Neutron optical test of the quantum commutator relation through imaginary weak values

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The canonical commutation relation is the hallmark of quantum theory and Heisenberg’s uncertainty relation is a direct consequence of it. But despite its fundamental role in quantum theory, surprisingly, its genuine direct experimental test has hitherto not been performed. In this article, we present a novel scheme to directly test the canonical commutation relation between two dichotomic observables, by exploiting the notion of weak measurement. The imaginary part of a suitably formulated weak value enables this direct test. The measurement of the weak value of a path-qubit observable in a neutron interferometer experiment is used to verify the approach. The experiment is realized using a newly developed technique in our neutron interferometric setup where the neutron’s spin/energy degree of freedom serves as ancilla.

I. INTRODUCTION

In his epoch-making paper “Quantum mechanical reinterpretation of kinematic and mechanical relations” in July 1925, Heisenberg’s \cite{1} put forward his breakthrough idea by introducing an entirely new representation of the position variable in terms of a set of transition amplitudes corresponding to atomic radiation. This led him to propose an unfamiliar rule of multiplication of two amplitudes in order to obtain correct intensities. It was immediately identified as matrix multiplication by Born. In the same year, by introducing a mathematically elegant language of matrices, Born and Jordan \cite{2} systematically formalized the Heisenberg’s ‘matrix mechanics’ and provided the world a one-line epitaph

\[ \hat{p}\hat{q} - \hat{q}\hat{p} = i\hbar \mathbb{1}, \tag{1} \]

where \( \hat{p} \) and \( \hat{q} \) are the matrix forms of position and momentum variables in classical mechanics. Eq. (1) is now widely known as canonical commutation relation and is the hallmark of quantum theory. The very next year, Schrödinger \cite{3} surprisingly put forward an alternative theory, coined as ‘wave mechanics’, in that the core element is the wave function \( \Psi \). In terms of interpretation and spirit, the wave mechanics greatly differs from matrix mechanics but produces equivalent quantum statistics.

Two years after his first breakthrough work, in another seminal paper, Heisenberg proposed \cite{1} his famous uncertainty relation \( \delta p \delta q \sim \hbar \) which he regarded as a direct mathematical consequence of the canonical commutation rule in Eq. (1). Here \( \delta p \) and \( \delta q \) are sort of uncertainties in momentum and position measurements respectively. Later, building upon Kennard’s \cite{5} idea of interpreting the uncertainties as standard deviation, Robertson\cite{6} generalized the Heisenberg’s preparation uncertainty relation for any two arbitrary observables \( A \) and \( B \), so that, \( \Delta A \Delta B \geq |\langle [A,B] \rangle|/2 \). In recent times, the distinction between preparation and measurement uncertainty relations has been made and many interesting new formulations have also been proposed \cite{7,8,9}. Quite a few of them have experimentally been tested \cite{10,11,12}. However, no clear consensus among physicists in this issue has been reached till date.

Although the uncertainty relations are regarded as direct (or indirect) consequences of relevant commutation relations, only a very few attempts have been made for direct test of commutation relations \cite{13,14,15}. We attribute this to the fact that the joint probability of two non-commuting observables does not exist in quantum theory and the product of such observables is in general not Hermitian. Hence, a direct test of commutation relation is nontrivial in experiment. Against this backdrop, in this work, we propose a theoretical scheme in which a single anomalous weak value enables a direct test of commutation relation between qubit observables. Further, we report an experimental test using neutron interferometry.

The weak value of an observable arises in a novel conditional measurement protocol \cite{22}, widely known as ‘weak measurement’. Consider a system prepared in a state \( |\psi_i\rangle \) (commonly known as pre-selected state) and \( A \) is the observable to be measured on the system. If the measurement interaction between the system and the apparatus is weak, the system state remains grossly undisturbed. After such weak interaction if a particular outcome \( |\psi_f\rangle \) is selected by sequentially performing a strong measurement, then the final pointer state yields the weak value, quantified by the formula

\[ \langle A \rangle_{w}^{\psi_i,\psi_f} = \frac{\langle \psi_f | A | \psi_i \rangle}{\langle \psi_f | \psi_f \rangle}. \tag{2} \]

Unlike expectation value \( \langle A \rangle \), the weak value \( \langle A \rangle_{w}^{\psi_i,\psi_f} \) can be beyond the range of eigenvalues and can even be complex. The physical interpretation and implications of complex and large weak values have been widely discussed in the literature (see, for example, \cite{30}).
The commutation relation is a fundamental principle in quantum mechanics, stating that certain pairs of observables cannot be simultaneously measured with arbitrary precision. In a qubit system, we experimentally performed the canonical commutation relation between two observables, say $\hat{A}$ and $\hat{B}$, satisfying

$$\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle \neq \gamma \text{ for all } |\psi\rangle$$

where $\gamma$ is a nonzero quantity. Since the product $\hat{A}\hat{B}$ may not be Hermitian in general, the traditional von Neumann quantum measurement scheme cannot be carried but the weak measurement sufficed for our purpose. Non-Hermitian observables may be measured using weak measurements. But our scheme here is direct and fundamentally different, as follows. If $\Pi^+_A = |+A\rangle\langle+ A|$ and $\Pi^-_B = |-B\rangle\langle+ B|$ are positive-eigenvalue projectors of $\hat{A}$ and $\hat{B}$ respectively, then by writing $\hat{A} = 2\Pi^+_A - 1$ and $\hat{B} = 2\Pi^-_B - 1$, one has

$$\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle = 4\langle \psi | \Pi^+_A \Pi^-_B | \psi \rangle - 4\langle \psi | \Pi^-_A \Pi^+_B | \psi \rangle$$

where $\langle \psi | \Pi^+_A \Pi^-_B | \psi \rangle$ is the weak value of the projector $\Pi^+_A$ for a post-selected state $|+B\rangle$ and pre-selected state $|\psi\rangle$ and $|\langle +B | \psi \rangle|^2$ is the probability of successful post-selection.

If $\hat{A} = \hat{\sigma}_x$ and $\hat{B} = \hat{\sigma}_y$ the commutation relation is

$$\langle \psi | [\hat{\sigma}_x, \hat{\sigma}_y] | \psi \rangle = 2i\langle \psi | \hat{\sigma}_y | \psi \rangle.$$  (4)

By writing $\hat{\sigma}_x = 2\Pi^+_x - 1$, $\hat{\sigma}_y = 2\Pi^+_y - 1$ and $\hat{\sigma}_z = 2\Pi^+_z - 1$ where $\Pi^+_x = |+x\rangle\langle+ x|$, $\Pi^+_y = |+y\rangle\langle+ y|$ and $\Pi^+_z = |+z\rangle\langle+ z|$ are the projectors with positive eigenvalue corresponding to the observables $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ respectively, from Eq. (4), we have

$$4|\langle +x | \psi \rangle|^2 \text{ Im} \left( \langle \Pi^+_x \psi | +x \psi \rangle \right) = |\langle +y | \psi \rangle|^2 - 1,$$  (5)

where $\langle \Pi^+_x \psi | +x \psi \rangle = \langle +x | \Pi^+_x \psi \rangle / \langle +x | \psi \rangle$ is the weak value of $\Pi^+_x$ given the pre- and post-selected states $|\psi\rangle$ and $|+x\rangle$. 

**THEORY**

Without loss of generality, consider two non-commuting qubit observables, say $\hat{A}$ and $\hat{B}$ satisfying
responding to the post-selected path state $\Pi_{w}^{\psi,+,\pi}$ is amenable to test the left-hand-side of the commutation relation in Eq. (4). For determining the quantity on the right-hand side one requires to separately measure the post-selected probability $|\langle +y|\psi \rangle|^2$.

**EXPERIMENT**

To determine the weak value a proven neutron interferometric setup [14,15], depicted in Fig. 1, is applied. The system is prepared (pre-selected) and post-selected in the path states:

$$|\psi_I(\chi)\rangle = \frac{1}{\sqrt{2}}(|I\rangle + e^{i\chi}|II\rangle)$$

$$|\psi_I\rangle \equiv |+x\rangle = \frac{1}{\sqrt{2}}(|I\rangle + |II\rangle),$$

where $|I\rangle$ and $|II\rangle$ are the path eigenstates of the Pauli observable $\sigma_\chi = |I\rangle\langle I| - |II\rangle\langle II|$. The imaginary part of the weak value of the path projector $\Pi_1$ is $\text{Im}(\langle \Pi_1 \psi,+,\pi \rangle) = \sin^2(\chi/2)/2$ and $|(+x|\psi(\chi))|^2 = 4\cos^2(\chi/2)$. The quantity on the right-hand side of Eq. (5) is determined by taking another post-selected state $|\psi_f\rangle \equiv |+y\rangle = (|I\rangle + i|II\rangle)/\sqrt{2}$, so that $2|(+y|\psi(\chi))|^2 - 1 = \sin^2 \chi$.

**Experimental setup**

The experiment was performed at the S18 interferometer beam line at the research reactor of the Institute Laue-Langevin (ILL), in Grenoble, France. The setup is depicted in Fig. 1. A monochromatic neutron beam with a wavelength of $\lambda = 1.92$ Å passes through a magnetic prism, which produces two divergent polarised beams with spin states $|z\rangle$ and $|+z\rangle$, respectively. For our purpose, only the beam with spin state $|+z\rangle$ is adjusted to fulfill the Bragg condition of the down-stream silicon perfect crystal interferometer of triple Laue geometry. The other beam is transmitted in forward direction and absorbed by a slab of cadmium to reduce the background noise in the detectors. The first crystal plate splits the beam into two paths $|I\rangle$ and $|II\rangle$. After passing through the first beam-splitter a phase shifter is inserted into the interferometer to tune the relative phase $\chi$ between path $I$ and path $II$ and prepares the pre-selected path state $|\psi_I(\chi)\rangle$. A resonance frequency (RF) spin manipulator is placed along the path $|I\rangle$ to implement the weak measurement. Both beams are recombined at the third interferometer plate.

Of the two outgoing beams of the interferometer, only the neutrons in the $O$ beam are selected for analysis corresponding to the post-selected path state $|\psi_f\rangle = |+x\rangle$. The transmitted $O$ beam passes through a direct current (DC) spin rotator and a polarizing CoTi multilayer array (henceforth, supermirror) to perform spin analysis in the $O$ beam. The refracted $H$ beam is used as a reference. The intensities in both outgoing beams are measured using $^3$He counting tubes with a high detection efficiency (over 97%). The detector for the $O$ beam has a diameter of 6 mm and is placed directly at the outgoing window of the supermirror. For the used wavelength a time resolution of 3 $\mu$s is achieved.

**Weak value extraction**

Given the pre- and post-selected path states $|\psi_I(\chi)\rangle$ and $|+x\rangle$, respectively, we performed the measurement of the weak value $\langle \Pi_1 | \psi,+,\pi \rangle$ of the path projector $\Pi_1 = |I\rangle\langle I|$. The weak interaction is implemented by introducing the RF spin manipulator in path $|I\rangle$. This can be represented by an unitary evolution as $U_{\text{int}} = \Pi_1 \otimes U_{\text{RF}}(t, \alpha, \omega, \delta) + \Pi_2$. Here, $\alpha$ is the spin-rotation angle, $\omega$ the frequency of the electromagnetic RF field and $\delta$ an arbitrary phase of this RF field. The parameter $\alpha$ is directly dependent on the magnetic field strength and a sufficiently small value of it warrants the required weak measurement criteria. In this experiment, $\alpha = \pi/9$ and $\omega = 60$ kHz were taken. As explained in detail in METHODS, the weak value $\langle \Pi_1 | \psi,+,\pi \rangle$ can be extracted from the time-dependent intensity at the $O$ detector that is given by

$$I(t) = \frac{1}{2}[\langle +x|\psi \rangle|^2 (1 + \alpha \text{Im}(e^{i\omega t}\langle \Pi_1 | \psi,+,\pi \rangle))].$$

By switching off the RF field, two post-selected probabilities $|\langle +x|\psi \rangle|^2$ and $|\langle +y|\psi(\chi)\rangle|^2$ are measured. The intensities in both outgoing beams are measured using $^3$He counting tubes with a high detection efficiency (over 97%). The detector for the $O$ beam has a diameter of 6 mm and is placed directly at the outgoing window of the supermirror. For the used wavelength a time resolution of 3 $\mu$s is achieved.

![Figure 2](image_url)  
Figure 2. Experimental results of the real and imaginary parts of weak value $\langle \Pi_1 | \psi,+,\pi \rangle$ are plotted against $\chi$. The solid dots denote the least-square fits to the data and the error bars represent unit standard deviation. The results successfully verify the theoretical predictions.
former resembles an empty interferogram, while the latter corresponds to an empty, but by \( \chi_\text{add} = \pi/2 \) shifted interferogram [49].

The weak value is in general a complex number and can be expressed in polar form \( \langle \Pi_1 \rangle_w^{x+y} = Ae^{i\varphi} \). The amplitude \( A \) and phase \( \varphi \) are retrieved from a sinusoidal fit to the time-dependent intensity. The reference phase \( \varphi_\text{ref} = 0 \) is obtained for the particular case, when pre- and post-selected states coincide, i.e., \( \chi = 0 \). In Fig. 3, we present the detailed analysis of the results of the weak measurement of \( \langle \Pi_1 \rangle_w \). For comparison, the experimental values of the real and the imaginary part of \( \langle \Pi_1 \rangle_w^{x+y} \) are separately plotted along with the theoretical predictions.

Verifying canonical commutation relation

The results of all three measurements, namely \( \langle +x|\psi_2(\chi)\rangle^2 \), \( \langle +y|\psi_2(\chi)\rangle^2 \) and \( \text{Im}(\langle \Pi_1 \rangle_w^{x+y}) \), are put together in Fig. 3. The left- and right-hand side related measurements of the commutation relation (Eq. (5)) are shown by orange and green data points respectively. The result is in good agreement with the relevant theoretical prediction (dotted blue). The error bars represent one standard deviation.

Figure 3. Experimental results of left-hand side (orange) and right-hand side (green) of the commutation relation in Eq. (5) are plotted as a function of phase-shift \( \chi \). The results are in excellent agreement with the relevant theoretical prediction (dotted blue). The error bars represent one standard deviation.

SUMMARY AND DISCUSSION

The results above confirm, that we have experimentally verified the canonical commutation relation by weak measurement of the path-qubit observable in a neutron interferometer. Accordingly, from the quantum foundational perspective, our experiment thus provides a genuine direct test of one of the fundamental tenets of quantum theory. Our test is as fundamental as the direct measurement of a quantum wave function by Lundeen et al. [39]. Heisenberg’s uncertainty relation is a direct consequence of the canonical commutation relation and several formulations of it have been tested experimentally [14, 15, 50, 52]. However, a genuine test of the canonical commutation relation has hitherto not been performed.

The direct test of the commutation relation boils down to the experimental determination of the imaginary part of a single weak value of a suitable path observable. The weak value has provided a multitude of practical and conceptual implications [23–44]. The imaginary part of the weak value is more counter-intuitive than its real counterpart. The latter can be operationally interpreted as an idealized conditioned average value of the concerned observable in the zero measurement disturbance limit. On the other hand, the imaginary part is associated with back-action on the system due to the measurement itself [31]. This explanation conceptually fits well with our test of the commutation relation through the imaginary part of the weak value. Finally, it would be interesting to test the traditional position-momentum uncertainty relation following the approach developed here. The experimental method presented here could be expanded to more path markers in the interferometer and several path weak values could be measured simultaneously. Applying it to a four-plate interferometer would even offer to install more than two path markers [53].

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METHODS

Detailed analysis of the intensity for the weak value extraction

Given the pre-selected state \( |\psi_1\rangle = \frac{1}{\sqrt{2}} (|I\rangle + e^{i\alpha} |II\rangle) \) and post-selected state \( |\psi_f\rangle = \frac{1}{\sqrt{2}} (|I\rangle + |II\rangle) \), to determine the weak value of path observable \( \Pi_1 \), an interaction of the resonance frequency (RF) spin manipulator in path \( |I\rangle \) is employed which can be represented by the unitary dynamics, denoted as \( U_{\text{int}} = \Pi_1 \otimes U_{\text{RF}}(t, \alpha, \omega, \delta) + \Pi_2 \). The spin manipulation leads to an energy-shift by the amount \( \Delta E = \pm \hbar \omega \) in the spin-flipped parts and can be written as an unitary operator:

\[
U_{\text{RF}}(t, \alpha, \omega, \delta) = \begin{pmatrix}
\cos\left(\frac{a}{2}\right) & \sin\left(\frac{a}{2}\right) e^{i(\omega t+\delta)} \\
-i\sin\left(\frac{a}{2}\right) e^{-i(\omega t+\delta)} & \cos\left(\frac{a}{2}\right)
\end{pmatrix},
\]

where \( \alpha, \omega \) and \( \delta \) are the spin-rotation angle, the frequency of the electromagnetic RF field and an arbitrary phase of this RF field. The energy-shifted contributions
appear due to the off-diagonal terms. The parameter $\alpha$ is related to magnetic field strength and for a small value of $\alpha$ becomes:

$$\lim_{\alpha \to 0} U_{\text{RF}} = 1 - \frac{\alpha}{2} \begin{pmatrix} 0 & e^{i(\omega t + \delta)} \\ e^{-i(\omega t + \delta)} & 0 \end{pmatrix}. \quad (9)$$

The total path-spin state of the in $\uparrow_z$-direction polarized neutrons after the first plate of the interferometer is

$$|\Psi\rangle = |\psi_1(\chi)\rangle \otimes |\uparrow_z\rangle. \quad (10)$$

Spin related states and projectors are indicated by the $\uparrow$ symbol. After the interaction with the spin-modulator in path $I$, the state evolves to $|\Psi\rangle' = U_{\text{int}} |\Psi\rangle$ and is subsequently projected onto the post-selected path state $|\psi_1\rangle$ at the third crystal plate

$$|\Psi\rangle' \to |\Psi''\rangle = \Pi_f U_{\text{int}} |\Psi\rangle. \quad (11)$$

where $\Pi_f = |\psi_f\rangle \langle \psi_f|$. Now, for our purpose, the spin analysis is performed for $|\uparrow_z\rangle$ which is implemented by the action of the DC coil and the supermirror (SM). The joint (un-normalized) path-spin state is projected onto $P_{\uparrow_z} \equiv |\uparrow_z\rangle \langle \uparrow_z|:

$$|\Psi''\rangle = P_{\uparrow_z} \Pi_f U_{\text{int}} |\Psi\rangle. \quad (12)$$

Using (9) we get the approximation in the limit for small $\alpha$ (neglecting contributions of order $O(\alpha^2)$ or higher):

$$|\Psi''\rangle \approx \frac{1}{\sqrt{2}} |\psi_f\rangle \otimes |\uparrow_z\rangle \langle \psi_f| \Pi_f e^{i(\omega t + \delta)} |\psi_1(\chi)\rangle. \quad (13)$$

Further rearrangement provides

$$|\Psi''\rangle = \frac{1}{\sqrt{2}} |\psi_f\rangle \otimes |\uparrow_z\rangle \langle \psi_f| \Pi_f e^{i(\omega t + \delta)} |\psi_1(\chi)\rangle. \quad (14)$$

The time-dependent intensity $I_{\text{ideal}}(t) = |\Psi''|^2$ in the ideal scenario can then be written as

$$I_{\text{ideal}}(t) = \frac{1}{2} |\langle \psi_f| \Pi_f e^{i(\omega t + \delta)} |\psi_1(\chi)\rangle|^2 \left( 1 + \alpha \left\{ \text{Im}(\Pi_1 \psi_1 e^{i(\omega t + \delta)}) \right\} \right), \quad (15)$$

which is dependent on $\chi$. With $|\psi_f\rangle \equiv |\uparrow_z\rangle$, $\Pi_1 \psi_1 + x = (1 + e^{i\chi})^{-1}$ can be extracted from the intensity analysis. By switching off the RF spin modulator ($\alpha = 0$) the signal for post-selected probabilities $|\langle \uparrow_x| \psi_1(\chi)\rangle|^2 = (1 + \cos \chi)/2$ and $|\langle \uparrow_y| \psi_1(\chi)\rangle|^2 = (1 + \sin \chi)/2$ in Eq. (5) can be measured. For the latter this is practically done by applying an additional phase shift of $\chi_{\text{add}} = \pi/2$ to the readings from the measurement of the empty interferogram.

**Data analysis**

In practical experiments, there will be inevitable loss of coherence in the signal. In order to account it the intensity has to be corrected by carefully taking into account the incoherent contributions. The amount of this incoherence can be accounted for by the contrast or fringe visibility parameter $\eta \in [0, 1]$.

$$I_{\text{real}} = \eta I_{\text{ideal}} + (1 - \eta) I_{\text{inc}}, \quad (16)$$

where $I_{\text{inc}}$ stands for the incoherent part of the intensity signal. By considering an empty interferometer, we quantified $I_{\text{ideal}}$ as

$$I_{\text{ideal}} \equiv |\langle \psi_f| \Pi_f e^{i(\omega t + \delta)} |\psi_1\rangle|^2 = \Pi_1 |\langle \psi_f| \Pi_f e^{i(\omega t + \delta)} |\psi_1\rangle|^2 = 1$$

$$\Pi_1 |\langle \psi_f| \Pi_f e^{i(\omega t + \delta)} |\psi_1\rangle|^2 + 2 \text{Re}(\Pi_1^* \Pi_2). \quad (17)$$

In an ideal scenario we expect full coherence of the interference term $2 \text{Re}(\Pi_1^* \Pi_2)$, i.e., $\eta = 1$. However, in real experiment, only partial coherence can be obtained, so that, in our experiment $I_{\text{ideal}}$ is quantified as

$$I_{\text{ideal}} \equiv |\Pi_1|^2 + |\Pi_2|^2 + 2 \eta \text{Re}(\Pi_1^* \Pi_2)$$

$$\eta = |\Pi_1|^2 + |\Pi_2|^2 + 2 \eta \text{Re}(\Pi_1^* \Pi_2). \quad (18)$$

Identifying the incoherent part as $I_{\text{inc}} = |\Psi_1|^2 + |\Psi_2|^2$, it is straightforward to see that (18) is equivalent to (16). The time-dependent intensity in our experiment can then be written as

$$I_{\text{real}}(t) = \frac{\eta}{2} |\langle \psi_f| \Pi_f e^{i(\omega t + \delta)} |\psi_1\rangle|^2 \left( 1 + \alpha \left\{ \text{Im}(\Pi_1 \psi_1 e^{i(\omega t + \delta)}) \right\} \right)$$

$$+ (1 - \eta) \left( \frac{1}{8} (1 - \alpha \sin(\omega t + \delta)) + \frac{1}{8} \right), \quad (19)$$

$|\Psi_1|^2$ and $|\Psi_2|^2$ are intensities for the isolated path $I$ and $II$, that have been measured separately to apply the correction. An example of such a signal is shown in Fig. 4. The visibility $\eta$ is extracted from a fit of (18) to the intensities of an empty interferometer, when $\chi$ is varied. In our measurements the average value of $\eta$ was 0.79. Now, the ideal intensity is the one that is the measured intensity minus contributions from independent intensities

$$I_{\text{ideal}}(t) = \frac{1}{\eta} \left( I_{\text{real}}(t) - (1 - \eta) \{I_1(t) + I_2(t)\} \right) \quad (20)$$

Figure 4. A typical time resolved intensity signal of the measurement campaign for the phase-shift $\chi = \pi/3$. 
Time-resolved signals have been recorded for several phase shifter settings in the range $0 \leq \chi \leq 2\pi$. These have been corrected according to (20), fitted and normalized and are shown in Fig. (5).

The weak value for the path projector is a complex number and can generally be written in the polar form $\langle \Pi_1 \rangle_w = A e^{i\varphi}$. So it follows from (15) that, its amplitude and phase can be obtained from sine fits to the time-dependent intensities:

$$I_{\text{ideal}}(t) = \frac{1}{2} |\langle \psi_f | \psi_i(\chi) \rangle|^2 \left(1 + \alpha \{ A \sin(\varphi - \omega t - \delta) \} \right).$$

Here $\delta$ is an arbitrary additional constant phase. The modulation in (21) show a reduced fringe visibility, too. The retrieved values for the amplitudes $A$ have therefore to be corrected with $\eta$, according to $A_w = A/\eta$ to obtain the amplitudes of the path projector weak values. At the phase shifter setting $\chi = 0$, the weak value $\langle \Pi_1 \rangle_w = (1 + e^{i\chi})^{-1}$ becomes real and exactly $1/2$. So the phase angle $\varphi$ is equal to zero. This serves as a reference in the measurement to retrieve the phase angle $\varphi$ for the other $\chi$ settings. A plot that collects the retrieved amplitudes $A_w$ and phases $\varphi$ for the weak value of the path projector $\langle \Pi_1 \rangle_w$ is shown in Fig. (6). The errors in all plots are from propagation of the standard deviations of the fit errors for the time dependent intensity and reference interferogram signals and the systematic effects due to uncertainties in the adjustment of the rotation angle $\alpha$.

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