Natural inflation mechanism in asymptotic noncommutative geometry

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The possibility of having an inflationary epoch within a noncommutative geometry approach to unifying gravity and the standard model is demonstrated. This inflationary phase occurs without the need to introduce ad hoc additional fields or potentials, rather it is a consequence of a nonminimal coupling between the geometry and the Higgs field.

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INTRODUCTION

It has long been known that despite its enormous success in explaining the expansion of the Universe, the origin of the cosmic microwave background radiation and the synthesis of light elements, the standard (hot big bang) cosmological model is plagued with a number of severe problems. More precisely, the hot big bang model is unable to address the big bang singularity, and it cannot explain the flatness of space, or the large-scale homogeneity and isotropy of space over causally disconnected regions. Thus, it has to admit particular initial conditions. In addition, it cannot explain the origin of initial inhomogeneities giving rise to the observed structure formation, neither can it account for the absence of dangerous relics, which would have been formed in the early universe according to the high energy particle physics models valid at those energy scales. Finally, the standard cosmological model is plagued by the vacuum energy (or cosmological constant) problem.

To address some of these shortcomings, it has been postulated that a period of accelerated expansion, called cosmological inflation [1], has proceeded the era of validity of the hot big bang model. Inflation has not only addressed successfully the problem of requiring particular initial conditions, but it has also succeeded in predicting the (almost) scale invariant spectrum of density perturbations that are measured in the cosmic microwave background temperature anisotropies. Although some questions have yet to be fully answered, such as the specifics of reheating [2] and the likelihood of the onset of inflation [3], the remarkable agreement with observation makes inflation an extremely attractive approach to understanding the early universe.

Unfortunately, it has proved difficult to naturally embed inflation within an underlying fundamental theory. Inflation most naturally occurs when the dynamics of the universe are dominated by the evolution of a scalar field, the inflaton, slowly rolling in its potential; the form of the potential defines the type of the inflationary model. There is only one scalar field within the standard model of particle physics, the Higgs field, and it is naturally hoped that this could play the rôle of the inflaton. However, it has been shown [4] that in order for the Higgs field to produce the correct amplitude of density perturbations, its mass would have to be some 11 orders of magnitude higher than the one required by particle physics. This conclusion was however reached using general relativistic cosmology and here we re-examine the calculation in the context of noncommutative geometry [5].

NONCOMMUTATIVE GEOMETRY

Motivation

Despite the long efforts, a unified theory of all interactions, including gravity, remains still lacking. The reason for this difficulty may have its origin in the different properties and underlying symmetries of the Einstein-Hilbert Lagrangian $L_{EH}$, and the Standard Model (SM) Lagrangian $L_{SM}$. Certainly, for physical processes much below the Planck scale ($\approx 10^{18}$ GeV), gravity can be safely considered as a classical theory. However, as energies approach the Planck scale, the quantum nature of space-time becomes apparent, and the Einstein-Hilbert action becomes an approximation. Moreover, at Planck scale, one expects all forces of nature (including gravity) to become unified. The structure of space-time at Planckian energies is one of the fundamental unanswered questions in physics today. At such scales, geometry can no longer be described in terms of the Riemannian geometry and General Relativity; one should search for a paradigm of geometry within the quantum framework. Such an attempt has been realised within the concept of NonCommutative Geometry (NCG).

To be more precise, considering the SM minimally coupled to gravity, the physical laws at sufficiently low energies can be described by the sum $L = L_{EH} + L_{SM}$. The symmetry group of $L_{EH}$ is, by the equivalence principle, the diffeomorphism group, $\text{Diff}(\mathcal{M})$, of the space-time manifold $\mathcal{M}$. However, the symmetry of the gauge theory in $L_{SM}$, is the group of local gauge transformations...
$G_{\text{SM}} = C^\infty(\mathcal{M}, U(1) \times SU(2) \times SU(3))$. Thus, considering the Lagrangian $\mathcal{L}$, the full symmetry group $G$ will be a semidirect product $G(\mathcal{M}) = G_{\text{SM}}(\mathcal{M}) \rtimes \text{Diff}(\mathcal{M})$. To argue that the whole theory is pure gravity on a space $\mathcal{M}$, one should find such a space for which $G = \text{Diff}(\mathcal{M})$. However, it is not possible to find such a space among ordinary manifolds, instead one needs to consider noncommutative spaces. The noncommutative space is a product $\mathcal{M} \times \mathcal{F}$ of an ordinary space-time manifold $\mathcal{M}$, by a finite (i.e., the algebra of coordinates on $\mathcal{M}$ is finite dimensional) noncommutative space $\mathcal{F}$.

To extract physical applications of NCG we will use its main idea, namely that all information about a physical system is contained within the algebra of functions, represented as operators in a Hilbert space, while the action and metric properties are encoded in a generalised Dirac operator. We will then look for a geometry (in the noncommutative sense, i.e., by specifying an algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ and a generalised Dirac operator $D$), such that the associated action function produces the SM of electroweak and strong interactions with all its refinements prescribed by experimental data $^1$.

There is a very simple noncommutative algebra $\mathcal{A}$, whose group of inner automorphisms $^2$ corresponds to the group of gauge transformations $G_{\text{SM}}(\mathcal{M})$, and it has a quotient that corresponds exactly to diffeomorphisms $^3$. The noncommutative algebra $\mathcal{A}$ is a direct sum $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with $\mathbb{C}$, $\mathbb{H}$, $M_3(\mathbb{C})$ denoting the algebra of complex numbers, quaternions, and $3 \times 3$ complex matrices, respectively. The algebra $\mathcal{A}$ corresponds to a finite space where the SM fermions and the Yukawa parameters determine the spectral geometry. The Hilbert space $\mathcal{H}$ is finite dimensional and admits the set of elementary fermions as a basis. The fermionic fields acquire mass through the spontaneous symmetry breaking produced by the Higgs field. The Standard Model of elementary particle physics provides an extraordinary example of a spectral triple $^4$ in the noncommutative setting $^5$. The exciting outcome of this theory is that the Higgs appears naturally as the “vector” boson of the internal noncommutative degrees of freedom.

In the past, the connection between strings and NCG has been investigated, while more recently connections between NCG and Loop Quantum Gravity are emerging. Given the plethora of very precise high energy physics data from astroparticle and cosmology, possibly also combined with the Large Hadron Collider, the geometry of space and the laws of physics at the Planck energy scale will not remain a mystery for much longer.

### Elements of NCG

Consider the extension of our smooth four-dimensional manifold $\mathcal{M}$, by taking the product of it with a discrete noncommuting manifold $\mathcal{F}$ of KO-homology dimension (i.e., the dimension modulo 8) equal $^4$ to 6. This internal space has dimension 6 to allow fermions to be simultaneously Weyl and chiral (as within string theory), whilst it is discrete to avoid the infinite tower of massive particles that are produced in string theory. The noncommutative nature of $\mathcal{F}$ is given by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involution of operators on the Hilbert space $\mathcal{H}$, which is essentially the algebra of coordinates, and $D$ is a self-adjoint unbounded operator $^5$ in $\mathcal{H}$, such that all commutators $[D, a]$ are bounded for $a \in \mathcal{A}$, and $(D - \lambda)^{-1}$ is compact for any $\lambda \notin \mathbb{R}$. The operator $D$ corresponds to the inverse line element of Riemannian geometry, whilst the commutator $[D, a], a \in \mathcal{A}$ will play the rôle of the differential quotient $d\phi/ds$, with $ds$ the unit of length.

By assuming that the algebra constructed in $\mathcal{M} \times \mathcal{F}$ is symplectic-unitary, the algebra $\mathcal{A}$ is restricted to be of the form $\mathcal{A} = M_k(\mathbb{H}) \oplus M_2(\mathbb{C})$, where $k = 2a$. The choice $k = 4$ is the first value that produces the correct number of fermions in each generation $^6$ (note however that the number of generations is an assumption in the theory). Finally, the Dirac operator $D$ connects $\mathcal{M}$ and $\mathcal{F}$ via the action functional, called spectral action, of the form $\text{Tr} (f(D/\Lambda))$, where $f$ is a test function (a smooth even function with fast decay at infinity) and $\Lambda$ is the cut-off energy scale, introduced so that $D/\Lambda$ becomes dimensionless $^6$. The expression $\text{Tr} (f(D/\Lambda))$ is taken as a natural spectral formulation of gravity, while it can be also used for spaces which are not Riemannian, and in particular for our choice of $\mathcal{M} \times \mathcal{F}$. Moreover, the spectral action has three main advantages. Firstly, when $f$ is a cut-off function (so, $f \geq 0$), the spectral action is just counting the number of eigenstates of $D$ in the interval $[-\Lambda, \Lambda]$, and secondly $\text{Tr} (f(D/\Lambda)) \geq 0$, namely it has the correct sign for a Euclidean action. Thirdly,

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$^1$ The self-adjoint operator in a Hilbert space $\mathcal{H}$ is the quantum analogue of the classical real variable. More precisely, complex and real variables, differentials and integrals have their corresponding analogues in the quantised calculus dictated by the noncommutative differential geometry.

$^2$ Corresponding in physics to internal symmetries.

$^3$ The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ encodes the geometry, given as a Hilbert space representation of the pair $(\mathcal{A}, D)$.

$^4$ The Standard Model with neutrino mixing favors the shift of dimension from the (familiar) 4 to 10 = 4 + 6 = 2 modulo 8 $^3$.

$^5$ The operator $D$ has a direct physical meaning; it is given by the Yukawa coupling matrix which encodes the masses of the 9 elementary fermions as well as the 4 mixing parameters of the Standard Model.

$^6$ It accounts only for the bosonic part of the model. The coupling with fermions is obtained by including an additional term, namely $\text{Tr}(f(D/\Lambda)) + (1/2)(J_\psi, D\psi)$, with $J$ the real structure on the spectral triple, and $\psi$ an element in the space $\mathcal{H}$, viewed as a classical fermion $^6$. 


the functional $\text{Tr} (f(D/\Lambda))$ is invariant under the unitary group of the Hilbert space $\mathcal{H}$.

Asymptotically, it can be shown that this approach leads to an effective four dimensional action that includes all the standard model particles, with the correct couplings, including the right-handed neutrinos as well as the see-saw mechanism. The gravitational and Higgs part of this action read [7]

$$
S_{\text{grav}} = \int \left( \frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \gamma_0 R^* R^* + \xi_0 |\mathbf{H}|^2 + \frac{1}{2} D_{\mu} \mathbf{H}^2 + V(|\mathbf{H}|) \right) \sqrt{g} d^4 x ,
$$

where $\mathbf{H}$ is the Higgs field, normalised to have a canonical kinetic term, the potential $V(|\mathbf{H}|) = \lambda_0 |\mathbf{H}|^4 - \mu_0^2 |\mathbf{H}|^2$, is the standard Higgs potential and the $\kappa_0^2, \alpha_0, \gamma_0, \lambda_0, \mu_0$ are specified in terms of the cut-off energy scale $\Lambda$, the couplings $a, b, c, d, e$, given by [7]

$$
a = \text{Tr}(Y_{11}^* Y_{11} + Y_{11}^* Y_{11} + 3(Y_{13}^* Y_{13} + Y_{13}^* Y_{13}^*)) ,
b = \text{Tr}(Y_{11}^* Y_{11})^2 + (Y_{11}^* Y_{11})^2 + 3(Y_{13}^* Y_{13})^2
+ 3(Y_{13}^* Y_{13}^*)^2 ,
c = \text{Tr}(Y_{R}^* Y_{R}) ,
d = \text{Tr}(Y_{R}^* Y_{R})^2 ,
e = \text{Tr}(Y_{R}^* Y_{R}^* Y_{11}^* Y_{11}) ,
$$

and the coefficients $f_k = \int_0^\infty f(v) v^{k-1} dv$ for $k > 0$ which is related to the coupling constants at unification and allows the action of the quaternions $\mathbb{H}$ to be expressed in terms of Pauli matrices as $q = f_0 + \sum_i f_i \sigma^i$. Note that the $Y$’s are used to classify the action of the Dirac operator and give the fermion and lepton masses, as well as lepton mixing, in the asymptotic version of the spectral action. The value of the coupling $\xi_0$ is set $\xi_0 = 1/12$. The couplings $a, \ldots, e$ are determined by the (unimodular) inner fluctuations of the metric.

In Ref. [10] we have shown that the equations of motion of the gravitational part of Eq. (1), in a homogeneous and isotropic background are exactly those of standard general relativity. Thus, background cosmology remains unchanged within this noncommutative approach to the standard model. We emphasise that this is the effect of the purely geometrical terms; the term $R^* R^*$ is topological and hence plays no rôle in dynamics, while the term $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ vanishes for homogeneous and isotropic metrics. Thus, we are left only with the standard Einstein-Hilbert term. It is important to remember however that inhomogeneous perturbations to this background will evolve differently from the equivalent classical system.

Equation (1) implies that, in addition to the cosmological constant term $\gamma_0$, which we neglect here, the geometry is nonminimally coupled to the Higgs field. In what follows, we investigate the consequences of this nonminimal coupling, with respect to the possibility of having naturally an inflationary scenario driven by the Higgs field. Remarkably, such modifications to the Einstein-Hilbert gravity have already been recently considered in the literature [4, 11]. In those studies, the nonminimal coupling was postulated, and then shown that the scale that sets the amplitude of perturbations during Higgs driven inflation is $\lambda_0/\xi_0^2$, rather than simply $\lambda_0$ as is the case without this additional nonminimal coupling. Indeed, this reduction in the amplitude of induced perturbations allows this Higgs field to satisfy the requirements of the standard model, as well as inflation simultaneously.

To be more specific, in Ref. [4] a conformal transformation of the metric was used, such that

$$
\left( \frac{1}{2\kappa_0^2} - \xi_0 |\mathbf{H}|^2 \right) \hat{R} \to -\frac{1}{2\kappa_0^2} \hat{R} .
$$

This leads to a noncanonical kinetic term for $|\mathbf{H}|$ which is removed via a re-definition of the field $|\mathbf{H}| \to |\chi|$ to give the Einstein frame action

$$
S_E = \int \left( \frac{1}{2\kappa_0^2} \hat{R} + \frac{1}{2} D_{\mu} \chi ||D^\rho \chi|| - U(\chi) \right) \sqrt{g} d^4 x ,
$$

where in the limit $|\mathbf{H}| \gg (\kappa_0 \sqrt{2\xi_0})^{-1}$, the potential $U(\chi)$, is given by

$$
U(\chi) \approx \frac{\lambda_0}{4\kappa_0^4 \xi_0^2} \left[ 1 - \exp \left( -\frac{2\chi_0}{\sqrt{6}\kappa_0} \right) \right] .
$$

It is the flatness of this potential that allows slow-roll inflation to occur. The above employed conformal transformation allows the system to be analysed in a standard manner. Note however that the effects of such a non-minimal coupling between the geometry and the Higgs field have been also investigated directly in the Jordan frame, i.e., without doing the conformal transformation [12].

Normalising the cosmic microwave background perturbations to the WMAP5 data [13], implies the requirement

$$
\xi_0 \approx 44700 \sqrt{\Lambda_0} ,
$$

which ensures that the Higgs field can produce inflation. Moreover, the spectral index $n_s \approx 0.97$ and the tensor-to-scalar ratio $r \approx 0.003$, are well within the WMAP5 limits. This conclusion is maintained under tree level [11] and one-loop [4] running of the couplings, provided the Higgs mass is in the, experimentally viable, range

$$
136.7 \text{ GeV} < m_{\text{H}} < 184.5 \text{ GeV} \text{ for } m_{\text{top}} = 171.2 \text{ GeV} .
$$

Note that two-loop calculations may lead to significant effects on the running of the Higgs potential [14, 17].

In the context of the noncommutative approach to the SM however, the couplings $\xi_0$ and $\lambda_0$ are not arbitrary.
Namely, since the action, Eq. (1), comes from an underlying theory, we have some control on the values of the couplings $\xi_0$ and $\lambda_0$. More precisely,

$$\xi_0 = \frac{1}{12} \quad \text{and} \quad \lambda_0 = \frac{\pi^2}{2 f_0} \frac{b}{a^2} . \quad (8)$$

Hence, within the noncommutative approach to the SM, for inflation to be naturally viable without the need to introduce additional nonstandard model fields, we need

$$\frac{b}{f_0 a^2} \approx 7.04 \times 10^{-13} , \quad (9)$$

where $a, b$ are defined in Eq. (2) and $f_0 = f(0)$, with $f_k$ defined as previously.

A detailed analysis of the running of these values with the cut-off scale would determine the energy scale at which inflation occurred. More precisely, one should compare the requirements, Eq. (9), so that the Higgs field can play the rôle of the inflaton, with the stemmings from the particle physics phenomenology of the SM. Unfortunately, the restrictions of the running of the couplings, found in the literature [7], have neglected the nonminimal coupling of the Higgs to the geometry, which as we have seen is indeed crucial for the inflation to be successful. Nevertheless, a back of the envelope calculation shows that since $b/a^2 \geq 1/4$ [7], a relation which is valid even for a large tau neutrino Yukawa coupling, Eq. (9) implies a severe constraint on $f_0$. Alternatively, since $(g_1^2 f_0)/(2\pi^2) = (1/4)$, one obtains equivalently a constraint on the gauge coupling $g_3$. Since $g_1^2 = g_2^2 = (5/3)g_3^2$ and at the unification scale $\Lambda \sim 1.1 \times 10^{17}\text{GeV}$, the three coupling constants are $\alpha_i(\Lambda) = g_i^2/(4\pi)$, we obtain

$$f_0 = \frac{\pi}{8\alpha_2(\Lambda)} \sim 18.45 , \quad (10)$$

which does not satisfy the requirement, Eq. (9), so that the Higgs field can play the rôle of the inflaton. More precisely, the constraint on $f_0$ so that inflation can be naturally incorporated here, reads $f_0 \sim 3.55 \times 10^{11}$.

Higgs inflation in the context of conventional cosmological models (as e.g., [4,11]) has been criticised [16], arguing that quantum corrections to the semi-classical approximation may no longer be neglected for such exotic inflationary models. However, this criticism is not applicable to the noncommutative approach employed here. More precisely, in conventional Higgs inflation there is a strong coupling, namely $\xi_0 \sim 10^4$ between the Higgs field and the Ricci curvature scalar. Thus, the effective theory ceases to be valid beyond a cut-off scale $m_{\text{Pic}}/\xi_0$, while one should know the Higgs potential profile for the field values relevant for inflation, namely $m_{\text{Pic}}/\sqrt{\xi_0}$, values which is much bigger than the cut-off. Clearly, this argument does not apply to the noncommutative Higgs drive inflation, since there $\xi_0 = 1/12$.

**CONCLUSIONS**

Considering the product of ordinary Euclidean space-time (i.e., space-time but with imaginary time) by a finite space (with the properties discussed above), a geometric interpretation of the experimentally confirmed effective low energy model of particle physics was given in Ref. [7].

Investigating cosmological consequences of this proposal, we have concluded that the Higgs field can play the rôle of the inflaton field within the noncommutative approach to the standard model, provided inflation will take place at a scale higher than the strong weak unification scheme, $10^{17}\text{GeV}$. In order to find the precise value of this scale, a detailed analysis of the running of the couplings above unification would be required. However, let us emphasise that the aim of this paper is simply to note that within the noncommutative geometry approach to unifying gravity and the Standard Model, it is possible to have an epoch of inflation sourced by the dynamics of the Higgs field.

In addition, this type of noncommutative inflation could have specific consequences that would discriminate it from alternative models. In particular, since the theory contains all of the Standard Model fields, along with their couplings to the Higgs field, which in this scenario plays the rôle of the inflaton, a quantitative investigation of reheating should be possible. More significantly, the cosmological evolution equations for inhomogeneous perturbations differs from those of the standard Friedmann-Lemaitre-Robertson-Walker cosmology [10]. This raises the possibility that signatures of this noncommutative inflation could be contained within the cosmic microwave background power spectrum.

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