Koashi-Winter relation for $\alpha$-Rényi entropies

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This work presents a generalization of the Koashi-Winter relation for $\alpha$-Rényi entropies. This result is based on the Rényi’s entropy version of quantum Jensen Shannon divergence. By means of this definition, a classical correlations quantifier $C_\alpha(\rho_{AB}) = \sup_{\xi_{AB}} Q_\alpha(\xi_{AB}^{MB})$ is proposed, where the optimization is taken over the ensembles $\xi_{AB}^{MB}$ created by the outputs of the local measurement process. The main result is applied to the capacity of a quantum classical channel over a tripartite pure state $\psi_{ABE}$, that is rated above in function of the probability of success to discriminate the states in the ensemble $\xi_{AB}^{MB}$, created by the local dephasing over partition $E$, and the asymptotic log generalized robustness of partition $AB$. Some analytical results are calculated for classical correlations and entanglement of formation.

I. INTRODUCTION

Quantum correlations are intrinsically related to quantum information theory, as resources for quantum information protocols $^1$. The quantification of quantum information rates is performed by a quantum version of entropic measures, especially by von Neumann entropy

$$S(\rho) = -\text{Tr}(\rho \log \rho),$$

and its related measures $^2$, although there are generalizations of von Neumann entropy, and its related measures, by means of Tsallis’ entropy $^3$ and Rényi’s entropy $^4$. The quantum version of Rényi’s entropy, in the context of quantum correlations and quantum information, has been a theme of intense investigation in the last few years $^5-^10$. This is the context in which this work is inserted.

The main result of this work is the generalization of the Koashi-Winter relation for Rényi’s entropy. This result is based on the Rényi’s entropy version of quantum Jensen Shannon Divergence (QJSD) $^8$. By means of Rényi’s QJSD and the distinguishability of quantum states, a quantifier of classical correlations is introduced: $C_\alpha$, for $\alpha \in (0, 1)$. The properties for $C_\alpha$ to be a quantifier of classical correlations are discussed and proved. From the main result, Rényi’s entanglement of formation for pure states is calculated, recovering Ref. $^17$. The values of $C_\alpha$ for pure states and classical correlated states are also calculated, recovering the classical Jensen Shannon Divergence. As an application of the main result, it is shown that the entanglement of formation, for $\alpha = 1/2$, is a convex roof of log generalized robustness. This final result is discussed in the context of channel capacity in asymptotic limit.

This paper is organized as follow. In Section $\text{II}$ some mathematical concepts about density matrix, the CPTP channels over density operators, the Schatten - $p$ norm for operators, and entropic measure for quantum systems are introduced. In Section $\text{III}$ the formalism of quantum correlations: quantumness of correlations and quantum entanglement is presented. The main results are presented in Section $\text{IV}$. In Section $\text{V}$ the results are applied for $\alpha = 1/2$ Rényi’s entropy, a relation between $C_\alpha$, the distinguishability of the states in the ensemble created by local measurement and the log robustness is obtained.

II. FORMALISM AND NOTATION

A. Density Operator

This work deals with finite dimensional Hilbert spaces. A given Hilbert space $H_N = \mathbb{C}^N$ is defined as a complex vector space. For a given Hilbert space there exists a dual space $H_N^\ast$, that maps $H$ to the complex numbers. For finite dimensional Hilbert spaces these two spaces are isomorphic, so $H_N^\ast = \mathbb{C}^N$. The space of linear transformations acting on a Hilbert space is denoted as $\mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$. A given linear transformation $A$ belongs to the space $\mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$, if $A : \mathbb{C}^N \to \mathbb{C}^M$. If $A$ is an operator, its space is denote as $\mathcal{L}(\mathbb{C}^N)$. The set of linear transformations on the Hilbert space $\mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$ is also a Hilbert space, therefore it is equipped with inner product. For two operators $M, N \in \mathcal{L}(\mathbb{C}^N)$, the inner product is defined as the Hermitian form:

$$\langle M, N \rangle = \text{Tr}(M^\dagger N).$$

As $\text{Tr}(M^\dagger N)$ is a finite number, the vector space $\mathcal{L}$ is often called the space of bound operators. Restricting the matrices in the positive cone of this space to be trace=1, it defines the set of density matrices $^18$. This set of operators is a vector space denoted as $\mathcal{D}(\mathbb{C}^N)$.

Definition 1. A linear positive operator $\rho \in \mathcal{D}(\mathbb{C}^N)$ is a density matrix, and represents the state of a quantum system, if it satisfies the following properties:
positive semi-definite: $\rho \geq 0$;

- Trace one: $\text{Tr}(\rho) = 1$.

A linear transformation $\Phi : \mathcal{L}(\mathbb{C}^N) \rightarrow \mathcal{L}(\mathbb{C}^M)$ represents a physical process if it is completely positive and preserves the trace. In other words, the transformation $\Phi$ satisfies:

- **Completely Positive**: Consider a composed system described by $\sigma_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$

  $$\mathbb{I}_A \otimes \Phi_B(\sigma_{AB}) \geq 0.$$  

- **Trace preserving**: For a given density matrix $\rho \in \mathcal{D}(\mathbb{C}^N)$

  $$\text{Tr}[\Phi(\rho)] = \text{Tr}[\rho] = 1$$

Satisfying these properties the transformations map a density matrix into another density matrix, named a Completely Positive and Trace Preserving (CPTP) channel. The set of CPTP channels is denoted as $\mathcal{P}(\mathbb{C}^N, \mathbb{C}^M)$.

Isometric transformations are linear transformations $V : \mathbb{C}^N \rightarrow \mathbb{C}^M$ satisfying:

$$V^\dagger V = \mathbb{I}_N,$$  

$\mathcal{U}(\mathbb{C}^N, \mathbb{C}^M)$ is the set of isometric operations. Isometries that map the space $\mathbb{C}^N$ on itself are named unitary operators. The set of unitary operations on $\mathbb{C}^N$ is denoted as $U \in \mathcal{U}(\mathbb{C}^N)$. A unitary operator $U \in \mathcal{U}(\mathbb{C}^N)$, satisfies $U^\dagger U = UU^\dagger = \mathbb{I}_N$. An isometric transformation preserves the inner product, and consequently the spectra of the operators.

An important CPTP channel for this work is the local measurement map $M_A \in \mathcal{P}(\mathbb{C}_A, \mathbb{C}_X)$. Where dim($\mathbb{C}_A$) = $|A|$ and dim($\mathbb{C}_X$) = $|X|$. For projective measurements, the measurement map is simply the dephasing operation in the measured orthonormal basis. Thus, considering a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ and a local projective measurement over $A$, the post-measurement state is:

$$M_A \otimes \mathbb{I}_B(\rho_{AB}) = \sum_x p_x |a_x\rangle \langle a_x| \otimes \rho_x^B,$$  

where $\{ |a_x\rangle \}_{1}^{[A]}$ is an orthonormal basis, and $p_x \rho_x^B = \text{Tr}_A(|a_x\rangle \langle a_x| \otimes \mathbb{I}_B\rho_{AB})$. The state $\rho_x^B$ is an output of the measurement process with probability $p_x$. For general measurement processes described by positive-operator valued measure (POVM), by Naimark’s dilation theorem the approach is the same as for projective measurements, the only difference is in the cardinality of the random variable $X = \{p_x\}^{[X]}_{x=1}$, that in this case, is equal to the number of POVM elements [19]. The local measurement outputs create an ensemble of quantum states $\xi_{AB}^{M} = \{p_x, \rho_x^B \}_{x=1}^{[X]}$.

### B. Schatten-p norm

The Schatten-p norm for operators is analogous to the $l_p$ norm for vectors.

**Definition 2.** Given a linear operator $A \in \mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$, the Schatten-p norm is defined as:

$$\|A\|_p = \left\{ \text{Tr}[(AA^\dagger)^{p/2}] \right\}^{1/p},$$  

where $p = [1, \infty)$.

The Schatten-p norm can be written as the $l_p$ norm of the spectral decomposition of the matrix $A$:

$$\|A\|_p = \left\{ \sum_k |\lambda(A)k|^p \right\}^{1/p},$$  

where $\{\lambda(A)k\}_k$ are the eigenvalues of $A$. As the Schatten norm only depends on the eigenvalues of the matrix, it is invariant under action of isometries.

### C. Quantum Entropies

Considering a random variable $X = \{p_x\}$, where $p_x \geq 0$ and $\sum_x p_x = 1$, its $\alpha$ - Renyi’s entropy is defined as [4]:

$$H_{\alpha}(X) = \frac{1}{1-\alpha} \log \sum_x p_x^\alpha.$$  

For $\alpha \rightarrow 1$ it is the Shannon entropy of $X$:

$$H_1(X) = -\sum_x p_x \log p_x.$$  

The quantum version of the Renyi’s entropy is defined as [20]:

**Definition 3 (Quantum $\alpha$ entropy).** Given a quantum state $\rho \in \mathcal{D}(\mathbb{C}^N)$, its $\alpha$ entropy is

$$S_{\alpha}(\rho) = \frac{1}{1-\alpha} \log \|\rho\|_\alpha^\alpha,$$  

where $\|\rho\|_\alpha$ is the Schatten norm, then $\|\rho\|_\alpha^\alpha = \text{Tr}(\rho^\alpha)$.

For quantum Renyi’s entropy, the following proposition is introduced.

**Proposition 4.** Renyi’s entropy is a concave function for $\alpha \in (0, 1)$:

$$S_{\alpha}(\sum_k p_k \rho_k) \geq \sum_k p_k S_{\alpha}(\rho_k)$$

The proof of this proposition was performed in Ref. [21], and a modern discussion about this issue can be found on
For a bipartite state $\rho_{AB} \in D(C_A \otimes C_B)$, classical correlations between A and B can be quantified by the amount of correlations extracted by means of local measurements:

$$J(A : B)_{\rho_{AB}} = \max_{I \otimes B \in P} I(A : X)_{I \otimes B} - I(A : B)_{\rho_{AB}},$$

where the optimization is taken over the set of local measurement maps $I \otimes B \in P(C_{AB}, C_X)$ and $I \otimes B(\rho_{AB}) = \sum_x p_x |x\rangle \langle x| \otimes \rho_{AB}$ is a quantum-classical state in the space $D(C_A \otimes C_X)$.

As the mutual information quantifies the total amount of correlations in the state, it is possible to define a quantifier of quantum correlations as the difference between the total correlations in the system, quantified by mutual information, and the classical correlations, measured by Eq.\,(17). This measure of quantumness of correlations is named quantum discord [29, 30]:

$$D(A : B)_{\rho_{AB}} = I(A : B)_{\rho_{AB}} - J(A : B)_{\rho_{AB}}.$$  

where $I(A : B)_{\rho_{AB}}$ is the von Neumann mutual information. The quantum discord quantifies the amount of information, that cannot be accessed via local measurements [31], therefore it measures the quantumness of correlations between A and B that cannot be recovered via a classical statistical inference process.

### B. Entanglement

A pure state $|\psi\rangle_{AB} \in C_A \otimes C_B$ is uncorrelated if it can be written as a tensor product of pure states of each partition:

$$|\psi\rangle_{AB} = |a\rangle \otimes |b\rangle,$$

where $|x\rangle \in C_X$, for $x = \{a, b\}$. By the definition of classical correlations in Eq.\,(16), a convex combination of product states can be quantum correlated. Therefore, taking convex combinations of non orthogonal states results in the notion of separable state [32].

**Definition 6 (Separable states).** Considering a composed system described by the state $\sigma \in D(C_A \otimes C_B)$, it is a separable state if and only if it can be written as:

$$\sigma_{AB} = \sum_{i,j} p_{i,j} |\psi_i\rangle \langle \psi_i| \otimes |\phi_i\rangle \langle \phi_i|,$$

where $|\psi_i\rangle_A \in C_A$ and $|\phi_i\rangle_B \in C_B$. 

III. QUANTUM CORRELATIONS

This section presents some definitions and discussions about quantum correlations. First the class of classical correlated state and the definition of a quantifier of classical correlations are presented. Next the concept of quantum entanglement is discussed. Finally the Koashi-Winter relation is introduced, which interplays classical correlations and quantum entanglement.

### A. Quantumness of Correlations

A composed state $\rho_{AB} \in D(C_A \otimes C_B)$ is said to be classically correlated if a local projective measurement $\Pi_A \otimes \Pi_B$, that commutes with the state, exists [27, 30]:

$$[\rho_{AB}, \Pi_A \otimes \Pi_B] = 0.$$  

The class of states that satisfies this equation is composed of states in the following form:

$$\rho_{AB} = \sum_{i=1}^{[A]} \sum_{j=1}^{[B]} p_{i,j} \Pi_i^A \otimes \Pi_j^B,$$

where $\sum_{i=1}^{[A]} \sum_{j=1}^{[B]} p_{i,j} = 1$, $p_{i,j} \geq 0$ and $\Pi_A \otimes \Pi_B \in \{\Pi_i^A \otimes \Pi_j^B\}_{i,j}$. The amount of classical correlations in a quantum state is measured by the capacity to extract information locally [31]. As the measurement process is a classical statistical inference, classical correlations can be quantified by the amount of correlations remaining in the system after a local measurement [28].

**Definition 5 (Classical Correlations).** For a bipartite density matrix $\rho_{AB} \in D(C_A \otimes C_B)$, classical correlations between A and B can be quantified by the amount of correlations extracted by means of local measurements:

$$I(A : B)_{\rho_{AB}} = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

where $\rho_A = \text{Tr}_B(\rho_{AB})$, the same for $\rho_B$. Mutual information is zero if the state is a product state $\rho_{AB} = \rho_A \otimes \rho_B$.

For classical quantum states $\rho_{AX} = \sum_x p_x \rho_x^A \otimes |x\rangle \langle x|$, where $X = \{p_x\}$ is a classical register, the mutual information is:

$$I(A : X)_{\rho_{AX}} = S(\rho_A) - \sum_x p_x S(\rho_x^A).$$

This function quantifies the distinguishability of states in the ensemble $\xi_A = \{p_x, \rho_x^A\}$, and is also named Jensen Shannon divergence [22], represented in this work as

$$Q(\xi_A) = S(\rho_A) - \sum_x p_x S(\rho_x^A).$$

For an ensemble of quantum state composed of two states, it is defined as the symmetric and smoothed version of Shannon relative entropy [22, 23]. It is related to the Bures distance and induces a metric for pure quantum states, related to the Fisher-Rao metric [24]. Holevo's Theorem states that quantity is the capacity of a given channel to transmit classical information [25, 26].
Note that the states \( \{ |\psi_i\rangle \}_i \) and \( \{ |\phi_j\rangle \}_j \) are, in general, not orthogonal states. If these sets are composed of orthogonal states, the state in Eq. (20) is classically correlated. Quantum entanglement is defined as the negation of Definition 6 [33]:

**Definition 7** (Entanglement). A composed state \( \rho_{AB} \in \mathcal{D}(\mathcal{C}_A \otimes \mathcal{C}_B) \) is entangled if it is not separable.

The amount of quantum entanglement of a bipartite system \( \rho_{AB} \in \mathcal{D}(\mathcal{C}_A \otimes \mathcal{C}_B) \) can be quantified by the entanglement of formation. The entanglement of formation is interpreted as the minimum amount of entangled pure states required to build \( \rho_{AB} \), by means of a convex combination [34].

**Definition 8** (Entanglement of Formation). Considering a quantum state \( \rho \in \mathcal{D}(\mathcal{C}_A \otimes \mathcal{C}_B) \), the entanglement of formation is defined as the convex roof:

\[
E_f(\rho) = \inf_{\xi} \sum_i p_i E_i(|\psi_i\rangle),
\]

where the minimization is performed over all ensembles \( \xi = \{ p_i, |\psi_i\rangle\}_i \), such that \( \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \), \( \sum_i p_i = 1 \) and \( p_i \geq 0 \).

The function \( E_i(|\psi_i\rangle) = S(\rho_i) \) is named entropy of entanglement [35, 36], and is the usual quantifier of entanglement for pure states. For pure states, entanglement of formation is equal to the classical correlations and quantum discord [37].

\[
E_f(|\psi_{AB}\rangle) = J(A : B) = D(A : B)_{\psi_{AB}} = S(\rho_A),
\]

where \( |\psi_{AB}\rangle \in \mathcal{D}(\mathcal{C}_A \otimes \mathcal{C}_B) \) is a pure state and \( \rho_A = \text{Tr}_B(\psi) \).

As in this work the interest is in Renyi’s entropies, the \( \alpha \) - Entanglement of Formation (EoF) is introduced [17]:

**Definition 9.** \( \alpha \) - Entanglement of formation of a bipartite state \( \rho_{AB} \in \mathcal{D}(\mathcal{C}_A \otimes \mathcal{C}_B) \) is defined as:

\[
E^\alpha_f(\rho_{AB}) = \inf_{\{ p_i, |\psi_i\rangle\}_i} \sum_i p_i S_\alpha(\text{Tr}_B(|\psi_i\rangle\langle\psi_i|_{AB})),
\]

for \( \rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|_{AB} \).

\( S_\alpha(\text{Tr}_B(|\psi_i\rangle\langle\psi_i|_{AB})) \) is the \( \alpha \) - entropy of entanglement, which is an entanglement monotone for \( \alpha \in (0, 1) \) [38, 39].

The Schur concavity of \( \alpha \) - entropy for \( \alpha \in (0, 1) \) guarantees that \( \alpha \) - entanglement of formation is a monotone function under the action of local operations and classical communication (LOCC) [17].

**C. Koashi - Winter relation**

Given a bipartite system \( \rho_{AB} \in \mathcal{D}(\mathcal{C}_A \otimes \mathcal{C}_B) \), and its purification \( |\psi\rangle_{ABE} \in \mathcal{C}_A \otimes \mathcal{C}_B \otimes \mathcal{C}_E \). The dimension of the global space is: \( \text{dim}(\mathcal{C}_{ABE}) = \text{dim}(A) \cdot \text{dim}(B) \cdot \text{rank}(\rho_{AB}) \). The purification creates quantum correlations between the system \( AB \) and the purification system \( E \), unless the state is already pure. The balance between the correlations of a tripartite purification is settled by the Koashi-Winter relation [40].

**Theorem 10** (Koashi-Winter relation). Considering \( \rho_{ABE} \in \mathcal{D}(\mathcal{C}_A \otimes \mathcal{C}_B \otimes \mathcal{C}_E) \) a pure state then:

\[
J(A : E)_{\rho_{ABE}} = S(\rho_A) - E_f(\rho_{AB}),
\]

where \( \rho_X = \text{Tr}_Y[\rho_{XY}] \).

The Koashi-Winter equation quantifies the amount of entanglement among \( A \) and \( B \), considering that the former is classically correlated with another system \( E \). This property is interesting as it is related with the monogamy of entanglement [11]. An analogous expression of the Koashi - Winter relation has been obtained for quantum discord and entanglement of formation [37]. From this new relation, the irreversibility of the entanglement distillation protocol and quantum discord are interplayed [42].

In this work a generalization of the Koashi - Winter relation is calculated for a class of Renyi’s entropies for the parameter \( \alpha \in (0, 1) \).

**IV. RESULTS**

First the \( \alpha \) Quantum Jensen Shannon divergence (QJSD) is introduced [51].

**Definition 11** (\( \alpha \) - Quantum Jensen Shannon divergence). Given a quantum ensemble \( \xi = \{ p_k, \rho_k \}_k \), for \( \rho_k \in \mathcal{D}(\mathcal{C}_N) \) the Renyi’s Quantum Jensen Shannon divergence, for \( \alpha \in (0, 1) \) is defined as:

\[
Q_\alpha(\xi) = S_\alpha(\sum_k p_k \rho_k) - \sum_k p_k S_\alpha(\rho_k),
\]

where

\[
S_\alpha(X) = \frac{1}{1 - \alpha} \log \|X\|_\alpha^\alpha,
\]

and \( \|\cdot\|_\alpha \) is the Schatten norm

\[
\|X\|_\alpha = \text{Tr}(X^\alpha),
\]

for any matrix \( X \in \mathcal{L}(\mathcal{C}_N) \).

A corollary of the concavity of the Renyi’s entropy is the positivity of the \( \alpha \) - QJSD function:

\[
Q_\alpha(\xi) \geq 0,
\]
for $\alpha \in (0, 1)$. The $\alpha$-QJSD is zero if the ensemble has cardinality equal to one ($M = 1$), and maximum if the states in the ensemble are pure and linearly independent. The $\alpha$-QJSD is a generalization of the QJSD [51], and quantifies the distinguishability between the states in the ensemble.

As aforementioned, local measurements create a quantum ensemble in the non-measured partition, composed by the output states. Given this property the function is defined:

**Definition 12.** For a bipartite state $\rho_{AB}$, one can define the function

$$C_\alpha(\rho_{AB}) = \sup_{\xi_{AB}^M} Q_\alpha(\xi_{AB}^M),$$

where $\xi_{AB}^M = \{p_x, \rho_A^x\}$ for $p_x \rho_A^x = \text{Tr}_B(1_A \otimes M_x \rho_{AB})$ and $(M_x)_x$ are elements of a POVM.

As discussed $Q_\alpha(\xi_{AB}^M)$ quantifies the distinguishability between the states in the ensemble $\xi_{AB}^M$, therefore $C_\alpha(\rho_{AB})$ measures the distinguishability between the states in the ensemble created by the measurement $M_B$ such that the states are the most distinguishable. For von Neumann entropy this quantity quantifies classical correlations of the state $\rho_{AB}$.

The relation of $C_\alpha(\rho_{AB})$ with quantum correlations in $\rho_{AB}$ is stated in the main result of this work, presented below.

**Theorem 13.** Considering a pure tripartite state $\psi_{ABE} = \rho_{ABE} \in D(C_A \otimes C_B \otimes C_E)$, the following equality holds:

$$C_\alpha(\rho_{AE}) = \sup_{\xi_{AE}^M} Q_\alpha(\xi_{AE}^M) = S_\alpha(\rho_A) - E^\alpha(\rho_{AB})$$  \hspace{1cm} (26)

Proof. There exists a set of orthogonal states $\{|l\rangle_{l=1}^{|E|}\}$ such that the states $\psi_{ABE}$ can be written as:

$$|\psi\rangle_{ABE} = \sum_{l} p_l |\phi_{l}\rangle_{AB} |l\rangle_E,$$

thus the reduced density matrix $\text{Tr}_E(\psi_{ABE}) = \rho_{AB} = \sum_{l} p_l |\phi_{l}\rangle_{AB} \langle\phi_{l}|$. Performing a measurement on subsystem $E$, such that the POVM measurement are rank-1 operators $\{M^E_x = |\mu_{x}\rangle\langle\mu_{x}|\}$, where $\sum_x M^E_x = 1$ and $M^E_x \geq 0$, the post-measurement state is:

$$\rho_{ABE'} = \mathbb{I}_{AB} \otimes M_E(\rho_{ABE})$$

$$= \sum_x \text{Tr}_E(\mathbb{I}_{AB} \otimes |\mu_{x}\rangle \langle\mu_{x}| \rho_{ABE}) \otimes |e_x\rangle \langle e_x|_{E'},$$

$$= \sum_x q_x |\psi_x\rangle_{AB} \langle\psi_x| \otimes |e_x\rangle \langle e_x|_{E'},$$

where $q_x |\psi_x\rangle_{AB} \langle\psi_x| = \text{Tr}_E(\mathbb{I}_{AB} \otimes |\mu_{x}\rangle \langle\mu_{x}| \rho_{ABE})$. As $\rho_{AB} = \text{Tr}(\rho_{ABE}) = \text{Tr}(\rho_{ABE'})$, it is clear that there exists a POVM such that:

$$q_x = p_x, \quad |\psi_x\rangle_{AB} = |\phi_x\rangle_{AB}.$$

In subsystem $AE'$ the post-measurement state is:

$$\rho_{AE'} = \text{Tr}_B(\rho_{ABE'})$$

$$= \sum_{x} q_x \text{Tr}_B(|\psi_x\rangle \langle AB| \otimes |e_x\rangle \langle e_x|_{E'}).$$

$$= \sum_{x} q_x \rho_A^x \otimes |e_x\rangle \langle e_x|_{E'},$$

where $\text{Tr}_B(|\psi_x\rangle \langle AB|) = \rho_A^x$. The state $\rho_{AE'}$ represents the ensemble of quantum state $\xi_{AE}^M = \{q_x, \rho_A^x\}$ prepared according to the random variable $X = \{q_x\}_x$. In this way calculating $\alpha$-QJSD for the ensemble $\xi_{AE}^M$:

$$Q_\alpha(\xi_{AE}^M) = S_\alpha(\rho_A) - \inf_{\xi_{AE}^M} \sum_x q_x S_\alpha(\text{Tr}_B(|\psi_x\rangle \langle AB|)).$$

However the ensemble is created by means of a measurement performed on $E$, thus one can rewrite the last term of the equation as

$$\inf_{\xi_{AE}^M} \sum_x q_x S_\alpha(\text{Tr}_B(|\psi_x\rangle \langle AB|))$$

$$= \inf_{\{q_x, |\psi_x\rangle\langle\psi_x|\}} \sum_x q_x S_\alpha(\text{Tr}_B(|\psi_x\rangle \langle AB|)).$$

As the state in $AB$, on average, does not change by the measurement on $E$, one can identify the right hand side of the equation as the $\alpha$-Renyi Entanglement of Formation:

$$E^\alpha_f(\rho_{AB}) = \inf_{\{q_x, |\psi_x\rangle\langle\psi_x|\}} \sum_x q_x S_\alpha(\text{Tr}_B(|\psi_x\rangle \langle AB|)).$$

From Eq. (26) it is possible to recover and generalize that, for pure states the entanglement of formation is equal to the von Neumann entropy of the reduced density matrix [17].

**Theorem 14 (Pure States).** Consider $\rho_{AB}$ a pure state, the $\alpha$-QJSD of the ensemble $\xi_{AE}^M$ is zero, therefore:

$$E^\alpha_f(\rho_{AB}) = S_\alpha(\rho_A).$$

Proof. The state $\rho_{AB}$ can be written in the Schmidt decomposition:

$$|\psi_{AB}\rangle = \sum_{l} c_l |a_l\rangle |b_l\rangle.$$
The purification of a pure state is just coupling another pure ancilla to:

$$\rho_{ABE} = |\psi_{AB}\rangle \langle \psi_{AB}| \otimes |0\rangle \langle 0|.$$  \hspace{1cm} (38)

Performing a measurement $\mathcal{M}_x$ with POVM elements $\{M^E_x\}_x$ on system $E$, the post-measurement states on subsystem $A$ are:

$$\rho^A_x = \frac{1}{p_x} \text{Tr}_E (\mathbb{I}_A \otimes M^E_x \rho_{AE}) = \frac{1}{p_x} \sum_l c_l \langle 0 | M_x | 0 \rangle |a_l\rangle \langle a_l|,$$

taking the partial trace over $B$ on Eq. (37) one realizes that:

$$\rho^A_x = \frac{1}{p_x} (0 | M_x | 0) \rho_A = \rho_A,$$

for every measurement map performed on $E$. Therefore the ensemble created by the local measurement is composed of only one single state $\xi^E_{AE} = \{1, \rho_A\}$, implying that Renyi’s QJSD of the ensemble is zero. \hfill \Box

The Renyi’s entropy generalization of the Koashi - Winter relation state that there is an interplay between the most distinguishable states of the ensemble created by the local measurement on the bipartite system, and the $\alpha$ EoF of the unmeasured system and the purification ancillary system. The standard KW relation relates classical correlations and the entanglement of formation of the unmeasured state and the purification ancillary system. Note that the function $C_\alpha(\rho_{AE}) = \sup_{\xi^E_{AE}} Q_\alpha(\xi^E_{AE})$ can be a quantifier of classical correlations. As discussed by Henderson and Vedral \[30\], the standard measure of classical correlations quantifies the amount of information accessed via local measurements on a bipartite system, that is equal to the distinguishability of the states of the ensemble created by the local measurement, quantified by the QJSD. In this way, the properties that $C_\alpha$ must satisfy to be a measure of classical correlations are now discussed, and some analytical results are obtained from this discussion.

It is possible to rewrite Eq. (20) changing the order of the labels $B \rightarrow E$:

$$C_\alpha(\rho_{AB}) = S_\alpha(\rho_A) - E^\alpha_f(\rho_{AE}).$$  \hspace{1cm} (39)

Then now the properties of the function $C_\alpha(\rho_{AB})$ are presented, for a given density operator $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, and state that it can be a quantifier of classical correlations of $\alpha \in (0, 1)$.

For a function of information to quantify correlations between quantum systems it must satisfy some important properties \[52\].

1. $C_\alpha(\rho_{AB}) = 0$ if and only if $\rho_{AB}$ is a product state;

2. $C_\alpha(\rho_{AB}) = C_\alpha(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger)$, for $U_X \in \mathcal{U}(\mathbb{C}_X)$ a unitary operation.

3. $C_\alpha(\rho_{AB}) \geq C_\alpha(\Phi_A \otimes \Phi_B(\rho_{AB}))$, for $\Phi_X$ a CPTP map.

The proof that $C_\alpha(\rho_{AB})$ satisfies these properties is performed in the sequence by the following theorems.

**Theorem 15 (Property 1).** Consider a state $\rho_{AB}$ and its post local measurement state $\rho_{AB} = \sum_x p_x \rho_A \otimes |x\rangle \langle x|_B'$, for the ensemble $\xi^E_{AB} = \{p_x, \rho^A_x\}$ the Renyi QJSD, maximized over all ensembles created by the local measurement, is zero if and only if $\rho_{AB}$ is a product state.

**Proof.** Given $\rho_{AB} = \rho_A \otimes \rho_B$ its purified state will also be a product state:

$$|\psi\rangle_{ABER} = |\phi\rangle_{AE} \otimes |\varphi\rangle_{BR} = \left( \sum_l \sqrt{a_l} |a_l\rangle_A |l\rangle_E \right) \otimes \left( \sum_k \sqrt{k} |b_k\rangle_B |k\rangle_R \right),$$

for $\rho_A = \sum_l a_l |a_l\rangle \langle a_l|$ and $\rho_B = \sum_k b_k |b_k\rangle \langle b_k|$. As shown in Proposition \[14\] $\alpha$ - entanglement of formation for pure state is equal to the Renyi’s entropy of the reduced density matrix:

$$E^\alpha_f(\rho_{AE}) = S_\alpha(\rho_A),$$  \hspace{1cm} (40)

therefore the $\alpha$-QJSD is zero. On the other hand, the ensemble $\xi^E_{AB}$ created by means of a local measurement $\mathcal{M}$ on $B$ the product state $\rho_{AB} = \rho_A \otimes \rho_B$:

$$\rho_{AB'} = \rho_A \otimes \mathcal{M}(\rho_B).$$

For every measurement performed over the subsystem $B$ the state in $A$ remains undisturbed:

$$\rho_A \otimes \mathcal{M}(\rho_B) = \rho_A \otimes \rho_{B'}.$$

Therefore the ensemble created in $A$ by means of this measurement over $B$ has just one element $\xi^E_{AB} = \{1, \rho_A\}$, which implies:

$$Q_\alpha(\xi^E_{AB}) = 0. \hspace{1cm} \text{(41)}$$

**Theorem 16 (Property 2).** $C_\alpha(\rho_{AB})$ is invariant under local unitary operations.

**Proof.** As Schatten-$p$ norm is invariant under unitary operations, then $\alpha$- QJSD is invariant under unitary operations:

$$Q(U \xi U^\dagger) = S_\alpha(U \left( \sum_k p_k \rho_k \right) U^\dagger) - \sum_k p_k S_\alpha(U \rho_k U^\dagger) = Q(\xi),$$

where $U \in \mathcal{U}(\mathbb{C}_1)$ is a unitary operation. As it is holds for every ensemble $\xi = \{p_k, \rho_k\}_{k=1}^M$. \hfill \Box
Theorem 17 (Property 3). For a bipartite state $\rho_{AB}$, the function

$$C_\alpha(\rho_{AB}) \geq C_\alpha(\tilde{\rho}_{AB}),$$

for $\alpha \in (0,1)$, where $\tilde{\rho}_{AB} = \Phi_A \otimes \Phi_B(\rho_{AB})$, and $\Phi_X \in \mathcal{P}(\mathbb{C}^X)$.

Proof. This comes from the fact that $\alpha$ - EoF is an entanglement monotone, and decreases under LOCC, for $\alpha \in (0,1)$.

An interesting analytical result obtained from Eq. (26) is that $C_\alpha(\rho_{AB})$ is equal to the entropy of entanglement for $\rho_{AB}$ a pure state.

Theorem 18 (Classical correlations of a pure state). For $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ a pure state $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$, it holds

$$C_\alpha(\rho_{AB}) = S(\rho_A),$$

(42)

where $\rho_A = Tr_B(|\psi\rangle\langle\psi|_{AB})$.

Proof. As $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$ is a pure state, its purification $|\phi\rangle_{ABE} = |\psi\rangle_{AB} \otimes |0\rangle_E$ is a product state in the space $\mathbb{C}_{AB} \otimes \mathbb{C}_E$, then $E_{\phi}(\rho_{AB}) = 0$, proving the statement.

For quantum states without quantum correlations it is expected that the amount of classical correlations is equal to a standard classical entropy. This is obtained in the next theorem.

Theorem 19 (Classically Correlated State). Considering $\rho_{AB}$ a classical correlated state:

$$\rho_{AB} = \sum_{x,y} p_{x,y} |a_x\rangle\langle a_x| \otimes |b_y\rangle\langle b_y|,$$

where $\{|a_x\rangle\}_{x=1}^{|A|}$ and $\{|b_y\rangle\}_{y=1}^{|B|}$ are orthonormal basis in $\mathbb{C}_A$ and $\mathbb{C}_B$ respectively, then:

$$C_\alpha(\rho_{AB}) = H_\alpha(X,Y) - H_\alpha(X|Y),$$

for $X = \{p_x = \sum_y p_{x,y} |a_x\rangle\}_{x=1}^{|A|}$, and analogous for $Y$.

Proof. Taking the local measurement over partition $B$, there exists a measurement operation $M_B$ that enables the classically correlated state invariant

$$\mathbb{I}_A \otimes M_B(\rho_{AB}) = \rho_{AB}.$$

The post-measurement ensemble of states is:

$$\xi_{AB}^{M_B} = \left\{ p_y \sum_x p(x|y) |a_x\rangle\langle a_x| \right\} \bigg|^B_{y=1},$$

where $p(x|y) = p_{x,y}/p_y$. Calculating $\alpha$ - QJSDF of $\xi_{AB}^{M_B}$:

$$Q_\alpha(\xi_{AB}^{M_B}) = S_\alpha(\sum_{x,y} p_{x,y} p(x|y) |a_x\rangle\langle a_x|) - \sum_y p_y S_\alpha(\sum_x p(x|y) |a_x\rangle\langle a_x|),$$

as:

$$S_\alpha(\sum_{x,y} p_{x,y} p(x|y) |a_x\rangle\langle a_x|) = H_\alpha(X,Y)$$

(45)

$$S_\alpha(\sum_x p(x) |a_x\rangle\langle a_x|) = H_\alpha(X|Y),$$

(46)

proving the proposition.

Remark 20. This definition for classical conditional entropy

$$H_\alpha(X|Y) = \frac{1}{1-\alpha} \sum_y p_y \log [\sum_x p(x|y)^\alpha],$$

does not satisfy the monotonicity under stochastic operations for every $\alpha \in (0,1)$ [3]. An interesting discussion about this issue can be found in Ref. [23], although it is not known if this is not monotone for every $\alpha \in (0,1)$.

V. KW - RELATION FOR LOG ROBUSTNESS

As an application of the results of this paper, an interesting measure of entanglement is the well known generalized robustness, which quantifies the amount of mixture with another state needed to destroy the entanglement of the system [54, 55]. Formally this is defined as:

Definition 21 (Generalized robustness). Consider an $n$-partite state $\rho \in \mathcal{D}(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n})$, generalized robustness of $\rho$ is defined as:

$$R_G(\rho) = \left\{ \min_{s \in \mathbb{R}_+} s : \exists \rho_s \ s.t. \ \frac{\rho + s \rho_s}{1 + s} \in \text{Sep} \right\},$$

(47)

where Sep is the set of separable states in $\mathcal{D}(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n})$.

The parameter $s$ is zero for separable states and finite for entangled states [54].

Another entanglement quantifier related to the generalized robustness of entanglement is the log - generalized robustness (LGR) defined as [50]:

$$LR_G(\rho) = \log_2(1 + R_G(\rho)),$$

(48)

where $R_G(\rho)$ is the generalized robustness of $\rho$. LGR is an entanglement monotone, sub-additive, non increasing under trace preserving separable operations, and an upper bound for the distillable entanglement [56]. It was also studied in the context of the resources theory of quantum entanglement [52, 59].

As an application of the main results in Eq. (26), it is possible to obtain that for $\alpha = 1/2$, Renyi's entanglement entropy of a pure state is equal to the LGR for the pure state.
Lemma 22. Considering a pure state $|\psi\rangle_{AB} \in \mathbb{C}_A \otimes \mathbb{C}_B$, the $\alpha = 1/2$ - entanglement entropy is defined equal to the log robustness:

$$S_{1/2}(\rho_A) = LR_g(\psi_{AB}),$$

where $\rho_A = \text{Tr}_B(\psi_{AB})$ and $\psi_A = |\psi\rangle \langle \psi|_{AB}$.

Proof. Considering the pure state in its Schmidt decomposition $|\psi\rangle_{AB} = \sum_i \sqrt{\mu_i} |a_i\rangle |b_i\rangle$, then its reduced density matrix is $\rho_A = \sum_i \mu_i |a_i\rangle \langle a_i|$. The $\alpha = 1/2$ entropy is simply:

$$S_{1/2}(\rho_A) = 2 \log \text{Tr}(\sqrt{\rho_A}) = 2 \log(\sum_i \sqrt{\mu_i}).$$

As bipartite pure state the generalized robustness is [39]:

$$R_G(\psi_{AB}) = \left( \sum_i \sqrt{\mu_i} \right)^2 - 1,$$

then by definition of LGR:

$$S_{1/2}(\rho_A) = LR_g(\psi_{AB}).$$

\hspace{1cm} \Box

As a direct corollary, it is possible to calculate that the $\alpha = 1/2$ - entanglement of formation is a convex roof version of the LGR: for a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$

$$E_{1/2}^{1/2}(\rho_{AB}) = \min_{\{p_i, \psi_i\}} \sum_i p_i LR_g(\psi_i),$$

where $\rho_{AB} = \sum_i p_i |\psi_i\rangle \langle \psi_i|_{AB}$.

Before introducing the main theorem of this section, some useful lemmas are proved.

Lemma 23. Given an ensemble of quantum states $\xi_A = \{p_x, \rho^A_x\}_{x=1}^{|X|}$, for $\rho^A_x \in \mathcal{D}(\mathbb{C}_A)$ and $|X|$ the cardinality of classical distribution $X = \{p_x\}$. The probability of success in distinguishing the states in the ensemble is defined as:

$$P_{\text{suc}}(X|A) = \sup_{\{E_x\}} \sum_x \text{Tr}(E_x \rho^A_x),$$

where $\{E_x\}_{x=1}^{|X|}$ is a set of POVM elements. It is rated by the $\alpha = 1/2$ entropy of $\rho_A = \sum_x p_x \rho_x^A$ as:

$$S_{1/2}(\rho_A) \geq - \log P_{\text{suc}}(X|A).$$

Proof. As $\alpha$ entropy monotonically increases for $\alpha \in (0, 1) \cup (1, 2]$ [57], then:

$$S_{1/2}(\rho_A) \geq S_{1/2}(\rho_{AX}),$$

where $\rho_{AX} = \sum_x p_x \rho_x^A \otimes |x\rangle \langle x|_X$. As discussed in Ref. [53]

$S_{1/2}$ is named $S_{\text{max}}$ or $\text{max}$ entropy. Therefore:

$$S_{1/2}(\rho_A) \geq S_{\text{max}}(\rho_{AX}) \geq S_{\text{min}}(X|A) \equiv - \log P_{\text{suc}}(X|A).$$

Where $S_{\text{min}}(X|A) = \max_{\sigma} \left\{ \|\sigma^{-1/2} \otimes \id_{\mathbb{C}_X} \sigma^{-1/2} \otimes \mathbb{I}_2 \|_1 \right\}$ [61]. Eq. (52) is Lemma 3.1.5 of Ref. [62] and Eq. (53) is Theorem 1 of Ref. [63].

\hspace{1cm} \Box

Lemma 24. Consider $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, the regularized $E_{1/2}^\infty(\rho_{AB})$ and $LR_g^\infty(\rho_{AB})$ defined respectively as:

$$E_{1/2}^\infty(\rho_{AB}) = \lim_{n \to \infty} \frac{E_{1/2}^{1/2}(\rho_{\otimes n}^{AB})}{n},$$

$$LR_g^\infty(\rho_{AB}) = \lim_{n \to \infty} \frac{LR_g(\rho_{\otimes n}^{AB})}{n},$$

the following equality holds:

$$E_{1/2}^\infty(\rho_{AB}) = LR_g^\infty(\rho_{AB}).$$

Proof. Consider the relative entropy of entanglement:

$$E_R(\rho) = \min_{\sigma \in \text{Sep}} S(\rho||\sigma),$$

where $S(\rho||\sigma) = - \text{Tr}(\sigma \log \rho) - S(\rho)$ is the relative entropy. For pure states $E_R(\psi) = S(\rho_A)$, where $\rho_A$ is the reduced density matrix of $\psi = |\psi\rangle \langle \psi|$. As demonstrated by Brandão and Plenio [57, 58]: $LR_g^\infty(\rho_{AB}) = E_R(\rho_{AB}) = E_C(\rho_{AB})$, where $E_C$ is the entanglement cost. It implies that for pure states:

$$LR_g^\infty(\psi_{AB}) = E_R(\psi_{AB}) = S(\rho_A),$$

which implies that $E_f^\infty(\rho_{AB}) = E_{1/2}^\infty(\rho_{AB})$. As proved in Ref. [53]: $E_f^\infty(\rho_{AB}) = E_C(\rho_{AB})$, where $E_f^\infty$ is the regularized entanglement of formation. Therefore the statement comes from:

$$LR_g^\infty(\rho_{AB}) = E_C(\rho_{AB}) = E_f^\infty(\rho_{AB}) = E_{1/2}^\infty(\rho_{AB}).$$

\hspace{1cm} \Box

As aforementioned, the function $Q_{\alpha}(\xi_{AE}^M)$, in analogy with QJSD, quantifies the distinguishability of the states in the ensemble. If the ensemble is generated by means of local measurements, its optimization over all local measurements quantifies classical correlations in the state $\rho_{AE}$. This concept is related to the channel capacity of a quantum - classical channel, where the capacity is rated by the HSW quantity [26, 64], that is the QJSD of the ensemble created by the quantum classical channel [51]. The following result provides a relation between the capacity of a quantum classical channel, a dephasing channel acting locally in a composed system, and the probability of success in discriminating the states in the output ensemble, depending on the entanglement with the purification ancillary system. It is considered that pure state $\psi_{AE}$ is shared, and information, encoding on $E$, is sent from $A$ to $B$ by a quantum classical channel. This ensemble is created by means of the optimal local measurement over $E$, considering that there may be many copies of the state.
Theorem 25. Consider a pure state $\rho_{ABE} \in D(\mathcal{C}_A \otimes \mathcal{C}_B \otimes \mathcal{C}_E)$, performing the optimal measurement over $E$ such that $C_{1/2}^{\infty}(\rho_{AE}) = \sup_{\xi_{AE}^M} Q_\alpha(\xi_{AE}^M)$ and $\xi_{AE}^M = \{p_x, \rho_x^A\}$ is the ensemble in $A$ created by the local measurement. $C_{1/2}^{\infty}(\rho_{AE})$ is rated below as:

$$C_{1/2}^{\infty}(\rho_{AE}) \geq -\log P_{\text{succ}}(X|A) - LR_\alpha^{\infty}(\rho_{AB}),$$

where $P_{\text{succ}}(X|A)$ is the probability of success in discriminating the states in the ensemble $\xi_{AE}^M$, and $LR_\alpha^{\infty}(\rho_{AB})$ is asymptotic log generalized robustness.

Proof. Given a regularized version of Eq. (20) for $\alpha = 1/2$

$$C_{1/2}^{\infty}(\rho_{AE}) = S_{1/2}^{\infty}(\rho_A) - E_{1/2}^{\infty}(\rho_{AB}),$$

where $f^{\infty}(\rho) = \lim_{n \to \infty} \frac{f^{\alpha^n}(\rho)}{n}$. Substituting Eq. (53), Eq. (54), and by linearity of the trace in definition of probability of success in Eq. (50), it proves the statement. \qed

VI. CONCLUSION

In this work a generalization of Koashi - Winter relation is presented by means of the Renyi’s entropic version of quantum Jensen Shannon divergence. From this generalization, some analytical results for quantifiers of classical and quantum correlations are presented. A $\alpha$ Renyi’s quantifier of classical correlations for $\alpha \in (0, 1)$ is also introduced. As an application of the main result, a lower bound for $C_{1/2}^{\infty}(\rho_{AE})$ is obtained, related to the discrimination of the states in the ensemble, created by the local measurement, and the asymptotic log robustness of entanglement. This result expresses the character of the quantifier of distinguishability of the states in the ensemble composed of the measurement output states, in contrast with its character as a correlation quantifier.

As a natural extension of these results one can define the Jensen Shannon divergence from the Sandwiched Renyi’s relative entropy [7]

$$QJSD_\alpha = D_\alpha(\rho_{AX}||\rho_A \otimes \rho_X),$$

where

$$D_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \log \left\{ \text{Tr}\left[ \left( \sigma^{1-\alpha} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right] \right\},$$

if $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$, otherwise it is not finite. This is known to be monotone decreasing [16]. From this definition of QJSD one can study its relation with $\alpha$ entanglement of formation defined in Ref. [12, 13], that is obtained from Sandwiched Renyi’s - relative entropy. Another interesting quantifier of correlations in this context is quantum discord, defined and explored in Ref. [13], obtained starting from the generalization of the Renyi’s conditional information [11]. Some application in quantum information protocols remain to be explored [9], and related to the generalization of Koashi - Winter relations.

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Conflict of Interest Disclosure

The author declares that there is no conflict of interest regarding the publication of this paper.

[1] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 465, 2537 (2009).
[2] M. M. Wilde, Quantum Information Theory (Cambridge University Press, 2013), ISBN 9781107034259, URL https://books.google.com.br/books?id=T36v2Sp7DnIC
[3] C. Tsallis, Journal of statistical physics 52, 479 (1988).
[4] A. Rényi et al., in Proceedings of the fourth Berkeley symposium on mathematical statistics and probability (1961), vol. 1, pp. 547–561.
[5] M. Mosonyi and F. Hiai, IEEE Transactions on Information Theory 57, 2474 (2011).
[6] A. Teixeira, A. Matos, and L. Antunes, IEEE Transactions on Information Theory 58, 4273 (2012), ISSN 0018-9448.
[7] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, Journal of Mathematical Physics 54, 122203 (2013).
[8] S. Fehr and S. Berens, IEEE Transactions on Information Theory 60, 6801 (2014).
[9] M. Mosonyi and T. Ogawa, Communications in Mathematical Physics 334, 1617 (2015), ISSN 1432-0916, URL http://dx.doi.org/10.1007/s00220-014-2248-x
[10] N. Datta and M. M. Wilde, Journal of Physics A: Mathematical and Theoretical 48, 505301 (2015).
[11] M. Berta, K. P. Seshadreesan, and M. M. Wilde, Journal of Mathematical Physics 56, 022205 (2015).
[12] M. Berta, K. P. Seshadreesan, and M. M. Wilde, Phys. Rev. A 91, 022333 (2015), URL https://link.aps.org/doi/10.1103/PhysRevA.91.022333
[13] K. P. Seshadreesan, M. Berta, and M. M. Wilde, Journal of Physics A: Mathematical and Theoretical 48, 395303 (2015).
[14] F. Dupuis and M. M. Wilde, Quantum Information Pro-
