Statistical Mechanical Approach to Error Exponents of Lossy Data Compression

Tadaaki Hosaka * and Yoshiyuki Kabashima †

Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, Nagatsuta-cho, Midori-ku, Yokohama 226-8502

We present herein a scheme by which to accurately evaluate the error exponents of a lossy data compression problem, which characterize average probabilities over a code ensemble of compression failure and success above or below a critical compression rate, respectively, utilizing the replica method (RM). Although the existing method used in information theory (IT) is, in practice, limited to ensembles of randomly constructed codes, the proposed RM-based approach can be applied to a wider class of ensembles. This approach reproduces the optimal expressions of the error exponents achieved by the random code ensembles, which are known in IT. In addition, the proposed framework is used to show that codes composed of non-monotonic perceptrons of a specific type can provide the optimal exponents in most cases, which is supported by numerical experiments.

KEYWORDS: error exponent, lossy data compression, replica method, random code ensemble, perceptron

1. Introduction

Recent research activities in the cross-disciplinary field that combines information theory (IT) and statistical mechanics (SM) have shown that the typical performance of various codes, such as error correction and compression codes, can be characterized as phase transitions between several phases representing the success or failure of coding when the length of messages \( M \) becomes infinite.\(^1\) However, for finite \( M \), probabilities of coding failure in the success phase and coding success in the failure phase do not vanish, and therefore, it is interesting to estimate the probabilities that those events occur.

For a reasonable code ensemble, the averages of those probabilities over the ensemble asymptotically scale with respect to \( M \) as \( \exp[-M\alpha] \). Here, \( \alpha (> 0) \) which characterizes the asymptotic behavior, is often termed the error exponent. The evaluation of \( \alpha \) is theoretically interesting and is of practical importance in the sense that the error exponents can be useful as one criterion in the case of assessing the coding performance for finite \( M \).

More recently, it has been shown that the replica method (RM) developed in SM can be used for accurate assessment of such an exponent for error correcting codes.\(^2,3\) Never-
theless, the proposed method relies on specific properties of error correcting codes, and the development of such techniques for other codes requires further investigation. Therefore, we herein provide a scheme by which to accurately evaluate the error exponents for lossy data compression problems of memoryless sources utilizing RM. The existing method used in IT has provided the optimal expressions of the error exponents. However, a precise assessment by the IT approach is, in practice, possible only for the ensembles of randomly constructed codes that exhibit optimal performance. In contrast, our SM-based approach can accurately evaluate the coding performance for a wider class of ensembles.

The paper is organized as follows. In the next section, we briefly review the concept of lossy data compression and the definition of the error exponents. In §3, a statistical mechanical approach for the assessment of the error exponents is introduced. In §4, this approach is applied to the random code ensemble (RCE). Although the exponents evaluated here characterize the asymptotic behavior of the average probabilities over the ensemble, this analysis successfully reproduces the known optimal exponents in IT literature by selecting the best code in that ensemble. We briefly discuss RM-based evaluation of the exponents of the best code, which reproduces a result that is identical to the analysis for the average case. In addition to being consistent with the existing IT results, a major advantage of the proposed method is the ability to accurately assess the exponents for suboptimal ensembles. This is demonstrated for a simple lossy compression problem of a binary memoryless source in §5. For this source, the error exponents are evaluated for a suboptimal code ensemble, composed of perceptrons, of practical codebook size using the developed RM-based approach. The validity of the assessment is also confirmed numerically. The final section is devoted to a summary.

2. Lossy Data Compression and Error Exponents

In this section, we present the notation used herein and briefly review the concept of lossy data compression of memoryless sources. Let us focus on a discrete message consisting of $M$ random variables $y = (y^1, y^2, \ldots, y^M)$ ($y^\mu \in Y = \{0, 1, \ldots, J - 1\}$), each component of which is assumed to be independently generated from an identical stationary distribution $P = (P(0), P(1), \ldots, P(J - 1))$. Although the arguments below are for sources of discrete messages, the newly developed scheme can be directly extended to the case of continuous memoryless sources, in which the error exponents are expressed identically by replacing summations and distribution functions with integrals and density functions, respectively.

The purpose of lossy data compression is to compress $y$ into a binary expression $s = (s_1, s_2, \ldots, s_N)$ ($s_i \in \{0, 1\}$), allowing a certain amount of distortion between the original message $y$ and the representative vector $\tilde{y} = (\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^M)$ ($\tilde{y}^\mu \in \tilde{Y} = \{0, 1, \ldots, L - 1\}$) when $\tilde{y}$ is retrieved from $s$. We deal herein with the distortion of single-letter fidelity criterion $d$ on $Y \times \tilde{Y}$, which is defined as $d(j, l) \geq 0$ ($j \in Y, l \in \tilde{Y}$) and $\min_{l \in \tilde{Y}}\{d(j, l)\} = 0$ ($\forall j \in Y$). For example, the distortions for Boolean messages $Y = \tilde{Y} = \{0, 1\}$ are frequently measured...
using the Hamming distance, $d(y, \tilde{y}) = \sum_{\mu=1}^{M} [1 - \delta_{y\mu, \tilde{y}\mu}] \geq 0$, where $\delta_{x,y}$ is 1 if $x = y$, and 0 otherwise.

A code $C$ is specified by a map $\tilde{y}(s; C) : s \rightarrow \tilde{y}$, which is used in the restoration phase. This reasonably determines the compression scheme as

$$s(y; C) = \arg\min_{s} \{d(y, \tilde{y}(s; C))\},$$

where $\arg\min_{s} \{\cdots\}$ represents the argument $s$ that minimizes $\cdots$. When $C$ is generated from a certain code ensemble, typical codes satisfy the fidelity criterion

$$\frac{1}{M} \min_{s} \{d(y, \tilde{y}(s; C))\} = \frac{1}{M} \min_{s} \left\{ \sum_{\mu=1}^{M} d(y^{\mu}, \tilde{y}^{\mu}(s; C)) \right\} < D,$$  

for a given permissible distortion $D$ and typical messages $y$ with probability 1 in the limit $M, N \rightarrow \infty$ keeping the coding rate $R \equiv N/M$ constant, if and only if $R$ is larger than a certain critical rate $R_c(D)$.

However, for finite $M$ and $N$, any code has a finite probability $P_F$ of breaking the fidelity (2) even for $R > R_c(D)$. Similarly, for $R < R_c(D)$, eq.(2) is satisfied with a certain probability $P_S$. For reasonable code ensembles, the averages of these probabilities are expected to decay exponentially with respect to $M$ when the message length $M$ is sufficiently large. Therefore, the two error exponents $\alpha_A(D, R) = \lim_{M \rightarrow \infty} -\frac{1}{M} \ln \langle P_F \rangle_C$ for $R > R_c(D)$ and $\alpha_B(D, R) = \lim_{M \rightarrow \infty} -\frac{1}{M} \ln \langle P_S \rangle_C$ for $R < R_c(D)$, where $\langle \cdots \rangle_C$ represents the average over the code ensemble, can be used to characterize the potential ability of the ensembles of finite message lengths. The development of a framework for evaluating these exponents utilizing RM is the primary goal of this paper.

3. Statistical Mechanical Approach to Error Exponents

3.1 Free energy as a lower-bound of distortion

Let us develop an analytical framework to assess the error exponents using RM. For this, we first regard the distortion function $d(y, \tilde{y}(s; C))$ as the Hamiltonian for the dynamical variable $s$, which also depends on predetermined variables $y$ and $C$. In the compression process, the optimal sequence is chosen as eq. (1). As the original message and the code are generated from a stationary distribution $P$ and the code ensemble, respectively, the resulting distortion (per bit) $\lambda(y, C) = \min_{s} \{M^{-1}d(y, \tilde{y}(s; C))\}$ is also expected to obey a certain distribution $P(\lambda, R)$.

In the thermodynamic limit, $P(\lambda, R)$ is expected to peak at the typical value $\lambda = D_t(R)$ and decay exponentially away from $D_t(R)$ as $P(\lambda, R) \sim \exp[-Mh(\lambda, R)]$. This indicates that $\langle P_F \rangle_C = \int_{D_t(R)}^{\infty} P(\lambda, R) d\lambda \sim P(D, R) \sim \exp[-Mh(D, R)]$ for $D > D_t(R)$ (or $R > R_c(D)$) and $\langle P_S \rangle_C = \int_{0}^{D_t(R)} P(\lambda, R) d\lambda \sim P(D, R) \sim \exp[-Mh(D, R)]$ for $D < D_t(R)$ (or $R < R_c(D)$). Therefore, we can express the error exponents $\alpha(D, R)$ using $h(D, R)$ for both cases (Fig. 1).
Fig. 1. Schematic profile of the distribution $P(\lambda, R)$. $D_t(R)$ indicates the typical value of the distortion $M^{-1} \min_s \{d(y, \tilde{y}(s;C))\}$. As $M, N \to \infty$ keeping $R = N/M$ fixed, the probability of compression failure $(P_F)_C$ for a given $D(> D_t(R))$, which is represented as the shadow area (a), tends toward zero. Similarly, the probability of compression success $(P_S)_C$ for a given $D(< D_t(R))$ is illustrated in figure (b). Here, the error exponents are defined for characterization of the decay rates of these average probabilities.

In order to assess the distribution $P(\lambda, R)$, we next utilize the inequality
\[ e^{-M\beta \lambda(y,C)} \leq \sum_s e^{-\beta d(y, \tilde{y}(s;C))} = Z(\beta; y, C) = e^{-M\beta f(\beta; y, C)}, \] (3)
which holds for any sets of $\beta > 0$, $y$ and $C$. The physical implication of this is that the ground state energy $\lambda(y, C)$ (per component) is lower bounded by the free energy $f(\beta; y, C)$ (per component) for an arbitrary temperature $\beta^{-1} > 0$. In particular, $f(\beta; y, C)$ agrees with $\lambda(y, C)$ in the zero temperature limit $\beta \to \infty$. This means that we can evaluate $P(\lambda, R)$ by first assessing the distribution of $f(\beta; y, C)$, $P(f; \beta)$, for general finite $\beta > 0$, and then taking the limit $\beta \to \infty$ afterward. Note that although most of the quantities appearing in this paper actually depend on the coding rate $R$, the dependency is not specified for some quantities such as $P(f; \beta), c(f, \beta)$, and $g(n, \beta)$.

3.2 Assessment of the error exponents from the moment of the partition function

$P(f; \beta)$ is also expected to peak at its typical value
\[ f_t(\beta) = \frac{1}{M\beta} \langle \ln Z(\beta; y, C) \rangle_{y, C} = -\lim_{n \to 0} \frac{1}{M\beta} \frac{\partial}{\partial n} \ln \langle Z^n(\beta; y, C) \rangle_{y, C}, \] (4)
and decay exponentially away from $f_t(\beta)$ as $P(f; \beta) \sim \exp[-Mc(f, \beta)]$ for large $M$. Here, we assume that $c(f; \beta) \geq 0$ is a convex downward function minimized to 0 at $f = f_t$. This implies that, for $\forall n \in \mathbb{R}$, the moment of the partition function $Z(\beta; y, C)$, $\langle Z^n(\beta; y, C) \rangle_{y, C}$, can be evaluated by the saddle point method as
\[ \langle Z^n(\beta; y, C) \rangle_{y, C} \approx \exp[-M\{n\beta f^* + c(f^*, \beta)\}], \] (5)
where \( \langle \cdots \rangle_{y,c} \) denotes the average over \( y \) and \( C \), and \( f^* \) represents the value at the saddle point, which leads to the Legendre transformation
\[
g(n, \beta) \equiv -\frac{\ln \langle Z^n(\beta; y, C) \rangle_{y,c}}{M} = \min_f \{ n \beta f + c(f, \beta) \}. \quad (6)
\]
Fig. 2 illustrates graphically the meaning of this evaluation. Given \( n \), the minimization in eq.(6) provides a condition for determining the dominant \( f \) as
\[
n = -\frac{1}{\beta} \frac{\partial c(f, \beta)}{\partial f}. \quad (7)
\]
For each \( \beta \), this can be solved pictorially by searching for the point on \( f \) at which the tangential slope of a function \( y = -\beta^{-1}c(f, \beta) \) agrees with \( n \). Since \( \beta \) is positive, \( y = -\beta^{-1}c(f, \beta) \) is a convex upward function. This indicates that \( n = 0 \), \( n > 0 \) and \( n < 0 \) correspond to the typical values \( f = f_t \), \( f < f_t \) and \( f > f_t \), respectively, which provides a useful clue for assessing the exponents.

Based on eq.(6), the exponent \( c(f, \beta) \) that characterizes the distribution of free energy \( P(f; \beta) \) can be assessed by the inverse Legendre transformation
\[
c(f, \beta) = \max_n \{-n \beta f + g(n, \beta)\}, \quad (8)
\]
where \( \max_x \{ \cdots \} \) denotes the maximization of \( \cdots \) with respect to \( x \), from \( g(n, \beta) \), which can be evaluated by using RM analytically extending expressions obtained for \( n \in \mathbb{N} \) to \( n \in \mathbb{R} \) if \( f \) is included in the support of \( P(f; \beta) \), which we assume below. This enables the evaluation of the error exponent \( \alpha(D, R) \), where \( D \) is assumed to be included in the support of \( P(\lambda, R) \) throughout this paper, as \( \alpha(D = f, R) = h(\lambda = f, R) = c(f, \beta \to \infty) \) taking the zero-temperature limit \( \beta \to \infty \). The extremum with respect to \( n \) in eq.(8) is characterized by the condition
\[
\frac{1}{\beta} \frac{\partial g(n, \beta)}{\partial n} = f, \quad (9)
\]
for a given \( f \), indicating that the exponent \( \alpha_{\{A,B\}}(D, R) \), which is an abbreviation denoting \( \alpha_A(D, R) \) and \( \alpha_B(D, R) \) for \( R > R_c(D) \) and \( R < R_c(D) \), respectively, can be assessed as
\[
\alpha_{\{A,B\}}(D, R) = \lim_{\beta \to \infty} c(f = D, \beta) = \lim_{\beta \to \infty} \left\{ -n \frac{\partial g(n, \beta)}{\partial n} + g(n, \beta) \right\}, \quad (10)
\]
where \( n \) in eq.(10) is a function of \( \beta \) that is determined by the condition
\[
\frac{1}{\beta} \frac{\partial g(n, \beta)}{\partial n} = D. \quad (11)
\]
Equations (10) and (11) constitute the basis of our approach.

It is necessary to mention two points here. First, \( \alpha_A(D, R) \) is evaluated for \( R > R_c(D) \), or \( D > D_t(R) \) for fixed \( R \), where the typical distortion \( D_t(R) \) can be evaluated as \( D_t(R) = \lim_{\beta \to \infty} f_t(\beta) \). Since Fig. 2 indicates that \( f > f_t \) corresponds to \( n < 0 \), \( n \) determined from
Fig. 2. Graphical scheme used to solve eq.(7), enabling the Legendre transformation of eq. (6) to be performed.

eq(11) becomes negative in the assessment of \( \alpha_A(D, R) \). Similarly, \( n > 0 \) is obtained for \( \alpha_B(D, R) \). Second, we assume that \( c(f, \beta) \) is a convex downward function of \( f \) for \( \forall \beta \), which may not hold in certain situations. In such cases, evaluation based on eqs. (10) and (11) provides the lower bounds of the error exponents due to the nature of the Legendre transformation.

4. Application to the Random Code Ensemble

4.1 The random code ensemble

In order to show that the assessment of the error exponents based on eqs.(10) and (11) is consistent with the existing results, we first apply this method to the random code ensemble (RCE), which has been reported extensively in IT literature.\(^6,7\)

The RCE is an ensemble that is characterized by the component-wise random construction of a map \( \tilde{y}(s; C) \) from \( s \) to representative sequences \( \tilde{y} \) following an identical distribution \( Q = (Q(0), Q(1), \ldots, Q(L-1)) \), as

\[
\text{Prob}\{\tilde{y}^\mu(s;C) = l\} = Q(l),
\]

where \( Q(l) \geq 0 \) \( (l = 0, 1, \ldots, L - 1) \) and \( \sum_{l \in \tilde{Y}} Q(l) = 1 \). The correspondence between \( s \) and \( \tilde{y}(s) \), termed a codebook, is known to both the compressor and the decompressor. The size of the codebook of the RCE grows as \( O(M \times 2^N) \), which makes compressing a given message computationally difficult when the message lengths \( N \) and \( M \) are large because, other than looking up the codebook, no compression method exists. This prevents the RCE from being practical. However, this ensemble exhibits optimal compression performance when appropriately tuned, and so analysis of the RCE is important for clarifying the theoretical limitations of the framework of lossy data compression.
4.2 The replica method: two replica symmetric solutions

Let us evaluate \( g(n, \beta) \) for RCE utilizing RM in order to assess eqs.(10) and (11). For this, we insert an identity \( 1 = \sum_{q_{ab}=0,1} \prod_{a>b} \delta(s_a, s_b - q_{ab}) \) \( (a, b = 1, 2, \ldots, n) \) into \( Z^n(\beta; y, C) \) for \( n \in \mathbb{N} \) and take the averages over \( y \) and \( C \), which yields

\[
g(n, \beta) = \text{extr}_{\{q_{ab} \in \{0,1\}\}} \left\{ -\frac{1}{M} \ln \left[ \frac{1}{M} \frac{\text{Tr} \left\{ e^{-\beta d(y, \tilde{y}(s^a))} \right\}}{y(\tilde{y}(s^1)\tilde{y}(s^2)\ldots\tilde{y}(s^n))} \times \prod_{a>b} \delta(s_a, s_b - q_{ab}) \right] \right\},
\]

where the summation \( \sum_{q_{ab}=0,1} \) is replaced with the extremization \( \text{extr}_{\{q_{ab} \in \{0,1\}\}} \), which is valid for \( M \to \infty \), and \( \langle \cdot \cdot \cdot \rangle_{y(\tilde{y}(s^1)\tilde{y}(s^2)\ldots\tilde{y}(s^n))} \) denotes the averages over the distributions \( P \) and \( \{Q(\tilde{y}(s^a))\} \) \( (a = 1, 2, \ldots, n) \).

In order to utilize this expression for real (and, more generally, complex) \( n \), we first employ the simplest replica symmetric (RS) ansatz \( q_{ab} = q \) \( (a > b = 1, 2, \ldots, n) \). The value of \( q \) is limited to only 0 or 1 in the current system, yielding two RS solutions:

\[
g_{RS1}(n, \beta) = -\ln \left[ \sum_{j \in Y} P(j) \left\{ \sum_{l \in Y} Q(l) e^{-\beta d(j,l)} \right\}^n \right] - nR \ln 2,
\]

and

\[
g_{RS2}(n, \beta) = -\ln \left[ \sum_{j \in Y} P(j) \sum_{l \in Y} Q(l) e^{-n \beta d(j,l)} \right] - R \ln 2,
\]

which correspond to \( q = 0 \) and 1, respectively.

4.3 Critical conditions and the frozen replica symmetry breaking solution

We now have two RS solutions: (14) and (15). These solutions, however, become invalid unless both of the following two conditions are satisfied, which signals the breakdown of the RS ansatz.

The first condition is regarding the local stability of the RS saddle point with respect to the infinitesimal disturbance for breaking the replica symmetry in order parameters, which is often termed the de Almeida-Thouless (AT) condition.\(^8\) However, such a disturbance is not allowed in the current system, since the order parameters \( q_{ab} = \delta(s_a, s_b) \) are discrete. Therefore, we expect that the AT stability is always satisfied for both of the solutions for the RCE, although the stability must be examined for other code ensembles.

The other condition is regarding the entropy of the dynamical variable \( s \). Equations (3) and (6) indicate that the equality

\[
s(n, \beta) = -\frac{\partial g(n, \beta)}{\partial n} + \frac{\beta}{n} \frac{\partial g(n, \beta)}{\partial \beta}
\]
\[ \frac{1}{M} \left\langle Z^n(\beta; y, C) \left\{ \ln Z(\beta; y, C) - \beta \frac{\partial \ln Z(\beta; y, C)}{\partial \beta} \right\} \right\rangle_{y, C} \]  

(16)

holds for any pairs of \( n \) and \( \beta > 0 \). Since \( \ln Z(\beta; y, C) - \beta \frac{\partial \ln Z(\beta; y, C)}{\partial \beta} \) represents the entropy of the discrete dynamical variable \( s \) given \( y \) and \( C \), eq. (16) must become non-negative as long as \( g(n, \beta) \) is correctly evaluated.

Substituting eq. (15) into eq. (16) yields \( s(n, \beta) = 0 \), which indicates that \( g_{RS2}(n, \beta) \) always critically satisfies this entropy condition. However, for \( g_{RS1}(n, \beta) \),

\[
- \frac{\partial g_{RS1}(n, \beta)}{\partial n} + \beta \frac{n}{\partial \beta} \frac{\partial g_{RS1}(n, \beta)}{\partial \beta} = 0,
\]

(19)

which guarantees that \( s(n, \beta) \) is non-negative (zero) for \( g_{1RSB}(n, \beta) \).

Equations (18) and (19) indicate that eq. (11) for \( g_{1RSB}(n, \beta) \) is reduced to a condition of \( g_{RS1}(n, \beta) \) as

\[ \frac{1}{\beta} \frac{\partial g_{1RSB}(n, \beta)}{\partial n} = \frac{1}{\beta^*} \frac{\partial g_{RS1}(n^*, \beta^*)}{\partial n^*} = D. \]

(20)

Equations (17), (19) and (20) indicate that the error exponents can be practically evaluated without using the 1RSB solution as

\[
\alpha_{A,B}(D, R) = \lim_{\beta \to \infty} \left\{ -n \frac{\partial g_{1RSB}(n, \beta)}{\partial n} + g_{1RSB}(n, \beta) \right\} = -n \frac{\partial g_{RS1}(n, \beta)}{\partial n} + g_{RS1}(n, \beta), \]

(21)

where \( n \) and \( \beta \) are determined by

\[ \frac{1}{\beta} \frac{\partial g_{RS1}(n, \beta)}{\partial n} = D, \]

(22)

\[ - \frac{\partial g_{RS1}(n, \beta)}{\partial n} + \beta \frac{\partial g_{RS1}(n, \beta)}{\partial \beta} = 0, \]

(23)

when \( g_{RS1}(n, \beta) \) is selected as the relevant solution, despite the fact that \( g_{RS1}(n, \beta) \) becomes invalid for \( \beta \to \infty \).
4.4 Assessment of the error exponents

We are now ready to evaluate $\alpha_{\{A,B\}}(D, R)$ for the RCE using the two RS solutions. We first consider the failure exponent $\alpha_A(D, R)$ for $R > R_\prec(D)$.

4.4.1 $\alpha_A(D, R)$

In order to assess this exponent, we must select the relevant solution from $g_{RS_1}(n, \beta)$ and $g_{RS_2}(n, \beta)$. Note that $g_{RS_2}(n, \beta)$ must not be relevant for $n \leq 0$ because this solution does not satisfy the trivial identity

$$\lim_{n \to 0} g(n, \beta) = -\lim_{n \to 0} \frac{1}{M} \ln \langle Z^n(\beta; y, C) \rangle = -\frac{1}{M} \ln \langle 1 \rangle = 0,$$

and therefore the analytic continuation of this solution from $n \in \mathbb{N}$ to $n < 0$ is not reliable. $\alpha_A(D, R)$ corresponds to $n \leq 0$, and therefore we adopt $g_{RS_1}(n, \beta)$ for the evaluation of $\alpha_A(D, R)$.

Inserting (14) into eqs.(22) and (23) yields

$$\sum_{j \in \tilde{Y}} U_1(j) \ln \left[ \sum_{l \in \tilde{Y}} Q(l) e^{-\beta d(j,l)} \right] + R \ln 2 + \beta D = 0,$$

$$\sum_{j \in \tilde{Y}} U_1(j) \sum_{l \in \tilde{Y}} V_1(l|j)d(j,l) = D,$$

where the probability distributions $U_1 = (U_1(0), U_1(1), \ldots, U_1(J-1))$ and $V_1 = \{V_1(l|j)\} (j \in \tilde{Y})$ are defined as

$$U_1(j) = \frac{P(j) \left\{ \sum_{l \in \tilde{Y}} Q(l) e^{-\beta d(j,l)} \right\}^n}{\sum_{j \in \tilde{Y}} P(j) \left\{ \sum_{l \in \tilde{Y}} Q(l) e^{-\beta d(j,l)} \right\}^n},$$

$$V_1(l|j) = \frac{Q(l) e^{-\beta d(j,l)}}{\sum_{l \in \tilde{Y}} Q(l) e^{-\beta d(j,l)}}.$$

Inserting eqs.(22), (14) and (25) into eq.(21) yields the following expression for the error exponent

$$\alpha_A(D, R) = \sum_{j \in \tilde{Y}} U_1(j) \ln \left[ \frac{U_1(j)}{P(j)} \right] = KL(U_1||P),$$

where $KL(\cdot||\cdot)$ is termed the Kullback-Leibler divergence.10

Equation (29) characterizes the average performance of the RCE specified by $Q$. Therefore, the performance can be improved by maximizing eq.(29) with respect to $Q$ under the constraint $\sum_l Q(l) = 1$ and $Q(l) \geq 0$, which reduces to

$$\sum_{j \in \tilde{Y}} U_1(j) \frac{e^{-\beta d(j,l)}}{\sum_{l \in \tilde{Y}} Q(l) e^{-\beta d(j,l)}} = 1 \implies Q(l) = \sum_{j \in \tilde{Y}} U_1(j) V_1(l|j), \quad \forall l \in \tilde{Y}.$$

The set of $n$, $\beta$ and $Q$ that optimizes the exponent given $D$ and $R$ can be searched by the following scheme, which is often termed the Arimoto-Blahut algorithm (ABA).10,12,13 We begin with initial conditions of $n(<0)$, $\beta(>0)$ and $Q$. Keeping $Q$ fixed, $n$ and $\beta$ are first
updated by solving eqs. (25) and (26) with respect to these variables, which yields $U_1$ and $V_1$ using eqs. (27) and (28). Next, $Q$ is updated from eq. (30) using the obtained $U_1$ and $V_1$. These procedures are iterated until $n$, $\beta$ and $Q$ converge, which is guaranteed by the convexity of the mutual information. Then, the optimized exponent for the given $D$ and $R$ is obtained by substituting the convergent solution into eq. (29).

In practice, it is much more convenient to deal with $n$ and $\beta$ as control parameters, rather than $D$ and $R$, for which $D$ and $R$ are easily obtained from eqs. (25) and (26), after solving $Q$ for the given $n$ and $\beta$ by simply iterating eqs. (27), (28) and (30). Inserting the optimal $U_1$, which is given by the solved $Q$ via eq. (27), into eq. (29) and varying $n$ and $\beta$, the $\alpha_A(D, R)$ surface is swept out.

4.4.2 $\alpha_B(D, R)$

Next, we turn to the success exponent $\alpha_B(D, R)$ for $R < R_c(D)$. Since we expect that $g(n, \beta)$ is analytic, except for a few possible singular points of $n$, $g_{RS1}(n, \beta)$ is likely to be relevant for $n \geq 0$ as well, at least in the vicinity of $n = 0$, because this solution is supposed to be relevant for $n \leq 0$. Then, the exponent is obtained as $\alpha_B(D, R) = KL(U_1||P)$, which is similar to $\alpha_A(D, R)$.

However, the validity of this expression in the present case must be examined because $g_{RS2}(n, \beta)$ can be relevant for $n > 0$. For this, we illustrate schematic profiles of $g_{RS1}(n, \beta)$ and $g_{RS2}(n, \beta)$ for a fixed $\beta$ in Fig. 3.

Equations (14) and (15) indicate that both $g_{RS1}(n, \beta)$ and $g_{RS2}(n, \beta)$, which intersect each other at $n = 1$ for $\forall \beta > 0$, are convex upward with respect to $n$. As a function of $n$, $g_{RS2}(n, \beta)$ increases monotonically. Although the first derivative of $g_{RS1}(n, \beta)$ can be both positive and negative, in accordance with eq. (22), only the region of positive slope need to be considered.

For $n \in \mathbb{N}$, the relevant solution of $g(n, \beta)$ can be chosen by selecting one of the lower values of the two RS solutions, following the criterion of the conventional saddle point method. For $n \notin \mathbb{N}$, RM relies on the assumption that an analytical expression of $g(n, \beta)$ that is relevant for a certain natural number $k$ is also relevant in the vicinity of $k$, unless the analyticity is lost. Since $g_{RS1}(n, \beta) = g_{RS2}(n, \beta)$ holds at $n = 1$, this implies that the selection of $g_{RS1}(n, \beta)$ for $n \geq 0$, which we tentatively adopted assuming that the analyticity of $g(n, \beta)$ is not broken between $n < 0$ and $n \geq 0$, is valid if

$$
\frac{\partial g_{RS2}(n, \beta)}{\partial n} \bigg|_{n=1} - \frac{\partial g_{RS1}(n, \beta)}{\partial n} \bigg|_{n=1} = \frac{1}{M} \sum_{j \in \mathcal{Y}} \sum_{l \in \mathcal{L}} \frac{P(j)}{Q(l)} e^{-\beta d(j, l)} (-\beta d(j, l))\sum_{l \in \mathcal{L}} \frac{Q(l)}{Q(l)} e^{-\beta d(j, l)}\ln\left\{\sum_{l \in \mathcal{L}} Q(l) e^{-\beta d(j, l)}\right\} + R \ln 2
$$

$$
= \left(\frac{\beta g_{RS1}(n, \beta)}{n} \frac{\partial}{\partial n} + \frac{\partial g_{RS1}(n, \beta)}{\partial \beta}\right) \bigg|_{n=1} > 0,
$$

(31)
Fig. 3. Schematic profiles of $g_{RS1}(n, \beta)$ and $g_{RS2}(n, \beta)$ for a fixed $\beta$. The two functions intersect at $n = 1$ and both functions are convex upward with respect to $n$. Whereas the first derivative of $g_{RS2}(n, \beta)$ is always positive, that of $g_{RS1}$ depends on $R, D, n$. RM assesses the value of $g(n, \beta)$ for $n \notin \mathbb{N}$ by analytically continuing the evaluation for $n \in \mathbb{N}$. This implies that the relevant solution for $n < 1$ is the smaller slope at $n = 1$ between $g_{RS1}(n, \beta)$ and $g_{RS2}(n, \beta)$ unless the analyticity is broken. Thus, the relevant solution is $g_{RS1}(n, \beta)$ and $g_{RS2}(n, \beta)$ for the cases of (a) and (b), respectively.

Let us denote the solution of eqs. (22) and (23) as $n = n_c$ and $\beta = \beta_c$, respectively. As

$$\left( -\frac{\partial g_{RS1}(n, \beta)}{\partial n} + \frac{\beta}{n} \frac{\partial g_{RS1}(n, \beta)}{\partial \beta} \right) \bigg|_{n=n_c, \beta=\beta_c} = 0,$$

holds, which corresponds to the situation illustrated in Fig. 3 (a).

For $g_{RS2}(n, \beta)$, eq. (11) is given as

$$\sum_{j \in Y} \sum_{l \in \tilde{Y}} V_2(l|j) U_2(j) d(j, l) = D,$$

where distributions $U_2 = \{U_2(0), U_2(1), \ldots, U_2(J - 1)\}$ and $V_2 = \{V_2(l|j)\}$ ($j \in Y, l \in \tilde{Y}$) are defined as

$$U_2(j) = \frac{P(j) \sum_{l \in \tilde{Y}} Q(l) e^{-\beta' d(j, l)}}{\sum_{j \in Y} \sum_{l \in \tilde{Y}} P(j) Q(l) e^{-\beta' d(j, l)}},$$

$$V_2(l|j) = \frac{Q(l) e^{-\beta' d(j, l)}}{\sum_{l \in \tilde{Y}} Q(l) e^{-\beta' d(j, l)}},$$

respectively, where $\beta' \equiv n \beta$. 

11/20
Note that the value of $\beta'$ determined from eq.(32) is kept invariant when $\beta$ tends toward infinity. In order to assess eq.(10) for $g_{RS2}(n, \beta)$, inserting eq.(32) yields the expression

$$\alpha_B(D, R) = KL(U_2 || P) + I - R \ln 2,$$

(35)

where $I$ is defined as

$$I = \sum_{j \in Y} \sum_{l \in \tilde{Y}} V_2(l|j) U_2(j) \ln \left[ \frac{V_2(l|j)}{Q(l)} \right].$$

(36)

In summary, the exponent $\alpha_B(D, R)$ for $R < R_c(D)$ is expressed as

$$\alpha_B(D, R) = \begin{cases} 
KL(U_1 || P), & \text{if } 0 < n_c < 1 \\
KL(U_2 || P) + I - R \ln 2, & \text{if } n_c > 1 
\end{cases}$$

(37)

for a given ensemble specified by $Q$.

Here, $\alpha_B(D, R)$ can be minimized with respect to the distribution $Q$ in a manner similar to that for $\alpha_B(D, R)$. Namely, we tentatively adopt the first expression of eq.(37), assuming that $g_{RS1}(n, \beta)$ is relevant, and employ ABA in order to obtain the optimal $n, \beta$ and $Q$. If the obtained solution of $n, n_c$, is smaller than 1, then the obtained expression is appropriate. Otherwise, we have to amend the solution using the second expression, which can be optimized by ABA as well. In this case, the convergent solution satisfies the relation $Q(l) = \sum_{j \in Y} V_2(l|j) U_2(j)$ for $\forall l \in \tilde{Y}$. This yields the expression of the optimized exponent as

$$\alpha_B(D, R) = KL(U_2 || P) + (R(U_2, D) - R) \ln 2,$$

(38)

where

$$R(U, D) = \min_{\sum_{j \in Y, l \in \tilde{Y}} V(l|j) U(j) \log_2 \left[ \frac{V(l|j)}{\sum_{j \in Y} V(l|j) U(j)} \right]} \sum_{j \in Y, l \in \tilde{Y}} V(l|j) U(j) \log_2 \left[ \frac{V(l|j)}{\sum_{j \in Y} V(l|j) U(j)} \right],$$

(39)

is termed the rate-distortion function, which represents the theoretically achievable limit of the compression rate for the information source $U$ when distortion up to $D$ is allowed in the limit $N, M \to \infty$.\(^{11}\)

4.5 Consistency with the IT literature

We obtained two expressions for the error exponents (29) and (37) using RM. In order to validate our results, we check for consistency with results in the IT literature.

4.5.1 $\alpha_A(D, R)$

We first examine $\alpha_A(D, R)$ for $R > R_c(D)$. In the IT literature, the exponent for the best code is provided\(^{4}\) as

$$\alpha_A^*(D, R) = \min_{U : R \leq R(U, D)} KL(U || P).$$

(40)

This minimization problem can be solved by the method of Lagrange multipliers. Intro-
ducing auxiliary variables \( z_1(\geq 0) \) and \( z_2(\leq 0) \) as
\[
\begin{align*}
z_1 &= \sum_{j \in Y} \sum_{l \in \tilde{Y}} V(l|j) U(j) \ln \left( \frac{V(l|j)}{Q(l)} \right) - R \ln 2, \\
z_2 &= \sum_{j \in Y} \sum_{l \in \tilde{Y}} V(l|j) U(j) d(j,l) - D,
\end{align*}
\]
where
\[
Q(l) = \sum_{j \in Y} V(l|j) U(j),
\]
eq(40) is converted to the minimization problem of
\[
J_A(U, V, z_1, z_2) = \sum_{j \in Y} U(j) \ln \left( \frac{U(j)}{P(j)} \right) \\
+ n_A \left\{ \sum_{j \in Y} \sum_{l \in \tilde{Y}} V(l|j) U(j) \ln \left( \frac{V(l|j)}{Q(l)} \right) - R \ln 2 - z_1 \right\} \\
+ \beta_A \left\{ \sum_{j \in Y} \sum_{l \in \tilde{Y}} V(l|j) U(j) d(j,l) - D - z_2 \right\} \\
+ \sum_{j \in Y} \nu(j) \left\{ \sum_{l \in \tilde{Y}} V(l|j) - 1 \right\} + \xi \left\{ \sum_{j \in Y} U(j) - 1 \right\},
\]
with respect to \( U, V, z_1 \), and \( z_2 \), where \( n_A, \beta_A, \nu(j) \) (\( j = 1, 2, \ldots, J - 1 \)), and \( \xi \) are Lagrange multipliers.

Note that if the minimum is achieved by an internal point \( z_1 > 0 \), \( \partial J_A / \partial z_1 = -n_A = 0 \); otherwise, the minimum is placed on the boundary \( z_1 = 0 \) and \( \partial J_A / \partial z_1 = -n_A \geq 0 \). Since the rate-distortion function \( R(U, D) \) decreases monotonically as \( D \) increases\(^{10}\) and \( R > R_c(D) \) is assumed, we cannot set \( U = P \). Furthermore, taking the convexity of the \( KL \) divergence into account, minimization (40) must be achieved on the boundary, which ensures that \( n_A < 0 \) \( (z_1 = 0) \). A similar argument holds for \( \beta_A \). According to the convexity of the mutual information, the rate-distortion function \( R(U, D) \) is determined by the distribution \( V \) on its boundary, which indicates \( \beta_A > 0 \) \( (z_2 = 0) \).

Minimizing eq.(44) with respect to \( V(l|j) \) provides
\[
V(l|j) = \frac{Q(l)e^{-\beta_A d(j,l)}}{\sum_{l \in \tilde{Y}} Q(l)e^{-\beta_A d(j,l)}}, \quad \forall j \in Y, l \in \tilde{Y},
\]
where the normalization constraint has been already factored into the equation. Similarly, minimization with respect to \( U(j) \) yields
\[
U(j) = \frac{P(j) \left\{ \sum_{l \in \tilde{Y}} Q(l)e^{-\beta_A d(j,l)} \right\}^{n_A}}{\sum_{j \in Y} P(j) \left\{ \sum_{l \in \tilde{Y}} Q(l)e^{-\beta_A d(j,l)} \right\}^{n_A}}, \quad \forall j \in Y.
\]

In practice, we can assess the optimal exponents using ABA to solve eqs.(46),(45) and (43) with respect to \( n_A < 0, \beta_A > 0 \) and \( Q \) under the constraint that the solutions should be
found on the boundary, which is represented as eqs. (41) and (42) \((z_1 = z_2 = 0)\). Identifying \(n_A\) and \(\beta_A\) with \(n\) and \(\beta\), respectively, this is nothing more than the information presented in the preceding section for obtaining the error exponent optimized with respect to the distribution \(Q\). Therefore, our SM-based framework is consistent with the result for \(\alpha^*_A(D,R)\) reported in the IT literature.

4.5.2 \(\alpha_B(D,R)\)

We next consider \(\alpha_B(D,R)\) for \(R < R_c(D)\). In the IT literature, the exponent for the best code for \(R < R_c(D)\) is given\(^5\) as

\[
\alpha^*_B(D,R) = \min_{U} KL(U\|P) + |R(U,D) - R|^+ \ln 2, \tag{47}
\]

where \(|x|^+ = x\) for \(x \geq 0\), and is 0 otherwise. This can be separately expressed as

\[
\min \left\{ \begin{aligned}
\min_{U : R \geq R(U,D)} & KL(U\|P), \\
\min_{U : R \leq R(U,D)} & KL(U\|P) + (R(U,D) - R) \ln 2.
\end{aligned} \right. \tag{48}
\]

\[
\min \left\{ \begin{aligned}
\min_{U : R \geq R(U,D)} & KL(U\|P), \\
\min_{U : R \leq R(U,D)} & KL(U\|P) + (R(U,D) - R) \ln 2.
\end{aligned} \right. \tag{49}
\]

As well as eq.(40), eq. (48) is converted to the minimization of

\[
J_{B1}(U,V,z_1,z_2) = \sum_{j \in Y} U(j) \ln \left[ \frac{U(j)}{P(j)} \right] + n_{B1} \left\{ \sum_{j \in Y} \sum_{l \in \tilde{Y}} V(l|j)U(j) \ln \left[ \frac{V(l|j)}{Q(l)} \right] - R \ln 2 - z_1 \right\} + \beta_{B1} \left\{ \sum_{j \in Y} \sum_{l \in \tilde{Y}} V(l|j)U(j)d(j,l) - D - z_2 \right\} + \sum_{j \in Y} \nu(j) \left\{ \sum_{l \in \tilde{Y}} V(l|j) - 1 \right\} + \xi \left\{ \sum_{j \in Y} U(j) - 1 \right\}, \tag{50}
\]

with respect to \(U, V, z_1, \) and \(z_2\). Although constraints \(R < R_c(D)\) and \(z_1 \leq 0\) are different from those for \(\alpha^*_A(D,R)\), the minimum is also achieved on the boundary in this case, which indicates \(n_{B1} > 0\) \((z_1 = 0, \partial J_{B1}/\partial z_1 = -n_{B1} < 0)\). This means that the distribution \(U\) and the conditional distribution \(V\) can be represented as

\[
U(j) = \frac{P(j) \{ \sum_{l \in \tilde{Y}} Q(l)e^{-\beta_{B1}d(j,l)} \}^{n_{B1}}}{\sum_{j \in Y} P(j) \{ \sum_{l \in \tilde{Y}} Q(l)e^{-\beta_{B1}d(j,l)} \}^{n_{B1}}}, \quad \forall j \in Y, \tag{51}
\]

\[
V(l|j) = \frac{Q(l)e^{-\beta_{B1}d(j,l)}}{\sum_{l \in \tilde{Y}} Q(l)e^{-\beta_{B1}d(j,l)}}, \quad \forall j \in Y, l \in \tilde{Y}, \tag{52}
\]

using the Lagrange multipliers \(n_{B1}\) and \(\beta_{B1}\).

Minimization (49) can be rewritten as

\[
\min_{U,V : \sum_{j \in Y} \sum_{l \in \tilde{Y}} V(l|j)U(j)d(j,l) \leq D, \ I \geq R \ln 2} KL(U\|P) + I - R \ln 2, \tag{53}
\]

where \(I\) is the mutual information expressed in eq.(36). This can also be solved by the method
of Lagrange multipliers, which yields the minimization of

\[ J_{B2}(U, V, z_1, z_2) = \sum_{j \in Y} U(j) \ln \left( \frac{U(j)}{P(j)} \right) + \sum_{j \in Y} \sum_{l \in \hat{Y}} V(l|j) U(j) \ln \left( \frac{V(l|j)}{Q(l)} \right) - R \ln 2 \]

\[ + n_{B2} \left\{ \sum_{j \in Y} \sum_{l \in \hat{Y}} V(l|j) U(j) \ln \left( \frac{V(l|j)}{Q(l)} \right) - R \ln 2 - z_1 \right\} \]

\[ + \beta_{B2} \left\{ \sum_{j \in Y} \sum_{l \in \hat{Y}} V(l|j) U(j) d(j, l) - D - z_2 \right\} \]

\[ + \sum_{j \in Y} \nu(j) \left\{ \sum_{l \in \hat{Y}} V(l|j) - 1 \right\} + \xi \left\{ \sum_{j \in Y} U(j) - 1 \right\}, \] (54)

where \( n_{B2}, \beta_{B2}, \{\nu(j)\} \) and \( \xi \) are Lagrange multipliers, and \( z_1 (\geq 0) \) and \( z_2 (\leq 0) \) are defined in eqs.(41) and (42). Note that \( n_{B2} \leq 0 \) and \( \beta_{B2} > 0 \), because \( KL(U||P) + I - R \ln 2 \) is not a convex function and we cannot exclude the possibility that \( n_{B2} = 0 \).

If the minimization (48) (or (50) ) is achieved for \( 0 < n_{B1} \leq 1 \), the minimization (49) ( or (54) ) is achieved by the same \( U \) and \( V \), by setting \( n_{B2} = n_{B1} - 1 (\leq 0) \). However, if \( n_{B1} > 1 \), no distributions that minimize (48) can simultaneously be the solution of eq.(49), which indicates that eq.(49) is achieved by a distribution \( U \) that satisfies \( R < R(U, D) \). In this case, \( n_{B2} \) must be zero, and therefore eq.(54) is reduced to

\[ J_{B2}(U, V, z_1, z_2) = \sum_{j \in Y} U(j) \ln \left( \frac{U(j)}{P(j)} \right) + \sum_{j \in Y} \sum_{l \in \hat{Y}} V(l|j) U(j) \ln \left( \frac{V(l|j)}{Q(l)} \right) - R \ln 2 \]

\[ + \sum_{j \in Y} \sum_{l \in \hat{Y}} V(l|j) U(j) d(j, l) - D - z_2 \]

\[ + \sum_{j \in Y} \nu(j) \left\{ \sum_{l \in \hat{Y}} V(l|j) - 1 \right\} + \xi \left\{ \sum_{j \in Y} U(j) - 1 \right\}. \] (55)

Differentiating eq.(55) with respect to \( V(l|j) \) and \( U(j) \) yields

\[ V(l|j) = \frac{Q(l) e^{-\beta_{B2} d(j, l)}}{\sum_{l \in \hat{Y}} Q(l) e^{-\beta_{B2} d(j, l)}}, \quad \forall j \in Y, l \in \hat{Y}, \] (56)

\[ U(j) = \frac{P(j) \sum_{l \in \hat{Y}} Q(l) e^{-\beta_{B2} d(j, l)}}{\sum_{j \in Y} P(j) \sum_{l \in \hat{Y}} Q(l) e^{-\beta_{B2} d(j, l)}}, \quad \forall j \in Y. \] (57)

Based on the above argument, the optimal exponent \( \alpha_{B}^{*}(D, R) \) is assessed by the following procedure. First, we employ ABA for the solution of minimization (48) with respect to \( n_{B1} > 0, \beta_{B1} > 0 \) and \( Q \) using eqs.(51), (52) and (43) under the constraints (41) and (42) (\( z_1 = z_2 = 0 \)). If the solved \( n_{B1} \) satisfies \( 0 < n_{B1} \leq 1 \), it is guaranteed that the obtained \( U \) and \( V \).
achieve the minimization (47). However, if the obtained $n_{B1}$ is greater than 1, this solution is not appropriate, because minimization (49) is not achieved. Therefore, we have to search for another solution using eqs.(57), (56) and (43) with $\beta_{B2} > 0$, under the constraint (42) ($z_2 = 0$), which can also be performed by ABA. In this case, the other constraint (41) ($z_1 > 0$) is always satisfied for $n_{B1} > 1$, which is confirmed by the fact that the minimization of

$$\min_{U: R \geq R(U,D)} KL(U\|P) + (R(U,D) - R) \ln 2$$

(58)

can be achieved by an internal point with respect to $z_1$ if and only if $0 < n_{B1} \leq 1$. This procedure is identical to that of the RM-based approach presented in a previous section. Therefore, the framework developed in this paper is consistent with the result for $\alpha_B^*(D, R)$ reported in the IT literature.

### 4.6 Discussion

Here, two points are worth noting. First, we have shown that the exponents assessed by the RM-based method become identical to those of the best code in the IT literature, when optimized with respect to the code ensemble. However, this may be somewhat curious because $\alpha_{\{A,B\}}(D, R)$ characterizes either the average of the compression failure or the success probability over a code ensemble, which implies that $\alpha_{\{A,B\}}(D, R)$ does not necessarily coincide with the exponent of the best code, even if the ensemble is optimized. In order to examine a possible difference in exponents between the average and optimal probabilities, we evaluated the exponents of the minimum failure probability $P_F^* = \lim_{t \to -\infty} \langle P_F^t(C,D) \rangle_C^{1/t}$ for $R > R_c(D)$ and the maximum success probability $P_s^* = \lim_{t \to +\infty} \langle P_s^t(C,D) \rangle_C^{1/t}$ for $R < R_c(D)$ for fixed ensembles, which reduced to the current calculations for the average probabilities. This means that in the RCE specified by $Q$, the performance of the best code is identical to that of typical codes in terms of the exponents, although differences may exist for ensembles of other types. Second, we may be able to apply the present framework to sources with memory, for which the optimal exponents have not been reported in the IT literature. This possibility is currently under investigation.

### 5. Application to a Sub-optimal Ensemble

In addition to consistency with the existing results, a major advantage of the proposed RM-based approach is its ability to accurately evaluate the exponents for a wider class of ensembles. Here, we demonstrate this ability for a lossy compression of a binary memoryless source, which is specified by $P = (P(0), P(1)) = (1 - p, p)$ where $0 < p < 1/2$.

Although RCEs exhibit the optimal performance, they are difficult to implement in practice because a storage of $O(M \times 2^N)$ is required in order to express the set of representative vectors $\tilde{g}(s)$. As a candidate to resolve this difficulty, we investigate the performance of a compression scheme which utilizes perceptrons having random connections.15
More specifically, we define a map from the compressed expression \( s \in \{+1, -1\}^N \) to the representative sequence \( \tilde{y}(s) \in \{0, 1\}^M \) as

\[
\tilde{y}^\mu(s) = f\left( \frac{1}{\sqrt{N}} s \cdot x^\mu \right), \quad (\mu = 1, 2, \ldots, M)
\]

for the specification of a code, where \( f(\cdot) \) is a function for which the output is limited to \( \{0, 1\} \) and \( x^\mu = 1, 2, \ldots, M \) are randomly predetermined \( N \)-dimensional vectors generated from an \( N \)-dimensional normal distribution \( P(x) = (\sqrt{2\pi})^{-N} \exp \left[ -|x|^2/2 \right] \). These vectors are known to the compressor and the decompressor, which act as the codebook. Here, for convenience, we introduce the alphabet \( \{+1, -1\} \), rather than the conventional alphabet \( \{0, 1\} \) with respect to the compressed sequence \( s \).

We employ the Hamming distortion

\[
d(y, \tilde{y}(s)) = \sum_{\mu=1}^{M} [1 - \delta_{y^\mu, \tilde{y}^\mu}] \]

to measure the fidelity of the representative sequences. Then, a lossy compression scheme can be defined on the basis of eq. (59) as follows:

**Compression:** For a given message \( y \), find a vector \( s \) that minimizes the distortion \( d(y, \tilde{y}(s)) \), where \( \tilde{y}(s) \) is the representative vector that is uniquely generated from \( s \) by eq. (59). The obtained \( s \) is adopted as the compressed expression.

**Decoding:** Given the compressed expression \( s \), the representative vector \( \tilde{y}(s) \) produced by eq. (59) yields an approximation of the original message.

Random selection of the connections naturally defines a code ensemble of this scheme.

Codes of this type may be preferred for practical implementation because the necessary storage cost is only \( O(M \times N) \). However, possible correlations between components of the representative vector may prevent the analysis of its performance by conventional methods in the IT literature. Nevertheless, the proposed RM-based approach makes it possible to accurately evaluate the performance of this ensemble using a recipe similar to the capacity analysis of perceptrons, which has been reported extensively over the last decade. In a previous paper, such an analysis indicated that a function \( f(u) = 1 \) for \(|u| < k \), and 0 otherwise, which offers optimal performance in the limit \( M, N \to \infty \) achieving the rate-distortion function of \( R(p, D) = H_2(p) - H_2(D) \) for this case, where \( H_2(x) = -x \log_2(x) - (1-x) \log_2(1-x) \) for \( 0 < x < 1 \) when \( k \) is adjusted such that \( 2 \int_k^\infty dz e^{-z^2/2}/\sqrt{2\pi} = \frac{1}{1-2p} D^* \), where \( D^* \) represents the lower bound of the Hamming distortion for a given compression rate \( R \), which is obtained from the inverse function of the rate distortion function, except for a very narrow AT instability region in the vicinity of \( p = 0.5 \).

The error exponents of this ensemble can be calculated by a procedure similar to that for the RCE. Taking the average of \( \left( \sum_s e^{-3d(y, \tilde{y}(s))} \right)^n \) with respect to the original message \( y \) and the connection vectors \( x^\mu = 1, 2, \ldots, M \) yields

\[
g(n, \beta) =
\]
where \( q_{ab} = \frac{s^a s^b}{N} \) for \( a > b = 1, 2, \ldots, n \),

\[
\Theta_k(u;1) = 1 - \Theta_k(u;0) = \begin{cases} 
1, & \text{for } |u| \leq k \\
0, & \text{otherwise,}
\end{cases}
\]  

(61)

and \( A = (\delta_{ab} + (1 - \delta_{ab})q_{ab}) \). This expression corresponds to eq.(13) for the RCE.

The mirror symmetry, \( f(-u) = f(u) \), of the transfer function yields a solution \( q_{ab} = q = 0 \) under the RS ansatz,\(^{15} \) which offers

\[
g_{RS1}(n, \beta) = -\ln \left[ p \left\{ 1 - \eta + \eta e^{-\beta} \right\}^n + (1 - p) \left\{ (1 - \eta)e^{-\beta} + \eta \right\}^n \right] - nR \ln 2,
\]

(62)

where \( \eta \) is defined as \( \eta = 1 - 2 \int_k^{\infty} dz e^{-z^2/2}/\sqrt{2\pi} \). In addition, there exists another RS solution

\[
g_{RS2}(n, \beta) = -\ln \left[ p \left\{ 1 - \eta + \eta e^{-\beta n} \right\} + (1 - p) \left\{ (1 - \eta)e^{-\beta n} + \eta \right\} \right] - R \ln 2,
\]

(63)

corresponding to \( q_{ab} = q = 1 \). Equations (62) and (63) coincide with eqs.(14) and (15) for the current source and the Hamming distortion, respectively.

Therefore, we can recycle the calculation for the RCE to examine the performance of the current ensemble, which indicates that the optimal error exponents can be obtained by adjusting the parameter \( k \) to the optimal value for each pair of \( D \) and \( R \) (such that \( 2 \int_k^{\infty} dz e^{-z^2/2}/\sqrt{2\pi} = \frac{1-D-p^*}{1-2D} \), where \( p^* \) satisfies the relation \( R = H_2(p^*) - H_2(D) \) for the rate \( R \) and the given permissible level \( D \) ) unless AT instability occurs for the above RS solutions. Note that for the current source, the optimal exponents can always be achieved by \( g_{RS1}(n, \beta) \). Here, \( g_{RS2}(n, \beta) \) becomes dominant in only suboptimal cases for \( \alpha_B(D, R) \) (Fig. 4(b) inset).

In order to justify the above analysis, we performed numerical experiments implementing the proposed scheme. As an exhaustive search was performed for compression, the system size was limited to \( N = 20 \). Fig. 4 shows the exponents averaged over the results from \( 5 \times 10^3 \sim 1 \times 10^6 \) experiments for the case of \( p = 0.2, R = 0.2 \). The white circles and triangles indicate data obtained by adjusting \( k \) so that the exponents are optimized for (a) \( D = 0.2 \) and (b) \( D = 0.0 \), respectively. The black circles and triangles indicate data obtained using \( k \simeq 0.136 \) so as to reproduce the rate-distortion relation, which implies that both exponents vanish at \( D^* \simeq 0.117 \). In Fig. 4(a), notice that the white symbols increase at \( D = 0.2 \) as \( N \) grows, whereas the black symbols decrease, approaching each theoretical prediction consistently. Fig. 4(b) shows that the white symbols are located below the black symbols at \( D = 0.0 \). In both figures, the discrepancies between the experimental data and the theoretical predictions are considered to be due to the finite size effect.
Fig. 4. Error exponents (a) $\alpha_A(D, R)$ and (b) $\alpha_B(D, R)$ for $p = 0.2$, $R = 0.2$. The solid and dashed curves indicate the optimal exponents $\alpha^*_A, B(D, R)$ and the exponents obtained for fixed $k(\approx 0.136)$ that realizes the rate-distortion relation $R(p, D) = H_2(p) - H_2(D)$, respectively. For $0 \leq D \lesssim 0.011$ in Fig. (b), the dashed curve is obtained from the solution $g_{RS2}(n, \beta)$, which dominates $g_{RS1}(n, \beta)$ in this region (Fig. (b) inset). The experimental data was obtained for (a) 5000-10000 trials for $N = 10, 20$ and (b) $10^6$ trials for $N = 4, 10$ through exhaustive search. The white circles and triangles represent the exponents optimized for (a) $D = 0.2$ and (b) $D = 0.0$, respectively, by adjusting $k$, and the black symbols indicate exponents for fixed $k(\approx 0.136)$.

6. Summary

In summary, we have developed a scheme by which to assess the error exponents of a lossy data compression problem using RM. The newly developed RM-based approach for the exponents corresponding to the average failure or success probabilities for the random code ensembles reproduces the optimal error exponents achieved by selecting the best code reported in the IT literature, which indicates that the performance of the best code is identical to that of typical codes in terms of error exponents. Furthermore, the proposed framework makes an accurate assessment of the coding performance possible for a wide class of code ensembles. Using this characteristic, we have shown that a lossy compression scheme based on a specific type of non-monotonic perceptron provides the optimized exponents in most cases, which has been supported numerically.

Evaluation of the error exponents of practical algorithms for lossy data compression is a subject for future study. In order to reduce the computational cost of the proposed coding scheme, the development of approximation algorithms by which to realize the compression phase using a perceptron is currently under way.

Acknowledgment

TH would like to thank T. Uematsu for his helpful comments. TH is a Research Fellow of the Japan Society for the Promotion of Science. The present study was supported in part by Grants-in-Aid (No. 164453 from the JSPS:TH) and (No. 14084206 from MEXT, Japan:YK).
References

1) H. Nishimori: *Statistical Physics of Spin Glasses and Information Processing* (Oxford University Press, Oxford, 2001).
2) Y. Kabashima, K. Nakamura and J van Mourik: *Phys. Rev. E* 66 (2002) 036125.
3) N. Skanzos, J van Mourik, D. Saad and Y. Kabashima: *J. Phys. A* 36 (2003) 11131.
4) K. Marton: *IEEE Trans. Inf. Theory* IT-20 (1974) 197.
5) I. Csiszár and J. Körner: *Information Theory, Coding Theorems for Discrete Memoryless Systems* (Academic Press, New York, 1981) p.158.
6) R. G. Gallager: *Information Theory and Reliable Communication* (Wiley, New York, 1968).
7) A. J. Viterbi and J. K. Omura: *Principles of Digital Communication and Coding* (McGraw-Hill Kogakush, Tokyo, 1979).
8) J. R. L. de Almeida and D. J. Thouless: *J. Phys. A* 11 (1977) 983.
9) D. J. Gross and M. Mézard: *Nucl. Phys. B* 240 (1984) 431.
10) T. M. Cover and J. A. Thomas: *Elements of Information Theory* (John Wiley & Sons, New York, 1991).
11) C. E. Shannon: *IRE Nat. Conv. Rec.* part 4 (1959) 142.
12) S. Arimoto: *IEEE Trans. Inf. Theory* IT-18 (1972) 14.
13) R. E. Blahut: *IEEE Trans. Inf. Theory* IT-18 (1972) 460.
14) K. Ogure and Y. Kabashima: *Prog. Theor. Phys.* 111 (2004) 661.
15) T. Hosaka, Y. Kabashima and H. Nishimori: *Phys. Rev. E* 66 (2002) 066126.
16) A. Engel and C. P. L. van den Broeck: *Statistical Mechanics of Learning* (Cambridge University Press, Cambridge, 2001).