On modular solutions of fractional weights for the Kaneko–Zagier equation for $\Gamma_0^* (2)$ and $\Gamma_0^* (3)$

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Abstract For the full modular group, Kaneko gave a modular form solution of fractional weights for a holomorphic modular differential equation of second order. In this paper, we give modular form solutions of fractional weights for a modular differential equation on the Fricke group of levels 2 and 3.

Keywords Modular form · Differential equations · The Heun series · Supersingular $j_N$-polynomial

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1 Introduction and preliminaries

In [4], Kaneko–Koike studied various solutions for the so-called Kaneko–Zagier equation:

$$f''(\tau) - \frac{k+1}{6} E_2(\tau) f'(\tau) + \frac{k(k+1)}{12} E'_2(\tau) f(\tau) = 0,$$

where $\tau = (2\pi i)^{-1} d/d\tau = q d/dq$, $q = e^{2\pi i \tau}$, $\tau$ a variable in the Poincaré upper-half plane, $k$ a fixed rational number, and $E_2(\tau)$ is the (quasimodular) Eisenstein series of weight 2 for $SL_2(\mathbb{Z})$ defined by

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \left( \sum_{d|n} d \right) q^n.$$
They gave modular forms expressed in terms of hypergeometric polynomials and quasimodular forms as its solution for weight $k$, where $k$ is integer or half-integer. In particular, Kaneko studied in [3] the modular form as its solution for weight one-fifth, which is closely related to certain models in conformal field theory.

From [5,6] and [11], we know that the Kaneko–Zagier equation for $\Gamma_0^*(N)(N = 2, 3)$

\[
\left(\frac{\eta^k}{\eta^{2k}}\right)' f''(\tau) - \frac{k + 1}{6 - N} E_{NA}(\tau) f'(\tau) + \frac{k(k + 1)}{2(6 - N)} E_{NA}'(\tau) f(\tau) = 0
\]

also has modular/quasimodular solutions similar to the case for $\text{SL}_2(\mathbb{Z})$, where the Fricke group of level $N$ ($N = 2, 3$) is defined by

\[
\Gamma_0^*(N) = \Gamma_0(N) \cup \Gamma_0(N) W_N,
\]

\[
W_N = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix},
\]

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \left| c \equiv 0 \pmod{N} \right. \right\},
\]

and $E_{NA}(\tau)$, the (quasimodular) Eisenstein series of weight 2 for $\Gamma_0^*(N)$, is defined by

\[
E_{NA}(\tau) = \frac{NE_2(N\tau) + E_2(\tau)}{N + 1}.
\]

In this paper, we give modular forms of a fractional weight as a solution of the Kaneko–Zagier equation for $\Gamma_0^*(N)(N = 2, 3)$. Hereafter, $N$ denotes the level 2 or 3.

For any complex numbers $v$ and $s$, we take $-\pi < \arg(v) \leq \pi$ and put $v^s = |v|^s e^{i s \arg(v)}$. Define

\[
\phi_1^{(2)}(\tau) = \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{1/3} \frac{\eta(2\tau)\eta(3\tau)^2}{\eta(\tau)\eta(6\tau)} = 1 + \frac{2}{3} q + \frac{8}{9} q^2 - \frac{50}{81} q^3 + \frac{74}{243} q^4 + \frac{320}{729} q^5 + \frac{1232}{6561} q^6 + \frac{7012}{19683} q^7 + \cdots,
\]

\[
\phi_2^{(2)}(\tau) = \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{1/3} \frac{\eta(6\tau)^2}{\eta(3\tau)} = q^{1/3} \left( 1 - \frac{1}{3} q + \frac{2}{9} q^2 - \frac{40}{81} q^3 + \frac{62}{243} q^4 - \frac{307}{729} q^5 + \frac{458}{6561} q^6 - \frac{19683}{4136} q^7 + \cdots \right),
\]

\[
\phi_1^{(3)}(\tau) = \left( \frac{\eta(\tau)}{\eta(3\tau)} \right)^{1/2} \frac{\eta(2\tau)^3\eta(3\tau)^2}{\eta(\tau)^2\eta(6\tau)} = 1 + \frac{3}{2} q + \frac{3}{8} q^2 + \frac{15}{16} q^3 + \frac{3}{128} q^4 - \frac{99}{256} q^5 + \frac{1671}{1024} q^6 + \frac{1383}{2048} q^7 + \cdots,
\]

\[
\phi_2^{(3)}(\tau) = \left( \frac{\eta(\tau)}{\eta(3\tau)} \right)^{1/2} \frac{\eta(6\tau)^3}{\eta(2\tau)} = q^{1/2} \left( 1 - \frac{1}{2} q + \frac{3}{8} q^2 + \frac{11}{16} q^3 + \frac{35}{128} q^4 - \frac{159}{256} q^5 + \frac{359}{1024} q^6 - \frac{573}{2048} q^7 + \cdots \right),
\]
where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function. Then, we find that $\phi_1^{(N)}(\tau)$ and $\phi_2^{(N)}(\tau)$ are holomorphic modular forms of weight $N/6$ for

$$
\Gamma(6) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(6) \mid a \equiv d \equiv 1, \ b \equiv 0 \pmod{6} \right\}
$$

using properties of $\eta(\tau)$ [7, §1.3 Theorem 1.7 and §2.3 Corollary 2.2]. Moreover, $(\phi_1^{(N)})^{6/N}$ and $(\phi_2^{(N)})^{6/N}$ are modular forms of weight 1 with the Legendre character $(\frac{\ast}{3})$ for $\Gamma_0(6)$.

**Remark 1** The following can be expressed in terms of theta series:

$$
\frac{\eta(2\tau)\eta(3\tau)^2}{\eta(\tau)\eta(6\tau)} = \sum_{n \in \mathbb{Z}} q^{(6n+1)^2/24}, \quad \frac{\eta(6\tau)^2}{\eta(3\tau)} = \sum_{n \in \mathbb{Z}} q^{3(4n+1)^2/8}.
$$

Note that if you find a solution $F(\phi_1^{(N)}, \phi_2^{(N)})$ of weight $k$ for Eq. $(\mathcal{D})^{(N)}_{k}$, you can get another solution $F(\phi_2^{(N)}, -\phi_1^{(N)}/N)$ immediately because the group $\Gamma_0(N)$ acts on the space of solutions as follows:

$$
\begin{pmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \end{pmatrix} | \frac{1}{3} \begin{bmatrix} 2\sqrt{2} & -3/\sqrt{2} \\ 3\sqrt{2} & -2\sqrt{2} \end{bmatrix} = \sqrt{2} e^{-\frac{\pi}{2}i} \begin{pmatrix} \phi_2^{(2)} \\ -\frac{1}{2} \phi_1^{(2)} \end{pmatrix},
$$

$$
\begin{pmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \end{pmatrix} | \frac{1}{3} \begin{bmatrix} -\sqrt{3} & -4/\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3} \end{bmatrix} = \sqrt{3} e^{-\frac{\pi}{2}i} \begin{pmatrix} \phi_2^{(3)} \\ -\frac{1}{3} \phi_1^{(3)} \end{pmatrix},
$$

where $F(X, Y)$ is a homogenous polynomial of two variables, and $|k[\cdot]|$ is a slash operator of weight $k$.

Finally, Heun’s local series $Hl$ is defined by

$$
Hl(a, w; \alpha, \beta, \gamma, \delta; x) = \sum_{n=0}^{\infty} c_n x^n,
$$

where the coefficients satisfy the recursion; $c_0 = 1, \ c_1 = \frac{w}{\alpha \gamma} c_0$, and

$$
c_{n+1} = \frac{n[(n-1+\gamma)(1+a)+a \delta + \varepsilon]+w}{(n+1)(n+\gamma)a} c_n - \frac{(n-1+\alpha)(n-1+\beta)}{(n+1)(n+\gamma)a} c_{n-1} \quad (n \geq 1),
$$

where $\gamma + \delta + \varepsilon = \alpha + \beta + 1$. This is a solution of Heun’s equation, which is the canonical form of a second-order linear differential equation with four regular singularities (cf. [10]):

$$
\frac{d^2 y}{dx^2} + \left( \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-a} \right) \frac{dy}{dx} + \frac{\alpha \beta x - w}{x(x-1)(x-a)} y = 0. \quad (1)
$$

In particular, $Hl$ is a polynomial when $\alpha$ or $\beta \in \mathbb{N}$.
2 Main result

Theorem 1 (1) Assume \( k = (4n + 1)/3 \) such that \( n = 0, 1, 2, \ldots, n \not\equiv 2 \pmod{3} \). Then Eq. \( (2)^{(2)}_k \) has a two-dimensional space of solutions in \( \mathbb{C}[\phi_1^{(2)}, \phi_2^{(2)}]_{w^t = k} \). Its generators are

\[
\phi_1^{(2)}(\tau)^{3k} Hl\left(-8, \frac{k(1-3k)}{2}; -k, \frac{1-3k}{4}, \frac{3-k}{4}, \frac{1-3k}{4}; 8 \frac{\phi_2^{(2)}(\tau)^3}{\phi_1^{(2)}(\tau)^3}\right) = 1 + O(q),
\]

\[
\phi_2^{(2)}(\tau)^{k+1} \phi_1^{(2)}(\tau)^{\frac{11k-1}{8}} Hl\left(-8, \frac{(k-7)(1-3k)}{16}; -k, \frac{1-3k}{4}, \frac{1-k}{2}, \frac{1-3k}{4}; 8 \frac{\phi_2^{(2)}(\tau)^3}{\phi_1^{(2)}(\tau)^3}\right) = q^{\frac{k+1}{12}} + O(q^{\frac{k+3}{12}}).
\]

(2) Assume \( k = (3n + 1)/2 \) such that \( n = 0, 1, 2, \ldots, n \not\equiv 1 \pmod{2} \). Then Eq. \( (2)^{(3)}_k \) has a two-dimensional space of solutions in \( \mathbb{C}[\phi_1^{(3)}, \phi_2^{(3)}]_{w^t = k} \). Its generators are

\[
\phi_1^{(3)}(\tau)^{2k} Hl\left(9, k(2k-1); -k, \frac{1-2k}{3}, \frac{2-k}{3}, \frac{1-2k}{3}; 9 \frac{\phi_2^{(3)}(\tau)^2}{\phi_1^{(3)}(\tau)^2}\right) = 1 + O(q),
\]

\[
\phi_2^{(3)}(\tau)^{\frac{k+1}{3}} \phi_1^{(3)}(\tau)^{\frac{5k-1}{3}} Hl\left(9, \frac{(k+10)(1-2k)}{9}; -k, \frac{1-2k}{3}, \frac{2-k}{3}, \frac{1-2k}{3}; 9 \frac{\phi_2^{(3)}(\tau)^3}{\phi_1^{(3)}(\tau)^2}\right) = q^{\frac{k+1}{6}} + O(q^{\frac{k+3}{6}}).
\]

Remark 2 By the conditions for weight \( k \), the Heun local series in the above theorem become polynomials.

3 Proof

We will prove the result only for the case of level 2. The case of level 3 can be treated in a similar manner. To prove this theorem, we need the following proposition.

Proposition 1 ([10, p.18]) If \( Hl(a, w; \alpha, \beta, \gamma, \delta; x) \) is a solution of the Heun differential equation (1), then \( x^{1-\gamma} Hl(a, w'; \alpha', \beta', \gamma', \delta; x) \) is also a solution of Eq. (1), where \( \alpha' = \alpha + 1 - \gamma, \beta' = \beta + 1 - \gamma, \gamma' = 2 - \gamma, w' = (a\delta + \varepsilon)(1 - \gamma) + w \).
Putting \( f(\tau)/\phi_1(\tau)^3 = g(\tau) \) and \( X = 8\phi_2(\tau)^3/\phi_1(\tau)^3 \), Eq. (\( \pi \)) can be transformed into

\[
\begin{align*}
g''(\tau) + & \left( \frac{1}{4} E_{2A}(\tau) + \frac{k}{32} (X^2 + 28X - 8)\phi_1(\tau)^6 \right) g'(\tau) \\
+ & \frac{k(3k-1)}{256} X(X-1)(X+2)(X+8)\phi_1(\tau)^{12} g(\tau) = 0.
\end{align*}
\]

Using the relation of derivatives between \( 2\pi i \tau \) and \( X \):

\[
\begin{align*}
g'(\tau) &= \frac{1}{8} X(X-1)(X+8)\phi_1(\tau)^6 \frac{dg}{dX}, \\
g''(\tau) &= \frac{1}{64} X^2(X-1)^2(X+8)^2\phi_1(\tau)^{12} \frac{d^2g}{dX^2} \\
&+ \frac{1}{256} X(X-1)(X+8)\phi_1(\tau)^6 \left( (5X^2 + 28X - 24)\phi_1(\tau)^6 - 8E_{2A}(\tau) \right) \frac{dg}{dX},
\end{align*}
\]

we have

\[
\frac{d^2g}{dX^2} + \left( \frac{1 - 3k}{4X} + \frac{3 - k}{4(X-1)} + \frac{1 - 3k}{4(X+8)} \right) \frac{dg}{dX} + \frac{k(3k-1)}{4} \frac{1}{X(X-1)(X+8)} g = 0.
\]

Comparing this equation with Eq. (1), we can obtain Heun’s solution. Using Proposition 1, we can obtain another solution, and the two solutions are polynomials, because \( \alpha \) or \( \beta \), \( \alpha' \) or \( \beta' \) \( \in \mathbb{N} \). Therefore, \( f = \phi_1(\tau)^3 \cdot g \) is a modular solution of \( (\pi)_k(\tau) \).

### 4 The relation to supersingular \( j_{NA} \)-polynomials

Koike defined in [9] (or see [2, 8, 11]) supersingular \( j_{NA} \)-polynomials. He proved that for the elliptic modular invariant \( j(\tau) \) over a finite field of characteristic \( p > 0 \) to be supersingular is equivalent to the Hauptmodul \( j_{NA}(\tau) \) for \( \Gamma_0^*(N) \) being supersingular. From his result with respect to a supersingular elliptic curve, we get the following definition.

**Definition 1** For a prime number \( p(\geq 5) \), we define the “supersingular \( j_{NA} \)-polynomials for \( \Gamma_0^*(N) \)” by

\[
\begin{align*}
S_p^{(2A)}(X) &:= X^{\delta_2}(X-256)^{\frac{1}{2}} \begin{cases} 
X^{m_2 \mathbb{F}\left( \frac{1}{5}, \frac{3}{5}, 1, \frac{256}{X} \right)} & p \equiv 1, 3 \pmod{8}, \\
X^{m_2 \mathbb{F}\left( \frac{1}{5}, \frac{3}{5}, 1, \frac{256}{X} \right)} & p \equiv 5, 7 \pmod{8},
\end{cases} \\
S_p^{(3A)}(X) &:= X^{\delta_3}(X-108)^{\delta_3} \begin{cases} 
X^{m_3 \mathbb{F}\left( \frac{1}{6}, \frac{1}{5}, 1, \frac{108}{X} \right)} & p \equiv 1 \pmod{6}, \\
X^{m_3 \mathbb{F}\left( \frac{1}{6}, \frac{1}{5}, 1, \frac{108}{X} \right)} & p \equiv 5 \pmod{6},
\end{cases}
\end{align*}
\]

where \( m_2 = \left[ \frac{p}{5} \right] \), \( p - 1 = 8m_2 + 2\delta_2 + 4\epsilon_2 \), \( m_3 = \left[ \frac{p}{3} \right] \), \( p - 1 = 6m_3 + 4\delta_3 \), \( \delta_2, \epsilon_2, \delta_3 \in \{0, 1\} \), and \( \mathbb{F}(\alpha, \beta, \gamma, x) \) is the hypergeometric series over a finite field of characteristic \( p > 0 \).
In this section, we present a certain conjecture about a reduction mod prime $p$ of Heun polynomials.

Let

$$j_{2A}(X_2) = \frac{(8 - 20X_2 - X_2^2)^4}{X_2(1 - X_2)^3(8 + X_2)^3}, \quad j_{3A}(X_3) = \frac{(X_3 + 3)^6}{X_3(1 - X_3)^2(9 - X_3)^2}$$

be the Hauptmodul for $\Gamma_0^*(N)$ expressed in terms of $X_2 = 8\left(\frac{\phi_2^{(2)}}{\phi_1^{(2)}}\right)^3$ and $X_3 = 9\left(\frac{\phi_3^{(3)}}{\phi_1^{(3)}}\right)^2$, and further let

$$T_{n}^{(2)}(X_2) = Hl\left(-8, -\frac{2n(4n + 1)}{3}; -\frac{4n + 1}{3}, -n, \frac{2 - n}{3}, -n; X_2\right)$$

and

$$T_{n}^{(3)}(X_3) = Hl\left(9, \frac{3n(3n + 1)}{2}; -\frac{3n + 1}{2}, -n, \frac{1 - n}{2}, -n; X_3\right)$$

be the Heun polynomials of degree $n (> 0)$.

**Conjecture 1** Let $p > 5$ be a prime. Then $T_{p-1}^{(N)}(X_N) \mod p$ is a “supersingular $X_N$-polynomial,” i.e., it is equal to $\prod_{Y_N \in \mathbb{F}_p} (X_N - Y_N)$, where $Y_N$ runs through those values for which the corresponding Hauptmodul $j_{NA}(Y_N)$ is supersingular.

### 5 The function like characters

From [4], we know that some solutions of the Kaneko–Zagier equation for $SL_2(\mathbb{Z})$ are closely related to the character for two-dimensional conformal field theory. Precisely, we can get the character from the solution of weight $k$ divided by $\eta(\tau)^2$.

From numerical examination, for the Fricke group of levels 2 and 3, we can get something like the character from the solution $f(\tau)$ of weight $k$ divided by $\Delta_1^{2A}(\tau)$, where $\Delta_2A = \eta(\tau)^8\eta(2\tau)^8$ and $\Delta_3A = \eta(\tau)^6\eta(3\tau)^6$ are cusp forms for $\Gamma_0(N)$. For example, in the case for $\Gamma_0^*(2)$, we get the following:

(a) For $k = 1/3$,

$$\phi_1^{(2)}/\Delta_2^{1/24} = \frac{1}{q^{1/24}} + q^{23/24} + 2q^{47/24} + q^{71/24} + 3q^{95/24} + 3q^{119/24} + 5q^{143/24} + 5q^{167/24} + 8q^{191/24} + \cdots$$

the number of partitions of $n$ in which no part appears more than twice and no two parts differ by 1.

(b) For $k = 2$,

$$\frac{(\phi_1^{(2)})^6 + 20(\phi_1^{(2)}\phi_2^{(2)})^3 - 8(\phi_2^{(2)})^6}{\Delta_2^{1/4}} = \frac{1}{q^{1/4}} + 26q^{3/4} + 79q^{7/4} + 326q^{11/4} + 755q^{15/4} + 2106q^{19/4} + \cdots$$

the McKay–Thompson series of class $8C$ for the Monster. (cf. [1])

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For several other weights $k$, we observe that each coefficient of $f(\tau)/\Delta_{NA}(\tau)^{k/(12-2N)}$ is a positive integer, where $f(\tau)$ is a modular solution of weight $k$ for $(\mathfrak{p})_k^{(N)}$. But we do not know to which the function corresponds and what are the properties of these functions.

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References

1. Conway, J.H., Norton, S.P.: Monstrous moonshine. Bull. London Math. Soc. 11(3), 308–339 (1979)
2. Ihara, Y.: Schwarzian equations. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 21, 97–118 (1974).
3. Kaneko, M.: On Modular Forms of Weight $(6n + 1)/5$ Satisfying a Certain Differential Equation. Number Theory Dev. Math., pp. 97–102. Springer, New York (2006).
4. Kaneko, M., Koike, M.: On modular forms arising from a differential equation of hypergeometric type. Ramanujan J. 7(1–3), 145–164 (2003)
5. Kaneko, M., Koike, M.: Quasimodular solutions of a differential equation of hypergeometric type. Galois theory and modular forms, Dev. Math., pp. 329–336. Kluwer Acad. Publ., Boston (2004).
6. Kaneko, M., Zagier, D.: Supersingular $j$-invariants, hypergeometric series, and Atkin’s orthogonal polynomials, Computational Perspectives on Number Theory (Chicago, IL, 1995), AMS/IP Stud. Adv. Math., vol. 7, pp. 97–126. Amer. Math. Soc., Providence, 1998.
7. Köhler, G., et al.: Eta Products and Theta Series Identities. Springer Monographs in Mathematics. Springer, Heidelberg (2011)
8. Koike, M.: Congruences between modular forms and functions and applications to the conjecture of Atkin. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 20, 129–169 (1973).
9. Koike, M.: On supersingular $j^2$-polynomials for $\Gamma_0^*(2)$, Preprint (2009).
10. Ronveaux A. (ed.): Heun’s Differential Equations, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1995, (With contributions by Arscott, F. M., Yu Slavyanov, S., Schmidt, D., Wolf, G., Maroni, P., Duval, A.).
11. Sakai, Y.: The Atkin orthogonal polynomials for the low-level Fricke groups and their application. Int. J. Number Theory 7(6), 1637–1661 (2011)