Stability of the pion and the pattern of chiral symmetry breaking

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We investigate the pressure of the pion, which should be equal to zero to ensure the stability of the pion, within the framework of the chiral quark model beyond the chiral limit. The pressure of the pion turns out to vanish nontrivially by the Gell-Mann-Oakes-Renner relation within the present framework. It implies that the stability of the pion might be deeply rooted in spontaneous chiral symmetry breaking. We also discuss physical quantities relevant to the energy-momentum tensor operator.

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1. The energy-momentum tensor form factors (EMTFFs) of hadrons, also known as the gravitational form factors, provide crucial information on how quarks and gluons are distributed to form a hadron.[1] On the one hand, it is almost impossible to get access to the EMTFFs experimentally, because they are probed by very weak graviton exchange. On the other hand, the EMTFFs are related to the moments of the generalized parton distributions (GPDs)[2–6]. They are identified as the hadronic matrix elements of a nonlocal flavor-singlet vector operator. These GPDs are experimentally accessible, for example, via deeply virtual Compton scattering or exclusive reactions.

The EMTFFs of the pion reveal explicitly its internal structure, in particular, in regard to spontaneous chiral symmetry breaking (S\text{\textit{\textdegree}}SB), since the pion, the lightest hadron, is a Goldstone boson arising from S\text{\textit{\textdegree}}SB. The diagonal space components of the energy-momentum tensor (EMT) are identified as the pressure of the pion which should vanish to ensure its stability. This stability condition of the pion should be satisfied by any approach of describing the structure of the pion. In this Letter, we want to show how the pressure of the pion is just expressed in terms of the Gell-Mann-Oakes-Renner (GOR) relation [7], based on the chiral quark model (\chi QM). The model can be constructed in such a way that chiral symmetry and its spontaneous breaking is realized by the nonlinear pion field without an elementary scalar field. \chi QM furnishes a simple but effective framework in investigating the structure of the pion. It has been shown to be successful in explaining various properties of the pion. The EMTFFs of the pion have been already studied within similar frameworks in the chiral limit. In this limit, the pressure of the pion trivially vanishes. When one turns on chiral symmetry breaking explicitly, prominent features emerge: The pion mass leads to a split of the two EMTFFs at zero momentum transfer and the stability of the pion is secured by the GOR relation. Both features are all deeply rooted in the pattern of explicit chiral symmetry breaking.

2. The isoscalar vector GPD of the pion is defined as the pionic matrix element of the nonlocal vector current [8, 9].

\begin{equation}
2\delta^{ab}H_{\gamma=0}^{a}(x, \xi, t) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{ix\lambda P \cdot n} \langle \pi^a(p') | \bar{\psi}(-\lambda n/2) \not\!\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\not\!\!\!\!\n
the EMT
\[
\langle \pi^a(p')|T_{\mu\nu}(0)|\pi^b(p) \rangle = \frac{\delta^{ab}}{2} \left[ (tg_{\mu\nu} - q_{\mu} q_{\nu})\Theta_1(t) + 2P_{\mu} P_{\nu}\Theta_2(t) \right],
\]
where \( T_{\mu\nu} \) denotes the quark part of the QCD EMT operator defined as
\[
T_{\mu\nu}(x) = \frac{1}{2} \tilde{\psi}(x)\gamma_{(\mu} \stackrel{i}{\leftrightarrow} \nabla_{\nu)}\psi(x).
\]
The form factors \( \Theta_1 \) and \( \Theta_2 \) are called the EMTFFs, which were already put forward by Pagels [1] decades ago. Compared with Eq. (2), we have the relations \( \Theta_1 = -4A_2^{L=0} \) and \( \Theta_2 = A_2^{L=0} \). The low-energy theorem tells that the EMTFFs should satisfy the following two conditions at the zero-momentum transfer: \( \Theta_2(0) = 1 \) and \( \Theta_1(0) - \Theta_2(0) = \mathcal{O}(m^2) \) [8, 10]. The first condition ensures that the pion mass is correctly reproduced, which is given by the the fully temporal component of the matrix element of the EMT
\[
\langle \pi^a(p)|T_{tt}(0)|\pi^b(p) \rangle |_{t=0} = -2m^2 \Theta_2(0) \delta^{ab},
\]
which gives the mass of the pion. On the other hand, the pressure of the pion is defined as the matrix element for the sum of the EMT spatial components
\[
\langle \pi^a(p)|T_{ii}(0)|\pi^b(p) \rangle |_{t=0} = \frac{3}{2} t \Theta_1(t)
\]
We will soon show that Eq. (3) will be related to the GOR relation.

We are now in a position to compute the EMTFFs within the framework of the \( \chi \)QM [11, 13]. It is known that the SU(2)_L \times SU(2)_R chiral symmetry is spontaneously broken to the vector subgroup SU(2)_V by the quark condensate, which gives rise to the Goldstone bosons that correspond to the homogeneous space SU(2)_L \times SU(2)_R\Sigma SU(2)_V known as the Goldstone-boson manifold. The Goldstone boson fields or the pion fields \( \Sigma \) satisfy the following transformation \( \Sigma \rightarrow \Sigma \mathcal{R} \) with the left-handed (right-handed) transformations \( L(R) \). Thus, having integrated out the quark fields, the effective chiral action in Euclidean space can be written as follows
\[
S_{\text{eff}} = -N_c \text{Tr} \log [i\theta + iM \Sigma P_L + iM \Sigma P_R + i\tilde{m}],
\]
where \( N_c \) is the number of colors and \( P_L(P_R) \) denote the projection operators defined as \( P_L = (1 - \gamma_5)/2 \) and \( P_R = (1 + \gamma_5)/2 \). The Tr stands for the functional trace over all involved spaces. The pseudo-Goldstone boson field \( \pi \) is represented non-linearly as
\[
\Sigma = \exp \left( \frac{i\tau \cdot \pi}{f_\pi} \right)
\]
with the Pauli matrices \( \tau_i \) and the pion decay constant \( f_\pi \). The \( M \) is the diagonal vacuum expectation value of the \( \Sigma \), which is called the dynamical quark mass that arises from \( \Sigma \chi \)SB. Since we consider explicitly the chiral symmetry breaking [11, 13], the current quark mass matrix \( \tilde{m} = \text{diag}(m_u, m_d) \) is introduced in the effective chiral action. Assuming isospin symmetry, we can define the current quark mass \( m \) as \( m = m_u = m_d \).

The quark EMT operator corresponding to Eq. (7) has the same form as Eq. (4). One can now straightforwardly compute the matrix elements of the EMT operator:
\[
\langle \pi^a(p_f)|T_{\mu\nu}(0)|\pi^b(p_i) \rangle = \delta^{ab}\frac{2N_c}{f_\pi} \int d^3k \sum_i F_i(k, p) \mu_{\nu} + (\mu \leftrightarrow \nu)
\]
where
\[
F_{a\mu
u} = \frac{-MMTk_{di}k_{dv}}{D_{b}D_{c}}
\]
\[
F_{b\mu
u} = \frac{2M^2k_{dv}}{D_{a}D_{b}D_{c}} \left[ -k_{am} \left( k_{bc} + M^2 \right) + k_{bm} \left( k_{ac} + M^2 \right) + k_{cm} \left( k_{ab} + M^2 \right) \right],
\]
with the internal quark momenta are defined as \( k_{am} = k_{a} - p_{\mu}/2 - q_{\mu}/2 \), \( k_{bm} = k_{a} + p_{\mu}/2 - q_{\mu}/2 \), and \( k_{cm} = k_{a} + p_{\mu}/2 + q_{\mu}/2 \). In addition, we have introduced abbreviations \( d^3k/d(2\pi)^4 \), \( M = m + M \), \( D_{i} = k_{i}^2 + M^2 \), \( k_{d} = k_{b} + k_{c} \), and \( k_{ij} = k_{i} \cdot k_{j} \) for convenience. It is interesting to observe that redundant quadratic divergences in Eq. (9) are canceled except for the term yielding the quark condensate. In the chiral limit, the quadratic divergent terms are all canceled out.

Selecting the diagonal spatial parts of the matrix elements at zero momentum transfer and summing them, we arrive at the following expression
\[
P = \langle \pi^a(p)|T_{ii}(0)|\pi^a(p) \rangle
\]
\[
= \frac{12N_c m M}{f_\pi^2} \int d^3l \left[ \frac{-l^2}{l^2 + M^2} \right] + \frac{12N_c M^2}{f_\pi^2} \int d^3l \int_0^1 dx \left[ \frac{-p^2 l^2}{l^2 + x(1 - x)p^2 + M^2} \right]^3
\]
In the chiral limit, the pressure trivially vanishes because of \( m = 0 \) and \( p^2 = -m^2_\pi = 0 \). When one switches on the explicit chiral symmetry breaking, Eq. (11) looks apparently persistent. However, the first term is pertinent to
the quark condensate in Euclidean space \(-i \langle \overline{\psi} \psi \rangle\), which is related to that in Minkowski space \(\langle \psi \psi \rangle = -i \langle \overline{\psi} \psi \rangle\), defined as

\[
i \langle \psi \psi \rangle = 8N_c \int \frac{d^4l}{(l^2 + M^2)^2}.
\] (12)

while the second term is proportional to the pion decay constant expressed as

\[
f_\pi^2 = 4N_c \int_0^1 dx \int \frac{M^2}{(l^2 + M^2 + x(1-x)p^2)^2}.
\] (13)

Equations (12) and (13) being used, Eq. (11) turns out to be related to the GOR relation

\[
\mathcal{P} = \frac{3M}{f_\pi^2 M} \left( m \langle \overline{\psi} \psi \rangle + m_\pi^2 f_\pi^2 \right),
\] (14)

which vanishes. Thus, the pressure of the pion beyond the chiral limit is still kept to be zero and consequently the stability of the pion is ensured. This is a remarkable result, since it gives a hint that the stability of the pion should be deeply rooted in \(S\chi SB\) and the pattern of explicit chiral symmetry breaking. The result of Eq. (14) is unique for the pion.

3. We now continue to compute and discuss the EMTFFs and the corresponding transverse charge densities. The EMTFFs are derived from the matrix element given in Eq. (11).

\[
\Theta_1(t) = 1 + 2 f_\pi \left[ t(4L_{11} + L_{12}) - 8m_\pi^2(L_{11} - L_{13}) \right]
\]

\[
\Theta_2(t) = 1 - \frac{2t}{f_\pi} L_{12}.
\] (17)

Expanding Eqs. (15) and (16) with respect to \(t\) or the chiral effective action (7) in curved space by the heat-kernel expansion [15, 16], we find the values of the LECs.
EMTFFs is defined as

\[
L_{11} = \frac{N_c}{192\pi^2} = 1.6 \times 10^{-3}, \\
L_{12} = -2L_{11} = -3.2 \times 10^{-3}, \\
L_{13} = -\frac{N_c}{96\pi^2} \frac{M}{B_0} \left(0, \frac{M^2}{\Lambda^2}\right) = 0.84 \times 10^{-3},
\]

where \(\Gamma(0, M^2/\Lambda^2)\) stands for the incomplete Gamma function with the cutoff mass \(\Lambda\) and \(B_0 = -<\bar{\psi}\psi>/f_\pi^2\).

The results of the LECs are in good agreement with those from \(\chi\)PT given as \(L_{11} = 1.4 \times 10^{-3}, L_{12} = -2.7 \times 10^{-3}\), and \(L_{13} = 0.9 \times 10^{-3}\) at the scale of 1 GeV. The identical expressions for \(L_{11}\) and \(L_{12}\) were already derived in Refs. [17][20], whereas the value of \(L_{13}\) is distinguished from those in them.

The transverse charge densities provide essential information on how the quarks inside a pion are distributed. Once the EMTFFs or the generalized form factors are given, one can derive the transverse charge density \(\rho_{20}(b)\) as a function of the impact parameter \(b\) in the transverse plane by carrying out the two-dimensional Fourier transform of them:

\[
\rho_{20}(b) = \int_0^{\infty} \frac{QdQ}{2\pi} J_0(bQ) \Theta_2(t),
\]

where \(b = \sqrt{b_x^2 + b_y^2}\). In Fig. 2 we illustrate the transverse charge density \(\rho_{20}(b)\) of the pion as a function of \(b\), which is spherically symmetric. With \(b_y\) fixed to be zero, we draw \(\rho_{20}(b)\) in Fig. 3 which shows that \(\rho_{20}(b)\) becomes singular as \(b_y\) approaches zero. This singular behavior is already known for the transverse charge density of the pion in the leading order [21][22].

The square of the transverse charge radius of the EMTFFs is defined as

\[
\langle b^2 \rangle_{20} = \int_0^{\infty} d^2b b^2 \rho_{20}(b).
\]

We derive the corresponding numerical results as \(\sqrt{\langle b^2 \rangle_{20}} = 0.270\) fm and 0.264 fm, respectively, in the chiral limit and with \(m_\pi = 140\) MeV. These values are approximately two times smaller than those of the transverse charge radius of the pion electromagnetic form factor. For example, we obtain \(\sqrt{\langle b^2 \rangle_{\text{00}}} = 0.535\) fm with the same pion mass, based on the results of Ref. [23], which is almost the same as the phenomenological one [24].

4. In this Letter, we investigated the energy-momentum tensor form factors or the generalized form factors of the pion, based on the chiral quark model with explicit flavor SU(3) symmetry breaking. We first showed that the pressure of the pion vanishes due to the Gell-Mann-Oakes-Renner relation, which is essential to keep the pion stable. We studied a difference between the two form factors \(\Theta_1(t)\) and \(\Theta_2(t)\) at \(Q^2 = 0\), which arises from the finite pion mass. We presented the results of the low-energy constants for the energy-momentum tensor form factors of the pion in comparison with those from chiral perturbation theory. Finally, we computed the corresponding transverse charge density of the pion electromagnetic form factor. The present result exhibits a singular behavior as the impact parameter approaches zero. The present transverse charge radius turns out to be approximately as twice as smaller than that of the pion electromagnetic form factor.

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