$J_1$-$J_2$ fractal studied by multirecursion tensor-network method

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We generalize a tensor-network algorithm to study the thermodynamic properties of self-similar spin lattices constructed on a square-lattice frame with two types of couplings, $J_1$ and $J_2$, chosen to transform a regular square lattice ($J_1 = J_2$) onto a fractal lattice if decreasing $J_2$ to zero (the fractally fully reconstructs when $J_2 = 0$). We modified the higher-order tensor renormalization group (HOTRG) algorithm for this purpose. Single-site measurements are performed by means of so-called impurity tensors. So far, only a single local tensor and uniform extension-contraction relations have been considered in HOTRG. We introduce 10 independent local tensors, each being extended and contracted by 15 different recursion relations. We applied the Ising model to the $J_1 - J_2$ planar fractal whose Hausdorff dimension at $J_2 = 0$ is $d^{(H)} = \ln 12/\ln 4 \approx 1.792$. The generalized tensor-network algorithm is applicable to a wide range of fractal patterns and is suitable for models without translational invariance.

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I. INTRODUCTION

The understanding of phase transitions and critical phenomena plays an important role in condensed-matter physics. Much of the research on phase transitions has been devoted to the regular lattices. The two-dimensional classical Ising model on the square lattice is exactly solvable; on the other hand, there are no exact solutions for the spin models on fractals, so these must be studied numerically, requiring significantly higher effort than for regular lattices.

Many condensed-matter systems can be characterized as fractal objects; for instance, percolation clusters, aggregates obtained from diffusion-limited growth processes, and adsorbent surfaces [1]. Earlier studies on fractals from the viewpoint of the renormalization flow were carried out by Gefen et al. [2–5]. One of the main results yields the fact that the short-range classical spin models on finitely ramified lattices exhibit no phase transition at nonzero temperature [1,6]. The explanation can come from the relation of the boundary length of a finite-size fractal to its linear size, which strongly resembles one-dimensional systems. However, the Ising model on the Sierpiński carpet exhibits a phase transition [7]. There have been many attempts to study the Ising model on the Sierpiński carpet numerically by Monte Carlo combined with the finite-size scaling method [8–12], including the Monte Carlo renormalization group method [13]. Significant progress has been made recently in understanding the phase transition and critical phenomena on fractal lattices.

Adaptation of the higher-order tensor renormalization group (HOTRG) for a fractal lattice with Hausdorff dimension $d^{(H)} = \ln 12/\ln 4 \approx 1.792$ was introduced in Ref. [14]. Therein, the numerical calculations were shown to be stable with respect to the bond dimension $D$. Based on the order parameter, the critical temperature was obtained, together with the critical magnetic exponent. The density matrix spectrum exhibits an exponential decay even at the critical point, which is possible to interpret as the system being less entangled because of the fractal geometry. The technical details of the methods used in [14] are presented in Ref. [15]. The hyperscaling hypothesis is briefly discussed as well. Preliminary results on two separate infinite series of fractal lattices are presented and discussed.

In Ref. [16], HOTRG was applied to the study of the transverse-field Ising model on the Sierpiński fractal with the Hausdorff dimension $\log_2 3 \approx 1.585$. Ground-state energy and magnetization were calculated and analyzed.

In Ref. [17], HOTRG was adapted to the classical Ising model on the Sierpiński carpet with the Hausdorff dimension $\log_2 8 \approx 1.8927$ using two types of local tensors. The position dependence of local thermodynamic functions was studied by employing impurity tensors, which were inserted at different locations on the fractal lattice. It was found that the critical exponent associated with the local spin polarization (spontaneous magnetization) varies by two orders of magnitude, depending on lattice location; however, the critical temperature $T_c$ was found to be positionally independent.

Compared with their regular lattice counterparts, the geometrical details, such as lacunarity and connectivity, are the distinct key features of fractal lattices. If we embed a fractal lattice into a regular lattice and treat the coupling on the bonds not covered by the fractal differently, we can continuously interpolate between fractal and regular lattices. In this paper, we explore the phase-transition phenomena on a particular family of lattices that we can continuously interpolate between the planar fractal lattice [14] with the Hausdorff dimension $d^{(H)} = \ln 12/\ln 4 \approx 1.792$ and the regular square lattice $d^{(H)} = 2$. The property of self-similarity (i.e., scale

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with 12 2 spin vertices connected via spin-spin coupling equal to \( J \) and the thin (red) lines represent the interactions with the 4 vertices located in the corners only, being surrounded by the \( J \) couplings (in red). Right: Extended cluster containing 162 vertices with 122 spin vertices connected via \( J \) (the thick bonds in black) and the remaining 112 (= 4 × 16 + 12 × 4) spin vertices with \( J \) coupling only (thin bonds in red).

invariance) of the lattice is preserved throughout the transformation. However, only the regular square lattice is fully translationally invariant.

Here, we develop a numerically stable technique, which can also be applicable to a family of fractal lattices with partial translational noninvariance. We call this family of lattices 1-J1,2 fractals, as our technique employs two types of couplings: \( J_1 \) for bonds creating the fractal (thick bonds in black, as depicted in Fig. 1) and the remaining couplings \( J_2 \) (thin bonds in red). A pure fractal lattice is obtained when specified bonds are cut by setting \( J_2 = 0 \), and a regular square lattice is recovered when \( J_1 = J_2 = 1 \).

In order to analyze the phase transitions for the spin models on the fractal, we generalized the extension scheme in HOTRG [18], as we have used in Refs. [14,15]. Rather than considering only one type of the local tensor with the uniform extension relation, we introduce several types of local tensors (in this case 10), with each one being extended by a different recursion relation. The recursion relations specify how to combine different tensors in order to extend the size of the fractal iteratively, as required by HOTRG. The recursion relations reflect the symmetry of the self-similar lattices at every scale (i.e., scale invariance) and are compatible with the framework of the renormalization group applied to tensor network states. The computational cost scales with the bond dimension in the same way as in the two-dimensional HOTRG with a constant-factor overhead.

Local observables, such as magnetization and energy, can be implemented by means of impurity tensors. For concreteness, we focus on the Ising model on the fractal lattice shown in Fig. 1, which was recently studied using a different approach [14,15] and is meant for comparison with the current approach, which has the potential to be applied to various types of fractals.

We expect that the critical behavior may substantially change as the lattice transforms from the regular lattice to a fractal one. For example, if comparing the Ising model on the fractal and square lattices [14,15], the numerically calculated critical temperatures \( T_c \) and associated magnetic critical exponents \( \beta \) differed significantly: \( T_c \approx 1.317 \), \( \beta \approx 0.0137 \) and \( T_c = 2/\ln(1 + \sqrt{2}) = 2.26919 \ldots \), \( \beta = 1/8 \approx 0.125 \), respectively.

Moreover, we observed no divergence of the specific heat at the critical temperature in the fractal-lattice Ising model, as it has to be on the regular lattice. Therefore, at some point during the transformation from the fractal to the regular lattice, the character of the phase transition must change from a weakly singular to the standard behavior, as known for the continuous (second-order) phase transition.

II. MODEL REPRESENTATION

First, we construct a square lattice with two types of bonds, the thick black (\( J_1 \)) and the thin red (\( J_2 \)), connecting only the nearest-neighbor lattice vertices, where the spins are located. Hence, the fractal structure (\( J_1 \)) and the remaining space (\( J_2 \)) are comprised of black and red bonds, respectively. The iterative structure of HOTRG follows an extension series of how to build up the (fractal) lattice. It starts from a unit cell made of a 4 × 4 grid of spin vertices, where 12 vertices consist of three or four bonds \( J_1 \) (in black) and the remaining 4 vertices consist of four bonds \( J_2 \) placed in the corners, as in Fig. 1 (left). After the initial 16-spin unit cell was copied 12 times, in the next iteration step, the identical pattern needs to be formed, as depicted for the 4 × 4 grid. It then becomes a 16 × 16 grid with the four corners, each made of the 16-spin vertices containing \( J_2 \) couplings only, in accord with Fig. 1 (right).

Notice that by disregarding the four corners, the expansion reduces to the process studied earlier [14].

Consider the \( J_1\)-\( J_2 \) fractal Ising model Hamiltonian,

\[
\mathcal{H} = -J_1 \sum_{\langle i,j \rangle_1} \sigma_i \sigma_j - J_2 \sum_{\langle i,j \rangle_2} \sigma_i \sigma_j - h \sum_i \sigma_i, \quad (1)
\]

where the Ising variable \( \sigma \) takes the value +1 or −1, the non-negative ferromagnetic couplings \( J_1, J_2 \), and \( h \) being the uniform external magnetic field. To distinguish the summation over \( J_1 \) and \( J_2 \) couplings, we separate the two sums and denote them \( \langle \cdot \rangle_1 \) and \( \langle \cdot \rangle_2 \), respectively. For brevity, we do not include the magnetic-field term \( h \) in the following. The two local Boltzmann weights (between two adjacent spins) are given by

\[
\mathcal{V}_h^{(\xi)}(\sigma_i, \sigma_j) = \exp \left( \frac{J_\xi}{k_BT} \sigma_i \sigma_j \right), \quad (2)
\]

where the superscript index \( \xi = \{1, 2\} \) specifies the two types of the coupling \( J_\xi \). Here, \( k_0 \) is the Boltzmann constant and \( T \) is the temperature. Without loss of generality, we will set \( k_B = 1 \) in what follows next. The partition function is expressed as

\[
\mathcal{Z} = \sum_{\{\sigma\}} \prod_{\langle i,j \rangle_\xi} \mathcal{V}_h^{(\xi)}(\sigma_i, \sigma_j), \quad (3)
\]
where the sum is taken over all spin configurations \([\sigma]\). Furthermore, the bond weights \(W_B^{(c)}(\sigma_i, \sigma_j)\) can be reexpressed in terms of the matrix factorization,

\[
W_B^{(c)}(\sigma_i, \sigma_j) = \sum_{x=0}^{1} W_i^{(c)} W_j^{(c)},
\]

(4)

where the first matrix index \(\xi_i = (1 - \sigma_i)/2\) takes the values 0 or 1 if \(\sigma_i = \pm 1\). Thus, the \(2 \times 2\) matrix \(W^{(c)}(\xi)\) has the elements

\[
W^{(c)}(\xi) = \begin{pmatrix}
\cosh J_{x}/k_B T & \sinh J_{x}/k_B T \\
\cosh J_{y}/k_B T & -\sinh J_{y}/k_B T
\end{pmatrix}.
\]

(5)

Finally, we are ready to represent the partition function as a (nonhomogeneous) tensor-network state at the \(n\)th iteration step \((n = 0, 1, 2, 3, \ldots)\),

\[
\mathcal{Z}_n = \text{Tr} \prod_i \mathcal{T}_i^{[k],n}_{\xi_i,\xi_j},
\]

(6)

with the position-dependent local tensor \(\mathcal{T}_i^{[k],n}\) specified at lattice site \(i\). Each local tensor also carries information on types of couplings \(J_1\) and \(J_2\), which is subindexed by integer \([k]\) at the vertex position \(i\). The four surrounded indices keep the fixed ordering so that \(x_i\) (points to the left), \(x_i'\) (right), \(y_i\) (up), and \(y_i'\) (down), i.e.,

\[
\mathcal{T}_i^{[k],n}_{\xi_i,\xi_j} = \sum_{\xi} W_i^{(c)} W_j^{(c)} W^{(c)}_{\xi_i},
\]

(7)

where \(c = 1, 2\) specifies the coupling constant \(J_c\) depending on the orientation (left, right, up, down), which is determined by the index ordering in the position-dependent local tensor \(\mathcal{T}\).

Notice that the local tensor can take up to \(2^4\) different configurations (four indices or legs of the two states). On the other hand, the tensor-network state of the current fractal structure is constructed by 10 types of the local tensors \(\mathcal{T}_i^{[k],n}\), where \(k = 1, 2, 3, \ldots, 10\), as in Table I (upper row). We initialize the local tensors at the zeroth iteration step \((n = 0)\) as

\[
\mathcal{T}_x^{[1],n=0} = \sum_{\sigma} W_{\sigma x}^{(1)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)} W_{\sigma y''}^{(1)}
\]

\[
\mathcal{T}_x^{[2],n=0} = \sum_{\sigma} W_{\sigma x}^{(1)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)} W_{\sigma y''}^{(1)}
\]

\[
\mathcal{T}_x^{[3],n=0} = \sum_{\sigma} W_{\sigma x}^{(2)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)}
\]

\[
\mathcal{T}_x^{[4],n=0} = \sum_{\sigma} W_{\sigma x}^{(2)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)}
\]

\[
\mathcal{T}_x^{[5],n=0} = \sum_{\sigma} W_{\sigma x}^{(2)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)}
\]

\[
\mathcal{T}_x^{[6],n=0} = \sum_{\sigma} W_{\sigma x}^{(2)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)}
\]

\[
\mathcal{T}_x^{[7],n=0} = \sum_{\sigma} W_{\sigma x}^{(2)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)}
\]

\[
\mathcal{T}_x^{[8],n=0} = \sum_{\sigma} W_{\sigma x}^{(2)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)}
\]

\[
\mathcal{T}_x^{[9],n=0} = \sum_{\sigma} W_{\sigma x}^{(2)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)}
\]

\[
\mathcal{T}_x^{[10],n=0} = \sum_{\sigma} W_{\sigma x}^{(2)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)}
\]

(8)

Coarse-graining procedure

To consistently define the iterative extension procedure, we need to extend each of the 10 local tensors according to the specific extension relation for the next iteration step \(n + 1\), as graphically summarized in Table I. After the extension procedure of the 10 extended tensors \(\mathcal{T}^{[k],n+1}\) is finalized, we will apply (renormalization) transformations to reduce the degrees of freedom of the expanded tensors, specified later. Multiple types of the local tensors enter the extension relation for each tensor. Therefore, all the extensions for the next iteration step \(n + 1\) need to be performed by means of those from the previous iteration step \(n\), which have to be kept simultaneously, until the entire update of the tensors is completed. (The tensor \(\mathcal{T}^{[10]}\) is a special case and is extended with the copies of itself.)

As a typical example, consider the extension of the tensor \(\mathcal{T}^{[1]}\) in detail. According to Table I, for obtaining the new tensor \(\mathcal{T}^{[1],n+1}\), we contract 16 tensors at step \(n + 1\) (four tensors of type \(k = 1\), two of type \(k = 2\), two of type \(k = 3\), two of type \(k = 4\), two of type \(k = 5\), and, finally, four of type \(k = 10\)) arranged onto a \(4 \times 4\) grid to satisfy the extended pattern \(k = 1\) for the next step \(n + 1\),

\[
\mathcal{T}_x^{[1],n+1} = \sum_{\sigma} W_{\sigma x}^{(1)} W_{\sigma y}^{(1)} W_{\sigma y'}^{(1)}
\]

(9)
FIG. 2. Graphical visualization of the extension and renormalization procedures to build up the tensor \( T^{[1,n+1]} \) out of the appropriate tensors \( T^{[s,n]} \). In total, 15 different projectors are introduced to perform four steps of the coarse graining in HOTRG. We stress the way of distinguishing the external projectors \( U_i, i = 1, 2, \ldots, 6 \), which perform the projections onto the external legs, from the internal projectors \( \tilde{U}_i, i = 1, 2, \ldots, 9 \), which perform the internal projections within the \( 4 \times 4 \) grid.

Analogous extension relations hold for the remaining tensors, as listed in Table I, and more details can be found in Appendix A.

III. RENORMALIZATION TRANSFORMATION

One step of the coarse-graining procedure defined by Eq. (9) increases the bond dimension as the fourth power. It is, thus, numerically inefficient to exactly contract all of the 16 tensors comprising one unit cell in just a single step. A simple way of introducing an efficient approximation is to perform four steps of the HOTRG-style coarse graining by contracting neighboring pairs of tensors at each step while specifying all the adjacent projectors, being properly matched, as depicted in Fig. 2. Notice Fig. 2, where we have introduced 15 different projectors, where six of the projectors \( U_i, i = 1, 2, \ldots, 6 \) perform projections onto the external (renormalized) tensor indices or legs, whereas nine of the projectors \( \tilde{U}_i, i = 1, 2, \ldots, 9 \) perform internal projections inside the \( 4 \times 4 \) tensor grid.

Now, let us demonstrate how the internal renormalization transformations \( \tilde{U}_1 \) and \( \tilde{U}_2 \) are calculated. By contracting the tensors \( T^{[s,n]} \) and \( T^{[1,n]} \) along the \( y \) axis, we define

\[
M^{[5,1,n]}_{x\times(y'\times y')} = \sum_i T^{[s,n]}_{x\times(y'\times y')} T^{[1,n]}_{x\times y' \times y'},
\]

where \( x = x_1 \otimes x_2 \) and \( x' = x'_1 \otimes x'_2 \). To truncate the tensor \( M^{[5,1,n]}_{x\times(y'\times y')} \) in accord with the higher-order singular-value decomposition (HOSVD) [19], the following matrix unfolding has to be prepared:

\[
M^{[5,1,n]}_{x\times(y'\times y')} = M^{[5,1,n]}_{x\times y'\times y'}.
\]

We then perform the singular-value decomposition (SVD),

\[
M^{[5,1,n]}_{x\times y'\times y'} = \tilde{U}_1 \tilde{\omega}_1 \tilde{V}_1^\dagger,
\]

where \( \tilde{U}_1 \) and \( \tilde{V}_1^\dagger \) are unitary matrices of the respective dimensions, and \( \tilde{\omega}_1 \) is a diagonal matrix with the non-negative singular values on its diagonal ordered in the decreasing order by convention.

To obtain the internal renormalization transformation \( \tilde{U}_2 \), we contract the tensors \( T^{[1,n]} \) and \( T^{[1,n]} \) along the \( y \) axis,

\[
M^{[3,1,n]}_{x\times y'\times y'} = \sum_i T^{[3,1,n]}_{x\times y' \times y'} T^{[1,n]}_{x\times y' \times y'},
\]

where \( x = x_1 \otimes x_2 \) and \( x' = x'_1 \otimes x'_2 \). To truncate the tensor \( M^{[3,1,n]}_{x\times y'\times y'} \) by HOSVD, the following matrix unfolding is prepared:

\[
M^{[3,1,n]}_{x\times(y'\times y')} = M^{[3,1,n]}_{x\times y'\times y'},
\]

Notice that this unfolding is different from Eq. (11), as we optimize the right side of the contracted tensor \( M^{[3,1,n]} \), as opposed to the left side. From SVD, we get

\[
M^{[3,1,n]}_{x\times(y'\times y')} = \tilde{U}_2 \tilde{\omega}_2 \tilde{V}_2^\dagger,
\]

where \( \tilde{U}_2 \) and \( \tilde{V}_2^\dagger \) are unitary matrices, and \( \tilde{\omega}_2 \) is a diagonal matrix with singular values (ordered decreasingly). The remaining projectors are calculated similarly.

The singular values obtained by SVD in Eqs. (12) or (15) can be used to calculate the entanglement entropy. Alternatively, we define the entanglement entropy using the singular values obtained from SVD applied directly to the tensor \( T^{[1,n]} \) unfolded into a matrix as \( T^{[1,n]}_{x\times y'\times y'} = T^{[1,n]}_{x\times y' \times y'} \). Having performed the SVD \( T^{[1,n]} = U \omega V^\dagger \), we calculate the entanglement entropy \( s \) as follows:

\[
s = -\text{Tr} \rho \ln \rho = - \sum_{\ell=1}^{D} \frac{\omega_\ell }{\Omega} \ln \frac{\omega_\ell }{\Omega},
\]

where \( \rho \) is a reduced density matrix and \( \Omega = \sum_{\ell=1}^{D} \omega_\ell \) is a normalization factor such that \( \text{Tr} \rho = 1 \).

The projectors \( U \) and \( \tilde{U} \) are obtained with the extension pattern \( (k = 1) \) corresponding to \( T^{[1]} \). However, we can consider applying these projectors uniformly to the remaining extension patterns with \( k > 1 \). With such a simple setup, the projectors’ consistency at the boundaries between all the unit cells is clearly satisfied. If necessary, the internal projectors \( \tilde{U} \) can be obtained from each extension pattern separately, which can increase numerical accuracy a bit, at the price of higher computational costs (by a constant factor at most). The external projectors \( U \), however, have to be uniform, and we need to decide which of the projectors to apply at the boundaries of the unit cells.

We have encountered some numerical instabilities when projecting the tensor patterns with \( k \geq 2 \) using the projectors obtained from \( T^{[1]} \). To improve the approximation scheme described above, we can introduce multiple sets of the external projectors. As a simple yet practical example, let us consider two sets of external projectors \( U_i^1, U_i^2 \) \((i = 1, 2, \ldots, 6)\), each containing six HOTRG isometries. The projectors \( U_i^1 \) are obtained as before (i.e., on \( T^{[1]} \)); however, \( U_i^2 \) are obtained on the homogeneous pattern defining \( T^{[10]} \). We can then use \( U^1 \) when truncating a thick leg (black) and \( U^2 \) when truncating a thin leg (red). For instance, the renormalization relations of
the tensors $T^{[1]}$, $T^{[2]}$, $T^{[6]}$, and $T^{[10]}$ for the external legs are straightforward,

$$T^{[1],n+1}_{x' x' y' y'} = x' \cdot y', \quad (17)$$

$$T^{[2],n+1}_{x' x' y' y'} = x' \cdot y', \quad (18)$$

$$T^{[6],n+1}_{x' x' y' y'} = x' \cdot y', \quad (19)$$

$$T^{[10],n+1}_{x' x' y' y'} = x' \cdot y', \quad (20)$$

where the projectors $U^1$ and $U^2$ are depicted by dashed thick (black) and thin (red) lines, respectively. The explicit form of Eqs. (17)–(20) is presented in Appendix B. For completeness, we list all the projection patterns in Appendix B as well.

Impurity tensors

Magnetization. We can define the impurity tensor $\tilde{T}^{n=0}$ by inserting a spin variable $\sigma = 1 - 2\xi$ into the local tensor with the pattern $T^{[1],n=0}$ as follows [cf. Eq. (8)]:

$$\tilde{T}^{n=0}_{y' y' y' y'} = \sum_{\xi} (1 - 2\xi) W^{(1)}_{\xi} W^{(1)}_{\xi} W^{(1)}_{\xi} W^{(1)}_{\xi}. \quad (21)$$

The extension of the impurity tensor, $\tilde{T}^{n} \rightarrow \tilde{T}^{n+1}$, is performed by taking an average over four central vertices in the extension pattern $T^{[1],n+1}$ [cf. Eq. (9)],

$$\tilde{T}^{n+1}_{x' x' x' x'} = \sum_{\xi} (1 - 2\xi) W^{(1)}_{\xi} W^{(1)}_{\xi} W^{(1)}_{\xi} W^{(1)}_{\xi}. \quad (22)$$

where the full circles represent the inserted impurities. The explicit form of Eq. (22) is presented in Appendix C1.

Bond energy. The bond energy is proportional to the correlation between two nearest-neighbor spins. We start by defining an initial impurity tensor, as we have done for the spontaneous magnetization [see Eq. (21)], and taking an average over the four bond energies at the central spins, as they correspond to the nearest-neighbor pairs. The first extension step ($n = 0$) is performed by taking an average over the four different neighboring pairs of the impurity tensors in the extension pattern $T^{[1],n+1}$,

$$\tilde{T}^{n=1}_{x' x' x' x'} = \sum_{\xi} (1 - 2\xi) W^{(1)}_{\xi} W^{(1)}_{\xi} W^{(1)}_{\xi} W^{(1)}_{\xi}. \quad (23)$$

When $n > 0$, we extend the bond-energy impurity tensor according to Eq. (22). The explicit form of Eq. (23) is presented in Appendix C2.

IV. NUMERICAL RESULTS

We are interested in the critical behavior of the $J_1$-$J_2$ fractal lattice in the regime between the pure fractal lattice (i.e., when $J_1 = 1$ and $J_2 = 0$) and the regular square lattice (i.e., when $J_1 = J_2 = 1$). From now on, we will set the fractal coupling to $J_1 = 1$, only changing $0 \leq J_2 \leq 1$.

Let us first analyze the spontaneous magnetization (see Fig. 3), where the phase-transition (critical) temperature $T_c$ continuously increases as $J_2$ increases. The power-law decay of the spontaneous magnetization, obtained from the impurity tensor $T^n$ at $T \leq T_c$ and the external magnetic field $h = 0$,
below the critical temperature, follows the scaling

\[ \langle \tilde{T}^n \rangle \propto (T_c - T)^\beta. \]  

(24)

It is important to stress the fact that the critical exponent \( \beta \) does not significantly vary within the entire interval of \( 0 \leq J_2 \leq 1 \) (see inset of Fig. 3). It remains almost identical to the case of the pure fractal lattice (\( J_2 = 0 \)), where \( \beta \approx 0.015 \), until the square lattice recovers (\( J_2 = 1 \)), where the exponent suddenly jumps to the expected value (exact value is \( \beta_{\text{square}} = 1/8 = 0.125 \)).

Likewise, the other critical exponent \( \delta \) associated with the induced magnetization at \( T = T_c \) and a nonzero external magnetic field \( 0 \leq h \leq 10^{-7} \),

\[ \langle \tilde{T}^n \rangle|_{T=T_c} \propto h^{1/\delta}, \]  

(25)

does not significantly vary within \( 0 \leq J_2 \leq 1 \) (see inset of Fig. 3). The value of \( \delta \) does not change from the pure fractal case \( \delta \approx 185 \) when \( J_2 = 0 \) until the square lattice recovers (\( J_2 = 1 \)), where \( \delta \) jumps to the exact value known for the square lattice, \( \delta_{\text{square}} = 15 \).

In the case of the pure fractal lattice (\( J_2 = 0 \)), it is sufficient to employ a moderate value of the bond dimension \( D \). Our numerical results do not change for \( D \geq 16 \). Analyzing the spontaneous magnetization, we found the critical temperature \( T_c \approx 1.31695 \) and the critical exponent \( \beta \approx 0.0153951 \), which does not change for \( D = (16, 18, 20, 22, 24) \).

When comparing the current results with the previous study [15], the critical temperature is almost identical (compare to \( T_c \approx 1.31717 \) (with \( D = 32 \)), which yields the relative difference of \( \sim 0.02\% \). On the other hand, the magnetic exponent exhibits a larger difference at \( D = 32 \), where \( \beta = 0.01388 \), thus yielding the relative difference of \( \sim 10\% \).

For \( J_2 = 0.99 \), we got \( T_c \approx 2.24964 \) and \( \beta \approx 0.021 \) (with \( D = 34 \)).

For \( J_2 = 1 \) (being the regular square lattice), we got \( T_c \approx 2.26919 \) and \( \beta \approx 0.128 \) (\( D = 34 \)), in full agreement with the exact solution \( T_c = 2/\ln(1 + \sqrt{2}) \approx 2.26919 \) and \( \beta = 1/8 = 0.125 \). To show the bond-dimension \( D \) dependence, we plot the \( \beta \) exponent for \( J_2 = 0.99 \) and \( J_2 = 1 \) in Fig. 4.

For \( J_2 = 1.01 \) (i.e., inverse fractal lattice), we got \( T_c \approx 2.28875 \) and \( \beta \approx 0.5 \) (\( D = 34 \)). The critical temperature \( T_c \) continues to rise, as we further increase \( J_2 > 1 \). However, the critical exponent \( \beta \) seems to be stabilized around the (mean-field) value \( \beta \approx 0.5 \) in the inverse fractal regime when \( J_2 > 1 \) (not shown, but confirmed up to \( J_2 = 1.1 \) at \( D = 32 \)).

If studying the magnetic-field response, we introduce a small \( h \) into the system at the critical point. For the regular square lattice (\( J_2 = 1 \)), the HORTG method results in the critical exponent \( \delta \) with the relative error of less than one percent (with \( D = 34 \)). For the pure fractal lattice, we obtained \( \delta \approx 185 \) (with \( D = 34 \)). Comparing them with the previous results [15], we found a relative difference of \( \sim 11\% \) (the previous study yielded \( \delta \approx 206 \) at \( D = 12 \)). Interestingly, close to the regular square lattice from the pure fractal side (\( J_2 = 0.99 \)), we obtained \( \delta \approx 129 \) (with \( D = 34 \)), whereas, from the inverse fractal side (\( J_2 = 1.01 \)), we found \( \delta \approx 5.1 \) (with \( D = 34 \)).

In order to observe the phase transition, we evaluate the specific heat \( c(T) = \frac{\partial}{\partial T} u(T) \) by deriving the bond energy \( u(T) = \langle \tilde{T}^n \rangle \) in accord with Eq. (23). The specific heat \( c(T) \) does not diverge at any value of \( J_2 < 1 \); see Fig. 5. We observe the divergence only in the case of the regular square lattice, i.e., when \( J_2 = 1 \). If \( 0 < J_2 < 1 \), the maximum of the specific heat \( c(T) \) does not correspond to the critical point. Instead, we have confirmed and numerically verified (see Refs. [14] and [15]) that the maximum of the first numerical derivative with respect to \( T \) corresponds to the critical point for all values of \( J_2 < 1 \), i.e.,

\[ T_c = \max_T \left\{ \frac{d}{dT} c(T) \right\}. \]  

(26)

We have also explored the vicinity of \( J_2 = 1 \) and considered the cases when \( J_2 = 0.99 \) and \( J_2 = 1.01 \). There, the singularity of \( c(T) \) at \( T_c \) appears only if \( J_2 = 1 \), as depicted in Fig. 6.

In order to determine the phase transition correctly, we analyze entanglement entropy \( s(T) \) introduced in Eq. (16),
18 iteration steps. We select two values temperature $[14,17]$. in the inflection point, which refers to the correct phase-transition

which usually achieves full numerical convergence after $n \approx 18$ iteration steps. We select two values $J_2 = 0.3$ and $J_2 = 0.5$ (at $D = 16$) to observe how the iteration steps $n$ affect the entanglement entropy $s(T)$, as plotted in Figs. 7 and 8, respectively. Surprisingly, the converged entanglement entropy exhibits a divergence at temperatures $T \approx 0.68$ ($J_2 = 0.3$) and $T \approx 1.13$ ($J_2 = 0.5$), which substantially differs from the expected fractal critical temperatures $T_c \approx 1.40405$ and $T_c \approx 1.5777$, respectively. We have numerically confirmed for additional values of $0 < J_2 < 1$ that the sharp maxima of the converged entanglement entropy $n > 18$, typically associated with the phase transition, occurs at

$$T = J_2 T_c^{(J_2=1)} = \frac{2J_2}{\ln(1 + \sqrt{2})},$$

FIG. 6. The specific heat $c(T)$ around phase transition $T_c$ if $J_2 = 0.99, J_2 = 1.0$, and $J_2 = 1.01$ (at $D = 24$).

which corresponds to the critical temperature of the regular square-lattice Ising model $T_c^{(J_2=1)}$ multiplied by the coupling $J_2$. The entanglement entropy $s(T)$ at $J_2 = 0.5$ exhibits an interesting behavior (see Fig. 8) after six iterations $T$ yield two peaks: the left peak around $T \approx 1.13$ and the right peak around the correct fractal critical temperature $T \approx 1.5777$ at $J_2 = 0.5$. If zooming in around the fractal critical temperature, the entanglement entropy becomes invariant in between $n = 3$ and $n = 9$. Such fixed-point behavior captures the correct phase transition of the fractal.

FIG. 7. Entanglement entropy $s(T)$ for $J_2 = 0.3$ ($D = 16$) measured at various iteration steps $n$. The left peak in $s(T)$ is converged after six iterations ($n \geq 6$) and is located at $T = J_2 T_c^{(J_2=1)} \approx 0.68$. The entanglement entropy is invariant at $3 \leq n \leq 9$ resulting in the critical point of the fractal at $T_c^{(J_2=0.3)} \approx 1.40405$. The values of $s(T)$ denoted by stars are not shown in the main plot for better visibility (for $n = 7, 8, 9$). Inset: the zoomed-in view, where the entropy has the fixed point at $T_c^{(J_2=0.3)}$.

FIG. 8. Entanglement entropy $s(T)$ for $J_2 = 0.5$ ($D = 16$). The left peak in $s(T)$ appears at $T = J_2 T_c^{(J_2=1)} \approx 0.68$. The entanglement entropy is invariant at $T_c^{(J_2=0.5)} \approx 1.5777$. Inset: the detail of the entropy fixed point.
V. CONCLUSIONS AND DISCUSSIONS

We have investigated the Ising model on a continuous family of planar self-similar lattices with two types of coupling strengths, $J_1$ and $J_2$. For this purpose, we have developed a modified HOTRG technique with multiple types of local tensors; each one is extended with specific coarse-graining patterns. We focused on the way the critical behavior of the Ising model on the self-similar lattices changes while transforming the lattice from the fractal pattern into the regular square lattice by the continuous change of the value of $J_2$ from zero to one (if fixing $J_1 = 1$). The critical temperature $T_c$, as captured by the local order parameter (the spontaneous magnetization) as well as the local bond energy (the internal energy), grows continuously from $T_c(J_2=0) \approx 1.317$ (correct value for the fractal lattice as shown in Refs. [14,15]) to $T_c(J_2=1) \approx 2.269$ (known exactly). Having analyzed the magnetic critical exponent $β$ when tuning $0 \leq J_2 \leq 1$, we did not observe a continuous change of the exponent in the interval 0.015 $\leq β_0 \leq 1$, as one would have naturally expected. Instead, we determined almost a constant value of $β_0 \approx 0.015$ on the entire interval $0 \leq J_2 < 1$, followed by a significant discontinuous jump if $J_2 = 1$, where $β_0 = 1$. The other magnetic critical exponent $δ$ exhibited similar singular behavior. We measured a constant value of $δ_0 \approx 185$ on the interval $0 \leq J_2 < 1$, followed by a significant discontinuous jump if $J_2 = 1$, where $δ_0 = 15$.

The specific heat $c(T)$ also followed qualitatively similar behavior as the magnetic exponent $β$. A sharp singularity appeared at $J_2 = 1$ only, whereas broadened maxima of $c(T)$ did not coincide with the correct $T_c$ determined from the spontaneous magnetization in Fig. 3. Instead, the sharp peaks of $\frac{d}{dT}c(T)$ referred to the correct $T_c$ (in accord with Ref. [14]).

To address the question of why we observed the discontinuity in $β$ when $J_2 = 1$ (including analogous behavior in the specific heat), we relate the answer to the fact that the $J_1$-$J_2$ model on the lattice becomes fully translationally invariant only if $J_1 = J_2$. Otherwise, the $J_1$-$J_2$ fractals possess a weaker type of symmetry, i.e., the scale invariance. Based on our numerical observations, we conjecture that there are three classes of the behavior of the $J_1$-$J_2$ fractals: (i) fractal-like when $J_1 < J_2$ (with $β \approx 0.02$), (ii) regular square lattice when $J_1 = J_2$ (with $β \approx 1/8$), and (iii) inverse fractal when $J_1 > J_2$ (with the mean-field value of $β \approx 0.5$).

The entanglement entropy $s(T)$ calculated from the singular values obtained as a byproduct of the SVD during the renormalization of the tensors captures the global behavior of the system. The entanglement entropy after around $n = 18$ iteration steps corresponds to the regular square lattice, whose critical point uniformly scales with the prefactor $J_2$ and information on the fractal structure is suppressed. However, when the number of the iteration step in HOTRG does not reach the full numerical convergence with spontaneously broken symmetry, $s(T)$ can capture two phase transitions: (i) the one occurring inside the fractal structure, which is in an agreement with the local quantities such as the spontaneous magnetization, and (ii) the phase transition taking place on the homogeneous part at the critical temperature $T = J_2T_c(J_2=1)$; cf. Eq. (27). The former case corresponds to the fixed-point-like behavior of the entanglement entropy $s(T)$ with respect to the iteration steps $n$.

Another interesting question arises: how to think about the dimensionality of the lattice, between the fractal and the regular square lattice. We propose that it might be meaningful to define an appropriate dimension, in this case, being a combination of both coupling strengths $J_1$ and $J_2$. A generalized Hausdorff dimension might have been defined,

$$d_1 = \frac{\ln(12J_1 + 4J_2)}{\ln 4}. \quad (28)$$

If considering the scaling of the boundary bonds, the other dimension could follow the expression

$$d_2 = 1 + \frac{\ln(2J_1 + 2J_2)}{\ln 4}. \quad (29)$$

Both formulas need to be adapted to the case when $J_2 = 0$. Then, irrespective of $J_1$, we need to recover $d_1 = \ln 12/\ln 4$ and $d_2 = 1.5$ (see Ref. [14]), provided that $J_1 \neq 0$.

The current study opens the door to many exciting directions for research. HOTRG can be applied to the study of the quantum Ising model on the $J_1$-$J_2$ fractals. Also, the extension to three dimensions of the correct classical spin model is possible, although computationally very complex and requiring more extensive computational resources. Moreover, the technique presented in this work is applicable to a variety of inhomogeneous lattices, including the Sierpiński carpet. In some cases, it might be desirable to employ different optimization scheme of the local tensors than used here; however, the basic idea of extending multiple types of the local tensors remains valid. Apart from the typical condensed-matter applications, our technique might inspire new data compression approaches, for example, in image processing.

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APPENDIX A: EXTENSION PATTERNS

Each of the 10 tensor types can be extended by means of a different extension relation (see Table 1). For instance, the extension formula for the tensor type $T^{[k=1]}$ is shown in Eq. (9).

Consider the extension of the tensor $T^{[1]}$ in detail. As seen from Table 1, to obtain the new tensor $T^{[1,\pi+1]}$, one needs to contract 16 tensors in total (four tensors of type 1, two of type 2, two of type 2, two of type 4, two of type 5, and, finally, four of type 10) arranged on a $4 \times 4$ grid according to the specified pattern

$$T^{[1,\pi+1]}_{\{i_1,i_2,i_3,i_4\} \{i_1',i_2',i_3',i_4'\}}(i_1,j_1,i_3,j_3)$$

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Here, we list all 10 extension relations in an abbreviated form where we omitted all the tensor indices except for the tensor-type index (in square brackets) and the iteration step \( n \) [the omitted tensor indices are identical to those in Eq. (A1) in all the remaining formulas]. The new tensors \( T^{[k],n+1} \) have been obtained from the preceding iteration step, out of the tensors \( T^{[k],n} \), where \( k = 1, 2, \ldots, 10 \). Each extension relation specifies the pattern of the 16 previously prepared tensors \( T^n \) on a 4 x 4 grid (on the right-hand side of the formulas below), which are needed to obtain the extended \( T^{n+1} \) tensors of each type (on the left-hand side),

\[
\begin{align*}
T^{[1],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t} \\
T^{[2],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t} \\
T^{[3],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t} \\
T^{[4],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t} \\
T^{[5],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t} \\
T^{[6],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t} \\
T^{[7],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t} \\
T^{[8],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t} \\
T^{[9],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t} \\
T^{[10],n+1} & \leftarrow \sum_{abcdjghi} \sum_{jk} \sum_{tnwux} \tau^{[10],a}_{t} \tau^{[10],b}_{j} \tau^{[10],c}_{k} \tau^{[10],d}_{n} \tau^{[10],e}_{w} \tau^{[10],f}_{t} \tau^{[10],g}_{j} \tau^{[10],h}_{k} \tau^{[10],i}_{n} \tau^{[10],x}_{u} \tau^{[10],y}_{w} \tau^{[10],z}_{t}
\end{align*}
\]
\[ \mathcal{T}^{[6],n+1} \leftrightarrow \sum \mathcal{T}^{[6],n+1} U \]
\[ \mathcal{T}^{[7],n+1} \leftrightarrow \sum \mathcal{T}^{[7],n+1} U \]
\[ \mathcal{T}^{[8],n+1} \leftrightarrow \sum \mathcal{T}^{[8],n+1} U \]
\[ \mathcal{T}^{[9],n+1} \leftrightarrow \sum \mathcal{T}^{[9],n+1} U \]
\[ \mathcal{T}^{[10],n+1} \leftrightarrow \sum \mathcal{T}^{[10],n+1} U \]

**APPENDIX C: EXTENSIONS OF IMPURITY TENSORS**

1. Magnetization

The extension of the impurity tensor \( \tilde{T}^{n} \) is performed by taking an average over four central spins of the impurity in the extension pattern \( \mathcal{T}^{[1]} \) [cf. Eq. (A1)],

\[
\tilde{T}^{n+1} = \frac{1}{4} \sum_{ijklm nop} \sum_{\text{c}} \left[ \cdots \right]
\]

where the letter \( \tilde{T} \) in bold is meant to stress the readability of the above formula.

2. Bond energy

The expression for the bond energy begins with the identical initial impurity tensor as we used in the spontaneous magnetization [see Eq. (21)]. However, the averaging over the bond energy, corresponding to the spin-pair, is necessary to be performed. Therefore, the first extension takes an average over the four different neighboring pairs of the impurities in the extension pattern \( \mathcal{T}^{[1]} \),

\[
\tilde{T}^{n+1} = \frac{1}{4} \sum \sum_{ijklm nop} \sum_{\text{c}} \left[ \cdots \right]
\]

If \( n > 0 \), we proceed in the bond-energy extension of the impurity tensor according to the relation defined for magnetization in Appendix C1.
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