FANO VARIETIES IN MORI FIBRE SPACES

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Abstract. We show that being a general fibre of a Mori fibre space is a rather restrictive condition for a Fano variety. In order to detect this property, we obtain two criteria (one sufficient and one necessary) for a Q-factorial Fano variety with rational singularities to be realised as a fibre of a Mori fibre space, which turn into a characterisation in the rigid case. As an application, we apply our criteria to solve the classically known smooth two-dimensional case, give an almost exhaustive answer for smooth threefolds and flag varieties and a further characterisation on the polytope in the Gorenstein toric case. An interesting connection with K-semistability is also investigated.

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1. Introduction

In this work we focus on a natural question which arises in the context of the classification of complex algebraic varieties and in the Minimal Model Program and tries to clarify the geography of Mori fibre spaces.
Question 1.1. Which Fano varieties can be realised as general fibres of a Mori fibre space?

Although every Fano variety of Picard number one is a Mori fibre space over a point, this work gives evidence about the restrictiveness of this condition for varieties of higher Picard rank.

The notion of “general fibre” will be clarified later in Section 2: the idea is to determine an open dense subset of the base, on which the fibre are “good enough” (cf. Definition 2.16).

Fano varieties, i.e. varieties with ample anti-canonical divisor, play an essential role in the birational classification of projective varieties with negative Kodaira dimension. This was established in low dimension in [Mor88]. The seminal work [BCHM10] shows that every $\mathbb{Q}$-factorial variety with log-terminal singularities and non-pseudo-effective canonical divisor is birational to a Mori fibre space (or simply MFS), i.e. to a contraction morphism with positive dimensional Fano fibres and relative Picard number one. In this work, we will also assume the existence of a dense open set over which the fibres of the MFS are $\mathbb{Q}$-factorial.

Since Mori fibre spaces arise as final products of an MMP, they have been widely studied in the last thirty years in the context of classification of higher dimensional varieties.

It is important to underline that distinct MFS’s can belong to the same birational class, already in the 2-dimensional case, e.g. elementary transformations between ruled surfaces. Relations between MFS’s within a birational class (the so called Sarkisov program) have been extensively studied by [Cor95] in low dimension. The same picture has been proved to endure in higher dimension in [HM13]: two MFS’s within the same birational class can be related via a sequence of very easy birational maps, called Sarkisov links.

Another interesting notion for MFS’s which appears in literature is birational rigidity (cf. [BCZ04]). Nonetheless, the geometric structure of MFS’s remains quite mysterious and very few explicit examples are known.

In this work we focus on the classification of the fibres of MFS’s rather than the total space.

The main results of this paper are the following criteria (cf. Theorem 3.1, Theorem 3.2 and Theorem 3.3).

Theorem 1.2.

- Sufficient criterion: A Fano variety $F$ such that
  \[ \text{NS}(F)^{\text{Aut}(F)}_{\mathbb{Q}} = \mathbb{Q}K_F \]
  can be realised as a general fibre of a MFS. Here the LHS is the invariant part of the Néron-Severi group under the action of $\text{Aut}(F)$.
  Moreover, an isotrivial MFS over a curve is constructed in this case.

- Necessary criterion: A Fano variety $F$ such that
  \[ \dim \text{NS}(F)^{\text{Mon}(F)}_{\mathbb{Q}} > 1, \]
  where $\text{Mon}(F)$ is the maximal subgroup of $\text{GL}(\text{NS}(F), \mathbb{Z})$ which preserves the birational data of $F$ (cf. Definition 2.10), cannot be realised as a general fibre of a MFS.

- Characterisation for rigid varieties: Assume $F$ is rigid. Then the sufficient criterion turns into a characterisation.
Our results rely upon a careful study of the monodromy action on MFS’s.
We can use our criteria to prove the following theorem (cf. Theorem 4.2 and Theorem 5.1).

**Theorem 1.3.**
- **Surfaces:** a smooth del Pezzo surface can be realised as the general fibre of a MFS if and only if it is not isomorphic to the blow-up of \( \mathbb{P}^2 \) in one or two points.
- **Threefolds:** the deformation type of a smooth Fano threefold \( F \) with \( \rho(F) \) strictly bigger than one which can be realised as a general fibre of a MFS is one of the 8 classes appearing in the table of Theorem 5.1.

Let us point out that cubic surfaces in \( \mathbb{P}^3 \) provide examples of varieties that do not satisfy our sufficient criterion but can be realised as a general fibre of a MFS.

To deal with Fano threefolds in Section 5, we give some ad hoc versions of the necessary criterion explicitly in terms of the birational geometry of \( F \).

**Remark 1.4.** The 2-dimensional case of Theorem 1.3 has been worked out in [Mor82, Theorem 3.5]. In Section 4 we give an alternative proof using our criteria, allowing bases of arbitrary dimension. Mori and Mukai classified all smooth Fano threefolds with Picard number bigger than one up to deformation into 88 classes in [MM82] and [MM03]; this shows how much the fibre-likeness condition is restrictive.

The second part of Theorem 1.3 can be deduced combining our criteria with the classification result [Pro13, Theorem 1.2]. In Section 5 we also work out this case without using Prokhorov’s work, but looking directly at the nef cone of \( F \).

On the positive side, we can give the following examples of varieties with high dimension and big Picard number which can be realised as a general fibre of a MFS.

**Theorem 1.5** (= Theorem 5.4). Take positive integers \( r, k, d, \) and \( n \) such that \( kd < n + 1 \). Let \( F \) be a smooth complete intersection of \( k \) divisors of multi-degree \( (d, d, \ldots, d) \) in the product of \( r \) copies of \( \mathbb{P}^n \). Then \( F \) can be realised as a general fibre of a MFS.

Section 6 deals with the toric case, which we can completely work out: we obtain the following combinatorial characterisation on the polytope.

**Theorem 1.6** (= Theorem 6.2). A terminal Gorenstein toric Fano variety \( F(\Delta) \) can be realised as a general fibre of a MFS if and only if \( \text{Aut}(\Delta) \) acts transitively on the vertexes of \( \Delta \).

**Remark 1.7.** The transitivity of this action can be verified computationally: as Table 6.5 shows, it is a rather restrictive condition. Nonetheless, the geometry of those polytopes admitting such action is far from being understood. We also start the classification of vertex-transitive polytopes and analyse a class of Fano varieties introduced by Klyachko in [Kly84] which verify our transitivity condition and generalise del Pezzo varieties (cf. Definition 6.24).

In section 7 we show that most flag varieties are not fibre-like.

We do not know if the fibre-likeness (i.e. the property of being realised as the general fibre of a MFS) is open in families. Our necessary criterion in Theorem 1.2 is invariant under flat deformation, while the sufficient criterion is closed in families, but it detects just a special kind of MFS’s: the _isotrivial_ ones.
It is important to underline that the fibre-likeness is supposed to be strictly related to $K$-stability for smooth Fano varieties. So far, we can prove the following result (cf. Corollaries 4.3 and 6.7).

**Corollary 1.8.**
- A smooth del Pezzo surface is $K$-stable if and only if it can be realised as a general fibre of a MFS;
- if a smooth toric Fano variety $F(\Delta)$ appears as the general fibre of a Mori fibre space, then it is $K$-stable.

Furthermore the potential fibre-like smooth threefolds are suspected to be $K$-stable. Inspired by [OO13], we propose the following question, which is the relative version of a conjecture by Odaka and Okada.

**Question 1.9.** Is it true that every smooth Fano variety which can be realised as a general fibre of a MFS is $K$-semistable?

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2. Preliminary results

2.1. **Monodromy Action and Deligne’s Theorem.** In this section we discuss some facts about monodromy on fibrations in Fano varieties. Where not specified, varieties are assumed to be $\mathbb{Q}$-factorial with rational singularities. Let us recall a basic definition.

**Definition 2.1.** A variety $F$ is said *Fano* if its anti-canonical divisor $-K_F$ is ample.

In this subsection, we will deal with a projective morphism between normal varieties

$$f : X \to Y,$$
such that $X$ is $\mathbb{Q}$-factorial with rational singularities and $-K_X$ is $f$-ample. We also assume that there is an open dense subset of $Y$ over which all fibres are $\mathbb{Q}$-factorial. Without this last assumption, the results of this subsection are just empty.

We consider two sheaves on $Y$. The first one is $R^2f_*\mathbb{Q}$; its fibre at $t$ is $H^2(F_t, \mathbb{Q})$. The second one is $R^1f_*\mathbb{G}_m \otimes \mathbb{Q}$; its fibre at $t$ is $\text{Pic}(F_t)_\mathbb{Q}$. The first Chern class defines a morphism

$$c: R^1f_*\mathbb{G}_m \otimes \mathbb{Q} \to R^2f_*\mathbb{Q}. $$

When the fibres have rational singularities, we can apply Kawamata-Viehweg vanishing Theorem to show that $c$ is an isomorphism. This makes the monodromy algebraic, and allows us to show our results.

If both $X$ and $Y$ are smooth, Sard’s Theorem shows the existence of an open dense subset $U$ of $Y$ where $f$ is a submersion. Using Ehresmann’s Theorem (cf. [Voi02, Proposition 9.3]), we can show that $f$ is a locally trivial fibration of topological spaces. The existence of an open dense subset $U_{\text{top}}$ of $Y$ where $f$ is a locally trivial topological fibration holds also in the singular case, by a delicate argument due to Verdier (cf. [Ver76, Corollary 5.1]). We will denote it with $U_{\text{top}}$, to simplify the notation. Unfortunately, the characterisation of $U_{\text{top}}$ is hard and it leads to the concept of equisingularity (cf. [Tei75]).

On $U_{\text{top}}$, the sheaf $R^2f_*\mathbb{Q}$ is a local system. Because of this, we have a monodromy action of $\pi_1(U_{\text{top}}, t)$ on the fibre $(R^2f_*\mathbb{Q})_t$. In this algebraic set-up, there is a more refined result.

**Theorem 2.2** ([KM92], [dFH11]). Let $U = U_f$ be the open dense subset of $Y$ where

1. $Y$ is smooth;
2. $f$ is flat;
3. the fibres of $f$ are $\mathbb{Q}$-factorial normal varieties with rational singularities.

Then the sheaf $R^1f_*\mathbb{G}_m \otimes \mathbb{Q}$ is a local system on $U$ with finite monodromy.

**Proof.** The first step is taken from [KM92, Proposition 2.2.5]. They introduce the sheaf $\mathcal{G}^N(X/U)$; they show it is a local system with finite monodromy and it is isomorphic to $R^1f_*\mathbb{G}_m \otimes \mathbb{Q}$ at a very general point of $U$. The second step is taken from [dFH11, Proposition 6.5]. The authors first show that $\mathcal{G}^N(X/U)$ is isomorphic to $R^1f_*\mathbb{G}_m \otimes \mathbb{Q}$ at the general point of $U$; to do this, Verdier’s result and the isomorphism (1) are essential. Then, they show that actually the isomorphism holds at every point of $U_f$.

Remark that in [dFH11] the base is a smooth curve and the Fano varieties involved have terminal singularities. Nonetheless, the same argument works for smooth bases of arbitrary dimension and rational singularities are enough to use the results in [KM92].

On $U_{\text{top}}$, the monodromy action on $R^2f_*\mathbb{Q}$ can be explicitly described as follows. Take the class of a loop $\gamma$ in $\pi_1(U, t)$. Pull back $X$ to the interval $I = [0, 1]$ via $\gamma$. Trivialise the family $X_I$. The identification between the fibre over 0 and the fibre over 1 is a homeomorphism of $F_I$: it gives the monodromy action. If we change the representative of $[\gamma]$ we may change the automorphism, but not its action on $H^2(F_I, \mathbb{Q})$. See [Voi02, Section 9.2.1] for more details. Let us introduce a special kind of fibration.
Definition 2.3 (Isotrivial fibration). A morphism 

\[ f : X_U \to U \]

is isotrivial if every point of \( U \) has an Euclidean neighbourhood over which \( f \) is a holomorphically trivial fibration.

A family of projective schemes is locally isotrivial if all fibres are isomorphic (cf. [Ser06, Proposition 2.6.10]). When we are dealing with isotrivial fibrations, the identification between the fibre over 0 and the fibre over 1 is given by a holomorphic automorphism of \( F_t \). This does not give us a map from \( \pi_1(U, t) \) to \( \text{Aut}(F_t) \), but it allows us to assume that the action of \( [\gamma] \) is induced by a (non-unique) element of \( \text{Aut}(F_t) \). We remark that isotriviality is a special condition. The only case when it is granted for free is when \( F_t \) is rigid.

Restricting a cohomology class to each fibre we have a morphism

\[ H^2(X, \mathbb{Q}) \to H^0(U, R^2 f_* \mathbb{Q}). \]

By evaluating the section at \( t \) we get an isomorphism

\[ H^0(U, R^2 f_* \mathbb{Q}) \to H^2(F_t, \mathbb{Q})^{\pi_1(U, t)}. \]

The following is a deep result by Deligne (cf. [Voi02, Theorem 16.24]).

Theorem 2.4. Assume that both \( X \) and \( Y \) are smooth. Then, the restriction morphism

\[ \rho : H^i(X, \mathbb{Q}) \to H^i(F_t, \mathbb{Q})^{\pi_1(U, t)} \]

is surjective.

Let us recall that Deligne’s Theorem does not hold with \( \mathbb{Z} \)-coefficients. In the case we are interested in, Deligne’s Theorem specialises to the following corollary.

Corollary 2.5. Let

\[ f : X \to Y \]

be a dominant morphism of smooth projective varieties. Assume that \(-K_X \) is ample. Take \( U \) as in Theorem 2.2. Then the restriction morphism

\[ \rho : \text{NS}(X)_{\mathbb{Q}} \to \text{NS}(F_t)_{\mathbb{Q}}^{\pi_1(U, t)} \]

is surjective.

Proof. The fibre \( F_t \) is a smooth Fano manifold, so

\[ H^2(F_t, \mathbb{Q}) \cong H^{1,1}(F_t, \mathbb{Q}) \cong \text{NS}(F_t)_{\mathbb{Q}}. \]

and this isomorphisms are \( \pi_1(U, t) \)-equivariant. We have to show that the rank of the map

\[ \rho : \text{NS}(X)_{\mathbb{Q}} \to \text{NS}(F_t)_{\mathbb{Q}} \]

equals the rank of the restriction map in cohomology

\[ R : H^2(X, \mathbb{Q}) \to H^2(F_t, \mathbb{Q}). \]

Let us call \( r \) the rank of this latter map. The image of the dual map

\[ R^\vee : H_2(F_t, \mathbb{Q}) \to H_2(X, \mathbb{Q}) \]

has rank \( r \) as well. Since \( F_t \) is Fano, we have

\[ N_1(F_t)_{\mathbb{Q}} \cong H_2(F_t, \mathbb{Q}). \]
So, the rank of $R^\vee$ equals the rank of the restriction map
$$\rho^\vee : N_1(F_t)_Q \to N_1(X)_Q.$$  
By the duality between $N_1$ and NS, we conclude that also the rank of $\rho$ is $r$. □

In the following, we generalise Corollary 2.5 to the singular case. This also gives us an independent proof of Deligne’s Theorem for the specific case we are interested in.

**Theorem 2.6.** Let

$$f : X \to Y$$

be a dominant morphism of projective normal varieties, where $X$ is $\mathbb{Q}$-factorial with rational singularities. Assume that the anti-canonical sheaf of $X$ is $f$-ample. Take $U = U_f$ as in Theorem 2.2. Then the restriction map

$$\rho : \text{NS}(X)_Q \to \text{NS}(F_t)^{\pi_1(U,t)}_Q$$

is surjective.

**Proof.** As remarked at the beginning of this section, the first Chern class map yields an isomorphism

$$R^1f_*\mathbb{G}_m \otimes \mathbb{Q} \to R^2f_*\mathbb{Q}$$

of sheaves of abelian groups on $U$. Let us denote by $\mathcal{F}$ one of these two isomorphic sheaves, which are locally constant by Theorem 2.2. We have a monodromy action of $\pi_1(U,t)$ on $\mathcal{F}_t = \text{NS}(F_t)_Q = H^2(F_t, \mathbb{Q})$. This action is finite by Theorem 2.2; let us denote by $G$ the finite group it factors through. Consider an étale Galois cover

$$p : V \to U,$$

which trivialises $p^*\mathcal{F}$. The Galois group of the cover is $G$ and it acts on $V$ and $V/G = U$. We can lift this action to an action on $X_V$, again $X_V/G = X_U$. By abuse of notation we denote again by $p$ the map from $X_V$ to $X_U$, and by $f$ the map from $X_V$ to $V$. Let $p^{-1}t = \{t_1, \ldots, t_k\}$ be the pre-images of $t$ in $V$. Denote by $F_i$ the fibre over $t_i$. The restriction $p_i$ of $p$ to $F_i$ is an isomorphism with $F_i$.

Let $g$ be an element of $G$. Write $g(t_1) = t_i$. Pick a divisor $D$ in $\text{NS}(F_1)$ and denote by $g(D)$ the monodromy action of $g$ on $D$. Via $p_1$, we can see $D$ as a divisor on $F_1$. Since $p^*\mathcal{F}$ is trivial, there exists a (non-unique) divisor $\tilde{D}$ in $\text{Pic}(X_V)$ which restricts to $D$ on $F_1$. We denote by $\tilde{D}_i$ the restriction of $\tilde{D}$ to $F_i$. The monodromy action on $\text{NS}(F_t)_Q$ is given by

$$g(D) = p_i^{-1}\ast \tilde{D}_i = p_i^{-1}\ast g\ast \tilde{D}_1,$$

where the second equality is due to $p_1 \circ g = p_1$. We remark that if $G$ fixes the isomorphism class of $\tilde{D}$ in $\text{Pic}(X_V)$, then $G$ fixes $D$ in $\text{NS}(F_t)_Q$. The converse requires a bit of care. We want to show that if $D$ is fixed by $G$, we can find a $G$-invariant $\tilde{D}$. Pick any $\tilde{D}$ in $\text{Pic}(X_V)$ which restricts to $D$ on $F_1$. Since $\tilde{D}$ is possibly not $G$-invariant, we consider the average in $\text{Pic}(X_V)_Q$:

$$E := \frac{1}{|G|} \sum_{g \in G} g\ast \tilde{D}.$$ 

Since $D$ is invariant under monodromy, the restriction of $g\ast \tilde{D}$ to $F_1$ is isomorphic to $D$ for any $g$ in $G$. We conclude that $E_1$ is isomorphic to $D$ as well. Moreover, $E$
is $G$-invariant. We conclude that the restriction $r_1$ to the fibre $F_1$ composed with $p_1$ defines a surjective morphism

$$r_1 : \text{Pic}(X_{V})^G_Q \to \text{Pic}(F_1)^G_Q.$$ 

We now recall that $X_{V}/G = X_U$, so we have

$$\text{Pic}(X_{V})^G_Q \cong \text{Pic}(X_U)_Q.$$ 

We have a surjective morphism

$$\text{Pic}(X_U)_Q \to \text{Pic}(F_1)^G_Q.$$ 

(this map is not in general injective: $f^* \text{Pic}(U)$ is in its the Kernel.) Since $X$ is $Q$-factorial, also the restriction morphism

$$\text{Pic}(X)_Q \to \text{Pic}(X_U)_Q$$ 

is surjective (given a Cartier divisor on $X_U$, its Zariski closure is a Weil divisor on $X$). The restriction commute with taking Chern classes, so we have the required surjection

$$\rho : \text{NS}(X)_Q \to \text{NS}(F_1)^G_Q.$$ 

\[\square\]

Remark 2.7 (Theorem 2.6 in arbitrary characteristic). The key ingredient in the proof of Theorem 2.6 in this purely algebraic setting is the isomorphism (2) given by the first Chern class, which is, in characteristic zero, a consequence of Kawamata-Viehweg vanishing Theorem.

Let us pick a variety $X$ over an algebraically closed field $k$ of arbitrary characteristic. Consider, for any prime $l \neq \text{char} k$ and $n$ positive integer, the Kummer exact sequence (in the étale topology):

$$0 \to \mu_{ln} \to G_m \xrightarrow{t_n} G_m \to 0.$$ 

Passing to cohomology, we obtain the following:

$$0 \to \text{Pic}(X)_{\mathbb{Z}/l^n\mathbb{Z}} \to H^2_{\text{ét}}(X, \mu_{ln}) \to \text{Br}(X)_{l^n} \to 0,$$

where $\text{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ is the (cohomological) Brauer group of $X$.

We can now take the inverse limit of (4) with respect to $n$ and obtain

$$0 \to \text{NS}(X)_{\mathbb{Z}_l} \to H^2_{\text{ét}}(X, \mathbb{Z}_l(1)) \to T_l \text{Br}(X) \to 0,$$

where $T_l \text{Br}(X) := \varprojlim \text{Br}(X)_{l^n}$. Furthermore we used the following fact:

$$\text{Pic}(X)_{\mathbb{Z}_l} \cong \text{NS}(X)_{\mathbb{Z}_l}.$$ 

We can recover the same result of Theorem 2.6 in this general setting, assuming that

$$\text{Br}(F_1) = 0.$$ 

This condition is known to hold, for instance, for rational surfaces in arbitrary characteristic. In this general setting, our analysis is in progress.
2.2. Monodromy and MMP. In this section we show that the monodromy preserves some information about the birational geometry of $F$ (a general reference on this topic is [dFH12]). Recall that, in our discussion, we are not assuming smoothness.

First of all, the monodromy preserves the intersection pairing. Indeed, it can be seen as an action on the algebra $H^*(F_t, \mathbb{Z})$. The class of the canonical divisor of $F_t$ is, by adjunction, the restriction of the canonical divisor of $X$; so it is preserved by the monodromy. Call $n$ the dimension of $F_t$. The $\mathbb{Q}$-valued bilinear form

$$b(A, B) := (K_{F_t})^{n-2} \cdot A \cdot B$$
onumber

on $\text{NS}(F_t)_{\mathbb{Q}}$ is equally preserved.

We remark that, in general, the Mori chambers are not preserved by the monodromy action. This is shown in [Tot12]. However, these cones are preserved under additional hypothesis. When the fibres are smooth, Wiśniewski proved in [Wis91] and [Wis09] that the nef cone is locally constant. In the singular case, [dFH11, Theorem 6.8] shows that movable cone is preserved. Moreover, [dFH11, Theorem 6.9] shows that the Mori chambers are locally constant in the following cases:

- 3-dimensional fibres;
- 4-dimensional and 1-canonical fibres;
- toric fibres.

In particular the nef cone is locally constant in these cases. In a more general set-up, we can give the following contribution.

**Theorem 2.8.** Keep notations as in Theorem 2.2. Up to shrinking $U_f$, the monodromy action preserves the nef cone of $F_t$.

**Proof.** By Theorem 2.2, we can take a finite étale cover $p : V \to U_f$ to trivialise the monodromy action. In particular, for every fibre $X_t$ of $f|_V$, the restriction map $N_1(X_t/V) \to N_1(X_t)$ is an isomorphism.

Since $-K_X$ is $f$-ample, by the cone theorem we know that $\overline{\text{NE}}(X_t/V)$ is rational polyhedral: in particular it is generated by a finite number of rays $R_1, \ldots, R_k$ and each ray is generated by the class of a curve $C_i$. Any $C_i$ is represented by an integral lattice point in $N_1(X_t/V)$.

By Kawamata’s rationality theorem, the primitive generators of $\overline{\text{NE}}(X_t/V)$ are integral lattice points that lie between the hyperplanes $H_0 = \{v \in N_1(X/Y) \mid K_X \cdot v = 0\}$ and $H_{2n} = \{v \in N_1(X/Y) \mid K_X \cdot v = 2n\}$, where $n = \dim X$. The number of integral points contained in $\overline{\text{NE}}(X_t/V)$ lying between $H_0$ and $H_{2n}$ is of course finite. Let us denote the set of such points by $C$.

Now, to any class in $C$ we can associate the variety

$$M := \bigcup_{\beta \in C} \text{Mor}(\mathbb{P}^1, X_t/V, \beta)$$

of morphisms from $\mathbb{P}^1$ to $X_t$, contracted by $X_t/V \to V$, whose images have class belonging to $C$. This is a quasi-projective scheme of finite type that comes equipped with a proper map to $\pi : M \to V$. Let $N$ be the union of those irreducible components of $M$ that do not dominate $V$ via $\pi$, and let $T = \pi (N)$. Let us remark that $T$ is a Zariski closed set of $V$, as $M$ is a variety of finite type proper over $V$.

Define $U := V \setminus T$ and $X := f^{-1}(U)$. The claim is that $p(U)$ is the Zariski open set we are looking for. In fact, consider the rational polyhedral cone $\overline{\text{NE}}(X_t/U)$. As above, the extremal rays are finite and their generators have bounded degree with
respect to $-K_X$. As every component of $M$ that does not belong to $N$ dominates $V$, then the classes that generate the extremal rays of $\overline{\text{NE}}(X_U/U)$ move over $U$. But this means that the restriction $N_i(X_U/U) \to N_i(X_U)$ identifies $\overline{\text{NE}}(X_U/U)$ and $\overline{\text{NE}}(X_U)$. The inclusion $\overline{\text{NE}}(X_U) \subset \overline{\text{NE}}(X_U/U)$ follows in fact directly from the definition, while the opposite one is a consequence of the last observation. □

Remark that our result does not provide any tool to characterise the open subset where the monodromy preserves the nef cone.

From now on, let $U$ be the open subset where the hypothesis of Theorem 2.2 holds and where the monodromy preserves the nef cone. Take an element $g$ of $\pi_1(U,t)$. Assume it exchanges two maximal faces $F_1$ and $F_2$ of the nef cone. These faces give contractions

$$\pi_i : F_i \to G_i,$$

which correspond to a first step of the MMP for $F$. The pull-back via $\pi_i$ identifies the nef cone of $G_i$ with the face $G_i$. The first step of such a run is one of the following three possibilities.

**Divisorial contraction:** $\pi : F \to G$ is birational, the exceptional locus is an irreducible divisor and all the curves in the fibres are numerically equivalent, i.e. $\rho(F/G) = 1$;

**Flipping contraction:** $\pi : F \to G$ is birational, the exceptional locus is of codimension at least 2 and $\rho(F/G) = 1$;

**Mori fibre contraction:** $\pi : F \to G$, $\dim F > \dim G$ and $\rho(F/G) = 1$.

The information about the type of map occurring in the MMP is encoded in the cohomology ring. Because of this, these types are preserved by the monodromy action, as stated in the following theorem.

**Theorem 2.9.** Keep notations as above. Assume that the monodromy action identifies two maximal faces $G_1$ and $G_2$ of the nef cone of $F$. Then, the two maps

$$\pi_1 : F \to G_1$$

$$\pi_2 : F \to G_2$$

correspond to the same kind of step of the MMP. In the case of the divisorial contraction, the monodromy preserves the exceptional divisor.

Moreover, the varieties $G_1$ and $G_2$ (and the morphisms $\pi_1$ and $\pi_2$) are deformation equivalent.

**Proof.** To prove the first part of the theorem, we do a case-by-case analysis.

**Divisorial contraction:** As $\pi_i : F \to G_i$ is birational, given a divisor $H$ in the relative interior of $\pi_i^* \text{Nef}(G_i)$, we have $(H^{\dim F}) > 0$. The exceptional locus is an irreducible divisor, call it $D_i$. It is clear that $D_i$ is the only effective divisor on $F$ such that $(H^{\dim F - 1} \cdot D_i) = 0$, for every $H$ in the relative interior of the corresponding facet. Moreover, we can characterise the dimension of $\pi_i(D_i)$ as the maximal integer $k$ such that $(H^k \cdot D_i) \neq 0$, i.e. the numerical dimension of the restriction of $H$ to $D_i$.

**Flipping contraction:** as $\pi_i : F \to G_i$ is birational, given a divisor $H$ in the relative interior of $\pi_i^* \text{Nef}(G_i)$, we have $(H^{\dim F}) \neq 0$. The smallness of $\pi_i$ is equivalent to the fact that for every effective divisor $E \in G_i$ we have $(H^{\dim F - 1} \cdot E) > 0$. Both these conditions are preserved by the monodromy action (as the effective cone is preserved).
Mori fibre contraction: as \(\dim F > \dim G_i\), given a divisor \(H\) in the relative interior of \(\pi^*_i \text{Nef}(G_i)\), we have \((H^{\dim F}) = 0\) and \(\dim G_i\) is the maximum integer such that \((H^k) \neq 0\). Hence, in this case, we know that also the dimension of the base of the fibration is preserved by monodromy.

Let us prove that there exists a flat deformation of \(G_1\) into \(G_2\). The monodromy action is finite, so after a finite étale cover \(p: V \rightarrow U\), we obtain a family

\[ f_p: X_V \rightarrow V \]

with trivial monodromy action on the fibres. This means that the restriction morphism \(\text{NS}(X_V)_Q \rightarrow \text{NS}(F')_Q\) is surjective for every fibre \(F'\). As \(\mathcal{G}_1\) and \(\mathcal{G}_2\) are two facets identified under the monodromy action on the family \(X_U \rightarrow U\), we can find two points \(t_1, t_2\) on \(V\) and a divisor \(H\) on \(X_V\) whose restrictions to \(X_{t_1}\) (resp. \(X_{t_2}\)) lie in the relative interior of \(\mathcal{G}_1\) (resp. \(\mathcal{G}_2\)).

Considering the variety \(\tilde{X} := \text{Proj}_{\mathcal{O}_V}(\bigoplus_{n \in \mathbb{N}} \pi_* \mathcal{O}_{X_V}(nH))\), we get a morphism relative to \(V\)

\[
\begin{align*}
X_V & \xrightarrow{g} \tilde{X} \\
\downarrow f_p & \downarrow \pi \\
V & \xrightarrow{\pi} \tilde{X}
\end{align*}
\]

The fibre of \(\pi\) over \(t_i\) is \(G_i\). The restriction of \(g\) over \(t_i\) is the contraction given by the face \(\mathcal{G}_i\). We are going to prove that \(\pi\) is flat. The base \(V\) is smooth and \(\tilde{X}\) is C-M, so it is enough to check that the fibres of \(\pi\) are equidimensional (cf. [Gro65, Proposition 6.1.5]). This was already shown in first part of the proof. We can conclude that \(G_1\) is deformation equivalent to \(G_2\).

Since both \(f_p\) and \(\pi\) are flat, also \(g\) is flat. This means that we actually have a flat deformation of the contraction given by \(\mathcal{G}_1\) to the contraction given by \(\mathcal{G}_2\). □

Remark that the second part of the proof works also if the faces are not maximal.

The previous discussion motivates the following definition.

**Definition 2.10** (The groups \(\text{HMon}\) and \(\text{Mon}\)). Let \(F\) be an \(n\)-dimensional Fano variety. We denote by \(\text{Aut}(F)^0\) the subgroup of \(\text{Aut}(F)\) which acts trivially on the Néron-Severi group. Then the group \(\text{HMon}(F)\) is defined as

\[ \text{HMon}(F) := \text{Aut}(F)/\text{Aut}(F)^0. \]

The group \(\text{Mon}(F)\) is defined as the maximal sub-group of \(\text{GL}(\text{NS}(F), \mathbb{Z})\) which preserves:

- the ray of \(K_F\);
- the \(\mathbb{Q}\)-valued bilinear form \(b(A, B) := (K_F)^{n-2} \cdot A \cdot B\);
- the nef cone;
- the type of step of the MMP associated to the facets of the nef cone and the exceptional divisor;
- the deformation type of the images of the maps defined by the faces of the nef cone.

**Remark 2.11.** The group \(\text{HMon}(F)\) is the sub-group of \(\text{Mon}(F)\) of elements induced by \(\text{Aut}(F)\).
Example 2.12. As an example, let us discuss the Picard rank 2 case: there are only two possibilities for $\text{Mon}(F)$: either trivial or order two; the latter occurs only if the ray of the canonical class is the bisector of the nef cone and the faces of the nef cone are equivalent from the point of view of the MMP.

Another easy example is for del Pezzo surfaces (cf. Section 4): the generic del Pezzo surface of degree 3 has no automorphisms ($\text{HMon}$ is then trivial in this case), but $\text{Mon}$ is certainly not trivial, as we will show in Section 3.

The group $\text{Mon}(F)$ is invariant under flat deformations which preserve the nef cone, whereas the group $\text{HMon}(F)$ can jump when we deform $F$. Let us state the definitive form of our result for further references.

Theorem 2.13. Let

$$f: X \to Y$$

be a dominant morphism of projective normal varieties, where $X$ is $\mathbb{Q}$-factorial with rational singularities and on an open dense subset of $Y$ the fibres of $f$ are $\mathbb{Q}$-factorial. Assume that the anti-canonical sheaf of $X$ is $f$-ample. Then, there exists a maximal open dense subset $U = U_f$ of $Y$ such that

- the morphism $f: X_U \to U$ is a flat family of $\mathbb{Q}$-factorial Fano varieties with rational singularities;
- for every $t$ in $U$, there is a monodromy action of $\pi_1(U, t)$ on $\text{NS}(F_t)_\mathbb{Q}$. This action factors through the finite group $\text{Mon}(F_t)$ defined above. If the fibration is isotrivial, the monodromy factors through $\text{HMon}(F_t)$.

Moreover, the restriction map

$$\rho: \text{NS}(X)_\mathbb{Q} \to \text{NS}(F_t)^{\pi_1(U, t)}_\mathbb{Q}$$

is surjective.

Remark 2.14. In the smooth case, this open set $U$ contains the set $U^{\text{top}}$ where $f$ is a submersion.

Since in the following we will focus on a specific class of Fano fibrations, let us recall a definition.

Definition 2.15. Let $f: X \to Y$ be a dominant projective morphism of normal varieties. Then $f$ is called a Mori Fibre Space (or simply MFS) if the following conditions are satisfied:

1. $X$ is $\mathbb{Q}$-factorial with rational singularities;
2. $f$ has connected fibres, with $\dim Y < \dim X$;
3. the relative Picard number of $f$ is one;
4. the anti-canonical sheaf of $X$ is $f$-ample.

Moreover, we assume one of the following:

1. there exists an open dense subset of $Y$ over which all fibres are $\mathbb{Q}$-factorial;
2. $X$ has terminal singularities.

Note that if $X$ has terminal singularities then, after restricting to the set of $Y$ where the fibration is flat and all fibres have terminal singularities, the set over which all fibres are $\mathbb{Q}$-factorial is open (cf. [KM92, Theorem 12.1.10]).

We can finally introduce the key notion for our purposes.
Definition 2.16 (Fibre-like). A Fano variety $F$ is said to be fibre-like if it can be realised as a fibre of a Mori Fibre Space $f : X \to Y$ over $U_f$, where $U_f$ is as in Theorem 2.13.

3. Criteria for Fibre-likeness

3.1. General Criteria. In this section we present two criteria, one sufficient and one necessary, which detect the fibre-likeness in a rather general setting. The necessary criterion is based on Theorem 2.13. When the Fano variety is rigid, we have a characterisation.

Theorem 3.1 (Sufficient Criterion). A Fano variety $F$ such that

$$\text{NS}(F)_{\mathbb{Q}}^{\text{Aut}(F)} = \mathbb{Q}K_F$$

is fibre-like.

Moreover, there exists a MFS such that the base $Y$ is a curve and the fibration is isotrivial.

Before giving the proof, let us remark that $K_F$ is always fixed by $\text{Aut}(F)$. In other words, we are asking that the subspace of $\text{NS}(F)_{\mathbb{Q}}$ fixed by $\text{Aut}(F)$ is minimal.

Proof. We know that $\text{HMon}(F)$ is finite: Mori theory tells us that the nef cone $\text{Nef}(F)$ of $F$ is rational polyhedral and $\text{HMon}(F)$ permutes its faces.

Pick a set of generators $[f_1], \ldots, [f_g]$ of $\text{HMon}(F)$. Call $G$ the sub-group of $\text{Aut}(F)$ generated by $f_1, \ldots, f_g$. Take a genus $g$ curve $C$ and denote by $a_i$ and $b_i$ the generators of its fundamental group. We know that the unique relation between the $a_i$’s and $b_i$’s is the one on the product of commutators:

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_g b_g a_g^{-1}b_g^{-1} = 1.$$ 

We define a surjective morphism

$$\rho : \pi_1(C, t) \to G,$$

$$a_i \mapsto f_i, \quad b_i \mapsto f_i^{-1}.$$ 

Let $\hat{C}$ be the universal cover of $C$, then we define

$$X := F \times \hat{C} / \pi_1(C, t),$$

where $\pi_1(C, t)$ acts on $F$ via $\rho$. The action is free, so $X$ has the same singularity as $F$. The natural projection

$$f : X \to C$$

is an isotrivial fibration with fibre isomorphic to $F$.

Let us show that $X$ is projective. The base $C$ is projective and the fibres are all projective, so first we can embed $X$ in a projective bundle $P$ over $C$ by the relative anticanonical bundle. Take a relatively ample line bundle over $P$, add the pull-back of an ample divisor on $C$. This line bundle will embed $X$ in a projective space.

To prove that

$$f : X \to C$$

is a MFS we need to show that the relative Picard number is 1. The Picard number of $C$ is 1, so we have to show that the Picard number of $X$ is 2. We consider the sequence

$$0 \to \text{NS}(C)_{\mathbb{Q}} \xrightarrow{f^*} \text{NS}(X)_{\mathbb{Q}} \xrightarrow{f^*} \text{NS}(F)_{\mathbb{Q}} \to 0.$$
If this sequence is exact, then the Picard number of \( X \) is 2 for dimensional reasons. Let us show the exactness.

The injectivity of \( f^* \) follows by the connectedness of the fibres and projection formula. The vector space \( \text{NS}(\mathbb{F}_t)_{\mathbb{Q}} \) is generated by \(-K_{\mathbb{F}_t}\). By adjunction \( \iota^* K_X = K_{\mathbb{F}_t} \), so \( \iota^* \) is surjective. We have \( \text{im} f^* \subseteq \ker \rho \). We need to show the remaining inclusion, which requires some care.

Pick a line bundle \( L \) on \( X \) such that \( \iota^* L = 0 \). Let \( q \) be a point of \( C \) different from \( t \). We want to show that \( L \) restricted to \( \mathbb{F}_q \) is trivial. First, we show that the Chern class \( c_1(L) \) is a section of the locally constant sheaf \( R^2 f_* \mathbb{Q} \); if it is zero on a neighbourhood of \( t \) then it is zero everywhere. Let \( U \) be a contractible neighbourhood of \( t \). The open subset \( V := f^{-1}U \) is a contractible neighbourhood of \( \mathbb{F}_t \). So \( c_1(L|_V) = 0 \). Every fibre is Fano. On a Fano variety a line bundle with trivial Chern class is trivial. Remark that we are working over \( \mathbb{Q} \). If we were working over \( \mathbb{Z} \) we should have taken care of the torsion.

We are now in the following situation: we have a flat morphism \( f: X \to C \) and a line bundle \( L \) on \( X \) which is trivial on every fibre. We want to show that it is the pull-back of a line bundle on \( C \). This is the so called see-saw principle (cf. [Mum08, Corollary 6, p. 54] or [KM92, Proposition 12.1.4]).

This concludes the proof. \( \square \)

We state now a necessary criterion for \( F \) to be fibre-like.

**Theorem 3.2** (Necessary Criterion). A Fano variety such that
\[
\dim \text{NS}(F)_{\mathbb{Q}}^{\text{Mon}(F)} > 1
\]
is not fibre-like.

**Proof.** We argue by contradiction. Let
\[
f: X \to Y
\]
be a Mori fibre space. By Theorem 2.13, there exists an open dense subset \( U = U_f \) of \( Y \) such that the map
\[
\rho: \text{NS}(X)_{\mathbb{Q}} \to \text{NS}(F)_{\mathbb{Q}}^{\pi_1(U_f)}
\]
is surjective, where \( F_f \) is a fibre over \( U \) isomorphic to \( F \). Let us show that \( \text{NS}(F)_{\mathbb{Q}}^{\pi_1(U_f)} \) is one dimensional. We first prove that sequence
\[
0 \to \text{NS}(Y)_{\mathbb{Q}} \xrightarrow{f^*} \text{NS}(X)_{\mathbb{Q}} \xrightarrow{\rho} \text{NS}(F)_{\mathbb{Q}}^{\pi_1(U_f)} \to 0.
\]
is exact. Since \( F \) is connected, the map \( f^* \) is injective on \( \text{NS}(Y)_{\mathbb{Q}} \). The inclusion \( \text{im} f^* \subseteq \ker \rho \) is standard. The map \( \rho \) is not the zero map. Now, we use that we are dealing with a Mori fibre space. The relative Picard number is one, so:
\[
\dim \text{NS}(X)_{\mathbb{Q}} = \dim \text{NS}(Y)_{\mathbb{Q}} + 1.
\]
We conclude that, for dimensional reasons, the sequence is exact and
\[
\dim \text{NS}(F)_{\mathbb{Q}}^{\pi_1(U_f)} = 1
\]
By Theorem 2.13, the monodromy action factors through \( \text{Mon}(F) \), so
\[
\text{NS}(F)_{\mathbb{Q}}^{\text{Mon}(F)} = \mathbb{Q} K_F.
\]
This contradicts our hypothesis. □

In the next section we will introduce a more handy versions of this criterion. Let us finish this section by considering the rigid case.

**Theorem 3.3 (Characterisation - Rigid case).** A rigid Fano variety is fibre-like if and only if

\[ \text{NS}(F)_{\mathbb{Q}}^{\text{Aut}(F)} = \mathbb{Q}K_F \]

In this case, an isotrivial MFS over a curve is constructed.

**Proof.** The “if” part is Theorem 3.1. The “only if” part follows from Theorem 3.2: just remark that if \( F \) is rigid the monodromy action factors through \( \text{HMon}(F) \) (cf. Theorem 2.13). □

If \( F \) is not rigid this characterisation is false. A counterexample is the del Pezzo surface of degree 3 (see Section 4).

### 3.2. Applications of the Necessary Criterion.

The group \( \text{Mon}(F) \), which was defined in 2.10, is in general difficult to describe. Roughly speaking, it can be thought as the group of symmetries of the nef cone preserving the birational geometry features. Taking this point of view, we can rephrase this criterion in terms of the birational geometry of \( F \). The idea is that, if the faces of the nef cone are different from the view point of birational geometry, then \( \text{NS}(F)_{\mathbb{Q}}^{\text{Mon}(F)} \) must be big. Let us give an easy example. Assume that \( F \) has Picard number 2. The nef cone has two faces, \( G_1 \) and \( G_2 \). Each face gives a contraction \( \pi_i : F \to G_i \).

**Corollary 3.4.** Keep notations as above. If 

\[ \dim G_1 \neq \dim G_2 \]

then \( F \) can not be a general fibre of a Mori fibre space.

**Proof.** The group \( \text{Mon}(F) \) can not exchange \( G_1 \) and \( G_2 \), so it is trivial. □

Case by case, one can cook up more refined versions of this corollary.

Let us give more examples.

**Corollary 3.5.** Let \( F \) be a Fano variety that is isomorphic in a unique way to the blowup of another deformation type of Fano variety \( G \). Then, \( F \) cannot be a fibre-like Fano.

In the statement of the proposition, the most important word is “unique”. In fact, the type of a primitive contraction and the deformation type of its image are preserved under the action of \( \text{Mon}(F) \). Hence, the uniqueness implies that the map must be preserved by such action.

**Proof of Corollary 3.5.** As we just pointed out, the face of \( \text{Nef}(F) \) corresponding to the pullback of \( \text{Nef}(G) \) is invariant by \( \text{Mon}(F) \). Hence, it is enough to show that on such face there is a fixed point and, consequently, a fixed one-dimensional subspace. As, in order to be a fibre-like Fano, the only subspace preserved by \( \text{Mon}(F) \) could be the span of \( K_F \) and this does not lay on the pullback of \( \text{Nef}(G) \), the required contradiction is immediate. As we explained above, if \( \text{Nef}(G) \) is stable by \( \text{Mon}(F) \), then the class of the exceptional divisor is fixed as well. □
The above criterion can be generalised quite easily. Let us explain how, by means of some examples.

**Example 3.6.** Let $F$ be a Fano manifold that possesses a unique Mori fibre contraction to a variety $G$, with $\dim G = k$. Then the face of the nef cone of $F$ corresponding to the nef cone of $G$ is stable under $\text{Mon}(F)$. In particular, the primitive generators of the extremal rays (in the lattice $\mathbb{N}^1(F) \subset \mathbb{N}^1_\mathbb{R}(F)$) of such a face are going to be permuted by $\text{Mon}(F)$. In particular their sum will be $\text{Mon}(F)$-invariant. Hence, $F$ cannot be fibre-like. This is the case, for example, of the projectivisation $F$ of the vector bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ over $F' := \mathbb{P}^1 \times \mathbb{P}^1$. Recall that $F$ is isomorphic to the blow-up of the cone over a smooth quadric in $\mathbb{P}^3$ with center the vertex. $\rho(F) = 3$ and the facets of $\text{Nef}(F)$ are given by the Mori Fibre contraction $F \to \mathbb{P}^1 \times \mathbb{P}^1$ and the two small contractions $F \to F_i$, $i = 1, 2$, given by contracting the two rulings of the exceptional copy of $\mathbb{P}^1 \times \mathbb{P}^1$.

The above analysis can be formalised into the following statement.

**Corollary 3.7.** Let $F$ be a Fano variety and assume that the nef cone of $F$ contains a facet $L$ corresponding to a certain variety $G$. Assume that for any other facet $H$ of the nef cone, the corresponding variety $H$ is not deformation equivalent to $G$. Then, $F$ cannot be a fibre-like Fano.

So far we have dealt with the case of a facet globally fixed by $\text{Mon}(F)$. What happens to facets that are translated around the nef cone?

Let $F$ be a Fano variety and $\mathcal{F}$ be a facet in the nef cone of $F$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be the facets corresponding to translates of $\mathcal{F}$ under $\text{Mon}(F)$. Again, for each of the facets $\mathcal{F}, \mathcal{F}_1, \ldots, \mathcal{F}_k$, let $L, L_1, \ldots, L_k$ be the sum of the extremal rays spanning the facet. It is clear, by our notation, and the finiteness of the $\text{Mon}(F)$-action that the $L_i$’s constitute the orbit of $L$ under the $\text{Mon}(F)$-action. Hence, $L + L_1 + \cdots + L_k$ is $\text{Mon}(F)$-invariant and, in order for $F$ to be fibre-like, it has to be a negative multiple of $K_F$, in particular it has to be ample.

When $\mathcal{F}$ corresponds to a divisorial contraction, then the same reasoning applies to show that the sum $E + E_1 + \cdots + E_k$ of the exceptional divisors relative to the different facets must be a multiple of $-K_F$, otherwise $F$ will not be fibre-like.

**Example 3.8.** Let $F$ be the blow-up of the cone over a smooth quadric $Q \subset \mathbb{P}^3$ with center the vertex and an elliptic curve $C$ on $Q$. There is a map $\pi : F \to Q$. The generic fibre of $\pi$ is $\mathbb{P}^1$, but over $C \subset Q$ the fibre is a chain of two $\mathbb{P}^1$’s intersecting in a point. $F$ has only two different divisorial contractions $\psi_i : F \to \mathbb{P}^1_q(\mathcal{O}_{\mathbb{P}^1_q} \oplus \mathcal{O}_{\mathbb{P}^1_q}(1, 1))$, $i = 1, 2$ given by contracting the two components of the fibres of $\pi$ over $C$. Hence, for $F$ to be fibre-like, the sum of the two exceptional divisors $E_1 + E_2$ must be ample. But this is not possible as $E_1 + E_2$ has intersection 0 with the fibres of $\pi$.

These criteria are enough to deal with most of the 3 dimensional Fano varieties (cf. Section 5). These kind of argument is also applied to flag varieties in Section 7.

4. The del Pezzo case

In this sections we focus on two-dimensional smooth Fano varieties. This case was studied in [Mor82]. We provide a proof using explicit constructions of Mori fibre spaces and applying our criteria. We start with some notation.
Definition 4.1. A del Pezzo surface $S$ is a smooth Fano variety of dimension two. A del Pezzo surface of degree $d := (K_S)^2$ obtained as the blow-up of $\mathbb{P}^2$ in $9 - d$ general points is denoted by $S_d$ ($1 \leq d \leq 8$).

We refer to [Kol96, Section III.3], for the general theory of del Pezzo surfaces. Combining Theorems 4.4, 4.7 and 4.8 we obtain the following result.

Theorem 4.2 (Mori). A del Pezzo surface $S$ is fibre-like if and only if it is isomorphic to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $S_d$, with $d \leq 6$.

By comparing our result with the classification of $K$-stable smooth del Pezzo surfaces, we obtain the following corollary.

Corollary 4.3. A smooth del Pezzo surface is $K$-stable if and only if it is fibre-like.

4.1. Ad hoc constructions. Before applying our criteria to surfaces, we give some explicit constructions of Mori fibre spaces whose general fibre is a del Pezzo of low degree ($1 \leq d \leq 4$).

Theorem 4.4. The following del Pezzo surfaces are fibre-like:

- $\mathbb{P}^2$,
- $\mathbb{P}^1 \times \mathbb{P}^1$,
- $S_d$, with $d = 3, 4$;
- the general $S_d$, with $d = 1, 2$.

The 2-dimensional projective space has a trivial structure of Mori fibre space: it is enough to consider the morphism $\mathbb{P}^2 \to \{pt\}$.

In the case of $\mathbb{P}^1 \times \mathbb{P}^1$, we consider a smooth hypersurface $X_2$ of degree $(2, m)$ with $m \geq 1$ in $\mathbb{P}^3 \times \mathbb{P}^1$. This gives a fibration in quadrics $f_2: X_2 \to \mathbb{P}^1$. Assuming $m \geq 1$, the ampleness of $X_2$ is guaranteed and we can apply Lefschetz hyperplane theorem to deduce that $\rho(X_2) = 2$. This implies that $f_2$ is a Mori fibre space, with the generic fibre isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This MFS is isotrivial over the open set over which $f_2$ is a submersion.

As a second example, we consider the case of cubics in $\mathbb{P}^3$. Fix a del Pezzo $S$ of degree 3. Consider a smooth hypersurface $X_3$ of degree $(3, m)$, with $m \geq 1$, in $\mathbb{P}^3 \times \mathbb{P}^1$. This gives a fibration in degree 3 del Pezzos $f_3: X_3 \to \mathbb{P}^1$. We can furthermore assume that one of the fibre is isomorphic to $S$. Arguing as before, we can show that this gives an example of MFS with a general fibre isomorphic to $S$. Remark that this MFS is far from being isotrivial, indeed there are just finitely many fibres isomorphic to $S$. The general fibre will be isomorphic to $S$ just as a topological space. Nonetheless, there will be finitely many singular fibres not even homeomorphic to $S$.

Let us now recall the following classical description (cf. [Kol96, Theorem III.3.5]).

Theorem 4.5. Every del Pezzo of degree $d$ can be realised as

- a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,2,3)$ if $d = 1$;
- a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1,1,1,2)$ if $d = 2$;
- a hypersurface of degree 3 in $\mathbb{P}^3$, if $d = 3$;
- a complete intersection of degree $(2,2)$ in $\mathbb{P}^4$ if $d = 4$. 

The degree 4 del Pezzo can be treated as the degree 3. With the other 2 cases there is an issue: classical Lefschetz hyperplane theorem does not hold, because the ambient space is not smooth. We can use the following generalisation (cf. [RS06]).

**Theorem 4.6** (Grothendieck-Lefschetz hyperplane theorem). *Let $Z$ be a normal irreducible projective variety and let $|H|$ be an ample base point free linear system on $Z$. Then for a general $X \in |H|$, the restriction map $\text{Cl}(Z) \to \text{Cl}(X)$ is an isomorphism, if $\dim Z \geq 4$.\*

We can now carry out again the same construction, but we need a genericity assumption on the surface (we think we could get rid of it by careful analysis of the singularities).

This method does not help for higher dimensional Fano varieties: if a Fano variety $F$ of dimension strictly greater than 2 is a complete intersection in an ambient space of Picard number one, then also the Picard number of $F$ is one. This already means that $F$ is fibre-like, so there is nothing to prove. However, this strategy could be used to study singular del Pezzo surfaces.

### 4.2. The general approach.

In this section we apply our criteria to del Pezzo surfaces.

**Theorem 4.7.** *The del Pezzo surfaces $S_7$ and $S_8$ are not fibre-like.*

**Proof.** In both cases the del Pezzo is rigid because a configuration of one or two points in $\mathbb{P}^2$ is unique up to projectivity, so we can use Theorem 3.3.

Let us first consider the blow up at one point. The automorphism group is given by all the projectivities which fix a marked point. This group preserves both the line bundle given by the hyperplane coming from $\mathbb{P}^2$ and the exceptional divisor.

Now we consider the blow up at two points. The automorphism group is given by all the projectivities which fix the marked points or exchange them. This group preserves the line bundle given by the hyperplane coming from $\mathbb{P}^2$ and the sum of the two exceptional divisors. \hfill $\square$

On the positive side, our sufficient criterion gives the following.

**Theorem 4.8.** *The following del Pezzo surfaces are fibre-like:*

- *(rigid case)* $\mathbb{P}^1 \times \mathbb{P}^1$ and $S_d$ with $d = 5, 6$;
- *(moduli case)* $S_d$ with $d = 1, 2, 4$.

**Proof.** Automorphisms groups of 2-dimensional Fanos have been widely studied (cf. [Koi88] and [DI09]), so we can easily apply Theorem 3.1.

$\mathbb{P}^1 \times \mathbb{P}^1$. The natural involution of the product $\mathbb{P}^1 \times \mathbb{P}^1$ guarantees that the invariant part of the Néron-Severi group has dimension 1.

$d = 1, 2$. Every del Pezzo surface obtained as the blow-up of $\mathbb{P}^2$ in 8 or 7 points comes with an involution: respectively the Bertini ($d = 1$) and Geiser ($d = 2$) involutions, denoted by $\iota_1$ and $\iota_2$, which emerge when realising $S_d$ as a 2 : 1 cover of $\mathbb{P}(1, 1, 2)$ (resp. $\mathbb{P}^2$) if $d = 1$ (resp. $d = 2$). The automorphism groups of these del Pezzos can be much more complicated, but these two involutions are enough to minimise the invariant Picard rank of $S_i$: for $i = 1, 2$ we have $\rho(S_i)^{\iota_i} = 1$.\*
The automorphism group of all the del Pezzo surfaces in this class contains a subgroup isomorphic to $\mathbb{Z}_4^2$. The quotient is the projective plane, so the invariant part of the Picard group is one dimensional.

$d = 5, 6$. The automorphism group of every del Pezzo of degree 5 is isomorphic to $S_5$; while $\text{Aut}(S_6) \cong \mathbb{P}GL(3; P_1, P_2, P_3) \rtimes \mathbb{Z}_2$. In particular, in both cases we can permute the exceptional divisors using projectivities from $\mathbb{P}^2$. The Cremona involution centred at 3 blown-up point is a regular automorphism of the surface. This Cremona transformation does not fix the sum of exceptional divisors, so the invariant part of the Picard group is one dimensional.

The generic del Pezzo of degree 3 has no non-trivial automorphisms (cf. [Seg42]). Because of this, we can not apply Theorem 3.1. Nonetheless, we showed in Corollary 4.4 that $S_3$ is fibre-like.

5. Fano threefolds

The results explained in the previous sections show that there are quite a few restrictions on the geometry of a smooth Fano variety $F$ to be fibre-like. We are interested in understanding how strong these restrictions are. As vague as this question may appear, drawing on the classification of smooth Fano threefolds due to Mori and Mukai (cf. [MM82] and [MM03]), we are able to show that in this context most threefolds cannot satisfy these rigid conditions.

We will refer to [MM82, Tables 2, 3, 4, 5] where a full description of the deformations type of Fano threefolds is given.

**Theorem 5.1.** Let $F$ be a smooth Fano threefold. Then, if $F$ is fibre-like, the deformation type of $F$ is one of the following:

| $n$ | $\rho(F)$ | $-K_F^3$ | Deformation type of $F$ |
|-----|------------|-----------|-------------------------|
| 1   | 2          | 12        | $F$ is a divisor of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. |
| 2   | 2          | 20        | $F$ is the blow-up of $\mathbb{P}^3$ with center a curve of degree 6 and genus 3 which is an intersection of cubics. Alternatively, $F$ is the intersection of three divisors of bidegree $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$. |
| 3   | 2          | 28        | $F$ is the blow-up of $Q \subset \mathbb{P}^4$ with center a twisted quartic, a smooth rational curve of degree 4 which spans $\mathbb{P}^4$. |
| 4   | 3          | 48        | $F$ is a divisor of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. |
| 5   | 3          | 12        | $F$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose branch locus is a divisor of tridegree $(1,1,1)$. |
| 6   | 3          | 30        | $F$ is the blowup of a smooth divisor of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ with center a curve $C$ of bidegree $(2,2)$ on it, such that $C \hookrightarrow W \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is an embedding for both both projections $\mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$. |
| 7   | 3          | 48        | $F = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. |
| 8   | 4          | 24        | $F$ is a divisor of multi degree $(1,1,1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. |

Moreover, varieties of deformation type no. 1, 4, 7 and 8 are fibre-like.
For the reader’s convenience, we will divide our analysis based on the Picard number of the Fano threefolds that we take into exam.

**Remark 5.2.** Let us remind the reader that the last column in each table presented in [MM82] enumerates all the possible ways a Fano threefold can be obtained from another Fano threefold by blowing up a curve. Alternatively, in the language of this section, all the facets of the nef cone corresponding to a divisorial contraction, in which the image of the exceptional divisor is a curve. Let us remark here that the facets are listed without taking into account the action of \( \text{Mon} \).

\( \rho(F) = 2 \). The nef cone cone of a Fano variety \( F \) of Picard number 2 is a rational polyhedral cone of the form \( \mathbb{R}_+ D_1 + \mathbb{R}_+ D_2 \), for \( D_1, D_2 \) two nef, semiample (integral) Cartier divisors on \( F \). In this representation, we always assume that the classes of the \( D_i \)'s are primitive in the Néron-Severi group.

**Remark 5.3.** As the nef cone is \( \text{Mon}(F) \)-invariant, and the only invariant subspace is the span of the anticanonical class, it follows that the sum of the primitive generators of the nef cone must be a multiple of the canonical class. In particular, when \( \rho(F) = 2 \), the span of the canonical class is simply the bisector of the nef cone, i.e.

\[
\lambda K_F \sim D_1 + D_2
\]

This is another useful condition: e.g. a divisor of type \((1, 2)\) contained in \( \mathbb{P}^2 \times \mathbb{P}^2 \) cannot be a general fibre in a Mori fibre space.

With reference to Table 2 of [MM82], using Corollary 3.7, we can immediately exclude the families corresponding to the following entries of the table:

- (1), (2), (3), (4), (5), (7), (8), (9), (10), (11), (13), (14),
- (15), (16), (17), (18), (19), (20), (22), (23), (25), (26),
- (27), (28), (29), (30), (31), (33), (34), (35), (36).

Using Remark 5.3, we can also exclude entry (24).

The variety corresponding to entry (32), i.e. a divisor of type \((1, 1)\) in \( \mathbb{P}^2 \times \mathbb{P}^2 \) (alternatively described as \( \mathbb{P}(\Omega_{\mathbb{P}^2}) \)) is of fibre-like type. In fact, up to the action of the automorphisms of \( \mathbb{P}^2 \times \mathbb{P}^2 \), such a Fano variety can be identified with the incidence correspondence

\[
\{(p, l) \mid p \in \mathbb{P}^2, l \in \mathbb{P}^2^*, p \in l\}.
\]

This is the flag variety of \( SL_3(C) \), which will be discussed in Proposition 7.1.

We consider divisors of bidegree \((2, 2)\) in \( \mathbb{P}^2 \times \mathbb{P}^2 \). To show that they are fibre-like, we construct some explicit Mori fibre spaces. Let \( \sigma \) be the involution of \( \mathbb{P}^2 \times \mathbb{P}^2 \) and \( \mathbb{P}^N := \mathbb{P}H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(2, 2)) \). Consider the incidence variety \( Z \) in \( \mathbb{P}^N \times \mathbb{P}^2 \times \mathbb{P}^2 \); it is a smooth divisor of degree \((1, 2, 2)\). We can apply Lefschetz’s Theorem to show that the Picard number of \( Z \) is 3. We have fibration a

\[
\pi: Z \to \mathbb{P}^n,
\]

whose relative Picard number is 2. The involution \( \sigma \) acts on this fibration. By taking the quotient, the relative Picard number decrease by one, so we obtain a Mori fibre space

\[
f: X \to Y.
\]

The singularities are finite quotient singularities, so certainly normal, \( \mathbb{Q} \)-factorial and rational. If \( F \) is a \((2, 2)\)-divisor which is not preserved by the involution, then
the action of $\sigma$ is free on a neighbourhood of $F$ in $Z$, so $F$ is a smooth fibre of $f$. If $F$ is preserved by $\sigma$, we conjugate $\sigma$ by a generic automorphism of $\mathbb{P}^2 \times \mathbb{P}^2$. In this way, we obtain a new involution $\sigma'$ which does not preserves $F$. We can now perform the same construction using $\sigma'$. Enhancing the notations, this argument proves the following result.

**Theorem 5.4.** Take positive integers $r, k, d$, and $n$ such that $kd < n+1$. Let $F$ be a smooth complete intersection of $k$ divisors of multi-degree $(d, \cdots, d)$ in the product of $r$ copies of $\mathbb{P}^n$. Then, $F$ is fibre-like.

Remark that $F$ has dimension $rn - k$ and Picard number $r$.

$\rho(X) = 3$. With reference to Table 3 of [MM82], using Corollary 3.7, we can immediately exclude the families corresponding to the following entries of the table:

(2), (4), (5), (6), (7), (8), (11), (12), (14), (15), (16),
(18), (20), (21), (22), (23), (24), (26), (28), (29), (30), (31).

**Remark 5.5.** Let $F$ be a Fano variety of Picard number 3. Suppose that the nef cone contains two facets for which the images of the corresponding contraction morphisms are deformation equivalent. Then, these may be identified by the action of $\text{Mon}(F)$. In particular, the primitive generators of the two facets are exchanged and their sum is then invariant. Hence it has to belong to the span of the canonical class, if $F$ is of fibre-like type.

When the two facets correspond to divisorial contractions, then the same holds true for the sum of the two exceptional divisors $E_i$, with $i = 1, 2$. In particular, $E_1 + E_2$ has to be ample.

So, using the previous remark, the following entries can be shown not to be of fibre-like type:

(3), (9), (10), (17), (19), (25).

$\rho(X) = 4$. With reference to Table 4 of [MM82] and using Corollary 3.7, it is immediate to see that we can exclude the families corresponding to the following entries of the table:

(3), (4), (5), (6), (8), (9), (10), (11).

Using the natural generalisation to Picard number 4 of Remark 5.5, we can exclude the following entries, too:

(2), (7), (12).

Divisors of multi degree $(1, 1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are treated in a similar way of divisors of degree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

$\rho(X) \geq 5$. In this case the only Fano threefolds are the following.

- $F$ is the blow-up of $Y$ - the blow up of a quadric $Q \subset \mathbb{P}^3$ along a conic on it - with center three exceptional lines of the blowing up $Y \to Q$; then the sums of the three exceptional divisors over the lines must be a (negative) multiple of $K_X$, in particular it has to be ample, which is clearly not true, as one can see by taking an exceptional line for the map $Y \to Q$ other than those already blown up.
• $F$ is the blow-up of $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1))$ with center two exceptional lines $l_1, l_2$ of the blow-up $\phi : Y \to \mathbb{P}^3$ such that $l_1$ and $l_2$ lie on the same irreducible component of the exceptional set of $\phi$; such $F$ is not of fibre-type by Proposition 3.5.

• Products

\[ \mathbb{P}^1 \times S_d, \ d \leq 6, \]

where $S_d$ is a del Pezzo of degree $d$. A quick analysis shows immediately that the projection onto the second factor must be $\text{Mon}(S_d)$-invariant as $\text{Nef}(\mathbb{P}^1 \times S_d) = \text{Nef}(\mathbb{P}^1) \times \text{Nef}(S_d)$. In particular, the projection onto the second factor $\mathbb{P}^1 \times S_d \to S_d$ is $\text{Mon}(S_d)$ invariant.

6. The Toric case

In this section we will deal with toric Gorenstein and $\mathbb{Q}$-factorial Fano varieties. We start recalling some notation and basic facts. For more details, see [CLS11], [Bat91] and [Cas03].

Let $N$ be a free abelian group of rank $n$ and set $N_\mathbb{Q} := N \otimes \mathbb{Q}$. Let $\Sigma \subset N_\mathbb{Q}$ be a fan of a $d$-dimensional toric Fano variety $F$ and let $\Delta$ be the polytope associated to the anti-canonical polarisation. Furthermore, $M$ will denote the dual of $N$.

The vertices of $\Delta$, denoted by $V(\Delta)$, are the generators of $\Sigma$. We denote by $Z(\sigma)$ the closure of the orbit corresponding to $\sigma \in \Sigma$, which is an irreducible invariant subvariety.

Let $A_1$ be the group of 1-cycles on $F$ modulo numerical equivalence and set $N_1 = A_1 \otimes \mathbb{Q}$. In $N_1$ we consider the Kleiman-Mori cone $\text{NE}(F)$ generated by the effective 1-cycles. There is the following basic exact sequence:

\begin{equation}
0 \to A_1(F) \to \mathbb{Z}^{V(\Delta)} \to N \to 0
\end{equation}

and dually

\begin{equation}
0 \to M \to \mathbb{Z}^{V(\Delta)} \to \text{NS}(F) \to 0.
\end{equation}

6.1. The General Criterion for Toric Fanos. In this subsection we apply our criteria to the toric case: Theorem 3.3 turns into a combinatorial characterisation on the polytope of the toric Fano variety. We start with a standard definition, keeping the same notation of the previous subsection.

Definition 6.1 (Vertex-transitive). A polytope $\Delta$ is called vertex-transitive if $\text{Aut}(\Delta)$ acts transitively on the vertices of $\Delta$. If $\Delta$ is associated to a toric Fano variety $F$, then $F$ is also called vertex-transitive.

Theorem 6.2 (Characterisation - Toric case). Let $F := F(\Delta)$ be a toric $\mathbb{Q}$-factorial Fano variety with terminal singularities. Assume that $\rho(F) \geq 2$. Then, $F$ is fibre-like if and only if it is vertex-transitive.

Remark 6.3. (Picard rank 1 and smooth case). It is clear that all Fano varieties with one-dimensional Néron-Severi are fibre-like. The only $d$-dimensional smooth Fano variety with $\rho = 1$ is the projective space, so our theorem can be stated in the smooth case without any assumption on the Picard rank. Nonetheless, the weighted projective space $\mathbb{P}(1,1,1,1,2)$ is Gorenstein, $\mathbb{Q}$-factorial, terminal and with Picard number one, but not vertex-transitive.
Proof. Any terminal Fano toric variety is rigid (cf. [dFH12, Corollary 4.6]), so we can apply Theorem 3.3. We have to show that $F$ is vertex-transitive if and only if $\dim \text{NS}(F)^G_Q = 1$. After tensoring by $Q$ the exact sequence (7), we obtain

$$0 \to M_Q \to Q^{V(\Delta)} \to \text{NS}(F(\Delta))_Q \to 0,$$

where $M_Q$ is the $d + 1$-dimensional $Q$-vector space containing the dual polytope of $\Delta$. The action of $\text{Aut}(F)$ on $\text{NS}(F)_Q$ can be identified with the action of $\text{Aut}(\Delta)$ on $M_Q$ and $Q^{V(\Delta)}$, and the previous sequence is equivariant (cf. [Cox95, Corollary 4.7]). We now make the following general observation.

**Lemma 6.4.** Let $G$ be a finite group and $S$ a finite $G$-set. Call $t$ the number of orbits of the action of $G$ on $S$. Assume that an exact sequence of $G$-modules over $Q$ is given:

$$0 \to A \to Q^S \to B \to 0.$$

Then

$$\dim B^G = t - \dim A^G.$$

Let now $G$ be $\text{Aut}(\Delta)$ and $t$ be the number of orbit of the action of $G$ on $V(\Delta)$. If $\dim M_Q^G = 0$, the result follows from the previous discussion. If this is not the case, we have to understand this invariant space.

Let $N_Q$ be the dual of $M_Q$, which contains $\Delta$. The following is the key proposition we need.

**Proposition 6.5.** Keep notations as above, in particular, assume $\rho(F) \geq 2$. Then, if $N_Q^G$ is not trivial, $G$ has at least $\dim N_Q^G + 2$ orbits on $V$.

Let us assume the proposition and finish the proof of the theorem. A representation of a finite group over $Q$ is isomorphic to its dual, so we have

$$\dim N_Q^G = \dim M_Q^G.$$ 

We want to show that if $M_Q^G$ is not trivial, then $F$ is neither vertex transitive nor fibre-like. The first statement follows directly from the proposition. For the second, remark that

$$\dim \text{NS}(F)^G_Q = t - \dim M_Q^G \geq 2$$

where the inequality again follows from the proposition. □

**Proof of Proposition 6.5.** Given a (possibly not maximal) face $F$ of $\Delta$ we denote by $V(F)$ its vertices. We think $N_Q^G$ as an affine space rather than a vector space. We proceed by induction on the dimension $k$ of $N_Q^G$.

Let us first assume that $N_Q^G$ is a line $L$ (i.e. $k = 1$). Then $L$ meets two distinct (possibly not maximal) faces $F_1$ and $F_2$ of $\Delta$. We obtain four invariant sets:

$$V(F_1) \setminus V(F_2), \quad V(F_2) \setminus V(F_1), \quad V(F_1 \cap F_2) \quad \text{and} \quad V \setminus (V(F_1) \cup V(F_2)).$$

Since we need to obtain three orbits, the only problematic case arises when both $V(F_1 \cap F_2)$ and $V \setminus (V(F_1) \cup V(F_2))$ are empty: we are going to show that at least one between $V(F_1)$ and $V(F_2)$ split in more than one orbit.

We use the following notation: $\# \{V(F_1)\} := m_1$ and $\# \{V(F_2)\} := m_2$, with $m_1 \geq m_2$, $H_1 \supset F_1$ and $H_2 \supset F_2$ the affine subspaces containing $F_1$ and $F_2$.

We know that $\dim H_i = m_i - 1$. We can assume that $m_1 + m_2 \geq d + 2$ because $\rho(F(\Delta)) \geq 2$. We consider now the affine space $T$ generated by $H_2$ and $L$. Using the Grassmann’s formula we deduce that $\dim(T \cap H_1) \geq 1$. 

Assume now that \( m_1 > m_2 \). Then the space \( T \) cannot contain the whole \( H_1 \). Moreover \( T \) meets two distinct faces of \( \mathcal{F}_1 \). This one is itself a simplex, so these two faces have non-empty intersection \( I \). The space \( T \) is globally (but not point-wise) fixed, so \( V(\mathcal{F}_1) \) splits in at least two orbits: \( V(I) \) and \( V(\mathcal{F}_1 \setminus I) \).

Assume now that \( m_1 = m_2 < d \). If the space \( T \) does not contain \( H_1 \), we can repeat the previous argument. If \( T \) contains \( H_1 \), we remark that the whole polytope is contained in \( T \). Contradiction.

If \( m_1 = m_2 = d \), let us assume by contradiction that we have 2 orbits. In this case the invariant part of the \( \overline{\text{NE}}(F(\Delta)) \) is just the ray of \(-K_{F(\Delta)}\). Since the image of all the vertices of an orbit is numerically equivalent to \(-\frac{1}{d}K_{F(\Delta)}\), we deduce that the index of our Fano variety is at least 2. A \( d \)-dimensional Fano variety with index 2 and Picard number \( d \) is, by a conjecture of Mukai proven in [Cas06], isomorphic to \((\mathbb{P}^1)^d\). The automorphism group of the polytope of \((\mathbb{P}^1)^d\) acts transitively on the vertices, so we have a contradiction. We have proved the first step of induction. Furthermore, in the same way one can see that the action of \( G \) on every simplicial polytope in \( N_\mathbb{Q} \) such that \( \dim N_\mathbb{Q}^G = 1 \) has at least 2 orbits.

For the inductive step we now assume the following: the action of \( G \) on every simplicial polytope in \( N_\mathbb{Q} \) such that \( \dim N_\mathbb{Q}^G = (k - 1) \) has at least \( k \) orbits.

Let \( N_\mathbb{Q}^G \) be a \( k \)-dimensional vector space \( H \). The intersection of \( H \) with \( \Delta \) is a \( (k - 1) \)-polytope. It intersects the interior of two distinct faces \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of \( \Delta \) of dimension at least \( k - 1 \).

First let us assume that the two faces have both dimension \( k - 1 \). This implies that \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) is fully contained in \( H \): this provides \( k \) orbits (= \#\{\( V(\mathcal{F}_1) \)\}). Another orbit is given by \( V(\mathcal{F}_2) \setminus V(\mathcal{F}_1) \). Since both faces are contained in \( H \), this implies the existence of an extra vertex of \( \Delta \) not contained in \( H \): this belongs to another orbit.

We now assume that \( \mathcal{F}_1 \) has dimension \( \text{at least} \ k \) and that \( H \) intersects it in its interior (otherwise, we can choose another face of strictly smaller dimension). By inductive hypothesis, we obtain \( k \) orbits. Since \( H \) intersects this face in its interior, it meets at least a facet \( \mathfrak{f} \) of \( \mathcal{F}_1 \): an extra orbit is then obtained choosing \( \mathcal{F}_2 \) to be the adjacent to \( \mathcal{F}_1 \) via \( \mathfrak{f} \). The only case in which \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) saturate the vertices of \( \Delta \) is when the Picard number is one, which is in contradiction with our assumptions and so we can find an extra vertex of \( \Delta \) which belongs to another orbit. The induction is proved.

\[ \square \]

So, our original problem is reduced, in the toric case, to the classification of vertex-transitive Fano varieties. We will start investigating these special varieties in the following subsection: as expected, it is a rather restrictive condition.

Furthermore, it is well-known that the symmetry of \( \Delta \) is strictly related to the \( K \)-stability, which is known to be equivalent to the existence of Kähler-Einstein metric (cf. [WZ04] and [BB13]) of the associated Fano variety. Mabuchi proved in [Mab87] the first result relating the \( K \)-stability with the triviality of the barycentre of \( \Delta \). The generality we need is achieved in the following theorem.

**Theorem 6.6 ([Ber12, Cor. 1.2]).** Let \( F(\Delta) \) be a Gorenstein, \( Q \)-factorial Fano toric variety. Then \( F \) is \( K \)-stable if and only if the barycentre of \( \Delta \) is the origin.
The proof of the previous result is analytic and passes through the existence of Kähler-Einstein metrics. In our setting, we can deduce the following necessary condition.

**Corollary 6.7.** Every smooth toric fibre-like Fano variety \( F(\Delta) \) is \( K \)-stable.

**Proof.** We just apply Theorem 6.2, since the barycentre of a vertex-transitive polytope is the origin. The corollary is now a consequence of Theorem 6.6. □

Let us remark that there are many smooth toric Fanos which are \( K \)-stable but not fibre-like. The previous corollary seems the relative version in the toric case of the following very general conjecture by Odaka and Okada.

**Conjecture 6.8** ([\( \text{OO13, Conj. 5.1} \)]). Any smooth Fano manifold \( X \) of Picard rank 1 is \( K \)-semistable.

### 6.2. Vertex-transitive Polytopes

In this subsection we investigate vertex-transitive Fano polytopes, which naturally appeared in the previous subsection. Our ambitious ultimate goal would be to classify these polytopes: the transitivity condition seems rather restrictive. We can currently obtain some partial results in this direction, which are enough to classify low dimensional vertex-transitive Fanos.

From now on, \( \Delta \) will denote a smooth Fano polytope of dimension \( d \) with \( m \) vertices and \( F \) will be the associated Fano variety.

The first interesting non-trivial class of vertex transitive Fanos are the del Pezzo varieties.

**Definition 6.9.** The \( d \)-dimensional del Pezzo variety \( V^d \) (with \( d \) even) is the smooth toric Fano whose associated polytope has vertices

\[
V(\Delta) = \{e_1, \ldots, e_d, -e_1, \ldots, -e_d, (e_1 + \ldots + e_d), -(e_1 + \ldots + e_d)\},
\]

where \( e_1, \ldots, e_d \) is the standard basis of \( \mathbb{N}_Q \).

**Remark 6.10.** In Section 4, del Pezzo surfaces denoted all the smooth Fano varieties of dimension 2 and this could cause confusion. Nonetheless both notations are standard. We just remark that the del Pezzo variety of dimension 2 is just \( S_6 \), in the notation of Section 4.

It is clear that projective spaces and del Pezzo varieties are vertex-transitive, but they are not the only one as we will see.

The following lemma is very useful for the classification of vertex-transitive polytopes.

**Lemma 6.11.** Let \( X(\Delta) \) be a vertex-transitive Fano toric variety such that \( X \cong Y \times Z \) for \( Y, Z \) toric varieties. Then there exists a vertex-transitive Fano toric variety \( W \) and positive integers \( r \) and \( s \) such that \( Y \cong W^r \) and \( Z \cong W^s \).

**Proof.** We may assume that \( Y \) and \( Z \) are not isomorphic to products of toric varieties; the general statement follows easily from this case. Let \( \Phi = \text{conv}(y_1, \ldots, y_n) \) and \( \Psi = \text{conv}(z_1, \ldots, z_m) \) be the polytopes of \( Y \) and \( Z \) respectively, so that \( \Delta = \text{conv}(\Phi \times \{0\}, \{0\} \times \Psi) \). Consider an element \( g \in \text{Aut}(\Delta) \) such that \( g((y_1, 0)) = (0, z_1) \). We claim that \( g(\Phi \times \{0\}) \subset \{0\} \times \Psi \), from which, by symmetry, it follows that \( \Phi \cong \Psi \) and then the statement. We may assume that \( g((y_i, 0)) = (0, z_i) \) for \( i = 1, \ldots, k \) with \( k \leq n \) and \( g((x_j, 0)) \in \Phi \times \{0\} \) for \( j = k + 1, \ldots, n \). Hence

\[
g(\Phi \times \{0\}) = \text{conv}((0, z_1), \ldots, (0, z_k), g(y_{k+1}, 0), \ldots, g(y_n, 0)).
\]
Since $Y$ is not isomorphic to a product, we conclude that $k = n$ and the claim is proven. □

We recall a classic result by Voskresenkii and Klyachko.

**Theorem 6.12** ([VK84, Theorem 6]). Let $F$ be a smooth toric Fano such that $\Delta$ is symmetric with respect to the origin. Then $F$ is isomorphic to the product of projective lines and del Pezzo varieties.

In this subsection, we also need some notation and results about primitive collections.

**Definition 6.13.** A subset $P \subset V(\Delta)$ is called a primitive collection if the cone generated by $P$ is not in $\Sigma$, while for each $x \in P$ the elements of $P \setminus \{x\}$ generate a cone in $\Sigma$.

For a primitive collection $P = \{x_1, \ldots, x_k\}$ denote by $\sigma(P)$ the (unique) minimal cone in $\Sigma$ such that $(x_1 + \ldots + x_k) \in \sigma(P)$. Let $y_1, \ldots, y_h$ be generators of $\sigma(P)$. Then we have

$$r(P): x_1 + \ldots + x_k = b_1 y_1 + \ldots + b_h y_h$$

where $b_i$ are positive integers.

**Definition 6.14.** The linear relation $r(P)$ is called the primitive relation of $P$ and the cone $\sigma(P)$ is called the focus of $P$. The integer $k$ is called the length of $r(P)$ and the degree of $P$ is defined as $\deg P = k - \sum b_i$.

Using the exact sequence (6) we have the following identification between $A_1$ and the group generated by relations among the vertexes of $\Delta$:

$$A_1(X) \cong \left\{ (b_x)_{x \in V(\Delta)} \in \text{Hom}(\mathbb{Z}^m, \mathbb{Z}) \mid \sum_{x \in V(\Delta)} b_x x = 0 \right\}.$$

**Remark 6.15.** By abuse of notation we denote with $r(P)$ also the cycle associated to $r(P)$ via the previous isomorphism. Note that $\deg P = -(K_F \cdot r(P))$ and so, since we are considering Fano varieties, any primitive relation has strictly positive degree.

Let $P$ a primitive collection such that $r(P)$ is extremal (i.e. it generates an extremal ray in $\overline{\text{NE}}(F)$). The exceptional locus of the associated contraction

$$\phi_P: F \to F_P$$

is $Z(\sigma(P))$. According to $\sigma(P)$, we obtain the different steps of the MMP.

**Divisorial contraction:** It appears when $\sigma(P)$ is a one-dimensional cone and the contracted divisor is precisely the one associated to the ray.

**Mori fibre space:** In this case $\sigma(P)$ is the origin.

**Flipping contraction:** This appears in all the other cases.

We will also need the following results.

**Lemma 6.16** ([Cas03, Prop. 4.3]). Let $\gamma \in \overline{\text{NE}}(F) \cap A_1(F)$ be a cycle such that $-(K_F \cdot \gamma) = 1$. Then $\gamma$ is extremal.
Theorem 6.17 ([Rei83, Theorem 2.4], [Cas03, Theorem 1.5]). Let $R$ be an extremal ray of $\text{NE}(F)$ and $\gamma \in R \cap \mathcal{A}_1(F)$ a primitive cycle. Then there exists a primitive collection $P = \{x_1, \ldots, x_k\}$ such that

$$\gamma = r(P) : x_1 + \ldots + x_k = b_1y_1 + \ldots b_hy_h.$$ 

For any cone $\nu = \langle z_1, \ldots, z_t \rangle$ such that $\{z_1, \ldots, z_t\} \cap \{x_1, \ldots, x_k, y_1, \ldots, y_h\} = \emptyset$ and $\langle y_1, \ldots, y_h \rangle + \nu \in \Sigma$, we have

$$\langle x_1, \ldots, x_i, \ldots, x_k, y_1, \ldots, y_h \rangle + \nu \in \Sigma$$

for all $i = 1, \ldots, h$.

Proposition 6.18 ([Cas03, Prop. 3.4]). Let $P$ a primitive extremal collection such that $\sigma(P) = \langle y_1, \ldots, y_h \rangle$. Then for any primitive collection $Q$ such that $P \cap Q \neq \emptyset$ and $Q \neq P$, the set $(Q \setminus P) \cup \{y_1, \ldots, y_h\}$ contains a primitive collection.

Remark 6.19. Let

$$a_1x_1 + \ldots + a_kx_k = b_1y_1 + \ldots + b_hy_h$$

be a relation among vertices of $\Delta$ with all $\{a_i\}$ and $\{b_j\}$ positive integers. Assume that $\sum a_i \geq \sum b_j$. Then, by Lemma 1.4 in [Cas03], $\langle x_1 \cdots x_k \rangle \notin \Sigma$.

We introduce the following classical definition.

Definition 6.20 ($K$-neighbourly polytope). A polytope $\Delta$ is called $k$-neighbourly if every set of $k$ vertices lies on a face of $\Delta$.

Proposition 6.21. Assume that $\Delta$ is smooth and vertex-transitive. Then either

1. $F(\Delta) = (\mathbb{P}^1)^d$ or $F(\Delta) = (V_k)^r$ for positive integers $r$ and $k$ or
2. $\Delta$ is 2-neighbourly.

Proof. Assume that $\Delta$ is not 2-neighbourly. This implies the existence of a primitive collection with two elements. As a first step, we want to prove that there is a primitive relation of the form $x + y = 0$. Assume by contradiction that there is no such relation and consider a primitive collection $P_1 = \{x_1, x_2\}$ with relation $R_1 : x_1 + x_2 = y_1$.

Acting with $\text{Aut}(\Delta)$ we obtain a family of primitive collections $P = \{P_i\}_{1 \leq i \leq r}$ and relations $\mathcal{R} = \{R_i\}_{1 \leq i \leq r}$ such that $m := \#\{V(\Delta)\}$ divides $r$, because of the transitivity of the action. Assume that the $P_i$’s are all disjoint. Then $2r = m$, which is impossible. So we may assume, by symmetry, that $P_2 = \{x_1, x_3\}$ and $R_2 : x_1 + x_3 = y_2$ with $x_2 \neq x_3$.

Combining $P_1$ and $P_2$ we get $x_2 + y_2 = x_3 + y_1$, from which we deduce that $\{x_2, y_2\}$ is also a primitive collection, by Remark 6.19. Write

$$S_1 : x_3 + y_1 = z_1 \quad \text{and} \quad S_2 : x_2 + y_2 = z_1.$$ 

By Lemma 6.16, $S_1$ and $S_2$ are both extremal. By Theorem 6.17, $\langle y_1, y_2 \rangle \in \Sigma$, which is impossible because

$$y_1 + y_2 = x_1 + z_1.$$ 

So there must be a primitive relation of the form $x + y = 0$. Acting with $\text{Aut}(\Delta)$ we obtain exactly $m/2$ such relations (it is easy to check that they are disjoint). In particular, by the vertex transitive condition, for any vertex $x$ there is a vertex $y$ such that $x + y = 0$. This means that $\Delta$ is symmetric. The result follows now by Theorem 6.12 and Lemma 6.11.
Proposition 6.22. Assume that $\Delta$ is smooth, vertex-transitive and 2-neighbourly. Then there exist an integer $k \geq 3$ and a set of primitive collections $\mathcal{P} = \{P_i\}_{i=1}^r$ such that $r = m/k$, $|P_i| = k$, $\sigma(P_i) = 0$ and $P_i \cap P_j = \emptyset$ for any $i \neq j$. Moreover, these are the only primitive relations with focus equal to zero.

If any of these relations is extremal, then $F(\Delta) = (\mathbb{P}^{k-1})^r$. On the other hand, if any of these relations is not extremal then $F$ does not admit any elementary contraction of fibre type.

Proof. By [Bat91, Proposition 3.2] we know that there exists a primitive collection $P_1$ such that $\sigma(P_1) = 0$. Set $k = |P_1|$. Note that $k \geq 3$, because $\Delta$ is 2-neighbourly.

Acting with $\text{Aut}(\Delta)$ we obtain a family of primitive collections $\mathcal{P} = \{P_i\}_{1 \leq i \leq r}$ such that $\sigma(P_i) = 0$ and $\cup P_i = V(\Delta)$. Assume that $P_i \cap P_j \neq \emptyset$ for some $i, j$. Assume $P_i = \{x_1, \ldots, x_k\}$ and $P_j = \{x_1, \ldots, x_h, y_{h+1}, \ldots, y_k\}$ with $y_k \neq x_i$ for any $s, t$. Then

$$x_{h+1} + \ldots + x_k = y_{h+1} + \ldots y_k,$$

which is impossible by Remark 6.19, because $x_{h+1}, \ldots, x_k$ generate a cone in $\Sigma$. It is immediate to check that there are no other primitive relations with focus equal to zero. The first part of the proposition is hence proven.

Assume now that one of these relations (and hence all by symmetry) is extremal. We claim that there are no other primitive collections. In fact, let $Q$ be a primitive collection such that $Q \notin \mathcal{P}$ and $Q$ has the minimal cardinality among the primitive collections not contained in $\mathcal{P}$. We may assume that $P_i \cap Q \neq \emptyset$. By Proposition 6.18, the set $(Q \setminus P_1)$ contains a primitive collection, which contradicts the minimality of $|Q|$.

This gives a complete description of the Mori cone of $X$. To conclude note that the index of $K_{\Delta}$ is $k$ and that $\dim F = d = (k - 1)r$ and $\rho(F) = r$ by [Bat91, Corollary 4.4]. Apply now Mukai’s conjecture (cf. [Cas06, Theorem 1]).

Finally, an extremal contraction of fibre type would be associated to an extremal primitive collection $P$ such that $\sigma(P) = 0$. Thus the conclusion is a consequence of what we have just proven.

Lemma 6.23. Assume that $\Delta$ is smooth, 2-neighbourly and vertex-transitive. Then there are no extremal relations of the form

$$x_1 + \ldots + x_k = b_1 y_1.$$

In particular $X$ does not admit any elementary divisorial contraction.

Proof. We argue by contradiction. Acting with $\text{Aut}(\Delta)$ we get another extremal primitive relation

$$x_1 + z_2 + \ldots + z_k = b_1 y_2$$

(we argue as in Proposition 6.21 to show that, moving the primitive collections with $\text{Aut}(\Delta)$, they cannot be all disjoint).

We treat the case $y_2 \notin \{x_2, \ldots, x_k, y_1\}$, the other one is similar.

We have

$$b_1 y_2 + x_2 + \ldots + x_k = b_1 y_1 + z_2 + \ldots + z_k.$$

Since $\Delta$ is 2-neighbourly, we know that $\langle y_1, y_2 \rangle$ is a cone of $\Sigma$. Applying Theorem 6.17 we get $\langle y_2, x_2, \ldots, x_k \rangle \in \Sigma$, which is impossible by Remark 6.19.

□
6.3. **Examples.** We now study a family of examples of vertex-transitive toric varieties, which are generalisations of del Pezzo varieties. They were first introduced in [Kly84] and also appeared in [VK84], although the dual polytopes of these toric varieties naturally appeared as a special class of transportation polytopes (the so-called central transportation polytopes), which have been extensively studied in optimisation problems (cf. [EKK81]).

We will adopt a different notation compared to the one introduced by Klyancho (cf. Remark 6.25): our choice is more convenient for our purposes of classification (cf. Remark 6.29).

**Definition 6.24** (Klyachko varieties). Let $k$ and $d$ be positive integers such that $d \geq 2$ and $(k - 1)d$. Fix a basis $e_1, \ldots, e_d$ of a lattice $N \cong \mathbb{Z}^d$. The Klyachko variety of order $k$ and dimension $d$ is the Fano toric variety $W^k_d$ associated to to the polytope $\Delta^k_d \subset N$ with vertices

$$V(\Delta^k_d) = \{e_1, e_2, \ldots, e_d, e_1 + \ldots + e_d, - (e_1 + \ldots + e_{k-1}), - (e_k + \ldots + e_{2k-2}), \ldots, - (e_d - k + 2 + \ldots + e_d),$$

$$- (e_1 + e_k + \ldots + e_{d-k+2}), - (e_2 + e_{k+1} + \ldots + e_{d-k+3}), \ldots,$$

$$- (e_{k-1} + e_{2k-2} + \ldots + e_d)\}.$$ 

Note that when $d$ is even, $W^2_d$ is the del Pezzo variety $V_d$.

**Remark 6.25.** In [VK84], the varieties $W^k_d$ are introduced with the notation $P_{m,n}$. We have the following relations between the indices:

$$d = (m - 1)(n - 1), \quad k = m \text{ (or } n\text{)}$$

(our definition of $k$ is consistent because of the isomorphisms proved in Lemma 6.26).

Assume that $W^k_d$ is smooth (cf. Proposition 6.28). Then we have the following interesting geometric interpretation.

The 1-dimensional cones of the fan of $W^k_d$ are exactly the 1-dimensional cones of the fan of the blow-up $Z^k_d$ of $(\mathbb{P}^{k-1})^\ell$ in $k$ invariant points. This implies that there exist a birational map between $W^k_d$ and $Z^k_d$ which is an isomorphism in codimension 1. Moreover, it can be shown that this map must factor through a sequence of flips, since $W^k_d$ is a smooth Fano: more precisely this map can be realised by a $K_{W^k_d}$-MMP with scaling.

Furthermore, the following isomorphisms hold.

**Lemma 6.26.** For any integers $d$ and $m$, we have $W^{d+1}_{md} \cong W^{m+1}_{md}$.

**Proof.** We can assume that $m \leq d$ and write down the vertices of $W^{d+1}_{md}$:

$$\{e_1, e_2, \ldots, e_{md}, e_1 + \ldots + e_{md},$$

$$- (e_1 + \ldots + e_d), - (e_{d+1} + \ldots + e_{2d}), \ldots, - (e_{m(d-1)+1} + \ldots + e_{md}),$$

$$- (e_1 + e_{d+1} + \ldots + e_{m(d-1)+1}), - (e_2 + e_{d+2} + \ldots + e_{m(d-1)+2}), \ldots,$$

$$- (e_d + e_{2d} + \ldots + e_{md})\}.$$ 

It is now clear that reordering the base vectors

$$e'_{mi+j} := e_{d(j-1)+i+1}$$

with $i \in \{0, \ldots, d\}$ and $j \in \{1, \ldots, m - 1\}$ gives the required isomorphism.
Lemma 6.27. The polytope $W^k_d$ is vertex-transitive, reflexive and terminal.

Proof. We start proving the transitivity by induction on the dimension $d$. Let us fix the index $k$ and consider the Klyachko variety with dimension $W^k_{k-1}$. Using Lemma 6.26 we deduce the isomorphism

$$W^k_{k-1} \cong W^2_{k-1},$$

where the RHS is a vertex-transitive del Pezzo variety. Assume now by induction that $W^t_i$ with $t < d$ is vertex transitive ($t$ has to be a multiple of $k-1$). We consider for every $i = 0, \ldots, d - k + 1$ the projections from $\Delta^k_d$ to the $(k-1)$-codimensional subspaces orthogonal to $\langle e_{i+1}, e_{i+2}, \ldots, e_{i+k-1} \rangle$.

By inductive hypothesis, the images of the polytope $\Delta^k_d$ via the projections are all vertex transitive polytopes isomorphic to $\Delta^k_{d-k+1}$. To deduce the transitivity of the whole polytope, we use the isomorphisms of $N_Q$ sopping the subspaces in (8).

Set now $W = W^k_d$ and $\Delta = \Delta^k_d$.

We prove that $\Delta$ is reflexive. It is enough to show that there are no lattice points lying between the affine hyperplane spanned by each facet of $\Delta^\ast \subset M_Q$ and its parallel through the origin. Since the action of $\text{Aut}(\Delta)$ induces a transitive action on the facets of $\Delta^\ast$, it is sufficient to consider just one facets of $\Delta^\ast$: it is obvious that $\{x_1 = -1\} \subset M_Q$ satisfies the property we want.

We now show that $W$ has terminal singularities, i.e. that $\Delta \cap N = V(\Delta) \cup \{0\}$. Assume by contradiction that there exists a non-zero $v \in \Delta \cap N$ such that $v \notin V(\Delta)$. We may assume that $v$ is not in the subspace $H$ generated by $e_1, \ldots, e_{k-1}$. Let $\pi_H$ the projection from $H$. Then $\Gamma = \pi_H(\Delta)$ is a Klyachko polytope, $\pi_H(v) \in \Gamma$ and $\pi_H(v) \notin V(\Gamma) \cup \{0\}$. Hence we can conclude by induction, noting that $W^2_d$ is terminal for any $d \geq 2$.

□

For any positive integers $d$ and $k$ we denote by $7_k$ the smallest non-negative integer $r$ such that $d \equiv r \mod k$.

The following result was is proved in [VK84]. Here we give a different proof using our notation.

Proposition 6.28. The variety $W^k_d$ is smooth if $\gcd(d-1, k) = 1$. If $\gcd(d-1, k) \neq 1$, then $W^k_d$ is not $\mathbb{Q}$-factorial.

In particular, if $k$ is a prime number then $W^k_d$ is smooth, unless $d \equiv 1 \mod k$.

Proof. For any $k \geq 2$ and $t \geq 1$ define the linear form

$$L_{k,t} = \sum_{i=0}^{k-3} (x_{t+ik} + x_{t+ik+1} + \ldots + x_{t+ik+k-2} - (k-1)x_{t+ik+(k-1)}).$$

We start noting that the polytope $\Delta^k_d$ for $d = (k-1)^2$ is not simplicial, since the hyperplane $\{L_{k,1} + x_d = 1\}$ supports a facet of $\Delta^k_d$ with turns out to have $k(k-1)$ vertices.

On the other hand we can see that for $d = k(k-1)$ the polytope $\Delta^k_d$ is smooth. In fact it is easy to see that any facet $F$ of $\Delta^k_d$ which contains the vertex $(1,1,\ldots,1)$
must contains at least \( f = (k - 1)(k - 2) + 1 \) elements of the standard basis. This implies that the supporting hyperplane
\[
\{ a_1 x_1 + \ldots + a_d x_d = 1 \}
\]
of \( F \) has exactly \( f \) coefficients equal to 1, \( k - 2 \) coefficients equal to \(-(k - 1)\) and \( k - 1 \) coefficients equal to 0 (as for the hyperplane \{\( L_{k,1} + x_k = 1 \}\}). One can also check that all these facets have vertices which form a basis of \( N \). By the transitivity of \( \text{Aut}(\Delta^k_d) \) we gain that \( W^k_d \) is smooth.

We now prove the general statement by induction on \( k \) and \( d \). The case \( k = 2 \) (and any \( d \)) and the case \( d = 2 \) are immediate.

Let \( \Delta^k_d \) be a Klyachko polytope with \( k, d \geq 3 \). If \( d < (k - 1)^2 \) then \( \Delta^k_d \cong \Delta^k_{d+1} \), where \( s = k - \frac{d}{k} \) and \( d = s(k - 1) \), so by induction on \( k \) we are done, because \( \gcd(d - 1, k) = \gcd(s + 1, k) = \gcd(s + 1, d - 1) \).

We have already seen how to deal with the cases \( d = (k - 1)^2 \) and \( d = k(k - 1) \).

If \( d > k(k - 1) \), then set \( t = d - k(k - 1) \) and consider the plane \( H \) spanned by \( \{e_{t+1}, e_{t+2}, \ldots, e_d\} \) and the associated projection \( \pi_H \) on the orthogonal space to \( H \). The image \( \pi_H(\Delta^k_d) \) is \( \Delta^k_t \) and \( \gcd(d - 1, k) = \gcd(t - 1, k) \). For any facet \( F \) of \( \Delta^k_t \) with supporting hyperplane \( \{ P(x_1, \ldots, x_t) = 1 \} \) we get a facet \( \mathbf{P} \) of \( \Delta^k_d \) supported by
\[
\{ P + L_{k,(t+1)} + (x_{d-k+1} + \ldots + x_{d-1} - (k - 1)x_d) = 1 \}.
\]
Note that \(|V(\overline{F})| = |V(F)| + k(k - 1)\), so if \( \Delta^k_t \) is not simplicial, then \( \Delta^k_d \) is not simplicial. In this way one can check that \( \Delta^k_d \) is smooth if and only if \( \Delta^k_t \) is smooth; the result follows now by induction on \( d \).

\[ \square \]

6.4. Low-dimensional case. In this subsection, we use our results to give a full classification of smooth vertex-transitive Fano polytopes in low dimension.

Let \( \Delta \) be a smooth Fano polytope of dimension \( d \). By [Cas06, Theorem 1] we know that \(|V(\Delta)| \leq 3d \) with equality if and only if \( d \) is even and \( X := X(\Delta) \cong (\mathbb{P}^2)^{d/2} \). We also note that for any extremal relation \( x_1 + \ldots + x_k = b_1 y_1 + \ldots + b_h y_h \), we have \( k + h \leq d + 1 \).

**Dim**(\( X \))=2. If \( \Delta \) is not 2-neighbourly, then by Proposition 6.21 we get \( X \cong \mathbb{P}^1 \times \mathbb{P}^1 \) or \( X \cong V_2 \).

Assume that \( \Delta \) is 2-neighbourly. Then there is an extremal collection \( P = \{ x_1, x_2, x_3 \} \) such that \( \sigma(P) = 0 \) and so, by Proposition 6.22, we get \( X \cong \mathbb{P}^2 \).

**Dim**(\( X \))=3. If \( \Delta \) is not 2-neighbourly, then by Proposition 6.21 we get \( X \cong (\mathbb{P}^1)^3 \).

Assume that \( \Delta \) is 2-neighbourly. By dimensional restriction the extremal relations could be of the form \( x_1 + x_2 + x_3 = 0 \) (which is impossible by Proposition 6.22), \( x_1 + x_2 + x_3 = y_1 \) (which is impossible by Lemma 6.23) or \( x_1 + x_2 + x_3 + x_4 = 0 \), which gives \( X \cong \mathbb{P}^3 \) by Proposition 6.22.

6.5. MAGMA computations. The following table collects the smooth Fano toric varieties (up to dimension 8) whose polytope is vertex-transitive. It has been obtained using the software MAGMA together with the Graded Ring Database \([GB^+]\) (for further details on the classification, cf. \([Ob07]\)). In the following table, the IDs of the Fano polytopes are the ones introduced in \([GB^+]\).
| Dimension | #Vertices | Description        | ID  |
|-----------|-----------|--------------------|-----|
| 2         | 6         | $V_2$              | 2   |
| 2         | 4         | $\mathbb{P}^1 \times \mathbb{P}^1$ | 4   |
| 2         | 3         | $\mathbb{P}^2$    | 5   |
| 3         | 6         | $(\mathbb{P}^1)^3$ | 21  |
| 3         | 4         | $\mathbb{P}^3$    | 23  |
| 4         | 10        | $V_4$              | 63  |
| 4         | 12        | $V_2 \times V_2$  | 100 |
| 4         | 8         | $(\mathbb{P}^1)^4$ | 142 |
| 4         | 6         | $\mathbb{P}^2 \times \mathbb{P}^2$ | 146 |
| 4         | 5         | $\mathbb{P}^4$    | 147 |
| 5         | 10        | $(\mathbb{P}^1)^5$ | 1003|
| 5         | 6         | $\mathbb{P}^5$    | 1013|
| 6         | 14        | $V_6$              | 1930|
| 6         | 12        | $W_6^3$            | 5817|
| 6         | 18        | $(V_2)^3$          | 7568|
| 6         | 12        | $(\mathbb{P}^1)^6$ | 8611|
| 6         | 9         | $(\mathbb{P}^2)^3$ | 8631|
| 6         | 8         | $(\mathbb{P}^3)^2$ | 8634|
| 6         | 7         | $\mathbb{P}^6$    | 8635|
| 7         | 14        | $(\mathbb{P}^1)^7$ | 80835|
| 7         | 8         | $\mathbb{P}^7$    | 80891|
| 8         | 18        | $V_8$              | 106303|
| 8         | 15        | $W_8^3$            | 277415|
| 8         | 20        | $(V_2)^2$          | 442179|
| 8         | 24        | $(V_2)^4$          | 790981|
| 8         | 12        | $\tilde{W}$       | 830429|
| 8         | 16        | $(\mathbb{P}^1)^8$ | 830635|
| 8         | 12        | $(\mathbb{P}^2)^4$ | 830767|
| 8         | 10        | $(\mathbb{P}^4)^2$ | 830782|
| 8         | 9         | $\mathbb{P}^8$    | 830783|

**Remark 6.29.** The 8-dimensional polytope denoted by $\tilde{W}$ is not a Klyachko variety. Nonetheless, following our notation, it can be considered as the candidate for a generalised Klyachko variety $W_8^4$. The study of this new class of examples is in progress.

We would like to finish off by stating the following speculation.

**Conjecture 6.30.** Let $\Delta$ be a smooth vertex-transitive polytope of dimension $d$. Assume that $d$ is an odd prime number. Then either $X(\Delta) = \mathbb{P}^d$ or $X(\Delta) = (\mathbb{P}^1)^d$.

### 7. Flag varieties

In this section we show that most flag varieties can not be fibres of a Mori fibre space. Given a semi-simple algebraic group $G$, the flag variety $F$ of $G$ is the quotient $G/B$, where $B$ is any Borel sub-group. All Borel sub-groups are conjugated, so $F$ does not depend on the choice of $B$. Flag varieties are known to be Fano varieties. As a warm-up, we prove the following.
Proposition 7.1. Take $G = SL_3(\mathbb{C})$. Then, the flag variety $F = G/B$ can be a fibre of an isotrivial Mori fibre space.

Proof. This flag variety is the moduli space of pairs line and plane $(L, P)$ in $\mathbb{C}^3$ such that $L \subset P$. It can be seen as a degree $(1, 1)$ divisor in $\mathbb{P}^3 \times (\mathbb{P}^3)^\vee$, so it has Picard number 2.

Pick a non-degenerate quadratic form $Q$ on $\mathbb{C}^3$. Consider the regular automorphism $F_Q : F \to F$. $(L, P) \mapsto (P^\perp, L^\perp)$

We want to study its action on $NS(F)$. The faces of the nef cone are given by the contractions to the Grassmannian of lines in $\mathbb{C}^3$ and to the Grassmannian of plane in $\mathbb{C}^3$. The morphism $F_Q^*$ exchanges these faces. We can thus apply Theorem 3.1. □

When $G = SL_2(\mathbb{C})$, the flag variety is $\mathbb{P}^1$. In general, we can prove the following result.

Theorem 7.2. Let $G$ be one of the following algebraic groups:

- $SL_n(\mathbb{C})$, $n \geq 4$;
- $SO_n(\mathbb{C})$, $n \geq 6$;
- $Sp_{2n}(\mathbb{C})$, $n \geq 3$.

Then, the flag variety $F = G/B$ is not fibre-like.

Proof. The geometry of $F$ can be described in term of the representation theory of $G$; we will take advantage of this description. A general reference is [Br]. Fix a set of simple positive roots for $G$

$$\Delta = \{\alpha_1, \cdots, \alpha_k\}.$$  

These roots are a basis of the weight lattice over $\mathbb{Q}$. To any weight we can associate a line bundle on $F := G/B$. This is an isomorphism between the weight lattice and $NS(F)$. On the weight lattice we have the Killing quadratic form $k$. Let $\omega_i$ be the hyperplane orthogonal to $\alpha_i$ with respect to $k$. Because of the Borel-Weil-Bott Theorem, these are the faces of the effective cone. Each divisor in the interior of the effective cone is very ample. By [Mat00, Theorem 0.1], the effective cone is equal to the nef cone. See also [Br].

Remark that in this case the nef cone is simplicial. The intersection of all the $\omega_i$ with $i \neq j$ is exactly $\alpha_j$. We conclude that the roots spans are the minimal faces of the nef cone. This means that $\text{Mon}(F)$ permutes the roots and $NS(F)$ is isomorphic to $\mathbb{Q}^\Delta$ as a $\text{Mon}(F)$-module. Because of Theorem 3.2, we need to show that

$$\dim NS(F)^{\text{Mon}(F)} > 1.$$  

This is equivalent to say that $\text{Mon}(F)$ does not act transitively on $\Delta$. To prove our theorem, we thus have to exhibit two roots that can not be exchanged by $\text{Mon}(F)$.

Each root $\alpha_i$ defines a contraction

$$\pi_i : F \to G/P_i,$$

where $P_i$ is a maximal parabolic subgroup. If $\text{Mon}(F)$ maps $\alpha_i$ to $\alpha_j$, then $G/P_i$ must be deformation equivalent to $G/P_j$. To conclude, it is enough to exhibit two examples of quotients $G/P_i$ and $G/P_j$ whose dimensions are different.
The quotient $G/P_i$ is a Grassmannian for $G = SL_n$, and Grassmannian of isotropic sub-spaces for $G = SO_n$ and $G = Sp_{2n}$. Our examples for $G/P_i$ and $G/P_j$ are the following:

- for $G = SL_n(\mathbb{C})$, with $n \geq 3$, take $G/P_i = Gr(1, \mathbb{C}^n)$ and $G/P_j = Gr(2, \mathbb{C}^n)$;
- for $G = SO_n(\mathbb{C})$, with $n \geq 6$, take $G/P_i = Gr_{iso}(1, \mathbb{C}^n)$, whose dimension is $n - 2$, and $G/P_j = Gr_{iso}(2, \mathbb{C}^n)$, whose dimension is $2(n - 2) - 3$;
- for $G = Sp_{2n}(\mathbb{C})$, with $n \geq 3$, take $G/P_i = Gr_{iso}(1, \mathbb{C}^{2n})$, whose dimension is $2n - 1$, and $G/P_i = Gr_{iso}(2, \mathbb{C}^n)$, whose dimension is $2(2n - 2) - 1$.

\[\square\]

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