On profinite groups in which centralizers have bounded rank

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Abstract. The article deals with profinite groups in which centralizers are of finite rank. For a positive integer $r$ we prove that if $G$ is a profinite group in which the centralizer of every nontrivial element has rank at most $r$, then $G$ is either a pro-$p$ group or a group of finite rank. Further, if $G$ is not virtually a pro-$p$ group, then $G$ is virtually of rank at most $r + 1$.

1. Introduction

There are several recent publications dealing with profinite groups in which centralizers have certain prescribed properties (cf. [12, 13, 15, 14]). In the present paper we are concerned with the following conjecture made in [15].

Conjecture 1.1. Let $G$ be a profinite group in which the centralizer of every nontrivial element has finite rank. Suppose that $G$ is not a pro-$p$ group. Then $G$ has finite rank.

Recall that a profinite group $G$ is said to have finite rank $r$ if every subgroup of $G$ can be generated by $r$ elements. Throughout the paper by a subgroup of a profinite group we mean a closed subgroup and we say that a subgroup is generated by some subset if it is topologically generated by that subset.

Our purpose is to establish the following theorem which provides a substantial evidence in favor of the above conjecture.

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Theorem 1.2. Let $r$ be a positive integer and $G$ a profinite group in which the centralizer of every nontrivial element has rank at most $r$. Then $G$ is either a pro-$p$ group or a group of finite rank.

We mention that in a free pro-$p$ group all centralizers are procyclic and therefore pro-$p$ groups satisfying the hypothesis of Theorem 1.2 in general need not be of finite rank. It is natural to suspect that in the latter case of the above theorem the rank of $G$ should be bounded in terms of $r$ only. The author is pretty much doubtful about this. On the other hand, we have the following result. Recall that a group is said to virtually have some property if it has a subgroup of finite index with that property.

Theorem 1.3. Let $G$ be a profinite group in which the centralizer of every nontrivial element has rank at most $r$. Then either $G$ is a virtually pro-$p$ group or $G$ is virtually of rank at most $r + 1$.

In the next section we collect some useful results needed for the proofs of the above theorems. In Section 3 we prove Theorem 1.2. Section 4 contains a proof of Theorem 1.3. Our notation throughout the article follows [9].

2. Preliminaries

Throughout the article automorphisms of a profinite group are assumed to be continuous. If $A$ is a group of automorphisms of a group $G$, the subgroup generated by elements of the form $g^{-1}g^\alpha$ with $g \in G$ and $\alpha \in A$ is denoted by $[G, A]$. It is well-known that the subgroup $[G, A]$ is an $A$-invariant normal subgroup in $G$. We also write $C_G(A)$ for the centralizer of $A$ in $G$.

The next lemma is a list of useful facts on coprime actions. Here $|K|$ means the order of a profinite group $K$ (see for example [9] for the definition of the order of a profinite group). For finite groups the lemma is well known (see for example [3] Ch. 5 and 6). For infinite profinite groups the lemma follows from the finite case and the inverse limit argument (see [9] Proposition 2.3.16) for a detailed proof of Part (iii). As usual, $\pi(G)$ denotes the set of prime divisors of the order of $G$.

Lemma 2.1. Let a profinite group $A$ act by automorphisms on a profinite group $G$ such that $(|G|, |A|) = 1$. Then

(i) $G = [G, A]C_G(A)$.
(ii) $[G, A, A] = [G, A]$.
(iii) $C_{G/N}(A) = NC_G(A)/N$ for any $A$-invariant normal subgroup $N$ of $G$. 
(iv) For each prime $q \in \pi(G)$ there is an $A$-invariant Sylow $q$-subgroup in $G$.

The following theorem is immediate from the corresponding results on finite groups obtained independently by Guralnick [4] and Lucchini [8]. For finite soluble groups the corresponding result was established by Kovacs [7].

**Theorem 2.2.** Let $r$ be a positive integer and $G$ a profinite group in which every Sylow subgroup has rank at most $r$. Then the rank of $G$ is at most $r + 1$.

In the sequel we will need the following theorem, due to Khukhro [5]. We use the expression “$(a, b, c \ldots)$-bounded” to mean “bounded from above by some function depending only on the parameters $a, b, c \ldots$”.

**Theorem 2.3.** Let $G$ be a finite nilpotent group with an automorphism $\alpha$ of prime order $q$ such that $C_G(\alpha)$ has rank $r$. Then $G$ contains a characteristic subgroup $N$ such that $N$ has $q$-bounded nilpotency class and $G/N$ has $(q, r)$-bounded rank.

The case $q = 2$ of the above theorem was established in [11] with a somewhat more precise statement, not even requiring the nilpotency of $G$. Using the routine inverse limit argument Theorem 2.3 can be extended to the case where $\alpha$ is a coprime automorphism of a profinite group $G$:

Let $G$ be a pronilpotent group admitting a coprime automorphism $\alpha$ of prime order $q$ such that $C_G(\alpha)$ has rank $r$. Then $G$ contains a characteristic subgroup $N$ such that $N$ has $q$-bounded nilpotency class and $G/N$ has $(q, r)$-bounded rank.

### 3. Proof of Theorem 1.2

The next lemma deals with a crucial case of Theorem 1.2.

**Lemma 3.1.** Let $r \geq 1$ and $G$ a profinite group in which every nontrivial element has centralizer of rank at most $r$. Assume that $G$ has a nontrivial proper normal subgroup $M$ and a subgroup $A$ such that $(|M|, |A|) = 1$ and $G = MA$. Then one of the following statements holds.

1. $A$ is infinite and virtually of rank at most $r$. In this case also $M$ has rank at most $r$.
2. $A$ is finite and the rank of $M$ is $(q, r)$-bounded, where $q$ is the smallest prime in $\pi(A)$.

In either case $G$ has finite rank.
Proof. If $M$ is finite the result is straightforward from Theorem 2.2 so without loss of generality we assume that $M$ is infinite. Suppose first that $A$ is infinite. Let $N$ be an open subgroup of $M$ that is normal in $G$. Then $A$ induces a finite group of automorphisms of $M/N$ and therefore $A$ contains an open normal subgroup $B$ that acts on $M/N$ trivially. By Lemma 2.1 (iii) we have $M = NC_M(B)$. It follows that $M/N$ has rank at most $r$. Since this holds for each open subgroup of $M$ that is normal in $G$ we conclude that $M$ has rank at most $r$. Moreover, since the subgroup $B$ has nontrivial centralizer in $M$, it follows that $B$ has rank at most $r$, whence $A$ is virtually of rank at most $r$.

Now consider the case where $A$ is finite. Choose $a \in A$ of order $q$. By Lemma 2.1 (iv) for each prime $p \in \pi(M)$ there is a Sylow $p$-subgroup $P$ of $M$ normalized by $a$. Observe that $a$ induces an automorphism of $P$ of order dividing $q$. Since $C_P(a)$ is of rank at most $r$, it follows from Khukhro’s Theorem 2.3 that $P$ has a characteristic subgroup $L$ such that $L$ is nilpotent and $P/L$ has $(q, r)$-bounded rank. Since $L$ is nilpotent, it has nontrivial centre. Clearly, $L$ is contained in the centralizer of any nontrivial element of its centre. The centralizer is of rank at most $r$ and so both $L$ and $P/L$ have bounded rank. We conclude that $P$ has $(q, r)$-bounded rank. Since this holds for each prime $p \in \pi(M)$, we deduce from Theorem 2.2 that $M$ has finite $(q, r)$-bounded rank, as required.

Proof of Theorem 1.2. Recall that $G$ is a profinite group in which the centralizer of every nontrivial element has rank at most $r$. We wish to show that $G$ is either a pro-$p$ group or a group of finite rank. Assume that $G$ is not a pro-$p$ group. Then there exists an open normal subgroup $N$ of $G$ such that $\pi(G/N)$ contains two different primes say $q_1$ and $q_2$. Fix any prime $p \in \pi(N)$ and a Sylow $p$-subgroup $P$ of $N$. Without loss of generality we may assume $q_1 \neq p$. By the Frattini argument, $G = N_G(P)N$. Therefore $N_G(P)$ contains a nontrivial $q_1$-element $a$. We make now use of Lemma 3.1 with $M = P$ and $A = \langle a \rangle$. Thus $r(P) \leq r$ or $r \leq f(r, q_1)$, where the function $f$ expresses the $(q_1, r)$-bound. Letting $d = \max\{r, f(q_1, r), f(q_2, r)\}$ we conclude that every Sylow subgroup of $N$ has rank at most $d$. Therefore, according to Theorem 2.2, $N$ has rank at most $d + 1$ and since $N$ is open, $G$ has finite rank.

4. Proof of Theorem 1.3

We start this section with general facts on finite groups of given rank. Recall that the Fitting height of a finite soluble group $G$ is the
length $h(G)$ of a shortest normal series of $G$ all of whose factors are nilpotent.

**Lemma 4.1.** Let $G$ be a finite soluble group of rank $r$. Then $h(G)$ is $r$-bounded.

**Proof.** Let $F = P_1 \times \cdots \times P_k$ be the Fitting subgroup of $G$, where $P_i$ are Sylow subgroups of $F$. Using [3, Theorem 6.1.6] we can pass to the quotient $G/\Phi(F)$ and without loss of generality assume that the subgroups $P_i$ are elementary abelian. For each $i$ the quotient $G/C_G(P_i)$ naturally embeds in the group of linear transformations of the $r$-dimensional linear space over the field with $p_i$ elements. By the well-known Zassenhaus theorem (see for example [10, Theorem 3.23]), the derived length of $G/C_G(P_i)$ is $r$-bounded. Let $j$ be the maximum of derived lengths of groups $G/C_G(P_i)$ for $i = 1, \ldots, k$. It follows that the $j$-th term of the derived series of $G$ is contained in $C_G(F)$. Taking into account that $C_G(F) \leq F$ ([3, Theorem 6.1.3]) we conclude that $G/F$ has derived length $j$. Since $j$ is $r$-bounded, the lemma is established. □

The nonsoluble length of a finite group $G$ is the least number of nonsoluble factors in a normal series all of whose factors are either soluble or direct products of nonabelian simple groups. We write $\lambda(G)$ for the nonsoluble length of $G$. It was shown in [6] that $\lambda(G)$ does not exceed the maximum Fitting height of soluble subgroups of $G$. Combining this with Lemma 4.1 we obtain

**Lemma 4.2.** Let $G$ be a finite group of rank $r$. Then $\lambda(G)$ is $r$-bounded.

Using the techniques developed by Wilson in [16] (in particular, Lemma 2 in [16]) the above results on finite groups of given rank can be extended to profinite groups. This enables us to deduce the following lemma.

**Lemma 4.3.** Let $G$ be a profinite group of finite rank. Then $G$ has a normal series of finite length

$$1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_s = G \quad (\ast)$$

all of whose factors are either pronilpotent or (topologically) isomorphic to Cartesian products of nonabelian finite simple groups.

We will now deduce that a profinite group of finite rank is virtually prosoluble.

**Lemma 4.4.** Let $G$ be a profinite group of finite rank. Then $G$ is virtually prosoluble.
Proof. Let 
\[ S = \prod_{i \in I} S_i \]
be a factor of the series (\(\ast\)), where \(S_i\) are nonabelian finite simple groups. Recall that the Sylow 2-subgroups of \(G\) have finite rank. Since all subgroups \(S_i\) have even order \(\mathbb{Z}/2\), it follows that \(S\) is in fact a product of only finitely many finite simple groups and hence \(S\) is finite. Thus, we conclude that the nonprosoluble factors of the series (\(\ast\)) are finite.

Let \(G_{j_1+1}/G_{j_1}, G_{j_2+1}/G_{j_2}, \ldots, G_{j_k+1}/G_{j_k}\) be the nonprosoluble factors of the series (\(\ast\)). Let \(H = \{ x \in G \mid [G_{j_i+1}, x] \leq G_{j_i} \text{ for } i = 1, \ldots, k \}\).

Thus \(H\) is the “centralizer” of the nonprosoluble factors of the series (\(\ast\)). It is straightforward that the subgroup \(H\) is prosoluble. Since the factors \(G_{j_i+1}/G_{j_i}\) are finite for all \(i\) we deduce that \(H\) is open in \(G\). Hence, \(G\) is virtually prosoluble, as required.

We will also require the following theorem which is immediate from [11, Theorem 5.7].

Theorem 4.5. Let \(G\) be a pro-p group of finite rank \(r\). Then \(\text{Aut}(G)\) is virtually a pro-p group.

By the Fitting height of a prosoluble group \(G\) we mean the length \(h(G)\) of a shortest series of normal subgroups all of whose factors are pronilpotent. Note that the parameter \(h(G)\) is finite if, and only if, \(G\) is an inverse limit of finite soluble groups of bounded Fitting height. Lemma 4.3 shows that if \(G\) is a prosoluble group of finite rank, then \(h(G)\) is finite. We write \(F(G)\) for the maximal pronilpotent normal subgroup of \(G\).

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Recall that \(G\) is a profinite group in which the centralizer of every nontrivial element has rank at most \(r\). We wish to show that either \(G\) is virtually a pro-p group for some prime \(p\) or \(G\) is virtually of rank at most \(r + 1\). Assume that \(G\) is not a pro-p group and so, by Theorem 1.2, \(G\) has finite rank. By Lemma 4.4, \(G\) has an open prosoluble normal subgroup with finite Fitting height. Since we wish to prove that \(G\) has a certain property virtually, without loss of generality we can assume that \(G\) is prosoluble and \(h = h(G)\) is finite. If \(G\) is pronilpotent, then \(G\) can be written as a direct product \(G = K \times L\), for some subgroups \(K\) and \(L\) such that \(|K|, |L| = 1\).

Since \(K\) and \(L\) centralize each other, it follows that both have rank at most \(r\) and thus \(G\) has rank at most \(r\) as well. We therefore assume
that $F(G)$ is a proper subgroup of $G$. Choose $q \in \pi(F(G))$ and let $Q_0$ be the Sylow $q$-subgroup of $F(G)$. Let $H$ be a Hall $q'$-subgroup of $G$. If $H$ is finite, then $G$ is virtually pro-$q$ and we are done. Therefore we assume that $H$ is infinite. Lemma 3.1 (applied with $M = Q_0$ and $A = H$) shows that the rank of $Q_0$ is at most $r$. Let $C = C_H(Q_0)$ In view of Theorem 4.5 $C$ is open in $H$. Therefore $G$ has an open normal subgroup whose intersection with $H$ is contained in $C$. We replace $G$ by that open subgroup and without loss of generality assume that $H = C$. Then for any prime $p \neq q$ the Sylow $p$-subgroups of $G$ have rank at most $r$. Since $G$ is prosoluble with finite Fitting height, $F(G)$ contains its centralizer $C_G(F(G))$. Thus, if $\pi(F(G)) = \{q\}$, we obtain a contradiction since $H$ centralizes $Q_0$. Otherwise, pick a prime $p \in \pi(F(G))$ such that $p \neq q$ and let $P$ be the Sylow $p$-subgroup of $F(G)$. Choose a Sylow $q$-subgroup $Q$ of $G$. Now Lemma 3.1 (applied with $M = P$ and $A = Q$) shows that $Q$ is virtually of rank at most $r$. Let $Q_1$ be an open subgroup of $Q$ of rank $\leq r$. We replace $G$ by an open normal subgroup whose intersection with $Q$ is contained in $Q_1$. Now all Sylow subgroups of $G$ have rank at most $r$. By Theorem 2.2 $G$ has rank at most $r + 1$. The proof is complete. 

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