A SUFFICIENT AND NECESSARY CONDITION FOR 
A-QUASIAFFINITY

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Abstract. We consider a homogeneous, constant rank differential operator $A$ and prove a characterisation theorem for $A$-quasiaffine functions in the spirit of Ball, Currie and Olver $[BCO81]$; i.e. functions such that

$$f(v) = \int_{T_N} f(v + \psi(y)) \, dy$$

for all $v$ and all $A$-free test functions $\psi$ with zero mean. This result is used to get a sufficient, but not necessary condition for the differential operator $A$, such that linearity along the characteristic cone of $A$ implies $A$-quasiaffinity. We show that this implication is true if $A$ admits a first order potential.

1. Introduction

1.1. $A$-quasiaffine functions. Let us consider a homogeneous differential operator $A: C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l)$ of order $k_A$, given by

$$(1.1) \quad Au = \sum_{|\alpha|=k_A} A_\alpha \partial_\alpha u,$$

where $A_\alpha \in \text{Lin}(\mathbb{R}^d, \mathbb{R}^l)$ are constant coefficients. We call $f: \mathbb{R}^d \to \mathbb{R}$ $A$-quasiaffine if for all $A$-free test functions on the $N$-torus, i.e. $\psi \in C^\infty(T_N, \mathbb{R}^d)$ with $A\psi = 0$ and $\int_{T_N} \psi = 0$, and all $v \in \mathbb{R}^d$

$$(1.2) \quad f(v) = \int_{T_N} f(v + \psi(y)) \, dy.$$ 

In this work, we prove a sufficient and necessary condition for a function $f$ to be $A$-quasiaffine depending on derivatives of $f$.

Let us shortly recall, that for differential operators as in $(1.1)$ and $\xi \in \mathbb{R}^N \setminus \{0\}$, the Fourier symbol $A[\xi]$ of $A$ is defined as

$$A[\xi] = \sum_{|\alpha|=k_A} A_\alpha \xi^\alpha \in \text{Lin}(\mathbb{R}^d, \mathbb{R}^l).$$

We assume that $A$ satisfies the following two conditions:

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(a) **Constant rank property:** The rank of the linear operator $A[\xi]$ is constant, i.e. there is $r \in \mathbb{N}$, such that for all $\xi \in \mathbb{R}^N \setminus \{0\}$

$$\dim \ker A[\xi] = r;$$

(b) **Spanning property:**

$$\text{span} \left\{ \bigcup_{\xi \in \mathbb{R}^N \setminus \{0\}} \ker A[\xi] \right\} = \mathbb{R}^d.$$ 

Recently, Raiţă proved another characterisation of constant rank operators \cite{Ra19}. Namely $A$ has constant rank if and only if it admits a potential $B$: $C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d)$, i.e. a differential operator $B$, such that its Fourier symbol $B[\xi]$ satisfies

$$\text{Im } B[\xi] = \ker A[\xi]$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$. Using this result, we are able to give several equivalent conditions of what it means to be an $A$-quasiaffine function.

**Theorem A.** Let $f: \mathbb{R}^d \to \mathbb{R}$ and let $A$ satisfy the constant rank property and the spanning property. Moreover, let $B$ be a potential of $A$. Then the following statements are equivalent.

(a) $f$ is $A$-quasiaffine;

(b) $f$ is a polynomial and $\forall x \in \mathbb{R}^d$, $\forall r \geq 2$, $\forall \xi_1, \ldots, \xi_r \in \mathbb{R}^d$ which are linear dependent and $\forall v_1, \ldots, v_r \in \mathbb{R}^d$ with $v_i \in \ker A[\xi_i]$ we have

$$D^r f(x)[v_1, \ldots, v_r] = 0;$$

Moreover, we have the following equivalent conditions (cf. \cite{GR19})

(c) $f$ is $C^1$ and the Euler-Lagrange equation

$$B^T (\nabla f(Bu)) = 0$$

is satisfied in the sense of distributions $\forall u \in C^k_{\text{bs}}(\overline{\Omega})$, i.e. for all $\varphi \in C^\infty_c(\Omega, \mathbb{R}^m)$ we have

$$\int_{\Omega} \nabla f(Bu) \cdot B \varphi = 0;$$

(d) The map $u \to f(u)$ is sequentially weak* continuous from $L^\infty(\Omega, \mathbb{R}^d) \cap \ker A$ to $L^\infty(\Omega, \mathbb{R}^d)$, i.e. if $u_n \in L^\infty(\Omega, \mathbb{R}^d)$ with $Au_n = 0$ and $u_n \rightharpoonup^* u$ in $L^\infty(\Omega, \mathbb{R}^d)$, then also $f(u_n) \rightharpoonup^* f(u)$ in $L^\infty(\Omega, \mathbb{R}^d)$;

(e) $f$ is a polynomial of degree $s \leq d$ and the map $u \to f(u)$ is sequentially weakly continuous from $L^s(\Omega, \mathbb{R}^d)$ to $D'(\Omega)$ (the space of distributions on $\Omega$), i.e. if $u_n \in L^s(\Omega, \mathbb{R}^d)$ with $Au_n = 0$ and $u_n \rightharpoonup u$ in $L^s(\Omega, \mathbb{R}^d)$, then

$$\lim_{n \to \infty} \int_{\Omega} f(u_n) \varphi = \int_{\Omega} f(u) \varphi \quad \forall \varphi \in C^\infty_c(\Omega).$$
We prove that condition (b) can in fact be weakened, indeed we may only consider 
$$2 \leq r \leq \min\{N, k_B\} + 1$$ instead of $$r \geq 2$$ (cf. Theorem 3.6). This can be used to show the following result.

**Theorem B.** Let $\mathcal{A}$ be a constant rank operator satisfying the spanning property and let $\mathcal{B}$ be a potential of $\mathcal{A}$. Suppose that $\mathcal{B}$ is of order one. Then $f: \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{A}$-quasiaffine if and only if $f$ is $\Lambda_{\mathcal{A}}$-affine, i.e. for all $v \in \mathbb{R}^d$ and $w \in \Lambda_{\mathcal{A}} = \bigcup_{\xi \in \mathbb{R}^N \setminus \{0\}} \ker \mathcal{A}[\xi]$ we have that 
$$t \mapsto f(v + tw)$$ is affine.

1.2. *Quasiaffinity and Null-Lagrangians.* Let us give a few historical remarks, before finishing the introduction with a few applications of $\mathcal{A}$-quasiaffinity. The study of $\mathcal{A}$-quasiaffine functions started with the operator $\mathcal{A} = \text{curl}$, i.e. with the potential operator $\mathcal{B} = \nabla: C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^N \times m)$. These curl-quasiaffine functions are also often called *Null-Lagrangians*. In this setting, it is a well-known result, that all curl-quasiaffine functions are linear combinations of minors of $N \times m$-matrices (e.g. [Mor66, Res67, CDKS06, Dac08]). BALL, CURRIE and OLIVER considered the potential operator of higher gradients $\mathcal{B} = \nabla^k$ and showed an analogue of Theorem A in this special setting [BCO81]. In particular, their work provides us with an example showing that the statement of Theorem B fails in the case, where the potential $\mathcal{B}$ has degree larger than one. We shall mention again that the equivalences (c)-(e) in Theorem A are treated in the general setting of constant rank operators $\mathcal{A}$ by GUERRA and RAIŢĂ in [GR19]. In particular (d) and (e) are examined in great detail.

1.3. *$\mathcal{A}$-quasiaffine functions and minimisation problems.* Let us outline a set of problems, where $\mathcal{A}$-quasiaffine functions play an important role. Consider the functional $I: L^p(\Omega, \mathbb{R}^d) \to [0, \infty)$, 

$$I(u) = \begin{cases} 
\int_{\Omega} f(u(y)) \, dy & \text{if } \mathcal{A}u = 0, \\
\infty & \text{else}, 
\end{cases}$$

(1.5)

A powerful tool to show that $I$ possesses a minimiser is the Direct Method. To get this method to work we need to show the following:

1. $I$ is bounded from below and there exists $u$, such that $I(u) < \infty$;
2. $I$ is weakly lower semi-continuous, i.e. if $u_n \weak u$, then $I(u) \leq \liminf I(u_n)$;
3. $I$ is coercive, i.e. for every $C > 0$ there is $R > 0$, such that $\|u\|_{L^p} \geq C$.

$\mathcal{A}$-quasiaffinity comes into play for (2) and (3). On the one hand, we note that if $f$ is $\mathcal{A}$-quasiaffine, the functional $I$ in (1.5) is already weakly continuous. Weak lower-semicontinuity of $I$ is equivalent to the notion of $A$-quasiconvexity [FM99], where we
relax (1.2) to
\[ f(v) \leq \int_{T_N} f(v + \psi(y)) \, dy \]
for all suitable test functions. It is, however, not so easy to show that a given function \( f \) is \( \mathcal{A} \)-quasiconvex. Therefore, one often studies functions of the form \( g(f(v)) \), where \( f : \mathbb{R}^d \to \mathbb{R}^e \) is a component-wise \( \mathcal{A} \)-quasiaffine function and \( g : \mathbb{R}^e \to \mathbb{R} \) is convex, which is, in the setting \( \mathcal{A} = \text{curl} \), referred to as polyconvexity (e.g. [AD92, Bal77]).

A short calculation shows that any function of the form \( g(f(\cdot)) \) for \( \mathcal{A} \)-quasiaffine \( f \) and convex \( g \) is already \( \mathcal{A} \)-quasiconvex.

On the other hand, \( \mathcal{A} \)-quasiaffine functions also can be used to consider non-standard coercivity conditions. Usually, we may just assume that
\[ f(v) \geq C_1 |v|^p - C_2. \]
Given some further restrictions on the problem, we may use a coercivity condition using \( \mathcal{A} \)-quasiaffine functions. As an example, consider \( \Omega \subset \mathbb{R}^N \) open and bounded,
\[ J : W^{1,N}(\Omega, \mathbb{R}^N), \ u_0 \in W^{1,N}(\Omega, \mathbb{R}^N), \ f : \mathbb{R}^{N \times N} \to [0, \infty) \text{ and} \]
\[ J(u) = \begin{cases} \int_{\Omega} f(\nabla u(y)) \, dy & \text{if } u - u_0 \in W^{1,N}_0(\Omega, \mathbb{R}^N) \\ \infty & \text{else,} \end{cases} \]

Then the (non-standard) coercivity condition including the quasiconvex function \( \det \) guarantees, that minimizing sequences of \( I \) are bounded in \( W^{1,N}(\Omega, \mathbb{R}^d) \). Moreover, if \( I \) is weakly lower-semicontinuous, this growth condition then implies existence of minimisers.

1.4. **Compensated Compactness.** As one might expect in view of Theorem (d) and (e) the notion of \( \mathcal{A} \)-quasiaffinity plays a crucial role in the theory of compensated compactness (e.g. [Mur78, DP85, Rin14, GR19]). In particular, the classical div-curl Lemma saying that if \( u_n, v_n \in L^2(\Omega, \mathbb{R}^d) \) satisfy \( \text{div} u_n = 0 \) and \( \text{curl} v_n = 0 \), then
\[ u_n \to u, \ v_n \to v \text{ in } L^2 \implies u_n \cdot v_n \to u \cdot v \text{ in the sense of distributions} \]
can be seen from the fact that \( u \cdot v \) is a div-curl quasiconvex function (cf. Section 4.3, [Mur78, Tar79, Mur81]) .

1.5. **Outline.** This paper is organised as follows. In Section 2 we recall basic facts about constant rank operators, their potentials and \( \mathcal{A} \)-quasiconvexity. We deal with \( \mathcal{A} \)-quasiaffine functions and prove Theorem 1 and Theorem 2 in Section 3. Section 4 presents a short coverage of examples of \( \mathcal{A} \)-quasiaffine functions for some well-known operators.
2. A-quasiconvex Functions

2.1. Notation. Denote by $e_1, \ldots, e_N$ the standard basis of $\mathbb{R}^N$. For $N \in \mathbb{N}$ let us define the $N$-torus as $[0, 1]^N$ with the usual identifications of faces. We may identify functions $u \in W^{k,p}(T_N, \mathbb{R}^d)$ with $\mathbb{Z}^N$-periodic functions in $W^{k,p}_{loc}(\mathbb{R}^N, \mathbb{R}^d)$. For $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, we write

$$
\partial_j u = \frac{\partial}{\partial x_j} u
$$

to denote partial derivatives. With $\nabla u \in L^p(\Omega, \mathbb{R}^{N \times d})$ we denote the gradient of $u$, a matrix consisting of the entries $\partial_j u_i$ and, likewise, $\nabla^r$ is the $r$-th gradient. In contrast to this, for $f : \mathbb{R}^d \to \mathbb{R}$, we denote by $D^r f$ the $r$-th derivative seen as a multilinear map from $(\mathbb{R}^d)^r$ to $\mathbb{R}$.

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_N)$ and for $\lambda = \sum_{i=1}^N \lambda_i e_i \in \mathbb{R}^N$ write

$$
|\alpha| := \sum_{i=1}^N \alpha_i, \quad \lambda^\alpha = \prod_{i=1}^N \lambda_i^{\alpha_i}.
$$

For $u \in C^{[\alpha]}(\mathbb{R}^N, \mathbb{R}^d)$ we write

$$
\partial^\alpha u = \partial_{\alpha_1} \cdots \partial_{\alpha_n} u.
$$

For a function space $X \subset L^1(T_N, \mathbb{R}^d)$, we define

$$
X_{\#} = \{ u \in X : \int_{T_N} u = 0 \}.
$$

2.2. Constant Rank Operators. Consider a differential operator $A : C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l)$ with constant coefficients given by

$$
Au = \sum_{|\alpha| = k_A} A_{\alpha} \partial^\alpha u,
$$

where $A_{\alpha} \in \text{Lin}(\mathbb{R}^d, \mathbb{R}^l)$. We denote by $A^*$ the adjoint operator of $A$,

i.e. $A^* : C^\infty(\mathbb{R}^N, \mathbb{R}^l) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d)$ and

$$
A^* v = (-1)^{k_A} \sum_{|\alpha| = k_A} A_{\alpha}^T \partial^\alpha u.
$$

We say that $u \in L^p(\mathbb{R}^N, \mathbb{R}^d) \cap \ker A$, if $u \in L^p(\mathbb{R}_N, \mathbb{R}^d)$ and for all $v \in C^\infty_c(\mathbb{R}^N, \mathbb{R}^d)$

$$
\int_{\mathbb{R}^N} u \cdot A^* v = 0.
$$

Likewise, we can also define what it means for $u \in L^p_{loc}(\mathbb{R}^N, \mathbb{R}^d)$ to be in $\ker A$ and hence also for $u \in L^p(T_N, \mathbb{R}^d)$. Let us recall some basic notions for the operator $A$ (c.f. [Mur78, Mur81, Tar79]).

Definition 2.1. Let $A$ be a differential operator as in (2.1).
(a) The Fourier symbol of the differential operator $\mathcal{A}$ is the map $\mathbb{R}^N \setminus \{0\} \to \text{Lin}(\mathbb{R}^d, \mathbb{R}^l)$ defined by

$$\mathcal{A}[\xi] = \sum_{|\alpha|=k_A} \xi^\alpha A_\alpha;$$

(b) $\mathcal{A}$ satisfies the constant rank property if there exists an $r \in \{0, \ldots, N\}$ such that

$$\dim \ker \mathcal{A}[\xi] = r;$$

(c) The characteristic cone $\Lambda_A \subset \mathbb{R}^d$ of $\mathcal{A}$ is defined by

$$\Lambda_A := \bigcup_{\xi \in \mathbb{R}^N \setminus \{0\}} \ker \mathcal{A}[\xi].$$

(d) $\mathcal{A}$ satisfies the spanning property, if the characteristic cone $\Lambda_A$ spans up $\mathbb{R}^d$.

In addition, we also consider a differential operator $\mathcal{B}: C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d)$ of order $k_B$, given by

$$\mathcal{B}u = \sum_{|\alpha|=k_B} B_\alpha \partial_\alpha u.$$  
Likewise, we define $\mathcal{B}^*$ and the cone $\Lambda_B$ for $\mathcal{B}$.

**Definition 2.2.** We call a differential operator $\mathcal{B}$ the potential of $\mathcal{A}$ if $\forall \xi \in \mathbb{R}^N \setminus \{0\}$ we have $\text{Im} \mathcal{B}[\xi] = \ker \mathcal{A}[\xi]$.

**Remark 2.3.** As it was pointed out in [GR19], the potential $\mathcal{B}$ is not unique, even if we fix the order $k_B$ of $\mathcal{B}$ and identify operators via homeomorphisms of the underlying space $\mathbb{R}^m$. Moreover, note that if $\mathcal{B}: C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d)$ is a potential of $\mathcal{A}$, then also

$$\mathcal{B} \circ \text{div}: C^\infty(\mathbb{R}^N, \mathbb{R}^m \otimes \mathbb{R}^N) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d)$$
is a potential of $\mathcal{A}$.

**Proposition 2.4.** Let $\mathcal{A}$ be a constant rank operator. The following statements are equivalent:

(a) $\mathcal{B}$ is the potential of $\mathcal{A}$;

(b) The following two properties hold

(b1) $\forall u \in L^2_{\text{loc}}(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ there exists $v \in W^{k_B,2}(T_N, \mathbb{R}^m)$ such that $\mathcal{B}v = u$;

(b2) $\forall v \in C^\infty(T_N, \mathbb{R}^m)$ we have $\mathcal{A}(\mathcal{B}v) = 0$;

(c) $\mathcal{A}^*$ is a potential of $\mathcal{B}^*$.

**Proof.** (a) $\iff$ (c) is just a purely algebraic calculation, using that $(\ker \mathcal{A}[\xi])^\perp = \text{Im} \mathcal{A}^*[\xi]$. We just need to verify (a) $\iff$ (b). To show that (a) implies (b), note
that we may write \( u \in L^2 \cap C^\infty(T_N, \mathbb{R}^d) \) as
\[
u(x) = \sum_{\lambda \in \mathbb{Z}^N} \hat{u}(\lambda)e^{-2\pi i x \cdot \lambda}.
\]
Note that \( u \in \ker \mathcal{A} \) if and only if \( \hat{u}(\lambda) \in \ker \mathcal{A}[\lambda] \), i.e. \( \hat{u}(\lambda) = \mathcal{B}[\lambda](\hat{v}(\lambda)) \) for some \( \hat{v}(\lambda) \in \mathbb{R}^d \). We may choose \( v \) in the orthogonal complement of \( \ker \mathcal{B}[\lambda] \). Hence, we may bound
\[
|\hat{v}(\lambda)| \leq C|\lambda|^{-k} \hat{u}(\lambda)
\]
and thus \( v \), defined by
\[
v(x) = \sum_{\lambda \in \mathbb{Z}^N} \hat{v}(\lambda)e^{-2\pi i x \cdot \lambda},
\]
is bounded in \( W^{k,2}(T_N, \mathbb{R}^d) \) and satisfies \( \mathcal{B}v = u \). \( \mathcal{A} \circ \mathcal{B} = 0 \) follows by a calculation using the Fourier transform. The converse direction that \( (b) \) implies \( (a) \) follows a very similar argument.

Remark 2.5. Note that condition \( (b2) \) can be generalised to domains \( \Omega \subset \mathbb{R}^N \) in general, i.e. \( (b2) \) holds for any \( v \in C^\infty(\Omega, \mathbb{R}^m) \) if \( \mathcal{B} \) is a potential of \( \mathcal{A} \). Condition \( (b1) \) cannot be extended to general domains \( \Omega \). In particular, this holds for the pair \( \mathcal{B} = \nabla, \mathcal{A} = \text{curl} \) on a set \( \Omega \) instead of \( T_N \) only if \( \Omega \) is simply connected.

RAIŢĂ showed the important equivalence of constant rank condition and existence of a potential \[ \text{Rai19, ARS21}. \]

Proposition 2.6. Let \( \mathcal{A} \) be a homogeneous differential operator as in (2.1). \( \mathcal{A} \) satisfies the constant rank property if and only if \( \mathcal{A} \) admits a potential \( \mathcal{B} \).

2.3. \( \mathcal{A} \)-quasiconvexity.

Definition 2.7. Let \( \mathcal{A} \) be a differential operator and \( f : \mathbb{R}^d \to \mathbb{R} \) a measurable function. \( f \) is called \( \mathcal{A} \)-quasiconvex if \( \forall x \in \mathbb{R}^d, \varphi \in C^\infty(T_N, \mathbb{R}^d) \) with \( \mathcal{A}\varphi = 0 \) we have
\[
f(x) \leq \int_{T_N} f(x + \varphi(y)) \, dy.
\]
We call \( f \) \( \mathcal{B} \)-potential-quasiconvex if \( \forall x \in \mathbb{R}^d, \forall \psi \in C^\infty_0(\Omega, \mathbb{R}^d) \) we have
\[
f(x) \leq \frac{1}{|\Omega|} \int_{\Omega} f(x + \mathcal{B}\psi(y)) \, dy.
\]

Definition 2.8. Let \( \Lambda \subset \mathbb{R}^d \) be a cone, i.e. \( t \in \mathbb{R}^+, v \in \Lambda \Rightarrow tv \in \Lambda \). We call \( f : \mathbb{R}^d \to \mathbb{R} \) \( \Lambda \)-convex, if \( \forall x \in \mathbb{R}^d, v \in \Lambda \) the function
\[
t \mapsto f(x + tv)
\]
is convex. We call \( f : \mathbb{R}^d \to \mathbb{R} \) \( \Lambda \)-affine if the above map is affine (\( f \) is \( \Lambda \)-convex and \( -f \) is \( \Lambda \)-convex).
Proposition 2.9. Let $A$ be a homogeneous differential operator satisfying the constant rank property and $B$ a potential of $A$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. The following statements are equivalent

(1) $f$ is $A$-quasiconvex.
(2) $f$ is $B$-potential quasiconvex.
(3) Let $Q = (0,1)^N$. Then for all $\varphi \in C_c^\infty(Q, \mathbb{R}^m)$ and for all $x \in \mathbb{R}^d$ we have

\[

f(x) \leq \int_Q f(x + B\varphi(y)) \, dy = 0.

\]

(4) For all $\varphi \in C_c^\infty(T_N, \mathbb{R}^m)$ and for all $x \in \mathbb{R}^d$ we have

\[

f(x) \leq \int_{T_N} f(x + B\varphi(y)) \, dy = 0.

\]

A proof of this statement (in the setting $B = \nabla$) can be found in [Müll99, Section 4.7]. Let us also mention following statement about equivalence of $A$-quasiconvexity and weak lower semicontinuity (c.f. [FM99]).

Proposition 2.10. Let $1 < p \leq \infty$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be continuous and

\[

0 \leq f(x,v) \leq C(1 + |v|^p), \quad \text{if } p < \infty.

\]

Then the functional $I : L^p(\Omega, \mathbb{R}^d) \to [0, \infty]$, defined by

\[

I(u) = \begin{cases} 
\int_\Omega f(x,u(x)) \, dy & \text{if } Au = 0 \text{ in the sense of distributions,} \\
\infty & \text{else},
\end{cases}
\]

is sequentially weakly lower-semicontinuous (weakly* if $p = \infty$) if and only if $f(x,\cdot)$ is $A$-quasiconvex for almost every $x \in \Omega$.

3. $A$-quasiaffine functions

In the following, $A$ is a homogeneous differential operator of the form (2.1), satisfying the constant rank property and the spanning property. We denote by $B$ its potential, which exists due to Proposition 2.6.

Definition 3.1. Let $M \in C(\mathbb{R}^d)$.

(a) We call $M$ $A$-quasiaffine if $M$ and $-M$ are $A$-quasiconvex.
(b) We call $M$ $B$-potential-quasiaffine if $M$ and $-M$ are $B$-potential-quasiconvex.

Proposition 3.2. Let $M : \mathbb{R}^d \to \mathbb{R}$ be continuous and let $B$ be a potential of $A$. The following statements are equivalent:

(a) $M$ is $A$-quasiaffine.
(b) For all \( u, v \in L^p(T_N, \mathbb{R}^d) \cap \ker A \) with \( p \geq d \) and \( \int_{T_N} u(y) \, dy = \int_{T_N} v(y) \, dy \)

we have

\[
\int_{T_N} M(u(y)) \, dy = \int_{T_N} M(v(y)) \, dy.
\]

(c) \( M \) is \( B \)-potential-quasiaffine.

This directly follows from Proposition 2.9 for the bound \( p \geq d \) in (b) we indeed also need Theorem 3.3 (c) and (f).

Note that we can already infer the following strong properties for \( A \)-quasiaffine functions using basic methods. One key point is that, for a differential operator \( A \) satisfying the constant rank and the spanning property, \( A \)-quasiaffine functions are already \( \Lambda_A \)-affine.

**Theorem 3.3.**

(a) Let \( f : \mathbb{R}^d \to \mathbb{R} \) be \( A \)-quasiconvex and continuous. Then \( f \) is also \( \Lambda_A \)-convex.

(b) Let \( f \in C^2(\mathbb{R}^d) \). Then \( f \) is \( \Lambda_A \)-convex if and only if for all \( x \in \mathbb{R}^d \) and \( v \in \Lambda_A \)

\[
D^2 f(x)[v, v] = \frac{\partial^2}{\partial t^2} f(x + tv) \geq 0.
\]

(c) Let \( M : \mathbb{R}^d \to \mathbb{R} \) be \( A \)-quasiaffine and continuous. Then \( M \) is also \( \Lambda_A \)-affine.

(d) Let \( M \in C^2(\mathbb{R}^d) \). Then \( f \) is \( \Lambda_A \)-affine if and only if for all \( x \in \mathbb{R}^d \) and \( v \in \Lambda_A \)

\[
D^2 M(x)[v, v] = \frac{\partial^2}{\partial t^2} M(x + tv) = 0.
\]

(e) Let \( M : \mathbb{R}^d \to \mathbb{R} \) be a polynomial of degree 2. Then \( M \) is \( A \)-quasiaffine if and only if \( M \) is \( \Lambda_A \)-affine.

(f) Any \( \Lambda_A \)-affine map is a polynomial of degree \( \leq d \).

(g) Any partial derivative of a \( \Lambda_A \)-affine map is also \( \Lambda_A \)-affine.

(h) A homogeneous polynomial \( M : \mathbb{R}^d \to \mathbb{R} \) of degree \( \geq 3 \) is \( \Lambda_A \)-affine if all its partial derivatives \( \partial_i M \) \( i \in \{1, \ldots, d\} \) are \( \Lambda_A \)-affine.

(i) There exists a basis of homogeneous polynomials of the space of \( \Lambda_A \)-affine maps.

**Proof.** (a) follows from considering test functions of the form \( ve^{-2\pi i \lambda x} \) for \( v \in \ker A[\lambda] \) (c.f. [FM99]). (b) follows from the classical equivalence of convexity and \( f''(x) \geq 0 \) for \( f \in C^2(\mathbb{R}) \). (c) and (d) then directly follow by Definition 3.1. (e) relies on Plancherel’s identity, which is valid for quadratic forms. In particular, as all affine functions are automatically \( A \)-quasiaffine, we may consider \( M \) to be 2-homogeneous. Then, using Plancherel’s identity, we find that

\[
\int_{T_N} M(u(y)) \, dy = \sum_{\lambda \in \mathbb{Z}^N} M(\hat{u}(\lambda)).
\]
As $M$ is homogeneous of degree 2 and $\hat{u}(\lambda) \in \Lambda_A$, it follows that $M(\hat{u}(\lambda)) = 0$ for $\lambda \neq 0$.

Ad $[f]$. Let now $v_1, ..., v_d$ be a basis of $\mathbb{R}^d$, which is contained in $\Lambda_A$. The existence of such a basis is ensured by the spanning property for $\mathcal{A}$. Denote by $\lambda_1(y), ..., \lambda_n(y)$ the coordinates with respect to this basis. We may write a $\Lambda_A$-affine function $f$ as

$$f(y) = \tilde{f}(\lambda_1, ..., \lambda_d).$$

Due to $\Lambda_A$-affinity, we know that the map

$$\lambda_i \mapsto \tilde{f}(\lambda_1, ..., \lambda_d).$$

is affine for fixed $i \in \{1, ..., d\}$ and fixed $\lambda_j j \neq i$. Hence, $\tilde{f}$ must be a polynomial in $\lambda_i$. In particular, as $\tilde{f}$ is affine in each $\lambda_i$, it has at most degree $d$.

The property $[g]$ follows from $[d]$. To see $[h]$ note that

$$D^2M(x)[v, v] = \int_0^1 D^3M(tx)[v, v, x] \, dt + D^2M(0)[v, v]$$

$$= \int_0^1 D^2 \left( \frac{\partial}{\partial x} M \right)(tx)[v, v] \, dt + D^2M(0)[v, v].$$

As $M$ is homogeneous of degree strictly larger than two, $D^2M(0) = 0$ and therefore $M$ is $\mathcal{A}$-quasiaffine.

For $[i]$ we use $[h]$. Write $f = \sum_{i=1}^d f_i$ for $i$-homogeneous polynomials $f_i$. We may consider $\tilde{f} = f - f_0 - f_1$, as $f_0$ and $f_1$ are affine and hence $\Lambda_A$-affine. Observe that then $\Lambda_A$-affinity yields $f(x) = 0$ for all $x \in \Lambda_A$. In particular, $f_i(x) = 0$ for all $i = 2, ..., d$ and $x \in \Lambda_A$.

But this implies $\Lambda_A$-affinity for $f_2$. Considering $\tilde{f} = \nabla(f - f_0 - f_1 - f_2)$, the statement $[h]$ and an inductive argument, we get that $f_0, ..., f_d$ are all already $\Lambda_A$-affine. Therefore, there must be a basis of homogeneous polynomials for $\Lambda_A$-affine maps. □

Remark 3.4. a) Due to Theorem 3.3 $[h]$, if there is $\Lambda_A$-affine polynomial $f$ of degree $k$, then there is also a $\Lambda_A$-affine polynomial of degree $k - 1$. In particular, the question of existence of non-affine $\Lambda_A$-affine functions reduces to the existence of quadratic $\Lambda_A$-affine functions. Recall that $\mathcal{A}$-quasiaffine functions are $\Lambda_A$-affine functions and the converse holds for quadratic functions. Hence, the existence of non-trivial $\mathcal{A}$-quasiaffine functions reduces to the existence of a quadratic function vanishing on $\Lambda_A$.

b) Šverák showed in [Š92], that the other direction in 3.3(a) is not true, i.e. there exist $\Lambda_A$-convex functions that are not $\mathcal{A}$-quasiconvex.

c) The converse implication in 3.3(c) is false, i.e. $\Lambda_A$-affinity does not imply $\mathcal{A}$-quasiconvexity (c.f. Lemma 3.8 [BCOS1]).
Let us now proof Theorem A. For this, we roughly follow the proof of this statement for $B = \nabla^k$ in [BCO81]. Note that by considering the potential $B$ of $A$, (b) in Theorem A is equivalent to

(b') $M$ is a polynomial and $\forall x \in \mathbb{R}^d$, $\forall r \geq 2$, $\forall \xi_1, \ldots, \xi_r \in \mathbb{R}^d$ which are linear dependent and $\forall w_1, \ldots, w_r \in \mathbb{R}^m$ we have

$$D^r M(x)[B[\xi_1](w_1), \ldots, B[\xi_r](w_r)] = 0.$$  

\textbf{Proof of Theorem A.} The validity of the implications (a) $\iff$ (d) $\iff$ (e) follows from Theorem 2.10. Indeed, for $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^d)$ (or $\varphi \in L^1$ for (d)), consider

$$I_\varphi(u) = \begin{cases} \int_\Omega M(u(x)) \cdot \varphi(x) \, dx & \text{if } A \underline{u} = 0 \\ \infty & \text{else.} \end{cases}$$

This functional is weakly continuous if and only if, for all $x \in \mathbb{R}^d$, the map $v \mapsto M(v) \cdot \varphi(x)$ is $A$-quasiaffine, which is equivalent to $A$-quasiaffinity of $M$ (for more details we refer to [GR10]).

We now prove (a) $\iff$ (e). If $M$ is $B$-potential-quasiaffine, then by Theorem 3.3 it is a polynomial and hence it is even $C^\infty$. Moreover, for all $u \in C^k(\bar{\Omega})$ and all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$ we have

$$0 = \frac{d}{dt} \left( \int_\Omega (M(Bu(y)) + tB\varphi(y)) \, dy \right) \bigg|_{t=0}$$

$$= \int_\Omega \frac{d}{dt} (M(Bu(y)) + tB\varphi(y)) \, dy \bigg|_{t=0}$$

$$= \int_\Omega DM(Bu(y)) \cdot B\varphi(y) \, dy.$$  

Thus, (1.4) holds in the sense of distributions if $M$ is $B$-potential-quasiaffine. The same calculation as in (3.3) also shows that, if (1.4) holds, then $M$ is $A$-quasiaffine.

It remains to show that (a) $\iff$ (b'). First, we prove the direction (a) $\Rightarrow$ (b').

If $r = 2$, note that $B[\lambda \xi] = \lambda^{k_2} B[\xi]$ for $\xi \in \mathbb{R}^N$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Hence, if $\xi_1$ and $\xi_2$ are linear dependent and nonzero, we may write $\xi_2 = \lambda \xi_1$ and

$$B[\xi_2](w_2) = B[\xi_1](\lambda^{k_2} w_2).$$

Therefore, we may only consider $\xi_1 = \xi_2 = \xi$. Thus,

$$D^2 M(x)[v_1, v_2] = D^2 M(x)[B[\xi](w_1), B[\xi](w_2)]$$

$$= \frac{1}{2} D^2 M(x)[B[\xi](w_1 + w_2), B[\xi](w_1 + w_2)]$$

$$- \frac{1}{2} D^2 M(x)[B[\xi](w_1), B[\xi](w_1)] - \frac{1}{2} D^2 M(x)[B[\xi](w_2), B[\xi](w_2)]$$

$$= 0.$$
We prove the statement for $r > 2$ by induction. Let (3.2) hold for some $r \in \mathbb{N}$. We consider linear dependent $\xi_1, \ldots, \xi_{r+1} \in \mathbb{R}^N$ and $w_1, \ldots, w_{r+1} \in \mathbb{R}^m$. First, suppose that $\xi_1, \ldots, \xi_r$ are already linear dependent. Then by induction hypothesis,

$$D^r M(x)[B[\xi_1](w_1), \ldots, B[\xi_r](w_r)] = 0 \quad \forall x \in \mathbb{R}^d.$$ 

Taking the derivative in direction $\mathbb{B}(\xi_{r+1})(w_{r+1})$, the result is also 0. Hence,

$$D^{r+1} M(x)[B[\xi_1](w_1), \ldots, B[\xi_{r+1}](w_{r+1})] = 0.$$ 

We may suppose that $\xi_{r+1}$ can be written as a linear combination of linear independent $\xi_1, \ldots, \xi_r \in \mathbb{R}^N \setminus \{0\}$. Due to the homogeneity of $B[\cdot](w)$ we may also suppose that

$$\xi_{r+1} = \xi_1 + \ldots + \xi_r.$$ 

Let $t_1, \ldots, t_r \in \mathbb{R}$ be real parameters. Define the function $\varphi \in C^\infty(T_N, \mathbb{R}^m)$ by

$$\varphi(y) := \begin{cases} 
\sum_{i=1}^{r+1} t_i w_i \cos(2\pi \xi_i \cdot y) & \text{if } k_B \text{ is even}, \\
\sum_{i=1}^{r+1} t_i w_i \sin(2\pi \xi_i \cdot y) & \text{if } k_B \text{ is odd}.
\end{cases}$$

For the sake of simplicity we shall consider the case $k_B = 2k$, the other case is rather similar.

Then, $B\varphi$ is given by

$$B\varphi(y) = (-4\pi^2)^k \sum_{i=1}^{r+1} t_i B[\xi_i](w_i) \cos(2\pi \xi_i \cdot y).$$

Now, $B$-potential-quasiaffinity means that

$$\int_{T_N} M(x + B\varphi) \, dy = M(x) \quad \forall x \in \mathbb{R}^d.$$ 

The left-hand side of (3.4) is a polynomial in $t_i$. The coefficient of $t_1 \cdot \ldots \cdot t_{r+1}$ is the constant $(-4\pi^2)^k$ times

$$\int_{T_N} D^{r+1} M(x)[B[\xi_1](w_1), \ldots, B[\xi_{r+1}](w_{r+1})] \cdot \cos(2\pi \xi_1 \cdot y) \cdot \ldots \cdot \cos(2\pi \xi_{r+1} \cdot y) \, dy$$

$$= D^{r+1} M(x)[B[\xi_1](w_1), \ldots, B[\xi_{r+1}](w_{r+1})]$$

$$\cdot \int_{[0,1]^N} \cos(2\pi \xi_1 \cdot y) \cdot \ldots \cdot \cos(2\pi \xi_r \cdot y) \cos(2\pi \sum_{i=1}^{r} \xi_i \cdot y) \, dy$$

$$= 2^{-r} D^{r+1} M(x)[B[\xi_1](w_1), \ldots, B[\xi_{r+1}](w_{r+1})].$$

To calculate the integral in this equation, we just use the addition theorem for cos and Fubini. As the coefficient of $t_1 \cdot \ldots \cdot t_{r+1}$ is 0 on the right-hand side of (3.4), we get the desired result.
For the direction \((b') \Rightarrow (a)\) we first claim that it suffices to show that for all \(r \geq 2\)

\[
\forall x \in \mathbb{R}^d, \quad \forall \varphi \in C^\infty(T_N, \mathbb{R}^m) \quad \text{for all } r \geq 2
\]

\[
(3.5) \quad \int_{T_N} D^r M(x)[B\varphi(y), ..., B\varphi(y)] = 0.
\]

Suppose that \((3.5)\) holds. We want to show \((a)\). Take arbitrary \(x \in \mathbb{R}^d\) and \(\varphi \in C^\infty(T_N, \mathbb{R}^m)\). Consider the Taylor series of \(M\) at the point \(x\) in the direction of \(B\varphi(y) \in \mathbb{R}^d\). As \(M\) is a polynomial of some degree \(s\), \(M\) equals its Taylor polynomial in \(x\) of degree \(s\), i.e.

\[
M(x + B\varphi(y)) = \sum_{r=0}^s \frac{1}{r!} D^r M(x)[B\varphi(y), ..., B\varphi(y)]
\]

Integrating over \(y \in T_N\), using \((b')\) and the fact that \(B\varphi\) has average 0, yields

\[
\int_{T_N} M(x + B\varphi(y)) \, dy = \sum_{r=0}^s \int_{T_N} \frac{1}{r!} D^r M(x)[B\varphi(y), ..., B\varphi(y)] \, dy
\]

\[
= \int_{T_N} M(x) \, dy + \int_{T_N} D M(x) \cdot B\varphi(y) \, dy
\]

\[
+ \sum_{r=2}^s \int_{T_N} \frac{1}{r!} D^r M(x)[B\varphi(y), ..., B\varphi(y)] \, dy
\]

\[
= \int_{T_N} M(x) \, dy = M(x).
\]

Thus, it suffices to prove \((3.5)\). To this end, we use the following formula:

If \(f_1, ... f_r \in C^0(T_N, \mathbb{R})\), then

\[
(3.6) \quad \int_{T_N} f_1(y) \cdot ... \cdot f_r(y) \, dy = \sum_{\xi_1, ..., \xi_r \in \mathbb{Z}^n} \hat{f}_1(\xi_1) \cdot \hat{f}_2(\xi_2) \cdot ... \cdot \hat{f}_{r-1}(\xi_{r-1}) \cdot \hat{f}_r \left( \xi_1 - \sum_{i=2}^{r-1} \xi_i \right).
\]

This equation can be derived by using Plancherel’s theorem once for \(f_1\) and \(f_2 \cdot ... \cdot f_r\) and then using a discrete version of the convolution formula, i.e.

\[
(f(\cdot) \cdot g(\cdot))(\xi_1) = \sum_{\xi_2 \in \mathbb{Z}^n} \hat{f}(\xi_2) \cdot \hat{g}(\xi_1 - \xi_2).
\]

Recall that \(D^r M(x)[\cdot, ... , \cdot]\) is a multilinear form (i.e. a homogenous polynomial in the entries). Therefore, we can use the identity \((3.6)\). Hence

\[
\int_{T_N} D^r M(x)[B\varphi(y), ..., B\varphi(y)]
\]

\[
= \sum_{i=1}^{r-1} \sum_{\xi_i \in \mathbb{Z}} D^r M(x) \left[ B[\xi_1](\hat{\varphi}(\xi_1)), ..., B[\xi_{r-1}](\hat{\varphi}(\xi_{r-1})), B[\xi_1 - \sum_{i=2}^{r-1} \xi_i](\hat{\varphi}(\xi_1 - \sum_{i=2}^{r-1} \xi_i)) \right]
\]

\[
= 0,
\]
as the vectors
\[ \xi_1, \ldots, \xi_{r-1}, \xi_1 - \sum_{i=2}^{r-1} \xi_i \]
are linear dependent. Each summand equals 0 due to condition (3.2) in (b'). We have shown the claim and therefore that (b') implies (a). \( \square \)

**Remark 3.5.** It shall be remarked, that Theorem A shows that \( \mathcal{A} \)-quasiaffinity is a local property; it can be verified by only considering gradients of \( f \) pointwise. As it is shown in [Kri99], this is not true for \( \mathcal{A} \)-quasiconvexity.

Let us now prove that the condition (b'), which is equivalent to Theorem A (b), can be slightly weakened to the following:

**Theorem 3.6.** Let \( \mathcal{B}: C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d) \) be a constant rank operator of order \( k_B \). Then \( M: \mathbb{R}^d \to \mathbb{R} \) is \( \mathcal{A} \)-quasiaffine if and only if for all \( 2 \leq r \leq \min\{k_B, N\} + 1 \), \( \xi_1, \ldots, \xi_r \in \mathbb{R}^N \setminus \{0\} \) linear dependent and \( w_1, \ldots, w_r \in \mathbb{R}^m \) we have

\[ D^r M(x)[\mathcal{B}[\xi_1](w_1), \ldots, \mathcal{B}[\xi_r](w_r)] = 0. \]  

(3.7)

**Remark 3.7.** If \( k_B = 1 \), then (3.7) only needs to hold for \( r = 2 \). In this case, (3.7) is equivalent to \( \Lambda_{\mathcal{A}} \)-affinity. Hence, as a special case of Theorem 3.6 we get the statement of Theorem B, that for first-order potentials, \( \mathcal{A} \)-quasiaffinity and hence \( \mathcal{B} \)-potential quasiaffinity are equivalent to \( \Lambda_{\mathcal{A}} \)-affinity.

**Proof.** We just need to prove that if equation (3.7) is true for \( 2 \leq r \leq \min\{k_B, N\} + 1 \), then it also holds for \( r \in \mathbb{N} \). Let us first deal with the case \( \min\{k_B, N\} = N \). Note that then for \( j > 2 \) and \( r = N + j \), there are \( N + 1 \) vectors \( \xi_i \), which are already linear dependent; say \( \xi_1, \ldots, \xi_{N+1} \) are linear dependent. Then,

\[ D^{N+1}(x)[\mathcal{B}[\xi_1](w_1), \ldots, \mathcal{B}[\xi_{N+1}](w_{N+1})] = 0. \]

Therefore, also

\[ D^{N+j}(x)[\mathcal{B}[\xi_1](w_1), \ldots, \mathcal{B}[\xi_{N+j}](w_{N+j})] = 0. \]

Suppose now that \( k_B \leq N \). If \( k_B = 1 \), then for all \( \xi_1, \xi_2 \in \mathbb{R}^N \setminus \{0\} \) and \( w \in \mathbb{R}^m \)

\[ \mathcal{B}[\xi_1 + \xi_2](w) = \mathcal{B}[\xi_1](w) + \mathcal{B}[\xi_2](w) \in \text{span}\{\mathcal{B}[\xi_1](w), \mathcal{B}[\xi_2](w)\}. \]

We prove an analogue of this statement for \( k_B > 1 \). Again, make the reductions from the proof of Theorem A. We just need to show that, for \( r > k_B + 1 \), \( \xi_1, \ldots, \xi_{r-1} \in \mathbb{R}^N \setminus \{0\} \) linear independent and \( w_1, \ldots, w_r \in \mathbb{R}^m \), we have

\[ D^r M(x)[\mathcal{B}[\xi_1](w_1), \ldots, \mathcal{B}[\xi_{r-1}](w_{r-1}), \mathcal{B}[\xi_1 + \ldots + \xi_{r-1}](w_r)] = 0. \]  

(3.8)
Claim: For $w \in \mathbb{R}^m$

\begin{equation}
B \left[ \sum_{i=1}^{r-1} \xi_i \right] (w) \in \text{span}_{\lambda \in I} \left\{ B \left[ \sum_{i=1}^{r-1} \lambda_i \xi_i \right] (w) \right\},
\end{equation}

where $r > k_B + 1$ and the set $I$ of coefficients is given by

$I = \{ \lambda \in \mathbb{R}^{r-1}: \lambda_i = 0 \text{ for some } i \in \{1, ..., r-1\} \}$

Suppose that (3.9) is proven. Then, for a finite index set $J \subset I$, we can write,

\begin{equation}
B \left[ \sum_{i=1}^{r-1} \xi_i \right] (w) = \sum_{\lambda \in J} B \left[ \sum_{i=1}^{r-1} \lambda_i \xi_i \right] (w)
\end{equation}

and use that, for each $\lambda \in J$, there is $i \in \{1, ..., r-1\}$ such that $\lambda_i = 0$. W.l.o.g. $i = 1$ for some fixed $\lambda \in J$. Then

\begin{equation}
D^r M(x) \left[ B[\xi_1](w_1), ..., B[\xi_{r-1}](w_{r-1}), B[\sum_{i=2}^{r-1} \lambda_i \xi_{r-1}](w_r) \right] \\
= \frac{\partial}{\partial B[\xi_1](w_1)} D^{r-1} M(x) \left[ B[\xi_2](w_2), ..., B[\xi_{r-1}](w_{r-1}), B[\sum_{i=2}^{r-1} \lambda_i \xi_{r-1}](w_r) \right].
\end{equation}

Note that we assume in 3.7 that the right-hand side is 0, whenever $r - 1 \leq k_B + 1$, i.e. $r \leq k_B + 2$. Assuming that (3.9) holds, one can prove (3.8) for all $r \in \mathbb{N}$ by an inductive argument.

It remains to prove the validity (3.9). Consider for $t_1, ..., t_{r-1}$ the polynomial

\begin{equation}
P(t_1, ..., t_{r-1}) = B \left[ \sum_{i=1}^{r-1} t_i \xi_i \right] (w_r).
\end{equation}

This polynomial has degree $k_B < r - 1$. Hence, in every monomial of $P$ of the form \( \prod_{i=1}^{r-1} t_i^{\alpha_i} \) there is at least one $j \in \{1, ..., r-1\}$, such that $\alpha_j = 0$. But we can recover the coefficients of these monomials by considering

\begin{equation}
B \left[ \sum_{i=1, i \neq j}^{r-1} t_i \xi_i \right] (w_r).
\end{equation}

In particular, we can recover these coefficients by taking linear combinations of $P(\lambda)$ for suitable $\lambda \in \{ \mu \in \mathbb{R}^{r-1}: \mu_j = 0 \} \subset I$. Therefore, (3.9) holds. This concludes the proof of Theorem 3.6. □

Theorem B is a special case of Theorem 3.6. In this setting, $k_B = 1$, i.e. $A$-quasiaffinity of $M$ is equivalent to the fact that

\begin{equation}
D^2 M(x)[B[\xi](w_1), B[\xi](w_2)] = 0.
\end{equation}
As it was already established in Theorem 3.3 (d), this is indeed equivalent to $\Lambda_{A^*}$-affinity of $M$.

Let us recall the Ball-Currie-Olver example showing that $A$-quasiaffinity does not follow if (3.7) does not hold for all $2 \leq r \leq \min\{k_B, N\} + 1$ (cf. [BCO81]). Let us consider the setting $k_B = 2$.

**Lemma 3.8** (Ball, Currie, Olver). **There is a first-order differential operator $A$ and a map $L : \mathbb{R}^d \to \mathbb{R}$ which is $\Lambda_A$-affine, but not $A$-quasiaffine.**

**Proof.** Consider the differential operator $B = \nabla^2$, i.e.

$$(\nabla^2 u)_{ijk} = \partial_i \partial_j u_k (i, j = 1, ..., N; k = 1, ..., m)$$

and $A$ the corresponding first order operator, such that $B$ is a potential of $A$ (cf. [Mey65]). The characteristic cone of $A$ is the space of tensors of the form

$$\lambda \otimes \lambda \otimes b : \lambda \in S^{N-1}, \ b \in \mathbb{R}^m.$$ 

Now choose $N = 2$ and $m = 3$ and consider the map $L$ defined via

$$(3.10) \quad L(\nabla^2 u) = \sum_{\sigma \in S_3} \text{sgn}(\sigma) \partial_x^2 u_{\sigma(1)} \partial_y^2 u_{\sigma(2)} \partial_y^2 u_{\sigma(3)}.$$ 

One can check that this is affine in $\Lambda_A$. On the other hand, one can check that for

$$u(x_1, x_2) = \begin{pmatrix} \cos(2\pi x_1) \\ \cos(2\pi x_2) \\ \cos(2\pi (x_1 + x_2)) \end{pmatrix},$$

we have

$$\int_{\mathcal{T}_N} L(u(x_1, x_2)) \, dx = -\frac{1}{4}. \quad \square$$

We have seen in Theorem 3.6 that the answer to the question, whether

$$f \text{ $\Lambda_A$-convex} \implies f \text{ A-quasiaffine}$$

really depends on the order of the operator $k_B$. We note that in view of the following lemmata, the minimal order of $k_B$ of the potential $B$ cannot be bounded in terms of the order of $A$. In view of Theorem 3.6 the differential condition on $M$ for being $A$-quasiaffine therefore depends only on the order of $B$ and not on the order of $A$.

**Lemma 3.9.** Let $B : C^\infty(\mathbb{R}^2, \mathbb{R}^m) \to C^\infty(\mathbb{R}^2, (\mathbb{R}^2)^k)$ be a differential operator such that

$$\text{Im } B[\xi] = \text{Im } \nabla^k[\xi] \quad \forall \xi \in \mathbb{R}^N \setminus \{0\},$$

where $\nabla^k : C^\infty(\mathbb{R}^2, \mathbb{R}) \to C^\infty(\mathbb{R}^2, (\mathbb{R}^2)^k)$. Then the operator $B$ is of order $k_B \geq k$. 

A SUFFICIENT AND NECESSARY CONDITION FOR $A$-QUASIAFFINITY

Proof. We note that
\[ \dim(\text{Im} \nabla^k[\xi]) = 1. \]
Consider $\xi_0 = e_1 + e_2$ and the coordinates of
\[ v_{1k} = \partial^k_1 u, \quad v_{2k} = \partial^k_2 u. \]
There exist $w \in \mathbb{R}^m$, such that
\[ (B[\xi_0](w))_{1k} \neq 0, \quad (B[\xi_0](w))_{2k} = 1. \]
Due to continuity of $B[-](w)$, there exists an open ball $B_r(\xi_0)$, such that, for all $\xi \in B_r(\xi_0)$,
\[ B[\xi](w) \neq 0. \]
In particular, as the dimension of the image of $\nabla^k[\xi]$ (and therefore also of the image of $B[\xi]$) is one, we then have, for all $\xi \in B_r(\xi_0)$,
\[ \xi^k_{1k}(B[\xi](w))_{1k} = \xi^k_{2k}(B[\xi](w))_{2k}. \]
Hence, $(B[\xi](v))_{1k}$ and $(B[\xi](v))_{2k}$ are polynomials of degree larger than $k$ in $\xi$. Therefore, $B$ has at least order $k$. □

Corollary 3.10. Let $N > 2$.

(a) For any $k \in \mathbb{N}$, there exists a first-order operator $A$, such that any potential $B$ of $A$ has order $k_B \geq k$.

(b) For any $k \in \mathbb{N}$, there exists a first-order operator $B$, such that any annihilator $A$ of $B$ (i.e. an operator $A$, such that $B$ is a potential of $A$) has order $k_A \geq k$.

Note that [a] follows directly from Lemma 3.9 and the result by Meyers, that $\nabla^k$ admits a first-order annihilator $A^k$ [Mey65]. [b] then follows from Proposition 2.4 [c]. In particular, $B = (A^k)^*$ is of first order and only admits annihilators of order $\geq k$.

4. EXAMPLES FOR $A$-QUASIAFFINE FUNCTIONS

In this section we discuss some examples and results for $A$-quasiaffine (or $B$-potential-quasiaffine functions) for explicit $A$.

4.1. $B = \nabla$ and $B = \nabla^k$. Consider the operator
\[ B = \nabla: C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^{N \times m}), \]
which is given by the coordinates
\[ (Bu)_{ij} = \partial_i u_j. \]
We have the following result in this special case (e.g. Mor66, Res67, CDKS06, Dac08).
Proposition 4.1. \( M : \mathbb{R}^{N \times m} \to \mathbb{R} \) is a \( \nabla \)-potential-quasiaffine function if and only if it is a linear combination of \( r \times r \) minors \((1 \leq r \leq \min\{m, N\})\).

As \( \text{curl} \) is the annihilator of \( \nabla \), we therefore have a basis of \( \text{curl} \)-quasiaffine functions. For higher-order gradients \( B = \nabla^k \), a characterisation is given in [BCO81]. Namely, a basis of \( \nabla^k \)-potential-quasiaffine functions are already \( \nabla \)-potential quasi-affine functions for the gradient acting on \( C^\infty(\mathbb{R}^N, \mathbb{R}^N \otimes_{\text{sym}} \ldots \otimes_{\text{sym}} \mathbb{R}^N) \).

4.2. The divergence operators on matrices. Consider the divergence operator acting on matrices, i.e.

\[
\text{div} : C^\infty(\mathbb{R}^N, \mathbb{R}^N \times \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d)
\]
defined by

\[
(\text{div } u)_i = \sum_{j=1}^N \partial_j u_{ij}.
\]

As, after a suitable rotation, the differential operator \( \text{div} \) equals \( \text{curl} \) for dimension \( N = 2 \), we just consider dimension \( N \geq 3 \). Note that the characteristic cone \( \Lambda_{\text{div}} \) is the space of matrices with rank \( \geq N - 1 \).

Proposition 4.2. Let \( N > 2 \) and \( \Lambda_2 \subset \mathbb{R}^N \times \mathbb{R}^d \) be the set of \( \mathbb{R}^{N \times d} \) matrices with rank less or equal to 2. Then \( f \) is \( \Lambda_2 \)-affine if and only if \( f \) is affine. Moreover, all \( \text{div} \)-quasiaffine functions are already affine.

Proof. Let \( e_{ij} \) be the standard basis of \( \mathbb{R}^{N \times d} \) matrices. We only prove that \( \Lambda_2 \)-affine functions \( M \) are already affine, the second follows directly from the observation, that these are contained in the characteristic cone of the divergence. To this end, note that \( \Lambda_2 \)-affinity of \( M \) implies that \( M \) is a polynomial and is the sum of some monomials \( P \). In particular, if a matrix \( A \) is decomposed as \( A = \sum_{i,j} A_{ij} e_{ij} \) for \( A_{ij} \in \mathbb{R} \), \( M(A) \) is a polynomial in \( A_{ij} \). Consider some matrix \( B \in \mathbb{R}^{N \times d} \) and the directions

\[
\lambda e_{kl} + \mu e_{ij}, \quad 1 \leq i, k \leq N, \quad 1 \leq l, j \leq d
\]

for \( \lambda, \mu \in \mathbb{R} \). The map

\[
(\lambda, \mu) \mapsto M(B + (\lambda e_{kl} + \mu e_{ij}))
\]
is affine. Hence, all coefficients containing of monomials \( P \), such that \( A_{ij}^2 \) or \( A_{ij} A_{kl} \) are factors of \( P \), vanish. Consequently, all coefficients of monomials with degree larger than one vanish. Therefore, \( M \) is already affine. \( \square \)

4.3. The div-curl Lemma and similar operators. Consider a constant rank operator \( \mathcal{A}_1 : C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l) \) and a potential \( \mathcal{B}_1 : C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d) \). Then we may consider the operator \( \mathcal{A} := (\mathcal{A}_1, \mathcal{B}_1) : C^\infty(\mathbb{R}^N, \mathbb{R}^d \times \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l \times \mathbb{R}^d) \).
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$\mathbb{R}^m$) defined by

$$(A_1, B_1^*)(u,v) = (A_1u, B_1^*v).$$

Note that we have

$$(\ker A_1[\xi])^\perp = \ker B_1^*[\xi] \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$  

Therefore, we obtain the following result.

**Proposition 4.3.** The map $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(a,b) = a \cdot b$$

is an $(A_1, B_1^*)$-quasiaffine map.

Note that this result has a lot of implications for weak convergence in the context of compensated compactness (e.g. [Mur78, Mur81, Tar79, DP85, Rin14, GR19]). In particular, if $u_n, v_n \in L^2(\Omega, \mathbb{R}^d)$ with $A_1u_n = 0$ and $B_1^*v_n = 0$, then due to the Characterisation Theorem we find that

$$u_n \rightharpoonup u, \quad v_n \rightharpoonup v \implies u_n \cdot v_n \rightharpoonup u \cdot v$$

in the sense of distributions.

The two most prominent examples are the following. On the one hand, the operators

$$A_1 = \text{curl}, \quad B_1^* = \text{div},$$

both acting on $\mathbb{R}^{N \times d}$-matrices are a well-known example, which is the initial example of compensated compactness. Another example of this type is

$$A_1 = \text{curl curl}^T, \quad B_1^* = \text{div}$$

on symmetric $N \times N$ matrices, considered in the context of linear elasticity (e.g. [CMO18]).

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**REFERENCES**

[AD92] J. Alibert and B. Dacorogna. An example of a quasiconvex function not polyconvex in dimension two. *Arch. Rat. Mech. Anal.*, 117:155–166, 1992.

[ARS21] A. Arroyo Rabasa and J. Simential. An elementary proof of the homological properties of constant-rank operators. *https://arxiv.org/abs/2107.05098*, 2021.

[Bal77] J. Ball. Convexity conditions and existence theorems in non-linear elasticity. *Arch. Rat. Mech. Anal.*, 63:337–403, 1977.

[BCO81] J. Ball, J. Currie, and P. Olver. Null Lagrangians, Weak Continuity, and Variational Problems of Arbitrary Order. *J. Func. Anal.*, 41:135–174, 1981.
S. Conti, G. Dolzmann, B. Kirchheim, and Müller. Sufficient conditions for the validity of the Cauchy-Born rule close to SO(n). *Journal of the European Mathematical Society*, 8:515–530, 2006.

S. Conti, S. Müller, and M. Ortiz. Data-driven problems in elasticity. *Arch. Rat. Mech. Anal.*, 229:79–123, 2018.

B. Dacorogna. *Direct Methods in the Calculus of Variations*. Springer-Verlag New York, 2 edition, 2008.

R. Di Perna. Compensated compactness and general systems of conservation laws. *Trans. Amer. Math. Soc.*, 292:383–420, 1985.

I. Fonseca and S. Müller. A-quasiconvexity, lower-semicontinuity and Young measures. *SIAM J. Math. Anal.*, 30(6):1355–1390, 1999.

A. Guerra and B. Raiță. Quasiconvexity, null Lagrangians, and Hardy space integrability under constant rank constraints. https://arxiv.org/abs/1909.03923, 2019.

J. Kristensen. On the non-locality of quasiconvexity. *Annales de l’Institut Henri Poincaré*, 6:1–13, 1999.

N. Meyers. Quasiconvexity and the lower semicontinuity of multiple variational integrals of any order. *Transactions of the American Mathematical Society*, 119(1):125–149, 1965.

C. Morrey. *Multiple Integrals in Calculus of Variations*. Springer, 1966.

S. Müller. Variational models for microstructure and phase transitions. In *Calculus of Variations and Geometric Evolution Problems: Lectures given at the 2nd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Cetraro, Italy, June 15–22, 1996*, Lecture Notes in Mathematics, pages 85–210. Springer, Berlin, Heidelberg, 1999.

F. Murat. Compacité par compensation. *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, IV. Ser, 5:489–507, 1978.

F. Murat. Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant. *Ann. Sc. Norm. Sup. Pisa*, 8:68–102, 1981.

B. Raiță. Potentials for A-quasiconvexity. *Calc. Var.*, 58:105, 2019.

Y. Reshetnyak. On the stability of conformal mappings in multidimensional spaces. *Sb. Math.*, 8:69–85, 1967.

F. Rindler. Directional oscillations, concentrations, and compensated compactness via microlocal compactness forms. *Arch. Ration. Mech. Anal.*, 215:1–63, 2014.

L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, volume 4, pages 136–212. Pitman Res. Notes Math, 1979.

V. Šverák. Rank-one convexity does not imply quasiconvexity. *Proc. Roy. Soc. Edinburgh*, 120:185–189, 1992.

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