Angular distribution of thrust axis with power-suppressed contribution in $e^+e^-$ annihilation

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Abstract: Structure function of $e^+e^- \rightarrow \text{hadrons}$ cross section proportional to the longitudinal part of the hadron tensor is power suppressed with respect to an event shape variable in the two-jet region. In the SCET framework, we study the event shape distribution for this structure function to NLL level of accuracy. As a result we obtain the angular distribution of hadron jets as a function of the thrust, in the two jet region. We further examine effects of non-perturbative hadronization corrections by adopting a shape function that reproduce the observed event shape distributions. Impacts of our findings on the electroweak measurements via the jet angular forward-backward asymmetry are discussed.

Keywords: QCD, SCET, thrust, angular distribution, forward-backward asymmetry.
1. Introduction

An intriguing feature of the Standard Model (SM) is the unification of weak and electromagnetic interactions. One of the most important predictions of the SM is that the interactions of all electroweak gauge bosons are determined by the electromagnetic coupling constant $\alpha$ and one additional parameter—the weak mixing parameter $\sin^2 \theta_W$.

Study of the $Z$ boson pole at $e^+e^-$ colliders provides the most accurate determination of the electroweak interaction parameters. The average value of $\sin^2 \theta_W$ found in experiments at LEP is $0.23153 \pm 0.00016$ (for a review and bibliography the reader is referred to [1]). An experimental method to measure $\sin^2 \theta_W$ is based on the fact that weak isospins for left-handed and right-handed fermions are not the same. This difference of coupling constants leads to various angular and polarization asymmetries. The most accurate measurement of $\sin^2 \theta_W$ comes from the forward-backward asymmetry with flavor tagging of the final-state quark. The plane orthogonal to the colliding beam lines divides the space of all direction into two hemispheres. The electron beam is pointing to the “forward” hemisphere. The asymmetry is defined as the relative difference in the numbers of events with a reference direction in the “forward” or “backward” hemispheres. Usually the thrust axis supplemented with the direction and charge of a prompt lepton from charm or bottom meson semi-leptonic decay is adopted as the reference direction.

Extraction of $\sin^2 \theta_W$ with high precision requires taking into account a large number of processes accompanying a basic $e^+e^- \rightarrow Z \rightarrow q\bar{q}$ process. One of them is QCD interaction in the final state. In order to reduce QCD radiation of additional partons, one has to implement experimental cuts which effectively select events in the two-jet region [2]. The event-shape variable thrust ($T$) can be used to establish such cuts [3]. Although, from theoretical point of view the thrust distribution is known to unprecedented level of accuracy [4], up to now the study of the angular distribution of the thrust axis depending on thrust value has not been done beyond the first nontrivial order of perturbation theory [5]. It was noted in Ref.[5] that an additional contribution to the event shape, which changes the angular distribution, is power suppressed in the $T \rightarrow 1$ limit.

The reason why the thrust axis is so stable in the two-jet region is the following: by definition, the thrust axis coincides with the total momentum of final state hadrons in a certain hemisphere, thus the multiple branchings, leaving secondary hadrons in the same hemisphere, do not change the thrust axis. For the same reason, it is rather hard to estimate the influence of multiple hadron radiation on the angular distribution in the strong two-jet limit using the present Monte-Carlo event generators. In this paper, we consider in detail the angular distribution of the thrust axis depending on thrust value taken in the two-jet region.

The paper is organized as follows. In the next section, we consider possible mechanisms for QCD radiations to have an effect on the angular distribution in the two-jet region. In Sect.3, using the method of expanding by regions, we find the perturbative correction to the structure function, which corresponds to the so-called longitudinal part of the hadronic tensor. In sections 4 and 5, we use the SCET (Soft Collinear Effective Theory) framework to study event shape for this structure function, which appears due to a local three-body
operator in the effective theory. Comparison with the existing experimental data for the angular distribution and the forward-backward asymmetry is left to the last section.

2. Mechanisms to change the angular distribution

We restrict our attention to the case where $Z$-boson is the only intermediate state in the processes $e^+e^- \rightarrow \text{hadrons}$. The thrust ($T$) dependent cross section for the process $e^+e^- \rightarrow \text{hadrons}$ is a contraction of the leptonic tensor and the hadronic one:

$$
\int_{1-\tau}^1 dT \frac{d\sigma}{d \cos \theta_T} dT = \frac{\alpha^2 \pi N_c}{2} \frac{Q^2}{(Q^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} L_{\mu\nu}(e) H_{\mu\nu}(\tau, n_T), \quad (2.1)
$$

where $Q^2 = (p_e^- + p_e^+)^2$, $n_e = p_e^-/|p_e^-|$, and $n_T$ is the direction of the thrust axis in the hemisphere which contains a quark. If one neglects the electroweak radiative corrections and the lepton masses, the leptonic tensor reads

$$
L_{\mu\nu}(e) = (g_{al}^2 + g_{vl}^2) g_{\perp\mu\nu}(n_e) - 2ig_{al}g_{vl}a_{\mu\nu}(n_e), \quad (2.2)
$$

while the hadronic tensor can be parameterized as follows

$$
H^{\mu\nu} = (g_{qv}^2 + g_{qa}^2) \left\{ F(\tau) g_{\perp\mu\nu}(n_T) + 2G(\tau) g_{\parallel\mu\nu}(n_T) \right\} + 2ig_{qv}g_{aq}K(\tau) a_{\mu\nu}(n_T), \quad (2.3)
$$

where the coupling constants have the following form

$$
g_{vl} = g_{al} = -g_{uv} = -\frac{1}{2 \sin 2\theta_W}, \quad g_{vl} = g_{al} \left( 1 - 4 \sin^2 \theta_W \right),
$$

$$
g_{vu} = -g_{al} \left( 1 - \frac{8}{3} \sin^2 \theta_W \right), \quad g_{vd} = g_{al} \left( 1 - \frac{4}{3} \sin^2 \theta_W \right). \quad (2.4)
$$

The tensors in Eqs. (2.2) and (2.3) are defined as follows:

$$
g_{\perp\mu\nu}(u) = -g^{\mu\nu} + \frac{n^{\mu}(u) n^{\nu}(u) + n^{\nu}(u) n^{\mu}(u)}{2}, \quad (2.5a)
$$

$$
g_{\parallel\mu\nu}(u) = \frac{1}{4} \left[ n^{\mu}(u) - n^{\mu}(u) \right] \left[ n^{\nu}(u) - n^{\nu}(u) \right], \quad (2.5b)
$$

$$
a_{\mu\nu}(u) = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} n_{\alpha}(u) n_{\beta+}(u), \quad (2.5c)
$$

where

$$
u^2 = 1, \quad n = (1, -u), \quad n_+ = (1, u). \quad (2.6)
$$

The expression (2.3) is the most general parametrization of the tensor which satisfies

$$
(n + n_+)_{\mu} H^{\mu\nu} = (n + n_+)_{\nu} H^{\mu\nu} = 0. \quad (2.7)
$$

Since we neglect the electron mass and since the hadronic tensor must be contracted with the leptonic one, only those structure functions that satisfy the condition (2.7) contribute to the cross section even for massive quarks.
Let the Z-boson be polarized along the electron beam direction in the \(e^+e^-\) collision rest frame, i.e., it is produced in the polarized state \(|1, m\rangle\) with \(m = \pm 1\). We then obtain the following angular distribution:

\[
\int_{1-\tau}^{1} \frac{d\sigma}{dT d\cos \theta_T} \sim (g_{voq}^2 + g_{aq}^2) \left[ F(\tau) \left(1 + \cos^2 \theta_T\right) + 2G(\tau) \sin^2 \theta_T \right] - m 2g_{voq}g_{aq} K(\tau) 2\cos \theta_T, \quad (2.8)
\]

where \(\cos \theta_T = \mathbf{n}_T \cdot \mathbf{n}_e\).

Let us consider a quark-antiquark \((q\bar{q})\) pair in the final state in the massless quark limit. In the center-of-mass frame, this pair should be produced in the spherical helicity state \(d^1_{\lambda,m}(\mathbf{n}_q) = (1 + \lambda m \cos \theta_q)/2\), where \(\theta_q\) is the angle between \(\mathbf{n}_e\) and the quark momentum \(p_q\), and \(\lambda=\pm 1\) is the projection of the total spin of the pair on the direction of \(p_q\). Violation of \(P\)-parity by interaction with the virtual boson implies different coupling constants for left-handed and right-handed fermions:

\[
g_{voq} - g_{aq}\gamma_5 = (g_{voq} + g_{aq}) \frac{1}{2}(1 - \gamma_5) + (g_{voq} - g_{aq}) \frac{1}{2}(1 + \gamma_5). \quad (2.9)
\]

It results in the following angular distribution for the primary \(q\bar{q}\)-pair:

\[
\frac{d\sigma}{d\cos \theta_q} \sim (g_{voq} + g_{aq})^2 |d^1_{1,m}|^2 + (g_{voq} - g_{aq})^2 |d^1_{-1,m}|^2
\]

\[
= \frac{1}{2} \left[ (g_{voq}^2 + g_{aq}^2) \left(1 + \cos^2 \theta_q\right) - m 2g_{voq}g_{aq} 2\cos \theta_q \right]. \quad (2.10)
\]

Assuming \(\theta_q \approx \theta_T\) and comparing Eq.(2.10) with the expression (2.8), we find that the following relations hold in the \(\tau \to 0\) limit:

\[
F(\tau) = K(\tau), \quad (2.11)
\]

\[
G(\tau) = 0. \quad (2.12)
\]

These relations are the consequence of the free parton approximation and they have the same nature as the known Callan-Gross relation \([6, 7]\) or the large recoil symmetry relation for heavy-to-light form factors \([8]\).

Additional radiation of high energy partons results in violation of the relations (2.11) and (2.12). Let us consider radiation of a single gluon with energy \(E_g \sim Q\). There are

\[\text{Figure 1: Three possible directions of the thrust axis in } e^+e^- \to q\bar{q}g \text{ processes.}\]
three different possibilities for the thrust axis to lie along the momenta of the final state partons (see Fig. 1). The topology (1b), where the thrust axis aligns the gluon momentum, can in principle be excluded from the analysis. To do this, simultaneous tagging of both flavored mesons is required. If one tags only one meson then the contribution of the topology (1c) should also be taken into account. Due to CP-invariance, this topology does not contribute to \( K(\tau) \) but gives the main contribution to \( F(\tau) \) and thereby violates the relation (2.11). In contrast, the topologies (1a, b) contributes mainly to \( G(\tau) \) and hence lead to the violation of the relation (2.12). In the present paper, we mainly consider the topologies (1a, b) such that \( |\theta_q - \theta_{\bar{q}}| \approx \pi \) in the \( \tau \to 0 \) limit. We will give a few comments about the topology (1c) in Sect. 4.

In the covariant perturbation theory, the amplitudes are obtained as the sum of the two Feynman diagrams, Figs. 2(a) and 2(b). For the sake of completeness, we present here the leading perturbative results for \( F(\tau) - K(\tau) \) and \( G(\tau) \):

\[
F_{\text{tree}}(\tau) - K_{\text{tree}}(\tau) = \frac{\alpha_s}{4\pi} C_F \left\{ \frac{2\pi^2}{3} + \frac{\tau (12\tau^2 + 17\tau - 45)}{\tau - 1} \right. \\
+ \left. \left( \frac{5}{2} - 8 \ln 2 - 4\tau - 2\tau^2 \right) \ln(1 - 2\tau) \right. \\
\left. + 2\tau(\tau + 2) \ln \tau + 8\ln (1 - \tau) \left[ \ln \tau - \ln(1 - 2\tau) + 6 \right] + 8 \left[ \text{Li}_2(\tau) - \text{Li}_2(2\tau - 1) \right] \right\}, \quad (2.13)
\]

\[
G_{\text{tree}}(\tau) = \frac{\alpha_s}{\pi} C_F \left\{ \tau - 4 \left[ \frac{\tau(2 - \tau)}{1 - \tau} + 2 \ln(1 - \tau) \right] \right\}, \quad (2.14)
\]

so that the power expansion near \( \tau = 0 \) takes the form:

\[
F_{\text{tree}}(\tau) - K_{\text{tree}}(\tau) = \frac{\alpha_s}{\pi} C_F \left[ \tau \ln \frac{1}{\tau} + O(\tau^2) \right], \quad (2.15)
\]

\[
G_{\text{tree}}(\tau) = G^{(0)}(\tau) + O(\tau^2) = \frac{\alpha_s}{\pi} C_F \left[ \tau + O(\tau^2) \right]. \quad (2.16)
\]

Below we give a qualitative explanation of the physical mechanisms which results in violation of the relations (2.11) and (2.12). Let us consider the the topology (1a), where the gluon is emitted along the quark momentum direction. In order to remind us of the kinematical configuration concerned, we draw the diagrams in Figs. 2(a) and 2(b) in such a way that the \( q, \bar{q}, g \) lines follow their momentum directions.

We find the old-fashioned perturbative theory (oPT) useful to identify two mechanisms that radiation affects the angular distribution. Using oPT, one can decompose each

\[\text{Figure 2: Feynman diagrams for the single gluon radiation process in hadronic Z-boson decays.}\]
covariant diagram (Fig. 2(a) or Fig. 2(b)) into the sum of two time-ordered diagrams shown in Fig. 3 and Fig. 4.

The first mechanism corresponds to the time ordering when additional partons are radiated off the primary $q\bar{q}$-pair after the Z-boson decay, as depicted in Fig. 3. The intermediate state involved is real but with the different energy $E_1$, and hence the angular distribution (2.10) is valid for the relative momentum of the primary $q\bar{q}$-pair. The only possibility for the subsequent radiation to change the thrust axis distribution is the following: after the splitting of a parton, one of the “child”-parton is radiated into the “wrong” thrust hemisphere, which is opposite to that of its “parent”-parton. For example, the $oPT$-decomposition of the diagram (2b) has such intermediate and final states. The main effect of such radiation is that the direction of the relative momentum of the real primary $q\bar{q}$-pair can differ from the thrust axis.

The second mechanism is when virtual states appears before the Z-boson decay, picking up the Z-boson from the state $|1, -1\rangle$, as depicted in Fig. 4.

Let us estimate the contribution of the first mechanism, assuming that $\theta_q - \theta_T$ misfit is of order $\tau^{1/2}$. For the sake of brevity, we assume that the Z-boson is produced in the state $|1, -1\rangle$. As one can see in Fig. 3(b), the primary $q\bar{q}$-pair is produced in states $d^1_{\pm 1, -1} (n_q)$, where $n_q = p_2/|p_2|$, while the thrust axis $n_T$ is along the momentum of the antiquark $p_1$, i.e. $n_T = -p_1/|p_1|$. To derive the distribution with respect to $\theta_T$, it is necessary to express $d^1_{\pm 1, -1} (n_q)$ through $d^1_{\lambda, -1} (n_T)$, that leads to the admixture of $d^1_{\lambda, -1} (n_T)$ state and therefore $G(\tau) = \pi/2$ as well as $F (\tau) = K (\tau)$ follow:

$$d\sigma \sim (g_{vq} + g_{aq})^2 |d^1_{-1, -1} (n_q)|^2 + (g_{vq} - g_{aq})^2 |d^1_{1, -1} (n_q)|^2$$

$$= \sum_{n,l=-1}^1 T_{nl} d^1_{n,-1} (n_T) d^1_{l,-1} (n_T), \quad (2.17)$$

with

$$T_{nl} = (g_{vq} + g_{aq})^2 D^{1}_{-1,n} D^{1}_{1,l} + (g_{vq} - g_{aq})^2 D^{1}_{1,n} D^{1}_{1,l}, \quad (2.18)$$

where $D^1_{\lambda,m}$ is the operator of finite rotations:

$$D^1_{\lambda,m} \equiv D^1_{\lambda,m}(\alpha, \theta_q, \gamma). \quad (2.19)$$

Here, $\theta_q$ is the angle between the vectors $n_T$ and $n_q$, $\alpha$ is the angle of rotation about $n_T$-axis transferring the vector $n_e \times n_T$ into $n_T \times n_q$, $\gamma$ is the angle of rotation about

![Figure 3](image1.png) \hspace{1cm} ![Figure 4](image2.png)

**Figure 3:** Two particle intermediate state in $oPT$.

**Figure 4:** Four particle intermediate state in $oPT$. 

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and a spinor $u$ projectors. To ascertain their geometrical meaning, we consider the following light-like 4-momenta relations:

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} T_{nl} = \left( g_{aq}^2 + g_{vq}^2 \right) \times \text{diag}\left( \frac{1 + \cos^2 \theta_{qT}}{2} + \frac{2g_{aq}g_{vq}}{g_{aq}^2 + g_{vq}^2} \cos \theta_{qT}, \sin^2 \theta_{qT}, \frac{1 + \cos^2 \theta_{qT}}{2} - \frac{2g_{aq}g_{vq}}{g_{aq}^2 + g_{vq}^2} \cos \theta_{qT} \right),$$

(2.20)

and the angular distribution of the thrust axis becomes

$$d\sigma \sim \frac{1}{2} \left[ \frac{1 + \cos^2 \theta_{qT}}{2} \left( g_{aq}^2 + g_{vq}^2 \right) \left( 1 + \cos^2 \theta_T \right) + \cos \theta_{qT} \left( 2g_{aq}g_{vq} \right) \left( 2 \cos \theta_T \right) \right]$$

$$+ \sin^2 \theta_{qT} \frac{1}{2} \left( g_{aq}^2 + g_{vq}^2 \right) \sin^2 \theta_T. \quad (2.21)$$

The coefficients before the distributions $1 + \cos^2 \theta_T$ and $2 \cos \theta_T$ are now different, but this difference is strongly suppressed in the collinear limit $\theta_{qT} \to 0$:

$$\left. \left. \frac{1 + \cos^2 \theta_{qT}}{2} \right|_{\theta_{qT} \to 0} = 1 + \frac{\theta_{qT}^4}{8} + \mathcal{O} \left( \theta_{qT}^6 \right). \quad (2.22)$$

Therefore, the corrections $2\cos \theta_T$ integrated over the region $\theta_{qT}^2 \sim \tau$ generate the following corrections to the relations $\left(2.11 \right), \left(2.12 \right)$:

$$F \left( \tau \right) - K \left( \tau \right) \sim \tau^3, \quad G \sim \tau^2. \quad (2.23)$$

To estimate the effect of virtual states depicted in Fig.\,4, let us consider the following projectors

$$\hat{P}_{(\pm)} \left( u \right) = \frac{\eta_+ \left( u \right) \eta_+ \left( u \right)}{4}, \quad \hat{P}_{(-)} \left( u \right) = \frac{\eta_+ \left( u \right) \eta_- \left( u \right)}{4},$$

(2.24)

where 4-vectors $n$ and $n_+$ are defined in Eq.(2.4). They satisfy the following projective relations:

$$\hat{P}^2_{(\pm)} = \hat{P}_{(\pm)}, \quad \hat{P}_{(\pm)}\hat{P}_{(\mp)} = 0, \quad \hat{P}_{(+)} + \hat{P}_{(-)} = 1. \quad (2.25)$$

To ascertain their geometrical meaning, we consider the following light-like 4-momenta

$$q = E \left( 1, v \right), \quad q' = E \left( 1, n \right),$$

(2.26)

and a spinor $u^{(\lambda)}_q$ such that $\eta \eta^{\lambda} = 0$ for a definite helicity $\lambda$. One can show that the wave functions $\hat{P}_{(\pm)} \left( n \right) u^{(\lambda)}_q$ correspond to the two states with the projection of the angular momentum $\pm \lambda$ on the axis $n$, but with the same helicity:

$$u^{(\lambda)}_q = \hat{P}_{(\pm)} \left( n \right) u^{(\lambda)}_q + \hat{P}_{(-)} \left( n \right) u^{(\lambda)}_q = d^{1/2}_{\lambda, \lambda} \left( \pi - \theta_{nv} \right) u^{(\lambda)}_{q'} + d^{1/2}_{\lambda, \lambda} \left( \theta_{nv} \right) u^{(\lambda)}_{q'}. \quad (2.27)$$

\textsuperscript{1}Here, we do not distinguish between $\tau^\alpha$ and $\tau^\alpha \ln \tau$, where the logarithmic part can appear due to integration over the gluon energy.
where \( \cos \theta_{nv} = \mathbf{n} \cdot \mathbf{v} \). Now let us consider the fermion propagator inside the Feynman diagram (2a). The total momentum of the quark and the gluon is

\[
p^\mu = p_2^\mu + p_3^\mu = (p \cdot n) \frac{n_+^\mu}{2} + (p \cdot n_+) \frac{n_-^\mu}{2}.
\]

(2.28)

Here and below we omit the arguments of \( n^\mu \) and \( n_+^\mu \), implying \( n^\mu (\mathbf{n}_T) \) and \( n_+^\mu (\mathbf{n}_T) \). Using the projectors \((2.24)\), one can split the propagator into two parts

\[
\frac{\hat{p}}{p^2 + i0} = \hat{P}_(-) \frac{\hat{p}}{p^2 + i0} \hat{P}_+ \hat{P}_(-) + \hat{P}_+ \frac{\hat{p}}{p^2 + i0} \hat{P}_-,
\]

(2.29)

which correspond to retarded and advanced propagations:

\[
\hat{P}_(-) \frac{\hat{p}}{p^2 + i0} \hat{P}_+ = \frac{p \cdot n}{p^2 + i0} \frac{\eta_+}{2} = \frac{1}{E_0 - E_1 + i0} \frac{\eta_+}{2},
\]

(2.30a)

\[
\hat{P}_+ \frac{\hat{p}}{p^2 + i0} \hat{P}_(-) = \frac{p \cdot n_+}{p^2 + i0} \frac{\eta_-}{2} = \frac{1}{E_0 - E_2 - i0} \frac{\eta_-}{2},
\]

(2.30b)

where the following simple relations were used:

\[
p^2 = (p \cdot n_+) (p \cdot n),
\]

(2.31)

\[
p \cdot n_+ = p^0 - |\mathbf{p}| = Q - (|\mathbf{p}_1| + |\mathbf{p}|) = E_0 - E_1,
\]

\[
-p \cdot n = -p^0 - |\mathbf{p}| = Q - (Q + |\mathbf{p}| + |\mathbf{p}_2| + |\mathbf{p}_3|) = E_0 - E_2.
\]

(2.32)

Here \( E_0 = Q \) is the energy of the initial state, \( E_1 = |\mathbf{p}_1| + |\mathbf{p}| \) and \( E_2 = Q + |\mathbf{p}| + |\mathbf{p}_2| + |\mathbf{p}_3| \) are the energies of the intermediate states depicted in Figs. (3) and (4) respectively. Taking into account the decomposition \((2.27)\), one can see that the term \((2.30a)\) corresponds to the quark propagating in the direction \( \mathbf{n}_T \) while the term \((2.30b)\) corresponds to the antiquark propagating in the direction \(-\mathbf{n}_T\). Therefore, we can conclude that in the first diagram (Fig 3) the Z-boson decays into the quark-antiquark pair with opposite helicities, i.e., from the \( d_{1_1,-1} \) state. In the second diagram (Fig.4), the Z-boson disappears being absorbed by the intermediate antiquark. Taking into account helicity conservation for massless quarks, we conclude that it is possible only from the state \( d_{1,-1} \). According to the energy conservation for the final state, i.e., \( |\mathbf{p}_1| + |\mathbf{p}_2| + |\mathbf{p}_3| = Q \), we find that \( E_0 - E_2 = -Q \), i.e., the propagator \((2.30b)\) does not depend on the kinematic configuration of the final state. Since the phase space for a definite thrust value is of order \( \tau \), the intermediate state shown in Fig. 3 leads to a contribution

\[
G(\tau) \sim \tau.
\]

(2.33)

This explains the tree-level result \((2.10)\).

The analysis presented above remains valid in the case of a multiparticle final state with primary \( q \) and \( q \) radiated into the opposite hemispheres. A QCD cascade starting from the quark-antiquark state as depicted in Fig. 3 cannot change the relations \((2.11)\) and \((2.12)\) significantly in the two jet region, where the corrections are suppressed by \( \tau^3 \) and \( \tau^2 \) respectively. In the small \( \tau \) region, the leading corrections to the relation \((2.12)\) come
from the process at short distances shown in Fig. 4, which gives rise to the contribution of order $\tau^{2.33}$. The proper consideration of violation of the relations (2.11) and (2.12) requires a consideration of a new type of jets. As it will be demonstrated below, those jets are initiated by two collinear partons produced at short distances, i.e., due to a local operator with more than two fields, so that all secondary collinear or soft particles are radiated coherently by these partons.

### 3. Perturbative corrections

If one has an integral over a scaleless domain whose integrand depends on a small external parameter $\lambda \ll 1$, then the method of expanding by regions gives an asymptotic expansion with respect to $\lambda$. The series may contain arbitrary non-integer powers of $\lambda$ as well as integer powers of $\ln \lambda$. Here, we outline the prescription used below without discussing details of the method, for which the reader is referred to the original studies [9, 10, 11].

The method utilizes the fact that the expansion of an integrand with respect to $\lambda$ may be invalid in a certain region where the integrand becomes singular. Analyzing the integrand singularities, one can establish the so-called power counting rules, according to which one can pick up a simplified singular behavior of the integrand by expanding it not only in the external parameters but also in integration variables. Using this method, one can represent the integral as a sum of integrals, such that each integrand is an expansion of the original one with respect to the power counting rules.

Let us demonstrate how to apply this method to calculate the leading perturbative correction to the structure function $F(\tau)$ in a region where $1 - T < \tau \ll 1$:

$$F(\tau) = \frac{4\pi}{(D - 2)N_c Q^2} \int d\rho_X \Theta \left( \sum_{h \in X} |p_h \cdot n_T| - (1 - \tau)Q \right) \times \sum_{\sigma, c} g_{\mu\nu}(n_T) \langle X | \hat{J}_\mu | 0 \rangle \star \langle X | \hat{J}_\nu | 0 \rangle,$$  

(3.1)

where $\hat{J}_\mu = \hat{\psi}_q \gamma_\mu \hat{\psi}_\bar{q}$ and $d\rho_X$ is the phase space of a final state $|X\rangle$:

$$d\rho_X = (2\pi)^D \delta^D \left( Q \frac{n + n_+}{2} - \sum_{h \in X} p_h \right) \prod_{h \in X} \frac{d^D p_h}{(2\pi)^D} \delta \left( p_h^2 \right) \Theta \left( p_h \cdot n + p_h \cdot n_+ \right)$$  

(3.3)

and $\sum_{\sigma, c}$ denotes the sum over spin and color states and $D = 4 - 2\epsilon$ is the space-time dimensionality.

First of all, we introduce a small parameter $\lambda$ such that $\lambda^2 \sim \tau$. We use the Sudakov decomposition to represent all real or virtual particle momenta:

$$p_h = (p_h \cdot n) \frac{n_+}{2} + (p_h \cdot n_+) \frac{n}{2} + p_{h\perp}.$$  

(3.4)
Power counting rules should estimate the components \((p_h \cdot n, p_h \cdot n_\perp, p_h \cdot n_+)\) in comparison with \(\lambda\). The regions of integrations and the corresponding power counting rules are presented in Table 1. There are two regions, namely, hard and soft ones, where all momentum components are of the same order. In the hard region one should expand an integrand with respect to \(\tau\) only and integrate over real or virtual particle momenta in dimensional regularization, ignoring any soft or collinear infrared singularities. The expansion in the soft region corresponds to an integrand approximation near the soft singularities. There are also \(r\)- and \(l\)-collinear regions, where symmetry among the space directions is strongly broken. These regions account for the collinear singularities.

The contribution of each region is gauge invariant by itself. In Feynman gauge, virtual corrections to the amplitude \(\langle \bar{q} q | \hat{J}_\mu | 0 \rangle\) contribute in the hard region only, and the corresponding contribution to \(F(\tau)\) has the form:

\[
F^{\text{hard}}_1(\tau) = C_F \left( \frac{Q^2}{\mu^2} \right)^{-\epsilon} \left( -\frac{4}{\epsilon^2} - \frac{6}{\epsilon} - \frac{7\pi^2}{3} + O(\epsilon) \right). \tag{3.5}
\]

In contrast to the virtual corrections, the real emissions give contributions in the collinear and soft regions:

\[
F^{l-\text{col}}_1(\tau) = F^{r-\text{col}}_1(\tau) = C_F \left( \frac{\tau Q^2}{\mu^2} \right)^{-\epsilon} \left( \frac{4}{\epsilon^2} + \frac{3}{\epsilon} - \pi^2 + 7 + O(\epsilon) \right) + O(\lambda^2), \tag{3.6}
\]

\[
F^{\text{soft}}_1(\tau) = C_F \left( \frac{\tau^2 Q^2}{\mu^2} \right)^{-\epsilon} \left( \frac{4}{\epsilon^2} + \frac{\pi^2}{3} + O(\epsilon) \right) + O(\lambda^2). \tag{3.7}
\]

The sum of all contributions is a well known result \cite{12}:

\[
F(\tau) = 1 + \frac{\alpha_s}{4\pi} C_F \left( -4 \ln^2 \frac{1}{\tau} + 6 \ln \frac{1}{\tau} - 2 + \frac{2\pi^2}{3} \right) + O(\alpha_s \tau) + O(\alpha_s^2). \tag{3.8}
\]

The singularities with respect to \(\epsilon\) and the \(\mu^2\)-dependence drop out of the sum of all the contributions \((3.5), (3.6)\) and \((3.7)\), which tests our use of the method of expanding by regions.

Let us apply the same method to calculate the perturbative result for \(G(\tau)\) function in the region \(1 - T < \tau \ll 1:\)

\[
G(\tau) = \frac{2\pi}{N_c Q^2} \int d\rho_x \sum_{\sigma, c} \left\langle X \left| \hat{J}_\| \right| 0 \right\rangle^* \left\langle X \left| \hat{J}_\| \right| 0 \right\rangle \Theta \left( \sum_{h \in X} |p_h \cdot n_T| - (1 - \tau)Q \right), \tag{3.9}
\]
where
\[
\hat{J}_\parallel = \hat{\psi}_q \left( \frac{\not{p} - \not{q}}{2} \right) \hat{\psi}_q.
\]

(3.10)

First, we consider the case when the thrust axis is aligned with the antiquark momentum \( p_1 = E_1 n \). The diagrams we would like to consider are depicted in Fig. 2. The momentum \( p_1 \) is \( l \)-collinear, the momenta \( p_2 \) and \( p_3 \) are \( r \)-collinear in accordance with the classification of Table 1. The corresponding amplitudes for the diagrams Figs. (2a) and (2b), which contribute to \( G(\tau) \), have the following form:

\[
\bar{u}(p_2) \tilde{V}^{\mu,a}_{(a)} \gamma^\mu p_2^\mu \left( \frac{\not{p}_2 - \not{q}}{2} \right) v(p_1). \\
\bar{u}(p_2) \tilde{V}^{\mu,a}_{(b)} \gamma^\mu p_2^\mu \left( \frac{\not{p}_2 - \not{q}}{2} \right) v(p_1). \\
(3.11)
\]

respectively. The sum of the amplitudes (3.11) has the following \( \lambda \)-expansion:

\[
\bar{u}(p_2) \left[ \tilde{V}^{\mu,a}_{(a)} + \tilde{V}^{\mu,a}_{(b)} \right] v(p_1) = \bar{u}(p_2) \left( 2g_{s}t^{a} \gamma^{\mu} \frac{p_2^{\mu} + p_3^{\mu}}{p_3 \cdot n + i0} \right) v(p_1) \left[ 1 + \frac{(p_2 + p_3)^2}{Q^2} \right] + O(\lambda^3).
\]

(3.12)

Using the light-cone gauge, such that the gluon propagator has the form:

\[
\int d^{D}x \ e^{-ip_{3} \cdot x} \langle 0 | T A^{\mu}_{a}(x) A^{\nu}_{b}(0) | 0 \rangle = i \frac{\delta_{ab}}{p_{3}^{2} + i0} \left( \gamma^{\mu} n^{\nu} + p_{3}^{\nu} n^{\mu} \right),
\]

(3.13)

and neglecting the power suppressed term \( (p_2 + p_3)^2/Q^2 \sim \lambda^2 \), we find that the following effective vertex

\[
\tilde{V}^{\mu,a}_{\text{eff}} = \tilde{V}^{\mu,a}_{(a)} + \tilde{V}^{\mu,a}_{(b)} = 2g_{s}t^{a} \left( \frac{\not{p}_2 + \not{p}_3}{4} \right) \gamma^{\mu} \left( \frac{\not{p}_2 + \not{p}_3}{4} \right) ;
\]

(3.14)

gives the leading contribution to \( G(\tau) \). The \( r \)- and \( l \)-collinear regions give equal contributions, and hence we find:

\[
G^{(0)}(\tau) = \frac{16\pi g_{s}^{2} C_{F}}{Q^{4}} \int \sum_{\sigma} \left| \bar{u}(p_2) \left( \frac{\not{p}_2 + \not{p}_3}{4} \right) \gamma^{\mu} \left( \frac{\not{p}_2 + \not{p}_3}{4} \right) v(p_1) \right|^{2} \Theta(\tau Q^{2} - p_{R}^{2}) \ d\rho_{3} \\
= \frac{\alpha_{s} C_{F}}{\pi} \tau + O(\tau^2),
\]

(3.15)

where \( p_{R}^{2} = (p_2 + p_3)^2 \), \( \epsilon(p_3) \) is the gluon polarization 4-vector \( \epsilon(p_3) \cdot p_3 = \epsilon(p_3) \cdot n = 0 \), and \( d\rho_{3} \) is the element of the three particle phase space (3.3).

The expression (3.14) remains valid in the light-cone gauge (3.13) even when the particles are off shell \( (p_1^2 \sim p_2^2 \sim p_3^2 \sim \lambda^2 Q^2) \), i.e., it appears as the internal part of the expansion of amplitudes with more than three collinear particles in the final state\(^2\).

The perturbative correction to \( G^{(0)}(\tau) \) has the form [13]:

\[
G(\tau) = G^{(0)}(\tau) \left[ 1 + \frac{\alpha_{s}}{4\pi} G^{(1)}(\tau) \right],
\]

(3.16)

\(^{2}\)See Ref. [13] for details.
where $G^{(1)}(\tau)$ contains hard, $r$-collinear, $l$-collinear and soft contributions:

$$
G^{(1)}(\tau) = -\frac{\beta_0}{\epsilon} + G^{(1)}_{\text{hard}}(Q^2, \mu^2) + G^{(1)}_{l-\text{col}}(\tau Q^2, \mu^2) + G^{(1)}_{r-\text{col}}(\tau Q^2, \mu^2) + G^{(1)}_{\text{soft}}(\tau^2 Q^2, \mu^2). 
$$

(3.17)

where the term $-\beta_0/\epsilon$ appears after $\alpha_s$ renormalization in the vertex (3.14). The hard contribution consists only of virtual corrections to the amplitudes (Fig.3), where the loop momentum is hard (Table 1):

$$
G^{(1)}_{\text{hard}}(Q^2, \mu^2) = \left(\frac{Q^2}{\mu^2}\right)^{-\epsilon} \left\{ -\frac{4C_F}{\epsilon^2} + \frac{1}{\epsilon} \left[ \left(\frac{2\pi^2}{3} - 4\right) C_A + \left(6 - \frac{4\pi^2}{3}\right) C_F \right] \right. 
$$

$$
\left. + C_A \left( -16 + \frac{2\pi^2}{3} + 16\zeta(3) \right) + C_F \left( 34 + \pi^2 - 32\zeta(3) \right) + O(\epsilon) \right\}. 
$$

(3.18)

As we shall see later, the hard contribution is by construction the matching coefficient of the weak current to the effective three-body operator, whose leading matrix element contains the vertex (3.14).

In contrast to Eq.(3.17), $G^{(1)}_{l-\text{col}}$ and $G^{(1)}_{r-\text{col}}$ contributions are different, because the leading amplitude contains two $r$-collinear particles and only one $l$-collinear particle. Additional $l$-collinear particle generates the same correction as Eq.(3.6) for $F(\tau)$, except for an additional integration over the invariant mass $p_R^2$ of the $r$-collinear particles:

$$
G^{(1)}_{l-\text{col}}(\tau Q^2, \mu^2) = \frac{4\pi}{\alpha_s \tau} \int_0^{\tau Q^2} F^{l-\text{col}} \left( \tau - \frac{p_R^2}{Q^2} \right) \frac{dp_R^2}{Q^2} 
$$

$$
= \left( \frac{\tau Q^2}{\mu^2} \right)^{-\epsilon} C_F \left\{ \frac{4}{\epsilon^2} + \frac{7}{\epsilon} + 14 - \pi^2 + O(\epsilon) \right\}. 
$$

(3.19)

The $r$-collinear contribution includes virtual corrections, so that the loop momentum is implied to be $r$-collinear (Table 1), as well as real radiation of three $r$-collinear and one $l$-collinear particles:

$$
G^{(1)}_{r-\text{col}}(\tau Q^2, \mu^2) = \left(\frac{\tau Q^2}{\mu^2}\right)^{-\epsilon} \left\{ \frac{4C_F}{\epsilon^2} + \frac{1}{\epsilon} \left[ \left(\frac{23}{3} - \frac{2\pi^2}{3}\right) C_A + \left(\frac{4\pi^2}{3} - 5\right) C_F - \frac{4N_t T_F}{3} \right] \right. 
$$

$$
\left. + C_A \left( \frac{641}{18} - \frac{2\pi^2}{3} - 22\zeta(3) \right) + C_F \left( \frac{95}{2} - \frac{\pi^2}{3} + 44\zeta(3) \right) - \frac{50}{9} N_t T_F + O(\epsilon) \right\}. 
$$

(3.20)

The soft radiation also has the same form as Eq.(3.6) for $F(\tau)$, but with an additional integration over the invariant mass $p_R^2$ of the $r$-collinear particles:

$$
G^{(1)}_{\text{soft}}(\tau^2 Q^2, \mu^2) = \frac{4\pi}{\alpha_s \tau} \int_0^{\tau^2 Q^2} F^{\text{soft}} \left( \tau - \frac{p_R^2}{Q^2} \right) \frac{dp_R^2}{Q^2} 
$$

$$
= \left(\frac{\tau^2 Q^2}{\mu^2}\right)^{-\epsilon} C_F \left\{ -\frac{4}{\epsilon^2} - \frac{8}{\epsilon} + \frac{\pi^2}{3} - 16 \right\}. 
$$

(3.21)
The singularities with respect to $\epsilon$ and the $\mu^2$-dependence drop out of the sum of all the contributions:

$$G^{(1)}(\tau) = -4C_F \ln^2 \frac{1}{\tau} + \ln \frac{1}{\tau} \left[ C_F \left( \frac{4\pi^2}{3} - 14 \right) + C_A \left( \frac{23}{3} - \frac{2\pi^2}{3} \right) - \frac{4}{3} T_F N_f \right]$$

$$+ C_F \left[ -\frac{31}{2} + 12\zeta(3) \right] + C_A \left[ \frac{353}{18} - 6\zeta(3) \right] - \frac{50}{9} T_F N_f. \quad (3.22)$$

It is interesting to compare this correction with that of the leading thrust distribution (3.8). The double logarithmic corrections are equal. It has a simple physical interpretation: resummation of double logarithms results in a statistical factor responsible for excluding some part of radiation which is out of the two-jet region, and hence this factor is insensitive to the jet structure. However, the single logarithms in Eq.(3.8) and Eq.(3.22) are different. In the case of $F(\tau)$, it gives a positive contribution because $6C_F \ln \tau^{-1} > 0$, but the correction to the $G(\tau)$ distribution (3.22) contains the logarithm $\ln \tau^{-1}$ with a negative coefficient:

$$C_F \left( \frac{4\pi^2}{3} - 14 \right) + C_A \left( \frac{23}{3} - \frac{2\pi^2}{3} \right) - \frac{4}{3} T_F N_f = 1 - \frac{2\pi^2}{9} < 0. \quad (3.23)$$

The nonlogarithmic correction to $G^{(1)}(\tau)$ is positive and larger than $C_F (2\pi^2/3 - 2)$ in Eq.(3.8). We will find in section 5 that these differences affect the event shape distributions $F(\tau)$ and $G(\tau)$ after resummation of large logarithmic corrections.

### 4. Factorization formulae

In this section we give the heuristic derivation of the factorization formula for $G(\tau)$, which allows us to sum all big logarithmic corrections to the NLL level of accuracy.

We first note that power counting rules imply a hierarchy of the components of integration momenta and hence a hierarchy of the components of fields. Instead of an expansion of integrands obtained in a full QFT, it is sometimes possible to introduce an effective theory such that its Feynman rules reproduce the expanded fundamental amplitudes. Field modes corresponding to different regions are associated with different fields of various effective theories. Although, in perturbative calculations, the effective theory framework does not provide new information, it turns out to be extremely efficient when one needs to establish factorization formulae and evolution equations for resummation of large logarithms. A lively presentation of these ideas is provided e.g. in Ref.[14], where, starting from the minimal set of assumptions about hadronic final states, the authors use the soft collinear effective theory (SCET) [15, 16] to derive the factorization formula for the distributions of a large class of infrared safe observables (angularities).

In order to construct the SCET Lagrangian for collinear quarks interacting with collinear and soft gluons, one has to split the quark field $\psi$ using the projectors (2.24):

$$\psi = \xi_n + \eta_n, \quad \xi_n = \hat{P}_{(+)} \psi, \quad \eta_n = \hat{P}_{(-)} \psi, \quad (4.1)$$

and split the gluon field into the collinear and soft parts:

$$A = A_{c,n} + A_s. \quad (4.2)$$
The key assumption is that all the collinear field operators $\xi_n, \eta_n$ and $A_{c,n}, A_s$ generate field modes with momenta obeying the power counting rules listed in Table I. Here collinear quanta corresponding to $A_{c,n}$ are $l$-collinear and soft quanta $A_s$ have soft momenta. This assumption implies that we restrict our consideration to the processes which can be described by the QCD fields that are smooth in certain directions. The power counting rules of Table I lead to the following hierarchy of the components of fields:

$$\xi_n \sim \lambda, \quad \eta_n \sim \lambda^2, \quad A_s \sim \lambda^2$$

$$n_+ \cdot A_{c,n} \sim 1, \quad A_{c,n \perp} \sim \lambda, \quad n \cdot A_{c,n} \sim \lambda^2,$$  (4.3)

where the Sudakov decomposition of the collinear field $A_{c,n}$ is used:

$$A_{c,n}^\mu = (n_+ \cdot A_{c,n}) \frac{n_\mu}{2} + (n \cdot A_{c,n}) \frac{n_\mu}{2} + A_{c,n \perp}.$$  (4.4)

Using the estimations (4.3), one can expand the QCD Lagrangian to all orders in $\lambda$. This expansion truncated to some order in such a way to preserve the invariance under the homogeneous gauge transformations corresponds to the SCET Lagrangian \([15, 16]\).

Here, we outline the result of Ref.\([14]\) for the thrust distribution:

$$F(\tau) = H(Q^2, \mu^2) \int dp_L^2 dp_R^2 dk J(p_L^2, \mu^2) J(p_R^2, \mu^2) S_T(k, \mu^2) \Theta(Q^2 \tau - p_L^2 - p_R^2 - Q k).$$  (4.5)

$H(Q^2, \mu^2)$ is the hard function or the square of the usual on-shell QCD Sudakov form factor. $J(p^2, \mu^2)$ is the jet function:

$$J(p^2, \mu^2) = \frac{1}{(p \cdot n_+)} N_c \frac{1}{2\pi} \Im \left[ i \int d^4 x e^{-ipx} \left\langle 0 \left| T \left\{ \xi_n'(x) W_n(x) \frac{p^+}{2} W_n^\dagger(0) \xi_n'(0) \right\} \right| 0 \right\rangle \right],$$  (4.6)

that is, up to an overall factor, the imaginary part of the QCD quark propagator in the light-cone gauge ($\xi_n'$ and and Wilson line $W_n$ are defined below in Eq.\((4.11)\)). The soft factor $S_T(k, \mu^2)$ is defined as follows:

$$S_T(k, \mu^2) = \sum_X \left| \left\langle X \right| Y_{n_+\perp}^\dagger Y_n \left| 0 \right\rangle \right|^2 \delta(k - n \cdot p_{X_L} - n_+ \cdot p_{X_R}).$$  (4.7)

$W$ and $Y$ are collinear and soft Wilson lines, respectively:

$$W_n(x) = P \exp \left[ ig_s \int_{-\infty}^0 ds n_+ \cdot A'_c(x + sn) \right],$$  (4.8)

$$Y_n(x) = P \exp \left[ ig_s \int_{-\infty}^0 ds n \cdot A'_s(x + sn) \right].$$  (4.9)

In the effective theory framework the vertex \((3.14)\) corresponds to the following operator:

$$O_3 = O_3R + O_3L, \quad O_3R = 2g_s^2 \xi_{n_+} A_{n_+,n_+} \xi_n, \quad O_3L = 2g_s \xi_{n_+} A_{n_+,n_+} \xi_n.$$  (4.10)
where $\xi_{n_i}$ and $A_{\perp,n_i}$ are fields from different SCET copies in the light-cone gauges of the type of Eq.\,(3.13) with the light-like vector $n_i$. In order to introduce the gauge invariant operators, one should replace the fields entering the operators \((4.10)\) to $\xi'$ and $A'_{\perp}$ in an arbitrary gauge, using the following relations:

$$
\xi = Y W^\dagger \xi', \quad g_s A_{c\perp} = Y \left( W^\dagger iD'_{c\perp} W - i\partial_{\perp} \right) Y^\dagger,
$$

\begin{equation}
(4.11)
\end{equation}

where fields without primes are in the corresponding light-cone gauge. Including a soft Wilson line $Y$ in the definitions \((4.11)\) allows one to decouple soft and collinear degrees of freedom in the leading order SCET Lagrangian \([13, 17]\). Using the expressions \((4.11)\) yields the following operators

$$
\mathcal{O}_3 = 2g_s \xi'_i \tilde{A}_{\perp,n_i} W_{n_i} Y^\dagger_{n_i} Y_n W^\dagger_{n} \xi'_n + 2g_s \xi'_i \tilde{W}_{n_i} Y_{n_i} W_n \tilde{A}_{\perp,n_i} \xi'_n,
$$

\begin{equation}
(4.12)
\end{equation}

where

$$
\tilde{A}_{\perp,n_i} = A'_{\perp,n_i} - \frac{i}{g_s} W_{n_i} \left[ \partial_{\perp}, W^\dagger_{n_i} \right].
$$

\begin{equation}
(4.13)
\end{equation}

The operator \((4.12)\) is in fact the operator $\mathcal{O}_3$ derived in Ref.\,[13] taken in the limit: $n_q \rightarrow n_+$, $n_q \rightarrow n$ and $n_g \rightarrow n$ or $n_g \rightarrow n_+$. Integration over hard modes gives the matching coefficient $C_H$ of the QCD operator $\hat{J}_\parallel$ \((3.10)\) onto the SCET operator $\mathcal{O}_3$ \((4.12)\):

$$
\hat{J}_\parallel \rightarrow C_H \left( Q^2, \mu^2 \right) \mathcal{O}_3.
$$

\begin{equation}
(4.14)
\end{equation}

The important point about the operator \((4.12)\) is that it is a local product of the $r$-, $l$-collinear and soft SCET operators. According to Ref.\,[14], this feature is the only requirement to establish a factorization formula for an angularity distribution. For the operator \((4.12)\), the thrust distribution takes the form:

$$
G (\tau) = 2 H_3 \left( Q^2, \mu^2 \right) \int dp_{R}^2 dp_{\perp}^2 dk \Sigma_\perp \left( p_R^2, \mu^2 \right) J \left( p_L^2, \mu^2 \right) \times S_T \left( k, \mu^2 \right) \Theta \left( Q^2 \tau - p_L^2 - p_R^2 - Qk \right),
$$

\begin{equation}
(4.15)
\end{equation}

where $S_T (k, \mu^2)$ is the same soft factor as defined in \((4.7)\), $J \left( p_L^2, \mu^2 \right)$ is the jet function defined in \((4.8)\), and $H_3 = |C_H|^2$. The new object in the formula \((4.15)\) is $\Sigma_\perp \left( p_R^2, \mu^2 \right)$:

$$
\Sigma_\perp \left( p^2, \mu^2 \right) = \frac{g_s^2}{(p \cdot n) Q^2 \Lambda^2 \pi} \frac{1}{N_c}
$$

\begin{equation}
\times \text{Im} \left[ i \int d^D x e^{-ipx} \left\langle 0 \left| T \left\{ \left( \xi_{n_i}^{\dagger} \tilde{A}_{\perp,n_i} W_{n_i} \right) \left( x \right) \frac{\gamma^\mu}{2} \left( W_{n_i}^{\dagger} \tilde{A}_{\perp,n_i} \xi_{n_i} \right) \left( 0 \right) \right\} \right| 0 \right\rangle \right],
\end{equation}

\begin{equation}
(4.16)
\end{equation}

which can be considered as the imaginary part of the quark “transverse” self energy projected onto $\eta$. In contrast to the jet function or the soft factor, whose leading expressions are $\delta$-functions, the tree level expression for $\Sigma_\perp \left( p_R^2, \mu^2 \right)$ is a smooth function:

$$
\Sigma_\perp \left( p_R^2, \mu^2 \right) = \frac{\alpha_s \left( \mu^2 \right) C_F}{4 \pi Q^2} \left( \frac{p_R^2}{4 \pi \mu^2} \right)^{D/2-2} \frac{2 \Gamma \left( D/2 \right)}{\Gamma \left( D-2 \right)}.
$$

\begin{equation}
(4.17)
\end{equation}
The most important property of the operator (4.12) is that it is a local product of the collinear fields. The reason is that dominant underlying process shown in Fig. 4 does not depend on the kinematics of the final state since the energy of the intermediate state \(E_2 = Q + |p| + |p_2| + |k|\) is equal to \(2Q\) due to energy conservation for the final state. It results in locality of the vertex (3.14) for the three-particle interactions. The essential consequence of the locality is that soft gluons are radiated off by just like the two-prong QCD antenna (4.7) for the leading contribution to the structure function \(F\); see Eq.(4.5).

An intuitive explanation of this fact is that soft gluons are coherently radiated off all collinear emitters and the factorized amplitude of the soft radiation does not depend on the fraction of the collinear momentum carried by a particular emitter. For example, using a noncovariant gauge, one can show that the amplitude of soft radiation off the \(r\)-collinear pair depicted in Fig. 2 is equivalent to the corresponding amplitude for a single quark:

\[
\begin{align*}
\hat{t}^a \hat{b} e^{(k) \cdot n_+} & \left( k \cdot n_+ \right) - i f^{abc} t^c e^{(k) \cdot n_+} = \hat{t}^a \hat{b} e^{(k) \cdot n_+} \left( k \cdot n_+ \right),
\end{align*}
\]

All those remarkable properties of the structure function \(G\) such as the locality of the corresponding effective operator and the universal soft factor can be hardly generalized to the difference \(F - K\), where the third topology (Fig. 1c) accounts for the leading contribution (2.15). In this case the soft radiation undoubtedly differs from that for the topologies (Fig. 1a, b) because one hemisphere contains the single high energy gluon only. Moreover, the part of the cross section contributing to \(F - K\) is singular in the region where the energy of the quark or antiquark tends to zero. This singularity, which is quite similar to that in the \(\gamma^* \gamma \rightarrow \pi^0\) process, leads to an additional \(\ln \tau\) in the difference \(F - K\) in Eq.(2.15). It indicates that the effective operator for \(F - K\) should be nonlocal in light-cone directions and the factorized objects should be different from those used in Eq.(4.15). We will discuss \(F - K\) contribution to single flavor tag measurements briefly in section 7.

5. Resummation of large logarithms

In order to perform integration in the factorization formula (4.15) we take the Laplace transform of each function entering Eq.(4.15) except for the hard coefficient function \(H_3(\mu^2)\),

\[
\begin{align*}
G(\tau) &= 2 H_3(Q^2, \mu^2) \frac{1}{2\pi i} \int_C \frac{d\nu}{\nu} e^{\nu Q^2 \tau} \sum_{sQ^2, \mu^2} \left( sQ^2, \mu^2 \right) j(sQ^2, \mu^2) s_T(sQ, \mu^2),
\end{align*}
\]

where \(s = \frac{1}{(\nu Q^2 e^{\gamma_E})}\) and

\[
\begin{align*}
j(sQ^2, \mu^2) &\equiv \int_0^\infty dp^2 e^{-\nu p^2} J(p^2, \mu^2), & s_T(sQ, \mu^2) &\equiv \int_0^\infty dk e^{-\nu Q k S_T(k, \mu^2)}, \\
\tilde{\Sigma}_\perp(sQ^2, \mu^2) &\equiv \int_0^\infty dp_R^2 e^{-\nu p_R^2} \sum_{sQ^2, \mu^2} (p_R^2, \mu^2).
\end{align*}
\]

Now we use the fact that the expression (4.1) does not depend on \(\mu^2\). We can exclude all collinear logarithms by setting \(\mu^2 = \tau Q^2 \sim \lambda^2\). In doing so, we can neglect all higher-order

\footnote{In fact, the same argument is the same as that for explaining the angular ordering in QCD sequential branching process (Ref.[20, 21, 22])}
corrections in \( \tilde{\Sigma}_\perp (sQ^2, \mu^2) \) and \( j (sQ^2, \mu^2) \) since all of them contribute either to the NLL level, or to the (pre-exponential) factor, which does not depend on \( \tau \) and can be found from the fixed-order result \( \langle 3.22 \rangle \). Therefore, we can replace in Eq. \( \langle 5.1 \rangle \) the jet function \( j (sQ^2, \mu^2) \) by unity and \( \tilde{\Sigma}_\perp (sQ^2, \mu^2) \) by \( \tilde{\Sigma}_\perp (sQ^2, \tau Q^2) \):

\[
\tilde{\Sigma}_\perp (sQ^2, \tau Q^2) = \int_0^\infty dp^2_{\tau} e^{-p^2_{\tau}} \Sigma_\perp^{(0)} (p^2_{\tau}, \tau Q^2) \bigg|_{D=4} = \frac{\alpha_s (\tau Q^2)}{2\pi} \frac{C_F}{\nu Q^2}. \tag{5.3}
\]

Thus, we obtain the following distribution:

\[
G (\tau) = \frac{2\alpha_s (\tau Q^2)}{\pi} C_F H_3 (Q^2, \tau Q^2) \frac{1}{2\pi i} \int_\nu^\infty \frac{d\nu}{\nu^2 Q^2} e^{\nu Q^2 \tau} s_T (sQ, \tau Q^2). \tag{5.4}
\]

Let us now consider the evolution equation for the hard coefficient functions and the soft factor \( \tilde{\Sigma}_\perp \):

\[
\frac{dH_i (Q^2, \mu^2)}{d\ln \mu^2} = \left\{ \Gamma_{\text{cusp}} \left[ \alpha_s (\mu^2) \right] \ln \frac{Q^2}{\mu^2} + \gamma_i \left[ \alpha_s (\mu^2) \right] \right\} H_i (Q^2, \mu^2),
\]

\[
\frac{ds_T (sQ, \mu^2)}{d\ln \mu^2} = \left\{ \Gamma_{\text{cusp}} \left[ \alpha_s (\mu^2) \right] \ln \frac{s^2 Q^2}{\mu^2} - \gamma_i \left[ \alpha_s (\mu^2) \right] \right\} s_T (sQ, \mu^2), \tag{5.5}
\]

where

\[
\Gamma_{\text{cusp}} (\alpha_s) = \frac{\alpha_s}{4\pi} \Gamma(0) + \left( \frac{\alpha_s}{4\pi} \right)^2 \Gamma(1) + \ldots,
\]

\[
\gamma_i (\alpha_s) = \frac{\alpha_s}{4\pi} \gamma_i(0) + \left( \frac{\alpha_s}{4\pi} \right)^2 \gamma_i(1) + \ldots. \tag{5.6}
\]

In the NLL accuracy, one needs the following expressions: two-loop \( \Gamma_{\text{cusp}}, \) two-loop \( \alpha_s (\mu^2) \), one-loop \( \gamma_i \):

\[
\Gamma(0) = 4C_F, \quad \Gamma(1) = \frac{4C_F}{9} \left[ C_A (67 - 3\pi^2) - 20 T_F N_f \right],
\]

\[
\gamma_H(0) = -6C_F, \quad \gamma_H(1) = \left( \frac{2\pi^2}{3} - 4 \right) C_A + \left( 6 - \frac{4\pi^2}{3} \right) C_F, \tag{5.7}
\]

\( \Gamma(1) \) is found in Ref. \( \langle 23 \rangle \), \( \gamma_H(0) \) and \( \gamma_H(1) \) can be found from the expressions \( \langle 3.5 \rangle \) and \( \langle 3.18 \rangle \), respectively. The initial conditions \( H_i(Q^2, Q^2) \) and \( s_T(sQ, sQ) \) for the equations \( \langle 5.3 \rangle \) contribute to the pre-exponential factor.

Let us, for a moment, omit all pre-exponential factors and set \( \mu^2 = \tau Q^2 \), thus the solutions of the equations \( \langle 5.3 \rangle \) can be represented as follows:

\[
H_i (Q^2, \tau Q^2) = \exp \left\{ \mathcal{F}_{H_i} [L, \alpha_s (Q^2)] \right\},
\]

\[
s_T (sQ, \tau Q^2) = \exp \left\{ \mathcal{F}_s [L, \tilde{L}, \alpha_s (Q^2)] \right\}, \tag{5.8}
\]

where

\[
\mathcal{F}_{H_i} [L, \alpha_s (Q^2)] = - \int_{Q^2}^{\tau Q^2} \frac{d\mu^2}{\mu^2} \left( \Gamma_{\text{cusp}} [\alpha_s (\tilde{\mu}^2)] \ln \frac{Q^2}{\tilde{\mu}^2} + \gamma_{H_i} [\alpha_s (\tilde{\mu}^2)] \right),
\]

\[
\mathcal{F}_s [L, \tilde{L}, \alpha_s (Q^2)] = \int_{s^2 Q^2}^{\tau Q^2} \frac{d\mu^2}{\mu^2} \left( \Gamma_{\text{cusp}} [\alpha_s (\tilde{\mu}^2)] \ln \frac{s^2 Q^2}{\tilde{\mu}^2} - \gamma_i [\alpha_s (\tilde{\mu}^2)] \right). \tag{5.9}
\]
and

\[ L = \ln \frac{1}{\tau}, \quad \bar{L} = \ln \frac{\tau}{e^{\gamma_E}} = \ln (\tau \nu Q^2). \]  

(5.10)

The method of calculating the integral transform in (5.4) is developed in Ref. [24]. It is based on the expansion of the function \( \mathcal{F}_s \left[ L, \bar{L}, \alpha_s (Q^2) \right] \) into the power series with respect to \( \bar{L} \) and using the simple formula

\[
\frac{1}{2\pi i} \int_C du \ln^k u e^{u-(1-g)\ln u} = \frac{d^k}{dg^k} \frac{1}{\Gamma (1 - g)}.
\]  

(5.11)

Using this method, we obtain the following result for the resummed distribution:

\[
G (\tau) = \left[ 1 + C_3 \alpha_s (Q^2) \right] \frac{\alpha_s (\tau Q^2)^2}{\pi} \frac{\tau C_F}{\Gamma [1 - g (L, \alpha_s)]} \exp \left\{ \mathcal{F}_H 3 (L, \alpha_s) + \mathcal{F}_s (L, 0, \alpha_s) \right\},
\]  

(5.12)

where \( \alpha_s = \alpha_s (Q^2) \) and

\[
g (L, \alpha_s) = \frac{\partial}{\partial L} \mathcal{F}_s \left( L, \bar{L}, \alpha_s \right) \bigg|_{\bar{L}=0}.
\]  

(5.13)

In the expression (5.12), we have restored the pre-exponential factor \( 1 + \alpha_s (Q^2) C_3 \), where

\[
C_3 = C_F \left[ -\frac{31}{2} + 12 \zeta (3) \right] + C_A \left[ \frac{353}{18} - 6 \zeta (3) \right] - \frac{50}{9} T_F N_f.
\]  

(5.14)

The corresponding expression for \( F (\tau) \) has the form:

\[
F (\tau) = \left[ 1 + C_2 \alpha_s \right] \frac{\alpha_s (\tau Q^2)^2}{\pi} \frac{\tau C_F}{\Gamma [1 - g (L, \alpha_s)]} \exp \left\{ \mathcal{F}_H 2 (L, \alpha_s) + \mathcal{F}_s (L, 0, \alpha_s) \right\},
\]  

(5.15)

where

\[
C_2 = C_F \left( -2 + \frac{2 \pi^2}{3} \right).
\]  

(5.16)

The SCET result (5.15) presented in Eq. (5.15) coincides with the result of Ref. [24].

Now one can compare the leading thrust distribution (5.13) and the power suppressed one (5.12):

\[
\frac{G (\tau)}{F (\tau)} = C_F \frac{\alpha_s (\tau Q^2)^2}{\pi} \tau \left[ 1 + \alpha_s (Q^2) (C_3 - C_2) \right] \frac{\exp \left\{ \mathcal{F}_H 3 [L, \alpha_s (Q^2)] - \mathcal{F}_H 2 [L, \alpha_s (Q^2)] \right\}}{1 - g [L, \alpha_s (Q^2)]},
\]  

(5.17)

Taking into account the explicit form of the exponents (5.9), we obtain

\[
\frac{G (\tau)}{F (\tau)} = G^{(0)} (\tau) e^{\omega (\tau)},
\]  

(5.18)

where

\[
\omega (\tau) = \frac{\gamma_{H3} (0) - \gamma_{H2} (0) - \beta_0}{\beta_0} \ln (1 - \lambda) - \ln [1 - g (L, \alpha_s)] + \alpha_s (C_3 - C_2),
\]  

(5.19)
with
\[ g(L, \alpha_s) = \frac{2 \Gamma_0}{\beta_0} \left[ \ln (1 - 2\lambda) - \ln (1 - \lambda) \right], \quad \lambda = \frac{\beta_0 \alpha_s}{4\pi} \ln \frac{1}{\tau}. \] (5.20)

Here the pre-exponential factor \( 1 + \alpha_s \left( Q^2 \right) \left( C_3 - C_2 \right) \) has also been exponentiated to the NLL level of accuracy. Since the soft factor in Eq. (4.13) is the same as the one in Eq. (4.5), it drops out of the ratio (5.17) almost completely. The resummation factor \( \exp [\omega(\tau)] \) is shown in Fig. 5.

Since the function \( G(\tau) \) gives the shape of the jets which have distinct angular distribution \( \sim \sin^2 \theta_T \), which is different from that of \( F(\tau) \), there is a possibility to measure \( G(\tau) \). Such analysis was performed by the OPAL collaboration [25]. The comparison of theoretical predictions with the OPAL data is presented in Fig. 6, where three curves are shown. The thin solid line corresponds to the perturbative result (2.14). As one can see from Eq. (3.15), \( dG(0)/d\tau \) tends to constant in the \( \tau \to 0 \) limit. The distribution improved by resummation (5.12) is drawn by the dashed line. The solid line present the prediction with nonperturbative effects as discussed below in Sect. 6.

Since the result (5.12) is valid in the region \( \tau \ll 1 \), we match the resummation factors with the perturbative result (2.14) so that all higher order corrections disappear when \( \tau \) tends to its maximal value for a three-jet configuration \( \tau_{\text{max}} = 1/3 \). The lack of multiplicity for the perturbative result (2.14) explains why the data exceeds the prediction in the region \( \tau \gtrsim 1/3 \), while the poor accuracy of the data in the region \( \tau \ll 1 \) does not allow one to

---

**Figure 5:** Resummation factor as a function of the thrust boundary \( \tau > 1 - T \).

**Figure 6:** Comparison of the OPAL data for the longitudinal thrust event shape with the theoretical predictions \( dG/d\tau \) at \( \tau = 1 - T \). The bars with short strokes represent systematic errors and those with long strokes are statistical errors. The thin solid curve shows \( O(\alpha_s) \) perturbative result, the dashed curve gives the LL+NLL prediction, while the solid curve is obtained after convoluting with the non-perturbative shape function discussed in section 6.
test the NLL effect specific for $G(\tau)$ against the background of the common Sudakov suppression.

The factorization scale $\mu^2$ presented in Eqs. (13), (15) formally separates collinear \textit{intra}-jet radiation and soft \textit{inter}-jet radiation. Since an infrared safe event shape does not depend on an explicit definition of jet, the factorization formulæ do not depend on this scale. This fact is very helpful in deriving the simple representation (5.4) for $G(\tau)$ where the role of the collinear scale is reduced to the renormalization scale of $\alpha_s$ and determination of the argument of the hard logarithms in Eq. (5.17). It is worth noticing that we use the same $\Gamma_{\text{cusp}}$ in the evolution equations (5.5) for all $H_i$, although, we test it only in the leading order (see Eq. (3.18)).

6. Nonperturbative correction

For the total thrust distribution, the resummation of large logarithms was performed to NLL accuracy in Ref. [24] by means of the NLL branching algorithm. The authors of Ref. [24] notice that in order to keep the algorithm in the perturbative regime, one has to introduce the infrared regulator for the argument of the running coupling constant. Alternatively, as it was demonstrated in Refs. [26, 27, 28], one may take into account non-perturbative (NP) effects by convoluting the resummed perturbative expression with a phenomenological shape function $u(\tau)$:

$$
\sigma_{\text{NP+PT}}(\tau) = \int_0^{\tau Q/\Lambda} d'\tau' \sigma_{\text{PT}}\left(\tau - \frac{\Lambda}{Q} \tau'\right) u(\tau'),
$$

(6.1)

where $\Lambda$ is a phenomenological soft scale characterizing the transition into the NP regime. Non-perturbative power corrections generated by this shape function were the subject of intensive experimental [29, 30, 31, 32, 33] and theoretical [34, 35, 28, 36] studies. Since the NP corrections appear mostly due to the radiation of soft partons, NP effects should firstly affect the generalized soft factor:

$$
S_T(\alpha, \beta; \mu^2) = \sum_X \left| \left\langle X \mid Y_{n+} Y_n \mid 0 \right\rangle \right|^2 \delta \left( \alpha - n \cdot p_{X_L} \right) \delta \left( \beta - n_+ \cdot p_{X_R} \right).
$$

(6.2)

It implies that NP effects are universal for the distributions containing the same factorized soft factor, as confirmed by the analysis performed in Ref. [37]. Since $F$ and $G$ structure functions contain the same soft factor, one can estimate the influence of NP effects on the ratio $G/(F+G)$ (see Eqs. (4.3) and (4.15)). A very simple but reasonably good parametrization for the shape function was found in Ref. [37]:

$$
u_K(x) = \frac{2}{\Gamma(3/2)} x^2 e^{-x^2}, \quad \Lambda = 0.7 \text{ GeV}.
$$

(6.3)

Using this parametrization, we find that the NP correction reduces to a simple shift of the cross section even in a region $\tau > 1 - \langle T \rangle = 0.066$, so that the relative correction is

$$
\frac{[G/(F+G)]_{\text{PT+NP}} - [G/(F+G)]_{\text{PT}}}{[G/(F+G)]_{\text{PT}}} = -0.22
$$

(6.4)
for $\tau = 1 - \langle T \rangle$ and increases for smaller $\tau$ (see Fig. 7 below).

Since the NP shape function (3.3) is normalized to unity, the NP correction to the structure function integrated over total domain, i.e. $\int d\tau dG/d\tau$, should vanish. However, the convolution with $u_K(x)$ shifts the point where $dG/d\tau = 0$. Thus, if we restrict the integration to the true value of $\tau_{\text{max}} = 1/2$, then the NP correction to $G(\tau_{\text{max}})$ would be very small. This statement is in agreement with the Monte-Carlo study of hadronization effects performed in Ref. [3], where some generators found no correction and some generators found a small positive correction to the inclusive combination $(F - 2G)/(F + 2G)$.

### 7. Forward-backward asymmetry

In order to suppress QCD corrections to the forward-backward (FB) asymmetry, one can reduce final state phase space to the two jet region, thereby suppressing gluon radiation. It has been studied in Ref.[4] how the experimental cuts bias the theoretical corrections. The event shape can also be used to select the events. Since the phase space of two $r$-collinear partons is of the order $\tau$ in the $\tau \to 0$ limit the corresponding correction decreases with $\tau$ (see Refs.[5, 3] and Fig. 7).

We define the FB asymmetry, which depends on the maximal thrust value ($T < 1 - \tau$), as follows

$$A(\tau) = \frac{\int_0^1 d\cos \theta T \ w(\theta T, \tau) - \int_{-1}^1 d\cos \theta T \ w(\theta T, \tau)}{\int_{-1}^1 d\cos \theta T \ w(\theta T, \tau)} = A^{(0)} \frac{K(\tau)}{F(\tau) + G(\tau)}, \quad (7.1)$$

Here

$$w(\theta, \tau) = \int_{1-\tau}^1 dT \frac{d\sigma}{d\cos \theta T dT}, \quad (7.2)$$

with $d\sigma(\tau)$ defined in Eq.(7.4), and $A^{(0)}$ is the tree-level asymmetry at $Q^2 = M_Z^2$:

$$A^{(0)} = \frac{3}{4} \frac{2g_{al}g_{vl}}{(g_{al}^2 + g_{vl}^2)} \frac{2g_{aq}g_{eq}}{(g_{aq}^2 + g_{eq}^2)}. \quad (7.3)$$

The effect of QCD radiative corrections can be characterized by the following quantity:

$$C(\tau) = 1 - \frac{A(\tau)}{A^{(0)}} = C^{(F-K)}(\tau) + C^{(G)}(\tau), \quad (7.4)$$

$$C^{(F-K)}(\tau) = \frac{F(\tau) - K(\tau)}{F(\tau) + G(\tau)}, \quad (7.5)$$

$$C^{(G)}(\tau) = \frac{G(\tau)}{F(\tau) + G(\tau)}. \quad (7.6)$$

The coefficients $C^{(F-K)}$ and $C^{(G)}$ vanish if the free parton model relations (2.11), (2.12) hold. The sum $F(\tau) + G(\tau)$ gives the integrated thrust cross section.

We show in Fig. 5 QCD predictions for the correction factor $C(\tau)$ which relates as in Eq.(7.4) the electroweak parameter and the observable FB asymmetry of $q\bar{q}$ jets whose thrust value is greater than $T > 1 - \tau$. Shown by thin solid lines are the tree-level results for $C^{(F-K)}$ and $C^{(G)}$. The correction factor $C^{(F-K)}$ arises from the three parton ($q\bar{q}g$)
configuration where both $q$ and $\bar{q}$ are in the same hemisphere along the thrust axis, as illustrated in Fig. 1(c). Those events cannot contribute to the P-odd function $K(\tau)$ and the observed asymmetry reduces by the factor $C^{(F-K)}$. As one can see from Eq. (2.15), the derivative of the function $F_{\text{tree}}(\tau) - K_{\text{tree}}(\tau)$ has logarithmic singularity in the two-jet limit $T \to 1$ (see Eq. (2.15)), which prevents us from introducing a local SCET operator for the function $F - K$. Because this region may be studied by parton shower models [38] that respect the exact matrix element [39] and because the contribution from the relevant jet configuration can in principle be removed by double-tag experiments, we concentrate our attention on the study of the correction factor $C^{(G)}$ in this report.

In contrast to $F - K$, the function $G(\tau)$ originates from the local three-body operator $\hat{O}_3$ of Eq. (4.10) in SCET, and we could show in the previous section that the leading soft singularities from this operator can be resummed to give the same Sudakov form factor as for the total distribution in the two-jet limit ($\tau = 1 - T \to 0$). Because this common Sudakov factor cancels in the ratio, the thin solid curve for $C^{(G)}$ in Fig. 7 may be regarded as the zeroth-order prediction of perturbative QCD with the leading log (LL) thrust distribution.

It is worth noting here that this interpretation does not hold for the $O(\alpha_s)$ curve for $C^{(F-K)}$ in Fig. 7, because the leading Sudakov factor differs from the $q\bar{q}$ events even in the $\tau \to 0$ limit.

In Fig. 7, our prediction for $C^{(G)}$ in the NLL level is shown by the dashed curve and thick solid curve is obtained after incorporating the non-perturbative effects by using the shape function (6.3). Since the NLL level resummation supplemented by the non-perturbative correction (6.3) reproduces the observed thrust distribution rather well (see Refs. [37, 36] and Fig. 8 below), one can directly compare the FB asymmetry of the observed jets in the two-jet region $1 - T < \tau \ll 1$ with the QCD prediction of Eq. (7.4) with the correction factor $C^{(G)}(\tau)$ depicted by the thick solid curve in Fig. 7, provided that the

**Figure 7:** Correction factors for the forward-backward asymmetry.
contribution from $F-K$ is reliably excluded by double-tagging and that the $\tau$-dependance of the observed asymmetry is consistent with the prediction. If one uses all the events with $T > \langle T \rangle = 0.934$ at $Q = m_Z$ \[40\], the QCD correction factor is estimated as

$$C^{(G)}(\tau) \bigg|_{\tau=1-\langle T \rangle=0.066} = 0.0024 \pm 0.0002,$$

(7.7)

in the massless quark limit, when the error is estimated by a quadratic sum of the uncertainty in $\alpha_s(M_Z) = 0.1184 \pm 0.0007$ \[11\] and the variation due to the different ansätze for the non-perturbative shape function found in Ref. \[37\] and Ref. \[42\].

In actual experimental analysis, one needs to correct for effects due to hadronization, detector acceptance and performances. They have been accounted for by making use of parton shower based event simulator, such as JETSET \[38\], which has been tuned to reproduce all the $e^+e^- \rightarrow \text{hadrons}$ data at collision c.m. energies between 30 and 200 GeV. The difficulty of the experimental analysis summarized in Ref. \[2\] for estimating the electroweak asymmetry parameter $A^{(0)}$ from the observed forward-backward asymmetry of charm and bottom quark jets may be traced back to the absence of the parton shower that describes the longitudinal structure function in Eq. \(3.9\).

It is worth noting here that the traditional scheme \[39\] to match the parton shower (that incorporates resummation of LL and NLL emissions) and the matrix elements (that give quantum mechanical correlations such as angular distributions) cannot simulate jets with correct angular distribution at the accuracy level of precision EW measurements. This is because in the leading order, the two-jet like events are matched to the two-parton matrix element which in $e^+e^- \rightarrow q\bar{q}$ contributes only to the transverse structure functions with $1 + \cos^2 \theta_T$ and $\cos \theta_T$ angular distributions. It is only for the three or more jet events the tree-level matrix elements contribute to the longitudinal structure function. Although the longitudinal contributions are power suppressed in the two-jet region, the accuracy required by the EW precision measurements may not allow us to neglect their contribution. In this work we have shown that they can be described as parton shower originated from the local three-body operator in SCET and that the resulting jet structure is similar but different from that of the jets from the $q\bar{q}$ operator. It is a usual consequence of color coherence that two collinear partons are indistinguishable for a large angle radiation and should be replaced by a single pseudo-particle in a jet clustering algorithm. The unusual point specific for power corrections is that the corresponding three-parton matrix element should be supplied by only two $\Delta_q$ Sudakov form factors and then combined with two jet-like parton showers. The short distance nature of longitudinal events requires modify the method to combine parton showers with matrix elements. A successful Monte-Carlo simulation of the QCD corrections to the FB asymmetry due to longitudinal structure function can be carried out only after such modification is performed.

Before closing this section let us discuss our predictions for the angular distribution of jets with a particular thrust value. The differential jet angular distribution can be
expressed as

\[ \frac{d\sigma}{d\cos\theta dT} = \frac{3}{4} \sigma^{(0)} \left[ F'(1-T) \frac{1 + \cos^2 \theta T}{2} + \frac{4}{3} A^{(0)} K' (1-T) \cos \theta T + G' (1-T) \sin^2 \theta T \right] \tag{7.8} \]

in terms of derivatives of the resummed functions at \( \tau = 1 - T \). Integration over \( \cos \theta T \) gives the thrust distribution:

\[ \frac{d\sigma}{dT} = \sigma^{(0)} \left[ F'(1-T) + G'(1-T) \right] \tag{7.9} \]

and the FB asymmetry of the jet with a thrust value of \( T \) is expressed as

\[ \tilde{A}_{FB}(T) = \frac{\int_{-1}^{1} d\cos \theta T \frac{d\sigma}{d\cos \theta dT} - \int_{-1}^{0} d\cos \theta T \frac{d\sigma}{d\cos \theta dT}}{\int_{-1}^{1} d\cos \theta T \frac{d\sigma}{d\cos \theta dT}} = A^{(0)} \frac{K'(1-T)}{F'(1-T) + G'(1-T)}. \tag{7.10} \]

The correction factor can also be defined for a given \( T \) value

\[ \tilde{A}_{FB}(T) = A^{(0)} \left[ 1 - \tilde{C}(T) \right] \tag{7.11} \]

with

\[ \tilde{C}(T) = \tilde{C}^{(F-K)}(T) + \tilde{C}^{(G)}(T), \tag{7.12} \]

\[ \tilde{C}^{(G)}(T) = \frac{F'(1-T) - K'(1-T)}{F'(1-T) + G'(1-T)}, \tag{7.13} \]

\[ \tilde{C}^{(G)}(T) = \frac{G'(1-T)}{F'(1-T) + G'(1-T)}. \tag{7.14} \]

As in the case for the asymmetry of jets with \( T > 1 - \tau \), in the two-jet region where the primary quark and antiquark have momenta in the opposite hemisphere (see Figs. 1(a) and (b)), we can safely assume that \( F(\tau) = K(\tau) \) and all the functions receive common non-perturbative corrections.

The thrust distribution (7.9) is shown as the dashed curve in Fig. 9 from the resummed expressions (5.12) and (5.15). The inclusion of the non-perturbative soft factor (6.3) shift the prediction to the solid curve. The LL+NLL prediction supplemented by the non-perturbative soft factor of Eq. (5.13) reproduces the observed thrust distribution in \( e^+e^- \) collisions at all energies between 14 and 206 GeV [4, 37], and in particular at \( \sqrt{s} = M_Z \) gives the mean value \( \langle T \rangle = 0.934 \) and the distribution is peaked at \( T_{\text{peak}} = 0.9794 \). Although our evaluation (3.16) of \( G'(\tau) \) can be regarded as a part of the NNLO correction to the thrust distribution, we don’t observe significant change in the total thrust distribution.

In Fig. 9, we show our prediction for the QCD correction to the thrust axis angular asymmetry, \( \tilde{C}^{(G)}(T) \), for the jets with a particular thrust value \( T \). Again, the dashed curve shows our LL+NLL order prediction and the solid curve is obtained after including the
non-perturbative soft factor of Eq. (6.3) for both $F$ and $G$ functions. The correction factor $\tilde{C}(T)$ decreases by the non-perturbative correction because it reduces the longitudinal distribution $G'(T)$ more strongly than $F'(T)$ at $T \approx 1$. We find

$$\tilde{C}(G)(T) = 0.0008 \quad \text{at} \quad T = T_{\text{peak}} = 0.9794,$$

(7.15)

$$\tilde{C}(G)(T) = 0.008 \quad \text{at} \quad T = \langle T \rangle = 0.934,$$

(7.16)

from Fig. 8. The correction factor is small but it can be a significant fraction of the error of the $b$-jet FB asymmetry which is as small as 1.7% on the Z-boson pole [1].

Two remarks on our predictions, Eqs.(7.7) and (7.16), are in order here: The first is on the validity of the non-perturbative corrections that lead to our predictions, and the second is on the additional correction $C(F-K)$ due to events where both quark and antiquark are emitted in the same hemisphere along to the thrust axis opposite the gluon jet direction, see Fig. 1(c).

Our prediction for the asymmetry correction factor $C(G)(\tau)$ and $\tilde{C}(G)(T)$ are obtained under the assumptions that both the transverse and longitudinal functions $F(\tau)$ and $G(\tau)$, respectively, receive common non-perturbative corrections via the soft factor Eq. (6.3), whose form has been chosen to reproduce the observed jet thrust distribution in $e^+e^- \rightarrow \text{hadrons}$ experiments. We believe that it is an excellent approximation in the two-jet ($T \rightarrow 1$) limit where collinear quarks and gluons radiates soft gluons and hadronizes coherently. Nevertheless, it is clear that the convolution with the shape function (6.3) does not exhaust all non-perturbative effects. Therefore, a careful study by using a shower MC program that incorporates the longitudinal radiation function $G(\tau)$ is desired for a quantitative estimate.

In this report we consider mainly the correction due to the longitudinal function $G(\tau)$ which arises from the three-jet configurations Figs. 1(a), (b) in the next-to-leading order, since the contribution to $F - K$ from the configuration Fig. 1(c) can be removed by requiring quark and antiquark momenta are in opposite hemispheres in double flavor-tag experiments. In practice, however, the double tagging condition leads to the reduction of the number of useful events by one order of magnitude and hence to the loss of accuracy.
of the measurement. We note here that probably the present PS program matched to the tree-level matrix element for three partons \[39\] can correctly account for those events where the quark and the antiquark are in the same thrust hemisphere. It is easy to show that the matrix element corresponding to the \( F - K \) combination for \( 1 - T \ll 1 \) is singular in the \( (p_q \cdot p_g) \to 0 \) limit, where \( p_q, p_g \) are the quark and gluon momenta, respectively. Due to this singularity, the logarithmically enhanced contribution comes from the phase-space region where the energy of the quark is much smaller than that of the antiquark. For a fixed thrust value such hierarchy leads to a large opening angle between \( q \) and \( \bar{q} \). This probably implies that one can neglect the interference between the radiation emitted along the quark or the antiquark momentum directions, i.e., three-jet configurations give a dominant contribution to \( F - K \) even for \( 1 - T \ll 1 \). However, the additional study of accompanied radiation is needed. Nevertheless, an extreme care is necessary for a reliable estimate of the error due to \( F - K \), since it is the dominant source of the correction to the FB asymmetry as shown in Fig. 7. It mimics the primary quark and antiquark jets in single flavor-tag experiments. It is not clear to us if all these points have been appropriately accounted for in the error analysis presented in Ref. \[3\].

8. Conclusion

The corrections to the angular distribution of the thrust axis considered in this paper are power suppressed with respect to the event shape variable in the two jet region. We identify two mechanisms that radiation affects the angular distribution in the event topology where the primary quark and antiquark are radiated into the opposite hemispheres. It is found that the short-distance process gives the leading contribution to the longitudinal cross section. Using SCET, we propose the factorization formula for the longitudinal cross section and perform the large logarithm resummation to the next-to-leading logarithmic level of accuracy. The factorization formula allows us to study the leading nonperturbative corrections to the longitudinal cross section. A part of the QCD corrections to the forward-backward asymmetry is the ratio of the longitudinal and total thrust cross sections. We find that the resummation and the nonperturbative effects result in additional suppression of this ratio (about 30% for the experimental mean value of the thrust). We observe that the short-distance nature of the leading correction yields potential problems with the Monte-Carlo simulation of the QCD corrections to the forward-backward asymmetry. We present estimates for the QCD corrections to the forward-backward asymmetry in the LL+NLL level including non-perturbative corrections. Underestimation of such corrections may be relevant for the discrepancy between the weak mixing parameter \( \sin^2 \theta_W \) extracted from the jet asymmetry data and the others. However, it can be found out only with a help of a new improved Monte-Carlo simulation.

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