Consistency of the Adaptive Multiple Importance Sampling

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Abstract

Among Monte Carlo techniques, the importance sampling requires fine tuning of a proposal distribution, which is now fluently resolved through iterative schemes. The Adaptive Multiple Importance Sampling (AMIS) of Cornuet et al. (2012) provides a significant improvement in stability and effective sample size due to the introduction of a recycling procedure. However, the consistency of the AMIS estimator remains largely open. In this work we prove the convergence of the AMIS, at a cost of a slight modification in the learning process. Contrary to Douc et al. (2007a), results are obtained here in the asymptotic regime where the number of iterations is going to infinity while the number of drawings per iteration is a fixed, but growing sequence of integers. Hence some of the results shed new light on adaptive population Monte Carlo algorithms in that last regime.

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1 Introduction

The aim of Monte Carlo techniques is to approximate a target distribution $\Pi(\cdot)$ on some space $\mathcal{X}$ with a weighted sample. Namely, they output a system of particles
\( X_e \in \mathcal{X} \) (indexed by \( e \in E \)), with their weights \( \omega_e \in [0; \infty) \). Then, the discrete measure
\[
\hat{\Pi}_E(\cdot) = \sum_{e \in E} \omega_e \delta_{X_e}(\cdot)
\]
serves as an approximation of the target \( \Pi(\cdot) \). And the Monte Carlo scheme is said to be consistent if, for a large class of functions \( \psi : \mathcal{X} \to \mathbb{R} \), the sum
\[
\int \psi(x)\hat{\Pi}_E(dx) = \sum_e \omega_e \psi(x_e)
\]
tends to the integral \( \Pi(\psi) = \int \psi(x)\Pi(dx) \) when the sample size (i.e., the cardinality of the index set \( E \)) tends to infinity. Let us assume that the target \( \Pi \) has a density \( \pi(\cdot) \) with respect to a reference measure \( dx \). Among the Monte Carlo methods, Importance Sampling (see Hesterberg, 1988, 1995; Ripley, 1987) consists in drawing \( X_e \) from a proposal probability \( Q \) with density \( q(\cdot) \) and then in weighting this draw with \( \omega_e = \pi(X_e)/q(X_e) \) to take into account the discordance between the sampling distribution and the target. The precision of the importance sampling approximation becomes very unsatisfactory when the importance weight \( \omega_e \) has high variance, see Owen and Zhou (2000). Such problematic cases arise when the proposal distribution \( Q \) is not well adapted to the target \( \Pi \) or when the space \( \mathcal{X} \) is of high dimensionality. Practically, the fine tuning of the proposal measure for a given target is difficult. However, there are adaptive techniques to learn the proposal probability sequentially (see Liu, 2008; Rubinstein and Kroese, 2004; Pennanen and Koivu, 2004). Among these adaptive methods, the PMC scheme of Cappé et al. (2004), generalised in the D-kernel paradigm (see Douc et al., 2007a,b; Cappé et al., 2008), aims at building a mixture distribution by minimising some optimality criterion (Kullback-Leibler, variance, ...). Given a parametric family of distributions \( Q(\theta) \) indexed by \( \theta \in \Theta \), Cappé et al. (2008) seek the optimal proposal \( Q(\theta^*) \) for a given target \( \Pi \) by estimating \( \theta^* \) sequentially on successive samples.

In many real problems where computing \( \pi(X_e) \) (hence the importance weight) is time consuming, recycling the successive samples generated during the learning process is essential. To this end, Cornuet et al. (2012) introduce the Adaptive Multiple Importance Sampling (AMIS), combining multiple importance sampling methods and adaptive techniques. The AMIS is a sequential scheme in the same vein as Cappé et al. (2008). During the learning process, the AMIS tries successive proposal distributions, say \( Q(\hat{\theta}_1), \ldots, Q(\hat{\theta}_T) \). Each stage of the iterative process estimates a better proposal \( Q(\hat{\theta}_{t+1}) \), by minimising a criterion such as, for instance, the Kullback-Leibler divergence between \( Q(\theta) \) and \( \Pi \). But the novelty of the AMIS is the following recycling procedure of all past simulations. At iteration \( t \), the AMIS has already produced \( t \) samples:
\[ X_1^1, \ldots, X_{N_1}^1 \sim Q(\widehat{\theta}_1), \]
\[ X_1^2, \ldots, X_{N_2}^2 \sim Q(\widehat{\theta}_2), \]
\[ \vdots \]
\[ X_1^t, \ldots, X_{N_t}^t \sim Q(\widehat{\theta}_t) \]

with respective sizes \( N_1, N_2, \ldots, N_t \). Then the scheme derives a new parameter \( \widehat{\theta}_{t+1} \) from all those past simulations. To that purpose, the weight of \( X_i^k \) \((k \leq t, i \leq N_k)\) is updated with

\[
\pi(X_i^k)/\left[ \sum_{\ell=1}^{t} \frac{N_\ell}{\Omega_\ell} q(X_i^k, \widehat{\theta}_\ell) \right],
\]

where \( \widehat{\theta}_1, \ldots, \widehat{\theta}_t \) are the parameters generated throughout the \( t \) past iterations, \( x \mapsto q(x, \theta) \) is the density of \( Q(\theta) \) with respect to the reference measure \( dx \) and \( \Omega_\ell = N_1 + N_2 + \cdots + N_\ell \) is the total number of past particles. The importance weight (1) is inspired by the techniques of Veach and Guibas (1995). Owen and Zhou (2000) popularise those techniques to merge several independent importance samples and even propose a more refined and stabilising alternative, named deterministic multiple mixture. On various numerical experiments where the target is the posterior distribution of some population genetics data sets, Cornuet et al. (2012) and Sirén et al. (2010) show considerable improvements of the AMIS in Effective Sampling Size (denoted further ESS, see Liu, 2008, chapter 2). In such settings where calculating \( \pi(X_i^k) \) is drastically time consuming, a recycling process makes sense. However, no proof of convergence had yet been provided in Cornuet et al. (2012). It is worth noting that the weight (1) introduces long memory dependence between the samples, and even a bias which was not controlled by theoretical results. The main purpose of this paper is to fill in this gap, and to prove the consistency of the algorithm at the cost of a slight modification in the adaptive process. We suggest learning the new parameter \( \theta_{t+1} \) on the last sample \( X_1^t, \ldots, X_{N_t}^t \) weighted with the classical \( \pi(X_i^k)/q(X_i^k, \widehat{\theta}_\ell) \) for all \( i = 1, \ldots, N_t \).

The only recycling procedure is in the final output that merges all the previously generated samples using (1).

In Douc et al. (2007a) for instance, the consistency of the adaptive population Monte Carlo schemes is proven assuming that the number of iterations, say \( T \), is fixed and that the number of simulations within each iteration, \( N = N_1 = N_2 = \ldots = N_T \).
\( \cdots = N_T \), goes to infinity. We decided to adopt a more realistic asymptotic setting in this paper. Contrary to these last results, the convergence of Theorem 4 holds when \( N_1, \ldots, N_T \) is a growing, but fixed sequence and \( T \) goes to infinity. Hence the proofs of Theorem 2 provide new insights on adaptive PMC in that last asymptotic regime. The convergence of \( \hat{\theta}_t \) to the target \( \theta^* \) relies on a weak law of large numbers on triangular arrays (see Chopin, 2004; Douc and Moulines, 2008; Cappé et al., 2005, Chapter 9), and a clever application of the Chebyshev inequality to obtain the almost sure consistency. The consistency of the final merging with weights given by (1) is not a straightforward consequence of asymptotic theorems. Its proof requires the introduction of a new weighting

\[
\pi(X^k_t) / q(X^k_t, \theta^*)
\]

that is more simple to study, although biased and non explicitly computable (because \( \theta^* \) is unknown). Under the set of assumptions given below, this last weighting scheme is consistent (see Proposition 11) and is comparable to the actual weighting given by (1), which yields the consistency proven in Theorem 4.

The paper is organised as follows. The modification of the original AMIS is detailed in Section 2. The main results are in Section 3. The proofs are in Sections 4 and 5. At last, in Section 6, we compare the performance of our new algorithm against the original AMIS and a scheme with a naive recycling strategy.

## 2 Modifications of the AMIS scheme

When compared with the recursive algorithm of Cappé et al. (2008), the novelty in the AMIS of Cornuet et al. (2012) lies in the recycling process at each iteration and in the final system which includes all particles generated during the \( T \) iterations of the adaptive process. Hence, the AMIS estimation of the integral \( \Pi(\psi) = \int \psi(x) \Pi(dx) \) is

\[
\hat{\Pi}_{AMIS}^{T}(\psi) = \frac{1}{\Omega_T} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \left[ \frac{\pi(X^t_i)}{\Omega_T^{-1} \sum_{k=1}^{T} N_k q(X^t_i, \hat{\theta}_k)} \right] \psi(X^t_i),
\]

based on the weights given in (1).

The scheme we propose relies on the same ideas: we fit the proposals assuming that the optimal proposal is at \( \theta = \theta^* \), whatever the criterion we believe in
Algorithm 1 Modified AMIS

Require: an initial parameter $\theta_1$ and increasing sample sizes $N_1, \ldots, N_T$.

1: for $t = 1 \rightarrow T$ do
2:   for $i = 1 \rightarrow N_t$ do
3:     draw $X_i^t$ from $Q(\theta_i)$
4:     compute $\omega_i^t = \pi(X_i^t)/q(X_i^t, \theta_i)$.
5:   end for
6:   compute $\theta_{t+1} = N_t^{-1} \sum_{i=1}^{N_t} \omega_i^t h(X_i^t)$
7: end for
8: set $\Omega_T = N_1 + \cdots + N_T$
9: for $t = 1 \rightarrow T$ do
10:   for $i = 1 \rightarrow N_t$ do
11:     update $\omega_i^t = \pi(X_i^t)/\Omega_T^{-1} \sum_{k=1}^{T} N_k q(X_i^t, \tilde{\theta}_k)$
12:  end for
13: end for
14: return $(X_1^1, \omega_1^1), \ldots, (X_{N_{t_1}}^1, \omega_{N_{t_1}}^1), \ldots, (X_T^1, \omega_T^1), \ldots, (X_{N_T}^T, \omega_{N_T}^T)$

(Kullback-Leibler divergence, variance criteria, moment fits, . . .). The only condition is that we are able to write the optimal parameter as $\theta^* = \int h(x) \Pi(dx)$, where $h$ is an explicitly known function, so that

$$\tilde{\theta}_{t+1} = \frac{1}{N_t} \sum_{i=1}^{N_t} \frac{\pi(X_i^t)}{q(X_i^t, \theta_i)} h(X_i^t).$$

(4)

Our process ends with the recycling of all past particles by updating all weights with (1). But, contrary to Cornuet et al. (2012), the calibration of the new parameter $\theta_{t+1}$ only considers the current sample $X_1^t, \ldots, X_{N_t}^t$. We hope improvements in the accuracy of $\theta_{t+1}$ against previous estimations by requiring that the sample size $N_t$ grows at each iteration. As a consequence, the influence of the first estimations $\tilde{\theta}_1, \ldots, \tilde{\theta}_{t-1}$ decreases significantly in the mixture of the denominator of the importance weight (1).

The modified AMIS is given in Algorithm 1. The learning process between lines 1 and 7 draws a sequence of samples from which it calibrates gradually the parameter $\theta$. The new value of the proposal’s parameter we compute at line 6 depends only on the last sample we have drawn. This is the only discrepancy from the original algorithm of Cornuet et al. (2012). Here, the recycling process
producing the final output is silently done by updating the weights of all produced samples between lines 9 and 13. Finally, we should note that, if calculating $\pi(x)$ is time consuming, the value computed at line 4 should be stored in memory to perform the update at line 11 during the recycling process.

3 Consistency results

We state here our main results (on the learning process in Paragraph 3.2, and on the final output in Paragraph 3.3). For the sake of clarity, their proofs are postponed to Sections 4 and 5. We begin with some hypothesis on the parametric family of proposals.

3.1 Assumptions on the proposals

We assume that the space $\Theta$ is a subset of the space $\mathbb{R}^d$ endowed with the Euclidian norm $\| \cdot \|$. The set $\mathcal{X}$ is a subset of a finite-dimensional vector space, equiped with a reference measure $dx$. All $Q(\theta)$ for $\theta \in \Theta$ and $\Pi$ are absolutely continuous with respect to the reference measure. They have densities $q(x, \theta)$ and $\pi(x)$ respectively. The minimal hypothesis for importance sampling schemes to provide consistent estimates is that $\Pi$ is absolutely continuous with respect to all proposals: $\forall \theta \in \Theta$, $\Pi \ll Q(\theta)$. Hence we assume that $q(x, \theta) = 0$ implies $\pi(x) = 0$.

Without loss of generality, we might assume that $\mathcal{X}$ and $\Theta$ are both open subsets of Euclidian spaces, and that both $\pi(x)$ and $q(x, \theta)$ are positive for all $x \in \mathcal{X}$, $\theta \in \Theta$. Besides we can note that $\theta_{t+1}$ is defined in (4) as a linear combination of (random) values of $h$, and the only fact we can safely affirm on the coefficients of this combination is that they are positive. Therefore, for the algorithm not to stop, any positive linear combination of elements of $\Theta$ should fall into $\Theta$. In particular, this implies that $\Theta$ cannot be a bounded subset of a Euclidian space.

We also impose some regularity conditions on the family of proposals $\{Q(\theta)\}_{\theta \in \Theta}$ which will ensure consistency of our procedure. For all $x \in \mathcal{X}$, $\theta \mapsto q(x, \theta)$ is continuous on $\Theta$, and the joint function $(x, \theta) \mapsto q(x, \theta)$ is lower semicontinuous on $\mathcal{X} \times \Theta$. Moreover, when $\theta \to \theta^*$, $q(\cdot, \theta)$ converges to $q(\cdot, \theta^*)$ uniformly over compact sets, i.e., for any compact subset $K$ of $\mathcal{X}$,

$$\|q(\cdot, \theta) - q(\cdot, \theta^*)\|_{K, \infty} := \sup \{ |q(x, \theta) - q(x, \theta^*)| : x \in K \}$$

converges to 0.
3.2 Consistency of the learning process

We focus here on the learnt $\hat{\theta}_t$ defined in (4) and show convergence to the optimal parameter $\theta^* = \int h(x)\pi(x)dx$. Sadly, the AMIS weight of a particle, see (3), is an average over the path in the parameter space being taken during the sequential algorithm. Weak consistency, i.e., convergence in probability, is not enough to control such averages, as there exists no Cesàro Lemma for the convergence in probability, see for instance Billingsley (1995), exercise 20.23 p. 272. Thus, we decided to rely on almost sure convergence. The challenge is to prove that the sequential algorithm do not accumulate Monte Carlo errors over iterations. Let us introduce the following class of functions.

**Definition 1.** A function $\psi : \mathcal{X} \rightarrow \mathbb{R}^d$ belongs to the class $\mathcal{G}^2(\mathbb{R}^d)$ if and only if $\int \pi^2(x)||\psi(x)||^2/q(x, \theta)d\pi(x)$ is finite for all $\theta$ in $\Theta$ and depends continuously on $\theta$.

We can interpret the integrability condition in the above definition as follow: the classical importance Monte Carlo algorithm that estimate $\Pi(\psi)$ has finite variance whatever the importance distribution in the parametric family. With such assumptions on the function $h$ used to learn the optimal parameter, and a condition on the sample sizes, we can show the following result.

**Theorem 2.** If $h \in \mathcal{G}^2(\mathbb{R}^d)$, the estimate $\hat{\theta}_t$ tends to $\theta^*$ in probability when $t \to \infty$. If, additionally, $\sum_1 1/N_t < \infty$, then $\hat{\theta}_t \to \theta^*$ almost surely.

The proof, which is in Section 4, consists of two parts. In the first part, we prove the convergence in probability using the weak law of large numbers on triangular arrays given in the Appendix. Afterwards, since $h \in \mathcal{G}^2(\mathbb{R}^d)$, conditionally on $\hat{\theta}_t$, the next estimate $\hat{\theta}_{t+1}$ is the sum of an iid and square integrable sample of size $N_t$. Thus, using the Chebyshev inequality artfully, and the weak law of large number, we obtain the almost sure convergence.

3.3 Consistency of the final recycling scheme

The remaining part of our results deals with the final output that merges all the samples. More precisely, Theorem 4 below says that the empirical sum of a function $\psi$ on the merged, re-weighted sample provides a consistent approximation of the integral $\Pi(\psi)$. The class of integrands $\psi \in L^1(\Pi)$ for which the above holds is determined by the following class of functions.
**Definition 3.** A function $\psi : \mathcal{X} \to \mathbb{R}$ belongs to the class $\mathcal{H}^2(\mathbb{R})$ if and only if the integral $\int [\pi(x)\psi(x)/q(x, \theta^*)]^2 q(x, \theta) dx$ is finite for all $\theta \in \Theta$ and is a function of $\theta$ that is continuous at $\theta = \theta^*$.

Likewise, the above class of functions might be interpreted in terms of quadratic moments. Note that, if $\psi$ is in $\mathcal{H}^2(\mathbb{R})$, then $\psi$ is in $L^1(\Pi)$. And finally, set

$$m_\varepsilon(x) := \inf\{q(x, \theta) : \theta \in B(\theta^*, \varepsilon)\}$$

(5)

where the infimum is actually attained because $\theta \mapsto q(x, \theta)$ is continuous, therefore positive on $\mathcal{X}$. We are now in a position to state the following strong consistency.

**Theorem 4.** Assume that $h \in \mathcal{G}^2(\mathbb{R}^d)$ and $\sum_t 1/N_t < \infty$. Moreover, assume that, for some $\varepsilon > 0$, $\psi(\cdot)q(\cdot, \theta^*)/m_\varepsilon(\cdot)$ is in $\mathcal{H}^2(\mathbb{R})$. Then, the sum over the final weighted sample $\bar{\Pi}^\text{AMIS}_T(\psi)$ given in (3) tends almost surely to $\int \psi(x)\pi(x) dx$ when $T \to \infty$.

The function $q(\cdot, \theta^*)/m_\varepsilon(\cdot)$ is larger than 1 on $\mathcal{X}$, and goes to 1 as $\varepsilon \to 0$. Hence the assumption that $\psi(\cdot)q(\cdot, \theta^*)/m_\varepsilon(\cdot)$ is in $\mathcal{H}^2(\mathbb{R})$ is a bit stronger than just $\psi$ is in $\mathcal{H}^2(\mathbb{R})$.

### 4 Proofs of the convergence during the learning process

This Section is devoted to the proof of Theorem 2. We first collect useful lemmas on the class $\mathcal{G}^2(\mathbb{R}^2)$, then rely on the weak law of large number of triangular arrays to obtain the convergence in probability. Finally, we prove the almost convergence of the learnt parameters.

#### 4.1 Technical results on the functions of class $\mathcal{G}^2(\mathbb{R}^d)$

The following lemma deals with the continuity condition imposed in $\mathcal{G}^2(\mathbb{R}^d)$. With this result, it becomes obvious that, if some function $\psi$ belongs to $\mathcal{G}^2(\mathbb{R}^d)$, and if $\varphi$ is another function such that $\|\varphi(x)\| \leq \|\psi(x)\|$ for all $x \in \mathcal{X}$, then $\varphi$ is also in $\mathcal{G}^2(\mathbb{R}^d)$.
Lemma 5. Assume that, for any $\theta \in \Theta$, the integral

$$v_{\psi}(\theta) := \int \pi^2(x)\|\psi(x)\|^2 / q(x, \theta) \, dx$$

is finite. Fix $\theta \in \Theta$. These conditions are equivalent: (i) the function $v_{\psi}$ is continuous at $\theta$; and (ii) when $\theta' \to \theta$,

$$\int \pi^2(x)\|\psi(x)\|^2 \left| \frac{1}{q(x, \theta')} - \frac{1}{q(x, \theta)} \right| \, dx \to 0.$$

Proof. Clearly, (ii) implies (i). It remains to show that (i) implies (ii), that is to say, if we assume (i), then, for any sequence $\theta_n$ that converges to $\theta$, $\lim_{n} A_n = 0$, where

$$A_n := \int \pi^2(x)\|\psi(x)\|^2 \left| \frac{1}{q(x, \theta_n)} - \frac{1}{q(x, \theta)} \right| \, dx.$$

To this aim, fix a random variable $X$ with distribution $\pi$ and set

$$Z_n := \pi(X)\|\psi(X)\|^2 / q(X, \theta_n), \quad Z = \pi(X)\|\psi(X)\|^2 / q(X, \theta)$$

so that $\mathbb{E}(Z_n) = \mathbb{E}[Z_n] = \int \pi^2(x)\|\psi(x)\|^2 / q(x, \theta_n) \, dx$. With the continuity conditions on the family $Q(\theta)$, $Z_n \to Z$ (almost) surely, and with (i), $\mathbb{E}[Z_n] \to \mathbb{E}[Z]$. Hence $Z_n$ is uniformly integrable and $A_n = \mathbb{E}[Z_n - Z]$ tends to 0. $\square$

In order to apply Theorem 15, we will also need the uniform integrability on compact set written below.

Lemma 6. If $\psi$ is in $\mathcal{C}^2(\mathbb{R}^d)$, on any compact set $K \subset \Theta$, we have

$$\lim_{\eta \to +\infty} \sup_{\theta \in K} \int \pi(x)\|\psi(x)\|^2 \left\{ \frac{\pi(x)\|\psi(x)\|}{q(x, \theta)} > \eta \right\} \, dx = 0.$$

Proof. Fix a compact set $K$. Since $1\{\|y\| > \eta\} \leq \eta^{-1}\|y\|$ for any $y \geq 0$, we have

$$\int \pi(x)\|\psi(x)\|^2 \left\{ \frac{\pi(x)\|\psi(x)\|}{q(x, \theta)} > \eta \right\} \, dx \leq \eta^{-1} \int \pi^2(x)\|\psi(x)\|^2 / q(x, \theta) \, dx.$$

The last integral depends continuously on $\theta$ and, consequently, is bounded by some finite $M$ on the compact set $K$. Therefore

$$\sup_{\theta \in K} \int \pi(x)\|\psi(x)\|^2 \left\{ \frac{\pi(x)\|\psi(x)\|}{q(x, \theta)} > \eta \right\} \, dx \leq \frac{M}{\eta},$$

and the desired result is proven. $\square$
4.2 Proof of weak consistency of Theorem 2

We define the $\sigma$-fields $F_t = \sigma(X_1^t, \ldots, X_{N_1}^t, \ldots, X_{N_t}^t)$ which form a filtration. We will derive the convergence applying Theorem 15 on the triangular array given by

$$V_{t,i} := \frac{\pi(X_i^t)}{N_t q(X_i^t, \theta_t)} h(X_i^t).$$

Indeed, we have $\hat{\theta}_{t+1} = \sum_{i=1}^{N_t} V_{t,i}$ The main difficulty here is in checking assumption (iii) of that theorem. Below, we begin by proving that $\hat{\theta}_t$ is tight, then we check that assumption (iii) of Theorem 15 is true, and conclude our proof.

**Tightness of $\hat{\theta}_t$.** We have

$$\|\hat{\theta}_{t+1}\| \leq \frac{1}{N_t} \sum_{i=1}^{N_t} \frac{\pi(X_i^t)}{q(X_i^t, \theta_t)} \|h(X_i^t)\|.$$  

Moreover

$$\mathbb{E} \left( \frac{\pi(X_i^t)}{q(X_i^t, \theta_t)} \|h(X_i^t)\| \bigg| F_t \right) = \int \pi(x) \|h(x)\| dx,$$

therefore $\mathbb{E}(\|\hat{\theta}_{t+1}\|) \leq \Pi(\|h\|)$. Then Markov’s inequality leads to

$$\lim_{c \to +\infty} \sup_{t} \mathbb{P}(\|\hat{\theta}_t\| > c) \leq \lim_{c \to +\infty} \frac{\Pi(\|h\|)}{c} = 0.$$  

Hence, the sequence $\{\hat{\theta}_t\}_{t \geq 1}$ is tight.

**Checking condition (iii) of Theorem 15.** We have to show that, for any $\eta > 0$, the following $S_t$ tends to 0 in probability, with

$$S_t := \sum_{i=1}^{N_t} \mathbb{E}[\|V_{t,i}\| \mathbb{I}\{\|V_{t,i}\| > \eta\} \bigg| F_t] = \int \pi(x) \|h(x)\| \mathbb{1}\{\frac{\pi(x)\|h(x)\|}{q(x, \theta_t)} > N_t\eta\} dx$$

$$= I(N_t\eta, \hat{\theta}_t) \text{ by setting } I(\eta, \theta) := \int \pi(x) \|h(x)\| \mathbb{1}\{\frac{\pi(x)\|h(x)\|}{q(x, \theta)} > \eta\} dx.$$  

Fix any $\varepsilon > 0$. Using tightness of $\hat{\theta}_t$ proven above, we might introduce a compact set $K_\varepsilon$ such that, for all $t \geq 1$, $\mathbb{P}(\hat{\theta}_t \notin K_\varepsilon) \leq \varepsilon$. On the event $\{\hat{\theta}_t \notin K_\varepsilon\}$, we
have \( I(N, \eta, \hat{\theta}_t) \leq I(0, \hat{\theta}_t) = \pi(||h||) \). Furthermore, on the event \( \{\hat{\theta}_t \in K_\varepsilon\} \),

\[
I(N, \eta, \hat{\theta}_t) \leq \sup_{\theta \in K_\varepsilon} \int \pi(x) ||h(x)|| \left\{ \frac{\pi(x)||h(x)||}{q(x, \theta)} > N_t \eta \right\} dx. \tag{6}
\]

Lemma 6 implies that the right-hand side of (6) goes to 0 since \( N_t \eta \) tends to infinity. Thus,

\[
\lim \sup \mathbb{E}(I(N, \eta, \hat{\theta}_t)) \leq \pi(||h||) \lim \sup \mathbb{P}(\hat{\theta}_t \not\in K_\varepsilon) \leq \pi(||h||) \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, we have proven that \( S_t \to 0 \) in \( L^1(\mathbb{P}) \) hence in probability, and condition (iii) is fulfilled.

**Conclusion.** We are now in a position to apply Theorem 15. Condition (i) is satisfied because, by construction, given \( F_t \), the random variables \( V_{t,i} \), for \( 1 \leq i \leq N_t \) are conditionally independent and \( \mathbb{E}[||V_{t,i}|| | F_t] < +\infty \). Condition (ii) is easy to check because, here,

\[
\sum_{i=1}^{N_t} \mathbb{E}[||V_{t,i}|| | F_t] = \int ||h(x)|| \pi(x) dx
\]

is not random (thus is a tight sequence). And condition (iii) has been proven above. Hence the conclusion of Theorem 15 is true and the proof is completed. \( \Box \)

### 4.3 Proof of the almost sure convergence of Theorem 2

**Bounding the conditional probabilities.** Fix \( \varepsilon > 0 \). Using that, conditionally on \( \hat{\theta}_t = \theta \), the \( X_i^j \)'s are iid from distribution \( Q(\theta) \), we have

\[
\text{Var}(||\hat{\theta}_{t+1} - \theta|| | \hat{\theta}_t = \theta) = \frac{1}{N_t} \int \left\{ \frac{\pi(x)}{q(x, \theta)} ||h(x)||^2 - 2 \left( \frac{\pi(x)}{q(x, \theta)} h(x), \theta^* \right) + ||\theta^*||^2 \right\} q(x, \theta) dx
\]

\[
= \frac{v(\theta) - ||\theta^*||^2}{N_t}
\]

where \( v(\theta) = \int \pi(x)^2 ||h(x)||^2 / q(x, \theta) dx \) is finite and continuous because \( h \in G^2(\mathbb{R}^d) \).
With the continuity assumption, \( v(\theta) - \|\theta^*\|^2 \) is bounded from above by some finite constant, \( K_\varepsilon \) say, on the compact ball, \( \bar{B}(\theta, \varepsilon) \) say, centered on \( \theta^* \), of radius \( \varepsilon \) and the conditional Chebyshev inequality gives, for all \( \theta \in \bar{B}(\theta^*, \varepsilon) \),

\[
P\left( \|\hat{\theta}_{t+1} - \theta^*\| > \varepsilon \big| \hat{\theta}_t = \theta \right) \leq \frac{K_\varepsilon}{\varepsilon^2 N_t}. \tag{7}
\]

Multiplying by \( 1 \{ \theta \in \bar{B}(\theta^*, \varepsilon) \} \) on both side of the above inequality and integrating over the distribution of \( \hat{\theta}_t \) leads to

\[
P\left( \|\hat{\theta}_{t+1} - \theta^*\| > \varepsilon \big| \|\hat{\theta}_t - \theta^*\| \leq \varepsilon \right) \leq \frac{K_\varepsilon}{\varepsilon^2 N_t}. \tag{8}
\]

**Proving the almost sure convergence.** Now, we recall that \( \hat{\theta}_t \) forms a (time-inhomogeneous) Markov chain. Thus, using (8),

\[
P\left( \bigcap_{t \geq T} \|\hat{\theta}_{t+1} - \theta^*\| \leq \varepsilon \right) = P\left( \|\hat{\theta}_{T+1} - \theta^*\| \leq \varepsilon \right) \prod_{t=1}^{T-1} P\left( \|\hat{\theta}_{t+1} - \theta^*\| \leq \varepsilon \big| \|\hat{\theta}_t - \theta^*\| \leq \varepsilon \right)
\]

\[
\geq P\left( \|\hat{\theta}_{T+1} - \theta^*\| \leq \varepsilon \right) \prod_{t=1}^{T-1} \left( 1 - \frac{K_\varepsilon}{\varepsilon^2 N_t} \right).
\]

And, when \( T' \to \infty \), we obtain

\[
P\left( \bigcap_{t \geq T} \|\hat{\theta}_{t+1} - \theta^*\| \leq \varepsilon \right) \geq P\left( \|\hat{\theta}_{T+1} - \theta^*\| \leq \varepsilon \right) \prod_{t \geq T+1} \left( 1 - \frac{K_\varepsilon}{\varepsilon^2 N_t} \right).
\]

Applying the logarithm on the product and classical results on series, because \( \Sigma, 1/N_t \) is finite, the infinite product \( \prod_{t}(1 - K_\varepsilon/\varepsilon^2 N_t) \) converges (that is to say the limit is finite and strictly positive). In particular, the remainder of the infinite product in the right hand side of the above inequality tends to 1 when \( T \to \infty \). Furthermore, because of the convergence in probability proven in Paragraph 4.3, \( P\left( \|\hat{\theta}_{T+1} - \theta^*\| \leq \varepsilon \right) \) tends also to 1 and thus

\[
\lim_{T \to \infty} P\left( \bigcap_{t \geq T} \|\hat{\theta}_{t+1} - \theta^*\| \leq \varepsilon \right) = 1.
\]

And we have proved that \( \lim sup_{T \to \infty} \|\hat{\theta}_T - \theta^*\| \leq \varepsilon \) almost surely. Since \( \varepsilon \) is arbitrary, this proves the desired almost sure convergence.
5 Proof of the convergence of the final recycling scheme

The last step is to prove Theorem 4, i.e., that the AMIS estimator

$$\hat{\Pi}_{AMIS}^T(\psi) = \frac{1}{\Omega_T} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \frac{\pi(X'_t)}{\Omega_T^{-1} \sum_{k=1}^{T} N_k q(X'_t, \theta_k)} \psi(X'_t),$$

which is the result of the unique recycling step of our scheme, is consistent for the integral $\Pi(\psi)$ for a large class of functions $\psi$. To this aim, we set

$$\hat{\Pi}_{T}^*(\psi) := \frac{1}{\Omega_T} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \frac{\pi(X'_t)}{q(X'_t, \theta^*)} \psi(X'_t),$$

(9) (which cannot be computed in practice because $\theta^*$ is unknown). Note that the auxiliary variable defined in (9) is a (weighted) average of the random variables $\hat{\pi}_t(\psi)$ given by

$$\hat{\pi}_t(\psi) := \frac{1}{N_t} \sum_{i=1}^{N_t} \frac{\pi(X'_t)}{q(X'_t, \theta^*)} \psi(X'_t).$$

We also define

$$I'_\psi(\theta) := \int \left[ \frac{\pi(x)\psi(x)}{q(x, \theta^*)} \right] q(x, \theta) dx$$

(10) which is the conditional expectation of $\hat{\pi}_t(\psi)$ knowing that $\hat{\theta}_t = \theta$.

The proof is organised as follows. After stating useful lemmas, we prove that the sequence of auxiliary variables are strongly consistent. We then show that the difference between our estimator and this auxiliary variable, namely $\hat{\Pi}_{AMIS}^T(\psi) - \hat{\Pi}_{T}^*(\psi)$, tends to 0 almost surely. Hence the consistency stated in Theorem 4.

5.1 Technical results on the function of class $\mathcal{H}^2(\mathbb{R})$

The proof of the following results, which deal with the continuity condition imposed in the definition of $\mathcal{G}^2(\mathbb{R})$, is left to reader, be very similar to the proof of Lemma 5. Likewise, this lemma implies that, if some function $\psi$ belongs to $\mathcal{H}^2(\mathbb{R})$, and if $\varphi$ is another function dominated by $\psi$, then $\varphi \in \mathcal{H}^2(\mathbb{R})$.

Lemma 7. Assume that, for any $\theta \in \Theta$, the integral

$$w_\psi(\theta) := \int \pi^2(x)\psi^2(x) q^2(x, \theta^*) q(x, \theta) dx$$

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is finite. These conditions are equivalent: (i) \( w_\psi \) is continuous at \( \theta^* \); and (ii) when \( \theta \to \theta^* \),

\[
\int \frac{\pi^2(x)\psi^2(x)}{q^2(x, \theta^*)}|q(x, \theta) - q(x, \theta^*)|\,dx \to 0.
\]

We also need the following to control the conditional expected values of \( \pi^*_t(\psi) \).

**Lemma 8.** If \( \psi \in \mathcal{H}^2(\mathbb{R}) \), then the integrals \( I^*_\psi(\theta) \) defined in (10) are well defined for all \( \theta \) and the map \( I^*_\psi \) is continuous at \( \theta = \theta^* \).

**Proof.** Fix any \( \theta \in \Theta \). If \( X_\theta \) is a random variable with distribution \( Q(\theta) \), then

\[
Y_\theta = \pi(X_\theta)(X_\theta)/q(X_\theta, \theta^*)
\]

is a random variable with distribution \( \mathcal{H}^2(\mathbb{R}) \). Hence \( Y_\theta \) is a \( L^1 \)-random variable and its expected value, namely \( I_\psi(\theta) \) is well defined.

Now, set \( g(x) = \pi(x)\psi(x)/q(x, \theta^*) \). We have \( |g(x)| \leq \max(1, g^2(x)) \), thus

\[
\int |g(x)||q(x, \theta) - q(x, \theta^*)|\,dx \leq \int |q(x, \theta) - q(x, \theta^*)|\,dx + \int g^2(x)|q(x, \theta) - q(x, \theta^*)|\,dx
\]

The first integral in this bound goes to 0 because of Scheffé’s Theorem, see e.g. Billingsley (1995), Theorem 16.12 p. 215. The second integral goes also to 0, because \( \psi \) is in \( \mathcal{H}^2(\mathbb{R}) \) and because of Lemma 7. Whence

\[
|I^*_\psi(\theta) - I^*_\psi(\theta^*)| \leq \int |g(x)||q(x, \theta) - q(x, \theta^*)|\,dx \to 0.
\]

\[\square\]

5.2 Convergence of some auxiliary variables

**Proposition 9.** Assume that \( h \in \mathcal{G}^2(\mathbb{R}^d) \), \( \sum 1/N_t \) is finite and \( \psi \in \mathcal{H}^2(\mathbb{R}) \). When \( t \to \infty \), \( (\pi_t^*(\psi) - I^*_\psi(\hat{\theta}_{t-1})) \) tends to 0 almost surely. Moreover, under those assumptions, \( I^*_\psi(\hat{\theta}_t) \to I^*_\psi(\theta^*) = \Pi(\psi) \).

**Proof.** The last result follows from Theorem 2 and continuity of \( I^*_\psi \) at \( \theta^* \) proven in Lemma 8. Adapting the proof written in Paragraph 4.2, we also have that \( (\pi_t^*(\psi) - I^*_\psi(\hat{\theta}_{t-1})) \) tends to 0 in probability

The rest of the proof is inspired from the part of the proof of Theorem 2 written in Paragraph 4.3. Fix \( \varepsilon > 0 \). Set \( \Delta_{t+1} = \pi_{t+1}^*(\psi) - I^*_\psi(\hat{\theta}_t) \), \( A_{t+1} = \{ |\Delta_{t+1}| \leq \varepsilon \} \cap \{ ||\hat{\theta}_{t+1} - \theta^*|| \leq \varepsilon \} \) and \( \hat{\Theta}_{t+1} \) the complementary event. We have

\[
\mathbb{E}[(\Delta_{t+1})^2]\hat{\theta}_t = \theta = \frac{1}{N_t} \left[ \int (\pi(x)\psi(x)/q(x, \theta^*))^2 q(x, \theta)\,dx - I^*(\theta)^2 \right].
\]
and the term between brackets is bounded from above by some finite $K'_\varepsilon$ for any $\theta \in \tilde{B}(\theta^*, \varepsilon)$ because $\psi \in H^2(\mathbb{R})$. Thus, for all $\theta \in \tilde{B}(\theta^*, \varepsilon)$,

$$
P(\tilde{A}_{t+1} \nmid \tilde{\theta}_t = \theta) \leq P(|\Delta_{t+1}| > \varepsilon \mid \tilde{\theta}_t = \theta) + P(\|\tilde{\theta}_{t+1} - \theta^*\| \geq \varepsilon \mid \tilde{\theta}_t = \theta) \leq K'_\varepsilon + K\varepsilon \frac{2N_t}{\varepsilon^2},$$

where we have used the Chebyshev inequality and resorted (7) to bound the first and second probabilities respectively. Hence $P(\tilde{A}_{t+1} \nmid \tilde{A}_t) \leq K/N_t$ for some finite $K$. Now, since $(\Delta_t, \tilde{\theta}_t)$ is a (time-inhomogeneous) Markov chain, we have

$$
P(\bigcap_{t=T}^{\infty} \tilde{A}_{t+1}) \geq P(A_{T+1}) \prod_{T=1}^{\infty} \left( 1 - \frac{K}{N_t} \right)$$

The above infinite product tends to 1 since $\sum_n 1/N_t$ is finite. And $P(A_{T+1}) \to 1$ because both $\Delta_{T+1}$ and $\|\tilde{\theta}_{T+1} - \theta^*\|$ tend to 0 in probability. \hfill \Box

We shall now recall that the auxiliary variable $\hat{\Pi}_T^*(\psi)$ is a weighted average of the $\hat{\pi}_t^*(\psi)$ for $t = 1, \ldots, T$ which are controlled by Proposition 9 proven above. The following Lemma is obvious, using Cesàro Lemma on sequence of (non random) vectors.

**Lemma 10.** Let $\{U_t\}$ be a sequence of random vectors and $U$ another random vector. If $\{b_t\}$ is a sequence of positive real numbers such that $B_t = b_1 + \ldots + b_t$, tends to infinity, then the event $\{U_t \to U\}$ is included in the event $\{B_t^{-1} \sum_{k=1}^{t} b_k U_k \to U_{\infty}\}$.

This Cesàro Lemma and Proposition 9 above leads to the following.

**Proposition 11.** Assume that $h \in G^2(\mathbb{R}^d)$, $\sum_t 1/N_t$ is finite and $\psi \in H^2(\mathbb{R})$. Then

$$
\hat{\Pi}_T^*(\psi) \to \Pi(\psi) \quad \text{almost surely.}
$$

**Proof.** Because of Proposition 9, $\hat{\Pi}_T^*(\psi)$ tends almost surely to $\Pi(\psi) = \int \psi(x)\pi(x)dx$. Applying Lemma 10 with $b_t = N_t$ yields the convergence of $\hat{\Pi}_T^*(\psi)$. \hfill \Box
5.3 Controlling the discrepancy between the AMIS estimator and the auxiliary variable

The convergence of $\hat{\Pi}_{AMIS}^T(\psi) - \Pi^*_T(\psi)$ towards 0 almost surely is proven in Proposition 13 below, whose proof relies on some preliminary result given in Lemma 12. To this end, we define the function $D_T(\cdot): \mathcal{X} \mapsto \mathbb{R}_+$ by

$$D_T(x) = \Omega_T^{-1} \sum_{k=1}^{T} N_k q(x, \hat{\theta}_k)$$

which appears in the denominator of the updated weight (1). Because of the consistency of the learning scheme proven in Section 4, we are able to show in the following lemma that this denominator resembles the denominator of the classical importance weight, when the proposal distribution is $Q(\theta^*)$.

**Lemma 12.** Let $K$ be a compact subset of $\Theta$. The event $\{\hat{\theta}_t \to \theta^*\}$ is included in the event where

$$\lim_{T \to +\infty} \left\| \frac{q(\cdot, \theta^*)}{D_T(\cdot)} - 1 \right\|_{K,\infty} = 0.$$

**Proof.** Denote by $m_{e,K}$ the infimum of $m_e(x)$ on $K$, where $m_e(\cdot)$ is the function defined in (5). Actually, $m_{e,K}$ is the infimum of the lower semicontinuous function $(x, \theta) \mapsto q(x, \theta)$ on the compact set $K \times \bar{B}(\theta^*, \varepsilon)$. Since a lower semicontinuous function attains its lower bound on any compact set, and $q(x, \theta) > 0$ for all $x$ and $\theta$, the infimum $m_{e,K}$ is positive.

Now fix a point of the probability space in the event $\{\hat{\theta}_t \to \theta^*\}$. There, there exists some $t_\varepsilon$ such that, for all $t > t_\varepsilon$, $\|\hat{\theta}_t - \theta^*\| < \varepsilon$. Hence, for all $T > t_\varepsilon$, and all $x \in \mathcal{X}$,

$$D_T(x) \geq \frac{1}{\Omega_T} \sum_{k=t_\varepsilon+1}^{T} N_k q(x, \theta_k) \geq \frac{\Omega_T - \Omega_e}{\Omega_T} m_e(x) \quad \text{where} \quad \Omega_e = \sum_{k=1}^{t_\varepsilon} N_k.$$

Therefore

$$\left| \frac{q(x, \theta^*)}{D_T(x)} - 1 \right| \leq \frac{\Omega_T}{(\Omega_T - \Omega_e)m_e(x)} \left| q(x, \theta^*) - D_T(x) \right|$$

$$\leq \frac{\Omega_T}{(\Omega_T - \Omega_e)m_{e,K}} \Omega_T^{-1} \sum_{t=1}^{T} N_k \left\| q(\cdot, \theta^*) - q(\cdot, \hat{\theta}_t) \right\|_{K,\infty}.$$

The bound in (11) is uniform on $K$ and goes to 0 using Lemma 10, which leads to the desired result. \qed
We can now state and prove the result controlling the difference between the AMIS estimator and the auxiliary variable \( \hat{\Pi}_T^\alpha(\psi) \).

**Proposition 13.** Assume that \( h \in \mathcal{C}^2(\mathbb{R}^d) \) and \( \sum_i 1/N_t < \infty \). Moreover, assume that, for some \( \varepsilon > 0 \), \( \psi(\cdot)q(\cdot, \theta^*)/m_\varepsilon(\cdot) \) is in \( \mathcal{C}^2(\mathbb{R}) \). Then

\[
\lim_{T \to +\infty} \Pi_T^{AMIS}(\psi) - \hat{\Pi}_T^\alpha(\psi) = 0 \quad \text{almost surely.}
\]

**Proof.** Fix \( \alpha > 0 \). The integral

\[
\int_{\mathcal{X} \setminus K} \left| \psi(x) \right| \frac{q(x, \theta^*)}{m_\varepsilon(x)} \pi(x) dx
\]

is finite because \( |\psi(\cdot)|q^*(\cdot)/m_\varepsilon(\cdot) \in \mathcal{C}^2(\mathbb{R}) \). Therefore we can find some compact subset \( K \) of \( \mathcal{X} \) such that

\[
\int_{\mathcal{X} \setminus K} \left| \psi(x) \right| \frac{q(x, \theta^*)}{m_\varepsilon(x)} \pi(x) dx < \alpha.
\]

Now, set \( \psi_1(x) := \psi(x)1\{x \in K\} \), \( \psi_2(x) := \psi(x)1\{x \notin K\} \) so that \( \psi(x) = \psi_1(x) + \psi_2(x) \). And consider the event

\[
E = \{ \hat{\theta}_T \to \theta^* \} \cap \{ \hat{\Pi}_T^\alpha(\psi_1) \to \Pi(\psi_1) \} \cap \{ \hat{\Pi}_T^\alpha(\psi_2) \to \Pi(\psi_2) \} \cap \{ \hat{\Pi}_T^\alpha(\varphi) \to \Pi(\varphi) \}.
\]

where \( \varphi(x) := \left| \psi_2(x) \right|q(x, \theta^*)/m_\varepsilon(x) \). With Theorem 2 and Proposition 11, this event is of probability 1. Moreover, note that, because of (5), \( q(x, \theta^*)/m_\varepsilon(x) \geq 1 \) and thus

\[
\Pi(\psi_2) \leq \Pi(\varphi) = \int_{\mathcal{X} \setminus K} \left| \psi_2(x) \right| \frac{q(x, \theta^*)}{m_\varepsilon(x)} \pi(x) dx < \alpha. \quad (12)
\]

Then, using linearity of the operators \( \Pi_T^{AMIS} \) and \( \hat{\Pi}_T^\alpha \), we have

\[
\left| \Pi_T^{AMIS}(\psi) - \hat{\Pi}_T^\alpha(\psi) \right| \leq \left| \Pi_T^{AMIS}(\psi_1) - \hat{\Pi}_T(\psi_1) \right| + \left| \Pi_T^{AMIS}(\psi_2) - \hat{\Pi}_T(\psi_2) \right| + \left| \hat{\Pi}_T^\alpha(\psi_2) - \hat{\Pi}_T^\alpha(\psi_2) \right|
\]

\[
\leq \left| \Pi_T^{AMIS}(\psi_1) - \hat{\Pi}_T(\psi_1) \right| + \left| \Pi_T^{AMIS}(\psi_2) - \hat{\Pi}_T(\psi_2) \right| + \left| \hat{\Pi}_T^\alpha(\psi_2) - \hat{\Pi}_T^\alpha(\psi_2) \right|. \quad (13)
\]

The first term in the right hand side of (13) can be controlled as follows:

\[
\Delta_T := \left| \Pi_T^{AMIS}(\psi_1) - \hat{\Pi}_T(\psi_1) \right| \leq \frac{1}{\Omega_T} \sum_{t=1}^{\infty} \sum_{i=1}^{N_t} \pi(X_t^i) \left| \psi_1(X_t^i) \right| \left| q^*(X_t^i, \theta^*) \right| \left| D_T^\alpha(\cdot) - 1 \right|_{\mathcal{K}_S, \infty}.
\]

\[
\leq \left| \frac{q^*(\cdot, \theta^*)}{D_T(\cdot)} - 1 \right|_{\mathcal{K}_S, \infty} \hat{\Pi}_T^\alpha(\psi_1).
\]

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On the event $E$, using Lemma 12, the first term of the last bound goes to 0, and $\hat{\Pi}_T^\ast(|\psi_1|) \rightarrow \Pi(|\psi_1|)$. Hence $\lim_T \Delta_T = 0$ on $E$.

On the event $E$, the second term of the right hand side of (13) can be bounded by

$$\hat{\Pi}_{AMIS}^T(|\psi_2|) \leq \frac{1}{\Omega_T} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \frac{\pi(X_i')|\psi_2(X_i')|}{q(X_i', \theta^*)} \frac{q(X_i', \theta^*)}{D_T(X_i')} \leq \frac{\Omega_T}{\Omega_T - \Omega_e} \hat{\Pi}_T^\ast(|\psi_2(\cdot)|) \frac{q(\cdot, \theta^*)}{m_e(\cdot)} = \frac{\Omega_T}{\Omega_T - \Omega_e} \hat{\Pi}_T^\ast(\varphi)$$

(14)

using the fact that, on $E$, there exists a $t_\epsilon$ such that, for all $t > t_\epsilon$, $\|\hat{\theta}_t - \theta^*\| < \epsilon$.

Hence, for all $T > t_\epsilon$, and all $x \in \mathcal{X}$,

$$D_T(x) \geq \frac{1}{\Omega_T} \sum_{k=t_\epsilon+1}^{T} N_k q(x, \theta_k) \geq \frac{\Omega_T - \Omega_e}{\Omega_T} m_e(x) \quad \text{where} \quad \Omega_e = \sum_{k=1}^{t_\epsilon} N_k.$$  

On the event $E$, $\hat{\Pi}_T^\ast(\varphi)$ converges to $\Pi(\varphi)$ which is smaller than $\alpha$ because of (12). Moreover, $(\Omega_T - \Omega_e)/\Omega_T \rightarrow 1$. Hence, on the event $E$,

$$\limsup_T \hat{\Pi}_{AMIS}^T(|\psi_2|) \leq \alpha$$

And, finally, on the event $E$, the third term of the right hand side of (13) converges to $\Pi(|\psi_2|)$ which is smaller than $\alpha$ using (12). Hence, on $E$,

$$\limsup_T \hat{\Pi}_T^\ast(|\psi_2|) \leq \alpha$$

Reporting in (13), we obtain that, on the event $E$ of probability 1,

$$\limsup_T \left| \hat{\Pi}_{AMIS}^T(\psi) - \hat{\Pi}_T^\ast(\psi) \right| \leq 2\alpha.$$  

Because $\alpha$ is arbitrary small, we have proven the desired result.

\[\square\]

### 5.4 Conclusion of the proof of Theorem 4

Proposition 11 gives the convergence of the auxiliary variable $\hat{\Pi}_T^\ast(\psi)$ towards the integral $\Pi(\psi)$ almost surely, while the discrepancy between the AMIS estimator and this auxiliary variable becomes negligible almost surely (Proposition 13). Whence the almost sure convergence of the estimator $\hat{\Pi}_{AMIS}^T(\psi)$ to the integral $\Pi(\psi)$, and then the proof of Theorem 4 is completed. \[\square\]
6 Numerical experiments

We provide here a detailed numerical example on which we have compared

(a) an iterative and adaptive algorithm learning $\theta$ with a naive recycling strategy at the end;

(b) the original AMIS of Cornuet et al. (2012) and

(c) our modified AMIS with its recycling step only at the end.

We refer the reader to Cornuet et al. (2012) for the original algorithm (b). The algorithm (c) is the main topic of this paper, and is described with great details in Section 2. At last, algorithm (a) follows the same code lines, but stops at line 7 and returns a merging of all past samples without updating the weights computed at line 4. In this example, the target is the posterior distribution when conducting a Bayesian analysis on a population genetic data set. It turns out that our algorithm was the most powerful for a given computational cost, that is to say, for a given number of simulations from proposals.

A Bayesian model in population genetics. This example comes from a population genetic problem. More precisely, we want to conduct a Bayesian analysis of a genetic data set $\mathcal{D}$ to infer mutation and migration rates in a parametric model. Assume that the species of interest is composed of two large populations at equilibrium, one on an island and the other one on the mainland. The parametric model we have used is coalescent based, and is detailed, for instance in Donnelly and Tavaré (1995) and Rousset and Leblois (2012). The genetic data come from two samples of individuals corresponding respectively to the two populations, genotyped at five independent microsatellite loci.

The model is composed of two populations, whose effective population sizes are both equal to 10000. We restrict the demographic scenario of this model to a symmetric migration between the two populations, and the migration rates are the supposed to be the same in both directions. We consider the mutation model SMM (Single Mutation Model). Our data set which we denote by $\mathcal{D}$ is simulated on five independent loci. At each locus we simulate the genotypes of individuals, using the software IBDSim of Leblois et al. (2009). For this data set, we set the mutation rate $x_{\text{mut}}$ to 2.3 and the symmetric migration rate $x_{\text{mig}}$ to 0.04. In this example, the likelihood of a data set $\ell(\mathcal{D}|x_{\text{mut}}, x_{\text{mig}})$ is the product of five integrals. Each integral represents the likelihood of the data set at a given locus. In this study,
we approximate these integrals through importance sampling methods. These approximations are provided by the software *Migraine* of Rousset and Leblois (2012).

We consider a uniform prior on the set \( \mathcal{X} = (10^{-1}, 10) \times (10^{-3}, 0.5) \), thus simplifying the expression of the posterior density to

\[
\pi(x_{\text{mut}}, x_{\text{mig}} | \mathcal{D}) \propto \ell(\mathcal{D} | x_{\text{mut}}, x_{\text{mig}}) 1_{\mathcal{X}} (x_{\text{mut}}, x_{\text{mig}}).
\]

Hence the target \( \pi(x) \) is the posterior distribution on a two dimensional parameter \( x = (x_{\text{mig}}, x_{\text{mut}}) \) when the prior is a non informative uniform distribution on some set \( \mathcal{X} \). This example is actually typical of situations where the density of the target \( \pi(x) \) is of high computational cost. With coalescent based models, the likelihood, thus the posterior density \( \pi(x) \), is an integral over a latent process that, by chance, is computed via importance sampling too, see De Iorio et al. (2005) and Rousset and Leblois (2012).
Tuning of the algorithms. The family of proposals in the three sequential algorithms is composed of bivariate Gaussian distributions, conditioned (or truncated) on the support $\mathcal{X}$ of the prior distribution. The parameter of the proposal is a four dimensional vector $\theta = (\mu_{\text{mig}}, \mu_{\text{mut}}, \sigma_{\text{mig}}^2, \sigma_{\text{mut}}^2)$, whose first two coordinates give the position of the mode and last two coordinates give the marginal variances of the diagonal covariance matrix. For each realization of a scheme, we set $T = 45$ and $N_t = N \times t$ where $N = 100$.

Results. Performances of the sampling algorithms were compared as follows. Instead of looking at the estimates of $\Pi(\psi)$ for various integrands $\psi$, we decided here to evaluate the outputs with distances between the discrete measure induced by the weighted final samples and the target. In Figure 1, we have represented the Cramer-von Mises, the $L^2$- and the $L^\infty$- distances between the empirical distribution function $\hat{F}_T(x)$ of the final weighted samples and the distribution function $F(x)$ of the target, i.e., the posterior distribution. Furthermore, since the density of the target cannot be written with a close formula in the concrete example, we shall describe how $F(x)$ was computed. Following an idea of R. Leblois and F. Rousset, estimates of $\pi(x)$ were computed for values of $x$ ranging a regular $500 \times 500$ grid of the support of the prior distribution. The estimation error was then decreased using a kriging model on $\pi(x)$, assuming regularity conditions of that posterior density. Of course, this sharp approximation comes at a much higher computational cost than any run of the Monte Carlo algorithms we compare here.

The results presented in Figure 1 exhibit a clear advantage to our modified AMIS, namely (c), in front of (a), the sequential scheme with a naive recycling and (b), the original AMIS, whatever the distance. Thus, modifying the original AMIS was not only a way to obtain the theoretical results of Section 3, but also a real improvement of the original algorithm. One of the reason that might explain this phenomenon is that the recycling scheme of the original AMIS introduces a bias on $\theta$ during the learning process which tends to accumulate, and thus is large enough to degrade the output quality when compared to our modified AMIS.

7 Conclusion and discussion

For a certain class of functions, we derived strong consistency of our modified AMIS. We proved a strong law of large numbers for a large class of integrands characterized by regularity conditions and for a general family of proposals. We
assumed that the size of the samples at each stage, namely $N_t$, tends to infinity rather quickly so that $\sum_t 1/N_t$ is finite. This condition might be unsatisfactory but is due to the fact that we only assumed that $\pi(X)||h(X)||/q(X, \theta)$ has a finite quadratic moment when $X \sim Q(\theta)$. In future research, we could try to relax the hypotheses on $N_t$, assuming that the above random variable has exponential moments and using a large deviation inequalities instead of the Chebyshev bound.

Besides, another route might be taken to prove theoretical results on the modified AMIS, based on Markovian arguments. Indeed, when the sample size does not vary between iterations during the learning process, i.e., when $N_1 = N_2 = \ldots = N_T = N$, the sequence of pairs $(X_{1:N}^t, \hat{\theta}_t)$ form a Markov chain. And the final sum in (3) might be traded using results on averages over a path of a Markov chain. But we have left this route for future works.

The present paper does not conduct a comprehensive numerical comparison between the original AMIS and our algorithm, because we focused here on theoretical results. But the numerical experiment presented above is a serious example exhibiting an advantage to our modification of the AMIS.

Finally two important, methodological issues have not been tackle in this theoretical work. The first one deals with the initialization of the AMIS. The original paper of Cornuet et al. (2012) proposed an answer based on a logistic sample when nothing is known on the target. We stress here that the starting distribution is of great practical consequence: for instance, if the first sample misses a mode of the target distribution, we have almost no chance to see it during the whole process. That was summed up by Cornuet et al. (2012) as the “what-you-get-is-what-you-see” nature of the AMIS. Likewise, a recurring numerical question on the AMIS concerns the allocation of the overall computational cost (given by the final system size $\Omega_T$). To optimize allocation, one could propose and study allocation strategies on different iterations via the sequence $N_1, \ldots, N_T$. We also believe that the winner in the competition between the original and the modified AMIS depends also on this allocation strategy.

Appendix – Weak law of large numbers on triangular arrays

The following result is a multidimensional generalisation of the weak law of large numbers given in Chapter 9 of Cappé et al. (2005). In the following, all random variables are assumed to be defined on a joint probability space $(\Xi, \mathcal{F}, \mathbb{P})$ and
\(\{N_t\}_{t \geq 1}\) denotes an increasing sequence of integers. In those theorem as well as throughout this paper, we used the notion of tightness of random variables that we recall here. (See Billingsley (1995), p. 336 for more details)

**Definition 14.** A sequence of random vectors \(\{U_n\}\) is tight if

\[
\lim_{\eta \to \infty} \sup_{n \geq 1} \mathbb{P}(\|U_n\| \geq \eta) = 0.
\]

We have the following law of large numbers.

**Theorem 15.** Let \(\{V_{t,i}\}_{1 \leq i \leq N_t}\) be a triangular array of random vectors on \(\mathbb{R}^d\), and let \(\{F_t\}_{t \geq 1}\) be a sequence of \(\sigma\)-fields. Assume that the following conditions hold true.

(i) The \(V_{t,i}\) for \(i = 1, \ldots, N_t\) are conditionally independent given \(F_t\), and for any \(t\) and \(i = 1, \ldots, N_t\), \(\mathbb{E}[\|V_{t,i}\| | F_t] < +\infty\).

(ii) The sequence \(\left\{ \sum_{i=1}^{N_t} \mathbb{E}[\|V_{t,i}\| | F_t] \right\}_{t \geq 1}\) is tight.

(iii) For any positive \(\eta\),

\[
\lim_{t \to +\infty} \sum_{i=1}^{N_t} \mathbb{E}[\|V_{t,i}\| \mathbb{1}_{\{\|V_{t,i}\| > \eta\}} | F_t] = 0 \text{ in probability.}
\]

Then,

\[
\lim_{t \to +\infty} \sum_{i=1}^{N_t} \left\{ V_{t,i} - \mathbb{E}[V_{t,i} | F_t] \right\} = 0 \text{ in probability.}
\]

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