One dimensional Newton’s equation with variable mass

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We revisit Newton’s equation of motion in one dimension when the moving particle has a variable mass \( m(x,t) \) depending both on position \( x \) and time \( t \). Geometrically the mass function is identified with one of the metric function in a 1 + 1-dimensional spacetime. As a reflection of the equivalence principle geodesics equation gives the Newton’s law of motion leaving the right hand side to be supplemented by the external forces. The resulting equation involves the speed of light so that our equation of motion addresses a wider scope than the customary classical mechanics. In the limit of infinite light speed which amounts to instantaneous interaction we recover the classical results.

I. INTRODUCTION

Numerous systems show that mass, which appears as a constant in the original Newton equation need not be constant. A rolling snowball, launched rocket, motion in a resistive medium etc., are just a few such examples that appear in classical physics. Recently, there has been an increasing interest in studying the position dependent mass (PDM) in classical systems at more sophisticated level \[1-6\]. Beside PDM some examples just cited within the context of non-constant masses involve cases of time dependent mass (TDM) as well. Herein, we wish to address both problems from a unified and geometrical point of view.

The invariance group of classical mechanics, the Galilei group is no more an invariance group in the presence of PDM. This reflects the fact that in the frames in relative motion the accelerations are no more equal. As mass depends on position such a situation creates different reaction forces to different observers. Under this condition Galilei group in its standard form doesn’t provide a symmetry any more. Clearly PDM gives rise to velocity dependent forces in the equation of motion. One such term familiar from the mechanics text-books is the Reyleigh’s dissipation function \[7\].

More generally the mass can be considered as a function of both position and time and the most general equation of motion can be derived from the Euler-Lagrange (EL) formalism. In the absence of an external force EL method serves well to derive the equation of motion. The physical meaning of Hamiltonian, however, is no more considered as an energy, or at least not a conserved energy and the Hamiltonian formalism is not well defined. In this paper we derive, more generally the one-dimensional Newton’s equation of motion from a geometrical principle. Namely, we consider a 1 + 1-dimensional (i.e. one space, one time) line element of the form \( ds^2 = c^2 dt^2 - m(x,t) dx^2 \), where \( m(x,t) \) will be interpreted as the mass function depending both on time \( t \) and position \( x \) and \( m_0 = m(x=0,t=0) = \text{const.} \). Whenever \( m(x,t) \) depends only on position our 1+1-spacetime reduces automatically to a flat space \( ds^2 = c^2 dt^2 - dX^2 \), upon a scaling of the coordinate \( x \). This yields the constant mass \( m_0 \) as an integration constant. In general, for \( m(x,t) \) we face a curved space whose geodesics equation will be identical with the one-dimensional Newton’s equation of motion. For this purpose the affinely parametrized geodesics equation with the proper time \( \tau \) (i.e. \( d\tau^2 = \frac{1}{c^2} ds^2 \)) must be transformed into the ordinary time \( t \).

The PDM geodesics equation takes the form \( m(x) \frac{d^2x}{d\tau^2} + \frac{1}{2} \frac{dm}{dx} \left( \frac{dx}{d\tau} \right)^2 = 0 \), which is well known from other approaches. By this formalism we derive the Newton’s equation from a covariant, geometrical approach in which physical mass emerges as a metric component of a curved space geometry. Our approach is in conform with the equivalence principle of general relativity which states the local equality of space curvature with the physical force acceleration. No doubt in the limit of a constant mass, i.e. \( m = m_0 = \text{constant} \), we recover the well known limit of \( m_0 \frac{d^2x}{d\tau^2} = 0 \), in the absence of an external force. In the presence of external forces of any kind what one should do is just to modify our geodesics Lagrangian with the supplementary terms.

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II. TIME AND POSITION DEPENDENT MASS

Let’s consider a one-dimensional classical system of mass \( M \) which is moving in positive \( x \)–direction with a speed of \( v \). Also we assume that there exists a time dependent force \( F(t) = f_0 \delta(t) \) (\( \delta(t) \) is the Dirac delta function) between two parts of the original system with masses \( M - m \) and \( m \). Here \( f_0 \) is a constant to be found. The task of the force \( F(t) \) is to divide the original system into two parts with masses \( m \) and \( M - m \) at \( t = 0 \) and the mass \( m \) comes to rest for \( t > 0 \) with respect to an observer at rest. The reaction force i.e. \( -F(t) \), acts on the other part of the system to make its speed increasing. Here we only accept the Newton’s second law for constant mass particle which for mass \( m \) reads,

\[
f_0 \delta(t) = m \frac{dv}{dt}
\]

and after integration becomes

\[
f_0 \int_{\epsilon}^{0} \delta(t) dt = m \int_{v}^{0} dv.
\]

We note that the entire process takes place at \( t = 0 \) and therefore \( \epsilon \) is a positive constant. The latter equation gives

\[
f_0 = -mv.
\]

A similar equation is also applicable for the second part of mass \( M - m \),

\[
-f_0 \delta(t) = (M - m) \frac{dv}{dt}
\]

which upon integration over the same time-interval yields

\[
-f_0 \int_{-\epsilon}^{\epsilon} \delta(t) dt = (M - m) \int_{v}^{v'} dv.
\]

Here \( v' \) is the speed of the mass \( M - m \) after \( t = 0 \) while \( v \) is its speed before \( t = 0 \). This equation gives

\[
-f_0 = (M - m) (v' - v)
\]

which eventually yields

\[
\left( \frac{M}{M - m} \right) v = v'.
\]

Eq. (7) is nothing but

\[
Mv = (M - m) v'
\]

which by recalling that the mass \( m \) is at rest shortly after \( t = 0 \), this is the linear momentum conservation. Having the velocities before and after \( t = 0 \), one can show that the total kinetic energy is not conserved i.e.,

\[
T_f - T_i = \left( \frac{m}{M - m} \right) \frac{1}{2} M v^2 \neq 0.
\]

Now let’s look at the problem from another perspective where due to a position dependent internal force \( F(x) = f_0 \delta(x) \), the same system (given above) is divided into two parts with masses \( M - m \) and \( m \) at \( x = 0 \). The function of this new force \( F(x) \) is to separate the mass \( m \) from the original system and bring it to rest wrt an observer at rest. The reaction force acts on the rest of the mass i.e. \( M - m \) to increase its speed to \( v' \) in the same direction. The equation of motion for the mass \( m \) is written as

\[
f_0 \delta(x) = m \frac{dv}{dt}
\]

and therefore

\[
f_0 \int_{-\epsilon}^{\epsilon} \delta(x) dx = m \int_{v}^{0} v dv,
\]
which admits
\[ f_0 = -\frac{1}{2}mv^2. \] (12)

A similar equation for \( M - m \) yields
\[ -f_0 \delta (x) = (M - m) \frac{dv}{dt} \] (13)
or equivalently
\[ -f_0 \int_{-\epsilon}^{\epsilon} \delta (x) \, dx = (M - m) \int_{v'}^{v} v \, dv. \] (14)
This equation yields
\[ v'^2 = \left( \frac{M}{M - m} \right) v^2. \] (15)

The latter equation is just the conservation of the kinetic energy while, unlike the time-dependent mass system, the linear momentum is not conserved. We note that having the mass \( m \) at rest after switching on the internal force \( F(x) \) is just for simplicity and in the general approach where \( m \) may get a non-zero velocity, the results remain the same.

In conclusion, in the absence of any external force term, in the case of time dependent mass classical system, the linear momentum is conserved but kinetic energy not. In contrast, if the mass of the classical system is position-dependent then the kinetic energy is conserved while the linear momentum not.

What we have shown above clearly suggests that in the absence of any external force term for a general time dependent mass particle one must use
\[ \frac{dp}{dt} = 0 \] (16)
in which \( p = M(t) \, v \), while for a general position-dependent mass particle
\[ \frac{dp}{dt} \neq 0 \] (17)
with \( p = M(x) \, v \) and instead one must use
\[ \frac{dT}{dt} = 0 \] (18)
or equivalently
\[ M(x) \ddot{x} + \frac{1}{2} \frac{M'(x)}{M(x)} \dot{x}^2 = 0. \] (19)

Here a prime "\( ' \)" denotes derivative wrt position \( x \). Eq. (19) can also be found by using the Hamiltonian method or Lagrangian formalism in which the Lagrangian is given by
\[ L = \frac{1}{2} M(x) \dot{x}^2 \] (20)
and the Euler-Lagrange equation reads
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}. \] (21)

Our simple examples given above can be easily extended into the general case with external forces. In such case if the mass is time dependent one should start with
\[ \frac{dp}{dt} = F_{ext} \] (22)
and for the particle with PDM the correct approach is introducing the Lagrangian $L = \frac{1}{2} M(x) \dot{x}^2 - V_{\text{ext}}(x)$ and then use the EL equation. We also add here that Eq. (22) is found if the departed mass comes to rest. This means that if the mass $m$ does not come to rest after the separation, we must consider $p$ the total linear momentum of the system. This is the method which is used to find the equation of a rocket.

Another remarkable point is that for a system with PDM usually we don’t talk about the departed mass, i.e., we only study the original system whose mass is decreasing or increasing. This implicitly implies that the mass difference (apart mass) is at rest and its linear momentum and kinetic energy both are zero.

Finally we refer to Ref. [6] where Eq. (3) (of [6]) has been found by considering the conservation of the linear momentum of a PDM particle in absence of an external force. As we shown above the correct equation is Eq. (19) and therefore by comparison, the proper form of the PDM function in Ref. [6] is given by

$$m(x) = \frac{m_0}{1 + \xi x^2},$$

in which $m_0$ has the meaning of a constant mass for $\xi = 0$.

III. A GEOMETRIC MODEL OF PDM

In this section we start with a $1 + 1$-dimensional curved spacetime in the form of

$$c^2 d\tau^2 = ds^2 = c^2 dt^2 - \frac{m(x,t)}{m_0} dx^2$$

or equivalently $d\tau^2 = dt^2 - \frac{m(x,t)}{\alpha_0} dx^2$ where $\alpha_0 = m_0 c^2 = \text{const}$. Here $m(x,t)$ is an arbitrary function of time and space while $m_0$ is a constant representing $m_0 = m(x = 0 = t)$ with identical dimension as $m(x,t)$. The geodesic equation of a free particle moving in this spacetime is given by

$$\frac{d^2 x_i}{d\tau^2} + \Gamma_{ab}^{i} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} = 0$$

in which $\Gamma_{ab}^{i}$ is the Christoffel symbol, $\tau$ is the proper time, $c$ is the speed of light and $i,a$ and $b$ run from 0 to 1. The explicit equations of motions are then given by

$$\frac{d^2 x}{d\tau^2} + \frac{m'}{2m} \left( \frac{dx}{d\tau} \right)^2 + \frac{\dot{m}}{m} \left( \frac{dt}{d\tau} \right) \left( \frac{dx}{d\tau} \right) = 0$$

and

$$\frac{d^2 t}{d\tau^2} + \frac{\dot{m}}{2\alpha_0} \left( \frac{dx}{d\tau} \right)^2 = 0.$$  

Eliminating proper time, one finds a single equation

$$m \frac{d^2 x}{dt^2} + \dot{m} \frac{dx}{dt} + \left( \frac{m'}{2} - \frac{m_0}{2\alpha_0} \right) \left( \frac{dx}{dt} \right)^2 = 0,$$

or equivalently

$$\frac{d}{dt} \left( m \dot{x} \right) = \frac{1}{2} \dot{x}^2 \left( m' + \frac{m_0}{\alpha_0} \right)$$

which follows directly from the variational principle $\delta \int \sqrt{1 - \frac{m(x,t)}{\alpha_0} \dot{x}^2} dt = 0$. First of all let’s consider $m_0$ to be a constant so that

$$m_0 \frac{d^2 x}{dt^2} = 0$$

which is the Newton’s second law for a free particle with mass $m$. Then we consider a time dependent mass which reads

$$m \frac{d^2 x}{dt^2} + \dot{m} \frac{dx}{dt} = \frac{m_0}{\alpha_0} \left( \frac{dx}{dt} \right)^3 = 0.$$
This equation in Newtonian limit (i.e. $c \rightarrow \infty$) becomes

$$m \frac{d^2 x}{dt^2} + m \frac{d x}{dt} = 0 \quad (32)$$

or equivalently

$$\frac{d (m \dot{x})}{dt} = 0 \quad (33)$$

which is the correct Newton’s second law for a free particle with time dependent mass. In these two cases one also observes that the usual definition of linear momentum i.e., $p = m \dot{x}$, implies that the linear momentum is conserved. As a matter of fact (32) admits a first integral given by

$$\frac{1}{p^2} - \frac{1}{p_0^2} = \frac{1}{\alpha_0} \left( \frac{1}{m} - \frac{1}{m_0} \right) \quad (34)$$

where $p_0$ and $m_0$ are constants of momentum and mass respectively. This states openly that both for $m = m_0$ and $c \rightarrow \infty$ limits we obtain $p = p_0$, i.e. conservation of momentum. Finally we consider the case in which the function $m$ is only a function of position which admits

$$m \frac{d^2 x}{dt^2} + m' \left( \frac{dx}{dt} \right)^2 = 0 \quad (35)$$

This is the correct equation of motion for a free particle whose mass is position dependent. We note that this equation can be written as

$$\frac{d (m \dot{x})}{dt} = \frac{m'}{2} \left( \frac{dx}{dt} \right)^2 \quad (36)$$

This clearly shows that the usual linear momentum is not conserved due to a force $\frac{m'}{2} \left( \frac{dx}{dt} \right)^2$ which has emerged from the PDM nature of the particle. In other words, the PDM changes the geometry of spacetime in such a way that the particle experiences a new geometrical force $F_G = \frac{m'}{2} \left( \frac{dx}{dt} \right)^2$. Furthermore, once the mass is only position dependent, the spacetime is not any more curved and by a simple transformation

$$\sqrt{\frac{m}{m_0}} dx = dX \quad (37)$$

one gets

$$ds^2 = c^2 dt^2 - dX^2 \quad (38)$$

The geodesic equation of the free particle in this transformed spacetime simply reads

$$\frac{d^2 X}{dt^2} = 0 \quad (39)$$

or equivalently

$$\frac{d}{dt} \sqrt{\frac{m(x)}{m_0}} \frac{dx}{dt} = 0 \quad (40)$$

This in turn implies

$$\frac{1}{2} m(x) \left( \frac{dx}{dt} \right)^2 = \text{const.} \quad (41)$$

which amounts to the conservation of kinetic energy. We note that

$$X = \int^x \sqrt{\frac{m(y)}{m_0}} dy \quad (42)$$
which for the case of the mass given in (23) one finds

\[ X = \frac{1}{\sqrt{\xi}} \sinh^{-1} \left( \sqrt{\xi} x \right). \]  

(43)

One easily observes that for \( \xi \to 0 \), one recovers, \( ds^2 = c^2 dt^2 - dx^2 \), which yields the standard Newton’s equation of motion. Let’s add that having external interaction would only lead to some external forces to the right side which is a trivial extension of the model. Finally we comment on the canonical formalism of our geometrical model. Fixing the square-root Lagrangian with the correct dimension of energy we have

\[ L(x, \dot{x}) = \sqrt{\alpha_0 \sqrt{1 - \frac{p^2}{m \alpha_0}}}. \]  

(44)

Canonical momentum is defined by \( p = \frac{\partial L}{\partial \dot{x}} \) and the usual Hamiltonian \( H = p \dot{x} - L \) gives

\[ H(x, p) = \alpha_0 \sqrt{1 - \frac{p^2}{m \alpha_0}}. \]  

(45)

It can be checked that \( H(x, p) \) and \( L(x, \dot{x}) \) satisfy the constraint

\[ H(x, p) L(x, \dot{x}) = 1 \]  

(46)

and the usual Hamiltonian equation, \( \dot{p} = \frac{\partial H}{\partial x} \), doesn’t give the correct equation of motion. This means that in such a formalism the standard Hamilton equation, without further modifications, fail to work while EL equation always works.

IV. CONCLUSION

Einstein’s general relativity is a geometric description of the universe and is known to extend Newton’s gravitational theory. Existence of the equivalence principle at the very heart of general relativity transcends physical theories from gravitation to the realm of all forces of nature. From this token the recently fashionable PDM / TDM in various physics theories is embedded into the geometric description of flat / curved spacetime. In such a model mass can naturally be identified with the metric function as the coefficient of spacetime element and as a result our model lies in between special relativity and classical mechanics. Geodesic equation is shown to yield the Newton’s equation of motion with variable mass in 1—space dimension correctly. When the mass is variable both in \( x \) and \( t \) the resulting geodesic equation (or Newton’s equation of motion) becomes rather involved since it gives velocity dependent forces. These forces are not put artificially but arise naturally as a result of our geometric description. The equation obtained from the variational principle are neither Galilean nor Lorentz invariant. Violation of these symmetries is all due to the PDM and TDM appearing in the Lagrangian. It has been shown that PDM gives conserved kinetic energy. As expected, taking mass as \( m(x, t) \) leaves us with no conserved quantities and a highly non-linear equation of motion. Undoubtedly, extending, this 1—dimensional toy model to 3—dimensional Newton’s equations with the effect of rotation taken into account requires much more effort.

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