Multiple inflation and the WMAP ‘glitches’

Paul Hunt and Subir Sarkar

Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

Observations of anisotropies in the cosmic microwave background by the Wilkinson Microwave Anisotropy Probe suggest the possibility of oscillations in the primordial curvature perturbation. Such deviations from the usually assumed scale-free spectrum were predicted in the multiple inflation model wherein ‘flat direction’ fields undergo rapid phase transitions due to the breaking of supersymmetry by the large vacuum energy driving inflation. This causes sudden changes in the mass of the (gravitationally coupled) inflaton and interrupts its slow roll. We calculate analytically the resulting modifications to the curvature perturbation and demonstrate how the oscillations arise.

I. INTRODUCTION

Despite the overall success of the standard ΛCDM inflationary model in matching the results from WMAP, it is striking that the goodness-of-fit statistics for the data are rather poor [1]. In particular, $\chi^2/\nu = 1432/1342$ for the fit to the TT spectrum which implies a probability of only $\sim 3\%$ for the best fit model to be correct. The lack of power at large scales relative to the standard ΛCDM model prediction has motivated several alternative proposals for the spectrum and nature of the primordial fluctuations [2, 31, 32, 33, 34, 35], however the uncertainties (and cosmic variance) at low multipoles are large and it has been argued that the observed low quadrupole (and octupole) are unlikely only at about a few per cent level [13, 14, 15]. In fact attempts to judge the goodness-of-fit by eye can be misleading since neighbouring $G_l$’s are correlated (being ‘pseudo-$G_l$’s evaluated on a ‘cut’ sky); taking this into account it is found [1] that the excess $\chi^2$ comes mainly from sharp features or ‘glitches’ in the power spectrum that the model is unable to fit. In particular there are glitches at $l \sim 120, 200$ and $340$ amongst the first and second acoustic peaks.

The WMAP team have noted that these glitches may have been caused by beam asymmetry, gravitational lensing of the CMB, non-Gaussianity in the noise maps, or the method of power spectrum reconstruction from the CMB maps [1]. However they have also considered the possibility that these glitches are due to features in the underlying primordial curvature perturbation spectrum [18], specifically in the ‘multiple inflation’ model [19] where sudden changes in the mass of the inflaton can generate characteristic localized oscillations in the spectrum. This is physically well motivated since spontaneous symmetry breaking can occur during inflation for ‘flat directions’ in supersymmetric theories and such fields are gravitationally coupled to the inflaton [19]. The resulting change in the potential of the inflaton field $\phi$ was modeled earlier as [20]

$$V(\phi) = \frac{1}{2} m_\phi^2 \phi^2 \left[ 1 + c_{\text{ampl}} \tanh \left( \frac{\phi - \phi_{\text{step}}}{d_{\text{grad}}} \right) \right],$$

which describes a standard ‘chaotic inflation’ potential with a step starting at $\phi_{\text{step}}$ with amplitude and gradient determined by the parameters $c_{\text{ampl}}$ and $d_{\text{grad}}$ respectively. This was shown to result in oscillations in the primordial curvature perturbation spectrum by numerically solving the governing Klein-Gordon equation [20]. The WMAP team found that the fit to the data improves if such oscillations are allowed for, with a reduction of $\chi^2$ by 10 for the model parameters $\phi_{\text{step}} = 15.5 M_P$, $c_{\text{ampl}} = 9.1 \times 10^{-4}$ and $d_{\text{grad}} = 1.4 \times 10^{-2} M_P$ [18].

We calculate the curvature perturbation spectrum from multiple inflation more realistically, taking into account the evolution of both fields as dictated by the inflationary dynamics [19], and using a WKB approximation [21], rather than numerical integration to solve the governing equation, in order to gain insight into how such oscillations are generated. In a subsequent publication we perform fits to the WMAP data with the cosmological parameters unconstrained, in order to investigate the sensitivity of their inferred values to such deviations from a scale-free spectrum [22].

The possibility that the WMAP glitches are due to oscillations in the primordial spectrum has also been considered in the context of inflationary models invoking ‘trans-Planckian’ physics [23, 24, 25, 26, 27]. Sharp features in the primordial spectrum may also have been generated by resonant particle production [28, 29, 30]. Several authors have attempted to reconstruct the curvature perturbation from the WMAP data and have noted possible features in its spectrum [2, 31, 32, 33, 34, 35]. It is clearly necessary to have an analytic formulation of the modification to the curvature perturbation spectrum by numerically solving the governing Klein-Gordon equation [20]. The WMAP team determined by the parameters $c$ which describes a standard ‘chaotic inflation’ potential with a step starting at $\phi_{\text{step}}$.

\[1\] However there is an unexpected alignment between the quadrupole and octupole [12, 13] and there appear to be systematic differences between the north and south ecliptic hemispheres [17], so this issue cannot be considered settled as yet.
usually assumed scale-free spectrum of the inflationary curvature perturbation, both in order to provide a link between CMB observables and the physics responsible for the glitches, and to distinguish between the various suggestions for such new physics.

II. MULTIPLE INFLATION

Supergravity (SUGRA) theories consist of a ‘visible sector’ and a ‘hidden sector’ coupled together gravitationally \[36, 37\]. The Standard Model particles are contained in the visible sector and supersymmetry (SUSY) breaking occurs in the hidden sector. Most SUGRA models of inflation have the inflaton situated in the hidden sector. This is because it is easier to protect the necessary flatness of the inflaton potential against radiative corrections there. The visible sector is usually assumed to be unimportant during inflation.

However this might not be true if one of the visible sector fields undergoes gauge symmetry breaking. Suitable candidates for this are the so-called flat direction fields which generically exist in supersymmetric theories (for a review, see ref.\[38\]). These are directions in field space in which the potential vanishes, in the limit of unbroken SUSY. During inflation in supergravity theories scalar fields typically receive a contribution of \(m^2 \sim \pm O(H^2)\) to their mass-squared due to SUSY breaking by the large vacuum energy driving inflation \[39, 40, 41\]. (For the inflaton this constitutes the notorious ‘η problem’ \[12\]; in common with most models of inflation we need to assume that some mechanism reduces the inflaton mass-squared by a factor of a least 20 in order to allow sufficient inflation to occur.) The SUSY-breaking mass term typically dominates the potential along the flat directions. If the mass-squared is negative at the origin in field space, the flat direction \(\rho\) will be stabilized at an intermediate scale vev of

\[\Sigma \sim (M_p^{n-4} m^2)^{1/(n-2)},\]  

by higher dimensional operators \(\propto \rho^n/M_p^{n-4}\) \[12\]. These appear in the potential after integrating out heavy degrees of freedom, because we are working in an effective field theory valid below some cut-off scale, which we take to be the reduced Planck mass: \(M_P \equiv (8\pi G_N)^{-1/2} \approx 2.4 \times 10^{18}\) GeV.

We assume that the inflationary era when the observed curvature perturbation was produced was preceded by a hot phase, so that all fields were initially in thermal equilibrium. The potential along the flat direction is then \[12\]

\[V(\rho, T) \simeq \begin{cases} C_1 T^2 \rho^2, & \text{for } \rho \ll T, \\ -m^2 \rho^2 + \frac{1}{16\pi^2} N_h(T) T^4 + \frac{2}{M_p^2}, & \text{for } T \ll \rho < \Sigma, \end{cases}\]

with a smooth interpolation at \(\rho \sim T\). Here we have included the 1-loop finite temperature correction to the potential \[12, 42\] with \(N_h(T)\) being the number of helicity states with mass much less than the temperature.

The finite temperature correction therefore creates a barrier of height \(O(T^4)\) in between the origin and \(\rho \sim T^2/m\). The tunneling rate through the barrier is negligible and so \(\rho\) is confined at the origin until the temperature, which falls rapidly during inflation, reaches \(T \sim m\) and the barrier disappears. The flat direction field \(\rho\) then undergoes a phase transition as it evolves to its global minimum at \(\Sigma\). This is governed by the usual equation of motion

\[\ddot{\rho} + 3H \dot{\rho} = -\frac{dV}{d\rho},\]

which has solutions

\[\rho \simeq \begin{cases} \rho_0 \exp \left[ \frac{3H_\Sigma}{2} \left( \sqrt{1 + \frac{8m^2}{T^2}} - 1 \right) \right], & (\rho) \ll \Sigma, \\ \Sigma + K_1 \exp \left( -\frac{3H_\Sigma}{2} \right) \sin \left[ \frac{3H_\Sigma \sqrt{(n-2) \frac{8m^2}{T^2}}}{n-1 + K_2} \right], & (\rho) \sim \Sigma. \end{cases}\]

Here we have taken

\[\frac{dV}{d\rho} \simeq \begin{cases} -2m^2 \rho, & (\rho) \ll \Sigma, \\ (\rho - \Sigma) \frac{d^2 V}{d\rho^2} \bigg|_{\rho=\Sigma}, & (\rho) \sim \Sigma. \end{cases}\]

Therefore \(\rho\) evolves exponentially towards its minimum, with most of the growth occurring over the last \(\sim 2\) e-folds \[19\], and then performs a few strongly damped oscillations about \(\Sigma\), as seen in Fig.\[10\]. (We assume that damping
FIG. 1: The evolution of the flat direction field $\rho$, corresponding to two different values for the change in the inflaton mass.

by particle creation as $\rho$ oscillates about its minimum is less important than the severe damping due to the large inflationary Hubble parameter evident in Fig. 1

After the phase transition the vacuum energy density driving inflation is reduced by a factor of $[1 - (\Sigma/M_P)^2]$. For $\Sigma$ significantly below the Planck scale this can be neglected, hence $H$ can be taken to be sensibly constant. A more important effect occurs due to the gravitational coupling between $\rho$ and the inflaton field $\phi$.

The particular background inflationary cosmology in which the phase transition occurs is not directly relevant for our work. For definiteness we take it to be a ‘new inflation’ potential with a cubic leading term which is a phenomenologically successful model with a naturally small inflaton mass [45, 46]. Then, for example, a term $\kappa \phi \phi^* \rho^2 / M_P^2$ in the Kähler potential which is allowed by symmetry (near $\phi \sim 0$) yields a coupling term $\lambda \phi^2 \rho^2 / 2$ in the inflaton potential, where $\lambda = \kappa m^2 / M_P$ [19]. With

$$V(\phi, \rho) = V_0 - c_3 \phi^3 + \frac{1}{2} \lambda \phi^2 \rho^2 + \ldots,$$

(7)

this causes the effective mass-squared $m_{\phi}^2 \equiv \frac{d^2 V}{d \phi^2}$ of the inflaton to change from $m_{\phi}^2 = -6c_3 \langle \phi \rangle$ before the phase transition, to $m_{\phi}^2 = -6c_3 \langle \phi \rangle + \lambda \Sigma^2$ afterwards. This will affect the curvature perturbation spectrum which is very sensitive to the mass of the inflaton.

In order to produce observable effects in the CMB or large-scale structure the phase transition must take place as cosmologically relevant scales are leaving the horizon. In supersymmetric theories there are many flat directions which can potentially undergo symmetry breaking during inflation, so it is not unlikely that a phase transition occurred in the $\sim 8$ e-folds which is sampled by observations [19], particularly since the observations suggest that (the last period of) inflation may not have lasted much longer than the minimum necessary to generate an universe as big as the present Hubble volume.

III. THE CURVATURE PERTURBATION

Some care is required in calculating the curvature perturbation spectrum for multiple inflation because we might a priori expect two-field inflation during the phase transition, when $\phi$ and $\rho$ evolve simultaneously. This might mean that both curvature as well as isocurvature perturbations are created, involving a significantly more complicated computation [47]. However, as we show below, our model of multiple inflation is effectively single-field inflation as far as the generation of curvature perturbations is concerned.

The curvature perturbation on comoving hypersurfaces $R$ is given by $R = \delta N$, where $N$ is the number of e-folds of expansion measured locally by a comoving observer passing from an initial, spatially flat, slice of space-time to a
final, comoving, slice [45]. The perturbations $\delta \phi_{\text{fts}}$ and $\delta \rho_{\text{fts}}$ are defined on the spatially flat time slice and the final comoving slice defines $R$. The first time slice is taken during inflation, soon after horizon crossing, and the second time slice is taken at the end of inflation, after $R$ has become constant. During the phase transition the slow-roll conditions in $\phi$ are no longer satisfied but we assume the end of inflation to be well after this epoch. Then we have

$$R = \frac{\partial N}{\partial \phi} \delta \phi_{\text{fts}} + \frac{\partial N}{\partial \rho} \delta \rho_{\text{fts}},$$

and so

$$P_R = \left( \frac{\partial N}{\partial \phi} \right)^2 P_\phi + \left( \frac{\partial N}{\partial \rho} \right)^2 P_\rho \simeq \left( \frac{dN}{d\phi} \right)^2 P_\phi,$$

which is just the single-field inflation expression for $P_R$. This is because the vacuum energy is dominated by $\phi$, which drives inflation, and $\rho$ has a negligible effect on the number of e-folds of inflation.

The curvature perturbation spectrum is usually calculated analytically using the slow-roll approximation to some chosen order in the slow-roll parameters [42]. This assumes the potential is relatively smooth, which is however inappropriate for multiple inflation. For example, for the standard chaotic inflation potential $V = \frac{1}{2} m^2 \phi^2$, the usual slow-roll expression gives

$$P_R^{1/2} = \frac{1}{2\sqrt{3\pi M^2_\text{P}}} \frac{V'^{3/2}}{dV/d\phi} \simeq \frac{1}{2\sqrt{3\pi M^2_\text{P}}} \frac{m^3}{m^2 \phi},$$

where all quantities are evaluated at horizon crossing. This gives a simple ‘step’ in the amplitude if the inflaton mass changes suddenly as in eq. (1), missing all the fine detail that we expect in the spectrum. (A discussion of the slow-roll approximation applied to a potential with a sudden change in its slope [49] rather than in its curvature can be found in ref. [50].)

Therefore we must resort to first principles and use the general formalism [51] for calculating the curvature perturbation spectrum. Instead of the slow-roll approximation we use the recently suggested WKB approximation [21, 52].

The metric describing scalar perturbations in a flat universe can be parameterized as

$$ds^2 = a^2 \left[ (1 + 2A) d\eta^2 - 2 \partial_i B_s d\eta dx^i - \{(1 - 2D_s) \delta_{ij} + 2 \partial_i \partial_j E_s\} dx^i dx^j \right].$$

We employ the gauge-invariant quantity

$$u = a \delta \phi^{(gi)} + z \Psi = zR,$$

and so

$$\delta \phi^{(gi)} = \delta \phi + \phi' \left( B_s - E_s' \right),$$

$$\Psi = D_s - \frac{a'}{a} \left( B_s - E_s \right),$$

are also gauge-invariant, $z = a \dot{\phi}/H$, and

$$R = D_s + H \frac{\delta \phi}{\dot{\phi}}.$$
The solutions of eq. (17) are governed by the relative sizes of \( k^2 \) and \( z''/z \). In de Sitter space \( z''/z = 2/\eta^2 \), therefore initially when \( k^2 \gg z''/z \) and the mode \( u_k \) is well inside the horizon, we have the flat spacetime solution

\[
u_k = \frac{1}{\sqrt{2k}} e^{-ik(\eta - \eta_0)},
\]

where \( \eta_0 \) is some arbitrary time at the beginning of inflation. In the opposite limit, \( k^2 \ll z''/z \), we have the solution

\[
u_k = A_k z + B_k z \int \eta \frac{d\mu}{z^2 (\mu)},
\]

when \( u_k \) is well outside the horizon. The first term on the right is the growing mode solution and the second is the decaying mode. Thus \( P_k \) at late times is equal to \( k^3 |A_k|^2 /2\pi^2 \).

We will use the WKB approximation to solve the mode equation, with eq. (14) as an initial condition. The WKB approximation cannot be directly applied to eq. (17) because the approximation breaks down on super-horizon scales. However, as noted in ref. [21], the WKB approximation becomes relevant if we switch to the variables

\[
x = \ln \left( \frac{H a}{k} \right),
\]

\[
U = e^{x/2} u_k.
\]

The transformed mode equation is then

\[
\frac{d^2 U}{dx^2} + Q(x) U = 0,
\]

where the effective frequency,

\[
Q(x) = \left(1 - \frac{z''}{zk^2}\right) e^{-2x} - \frac{1}{4},
\]

generally decreases with time during inflation, passing through zero at least once.\(^2\) In de Sitter space, this is

\[
Q(x) = \left(\frac{k}{aH}\right)^2 - \frac{9}{4}.\]

In the WKB formalism \[58\] we expand \( U \) as an asymptotic series

\[
U(x) = \exp \left[ \sum_{n=0}^{\infty} S_n(x) \right], \quad S_n(x) \gg S_{n+1}(x).
\]

Substituting this into eq. (23) results in the following series of equations

\[
S_0^2 = -Q,\]

\[
2S_0' S_1' + S_0'' = 0,\]

\[
2S_0'' S_n' + S_n'' + \sum_{j=1}^{n-1} S_j' S_{n-j}' = 0, \quad \text{for } n \geq 2,
\]

where the primes indicate derivatives with respect to \( x \).

This gives the first few terms as

\[
\begin{cases}
S_0 = \pm i \int_{x_0}^{x} Q^{1/2} dy, \\
S_1 = -\frac{1}{4} \ln Q, \\
S_2 = \pm i \int_{x_0}^{x} \left(-\frac{Q''}{8Q^{7/2}} + \frac{5Q^2}{32Q^{11/2}}\right) dy, \\
S_3 = \frac{Q''}{16Q^2} - \frac{5Q^2}{64Q^3},
\end{cases}
\]

\(^2\) \( Q \) is related to the variable \( g_a \) of Refs. [56, 57] by \( Q = -g_a \eta^2 \).
\[
Q < 0 \quad \begin{cases}
S_0 = \pm \int^x (-Q)^{1/2} \, dy, \\
S_1 = -\frac{1}{4} \ln (-Q), \\
S_2 = \pm \int^x \left[ -\frac{Q''}{8(-Q)^{3/2}} + \frac{5Q'^2}{32(-Q)^{3/2}} \right] \, dy, \\
S_3 = \frac{Q''}{16Q^2} - \frac{5Q'^2}{64Q^2}.
\end{cases}
\] (31)

Using only the first two terms in the expansion and neglecting the rest gives the 1st-order WKB approximation (for \(Q > 0\))

\[
U_1 = \frac{A}{Q^{1/4}(x)} \exp \left[ i \int_{x_i}^x Q^{1/2}(y) \, dy \right] + \frac{B}{Q^{1/4}(x)} \exp \left[ -i \int_{x_i}^x Q^{1/2}(y) \, dy \right],
\] (32)

where \(A\) and \(B\) are constants. At early times \(Q \simeq e^{-2x}\) which means that the WKB solution will match eq. (32) if \(A = 0\) and \(B = 1/\sqrt{2k}\).

The solutions become inaccurate close to the roots (‘turning points’) \(x_\ast\) of \(Q(x_\ast) = 0\). To continue the solution through a turning point we therefore need to match the WKB solutions valid to the right and left of the turning point, with a solution valid in the neighbourhood of the turning point. We patch the solutions together using asymptotic matching.

We consider the case of a single turning point and divide the \(x\)-axis into three regions. Region I is defined as the region where \(x < x_\ast, \ Q > 0\) and the WKB approximation is valid. Region II is the neighbourhood of \(x_\ast\) where \(Q\) can be approximated by its tangent at \(x_\ast\). Region III is where \(x > x_\ast, \ Q < 0\), and the WKB solution is accurate. We assume there is an overlap between regions I and II, and also between regions II and III.

In region II we have

\[
Q \simeq -\alpha X, \quad X \equiv x - x_\ast, \quad \alpha \equiv -\frac{dQ}{dx} \bigg|_{x=x_\ast} > 0. \quad (33)
\]

We rewrite the integration limits of \(U_1\) (32) as

\[
\int_{x_i}^x Q^{1/2} \, dy = \int_{x_i}^{x_\ast} Q^{1/2} \, dy + \int_{x_\ast}^x Q^{1/2} \, dy \equiv \Gamma + \int_{x_\ast}^x Q^{1/2} \, dy. \quad (34)
\]

Then in the overlap between regions I and II, \(U_1\) becomes

\[
U_1 \simeq \frac{\alpha^{-1/4}}{\sqrt{2k}} (-X)^{-1/4} \exp \left[ -i \left\{ -\frac{2}{3} (-X)^{3/2} + \Gamma \right\} \right].
\]

With \(Q \simeq -\alpha X\), the solution to eq. (32) is

\[
U_{II} \simeq CAi \left( \alpha^{1/3} X \right) + DBi \left( \alpha^{1/3} X \right), \quad (36)
\]

where \(Ai\) and \(Bi\) are Airy functions of the first and second kinds and \(C\) and \(D\) are constants. For \(X \ll 0\), \(U_{II}\) becomes

\[
U_{II} \simeq \frac{\alpha^{-1/12}}{2\sqrt{\pi}} (-X)^{-1/4} \left\{ \left( D - iC \right) \exp \left\{ i \left( \frac{2}{3} (-X)^{3/2} + \frac{\pi}{4} \right) \right\} \\
+ \left( D + iC \right) \exp \left\{ -i \left( \frac{2}{3} (-X)^{3/2} + \frac{\pi}{4} \right) \right\} \right\}. \quad (37)
\]

Requiring this to match \(U_1\) (33) in the overlap region fixes the coefficients

\[
C = iD, \quad D = \sqrt{\frac{\pi}{2k}} \alpha^{-1/6} \exp \left[ -i \left( \Gamma + \frac{\pi}{4} \right) \right]. \quad (38)
\]

For \(X \gg 0\) we have

\[
U_{II} \simeq \frac{\alpha^{-1/12}}{\sqrt{\pi}} X^{-1/4} \left[ \frac{C}{2} \exp \left( -\frac{2}{3} \alpha^{1/2} X^{3/2} \right) + D \exp \left( \frac{2}{3} \alpha^{1/2} X^{3/2} \right) \right]. \quad (39)
\]
In region III, the 1st-order WKB solution is

\[ U_{III} = \frac{F}{[-Q(x)]^{1/4}} \exp \left\{ \int_{x_*}^{x} [-Q(y)]^{1/2} \, dy \right\} + \frac{G}{[-Q(x)]^{1/4}} \exp \left\{ - \int_{x_*}^{x} [-Q(y)]^{1/2} \, dy \right\}. \] (40)

When \( Q \approx -\alpha X \), this becomes

\[ U_{III} \approx \alpha^{-1/4} X^{-1/4} \left[ F \exp \left( \frac{2}{3} \alpha^{1/2} X^{3/2} \right) + G \exp \left( -\frac{2}{3} \alpha^{1/2} X^{3/2} \right) \right], \] (41)

which matches \( U_{II} \) in the second overlap region if

\[ F = \frac{\alpha^{1/6}}{\sqrt{\pi}} D, \quad G = \frac{\alpha^{1/6}}{2\sqrt{\pi}} C. \] (42)

Substituting \( U_{III} \) into eq. (18) gives finally the curvature perturbation spectrum

\[ T^{1/2}_R (k) = \frac{H^2}{2\pi \phi} \left( \frac{k}{aH} \right)^{3/2} \left. [-Q(x)]^{-1/4} \exp \left\{ \int_{x_*}^{x} [-Q(y)]^{1/2} \, dy \right\} \right|_{x=x_i}. \] (43)

where \( x_i \) is the value of \( x \) at some late time well after horizon crossing when \( R \) is constant, and we have neglected the decaying mode proportional to \( G \) which is negligible at late times.

We now consider the 2nd-order WKB approximation where the first three terms in the expansion (26) are retained. At early times the WKB solution is

\[ U_1 = \frac{1}{\sqrt{2kQ^{1/4}}} \exp \left\{ -i \int_{x_i}^{x_*} \left( \frac{Q^{1/2}}{8Q^{3/2}} - \frac{Q''}{32Q^{3/2}} \right) \, dy \right\}, \]
\[ = \frac{1}{\sqrt{2kQ^{1/4}}} \exp \left\{ -i \left( \int_{x_i}^{x_*} Q^{1/2} \, dy - \frac{5Q'}{48Q^{3/2}} \int_{x_i}^{x_*} Q'' \, dy \right) \right\}, \] (44)

where the second line follows after an integration by parts. As before we rewrite the integration limits as

\[ \int_{x_i}^{x_*} Q^{1/2} \, dy - \frac{Q''}{48Q^{3/2}} \int_{x_i}^{x_*} \, dy = \int_{x_i}^{x_*} Q^{1/2} \, dy - \int_{x_i}^{x_* - \epsilon} Q'' \, dy + \int_{x_* - \epsilon}^{x} Q^{1/2} \, dy - \int_{x_* - \epsilon}^{x} Q'' \, dy, \]
\[ \equiv \Upsilon + \int_{x_*}^{x} Q^{1/2} \, dy - \int_{x_* - \epsilon}^{x} Q'' \, dy. \] (45)

Here \( \epsilon \) is a small parameter introduced to regularize the divergent integral \( S_2 \). This time, region II is the neighbourhood of \( x_* \) where

\[ Q \approx -\alpha X + \beta X^2, \quad \beta = \frac{1}{2} \frac{d^2Q}{dx^2} \bigg|_{x=x_*.} \] (46)

Then in the overlap of regions I and II

\[ U_1 \approx \alpha^{-1/4} \sqrt{2k} (-X)^{-1/4} \left( 1 + \frac{\beta X}{4\alpha} \right) \exp \left\{ -i \left\{ \frac{2}{3} (-X)^{3/2} - \frac{\beta}{5\alpha^{1/2}} (-X)^{3/2} + \frac{5}{48\alpha^{1/2}} (-X)^{-3/2} - \frac{\beta}{12\alpha^{3/2}} (-X)^{-1/2} + \Upsilon \right\} \right\}. \] (47)
With \( Q \approx -\alpha X + \beta X^2 \), eq. [23] has the approximate solution

\[
U_{II} \approx K \left( 1 + \frac{\beta X}{5\alpha} \right) \text{Ai} \left[ \alpha^{1/3} X \left( 1 - \frac{\beta X}{5\alpha} \right) \right] + L \left( 1 + \frac{\beta X}{5\alpha} \right) \text{Bi} \left[ \alpha^{1/3} X \left( 1 - \frac{\beta X}{5\alpha} \right) \right].
\]

For \( X \ll 0 \),

\[
U_{II} \approx \frac{\alpha^{-1/12}}{2\sqrt{\pi}} (-X)^{-1/4} \left( 1 + \frac{\beta X}{4\alpha} \right) \times \left[ (L - iK) \exp \left\{ i \left( \frac{2}{3} (-X)^{3/2} + \frac{\beta}{5\alpha^{1/2}} (-X)^{5/2} + \frac{\pi}{4} \right) \right\} 
+ (L + iK) \exp \left\{ -i \left( \frac{2}{3} (-X)^{3/2} + \frac{\beta}{5\alpha^{1/2}} (-X)^{5/2} + \frac{\pi}{4} \right) \right\} \right],
\]

which matches \( U_I \) in the overlap region provided that

\[
K = iL, \quad L = \sqrt{\frac{\pi}{2k}} \alpha^{-1/6} \exp \left[ -i \left( Y + \frac{\pi}{4} - \frac{\beta}{12\alpha^{3/2}} \varepsilon^{-1/2} \right) \right].
\]

When \( X \gg 0 \), we have

\[
U_{II} \approx \frac{\alpha^{-1/12}}{\sqrt{\pi}} X^{-1/4} \left( 1 + \frac{\beta X}{4\alpha} \right) \left[ \frac{K}{2} \exp \left( -\frac{2}{3} \alpha^{1/2} X^{3/2} \right)
+ \frac{\beta}{5\alpha^{1/2}} X^{5/2} \right] + \text{L exp} \left( \frac{2}{3} \alpha^{1/2} X^{3/2} - \frac{\beta}{5\alpha^{1/2}} X^{5/2} \right).
\]

The 2nd-order WKB solution in region III is

\[
U_{III} = \frac{M}{(-Q)^{1/4}} \exp \left[ \int_{x_*}^x \left\{ (-Q)^{1/2} dy - \frac{Q''}{8 (-Q)^{3/2}} + \frac{5Q'^2}{32 (-Q)^{3/2}} \right\} dy \right] 
+ \frac{N}{(-Q)^{1/4}} \exp \left[ -\int_{x_*}^x \left\{ (-Q)^{1/2} dy - \frac{Q''}{8 (-Q)^{3/2}} + \frac{5Q'^2}{32 (-Q)^{3/2}} \right\} dy \right],
\]

\[
= \frac{M}{(-Q)^{1/4}} \exp \left[ \int_{x_*}^x (-Q)^{1/2} dy - \frac{5Q'}{48 (-Q)^{3/2}} - \int_{x_*+\varepsilon}^x \frac{Q''}{48 (-Q)^{3/2}} dy \right] 
+ \frac{N}{(-Q)^{1/4}} \exp \left[ -\int_{x_*}^x (-Q)^{1/2} dy + \frac{5Q'}{48 (-Q)^{3/2}} + \int_{x_*+\varepsilon}^x \frac{Q''}{48 (-Q)^{3/2}} dy \right].
\]

With \( Q \approx -\alpha X + \beta X^2 \), this becomes

\[
U_{III} \approx \alpha^{-1/4} X^{-1/4} \left( 1 + \frac{\beta X}{4\alpha} \right) \left[ M \exp \left( \frac{2}{3} \alpha^{1/2} X^{3/2} - \frac{\beta}{5\alpha^{1/2}} X^{5/2} \right)
+ \frac{5}{48\alpha^{1/2}} X^{-3/2} + \frac{\beta}{12\alpha^{3/2}} X^{-1/2} - \frac{\beta}{12\alpha^{3/2}} \varepsilon^{-1/2} \right]
+ N \exp \left( \frac{2}{3} \alpha^{1/2} X^{3/2} + \frac{\beta}{5\alpha^{1/2}} X^{5/2} - \frac{5}{48\alpha^{1/2}} X^{-3/2}
- \frac{\beta}{12\alpha^{3/2}} X^{-1/2} + \frac{\beta}{12\alpha^{3/2}} \varepsilon^{-1/2} \right).
\]

Hence if

\[
M = \frac{\alpha^{1/6}}{\sqrt{\pi}} \exp \left( \frac{\beta}{12\alpha^{3/2}} \varepsilon^{-1/2} \right) L,
\]

\[
N = \frac{\alpha^{1/6}}{2\sqrt{\pi}} \exp \left( -\frac{\beta}{12\alpha^{3/2}} \varepsilon^{-1/2} \right) K,
\]
then $U_{11}$ matches $U_{11}$. Therefore in the 2nd-order approximation

$$\mathcal{P}_R^{1/2}(k) = \frac{H^2}{2\pi \phi} \left| \frac{k}{aH} \right|^{3/2} \exp \left( \frac{\beta}{12a^{3/2}} \varepsilon^{-1/2} \right) \left[ -Q(x_f) \right]^{-1/4} \times \exp \left[ \int_{x_f}^{x_\ast + \varepsilon} (-Q)^{1/2} dy \right] - \frac{5Q'}{48 (-Q)^{3/2}} \int_{x_f}^{x_\ast + \varepsilon} \frac{Q''}{48 (-Q)^{3/2}} dy \right] . \quad (55)$$

A weak dependence on the parameter $\varepsilon$ arises because of the incomplete cancellation between $\exp \left( \frac{\beta}{12a^{3/2}} \varepsilon^{-1/2} \right)$ and $\exp \left[ \int_{x_f}^{x_\ast + \varepsilon} \frac{Q''}{48 (-Q)^{3/2}} dy \right]$ above. This is because we have truncated the expansion of $Q$ at 2nd-order (see eq. (56) for computational convenience.

### IV. RESULTS

During the phase transition, the scalar potential can be written as

$$V(\phi, \rho) = V_0 - c_3 \phi^3 - m^2 \rho^2 + \frac{1}{2} \lambda \phi^2 \rho^2 + \frac{\gamma}{M_p^2} \rho^n . \quad (56)$$

Then the change in the inflaton mass-squared after the phase transition is

$$\Delta m^2_\phi = \lambda \Sigma^2, \quad \Sigma = \left[ \frac{M_p^{n-4}}{n\gamma} (2m^2 - \lambda \phi^2) \right]^{1/(n-2)} \approx \left( \frac{2m^2 M_p^{n-4}}{n\gamma} \right)^{1/(n-2)} . \quad (57)$$

The equations of motion are

$$\ddot{\phi} + 3H \dot{\phi} = -\frac{\partial V}{\partial \phi} = (3c_3 \phi - \lambda \rho^2) \phi, \quad (58)$$

$$\ddot{\rho} + 3H \dot{\rho} = -\frac{\partial V}{\partial \rho} = \left( 2m^2 - \lambda \phi^2 - \frac{n\gamma}{M_p^2} \rho^{n-2} \right) \rho . \quad (59)$$

Using these gives

$$\frac{\dot{a}}{a} = a^2 \left( 2H^2 + 6c_3 \phi - \lambda \rho^2 - \frac{2\lambda \rho \dot{\phi}}{\phi} \right), \quad (60)$$

which is shown in Fig. 2 taking $m_\phi^2 \approx 0.05 H^2$. The effective frequency of eq. (24) is then obtained to be

$$Q = \frac{1}{H^2} \left( \frac{k^2}{a^2} - \frac{9}{4} H^2 - 6c_3 \phi + \lambda \rho^2 + \frac{2\lambda \rho \dot{\phi}}{\phi} \right) . \quad (61)$$

We calculated the WKB solutions using $\phi(t)$ and $\rho(t)$ found by numerical solution of eqs. (58) and (59). An example of a 1st-order WKB solution is shown in Fig. 3, while Fig. 4 shows the power spectrum obtained by numerically integrating eq. (17) mode by mode, together with the 1st-order WKB approximation (49), for two different values of $\Delta m^2_\phi$. The spectrum $\mathcal{P}_R$ has the form of a positive step with superimposed damped oscillations and a spectral index $n - 1 = \frac{\partial \ln P}{\partial \ln k}$ of $n \approx 0.961$. The 1st-order WKB spectra are too small by about 10% at large and small scales. They do exhibit oscillations but these are slightly out of phase. We can understand the spectrum as follows.

The equation $Q = 0$ can be rewritten as

$$k = s(t), \quad s(t) \equiv \frac{\sqrt{\dot{z}^2 - a^2 H^2}}{a} . \quad (62)$$

Therefore $\ell = a/s$ defines a new length scale in inflationary cosmology — the ‘potential scale’; it is $\ell$, not the horizon scale $H^{-1}$ as is commonly stated in the literature, that controls the form of the mode functions [52]. When a mode is inside the potential scale it has the oscillatory behaviour of eq. (56) and when it is outside it has the exponential behaviour of eq. (52). In de Sitter space we have $\ell = 2/3H^{-1}$ and so the two scales are nearly equal for slow-roll inflation. However, they behave quite differently during the phase transition, with $\ell$ undergoing oscillations.
FIG. 2: The evolution of $z''/z$ during the phase transition, for two different values of the change in the inflaton mass.

FIG. 3: Comparison of the 1st-order WKB solution (36) with $u_k$ calculated numerically for $k = 6.7 \times 10^{-3}$ h Mpc$^{-1}$. The turning point is at $x_* \simeq 8.64$ and $\Delta m^2_\phi = 0.1 m^2_\phi$. Note that the solutions $U_I$ and $U_{III}$ diverge at $x = x_*$ but $U_{II}$ is continuous and close to the exact solution.

The oscillations in the curvature perturbation spectrum occur on length scales which cross the ‘potential scale’ as it is oscillating. The maxima of the oscillations in $P_R$ correspond to scales which cross at the minima of the potential scale. These modes begin their exponential growth early and so have increased by a larger amount at $x_f$. Similarly modes which cross at maxima of $\ell$ give rise to minima of $P_R$. The amplitude of the oscillations of the spectrum decreases with wavenumber because the oscillations of $z''/z$ are damped.

Modes which cross the potential scale well before the phase transition are growing according to $u_k \propto z$ by the time of the phase transition, and so $P_R$ is unchanged on large scales. Modes which cross the potential scale well after the
FIG. 4: Comparison of the 1st-order WKB approximation with the numerically calculated exact spectrum, for two different values of the change in the inflaton mass.

phase transition are insensitive to the variation of $z''/z$ at the time of the phase transition, because $k^2 \gg z''/z$ then. However, they cross outside the potential scale slightly later than would be the case if there were no phase transition, because $\ell$ has increased from $(9H^2 + 6c_3\phi)^{-1/2}$ to $(9H^2 + 6c_3\phi - \lambda\Sigma^2)^{-1/2}$. Therefore for fixed $x_f$ the modes will have undergone less growth. On the other hand, the introduction of the phase transition causes an even larger decrease in the value of $\dot{\phi}_f$ and so $P_R$ increases on small scales. Consequently there is a step in the spectrum. The factor by which the phase transition causes the spectrum to increase is given using the 1st-order WKB approximation as

$$
\frac{P_R^{\text{PT}}(k)}{P_R^{\text{NPT}}(k)} = \left[ \frac{3c_3 \left( \phi_f^{\text{NPT}} \right)^2}{3c_3 \phi_f^{\text{PT}} - \lambda\Sigma^2} \frac{\phi_f^{\text{PT}}}{\phi_f^{\text{PT}}} \right]^{1/2} \left[ \frac{Q^{\text{NPT}}(x_f)}{Q^{\text{PT}}(x_f)} \right]^{1/2} \times \exp \left\{ 2 \int_{x_f^{\text{PT}}}^{x_f} [-Q^{\text{PT}}(y)]^{1/2} dy - 2 \int_{x_f^{\text{NPT}}}^{x_f} [-Q^{\text{NPT}}(y)]^{1/2} dy \right\}. \tag{63}
$$

Here the superscripts ‘PT’ and ‘NPT’ denote quantities evaluated with and without the phase transition, and we have used the slow-roll expression $\dot{\phi}_f \simeq -\frac{3}{2H} \frac{\partial V}{\partial \phi}$ which is applicable at late times. The WKB expression (63) is surprisingly accurate as can be seen from Fig. 5.

As shown in Fig. 6, the spectrum calculated using the 2nd-order WKB approximation has an extra dip (at $k \approx 5 \times 10^{-3}$ h Mpc$^{-1}$ in this example) and is too large by $\sim 10\%$ at small and large scales. The 2nd-order solution is also weakly dependent on $\epsilon$ as seen in Fig. 7, we noted earlier that this is due to the incomplete cancellation between $\exp \left( \frac{3}{12c_3\phi_f^{\text{PT}}} \right)^{-1/2}$ and $\int_{x_f^{\text{PT}}}^{x_f} \frac{Q''}{48(-Q)^{3/2}} dy$ in eq. (55). Overall the 2nd-order WKB approximation is in fact less accurate than the 1st-order approximation because the asymptotic matching is less successful. As can be seen from Fig. 8 the 2nd-order solution is indeed more accurate that the 1st-order one away from the turning point, but is less accurate close to $x_*$ where it is more divergent. Hence the 2nd-order solution is less accurate in region II, consequently eq. (48) which is matched to it there is less accurate than eq. (36) which matches the 1st-order solution. The overlap between regions I and II, and between regions II and III is reduced, so we have the seemingly paradoxical result that the 1st-order approximation is superior to the 2nd-order one.
FIG. 5: Comparison of the numerical result for the factor by which $P_R$ increases due to the phase transition with the 1st-order WKB expression, taking $k = 10 \ h \ Mpc^{-1}$.

FIG. 6: Comparison of the 2nd-order WKB approximation with the numerically calculated exact spectrum, for two different values of the change in the inflaton mass.

V. CONCLUSIONS

It is usually assumed that the curvature spectrum from primordial inflation is scale-free, especially when fitting cosmological models to data on CMB anisotropies and large-scale structure. This is indeed the case for simple models of the inflaton where its potential is assumed to be dominated by a (fine-tuned) single polynomial term in the region where the observed fluctuations are generated. However it must be emphasised that these are ‘toy’ models and the
actual physics behind inflation is likely to be more complicated [59, 60, 61]. In particular when one attempts to construct realistic models of inflation in the context of supergravity or string theory, more than one field is typically involved and their mutual interactions are unlikely to yield the usually assumed smooth potential. Scalar fields other than the inflaton can exhibit non-trivial dynamics during inflation and affect the slow-roll evolution of the inflaton. In particular, ‘flat direction’ fields, which are generic in such models, may undergo symmetry-breaking phase transitions and, by virtue of their gravitational coupling to the inflaton, induce sudden changes in its mass, resulting in multiple bursts of slow-roll inflation punctuated by such phase transitions [19].

We have calculated analytically the effect of such sudden changes in the inflaton mass on the curvature perturbation, using a WKB method [21] to solve the governing Klein-Gordon equation without recourse to any slow-roll approx-
imitations. Although the analytic calculation does not yet provide an exact match to the exact numerical solution, it does capture the characteristic oscillations that are generated in the spectrum [20, 19] and provides considerable insight into the physics behind this phenomenon. There may be scope for developing more accurate calculational techniques in this context [24, 57]. It should be of course borne in mind that caution is required in using the standard calculational framework for the inflationary metric perturbation for the case of a time-dependent effective potential [62, 63]. Nevertheless when the scalar fields involved are weakly coupled and the time scales involved exceed the Hubble time, the usual results from perturbation theory ought to be valid.

Given that the precision WMAP data does show preliminary evidence for such oscillations [18], we are now investigating what the data imply for the physics of the multiple inflation model and indeed whether the estimation of cosmological parameters from the data is significantly affected when the primordial spectrum is not scale-free [22].

Acknowledgments

We are grateful to Jenni Adams, David Lyth, Graham Ross and Alexei Starobinsky for discussions and to the anonymous Referee for a critical and helpful report.

[1] D. N. Spergel et al., Astrophys. J. Suppl. 148, 175 (2003) [arXiv:astro-ph/0302209].
[2] S. L. Bridle, A. M. Lewis, J. Weller and G. Efstathiou, Mon. Not. Roy. Astron. Soc. 342, L72 (2003) [arXiv:astro-ph/0302306].
[3] C. R. Contaldi, M. Peloso, L. Kofman and A. Linde, JCAP 0307, 002 (2003) [arXiv:astro-ph/0303636].
[4] A. Blanchard, M. Douspis, M. Rowan-Robinson and S. Sarkar, Astron. Astrophys. 412, 35 (2003) [arXiv:astro-ph/0304237].
[5] J. M. Cline, P. Crotty and J. Lesgourgues, JCAP 0309, 010 (2003) [arXiv:astro-ph/0304558].
[6] B. Feng and X. Zhang, Phys. Lett. B 570, 145 (2003) [arXiv:astro-ph/0305020].
[7] M. Kawasaki and F. Takahashi, Phys. Lett. B 570, 151 (2003) [arXiv:hep-ph/0305319].
[8] M. Barko-Gil, K. Freese and L. Mersini-Houghton, Phys. Rev. D 68, 123514 (2003) [arXiv:astro-ph/0306289].
[9] N. Kaloper and M. Kaplinghat, Phys. Rev. D 68, 123522 (2003) [arXiv:hep-th/0307016].
[10] S. Tsujikawa, M. Hattori and R. Brandenberger, Phys. Lett. B 574, 141 (2003) [arXiv:astro-ph/0308169].
[11] T. Moroi and T. Takahashi, Phys. Rev. Lett. 92, 091301 (2004) [arXiv:astro-ph/0308208].
[12] Y. S. Piao, B. Feng and X. Zhang, Phys. Rev. D 69, 103520 (2004) [arXiv:hep-th/0310206].
[13] E. Gaztanaga, J. Wagg, T. Multamaki, A. Montana and D. H. Hughes, Mon. Not. Roy. Astron. Soc. 346, 47 (2003) [arXiv:astro-ph/0304178].
[14] G. Efstathiou, Mon. Not. Roy. Astron. Soc. 346, L26 (2003) [arXiv:astro-ph/0306431].
[15] A. de Oliveira-Costa, M. Tegmark, M. Zaldarriaga and A. Hamilton, Phys. Rev. D 69, 063516 (2004) [arXiv:astro-ph/0307282].
[16] D. J. Schwarz, G. D. Starkman, D. Huterer and C. J. Copi, Astrophys. J. 607, 1 (2004) [arXiv:astro-ph/0403535].
[17] H. K. Eriksen, F. K. Hansen, A. J. Banday, K. M. Gorski and P. B. Lilje, Astrophys. J. 605, 14 (2004) [arXiv:astro-ph/0307507].
[18] H. V. Peiris et al., Astrophys. J. Suppl. 148, 213 (2003) [arXiv:astro-ph/0302225].
[19] J. A. Adams, G. G. Ross and S. Sarkar, Nucl. Phys. B 503, 405 (1997) [arXiv:hep-th/9704286].
[20] J. Adams, B. Cresswell and R. Easther, Phys. Rev. D 64, 123514 (2001) [arXiv:astro-ph/0102236].
[21] J. Martin and D. J. Schwarz, Phys. Rev. D 67, 083512 (2003) [arXiv:astro-ph/0210090].
[22] M. Douspis, P. Hunt and S. Sarkar, in preparation.
[23] X. Wang, B. Feng and M. Li, arXiv:astro-ph/0407524.
[24] C. P. Burgess, J. M. Cline, F. Lemieux and R. Holman, JHEP 0302, 048 (2003) [arXiv:hep-th/0210233].
[25] J. Martin and C. Ringeval, Phys. Rev. D 69, 083515 (2004) [arXiv:astro-ph/0310382].
[26] T. Okamoto and E. A. Lim, Phys. Rev. D 69, 083519 (2004) [arXiv:astro-ph/0312284].
[27] J. Martin and C. Ringeval, Phys. Rev. D 69, 127303 (2004) [arXiv:astro-ph/0406209].
[28] D. J. H. Chung, E. W. Kolb, A. Riotto and I. I. Tkachev, Phys. Rev. D 62 (2000) 043508 [arXiv:hep-ph/9910437].
[29] O. Elgaroy, S. Hannestad and T. Haugboelle, JCAP 0309, 008 [arXiv:astro-ph/0306229].
[30] M. Kawasaki, F. Takahashi and T. Takahashi, arXiv:astro-ph/0407631.
[31] P. Mukherjee and Y. Wang, Astrophys. J. 599, 1 (2003) [arXiv:astro-ph/0303211].
[32] S. Hannestad, JCAP 0404, 002 (2004) [arXiv:astro-ph/0311191].
[33] A. Shafieloo and T. Souradeep, Phys. Rev. D 70, 043523 (2004) [arXiv:astro-ph/0312174].
[34] N. Kogo, M. Matsumiya, M. Sasaki and J. Yokoyama, Astrophys. J. 607, 32 (2004) [arXiv:astro-ph/0309662].
[35] D. Tocchini-Valentini, M. Douspis and J. Silk, arXiv:astro-ph/0402583.
[36] H. P. Nilles, Phys. Rept. 110, 1 (1984).
[37] D. J. H. Chung, L. L. Everett, G. L. Kane, S. F. King, J. Lykken and L. T. Wang, arXiv:hep-ph/0312378.
[38] K. Enqvist and A. Mazumdar, Phys. Rept. 380, 99 (2003) [arXiv:hep-ph/0209244].
[39] G. D. Coughlan, R. Holman, P. Ramond and G. G. Ross, Phys. Lett. B 140, 44 (1984).
[40] E. J. Copeland, A. R. Liddle, D. H. Lyth, E. D. Stewart and D. Wands, Phys. Rev. D 49, 6410 (1994) [arXiv:hep-ph/9401011].
[41] M. Dine, L. Randall and S. Thomas, Phys. Rev. Lett. 75, 398 (1995) [arXiv:hep-ph/9503303].
[42] D. H. Lyth and A. Riotto, Phys. Rept. 314, 1 (1999) [arXiv:hep-ph/9807278].
[43] K. Yamamoto, Phys. Lett. B 168, 341 (1986).
[44] T. Barreiro, E. J. Copeland, D. H. Lyth and T. Prokopec, Phys. Rev. D 54, 1379 (1996) [arXiv:hep-ph/9602263].
[45] G. G. Ross and S. Sarkar, Nucl. Phys. B 461, 597 (1996) [arXiv:hep-ph/9506283].
[46] J. A. Adams, G. G. Ross and S. Sarkar, Phys. Lett. B 391, 271 (1997) [arXiv:hep-ph/9608336].
[47] J. Garcia-Bellido and D. Wands, Phys. Rev. D 53, 5437 (1996) [arXiv:astro-ph/9511029].
[48] M. Sasaki and E. D. Stewart, Prog. Theor. Phys. 95, 71 (1996) [arXiv:astro-ph/9507001].
[49] A. A. Starobinsky, JETP Lett. 55, 489 (1992) [Pisma Zh. Eksp. Teor. Fiz. 55, 477 (1992)].
[50] S. M. Leach, M. Sasaki, D. Wands and A. R. Liddle, Phys. Rev. D 64, 023512 (2001) [arXiv:astro-ph/0101406].
[51] E. D. Stewart and D. H. Lyth, Phys. Lett. B 302, 171 (1993) [arXiv:gr-qc/9302019].
[52] J. Martin, arXiv:astro-ph/0312492.
[53] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. 215, 203 (1992).
[54] K. A. Malik, arXiv:astro-ph/0101563.
[55] A. Riotto, arXiv:hep-ph/0210162.
[56] S. Habib, K. Heitmann, G. Jungman and C. Molina-Paris, Phys. Rev. Lett. 89, 281301 (2002) [arXiv:astro-ph/0208443].
[57] S. Habib, A. Heinen, K. Heitmann, G. Jungman and C. Molina-Paris, arXiv:astro-ph/0406134.
[58] C. M. Bender and S. A. Orszag, “Advanced mathematical Methods for Scientists and Engineers”, (McGraw-Hill, 1978).
[59] F. Quevedo, Class. Quant. Grav. 19, 5721 (2002) [arXiv:hep-th/0210292].
[60] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAllister and S. P. Trivedi, JCAP 0310, 013 (2003) arXiv:hep-th/0308055.
[61] C. P. Burgess, arXiv:hep-th/0408037.
[62] F. Cooper, S. Habib, Y. Kluger and E. Mottola, Phys. Rev. D 55, 6471 (1997) [arXiv:hep-ph/9610345].
[63] D. Boyanovsky, D. Cormier, H. J. de Vega, R. Holman and S. P. Kumar, Phys. Rev. D 57, 2166 (1998) arXiv:hep-ph/9709232.