\end{bmatrix}, \quad a = 1, \ldots, m - p, i, j = 1, \ldots, n - q, \mod \mathcal{D},
\]

the matrix consisting of the coefficients of the Lie brackets of graded vector fields \( \{ \delta / \delta x_j, \delta / \delta \eta_{\nu} \} \) at the point \( x \), has rank \( n - q \), i.e., \( n - q = \text{rank} \mathcal{D}(x) \), and this matrix is even. Hence associated with \( F \) is the graded vector field, represented by the matrix \( \mathcal{D}(x) \), \mod \mathcal{D}, \)

relative to the above basis. It is clear that \( \text{rank} F(x) = (m - p, n - q) \).
Conversely, suppose that \( x \in \mathcal{M} \) and \( \text{rank}(F(x)) = (m - p, n - q) \) on \( \mathcal{M} \). Let \( \{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \eta_j}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial \zeta_l} \} \) be a basis of \( \text{Der}\mathcal{O}_\mathcal{M}(U) \) in a coordinate neighborhood \( U \) of \( x \). Consider the coefficient matrix of the graded vector fields \( F(x) \) and \( F'(x) \), which is even and denoted by
\[
\begin{bmatrix}
F_{ij}^a(x) & F_{ij}^a(x)\\
F_{ji}^b(x) & F_{ji}^b(x)
\end{bmatrix}
\]
\[ (3.2) \]

Note that its rank is \( m - p, \) otherwise \( F \) would not be a map of the given rank. Thus there are two non-negative integers \( r \) and \( s \) such that \( r + s = m - p \) and \( \text{rank}(F_{ij}^a(x)) = r, \text{rank}(F_{ji}^b(x)) = s \). Hence we may assume that the submatrices \( G = (F_{ij}^a(x)), 1 \leq i, j < r \) and \( J = (F_{ji}^b(x)), 1 \leq \alpha, \beta < s \), are both invertible. Therefore, the submatrix,
\[
\begin{bmatrix}
G & H \\
I & J
\end{bmatrix}
\]
is invertible.

Similarly, consider the coefficient matrix of the graded vector fields \( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \eta_j} \), which is odd and its rank is \( n - q \). We denote it by
\[
\begin{bmatrix}
F_{ia}^a(x) & F_{ia}^a(x)\\
F_{ai}^b(x) & F_{ai}^b(x)
\end{bmatrix}
\]
\[ (3.3) \]

Since \( \text{rank}(F_{ia}^a(x)) = n - q \), we may assume that the submatrix \( (F_{ia}^a(x)), 1 \leq a < m - p, 1 \leq \alpha < n - q \), is invertible. We thus consider
\[
\begin{bmatrix}
F_{ia}^a(x) & F_{ia}^a(x)\\
F_{ai}^b(x) & F_{ai}^b(x)
\end{bmatrix}
\]
\[ (3.4) \]

Given the matrices \( (3.3) \) and \( (3.4) \), we may change the generators of \( \text{Der}\mathcal{O}_\mathcal{M} \) to \( \{ \delta/\delta x_i, \delta/\delta \eta_j, Y_b, Z_v \}, b = 1, \ldots, m - p; v = 1, \ldots, n - q \), where \( Y_b \in \{ \frac{\delta}{\delta x_i}, \frac{\delta}{\delta \eta_j}, \frac{\delta}{\delta \eta_{j'}}, \frac{\delta}{\delta \eta_{j''}} \} \), with local coefficients \( (F_{ij}^a(x)) \) or \( (F_{ia}^a(x)) \) of the matrix \( (3.3) \) and \( Z_v \in \{ \frac{\delta}{\delta x_i}, \frac{\delta}{\delta \eta_j}, \frac{\delta}{\delta \eta_{j'}}, \frac{\delta}{\delta \eta_{j''}} \} \), with local coefficients \( (F_{ia}^a(x)) \) of the matrix \( (3.3) \). Thus \( \mathcal{D} \) is bracket-generating of step 2.

By using Theorem \( (3.1) \) we can easily prove the following theorems.

**Theorem 3.2.** Let \( \mathcal{M} \) be an \( (m, n) \) dimensional graded manifold. Suppose that \( \mathcal{D} \) is a distribution of rank \( (m - 1, n) \). Then \( \mathcal{D} \) is bracket-generating of step 2, if and only if, for the linear map \( F = F_0 + F_1 \) associated to \( \mathcal{D} \), \( F_0 \neq 0 \) on \( \mathcal{M} \).

**Proof.** Since \( \text{rank}(\mathcal{D})(z) = (m - 1, n) \), there exist a coordinate system \( (x_i, t, \eta_j), i = 1, \ldots, m - 1, \mu = 1, \ldots, n \), defined in a neighborhood \( U \) of \( z \), such that \( \mathcal{D} \) is locally given by \( \{ \delta/\delta x_i, \delta/\delta \eta_j \} \) and \( \{ \delta/\delta x_i, \delta/\delta \eta_j, \partial/\partial t \} \) is a local basis for \( \text{Der}\mathcal{O}_\mathcal{M} \). Therefore, according to the Theorem \( (3.1) \), the coefficient matrix,
\[
\begin{bmatrix}
D_{ij}^a(x) & D_{ij}^b(x) \\
D_{ji}^a(x) & D_{ji}^b(x)
\end{bmatrix}
\]
has the rank \( 1 \). Hence \( F_0 \neq 0 \).

**Theorem 3.3.** Let \( \mathcal{M} \) be an \( (m, n) \) dimensional graded manifold. Suppose that \( \mathcal{D} \) is a distribution of rank \( (m - 1, n) \). Then \( \mathcal{D} \) is bracket-generating of step 2, if and only if, for the linear map \( F = F_0 + F_1 \) associated to \( \mathcal{D} \), \( F_1 \neq 0 \) on \( \mathcal{M} \).

**Proof.** Since \( \text{rank}(\mathcal{D})(z) = (m - 1, n) \), there exist a coordinate system \( (x_i, \eta_j, \theta), i = 1, \ldots, m, \mu = 1, \ldots, n - 1 \), defined in a neighborhood \( U \) of \( z \), such that \( \mathcal{D} \) is locally given by \( \{ \delta/\delta x_i, \delta/\delta \eta_j \} \) and \( \{ \delta/\delta x_i, \delta/\delta \eta_j, \partial/\partial \theta \} \) is a local basis for \( \text{Der}\mathcal{O}_\mathcal{M} \). Therefore, according to the Theorem \( (3.1) \), the coefficient matrix,
\[
\begin{bmatrix}
D_{ia}^a(x) \\
D_{ia}^b(x)
\end{bmatrix}
\]
has the rank \( n \). Hence \( F_1 \neq 0 \).

**Theorem 3.4.** Let \( \mathcal{M} \) be an \( (m, n) \) dimensional graded manifold. Suppose that \( \mathcal{D} \) is a distribution of rank \( (0, n) \). Then \( \mathcal{D} \) is bracket-generating of step 2, if and only if, for the linear map \( F = F_0 + F_1 \) associated to \( \mathcal{D} \), \( \text{rank}(F_0) = m \) on \( \mathcal{M} \).

**Proof.** The details are the same as those given in the proof of Theorem \( (3.1) \).

**Example 3.5.** Consider the graded manifold \( \mathcal{M} = \mathbb{R}^3 \). Let \( (x_i, t, \eta_j), i = 1, \ldots, 2 \) be local supercoordinates on a coordinate neighborhood \( U \) of \( x \in \mathbb{R}^3 \). Suppose that \( \mathcal{D} \) is the distribution spanned by \( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \) and \( \frac{\partial}{\partial \eta_1} \) where
\[
\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \eta} + \theta \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial t}.
\]
A simple calculation shows that \( \{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \eta} \} \) is a basis of \( \text{Der}\mathcal{O}_\mathcal{M}(U) \). Thus \( \mathcal{D} \) is bracket-generating of step 2.
Example 3.6. Consider the graded manifold $\mathcal{M} = \mathbb{R}^{4|4}$. Let $(x, \eta_1, \eta_2, \eta_3, \eta_4)$ be local supercoordinates on a neighborhood $U$ of $x \in \mathbb{R}^4$.
Suppose that $\mathcal{D}$ is the distribution (see [7]) spanned by $\frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_2}, \frac{\partial}{\partial \eta_3}$, and $\frac{\partial}{\partial \eta_4}$, where

\[
\begin{align*}
\frac{\partial}{\partial \eta_1} &= \frac{\partial}{\partial \eta_1} - i_{\eta_4}\frac{\partial}{\partial \eta_2} - i_{\eta_4}\frac{\partial}{\partial \eta_3} - i_{\eta_4}\frac{\partial}{\partial \eta_4}, \\
\frac{\partial}{\partial \eta_2} &= \frac{\partial}{\partial \eta_2} - i_{\eta_1}\frac{\partial}{\partial \eta_3} - i_{\eta_1}\frac{\partial}{\partial \eta_4} + i_{\eta_1}\frac{\partial}{\partial \eta_4}, \\
\frac{\partial}{\partial \eta_3} &= \frac{\partial}{\partial \eta_3} - i_{\eta_1}\frac{\partial}{\partial \eta_2} + i_{\eta_1}\frac{\partial}{\partial \eta_4} - i_{\eta_1}\frac{\partial}{\partial \eta_4}, \\
\frac{\partial}{\partial \eta_4} &= \frac{\partial}{\partial \eta_4} - i_{\eta_1}\frac{\partial}{\partial \eta_1} + i_{\eta_1}\frac{\partial}{\partial \eta_3} + i_{\eta_1}\frac{\partial}{\partial \eta_3}.
\end{align*}
\]

Here $i = \sqrt{-1}$. Thus the vector fields $[\frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_2}, \frac{\partial}{\partial \eta_3}, \frac{\partial}{\partial \eta_4}]$, and $[\frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_2}, \frac{\partial}{\partial \eta_3}, \frac{\partial}{\partial \eta_4}]$ are zero and

\[
\begin{align*}
[\frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_2}] &= -2i\frac{\partial}{\partial x_1} - 2i\frac{\partial}{\partial x_4}, \\
[\frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_3}] &= -2i\frac{\partial}{\partial x_1} - 2i\frac{\partial}{\partial x_4}, \\
[\frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_4}] &= -2i\frac{\partial}{\partial x_1} + 2i\frac{\partial}{\partial x_4}.
\end{align*}
\]

In the notation used in Theorem 3.1, all of the entries $D_{ij}^a$, $D_{ij}^{a\nu}$, $D_{ij}^{a\mu}$, $D_{ij}^{a\nu\mu}$, $D_{ij}^{a\mu\nu}$ of the coefficient matrix except $D_{ij}^{a\nu\mu}$ are zero and

\[
D_{ij}^{a\nu\mu} = \begin{bmatrix}
0 & -2i & 0 & -2i & 0 \\
0 & 0 & -2i & -2i & 0 \\
0 & 0 & -2 & 0 & 0 \\
0 & -2i & 0 & 0 & -2i
\end{bmatrix}.
\]

So we have $\text{rank}(D_{ij}^{a\nu\mu}) = 4$, and we conclude from Corollary 3.4, that $\mathcal{D}$ is a bracket-generating distribution of step 2. By calculation we have

\[
\begin{align*}
1 \delta_{\eta_1} + 4[(\frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_2}) + (\frac{\partial}{\partial \eta_3} + \frac{\partial}{\partial \eta_4}) - 2\frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_3}] = \frac{\partial}{\partial x_1}, \\
1 \delta_{\eta_2} + 4[(\frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_2}) + (\frac{\partial}{\partial \eta_3} + \frac{\partial}{\partial \eta_4}) - 2\frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_3}] = \frac{\partial}{\partial x_2}, \\
1 \delta_{\eta_3} + 4[(\frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_2}) - (\frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_3})] = \frac{\partial}{\partial x_3}, \\
1 \delta_{\eta_4} + 4[(\frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_2}) - (\frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_3})] = \frac{\partial}{\partial x_4}.
\end{align*}
\]

Example 3.7. Let $\mathcal{M} = \mathbb{R}^{3|1}$ equipped with local supercoordinates $(x_1, x_2, x_3, \eta)$ and $\mathcal{D}$ be the distribution spanned by $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \eta}\}$. In this case we have

\[
\begin{align*}
\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \eta} = 0, \\
\frac{\partial}{\partial x_2} = 2x_1 \frac{\partial}{\partial x_3}, \\
\frac{\partial}{\partial x_3} = 2 \frac{\partial}{\partial x_3}.
\end{align*}
\]

We conclude from Corollary 3.2 that $\mathcal{D}$ is not bracket-generating of step 2 on the whole $\mathbb{R}^{3|1}$. It is bracket-generating of step 3.

Example 3.8. Let $\mathcal{M} = \mathbb{R}^{1|2}$ equipped with local supercoordinates $(x, \eta_1, \eta_2)$ and $\mathcal{D}$ be the distribution spanned by $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_2}\}$. Then $\frac{\partial}{\partial \eta_1} = \frac{\partial}{\partial \eta_2}$ and from Corollary 3.3, we see that $\mathcal{D}$ is bracket-generating of step 2 on $\mathbb{R}^{1|2}$.

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