Determination of the b-mass using renormalon cancellation

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Abstract

Methods of Borel integration to calculate the binding ground energies and mass of \( b\bar{b}\) quarkonia are presented. The methods take into account the leading infrared renormalon structure of the "soft" binding energy \( E(s) \) and of the quark pole masses \( m_q \), and the property that the contributions of these singularities in \( M(s) = 2m_q + E(s) \) cancel. The resummation formalisms are applied to quantities which do not involve renormalon ambiguity, such as \( \overline{\text{MS}} \) mass \( \overline{m}_q \) and \( \alpha_s(\mu) \).

1 Introduction

The calculation of binding energies, masses of heavy quarkonia \( q\bar{q} \) and another physical parameter using renormalon method has attracted the attention recently. The calculations, based on perturbative expansions, are primarily due to the knowledge of up to \( N^2\text{LO} \) term (\( \sim \alpha_s^3 \)) of the static quark-antiquark potential \( V(r) \) and partial knowledge of the \( N^3\text{LO} \) term there, and the ultrasoft gluon contributions to a corresponding effective theory \( N^3\text{LO} \) Hamiltonian; and the knowledge of the pole mass \( m_q \) up to order \( \sim \alpha_s^3 \).

Another impetus in these calculations was given by the observation of the fact that the contributions of the leading infrared (IR) renormalon singularities (at \( b = 1/2 \)) of the pole mass \( m_q \) and of the static potential \( V(r) \) cancel in the sum \( 2m_q + V(r) \). Consequently, this cancellation effect must be present also in the total quarkonium mass \( M = 2m_q + E_{q\bar{q}} \), or more precisely, in \( M(s) = 2m_q + E(s) \) where \( E(s) \) is the hard+soft part of the binding energy, i.e., the part which includes the contributions of relative quark-antiquark momenta \( |k^0|, |k| \geq m_q\alpha_s \), i.e., soft/potential scales (predominant) and higher hard scales (smaller contributions). In addition, the binding energy has contribution \( E_{q\bar{q}}(us) \) from the ultrasoft momenta regime \( |k^0|, |k| \sim m_q\alpha_s^2 \). The ultrasoft contribution is not related to the \( b = 1/2 \) renormalon singularity, since this singularity has to do with the behavior of theory in the region which includes the hard (\( \sim m_q \)) and soft/potential (\( \sim m_q\alpha_s \)) scales.

This contribution is based on ref. \[13\] and we present numerical calculation of the binding ground energies \( E_{b\bar{b}} \) (separately the \( s \) and the \( us \) parts) and the mass \( 2m_b + E_{b\bar{b}} \) of the heavy \( b\bar{b} \) system, by taking into account the leading IR renormalon structure of \( m_b \) and \( E_{b\bar{b}}(s) \), and combining some features as: (a) the mass that we use in the perturbation expansions is a renormalon-free mass \([10]\,[11]\,[14]\,[15]\,[16]\); (b) Borel integrations \[12\] are used to perform resummations.

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Before resummations we perform separation of the soft/potential ($s$) and ultrasoft ($us$) part of the binding energies, and apply the renormalon-based Borel resummation only to the $s$ part. The renormalization scales used in the Borel resumptions are $\mu_h \sim m_q$ (hard scale) for $2m_q$, and $m_q\alpha_s \leq m_s < m_q$ for $E_{q\bar{q}}(s)$. The term corresponding to $E_{q\bar{q}}(us)$ is evaluated at $\mu_{us} \sim m_q\alpha_s^2$ whenever perturbatively possible. Further, the Borel resumptions are performed in three different ways: (a) using a slightly extended version of the full bilocal expansion Refs. [17, 12]; (b) using a new “$\sigma$-regularized” full bilocal expansion introduced in the ref. [13]; (c) using the form of the Borel transform where the leading IR renormalon structure is a common factor of the transform [18, 19] (we call it $R$-method). The Borel integrations for both $m_q$ and $E_{q\bar{q}}(s)$ are performed by the same prescription (generalized principal value PV [20, 18, 19, 21]) so as to ensure the numerical cancellation of the renormalon contributions in the sum $2m_q + E_{q\bar{q}}(s)$. Furthermore, we demonstrate numerically that in the latter sum the residues at the renormalons are really consistent with the aforementioned cancellation when a reasonable factorization scale parameter for the $s$-$us$ separation is used, while they become inconsistent with the aforementioned cancellation when no such separation is used. The numerical results allow us to extract the mass $\overline{m}_b$ from the known $\Upsilon(1S)$ mass of the $b\bar{b}$ system.

2 Pole mass

Here we calculate the pole mass $m_q$ in terms of the $\overline{\text{MS}}$ renormalon-free mass $\overline{m}_q \equiv \overline{m}_q(\mu = \overline{m}_q)$ and of $\alpha_s(\mu, \overline{\text{MS}})$, using elements of the renormalization group approach of Ref. [11] and the bilocal expansion method Refs. [17, 12]. The ratio $m_q/\overline{m}_q$ has, divergent, perturbation expansion in $\overline{\text{MS}}$ scheme which is at present known to order $\sim \alpha_s^3$ (Ref. [6] for $\sim \alpha_s^2$; [7] for $\sim \alpha_s^3$). Let us define

$$S \equiv \frac{m_q}{\overline{m}_q} - 1 = \frac{4}{3} a(\mu) \left[1 + a(\mu) r_1(\mu) + a^2(\mu) r_2(\mu) + O(a^3)\right],$$

$$r_1(\mu) = \kappa_1 + \beta_0 L_m(\mu),$$

$$r_2(\mu) = \kappa_2 + (2\kappa_1\beta_0 + \beta_1) L_m(\mu) + \beta_0^2 L_m^2(\mu),$$

$$\left(\frac{4}{3}\right)\kappa_1 = 6.248\beta_0 - 3.739,$$

$$\left(\frac{4}{3}\right)\kappa_2 = 23.497\beta_0^2 + 6.248\beta_1 + 1.019\beta_0 - 29.94,$$

where $L_m = \ln(\mu^2/\overline{m}_q^2)$, while $\beta_0 = (11 - 2n_f/3)/4$ and $\beta_1 = (102 - 38n_f/3)/16$ are the renormalization scheme independent coefficients with $n_f = n_l$ being the number of light active flavors (quarks with masses lighter than $m_q$). The natural renormalization scale here is $\mu = \mu_h \sim m_q$ (hard scale).

Let us consider $S$ as a function of the running coupling $a(Q) = \alpha(q)/\pi$, and the perturbation expansion for $S$ reads:

$$S(a(Q)) = \sum_{n=0}^{\infty} s_n a^n,$$

which has to be summed. Then, one defines the Borel transform as

$$B(b, Q) = \sum_{n=0}^{\infty} s_n \frac{b^n}{n!}.$$

This series has a finite radius of convergence in the $b$-plane, and we introduce the Borel function $\tilde{S}(a)$ (Borel Integral) corresponding to $S(a)$ as

$$\tilde{S}(a) = \int_0^{\infty} db B(b, Q) e^{-b/a}$$

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and in our case $B_S(b)$ is known to order $\sim b^2$

$$B_S(b; \mu) = \frac{4}{3} \left[ 1 + \frac{r_1(\mu)}{1!\beta_0} b + \frac{r_2}{2!\beta_0} b^2 + \mathcal{O}(b^3) \right]. \quad (9)$$

It has renormalon singularities at $b = 1/2, 3/2, 2, \ldots, -1, -2, \ldots \,$[14] [22] [23]. The behavior of $B_S$ near the leading IR renormalon singularity $b = 1/2$ is determined by the resulting renormalon ambiguity of $m_q$ which has to have the dimensions of energy and should be renormalization scale and scheme independent – the only such QCD scale being $\text{const} \times \Lambda_{QCD}$ [24]. The Stevenson scale $\Lambda$ [25] can be obtained in terms of the strong coupling parameter $a(\mu; c_2, c_3, \ldots) = \alpha_s(\mu; c_2, c_3, \ldots)/\pi$, where $c_j = \beta_j/\beta_0$ ($j \geq 2$) are the parameters characterizing the renormalization scheme, by solving the renormalization group equation (RGE) [25].

$$\frac{da(\mu)}{d \ln \mu^2} = -\beta_0 a^2(\mu)(1 + c_1 a(\mu) + c_2 a^2(\mu) + \cdots) \Rightarrow$$

$$\ln \left( \frac{\Lambda^2}{\mu^2} \right) = \frac{1}{\beta_0} \int_0^{a(\mu)} dx \left[ \frac{1}{x^2(1 + c_1 x + c_2 x^2 + \cdots)} - \frac{1}{x^2(1 + c_1 x)} \right]$$

$$- \frac{1}{\beta_0 a(\mu)} + \frac{c_1}{\beta_0} \ln \left( \frac{1 + c_1 a(\mu)}{c_1 a(\mu)} \right) \Rightarrow \Lambda = \mu \exp \left( -\frac{1}{2\beta_0 a(\mu)} \right) \left( \frac{1 + c_1 a(\mu)}{c_1 a(\mu)} \right)^{\nu}$$

$$\exp \left[ -\frac{1}{2\beta_0} \int_0^{a(\mu)} dx \frac{(c_2 + c_3 x + c_4 x^2 + \cdots)}{(1 + c_1 x)(1 + c_1 x + c_2 x^2 + \cdots)} \right]$$

$$\text{where } \nu = c_1/(2\beta_0) = \beta_1/(2\beta_0^2). \text{ Expansion of expression } [12] \text{ in powers of } a(\mu) \text{ then gives}$$

$$\Lambda = \mu \exp \left( -\frac{1}{2\beta_0 a(\mu)} \right) a(\mu)^{-\nu} c_1^{-\nu} \left[ 1 + \sum_{k=1}^{\infty} \bar{r}_k a^k(\mu) \right], \quad (13)$$

$$\text{where}$$

$$\bar{r}_1 = \frac{(c_1 - c_2)}{2\beta_0}, \quad \bar{r}_2 = \frac{1}{8\beta_0^2} \left[ (c_1^2 - c_2)^2 - 2\beta_0 (c_1^3 - 2c_1c_2 + c_3) \right],$$

$$\bar{r}_3 = \frac{1}{48\beta_0^3} \left[ (c_1^2 - c_2)^3 - 6\beta_0 (c_1^4 - c_2) (c_1^4 - 2c_1c_2 + c_3) \right.$$

$$+ 8\beta_0^2 (c_1^4 - 3c_1^2c_2 + c_2^2 + 2c_1c_3 - c_4) \right]. \quad (14)$$

The singular part of the Borel transform $B_S(b)$ around $b = 1/2$ must have the form

$$B_S(b; \mu) = N_m \pi \frac{\mu}{m_q} \left[ \frac{1}{(1 - 2b)^{1+\nu}} \left[ 1 + \sum_{k=1}^{\infty} \bar{c}_k (1 - 2b)^k \right] \right] + B_S^{(an.)}(b; \mu), \quad (15)$$

$$\bar{c}_1 = \frac{\bar{r}_1}{(2\beta_0)^{\nu}}, \quad \bar{c}_2 = \frac{\bar{r}_2}{(2\beta_0)^2\nu(\nu - 1)}; \quad \bar{c}_3 = \frac{\bar{r}_3}{(2\beta_0)^3\nu(\nu - 1)(\nu - 2)}, \quad (16)$$

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and $B_S^{(an.)}(b; \mu)$ is analytic on the disk $|b| < 1$. The $\overline{MS}$ coefficients for $n_f = 4$: $c_1 = 1.5400$, $c_2 = 3.0476$ and $c_3 = 15.0660$ are already known [26, 27], but for $c_4$ we have only estimates [28, 29] obtained by Padé-related methods. We consider $c_4 = 40 \pm 60$. Thus, $\tilde{c}_j$ can be obtained: $\tilde{c}_1 = -0.1054$, $\tilde{c}_2 = 0.2736$ and $\tilde{c}_3 = 0.01 \pm 0.17$.

The bilocal method [17] consists of taking in the expansion (15) for the analytic part the polynomial in powers of $b$, so that the expansion of $B_S$ around $b = 0$ agrees with expansion (9). For that, the residue parameter $N_m$ in Eq. (15) has to be determined. Using Refs. [31]:

$$N_m = \frac{\overline{m}_q}{\mu} \frac{1}{\pi} R_S(b = 1/2) ,$$

and according to (15)

$$R_S(b; \mu) \equiv (1 - 2b)^{1+\nu} B_S(b; \mu) .$$

Then, the $N_m$ is estimated (see Ref. [31]), using $R_S(b)$ TPS and Padé approximation $[1/1]$

$$N_m(n_f = 4) = 0.555 \pm 0.020 ,$$

The bilocal expansion (15) has then for the analytic part the polynomial

$$B_S^{(an.)}(b; \mu) = h_0^{(m)} + \frac{h_1^{(m)}}{1! \beta_0} b + \frac{h_2^{(m)}}{2! \beta_0^2} b^2 ,$$

where, by convention, $r_0 = \tilde{c}_0 = 1$. Then, the bilocal formula, is

$$B_S(b; \mu)^{(biloc.)} = N_m \frac{\mu}{\overline{m}_q} \frac{1}{(1 - 2b)^{1+\nu}} \left[ 1 + \sum_{k=1}^{\tilde{c}_k} (1 - 2b) b^k \right] + \sum_{k=0}^{2} \frac{h_k^{(m)}}{k! \beta_0^k} b^k .$$

Applying the (generalized) principal value (PV) prescription for the Borel integration

$$S(a) = \frac{1}{\beta_0} \text{Re} \int_{\pm i \epsilon}^{\pm i \epsilon} \text{d} b \ \text{exp} \left( -\frac{b}{\beta_0 a(\mu)} \right) B_S(b; \mu) ,$$

we obtain the pole mass $m_q$ in terms of the mass $\overline{m}_q$. The numerical integration is performed, using the Cauchy theorem (Refs. [19]).

In Figs. 1 (a) we present the resulting (PV) pole masses of the $b$, as function of the renormalization scale $\mu$. The spurious $\mu$-dependence is very weak. In addition, results of another method (“R”-method) are presented in Figs. 1 (a), with the $\mu$-dependence stronger in the low-$\mu$ region ($\mu / \overline{m}_q < 1$). The R-method (Refs. [19, 19]) consists in the Borel integration of the function

$$S = \frac{1}{\beta_0} \text{Re} \int_{\pm i \epsilon}^{\pm i \epsilon} \text{d} b \ \text{exp} \left( -\frac{b}{\beta_0 a(\mu)} \right) \frac{R_S(b; \mu)}{(1 - 2b)^{1+\nu}} ,$$

where for $R_S(b)$ the corresponding (NNLO) TPS is used. When we take $\overline{m}_b = 4.23$ GeV and we vary the values of the residue parameter $N_m$, the bilocal method gives, at $\mu / \overline{m}_q = 1$, variation $\delta m_b = \mp 3$ MeV. When
the central values of $N_m$ are used, the variation of the obtained values of $m_q$ with $\mu$, when $\mu/\overline{m}_q$ grows from 1.0 to 1.5, is about 5 MeV for $m_b$ (for $R$-method: 4 MeV). When $c_4$ is varied, the variation is about $\mp 2$ and $\mp 1$ MeV for $m_b$. The uncertainty in $\alpha_s$ can be taken as $\alpha_s(M_Z) = 0.1192 \pm 0.0015$. This uncertainty is by far the major source in the variation of the pole masses: $(\delta m_b)_{\alpha_s} = (135)_{+148}^{137} - 150\text{ MeV for bilocal method (})^7$. The uncertainty in $\alpha_s$ is by far the major source in the variation of the pole masses: $(\delta m_b)_{\alpha_s} = (135)_{+148}^{137} - 150\text{ MeV for bilocal method (})^7$.

The natural renormalization scale $\mu$ here is a hard scale $\mu \sim \overline{m}_q$, and will be denoted later in this work as $\mu_m$ in order to distinguish if from the “soft” renormalization scale $\mu$ used in the analogous renormalon-based resummations of the (hard+)soft binding energy $E_{q\bar{q}}(s)$ ($\overline{m}_q > \mu \geq \overline{m}_q \alpha_s$) in Sec. 4. The fact that the two renormalization scales are different does not affect the mechanism of the (b = 1/2) renormalon cancellation in the bilocal calculations of the meson mass ($2m_q + E_{q\bar{q}}(s)$), because the renormalon ambiguity in each of the two terms is renormalization scale independent $\sim \Lambda$, as seen by Eqs. (15)–(19). On the other hand, if $R$-type methods (24) [cf. also Eq. (17)] are applied for the resummations of $2m_q$ and $E_{q\bar{q}}(s)$, the renormalon ambiguities are renormalization scale independent in the approximation of the one-loop RGE running, and the renormalon cancellation is true at this one–loop level.

3 Separation of the soft and ultrasoft contributions

The perturbation expansion of the (hard + soft + ultrasoft) binding energy $E_{q\bar{q}}$ of the $q\bar{q}$ heavy quarkonium vector ($S = 1$) or scalar ($S = 0$) ground state ($n = 1, \ell = 0$) up to the $N_c^3$LO $O(m_q \alpha_s^5)$ was given in [33]. The reference mass scale used was the pole mass $m_q$. The ground state energy expansion has the form

$$E_{q\bar{q}} = -\frac{4}{9} m_q \pi^2 a^2(\mu) \left\{ 1 + a(\mu) \left[ k_{1,0} + k_{1,1} L_p(\mu) \right] + a^2(\mu) \left[ k_{2,0} + k_{2,1} L_p(\mu) + k_{2,2} L_p^2(\mu) \right] + a^3(\mu) \left[ k_{3,0} + k_{3,1} L_p(\mu) + k_{3,2} L_p^2(\mu) + k_{3,3} L_p^3(\mu) \right] + O(a^4) \right\}$$

(25)

where

$$L_p(\mu) = \ln \left( \frac{\mu}{\frac{4}{9} m_q \pi a(\mu)} \right).$$

(26)

The coefficients $k_{i,j}$ of perturbation expansion (24) of the quarkonium ($n = 1; \ell = 0; S = 1$ or 0) are given below. [34] [35] [36] [37], [38] – For $N_c = 3$:

$$k_{1,1} = 4 \beta_0, \quad k_{1,0} = \left( \frac{97}{6} - \frac{11}{9} n_f \right),$$

(27)

$$k_{2,2} = 12 \beta_0^2, \quad k_{2,1} = \frac{927}{4} - \frac{193}{6} n_f + n_f^2,$$

(28)

$$k_{2,0} = 361.342 - 40.9649 n_f + 1.16286 n_f^2 - 11.6973 S(S + 1),$$

(29)

$$k_{3,3} = 32 \beta_0^3, \quad k_{3,2} = 4521 + \frac{10955}{24} n_f + \frac{1027}{36} n_f^2 - \frac{5}{9} n_f^3,$$

(30)

$$k_{3,1} = 7242.3 - 1243.95 n_f + 69.1066 n_f^2 - 1.21714 n_f^3$$

$$+ \frac{\pi^2}{2592} (-67584 + 4096 n_f) S(S + 1),$$

(31)
$$k_{3,0} = \left[ (7839.82 - 1223.68 n_f + 69.4508 n_f^2 - 1.21475 n_f^3) \\
+ (-109.05 + 4.06858 n_f) S(S+1) \\
- \frac{\pi^2}{18} (-1089 + 112 S(S+1)) \ln (a(\mu) + 2\frac{a_3}{4^3}) \right],$$

(32)

Here, $a_3$ have been estimated in Ref. 31, obtained from the condition of renormalon cancellation in the sum $(2m_q + V_{q\bar{q}}(r))$

$$\frac{1}{4^3} a_3(n_f=4) \approx 86. \pm 23.$$  

(33)

The coefficients $k_{i,j}$ in the expansion 29 originate from quantum effects from various scale regimes of the participating particles: (a) the hard scales ($\sim m_q$); (b) the soft and potential scales where the three momenta are $|\mathbf{q}| \sim m_q \alpha_s$ ($|\mathbf{q}|^0 \sim m_q \alpha_s$ in the soft and $|\mathbf{q}|^0 \sim m_q \alpha_s^2$ in the potential regime); (c) ultrasoft scales where $|\mathbf{q}|^0$ and $|\mathbf{q}|$ are both $\sim m_q \alpha_s$. The coefficients are dominated by the soft scales; the hard scales start contributing at the NNLO 3 and are numerically smaller. For this reason, we will usually refer to the combined soft and hard regime contributions to the binding energy as simply soft ($s$) contribution $E_{\bar{q}q}(s)$. Strictly speaking, it is only the pure soft regime that contributes to the $b = 1/2$ renormalon. However, for simplicity, we will resum the hard+soft contributions $E_{\bar{q}q}(s)$ together. This will pose no problem, since the hard regime, being clearly perturbative, is not expected to deteriorate the convergence properties of the series for $E_{\bar{q}q}(s)$.

The natural renormalization scale $\mu$ in the resummations of $E_{\bar{q}q}(s)$ is expected to be closer to the soft scale ($m_q \leq \alpha_s \mu < m_q$).

On the other hand, the N$^3$LO coefficient $k_{3,0}$ obtains additional contributions from the from the ultrasoft (us) regime. The leading ultrasoft contribution comes from the exchange of an ultrasoft gluon in the heavy quarkonium 5 7. It consists of two parts:

1. The retarded part, which cannot be interpreted in terms of an instantaneous interaction

$$\frac{1}{\pi^3} k_{3,0}(us, \text{ret.}) = -\frac{2}{3 \pi} \left( \frac{4}{3} \right)^2 L_1^E \approx +41.014,$$  

(34)

where $L_1^E \approx -81.538$ is the QCD Bethe logarithm - see Refs. 5 8.

2. The non-retarded part can be calculated as expectation value of the $us$ effective Hamiltonian $\mathcal{H}^{us}$ in the Coulomb (i.e., leading order) ground state $|1\rangle$, where $\mathcal{H}^{us}$ (in momentum space) was derived in Refs. 5 8. Direct calculation of the expectation value, here in coordinate space, then gives:

$$\frac{1}{\pi^3} k_{3,0}(us, \text{nonret.}) = -\frac{9}{4 \pi^2} \frac{1}{m_q a^3(\mu)} (1|\mathcal{H}^{us}|1) = \frac{2}{\pi^3 m_q a^4(\mu)} \left\{ \frac{1}{2} \ln \frac{\mu^2}{(E_1)^2} + \frac{5}{6} - \ln 2 \right\} \left\{ -\frac{27\pi^2}{8} a^3(\mu) \langle 1 \mid \frac{1}{r} \mid 1 \rangle \right\}$$

$$-17\pi^2 a^2(\mu) m_q \langle 1 \mid \frac{1}{r^2} \mid 1 \rangle + \frac{4\pi^2 a(\mu)}{3 m_q} \langle 1 \mid \delta(\mathbf{r}) \mid 1 \rangle$$

$$+ 3\pi \frac{a(\mu)}{m_q} \langle 1 \mid \{\Delta_r, \frac{1}{r} \} \mid 1 \rangle \right\},$$

(35)

$$= -14.196 \left[ \ln \left( \frac{\mu_f}{m_q \alpha_s^2(\mu)} \right) + 0.9511 \right],$$

(36)
Here, \( E_1^C = -(4/9)m_q \alpha_s^2(\mu) \) is the Coulomb energy of the state \(|1\rangle\), and \( \mu_f \) is the factorization energy between the soft (\( \sim m_q \alpha_s \)) and ultrasoft (\( \sim m_q \alpha_s^2 \)) scale.

The \( s-us \) factorization scale \( \mu_f \) can be estimated as being roughly in the middle between the \( s \) and \( us \) energies on the logarithmic scale \[31\]

\[
\mu_f \approx (E_S E_{US})^{1/2} = \kappa m_q \alpha_s(\mu_s)^{3/2},
\]

where \( \kappa \approx 1 \) and \( \mu_s \approx E_S(\mu) \). Therefore, the ultrasoft part of the N³LO coefficient \( k_{3,0} \) can be rewritten, by Eqs. \[31\], \[2\] and \[37\], in terms of the \( s-us \) parameter \( \kappa \) as

\[
\frac{1}{\pi} k_{3,0}(us) = 27.512 + 7.098 \ln(\alpha_s(\mu_s)) - 14.196 \ln(\kappa).
\]

The soft scale \( \mu_s \) appearing here will be fixed by the condition \( \mu_s = (4/3)m_q \alpha_s(\mu_s) \).

The formal perturbation expansions for the separate soft and ultrasoft parts of the ground state binding energy \[25\] are then

\[
E_{qq}(s) = -\frac{4}{9} m_q \pi^2 a^2(\mu) \left\{ 1 + \sum_{i=1}^{2} a_i(\mu) \sum_{j=0}^{i} k_{i,j} L_p(\mu)^j \right. \\
\left. + a^3(\mu) \sum_{j=1}^{3} k_{3,j} + a^3(\mu) [k_{3,0} - k_{3,0}(us)] + O(a^4) \right\},
\]

\[
E_{qq}(us) = -\frac{4}{9} m_q \pi^2 a^2(\mu) \left\{ a^3(\mu) k_{3,0}(us) + O(a^4) \right\}.
\]

The energy \( E_{qq}(s) \) \[39\] contains the leading IR renormalon effects, and \( E_{qq}(us) \) \[31\] does not. In these expressions, the common factor is the soft scale \( \mu_q(\mu) = (4/3)m_q \alpha_s(\mu) \) which is also present as the reference scale in the logarithms \( L_p(\mu) = \ln(\mu/\mu_p(\mu)) \) appearing with the coefficients \( k_{i,j} \) (when \( j \geq 1 \)) in Eqs. \[25\].

We will re-express \( m_q \) everywhere in \( E_{qq} \) with the renormalon-free mass \( \overline{m}_q \), and will consider the dimensionless soft-energy quantity \( E_{qq}(s)/\overline{m}_q \).

Thus, we will divide the soft binding energy with the quantity \( \overline{m}(\overline{\mu}) = (4/3)\overline{m}_q \alpha_s(\overline{\mu}) \), where \( \overline{\mu} \) can be any soft scale. We will fix this scale by the condition \( \overline{\mu} = (4/3)\overline{m}_q \alpha_s(\overline{\mu}) \) (\( \Rightarrow \overline{\mu} = \mu_s \)). Further, in the logarithms \( L_p(\mu) \) we express the pole mass \( m_q \) in terms of \( \overline{m}_q \) and powers of \( a(\mu) \) (cf. Sec. \[2\]), and the powers of logarithms \( \ln^k[a(\overline{\mu})] \) we re-express in terms of \( \ln^k[a(\overline{\mu})] \). This then results in the following soft binding energy quantity \( F(s) \) to be resummed

\[
F(s) = -\frac{9}{4\pi^2} \frac{E_{qq}(s)}{\overline{m}_q a(\overline{\mu})} = a(\mu) \left[ 1 + a(\mu) f_1 + a^2(\mu) f_2 + a^3(\mu) f_3 + O(a^4) \right],
\]

where the coefficients \( f_j \) depend on \( \ln a(\overline{\mu}) \) and on three scales: the renormalization scale \( \mu (\geq m_q \alpha_s) \), the (fixed) soft scale \( \overline{\mu} \), and \( \overline{m}_q \). The coefficient \( f_3 \) depends, in addition, on the parameters \( \kappa \) \[37\], \[39\], \( \mu_s \), and \( \alpha_3 \) \[35\]. The coefficients \( f_j \) are written explicitly in Appendix \[A\]. The \( b = 1/2 \) renormalon in the quantity \( F(s) \) is then of the type of the renormalon of the pole mass \( m_q \) discussed in the previous Sec. \[2\].

However, if we divided in Eq. \[31\] by \( m_q \) instead of \( \overline{m}_q \) and at the same time used in the resulting \( f_j \)-coefficients \( \ln m_q \), the numerical resumptions of \( F(s) \) by methods of Sec. \[4\] would give us values for
\[ E_{q\bar{q}}(s) \text{ different usually by not more than } O(10^3 \text{MeV}) \text{ (we checked this numerically). We will briefly refer to these approaches later in this Section as “pole mass” approaches. A version of such pole mass bilocal approach was applied in Ref. [12] for resummation of the unseparated } E_{q\bar{q}}(s). \]

The ultrasoft part [10], on the other hand, has no \( b = 1/2 \) renormalon. The mass scale used there should also be renormalon free (\( \overline{m} \)). The renormalization scale \( \mu \) there should be adjusted downward to the typical \( us \) scale of the associated process \( \mu \rightarrow \mu_{us} \sim m_q \alpha_s^2 \) in order to come closer to a realistic estimate

\[ E_{q\bar{q}}(us) \approx -\frac{4}{9} \overline{m}_q \pi^2 k_{3,0}(us) a^5(\mu_{us}) \, . \]  

(42)

4 Mass \( \overline{m}_b \) from the known mass of the vector \( b\bar{b} \) \([\Upsilon(1S)]\)

The soft binding energy quantity to be resummed is \( F(s) \) of Eq. [11]. However, in the \( N^3\text{LO} \) coefficient \( f_q \) we have dependence on \( a_3 \), and on the \( s-us \) factorization scale parameter \( \kappa \sim 1 \). Then, the value of \( \kappa \) is obtained requiring that the residue parameter values be reproduced from the Borel transform of the soft binding energy quantity \( F(s) \) of Eq. [11].

Similarly as in Eq. [15], we have

\[ B_{F(s)}(b; \mu) = N_m \frac{9}{2 \pi} \frac{\mu}{\overline{m}_q a(\mu)} \left[ \frac{1}{\left( 1 - 2b \right)^{1+\nu}} \right] + B_{F(s)}^{\text{an.}}(b; \mu) \]  

(43)

where the factor in front of the singular part was determined by the condition of renormalon cancellation of the sum \( 2m_q + E_{q\bar{q}}(s) \). We now define in analogy with Eq. [15]

\[ R_{F(s)}(b; \mu; \mu_f) = (1 - 2b)^{1+\nu} B_{F(s)}(b; \mu; \mu_f) \, . \]  

(44)

Here we denoted, explicitly the dependence on the factorization scale \( \mu_f \) and

\[ N_m = \frac{2 \pi \overline{m}_q a(\mu_f)}{9} R_{F(s)}(b; \mu; \mu_f) \big|_{b=1/2} \, . \]  

(45)

Theoretically, \( R_{F(s)}(b) \) should be a function with only a weak singularity (cut) at \( b = 1/2 \), and the nearest pole at \( b = 3/2 \) ( [13] ). Resummations such as Padé approximations (PA’s) are applied, then \( R_{F(s)}[2/1](b) \) has physically acceptable pole structure \( |b_{\text{pole}}| \geq 1 \) for most of the values of \( \mu \geq m_q \alpha_s \) and \( \kappa \sim 1 \). and the residue parameter \( N_m \) is reasonably stable under the variation of \( \mu \).

In Fig. 2(a) we show the dependence of \( N_m \) on \( \kappa \), at a typical \( \mu \) value \( \mu = 3 \, \text{GeV} \), for the \( b\bar{b} \) system. The know value of \( N_m \) is obtained by the \( R_{F(s)}[2/1](b=1/2) \) expression at \( \kappa \approx 0.59 \). In Fig. 2(b) we present, for \( \kappa = 0.59 \), the dependence of calculated \( N_m \) on the renormalization scale \( \mu \). There, we include also the \( (2/1) \)-resummed) curve for the case when no separation of the \( s \) and \( us \) parts of the energy is performed. In that case, the obtained values of \( N_m \) are unacceptable. The other values of the input parameters are chosen to have the \( b\bar{b} \) “central” values: \( a_3 / 4^3 = 86; \overline{m}_b = 4.23 \, \text{GeV}; \bar{\mu} = 1.825 \, \text{GeV} \) \( (\approx \mu_s) \) and \( \alpha_s(\bar{\mu}; n_f = 4) = 0.3263 \approx 0.3262 \) \( (\approx \alpha_s(\mu_s; n_f = 4) = 0.326) \) from: \( \alpha_s(M_Z) = 0.1192 \). For the RGE running, we use four-loop \( \overline{\text{MS}} \) \( \beta \)-function (TPS).

\[ \text{Variation } N_m = 0.555 \pm 0.020 \text{ [}\bar{b}\bar{b}\text{] implies } \kappa = 0.59 \pm 0.19. \, \text{If, on the other hand, } a_3 \text{ parameter is varied, then for } \bar{b}\bar{b} \kappa = 0.59 \pm 0.06. \, \text{Thus, the value of } s-us \text{ factorization scale parameter } \kappa \text{ is influenced largely by the allowed values of the renormalon residue parameter, and significantly less by the allowed values} \]
$a_3$ of the N$^3$LO coefficient of the static $q\bar{q}$ potential. Therefore, we will consider the variations of $N_m$ and of $\kappa$ to be related by a one-to-one relation, while the variations of $a_3$ will be considered as independent.

In this way, we have the following value for the s-us factorization scale parameter:

$$N_m = 0.555 \pm 0.020 \Rightarrow \kappa = 0.59 \pm 0.19 \quad (n_f = 4, \; S = 1)$$

(46)

and thus we obtain the N$^3$LO TPS \[11\] for the soft part of the ground binding energy. Now with the value of $\kappa$, we can perform the resummation of the soft part of the ground binding energy. The full bilocal method \[17\] \[12\] can be performed as in Sec. 2, Eqs. (22) and (23). Therefore

$$B_{F(s)}^{(\text{biloc.})}(b; \mu) = N_m \frac{9}{2\pi} \frac{\mu}{m_q a(\mu)} \frac{1}{(1-2b)^{1+\nu} + 1} \sum_{k=0}^{3} \tilde{c}_k (1-2b)^k + \sum_{k=0}^{3} \frac{h_k}{k! \beta_0^k} b^k ,$$

(47)

where the coefficients $h_k$ in the expansion of the analytic part are known up to order $k = 3$

$$h_k = f_k - N_m \frac{9}{2\pi} \frac{\mu}{m_q a(\mu)} (2b_0)^k \sum_{n=0}^{3} \tilde{c}_n \frac{\Gamma(\nu+k+1-n)}{\Gamma(\nu+1-n)} \quad (k = 0, 1, 2, 3).$$

(48)

The result has spurious $\mu$-dependence, and for the of the Borel transformation turns out to have a problematic behavior: the obtained pole of $B_{F(s)}^{(\text{an.})}[2/1](b)$ is unacceptably small in size: $|b_{\text{pole}}| \leq 1/2$. Theoretically, $B_{F(s)}^{(\text{an.})}(b)$ should have the nearest pole at $b = 3/2$ \[35\]. The reason for this problem appears to lie in the specific truncated form of the singular part taken in the bilocal method. While the latter part describes well the behavior of the transform near $b = 1/2$, it influences apparently strongly the coefficients $h_k$ and thus the analytic part, so that no reliable resummation of that part (apart from TPS) can be done. This problem can be alleviated by introducing a “form” factor which suppresses that part away from $b \approx 1/2$, but keeps it unchanged at $b \approx 1/2$. We choose a Gaussian type of function, and the “$\sigma$-regularized” bilocal expressions for the Borel transform is

$$B_{F(s)}^{(\sigma)}(b; \mu) = N_m \frac{9}{2\pi} \frac{\mu}{m_q a(\mu)} \frac{1}{(1-2b)^{1+\nu} + 1} \tilde{c}_1 (1-2b)$$

$$+ \tilde{c}_2 + \frac{1}{8\sigma^2} (1-2b)^2 + \left( \tilde{c}_3 + \frac{\tilde{c}_1}{8\sigma^2} (1-2b)^3 \right)$$

$$\times \exp \left[ -\frac{1}{8\sigma^2} (1-2b)^2 \right] + \sum_{k=0}^{3} \frac{1}{k! \beta_0^k} h_k^{(\sigma)} b^k .$$

(49)

The corrective terms $1/(8\sigma^2)$ and $\tilde{c}_1/(8\sigma^2)$ in the coefficients of Eq. (10) appear to ensure the correct known behavior of the renormalon part up to order $\sim (1-2b)^{-\nu+2}$. Numerical analysis indicate that $\sigma$ is between zero and one. Namely, when $\sigma$ decreases from $\sigma = \infty$ to about $\sigma \approx 0.3-0.4$, the value of the pole of the $[2/1]$ Padé-resummed analytic part $B_{F(s)}^{(\text{an.})}(b)$ of Eq. (10) gradually turns acceptable ($|b_{\text{pole}}| > 1$) and rather stable when the renormalization scale $\mu$ varies in the interval $[m_q \alpha_s, m_q]$ (except close to $\mu \approx m_q \alpha_s$). When the value of $\sigma$ falls below 0.3, the analytic part starts showing erratic behavior again and the Borel resummation gives significantly differing results with the TPS- and the Padé-evaluated analytic parts. On these grounds, the obtained optimal $\sigma$ turn out to be

$$\sigma = 0.36 \pm 0.03 \quad (n_f = 4, \; S = 1).$$

(50)
In Fig. 4(a) we present the Borel-resummed soft part of ground state energy for the bottomium \((S = 1)\), as a function of the \(\sigma\) parameter of method \([19]\).

Finally, the results for the soft binding energy \(E^{(s)}_{bb}(s)\) of the ground state of bottomium using the \(R\)-method \([18, 19]\), where we resum the function \(R_{(F)}(s; \mu)\) \([18]\) and then employ the (PV) Borel resummation as written in Eq. \(22\) (with \(R_{(F)}\) instead of \(R_{P}\) there), as functions of the renormalization scale \(\mu\), are presented in Fig. 4(a). We observe from the Figure that the bilocal “\(\sigma\)-regularized” method \([18, 19]\) \((\sigma = 0.36)\) gives the TPS and PA results closer to each other. The methods \(\sigma\)-TPS, \(\sigma\)-PA, and \(R\)-PA give similar results in the entire presented \(\mu\)-interval. \(R\)-TPS appears to fail at low \(\mu\) \((\approx 1-2 \text{ GeV})\). In Fig. 4(b) we include, for comparison, the simple TPS evaluation of \(E^{(s)}_{bb}(s)\), according to formula [cf. Eq. (41)]

\[
F(s)^{\text{(TPS)}} = -\frac{9}{4\pi} \frac{1}{m_b\alpha_s(\mu)} E_{\bar{q}q}(s) = a(\mu) \left[1 + a(\mu)f_1 + a^2(\mu)f_2 + a^3(\mu)f_3\right],
\]

where for N\(^2\)LO TPS case we take \(f_3 = 0\). In Fig. 4(b) the same input parameters are used as in Fig. 4(a).

We see that the perturbation series shows strongly divergent behavior already at N\(^3\)LO. In this Figure, we also included the “perturbative” ultrasoft part \(E^{(p)}_{bb}(us; \mu)\) calculated according to \([\text{see Eqs. (38) and (40)}]\).

\[
F^{(p)}(us) \equiv -\frac{9}{4\pi} \frac{1}{m_b\alpha_s(\mu)} E^{(p)}_{\bar{q}q}(us; \mu) = k_{3,0}a^4(\mu).
\]

### 4.1 Extraction of bottom mass

The estimate of the perturbative part is given in Eq. \(42\), where it was essential to take for the renormalization scale a \(us\) scale \(\mu \sim \mu_{us} \sim m_q\alpha_s^2\).

For the bottomonium case, this scale is below 1 GeV, the energy at which we cannot determine perturbatively \(\alpha_s(\mu)\). This indicates that in the bottomonium the \(us\) part of the binding energy has an appreciable nonperturbative part. The lowest energy at which we can still determine perturbatively \(\alpha_s\) is \(\mu \approx 1.5-2.0\) GeV, giving \(\alpha_s(\mu) \approx 0.30 - 0.35\). Although this is a soft scale for \(bb\), we will use this also as an ultrasoft scale. Then by Eq. \(42\)

\[
E^{(p)}_{bb}(us) \approx -\frac{4}{9\pi} k_{3,0}^2 a^5(\mu_{us}) \approx (-150 \pm 100) \text{ MeV}.
\]

The nonperturbative contribution coming from the gluonic condensate is given by \([39]\)

\[
E^{(np)}_{bb}(us) \approx \frac{624}{425} \left(\frac{4}{3}\right)^4 \frac{1}{m_b^2 \alpha_s(\mu_{us})} = 0.009 \pm 0.007 \text{ GeV}^4\]

where we used \(\overline{m}_b = 4.2\) GeV, and the value of the gluon condensate \(\langle G_{\mu\nu} G^{\mu\nu}\rangle = 0.009\) GeV\(^4\) \([39]\). Eqs. \(43\) and \(44\) give

\[
E^{(p+np)}_{bb} \approx (-100 \pm 106) \text{ MeV},
\]

The finite charm mass contributions has been calculated in Ref. \([11]\) (Refs. \([6, 42, 43]\)). The contribution to the mass \(M_T(1S) = (2m_b + E_{bb})\) is

\[
\delta M_T(1S, m_c \neq 0) \approx 25 \pm 10 \text{ MeV}.
\]

The estimates \([50, 51]\) then give a rough estimate of the \(us\) and \(m_c \neq 0\) contributions to the bottomonium mass \(\delta M_T(1S; us + m_c) \approx (-75 \pm 106) \text{ MeV}\). The mass of the \(\Upsilon(1S)\) vector bottomonium ground state is well
measured $M_T(1S) = 9460$ MeV with virtually no uncertainty \[11\]. Therefore, the pure perturbative “soft” mass is

$$M_T(1S; s) = 2m_b + E_{\overline{b\bar{s}}}(s) = 9535 \mp 106 \text{ MeV},$$

where the uncertainty is dominated by the uncertainty of the $us$ regime contribution. Our numerical results for $E_{\overline{b\bar{s}}}(s)$ and for $m_b$ allow us, by varying the input value of $\overline{m}_b$, to adjust the sum $2m_b + E_{\overline{b\bar{s}}}(s)$ to the value given in Eq. (57). For the soft binding energy we apply the “perturbative” ultrasoft energy part at the corresponding low renormalization energy (1.5–2 GeV). Eqs. (59) and (65). If we had not separated the (“perturbative”) ultrasoft from the soft part of the binding energy, the use of the common renormalization energy scale $\mu \approx 3$ GeV in the resummation then would have given us the central value of $E_{\overline{b\bar{s}}}(us)$ by about +100 MeV higher. Then the extracted value of $\overline{m}_b$ would have gone down by about 46 MeV, giving the value $\overline{m}_b \approx 4.195 \pm 0.068$, with the central value close to that of L03 in Table 2. On the other hand, that latter value is quite clearly lower than the value PS02 in Table 2 by about 150 MeV, principally because of the $b = 1/2$ renormalon effect which were taken into account here and in Ref. [12]. Thus, the renormalon effect brings down the extracted central value of $\overline{m}_b$ by about 150 MeV, but the separate evaluation of the ultrasoft contribution brings it up by about 50 MeV.

5 Comparisons and conclusions

Our result for the mass $\overline{m}_b = 4.231 \pm 0.068$ GeV, is compared in Table 2 with recently values of $\overline{m}_b$. The only input parameter common to all these methods is $\alpha_s$. Our central value was $\alpha_s(m_T) = 0.3254 \Rightarrow \alpha_s(M_Z) = 0.1192$ since such $\alpha_s(M_Z)$, or similar values follow from the (nonstrange) semihadronic $\tau$ decay data which are very precise \[16\]. On the other hand, the world average as of September 2002 is $\alpha_s(M_Z) = 0.1183 \pm 0.0027$ \[17\]. Most of the authors during the last four years used central value $\alpha_s(M_Z) \approx 0.118$. Therefore, for comparisons, we convert our results \[18\] to this central value of $\alpha_s$ – more specifically, from $\alpha_s(M_Z) = 0.1192 \pm 0.0015$ to $0.1180 \pm 0.0015$. This can be easily done by inspecting in Table 3 the column under $\alpha_s$, an increase in the central values of $\overline{m}_b$ by 11, 12, 8 and 10 MeV, respectively. This gives the average 10 MeV higher than in Eq. (62)

$$\overline{m}_b = 4.241 \pm 0.068 \text{ GeV} \quad \text{average when: } \alpha_s(M_Z) = 0.1180 \pm 0.0015.$$  

There are two important numerical effects in our result. The first is the separate evaluation of the “perturbative” ultrasoft energy part at the corresponding low renormalization energy (1.5–2 GeV). Eqs. (42) and (53). If we had not separated the (“perturbative”) ultrasoft from the soft part of the binding energy, the use of the common renormalization energy scale $\mu \approx 3$ GeV in the resummation then would have given us the central value of $E_{\overline{b\bar{s}}}(us)$ by about +100 MeV higher. Then the extracted value of $\overline{m}_b$ would have gone down by about 46 MeV, giving the value $\overline{m}_b \approx 4.195 \pm 0.068$, with the central value close to that of L03 in Table 2. On the other hand, that latter value is quite clearly lower than the value PS02 in Table 2 by about 150 MeV, principally because of the $b = 1/2$ renormalon effect which were taken into account here and in Ref. [12]. Thus, the renormalon effect brings down the extracted central value of $\overline{m}_b$ by about 150 MeV, but the separate evaluation of the ultrasoft contribution brings it up by about 50 MeV.
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A Coefficients for the expansion of the soft binding energy

We write down here the explicit coefficients $f_j$ of the expansion [11] for the soft part of the ground state binding energy. The logarithms appearing in these expressions involve three scales $[\mu, \bar{\mu}, \mu_q]$ and $\bar{\mu} = (4/3)\mu_q\alpha_s(\bar{\mu})$

$$L_1 = \ln \left( \frac{\mu_q}{\bar{\mu}} \right), \quad L_2 = \ln \left( \frac{\mu_q}{\mu} \right), \quad L_\mu = \ln \left( \frac{\mu_q}{\mu} \right).$$

The coefficients $f_j$ are

$$f_1 = \frac{1}{2} (35 + 22L_1 - 11L_\mu - 11L_2) + \frac{1}{9} (-11 - 6L_1 + 3L_\mu + 3L_2) n_f. \quad (65)$$

$$f_2 = f_2^{(0)} + f_2^{(1)} n_f + f_2^{(2)} n_f^2,$$  \hspace{2cm} (66)

$$f_2^{(0)} = \left[ 381.67 + 90.75 L_1^2 + 30.25 L_\mu^2 + L_1 (246.42 - 121 L_\mu - 60.5 L_2) - 48.5 L_2 
+ L_\mu (-205.25 + 60.5 L_2) - 11.697 S(S+1) \right],$$

$$f_2^{(1)} = \left[ -42.7469 - 11 L_1^2 - 3.6667 L_\mu^2 + L_\mu (26.6944 - 7.3333 L_2) + 6.8056 L_2 
+ L_1 (-33.0556 + 14.6667 L_\mu + 7.3333 L_2) \right],$$

$$f_2^{(2)} = \left[ 1.16286 + (3/9) L_1^2 + (1/9) L_\mu^2 + L_\mu (1 - (4/9) L_\mu - (2/9) L_2) 
+ L_\mu (-0.81482 + (2/9) L_2 - 0.18518 L_2) \right].$$

$$f_3 = f_3^{(0)} + f_3^{(1)} n_f + f_3^{(2)} n_f^2 + f_3^{(3)} n_f^3,$$  \hspace{2cm} (67)

$$f_3^{(0)} = \left[ 6726.11 + 665.5 L_1^3 - 166.375 L_\mu (40.802 + (-10.599 + L_\mu)L_\mu) 
+ L_1^2 (2381.5 - 1497.38 L_\mu - 499.125 L_2) - 871.429 L_2 
- 499.125 (-1.884 + L_\mu)L_\mu L_2 - 201.438 L_2^2 + L_1 (7457.17 
- 497.292 L_2 + L_\mu (-3436.38 + 998.25 L_\mu + 998.25 L_2)) 
- 257.341 (0.2112 + L_1 - 0.75 L_\mu - 0.25 L_2) S(S+1) 
- 61.4109 (-6.1394 + S(S+1)) \ln (\alpha_s(\mu_s)) + 440.172 \ln (\kappa) + 2 \alpha_s / 4 \right],$$

$$f_3^{(1)} = \left[ -1274.33 - 1277.92 L_1 - 471.125 L_1^2 - 121 L_1^3 + 1182.32 L_\mu 
+ 843.667 L_1 L_\mu + 272.25 L_2^2 L_\mu - 335.813 L_\mu^2 - 181.5 L_1 L_\mu^2 
+ 30.25 L_\mu^3 + 124.501 L_2 + 108.361 L_1 L_2 + 90.75 L_1^2 L_2 
- 186.708 L_\mu L_2 - 181.5 L_1 L_\mu L_2 + 90.75 L_\mu^2 L_2 + 36.729 L_2^2 \right].$$
\begin{align*}
\mathcal{f}_3^{(2)} &= \left[ 70.8992 + 70.2453L_1 + 28.9722L_2^2 + 7.3333L_3 - 65.9925L_\mu \\
&\quad - 51.6667L_1L_\mu - 16.5L_3L_\mu + 20.5972L_\mu^2 + 11L_1L_\mu^2 - 1.8333L_\mu^3 \\
&\quad - 5.1939L_2 - 6.5741L_1L_2 - 5.5L_2^2L_\mu + 10.9167L_\mu L_2 \\
&\quad + 11L_1L_\mu L_2 - 5.5L_\mu^2L_2 - 2.0972L_3 \right], \\
\mathcal{f}_3^{(3)} &= \left[ -1.21475 - 1.21714L_1 - (5/9)L_1^2 - 0.14815L_1^3 + 1.16286L_\mu \\
&\quad + L_1L_\mu + (1/3)L_1^2L_\mu - 0.40741L_\mu^2L_\mu - (2/9)L_1 L_\mu^2 + 0.3704L_3^3 \\
&\quad + 0.05429L_2 + (1/9)L_1L_2 + (1/9)L_1^2L_2 \\
&\quad - 0.18518L_\mu L_2 - (2/9)L_1L_\mu L_2 + (1/9)L_\mu^2L_2 + 0.03704L_3^2 \right].
\end{align*}

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Figure 1: The (PV) pole mass of the bottom quark, as function of the renormalization scale $\mu$. The input parameters used were $m_b = 4.23$ GeV and the residue parameter values Eq. (19). The reference value for $\alpha_s$ (in $\overline{\text{MS}}$) was taken to be $\alpha_s(m_\tau) = 0.3254$ (32) corresponding to $\alpha_s(M_Z) = 0.1192$. 

$m_b = 4.23$ GeV  
$\alpha_s(M_Z) = 0.1192$
Figure 2: The residue parameter value $N_m$ as calculated from the soft part of the binding energy of the bottomonium according to Eq. (45), (a) as a function of the $s$-$u_s$ factorization scale parameter $\kappa$ (37), at $\mu = 3$ GeV; (b) as a function of the renormalization scale $\mu$, at $\kappa = 0.59$. Further explanations given in the text. In Fig. (a), the known values of $N_m$ are denoted as dotted horizontal lines.

Figure 3: (a) Soft part of the ground state binding energy of $b\bar{b}$, evaluated with the (PV) Borel-resummed expression (49), as a function of the method parameter $\sigma$. Details are given in the text.
Figure 4: (a) Soft part $E_{bb}(s)$ of the ground state binding energy of $b\bar{b}$, evaluated with four different methods involving (PV) Borel resummation, as functions of the renormalization scale $\mu$. Details are given in the text. In Fig. (b) the simple TPS results for $E_{bb}(s)$ are included [Eq. (51)], as well as the “perturbative” ultrasoft part $E_{bb}^{(p)}(us;\mu)$ [Eq. (52)].

Table 1: The separate uncertainties $\delta m_b$ (in MeV) for the extracted value of $m_b$ from various sources: 1.) $us$ $[\delta E_{bb}(us)^{p+np} = -100 \pm 106$ MeV]; 2.) $\mu = 3 \pm 1$ GeV; 3.) $\mu_m = m_b(1 \pm 0.5)$; 4.) $\alpha_s(m_T) = 0.3254 \pm 0.0125$ [$\alpha_s(M_Z) = 0.1192 \pm 0.0015$]; 5.) $N_m = 0.555 \pm 0.020$ [$\kappa = 0.59 \pm 0.19$]; 6.) $a_3/4^3 = 86.23 \pm 3$; 7.) $c_4 = 40.60 \pm 0.03$; 8. $\sigma = 0.36 \pm 0.03; 9. m_c \neq 0 (\delta M_T(m_c \neq 0) = \pm 10$ MeV).

|     | $us$ | $\mu$ | $\mu_m$ | $\alpha_s$ | $N_m$ | $a_3$ | $c_4$ | $\sigma$ | $m_c$ |
|-----|------|-------|---------|------------|-------|-------|-------|----------|-------|
| $\sigma$-TPS | -49  | +9    | -4      | -13        | -3    | +2    | -8    | +4       | -5    |
|     | +49  | -13   | +2      | +14        | +2    | -2    | -8    | -9       | +5    |
| $\sigma$-PA  | -49  | +13   | -4      | -15        | -3    | +1    | -5    | +5       | -5    |
|     | +49  | -20   | +2      | +15        | +2    | -1    | +2    | +9       | +5    |
| R-TPS      | -50  | -4    | +4      | -8         | -9    | -3    | 0     | 0        | -5    |
|     | +50  | +45   | -40     | +10        | +11   | +3    | 0     | 0        | +5    |
| R-PA       | -49  | +3    | +4      | -11        | -4    | -2    | 0     | 0        | -5    |
|     | +49  | -20   | -40     | +12        | +4    | +2    | 0     | 0        | +5    |
Table 2: Recently obtained values of (MS) $m_b$ mass obtained from $\Upsilon$ sum rules or from spectrum of the $\Upsilon(1S)$ resonance. Wherever needed (33 [12]), the central mass values were adjusted to the common input central value $\alpha_s(M_Z) = 0.118$.

| reference | method            | order | $m_b(m_b)$ (GeV) |
|-----------|------------------|-------|-----------------|
| PP98 [37] | $\Upsilon$ sum rules | NNLO  | 4.21 ± 0.11     |
| MY98 [36] | $\Upsilon$ sum rules | NNLO  | 4.20 ± 0.10     |
| BS99 [48] | $\Upsilon$ sum rules | NNLO  | 4.25 ± 0.08     |
| H00 [48]  | $\Upsilon$ sum rules | NNLO  | 4.17 ± 0.05     |
| KS01 [49] | $\Upsilon$ sum rules | NNLO  | 4.209 ± 0.050   |
| CH02 [50] | $\Upsilon$ sum rules | NNLO  | 4.20 ± 0.09     |
| E02 [51]  | $\Upsilon$ sum rules | NNLO  | 4.24 ± 0.10     |
| P01 [11]  | spectrum, $\Upsilon(1S)$ | NNLO  | 4.210 ± 0.090 ± 0.025 |
| BSV01 [41] | spectrum, $\Upsilon(1S)$ | NNLO  | 4.190 ± 0.020 ± 0.025 |
| PS02 [33] | spectrum, $\Upsilon(1S)$ | N^3LO | 4.349 ± 0.070   |
| L03 [12]  | spectrum, $\Upsilon(1S)$ | N^3LO | 4.19 ± 0.04     |
| this work, Eq. (63) | spectrum, $\Upsilon(1S)$ | N^3LO | 4.241 ± 0.070   |