Lumped Mass Model for Flexible Cable: A Review

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Abstract. Model of a flexible cable plays a great role in the applications of underwater towing system, mooring line system, etc. A complete mathematical model is a basis to accurately describe the dynamic characteristics of flexible cable. This article presents a state of art comprehensive review of lumped mass model. First, the mathematical principle of the lumped mass model, including geometric simplification, force analysis, and solution, are systematically summarized. Second, the dynamic response of cable under different boundary conditions is introduced, and boundary conditions are categorized. Third, the numerical solution of the lumped mass model are assessed. Finally, based on the application of guide dog robot, the authors discussed the research future and trends. The authors aim to realize human-robot interaction through a flexible cable.

1. Introduction
Based on the author's survey, a simplified model is often adopted in analysis. In a simplified model, the flexible cable is a structure with a large slenderness ratio. Usually, it is coupled with the characteristics of small damping, high tensile strength, good flexibility, and certain load-bearing capacity under tension. Due to these features, the flexible cable has been widely applied in engineering fields. The ability to describe the dynamic properties of the flexible cable objectively and accurately is a priority in the research of flexible cable.

The dynamic modeling of the flexible cable is mainly divided into two categories: the discretization model and the continuum model. The continuum model was born in the early stage of cable dynamics research, based on the theory of continuum mechanics [1-3]. However, since this modeling method didn't take non-linear dynamic characteristics such as hysteresis, damping, and friction into consideration, it can only focus on theoretical analysis. Many scholars have begun to study the discretization of the flexible cable into segments or node units, guaranteeing a comprehensive description of the nonlinear dynamic characteristics of flexible cables. According to discrete thinking, the lumped-mass method divides the cable into several segments, which are connected by nodes with masses. Segments are massless and are considered rigid or elastic elastomers. The distributed load on the flexible body is equivalent to the nodes, which can simulate the bending of the flexible cable and describe the geometric shape with the divided line. It is suitable for the complex flexible mechanical system for it the advantages of small calculation and fast simulation speed. Nowadays, the lumped mass model has been widely applied in flexible cable systems such as underwater cable [4-6], space tether [7, 8], etc.
Lumped mass method, firstly proposed by Walton [9], is often used in the scene of flexible cable dynamic analysis for its clear physical meaning, a simple algorithm, and easy-programming. Therefore, it is necessary to review the lumped mass method objectively.

2. Mathematical Model

2.1. Simplification of Model
The flexible cable is divided into \( n \) elements, which are equivalent to the combination of spring and damper. The weight of each element is focused on both ends of the cable section, and there are \( n + 1 \) nodes in total. Except for the two ends, the lumped mass on the node is taken as the sum of half of the mass of adjacent cable segment elements. Assume that the mass of each cable element is \( m \), the lumped mass \( m_i \) can be described as:

\[
m_i = \begin{cases} 
0.5m, & i = 0, n \\
m, & i = 1, 2, ..., n - 1 
\end{cases}
\]  

(1)

![Figure 1. A lumped mass cable model arranged in the \( xyz \) frame.](image)

2.2. Force analysis of Each Node
The bottom of the node is set as the origin of the inertial reference frame \( \{x, y, z\} \). The lumped mass model assumes that the flexible cable is composed of a finite number of elements, which are connected by massless elastic elements and define the elastic force. It also assumes that all forces on the flexible cable are concentrated on these elements. According to Newton's second law, the following equations of motion for each node can be established:

\[
M_i \ddot{x}_i = F_i
\]  

(2)

Where \( \ddot{x}_i \) represents the acceleration of the \( i \)th node, \( M_i \) is the mass matrix, \( F_i \) (including external force and internal force) refers to all forces acting on the node \( i \). Among them, internal force refers to the elastic force and damping force between the nodes while external force includes buoyancy, hydrodynamic loads, etc. in the underwater application.

2.3. Solution Process
The simultaneous governing equ (2), plus the given initial conditions, and by \( v = dv/dt \), the complete partial differential equations are obtained as equ (3), then the motion state of each node could be solved by numerical solution method.
2.4. Runge-Kutta Method

To form a number of simplified node models for calculation, the flexible cable with overall mass is divided into several segments. The calculation accuracy should be improved as much as possible and the error should be reduced.

In order to improve calculation accuracy and avoid the need to calculate a large number of derivatives and high-order derivatives in other numerical methods, the Runge-Kutta method is often used, that is, using the appropriate linear combination of function values at several points to replace the derivatives or high-order derivatives at the certain point \((x_n, y_n)\). It can improve the accuracy of the calculation effectively.

The fourth-order Runge-Kutta scheme is shown as follows:

For the ordinary differential equations with the initial value (boundary value):

\[
\begin{align*}
F(x, y, y', y'') &= 0 \\
y(a) &= C_1, y'(a) = C_2
\end{align*}
\]

Set \(z = y'\),

Then the original equation is changed into:

\[
\begin{align*}
y' &= z \\
F(x, y, y', z') &= 0 \\
y(a) &= C_1, z(a) = C_2
\end{align*}
\]

When the fourth order Runge-Kutta scheme is applied to equation (3), the following results are obtained:

\[
\begin{align*}
x_{i+1} &= x_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
x'_i &= x'_i + \frac{h}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\
k_1 &= x'_i \\
l_1 &= (1 + \frac{r_i}{a})x_i + 1 \\
k_2 &= x'_i + \frac{h}{2}l_1 \\
l_2 &= [1 + (t_a + \frac{h}{2})^2](x_i + \frac{h}{2}k_1) + 1 \\
k_3 &= x'_i + \frac{h}{2}l_2 \\
l_3 &= [1 + (t_a + \frac{h}{2})^2](x_i + \frac{h}{2}k_2) + 1 \\
k_4 &= x'_i + hl_3 \\
l_4 &= [1 + (t_a + h)^2](x_i + h k_4) + 1 \\
x_{i0} &= C_1, x'_{i0} = C_2
\end{align*}
\]
3. Research Focus

3.1. Dynamic Response under Complex Conditions
The force analysis of cable nodes will inevitably be influenced by the diverse external incentives, especially in the underwater application. In a single-point mooring submerged buoy system, the hydraulic resistance in each node was calculated by the Morrison equation for establishing the dynamic model [10]. Besides, additional mass, tension, and buoyancy are usually considered. The dynamic response of the flexible cable has obtained enough attention because the impact load of the cable has a great influence on the safety of the system. It is mainly influenced by boundary conditions. The research conducted by Zhao et al. [11] indicated that the dynamic tension of deepwater cable is mainly caused by the motion of the floating body at the upper end and the mooring tension is nonlinear with the increase of current velocity. According to [12], in deep water deployment of offshore structures with time-varying deployment length, deployment speed is not the dominant factor of dynamic force. Instead, they found that the wave heights have a noticeable effect on dynamic tension where hazardous condition tends to appear as the wave height increases. To minimize the cable force effect, the two moving objects connected by the cable should move at a similar speed. In [13], the motion simulation of UUV and USV connected by underwater cable is carried out. The researchers found that when there is a velocity difference between UUV and USV, the force generated by the underwater cable will affect the X-Y plane motion.

3.2. Boundary Condition
For a flexible cable, its boundary condition mainly refers to the motion state of both ends, including velocity and displacement. In engineering applications, the motion state of the cable tends to be coupled with the moving object it connected, we called it coupling boundary conditions. The common boundary conditions of flexible cable include one fixed end and one end excited by motion. Tang et al. [9] have researched the snap tension of deepwater mooring where the lower end was fixed and the upper end was excited horizontally (Figure. 2a). It revealed that the excitation frequency and amplitude imposed a great influence on the dynamic tension of the mooring line. Xiong et al. [14] presented a mooring system with a fixed and free boundary (Figure. 2b). They considered the soil-chain interaction for the mooring line by setting the soil resistance formula, which means that a boundary condition is fixed and the tension response is related to the safety of the mooring system. In [15], for the free end without dragging the body, the end was regarded as a node directly (Figure. 2c). Besides, it is also covered that the cable with moving and free boundary: Wang et al. [16] treated cable and surface towing ship as a whole and the tension of the cable was integrated into the motion control equation as the boundary condition of dynamic coupling (Figure. 2d). The lumped mass model of the towed system connected by two moving boundary conditions (two vehicles) was established for usage in long-time-domain simulations to accurately captures the dynamics [17].

![Figure 2. Lumped mass model with different boundary conditions.](image-url)
Not only the boundary condition influences the dynamic response, but also it runs conversely. In the process of submarine cable laying, with the navigation of the construction ship, the cable has been partially laid. The length and tension of the cable can determine the boundary conditions of the submarine cable in the seabed and seawater [5].

The length of the flexible cable system in normal operation is fixed, while in the process of retraction and release, its length will change gradually with time (Figure 3). In subsea equipment lifting installation, the boundary condition was related to the lowering velocity of the cable [18]. In research [19, 20], set the velocity of the node connected to the pulley is $v_{rp}$, the cable length at any time $t$ can be determined by:

$$S = \begin{cases} 
    s + \frac{v_{rp}}{1 + \varepsilon} dt, \text{deployment} \\
    s - \frac{v_{rp}}{1 + \varepsilon} dt, \text{retrial}
\end{cases}$$

Where, the $\varepsilon$ represents the function of tension of node N and changes with time. But in absolute quantity, the strain is very small, so its influence can be ignored. The boundary condition of the end node can be expressed as:

$$\mathbf{x}_N = \mathbf{x}_{pilley}, \dot{\mathbf{x}}_N = v_{rp} \mathbf{\tau}_{rp}$$

Where, $\mathbf{\tau}_{rp}$ is the tangent vector of the cable end node. In numerical solution, the tangent vector is obtained by difference, so the tangent vector at the end node N can not be obtained directly. Instead, it can be replaced by that at the N-1/2 node due to only half a step error [21].

3.3. Numerical Solution
When solving the partial differential equation of lumped mass model, the solution accuracy should firstly be considered. Theoretically, the more the segment number is, the more accurate the calculation results should be. However, the amount of calculation will increase as the node number increase. Zhu et al. [22] have done detailed research on the influence of the segment number on tension. It revealed that the variation amplitude of top tension will improve as the nodes number increase.

Besides, the stability of the numerical solution is especially considered. The stability of the numerical solution not only depends on the control equation but also on the numerical algorithm. To obtain the stability of the difference scheme of the governing equation, the Howbolt method is used to calculate the dynamic response of the mooring line due to its unconditional stability [23]. Nowadays, fourth-order Runge-Kutta (RK4) method is the most commonly used numerical algorithm for the dynamic equation of cable lumped mass model. Wang [21, 24] proposed that when the explicit RK4 method is used to solve the equation, the allowable time step $\Delta t$ is directly proportional to the space step $\Delta s$.

The stability condition of the numerical solution was given:
Since the lumped mass model using the RK4 numerical method is conditionally stable, the time step $\Delta t$ must satisfy the stability condition and its value is usually very small, generally in ms level. However, it inevitably brings huge computation work. To solve the contradiction between model stability and huge calculation, Zhu et al. [22] proposed a simplified algorithm where the redundant computation (including gravity, buoyancy, tension, additional mass, etc.) acting on the $(i-1/2)$th node was removed and only calculation on the $(i+1/2)$th node would be run. It was proved 30% calculation time was saved compared with before optimization.

 Compared with the thin bar model, the lumped mass model has some defects in the convergence rate. For example, when a flexible cable is divided into several segments for calculation, the thin bar model may be divided into about 25 segments to converge, while the lumped mass method may be divided into 100 segments without convergence [25]. It indicates that the lumped mass model presents a relatively low calculation efficiency.

4. Conclusion

A description of the lumped mass model of the flexible cable has been elaborated in this paper. According to the discretization thinking, the flexible cable is divided into finite mass nodes, then the force analysis can be conducted on each node by Newton's second law. The differential equation will be solved by a numerical solution to capture the motion state of the cable. The focus is to study the dynamic response of the flexible cable with one end fixed and one end excited by motion as the boundary condition in a specific application such as an underwater towing system. Besides, the diverse boundary condition of different motion states has been introduced, coupled with the deployment/retrieval of cable. Finally, the numerical solution, especially the calculation stability of the RK4 method is elaborated. It is expected to elicit some inspiration in the field of the dynamic modeling of the flexible cable.

Based on the lumped mass model of the flexible cable, we are committed to the research of physical human-robot interaction based on the flexible cable, that is, the motion state of blind people’s hand and the guide dog robot is taken as the boundary condition of the cable to explore the dynamic response of the cable in different motion states of the blind people. The boundary conditions of the two moving ends have been studied based on the rolling-foot model and the dual motor drive model. It is expected to apply the lumped mass model in the field of human-robot interaction.

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