An Analytic Study of the Phase Structure of Lattice QCD with Wilson Fermions at Infinitely Strong Coupling.

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Abstract

The phase structure of lattice QCD with two flavors and Wilson fermions is studied analytically. At $\beta = 0$ we obtain rigorous lower and upper bounds for the critical hopping parameter $k_c(0)$ from a convergent hopping parameter expansion to infinite order. The result supports the value $k_c(0) = \frac{1}{4}$ observed in Monte Carlo simulations.
1 Introduction

The outstanding properties of QCD are asymptotic freedom and confinement. Although a proof that QCD is confining does not exist, there are strong indications that QCD has this property and we assume it to be true in the following. The coexistence of asymptotic freedom and confinement means that the effective quark-gluon and gluon-gluon coupling changes continuously from weak coupling at short distance to strong coupling at long distance. At long distance quarks and gluons are confined inside color singlet hadrons. All present known strong interaction phenomena support the confining picture. In addition hadrons built from the two lightest quarks exhibit an approximate flavour symmetry which is spontaneously broken. Due to the confinement, a perturbative description of these phenomena is not possible. The description of the long distance strong forces requires non-perturbative methods. Lattice QCD formulates the theory non-perturbatively and allows us to test confinement and to determine numerically properties of low energy hadrons.

Some useful qualitative informations in lattice QCD can be obtained in the infinitely strong bare gauge coupling limit $\beta = 0$. At infinite gauge coupling confinement and spontaneous breakdown of chiral flavour symmetry are manifest. The spontaneous chiral symmetry breakdown and the emergence of massless Goldstone bosons explain the lightness of the pion. The small but non vanishing mass of the pion is due to the non exactness of the flavour symmetry due to the small but finite light quark masses.

In the past many Monte Carlo simulations of lattice QCD with Wilson fermions has shown the existence of a critical line $k_c(\beta)$ in the $k-\beta$ plane, where the mass of the Goldstone boson vanishes. For any given coupling $\beta$ there exists a critical hopping parameter $k_c(\beta)$. The continuum limit is approached along this line which ends at $k_c(\infty) = \frac{1}{8}$. In the strong coupling regime, at $\beta = 0$, Monte Carlo simulations indicate that $k_c(0) = \frac{1}{4}$. Some strong coupling expansion analysis give also some good argument to support this feature [1, 2].

Up to now there is not a rigorous proof that $k_c(0) = \frac{1}{4}$ for Wilson fermions. On the other hand, at infinite gauge coupling an important simplification of the system takes place which makes feasible a convergent hopping parameter expansion analysis to infinite order. In fact, at $\beta = 0$ all gauge variables on different links are independent and can be integrated out explicitly, leaving us with a pure fermion problem which is however still not trivial.

In this work we analyse rigorously the behaviour of the Goldstone boson spectrum at $\beta = 0$ with the method of the hopping parameter expansion. Our analysis is based on a random walk representation of the meson propagators [3]. In particular this allows us a rigorous estimation of the mass of the pseudoscalar isotriplet and of $k_c(0)$. A full hopping parameter expansion to infinite order of perturbation is a very difficult problem because there are no known general methods to identify all diagrammatic contributions to the expansion for a given order of the perturbation expansion. However, there is an easy way to determine an upper and a lower bound to $k_c(0)$ to infinite order of the perturbation expansion by neglecting and overcounting, respectively, a class of diagrammatic contributions of the expansion which are very difficult to be identified. This simplification leads to an upper
and a lower estimation of the mass of the Goldstone boson (the pseudoscalar isotriplet) expressed simply by geometric series of the hopping parameter $k$. These bounds allow us to determine bounds on the critical hopping parameter $k_c(0)$ from the condition that this mass has to vanish.

2 The Wilson lattice action

There is a SU(3) color matrix $U(b)$ in the fundamental representation defined on each oriented lattice bond $b$. Our convention is that

$$U(-b) = U^\dagger(b).$$

(1)

An oriented path $\omega$ on the lattice is a set of bonds

$$\omega = b_1 \cup b_2 \cup \ldots \cup b_n$$

(2)

such that the endpoint of $b_i$ is the start point of $b_{i+1}$ for $1 \leq i \leq n - 1$. We can associate a SU(3) color matrix with $\omega$ by defining the path ordered product

$$U(\omega) = U(b_1)U(b_2)\ldots U(b_n)$$

(3)

The spin matrices are defined in terms of $\gamma$-matrices by

$$\Gamma(b) = \begin{cases} \Gamma^\mu = r + \gamma^\mu & \text{if } b \text{ in } +\mu \text{ direction} \\ \Gamma^\mu = r - \gamma^\mu & \text{if } b \text{ in } -\mu \text{ direction} \end{cases}$$

(4)

The $\gamma$-matrices are hermitian 4x4 matrices, satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\delta^\mu\nu$.

The Wilson action is then defined on a lattice $\Lambda$ by

$$S = \beta \sum_{p \subset \Lambda} \text{Tr}U(\partial p) - k \sum_{\langle xy \rangle} \bar{\psi}(x)\Gamma(b)U(b)\psi(y) - \sum_{x \in \Lambda} \bar{\psi}(x)\psi(x)$$

(5)

where $p$ represents a plaquette. The quark fields are represented by the anti-commuting variables $\psi(x)$ and $\bar{\psi}(x)$ which transform under the 3 and $\bar{3}$ representation of color. We consider only two flavors.

3 The meson propagator

In order to evaluate masses of mesons we have to consider the propagator of these states. The masses are obtained by studying the long distance behaviour of these propagators. A propagator of a meson is given by

$$\langle M(x)^\dagger M(y) \rangle = \frac{1}{Z} \int [d\bar{\psi}][d\psi][dU]M(x)^\dagger M(y)e^{-S(U,\bar{\psi},\psi)}$$

(6)

where $Z$ is the partition functional and $M(x) = \bar{\psi}(x)M\psi(x)$ represents a meson operator ($M$ is some matrix in spin, flavour and color space). The hopping parameter expansion
is obtained by breaking the action $S$ into two parts: the unperturbed part $S_0$ and the perturbation $S_I$.

$$S_0 = -\sum_{x \in \Lambda} \bar{\psi}(x)\psi(x)$$
$$S_I = -k \sum_{b=(xy)} \bar{\psi}(x)\Gamma(b)U(b)\psi(x).$$

(7)

Here the plaquette term is not present because we are at $\beta = 0$. Expanding the exponential in eq. (6) in term of the perturbation and integrating out the gauge degrees of freedom $U$ we obtain an expression for the meson propagator

$$\langle M(x)^\dagger M(y) \rangle = \sum_{\omega^+:x\rightarrow y} C_M \times Tr \left( \left( M^\dagger \Gamma(\omega^+)M\Gamma(\omega^-) \right) \times (-1)^L \kappa^{\omega^+ \cup \omega^-} \right) \equiv$$
$$\equiv \sum_{\omega^+:x\rightarrow y} C_M \times A(\omega^+ \cup \omega^-)$$

(8)

where $C_M$ is some overall constant amplitude which does not affect the meson mass, $\omega^+$ is a path on the lattice, $\omega^- = -\omega^+$ is a path in the opposite direction of $\omega^+$ and $L$ is the number of closed loops formed by the path $\omega^+ \cup \omega^-$. Here $\Gamma(\omega)$ and $U(\omega)$ represent the ordered products of $\Gamma(b)$ and $U(b)$ on the bonds $b \in \omega$, respectively. We do not give a proof of (8) since it is standard [3, 4].

4 Meson masses from the hopping parameter expansion

In order to compute the meson masses we consider the static propagator

$$G_M^M(t) = \sum_{\vec{x}} \langle M((0,\vec{x}))^\dagger M((t,\vec{x})) \rangle.$$

(9)

The lowest order diagram representing a static propagator is a double fermion path $\omega_0$ in the time direction (Fig.1). Since there is a non-zero probability for a transition from the lowest order state to a more complicate state, the full propagator is given by a sequence of exited states connected by lowest order states (Fig.2). Each double path $\omega \neq \omega_0$ can be viewed as a space-time process contributing to the excitation. These excitations renormalize the mass of the static propagator.

The unrenormalized mass of a meson can be found from the properties of the lowest order diagram of the static propagator which is given by

$$G_0^M(t) = C_M \times \exp(-m_0 t) = C_M \times (4k^2)^t.$$

(10)

for $r = 1$ in (4). This tells us that the unrenormalized mass is $m_0 = -log(4k^2)$. Having computed the unrenormalized static propagator, we will now compute the renormalization produced by intermediate exited states. Fig. 3 shows us such an excitation, with initial point $w$ and final point $z$, where the static unrenormalized propagator arrives and departs.
For fixed \(w\) and \(z\), we will sum over all intermediate exited states to obtain a total weight for the event. We denote this weight by \(D^{M}(w, z)\). The full propagator takes the form

\[
\langle M(x)\dagger M(y) \rangle = \sum_{w_i, z_i (i=1...n)} G_{0}^{M}(x, w_1)D^{M}(w_1, z_1)G_{0}^{M}(z_1, w_2)D^{M}(w_2, z_2) \times \\
\times \ldots \times D^{M}(w_n, z_n)G_{0}^{M}(z_n, y) \tag{11}
\]

where the points \(w_i\) and \(z_i\) are required to lie between \(x\) and \(y\) and to be ordered so that each \(z_i\) is "later" than \(w_i\), which is itself "later" than \(z_{i-1}\). This last equation represents the picture in Fig. 2.

In order to find the renormalized mass of the propagator, we consider the full static propagator

\[
G^{M}(t) = \sum_{x,y} \langle M(x)\dagger M(y) \rangle \bigg|_{y_0-x_0=t} \tag{12}
\]

where \(x_0\) and \(y_0\) represent the time component of the lattice points \(x\) and \(y\), respectively. Similarly we define

\[
\tilde{D}^{M}(t) = (G_{0}^{M}(t))^{-1} \sum_{x,y} D^{M}(x,y) \bigg|_{y_0-x_0=t} \tag{13}
\]

Here we have normalized the excitation relative to a static unrenormalized propagator over the time interval \(t\). From this definition and eq. (11) we obtain

\[
G^{M}(t) = e^{-m_0 t} \sum_{s_1, t_1 (i=1...n)} \tilde{D}^{M}(t_1 - s_1) \ldots \tilde{D}^{M}(t_n - s_n) = e^{-m_0 t} e^{p_M t} \tag{14}
\]

where \(t_i\) and \(s_i\) are the corresponding time coordinates of \(z_i\) and \(w_i\). The exponential representation of (14) is up to irrelevant boundary effect exact. So the renormalized mass \(m_M\) is

\[
m_M = m_0 - p_M. \tag{15}
\]

The leading order of \(p_M\) can just be written in the form

\[
p_M = \sum_{z=(t'\geq 0, \vec{z})} (G_{0}^{M}(t'))^{-1}D^{M}(0, z) = \sum_{z=(t'\geq 0, \vec{z})} (4k^2)^{-t'}D^{M}(0, z) \tag{16}
\]

To proof this last equation we have to insert it into the last term of (14) and to expand the exponential function containing \(p_M\), then we have to compare the result with the first term of (14). The excitation term \(D^{M}(w, z)\) is defined starting from eq. (8) in the following way:

\[
D^{M}(w, z) = \sum_{\omega: w \rightarrow z \rightarrow w} \frac{A(\omega)}{C_M} \tag{17}
\]

where the sum is over all closed paths \(\omega = \omega^+ \cup \omega^-\) from \(w\) to \(z\) and return. \(A(\omega)\) denotes the amplitude of an intermediate excitation diagram and \(C_M\) is the overall constant of eq. (8) and (10).
5 Goldstone boson mass

The critical hopping parameter $k_c(0)$ can be evaluated from the condition that the mass of the Goldstone boson (the pseudoscalar isotriplet) vanishes. The mass of the Goldstone boson is expressed in terms of a hopping parameter expansion series by eq. (15-17). The Goldstone boson is characterised by the pseudoscalar isotriplet bound state matrix $M = M_{Gb} \equiv \gamma_5 T^I$ for $I = 1, 2, 3$, where $T^I$ are the isospin SU(2) generators.

To evaluate the correction to the unrenormalized mass $m_0$ due to the excitation $p_M$ we have to identify all closed paths $\omega$ from the point 0 to the point $z$ and to sum over all points $z = (t \geq 0, \vec{z})$ with positive time in eq. (16). An excitation is a closed path. At $\beta = 0$ we do not have plaquettes in the expansion because the plaquette expectation value vanishes, therefore an excitation is a closed path consisting of a double line $\omega = \omega^+ \cup \omega^-$ with $\omega^- = -\omega^+$ composed by a set of connected bonds

$$\omega = \omega^+ \cup \omega^- = \left[ \bigcup_{b_i \in \omega^+} b_i \right] \cup \left[ \bigcup_{b_i \in \omega^+} (-b_i) \right]. \quad (18)$$

An excitation has to be irreducible, which means that cutting a double bond $b \cup -b$ of the path $\omega$ in the time direction, the resulting two paths have some time overlap. Reducible excitations can be split into several time separated excitations, therefore to avoid overcounting we have to consider only irreducible excitations.

Using the following two properties for $r = 1$ of the gamma matrices $\Gamma(b)\Gamma(b) = 2\Gamma(b)$ and $\Gamma(b)\Gamma(-b) = 0$ the reader can easily convince himself that the amplitude $A(\omega)$ of an excitation of the Goldstone boson can be expressed by a product of terms depending on the order of the bonds which compose the path $\omega$

$$A(\omega = \omega^+ \cup \omega^-) = C_{Mb} \times A_0 \times \prod_{b_i \subset \omega^+} F(b_i, b_{i+1}) k^2 \quad (19)$$

where $A_0$ and $C_{Mb}$ are some constants and

$$F(b_i, b_{i+1}) = \begin{cases} 
0 & \text{if } b_{i+1} = -b_i \\
2 & \text{if } b_{i+1} \perp b_i \\
4 & \text{if } b_{i+1} \parallel b_i 
\end{cases} \quad (20)$$

The hopping parameters is squared in (19) because the product is only on the half path $\omega^+$ and all the bonds of $\omega^- = -\omega^+$ are paired with bonds in $\omega^+$.

There are two classes of irreducible paths $\omega$. The first contains paths with all bonds $b_i$ perpendicular to the time direction. These paths are trivially irreducible, because they have no bonds in the time direction. It is very easy to identify and count them. The second class contains irreducible paths which have some bond parallel to the time direction (Fig. 4). These paths are very difficult to be counted because they are irreducible and therefore they can not be decomposed into a product of spatial excitations separated in time and time-like double bonds.

Because the function $F$ in (20) is positive we can however undercount and overcount the irreducible paths contributing to the sum in (17), by neglecting all paths of the second class and, respectively, considering all possible paths (reducible and irreducible) with bonds in
all directions. We obtain in this way an underestimation and an overestimation of $p_{M_{Gb}}$:

$$p_{M_{Gb}}^{\text{under}} = A_0 \times \sum_{S' \subset S} \prod_{b_i \in S'} C(b_i, b_{i+1}) F(b_i, b_{i+1}) k^2$$

$$p_{M_{Gb}}^{\text{over}} = A_0 \times \sum_{S' \subset S} \sum_{T' \subset T} \prod_{b_i \in S' \cup T'} C(b_i, b_{i+1}) F(b_i, b_{i+1}) k^2$$

where the set $S$ represents all bonds perpendicular to the time direction and the set $T$ all bonds parallel to the time direction. The term $(4k^2)^{-t'}$ in the sum (16) is not present in (21) because there is no time extension in the sum in (21). In (22) it is implicitly present in the overestimated form $(k^2)^{-t'}$. The mapping $C(b_i, b_{i+1})$ controls if the sequence of bonds forms a connected path

$$C(b_i, b_{i+1}) = \begin{cases} 0 & \text{if } b_{i+1} \text{ and } b_i \text{ are sequentially disconnected} \\ 1 & \text{if } b_{i+1} \text{ and } b_i \text{ are sequentially connected} \end{cases} \quad (23)$$

We define two bonds $b_i$ and $b_{i+1}$ to be sequentially connected if the end point of $b_i$ is the start point of $b_{i+1}$. It is easy to see that the set of all possible paths which contribute non-trivially to eq. (21-22) forms a 3-dimensional and, respectively, a 4-dimensional tree. The number of branches of these trees counts the number of different paths. For a $n$-dimensional tree each bond can be chosen perpendicular or parallel or antiparallel to the previous one. For all these possibilities the function $F$ we obtain for (21) and (22) two geometrical series

$$p_{M_{Gb}}^{\text{under}} = A_0 \sum_{n=1}^{\infty} (4(d-1)k^2)^n$$

$$p_{M_{Gb}}^{\text{over}} = A_0 \sum_{n=1}^{\infty} (4dk^2)^n$$

where $d=4$ is the dimension of the space-time lattice and $A_0 = \frac{1}{4}$ is evaluated explicitly from eq. (8).

The final result is an over and upper bound on the Goldstone boson mass. From eq. (15) we obtain

$$- \log(4k^2) - \frac{dk^2}{1 - 4dk^2} \leq m_{M_{Gb}}(k, \beta = 0) \leq - \log(4k^2) - \frac{(d-1)k^2}{1 - 4(d-1)k^2} \quad (26)$$

From the condition $m_{M_{Gb}} = 0$ we obtain two bounds on $k_c(\beta = 0)$

$$0.231 < k_c(\beta = 0) < 0.264. \quad (27)$$

This rigorous result supports the observed value in lattice Monte Carlo simulations.

The same method can be used for finding an upper and a lower bound on the masses of the

\footnote{We recall that considering in the sum (17) a reducible paths has the effect of overcounting some irreducible paths.}

\footnote{We consider, for example, eq. (22). For a given bond $b_i$ we can choose $b_{i+1}$ perpendicular ($2(d-1)$ possibilities) or parallel (1 possibility) or antiparallel (1 possibility) to $b_i$. Using the value of the function $F$ we obtain for these two bonds: $2 \times 2(d-1) + 4 \times 1 + 0 \times 1 = 4d$. Eq. (21) is analogous to eq. (22).}
\( \rho \) and \( \omega \) mesons. The \( \rho \) and \( \omega \) mesons are characterised by the matrices \( M_\rho = \gamma^j T^I \) (\( j = 1, 2, 3 \)) and \( M_\omega = \gamma^j \) (\( j = 1, 2, 3 \)), respectively. One has to find the corresponding functions \( F \) in eq. (20). For other mesons the method can not be applied because the function \( F \) vanishes.

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Figure Caption

1. A static unrenormalized propagator.

2. An excitation of the static unrenormalized propagator.

3. A close up of an intermediate excitation.

4. A typical irreducible excitation with some bond in the time direction.
This figure "fig1-1.png" is available in "png" format from:

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