Strings and Noncommutativity

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1 Abstract

In these talks we review some of the recent results on open strings and noncommutative gauge theories, starting from the early calculations of open strings in a constant electromagnetic background. We discuss both the neutral string and the charged string. In the latter case, the scaling limit that leads to noncommutative abelian gauge theory can be generalized to a scaling limit in which multiple noncommutativity parameters enter. Our approach corresponds to expanding a theory with $U(N)$ Chan-Paton factors around a background $U(1)^N$ gauge field with different magnetic fields in each $U(1)$. This scaling limit can be interpreted in terms of a matrix model. We also describe an open string model with a time-dependent noncommutativity parameter. This model is the open string version of a WZW model based on a non-semi-simple group. It has a time-dependent background, and a spacetime metric of the plane wave type supported by a Neveu-Schwarz two-form potential.

2 Introduction

There are many ways how string theory seems to challenge our current understanding of spacetime. Probably, string dualities are the most striking example of this statement, not to mention supersymmetry and extra dimensions. More recently, a new fact has emerged, the noncommutativity of spacetime variables at the string scale, a fact that obviously might have important implications for the structure of spacetime. Noncommutativity originally emerged in the context of open strings, starting from the treatment of open string field theory in [1]. More recently, it reappeared in the context of matrix theory compactified on a torus [2, 3]. Finally, it showed up in the low energy description of strings in an electromagnetic background [4, 5]. In these notes we will concentrate in the last approach. Our interest is to investigate the emergence of noncommutativity in
various open string models, and compute the noncommutativity parameters there. The question behind our investigation is to see what novel noncommutative space geometries can emerge from string theory. We will go from the single noncommutativity parameter of the neutral string to the multiple noncommutative parameters of the open string in a $U(N)$ background, and finally to the time-dependent noncommutativity parameters in a model with a time-dependent background.

A very interesting question of course is what is the generalization of the Moyal-Weyl product corresponding to more general noncommutativity situations. A related question is what is the corresponding Born-Infeld action. In the $U(1)$ case the noncommutativity can be seen to emerge directly from a rewriting/reinterpretation of the Born-Infeld action. How does that work in the non-abelian case? Is there an equivalent in the time-dependent case? These are still open questions, and we will not have much to say about them.

3 Neutral String

The propagator for the open neutral string in a slowly varying background $U(1)$ gauge field was computed in [5] by solving the equations of motion

$$\Box < x^i(z, \bar{z}) x^j(\zeta, \bar{\zeta}) > = -2\pi\alpha' \delta^2(z - \zeta) G^{ij}$$

with the boundary conditions

$$(\partial_z - \partial_{\bar{z}}) < x^i(z, \bar{z}) x^j(\zeta, \bar{\zeta}) > + 2\pi\alpha' g^{ik} B_{kl}(\partial_z + \partial_{\bar{z}}) < x^l(z, \bar{z}) x^j(\zeta, \bar{\zeta}) > |_{z=\bar{z}} = 0$$

where $\Box \equiv 4\partial_z \partial_{\bar{z}}$. Here the open string worldsheet $\Sigma$ will denote the disc with Euclidean metric $\gamma^{\alpha\beta} = \delta^{\alpha\beta}$, and the complex coordinates $z, \bar{z}$ are related to the original strip coordinates $\sigma, \tau$ with $\tau$ rotated to $t \equiv i \tau$ and $-\infty \leq t \leq \infty, 0 \leq \sigma \leq \pi$, by $z \equiv e^{t+i\sigma}, \bar{z} \equiv e^{t-i\sigma}$ with $\text{Im} z \geq 0$.

The propagator was found to be

$$< x^i(z, \bar{z}) x^j(\zeta, \bar{\zeta}) > = -\alpha' \left[\frac{1}{2} g^{ij} \ln(z - \zeta) + \frac{1}{2} g^{ij} \ln(\bar{z} - \bar{\zeta})\right]$$

$$+ (-\frac{1}{2} g^{ij} + G^{ij} + \frac{\theta^{ij}}{2\pi\alpha'}) \ln(z - \zeta)$$

$$+ (-\frac{1}{2} g^{ij} + G^{ij} - \frac{\theta^{ij}}{2\pi\alpha'}) \ln(\bar{z} - \bar{\zeta}) - \frac{i}{2\alpha'} \theta^{ij}$$

where

$$G^{ij} = [(g + 2\pi\alpha B)^{-1} g (g - 2\pi\alpha B)^{-1}]^{ij}$$

$$G_{ij} \equiv g_{ij} - (2\pi\alpha')^2 (Bg^{-1}B)_{ij} = [(g - 2\pi\alpha B) g^{-1} (g + 2\pi\alpha B)]_{ij}$$

$$\theta^{ij} \equiv -(2\pi\alpha')^2 [(g + 2\pi\alpha B)^{-1} B (g - 2\pi\alpha B)^{-1}]^{ij}.$$
In [4] the above propagator was used to compute the “equal time” commutator of the string operators on the boundary via a short distance expansion procedure [6], and to define a noncommutativity parameter

\[ [X^i, X^j] = i\theta^{ij}, \quad (3.5) \]

where \( \theta^{ij} \) is given in (3.4). This shows that the \( X^i \) are coordinates in a noncommutative space with noncommutativity parameter \( \theta \). Indeed in [4] it was also shown that, in the scaling limit \( \alpha' = \sqrt{\epsilon} \to 0 \) and \( g_{ij} = \epsilon \to 0 \), a noncommutative gauge theory emerged with field strength

\[ \hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i\hat{A}_i \star \hat{A}_j + i\hat{A}_j \star \hat{A}_i \quad (3.6) \]

where \( \hat{A}_i \) is related to an ordinary gauge potential \( A_i \) via the relation

\[ \hat{F} = \frac{1}{1 + F \theta} F, \quad (3.7) \]

known as the Seiberg-Witten map [4]. In (3.6) the \( \star \) product is the Moyal-Weyl product

\[ (f \star g)(x) = e^{i\theta^{ij} \partial_i \partial_j} f(x)g(x') |_{x=x'} \quad (3.8) \]

The relation between this noncommutative field strength and the commutative one we started from is encapsulated in the Born-Infeld action \( \sqrt{\det(g + 2\pi\alpha'(B + F))} \), the effective target space action for an open string with slowly varying background fields [7, 5]. It turns out that

\[ \sqrt{\det(g + 2\pi\alpha'(B + F))} \sim \sqrt{\det(G + 2\pi\alpha'\hat{F})} \quad (3.9) \]

where \( g_{ij} \) is the ‘closed string metric’ and \( G_{ij} \) is the ‘open string metric’ which appear in (3.4).

### 4 Charged Open String

We will now find the noncommutativity parameter in the charged open string case [9]. Rather than studying the scaling limit of an open string theory in a fully non-abelian \( U(N) \) background (which is still difficult), we will restrict ourselves to backgrounds that reside in the \( U(1)^N \) Cartan subgroup, but with different background \( U(1) \) fields on each brane. So we will discuss the charged string, i.e. a string with different \( U(1) \) backgrounds at each end. We consider magnetic backgrounds with constant field strength. In this case we have the exact mode expansion to start with, originally derived in [5] and [8], and we use it to compute the propagator on the disk. Starting from this propagator, we repeat the argument in [6] to compute two noncommutativity parameters, one at each end of the string. This emphasizes the interpretation that the noncommutativity of the
D-brane worldvolume in the presence of a background B-field along the brane is really a property of the endpoint of the string, rather than a feature of the worldvolume itself. For N D-branes, there are N different noncommutativity parameters, even when the D-branes are coincident. Hence, the short distance behavior of the operator products of tachyon vertex operators which are inserted on the boundaries defines star products with different noncommutativity parameters. These enter into the computation of the scattering amplitudes of the scaling limit theory.

The worldsheet action for the “charged” string with different magnetic fields at each end is

$$ S = \frac{1}{4\pi\alpha'} \int_\Sigma g_{\mu\nu} \partial_\sigma X^\mu \partial^\nu X - \frac{i}{2} \int_{-\infty}^{\infty} dt \left( B^{(1)}_{ij} X^i \partial_t X^j |_{\sigma=0} + B^{(2)}_{ij} X^i \partial_t X^j |_{\sigma=\pi} \right). \tag{4.10} $$

Here $0 \leq \mu, \nu \leq 25$ and the open strings end on Dp-branes in the $(0,i)$ directions for $1 \leq i \leq p$. Variation of (4.10) gives the equations of motion for the worldsheet field

$$ (\partial_{\sigma}^2 + \partial_t^2) X^\mu(\sigma,t) = 0 \tag{4.11} $$

together with the boundary conditions at each end of the string

$$ g_{ij} \partial_\sigma X^j + 2\pi i \alpha' B^{(1)}_{ij} \partial_t X^j |_{\sigma=0} = 0 $$
$$ g_{ij} \partial_\sigma X^j - 2\pi i \alpha' B^{(2)}_{ij} \partial_t X^j |_{\sigma=\pi} = 0 \tag{4.12} $$

and

$$ \partial_\sigma X^0 |_{\sigma=0,\sigma=\pi} = 0 $$
$$ X^I |_{\sigma=0} = 0, \quad p + 1 \leq I \leq 25. \tag{4.13} $$

Note that the worldsheet action for the neutral string, whose propagator in the directions along the Dp-brane is given in (3.3), is a special case of (4.10) with $B^{(1)}_{ij} = -B^{(2)}_{ij}$. For simplicity, we now specialize to the case where the ends of the string live on D2-branes, i.e. $i = 1, 2$. In this case the magnetic fields have only one component and we relabel them as $B^{(1)}_{12} = q_1 B_{12}$ and $B^{(2)}_{12} = q_2 B_{12}$, where $q_1 + q_2 \neq 0$.

In a basis given by $X^\pm(\sigma,t) \equiv X^1(\sigma,t) \pm i X^2(\sigma,t)$, the charged string normal mode expansion [5, 8] can be written as

$$ X^+(z,\bar{z}) = x^+ + \frac{i}{2} \sqrt{2\alpha'} \sum_{r \in \mathbb{Z}+A} \frac{a_r}{r} (z^{-r} + \bar{z}^{-r}) - \frac{i}{2} \sqrt{2\alpha'} B \sum_{r \in \mathbb{Z}+A} \frac{a_r}{r} (z^{-r} - \bar{z}^{-r}) \tag{4.14} $$

$$ X^-(z,\bar{z}) = x^- + \frac{i}{2} \sqrt{2\alpha'} \sum_{s \in \mathbb{Z}-A} \frac{\tilde{a}_s}{s} (z^{-s} + \bar{z}^{-s}) + \frac{i}{2} \sqrt{2\alpha'} B \sum_{s \in \mathbb{Z}-A} \frac{\tilde{a}_s}{s} (z^{-s} - \bar{z}^{-s}). \tag{4.15} $$

4
with commutation relations for the zero modes given by

\[[a_r, \tilde{a}_s] = 2G r \delta_{r,-s}; \quad [a_r, a_{r'}] = 0 = [\tilde{a}_s, \tilde{a}_{s'}];\]

\[[x^+, x^-] = -\frac{2}{(q_1 + q_2)B_{12}}; \quad [a_r, x^\pm] = 0 = [\tilde{a}_s, x^\pm]. \quad (4.16)\]

Here \(g_{ij} = g^{-1}\delta_{ij}, G_{ij} = G^{-1}\delta_{ij}\), where now

\[G = \frac{g}{1 + B^2}, \quad B = gq_1 2\pi \alpha' B_{12}, \quad A = \frac{1}{\pi}(\arctan B + \arctan \frac{q_2}{q_1} B). \quad (4.17)\]

Unlike the neutral string case, oscillators \(a_r, \tilde{a}_s\) are non-integrally moded, since \(r = n + A\), \(s = n - A\), for \(n \in \mathbb{Z}\). We use the mode expansions computed above to compute the charged propagator. The details of this calculation can be found in \([9]\). For \(|z| > |\zeta|\), we find it is given by

\[<X^+(z, \bar{z})X^-(\zeta, \bar{\zeta}) > \equiv \langle \beta |X^+(z, \bar{z})X^-(\zeta, \bar{\zeta})| \alpha \rangle \]

\[= -\frac{2\alpha' \pi g}{B + \frac{q_2}{q_1} B} - 2\alpha' G \frac{1}{A}(\zeta^A + \bar{\zeta}^A - 1) + 2i\alpha' GB \frac{1}{A}(\zeta^A - \bar{\zeta}^A)\]

\[+\alpha' G \left[ f\left(\frac{\zeta}{z}\right) + f\left(\frac{\bar{\zeta}}{\bar{z}}\right) + f\left(\frac{\zeta}{\bar{z}}\right) + f\left(\frac{\bar{\zeta}}{z}\right) \right]\]

\[+\alpha' GB^2 \left[ f\left(\frac{\zeta}{z}\right) + f\left(\frac{\bar{\zeta}}{\bar{z}}\right) - f\left(\frac{\zeta}{\bar{z}}\right) - f\left(\frac{\bar{\zeta}}{z}\right) \right]\]

\[+2i\alpha' GB \left[ -f\left(\frac{\zeta}{z}\right) + f\left(\frac{\bar{\zeta}}{\bar{z}}\right) \right] \quad (4.18)\]

where

\[f(\rho) \equiv \sum_{r = n + A; n \geq 0}^{\rho^r} r; \quad \lim_{A \to 0} f(\rho) = -\ln(1 - \rho) + \lim_{A \to 0} \frac{\rho^A}{A}. \quad (4.19)\]

As a first step toward computing the commutator on the boundary, we need to compute the propagator on the boundary. We distinguish the two boundary regions of the open string disk as follows. On the boundary \(\sigma = 0\), we have \(z = |z| = \tau\) and \(\zeta = |\zeta| = \tau'\) so \(\tau, \tau' > 0\); while on \(\sigma = \pi\), then \(z = |z|e^{i\pi} = \tau\) and \(\zeta = |\zeta|e^{i\pi} = \tau'\) so here \(\tau, \tau' < 0\).

For \(|z| > |\zeta|\), and on the boundary \(\sigma = 0\), the propagator is

\[<X^+(z, \bar{z})X^-(\zeta, \bar{\zeta}) > \mid_{\sigma = 0} = \frac{-2\alpha' \pi g}{B + \frac{q_2}{q_1} B} + 4\alpha' G \sum_{n=0}^{\infty} \frac{1}{n + A} \zeta^{n+A}\]

\[+\frac{2\alpha' G}{A} - \frac{4\alpha' G}{A} \zeta^A, \quad (4.20)\]
< X^-(z, \bar{z}) X^+(\zeta, \bar{\zeta}) > |_{\sigma=0} = \frac{2\alpha' \pi g}{B + \frac{q_2}{q_1} B} + 4\alpha' G \sum_{n=1}^{\infty} \frac{1}{n - A} (\bar{\zeta})^{n-A} + \frac{2\alpha' G}{A} - \frac{4\alpha' G}{A} z^A. \quad (4.21)

By using the above equations, we compute the commutator at the \( \sigma = 0 \) end of the string.

\[
[X^+(\tau), X^-(\tau)] = T(X^+(\tau) X^-(\tau^-) - X^+(\tau^-) X^-) \]
\[
\equiv \lim_{\epsilon \rightarrow 0} (< X^+(\tau) X^- (\tau - \epsilon) > - < X^-(\tau + \epsilon) X^+(\tau) >), \quad \text{(for } \epsilon > 0) \]
\[
= \lim_{\epsilon \rightarrow 0} \left( \frac{-4\alpha' \pi g}{B + \frac{q_2}{q_1} B} + 4\alpha' G \sum_{n=0}^{\infty} \frac{1}{n + A} (\frac{\tau-\epsilon}{\tau})^{n+A} - \sum_{n=1}^{\infty} \frac{1}{n - A} (\frac{\tau+\epsilon}{\tau})^{n-A} \right) \]
\[
- \frac{4\alpha' G}{A} (\tau - \epsilon)^A + \frac{4\alpha' G}{A} (\tau + \epsilon)^A \]
\[
= \frac{-4\alpha' \pi g}{B + \frac{q_2}{q_1} B} + 4\alpha' G \pi \cot \pi A \]
\[
= -4\alpha' \pi (g)^2 \frac{q_2 \pi \alpha' B_{12}}{1 + (g)^2 (q_2 \pi \alpha' B_{12})^2} \]
\[
= 2\Theta^{12}. \quad (4.22)
\]

Notice that \( \Theta^{12} \) is the same expression that appears in the neutral string. The analogous calculation at \( \sigma = \pi \) provides a different commutator.

\[
[X^+(\tau), X^-(\tau)] = T(X^+(\tau) X^-(\tau^-) - X^+(\tau^-) X^-) \]
\[
\equiv \lim_{\epsilon \rightarrow 0} (< X^+(\tau) X^- (\tau + \epsilon) > - < X^-(\tau - \epsilon) X^+(\tau) >), \quad \text{(for } \epsilon > 0) \]
\[
= -4\alpha' \pi (g)^2 \frac{q_2 \pi \alpha' B_{12}}{1 + (g)^2 (q_2 \pi \alpha' B_{12})^2} \]
\[
= 2\tilde{\Theta}^{12}. \quad (4.23)
\]

In the limit \( q_1 \rightarrow -q_2 \), then \( \tilde{\Theta}^{12} \rightarrow -\Theta^{12} \). Indeed in the neutral string case, where both ends of the string are on the same D-brane, the noncommutativity parameter at one end of the string is equal to minus that of the other end. For \( U(N) \) Chan-Paton factors, the background magnetic fields can take on \( N \) possible values, giving rise to \( N \) noncommutativity parameters.

As in [4], one can show that in the scaling limit \( (\alpha' \rightarrow 0, \text{keeping } G \text{ and } \Theta^{12} \text{ fixed}) \), the operator product expansion reduces to the star product [9]

\[
e^{ip \cdot X(\tau)} e^{iq \cdot X(\tau')} \sim e^{-\Theta^{12}(p_+ q_+ - p_- q_-)} : e^{i(p+q) \cdot X(\tau')} : \\
\equiv e^{ip \cdot X(\tau')} e^{iq \cdot X(\tau')} . \quad (4.24)
\]

For \( \sigma = \pi \) the same equations will hold with \( \Theta^{12} \) replaced by \( \tilde{\Theta}^{12} \).
The same results about noncommutativity parameters can be derived by starting with
the charged string propagator on the annulus, as calculated in [10] from a charged string
annulus propagator. Since the result is a short distance effect, it is independent of the
topology of the worldsheet.

5 The Spectrum in the Scaling Limit and a Matrix
Model

Our explicit knowledge of the mode expansion allows us to compute the spectrum of the
gauge theory obtained in the scaling limit, and compare it to one predicted by the matrix
model. In addition to the $U(1)^N$ massless gauge bosons, we find charged vector states
that survive for each Landau level. The states of the limiting non-abelian noncommutative
gauge theory are no longer massless, but rather tachyonic or massive.

As in [4], we consider the scaling limit $g^{-1} \to \epsilon$ and $\alpha' \to \sqrt{\epsilon}$, for $\epsilon \to 0$. Actually
this limit means letting the dimensionless quantity $\alpha' B_{12} \to \sqrt{\epsilon}$, while keeping $B_{12}$ fixed.
Then we have that the noncommutativity parameters are finite in the scaling limit and
are given by $\Theta^{12} \to (-q_1 B_{12})^{-1}$, $\tilde{\Theta}^{12} \to (-q_2 B_{12})^{-1}$.

One can see from (4.17) that $\tan \pi A = \frac{B + \frac{\alpha'}{2\pi q_1 B_{12}}}{1 - \frac{\alpha'}{2\pi q_2 B_{12}}}$. In the scaling limit,

$$\tan \pi A \to -\frac{(q_1 + q_2)\sqrt{\epsilon}}{2\pi q_1 q_2} \quad A \to -\frac{(q_1 + q_2)\sqrt{\epsilon}}{2\pi q_1 q_2}.$$  \hspace{1cm} (5.25)

Using these formulae, one finds [9] that the mass formulae for the two polarization states
of the charged vectors are finite in the scaling limit and behave as

$$\text{mass}^2 = -\frac{1}{2\alpha'} A(1 + A) \to \frac{1}{2} \frac{(q_1 + q_2) B_{12}}{2\pi^2 q_1 q_2} = \frac{(q_1 + q_2)G}{2q_2 \Theta^{12}},$$  \hspace{1cm} \hspace{1cm} (5.26)

$$\text{mass}^2 = \frac{1}{2\alpha'} A(3 - A) \to -\frac{3}{2} \frac{(q_1 + q_2) B_{12}}{2\pi^2 q_1 q_2 \Theta^{12}} = \frac{3(q_1 + q_2)G}{2q_2 \Theta^{12}}.$$  \hspace{1cm} \hspace{1cm} (5.26)

Also in the spectrum are the charged vectors at different Landau levels. For each charged
boson, the two polarizations have different masses. They differ from those of (5.26) by
integer multiples of $\frac{2(m+1)G}{q_1 q_2 G}$. (Notice that each Landau level has infinite degeneracy).
All other states in the charged string spectrum become infinitely heavy and decouple in
the scaling limit. So the complete spectrum of our $U(1)^N$ noncommutative field theory,
which is derived from $N$ neutral and $N^2 - N$ charged string sectors, is described by $N$
massless neutral gluons and the charged vectors above.

We have seen that our theory, which is a $U(2)$ gauge theory expanded around a
$U(1) \times U(1)$ background, has a scaling limit. The next question is to find the corresponding
action.
It turns out that the noncommutative gauge theory action for the $U(1) \times U(1)$ case can be described by a matrix model. A matrix theory action in terms of two infinite-dimensional matrices $X^1, X^2$ is

$$\mathcal{S} = -\frac{1}{2g_{YM}^2} \text{Tr}_\mathcal{H}[X^1, X^2]^2$$

where $X^1, X^2$ are some operators acting in a Hilbert space $\mathcal{H}$. Its equations of motion are

$$\delta_{kl}[X^k, [X^\ell, X^j]] = 0.$$

A solution \cite{11} is $X^i_{\text{sol}}$ given by a $2 \times 2$ block matrix of the form

$$X^i_{\text{sol}} = \begin{pmatrix} y^i & 0 \\ 0 & z^i \end{pmatrix}$$

where $y, z$ satisfy the Heisenberg algebras

$$[y^i, y^j] = i\theta_1^{ij} \mathbf{1},$$
$$[z^i, z^j] = i\theta_2^{ij} \mathbf{1},$$

i.e. $X^1_{\text{sol}} + iX^2_{\text{sol}} = \begin{pmatrix} \sqrt{2\theta_1 a_1} & 0 \\ 0 & \sqrt{2\theta_2 a_2} \end{pmatrix}$. Above, $a_1, a_2$ are the creation and annihilation operators in the Hilbert space $\mathcal{H}$. Then it can be shown \cite{11} by expanding the equations of motions around $X^i = X^i_{\text{sol}} + y^i$ and computing the coefficients of the terms quadratic in $y^i$, that the spectrum of fluctuations about this classical solution coincides with the $\alpha' \to 0$ limit of the string spectrum described above. One needs to identify

$$\theta_1 - \theta_2 = \frac{(q_1 + q_2)G}{q_2 \Theta^{12}},$$

where the RHS follows the notation from \cite{5720}.

6 Time-dependent Background

In most of the examples currently known, the noncommutativity parameter is constant. An obvious task is to look for time-dependent noncommutativity parameters. In \cite{13} we studied an open string model, whose target space has a plane wave metric supported by a time-dependent Neveu-Schwarz two-form potential. This background was studied in \cite{12} for closed strings, while in \cite{13} we looked at the open string version. We quantized the sigma-model in light-cone gauge, computed the worldsheet propagator, and used it to derive a time-dependent noncommutativity parameter. Indeed, for large values of the time parameter, this model resembles a neutral string in a constant background $B$ field, hence it is a good candidate for spacetime noncommutativity.
We do not solve the exact open mode expansion in closed form, but we compute it as a power series in a suitable parameter \( \mu \). This expansion is adequate to show noncommutativity. To do that, we follow the same strategy adopted for the neutral and charged string. We compute a mode expansion, derive from it a worldsheet propagator on the disk, and evaluate the commutator on the boundaries to find a time-dependent noncommutativity parameter.

The worldsheet action coupling a string to a general metric and background Neveu-Schwarz field is

\[
S = \int_{\Sigma} d\tau d\sigma \left[ \sqrt{-\gamma} \gamma^{\alpha\beta} G_{MN} \partial_\alpha X^M \partial_\beta X^N + B_{MN} \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \right]
\]

(6.27)

where we choose the string worldsheet \( \Sigma \) with Lorentz signature, and have rescaled the scalar worldsheet fields by \( (2\sqrt{\pi\alpha'})^{-1} \) so that the \( X^M \) are dimensionless. We consider the time-dependent background provided by the model based on a non-semi-simple group discussed in \([12]\), and adopt the same notation, with \( X^M = (a_1, a_2, u, v) \), and \( u \) being identified with the time in the target space.

The background field \( G_{MN} \) and \( B_{MN} \) are given by

\[
G_{MN} = \begin{pmatrix}
1 & 0 & \frac{a_2}{2} & 0 \\
0 & 1 & -\frac{a_1}{2} & 0 \\
\frac{a_2}{2} & -\frac{a_1}{2} & b & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad B_{MN} = \begin{pmatrix}
0 & u & 0 & 0 \\
-u & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(6.28)

The Lorentz signature target space metric \( G_{MN} \) can be recognized as a plane wave metric \([12]\). The background is time-dependent because of the \( u \)-dependence of \( B_{12} \). In \([12]\) it was shown that this model is exactly conformally invariant (i.e. to all orders in \( \alpha' \)) by checking the one-loop \( \beta \) function equations for the closed string backgrounds, and then proving that there were no higher loop contributions.

Here, since we are interested in noncommutativity, we consider open string boundary conditions. This case is not conformally invariant. \(^1\) The background (6.28) satisfies the the Born-Infeld field equations for \( N \neq u \),

\[
(D_M F_{NL})(1 - F^2)^{-1}_{LM} = 0
\]

(6.29)

where \( (1 - F^2)^{-1}_{LM} = (1 + F)^{-1}_{LP} \gamma_{PN} (1 - F)^{-1}_{NM} \) and \( (1 - F)_{MN} \equiv G_{MN} - 2\pi\alpha'F_{MN} \). In our case \( F_{MN} = B_{MN} \). Using (6.28), one can check that the nonvanishing components of the Ricci tensor and affine connections are \( R_{uu} = -\frac{1}{2} \), \( \Gamma^i_{uj} = \frac{1}{2} \epsilon^i_j \), \( \Gamma^u_{ui} = -\frac{a'_i}{4} \). It follows that \((D_M F_{NL})(1 - F^2)^{-1}_{LM} = \epsilon_{ij}(1 - F^2)^{-1}_{ju} = 0\), for \( N \neq u \). But

\[
(D_M F_{uL})(1 - F^2)^{-1}_{LM} = -\frac{u}{1 + (2\pi\alpha'u)^2}.
\]

(6.30)

\(^1\)A conformally invariant version of (6.28) is studied in \([14]\), but its noncommutativity parameter although non-constant, is not time-dependent.
In terms of the $a_i, u, v$ variables, the sigma model action is

$$S = \int_{\Sigma} d\tau d\sigma \left[ \sqrt{-\gamma} \gamma^{\alpha\beta} (\partial_{\alpha} a^i \partial_{\beta} a^i + 2\partial_{\alpha} u \partial_{\beta} v + b \partial_{\alpha} u \partial_{\beta} u + \epsilon_{ij} \partial_{\alpha} u \partial_{\beta} a^i a^j + e^{\alpha\beta} \epsilon_{ij} \partial_{\alpha} a^i \partial_{\beta} a^j) \right].$$

(6.31)

Although our background is not conformally invariant, we will consider a light-cone version of the sigma model in order to study open string propagators in a B-field with linear time dependence. We identify the target space time $u$ with the worldsheet time $\tau$ via $u = \mu \tau$, where $\mu$ is a dimensionless parameter. The equations of motion and boundary conditions for the transverse fields $a^i$ written in terms of $X \equiv a^1 + ia^2$ and $\tilde{X} \equiv a^1 - ia^2$ become:

$$\Box X - i\mu(\partial_{\sigma} X - \partial_{\tau} X) = 0, \quad \Box \tilde{X} + i\mu(\partial_{\sigma} \tilde{X} - \partial_{\tau} \tilde{X}) = 0,$$

$$[\partial_{\sigma} X + i\mu \tau \partial_{\tau} X]_{\sigma=0,\pi} = 0, \quad [\partial_{\sigma} \tilde{X} - i\mu \tau \partial_{\tau} \tilde{X}]_{\sigma=0,\pi} = 0,$$

(6.32)

where $\Box \equiv -\partial^2_{\sigma} + \partial^2_{\tau} = 4z\bar{z}\partial_z \partial_{\bar{z}}$.

For large $\tau$ (so that $\tau$ can be considered constant), notice the similarity of the boundary condition in (6.32) with the boundary condition for an open string in a background $B$ field. Since in the latter case the noncommutativity parameter is proportional to the background, this suggests we should expect here a noncommutativity parameter which depends on time.

The solution of (6.32) is given by the normal mode expansion for the transverse coordinates $X$ and $\tilde{X}$, to first order in $\mu$:

$$X(\sigma, \tau) = x_0 + a_0[\tau + \mu(-i\tau \sigma + \frac{i}{2} \tau^2)] + \sum_{n \neq 0} a_n e^{-in\tau} \left[ i \frac{1}{n} \cos n\sigma + \mu((-\frac{1}{2n^2} - i \frac{\tau}{n}) \sin n\sigma + (\frac{i}{2n^2} + \frac{(\sigma - \tau)}{2n}) \cos n\sigma) \right]$$

(6.33)

$$\tilde{X}(\sigma, \tau) = \bar{x}_0 + \bar{a}_0[\tau - \mu(-i\tau \sigma + \frac{i}{2} \tau^2)] + \sum_{n \neq 0} \bar{a}_n e^{-in\tau} \left[ i \frac{1}{n} \cos n\sigma - \mu((-\frac{1}{2n^2} - i \frac{\tau}{n}) \sin n\sigma + (\frac{i}{2n^2} + \frac{(\sigma - \tau)}{2n}) \cos n\sigma) \right]$$

(6.34)

7 Time-dependent Noncommutativity

Having found a mode expansion, the propagator on the disk can be computed along the lines of [9]. We will use the notations $z = e^{i(\tau + \sigma)}$, $\bar{z} = e^{i(\tau - \sigma)}$, $\zeta = e^{i(\tau' + \sigma')}$ and $\bar{\zeta} = e^{i(\tau' - \sigma')}$. Then for $|z| > |\zeta|$, the propagator to order $\mu$ is
\[ < X(z, \bar{z}) \bar{X}(\zeta, \bar{\zeta}) > = -i 4\alpha' (\tau + \mu(-i\tau + \frac{i}{2}\tau^2)) + 4\alpha' \sum_{n=1}^{\infty} e^{-\text{in}(\tau-\tau')} [\frac{1}{n} \cos n\sigma \cos n\sigma'] \]

\[ + i\mu \cos n\sigma'(\frac{1}{2n^2} + \frac{i\tau}{n}) \sin n\sigma - (\frac{i}{2n^2} + \frac{(\sigma - \tau)}{2n}) \cos n\sigma \]

\[ - i\mu \cos n\sigma'(\frac{1}{2n^2} - \frac{i\tau}{n}) \sin n\sigma' + (\frac{i}{2n^2} - \frac{(\sigma' - \tau')}{2n}) \cos n\sigma' \]

\[ - \frac{\mu}{n^2} \cos n\sigma \cos n\sigma'] \]

\[ + \mu(c_1 \tau + c_0). \quad (7.35) \]

We are free to add the function \( \mu(c_1 \tau + c_0) \) to the expression since it does not affect the equation of motion or the boundary condition for the propagator to first order in \( \mu \). For \( |z| > |\zeta| \), the expression for \( < \bar{X}(z, \bar{z}) X(\zeta, \bar{\zeta}) > \) is given by letting \( \mu \rightarrow -\mu \) in the above propagator. In the \( \mu \rightarrow 0 \) limit, these propagators reduce to the open bosonic string propagator \( \lim_{\mu \rightarrow 0} < X(z, \bar{z}) \bar{X}(\zeta, \bar{\zeta}) >= -2\alpha'(\ln |z - \zeta| + \ln |z - \bar{\zeta}|) \).

We evaluate the propagator on the worldsheet boundary at \( \sigma = 0 \) and \( \sigma = \pi \). We denote the points on the boundary at \( \sigma = 0 \) with \( z = e^{i\tau} \equiv T \), and \( \zeta = e^{i\tau'} = T' \). We get on the \( \sigma = 0 \) boundary

\[ < X(z, \bar{z}) \bar{X}(\zeta, \bar{\zeta}) > |_{\sigma=0} = -i 4\alpha' (\tau + \mu\frac{i}{2}\tau^2) + \mu(c_1 \tau + c_0) \]

\[ - 4\alpha' \ln(1 - e^{-i(\tau - \tau')}) - 2\alpha' i(\tau - \tau') \ln(1 - e^{-i(\tau - \tau')}) \]

\[ = -4\alpha' \ln(T - T') \]

\[ + \mu(-2\alpha' \ln^2 T - 2\alpha' \ln(\frac{T}{T'}) \ln(1 - T') + c_0)) \]

\[ < \bar{X}(z, \bar{z}) X(\zeta, \bar{\zeta}) > |_{\sigma=0} = -i 4\alpha' (\tau - \mu\frac{i}{2}\tau^2) - \mu(c_1 \tau + c_0) \]

\[ - 4\alpha' \ln(1 - e^{-i(\tau - \tau')}) + 2\alpha' i(\tau - \tau') \ln(1 - e^{-i(\tau - \tau')}) \quad (7.36) \]

Then at \( \sigma = 0 \) the commutator is

\[ [X(T), \bar{X}(T)] = T(X(T) \bar{X}(T) - X(T) \bar{X}(T^+)) \]

\[ \equiv \lim_{\epsilon \rightarrow 0} (X(T) \bar{X}(T - \epsilon) > - < X(T + \epsilon) X(T) >), \quad \text{(for } \epsilon > 0) \]

\[ = \mu(-4i\alpha'(\pi \ln T - i \ln^2 T) \]

\[ = \mu 4\alpha'(\pi \tau + \tau^2) \equiv \Theta, \quad (7.37) \]

where we chose \( c_1 = 2\pi\alpha' \), \( c_0 = 0 \), and use \( \lim_{\epsilon \rightarrow 0} (\ln(1 + \epsilon) \ln \epsilon) = 0 \). The noncommutativity parameter \( \Theta \) is time-dependent.
Similarly, at $\sigma = \pi$ the commutator is

\[
[X(\mathcal{T}), \tilde{X}(\mathcal{T})] = T(X(\mathcal{T}) \tilde{X}(\mathcal{T}^-) - X(\mathcal{T})\tilde{X}(\mathcal{T}^+)) \\
\equiv \lim_{\epsilon \to 0} (\langle X(\mathcal{T}) \tilde{X}(\mathcal{T} + \epsilon) \rangle - \langle \tilde{X}(\mathcal{T} - \epsilon) X(\mathcal{T}) \rangle), \quad (\text{for } \epsilon > 0) \\
= (-i4\alpha') \mu [-\pi \ln \mathcal{T} - i \ln^2 \mathcal{T}] \\
= \mu 4\alpha' (-\pi\tau + \tau^2) \quad (7.38)
\]
Thus for small $\mu$, we have

\[
\Theta = \mu 4\alpha' (\pi\tau + \tau^2) \quad \text{at } \sigma = 0, \\
\Theta = \mu 4\alpha' (-\pi\tau + \tau^2) \quad \text{at } \sigma = \pi. \quad (7.39)
\]
For small $\tau$, the theta parameter at the $\sigma = 0$ end of the string is minus that at the $\sigma = \pi$ end. This is the case for the neutral string in a constant background $B$ field as well. In fact, although we have worked only to lowest order in $\mu$, we can see directly from the equations of motion and boundary conditions (in $z$, $\bar{z}$) variables in the limit of large $z$, \textit{i.e.} large $i\tau$, a limit for which $z^{-1} \to 0$, that the system reduces to the neutral string with the identification $-\mu\tau = B$, a constant. (In the large $\tau$ limit, we note that $\ln |z|$ is approximately constant, in the sense that it is changing slowly, \textit{i.e.} its derivative $|z|^{-1}$ is small. Therefore, for large $\tau$ the noncommutativity parameter becomes constant, and our model is similar to the neutral string.) For large $\tau$, using the neutral string expressions, we find the noncommutativity parameter be time-dependent

\[
\Theta = -4\alpha' \pi B = 4\alpha' \mu \pi \tau \quad \text{at } \sigma = 0, \\
\Theta = 4\alpha' \pi B = 4\alpha' \mu \pi \tau \quad \text{at } \sigma = \pi. \quad (7.40)
\]
We have shown that our model exhibits noncommutativity for both small and large $\tau$. The expectation is that the model will remain noncommutative with a time-dependent noncommutativity parameter for all times.

8 Conclusions

The noncommutativity properties of string theories are interesting at many levels. In string theory, noncommutativity has led to new insights and new techniques. From a mathematical point of view, noncommutativity in string theory is interesting because of its connection to noncommutative geometries and noncommutative algebras. Many open questions remain. One is what is the generalization of the Moyal-Weyl product in the case of multiple noncommutativity parameters, as in the case of non-abelian gauge theories.
What is the generalization of the map in (3.7)? Partial answers have been attempted in [15, 16, 17], but much more needs to be understood. In fact, we might be just at the beginning of our understanding of the relation between strings and noncommutativity. We suspect that the inclusion of additional terms in the worldsheet action, such as mass terms or Ramond backgrounds [18], will lead to novel types of noncommutativities. Probably there is a sort of “noncommutative democracy” among the various pieces of the worldsheet action. This set of noncommutativity parameters, one for each of the additional terms in the worldsheet action, could be reflected in the interpretation of the open string field theory star product as a continuous tensor product of Moyal products [19].

Finally, the most interesting aspect is that spacetime noncommutativity discussed above offers an insight on the structure of spacetime, as it implies a spacetime uncertainty principle

\[ \Delta X^\mu \Delta X^\nu \geq \alpha'. \]

This relation suggests the existence of a spacetime cutoff at the Planck scale, and hence possible deviations from the smoothness of spacetime at small distances. It might well be that these aspects of Planck scale physics lead to distinctive signatures observable in cosmology. In particular, they could leave their mark on the spectrum of density fluctuations in the early universe, and modify the inflationary perturbation spectrum.

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