BRST superspace and auxiliary fields for $\mathcal{N}=1$ supersymmetric Yang-Mills theory

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Abstract

We use a Becchi-Rouet-Stora-Tyutin (BRST) superspace approach to formulate off-shell nilpotent BRST and anti-BRST transformations in four dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory. The method is based on the possibility of introducing auxiliary fields through the supersymmetric transformations of the superpartener of the gauge potential associated to a supersymmetric Yang-Mills connection. These fields are required to achieve the off-shell nilpotency of the BRST and anti-BRST operators. We also show how this off-shell structure is used to build the BRST and anti–BRST invariant gauge-fixing quantum action.

Keywords: Supersymmetric gauge theories; An off-shell nilpotent BRST and anti-BRST algebra.

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1 Introduction

An on-shell quantization of general gauge theories which are reductible and/or whose classical gauge algebra is not closed (for a review see e.g. Ref. [1]), can be successfully performed by the Batalin-Vilkovisky (BV) formalism [2].

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The BV formalism is a very general covariant Lagrangian approach which overcomes the need of closed classical algebra by a suitable construction of BRST operator. The construction is realized by introducing a set of new antifields besides the fields occurring in the theory. The elimination of these antifields at the quantum level via a gauge-fixing procedure leads to the quantum theory in which effective BRST transformations are nilpotent only on-shell.

Another possibility for quantizing reducible and/or open gauge theories consists to introduce a set of auxiliary fields as in supersymmetric theories, such as the Wess-Zumino model for which it is only with auxiliary fields that one can obtain a tensor calculus [3], or in topological antisymmetric tensor gauge theories, the so-called BF theories [4]. The introduction of auxiliary fields are needed to close the gauge algebra, and then provides an efficient way to use the usual Faddeev-Popov quantization method [5].

On the other hand, it is well known that in the superspace formalism one can naturally introduce a set of auxiliary fields that gives rise to the construction of the off-shell BRST invariant quantum action. Indeed, as shown in Ref. [4] for the case of 4D non-Abelian BF theory and Ref. [6] for the case of the simple supergravity where the classical gauge algebra is open [3], the superspace formalism has been used in order to realize the quantization of such theories. It leads to introduce the minimal set of auxiliary fields ensuring the off-shell invariance of the quantum action.

In the context of superspace formalism, the gauge field, the ghost and anti-ghost fields in gauge theories can be incorporated into a natural gauge superconnection by extending spacetime to a (4, 2)−dimensional superspace [7]. In this framework, the BRST and anti-BRST transformations can be derived by imposing horizontality conditions on the supercurvature associated to a superconnection on a superspace.

Let us mention that in the case of supersymmetric theories the corresponding supersymmetric algebra close only modulo equations of motion [8, 9]. So, the appropriate framework to quantize such theories is the BV one. Indeed, in Ref. [10] it has been discussed how to realize the quantization of supersymmetric systems in BV approach.

As shown in [11, 12], a superfield description of the BV quantization method can be realized by simply introducing superfields whose lowest components coincide with the usual fields in the BV formalism. Superfield method has provided a powerful tool for producing supersymmetric field equations for any degree of supersymmetry. In [8, 13], one has also estab-
lished an off-shell superfield formulation of four dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. Considerably more involved off-shell superfield formulations are also available for $\mathcal{N} = 2$ in terms of harmonic and analytic superspace [14], while the off-shell formulation of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with non-Abelian gauge group $SU(N)$ is not available in terms of unconstrained fields because it would require the introduction of an infinite set of auxiliary fields [15].

The purpose of the present paper is to derive, in the framework of BRST superspace\(^1\), the off-shell nilpotent version of the BRST and anti-BRST transformations for $\mathcal{N} = 1$, $D = 4$ supersymmetric Yang-Mills theory ($\text{SYM}_4$), in analogy to what is realized for the case of simple supergravity [4] and the four-dimensional non-Abelian BF theory [6]. The Action for the $\mathcal{N} = 1$ supersymmetric Yang-Mills in four dimensions is given by

$$S_0 = \int dx^4 \left( \frac{-1}{2g^2} tr F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{8\pi^2} tr F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{i}{2} tr \overline{\lambda} \sigma^\mu D_\mu \lambda \right), \quad (1)$$

where $(A_\mu, \lambda)$ is the gauge multiplet, $g$ is the gauge coupling, $\theta$ is the instanton angle, the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ is the dual of $F$, and $D_\mu = \partial_\mu + [A_\mu]$. By construction, the Action (1) is invariant under the supersymmetry transformations

$$\begin{align*}
\delta_\xi A_\mu &= i \xi \overline{\sigma}_\mu \lambda - i \overline{\lambda} \sigma_\mu \xi, \\
\delta_\xi \lambda &= \sigma^{\mu\nu} F_{\mu\nu} \xi,
\end{align*} \quad (2)$$

where $\xi$ is a spin 1/2 valued infinitesimal supersymmetry parameter.

Our paper is organized as follows. In Section 2, the BRST superspace approach and horizontality conditions for $\mathcal{N} = 1$, $D = 4$ supersymmetric Yang-Mills theory are discussed. In Section 3, we introduce the various fields and derive the off-shell nilpotent BRST and anti-BRST transformations. The construction of the BRST-invariant quantum action for $\mathcal{N} = 1$ super Yang-Mills theory in terms of this off-shell structure is described in Section 4. The obtained quantum action permits us to see that the extrafields are nondynamical, auxiliary fields. Section 5 is devoted to concluding remarks.

\(^1\)Here, we call the superspace obtained by enlarging spacetime with two ordinary anticommuting coordinates BRST superspace, in order to distinguish it from superspace of supersymmetric theories.
2 BRST superspace

Let $\Phi$ be a super Yang-Mills connection on a $(4, 2)$-dimensional BRST superspace with coordinates $z = (z^M) = (x^\mu, \theta^\alpha)$, where $(x^\mu)_{\mu=1,...,4}$ are the coordinates of the spacetime manifolds and $(\theta^\alpha)_{\alpha=1,2}$ are ordinary anticommuting variables. Acting the exterior covariant superdifferential $D$ on $\Phi$ we obtain the supercurvature $\Omega$ satisfying the structure equation, $\Omega = d\Phi + \frac{1}{2}[\Phi, \Phi]$, and the Bianchi identity, $d\Omega + [\Phi, \Omega] = 0$. The superconnection $\Phi$ as 1-superform on the BRST superspace can be written as

$$\Phi = dz^M(\Phi^i_M I_i + \Phi^\mu_M P_\mu + \Phi^\alpha_M Q_\alpha),$$

(3)

where $\{I_i\}_{i=1,...,d}$, and $\{P_\mu, Q_\alpha\}_{\mu=1,...,4; \alpha=1,...,4}$ are the generators of the internal symmetry group $(G)$ and the $\mathcal{N} = 1$ supersymmetric group $(SG)$ respectively. They satisfy the following commutation relations

$$[I_i, I_j] = f^{k}_{ij} I_k,$$
$$[I_i, P_\mu] = [Q_\alpha, P_\mu] = [P_\nu, P_\mu] = 0,$$
$$[Q_\alpha, Q_\beta] = 2(\gamma^\mu)_{\alpha\beta} P_\mu,$$
$$[I_i, Q_\alpha] = b_i^* Q_\alpha,$$

(4)

where $\{\gamma^\mu\}_{\mu=1,...,4}$ are the Dirac matrices in the Weyl basis, $b_i^* = b_i$ for $\alpha = 1, 2$ and $b_i^* = -b_i$ for $\alpha = 3, 4$ giving the representation of the internal symmetry of $Q_\alpha$ and $[,]$ the graded Lie bracket. Let us mention that the supersymmetric generators $\{Q_\alpha\}$ are given in the Majorana representations [9]. Note that the Grassmann degrees of the superfield components of $\Phi$ are given by $|\Phi^i_M | = m$, $|\Phi^\mu_M | = m + 1$ (mod 2), where $m = |z^M |$ ($m = 0$ for $M = $ and $m = 1$ for $M = \alpha$), since $\Phi$ is an even 1-superform.

However, we assign to the anticommuting coordinates $\theta^1$ and $\theta^2$ the ghost numbers $(-1)$ and $(+1)$ respectively, and ghost number zero for an even quantity: either a coordinate, a superform or a generator. These rules permit us to determine the ghost numbers of the superfields $(\Phi^i_\mu, \Phi^\nu_\mu, \Phi^\alpha_\mu, \Phi^i_1, \Phi^i_2, \Phi^\nu_1, \Phi^\nu_2, \Phi^\alpha_1, \Phi^\alpha_2)$ which are given by $(0, 0, 0, +1, -1, +1, -1, +1, -1)$.

Upon expressing the supercurvature $\Omega$ as

$$\Omega = \frac{1}{2} dz^N \wedge dz^M \Omega_{MN} = \frac{1}{2} dz^N \wedge dz^M (\Omega^i_M I_i + \Omega^\mu_M P_\mu + \Omega^\alpha_M Q_\alpha),$$

(5)
we find from the structure equation
\[ \Omega_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu + [\Phi_\mu, \Phi_\nu], \]
\[ \Omega_{\mu\alpha} = \partial_\mu \Phi_\alpha - \partial_\alpha \Phi_\mu + [\Phi_\mu, \Phi_\alpha], \]
\[ \Omega_{\alpha\beta} = \partial_\alpha \Phi_\beta + \partial_\beta \Phi_\alpha + [\Phi_\alpha, \Phi_\beta]. \]

Similarly, the Bianchi identity becomes
\[ D_\mu \Omega_{\nu\kappa} + D_\kappa \Omega_{\mu\nu} + D_\nu \Omega_{\kappa\mu} = 0, \]
\[ D_\alpha \Omega_{\mu\nu} - D_\nu \Omega_{\mu\alpha} + D_\mu \Omega_{\nu\alpha} = 0, \]
\[ D_\alpha \Omega_{\beta\gamma} + D_\beta \Omega_{\alpha\gamma} + D_\gamma \Omega_{\alpha\beta} = 0, \]
\[ D_\mu \Omega_{\alpha\beta} - D_\alpha \Omega_{\mu\beta} - D_\beta \Omega_{\mu\alpha} = 0, \]

where \( D_M = \partial_M + [\Phi_M, .] \) is the \( M \) covariant superderivative. Now, we shall search for the constraints to the supercurvature \( \Omega \) in which the consistency with the Bianchi identities (7) is ensured. This requirement insures then the off-shell nilpotency of the BRST and anti-BRST algebra. The full set of supercurvature constraints turns out to be given by
\[ \Omega_{\mu\alpha} = 0, \quad \Omega_{\alpha\beta} = 0. \]

It is easy to check the consistency of this set of supercurvature constraints through an analysis of the Bianchi identities. Indeed, we remark that identities (7c) and (7d) are automatically satisfied because of the constraints (8) while identities (7a) and (7b) yield a further restriction on supercurvature \( \Omega \)
\[ F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + [A_\mu, A_\nu]^i, \]
\[ \Omega^\kappa_{\mu\nu} = 0, \quad \Omega^\alpha_{\mu\nu} = 0. \]

At this point, let us mention that the consistency of the horizontability conditions (8) and (9) with the Bianchi identities (7), as we will see later, guarantees automatically the off-shell nilpotency of the BRST and anti-BRST transformations on all the fields belonging to \( \mathcal{N} = 1 \) super Yang-Mills theory.
3 Auxiliary fields

In order to derive the off-shell BRST structure of $N = 1$ super Yang-Mills theory using the above BRST superspace formalism, it is necessary to give the geometrical interpretation of the fields occurring in the quantization of such theory. Besides the gauge potential $\Phi^i_\mu| = A^i_\mu$, there exists the superpartner $\Phi^i_\alpha| = \lambda^i_\alpha$ of $A^i_\mu$ which is introduced via the field redefinition

$$\Phi^i_\rho| = -\frac{1}{4} \left( \gamma^\rho_{\sigma\tau} \right) \phi \Phi^i_\mu|, \Phi^i_\mu| = 0$$

representing the fact that the gauge potential associated to the translation will not exist in the super Yang-Mills theory and an auxiliary\(^2\) real field $\Phi^i_\mu| = \Lambda^i$ which is required for the construction of off-shell nilpotent BRST and anti-BRST transformations and introduced via the supersymmetric transformations of the superpartner of the gauge potential

$$\Phi^i = \delta^\rho_\sigma / 4 \left[ Q_\rho, \Phi^i_\sigma \right].$$

Furthermore, we introduce the following now: $\Phi^i_1| = c^i_1$ is the ghost for Yang-Mills symmetry, $\Phi^i_2| = c^i_2$ is the antighost of $c^i_1$, $B^i = \partial_1 \Phi^i_2|$ is the associated auxiliary field, $\Phi^\alpha_1| = \chi^\alpha_1$ is the supersymmetric ghost, $\Phi^\alpha_2| = \overline{\chi}^\alpha_2$, is the antighost of $\chi^\alpha_1$, $G^\alpha = \partial_1 \Phi^\alpha_2|$ is the associated auxiliary field, $\Phi^\mu_1| = \xi^\mu_1$ is the translation symmetry ghost, $\Phi^\mu_2| = \overline{\xi}^\mu_2$, is the antighost of $\xi^\mu_1$ and $E^\mu = \partial_1 \Phi^\mu_2|$ is the associated auxiliary field. Let us mention that the symmetry ghosts antighosts $\chi^\alpha_\rho$ are commuting fields while the others $c^i_\rho$ and $\xi^\mu_\rho$ are anticommuting. Note that the symbol “ $|$ ” indicates that the superfield is evaluated at $\theta^\alpha = 0$. We also realize the usual identifications: $Q_\alpha (X_1) = \partial_\alpha X_1|$, where $X$ is any superfield $Q = Q_1$ and $(\overline{Q} = Q_2)$ is the BRST (anti-BRST) operator.

The action of the $N = 1$ supersymmetric generators $\{P_\mu, Q_\alpha\}_{\mu=1,\ldots,4;\alpha=1,\ldots,4}$ on these fields is given by

$$
\begin{align*}
[Q_\alpha, X] &= \partial_\alpha X, \\
[P_\mu, X] &= \partial_\mu X, \\
[Q_\alpha, A^i_\mu] &= -\left( \gamma^\mu_{\alpha\beta} \right) \lambda^\beta_1, \\
[Q_\alpha, \lambda^\beta_1] &= -\frac{1}{2} \left( \sigma^\mu_{\alpha\beta} \right) F^\beta_\mu + \delta^\beta_\alpha \Lambda^i,
\end{align*}
$$

\(^2\)By definition an auxiliary field does not describe an independent degree of freedom; its equation of motion is algebraic.
\[
\begin{align*}
[Q_\alpha, c^i_\rho] &= 2(\gamma^\mu)_{\alpha\beta}\chi_\rho^{\beta}A^i_\mu, \\
[Q_\alpha, \Lambda^i] &= \gamma^\mu D_\mu\chi_\alpha^{\beta}, \\
[Q_\alpha, F^i_{\mu\nu}] &= (\gamma_\mu)_{\alpha\beta}(D_\nu\lambda)^{\beta i} - (\gamma_\nu)_{\alpha\beta}(D_\mu\lambda)^{\beta i},
\end{align*}
\]

where \(D_\mu = \partial_\mu + [A_\mu, \cdot]\) and \(X\) any fields.

It is worthwhile to mention that we are interested in our present investigation on the global supersymmetric transformations, so that the parameters of the \(\mathcal{N} = 1\) supersymmetric and translation groups must be space-time constant, i.e.

\[
\begin{align*}
\partial_\mu \chi_\alpha^\rho &= 0, \\
\partial_\mu \xi_\alpha^\nu &= 0.
\end{align*}
\]

Using the above identifications with (12) and inserting the constraints (8) and (9) into the Eqs. (6) and (7b), we obtain

\[
\begin{align*}
\partial_\alpha \Phi^i_{\mu|} &= (D_\mu c_\alpha)^i - \xi_\alpha^\nu [P_\nu, A^i_\mu] - \chi_\alpha^\rho [Q_\rho, A^i_\mu], \\
\partial_\alpha \Phi^i_{\beta|} + \partial_\beta \Phi^i_{\alpha|} &= -[c_\alpha, c_\beta]^i - \xi_\alpha^\nu [P_\nu, c^i_\beta] - \chi_\alpha^\rho [Q_\rho, c^i_\beta], \\
-\xi_\beta^\nu [P_\nu, c^i_\alpha] - \chi_\beta^\rho [Q_\rho, c^i_\alpha], \\
\partial_\alpha \Phi^i_{\beta|} + \partial_\beta \Phi^i_{\alpha|} &= 0, \\
\partial_\alpha \Phi^i_{\mu|} + \partial_\beta \Phi^i_{\nu|} &= -2\chi_\alpha^\rho(\gamma^\nu)_\rho\chi_\beta^\nu, \\
\partial_\alpha \Phi^i_{\nu|} &= -[\lambda^\kappa, c_\alpha]^i - \xi_\alpha^\nu [P_\nu, \Lambda^i] - \chi_\alpha^\rho [Q_\rho, \Lambda^i] + \chi_\alpha^\rho \Lambda^i, \\
\partial_\alpha \Omega_{\mu\nu|} &= -[c_\alpha, F^i_{\mu\nu}] - \xi_\alpha^\nu [P_\tau, F^i_{\mu\nu}] - \chi_\alpha^\rho [Q_\rho, F^i_{\mu\nu}], \\
\chi_\alpha^\rho (\partial_\beta \varphi^i_\alpha) + \chi_\beta^\rho (\partial_\alpha \varphi^i_\beta) &= -\chi_\alpha^\rho \left( [c_\alpha, \Lambda^i] + \xi_\alpha^\nu [P_\nu, \Lambda^i] + \chi_\alpha^\rho [Q_\rho, \Lambda^i] \right) \\
&- \chi_\beta^\rho \left( [c_\alpha, \Lambda^i] + \xi_\alpha^\nu [P_\nu, \Lambda^i] + \chi_\alpha^\rho [Q_\rho, \Lambda^i] \right), \\
\partial_\alpha \Phi^i_{\mu|} &= (\gamma^\mu)^{\sigma\alpha}/4 [Q_\sigma, \partial_\alpha \Phi^i_{\mu|}], \\
\partial_\alpha \Phi^i &= -\delta^\rho_\sigma/4 [Q_\rho, \Phi^i_\sigma].
\end{align*}
\]

Inserting Eq. (11) into (13), and evaluating these at \(\theta^\alpha = 0\), we find the following BRST transformations
\[ QA^i_\mu = D_\mu c^i - \xi^\rho \partial_\rho A^i_\mu + \chi_\gamma \mu A^i, \]

\[ Q\lambda^i_\alpha = -f^i_{jk} c^j \lambda^k_\alpha - \xi^\rho \partial_\rho \lambda^i_\alpha + \frac{1}{2}(\chi \sigma^{\mu \nu})_\alpha F^i_{\mu \nu} + \chi_\alpha \Lambda^i, \]

\[ Qc^i = -\frac{1}{2}f^i_{jk} c^j c^k - \xi^\rho \partial_\rho c^i + \chi_\gamma^\mu c A^i_\mu, \]

\[ Q\xi^\rho = -\chi_\gamma^\rho \chi, \]

\[ Q\Lambda^i = -f^i_{jk} c^j \Lambda^k - \xi^\rho \partial_\rho \Lambda^i - \chi_\gamma^\mu D_\mu \Lambda^i, \]

\[ QF^i_{\mu \nu} = -f^i_{jk} c^j F^k_{\mu \nu} - \xi^\rho \partial_\rho F^i_{\mu \nu} - \chi^\rho \left\{ (\gamma_\mu)_{\rho \sigma} (D_\nu \lambda)^{\sigma i} - (\gamma_\nu)_{\rho \sigma} (D_\mu \lambda)^{\sigma i} \right\} \]

\[ Q\chi^\alpha = 0, \]

\[ QB^i = 0, \]

\[ Q\xi^\mu = E^\mu, \]

\[ QE^\mu = 0, \]

\[ Q\Lambda^i = G^\alpha, \]

\[ QG^\alpha = 0, \] (14)

and also the anti-BRST transformations, which can be derived from (14) by the following mirror symmetry of the ghost numbers given by: \( X \to X \) if \( X = A^i_\mu, \lambda^i_\alpha, \Lambda^i, \) \( X \to \overline{X} \) if \( X = Q, c^i, B^i, \xi^\mu, E^\mu, \chi^\alpha, G^\alpha \), and \( X = \overline{X} \) where

\[ B^i + \overline{B}^i = f^i_{jk} c^j \overline{c}^k - \xi^\rho \partial_\rho c^i - \overline{\xi}^\rho \partial_\rho \overline{c}^i - \chi_\gamma^\mu \overline{c} A^i_\mu - \overline{\chi}_\gamma^\mu \overline{A}^i_\mu, \] (15)

\[ E^\mu + \overline{E}^\mu = 2\chi_\gamma^\mu \overline{c}, \]

\[ G^\alpha + \overline{G}^\alpha = 0. \] (16)

Let us note that the introduction of an auxiliary real field \( \Lambda^i \) besides the fields present in quantized \( \mathcal{N} = 1 \) super Yang-Mills theory in four-dimensions, guarantees automatically the off-shell nilpotency of the \( \{Q, \overline{Q}\}\)-algebra and make easier then, as we will see in the next Section, the gauge-fixing process.

### 4 Quantum action

In the present section, we show how to construct in the context of our procedure a BRST-invariant quantum action for \( \mathcal{N} = 1 \) super Yang-Mills theory.
as the lowest component of a quantum superaction. To this purpose, let us recall that the gauge-fixing superaction similar to that obtained in the case of Yang-Mills theories [16, 17] and gauge-affine gravity [18] is given by

\[ S_{sgf} = \int d^4x L_{sgf}, \]
\[ L_{sgf} = (\partial_1 \Phi_2)(\partial^\mu \Phi_\mu) + (\partial^\mu \Phi_2)(\partial_1 \Phi_\mu) + (\partial_1 \Phi_2)(\partial_1 \Phi_\mu). \] (17)

We note first that in the case of Yang-Mills theory the superaction involve a Lorentz gauge [19] given by

\[ \partial_\mu \Phi_\mu = 0. \] (18)

In the case of super Yang-Mills theory we shall choose a supersymmetric gauge-fixing which is the extension of the Lorentz gauge. This gauge fixing can be obtained from (18) by using the following substitution

\[ \Phi_\mu \rightarrow \tilde{\Phi}_\mu = \Phi_\mu + [\partial_\mu \Phi^\alpha, Q_\alpha]. \] (19)

Now, it is easy to see that the gauge-fixing superaction (17) can be put in the following form

\[ S_{sgf} = \int d^4x (\partial_1 \Phi_2)(\partial^\mu \tilde{\Phi}_\mu) + (\partial^\mu \Phi_2)(\partial_1 \tilde{\Phi}_\mu). \] (20)

To determine the gauge-fixing action \( S_{gf} \) as the lowest component of the gauge-fixing superaction \( S_{gf} = S_{sgf} \), we impose the following rules

\[ Tr(I^m I^n) = \delta^m_n, \]
\[ Tr([Q_\alpha, Q_\beta]) = 2\gamma^\mu_{\alpha\beta} \partial_\mu, \]
\[ Tr(P^2) = 0. \] (21)

These rules permit us to compute the trace of each term in (20). Indeed, from (21) it is easy to put the gauge-fixing action \( S_{gf} \) in the form

\[ S_{gf} = S_{sgf} = \int d^4x (B^\mu A_\mu + 2b^*_j G(\gamma^\mu \partial_\mu \Box \lambda^j) \]
\[ + (\partial^\mu \tau) \{ D_\mu c + \xi^\mu \partial_\mu A_\mu + \chi \gamma_\mu \lambda \}) \]
\[ - 2b^*_j (\partial^\mu \chi) \gamma^\nu \partial_\nu \partial_\mu \left\{ f_{ik} \lambda^k c^i + \xi^\tau \partial_\tau \lambda^j + \frac{1}{2} \chi \sigma^{\tau \nu} F_{\tau \nu} + \chi D^j \right\}. \] (22)
On the other hand, the presence of the extrafields breaks the invariance of the classical action (1). In fact, the only terms which may contribute to the $Q$-variation of the classical action $S_0$ are those containing the extrafield $\Lambda^i$. This follows from the fact that the BRST transformations up to terms $\Lambda^i$ represent the $\mathcal{N} = 1$ super Yang-Mills transformations expressed à la BRST. A simple calculation with the help of the BRST transformations (14) leads to

$$QS_0 = \Lambda^i \gamma^\mu D_\mu \Lambda_i.$$  

(23)

Thus the classical action $S_0$ is not BRST-invariant, and in order to find the BRST-invariant extension $S_{inv}$ of the classical action, we shall add to $S_0$ a term $\tilde{S}_0$ so that

$$Q(S_0 + \tilde{S}_0) = 0.$$  

(24)

To this end, we propose to write $\tilde{S}_0$, which define the extension action for the auxiliary field $\Lambda^i$ as follows

$$\tilde{S}_0 = -\frac{1}{2} \Lambda^i \Lambda_i.$$  

(25)

Then, it is quite easy to show that $Q(S_0) = -Q(\tilde{S}_0)$ by a direct calculation with the help of the transformations (14).

Having found the BRST-invariant extension action $S_{inv}$ we now write the full off-shell BRST-invariant quantum action $S_q$ by adding to the $Q$-invariant action $S_{inv} = S_0 + \tilde{S}_0$ the $Q$-gauge-fixing action $S_{gf}$

$$S_q = S_0 + \tilde{S}_0 + S_{gf}.$$  

(26)

Let us mention that the quantum action (26) and the off-shell BRST transformations (14) obtained in the previous section are equivalent to those proposed in [10], where the Batalin-Vilkovisky formalism has been considered to close the supersymmetric algebras without relying on the unusual auxiliary fields.

It is worth noting that the quantum action (26) allows us to see that the auxiliary field $\Lambda^i$ does not propagate, as its equation of motion is a constraint

$$\frac{\delta S_q}{\delta \Lambda^i} = -\Lambda^i + 2 b^*_i (\chi \gamma^\mu \partial_\mu \Box \chi) = 0.$$  

(27)

Thus the essential role of the nondynamical auxiliary field $\Lambda^i$ is to close the BRST and anti-BRST algebra off-shell.

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The elimination of the auxiliary field $\Lambda^i$ by means of its equation of motion (27) leads to the same gauge-fixed theory with on-shell nilpotent BRST transformations obtained in the context of BV formalism [10] as well as in the framework of the superfibre bundle approach [16].

Moreover, in our formalism we have also introduced an anti-BRST operator $\overline{Q}$ and it is important to realize that both the BRST symmetry and anti-BRST symmetry can be taken into account on an equal footing. To this end, we simply use the fact that there is a complete duality, with respect to the mirror symmetry of the ghost number, between the $Q$- and $\overline{Q}$-transformations. So, the $\overline{Q}$-variation of the classical action $S_0$ is given by

$$QS_0 = \overline{\Lambda}^i \gamma^\mu D_\mu \overline{\rho}_i$$

where $\overline{\Lambda}^i$ represents an auxiliary field in the context of $\overline{Q}$-symmetry. Using however the $Q$-transformations of the auxiliary field (see Eqs. (14) with the mirror symmetry), we obtain that the $Q$-invariant action $S_{inv} = S_0 + \tilde{S}_0$ is also $\overline{Q}$-invariant. Furthermore, the $Q$-gauge-fixing action can be also written as in Yang-Mills theories in $\overline{Q}$-form. At this point, we remark that the mirror symmetry allows to replace, in particular the auxiliary field $\Lambda^i$ in the context of $Q$-symmetry by $\overline{\Lambda}^i$. Therefore the full off-shell BRST-invariant quantum action $S_q = S_0 + \tilde{S}_0 + S_{gf}$ is also an off-shell anti-BRST-invariant quantum action.

## 5 Conclusion

In this paper we have presented a BRST superspace approach in order to perform the quantization of the four dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory as model where the classical gauge algebra is not closed. The construction is entirely based on the possibility of introducing a set of auxiliary fields via the supersymmetric transformations of the superpartner of the gauge potential associated to a super Yang-Mills connection. The gauge fields and their associated ghost and antighost fields occurring in quantized four dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory have been described through a super Yang-Mills connection, whereas the extrafields coming from the supersymmetric transformations are required to achieve the off-shell nilpotency of the BRST and anti-BRST operators. The minimal set of
extrafields is defined after having imposed constraints on the supercurvature in which the consistency with the Bianchi identities is guaranteed.

Furthermore, we have performed a direct construction of the BRST invariant extension of the classical action for $\mathcal{N} = 1$, 4 $D$ supersymmetric Yang-Mills theory in analogy with what it is realized in simple supergravity [6] and four-dimensional BF theories [4]. The obtained quantum action allows us to see that the extrafields enjoy the auxiliary freedom, i.e. their auxiliary fields do not propagate, being vanishing on-shell. The elimination of these auxiliary fields using the solution of their equations of motion permits us to recover the standard quantum action with on-shell nilpotent BRST symmetry. The transformations of this minimal set of auxiliary fields and the obtained BRST invariant action agree with the standard results. By using the mirror symmetry between the BRST and anti-BRST transformations, we can see that the BRST invariant action is also anti-BRST invariant. Therefore the full quantum action is BRST and anti-BRST invariant, since the gauge-fixing action can be written as in the Yang-Mills case in BRST as well as anti-BRST exact form, due to the off-shell nilpotency of the BRST-anti-BRST algebra.

Let us note, that the Batalin-Vilkovisky formalism can be used to obtain the on-shell BRST invariant gauge fixed action for $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in four dimensions without requiring the set of auxiliary fields but by extending the fields in the theory to include the so-called anti-fields [10].

Finally, we should mention that the BRST superspace formalism presented in this paper was successfully applied to several interesting theories: simple supergravity [6] and 4$D$ non-Abelian BF theory where the symmetry is reducible [4]. Such formalism permit us to determine the off-shell nilpotent BRST and anti-BRST algebra for gauge theories. In particular, it gives another possibility leading to the minimal set of auxiliary fields. Thus, it would be a very nice endeavour to use this basic idea to study the structure of auxiliary fields in other gauge theories. These are some of the issues that are under investigation at the moment.

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