Baryons from Quarks in the 1/N Expansion

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Abstract

We present a diagrammatic analysis of baryons in the 1/N expansion, where N is the number of QCD colors. We use this method to show that there are an infinite number of degenerate baryon states in the large-N limit. We also show that forward matrix elements of quark bilinear operators satisfy the static quark-model relations in this limit, and enumerate the corrections to these relations to all orders in 1/N. These results hold for any number of light quark flavors, and the methods used can be extended to arbitrary operators. Our results imply that for two flavors, the quark-model relations for the axial currents and magnetic moments get corrections of order 1/N^2. For three or more flavors, the results are more complicated, and corrections are generically of order 1/N. We write an explicit effective lagrangian which can be used to carry out chiral perturbation theory calculations in the 1/N expansion. Finally, we compare our results to what is expected from a chiral constituent quark model.

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1. Introduction

It is well-known that QCD has a non-trivial limit as $N \to \infty$, where $N$ is the number of colors [1]. This limit can be defined by writing the QCD covariant derivative as

$$D_\mu = \partial_\mu + \frac{i g}{\sqrt{N}} A_\mu,$$

where $A_\mu$ is the gluon field, and letting $N \to \infty$ holding $g$ fixed. QCD in the large-$N$ limit is believed to share many features with QCD at $N = 3$, including confinement and chiral symmetry breaking. (It can be shown that confinement implies chiral symmetry breaking in the large-$N$ limit of QCD [2].) Therefore, an expansion in $1/N$ around the large-$N$ limit of QCD can be expected to give valuable insights into non-perturbative QCD phenomena.

The properties of baryons in the large-$N$ limit were first discussed in detail by Witten [3], who argued that baryons can be described by Hartree–Fock equations in this limit. This approach gave a consistent picture of baryons, and showed how various physical quantities scale with $N$. Most of the subsequent work on baryons in the large-$N$ limit was done in the context of the Skyrme model [4], which has been argued to be exact as $N \to \infty$ [5]. The baryon spectrum in the Skyrme model consists of an infinite number of degenerate states. It can also be shown that the axial current matrix elements obey the static quark-model relations in the large-$N$ limit of the Skyrme model [6]. This result was extended to all operators in ref. [7]. Another approach was initiated by Gervais and Sakita [8], who showed that consistency of chiral perturbation theory with the large-$N$ limit implies the quark-model relations for the axial current matrix elements in the large-$N$ limit. These methods were recently rediscovered and extended by Dashen and Manohar [9] and Jenkins [10], who considered $1/N$ corrections in this approach.

In this paper, we will use diagrammatic and group-theoretic methods to analyze baryons in the $1/N$ expansion. Using these methods, we derive the quark-model relations directly from QCD in a way which makes clear the role of quarks as the constituents of baryons. These methods allow us to enumerate the corrections to these relations to all orders in the $1/N$ expansion for the baryon masses and matrix elements of quark bilinear operators for any number of light quark flavors. (The methods used can be easily extended to general matrix elements.) For 2 flavors, our results imply that the quark-model relations for the axial currents and magnetic moments get corrections of order $1/N^2$. (For the non-singlet axial currents, this result was obtained in ref. [9].) For 3 or more flavors, the results are more complicated, since the baryon representations depend on $N$ in this case. We find that the corrections to the quark-model relations are generically of order $1/N$.

Combining these results with current algebra ideas, we then write an effective lagrangian which can be used to carry out a simultaneous chiral and $1/N$ expansion for
processes involving baryons and soft pseudo Nambu–Goldstone bosons. This lagrangian contains couplings that grow with $N$, so that it is not manifest that its predictions are physically sensible. We have verified the consistency of the predictions of this lagrangian in some non-trivial cases (including all cases considered in refs. [8][9][10]), but we have not proved it in general. However, we argue that this effective lagrangian is the most general one with the right field content and symmetry structure in the large-$N$ limit, and we therefore believe that it must be consistent on general grounds.

We conclude with some remarks comparing our results to what is expected from a non-relativistic constituent quark model.

2. $N$-Counting Rules for Baryons

In this section, we derive our main results on the structure of baryon masses and matrix elements in the $1/N$ expansion. We begin by giving a rough idea of the nature of our argument. Consider a baryon matrix element of the form $\langle B'|T\hat{O}_1\cdots\hat{O}_n|B\rangle$. The operators $\hat{O}_1,\ldots,\hat{O}_n$ can change the spin and flavor of the quarks in the initial baryon consistent with invariance under the flavor symmetry $SU(N_F)$. If the velocities of the initial and final baryon states are equal, then the spin and flavor changes must also be invariant under $SU(2)_v$, the group of spatial rotations in the common baryon rest frame. It turns out that invariance under the combined $SU(2)_v \times SU(N_F)$ symmetry is especially restrictive when $N$ is large for states with total spin $J \ll N$: the amplitude to change the spin or flavor of a quark is suppressed by $1/N$ for such states. This combinatoric result is responsible for the appearance of the static quark-model relations in the large-$N$ limit for forward baryon matrix elements.

To make this argument precise, our strategy is to write a diagrammatic expansion for the quantities we wish to study, and then find properties of this expansion that hold to all orders in $\mathcal{g}$. This method is not strictly rigorous, since it is not known whether the quantities in question can actually be obtained by “summing all diagrams.” However, the results derived here depend only on the $N$-counting of the diagrammatic expansion, which we believe to be reliable.

We consider the QCD states

$$|\mathcal{B}_0\rangle = \mathcal{B}^{a_1\alpha_1\cdots a_N\alpha_N}e^{A_1\cdots A_N}a_a a_{\alpha_1} a_{A_1} \cdots a_a a_{\alpha_N} a_{A_N}|0\rangle,$$

where $a_1,\ldots,a_N = 1,\ldots,N_F$ are flavor indices, $\alpha_1,\ldots,\alpha_N = \uparrow,\downarrow$ are spin indices, and $A_1,\ldots,A_N = 1,\ldots,N$ are color indices. The operators $a_a$ in eq. (2) create a quark with definite flavor, spin, and color, in a perturbative 1-quark state $|\phi\rangle$. We take $|\phi\rangle$ to
have average momentum zero and momentum uncertainty of order $\Lambda_{\text{QCD}}$. Our results are insensitive to the precise nature of the state $|\phi\rangle$. The important feature of the state $|B_0\rangle$ for our analysis is that it has the right quantum numbers to be a 1-baryon state at rest: it is a color singlet, has unit baryon number, and has angular momentum and flavor quantum numbers determined by the tensor $\mathcal{B}$.

The tensor $\mathcal{B}$ in eq. (2) is symmetric under combined interchange of spin and flavor indices, $a_1\alpha_1 \leftrightarrow a_2\alpha_2$, etc. $\mathcal{B}$ may be thought of as a quark-model wave function describing states with spin $J = \frac{1}{2}, \ldots, \frac{1}{2}N$ (for $N$ odd). The Young tableaux of the $SU(N_F)$ representation with spin $J$ is shown in fig. (1a). We will assume that the lowest-lying baryon states in QCD have these quantum numbers for arbitrarily large $N$, with the masses of the baryons increasing with $J$. This assumption is supported by the Skyrme model, and of course is true for $N = 3$.

Under time evolution, the state $|B_0\rangle$ projects onto the state $|\mathcal{B}\rangle$, where $|\mathcal{B}\rangle$ is the minimal energy eigenstate of the full hamiltonian with a nonzero overlap with $|B_0\rangle$:†

$$\hat{U}(0,-T)|B_0\rangle \longrightarrow |\mathcal{B}\rangle \langle \mathcal{B}|B_0\rangle e^{-iE_B T} + \cdots, \tag{3}$$

$$\langle B_0|\hat{U}(T,0) \longrightarrow \langle B_0|B\rangle |\mathcal{B}\rangle e^{-iE_B T} + \cdots,$$

as $T \rightarrow -i\infty$, where the ellipses denote terms which are exponentially suppressed. (The analytic continuation is the usual Wick rotation, which is justified by the assumption that the only singularities of the time-evolution operator are those required by unitarity and crossing.) Eq. (3) allows us to write matrix elements involving the state $|\mathcal{B}\rangle$ as a sum of diagrams with external states of the form $|B_0\rangle$. In our approach the state $|B_0\rangle$ plays a role similar to that of the perturbative vacuum, where a formula like eq. (3) is used to derive a diagrammatic expansion for time-ordered products in the interacting vacuum.

If we choose the tensor $\mathcal{B}$ in eq. (2) to have definite spin–flavor quantum numbers, then by the assumptions above, there is a 1-baryon state with the same quantum numbers as $|B_0\rangle$. We would like to identify $|\mathcal{B}\rangle$ with this 1-baryon state, but we must ensure that there is no state with lower energy with the same quantum numbers. Since $|\mathcal{B}\rangle$ has unit baryon number, the candidates for this state contain a single baryon and any number (possibly zero) of mesons, glueballs, and exotics. For example, if $|\mathcal{B}\rangle$ has spin $J$, we can consider states consisting of a spin-$(J - 1)$ baryon and a meson. Since mesons have masses which are order 1 in the $1/N$ expansion, this state will have more energy than the 1-baryon state

† For notational simplicity, we work in finite volume; the spectrum of states is then discrete, and we take all states to be normalized to unity. If $|\mathcal{B}\rangle$ is a 1-baryon state with zero momentum, the overlap $\langle \mathcal{B}|B_0\rangle$ goes to zero in the infinite volume limit as $1/\sqrt{V}$. Because all of our results are independent of $\langle \mathcal{B}|B_0\rangle$, this is harmless.
provided that \( M_J - M_{J-1} \to 0 \) as \( N \to \infty \). Generalizing these considerations, we see that the method we are using allows us to study those states which become degenerate with the lowest-lying baryon states (assumed to be the \( J = \frac{1}{2} \) multiplet) in the large-\( N \) limit. We already know from previous work that there are an infinite number of such states in the large-\( N \) limit; this result will also emerge self-consistently from our analysis.

2.1. Masses

In this subsection, we derive our main results on the baryon masses. We will work in the limit where the current quark mass differences and electromagnetism are turned off, so that the theory has an exact \( SU(N_F) \) flavor symmetry. In this limit, the baryon spectrum consists of multiplets with spin \( J = \frac{1}{2}, \ldots, \frac{1}{2}N \) with the baryons degenerate in each multiplet. (We will show later how to treat \( SU(N_F) \) breaking perturbatively using chiral perturbation theory.)

Consider the quantity

\[
Z \equiv \langle B_0 | e^{-i\hat{H}T} | B_0 \rangle, \tag{4}
\]

where \( \hat{H} \) is the QCD hamiltonian. Using eq. (3), we have

\[
\frac{1}{T} \ln Z \longrightarrow -iE_B,
\]

as \( T \to -i\infty \). To define a diagrammatic expansion for \( Z \), we write the hamiltonian as \( \hat{H} = \hat{H}_0 + \delta\hat{H} \). We choose \( \hat{H}_0 \) to be a 1-body operator such that \( |B_0\rangle \) is an eigenstate; consequently, \( \delta\hat{H} \) will in general contain 1-body “interactions,” but these do not affect the \( N \)-counting of the expansion. Once again, our results are independent of the precise form of \( \hat{H}_0 \) and \( \delta\hat{H} \). We then write

\[
Z = e^{-iE_0T} \langle B_0 | \hat{U}_I(\frac{1}{2}T, -\frac{1}{2}T) | B_0 \rangle, \tag{6}
\]

where \( \hat{H}_0|B_0\rangle = E_0|B_0\rangle \), and

\[
\hat{U}_I(t_1, t_0) \equiv T \exp \left[ -i \int_{t_0}^{t_1} dt \delta\hat{H}_I(t) \right], \quad \hat{H}_I(t) \equiv e^{i\hat{H}_0t} \delta\hat{H} e^{-i\hat{H}_0t}, \tag{7}
\]

† One might worry about states containing a baryon and soft pions in the chiral limit. However, we can add \( SU(N_F) \)-preserving current quark mass term to make the pions massive and remove the degeneracy of these states with the one-baryon states. The smoothness of the chiral limit then allows us to extrapolate our results to zero current quark masses.
Eq. (6) can be expanded as a sum of Feynman diagrams using a simple adaptation of standard techniques. Each term in the expansion of eq. (6) in powers of $\delta \hat{H}$ consists of a time-ordered product of operators. Using Wick’s theorem, this can be written as a sum of all possible normal orderings and contractions. The terms in which all operators are contracted are just the usual “vacuum” graphs which sum to give the vacuum energy. However, because the state $|\mathcal{B}_0\rangle$ contains particles, the normal-ordered terms can also contribute. We represent these contributions diagrammatically by associating a dashed line for each external quark creation or annihilation operator, as shown in fig. (2b,c). In each such term, it is understood that all creation and annihilation operators are normal ordered.

In this diagrammatic expansion, the mass of the state $|\mathcal{B}\rangle$ is given by

$$M_\mathcal{B} \equiv E_\mathcal{B} - E_{\text{vac}} = \sum_r c_r \langle \mathcal{B}_0 | \hat{O}^{(r)} | \mathcal{B}_0 \rangle,$$

(8)

where $E_{\text{vac}}$ is the vacuum energy and $\hat{O}^{(r)}$ is an $r$-body operator of the form

$$\hat{O}^{(r)} = X_{b_1,\beta_1 \cdots b_r,\beta_r}^{a_1,\alpha_1 \cdots a_r,\alpha_r} \hat{a}_{a_1\alpha_1}^{\dagger} \hat{a}_{b_1\beta_1} \cdots \hat{a}_{a_r\alpha_r}^{\dagger} \hat{a}_{b_r\beta_r}.$$  

(9)

The external creation and annihilation operators again create a quark with definite spin, flavor, and color, in the 1-quark state $|\phi\rangle$ used to define the state $|\mathcal{B}_0\rangle$. (We can take $|\phi\rangle$ to be one of a complete basis of 1-quark states, and write all contributions in terms of creation and annihilation operators in this basis. Operators which annihilate quarks in 1-quark states other than $|\phi\rangle$ then vanish on states of the form $|\mathcal{B}_0\rangle$.) To see that eq. (9) is the most general form of a color-singlet $r$-body operator, note that operators involving color $\epsilon$ symbols can always be reduced to this form by using the identity

$$\epsilon^{A_1 \cdots A_N} \epsilon_{B_1 \cdots B_N} = \det \Delta, \quad \text{where} \quad \Delta_{st} \equiv \delta_{B_t}^{A_s} \quad (s, t = 1, \ldots, N).$$

(10)

Since QCD conserves angular momentum and flavor, the operators $\hat{O}^{(r)}$ in eq. (8) must be singlets under $SU(2)_v \times SU(N_F)$, where $SU(2)_v$ denotes the group of spatial rotations in the baryon rest frame. The classification of these operators is a group-theoretical problem which is solved in appendix A. Since the results concern only the spin and flavor structure, it is convenient to express the results in terms of a spin–flavor Fock space: we define

$$|\mathcal{B}\rangle \equiv \mathcal{B}^{a_1\alpha_1 \cdots a_N\alpha_N} \alpha_1^{\dagger} \alpha_2^{\dagger} \cdots \alpha_N^{\dagger} |0\rangle.$$  

(11)

Because the baryon wavefunction is symmetric under $a_1\alpha_1 \leftrightarrow a_2\alpha_2$ etc., the creation operators $\alpha_i^{\dagger}$ are bosonic. We use the “curved bra–ket” notation to distinguish these states.
from QCD states. Given any QCD operator $\hat{O}$ appearing in eq. (8), we can construct an operator in the spin–flavor Fock space such that $\langle B_0 | \hat{O} | B_0 \rangle \propto (B|O|B)$, so we can write

$$M_B \equiv E_B - E_{\text{vac}} = \sum_r c_B (B|O^{(r)}|B),$$

(12)

More details are given in the appendix.

The result of the classification is that an arbitrary $r$-body operator appearing in eq. (12) can be written as a linear combination of

$$\{1\}^{r-2s}\{J^2\}^s.$$  

(13)

Here $\{1\}$ is the quark number operator and $\{J^2\}$ is the usual angular momentum Casimir operator:

$$\{1\}|J\rangle = N|J\rangle, \quad \{J^2\}|J\rangle = J(J+1)|J\rangle,$$

(14)

where $|J\rangle$ denotes any state $|B\rangle$ with total spin $J$. We emphasize that this result holds for any number of flavors $N_F \geq 2$. This result can be understood heuristically by noting that $\{J^2\}$ is a Casimir operator which suffices to label the possible representations of $|B\rangle$, since the flavor representations are determined by $J$ (see fig. (1a)). Of course, we can eliminate the operator $\{1\}$ using the relation $\{1\} = N$ which holds on states with baryon number 1.

One way to determine the $N$ dependence of the coefficients in eq. (8) is to use the fact that the baryon mass is order $N$ [3]. Because we assume that the baryon masses increase with $J$, $M_B$ must be greater than the mass of the $J = \frac{1}{2}$ baryon multiplet, which is of order $N$. On the other hand, $M_B$ can be no greater than the mass of the 1-baryon state with the same quantum numbers, which is also of order $N$. Therefore, $M_B \sim N$ for all states $|B\rangle$, and each term in eq. (8) must be at most of order $N$. This implies

$$M_B = M_0 + \sum_s c_s (B) \{J^2\}^s |B\rangle, \quad c_s \leq \frac{1}{N^{2s-1}} + \cdots,$$

(15)

where $M_0 \sim N$ and the ellipses denote terms higher order in $1/N$. Since there seems to be no reason to suppose otherwise, we will assume that the inequalities in eq. (15) are saturated.

Eq. (15) is our main result for the baryon masses. We can immediately see that the masses of states with $J \sim 1$ become degenerate in the large-$N$ limit. This allows us to conclude that the states with $J \sim 1$ are in fact 1-baryon states, as discussed above. The fact that the baryon spectrum consists of an infinite tower of degenerate states in the large-$N$ limit is a well-known result of the Skyrme model, and is also a consequence of
the methods of ref. [8]. We also see that the leading mass differences between states with 
$J \sim 1$ are given by

$$M_J - M_{J'} = \frac{\mu}{N} [J(J + 1) - J'(J' + 1)].$$  (16)

This result was derived recently in ref. [10] by demanding consistency of chiral perturbation
theory in the large-$N$ limit. We have extended these results by classifying the corrections
to all orders in $1/N$. A qualitative picture of the baryon spectrum for large values of $N$ is
shown in fig. (1b).

The previous discussion relied on the results of ref. [3], and does not make use of the
full power of the diagrammatic approach. We now consider the diagrammatic expansion
in more detail and show that eq. (15) can be derived entirely within this approach.

We begin by showing that $\ln Z$ can be expressed as a sum of connected diagrams.
The expansion of $Z$ clearly includes disconnected diagrams such as the ones shown in fig.
(2c). A diagram with $n$ disconnected components is proportional to $T^n$, where $T$ is the
time extent. This can be most easily seen in position space where there is one overall
time integration for each disconnected component ranging from $-\frac{1}{2}T$ to $\frac{1}{2}T$. However,
since $\ln Z = -iE_{\text{vac}}T$, we know that the disconnected contributions must exponentiate in
some way. For the vacuum graphs (i.e. those with no external creation or annihilation
operators), this exponentiation is very simple: it is a standard result in quantum field
theory that the disconnected diagrams have the right combinatoric factors to combine into
an exponential. The graphs with external creation and annihilation operators have the
same combinatoric factors, so we can write

$$\ln Z = -iE_{\text{vac}}T + \ln \langle B_0|e^{\Delta}|B_0 \rangle,$$  (17)

where $E_{\text{vac}}$ is the sum of all connected vacuum graphs, and $\Delta$ is the sum of connected
diagrams with external creation and annihilation operators. If $|B_0\rangle$ is a state with definite
total spin, then it is an eigenstate of any $SU(2)_c \times SU(N_F)$ singlet operator of the form
of eq. (9), so that

$$\ln \langle B_0|e^{\Delta}|B_0 \rangle = \langle B_0| \ln: e^{\Delta}: |B_0 \rangle
= \langle B_0| \Delta |B_0 \rangle + \frac{1}{2} \langle B_0| \Delta^2 - \langle \Delta \rangle^2 |B_0 \rangle + \cdots$$  (18)

We see that the disconnected diagrams with external creation and annihilation operators
do not exponentiate directly.

However, it is not hard to see that the right-hand side of eq. (18) is a sum of connected
diagrams of a new type. Consider the $O(\Delta^n)$ terms in the expansion of $\ln Z$. By anticommuting
the quark operators, each $O(\Delta^n)$ term can be written in a standard form as $(\Delta^n)$
terms with anticommutators. The relevant anticommutator is \( S_H(x - y) \equiv \{ \psi^-(x), \bar{\psi}^+(y) \} \), where \( \psi \) is the quark field and we use \( \pm \) to indicate the creation/annihilation part of the field. \( S_H \) is a Green’s function, and is denoted diagrammatically by a dashed line. (Since it appears only in internal lines, there can be no confusion with the external creation and annihilation operators.) Physically, \( S_H \) is a correction to the propagator required to properly describe the propagation of “holes” (the absence of a quark in the initial state). A typical diagram involving hole propagators is shown in fig. (2d). We now say that a diagram in this standard form is “connected” if it is connected by either Feynman propagators or hole propagators. A graph with \( n \) connected components is proportional to \( T^n \), so only the connected graphs can contribute to \( \ln Z \); the disconnected diagrams must cancel. It is easy to see explicitly that this happens in the \( O(\Delta^2) \) term in eq. (18). This expansion in terms of connected diagrams with hole propagators is a generalization of the connected cluster expansion for Hartree–Fock perturbation theory which is well known in many-body physics [11].

The \( N \)-counting of diagrams is now very simple: we associate one power of \( 1/\sqrt{N} \) to each gluon vertex and one power of \( N \) to each internal color loop. (Color loops are most easily counted using the double-line notation [1].) In this way, one can see that the diagrammatic expansion for the mass of the state \( |\mathcal{B}\rangle \) can be written

\[
M_B = \sum_r c_r (\mathcal{O}^{(r)} | \mathcal{B} \rangle, \quad c_r \lesssim \frac{1}{N^{r-1}}, \tag{19}
\]

where \( \mathcal{O}^{(r)} \) is an \( r \)-body operator in the spin–flavor Fock space. Note that

\[
\{1\} = \alpha_{a\alpha}^\dagger \alpha^\alpha = 1\text{-body operator}, \tag{20}
\]

\[
\{J^2\} = \{J_j\} \{J_j\} = 2\text{-body operator}, \tag{21}
\]

where \( \{J_j\} \equiv \alpha_{a\alpha}^\dagger (J_j)^\alpha a^\alpha \). Eq. (19) therefore gives rise to the results for the masses quoted above.

2.2. Matrix Elements

We now extend the techniques described above to operator matrix elements. Using eq. (3), we can write

\[
\langle \mathcal{B}' | T \hat{O}_1(t_1) \cdots \hat{O}_n(t_n) | \mathcal{B} \rangle = \frac{\langle \mathcal{B}'_0 | T \hat{O}_1(t_1) \cdots \hat{O}_n(t_n) \hat{U}_I | \mathcal{B}_0 \rangle}{\langle \mathcal{B}'_0 | \hat{U}_I | \mathcal{B}'_0 \rangle^{1/2} \langle \mathcal{B}_0 | \hat{U}_I | \mathcal{B}_0 \rangle^{1/2}}. \tag{22}
\]

\[\dagger\] We choose the phases of \( |\mathcal{B}\rangle \) and \( |\mathcal{B}'\rangle \) so that \( \langle \mathcal{B}_0 | \mathcal{B} \rangle \) and \( \langle \mathcal{B}'_0 | \mathcal{B}' \rangle \) are real.
Here, $\hat{O}_1, \ldots, \hat{O}_n$ are Heisenberg-picture operators, $\hat{O}_I, \ldots, \hat{O}_I$ are the corresponding interaction-picture operators, and we use the abbreviation $\hat{U}_I \equiv \hat{U}_I(\frac{1}{2}T, -\frac{1}{2}T)$. The state $|\mathcal{B}'_0\rangle$ has the same form as eq. (2), possibly with different spin and flavor quantum numbers. (In particular, it is defined in terms of the same 1-quark state $|\phi\rangle$ used to define $|\mathcal{B}_0\rangle$.) The times $t_1, \ldots, t_n$ in eq. (22) must be lie within a fixed finite time interval as $T \to \infty$ in order for the states $|\mathcal{B}_0\rangle$ and $|\mathcal{B}'_0\rangle$ to project onto the corresponding interacting states $|\mathcal{B}\rangle$ and $|\mathcal{B}'\rangle$.

We now consider the diagrammatic expansion for the right-hand side of eq. (22). The numerator can be written as the sum of diagrams with insertions of the operators $\hat{O}_1, \ldots, \hat{O}_n$. The denominator can be written as a sum of diagrams with combinatoric factors obtained by expanding out the square roots. The numerator and denominator can be combined into a sum of diagrams involving hole propagators in a manner similar to that of eq. (18). The diagrammatic expansion therefore gives

$$
\langle \mathcal{B}' | T \hat{O}_1(\vec{p}_1, t_1) \cdots \hat{O}_n(\vec{p}_n, t_n) | \mathcal{B} \rangle = \sum_{\ell} F_{\ell}(\vec{p}_1, t_1, \ldots, \vec{p}_n, t_n) \langle \mathcal{B}_0 | \hat{O}_{\ell}^{(r_\ell)} | \mathcal{B}_0 \rangle,
$$

where we have explicitly indicated the dependence on the 3-momenta of the operators. Each term in this series can be thought of as a form factor. The coefficient $F_{\ell}$ contains all of the kinematic information of the form factor. $\hat{O}_{\ell}^{(r_\ell)}$ is an $r_\ell$-body operator of the form of eq. (9) (with no kinematic dependence) which gives the spin and flavor dependence of the form factor.

We now argue that only connected diagrams contribute to the right-hand side of eq. (23). First, we note that we can discard any graph in which the insertions of the operators $\hat{O}_1, \ldots, \hat{O}_n$ are in different connected components by giving the operators definite 3-momenta and restricting attention to “unexceptional” kinematic regions where no proper subset of the 3-momenta sum to zero. A graph with $n$ connected components is therefore proportional to $T^{n-1}$, since the overall time integrations for the $n-1$ connected components which do not contain the operator insertions can range from $-\frac{1}{2}T$ to $\frac{1}{2}T$; the connected component containing the operators has no overall time integral because the operators act at definite times. However, because $\langle \mathcal{B}' | T \hat{O}_1(t_1) \cdots \hat{O}_n(t_n) | \mathcal{B} \rangle \sim T^0$, only the connected diagrams contribute.

By counting powers of $N$ in connected graphs, we find that the coefficient of an $r_\ell$-body operator in eq. (23) is

$$
F_{\ell} \lesssim \frac{1}{N^{r_\ell+L-1}},
$$

where $L$ is the minimum number of quark loops in graphs that contribute to the matrix element. These results are valid as long as the momenta of the operators are held fixed.
as $N \to \infty$. In this case, the initial and final baryon velocity are the same in the large-$N$ limit. If we attempt to describe processes in which the baryon velocity changes, some of the momenta of the operators must become large with $N$. In this case, there is additional $N$ dependence coming from the overlap integrals between the initial and final states and the matrix elements are suppressed by $e^{-N}$ [3].

As was the case for the baryon masses, we can now obtain interesting results from a group-theoretic classification of the operators that appear on the right-hand side of eq. (23). In terms of the spin–flavor Fock space, we have

$$
\langle B'|T\hat{O}_1(p_1,t_1)\cdots\hat{O}_n(p_n,t_n)|B\rangle = \sum_\ell F_\ell(p_1,t_1,\ldots,p_n,t_n)(B|O_\ell^{(r)}|B).
$$

(25)

Note that the coefficients $F_\ell$ may have $SU(2)_v$ indices coming from non-trivial dependence on the angles of the momenta $p_1,\ldots,p_n$, so $O_\ell$ does not necessarily have the same quantum numbers as the product of operators on the left-hand side.

The matrix elements of most interest are forward matrix elements of a single quark bilinear of the form $\bar{\psi}\Gamma W\psi$, where $\psi$ is the quark field, $W$ is a flavor matrix, and $\Gamma$ is a Dirac spinor matrix. For such matrix elements, the operators $O_\ell$ transform under $SU(2)_v \times SU(N_F)$ the same way as the tensor $W^a_\alpha \Gamma^\alpha$. Here, $\Gamma^\alpha$ is the static projection of $\Gamma$, i.e.

$$
(\gamma_\beta)^\alpha_\beta = (J_j)^\alpha_\beta, \quad (\gamma_0)^\alpha_\beta = \delta^\alpha_\beta,
$$

(26)

etc. As shown in appendix A, the most general operator with these quantum numbers is a combination of

$$
\{W\Gamma\}, \quad \text{tr}(W)\{\Gamma\}, \quad \{W\}\{\Gamma\}, \quad \{WJ_j\}\{J_j\}\{\Gamma\},
$$

(27)

multiplied by powers of $\{J^2\}$ and $\{1\}$. Here,

$$
\{W\Gamma\} \equiv \alpha^i_a W^a_\beta \Gamma^\alpha_\beta a^b_b,
$$

(28)

etc. This classification holds for any number of flavors $N_F \geq 2$. The operator $\{W\Gamma\}$ gives rise to the quark model predictions for the matrix elements, which obey the well-known $SU(2N_F)$ “spin–flavor” relations. The methods used in the appendix for classifying operators can be easily extended to arbitrary operators.

The $N$ dependence of the coefficients of the operators in eq. (27) can be read off from the rule in eq. (24). The operator $\text{tr}(W)\{\Gamma\}$ has contributions from diagrams involving at least one quark loop, and so has coefficient $1/N$. The other operators do not require quark loops, and so have coefficients $1/N^{r-1}$ for an $r$-body operator.
For \( N_F > 2 \), the \( N \)-dependence of the matrix elements is somewhat complicated due to the fact that the size of the \( SU(2N_F) \) representation with given \( J^2 \) grows with \( N \) (see fig 1a). We will therefore begin by discussing the case \( N_F = 2 \), where the baryon states are characterized by \( I = J = \frac{1}{2}, \frac{3}{2}, \ldots \). In this case, direct calculation shows that
\[
\{ W J_j \} \{ J_j \} = \frac{N + 2}{4} \{ W \},
\]
for \( \text{tr} W = 0 \). Therefore, the last operator in eq. (27) is redundant.† (This result does not hold for \( N_F > 2 \).)

We now consider the \( N \) dependence of matrix elements of the operators in states with \( J \sim 1 \). If both \( W \) and \( \Gamma \) are equal to the unit matrix (e.g. for the flavor singlet scalar density), then the matrix element is clearly of order \( N \). If one of the matrices \( W \) and \( \Gamma \) is the unit matrix and the other is a traceless matrix (e.g. for the flavor non-singlet vector charge), the matrix element is of order 1, since \( \{ W \Gamma \} \) is an \( SU(2) \times SU(N_F) \) generator in this case. The only case which is not immediately obvious is when both \( W \) and \( \Gamma \) are traceless, corresponding to the flavor non-singlet axial current. Direct calculation shows that the matrix element is of order \( N \) in this case.

These results, along with the order of the leading corrections to the large-\( N \) limit, are shown in table 1. Note that in every case, the quark-model result corresponding to the operator \( \{ W \Gamma \} \) is exact in the large-\( N \) limit, and the corrections start at order \( 1/N^2 \).

| \( \{ WT \} \) | \( W, \Gamma = 1 \) | \( \text{tr} W = 0, \Gamma = 1 \) | \( W = 1, \text{tr} \Gamma = 0 \) | \( \text{tr} W, \text{tr} \Gamma = 0 \) |
|---|---|---|---|---|
| \( \text{tr}(W)\{ \Gamma \} / N \) | \( N \) | 1 | 1 | \( N \) |
| \( \{ W \} \{ \Gamma \} / N + \text{h.c.} \) | \( N^* \) | 0 | 1/\( N^* \) | 0 |
| \( \{ WT \}{J^2} / N^2 + \text{h.c.} \) | \( 1/N \) | \( 1/N^2 \) | \( 1/N^2 \) | \( 1/N \) |

Table 1. Leading corrections to quark bilinear matrix elements for 2 light quark flavors. The corrections marked with a * are proportional to the lowest-order values.

We now consider briefly the extension of these results to the case \( N_F = 3 \). In this case, the \( SU(N_F) \) representations corresponding to spin \( J \sim 1 \) contain \( \sim N \) states. We thank A. V. Manohar for pointing out this redundancy, which was missed in the initial version of this work. We also thank A. Cohen for discussions on this point, and on the \( N \)-dependence of amplitudes involving baryons with finite strangeness in the 3-flavor case.
identify the physical baryon states with the large-$N$ states that have the same isospin and strangeness. The $N$-dependence of the matrix elements is now more complicated due to the dependence on the strangeness $S$ of the baryon states. For example, the $\Delta S = 1$ semileptonic hyperon decays proceed through matrix elements of the QCD operator

$$\hat{O}_{\Delta S=1}^{\mu} = \bar{\psi}_L \gamma^\mu X \psi_L,$$

where

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (31)

In this case, the corresponding 1-body operator in the spin–flavor Fock space is $\{XJ_j\}$. This has matrix elements of order $\sqrt{|S|N}$ between states with strangeness $S$, since it removes a quark in a state with occupation number $\sim S$ and places it into a state with occupation number $\sim N$. The computation of these matrix elements is rather complicated, and must be handled on a case-by-case basis. Generically, we find that corrections to the quark-model relations which depend on $SU(3)_F$ symmetry (and not only isospin symmetry alone) are of order $1/N$. These issues will be discussed more fully in a future work, where we will also consider in detail whether the $1/N$ expansion is quantitatively relevant for $N = 3$.

We close this section with some general observations on these results. The quark-model relations we have derived in the large-$N$ limit are often said to be the consequence of an $SU(2N_F)$ spin–flavor symmetry. It is therefore natural to ask whether there is any sense in which QCD possesses such a symmetry in the large-$N$ limit. Because any $SU(2N_F)$ symmetry should contain the $SU(N_F)$ flavor symmetry of the light quarks, all particles with $SU(N_F)$ quantum numbers must transform non-trivially under $SU(2N_F)$.\(^\dagger\)(This is in contrast with heavy-quark symmetry, which commutes with the chiral symmetry of the light quark flavors.) However, it is clear that the observed mesons (for example) do not lie in $SU(6)$ multiplets. Because of this difficulty, we do not know of any sense in which the quark-model relations are the manifestation of an extra symmetry in the large-$N$ limit.

Finally, we note that it may seem paradoxical that “non-relativistic” relations such as the quark-model relations can arise in a relativistic system of quarks and gluons such as the one we have been studying. However, the quark-model relations we have derived

\(^\dagger\) If all particles with $SU(N_F)$ quantum numbers decoupled from the baryons in the large-$N$ limit, then the $SU(2N_F)$ symmetry could arise as a superselection rule in this limit. However, this does not happen. For example, the pion–nucleon scattering amplitude is order 1 in the $1/N$ expansion (see eq. (46) below).
are non-relativistic only in the sense that Lorentz invariance implies that they can hold
in at most one reference frame. In our approach, this frame is picked out by the baryon’s
velocity, and the resulting picture is completely Lorentz covariant.
3. Chiral Perturbation Theory

Using the results for operator matrix elements derived above, we can write down an effective lagrangian which describes the interactions of baryons with soft pions in the $1/N$ expansion.

We begin by discussing the baryons. Because the baryon mass is of order $N$, we can describe the baryons using a heavy-particle effective field theory [12]. We write the baryon momentum as

$$P = M_0 v + k,$$

(32)

where $M_0 \sim N$ is a baryon mass and $v$ is a 4-velocity ($v^2 = 1$) which defines the baryon rest frame. We then write an effective field theory in terms of baryon fields whose momentum modes are the residual momenta $k$. The reason that this effective theory is useful is that all kinematic dependence on $M_0$ is removed.

The Lorentz structure also simplifies considerably in a heavy-particle effective theory. All Lorentz invariants in the spin-$1/2$ representation can be constructed from $v^\mu$, $\epsilon^{\mu\nu\lambda\rho}$, and the spin matrix

$$s^\mu \equiv \frac{1}{2}(\gamma^\mu - v^\mu \gamma^5).$$

(33)

We find it convenient to simply work in the rest frame defined by $v$ and use a non-relativistic notation.

To describe the baryons with spin $1/2, 3/2, \ldots$, we use the baryon field $B^{a_1\alpha_1 \cdots a_N\alpha_N}(x)$. (In a relativistic notation, these fields would satisfy $N$ constraints of the form $P_+ B = 0$, where the particle projection operator $P_+ \equiv \frac{1}{2}(1 + \gamma^5)$ can act on any of the spinor indices.) The heavy field $B$ is defined using a common mass $M_0$ for all of the baryons, which we choose to be the mass of the $J = 1/2$ multiplet in the $SU(N_F)$ limit. With this choice, $M_0$ is close to the physical mass of baryons with $J \sim 1$.

The large-$N$ relations derived in the previous section hold only for baryons with $J \sim 1$, while the effective lagrangian we are describing also includes baryon fields with $J \sim N$. Because these states have masses $M_0 + O(N)$, their effects on the dynamics of the states with $J \sim 1$ will be local (on the scale of external momenta which are order 1 in the $1/N$ expansion) and suppressed by powers of $1/N$. (We do not want to consider an effective theory with these states integrated out, since integrating out a subset of the baryon states violates the spin–flavor relations explicitly.) Furthermore, baryons with $J \sim N$ will only appear in intermediate states if we consider the insertion of operators with spin $\sim N$ or if we carry consider diagrams involving $\sim N$ vertices. However, it is still important to know whether these contributions, when they appear, signal a breakdown of the $1/N$ expansion as defined by this effective lagrangian. If the contributions from baryons with $J \sim N$ have
the same form as the tree-level terms, then these contributions are harmless. While we believe this to be the case, we have no formal proof. We will see below that the consistency of the effective lagrangian we are describing is non-trivial even at low orders in the $1/N$ expansion.

We now discuss the light mesons. The main new feature of chiral dynamics in the large-$N$ limit is that the effects of the axial anomaly are subleading in $1/N$, so that the pattern of spontaneous symmetry breakdown in the large-$N$ limit is \[ SU(N_F)_L \times U(1)_L \times SU(N_F)_R \times U(1)_R \rightarrow SU(N_F)_{L+R} \times U(1)_{L+R}. \] (34)

Therefore, in addition to the usual $N^2_F - 1$ Nambu–Goldstone bosons (“pions”), there is an extra Nambu–Goldstone boson in the large-$N$ limit, which can be identified with the $\eta'$. At higher orders in $1/N$, $U(1)_{L-R}$ is broken explicitly. In this section, we restrict ourselves to making some remarks on chiral perturbation theory which do not involve the $\eta'$ in an essential way, so we will ignore the presence of the light $\eta'$ in the following discussion.

The $N^2_F - 1$ pions are described by a field

$$\xi(x) = e^{i\Pi(x)/f_\pi},$$

which is taken to transform under $SU(N_F)_L \times SU(N_F)_R$ as

$$\xi \mapsto L\xi U^{\dagger} = U\xi R^{\dagger},$$

where this equation implicitly defines $U$ as a function of $L$, $R$, and $\xi$. The effective lagrangian is most conveniently written in terms of the hermitian fields

$$V_\mu \equiv \frac{i}{2} (\xi \partial_\mu \xi^{\dagger} + \xi^{\dagger} \partial_\mu \xi), \quad A_\mu \equiv \frac{i}{2} (\xi \partial_\mu \xi^{\dagger} - \xi^{\dagger} \partial_\mu \xi),$$

transforming as

$$V_\mu \mapsto UV_\mu U^{\dagger} - i\partial_\mu UU^{\dagger}, \quad A_\mu \mapsto UA_\mu U^{\dagger}.$$ (38)

With these definitions, it is easy to write down the effective lagrangian. In order to keep track of the tensor algebra, it is convenient to once again utilize the spin–flavor Fock space. For example, we write the heavy field $\mathcal{B}$ in terms of

$$|\mathcal{B}\rangle \equiv B^{a_1\alpha_1 \cdots a_N\alpha_N} \alpha_{a_1\alpha_1}^{\dagger} \cdots \alpha_{a_N\alpha_N}^{\dagger} |0\rangle.$$ (39)

The chiral covariant derivative acting on baryon fields can then be written as

$$\nabla_\mu |\mathcal{B}\rangle = (\partial_\mu - i\{V_\mu\}) |\mathcal{B}\rangle,$$ (40)
and the first few terms in the effective lagrangian involving the baryon fields are

\[ \mathcal{L}_{\text{eff}} = \langle B | i \nabla_0 | B \rangle + g \langle B | \{ A \cdot s \} | B \rangle + c \langle B | \{ m \} | B \rangle + \cdots, \]  

(41)

where \( m \) is the quark mass matrix spurion, and

\[ \{ A \cdot s \} \equiv (s^j)_\beta^\alpha (A_j)_{a}^a \alpha^\beta \alpha^\dagger, \quad \{ m \} \equiv m_a^i \alpha^\dagger \alpha^\alpha \alpha^\beta. \]  

(42)

To find the \( N \) dependence of the coefficients \( g \) and \( c \) in eq. (41), we use the fact that the terms in the effective lagrangian can be related to matrix elements of QCD operators. The term proportional to \( g \) gives rise to a contribution to the matrix element of the axial current, while the term proportional to \( c \) gives rise to a contribution to the matrix element of the scalar density. Comparing with eq. (24), we see that \( g \) and \( c \) are both of order 1 in the \( 1/N \) expansion. In general, it is not hard to see that a general term in the effective lagrangian involving the baryon fields will have the form

\[ \frac{1}{N^{r_1-1}} \langle B | r\text{-body operator} | B \rangle, \]  

(43)

where the \( r \)-body operator in the spin–flavor Fock space is constructed out of the fields and spurions in the effective lagrangian. This form ensures that the \( N \)-counting of the effective lagrangian is the same as that found in the previous section.

We now consider the question of the consistency of this effective lagrangian. Suppose we form a graph from two terms in the effective lagrangian, corresponding to an \( r_1 \)- and an \( r_2 \)-body operator. The graph will therefore have the form \( 1/N^{r_1+r_2-2} \) times an \((r_1+r_2)\)-body operator, apparently violating the \( N \)-counting of the previous section. Such contributions can only be consistent if the \((r_1+r_2)\)-body operator has a special form, so that it can be written as a fewer-body operator. We will show that this is in fact what happens in some examples, but we have not been able to prove the consistency of this effective lagrangian in general.

We first consider baryon-pion scattering at low energies. For simplicity, we work in the chiral limit. The three graphs that contribute are shown in fig. (3). The graph in fig. (3c) gives a contribution of order \( 1/N \) to the amplitude,\(^\dagger\) while the graphs in fig. (3a,b) lead to an amplitude \([8][9]\)

\[ A(B^{\pi_1^A} \rightarrow B'^{\pi^B_2}) \propto \frac{q_1^j q_2^k}{q_0^j f_\pi^2} \langle B' \{ [X^{A_j}, X^{B_k}] | B \rangle + \cdots, \]  

(44)

\(^\dagger\) We work with non-relativistically normalized states, so that there is no dependence on the mass in the normalization of amplitudes.
where
\[ X^jA \equiv (\lambda^A)_b^a (J^j)_b^0 \alpha_{\alpha} \alpha^a_{\alpha}, \quad (45) \]
is the axial current in the spin–flavor Fock space corresponding to the pion \( \pi^A \). Because the initial and final baryons are on shell, their residual momenta are \( O(1/N) \), and can be neglected. Since \( (B'|X^{A_j}|B) \sim N \) and \( f_\pi \sim \sqrt{N} \) [3], one naively expects \( \mathcal{A} \sim N \), which would violate unitarity for any fixed kinematics as \( N \to \infty \). However the commutator of two axial currents is a vector current, so for baryons with \( J \sim 1 \) the term shown in eq. (44) is of order \( 1/N \). If we include the mass splitting between baryons with different spins, we obtain a contribution to the amplitude proportional to
\[ \frac{1}{f_\pi} \frac{1}{N} (B'|X^{A_j}\{J^2\}X^{Bk}\{J^2\}X^{A_j}|B) \sim 1, \quad (46) \]
and so the full amplitude is actually of order 1 in the \( 1/N \) expansion.$^\dagger$

Similarly, we can consider scattering processes involving more pions. The graphs shown in fig. (4) give rise to a scattering amplitude
\[ \mathcal{A}(B_{\pi_1}^{A_1} \to B'_{\pi_2}^{B_2} \pi_3^{C}) \propto \frac{q_1^j q_2^k q_3^\ell}{q_1^0 q_2^0 q_3^0 f_\pi^3} \left[ q_2^0 (B'|[X^{A_j}, [X^{Bk}, X^{C\ell}]])|B) \right. \]
\[ - q_1^0 (B'|[X^{Bk}, [X^{C\ell}, X^{A_j}]])|B) \bigg] + \ldots. \quad (47) \]
The naïve estimate of this term is \( \mathcal{A} \sim N^{3/2} \), but the double commutators lead to matrix elements of the form \( (B'|X|B) \sim N \), so that in fact \( \mathcal{A} \sim 1/\sqrt{N} \) [9]. The finiteness of this amplitude in the large-\( N \) limit was used in ref. [9] to argue that there are no \( 1/N \) corrections to the axial current matrix element. It is easy to check that the corrections to the axial current that we have enumerated give rise to corrections to this amplitude which are consistent in the large-\( N \) limit.

In both of the cases considered above, products of operators that naïvely violate the \( N \)-counting rules had a commutator structure that allowed them to be expressed in terms of operators with fewer creation and annihilation operators. As argued above, a mechanism of this sort is required for consistency in the large-\( N \) limit. This commutator structure has its origin in the antisymmetry of tree amplitudes under crossing. For loop amplitudes, the “\( i\epsilon \)” part of the baryon propagator ruins this antisymmetry, and amplitudes are no longer proportional to commutators. For example, if the pions have a common mass \( m_\pi \), the baryon masses have a 1-loop contribution [10]
\[ \Delta M_B \propto \frac{m_\pi^3}{16\pi f_\pi^2} (B|X^{A_j}X^{A_j}|B) \propto \frac{m_\pi^3}{16\pi f_\pi^2} \left[ N^2 + J(J + 1) \right] \quad (48) \]

$^\dagger$ We thank A. V. Manohar for pointing this out to us.
for a baryon with spin $J$ (fig. (5)). The leading term grows with $N$, but it is a harmless constant mass shift consistent with the $N$-counting results given in table 1. The mass difference between baryons with different spin is of the form derived in section 1.

We have verified that all of the examples considered in refs. [8][9][10] give consistent results using our approach. However, other than the remarks made above, we do not have any general insight into the cancellation mechanism at work. However, we believe that the effective lagrangian we have written down is the most general one compatible with the symmetries and $N$-counting structure of the large-$N$ limit. We believe that any effective lagrangian which is defined by its field content and symmetry structure is consistent as long as all allowed terms are included. (This “theorem” is advocated by Weinberg as the foundation of the method of effective lagrangians [14].) If this is correct, the effective lagrangian we have described provides an explicit and consistent description of pion–baryon interactions at low energies in the $1/N$ expansion.

4. The Non-relativistic Constituent Quark Model

Traditionally, $SU(6)$ relations have been justified by appealing to a non-relativistic constituent quark model (NRCQM) in which the $SU(6)$ symmetry arises because the constituent quarks are heavy. In this section, we make some brief remarks comparing our results to what is expected from such a model. We include this discussion to point out that the pattern of corrections to $SU(6)$ relations we have derived above is very different from what is expected from a NRCQM.

The $N$-counting of the NRCQM is identical to that of QCD, and so $SU(6)$-violating amplitudes will be suppressed by powers of $1/N$ in the NRCQM. However, when we discuss the NRCQM below, we will not treat $1/N$ as a small parameter, since the point of our discussion is whether the $SU(6)$ symmetry can be explained solely in terms of the heaviness of the constituent quarks.

The NRCQM can be formulated by assuming that below the QCD chiral symmetry breaking scale but above the confinement scale, QCD dynamics can be approximated by an effective lagrangian containing constituent quarks and weakly-interacting gluons [15]. The constituent quarks are to be viewed as composite objects whose properties are the result of non-perturbative QCD dynamics at the chiral symmetry breaking scale. The most important difference between the constituent quarks and the “current” quarks that appear in the fundamental QCD lagrangian is that the constituent quarks have a flavor-independent mass $M_Q \sim \Lambda_{\text{QCD}}$ due to chiral symmetry breaking. Because chiral symmetry is broken spontaneously, the theory also contains 8 “pions” in the chiral limit.

In this model, it is easy to see how the $SU(6)$ spin–flavor symmetry arises. If we
assume that the effective gluon coupling $\alpha_s$ is weak, then the constituent quarks inside a baryon will be nonrelativistic with binding energy $E_B \sim \alpha_s^2 M_Q \ll M_Q$. The interactions of the constituent quarks at energies of order $E_B$ therefore exhibit an $SU(6)$ spin–flavor symmetry exactly analogous to the spin–flavor symmetry discussed recently for $b$ and $c$ quarks \cite{16}. This $SU(6)$ symmetry will give rise to the same relations for baryon masses and matrix elements derived above in the large-$N$ limit of QCD.

However, it should be emphasized that the picture of QCD dynamics contained in the NRCQM has some serious drawbacks when taken seriously. First of all, if the constituent quarks are heavy and weakly interacting, there should be weakly bound states with the same quantum numbers as the pions with masses $\simeq \frac{2}{3}$ of a baryon mass. Furthermore, because of the $SU(6)$ symmetry, these states should lie in an $SU(6)$ multiplet with a nonet of vector mesons. Identifying these states with observed mesons is rather problematic. Another difficulty with the constituent quark model is that it leads us to expect hadron form factors to deviate significantly from point-like behavior for momentum transfers of order $E_B \ll M_Q \simeq 300$ MeV. Instead, we find that baryon form factors vary on a scale of order $m_{\rho} \simeq 770$ MeV. Finally, the gluons which couple the constituent quarks confine at some scale below $E_B$ (since the gluons are supposed to be weakly interacting at that scale), so we expect glueball states with mass $\ll M_Q$. It is not clear whether any of these objections is fatal, but we want to emphasize that the validity of the NRCQM should not be taken for granted, despite its successes.

Leaving these problems aside, we want to compare the expected pattern of corrections to the $SU(6)$ relations from the NRCQM to what we found for the $1/N$ corrections in QCD. This is easy to do if we write an effective lagrangian for the constituent quarks treated as heavy fields:

$$\mathcal{L}_{\text{NRCQM}} = \bar{Q} i\gamma_{\mu} \cdot D Q + g_Q \bar{Q} A \cdot s Q + \frac{b}{2M_Q} \bar{Q} \sigma_{\mu\nu} s^\nu \tilde{G}_{\mu\nu} Q + O(1/M_Q^2).$$

(49)

Here $Q$ is the constituent quark field, $D_\mu$ is the gluon covariant derivative, and $\tilde{G}_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda\rho} G^{\lambda\rho}$ is the dual of the gluon field strength. The $SU(6)$ symmetry is violated by the pion couplings and by the color magnetic moment term. Pion loops give rise to $SU(6)$-breaking corrections to matrix elements which vanish at zero momentum in the chiral limit. When the current quark masses are turned on, the pion loops give $SU(6)$-breaking corrections at zero momentum proportional to powers of the current quark masses. The $1/M_Q$ terms give rise to $SU(6)$-breaking corrections at zero momentum. These corrections will be proportional to powers of the $SU(6)$-breaking spurion $s^\mu/M_Q$; only even powers of this spurion can appear in corrections by parity invariance. Thus, all of the $SU(6)$ relations for matrix elements and baryon masses are expected to have $1/M_Q^2$ corrections.
in the NRCQM. This pattern of corrections is very different from that found in the $1/N$ expansion of QCD, where quark-model relations can have corrections which are either $1/N$ or $1/N^2$.

5. Conclusions

In this paper we have developed a diagrammatic expansion for baryons in the large-$N$ limit. This method gives a simple recipe for determining the $N$-dependence of physical quantities. Using these techniques, we have shown that forward matrix elements of quark bilinears obey $SU(2N_F)$ spin–flavor relations. Our approach makes a direct connection between the emergence of these relations and the spin–flavor properties of the baryon wavefunctions, which we feel is very attractive.

On a more practical level, our methods allow the enumeration of the $1/N$ corrections to the large-$N$ results to all orders in $1/N$, completing previous partial results on the classification of $1/N$ corrections. We have also written down explicitly an effective lagrangian which describes the interactions of baryons and pions at low energies in the $1/N$ expansion. The consistency of this lagrangian in the large-$N$ limit is not manifest, because some of the couplings grow with $N$. While we have not been able to prove consistency in general, we have verified it in a number of cases and given general arguments indicating that this effective lagrangian must in fact be correct.

After completing this paper, we received ref. [17], which considers baryon masses and matrix elements using the Hartree–Fock approach of ref. [3]. We also received ref. [18] which develops further the approach of refs. [8][9].

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Appendix A. Classification of Operators

In this appendix, we consider the classification of few-body operators with given transformation properties under $SU(2)_v \times SU(N_F)$. Since only the spin and flavor quantum numbers are relevant for our discussion, it is useful to use a notation in which only spin and flavor indices appear. To do this, we note that the fact that the states $|\mathcal{B}_0\rangle$ are antisymmetrized in color allows us to treat the quarks as colorless bosons. To make this precise, we make use of the identity

$$a_c^{\dagger} a^b_{\gamma} A |\mathcal{B}_0\rangle = B^{a_1\alpha_1 \cdots a_N\alpha_N} e^{A_1 \cdots A_N} \sum_{s=1}^{N} \delta^b_{a_s} \delta^\beta_{\alpha_s} a^{\dagger}_{a_1\alpha_1 A_1} \cdots a^{\dagger}_{c\gamma A_s} \cdots a^{\dagger}_{a_N\alpha_N A_N} |0\rangle.$$  \hspace{1cm} (50)

Exactly the same algebraic relation is satisfied by bosonic creation and annihilation operators carrying no color. Specifically, we define

$$|\mathcal{B}\rangle \equiv B^{a_1\alpha_1 \cdots a_N\alpha_N} a^{\dagger}_{a_1\alpha_1} \cdots a^{\dagger}_{a_N\alpha_N} |0\rangle,$$  \hspace{1cm} (51)

where the $\alpha^{\dagger}$'s are bosonic creation operators which act on a spin–flavor Fock space. We use the “curved bra–ket” notation to distinguish these states from QCD states. We then have

$$\alpha^{\dagger}_{c\gamma} a^b_{\beta} |\mathcal{B}\rangle = B^{a_1\alpha_1 \cdots a_N\alpha_N} \sum_{s=1}^{N} \delta^b_{a_s} \delta^\beta_{\alpha_s} a^{\dagger}_{a_1\alpha_1} \cdots a^{\dagger}_{c\gamma} \cdots a^{\dagger}_{a_N\alpha_N} |0\rangle.$$  \hspace{1cm} (52)

Comparison of eqs. (50) and (52) shows that

$$\langle \mathcal{B}_0 | \hat{O}^{(r)} | \mathcal{B}_0 \rangle \propto (\mathcal{B}|\hat{O}^{(r)}|\mathcal{B}),$$  \hspace{1cm} (53)

where we use the caret notation to denote QCD operators, and

$$\hat{O}^{(r)} \equiv X^{b_1\beta_1 \cdots b_r\beta_r} a^{\dagger}_{a_1\alpha_1} \cdots a^{\dagger}_{a_r\alpha_r} a^{\dagger}_{b_1\beta_1} \cdots a^{\dagger}_{b_r\beta_r}.$$  \hspace{1cm} (54)

Our job is now to classify the general $r$-body operators with specified $SU(2)_v \times SU(N_F)$ transformation properties. First note that the most general $r$-body operator can be written as

$$\hat{O}^{(r)} = X^{a_1\alpha_1 \cdots a_r\alpha_r} a^{\dagger}_{a_1\alpha_1} \cdots a^{\dagger}_{a_r\alpha_r} a^{\dagger}_{b_1\beta_1} \cdots a^{\dagger}_{b_r\beta_r} + \cdots,$$  \hspace{1cm} (55)

where the omitted terms arise from the re-ordering of the creation and annihilation operators. These omitted terms are irrelevant for the classification of the operators, since they are $s$-body operators, with $s < r$. We can therefore classify the operators inductively, starting with 1-body operators. We will make frequent use of this simplification in this appendix.
With the ordering chosen above, the creation operators project out the part of \( X \) which is symmetric under simultaneous interchange of spin and flavor indices \( a_1 \alpha_1 \leftrightarrow a_2 \alpha_2 \), etc. For notational simplicity, we will consider the case of 3 flavors, although the results are valid for an arbitrary number of flavors.

6.1. Singlets

In this subsection, we classify the \( SU(2)_v \times SU(3) \) singlet \( r \)-body operators. We proceed inductively in \( r \). It is clear that the only 1-body singlet operator is \( \alpha_1^\dagger a_\alpha \alpha a_\alpha \).

Now suppose that we have classified all of the singlet operators up to the \((r-1)\)-body operators, and consider the \( r \)-body operators. The tensor \( X \) in eq. (55) must be a linear combination of products of the \( SU(3) \) and \( SU(2)_v \) invariant tensors \( \delta^a_b \), \( \epsilon^{abc} \), \( \delta^\alpha_\beta \), and \( \epsilon^{\alpha\beta} \). First we note that the spin and flavor \( \epsilon \) symbols always appear in pairs and can thus be replaced by Kronecker \( \delta \)'s using eq. (10). Therefore, the tensor \( X \) in eq. (55) can be written as a linear combination of terms of the form

\[
X_{b_1 \beta_1 \ldots b_r \beta_r}^{a_1 \alpha_1 \cdots a_r \alpha_r} = \delta_{b_1}^{a_1} \cdots \delta_{b_r}^{a_r} Y_{\beta_1 \cdots \beta_r}^{\alpha_1 \cdots \alpha_r},
\]  

(56)

where we have used the symmetry properties of the creation and annihilation operators to put the flavor indices in canonical order. The tensor \( Y \) is an \( SU(2)_v \) singlet, and can therefore be written as a linear combination of tensors of the form

\[
Y_{\alpha_1 \cdots \alpha_r}^{\beta_1 \cdots \beta_r} = \delta_{\beta_{\sigma_1}}^{\alpha_1} \cdots \delta_{\beta_{\sigma_r}}^{\alpha_r},
\]  

(57)

where \( \sigma \) is a permutation of 1, \ldots, \( r \). (We have used up our freedom to re-order the indices in writing eq. (56).)

We can rewrite \( SU(2)_v \) tensors of the form eq. (57) by writing the permutation \( \sigma \) as a product of simple interchanges. In this way, \( Y \) can be written as a product of the tensors

\[
S_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} \equiv \delta_{\beta_1}^{\alpha_1} \delta_{\beta_2}^{\alpha_2}.
\]  

(58)

We can then use the identity

\[
(J_j)_{\beta_1}^{\alpha_1} (J_j)_{\beta_2}^{\alpha_2} = \frac{1}{2} [S_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} - \frac{1}{2} \delta_{\beta_1}^{\alpha_1} \delta_{\beta_2}^{\alpha_2}]
\]  

(59)

to write \( Y \) as a linear combination of products of \( SU(2)_v \) generators and the identity matrix. Finally, some simple \( SU(2) \) algebra gives us an expression for \( O^{(r)} \) as a linear
combination of products of \( \{J_j\} \)'s and \( \{1\} \)'s (the number operator), which can be reduced to the form \( \{1\}^{r-2s} \{J^2\}^s \), where we have used the notation
\[
\{J_j\} \equiv \alpha_{\alpha a}^\dagger (J_j)^{\alpha \alpha} a^\beta ,
\]
and \( \{J^2\} \equiv \{J_j\}\{J_j\} \).

To understand these last steps in detail, consider a simple example which can be easily extended to the general case:
\[
Y_{\alpha_1 \beta_1 \beta_2 \beta_3}^{\gamma_1 \gamma_2 \gamma_3} \equiv \delta_{\beta_2}^{\alpha_1} \delta_{\beta_3}^{\alpha_2} \delta_{\beta_1}^{\alpha_3} = S_{\beta_1}^{\gamma_1} S_{\beta_2}^{\gamma_2} S_{\beta_3}^{\gamma_3} .
\]
Using eq. (59), we can write the corresponding 3-body operator as
\[
O^{(3)} = 4\{J_j\}\{J_k J_j\}\{J_{k}\} - 2\{1\}\{J^2\} + \frac{1}{4} \{1\}^3 + \cdots .
\]
The ellipses in eq. (62) indicate 1- and 2-body operators obtained by re-ordering the creation and annihilation operators. Using the identity
\[
J_j J_k = \frac{1}{4} \delta_{jk} + i \frac{1}{2} \epsilon_{jkl} J_{\ell},
\]
we obtain
\[
O^{(3)} = -\{1\}\{J^2\} + 2i \epsilon_{jkl} \{J_j\}\{J_k\}\{J_{\ell}\} + \frac{1}{4} \{1\}^3 + \cdots .
\]
Finally, we use the identity
\[
\epsilon_{jkl} \{J_k\}\{J_{\ell}\} = \frac{1}{2} \epsilon_{jkl} \{[J_k], [J_{\ell}]\} = i \{J_j\}
\]
to write
\[
O^{(3)} = -3\{1\}\{J^2\} + \frac{1}{4} \{1\}^3 + \cdots .
\]
These steps can be repeated for an arbitrary tensor \( Y \) of the form of eq. (57). After writing \( Y \) as a product of the \( S \) tensors defined in eq. (58), we use the identity eq. (59) to write \( O^{(r)} \) as a linear combination of products of \( \{1\} \) and operators of the form \( \{J \cdots J\} \), where the indices on the \( J \)'s are contracted using \( \delta_{ij} \)'s. Using eq. (63), we reduce this to products of \( \{1\} \)’s and \( \{J_j\} \)'s, contracted with \( \delta_{jk} \)'s and \( \epsilon_{jkl} \)'s. We then use eq. (10) to eliminate the \( \epsilon_{jkl} \)'s in pairs in favor of \( \delta_{jk} \)'s, and then apply eq. (65) to eliminate the last \( \epsilon_{jkl} \) (if any). In this way, we obtain products of \( \{1\} \)’s and \( \{J_j\} \)'s contracted with \( \delta_{jk} \)'s, which can be written as \( \{1\}^{r-2s}(\{J^2\})^s \) up to operators with fewer creation and annihilation operators. Thus, we have shown that the most general \( r \)-body operator is a linear combination of operators of the form
\[
\{1\}^{r-2s} \{J^2\}^s .
\]
6.2. Quark Bilinears

In this subsection, we classify the possible operators with the quantum numbers of the tensor $W^a_b \Gamma^\alpha_\beta$, corresponding to the matrix elements of a quark bilinear. An arbitrary quark bilinear operator can be written as a linear combination of operators of this form. Our argument is very similar to the one presented above for singlets, and so we will be somewhat terse.

We again proceed inductively. We begin by considering the case $\text{tr} W = \text{tr} \Gamma = 0$. In this case, the only effective 2-body operator with the right quantum numbers is

$$O^{(2)} \equiv \alpha^+_a a \Gamma^\alpha_\beta b.$$  \hfill (68)

Now assume we have classified all operators up to $(r-1)$-body operators, and consider the $r$-body operators. A general $r$-body tensor of the form of eq. (55) with the right quantum numbers can be written as a linear combination of tensors of the form

$$X^{a_1 \alpha_1 \cdots a_r \alpha_r}_{b_1 \beta_1 \cdots b_r \beta_r} = W^{b'_1 \Gamma^\beta_1}_{a'_1 \alpha_1 \cdots a_r \alpha_r} \cdot \cdots \cdot \cdot Y^{a_1 \cdots \alpha_s}_{\beta_1 \cdots \beta_t}$$  \hfill (69)

where $Y$ is an $SU(2)_v \times SU(3)$ invariant tensor. It must therefore consist of combinations of $\delta^a_b$'s, $\delta^\alpha_\beta$'s, and $\epsilon$ symbols for the groups $SU(2)_v$ and $SU(3)$. However, since $Y$ has an equal number of upper and lower indices for both groups, the $\epsilon$ tensors must appear in pairs, and we can use eq. (10) to eliminate them in favor of combinations of $\delta$'s. The tensor in eq. (69) thus reduces to a linear combination of the form

$$X^{a_1 \alpha_1 \cdots a_r \alpha_r}_{b_1 \beta_1 \cdots b_r \beta_r} = W^{b'_1 \Gamma^\beta_1}_{a'_1 \alpha_1 \cdots a_r \alpha_r} \cdot \cdots \cdot \cdot Y^{a_1 \cdots \alpha_s}_{\beta_1 \cdots \beta_t}$$  \hfill (70)

where $s, t = 1, \cdots, r$, and the caret denotes omission. $\tilde{Y}$ is an $SU(2)_v$ invariant tensor, formed from a linear combination of products of $\delta^\alpha_\beta$'s, where the lower indices are permuted relative to their canonical order. Going through the same steps as for the singlet operators, we can write this tensor in terms of $\delta^a_b$'s and $J_j$'s with indices in canonical order. In this way, an arbitrary $r$-body operator can be written as a linear combination of operators of the form

$$O^{(r)} = \{ W J \cdots J \Gamma J \cdots J \} \{ J \cdots J \} \cdots \{ J \cdots J \} \{ 1 \} \cdots \{ 1 \} + \cdots,$$  \hfill (71)

where the $J$'s are contracted using $\delta_{jk}$'s, $\{ W \Gamma \} \equiv \alpha^+_a a \Gamma^\alpha_\beta b$, and we have discarded all lower-body operators.
We now use the identity eq. (63) to reduce this to a product of \{1\}'s and \{J_j\}'s contracted with \(\delta_{jk}\)'s and \(\epsilon_{jkl}\)'s. If \(\Gamma = J_j\), the operator is a linear combination of

\[
O^{(r)} \sim \{W J_j\} \{J\} \cdots \{J\},
\]

(72)

\[
\{W\} \{J_j\} \{J\} \cdots \{J\},
\]

(73)

\[
\{W J\} \{J_j\} \{J\} \cdots \{J\},
\]

(74)

\[
\epsilon_{j\cdots} \{W\} \{J\} \cdots \{J\},
\]

(75)

\[
\epsilon_{j\cdots} \{W J\} \{J\} \cdots \{J\}.
\]

(76)

The unlabeled \(J\)'s in eqs. (72)–(76) are contracted using \(\delta_{jk}\) and \(\epsilon_{jkl}\). We have dropped all factors of the operator \(\{1\}\), since \(\{1\} = N\) on the states we are interested in, and these factors therefore do not affect the \(N\)-counting (see eq. (24)).

We now eliminate the \(\epsilon\) symbols pairwise using the identity eq. (10). This leaves at most one \(\epsilon\) symbol. The identity eq. (65) can be used to eliminate the final \(\epsilon\) symbol (if any) from eqs. (72)–(75). The only non-trivial way an \(\epsilon\) can appear in eq. (76) is as a factor of

\[
\epsilon_{jkl} \{W J_k\} \{J_l\} = i \left[ \{W J_j\}, \{J^2\} \right] + \cdots,
\]

(77)

where the ellipses denote 1-body operators. However, the right-hand side of eq. (77) can be seen to have the opposite transformation property as \(\{WT\}\) under time reversal, so this operator cannot appear.

In this way, we find that the most general \(r\)-body operator with the quantum numbers of \(WT\) is a linear combination of the operators

\[
\{WT\}, \quad \{W\} \{\Gamma\}, \quad \{W J_j\} \{J_j\} \{\Gamma\},
\]

(78)

multiplied by an \(SU(2)_v \times SU(3)\) singlet operator. (We may need to add or subtract the hermitian conjugate from the operators in eq. (78) so that the effective operator has the same hermiticity properties as the corresponding QCD operator.)

It remains only to consider the case where \(W = 1\) and/or \(\Gamma = 1\). Repeating the above argument with trivial modifications shows that eq. (78) gives the most general corrections in this case as well.
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Figure Captions

Fig. 1. (a) The Young tableaux describing the $SU(2)_c \times SU(N_F)$ representation of the baryon states with spin $J$. (b) The spectrum of baryon states in the large-$N$ limit. The mass splitting $\Delta M$ between adjacent states with spin $J \sim 1$ is of order $1/N$, while for $J \sim \sqrt{N}$, $\Delta M \sim 1$; our methods do not allow us to study baryon states with $J \sim N$.

Fig. 2. Typical diagrams contributing to the diagrammatic expansion of $Z$ in eq. (4). (a) A “vacuum” graph arising from the fully contracted part of the Wick expansion. (b) A 3-body operator arising from the normal-ordered terms in the Wick expansion. The external creation and annihilation operators are denoted by dashed lines. The solid internal lines denote the usual quark propagators. (c) A disconnected diagram with external annihilation and creation operators. The entire contribution is normal-ordered. (d) A typical diagram involving the “hole” propagator $S_H(x-y)$ (denoted by an internal dashed line).

Fig. 3. The leading contributions to the amplitude for the process $B\pi \to B'\pi$. The diagrams (a) and (b) are proportional to axial current matrix elements (see eq. (44)). Diagram (c) is proportional to the matrix element of a vector current, and is of order $1/N$ for baryons with $J \sim 1$.

Fig. 4. Contributions to the amplitude for $B\pi \to B'\pi\pi$ which are proportional to matrix elements of three axial currents.

Fig. 5. A 1-loop diagram contributing to the baryon masses.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9310369v3
$J \sim N^{1/2}$

$\Delta M \sim 1$

$J \sim 1$

$\Delta M \sim 1/N$

(1b)
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9310369v3
This figure "fig1-3.png" is available in "png" format from:

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