Commutative Information Algebras: Representation and Duality Theory

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Abstract

Information algebras arise from the idea that information comes in pieces which can be aggregated or combined into new pieces, that information refers to questions and that from any piece of information, the part relevant to a given question can be extracted. This leads to a certain type of algebraic structures, basically semilattices endowed with additional unary operations. These operations essentially are (dual) existential quantifiers on the underlying semilattice. The archetypical instances of such algebras are semilattices of subsets of some universe, together with the saturation operators associated with a family of equivalence relations on this universe. Such algebras will
be called *set algebras* in our context. Our first result is a basic representation theorem: Every abstract information algebra is isomorphic to a set algebra. When it comes to combine pieces of information, the idea to model the logical connectives *and*, *or* or *not* is quite natural. Accordingly, we are especially interested in information algebras where the underlying semilattice is a lattice, typically distributive or even Boolean. A major part of this paper is therefore devoted to developing explicitly a full-fledged natural duality theory - in the sense of (Clark, 1998) - extending Stone resp. Priestley duality in a suitable way in order to take into account the additional operations.
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1 Introduction and Overview

Information algebras arise from the idea that information comes in pieces which can be aggregated or combined into new pieces, that information refers
1 INTRODUCTION AND OVERVIEW

to questions and that from any piece of information, the part relevant to a
given question can be extracted. This view leads to two different, but es-
tentially equivalent types of algebraic structures, domain-free and labeled
information algebras [Kohlas, 2003; Kohlas & Schmid, 2014]. Archetypical
instances of such algebras are so-called set algebras (for the domain-free
version), resp. relational algebras connected to relational database theory (for
the labeled version). In both instances questions are represented by the sets
of all their possible answers, and pieces of information are thought of as cer-
tain sets of possible answers, giving a precise meaning to the elements of
information algebras. This paper will deal with domain-free type of informa-
tion information algebras exclusively.

The natural question is therefore whether and to what extent abstract in-
formation algebras are isomorphic to such set algebras. Partial answers
were given in [Kohlas, 2003]. Here, we want to address the problem more
systematically. The problem is similar to representation problems in lat-
tice theory, where Boolean algebras or distributive lattices are shown to be
isomorphic to subset algebras resp. lattices of certain topological spaces
(Davey & Priestley, 2002). A substantial part of this paper is motivated by
the classical duality theories for Boolean algebras resp. distributive lattices,
and we extend Stone resp. Priestley duality to domain-free information al-
gebras.

Commutative domain-free information algebras are introduced in Section 2.
The notion of a set (information) algebra is defined, and a few illustrative
examples of such algebras are given. For a more complete presentation of
information algebras we refer to [Kohlas, 2003] and for more examples to
[Pouly & Kohlas, 2011]. A parallel representation theory for the associated
so-called labeled information algebras must be postponed; for some partial
results see [Kohlas, 2003]. An information algebra induces a partial order on
its elements, reflecting the information contents of the pieces of information.
We show that so-called truncated up-sets (relative to this order) of elements
of an information algebra may be used to construct a set algebra isomorphic
to the given algebra, providing a first general representation theorem.

To see set algebras at work, we consider in Section 3 atomic or atomistic
information algebras. Such algebras have a very natural representation as
set algebras consisting of sets of atoms, loosely speaking, of maximally infor-
mative pieces of information. This representation could be used directly to
develop a representation theory for information algebras based on a Boolean algebra, since maximal ideals in such algebras are atoms in the ideal completion of the underlying Boolean algebra, resulting in an extension of Stone’s representation theory for Boolean algebras. We do not elaborate this approach, since the Boolean case is subsumed in that of information algebras based on distributive lattices, to be considered in full generality in the following Section 4.

The treatment of quantifiers on distributive lattices by (Cignoli, 1991), generalizing Halmos’ theory of monadic Boolean algebras, will provide the basis for a representation theory of information algebras based on distributive lattices, extending and generalizing the Boolean case. We will show in Section 4 that, in fact, there is a full-fledged natural duality in the sense of (Clark, 1998) between the categories of commutative domain-free information algebras based on distributive lattices with morphisms as defined in Subsection 2.2 on one side and Priestley spaces equipped with a semigroup of commuting and separating equivalences and morphisms as defined in Subsection 4.4 on the other. In Subsection 4.6 we consider the special case of information algebras based on Boolean lattices. Finally, in Subsection 4.7 we look in some detail at information algebras carried by finite distributive lattices. It turns out that this class is as close to an elementary class in the sense of first order logic as one can possibly get.

For the sake of completeness, it should be noted that embedding information algebras into set algebras is not the only way to model information algebras with sets. Already the ideal completion of an information algebra embeds the information algebra into an algebra of sets, namely the algebra of its ideals. But this is not a set algebra in the strict sense used in this paper. Also, it is well known that information algebras are closely related to information systems (in the sense of domain theory), see (Kohlas, 2003). Again, this yields not a representation theory in the sense considered here. As mentioned, most of the results contained in this paper should have, in some way or another, a counterpart in the labeled version of information algebras. This is a subject still to be worked out.

From the point of view of universal algebra, information algebras as considered in this paper can be seen as semilattices endowed with a family of (dual) existential quantifiers which form a commutative, idempotent semigroup with respect to composition. Many examples of such structures can be found in al-
2 Domain-free Information Algebras

2.1 Structures

We will define a type of algebra describing the interaction between “pieces of information” and “questions” as discussed in the introduction.

Defining operations

Beginning with “pieces of information”, let $\Phi$ be an abstract set whose elements are thought to represent such pieces, denoted by lower case Greek letters. We assume that $\Phi$ is equipped with a binary operation $\cdot$:

$Combination: \cdot : \Phi \times \Phi \rightarrow \Phi.$

For $\phi, \psi \in \Phi$, the element $\phi \cdot \psi$ represents the aggregation of the pieces of information represented by $\phi$ resp. $\psi$. Mimicking the intuitive properties of “aggregation”, combination is assumed to be associative, commutative and idempotent. Additionally, we assume that there exist in $\Phi$ a unit or neutral element 1 and a null element 0 satisfying $1 \cdot \phi = \phi \cdot 1 = \phi$ resp. $0 \cdot \phi = \phi \cdot 0 = 0$. 1 represents vacuous information which does not change any other information under combination. 0 represents contradiction and
destroys any other information. Summing up, \((\Phi; \cdot, 1, 0)\) is a commutative idempotent semigroup with a neutral resp. null element.

Turning to “questions”, we think of an abstract set \(Q\) whose elements represent such questions. Elements of \(Q\) will typically be denoted by \(x, y, z, \ldots\) etc.. In view of the discussion in the introduction, we will not deal with the questions \(x \in Q\) themselves, but represent them, for each \(x \in Q\), by a unary operation \(\epsilon_x : \Phi \rightarrow \Phi\) which extracts, from every \(\phi \in \Phi\), the piece of information \(\epsilon_x(\phi)\) which is relevant to question \(x\) (this amounts to replacing \(x\) by the graph of the map \(\epsilon_x\)):

\[
\text{Extraction: } \epsilon_x : \Phi \rightarrow \Phi.
\]

The set of all such operations will be denoted by \(E(\Phi, Q)\) or just \(E\) if \(\Phi\) and \(Q\) are clear from the context.

The members of \(E\) will be required to satisfy all \(x\)-\(y\)-instances (for \(x, y \in Q\)) of the following conditions:

1. \(\epsilon_x(0) = 0\) \hspace{1cm} (N)
2. \(\phi \cdot \epsilon_x(\phi) = \phi\), for all \(\phi \in \Phi\) \hspace{1cm} (A)
3. \(\epsilon_x(\epsilon_x(\phi) \cdot \psi) = \epsilon_x(\phi) \cdot \epsilon_x(\psi)\), for all \(\phi, \psi \in \Phi\) \hspace{1cm} (Q)

(N) says that contradiction cannot be eliminated by extraction. (A) states that information extracted from \(\phi\) is contained in \(\phi\). The crucial condition is (Q) as we shall see. Operations \(\epsilon : \Phi \rightarrow \Phi\) satisfying (N), (A) and (Q) will be called extraction operators.

At this point, we impose an additional condition on extraction operators, defining the scope of this paper: We require that the order of successive extractions does not matter, that is,

4. \(\epsilon_x(\epsilon_y(\phi)) = \epsilon_y(\epsilon_x(\phi))\), for all \(\phi \in \Phi\) and \(x, y \in Q\) \hspace{1cm} (C).

Structures \((\Phi, E)\) with \((\Phi; \cdot, 0, 1)\) a commutative idempotent semigroup with null and unit and \(E\) satisfying conditions 1. to 4. will be called commutative domain-free information algebras.
Lemma 2.1. Let $(\Phi, E)$ a commutative domain-free information algebra. Then
5. $\epsilon_x(\epsilon_x(\phi)) = \epsilon_x(\phi)$, for all $\phi \in \Phi$ and $x \in Q$. (I)

Proof. Note that $\epsilon_x(1) = 1 \cdot \epsilon_x(1) = 1$ by (A). Using (Q), we get $\epsilon_x(\epsilon_x(\phi)) = \epsilon_x(\epsilon_x(\phi) \cdot 1) = \epsilon_x(\phi) \cdot \epsilon_x(1) = \epsilon_x(\phi) \cdot 1 = \epsilon_x(\phi)$. □

Note that (I) already follows from (A) and (Q).

Lemma 2.2. If $E$ satisfies all instances of (N), (A), (C) and (Q), then so does $E \cup \{\epsilon_x \circ \epsilon_y\}$, for any $\epsilon_x, \epsilon_y \in E$.

Proof. (N) is obvious. For (A),

\[
\begin{align*}
\phi \cdot \epsilon_x \circ \epsilon_y(\phi) \\
= (\phi \cdot \epsilon_x(\phi)) \cdot \epsilon_x \circ \epsilon_y(\phi) & \text{ by (A)} \\
= \phi \cdot (\epsilon_x(\phi) \cdot \epsilon_y \circ \epsilon_x(\phi)) & \text{ by (C)} \\
= \phi \cdot \epsilon_x(\phi) & \text{ by (A)} \\
= \phi & \text{ by (A)}
\end{align*}
\]

For (Q),

\[
\begin{align*}
\epsilon_x \circ \epsilon_y(\epsilon_x \circ \epsilon_y(\phi) \cdot \psi) \\
= \epsilon_y \circ \epsilon_x(\epsilon_x \circ \epsilon_y(\phi) \cdot \psi) & \text{ by (C)} \\
= \epsilon_y(\epsilon_x \circ \epsilon_y(\phi) \cdot \epsilon_x(\psi)) & \text{ by (Q)} \\
= \epsilon_y(\epsilon_y \circ \epsilon_x(\phi) \cdot \epsilon_x(\psi)) & \text{ by (C)} \\
= \epsilon_y \circ \epsilon_x(\phi) \cdot \epsilon_y \circ \epsilon_x(\psi) & \text{ by (Q)} \\
= \epsilon_x \circ \epsilon_y(\phi) \cdot \epsilon_x \circ \epsilon_y(\psi) & \text{ by (C)}
\end{align*}
\]

For (C), $(\epsilon_x \circ \epsilon_y) \circ \epsilon_z = \epsilon_z \circ (\epsilon_x \circ \epsilon_y)$ using associativity of composition and (C) for $E$. □

Corollary 2.3. If $E$ satisfies all instances of (N), (A), (C), (Q) and (I) then so does $E^\circ$, the closure of $E$ under composition $\circ$. Obviously, $(E^\circ, \circ)$ is the least idempotent semigroup satisfying (N), (A),(C) and (Q) containing $E$ as a subset.
In order to develop a meaningful algebraic theory of commutative information algebras avoiding partial morphisms, we assume henceforth, based on Corollary 2.3, that $E$ is closed under composition.

Altogether, we have set up a two-sorted algebra $A = (\Phi, \cdot, 1, 0; E, \circ)$. Such algebras will be called \textit{commutative domain-free information algebras} or, for short, \textit{CDF information algebras}. However, in order to avoid cluttering the paper with a plethora of CDF’s, we agree that "information algebra" will mean "commutative domain-free information algebra" if not explicitly stated otherwise.

We introduce the following notation: Write $\Phi$ for the commutative semigroup $(\Phi; \cdot, 1, 0)$, $E$ for the commutative semigroup $(E; \circ)$ and finally $A = (\Phi; E)$. More generally, if $S$ is any set carrying some structure, we will write $S$ for the set equipped with the type of structure under consideration.

\textit{Introducing order on $\Phi$}

It is well-known that any idempotent commutative semigroup may be equipped with a compatible order relation in exactly two ways. Explicitly, in $(\Phi; \cdot)$ we may define an order $\leq_1$ by $\phi \leq_1 \psi \iff \phi \cdot \psi = \phi$ respectively by $\phi \leq_2 \psi \iff \phi \cdot \psi = \psi$. For $\Phi$, we will use $\leq_2$:

\textbf{Definition 2.4.} Information order: For $\phi, \psi \in \Phi$, we put $\phi \leq \psi$ iff $\phi \cdot \psi = \psi$.

This is appropriate since $\phi \cdot \psi = \psi$, in a natural way, means that $\phi$ is less informative than $\psi$. It is easy to check that in the ordered set $(\Phi; \leq)$ the combination $\phi \cdot \psi$ is in fact the \textit{supremum} of $\phi$ and $\psi$, that is, $\phi \cdot \psi = \text{sup}_{\leq} \{\phi, \psi\}$. We use $\leq$ to define a binary operation $\lor : \Phi \times \Phi \rightarrow \Phi$ (called join) on $\Phi$ by

$$\phi \lor \psi := \text{sup}_{\leq} \{\phi, \psi\} \quad (= \phi \cdot \psi).$$

This turns $\Phi$ into a join-semilattice $\Phi = (\Phi; \lor, 1, 0)$ with least element 1 and greatest element 0, which neatly reflects the fact that contradiction 0 dominates every piece of information, and that the vacuous information 1 is contained in every piece of information.
The interplay between the *alter egos* of $\Phi$ as *semigroup* resp. *semilattice* resp. *ordered set* will prove to be very fruitful. So combination will be denoted by both $\cdot$ and $\lor$ in order to indicate which aspect is prevalent in a given context.

Using $\leq$ and $\lor$, the conditions (N), (A) and $Q$ for an extraction operator may be rewritten as follows:

1. $\epsilon(0) = 0$,
2. $\epsilon(\phi) \leq \phi$ for all $\phi \in \Phi$,
3. $\epsilon(\epsilon(\phi) \lor \psi) = \epsilon(\phi) \lor \epsilon(\psi)$, for all $\phi, \psi \in \Phi$.

An operator $\epsilon$ on a semilattice $(\Phi; \land, 0)$ satisfying these three conditions is called an *existential quantifier* in algebraic logic. However, it must be noted that in the relevant literature rather the order relation $\leq_1$ is used to define an existential quantifier. This gives rise to a meet-semilattice $(\Phi; \land, 0)$ with $\phi \land \psi := \inf_{\leq_1}\{\phi, \psi\}$ and least element 0. The three conditions then read $\epsilon(0) = 0$, $\epsilon(\phi) \geq \phi$ and $\epsilon(\epsilon(\phi) \land \psi) = \epsilon(\phi) \land \epsilon(\psi)$. We could have called our variant a “dual existential quantifier” with the risk of cluttering the paper with a plethora of “duals” – from which we shrank back. In any case, our choice of $\leq_2$ over $\leq_1$ is amply justified by the natural order between pieces of information.

**Lemma 2.5.** An extraction operator preserves (information) order.

**Proof.** Assume $\phi \leq \psi$. Since $\epsilon(\phi) \leq \phi$, we have $\epsilon(\phi) \leq \psi$, thus $\epsilon(\phi) \cdot \psi = \psi$ and $\epsilon(\epsilon(\phi) \cdot \psi) = \epsilon(\psi)$. Using (Q), we obtain $\epsilon(\phi) \cdot \epsilon(\psi) = \epsilon(\psi)$, that is, $\epsilon(\phi) \leq \epsilon(\psi)$. \hfill $\square$

### 2.2 Homomorphisms and Subalgebras

Let $\mathcal{A} = (\Phi; E)$ and $\mathcal{B} = (\Psi; D)$ any two information algebras. We do not notationally distinguish between the operations in the two algebras as their meaning will be clear from the context.

**Definition 2.6.** A pair $(f, g)$ of maps $f : \Phi \to \Psi$, $g : E \to D$ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ iff...
1. \( f(\phi \cdot \psi) = f(\phi) \cdot f(\psi) \) for all \( \phi, \psi \in \Phi \),

2. \( f(0) = 0 \) and \( f(1) = 1 \).

3. \( g(\epsilon \circ \eta) = g(\epsilon) \circ g(\eta) \) for all \( \epsilon, \eta \in E \).

4. \( f(\epsilon(\phi)) = g(\epsilon(f(\phi))) \) for all \( \phi \in \Phi \) and \( \epsilon \in E \).

Note that in a homomorphism \((f, g)\) the map \( f \) is order-preserving.

**Lemma 2.7.** If for a homomorphism \((f, g)\) both \( f^{-1} \) and \( g^{-1} \) exist, then \((f^{-1}, g^{-1})\) also satisfies condition 2.6.(4).

**Proof.** Let \((\psi, \delta) \in (\Psi, D)\). Then \( \psi = f(\phi) \) and \( \delta = g(\epsilon) \) for an unique pair \((\phi, \epsilon) \in (\Phi, E)\). Now \( f(\epsilon(\phi)) = g(\epsilon)(f(\phi)) \) by assumption. Applying \( f^{-1} \) on both sides, we obtain \( \epsilon(\phi) = f^{-1}(g(\epsilon)(f(\phi))) \) or \( g^{-1}(\delta)(f^{-1}(\psi)) = f^{-1}(\delta(\psi)) \). \( \square \)

**Corollary 2.8.** A homomorphism \((f, g)\) is an isomorphism iff both \( f \) and \( g \) are bijective.

If the maps \( f \) and \( g \) both are one-to-one, then \( A \) is said to be embedded into \( B \). If \( \Phi \subseteq \Psi \) and \( E \subseteq D \) are such that

(i) \( \Phi \) is closed under the combination operation of \( \Psi \) and contains the neutral and null elements of \( \Psi \),

(ii) \( E \) is closed under the composition operation of \( D \), and

(iii) for all \( \eta \in E \) and \( \phi \in \Phi \), the element \( \eta(\phi) \) belongs to \( \Phi \),

then \( A \) is called a subalgebra of \( B \). Clearly, then, the pair of the identity maps of \( \Phi \) and \( E \) into \( \Psi \) resp. \( D \) is an embedding of \( A \) into \( B \). Also, if \((f, g)\) is a homomorphism from \( A \) into \( B \), then the image \((f(\Phi), g(E))\) of \( A \) is a subalgebra of \( B \).

As an example and for further reference, consider an arbitrary but fixed \( \eta \in D \) and let \( \eta \Psi = \{\eta \psi : \psi \in \Psi\} \). We have \( \eta 1 = 1 \), \( \eta 0 = 0 \) and \( \eta \psi \cdot \eta \psi' = \eta(\eta \psi \cdot \eta \psi') \) by (Q), so \( \eta \Psi \) is closed under the operations of \( \Psi \), making it a substructure \( \eta \Psi \) of \( \Psi \).
Lemma 2.9. \((\eta \Psi; D)\) is a subalgebra of \((\Psi; D)\).

Proof. (i) above is satisfied as just shown, (ii) is vacuously true, and for (iii) observe that \(\eta'(\eta\psi) = \eta(\eta'\psi) \in \eta\Psi\) by (C) for any \(\eta' \in D\).

In the sequel we are particularly interested in homomorphisms, embeddings and isomorphisms between an arbitrary information algebra \(\mathcal{A}\) and so-called set algebras, to be defined in the following section. Such embeddings and isomorphisms will be called representations of \(\mathcal{A}\).

Finally, we remark that from a category-theoretic point of view other types of morphisms may be more appropriate, see e.g. [Kohlas & Schmid, 2014] for Cartesian-closed categories of information algebras.

### 2.3 Set Algebras

So far, the set \(\Phi\) of pieces of information as well as the set \(Q\) of questions have been arbitrary abstract sets, subject only to the conditions specified for composition and extraction. We will now define a special type of information algebras - to be called set algebras - where the elements of these sets have an internal structure, described by set-theoretical constructs over some base set \(U\) \((U \neq \emptyset)\). The power set of \(U\) will be denoted by \(P(U)\).

We may equip \(P(U)\) with a lattice structure in the obvious way. To be precise, let \(\mathcal{P}(U) := (P(U); \cap, \cup, \emptyset, U)\) the bounded distributive lattice with carrier \(P(U)\), set intersection as meet, set union as join and with \(\emptyset\) resp. \(U\) as least rep. greatest elements. Due to our use of information order, we will mostly be concerned with the order dual \(\mathcal{P}^d(U)\) of \(\mathcal{P}(U)\), and especially with \((\cap, U, \emptyset)\)-reducts of the latter.

The basic idea is to consider \(U\) as a set of possible worlds. Questions \(x \in Q\) will then be modelled by equivalence relations \(\equiv_x\) on \(U\), the idea being that for \(u, u' \in U\) we have \(u \equiv_x u'\) iff question \(x\) has the same answer in the worlds \(u\) resp. \(u'\).

**Equivalences and saturation operators**

It is useful for our purposes to examine, in some detail, the set \(\text{Eq}(U)\) of all equivalence relations on \(U\). Recall that any equivalence \(\Theta \in \text{Eq}(U)\) has
an alter ego as a partition of \( U \) into pairwise disjoint nonempty sets, the equivalence classes of \( \Theta \) or, for short, \( \Theta \)-blocks. A \( \Theta \)-block thus contains, with any \( u \in B \), all \( u' \in U \) satisfying \((u, u') \in \Theta \). Abusing notation to the limit, we also write \( B \in \Theta \) to indicate that \( B \) is a \( \Theta \)-block, and \( u\Theta u' \) instead of \((u, u') \in \Theta \).

To every \( \Theta \in Eq(U) \) we associate a saturation operator \( \sigma_\Theta : P(U) \to P(U) \) defined by \( \sigma_\Theta(X) = \bigcup\{B : B \in \Theta \text{ and } B \cap X \neq \emptyset\} \), for any subset \( X \subseteq U \). Accordingly, a set \( X \subseteq U \) will be called \( \sigma_\Theta \)-saturated iff \( \sigma_\Theta(X) = X \).

The following properties of saturation operators will be crucial for our purposes:

**Lemma 2.10.** Let \( \Theta \in Eq(U) \) with associated saturation operator \( \sigma_\Theta \). Then for all \( X, Y \subseteq U \):

1. \( \sigma_\Theta(\emptyset) = \emptyset \),
2. \( X \subseteq \sigma_\Theta(X) \),
3. \( X \subseteq Y \) implies \( \sigma_\Theta(X) \subseteq \sigma_\Theta(Y) \),
4. \( X = \sigma_\Theta(X) \) and \( Y = \sigma_\Theta(Y) \) jointly imply \( X \cap Y = \sigma_\Theta(X \cap Y) \),
5. \( \sigma_\Theta(\sigma_\Theta(X) \cap Y) = \sigma_\Theta(X) \cap \sigma_\Theta(Y) \),
6. \( \sigma_\Theta(X \cup Y) = \sigma_\Theta(X) \cup \sigma_\Theta(Y) \).

**Proof.** For 1., we have \( \sigma_\Theta(\emptyset) = \bigcup\{B \in \Theta : B \cap \emptyset \neq \emptyset\} = \emptyset \).

Items 2. and 3. are obvious.

For 4., observe that \( X = \sigma_\Theta(X) \) iff \( X \) is a set union of whole \( \Theta \)-blocks, and that for two \( \Theta \)-blocks \( B_1 \) and \( B_2 \), either \( B_1 \cap B_2 = \emptyset \) or \( B_1 = B_2 \).

For 5., observe that \( \sigma_\Theta(X \cap Y) \subseteq \sigma_\Theta(X) \cap \sigma_\Theta(Y) \), so \( \sigma_\Theta(\sigma_\Theta(X) \cap Y) \subseteq \sigma_\Theta(\sigma_\Theta(X) \cap \sigma_\Theta(Y)) = \sigma_\Theta(X) \cap \sigma_\Theta(Y) \) by 3. and 4. For the reverse inclusion, we have \( \sigma_\Theta(X \cap Y) = \bigcup\{B \in \Theta : B \cap X \neq \emptyset \neq B \cap Y\} \). Obviously, for each such \( B \) we have \( B \cap \sigma_\Theta(X) = B \), so \( B \cap \sigma_\Theta(X) \cap Y \neq \emptyset \) and \( B \) participates in the union of all \( B' \in P \) forming \( \sigma_\Theta(\sigma_\Theta(X) \cap Y) \). So \( \sigma_\Theta(X \cap \sigma_\Theta(Y) \subseteq \sigma_\Theta(\sigma_\Theta(X) \cap Y) \).

Finally, 6. is immediate. \( \square \)

**Corollary 2.11.** \( \sigma_\Theta \) is an extraction operator on the \((\cap, \emptyset)\)-reduct of \( P(U)^d \), for any \( \Theta \in Eq(U) \).
Proof. Items 1., 2. and 5. in Lemma 2.10 are just conditions (N), (A) and (Q) for an extraction operator, in their semilattice version.

Recall that the relational product \( \ast \) of two binary relations \( R, S \subseteq U \times U \) is given by \( R \ast S = \{(u, u') \in U \times U : \text{there exists } v \in U \text{ such that } uRvSu'\} \). In general, \( Eq(U) \) is not closed under \( \ast \). In fact, we have

Lemma 2.12. Given \( \Theta, \Gamma \in Eq(U) \), their relational product \( \Theta \ast \Gamma \) belongs to \( Eq(U) \) iff \( \Theta \ast \Gamma = \Gamma \ast \Theta \).

Proof. Since \( u\Theta u\Gamma u \) for all \( u \in U \), \( \Theta \ast \Gamma \) is reflexive. Now \( u\Theta \ast \Gamma u' \) iff for some \( v \in U \) we have \( u\Theta v\Gamma u' \). So \( \Theta \ast \Gamma \) is symmetric iff \( u\Theta v\Gamma u' \) implies the existence of \( w \in U \) such that \( u\Theta w\Gamma u \) for all \( u, u' \in U \). So \( \Theta \ast \Gamma \subseteq \Gamma \ast \Theta \). The reverse inclusion is obtained in the same way and we have that \( \Theta \ast \Gamma \subseteq \Theta \ast \Gamma \). It remains to establish transitivity of \( \Theta \ast \Gamma \). Assume \( u\Theta \ast \Gamma w \) and \( w\Theta \ast \Gamma u' \). So there are \( x, y \in U \) such that \( u\Theta x\Gamma w\Theta y\Gamma u' \). So \( x\Gamma \ast y \) and using \( \Gamma \ast \Theta \) we find \( w' \in U \) such that \( x\Theta w'\Gamma y \). Putting all together we have \( u\Theta x\Theta w'\Gamma y \) and by transitivity \( u\Theta v\Gamma u' \), that is, \( u\Theta \ast \Gamma u' \) as desired.

Equivalences \( \Theta, \Gamma \) satisfying \( \Theta \ast \Gamma = \Gamma \ast \Theta \) will be called commuting. The following lemma collects some properties of commuting equivalences:

Lemma 2.13. Assume \( \Theta, \Gamma \in Eq(U) \) commute. Then

1. \( \Theta \ast \Gamma \) is the least equivalence relation on \( U \) containing \( \Theta \) and \( \Gamma \) (as subsets of \( U \times U \)),
2. \( \sigma_{\Theta \ast \Gamma} = \sigma_{\Theta} \circ \sigma_{\Gamma} \),
3. \( \sigma_{\Theta} \circ \sigma_{\Gamma} = \sigma_{\Gamma} \circ \sigma_{\Theta} \).

Proof. 1. Assume \( u\Theta u' \). Then \( u\Theta u'\Gamma u' \) and thus \( u\Theta \ast \Gamma u' \), so \( \Theta \subseteq \Theta \ast \Gamma \), and analogously \( \Gamma \subseteq \Theta \ast \Gamma \). Conversely, let \( \Theta, \Gamma \subseteq \Lambda \subseteq Eq(U) \) and and assume \( u\Theta \ast \Gamma u' \). This means that \( u\Theta v\Gamma u' \) for some \( v \in U \). Hence \( u\Lambda v\Lambda u' \) and so \( u\Lambda u' \).

2. Let \( X \subseteq U \). Then \( u \in \sigma_{\Theta \ast \Gamma}(X) \) iff there exists \( x \in X \) such that \( u\Theta \ast \Gamma x \). Now \( u\Theta \ast \Gamma x \) iff there exists \( v \in U \) such that \( u\Theta v\Gamma x \). But this is equivalent with \( u \in \sigma_{\Theta}(\sigma_{\Gamma}(X)) \).

3. By 2. since \( \Theta, \Gamma \) commute.
Call a subset $T \subseteq Eq(U)$ $\star$-closed iff $\Theta, \Gamma \in T$ implies $\Theta \star \Gamma \in T$.

**Lemma 2.14.** A subset $T \subseteq Eq(U)$ is $\star$-closed iff $T = (T; \star|_T)$ is a commutative idempotent semigroup.

**Proof.** By Lemma 2.12 $T$ is $\star$-closed iff $\star|_T$ is commutative. Moreover, $\star$ is associative and idempotent on the whole of $Eq(U)$.

We will refer to such semigroups shortly as $\star$-semigroups in $Eq(U)$. For an arbitrary such $\star$-semigroup $T$ let $Sat(T) = \{ \sigma_\Theta : \Theta \in T \}$ and put $Sat(T) = (Sat(T); \circ)$.

**Proposition 2.15.** $Sat(T)$ is a commutative idempotent semigroup isomorphic to $T$.

**Proof.** The map $\Theta \mapsto \sigma_\Theta$ from $T$ to $Sat(T)$ is one-to-one and onto as $\Theta$ may be recovered from $\sigma_\Theta$ by $u\Theta u'$ iff $u' \in \sigma_\Theta(\{u\})$. It is a semigroup homomorphism by Lemma 2.13.

Our interest in $\star$-semigroups and their associated semigroups of saturation operators is based on the following case:

Let $(\Phi, E)$ be any information algebra. For $\epsilon \in E$ define an equivalence relation $\equiv_\epsilon$ in $Eq(\Phi)$ by $\phi \equiv_\epsilon \psi$ if $\epsilon(\phi) = \epsilon(\psi)$, that is, $\equiv_\epsilon$ is $\ker \epsilon$, the kernel of $\epsilon$.

**Theorem 2.16.** For any $\epsilon, \eta \in E$, we have $\ker \epsilon \star \ker \eta = \ker (\epsilon \circ \eta)$, that is, $\mathcal{E} = (\{ \ker \epsilon : \epsilon \in E \}, \star)$ is a $\star$-semigroup in $Eq(\Phi)$.

**Proof.** Assume first that $(\phi, \psi) \in \ker \epsilon \star \ker \eta$. So there is $\chi \in \Phi$ such that $\epsilon \phi = \epsilon \chi$ and $\eta \chi = \eta \psi$. Now $\eta \epsilon \phi \overset{\text{by ass.}}{=} \eta \epsilon \chi \overset{(C)}{=} \epsilon \eta \chi \overset{\text{by ass.}}{=} \epsilon \eta \psi = \eta \epsilon \psi$. It follows that $(\phi, \psi) \in \ker \eta \circ \epsilon = \ker \epsilon \circ \eta$.

Conversely, assume that $(\phi, \psi) \in \ker (\epsilon \circ \eta)$. So $\epsilon \eta \phi = \epsilon \eta \psi$ and using (C), we obtain

$$\eta \epsilon \phi = \epsilon \eta \phi = \epsilon \eta \psi = \eta \epsilon \psi.$$  \hspace{1cm} (1)

Put $\xi := \epsilon \phi \cdot \eta \psi$. Then $\epsilon \xi = \epsilon (\epsilon \phi \cdot \eta \psi) \overset{(1),(Q)}{=} \epsilon \phi \cdot \eta \psi = \epsilon \phi \cdot \epsilon \eta \phi \overset{(A)}{=} \epsilon \phi$. Similarly, one obtains $\eta \xi = \eta \psi$. So $(\phi, \psi) \in \ker \epsilon \star \ker \eta$. \hfill $\square$
Construction of set algebras

We will construct an information algebra SetAlg(Φ, T) based on a join-subsemilattice Φ of $P^d(U)$ containing $U$ and $\emptyset$, and a $\star$-semigroup $\mathcal{T}$ in Eq(U).

This means that Φ is closed under ordinary set intersection and that the semilattice operation $\lor$ on Φ is given by $\phi \lor \psi = \phi \cap \psi = \phi \cdot \psi$, for all $\phi, \psi \subseteq U$ belonging to Φ. The corresponding order $\leq$ on Φ is then $\phi \leq \psi$ iff $\phi \supseteq \psi$, and $U \leq \phi \leq \emptyset$ for all $\phi \in \Phi$.

Note that for any $\star$-semigroup $\mathcal{T}$ in Eq(U) we have $\sigma_\Theta(U) = U$ and $\sigma_\Theta(\emptyset) = \emptyset$ for all $\sigma_\Theta \in \text{Sat}(\mathcal{T})$. $\mathcal{T}$ will be called Φ-compatible (or just compatible, if Φ is clear from the context) iff Φ is closed under all $\sigma_\Theta \in \text{Sat}(\mathcal{T})$, that is, $\sigma_\Theta(\phi) \in \Phi$ for all $\phi \in \Phi$, $\Theta \in \mathcal{T}$.

Define $\text{Sat}_\Phi(\mathcal{T}) = \{ \sigma_\Theta | \Theta \in \mathcal{T} \}$, and $\text{Sat}_\Phi = (\text{Sat}_\Phi(\mathcal{T}), \circ)$.

**Theorem 2.17.** Let $\Phi$ be a (\cap, U, $\emptyset$)-subsemilattice of $P^d(U)$ containing $U$ and $\emptyset$, and $\mathcal{T}$ a Φ-compatible $\star$-semigroup in Eq(U). Then SetAlg(Φ, $\mathcal{T}$) := $(\Phi; \text{Sat}_\Phi)$ is an information algebra.

**Proof.** Corollary 2.11, Lemma 2.13 and Proposition 2.15.

The algebras described by Theorem 2.17 will be called set algebras in the sequel. As it will turn out, they are the archetypes of information algebras.

A special type of set algebras

Any set algebra SetAlg(Φ, $\mathcal{T}$) is forced by definition to contain, as members of Φ, many sets which are unions of blocks of some equivalence $\Theta \in \mathcal{T}$. Is it possible to have a set algebra where Φ consists precisely of all possible unions of this type? The following proposition shows that is the case iff $\mathcal{T}$ satisfies a simple property:

**Theorem 2.18.** Let $\mathcal{T}$ be a $\star$-semigroup in Eq(U) and put $\Phi = \{ X \subseteq U : X = \bigcup_i B_i \text{ where } B_i \in \Theta \text{ for some } \Theta \in \mathcal{T} \}$. Then SetAlg(Φ, $\mathcal{T}$) is a set algebra if and only if $\mathcal{T}$ is downwards directed in Eq(U) ordered by standard set inclusion.
Proof. We only need to show that $\Phi$ is a join-subsemilattice of $P^d(U)$ containing $U$ and $\emptyset$. Now $\emptyset \in \Phi$ as the set union of the empty collection of blocks from any $\Theta \in \mathcal{T}$, and $U \in \Phi$ as the set union of all blocks of any $\Theta \in \mathcal{T}$. It remains to show that $\Phi$ is closed under set intersection. Then

$$
\bigcup_i B_i \cap \bigcup_j B'_j = \bigcup_{i,j} (B_i \cap B'_j)
$$

for blocks $B_i \in \Theta, B'_j \in \Theta'$, so it suffices to show that $B_i \cap B'_j$ is a union of blocks of some $\Theta'' \in \mathcal{T}$ for all $i,j$. This is clearly the case exactly if for any $\Theta, \Theta' \in \mathcal{T}$ there exists $\Theta'' \in \mathcal{T}$ such that $\Theta'' \subseteq \Theta$ and $\Theta'' \subseteq \Theta'$.

\[\Box\]

2.4 A general representation theorem

If $(X, \leq)$ is any ordered set, $V \subseteq X$ is called an up-set if $v \in V$ and $v \leq x$ jointly imply that $x \in V$. For any $x \in V$, the principal up-set generated by $x$ is given by $
abla x = \{ y \in V : x \leq y \}$. We write $U_p(X)$ for the collection of all principal up-sets in $X$, considered as an ordered set under ordinary set inclusion.

In the following, $(\Phi, E)$ will denote an arbitrary but fixed information algebra. Put $\Phi_0 := \Phi \setminus \{0\}$. Our aim is to construct a set algebra based on the universe $\Phi_0$.

For any $\phi, \psi \in \Phi_0$, we have $\nabla \phi \cap \nabla \psi = \nabla (\phi \lor \psi)$ if $\phi \lor \psi \in \Phi_0$, and

$$
\nabla \phi \cap \nabla \psi = \emptyset
$$

if $\phi \lor \psi = 0$. Let $U^+_p(\Phi_0) := U_p(\Phi_0) \cup \{\emptyset\}$.

It follows that $U^+_p(\Phi_0) = (U^+_p(\Phi_0); \cap, \Phi_0, \emptyset)$ is a $(\cap, \Phi_0, \emptyset)$-subsemilattice of $P^d(\Phi_0)$.

While $\Phi_0$ is not be closed under the join operation of $\Phi$ (unless $\Phi_0$ happens to contain a greatest element), it is closed under all extractions $\epsilon \in E$ since $\epsilon \phi = 0$ implies $\phi = 0$ as $\epsilon \phi \leq \phi$. Similarly, if $\epsilon \phi = \epsilon 0$ for some $\phi \in \Phi$ and $\epsilon \in E$, then $\epsilon \phi = 0$ and again $\phi = 0$. In other words, the $\equiv_\epsilon$-class of 0 is $\{0\}$, where $\equiv_\epsilon$ is infix for the kernel $ker \epsilon$ of $\epsilon$. It follows that $\Phi_0$ is also closed under $ker \epsilon$. Abusing notation in a trivial way, we do not distinguish between $\epsilon$ resp. $\equiv_\epsilon$ and their restrictions to $\Phi_0$. Let $\mathcal{E} = \{ ker \epsilon : \epsilon \in E \}$. So $\mathcal{E}$ is a $\star$-semigroup in $Eq(\Phi_0)$ by Theorem 2.16. Denote the saturation operator on $P^d(\Phi_0)$ associated with $\equiv_\epsilon$ by $\sigma_\epsilon$, and put $Sat(\mathcal{E}) := \{ \sigma_\epsilon : \epsilon \in E \}$.

Lemma 2.19. $U_p(\Phi_0)$ is closed under all $\sigma_\epsilon \in Sat(\mathcal{E})$. 


Proof. We have to show that that $$\sigma_\epsilon(\uparrow \phi) \in \mathcal{U}_p(\Phi_0)$$ for all $$\uparrow \phi \in \mathcal{U}_p(\Phi_0)$$. We have $$\phi \equiv_\epsilon \epsilon(\phi)$$ since $$\epsilon(\phi) = \epsilon(\epsilon(\phi))$$. Let $$\psi \geq \epsilon(\phi)$$ and consider $$\chi = \phi \lor \epsilon(\psi) \in \uparrow \phi$$: Using (Q), we obtain $$\epsilon(\chi) = \epsilon(\phi \lor \epsilon(\psi)) = \epsilon(\phi) \lor \epsilon(\psi)$$. But $$\psi \geq \epsilon(\phi)$$ implies $$\epsilon(\psi) \geq \epsilon(\phi)$$, so we get $$\epsilon(\chi) = \epsilon(\psi)$$, that is, $$\chi \equiv_\epsilon \psi$$ and thus $$\psi \in \sigma_\epsilon(\uparrow \phi)$$. Conversely, if $$\chi \geq \phi$$ and $$\chi \equiv_\epsilon \psi$$, then $$\epsilon(\psi) = \epsilon(\chi) \geq \epsilon(\phi)$$. Summing up, we obtain

$$\sigma_\epsilon(\uparrow \phi) = \uparrow (\epsilon(\phi)) \tag{2}$$

so indeed $$\sigma_\epsilon(\uparrow \phi) \in \mathcal{U}_p(\Phi_0)$$. □

Put $$\text{Sat}(E) := (\text{Sat}(E), \circ)$$. Consequently,

Lemma 2.20. $$(\mathcal{U}_p^+(\Phi_0); \text{Sat}(E))$$ is a set algebra, to be called the principal up-set algebra associated with $$(\Phi; E)$$.

Consider the maps $$i : \Phi \rightarrow \mathcal{U}_p^+(\Phi_0)$$ given by $$\phi \mapsto \uparrow \phi$$ for $$\phi \in \Phi_0$$ and $$i_0 = \emptyset$$, resp. $$j : E \rightarrow \text{Sat}(E)$$ given by $$\epsilon \mapsto \sigma_\epsilon$$.

Lemma 2.21. The pair $$(i, j)$$ provides an isomorphism between $$(\Phi; E)$$ and $$(\mathcal{U}_p^+(\Phi_0); \text{Sat}(E))$$.

Proof. Note first that both $$i$$ and $$j$$ obviously are one-to-one and onto. Further, $$i$$ preserves $$\lor$$, 1 and 0: If $$\phi \lor \psi \neq 0 \in \Phi$$, then $$\uparrow (\phi \lor \psi) = \uparrow \phi \cap \uparrow \psi$$; if $$\phi \lor \psi = 0 \in \Phi$$ then $$\uparrow \phi \cap \uparrow \psi = \emptyset$$. Further, $$\uparrow 1 = \Phi_0$$, and $$i_0 = \emptyset$$ by definition.

Also, $$i$$ and $$j$$ satisfy Def. 2.6.(4): By (2) above, we have $$j(\epsilon)(i \phi) = \sigma_\epsilon(\uparrow \phi) = \uparrow (\epsilon(\phi)) = i(\epsilon(\phi))$$ for $$\phi \in \Phi_0$$, and $$j(\epsilon)(i\emptyset) = j(\epsilon)(\emptyset) = \sigma_\epsilon(\emptyset) = \emptyset = i(\epsilon)(\emptyset) = i(0)$$.

Finally, $$j$$ preserves $$\circ$$: We have to show that $$j(\epsilon \circ \eta)(\uparrow \phi) = (j(\epsilon) \circ j(\eta))(\uparrow \phi)$$ for all $$\uparrow \phi \in \mathcal{U}_p(\Phi_0)$$. Observe first that the least element of $$\epsilon(\uparrow \eta(\phi))$$ is $$\epsilon(\eta(\phi))$$ (since $$\epsilon$$ is order-preserving). Hence $$\uparrow \epsilon(\uparrow \eta(\phi)) = \uparrow \epsilon(\eta(\phi))$$, that is, $$\sigma_\epsilon(\sigma_\eta(\uparrow \phi)) = \sigma_{\epsilon \circ \eta}(\uparrow \phi)$$, using (2). The case $$\phi = 0$$ is trivial. □

Theorem 2.22. Every information algebra is isomorphic to a set algebra, more precisely, to its principal up-set algebra.
The semilattice part of this result is not very surprising, since any ordered set is order-isomorphic to the collection of all its principal up-sets. The point of working with \( \Phi_0 \) instead of \( \Phi \) is to have \( \emptyset \) as the image of \( 0 \in \Phi \) (instead of \( \uparrow 0 \)). So the main content of Thm. 2.22 is that the extraction part of any information algebra may be modeled by (the saturation operators of) a \( * \)-semigroup of compatible equivalence relations on the underlying semilattice.

Note that using \( \Phi_0 \) in order to obtain a set algebra representation is not compulsory. Indeed, much of the rest of this paper is devoted to showing that using other base sets, possibly equipped with additional structure, will produce representations offering deeper insight into the properties of not only the information algebras concerned, but also of their morphisms.

2.5 Examples

Algebra of Strings

Consider a finite alphabet \( \Sigma \), the set \( \Sigma^* \) of finite strings over \( \Sigma \), including the empty string \( \epsilon \), and the set \( \Sigma^\omega \) of infinite strings over \( \Sigma \). Let \( \Sigma^{**} = \Sigma^* \cup \Sigma^\omega \cup \{0\} \), where 0 is a symbol not contained in \( \Sigma \). For two strings \( r, s \in \Sigma^{**} \), define \( r \leq s \), if \( r \) is a prefix of \( s \) or if \( s = 0 \). The empty string is a prefix of any string. Define a combination operation \( \cdot \) in \( \Sigma^{**} \) as follows:

\[
r \cdot s = \begin{cases} 
  s, & \text{if } r \leq s, \\
  r & \text{if } s \leq r, \\
  0 & \text{otherwise}
\end{cases}
\]

Clearly, \( \Sigma^{**} = (\Sigma^{**}, \cdot, \epsilon, 0) \) is a commutative idempotent semigroup with \( \epsilon \) as unit element and 0 as null element of combination. For extraction, we define operators \( \epsilon_n \) for any \( n \in \mathbb{N} \) and also for \( n = \infty \). Let \( \epsilon_n(s) \) be the prefix of length \( n \) of string \( s \), if the length of \( s \) is at least \( n \), and let \( \epsilon_n(s) = s \) otherwise. In particular, define \( \epsilon_\infty(s) = s \) for any string \( s \) and \( \epsilon_n(0) = 0 \) for any \( n \). It is easy to verify that any \( \epsilon_n \) maps \( \Sigma^{**} \) into itself, and that it satisfies conditions (N), (A), and (Q) for an extraction operator. Moreover, \( \mathcal{E} = (\{\epsilon_n : n \in \mathbb{N} \cup \{\infty\}\}, \circ) \) is a commutative and idempotent semigroup under composition \( \circ \) of maps. It follows that the so-called string algebra \( (\Sigma^{**}, \mathcal{E}) \) is an instance of a information algebra.
Multivariate Algebras

In many applications a set of variables is considered and the information one is interested in concerns the values of certain groups of variables, similar to ordinary relational algebra in database theory (see (Kohlas, 2003) for more general relational information algebras). So, consider a countable family of variables $X = \{X_i : i \in \mathbb{N}\}$, and let $V_i$ denote the set of possible values of the variable $X_i$. For a subset $s \subseteq X$ of variables consider

$$V_s = \prod_{X_i \in s} V_i$$

as the set of possible answers relative to $s$. Let

$$V_\omega = \prod_{i=1}^{\infty} V_i,$$

and put $\Phi = P(V_\omega)$, the powerset of $V_\omega$. Note that the elements of $V_\omega$ are the sequences $t = (t_1, t_2, \ldots)$ with $t_i \in V_i$. An element $\phi \in \Phi$ may be interpreted as a piece information, which states that a generic element $t \in V_\omega$ belongs to the set $\phi$. Within $\Phi$ we define combination by set intersection, which represents aggregation of information:

$$\phi \cdot \psi = \phi \cap \psi.$$

Equipped with this operation, $\Phi$ becomes an idempotent commutative semigroup $\Phi$ with least element $V_\omega$ and greatest element $\emptyset$ under the associated information order (given by $\psi \leq \phi$ iff $\phi \subseteq \psi$). The smaller the subset representing a piece of information about elements of $V_\omega$ is, the more information it contains.

Let $s$ be any subset of $X$. Define, for any sequence $t$ in $V_\omega$, its restriction to $s$, denoted by $t|s$, as follows: If $s = \{X_{i_1}, X_{i_2}, \ldots\}$, then $t|s = (t_{i_1}, t_{i_2}, \ldots)$. Also, define an equivalence relation $\equiv_s$ in $V_\omega$ by

$$t \equiv_s t' \text{ iff } t|s = t'|s.$$

It is easy to see that the relational product $\equiv_s \star \equiv_{s'}$ is $\equiv_{s \cap s'}$. It follows that any two of such equivalence relations commute, and thus so do their associated saturation operators. Let $\mathcal{E} = \{\equiv_s : s \subseteq X\}$, $\text{Sat}(\mathcal{E})$ be the
set of all saturation operators $\sigma_s$ associated with $\equiv_s$ for $s \subseteq X$ and finally 
$\text{Sat}(E) = (\text{Sat}(E), \circ)$.

It is immediate that $\sigma_s$ maps $\Phi$ into $\Phi$ and that $\sigma_s(\emptyset) = \emptyset$ for all $s \subseteq X$. So 
$(\Phi; \text{Sat}(E))$ is an information algebra - in fact a set algebra - by Theorem 2.17. It is commonly called the 
multivariate algebra; also, the sets $\sigma_s(\phi)$ are called cylindric over $s$.

The multivariate algebra is an information algebra closely related to relational algebras as used in relational database systems (see (Kohlas, 2003; Kohlas & Schmid, 2014)). We may also consider the set $\Phi'$ consisting of all 
subsets of $V_\omega$ which are cylindric over some finite $s \subseteq X$ (plus $V_\omega$) and limit ourselves to operators $\sigma_s$ for finite $s$. The resulting system $(\Phi'; \text{Sat}(E))$ is 
an information algebra, in fact a subalgebra of $(\Phi; \text{Sat}(E))$ (this is the case since the set intersection of sets cylindric over $r$ respectively $s$ is cylindric over $r \cup s$).

**Lattice-Valued Algebras**

Similar to the multivariate model consider a finite family of variables $X_i$, 
$i = 1, \ldots, n$ with variable $X_i$ taking values in a finite set $V_i$, and let $V$ be 
the cartesian product $V_1 \times \cdots \times V_n$. Further let $\Lambda$ be a bounded distributive lattice 
with greatest element $\top$ and smallest element $\bot$. Consider the set $\Phi$ of all maps $\phi : V \to \Lambda$, and define an operation of combination $\phi \cdot \psi$ on $\Phi$ by 

$$(\phi \cdot \psi)(t) = \phi(t) \land \psi(t) \text{ for all } t \in V.$$ 

Obviously, this defines an idempotent semigroup $\Phi$ with unit element 1 given 
by $1(t) = \top$ for all $t \in \Lambda$ and null element 0 by $0(t) = \bot$ for all $t$. Note that 
under the information order we have $\phi \leq \psi$ in $\Phi$ iff $\phi(t) \geq \psi(t)$ for all $t \in V$.

For any subset $s$ of the index set $r = \{1, \ldots, n\}$ we introduce an operator $\epsilon_s$ 
mapping $\Phi$ into $\Phi$, defined by 

$$\epsilon_s(\phi)(t) = \bigvee \{\phi(u_1, \ldots, u_n) : u_i = t_i \text{ for } i \in s \text{ and } u_i \in V_i \text{ for } i \notin s\}$$

Using the distributivity of the lattice $\Lambda$ it is easy to verify that all of these operators $\epsilon_s$ are extraction operators on $\Phi$. Further, the set $E = \{\epsilon_s : s \subseteq r\}$ 
is a commutative, idempotent semigroup under composition. Thus $(\Phi; E)$ is 
an information algebra; in fact it is a distributive information algebra as will
be considered in Section 4. Also, it is a particular case of a semiring induced valuation algebra as considered in (\ref{9}).

There are many more instances of information algebras related to algebraic logic, graph theory, linear algebra and convex sets, and other topics as well. We refer to (\cite{Kohlas2003, Pouly2011}) for further examples.

**Ideal Completions**

In an algebra \( (\Phi; E) \), a *consistent* set of pieces of information \( I \) is a nonempty subset \( I \subseteq \Phi \) such that (i) with any element \( \phi \in I \) also all elements \( \psi \leq \phi \) implied by \( \phi \) (or contained in \( \phi \)) belong to \( I \), and (ii) with any two elements \( \phi, \psi \in I \) also their combination \( \phi \cdot \psi \) belongs to \( I \). Such sets are just *ideals* in the context of the semilattice \( \Phi \). Ideals not equal to \( \Phi \) are called proper. Consistent sets (also called *theories*) may also be thought of as pieces of information. In fact, we may define among them operations of combination and extraction.

Let \( I(\Phi) \) denote the family of all ideals contained in \( \Phi \). We define the following two operations for ideals \( I_1, I_2, I \in I(\Phi) \):

1. **Combination:** \( I_1 \cdot I_2 = \{ \phi \in \Phi : \phi \leq \phi_1 \cdot \phi_2 \text{ for some } \phi_1 \in I_1, \phi_2 \in I_2 \} \),

2. **Extraction:** \( \hat{\epsilon}(I) = \{ \phi \in \Phi : \phi \leq \epsilon(\psi) \text{ for some } \psi \in I \} \).

Let \( \hat{E} = \{ \hat{\epsilon}; \epsilon \in E \} \), \( \hat{E} = (\hat{E}, \circ) \) and \( I(\Phi) = (I(\Phi), \cdot, \{1\}, \Phi) \). It is not hard to check that \( (I(\Phi); \hat{E}) \) is an information algebra \cite{Kohlas2003, KohlasSchmid2014}, called the *ideal completion* of \( (\Phi; E) \). Moreover, \( (\Phi; E) \) embeds into \( (I(\Phi); \hat{E}) \) by the pair of maps \( \phi \mapsto \downset{\phi} \) and \( \epsilon \mapsto \hat{\epsilon} \), where the *down-set* \( \downset{\phi} = \{ \psi : \psi \leq \phi \} \) is the principal ideal generated by \( \phi \). Ideal completions play an important role for the discussion of compact information algebras, see \cite{Kohlas2003}. It is well-known that \( I(\Phi) \), ordered by ordinary set inclusion, is a complete lattice.

# 3 Atomic Algebras

In many examples of information algebras there exist maximally informative elements. The concept of such elements is captured by the notion of an atom. Here is the formal definition:
Definition 3.1. Let $A = (\Phi; E)$. An element $\alpha \in \Phi$ is called an atom, if

1. $\alpha \neq 0$,
2. If $\phi \in \Phi$, then $\alpha \leq \phi$ implies either $\alpha = \phi$ or $\phi = 0$.

The following lemma lists some properties of atoms:

**Lemma 3.2.** Let $A = (\Phi; E)$.

1. If $\alpha$ is an atom and $\phi \in \Phi$, then either $\alpha \cdot \phi = \alpha$ or $\alpha \cdot \phi = 0$.
2. If $\alpha$ is an atom and $\phi \in \Phi$, then either $\phi \leq \alpha$ or $\alpha \cdot \phi = 0$.
3. If $\alpha$ and $\beta$ are atoms, then either $\alpha = \beta$ or $\alpha \cdot \beta = 0$.

Proof. Let $\alpha$ be an atom and $\phi \in \Phi$. Then $\alpha \leq \alpha \cdot \phi$. Since $\alpha$ is an atom we have either $\alpha \cdot \phi = \alpha$ or $\alpha \cdot \phi = 0$. In the first case $\phi \leq \alpha$. This proves the first two items.

Assume $\alpha$ and $\beta$ are atoms. Then $\alpha \leq \alpha \cdot \beta$, hence either $\alpha \cdot \beta = 0$ or $\alpha = \alpha \cdot \beta$, which means that $\beta \leq \alpha$, thus $\alpha = \beta$. Q.E.D.

Let $At(\Phi)$ (or just $At\Phi$ if it improves readability) denote the set of all atoms of $\Phi$. If $\phi \leq \alpha$, this means that $\alpha$ implies $\phi$. Let $At(\phi) = \{\alpha \in At\Phi : \phi \leq \alpha\}$ be the set of all atoms implying $\phi$. We define different types of information algebras, depending on the occurrence of atoms:

**Definition 3.3.** Let $A = (\Phi; E)$.

1. $A$ is called atomic if $At(\phi) \neq \emptyset$ for all $0 \neq \phi \in \Phi$.
2. $A$ is called atomistic if $\phi = \inf At(\phi)$ for all $0 \neq \phi \in \Phi$.
3. $A$ is called completely atomistic, if it is atomistic and if for all $\emptyset \neq A \subseteq At\Phi$ there exists $\phi \in \Phi$ such that $A = At(\phi)$.

\[ ^{1}\text{We remark that in order theory an atom usually is a minimal, not a maximal element. The present concept corresponds, in a natural way, to our use of information order.} \]
If $A$ is an atomic information algebra, we will construct an associated set algebra (see Section 2.3) based on the set of atoms $At(\Phi)$. Recall (see Lemma 2.9) that $\epsilon A := (\epsilon \Phi; E)$ is a subalgebra of $A$.

**Lemma 3.4.** If $A$ is atomic and $\epsilon \in E$, then $\epsilon A$ is atomic and $At(\epsilon \Phi) = \{\epsilon \phi : \phi \in At(\Phi)\}$.

**Proof.** Let $\alpha \in At(\Phi)$. Then $\alpha \neq 0$ and thus $\epsilon \alpha \neq 0$. Assume $\epsilon \alpha \leq \epsilon \phi$ for some $\phi \in \Phi$. Then $\epsilon (\alpha \cdot \epsilon \phi) = \epsilon \alpha \cdot \epsilon \phi = \epsilon \phi$. Since $\alpha$ is an atom in $\Phi$, we have either $\alpha \cdot \epsilon \phi = \alpha$ or $\alpha \cdot \epsilon \phi = 0$. In the first case, $\epsilon \phi = \epsilon (\alpha \cdot \epsilon \phi) = \epsilon \alpha$, in the second, $\epsilon \phi = \epsilon (\alpha \cdot \epsilon \phi) = \epsilon 0 = 0$. So $\epsilon \phi$ is an atom in $\epsilon \Phi$.

Conversely, assume $0 \neq \epsilon \phi \in \epsilon \Phi$. Since $A$ is atomic, there exists $\xi \in At(\Phi)$ such that $\epsilon \phi \leq \xi$ and thus $\epsilon \phi \leq \epsilon \xi$. As shown above, $\epsilon \xi$ is an atom in $\epsilon \Phi$, so $\epsilon A$ is atomic. If $\epsilon \phi$ is an atom in $\epsilon \Phi$ itself, then obviously $\epsilon \phi = \epsilon \xi$ and thus $\epsilon \phi \in \epsilon At(\Phi)$.

Let $\equiv'_{\epsilon}$ be the restriction of $\equiv_{\epsilon}$ (that is, $\ker \epsilon$) to $At(\Phi)$.

**Lemma 3.5.** Assume $A$ is atomic. Then $\equiv'_{\epsilon}$ and $\equiv'_{\eta}$ commute for $\epsilon, \eta \in E$.

**Proof.** Let $\alpha, \beta \in At(\Phi)$ and assume $\alpha \equiv'_{\epsilon} \gamma$ and $\gamma \equiv'_{\eta} \beta$. So there exists $\gamma \in At(\Phi)$ such that $\epsilon \gamma \leq \xi$ and thus $\epsilon \phi \leq \epsilon \xi$. As shown above, $\epsilon \xi$ is an atom in $\epsilon \Phi$, so $\epsilon A$ is atomic. If $\epsilon \phi$ is an atom in $\epsilon \Phi$ itself, then obviously $\epsilon \phi = \epsilon \xi$ and thus $\epsilon \phi \in \epsilon At(\Phi)$.

Assume $A$ is atomic and put $\mathcal{E}' := \{\equiv'_{\epsilon} : \epsilon \in E\}$, and let $Sat(\mathcal{E}')$ be the set of all of all saturation operators associated with the equivalences $\equiv'_{\epsilon}$. The preceding lemma shows that $(\mathcal{E}', \ast)$ is a $\ast$-semigroup $\mathcal{E}'$, so $Sat(\mathcal{E}') := (Sat(\mathcal{E}'), \circ)$ is a commutative idempotent semigroup by Prop. 2.15. Let $P_{\text{red}}(At(\Phi))$ be the $(\cap, At(\Phi), \emptyset)$-reduct of $P(At(\Phi))$. The operators $\sigma'_{\epsilon}$ map $P(At(\Phi))$ into itself by definition, and $\sigma'_{\epsilon}(At(\Phi)) = At(\Phi)$, so we have a set algebra based on $At(\Phi)$ at hand.
Theorem 3.6. Let $\mathcal{A}$ be atomic. Then $(P^{d}_{\text{red}}(\text{At}(\Phi)); \text{Sat}(\mathcal{E}'))$ is an information algebra isomorphic to $\text{SetAlg}(P^{d}_{\text{red}}(\text{At}(\Phi)); \mathcal{E}')$.

This is the type of set algebras into which atomistic information algebras may be embedded or to which completely atomistic information algebras are isomorphic. In fact, we have the following representation theorem:

Theorem 3.7. Let $\mathcal{A} = (\Phi; \mathcal{E})$ be an atomic information algebra. Then the pair of maps $\text{at}: \phi \mapsto \text{At}(\phi)$ and $j: \epsilon \mapsto \sigma'_{\epsilon}$ defines a homomorphism from $\mathcal{A}$ to $(P^{d}_{\text{red}}(\text{At}(\Phi)); \text{Sat}(\mathcal{E}'))$. If $\mathcal{A}$ is atomistic, the pair $(\text{at}, j)$ is an embedding, and if $\mathcal{A}$ is completely atomistic, it is an isomorphism.

Proof. We verify that

1. $\text{At}(\phi \cdot \psi) = \text{At}(\phi) \cap \text{At}(\psi)$, $\text{At}(1) = \text{At}(\Phi)$, and $\text{At}(0) = \emptyset$,
2. $\sigma'_{\epsilon \cdot \eta} = \sigma'_{\epsilon} \circ \sigma'_{\eta}$, for all $\epsilon, \eta \in E$
3. $\sigma'_{\epsilon} \text{At}(\phi) = \text{At}(\epsilon \phi)$.

For 1.: $\text{At}(1) = \text{At}(\Phi)$ and $\text{At}(0) = \emptyset$ are obvious. Since the algebra is atomic, $\text{At}(\phi) \neq \emptyset$ if $\phi \neq 0$. Assume $\phi \cdot \psi \neq 0$ and let $\alpha \in \text{At}(\phi \cdot \psi)$, thus $\phi, \psi \leq \phi \cdot \psi \leq \alpha$ and $\alpha \in \text{At}(\phi) \cap \text{At}(\psi)$. Conversely, let $\alpha \in \text{At}(\phi) \cap \text{At}(\psi)$. Then $\phi, \psi \leq \alpha$, hence $\phi \cdot \psi \leq \alpha$ and therefore $\alpha \in \text{At}(\phi \cdot \psi)$. This shows that $\text{At}(\phi \cdot \psi) = \text{At}(\phi) \cap \text{At}(\psi)$. If $\phi \cdot \psi = 0$, then $\text{At}(\phi \cdot \psi) = \emptyset$ and $\text{At}(\phi) \cap \text{At}(\psi) = \emptyset$.

For 2.: Theorem 2.16 and Lemma 2.13

For 3. Assume first that $\alpha \in \sigma'_{\epsilon} \text{At}(\phi)$. So there exists $\beta \in \text{At}(\phi)$ such that $\epsilon \alpha = \epsilon \beta$. $\beta \in \text{At}(\phi)$ implies $\phi \leq \beta$, so $\epsilon \phi \leq \epsilon \beta = \epsilon \alpha \leq \alpha$ and thus $\alpha \in \text{At}(\epsilon \phi)$.

Conversely, let $\alpha \in \text{At}(\epsilon \phi)$, thus $\epsilon \phi \leq \alpha$. Recall that $\phi \leq \epsilon(\alpha) \cdot \phi$. We claim that $\epsilon \alpha \cdot \phi \neq 0$. Indeed, otherwise we would have $\epsilon(\alpha \cdot \phi) = \epsilon \alpha \cdot \epsilon \phi = \epsilon(\epsilon \alpha \cdot \phi) \circ \epsilon \phi = 0$, implying $\alpha \cdot \epsilon \phi = 0$ and contradicting $\alpha \in \text{At}(\epsilon \phi)$. So there exists $\beta \in \text{At}(\epsilon \alpha \cdot \phi)$, and thus $\phi \leq \epsilon \alpha \cdot \phi \leq \beta$. We conclude that $\beta \in \text{At}(\phi)$.

Further $\epsilon(\epsilon \alpha \cdot \phi) = \epsilon \alpha \cdot \epsilon \phi \leq \epsilon \beta$, hence $\epsilon \alpha \cdot \epsilon \beta \cdot \epsilon \phi = \epsilon \beta$. This implies $\epsilon \alpha \cdot \epsilon \beta \neq 0$. Since $\epsilon \alpha \cdot \epsilon \beta = \epsilon(\alpha \cdot \epsilon \beta)$ we conclude that $\alpha \cdot \epsilon \beta \neq 0$, hence $\epsilon \beta \leq \alpha$ since $\alpha$ is an atom. We infer that $\epsilon \beta \leq \epsilon \alpha$. 


Proceed in the same way from $\epsilon \alpha \cdot \epsilon \beta = \epsilon (\epsilon \alpha \cdot \beta)$ in order to obtain $\epsilon \alpha \leq \epsilon \beta$, and so finally $\epsilon \alpha = \epsilon \beta$. But this means that $\alpha \in \sigma'_\epsilon \text{At}(\phi)$ and so $\sigma'_\epsilon \text{At}(\phi) = \text{At}(\epsilon \phi)$ as claimed.

Finally, the map $j : \epsilon \mapsto \sigma'_\epsilon$ is bijective by construction. The map $\text{at} : \phi \mapsto \text{At}(\phi)$ is obviously one-to-one whenever $\text{A}$ is atomistic, and even onto if $\text{A}$ is completely atomistic, concluding the proof.

$\square$

For completely atomistic information algebra there is a much stronger result:.

**Theorem 3.8.** Let $\text{A} = (\Phi; E)$ be a completely atomistic information algebra. Then $\Phi$ is a complete Boolean lattice, and the map $\text{at} : \phi \mapsto \text{At}(\phi)$ preserves arbitrary joins and meets (in the information order) as well as complements.

**Proof.** Let $X$ be any subset of $\Phi$ and define

$$A_X = \bigcap_{\phi \in X} \text{At}(\phi).$$

Assume $A_X \neq \emptyset$. Since the algebra is completely atomistic, there exists $\psi \in \Phi$ such that $A_X = \text{At}(\psi)$ and $\psi = \bigwedge A_X$. For any $\alpha \in A_X$ and $\phi \in X$ we have $\phi \leq \alpha$, therefore $\phi \leq \bigwedge A_X$ which shows that $\bigwedge A_X$ is an upper bound of $X$. Let $\chi$ be any other such upper bound. Then $\text{At}(\chi) \subseteq \text{At}(\phi)$ for all $\phi \in X$, hence $\alpha \in \text{At}(\chi)$ implies $\alpha \in A_X$, and therefore $\chi = \bigwedge \text{At}(\chi) \geq \bigwedge A_X$. It follows $\bigwedge A_X$ is the supremum of $X$, that is, $\bigvee X = \bigwedge A_X$. Consequently,

$$\text{At}(\bigvee X) = \bigcap_{\phi \in X} \text{At}(\phi).$$

If $A_X = \emptyset$, then $\bigvee X = 0$ and $\text{At}(0) = \emptyset$. So join (in the information order) is preserved.

Consider $\phi \in \Phi$ and define $\text{At}^c(\phi) := \text{At} \Phi \setminus \text{At}(\phi)$. Since $\text{A}$ is atomistic, $\psi = \bigwedge \text{At}^c(\phi)$ exists and belongs to $\Phi$. Moreover, $\text{At}(\psi) = \text{At}^c(\phi)$. We know
that $At(\phi \cdot \psi) = At(\phi) \cap At(\psi)$, $At(1) = At\Phi$, and $At(0) = \emptyset$ (see the proof of 3.7 item 1). Hence

$$\phi \lor \psi = \bigwedge At(\phi \cdot \psi) = \bigwedge (At(\phi) \cap At(\psi)) =$$
$$= \bigwedge (At(\phi) \cap At^c(\phi)) = \bigwedge \emptyset = 0$$

Not unexpectedly, it is true that $At(\phi \wedge \psi) = At(\phi) \cup At(\psi)$, but this is where we need $A$ to be completely atomistic. Indeed, putting $A := At(\phi) \cup At(\psi)$, then certainly $A \subseteq At(\phi \wedge \psi)$, and there exists $\xi$ such that $A = At(\xi)$ and $\xi = \bigwedge A$. It is clear that $\xi \leq \phi$ and $\xi \leq \psi$ since $A$ is atomistic. If now $\chi \leq \phi$ and $\chi \leq \psi$ for some $\chi$, then certainly $At(\chi) \supseteq A$ and - using "atomistic" in the reverse way - we obtain $\xi = \phi \wedge \psi$ and

$$\phi \wedge \psi = \bigwedge (At(\phi) \cup At^c(\phi)) = \bigwedge At(\Phi) = 1$$

So $\psi =: \phi^c$ is the complement of $\phi$ and $At(\phi^c) = At^c(\phi)$.

The map $at$ thus preserves arbitrary joins and complements and consequently also arbitrary meets, completing the proof.

As an illustration we consider string algebras (Section 2.5): The infinite strings in $\Sigma^\omega$ are the atoms of this algebra. If $s$ is a finite string, then the atoms in $At(s)$ are all infinite strings with $s$ as a prefix. These algebras are atomistic as any string $s$ is the infimum of the set of infinite strings with $s$ as a prefix. But it is not completely atomistic, since there are sets $A$ of infinite strings which do not arise as the set of all atoms over some string $s$, namely sets $A$ containing strings with different prefixes. Thus, the algebra of strings is embedded into the set algebra of its atoms $(P_{ed}(\Sigma^\omega), \mathcal{E}')$ (Theorem 3.7) by the map $s \mapsto At(s)$ where, for any $n$, the saturation operator $\sigma_n^d$ maps a set $S$ of infinite strings into the set of all infinite strings which have a common prefix of length $n$ with some string from $S$ (compare this representation of the string algebra by sets of infinite strings with the representation of the same algebra by truncated up-sets of arbitrary strings in Section 2.4).
4 Distributive Information Algebras

In this section we consider information algebras where \( \Phi \) is a distributive lattice. For distributive lattices, there is a well established representation and duality theory, the so-called Priestley duality theory, generalizing Stone duality for Boolean algebras, see (Davey & Priestley, 2002). It will be the base for developing a corresponding representation resp. duality theory for information algebras based on distributive lattices. Moreover, Cignoli studied existential quantifiers on distributive lattices in (Cignoli, 1991). His results are exactly what we need to extend the representation theory of distributive lattices to information algebras carried by a distributive lattice.

**Definition 4.1.** An information algebra \( \mathfrak{A} = (\Phi; E) \) is called distributive iff

(i) in \( \Phi \) the infimum (relative to the information order) \( \phi \land \psi \) exists for all \( \phi, \psi \in \Phi \), making \( (\Phi; \cdot, \land, 1, 0) \) a lattice,

(ii) \( (\Phi; \cdot, \land, 1, 0) \) is distributive and

(iii) \( \epsilon(\phi \land \psi) = \epsilon\phi \land \epsilon\psi \) for all \( \epsilon \in E \) and \( \phi, \psi \in \Phi \).

We denote by \( \mathbb{D} \) the category of all distributive information together with homomorphisms \( (f, g) \) according to Def. 2.6 but subject to the additional condition that \( f \) is also meet-preserving. In particular, the restriction \( \epsilon|_{\Phi} := \epsilon_{\cdot} \) of \( \epsilon \) to \( \epsilon\Phi \) is such a morphism while \( \epsilon \) is not, in general. This makes \( \epsilon\Phi \) a sublattice of \( \Phi \) - a fact that will play a central rôle.

4.1 Adapting Priestley theory to \( \mathbb{D} \)

We consider first an arbitrary distributive lattice \( K = (K; \lor, \land, 0, 1) \) in its standard order where 0 is the least and 1 the greatest element. The reader is referred to (Davey & Priestley, 2002) for background and for proofs of facts stated below without justification.

**Definition 4.2.** (i) An ideal \( I \) in \( K \) is a nonempty down-set \( I \subseteq K \) which is closed under join (cf. “Ideal Completions” in Section 2.6). \( I \) is called prime, if \( I \neq K \) and whenever \( x \land y \in I \), then either \( x \in I \) or \( y \in I \).
(ii) A filter $F$ in $K$ is a nonempty up-set $F \subseteq K$ which is closed under meet. $F$ is called prime, if $F \neq K$ and whenever $x \lor y \in F$, then either $x \in F$ or $y \in F$.

It is easy to check that a subset $I \subseteq K$ is a prime ideal iff $K \setminus I$ is a prime filter, and vice versa.

**Definition 4.3.** An ideal $I \subseteq K$ of is called maximal, if $I \neq K$, and whenever $I \subseteq J$ for some ideal $J \subseteq K$, then either $J = I$ or $J = K$.

It is easy to check that in an arbitrary bounded distributive lattice $K$ every maximal ideal is prime (the converse is not true in, in general - cf. Lemma [1.23]).

The existence of prime ideals (resp. filters) in arbitrary bounded distributive lattices is not trivial and needs some form of a set existence axiom like the Axiom of Choice (AC) or weaker forms thereof. This is not a real issue except for set theorists. So we take (AC) for granted without reservation and use it in the following (weak) version, tailor-made for our purpose, and labelled as (DPI) in [Davey & Priestley, 2002]:

(DPI) In a bounded distributive lattice $K$, let $J$ be an ideal and $G$ a filter such that $J \cap G = \emptyset$. Then there exist a prime ideal $I \supseteq J$ and a prime filter $F \supseteq G$ such that $I \cap F = \emptyset$ (one may take $F = K \setminus I$, of course).

Let $X(K)$ (or $XK$ to improve readability of formulas) denote the set of all prime ideals of an arbitrary bounded distributive lattice $K = (K; \lor, \land, 0, 1)$, and put $W_u = \{I \in XK : u \notin I\}$ for $u \in K$. Further, let $K^- = \{W_u : u \in K\}$. We have $W_u \cup W_v = W_{u \lor v}$ by the ideal property of the members of $XK$ and $W_u \cap W_v = W_{u \land v}$ by their primeness. Also, $W_0 = \emptyset$ and $W_1 = XK$. It follows that $K^- = (K^-; \cup, \land, \emptyset, K)$ is a sublattice of the power set lattice $P(XK)$.

The basic fact underlying Priestley duality theory is that $K$ is isomorphic with $K^-$, the isomorphism being given by $u \mapsto W_u$ for all $u \in K$.

We have to adjust the development to our use of information order: In a set algebra the order is the reverse of set inclusion, and combination - that is: join - in the information algebra should be represented by set intersection. Technically, this means replacing $K$ by its order dual $K^d$ and adjusting the isomorphism described above in order to have $\emptyset$ as the greatest and $XK$ as the least element. Consider a prime ideal $I \subseteq K$ and $u \in K$, $u \notin I$: In $K^d$, $I$
is prime filter not containing \( u \) and thus \( K \setminus I \) a prime ideal containing \( u \). So we put \( X_u = \{ I \in XK; u \in I \} \) and \( K^+ = \{ X_u; u \in K \} \). We have \( X_u \cap X_v = X_{u \wedge v} \) by the ideal property and \( X_u \cup X_v = X_{u \vee v} \) by primeness, moreover \( X_0 = XK \) and \( X_1 = \emptyset \). So \( K^d \) is isomorphic with \( K^+ = (K^+; \cap, \cup, KX, \emptyset) \), a sublattice of the dual powerset lattice \( \mathcal{P}(XK)^d \), the isomorphism being given by \( u \mapsto X_u \) for all \( u \in K \).

So far, we have a representation of an arbitrary bounded distributive lattice (resp. its order dual) as a lattice of sets with set intersection and union as operations. However, we don’t have, at this point, much insight into the nature of the representing sets \( X_u \) (resp. \( W_u \)). Introducing a suitable topology on \( XK \) will provide a very satisfactory solution. Focussing on \( K^+ \), we consider the family of sets

\[ \mathcal{B}_K = \{ X_u \cap (XK \setminus X_v); u, v \in K \} \]

which is clearly closed under finite set intersection and thus may serve as an (open) base for a topology \( \mathfrak{T}_K \) on \( XK \), that is, \( \mathfrak{T}_K \) is the collection of all set unions of members of \( \mathcal{B}_K \). \( \mathfrak{T}_K \) is compact and Hausdorff. Compactness implies that the clopen (simultaneously open and closed) sets of \( \mathfrak{T}_K \) are precisely the members of \( \mathcal{B}_K \). In order to characterize the sets \( X_u \) within the space \( (XK, \mathfrak{T}_K) \), we use the natural order on \( XK \) given by ordinary set inclusion between prime ideals: The sets \( X_u \) for some \( u \in K \) are precisely the clopen up-sets of the ordered space \( XK = (XK, \mathfrak{T}_K, \subseteq) \). Assuming (DPI) one shows that for \( I, I' \in XK \) satisfying \( I \not\subseteq I' \) there exists a clopen subset \( U \subseteq XK \) such that \( I \in U \) but \( I' \not\in U \). This means that the ordered space \( XK \) is totally order-disconnected. A compact totally order-disconnected ordered space is commonly referred to as a Priestley space.

Going back to distributive information algebras \( (\Phi; E) \) we obtain a representation theorem for their lattice parts: \( \Phi \) is isomorphic with the lattice of all clopen up-sets of the Priestley space \( X\Phi \), the isomorphism being given by \( \phi \mapsto X_\phi \) for all \( \phi \in \Phi \).

Mapping \( \phi \in \Phi \) to the set of prime ideals containing it, instead of the set of prime ideals excluding it makes sense from the information-theoretic point of view: Prime ideals are consistent complete theories or collections of information elements. Indeed, as ideals they are consistent in the sense that they contain with any element all elements implied by it and with any two elements also their combination. Moreover, they are complete theories in
the sense that if they contain $\phi \land \psi$, they must contain $\phi$ or $\psi$. So the map $\phi \mapsto X_\phi$ assigns to $\phi$ all consistent and complete theories $I$ which are consistent with $\phi$ (that is, contain $\phi$).

### 4.2 “Up-side down” Priestley duality in a nutshell

To facilitate the discussion, we introduce the following notation: For any distributive lattice $\Phi = (\Phi; \cdot, \land, 1, 0)$, let $X\Phi$ be the ordered topological space $X\Phi = (X\Phi, \preceq)$. On the other hand, for any compact totally order-disconnected topological space $\overline{Y} := (Y, \preceq, \leq)$ let $L\overline{Y}$ (or $LY$ to improve readability of formulas) be the collection of all clopen up-sets of $\overline{Y}$, and $L\overline{Y}$ the sublattice of the dual power set lattice $P(\overline{Y})^d$ induced by $LY$. So the representation for the lattice part $\Phi$ of a distributive information algebra $(\Phi; E)$ obtained above takes the simple form $\Phi \sim = L\overline{X\Phi}$. This isomorphism is given explicitly as $\kappa\Phi: \phi \in \Phi \mapsto X_\phi = \{I \in X\Phi: \phi \in I\}$ for all $\phi \in \Phi$.

Consider any abstract Priestley space $\overline{Y}$, and for any $p \in Y$, let $L_p = \{U \in L\overline{Y}: p \in U\}$. It is easy to check that $L_p$ is a prime ideal in $L\overline{Y}$. Define a map $\lambda\overline{Y}: \overline{Y} \mapsto XL\overline{Y}$ by $\lambda\overline{Y}(p) = L_p$. Priestley duality shows that $\lambda\overline{Y}$ is in fact an order-homeomorphism between the spaces $\overline{Y}$ and $XL\overline{Y}$, so $\overline{Y} \cong XL\overline{Y}$ as Priestley spaces.

Summing up, this establishes a bijective correspondence between bounded distributive lattices on one hand and Priestley spaces, on the other - in fact, essentially the object part of a full categorical equivalence.

Turning to morphisms, consider first two bounded distributive lattices $\Phi$ and $\Psi$ and let $Hom(\Phi, \Psi)$ be the set of all 1-0-preserving lattice homomorphisms from $\Phi$ to $\Psi$. Similarly, for two Priestley spaces $\overline{Y}$ and $\overline{Z}$, let $Hom(\overline{Y}, \overline{Z})$ be the set of all continuous order-preserving maps from $\overline{Y}$ to $\overline{Z}$.

For $f \in Hom(\Phi, \Psi)$ define $Xf \in Hom(X\Psi, X\Phi)$ by $Xf : I \in X\Psi \mapsto f^{-1}(I) \in X\Phi$.

For $\alpha \in Hom(\overline{Y}, \overline{Z})$ define $L\alpha \in Hom(L\overline{Z}, L\overline{Y})$ by $L\alpha : V \in L\overline{Z} \mapsto \alpha^{-1}(V) \in L\overline{Y}$.

The maps $f \mapsto Xf$ resp. $\alpha \mapsto L\alpha$ provide bijections from $Hom(\Phi, \Psi)$ to $Hom(X\Psi, X\Phi)$ resp. from $Hom(\overline{Y}, \overline{Z})$ to $Hom(L\overline{Z}, L\overline{Y})$. 
f \in \text{Hom}(\Phi, \Psi)$ is one-to-one iff $Xf \in \text{Hom}(X\Psi, X\Phi)$ is onto, and $f \in \text{Hom}(\Phi, \Psi)$ is onto iff $Xf \in \text{Hom}(X\Psi, X\Phi)$ is an order embedding.

Finally $LXf : LX\Phi \to LX\Psi$ satisfies $LXf \circ \kappa_\Phi = f \circ \kappa_\Psi$. In particular, we have $LXf(X\phi) = X_{f(\phi)}$ for all $\phi \in \Phi$. Similarly, $XL\alpha : XLY \to XLZ$ satisfies $XL\alpha \circ \lambda_Y = \lambda_Z \circ \alpha$. Again, one has $XL\alpha(L\mu) = L_{\alpha(p)}$ for all $p \in Y$.

### 4.3 Extraction operators

Consider any distributive information algebra $A = (\Phi; E)$. The key observation is that for any $\epsilon \in E$, the restriction $\epsilon_r := \epsilon|_{\Phi}$ of $\epsilon$ to $\Phi$ is a 1-0-preserving lattice homomorphism while $\epsilon$ is not, in general. So we may use Priestley duality to model $\epsilon_r$ in Priestley spaces. Our exposition is based on the results of (Vrancken-Mawet, 1984) and Cignoli (Cignoli, 1991). Clearly, $\epsilon_r : \epsilon\Phi \to \Phi$ is a one-to-one embedding of $\epsilon\Phi$ into $\Phi$. So $X\epsilon_r = \epsilon_r^{-1} : X(\epsilon\Phi) \to X(\epsilon\Phi)$ is onto, continuous and order-preserving. Thus $LX\epsilon_r = (\epsilon_r^{-1})^{-1} : LX(\epsilon\Phi) \to LX(\epsilon\Phi)$ is a lattice embedding. Note that $\epsilon_r^{-1}$ takes a prime ideal $I \in X\Phi$ to $I \cap \epsilon\Phi$ which is a prime ideal in $\epsilon\Phi$. Also, $(\epsilon_r^{-1})^{-1}$ takes a clopen up-set $U \in L(X(\epsilon\Phi))$ to $\{I \in X\Phi ; I \cap \epsilon\Phi \subseteq U\}$, this latter being a clopen up-set since $\epsilon_r^{-1}$ is continuous and order-preserving. Since $LX\epsilon_r \circ \kappa_\epsilon\Phi = \kappa_\Phi \circ \epsilon_r$, we see that $LX\epsilon_r$ takes $X'_{\epsilon\phi} := \{I' \in X(\epsilon\Phi) : \epsilon\phi \in I'\}$ to $X_{\epsilon\phi} = \{I \in X\Phi : \epsilon\phi \in I\}$ for all $\phi \in \Phi$.

Consider the kernel of $X\epsilon_r$, that is $\ker X\epsilon_r = \{(I, I') \in X\Phi \times X\Phi : I \cap \epsilon\Phi = I' \cap \epsilon\Phi\}$, shortly denoted by $\cong_\epsilon$. We are interested in the saturation operator $\varsigma_\epsilon$ associated with $\cong_\epsilon$ which for any subset $U \subseteq X\Phi$ returns $\{I' \in X\Phi : I' \cong_\epsilon I \text{ for some } I \in U\}$, and particularly in the restriction of $\varsigma_\epsilon$ to $L(X\Phi)$. Sets $U$ satisfying $U = \varsigma_\epsilon(U)$ will be called $\epsilon$-saturated. We write $L^\epsilon(X\Phi)$ for the family of all $\epsilon$-saturated sets in $L(X\Phi)$.

The key fact we need is the following lemma which is part of Thm. 2.2 in (Cignoli, 1991).

**Lemma 4.4** (Cignoli). For any $I \in X\Phi$ containing $\epsilon\phi \in \Phi$, there exists $I' \in X\Phi$ such that $I' \cong_\epsilon I$ and $\phi \in I'$.

Recall that the members of $L(X\Phi)$ are exactly the sets $X_{\phi}$ for $\phi \in \Phi$. We have
Corollary 4.5. For all \( \phi \in \Phi \), \( \varsigma_\phi X_\phi = X_\phi \).

Proof. We first show that \( X_\phi \) is \( \varsigma_\phi \)-saturated for all \( \phi \in \Phi \). Indeed, let \( I \in X_\phi \) and \( I' \cong_\epsilon I \). Now \( \epsilon \phi \in I \cap e \Phi = I' \cap e \Phi \) and so \( \epsilon \phi \in I' \), that is, \( I' \in X_\phi \) and thus \( \varsigma_\phi(X_\phi) = X_\phi \).

Moreover, \( X_\phi \subseteq X_\phi \) since \( e \Phi \leq \phi \), so \( \varsigma_\phi(X_\phi) \subseteq \varsigma_\phi(X_\phi) = X_\phi \). Let \( I \in X_\phi \). By Lemma 4.4 we find \( I' \in X_\phi \) such that \( I' \cap e \Phi = I \cap e \Phi \), implying \( I \in \varsigma_\phi(X_\phi) \) which shows that \( X_\phi \subseteq \varsigma_\phi(X_\phi) \). This implies \( \varsigma_\phi(X_\phi) \subseteq \varsigma_\phi(X_\phi) \) so finally \( \varsigma_\phi(X_\phi) = \varsigma_\phi(X_\phi) = X_\phi \). \( \square \)

The following proposition collects the main properties of the saturation operator \( \varsigma_\epsilon \):

Proposition 4.6. Let \( \cong_\epsilon, \varsigma_\epsilon \) and \( L'(X \Phi) \) be given as described above. Then:

(i) \( \varsigma_\epsilon \) maps \( L(X \Phi) \) into \( L(X \Phi) \).

(ii) The members of \( L'(X \Phi) \) are exactly the sets \( X_\phi \) for \( \phi \in \Phi \).

(iii) If \( I, I' \in X \Phi \) and \( I \not\cong_\epsilon I' \), there is \( U \in L'(X \Phi) \) containing exactly one of \( I, I' \).

(iv) \( \varsigma_\epsilon \) is an extraction operator on \( L X \Phi \).

(v) \( L'(X \Phi) \) endowed with the operations inherited from \( L X \Phi \) is a sublattice \( L' X \Phi \) of \( L X \Phi \), and \( L' X \Phi \cong e \Phi \).

(vi) For any \( I, I' \in X \Phi \), we have \( I \cap e \Phi \subseteq I' \cap e \Phi \) iff for all \( U \in L'(X \Phi) \), \( I \in U \) implies \( I' \in U \).

Proof. Ad (i): Follows directly from Lemma 4.5. Ad (ii): \( \varsigma_\epsilon X_\phi = X_\phi \) iff \( X_\phi = X_\phi \) iff \( \epsilon \phi = \phi \). Ad (iii): Let \( I \not\cong_\epsilon I' \), thus \( I \cap e \Phi \neq I' \cap e \Phi \). Assume w.l.o.g. that \( I \cap e \Phi \notin I' \cap e \Phi \). So there exists \( \epsilon \phi \in I \cap e \Phi \) such that \( \epsilon \phi \notin I \cap e \Phi \). This means \( I \in X_\phi \) but \( I \notin X_\phi \). Ad (iv): Lemma 2.10. Ad (v): Reformulates the description of the map \( L X \epsilon_r \) given above in terms of \( \varsigma_\epsilon \)-saturated sets. Ad (vi): Using (ii), the assertion becomes \( I \cap e \Phi \subseteq I' \cap e \Phi \) iff for all \( X_\phi, I \in X_\phi \) implies \( I' \in X_\phi \) iff for all \( \epsilon \phi, \epsilon \phi \in I \) implies \( \epsilon \phi \in I' \) which is the same as \( I \cap e \Phi \subseteq I' \cap e \Phi \). \( \square \)

Looking at the other end of the sought duality between algebras and spaces, the obvious question is now how to characterize equivalence relations \( \Theta \) on
a Priestley space \( Y \) such that the associated saturation operators \( \sigma_\Theta \) induce quantifiers on the lattice \( \mathbb{L}_Y \). Obviously, this requires that \( \sigma_\Theta \) maps \( L^\Theta Y \) into \( L^\Theta Y \) (corresponding to Prop. 4.6 (i)). For any \( p \in Y \), let \( [p]_\Theta \) be the \( \Theta \)-class of \( p \), and put \( Y/\Theta : = \{ [p]_\Theta : p \in Y \} \). Write \( L^\Theta(Y) \) for the collection of all \( \sigma_\Theta \)-saturated clopen up-sets of \( Y \), and \( L^\Theta(Y) \) for the corresponding sublattice of \( \mathbb{L}_Y \).

We want to equip \( Y/\Theta \) with an order and a topology making it a Priestley space such that the canonical projection \( \pi_\Theta : Y \rightarrow Y/\Theta \) is continuous and order-preserving. Imitating Prop. 4.6 (vi), tentatively define \( [p]_\Theta \leq [q]_\Theta \) iff for all \( U \in L^\Theta(Y) \), \( p \in U \) implies \( q \in U \). It is obvious that \( \leq_\Theta \) is reflexive and transitive.

The key fact we need here is contained in the following lemma, which is part of Lemma 1.6 in (Vrancken-Mawet, 1984).

**Lemma 4.7** (Vrancken-Mawet). A Priestley structure on \( Y/\Theta \) making \( \pi_\Theta : Y \rightarrow Y/\Theta \) continuous and order-preserving exists exactly if \( \leq_\Theta \) is antisymmetric and thus an order. If this is the case, the sought topology on \( Y/\Theta \) is uniquely determined as the quotient topology relative to \( Y \) and \( \pi_\Theta \).

Consider \( p,q \in Y \) such that \( (p,q) \notin \Theta \), that is, \( [p]_\Theta \neq [q]_\Theta \). Now \( \Theta \) is antisymmetric iff this implies \( [p] \not\leq_\Theta [q] \) or \( [q] \not\leq_\Theta [p] \). According to the definition of \( \leq_\Theta \) this means that there exists \( U \in L^\Theta(Y) \) containing exactly one of \( p \) and \( q \) (note that this corresponds to Prop. 4.6 (iii)).

**Definition 4.8.** An equivalence \( \Theta \) on a Priestley space \( Y \) is separating iff (i) \( \sigma_\Theta \) maps \( L(Y) \) into \( L(Y) \) and (ii) for any \( p,q \in Y \) with \( (p,q) \notin \Theta \) there exists \( U \in L^\Theta(Y) \) containing exactly one of \( p \) and \( q \).

Note that the equivalences \( \sim_\epsilon \) considered in Prop. 4.6 are kernels and separating. It remains to see that all separating equivalences on a Priestley space arise as kernels of this type.

Assume that a Priestley \( Y \) space carries a separating equivalence relation \( \Theta \), so that \( Y/\Theta \) ordered by \( \leq_\Theta \) and equipped by the quotient topology relative to \( \pi_\Theta \) is a Priestley space \( Y/\Theta \), and \( \pi_\Theta \) is order-preserving and continuous. Putting Priestley duality to work, we see that \( L\pi_\Theta = \pi_\Theta^{-1} \) takes \( U \in L(Y/\Theta) \) to \( \pi_\Theta^{-1}(U) \in L^\Theta(Y) \subseteq L^\Theta(Y) \), is one-to-one and thus provides a lattice isomorphism between \( L(Y/\Theta) \) and \( L^\Theta(Y) \). Note that \( \pi_\Theta^{-1} \pi_\Theta \) is an extraction operator on \( \mathbb{L}_Y \) with image \( L^\Theta(Y) \).
Now $XL\pi\Theta = (\pi^{-1}_\Theta)\pi^{-1}$ takes a prime ideal $I \in XL(Y)$ to $I \cap L^\Theta(Y) \in XL(Y/\Theta)$ and is onto $XL(Y/\Theta)$. Moreover, $ker XL\pi\Theta = \{(I, I') \in XL(Y) \times XL(Y) : I \cap L^\Theta(Y) = I' \cap L^\Theta(Y)\}$. But the prime ideals in $XL(Y)$ are exactly the sets $L_p = \{U \in L(Y) : p \in U\}$. So we obtain

**Lemma 4.9.** $(p, q) \in \Theta$ iff $(L_p, L_q) \in ker XL\pi\Theta$.

**Proof.** We have $(L_p, L_q) \in ker XL\pi\Theta$ iff $L_p \cap L^\Theta = L_q \cap L^\Theta$ iff $(p, q) \in \Theta$ since $\Theta$ is separating. \(\Box\)

Since $Y \cong XL(Y)$ as Priestley spaces, we conclude that $\Theta$ indeed corresponds to the kernel of $XL\pi\Theta$ under this isomorphism.

**Corollary 4.10.** There is a bijective correspondence between meet-preserving extraction operators on a bounded distributive lattice and separating equivalence relations on its Priestley space.

It should be noted (see (Cignoli, 1991)) that condition (ii) in Def. 4.8 could be replaced by requiring the equivalence classes of $\Theta$ to be topologically closed in $Y$. While undoubtedly more elegant, this approach does not immediately reveal how the condition actually is put to work.

### 4.4 $Q$-Priestley spaces

In order to extend Priestley duality theory of distributive lattices to lattices with a quantifier, (Cignoli, 1991) introduced the concept of a $Q$-space. We will extend this concept further in order to obtain, in the end, a full duality theory for distributive information algebras. In view of Cor. 4.10 the dual object of a distributive information algebra $A = (\Phi, E)$ should obviously be a Priestley space equipped with a collection of commuting separating equivalence relations.

**Lemma 4.11.** $\varsigma_\epsilon \circ \varsigma_\eta = \varsigma_{\epsilon \eta}$ for any $\epsilon, \eta \in E$.

**Proof.** By Corollary 4.4 we have $\varsigma_\epsilon X_\phi = X_{\epsilon \phi}$. So $(\varsigma_\epsilon \circ \varsigma_\eta) X_\phi = \varsigma_\epsilon (\varsigma_\eta X_\phi) = \varsigma_\epsilon X_{\eta \phi} = X_{\epsilon \eta \phi} = \varsigma_{\epsilon \eta} X_\phi$. \(\Box\)

**Lemma 4.12.** If $\epsilon \neq \eta$ in $E$, then $\varsigma_\epsilon \neq \varsigma_\eta$. 
Proof. If \( \epsilon \neq \eta \), there is \( \phi \in \Phi \) such that \( \epsilon \phi \neq \eta \phi \). By (DPI), we find \( I \in X\Phi \) such that w.l.o.g. \( \epsilon \phi \in I \) but \( \eta \phi \notin I \). This means that \( I \in X_{\epsilon \phi} = \varsigma_{\epsilon}X_{\phi} \) but \( I \notin X_{\eta \phi} = \varsigma_{\eta}X_{\phi} \) using Lemma 4.11, so \( \varsigma_{\epsilon}X_{\phi} \neq \varsigma_{\eta}X_{\phi} \).

Note: \( X_{\phi} \) used in the proof above is a member of \( L(X\Phi) \). This means that even the restrictions \( \varsigma_{\epsilon}|_{L(X\Phi)} \) and \( \varsigma_{\eta}|_{L(X\Phi)} \) differ whenever \( \epsilon \neq \eta \).

We extend the \( X \)-\( L \)-machinery in order to include extraction and define \(XE\) (or just \(XE \) for short) for \( A = (\Phi; E) \) by \( XE = \{ \approx \varepsilon : \varepsilon \in E \} \) and \( XE := (XE,\ast) \). Also, let \( Sat_{L(X\Phi)}(XE) := \{ \varsigma_{\epsilon}|_{L(X\Phi)} : \epsilon \in E \} \) and \( Sat_{L(X\Phi)}(XE) := (Sat_{L(X\Phi)}(XE),\circ) \).

Theorem 4.13. \( E, Sat_{L(X\Phi)}(XE) \) and \( XE \) are isomorphic as semigroups.

Proof. Lemma 4.11 and Lemma 4.12 for the first isomorphism, and Prop. 2.15 for the second.

Definition 4.14. **Q-Priestly Spaces**: A Q-Priestley space is a pair \( (Y, \Upsilon) \) consisting of a Priestley space \( Y \) and a \( \ast \)-semigroup \( \Upsilon \) in \( Eq(Y) \) consisting of separating equivalence relations.

Given a Q-Priestley space \( (Y, \Upsilon) \), we extend notation again and write \( LT := \{ \sigma_{\Theta}|_{LY} : \Theta \in \Upsilon \} = Sat_{LY}(\Upsilon) \), and \( LT := (LT,\circ) \).

The task at hand is to find the appropriate morphisms between Q-Priestley spaces with the objective of obtaining a full duality between distributive information algebras with their algebra homomorphisms and Q-Priestley spaces with the morphisms sought after.

So let \( A = (\Phi; E) \) and \( B = (\Psi; D) \) be two distributive information algebras. Assume \( (f, g) : A \rightarrow B \) is an information algebra homomorphism, which means that \( f \) is lattice homomorphism and \( g \) a semigroup homomorphism subject to the compatibility condition Def. 2.6.(4). Going to spaces, we have \( Xf = f^{-1} : X\Psi \rightarrow X\Phi \) for the lattice part. For the extraction part, the canonical map naturally associated with \( g \) is \( Xg : XE \rightarrow XD \) given by \( Xg(\approx \epsilon) := \approx g(\epsilon) \).

Lemma 4.15. \( Xg \) is a semigroup homomorphism between the \( \ast \)-semigroups \( XE \) and \( XD \).
Proof. We have $Xg(\cong\epsilon \star \cong\epsilon') = Xg(\cong\epsilon \circ \epsilon') = \cong g(\epsilon \circ \epsilon') = \cong g \star \cong g' = Xg(\cong\epsilon) \star Xg(\cong\epsilon')$.

Proceeding in the obvious way, define $LXg : L(XE) \rightarrow L(XD)$ by $LXg(\varsigma) := \varsigma \circ g(\epsilon)$.

**Lemma 4.16.** $LXg$ is semigroup homomorphism between the semigroups $LXE := (L(XE), \circ)$ and $LXD := (L(XD), \circ)$.

**Proof.** Theorem 4.13

Next, we will show that the pair of maps $(LXf, LXg)$ is an information algebra homomorphism from the algebra $LXA := (LX\Phi, LX\Phi)$ to the algebra $LXB := (LX\Psi, LX\Phi)$, which means that $(LXf, LXg)$ satisfies the compatibility condition Def. 2.6.(4). Let $U \in L(X\Phi)$. So $U = X\phi$ for some uniquely determined $\phi \in \Phi$. Now

$$LXf(\varsigma \circ X\phi) = LXf(X\phi(\epsilon)) = Xf(\epsilon(\phi)).$$  (3)

On the other hand,

$$LXg(\varsigma)(LXf(X\phi)) = \varsigma \circ g(\epsilon)(Xf(\phi)) = Xg(\epsilon)(f(\phi)).$$  (4)

Hence $LXf(\varsigma \circ X\phi) = LXg(\varsigma)(LXf(X\phi))$ iff $Xf(\epsilon(\phi)) = Xg(\epsilon)(f(\phi))$.

**Proposition 4.17.** $(LXf, LXg)$ satisfies Def. 2.6.(4) iff $(f, g)$ so does.

Turning to spaces, a morphism $(\alpha, \omega)$ from a $Q$-Priestley space $(Y, T)$ to a $Q$-Priestley space $(Z, G)$ should obviously be a pair $(\alpha, \omega)$ consisting of a continuous order-preserving map $\alpha : Y \rightarrow Z$ and $\star$-homomorphism $\omega : G \rightarrow T$. Define $L\omega : LG \rightarrow LT$ by $L\omega(\sigma) := \sigma \omega$. $L\omega$ is a $\circ$-homomorphism by Prop. 2.15.

Obviously, we want $(L\alpha, L\omega)$ to be an algebra homomorphism from $LZ$ to $LY$. This is the case exactly iff $(L\alpha, L\omega)$ satisfies Def. 2.6.(4), explicitly,

$$L\alpha(\sigma(\Gamma)(V)) = \sigma(\omega(\alpha(V)))$$  (5)

for all $V \in LZ$ and $\Gamma \in G$. 

Finally, put $\simeq_{\Gamma} := \ker \sigma_{\Gamma}^{-1}$ for $\Gamma \in G$ and let $X_L G := \{ \simeq_{\Gamma}: \sigma_{\Gamma} \in L G \}$ resp. $X_L T := \{ \simeq_{\Theta}: \sigma_{\Theta} \in L T \}$. Define a map $X_L \omega : X_L G \to X_L T$ by $X_L \omega (\simeq_{\Gamma}) := \simeq_{\omega_{\Gamma}}$ for all $\sigma_{\Gamma} \in L G$. $X_L \omega$ is a $*$-homomorphism by Lemma 4.9.

Note that (3) and (4) together just say that $X f =: \alpha$ and $X g =: \omega$ satisfy (5). So (5) is indeed the correct $Q$-Priestley space analogue of the algebra compatibility condition Def. 2.6.(4) and we formally define

**Definition 4.18.** A $Q$-morphism $(\alpha, \omega)$ from a $Q$-Priestley space $(Y, T)$ to a $Q$-Priestley space $(Z, G)$ is a pair $(\alpha, \omega)$ consisting of a continuous order-preserving map $\alpha : Y \to Z$ and $*$-homomorphism $\omega : G \to T$ satisfying $L \alpha (\sigma_{\Gamma}(V)) = \sigma_{\omega_{\Gamma}}(L \alpha(V))$ for all $V \in LZ$ and $\Gamma \in G$.

**4.5 Representation and Duality**

Remember that $D$ stands for the category of all distributive information algebras with CDF homomorphisms, and write $Q$ for the category of all $Q$-Priestley spaces with $Q$-morphisms. A full duality between $D$ and $Q$ will be established by two commutative diagrams generalizing these given in (Davey & Priestley, 2002) for the categories of distributive bounded lattices with 0-1-preserving lattice homomorphisms and Priestley spaces with continuous order-preserving maps.

We start with the algebra point of view where the definition of an isomorphism is the natural one (see section 2.2).

For any information algebra $A = (\Phi, E)$, we have - by upside down Priestley duality - a natural 1-0-preserving lattice isomorphism $\kappa_{\Phi} : \Phi \to LX \Phi$, given by $\kappa_{\Phi}(\phi) = X_{\phi}$ for all $\phi \in \Phi$ (see section 4.2). For the extraction part, define a map $\kappa_E : E \to LX E$ by $\kappa_E(\epsilon) := \varsigma_{\epsilon}$, which is a semigroup isomorphism by Thm. 4.13. It remains to show that $(\kappa_{\Phi}, \kappa_E)$ satisfies Def. 2.6.(4): We have $\kappa_E(\epsilon(\phi)) = X_{\epsilon(\phi)} = \varsigma_{\epsilon}(X_{\phi}) = \kappa_E(\epsilon)(\kappa_{\Phi}(\phi))$ by Corollary 4.5, so $(\kappa_{\Phi}, \kappa_E)$ is indeed an isomorphism of CDF information algebras by Corollary 2.8. The same is true for $(\kappa_{\Phi}, \kappa_D)$. $(LX f, LX g)$ is an algebra homomorphism by Prop. 4.17 so the following diagram is commutative, providing the algebra half of sought duality:
Ignoring the horizontal arrows in the preceding diagram, we obtain a general representation theorem:

**Theorem 4.19** (Representation Theorem). Any distributive CFD information algebra \((\Phi; E)\) is isomorphic with the set algebra \((\text{LX}\Phi; \text{LX}E)\).

For the space analogue, we need a workable description of \(Q\)-isomorphisms, taking over the rôle of Corollary 2.8. Such is provided by an appropriate extension of Corollary 2.9 in (Cignoli, 1991):

**Lemma 4.20** (Cignoli). A \(Q\)-morphism \((\alpha, \omega) : (Y, T) \rightarrow (Z, G)\) is a \(Q\)-isomorphism iff \(\alpha\) is an order-homeomorphism, \(\omega\) is a semigroup isomorphism, and for all \(\Theta \in T\) and all \(p, q \in Y\) we have \((p, q) \in \Theta\) iff \((\alpha(p), \alpha(q)) \in \omega\Gamma\).

For any \(Q\)-Priestley space \((Y, T)\), we have - by up-side down Priestley duality - a natural order homeomorphism \(\lambda_Y : Y \rightarrow \text{LX}Y\) given by \(\lambda_Y(p) = Lp\) for all \(p \in Y\) (see section 1.3). For the extraction part, define a map \(\lambda_T : T \rightarrow \text{LX}T\) by \(\lambda_T(\Theta) := \simeq_\Theta\), which is a semigroup isomorphism (cf. the proof of Thm. 1.3). Consider \(p, q \in Y\). We have \(\lambda_Y(p) \simeq \lambda_Y(q)\) iff \(Lp \simeq Lq\) iff \(\sigma_\Theta^{-1}(Lp) = \sigma_\Theta^{-1}(Lq)\). Now \(\sigma_\Theta^{-1}(Lp) = \{U \in LY : \sigma_\Theta(U) \in Lp\} = \{U \in LY : p \in \sigma_\Theta(U)\}\). So \(Lp \simeq Lq\) iff \(\{U \in LY : p \in \sigma_\Theta(U)\} = \{U \in LY : q \in \sigma_\Theta(U)\}\). This is equivalent with \((p, q) \in \Theta\) since \(\Theta\) is separating (cf. Lemma 1.3). By Cignoli’s Lemma above it follows that \((\lambda_Y, \lambda_T)\) is a \(Q\)-isomorphism - and with that, also \((\lambda_Y^{-1}, \lambda_T^{-1})\). The same goes for \((\lambda_Z, \lambda_G)\) and \((\lambda_Z^{-1}, \lambda_G^{-1})\), of course. The diagram below is commutative by construction, so - using these isomorphisms - we see that \((\alpha, \omega)\) satisfies (5) iff \((\text{LX}\alpha, \text{LX}\omega)\) so does, establishing the space half of the sought duality.
So the two commutative diagrams together establish

**Theorem 4.21.** The functors \( X \) and \( L \) induce a full duality between the categories \( \mathbb{D} \) and \( \mathbb{Q} \).

So distributive information algebras and Q-Priestley spaces are two sides of the same coin.

### 4.6 Boolean Information Algebras

Consider a bounded distributive lattice \( \Phi = (\Phi; \cdot, \wedge, 1, 0) \). For \( \phi \in \Phi \), an element \( \psi \in \Phi \) is a complement of \( \phi \) iff \( \phi \cdot \psi = 0 \) and \( \phi \wedge \psi = 1 \). Using distributivity, it is not hard to see that complements, whenever they exist in \( \Phi \), are uniquely determined. If every \( \phi \in \Phi \) has a (unique) complement, the \( \Phi \) is called complemented. A complemented distributive lattice is commonly referred to as a Boolean lattice.

This must be distinguished from a Boolean algebra which is Boolean lattice where the operation \( \phi \mapsto \phi^c \) with \( \phi^c \) the complement of \( \phi \) is a fundamental operation. Boolean algebras thus are structures of type \((\cdot, \wedge, ^c, 1, 0)\). Using distributivity, it is easy to check that a 1-0-preserving lattice homomorphism between Boolean lattices automatically also preserves complements. There is more:

**Lemma 4.22.** Let \( \underline{A} = (\Phi; E) \) be information algebra with \( \Phi \) a Boolean lattice. Then \( \underline{A} \) is distributive information algebra in the sense of Def. [4.1].

**Proof.** It suffices to show that item (iii) of Def. [4.1] is satisfied. Recall that for \( \phi, \psi \in \Phi \) we have \( \psi \leq \phi \) iff \( \phi \cdot \psi^c = 0 \) (*) in any Boolean lattice. Consider any \( \phi, \psi \in \Phi \) and put \( \eta = \phi \wedge \psi \). Then \( \eta \leq \phi, \psi \) implies \( \epsilon(\eta) \leq \epsilon(\phi) \) and \( \epsilon(\eta) \leq \epsilon(\psi) \). Hence \( \epsilon(\eta) \) is a lower bound of \( \epsilon(\phi) \) and \( \epsilon(\psi) \).
Let $\chi$ be another lower bound of $\epsilon(\phi)$ and $\epsilon(\psi)$. Then by (*) above, $\epsilon(\phi) \cdot \chi^c = 0$ and $\epsilon(\psi) \cdot \chi^c = 0$. It follows that

$$0 = \epsilon(0) = \epsilon(\phi) \cdot \chi^c = \epsilon(\phi) \cdot \epsilon(\chi^c) = \epsilon(\phi \cdot \epsilon(\chi^c)).$$

This implies $\phi \cdot \epsilon(\chi^c) = 0$. In the same way we obtain $\psi \cdot \epsilon(\chi^c) = 0$. Using distributivity and remembering that combination is join, we get

$$0 = (\phi \cdot \epsilon(\chi^c)) \land (\psi \cdot \epsilon(\chi^c)) = (\phi \land \psi) \cdot \epsilon(\chi^c) = \eta \cdot \epsilon(\chi^c).$$

It follows that

$$0 = \epsilon(0) = \epsilon(\eta \cdot \epsilon(\chi^c)) = \epsilon(\eta) \cdot \epsilon(\chi^c) = \epsilon(\eta) \cdot \epsilon(\chi^c),$$

hence $\epsilon(\eta) \cdot \chi^c = 0$. But this implies $\chi \leq \epsilon(\eta)$ by (*) and $\epsilon(\eta)$ is thus the greatest lower bound of $\epsilon(\phi)$ and $\epsilon(\psi)$, that is, $\epsilon(\phi \land \psi) = \epsilon(\phi) \land \epsilon(\psi)$ as claimed. \qed

Accordingly, we define a Boolean information to be an information algebra $\mathfrak{A} = (\Phi; E)$ where $\Phi$ is a Boolean lattice.

**Lemma 4.23.** In a Boolean lattice $\Phi$, prime ideals are maximal.

**Proof.** Let $I \subseteq \Phi$ be a prime ideal, and $\phi \not\in I$. Now $\phi \land \phi^c = 1 \in I$, so $\phi^c \in I$ by primeness of $I$. Let $I'$ be the ideal generated by $I \cup \{\phi\}$ in $\Phi$. Then $\phi, \phi^c \in I$ and thus $1 = \phi \cdot \phi^c \in I'$, which implies $I' = \Phi$. So $I$ is maximal as claimed. \qed

**Corollary 4.24.** The Priestley space $X\Phi$ of a Boolean information algebra $\mathfrak{A} = (\Phi; E)$ carries the trivial order.

Let $\mathfrak{A} = (\Phi; E)$ be any Boolean information algebra. Then $X\Phi$ is just a compact Hausdorff space such that for $I, I' \in X\Phi$ satisfying $I \neq I'$ there exists a clopen subset $U \subseteq X\Phi$ with $I \in U$ but $I' \not\in U$. This latter property is called total disconnectedness, and compact Hausdorff totally disconnected spaces are better known as Stone spaces. Since there is no order to be preserved, the appropriate morphisms between Stone spaces are just continuous maps. Turning to extraction, an equivalence $\Theta$ on a Stone space $\underline{Y}$ will be called separating iff $\sigma_\Theta$ maps clopen subsets of $Y$ to clopen subsets, and for any $p, q \in Y$ with $(p, q) \not\in \Theta$ there exists a clopen subset $U \subseteq Y$ containing...
exactly one of \( p \) and \( q \) (cf. Def. 4.8). Mimicking Def. 4.14, we say that \( Q \)-Stone space is a pair \((Y, \mathcal{T})\) consisting of a Stone space \( Y \) and a \(*\)-semigroup \( \mathcal{T} \) in \( Eq(Y) \) consisting of separating equivalence relations. Finally, let \( \mathbb{B} \) the category of Boolean information algebras with CDF homomorphisms, and \( QS \) that of \( Q \)-Stone spaces with \( Q \)-morphisms. It immediately follows that

**Theorem 4.25.** The functors \( X \) and \( L \) induce a full duality between the categories \( \mathbb{B} \) and \( QS \).

Remember that 1-0-preserving lattice homomorphisms between Boolean lattices also preserve complements. So we could substitute "Boolean lattice" by "Boolean algebra" in the preceding discussion since introducing complementation as an additional fundamental operation does not interfere with extraction.

### 4.7 Finite Distributive Information Algebras

In the preceding subsection, order was trivial on the Priestley space \( X\Phi \) associated with a Boolean information algebra. A similar situation arises if we consider a distributive information algebra \( A = (\Phi; E) \) where \( \Phi \) is finite: Here the topology of the Priestley space \( X\Phi \) is trivial - more precisely: discrete -, being Hausdorff. In plainer terms, \( X\Phi \) is just a finite (partially) ordered set \((H, \leq)\). Turning to extraction, \( E \) is obviously finite and so \( XE \) is a finite set of equivalence relations on \( H \), closed under \(*\) - hence pairwise commuting by Lemma 2.14 - and subject to the two conditions of Def. 4.8 characterizing separating equivalences.

The point here is that \( H \) may be identified with a subset of \( \Phi \), which decreases the set-theoretical complexity of the members of \( X\Phi \). Indeed, \( \Phi \) being finite, the ideals in \( \Phi \) are precisely the principal down-sets \( I_\phi = \downarrow \phi = \{ \psi : \psi \leq \phi \} \) for \( \phi \in \Phi \). Call an element \( \phi \in \Phi \) meet-irreducible iff \( \phi = \psi_1 \land \psi_2 \) for some \( \psi_1, \psi_2 \in \Phi \) implies that \( \phi = \psi_1 \) or \( \phi = \psi_2 \) (equivalently, iff \( \phi \) has exactly one upper neighbor in the order of \( \Phi \)).

**Lemma 4.26.** \( I_\phi \) is prime iff \( \phi \) is meet-irreducible.

**Proof.** If \( \phi = \psi_1 \land \psi_2 \) and \( \psi_1 \neq \phi \neq \psi_2 \), then \( I_\phi \) is clearly not prime. So assume \( \phi \) is meet-irreducible and \( \psi_1 \land \psi_2 \in I \). Then \( \psi_1 \land \psi_2 \in I_\phi \), that is,
ψ₁ ∧ ψ₂ ≤ φ. Thus (ψ₁ ∧ ψ₂) ∨ φ = φ = (ψ₁ ∨ φ) ∧ (ψ₂ ∨ φ), using distributivity, and so φ = ψ₁ ∨ φ or φ = ψ₂ ∨ φ. But this means ψ₁ ≤ φ or ψ₂ ≤ φ, that is, ψ₁ ∈ I₀ or ψ₂ ∈ I₀.

Let M(Φ) be the set of all meet-irreducibles of Φ. Obviously, φ ∈ I₀ iff φ ≤ µ. So X₀ = \{ I ∈ XΦ : φ ∈ I \} may be identified with \{ µ ∈ M(Φ) : φ ≤ µ \} = ↑ φ ∩ M(Φ). So the ordered set (H, ≤) at hand may be concretized as U(M(Φ), ⊆) the final result is

**Proposition 4.27.** The map φ ∈ Φ ↦ ↑ φ ∩ M(Φ) provides a lattice isomorphism between Φ and the lattice of all up-sets in M(Φ), a sublattice of the dual power set lattice P(XΦ)ᵈ.

For a detailed account, the reader is referred to [Davey & Priestley, 2002].

Focussing on the object part of the duality between distributive information algebras and their representing structures, we are left with pairs (≤, T) where ≤ is an order on a finite set H and T = \{ Θ₁,...,Θₖ \} a bunch of equivalence relations on H which is closed under ⋆. The latter must be separating as specified in Def. [L5] that is, (i) the closure operator σₘ, associated with Θᵢ, takes up-sets to up-sets, and (ii) whenever x, y ∈ H, Θᵢ ∈ T and (x, y) /∈ Θᵢ, then there exists a Θᵢ-saturated up-set V ⊆ H containing exactly one of x, y.

To enhance readability, we abbreviate σₘ by σᵢ whenever appropriate.

Our goal is to describe such structures - rather informally - by sentences of a first-order language Λₖ with equality containing a binary relation symbol ≤ and a finite number of binary relation symbols Θ₁,...,Θₖ. It is straightforward how to express by Λₖ-sentences that ≤ is an order relation on H and that the Θᵢ are equivalence relations on H. As an example, the sentence Cᵢⱼ below expresses that Θᵢ and Θⱼ commute:

Cᵢⱼ : ∀ xuy∃ u′(xΘᵢuΘⱼy → Θⱼu′Θᵢy).

For condition [L5i], remember that any up-set U ⊆ H is a set union of principal up-sets ↑ x with x ∈ H, so it will do to enforce that σᵢ(↑ x) is an up-set for all x ∈ H. Put

A₁ : ∀ xuyv∃ y′((x ≤ yΘᵢu ≤ v) → (x ≤ y′Θᵢv)).

**Claim 4.28.** σᵢ(↑ x) is an up-set iff (H, ≤, T) satisfies A₁ for all x ∈ H and all Θᵢ ∈ T.
Proof. The formula just says if \( u \) is in the \( \Theta_i \)-class of some \( y \in \uparrow x \) and \( v \geq u \), then \( v \) is in the \( \Theta_i \)-class of some \( y' \in \uparrow x \), making it a member of \( \sigma_i(\uparrow x) \). \( \square \)

For condition 4.8(ii), observe that \( \sigma_i(\uparrow x) \) is obviously the least \( \Theta_i \)-saturated up-set containing \( x \), assuming \( L_i \). Consequently, if \( x \) and \( y \) can be separated by any \( \Theta_i \)-saturated up-set, they can be separated by \( \sigma_i(\uparrow x) \) or \( \sigma_i(\uparrow y) \). So we have to rule out the possibility that simultaneously \( x \in \sigma_i(\uparrow y) \) and \( y \in \sigma_i(\uparrow x) \), whenever \((x, y) \notin \Theta_i \). This is exactly what the following sentence does:

\[
B_i : \forall xyx'y'((x \leq x' \Theta_i y \ & \ y \leq y' \Theta_i x) \rightarrow x \Theta_i y).
\]

Summing up, we have

**Proposition 4.29.** The dual objects of finite distributive information algebras are structures \((H; \leq, \mathcal{T})\) where \( H \) is finite, \( \leq \) is an order on \( H \) and \( \mathcal{T} \) is a set of equivalence relations on \( H \) satisfying conditions \( A_i \), \( B_i \) and \( C_{ij} \) for all \( \Theta_i, \Theta_j \in \mathcal{T} \).

This amounts to a first-order description of the dual objects of finite distributive information algebras. However, \( \Lambda_k \) cannot express the property of \( \mathcal{T} \) being closed under \( * \). We will address this problem below. The existence of \( * \)-closed subsets \( \mathcal{T} \subseteq Eq(H) \) consisting of separating equivalences on an arbitrary ordered set \((H; \leq)\) is trivial: Let \( \Delta \) be the identity relation on \( H \), and \( \nabla \) the all-relation \( \nabla = H \times H \). It is straightforward to see that both \( \Delta \) and \( \nabla \) (trivially, since there is nothing to separate) are separating and that \( \Delta * \nabla = \nabla = \nabla * \Delta \), so the answer is yes. The right question at this place is to ask for nontrivial such \( \mathcal{T} \), meaning \( \mathcal{T} \supseteq \{\Delta, \nabla\} \).

**Lemma 4.30.** On any ordered set \((H, \leq)\) with \(|H| \geq 2\) there exists a non-trivial separating equivalence \( \Theta \neq \Delta. \nabla \).

*Proof.* Pick any up-set \( U = \uparrow x \neq H \) and define an equivalence \( \Theta \) on \( H \) with the blocks \( U \) and all singletons \( \{y\} \) with \( y \notin \uparrow x \). Consider an arbitrary principal up-set \( V = \uparrow y \subseteq H \). Now if \( V \cap U = \emptyset \), then obviously \( \sigma_{\Theta}(V) = V \), and if \( V \cap U \neq \emptyset \), then \( \sigma_{\Theta}(V \cap U) = U \) and \( \sigma_{\Theta}(V \setminus U) = V \setminus U \), hence \( \sigma_{\Theta}(V) = V \cup U \), a (not necessarily principal) up-set. So the first part of the separation condition is satisfied.

For the second part, consider \( y, y' \in H \) such that \((y, y') \notin \Theta \). If \( y \in U \) and \( y' \notin U \), then \( U \) will do the job. So suppose \( y, y' \in H \setminus U \). Since \( y \neq y' \) we
have \( y \not\leq y' \) or \( y' \not\leq y \). Then, borrowing the above argument, \( \uparrow y' \cup U \) is a \( \Theta \)-closed up-set containing \( y' \) but not \( y \).

**Corollary 4.31.** On any ordered set \( (H, \leq) \) with \( |H| \geq 2 \) there exists a nontrivial \( \star \)-closed subset \( T \subseteq Eq(H) \) consisting of separating equivalences.

**Proof.** Take \( T = \{ \Theta, \Delta, \nabla \} \) with \( \Theta \) as constructed in Lemma 4.30.

Remember that Cor. 2.3 allowed us to restrict our attention to information algebras where the set of all extraction operations is closed under composition, which corresponds to \( T \) being \( \star \)-closed. There was a good reason to do so: Otherwise, in Def. 2.6, the \( g \)-half of a homomorphism \((f, g)\) would become a partial operation which is highly undesirable. So we have a closer look at how \( \star \) interacts with separating equivalences. Assume \( \Theta_i, \Theta_j \in T \) are separating.

In order to satisfy Def. 4.8(i), the closure operator \( \sigma_{\Theta_i \star \Theta_j} \) must take up-sets to up-sets. Since \( \sigma_{\Theta_i \star \Theta_j} = \sigma_i \circ \sigma_j \) by Lemma 2.13, this is obvious.

Def. 4.8(ii) for \( \Theta_i \star \Theta_j \) is harder to enforce. We need need a stronger form of \( B_i \) ensuring that whenever \( x, y \in H \) and \( (x, y) \not\in \Theta_i \star \Theta_j \), then there exists a \( \Theta_i \star \Theta_j \)-saturated up-set \( V \subseteq H \) containing exactly one of \( x, y \). Now, assuming \( A_i \) and \( A_j \), we have \( y \in \sigma_j \sigma_i(\uparrow x) \) iff \( x \leq x' \Theta_i u \leq u' \Theta_j y \) for some \( x', u, u' \in H \), and \( x \in \sigma_j \sigma_i(\uparrow y) \) iff \( y \leq y' \Theta_i v \leq v' \Theta_j x \) for some \( y', v, v' \in H \). The following formula rules out the possibility of having \( y \in \sigma_j \sigma_i(\uparrow x) \) and \( x \in \sigma_j \sigma_i(\uparrow y) \) simultaneously whenever \( (x, y) \not\in \Theta_i \star \Theta_j \):

\[
B_{ij} : \forall x, y, x', u, u', v, v' \exists z ((x \leq x' \Theta_i u \leq u' \Theta_j y) \land (y \leq y' \Theta_i v \leq v' \Theta_j x)) \rightarrow x \Theta_i z \Theta_j y).
\]

Note that if \( \Theta_i = \Theta_j \), then putting \( u = u' \), \( v = v' \) and \( z = x \) or \( z = y \) reduces \( B_{ij} \) to \( B_i \), so \( B_{ij} \) indeed contains \( B_i \).

In our original definition of an information algebra, the set \( E \) of extraction operators was not supposed to be closed under composition. Let us call, for convenience, an algebra \( \underline{A} = (\Phi; E) \) *partial* if this is not necessarily the case. Then the dual structures of finite distributive partial information algebras are exactly the structures \((H; \leq, T)\) where \( H \) is finite, \( \leq \) is an order on \( H \) and \( T \) is a (finite) set of equivalence relations on \( H \) satisfying conditions \( A_i, B_{ij} \) and \( C_{ij} \) for all \( \Theta_i, \Theta_j \in T \), but with \( T \) not necessarily closed under \( \star \). Write \( T^\star \) for the closure of \( T \) under \( \star \), then obviously \((H; \leq, T^\star)\) will be, by
Cor. 2.3. The dual of an "ordinary distributive information algebra. Since obviously $T^{**} = T^*$, we obtain

**Proposition 4.32.** $(H; \leq, T)$, where $H$ is finite, $\leq$ is an order on $H$ and $T$ is a (finite) set of equivalence relations on $H$, is the dual structure of a finite distributive information algebra iff there is subset $G \subseteq T$ satisfying $A_i$, $B_{ij}$ and $C_{ij}$ such that $T = G^*$.

Consider a first order language $\Lambda$ containing $\leq$ and countably many relation symbols $\Theta_1, \Theta_2, \ldots$. We obtain

**Theorem 4.33.** Then the class of all finite partial distributive information algebras is (relatively) $\Lambda$-elementary, and any finite distributive information arises as the $\star$-closure of a (generally non-unique) member of this class.

## 5 Summary

We considered the intuitive notion of a "piece of information" not by giving a precise definition of it, but by precisely specifying the rules which should - again intuitively - govern their properties. The basic idea is that "pieces of information" must be able to be combined, and that this combination does not depend on the order in which the pieces under consideration are put into the combination. This is modeled algebraically by a commutative idempotent semigroup $(\Phi, \cdot)$ of "pieces of information" containing a unit $1$ which doesn't change any piece of information when combined with it, and a zero $0$ (representing contradiction) which outputs $0$ when combined with any piece of information.

On other hand, pieces of information are obtained when one asks questions from an abstract set $Q$ of questions, and given a piece of information as an answer, one should be able to extract from this piece the information relevant to the question asked. This defines, for each question, an unary operation from pieces of information to pieces of information. At this point, we make a crucial assumption: We stipulate that, when two question are asked in succession, the information obtained does not depend on order of the questions. This clearly delimits the scope of algebraic theory we developed, but as the literature cited shows, a plethora of important examples falls in this category (see also Subsection 2.5). In order to obtain algebras without partial
operations, we also stipulate at this point that the set of these operations be closed under composition. All said and done, we end up with with a second commutative idempotent semigroup \((E, \circ)\) of so-called extraction operators (on \(\Phi\)) indexed by \(Q\).

Commutative idempotent semigroups may be equipped with a compatible order structure in exactly two ways. To stay in accordance with the existing literature, we opted - for \(\Phi\) - for the one making 0 the greatest and 1 the least element, which turns \(\Phi\) into a bounded join-semilattice. This order was referred to as the information order. It turned out that extraction operators preserve the information order and that the defining properties of extraction operators may be expressed in order-theoretic terms. This made clear that extraction operators are duals of existential quantifiers as considered in algebraic logic.

So far, the set \(\Phi\) of pieces of information as well as the set \(Q\) of questions were arbitrary abstract sets, subject only to the conditions specified for composition and extraction. We proceeded by giving them an internal structure as specific set-theoretical constructs over a (non-empty) base set \(U\), best thought of as a set of possible worlds. Questions \(x \in Q\) were then be modelled by equivalence relations \(\equiv_x\) on \(U\), the idea being that for \(u, u' \in U\) we have \(u \equiv_x u'\) iff question \(x\) has the same answer in the worlds \(u\) resp. \(u'\). The point then was to model pieces of information as semilattices of subsets of \(U\), and extraction operators as the saturation operators associated with the equivalence relations \(\equiv_x\) on \(U\) for \(x \in Q\). This led to a type of information algebra called set algebra, for lack of a better term.

The rest of the paper is concerned with representations of abstract information algebras, and with duality theory in the sense of the book ”Natural Dualities for the Working Algebraist” by David Clark and Brian Davey, putting information algebras into the context of classical dualities like Stone resp. Priestley duality for Boolean algebras resp. distributive lattices. First, we showed that any information algebra in our sense may be represented by a set algebra as mentioned above. Then, we obtained a direct representation of information algebras containing enough ”maximally informative” in terms of Boolean algebras, and finally we showed that the category of information algebras based on a distributive lattice is fully dual - modelling objects as well as morphisms - to a category of certain topological spaces equipped with appropriate equivalence relations.
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