Fermionic One-Loop Corrections to Soliton Energies in 1+1 Dimensions

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Abstract

We demonstrate an unambiguous and robust method for computing fermionic corrections to the energies of classical background field configurations. We consider the particular case of a sequence of background field configurations that interpolates continuously between the trivial vacuum and a widely separated soliton/antisoliton pair in 1+1 dimensions. Working in the continuum, we use phase shifts, the Born approximation, and Levinson’s theorem to avoid ambiguities of renormalization procedure and boundary conditions. We carry out the calculation analytically at both ends of the interpolation and numerically in between, and show how the relevant physical quantities vary continuously. In the process, we elucidate properties of the fermionic phase shifts and zero modes.
I. INTRODUCTION

1+1 dimensional models have pointed out many subtleties in the computation of one-loop fermionic quantum corrections to the energies of classical field configurations. Many works, usually in the context of supersymmetric theories [1], have reached conflicting conclusions about the correct boundary conditions, regularization and renormalization procedures, and bound state and continuum contributions. Since the one-loop energy is given by the cancellation of a divergent sum of zero-point energies against a divergent counterterm, one must be extremely careful to fix the counterterm with a definite renormalization scheme in order to obtain a physically meaningful result.

In this paper we extend the methods of Ref. [2,3] to this case. We again find that 1+1 dimensional models provide an excellent testing ground for our methods, and can give considerable insight into the causes and resolutions of the disagreements among the previous results. As in our previous works, we will use a continuum formalism based on phase shifts, their Born approximations, and Levinson’s theorem. Relying on such physical quantities results in a procedure that is both analytically unambiguous and numerically robust. As a result, we will be able to compare numerical results in generic, but not exactly soluble cases to analytic results in specific, exactly soluble cases.

We will consider a Majorana fermion coupled to a real scalar background field, with a definite choice of bosonic and fermionic potentials, though our method applies equally well for any other choice of potentials, as well as for complex scalars and Dirac fermions (including cases without C or CP invariance). We will consider a continuous sequence of scalar field configurations that interpolates between a trivial background and a widely separated soliton/antisoliton pair, always with the same, trivial field configuration as \( x \to \pm \infty \). In this way we can unambiguously track physical quantities such as phase shifts and bound state energies, staying in the trivial sector of the theory throughout the process. In so doing, we obtain results numerically that continuously approach the results obtained analytically in Ref. [4].

In §II we develop the methods we will need to compute phase shifts, which in turn allow us to compute the renormalized one-loop energy as a sum over zero-point energies \(-\frac{1}{2} \omega\). In §III we apply these methods to the specific sequence of field configurations we have chosen and obtain numerical and analytical results. In §IV we summarize our results, and connect them to previous and future works. The Appendix contains techniques parallel to those of §II but for a single soliton instead of a soliton/antisoliton pair.

II. FERMIONIC SMALL OSCILLATIONS AND PHASE SHIFTS

We consider a Majorana fermion \( \Psi \) interacting with a scalar background field \( \phi \), with the classical Lagrangian density

\[
\mathcal{L} = \frac{m^2}{2\lambda} \left( i\bar{\Psi} \frac{\partial}{\partial x} \Psi - m\phi \bar{\Psi} \Psi + \mathcal{L}_\phi \right)
\]

where \( \mathcal{L}_\phi \) is the Lagrangian density for the \( \phi \) background field. As discussed in the Introduction, we can treat any bosonic potential. To be specific, we choose
\[ \mathcal{L}_\phi = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{8}(\phi^2 - 1)^2 \]  \hspace{1cm} (2)

which has classical soliton (antisoliton) solutions satisfying

\[ \phi'(x) = \pm \frac{m}{2}(\phi^2 - 1) \]  \hspace{1cm} (3)

giving

\[ \phi(x) = \pm \tanh \frac{mx}{2}. \]  \hspace{1cm} (4)

For the most part, we will be treating \( \phi \) as a classical background, so that its dynamics will be unimportant. However, we note that choosing the Lagrangian density of Eq. (2) causes the bosonic and fermionic degrees of freedom to be related by supersymmetry for a background field that is a solution to the equations of motion (such as a single soliton or an infinitely separated soliton/antisoliton pair). In what follows, we will find it instructive to compare the bosonic and fermionic small oscillations spectra. The supersymmetric model is discussed in more detail in Ref. [4].

The one-loop corrections to the energy due to fermionic fluctuations will be given by the appropriately renormalized sum of the zero-point energies, \(-\frac{1}{2}\omega\), of the fermionic small oscillations. The spectrum of fermionic small oscillations in a background \( \phi_0(x) \) is given by the Dirac equation,

\[ \gamma^0 \left( -i\gamma^1 \frac{d}{dx} + V_F(x) \right) \psi_k(x) = \omega_k^F \psi_k(x) \]  \hspace{1cm} (5)

where \( V_F(x) = m\phi_0(x) \) is the fermionic potential and \( k = \pm \sqrt{\omega^2 - m^2} \) is the momentum labeling the scattering states.

We will choose the convention \( \gamma^0 = \sigma_2 \) and \( \gamma^1 = i\sigma_3 \) so that the Majorana condition becomes simply \( \Psi^* = \Psi \). We note that since our Lagrangian is \( CP \) invariant, all of our results for the spectrum of a Majorana fermion can be extended to a Dirac fermion simply by doubling.

Our primary tool for analyzing the small oscillations will be the phase shifts. We can solve for the phase shifts of any fermionic potential \( V_F(x) \) that satisfies \( V_F(x) = V_F(-x) \) and \( V_F(x) \to m \) as \( x \to \pm \infty \) by generalizing the methods of Ref. [2] to fermions. This form will be useful for considering our example of a sequence of background field configurations that continuously interpolates between the trivial vacuum and a widely separated soliton-antisoliton pair. In the limit of infinite separation, the phase shift for the pair goes to twice the result for a single soliton. For comparison, we also do the computation for a single soliton directly in the Appendix.

We define the parity operator acting on fermionic states as \( P = \Pi \gamma^0 \), where \( \Pi \) sends \( x \to -x \). \( P \) commutes with the Hamiltonian, so parity is a good quantum number. As a result, we can separate the small oscillations into positive and negative parity channels, now restricted to \( k > 0 \) in the continuum.

We parameterize the fermion wavefunction by

\[ \psi(x) = \left( \frac{e^{i\nu(x)}}{i e^{i\zeta(x)} e^{i\theta}} \right) e^{ikx} \]  \hspace{1cm} (6)
where \( \theta = \tan^{-1} \frac{k}{m} \) and \( \nu \) and \( \zeta \) are complex functions of \( x \). We then find the phase shift in each channel \( \delta^\pm(k) \) by solving Eq. (5) subject to the boundary conditions

\[
\psi^\pm(0) \propto \left( \frac{1}{1 \pm i} \right). 
\]

We obtain

\[
\begin{align*}
\delta^+(k) &= -\text{Re} \nu(0) + \frac{\theta}{2} + \frac{1}{2i} \log \frac{Y - 1}{1 - Y^*} \\
\delta^-(k) &= -\text{Re} \nu(0) + \frac{\theta}{2} + \frac{1}{2i} \log \frac{1 + Y}{1 + Y^*}
\end{align*}
\]

where \( Y = \frac{1}{\omega}(V_F(0) - ik + i\nu'(0)^*) \) and \( \nu(x) \) satisfies

\[
-\nu(x)'' + \nu'(x)^2 + 2k\nu'(x) + V_F(x)^2 - V_F(x)' - m^2 = 0
\]

with the boundary condition that \( \nu(x) \) and \( \nu'(x) \) vanish at infinity. The total phase shift is given by summing the phase shifts in each channel, \( \delta_F(k) = \delta^+(k) + \delta^-(k) \).

Our calculation also requires us to find the bound state energies, which we do by solving Eq. (5) using ordinary “shooting” methods. However, once we know the phase shifts, Levinson’s theorem tells us how many bound states there will be. It works exactly the same way as in the bosonic case [4,5]: In the odd parity channel, the number of bound states \( n_- \) is given by

\[
\delta^-(0) = \pi n_-
\]

while in the even parity channel the number of bound states \( n_+ \) is given by

\[
\delta^+(0) = \pi (n_+ - \frac{1}{2}).
\]

One can derive this result either by the same Jost function methods used in the boson case, or by observing that at small \( k \), the nonrelativistic approximation becomes valid so the bosonic results carry over directly. For a particular potential there may exist a \( k = 0 \) state in either of the two channels whose Dirac wavefunction goes to a constant spinor as \( x \to \pm \infty \). (Generically, for \( k = 0 \) the components of the Dirac wavefunction go to straight lines as \( x \to \pm \infty \), but not lines with zero slope.) Just as in the bosonic case, such threshold states (which we term “half bound states”) should be counted with a factor of \( \frac{1}{2} \) in Levinson’s theorem.

Given the phase shifts and bound state energies, we can calculate the one-loop fermionic correction to the energy. We work in a simple renormalization scheme in which we add a counterterm proportional to \( \phi^2 - 1 \), and perform no further renormalizations. The counterterm is fixed by requiring that the tadpole graph cancel. We must sum the zero-point energies \(-\frac{1}{2} \omega\) of the small oscillations, subtract the same sum for the free case, and add the contribution of the counterterm. As in Ref. [2–4], we use the density of states

\[
\rho(k) = \rho_0(k) + \frac{1}{\pi} \frac{d\delta_F(k)}{dk}
\]
to write

$$\Delta H = -\frac{1}{2} \sum_j \omega_j - \int_0^\infty \frac{dk}{2\pi} \frac{d\delta_F(k)}{dk} + \Delta H_{ct}$$

where \(\omega = \sqrt{k^2 + m^2}\) and the sum over \(j\) counts bound states with appropriate factors of \(\frac{1}{2}\) as discussed above. Next, we use Eq. (10) and Eq. (11) to rewrite this expression as

$$\Delta H = -\frac{1}{2} \sum_j (\omega_j - m) - \int_0^\infty \frac{dk}{2\pi} (\omega - m) \frac{d\delta_F(k)}{dk} + \Delta H_{ct}.$$  

The (divergent) contribution from the tadpole graph is given by the leading Born approximation to \(\delta_F\). Since we have chosen the counterterm to exactly cancel the tadpole graph, we can rewrite the counterterm contribution in terms of the Born approximation, giving

$$\Delta H = -\frac{1}{2} \sum_j (\omega_j - m) - \int_0^\infty \frac{dk}{2\pi} (\omega - m) \frac{d\delta_F(k)}{dk} (\delta_F(k) - \delta_F^{(1)}(k))$$

where the subtraction \(\delta_F^{(1)}(k)\) consists of the first Born approximation to \(\delta_F(k)\) plus the part of the second Born approximation related by \(\phi \to -\phi\) symmetry,

$$\delta^{(1)}(k) = -\frac{1}{k} \int \left( V_F(x)^2 - m^2 \right) dx$$

which can also be obtained numerically by iterating Eq. (9). As expected, it is indeed proportional to \(\phi^2 - 1\).

III. NUMERICAL RESULTS

We study a sequence of background fields labeled by a parameter \(x_0\) that continuously interpolates from the trivial vacuum \(\phi(x) = 1\) at \(x_0 = 0\) to a widely separated soliton/antisoliton pair at \(x_0 \to \infty\),

$$\phi(x, x_0) = 1 + \tanh \frac{m(x - x_0)}{2} - \tanh \frac{m(x + x_0)}{2}.$$  

Fig. 1 shows the fermionic bound state energies as a function of \(x_0\). In the limit \(x_0 \to \infty\) two bound states approach energy \(m\sqrt{\frac{3}{2}}\). These are simply the odd and even parity combinations of the single soliton bound state at \(m\sqrt{\frac{3}{2}}\). The third (positive parity) bound state approaches \(\omega = 0\) where the single soliton also has a bound state, but there is only one such state (with \(\omega > 0\)). This is consistent with the result obtained analytically in Ref. \[4\]. As emphasized in Ref. \[4\], for a single soliton we must thus count the zero mode with a factor of \(\frac{1}{2}\).

For any finite \(x_0\), Levinson’s theorem holds without subtleties; there are no states that require factors of \(\frac{1}{2}\). Thus for a large but finite \(x_0\), we find

$$\delta^+(0) = \frac{3\pi}{2}$$
Fig. 1. Fermion bound state energies as a function of $mx_0$ in units of $m$. Even parity states are denoted with $\diamondsuit$ and odd parity states with $\pm$.

$$\delta^-(0) = \pi$$

(18)

consistent with having two positive parity and one negative parity bound states (see Fig. 1). In the limit $x_0 \to \infty$, an even parity “half-bound” threshold state enters the spectrum at $\omega = m$ just as in the bosonic case [2]. Also, in this limit, the lowest (positive parity) mode approaches $\omega = 0$, and is only counted as a $\frac{1}{2}$ as described above. Finally, a negative parity mode enters the spectrum from below, also to be weighted by $\frac{1}{2}$.

Thus the net change is to add half of a negative parity state, which via Levinson’s theorem requires the phase shift $\delta^-(0)$ to jump from $\pi$ to $\frac{3\pi}{2}$ as $x_0 \to \infty$. This jump occurs by the same continuous but nonuniform process as in all cases where a new state gets bound, which is illustrated in Fig. 2. Just as in the bosonic case, in the limit of infinite separation the potential we have chosen becomes reflectionless, which requires $\delta^+(k) = \delta^-(k)$.

To contrast the behavior of the zero modes, Fig. 3 shows $\omega^2$ for the bound states of the bosonic small oscillations [2]. Because we have chosen the bosonic potential to be consistent with supersymmetry, the bosonic and fermionic spectra are related. Again as $x_0 \to \infty$ the bound state energies approach those of the single soliton, and the wavefunctions are formed from the odd and even combinations of the wavefunctions for the single soliton. However, we see that in the boson case both the mode at $m\frac{\sqrt{3}}{2}$ and the mode at $\omega = 0$ are doubled, so there is no factor of $\frac{1}{2}$ in counting the bosonic zero modes. For $x_0$ above approximately $0.75/m$, the system is unstable with respect to small oscillations in one direction in field space. This property is reflected in the lowest bound state having $\omega^2 < 0$ [2]. Just as in the fermionic case, in the limit $x_0 \to \infty$, a “half-bound” state enters the spectrum at $\omega = m$, and the phase shifts in the two channels become equal as the potential becomes reflectionless.

Fig. 4 shows the values of $\Delta H$ we obtain from Eq. (15) as a function of $x_0$. In the $x_0 \to \infty$ limit, we can also do the calculation analytically [4]. We find a phase shift
FIG. 2. Negative-parity phase shifts as functions of $k/m$ for $x_0 = 2.0, 3.0, 4.0, 5.0, 6.0, \text{ and } 8.0$. For any finite separation, the phase shift is equal to $\pi$ at $k = 0$, but as $x_0$ gets larger, the phase shift ascends more and more steeply to $\frac{3\pi}{2}$.

FIG. 3. Boson bound state squared energies as a function of $mx_0$ in units of $m$. Symmetric states are denoted with ♦ and antisymmetric states with +.
\[ \delta_F(k) = 4 \tan^{-1} \frac{m}{2k} + 2 \tan^{-1} \frac{m}{k} \]  

with Born approximation

\[ \delta_F^{(1)}(k) = \frac{4m}{k} \]  

and contributions from two bound states at \( \omega = m\sqrt{\frac{3}{2}} \) and one at \( \omega = m \). We thus find

\[ \Delta H = 2m \left( \frac{1}{\pi} - \frac{1}{4\sqrt{3}} \right) \]

which agrees with the numerical calculation. As shown in Ref. [2], the bosonic contribution to the energy is \( 2m \left( \frac{1}{4\sqrt{3}} - \frac{3}{2\pi} \right) \) in this limit, giving a total of \( -\frac{m}{\pi} \) for the soliton/antisoliton pair in the supersymmetric model, in agreement with Ref. [4].

**DISCUSSION**

We have shown how to compute one-loop fermionic contributions to the energies of symmetric background field configurations. Taking these methods together with those of Ref. [2], one can calculate the energy of a localized field configuration in any model of scalars and spinors in 1+1 dimensions. This procedure implements standard renormalization conditions without ambiguity and lends itself to robust and efficient numerical calculation. A natural extension of this work is to treat translationally invariant classical field configurations. Long ago, Christ and Lee showed how to quantize the small fluctuations about momentum eigenstates formed by taking superpositions of *solutions* to the classical equations of motion at different positions [6]. New problems arise when studying the effects of translation.
invariance upon fluctuations about time-independent classical field configurations that are not solutions to the classical equations of motion. Work is underway to extend the above techniques to the nonlocal potentials encountered in this case [7].

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APPENDIX: PROPERTIES OF A SINGLE SOLITON

As a check on the calculations presented in the body of the paper, we present the properties (phase shifts, eigenenergies, etc.) for a single soliton directly. In this case, we have a potential that satisfies $V_F(x) = -V_F(-x)$ with $V_F(x) \to \pm m$ as $x \to \pm \infty$. We will use the bosonic potentials that we obtain by squaring the Dirac equation to obtain a result that agrees with the previous method in the limit of a widely separated soliton/antisoliton pair. We find

$\begin{pmatrix} -\frac{d^2}{dx^2} + V_B(x) & 0 \\ 0 & -\frac{d^2}{dx^2} + \tilde{V}_B(x) \end{pmatrix} \psi_k(x) = k^2 \psi_k(x) \tag{22}$

where

$V_B(x) = V_F(x)^2 - V'_F(x) - m^2 \tag{23}$

and

$\tilde{V}_B(x) = V(x)^2 + V'_F(x) - m^2 \tag{24}$

are bosonic potentials.

Since $V_F(x)$ is antisymmetric, both $V_B(x)$ and $\tilde{V}_B(x)$ are symmetric. We decompose their solutions (as bosonic potentials) into symmetric and antisymmetric channels. For $x \to \pm \infty$, we have

$\eta^S_k(x) = \cos(kx \pm \delta^S(k))$

$\eta^A_k(x) = \sin(kx \pm \delta^A(k))$

$\tilde{\eta}^S_k(x) = i \cos(kx \pm \tilde{\delta}^S(k))$

$\tilde{\eta}^A_k(x) = -i \sin(kx \pm \tilde{\delta}^A(k)) \tag{25}$

where the arbitrary factors of $\pm i$ are inserted for convenience later. By the (first-order) Dirac equation, for all $x$,

$\omega_k \tilde{\eta}^S_k(x) = i \left( \frac{d}{dx} + V_F(x) \right) \eta^A_k(x)$

$\omega_k \eta^A_k(x) = i \left( \frac{d}{dx} - V_F(x) \right) \tilde{\eta}^S_k(x) \tag{26}$
and
\[
\omega_k \tilde{\eta}_k^A(x) = i \left( \frac{d}{dx} + V_F(x) \right) \tilde{\eta}_k^S(x)
\]
\[
\omega_k \eta_k^S(x) = i \left( \frac{d}{dx} - V_F(x) \right) \eta_k^A(x)
\]
so that the solutions to the Dirac equation are
\[
\psi_k^+(x) = \begin{pmatrix} \eta_k^S \\ \tilde{\eta}_k^A \end{pmatrix}
\]
and
\[
\psi_k^-(x) = \begin{pmatrix} \eta_k^A \\ \tilde{\eta}_k^S \end{pmatrix}.
\]

The phase relation between the upper and lower components of these wavefunctions must be different as \( x \to \pm \infty \) since the mass term has opposite signs in these two limits.

Putting this all together gives, as \( x \to \pm \infty \),
\[
\cos(kx \pm \delta^S(k)) = \frac{1}{\omega_k} \left( \frac{d}{dx} - V_F(x) \right) \sin(kx \pm \tilde{\delta}^A(k)) = \frac{1}{\omega_k} \left( k \cos(kx \pm \tilde{\delta}^A(k)) \mp m \sin(kx \pm \tilde{\delta}^A(k)) \right) = \mp \sin(kx \pm \tilde{\delta}^A(k) \mp \theta) = \cos(kx \pm \tilde{\delta}^A(k) \mp \theta \pm \frac{\pi}{2})
\]
and thus
\[
\delta^S(k) = \tilde{\delta}^A(k) + \tan^{-1} \frac{m}{k}
\]
and similarly
\[
\delta^A(k) = \tilde{\delta}^S(k) + \tan^{-1} \frac{m}{k}.
\]

The fermion phase shift in each channel is given by the average of the bosonic phase shifts of the two components
\[
\delta^+(k) = \frac{1}{2}(\delta^S(k) + \delta^A(k)) = \delta^S(k) - \frac{1}{2} \tan^{-1} \frac{m}{k}
\]
\[
\delta^-(k) = \frac{1}{2}(\delta^A(k) + \delta^S(k)) = \delta^A(k) - \frac{1}{2} \tan^{-1} \frac{m}{k}
\]
so that
\[
\delta_F(k) = \delta^+(k) + \delta^-(k) = \delta^S(k) + \delta^A(k) - \tan^{-1} \frac{m}{k}.
\]

We note that in this case there is always a zero-energy bound state given by
\[ \psi(x) = \left( e^{-\int_0^x V_F(y) dy} \right). \] (35)

Because it is at zero energy, this state is shared with the antiparticle spectrum and thus only contributes with a weight of \( \frac{1}{2} \) to Levinson’s theorem and the sum over small oscillations, as we found by following a soliton/antisoliton pair as a function of separation. The extra \( \tan^{-1} \frac{m}{k} \) (which goes to \( \frac{\pi}{2} \) as \( k \to 0 \)) is thus necessary to connect the fermionic phase shift to bosonic phase shifts, which have no such factors of \( \frac{1}{2} \) associated with their zero modes.
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