Construction of quantum dark soliton in one-dimensional Bose gas

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Abstract

Dark soliton solutions in the one-dimensional classical nonlinear Schrödinger equation have been considered to be related to the yrast states corresponding to the type-II excitations in the Lieb–Liniger model. However, the relation is nontrivial and remains unclear because a dark soliton localized in space breaks the translation symmetry, while yrast states are translationally invariant. In this work, we construct a symmetry-broken quantum soliton state and investigate the relation to the yrast states. By interpreting a quantum dark soliton as a Bose–Einstein condensation to the wave function of a classical dark soliton, we find that the quantum soliton state has a large weight only on the yrast states, which is analytically proved in the free-boson limit and numerically verified in the weak-coupling regime. By extending these results, we derive a parameter-free expression of a quantum soliton state that is written as a superposition of yrast states with Gaussian weights. The density profile of this quantum soliton state excellently agrees to that of the classical dark soliton. The dynamics of a quantum dark soliton are also studied, and it turns out that the density profile of a dark soliton decays, but the decay time increases as the inverse of the coupling constant in the weak-coupling limit.

Keywords: ultracold gases, Bose gases, dark solitons

(Some figures may appear in colour only in the online journal)

1. Introduction

Ultracold bosonic atoms form a Bose–Einstein condensate (BEC). All the particles in the condensate can be well described by a single wave function, which is called a ‘macroscopic wave function’ \cite{1,2}. In one-dimensional infinite systems, the BEC in the strict sense is impossible since it is unstable against long-wavelength quantum fluctuations. However, in a weak-interaction regime, it forms a quasi-condensate \cite{3,4}, in which the concept of the macroscopic wave function is still valid.

The macroscopic wave function obeys the classical nonlinear Schrödinger/Gross–Pitaevskii equation \cite{5,6}, which may possess solitary wave solutions whose shape do not change in the time evolution. Indeed, in one dimension, the nonlinear Schrödinger equation possesses bright \cite{7} and dark \cite{8} soliton solutions for attractive and repulsive interactions, respectively. Solitons have also been experimentally generated by the phase imprinting method \cite{9,10}.

The one-dimensional classical nonlinear Schrödinger equation corresponds to the classical field approximation of the Lieb–Liniger model \cite{11}, which is a representative of quantum integrable models. Thus, it has been desired to understand the relation between many-body energy eigenstates of the Lieb–Liniger Hamiltonian and the classical soliton solutions of the nonlinear Schrödinger equation.

The problem is that in the periodic boundary condition, an individual energy eigenstate has a flat single-particle probability density due to the translation invariance, and hence, we should consider some nontrivial superposition of energy eigenstates to construct a symmetry-broken state with a solitonic density profile. As for the attractive interactions, such a quantum-classical correspondence has been established. The attractive
Lieb–Liniger model has bound states with momenta \( \{P\} \), and it has been analytically shown that a quantum state corresponding to a classical bright soliton at position \( x \) can be constructed by performing the Fourier transform of those bound states [12].

On the other hand, this problem has not been settled in a satisfactory manner for repulsive interactions. It has been argued that [13–21] a set of yrast states (energy eigenstates corresponding to Lieb’s type-II excitations), each of which is the energy eigenstate with the lowest energy at a given momentum, in the Lieb–Liniger model is related to the family of classical dark solitons with momenta \( P \in [-\pi\rho_0, \pi\rho_0] \), where \( \rho_0 \) is the particle number density. Indeed, the dispersion relation of yrast states is similar to that of the classical dark solitons in the weak-coupling regime [13]. As pointed out above, however, a single energy eigenstate cannot represent a dark soliton at a fixed position. It is not at all obvious how one can construct a many-body symmetry-broken state that corresponds to a classical dark soliton.

Sato et al [22, 23] tried to construct such a quantum many-body state guided by an analogy with the case of attractive interactions. They considered the Fourier transform of yrast states \( \{N, P\} \) as an \( N \)-particle quantum dark soliton state at position \( x \), i.e. \( |N, x\rangle = \sum_{P} e^{iP(x-L^2/2)}|N, P\rangle / \sqrt{N} \), where \( L \) is the length of the system size, the sum is taken over \( P = 2\pi M / L \) with integer \( M \in [0, N - 1] \), and numerically found that the density profile in this state is similar to that in a classical dark soliton solution \( \varphi_{\rho_0}(x - X) \) with a certain momentum \( P_0 \) at position \( X \),

\[
\langle N, X | \hat{\psi}^\dagger(x) \hat{\psi}(y) | N, X \rangle \approx |\varphi_{\rho_0}(x - X)|^2 .
\] (1)

Here, \( \hat{\psi}(x) \) and \( \hat{\psi}^\dagger(x) \) are annihilation and creation operators of a boson at \( x \), which satisfies the commutation relations \( [\hat{\psi}(x), \hat{\psi}^\dagger(y)] = \delta(x - y) \). However, their construction is heuristic and there is no theoretical justification to consider the Fourier transform of \( \{N, P\} \). More seriously, this method only reproduces a classical dark soliton at a certain momentum \( P_0 \). It remains unclear how one can construct a quantum dark soliton state with a different momentum \( P \neq P_0 \), e.g. a completely ‘black’ soliton with \( P = \pi\rho_0 \).

In this work, we study the quantum-classical correspondence of dark solitons, and construct a many-body quantum dark soliton state. We start from the idea that a quantum soliton state should be interpreted as the Bose–Einstein condensation to the ‘single-particle dark soliton state’. It turns out that this BEC-like state is approximately expressed as a superposition of yrast states, which is concluded analytically in the free-boson limit, and checked numerically in a weak but finite coupling constant. Based on this result, we propose a construction of the quantum dark soliton state by superposing yrast states. It turns out that the quantum dark soliton is not obtained by the Fourier transform, but by the Gaussian superposition of type-II excitations. The center of the Gaussian distribution determines the velocity of the dark soliton, and the width is found to be proportional to \( c^{1/4} \) (see equation (19) below). The density profile of the quantum dark soliton state constructed in this way shows an excellent agreement with that of the classical dark soliton solution.

Moreover, it turns out that this state has a lifetime longer than the soliton state constructed previously [22, 23].

2. Setup

We consider interacting bosons in a one-dimensional ring (i.e. the periodic boundary condition is imposed). Such a system is described by the Lieb–Liniger model

\[
\hat{H} = \int_{-L/2}^{L/2} dx \left( -\psi^\dagger(x) \partial_x^2 \psi(x) + c \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) \right)
\] (2)

with \( c > 0 \) (i.e. repulsive interactions). The number of particles is given by \( N = \int_{-L/2}^{L/2} dx \psi^\dagger(x) \psi(x) \), and the average number density is denoted by \( \rho_0 = N/L \).

In the Lieb–Liniger model, the Bethe ansatz offers exact energy eigenstates and eigenvalues [11]. Each energy eigenstate \( \{|I\}_N \rangle \) is characterized by a set of quantum numbers \( I_1 < I_2 < \ldots < I_N \), where \( I_i \) is integer for odd \( N \) and half-odd integer for even \( N \). For a given \( |I\rangle \), a set of quasi-momenta \( k_1 < k_2 < \ldots < k_N \) is obtained by the following Bethe ansatz equations:

\[
k_j = \frac{2\pi I_j - \sum_{i=1}^N \arctan \left( \frac{k_i - k_j}{c} \right)}{L} .
\] (3)

The total momentum \( P \) and the energy eigenvalue \( E \) are obtained by \( P = \sum_{j=1}^N k_j \) and \( E = \sum_{j=1}^N \epsilon_k^2 \).

The ground state corresponds to the quantum numbers \( \{I\} = \{-(N-1)/2, -(N-1)/2 + 1, \ldots, (N-1)/2\} \). Yrast states are obtained by removing one of \( I_j \) and adding \((N-1)/2 + 1\) (or \(-(N-1)/2 - 1\)), whose energy spectrum in a large finite system has been obtained [24]. It has been argued that yrast states correspond to the dark solitons since the dispersion relations coincide with each other in the weak-coupling limit after the thermodynamic limit [13].

The classical nonlinear Schrödinger equation is obtained as the classical limit of the Heisenberg equation for \( \psi(x) \); we replace the quantum field operator \( \hat{\psi}(x) \) and \( \hat{\psi}^\dagger(x) \) by the classical field \( \psi(x) \) and \( \psi^*(x) \), respectively, where \( \psi^* \) is the complex conjugate of \( \psi \):

\[
\partial_t \psi(x) = -\partial_x^2 \psi(x) + 2c \psi^*(x) \psi(x) \psi(x) - \mu \psi(x) ,
\] (4)

where \( \mu \) is the chemical potential, which is given by \( 2\rho_0 c \) in the thermodynamic limit, but there is a correction in a finite system [23]. Equation (4) has dark soliton solutions \( \varphi(x - vt) \), where \( P \) is the total momentum of a soliton and \( v \) is the velocity that depends on \( P \) through \( v = -dE/dP \). It is noted that \( |P| \leq \pi\rho_0 \) and \( |v| \leq |v_c| \), where \( v_c \) is called the critical velocity. In the thermodynamic limit, \( v_c = 2\sqrt{\rho_0 c} \), but there is a correction in a finite-size system [23]. The absolute square of \( \varphi(x - x_0) \) corresponds to the particle number density of a dark soliton localized at \( x_0 \), and hence the normalization is given by \( \int_{-L/2}^{L/2} dx \varphi^2(x) = N \).

An explicit expression of \( \varphi(x) \) in a finite system is complicated but given in [23]. In the thermodynamic limit,
\[ \varphi_{p}^\infty(x) \] is given by
\[ \varphi_{p}^\infty(x) = \sqrt{\rho_0} \left[ \gamma \tanh(\sqrt{\rho_0} x) + i \frac{\nu}{\sqrt{\nu^2 - 1}} \right], \] (5)
where \( \gamma = \sqrt{1 - \nu^2/\nu_c^2} \) and \( \nu \) is related to \( P \) by the equation
\[ P(\nu) = 2\rho_0 \left\{ \frac{\pi}{2} - \left[ \frac{\nu}{\nu_c} + \arcsin\left( \frac{\nu}{\nu_c} \right) \right] \right\} \] (6)
for \( \nu \geq 0 \), and \( P(-\nu) = -P(\nu) \) [13]. In particular, the dark soliton at rest \( (\nu = 0) \) corresponds to \( P = \pi\rho_0 \) and its wave function is given by
\[ \varphi_{\pi\rho_0}^\infty(x) = \sqrt{\rho_0} \tanh(\sqrt{\rho_0} x), \] (7)
which is completely ‘black’, i.e. \( \varphi_{\pi\rho_0}^\infty(0) = 0 \).

3. Quantum soliton state

The success of the classical field approximation in BECs stems from the fact that it is a good picture that a macroscopically large number of particles occupy the same single-particle state with a wave function \( \varphi(x) \). If all the particles are in this state, the corresponding \( N \)-particle state is given by
\[ \frac{1}{\sqrt{N!}} \left( \int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi L/2}} dx \varphi_p(x - X) \right)^N |\Omega\rangle, \]
where \( |\Omega\rangle \) denotes the vacuum. It is then natural to guess that an \( N \)-particle quantum soliton state corresponding to a classical dark soliton solution \( \varphi_p(x - X) \) is given by the following BEC state:
\[ |N, X; P\rangle = \frac{1}{\sqrt{N!}} \left( \int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi L/2}} dx \varphi_p(x - X) \right)^N |\Omega\rangle. \] (8)

This quantum state has a desired property; it exactly reproduces the classical dark soliton density profile by the overlap
\[ \langle N, X; P | \hat{\varphi}^\dagger(x) \hat{\varphi}(x) | N, X; P \rangle_{\text{BEC}} = | \varphi_p(x - X) |^2, \] (9)
as well as the wave function itself as
\[ \hat{\varphi}(x)|N, X; P\rangle = \varphi_p(x - X)|N - 1, X; P\rangle. \] (10)

Since the BEC state is a product state, it is regarded as a semiclassical many-body state that corresponds to the macroscopic wave function \( \varphi_p(x) \).

An important problem is to figure out the relation between \( |N, X; P\rangle \) and the energy eigenstates \( |I_j\rangle_N \) of the Lieb–Liniger model; we can always write
\[ |N, X; P\rangle = \sum_{(I)} C_{(I)} |I_j\rangle_N, \] (11)
but we want to understand the structure of expansion coefficients \( C_{(I)} = \langle I_j\rangle_N |N, X; P\rangle \).

First, we numerically calculate overlaps by using the determinant formulas for form factors [25–29]. For simplicity, we focus on the quantum soliton state with \( P = \pi\rho_0 \). Overlaps for \( N = L = 8 \) (\( \rho_0 = 1 \)) and \( c = 0.1 \) are presented in figure 1. Since the yrast state \( |N, P\rangle_{\text{yr}} \) with the momentum \( P \) corresponds to the eigenstate with the lowest energy eigenvalue for the fixed momentum \( P \), figure 1 shows that the quantum soliton state has weights concentrated on yrast states.

Next, we analytically calculate the expansion coefficients in the free-boson limit, i.e. \( c \to 0 \) at a fixed \( \lambda^2 \). In this limit, classical dark soliton solutions \( \{ \varphi_p(x) \} \) with \( 0 \leq P \leq \pi\rho_0 \) become
\[ \varphi^\text{free}_p(x) = \sqrt{\rho_0} \left( 1 - \frac{P}{2\pi\rho_0} - \frac{P^2}{2(2\pi\rho_0)^2} \right), \] (12)

which can be checked by taking the free-boson limit of the explicit expression of \( \varphi^\infty_p(x) \) given in [23]. On the other hand, the free-boson limit of yrast states \( |N, P\rangle_{\text{yr}} \) yields
\[ |N, P^\text{free}_{\text{yr}}\rangle = |n_0 = N - M, n_{2\pi/L} = M\rangle, \] (13)
where \( M \) is an integer given by \( P = 2\pi M/L \), and the right-hand side means the state in which \( N - M \) particles occupy the mode with the wave number \( k = 0 \) and \( M \) particles occupy the mode with \( k = 2\pi/L \). By using equations (8), (12), and (13), we can analytically calculate the expansion coefficients. The result is that expansion coefficients are nonzero only for yrast eigenstates \( \{ |N, P^\text{free}_{\text{yr}}\rangle \} \), and the quantum soliton state is expanded solely by yrast states as
\[ |N, X; P\rangle_{\text{free}} = \sum_{P^\prime} e^{iP^\prime(X-L/2)} e^{-iP^\prime(n_0+P^\prime/2)} |N, P^\prime_{\text{yr}}\rangle_{\text{free}}, \] (14)
where the sum over \( P^\prime \) is taken for \( P^\prime = 2\pi M/L \) with \( M = 0, 1, \ldots, N \). The real function \( g^\text{free}_{P^\prime}(P^\prime) \) is given for large \( N \) by
\[ g^\text{free}_{P^\prime}(P^\prime) \approx \left( 1 - \frac{P^\prime}{2\pi\rho_0} \right) \ln \left( \frac{2\pi\rho_0 - P^\prime}{2\pi\rho_0} \right) + \frac{P^\prime}{2\pi\rho_0} \ln \frac{P^\prime}{P}. \] (15)

It should be noted that equations (14) and (15) have already been obtained in [18].

6 It should be noted that the free-boson limit is different from the weak-coupling limit \( c \to 0 \) after the thermodynamic limit.

7 This state is indeed the eigenstate with the lowest energy for a given momentum \( P \), and is therefore interpreted as an yrast state.
The function $g_P^{\text{free}}(P')$ takes the minimum value at $P' = P$ and is expanded as

$$g_P^{\text{free}}(P') \approx \frac{(P' - P)^2}{2\sigma_{\text{free}}^2}$$

with

$$\sigma_{\text{free}}^2 = \frac{1}{N}p(2\pi\rho_0 - P) \propto \frac{1}{N}.$$  (17)

The soliton BEC state $|N, X; P\rangle$ is therefore expressed as a superposition of yrast states $|N, P^{\text{yr}}\rangle$ with a Gaussian weight of mean $P$ and variance $\sigma_P^2$. This simple calculation in the free boson limit indicates the importance of the Gaussian weight for constructing a quantum soliton state.

This result for the free-boson limit should be relevant for interacting case $c > 0$. Even when the interaction is finite, dark solitons reduce to pure plane waves (noninteracting bosons) in the sound-speed limit, which corresponds to infinitely shallow solitons (with infinitely small $P$) [30]. It is therefore expected that the free-boson limit is connected to the sound-speed limit for finite interactions. The Gaussian superposition of yrast states would also be relevant.

4. Gaussian superposition of yrast states

From the numerical result for $c > 0$ and the analytical result for the free-boson limit, it is expected that a quantum soliton state $|N, X; P\rangle$ is generically expressed as a superposition of yrast states (not restricted to the sound-speed limit mentioned above). By assuming it, we can find how yrast states should be superposed to construct a quantum soliton state. In the thermodynamic limit with a fixed $c$, the mean and the variance of the total momentum operator $\hat{P} = \int_{-L/2}^{L/2} dx \hat{\psi}^+(x)(-i\partial_x)\hat{\psi}(x)$ are given by

$$\lim_{N \to \infty} \langle N, X; \hat{P}\rangle|N, X; P\rangle = P$$

and

$$\lim_{N \to \infty} \langle N, X; (\hat{P} - P)^2\rangle|N, X; P\rangle = \frac{4}{3}\gamma^2\rho_0 \sqrt{\rho_0 c} \equiv \sigma_P^2,$$

respectively (the definition of $\gamma$ is given below equation (5)). In the BEC state $|N, X; P\rangle$, the momenta $\{p_j\}_{j=1}^N$ of $N$ particles can be considered to be independent random variables, and thus the central limit theorem implies that the total momentum $\sum_{j=1}^N p_j$ has a Gaussian distribution. Therefore, if the quantum soliton state consists of the yrast states, the former is given by a Gaussian superposition of the latter for large system sizes:

$$|N, X; P\rangle \approx \mathcal{N}^{-1/2}\sum_{P^{\text{yr}}} e^{iP(X-L/2)}e^{-g_P^{\text{yr}}(P)/2}|N, P^{\text{yr}}\rangle,$$

where

$$g_P(P') = \frac{(P' - P)^2}{2\sigma_P^2}$$

and $\mathcal{N}$ is a normalization constant, and the sum is taken over $P' = 2\pi M/L$ with $M = 0, 1, \ldots, N$.

Equation (20) is greatly simplified compared to equation (8) since the former is restricted to the yrast states. It is therefore possible to calculate some observables using the Bethe ansatz method for large system sizes. In figure 2, we
const. P

P
c

g

Figure 3. Comparison of the density profile of the classical dark soliton and that of the quantum dark soliton states constructed by the Gaussian superposition of the yrast states for \( N = 100, \rho_0 = 1, \) and \( c = 1. \) (Top) the black soliton with \( P = \pi \rho_0. \) (Bottom) the gray soliton with \( P = (\pi/2 - 1)\rho_0. \)

We compare the density profile in the quantum soliton state of equation (20), \( \langle N, 0, P \rangle \hat{\psi}^\dagger(x) \hat{\psi}(x) \rangle_{N, 0, P} \) with its classical counterpart \( |\varphi_P(x)|^2 \) for \( P = \pi \rho_0 \) (black soliton) and \( P = (\pi/2 - 1)\rho_0 \) (gray soliton) for \( N = L = 100 \) and \( c = 0.01. \) Quantum and classical solitons excellently agree with each other.

Here we emphasize that the quantum soliton states are completely determined by equation (20) as a superposition of yrast states. The quantum dark soliton constructed in [22] corresponds to the uniform superposition, i.e. \( g_p(P') = \) const., but our result shows that the Gaussian weight is important in constructing a quantum dark soliton, which is different from the case of blight solitons [12]. By using equation (20), we can construct a family of dark solitons with varying \( P, \) where \( P \) determines the depth of the soliton. This is an advantage of equation (20) compared to the uniform superposition introduced in [22]; the uniform superposition of the yrast states cannot control the depth of the soliton. Figure 2 shows that the Gaussian superpositions indeed reproduce dark solitons with arbitrary depth i.e. arbitrary \( P, \) while the uniform superposition does not.

It should be noted that the idea of the Gaussian superposition of yrast states has already been found in a recent work [20]. In that work, the Gaussian form was assumed and the Gaussian width \( \sigma_P \) was just a free parameter. In contrast, we derived the Gaussian form by applying the central limit theorem to the quantum soliton state equation (8) and obtained an analytical expression of \( \sigma_P, \) and the Gaussian weights to yrast states are completely determined without any free parameter.

We emphasize that a Gaussian superposition of yrast states does not reproduce the density profile of a classical soliton in the deep quantum regime. For \( P = \pi \rho_0 \) (the black soliton) and \( P = (\pi/2 - 1)\rho_0 \) (gray soliton), we compare the density profile calculated from the Gaussian superposition of yrast states with that of the classical soliton for \( N = L = 100 \) and \( c = 1 \) in figure 3. From the figure, it turns out that there are deviations between the classical solitons and the quantum solitons which are constructed by the Gaussian superposition of yrast states. On the other hand, the soliton-BEC state (8) always reproduces exactly the same particle density and phase as the classical soliton, as we explained in equations (9) and (10). A Gaussian superposition of yrast states should thus be regarded as an approximation of the BEC state. Indeed, as demonstrated in figure 1, the BEC state has a small but finite overlap with energy eigenstates that do not belong to yrast states for \( N = L = 8 \) and \( c = 0.1. \) If we go into the deep quantum regime, the BEC state (8) would provide us a better starting point to consider the correspondence between classical solitons and quantum solitons.

5. Time evolution

Since energy eigenvalues are obtained by solving equation (3) and using \( E = \sum_{j=1}^N k_j^2, \) we can compute the dynamics of the density profile in a numerically exact manner. Figure 4 shows the time evolution of the expectation value of \( \hat{\psi}^\dagger(x - vt) \hat{\psi}(x - vt), \) i.e. the density in the moving frame at the soliton velocity, starting from a quantum soliton state.
In the moving frame at the soliton velocity $v$, the first term $v \Delta P$ vanishes, so we have $\Delta E \approx |\langle \Delta P^2/2 \rangle | v/\partial P|$. By using equations (6) and (19) we obtain $\Delta E \approx \gamma^2 \rho_0 c/\gamma$, and hence $\tau \approx 3\pi/(2\gamma^2 \rho_0 c)$. The decay time of a quantum dark soliton is inversely proportional to $c$ in the weak-coupling regime, $\tau \sim 1/c$, which is confirmed numerically and consistent with [23].

It is noted that the dependence of $\tau \sim 1/c$ has been reported by Sato et al [23], but quantitatively, the decay time of our quantum soliton state is much longer than that of the dark soliton state constructed in [23]. Following [23], if the decay time is defined by the time when the smallest value of the density notch reaches the value of 0.5, the decay time for $c = 0.01$ is about 600 in our dark soliton state, while it is about 100 in the dark soliton state proposed in [23]. This fact implies that the precise form of the Gaussian weight is important in dynamics.

6. Relationship between our method and measurements of particle positions method

A remaining open problem is to understand the relation between the quantum soliton state constructed in this work, i.e. equations (8) or (20), and the recent theoretical observation by Syrwid and Sacha [17] that a dark soliton emerges in successive measurements of particle positions starting from a single yrast state. Successive measurements of particle positions were originally considered in the context of interference of two independent BECs to mimic a simultaneous measurement of particle positions [32]. The result by Syrwid and Sacha indicates that an yrast state should be interpreted as a state in which dark solitons are present but their positions are uncertain. Although the expectation value of $\hat{\psi}^\dagger (x) \hat{\psi}(x)$ in a single yrast state is uniform, a single simultaneous measurement of all the positions yields a soliton density profile.

Let us describe their method in detail. We set a certain $N$-particle initial state $|\psi_N\rangle$. The probability density distribution $\rho_1(x_1)$ of finding a particle at $x_1$ is given by

$$\rho_1(x_1) = \frac{1}{N} \langle \psi_N | \hat{\psi}^\dagger (x_1) \hat{\psi}(x_1) | \psi_N \rangle.$$  

After the measurement, the detected particle is removed. Thus, after the first measurement, $(N-1)$-particle state is given by

$$|\psi_{N-1}\rangle = \frac{\psi(x_1)|\psi_N\rangle}{\sqrt{\langle \psi_N | \hat{\psi}^\dagger (x_1) \hat{\psi}(x_1) | \psi_N \rangle}}.$$  

We again perform the measurement of a particle position. The probability density distribution $\rho_2(x_2)$ of finding the second particle at $x_2$ is given by

$$\rho_2(x_2) = \frac{1}{N-1} \langle \psi_{N-1} | \hat{\psi}^\dagger (x_2) \hat{\psi}(x_2) | \psi_{N-1} \rangle.$$  

By removing the detected particle, we have the following $(N-2)$-particle state after the second measurement:

$$|\psi_{N-2}\rangle = \frac{\psi(x_2)|\psi_{N-1}\rangle}{\sqrt{\langle \psi_{N-1} | \hat{\psi}^\dagger (x_2) \hat{\psi}(x_2) | \psi_{N-1} \rangle}}.$$  

By repeating this process $K$ times, we obtain the following quantum state with the $N-K$ particles:

$$|\psi_{N-K}\rangle \propto \psi(x_K) |\psi_{N-K-1}\rangle \ldots |\psi(x_1)\rangle |\psi_N\rangle,$$

where $x_i$ is the outcome of $i$th measurement of the particle position. It has been numerically shown in [17] that if the initial state $|\psi_N\rangle$ is set to be a single yrast state with momentum $P$, the dark soliton with the same momentum emerges in the post-measurement state. In particular, the state of the last particle $|\psi_1\rangle$ represents a dark soliton; the wave function $\langle x | \psi_1 \rangle$ coincides with that of the classical dark soliton with momentum $P$ at a certain position $X$ that depends on the measurement outcomes $\{x_1, x_2, \ldots, x_{N-1}\}$.

The soliton-BEC state of equation (8) is also important in this context. By using equation (10), we have $\hat{\psi}(x)|N, X; P\rangle = \varphi_p(x - X)|N-1, X; P\rangle \propto |N-1, X; P\rangle$ for any $x$. This relation leads us to the conclusion that if the soliton-BEC state $|N, X; P\rangle$ is chosen as an initial state, the state after $K$ measurements of particle positions is identical to $|N-K, X; P\rangle$ (up to the global phase) with probability one. In other words, the sequence of the soliton-BEC states $\{N^\prime, X; P\prime\}_{N^\prime = 1}$ is an exact solution of the measurement dynamics.

In the free-boson ($c = 0$) and black-soliton ($P = \pi \rho_0$) case, the relation is clearer; it can be proved that when the pre-measurement state $|\psi_N\rangle$ is set to $|N, P = \pi \rho_0 \rangle$, the wave function of the last particle $\langle x | \psi_1 \rangle$ is indeed proportional to $\varphi_{\pi \rho_0}^\text{free}(x - X) = X = x + \frac{L}{2}$, where $x = \frac{1}{N-1} \sum_{i=1}^{N-1} x_i$ [19]. In other words, the black soliton emerges after the successive measurements of particle positions, and its position depends on the particular realization of the measurement outcomes. Moreover, it is analytically shown that not only the state of the last particle $|\psi_1\rangle$, but also the state $|\psi_{N-K}\rangle$ after $K$ measurements with $1 \leq N - K \ll N$ is already very close to the quantum soliton-BEC state $|N-K, X; P = \pi \rho_0 \rangle$ with probability very close to one, whose proof is provided in appendix. Figure 5 shows the normalized density distribution for $N-K$ particle state after the $K$th measurement of the particle position. We can see that the dark soliton density profile emerges after measurements of particle positions, although the density profile in the yrast state ($K = 0$, thin solid line in figure 5) is uniform.
It should be however emphasized that it is still open to prove the corresponding result for a small but finite coupling $c > 0$.

7. Conclusion and discussion

In this work, we have discussed the property of a quantum soliton state given by equation (8), which is interpreted as a Bose–Einstein condensation to a single-particle wave function $\varphi_p(x - x_0)/\sqrt{N}$, where $\varphi_p(x)$ is the dark soliton solution of the classical nonlinear Schrödinger equation. It has turned out that this quantum soliton state almost consists of the yrast states, which is confirmed numerically for small $c > 0$ and analytically for $c = 0$. This result offers a direct confirmation that a classical dark soliton with a localized position corresponds to a superposition of yrast states of the Lieb–Liniger model. In addition, we have revealed that a quantum soliton state $|N, X; P\rangle$ is well approximated by the state $|N, X; P\rangle_{\text{yrast}}$ that has a Gaussian weight on each yrast state $|N, P\rangle_{\text{yrast}}$ with mean $\langle P\rangle = P$ and variance $\sigma_p^2 = 4\gamma^2\rho_0\sqrt{\rho_0}c/3$ in the weak-coupling regime. Numerical calculations show excellent agreements between the density profile of the Gaussian superposition of the yrast states and that of the classical dark soliton. We have also discussed dynamics of a quantum dark soliton, and it has been shown that the decay time is proportional to $c^{-1}$.

We emphasize the value of a completely analytical expression of quantum dark solitons as a superposition of the yrast states. Firstly, a Gaussian superposition of yrast states was conjectured in an earlier work [20], but the Gaussian weight has been derived in our work from the soliton-BEC state (8). Secondy, in [20], the Gaussian width $\sigma_p$ was a free parameter, but we have obtained an analytic expression of $\sigma_p$.

As we have seen in section 5, the lifetime of our quantum soliton is much longer than that of a soliton-like state constructed in [22, 23] as a uniform superposition of yrast states. It means that weights to yrast states also affect the dynamics of a dark soliton. Therefore, our analytic expression of the quantum soliton state is also important in investigating dynamical properties of dark solitons beyond the classical regime.

A quantum dark soliton state constructed by our method is particularly relevant near the classical limit (weak-interaction limit), and some quantum nature would be lost in our construction. It is still an open problem whether we can extend the notion of ‘quantum soliton state’ to a deep quantum regime. In other words, it has not yet been obvious whether there exists a quantum soliton state that has a lifetime much longer than that of the soliton state constructed in this paper.

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Appendix. Post-measurement state in the free-boson case

In the free-boson case, it has been shown in [19] that the single-particle wave function after the measurement of the positions of $N - 1$ particles coincides with the wave function of the classical dark soliton with $P = \pi\rho_0$ (that corresponds to the black soliton) when the pre-measurement state is set to be an yrast state with the same momentum $|N, P = \pi\rho_0\rangle_{\text{free}}$.

In this appendix, we show that under the same setting, the state after the measurement of $K$ particle positions approximately coincides with the quantum soliton state $|N - K, X; P = \pi\rho_0\rangle_{\text{free}}$ with probability close to one for large $N$ if $N - K \ll N$.

For simplicity, we consider the case of even $N$. The pre-measurement state $|\psi_N\rangle$ is given by the yrast state with $P = \pi\rho_0$, i.e.

$$|\psi_N\rangle = |N, P = \pi\rho_0\rangle_{\text{yrast}} = |n_0 = N/2, n_{2\pi/L} = N/2\rangle.$$  \hspace{1cm} (A1)

The state after the positions of $K = N - n$ particles is given by

$$|\psi_n\rangle \propto \psi(x_{n-1})\ldots\psi(x_2)\psi(x_1)|\psi_N\rangle \equiv \sum_{m=0}^{n} \alpha_m |n - m, m\rangle,$$ \hspace{1cm} (A2)
where \([n - m, m] \) is a shorthand notation of \([n_0 = n - m, n_{2z,L} = m] \). An explicit calculation yields
\[
\Omega_m \propto \sum_{1 \leq i < j < \ldots < k_{2z,L} \leq N-n} \\
\times \sqrt{(N-M)(N-M-1) \ldots (n-m+1)} \\
\times \sqrt{(N/2)(N/2-1) \ldots (m+1)} e^{\frac{\lambda}{2}(u_{i1}^2 + \ldots + u_{k_{2z,L}}^2)} \\
\times \frac{Z_m}{(n-m)!m!},
\]
(A3)
where \( Z_m \) is defined by
\[
Z_m = \sum_{1 \leq i < j < \ldots < k_{2z,L} \leq N-n} e^{\frac{\lambda}{2}(u_{i1}^2 + \ldots + u_{k_{2z,L}}^2)}. \quad \text{(A4)}
\]
By introducing auxiliary binary variables \([s_i]_{i=1,2,\ldots,N-n} \) and the Kronecker delta \( \delta(a, b) \), we obtain
\[
Z_m = \sum_{s_{1z}} \ldots e^{\frac{\lambda}{2} \sum_{i=j} s_{ij}(\sum_{i=1}^{N-n} s_{ij} - \frac{N}{2} - m)} \\
\times \delta(\lambda) \sum_{s_{1z}} \ldots e^{\frac{\lambda}{2} \sum_{i=j} s_{ij}(\sum_{i=1}^{N-n} s_{ij} - \frac{N}{2} - m)} \\
= \int d\lambda e^{Xm} e^{-Nf(\lambda)},
\]
(A5)
where
\[
f(\lambda) = \frac{i\lambda}{2} - \frac{1}{N} \sum_{j=1}^{N-n} \ln[1 + e^{(\lambda - \frac{2\pi}{L})j}].
\]
We apply the saddle-point method to equation (A5). Since the saddle point \( \lambda^* \) is independent of \( m \), we have, by neglecting proportional constant that is independent of \( m \),
\[
Z_m \propto e^{\lambda^* m}.
\]
(A7)
The saddle-point equation \( df(\lambda)/d\lambda = 0 \) yields for large \( N \)
\[
N^{-1} \sum_{i=1}^{N} \tan \left[ \frac{1}{2} (\lambda^* + \frac{2\pi}{L} x_i) \right] = 0.
\]
(A8)
Because of the symmetry, we expect that the distribution of the measurement outcomes \( \{x_i\} \) is almost symmetric around its mean value \( \bar{x} = \sum_{i=1}^{N-n} x_i/(N-m) \), and therefore
\[
\lambda^* = \frac{2\pi}{L} \bar{x}.
\]
(A9)
By substituting it to equation (A7), we have
\[
Z_m \propto e^{i\frac{\pi}{2} \bar{x} m}.
\]
(A10)
From equations (A2) and (A3), we obtain
\[
|\psi_m\rangle \propto \frac{1}{\sqrt{(n-m)!m!}} \sum_{m=0}^{n-m} e^{i\frac{\pi}{2} \bar{x} m} [n - m, m]. \quad \text{(A11)}
\]
Let us compare equation (A11) with the quantum soliton state (8). By substituting equation (12) into equation (8), we find that the soliton-BEC state for free bosons is given by
\[
|n, X; P = \pi \rho_0^{\text{free}} \rangle \propto \sqrt{\frac{n!}{2^{n/2} \pi^{n/2}}} \sum_{m=0}^{n-m} e^{i\frac{\pi}{2} \frac{(X-\frac{1}{2})m}{m!(n-m)!}} [n - m, m] \\
\times \frac{e^{i\frac{\pi}{2} \frac{(X-\frac{1}{2})m}{m!(n-m)!}} [n - m, m].}
\]
(A12)
Comparing it with equation (A11), we obtain the desired result
\[
|\psi_m\rangle = |n, X = \bar{x} + L/2; P = \pi \rho_0 |.
\]
(A13)
In this way, if \( n \ll N \) (this condition is necessary to apply the saddle-point method), the post-measurement state is almost certainly given by the soliton-BEC state for large \( N \).

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