Correlation Functions of a Conformal Field Theory in Three Dimensions

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Abstract

We derive explicit forms of the two–point correlation functions of the $O(N)$ non-linear sigma model at the critical point, in the large $N$ limit, on various three dimensional manifolds of constant curvature. The two–point correlation function, $G(x, y)$, is the only $n$-point correlation function which survives in this limit. We analyze the short distance and long distance behaviour of $G(x, y)$. It is shown that $G(x, y)$ decays exponentially with the Riemannian distance on the spaces $R^2 \times S^1$, $S^1 \times S^1 \times R$, $S^2 \times R$, $H^2 \times R$. The decay on $R^3$ is of course a power law. We show that the scale for the correlation length is given by the geometry of the space and therefore the long distance behaviour of the critical correlation function is not necessarily a power law even though the manifold is of infinite extent in all directions; this is the case of the hyperbolic space where the radius of curvature plays the role of a scale parameter. We also verify that the scalar field in this theory is a primary field with weight $\delta = -\frac{1}{2}$; we illustrate this using the example of the manifold $S^2 \times R$ whose metric is conformally equivalent to that of $R^3 - \{0\}$ up to a reparametrization.

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1. Introduction

In a previous paper \[1\] we study the $O(N)$ non-linear sigma model in three dimensions, on manifolds of constant (positive, negative and zero) curvature, of the type $\Sigma \times R$ or $\Sigma \times S^1$, where $\Sigma$ is a two dimensional surface. We study the model in the $1/N$ expansion, and show that it is a conformally invariant theory in the lowest order, at the ultra–violet stable fixed point. We also find the critical values of the physical mass, the vacuum expectation of the field (spontaneous magnetization) and the free energy.

In this paper we address the problem of the two–point correlation function (Green’s function) of the theory, for the manifolds mentioned above.

We give here a brief summary of the $O(N)$ non-linear sigma model in the large $N$ limit on three dimensional curved spaces. For a detailed description we refer the reader to \[1\].

The regularized euclidean partition function of the $O(N)$ non-linear sigma model in three dimensions, in the presence of a background metric, $g_{\mu\nu}(x)$, can be written as,

$$ Z[g, \Lambda, \lambda(\Lambda)] = \int D[\phi] D[\sigma] \exp \{- \int d^3x \sqrt{g} \left[ \frac{1}{2} \phi^i(-\Box_g + \sigma) \phi_i - \frac{\sigma \Lambda^2}{2\lambda(\Lambda)} \right] \}$$  \hspace{1cm} (1)

where $i = 1, 2, \cdots, N$; $\lambda$ is the coupling constant and $\Lambda$ is the ultraviolet cut-off introduced to regularize the theory; $D[\phi] = \prod_{|k| < \Lambda} d\phi(k)$ and similarly $D[\sigma]$.

$\Box_g$ is the conformal laplacian: $-\Box_g = -\Delta_g + \xi R$, where $R$ denotes the Ricci scalar and $\xi = \frac{d-2}{4(d-1)}$; $d$ is the dimension of the manifold. The constraint on the $\phi$ fields, $\phi^i(x)\phi_i(x) = 1$, has been implemented by a Lagrange multiplier, in the form of an auxiliary field $\sigma$ (the canonical dimension of $\sigma$ in mass units is $[\sigma] = 2$).

Under the conformal transformation of the metric, $g_{\mu\nu}(x) \rightarrow e^{2f(x)}g_{\mu\nu}(x)$ with $\phi(x) \rightarrow e^{(2-d)f(x)}\phi(x)$ and $\sigma(x) \rightarrow e^{-2f(x)}\sigma(x)$ only the part of the classical action which is quadratic in $\phi^i$ is conformally invariant, but the quantum theory has a non-trivial fixed point at which $Z$ is conformally invariant.

In \[1\] the model is studied in the leading order of the $1/N$ expansion. For this purpose, we redefine $(N - 1)\lambda(\Lambda)$ as $\lambda(\Lambda)$ (kept fixed as $N \rightarrow \infty$); also, we rescale the $\phi$ field to $\sqrt{N - 1} \phi$, we integrate out the first $N - 1$ components of the $\phi$ field (which is always possible on spaces of constant curvature) and the partition function is finally rewritten as

$$ Z[g, \Lambda, \lambda(\Lambda)] = \int D[\phi_N] D[\sigma] \exp \left\{ -\frac{(N-1)}{2} \left[ \text{Tr} \log(-\Box_g + \sigma) \right] \right\} + \int d^3x \sqrt{g}[\phi_N(-\Box_g + \sigma) \phi_N - \frac{\Lambda}{\lambda(\Lambda)} \sigma(x)] \right\}. \hspace{1cm} (2)$$
The above expression is evaluated in the large $N$ limit at the uniform saddle point:

\[
\langle \sigma \rangle = m^2 \quad \langle \phi_N \rangle = b,
\]

where $m^2$ and $b$ are constants representing respectively the physical mass and the spontaneous magnetization.

They are the solutions to the following 'gap equations', which are obtained by extremizing the action with respect to $\phi_N(x)$ keeping $\sigma(x)$ fixed and vice-versa:

\[
(-\Box_g + m^2)b = 0 \quad \Lambda \frac{\lambda(\Lambda)}{G(\lambda)} - G(\lambda, \lambda; m^2, g) = b^2; \tag{3}
\]

here $G(x, y; m^2, g)$ is the two-point Green’s function of the $\phi$ fields at the saddle point, defined as

\[
(-\Box_g + m^2)G(x, y, m^2, g) = \frac{1}{\sqrt{g}}\delta(x, y) \tag{4}
\]

where $\sqrt{g}$ denotes the square root of the determinant of the metric; also, the Green’s function is formally represented as

\[
G(x, y, m^2, g) = \langle x | (-\Box_g + m^2)^{-1} | y \rangle \tag{5}
\]

where $|x\rangle$, $|y\rangle$ are position eigenstates.

We will be interested in solving (4), for various geometries at the non-trivial critical point $\lambda = \lambda_c$. The value of $m$ we would be using would be the critical value given by the gap equations at $\lambda_c$. For a derivation of the critical values of $m$ (and $b$) we refer to [1]. Here, we merely summarize the results from [1] as we will be using the critical value of $m$, $m_c$, for our calculations.

| Manifold          | $m_c$       | $b_c$       |
|-------------------|-------------|-------------|
| $R^3$             | zero        | zero        |
| $R^2 \times S^1_\beta$ | $2\log_\beta (1+\sqrt{5}/2)$ | zero        |
| $S^1 \times S^1 \times R$ | $\neq 0$ | zero        |
| $S^2 \times R$    | zero        | zero        |
| $H^2 \rho \times R$ | $1/2\rho$   | $\neq 0$    |
Solutions to (4) can be written in terms of the heat kernel of the operator $-\Box_g + m^2$, which we will denote by $h(t; x, y, m^2, g)$:

$$G(x, y, m^2, g) = \int_0^\infty dt \ h(t; x, y, m^2, g)$$

where $h(t; x, y, m^2, g) = \langle x | e^{-(\Box_g + m^2)t} | y \rangle$. The heat kernel is determined by the heat equation

$$(\Box_g + m^2)h(t; x, y, m^2, g) = -\frac{\partial}{\partial t}h(t; x, y, m^2, g)$$

with the boundary condition

$$h(0; x, y, m^2, g) = \frac{1}{\sqrt{g}}\delta(x, y).$$

As well known this equation is solved by

$$h(t; x, y, m^2, g) = \sum_n e^{-\lambda_n t} \psi_n^*(x) \psi_n(y)$$

where $\lambda_n$ are the eigenvalues of $-\Box_g + m^2$ and $\psi_n$ are the eigenstates, with $\psi_n(x) = \langle x | \psi_n \rangle$; the sum is understood to take into account the multiplicity of the eigenvalues. When the spectrum is continuous the sum is replaced by an integral and the multiplicity by the density of states. In this paper we find the heat kernel (7), and consequently the Green’s function (6), on various manifolds of constant curvature.

We note that, to the leading order in the $\frac{1}{N}$ expansion of the generating functional, only the two–point correlation function survives, the higher $n$-point correlation functions being subleading in the expansion parameter $\frac{1}{N}$. This can be seen by coupling a source current $J$ to the field $\phi_N$ in the action in (2): it is easy to check that the $n$–point correlation function of the field $\phi_N$, $\langle \phi_N(x_1)\phi_N(x_2)\cdots\phi_N(x_n) \rangle$ is of order $\frac{1}{\sqrt{N}}$ with respect to the $(n-1)$–point correlation function. Therefore it is sufficient to study the one–point (spontaneous magnetization) and two–point correlation functions for the large $N$ theory.

2. Manifolds with zero curvature

We consider the following examples of flat spaces: $R^3$, $R^2 \times S^1$, $S^1 \times S^1 \times R$. The first example ($R^3$), being a well known one, is sketched for future comparisons. For $R^2 \times S^1$, some results are also known (2), the Ricci scalar being zero.
i) The euclidean space \( \mathbb{R}^3 \)

The eigenvalues of the Laplacian \( -\Delta_{\mathbb{R}^3} \) are given by \( k^2 \), where \( k \) takes values on the real line, so that the heat kernel of \( -\Delta_{\mathbb{R}^3} + m^2 \) is

\[
h_{\mathbb{R}^3}(t; x, y, m^2, g) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{-(k^2+m^2)t} e^{ik(x-y)}
\]

\[
= e^{-\frac{|x-y|^2}{4t}} e^{-m^2t}.
\]

The two-point Green’s function is then:

\[
G_{\mathbb{R}^3}(x, y, m^2, g) = \int_0^\infty dt \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{\frac{3}{2}}} e^{-m^2t}.
\]

At the critical point, \( m = 0 \), namely the correlation length (inverse of the square root of the smallest eigenvalue of \( -\Delta_g + m^2 \)) diverges, and we recover the expected result that at the phase transition the two-point correlation function has a power law behaviour, \( |\bar{x} - \bar{y}|^\alpha \), with \( \alpha = -1 \) and there is long range order in the system.

ii) \( \mathbb{R}^2 \times S^1 \)

For this and other cases where the manifold \( M \) is a product of two, say \( M = A \times B \), it is convenient to recall that the heat kernel of an operator on \( M \) is expressible as the product of heat kernels on the spaces \( A \) and \( B \). From (8) we have indeed

\[
h(t; x, y, m^2, g) = \sum_{n,k} e^{-(a_n+b_k+m^2)t} \psi^*_n(x)\psi_k(y)
\]

being \( \Box_j = \Box^A_j + \Box^B_j \) and \( a_n, b_k \) the eigenvalues of \( \Box^A_j \) and \( \Box^B_j \) respectively. Moreover the eigenvectors of \( \Box_j \) can be written as tensor products of eigenvectors of \( \Box^A_j \) and \( \Box^B_j \) (\( |\psi_{n,k}^A| = |\psi_n^A\rangle \otimes |\psi_k^B\rangle \)), and the position eigenstates on \( M \) as tensor products of position eigenstates on the two spaces \( (|x\rangle = |x_A\rangle \otimes |x_B\rangle) \). We have then

\[
h(t; x, y, m^2, g) = \sum_{n,k} e^{-(a_n+b_k+m^2)t} \psi^*_n(x_A)\psi^A_n(y_A)\psi^B_k(x_B)\psi_k(y_B)
\]

\[
= h_A(t; x_A, y_A, g_A)h_B(t; x_B, y_B, g_B) e^{-m^2t}.
\]

In the case of \( \mathbb{R}^2 \times S^1 \) (10) becomes

\[
h_{\mathbb{R}^2 \times S^1}(t; x, y, m^2, g) = h_{\mathbb{R}^2}(t; \bar{x}, \bar{y}, g_{\mathbb{R}^2})h_{S^1}(t; \theta, \theta', g_{S^1}) e^{-m^2t}
\]
where \( \bar{x}, \bar{y} \), are coordinates on \( R^2 \), \( \theta, \theta' \) are coordinates on \( S^1 \). The critical value of \( m \) in this case is non-zero: in [8] \( m \) was found to be \( \frac{2}{\beta} \log \left( \frac{1+\sqrt{\gamma}}{2} \right) \), where \( \beta \) is the radius of the circle (see also [1]). The heat kernels for \(-\Delta_{R^2}\) are respectively

\[
h_{R^2}(t; \bar{x}, \bar{y}, g_{R^2}) = \int_{-\infty}^{\infty} \frac{k^2}{(2\pi)^2} e^{-(k^2 + m^2)t} e^{ik(x-y)} = \frac{e^{-\frac{\bar{x}^2}{4t}}}{4\pi t}
\]

\[
h_{S^1}(t; \theta, \theta', g_{S^1}) = \frac{1}{\beta} \sum_{\infty} e^{-\frac{4\pi^2\bar{x}^2}{\beta^2}} \psi_\lambda^*(\theta)\psi_n(\theta')
\]

(12)

where \( k^2 \) are the eigenvalues for \(-\Delta_{R^2}\), \( \omega_n = \frac{4\pi^2n^2}{\beta^2} \) and \( \psi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2n}} \) are respectively the eigenvalues and the eigenfunctions for \(-\Delta_{S^1}\). Substituting in (8) we obtain

\[
G(x,y) = \frac{1}{\beta} \int_{0}^{\infty} dt \frac{e^{-\frac{\bar{x}^2}{4t}}-m^2t}{4\pi t} \sum_{\infty} e^{-\frac{4\pi^2\bar{x}^2}{\beta^2}} \psi_\lambda^*(\theta)\psi_n(\theta').
\]

(13)

Long range correlations, if any, could occur only along the \( R^2 \) direction, therefore we may assume the angular co-ordinates of \( x \) and \( y \) to be the same, which will imply \( \psi_\lambda^*(\theta)\psi_n(\theta')|_{\theta=\theta'} = \frac{1}{2\pi} \). We use the Poisson sum formula

\[
\frac{1}{2\pi} \sum_{\infty} e^{-\frac{4\pi^2\bar{x}^2}{\beta^2}t} = \frac{\beta}{(4\pi t)^\frac{1}{2}} + \frac{2\beta}{(4\pi t)^\frac{1}{2}} \sum_{\infty} e^{-\frac{4\pi^2\bar{x}^2}{\beta^2}}
\]

(14)

to be able to do the integral in \( t \) first. The integral over \( t \) can be performed by using the following standard result [1]

\[
\int_{0}^{\infty} dt \ t^{\nu-1} e^{-(\frac{x}{\sqrt{t}}+\gamma t)} = 2 \left( \frac{\sigma}{\gamma} \right)^\frac{\nu}{2} \mathcal{K}_\nu(2\sqrt{\sigma\gamma}); \quad \text{Re}\sigma > 0, \text{Re}\gamma > 0,
\]

(15)

where \( \mathcal{K}_\nu \) is the MacDonald’s function. We find

\[
G_{R^2 \times S^1}(x,y,m^2,g) = \frac{1}{4\pi} \frac{e^{-m|\bar{x} - \bar{y}|}}{|\bar{x} - \bar{y}|} + \frac{1}{2\pi} \sum_{1} \frac{e^{-m\sqrt{(|\bar{x} - \bar{y}|^2 + n^2\beta^2)}}}{(|\bar{x} - \bar{y}|^2 + n^2\beta^2)^\frac{1}{2}}
\]

(16)

where we used

\[
\mathcal{K}_\frac{1}{2}(x) = \left( \frac{\pi}{2x} \right)^\frac{1}{4} e^{-x}.
\]

(17)

In the limit \(|\bar{x} - \bar{y}| \to \infty\), \( \beta \) being finite, the sum in (16) can be approximated by an integral which can be evaluated using the standard result [1]:

\[
\int_{|\bar{x} - \bar{y}|}^{\infty} du \frac{e^{-mu}}{(u^2 - |\bar{x} - \bar{y}|^2)^\frac{1}{2}} = \mathcal{K}_0(m|\bar{x} - \bar{y}|).
\]

(18)

Moreover, using the asymptotic expression of the MacDonald’s function for large \(|\bar{x} - \bar{y}|\)

\[
\mathcal{K}_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \quad z \to \infty.
\]

(19)
the Green’s function takes the following form when $|\bar{x} - \bar{y}| \to \infty$:

$$G(x, y) \xrightarrow{|\bar{x} - \bar{y}| \to \infty} \frac{1}{4\pi} \frac{e^{-m|\bar{x} - \bar{y}|}}{|\bar{x} - \bar{y}|} + \frac{1}{\sqrt{8\pi \beta}} \frac{e^{-m|\bar{x} - \bar{y}|}}{\sqrt{m|\bar{x} - \bar{y}|}}$$  \hspace{1cm} (20)

We see therefore that the correlation function decays exponentially for large $|\bar{x} - \bar{y}|$. The correlation length is in this case finite, due to the finite size of the manifold in the $S^1$ direction (we will see that a finite correlation length at criticality is not always connected to compactness of the manifold in some directions).

For small values of $|\bar{x} - \bar{y}|$, we should recover the result on $R^3$, at criticality.

$$G(x, y) \xrightarrow{|\bar{x} - \bar{y}| \to 0} \frac{1}{4\pi} \left[ \frac{1}{|\bar{x} - \bar{y}|} - m \right] + \frac{1}{2\pi} \sum_{n=1}^{\infty} e^{-m\beta n} n\beta.$$

This on simplification reduces to

$$G(x, y) \xrightarrow{|\bar{x} - \bar{y}| \to \infty} \frac{1}{4\pi |\bar{x} - \bar{y}|} + \log 2 \sinh \frac{m\beta}{2};$$

but, $\log 2 \sinh \frac{m\beta}{2} = 0$ for the critical value of $m$. Therefore the short distance behaviour of the Green’s function on $R^2 \times S^1$ is

$$G(x, y) \xrightarrow{|\bar{x} - \bar{y}| \to 0} \frac{1}{4\pi |\bar{x} - \bar{y}|}$$  \hspace{1cm} (21)

which is the same as in $R^3$, where the Green’s function at criticality is given by (8) with $m = 0$.

**iii) $S^1 \times S^1 \times R$**

For simplicity, we consider the circles to have the same radius, $\rho$. Let the coordinates on $S^1 \times S^1$ be denoted by $(\theta_1, \theta_2)$ and the coordinate on $R$ by $x_0$. The Ricci scalar is zero on this space so that $-\Box_g = -\Delta_{S^1 \times S^1} - \Delta_R$. From (10) we have

$$h_{S^1 \times S^1 \times R}(t; x, y, m^2, g) = h_{S^1}(t; \theta_1, \theta_1') h_{S^1}(t; \theta_2, \theta_2') h_R(t; x_0, y_0) e^{-m^2 t}$$  \hspace{1cm} (22)

The heat kernel of $-\Delta_{S^1}$ has been given in (12) while the heat kernel of $-\Delta_R$ is

$$h_R(t; x_0, y_0) = \int_{-\infty}^{\infty} dk \frac{e^{-(k^2+m^2)t} e^{ik|x_0-y_0|}}{4\pi t^{1/2}}$$

As in the previous case, we fix the angular separation between the points $x$ and $y$ to be zero ($\theta_1 = \theta_1', \theta_2 = \theta_2'$), that is, we are looking for the behaviour of the two–point correlation function along the $R$ direction. Substituting (22) in (3) we have
\[ G(x, y) = \frac{1}{4\pi^2 \rho^2} \int_0^\infty dt \frac{e^{-\frac{|x_0-y_0|^2}{4\rho^2 t}}}{\sqrt{4\pi t}} \sum_{p,q=-\infty}^{\infty} e^{-\frac{4\pi^2}{\rho^2} (p^2 + q^2) t} e^{-m^2 t}. \]  

(24)

On using the Poisson sum formula (14), the two–point correlation function is

\[ G(x, y) = \sum_{p,q} \int_0^\infty \frac{dt}{(4\pi)^2} t^{-\frac{3}{2}} e^{-m^2 t} e^{-(p^2 + q^2) \frac{2\pi^2}{\rho^2} t} e^{-\frac{|x_0-y_0|^2}{4t}}. \]  

(25)

The modes \( p, q = 0 \), \( \{ p = 0, q = 1 \} \) and \( \{ p = 1, q = 0 \} \) can be separated and the integrals over \( t \) can be performed using (13) and the resulting expressions simplified using (17) to give

\[ G_{S^1 \times S^1 \times R}(x, y, m^2, g) = \frac{1}{4\pi} \frac{e^{-m|x_0-y_0|}}{|x_0 - y_0|} + \frac{1}{\pi} \sum_{p=1}^{\infty} \frac{e^{-m\sqrt{p^2 \rho^2 + |x_0-y_0|^2}}}{\sqrt{p^2 \rho^2 + |x_0-y_0|^2}} \]

\[ + \frac{1}{\pi} \sum_{p=1}^{\infty} \frac{e^{-m\sqrt{p^2 \rho^2 + q^2 \rho^2 + |x_0-y_0|^2}}}{\sqrt{p^2 \rho^2 + q^2 \rho^2 + |x_0-y_0|^2}}. \]  

(26)

The critical value of \( m \) was found in [1] to be non-zero and is the solution to the gap equation

\[ -\frac{m\rho}{4} + \sum_{p=1}^{\infty} \frac{e^{-mp^2}}{p} + \sum_{p,q=1}^{\infty} \frac{e^{-m\sqrt{p^2 + q^2}}}{\sqrt{p^2 + q^2}} = 0. \]  

(27)

We can now analyze the behaviour of the Green’s function in the limits \( |x_0 - y_0| \to \infty \) and \( |x_0 - y_0| \to 0 \). When \( |x_0 - y_0| \to \infty \) (\( \rho \) being finite), we again approximate the sums in (26) with the corresponding integrals which can be performed to give

\[ G(x, y) \rightarrow_{|x_0-y_0| \to \infty} \frac{1}{4\pi} \frac{e^{-m|x_0-y_0|}}{|x_0-y_0|} + \frac{1}{\sqrt{2\pi}} \frac{e^{-m|x_0-y_0|}}{\rho \sqrt{m|x_0-y_0|}} + \frac{1}{2m\rho^2} e^{-m|x_0-y_0|}. \]  

(28)

showing that the correlation function decays exponentially and the correlation length is finite at criticality along the \( R \) direction. This again is due to the finite size effect of the torus.

In the limit \( |x_0 - y_0| \to 0 \),

\[ G(x, y) \rightarrow_{|x_0-y_0| \to 0} \frac{1}{4\pi} \left[ \frac{1}{|x_0-y_0|} - m \right] + \frac{1}{\pi\rho} \sum_{p=1}^{\infty} \frac{e^{-mp^2}}{p} + \frac{1}{\pi\rho} \sum_{p,q=1}^{\infty} \frac{e^{-mp\sqrt{p^2 + q^2}}}{\sqrt{p^2 + q^2}}. \]  

(29)

At the critical point, on using the gap equation (27), this expression reduces to

\[ G(x, y) \rightarrow_{|x_0-y_0| \to 0} \frac{1}{4\pi|x_0-y_0|}. \]  

(30)
which is the correct behaviour of the Green’s function in this limit as it is the same as that on $R^3$ (\(m = 0\)), when the separation between the angular co-ordinates of the points $x$ and $y$ is zero.

3. **A manifold with positive curvature: $S^2 \times R$**

This example is of particular interest as it provides us with a setting to describe what a primary field is in this three dimensional context.

We indicate with $(\theta, \phi)$ the coordinate on the sphere and with $x_0$ the coordinate on the real line. $\rho$ is the radius of the sphere. The conformal Laplacian on the sphere is $-\Box_{S^2} = -\Delta_{S^2} + \frac{1}{\rho^2}R$ where $R = \frac{2}{\rho^2}$. The eigenvalues of $-\Box_{S^2}$ are $(l + \frac{1}{2})^2$ with degeneracy $(2l + 1)$ where $l = 0, 1, 2, \cdots, \infty$ and the eigenfunctions are the spherical harmonics denoted by $Y^m_l(\theta, \phi)$, $m = -l, -l + 1, \cdots, l$. The heat kernel of $-\Box_{S^2}$ is then

$$h_{S^2}(t, \bar{x}, \bar{y}) = \frac{1}{\rho^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-\frac{(l + \frac{1}{2})^2}{\rho^2}t} Y^m_l(\theta, \phi) Y^m_l(\theta', \phi')$$

while the heat kernel of $-\Delta_R$ is given by (23). The critical value of the mass is zero \([1]\) for this case. In order to simplify the problem, let us look for the heat kernel and hence the Green’s function when the angular separation $\theta - \theta'$ and $\phi - \phi'$ between the points $x$ and $y$ is zero. Again, we can do this without loss of generality as it is only meaningful to look for the presence of long range correlations along the $R$ direction.

Recalling that

$$\sum_{m=-l}^{l} Y^m_l(\theta, \phi) Y^m_l(\theta, \phi) = \frac{2l + 1}{4\pi},$$

and substituting (23) and (31) in (6), we find the Green’s function to be

$$G(x, y) = \frac{1}{4\pi^2 \rho^2} \int_0^\infty dt \ t^{-\frac{3}{2}} e^{-\frac{(x_0 - y_0)^2}{4\rho^2t}} \sum_{l=0}^{\infty} (l + \frac{1}{2}) e^{-\frac{(l + \frac{1}{2})^2}{\rho^2}t}. \quad (33)$$

We now use an extension of the Poisson sum formula to this case \([1]\),

$$\frac{1}{2\pi} \sum_{l=\frac{1}{2}}^{\infty} le^{-\frac{t^2}{4\rho^2}} = \frac{\rho^2}{(4\pi t)^{\frac{3}{2}}} \int_{-\infty}^{\infty} dz \ (\frac{z^2}{2\rho} \csc \frac{z}{2\rho} - 1)e^{-\frac{z^2}{4\pi t}} + \frac{\rho^2}{4\pi t}, \quad (34)$$

to rewrite $G(x, y)$ as

$$G(x, y) = \frac{1}{8\pi^2} \int_0^\infty dt \ e^{-\frac{(x_0 - y_0)^2}{4\pi t}} \left[ t^{-\frac{3}{2}} + \frac{t^{-2}}{\sqrt{4\pi}} \int_{-\infty}^{\infty} dz \ (\frac{z}{2\rho} \csc \frac{z}{2\rho} - 1)e^{-\frac{z^2}{4\pi t}} \right]. \quad (35)$$

The integral over $t$ can be performed easily and the Green’s function simplifies to

$$G(x, y) = \frac{1}{4\pi} \left[ \frac{1}{|x_0 - y_0|} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dz}{z^2 + |x_0 - y_0|^2} \ (\frac{z}{2\rho} \csc \frac{z}{2\rho} - 1) \right]. \quad (36)$$
The integral over $z$ can now be performed by going to the complex plane and the final expression for the Green’s function is

$$G_{S^2 \times R}(x, y, g) = \frac{1}{8\pi \rho} \operatorname{cosech} \frac{|x_0 - y_0|}{2\rho}. \quad (37)$$

When $|x_0 - y_0| \to \infty$,

$$G(x, y) \to_{|x_0 - y_0|\to \infty} \frac{1}{4\pi \rho} e^{-\frac{|x_0 - y_0|}{2\rho}}. \quad (38)$$

There is no long range correlation even along the $R$ direction at the critical point. Also, when $|x_0 - y_0| \to 0$,

$$G(x, y) \to_{|x_0 - y_0|\to 0} \frac{1}{4\pi |x_0 - y_0|} \quad (39)$$

which is the flat space limit as one should expect.

Let us now recall the definition of a primary field as given in [3], [1]. A primary field is one whose correlation functions transform homogeneously under a conformal transformation:

$$<\phi(x_1) \cdots \phi(x_n)>_{\mathcal{L}_g} = e^{\delta(f(x_1) + \cdots + f(x_n))} <\phi(x_1) \cdots \phi(x_n)>_g. \quad (40)$$

where $\delta$ is the conformal weight. The field $\phi$ of the $O(N)$ sigma model is an example of such a primary field. The example on $S^2 \times R$ is particularly suitable for illustrating this fact because the metric on this space is conformally equivalent to that on $R^3 - \{0\}$ after a reparametrization. This can be easily checked: let the metric on $S^2 \times R$ be denoted by $\tilde{g}_{\mu\nu}$ and that on $R^3 - \{0\}$ by $g_{\mu\nu}$. The line element on $S^2 \times R$ is,

$$ds^2_{S^2 \times R} = \rho^2 (du^2 + d\Omega^2)$$

where, $u$ is the co-ordinate on $R$ and $d\Omega$ is the solid angle. The line element on $R^3 - \{0\}$ in spherical polar co-ordinates is,

$$ds^2_{R^3 - \{0\}} = dr^2 + r^2 d\Omega^2. \quad (41)$$

We therefore see that the metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are related by a conformal transformation, $\tilde{g}_{\mu\nu} = e^{2f} g_{\mu\nu}$ with $f(x) = -u$. Thus the manifolds $S^2 \times R$ and $R^3 - \{0\}$ are conformally equivalent. We see that $\frac{|x_0 - y_0|}{\rho} = |u - v|$ where $u$ and $v$ are dimensionless, real–valued quantities. In terms of the dimensionless variables $u$, $v$

$$G_{\tilde{g}}(x, y) = \frac{1}{8\pi \rho} \operatorname{cosech} \frac{|u - v|}{2}. \quad (41)$$
Equation (40) gives a general definition of a primary field. The specific case of this, when we look at the two-point correlation function, can be easily derived and the weights fixed. The two-point correlation function $G_g(x, y)$ is the solution to the equation

$$(-\Box_g + m^2)G_g(x, y) = \frac{1}{\sqrt{g}}\delta(x - y)$$

(42)

and $G_{\tilde{g}}(x, y)$ is the solution to

$$(-\Box_{\tilde{g}} + \tilde{m}^2)G_{\tilde{g}}(x, y) = \frac{1}{\sqrt{\tilde{g}}}\delta(x - y).$$

(43)

Under the conformal transformation of the metric, $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{-2u}g_{\mu\nu}$, we have

$$\sqrt{\tilde{g}} = e^{-3u}\sqrt{g}$$

and

$$(-\Box_g + m^2) \rightarrow (-\Box_{\tilde{g}} + \tilde{m}^2)$$

(44)

where

$$(-\Box_{\tilde{g}} + \tilde{m}^2) = e^{\frac{5u}{2}}(-\Box_g + m^2) e^{-\frac{3u}{2}}.$$  

(45)

(See reference [6] for the transformation property of the conformal Laplacian.) Using all this in (43) we get the transformation property for the Green’s function:

$$e^{-\frac{u}{2}}G_{\tilde{g}}(x, y) e^{-\frac{u}{2}} = G_g(x, y).$$

(46)

$$e^{-\frac{u}{2}}G_{\tilde{g}}(x, y) e^{-\frac{u}{2}} = \frac{1}{4\pi|\rho - \rho'|}$$

(47)

where $\rho = \rho e^u$ and $\rho' = \rho e^v$. The right hand side is nothing but the Green’s function on $R^3 - \{0\}$ when the angular separation between two points $x$ and $y$ is zero. This shows that the $\phi$ fields are primary fields and transform with weight $\delta = -\frac{1}{2}$.

4. A manifold with negative curvature: $H^2 \times R$

We consider a space which is a product of two non-compact manifolds, $H^2$ being the surface of a three dimensional hyperboloid, $R$ the real line. We denote the coordinate on $R$ by $x_0$ and parametrize $H^2$ as

$$H^2 = \{\bar{x} = (x_1, x_2), \ x_1 \in R, \ 0 < x_2 < \infty\}$$

with line element and Laplacian given respectively by

$$ds^2 = \frac{\rho^2}{x_2^2}(dx_1^2 + dx_2^2)$$

$$\Delta_{H^2} = \frac{x_2^2}{\rho^2}(\partial_{x_1}^2 + \partial_{x_2}^2)$$

(48)
where $\rho$ is a constant positive parameter. The scalar curvature for $H^2$ (hence for the product manifold) is $R = -\frac{2}{\rho^2}$; therefore $\xi R = -\frac{1}{4}\rho^2$. At the critical point the value of $m^2$ is $m^2_c = \frac{1}{4\rho^2}$, which exactly cancels with $\xi R$. We have then:

$$-\Box_g + m^2_c = -\Delta_R - \Delta_{H^2} + \xi R + m^2_c = -\Delta_R - \Delta_{H^2}. \quad (49)$$

This implies that the heat kernel of $-\Box_g + m^2_c$ is just the product of the heat kernel of $-\Delta_R$, given by (23), and the heat kernel \[7\] of $-\Delta_{H^2}$ which is

$$h_{H^2}(t; \bar{x}, \bar{y}, g_{H^2}) = \rho \frac{2\pi e^{-\frac{d^2}{4\rho^2}}}{(4\pi t)^{\frac{3}{2}}} \int_{\frac{1}{2}}^{\infty} \frac{\tau e^{-\frac{\tau^2}{4\rho^2}}}{\sqrt{\cosh \tau - \cosh \frac{d}{\rho}}} d\tau; \quad (50)$$

d is the geodesic distance on $H^2$ defined as

$$\cosh \frac{d}{\rho}(\bar{x}, \bar{y}) = 1 + \frac{|\bar{x} - \bar{y}|^2}{2\bar{x}\bar{y}}. \quad (51)$$

Substituting (23) and (50) in (6) we get

$$G(x, y) = \frac{\rho}{8\sqrt{2}\pi^2} \int_{\frac{1}{2}}^{\infty} d\tau \frac{\tau}{\sqrt{\cosh \tau - \cosh \frac{d}{\rho}}} \int_0^\infty dt \; t^{-\frac{1}{2}} e^{-\frac{|x_0 - y_0|^2 - \tau^2}{4t}} e^{-\frac{\tau}{4\rho^2}};$$

the integral in $t$ can be performed using (19) and we find

$$G_{H^2 \times R}(x, y, g) = \frac{1}{4\sqrt{2}\pi^2\rho} \int_{\frac{1}{2}}^{\infty} d\tau \frac{\mathcal{K}_1(\frac{1}{2}\sqrt{\tau^2 + \frac{|x_0 - y_0|^2}{\rho^2}})}{\sqrt{\cosh \tau - \cosh \frac{d}{\rho}}} \frac{\tau}{\sqrt{\tau^2 + \frac{|x_0 - y_0|^2}{\rho^2}}}; \quad (52)$$

where $\mathcal{K}_1(z)$ is a MacDonald’s function.

The large distance behaviour of the two-point correlation function can be analyzed in the $H^2$ and $R$ directions separately. For $d(\bar{x}, \bar{y}) \to \infty$, $|x_0 - y_0|$ remaining finite, $\tau$ is also large (being $\tau \geq d$), so that we can approximate the MacDonald’s function by its asymptotic expression given in (19). We have

$$\mathcal{K}_1\left(\frac{1}{2}\sqrt{\tau^2 + \frac{|x_0 - y_0|^2}{\rho^2}}\right) \sim \mathcal{K}_1\left(\frac{\tau}{2} \frac{\rho}{\sqrt{\tau^2 + \frac{|x_0 - y_0|^2}{\rho^2}}} \right) \sim \frac{1}{\tau} e^{-\frac{\pi}{\tau}};$$

in this approximation the integral in $\tau$ in (52) can be performed and we find

$$G(x, y) \to \frac{1}{4\pi^2\rho} e^{-\frac{d}{2\rho}} \mathcal{K}_0\left(\frac{d}{2\rho}\right); \quad (53)$$

on using again (19), we finally obtain

$$G(x, y) \to \frac{1}{4\pi\rho} \frac{e^{-\frac{d}{\rho}}}{\sqrt{\frac{d}{\rho}}}. \quad (54)$$
Let us analyze the other limit, \( |x_0 - y_0| \to \infty \) keeping \( d \) fixed, in particular we choose it to be zero. In this case, we can still use the large \( z \) approximation for \( K_\nu(z) \) and (52) becomes

\[
G(x, y) \xrightarrow{|x_0 - y_0| \to \infty} \frac{1}{8\pi^2} \rho \int_{|x_0 - y_0|}^{\infty} d\tau \frac{e^{-\frac{\tau}{2}}}{\sqrt{\tau} \sinh \frac{\tau}{2} \sqrt{\tau^2 - \frac{|x_0 - y_0|^2}{\rho^2}}}
\]

\[
\approx \frac{1}{8\pi^2} K_\frac{1}{2}(\frac{1}{4\rho}|x_0 - y_0|)
\]

and using (19)

\[
G(x, y) \xrightarrow{|x_0 - y_0| \to \infty} \frac{1}{4\pi} \frac{e^{-\frac{|x_0 - y_0|}{2\rho}}}{|x_0 - y_0|}.
\]

In both the cases analyzed \( (d \to \infty \text{ and } |x_0 - y_0| \to \infty) \), the large distance behaviour of the two–point correlation function is not a power law, that is the correlation length is not infinite, as we might have expected from the manifold being a non-compact one in all the directions and from the fact that there is a non-zero spontaneous magnetization \( b \) as we showed in [1]. We find instead an exponential decay where the finite correlation length is proportional to \( \rho \). We conclude that the finiteness of the correlation length, which is generally associated with the finite size of the manifold, is more precisely connected to the presence of a scale in the theory, in this case the scale being the radius of curvature of the manifold.

Let us finally analyze the short distance behaviour of the two–point Green’s function (52). For simplicity we consider separately the two limits \( d \to 0, \ |x_0 - y_0| = 0 \) and \( |x_0 - y_0| \to 0, \ d = 0 \).

For \( d \to 0, \ |x_0 - y_0| = 0 \), (52) can be approximated as

\[
G(x, y) \xrightarrow{d \to 0} \frac{1}{8\pi^2} \rho \int_{\frac{1}{2}}^{\infty} d\tau \frac{K_1(\frac{1}{2}\tau)}{\sinh \frac{\tau}{2}}.
\]

The integrand is very rapidly converging to zero as \( \tau \) increases. We can split the integral in two parts and use two different approximations for the MacDonald’s function:

\[
K_\nu(x) \sim 2^{\nu-1}(\nu - 1)!x^{-\nu}, \quad \nu > 0
\]

valid in the region \( x \leq 1 \), and (19) which is valid for \( x > 1 \). We have

\[
G(x, y) \xrightarrow{d \to 0} \frac{1}{8\pi^2} \rho \left\{ \int_{\frac{1}{2}}^{1} d\tau \frac{K_1(\frac{1}{2}\tau)}{\sinh \frac{\tau}{2}} + \int_{1}^{\infty} d\tau \frac{K_1(\frac{1}{2}\tau)}{\sinh \frac{\tau}{2}} \right\};
\]

on substituting (58) in the first integral and (19) in the second one we find

\[
G(x, y) \xrightarrow{d \to 0} \frac{1}{2\pi^2} \rho \left\{ \left( \frac{d}{\rho} \right)^{-1} + \text{finite terms} \right\}.
\]
Similarly, for $|x_0 - y_0| \to 0$, $d = 0$, (52) can be approximated as
\[
G(x, y) \to \frac{1}{8\pi^2 \rho} \int_0^\infty d\tau \frac{K_1(\frac{\tau}{2})}{\sinh \frac{\tau}{2}} \frac{\tau}{\sqrt{\tau^2 + \frac{|x_0 - y_0|^2}{\rho^2}}} 
\]
where $\tau' = \sqrt{\tau^2 + \frac{|x_0 - y_0|^2}{\rho^2}}$. This integral is identical to (57), therefore it can be evaluated in the same way yielding
\[
G(x, y) \to \frac{1}{8\pi^2 \rho} \int_0^\infty d\tau' \frac{K_1(\frac{\tau'}{2})}{\sinh \frac{\tau'}{2}}. 
\]
(61)

Both the results (60) and (62) confirm the expected flat space behaviour of the two–point Green’s function for short distances.

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