1 A conformally invariant generalization of string theory to higher-dimensional objects

The remarkable achievements of string and superstring theories are well known: evaluation of the space-time dimension, fixing a particular gauge group, inclusion of gravity into a unified scheme etc. [1]. These achievements stimulate an interest in studies of geometric objects of higher dimension, such as membranes or p-branes. It is known, however, that [5] in standard membrane theories the absence of conformal invariance precludes the usage of string-theoretical methods. For instance, the requirement that conformal invariance should be preserved at the quantum level leads, in string theory, to fixing the space-time dimension [2]. There are also other arguments [3] in favour of the requirement that a physical field theory should be conformally invariant, at least at the classical level.

On this basis, we have previously suggested a conformally invariant generalization of string theory to higher-dimensional objects [4]. This paper, aimed at further realization of this approach, is devoted to obtaining and investigation of Hamiltonian equations and constraint equations of the theory under consideration. This idea was originally suggested as a quantum theory by analogy with string theory. However, a further analysis has shown the necessity of an initial classical analysis of this theory. It has turned out that even the classical level of the theory contains results of interest related to gravitation theory and p-brane theory. The action that serves as a basis for the suggested theory, being a generalization of string and p-brane theory, is simultaneously a certain generalization of Einstein’s general relativity. We suggest that general relativity should be considered as a special case of a conformally invariant sigma model, appearing as a result of conformal symmetry violation.

This paper is devoted to foundation and analysis of the above ideas in the classical case.

The recent development of multidimensional theories have been, to a large extent, related to the so-called branes. In this theory [6–9], the observable Universe is considered as a surface (brane) embedded in a higher-dimensional space-time. It is hoped that this approach can lead to a success in solving the fundamental problem of the hierarchy of physical coupling constants and the cosmological constant problem. The hierarchy problem lies in the existence of a huge difference between the electroweak energy scale of about 1 TeV and the gravitational energy scale of the order of $10^{19}$ GeV. Besides, the energy density related to the cosmological constant should be about 120 orders of magnitude smaller than the possible energy density values for the known models of quantum theory of the weak and strong interactions.

In the theory suggested, the gravitational constant is related to the dynamic characteristics of the model, and it is obtained in multidimensional space-time due to localization of solutions to nonlinear equations, by analogy with the Higgs effect in gauge field theory.

1.1 The Lagrangian approach

The action for a membrane (or p-brane) does not admit conformal transformations, and these models do not possess a natural candidate for the role of an anomalous symmetry like conformal symmetry in string theory. To circumvent this difficulty without abandoning the string-theoretical ideology, we suggest the following generalization of string theory:

$$S = \frac{1}{w} \int \left\{ -\frac{1}{2}(\nabla_{\nu} X, \nabla^\nu X) + \xi R(X, X) + \Lambda(X, X) \right\} \sqrt{-g} d^{p+1} \sigma,$$

(1)

where we use the notations:

$$(X, X) = X^A X^B \eta_{AB},$$

$$(\nabla_{\nu} X, \nabla^\nu X) = \nabla_{\nu} X^A \nabla_{\nu} X^B \eta^A_{\mu} \eta^B_{\nu},$$

$$\rho = (p+1)/(p-1).$$

In the action (1), the functions $X^A = X^A(\sigma^\mu)$, with $A, B = 1, 2, \ldots, D; \mu, \nu = 0, 1, \ldots, p$, map the $n = 2$
Let us point out the important fact that for strings \((p = 1)\) the general solution to Eqs. (5) has the form

\[
B g_{\mu\nu} = (\nabla_\mu X, \nabla_\nu X), \quad \mu, \nu = 0, p, \tag{10}
\]

where \(B\) is an arbitrary function. Thus the original metric \(g_{\mu\nu}\) is connected by a conformal transformation with the induced metric \((\nabla_\mu X, \nabla_\nu X)\). Unfortunately, in the general case \(p > 1\), for Eqs. (11)–(13), the solution (10) is not a general solution. The problem of connection between the metric of the manifold \(\Pi^p\) with the metric induced by the solutions \(X^A = X^A(\sigma^I)\), as well as that of a physical interpretation of this connection, have not been solved for an arbitrary dimension.

In what follows, we will consider some special solutions to Eqs. (11)–(13), being of interest for physics.

### 1.2 Hamiltonian formalism

To pass over to the Hamiltonian formalism, we make, in the action (1), a \((p + 1)\)-partition. Employing the results of Refs. [16, 17], we introduce the parameters \(N\) and \(\tau\), the “lapse” or “shift” functions, and the metric functions of the \(p\)-dimensional geometry \(h_{ij}\), where \(i = 1, \ldots, p\):

\[
g_{00} = N_s N^s - N^2, \quad g_{0i} = N_i, \quad g_{ij} = h_{ij}, \quad \sqrt{-g} = N \sqrt{\hat{h}}.
\]

Then, taking into account the results of Ref. [16], we can present the scalar curvature in the form

\[
\frac{n}{\sqrt{\hat{h}}} R = R - (\text{Sp} \hat{K})^2 + \text{Sp}(\hat{K}^2) - \frac{2}{N \sqrt{\hat{h}}} \partial_a [N \sqrt{\hat{h}} n^a \text{Sp} \hat{K} + \nabla_\alpha n^\alpha)]. \tag{11}
\]

Here \(R\) is the scalar curvature calculated for the metric \(h_{ij}\), \(a^a = \nabla_\beta n^\alpha n^\beta\) is the \((p + 1)\)-dimensional acceleration of an observer moving along a timelike normal \(\vec{n}\) to consecutive sections. The space-time \(\Pi^p\) is assumed to be foliated into a one-parameter family of spacelike hypersurfaces with the parameter \(t\). The quantity \(\hat{K}\) is the extrinsic curvature tensor of the spacelike sections:

\[
n_a = \{-N, 0\}, \quad (\vec{n} \cdot \vec{n}) = -1, \tag{12}
\]

\[
\text{Sp} \hat{K} = h^{ij} K_{ij}, \quad \text{Sp} \hat{K}^2 = K_{ij} K^{ij},
\]

\[
K_{ij} = \frac{1}{2N} (D_i N_j + D_j N_i - \partial_i h_{ij}), \tag{13}
\]

where \(D_i\) is a covariant derivative calculated with the metric \(h_{ij}\).

Let us now pass over to a description in terms of the phase-space variables, i.e., the generalized coordinates and momenta:

\[
\{q_I\} = \{X^A, h_{ik}, N, N_i\}, \quad \{p_I\} = \{P_A, \Pi^k, p_N, p_N\}, \tag{14}
\]

where

\[
p_I = \frac{\delta L}{\delta (\partial_I q_I)}.
\]
where $L$ is the Lagrangian corresponding to the action (1). We obtain as a result:

\[
P = \frac{\sqrt{\hbar}}{NW}[(X, X)(Sp \hat{K} \cdot h_{ik} - K^{ik})] + \frac{1}{N}h^{ik}(\partial_i(X, X) - N^iD_i(X, X)),
\]

the remaining momenta are zero. In the conformal transformations (3), the phase variables are transformed as follows:

\[
h_{ik} \mapsto e^{2\phi}h_{ik}, \quad X \mapsto e^{4\xi p\phi}X, \quad N \mapsto e^{\phi}N, \quad N_i \mapsto e^{\phi}N_i, \quad P \mapsto e^{-4\xi p\phi}P, \quad \Pi^{ik} \mapsto e^{-2\phi}\Pi^{ik}.
\]

As follows from (10), there is a constraint between $P$ and $\Pi^{ik}$. This constraint may be written as

\[
M = 2Sp\tilde{\Pi} + 4\xi p(P, X) = 0.
\]

Integrating by parts and rejecting terms with a full divergence, one can write the action (1) in terms of the canonical variables as

\[
S = \int [\dot{h}_{ik}\Pi^{ik} + (P, \dot{X}) - NH_0 - N_iH^i
\]

\[
- \lambda_M M - \partial_iQ^i]\dot{d}^{i+1}\sigma,
\]

where

\[
H_0 = \frac{w}{\sqrt{\hbar}(X, X)}[Sp(\tilde{\Pi}) - \frac{1}{p}(SP\tilde{\Pi})^2] + \frac{w}{2\sqrt{\hbar}}P^2
\]

\[
+ \frac{\sqrt{\hbar}}{w}\left[-\xi(X, X)R + \frac{1}{2}(D_sX, D^sX)
\]

\[
+ 2\xi\Delta (X, X) - \Lambda(X, X)\right],
\]

\[
H^i = (P, D^jX) - 2D_j\Pi^{il},
\]

\[
Q^i = \frac{2\sqrt{\hbar}\xi}{w}[(X, X)D^iN - ND^i(X, X)]
\]

\[
+ 2N_k\Pi^{ki} + \frac{N_i}{4}\left[\frac{1}{\xi p}Sp\tilde{\Pi} + \frac{1}{wN}E\right].
\]

Here we have used the notations

\[
E = \partial_i(X, X) - N^sD_s(X, X),
\]

\[
\Delta = h^{ik}D_iD_k \quad \text{and} \quad (D_sX, D^sX) = (D_iX, D_jX)h^{ij}.
\]

The function $\lambda_M$ is arbitrary. This is related to the impossibility of resolving the velocity $\lambda_M = \partial_i(X, X)$ in terms of the momenta. It can be shown that

\[
p_E = \frac{\delta L}{\delta \lambda E} = \frac{M}{8\xi p(X, X)}.
\]

The constraint (17) has appeared because of the invariance of the theory with respect to (16). To take this fact into account explicitly, let us transform the integrand in (18) according to (19) for $\phi = \psi$ and introduce the field $\psi$. Then the expression (18) takes the form

\[
S = \int [\dot{h}_{ik}\Pi^{ik} + (P, \dot{X}) - NH_0 - N_iH^i
\]

\[
- \lambda_M M - \partial_iQ^i]\dot{d}^{i+1}\sigma.
\]

That is, we could use, instead of $\lambda_M$, the field $\psi(\sigma)$. The corresponding momentum is $p_\psi = M$. The divergence term in Eq. (23) does not affect the equations of motion but affects the boundary conditions. After the transformations (10), the quantities $Q^i$ turn into $Q^i$, where

\[
Q^i = Q^i + (X, X)\frac{\sqrt{\hbar}}{w} \times \left[2\xi pND^i\psi + \frac{\xi p Ni}{N}(\psi - N_sD\psi)\right].
\]

The latter expression implies that, taking into account the boundary effects, we shall obtain certain boundary conditions applied to the function $\psi$, violating the invariance of the theory with respect to (19). It is probably reasonable to omit the divergence term (24) from the action, replacing the original Lagrangian density with $L + \partial_iQ^i$. An argument in favour of such a replacement is that for $(X, X) = \text{const}$ and $X^s = \text{const}$, the action acquires the Einstein form. In the construction of the Hamiltonian formalism for Einstein’s theory, such terms are omitted [17].

The conditions that the primary constraints are conserved in time,

\[
\Phi_1^{(1)}: \quad p_\mu \equiv \{p_N, p_N^i\} = 0, \quad p_\psi - M = 0,
\]

with the Hamiltonian constructed in the standard way [17],

\[
H = \partial_iQ^i + NH_0 + N_iH^i + \lambda_0M
\]

and the extended Hamiltonian

\[
H^1 = H + \lambda_1\Phi_1^{(1)},
\]

do not allow one to determine the functions $\lambda_1$, but there emerge secondary constraints:

\[
\Phi_2^{(2)}: \quad H^\nu \equiv \{H_0, H^1, M\} = 0.
\]

Consider the conservation conditions for the constraints (27). To do so, it is necessary to calculate the Poisson brackets:

\[
[\Phi_K, \Phi_J] \equiv \frac{\delta\Phi_K}{\delta q_I} \frac{\partial p_J}{\partial d^I} - \frac{\delta\Phi_K}{\delta p^I} \frac{\partial d_J}{\delta q_I}.
\]

After cumbersome calculations, it can be shown that if the appearing divergence terms, leading to surface integrals, vanish, then the constraint conservation conditions do not allow determining the functions $\lambda_1$ and do not lead to new constraints. All constraints are thus first-class constraints.
The equations of motion \( \dot{q} = [q, H] \) have the following form:

\[
\dot{X} = \frac{N w}{\sqrt{\hbar}} P + N^s D_s X + 4 \xi \rho \lambda_M X, \tag{28}
\]

\[
\dot{h}_{ik} = \frac{2 w N}{\xi \sqrt{\hbar}(X, X)} [\Pi_{ik} - \frac{1}{p} (\text{Sp} \Pi) h_{ik}] + D_i (\dot{N}_k) + 2 \lambda M h_{ik}, \tag{29}
\]

\[
\dot{P} = \frac{2 w N}{\xi \sqrt{\hbar}(X, X)^2} [\text{Sp}(\Pi \Pi) - \frac{1}{p} (\text{Sp} \Pi)^2] X
+ \frac{2 \xi \sqrt{\hbar}}{w} [R N - 2 \Delta N] X + \frac{\sqrt{\hbar}}{w} (N \Delta X + D^s N D_s X)
+ 2 \rho N \sqrt{\hbar} (\Delta (X, X) X + D_s (N^s P) - 4 \xi \rho \lambda_M P), \tag{30}
\]

\[
\dot{\Pi}_{ik} = - \frac{2 w N}{\xi \sqrt{\hbar}(X, X)} [\Pi^l_m \Pi^{mk} - \frac{1}{p} \text{Sp}(\Pi \Pi) h_{ik}]
+ \frac{N}{\sqrt{\hbar}} \{ \frac{w}{\xi} \text{Sp}(\Pi \Pi) - \frac{1}{p} (\text{Sp} \Pi)^2 \} + \frac{w}{2} P^2 \}
+ \frac{\xi \sqrt{\hbar}}{w} [N D^l D^k (X, X) - N R_{ik} + (X, X) D^l D^k N]
- h_{ik} (X, X) \Delta N + D_s (X, X) D^s N \}
+ \frac{N \sqrt{\hbar}}{2 w} (D^s X, D_k X) - \epsilon^{sk} + \frac{1}{2} P N (D^l) X
- 2 \lambda M \Pi_{ik} - \frac{N w}{2 \sqrt{\hbar}(X, X)} h_{ik} H_0, \tag{31}
\]

where

\[
\epsilon^{sk} = \sqrt{\hbar} D_s \left( \frac{1}{\sqrt{\hbar}} (N^s (\Pi^k)^s - N^s \Pi^{ik}) \right)
\]

In what follows, we will put the constant \( w \) equal to unity. If, instead of the indefinite coefficient \( \lambda_M \), we introduce the field \( \psi \), we should make the following substitution in the equations of motion:

\[
\lambda_M = \lambda_0 - \psi + \psi N^s, \tag{32}
\]

where \( \lambda_0 \) is an arbitrary function of the phase variables. This function may be chosen to be equal to zero, which simply re-defines the function \( \psi \). Then, using the substitutions

\[
h_{ik} = e^{-2 \psi} \tilde{h}_{ik}, \quad X = e^{-4 \xi \rho \psi} \tilde{X}, \quad N = e^{-\psi} \tilde{N}, \quad N_i = e^{-\psi} \tilde{N}_i, \quad P = e^{4 \xi \rho \psi} \tilde{P}, \quad \Pi_{ik} = e^{2 \psi} \tilde{\Pi}_{ik},
\]

one can exclude the field \( \psi \) from Eqs. (28)–(31) and pass over to the conformally invariant canonical variables \( \{ \tilde{q}_i, \tilde{p}^j \} \), which is equivalent to putting \( \lambda_M = 0 \) in the equations. However, for studying different gauge conditions, it is more convenient to preserve the arbitrariness in choosing the function \( \lambda \). To impose the canonical gauge, it is necessary to impose \( 2p + 4 \) supplementary conditions, according to the number of first-class constraints. We will consider as such constraints the class of additional conditions \( \Phi_G \) of the form

\[
N = \tilde{N}, \quad N_i = \tilde{N}_i, \quad \lambda = 0, \quad \chi^u = 0, \quad F = 0, \tag{34}
\]

where \( \chi^u \) are \( p + 1 \) functions of the phase variables \( h_{ik} \) and \( X^A \), while \( F \) is a function of the phase variables \( h_{ik}, X^A, P, \Pi^{ik} \). These functions are chosen in such a way that \( \det \{ \tilde{\Phi}, \tilde{\Phi} \} \neq 0 \), where \( \tilde{\Phi} = \{ \Phi, \Phi_G \} \). Let us denote \( H^v = \{ H_0, H^i, M \} \) and \( \chi_u = \{ \chi_u, F \} \), then, in the case under consideration,

\[
\det \{ \tilde{\Phi}, \tilde{\Phi} \} = (\det[H^v, \chi_u])^2 (|\lambda, p|)^2. \tag{35}
\]

A gauge, related to a choice of the function \( F \), violates the conformal symmetry and determines a “representative” from each class of conformally equivalent metrics. To reach comprehension of the different kinds of gauge conditions, let us consider some consequences of the constraint equations (27) and the equations of motion (28), (31). Let us define the conformally invariant tensor \( \Theta_{ik} \) which is traceless on the surface of the constraints:

\[
\Theta_{ik} \equiv 2 \xi \sqrt{\hbar} N \left[ D_i D_k (X, X) - \frac{h_{ik}}{p} \Delta (X, X) \right]
- \frac{h_{ik}}{p} \Delta N \left( X, X \right) \right.
+ 2 \xi \sqrt{\hbar} \left( X, X \right) R_{ik} - \frac{h_{ik}}{p} \Delta N \right]
+ N \sqrt{\hbar} \left( (D_i, D_k) X - \frac{h_{ik}}{p} (D_s X, D^s X) \right]
+ \frac{h_{ik}}{p} H_0. \tag{36}
\]

Then, using Eqs. (28) and (31) as well as the definitions (19), (20) and (17), it is easy to prove the following identity:

\[
\partial_k \left[ \Pi^l_k - \frac{\delta^l_k}{p} \Pi^l_0 \right] - D_s \left[ N^s (\Pi^l_k - \frac{\delta^l_k}{p} \Pi^l_0) \right]
= F^l_k + \frac{1}{2} \Theta^l_k + \frac{1}{2} N^s (H^s) h_{sk} - \frac{\delta^l_k}{p} N^s H_s, \tag{37}
\]

where

\[
F^l_k = \Pi^l_k D_k N^s - \Pi^l_k D_s N^l.
\]

The latter equation is equivalent to Eq. (31) provided the conditions (28)–(30) hold.

### 2 Induced gravity

Let us call the “partial embedding” condition the choice of the supplementary conditions \( \Phi_G \) obtained from the requirement

\[
h_{ik} = B(D_i X, D_k X), \quad l, k = 1, p, \tag{38}
\]

where \( B \) is a certain function. Thus the metric \( h_{ik} \), entering into the original action, is connected with the induced metric \( (\nabla_i X, \nabla_k X) \) by a conformal mapping.
In the string case, the general solution to the constraining equations has the form

\[ Bg_{\mu\nu} = (\nabla_{\mu} X, \nabla_{\nu} X), \quad \mu, \nu = 0, p, \]  

(39)

where \( B \) is an arbitrary function. This solution follows from (36) and (37) if one puts the momenta \( \Pi^i \) and the parameter \( \xi \) equal to zero. For an arbitrary dimension, Eq. (39) (for \( B = 1 \)) determines the condition of full embedding of the manifold \( \Pi \) into the space-time \( M \). This equation, written in the \( p + 1 \)-partition formalism, is equivalent to Eq. (38) and the equations

\[
\begin{align*}
(\dot{X}, \dot{X}) &= B(N_i N^s - N^2), \\
(D_t X, X) &= B N_t, \quad B \equiv (D_t X, D^t X)/p = 1.
\end{align*}
\]

(40)

Thus we can consider two kinds of solutions corresponding to “partial” or full embedding. In the first case, the validity of Eq. (38) (for \( B = 1 \)), as well as for strings, would permit one to interpret the fields \( X^A \) as the conventional coordinates of a \( d \)-object \( \Gamma \) in the space-time \( M \). In other words, this means that, from each class \( \Gamma \) of conformally equivalent manifolds, it is possible to choose at least one “representative” \( \Gamma_0 \), such that the functions \( X^A \) perform embedding of \( \Gamma_0 \) into the surrounding space \( M \). Here, conformally equivalent manifolds are understood as manifolds whose metrics are connected with the reparametrization invariance and the conformal invariance (41).

An invalidity of the relations (40), if (38) is valid, leads to some difficulties in the physical interpretation. If, by analogy with string theory, the space-time \( M \) is considered as physical space-time, then the coincidence between the original metric \( h_{ik} \) and the induced metric \((D_t X, D_k X)\) makes the theory transparent, making it possible to interpret the solutions \( X^A = X^A(t, \sigma) \) as an embedding of a \( p \)-dimensional object \( \Pi \) into the physical space-time \( M \). However, a non-coincidence between the “lapse” or “shift” functions of the original manifold \( \Pi \) and the \( p + 1 \)-dimensional “world history” manifold of the object \( \Gamma \) poses a question on the physical meaning of the original functions \( N \) and \( N_t \). To answer this question, one can try to invoke the ideas of the Kaluza-Klein theory. We, however, put forward a conjecture according to which it is possible, at the expense of a choice of the corresponding reference frame and conformal gauge, and maybe also the dimension \( D \), to achieve the validity of the conditions of full embedding of the whole manifold \( \Pi \) into the space-time \( M \). Eqs. (38) and (37) determine \( \frac{p(p+1)}{2} - 1 = k_p \) equations, while the number of arbitrary functions, determining the constraints \( \Phi_G \), is equal to \( 2p + 4 \). It is necessary to specify \( p + 2 \) functions \( N, N_t, \lambda_M \). Besides, according to the number of first-class conditions, we should impose \( p + 2 \) supplementary conditions. One can try to impose the latter relations by requiring the \( p + 2 \) conditions (16) to be valid. As follows from Eqs. (38) and (37), to fulfill the “partial embedding” conditions (38), the following equations should hold:

\[
\begin{align*}
\partial_t \left[ \Pi^i_k - \frac{\delta}{\delta \Pi^s} \right] - D_s \left[ N^s (\Pi^l_k - \frac{\delta}{\delta \Pi^s}) \right] - \frac{1}{2} F^i_k &= 0, \\
(\dot{X}, \dot{X}) &= B(N_i N^s - N^2), \\
(D_t X, X) &= B N_t, \quad B \equiv (D_t X, D^t X)/p = 1.
\end{align*}
\]

(41)

If Eq. (41) holds, then (38) follows from (36) and (37). The \( p + 2 \) functions \( N, N_t, \lambda_M \) should be chosen in such a way that, due to this choice, Eqs. (41) hold. The number of these functions for the dimensions \( p = 2 \) and \( p = 3 \) is 4 and 5, respectively, while the number of equations (41) (the number \( k_p \)) is 2 and 5, respectively. This simple counting of the degrees of freedom shows that, in the cases of interest \( p = 2 \) and \( p = 3 \), it is possible to choose a full embedding gauge.

In the most general case, it can be proved that, to satisfy the full embedding conditions (38) (for \( B = 1 \)), it is necessary that the following equations hold:

\[ \left[ \frac{n}{2} R + \rho(X, X) - 1 \right] \partial_\mu (X, X) = 0, \]

(42)

where \( \mu = 0, p \). The simplest proof can be performed with the aid of the generally covariant equations (4). They are equivalent to the Hamiltonian equations (28)–(30). Acting with the covariant derivative \( \nabla_\nu \) on the relation (41) and contracting different pairs of indices, we obtain the equations

\[
\begin{align*}
(\nabla_\nu X, \nabla_\sigma X) + (\nabla_\mu \nabla_\nu \nabla_\sigma X) &= \nabla_\nu B, \\
2(\nabla_{\mu}^n X, \nabla_{\nu} \gamma X) &= \nabla_{\gamma} B(p + 1),
\end{align*}
\]

(43, 44)

where \( \mu, \nu, \gamma = 0, p \). From the latter equations combined with (4), we obtain Eq. (42). In the derivation, we taking into account the covariant constancy of the metric tensor \( g_{\mu\nu} \) and that \( B = 1 \). Thus, as follows from (42), we can use two kinds of supplementary gauge conditions agreeing with the full embedding condition:

\[
\begin{align*}
(1) \quad F &\equiv (X, X) - C = 0, \quad C = \text{const}; \\
(2) \quad F &\equiv \xi (X, X) - \rho(X, X)^{p-1} = 0.
\end{align*}
\]

(45, 46)

As follows from (3) and (45), in the first case the constraint equations are similar to Einstein’s equations with the canonical energy-momentum tensor (6) and the effective gravitational constant \( G_\epsilon \). For the second case, one cannot exclude solutions in which \( G_\epsilon \) is variable and coordinate-dependent.

2.1 Model theory

The equations obtained, like Einstein’s equations, are strongly nonlinear and cannot be solved in a general form. However, the existing additional conformal symmetry simplifies the search for solutions of these equations. In this section, we simplify the equations obtained by restricting the class of metrics under consideration. Paying more attention to the dimension \( n = 4 \), let us
consider a model problem with the metric tensor $h_{lk}$ chosen in the form
\[ h_{lk} = b^2 \omega_{lk}, \quad (47) \]
where $b^2 = b^2(t, \sigma)$ is an arbitrary function and $\omega_{lk}$ is some fixed metric. It will be essential for what follows that the functions $\omega_{lk}$ are time-independent: $\dot{\omega}_{lk} = 0$. In the two-dimensional case, the following relations always hold:
\[ \ddot{\omega}_{lk} \dot{R}_{lk} - \frac{\dot{R}}{p} \omega_{lk} = 0. \quad (48) \]
The index $\omega$ means that the corresponding quantities are calculated for the metric coefficients $\omega_{lk}$. For instance, $\omega_{lk}$ may be chosen to be the metric of a constant-curvature space. Then,
\[ \ddot{\omega}_{lk} = k_0 (p - 1) \omega_{lk} = -8k_0 p \omega_{lk}, \quad (49) \]

$k_0 = \{0, 1, -1\}$ for surfaces of zero, positive and negative curvature, respectively.

2.2 The “conformal time” gauge

Let us impose the following conditions on the “lapse” and “shift” functions:
\[ N^2 = b^2, \quad N_i = 0. \quad (50) \]

After taking the trace of Eq. (29), it follows:
\[ \lambda_M = \frac{\dot{b}}{b} - \dot{\psi} \implies \Pi_{ik} = \frac{1}{p} (\dot{Sp} \Pi) h_{lk}. \quad (51) \]

With the constraint $M = 0$, we obtain
\[ \Pi_{lk} = -2\xi (P, X) h_{lk}. \quad (52) \]

Consider the gauge condition (45)
\[ (X, X) - C = 0, \quad C = \text{const} \neq 0. \quad (53) \]

Then, substituting $X = gb^{4p}$, from (28) and (30) we obtain the equations for the field $g^A$ components:
\[ \ddot{g} - \Delta g + 2\xi \dot{g} R = 2\rho \Lambda C^p - 1 b^2 g. \quad (54) \]

The constraint equations $H_0 = 0$ and $H_i = 0$ may be brought to the form
\[ \dot{X}, \dot{X} = -(4\xi p \lambda_M)^2 C + b^2 (2\xi CR + 2\Lambda C^p - B d), \quad (55) \]
\[ \dot{X}, D_i X = 16\xi^2 p CD_l (\lambda_M / N). \quad (56) \]

Eqs. (41) are reduced to the following ones:
\[ 8\xi p C \left( -D_t D_k \psi + D_l \psi D_k \psi - \frac{h_{lk}}{p} (-\Delta \psi + D_s \psi D^s \psi) \right) + C (\ddot{\omega}_{lk} - \frac{\omega_{lk}}{\dot{R}}) = 0. \quad (57) \]

Using (55), (56) and the consequence of Eq. (53)
\[ (\dot{X}, \dot{X}) = -(\ddot{X}, X), \]

we obtain an equation for finding the conformal factor:
\[ \dot{\psi} + 4\xi p \dot{\psi}^2 - \frac{1}{2p} \dot{\psi} ^2 (R + 2\Delta \psi) \]
\[ - 2D_s \psi D^s \psi - 2(p + 1)q = 0. \quad (58) \]

The scalar curvature may be expressed in terms of the function $\psi$:
\[ R = \dot{R} b^{-2} + 8\xi p [ -2\Delta \psi - (p - 2)\psi_s \psi^s ]. \quad (59) \]

If one requires that the first and the second relations of the condition (40) hold, this leads to the equations
\[ -8\xi p \dot{\psi}^2 + b^2 (R + 16\xi pq) = 0, \quad (60) \]
\[ D_t \psi = -\dot{\psi} D_t \psi, \quad (61) \]

where
\[ q = \frac{1}{4\xi C} \left( B + \Lambda C^p \frac{1}{4p} \right). \]

It can be shown that (58) is a differential consequence of (60) and (61). Then Eq. (58) can be brought to the form
\[ p \ddot{\psi} + b^2 (-2q + \Delta \psi - D_s \psi D^s \psi) = 0, \quad (62) \]

or, in terms of the metric $\omega$,
\[ p \ddot{\psi} + 2q b^2 + \Delta \psi + 8\xi p D_s \psi D^s \psi = 0, \quad (63) \]

Thus the function $\psi$ is found by solving Eqs. (60), (61) and (57).

Then the functions $X^A$ are determined by Eq. (54). If we write Eq. (51) directly in terms of the variables $X^A$, we obtain a linear equation with respect to $X^A$. This equation, with (58), may be written as
\[ \ddot{X} - 8\xi p \dot{\psi} \dot{X} - \Delta X + 8\xi p D_s XD^s \psi - \frac{p + 1}{C} b^2 X = 0. \quad (64) \]

Solutions of the latter equations should satisfy the remaining constraint equations which have the form
\[ \dot{X}, \dot{X} = -b^2, \quad (X, X) = C, \quad (65) \]
\[ \dot{X}, D_i X = 0. \quad (D_k X, D_i X) = h_{kl}. \quad (66) \]

Let us present some special solutions to the equations obtained for the case of conformally flat manifolds. Let us first consider a flat $p$-dimensional model:
\[ \ddot{R} = 0. \]
Let the metric matrix (ω_{ik}) be a unit matrix. A solution to Eqs. (59), (61) and (67) has the form
\[ b^{-2} = e^{2\psi} = \left[ \frac{u_0}{2} \left( (r^2 - t^2) + mt + n_i \sigma^i + l_0 \right) \right]^2, \] (67)
where
\[ r^2 = \sum_{i=1}^{p} (\sigma^i)^2, \]
u_0, m, n_i, l_0 being integration constants satisfying the condition \( m^2 + 2a_0l_0 - n_i n_i = 2q/p. \) Here \( t \) is the time coordinate, and the \( p \)-dimensional coordinates may be interpreted as the conventional Cartesian coordinates. It can be verified by a direct inspection that the functions \( f_0 = a_0 \psi \) and \( f_1 = a_1 D_1 \psi \) (where \( a_0, a_1 = \text{const} \)) are special solutions to Eq. (67). Using this, let us build solutions which also satisfy Eqs. (65)–(66). Probably, there can be many such solutions. But we will here seek solutions with a minimal set of fields \( X^A \). With this approach, we consider solutions which describe an embedding of the manifold \( \Pi \) into the 5-dimensional space \( M \). Calculations show that the solutions linear in the functions \( f_0 \) and \( f_1 \) satisfy Eqs. (62)–(67) with the following values of the constants:
\[ q = \frac{p}{2C} \implies C^p = \frac{(p-1)(p+3)}{8\Lambda}, \] (68)
\[ a_0^2 = a_1^2 = \frac{1}{u_0^2} = Cc_0, \] (69)
where \( c_0 = 2l_0/u_0 \). Without losing generality in the solution (67), we put \( m = n_i = 0 \), which simply corresponds to a parallel transport. Then the scale factor (67) is rewritten in the form
\[ b^2 = \frac{4a_0^2}{(r^2 - t^2 + c_0)^2}. \] (70)
\[ \text{(69)} \]
follows that all solutions split into two types:
\begin{enumerate}
  \item \( C > 0, \ c_0 > 0; \)
  \item \( C < 0, \ c_0 < 0. \)
\end{enumerate}

To embed the manifold \( \Pi \) into \( M \), it turns out to be convenient (see [18]) that, for the second type of solutions, the metric signature in \( M \) be \((-+, +, +, +, +, -)\). For the first type it should be \((-+, +, +, +, +, +)\). If we define \(|c_0| = g_0^2\), then the solution have the following form:
\begin{align*}
X^0 &= \sqrt{\frac{C}{t^2 - r^2}} \frac{2g_0}{t^2 - r^2 - g_0^2}, \\
X^i &= \sqrt{\frac{C}{t^2 - r^2}} \frac{2g_0}{t^2 - r^2 - g_0^2}, \\
X^4 &= \sqrt{\frac{C}{t^2 - r^2 + g_0^2}} \frac{2g_0}{t^2 - r^2 + g_0^2},
\end{align*}
(71)
for the first type and
\begin{align*}
X^0 &= \sqrt{\frac{C}{t^2 - r^2 + g_0^2}} \frac{2g_0}{t^2 - r^2 + g_0^2}, \\
X^i &= \sqrt{\frac{C}{t^2 - r^2 + g_0^2}} \frac{2g_0}{t^2 - r^2 + g_0^2}, \\
X^4 &= \sqrt{\frac{C}{t^2 - r^2 + g_0^2}} \frac{2g_0}{t^2 - r^2 + g_0^2},
\end{align*}
(72)
for the second type.

To study the global properties of the manifold, let us study its boundaries. For the second type of solutions, consider a range \( W_{\eta} \) specified by the following constraints on the coordinate variables:
\[-|\eta| \leq (t+r) \geq |\eta|, \quad -|\eta| \leq (t-r) \geq |\eta|. \] (73)
Let us introduce the new coordinate \((\eta, \chi)\) instead of \((t, r)\):
\[ t + r = g_0 \tanh \left( \frac{1}{2}(\eta + \chi) \right), \]
\[ t - r = g_0 \tanh \left( \frac{1}{2}(\eta - \chi) \right). \] (74)
In the new coordinates, using a conformal transformation, the metric may be brought to a form exactly coinciding with the open anti-de Sitter space metric
\[ ds^2 = a^2(\eta)[d\eta^2 - (d\chi)^2 - K(\chi)d\Omega^2], \] (75)
where
\[ a^2(\eta) = \frac{C}{\cosh^2(\eta + \eta_0)}, \quad K(\chi) = \sin^2 \chi, \]
and \( d\Omega^2 \) is the metric form of a \((p-1)\)-sphere of unit radius, expressed in spherical coordinates.

In a similar way, for the first type of solutions, the whole range \( W_{l1} : -\infty < t < +\infty, \ -\infty < r < +\infty \) may be mapped into a part of the compact (closed) de Sitter space.

To this end, we introduce new coordinates by the relations
\[ t + r = g_0 \tan \left( \frac{1}{2}(\eta + \eta_0 + \chi) \right), \]
\[ t - r = g_0 \tan \left( \frac{1}{2}(\eta + \eta_0 - \chi) \right), \] (76)
with \( \eta_0 = \text{const} \). The metric has the form (75), where
\[ a^2(\eta) = \frac{C}{\cos^2(\eta + \eta_0)}, \quad K(\chi) = \sin^2 \chi, \] (77)

The functions \( X^A \) are scalars with respect to the above coordinate transformations and may be rewritten in the new coordinates. For instance, for de Sitter space,
\[ X^0 = \sqrt{C} \tan(\eta + \eta_0), \quad X^a = \frac{\sqrt{C}}{\cos(\eta + \eta_0)} k^a, \] (78)
where \( k^a \) are the embedding functions of a \( p \)-dimensional sphere. For dimension \( p = 3 \), these functions are
\[ k^1 = \sin \chi \sin \theta \cos \phi, \quad k^2 = \sin \chi \sin \theta \sin \phi, \quad k^3 = \sin \chi \cos \theta, \quad k^4 = \cos \chi. \] (79)
In this case, the metric form is
\[ ds^2 = a^2(\eta)[d\eta^2 - d\chi^2 - \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)], \] (80)
which corresponds to the Robertson-Walker metric describing the Friedmann cosmological models.

The solutions for anti-de Sitter space are obtained from the above equations if one makes there the following substitution:
\[ \sin \chi \mapsto \sinh \chi, \quad \cos \chi \mapsto \cosh \chi, \quad \cos \eta \mapsto \cosh \eta. \]
2.3 Induced gravity as a result of a spontaneous violation of the conformal invariance

In addition to considering different gauge conditions, let us note that the field equations and constraint equations may also be studiedly the canonical gauge. To do so, using the substitution $X = b^{4/p} g$, without imposing the supplementary condition (53), one can entirely exclude the field $\lambda_M$ in the Hamiltonian equations if the condition (50) is valid. In this case, the following equations are obtained:

$$\ddot{g} = \Delta g + 2\xi g \dot{R} + 2p\Lambda g Z^{p-1}, \quad Z \equiv (g, g), \quad (81)$$

$$\frac{1}{2\xi}(Dg, Dg) - B_g \omega_{lk} + D_l D_k Z - \frac{1}{p} \Delta Z \omega_{lk} - \left[ \dot{\omega}_{lk} - \frac{1}{p} \dot{\omega} \omega_{lk} \right] Z = 0, \quad (82)$$

$$(\dot{g}, \dot{g}) + B_g \rho + 4\xi \Delta Z - 2\xi Z \dot{R} - 2\Lambda Z^{p} = 0,$$

$$B_g \equiv \frac{1}{p} \omega_{lk}(Dg, D_k g), \quad (83)$$

$$D_l \dot{Z} + \frac{1}{2\xi}(\dot{g}, D_l g) = 0. \quad (84)$$

The latter two equations are equivalent to the constraint equations $H_0, H_1$. As their consequence, we obtain an equation for the function $Z$:

$$\ddot{Z} = (1 - 8\xi) \Delta Z + 8\xi \dot{Z} \dot{R} - \frac{1}{\xi} \Lambda Z^{p} - 4dB_g, \quad (85)$$

Eq. (11) has the form

$$D_l D_k Z - \frac{1}{p} \Delta Z \omega_{lk} - \left[ \dot{\omega}_{lk} - \frac{1}{p} \dot{\omega} \omega_{lk} \right] Z = 0, \quad (86)$$

We will seek special solutions to Eqs. (81)–(85) for the dimension $p = 3$, when the metric $\omega_{lk}$ is determined by the 3-dimensional part of the linear element of an open-type Robertson-Walker space-time. We seek solutions in the form

$$g_0 = u_0(\eta), \quad g^a = u(\eta)k^a(\sigma), \quad a = 1, 2, 3, 4, \quad (87)$$

and, doing so, we do not require that the full embedding conditions (10) should hold. Then, for the functions $u_0(\eta)$ and $u(\eta)$ we obtain

$$\dot{u}^2 + 4k u^2 - 8k\Lambda \int (u^2 - ku_0^2)udu - 2H = 0, \quad (88)$$

$$H = \text{const}, \quad (89)$$

The last two equation with respect to the variables $u_0(\eta)$ and $u(\eta))$ may be considered as a dynamic system with the potential energy

$$U(u, u_0) = -k\Lambda u^2(u^2 + 2u_0^2) + \Lambda u_0^4 + k(2u^2 - u_0^2)/2$$

and zero total energy. Integrating by parts and summing, we can obtain

$$\dot{u}^2 + \dot{u}_0^2 + 2U = 0.$$

A further study shows that the previously found solution, describing an open de Sitter space, is a stable exceptional solution to Eqs. (88)–(89). In the present formulation, the following solution corresponds to the one obtained above:

$$u = \frac{r^2}{(\cosh \eta)^2}, \quad u_0 = \frac{r^2 \tanh \eta}{\cosh \eta}, \quad r^2 = \sqrt{\frac{3}{2|\Lambda|}}, \quad (90)$$

Using the terminology of the qualitative theory of differential equations, the singular point $u = 0$, $u_0 = 0$ is unstable. There are no other static points. Meanwhile, the solution (90), being a separatrix in the phase space of the variables $u_0(\eta)$ and $u(\eta)$, minimizes the total energy.

From this point of view, it is of interest to invoke the Higgs mechanism to obtain the constraints (53). The fields $X^A$, being coordinates of the space $M$, may play the role of Higgs’ fields in Grand Unification models. On the other hand, from the viewpoint of the Hamiltonian formalism considered above, solutions with a broken symmetry may be treated as a particular choice of the gauge.

2.4 The hierarchy of coupling constants

As has been already noted above, we here obtain equations similar to the Einstein equations with an effective gravitational constant, see [9]. Indeed, in case $(X, X) = C$ and if

$$g_{\mu\nu} = (\nabla_\mu X, \nabla_\nu X), \quad \mu, \nu = \sigma, p, \quad (91)$$

Eqs. (8) take the form

$$G_{\alpha\beta} = 8\pi G_T X_\alpha X_\beta + \Lambda e_\beta, \quad (92)$$

where $G_T$ is given by Eq. (5), while the cosmological constant is

$$\Lambda_0 = \frac{1}{2\xi C}(-1 + \Lambda C^2). \quad (93)$$

From the solution (65) and (83), we find that

$$\Lambda = \frac{3}{2C^2}, \quad \Lambda_0 = \frac{3}{C}. \quad (94)$$

In a closed model, the constant $C$ satisfies the equation

$$- (X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = C. \quad (95)$$

Then $\sqrt{C}$ characterizes the size of the observed part of the Universe. In the solution (77) (for $\eta_0 = \pi/2$), we pass over to the proper time $t$ and obtain:

$$a(t) = \frac{\sqrt{C} \cosh(t/\sqrt{C})}{\sqrt{C}}, \quad H \equiv \frac{\dot{a}}{a} = \sqrt{C} \tanh(t/\sqrt{C}). \quad (96)$$
Suppose that the Hubble “constant” \( H \sim (3 \cdot 10^{17})^{-1}c^{-1} \) (in the Planck units, \( \hbar = 1 \) and \( c = 1 \)) and that our epoch corresponds to the time \( t \approx \sqrt{C} \), then we obtain the value of \( C \): \( \sqrt{C} \approx 7.2 \cdot 10^{47} \text{ cm} \sim 10^{28} \text{ cm} \). Substituting this value into (94), we find \( \Lambda_e \sim 10^{-56} \text{ cm}^{-2} \), or, the same in energy units, \( \Lambda_e \sim 10^{-46} \text{ GeV}^4 \). This result confirms the existence of a nonzero cosmological constant \( \Lambda \), which is also in agreement with the observational data, see, e.g., Ref. [19].

Equating the expression (1) to \( 1/M_p^2 \sim 10^{-66} \text{ cm}^2 \), we find that \( w_0 \sim 4 \cdot 10^{-10} \text{ cm}^4 \). The parameter \( w_0 \) also corresponds to distances of the order \( l_w = \sqrt{w_0} \sim 0.05 \text{ mm} \). To explain the nature of the emerging scale, one can invoke the Randall-Sundrum conjecture [6], where the existence of extra dimensions \( (n > 4) \) is supposed, with a sufficiently small size \( (l < 0.2 \text{ mm}) \) for being in agreement with the experimental data.

In conclusion, let us note that if we consider the action obtained from (11) by adding to it the Einstein term \((1/G)R\), which violates the conformal invariance of the equations, then there emerges the effective gravitational “constant” \( G_e = wG(w + 2G(X,X))^{-1} \). As is shown in Ref. [14], this leads to an instability of cosmological solutions for \( G_e \rightarrow 0 \). This result is one of the arguments in favour of consideration of an initially conformally invariant theory of gravity; this invariance will then be probably violated due to quantum effects.

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