On $H$C-subgroups of a finite group$^{*\dagger}$

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Abstract

A subgroup $H$ of a finite group $G$ is said to be an $H$C-subgroup of $G$ if there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$. In this paper, we investigate the structure of a finite group $G$ under the assumption that certain subgroups of $G$ of arbitrary prime power order are $H$C-subgroups of $G$.

Key words: $H$-subgroups; $H$C-subgroups; $p$-nilpotent group; nilpotent group; supersolvable group.

1 Introduction

Throughout this paper, all groups considered are finite. $G$ always denotes a group, $p$ denotes a prime, and $|G|_p$ denotes the order of Sylow $p$-subgroups of $G$. A class of groups $\mathcal{F}$ is called a formation if $\mathcal{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. All unexplained notation and terminology are standard, as in [7,10,12].

Recall that a subgroup $H$ of $G$ is said to be an $H$-subgroup of $G$ if $H^g \cap N_G(H) \leq H$ for all $g \in G$. This concept was introduced by Goldschmidt in [9] and Bianchi et al. in [5]. It is easy to see that normal subgroups, Sylow subgroups and self-normalizing subgroups of $G$ are all $H$-subgroups of $G$. Csörgö, Herzog [6] and Asaad [11] further investigated the influence of $H$-subgroups on the structure of a finite group.

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Besides, Y. Wang [20] introduced the concept of c-normal subgroups. A subgroup $H$ of $G$ is said to be c-normal in $G$ if there exists a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_G$, where $H_G$ is the largest normal subgroup of $G$ contained in $H$. The properties of c-normal subgroups have been studied by many authors, see for example, [3, 4, 14, 15].

Recently, some attempts were made to give a generalization of both c-normal subgroups and $H$-subgroups. In [2], M. Asaad et al. introduced the concept of weakly $H$-subgroups: a subgroup $H$ of $G$ is called an $HC$-subgroup of $G$ if there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is an $H$-subgroup of $G$. Meanwhile, X. Wei and X. Guo [21] introduced the concept of $HC$-subgroups: a subgroup $H$ of $G$ is said to be an $HC$-subgroup of $G$ if there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$. It is easy to see that every weakly $H$-subgroup of $G$ is an $HC$-subgroup of $G$.

In [21], the authors gave some conditions on maximal subgroups or minimal subgroups of Sylow subgroups, which are sufficient to guarantee a group to be $p$-nilpotent or supersolvable. In this paper, we continue to investigate the structure of a group $G$ under the assumption that certain subgroups of $G$ of arbitrary prime power order are $HC$-subgroups of $G$. New characterizations of some classes of finite groups are obtained.

2 Preliminaries

Lemma 2.1. Suppose that $H$ is an $\mathcal{H}$-subgroup of $G$.

1. ( [5, Theorem 6(2)]) If $H$ is subnormal in $G$, then $H$ is normal in $G$.
2. ( [5, Lemma 7(2)]) If $H \leq K \subseteq G$, then $H$ is an $\mathcal{H}$-subgroup of $K$.
3. ( [5, Lemma 2(1)]) If $N \leq H$ and $N \trianglelefteq G$, then $H$ is an $\mathcal{H}$-subgroup of $G$ if and only if $H/N$ is an $\mathcal{H}$-subgroup of $G/N$.
4. ( [5, Theorem 6(3)]) If $N \leq G$ and $N \leq N_G(H)$, then $N_G(HN) = N_G(H)$ and $HN$ is an $\mathcal{H}$-subgroup of $G$.

Lemma 2.2. Let $H$ and $K$ be subgroups of $G$, and $N \trianglelefteq G$.

1. ( [21, Lemma 2.3(1)]) If $H \leq K$ and $H$ is an $HC$-subgroup of $G$, then $H$ is an $HC$-subgroup of $K$.
2. ( [21, Lemma 2.3(2)]) If $N \leq H$, then $H$ is an $HC$-subgroup of $G$ if and only if $H/N$ is an $HC$-subgroup of $G/N$.
3. ( [21, Lemma 2.4]) If $H$ is a $p$-group with $(p, |N|) = 1$ and $H$ is an $HC$-subgroup of $G$, then $HN$ is an $HC$-subgroup of $G$ and $HN/N$ is an $HC$-subgroup of $G/N$.

Lemma 2.3. [21, Theorem 3.7] Let $p$ be the smallest prime dividing $|G|$ and let $P$ be a Sylow $p$-subgroup of $G$. Then $G$ is $p$-nilpotent if every maximal subgroup of $P$ is an $HC$-subgroup of $G$. 

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Lemma 2.4. [9 Corollary B3] Suppose that $S$ is a 2-subgroup of $G$ such that $S$ is an $\mathcal{H}$-subgroup of $G$ and $N_G(S)/C_G(S)$ is a 2-group. Then $S$ is a Sylow 2-subgroup of $S^G$.

Lemma 2.5. [16 Theorem 1] Let $P$ be a Sylow $p$-subgroup of $G$. Then the following two statements are true:

1. If $p$ is odd and every minimal subgroup of $P$ lies in $Z(N_G(P))$, then $G$ is $p$-nilpotent.

2. If $p = 2$ and every cyclic subgroup of $P$ of order 2 or 4 is quasi-normal in $N_G(P)$, then $G$ is 2-nilpotent.

Let $F^*(G)$ denote the generalized Fitting subgroup of $G$, that is, the largest normal quasinilpotent subgroup of $G$. The following basic facts can be found in [13 Chapter X].

Lemma 2.6. (1) If $N$ is a normal subgroup of $G$, then $F^*(N) = N \cap F^*(G)$.

(2) $F(G) \leq F^*(G) = F^*(F^*(G))$. If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.

(3) $C_G(F^*(G)) \leq F(G)$.

(4) If $G > 1$, then $F^*(G) > 1$. In fact, $F^*(G)/F(G) = \text{soc}(F(G)C_G(F(G))/F(G))$.

Lemma 2.7. [21 Lemma 2.5] Let $K$ be a normal subgroup of $G$ and let $H$ be a normal subgroup of $K$. If $H$ is an $\mathcal{H}C$-subgroup of $G$, then $H$ is c-normal in $G$.

Lemma 2.8. Let $H$ be a $p$-subgroup of $G$. If $H$ is an $\mathcal{H}C$-subgroup of $G$ and $H$ is not an $\mathcal{H}$-subgroup of $G$, then $G$ has a normal subgroup $M$ such that $|G : M| = p$ and $G = HM$.

Proof. By hypothesis, $G$ has a normal subgroup $T$ such that $G = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$. Since $H$ is not an $\mathcal{H}$-subgroup of $G$, we have that $T < G$. Hence $G/T$ is a $p$-group, and so $G$ has a normal subgroup $M$ containing $T$ such that $|G : M| = p$ and $G = HM$. □

Lemma 2.9. [11 Lemma 2.9] Let $\mathfrak{F}$ be a saturated formation containing all supersolvable groups and let $G$ be a group with a normal subgroup $E$ such that $G/E \in \mathfrak{F}$. If $E$ is cyclic, then $G \in \mathfrak{F}$.

Lemma 2.10. Let $\mathfrak{F}$ be a saturated formation containing all supersolvable groups. Suppose that $M$ is a subgroup of $G$ such that $|G : M| = p$, $F(G) \notin M$ and $M \in \mathfrak{F}$. Then $G \in \mathfrak{F}$.

Proof. If $\Phi(G) > 1$, then it is easy to see that $G/\Phi(G)$ satisfies the hypothesis of the lemma, and so $G/\Phi(G) \in \mathfrak{F}$ by induction. This implies that $G \in \mathfrak{F}$. We may, therefore, assume that $\Phi(G) = 1$. Then $F(G) = N_1 \times N_2 \cdots \times N_t$, where $N_i$ ($i = 1, \ldots, t$) is a solvable minimal normal subgroup of $G$. Since $F(G) \notin M$, there exists a solvable minimal normal subgroup $N_i$ of $G$ such that $N_i \notin M$. Then clearly, $G = N_i M$, and so $N_i \cap M = 1$. Therefore, $|N_i| = |G : M| = p$ and $G/N_i \cong M \in \mathfrak{F}$. It follows from Lemma 2.9 that $G \in \mathfrak{F}$. □
Lemma 2.11. Let $H$ be an $\mathcal{H}C$-subgroup of $G$. If $L/\Phi(L)$ is a chief factor of $G$ and $H \leq L$, then $H$ is an $\mathcal{H}$-subgroup of $G$.

Proof. By hypothesis, there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$. Since $L/\Phi(L)$ is a chief factor of $G$, either $(L \cap T)\Phi(L)/\Phi(L) = 1$ or $(N \cap T)\Phi(L)/\Phi(L) = L/\Phi(L)$. In the former case, $L = H(L \cap T) = H$. This implies that $H \trianglelefteq G$, and so $H$ is an $\mathcal{H}$-subgroup of $G$. In the latter case, $(L \cap T)\Phi(L) = L$, and so $T = G$. This also implies that $H$ is an $\mathcal{H}$-subgroup of $G$. \hfill $\Box$

Lemma 2.12. [18, Lemma 2.8] Let $P$ be a normal $p$-subgroup of $G$ contained in $Z_\infty(G)$. Then $O_p^p(G) \leq C_G(P)$.

Lemma 2.13. [8, Lemma 2.4] Let $P$ be a $p$-group. If $\alpha$ is a $p'$-automorphism of $P$ which centralizes $\Omega_1(P)$, then $\alpha = 1$ unless $P$ is a non-abelian $2$-group. If $[\alpha, \Omega_2(P)] = 1$, then $\alpha = 1$ without restriction.

3 Main results

Theorem 3.1. Let $p$ be the smallest prime divisor of $|G|$ and let $P$ be a Sylow $p$-subgroup of $G$. Suppose that $P$ is cyclic or $P$ has a subgroup $D$ with $1 < |D| < |P|$ such that every subgroup $H$ of $P$ of order $|D|$ is an $\mathcal{H}C$-subgroup of $G$. When $p = 2$ and $|P : D| > 2$, suppose further that $H$ is an $\mathcal{H}C$-subgroup of $G$ if there exists $D_1 \leq H \leq P$ such that $2|D_1| = |D|$ and $H/D_1$ is a cyclic group of order $4$. Then $G$ is $p$-nilpotent.

Proof. Suppose that the result is false and let $G$ be a counterexample of minimal order. Then we proceed via the following steps.

(1) $P$ is not cyclic and $|P : D| > p$.

If $P$ is cyclic, then by [19] (10.1.9), $G$ is $p$-nilpotent, a contradiction. Suppose that $|P : D| = p$. Then every maximal subgroup of $P$ is an $\mathcal{H}C$-subgroup of $G$. Hence by Lemma 2.3, $G$ is $p$-nilpotent, also a contradiction.

(2) Every proper subgroup of $G$ containing $P$ is $p$-nilpotent.

Let $V$ be any proper subgroup of $G$ containing $P$. Then by Lemma 2.2(1), $V$ satisfies the hypothesis of the theorem. By the choice of $G$, $V$ is $p$-nilpotent. Thus (2) follows.

(3) $O_{p'}(G) = 1$.

If not, then by Lemma 2.2(3), $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. By the choice of $G$, $G/O_{p'}(G)$ is $p$-nilpotent, and so $G$ is $p$-nilpotent, which is impossible.

(4) $G$ is not a non-abelian simple group.
Assume that $G$ is a non-abelian simple group. Then by Feit-Thompson’s Theorem, we have that $p = 2$. Let $H$ be a subgroup of $P$ of order $|D|$. Then clearly, $H$ is an $\mathcal{H}$-subgroup of $G$. Hence by Lemma 2.1(2), $H$ is an $\mathcal{H}$-subgroup of $P$, and thus $H \leq P$ by Lemma 2.1(1). It follows from (2) that $N_G(H)$ is 2-nilpotent for $H \not\leq G$, and so $N_G(H)/C_G(H)$ is a 2-group. By Lemma 2.4, $H$ is a Sylow 2-subgroup of $G$, a contradiction. Therefore, $G$ is not a non-abelian simple group.

(5) $O_p(G) > 1$, and every proper normal subgroup of $G$ is contained in $O_p(G)$.

Let $L$ be a proper normal subgroup of $G$. Then we only need to prove that $L$ is a $p$-group. By (3), $p | |L|$. If $|L|_p > |D|$, then $L$ satisfies the hypothesis of the theorem by Lemma 2.2(1). Hence $L$ is $p$-nilpotent due to the choice of $G$. It follows from (3) that $L$ is a $p$-group. Now consider that $|L|_p \leq |D|$. Then there exists a normal subgroup $K$ of $P$ such that $P \cap L \leq K$ and $|K| = p|D|$. This induces that $|LK|_p = |K| = p|D|$, and so $K$ is a Sylow $p$-subgroup of $LK$. If $LK = G$, then $|P| = |K| = p|D|$, which contradicts (1). Thus $LK < G$. By Lemma 2.2(1), $LK$ satisfies the hypothesis of the theorem. Then by the choice of $G$, $LK$ is $p$-nilpotent, and so is $L$. Hence $L$ is a $p$-group by (3), and consequently (5) holds.

(6) Every $\mathcal{H}C$-subgroup of $G$ contained in $P$ is an $\mathcal{H}$-subgroup of $G$.

Let $V$ be any $\mathcal{H}C$-subgroup of $G$ contained in $P$. Then there exists a normal subgroup $T$ of $G$ such that $G = VT$ and $V^g \cap N_T(V) \leq V$ for all $g \in G$. By (5), since $G$ is not a $p$-group, we have that $T = G$. Therefore, $V$ is an $\mathcal{H}$-subgroup of $G$.

(7) Let $N$ be a minimal normal subgroup of $G$ contained in $O_p(G)$. Then $|N| \leq |D|$ and $G/N$ is $p$-nilpotent.

If $|N| > |D|$, then there exists a subgroup $H$ of $N$ of order $|D|$ such that $H$ is an $\mathcal{H}$-subgroup of $G$ by (6). It follows from Lemma 2.1(1) that $H$ is normal in $G$, which is impossible. Hence $|N| \leq |D|$. First suppose that $|N| < |D|$. Then by (6) and Lemma 2.1(3), $G/N$ satisfies the hypothesis of the theorem. By the choice of $G$, $G/N$ is $p$-nilpotent.

Now consider that $|N| = |D|$. We claim that every cyclic subgroup of $P/N$ of order prime or 4 (when $p = 2$) is normal in $N_G(P)/N$. Let $X/N$ be a subgroup of $P/N$ of order $p$. If $N \leq \Phi(X)$, then $X$ is cyclic, and so is $N$. This implies that $|N| = |D| = p$. Then by (6), every cyclic subgroup of $P$ of order $p$ or 4 (when $p = 2$) is an $\mathcal{H}$-subgroup of $G$. By Lemmas 2.1(1) and 2.1(2), every cyclic subgroup of $P$ of order $p$ or 4 (when $p = 2$) is normal in $N_G(P)$. Since $p$ is the smallest prime divisor of $|G|$, every minimal subgroup of $P$ lies in $Z(N_G(P))$. Hence by Lemma 2.5, $G$ is $p$-nilpotent, a contradiction. Thus $N \not\leq \Phi(X)$, and so $X$ has a maximal subgroup $S$ such that $X = SN$. Since $|S| = |N| = |D|$, $S$ is an $\mathcal{H}$-subgroup of $G$ by (6). By Lemmas 2.1(1) and 2.1(2), $S \leq N_G(P)$, and thus $X/N = SN/N \leq N_G(P)/N$. This shows that the claim holds when $p$ is odd. Consider that $p = 2$. Then by (1), $|P : D| > 2$. Let $Y/N$ be a cyclic subgroup of $P/N$ of order 4. If $N \leq \Phi(Y)$, then $Y$ is cyclic. This implies that $|N| = |D| = 2$, a contradiction. Thus $N \not\leq \Phi(Y)$, and
so $Y$ has a maximal subgroup $U$ such that $Y = UN$. Clearly, $|U| = 2|D|$. Since $U/U \cap N \cong Y/N$ is a cyclic group of order 4, by hypothesis and (6), $U$ is an $\mathcal{H}$-subgroup of $G$. A similar discussion as above shows that $Y/N = UN/N \leq N_G(P)/N$. Hence the claim holds when $p = 2$. Since $p$ is the smallest prime divisor of $|G|$, every minimal subgroup of $P/N$ lies in $Z(N_G(P)/N)$. Therefore, $G/N$ is $p$-nilpotent by Lemma 2.5.

(8) Final contradiction.

By (7), $G/N$ is $p$-nilpotent. Then $G$ has a normal subgroup $M$ of $G$ such that $|G : M| = p$. By (1), $|M| > |D|$. Then by (6) and Lemma 2.1(2), $M$ satisfies the hypothesis of the theorem. Hence $M$ is $p$-nilpotent due to the choice of $G$, and so $G$ is $p$-nilpotent. The final contradiction completes the proof. \hfill \Box

The following corollary can be deduced immediately from Lemma 2.2(3) and Theorem 3.1.

**Corollary 3.2.** Suppose that every noncyclic Sylow subgroup $P$ (if exists) of $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ of order $|D|$ is an $\mathcal{H}C$-subgroup of $G$. When $P$ is a Sylow 2-subgroup of $G$ and $|P : D| > 2$, suppose further that $H$ is an $\mathcal{H}C$-subgroup of $G$ if there exists $D_1 \leq H \leq P$ such that $2|D_1| = |D|$ and $H/D_1$ is a cyclic group of order 4. Then $G$ has a Sylow tower of supersolvable type.

**Theorem 3.3.** Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and let $G$ be a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup $P$ (if exists) of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ of order $|D|$ is an $\mathcal{H}C$-subgroup of $G$. When $P$ is a Sylow 2-subgroup of $F^*(E)$ and $|P : D| > 2$, suppose further that $H$ is an $\mathcal{H}C$-subgroup of $G$ if there exists $D_1 \leq H \leq P$ such that $2|D_1| = |D|$ and $H/D_1$ is a cyclic group of order 4. Then $G \in \mathcal{F}$.

**Proof.** Suppose that the result is false and let $(G, E)$ be a counterexample such that $|G| + |E|$ is minimal. Then we proceed via the following steps.

1. $F^*(E) = F(E)$.

By Lemma 2.2(1) and Corollary 3.2, $F^*(E)$ has a Sylow tower of supersolvable type, and so $F^*(E)$ is solvable. It follows from Lemma 2.6(2) that $F^*(E) = F(E)$.

2. There exist a noncyclic Sylow $p$-subgroup $P$ of $F(E)$ and a subgroup $H$ of $P$ of order $|D|$ or $2|D|$ (when $p = 2$, and there exists $D_1 \leq H \leq P$ such that $2|D_1| = |D|$ and $H/D_1$ is a cyclic group of order 4) such that $|P : D| > p$ and $H$ is not an $\mathcal{H}$-subgroup of $G$.

Suppose that for every prime divisor $p$ of $|F(E)|$ and every noncyclic Sylow $p$-subgroup $P$ of $F(E)$, either $|P : D| = p$ or all subgroups $H$ of $P$ of order $|D|$ or $2|D|$ (when $p = 2$, $|P : D| > 2$ and there exists $D_1 \leq H \leq P$ such that $2|D_1| = |D|$.
and \( H/D_1 \) is a cyclic group of order 4) are \( \mathcal{H} \)-subgroups of \( G \). In the former case, by Lemma 2.7, all subgroups \( H \) of \( P \) of order \( |D| \) are c-normal in \( G \). In the latter case, by Lemma 2.1(1), all subgroups \( H \) of \( P \) of order \( |D| \) or \( 2|D| \) (when \( p = 2 \), \( |P : D| > 2 \) and there exists \( D_1 \leq H \leq P \) such that \( 2|D_1| = |D| \) and \( H/D_1 \) is a cyclic group of order 4) are normal in \( G \). Hence by [17] Theorem 1.4], \( G \in \mathfrak{F} \), a contradiction. Thus (2) holds.

(3) Final contradiction.

By hypothesis and (2), \( H \) is an \( \mathcal{H} \)-subgroup of \( G \) and \( H \) is not an \( \mathcal{H} \)-subgroup of \( G \). Hence \( G \) has a normal subgroup \( M \) such that \( |G : M| = p \) and \( G = HM \) by Lemma 2.8. Since \( G = HM = EM \), we have that \( M/E \cap M \cong G/E \in \mathfrak{F} \).

By Lemma 2.6(1), \( F^*(E \cap M) = F^*(E) \cap M = F(E) \cap M \). Note that \( |F(E) : F(E) \cap M| = |G : M| = p \) and \( |F(E) \cap M_p| > |D| \) by (2). Then clearly, \( (M, E \cap M) \) satisfies the hypothesis of the theorem by Lemma 2.2(1). By the choice of \( (G, E) \), \( M \in \mathfrak{F} \). It follows from Lemma 2.10 that \( G \in \mathfrak{F} \). The final contradiction completes the proof.

\[ \square \]

**Theorem 3.4.** Let \( \mathfrak{F} \) be a saturated formation containing all supersolvable groups and let \( G \) be a group with a normal subgroup \( E \) such that \( G/E \in \mathfrak{F} \). Suppose that every noncyclic Sylow subgroup \( P \) (if exists) of \( E \) has a subgroup \( D \) such that \( 1 < |D| < |P| \) and every subgroup \( H \) of \( P \) of order \( |D| \) is an \( \mathcal{H} \)-subgroup of \( G \). When \( P \) is a Sylow 2-subgroup of \( E \) and \( |P : D| > 2 \), suppose further that \( H \) is an \( \mathcal{H} \)-subgroup of \( G \) if there exists \( D_1 \leq H \leq P \) such that \( 2|D_1| = |D| \) and \( H/D_1 \) is a cyclic group of order 4. Then \( G \in \mathfrak{F} \).

**Proof.** Suppose that the result is false and let \( (G, E) \) be a counterexample such that \( |G| + |E| \) is minimal. By Lemma 2.2(1) and Corollary 3.2, we see that \( E \) has a Sylow tower of supersolvable type. Without loss of generality, let \( p \) be the largest prime divisor of \( |E| \). Then \( P \leq G \). By Lemma 2.2(3), \( (G/P, E/P) \) satisfies the hypothesis of the theorem. Then the choice of \( (G, E) \) implies that \( G/P \in \mathfrak{F} \). Hence \( (G, P) \) satisfies the hypothesis of Theorem 3.3, and so \( G \in \mathfrak{F} \).

\[ \square \]

**Theorem 3.5.** Let \( E \) be a normal subgroup of \( G \) such that \( G/E \) is nilpotent. Suppose that every minimal subgroup of \( E \) is contained in \( Z_\infty(G) \), and every cyclic subgroup of \( E \) of order 4 is an \( \mathcal{H} \)-subgroup of \( G \). Then \( G \) is nilpotent.

**Proof.** Assume that the result is false and let \( (G, E) \) be a counterexample such that \( |G| + |E| \) is minimal. Then we prove the theorem via the following steps.

(1) \( G \) is a minimal nonnilpotent group, that is, \( G = P \rtimes Q \), where \( P \) is a normal Sylow \( p \)-subgroup of \( G \) and \( Q \) is a nonnormal cyclic Sylow \( q \)-subgroup of \( G \) for some prime \( q \neq p \); \( P/\Phi(P) \) is a chief factor of \( G \); \( exp(P) = p \) when \( p > 2 \) and \( exp(P) \) is at most 4 when \( p = 2 \).

Let \( K \) be any proper subgroup of \( G \). Then \( K/E \cap K \cong EK/E \leq G/E \) is nilpotent, and every minimal subgroup of \( E \cap K \) is contained in \( Z_\infty(G) \cap K \leq Z_\infty(K) \).
By hypothesis, every cyclic subgroup of $E \cap K$ of order 4 is an $\mathcal{H}C$-subgroup of $G$. Thus by Lemma 2.2(1), every cyclic subgroup of $E \cap K$ of order 4 is an $\mathcal{H}C$-subgroup of $K$. Hence $(K, E \cap K)$ satisfies the hypothesis of the theorem. Then the choice of $(G, E)$ implies that $K$ is nilpotent. Hence $G$ is a minimal nonnilpotent group, and so (1) holds by [12, Chapter III, Satz 5.2].

(2) $P \leq E$.

If not, then $P \cap E < P$, and so $(P \cap E)Q < G$. By (1), $(P \cap E)Q$ is nilpotent. This implies that $Q \leq (P \cap E)Q$. Since $G/P \cap E \leq G/P \times G/E$ is nilpotent, $(P \cap E)Q \leq G$, and thus $Q \leq G$, a contradiction.

(3) Final contradiction.

If $\exp(P) = p$, then $P \leq Z_\infty(G)$, and so $G$ is nilpotent, which is impossible. Hence we may assume that $p = 2$ and $\exp(P) = 4$. Then by Lemma 2.11, every cyclic subgroup of $P$ of order 4 is an $\mathcal{H}$-subgroup of $G$, and so every cyclic subgroup of $P$ of order 4 is normal in $G$ by Lemma 2.1(1). Take an element $x \in P \setminus \Phi(P)$. Since $P/\Phi(P)$ is a chief factor of $G$, $P = \langle x \rangle \Phi(P) = \langle x \rangle^G$. If $x$ is of order 2, then $P = \langle x \rangle^G \leq Z_\infty(G)$, a contradiction. Now assume that $x$ is of order 4. Then $\langle x \rangle \leq G$, and so $P = \langle x \rangle$ is cyclic. By [19, (10.1.9)], $G$ is 2-nilpotent, and so $Q \leq G$. This is the final contradiction. \qed

**Theorem 3.6.** Let $E$ be a normal subgroup of $G$ such that $G/E$ is nilpotent. Suppose that every minimal subgroup of $F^*(E)$ is contained in $Z_\infty(G)$, and every cyclic subgroup of $F^*(E)$ of order 4 is an $\mathcal{H}C$-subgroup of $G$. Then $G$ is nilpotent.

**Proof.** Assume that the result is false and let $(G, E)$ be a counterexample such that $|G| + |E|$ is minimal. Then we prove the theorem via the following steps.

(1) Every proper normal subgroup of $G$ is nilpotent.

Let $K$ be any proper normal subgroup of $G$. Then $K/E \cap K \cong EK/E \leq G/E$ is nilpotent. By Lemma 2.6(1), $F^*(E \cap K) = F^*(E) \cap K$. Hence by Lemma 2.2(1), $(K, E \cap K)$ satisfies the hypothesis of the theorem. The the choice of $(G, E)$ implies that $K$ is nilpotent.

(2) $E = G = \gamma_\infty(G)$ and $F^*(G) = F(G) < G$, where $\gamma_\infty(G)$ is the nilpotent residual of $G$.

If $E < G$, then $E$ is nilpotent by (1), and so $F^*(E) = F(E) = E$. By Theorem 3.5, $G$ is nilpotent, a contradiction. Thus $E = G$. Now suppose that $F^*(G) = G$. Then by Theorem 3.5 again, $G$ is nilpotent, which is impossible. Hence $F^*(G) < G$, and $F^*(G) = F(G)$ by (1). If $\gamma_\infty(G) < G$, then by (1), $\gamma_\infty(G) \leq F(G)$, and so $G/F(G)$ is nilpotent. It follows from Theorem 3.5 that $G$ is nilpotent, a contradiction. Thus $\gamma_\infty(G) = G$.

(3) every cyclic subgroup of $F(G)$ of order 4 is contained in $Z(G)$.

By hypothesis and (2), every cyclic subgroup $H$ of $F(G)$ of order 4 is an $\mathcal{H}C$-subgroup of $G$. Then there exists a normal subgroup $T$ of $G$ such that $G = HT$.
and \( H^g \cap N_T(H) \leq H \) for all \( g \in G \). If \( T < G \), then \( T \leq F(G) \) by (1), and thereby \( F(G) = G \), a contradiction. Hence \( T = G \), and so \( H \) is an \( \mathcal{H} \)-subgroup of \( G \). By Lemma 2.1(1), \( H \leq G \). This implies that \( G/C_G(H) \) is abelian. Then by (2), \( C_G(H) = \gamma_\infty(G) = G \), and so \( H \leq Z(G) \). Thus (3) holds.

(4) Final contradiction.

Let \( p \) be any prime divisor of \(|F(G)|\) and let \( P \) be the Sylow \( p \)-subgroup of \( F(G) \). Then \( P \leq G \). If \( p \) is odd, then by hypothesis, \( \Omega_1(P) \leq Z_\infty(G) \). It follows from Lemma 2.12 that \( O^p(G) \leq C_G(\Omega_1(P)) \), and so \( O^p(G) \leq C_G(P) \) by Lemma 2.13. Then by (2), \( C_G(P) = \gamma_\infty(G) = G \). Now consider that \( p = 2 \). Then by hypothesis and (3), \( \Omega_2(P) \leq Z_\infty(G) \). A similar discussion as above also shows that \( C_G(P) = G \). Therefore, we have that \( C_G(F(G)) = G \), which contradicts the fact that \( C_G(F(G)) \leq F(G) \) by (2) and Lemma 2.6(3). The proof is thus completed.

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