LETTER

Collective states of interacting $D(D_3)$ non-Abelian anyons

Peter E Finch and Holger Frahm
Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany
E-mail: peter.finch@itp.uni-hannover.de and frahm@itp.uni-hannover.de

Received 4 January 2012
Accepted 14 April 2012
Published 3 May 2012

Online at stacks.iop.org/JSTAT/2012/L05001
doi:10.1088/1742-5468/2012/05/L05001

Abstract. We study the finite size spectra of integrable quantum chains of interacting non-Abelian anyons constructed using the Drinfeld double of the dihedral group $D_3$. The gapless low energy modes are identified as the direct product of two conformal field theories which can be decomposed according to the residual symmetries of the chains subject to periodic boundary conditions.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz), conformal field theory (theory), finite-size scaling

ArXiv ePrint: 1108.3228
Ground state degeneracies in a many particle system may be an indication for the presence of topological order without a local order parameter. Well known examples for such topological quantum liquids are the fractional Hall states [1]. Another class of systems where transitions between quantum phases driven by topology have been proposed are two-dimensional frustrated quantum magnets [2]–[4]. Possible realizations for these spin-liquid states are certain iridium compounds [5]. As a first step towards the characterization of the different phases in a given model the quasi-particles of the corresponding theory need to be identified. In a (2 + 1)-dimensional topological quantum liquid these collective excitations can be described as defects in a planar gauge theory in a broken symmetry phase with finite residual gauge group $H$ [6]. The quasi-particles in these systems are irreducible representations (irreps) of the Drinfeld double $D(H)$ labelled by their flux, i.e. an element of $h \in H$, and their topological charge determined by the transformation properties under the residual global symmetry commuting with the flux $h$. States with several quasi-particles can be manipulated by two operations: (1) fusion is determined by the decomposition of product states into irreps of the Drinfeld double; (2) interchange of two constituents in this state may reveal non-trivial anyonic statistics corresponding to a representation of the braid group. Hence, representations of Drinfeld doubles allow for the description of particles with anyonic statistics. Apart from the appearance of a phase factor, as in the case of Abelian anyons, braiding may correspond to a unitary rotation of the wavefunction in a degenerate manifold. The fact that these non-Abelian anyons are protected by their topological charge makes them potentially interesting as resources for quantum computation [7,8].

This picture provides a framework for the investigation of topological phases and quantum phase transitions [9]. Microscopic lattice models satisfying these constraints have been obtained for the non-Abelian degrees of freedom in $su(2)_k$ Chern Simons theories, e.g. Ising [4] and Fibonacci anyons [10]–[12]. Local interactions between these objects can be varied to favour certain fusion channels. This allows us to explore the phase diagram of these systems and to study their critical properties near quantum phase transitions. This approach works particularly well for quasi-one-dimensional anyonic models such as chains or ladders where powerful numerical and analytical methods are available. At the quantum phase transition the low energy effective theory of these systems is expected to be a conformal field theory (CFT) and the universality class is determined by the central charge of the underlying Virasoro algebra. At the same time anyonic chains can be seen as realizations for the interface between phases with different topological order [13]–[16] and the CFT determines the properties of gapless edge modes propagating along these interfaces.

In this letter we study the critical behaviour of a system of interacting anyons within an integrable quantum chain model which is constructed directly using the algebraic structure of the gauge theory with the dihedral group of order 6 as its gauge group, i.e. the Drinfeld double $D(D_3)$. The general representations of the Drinfeld doubles of finite group algebras are well known [17]: for $D(D_3)$ one has two one-dimensional irreducible representations $\pi_1^{\pm}$, four two-dimensional ones $\pi_2^{(a,b)}$, $(a,b) = (0,1)$ or $(1,b)$ with $b = 0,1,2$, and two three-dimensional ones $\pi_3^{\pm}$. This leads to an increased number of fusion channels in $D(D_3)$ models as compared to the $su(2)_k$ anyon chains, see e.g. [7,18] in the context of 2D lattice models. On the other hand $D(D_3)$ forms a quasi-triangular Hopf algebra which provides it with a natural tensor product structure; the so-called quantum
dimensions of these anyons are integers. Therefore we can consider a chain of length $L$ with a $D(D_3)$ anyon with three internal states represented by a copy of $\pi^+_3$ on each site. This spin chain is a particular limit of the spin 1 Fateev–Zamolodchikov model [19] and differs from the corresponding anyonic chain in the boundary conditions. For periodic boundary conditions the resulting integrable model is given by a one parametric Hamiltonian with interactions between spins on neighbouring sites [20]

$$H_\theta = \sum_{i=1}^{L} \cos \theta \ h^{(1)}_{i,i+1} + \sin \theta \ h^{(2)}_{i,i+1}. \quad (1)$$

In terms of the projection operators on irreps [21] appearing in the tensor product $\pi^+_3 \otimes \pi^+_3$ the local Hamiltonian can be expressed as

$$h^{(1)}_{i,i+1} = \frac{2\sqrt{3}}{3} p^+_1 - \frac{\sqrt{3}}{3} p^+_2. \quad (2)$$

The operator $h^{(2)} = PH^{(1)}P = (h^{(1)})^\dagger$ can be obtained either by a permutation $P$ of the spins on the neighbouring sites or by complex conjugation; with the exception of $p^{(1,3)}_2$ the projection operators are invariant under these operations. Depending on the parameter $\theta$ the interactions in (1) favour different channels for the fusion of spins on neighbouring sites, e.g. the vacuum channel $\pi^+_1$ for $\pi < \theta < 3\pi/2$.

By construction the local Hamiltonians have the full $D(D_3)$ symmetry. This symmetry, however, is broken by imposing periodic boundary conditions in the global Hamiltonian (1), see [20]. Based on the representation theory of $D(D_3)$ the tensor product $(\pi^+_3)^\otimes L$ can be decomposed into a sum of one- and two-dimensional irreps for even $L$ (as in equation (2) for $L = 2$). For the periodic chain (1) only a partial decomposition of the Hilbert space is possible. Below we shall use this fact to assign states to (sums of) irreps $\pi^+_1 \oplus \pi^-_1$, $\pi^{(0,1)}_2 \oplus \pi^{(1,0)}_2$, or $\pi^{(1,1)}_2 \oplus \pi^{(1,2)}_3$. For odd $L$ only the three-dimensional irreps $\pi^+_3$ appear in the tensor product $(\pi^+_3)^\otimes L$. In the spectrum of the periodic chain these irreps cannot be separated but we find a connection between the eigenstates and their transformation properties under the residual $D_3$ symmetry, namely the action of the rotation $\sigma$.

For the analysis of the spectrum of the quantum chain (1) we use the property that its Hamiltonian can be split into two commuting ones with identical spectra; as a consequence of the decomposition (2) the local Hamiltonians $h^{(1,2)}$ commute as do the global ones (1), $[H_\theta, H_\pi] = 0$. Furthermore, $H_0 \equiv \sum_i h^{(1)}_{i,i+1}$ and $H_{\pi/2} \equiv \sum_i h^{(2)}_{i,i+1}$ are related by a spatial inversion and have the same eigenvalues albeit with opposite momentum. These properties can be traced back to the underlying two-parameter transfer matrix of this model and its symmetries [20, 22]. The eigenvalues of $H_0 = -H_{\pi}$ can be parametrized by $L$ complex rapidities $x_j$ ($\omega = \exp(2\pi i/3)$)

$$E(X = \{x_j\}) = \sum_{j=1}^{L} \frac{1}{\epsilon^{x_j}} \ - \frac{\sqrt{3}}{3} \omega L \quad (3)$$

Solving the Bethe equations ($j = 1, \ldots, L$):

$$(-1)^{L+1} \left( \frac{1 + (i/\omega)e^{x_j}}{1 - i/\omega e^{x_j}} \right)^L = \prod_{k=1}^{L} \frac{e^{x_k} - \omega e^{x_j}}{e^{x_k} - e^{x_j}}. \quad (4)$$

doi:10.1088/1742-5468/2012/05/L05001
The spectrum of the quantum chain (1) is given by pairs of solutions to these equations corresponding to energies \( E_{(\alpha,\beta)}(\theta) = \cos \theta E(X_\alpha) + \sin \theta E(X_\beta) \) and, similarly, momenta \( p = p(X_\alpha) - p(X_\beta) \), provided that the combination \((\alpha, \beta)\) satisfies the pairing rules discussed below.

In the thermodynamic limit, \( L \to \infty \), all finite solutions of equations (4) can be grouped into three types of so-called strings [20, 23]; \( \pm \)-strings correspond to solutions \( \text{Im}(x_j) = 0, \pi \) and \( 2 \)-strings are complex conjugate pairs of rapidities \( x_j,_{\pm} = \tilde{x}_j \pm i2\pi/3 \) with real centre \( \tilde{x}_j \). For finite chains this classification continues to work very well for the ground states and low energy excitations of \( \pm H_0 \). At higher energies the \( 2 \)-strings become deformed, i.e. have an imaginary part different from \( \pm 2\pi/3 \). We have numerically diagonalized the transfer matrix for chains of up to \( L = 10 \) sites and found that (taking into account this deformation) all eigenstates can be classified this way. Denoting the number of string solutions by \( N_\pm, N_2 \) we find \( N_+ + N_- + 2N_2 = L - n_{+\infty} - n_{-\infty} \) where \( n_{\pm\infty} \in \{0, 1\} \) is the number of Bethe roots at \( x = \pm \infty \). We also find that there exist only \( 4 \times 3^{L/2-1} \) (\( 2 \times 3^{(L-1)/2} \)) different root configurations solving (4) for even (odd) length chains. This implies that the spectrum \( H_\pi \) displays massive degeneracies (exponential in \( L \)) arising from level crossings when \( \theta \) is a multiple of \( \pi/2 \). For generic values of \( \theta \) the degeneracies of the level \( E_{(\alpha,\beta)}(\theta) \) are lifted up to a remaining ‘pairing multiplicity’ of its components \( \alpha \) and \( \beta \). Due to the pairing mechanism these components can be discussed separately. For the analysis of the low energy spectrum of (1) this amounts to the identification of the ground state and low lying excitations of \( H_\pi = -H_0 \) and \( H_0 \).

The ground state energy of \( H_\pi \) has been computed in [20]. For even length lattices its root configuration is given by a distribution of \( N_2 = L/2 \) \( 2 \)-strings. In the thermodynamic limit density functions for these strings can be introduced allowing the energy density \( \epsilon_\infty \) to be computed. Here we have extended the analysis of the Bethe equations to excitations close to this state: in the corresponding configurations one or more of the \( 2 \)-strings are replaced by \( \pm \)-strings and/or Bethe roots at \( \pm \infty \). The spectrum of these excitations has a linear dispersion with Fermi velocity \( v_F \) allowing identification of the CFT for the low energy modes from the finite size scaling behaviour of the energies for large but finite \( L \) [24]. At the critical point the ground state energy of a \( 1+1 \)-dimensional quantum system scales as \( E_0(L) - L\epsilon_\infty = -(\pi v_F/6L) c \) where \( c \) is the universal central charge of the underlying Virasoro algebra. From the energy and momentum of low lying excitations in the finite system

\[
E(L) - E_0(L) = \frac{2\pi v_F}{L} (X + n + \bar{n}), \quad p(L) - p_\infty = \frac{2\pi}{L} (s + n - \bar{n}) \quad (5)
\]

the scaling dimensions \( X = h + \bar{h} \) and conformal spins \( s = h - \bar{h} \) of the primary fields in the theory can be determined \((n, \bar{n} \) are non-negative integers).

For \( H_\pi \) the product \( v_{F,x} c \) had been determined to be 12/5 previously [20]. Using Bethe ansatz methods and comparing the observed structure of the low energy spectrum with (5) we find the Fermi velocity for this sector to be \( v_{F,x} = 3 \). Hence the central charge of the effective field theory for the low energy degrees of freedom in \( H_\pi \) is \( c = 4/5 \). This sector of the model is in the universality class of the minimal model \( \mathcal{M}_{(5,6)} \); the conformal weights \( h, \bar{h} \) of the primary fields can take the rational values from the Kac table \( h_{pq} = ((6p-5q)^2-1)/120, 1 \leq q \leq p < 5 \). The operator content of a given realization of the CFT is constrained further by modular invariance of the partition function, locality of the physical fields and boundary conditions [25, 26].
the same sector with respect to the residual components. From our numerical analysis of the spectrum we find that only operators in Momentum and spin of a physical state are given by the difference between that of its two \( \pi \) \( \tilde{\pi} \) with fixed \( \Phi(D) \) \( \Phi(H) \) \( \Phi(h, \tilde{h}) \) are the predictions from the \( M(5,6) \) minimal model. We have also indicated the \( D(D_3) \) sector in which the state appears and its pairing multiplicity. The operator content of the sector \( \pi_2^{(1,0)} \) is obtained from that of \( \pi_2^{(0,1)} \) by interchanging \( h \) and \( \tilde{h} \).

### Table 1. Scaling dimensions \( X^\text{num} \) extrapolated from the finite size behaviour of the ground state and low energy excitations of \( H_\pi \) for even \( L \). \( (h, \tilde{h}) \) are the predictions from the \( M(5,6) \) minimal model. We have also indicated the \( D(D_3) \) sector in which the state appears and its pairing multiplicity. The operator content of the sector \( \pi_2^{(1,0)} \) is obtained from that of \( \pi_2^{(0,1)} \) by interchanging \( h \) and \( \tilde{h} \).

| \( D(D_3) \) | \( X^\text{num} \) | \( (h, \tilde{h}) \) | Spin | Pairing mult. |
|----------------|------------------|------------------|------|---------------|
| \( \pi_1^+ \oplus \pi_1^- \) | 0.000000(1) | (0, 0) | 0 | 1 |
| | 0.801(3) | (\( \frac{2}{3} \), \( \frac{2}{3} \)) | 0 | 1 |
| | 1.80(1) | (\( \frac{1}{3}, \frac{2}{3} \), \( \frac{2}{3}, \frac{1}{3} \)) | \( \pm 1 \) | 1 |
| \( \pi_2^{(0,1)} \) | 0.4668(2) | (\( \frac{1}{15}, \frac{2}{15} \)) | \( -\frac{1}{4} \) | 2 |
| | 0.666666(1) | (\( \frac{5}{9}, 0 \)) | \( \frac{2}{3} \) | 2 |
| \( \pi_2^{(1,1)} \oplus \pi_2^{(1,2)} \) | 0.133334(6) | (\( \frac{1}{15}, \frac{11}{15} \)) | 0 | 4 |
| | 1.333333(3) | (\( \frac{2}{3}, \frac{2}{3} \)) | 0 | 4 |

### Table 2. As table 1 for odd \( L \). Symmetry is classified by the action of the \( D_3 \) rotation \( \sigma \).

| \( \sigma \) | \( X^\text{num} \) | \( (h, \tilde{h}) \) | Spin | Pairing mult. |
|---------------|------------------|------------------|------|---------------|
| 1 | 0.125000(5) | (0, \( \frac{1}{2} \)) | \( -\frac{1}{8} \) | 1 |
| | 0.42502(2) | (\( \frac{2}{3}, \frac{1}{3} \)) | \( \frac{3}{8} \) | 1 |
| | 0.92490(6) | (\( \frac{4}{9}, \frac{4}{9} \)) | \( -\frac{1}{8} \) | 1 |
| | 1.625000(1) | (0, \( \frac{13}{8} \)) | \( -\frac{13}{8} \) | 1 |
| \( \omega, \omega^{-1} \) | 0.091665(2) | (\( \frac{1}{15}, \frac{11}{15} \)) | \( \frac{1}{27} \) | 2 |
| | 0.59168(7) | (\( \frac{1}{15}, \frac{2}{3} \)) | \( -\frac{1}{11} \) | 2 |
| | 0.791667(1) | (\( \frac{2}{3}, \frac{1}{3} \)) | \( \frac{11}{27} \) | 2 |

In tables 1 and 2 we present numerical results for the scaling dimensions identified in the excitation spectrum of \( H_\pi \) together with the conformal predictions. For the states given we have solved the Bethe equations up to a minimum of 40 sites, although in general over 100 sites were considered when possible. Also listed are the residual symmetry sectors in which the levels appear. For \( L \) even the low energy spectrum of \( H_\pi \) for a given symmetry coincides with that of the three-state Potts model subject to cyclic boundary conditions with fixed \( Z_3 \) charge \([25, 27]\) containing spin \( \frac{1}{3} \) parafermions in the sectors \( \pi_2^{(0,1)} \) and \( \pi_2^{(1,0)} \). For \( L \) odd ‘twist operators’ with conformal weights \( h_{pq} \) with \( q \) even appear in the (anti)holomorphic sector. The ground state is not invariant under a \( D_3 \) rotation.

Note that in spite of the appearance of fields \( \Phi(h, \tilde{h}) \) with conformal spin \( s = h - \tilde{h} \notin \mathbb{Z}/2 \) physical operators in the theory are local; they are direct products of primary fields \( \Phi(h_\alpha, \tilde{h}_\alpha)\Phi(h_\beta, \tilde{h}_\beta)^* \) in one-to-one correspondence to the energy levels \( E_{(\alpha, \beta)} \) of (1). Momentum and spin of a physical state are given by the difference between that of its two components. From our numerical analysis of the spectrum we find that only operators in the same sector with respect to the residual \( D(D_3) \)-symmetry (the action of the rotation \( \sigma \)) pair for \( L \) even (odd). With this rule the total spin of a physical field is either integer.
or half-integer. For \(-\pi < \theta < -\pi/2\) the low energy states are obtained by pairing of the states listed in tables 1 and 2. In the \(\pi_2^{(0,1)}\) sector of the model for even \(L\) the physical fields allowed by the pairing rules carry spin 0 or 1, for odd \(L\) fields with spin 0, \(\frac{1}{2}\), 1 are possible. Note that the observed number of different root configurations together with the pairing multiplicity indicated in the tables yields the total number of 3\(^L\) states of the quantum chain.

The ground state of \(+H_0\) is given by a solution of the Bethe equations (4) with \(N_+ = L/4+\)-strings and \(N_- = 3L/4--\)-strings [20]. It is realized for lattices of length \(L = 0 \pmod{4}\) and its energy scales as \(E_0(L) - L\epsilon_\infty = -3\pi/12L\) [20]. The root configurations for the lowest excitations differ from this one by the replacement of one or two of the strings by roots at \(\pm \infty\). For excitations at higher energies 2-strings have to be taken into account. Following [28]-[31] the lowest finite size energy gaps are found to be

\[
\frac{L\Delta E(\Delta N_\pm, Q_{\pm})}{2\pi} = \frac{1}{4} \left( (\Delta N_\pm)^2 - \Delta N_+ \Delta N_- + (\Delta N_-)^2 \right) + \frac{3}{4} \left( (Q_\pm)^2 + Q_+ Q_- + (Q_-)^2 \right).
\]

For the lattice model the numbers \(\Delta N_\pm\) are related to the change in the number of \(\pm\)-strings as compared to the ground state, i.e. take values \(\mp L/4 \pmod{1}\). They are further constrained by the fact that the total number of Bethe roots has to be \(L\). \(Q_{\pm}\) can take values \(Q_\pm \cong -\Delta N_\pm + \frac{3}{4}(n_\infty + n_- - n_+) \pmod{1}\). We can determine the Fermi velocity of low lying excitations in this sector as before finding \(v_{F,0} = 3/2\).

The effective field theory for this part of the spectrum is a CFT with central charge \(c = 1\). The field content of the theory is obtained from the finite size spectrum (6) subject to the constraints mentioned. It can be identified with that of a \(Z_4\) parafermionic theory [32,33], see tables 3–5. In particular, the finite size gap of the lowest states for \(\ell = L \pmod{4} \neq 0\) is determined by an (anti-)chiral \(Z_{k=1}\) spin field with conformal weight \(h_{\ell} = \ell(k-\ell)/(2k(k+2))\).

As in the spectrum of \(H_{\pi}\), we find states with fractional conformal spin. The physical fields obtained after application of the pairing rules discussed above, however, have integer spin for \(0 < \theta < \pi/2\) and \(L = 0 \pmod{4}\). For lattices of length \(L = 2 \pmod{4}\) the spins can take integer or half-integer values while we find \(Z_4\) parafermions with quarter spin in the spectrum of chains with \(L\) odd. For \(\theta \in (\pi/2, \pi)\) or \((-\pi/2, 0)\) the operators from the minimal model \(\mathcal{M}_{5,6}\) and the \(Z_4\) CFT are paired. In this regime parafermionic fields with quarter spin are present for \(L = 2 \pmod{4}\).
Table 4. As table 3 but for \( L = 2 \pmod{4} \).

| \( D(D_3) \) | \( X_0^{num} \) | \( (h, \bar{h}) \) | Spin | Pairing mult. |
|----------------|----------------|----------------|------|--------------|
| \( \pi_1^1 \oplus \pi_1^1 \) | 0.750000 | \( (0, \frac{1}{4}) \times 2, (\frac{1}{2}, 0) \times 2 \) | \( \pm \frac{3}{2} \) | 1 |
| \( \pi_2^{(0,1)} \) | 0.083333 | \( (0, \frac{1}{12}) \) | \(-\frac{1}{2} \) | 2 |
| \( 1.083333 \) | \( (\frac{1}{4}, \frac{1}{4}) \) | \( \pm \frac{1}{4} \) | 2 |
| \( \pi_2^{(1,1)} \oplus \pi_2^{(1,2)} \) | 0.416667 | \( (\frac{1}{12}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{12}) \) | \( \pm \frac{1}{4} \) | 4 |

Table 5. As table 3 for \( L \) odd. Symmetry is classified by the action of the \( D_3 \) rotation \( \sigma \).

| \( \sigma \) | \( X_0^{num} \) | \( (h, \bar{h}) \) | Spin | Pairing mult. |
|--------------|----------------|----------------|------|--------------|
| 1 | 0.062500 | \( (\frac{1}{16}, 0) \) | \( \frac{1}{16} \) | 1 |
| 0.562500 | \( (\frac{9}{16}, 0) \) | \( \frac{9}{16} \) | 1 |
| 0.812500 | \( (\frac{1}{16}, \frac{3}{4}) \) | \( -\frac{1}{16} \) | 1 |
| \( \omega, \omega^{-1} \) | 0.145833 | \( (\frac{1}{12}, \frac{1}{12}) \) | \(-\frac{1}{12} \) | 2 |
| 0.395833 | \( (\frac{1}{12}, \frac{1}{3}) \) | \(-\frac{1}{12} \) | 2 |
| 0.645833 | \( (\frac{1}{12}, \frac{1}{12}) \) | \( \frac{13}{48} \) | 2 |

Both conformal field theories appearing in the low energy sector of the model are connected with the \( sl(2) \) affine algebra by coset constructions. Following the recent discussion of edge modes between different quantum Hall states [13]–[16] this allows us to view each of them as a description for the boundary between a topological fluid nucleating within a surrounding non-Abelian \( su(2)_4 \) liquid; depending on the sign of the interaction the intervening phase is characterized by the topological properties of \( su(2)_3 \times su(2)_1 \) or an Abelian \( U(1) \). As in the quantum Hall systems, the quantum critical point is protected by topological symmetries against local perturbations. This becomes manifest in the small corrections to scaling (5) due to deviations of the quantum chain (1) from the CFT fixed point Hamiltonian: the subleading \( L \)-dependence of the ground state energies indicates that these deviations are generated by the presence of an irrelevant operator with scaling dimension \( X = 3.82(3) \) in \( H_\sigma \) (\( X = 4.00(3) \) in \( H_0 \)).

For the full model (1) we have utilized its exact solution to identify the low energy effective theory for the critical phases in terms of the direct product of two CFTs. In the context of interfaces between different topological quantum liquids this provides a description of boundaries supporting pairs of gapless modes. The observed pairing in the spectrum of the quantum chain implies that the interaction between these edge modes has to be of purely topological nature. Another consequence of this interaction is the possible existence of more general combinations of Virasoro characters in the low energy spectrum than those present in the known off-diagonal modular invariants for the minimal model \( M_{(5,6)} \) or the \( Z_4 \) parafermionic model alone [25, 26, 33]. For an explicit construction of these new invariants, further studies of the quantum chain (1) and the related \( D(D_n) \) models [22] are required, including their extensions to different boundary conditions. This will yield the insights necessary for the application of the present results to a description of the gapless modes on the boundary between different topological bulk...
Collective states of interacting $D(D_3)$ non-Abelian anyons

phases realized in condensates of interacting $D(D_3)$ anyons including the identification of the non-Abelian degrees of freedom in the latter. These explicit realizations within a class of exactly solvable lattice models will provide further support for the CFT description of such interfaces.

Finally, we note that for the integrable cases of braided boundaries or free ends [20] the full $D(D_n)$ symmetry is restored and the models are equivalent to those obtained in the anyon formulation based on a fusion path along the chain. This equivalence can be used to embed concepts such as topological symmetry into the framework of integrable systems allowing for a different perspective to study possible instabilities of anyonic systems against local perturbations.

We thank Michael Flohr for useful discussions. This work has been supported by a grant from the Deutsche Forschungsgemeinschaft.

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Collective states of interacting $D(D_3)$ non-Abelian anyons

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doi:10.1088/1742-5468/2012/05/L05001