CR-quadrics with a symmetry property

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1. Introduction

A well studied class of CR-submanifolds in a complex linear space consists of the quadrics, that is, of real quadratic submanifolds of the form

\[ Q = \{ (w, z) \in \mathbb{C}^{k+n} : w + \overline{w} = h(z, z) \}, \]

where \( h : \mathbb{C}^{n} \times \mathbb{C}^{n} \rightarrow \mathbb{C}^{k} \) is a non-degenerate hermitian form such that the image of \( h \) spans all of \( \mathbb{C}^{k} \). It is well known and easy to see that the group \( \text{Aff}(Q) \) of all (complex affine) transformations leaving \( Q \) invariant acts transitively on \( Q \). Besides these global CR-automorphisms of \( Q \), in general there exist also non-affine local CR-automorphisms (between domains) of \( Q \), which cannot be extended to global CR-automorphisms of \( Q \). By [1] every such local (smooth) CR-isomorphism is real-analytic and by [10] extends to a birational transformation of \( \mathbb{C}^{k+n} \). All birational transformations obtained this way generate a group that we denote by Bir\((Q)\) in the following. It is by no means evident, but shown in [6], that \( g(Q \cap \text{reg}(g)) = Q \cap \text{reg}(g^{-1}) \) holds, where \( \text{reg}(g) \subset \mathbb{C}^{k+n} \) is the Zariski open subset of all regular points of the birational transformation \( g \) with \( Q \cap \text{reg}(g) \) being a dense domain in \( Q \).

Up to CR-isomorphism there exists a unique homogeneous real-analytic CR-manifold \( \hat{Q} \) containing \( Q \) as open dense CR-submanifold in such a way that every CR-isomorphism between domains in \( Q \) extends to a global CR-automorphism of \( \hat{Q} \). In particular, the group Bir\((Q)\) can be canonically identified with the CR-automorphism group \( \text{Aut}(\hat{Q}) \) of the extended quadric \( \hat{Q} \). The group \( G := \text{Aut}(\hat{Q}) \) has no center and can be realized as closed subgroup \( G \subset \text{SL}_N(\mathbb{C}) \) for some integer \( N \geq 2 \) is such a way that \( \hat{Q} \) is a \( G \)-orbit in the complex projective space \( Z := \mathbb{P}(\mathbb{C}^N) \). In fact, there are complex-algebraic subvarieties \( A \subset B \subset Z \) such that the \( G \)-orbit \( \hat{Q} \) is a closed CR-submanifold of \( Z \setminus A \) and \( Q = \hat{Q} \setminus B \) (for all this and further details compare [6]). In particular, the group \( G \) inherits a Lie group structure from \( \text{SL}_N(\mathbb{C}) \) and with it is a transitive real-analytic transformation group on \( \hat{Q} \). It can be shown that \( G \) has only finitely many connected components.

A convenient method for the study of a Lie group action is given by the associated infinitesimal action in terms of vector fields. For every domain \( D \subset Q \) let \( \mathfrak{h} \mathfrak{o} \mathfrak{l}(D) \) be the real Lie algebra of all real-analytic infinitesimal CR-transformations on \( D \) (that is of all real-analytic vector fields on \( D \) whose local flows consist of CR-transformations). By [2] every vector field in \( \mathfrak{h} \mathfrak{o} \mathfrak{l}(D) \) extends to a (complex) polynomial vector field of degree \( \leq 2 \) on \( \mathbb{C}^{k+n} \). This implies, in particular, that \( \mathfrak{g} := \mathfrak{h} \mathfrak{o} \mathfrak{l}(Q) \) has finite dimension and that for every domain \( D \subset Q \) the restriction operator \( \mathfrak{g} \rightarrow \mathfrak{h} \mathfrak{o} \mathfrak{l}(D) \) is an isomorphism of real Lie algebras. Furthermore, \( \mathfrak{g} \) is in a canonical way the Lie algebra of the Lie group \( G = \text{Aut}(\hat{Q}) \). The vector field \( \zeta := 2w \frac{\partial}{\partial w} + z \frac{\partial}{\partial z} \in \mathfrak{g} \) gives a canonical grading

\begin{equation}
\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2
\end{equation}

into the \( \text{ad}(\zeta) \)-eigenspaces, where the Lie algebra \( \text{aff}(Q) := g^{-2} \oplus g^{-1} \oplus g^0 \subset \mathfrak{g} \) is the Lie subalgebra of the subgroup \( \text{Aff}(Q) \subset G \). The affine subalgebra \( \text{aff}(Q) \) has an explicit description in terms of the hermitian
form $h$, see (2.4), whereas it seems to be unknown how big the nilpotent Lie subalgebra $\mathfrak{g}^+ := \mathfrak{g}^1 \oplus \mathfrak{g}^2$ can be in terms of $k, n$ (it can definitely be zero). Once $\mathfrak{g}^+$ is explicitly known also the nilpotent closed subgroup $G^+ := \exp(\mathfrak{g}^+) \subset G$ is explicitly known - indeed, $\mathfrak{g}^+$ is ad-nilpotent so that $\exp: \mathfrak{g}^+ \rightarrow G^+$ is a polynomial homeomorphism. The group $G$ is generated by the connected subset $\exp(\mathfrak{g})$ together with the linear subgroup $GL(Q) \subset \text{Aff}(Q)$. The Lie algebras $\mathfrak{g}^k$ and hence $\mathfrak{g}$ itself can be explicitly determined by solving certain linear equations, a comparatively much easier task than the explicit determination of the linear group $GL(Q)$, where high order polynomial equations have to be solved.

The intention of this short note is to discuss several classes of examples for $Q$ as above where $\mathfrak{g}^+ = \mathfrak{g}^1 \oplus \mathfrak{g}^2$ is 'big' and has a simple description. For that we introduce the Property (S): There exists a transformation $\gamma = \gamma^{-1} \in G = \text{Aut}(Q) \cong \text{Bir}(Q)$ such that $\text{Ad}(\gamma)(\zeta) = -\zeta$ for the adjoint representation $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$. Then, if such a $\gamma$ is obtained in a concrete situation (by guess-work or any other form of computation) the obvious formula $\mathfrak{g}^k = \text{Ad}(\gamma)(\mathfrak{g}^{-k})$ allows to immediately write down $\mathfrak{g}^+ = \mathfrak{g}^1 \oplus \mathfrak{g}^2$ explicitly. The same holds with $G^+ = \gamma H \gamma$, where $H := \exp(\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1})$ is the Heisenberg group. As indicated above, the determination of $GL(Q)$ is more involved.

The paper is organized as follows: After the necessary preliminaries in Section 2 we introduce in Section 3 the symmetry Property (S) and present with Example 3.4 our basic class of quadrics having this property. In Section 4 we obtain for every $Q$ from the basic class by tensoring with an arbitrary unital (associative) complex *-algebra $A$ of finite dimension a new quadric $Q(A)$ that also has Property (S). In the final section we briefly explain how from the classification of irreducible bounded symmetric domains of non-tube type further quadrics with Property (S) can be obtained.

2. Preliminaries

Let $W, Z$ be complex vector spaces of finite positive dimension. Suppose that on $W$ a conjugation $w \mapsto \overline{w}$ is given and put $V := \{w \in W : \overline{w} = w\}$. Then for every sesquilinear form $h: Z \times Z \rightarrow W$ (complex linear in the first and conjugate linear in the second variable) the real-algebraic subset

$$Q = Q_h = \{(w, z) \in W \times Z : w + \overline{w} = h(z, z)\}$$

is called a standard quadric in the following, provided

(i) $h(z, z') = 0$ for all $z' \in Z$ implies $z = 0$ (non-degeneracy),

(ii) $V$ is the linear span over $\mathbb{R}$ of all vectors $h(z, z), z \in Z$ (minimality).

It is clear that $Q$ is invariant under the two 1-parameter groups of linear transformations

$$(w, z) \mapsto (e^{2t}w, e^t z) \text{ and } (w, z) \mapsto (w, e^{it}z), \quad t \in \mathbb{R}.$$

Therefore $\mathfrak{g} := \mathfrak{hol}(Q)$ contains the commuting linear vector fields

$$\zeta := 2w \partial/\partial w + z \partial/\partial z \quad \text{and} \quad \chi := iz \partial/\partial z,$$

and $\mathfrak{g} + i \mathfrak{g}$ contains the Euler field $\eta = (\zeta - i\chi)/2$.

Denote by $\mathfrak{P}$ the complex Lie algebra of all (complex) polynomial vector fields on $E := W \oplus Z$. The vector field $\zeta \in \mathfrak{g} \subset \mathfrak{P}$ induces a $\mathbb{Z}$-grading

$$\mathfrak{P} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{P}^k, \quad [\mathfrak{P}^k, \mathfrak{P}^\ell] \subset \mathfrak{P}^{k+\ell} \text{ for } \mathfrak{P}^k := \{\xi \in \mathfrak{P} : [\zeta, \xi] = k\xi\}$$
of \( \mathfrak{P} \) with \( \mathfrak{P}^k = 0 \) if \( k < -2 \) and induces also the grading (1.1) with \( g^k := \mathfrak{g} \cap \mathfrak{P}^k \), compare [2]. The subalgebra \( \mathfrak{g}^{-2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^2 \) is the kernel of \( \text{ad}(\chi) \), while the restriction of \( \text{ad}(\chi) \) to the invariant subspaces \( \mathfrak{g}^{-1}, \mathfrak{g}^1 \) has the eigenvalues \( \pm i \). The following is well known and easily verified:

\[
\begin{align*}
\mathfrak{g}^{-2} &= \{ a \partial/\partial w : a \in i \mathbb{V} \}, \\
\mathfrak{g}^{-1} &= \{ h(z, c) \partial/\partial w + c \partial/\partial z : c \in \mathbb{Z} \}, \\
\mathfrak{g}^0 &= \{ aw \partial/\partial w + bz \partial/\partial z : a \in \mathfrak{gl}(V), b \in \mathfrak{gl}(Z) \text{ with } ah(z, z) = h(bz, z) + h(z, bz) \}.
\end{align*}
\]

The derived algebra \( \mathfrak{d} := [\mathfrak{g}, \mathfrak{g}] \) is an ideal in \( \mathfrak{g} \) and has the grading \( \mathfrak{d} = \bigoplus_{|k| \leq 2} \mathfrak{g}^k \), where

\[
\mathfrak{d}^k = \begin{cases} 
\mathfrak{g}^k & k \neq 0 \\
[\mathfrak{g}^{-2}, \mathfrak{g}^2] + [\mathfrak{g}^{-1}, \mathfrak{g}^1] + [\mathfrak{g}^0, \mathfrak{g}^0] & k = 0.
\end{cases}
\]

Furthermore, \( \mathfrak{g}^{-} := \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \) and \( \mathfrak{g}^{+} := \mathfrak{g}^1 \oplus \mathfrak{g}^2 \) are nilpotent Lie algebras of step 2.

Let \( \tilde{Q} \) be the extended quadric and \( G := \text{Aut}(\tilde{Q}) \), compare Section 1. Then \( \mathfrak{g} = \mathfrak{hol}(Q) \) can also be identified canonically with \( \mathfrak{hol}(\tilde{Q}) \). Consider the following subgroups of \( G \)

\[
\begin{align*}
G_0 &= \{ g \in G : g(0) = 0 \} \\
G^\pm &= \text{exp}(\mathfrak{g}^\pm) \\
\text{GL}(Q) &= \{ g \in \text{GL}(E) : g(Q) = Q \} \\
\text{Aff}(Q) &= \{ g \in \text{Aff}(E) : g(Q) = Q \} = \text{GL}(Q) \times G^-
\end{align*}
\]

with \( \text{Aff}(E) \) the group of all complex affine automorphisms of \( E \). Then

\[
\text{GL}(Q) = \{(f \times g) \in \text{GL}(V) \times \text{GL}(Z) \subset \text{GL}(E) : fh(z, z) = h(gz, gz)\}
\]

is a real algebraic subgroup of \( \text{GL}(E) \) with Lie algebra \( \mathfrak{gl}(Q) := \mathfrak{g}^0 \subset \mathfrak{gl}(E) \) and, in particular, has only finitely many connected components. Every \( G^\pm \) is a connected nilpotent closed subgroup of \( G \) with Lie algebra \( \mathfrak{g}^\pm \). For instance, \( G^- = \text{exp}(\mathfrak{g}^-) \text{exp}(\mathfrak{g}^1) \) is the group of all affine transformations of the form

\[
(w, z) \mapsto (w + a + h(z, b), z + b), \quad (a, b) \in Q,
\]

which acts simply transitively on \( Q \) and is called the Heisenberg group.

Every \( g \in G = \text{Aut}(\tilde{Q}) \) acts on its Lie algebra \( \mathfrak{g} \) by \( \text{Ad}(g) \in \text{Aut}(\mathfrak{g}) \), here given in terms of vector fields by

\[
\text{Ad}(g)(f(z) \partial/\partial z) = h(z) \partial/\partial z \quad \text{with} \quad h(g(z)) = g'(z)(f(z)),
\]

where \( g'(z) \in \text{End}(E) \) is the derivative of \( g \) at \( z \). The group \( G \) has no center, that is, the group homomorphism \( \text{Ad} : G \to \text{Aut}(\mathfrak{g}) \) is injective.

The following result will not be used later but may be of independent interest.

**2.7 Lemma.** For every standard quadric \( Q \) the extended quadric \( \tilde{Q} \) is simply connected.

**Proof.** Denote by \( \pi : \tilde{Q} \to \tilde{Q} \) the universal covering of \( \tilde{Q} \). Then by [6] there exists a complex manifold \( X \), containing \( \tilde{Q} \) as generic real-analytic submanifold, together with a complex-analytic subset \( A \subset X \) such that \( Q = \tilde{Q} \setminus A \). There exists a complex manifold \( \tilde{X} \), containing \( \tilde{Q} \) as generic real-analytic submanifold, in such a way that \( \pi \) extends to a holomorphic map \( \pi : \tilde{X} \to X \). This implies that \( A := \pi^{-1}(A) \) is complex-analytic in \( \tilde{X} \) and hence that \( \pi^{-1}(Q) = \tilde{Q} \setminus \tilde{A} \) is connected by Lemma 2.2 in [5]. Since \( Q \) is simply connected the covering map \( \pi : \tilde{Q} \to \tilde{Q} \) must be a homeomorphism. 

\[\Box\]
3. The symmetry property

With the notation of Section 2 fix a standard quadric \( Q \) and consider the following symmetry property:

**Property (S)** There exists an automorphism \( \gamma = \gamma^{-1} \in G = \text{Aut}(\hat{Q}) \) with \( \text{Ad}(\gamma)(\zeta) = -\zeta \).

We call \( \gamma \) also a symmetry of the quadric \( Q \). There may not exist a fixed point of \( \gamma \) in the extended quadric \( \hat{Q} \), insofar \( \gamma \) is not necessarily a CR-symmetry of \( \hat{Q} \) in the sense of [7].

If the symmetry property (S) is satisfied with \( \gamma \) then \( \text{Ad}(\gamma) \in \text{Aut}(\mathfrak{g}) \) permutes the eigenspaces of \( \text{ad}(\zeta) \) in \( \mathfrak{g} \), more precisely,

\[
\text{Ad}(\gamma)(\mathfrak{g}^k) = \mathfrak{g}^{-k} \quad \text{for all } k, \quad \text{in particular,}
\dim(\mathfrak{g}^k) = \dim(\mathfrak{g}^{-k}) \quad \text{for all } k \quad \text{and} \quad [\mathfrak{g}^1, \mathfrak{g}^1] = \mathfrak{g}^2.
\]

The symmetry \( \gamma \) is not uniquely determined, every \( g\gamma g^{-1} \) with \( g \in \text{GL}(Q) \) is also a symmetry of \( Q \). Using (3.1) the spaces \( \mathfrak{g}^1 \) and \( \mathfrak{g}^2 \) can be explicitly computed from (2.4). In Section 4 we given an example of this method. This also works on the group level. Indeed, the inner automorphism \( \text{Int}(\gamma) \) of \( G \) defined by \( g \mapsto g\gamma g^{-1} \) (note that \( \gamma^{-1} = \gamma \) by definition) satisfies

\[
G^+ = \gamma G^{-}\gamma \quad \text{and} \quad G_0 = \text{GL}(Q) \ltimes G^+ = \gamma \text{Aff}(Q) \gamma.
\]

As a consequence we state

**3.3 Proposition.** The group \( G \) is generated by the subgroup \( \text{Aff}(Q) \) and \( \gamma \).

In the following we give some examples of standard quadrics having a symmetry. We start with the case of hyperquadrics.

**3.4 Example.** Suppose that in \( \mathbb{C}^{n+1} \) with coordinates \( (w, z_1, \ldots, z_n) \) the quadric \( Q \) is given by

\[
Q = \left\{ (w, z) \in \mathbb{C}^{n+1} : w + \overline{w} = \sum_{1 \leq j \leq k} |z_j|^2 - \sum_{k < j \leq n} |z_j|^2 \right\},
\]

where \( 0 \leq k \leq n \) is a fixed integer. Then \( (w, z) \mapsto (w^{-1}, -w^{-1}z) \) defines a symmetry \( \gamma \) of \( Q \). Obviously, there is a fixed point \( (1, z) \in Q \) of \( \gamma \), provided \( k > 0 \) (in case \( k = 0 \) the symmetry \( (w, z) \mapsto (w^{-1}, -w^{-1}z) \) would have a fixed point \( (-1, z) \in Q \). The Lie algebra \( \mathfrak{so}(Q) \) is isomorphic to \( \mathfrak{su}(p, q) \) with \( p = k+1 \) and \( q = n+1-k \). Clearly, replacing \( k \) by \( n-k \) gives a linearly equivalent quadric.

This example can be generalized to higher codimensions. For every matrix \( w \) we denote by \( w^* \) its adjoint (conjugate transpose).

**3.5 Example.** Let \( m, n \geq 1 \) be fixed integers. Then, for every hermitian matrix \( \beta \in \text{GL}_n(\mathbb{C}) \),

\[
Q := \left\{ (w, z) \in \mathbb{C}^{m \times n} \times \mathbb{C}^{m \times n} : w + w^* = z\beta z^* \right\}
\]

is a standard quadric with CR-codimension \( m^2 \), and \( (w, z) \mapsto (w^{-1}, -w^{-1}z) \) defines a symmetry \( \gamma \) of \( Q \). In case \( n < m \) there is no fixed point of \( \gamma \) in \( \hat{Q} \). On the other hand, in case \( n \geq m \) there exists a fixed point \( (1, z) \in \hat{Q} \), provided \( k \geq m \) for the number \( k \) of positive eigenvalues of \( \beta \). The Lie algebra \( \mathfrak{so}(Q) \) is isomorphic to \( \mathfrak{su}(p, q) \) with \( p = k+m \) and \( q = n+m-k \). The extended quadric \( \hat{Q} \) can be realized as follows: Choose on \( \mathbb{C}^{2m+n} \) a non-degenerate hermitian form \( \psi \) of type \( (p, q) \) and denote by \( \mathbb{G} \) the Grassmannian of all linear \( m \)-spaces in \( \mathbb{C}^{p+q} \). Then the compact real-analytic submanifold \( S := \{ L \in \mathbb{G} : \psi|_L = 0 \} \) of \( \mathbb{G} \) is CR-isomorphic to \( \hat{Q} \). In fact, \( S \) is the unique closed orbit of the unitary group \( \text{SU}(\psi) \cong \text{SU}(p, q) \) acting on the Grassmannian. For \( m = 1 \) we get back Example 3.4 (up to an affine transformation) and \( \mathbb{G} \) is the complex projective space \( \mathbb{P}(\mathbb{C}^{n+2}) \).

It is obvious that for every pair \( Q', Q'' \) of standard quadrics with Property (S) also the direct product \( Q := Q' \times Q'' \) is a standard quadric with Property (S). In the next section we describe a more interesting method to produce new standard quadrics out of known ones.
4. Tensoring with *-algebras

In the following we use the notion of a *-algebra, that is, a complex associative algebra $A$ with product $(w, z) \mapsto wz$ and (conjugate linear) involution $z \mapsto z^*$ satisfying $z^{**} = z$ and $(zw)^* = w^*z^*$ for all $w, z \in A$. Then the self-adjoint part $A_{sa} := \{ z \in A : z^* = z \}$ is a real Jordan subalgebra with respect to the anti-commutator product $x \circ y := (xy + yx)/2$ and $iA_{sa}$ is a real Lie subalgebra with respect to the commutator product $[x, y] = xy - yx$. Clearly $A = A_{sa} \oplus iA_{sa}$, and $A_{sa}$ is an associative subalgebra of $A$ only if $A$ is commutative.

Here we assume without further notice that every *-algebra has finite dimension. In addition we also assume that $A$ has a unit $e$, which then is contained in $A_{sa}$. The subgroup $G(A) \subset A$ of all invertible elements is Zariski open in $A$ and a connected complex Lie group. In particular, $G(A) \subset GL(A)$ is generated by the image of the exponential map $\exp : A \to G(A)$. In case $A$ is commutative, $\exp$ is a surjective group homomorphism as well as a locally biholomorphic covering map.

Denote by

$$\text{Aut}(A, \star) := \{ g \in GL(A) : g(ac^*) = g(a)g(c)^* \}$$

the *-algebra automorphism group of $A$. Then $\text{Aut}(A, \star)$ is a real linear algebraic subgroup of $GL(A)$ leaving $A_{sa}$ invariant and has Lie algebra

$$\mathfrak{der}(A, \star) := \{ \delta \in \mathfrak{gl}(A) : \delta(ac^*) = \delta(a)c^* + a\delta(c)^* \}.$$

In case that $A$ is commutative, $\text{Aut}(A, \star)$ can be identified with the real algebra automorphism group of the associative algebra $A_{sa}$ and $\mathfrak{der}(A, \star)$ can be identified with the derivation algebra of $A_{sa}$.

For every integer $m \geq 1$ the matrix algebra $M := \mathfrak{M}^{m \times m}$ is a unital *-algebra with respect to the usual adjoint $\star$ as involution, and $M$ is even simple as complex algebra. Every other involution $\star$ on $M$ is of the form $z^* = \alpha z^* \alpha^{-1}$ for some $\alpha = \alpha^* \in GL_m(\mathbb{C})$. Also, the product algebra $M \times M$ becomes a simple *-algebra with respect to the involution $(a, b)^* := (b^*, a^*)$. By Wedderburn’s Theorem [8] every semi-simple complex unital algebra $A$ (of finite dimension) is a unique direct product $A = \prod_{j \in J} A_j$ of simple unital algebras $A_j \cong \mathfrak{M}^{m_j \times m_j}$. To every involution $\star$ of $A$, making it to a *-algebra, there is an involution $j \mapsto j^*$ of the index set $J$ such that $(A_j)^* = A_{j^*}$ for every $j \in J$. Then, choosing a minimal subset $K \subset J$ with $J = K \cup K^*$, we get the representation $A = \prod_{j \in K} (A_j + A_{j^*})$ as direct product of simple *-algebras.

In general the algebra $A$ has a radical (the intersection of all maximal left ideals in $A$). Every involution of $A$ leaves $\text{rad}(A)$ invariant and makes the semi-simple algebra $A/\text{rad}(A)$ to a *-algebra. It can be shown that even in the commutative case there are uncountably many isomorphism classes of (finite dimensional) *-algebras.

Now consider for given $W, Z, h$ the standard quadric $Q = Q_h$ defined as in (2.1) and fix a unital *-algebra $A$. Put $V := \{ w \in W : \overline{w} = w \}$ and also

$$W := W \otimes_{\mathbb{C}} A, \quad Z := W \otimes_{\mathbb{C}} A \quad \text{and} \quad V := V \otimes_{\mathbb{R}} A_{sa}. \quad (4.1)$$

Then there exists a unique conjugation $w \mapsto \overline{w}$ on $W$ such that $\overline{w} = \overline{w} \otimes a^*$ for every $w = w \otimes a \in W$. The fixed point set of this conjugation is $V$, considered in a canonical way as $\mathbb{R}$-linear subspace of $W$. Also there exists a unique hermitian form $h : Z \times Z \to W$ satisfying $h(x \otimes a, y \otimes b) = h(x, y) \otimes ab^*$

defining the quadric

$$Q := Q(A) := \{ (w, z) \in W \times Z : w + \overline{w} = h(z, z) \}.$$
If we denote by \( e \in A \) its unit we realize \( V, W, Z \) as linear subspaces of \( V, W, Z \) by identifying every \( w \in W, z \in Z \) with \( w \otimes e, z \otimes e \) respectively. In this sense \( Q = Q \cap (W \times Z) \subset W \times Z \). Furthermore, for every pair \( A, B \) of unital \( * \)-algebras we have \( Q(A \times B) \cong Q(A) \times Q(B) \) and \( Q(A)(B) \cong Q(A \otimes B) \), where product and involution on \( A \otimes B \) are uniquely determined by \( (a \otimes b)(c \otimes d) = (ac \otimes bd) \) and \((a \otimes b)^* = (a^* \otimes b^*)\).

**Tensored Example 3.4:** We consider Example 3.4 in the lowest possible dimension \( n = 1 \), that is without loss of generality,

\[
Q = \{(w, z) \in \mathbb{C}^2 : w + \overline{w} = z\overline{z}\}
\]

is the Heisenberg sphere in dimension 2. With \( A \) a fixed \( * \)-algebra of complex dimension \( d \) then

\[
(4.1) \quad Q = Q(A) = \{(w, z) \in A^2 : w + w^* = zw^*\}
\]

is a standard quadric of CR-codimension \( d \), and a symmetry is given by \( \gamma(w, z) = (w^{-1}, w^{-1}z) \).

**4.2 Proposition.** For \( Q \) in (4.1) and \( \text{hol}(Q) = g^{-2} \oplus g^{-1} \oplus g^0 \oplus g^1 \oplus g^2 \) we have

\[
\begin{align*}
g^{-2} &= \{a \partial/\partial w : a \in iA_{sa}\} \\
g^{-1} &= \{zc^* \partial/\partial w + c \partial/\partial z : c \in A\} \\
g^0 &= \{(aw + wa^*) \partial/\partial w + az \partial/\partial z : a \in A\} \times \{(\delta(w) \partial/\partial w + \delta(z) \partial/\partial z : \delta \in \text{der}(A,*)\} \\
g^1 &= \{zc^* w \partial/\partial w + (z^*c^* + wc) \partial/\partial z : c \in A\} \\
g^2 &= \{waw \partial/\partial w + waz \partial/\partial z : a \in iA_{sa}\}.
\end{align*}
\]

**Proof.** \( g^k \) for \( k < 0 \) is (2.4) and for \( k > 0 \) follows immediately by applying (3.1) to \( g^{-1}, g^{-2} \). For \( k = 0 \) the claim is a direct consequence of the following proposition.

**4.3 Proposition.** For \( Q \) in (4.1) and the subgroups \( \text{GL}(Q) \), \( G^+ \) of \( G = \text{Aut}(Q) \) as defined in Section 2 we have, where \( e \in A \) is the unit:

\[
\begin{align*}
\text{GL}(Q) &= \{(w, z) \mapsto (awa^*, az) : a \in G(A)\} \times \{g \times g : g \in \text{Aut}(A,*)\} \\
G^+ &= \{(w, z) \mapsto (e + wa + zc^*)^{-1}(w, z + wc) : (a, c) \in Q\}.
\end{align*}
\]

**Proof.** The second equation follows immediately with (3.2) applied to (2.6).

Next consider an arbitrary \( \varphi \in \text{GL}(Q) \). Then \( \varphi = f \times g \in \text{GL}(A_{sa}) \times \text{GL}(A) \) with \( f(xy^*) = (gx)(gy)^* \) for all \( x, y \in A \) by (2.5). We claim that \( g(e) \) is invertible in \( A \) (compare also Lemma 2.5 in [3]): Indeed, for \( x = e \) we have \( f(y) = (ge)(gy)^* \), and choosing \( y \) with \( f(y) = e \) we find \( e = (ge)(gy)^* \), that is, \( g(e) \in G(A) \).

Obviously the group \( K := \{(w, z) \mapsto (awa^*, az) : a \in G(A)\} \) is contained in \( \text{GL}(Q) \) and the group \( \{\beta \in \text{GL}(A) : (\alpha \times \beta) \in K \text{ for some } \alpha \in \text{GL}(A_{sa})\} \) acts simply transitively on \( G(A) \). We may therefore assume that \( g(e) = e \) for \( \varphi = f \times g \). For every \( x \in A_{sa} \), then \( f(x) = f(xe^*) = (gx)(ge)^* = gx \), that is \( f = g \). Finally, \( f(xy^*) = (gx)(gy)^* \) implies \( g \in \text{Aut}(A,*) \). This implies that \( K \) is normal in \( \text{GL}(Q) \) and that \( \text{GL}(Q) \) is a semi-direct product of the claimed form.

In [3] the notion of a Real Associative Quadric (RAQ for short) has been introduced and for every quadric \( Q \) of this type the subgroups \( \text{GL}(Q) \) and \( G^+ \) of \( \text{Aut}(Q) \) have been explicitly described. In our terminology, the RAQs are just the tensored quadrics \( Q(A) \) of type (4.1), where \( Q \subset \mathbb{C}^2 \) is the Heisenberg quadric in dimension 2 and \( A \) is a commutative \( * \)-algebra. In this special case of a commutative \( * \)-algebra \( A \) the group \( \text{Aut}(A,*) \) obviously can also be written in the following way:

\[
\text{Aut}(A,*) = \text{Aut}(R) \quad \text{for the commutative real algebra} \quad R = A_{sa}.
\]
For every \( a \in R \) we denote by \( L(a) \in \text{End}(R) \) the left multiplication \( x \mapsto ax \) and identify the algebra \( R \) with its image \( L(R) \subset \text{End}(R) \). Then the normalizer \( \{ g \in \text{GL}(R) : g L(R) = L(R) g \} \) is canonically isomorphic to \( \text{Aut}(R) \). Via this identification, for the commutative case the descriptions of \( \text{GL}(Q) \) and \( G^+ \) in Proposition 4.3 occur already in [3].

**Tensored Example 3.5:** For \( m, n \geq 1 \) and hermitian matrix \( \beta \in \text{GL}_n(\mathbb{C}) \) as in Example 3.5 we consider the tensored quadric \( Q = Q(A) \), where \( A \) is an arbitrary \(*\)-algebra, that is,

\[
Q = \{ (w, z) \in A^{m \times m} \times A^{m \times n} : w + w^* = z \beta z^* \},
\]

where for every matrix \( a = (a_{jk}) \in A^{r \times s} \) the adjoint \( a^* \in A^{s \times r} \) is the matrix \( (a_{kj})^* \). Then \( Q \) is a standard quadric and, as before, \( (w, z) \mapsto (w^{-1}, w^{-1} z) \) defines a symmetry. It is seen immediately that for \( G^+ \) and \( g = \text{hol}(Q) \) we have

\[
G^+ = \{ (w, z) \mapsto (e + wa + z\beta^* z^{-1})^{-1}(w, z + wc) : (a, c) \in Q \},
\]

\[
g^{-2} = \{ a \partial_{\partial w} : a \in A^{m \times m}, a + a^* = 0 \},
\]

\[
g^{-1} = \{ z\beta c^* \partial_{\partial w} + c \partial_{\partial z} : c \in A^{m \times n} \},
\]

\[
g^1 = \{ z\beta c^* w \partial_{\partial w} + (z\beta c^* z + wc) \partial_{\partial z} : c \in A^{m \times n} \},
\]

\[
g^2 = \{ wau \partial_{\partial w} + waz \partial_{\partial z} : a \in A^{m \times m}, a + a^* = 0 \},
\]

and a simple check reveals \( [g^j, g^k] = g^{j+k} \) for all \( j, k \in \{ 0, \pm 1, \pm 2 \} \) with \( j + k \neq 0 \).

More involved is the linear group \( \text{GL}(Q) \). Notice that the group \( \text{GL}_n(A) \) acts by matrix multiplication from the right on \( A^{m \times n} \). A subgroup is the \( \beta \)-unitary group

\[
U_\beta(A) := \{ u \in \text{GL}_n(A) : u\beta u^* = \beta \} \quad \text{with Lie algebra} \quad u_\beta(A) := \{ u \in A^{n \times n} : u\beta + \beta u^* = 0 \}.
\]

For every matrix space \( A^{r \times s} \) we have the embedding \( \text{End}(A) \hookrightarrow \text{End}(A^{r \times s}) \) given by \( f a = (f(a_{jk})) \) for every \( a = (a_{jk}) \in A^{r \times s} \). Then a simple computation gives that

\[
B := \{ (w, z) \mapsto (a(g \cdot w)a^*, a(g \cdot z)u) : a \in \text{GL}_m(A), u \in U_\beta(A), g \in \text{Aut}(A, \ast) \}
\]

is a subgroup of \( \text{GL}(Q) \). In case \( A \) is a semi-simple, up to a permutation of its simple factors, every element of \( \text{Aut}(A, \ast) \) is an inner \(*\)-automorphism of the form \( a \mapsto wa u^* \) with \( u^* = u^{-1} \in G(A) \). In particular, \( \text{Aut}(\mathbb{C}, \ast) \) is the trivial group and it can be seen that \( B = \text{GL}(Q) \) holds in case \( A = \mathbb{C} \). Also \( B = \text{GL}(Q) \) in case \( m = n = 1 \) by Proposition 4.3.

We discuss briefly another local realization of the above tensored quadrics \( Q = Q(A) \): Fix integers \( r > m \geq 1 \) and a hermitian matrix \( \alpha \in \text{GL}_r(\mathbb{C}) \) having at least \( m \) positive eigenvalues. For fixed \(*\)-algebra \( A \) put \( E := A^{m \times r} \) and let \( 1 \in \text{GL}_m(A) \) be the unit matrix. Then we call

\[
S := \{ z \in E : z \alpha z^* = 1 \}
\]

a **generalized sphere**. Without loss of generality we may assume that \( \alpha = 1 \times \beta \in \text{GL}_m(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) \) for \( n := r - m \) and some hermitian matrix \( \beta \in \text{GL}_n(\mathbb{C}) \). Now put \( W := A^{m \times m}, Z := A^{m \times n} \). Then \( E = W \oplus Z \) in a canonical way and the quadric \( Q := \{(x, y) \in W \times Z : x + x^* y^{-1} = y \beta y^* \} \) is locally CR-isomorphic to \( S \). Indeed, consider the Cayley transformation \( \kappa \in \text{Bir}(E) \) defined on \( E \) by

\[
(4.4) \quad \kappa(x, y) = (1 - x)^{-1}(1 + x, \sqrt{2}y).
\]
Then \( \kappa^{-1}(x,y) = (x+1)^{-1}(x-1, \sqrt{2}y) \) and a simple computation shows that the birational transformation \( \kappa \) gives a CR-isomorphism

\[
\kappa : S \cap \text{reg}(\kappa) \to Q \cap \text{reg}(\kappa^{-1}).
\]

Since \( Q \cap \text{reg}(\kappa^{-1}) \) is connected by Lemma 2.2 in [5], also \( S \cap \text{reg}(\kappa) \) is connected. Consequently \( S \) is a connected generic real-analytic CR-submanifold of \( E \). Furthermore, \( \kappa \) induces an isomorphism between the real Lie algebras \( g = \mathfrak{hol}(Q) \) and \( \mathfrak{s} = \mathfrak{hol}(S) \). Since \( \kappa = \exp(\xi) \) for some \( \xi \in \mathfrak{l} := \mathfrak{g} + i\mathfrak{g} \), we have also \( \mathfrak{l} = \mathfrak{s} + i\mathfrak{s} \) and every vector field in \( \mathfrak{s} \) is polynomial of degree \( \leq 2 \) on \( E \). For every \( a \in E \) application of the vector field \( \xi := (a - za\sigma^* z) \partial/\partial z \) on \( E \) to the defining equation for \( S \) gives

\[
\xi(zaz^* - 1) = (a - za\sigma^* z)az^* + za(\sigma^* - a^*aaz^*) = (1 - za\sigma^*)aaaz^* + za\sigma^*(1 - za\sigma^*).
\]

This shows that \( \xi \) is tangent to \( S \) and we get the decomposition

\[
s = \mathfrak{k} \oplus \mathfrak{p} \quad \text{with} \quad \mathfrak{k} := \{ \xi \in \mathfrak{s} : \xi_0 = 0 \} \quad \text{and} \quad \mathfrak{p} := \{(a - za\sigma^* z) \partial/\partial z : a \in E \}.
\]

Notice that the evaluation map \( e_0 : \mathfrak{s} \to E \) at the origin induces an \( \mathbb{R} \)-linear isomorphism of \( \mathfrak{p} \) onto \( E \). The Lie subalgebra \( \mathfrak{k} \) contains the multiple \( \delta := iz\partial/\partial z \) of the Euler field, and \( \mathfrak{p} \) resp. \( \mathfrak{p} \) are the 0- resp. \( -1 \)-eigenspaces of \( (ad \delta)^2 \) in \( \mathfrak{s} \). Also, every vector field in \( \mathfrak{k} \) is linear, and \( \mathfrak{t} = \mathfrak{gl}(\mathfrak{s}) \) is the Lie algebra of the linear algebraic group \( K := \text{GL}(\mathfrak{s}) = \{ g \in \text{GL}(E) : g(S) = S \} \). It is clear that the group \( U := U_1 \times U_\alpha \subset \text{GL}_m(A) \times \text{GL}_r(A) \) acts linearly on \( S \) via \( z \mapsto uzv^* \), and it can be seen that there is an open orbit in \( S \) for this action.

5. Links with bounded symmetric domains

For the special case \( \beta = 1 \) in Example 3.5 the quadric \( Q \) coincides with the \( \check{\text{Silov}} \) boundary \( \check{D} \) of the symmetric Siegel domain

\[
D := \{ (w,z) \in \mathfrak{C}^{m \times m} \times \mathfrak{C}^{m \times n} : (w + w^* - zz^*) > 0 \},
\]

which is also the interior of the convex hull of \( Q \). The inverse Cayley transform \( \kappa^{-1} \), see (4.4), maps the Siegel domain \( D \) biholomorphically onto the bounded symmetric domain

\[
B := \{ (w,z) \in \mathfrak{C}^{m \times m} \times \mathfrak{C}^{m \times n} : (1 - ww^* - zz^*) > 0 \}
\]

and the quadric \( Q = \check{D} \) to an open dense part of the (compact) \( \check{\text{Silov}} \) boundary

\[
\check{B} = \{ (w,z) \in \mathfrak{C}^{m \times m} \times \mathfrak{C}^{m \times n} : (ww^* + zz^*) = 1 \}
\]

(called a generalized sphere above) of \( B \). The extended quadric \( \check{Q} \) is CR-isomorphic to \( \check{B} \).

Now consider the symmetry \( \sigma \in \text{Aut}(D) \) of \( D \) at the point \( e := (1, 0) \in D \). Then \( \sigma = \kappa \circ (-\text{id}) \circ \kappa^{-1} \) with \( -\text{id} \) being the symmetry of \( B \) at the origin. A simple computation shows \( \sigma(w,z) = (w^{-1}, -w^{-1}z) \). But \( \sigma \) extends to a symmetry of \( Q \) in the sense of Section 3 and is essentially the same (up to the sign in the second variable) as the symmetry \( \gamma \) we used before. Besides these bounded symmetric domains of type \( I \) and their variations from Example 3.5 there are two more non-tube types of irreducible bounded symmetric domains, all leading to symmetric standard quadrics, namely those of types \( II \) and \( V \) (for this and further details in the following see [9]).

**Type II** Fix an even integer \( m \geq 4 \) and let \( j := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{GL}_m(\mathbb{Z}) \). Also put

\[
W := \{ w \in \mathfrak{C}^{m \times m} : w^j = w \} \quad \text{and} \quad Z := \mathfrak{C}^{m \times 1},
\]
where \( w^J := jw'j^{-1} \) and \( w' \) is the transpose of \( w \). Then \( W \) is a complex unital Jordan subalgebra of \( \Phi_{m \times m}^{\oplus} \) with \( W = W^* \).

\[
Q := \{(w, z) \in W \oplus Z : w + w^* = zz^* + (zz^*)^J\},
\]

is a standard quadric with symmetry \( \sigma(w, z) = (w^{-1}, -w^{-1}z) \), and \( \mathfrak{hol}(Q) \cong \mathfrak{so}^*(2m+2) \).

**Type V** Let \( \Phi \) be the real Cayley division algebra and \( x \mapsto \sigma \) its canonical (real) involution. \( \Phi \) is an alternative algebra of dimension 8 over \( \mathbb{R} \) with \( \sigma = x \) if and only if \( x \in \mathbb{R} \cdot 1 \). Denote by \( \Phi^c \) the complexification of \( \Phi \) and extend the involution of \( \Phi \) to a conjugate linear involution \( z \mapsto \bar{z} \) of the complex alternative algebra \( \Phi^c \). Then \( \bar{wz} = \bar{z} \bar{w} \) for all \( w, z \in \Phi^c \) and

\[
Q := \{(w, z) \in \Phi^c \times \Phi^c : w + \bar{w} = zz^*\}
\]

is a standard quadric of CR-codimension 8 in \( \Phi^{16} \). Again, \( \sigma(w, z) = (w^{-1}, -w^{-1}z) \) is a symmetry of \( Q \). Notice that for every \( w, z \in \Phi^c \) with \( w \) invertible there is a unital associative complex subalgebra of \( \Phi^c \) containing \( w, w^{-1} \) and \( z \). Furthermore, \( \mathfrak{hol}(Q) \) is isomorphic to the exceptional real Lie algebra \( \mathfrak{e}_6(-14) \).

**References**

1. Baouendi, M.S., Jacobowitz, H., Treves F.: On the analyticity of CR mappings. Ann. of Math. 122, 365-400 (1985).
2. Beloshapka, V.: On holomorphic transformations of a quadric. Math. USSR Sb. 72, 189-205 (1992).
3. Ezhov, V., Schmalz, G.: A Matrix Poincaré Formula for Holomorphic Automorphisms of Quadrics of Higher Codimension. Real Associate Quadrics. J. Geom. Analysis 8, 27-41 (1998).
4. Ezhov, V., Schmalz, G.: Automorphisms of Nondegenerate CR-Quadrics and Siegel Domains. Explicit Description. J. Geom. Analysis 11, 441-467 (2001).
5. Fels, G., Kaup, W.: Local tube realizations of CR-manifolds and maximal abelian subalgebras. arXiv 0810.2019.
6. Isaev, A., Kaup, W.: Regularization of Local CR-Automorphisms of Real-Analytic CR-manifolds. arXiv:0906.3079.
7. Kaup W., Zaitsev, D.: On Symmetric Cauchy-Riemann Manifolds. Adv. Math. 149, 145-181 (2000).
8. Lang, S: Algebra. Graduate Texts in Mathematics 211, Springer 2005.
9. Loos, O.: Bounded symmetric domains and Jordan pairs, Mathematical Lectures. Irvine: University of California at Irvine 1977.
10. Tumanov, A., Finite-dimensionality of the group of CR automorphisms of a standard CR manifold, and proper holomorphic mappings of Siegel domains. Math. USSR. Izv. 32, 655–662 (1989).