Spontaneous Symmetry Breaking of $\phi^4_{1+1}$ in Light-Front Field Theory

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Abstract

We study spontaneous symmetry breaking in $(\phi^4)_{1+1}$ using the light-front formulation of the field theory. Since the physical vacuum is always the same as the perturbative vacuum in light-front field theory the fields must develop a vacuum expectation value through the zero-mode components of the field. We solve the nonlinear operator equation for the zero-mode in the one-mode approximation. We find that spontaneous symmetry breaking occurs at $\lambda_{\text{critical}} = 4\pi \left(3 + \sqrt{3}\right)$, which is consistent with the value $\lambda_{\text{critical}} = 54.27$ obtained in the equal time theory. We calculate the value of the vacuum expectation value as a function of the coupling constant in the broken phase both numerically and analytically using the $\delta$ expansion. We find two equivalent broken phases. Finally we show that the energy levels of the system have the expected behavior within the broken phase.
**I. Introduction**

It was Dirac [1] who first recognized that different generators of the Poincaré group could be used as a Hamiltonian for the purpose of quantizing a field theory. He showed that, among these, light-front quantization was unique. In light-front quantization the Hamiltonian is \( p^- = (p^0 - p^z) / \sqrt{2} \), and \( p^+ = (p^0 + p^z) / \sqrt{2} \) is the third or “longitudinal” component of momentum. The longitudinal momentum has a positive semi-definite spectrum, and massive excitations cannot mix with the vacuum. Therefore, the bare Fock space vacuum is an eigenstate of the full Hamiltonian. Since theories such as QCD exhibit spontaneous symmetry breaking, spontaneous symmetry breaking must appear through some other mechanism in the context of light-front field theory. We will see that spontaneous symmetry breaking occurs in light-front theory because the field includes a zero-mode that is not an independent degree of freedom. This mode is a complicated operator-valued function of all other modes in the theory and may lead to a non-zero vacuum expectation value (VEV).

In order to investigate the zero-mode in light-front field theory, we will consider a discretized \( \phi^4 \) field theory in two space-time dimensions. The formal constraint equation obtained from the Dirac-Bergmann quantization procedure relates the zero-modes to all the other modes in the problem [2]. This equation is, however, most easily obtained by integrating the equation of motion [3]. It is clearly very difficult to solve and, to date, only approximate qualitative solutions have been generated [4]. The procedure that generates the constraint equation does not specify an operator ordering. We will use a symmetric operator-ordering prescription [5]. In order to render the problem tractable, we will truncate the Fock space to include only the lowest-energy mode. We will see that for weak coupling the theory has only the trivial solution for the zero-mode. However, as we increase the coupling we reach a critical coupling where a pair of nontrivial solutions to the constraint equation appear.

Section II presents a simple derivation of the zero-mode constraint equation in the classical case. Section III briefly discusses quantization and mass renormalization of the theory. In Section IV we discuss the asymptotic behavior of the zero-mode in the large-particle-number sectors. From the asymptotic behavior of the constraint equation we show that the theory has a critical coupling. In Section V we use the \( \delta \) expansion to study the solution branches away from the critical point. Section VI present some numerical solutions to the equation. We show that the theory has a
critical point predicted by the asymptotic behavior of the constraint equation. The two nontrivial solutions are plotted as functions of the coupling and the VEV along with the δ expansion solution. We also study the behavior of matrix elements of the zero-mode near the critical curves and the energy eigenvalue as a function of the coupling. Finally in Section VI we discuss our results and the remaining work that is needed on this problem.

II. The Classical Case

The details of the Dirac-Bergmann prescription and its application to the system considered in this paper are discussed elsewhere in the literature [2, 4]. In terms of light-front coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$. For a classical field the $(\phi^4)_{1+1}$ Lagrangian is

$$\mathcal{L} = \partial_+ \phi \partial_- \phi - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 .$$

We put the system in a box of length $d$ and impose periodic boundary conditions. For most of our discussion we work in momentum space. We define $q_k$ by

$$\phi(x) = \frac{1}{\sqrt{d}} \sum_n q_n(x^+) e^{ik_n^+ x^-},$$

where $k_n^+ = 2\pi n/d$ and the summations run over all integers unless otherwise noted.

Next, we introduce some notation and separate out the zero-mode. We define

$$\Sigma_n = \frac{1}{n!} \sum_{i_1, i_2, \ldots, i_n \neq 0} q_{i_1} q_{i_2} \cdots q_{i_n} \delta_{i_1 + i_2 + \cdots + i_n, 0} .$$

Using the Dirac-Bergmann prescription, one can find the canonical Hamiltonian

$$P^- = \frac{\mu^2 q_0^2}{2} + \mu^2 \Sigma_2 + \frac{\lambda q_0^4}{4!d} + \frac{\lambda q_0^2 \Sigma_2}{2!d} + \frac{\lambda \Sigma_3}{d} + \frac{\lambda \Sigma_4}{d} .$$

Following the Dirac-Bergmann prescription, we identify first-class constraints which define the conjugate momenta

$$0 = p_n - ik_n^+ q_{-n} ,$$

3
and a secondary constraint,
\[ 0 = \mu^2 q_0 + \lambda q_0^3 + \frac{\lambda q_0 \Sigma_2}{d} + \frac{\lambda \Sigma_3}{d}, \tag{2.6} \]
which determines the “zero-mode” \( q_0 \). This result can also be obtained by integrating the equations of motion. One can calculate the Dirac brackets between the coordinates \( q_n \),
\[ [q_m, q_n] = \frac{d}{4\pi i m} \delta_{m+n,0}, \quad m, n \neq 0, \tag{2.7} \]
\[ [q_0, q_n] = \frac{\lambda}{4\pi i m} \sum_{k,l} q_k q_l \delta_{k+l,n}, \quad n \neq 0 \tag{2.8} \]
and with the conjugate momenta,
\[ [q_m, p_n] = \frac{\delta_{n,m}}{2}, \quad m, n \neq 0. \tag{2.9} \]
The total longitudinal momentum is given by
\[ P^+ = 2 \sum_n \left( k_n^+ \right)^2 q_n q_{-n}. \tag{2.10} \]
One can show that \( P^+ \) has vanishing Dirac brackets with \( \Sigma_n, q_0, \) and \( P^- \):
\[ [P^+, \Sigma_n] = [P^+, q_0] = [P^+, P^-] = 0. \tag{2.11} \]

III. Canonical Quantization

To quantize the system one replaces the classical fields with corresponding field operators. One uses commutators instead of Dirac brackets and inserts a factor of \( i \). One must choose a regularization and an operator-ordering prescription in order to make the system well-defined.

We begin by defining creation and annihilation operators \( a_k^\dagger \) and \( a_k \),
\[ q_k = \sqrt{\frac{d}{4\pi |k|}} a_k, \quad a_k = a_{-k}^\dagger, \quad k \neq 0, \tag{3.1} \]
which satisfy the usual commutation relations

\[ [a_k, a_l] = 0, \quad [a^*_k, a^*_l] = 0, \quad [a_k, a^*_l] = \delta_{k,l}, \quad k, l \neq 0. \tag{3.2} \]

We also define

\[ q_0 = \sqrt{\frac{d}{4\pi}} a_0. \tag{3.3} \]

Very general arguments suggest that the Hamiltonian should be symmetric ordered [5]. This choice of operator ordering produces tadpoles which we eliminate by adding an overall constant and a mass counterterm to the Hamiltonian:

\[ -\frac{\mu^2 d}{8\pi} \sum_{l>0} \frac{1}{l} + \frac{\lambda d}{128\pi^2} \left( \sum_{l>0} \frac{1}{l} \right)^2 - \frac{\lambda d}{64\pi^2} \left( \sum_{l>0} \frac{1}{l} \right) \sum_k \frac{a_k a_{-k}}{l}. \tag{3.4} \]

Then, the quantum Hamiltonian is

\[ P^- = \frac{\mu^2 da^2_0}{8\pi} + \mu^2 \Sigma^2 + \frac{\lambda d}{4!16\pi^2} \sum_{k,l,m,n} \frac{a_k a_l a_m a_n \delta_{k+l+m+n,0}}{\sqrt{klmn}} \]

\[ + \frac{\lambda d}{128\pi^2} \left( \sum_{l>0} \frac{1}{l} \right)^2 - \frac{\lambda d}{64\pi^2} \left( \sum_{l>0} \frac{1}{l} \right) \sum_k \frac{a_k a_{-k}}{l}. \tag{3.5} \]

Note that the constraint equation for the zero-mode is obtained by taking a derivative of \( P^- \) with respect to \( a_0 \). Consequently, it is natural in the quantum case to order symmetric order the constraint equation:

\[ 0 = ga_0 + a^3_0 + 2a_0 \Sigma_2 + 2\Sigma_2 a_0 + \sum_{k>0} \frac{a_k a_0 a_{-k} + a_{-k} a_0 a_k - a_0}{k} + 6\Sigma_3, \tag{3.6} \]

where \( g = 24\pi\mu^2/\lambda \). This equation implies that in matrix form \( a_0 \) is real and symmetric. Moreover, it is block diagonal in states with equal \( P^+ \) eigenvalues. Using the constraint equation, we define a rescaled Hamiltonian \( H \):
IV. One Mode, Many Particles

It is reasonable to assume that the lowest-energy mode will be the most important one in the constraint equation (3.6). We therefore study the case where the zero-mode is just a function of the lowest-energy mode. In this case, the zero-mode is diagonal and can be written as

\[ a_0 = f_0 |0\rangle \langle 0| + \sum_{k>0} f_k |k\rangle \langle k| . \]  

(4.1)

Equivalently, one can think of the zero-mode as an operator-valued function of the number operator \( N = a^\dagger a \). The VEV is given by

\[ \langle 0| \phi |0\rangle = \frac{1}{\sqrt{4\pi}} \langle 0| a_0 |0\rangle = \frac{1}{\sqrt{4\pi}} f_0 . \]  

(4.2)

Substituting (4.1) into the constraint equation (3.6) and sandwiching the constraint equation between Fock states, we get a recursion relation for \( f_n \):

\[ 0 = g f_n + f_n^3 + (4n-1) f_n + (n+1) f_{n+1} + n f_{n-1} . \]  

(4.3)

If we take \( f_0 = 0 \) and we assume that \( n f_{n-1} \) evaluated at \( n = 0 \) vanishes \[3\], then we see that all the \( f_k \)'s are zero. This corresponds to the unbroken phase, \( a_0 = 0 \).

Our objective is to determine whether spontaneous symmetry breaking occurs and a nonzero solution for \( a_0 \) appears as we increase \( \lambda \) (decrease \( g \)).

We begin our analysis of Eq. (4.3) by finding its asymptotic behavior for large \( n \). If \( f_n \gg 1 \) in this limit, then the \( f_n^3 \) term will dominate and

\[ H = \frac{96\pi^2}{\lambda d} P^- = g \Sigma_2 + g \Sigma_4 - \frac{a_0^4}{4} - \frac{a_0 \Sigma_2 a_0}{2} \]

\[ + \frac{1}{4} \sum_{j,k,l \neq 0} a_j a_0 a_k a_l + a_j a_k a_0 a_l \frac{\delta_{j+k+l,0}}{\sqrt{jkl}} \]

\[ + \frac{1}{4} \sum_{k>0} a_{k-1} a_k a_k - a_{k-1} a_k^2 - a_0^2 . \]

(3.7)
\begin{equation}
    f_{n+1} \sim \frac{f_n^3}{n},
\end{equation}

from which we deduce that

\begin{equation}
    \lim_{n \to \infty} f_n \sim (-1)^n \exp(3^n \text{constant}).
\end{equation}

We now argue that we reject this rapidly growing solution. The part of the Hamiltonian (3.7) corresponding to the lowest mode is diagonal in N and the zero-modes will affect all of the energy levels. If \( f_n \) is large for large \( n \) then the high energy levels will be strongly affected. The paradigm for spontaneous symmetry breaking is the symmetric double well which has a ground state localized in either well rather than at the symmetry point. This paradigm indicates that the behavior of the system is unaffected by the barrier for energies far above the barrier separating the wells. Hence, we only seek solutions where \( f_n \) is small for large \( n \). This is the central condition that will be used to determine the critical couplings in all the subsequent calculations. We therefore neglect the \( f_n^3 \) term for large \( n \); it is the terms linear in \( f_n \) that dominate, giving

\begin{equation}
    f_{n+1} + 4f_n + f_{n-1} = 0.
\end{equation}

There are two solutions to this equation:

\begin{equation}
    f_n \propto (\sqrt{3} \pm 2)^n.
\end{equation}

We reject the plus solution because it grows with \( n \). Dropping the cubic term from (4.6) we define the generating function

\begin{equation}
    F(z) = \sum_{n=0}^{\infty} f_n z^n.
\end{equation}

If \( f_n \) goes like \((\sqrt{3} - 2)^n\) then the radius of convergence of \( F(z) \) is \( 2 + \sqrt{3} \) and we expect \( F(z) \) to be singular at \( |z| = 2 + \sqrt{3} \). Similarly, if \( f_n \sim (\sqrt{3} + 2)^n \), then we expect \( F(z) \) to be singular at \( |z| = 2 - \sqrt{3} \).
The function $F(z)$ satisfies the differential equation

$$
\frac{1}{F(z)} \frac{dF(z)}{dz} = -\frac{g - 1 + z}{z^2 + 4z + 1},
$$

whose solution is

$$
\frac{F(z)}{F(0)} = \left(\frac{z + 2 - \sqrt{3}}{2 - \sqrt{3}}\right)^{\frac{\sqrt{3} - 3 + g}{2\sqrt{3}}} \left(\frac{z + 2 + \sqrt{3}}{2 + \sqrt{3}}\right)^{-\frac{\sqrt{3} - 3 - g}{2\sqrt{3}}}.
$$

Note that this solution for $F(z)$ has singularities at the expected values of $z$. If we want $f_n$ to have the asymptotic behavior $\left(\sqrt{3} - 2\right)^n$ for large $n$, then we must eliminate the branch point of $F(z)$ at $|z| = 2 - \sqrt{3}$. This gives the condition

$$
-\frac{\sqrt{3} - 3 + g}{2\sqrt{3}} = K, \quad K = 0, 1, 2.
$$

Only $K = 0$ gives $g > 0$. Therefore, we find a critical coupling

$$
g_{critical} = 3 - \sqrt{3}.
$$

We conclude that there is a critical value of the coupling constant for which there is a nonzero value for $a_0$ and a solution to the linearized equation for $f_n$ exists that does not grow rapidly for large $n$. We can compare our critical value of $g$ with that obtained in the equal-time formulation. Chang finds that $\lambda_{critical} = 54.3$ which differs from our result

$$
\lambda_{critical} = 4\pi \left(3 + \sqrt{3}\right) \approx 59.5
$$

by about 10%.

Of course, we need to determine if there is just an isolated critical point or if there is a continuous range of values of $g < g_{critical}$ for which $a_0$ has nontrivial solutions. This requires that we investigate the full nonlinear equation. Away from $a_0 = 0$ where the $a_0^3$ term can make substantial contributions we will use both the $\delta$-expansion and numerical methods to answer this question. We denote the values of $f_0$ vs. $g$ that satisfy (4.3) the “critical curve”.

8
V. The $\delta$-expansion

The $\delta$-expansion is a powerful perturbative technique for linearizing nonlinear problems. It has been shown to be an accurate technique for solving problems in differential equations, quantum mechanics, and quantum field theory [7].

We rewrite Eq. (4.3) as

$$(g - 1 + 4n) f_n + f_{n+1}^{1+2\delta} + (n + 1) f_{n+1} + n f_{n-1} = 0.$$  \hspace{1cm} (5.1)

Setting $\delta = 0$ gives the linear finite difference equation which is the zeroth-order approximation in the $\delta$-expansion. One then expands in powers of $\delta$ about $\delta = 0$. One recovers the problem of interest at $\delta = 1$. Expanding about $\delta = 0$ we have

$$g = g^{(0)} + \delta g^{(1)} + \ldots, \quad f_n = f_n^{(0)} + \delta f_n^{(1)} + \ldots, \quad n = 1, 2, \ldots$$  \hspace{1cm} (5.2)

and

$$f_{n+1}^{1+2\delta} = f_n \left( 1 + \delta \ln f_n^2 + \ldots \right) = f_n^{(0)} + \delta \left( f_n^{(0)} + f_n^{(0)} \ln f_n^{(0)}^2 \right) + \ldots .$$  \hspace{1cm} (5.3)

Substituting this into equation (4.1) we find, to zeroth order in $\delta$,

$$(g^{(0)} + 4n) f_n^{(0)} + (n + 1) f_{n+1}^{(0)} + n f_{n-1}^{(0)} = 0 .$$  \hspace{1cm} (5.4)

This zeroth-order equation is the same equation as that studied in Section IV with $g$ displaced by one. As discussed before, we can determine the solution for large $n$. One then finds that

$$f_n^{(0)} = f_0 \left( \sqrt{3} - 2 \right)^n , \quad g^{(0)} = 2 - \sqrt{3} .$$  \hspace{1cm} (5.5)

To first order in $\delta$ we obtain an inhomogeneous second-order finite difference equation:

$$(g^{(1)} + 4n) f_n^{(1)} + (n + 1) f_{n+1}^{(1)} + n f_{n-1}^{(1)} = -f_n^{(0)} \left( \ln f_n^{(0)}^2 + g^{(1)} \right) .$$  \hspace{1cm} (5.6)
This equation can also be solved exactly. We obtain

\[
f_n^{(1)} = -\frac{(\sqrt{3} - 2)^n f_0}{\sqrt{3}} \ln (2 + \sqrt{3}) + (\sqrt{3} - 2)^n \left[ \ln f_0^2 + g^{(1)} + \frac{1}{\sqrt{3}} \frac{\ln (2 - \sqrt{3})}{2 + \sqrt{3}} \right] \\
\times \left\{ (2 + \sqrt{3})^2 \sum_{p=1}^{n} \frac{(2 + \sqrt{3})^{2p}}{p} - \sum_{p=1}^{n} \frac{1}{p} \right\} \frac{f_0}{2\sqrt{3}}. \quad (5.7)
\]

The second term in (5.7) grows with \( n \) so we demand that its coefficient vanishes, which gives

\[
g^{(1)} = -\ln f_0^2 + \frac{1}{\sqrt{3}} \ln (2 + \sqrt{3}) (2 - \sqrt{3}), \quad (5.8)
\]

\[
f_n^{(1)} = -\frac{f_0}{\sqrt{3}} (\sqrt{3} - 2)^n \ln (2 + \sqrt{3}). \quad (5.9)
\]

These results are plotted in Fig. 1. The dashed line is a plot of the critical curve for \( \delta = 1 \),

\[
g = (2 - \sqrt{3}) \left( 1 + \frac{1}{\sqrt{3}} \ln (2 + \sqrt{3}) \right) - \ln f_0^2, \quad (5.10)
\]

away from \( f_0 = 0 \). The expansion behaves badly near \( f_0 = 0 \) because of the \( \ln f_0^2 \) term in the expansion. The \( \delta \)-expansion analysis clearly shows that there is a critical curve and not merely a critical point.

VI. Numerical Solution

We can study this critical curve in detail by looking for numerical solutions to Eq. (4.3). The method used here is to write Eq. (4.3) as a set of \( M \) simultaneous equations:

\[
0 = (g - 1) f_0 + f_0^3 + f_1, \\
0 = (g + 3) f_1 + f_1^3 + 2 f_2 + f_0, \\
0 = (g + 7) f_2 + f_2^3 + 3 f_3 + 2 f_1, \\
\vdots \\
0 = (g + 4M - 1) f_M + f_M^3 + M f_{M-1}.
\]
In the $M$th equation $f_{M+1}$ is set to zero. Since we seek a solution where $f_n$ is decreasing with $n$, this is a good approximation. We then pick a value of $g$ and look for real solutions for $f_0, f_1, \ldots, f_M$. We find that for $g > 3 - \sqrt{3}$ the only real solution is $f_n = 0$ for all $n$. For $g$ less than $3 - \sqrt{3}$ there are two additional solutions and near the critical point $|f_0|$ is small and

$$f_n \approx f_0 (2 - \sqrt{3})^n$$

(6.2)

As $g$ decreases ($\lambda$ increases), the solution for $|f_0|$ increases. The critical curve is indicated by the solid line in Fig. 1. The solution of (6.1) converges quite rapidly with $M$. The critical curve is approximately parabolic in shape:

$$g = 3 - \sqrt{3} - 0.9177 f_0^2$$

(6.3)

For a given value of $f_0$ and $g$ Eq. (4.2) can be used to calculate all values of $f_n$.

It is interesting to study the behavior of the constraint equation (4.3) away from the critical curve. In Fig. 2 we plot $|f_n|$ as a function of $n$ and $f_0$ for $g = 1.2$. We see that, as $n$ becomes large, all the $|f_n|$ increase and as $f_0$ approaches the critical curve, which is at $f_0 \approx 0.2700$ for $g = 1.2$, all the $|f_n|$’s decrease rapidly. As $f_0$ increases beyond the critical curve the $|f_n|$’s increase rapidly once again. The fact that $|f_n|$ increases rapidly on both sides of the critical curve is a manifestation of the nonlinearity in (4.3).

We can also study the eigenvalues of the Hamiltonian (3.7) for this one-mode problem. The Hamiltonian is diagonal in the number operator $N$ so the energy eigenstates are just the eigenstates of $N$. Thus,

$$\langle n | H | n \rangle = \frac{3}{2} n(n-1) + ng - \frac{f_n^4}{4} - \frac{2n+1}{4} f_n^2 + \frac{n+1}{4} f_{n+1}^2 + \frac{n}{4} f_{n-1}^2$$

(6.4)

In Fig. 3, the dashed lines show the first few eigenvalues as a function of $g$ without the zero-mode. Observe that the vacuum is at zero for all $g$. When we include the zero-mode, the energy levels shift as shown by the solid curves. The vacuum is at zero energy for $g > g_{\text{critical}}$ but at $g = g_{\text{critical}}$ there is a phase transition and the energy decreases below zero as $g$ is decreased. We also see that for $g < g_{\text{critical}}$ all the higher
energy level increase above the value they had without the zero-mode. The higher levels change very little, as our paradigm would suggest, because $f_n$ is small for large $n$.

VII. Discussion

In the context of $\phi^4_{1+1}$ field theory on the light front, our paradigm for spontaneous symmetry breaking suggests that spontaneous symmetry breaking occurs when fields can develop a zero-mode. This zero-mode gives rise to a nonzero VEV and the full vacuum remains the perturbative vacuum. In the broken phase, the theory behaves exactly as expected and the numerical value of the critical coupling $\lambda_{\text{critical}} = 4\pi \left(3 + \sqrt{3}\right)$ agrees well with the value obtained in the equal-time theory $\lambda_{\text{critical}} = 54.27$ \[9, 10\].

Spontaneous symmetry breaking is simpler to understand in light-front field theory because the entire effect comes from one mode. However, the problem of solving for this one mode is quite difficult. In the literature, it has been suggested that a direct solution of the zero-mode problem may be intractable \[8\]. We hope to have convinced the reader that this is not true.

We interpret the existence of more than one solution to the constraint equation to be spontaneous symmetry breaking. However, this spontaneous symmetry breaking is non-dynamical. We have no motivation to choose one solution to the constraint equation over the others. The conventional argument that when the system is coupled to a heat bath it will dynamically pick the lowest energy state to be the vacuum does not apply here.

Many problems remain to be addressed. A more complete solution of the zero-mode problem including all the oscillators would be very interesting, if only to confirm our results. In reference \[4\], the authors retain the one particle and, implicitly, some of the two particle states for all of the modes and find a solution for the critical coupling $\lambda_{\text{critical}} = 4\pi (3.184 \ldots)$. Also, some questions regarding operator ordering still remain. When we chose a quantum Hamiltonian, we demanded that it be symmetrically ordered and we treated the zero-mode as an ordinary field operator. However, $q_0$ is not an ordinary field operator and can, in principle, be written in terms of the other field operators $a_0 = c_0 + \sum c_k a_k^\dagger a_k + \ldots$. In this sense, the Hamiltonian we chose is not really symmetrically ordered after all. It is unclear whether this is a problem.
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Figure Captions

Figure 1. $g = 24\pi \mu^2/\lambda$ vs. $f_0 = \sqrt{4\pi} \text{VEV}$. The solid curve is the critical curve obtained from numerical solution of (6.1) with $M = 10$. The dashed curve is the critical curve obtained from the first-order $\delta$-expansion.

Figure 2. $|f_n|$ as a function of $n$ and $f_0$ for $g = 1.2$ from the numerical solution of (6.1) with $M = 10$.

Figure 3. The lowest three energy eigenvalues as a function of $g$ from the numerical solution of (6.1) with $M = 10$. The dashed line is the symmetric solution $f_0 = 0$ and the solid line is the solution with $f_0 \neq 0$ for $g < g_{\text{critical}}$. 