ON SWITALA’S MATLIS DUALITY FOR $\mathcal{D}$-MODULES

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Abstract. We show that N. Switala’s Matlis duality for $\mathcal{D}$-modules yields a generalization of a result of R. Hartshorne and C. Polini.

1. Introduction

Let $(R, m)$ be a commutative Noetherian complete equicharacteristic local ring and let $\mathcal{D} = \mathcal{D}(R, k)$ be the ring of $k$-linear differential operators of $R$, where $k \subset R$ is a fixed coefficient field of $R$ ($k$ exists as $R$ is complete).

Let $E$ be the injective hull of $R/m \cong k$ in the category of $R$-modules. Matlis duality $D(-) = \text{Hom}_R(-, E)$ is a contravariant exact functor from $R$-modules to $R$-modules. N. Switala [2, Section 4] has shown that if $M$ is a left (resp. right) $\mathcal{D}$-module, then $D(M)$ carries the structure of a right (resp. left) $\mathcal{D}$-module. In this short note our main result is the following.

Theorem 1.1. Let $M$ be a left $\mathcal{D}$-module and let $N$ be a right $\mathcal{D}$-module. There is a $k$-linear bijection

$$\text{Ext}^i_{\mathcal{D}^{-}\text{mod}}(M, D(N)) \cong \text{Ext}^i_{\text{mod}-\mathcal{D}}(N, D(M))$$

which is functorial in $M$ and $N$, where $\mathcal{D}^{-}\text{mod}$ and $\text{mod}-\mathcal{D}$ denote the categories of, respectively, left and right $\mathcal{D}$-modules.

R. Hartshorne and C. Polini proved the following important result [1, 5.2].

Theorem 1.2. Notation being as above, assume the characteristic of $k$ is 0 and $R = k[[x_1, \ldots, x_n]]$ is the ring of formal power series in $n$ variables over $k$. Let $M$ be a left $\mathcal{D}$-module and let $H^i_{\text{dR}}(M)$ denote the $i$-th de Rham cohomology group of $M$. If $M$ is holonomic, then

$$\dim_k H^i_{\text{dR}}(M) = \dim_k \text{Hom}_{\mathcal{D}^{-}\text{mod}}(M, E).$$

As an application of our Theorem 1.1 we show that Switala’s Matlis duality theory yields the following generalization of Theorem 1.2.

Theorem 1.3. Notation being as in Theorem 1.2 for every $i$

$$\dim_k H^{n-i}_{\text{dR}}(M) = \dim_k \text{Ext}^i_{\mathcal{D}^{-}\text{mod}}(M, E)$$

Theorem 1.2 is the $i = 0$ case of Theorem 1.3.

2000 Mathematics Subject Classification. Primary 13H05, 13N10, 16S32.

Key words and phrases. complete local ring, holonomic module, de Rham cohomology. NSF for support through grant DMS-1500264 is gratefully acknowledged.
2. A review of Switala’s Matlis duality

Let $L$ be an $R$-module. The $k$-module $\text{Hom}_k(L, k)$ has a structure of $R$-module as follows: if $f : L \to k$ is a $k$-linear map and $r \in R$, the map $g = rf : L \to k$ is defined by $g(x) = f(rx)$ for every $x \in L$. The following definition is equivalent to but somewhat simpler than Switala’s original definition \cite{2, Definitions 3.13, 3.14(c)].

**Definition 2.1.** A $k$-linear map $f : L \to k$ is $\Sigma$-continuous if for every $x \in L$ there is some $i \geq 0$ so that $f(m^ix) = 0$, where $m \subset R$ is the maximal ideal. The set of all $\Sigma$-continuous maps $L \to k$ is denoted by $D^\Sigma(L)$.

If $f : L \to k$ is $\Sigma$-continuous, then so is $rf : L \to k$ because $rm^i \subset m^i$ and therefore $f(m^ix) = 0$ implies $f(rm^ix) = 0$. Thus $D^\Sigma(L)$ is an $R$-module homomorphism of $\text{Hom}_k(L, k)$.

If $\phi : L' \to L$ is an $R$-module homomorphism and $f : L \to k$ is $\Sigma$-continuous, then the composition $f \circ \phi : L' \to k$ also is $\Sigma$-continuous. This defines a map $D^\Sigma(\phi) : D^\Sigma(L) \to D^\Sigma(L')$ that sends $f \in D^\Sigma(L)$ to $f \circ \phi \in D^\Sigma(L')$. It is not hard to see that $D^\Sigma(\phi)$ is an $R$-module homomorphism.

Thus one gets a contravariant functor

$$D^\Sigma : R^{\text{-mod}} \to R^{\text{-mod}}$$

which takes every $R$-module $L$ to $D^\Sigma(L)$ and every $R$-module homomorphism $\phi : L \to L'$ to $D^\Sigma(\phi) : D^\Sigma(L') \to D^\Sigma(L)$.

Switala has shown \cite{2, Theorem 3.15} that there is an isomorphism of functors

$$D^\Sigma \cong D$$

where $D = \text{Hom}_R(-, E)$ is the classical Matlis duality functor. Thus as long as one sticks to $R$-modules and $R$-module homomorphisms, $D^\Sigma$ does not give anything new compared to $D$.

But $D^\Sigma$ makes it possible to dualize more maps between $R$-modules, than just $R$-module homomorphisms. For example, assume $L$ is a (left or right) $D$-module, $\delta \in D$ is a differential operator and $\delta : L \to L$ is the action of $\delta$ on $L$. Switala shows that if $f : L \to k$ is $\Sigma$-continuous, then the composition $f \circ \phi : L \to k$ also is $\Sigma$-continuous. This makes it possible to define the action of $\delta$ on $D^\Sigma(L)$, namely, $\delta(f) = f \circ \phi$ for every $f \in D^\Sigma$ \cite{2, Corollary 4.10].

This definition of the action of differential operators on $D^\Sigma$ gives $D^\Sigma$ a structure of $D$-module. More precisely, if $L$ is a left $D$-module, then $D^\Sigma(L)$ becomes a right $D$-module and if $L$ is a right $D$-module, then $D^\Sigma(L)$ becomes a left $D$-module and if $\phi : L \to L'$ is a map of left (resp. right) $D$-modules, then the map $D^\Sigma(\phi) : D^\Sigma(L') \to D^\Sigma(L)$ is a map of right (resp. left) $D$-modules. Thus $D^\Sigma$ induces two contravariant exact functors on $D$-modules:

$$D^\Sigma : D^{\text{-mod}} \to \text{mod-}D$$
$$D^\Sigma : \text{mod-}D \to D^{\text{-mod}},$$

where $D^{\text{-mod}}$ and $\text{mod-}D$ are the categories of left and right $D$-modules respectively \cite{2, Proposition 4.11].
3. Proof of Theorem 1.1

Given the $R$-module structures on $\text{Hom}_R(M, k)$ and $\text{Hom}_R(N, k)$ defined in the beginning of the preceding section, there are $R$-linear maps

$$\tau(M, N) : \text{Hom}_R(M, \text{Hom}_k(N, k)) \to \text{Hom}_R(N, \text{Hom}_k(M, k))$$

$$\tau(N, M) : \text{Hom}_R(N, \text{Hom}_k(M, k)) \to \text{Hom}_R(M, \text{Hom}_k(N, k))$$

defined as follows. If $f \in \text{Hom}_R(M, \text{Hom}_k(N, k))$, then $g = \tau(M, N)(f) \in \text{Hom}_R(N, \text{Hom}_k(M, k))$ is defined by the formula

$$(1) \quad g(y)(x) = f(x)(y)$$

for every $x \in M$ and $y \in N$, and, similarly, if $g \in \text{Hom}_k(N, \text{Hom}_R(M, k))$, then $f = \tau(N, M)(g) \in \text{Hom}_R(M, \text{Hom}_k(N, k))$ is defined by the same formula (1). Clearly, $g = \tau(M, N)(f)$ if and only if $f = \tau(N, M)(g)$, i.e. the two maps $\tau(M, N)$ and $\tau(N, M)$ are inverses of each other and therefore establish an $R$-linear bijection

$$\text{Hom}_R(M, \text{Hom}_k(N, k)) \cong \text{Hom}_R(N, \text{Hom}_k(M, k)).$$

Let $f \in \text{Hom}_R(M, \text{Hom}_k(N, k))$ and $g \in \text{Hom}_R(N, \text{Hom}_k(M, k))$ correspond to each under this bijection, i.e. $g = \tau(M, N)(f)$ and $f = \tau(N, M)(g)$. We claim that $f(x) : N \to k$ is $\Sigma$-continuous for every $x \in M$ if and only if $g(y) : M \to k$ is $\Sigma$-continuous for every $y \in N$. Indeed, assume $f(x)$ is $\Sigma$-continuous for every $x \in M$. Pick any $x \in M$ and $y \in N$. Since $f(x) \in D^\Sigma(N)$, there exists $i$ such that $f(x)(m^i y) = 0$. Let $r \in m^i$. By definition, $(rf(x))(y) = f(x)(ry)$, hence $(rf(x))(y) = 0$. Since $f$ is a map of $R$-modules, $rf(x) = f(rx)$. Hence $f(rx)(y) = 0$, i.e. $g(y)(rx) = 0$. Since this holds for every $r \in m^i$, the map $g(y)$ is $\Sigma$-continuous for every $y$. A similar argument shows that if $g(y)$ is $\Sigma$-continuous for every $y$, then $f(x)$ is $\Sigma$-continuous for every $x \in M$. This proves the claim.

Thus the maps $\tau(M, N)$ and $\tau(N, M)$ (once they are restricted to the $R$-submodules $\text{Hom}_R(M, D^\Sigma(N))$ of $\text{Hom}_R(M, \text{Hom}_k(N, k))$ and $\text{Hom}_R(N, D^\Sigma(M))$ of $\text{Hom}_R(N, \text{Hom}_k(M, k)))$ establish an $R$-linear bijection

$$\text{Hom}_R(M, D^\Sigma(N)) \cong \text{Hom}_R(N, D^\Sigma(M)).$$

Let $f \in \text{Hom}_R(M, D^\Sigma(N))$ and $g \in \text{Hom}_R(N, D^\Sigma(M))$ correspond to each under this bijection, i.e. $g = \tau(M, N)(f)$ and $f = \tau(N, M)(g)$. We claim that $f$ is a map of left $D$-modules if and only if $g$ is a map of right $D$-modules. Indeed, $f$ is a map of left $D$-modules if and only if $f(\delta x) = \delta(f(x))$ for every $x \in M$ and every $\delta \in D$. Set $h = f(x) : N \to k$. By definition $\delta(h) : N \to k$ is defined by $(\delta(h))(y) = h(y\delta)$ for every $y \in N$. Hence $f$ is a map of left $D$-modules if and only if

$$f(\delta x)(y) = f(x)(y\delta)$$

for every $x \in M, y \in N$ and $\delta \in D$. A similar argument shows that $g$ is a map of right $D$-modules if and only if

$$g(y\delta)(x) = g(y)(\delta x).$$
for all \(x \in M, y \in N\) and \(\delta \in D\). Since by formula (1), 
\[f(\delta x)(y) = g(y)(\delta x)\] 
and \(f(x)(y\delta) = g(y\delta)(x)\), the claim is proven.

Thus the maps \(\tau(M, N)\) and \(\tau(N, M)\) (after restriction to the \(k\)-submodules 
\(\text{Hom}_{D-\text{mod}}(M, D^\Sigma(N))\) of \(\text{Hom}_{R}(M, D^\Sigma(N))\) and \(\text{Hom}_{\text{mod}-D}(N, D^\Sigma(M))\) of \(\text{Hom}_{R}(N, D^\Sigma(M))\)) establish a \(k\)-linear bijection

\[
\text{Hom}_{D-\text{mod}}(M, D^\Sigma(N)) \cong \text{Hom}_{\text{mod}-D}(N, D^\Sigma(M)).
\]

Since this bijection is clearly functorial in \(M\) and \(N\), the \(i = 0\) case of the theorem is proven.

We need the following lemma.

**Lemma 3.1.** If \(P\) is a projective left (resp. right) \(D\)-module, then \(D(P)\) is an injective right (resp. left) \(D\)-modules.

**Proof of the lemma.** Assume \(P\) is projective left \(D\)-module. Let

\[0 \to N' \to N \to N'' \to 0\]

be an exact sequence of right \(D\)-modules. The induced sequence

\[0 \to \text{Hom}_{\text{mod}-D}(N'', D(P)) \to \text{Hom}_{\text{mod}-D}(N, D(P)) \to \text{Hom}_{\text{mod}-D}(N', D(P)) \to 0\]

is exact because by the \(i = 0\) case of the theorem that has just been proven, this sequence is isomorphic to the sequence

\[0 \to \text{Hom}_{\text{mod}-D}(P, D(N'')) \to \text{Hom}_{\text{mod}-D}(P, D(N)) \to \text{Hom}_{\text{mod}-D}(P, D(N')) \to 0\]

which is exact since \(P\) is projective and \(D\) is an exact functor. Therefore \(D(P)\) is injective.

For a projective right \(D\)-module \(P\) the proof is the same except "left" must be replaced by "right" and vice versa. \(\square\)

We continue with the proof of Theorem \(\text{[1]}\) Let

\[\cdots \to P_1 \to P_0 \to M \to 0\]

be a projective resolution of \(M\) in the category of left \(D\)-modules. The \(k\)-module \(\text{Ext}^1_{D-\text{mod}}(M, D(N))\) is the \(i\)th cohomology of the induced complex

\[0 \to \text{Hom}_{D-\text{mod}}(P_0, D(N)) \to \text{Hom}_{D-\text{mod}}(P_1, D(N)) \to \cdots\]

By the \(i = 0\) case of the theorem that has already been proven, this complex is isomorphic to the complex

\[0 \to \text{Hom}_{\text{mod}-D}(N, D(P_0)) \to \text{Hom}_{\text{mod}-D}(N, D(P_1)) \to \cdots\]

The \(k\)-module \(\text{Ext}^i(N, D(M))\) is the \(i\)-th cohomology of this complex since by Lemma \(\text{[5.1]}\) \(D(P^\bullet)\) is an injective resolution of \(D(M)\). This completes the proof of Theorem \(\text{[1]}\). \(\square\)

The natural action of \(D\) on \(R\) makes \(R\) a left \(D\)-module. Hence by Switala’s Matlis duality, \(E = D(R)\) is a right \(D\)-module.
Corollary 3.2. If $N$ is a right $\mathcal{D}$-module, there exist $k$-linear bijections
\[
\text{Ext}_\mathcal{D}^i(R, D(N)) \cong \text{Ext}_{-\mathcal{D}}^i(N, E)
\]
that are functorial in $N$.

Proof. This is immediate from Theorem 1.1 upon setting $M = R$. \qed

4. Proof of Theorem 1.3

Throughout this section the field $k$ has characteristic 0 and $R$ is regular, i.e. $R = k[[x_1, \ldots, x_n]]$ is the ring of formal power series in $n$ variables over $k$. Let
\[
\partial_i = \frac{\partial}{\partial x_i} : R \to R
\]
be the $k$-linear partial differentiation with respect to $x_i$. The ring $\mathcal{D}$ is the free left (as well as right) $R$-module on the monomials $\partial_1^{a_1} \cdots \partial_n^{a_n}$. There is a ring anti-automorphism
\[
t : \mathcal{D} \to \mathcal{D}
\]
(that is an isomorphism of the underlying additive groups that reverses the order of multiplication, i.e. $t(\delta_1 \delta_2) = t(\delta_2) t(\delta_1)$ for all $\delta_1, \delta_2 \in \mathcal{D}$) defined by
\[
t(r \partial_1^{a_1} \cdots \partial_n^{a_n}) = (-\partial_1)^{a_1} \cdots (-\partial_n)^{a_n} r
\]
for all $r \in R$. This anti-automorphism $t$ is called transposition.

We can view a right $\mathcal{D}$-module $M$ as a left $\mathcal{D}$-module $t(M)$ via this transposition anti-automorphism [2, Definition 4.14], i.e.
\[
\delta \ast x = xt(\delta)
\]
for every $x \in M$ and $\delta \in \mathcal{D}$ where $\ast$ denotes the left $\mathcal{D}$-action on $t(M)$. Thus $t(M)$ has the same elements and the same $R$-module structure as $M$ but the $\mathcal{D}$-module structure is different. If $\phi : M \to M'$ is a map of right $\mathcal{D}$-modules, the map $t(\phi) : t(M) \to t(M')$, which is the same map as $\phi$ on the underlying sets, is a map of left $\mathcal{D}$-modules.

The resulting functor from right $\mathcal{D}$-modules to left $\mathcal{D}$-modules that sends a right $\mathcal{D}$-module $M$ to the left $\mathcal{D}$-module $t(M)$ and sends a map of right $\mathcal{D}$-modules $\phi : M \to M'$ to the map of left $\mathcal{D}$-modules $t(\phi) : t(M) \to t(M')$ is covariant, exact, induces bijections on Hom-sets (i.e. $\text{Hom}_{-\mathcal{D}}(M, M') \cong \text{Hom}_{\mathcal{D}}(t(M), t(M'))$) for all right $\mathcal{D}$-modules $M, M'$ and therefore takes projectives to projectives and injectives to injectives. Therefore, for all right $\mathcal{D}$-modules $M, M'$ there exist functorial $k$-linear bijections
\[
(2) \quad \text{Ext}_{-\mathcal{D}}^i(M, M') \cong \text{Ext}_\mathcal{D}^i(t(M), t(M')).
\]

In view of all this, in the case that $\text{char} k = 0$ and $R = k[[x_1, \ldots, x_n]]$ we are going to view all right $\mathcal{D}$-modules $M$ as left $\mathcal{D}$-modules $t(M)$ and by abuse of notation, will denote $t(M)$ by $M$. For example, if $M$ is a left $\mathcal{D}$-module, then $t(D(M))$ also is a left $\mathcal{D}$-module that we are going to denote simply $D(M)$. This convention is used in the statements of Theorems 1.2 and 1.3 where the left $\mathcal{D}$-module $t(D(R)) = t(E)$ is denoted by $E$. With
this notational convention and taking into account the $k$-linear bijections \[2\] and the fact that the functors $D$ and $t$ commute, Corollary \[3.2\] in the case that $R = k[[x_1, \ldots, x_n]]$ and char $k = 0$ takes the following form.

**Corollary 4.1.** If $M$ is a left $\mathcal{D}$-module, there exist $k$-linear bijections

\[
\text{Ext}^i_{\mathcal{D} \text{-mod}}(R, D(M)) \cong \text{Ext}^i_{\mathcal{D} \text{-mod}}(M, E)
\]

that are functorial in $M$.

For the rest of the paper by a $\mathcal{D}$-module we mean a left $\mathcal{D}$-module.

Let $M$ be a $\mathcal{D}$-module. The de Rham complex $\Omega^\bullet(M)$ of $M$ is the complex

\[0 \to \Omega^0(M) \to \Omega^1(M) \to \cdots \to \Omega^n(M) \to 0\]

where $\Omega^s(M) = \oplus M_{i_1, \ldots, i_s}$ is the direct sum of copies of $M$ indexed by all the $s$-tuples $(i_1, \ldots, i_s)$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ and the differential $d_s : \Omega^s(M) \to \Omega^{s+1}(M)$ is given by

\[d_s(x)_{i_1, \ldots, i_{s+1}} = \Sigma_j (-1)^j \partial_j (x_{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{s+1}})\]

for every $x \in \Omega^s(M)$ where $d_s(x)_{i_1, \ldots, i_{s+1}}$ and $x_{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{s+1}}$ denote the components of $d_s(x)$ and $x$ in, respectively, $M_{i_1, \ldots, i_{s+1}}$ and $M_{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{s+1}}$.

The $i$-th de Rham cohomology group of $M$, denoted $H^i_{dR}(M)$, is by definition the $i$-th cohomology group of the de Rham complex $\Omega^\bullet(M)$. The following theorem is well-known; we provide a proof for the convenience of the reader since we do not know a suitable reference.

**Theorem 4.2.** Let $M$ be a $\mathcal{D}$-module. There is a $k$-linear isomorphism

\[H^i_{dR}(M) \cong \text{Ext}^i_{\mathcal{D} \text{-mod}}(R, M)\]

**Proof.** Let $\Delta^\bullet$ be the complex

\[0 \to \Delta_n \to \Delta_{n-1} \to \cdots \to \Delta_1 \to \Delta_0 \to 0\]

where $\Delta_s = \oplus \mathcal{D}_{i_1, \ldots, i_s}$ is the direct sum of copies of $\mathcal{D}$ indexed by all the $s$-tuples $(i_1, \ldots, i_s)$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ and the differential $d_s : \Delta_s \to \Delta_{s-1}$ is given by

\[d_s(x) = \Sigma_j (-1)^j (x\partial_j)_{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_s}\]

for every $x \in \Delta_{i_1, \ldots, i_s} \subset \Delta_s$ where $(x\partial_j)_{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_s}$ denotes the element $x\partial_j \in \mathcal{D}_{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_s} \subset \Delta_{s-1}$.

The complex $\Delta^\bullet$ is a finite free resolution of $R$ in the category of left $\mathcal{D}$-modules. Therefore $\text{Ext}^i_{\mathcal{D} \text{-mod}}(R, M)$ is the $i$th cohomology group of the complex $\text{Hom}_{\mathcal{D} \text{-mod}}(\Delta^\bullet, M)$, i.e. the complex

\[0 \to \text{Hom}_{\mathcal{D} \text{-mod}}(\Delta_0, M) \to \text{Hom}_{\mathcal{D} \text{-mod}}(\Delta_1, M) \to \cdots \to \text{Hom}_{\mathcal{D} \text{-mod}}(\Delta_n, M) \to 0\]

The map $\phi : \mathcal{D} \xrightarrow{x \mapsto x\partial_j} \mathcal{D}$ which is the right multiplication by $\partial_j$ is a map of left $\mathcal{D}$-modules. The induced map $\text{Hom}_{\mathcal{D} \text{-mod}}(\phi, M)$ is nothing but the left multiplication by $\partial_j$ on $M$, i.e. $M \xrightarrow{x \mapsto x\partial_j} M$. Therefore the complex $\text{Hom}_{\mathcal{D} \text{-mod}}(\Delta^\bullet, M)$ is nothing but the de Rham complex $\Omega^\bullet(M)$ of $M$. \[\square\]
A key result of N. Switala’s theory is the following [2, 5.1].

**Theorem 4.3.** If $M$ is a holonomic $D$-module, then for all $i$ there are isomorphisms $(H^i_{dR}(M))' \cong H^{n-i}_{dR}(D(M))$, where $(-)' = \text{Hom}_k(-, k)$.

An important result of van den Essen [4, Proposition 2.2] says

**Theorem 4.4.** If $M$ is a holonomic $D$-module, then $H^i_{dR}(M)$ is a finite-dimensional $k$-vector space.

Corollary 4.1 implies that $\dim_k \text{Ext}^i_{D\text{-mod}}(M, E)$ equals $\dim_k \text{Ext}^i_{D\text{-mod}}(R, D(M))$ which by Theorem 4.2 equals $\dim_k H^i_{dR}(D(M))$ which by Theorems 4.3 and 4.4 equals $\dim_k H^{n-i}_{dR}(M)$. This completes the proof of Theorem 1.3. □

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