ALL WHAT I WANTED TO KNOW ABOUT
LANGLANDS PROGRAM AND WAS AFRAID TO
ASK.

BY MICHÈLE VERGNE

Contents
1. Sources and short description 2
2. The importance of automorphic forms 4
  2.1. Classical automorphic forms 4
  2.2. Abelian reciprocity law 8
  2.3. Artin L-functions and Langlands conjectural reciprocity law 11
3. Automorphic representations 14
  3.1. The use of adeles 14
  3.2. A little more of representation theory 15
  3.3. From classical automorphic forms to automorphic representations 15
  3.4. Hecke algebras 16
  3.5. Local Langlands conjecture 17
  3.6. Global Langlands conjecture on function fields 18
4. Functoriality 19
  4.1. The problem of liftings 19
  4.2. Trace formula 19
  4.3. Transfer and the fundamental lemma 22
5. Geometric Langlands correspondence 27
References 32

Date: October 2005.
1. Sources and short description

My notes are extracted from informal conversations with Gérard Laumon, Eric Vasserot and Jean-Louis Waldspurger and from the following articles.

Stephen Gelbart: *An elementary introduction to the Langlands program*. Bulletin of the American Mathematical Society, 10, 1984, pp 177-215.

Several articles (E. Kowalski, Gelbart, Gaitsgory) from the book: *An introduction to the Langlands program*. Editors: Bernstein-Gelbart; Birkhauser (2003).

Thomas Hales: *A statement of the fundamental lemma*. Arkiv RT/031227 (2003).

Gerard Laumon *Travaux de Frenkel,Gaitsgory et Vilonen sur la correspondance de Drinfeld-Langlands*. Séminaire Bourbaki Juin 2002

I refer to these articles to get more complete bibliography on this subject and more details.

Here is the description of these notes. I will try to motivate the “Langlands program” by discussing first $L$-functions attached to representations of the Galois group of number fields. Then I will discuss the Langlands program and its conjectural implications of functoriality. I will discuss a simple example of the fundamental lemma, as a combinatorics problem of counting lattices. Then I will outline the Langlands-Drinfeld program for the function field of a complex curve. Here this is very very sketchy.

I give here only the definitions I understood. So when I write: “it is more complicated”, it means I do not understand. I oversimplified many definitions and conjectures, some willingly, and probably many unwillingly. Furthermore, I will only discuss here representations in characteristic 0, so that I will not touch upon the recent developments on modular representations, modular Serre conjecture, etc, . . . .

Section 2 describes some historical motivations to the Langlands program.

Section 3 gives the main definitions.

The last two sections “Functoriality” and “Geometric Langlands correspondence” are independent. Here I have only be trying to give some statements and some simple examples. No indications of techniques are given. Thus probably, the beauty of the works of researchers in this field will not be fully apparent, as the interplay between representation
theory and algebraic geometry is most remarkable, but I felt too in-
competent on these domains (Hitchin moduli spaces, perverse sheaves,
etc..) to write something not totally nonsensical.
2. The Importance of Automorphic Forms

2.1. Classical automorphic forms. What is a (classical) automorphic form? Roughly speaking, this is a function $f(z)$ on the upper-half plane $\mathcal{H} := \{z = x + iy; y > 0\}$ of the form

$$f(z) := \sum_{n \geq 0} a_n e^{2\pi n z/N}$$

($N$ a positive integer) and such that it takes almost the same form (I suppose this is why it has the name automorphic) when changing $z$ to $-1/z$:

$$f(-1/z) = \text{constant} \cdot z^k f(z),$$

for some $k$.

Remark that $f$ is periodic of period $N$: $f(z + N) = f(z)$. The integer $k$ will be called the weight ($k$ can be a half integer, but we will mostly restrict ourselves to integer weights). The function $|e^{2\pi n z/N}| = e^{-2\pi n y/N}$ decreases very rapidly as a function of $n$ when $y = \text{Im}(z) > 0$ and $n > 0$. Provided the coefficients $a_n$ have a reasonable growth (for example polynomial in $n$), the series is indeed convergent.

For example, the Theta series $\Theta(z) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z}$ satisfies $\Theta(-1/z) = (-iz)^{1/2} \Theta(z)$, as follows from Poisson formula. Thus it is an automorphic form of weight $1/2$.

Let us give more precise definitions. Let $G = SL(2, \mathbb{R})$ be the group of holomorphic transformations of the upper-half plane $\mathcal{H}$:

$$G := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; (a, b, c, d) \text{ reals, } ad - bc = 1 \right\}.$$ 

The corresponding transformation is $g : z \rightarrow (az + b)/(cz + d)$.

Let $N$ be a positive integer, let $\mathbb{Z}/N\mathbb{Z}$ be the ring of integers mod $N$ and $(\mathbb{Z}/N\mathbb{Z})^*$ the multiplicative group of invertible elements of this ring. Let $\chi$ a character of $(\mathbb{Z}/N\mathbb{Z})^*$ and $k$ an integer or half-integer. We extend $\chi$ to a periodic function of period $N$ of $\mathbb{Z}$ by setting $\chi(u) = 0$ if $u$ is not relatively prime to $N$ while $\chi(u) = \chi(mod(u, N))$ otherwise, so that $\chi(nm) = \chi(n)\chi(m)$. Such a $\chi$ is called a primitive Dirichlet character of level $N$ if $N$ is the smallest period. Most of the time in this introduction, we can think that $N = 1$ and $\chi = 1$ (integers prime to 1 form the empty set, thus $\chi = 1$ (Hum!); anyway this is the convention).

Consider the discrete subgroup of $SL(2, \mathbb{R})$:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; (a, b, c, d) \text{ integers, } ad - bc = 1, c \equiv 0 \text{ mod } N \right\}.$$
Remark that $\Gamma_0(N)$ contains the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ producing the transformation $z \mapsto z + 1$ of $\mathcal{H}$.

If $N = 1$, the group $\Gamma_0(1)$ is simply $SL(2, \mathbb{Z})$.

**Definition 1.** Let $k$ be an integer, $N$ an integer and $\chi$ a character. The space $M(N, \chi, k)$ is (with some analytic restrictions) the space of functions $f(z)$ of $z \in \mathcal{H}$ of the form

$$f(z) := \sum_{n \geq 0} a_n e^{2i\pi nz}$$

and such that:

$$f((az + b)/(cz + d)) = \chi(a)(cz + d)^k f(z)$$

for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma_0(N)$. The integer $N$ is called the level, $k$ the weight.

Remark that we decided that our automorphic form $f$ should be periodic of period 1: with respect to the preceding informal definition, where $f$ was periodic of period $N$, we made the change of variable $z \mapsto z/N$. It is customary to write $q = e^{2i\pi z}$, so that

$$f = \sum_{n \geq 0} a_n q^n.$$

If $a_0 = 0$, then $f$ is said to be cuspidal at $\infty$: it vanishes on the cusp $\infty$ of the domain $\Gamma_0(N) \backslash \mathcal{H}$.

An example is $\Delta \in M(1, 1, 12)$ with

$$\Delta := q \prod_{n \geq 1} (1 - q^n)^{24}.$$

The space $M(N, \chi, k)$ is a finite dimensional vector space and its dimension is known. For example, $M(1, 1, 12) = \mathbb{C}\Delta$ and $M(1, 1, k) = 0$ if $k < 12$.

To a cusp form $f = \sum_{n > 0} a_n e^{2i\pi nz}$, one associates its $L$ series

$$L(s, f) = \sum_{n > 0} a_n n^{-s} = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(iy) y^s dy.$$

It is particularly nice to have an Euler product formula for $L(s, f)$ similar to the formula:

$$\zeta(s) = \prod_{p: \text{primes}} \frac{1}{1 - \frac{1}{p^s}}$$

for the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$. 
**Definition 2.** Let $\chi$ be a Dirichlet character. We say that $f = \sum_{n>0} a_n e^{2i\pi nz}$ is a newform of weight $k$ and level $N$ if:

1) $$L(s, f) = \prod_{p: \text{primes}} \frac{1}{1 - a_p p^{-s} + \chi(p) p^{k-1} p^{-2s}}$$

and

2) $z^k f(-1/Nz)$ is proportional to $\tilde{f}(z) = \sum_{n>0} \overline{a_n} e^{2i\pi nz}$.

We denote by $S(N, k, \chi)$ the space of such newforms.

Note that the coefficients $a_n$ of the expansion $f(z) = \sum_{n \geq 1} a_n e^{2i\pi nz}$ are entirely determined by the $a_p$ for $p$ prime and $a_1$.

I will ignore the description of the coefficient of proportionality in condition 2) above, although it has a great importance in the theory as the “epsilon” factor.

Using eventually lower levels, Hecke showed (with the help of operators later called Hecke operators, generating the so-called Hecke algebra) that every cusp form is a linear combination of newforms.

Recall the definition of Hecke operators. Let $p$ be a prime number and consider the set $D(p) = \{ \gamma \in GL(2, \mathbb{Z}); \det(\gamma) = p \}$. It is easy that every integral 2 by 2 matrix $g$ with integral coefficients of determinant $p$ can be written as $g_1 \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right) g_2$, where $g_1, g_2$ are integral matrices of determinant 1. In other words $D(p)$ is a double coset for the action of $SL(2, \mathbb{Z})$ by left and right translation.

$$D(p) = SL(2, \mathbb{Z})( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} )SL(2, \mathbb{Z}).$$

It $f$ is an automorphic form of even weight $k$ (for $SL(2, \mathbb{Z})$), then

$$(H_p f)(z) = p^{-1} \left( \sum_{u=0}^{p-1} f(z + u/p) + p^k f(pz) \right)$$

is again an automorphic form of weight $k$ and the operators $H_p$ commute for all primes $p$.

The definition of $H_p$ is natural when interpreting an automorphic form as a function on $SL(2, \mathbb{R})$. If $f$ is an automorphic form of even weight $k$ (for $SL(2, \mathbb{Z})$), the function

$$(Af)(g) = (a - ci)^{-k} f(g^{-1} \cdot i) = (a - ci)^{-k} f((di - b)/(ci + b))$$
is a function on $SL(2, \mathbb{R})$ invariant by right translation by $SL(2, \mathbb{Z})$.
Consider the double coset $\tilde{D}(p)$
$$\tilde{D}(p) = SL(2, \mathbb{Z}) \left( \begin{array}{cc} p^{1/2} & 0 \\ 0 & p^{-1/2} \end{array} \right) SL(2, \mathbb{Z})$$
in $SL(2, \mathbb{R})$. As $\tilde{D}(p)$ is left and right invariant by $SL(2, \mathbb{Z})$, if $F$ is a function of $SL(2, \mathbb{R})$ invariant by right translation by $SL(2, \mathbb{Z})$, then
$$(\mathcal{H}_p F)(g) = \frac{1}{p} \sum_{\gamma \in \tilde{D}(p)/SL(2, \mathbb{Z})} F(g\gamma)$$
is still right invariant by $SL(2, \mathbb{Z})$.

The equality
$$p^{k/2} \mathcal{H}_p A f = A(H_p f)$$
provides a natural interpretation of $H_p$ as summing a function over the finite $SL(2, \mathbb{Z})$-orbit $\tilde{D}(p)/SL(2, \mathbb{Z})$ in $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$.

Let us check this equality. Let $F = Af$ and compute $F(g(x, y)\gamma(u))$ with
$$g(x, y) = \left( \begin{array}{cc} y^{-1/2} & -xy^{-1/2} \\ 0 & y^{1/2} \end{array} \right), \quad \gamma(u) = \left( \begin{array}{cc} 1 & -u \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} p^{1/2} & 0 \\ 0 & p^{-1/2} \end{array} \right).$$
We obtain
$$p^{k/2} F(g(x, y)\gamma(u)) = y^{k/2} f\left( \frac{z + u}{p} \right).$$
If
$$\gamma_0 = \left( \begin{array}{cc} p^{-1/2} & 0 \\ 0 & p^{1/2} \end{array} \right),$$
we obtain
$$p^{k/2} F(g(x, y)\gamma_0) = p^{k} y^{k/2} f(pz).$$
The equality $p^{k/2} \mathcal{H}_p A f = A(H_p f)$ follows from the fact that the elements $\gamma(u), u = 0, \ldots, (p - 1)$ and $\gamma_0$ are the representatives of the finite set $\tilde{D}(p)/SL(2, \mathbb{Z})$.

A newform is an eigenvector for all operators $H_p$. Furthermore, the Fourier coefficient $a_p$ of a newform (with $a_1 = 1$) is exactly the eigenvalue of $H_p$. We record this important fact in a proposition.

**Proposition 3.** If $f$ is a newform with $a_1 = 1$, then
$$H_p f = a_p f.$$
The automorphic property of \( f \in S(N, k, \chi) \) translates immediately to a relation between \( L(s, f) \) and \( L(k - s, \tilde{f}) \): “proof”: change \( y \) in \( 1/Ny \) in the integration formula. This gives for example the proof of the functional equation for the Riemann Zeta function from the \( \Theta \) series automorphic property. More precisely, define for a newform \( f \),

\[
\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f). 
\]

Then one obtain

\[
\Lambda(s, f) = \epsilon(f)N^{k/2-s}\Lambda(k - s, \tilde{f})
\]

with a constant factor \( \epsilon(f) \), the “\( \epsilon \)” factor.

The hope of the “automorphic gang” is that any function \( L \) (defined as a nice looking product over primes) arising in the world of mathematics is the \( L \) function of an automorphic form, in short \( L \) is modular. This optimistic assumption had already many successes. Recall for example that the word “END” appears on Wiles proof of Fermat’s theorem after the phrase: If there is a solution of Fermat equation, then there is a newform of weight 2 and level 2 but there is no such forms. END.

2.2. Abelian reciprocity law.

2.2.1. A few notations. Let \( G \) be a group.

Two elements \( g \) and \( g' \) are conjugated in \( G \) if there exists \( u \in G \) such that \( g' = ugu^{-1} \). If \( G \) is abelian (commutative), then \( g' = g \). Otherwise, the conjugacy class \( O_g \) of an element \( g \) in a group \( G \) is the subset

\[
O_g := \{ ugu^{-1}, u \in G \}
\]

of all the conjugated elements to \( g \) in \( G \).

A representation \( \sigma \) of a group \( G \) (finite or infinite) associates to \( g \in G \) a linear transformation \( \sigma(g) \) of a vector space \( V \). We must have \( \sigma(1) = 1d_V \) and \( \sigma(g_1g_2) = \sigma(g_1)\sigma(g_2) \). In this report, our spaces \( V \) will be complex vector spaces.

If \( V \) is one dimensional, then \( \sigma \) is a character of \( G \). In general, \( \sigma(g) \) is represented by a matrix in a basis of \( V \). We can take (at least when \( V \) is finite dimensional) its trace \( \text{Trace}_V(\sigma(g)) \) and its characteristic polynomial \( \text{det}_V(1 - \sigma(g)) \). Remark that the trace and characteristic polynomial take same value on two conjugated elements \( g, g' \).

Most of the time, if \( V \) is a vector space provided with an action of \( G \), \( v \) a vector of \( V \) and \( g \) an element of \( G \), we note \( \sigma(g) \cdot v \) simply by \( g \cdot v \).

If \( V \) is a representation of \( G \) and \( H \) a subgroup of \( G \), we define the space \( V^H \) of \( H \) invariants by
\[ V^H := \{ v \in V; h \cdot v = v \text{ for all } h \in H \}. \]

Let \( p \) be a prime and \( F \) a finite extension of the field \( \mathbb{Z}/\mathbb{Z}p \). Then \( x \mapsto x^p \) is an automorphism of \( F \), called the Frobenius \( \text{Fr} \): the fixed elements of \( \text{Fr} \) is the ground field \( \mathbb{Z}/p\mathbb{Z} \) (\( a^p = a \mod p \) for all \( a \in \mathbb{Z} \)).

2.2.2. Abelian reciprocity law. The motivations of Langlands program come from number theory.

Let \( E \) be a number field: a finite extension of \( \mathbb{Q} \). We consider the ring \( O_E \) of integers of \( E \): that is the set of solutions \( x \) of an equation \( x^n + \sum_{i<n} a_i x^i = 0 \) with \( a_i \in \mathbb{Z} \). A prime \( p \in \mathbb{Z} \) gives rise to an ideal \( (p) = O_E p \) of \( O_E \). Assume \( E \) is a Galois extension, with Galois group \( G \). The ideal \( (p) \) factors as \( (\prod_i P_i)^e \) where \( P_i \) are different prime ideals in \( O_E \) and \( e \) is called the ramification index of \( p \). If \( e = 1 \), then \( p \) is said to be unramified. The prime ideals \( P_i \) are said to be above \( p \). Assume \( p \) is unramified (all \( p \), but a finite number), and choose \( P \) above \( p \). Then there exists a unique element \( \text{Fr}_P \) of the Galois group \( G \) such that

- The element \( \text{Fr}_P \) leaves stable the prime \( P \) above \( p \).
- \( \text{Fr}_P \) induces the Frobenius transformation in the field \( O_E/P \) extension of \( \mathbb{Z}/p\mathbb{Z} \).

For example, if \( p \) splits completely (number of \( P \) above \( p \)=degree of the extension), then \( \text{Fr}_P = I \).

For \( p \) unramified, when the prime \( P \) varies above \( p \), the Frobenius element \( \text{Fr}_P \in G \) are in the same conjugacy class, and sometimes we will write \( \text{Fr}_p \) for an element in the conjugacy class of \( \text{Fr}_P \). If \( G \) is abelian, \( \text{Fr}_p \) is well defined.

If \( p \) is ramified, choose a prime \( P \) above \( p \), let \( D_P \) the stabilizer of the prime ideal \( P \) in the Galois group \( G \). The group \( D_P \) is called the decomposition group. There is a morphism of \( D_P \) onto the Galois group of \( O_E/P \) with kernel \( I_P \), the inertia group. The cardinality of \( I_P \) is the ramification index \( e \). Now \( \text{Fr}_P \) is still defined as the element producing the Frobenius transformation in the field \( O_E/P \) extension of \( \mathbb{Z}/p\mathbb{Z} \), but now it leaves in \( D_P/I_P \).

- **Example** Take the example \( E = \mathbb{Q}[\sqrt{-1}] = \mathbb{Q}[i] \) with ring of integers \( O_E = \mathbb{Z} + \mathbb{Z}i \). The Galois group is \( G = \mathbb{Z}/2\mathbb{Z} \), generated by the complex conjugation \( i \mapsto -i \). If \( p \neq 2 \) is the sum of 2 squares \( p = n^2 + m^2 \), then \( (p) = (n+mi)(n-mi) \), thus \( p \) splits completely and \( \text{Fr}_p = I \).
If \( p \neq 2 \) is not a sum of two squares, then the extension \( O_E/O_{E^p} \) is an extension of degree 2 and \( \text{Fr}_p = \text{Conjugation} \).

We have \((2) = (1 + i)(-i(1 + i))\), so that \((2) = (\mathbb{Z}[i])(1 + i)^2\), and 2 is the only ramified prime.

In this example, \( \text{Fr}_p = I \) is equivalent to \( p = 1 \mod 4 \).

It is very important to understand \( \text{Fr}_p \) as we want to understand the “factorization” of primes.

The abelian reciprocity law is the following.

**Theorem 4. Abelian reciprocity law (KRONECKER-WEBER).**

Let \( E \) be a Galois extension of \( \mathbb{Q} \) with Galois group \( G \). Let \( \sigma : G \rightarrow \mathbb{C}^* \) a character of the Galois group. Then there exists an integer \( N_\sigma \) (the conductor) and a primitive Dirichlet character \( \chi_\sigma \) of \( \mathbb{Z} \) of level \( N_\sigma \) such that

\[
\sigma(\text{Fr}_p) = \chi_\sigma(p)
\]

for all unramified \( p \).

**Example**

Take \( \mathbb{Q}(i) \) with \( G = \{1, \text{conjugation}\} \) as Galois group. Thus for the character \( \sigma : G \mapsto \{1, -1\} \subset \mathbb{C}^* \), sending complex conjugation to \((-1) \in \mathbb{C}^* \), we have \( \sigma(\text{Fr}_p) = \chi_\sigma(p) \) where \( \chi_\sigma \) is the primitive Dirichlet character mod 4 defined by \( \chi_\sigma(n) = (-1)^{(n-1)/2} \), when \( n \) is odd.

Let \( E \) be an abelian extension of \( \mathbb{Q} \) and \( \sigma \) a character of \( G \). To each prime integer \( p \), introduce the \( L \)-factor \( L_p(s, \sigma) \). When \( p \) is unramified:

\[
L_p(s, \sigma) = \frac{1}{1 - \sigma(\text{Fr}_p)p^{-s}}.
\]

For ramified \( p \), define

\[
L_p(s, \sigma) = 1.
\]

Then define the \( L \)-function

\[
L(s, \sigma) = \prod_{p \text{; primes}} L_p(s, \sigma).
\]

From the Abelian reciprocity law, we see that

\[
L(s, \sigma) = \sum_{n \geq 1} \chi_\sigma(n)n^{-s}
\]

and again Poisson formula shows that \( L(s, \sigma) \) is entire, if \( \chi \neq 1 \), and has some functional equation.

To summarize:

The abelian reciprocity law relates the Galois group of an abelian extension to the groups \((\mathbb{Z}/N\mathbb{Z})^*\) which are the Galois group of the
cyclo
tomic field $\mathbb{Q}(\mu_N)$ obtained by adding to $\mathbb{Q}$ a primitive $N$-th root of unity $\mu_N$. It has as consequence that any abelian extension can be imbedded in a cyclotomic field:

*Example:* $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\mu_8)$ as $\sqrt{2} = e^{i\pi/4} + e^{-i\pi/4}$.

### 2.3. Artin $L$-functions and Langlands conjectural reciprocity law

#### 2.3.1. Artin $L$ functions.

Langlands aim was to formulate and eventually prove a “non-abelian” reciprocity law.

Now consider representations $\sigma$ of $G$, where $G$ is the Galois group of a Galois extension of $\mathbb{Q}$ in a finite dimensional complex vector space $V$ of dimension $n$. Thus each element $g \in G$ is represented by a $n \times n$ complex matrix.

To each prime integer $p$, introduce the local $L$-factor $L_p(s, \sigma)$, as follows. Take a prime $P$ above $p$.

If $p$ is unramified, define

$$L_p(s, \sigma) = \frac{1}{\det(1 - \sigma(Fr_P)p^{-s})}.$$

If $p$ is ramified, we consider the space $V^{I_P}$ of invariants of $V$ under the inertia group $I_P$. Then we can define

$$L_p(s, \sigma) = \frac{1}{\det(1 - \sigma(Fr_P)p^{-s})}$$

where the transformation $Fr_P$ acts on the space of $I_P$ invariants.

The definition of the local $L$-factor does not depend of the choice of $P$ above $p$.

Artin defined

$$L(s, \sigma) = \prod_{p: \text{primes}} L_p(s, \sigma).$$

If $\sigma$ is of dimension 2, and $p$ is unramified, the element $Fr_p$ (defined only up to conjugacy) of $G$ is send to a matrix $\sigma(Fr_p)$ with two eigenvalues $\alpha_p, \beta_p$. The local $L$ factor associated by Artin to $\sigma$ is thus

$$L_p(s, \sigma) = \frac{1}{\det(1 - \sigma(Fr_p)p^{-s})} = \frac{1}{1 - \text{Trace}(\sigma(Fr_p))p^{-s} + \det(\sigma(Fr_p))p^{-2s}}.$$

Remark that $g \mapsto \det(\sigma(g))$ is a one dimensional character of $G$, so that by the Abelian reciprocity theorem, we already have the existence of a Dirichlet character $\chi$ of level $N$ such that

$$\det(\sigma(Fr_p)) = \chi(p).$$
Thus
\[ L_p(s, \sigma) = \frac{1}{1 - (\alpha_p + \beta_p)p^{-s} + \chi(p)p^{-2s}}. \]

Artin conjectured that \( L(s, \sigma) \) is entire, when \( \sigma \) is irreducible.

Let \( \sigma \) be an odd representation of \( G \) (help ?). The conjecture of Artin (still unproved) can be restated by saying that \( L(s, \sigma) \) is automorphic: more precisely there exists a newform \( f \in S(N, \chi, 1) \) of level \( N \), weight one and Dirichlet Character \( \chi \) such that
\[ L(s, \sigma) = L(s, f). \]

In short, the traces \( \text{Trace}(\sigma(Fr_p)) \) of the Frobenius elements should be the Fourier coefficients \( a_p \) of a newform \( f = \sum_{n \geq 1} a_n e^{2\pi i n z} \) for all unramified prime \( p \).

2.3.2. Langlands reciprocity conjecture. Let now \( \sigma \) a representation of the Galois group in \( \mathbb{C}^n \). Langlands formulated the conjecture that \( L(s, \sigma) \) is the \( L \)-function associated to an automorphic representation of \( GL(n, \mathbb{A}) \) where \( \mathbb{A} \) is the ring of adeles of \( \mathbb{Q} \). Roughly speaking representations \( \sigma \) of degree \( n \) of the absolute Galois group \( G(\overline{\mathbb{Q}}/\mathbb{Q}) \) parametrize some automorphic representations of \( GL(n, \mathbb{A}) \).

We give more details later.

Langlands-Tunnell (7, 9) have proved that the Artin \( L \)-function for representations of the Galois group on \( \mathbb{C}^2 \) with image a solvable group are \( L \)-functions of automorphic forms. Already to settle this case, base change, the trace formula, the fundamental lemma, the lifting of automorphic representations have to be established for non trivial cases. The Langlands program has taken a life of its own since then, and many results have been proved, as parts of the Langlands original “program” or inspired by it. Indeed, many natural problems: functoriality, base change, local correspondance arise from this dictionary (representations of the Galois group = automorphic forms). It is clear that now the study of automorphic forms is a central topic in mathematics, with interconnections with algebraic geometry, arithmetic geometry, representations of quantum groups, etc...

Up to now, we discussed the field \( \mathbb{Q} \). It is also important to have the same theory for any number field. For example, Langlands proof of the Artin’s conjecture for the solvable subgroup \( A_4 \) (\( A_4 \subset PGL(2, \mathbb{C}) = SO(3, \mathbb{C}) \) embedded as the symmetry group of the tetrahedron) uses a composition series leading to study cubic extensions \( E \) of \( \mathbb{Q} \) and the corresponding base change.
So let $E$ be a number field. Similarly a representation of the Galois group $G(E/E)$ of dimension $n$ should lead to an automorphic representation of $GL(n, \mathbb{A}_E)$, where now $\mathbb{A}_E$ is defined using all completions of $E$. If $E$ is a number field, then clearly $G(\overline{\mathbb{Q}}/\mathbb{Q})$ has an homomorphism into the finite group $G(E/\mathbb{Q})$ with kernel $G(\mathbb{Q}/E)$. Recall that if $K$ is a given finite group, it is unknown if $K$ is a Galois group of a number field. So the idea to study representations of $G(\overline{\mathbb{Q}}/\mathbb{Q})$ with finite image may be a tool to understand the possible Galois groups.

2.3.3. Arithmetic varieties and Langlands conjecture. (very sketchy)

There are many representations of the Galois group $G = G(\overline{\mathbb{Q}}/\mathbb{Q})$ occuring in “nature”.

Let $X$ be an arithmetic variety: the set of solutions of equations defined over $\mathbb{Z}$. There are very natural representations of the absolute Galois group $G$ associated to $\ell$-adic cohomology groups of $X$ (these representations do not factor through finite groups and do not have finite images in $GL(n, \mathbb{C})$), but with the same formulae as in the preceding section, they give rise to $L$-functions (called motivic $L$-functions). Furthermore if $X$ is provided with an action of a group $S$, then the $\ell$-adic cohomology groups $H^i_\ell$ are provided with a representation of $S \times G$.

For example, let $X := \{y^2 = 4x^3 - Ax - B, A, B \in \mathbb{Z}, A^3 - 27B^2 \neq 0\}$ a smooth elliptic curve, then its first $\ell$-adic cohomology group is a vector space of dimension 2 over $\overline{\mathbb{Q}}_\ell$ and this representation of dimension 2 of $G$ gives rise to the $L$-function attached by Hasse-Weil to $E$, described “concretely” as follows: we consider the number of points $N(p)$ of $X$ in the finite field $\mathbb{Z}/p\mathbb{Z}$. Then for good primes $p$, the local $L$ factor is $\frac{1}{1-a_p+1+(-1)^{g+1}}$, with $N(p) = 1 - a_p + p$. The Shimura-Taniyama-Weil conjecture, proved by Wiles and Taylor (+ Diamond, Conrad, Breuil), says that these $L$-functions are modular.

Another example. Let $X = X_0(N)$ be the Borel-Bailey compactification of $\Gamma_0(N)\backslash \mathcal{H}$, then $X$ is an arithmetic curve of genus $g$, called the modular curve. Thus there are exactly $g$ newforms of weight 2 and character $\chi = 1$. The $L$-function attached to the first $\ell$-adic cohomology group $H^1_\ell$ (a vector space of dimension $2g$) is the product of the $L$ functions $L(s, f_i)$ where $f_1, f_2, \ldots, f_g$ are the $g$ newforms. Here for each prime $p$, the Hecke operator $H_p$ as well as the Frobenius element $Fr_p$ acts on $H^1_\ell$ and have the same eigenvalues. This gives “the” explanation of the correspondence between automorphic forms and some representations of $G$. 
Higher dimensional analogues of the modular curves are Shimura varieties (for the symplectic group $\text{Sp}(n,\mathbb{Z})\backslash \text{Sp}(n,\mathbb{R})/U(n)$). Unfortunately for $GL(n)$ and $n > 2$, there is no analogue of the Shimura varieties.

3. Automorphic representations

3.1. The use of adeles. Interpreting the $L$ function of Artin associated to representations $\sigma$ of $G$ in $GL(n,\mathbb{C})$ needs the notion of automorphic representation. If $n = 2$, there is a nice theory (classical) of automorphic forms as part of the function theory of the upper half plane. But to generalize it to $GL(n,\mathbb{R})$, it is easier to use adeles. The factorisation of the $L$ function over primes will then have natural interpretations, etc...

If $p$ is a prime number, a $p$-adic integer is a series $\alpha = \sum_{n=0}^{\infty} a_p p^n$ with $a_p$ an integer between $(0, p - 1)$. Thus

$$\mathbb{Z}_p := \{ \alpha = \sum_{i=0}^{\infty} a_i p^i, 0 \leq a_i < p \}$$

and the ring of fractions of $\mathbb{Z}_p$ is naturally identified to

$$\mathbb{Q}_p = \{ \alpha = \sum_{i=i_0}^{\infty} a_i p^i, 0 \leq a_i < p \}$$

where now $i_0$ can be negative.

As $\mathbb{Z}$ is naturally embedded in $\mathbb{Z}_p$ by writing an integer in base $p$, the ring $\mathbb{Q}$ is embedded in $\mathbb{Q}_p$.

The order of an element $\alpha$ in $\mathbb{Q}_p$, $\alpha$ non zero, is the smallest $i$ with $a_i \neq 0$ and is denoted by $\text{val}(\alpha)$. The integer $p$ has order 1, and called the uniformizer. The group $\mathbb{Z}_p^*$ is exactly the set of elements of order 0 in $\mathbb{Q}_p$ and $\mathbb{Q}_p^*/\mathbb{Z}_p^* = \mathbb{Z}$, via the map $\text{val}$.

The topology on $\mathbb{Q}_p$ is as follows. A sequence $x_n$ converges to 0, if the orders of the elements $x_n$ tends to $\infty$ (more and more divisible by $p$). Thus it is clear that $\mathbb{Z}_p$ is compact and open. The normalized additive Haar measure of $\mathbb{Q}_p$ gives mass 1 to $\mathbb{Z}_p$. Thus on $\mathbb{Q}_p/\mathbb{Z}_p$, integrating means counting. For example, when $r > 0$, the measure of the set $\{ u, \text{val}(u) \geq -r \}$ modulo $\mathbb{Z}_p$ is $p^r$. The Haar measure on the locally compact multiplicative group $\mathbb{Q}_p^*$ is normalized by giving mass 1 to $\mathbb{Z}_p^*$. Thus integration on $\mathbb{Q}_p^*/\mathbb{Z}_p^* = \mathbb{Z}$ means counting.

The adele ring $\mathbb{A}$ is the ring $\mathbb{R} \times \prod_{\{ p: \text{primes} \}} \mathbb{Q}_p$ where we assume that almost all components $\alpha_p$ of an element $\alpha = (\alpha_p)$ are in $\mathbb{Z}_p$. 
Then \( \mathbb{Q} \) is embedded diagonally in \( A \):

\[
\alpha \mapsto (\alpha, \alpha, \alpha, \ldots)
\]

Let \( A^* \) be the group of invertible elements of \( A \). Clearly, we can multiply an element \( a \in A^* \) by some element \( \alpha \in \mathbb{Q}^* \) so that \( \alpha a \) is in \( \mathbb{R}^+ \times \prod_p \mathbb{Z}_p^* \). Thus

\[
A^* = \mathbb{Q}^* \times \mathbb{R}^+ \times \prod_p \mathbb{Z}_p^*.
\]

3.2. **A little more of representation theory.** If \( G \) is a (locally compact) group, a very natural representation of \( G \) is the regular representation of \( G \). It acts on the space \( L^2(G) \) of functions (square integrable) on \( G \) by

\[
R(g)f(u) = f(g^{-1}u).
\]

In particular if \( f \) is a function in \( L^2(G) \), the closed subspace \( \text{Translate}(f) \) generated by linear combinations of translates of the function \( f \) is a \( G \)-invariant subspace of \( L^2(G) \).

More generally, if \( \Gamma \) is a subgroup of \( G \), the space \( L^2(G/\Gamma) \) of functions on \( G \) such that \( f(g\gamma) = f(g) \) for all \( g \in G, \gamma \in \Gamma \) is the “most natural” way to construct representations of \( G \). If \( f \) is in \( L^2(G/\Gamma) \), the space \( \text{Translate}(f) \) is a subspace of \( L^2(G/\Gamma) \). The group \( G \) acts on \( L^2(G/\Gamma) \) by left translations \( L(g_0)f(g) = f(g_0^{-1}g) \). The corresponding representation of \( G \) in \( L^2(G/\Gamma) \) is called a quasi regular representation (or a permutation representation) and is denoted by \( \text{Ind}^G_\Gamma \).

A representation \( \pi \) of \( G \) in a Hilbert space \( V \) is irreducible, if \( V \) does not admit (closed) non trivial invariant subspaces.

3.3. **From classical automorphic forms to automorphic representations.** Here we consider first \( GL(2) \). We write \( V := \{ p; \text{prime integers}\} \cup \{ \infty \} \), the set of valuations. The local groups are \( GL(2, \mathbb{Q}_p) \) for \( p \) a prime and \( GL(2, \mathbb{R}) \) for \( v = \infty \).

We write \( K_p := GL(2, \mathbb{Z}_p) \) (in particular \( \det(g) \in \mathbb{Z}_p^* \), for \( g \in GL(2, \mathbb{Z}_p) \)). An element \( g \) of the group \( GL(2, \mathbb{A}) \) is a family \( (g_v)_{v \in V} \) where for all prime \( p \), except a finite number, \( g_p \) is in \( K_p = GL(2, \mathbb{Z}_p) \). Similarly we can send \( GL(2, \mathbb{Q}) \) in \( GL(2, \mathbb{A}) \) where it becomes a discrete subgroup. We write \( K_0 = \prod_{p<\infty} K_p \). The center of \( GL(2, \mathbb{A}) \) is

\[
Z_{\mathbb{A}} = \{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}; a \in \mathbb{A}^* \}.
\]

Then

\[
K_0 \backslash GL(2, \mathbb{A})/GL(2, \mathbb{Q})Z_{\mathbb{A}} = SL(2, \mathbb{R})/SL(2, \mathbb{Z}).
\]
If \( f \) is a classical automorphic form on \( \mathcal{H} \) (for \( SL(2, \mathbb{Z}) \)) of weight \( k \), it is easy to see that \( \phi(g) = f(g^{-1} i)(ci + d)^k \) is a function on \( SL(2, \mathbb{R})/SL(2, \mathbb{Z}) \). Here \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then given a classical newform \( f \) (for \( SL(2, \mathbb{Z}) \)) on \( \mathcal{H} \) of weight \( k \), there exists a unique function \( F \) in \( L^2(GL(2, A)/GL(2, \mathbb{Q})Z_A) \) such that \( \phi(g) \) coincide with \( F \) on \( SL(2, \mathbb{R}) \) and such that \( F \) is invariant by left translations by all subgroups \( K_p \) for primes \( p \). Then the function \( F \) is an element of \( L^2(GL(2, A)/GL(2, \mathbb{Q})) \) and the space \( \text{Translate}(F) \) span an irreducible subspace in \( L^2(GL(2, A)/GL(2, \mathbb{Q})Z_A) \): this is by definition an automorphic representation of \( GL(2, A) \).

We summarize: A newform \( f \) (for \( SL(2, \mathbb{Z}) \)) is the same thing as a function \( F \) on \( GL(2, A)/GL(2, \mathbb{Q}) \) such that \( F \) is a decomposable vector \( F = \otimes_{v \in V} F_v \), and \( F_v \) is left invariant invariant by all compact groups \( GL(2, \mathbb{Z}_p) \) for all primes \( p \).

This definition can easily be generalized to any integer \( n \). We define \( GL(n, A), GL(n, \mathbb{Q}) \) as before, with embedding \( GL(n, \mathbb{Q}) \mapsto GL(n, A) \).

**Definition 5.** An automorphic cuspidal representation \( \pi \) of \( GL(n, A) \) is an irreducible subrepresentation of \( L^2(GL(n, A)/GL(n, \mathbb{Q})Z_A) \). Furthermore the representation must satisfy a certain “cuspidal” condition.

In fact, the representation \( \pi \) is necessarily given as a product \( \prod_{v \in V} \pi_v \) where \( \pi_v \) are irreducible representations of \( GL(n, \mathbb{Q}_v) \). There is a \( L \)-function associated to \( \pi \). Indeed for almost all primes \( p \), the representation \( \pi_p \) has a fixed vector under the maximal compact group \( K_p \) and gives rise to a representation of the Hecke algebra \( H_p \) in \( \mathbb{C} \), and we will see (next subsection) that this gives rise to a \( L \)-factor.

There are similar definitions for number fields of \( GL(n, \mathbb{A}_E) \), etc.

**General reciprocity law: Langlands conjecture.**

Let \( E \) be a finite extension of \( \mathbb{Q} \) with Galois group \( G \) and \( \sigma \) be an irreducible representation of \( G \) in \( \mathbb{C}^n \). Then there exists an automorphic cuspidal representation \( \pi_\sigma \) of \( GL(n, \mathbb{A}_E) \) such that

\[
L(s, \sigma) = L(s, \pi_\sigma).
\]

Recall that this conjecture is open even for \( n = 2 \) and \( E = \mathbb{Q} \).

### 3.4. Hecke algebras

A common tool for studying automorphic representations is the Hecke algebra. We already discussed this in the case of classical automorphic form.

Let \( G \) be a locally compact group and \( K \) a compact subgroup. The algebra \( H(G, K) \) is the algebra of (compactly supported) functions on
G invariant by left and right translations. This is an algebra under convolution:

$$(\phi_1 \ast \phi_2)(g) = \int_{g_1, g_2; g_1g_2 = g} \phi(g_1)\phi(g_2)$$

Clearly if $(\pi, V)$ is a representation of $G$, the action of the operators $\pi(\phi)$ (Definition 8) for $\phi \in \mathcal{H}(G, K)$ leaves stable the space $V^K$ of $K$-invariant vectors in $V$.

Let $G = GL(n, \mathbb{Q}_p)$ and $K = K_p = GL(n, \mathbb{Z}_p)$. Then the algebra $\mathcal{H}(G, K)$ is called the Hecke algebra. It has the following description

$$\mathcal{H}(G, K) = \mathbb{C}[T_1, T_2, \ldots, T_n, T_{-1}]$$

where $T_i$ is the characteristic function of the double coset of the matrix

$$h^i_p = \begin{pmatrix}
p & 0 & 0 & 0 & 0 \\
0 & p & 0 & 0 & 0 \\
0 & 0 & p & 0 & 0 \\
0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

with $i$ elements $p$ on the diagonal and the remaining $(n - p)$ entries equal to 1.

By definition, an unramified irreducible representation of $GL(n, \mathbb{Q}_p)$ is a representation having a fixed vector by $K_p$. The following proposition follows.

**Proposition 6. (Satake)** If $\pi$ is a unramified irreducible representation of $GL(n, \mathbb{Q}_p)$, then the space of $K$-fixed vectors is 1-dimensional and gives rises to a one dimensional $\chi$ character of $\mathcal{H}(G, K)$.

The local $L$-factor attached to the representation $\pi$ is by definition

$$L_p(s, \pi) = \prod_{i=1}^{n} \frac{1}{1 - a_i p^{-s}}$$

where the $i$-th symmetric function $s_i(a)$ of the $a_i$ is equal to $\chi(h^i_p)$ (up to some power of $p$).

The conclusion is:

Let $\mathcal{F} = \otimes_v \mathcal{F}_v$ be a decomposable vector in $L^2(GL(n, \mathbb{A})/GL(n, \mathbb{Q}))$ which generates an automorphic representation. Assume that at $p$, the function $\mathcal{F}_p$ is left invariant by $K_p$ ($p$ unramified for almost all $p$), then the function $\mathcal{F}_p$ on $GL(n, \mathbb{Q}_p)/GL(n, \mathbb{Z}_p)$ is an eigenvector for the action of the Hecke algebra $\mathcal{H}(GL(n, \mathbb{Q}_p), K_p)$.

### 3.5. Local Langlands conjecture

Let $G$ be the absolute Galois group $G(\mathbb{Q}/\mathbb{Q})$. The preceding conjecture says that there is a map from representations of $G$ of degree $n$ to automorphic representations
of $GL(n, \mathbb{A})$. It suggests that at the local level, irreducible representations $\pi_v$ of $GL(n, \mathbb{Q}_v)$ should have something to do with the set $Rep_n$ of representations of degree $n$ of the Galois group $G(\overline{\mathbb{Q}_v}/\mathbb{Q}_v)$. In fact, the right group to take is the Weil group $W_v$ (strongly related to the Galois group). This local correspondence has been proved.

**Theorem 7. Local Langlands correspondence** (Harris-Taylor [4]; simplified by Henniart[5]) Let $W_p$ the local Weil group. There exists a bijective map

$$Langlands : Rep_n(W_p) \rightarrow Irr(GL(n, \mathbb{Q}_p))$$

such that:

$$L_p(s, \sigma) = L_p(s, Langlands(\sigma))$$

$$\epsilon_p(s, \sigma) = \epsilon_p(s, Langlands(\sigma))$$

Note that the Langlands conjecture makes sense for any reductive group $G$. For this, we need to introduce $L$ groups. I will not do it here. The local Langlands conjecture for general reductive groups remains open.

The archimedean local Langlands conjecture was proven by Langlands.

3.6. **Global Langlands conjecture on function fields.** (very sketchy)

Let $p$ be a prime, and let $F_p = \mathbb{Z}/p\mathbb{Z}$ be the finite field with $p$ elements. As emphasized by Weil, there is a complete analogy between number fields and finite extensions of the field $F_p(t)$. Such an extensions is the field of rational functions on an algebraic curve $C$ defined over $F_p$.

- A Number field $F$ $\mapsto$ A finite extension of $F_p(t)$ $\mapsto$ A curve $C$
- A prime number $\mapsto$ An irreducible polynomial in $F_p[T]$ $\mapsto$ Points in $C$

The adele ring can be defined, and the theory of automorphic forms have been developed for any global field (number fields or function fields).

The Langlands program can be formulated. In fact, in this case, there are more geometric tools. In particular, Drinfeld constructed an arithmetic variety $X$ (chtoucas) over $C \times C$, where Hecke operators and Frobenius operators acts and could prove that they have same eigenvalues in the case of $GL(2)$. Lafforgue has obtained the proof of the **Global Langlands correspondence** over function fields in characteristic $p$ for $GL(n)$. 
In the last section, we will discuss very briefly “geometric Langlands correspondence” for a function field of characteristic 0.

4. Functoriality

4.1. The problem of liftings. The Langlands dictionary: “(Galois representations)= (automorphic representations)” suggests that some “trivial operations” in one side (restrictions of representations) have a counterpart on the other side (lifting of automorphic forms).

For example, existence of base change and liftings are conjectured to exist from this correspondence. Let us explain.

Base Change. If $E$ is a Galois extension of $\mathbb{Q}$, there is a map from $G(\mathbb{Q}|E)$ to $G(\mathbb{Q}|\mathbb{Q})$, so that a representation of $G(\mathbb{Q}|\mathbb{Q})$ gives us a representation of $G(\mathbb{Q}|E)$. Thus there should be a base change (denoted $BC$) from automorphic representations $\pi$ of $GL(n, \mathbb{A}_\mathbb{Q})$ to automorphic representations $BC(\pi)$ of $GL(n, \mathbb{A}_E)$ (and such that the action of $G(E|\mathbb{Q})$ on $GL(n, \mathbb{A}_E)$ leaves fixed the isomorphism class of $BC(\pi)$).

Liftings. Consider a homomorphism $h : GL(n, \mathbb{C}) \to GL(m, \mathbb{C})$. For example, if $g \in GL(n, \mathbb{C})$, then $g$ acts on the space $S^h(\mathbb{C}^n)$ of homogeneous polynomials of degree $h$ in $n$ variables by $(g \cdot P)(u) = P(g^{-1}u)$. If $n = 2$, and $h = 2$, from a representation of $G$ in $\mathbb{C}^2 = \{(x_1e_1 + x_2e_2)\}$ we obtain a representation of $G$ in $\mathbb{C}^3$ (basis $x_1^2, x_1x_2, x_2^2$). These representations are referred as the symmetric tensors.

If $\sigma$ is a representation of $G$ in $GL(n, \mathbb{C})$, composing with $h$, we obtain a representation of $G$ in $GL(m, \mathbb{C})$. Thus according to the Langlands dictionary, there should be a map $\text{Lift}_h$ associating to $\pi$ an automorphic representation of $GL(n, \mathbb{A})$ an automorphic representation $\text{Lift}_h(\pi)$ of $GL(m, \mathbb{A})$. In other words, there should be a lifting of automorphic representations, with of course correspondence of the local factors.

The lifting corresponding to the symmetric tensor $GL(2, \mathbb{C}) \to GL(3, \mathbb{C})$ has been established by Gelbart-Jacquet and Piatetskii-Shapiro. These liftings are highly non trivial. If the lifts for any symmetric tensor of a representation of $GL(2)$ was constructed, as well as general base change for $GL(2)$, then Artin conjecture would follow. We know a priori what the lift produces at the level of $L$ functions, but it is quite difficult to understand what is the lift for representations. Up to now, tools are “special”: for example, use of tensor products of the Weil representations, of the trace formula (with use of the fundamental lemma), etc...

4.2. Trace formula.
4.2.1. Baby version of the trace formula. Let $G$ be a finite group, and \( \phi \) be a function on $G$. Let $g$ be an element of $G$. The orbital integral $< O_g, \phi >$ of $\phi$ is the sum of the values of $\phi$ on the conjugacy class $O_g$ of $g$:

$$< O_g, \phi > = \sum_{g' \in O_g} \phi(g').$$

Let $\pi$ be a finite dimensional representation of $G$. Then the most important invariant of $\pi$ (this determines $\pi$) is its “character”: This is the function $g \mapsto \text{Trace}(\pi(g))$. Remark that this function is constant over conjugacy classes, so that the representation $\pi$ is completely determined by the value of its characters over the set of conjugacy classes.

**Example**: the regular representation $R$ of a finite group $G$: all elements of $g$ except the identity shifts the elements of the group, so that $\text{Trace}(R(g)) = 0$ if $g \neq 1$ while $\text{Trace}(R(1)) = |G|$.

It is useful to introduce the trace as a “distribution”, that is a linear form on functions on $G$:

$$< \text{Trace} \pi, \phi > = \frac{1}{|G|} \sum_{g \in G} \phi(g) \text{Trace}(\pi(g)).$$

**Example.** We have

$$\text{Trace}_{L^2(G)}(g) = \text{Dirac}_1(g).$$

Let $\Gamma$ be a subgroup of $G$ and let us consider the quasi regular representation $\pi = \text{Ind}^G_\Gamma 1$. To compute the trace of this representation on an element $g \in G$, we need to find the fixed classes $w$: $gw\Gamma = w\Gamma$ of the action of $g$ on $G/\Gamma$. That is $w^{-1}gw \in \Gamma$. We obtain:

$$\sum_{g} \phi(g)(\text{Trace} \pi)(g) = \sum_{g} |\{w \in G/\Gamma; wgw^{-1} \in \Gamma\}| \phi(g).$$

Writing $g = w\gamma w^{-1}$ for some $\gamma$ in $\Gamma$ and changing variables, we obtain:

$$< \text{Trace} (\text{Ind}^G_\Gamma 1), \phi > = \sum_{\gamma \in \Gamma/\sim} c(\gamma) < O_\gamma, \phi >$$

where the equivalence sign denote the action of $\Gamma$ on $\gamma$ by conjugation, and the constant $c(\gamma) = \frac{|G(\gamma)|}{|\Gamma(\gamma)|}$ is the quotient of the cardinal of the stabilizers of $\gamma$ in $G$ and in $\Gamma$.

If now we decompose the quasi-regular representation

$$\text{Ind}^G_\Gamma 1 = \bigoplus m_i \pi_i$$
in sum of irreducible representations $\pi_i$ with multiplicities $m_i$, we obtain
\[
\sum_i m_i < \text{Trace}(\pi_i), \phi > = \sum_{\gamma \in \Gamma/\sim} c(\gamma) < O_\gamma, \phi >.
\]
This is the baby version of the trace formula.

4.2.2. Teenager version of the trace formula. The preceding definitions make sense for a locally compact group $G$ with Haar measure $dg$.

**Definition 8.** If $\phi$ is a function on $G$, and $\pi$ a representation of $G$ in a Hilbert space $V$, the operator $\pi(\phi)$ is the operator (provided integrals are convergent)
\[
\pi(\phi) = \int_G \phi(g)\pi(g)dg.
\]
A representation $\pi$ in a Hilbert space $V$ may have a distributional trace:
\[
< \text{Trace} \pi, \phi > = \text{Trace}_V(\pi(\phi)) = \text{Trace}_V(\int_G \phi(g)\pi(g)dg).
\]

**Definition 9.** Let $O_\gamma$ be a closed orbit in $G$. Orbital integrals may also (in good cases) be defined as distributions:
\[
< O_\gamma, \phi > = \int_{O_\gamma} \phi(u)d_\gamma u
\]
with respect to an invariant measure $d_\gamma u$ on $O_\gamma$.

More generally, for a packet $\{\gamma\}$ of conjugacy classes, $< O_{\{\gamma\}}, \phi >$ is a sum (eventually weighted) sum of orbital integrals in the packet.

Thus we have two important sets of invariant distributions on the locally compact group $G$: the characters and the orbital integrals. One set is on the side of “representation theory”, the other one is on the side of “geometry”. Harmonic analysis on $G$ mainly consists on understanding how to write an invariant distribution in one set in function of the other set.

Let $\Gamma$ be a discrete subgroup with compact quotient $G/\Gamma$, then the representation $\pi := \text{Ind}_{\Gamma_0}^{\Gamma} 1$ has a trace and
\[
< \text{Trace} \pi, \phi > = \sum_{\gamma \in \Gamma/\sim} c(\gamma) < O_\gamma, \phi >
\]
leading to the relation
\[
(2) \quad \sum_i m_i < \text{Trace} \pi_i, \phi > = \sum_{\gamma \in \Gamma/\sim} c(\gamma) < O_\gamma, \phi >,
\]
if $\pi = \oplus m_i \pi_i$. The left hand side is the representation side, while the left hand side is the geometric side.

**Example:** $\mathbb{R}/\mathbb{Z}$.

On the representation side, we decompose $L^2(\mathbb{R}/\mathbb{Z})$ in a discrete sum of representations using Fourier series, while on the geometric side we just take the value of $\phi$ at integers. Thus we obtain

$$\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} e^{2\pi i nx} \phi(x) dx = \sum_{n \in \mathbb{Z}} \phi(n).$$

This is Poisson formula. We have to make some reasonable assumptions on $\phi$ for the Poisson formula to hold: for example, if $\phi$ in the Schwartz space, this is of course true.

Arthur-Selberg trace formula is the generalization of the simple formula (2) to cases of non compact quotients as $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$. It is a main tool in automorphic forms. It allows to deduce from geometric statements the existence of some wanted representations. Let us very roughly explain how it is used in lifting questions.

### 4.3. Transfer and the fundamental lemma.

Now let $G_1$ and $G_2$ be two locally compact groups. Assume that there is a natural map $N := Conj(G_1) \to Conj(G_2)$ (or more generally a map between packets of conjugacy classes). For example, let $G_1$ be a subgroup of $G_2$. Then we associate to a conjugacy class $O$ of $G_2$ the packet $O \cap G_1$ of conjugacy class of $G_1$. However in general the groups $G_1, G_2$ do not need to be related.

**Example.** Let $G_1 = SU(2)$ and $G_2 = SL(2,\mathbb{R})$. Conjugacy classes of $G_1$ are classified by the matrices in

$$T := \{ g(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \}. $$

We associate to the conjugacy class of $g(\theta)$ in $SU(2)$ the conjugacy class in $SL(2,\mathbb{R})$ of the same matrix $g(\theta)$ which is also in $SL(2,\mathbb{R})$ (more precisely, the packet of the two conjugacy classes of $g(\theta)$ and $g(-\theta)$).

Neither of the groups $SL(2,\mathbb{R})$ and $SU(2)$ is a subgroup of the other. In Langlands terminology, $SL(2,\mathbb{R})$ is an endoscopic group of $SU(2)$, and they share the same Cartan subgroup $T$.

Let $G_1, G_2$ be reductive groups over a local field with a map $N : Conj(G_1) \mapsto Conj(G_2)$. Let $\pi$ be an irreducible representation of $G_1$. Then the character of $\pi$ is a distribution, which is is given on a dense open set (the union of the closed conjugacy classes of maximal dimension) by integration against a smooth function. If $\pi$ is an irreducible
representation of $G_1$, we say that a representation $\pi'$ (or a finite sum of irreducible representations of $G_2$) is the lift of $\pi$ if the character of $\pi'$ coincides with the character of $\pi$ on matched conjugacy classes up to a “transfer factor”.

**Example**

In the example of SU(2), given the finite dimensional irreducible representation $\pi_d$ of SU(2) of dimension $d + 1$ with character

$$\text{Trace} \pi_d(g(\theta)) = \sum_{|j| \leq d, j \equiv d \mod 2} e^{ij\theta}$$

we can find a sum $\pi'_d$ of two infinite dimensional irreducible representations of SL(2, $\mathbb{R}$) such that

$$\text{Trace} \pi'_d(g(\theta)) = \sum_{|j| > d, j \equiv d \mod 2} e^{ij\theta}$$

As distributions, these characters (up to sign) coincide on the conjugacy classes $g(\theta), \theta \notin \{0, \pi\}$. Indeed (for $d$ even)

$$\text{Trace}(\pi_d)(g(\theta)) + \text{Trace}(\pi'_d)(g(\theta)) = \sum_{a \in 2\mathbb{Z}} e^{ia\theta} = \delta_0(\theta) + \delta_\pi(\theta).$$

For reductive groups $G_1, G_2$ over $\mathbb{Q}_p$, we see that there may be a relation between conjugacy classes if $G_1$ and $G_2$ share a common subgroup $T$. More exactly, let $T$ be an abelian group with two homomorphisms in $G_1, G_2$ with images a maximal abelian subgroup of semi-simple elements of $G_1$ (respectively $G_2$) (Cartan subgroups of $G_1, G_2$). Then an element $\gamma$ of $T$ give rise to conjugacy class $O_{G_1,1}\gamma$ and $O_{G_2,1}\gamma$ in $G_1$ and in $G_2$ (or packets). The transfer conjecture says (roughly) that if $f_1$ is a function on $G_1$, there exists a function $f_2$ on $G_2$ such that the orbital integrals of $f_1$ on $O_{G_1,1}\gamma$ coincide with the orbital integral of $f_2$ on $O_{G_2,1}\gamma$ (some factors are needed) for all $\gamma \in T$. One particular case is as follows. Assume $K_1 = G_1(\mathbb{Z}_p), K_2 = G_2(\mathbb{Z}_p)$ are maximal compact subgroups of $G_1, G_2$. Then one hope that the orbital integrals of the characteristic functions $1_{K_1}$ and $1_{K_2}$ are related on $O_{G_1,1}\gamma$ and $O_{G_2,1}\gamma$. In short $1_{K_1}$ is the transfer of $1_{K_2}$ (and Waldspurger proved that transfer of other functions follows automatically). This is the fundamental Lemma: (still a conjecture)

**Fundamental Lemma (Very roughly)**

$$< O_{G_1,1}\gamma, 1_{K_1} > = \text{(transfer constant)}* < O_{G_2,1}\gamma, 1_{K_2} >.$$

The precise statements have been formulated by Langlands-Shelstad. This “matching of orbital integrals” is one of the tools to obtain liftings
of automorphic forms. Indeed this equality (introduced by Labesse-Langlands) allows to compare the contributions of spherical functions in the trace formula at almost all places.

We give some explanations only in the case of linear groups and unitary groups, and we give a simplified version.

4.3.1. Fundamental lemma and counting lattices. I will state the fundamental lemma for the example of the linear group. It is equivalent to a problem of counting lattices stable under a transformation $\gamma$. In the next subsection, I will give the calculation of Labesse-Langlands in a case where it can be done “by hand”.

Let $F$ be a local field, and $E = \bigoplus_{i=1}^{n} F e_i$ be the standard $n$ dimensional vector space over $F$, with standard $O_F$-lattice $L_0 = \bigoplus_{i=1}^{n} O_F e_i$. Then the set $GL(n, F)/GL(n, O_F)$ is in bijection with the set $L$ of lattices $L$ over $O_F$: $L = \bigoplus_{i=1}^{n} O_F A_i$, with $A_i$ independent vectors in $E$. We denote $GL(n, O_F)$ by $K$ as usual. We further decompose $L$ in the unions of $L(r)$ where $r$ is an integer (mod $n$) and

$L(r) = \{ L \in L; \text{length}(L/L \cap L_0) - \text{length}(L_0/L \cap L_0) = r \mod n \}$.

Example: Let us consider $E = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$. If

$L(x, y) = \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p (xe_1 + ye_2),$

then we obtain all lattices up to homotheties ($L \mapsto \lambda L$), when $y$ varies in $\mathbb{Q}_p^* / \mathbb{Z}_p^*$, and $x$ in $\mathbb{Q}_p / \mathbb{Z}_p$. If $\text{val}(y)$ is even, then $L(x, y)$ is in $L(0)$, while if $\text{val}(y)$ is odd, $L(x, y)$ is in $L(1)$.

Let $\gamma \in GL(n, F)$ be an element such that its characteristic polynomial is irreducible. Let $L(\gamma)$ be the set of lattices stable by $\gamma$

$L(\gamma) := \{ L \in L; \gamma(L) = L \}$.

Then this number, modulo homothetic lattices, is finite and (for an adequate measure normalization), the orbital integral just counts the number of elements in $L(\gamma)$, modulo homotheties.

$\langle O_\gamma, 1_K \rangle = \text{cardinal}(L(\gamma)/Z),$

where $Z$ denotes the action of $\mathbb{Z}$ by homotheties $L \mapsto \lambda L$.

Example: Let $E = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ (with $p \neq 2$). Let $\gamma \in GL(2, \mathbb{Z}_p)$ of the form

$\gamma := \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix}$
where $\delta \in \mathbb{Z}_p^*$ is not a square, $a \in \mathbb{Z}_p^*$ and $\text{val}(b) > 0$. Then the characteristic polynomial of $\gamma$ is irreducible (here $b \neq 0$). It is easy to see that $L(x, y)$ is stable by $\gamma$, if and only if

$$-\text{val}(b) \leq \text{val}(y) \leq \text{val}(b)$$

(3)

$$\frac{1}{2}(\text{val}(y) - \text{val}(b)) \leq \text{val}(x)$$

(4)

As $y$ varies in $\mathbb{Q}_p^*/\mathbb{Z}_p^* \sim \mathbb{Z}$, and $x$ in $\mathbb{Q}_p/\mathbb{Z}_p$, we see that the set $L(\gamma)$ modulo homotheties is finite.

To formulate the fundamental lemma, we need to introduce twisted integrals. Instead of giving the integral definition, we just give what it computes:

**Lemma 10.** Let $\kappa$ an integer $0 \leq \kappa \leq (n - 1)$. The twisted orbital integral $< O_{\gamma, \kappa}, 1_K >$ is equal to

$$< O_{\gamma, \kappa}, 1_K > = \sum_{r=0}^{n-1} e^{2i\pi \kappa/n} \text{cardinal}((L(\gamma) \cap L(r))/\mathbb{Z}).$$

If $\kappa = 0$, we obtain just the usual orbital integral $< O_{\gamma, 0}, 1_K >$.

Let $F'$ be an extension of $F$ of degree $d$, where $n = dm$. It is clear that an element of $GL(m, F')$ gives rise to an element of $GL(n, F)$ (a vector space of dimension $m$ over $F'$ is of dimension $n = md$ over $F$). Let $\xi : GL(m, F') \rightarrow GL(n, F)$ be the corresponding homomorphism. Let $\gamma' \in GL(m, F')$ such that $\gamma = \xi(\gamma')$ has an irreducible characteristic polynomial. We denote by $G_1 = GL(m, F')$ and by $G_2 = GL(n, F)$, the compact groups $K_1$ and $K_2$ being as usual. Then the fundamental lemma asserts

**Fundamental Lemma for linear groups**

$$< O_{\gamma', 1_K}, G_1 > = \Delta(\gamma) < O_{\gamma', 1_K}, G_2 >.$$

Here $\gamma = \xi(\gamma')$ and is assumed to have an irreducible characteristic polynomial. The factor $\Delta(\gamma)$ is an explicit function of the eigenvalues of $\gamma$ and is called the transfer factor.

This “lemma” has been proved by Waldspurger [10].

**4.3.2. Labesse-Langlands simplest example.** The following case is the simplest case of the cases encountered by Labesse-Langlands. We will see that the computation is possible to do by hand, but relies already on wonderful cancellations.

Let $n = 2, d = 2, m = 1, G_1 = GL(2, \mathbb{Q}_p)$ and $G_2 = GL(1, F')$, where $F'$ is a quadratic extension of $\mathbb{Q}_p$. 
Thus consider $F' = \mathbb{Q}_p(\sqrt{\delta})$ where $\delta \in \mathbb{Z}_p^*$ is not a square. Then for $\gamma' = a + b\sqrt{\delta}$ non zero in $F'$, the element $\xi(\gamma') \in GL(2, \mathbb{Q}_p)$ is equal to $\gamma := \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix}$.

Then the fundamental lemma asserts that

$$< O_{G_{\gamma',1}}^{G_1}, 1_{K_1} > = (-p)^{\text{val}(b)} < 1_{O_{F'}^*}, \gamma' > .$$

Let us prove the fundamental lemma above: The first member is 1 or 0 according to the fact that $\gamma' \in O_{F'}^*$ or not. It is easy to see that if $\gamma'$ is not in $O_{F'}^*$, there is no invariant lattice under $\gamma$, so that the formula is true in this case. Assume now that $\gamma$ is in $O_{F'}^*$ and that $\gamma = a + b\sqrt{\delta}$ with $a, b \in \mathbb{Z}_p$, $\text{val}(b) > 0$ and $\text{val}(a) = 0$. From the preceding description of all the lattices $L(x, y)$ stable by $\gamma$, we have first to count for $\text{val}(y)$ fixed, the number $q(y)$ of $x \in \mathbb{Q}_p/\mathbb{Z}_p$ with $\text{val}(x) \geq \frac{1}{2}(\text{val}(y) - \text{val}(b))$.

As $\text{val}(x)$ is an integer, this is $p^s$ where $s$ is $\frac{1}{2}(\text{val}(y) - \text{val}(b))$ if $\text{val}(y)$ and $\text{val}(b)$ have same parity, or $\frac{1}{2}(\text{val}(y) + 1 - \text{val}(b))$ if $\text{val}(y)$ and $\text{val}(b)$ do not have the same parity. When we vary $\text{val}(y)$ from $\text{val}(b)$ to $-\text{val}(b)$, this number $q(y)$ remains the same for 2 consecutive values of $\text{val}(y)$.

As we compute twisted orbitals, we take the alternate sum of the numbers $q(y)$. It follows that all terms cancel except for the last possible value of $\text{val}(y) = -\text{val}(b)$, and the formula follows.

4.3.3. Unitary groups. Assume now $F'$ is a non ramified quadratic extension of $F$, and let $E' = \oplus_{i=1}^n F'e_i$. Let $J$ be a matrix written as

$$J = \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & c_n \end{pmatrix} \right\}$$

where $c_i \in O_{F'}^*$. Denote $x \mapsto \overline{x}$ the conjugation in $F'$. Consider the unitary group

$$G = U(n, J, F) = \{ M \in GL(n, F'); MJM^t = J \}$$

and let $K = GL(n, O_{F'}) \cap G$. The matrix $J$ determines an hermitian form $q_J$ on $E'$, and $U(n, J, F)$ leaves this form invariant.
Take
\[ \gamma = \{ \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & \ast & 0 & 0 \\ 0 & 0 & 0 & \ast & 0 \\ 0 & 0 & 0 & 0 & \gamma_n \end{pmatrix} \} \]
where \( \gamma_i \in F' \) satisfies \( \gamma_i \overline{\gamma_i} = 1 \).

Then the orbital integral \( \langle O^G_{\gamma}, 1_K \rangle \) is the number of \( O_{F'} \)-lattices \( L \subset E' \) stable by \( \gamma \) and self-dual with respect to the quadratic form \( q_J \).

Let us consider the packet \( \{ \gamma \} \) of conjugacy classes of elements \( \gamma' \) of \( U(n, J, F) \) conjugated to \( \gamma \) in \( GL(n, F') \). There is a similar twist factor \( \kappa \) and we can consider the corresponding sum of orbital integrals is denoted by \( \langle O^G_{\gamma, \kappa}, 1_K \rangle \). The sum (without weights \( \kappa \)) over \( \{ \gamma \} \) is called the stable orbital integral and denoted by \( \langle SO^G_{\gamma}, 1_K \rangle \).

Ngo Bao Chau and Laumon proved the fundamental lemma for stable orbital integrals: that is the stable orbital integral \( \langle SO^G_{\gamma}, 1_K \rangle \) is compared to the similar twisted packet \( \langle O^G_{\gamma, \kappa}, 1_K \rangle \). Most remarkably, the proof uses equivariant cohomology of Hitchin’s moduli space.

The proof of the fundamental lemma should lead to advances on \( L \)-functions attached to Shimura varieties (higher cases of the modular curves), etc...

5. Geometric Langlands correspondence

A geometric analogue of the Langlands program has been formulated by Drinfeld. Here a finite extension \( E \) of \( F_q(t) \) is replaced by the field of meromorphic functions \( \mathcal{M}(C) \) over a complex compact curve \( C \).

Let us first explain the analogy of the geometric program with the number field program.

Let \( P_1(C) = \{ [z_1, z_2] \} / \mathbb{C}^* \). The ring of rational functions on \( P_1(\mathbb{C}) \) is \( \mathbb{C}(T) \) with \( T = z_1/z_2 \). A finite extension of \( \mathbb{C}(T) \) is the same thing as a covering of \( P_1(\mathbb{C}) \). Namely if \( m : C \to P_1(\mathbb{C}) \) is a finite cover, we embed \( \mathbb{C}(T) \) into \( \mathcal{M}(C) \), the function field over \( C \), by the map \( m^* \). Thus, the analogue of a number field will be the field \( \mathcal{M}(C) \) of functions on a complex curve \( C \). A place of \( \mathcal{M}(C) \) is simply a point \( a \) of \( C \), with corresponding local field \( F_a \) Laurent series in \( (z - a) \). (Here
$z$ is a local coordinate around $a$). We see that this local field is very analogous to $\mathbb{Q}_p$, which are “Laurent series” in $p$.

Example If $p_1, p_2, p_3$ are distinct primes, then the analogue of $F := \mathbb{Q}(\sqrt[p_1p_2p_3]{})$ will be the field of functions on the elliptic curve $C := \{y^2 = (z - a_1)(z - a_2)(z - a_3)\}$. Remark that the cover $(y, z) \mapsto z$ has two points, except at $a_1, a_2, a_3$. The only points where this cover is ramified are the points $a_1, a_2, a_3$. Similarly primes $p$ different from $p_1, p_2, p_3$ (and 2) are unramified in $\mathbb{Q}(\sqrt[p_1p_2p_3]{})$.

So let $C$ be a complex curve and $\mathcal{M}(C)$ its function field. We will consider only the “unramified” case: the fundamental group $G := \pi(C)$ of the curve $C$ is a quotient of the absolute Galois group and we will consider only representations of this quotient group. This corresponds to the Galois group of unramified covers of $C$. Indeed let $Y$ be an unramified cover of $C$, then $\mathcal{M}(Y)$ is an extension of $\mathcal{M}(C)$. A deck transformation $Y \to Y$ induces a transformation of $\mathcal{M}(Y)$ which is the identity on $\mathcal{M}(C)$, thus the Galois group $\text{Galois}(\mathcal{M}(Y)/\mathcal{M}(C))$ is the group of deck transformations. Now, as $Y$ is unramified, an element of $g$ induces a deck transformation on $Y$. Thus, we have a morphism from $G$ to Galois groups $G(\mathcal{M}(Y)/\mathcal{M}(C))$ for the unramified cover $Y \to C$. In the following, only representations of $G = \pi(C)$ will be considered, and this corresponds (in the dictionary below) to automorphic representations everywhere non ramified (thus there exists a canonical vector in this representation).

Here is the dictionary of analogies for this situation (the unramified case).

- **1**
  
  (quotient of the) Galois group of a number field $F$
  
  $\mapsto$
  
  Fundamental group of $C$

- **2**
  
  Representation of the Galois group
  
  $\mapsto$
  
  Local system on $C$

- **3**
  
  The double coset defined by $g = (g_v)_{v \in V}$ in $GL(n, O_F) \backslash GL(n, \mathbb{A}_F)/GL(n, F)$
  
  $\mapsto$
  
  A vector bundle of rank $n$ on $C$.

- **4**
\[ GL(n, O_F) \backslash GL(n, \mathbb{A}_F) / GL(n, F) \]

the moduli space \( \text{Bun}_n \) of rank \( n \) vector bundles over \( C \).

\[ GL(n, O_F) \backslash GL(n, \mathbb{A}_F) / GL(n, F) \]

\[ \text{A perverse sheaf } F \text{ on } \text{Bun}_n. \]

\[ \text{A (nice) function } f \text{ on } GL(n, O_F) \backslash GL(n, \mathbb{A}_F) / GL(n, F) \]

\[ \text{A Hecke eigensheaf } F \text{ on } \text{Bun}_n. \]

\[ \text{An eigenfunction of the Hecke operators} \]

\[ \text{A Hecke eigensheaf } F \text{ on } \text{Bun}_n. \]

Let me explain what I understood of this dictionary.

- **Point 2.** A representation of the fundamental group \( \sigma : G \mapsto GL(n, \mathbb{C}) \) leads to a flat vector bundle over \( C \). This is called a local system \( E \) over \( C \). Thus we know very well the geometric analogue of representations of the Galois group. These are local systems \( E \) on \( C \).

- **Point 3 and 4.** At a place \( a \) of \( C \), we consider a coordinate \( z \), and we identify \( GL(n, \mathcal{M}(C)_a) \) with \( GL(n, \mathbb{C}((z))) \). Let \( V \to C \) be an holomorphic vector bundle of rank \( n \) and choose \( S_1, S_2, \ldots, S_n \) meromorphic sections of \( V \) generically independent. At each point \( a \) in \( C \), we take a local trivialisation of \( V_a = \mathbb{C}^n \) via holomorphic sections \( s_1, s_2, \ldots, s_n \) (defined at \( a \)). We obtain an element \( g_a(z) \in GL(n, \mathcal{M}(C)_a) \) by writing \( s_i(z) = \sum_j g_i^j(z)S_j(z) \). Thus an holomorphic vector bundle \( V \) of rank \( n \) gives, via its transition functions, an element \( g = (g_a(z))_{a \in C} \) with \( g_a(z) \in GL(n, \mathbb{C}((z))) \). Thus the analogue of the space \( GL(n, O_F) \backslash GL(n, \mathbb{A}_F) / GL(n, F) \) is the space \( \text{Bun}_n \) of equivalence classes of holomorphic vector bundles over \( C \) of rank \( n \). The space \( \text{Bun}_n \) is a “stack”, but I will employ anyway terms of “varieties”. Anyway, all this is very vague for me. The space \( \text{Bun}_n \) is not connected. It is the union of connected components \( \text{Bun}_n^d \) of vector bundles of degree \( d \).

- **Points 5 and 6.**
Now, it is more difficult (for me) to understand what is the analogue of the “automorphic side”, that is what is the analogue of an automorphic representation. Recall that in the case where $\pi$ was an irreducible automorphic representation, we could singled out in $\pi$ a particular function $f = \otimes_v f_v$ in $\pi \subset L^2(GL(n, A_F)/GL(n, F))$, at least at all unramified places, by saying that $f_p$ was the fixed vector under $K_p$ (normalized to be 1 at 1). This vector $f_p$ was automatically an eigenvector for the Hecke operators.

Now, in the geometric context, the analogue of a function on a space $X$ is a perverse sheaf on $X$. Indeed consider the case where $X$ is finite. A sheaf $\mathcal{F}$ of $X$ is just a collection of vector space $\mathcal{F}_x$. We consider sheaves up to isomorphisms. Thus $\mathcal{F}$ is completely determined by the function $x \mapsto \dim(\mathcal{F}_x)$. Remark that this operation commutes with the 6 operations, in the Grothendieck dictionary: If we consider the tensor product $\mathcal{F} \otimes \mathcal{G}$ of “sheaves”, then the corresponding function is $f(x)g(x)$. If $X \subset Y$, we extend $\mathcal{F}$ on $Y$ by $\mathcal{F}_y = 0$ if $y \notin X$, this corresponds to extending $f$ by 0. Similarly for pushforward $X \mapsto Y$, the function $\pi_*(f)(y) = \sum_{x; \pi(x) = y} f(x)$ commutes with pushforward of sheaves $(\pi_* \mathcal{F})_y = \otimes_{x; \pi(x) = y} \mathcal{F}_x$. When $X$ is a variety defined over a finite field $F_q$, and $\mathcal{F}$ a sheaf on $X$, and $x$ a point of $X(F_q)$, the Frobenius element $F_{r_q}$ acts on the cohomology groups $H_i(x, \mathcal{F})$, and we obtain a function on $X(F_q)$, by taking the trace of the action of the Frobenius.

If $X$ is a complex variety, an irreducible perverse sheaf is completely determined by a locally closed complex subvariety $Y$ of $X$, and a local system on $Y$, and operations on perverse sheaves of extensions, pushforward, etc... are well defined.

Finally, we discuss the notion of Hecke eigensheaves.

For automorphic forms, at each prime $p$, consider the subspace $D^1_p$ of $GL(n, \mathbb{Q}_p)/GL(n, \mathbb{Z}_p) \times GL(n, \mathbb{Q}_p)/GL(n, \mathbb{Z}_p)$ consisting of elements $\{g_1, g_2\}$ with $g_1 = h^i_p g_2$ where

$$h^i_p = \begin{pmatrix} p & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Now at each place $a$ of $C$, we consider the following subset

$$\text{Hecke}_a^i \subset \text{Bun}_n \times \text{Bun}_n$$
of vector bundles \((V, V')\) with a map \(V \to V'\) isomorphism, except at the point \(a\) of \(C\), where it is locally given the matrix (analogue of the Hecke matrix)

\[
\begin{pmatrix}
(z - a) & 0 & 0 & 0 & 0 \\
0 & (z - a) & 0 & 0 & 0 \\
0 & 0 & (z - a) & 0 & 0 \\
0 & 0 & 0 & \ast & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

We now vary the base point \(a\), and obtain a subvariety

\[H^i \subset C \times \text{Bun}_n \times \text{Bun}_n\]

Let us denote by \(p_1(a, V, V') = V'\) and \(p_2(a, V, V') = (a, V)\). Thus \(H^i\) operates on perverse sheafs on \(\text{Bun}_n\) by

**Definition 11.**

\[H^i(\mathcal{F}) = (p_2)_\ast(p_1^\ast(\mathcal{F})).\]

Thus from a perverse sheaf on \(\text{Bun}_n\), we obtain a perverse sheaf on \(C \times \text{Bun}_n\).

We are now ready to state what should be the analogue of the Langlands correspondence.

Let \(E\) be a local system on \(C\) (a representation of the Galois group) of rank \(n\). The classical Hecke operators were operators on functions. The equation \(H_p f = a_p f\) saying that \(f\) is an eigenfunction of \(H_p\) is translated by the equation on sheaves: \(H^1 \mathcal{F} = E \otimes \mathcal{F}\).

**Theorem 12.** For each irreducible local system \(E\) on \(C\) of rank \(n\), there exists an irreducible (on each component \(\text{Bun}_d\)) perverse sheaf \(\text{Aut}_E\) (the automorphic sheaf attached to \(E\)) on \(\text{Bun}_n\) which is a Hecke eigensheaf:

\[H^i(\text{Aut}_E) = \wedge^i(E) \otimes \text{Aut}_E.\]

When \(n = 1\), the theory is very simple and due to Rosenlicht, Lang and Serre. Indeed \(\text{Bun}_n\), for \(n = 1\), is just the Picard group \(\text{Pic}(C)\) of line bundles on \(C\). One of the connected component of \(\text{Pic}(C)\) is the Jacobian variety \(\text{Jac}(C)\) of \(C\). If \(g\) is the genus of \(C\), then the fundamental group of the Jacobian of \(C\) (a complex torus of dimension \(g\)) is isomorphic to \(H_1(C, \mathbb{Z}) = \pi(C) / \text{commutators}\). Thus there is an equivalence between local systems of rank 1 on \(C\) or on \(\text{Jac}(C)\). Thus a local system on \(E\) gives rise to a sheaf \(\text{Aut}_E\) on \(\text{Pic}_d(C)\). It is possible here to describe how \(\text{Aut}_E\) looks like. Consider the restriction
of Aut$_E$ on Pic$_d(X)$ the space of line bundles of degree $d$. If $\mathcal{L}$ is a line bundle on $C$ of degree $d$, an section $s \in H^0(C, \mathcal{L})$ gives us $d$ points on $C$ where $s$ vanishes. Thus we obtain a map $p$ from the fiber bundle over Pic$(C)$, with fiber at $\mathcal{L}$ the projective space $[H^0(C, \mathcal{L})]$ to $(C \times C \times \cdots \times C)/\text{Permutations}$. The sheaf Aut$_E$ is the unique local system on Pic$(C)$ such that its pull back is $\otimes^d E$ on $C^d/\text{permutations}$. Its existence is deduced from the fact that fibers (projective spaces) are simply connected.

The construction of the automorphic bundle Aut$_E$ for a local system $E$ of rank $n$ requires much more subtle arguments. It takes its grounds in works of Laumon, Frenkel-Gaitsgory-Vilonen with the final step established by Gaitsgory.

References

[1] V.G. Drinfeld, Langlands conjecture for GL(2) over function fields, Proceedings of the International Congress of Mathematicians, Helsinki, (1978).
[2] E. Frenkel, D. Gaitsgory, K. Vilonen On the geometric Langlands conjecture. J. Amer. Math. Soc. 15, (2002), 367-417.
[3] D. Gaitsgory On a vanishing conjecture appearing in the geometric Langlands correspondence (2002)
[4] M. Harris and R. Taylor The geometry and cohomology of certain simple Shimura varieties Annals of Math Studies 151. Princeton University Press (2001).
[5] G. Henniart Une preuve simple de la correspondance de Langlands pour GL(n) sur un corps $p$-adique. 139. Inventiones Mathematicae 113. (2000), 439-455.
[6] L. Lafforgue Choutcas de Drinfeld et correspondance de Langlands Inventiones Mathematicae 147 (2002), 1-241.
[7] R. Langlands Base change for GL(2) Annals of Math. Studies, vol 96, Princeton University Press, Princeton, NJ, 1980.
[8] G. Laumon and Ngo Bao Chau The fundamental lemma for unitary groups To appear Annals of Mathematics
[9] J. Tunnell, Artin’s conjecture for representations of octahedral type, Bull. Amer. Math. Soc. 5, (1981), 173-175.
[10] J.L. Waldspurger Sur les intériales orbitales tordues pour les groupes linéaires: un lemme fondamental. Canad. J. Math. 43 (1991), 852-896

Centre de Mathématiques Laurent Schwartz, 91128 Palaiseau, France; Institut de Mathématiques de Jussieu, Théorie des Groupes, Case 7012, 2 Place Jussieu, 75251 Paris Cedex 05, France

E-mail address: vergne@math.polytechnique.fr, vergne@math.jussieu.fr