On Finite Noncommutativity in Quantum Field Theory

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Abstract

We consider various modifications of the Weyl-Moyal star-product, in order to obtain a finite range of nonlocality. The basic requirements are to preserve the commutation relations of the coordinates as well as the associativity of the new product. We show that a modification of the differential representation of the Weyl-Moyal star-product by an exponential function of derivatives will not lead to a finite range of nonlocality. We also modify the integral kernel of the star-product introducing a Gaussian damping, but find a nonassociative product which remains infinitely nonlocal. We are therefore led to propose that the Weyl-Moyal product should be modified by a cutoff like function, in order to remove the infinite nonlocality of the product. We provide such a product, but it appears that one has to abandon the possibility of analytic calculation with the new product.

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1 Introduction

In a noncommutative field theory the space-time coordinates $x_\mu$ are promoted to the status of operators $\hat{x}_\mu$ which are characterized by the commutation relation

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}.$$ (1)

The commutation relation (1) is motivated by the works [1] where $\theta_{\mu\nu}$ is taken as a tensor and [2], where $\theta_{\mu\nu}$ is taken as a constant antisymmetric matrix. In [1], it was shown that the commutation relation (1) can follow from a high energy gedanken experiment where, as the energy is raised, the formation of black holes prevents smaller distances than the diameter of the black hole to be measured. Then space-time can be characterized by uncertainty relations among coordinate operators, which can straightforwardly be interpreted as noncommutativity of space-time coordinates. In [2], the commutation relation (1) follows as a low-energy limit of open string theory in a background $B$-field.

Although noncommutative field theories have a solid motivation in the works [1] and [2], they exhibit certain problematic properties such as the UV/IR mixing effect [3] and the apparent violation of Lorentz invariance by the commutator (1). The problem of Lorentz noninvariance does nevertheless not result in a need to change the representations of the Poincaré group [1] and consequently, the classification of particles according to their spin and mass in commutative quantum field theory can be taken over to noncommutative quantum field theory without change. This justification is very important for the usage of Lorentz invariant quantities and representations of the Poincaré symmetry within noncommutative field theory. Therefore one might say that the problem of Lorentz noninvariance, within noncommutative field theory, is presently a smaller problem than the one of UV/IR mixing. Indeed, the UV/IR mixing effect has yet to be included in a theory, i.e. so that it does not spoil renormalizability, or alternatively by some unknown mechanism, completely removed from the theory. One can generally argue that the UV/IR mixing effect is a result of the infinite nonlocality of the commutator (1) and that restricting the range of it, should result in a theory without UV/IR mixing.
Work along these lines has been done in [5] where it was attempted to come to terms with the infinite nonlocality of noncommutative models of space-time by introducing a support for the noncommutativity parameter \( \theta \) inside a specific range. This, together with an appropriately chosen deformation of the states of the theory, results in a finite range for microcausality as it reduces to the microcausality of commutative quantum field theory outside the support of \( \theta \). However, this approach makes it difficult to construct an interaction that would remain nonlocal on a finite range and moreover, the definition of observables that respect this type of microcausality becomes nontrivial. In addition, the choice of the deformation of the states is highly nonunique. The present work differs from the approach in [5], in that no support for \( \theta \) is introduced, and were we able to construct a product under our requirements, there would be no problem of introducing interactions or observables into the theory.

The work is organized as follows. We begin by giving the representations of the star-product most useful for this work in section 2. We then move on to modify the star product with a Gaussian damping term and discuss the possibility of modifying the differential representation of the star-product. This analysis leads us to propose a modification of the integral-kernel of the Weyl-Moyal star-product by a Heaviside stepfunction cutoff in section 3. In section 4 we make our concluding remarks.

## 2 Representations of the star-product

In the usual approach to noncommutative quantum field theory we assume the commutator

\[
\left[ \hat{x}_\mu, \hat{x}_\nu \right] = i \theta_{\mu\nu},
\]  

(2)

where \( \theta_{\mu\nu} \) is taken to be a constant and antisymmetric matrix. However, because of the loss of unitarity [6] and causality [7, 8], in theories where time and space do not commute, one often considers theories with only space-space noncommutativity. This will also be the approach in this work. Additionally, since \( \theta_{\mu\nu} \) can always be transformed into a frame where only four
distinct components of the antisymmetric matrix survive and only two of them are independent, we take $\theta_{\mu\nu}$ to be given by

$$\theta_{23} = \theta \neq 0, \quad \theta_{12} = \theta_{13} = \theta_{0i} = 0,$$

so that the symmetry of space-time is $O(1,1) \times SO(2)$. The standard way to realize the commutator (2) is via the Weyl-Moyal star-product

$$(fg)(\hat{x}) \rightarrow (f \ast_W g)(x) = \exp \left[ \frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right] f(x)g(y) \bigg|_{x=y}. \quad (4)$$

That is, products between functions of $x$ are now taken with respect to the star-product (4).

Another way is to use the coherent-state basis. This corresponds to optimal localization in the noncommutative plane and leads to the Wick-Voros star-product,

$$(f \ast_V g)(x) = e^{\frac{i}{2} \theta_{ij} \partial_x^i \partial_y^j} f(x)g(y) \bigg|_{y=x}, \quad (5)$$

where $\partial_x \wedge \partial_y = \frac{\partial}{\partial x_2} \frac{\partial}{\partial y_3} - \frac{\partial}{\partial x_3} \frac{\partial}{\partial y_2}$ and $\partial_x \partial_y = \frac{\partial}{\partial x_2} \frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_3} \frac{\partial}{\partial y_3}$.

These star-products give isomorphic representations of the algebra of noncommutative fields. In the extensively studied $\ast_W$-case one encounters nonlocality-related problems such as UV/IR-mixing and acausality, arising from the fact that the product is infinitely nonlocal. This is clearly seen from the integral representation of the product:

$$f(x) \ast_W g(x) = \int d^Dz \, d^Dy \frac{1}{\pi^D \det \theta} \exp[2i(x\theta^{-1}y + y\theta^{-1}z + z\theta^{-1}x)] f(y)g(z), \quad (6)$$

which receives contributions from $f(y)$ and $g(z)$ for all values of $y$ and $z$. Thus, for example the matrix elements of the commutator

$$[\phi(x) \ast \phi(x) \; ; \; \phi(y) \ast \phi(y) \; ;],$$

vanish in general only for $((x_0-y_0)^2-(x_1-y_1)^2) < 0$, corresponding to the light-wedge causality condition [8, 9].

In the case of $\ast_V$ these problems still persist. It is notable that the different form of the star-product leads to damping factors in the Green functions. For example the propagator of
a free scalar field has a damping factor that makes the Green function finite:

\[ G^{(0)}(x, y) = \frac{1}{(2\pi)^3} \int d^3k e^{i k(x-y)} e^{-\frac{\theta}{2}k^2} \quad (8) \]

However, in perturbation theory, the damping factors in the propagators are cancelled by opposite factors from the vertices, so that UV divergences and UV/IR mixing still appear \[10\].

### 3 Modifying the star product

The right hand side of the commutator (2) has to be a constant in order to preserve translational invariance. That is why we will adopt the view that we cannot modify the r.h.s. to obtain noncommutativity of a finite range, since any \(x\)-dependent function would break translational invariance. Nevertheless, we may note that similar considerations of modifying the star-product to obtain a new associative product when \(\theta\) is a function of the position \(x\), have been made in \[11\].

#### 3.1 Modified star-product

Instead of modifying the commutator itself, let us now consider modifying the star-product. We will generally require that our modified star-product respects the commutator (2) and remains associative. We will also assume, for the sake of simplicity, that the noncommutativity is restricted to a plane as in (3).

Let us begin by introducing some damping into the integrand of (6). For example an exponential damping, \(\exp[-l^{-2}(x-y)^2 - l^{-2}(x-z)^2]\), where \(l\) has a finite value and represents the scale for the reach of the noncommutativity.
3.1.1 Gaussian damping

In the case of a Gaussian damping of the star-product, we can, using (6), define the modified star-product as

\[ f(x) \star' g(x) := \int d^2z \, d^2y \, \frac{1}{\pi^2 \det \theta} \exp\left\{ \frac{2i}{\theta} (x \wedge y + y \wedge z + z \wedge x) \right\} \exp\left\{ -\frac{1}{\theta} \left( (x - y)^2 + (x - z)^2 \right) \right\} f(y) g(z), \]

where we have denoted \((x_2, x_3)\) and \((y_2, y_3)\) by \(x\) and \(y\) respectively and taken \(l^2 = \theta\). Now we should check our requirements stated at the beginning of section 3.1, e.g. to what extent it realizes the commutator (2) and whether it gives an associative product.

For this kind of a modification of the star-product we have the following results:

1. The product of two plane waves is up to a multiplicative constant

\[ e^{ix \cdot k} \star' e^{ix \cdot q} = e^{ix \cdot (k+q)} e^{-\frac{i}{2} \theta k \wedge q} e^{-\frac{i \theta}{4} (k^2 + q^2)}. \]

This is to be contrasted with the standard star-product case,

\[ e^{ix \cdot k} \star W e^{ix \cdot q} = e^{ix \cdot (k+q)} e^{-i \frac{\theta}{2} k \wedge q}, \]

and the Wick-Voros case:

\[ e^{ix \cdot k} \star V e^{ix \cdot q} = e^{ix \cdot (k+q)} e^{-\frac{i \theta}{2} k \wedge q} e^{-\frac{i \theta}{2} k \cdot q}. \]

2. By considering the product of three plane waves

\[ e^{ix \cdot p} \star' \left( e^{ix \cdot k} \star' e^{ix \cdot q} \right) = e^{ix \cdot (p+k+q)} e^{-\frac{i \theta}{2} p \wedge (k+q)} e^{-\frac{i \theta}{4} (p^2 + (k+q)^2)} \left( e^{-\frac{i \theta}{2} k \wedge q} e^{-\frac{i \theta}{4} (k^2 + q^2)} \right) \]

\[ \neq \left( e^{ix \cdot p} \star' e^{ix \cdot k} \right) \star' e^{ix \cdot q}, \]

we see that the product is nonassociative.

3. It realizes the commutator \([x_2, x_3]_{\star'} = i\theta\).

Equation (10) suggests that the product can be written as an exponential of derivatives as

\[ (f \star_m g)(x) = e^{i\theta \partial_x \wedge \partial_y + \frac{\theta}{4} (\partial_x^2 + \partial_y^2)} f(x) g(y) \bigg|_{y=x}, \]
where $*^m$ stands for a modified star-product. This resembles the Wick-Voros product \( \text{\textcircled{5}} \). However, the Wick-Voros product is associative and it seems that the non-associativity of \( \text{\textcircled{14}} \) arises from the second order derivatives with respect to the same variable in the exponent of \( \text{\textcircled{14}} \). We will show that this is indeed the case in section 3.1.2.

In order to see the effect of the Gaussian damping to microcausality, we calculate the Equal Time Commutation Relation (ETCR)

\[
< 0| [\phi(x) * \phi(x) : ; \phi(y) * \phi(y) :] |p, p' > .
\] (15)

In the case of $\theta_{0i} = 0$, the ETCR vanishes as in the case of the usual star-product. So let us suppose a 1+1 dimensional space-time with $\theta_{01} = \theta > 0$, and define the $*'$ product similarly to the $\theta_{0i} = 0$ case, by:

\[
f(x) *' g(x) := \int d^2z d^2y \frac{1}{\pi^2 \det \theta} \exp \left[ \frac{2i}{\theta} (x \wedge y + y \wedge z + z \wedge x) \right] \exp \left[ -\frac{1}{\theta} ((x - y)^2 + (x - z)^2) \right] f(y) g(z),
\] (16)

with this definition the ETCR for a Euclidean signature in the damping factors $(x - y)^2 := (x^0 - y^0)^2 + (x^1 - y^1)^2$ and $(x - z)^2 := (x^0 - z^0)^2 + (x^1 - z^1)^2$ becomes, up to a constant factor,

\[
(e^{-ip'y-ipx} + e^{-ip'z-ipy}) \int d^2k e^{-\frac{i}{2} (2k^2 + p^2 + p'^2)} \cos(\frac{\theta}{2} p \wedge k) \cos(\frac{\theta}{2} p' \wedge k) \sin(k(y - x)),
\] (17)

where the sign in the Gaussian factor changes to + for a Minkowskian signature in the damping factor. For the Minkowskian signature the product may however, not be well defined, as the Gaussian exponential blows up for $y^2 \rightarrow -\infty$ or $z^2 \rightarrow -\infty$ in (16) and we therefore consider only the Euclidean case here.

From the form (17) we see that the ETCR certainly does not vanish even for large $(y - x)^2$. This was to be expected, because of the infinite tails of the Gaussian distribution. That is, although small, the Gaussian distribution has some value even at very large $x$ and vanishes strictly first at infinity. This suggests that a modification of the star-product, that would produce a strictly vanishing ETCR for $(x - y)^2 < -l^2$, should involve some sort of cutoff.
For example a step-function type of cutoff. We will explore this possibility in section 3.1.3.

An apparent drawback of the star-product modified by a Gaussian damping is the loss of associativity, due to which we cannot use it straightforwardly to define a sensible field theory in noncommutative space-time. It would nonetheless be interesting to know if the nonassociativity of this particular product could have been expected from a geometrical interpretation of the associativity of star-products, as in [12].

### 3.1.2 An exponential modification of the star-product

If we define some modification of the Weyl-Moyal product as

\[
(f \ast_m g)(x) = e^{\frac{i}{2} \partial_x \wedge \partial_y + F(\partial_x, \partial_y) f(x)g(y)}\bigg|_{y=x}, \tag{18}
\]

where \(F(\partial_x, \partial_y)\) is an arbitrary function of the differentials \(\partial_x\) and \(\partial_y\) that contains \(\theta\) to first order. Then the requirement of associativity can be reduced to the equation

\[
\left[ F(\partial_y, \partial_z) f(y) g(z) h(z) + f(x) F(\partial_y, \partial_z) g(y) h(z) = F(\partial_y, \partial_z) f(y) g(z) h(x) + F(\partial_y, \partial_z) f(y) g(y) h(z) \right] \bigg|_{y=z=x}, \tag{19}
\]

which is the equation of associativity for the first order terms in the expansion of the exponent of the product \(\ast_m\) in (18). We can easily see from here that only in the case that the derivative operator \(F(\partial_y, \partial_z)\) is linear in both \(\partial_y\) and \(\partial_z\), but does not contain higher order derivatives of the same variable, the equation is satisfied. This clearly shows that any attempt to modify the differential representation of the Weyl-Moyal star-product by an exponential function of derivatives, must be of the form

\[
(f \ast_m g)(x) = e^{\frac{i}{2} \partial_x \wedge \partial_y + a^{ij} \partial_i \partial_j + b^i \partial_i + c^i \partial_i} f(x)g(y)\bigg|_{y=x}, \tag{20}
\]

to remain associative. Here, \(a^{ij}, b^i\) and \(c^i\) are constants, independent of \(y, z\) or \(x\). If we also require that the product produces the commutator (2), we are led to the requirement

\[
a^{ij} = a^{ji}, \quad b^i = c^i. \tag{21}
\]
If we calculate the product of two plane waves using (20) with the requirement (21), we obtain
\[e^{ik\cdot x} e^{iq\cdot x} = e^{ik\cdot(x+b)+iq\cdot(x+b)} e^{-\frac{1}{4} q^2 k^2} e^{-a^{ij} k_i q_j}, \quad (22)\]
From this it can be seen that the terms with \(b^i \neq 0\), only correspond to a translation of the coordinates \(x\) by a constant vector \(b^i\), and therefore we will omit them from now on. The \(a^{ij}\)-part of the product can on the other hand be recognized as the Wick-Voros product in the particular case: \(a^{ij} = \frac{\theta}{2} \delta^{ij}\). Therefore we should check whether this modified star-product can change the properties of microcausality in noncommutative quantum field theory, should we be able to choose \(a^{ij}\) appropriately.

The ETCR for the star-product (20) with \(b^i = c^i = 0\) and \(a^{ij} = a^{ji}\) yields
\[(e^{-ip'y-ipx} + e^{-ip'x-ipy}) \int d^2k \ e^{-a^{ij}(p'_i + p_j)k_j} \cos(\frac{\theta}{2} p^i \wedge k) \cos(\frac{\theta}{2} p'_i \wedge k) \sin(k(y-x)). \quad (23)\]
Clearly, there is no way of making the expression (23) vanish for any \((y-x)^2\) no matter how we choose \(a^{ij}\). This was to be expected, knowing that the Wick-Voros product does not change the locality properties of field theory and it is a special case of (20) with \(a^{ij} = \frac{\theta}{2} \delta^{ij}\) and \(b^i = c^i = 0\). At best, an appropriately chosen \(a^{ij}\) could introduce some small suppression into the ETCR, but since it would not solve the UV/IR problem [13], we will not investigate the properties of (20) any further. What is appealing with the product (20) under the requirement (21), is that it is associative and satisfies the commutator [2].

3.1.3 Step function

If we want to obtain strict finite locality at some range, it seems that we must introduce a cutoff-like function into the integral representation of the star-product (4), as discussed in section 3.1.1. We can define such a star-product e.g. by
\[f(x) \ast'' g(x) := \int d^2z \ d^2y \ \frac{1}{\pi^2 \det\theta} \exp\left[\frac{2i}{\theta} (x \wedge y + y \wedge z + z \wedge x)\right] \Theta(l^2 - (x-y)^2) \Theta(l^2 - (x-z)^2) f(y) g(z), \quad (24)\]
This definition is justified by that it respects the noncommutative symmetry of space-time $O(1, 1) \times SO(2)$. Analytical calculations are however very restricted with the step-function cutoff and we cannot check for its associativity, let alone calculate the commutator (2) for the modification (24). Therefore we will be content with presenting the product (24) as a curiosity that might satisfy all of our requirements.

4 Concluding remarks and discussion

It has been shown here that the introduction of an extra exponential of derivatives into the differential representation of the Weyl-Moyal star-product cannot remove the infinite nonlocality, inherent in noncommutative theories using the Weyl-Moyal star-product. This result is similar to the one found in [13], where it was shown that UV/IR mixing will remain, in theories with a more general star-product of the form

$$f \star g = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^d p d^d q e^{ip \cdot \tilde{x}} \tilde{f}(q) \tilde{g}(p - q)e^{\alpha(p, q)}.$$ (25)

This product is required to remain associative, translationally invariant and satisfy the requirement of the integral being a trace. That is,

$$\int d^d x f \star g = \int d^d x d^d p d^d q e^{\alpha(p, q)} e^{ip \cdot \tilde{x}} \tilde{f}(q) \tilde{g}(p - q)$$

$$= \int d^d q e^{\alpha(0, q)} \tilde{f}(q) \tilde{g}(-q).$$ (26)

Moreover, because the nonassociative Gaussian product (9) does not remove the infinite nonlocality of noncommutative quantum field theory, we thereby expect no analytical function in place of the Gaussian to do so. This leads one to propose that a possible way out could be placing a cutoff into the integral representation of the Weyl-Moyal star-product. However, the very simple proposal (24), loses much, if not all possibility of analytical calculation.

A different possibility out of the UV/IR problem could be provided by star-products on compact spaces, such as the fuzzy sphere [14]. Since the standard cosmological model provides
a universe with a finite age and a finite size, it could be thought that spacetime is compact. On
the other hand, should spacetime be noncommutative and compact, the UV/IR problem could
be solved but we would have no better understanding of the nonvanishing ETCR. That is why
the construction of a finite star-product in noncommutative field theories, if possible, would
enjoy a two-fold merit. It would provide a way out of both the UV/IR mixing problem and the
nonvanishing ETCR. However, it should also be noted that were we able to modify the Weyl-
Moyal star-product to become nonlocal at a finite range, the connection of noncommutative field
theory to string theory would become vague. This follows because the Weyl-Moyal star-product
with its infinite nonlocality is exactly the product found in the analysis of [2].

An additional remark we can make, regarding the construction of noncommutative field
theories with a finite range of noncommutativity, concerns the Wightman-Vladimirov-Petrina
(WVP) theorem [15, 16]. A discussion of the implications of this theorem to noncommutative
quantum field theory can also be found in [17]. The WVP-theorem is as follows: field theory
can be found in [21] If the commutator of observables in an ordinary quantum field theory
satisfies a nonlocal causality condition of the form

\[ [\hat{O}(x), \hat{O}(y)] = 0, \quad \text{for} \quad z_0^2 - \sum_i z_i^2 < -l^2, \quad z = x - y, \quad (27) \]

with \( l \) a finite constant (fundamental) length, then this implies the local microcausality condition

\[ [\hat{O}(x), \hat{O}(y)] = 0, \quad \text{for} \quad z_0^2 - \sum_i z_i^2 < 0, \quad (28) \]

for all of spacetime. At equal time, the relation (27) is given by

\[ [\hat{O}(x), \hat{O}(y)] \bigg|_{x_0=y_0} = 0, \quad \text{for} \quad \sum_i z_i^2 > l^2, \quad z_i = x_i - y_i. \quad (29) \]

At a quick glance one might think that this theorem relates to our desire of constructing
finite range noncommutativity, since the Wightman functions and axioms can be adapted to
the noncommutative case [18, 19]. However, the finite noncommutativity we wish to obtain is
of a different kind. We wish to have the two conditions:

\[
\text{when } (\vec{x}_m - \vec{y}_m)^2 < l^2, \text{ to have } [\hat{O}(x), \hat{O}(y)] = 0 \text{ for } (x_0 - y_0)^2 - (x_1 - y_1)^2 \leq 0, \quad (30)
\]

and

\[
\text{when } (\vec{x}_m - \vec{y}_m)^2 > l^2, \text{ to have } [\hat{O}(x), \hat{O}(y)] = 0 \text{ for } (x - y)^2 < 0. \quad (31)
\]

Here, $\vec{x}_m = (x_2, x_3)$ and similarly for $\vec{y}_m$, i.e. the coordinates in the noncommutative plane.

Condition (30) is the condition for finite light-wedge causality, and gives us nonlocality within the range of $l$. Condition (31) is the usual causality condition outside of the range $l$ of the finite noncommutativity. It can easily be seen that the two conditions (30) and (31) cannot produce together the condition (27) and thus the WVP-theorem is unrelated to the kind of finite noncommutativity we have been contemplating in this work.

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In a noncommutative field theory the space-time coordinates $x_\mu$ are promoted to the status of operators $\hat{x}_\mu$ which are characterized by the commutation relation

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}.$$  \hspace{1cm} (1)

The commutation relation (1) is motivated by the works [1] where $\theta_{\mu\nu}$ is taken as a tensor and [2], where $\theta_{\mu\nu}$ is taken as a constant antisymmetric matrix. In [1], it was shown that the commutation relation (1) can follow from a high energy gedanken experiment where, as the energy is raised, the formation of black holes prevents smaller distances than the diameter of the black hole to be measured. Then space-time can be characterized by uncertainty relations among coordinate operators, which can straightforwardly be interpreted as noncommutativity of space-time coordinates. In [2], the commutation relation (1) follows as a low-energy limit of open string theory in a background $B$-field.

Although noncommutative field theories have a solid motivation in the works [1] and [2], they exhibit certain problematic properties such as the UV/IR mixing effect [3] and the apparent violation of Lorentz invariance by the commutator (1). The problem of Lorentz noninvariance does nevertheless not result in a need to change the representations of the Poincaré group [4] and consequently, the classification of particles according to their spin and mass in commutative quantum field theory can be taken over to noncommutative quantum field theory without change. This justification is very important for the usage of Lorentz invariant quantities and representations of the Poincaré symmetry within noncommutative field theory. Therefore one might say that the problem of Lorentz noninvariance, within noncommutative field theory, is presently a smaller problem than the one of UV/IR mixing. Indeed, the UV/IR mixing effect has yet to be included in a theory, i.e. so that it does not spoil renormalizability, or alternatively by some unknown mechanism, completely removed from the theory. One can generally argue that the UV/IR mixing effect is a result of the infinite nonlocality of the commutator (1) and that restricting the range of it, should result in a theory without UV/IR mixing.
Work along these lines has been done in [5] where it was attempted to come to terms with the infinite nonlocality of noncommutative models of space-time by introducing a support for the noncommutativity parameter $\theta$ inside a specific range. This, together with an appropriately chosen deformation of the states of the theory, results in a finite range for microcausality as it reduces to the microcausality of commutative quantum field theory outside the support of $\theta$. However, this approach makes it difficult to construct an interaction that would remain nonlocal on a finite range and moreover, the definition of observables that respect this type of microcausality becomes nontrivial. In addition, the choice of the deformation of the states is highly nonunique. The present work differs from the approach in [5], in that no support for $\theta$ is introduced, and were we able to construct a product under our requirements, there would be no problem of introducing interactions or observables into the theory.

The work is organized as follows. We begin by giving the representations of the star-product most useful for this work in section 2. We then move on to modify the star product with a Gaussian damping term and discuss the possibility of modifying the differential representation of the star-product. This analysis leads us to propose a modification of the integral-kernel of the Weyl-Moyal star-product by a Heaviside stepfunction cutoff in section 3. In section 4 we make our concluding remarks.

2 Representations of the star-product

In the usual approach to noncommutative quantum field theory we assume the commutator

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu},$$

where $\theta_{\mu\nu}$ is taken to be a constant and antisymmetric matrix. However, because of the loss of unitarity [6] and causality [7, 8], in theories where time and space do not commute, one often considers theories with only space-space noncommutativity. This will also be the approach in this work. Additionally, since $\theta_{\mu\nu}$ can always be transformed into a frame where only four
distinct components of the antisymmetric matrix survive and only two of them are independent, we take $\theta_{\mu\nu}$ to be given by

$$\theta_{23} = \theta \neq 0, \quad \theta_{12} = \theta_{13} = \theta_{0i} = 0,$$

so that the symmetry of space-time is $O(1, 1) \times SO(2)$. The standard way to realize the commutator (2) is via the Weyl-Moyal star-product

$$(fg)(\hat{x}) \mapsto (f \ast_W g)(x) = \exp \left[ \frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right] f(x)g(y) \bigg|_{x=y}.$$  

(4)

That is, products between functions of $x$ are now taken with respect to the star-product (4).

Another way is to use the coherent-state basis. This corresponds to optimal localization in the noncommutative plane and leads to the Wick-Voros star-product,

$$(f \ast_V g)(x) = e^{i \theta_{xy}} f(x)g(y) \bigg|_{y=x},$$  

(5)

where $\partial_x \neq \partial_y = \partial_{x_1} - \partial_{y_1}$ and $\partial_x \partial_y = \partial_{x_2} \partial_{y_2} + \partial_{x_3} \partial_{y_3}$.

These star-products give isomorphic representations of the algebra of noncommutative fields. In the extensively studied $\ast_W$-case one encounters nonlocality-related problems such as UV/IR-mixing and acausality, arising from the fact that the product is infinitely nonlocal. This is clearly seen from the integral representation of the product:

$$f(x) \ast_W g(x) = \int d^D z \, d^D y \, \frac{1}{\pi^D \det \theta} \exp[2i(x\theta^{-1}y + y\theta^{-1}z + z\theta^{-1}x)] f(y)g(z),$$  

(6)

which receives contributions from $f(y)$ and $g(z)$ for all values of $y$ and $z$. Thus, for example the matrix elements of the commutator

$$[\phi(x) \ast \phi(x) : ; \phi(y) \ast \phi(y) :],$$  

vanish in general only for $((x_0 - y_0)^2 - (x_1 - y_1)^2) < 0$, corresponding to the light-wedge causality condition [8, 9].

In the case of $\ast_V$ these problems still persist. It is notable that the different form of the star-product leads to damping factors in the Green functions. For example the propagator of
a free scalar field has a damping factor that makes the Green function finite:

\[ G^{(0)}(x, y) = \frac{1}{(2\pi)^3} \int d^3 k e^{ik(x-y)} e^{-\frac{\theta}{2}k^2} \] (8)

However, in perturbation theory, the damping factors in the propagators are cancelled by opposite factors from the vertices, so that UV divergences and UV/IR mixing still appear [10].

3 Modifying the star product

The right hand side of the commutator (2) has to be a constant in order to preserve translational invariance. That is why we will adopt the view that we cannot modify the r.h.s. to obtain noncommutativity of a finite range, since any \( x \)-dependent function would break translational invariance. Nevertheless, we may note that similar considerations of modifying the star-product to obtain a new associative product when \( \theta \) is a function of the position \( x \), have been made in [11].

3.1 Modified star-product

Instead of modifying the commutator itself, let us now consider modifying the star-product. We will generally require that our modified star-product respects the commutator (2) and remains associative. We will also assume, for the sake of simplicity, that the noncommutativity is restricted to a plane as in (3).

Let us begin by introducing some damping into the integrand of (6). For example an exponential damping, \( \exp[-l^{-2}(x-y)^2 - l^{-2}(x-z)^2] \), where \( l \) has a finite value and represents the scale for the reach of the noncommutativity.
3.1.1 Gaussian damping

In the case of a Gaussian damping of the star-product, we can, using (6), define the modified star-product as

\[ f(x) \star'_g(x) := \int d^2z \int d^2y \frac{1}{\pi^2 \det \theta} \exp \left[ \frac{2i}{\theta} (x \wedge y + y \wedge z + z \wedge x) \right] \exp \left[ -\frac{1}{\theta} \left( (x - y)^2 + (x - z)^2 \right) \right] f(y) g(z), \] (9)

where we have denoted \((x_2, x_3)\) and \((y_2, y_3)\) by \(x\) and \(y\) respectively and taken \(l^2 = \theta\). Now we should check our requirements stated at the beginning of section 3.1, e.g. to what extent it realizes the commutator (2) and whether it gives an associative product.

For this kind of a modification of the star-product we have the following results:

1. The product of two plane waves is up to a multiplicative constant

\[ e^{ix \cdot k} \star^' e^{ix \cdot q} = e^{ix \cdot (k+q)} e^{-\frac{\theta}{2} k \wedge q} e^{\frac{\theta}{4} (k^2 + q^2)}. \] (10)

This is to be contrasted with the standard star-product case,

\[ e^{ix \cdot k} \star W e^{ix \cdot q} = e^{ix \cdot (k+q)} e^{-\frac{2\theta}{4} k \wedge q}, \] (11)

and the Wick-Voros case:

\[ e^{ix \cdot k} \star^W e^{ix \cdot q} = e^{ix \cdot (k+q)} e^{-\frac{\theta}{4} k \wedge q} e^{\frac{\theta}{4} k \cdot q}. \] (12)

2. By considering the product of three plane waves

\[ e^{ix \cdot p} \star^' (e^{ix \cdot k} \star^' e^{ix \cdot q}) = e^{ix \cdot (p+k+q)} e^{-\frac{\theta}{2} p \wedge (k+q)} e^{\frac{\theta}{4} (p^2 + (k+q)^2)} \left( e^{-\frac{\theta}{4} k \wedge q} e^{-\frac{\theta}{4} (k^2 + q^2)} \right) \]

\[ \neq (e^{ix \cdot p} \star^' e^{ix \cdot k}) \star^' e^{ix \cdot q}, \] (13)

we see that the product is nonassociative.

3. It realizes the commutator \([x_2, x_3]_{\star'} = i\theta\).

Equation (10) suggests that the product can be written as an exponential of derivatives as

\[ (f \star_m g)(x) = e^{ig \partial_x \wedge \partial_y + i(\partial_x^2 + \partial_y^2)} f(x) g(y) \bigg|_{y=x}, \] (14)
where \(*_m\) stands for a modified star-product. This resembles the Wick-Voros product. However, the Wick-Voros product is associative and it seems that the non-associativity of arises from the second order derivatives with respect to the same variable in the exponent of . We will show that this is indeed the case in section 3.1.2.

In order to see the effect of the Gaussian damping to microcausality, we calculate the Equal Time Commutation Relation (ETCR)

\[ < 0| [: \phi(x) \ast' \phi(x) :, \phi(y) \ast' \phi(y) :]|p, p'> . \]  \tag{15}

In the case of \(\theta_{0i} = 0\), the ETCR vanishes as in the case of the usual star-product. So let us suppose a 1+1 dimensional space-time with \(\theta_{01} = \theta > 0\), and define the \(\ast'\) product similarly to the \(\theta_{0i} = 0\) case, by:

\[ f(x) \ast' g(x) := \int d^2 z d^2 y \frac{1}{\pi^2 \det \theta} \exp \left[ \frac{2}{\theta} (x \wedge y + y \wedge z + z \wedge x) \right] \exp \left[ -\frac{1}{\theta} (y - x)^2 + (x - z)^2 \right] f(y) g(z), \]  \tag{16}

with this definition the ETCR for a Euclidean signature in the damping factors \((x - y)^2 := (x^0 - y^0)^2 + (x^1 - y^1)^2\) and \((x - z)^2 := (x^0 - z^0)^2 + (x^1 - z^1)^2\) becomes, up to a constant factor,

\[ (e^{-ip'y - ipx} + e^{-ip'z - ipy}) \int d^2 k \ e^{-\frac{4}{\theta} (2k^2 + p^2 + p'^2)} \cos \left( \frac{\theta}{2} p \wedge k \right) \cos \left( \frac{\theta}{2} p' \wedge k \right) \sin (k(y - x)), \]  \tag{17}

where the sign in the Gaussian factor changes to + for a Minkowskian signature in the damping factor. For the Minkowskian signature the product may however, not be well defined, as the Gaussian exponential blows up for \(y^2 \rightarrow -\infty\) or \(z^2 \rightarrow -\infty\) in (16) and we therefore consider only the Euclidean case here.

From the form (17) we see that the ETCR certainly does not vanish even for large \((y - x)^2\). This was to be expected, because of the infinite tails of the Gaussian distribution. That is, although small, the Gaussian distribution has some value even at very large \(x\) and vanishes strictly first at infinity. This suggests that a modification of the star-product, that would produce a strictly vanishing ETCR for \((x - y)^2 < -l^2\), should involve some sort of cutoff.
For example a step-function type of cutoff. We will explore this possibility in section 3.1.3. An apparent drawback of the star-product modified by a Gaussian damping is the loss of associativity, due to which we cannot use it straightforwardly to define a sensible field theory in noncommutative space-time. It would nonetheless be interesting to know if the nonassociativity of this particular product could have been expected from a geometrical interpretation of the associativity of star-products, as in [12].

3.1.2 An exponential modification of the star-product

If we define some modification of the Weyl-Moyal product as

\[(f *_m g)(x) = e^{\frac{i\theta}{2} \partial_x \wedge \partial_y + F(\partial_x, \partial_y) f(x) g(y)} \bigg|_{y=x}, \] (18)

where \(F(\partial_x, \partial_y)\) is an arbitrary function of the differentials \(\partial_x\) and \(\partial_y\) that contains \(\theta\) to first order, then the requirement of associativity can be reduced to the equation

\[
\left[ F(\partial_y, \partial_z) f(y) g(z) h(z) + f(x) F(\partial_y, \partial_z) g(y) h(z) = F(\partial_y, \partial_z) f(y) g(z) h(x) + F(\partial_y, \partial_z) f(y) g(y) h(z) \right] \bigg|_{y=z=x},
\] (19)

which is the equation of associativity for the first order terms in the expansion of the exponent of the product \(*_m\) in (18). We can easily see from here that only in the case that the derivative operator \(F(\partial_y, \partial_z)\) is linear in both \(\partial_y\) and \(\partial_z\), but does not contain higher order derivatives of the same variable, the equation is satisfied. This clearly shows that any attempt to modify the differential representation of the Weyl-Moyal star-product by an exponential function of derivatives, must be of the form

\[(f *_m g)(x) = e^{\frac{i\theta}{2} \partial_x \wedge \partial_y + a^{ij} \partial_i \partial_y^j + b^i \partial_i + c^i \partial_y^i} f(x) g(y) \bigg|_{y=x}, \] (20)

to remain associative. Here, \(a^{ij}, b^i\) and \(c^i\) are constants, independent of \(y, z\) or \(x\). If we also require that the product produces the commutator [2], we are led to the requirement

\[a^{ij} = a^{ji}, \quad b^i = c^i.\] (21)
If we calculate the product of two plane waves using (20) with the requirement (21), we obtain
\[ e^{ik\cdot x}e^{iq\cdot x} = e^{ik(x+b)+iq(x+b)}e^{-\frac{i}{2}gk\wedge q}e^{-a^{ij}k_ik_j}. \]
(22)

From this it can be seen that the terms with \( b^i \neq 0 \), only correspond to a translation of the coordinates \( x \) by a constant vector \( b^i \), and therefore we will omit them from now on. The \( a^{ij} \)-part of the product can on the other hand be recognized as the Wick-Voros product in the particular case: \( a^{ij} = \frac{\theta}{2}\delta^{ij} \). Therefore, we should check whether this modified star-product can change the properties of microcausality in noncommutative quantum field theory, should we be able to choose \( a^{ij} \) appropriately.

The ETCR for the star-product (20) with \( b^i = c^i = 0 \) and \( a^{ij} = a^{ji} \) yields
\[ (e^{-ip'y-ipx} + e^{-ip'x-ipy}) \int d^2k e^{-a^{ij}(p'_i+p_i)k_j} \cos(\frac{\theta}{2}p \wedge k) \cos(\frac{\theta}{2}p' \wedge k) \sin(k(y-x)). \]
(23)

Clearly, there is no way of making the expression (23) vanish for any \((y-x)^2\) no matter how we choose \( a^{ij} \). This was to be expected, knowing that the Wick-Voros product does not change the locality properties of field theory and it is a special case of (20) with \( a^{ij} = \frac{\theta}{2}\delta^{ij} \) and \( b^i = c^i = 0 \).

At best, an appropriately chosen \( a^{ij} \) could introduce some small suppression into the ETCR, but since it would not solve the UV/IR problem \([13]\), we will not investigate the properties of (20) any further. What is appealing with the product (20) under the requirement (21), is that it is associative and satisfies the commutator (2).

3.1.3 Step function

If we want to obtain strict finite locality at some range, it seems that we must introduce a cutoff-like function into the integral representation of the star-product (4), as discussed in section 3.1.1. We can define such a star-product e.g. by
\[ f(x) \star g(x) := \int d^2z d^2y \frac{1}{x'x'y'z'} \exp[\frac{2i}{g}(x \wedge y + y \wedge z + z \wedge x)] \Theta(l^2 - (x-y)^2) \Theta(l^2 - (x-z)^2) f(y) g(z), \]
(24)
This definition is justified by that it respects the noncommutative symmetry of space-time $O(1,1) \times SO(2)$. Analytical calculations are however very restricted with the step-function cutoff and we cannot check for its associativity, let alone calculate the commutator (2) for the modification (24). Therefore we will be content with presenting the product (24) as a curiosity that might satisfy all of our requirements.

4 Concluding remarks and discussion

It has been shown here that the introduction of an extra exponential of derivatives into the differential representation of the Weyl-Moyal star-product cannot remove the infinite nonlocality, inherent in noncommutative theories using the Weyl-Moyal star-product. This result is similar to the one found in [13], where it was shown that UV/IR mixing will remain, in theories with a more general star-product of the form

$$f \star g = \frac{1}{(2\pi)^2} \int d^d p d^d q e^{ip \cdot x} \tilde{f}(q) \tilde{g}(p-q) e^{\alpha(p,q)}. \quad (25)$$

This product is required to remain associative, translationally invariant and satisfy the requirement of the integral being a trace. That is,

$$\int d^d x f \star g = \int d^d x d^d p d^d q e^{\alpha(p,q)} e^{ip \cdot x} \tilde{f}(q) \tilde{g}(p-q)$$

$$= \int d^d q e^{\alpha(0,q)} \tilde{f}(q) \tilde{g}(-q). \quad (26)$$

Moreover, because the nonassociative Gaussian product (9) does not remove the infinite nonlocality of noncommutative quantum field theory, we thereby expect no analytical function in place of the Gaussian to do so. This leads one to propose that a possible way out could be placing a cutoff into the integral representation of the Weyl-Moyal star-product. However, the very simple proposal (24), loses much, if not all possibility of analytical calculation.

A different possibility out of the UV/IR problem could be provided by star-products on compact spaces, such as the fuzzy sphere [14]. Since the standard cosmological model provides
a universe with a finite age and a finite size, it could be thought that space-time is compact. On the other hand, should space-time be noncommutative and compact, the UV/IR problem could be solved but we would have no better understanding of the nonvanishing ETCR. That is why the construction of a finite star-product in noncommutative field theories, if possible, would enjoy a two-fold merit. It would provide a way out of both the UV/IR mixing problem and the nonvanishing ETCR. However, it should also be noted that were we able to modify the Weyl-Moyal star-product to become nonlocal at a finite range, the connection of noncommutative field theory to string theory would become vague. This follows because the Weyl-Moyal star-product with its infinite nonlocality is exactly the product found in the analysis of [2].

An additional remark we can make, regarding the construction of noncommutative field theories with a finite range of noncommutativity, concerns the Wightman-Vladimirov-Petrina (WVP) theorem [15, 16]. A discussion of the implications of this theorem to noncommutative quantum field theory can also be found in [17]. The WVP-theorem is as follows: If the commutator of observables in an ordinary quantum field theory satisfies a nonlocal causality condition of the form

\[
[\hat{O}(x), \hat{O}(y)] = 0, \quad \text{for} \quad z_0^2 - \sum_i z_i^2 < -l^2, \quad z = x - y,
\]

with \( l \) a finite constant (fundamental) length, then this implies the local microcausality condition

\[
[\hat{O}(x), \hat{O}(y)] = 0, \quad \text{for} \quad z_0^2 - \sum_i z_i^2 < 0,
\]

for all of space-time. At equal time, the relation (27) is given by

\[
[\hat{O}(x), \hat{O}(y)]|_{x_0 = y_0} = 0, \quad \text{for} \quad \sum_i z_i^2 > l^2, \quad z_i = x_i - y_i.
\]

At a quick glance one might think that this theorem relates to our desire of constructing finite range noncommutativity, since the Wightman functions and axioms can be adapted to the noncommutative case [18, 19]. However, the finite noncommutativity we wish to obtain is
of a different kind. We wish to have the two conditions:

when \((\vec{x}_m - \vec{y}_m)^2 < l^2\), to have \([\hat{O}(x), \hat{O}(y)] = 0\) for \((x_0 - y_0)^2 - (x_1 - y_1)^2 \leq 0\), \((30)\)

and

when \((\vec{x}_m - \vec{y}_m)^2 > l^2\), to have \([\hat{O}(x), \hat{O}(y)] = 0\) for \((x - y)^2 < 0\). \((31)\)

Here, \(\vec{x}_m = (x_2, x_3)\) and similarly for \(\vec{y}_m\), i.e. the coordinates in the noncommutative plane.

Condition \((30)\) is the condition for finite light-wedge causality, and gives us nonlocality within the range of \(l\). Condition \((31)\) is the usual causality condition outside of the range \(l\) of the finite noncommutativity. It can easily be seen that the two conditions \((30)\) and \((31)\) cannot produce together the condition \((27)\) and thus the WVP-theorem is unrelated to the kind of finite noncommutativity we have been contemplating in this work.

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