Interferometric visibility of single-lens models:
The thin-arcs approximation

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Received 13 October 2021 / Accepted 26 June 2023

ABSTRACT

Long baseline interferometry of microlensing events can resolve the individual images of the source produced by the lens, which combined with the modelling of the microlensing light curve, leads to the exact lens mass and distance. Interferometric observations thus offer a unique opportunity to constrain the mass of exoplanets detected by microlensing, and to precisely measure the mass of distant isolated objects such as stars and brown dwarfs, and of stellar remnants such as white dwarfs, neutron stars, and stellar black holes. Having accurate models and reliable numerical methods is of particular importance as the number of targets is expected to increase significantly in the near future. In this work we discuss the different approaches to calculating the fringe complex visibility for the important case of a single lens. We propose a robust integration scheme to calculate the exact visibility, and introduce a novel approximation, which we call the ‘thin-arcs approximation’, which can be applied over a wide range of lens–source separations. We find that this approximation runs six to ten times faster than the exact calculation, depending on the characteristics of the event and the required accuracy. This approximation provides accurate results for microlensing events of medium to high magnification observed around the peak (i.e. a large fraction of potential observational targets).

Key words. gravitational lensing: micro – techniques: interferometric – methods: numerical

1. Introduction

Measuring the mass of isolated objects in our Milky Way is a major challenge in astrophysics, whether it be the mass of stars, brown dwarfs, or stellar remnants such as white dwarfs, neutron stars, or stellar black holes. Few observational techniques allow such measurements to be made with high precision and/or independently of assumptions about the structure of the targeted objects. Gravitational microlensing (Paczynski 1986), based on the deflection of light rays by a lensing body transiting the line of sight of a distant star, provides the solution of choice for measuring the mass of such isolated lenses. The technique allows us to probe objects, intrinsically luminous or not, up to Galactic scales and independently of the light emitted by the lens itself.

Microlensing affects the shape and the number of images of the source star, which results in a global enhancement of the total flux received by the observer. As the individual images produced by the microlens cannot be separated by classical telescopes, what is usually measured is the increase (or magnification) in the flux of the source star as a function of time. Nevertheless, when for bright-enough microlensing events, the lensed images can in principle be resolved with long baseline interferometers (Delplancke et al. 2001; Dalal & Lane 2003; Rattenbury & Mao 2006; Cassan & Ranc 2016) since their typical separation is of the order of a milliarcsecond (i.e. within the reach of interferometers with baselines of ~40–100 metres. A first series of successful interferometric observations was recently made with the Very Large Telescope Interferometer (ESO/VLTI) on microlensing events TCP J05074264+2447555 ‘Kojima-1’ (Dong et al. 2019) and Gaia19bld (Cassan et al. 2022).

To measure the mass of the lens, two quantities must be derived from the observations. The first is \( \pi_E \), the microlensing parallax; the second is \( \theta_E \), the angular Einstein ring radius, which is the angular radius of the ring-like image of the source were it to be perfectly aligned with the lens. The mass follows from

\[
M = \theta_E/k\pi_E, \quad \text{where} \quad k = 8.144 \text{ mas}/M_\odot \quad \text{(Gould 2000)}.
\]

Ground-based observations can access \( \pi_E \) for long-lasting microlensing events, for which the transverse motion of the Earth is significant enough to allow a good parallax measurement, while for shorter microlensing events space-based parallax is in general required to provide a different vantage point from Earth. Classically, \( \theta_E \) can be estimated from the photometric light curve if the spatial extension of source reveals itself by producing noticeable deviations in the light curve and if the source star is well characterised; in the case of bright microlenses, high-resolution adaptive-optics imaging can also access \( \theta_E \) typically 5–10 yr after the microlensing event is over, when the background star and the microlens can be resolved individually. As for long baseline interferometry, it provides a direct measurement of \( \theta_E \) by resolving the split images of the source star and measuring their angular separations. An additional constraint on \( \pi_E \) can be obtained when times series observations are performed as they allow the direction of the relative motion between the lens and the source to be measured (Cassan et al. 2022).

The modelling of interferometric data requires both robust and efficient numerical methods to compute the microlensing models, with a good control on numerical errors, in order to calculate the wide range of models typically required by Markov chain Monte Carlo (MCMC) algorithms. In this work we discuss in detail the case of single-lens models. In Sect. 2, we...
propose a new and more efficient approach for the exact calculation of the complex interferometric visibility than exists in the literature; we also establish a new approximation, called the thin-arcs approximation, which runs six to ten times faster than the exact calculation, and should apply to a large fraction of potential observational targets. In Sect. 3, we illustrate and discuss the domain of validity of the thin-arcs approximation by comparing it to the exact calculation, and also to the point-source approximation. We discuss the possible shortcomings of ill-defined parametrisations, and advocate for suitable sets of parameters that depend on the characteristics of the observed microlensing event. Finally, in Sect. 4 we summarise the main results and discuss the perspectives of optical/infrared long baseline interferometric observations of microlensing events, in particular in the context of recent developments of the ESO VLTI/GRAVITY instrument.

2. Visibility of reference single-lens models

2.1. Key concepts and equations

The lens equation relates the angular position of the background source star to that of its multiple images. If we set up a Cartesian frame of reference (O, x, y) with axes fixed in the plane of the sky (e.g. north, east) and if we choose the lens to be at the centre of the coordinate system, the complex lens equation for an isolated massive body reads

\[ \zeta = z - \frac{1}{\xi}, \]

where \( \xi \) is the affix of the (point-like) centre of the source, \( z \) the affix of one of the individual point-like images, and \( \bar{z} \) the complex conjugate of \( z \). For the lens equation to be correct, the quantities \( \zeta \) and \( \bar{z} \) are further normalised by \( \theta_E \), the Einstein angular ring radius (Einstein 1936), which is a function of the lens mass \( M \), the observer-lens distance \( D_L \), and the observer-source distance \( D_S \) through

\[ \theta_E \equiv \sqrt{\frac{4GM}{c^2}} \left( \frac{D_S - D_L}{D_S D_L} \right). \]

with \( c \) is the speed of light and \( G \) the gravitational constant. The typical separation of the images is of the order of \( \theta_E \); when the source, lens, and observer are perfectly aligned, the image is seen as a perfect ring-shaped image, called an Einstein ring. For a given position \( \zeta = u_1 e^{i\delta} \) of the source centre \( S \) (Fig. 1), the single lens Eq. (1) is easily solved by writing \( z = re^{i\delta} \), with \( r \) solution of \( r^2 - u_1 r - 1 = 0 \). This yields two solutions for the images, \( \zeta^{\pm} \equiv r^{\pm} e^{i\delta} \),

\[ r^{\pm} = \frac{u_1 \pm \sqrt{u_1^2 + 4}}{2}. \]

If we assume \( u_1 > 0 \), the image with \( r^{(+)} > 0 \) is the major image (\( I^{(+)} \) in Fig. 1) and that with \( r^{(-)} < 0 \) is the minor image (\( I^{(-)} \) in the figure). As \( \zeta \), \( \zeta^{(+)} \), and \( \zeta^{(-)} \) have the same argument, \( S \), \( I^{(+)} \), and \( I^{(-)} \) are aligned together with the lens, as shown in Fig. 1. If the lens is perfectly aligned with the lens (i.e. \( \xi = 0 \)), then Eq. (1) yields \( |\zeta| = 1 \) and the image is an Einstein ring of physical angular radius \( \theta_E \). To avoid any confusion in the units in the angular quantities we discuss here, in the following we assign a subscript ‘E’ to all angular coordinates expressed in \( \theta_E \) units. Hence, we write \( \zeta \equiv \xi_E + i\eta_E \) and \( z \equiv x_E + i y_E \), with

\[ (x_E, y_E) = (x, y)/\theta_E, \]

where \( (x, y) \) are expressed in radians and \( (x_E, y_E) \) in \( \theta_E \) units.

The complex (fringe) visibility measured by the interferometer is the Fourier transform of the spatial distribution of light \( I(x_E, y_E) \) in the plane of the sky (more precisely here, the surface brightness of the images), or

\[ \mathcal{F}[I](u_E, v_E) \equiv \int_{\mathbb{R}^2} I(x_E, y_E) e^{-2\pi i (u_E x_E + v_E y_E)} \, dx_E dy_E, \]

where \( (u_E, v_E) \) are the conjugate coordinates of \( (x_E, y_E) \). The latter are thus expressed as

\[ (u_E, v_E) = (u, v) \times \theta_E, \]

where \( (u, v) \) are the usual spatial frequencies, related to the projected baselines \( B_x \) and \( B_y \) (respectively in the (Ox) and (Oy) directions) by \( u = B_x / \lambda \) and \( v = B_y / \lambda \), with \( \lambda \) the wavelength of observations; \( (u, v) \) are expressed in radians and \( (u_E, v_E) \) in \( \theta_E^{-1} \) units (Cassan & Rane 2016). Hereafter, we also call ‘visibility’ the quantity

\[ V_E(u_E, v_E) \equiv \frac{\mathcal{F}[I](u_E, v_E)}{\mathcal{F}[I](0, 0)}. \]
which is a normalised version of Eq. (5), as the term in the denominator is the total flux. The squared modulus of the visibility, \( |V_E|^2 \), and its phase is \( \phi = \text{arg} \, V_E \).

We now consider a circular uniformly bright source of constant surface brightness \( I(x_E, y_E) = I_S \), lensed by an isolated massive body (limb-darkened sources are treated in Sect. 2.5). As \( I_S \) cancels out in the expression of \( V_E \) in Eq. (7), it is convenient to use the quantity \( \Phi \equiv \mathcal{F}[I]/I_S \) (which we also call ‘visibility’ for simplicity); for the lensed images, we then have

\[
\Phi_{\mu l}(u_E, v_E) = \int_I e^{-2\pi i (u_E x_E + v_E y_E)} \, dx \, dy,
\]

where the subscript \( \mu \) stands for microlensed. The integration is performed within the boundaries \( I \) of the (multiple) lensed images. These images are elongated around the Einstein ring, as shown in Fig. 1 for the \( S_2 \) and \( dS_1 \) sources.

In addition to the lensed images of the source, bright stars in the line of sight, although not magnified by the lens, may also be considered as contributing ‘blend’ stars of total visibility \( \Phi_B \). In particular, if the lens itself is bright enough, it may indeed contribute to a blend term \( \Phi_B \). Other stars than the lens are unlikely to be involved, even in crowded fields in the Galactic bulge region, because most stars in the immediate vicinity of the lens are faint. We define the blending factor of individual blend star \( k \) as the ratio

\[
g_k \equiv F_{B_k}/F_S,
\]

where \( F_{B_k} \) is the flux of blend star \( k \) and \( F_S = I_S S \) the flux of the source when it is not lensed, with \( S = \pi \rho^2 \) and \( \rho \) the radius of the source in \( \theta_0 \) units. We justify that \( I_S \) is used to calculate the flux of both the source and its lensed images, as gravitational lensing has the important property of preserving surface brightness when forming the lensed images. For a given blend star \( k \) of surface \( S_k \) (in \( \theta_0^2 \) units), the star’s (constant) surface brightness reads \( I_{B_k} \equiv F_{B_k}/S_k = g_k S I_S / S \). As in general these stars are not resolved by the interferometer (including the lens); a typical solar-mass lens at 4 kpc has an angular diameter of about 2 mas, \( S_k \) can be considered infinitely small and we can write \( I_{B_k} = g_k S \delta(x_E - x_{E,k}) \delta(y_E - y_{E,k}) \), with \( \delta \) the Dirac distribution and \( (x_{E,k}, y_{E,k}) \) the coordinates of star \( k \). Considering all blend stars, the surface brightness reads

\[
I'(x_{E}, y_{E}) = \sum_k I_{B_k} = I_S \pi \rho^2 \sum_k g_k \delta(x_E - x_{E,k}) \delta(y_E - y_{E,k}),
\]

so that the blend visibility is given by

\[
\Phi_B(u_E, v_E) = \pi \rho^2 \sum_k g_k e^{-2\pi i (u_E x_{E,k} + v_E y_{E,k})}.
\]

In particular, a bright lens (in the centre of the frame) would contribute to \( \Phi_L = g_L \pi \rho^2 \), with \( g_L = F_L / F_S \) the blend-to-source flux ratio. The overall visibility is expressed as

\[
V_E(u_E, v_E) = \Phi_L + \Phi_B = \Phi_{\text{blend}} + \Phi_B,
\]

where \( \Phi_{\text{blend}} \equiv \Phi_L(0, 0) \) and \( \Phi_B \equiv \Phi_B(0, 0) \). When three or more baselines are involved, the bispectrum \( B_{E,1,2,3} \) and closure phase \( \phi_{E,T3} \) of each triangle of baselines are given by

\[
\Phi_{E,T3} = \Phi_{E_{1,2,3}} = \Phi_{E_{1,2}} \times \Phi_{E_{1,3}} \times \Phi_{E_{2,3}},
\]

where \( (u_{E,1}, v_{E,1}) = -(u_{E,2}, v_{E,2}) = (u_{E,3}, v_{E,3}) \), or an equivalent formula considering that \( V_E(−u_{E,2}, −v_{E,2}) = \overline{V_E(u_{E,2}, v_{E,2})} \).

For an interferometer observing in the \( H \) band (\( \lambda = 1.65 \mu \text{m} \)) with projected baseline \( B = 100 \) m along the \( x \)-axis, and typical values of \( \theta_0 \) of 0.5, 1, 1.5, and 2 mas, the interferometer probes \( u_B \)-values of 0.15, 0.30, 0.45, and 0.60 (in \( \theta_0^{-1} \) units). As the angular separation of the major and minor images is \( \sim 2 \times \theta_0 \) (see Fig. 1), we expect the visibility to be modulated with a period of \( \sim 0.5 \) along \( u_B \), with a first minimum at \( |u_B| \approx 0.25 \) (in \( \theta_0^{-1} \) units). This means that the provided microlensing event is bright enough to be observed, the lensed images can be resolved in most cases, and the value of \( \theta_0 \) measured.

In the following sections, we derive suitable formulae to compute efficiently Eq. (8) for three reference single-lens models: the point-source approximation, the exact formula, and a novel thin-arcs approximation, for uniform and limb-darkened sources.

### 2.2. Point-source approximation

The visibility for a single lens in the point-source approximation was first studied by Delplancke et al. (2001). The resulting major and minor point-like images can then be modelled by Dirac distributions, respectively located at \( r^{(1)} \) and \( r^{(−1)} \), from Eq. (3) and Fig. 1, and weighted by their individual magnification factor \( \mu^{(1)} \) and \( \mu^{(−1)} \) given by

\[
\mu^{(±)} = \frac{2}{2 + \frac{u_i^2 + 2}{u_i^2 + 4}}.
\]

When there is no source of blend (Delplancke et al. 2001), the squared visibility reads

\[
|V_E|^2 = \frac{\mu^{(+) e^{-2\pi i u_{E}^{(+)}} + \mu^{(−)} e^{-2\pi i u_{E}^{(−)}}}}{\mu^{(+) + \mu^{(−)}}} = 1 + R^2 + 2 R \cos \left( \frac{2 \pi u_{E}^{(1)} u_{E}^{(−)} + 4}{1 + R^2} \right),
\]

where \( R = \mu^{(+)}/\mu^{(−)} \) is the ratio of the magnification of the major image to that of the minor image. The quantity \( |V_E|^2 \) is a sinusoidal function along the \((O\gamma_E)\) direction, and is invariant along the \((O\chi_E)\) direction. The modulation has a period of \( T = 1/(u_{E}^{(1)} + 1/4)^2 \) and the squared visibility oscillates between \( (\frac{4}{\pi^2}) \) and 1. If \( u_1 \) is not too large (i.e. \( u_1 \leq 0.5 \)), the periodicity is approximately constant and equal to \( T = 0.5 \). The amplitude is largest when \( u_1 \) is small, as \( \mu^{(+)}/\mu^{(−)} \) are both approximated by \( 1/2u_1 \) so that \( R \approx 1 \). However, we show in Sect. 3 that the point-source model provides in this case a poor approximation of the visibility as the lens strongly distorts the images.

In the general case when one or several unrelated objects \( k \) contribute to the blending flux (Sect. 2.1), the resulting complex visibility reads

\[
V_E = \frac{\mu^{(+) e^{-2\pi i u_{E}^{(+)}} + \mu^{(−)} e^{-2\pi i u_{E}^{(−)}}}}{\mu^{(+) + \mu^{(−)}}} = \sum_k g_k e^{-2\pi i (x_{E,k} x_{E,E} + y_{E,k} y_{E,E})},
\]

where \( (x_{E,k}, y_{E,k}) \) are the location of the blend sources on the plane on the sky in \( \theta_0 \) units, and \( g_k \) their blending factors as defined by Eq. (9).
Finally, we note that a single lens event will always lead to a strong interferometric signal, even for microlensing events with relatively low peak magnification or observed far from the peak of the light curve. From the definition of $R$ in Eqs. (16) and (15), it is possible to compute the maximum amplitude of the oscillation of $V^2_{\Omega}$, which is equal to $S = 1 - (\frac{R}{2})^2 = 4/(u^2 + 1)^2$. Even for an unrealistic observation at very low magnification (e.g. 1.4; $u_1 = 0.9$), we have $S > 0.5$, and for a more standard value of $u_1 < 0.2$ (magnification >5), the contrast reaches values greater than $S > 0.96$.

2.3. Exact formula for an extended source star

The visibility and closure phase for an extended-source single-lens model was first studied by Rattenbury & Mao (2006). As the images are elongated along the Einstein ring (Fig. 1) with analytically well-defined contours (Sect. 2.1), the authors proposed computing numerically the visibility defined in Eq. (5) by a line integral along the outer boundary of each of the major and minor images.

Here, however, we follow a different route and set up another approach to compute Eq. (5), motivated by different arguments. Firstly, our tests have shown that while the line integration scheme works well when the images are reasonably elongated (i.e. at low to medium magnifications), when the images take the form of thin arcs the parametrisation of the contours is far from optimal, as the individual points defining the images contours become strongly unevenly spaced. To achieve reasonable accuracy, the points on the contours need to be resampled, and in any case the number of contour points must be significantly increased. This operation is mandatory, as the line integral basically operates a subtraction between the wedge-shaped area subtended by the outer boundary and that subtended by the inner boundary, which are both of the order of $\varphi/2$ (as they are located at $r \gg 1$), where $\varphi$ is the opening angle defined in Fig. 2. In contrast, when $\varphi \ll 1$ the area enclosed in a given image is of the order of $\varphi^2$, and the fractional difference corresponding to the searched visibility is of the order of $\varphi^2/(\varphi/2) = 2\varphi \ll 1$. Hence, an accurate visibility requires a very high accuracy on the line integrals, in particular in the portions where the contours are not well sampled by the parametrisation. The approach we derive below is both robust and computationally much more efficient, and allows a precise control on the final accuracy. This can be achieved by calculating Eq. (5) as a two-dimensional integral in polar coordinates, as we detail below.

We again let $u_1 > 0$ be the ordinate of the centre of the source $S$ along the $y$-axis, and $\rho$ the source radius (both in $\theta_E$ units), as shown in Fig. 3. We again assume that the source is uniformly bright (and also the images, since surface brightness is preserved). We let $\theta$ be the angle of the usual polar coordinates, and $u_\pm$ and $u_\mp$ the radii where the radial black line in Fig. 3 intersects the upper and lower contours of the source. From geometrical considerations, these are given by

$$u_\pm = u_1 \cos \beta \pm \sqrt{\rho^2 - u_1^2 \sin^2 \beta}, \quad (18)$$

where $\beta \equiv \theta - \pi/2$. If $\rho < u_1$, we restrict $\beta$ to vary between $-\arcsin \eta_1$ and $\arcsin \eta_1$ (left panel of Fig. 3), where

$$\eta_1 \equiv \rho/u_1. \quad (19)$$

Otherwise, if $0 < u_1 < \rho$, we limit $\beta$ to vary between $-\pi/2$ and $\pi/2$ (right panel). These choices are sufficient to parametrise the two arc-shaped images (or the ring) as we obtain two points above the horizontal axis for the major image $I^{(\varphi)}$, $I^{(+)}$, and $I^{(-)}$, and two points below for the minor image $I^{(\varphi)}$, $I^{(+)}$, and $I^{(-)}$, given by

$$\begin{align*}
I^{(+)}_{\pm} &= \frac{1}{2} \left( u_\pm + \sqrt{u_\pm^2 + 4} \right), \\
I^{(-)}_{\pm} &= \frac{1}{2} \left( u_\pm - \sqrt{u_\pm^2 + 4} \right), \\
\end{align*} \quad (20)$$

from Eq. (3). In all cases, $\rho_{\pm} < \rho_{+} < \rho_{-} < \rho_{\mp}$. If $u_1 = 0$, the ring is perfectly symmetric (Einstein ring) as $u_\pm = \pm \rho$, so that $\rho_{\pm} = -\rho_{\mp}$ and $\rho_{\pm} = -\rho_{\mp}$. To calculate the visibility, we further perform the integration of Eq. (8) in polar coordinates,

$$\Phi_{\mu\ell} = \int_{-\infty}^{\infty} e^{-i2\pi\Omega(u_\mu \cos \theta + u_\ell \sin \theta)} r dr d\theta$$

where

$$\Omega \equiv u_\mu \cos \theta + u_\ell \sin \theta. \quad (22)$$

We also decompose the full integral into two separates integrals, one for the image above the horizontal axis (elongated image or half-ring), $\Phi^{(+)}$, and one for the image below, $\Phi^{(-)}$, with

$$\Phi_{\mu\ell} = \Phi^{(+)} + \Phi^{(-)}. \quad (23)$$
We first consider the case $\rho < u_1$ (or $0 < \eta_1 < 1$). Since $0 < r_{+}^{(1)} < r_{+}^{(2)}$ and $\pi/2 - \arcsin \eta_1 \leq \theta \leq \pi/2 + \arcsin \eta_1$, we can write the integral Eq. (21) for the major image as

$$\Phi^{(+)i} = \int_{\pi/2 - \arcsin \eta_1}^{\pi/2 + \arcsin \eta_1} \left( \int_{r_{+}^{(1)}}^{r_{+}^{(2)}} r e^{-i2\pi \Omega r} dr \right) d\theta$$

$$= \left[ \frac{1}{2} \left( r_{+}^{(2)} - r_{+}^{(1)} \right) \right]_{\theta = \pi/2}^{\theta = \pi/2 + \arcsin \eta_1}$$

$$- \Omega \int_{\pi/2 - \arcsin \eta_1}^{\pi/2 + \arcsin \eta_1} \int_{r_{+}^{(1)}}^{r_{+}^{(2)}} r e^{-i2\pi \Omega r} dr d\theta,$$

where we change to variable $\beta$ in the second line,

$$\Omega = -u_1 \sin \beta + v_\perp \cos \beta$$

and

$$R(r_1, r_2) = \int_{r_1}^{r_2} r e^{-i2\pi \Omega r} dr$$

$$= \left[ \frac{1}{2} \left( r_{+}^{(2)} - r_{+}^{(1)} \right) \right]_{\theta = \pi/2}^{\theta = \pi/2 + \arcsin \eta_1}$$

otherwise.

The first terms of the series expansion (with respect to $\Omega$) of the expression inside the brackets are

$$e^{-i2\pi \Omega r} (1 + i2\pi \Omega r) \approx \frac{r^2}{2} + \frac{-i}{4\pi^2 \Omega^2} - \frac{1}{2} \pi r^3 \Omega - \frac{1}{2} \pi r^4 \Omega^2.$$

When $\Omega \ll 1$, the term $1/4\pi^2 \Omega^2$ becomes large, possibly generating numerical issues, although theoretically this term cancels out in the difference Eq. (26). Hence, the formula $(r_{+}^{2} - r_{+}^{1})/2$ may be used for values of $\Omega$ below a threshold of typically $\Omega \sim 10^{-3}$ if we want the second term of the series (in $\Omega r^3$) to contribute no more than $\sim 10^{-3}$ times the term in $\Delta r^2$.

We proceed in a similar way for the minor image, but this time with $r_{-}^{(1)} < r_{-}^{(2)} < 0$ and $-\pi/2 - \arcsin \eta_1 \leq \theta \leq -\pi/2 + \arcsin \eta_1$, so that the integral reads

$$\Phi^{(-)i} = \int_{-\pi/2 - \arcsin \eta_1}^{-\pi/2 + \arcsin \eta_1} \left( \int_{r_{-}^{(1)}}^{r_{-}^{(2)}} r e^{-i2\pi \Omega r} dr \right) d\theta$$

$$= \left[ \frac{1}{2} \left( r_{-}^{(2)} - r_{-}^{(1)} \right) \right]_{\theta = -\pi/2}^{-\theta = -\pi/2 + \arcsin \eta_1}$$

after changing the variable to $\beta' = \beta + \pi/2$, and introducing $\Omega' = -u_1 \sin \beta' + v_\perp \cos \beta' = -[u_1 \cos(\beta' - \pi/2) + v_\perp \sin(\beta' - \pi/2)] = -\Omega$. As $\beta'$ is a dummy variable, we call it $\beta$ and write

$$\Phi^{(-)i} = \int_{-\pi/2 - \arcsin \eta_1}^{-\pi/2 + \arcsin \eta_1} \left( \int_{r_{-}^{(1)}}^{r_{-}^{(2)}} r e^{-i2\pi \Omega' r} dr \right) d\beta$$

$$= \left[ \frac{1}{2} \left( r_{-}^{(2)} - r_{-}^{(1)} \right) \right]_{\beta = -\pi/2}^{-\beta = -\pi/2 + \arcsin \eta_1}$$

by changing variable $r$ to $-r$ and inverting the boundaries of the integral, so that

$$\Phi^{(-)i} = \int_{-\pi/2 - \arcsin \eta_1}^{-\pi/2 + \arcsin \eta_1} \left( - \int_{r_{-}^{(2)}}^{r_{-}^{(1)}} r e^{-i2\pi \Omega' r} dr \right) d\beta$$

We now examine the case $0 < u_1 < \rho$ (or $\eta_1 > 1$). In this situation the lens lies inside the source, and there is a single ring-like image. The full ring can be drawn by varying $\theta$ from $0$ to $\pi$, as $r_{+}^{(1)}$ and $r_{+}^{(2)}$ draw the half-ring above the horizontal axis and $r_{-}^{(1)}$ and $r_{-}^{(2)}$ the half ring below it. Hence, the calculation...
is exactly the same as for \( \rho < u_1 \); the only difference is that the integration is now performed between \(-\pi/2 \leq \beta \leq \pi/2\).

In summary, for all values of \( u_1 > 0 \) and \( \rho > 0 \) (i.e. \( \eta_1 > 0 \)) we have

\[
\Phi_{\beta m} = \int_{-\beta_m}^{\beta_m} \left[ R\left( r^{(+)}, r^{(+)} \right) - R\left( r^{(-)}, r^{(-)} \right) \right] \, d\beta, \tag{31}
\]

where

\[
\beta_m \equiv \arcsin \left[ \min \left( \eta_1, 1 \right) \right]. \tag{32}
\]

When \( u_1 = 0 \) (Einstein ring), this formula still holds with \( \beta_m = \pi/2 \). In that case the visibility has no imaginary part, which is expected from the symmetry of the ring image.

2.4. The thin-arcs approximation

We consider a common case in practice where the source has a small radius \( \rho \) compared to \( \theta_0 \) (typically, \( \rho \ll 0.1 \)) and passes the lens at small impact parameter, typically \( u_1 \ll 0.2 \) (which corresponds to a point-source magnification at a peak of about 5).

This situation is illustrated in Fig. 2 for two sources of radii \( S_1 \) and \( S_2 \). In the figure it is clear that if the major and minor images are resolved by the interferometer along the \((\Omega_{E})\) axis (typical angular separation of \( 2 \times \theta_0 \), cf. Fig. 2), they have a good chance to be resolved along the \((x_{E})\) axis as well (typical angular separation of \( 2 \times \theta_0 \), where ‘\( a \)’ stands for arcs).

Since the opening angle of the images is given by \( \varphi = 2 \arcsin \eta_1 \) (with \( \eta_1 = \rho/u_1 < 1 \), and \( \varphi = \pi \) for a ring-like image), we have \( \theta_a = \eta_1 \theta_0 \) (or \( \theta_a = \theta_0 \) for a ring). Values of \( \eta_1 > 0.5 \) are easily reached for the range of values of \( u_1 \) we discussed above. Hence, we expect this situation to be common in observed microlensing events.

While the major and minor images are resolved in their individual elongations (\( \sim 2\theta_0 \)) and mutual separation (\( \sim 2\theta_0 \)), on the contrary their thickness (\( \delta \) in Fig. 2) will most certainly never be resolved by current interferometric facilities: a value of \( \delta \) of the order of \( \rho \) requires reaching, at best, typical angular resolutions of \( \delta = \rho \theta_0 < 10 \mu\text{as} \). Measuring the thickness of the arcs, however, is not in itself a requirement for measuring \( \theta_0 \), as the thickness and the extension of the arcs are directly related by the model. It simply means that the source size \( \rho \) is not measured directly by interferometry, but it does not matter in practice as \( \rho \) is usually obtained from the modelling of the photometric light curve. It thus appears natural to investigate the possibility of an approximation formula for the visibility that does not directly depend on \( \rho \). A second argument for it is that, as shown in Fig. 2, the two sources of different radii but with same opening angle \( \varphi \) (i.e. the same value of \( \eta_1 \)) are difficult to distinguish from an interferometric point of view, as the displacement is again of the order of \( \rho \theta_0 \). Hence, a natural model parameter for the sought-after approximation is \( \eta_1 \) (i.e. the ratio of \( \rho/u_1 \) instead of the parameters \( \rho \) and \( u_1 \) individually). Conversely, if \( \rho \) or \( u_1 \) are used in the situation of arc-shaped images (using the exact formula given in Sect. 2.3), we expect these parameters to be strongly correlated (if not degenerate), which may alter the smooth running of the fitting process when modelling interferometric data.

To further illustrate this aspect, Fig. 4 shows the difference in squared visibility \( \Delta V_{E}[\rho] \) between a model computed for parameters \( \rho = 0.05 \) and \( u_1 = 8.3 \times 10^{-2} \), and a reference model obtained with \( \rho = 0.001 \) and \( u_1 = 1.7 \times 10^{-3} \), so that both models have same parameter \( \eta_1 = 0.6 \). It appears from the figure that while the source radius is multiplied by a factor of 50 between the two models, the squared visibility is not changed by more than \( 6 \times 10^{-3} \) (for a maximum excursion between 0 and 1).

To establish a suitable approximation, which we call the thin-arcs approximation, we note that when both \( \rho \) and \( u_1 \) are small (see Sect. 3.1), \( u_2 \) are small as well so that we can expand \( r^{(\pm)} \) in the first order in \( u_2 \):

\[
\begin{aligned}
& r^{(+)}(\pm) \approx 1 + \frac{u_2}{2} \\
& r^{(-)}(\pm) \approx -1 + \frac{u_2}{2}.
\end{aligned} \tag{33}
\]

Calculating \( R \) in Eq. (26) for \( \Omega \geq 0 \) yields

\[
\begin{aligned}
& R\left( r^{(+)}, r^{(+)} \right) = e^{-2\Omega} \sqrt{\rho^2 - u_1^2 \sin^2 \beta} \\
& R\left( r^{(-)}, r^{(-)} \right) = -e^{2\Omega} \sqrt{\rho^2 - u_1^2 \sin^2 \beta},
\end{aligned} \tag{34}
\]

which allows us to write, in Eq. (31),

\[
R\left( r^{(+)}, r^{(+)} \right) - R\left( r^{(-)}, r^{(-)} \right) = 2 \cos \left( 2\Omega \theta_0 \right) \sqrt{\rho^2 - u_1^2 \sin^2 \beta}. \tag{35}
\]

From Eq. (25), the real part of \( \cos \left( 2\Omega \right) \) is the product \( \cos(2\Omega \theta_0 \sin \beta) \times \cos(2\Omega \theta_0 \cos \beta) \), while its imaginary part reads \(-\sin(2\Omega \theta_0 \sin \beta) \times \sin(2\Omega \theta_0 \cos \beta) \). As we integrate from \(-\beta_m \) to \( \beta_m \), the imaginary part cancels out while the real part is doubled. Finally, for \( u_1 > 0 \) the visibility reads\(^1\)

\[
\Phi_{\beta m} = 4u_1 \int_{-\beta_m}^{\beta_m} f(\beta) \, d\beta, \tag{36}
\]

where again \( \beta_m = \arcsin \left[ \min \left( \eta_1, 1 \right) \right] \) and

\[
f(\beta) = \cos(2\Omega \theta_0 \sin \beta) \cos(2\Omega \theta_0 \cos \beta) \sqrt{\eta_1^2 - \sin^2 \beta}, \tag{37}
\]

\(^1\) The denominator of \( \frac{V_E}{V_0} \) in Eq. (12), \( \Phi_{\beta m} = F_{\beta m}(0, 0) \), can be written as \( \Phi_{\beta m} = 4\rho E(\beta_m, \eta_1^2) \), where \( E(\phi, m) = \int_0^\phi \left( 1 - m \sin^2 \theta \right)^{1/2} \, d\theta \) is the incomplete elliptic integral of the second kind. The total source flux magnification \( A \) being the ratio of the total area of the images \( \Phi_{\beta m} \) to the area of the source, \( \pi r^2 \), we obtain \( A = (4/\pi\rho)E(\beta_m, \eta_1^2) \), which is the approximation derived by Yoo et al. (2004), though with a slightly different definition of \( E \) and \( z = 1/\eta_1 \).
while for $u_1 = 0$ (Einstein ring),

$$\Phi_{ml} = 4\rho \int_0^\pi \cos(2\pi u_0 \sin \beta) \cos(2\pi u_0 \cos \beta) \, d\beta.$$  \hspace{1cm} (38)

As expected, the integrand Eq. (37) does not depend on $\rho$ and $u_1$ individually, but on their ratio $\eta_1$ (for the perfect Einstein ring, the integrand Eq. (38) does not depend of any of these parameters). In both cases (assuming no blend stars, or $\Phi_{ml} = 0$), the factors $4u_1$ or $4\rho$ cancel out in Eq. (12), so that the visibility $V_E$ depends on $\eta_1$ only for $u_1 > 0$. It is noteworthy that in the thin-arcs approximation, $\phi = \arg \Phi_{ml}$ can take only two values, 0 and $\pi$. It means that the bispectrum $B_{E,1,2,3}$ in Eq. (13) is also real, and the closure phases $\Phi_{E,1,2,3}$ in Eq. (14) are 0 or $\pi$. As $u_1$ increases, the exact value of $\Phi_{E,1,2,3}$ starts to differ from these two values, and should be calculated with the exact formula derived in Sect. 2.3.

Our numerical simulations show that the thin-arcs approximation speeds up the computation by a factor of 6 to 10 (depending on the specific configuration of the images) compared to the exact formula, under a common implementation in Python using the scipy/romberg integration scheme and a given achieved accuracy ($5 \times 10^{-5}$) on both the real and imaginary parts of $V_E$. As we further discuss in Sect. 3.1, the domain of validity of the thin-arcs approximation is wide. Considering $|V_E|^2$, for the usual values of $\rho$ and typical excursions in the $u_{E,1,2,3}$-plane, the point-source approximation can be linked with the thin-arcs approximation without having to use the exact formula at all.

### 2.5. Stellar limb darkening

The most convenient way to treat limb-darkening effects is to decompose the source (assumed to be a disk) into $N$ concentric annuli of inner and outer radii $\rho_{k-1} < \rho_k$ $(1 \leq k \leq N)$ of constant surface brightness $I_k$, with $\rho_0 = 0$ and $\rho_N = \rho$. The visibility is then simply calculated as

$$\Phi_{ml} = \sum_{k=1}^N k \left( \Phi_{ml,k} - \Phi_{ml,k-1} \right),$$  \hspace{1cm} (39)

where $\Phi_{ml,k}$ is computed for a source of radius $\rho_k$ (or equivalently for the thin-arcs approximation, $\eta_{1,k}$, with $\eta_{1,0} = 0$ and $\eta_{1,N} = \eta_1$). As limb-darkening affects the border of the disk of the source star, it will affect the ends of the arc-shaped images and will contribute as a correction only to the visibility.

In any case, a linear limb-darkening law will always provide a suitable description of the source’s limb darkening. Adopting the (micro)relensing convention that the limb-darkening law is normalised to total unit flux yields $I_k = 1 - \Gamma \left(1 - \frac{1}{2}(1 - r_k^2)^{1/2}\right)$, where $\Gamma$ is the linear limb-darkening coefficient ($0 \leq \Gamma \leq 1$), and where $r_k = \rho_k/\rho = \eta_{1,k}/\eta_1$. The coefficient $\Gamma$ is related to the more usual coefficient $a$ by $a = 3\Gamma/(2 + \Gamma)$. The choice of the particular set of values $r_k$ (or $\eta_{1,k}$) can be optimised to minimise $N$ (e.g. with a linear sampling of $I_k$ between its maximum and minimum values from centre to limb, respectively $(1 + \Gamma/2)$ and $(1 - \Gamma)$).

### 3. Application

#### 3.1. Examples and discussion

Typical examples of visibilities ($|V_E|^2$ and $\phi = \arg V_E$) are shown in Figs. A.1 to A.6. All the figures were calculated for the same value of $\rho = 0.03$, slightly above the typical values to challenge the approximations derived in the previous sections. From Figs. A.1 to A.5 the distance of the source to the lens is decreased from $u_1 = 0.6$ to 0.032, and Fig. A.6 is a perfect Einstein ring ($u_1 = 0$). For each figure the upper panels show, on the left, the positions and shapes of the source and the images with the lens in the centre and, on the right, a three-dimensional view of the squared visibility in the $u_{E,1,2,3}$-plane. The middle panels show the squared visibility $|V_E|^2$ (left plot) and the phase $\phi$ (right plot). The bottom panels show the difference in squared visibility between either the point-source approximation (left) or the thin arcs approximation (right) and the exact calculation. For reference, the colour scale for these plots is set to saturate at $\Delta|V_E|^2 \leq 0.1$ (negative values in blue, positive in red) as they are typical minimum instrumental values of error bars on $|V_E|^2$.

For relatively large source-lens separations, such as $u_1 = 0.6$ in Fig. A.1, the images are just slightly elongated, and the point source provides a good approximation to the visibility. The minimum squared visibility differs from 0 and the phase spans a range of values because the two images have different magnifications. The thin-arcs approximation does not provide a good approximation in that case. When $u_1 = 0.3$ (Fig. A.2), the point-source approximation still holds for typical $(u_{E,1,2,3})$ values probed by the interferometer (data points expected at best between the two dashed circles), and the thin-arcs approximation starts to provide a fair approximation within the inner dashed circle.

When $u_1 \lesssim 0.2$ (peak point-source magnification of about 5), the situation is reversed, as seen in Fig. A.3 for $u_1 = 0.1$. The thin-arcs approximation now provides a very good approximation to the squared visibility ($\leq 10^{-2}$) even outside the outer dashed circle. The point-source approximation is no longer a suitable model. In Fig. A.4 ($u_1 = 0.05$) and Fig. A.5 ($u_1 = 0.032$), the squared visibility progressively takes a circular shape, while the phase does not differ more than a few degrees from 0 or 180 deg. Finally, in Fig. A.6, the image is a perfect ring ($u_1 = 0$) and the thin-arcs approximation is as accurate as the exact calculation. The phase takes only the two values 0 and 180 deg.

In the situation described here ($\rho = 0.03$), the transition from the point-source to the thin-arcs approximation appears smooth for $|V_E|^2$, and unless we have to model interferometric data with very small error bars, it appears unnecessary to perform the exact calculation. The closure phase, however, still requires using the exact calculation, but its value is not expected to deviate more than a few degrees from 0. Finally, for smaller values of $\rho$, the thin-arcs approximation gives even better results for $|V_E|^2$, which justifies its use for a wide range of single-lens parameters.

#### 3.2. Practical modelling

In this section, we study possible strategies for fitting interferometric data to single-lens models, and we discuss suitable choices of model parameters for single-epoch or time-series data. In the following, we assume that the limb-darkening coefficient $\Gamma$ of the source can be estimated independently (e.g. from a colour-magnitude diagram). The main parameters we discuss below are shown in Fig. 5.

We first consider individual interferometric epochs. We first insist that the images are almost static during the time of an interferometric exposure (~10 min). To compute the visibility, the point-source approximation requires $u_1$ as parameter, the exact formula ($u_1, \rho$), and the thin-arcs approximation $\eta_1$. To fit the interferometric data, in all cases we must add $\theta_E$ as a parameter (orientation on the sky of the images) as well as $\theta_E$ (to convert Einstein units into radians). If the lens is luminous, an extra
4. Summary and perspectives

In this work, we first reviewed the main concepts and general formulae of interferometric microlensing, and detailed the equations useful for treating the case of a single lens. We recalled the well-known visibility formula for a point source, and then treated the case of an extended source, for which we proposed a new approach for the calculation of the visibility, allowing a robust and numerically efficient calculation.

This formalism allowed us to establish a new approximation, which we called the thin-arcs approximation, and which applies to microlensing events of medium or higher magnification observed around the peak (i.e. a large fraction of potential observational targets). We demonstrated that the computation time using this approximation is 6–10 times faster than with the exact formula, and applies over a wide range of lens-source separations. It even turns out that a direct transition from the point-source to the thin-arcs approximations is possible in many situations, without having to calculate the visibility with the exact formula.

Accurate models and reliable numerical methods are of particular importance as the number of targets is expected to increase significantly in the near future. Based on a 4 yr of statistics of microlensing events alerted by the OGLE collaboration (2011–2014, about 7000 events), Cassan & Ranc (2016) found that the number of potential interferometric targets N scales as \( \sim 10^{0.4x\Delta m^*} \) with the event’s peak magnitude \( m_K \) (obtained from a linear regression of the event’s count, right panel of Fig. 3 in Cassan & Ranc 2016); in other words, a gain in ~2.4 magnitudes in the instrument sensitivity results in about ten times more potential microlensing targets. Pushing the limiting magnitude of current or new-generation interferometers will therefore have a huge impact on the field.

Until recently, interferometric facilities suffered from a lack of sensitivity, limiting the pool of observable microlensing targets to the very bright tip of the distribution. Observations like those obtained for Gaia19bld at the VLTI (peak magnitude of \( H = 6.2 \), just above the PIONIER instrument’s limiting magnitude of \( H = 7.5 \), see Ext. Data. Figure 1 in Cassan et al. 2022) hence remained exceptional. However, the latest improvements in interferometric instruments are going to be a ‘game-changer’, in particular for the ESO GRAVITY instrument at the VLTI. The new dual-field wide mode now available for GRAVITY, which uses a close and brighter star in the vicinity of the target, will allow us, for the first time, to unambiguously find isolated stellar black holes, and measure their masses with an exquisite precision.

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Appendix A: Additional figures

Fig. A.1. Plots for $\rho = 0.03$ and $u_1 = 0.6$. Upper panels: Shown in the plot on the left are the lens (black dot in the centre), the (unseen) extended source (orange disk), and its two lensed images (red and blue arc-shaped images), displayed in $(x_E, y_E)$ coordinates which are normalised by the angular Einstein ring radius $\theta_E$. The plot on the right displays a three-dimensional view of the squared visibility $|V_E|^2$ in the Einstein $u_E v_E$-plane, normalised by $\theta_{E}^{-1}$. Middle panels: Shown in the plot on the left is a contour plot of $|V_E|^2$, while the plot on the right shows the phase of the complex visibility $\phi = \arg V_E$ (the thin white line is a visualisation artefact when $\phi$ jumps from $-\pi$ to $\pi$, or vice versa). Lower panels: Difference in squared visibility $\Delta |V_E|^2$ between either the point-source (left) or the thin-arcs (right) approximation and the exact calculation. The colours saturate for a difference of $\pm 0.1$. The inner dashed circle marks the typical angular resolution (radius 0.25), and the outer circle (radius 0.5) twice the typical resolution.
Fig. A.2. Same as Fig. A.1, but for $u_1 = 0.3$. 
Fig. A.3. Same as Fig. A.1, but for $u_1 = 0.1$. 
Fig. A.4. Same as Fig. A.1, but for \( u_1 = 0.05 \).
Fig. A.5. Same as Fig. A.1, but for $u_1 = 0.032$. 
Fig. A.6. Same as Fig. A.1, but for $u_1 = 0$. In this case the point-source approximation yields a circle instead of two point-like images and is not shown here.