Possible dynamics of Josephson junction arrays connected to high-Q tank circuit

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Abstract. Serial Josephson junction arrays connected to high-Q tank circuit are analysed and discussed. Numerical simulation of the systems shows two possible oscillation modes above resonance frequency when the tank impedance is capacitive. These are the inphase oscillation mode and the collectively antiphase oscillation mode. This fact is responsible for complicate switching dynamics, which can block parametric resonance excitation. Stability domains of the modes depend on McCumber parameter of the junctions. Increase in this parameter is favourable to the inphase mode. When the inphased oscillations are feasible, the resonance peak-like peculiarities on IV curve and Shapiro steps can be described with the analytic theory derived earlier for one-junction case and extended over the array systems.

1. Introduction
Using Josephson junction arrays instead of single junctions can be beneficial for many applications [1-3] including the ones when the array has to be connected to high-Q tank circuit. Besides, similar natural structures such as high-temperature superconductor mesas with stacks of intrinsic Josephson-junctions are of great research interest [4-9]. For a start, one can restrict oneself to lumped Josephson elements and use equivalent circuit shown in Fig. 1, where the external microwave irradiation is described with rf current source.

Josephson junction is a unique device with properties of both the strongly nonlinear reactive element (due to superconducting current component) and the active device generating ac voltage in resistive state [2, 10, 11]. Therefore, both the force and parametric resonances can be observed, when

![Fig. 1. Equivalent circuit describing serial array of Josephson junction coupled to a high-quality parallel resonance circuit (framed by dotted line) under external rf signal. Here $C_0$ is a capacitor block, $I$ and $\tilde{I}$ are dc bias current and rf current, respectively.](image)
Josephson junction is connected to high-Q tank circuit [11-13]. The resonances manifest themselves in autonomous state (when no external rf signal is applied) through sharp peak-like peculiarities on IV curve as shown in Fig. 2a (calculated in [13]) at \( \omega = \omega_0 \equiv \Omega_0 / \Omega_C = 0.25 \), where \( \Omega_0 \) is resonance frequency and \( \Omega_C \equiv 2 \pi V_C / \Phi_0 \) is characteristic Josephson oscillation frequency; \( V_C = I_c R_N \) is characteristic voltage of the junction with critical current \( I_c \) and normal resistance \( R_N \).

The only main parametric resonance (at Josephson oscillation frequency \( \omega = 2 \omega_0 \)) can be softly excited, however, when \( \omega_0 < 0.5 \) [11-13]. The height of the resonance peak shows bell-shape-like dependence on \( \omega_0 \) as shown in Fig. 2b. For higher subharmonics (\( \omega = m \omega_0 \), \( m > 2 \)), the soft excitation is not possible at all although the rigid excitation is possible, however, at lower \( \omega_0 \) (the upper allowable value of \( \omega_0 \) decreases with \( m \)).

If external rf signal with frequency \( \omega \) close to \( \omega_0 \) is applied to the system, phase-locking effect manifests itself through set of big Shapiro steps [14] on IV curve at \( V / V_C = n \omega \) even at low amplitude of the applied microwave signal as shown for example in Fig. 3 (see also [10-13]).

When Q-factor is high enough, both the force and parametric resonances and Shapiro steps at \( \omega \) close to \( \omega_0 \) can be analyzed analytically [12, 13].

In this paper, we consider the distinctions, which appear when a serial array of Josephson junctions is connected to high-Q tank circuit. Due to additional degrees of freedom, the system shows much more complicated dynamics as compared to single junction. Beyond resonance regions, Josephson junctions can oscillate not only in phase, but also with some difference in their phases. This fact leads to more complicated behavior of the array systems as compared to the single-junction one.

**Fig. 2.** (a) IV curve of single Josephson junction connected to a high-quality tank circuit with resonance frequency \( \omega_0 = 0.25 \), \( Q \approx 10 \) in autonomous state, when no external rf signal is applied, as calculated in [13]. (b) Dependence of the parametric resonance peak height (solid line) and amplitude \( x_2 \) of the oscillating phase component on resonance frequency \( \omega_0 \) [13].

**Fig. 3.** The big first Shapiro step (bold red line) formed on the force resonance peak in IV curve of the junction connected to high-Q tank circuit at small detuning of the external rf signal frequency from the resonance frequency \( \omega_0 = 0.5 \). The small steps at lower frequencies \( \omega \approx 0.225 \) and \( \omega \approx 0.15 \) are subharmonic 1/2 and 1/3 Shapiro steps.
2. Analytic theory for in-phase mode of the junction array

When it is supposed that the in-phase oscillation mode holds, both the resonance peculiarities in the array IV-curve and Shapiro steps caused by an external signal with frequency \( \omega \approx \omega_0 \) can be calculated using analytic theory developed earlier for one Josephson junction connected to the high-Q tank circuit [13]. In fact, each of the junctions oscillating in-phase should obey the same equation coinciding with the one for one-junction system, but with some change in resonance frequency (yielded from the junction capacitances).

As it did in [13], one can use Resistively Shunted Josephson-junction (RSJ) model [2, 10] and standard dimensionless quantities: electric current \( i = I/I_c \), voltage \( v = V/V_c \equiv d\phi/d\tau \), time \( \tau = \Omega_c t \), frequency \( \omega = \Omega/\Omega_c \), McCumber parameter \( \beta = \Omega_c R_N C_j = 2\pi I_c R_N^2 C_j/\Phi_0 \) (normalized junction capacitance \( C_j \)), normalized value of the tank capacitance \( (C) \), block capacitance \( (C_b) \) and tank inductance \( (L) \), \( \beta_e = \Omega_c R_N C, \beta_{e0} = \Omega_c R_N C_0, I_c = 2\pi I_c/\Phi_0 \), where \( I_c \) is Josephson junction critical current, \( V_c = I_c R_N \) is characteristic voltage, \( R_N \) is Josephson junction normal resistance, \( \Omega_c = 2\pi I_c R_N/\Phi_0 \) is characteristic frequency, \( \Phi_0 = \hbar/2e \) is magnetic flux quantum, and \( \phi \) is Josephson junction phase.

In case of the high-Q circuit, voltage \( \varphi \) across the array can contain only two components which are dc voltage \( v \) and a sinusoidal voltage component oscillating with frequency \( \omega \) close to the resonance one \( (\omega_0) \). The oscillating component can be induced by the running Josephson phase or by an external signal (if applied) with frequency \( \omega \approx \omega_0 \). Therefore, the phase and voltage across each junction in the in-phased oscillation state can be written as follows:

\[
\varphi = v + x \sin(\omega t) + \chi, \quad (1)
\]

\[
\dot{\varphi} = \dot{v} + x \omega \cos(\omega t), \quad (2)
\]

where \( v \) is dc voltage, \( x \) is amplitude of the ac component, and \( \chi \) is a phase constant.

Next, taking into account an applied ac current

\[
\dot{I} = b \cos(\omega t - \gamma), \quad (3)
\]

where \( \gamma \) is phase constant, and repeating the calculations given in [13], one can come to resulting equation set:

\[
i \equiv n\omega + i_n, \quad (4)
\]

\[
i_n = (-1)^n j_n(x) \sin(\chi), \quad (5)
\]

\[
x\omega + (-1)^{n+1} [ j_{n-1}(x) + j_{n+1}(x)] \sin(\chi) = b \cos(\gamma), \quad (6)
\]

\[
-2Q \xi x \omega_0 + (-1)^{n+1} [ j_{n-1}(x) - j_{n+1}(x)] \cos(\chi) = b \sin(\gamma), \quad (7)
\]

where \( \xi = (\omega - \omega_0)/\omega_0 \) is frequency detuning, and \( j_n(\cdot) \) is Bessel functions of the first kind of order \( n \). Both the shift in resonance frequency and resulting quality factor \( Q \) depend on number \( N \) of Josephson junctions:

\[
\omega_0 = [(\beta_e + \beta/N)I_c]^{-1/2}, \quad (8)
\]

\[
Q = N[(\beta_e + \beta/N)/I_c]^{1/2}. \quad (9)
\]

Beyond peculiarities, IV-curve approaches linear law (at \( Q \gg 1 \)):

\[
v = i, \quad (10)
\]

where \( i \) is dc bias current. The peculiarities (resonance peaks and Shapiro steps) arised at \( v = n\omega \) are described by dc current \( i_n \).

At \( b = 0 \), equation set (4)-(7) describes both the force and parametric resonance peculiarities, while at \( b \neq 0 \) these equations describe Shapiro steps \( (v = n\omega, n = 1, 2, 3, \ldots) \). To find heights of possible resonance peaks in IV-curve and resonance phase amplitude \( x \) in free-running mode, one ought to set \( \omega = \omega_0, \xi = 0 \) at \( b = 0 \). Then, it becomes evident that angle \( \chi \) should be set equal to \( -\pi/2 \) at odd number \( n \) to meet \( \cos(\chi) = 0, \sin(\chi) = -1 \) or equal to \( +\pi/2 \) at even number \( n \) to meet \( \cos(\chi) = 0, \sin(\chi) = +1 \). Thus, one comes to the following equations, which allow deriving both the peak height and resonance amplitude of the phase oscillations:
\[ x\omega_0 = f_{n-1}(x) + f_{n+1}(x), \quad (11) \]
\[ (i_n)_{\text{res}} = f_n(x). \quad (12) \]

At \( n = 1 \), these equations describe standard force resonance caused by \textit{ac} superconducting current component applied to the parallel oscillatory circuit, while at \( n = 2, 3, \ldots \) the equation set describes parametric resonance oscillations supported through parametric mechanism of the energy delivering. As in case of the single-junction system, the parametric resonance oscillations can be softly excited only at \( n = 2 \) and \( \omega_0 < 0.5 \) within some small range of the frequency detuning \( \xi \):

\[ (1 + 4Q^2\xi^2)\omega_0 < 0.5, \quad (13) \]

while at \( n > 2 \) they can be obtained only with rigid excitation [13].

3. Numerical simulation

As a rule, resonance peculiarities have to be approached (in simulation and experimental study) with decrease in the Josephson oscillation frequency by decrease in the bias current. In fact, the array connected to high-\( Q \) LC circuit switches from superconductive branch in IV-curve to the resistive state where Josephson oscillations frequency is much higher than \( \omega_0 \) and \( 2\omega_0 \), for example, as shown in Fig. 2a, and hence the array gets loaded with capacitive impedance \( z(\omega) \approx (i\omega\beta_e)^{-1} \). Therefore, to understand better the system behavior, numerical simulation was performed for two cases when the arrays of two to four Josephson junctions connected to a single capacitor \( C \) having normalized value \( \beta_e = 40 \) and to the parallel \( LC \)-circuit (see Fig. 1) with the same capacitance, resonance frequency \( \omega_0 = 0.25 \) (allowing parametric resonance excitation), and quality factor \( Q \approx 10 \).

3.1. Two oscillation modes

Numerical simulation shows that IV-curves of the series Josephson junction arrays connected to a capacitor can have two resistive branches as shown in Fig. 4 for two- and three-junction arrays, where the junctions oscillate in phase or in antiphase, respectively. The phase modes are locked with corresponding \textit{ac} current flowing through the external capacitor \( C \) and being looped through the array. In case of the in-phase mode, the locking current has the fundamental tone frequency \( \omega \) and amplitude close to the critical current value at \( \beta_e \gg 1 \), while the voltage amplitude is very small. The other mode is shown in Fig. 5 for the arrays of two junctions (purely antiphased oscillations) and three junctions (collectively antiphased oscillations with phase shift \( 2\pi/3 \)). In this case, fundamental voltage harmonics of the junction oscillations do not “see” the connected capacitor and therefore amplitude of

![Fig. 4. IV-curves of the two- and three- junction arrays (left and right, respectively) connected to capacitor \( C \) with normalized value \( \beta_e = 40 \) at different intrinsic capacitances of the junctions: \( \beta = 0, 0.01, 0.02, 0.05, 0.1 \). Resistive branches A and B correspond oscillation modes in phase and anti-phase, respectively.](image-url)
the voltage oscillations keep high ($V/V_c \sim 1$) although their sum (voltage across the array) is of small amplitude. The ac current locking such a mode has harmonic frequency ($2\omega$ in the two-junction array or $3\omega$ in the three-junction array). As far as decrease in Josephson oscillation frequency is favorable for the harmonic component existence, stability of the antiphase mode is higher at lower frequencies. In line with the fact, increase of the junction capacitances reduces upper frequency of the antiphase mode occurrence and extend frequency domain of the other, inphase, mode down to much lower frequencies, although the extension is nonmonotonic. After rapid drop with $\beta$ at small values of the parameter (as can see from Fig. 4), the lower frequency margin shows some return at $\beta \sim 1$ and then permanent reducing with $\beta$.

3.2. Resonance excitation and missing

When a serial array of Josephson junctions is connected to a high-Q $LC$-circuit, the existence of two oscillation modes can lead to intricate switching dynamics and even to missing the softly excited parametric resonance. The latter is evident from Fig. 6 presenting IV-curves for the serial arrays consisting of three and four overdamped ($\beta = 0$) Josephson junctions. In both cases, this resonance peak (at $\omega = 2\omega_0 = 0.5$) is not observed. Being in superconducting state, the system switches always to the antiphase oscillation mode incapable of providing the parametric resonance excitation. This can
be evident from the fact that frequency $\omega \equiv v = 0.5$ is achieved at the bias current value $i > 1$, which is higher than the vertex position of the implicit resonance current peak $i = 2\omega_0 + J_2(x) \approx 0.96$ given by (4) and (5). As for the inphase mode, its stability gets broken at frequency much higher than $2\omega_0$ and the only antiphase mode can exist below the cut-off frequency. In both IV-curves, we can see switches to the force resonance peak with decrease in bias current, however, in different ways. In case of the 4-junction array, its resistive branch goes down and achieves resonance voltage $v = \omega_0 = 0.25$, but at the bias current value exceeding vertex of the force resonance peak. Therefore, the oscillations excited in the tank circuit cause the system to switch at first to the rigidly excited resonance state at $\omega = (3/2)\omega_0$ and then to the force resonance peak with the further decrease of the bias current. Somewhat different scenario can be seen for 3-junction array. Its resistive branch bends with the bias current decrease and does not approach the resonance voltage. The curvature results from changes in the junction oscillation dynamics with the bias current decrease when at first a 1/2 subharmonic component appears (knee at $i \approx 1.2$) and then some chaotic-like beats start (knee at $i \approx 0.9$). When stability margin of the oscillation mode is reached, the system switches to the force resonance peak.

Fig. 7 shows IV-curves of the 4- and 3-junction arrays at the junction McCumber parameter $\beta = 1$, when the upper possible frequency $\omega$ of the antiphase mode is less than 1, and the inphase mode frequency domain extends down to 0.76 and 0.8, respectively. Both systems switch with the bias current decrease from inphase mode firstly to their antiphase modes with frequency close to $2\omega_0$. In case of the 4-junction array, this frequency enters the narrow range of the parametric resonance excitation and therefore the running resonance excitation resets the junction oscillations into inphase mode in the resonance peculiarity in IV-curve. This process does not occur in the 3-junction system after switching into antiphase mode, since the obtained oscillation frequency does not enter the narrow resonance excitation range. Therefore, the antiphase mode keeps in some range of bias current and then the system switches to the force resonance peculiarity when stability margin of this mode is reached with decrease in bias current. A back switch from the force resonance peak goes always to the other, parametric, resonance peak due to the resonance oscillations existing in LC-circuit.

4. Conclusion

Two different modes of the Josephson junction oscillations in the serial arrays connected to a capacitor and to a high-Q resonator were studied by numerical simulation of the arrays of two to four junctions. Existence of the inphase and antiphase modes in the array systems with high-Q tank circuits
leads to their complicate behavior resulting in hysteretic loops in IV-curves, existence and missing some resonance peculiarities. Intrinsic junction capacitances strongly influence frequency domains of the two modes and hence behavior and resulting characteristics of the systems.

In the case when inphase mode is induced, including case when the state is achieved with application of an external rf signal with frequency close to the resonance one, both the force and parametric resonance peaks in IV curve, as well as Shapiro steps can be described using analytic theory which was derived earlier for one-junction system and has been extended over the multi-junction systems.

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