THE SCHUR PROBLEM AND BLOCK OPERATOR CMV MATRICES

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ABSTRACT. The CMV matrices and their sub-matrices are applied to the description of all solutions to the Schur interpolation problem for contractive analytic operator-valued functions in the unit disk (the Schur class functions).

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1. Introduction

In what follows the class of all continuous linear operators defined on a complex Hilbert space $H_1$ and taking values in a complex Hilbert space $H_2$ is denoted by $L(H_1, H_2)$ and $L(H) := L(H, H)$. All infinite dimensional Hilbert spaces are supposed to be separable. We denote by $I_H$ the identity operator in a Hilbert space $H$ and omit the symbol $H$ in these notations if there is no danger of confusion; by $P_L$ the orthogonal projection onto the subspace (the closed linear manifold) $L$. The notation $T | L$ means the restriction of a linear operator $T$ on the set $L$. The range and the null-space of a linear operator $T$ are denoted by $\text{ran} T$ and $\ker T$, respectively. We use the standard symbols $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{N}_0$ for the sets of complex numbers, positive integers, and nonnegative integers, respectively. An operator $T \in L(H_1, H_2)$ is said to be

- contractive if $\|T\| \leq 1$;
- isometric if $\|Tf\| = \|f\|$ for all $f \in H_1 \iff T^* T = I_{H_1}$;
- co-isometric if $T^*$ is isometric $\iff TT^* = I_{H_2}$;
- unitary if it is both isometric and co-isometric.

Given a contraction $T \in L(H_1, H_2)$, the operators $D_T := (I_{H_1} - T^* T)^{1/2}$ and $D_{T^*} := (I_{H_2} - TT^*)^{1/2}$ are called the defect operators of $T$, and the subspaces $\mathcal{D}_T = \overline{\text{ran} D_T}$, $\mathcal{D}_{T^*} = \overline{\text{ran} D_{T^*}}$ the defect subspaces of $T$. The defect operators satisfy the relations $TD_T = D_{T^*} T$, $T^* D_{T^*} = D_T T^*$. Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert spaces. The Schur class $S(\mathcal{M}, \mathcal{N})$ is the set of all functions $\Theta(z)$ with values in $L(\mathcal{M}, \mathcal{N})$, holomorphic in the unit disk

$$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$$

and such that $\|\Theta(z)\| \leq 1$ for all $z \in \mathbb{D}$. Let $\Theta$ be holomorphic in $\mathbb{D}$ operator valued function acting between Hilbert spaces $\mathcal{M}$ and $\mathcal{N}$ and let

$$\Theta(z) = \sum_{n=0}^{\infty} z^n C_n, \quad z \in \mathbb{D}, \quad C_n \in L(\mathcal{M}, \mathcal{N}), \quad n \geq 0$$

be the Taylor expansion of $\Theta$. Consider the lower triangular (analytic) Toeplitz matrix

$$T_{\Theta} := \begin{bmatrix} C_0 & 0 & 0 & 0 & \cdots & \cdots \\ C_1 & C_0 & 0 & 0 & \cdots & \cdots \\ C_2 & C_1 & C_0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Let $\mathcal{H}$ be a separable Hilbert space and let $\ell_2(\mathcal{H})$ be a Hilbert space of all vectors $\vec{f} = [f_0, f_1, \ldots]^T$, $f_k \in \mathcal{H}, k \in \mathbb{N}_0$, $\sum_{k=0}^{\infty} \|f_k\|^2_{\mathcal{H}} < \infty$, with the inner product $(\vec{f}, \vec{g}) = \sum_{k=0}^{\infty} (f_k, g_k)_{\mathcal{H}}$. As is well known \cite{14, 23}

$$\Theta \in S(\mathcal{M}, \mathcal{N}) \iff T_{\Theta} \in L(\ell_2(\mathcal{M}), \ell_2(\mathcal{N})) \text{ is a contraction.}$$

Set for $n \in \mathbb{N}_0$

$$\mathcal{M}^{n+1} = \underbrace{\mathcal{M} \oplus \mathcal{M} \oplus \cdots \oplus \mathcal{M}}_{n+1}, \quad \mathcal{N}^{n+1} = \underbrace{\mathcal{N} \oplus \mathcal{N} \oplus \cdots \oplus \mathcal{N}}_{n+1}.$$
Clearly, if \( T_\Theta \) is a contraction, then the operator \( T_{\Theta,n} \in L(\mathcal{M}^{n+1}, \mathcal{M}^{n+1}) \) given by the block operator matrix

\[
T_{\Theta,n} := \begin{bmatrix}
C_0 & 0 & 0 & \ldots & 0 \\
C_1 & C_0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_n & C_{n-1} & C_{n-2} & \ldots & C_0
\end{bmatrix}
\]

is a contraction for each \( n \).

The following interpolation problem is called the Schur problem:

Let \( \mathcal{M} \) and \( \mathcal{N} \) be Hilbert spaces.
Given operators \( C_k \in L(\mathcal{M}, \mathcal{N}) \), \( k = 0, 1, \ldots, N \),
does there exist \( \Theta \in S(\mathcal{M}, \mathcal{N}) \) such that \( C_k \) are the first
Taylor coefficients of \( \Theta \), i.e.,
\[
\frac{\Theta^{(k)}(0)}{k!} = C_k \quad \text{for} \quad k = 0, 1, \ldots, N?
\]

If such functions exist, describe all of them.

The Schur problem is often called the Carathéodory or the Carathéodory-Fejér problem. This problem was posed and solved by I. Schur in [32] for scalar case (\( \mathcal{M} = \mathcal{N} = \mathbb{C} \)) and later this and other interpolation problems for matrix and operator cases attracted attention of many authors, which used various methods to solve them, see [1, 13, 14, 19, 21, 22, 23, 24, 25, 27, 28, 29, 31] and references therein. It is proved in [32] for \( \mathcal{M} = \mathcal{N} = \mathbb{C} \) that solutions of the interpolation problem (1.1) exist if and only if the lower triangular Toeplitz matrix

\[
T_N := \begin{bmatrix}
C_0 & 0 & 0 & \ldots & 0 \\
C_1 & C_0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_N & C_{N-1} & C_{N-2} & \ldots & C_0
\end{bmatrix}
\]

is a contraction in \( \mathbb{C}^{N+1} \) with the standard inner product, i.e., \( I - T_N^* T_N \geq 0 \). The uniqueness holds if and only if
\[
\det(I - T_N^* T_N) = 0.
\]

In the non-uniqueness case a descriptions of all solutions in [32] is given in the form of fractional-linear transformation

\[
\Theta(z) = \frac{e_N(z) E(z) + f_N(z)}{g_N(z) E(z) + h_N(z)},
\]

were \( E \) is an arbitrary scalar Schur class function and
\[
e_N(z), \ f_N(z), \ g_N(z), \ h_N(z)
\]
are polynomials. The approach proposed by Schur is based on the transformation

\[
S \ni f \mapsto \varphi(z) := \frac{f(z) - f(0)}{z(1 - \overline{f(0)} f(z))} \in S, \ z \in \mathbb{D}.
\]
The successive application of this transform to \( f \in S \)
\[
f_0(z) = f(z), \ldots, f_k(z) = \frac{f_k(z) - f_k(0)}{z(1 - \overline{f_k(0)} f_k(z))} \ldots
\]
is called nowadays the Schur algorithm. If \( f(z) \) is not a finite Blaschke product, then \( \{f_k\} \) is an infinite sequence of Schur functions called the associated functions.
and none of them is a finite Blaschke product. The numbers $\gamma_k := f_k(0)$ are called the Schur parameters. Note that

$$f_k(z) = \frac{\gamma_k + zf_{k+1}(z)}{1 + \gamma_k z f_{k+1}(z)} = \gamma_k + \frac{zf_{k+1}(z)}{1 + \gamma_k z f_{k+1}(z)}, \quad k \in \mathbb{N}_0.$$  

In the case when

$$f(z) = e^{i\varphi} \prod_{k=1}^{l} \frac{z - z_k}{1 - \bar{z}_k z},$$

is a finite Blaschke product of order $l$, the Schur algorithm terminates at the $l$-th step, i.e., the sequence of Schur parameters $\{\gamma_n\}_{k=0}^{l}$ is finite, $|\gamma_k| < 1$ for $n = 0, 1, \ldots, l - 1$, and $|\gamma_l| = 1$. Schur proved that $f$ can be uniquely recovered from $\{\gamma_k\}$, i.e., there is one-to-one correspondence between Schur class functions and their Schur parameters.

The coefficient matrix in (1.3)

$$W_N(z) = \begin{bmatrix} e_N(z) & g_N(z) \\ f_N(z) & h_N(z) \end{bmatrix}$$

can be calculated inductively ($N + 1$ steps) by means of Schur parameters $\gamma_0, \gamma_1, \ldots, \gamma_N$, obtained by the Schur algorithm which starts with

$$f_0(z) = C_0 + C_1 z + \cdots + C_N z^N.$$  

In the case of uniqueness the solution $\Theta$ can be obtained as follows [32]:

- find $m \in \mathbb{N}_0$, $m \leq N$, such that
  $$\det(I - T^{*m}_{m-1} T_{m-1}) \neq 0 \quad \text{and} \quad \det(I - T^{*m} T_m) = 0,$$

- calculate
  $$\Theta_{m-1}(z) = \frac{\gamma_{m-1} + z\gamma_m}{1 + z\gamma_{m-1}\gamma_m}, \ldots, \Theta_0 = \frac{\gamma_0 + z\Theta_1(z)}{1 + z\gamma_0 \Theta_1(z)}, \quad z \in \mathbb{D}$$

- the function $\Theta(z) = \Theta_0(z)$, $z \in \mathbb{D}$ is the solution.

For matrix and operator cases parameterizations similar to (1.3) can be found in [11, 13, 21, 23, 24]. As has been mentioned before, different methods were used. The existence criteria in the operator case is similar to the scalar one: the Toeplitz block operator matrix $T_N$ given by (1.2) is a contraction, acting form $\mathcal{M}_N^{N+1}$ into $\mathcal{N}_N^{N+1}$.

In the present paper a new approach to a parametrization of all solutions to the Schur problem for operator valued functions is suggested. We essentially use the properties of the block operator CMV (Cantero–Moral–Velázquez) matrices [6] and their sub-matrices. The scalar CMV matrices ($\mathcal{M} = \mathcal{N} = \mathbb{C}$) appeared in the theory of scalar orthogonal polynomials on the unit circle $T = \{z \in \mathbb{C} : |z| = 1\}$ [16, 35]. In [6] the block operator CMV matrices were defined, studied, and applied to the theory of conservative discrete time-invariant linear systems and to the dilation theory. Block operator CMV matrices are built by means of an arbitrary choice sequence and, in particular, by means of the Schur parameters of an arbitrary $\Theta \in \mathcal{S}^{\mathcal{M}, \mathcal{N}}$. It is established in [6] that the simple conservative systems associated with block operator CMV matrix are realizations of given function from $\mathcal{S}^{\mathcal{M}, \mathcal{N}}$, while the truncated block operator CMV matrices are the models of completely non-unitary contractions [6, 8]. We will use finite sub-CMV matrices which take
three-diagonal block operator form and which are constructed by means of the finite choice sequences associated with the data of solvable Schur problems. It should be mentioned that in our paper [9] the similar approach, using three-diagonal matrices and the corresponding resolvent formulas, has been applied to the descriptions of all solutions to the operator truncated Hamburger moment problem.

The paper is organized as follows. In Section 2 we describe the Schur algorithm for operator-valued Schur class functions and the relationships between the Schur parameters, the Taylor coefficients, the Krein shorted operators, and lower triangular Toeplitz matrices. In Section 3 we recall (see [14]) the constructions of block operator CMV and truncated CMV matrices for a choice sequence \( \{ \Gamma_n \} \) in the case \( D_{\Gamma_n} \neq \{ 0 \}, D_{\Gamma^*_n} \neq \{ 0 \} \) for all \( k \). In Appendix A we present the explicit form of the block operator CMV and truncated CMV matrices in the cases when \( \Gamma_m \) is isometry, co-isometry, unitary for some \( m \). The core of the paper is Section 4 where we derive useful resolvent formulas (Proposition 4.3 and Theorem 4.4) and, using the three-diagonal finite sub-matrices of truncated CMV matrices, we establish explicit connections between the function \( \Theta \in S(\mathcal{M}, \mathcal{N}) \) and the functions \( \Theta_k \) associated with \( \Theta \) in accordance with the Schur algorithm (Theorems 4.6, 4.9). In Section 5 we apply results obtained in Section 4 to the parametrization of all solutions to the Schur problem.

2. Preliminaries

2.1. The Schur algorithm for operator-valued functions. The Schur algorithm for matrix and operator valued Schur class functions has been considered in [20, 21, 14, 18, 19, 23]. It is based on the following theorem which goes back to [33, 34]:

**Theorem 2.1.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be Hilbert spaces and let the function \( \Theta(z) \) be from the Schur class \( S(\mathcal{M}, \mathcal{N}) \). Then there exists a function \( Z(z) \) from the Schur class \( S(D_{\Theta(0)}, D_{\Theta^*(0)}) \) such that

\[
\Theta(z) = \Theta(0) + D_{\Theta(0)} Z(z) (I + \Theta^*(0) Z(z))^{-1} D_{\Theta(0)}, \quad z \in \mathbb{D}.
\]

The representation (2.1) of a function \( \Theta(z) \) from the Schur class is called the M"{o}bius representation of \( \Theta(z) \) and the function \( Z(z) \) is called the M"{o}bius parameter of \( \Theta(z) \). Clearly, \( Z(0) = 0 \) and from Schwartz’s lemma one obtains that

\[
z^{-1} Z(z) \in S(D_{\Theta(0)}, D_{\Theta^*(0)}).
\]

**The operator Schur’s algorithm** [14]. For \( \Theta \in S(\mathcal{M}, \mathcal{N}) \) put \( \Theta_0(z) = \Theta(z) \) and let \( Z_0(z) \) be the M"{o}bius parameter of \( \Theta \). Define

\[
\Gamma_0 = \Theta(0), \quad \Theta_1(z) = z^{-1} Z_0(z) \in S(D_{\Gamma_0}, D_{\Gamma^*_0}), \quad \Gamma_1 = \Theta_1(0) = Z'_0(0).
\]

If \( \Theta_0(z), \ldots, \Theta_k(z) \) and \( \Gamma_0, \ldots, \Gamma_k \) already defined, then let \( Z_{k+1} \in S(D_{\Gamma_k}, D_{\Gamma^*_k}) \) be the M"{o}bius parameter of \( \Theta_k \). Put

\[
\Theta_{k+1}(z) = z^{-1} Z_{k+1}(z), \quad \Gamma_{k+1} = \Theta_{k+1}(0).
\]

The contractions \( \Gamma_0 \in L(\mathcal{M}, \mathcal{N}), \Gamma_k \in L(D_{\Gamma_{k-1}}, D_{\Gamma^*_{k-1}}), k = 1, 2, \ldots \) are called the Schur parameters of \( \Theta \) and the function \( \Theta_k \in S(D_{\Gamma_{k-1}}, D_{\Gamma^*_{k-1}}) \) is called the \( k \)-th associated function. Thus,

\[
\Theta_k(z) = \Gamma_k + z D_{\Gamma^*_k} \Theta_{k+1}(z) (I + z \Gamma^*_k \Theta_{k+1}(z))^{-1} D_{\Gamma_k}
\]

\[
= \Gamma_k + z D_{\Gamma^*_k} (I + z \Theta_{k+1}(z) \Gamma^*_k)^{-1} \Theta_{k+1}(z) D_{\Gamma_k}, \quad z \in \mathbb{D},
\]
and
\[ \Theta_{k+1}(z) \mid \text{ran } D_{\Gamma_k} = z^{-1} D_{\Gamma_k}^\ast (I - \Theta_k(z) \Gamma_k^\ast)^{-1} (\Theta_k(z) - \Gamma_k) D_{\Gamma_k}^{-1} \mid \text{ran } D_{\Gamma_k}. \]

Clearly, the sequence of Schur parameters \( \{\Gamma_k\} \) is infinite if and only if the operators \( \Gamma_k \) are non-unitary. The sequence of Schur parameters consists of finite number of operators \( \Gamma_0, \Gamma_1, \ldots, \Gamma_N \) if and only if \( \Gamma_N \in \text{L}(\mathfrak{D}_{\Gamma_{N-1}}, \mathfrak{D}_{\Gamma_{N-1}}) \) is unitary. If \( \Gamma_N \) is non-unitary but isometric (respect., co-isometric), then \( \Gamma_k = 0 \in \text{L}(0, \mathfrak{D}_{\Gamma_N}) \) (respect., \( \Gamma_k = 0 \in \text{L}(\mathfrak{D}_{\Gamma_N}, 0) \)) for all \( k > N \). The following theorem [14] is the operator generalization of Schur’s result.

**Theorem 2.2.** There is a one-to-one correspondence between the Schur class \( \mathcal{S}(\mathcal{M}, \mathcal{N}) \) and the set of all sequences of contractions \( \{\Gamma_k\}_{k \geq 0} \) such that
\[ (2.2) \quad \Gamma_0 \in \text{L}(\mathcal{M}, \mathcal{N}), \quad \Gamma_k \in \text{L}(\mathfrak{D}_{\Gamma_{k-1}}, \mathfrak{D}_{\Gamma_{k-1}}), \quad k \geq 1. \]

Notice that a sequence of contractions of the form (2.2) is called the *choice sequence* [17]. There are connections, established in [18], between the Taylor coefficients \( \{C_n\}_{n \geq 0} \) and Schur parameters of \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \). These connections are given by the relations
\[ C_0 = \Gamma_0, \quad C_n = \text{formula}_n(\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}) + D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_{n-1}} \Gamma_n D_{\Gamma_{n-1}} \cdots D_{\Gamma_0}, \quad n \geq 1, \]
where *formula* \(_n(\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}) \) is some expression, depending on \( \Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1} \).

Let now \( \{\Gamma_k\}_{k=0}^\infty \) be a sequence of operators from \( \text{L}(\mathcal{M}, \mathcal{N}) \). Then (Theorem 2.1) there is a one-to-one correspondence between the set of contractions
\[ T_\infty := \begin{bmatrix} C_0 & 0 & 0 & 0 & 0 & \ldots \\ C_1 & C_0 & 0 & 0 & 0 & \ldots \\ C_2 & C_1 & C_0 & 0 & 0 & \ldots \\ C_3 & C_2 & C_1 & C_0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} : \ell_2(\mathcal{M}) \rightarrow \ell_2(\mathcal{N}) \]
and the set of all choice sequences \( \Gamma_0 \in \text{L}(\mathcal{M}, \mathcal{N}), \Gamma_k \in \text{L}(\mathfrak{D}_{\Gamma_{k-1}}, \mathfrak{D}_{\Gamma_{k-1}}), \quad k = 1, \ldots. \)

The connections between \( \{C_k\} \) and \( \{\Gamma_k\} \) are also given by (2.3). The operators \( \{\Gamma_k\} \) can be successively defined [14] proof of Theorem 2.1], using parametrization of contractive block-operator matrices, from the matrices
\[ T_0 = C_0 = \Gamma_0, \quad T_1 = \begin{bmatrix} C_0 & 0 \\ C_1 & C_0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} C_0 & 0 & 0 \\ C_1 & C_0 & 0 \\ C_2 & C_1 & C_0 \end{bmatrix}, \ldots. \]

Moreover, \( T_\infty = T_0, \quad \Theta(\lambda) = \sum_{n=0}^\infty \lambda^n C_n, \quad \lambda \in \mathbb{D}, \) and \( \{\Gamma_k\}_{k \geq 0} \) are the Schur parameters of \( \Theta \) [14] Proposition 2.2]. Put
\[ \widetilde{\Theta}(\lambda) := \Theta^\ast(\lambda), \quad |\lambda| < 1. \]

Then \( \widetilde{\Theta}(\lambda) = \sum_{n=0}^\infty \lambda^n C_n^\ast \). Clearly, if \( \{\Gamma_0, \Gamma_1, \ldots\} \) are the Schur parameters of \( \Theta \), then \( \{\Gamma_0^\ast, \Gamma_1^\ast, \ldots\} \) are the Schur parameters of \( \widetilde{\Theta} \).
Besides $T_N$ we will consider the following lower triangular block operator matrices from $L(M^{N+1}, \mathfrak{M}^{N+1})$:

\[
\mathcal{T}_N = \mathcal{T}_N(C_0, C_1, \ldots, C_N) := \begin{bmatrix}
C_0^* & 0 & 0 & 0 \\
C_1^* & C_0^* & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
C_N^* & C_{N-1}^* & \ldots & C_0^*
\end{bmatrix}.
\]

If $T_N$ of the form (1.2) is a contraction, then operators $\{C_k\}_{k=0}^N$ are said to be the Schur sequence [21].

2.2. The Krein shorted operator and Toeplitz matrices. For every non-negative bounded operator $S$ in the Hilbert space $H$ and every subspace $K \subset H$ M.G. Krein [30] defined the operator $S_K$ by the relation

\[
S_K = \max \{ Z \in L(H) : 0 \leq Z \leq S, \text{ran} Z \subseteq K \}.
\]

The equivalent definition is:

\[
(S_K f, f) = \inf_{\varphi \in K^\perp} \{(S f + \varphi, f + \varphi)\}, \quad f \in H.
\]

Here $K^\perp := H \ominus K$. The properties of $S_K$, were studied by M. Krein and by other authors (see [4] and references therein). $S_K$ is called the shorted operator (see [2, 3]). Let the subspace $\Omega$ be defined as follows

\[
\Omega = \{ f \in \text{ran} S : S^{1/2} f \in K \} = \text{ran} S \ominus S^{1/2} K^\perp.
\]

It is proved in [30] that $S_K$ takes the form

\[
S_K = S^{1/2} P_\Omega S^{1/2}.
\]

Hence, $\ker S_K \supseteq K^\perp$. Moreover [30],

\[
\text{ran} S_K^{1/2} = \text{ran} S^{1/2} \cap K.
\]

It follows that

\[
S_K = 0 \iff \text{ran} S^{1/2} \cap K = \{0\}.
\]

We identify $\mathfrak{M}$ (or, respectively) with the subspace

\[
\mathfrak{M} \oplus \{0\} \oplus \{0\} \oplus \cdots \oplus \{0\}
\]

in $\mathfrak{M}^{n+1}$ ($\mathfrak{M}^{n+1}$), and with $\mathfrak{M} \oplus \bigoplus_{k=1}^{n} \{0\}$ (or $\mathfrak{M} \oplus \bigoplus_{k=1}^{\infty} \{0\}$) in $l_2(\mathfrak{M})$ ($l_2(\mathfrak{M})$). The next relations are established in [7].

**Theorem 2.3.** Let $\Theta \in S(\mathfrak{M}, \mathfrak{M})$ and let $\{\Gamma_0, \Gamma_1, \cdots\}$ be the Schur parameters of $\Theta$. Then for each $n$ the relations

\[
\left(\begin{array}{c}
D_{\Theta_{\mathfrak{M}, n}}^2
\end{array}\right)^{\mathfrak{M}} = D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_{n-1}} D_{\Gamma_n} \cdots D_{\Gamma_1} D_{\Gamma_0} P_{\mathfrak{M}}
\]

\[
\left(\begin{array}{c}
D_{\Theta_{\mathfrak{M}, n}}^2
\end{array}\right)^{\mathfrak{M}} = D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_{n-1}} D_{\Gamma_n} \cdots D_{\Gamma_1} D_{\Gamma_0} \quad P_{\mathfrak{M}},
\]

hold. Moreover

\[
(D_{\Theta_{\mathfrak{M}, n}}^2)^{\mathfrak{M}} = s - \lim_{n \to \infty} \left(D_{\Gamma_0} D_{\Gamma_1} \cdots D_{\Gamma_{n-1}} D_{\Gamma_n} \cdots D_{\Gamma_1} D_{\Gamma_0} \right) P_{\mathfrak{M}},
\]


\[
\left( D^2_{T_n} \right)_{\mathfrak{M}} = s - \lim_{n \to \infty} \left( D_{T_n} D_{T_{n-1}} \cdots D_{T_1} D^{2}_{T_0} D_{T_{n-1}} \cdots D_{T_1} D_{T_0} \right) P_{\mathfrak{N}}.
\]

By means of relations (2.3) contractions \( T_0, T_1, \ldots, T_N \) determine choice parameters
\[
\Gamma_0 := C_0, \Gamma_1 \in \mathcal{L}(\mathcal{D}_{T_0}, \mathcal{D}_{T_0}), \ldots, \Gamma_N \in \mathcal{L}(\mathcal{D}_{T_{N-1}}, \mathcal{D}_{T_{N-1}})
\]
and for operators \( T_n \) and \( \tilde{T}_n \) \( (n \leq N) \) the relations [21]:
\[\begin{align*}
(D^2_{T_n})_{\mathfrak{M}} &= D_{T_n} D_{T_{n-1}} \cdots D_{T_1} D^2_{T_0} D_{T_{n-1}} \cdots D_{T_1} D_{T_0} P_{\mathfrak{M}}, \\
(D^2_{\tilde{T}_n})_{\mathfrak{M}} &= D_{\tilde{T}_n} D_{\tilde{T}_{n-1}} \cdots D_{T_1} D^2_{T_0} D_{T_{n-1}} \cdots D_{T_1} D_{T_0} P_{\mathfrak{M}}.
\end{align*}\]

hold true. The next result can be found in [14, Theorem 2.6].

**Theorem 2.4.** Consider a solvable Schur problem with the data
\[
C_0, \ldots, C_N \in \mathcal{L}(\mathfrak{M}, \mathfrak{N}).
\]
Then the solution is unique if and only if the corresponding choice parameters \( \{\Gamma_n\}_{n=0}^N \), determined by the operator \( T_N \), satisfy the condition: one of \( \Gamma_n, 0 \leq n \leq N \) is an isometry or a co-isometry.

The next statement is established in [21].

**Theorem 2.5.** Let the data \( C_0, C_1, \ldots, C_N \in \mathcal{L}(\mathfrak{M}, \mathfrak{N}) \) be the Schur sequence. Then the following statements are equivalent
\[\begin{align*}
(i) \quad & \text{the Schur problem has a unique solution;} \\
(ii) \quad & \text{either} (D^2_{T_n})_{\mathfrak{M}} = 0 \text{ or } (D^2_{\tilde{T}_n})_{\mathfrak{M}} = 0; \\
(iii) \quad & \text{either} \mathfrak{M} \cap \text{ran } D_{T_n} = \{0\} \text{ or } \mathfrak{M} \cap \text{ran } D_{\tilde{T}_n} = \{0\}.
\end{align*}\]

2.3. **The Schur sequences and the Schur parameters.** Let the Schur sequence \( C_0, \ldots, C_N \subset \mathcal{L}(\mathfrak{M}, \mathfrak{N}) \), satisfying the conditions
\[D^2_{T_n})_{\mathfrak{M}} \neq 0, \quad (D^2_{\tilde{T}_n})_{\mathfrak{M}} \neq 0,
\]
be given. Then the corresponding Schur parameters
\[
\Gamma_0 \in \mathcal{L}(\mathfrak{M}, \mathfrak{N}), \ldots, \Gamma_N \in \mathcal{L}(\mathcal{D}_{T_{N-1}}, \mathcal{D}_{T_{N-1}}),
\]
satisfy the conditions
\[\mathcal{D}_{T_n} \neq \{0\}, \quad \mathcal{D}_{\Gamma_n} \neq \{0\}.
\]

One more way to find \( \{\Gamma_k\}_{k=0}^N \) goes back to Schur [32] (see also [23, page 448]). We shall describe it. Let \( \Theta \in \mathcal{S}(\mathfrak{M}, \mathfrak{N}) \) be a solution to the Schur problem with data \( C_0, \ldots, C_N \). By definition \( \Gamma_0 = C_0 \). Due to (2.5) and (2.6) we have \( \mathcal{D}_{T_0} \neq \{0\} \) and \( \mathcal{D}_{T_0} \neq \{0\} \). If \( \Theta_1 \in \mathcal{S}(\mathcal{D}_{T_0}, \mathcal{D}_{T_0}) \) is the first function associated with \( \Theta \) and if
\[F_1(z) := z(I + z\Theta_1(z)\Gamma_0^{-1})^{-1}\Theta_1(z), \quad z \in \mathbb{D},
\]
then the Taylor expansion
\[F_1(z) = \sum_{k=1}^{\infty} z^k B^{(1)}_k, \quad B_k \in \mathcal{L}(\mathcal{D}_{T_0}, \mathcal{D}_{T_0}), \quad k \geq 1,
\]
and the relation \( \Theta(z) - \Gamma_0 = D_{T_0} F_1(z)D_{T_0} \) lead to the equalities
\[C_k = D_{T_0} B^{(1)}_k D_{T_0}, \quad k \geq 1.
\]
Let
\[ \Theta_1(z) = \sum_{k=0}^{\infty} z^k C_k^{(1)}, \quad C_k^{(1)} \in \mathbb{L}([\Gamma_0, \Gamma_0^*]), \quad k \geq 0 \]
be the Taylor expansion of \( \Theta_1 \). Then from (2.8) we get the equalities
\[ \begin{align*}
C_0^{(1)} &= B_1^{(1)}, \\
C_1^{(1)} &= B_2^{(1)} + C_0^{(1)} \Gamma_0^* B_1^{(1)}, \\
& \quad \vdots \\
C_{N-1}^{(1)} &= B_N^{(1)} + C_0^{(1)} \Gamma_0^* B_{N-1}^{(1)} + C_1^{(1)} \Gamma_0^* B_{N-2}^{(1)} + \cdots + C_{N-2}^{(1)} \Gamma_0^* B_1^{(1)}.
\end{align*} \]
(2.9)
The system (2.9) can be rewritten as follows
\[ \begin{align*}
B_1^{(1)} &= C_0^{(1)}, \\
B_2^{(1)} &= C_1^{(1)} - C_0^{(1)} \Gamma_0^* B_1^{(1)}, \\
& \quad \vdots \\
B_N^{(1)} &= C_{N-1}^{(1)} - C_0^{(1)} \Gamma_0^* B_{N-1}^{(1)} - C_1^{(1)} \Gamma_0^* B_{N-2}^{(1)} - \cdots - C_{N-2}^{(1)} \Gamma_0^* B_1^{(1)}.
\end{align*} \]
(2.10)
It follows that the data \( C_0, C_1, \ldots, C_N \) produce \( \Gamma_0, B_1^{(1)}, \ldots, B_N^{(1)} \) and then
\[ \Gamma_1 = C_0^{(1)}, C_1^{(1)}, \ldots, C_{N-1}^{(1)} \].

Arguing similarly for \( \Theta_2, \ldots, \Theta_N \) we get \( \Gamma_2 = C_0^{(2)} = \Theta_2(0), \ldots, \Gamma_N = C_0^{(N)} = \Theta_N(0) \) and
\[ \mathcal{D}_{\Gamma_j} \neq \{0\}, \quad \mathcal{D}_{\Gamma_j^*} \neq \{0\}, \quad j = 1, \ldots, N \].

Conversely, given the choice sequence \( \Gamma_0, \ldots, \Gamma_N \) satisfying (2.7). Then
\[ C_0^{(N)} = \Gamma_N, \quad B_1^{(N)} = C_0^{(N)}, \quad C_1^{(N-1)} = D_{\Gamma_{N-1}} B_1^{(N)} D_{\Gamma_{N-1}}, \quad C_0^{(N-1)} = \Gamma_{N-1} . \]
Using equation of the type (2.10) we find \( B_1^{(N-1)} \) and \( B_2^{(N-1)} \) and then find
\[ C_1^{(N-2)} = D_{\Gamma_{N-2}} B_1^{(N-1)} D_{\Gamma_{N-2}}, \quad C_2^{(N-2)} = D_{\Gamma_{N-2}} B_2^{(N-1)} D_{\Gamma_{N-2}} . \]
Finally we find the Schur sequence \( C_0^{(0)}, \ldots, C_N^{(0)} \). Clearly it satisfies (2.6).

Notice that in [5, Theorem 2.2] it is proved that if \( \Theta, \hat{\Theta} \in \mathcal{S}(\mathfrak{M}, \mathfrak{N}), \{\Gamma_k\}, \{\hat{\Gamma}_k\} \) are the Schur parameters of \( \Theta \) and \( \hat{\Theta} \), correspondingly, and
\[ \Gamma_0 = \hat{\Gamma}_0, \ldots, \Gamma_N = \hat{\Gamma}_N, \]
then
\[ ||\Theta(z) - \hat{\Theta}(z)|| = o(|z|^{N+1}), \quad z \to 0. \]
Suppose now that the Schur sequence \( \{C_k\}_k^{N} \subset \mathbb{L}(\mathfrak{M}, \mathfrak{N}) \) is such that
\[ (D_{\Gamma_N}^2)_{\mathfrak{M}} = 0. \]
Then by Theorem 2.3 the Schur problem has a unique solution \( \Theta \). From (2.4) it follows that there is a number \( p, p \leq N \) such that \( (D_{\Gamma_{p-1}}^2)_{\mathfrak{M}} \neq 0 \) but \( (D_{\Gamma_{p}}^2)_{\mathfrak{M}} = 0 \), i.e., the Schur parameters of \( \Theta \) are
\[ \Gamma_0, \ldots, \Gamma_{p-1}, \Gamma_p, \mathcal{D}_{\Gamma_{p}} = 0, \quad \mathcal{D}_{\Gamma_{p}^*} \neq \{0\}, \quad \Gamma_{p+l} = 0 \in \mathbb{L}(0, \mathcal{D}_{\Gamma_{p}^*}), \quad l \geq 1. \]
The operators \( \Gamma_0, \ldots, \Gamma_p \) can be found by the procedure described above.

In the case \( (D_{\Gamma_N}^2)_{\mathfrak{M}} = 0 \) we proceed similarly.
3. Block operator CMV matrices

Let 
\[ \Gamma_0 \in L(\mathcal{M}, \mathcal{M}), \Gamma_k \in L(\mathcal{D}_{\Gamma_{k-1}}, \mathcal{D}_{\Gamma_k^*}), \ k \in \mathbb{N} \]

be a choice sequence. There are two unitary and unitarily equivalent block operator CMV matrices corresponding to the choice sequence \( \{\Gamma_n\} \) \[ \text{[6]} \]. We briefly describe their constructions and their forms for the case \( \mathcal{D}_{\Gamma_k} \neq \{0\}, \mathcal{D}_{\Gamma_k^*} \neq \{0\} \) for all \( k \), i.e., \( \mathcal{D}_{\Gamma_k} \neq \{0\}, \mathcal{D}_{\Gamma_k^*} \neq \{0\} \) for each \( k \). The rest cases we consider in detail in Appendix A.

3.1. CMV matrices. Recall that if \( \{\mathcal{H}_k\}_{k=1}^{\infty} \) be a given sequence of Hilbert spaces, then

\[ H_N = \bigoplus_{k=1}^{N} \mathcal{H}_k \]

is the Hilbert space with the inner product \( \langle f, g \rangle = \sum_{k=1}^{N} \langle f_k, g_k \rangle_{\mathcal{H}_k} \) for \( f = (f_1, \ldots, f_N)^T \) and \( g = (g_1, \ldots, g_N)^T \), \( f_k, g_k \in \mathcal{H}_k, \ k = 1, \ldots, N \) and the norm \( ||f||^2 = \sum_{k=1}^{N} ||f_k||_{\mathcal{H}_k}^2 \).

The Hilbert space

\[ H_\infty = \bigoplus_{k=1}^{\infty} \mathcal{H}_k \]

consists of all vectors of the form \( f = (f_1, f_2, \ldots)^T, f_k \in \mathcal{H}_k, \ k = 1, 2, \ldots, \) such that

\[ ||f||^2 = \sum_{k=1}^{\infty} ||f_k||_{\mathcal{H}_k}^2 < \infty. \]

The inner product is given by \( \langle f, g \rangle = \sum_{k=1}^{\infty} \langle f_k, g_k \rangle_{\mathcal{H}_k} \).

Define the Hilbert spaces

\[ \mathcal{H}_0 = \mathcal{H}_0(\{\Gamma_k\}_{k=0}^{\infty}) := \bigoplus_{k \geq 0} \mathcal{D}_{\Gamma_{2k}} \quad \text{and} \quad \mathcal{H}_0 = \mathcal{H}_0(\{\Gamma_k\}_{k \geq 0}) := \bigoplus_{k \geq 0} \mathcal{D}_{\Gamma_{2k}^*}. \]

From these definitions it follows, that

\[ \tilde{\mathcal{H}}_0(\{\Gamma_k^*\}_{k \geq 0}) = \mathcal{H}_0(\{\Gamma_k\}_{k \geq 0}), \ \mathcal{H}_0(\{\Gamma_k^*\}_{k \geq 0}) = \tilde{\mathcal{H}}_0(\{\Gamma_k\}_{k \geq 0}). \]

The spaces \( \mathcal{M} \bigoplus \mathcal{H}_0 \) and \( \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0 \) we represent in the form

\[ \mathcal{M} \bigoplus \mathcal{H}_0 = \mathcal{M} \bigoplus \mathcal{D}_{\Gamma_0^*} \bigoplus \mathcal{D}_{\Gamma_{2k-1}} \quad \text{and} \quad \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0 = \mathcal{M} \bigoplus \mathcal{D}_{\Gamma_0} \bigoplus \mathcal{D}_{\Gamma_{2k}}. \]

Let

\[ J_{\Gamma_0} = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*} \\ D_{\Gamma_0} & -\Gamma_0 \end{bmatrix} : \mathcal{M} \bigoplus \mathcal{H}_0 \rightarrow \mathcal{M} \bigoplus \mathcal{H}_0, \ J_{\Gamma_k} = \begin{bmatrix} \Gamma_k & D_{\Gamma_k^*} \\ D_{\Gamma_k} & -\Gamma_k \end{bmatrix} : \mathcal{M} \bigoplus \mathcal{H}_0 \rightarrow \mathcal{M} \bigoplus \mathcal{H}_0, \ k \in \mathbb{N}. \]
be the elementary rotations. Define the following unitary operators
\[ \mathcal{M}_0 = \mathcal{M}_0(\{\Gamma_k\}_{k \geq 0}) := I_{2\mathbb{R}} \bigoplus_{k \geq 1} J_{\Gamma_{2k-1}} : \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0 \to \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0, \]
(3.1) \[ \tilde{\mathcal{M}}_0 = \tilde{\mathcal{M}}_0(\{\Gamma_k\}_{k \geq 0}) := I_{\mathbb{R}} \bigoplus_{k \geq 1} J_{\Gamma_{2k-1}} : \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0 \to \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0, \]
\[ \mathcal{L}_0 = \mathcal{L}_0(\{\Gamma_k\}_{k \geq 0}) := J_{\Gamma_0} \bigoplus_{k \geq 1} J_{\Gamma_{2k-1}} : \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0 \to \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0. \]

Observe that \( (\mathcal{L}_0(\{\Gamma_k\}_{k \geq 0}))^* = \mathcal{L}_0(\{\Gamma_k^*\}_{k \geq 0}) \). Let
\[ \mathcal{V}_0 = \mathcal{V}_0(\{\Gamma_k\}_{k \geq 0}) := \bigoplus_{k \geq 1} J_{\Gamma_{2k-1}} : \tilde{\mathcal{H}}_0 \to \tilde{\mathcal{H}}_0. \]

Clearly, the operator \( \mathcal{V}_0 \) is unitary and
\[ \mathcal{M}_0 = I_{2\mathbb{R}} \bigoplus \mathcal{V}_0, \quad \tilde{\mathcal{M}}_0 = I_{\mathbb{R}} \bigoplus \mathcal{V}_0. \]

It follows that \( (\tilde{\mathcal{M}}_0(\{\Gamma_k\}_{k \geq 0}))^* = \mathcal{M}_0(\{\Gamma_k^*\}_{k \geq 0}), (\mathcal{M}_0(\{\Gamma_k\}_{k \geq 0}))^* = \tilde{\mathcal{M}}_0(\{\Gamma_k^*\}_{k \geq 0}). \)

Finally, define the unitary operators
\[ \mathcal{U}_0 = \mathcal{U}_0(\{\Gamma_k\}_{k \geq 0}) := \mathcal{L}_0 \mathcal{M}_0 : \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0 \to \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0, \]
(3.4) \[ \tilde{\mathcal{U}}_0 = \tilde{\mathcal{U}}_0(\{\Gamma_k\}_{k \geq 0}) := \tilde{\mathcal{M}}_0 \mathcal{L}_0 : \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0 \to \mathcal{M} \bigoplus \tilde{\mathcal{H}}_0. \]

By calculations we get
\[ \mathcal{U}_0 = \begin{bmatrix} \Gamma_0 & D_{\Gamma_3} \Gamma_4 & D_{\Gamma_5} \Gamma_7 & 0 & 0 & 0 & 0 & 0 & \ldots \\ D_{\Gamma_0} & -\Gamma_0 \Gamma_3 & -\Gamma_0 \Gamma_5 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & \Gamma_2 \Gamma_4 & -\Gamma_2 \Gamma_5 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & \Gamma_2 \Gamma_4 & -\Gamma_2 \Gamma_5 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & \Gamma_2 \Gamma_4 & -\Gamma_2 \Gamma_5 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \Gamma_4 \Gamma_5 & -\Gamma_4 \Gamma_6 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \Gamma_4 \Gamma_5 & -\Gamma_4 \Gamma_6 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \Gamma_4 \Gamma_5 & -\Gamma_4 \Gamma_6 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \Gamma_4 \Gamma_5 & -\Gamma_4 \Gamma_6 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \]
(3.5)
\[ \tilde{\mathcal{U}}_0 = \begin{bmatrix} \Gamma_0 & D_{\Gamma_5} \Gamma_3 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\Gamma_1 \Gamma_0 & -\Gamma_1 \Gamma_3 & \Gamma_1 \Gamma_5 & 0 & 0 & 0 & 0 & \ldots \\
D_{\Gamma_1} \Gamma_0 & -\Gamma_1 \Gamma_3 & -\Gamma_1 \Gamma_5 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & \Gamma_3 \Gamma_5 & -\Gamma_3 \Gamma_6 & 0 & 0 & 0 & \ldots \\
0 & 0 & \Gamma_3 \Gamma_5 & -\Gamma_3 \Gamma_6 & 0 & 0 & 0 & \ldots \\
0 & 0 & \Gamma_3 \Gamma_5 & -\Gamma_3 \Gamma_6 & 0 & 0 & 0 & \ldots \\
0 & 0 & \Gamma_3 \Gamma_5 & -\Gamma_3 \Gamma_6 & 0 & 0 & 0 & \ldots \\
0 & 0 & \Gamma_3 \Gamma_5 & -\Gamma_3 \Gamma_6 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \]
(3.6)

The block operator matrices \( \mathcal{U}_0 \) and \( \tilde{\mathcal{U}}_0 \) are called \( \mathbb{R} \) block operator CMV matrices. Observe that
\[ \tilde{\mathcal{M}}_0 \mathcal{U}_0 = \tilde{\mathcal{U}}_0 \mathcal{M}_0, \]
(3.7) \[ (\mathcal{U}_0(\{\Gamma_k\}_{k \geq 0}))^* = \tilde{\mathcal{U}}_0(\{\Gamma_k^*\}_{k \geq 0}), \quad (\tilde{\mathcal{U}}_0(\{\Gamma_k\}_{k \geq 0}))^* = \mathcal{U}_0(\{\Gamma_k^*\}_{k \geq 0}). \]

Therefore the matrix \( \tilde{\mathcal{U}}_0 \) can be obtained from \( \mathcal{U}_0 \) by passing to the adjoint \( \mathcal{U}_0^* \) and then by replacing \( \Gamma_k \) (respect., \( \Gamma_k^* \)) by \( \Gamma_k^* \) (respect., \( \Gamma_k \)) for all \( n \). In the case when the choice sequence consists of complex numbers from the unit disk the matrix \( \tilde{\mathcal{U}}_0 \) is the transpose to \( \mathcal{U}_0 \), i.e., \( \tilde{\mathcal{U}}_0 = \mathcal{U}_0^* \).
Remark 3.1. The three-diagonal block form of the CMV matrices with scalar entries has been established in [15].

3.2. Truncated block operator CMV matrices. Define two contractions

\[ T_0 = T_0(\{ \Gamma_k \}_{k \geq 0}) := P_{\mathcal{B}_0} \mathcal{U}_0 | \mathcal{F}_0 : \mathcal{F}_0 \to \mathcal{F}_0, \]

\[ \widetilde{T}_0 = \widetilde{T}_0(\{ \Gamma_k \}_{k \geq 0}) := P_{\mathcal{B}_0} \mathcal{U}_\widetilde{0} | \mathcal{F}_0 : \mathcal{F}_0 \to \mathcal{F}_0. \]

The operators $T_0$ and $\widetilde{T}_0$ take on the three-diagonal block operator matrix forms

\[ T_0 = \begin{bmatrix} B_1 & C_1 & 0 & 0 & 0 & \cdots \\ A_1 & B_2 & C_2 & 0 & 0 & \cdots \\ 0 & A_2 & B_3 & C_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \end{bmatrix}, \quad \widetilde{T}_0 = \begin{bmatrix} \widetilde{B}_1 & \widetilde{C}_1 & 0 & 0 & 0 & \cdots \\ \widetilde{A}_1 & \widetilde{B}_2 & \widetilde{C}_2 & 0 & 0 & \cdots \\ 0 & \widetilde{A}_2 & \widetilde{B}_3 & \widetilde{C}_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \end{bmatrix}, \]

where $A_n, B_n, C_n$, and $\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n$ are given by (3.8) and (3.9). Since the matrices $T_0$ and $\widetilde{T}_0$ are obtained from $\mathcal{U}_0$ and $\mathcal{U}_\widetilde{0}$ by deleting the first rows and the first columns,
we will call them truncated block operator CMV matrices. Observe that from the definitions of $L_0$, $M_0$, $M_0$, $T_0$, and $\widetilde{T}_0$ it follows that $T_0$ and $\widetilde{T}_0$ are products of two block-diagonal matrices

$$\begin{align*}
(3.12) \quad & T_0 = T_0((\Gamma_n)_{n \geq 0}) = (-\Gamma_0^* \oplus J_{\Gamma_2} \oplus \ldots \oplus J_{\Gamma_{2n}} \oplus \ldots) \times
\quad & (J_{\Gamma_1} \oplus J_{\Gamma_3} \oplus \ldots \oplus J_{\Gamma_{2k-1}} \oplus \ldots) \\
(3.13) \quad & \widetilde{T}_0 = \widetilde{T}_0((\Gamma_k)_{k \geq 0}) = (J_{\Gamma_1} \oplus J_{\Gamma_3} \oplus \ldots \oplus J_{\Gamma_{2k-1}} \oplus \ldots) \times
\quad & (-\Gamma_0^* \oplus J_{\Gamma_2} \oplus \ldots \oplus J_{\Gamma_{2k}} \oplus \ldots).
\end{align*}$$

In particular, it follows that $(T_0((\Gamma_k)_{k \geq 0}))^* = \widetilde{T}_0((\Gamma_k^*)_{k \geq 0})$. From (3.12) and (3.13) we have $V_0 T_0 = \widetilde{T}_0 V_0$, where the unitary operator $V_0$ is defined by (3.2). Therefore, the operators $T_0$ and $\widetilde{T}_0$ are unitarily equivalent, in particular the following equalities

$$\begin{align*}
(3.14) \quad & \tilde{E}_k J_{\Gamma_{2k-1}} = J_{\Gamma_{2k-1}} E_k, \quad \tilde{A}_k J_{\Gamma_{2k-1}} = J_{\Gamma_{2k-1}} A_k,
\end{align*}$$

hold true.

**Proposition 3.2.** Let $\Theta \in S(\mathfrak{M}, \mathfrak{R})$ and let $\{\Gamma_k\}_{k \geq 0}$ be the Schur parameters of $\Theta$. Suppose $\Gamma_k$ is neither isometric nor co-isometric for each $k$. Let the function $\Omega \in S(\mathfrak{R}, \mathfrak{L})$ coincides with $\Theta$ in the sense of [15] and let $\{G_k\}_{k \geq 0}$ be the Schur parameters of $\Omega$. Then truncated block operator CMV matrices $T_0((\Gamma_k)_{k \geq 0})$ and $\widetilde{T}_0((\Gamma_k)_{k \geq 0})$ (respect., $\widetilde{T}_0((\Gamma_k^*)_{k \geq 0})$ and $\widetilde{T}_0((\Gamma_k^*)_{k \geq 0})$) are unitarily equivalent.

### 3.3. Simple conservative realizations of the Schur class function by means of its Schur parameters.

Set

$$\begin{align*}
\hat{G}_0 &= \hat{G}_0((\Gamma_k)_{k \geq 0}) := \begin{bmatrix} D_{\Gamma_0^*} \Gamma_1 & D_{\Gamma_0^*} \Gamma_3 & 0 & 0 & \ldots \\ D_{\Gamma_0^*} \Gamma_1 & D_{\Gamma_0^*} \Gamma_3 & 0 & \ldots \\ 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots
\end{bmatrix} \in \mathbb{L}(\mathfrak{S}_0, \mathfrak{M}),
\hat{G}_0 &= \hat{G}_0((\Gamma_k)_{k \geq 0}) := \begin{bmatrix} D_{\Gamma_0^*} \Gamma_1 & D_{\Gamma_0^*} \Gamma_3 & 0 & 0 & \ldots \\ D_{\Gamma_0^*} \Gamma_1 & D_{\Gamma_0^*} \Gamma_3 & 0 & \ldots \\ \Gamma_1 & 0 & \ldots \end{bmatrix} \in \mathbb{L}(\widetilde{\mathfrak{S}}_0, \mathfrak{M}),
\hat{F}_0 = \hat{F}_0((\Gamma_k)_{k \geq 0}) := \begin{bmatrix} D_{\Gamma_0} \\ 0 \\ \vdots \end{bmatrix} \in \mathbb{L}(\mathfrak{R}, \mathfrak{S}_0), \quad \hat{F}_0 = \hat{F}_0((\Gamma_k)_{k \geq 0}) := \begin{bmatrix} \Gamma_1 D_{\Gamma_0} \\ D_{\Gamma_0} \Gamma_1 & D_{\Gamma_0} \Gamma_3 & 0 \end{bmatrix} \in \mathbb{L}(\mathfrak{S}_0, \mathfrak{M}).
\end{align*}$$

Then the operators $U_0$ and $\tilde{U}_0$ defined by (3.14) and taking the form (3.5) and (3.6) can be represented by $2 \times 2$ block operator matrices

$$\begin{align*}
U_0 &= \begin{bmatrix} \Gamma_0 & \hat{G}_0 \\ \hat{F}_0 & \tilde{G}_0 \end{bmatrix} : \mathfrak{M} \oplus \to \mathfrak{R}, \quad \tilde{U}_0 &= \begin{bmatrix} \Gamma_0 & \hat{G}_0 \\ \hat{F}_0 & \tilde{G}_0 \end{bmatrix} : \mathfrak{M} \oplus \to \mathfrak{R}.
\end{align*}$$

Recall [10] that the discrete time-invariant system

$$\Sigma = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathfrak{M}, \mathfrak{R}, \mathfrak{H} \right\}$$

is called conservative if the operator

$$U = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \mathfrak{M} \to \mathfrak{R}$$
is unitary. The function
\[ \Theta_\Sigma(z) = D + zC(I_H - zA)^{-1}B \]
is called the transfer function of the system \( \Sigma \) [10]. Define the following conservative systems:
\[
\begin{align*}
\zeta_0 &= \left\{ \begin{array}{ccc}
\Gamma_0 & \mathcal{G}_0 & \mathcal{M}, \mathcal{N}, \mathcal{H}_0 \end{array} \right\} = \{ \mathcal{U}_0(\{ \Gamma_k \}_{k \geq 0}); \mathcal{M}, \mathcal{N}, \mathcal{H}_0(\{ \Gamma_k \}_{k \geq 0}) \}, \\
\tilde{\zeta}_0 &= \left\{ \begin{array}{ccc}
\Gamma_0 & \tilde{\mathcal{G}}_0 & \mathcal{M}, \mathcal{N}, \tilde{\mathcal{H}}_0 \end{array} \right\} = \{ \tilde{\mathcal{U}}_0(\{ \Gamma_k \}_{k \geq 0}); \mathcal{M}, \mathcal{N}, \tilde{\mathcal{H}}_0(\{ \Gamma_k \}_{k \geq 0}) \}.
\end{align*}
\]
Equalities [3.3] and [3.7] yield that systems \( \zeta_0 \) and \( \tilde{\zeta}_0 \) are unitarily equivalent. Hence, \( \zeta_0 \) and \( \tilde{\zeta}_0 \) have equal transfer functions [10]. Let \( \Theta(z) \) be the transfer function of \( \zeta_0 \). Then
\[ \Theta(z) = \Gamma_0 + z\mathcal{G}_0(I_{\mathcal{H}} - z\mathcal{N})^{-1}\mathcal{F}_0 = \Gamma_0 + z\tilde{\mathcal{G}}_0(I_{\tilde{\mathcal{H}}} - z\tilde{\mathcal{N}}_0)^{-1}\tilde{\mathcal{F}}_0. \]
Using expressions for CMV matrices \( \mathcal{U}_0 \) and \( \tilde{\mathcal{U}}_0 \) we obtain
\[
\begin{align*}
\Theta(z) &= \Gamma_0 + zD_{\mathcal{G}}^* \left[ \begin{array}{c}
\Gamma_1 \\
D_{\mathcal{F}} \end{array} \right] \left( P_{\mathcal{H}_1}(I_{\mathcal{H}} - z\mathcal{N})^{-1} \mathcal{D}_{\mathcal{G}} \right) \left[ \begin{array}{c}
\Gamma_1 \\
D_{\mathcal{F}} \end{array} \right] D_{\mathcal{G}}.
\end{align*}
\]
The next theorem has been established in [6], using conservative realizations of the Schur algorithm obtained in [5].

**Theorem 3.3.** [10] 1) The unitarily equivalent conservative systems \( \zeta_0 \) and \( \tilde{\zeta}_0 \) are simple and the Schur parameters of the transfer function \( \Theta \) of \( \zeta_0 \) and \( \tilde{\zeta}_0 \) are \( \{ \Gamma_n \}_{n \geq 0} \).

2) Let \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) and let \( \{ \Gamma_n \}_{n \geq 0} \) be the Schur parameters of \( \Theta \). Then the systems \( \{ 3.15 \} \) are simple conservative realizations of \( \Theta \).

4. Connections between \( \Theta \) and \( \Theta_k \)

In the sequel we need some sub-matrices of block operator CMV matrices, which will play essential role in the parametrization of all solutions to the Schur problem.

Suppose
\[ \Gamma_0 \in \mathcal{L}(\mathcal{M}, \mathcal{N}), \Gamma_1 \in \mathcal{L}(\mathcal{D}_{\mathcal{G}}, \mathcal{D}_{\mathcal{F}}), \ldots \]
is a choice sequence and
\[
\mathcal{D}_{\mathcal{G}_{2n+1}} \neq \{ 0 \}, \mathcal{D}_{\mathcal{F}_{2n+1}} \neq \{ 0 \}
\]
for some \( n \). Define the Hilbert spaces
\[
\begin{align*}
\mathcal{H}_k &= \bigoplus_{i=1}^{2k} \mathcal{H}_{2k-2}, \quad \tilde{\mathcal{H}}_k &= \bigoplus_{i=1}^{2k} \mathcal{H}_{2k-2}, \quad k = 1, \ldots, n + 1, \\
\mathcal{K}_n &= \bigoplus_{k=1}^{n+1} \mathcal{H}_k, \quad \tilde{\mathcal{K}}_n &= \bigoplus_{k=1}^{n+1} \tilde{\mathcal{H}}_k.
\end{align*}
\]
4.1. Sub-matrices $S_n$ and $\tilde{S}_n$ of block operator CMV matrices.

**Proposition 4.1.** Let (4.1) holds true. Then the operators $S_n$ and $\tilde{S}_n$ given by three-diagonal block operators matrices

\[ S_n = S_n (\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+2}) := \begin{bmatrix} B_1 & C_1 & 0 & \cdots & 0 \\ A_1 & B_2 & C_2 & \cdots & 0 \\ & \vdots & \vdots & \ddots & \vdots \\ & \vdots & \vdots & \ddots & B_{n+1} \end{bmatrix}, \]

and acting in the Hilbert spaces $\mathcal{K}_n$ and $\tilde{\mathcal{K}}_n$, respectively, are unitarily equivalent contractions and

\[ \left( \tilde{S}_n (\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+2}) \right)^* = S_n (\Gamma_0^*, \Gamma_1^*, \ldots, \Gamma_{2n+2}^*). \]

**Proof.** Since $S_n = P_{\mathcal{K}_n} T_0 |_{\mathcal{K}_n}$, $\tilde{S}_n = P_{\tilde{\mathcal{K}}_n} \tilde{T}_0 |_{\tilde{\mathcal{K}}_n}$, the operators $S_n$ and $\tilde{S}_n$ are contractions and can be represented as products

\[ S_n = \mathcal{W}_n \mathcal{V}_n, \quad \tilde{S}_n = \mathcal{V}_n \mathcal{W}_n, \]

where

\[ \mathcal{V}_n = \bigoplus_{k=1}^{n+1} J_{\Gamma_{2k-1}} : \mathcal{K}_n \to \tilde{\mathcal{K}}_n, \]

\[ \mathcal{W}_n = -\Gamma_0^* \bigoplus \bigoplus_{k=1}^{n} J_{\Gamma_{2k}} \oplus \Gamma_{2n+2} : \tilde{\mathcal{K}}_n \to \mathcal{K}_n. \]

In (4.8) it is convenient to represent $\tilde{\mathcal{K}}_n$ as

\[ \tilde{\mathcal{K}}_n = \mathcal{D}_{\Gamma_0^*} \oplus \bigoplus_{k=1}^{n} \mathcal{D}_{\Gamma_{2k-1}} \oplus \mathcal{D}_{\Gamma_{2k}} \oplus \mathcal{D}_{\Gamma_{2n+1}}. \]

Then from (3.13) follows the equality $\mathcal{V}_n S_n = \tilde{S}_n \mathcal{V}_n$. Since $\mathcal{V}_n$ unitarily maps $\mathcal{K}_n$ onto $\tilde{\mathcal{K}}_n$, the operators $S_n$ and $\tilde{S}_n$ are unitarily equivalent. Relation (4.5) follows from (3.8) and (3.9). $\square$

4.2. The matrix $S_{n,0}$. It should be mentioned that

\[ B_{n+1} = \begin{bmatrix} -\Gamma_{2n}^* & -\Gamma_{2n}^* \Gamma_{2n+1}^* & -\Gamma_{2n+2}^* \Gamma_{2n+1}^* \\ \Gamma_{2n+2} \Gamma_{2n+1}^* & -\Gamma_{2n+2} \Gamma_{2n+1}^* \Gamma_{2n+2} \Gamma_{2n+1}^* \\ -\Gamma_{2n+1} \Gamma_{2n} & D_{\Gamma_{2n+1}} \Gamma_{2n+2} \\ -D_{\Gamma_{2n+1}} \Gamma_{2n} & \Gamma_{2n+1} \Gamma_{2n+2} \end{bmatrix} \in \mathcal{L}(\mathcal{H}_{n+1}), \]

\[ \tilde{B}_{n+1} = \begin{bmatrix} -\Gamma_{2n}^* & -\Gamma_{2n}^* \Gamma_{2n+1}^* & -\Gamma_{2n+2}^* \Gamma_{2n+1}^* \\ \Gamma_{2n+2} \Gamma_{2n+1}^* & -\Gamma_{2n+2} \Gamma_{2n+1}^* \Gamma_{2n+2} \Gamma_{2n+1}^* \\ -\Gamma_{2n+1} \Gamma_{2n} & D_{\Gamma_{2n+1}} \Gamma_{2n+2} \\ -D_{\Gamma_{2n+1}} \Gamma_{2n} & \Gamma_{2n+1} \Gamma_{2n+2} \end{bmatrix} \in \mathcal{L}(\tilde{\mathcal{H}}_{n+1}); \]
the entries $B_{n+1}$ and $\tilde{B}_{n+1}$ admits the representations

\[
B_{n+1} = \begin{bmatrix}
-\Gamma_n\Gamma_{n+2} & -\Gamma_n^*D\Gamma_{n+2} \\
\Gamma_n & 0
\end{bmatrix} + \begin{bmatrix}
\Gamma_n & 0 \\
0 & 0
\end{bmatrix}
\]

\[
\tilde{B}_{n+1} = \begin{bmatrix}
-\Gamma_{n+1}\Gamma_n & D\Gamma_{n+1} & \Gamma_{n+2} \\
-D\Gamma_{n+1} & -\Gamma_n & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

**Proposition 4.2.** Let

\[
\Gamma_0 \in L(\mathcal{M}), \Gamma_1 \in L(\mathcal{O}, \mathcal{O}) \ldots , \Gamma_{2n+1} \in L(\mathcal{O}, \mathcal{O})
\]

be a finite choice sequence consisting of neither isometric nor co-isometric operators. Then for each contraction $\Gamma \in L(\mathcal{O}, \mathcal{O})$ the operators

\[
S_{n, \Gamma} := S_n(\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+1}, \Gamma), \quad \tilde{S}_{n, \Gamma} := \tilde{S}_n(\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+1}, \Gamma),
\]

are unitarily equivalent contractions.

**Proof.** Take an arbitrary $S \in S(\mathcal{O}, \mathcal{O})$ with $S(0) = \Gamma$. If

\[
\gamma_0 = \Gamma, \gamma_1, \gamma_2, \ldots
\]

are the Schur parameters of $S$, then

\[
\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+1}, \Gamma,
\]

is the choice sequence. Applying Proposition 4.1 we arrive at the statement of the proposition. \hfill $\Box$

Let the finite choice sequence

\[
\Gamma_0 \in L(\mathcal{M}), \Gamma_1 \in L(\mathcal{O}, \mathcal{O}) \ldots , \Gamma_{2n+1} \in L(\mathcal{O}, \mathcal{O})
\]

be given and let $\Gamma \in L(\mathcal{O}, \mathcal{O})$ be a contraction. Set

\[
B_{n+1, 0} := \begin{bmatrix}
-\Gamma_n\Gamma_{n+1} & -\Gamma_n^*D\Gamma_{n+1} \\
0 & 0
\end{bmatrix} \in L(\mathcal{H}_{n+1}),
\]

\[
\tilde{B}_{n+1, 0} := \begin{bmatrix}
-\Gamma_{n+1}\Gamma_n & D\Gamma_{n+1} & 0 \\
-D\Gamma_{n+1} & \Gamma_n & 0
\end{bmatrix} \in L(\tilde{\mathcal{H}}_{n+1})
\]

(4.9) \quad \mathcal{S}_{n, 0} := S_n(\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+1}, 0), \quad \tilde{\mathcal{S}}_{n, 0} := \tilde{S}_n(\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+1}, 0),
where \(0 \in \mathbf{L}(\mathcal{D}_{\Gamma_{2n+1}}, \mathcal{D}_{\Gamma_{2n+1}^*})\). Then

\[
\mathcal{S}_{n,0} = \begin{bmatrix}
B_1 & C_1 & 0 & 0 & \cdots & 0 \\
A_1 & B_2 & C_2 & 0 & \cdots & 0 \\
& & \ddots & & & \vdots \\
& & & \ddots & & \vdots \\
& & & & \ddots & \ddots \\
& & & & & A_n & B_{n+1,0}
\end{bmatrix},
\tilde{\mathcal{S}}_{n,0} = \begin{bmatrix}
\tilde{B}_1 & \tilde{C}_1 & 0 & 0 & \cdots & 0 \\
\tilde{A}_1 & \tilde{B}_2 & \tilde{C}_2 & 0 & \cdots & 0 \\
& & \ddots & & & \vdots \\
& & & \ddots & & \vdots \\
& & & & \ddots & \ddots \\
& & & & & \tilde{A}_n & \tilde{B}_{n+1,0}
\end{bmatrix},
\]

(4.10)

\[
S_{n,\Gamma} = \mathcal{S}_{n,0} + j_{n+1}\Gamma [D_{\Gamma_{2n+1}} - \Gamma_{2n+1}^*] P_{\mathcal{H}_{n+1}},
\]

(4.11)

where \(j_{n+1}\) is the embedding operator from \(\mathcal{D}_{\Gamma_{2n+1}}\) into \(\mathcal{K}_n\), \(\tilde{j}_{n+1}\) is the embedding operator from \(\tilde{\mathcal{H}}_{n+1}\) into \(\tilde{\mathcal{K}}_n\). Observe that the block operator matrices \(\mathcal{S}_{n,0}\) and \(\tilde{\mathcal{S}}_{n,0}\) can be obtained from truncated CMV matrices \(\mathcal{F}_0\) and \(\tilde{\mathcal{F}}_0\) corresponding to the infinite choice sequence

\[
\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n}, \Gamma_{2n+1}, 0, 0, \ldots,
\]

where \(0 \in \mathbf{L}(\mathcal{D}_{\Gamma_{2n+1}}, \mathcal{D}_{\Gamma_{2n+1}^*})\). Let

(4.12)

\[
\mathcal{W}_{n,0} = -\Gamma_0^0 \bigoplus \bigoplus_{k=1}^{n} \mathbf{J}_{2k} \bigoplus 0 : \mathcal{K}_n \rightarrow \mathcal{K}_n.
\]

Due to (4.8), (4.12), and (4.6), the operators \(\mathcal{S}_{n,0}\) and \(\tilde{\mathcal{S}}_{n,0}\) admit factorizations

(4.13)

\[
\mathcal{S}_{n,0} = \mathcal{W}_{n,0} \mathcal{V}_{n} = \left( -\Gamma_0^0 \bigoplus \bigoplus_{k=1}^{n} \mathbf{J}_{2k} \bigoplus 0 \right) \times \left( \bigoplus_{k=1}^{n+1} \mathbf{J}_{2k-1} \right),
\]

\[
\tilde{\mathcal{S}}_{n,0} = \mathcal{V}_{n} \mathcal{W}_{n,0} = \left( \bigoplus_{k=1}^{n+1} \mathbf{J}_{2k-1} \right) \times \left( -\Gamma_0^0 \bigoplus \bigoplus_{k=1}^{n} \mathbf{J}_{2k} \bigoplus 0 \right).
\]

Notice that

\[
P_{\mathcal{D}_{\Gamma_{2n+1}}} \mathcal{S}_{n,0} = 0, \quad \tilde{\mathcal{S}}_{n,0} | \mathcal{D}_{\Gamma_{2n+1}} = 0.
\]

Our next goal is to express for \(z \in \mathbb{D}\):

1. the resolvent \((I_{\mathcal{K}_n} - z\mathcal{S}_{n,\Gamma})^{-1}\) through the resolvent \((I_{\mathcal{K}_n} - z\mathcal{S}_{n,0})^{-1}\),
2. the resolvent \((I_{\tilde{\mathcal{K}}_n} - z\tilde{\mathcal{S}}_{n,\Gamma})^{-1}\) through \((I_{\tilde{\mathcal{K}}_n} - z\tilde{\mathcal{S}}_{n,0})^{-1}\).

**Proposition 4.3.** If \(|z| < 1\), then

\[
(I_{\mathcal{K}_n} - z\mathcal{S}_{n,\Gamma})^{-1} = (I_{\mathcal{K}_n} - z\mathcal{S}_{n,0})^{-1} + z \left( (I_{\mathcal{K}_n} - z\mathcal{S}_{n,0})^{-1} \big| \mathcal{D}_{\Gamma_{2n+1}} \right) \times
\]

\[
\left( I_{\mathcal{D}_{\Gamma_{2n+1}}} - z \left[ D_{\Gamma_{2n+1}} - \Gamma_{2n+1}^* \right] P_{\mathcal{H}_{n+1}} (I_{\mathcal{K}_n} - z\mathcal{S}_{n,0})^{-1} \big| \mathcal{D}_{\Gamma_{2n+1}}^* \right)^{-1} \times
\]

\[
\Gamma \left[ D_{\Gamma_{2n+1}} - \Gamma_{2n+1}^* \right] P_{\mathcal{H}_{n+1}} (I_{\mathcal{K}_n} - z\mathcal{S}_{n,0})^{-1},
\]

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\[
\left(I_{\tilde{K}_n} - z\tilde{S}_{n,\Gamma}\right)^{-1} = \left(I_{\tilde{K}_n} - z\tilde{S}_{n,0}\right)^{-1} \\
+ z\left(I_{\tilde{K}_n} - z\tilde{S}_{n,0}\right)^{-1} \left(\begin{array}{c|c}
D_{\Gamma_{2n+1}} & I_{\tilde{K}_n} \\
-\Gamma_{2n+1}^* & I_{\tilde{K}_n} 
\end{array}\right) \Gamma \times \\
\left(\begin{array}{c|c}
D_{\Gamma_{2n+1}} & I_{\tilde{K}_n} \\
-\Gamma_{2n+1}^* & I_{\tilde{K}_n} 
\end{array}\right)^{-1}
\]

\[
\left(I_{\mathbb{D}_{\Gamma_{2n+1}}} - z\left(P_{\mathbb{D}_{\Gamma_{2n+1}}} I_{\tilde{K}_n} - z\tilde{S}_{n,0}\right)^{-1} \left(\begin{array}{c|c}
D_{\Gamma_{2n+1}} & I_{\tilde{K}_n} \\
-\Gamma_{2n+1}^* & I_{\tilde{K}_n} 
\end{array}\right) \Gamma \right)^{-1}
\]

\[P_{\mathbb{D}_{\Gamma_{2n+1}}} \left(I_{\tilde{K}_n} - z\tilde{S}_{n,0}\right)^{-1}.
\]

Proof. Let \(A\) and \(B\) be bounded operators in the Hilbert space \(H\). Suppose that \(-1 \in \rho(A) \cap \rho(B)\). Denote \(S := A - B\), i.e.,

\[A = B + S.
\]

Then

\[I + A = (I + B)(I + (I + B)^{-1} S),
\]

where \(I = I_H\). It follows that \((I + (I + B)^{-1} S)^{-1} \in \mathcal{L}(H)\) and

\[(I + A)^{-1} = (I + (I + B)^{-1} S)^{-1} (I + B)^{-1}.
\]

On the other hand

\[(I + A)^{-1} - (I + B)^{-1} = -(I + B)^{-1} S (I + A)^{-1}.
\]

Hence

\[(I + A)^{-1} = (I + B)^{-1} - (I + B)^{-1} S (I + (I + B)^{-1} S)^{-1} (I + B)^{-1}.
\]

Similarly

\[(I + A)^{-1} = (I + B)^{-1} - (I + B)^{-1} (I + S(I + B)^{-1})^{-1} S(I + B)^{-1}.
\]

If \(\mathfrak{R}\) is a proper subspace in \(H\) and \(\text{ran } S \subseteq \mathfrak{R}\), then

\[(I + A)^{-1} = (I + B)^{-1} - (I + B)^{-1} (I_{\mathfrak{R}} + S(I + B)^{-1} \mathfrak{R})^{-1} S(I + B)^{-1}.
\]

If \(\ker S \subseteq H \ominus \mathfrak{R}\), then \(S = SP_{\mathfrak{R}}\) and

\[(I + A)^{-1} = (I + B)^{-1} - (I + B)^{-1} S (I_{\mathfrak{R}} + P_{\mathfrak{R}}(I + B)^{-1} S)^{-1} P_{\mathfrak{R}}(I + B)^{-1}.
\]

Applying the latter equalities to \(A = -z\tilde{S}_{n,\Gamma}\) and \(A = -z\tilde{S}_{n,0}\) and \(B = -z\tilde{S}_{n,0}\) and using (4.10) and (4.11) we get formulas in the proposition. \(\square\)

As it is well-known, the resolvent \(R_T(\lambda) = (T - \lambda I)^{-1}\) of a block operator matrix

\[T = \begin{pmatrix} B & D \\ C & A \end{pmatrix} : \begin{array}{c} \mathfrak{R} \\ \mathcal{H} \end{array} \rightarrow \begin{array}{c} \mathfrak{R} \\ \mathcal{H} \end{array} \]

takes the form (the Schur-Frobenius formula)

\[(4.14) \quad R_T(\lambda) = \begin{pmatrix} -V^{-1}(\lambda) & V^{-1}(\lambda)DR_A(\lambda) \\ R_A(\lambda)CV^{-1}(\lambda) & R_A(\lambda) \left(I_{\mathcal{H}} - CV^{-1}(\lambda)DR_A(\lambda)\right) \end{pmatrix}
\]

for \(\lambda \in \rho(T) \cap \rho(A)\),

where

\[V(\lambda) := \lambda I_{\mathfrak{R}} - B + DR_A(\lambda)C, \quad \lambda \in \rho(A),
\]

and \(I = I_{\mathfrak{R} \oplus \mathcal{H}}\). It follows that

\[P_{\mathfrak{R}} R_T(\lambda) |_{\mathfrak{R}} = -(\lambda I_{\mathfrak{R}} - B + DR_A(\lambda)C)^{-1}.
\]
In order to prove this theorem it is necessary to consider the following cases 1) $\Gamma_2$ and $\Gamma_3$ are truncated CMV matrices corresponding to the CMV matrices $\Gamma_2$ and $\Gamma_3$. Then for each $z \in \mathbb{D}$ one has

$$\begin{align*}
(4.15) & \quad P_{\mathcal{K}_n}(I_n - z\mathcal{T}_0)^{-1} | \mathcal{K}_n = (I_{\mathcal{K}_n} - z\mathcal{S}_n,\theta_{2n+2}(z))^{-1} | \mathcal{K}_n, \\
(4.16) & \quad P_{\mathcal{K}_n}(I_\mathcal{K} - z\mathcal{T}_0)^{-1} | \mathcal{K}_n = (I_{\mathcal{K}_n} - z\mathcal{S}_n,\theta_{2n+2}(z))^{-1} | \mathcal{K}_n,
\end{align*}$$

where $\theta_{2n+2}$ is the function associated with $\Theta$ in accordance with the Schur algorithm.

**Proof.** The CMV matrices $\mathcal{T}_0$ and $\mathcal{T}_0'$ can be represented as follows

$$\mathcal{T}_0 = \begin{bmatrix} \mathcal{S}_n & \mathcal{Q}_n \\ \mathcal{D}_n & \mathcal{T}' \end{bmatrix} : \oplus \to \oplus, \quad \bar{\mathcal{T}}_0 = \begin{bmatrix} \bar{\mathcal{S}}_n & \bar{\mathcal{Q}}_n \\ \bar{\mathcal{D}}_n & \bar{\mathcal{T}}' \end{bmatrix} : \oplus \to \oplus,$$

where

$$\begin{align*}
\mathcal{S}' & = \mathcal{S}'(\{\Gamma_k\}_{k \geq 2n+2}) = \mathcal{S}_0 \oplus \mathcal{K}_n, \\
\bar{\mathcal{S}}' & = \bar{\mathcal{S}}'(\{\Gamma_k\}_{k \geq 2n+2}) = \bar{\mathcal{S}}_0 \oplus \bar{\mathcal{K}}_n,
\end{align*}$$

and

$$\mathcal{T}' = \mathcal{T}_0(\{\Gamma_k\}_{k \geq 2n+2}), \quad \bar{\mathcal{T}}' = \bar{\mathcal{T}}_0(\{\Gamma_k\}_{k \geq 2n+2}).$$

are truncated CMV matrices corresponding to the CMV matrices

$$\mathcal{U}_{2n+2} = \mathcal{U}_{2n+2}(\{\Gamma_k\}_{k \geq 2n+2}) \quad \text{and} \quad \bar{\mathcal{U}}_{2n+2} = \bar{\mathcal{U}}_{2n+2}(\{\Gamma_k\}_{k \geq 2n+2}).$$

In order to prove this theorem it is necessary to consider the following cases 1) $\Gamma_{2n+2}$ and $\Gamma_{2n+3}$ are neither isometric nor co-isometric, 2) $\Gamma_{2n+3}$ is isometric, 3) $\Gamma_{2n+3}$ is co-isometric, 4) $\Gamma_{2n+3}$ is unitary, 5) $\Gamma_{2n+2}$ is isometric, 6) $\Gamma_{2n+2}$ is co-isometric, 7) $\Gamma_{2n+2}$ is unitary. We consider only the cases 1), 2), and 6) leaving the rest for a reader.

**The operators $\Gamma_{2n+2}$ and $\Gamma_{2n+3}$ are neither isometric nor co-isometric.**

In this case the entries in (4.17) take the form

$$\begin{align*}
\mathcal{D}_n & = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_{n+1} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \end{bmatrix} : \mathcal{K}_n \to \mathcal{S}', \quad \mathcal{Q}_n = \begin{bmatrix} 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots \end{bmatrix} : \mathcal{S}' \to \mathcal{K}_n, \\
\bar{\mathcal{D}}_n & = \begin{bmatrix} 0 & 0 & \cdots & 0 & \bar{A}_{n+1} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \end{bmatrix} : \bar{\mathcal{K}}_n \to \bar{\mathcal{S}}', \quad \bar{\mathcal{Q}}_n = \begin{bmatrix} 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots \end{bmatrix} : \bar{\mathcal{S}}' \to \bar{\mathcal{K}}_n
\end{align*}$$
Observe that from (3.10) follows the equality

$$\Theta_{2n+2}(z) = \left( \Gamma_{2n+2} + zD_{2n+2} \begin{bmatrix} \Gamma_{2n+3} & D_{2n+3} \end{bmatrix} \right) \times \left( \begin{bmatrix} P_{H_{n+2}}(I_B - zT')^{-1} \mid \mathcal{D}_{2n+2} \end{bmatrix} D_{2n+2} \right) \mid \mathcal{D}_{2n+1}.$$  

From the Schur-Frobenius formula (4.14) we have

$$P_{\mathcal{K}_n}(I_B - zT_0)^{-1} \mid \mathcal{K}_n = \left( I_{\mathcal{K}_n} - z \left( S_n + zQ_n(I_{B'} - zT')^{-1} \mathcal{D}_n \right) \right)^{-1}.$$  

Since

$$C_{n+1} = \begin{bmatrix} 0 & 0 & D_{2n+2} \Gamma_{2n+3} & D_{2n+2} D_{2n+3} \end{bmatrix} \begin{bmatrix} \mathcal{D}_{2n+2} \end{bmatrix} = H_{n+2} \rightarrow \mathcal{D}_{2n+2} = H_{n+1},$$  

we get

$$Q_n(I_{B'} - zT')^{-1} \mathcal{D}_n = D_{2n+2} \Gamma_{2n+3} \begin{bmatrix} \Gamma_{2n+3} & D_{2n+3} \end{bmatrix} \left( P_{H_{n+2}}(I_B - zT')^{-1} \mid \mathcal{D}_{2n+2} \right) \left( D_{2n+2} \left[ D_{2n+1} - \Gamma_{2n+1} \right] P_{H_{n+1}} \right).$$  

Now using the Schur-Frobenius formula, we arrive at (4.15). Similarly (4.16) can be proved.

The operator $\Gamma_{2n+3}$ is isometric. In this case $\mathcal{D}_{2n+3} = 0$. We will prove (4.10). One can see that

$$\tilde{y}' = \mathcal{D}_{2n+2} \Gamma_{2n+3} \mathcal{D}_{2n+3} \mathcal{D}_{2n+3} \mathcal{D}_{2n+3} \mathcal{D}_{2n+3} \cdots,$$

and in the matrix representation (4.17) the entries $\tilde{D}_n$, $\tilde{Q}_n$, and $\tilde{T}'$ take the form (see Appendix A)

$$\tilde{D}_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & \Gamma_{2n+3} D_{2n+2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \end{bmatrix}, \quad \tilde{Q}_n = \begin{bmatrix} 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots \\ D_{2n+2} \Gamma_{2n+3} D_{2n+2} & \cdots & \cdots \\ -\Gamma_{2n+1} D_{2n+2} & 0 & \cdots \end{bmatrix}.$$
\[\bar{T} = \bar{\theta}_0 \{ \Gamma_k \}_{k \geq 2n+2} = \begin{bmatrix} -\Gamma_{2n+3} \Gamma_{2n+2}^* & I_{D_{\Gamma_{2n+3}}}^* & 0 & 0 & 0 & \ldots \\ 0 & 0 & I_{D_{\Gamma_{2n+3}}} & 0 & 0 & \ldots \\ 0 & 0 & 0 & I_{D_{\Gamma_{2n+3}}} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.\]

We note that since \(\Gamma_{2n+3}\) is isometric operator from \(L(D_{\Gamma_{2n+3}}, D_{\Gamma_{2n+3}}^*)\), the function \(\Theta_{2n+2} \in S(D_{\Gamma_{2n+3}}, D_{\Gamma_{2n+3}}^*)\) is of the form

\[\Theta_{2n+2}(z) = \Gamma_{2n+2} + zD_{\Gamma_{2n+2}}^* \left( I_{D_{\Gamma_{2n+2}}} + z\Gamma_{2n+3}^* \right) \left( I_{D_{\Gamma_{2n+2}}}^* + z\Gamma_{2n+3} \right)^{-1} \Gamma_{2n+3}D_{\Gamma_{2n+2}}.\]

The CMV matrix corresponding to \(\Gamma_{2n+2}, \Gamma_{2n+3}, 0 \in L(0, D_{\Gamma_{2n+3}}), \ldots\) is

\[\bar{\theta}_0 \{ \{ \Gamma_k \}_{k \geq 2n+2} \} = \begin{bmatrix} \Gamma_{2n+2} & D_{\Gamma_{2n+2}} & 0 & 0 & 0 & \ldots \\ \Gamma_{2n+3}D_{\Gamma_{2n+2}} & -\Gamma_{2n+3} & I_{D_{\Gamma_{2n+3}}} & 0 & 0 & \ldots \\ 0 & 0 & 0 & I_{D_{\Gamma_{2n+3}}} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.
\]

Hence, we get

\[\Theta_{2n+2}(z) = \left( \Gamma_{2n+2} + zD_{\Gamma_{2n+2}}^* \left( I_{D_{\Gamma_{2n+2}}} + z\bar{T}^* \right)^{-1} \right) \left( I_{D_{\Gamma_{2n+2}}}^* + z\bar{T} \right)^{-1} \Gamma_{2n+3}D_{\Gamma_{2n+2}}.\]

\[\bar{S}_n + z\bar{Q}_n \left( I_{\bar{J}} - z\bar{T}^* \right)^{-1} \bar{D}_n = \bar{S}_n, + \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \Gamma_{2n+2} P_{D_{\Gamma_{2n+1}}} + z \left[ \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right] D_{\Gamma_{2n+2}} \left( P_{D_{\Gamma_{2n+2}}} \left( I_{\bar{J}} - z\bar{T}^* \right)^{-1} \right) D_{\Gamma_{2n+2}}^* \frac{D_{\Gamma_{2n+2}}^{\perp}}{-\Gamma_{2n+1}} \Gamma_{2n+3}D_{\Gamma_{2n+2}} P_{D_{\Gamma_{2n+1}}} = \bar{S}_n, + zQ_n \left( I_{\bar{J}} - z\bar{T} \right)^{-1} D_n = \bar{S}_n, + \Gamma_{2n+2} \left[ \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right] P_{H_{n+1}} = \bar{S}_n, + zQ_n \left( I_{\bar{J}} - z\bar{T} \right)^{-1} D_n = \bar{S}_n, + \Gamma_{2n+2} \left[ \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right] P_{H_{n+1}} = \bar{S}_n, + \Gamma_{2n+2} \left( \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right) \Theta_{2n+2}(z) = \bar{S}_n, + \Gamma_{2n+2} \left( \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right) \Theta_{2n+2}(z) = \bar{S}_n, + \Gamma_{2n+2} \left( \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right) \Theta_{2n+2}(z) = \bar{S}_n, + \Gamma_{2n+2} \left( \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right) \Theta_{2n+2}(z) = \bar{S}_n, + \Gamma_{2n+2} \left( \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right) \Theta_{2n+2}(z) = \bar{S}_n, + \Gamma_{2n+2} \left( \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right) \Theta_{2n+2}(z) = \bar{S}_n, + \Gamma_{2n+2} \left( \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right) \Theta_{2n+2}(z) = \bar{S}_n, + \Gamma_{2n+2} \left( \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right) \Theta_{2n+2}(z).\]

The operator \(\Gamma_{2n+2}\) is co-isometric. Now \(\Theta_{2n+2}(z) = \Gamma_{2n+2}\) for all \(z \in \mathbb{D}\),

\[\bar{J} = \mathbb{D}_{\Gamma_{2n+2}} + \mathbb{D}_{\Gamma_{2n+2}}^* + \ldots; \]

\(Q_n = 0 : \bar{J} \rightarrow \mathbb{K}_n\). Therefore, \(Q_n \left( I_{\bar{J}} - z\bar{T} \right)^{-1} D_n = 0\) and

\[\bar{S}_n + zQ_n \left( I_{\bar{J}} - z\bar{T} \right)^{-1} D_n = \bar{S}_n, + \Gamma_{2n+2} \left[ \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right] P_{H_{n+1}} = \bar{S}_n, + \Gamma_{2n+2} \left( \frac{D_{\Gamma_{2n+1}}^{\perp}}{-\Gamma_{2n+1}} \right) \Theta_{2n+2}(z).\]

4.3. Connection between \(\Theta\) and \(\Theta_{2n+2}\). Applying Proposition 4.3 for fixed \(z \in \mathbb{D}\) and \(\Gamma = \Theta_{2n+2}(z)\) we get

\[4.18 \quad (I_{K_n} - zS_n, \Theta_{2n+2}(z))^{-1} = (I_{K_n} - zS_n, 0)^{-1} + \]

\[z \left( (I_{K_n} - zS_n, 0)^{-1} | D_{\Gamma_{2n+1}} \right) \times\]

\[\left( I_{D_{\Gamma_{2n+1}}} - z\Theta_{2n+2}(z) \left[ D_{\Gamma_{2n+1}} - \Gamma_{2n+1} \right] \left( P_{H_{n+1}} \left( I_{K_n} - zS_n, 0 \right)^{-1} | D_{\Gamma_{2n+1}} \right) \right)^{-1} \times \Theta_{2n+2}(z) \left[ D_{\Gamma_{2n+1}} - \Gamma_{2n+1} \right] P_{H_{n+1}} \left( I_{K_n} - zS_n, 0 \right)^{-1},\]
Define the following operator functions in $D$:

$$\Theta_n^{(0)}(z) := \Gamma_0 + z D_{\Gamma_0}^* \begin{bmatrix} \Gamma_1 & D_{\Gamma_1} \\ P_{H_1} (I_{\mathcal{K}_n} - z S_{n,0})^{-1} & D_{\Gamma_0} \end{bmatrix} D_{\Gamma_0}$$

$$A_n(z) := z \begin{bmatrix} D_{\Gamma_0} & -\Gamma_{2n+1}^\ast \end{bmatrix} \begin{bmatrix} \Gamma_1 & D_{\Gamma_1} \\ P_{H_1} (I_{\mathcal{K}_n} - z S_{n,0})^{-1} & D_{\Gamma_0} \end{bmatrix} D_{\Gamma_0}$$

$$B_n(z) := z \begin{bmatrix} D_{\Gamma_0} & -\Gamma_{2n+1}^\ast \end{bmatrix} \begin{bmatrix} \Gamma_1 & D_{\Gamma_1} \\ P_{H_1} (I_{\mathcal{K}_n} - z S_{n,0})^{-1} & D_{\Gamma_0} \end{bmatrix} D_{\Gamma_0}$$

$$C_n(z) := z D_{\Gamma_0}^* \begin{bmatrix} \Gamma_1 & D_{\Gamma_1} \\ P_{H_1} (I_{\mathcal{K}_n} - z S_{n,0})^{-1} & D_{\Gamma_0} \end{bmatrix} D_{\Gamma_0}$$

Consider the following discrete time-invariant systems:

$$\tau_n = \left\{ \begin{bmatrix} N_n & M_n \\ L_n & S_{n,0} \end{bmatrix} : \mathcal{M} \oplus \mathcal{D}_{\mathcal{G}_{2n+1}}, \mathcal{N} \oplus \mathcal{D}_{\mathcal{G}_{2n+1}}, \mathcal{K}_n \right\}$$

and

$$\check{\tau}_n = \left\{ \begin{bmatrix} \bar{N}_n & \bar{M}_n \\ \bar{L}_n & \bar{S}_{n,0} \end{bmatrix} : \mathcal{M} \oplus \mathcal{D}_{\mathcal{G}_{2n+1}}, \mathcal{N} \oplus \mathcal{D}_{\mathcal{G}_{2n+1}}, \mathcal{K}_n \right\}.$$
where
\[ N_n = \tilde{N}_n = \begin{bmatrix} \Gamma_0 & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{M} \oplus \mathcal{O}_{2n+1} \rightarrow \mathcal{N} \oplus \mathcal{O}_{2n+1}, \]

\[ M_n f = \begin{bmatrix} D_{\Gamma_0} \Gamma_1 & D_{\Gamma_0} \Gamma_1 \\ D_{\Gamma_0} \Gamma_1 & -\Gamma_2 \Gamma_1 \end{bmatrix} P_{\mathcal{H}_1} f + [D_{\Gamma_2} \Gamma_2, -\Gamma_2 \Gamma_1] P_{\mathcal{H}_1} f \in \mathcal{N} \oplus \mathcal{O}_{2n+1}, \quad f \in \mathcal{K}_n, \]

\[ L_n \varphi = D_{\Gamma_0} P_{\mathcal{M}} \varphi + P_{\mathcal{O}_{2n+1}} f \in \mathcal{O}_{2n+1}, \quad f \in \tilde{\mathcal{K}}_n, \]

Then from (4.20), (4.21), (4.22), and (4.23) it follows that \( \mathcal{Q}(z) \) and \( \tilde{\mathcal{Q}}(z) \) are the transfer functions of the systems \( \tau_n \) and \( \tilde{\tau}_n \), respectively.

**Proposition 4.5.** The discrete time-invariant systems \( \tau_n \) and \( \tilde{\tau}_n \) are conservative and unitary equivalent. Therefore,

\[ \mathcal{Q}_n = \tilde{\mathcal{Q}}_n \in \mathcal{S} \left( \mathcal{M} \oplus \mathcal{O}_{2n+1}, \mathcal{N} \oplus \mathcal{O}_{2n+1} \right). \]

**Proof.** The statements follow from definitions of \( \tau_n \) and \( \tilde{\tau}_n \), equalities (4.6), (4.7), (4.12). One can verify that

\[ \tilde{M}_n V_n = M_n, \quad V_n L_n = \tilde{L}_n, \quad V_n S_n, 0 = \tilde{S}_n, 0 V_n, \]

where \( V_n \) is given by (4.7). This means that \( \tau_n \) and \( \tilde{\tau}_n \) are unitary equivalent. \( \Box \)

Proposition 4.5 yields the equalities

\[ \Theta_n^{(0)}(z) = \tilde{\Theta}_n^{(0)}(z), \quad A_n(z) = \tilde{A}_n(z), \quad B_n(z) = \tilde{B}_n(z), \quad C_n(z) = \tilde{C}_n(z), \quad z \in \mathbb{D}. \]

Since \( B_n(0) = 0 \), \( C_n(0) = 0 \), and \( A_n(0) = 0 \), we get

\[ ||B_n(z)|| \leq |z|, \quad ||C_n(z)|| \leq |z|, \quad ||A_n(z)|| \leq |z|, \quad z \in \mathbb{D}. \]

**Theorem 4.6.** Let \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) and let \( \{\Gamma_k\}_{k \geq 0} \) be its Schur parameters. Suppose that \( \mathcal{O}_{2n+1} \neq \{0\} \) and \( \mathcal{O}_{2n+2} \neq \{0\} \) for some \( n \). Then the functions \( \Theta \) and \( \Theta_{2n+2} \) are connected by the relations

\[ \Theta(z) = \Theta_n^{(0)}(z) + C_n(z) \left( I_{\mathcal{O}_{2n+1}} - \Theta_{2n+2}(z) A_n(z) \right)^{-1} \Theta_{2n+2}(z) B_n(z) \]

\[ = \Theta_n^{(0)}(z) + C_n(z) \Theta_{2n+2}(z) \left( I_{\mathcal{O}_{2n+1}} - A_n(z) \Theta_{2n+2}(z) \right)^{-1} B_n(z), \]

where the entries of the Schur class function \( \mathcal{Q}_n(z) = \begin{bmatrix} \Theta_n^{(0)}(z) & C_n(z) \\ B_n(z) & A_n(z) \end{bmatrix} \) are given by (4.20).
Theorem 4.7. Let

\[ \Gamma_0 \in L(\mathcal{M}, \mathcal{N}), \Gamma_1, \ldots, \Gamma_{2n+1} \]

be a choice sequence. Suppose \( \mathcal{D}_{\Gamma_{2n+1}} \neq \{0\}, \mathcal{D}_{\Gamma_{2n+1}} \neq \{0\} \). Then the formula

\[
\Theta(z) = \Theta_n^{(0)}(z) + C_n(z) \mathcal{E}(z) \left( I_{\mathcal{D}_{\Gamma_{2n+1}}} - A_n(z) \mathcal{E}(z) \right)^{-1} B_n(z), \quad z \in \mathbb{D}
\]

gives a one-to-one correspondence between all functions \( \mathcal{E} \in \mathcal{S}(\mathcal{D}_{\Gamma_{2n+1}}, \mathcal{D}_{\Gamma_{2n+1}}) \) and all functions \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) having given choice sequence \( \Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+1} \) as their first \( 2n + 2 \) Schur parameters. Moreover the Schur parameters of the function \( \Theta \) given by (4.24) are

\[
\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+1}, \gamma_0^{(E)}, \gamma_1^{(E)}, \ldots,
\]

where \( \gamma_0^{(E)} \in L(\mathcal{D}_{\Gamma_{2n+1}}, \mathcal{D}_{\Gamma_{2n+1}}) \) is given by equality (4.24).

Proof. Assume \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) has \( \Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+1} \) as its first \( 2n + 2 \) Schur parameters and let \( \Gamma_{2n+2}, \ldots \) are the rest Schur parameters of \( \Theta \). Denote \( \mathcal{E} \) the function from \( \mathcal{S}(\mathcal{D}_{\Gamma_{2n+1}}, \mathcal{D}_{\Gamma_{2n+1}}) \) with the Schur parameters \( \Gamma_{2n+2}, \ldots \). Then \( \Theta_{2n+2}(z) = \mathcal{E}(z) \) for all \( z \in \mathbb{D} \). Here \( \Theta_{2n+2} \) is the function associated with \( \Theta \) in accordance with the Schur algorithm. Then constructing the block operator matrix \( \mathcal{S}_{n,0} \) by means of (4.13), the function \( Q_n(z) \) of the form (4.21), and applying Theorem 4.6 we get equality (4.24).

Conversely, suppose \( \mathcal{E} \in \mathcal{S}(\mathcal{D}_{\Gamma_{2n+1}}, \mathcal{D}_{\Gamma_{2n+1}}) \) is given. Let

\[
\gamma_0^{(E)} \in L(\mathcal{D}_{\Gamma_{2n+1}}, \mathcal{D}_{\Gamma_{2n+1}}), \gamma_1^{(E)}, \ldots,
\]

be the Schur parameters of \( \mathcal{E} \). Let \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) be the function with the Schur parameters

\[
\Gamma_0, \ldots, \Gamma_{2n+1}, \gamma_0^{(E)}, \ldots,
\]

Then \( \Theta_{2n+2}(z) = \mathcal{E}(z) \) for all \( z \in \mathbb{D} \) and by Theorem 4.6 the functions \( \mathcal{E}(z) \) and \( \Theta(z) \) are connected by (4.24).

Observe that the function \( \Theta_n^{(0)} \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) corresponds to the parameter

\[
\mathcal{E} \equiv 0 \in \mathcal{S}(\mathcal{D}_{\Gamma_{2n+1}}, \mathcal{D}_{\Gamma_{2n+1}}),
\]

i.e., the Schur parameters of \( \Theta_n^{(0)} \) are \( \Gamma_0, \ldots, \Gamma_{2n+1}, 0, 0, \ldots, \).

Corollary 4.8. Let

\[ \Gamma_0 \in L(\mathcal{M}, \mathcal{N}), \Gamma_1, \ldots, \Gamma_{2n+1}, \Gamma_{2n+2} \]

be a choice sequence. Suppose \( \Gamma_{2n+2} \) is either isometry or co-isometry. Then

\[
\Theta(z) = \Theta_n^{(0)}(z) + C_n(z) \Gamma_{2n+2} \left( I_{\mathcal{D}_{\Gamma_{2n+2}}} - A_n(z) \Gamma_{2n+2} \right)^{-1} B_n(z), \quad z \in \mathbb{D}
\]

is a unique function from \( \mathcal{S}(\mathcal{M}, \mathcal{N}) \) having \( \Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+2} \) as its first \( 2n + 2 \) Schur parameters.
4.3.1. CMV block operator matrices and the coupling of conservative systems. Let \( \{\Gamma_k\}_{k \geq 0} \) be a choice sequence, \( \Gamma_0 \in \text{L}(\mathcal{M}, \mathcal{N}) \). Suppose condition (4.11). Let \( \mathcal{U}_0 = \mathcal{U}_0(\Gamma_0) \) be CMV block operator matrix. The unitary operator \( \mathcal{U}_0 \) acts from the Hilbert space \( \mathcal{H} \oplus \mathcal{H}_0 \) onto the Hilbert space \( \mathcal{M} \oplus \mathcal{N} \), where \( \mathcal{H}_0 \) is a Hilbert space constructed by means of defect spaces \( \{\mathcal{D}_{\Gamma_k}, \mathcal{D}_{{\Gamma_k}^{-1}}\}_{k \geq 0} \) (see Section 3 and Appendix). The operator \( \mathcal{U}_0 \) takes the form

\[
\mathcal{U}_0 = \begin{bmatrix}
\Gamma_0 & D_{\Gamma_0} \Gamma_1 & D_{\Gamma_0} \Gamma_2 & 0 & \ldots \\
D_{\Gamma_0} & D_{\Gamma_0} \Gamma_2 & 0 & & \\
\vdots & & & & \\
\end{bmatrix},
\]

where \( \mathcal{T}_0 \) is truncated CMV matrix. Let the Hilbert space \( \mathcal{H} \), be CMV block operator matrix. The unitary operator \( \Psi_n = \begin{bmatrix} N_n & M_n \\ L_n & S_{n,0} \end{bmatrix} \) is of the form

\[
\begin{pmatrix}
\Gamma_0 & 0 & 0 & D_{\Gamma_0} \Gamma_1 & D_{\Gamma_0} \Gamma_2 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & D_{\Gamma_0} \Gamma_2 & -\Gamma_2 \Gamma_1 \\
\end{pmatrix}
\]

Consider also the CMV matrix \( \mathcal{U}_{2n+2}(\{\Gamma_k\}_{k \geq 2n+2}) \). The precise form of \( \mathcal{U}_{2n+2} \) depends on the cases 1)–7) mentioned in the proof of Theorem 4.4. In particular, if both operators \( \Gamma_{2n+2} \) and \( \Gamma_{2n+3} \) are neither isometric nor co-isometric, then

\[
\mathcal{U}_{2n+2} = \begin{bmatrix}
\Gamma_{2n+2} & D_{\Gamma_{2n+2}} \Gamma_{2n+3} & 0 & \ldots & 0 \\
D_{\Gamma_{2n+2}} & D_{\Gamma_{2n+2}} \Gamma_{2n+3} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\mathcal{T}_{2n+2} & & & & \\
\end{bmatrix},
\]

where \( \mathcal{T}_{2n+2} = \mathcal{T}' \) is the truncated CMV matrix related to \( \mathcal{U}_{2n+2} \). Let

\[
\zeta_{2n+2} = \begin{bmatrix} \mathcal{U}_{2n+2}; \mathcal{D}_{\Gamma_{2n+2}}, \mathcal{D}_{{\Gamma_{2n+2}}^{-1}}, \mathcal{H}_{2n+2} \end{bmatrix}
\]

be the corresponding conservative system, \( \mathcal{H}_{2n+2} = \mathcal{H}_{2n+2}(\{\Gamma_k\}_{k \geq 2n+2}) = \mathcal{H}_0 \) (see (4.11)), \( \mathcal{H}_0 = \mathcal{K}_n \oplus \mathcal{H}_{2n+2} \). The truncated CMV block operator matrix \( \mathcal{T}_0 \) with
respect to the decomposition \( \mathcal{H}_0 = \mathcal{K}_n \oplus \mathcal{H}_{2n+2} \) takes the form (4.17):

\[
\mathcal{T}_0 = \begin{bmatrix}
\mathcal{S}_n & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots \\
0 & 0 & \mathcal{A}_{n+1} & \mathcal{C}_{n+1} & 0 & \ldots \\
0 & \ldots & 0 & \mathcal{A}_{n+1} & \mathcal{C}_{n+1} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

The notion of the coupling of unitary colligations (conservative systems) can be found in \([27, 28, 31]\). By straightforward calculations one can verify that the equality

\[
\text{This means that the equality}
\]

\[
\begin{pmatrix} n \\ h_0' \\ h_1' \end{pmatrix} = U_0 \begin{pmatrix} m \\ h_0 \\ h_1 \end{pmatrix}, \quad m \in \mathcal{M}, \; n \in \mathcal{N}, \; h_0, h_0', h_1, h_1' \in \mathcal{H}_{2n+2}
\]

holds if

\[
\Psi_n \begin{pmatrix} m \\ \gamma \\ h_0 \end{pmatrix} = \begin{pmatrix} n \\ \gamma \\ h_0' \end{pmatrix} \quad \text{and} \quad U_{2n+2} \begin{pmatrix} \gamma \\ h_1 \end{pmatrix} = \begin{pmatrix} \gamma \ast \\ h_1 \end{pmatrix}
\]

for some \( \gamma \ast \in \mathcal{D}_{\Gamma_{2n+1}} \) and \( \gamma \in \mathcal{D}_{\Gamma_{2n+1}} \). Notice that

\[
\gamma = \left(D_{\Gamma_{2n+1}} P_{\mathcal{D}_{\Gamma_{2n}}} - \Gamma_{2n+1}' P_{\mathcal{D}_{\Gamma_{2n+1}}'} \right) h_0,
\]

\[
\gamma \ast = \Gamma_{2n+2} \left(D_{\Gamma_{2n+2}} P_{\mathcal{D}_{\Gamma_{2n}}} - \Gamma_{2n+1}' P_{\mathcal{D}_{\Gamma_{2n+1}}'} \right) h_0
\]

\[
+ D_{\Gamma_{2n+2}} \left( \Gamma_{2n+3} P_{\mathcal{D}_{\Gamma_{2n+3}}} + D_{\Gamma_{2n+3}} P_{\mathcal{D}_{\Gamma_{2n+3}}'} \right) h_1.
\]

As it is established in \([12, 27, 29, 28]\) if a conservative system \( \Sigma \) is the coupling of certain universal conservative system \( \Sigma_0 \) and a conservative system \( \Sigma' \), then the transfer functions \( \Theta_{\Sigma} \) and \( \Theta_{\Sigma'} \) of the systems \( \Sigma \) and \( \Sigma' \), respectively, are connected by the relation

\[
\Theta_{\Sigma}(z) = a_{11}(z) + a_{12}(z) (I - \Theta_{\Sigma'}(z) a_{22}(z))^{-1} \Theta_{\Sigma'}(z) a_{21}(z), \quad z \in \mathbb{D},
\]

where

\[
\Theta_{\Sigma_0}(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}
\]

is the transfer function of the universal system \( \Sigma_0 \). Thus, relations in Theorem 4.6 are also a consequence of the facts that \( \mathcal{Q} \) is the coupling of \( \tau_n \) and \( \zeta_{2n+2} \) and the unitary equivalence of the systems \( \tau_n \) and \( \tilde{\tau}_n \) (see Proposition 4.5).

4.4. The matrix \( \mathcal{S}_{n,0} \) and connection between \( \Theta \) and \( \Theta_{2n+1} \). Now we established a connection between \( \Theta \) and \( \Theta_{2n+1} \). Suppose \( \{\Gamma_k\} \) are the Schur parameters of \( \Theta \in \mathcal{S}(\mathcal{M}, \mathcal{N}) \) and \( \mathcal{D}_{\Gamma_{2n}} \neq \{0\}, \; \mathcal{D}_{\Gamma_{2n+1}} \neq \{0\} \). Then the operators (the choice sequence)

\[
\Gamma_0 := 0 \in \mathcal{L}(\mathcal{M}, \mathcal{N}), \; \Gamma_1 := \Gamma_0 \in \mathcal{L}(\mathcal{M}, \mathcal{N}), \; \Gamma_2 := \Gamma_1 \in \mathcal{L}(\mathcal{D}_{\Gamma_0}, \mathcal{D}_{\Gamma_1}), \ldots
\]
are the Schur parameters of the function \( \hat{\Theta}(z) = z\Theta(z) \in S(\mathcal{M}, \mathcal{N}) \). So, \( \hat{\Gamma}_l := \Gamma_{l-1}, \ l \geq 1 \). Let

\[
\hat{\mathcal{H}}_1 = \mathcal{M} \oplus \bigoplus_{\mathcal{D}_{\Gamma_{2k-2}}}, \quad \hat{\mathcal{H}}_k := \mathcal{D}_{\Gamma_{2k-1}}^* \oplus \mathcal{D}_{\Gamma_{2k-2}}^*, \ k = 2, \ldots, n + 1,
\]

\[
\hat{\mathcal{K}}_n := \bigoplus_{k=1}^{n+1} \hat{\mathcal{H}}_k.
\]

Now we can apply the approach of Subsection 4.3. Define the operator \( \hat{\mathcal{S}}_{n,0} \) in accordance with (4.9)

\[
\hat{\mathcal{S}}_{n,0} := \mathcal{S}_n \left( \hat{\Gamma}_0, \hat{\Gamma}_1, \ldots, \hat{\Gamma}_{2n+1}, 0 \right) = \mathcal{S}_n \left( 0, \Gamma_0, \Gamma_1, \ldots, \Gamma_{2n}, 0 \right),
\]

i.e.,

\[
\hat{\mathcal{S}}_{n,0} = \left( -\hat{\Gamma}_0^* \bigoplus_{k=1}^{n} \mathcal{J}_{\mathcal{F}_{2k}} \bigoplus 0 \right) \times \left( \bigoplus_{k=1}^{n+1} \mathcal{J}_{\mathcal{F}_{2k-1}} \right)
\]

\[
= \left( 0 \bigoplus_{k=1}^{n} \mathcal{J}_{\mathcal{F}_{2k-1}} \bigoplus 0 \right) \times \left( \bigoplus_{k=1}^{n+1} \mathcal{J}_{\mathcal{F}_{2k-2}} \right).
\]

Then construct the function

\[
\hat{\mathcal{Q}}_n(z) = \begin{bmatrix} \hat{\Theta}_n^{(0)}(z) \\ \hat{\mathcal{B}}_n(z) \\ \hat{\mathcal{A}}_n(z) \end{bmatrix} : \mathcal{M} \oplus \mathcal{D}_{\Gamma_{2n}} \longrightarrow \mathcal{M} \oplus \mathcal{D}_{\Gamma_{2n}}
\]

in accordance with (4.20) and (4.21). We have

\[
(4.26)
\]

\[
\begin{align*}
\hat{\mathcal{Q}}_n^{(0)}(z) & := z \left[ \Gamma_0 \ D_{\Gamma_0^*} \right] \left( \mathcal{P}_{\hat{\mathcal{H}}_1} \left( I_{\hat{\mathcal{K}}_n} - z\hat{\mathcal{S}}_{n,0} \right)^{-1} \mid \mathcal{M} \right) \\
\hat{\mathcal{A}}_n(z) & := z \left[ D_{\Gamma_{2n}} \ -\Gamma_{2n}^* \right] \left( \mathcal{P}_{\hat{\mathcal{H}}_{n+1}} \left( I_{\hat{\mathcal{K}}_n} - z\hat{\mathcal{S}}_{n,0} \right)^{-1} \mid \mathcal{D}_{\Gamma_{2n}} \right) \\
\hat{\mathcal{B}}_n(z) & := z \left[ D_{\Gamma_{2n}} \ -\Gamma_{2n}^* \right] \left( \mathcal{P}_{\hat{\mathcal{H}}_{n+1}} \left( I_{\hat{\mathcal{K}}_n} - z\hat{\mathcal{S}}_{n,0} \right)^{-1} \mid \mathcal{M} \right) \\
\hat{\mathcal{C}}_n(z) & := z \left[ \Gamma_0 \ D_{\Gamma_0^*} \right] \left( \mathcal{P}_{\hat{\mathcal{H}}_1} \left( I_{\hat{\mathcal{K}}_n} - z\hat{\mathcal{S}}_{n,0} \right)^{-1} \mid \mathcal{D}_{\Gamma_{2n}} \right)
\end{align*}
\]

Due to Proposition 4.5, the function \( \hat{\mathcal{Q}}_n \) belongs to the Schur class \( S(\mathcal{M} \oplus \mathcal{D}_{\Gamma_{2n}}, \mathcal{N} \oplus \mathcal{D}_{\Gamma_{2n}}) \).

Since \( ||\hat{\mathcal{Q}}_n(z)|| \leq 1 \) for all \( z \in \mathbb{D} \) and \( \hat{\mathcal{Q}}_n(0) = 0 \), by Schwarz’s lemma for the function

\[
\hat{q}_n(z) := z^{-1} \hat{\mathcal{Q}}_n(z) = \begin{bmatrix} z^{-1} \hat{\Theta}_n^{(0)}(z) & z^{-1} \hat{\mathcal{C}}_n(z) \\ z^{-1} \hat{\mathcal{B}}_n(z) & z^{-1} \hat{\mathcal{A}}_n(z) \end{bmatrix} : \mathcal{M} \oplus \mathcal{D}_{\Gamma_{2n}} \longrightarrow \mathcal{M} \oplus \mathcal{D}_{\Gamma_{2n}}
\]
we obtain \(|q_n(z)|| \leq 1, z \in \mathbb{D}\). Clearly, the function

\[ q_n(z) := \begin{bmatrix} z^{-1} \hat{\Theta}_n(z) & z^{-1} \hat{\Theta}_n(z) \\ \hat{B}_n(z) & \hat{A}_n(z) \end{bmatrix} : \mathbb{M} \oplus \mathbb{M} \rightarrow \mathbb{M} \oplus \mathbb{M}, \]

is also from the Schur class. Set

\[ \rho_n^{(0)} := z^{-1} \hat{\Theta}_n^{(0)}(z), c_n(z) := z^{-1} \hat{\Theta}_n(z), a_n(z) := \hat{A}_n(z), b_n(z) := \hat{B}_n(z), z \in \mathbb{D}. \]

So,

\[
\begin{cases}
\theta_n^{(0)}(z) := [\Gamma_0 \ D_{\Gamma_0^\prime}] \begin{pmatrix} P_{\mathbb{M}_n} \left( I_{\mathbb{K}_n} - z \hat{S}_{n,0} \right) \end{pmatrix}^{-1} | \mathbb{M} \\
\Gamma_0 \ D_{\Gamma_0^\prime} & \mathbb{L}(\mathbb{M}, \mathbb{M}), \\

a_n(z) := z \left[ D_{\Gamma_2n} - \Gamma_{2n}^* \right] \begin{pmatrix} P_{\mathbb{K}_n} \left( I_{\mathbb{K}_n} - z \hat{S}_{n,0} \right) \end{pmatrix}^{-1} | \mathbb{M} \\
\Gamma_0 \ D_{\Gamma_0^\prime} & \mathbb{L}(\mathbb{M}, \mathbb{M}), \\
b_n(z) := z \left[ D_{\Gamma_2n} - \Gamma_{2n}^* \right] \begin{pmatrix} P_{\mathbb{K}_n} \left( I_{\mathbb{K}_n} - z \hat{S}_{n,0} \right) \end{pmatrix}^{-1} | \mathbb{M} \\
\Gamma_0 \ D_{\Gamma_0^\prime} & \mathbb{L}(\mathbb{M}, \mathbb{M}), \\
c_n(z) := [\Gamma_0 \ D_{\Gamma_0^\prime}] \begin{pmatrix} P_{\mathbb{M}_n} \left( I_{\mathbb{K}_n} - z \hat{S}_{n,0} \right) \end{pmatrix}^{-1} | \mathbb{M} \\
\Gamma_0 \ D_{\Gamma_0^\prime} & \mathbb{L}(\mathbb{M}, \mathbb{M}).
\end{cases}
\]

**Theorem 4.9.** Let \( \Theta \in \mathbf{S}(\mathbb{M}, \mathbb{M}) \) and let \( \{\Gamma_n\}_{n \geq 0} \) be its Schur parameters. Suppose that \( \mathbb{D}_{\Gamma_{2n}} \neq \{0\} \) and \( \mathbb{D}_{\Gamma_{2n}^\prime} \neq \{0\} \). Then the functions \( \Theta \) and \( \Theta_{2n+1} \) are connected by the relations

\[
\Theta(z) = \theta_n^{(0)}(z) + c_n(z) \left( I_{\mathbb{D}_{\Gamma_{2n}}} - \Theta_{2n+1}(z) a_n(z) \right)^{-1} \Theta_{2n+1}(z) b_n(z) \\
= \theta_n^{(0)}(z) + c_n(z) \Theta_{2n+1}(z) \left( I_{\mathbb{D}_{\Gamma_{2n}}} - a_n(z) \Theta_{2n+1}(z) \right)^{-1} b_n(z), z \in \mathbb{D},
\]

where the entries of the Schur class function

\[ q_n(z) = \begin{bmatrix} \theta_n^{(0)}(z) & c_n(z) \\ b_n(z) & a_n(z) \end{bmatrix} : \mathbb{M} \oplus \mathbb{M} \rightarrow \mathbb{M} \oplus \mathbb{M}, \]

are given by \( (4.28) \).

**Proof.** Taking into account that \( \Theta_{2n+1}(z) = \hat{\Theta}_{2n+2}(z), z \in \mathbb{D} \) and applying Theorem 4.6 we obtain

\[ \Theta(z) = z \Theta(z) = \hat{\Theta}_n^{(0)}(z) + \hat{\Theta}_n(z) \left( I_{\mathbb{D}_{\Gamma_{2n}}} - z \hat{\Theta}_{2n+2}(z) \hat{A}_n(z) \right)^{-1} \hat{\Theta}_{2n+2}(z) \hat{B}_n(z) \\
= \hat{\Theta}_n^{(0)}(z) + \hat{\Theta}_n(z) \hat{\Theta}_{2n+2}(z) \left( I_{\mathbb{D}_{\Gamma_{2n}}} - z \hat{A}_n(z) \hat{\Theta}_{2n+2}(z) \right)^{-1} \hat{B}_n(z). \]

Then from \( (4.26), (4.27), \) and \( (4.28) \) we get \( (4.29) \). \( \Box \)

**Theorem 4.10.** Let

\[ \Gamma_0 \in \mathbf{L}(\mathbb{M}, \mathbb{M}), \Gamma_1, \ldots, \Gamma_{2n} \]

be a choice sequence. Suppose \( \mathbb{D}_{\Gamma_{2n}} \neq \{0\}, \mathbb{D}_{\Gamma_{2n}^\prime} \neq \{0\} \). Then the formula

\[
\Theta(z) = \theta_n^{(0)}(z) + c_n(z) \mathcal{E}(z) \left( I_{\mathbb{D}_{\Gamma_{2n}}} - a_n(z) \mathcal{E}(z) \right)^{-1} b_n(z), z \in \mathbb{D}
\]

gives a one-to-one correspondence between all functions \( \mathcal{E} \in \mathbf{S}(\mathbb{D}_{\Gamma_{2n}}, \mathbb{D}_{\Gamma_{2n}^\prime}) \) and all functions \( \Theta \in \mathbf{S}(\mathbb{M}, \mathbb{M}) \) having given choice sequence \( \Gamma_0, \Gamma_1, \ldots, \Gamma_{2n} \) as their first
The formula (4.24) gives all solutions to the Schur problem. Moreover the Schur parameters of the function $\Theta$ given by (4.30) are
\[
\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n}, \gamma_0^{(\varepsilon)}, \gamma_1^{(\varepsilon)}, \ldots,
\]
where $\gamma_0^{(\varepsilon)} \in L(D_{\Gamma_2}, D_{\Gamma_2}',) \gamma_1^{(\varepsilon)}, \ldots$ are the Schur parameters of $E(z)$.

Corollary 4.11. Let
\[
\Gamma_0 \in L(\{M, N\}), \Gamma_1, \ldots, \Gamma_{2n+1}
\]
be a choice sequence. Suppose $\Gamma_{2n+1}$ is either isometry or co-isometry. Then
\[
\Theta(z) = \theta_n^{(0)}(z) + c_n(z)\Gamma_{2n+1} (I_{D_{\Gamma_2}} - a_n(z)\Gamma_{2n+1})^{-1} b_n(z), z \in D
\]
is a unique function from $S(M, N)$ having $\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n+1}$ as its first $2n+1$ Schur parameters.

5. Descriptions of all solutions to the Schur problem

We describe the algorithm for the solutions to the Schur problem involving sub-matrices of block-operator CMV matrices.

Let the Schur sequence $C_0, \ldots, C_N \in L(M, N)$ be given. Calculate $(D_{T_N}^2)_{MR}$ and $(D_{T_N}^2)_{NI}$, where $T_N$ and $\bar{T}_N$ are the Toeplitz matrices of the form (4.2) and (2.4).

Suppose $(D_{T_N}^2)_{MR} \neq 0$ and $(D_{T_N}^2)_{NI} \neq 0$. Find the choice sequence
\[
\Gamma_0 = C_0, \Gamma_1, \ldots, \Gamma_N,
\]
corresponding to the data $\{C_k\}_{k=0}^N$ (see Subsection 2.3). Then any two solutions of the Schur problem differ by the Schur parameters, which start with the number $N + 1$.

If $N = 2n + 1$, find the matrix $S_{n,0}$ (see (4.9)) and calculate the functions (4.20). The formula (4.24) gives all solutions to the Schur problem.

If $N = 2n$, then calculate the matrix $\tilde{S}_{n,0}$ constructed by means of the choice sequence
\[
0 \in L(M, N), \Gamma_0, \ldots, \Gamma_N,
\]
and calculate the functions (4.28). The formula (4.30) gives all solutions.

Thus, all solutions are given by the fractional linear transformation
\[
\Theta(z) = \Theta_n^{(0)}(z) + C_N(z)\varepsilon(z) \left( I_{D_{\Gamma_N}} - A_N(z)\varepsilon(z) \right)^{-1} B_N(z)
\]
\[
= \Theta_n^{(0)}(z) + C_N(z) \left( I_{D_{\Gamma_N}} - \varepsilon(z)A_N(z) \right)^{-1} \varepsilon(z)B_N(z),
\]
where $\varepsilon(z)$ is an arbitrary function from $S(D_{\Gamma_N}, D_{\Gamma_N})$.

\[
\Theta_n^{(0)} = \left\{ \begin{array}{ll}
\Gamma_0 + zD_{\Gamma_0} \left[ \Gamma_1 \ D_{\Gamma_1} \right] \left( P_{\mathcal{H}_1} (I_{\mathcal{K}_n} - z\hat{S}_{n,0})^{-1} \mathcal{M} \right) D_{\Gamma_0}, & N = 2n + 1 \\
\Gamma_0 \ D_{\Gamma_0} \left[ P_{\mathcal{H}_1} (I_{\mathcal{K}_n} - z\hat{S}_{n,0})^{-1} \mathcal{M} \right], & N = 2n
\end{array} \right.,
\]
\[
C_N(z) = \left\{ \begin{array}{ll}
zD_{\Gamma_0} \left[ \Gamma_1 \ D_{\Gamma_1} \right] \left( P_{\mathcal{H}_1} (I_{\mathcal{K}_n} - z\hat{S}_{n,0})^{-1} \mathcal{D}_{\Gamma_{N+1}} \right), & N = 2n + 1 \\
\Gamma_0 \ D_{\Gamma_0} \left[ P_{\mathcal{H}_1} \left( I_{\mathcal{K}_n} - z\hat{S}_{n,0} \right)^{-1} \mathcal{D}_{\Gamma_{2n}} \right], & N = 2n
\end{array} \right.,
\]
The function $\Theta$ is the transfer function of the simple conservative systems connected by \(\cdot\) and truncated CMV matrices. In particular we revise some misprints in [6].

Let $\Gamma_n$ be the Schur parameters of the function $\Theta^{(0)}_n$, corresponding to the parameter $E \equiv 0 \in L(D_{\Gamma_n}, D_{\Gamma_n}^*)$ in [5].

**Parametrization (5.1)** is similar to known parameterizations [11], [14], [22], [23], [24], [25], [26] which are obtained by another methods.

Then using (4.25) for $p = 2n + 2$ or (4.31) for $p = 2n + 1$ we get

\[
\Theta(z) = \Theta_{p-1}^{(0)}(z) + C_{p-1}(z) \left(I_{D_{\Gamma_{p-1}}} - A_{p-1}(z)I_p\right)^{-1}B_{p-1}(z)
\]

The case (4.25) $\Theta_E = 0$ is similar to the previous one.

**APPENDIX A. SPECIAL CASES OF BLOCK OPERATOR CMV MATRICES**

Let $\{\Gamma_n\}$ be the Schur parameters of the function $\Theta \in S(M, 0)$. Suppose $\Gamma_m$ is an isometry (respect., co-isometry, unitary) for some $m \geq 0$. Then $\Theta_m(z) = \Gamma_m$ for all $z \in \mathbb{D}$ and

\[
\Theta_m(z) = \Gamma_m(z) + zD_{\Gamma_m^{-1}}\Gamma_m(I_{D_{\Gamma_m^{-1}}} + z\Gamma_m(z))^{-1}D_{\Gamma_m^{-1}},
\]

\[
\Theta(z) = \Gamma_0 + zD_{\Gamma_0}\Theta_1(z)(I_{D_{\Gamma_0}} + z\Theta_1(z))^{-1}D_{\Gamma_0}, z \in \mathbb{D}.
\]

The function $\Theta$ is the transfer function of the simple conservative systems constructed by means of its Schur parameters $\{\Gamma_n\}$ and the corresponding block operator CMV matrices $U_0$ and $U_0$ [6]. Here we present the explicit form of block operator CMV and truncated CMV matrices. In particular we revise some misprints in [6]. Notice that if $\Gamma_m$ is isometric (respect., co-isometric), then

1. $D_{\Gamma_n}^* = D_{\Gamma_n}, D_{\Gamma_n} = I_{D_{\Gamma_n}}, \Gamma_n = 0 : \{0\} \to D_{\Gamma_n}$ for $n > m$ (respect., $D_{\Gamma_n} = D_{\Gamma_n}, D_{\Gamma_n} = I_{D_{\Gamma_n}}, \Gamma_n = 0 : D_{\Gamma_n} \to \{0\}$ for $n > m$);

2. in the definitions of the state spaces $S_0 = \delta_0(\{\Gamma_n\}_{n \geq 0})$ and $\tilde{S}_0 = \tilde{\delta}_0(\{\Gamma_n\}_{n \geq 0})$ we replace $D_{\Gamma_n}$ with $\{0\}$ (respect., $D_{\Gamma_n}$ with $\{0\}$) for $n \geq m$, and $D_{\Gamma_n}$ by $D_{\Gamma_n}$ (respect., $D_{\Gamma_n}$ by $D_{\Gamma_n}$) for $n > m$.

The case $D_{\Gamma_0}^* \equiv 0$ is similar to the previous one.

The function $\Theta$ is the transfer function of the simple conservative systems constructed by means of its Schur parameters $\{\Gamma_n\}$ and the corresponding block operator CMV matrices $U_0$ and $U_0$ [6]. Here we present the explicit form of block operator CMV and truncated CMV matrices. In particular we revise some misprints in [6]. Notice that if $\Gamma_m$ is isometric (respect., co-isometric), then

1. $D_{\Gamma_n}^* = D_{\Gamma_n}, D_{\Gamma_n} = I_{D_{\Gamma_n}}, \Gamma_n = 0 : \{0\} \to D_{\Gamma_n}$ for $n > m$ (respect., $D_{\Gamma_n} = D_{\Gamma_n}, D_{\Gamma_n} = I_{D_{\Gamma_n}}, \Gamma_n = 0 : D_{\Gamma_n} \to \{0\}$ for $n > m$);

2. in the definitions of the state spaces $S_0 = \delta_0(\{\Gamma_n\}_{n \geq 0})$ and $\tilde{S}_0 = \tilde{\delta}_0(\{\Gamma_n\}_{n \geq 0})$ we replace $D_{\Gamma_n}$ with $\{0\}$ (respect., $D_{\Gamma_n}$ with $\{0\}$) for $n \geq m$, and $D_{\Gamma_n}$ by $D_{\Gamma_n}$ (respect., $D_{\Gamma_n}$ by $D_{\Gamma_n}$) for $n > m$.
U \text{tmatrices are five block-diagonal. In the case when the operator } \Gamma \text{ holds true. The operators given by truncated block operator CMV matrices are finite and otherwise they are semi-infinite.}

(3) the corresponding unitary elementary rotation takes the row (respect., the column) form, i.e.,

\begin{align*}
J_{10}^{(r)} &= \begin{bmatrix} \Gamma_0 & I_{D_{\Gamma_0}^*} \\ D_{\Gamma_0} & 0 \end{bmatrix} : \mathcal{M} \oplus D_{\Gamma_0} \to \mathcal{M} \\
J_{1}^{(c)} &= \begin{bmatrix} \Gamma_1 & I_{D_{\Gamma_1}^*} \\ D_{\Gamma_1} & 0 \end{bmatrix} : \mathcal{M} \oplus D_{\Gamma_1} \to \mathcal{M} \oplus D_{\Gamma_1},
\end{align*}

\begin{align*}
J_{1}^{(r)} &= \begin{bmatrix} \Gamma_1 & I_{D_{\Gamma_1}^*} \\ D_{\Gamma_1} & 0 \end{bmatrix} : \mathcal{M} \oplus D_{\Gamma_1} \to \mathcal{M} \\
J_{1}^{(c)} &= \begin{bmatrix} \Gamma_1 & I_{D_{\Gamma_1}^*} \\ D_{\Gamma_1} & 0 \end{bmatrix} : \mathcal{M} \oplus D_{\Gamma_1} \to \mathcal{M} \oplus D_{\Gamma_1}, \quad m \geq 1.
\end{align*}

Therefore, in definitions (3.11) of the block diagonal operator matrices

\mathcal{L}_0 = \mathcal{L}_0(\{\Gamma_n\}_{n \geq 0}), \mathcal{M}_0 = \mathcal{M}_0(\{\Gamma_n\}_{n \geq 0}), \text{ and } \tilde{\mathcal{M}}_0 = \tilde{\mathcal{M}}_0(\{\Gamma_n\}_{n \geq 0})

we will replace

- \( J_{\Gamma_m} \) by \( J_{\Gamma_m}^{(r)} \) and \( J_{\Gamma_m} \) by \( I_{D_{\Gamma_m}^*} \) for \( n > m \), when \( \Gamma_m \) is isometry,
- \( J_{\Gamma_m} \) by \( J_{\Gamma_m}^{(c)} \) and \( J_{\Gamma_m} \) by \( I_{D_{\Gamma_m}^*} \) for \( n > m \), when \( \Gamma_m \) is co-isometry,
- \( J_{\Gamma_m} \) by \( \Gamma_m \), when \( \Gamma_m \) is unitary.

In all these cases the block operators CMV matrices \( \mathcal{U}_0 = \mathcal{U}_0(\{\Gamma_n\}_{n \geq 0}) \) and \( \tilde{\mathcal{U}}_0 = \tilde{\mathcal{U}}_0(\{\Gamma_n\}_{n \geq 0}) \) are defined by means the products \( \mathcal{U}_0 = \mathcal{L}_0 \mathcal{M}_0, \mathcal{U}_0 = \mathcal{M}_0 \mathcal{L}_0 \). These matrices are five block-diagonal. In the case when the operator \( \Gamma_m \) is unitary the block operator CMV matrices \( \mathcal{U}_0 \) and \( \tilde{\mathcal{U}}_0 \) are finite and otherwise they are semi-infinite.

As before the truncated block operator CMV matrices \( \mathcal{T}_0 = \mathcal{T}_0(\{\Gamma_n\}_{n \geq 0}) \) and \( \tilde{\mathcal{T}}_0 = \tilde{\mathcal{T}}_0(\{\Gamma_n\}_{n \geq 0}) \) are defined by (3.10) and (3.11), i.e.,

\[ \mathcal{T}_0 = P_{\mathcal{H}_0} \mathcal{U}_0 |_{\mathcal{H}_0}, \quad \tilde{\mathcal{T}}_0 = P_{\mathcal{H}_0} \tilde{\mathcal{U}}_0 |_{\mathcal{H}_0}. \]

The operators \( \mathcal{T}_0 \) and \( \tilde{\mathcal{T}}_0 \) are unitarily equivalent completely non-unitary contractions and Proposition (3.2) hold true. The operators given by truncated block operator CMV matrices \( \mathcal{T}_m \) and \( \tilde{\mathcal{T}}_m \) obtaining from \( \mathcal{U}_0 \) and \( \tilde{\mathcal{U}}_0 \) by deleting first \( m + 1 \) rows and \( m + 1 \) columns are

- co-shifts of the form

\[ \mathcal{T}_m = \tilde{\mathcal{T}}_m = \begin{bmatrix}
0 & I_{D_{\Gamma_m}^*} & 0 & 0 & \cdots \\
0 & 0 & I_{D_{\Gamma_m}^*} & 0 & \cdots \\
0 & 0 & 0 & I_{D_{\Gamma_m}^*} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} : \mathcal{D}_{\Gamma_m^*} \oplus \mathcal{D}_{\Gamma_m} \to \mathcal{D}_{\Gamma_m} \oplus \mathcal{D}_{\Gamma_m^*}, \]

when \( \Gamma_m \) is isometry,
the unilateral shifts of the form
\[ T_m = \tilde{T}_m = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots \\
I_{\mathcal{D}_{\Gamma_m}} & 0 & 0 & 0 & \ldots \\
0 & I_{\mathcal{D}_{\Gamma_m}} & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \]
\[ \mathcal{D}_{\Gamma_m} \oplus \mathcal{D}_{\Gamma_m} \oplus \mathcal{D}_{\Gamma_m} \oplus \mathcal{D}_{\Gamma_m} \oplus \ldots, \]
when \( \Gamma_m \) is co-isometry.

One can see that Proposition 3.2 remains true.

The conservative systems
\[ \zeta_0 = \{U_0; \mathcal{M}, \mathcal{N}, \tilde{\mathcal{N}}_0\}, \quad \tilde{\zeta}_0 = \{\tilde{U}_0; \mathcal{M}, \mathcal{N}, \tilde{\mathcal{N}}_0\}. \]
are simple and unitarily equivalent and, moreover, Theorem 3.3 remains valid.

In order to obtain precise forms of \( U_0, \tilde{U}_0, T_0, \) and \( \tilde{T}_0 \) one can consider the following cases:

1. \( \Gamma_{2N} \) is isometric (co-isometric) for some \( N \),
2. \( \Gamma_{2N+1} \) is isometric (co-isometric) for some \( N \),
3. the operator \( \Gamma_{2N} \) is unitary for some \( N \),
4. the operator \( \Gamma_{2N+1} \) is unitary for some \( N \).

In the following we consider all these situations and will give the forms of truncated CMV matrices. We use the sub-matrices defined by (4.3) and (4.4).

A.1. \( \Gamma_{2N} \) is isometric. Define
\[ \tilde{\mathcal{N}}_0 = \mathcal{D}_{\Gamma_0} \oplus \mathcal{D}_{\Gamma_0} \oplus \ldots, \quad \text{if } N = 0, \]
\[ \tilde{\mathcal{N}}_0 = \bigoplus_{n=0}^{N-1} \mathcal{D}_{\Gamma_{2n}} \oplus \mathcal{D}_{\Gamma_{2n}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{2n}} \oplus \ldots, \]
\[ \tilde{\mathcal{N}}_0 = \bigoplus_{n=0}^{N-1} \mathcal{D}_{\Gamma_{2n}} \oplus \mathcal{D}_{\Gamma_{2n}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{2n}} \oplus \ldots, \quad N \geq 1. \]

Define the unitary operators
\[ \mathcal{M}_0 = I_{\mathcal{M} \oplus \tilde{\mathcal{N}}_0}, \quad \tilde{\mathcal{M}}_0 = I_{\mathcal{M} \oplus \tilde{\mathcal{N}}_0}, \quad N = 0, \]
\[ \mathcal{M}_0 = I_{\mathcal{M} \oplus \bigoplus_{n=1}^N \mathcal{J}_{\Gamma_{2n-1}}} \oplus I_{\mathcal{D}_{\Gamma_{2N}}} \oplus I_{\mathcal{D}_{\Gamma_{2N}}} \oplus \ldots : \mathcal{M} \oplus \tilde{\mathcal{N}}_0 \to \mathcal{M} \oplus \tilde{\mathcal{N}}_0 \]
\[ \tilde{\mathcal{M}}_0 = I_{\mathcal{M} \oplus \bigoplus_{n=1}^N \mathcal{J}_{\Gamma_{2n-1}}} \oplus I_{\mathcal{D}_{\Gamma_{2N}}} \oplus I_{\mathcal{D}_{\Gamma_{2N}}} \oplus \ldots : \mathcal{N} \oplus \tilde{\mathcal{N}}_0 \to \mathcal{N} \oplus \tilde{\mathcal{N}}_0, \quad N \geq 1. \]

The unitary operator \( \mathcal{L}_0 : \mathcal{M} \oplus \tilde{\mathcal{N}}_0 \to \mathcal{N} \oplus \tilde{\mathcal{N}}_0 \) is defined as follows
\[ \mathcal{L}_0 = \begin{cases}
\mathcal{J}_{\Gamma_0} \oplus I_{\mathcal{D}_{\Gamma_0}} \oplus I_{\mathcal{D}_{\Gamma_0}} \oplus \ldots, & \text{if } N = 0, \\
\mathcal{J}_{\Gamma_0} \oplus \bigoplus_{n=1}^{N-1} \mathcal{J}_{\Gamma_{2n}} \oplus I_{\mathcal{D}_{\Gamma_{2N}}} \oplus I_{\mathcal{D}_{\Gamma_{2N}}} \oplus \ldots, & \text{if } N = 1, \\
\mathcal{J}_{\Gamma_0} \oplus \bigoplus_{n=1}^{N-1} \mathcal{J}_{\Gamma_{2n}} \oplus I_{\mathcal{D}_{\Gamma_{2N}}} \oplus I_{\mathcal{D}_{\Gamma_{2N}}} \oplus \ldots, & \text{if } N \geq 2.
\end{cases} \]
Define $U_0 = \mathcal{L}_0 \mathcal{M}_0, \tilde{U}_0 = \tilde{\mathcal{M}}_0 \mathcal{L}_0$. In particular, if the operator $\Gamma_0$ is isometric, then

$$
U_0 = \tilde{U}_0 = \begin{bmatrix}
\Gamma_0 & I_{\mathcal{D}_{\Gamma_0}} & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & I_{\mathcal{D}_{\Gamma_0}} & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & I_{\mathcal{D}_{\Gamma_0}} & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & I_{\mathcal{D}_{\Gamma_0}} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}.
$$

The block operator truncation CMV matrices $T_0$ and $\tilde{T}_0$ are the products (for $N \geq 1$):

$$
T_0 = \left(-\Gamma_0^* \oplus \left(\bigoplus_{n=1}^{N-1} J_{\Gamma_2} \right) \oplus J_{\Gamma_2}^{(r)} \oplus I_{\mathcal{D}_{\Gamma_2}} \oplus I_{2n} \oplus I_{2n} \oplus \ldots \right)
\begin{pmatrix}
0 & N_{1,1} & \ldots & \\
N_{2,1} & 0 & \ldots & \\
\vdots & \vdots & \ddots & \\
N_{N,1} & \ldots & 0 & 0
\end{pmatrix}
\left(-\Gamma_0^* \oplus \left(\bigoplus_{n=1}^{N-1} J_{\Gamma_2} \right) \oplus J_{\Gamma_2}^{(r)} \oplus I_{\mathcal{D}_{\Gamma_2}} \oplus I_{2n} \oplus I_{2n} \oplus \ldots \right)
$$

Calculations give

$$
S_{N-1} = \begin{bmatrix}
0 & 0 & \ldots & \\
\vdots & \vdots & \ddots & \\
0 & 0 & \ldots & \\
I_{\mathcal{D}_{\Gamma_2}} & 0 & \ldots & \\
0 & I_{\mathcal{D}_{\Gamma_2}} & 0 & \ldots & \\
0 & 0 & I_{\mathcal{D}_{\Gamma_2}} & 0 & \ldots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix},
\tilde{S}_{N-1} = \begin{bmatrix}
0 & 0 & \ldots & \\
\vdots & \vdots & \ddots & \\
0 & 0 & \ldots & \\
D_{\Gamma_2} & 0 & \ldots & \\
0 & 0 & \ldots & \\
0 & I_{\mathcal{D}_{\Gamma_2}} & 0 & \ldots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}.
$$

A.2. $\Gamma_{2N}$ is co-isometric. Then $\Gamma_n = 0$, $\mathcal{D}_{\Gamma_n} = \mathcal{D}_{\Gamma_{2N}}$, $D_{\Gamma_n} = I_{\mathcal{D}_{\Gamma_{2N}}}$ for $n > 2N$. Define

$$
\mathcal{H}_0 = \mathcal{H}_0 = \bigoplus_{n=0}^{\infty} \mathcal{D}_{\Gamma_n}, \text{ if } N = 0,
$$

$$
\mathcal{H}_0 = \left(\bigoplus_{n=0}^{N-1} \mathcal{D}_{\Gamma_2} \right) \oplus \mathcal{D}_{\Gamma_2} \oplus \mathcal{D}_{\Gamma_2} \oplus \ldots \oplus \mathcal{D}_{\Gamma_2} \oplus \ldots,
$$

$$
\mathcal{H}_0 = \left(\bigoplus_{n=0}^{N-1} \mathcal{D}_{\Gamma_2} \right) \oplus \mathcal{D}_{\Gamma_2} \oplus \mathcal{D}_{\Gamma_2} \oplus \ldots \oplus \mathcal{D}_{\Gamma_2} \oplus \ldots, \text{ if } N \geq 1.
$$
Define the unitary operators $L_0 : \mathcal{H} \oplus \tilde{\mathcal{H}}_0 \to \mathcal{H} \oplus \tilde{\mathcal{H}}_0$, $M_0 : \mathcal{H} \oplus \tilde{\mathcal{H}}_0 \to \mathcal{H} \oplus \tilde{\mathcal{H}}_0$, and $\tilde{M}_0 : \mathcal{H} \oplus \tilde{\mathcal{H}}_0 \to \mathcal{H} \oplus \tilde{\mathcal{H}}_0$ as follows

\[
L_0 = J^{(c)}_0 \oplus I_{D_{T_0}} \oplus I_{D_{T_0}} \oplus \cdots \text{ if } N = 0,
\]
\[
L_0 = J_0 \oplus \left( \oplus_{n=0}^{N-1} J_{T_2n} \oplus J^{(c)}_2 \oplus I_{D_{T_2N}} \oplus I_{D_{T_2N}} \oplus \cdots \right) \text{ if } N \geq 1,
\]
\[
M_0 = I_{2N} \oplus I_{D_{T_0}} \oplus I_{D_{T_0}} \oplus \cdots \text{ if } N = 0,
\]
\[
M_0 = I_{2N} \oplus \left( \bigoplus_{n=1}^N J_{T_2n} \right) \oplus I_{D_{T_2N}} \oplus I_{D_{T_2N}} \oplus \cdots \text{ if } N \geq 1,
\]
\[
\tilde{M}_0 = I_{2N} \oplus \left( \bigoplus_{n=1}^N J_{T_2n} \right) \oplus I_{D_{T_2N}} \oplus I_{D_{T_2N}} \oplus \cdots \text{ if } N \geq 1.
\]

Finally define $U_0 = L_0 M_0$ and $\tilde{U}_0 = \tilde{M}_0 L_0$.

In particular, if the operator $G_0$ is co-isometric, then $\tilde{\mathcal{H}}_0 = \tilde{\mathcal{H}}_0 = \mathcal{D}_{G_0} \oplus \mathcal{D}_{G_0} \oplus \cdots$,

\[
M_0 = I_{2N} \oplus \mathcal{H}_0, \quad \tilde{M}_0 = I_{2N} \oplus \tilde{\mathcal{H}}_0, \quad L_0 = J^{(c)}_0 \oplus I_{D_{T_0}} \oplus I_{D_{T_0}} \oplus \cdots,
\]

\[
U_0 = \tilde{U}_0 = \begin{bmatrix}
\Gamma_0 & 0 & 0 & 0 & \cdots \\
D_{T_0} & 0 & 0 & 0 & \cdots \\
0 & I_{D_{T_0}} & 0 & 0 & \cdots \\
0 & 0 & I_{D_{T_0}} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

If $N \geq 1$, then truncated CMV matrices $T_0$ and $\tilde{T}_0$ are of the form

\[
T_0 = \begin{bmatrix}
S_{N-1} & 0 \\
0 & D_{T_2N} D_{T_2N-1} & -D_{T_2N} & \cdots & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots & D_{T_2N} & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

\[
\tilde{T}_0 = \begin{bmatrix}
\tilde{S}_{N-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & D_{T_2N} \cdots & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & D_{T_2N} \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

A.3. $\Gamma_{2N+1}$ is isometric. In this case $\Gamma_n = 0$, $D_{T_2n} = D_{T_2N+1}$, $D_{T_2n} = D_{T_2N+1}$ for $n > 2N + 1$. Define

\[
\mathcal{H}_0 = \left( \bigoplus_{n=0}^N \mathcal{D}_{T_2n} \right) \oplus \mathcal{D}_{T_2N+1} \oplus \mathcal{D}_{T_2N+1} \oplus \cdots \oplus \mathcal{D}_{T_2N+1} \oplus \cdots,
\]

\[
\tilde{\mathcal{H}}_0 = \mathcal{D}_{T_2n} \oplus \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_1} \oplus \cdots \mathcal{D}_{T_1} \oplus \cdots, \text{ if } N = 0,
\]

\[
\tilde{\mathcal{H}}_0 = \left( \bigoplus_{n=0}^{N-1} \mathcal{D}_{T_2n} \right) \oplus \mathcal{D}_{T_2N} \oplus \mathcal{D}_{T_2N+1} \oplus \cdots \oplus \mathcal{D}_{T_2N+1} \oplus \cdots, \text{ if } N \geq 1.
\]
Define the unitary operators

\[
\mathcal{M}_0 = I_{3\mathbb{N}} \bigoplus J^{(r)}_{\Gamma_1} \oplus I_{D_{\Gamma_1}} \oplus I_{D_{\Gamma_1}} \oplus \ldots (N = 0),
\]

\[
\mathcal{M}_0 = I_{3\mathbb{N}} \bigoplus \left( \bigoplus_{n=1}^{N} J_{\Gamma_{2n}} \right) \bigoplus J^{(r)}_{\Gamma_{2N+1}} \oplus I_{D_{\Gamma_{2N+1}}} \oplus I_{D_{\Gamma_{2N+1}}} \oplus \ldots (N \geq 1),
\]

\[
\mathcal{L}_0 = J_{\Gamma_0} \oplus I_{D_{\Gamma_1}} \oplus I_{D_{\Gamma_1}} \oplus \ldots (N = 0),
\]

\[
\mathcal{L}_0 = J_{\Gamma_0} \bigoplus \left( \bigoplus_{n=1}^{N} J_{\Gamma_{2n}} \right) \bigoplus I_{D_{\Gamma_{2N+1}}} \oplus I_{D_{\Gamma_{2N+1}}} \oplus \ldots (N \geq 1),
\]

\[
\bar{\mathcal{M}}_0 = I_{3\mathbb{N}} \bigoplus \left( \bigoplus_{n=1}^{N} J^{(r)}_{\Gamma_{2n}} \right) \bigoplus J^{(r)}_{\Gamma_{2N+1}} \oplus I_{D_{\Gamma_{2N+1}}} \oplus I_{D_{\Gamma_{2N+1}}} \oplus \ldots (N \geq 1).
\]

Define \( U_0 = L_0M_0 \) and \( \bar{U}_0 = \bar{M}_0\bar{L}_0 \).

If the operator \( \Gamma_1 \) is isometric, then

\[
\mathcal{H}_0 = \mathcal{D}_{\Gamma_0} \bigoplus \mathcal{D}_{\Gamma_1} \bigoplus \mathcal{D}_{\Gamma_1} \bigoplus \ldots \mathcal{D}_{\Gamma_1} \bigoplus \ldots,
\]

\[
\bar{\mathcal{H}}_0 = \mathcal{D}_{\Gamma_0} \bigoplus \mathcal{D}_{\Gamma_1} \bigoplus \mathcal{D}_{\Gamma_1} \bigoplus \ldots \mathcal{D}_{\Gamma_1} \bigoplus \ldots,
\]

\[
U_0 = \begin{bmatrix}
\Gamma_0 & D_{\Gamma_0} & 0 & 0 & 0 & 0 & \ldots \\
D_{\Gamma_0} & -\Gamma_0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},
\]

\[
\bar{U}_0 = \begin{bmatrix}
\Gamma_0 & D_{\Gamma_0} & 0 & 0 & 0 & 0 & \ldots \\
\Gamma_1 & D_{\Gamma_0} & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.
\]

If \( N \geq 1 \), then truncated CMV matrices \( \mathcal{T}_0 \) and \( \bar{T}_0 \) take the form

\[
\mathcal{T}_0 = \begin{bmatrix}
S_{N-1} & 0 & 0 & 0 & \ldots \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & \ddots \\
0 & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

\[
-D_{\Gamma_{2N}}D_{\Gamma_{2N-1}} \begin{bmatrix}
D_{\Gamma_{2N}}^* & -D_{\Gamma_{2N}}^* & \Gamma_{2N+1} & D_{\Gamma_{2N+1}} & 0 & \ldots \\
D_{\Gamma_{2N}} & -\Gamma_{2N+1} & -\Gamma_{2N} & 0 & 0 & \ldots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}.
\]
\[ \mathcal{U}_0 = \mathcal{L}_0 \mathcal{M}_0 \quad \text{and} \quad \mathcal{U}_0 = \mathcal{M}_0 \mathcal{L}_0. \]

If \( \Gamma_1 \) is co-isometric, then

\[
\mathcal{U}_0 = \begin{bmatrix}
\Gamma_0 \\
D_{\Gamma_0} G_{\Gamma_1} \\
0 \\
0 \\
0 \\
0 \\
\vdots \\
\end{bmatrix}, \quad \mathcal{T}_0 = \begin{bmatrix}
\Gamma_0 \\
D_{\Gamma_0} G_{\Gamma_1} \\
0 \\
0 \\
0 \\
0 \\
\vdots \\
\end{bmatrix}
\]

If \( N \geq 1 \), then truncated CMV matrices \( \mathcal{T}_0 \) and \( \mathcal{U}_0 \) in this case take the form
\[ T_0 = \begin{bmatrix} S_{N-1} & 0 & 0 & \cdots & 0 \\ D_{r2N}D_{r2N-1} & -D_{r2N}^*G_{2N-1}^* & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \]

\[ \tilde{T}_0 = \begin{bmatrix} S_{N-1} & 0 & 0 & \cdots & 0 \\ -G_{2N+1}^*G_{2N}^* & 0 & 0 & \cdots & 0 \\ -D_{r2N+1}G_{2N+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \]

A.5. \( \Gamma_{2N} \) is unitary. In this case

\[ \mathcal{S}_0 = \bigoplus_{n=0}^{N-1} \mathcal{D}_{r2n} = \mathcal{G}_{2n}, \quad \mathcal{S}_0 = \bigoplus_{n=0}^{N-1} \mathcal{D}_{r2n} = \mathcal{G}_{2n}, \]

\[ \mathcal{U}_0 = (J_{r0} \oplus J_{r2} \oplus \cdots \oplus J_{r2(N-1)} \oplus \Gamma_{2N}) \times (I_{r0} \oplus J_{r1} \oplus \cdots \oplus J_{r2N-1}), \]

\[ \tilde{\mathcal{U}}_0 = (I_{r0} \oplus J_{r1} \oplus \cdots \oplus J_{r2N-1}) \times (J_{r0} \oplus J_{r2} \oplus \cdots \oplus J_{r2(N-1)} \oplus \Gamma_{2N}). \]

If \( N \geq 1 \), then

\[ T_0 = S_{N-1}, \quad \tilde{T}_0 = S_{N-1}. \]

A.6. \( \Gamma_{2N+1} \) is unitary. Then

\[ \mathcal{S}_0 = \mathcal{D}_{r0} = \mathcal{D}_{r0}, \quad \tilde{\mathcal{S}}_0 = \bigoplus_{n=0}^{N-1} \mathcal{D}_{r2n+1} = \mathcal{G}_{2n+1}, \]

\[ \mathcal{U}_0 = (I_{r0} \oplus J_{r1} \oplus \cdots \oplus J_{r2N}) \times (I_{r0} \oplus J_{r1} \oplus \cdots \oplus J_{r2N-1} \oplus \Gamma_{2N+1}), \]

\[ \tilde{\mathcal{U}}_0 = (I_{r0} \oplus J_{r1} \oplus \cdots \oplus J_{r2N-1} \oplus \Gamma_{2N+1}) \times (J_{r0} \oplus J_{r2} \oplus \cdots \oplus J_{r2N}), \quad N \geq 1. \]

\[ \mathcal{U}_0 = \begin{bmatrix} \Gamma_0 & D_{r0}^* \\ D_{r0} & -\Gamma_0^* \end{bmatrix} = \begin{bmatrix} \Gamma_0 & D_{r0}^* \Gamma_1 \\ D_{r0} & -\Gamma_0^* \Gamma_1 \end{bmatrix}, \]

\[ \tilde{\mathcal{U}}_0 = \begin{bmatrix} I_{r1} & 0 \\ 0 & \Gamma_1 \end{bmatrix} = \begin{bmatrix} \Gamma_0 & D_{r0}^* \Gamma_1 \\ \Gamma_1 & -\Gamma_0^* \Gamma_1 \end{bmatrix} \]

if \( N = 0 \).
In this case if $N \geq 1$, then

$$\mathcal{T}_0 = \begin{bmatrix} 0 & \tilde{S}_{N-1} & 0 & \cdots & 0 \\ D_{12N}^* & D_{12N-1} & -D_{12N}^* & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{S}_1 \end{bmatrix}.$$

In particular if $N = 1$ ($\Gamma_3$ is unitary), then

$$\mathcal{U}_0 = \begin{bmatrix} \Gamma_0 & D_{12}^* \Gamma_1 & D_{12}^* & 0 \\ \Gamma_0 & -\Gamma_2^* \Gamma_1 & -\Gamma_2^* & 0 \\ 0 & \Gamma_2 \Gamma_1^* & \Gamma_2 \Gamma_1 & \Gamma_3 \\ 0 & 0 & -\Gamma_2^* \Gamma_1 & \Gamma_3 \\ 0 & 0 & 0 & \Gamma_3 \end{bmatrix}.$$

$$\tilde{\mathcal{U}}_0 = \begin{bmatrix} \Gamma_0 & D_{12}^* & 0 & 0 \\ \Gamma_0 & -\Gamma_1^* \Gamma_0 & D_{12}^* \Gamma_2 & D_{12}^* \Gamma_2^* \\ \Gamma_0 & -\Gamma_1^* \Gamma_0 & \Gamma_3 \Gamma_2 & -\Gamma_3 \Gamma_2^* \\ 0 & 0 & \Gamma_3 \Gamma_2 & \Gamma_3 \Gamma_2^* \\ 0 & 0 & 0 & \Gamma_3 \end{bmatrix}.$$

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