HOMOGENEOUS METRIC ANR COMPACTA ARE DIMENSIONALLY FULL-VALUED

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Abstract. One of the problems accompanying the famous Bing-Borsuk conjecture \cite{3} is whether homogeneous metric ANR compacta are dimensionally full-valued. In the present paper we provide a positive answer to that problem.

1. Introduction

The Bing-Borsuk conjecture \cite{3} asserts that a homogeneous Euclidean neighborhood retract is a topological manifold. In the present paper we prove that one of the problems accompanying this conjecture (whether homogeneous metric ANR’s are dimensionally full-valued) has a positive solution. Our proofs are based on Theorem 1.1 below stating that the local cohomological structure of any \(n\)-dimensional homogeneous metric ANR compactum to some extend is similar to the local structure of \(\mathbb{R}^n\).

Everywhere in this paper by a space we mean a homogeneous metric ANR compactum \(X\) with \(\dim_G X = n\), where \(G\) is a fixed countable abelian group and \(n \geq 2\). Reduced Čech homology \(H_n(X; G)\) and cohomology groups \(H^n(X; G)\) with coefficient from \(G\) are considered everywhere below. Suppose \((K, A)\) is a pair of closed subsets of a space \(X\) with \(A \subset K\). Following \cite{3}, we say that \(K\) is an \(n\)-homology membrane spanned on \(A\) for an element \(\gamma \in H_n(A; G)\) provided \(\gamma\) is homologous to zero in \(K\), but not homologous to zero in any proper closed subset of \(K\) containing \(A\). Similarly, \(K\) is said to be an \(n\)-cohomology membrane spanned on \(A\) for an element \(\gamma \in H^n(A; G)\) if \(\gamma\) is not extendable over \(K\), but it is extendable over every proper closed subset of \(K\) containing \(A\). Here, \(\gamma \in H^n(A; G)\) is not extendable over \(K\) means that \(\gamma\) is not contained in the image \(j^n_{K,A}(H^n(K; G))\), where

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\end{itemize}
$j_{K,A}^n : H^n(K;G) \to H^n(A;G)$ is the homomorphism generated by the inclusion $A \hookrightarrow K$.

We note the following simple fact, which will be used in this paper and follows from Zorn’s lemma and the continuity of Čech cohomology \[20\]: If $A$ is a closed subset of a compact space $X$ and $\gamma$ is an element of $H^n(A;G)$ not extendable over $X$, then there exists an $n$-cohomology membrane for $\gamma$ spanned on $A$.

We also say that a closed set $A \subset X$ is a cohomological carrier of a non-zero element $\alpha \in H^n(A;G)$ if $j_{A,B}^n(\alpha) = 0$ for every proper closed subset $B \subset A$. If $H^n(X;G) \neq 0$, but $H^n(B;G) = 0$ for every closed proper subset $B \subset X$, then $X$ is called an $(n,G)$-bubble.

Here is our main result:

**Theorem 1.1.** Let $X$ be a homogeneous metric ANR compactum $X$ with $\dim_G X = n$, where $G$ is a countable Abelian group and $n \geq 2$. Then every point $x$ of $X$ has a basis $\mathcal{B}_x$ of open sets $U \subset X$ satisfying the following conditions:

1. $\overline{U}$ is an $(n-1)$-cohomology membrane spanned on $\text{bd} \overline{U}$ for any non-zero $\gamma \in H^{n-1}(\text{bd} \overline{U};G)$;
2. $H^{n-1}(\overline{U};G) = 0$ and $X \setminus \overline{U}$ is connected;
3. $H^{n-1}(\text{bd} \overline{U};G)$ is a non-trivial finitely generated group;
4. $\dim_G \text{bd} \overline{U} = n-1$ and $\text{bd} \overline{U}$ is an $(n-1,G)$-bubble.

**Corollary 1.2.** Any homogeneous metric ANR compactum $X$ with $\dim_G X = n$ has the following property $K(n)$: If a proper closed subset $K \subset X$ is an $(n-1)$-cohomology membrane spanned on $A$ for some $\gamma \in H^{n-1}(A;G)$, then $(K \setminus A) \cap X \setminus K = \emptyset$.

The property $K(n)$ implies the invariance of domain for homogeneous ANRs (see \[15\] and \[17\] for homogeneous or locally homogeneous ANR spaces $X$ with $\dim X = n$).

**Corollary 1.3.** Let $X$ be as in Theorem 1.1 and $f : U \to X$ be an injective map, where $U \subset X$ is open. Then $f(U)$ is also open in $X$.

Recall that a compactum $X$ is said to be dimensionally full-valued if $\dim(X \times Y) = \dim X + \dim Y$ for any compact space $Y$, or equivalently, $\dim_G X = \dim_{\mathbb{Z}} X$ for any abelian group $G$. Recent work of Bryant \[5\] was believed to provide a positive answer to the question whether any homogeneous metric ANR is dimensionally full-valued, but Bryant discovered a gap in the proof of one of the theorems from \[5\]. The question whether $\dim(X \times Y) = \dim X + \dim Y$ if both $X$ and $Y$ are homogeneous compact ANRs was raised in \[7\] and \[11\]. Theorem 1.4 below shows that both questions have a positive answer.
Theorem 1.4. Every homogeneous metric ANR-compactum is dimensionally full-valued.

2. Some preliminary results

In this section, if not stated otherwise, $X$ always denotes a homogeneous metric ANR compactum with $\dim_G X = n$, $n \geq 2$. If $H^n(X; G) \neq 0$, then $H^n(B; G) = 0$ for all proper closed subsets $B$ of $X$, see [22]. Obviously, this is true when $H^n(X; G) = 0$. Therefore, we can assume that all proper closed subsets of $X$ have trivial $n$-cohomology groups.

We begin with the following analogue of Theorem 8.1 from [3].

Proposition 2.1. Let $A \subset X$ be a closed set and $K$ be an $(n - 1)$-cohomology membrane spanned on $A$ for some $\gamma \in H^{n-1}(A; G)$. Then $(K \setminus A) \cap \overline{X \setminus K} = \emptyset$ provided $K$ is contractible in a proper subset of $X$.

Proof. It is easily seen that $K$ can be assumed to be a subset of a component of $X$. Since each component is also homogeneous ANR, we can suppose that $X$ is connected.

According to the duality between homology and cohomology for countable groups [12, viii 4G]), for any compact metric space $Y$ the groups $H_{n-1}(Y, G^*)$ and $H^{n-1}(Y; G)^*$ are isomorphic, where $G^*$ and $H^{n-1}(Y; G)^*$ denote the character groups of $G$ and $H^{n-1}(Y; G)$, respectively. Here $H^{n-1}(Y; G)$ and $G$ are considered as discrete groups. Using this duality, we can show that $K$ is an $(n - 1)$-homology membrane for some $\beta \in H_{n-1}(A, G^*)$ spanned on $A$. Indeed, consider the homomorphism $j_{K,A}^{n-1} : H^{n-1}(K; G) \to H^{n-1}(A; G)$. Since $\gamma$ is not extendable over $K$, $\gamma \notin G_A = j_{K,A}^{n-1}(H^{n-1}(K; G))$. Considering $H^{n-1}(A; G)$ as a discrete group, we can find a character $\beta : H^{n-1}(A; G) \to S^1$ such that $\beta(\gamma) \neq e$ and $\beta(G_A) = \exp e$, where $e$ is the unit of $S^1$. On the other hand, $\gamma$ is extendable over every proper closed subset $B$ of $K$ which contains $A$. Therefore, $\gamma$ is contained in the image of $j_{B,A}^{n-1} : H^{n-1}(B; G) \to H^{n-1}(A; G)$ for any such $B$. Then the composition $j_{K,A}^{n-1} \circ \beta$ is the trivial character of $H^{n-1}(K; G)$, while the composition $j_{B,A}^{n-1} \circ \beta$ is non-trivial for any proper closed subset $B$ of $K$ containing $A$. So, $\beta$ is homologous to zero in $K$, but not homologous to zero in any proper closed subset of $K$ containing $A$. Hence, $K$ is an $(n - 1)$-homology membrane for $\beta$ spanned on $A$.

Now, assume that $(K \setminus A) \cap \overline{X \setminus K} \neq \emptyset$. Then following the proof of Theorem 16.1 from [6] (see also [3, Theorem 8.1]), we can find a
proper closed subset $\Gamma$ of $X$ and a non-zero element $\alpha \in H_n(\Gamma; G^*)$. This means that $H^n(\Gamma; G) \neq 0$, a contradiction. □

Since the Bing-Borsuk result used in the proof of Proposition 2.1 was established for locally homogeneous spaces, Proposition 2.1 remains valid for locally homogeneous spaces $X$ such that $H^n(A; G) \neq 0$.

**Corollary 2.2.** Let $Z \subset X$ be a closed set. Then $\dim_G Z = n$ if and only if $Z$ has a non-empty interior in $X$. Moreover, $\dim_G F = n-1$ for every non-empty, closed and nowhere dense subset $F \subset X$ separating $X$.

**Proof.** The first statement was established by Seidel in [17] for the covering dimension. His arguments can be modified for $\dim_G$. If $\dim_G Z = n$, we may assume that $Z$ is contractible in proper closed subset of $X$ (this can be done because $X$ is locally contractible and $\dim_G$ satisfies the countable sum theorem). Since $\dim_G Z = n$, there exists a closed set $A \subset Z$ such that $H^n(Z, A; G) \neq 0$. On the other hand, $H^n(Z; G) = 0$ (as a proper closed subset of $X$). So, according to the exact sequence

$$H^{n-1}(Z; G) \xrightarrow{j_{Z,A}^{n-1}} H^{n-1}(A; G) \xrightarrow{\delta} H^n(Z, A; G) \to 0$$

there exists $\gamma \in H^{n-1}(A; G)$ not extendable over $Z$. Hence, as it was noted above, we can find a closed subset $K$ of $Z$ such that $K$ is an $(n-1)$-cohomological membrane for $\gamma$ spanned on $A$. So, by Proposition 2.1 $(K \setminus A) \cap \overline{X \setminus K} = \emptyset$. This means that $K \setminus A$ is open in $X$, and it is contained in $Z$. The other direction follows because $X$ is homogeneous and contains arbitrary small open sets $U$ with $\dim_G U = n$.

According to the first part of this corollary, $\dim_G F \leq n-1$ provided $F$ is nowhere dense. On the other hand, every non-empty separator of $X$ is of dimension $\geq n-1$, see [13]. Therefore, $\dim_G F = n-1$. □

**Corollary 2.3.** Let $A \subset X$ be a closed subset and $K$ an $(n-1)$-cohomology membrane for some $\gamma \in H^{n-1}(A; G)$ spanned on $A$. If $K$ is contractible in a proper closed subset of $X$, then $K \setminus A$ is a connected open subset of $X$.

**Proof.** By Proposition 2.1, $(K \setminus A) \cap \overline{X \setminus K} = \emptyset$. This implies that $K \setminus A$ is an open set in $X$. Suppose $K \setminus A$ is the union of two non-empty, disjoint open sets $U$ and $V$. Then $K \setminus U$ and $K \setminus V$ are closed proper subsets of $K$ such that $(K \setminus U) \cap (K \setminus V) \subset A$. Hence, $\gamma$ is extendable over each of these sets and, because $A$ contains their common part, $\gamma$
is extendable over $K$. The last conclusion contradicts the fact that $K$ is $(n - 1)$-cohomology membrane for $\gamma$. \hfill $\Box$

**Proposition 2.4.** Let $A \subset X$ be a closed subset of an arbitrary compactum $X$ such that $X \setminus A = U \cup V$, where $U$ and $V$ are disjoint open subsets of $X$. Then any $(n - 1)$-cohomology membrane in $X$ spanned on $A$ is contained either in $U \cup A$ or in $V \cup A$.

**Proof.** Suppose that a closed set $K \subset X$ is an $(n - 1)$-cohomology membrane spanned on $A$ for some $\gamma \in H^{n-1}(A;G)$. Consider the following diagram, where $B$ is a proper closed subset of $K$ containing $A$ and the vertical arrows are generated by the corresponding inclusions:

\[
\begin{array}{cccc}
H^{n-1}(K;G) & \to & H^{n-1}(A;G) & \to & H^n(K,A;G) \\
\downarrow i_1 & & \downarrow i_2 & & \downarrow i_B \\
H^{n-1}(B;G) & \to & H^{n-1}(A;G) & \to & H^n(B,A;G).
\end{array}
\]

Because $\gamma \not\in j_{K,A}^{n-1}(H^{n-1}(K;G))$, $\beta = \delta_1(\gamma)$ is a non-trivial element of $H^n(K,A;G)$. On the other hand, since $\gamma$ is extendable over $B$, $i_2(\gamma)$ belongs to $j_{B,A}^{n-1}(H^{n-1}(B;G))$. So, $\delta_2(i_2(\gamma)) = i_B(\beta) = 0$. In this way we show the existence of a non-zero element $\beta \in H^n(K,A;G)$ such that the image of $\beta$ under the homomorphism $i_B$ is trivial for every proper closed subset $B \subset K$ containing $A$.

Assume now that both $K_1 = K \cap (U \cup A)$ and $K_2 = K \cap (V \cup A)$ are proper non-empty subsets of $K$ ($U \cup A$ and $V \cup A$ are closed sets, so are $K_1$ and $K_2$). Since $A = K_1 \cap K_2$, we have the Mayer-Vietoris sequence (the group $G$ is not shown)

\[
H^{n-1}(A,A) \to H^n(K,A) \to \varphi H^n(K_1,A) \oplus H^n(K_2,A).
\]

Because $H^{n-1}(A,A;G) = 0$ and $\beta$ is a non-zero element of $H^n(K,A;G)$ with $\varphi(\beta) = (i_{K_1}(\beta), i_{K_2}(\beta)) = 0$, we have a contradiction. \hfill $\Box$

**Lemma 2.5.** Let $X$ be an arbitrary compactum and $A \subset X$ be a carrier for a non-zero element $\gamma \in H^{n-1}(A;G)$ with $\dim_G A \leq n - 1$, $n \geq 2$. Then $A$ is connected.

**Proof.** Suppose $A$ is not connected, so $A$ is the union of two closed disjoint non-empty sets $A_1$ and $A_2$. Then $H^{n-1}(A;G)$ is isomorphic to $H^{n-1}(A_1;G) \oplus H^{n-1}(A_2;G)$ and $\gamma$ is identified with the pair $(\gamma_1, \gamma_2)$, where $\gamma_i = j_{A_i}^{n-1}(\gamma)$, $i = 1, 2$. Because $A$ is a carrier of $\gamma$ and $A_i$ are proper closed non-empty subsets of $A$, $\gamma_1 = \gamma_2 = 0$. So, $\gamma = 0$, a contradiction. \hfill $\Box$
We say that a subset $A$ of a space $Z$ is an $(n-1,G)$-bubble with respect to a subgroup $L \subset H^{n-1}(Z;G)$ if the group $j_{Z,A}^{n-1}(L) \subset H^{n-1}(A;G)$ is non-trivial, but $j_{Z,B}^{n-1}(L) \subset H^{n-1}(B;G)$ is trivial for any closed proper subset $B \subset A$.

**Lemma 2.6.** If $A$ is a closed subset of an arbitrary compactum $X$ and $L \subset H^{n-1}(A;G)$ is a non-trivial and finitely generated subgroup, then $A$ contains a non-empty closed subset $F$ such that $F$ is an $(n-1,G)$-bubble with respect to $L$.

**Proof.** We are going to use an induction with respect to the number of generators of $L$. If $L$ has one generator $\gamma$, we just take a closed set $F \subset A$, which is a carrier for $\gamma$. Then $\beta = j_{A,F}^{n-1}(\gamma)$ and $\beta_B = j_{A,B}^{n-1}(\gamma)$ are generators, respectively, of $j_{A,F}^{n-1}(L) \subset H^{n-1}(F;G)$ and $j_{A,B}^{n-1}(L) \subset H^{n-1}(B;G)$ for any closed set $B \subset A$. So, $j_{A,B}^{n-1}(L) = 0$ for every proper closed subset $B$ of $F$ because $j_{A,B}^{n-1}(\gamma) = j_{F,B}^{n-1}(\beta) = 0$. Hence, $F$ is an $(n-1,G)$-bubble with respect to $L$. Suppose our lemma is true for any such set $A$ and a subgroup $L \subset H^{n-1}(A;G)$ with $\leq k$ generators. In case $L$ has $k+1$ generators $\gamma_1, \ldots, \gamma_k, \gamma_{k+1}$, we first take a closed non-empty set $F_1 \subset A$, which is a carrier for $\gamma_1$. So, $j_{A,B}^{n-1}(\gamma_1) = 0$ for any proper closed subset $B$ of $F_1$. If $H^{n-1}(B;G) = 0$ for all closed $B \subsetneq F_1$, then $F_1$ is as required. If $j_{A,B^*}^{n-1}(L) \neq 0$ for some closed proper set $B^* \subset F_1$, then $j_{A,B}^{n-1}(L)$ is generated by the set $\{j_{A,B}^{n-1}(\gamma_i) : i = 2, 3, \ldots, k+1\}$. Obviously $B^*$ satisfies the hypotheses of the lemma, so according to our inductive assumption, there exists a closed non-empty set $F \subset B^*$ being an $(n-1,G)$-bubble in $B^*$ with respect to $j_{A,B}^{n-1}(L)$. Consequently, $F$ is an $(n-1,G)$-bubble in $A$ with respect to $L$. \[\square\]

**Lemma 2.7.** Let $F, W$ be a closed and open, respectively, subsets of $X$ such that $F$ is contractible in $W$ and $\overline{W}$ is contractible in a proper closed subset of $X$. If $\overline{W}$ an $(n-1)$-cohomology membrane spanned on $bd\overline{W}$ for some non-zero $\alpha \in H^{n-1}(bd\overline{W};G)$, then the following conditions are equivalent:

- $F$ separates $X$;
- $F$ separates $W$;
- $H^{n-1}(F;G) \neq 0$.

**Proof.** Obviously, if $F$ separates $X$, then it separates $W$. Since $\overline{W}$ is an $(n-1)$-cohomology membrane spanned on $bd\overline{W}$ for $\alpha \in H^{n-1}(bd\overline{W};G)$, $\alpha$ (considered as a map from $bd\overline{W}$ to the Eilenberg-MacLane complex $K(G,n-1)$) is not extendable over $\overline{W}$ but it is extendable over any proper closed subset of $\overline{W}$. Hence, by [21] Proposition 2.10, the couple $(\overline{W}, bd\overline{W})$ is a strong $K^n_G$-manifold (see [21] for the definition of a strong[$\overline{?}$]}
Lemma 2.8. Let $X$ be an arbitrary compactum with $H^n(B;G) = 0$ for any closed proper subset $B \subset X$. Suppose $U \subset X$ is open and $P \nsubseteq X$ is closed such that $\overline{U} \subset P$ and $H^{n-1}(bd\overline{U};G)$ contains elements not extendable over $\overline{U}$. Then, there exists $\gamma \in H^{n-1}(bd\overline{U};G) \setminus L$ extendable over $P \setminus U$, where $L = j_{P\setminus U,bd\overline{U}}^{-1}(H^{n-1}(\overline{U};G))$. Moreover, if $L = 0$, then every $\gamma \in H^{n-1}(bd\overline{U};G)$ is extendable over $P \setminus U$.

Proof. Indeed, since $H^{n-1}(bd\overline{U};G)$ contains elements not extendable over $\overline{U}$, $L$ is a proper subgroup of $H^{n-1}(bd\overline{U};G)$. Consider the homomorphism $j_{P\setminus U,bd\overline{U}}^{-1}: H^{n-1}(P \setminus U;G) \rightarrow H^{n-1}(bd\overline{U};G)$. It suffices to show that the image of $H^{n-1}(P \setminus U;G)$ under $j_{P\setminus U,bd\overline{U}}^{-1}$ is not contained in $L$. To this end, suppose $j_{P\setminus U,bd\overline{U}}^{-1}(H^{n-1}(P \setminus U;G)) \subset L$. Consider the Mayer-Vietoris exact sequence, where $A = P \setminus U$ and $\varphi(\gamma_1, \gamma_2) = j_{A,bd\overline{U}}^{-1}(\gamma_2) - j_{U,bd\overline{U}}^{-1}(\gamma_1)$ for $\gamma_1 \in H^{n-1}(\overline{U};G)$, $\gamma_2 \in H^{n-1}(A;G)$:

$$H^{n-1}(\overline{U};G) \oplus H^{n-1}(A;G) \xrightarrow{\varphi} H^{n-1}(bd\overline{U};G) \xrightarrow{\Delta} H^n(P;G) \rightarrow$$

Obviously, $L_U = \varphi(H^{n-1}(\overline{U};G) \oplus H^{n-1}(A;G)) \subset L$. Consequently, any $\gamma \in H^{n-1}(bd\overline{U};G) \setminus L$ is not contained in $L_U$. Hence, $\Delta(\gamma) \neq 0$ for all $\gamma \in H^{n-1}(bd\overline{U};G) \setminus L$. So, $H^n(P;G) \neq 0$, a contradiction (recall that the $n$-th cohomology groups of all proper closed sets in $X$ are trivial).

If $L = 0$, then $j_{U,bd\overline{U}}^{-1}(\gamma_1) = 0$ for all $\gamma_1 \in H^{n-1}(\overline{U};G)$, so $\varphi(\gamma_1, \gamma_2) = j_{A,bd\overline{U}}^{-1}(\gamma_2)$. Since $\Delta(H^{n-1}(bd\overline{U};G)) = 0$, we obtain that for any $\gamma \in H^{n-1}(bd\overline{U};G)$ there exist $\gamma_1 \in H^{n-1}(\overline{U};G)$ and $\gamma_2 \in H^{n-1}(A;G)$ such that $\varphi(\gamma_1, \gamma_2) = \gamma$. Hence, $\gamma = j_{A,bd\overline{U}}^{-1}(\gamma_2)$, which means that $\gamma$ is extendable over $A$. This completes the proof.

Lemma 2.9. If $U \subset X$ is a connected open set and $\overline{U}$ is contractible in a proper closed subset of $X$, then $\overline{U}$ is an $(n-1)$-cohomology membrane spanned on $bd\overline{U}$ for every $\gamma \in H^{n-1}(bd\overline{U};G)$ not extendable over $\overline{U}$. 

$K^\infty_2$-manifold). Then, according to [21, Theorem 3.3], $H^{n-1}(F;G) \neq 0$ provided $F$ separates $W$. Finally, suppose $H^{n-1}(F;G) \neq 0$. Because $F$ is contractible in $W$, any non-zero element $\gamma \in H^{n-1}(F;G)$ is not extendable over $\overline{W}$ (otherwise $\gamma$, considered as a map from $F$ to $K(G, n-1)$, would be homotopic to a constant). This yields the existence of an $(n-1)$-cohomology membrane $K_\gamma \subset \overline{W}$ for $\gamma$ spanned on $F$. Because $\overline{W}$ is contractible in a proper closed subset of $X$, so is $K_\gamma$. Hence, by Proposition 2.1, $(K_\gamma \setminus F) \cap X \setminus K_\gamma = \emptyset$. The last equality implies that $F$ separates $X$. 

□
Proof. Observe first that $U$ is dense in $V = \text{Int}(\overline{U})$, so $V$ is also connected. Let $\gamma$ be an element of $H^{n-1}(\text{bd}U; G)$ not extendable over $\overline{U}$. Then there exists a closed subset $K \subset \overline{U}$ such that $K$ is an $(n - 1)$-cohomology membrane for $\gamma$ spanned on $\text{bd} U$. Since $K$ is contractible in a proper closed subset of $X$ (as a subset of $\overline{U}$), by Proposition 2.1 $(K \setminus \text{bd}U) \cap X \setminus K = \emptyset$. Hence, $K \setminus \text{bd}U$ is open in $X$. This implies that $K = \overline{U}$, otherwise $V$ would be the union of the non-empty disjoint open sets $V \setminus K$ and $(K \setminus \text{bd}U) \cap V$. Therefore, $\overline{U}$ is an $(n - 1)$-cohomology membrane spanned on $\text{bd}U$ for $\gamma$. \qed

3. Proof of Theorem 1.1 and Corollaries 1.2 - 1.3

We consider the following properties of a space $X$ and open subsets $U \subset X$, where $\text{bd}U$ denotes the boundary of $U$.

$M_1(n)$: $\dim_G \text{bd}U = n - 1$, $H^{n-1}(\text{bd}U; G) \neq 0$ and there exists $\gamma \in H^{n-1}(\text{bd}U; G)$ not extendable over $\overline{U}$.

$M_2(n)$: $H^{n-1}(\overline{U}; G) = 0$ and $\overline{U}$ is an $(n - 1)$-cohomology membrane spanned on $\text{bd}U$ for any non-zero $\gamma \in H^{n-1}(\text{bd}U; G)$.

$M_3(n)$: $H^{n-1}(\text{bd}U; G)$ is finitely generated and $\text{bd}U$ is an $(n - 1, G)$-bubble.

Proof of Theorem 1.1. As in the proof of Proposition 2.1, we may suppose that $X$ is connected and $H^n(C; G) = 0$ for any closed proper subset $C$ of $X$. Moreover, we equip $X$ with a convex metric $d$ generating its topology (such a metric exists, see [2]). According to [14], there exists a closed subset $Y \subset X$ and its open subset $D \subset Y$ with the following properties: $\dim_G Y = n$ and any $y \in D$ has sufficiently small neighborhoods $U_y$ in $Y$ such that the homomorphism $j^{n-1}_{U_y, \text{bd}U_y}$ is not surjective. Because $Y$ has a non-empty interior in $X$ (by Corollary 2.2), there exists a point $x \in \text{Int}(Y) \cap D$, its open connected neighborhood $W_x$ in $X$ and an element $\alpha_x \in H^{n-1}(\text{bd}W_x; G)$ such that $\alpha_x$ is not extendable over $\overline{W}_x$. We can suppose that $\overline{W}_x$ is contractible in a proper closed subset of $X$. So, by Lemma 2.9, $\overline{W}_x (n - 1)$-cohomology membrane for $\alpha_x$ spanned on $\text{bd} W_x$.

We define $\mathcal{B}_x'$ to be the family of all open connected subsets $U \subset X$ containing $x$ such that $U = \text{Int}(\overline{U})$ and $\overline{U}$ is contractible in $W_x$. Because $X$ is locally contractible, $\mathcal{B}_x'$ is a local base at $x$. Moreover, $\text{bd} U = \text{bd} \overline{U}$ for all $U \in \mathcal{B}_x'$.

Claim 1. Every $U \in \mathcal{B}_x'$ has the following properties:

(i) $\overline{U}$ is an $(n - 1)$-cohomology membrane for some element of $H^{n-1}(\text{bd}U; G)$ spanned on $\text{bd}U$;
(ii) the group \( L_U = j_{W_x \setminus U, bdU}^{n-1}(H^{n-1}(W_x \setminus U; G)) \subset H^{n-1}(bdU; G) \) is non-trivial and finitely generated;

(iii) the group \( H^{n-1}(bdU; G) \) is finitely generated provided the homomorphism \( j_{W, bdU}^{n-1} \) is trivial.

We fix \( U \in B_x' \) and a non-zero element \( \alpha_x \in H^{n-1}(bdW_x; G) \) such that \( W_x \) is an \((n-1)\)-cohomology membrane for \( \alpha_x \) spanned on \( bdW_x \). Then \( \alpha_x \) is not extendable over \( W_x \) but it is extendable over every closed proper subset of \( W_x \). Next, extend \( \alpha_x \) to an element \( \tilde{\alpha}_x \in H^{n-1}(W_x \setminus U; G) \). Obviously, \( bdU \subset W_x \setminus U \). Hence, the element \( \gamma_U = j_{W_x \setminus U, bdU}^{n-1}(\tilde{\alpha}_x) \in H^{n-1}(bdU; G) \) is not extendable over \( \overline{U} \) (otherwise \( \alpha_x \) would be extendable over \( W_x \)), in particular \( \gamma_U \neq 0 \). Since \( U \) is connected, by Lemma 2.9, \( U \) is an \((n-1)\)-cohomology membrane for \( \gamma_U \) spanned on \( bdU \).

To prove the second item \((ii)\), let \( U_0 \) be an open subset of \( X \) with \( \overline{U}_0 \subset U \). Since \( \gamma_U \in L_U \) and \( \gamma_U \neq 0 \), \( L_U \neq 0 \). For any \( \gamma \in L_U \) there are two possibilities: either \( \gamma \) is extendable over \( \overline{U} \) or it is not extendable over \( \overline{U} \). If \( \gamma \) is not extendable over \( \overline{U} \), then \( \overline{U} \) is an \((n-1)\)-cohomology membrane for \( \gamma \) spanned on \( bdU \) (Lemma 2.9). Hence, \( \gamma \) is extendable over the set \( \overline{U} \setminus U_0 \). So, every \( \gamma \in L_U \) is extendable over the set \( W_x \setminus U_0 \), which is closed in \( X \) and contains \( bdU \) in its interior. Therefore, by [4, Theorem 17.4, p.127], \( L_U \) is finitely generated. If \( j_{U, bdU}^{n-1}(H^{n-1}(U; G)) = 0 \), then every \( \gamma \in H^{n-1}(bdU; G) \) is extendable over \( W_x \setminus U \), see Lemma 2.8. Hence, \( H^{n-1}(bdU; G) \subset L_U \), and item \((ii)\) yields item \((iii)\).

According to Corollary 2.2, \( \dim_G bdU = n-1 \) for all such \( U \in B_x' \). Hence, by Claim 1, each \( U \in B_x' \) has \( M_1(n) \).

Let \( B_x'' \) be the family of all \( U \in B_x' \) satisfying the following condition: \( bdU \) contains a continuum \( F_U \) such that \( X \setminus F_U \) has exactly two components.

**Claim 2.** \( B_x'' \) is a local base at \( x \).

We fix \( W_0 \in B_x' \) and for every \( \delta > 0 \) denote by \( B(x, \delta) \) the open ball in \( X \) with center \( x \) and radius \( \delta \). There exists \( \varepsilon_x > 0 \) such that \( B(x, \delta) \subset W_0 \) for all \( \delta \leq \varepsilon_x \). Since \( d \) is a convex metric, each \( B(x, \delta) \) is a connected open set such that \( Int(B(x, \delta)) = B(x, \delta) \). Because \( W_0 \) is contractible in \( W_x \), so is \( B(x, \delta) \). Hence, all \( U_\delta = B(x, \delta), \delta \leq \varepsilon_x \), belong to \( B_x' \). Consequently, by Claim 1, the groups \( L_\delta = j_{W_x \setminus U_\delta, bdU_\delta}^{n-1}(H^{n-1}(W_x \setminus U_\delta; G)) \) are finitely generated. Then, by Lemma 2.6, there exists a closed non-empty set \( F_\delta \subset bdU_\delta \) with \( F_\delta \) being an \((n-1; G)\)-bubble.
with respect to $L_\delta$. Because $F_\delta$ is a carrier for any $\gamma \in L_\delta$, Lemma 2.5 yields that each $F_\delta$ is a continuum. Let us show that the family $\{F_\delta : \delta \leq \varepsilon_x\}$ is uncountable. Since the function $f : X \to \mathbb{R}$, $f(y) = d(x, y)$, is continuous and $W_0$ is connected, $f(W_0)$ is an interval containing $[0, \varepsilon_x]$ and $f^{-1}([0, \varepsilon_x)) = B(x, \varepsilon_x) \subset W_0$. So, $f^{-1}(\delta) = bdU_\delta \neq \emptyset$ for all $\delta \leq \varepsilon_x$. Hence, the family $\{F_\delta : \delta \leq \varepsilon_x\}$ is indeed uncountable and consist of disjoint continua. Moreover, $H^{n-1}(F_\delta; G) \neq 0$ and, according to Lemma 2.7, $F_\delta$ separates $X$. So, each $X \setminus F_\delta$ has at least two components. Then, by [8, Theorem 8], there exists $\delta_0 \leq \varepsilon_x$ such that $X \setminus F_{\delta_0}$ has exactly two components. Therefore, $U_{\delta_0} = B(x, \delta_0) \in \mathcal{B}_x$ and it is contained in $W_0$. This completes the proof of Claim 2.

Now, let $\mathcal{B}_x$ be the subfamily of all $U \in \mathcal{B}_x''$ such that both $U$ and $X \setminus \overline{U}$ are connected.

Claim 3. $\mathcal{B}_x$ is a local base at $x$.

We fix $U_0 \in \mathcal{B}_x''$ with $U_0 \subset W_x$ and choose $W_1 \in \mathcal{B}_x''$ such that $\overline{W}_1 \subset U_0$ (this is possible because $\mathcal{B}_x''$ is a local base at $x$). Let $\varepsilon = d(\overline{W}_1, X \setminus U_0)$. According to the Effros’ theorem [10], there is $\eta > 0$ such that if $y, z \in X$ with $d(y, z) < \eta$, then $h(y) = z$ for some homeomorphism $h : X \to X$, which is $\varepsilon$-close to the identity on $X$. Now, take $U, W \in \mathcal{B}_x''$ such that $\overline{U}$ is contractible in $W$, $\overline{W} \subset W_1$ and $\text{diam}(\overline{W}) < \eta$. There exists a continuum $F_U \subset bdU$ such that $X \setminus F_U$ has exactly two components and $F_U$ is an $(n - 1, G)$-bubble with respect to the group $L_U = j^{n-1}_{\overline{W}_1 \setminus U, bdU}(H^{n-1}(\overline{W}_1 \setminus U; G))$ (see the proof of Claim 2). If $F_U = bdU$, we are done because $U \subset U_0$ and both $U$ and $X \setminus \overline{U}$ are connected. Suppose that $F_U$ is a proper subset of $bdU$. Then there exists $\gamma \in L_U$ such that $\beta = j^{n-1}_{bdU, F_U}(\gamma) \neq 0$. Because $F_U$ separates $X$, it also separates $\overline{W}$. So, $\overline{W} \setminus F_U = V_1 \cup V_2$ for some open, non-empty disjoint subsets $V_1, V_2 \subset \overline{W}$. Since $U$ is a connected subset of $\overline{W} \setminus F_U$, $U$ is contained in one of the sets $V_1, V_2$, say $U \subset V_1$. Hence, $F_U \cup V_2 \subset \overline{W} \setminus U$. Observe that $\gamma \in L_U$ implies $\gamma$ is extendable over $\overline{W_x} \setminus U$. Consequently, $\beta$ is also extendable over $\overline{W_x} \setminus U$, in particular $\beta$ is extendable over $F_U \cup V_2$. On the other hand, $F_U$ (as a subset of $\overline{U}$) is contractible in $\overline{W}$, so $\beta$ is not extendable over $\overline{W}$ (otherwise $\beta$ would be zero). Thus, since $(F_U \cup V_1) \cap (F_U \cup V_2) = F_U$, $\beta$ is not extendable over $F_U \cup \overline{V}_1$. Let $\beta' = j^{n-1}_{F_U, F'}(\beta)$, where $F' = \overline{V}_1 \cap F_U$ (observe that $F' \neq \emptyset$ because $\overline{W}$ is connected). If $F'$ is a proper subset of $F_U$, then $\beta' = 0$ (recall that $j^{n-1}_{bdU, F'}(\gamma) = \beta'$ and $F_U$ being a carrier for any element of $L_U$ yields $j^{n-1}_{bdU, Q}(L_U) = 0$ for any proper closed subset $Q$ of $F_U$). So, $\beta'$ would be extendable over $\overline{V}_1$, which yields $\beta$ is extendable
over $F_U \cup \overline{V}_1$, a contradiction. Therefore, $F' = F_U \subset \overline{V}_1$ and $\beta$ is not extendable over $\overline{V}_1$. Consequently, there exists an $(n-1)$-cohomology membrane $P_{\beta} \subset \overline{V}_1$ for $\beta$ spanned on $F_U$. By Corollary 2.3, $V = P_{\beta} \setminus F_U$ is a connected open set in $X$ whose boundary, according to Proposition 2.1, is the set $F'' = X \setminus P_{\beta} \cap \overline{P_{\beta} \setminus F_U} \subset F_U$ (we can apply Proposition 2.1 and Corollary 2.3 because $P_{\beta}$, as a subset of $\overline{W}$, is contractible in a proper closed subset of $X$). As above, using that $\beta$ is not extendable over $P_{\beta}$ and $J_{bdU;Q}(L_U) = 0$ for any proper closed subset $Q \subset F_U$, we can show that $F'' = F_U$ and $bd\overline{V} = F_U$. Summarizing the properties of $V$, we have that $\overline{V}$ is contractible in $W_\varepsilon$ (because $\overline{V} \subset W \subset W_1$), $V = \text{Int}(\overline{V})$ (because $F_U = bd\overline{V}$) and $V$ is connected. Moreover, since $X \setminus F_U$ is the union of the open disjoint non-empty sets $V$, $X \setminus P_{\beta}$ with $V$ being connected and $X \setminus F_U$ has exactly two components, $X \setminus \overline{V}$ is also connected. If $V$ does not contain $x$, we take a point $y \in V$ and a homeomorphism $h$ on $X$ such that $h(y) = x$ and $d(z, h(z)) < \varepsilon$ for all $z \in X$. Such a homeomorphism exists because $\text{diam}(\overline{V}) < \eta$ and $x, y \in \overline{W}$. Then $h(V) \subset U_0$ (this inclusion follows from the choice of $\varepsilon$ and the fact that $h$ is $\varepsilon$-close to the identity on $X$). So $h(V)$ is contractible in $W_\varepsilon$ and both $h(V)$ and $X \setminus h(V)$ are connected. Consequently, $h(V) \in \mathcal{B}_x$, and we conclude that $\mathcal{B}_x$ is a base at the point $x$. This completes the proof of Claim 3.

Claim 4. Every $U \in \mathcal{B}_x$ has the properties $M_2(n)$ and $M_3(n)$.

Since each $\overline{U}$ is contractible in the set $\overline{W}_x$, for any non-zero element $\gamma \in H^{n-1}(bdU;G)$ there exists an $(n-1)$-cohomology membrane $P_\gamma \subset \overline{W}_x$ for $\gamma$ spanned on $bdU$. Because $P_\gamma$, as a subset of $\overline{W}_x$, is contractible in a proper closed subset of $X$, $P_\gamma \setminus bdU$ is open in $X$ (by Proposition 2.1). According to Proposition 2.4, $P_\gamma$ is contained either in $\overline{U}$ or in $X \setminus U$. First, consider the case $P_\gamma$ is contained in $X \setminus U$. Since $X \setminus \overline{U}$ is connected, we obtain a contradiction because $P_\gamma \setminus bdU$ and $X \setminus (P_\gamma \cup \overline{U})$ are non-empty open disjoint sets whose union is $X \setminus \overline{U}$. Therefore, $P_\gamma \subset \overline{U}$, and $\gamma$ is not extendable over $\overline{U}$. Hence, by Lemma 2.9, $\overline{U}$ is an $(n-1)$-cohomology membrane for $\gamma$ spanned on $bdU$. Moreover, $\overline{U}$ does not separate $X$ (recall that $X \setminus \overline{U}$ is connected). Then, by Lemma 2.7, $H^{n-1}(\overline{U};G) = 0$. So, $U \in M_2(n)$.

We actually proved in the previous paragraph that the homomorphism $j_{\overline{U};bdU}^{-1}$ is trivial for any $U \in \mathcal{B}_x$. Thus, by Lemma 2.8 we have $H^{n-1}(bdU;G) = j_{\overline{W}_x;U;bdU}^{-1}(H^{n-1}(\overline{W}_x \setminus U;G))$ and, by Claim 1(ii), $H^{n-1}(bdU;G)$ is finitely generated. Suppose there exists a proper
closed subset \( F \subset \text{bd}U \) and a non-trivial element \( \alpha \in H^{n-1}(F; G) \). Observe that \( \alpha \) is not extendable over \( \overline{U} \) because \( H^{n-1}(\overline{U}; G) = 0 \). Hence, there is an \((n-1)\)-cohomology membrane \( K_\alpha \subset \overline{U} \) for \( \alpha \) spanned on \( F \). Because \( K_\alpha \setminus F \) is open in \( X \) (Proposition 2.1) and \( U \) is connected, it follows from the proof of Lemma 2.9 that \( K_\alpha = \overline{U} \). Finally, according to Proposition 2.1, we have \((K_\alpha \setminus F) \cap X \setminus K_\alpha = \emptyset \). On the other hand, any point from \( \text{bd}U \setminus F \) belongs to \((K_\alpha \setminus F) \cap X \setminus K_\alpha \), a contradiction. Therefore, \( \text{bd}U \) is an \((n-1, G)\)-bubble and \( U \in M_3(n) \).

Combining the above claims, we obtain the proof of Theorem 1.1. □

**Proof of Corollary 1.2.** Suppose \( K \) is a proper closed subset of \( X \), which is an \((n-1)\)-cohomology membrane spanned on \( A \) for some \( \gamma \in H^{n-1}(A; G) \), but there exists a point \( a \in (K \setminus A) \cap \overline{X \setminus K} \). Take a neighborhood \( U \in B_a \) such that \( \overline{U} \cap A = \emptyset \). Since \( K \setminus U \) is a closed proper subset of \( K \) containing \( A \), \( \gamma \) is extendable over \( K \setminus U \). So, there exists \( \beta \in H^{n-1}(K \setminus U; G) \) with \( j_{K \setminus U, A}^{n-1}(\beta) = \gamma \). Then \( \beta_1 = j_{K \setminus U, \text{bd}U \cap K}^{n-1}(\beta) \) is a non-zero element of \( H^{n-1}(\text{bd}U \cap K; G) \) (otherwise \( \beta_1 \) would be extendable over \( \overline{U} \cap K \), and hence, \( \gamma \) would be extendable over \( K \)). Since \( \dim_G \text{bd}U = n - 1 \), there exists a non-zero element \( \alpha \in H^{n-1}(\text{bd}U; G) \) extending \( \beta_1 \). Because \( U \) has property \( M_3(n) \), \( \overline{U} \) is an \((n-1)\)-cohomology membrane for \( \alpha \) spanned on \( \text{bd}U \). Finally, using that \( \text{bd}U \cup (\overline{U} \cap K) \) is a closed proper subset of \( \overline{U} \), we extend \( \alpha \) over \( \text{bd}U \cup (\overline{U} \cap K) \). This yields that \( \beta_1 \) is extendable over \( \overline{U} \cap K \), a contradiction. Hence \((K \setminus A) \cap \overline{X \setminus K} = \emptyset \). □

**Proof of Corollary 1.3.** Take a point \( y \in V = f(X) \) and let \( x = f^{-1}(y) \). Choose a connected open set \( W \in B_x \) such that \( \overline{W} \subset U \) and \( \overline{W} \) is an \((n-1)\)-cohomology membrane for some \( \gamma \in H^{n-1}(\text{bd}W; G) \) spanned on \( \text{bd}W \). Then \( f(\overline{W}) \) is homeomorphic to \( \overline{W} \), so it is an \((n-1)\)-cohomology membrane for \( (f^*)^{-1}(\gamma) \in H^{n-1}(f(\text{bd}W); G) \) spanned on \( f(\text{bd}W) \). Since \( X \in K(n) \), \( f(W) \setminus f(\text{bd}W) \) does not intersect \( X \setminus f(\overline{W}) \). This means that \( f(\overline{W}) \setminus f(\text{bd}W) \) is an open set in \( X \), which contains \( y \) and is contained in \( V \). So, \( V \) is also open. □

4. Proof of Theorem 1.4

Suppose \( X \) is a homogeneous compact metric ANR-space of dimension \( n \). According to Theorem 1.1, every point \( x \in X \) has a basis \( B_x = \{U_k\}_{k \geq 1} \) consisting of open sets such that \( \overline{U}_1 \) is a proper subset of \( X \), \( \overline{U}_{k+1} \subset U_k \), \( H^{n-1}(\overline{U}_k) = H^n(\overline{U}_k) = 0 \) and \( H^{n-1}(\text{bd}U_k) \) is a non-trivial finitely generated group (everywhere in this section the coefficient group \( \mathbb{Z} \) in all homology and cohomology groups is suppressed).
Let \( \hat{H}_n \) denote the exact homology (see [16], [18]) and \( Q_1 = Q/\mathbb{Z} \), where \( Q \) is the group of rational numbers. Note that for any group \( G \) and a closed set \( A \subset X \) we have \( \hat{H}_n(X, A; G) \simeq H_n(X, A; G) \) provided \( \dim X = n \) [18].

**Lemma 4.1.** All groups \( \hat{H}_{n-1}(bd\overline{U}_k; Q_1) \), \( k \geq 2 \), are non-trivial.

**Proof.** For any abelian group \( G \) the homological dimension \( h \dim_G Y \) of a space \( Y \) is the greatest integer \( m \) such that \( H_m(Y, A; G) \neq 0 \) for some closed \( A \subset Y \) (if there is no such \( m \), then \( h \dim_G Y = \infty \)). It is well known [1] that for a finite dimensional metric compacta \( Y \) we have \( h \dim_{Q_1} Y = \dim Y \). Moreover, if \( h \dim_G Y < \infty \), then \( h \dim_G Y \) is the greatest \( m \) such that the local homology group \( \hat{H}_m(Y, Y \setminus y; G) = \lim_{n \to \infty} \hat{H}_m(Y, Y \setminus V; G) \) is not trivial for some \( y \in Y \) [19]. Therefore, there exists \( x \in X \) such that \( \hat{H}_n(X, X \setminus x; Q_1) = \lim_{n \to \infty} \hat{H}_n(X, X \setminus U_k; Q_1) \neq 0 \).

So, by the excision axiom, we may assume that all groups \( \hat{H}_n(\overline{U}_1, \overline{U}_1 \setminus U_k; Q_1), k \geq 2 \), are non-trivial. For every \( k \geq 1 \) we have the following exact sequences, see [18]:

\[
0 \to \Ext(H^n(\overline{U}_k), Q_1) \to \hat{H}_{n-1}(\overline{U}_k; Q_1) \to \Hom(H^{n-1}(\overline{U}_k), Q_1) \to 0
\]

and

\[
0 \to \Ext(H^{n+1}(\overline{U}_k), Q_1) \to \hat{H}_n(\overline{U}_k; Q_1) \to \Hom(H^n(\overline{U}_k), Q_1) \to 0.
\]

Since \( H^{n-1}(\overline{U}_k) = H^n(\overline{U}_k) = H^{n+1}(\overline{U}_k) = 0 \), \( \hat{H}_{n-1}(\overline{U}_k, Q_1) = 0 \) and \( \hat{H}_n(\overline{U}_k, Q_1) = 0 \). Hence, it follows from the Mayer-Vietoris sequence (the coefficient group \( Q_1 \) is suppressed)

\[
\to \hat{H}_n(\overline{U}_1) \to \hat{H}_{n-1}(bd\overline{U}_k) \to \hat{H}_{n-1}(\overline{U}_1 \setminus U_k) \oplus \hat{H}_{n-1}(\overline{U}_k) \to \hat{H}_{n-1}(\overline{U}_1) \to
\]

that \( \hat{H}_{n-1}(bd\overline{U}_k; Q_1) \simeq \hat{H}_{n-1}(\overline{U}_1 \setminus U_k; Q_1) \). Similarly, the exact sequence

\[
\to \hat{H}_n(\overline{U}_1) \to \hat{H}_n(\overline{U}_1, \overline{U}_1 \setminus U_k) \to \hat{H}_n(\overline{U}_1 \setminus U_k) \to \hat{H}_{n-1}(\overline{U}_1) \to
\]

yields \( \hat{H}_n(\overline{U}_1, \overline{U}_1 \setminus U_k; Q_1) \simeq \hat{H}_{n-1}(\overline{U}_1 \setminus U_k; Q_1) \). Therefore, for all \( k \geq 2 \) the groups \( \hat{H}_{n-1}(bd\overline{U}_k; Q_1) \) and \( \hat{H}_n(\overline{U}_1, \overline{U}_1 \setminus U_k; Q_1) \) are non-trivial and isomorphic to each other. \( \square \)

**Lemma 4.2.** Suppose some \( \mathcal{B}_x \) contains an open set \( U \subset X \) such that the group \( H^{n-1}(bd\overline{U}) \) is finite. Then \( \dim X - \dim_{\mathbb{Q}} X \leq 1 \).

**Proof.** Suppose \( \dim X - \dim_{\mathbb{Q}} X \geq 2 \), so \( \dim_{\mathbb{Q}} X \leq n - 2 \). Then, since the interior of \( bd\overline{U} \) is empty, Corollary 2.2 implies \( \dim_{\mathbb{Q}} bd\overline{U} \leq n - 3 \). Hence, \( H^{n-2}(bd\overline{U}; Q) = 0 \). Consider the exact sequence

\[
0 \to \Ext(H^n(bd\overline{U}), Q) \to \hat{H}_{n-1}(bd\overline{U}; Q) \to \Hom(H^{n-1}(bd\overline{U}), Q) \to 0
\]
Since \( \dim \partial U = n - 1 \), \( H^n(\partial U) = 0 \). The group \( \text{Hom}(H^{n-1}(\partial U), \mathbb{Q}) \) is also trivial because \( H^{n-1}(\partial U) \) is torsion (as a finite group). Thus, \( \hat{H}_{n-1}(\partial U; \mathbb{Q}) = 0 \). Next, the exact sequence
\[
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}_1 \to 0
\]
yields the exact homology sequence (see [16])
\[
0 \to \hat{H}_{n-1}(\partial U; \mathbb{Q}) \to \hat{H}_{n-1}(\partial U; \mathbb{Q}_1) \to \hat{H}_{n-2}(\partial U) \to
\]
According to Lemma 4.1, \( \hat{H}_{n-1}(\partial U; \mathbb{Q}_1) \neq 0 \), so \( \hat{H}_{n-2}(\partial U) \neq 0 \). Then, it follows from the exact sequence
\[
0 \to \text{Ext}(H^{n-1}(\partial U), \mathbb{Z}) \to \hat{H}_{n-2}(\partial U) \to \text{Hom}(H^{n-2}(\partial U), \mathbb{Z}) \to 0
\]
that there exists a non-trivial homomorphism \( \varphi : H^{n-2}(\partial U) \to \mathbb{Z} \). This means that \( H^{n-2}(\partial U) \) contains an isomorphic copy of \( \mathbb{Z} \) as a direct summand. Therefore, \( H^{n-2}(\partial U) \otimes \mathbb{Q} \neq 0 \), and the universal coefficients formula
\[
0 \to H^{n-2}(\partial U) \otimes \mathbb{Q} \to H^{n-2}(\partial U; \mathbb{Q}) \to
\]
implies \( H^{n-2}(\partial U; \mathbb{Q}) \neq 0 \), a contradiction. \( \square \)

**Proof of Theorem 1.4.** It is well known that \( \dim_{\mathbb{Q}} Y \leq \dim_{\mathbb{G}} Y \) for any ANR compactum \( Y \) and any abelian group \( \mathbb{G} \), see [9, Theorem 12.3]. So, it suffices to show that \( \dim_{\mathbb{Q}} X = n \). Striving for a contradiction, assume that \( \dim_{\mathbb{Q}} X \leq n - 1 \), and consider \( X^2 \). Obviously, \( X^2 \) is also a homogeneous ANR-compactum. Moreover, \( \dim X^2 = 2n \) and \( \dim_{\mathbb{Q}} X^2 \leq 2n - 2 \), see Proposition 3.3 and Corollary 12.4 from [9]. Since \( \dim X^2 - \dim_{\mathbb{Q}} X^2 \geq 2 \), it follows from Lemma 4.2 and Theorem 1.1 that every point of \( X^2 \) has a basis of open sets \( W_k \subset X^2 \) such that \( \dim \partial W_k = 2n - 1 \) and each group \( H^{2n-1}(\partial W_k) \) is finitely generated and contains elements of infinite order. Hence, by the universal coefficients formula, \( H^{2n-1}(\partial W_k; \mathbb{Q}) \neq 0 \). So, \( \dim_{\mathbb{Q}} \partial W_k = 2n - 1 \). This implies that \( \dim_{\mathbb{Q}} X^2 \geq \dim_{\mathbb{Q}} \partial W_k = 2n - 1 \), a contradiction. Thus, \( X \) is dimensionally full-valued. \( \square \)

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