Noise-enhanced spontaneous chaos in semiconductor superlattices at room temperature

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Abstract – Physical systems exhibiting fast spontaneous chaotic oscillations are used to generate high-quality true random sequences in random number generators. The concept of using fast practical entropy sources to produce true random sequences is crucial to make storage and transfer of data more secure at very high speeds. While the first high-speed devices were chaotic semiconductor lasers, the discovery of spontaneous chaos in semiconductor superlattices at room temperature provides a valuable nanotechnology alternative. Spontaneous chaos was observed in 1996 experiments at temperatures below liquid nitrogen. Here we show spontaneous chaos at room temperature appears in idealized superlattices for voltage ranges where sharp transitions between different oscillation modes occur. Internal and external noises broaden these voltage ranges and enhance the sensitivity to initial conditions in the superlattice snail-shaped chaotic attractor thereby rendering spontaneous chaos more robust.

Introduction. – Spontaneously chaotic semiconductor superlattice (SL) devices (sketched in figs. 1(a) and (b)) may become the key to achieve fast true random number generators (RNGs) [1–5]. The latter are crucial to secure fast and safe data storage and transmission [6–8], stochastic modeling [9], and Monte Carlo simulations [10]. In ideal spontaneously chaotic systems without noise/uncertainty, with exactly the same initial and boundary conditions, a chaotic RNG would generate the same random number sequence at each repetition of the whole process. Thus ideal deterministic chaotic RNGs correspond to pseudo-random number generators and do not increase information entropy. The randomness in a practical RNG will not be more than that provided by the noise sources therein. However, the chaotic system is sensitive to all the changes in the initial, boundary and bulk conditions and possible noises. Thus it may provide more randomness than just the amplification of a simple noise source, because it will generate a cumulative randomness resulting from all these noises, similarly to the situation when we multiply independent noises of zero mean. We show here that internal noise enhances spontaneous chaos at room temperature arising from sharp transitions between different oscillatory modes in SLs. It also induces chaos in voltage intervals close to those corresponding to barely noticeable deterministic chaotic attractors. Thus noise widens the voltage range of spontaneous chaos and it increases the sensitivity of the chaotic attractor to initial conditions thereby improving its speed as a random bit generator. This noise-enhanced chaos, already demonstrated in simple dynamical systems [11,12], is another paradigm of the constructive role of noise similar to stochastic resonance (noise-induced escape from one attractor to another resonates with external force [13]) and coherence resonance (noise-induced oscillations in excitable systems [14]).

The mechanisms that produce spontaneous chaos in SLs need to be understood to improve design of superlattice-based fast RNGs. Spontaneous chaos at room temperature appears in SLs whose configuration and aluminum content in the barriers are tuned to suppress leakage current through the X valley of the barrier [5,15]. Below liquid nitrogen temperatures, spontaneous chaos was observed in SLs with AlAs barriers in 1996 [16], but leakage currents through their X valley suppressed all oscillations (chaotic or not) for higher temperatures [17,18]. A generic mechanism to induce spontaneous chaos in bulk semiconductors [19] and in SLs at ultra low temperatures [20,21] is
based on wave front dynamics. For appropriate\nvalues of the injecting contact conductivity (such that the\ncritical current is close to the one at which accumula-
tion and depletion layers move at the same veloc-
ity \cite{[19,20,22]}), electric field domains formed by pairs of\nincreasing and decreasing wave fronts are randomly trig-
gered at the injecting contact. The relation of this theo-
retical mechanism for spontaneous chaos to experiments\nremains unclear \cite{16}. In contrast to this situation, the-
oretical predictions of driven chaos under ac + dc volt-
age bias based on a simple model \cite{23} have been observed in\nexperiments \cite{24}. At room temperature, wave fronts\nare not sharp and therefore it is not obvious that many\ncoexist simultaneously on a 50-period SL as do the\nsharp wave fronts seen in numerical simulations at ultra\nlow temperatures \cite{20}. We have found a different mecha-
nism that produces spontaneous chaos near voltage regions\nwhere sharp transitions between two different oscillation\nmodes occur. Noise enhances this very weak deterministic\nchaos.

Weakly coupled SLs may act as spatially discrete ex-
citable or oscillatory systems with local coupling between\nquantum wells, produced by the inter-well resonant tun-
eling current and the Poisson equation, and global cou-
ping due to voltage bias \cite{22}. The dynamics of charge\ndipoles and monopoles play a crucial role in both the ex-
citability of a stable stationary state and in generating sta-
ble self-sustained oscillations of the current \cite{22,25}. Both\nbehaviors may appear in different voltage ranges for the\nsame device.

**Model.** – In our simulations, we have included intrin-
sic noise in the usual sequential tunneling model of elec-
tron transport in a weakly coupled n-doped SL and also

\[
\frac{\varepsilon}{\lambda} \frac{dF_i}{dt} + J_{i \rightarrow i+1} + \xi(t) = J(t),
\]

\[
J_{i \rightarrow i+1} = \frac{em_i}{l} v^{(f)}(F_i) - J_{i \rightarrow i+1}(F_i, n_{i+1}, T),
\]

\[
n_i = N_D + \frac{\varepsilon}{\lambda}(F_i - F_{i-1}),
\]

\[
J_{i \rightarrow i+1}(F_i, n_{i+1}, T) = \frac{em^* k_B T}{\pi \hbar^2} v^{(f)}(F_i)
\]

\[
i \ln \left[ 1 + \frac{e^{v^{(f)}(F_i)}}{\left( \frac{e^{v^{(f)}(F_i)} - 1}{\varepsilon} \right) \sigma} \right],
\]

\[
\langle \xi_i(t) \xi_j(t') \rangle = \frac{e}{A} \frac{ev^{(f)}(F_i)}{l} n_i + J_{i \rightarrow i+1}(F_i, n_{i+1}, T)
\]

\[
+ 2J_{i \rightarrow i+1}(F_i, n_{i}, T) \delta_{ij} \delta(t - t'),
\]

\[
J_{0 \rightarrow 1} = \sigma_0 F_0, \quad J_{N \rightarrow N+1} = \sigma_0 \frac{n_N N_D}{F_N},
\]

\[
\sum_{i=1}^{N} F_i = \frac{V + \eta(t)}{J}.
\]

Here \(i = 1, \ldots, N\) (\(N = 50\) is the number of SL peri-
dods \cite{15}), and \(J_{i \rightarrow i+1}, \xi(t), J(t)\) are the tunneling cur-
cent density from well \(i\) to well \(i + 1\), the corresponding\nfluctuating current density, and the total current density,\nrespectively. The tunneling current density in eq. (2) from\nwell \(i\) to \(i + 1\) depends on the electric field \(F_i\) in well \(i\) and\nthe two-dimensional electron (2D) densities in the corre-
sponding wells, \(n_i\) and \(n_{i+1}\). The forward velocity \(v^{(f)}(F_i)\)\nis a function given in \cite{22,26} with peaks corresponding to\nthree energy levels at 53, 207, and 440 meV calculated by\nsolving a Kronig-Penney model for the SL configuration\nof refs. \cite{5,15,27}. The level broadenings due to scatter-
ing are 2.5, 8 and 24 meV for the three energy levels \cite{26}.\nThe equivalent 2D doping density due to the doping of the\ncentral part of the quantum well is \(N_D = 6 \times 10^{10} \text{ cm}^{-2}\).\nAlso \(m^* = (0.063 + 0.083x) m_e = 0.1 m_e \) (for \(x = 0.45\),\n\(-c < 0, A = s^2 \text{ with } s = 1.2 \text{ mm}, l_b = 4 \text{ nm}, l_w = 7 \text{ nm},\nl = l_b + l_w, \varepsilon = l/l_{nw} = \left( \frac{l_{nw}}{l_{nw}^2} + \frac{l_b}{l_{nw}} \right), \varepsilon_b = 10.9 \varepsilon_0, \varepsilon_w = 12.9 \varepsilon_0,\n\varepsilon_0, k_B, T, V, \sigma_0\) are the effective electron mass,\nthe electron charge, the SL cross-section, the side length of\na square mesa, the (Al,Ga)As barrier thickness, the\nGaAs well thickness, the SL period, the SL permittivity,\nthe barrier permittivity, the well permittivity, the dielec-
tric constant of the vacuum, the Boltzmann constant, the\nlattice temperature, the dc voltage, and the contact con-
ductivity, respectively. The fluctuating currents are in-
dependent identically distributed (i.i.d.) zero-mean white\noptics. Although these noises are i.i.d., the strongly non-
linear character of the system causes the noise sources to\ninfluence each other. The first two terms in their correla-
tions in eq. (5) are the SL shot noise \cite{28}, and the last term\nis an effective thermal noise similar to that describing cur-
rent fluctuations in the bulk Gunn effect \cite{25,29}. The total\ncurrent \(J(t)\) can be calculated from eq. (1) and the bias\ncondition in eq. (7), thereby providing effective nonlocal

Fig. 1: (Colour on-line) Sketch of a mesa-shaped semiconductor\nsuperlattice device having 1.2 mm side and 1.5 µm thickness.\n(a) dc voltage-biased superlattice consisting of two contact re-
regions of about 0.5 µm width and 50 periods formed by two\nsemiconductor layers of different bandgaps as depicted in (b).
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![Graphs](image.png)

**Fig. 2**: (Colour on-line) (a) Current response to voltage up-sweep and then down-sweep of the deterministic system showing that the self-oscillations start as a subcritical Hopf bifurcation from the stationary state. (b) Fourier spectrum of the current vs. voltage for zero noise. (c) Largest Lyapunov exponent vs. voltage. The LLE curves have the same shape for the deterministic and stochastic cases which cannot be appreciated due to their very different scales: Noise increases LLEs from 0.0004 up to 0.028.

**equations of motion.** \( \eta(t) \) is the unavoidable noise of the voltage source characterized as an independent zero-mean Gaussian white noise with 0.28 mV standard deviation.

**Results.** – We have solved the stochastic model given by eqs. (1)–(7) for the SL of refs. [5,15,27] at 300 K using a standard stochastic Euler method. To calculate the largest Lyapunov exponent (LLE), we have simultaneously integrated all perturbed and unperturbed trajectories and used the Benettin et al. algorithm [30]. LLE calculations with the Gao et al. algorithm [12,31] give the same results. The deterministic (noiseless) system exhibits self-sustained oscillations in two voltage intervals or plateaus. In the first plateau (corresponding to the one reported in [5]), large-amplitude current oscillations appear as a subcritical Hopf bifurcation from the stationary state, as shown in fig. 2(a). They are caused by the repeated creation of field pulses (charge dipoles) that dissolve before arriving at the collector. For larger voltages, there is a transition from large amplitude and frequency to smaller amplitude and frequency current oscillations. The transition to a two-frequency oscillation of richer harmonic content is clearly observable in the Fourier spectrum of fig. 2(b). In the transition region, a new frequency appears and fig. 2(c) shows that the LLE becomes positive indicating sensitivity to initial conditions and chaos. Direct routes to chaos from two-frequency quasi-periodic attractors have been shown in simpler dynamical systems [32]. Figure 2(c) shows that the deterministic system is weakly chaotic in a narrow voltage interval and that internal and external noises enhance chaos for these voltages. For larger voltages the oscillation again has a single dominant frequency. For the stochastic system, fig. 3(a) and (b) display the electric field profile and the current traces vs. time at a voltage for which the LLE is positive. In addition, noise enriches the content of the power spectrum as shown by fig. 3(c). That two oscillatory modes are present in the observed spontaneous chaotic oscillations is commented in ref. [27], whose authors identify them as the dipole motion mode and the well-to-well hopping mode. In fig. 3(a), they should correspond to dipoles reaching the collector in their motion and to the confined dipole motion, respectively.

Internal and external noises affect nonlinear charge transport in SLs in ways that are currently being explored. Noise-induced current switching between stable stationary states has been reported in a weakly coupled GaAs/AlAs SL that acts as an excitable system [33]. Noise-induced coherence resonance predicted in [34] has been observed in another GaAs/AlAs SL at 77 K [35]. Here we propose the noise-enhanced-chaos mechanism to explain for the first time spontaneous chaotic oscillations in a SL at room temperature and confirm it by numerical simulation. Essentially, the noise in a SL may convert regions where the deterministic description of the system presents sharp transitions between different oscillation modes into a chaotic attractor. The chaotic attractor may exist in a narrow region of the deterministic system, but the noise then enforces and changes it into the chaotic attractor of higher fractal dimension shown in figs. 4(a) and (b). (The multifractal dimension has been calculated using the methods explained in chapt. 9 of ref. [36] and in refs. [37,38]). We have checked that noise enhances chaos if its level falls within a narrow range, as indicated in ref. [12] for simpler systems. Too small a noise does
not induce chaos whereas too large a noise swamps spontaneous chaos. Far from the transition, each oscillatory mode is globally stable, and the noise cannot induce an attractor with extreme sensitivity to initial conditions quantified by a positive LLE.

At the voltage corresponding to the maximum value of the LLE in fig. 2(c), the fluctuations create a snail-shaped *multifractal* chaotic attractor out of the above-mentioned small- and large-amplitude oscillatory modes, as shown in fig. 4(a). The current trace shows several irregularly separated large and smaller spikes that indicate nucleation of a new dipole at the injector. Large (small) spikes occur when a dipole disappears before (after) reaching the collector. When a dipole dies before reaching the collector, the corresponding electric field at the injector (and, by (6), the total current) is larger than that corresponding to a dipole that reaches the collector. Thus, two different oscillation modes are observable in the chaotic attractor: injector-to-collector dipole motion and dipole motion from injector to premature annihilation inside the SL. The latter corresponds to the well-to-well hopping mode postulated in ref. [27]. The inter-spike intervals are similar but never repeat themselves and tend to produce a snail-shaped attractor in the phase plane of the field at two separated wells as depicted in fig. 4(a).

We have predicted that spontaneous chaos appears at room temperature dc voltage-biased SLs and is strongly enhanced by internal and external noises. Time periodic current oscillations appear as subcritical Hopf bifurcations for voltages on the first plateau and we have checked that they do occur as supercritical Hopf bifurcations on the second plateau. In both plateaus, a second oscillation frequency appears after the Hopf bifurcations and spontaneous chaos occurs near the corresponding transition voltage. Unavoidable internal and external noises enhance chaos: LLE and multifractal dimension of attractors increase with respect to the deterministic case. It is known that adding external noise may suppress or enhance the effects of internal noise on a given system [39]. In our
system, increasing the complexity of the chaotic attractor by adding controlled sources of external noise may speed up random number generation. This is what we find for an ideal model of perfect SLs with sequential tunneling mechanism.

Our results differ from experimental reports [5] in two aspects. Firstly, the voltage intervals for spontaneous chaos seem wider in experiments (0.3 V instead of 0.13 V) and the oscillations are more irregular than ours. Secondly, the oscillation frequencies are about 8 times larger in the experiments and the mean current density is about 26 times larger (260 instead of 10 A/cm²). However, the first plateau is located at comparable voltages in theory and experiments once we subtract the voltage at the series resistance (3 V in the experiments, negligible in the model). Both discrepancies have to do with the limitations of our model. Real samples have fluctuations in the aluminium density at the barriers, the width of barriers and wells and the well doping density. These effects could be included in our model by adding time-independent but well-dependent random variables to the tunneling current and to the doping density. This means that the functional form of the tunneling current density may vary from SL well to well. This in turn may alter and enrich charge transport in the discrete model of eqs. (1)–(3).

In conclusion, we have explained for the first time spontaneous chaos in semiconductor superlattices at room temperature as being associated to sharp transitions between oscillatory modes. Internal and external noises enhance spontaneous chaos in these devices. Increasing the complexity of the chaotic attractor by adding controlled sources of external noise may speed up random number generation. Our work paves the way for the design and optimization of superlattice-based devices that exploit spontaneous chaos to generate truly random numbers at high speed [5]. Such random number generators are of paramount importance in encryption systems for data storage and transmission [6–8], stochastic modeling [9], and Monte Carlo simulation [10].

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Appendix: internal noise. -- We have considered that the internal noise is due to shot and thermal noise [41]. Shot/partition noise for our discrete model of SL transport is due to charge quantisation when electrons cross SL barriers [41], as indicated in ref. [28], and in ref. [42] for a somewhat simpler discrete transport model. We consider current fluctuations associated to dissipation due to electron diffusion and model them by Landau-Lifshitz fluctuating hydrodynamics [29,43] adapted to SLs. In this way, our thermal noise is based on general principles of local equilibrium and detailed balance and we avoid more specific noise modeling. Current fluctuations are due to electron diffusion in the discrete model of eqs. (1)–(3). The latter can be written as a discrete drift-diffusion current density,

\[
J_{i \rightarrow i+1} = \frac{\epsilon m_e}{t} v_{i}(F_i) - J_{i \rightarrow i+1}^-(F_i, n_i, T)
\]

\[
- [J_{i \rightarrow i+1}^-(F_i, n_{i+1}, T) - J_{i \rightarrow i+1}^-(F_i, n_i, T)],
\]

(A.1)

where the last two terms correspond to electron diffusion. According to fluctuating hydrodynamics [29,43], the current density fluctuations are zero-mean white noises with correlation \( \frac{2}{t} J_{i \rightarrow i+1}^- (F_i, n_i, T) \), proportional to the diffusion current in eq. (4), [29,43]. This is similar to fluctuations in Gunn diodes where the fluctuation of the diffusion current is proportional to the diffusion coefficient times the electron density [29] instead of the expression (5).

Numerical issues. -- To solve the Ito stochastic differential equations of the model (1)–(7), we have used a standard stochastic Euler-Maruyama method [44] with 100 fs time steps. To calculate the LLE, we take an arbitrary initial perturbation vector \( d_0 \) and simultaneously integrate both perturbed and unperturbed trajectories during 10000 ns. The LLE is given by

\[
\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \|d(t)\|/\|d_0\|,
\]

where \( \|d(t)\| \) is the distance in phase space between the perturbed and unperturbed trajectories. The perturbation will need to be renormalized from time to time in order to prevent accumulative numerical rounding errors. To this end, we use the Benettin et al. algorithm [30] with a renormalization period of \( \tau_r = 1 \) ns. Then

\[
\lambda = \lim_{k \to \infty} \frac{1}{k \tau_r} \sum_{j=1}^{k} \ln \|d_j\|/\|d_0\|,
\]

where \( d_j \) is the perturbation vector during the \( j \)-th renormalization period.

We have also calculated the LLE by the quite different Gao et al. algorithm [12,31]. In this method, numerical simulation yields the current \( J(t) \) sampled every \( \delta t \approx 1 \) ns, which produces 4000 values of the current. Then we calculate \( m \)-dimensional vectors \( J(t_i), \ldots, J(t_i - (m-1) \delta t) \), where \( m \) is the smallest integer equal to or larger than twice the box-counting fractal dimension and, for different \( k \), the average

\[
\Lambda(k) = \left\langle \ln \left[ \frac{\|J_{i+k} - J_{i+k+1}\|}{\|J_i - J_{i+1}\|} \right] \right\rangle,
\]

with \( \|J_i - J_j\| < r \). The function \( \Lambda(k) \) is linear over an extended interval and its slope (written in nondimensional...
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