Counting Quiver Representations over Finite Fields Via Graph Enumeration

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Abstract

Let $\Gamma$ be a quiver on $n$ vertices $v_1, v_2, \ldots, v_n$ with $g_{ij}$ edges between $v_i$ and $v_j$, and let $\alpha \in \mathbb{N}^n$. Hua gave a formula for $A_\Gamma(\alpha, q)$, the number of isomorphism classes of absolutely indecomposable representations of $\Gamma$ over the finite field $\mathbb{F}_q$ with dimension vector $\alpha$. Kac showed that $A_\Gamma(\alpha, q)$ is a polynomial in $q$ with integer coefficients. Using Hua’s formula, we show that for each integer $s \geq 0$, the $s$-th derivative of $A_\Gamma(\alpha, q)$ with respect to $q$, when evaluated at $q = 1$, is a polynomial in the variables $g_{ij}$, and we compute the highest degree terms in this polynomial. Our formulas for these coefficients depend on the enumeration of certain families of connected graphs.

1 Introduction

Let $\Gamma$ be a quiver on $n$ vertices $v_1, v_2, \ldots, v_n$ with $g_{ij}$ edges between vertices $v_i$ and $v_j$ for $1 \leq i \leq j \leq n$. All of the following results are independent of the orientation of these edges. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ (throughout the paper, vectors will be represented by boldface symbols). We are interested in $A_\Gamma(\alpha, q)$, the number of isomorphism classes of absolutely indecomposable representations of $\Gamma$ over the finite field $\mathbb{F}_q$ with dimension vector $\alpha$. Kac [5] proved that $A_\Gamma(\alpha, q)$ is a polynomial in $q$ with integer coefficients and that it is independent of the orientation of $\Gamma$. He conjectured that the coefficients of $A_\Gamma(\alpha, q)$ are non-negative and that if $\Gamma$ has no loops, then the constant term of $A_\Gamma(\alpha, q)$ is equal to the multiplicity of $\alpha$ in the Kac-Moody algebra defined by $\Gamma$. Both conjectures are true for quivers of finite and tame type (see Crawley-Boevey and Van den Bergh [1]); a proof of the multiplicity statement in Kac’s conjectures for general quivers was recently announced by Hausel [2].

Our goal is to understand $A_\Gamma(\alpha, 1)$, and more generally $\left. \left( \frac{d^s}{dq^s} A_\Gamma(\alpha, q) \right) \right|_{q=1}$, as a function of the variables $g_{ij}$. The main impetus for studying $A_\Gamma(\alpha, 1)$ comes from the work of Hausel
and Rodriguez-Villegas [3]. They show that when $\Gamma$ is the quiver $S_g$ consisting of one vertex $v$ with $g$ self-loops, $A_{S_g}(\alpha, 1)$ (where $\alpha = \alpha \in \mathbb{Z}$) is (conjecturally) the dimension of the middle cohomology group of a character variety parameterizing certain representations of the fundamental group of a closed genus-$g$ Riemann surface to $\text{GL}_n(\mathbb{C})$.

One can imagine that specializing to $q = 1$ will relate $A_\Gamma(\alpha, q)$ to counting representations of $\Gamma$ in the category of finite sets; this hope follows a well-known philosophy about the significance of letting $q \to 1$ in formulas that depend on a finite field $\mathbb{F}_q$, although it seems hard to make this philosophy precise. In this paper we show in Theorems 6.3 and 7.1 that $\left(\frac{d}{dq} A_\Gamma(\alpha, q)\right)\bigg|_{q=1}$ is a polynomial in the variables $g_{ij}$, and we give a formula for its leading coefficients. This formula relies on the number of connected graphs in a family determined by $\Gamma$ and on Stirling numbers of the second kind, which arise from derivatives of $q$-binomial coefficients. The description of the graphs in question is given prior to Theorem 4.3 and all necessary information about Stirling numbers and $q$-binomial coefficients is given in Appendix A. Unfortunately, our proofs of Theorems 6.3 and 7.1 do not give any conceptual indication as to why our results should involve the enumeration of connected graphs.

To illustrate the type of result found in this paper, consider $\Gamma = S_g$. Using a formula of Hua [4, Theorem 4.6] for $A_\Gamma(\alpha, q)$, which we will present in Section 2 and which is our starting point for the results in this paper, we can compute the polynomial $A_{S_g}(\alpha, q)$ for small $\alpha$ and $g$. These computations are displayed in the following table:

| $A_{S_g}(\alpha, q)$ | $g = 1$ | $g = 2$ | $g = 3$ | $g = 4$ |
|----------------------|---------|---------|---------|---------|
| $\alpha = 1$         | $q$     | $q^2$   | $q^3$   | $q^4$   |
| $\alpha = 2$         | $q$     | $q^5 + q^3$ | $q^9 + q^7 + q^5$ | $q^{13} + q^{11} + q^9 + q^7$ |
| $\alpha = 3$         | $q$     | $q^{10} + q^8 + q^7 + \cdots$ | $q^{19} + q^{17} + q^{16} + \cdots$ | $q^{28} + q^{26} + q^{25} + \cdots$ |
| $\alpha = 4$         | $q$     | $q^{17} + q^{15} + q^{14} + \cdots$ | $q^{33} + q^{31} + q^{30} + \cdots$ | $q^{49} + q^{47} + q^{46} + \cdots$ |
| $\alpha = 5$         | $q$     | $q^{26} + q^{24} + q^{23} + \cdots$ | $q^{51} + q^{49} + q^{48} + \cdots$ | $q^{76} + q^{74} + q^{73} + \cdots$ |
| $\alpha = 6$         | $q$     | $q^{37} + q^{35} + q^{34} + \cdots$ | $q^{73} + q^{71} + q^{70} + \cdots$ | $q^{109} + q^{107} + q^{106} + \cdots$ |

Evaluating each polynomial at $q = 1$ gives the following values for $A_{S_g}(\alpha, 1)$:

| $A_{S_g}(\alpha, 1)$ | $g = 1$ | $g = 2$ | $g = 3$ | $g = 4$ | $g = 5$ | $g = 6$ |
|----------------------|---------|---------|---------|---------|---------|---------|
| $\alpha = 1$         | 1       | 1       | 1       | 1       | 1       | 1       |
| $\alpha = 2$         | 1       | 2       | 3       | 4       | 5       | 6       |
| $\alpha = 3$         | 1       | 6       | 15      | 28      | 45      | 66      |
| $\alpha = 4$         | 1       | 22      | 95      | 252     | 525     | 946     |
| $\alpha = 5$         | 1       | 95      | 710     | 2674    | 7215    | 15961   |
| $\alpha = 6$         | 1       | 449     | 5856    | 31374   | 109707  | 298023  |

Fitting each row of the above table to a polynomial gives empirical evidence that the next
This suggests that $A_{S_g}(\alpha, 1)$ is a polynomial in $g$ of degree $\alpha - 1$ with leading coefficient $2^{\alpha} - 1/\alpha$. We prove this and a generalization to all quivers in Theorem 6.3 below. Theorem 7.1 offers a similar result for any derivative (with respect to $q$) of $A_{\Gamma} (\alpha, q)$ evaluated at $q = 1$.

The fact that the leading coefficient of $A_{S_g}(\alpha, 1)$ equals $2^{\alpha} - 1/\alpha$ was mentioned (without proof) in [3, Remark 4.4.6]. As mentioned above, in the context of that paper, $S_g$ corresponds to a closed Riemann surface of genus $g$ and it seems more appropriate to use its Euler characteristic $2g - 2$ instead of $g$ as a variable. One possibly telling feature of this choice is that the factor $2^{\alpha} - 1$ in the leading coefficient disappears, though we do not know of a similar approach for the general case. Finally, we note that $\alpha^{\alpha - 2}$ appears in the formula for the leading coefficient of $A_{S_g}(\alpha, 1)$ because $\alpha^{\alpha - 2}$ is the number of trees on $\alpha$ labeled vertices by Cayley’s Theorem. As indicated above, for other quivers, the leading coefficient formula involves the enumeration of other families of graphs.

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**2 Hua’s Formula**

We begin with a presentation of Hua’s formula for $A_{\Gamma} (\alpha, q)$. Let $T = (T_1, T_2, \ldots, T_n)$ be a vector of indeterminates. Let $P$ denote the set of all integer partitions, including the unique partition of 0. If $\lambda$ and $\mu$ are partitions with transposes $\lambda'$ and $\mu'$ respectively, let

$$\langle \lambda, \mu \rangle := \sum_{1 \leq i} \lambda'_i \mu'_i.$$ 

Also, let

$$b_\lambda(q) := \prod_{1 \leq i} \prod_{1 \leq j \leq n_i} (1 - q^j),$$

where $\lambda$ has $n_i$ parts of size $i$ for each $i$. As a notational convenience, we will write monomials as a vector with a vector exponent, as in $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$. If $\lambda$ is a partition or a composition, let $|\lambda|$ denote the sum of the parts of $\lambda$. 

| $A_{S_g}(\alpha, 1)$ | 
|---------------------|
| $\alpha = 1$ | 1 |
| $\alpha = 2$ | $\binom{2}{1}$ |
| $\alpha = 3$ | $4 \binom{3}{2} + \binom{3}{1}$ |
| $\alpha = 4$ | $32 \binom{4}{2} + 20 \binom{4}{1}$ |
| $\alpha = 5$ | $400 \binom{5}{4} + 428 \binom{5}{2} + 93 \binom{5}{1}$ |
| $\alpha = 6$ | $6912 \binom{6}{3} + 10640 \binom{6}{3} + 4512 \binom{6}{4} + 447 \binom{6}{2} + \binom{6}{1}$ |

This suggests that $A_{S_g}(\alpha, 1)$ is a polynomial in $g$ of degree $\alpha - 1$ with leading coefficient $2^{\alpha} - 1/\alpha$.
Finally, define the function $P_T(T, q)$ by

$$P_T(T, q) := \sum_{\lambda^1, \ldots, \lambda^n \in \mathcal{P}} \prod_{1 \leq i < j \leq n} q^{q^{\lambda_i^j \lambda_j^i}} \prod_{1 \leq i \leq n} T_i^{\lambda_i^1} \ldots T_n^{\lambda_i^n} \tag{1}$$

and the function $H_T(\alpha, q)$ implicitly by

$$\log P_T(T, q) = \sum_{\alpha \leq \alpha \in \mathbb{N}^n} \frac{H_T(\alpha, q)}{\alpha} \varsigma^\alpha, \tag{2}$$

where $\alpha = \gcd(\alpha_1, \ldots, \alpha_n)$. Hua expresses $A_T(\alpha, q)$ in terms of $H_T(\alpha, q)$.

**Theorem 2.1** (Hua [4, Theorem 4.6]).

$$A_T(\alpha, q) = \frac{q - 1}{\alpha} \sum_{d|\alpha} \mu(d) H_T(\alpha/d, q^d). \tag{3}$$

### 3 A Deformation of $A_T(\alpha, q)$

Although we want to understand $A_T(\alpha, 1)$, we cannot use Equations (1), (2), and (3) directly, since the summands in $P_T(T, q)$ have poles at $q = 1$. We will proceed instead by introducing extra variables, computing certain limits as $q$ approaches 1, and then specializing the results. This section analyzes $A_T(\alpha, u, q)$, a generalization of $A_T(\alpha, q)$, while Sections 6 and 7 apply the results to $A_T(\alpha, q)$.

In what follows, vectors $u \in \mathbb{N}^{n(n+1)/2}$ will have components $u_{ij}$ for $1 \leq i \leq j \leq n$, and for $\ell \in \mathbb{N}^n$ we let $u^\ell := \prod_{1 \leq i < j \leq n} u_{ij}^{\ell_{ij}}$. Let $u \in \mathbb{N}^{n(n+1)/2}$. Define functions $P_T(T, u, q)$, $H_T(\alpha, u, q)$, and $A_T(\alpha, u, q)$ by the formulas

$$P_T(T, u, q) := \sum_{\lambda^1, \ldots, \lambda^n \in \mathcal{P}} \prod_{1 \leq i < j \leq n} q^{u_{ij}^{\lambda_i^j \lambda_j^i}} \prod_{1 \leq i \leq n} T_i^{\lambda_i^1} \ldots T_n^{\lambda_i^n} \tag{4}$$

$$\log P_T(T, u, q) := \sum_{\alpha \leq \alpha \in \mathbb{N}^n} \frac{H_T(\alpha, u, q)}{\alpha} \varsigma^\alpha, \tag{5}$$

$$A_T(\alpha, u, q) := \frac{q - 1}{\alpha} \sum_{d|\alpha} \mu(d) H_T(\alpha/d, u^d, q^d). \tag{6}$$

Observe that $P_T(T, u, q)$, $H_T(\alpha, u, q)$, and $A_T(\alpha, u, q)$ specialize to $P_T(T, q)$, $H_T(\alpha, q)$, and $A_T(\alpha, q)$ respectively when $u_{ij} = q^{\lambda_i^j}$ for $1 \leq i \leq j \leq n$. However, $A_T(\alpha, u, q)$ typically is not a polynomial in $q$ even though $A_T(\alpha, q)$ is. For $\ell \in \mathbb{N}^n$ let $\ell! := \ell_1! \cdots \ell_n!$ and for $u \in \mathbb{N}^{n(n+1)/2}$ let $u! := u_{11}! \cdots u_{jj}! \cdots u_{nn}!$. Our first result computes a limit involving $A_T(\alpha, u, q)$.

**Proposition 3.1.**

$$\lim_{q \to 1} (q - 1)^{|\alpha|-1} A_T(\alpha, u, q) = \text{[the coefficient of } T^\alpha \text{ in } \sum_{\ell \in \mathbb{N}^n} \frac{T^\ell}{\ell!}, \tag{7}$$
Proof. We begin with a limit involving \( P_T(T, u, q) \). By the definition of \( P_T(T, u, q) \),

\[
\lim_{q \to 1} P_T((q - 1)T, u, q) = \lim_{q \to 1} \sum_{\lambda^1, \ldots, \lambda^n \in \mathcal{P}} \prod_{1 \leq i < j \leq n} \mu_{i,j}^{(\lambda^i, \lambda^j)} \prod_{1 \leq i \leq n} q^{(\lambda^i, \lambda^j)} b_\lambda(q^{-1}) (q - 1)^{|\lambda^1| + \cdots + |\lambda^n|} T_1^{\lambda_1^1} \cdots T_n^{\lambda_n^n}.
\]

The quantity \( b_\lambda(q^{-1}) \) has a zero at \( q = 1 \) of multiplicity \( \ell(\lambda) \) (the number of parts of \( \lambda \)). Thus each summand on the right-hand side of Equation (8) has limit 0 as \( q \to 1 \) unless \( \lambda^i = (1^{\ell_i}) \) for \( 1 \leq i \leq n \) for some \( \ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n \). In this case,

\[
\lim_{q \to 1} \frac{(q - 1)^{\ell_i}}{b_{(1^{\ell_i})}(q^{-1})} = \frac{1}{\ell_i!}.
\]

Therefore

\[
\lim_{q \to 1} P_T((q - 1)T, u, q) = \sum_{\ell \in \mathbb{N}^n} u^\ell \frac{T^\ell}{\ell!},
\]

since

\[
\langle (1^{\ell_i}), (1^{\ell_j}) \rangle = \ell_i \ell_j.
\]

Combining this limit with Equation (5), the defining equation for \( H_T(\alpha, u, q) \), shows that

\[
\lim_{q \to 1} \sum_{0 \neq \alpha \in \mathbb{N}^n} \frac{H_T(\alpha, u, q)}{\alpha} (q - 1)^{|\alpha|} T^\alpha = \log \sum_{\ell \in \mathbb{N}^n} u^\ell \frac{T^\ell}{\ell!}.
\]

In particular, this proves that

\[
\lim_{q \to 1} (q - 1)^{|\alpha|} H_T(\alpha, u, q)
\]

exists and therefore

\[
\lim_{q \to 1} (q - 1)^{|\alpha|} H_T(\alpha / d, u, q) = 0
\]

if \( d \) is greater than 1 and divides \( \alpha \).

Equations (6), (10), (11), and (12) show that

\[
\lim_{q \to 1} (q - 1)^{|\alpha| - 1} A_T(\alpha, u, q) = \lim_{q \to 1} \frac{(q - 1)^{|\alpha|}}{\alpha} \sum_{d | \alpha} \mu(d) H_T(\alpha / d, u^d, q^d)
\]

\[
= \lim_{q \to 1} \frac{(q - 1)^{|\alpha|}}{\alpha} H_T(\alpha, u, q)
\]

\[
= \text{the coefficient of } T^\alpha \text{ in } \log \sum_{\ell \in \mathbb{N}^n} u^\ell \frac{T^\ell}{\ell!}.
\]
4 Multivariate Exponential Formula

The limit in Proposition 3.1, namely Equation (7), can be rewritten using a multivariate version of the Exponential Formula applied to the enumeration of graphs. Our presentation of the Multivariate Exponential Formula (Theorem 4.1) follows that of the usual Exponential Formula found in Stanley [7, Section 5.1].

Let $K$ be any field and let $X = (X_1, X_2, \ldots, X_n)$ be a vector of indeterminates. Given a function $f : \mathbb{N}^n \to K$, define

$$E_f(X) := \sum_{\ell \in \mathbb{N}^n} f(\ell_1, \ldots, \ell_n) \frac{X^\ell}{\ell!}. \quad (13)$$

Similarly, given a function $f : \mathbb{N}^n \setminus \{0\} \to K$, define

$$E_f(X) := \sum_{\ell \in \mathbb{N}^n \setminus \{0\}} f(\ell_1, \ldots, \ell_n) \frac{X^\ell}{\ell!}. \quad (14)$$

Also, if $V_1, \ldots, V_n$ are finite (disjoint) sets, let $\Pi(V_1, \ldots, V_n)$ denote the set of set partitions of $V_1 \cup \cdots \cup V_n$. If $\pi \in \Pi(V_1, \ldots, V_n)$, let $\ell(\pi)$ equal the number of blocks in $\pi$ and let $\pi_1, \ldots, \pi_{\ell(\pi)}$ denote the blocks of $\pi$.

**Theorem 4.1** (Multivariate Exponential Formula). Let $f : \mathbb{N}^n \setminus \{0\} \to K$. Define a function $g : \mathbb{N}^n \to K$ by $g(0, \ldots, 0) := 1$ and by

$$g(|V_1|, \ldots, |V_n|) := \sum_{\pi \in \Pi(V_1, \ldots, V_n)} \prod_{i=1}^{\ell(\pi)} f(|\pi_i \cap V_1|, \ldots, |\pi_i \cap V_n|) \quad (15)$$

if at least one of the sets $V_i$ is nonempty. Then $\log E_g(X) = E_f(X)$.

**Proof.** Fix a positive integer $k$ and define the function $g_k : \mathbb{N}^n \setminus \{0\} \to K$ by

$$g_k(|V_1|, \ldots, |V_n|) := \sum_{\pi \in \Pi(V_1, \ldots, V_n)} \prod_{\ell(\pi) = k}^{\ell(\pi) = k} f(|\pi_i \cap V_1|, \ldots, |\pi_i \cap V_n|).$$

Since $\pi_1, \ldots, \pi_k$ are non-empty, they are all distinct and there are $k!$ ways of ordering them. Thus

$$E_{g_k}(X) = \frac{1}{k!} E_f(X)^k.$$

Then

$$E_g(X) = 1 + \sum_{k=1}^{\infty} E_{g_k}(X)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} E_f(X)^k$$

$$= e^{E_f(X)},$$

proving the theorem. \qed
To apply Theorem 4.1 to the enumeration of graphs, we must introduce some more notation. If \( \ell \in \mathbb{N}^n \), let \( \mathcal{G}^{\ell} \) be the set of graphs on the vertices \( v_1, v_2, \ldots, v_{|\ell|} \). Let \( V_1 := \{v_1, \ldots, v_{\ell_1}\}, V_2 := \{v_{\ell_1+1}, \ldots, v_{\ell_2}\}, \ldots, V_n := \{v_{\ell_{n-1}+1}, \ldots, v_{\ell_n}\} \). If \( k \in \mathbb{N}^{(n+1)/2} \), then let \( \mathcal{G}^{\ell}_k \) be the set of graphs in \( \mathcal{G}^{\ell} \) that have \( k_{ij} \) edges between \( V_i \) and \( V_j \) for \( 1 \leq i \leq j \leq n \) and let \( \mathcal{G}^{\ell}_k \) be the number of connected graphs in \( \mathcal{G}^{\ell}_k \). Since a connected graph in \( \mathcal{G}^{\ell} \) must have at least \(|\ell| - 1\) edges, the following proposition is clear.

**Proposition 4.2.** If \(|k| < |\ell| - 1\), then \( \mathcal{G}^{\ell}_k = 0 \).

Now let \( x = (x_{11}, \ldots, x_{ij}, \ldots, x_{nn}) \) be a vector of \( n(n+1)/2 \) indeterminates, where \( 1 \leq i \leq j \leq n \), and define the weight of \( G \in \mathcal{G}^{\ell}_k \) to be \( x_k := \prod_{1 \leq i \leq j \leq n} x_{ij}^{k_{ij}} \).

**Theorem 4.3.**

\[
\log \left( \sum_{\ell \in \mathbb{N}^n} \left( \prod_{1 \leq i < j \leq n} (1 + x_{ij})^{\ell_i \ell_j} \right) \left( \prod_{1 \leq i \leq n} (1 + x_{ii})^{\ell_i/2} \right) \frac{X^\ell}{\ell!} \right) = \sum_{0 \neq \alpha \in \mathbb{N}^n} \sum_{k \in \mathbb{N}^{(n+1)/2}} G_{\alpha}^k x^k \frac{X^\alpha}{\alpha!},
\]

where \( G_{\alpha}^k \) is the number of connected graphs in \( \mathcal{G}_{\alpha}^k \).

**Proof.** Let \( K = \mathbb{C}(x_{11}, \ldots, x_{ij}, \ldots, x_{nn}) \). Let

\[
f : \mathbb{N}^n \setminus \{0\} \to K \quad (\alpha_1, \ldots, \alpha_n) \mapsto \sum_{k \in \mathbb{N}^{(n+1)/2}} G_{\alpha}^k x^k
g : \mathbb{N}^n \to K \quad (\ell_1, \ldots, \ell_n) \mapsto \prod_{1 \leq i \leq j \leq n} (1 + x_{ij})^{\ell_i \ell_j} \prod_{1 \leq i \leq n} (1 + x_{ii})^{\ell_i/2},
\]

The sum of the weights of the connected graphs in \( \mathcal{G}_{\alpha}^k \) equals \( f(\alpha_1, \ldots, \alpha_n) \). The sum of the weights of the graphs in \( \mathcal{G}^{\ell} \) equals \( g(\ell_1, \ldots, \ell_n) \). A graph in \( \mathcal{G}^{\ell} \) is obtained by choosing a partition of the vertex set and choosing a connected graph structure for each block of the partition (and the weight of such a graph is the product of the weights of its connected components). It is clear that \( f \) and \( g \) satisfy Equation (15), so Theorem 4.1 proves the desired result. \( \square \)

Perhaps Theorem 4.1 and its application in Theorem 4.3 to the enumeration of graphs are known, but we do not know of any reference and hence have included the proofs. Presumably there is no explicit formula for \( G_{\alpha}^k \) in general. However, when \(|k| = |\alpha| - 1\) (that is, when the connected graphs in \( \mathcal{G}_{\alpha}^k \) are trees), certain sums of the numbers \( G_{\alpha}^k \) can be computed by the methods in Knuth [6].
5 $A_\Gamma(\alpha, u, q)$ and Connected Graphs

To understand Equation (7) better, we obtain a corollary of Theorem 4.3 by rewriting Equation (16) with the substitutions $1 + x_{ij} = u_{ij} (1 \leq i < j \leq n)$, $1 + x_{ii} = u_{ii}^2 (1 \leq i \leq n)$, and $X_i = u_{ii}T_i (1 \leq i \leq n)$.

Corollary 5.1.

$$\log \sum_{\ell \in \mathbb{N}^n} u^{\ell} \frac{T^{\ell}}{\ell!}$$

$$= \sum_{\alpha \in \mathbb{N}^n} \sum_{k \in \mathbb{N}^{n(n+1)/2}} G_\alpha^k \left( \prod_{1 \leq i<j \leq n} (u_{ij} - 1)^{k_{ij}} \right) \left( \prod_{1 \leq i \leq n} (u_{ii}^2 - 1)^{k_{ii}} \right) \left( \prod_{1 \leq i \leq n} u_{ii}^{\alpha_i} \right) \frac{T^\alpha}{\alpha!}.$$

This allows us to rewrite the result of Proposition 3.1. For each $k \in \mathbb{N}^{n(n+1)/2}$, let

$$S_k = \left\{ p \in \mathbb{N}^{n(n+1)/2} : k_{ii} \geq p_{ii} \text{ for } 1 \leq i \leq n, \ k_{ij} = p_{ij} \text{ for } 1 \leq i < j \leq n \right\}.$$

Proposition 5.2. For each $k \in \mathbb{N}^{n(n+1)/2}$,

$$\left[ \text{the coefficient of } (u - 1)^k \text{ in } \right] \lim_{q \to 1} (q - 1)^{|\alpha| - 1} A_\Gamma(\alpha, u, q)$$

$$= \frac{1}{\alpha!} \sum_{p \in S_k} c_{kp} G_\alpha^p,$$

where

$$c_{kp}^\alpha := \prod_{1 \leq i \leq n} \left( \sum_{j=0}^{\infty} \binom{p_{ii}}{j} \left( \frac{\alpha_i}{k_{ii} - p_{ii} - j} \right) 2^{p_{ii} - j} \right)^2$$

and

$$(u - 1)^k := \prod_{1 \leq i < j \leq n} (u_{ij} - 1)^{k_{ij}}.$$

In particular, if $|k| = |\alpha| - 1$, then

$$\left[ \text{the coefficient of } (u - 1)^k \text{ in } \right] \lim_{q \to 1} (q - 1)^{|\alpha| - 1} A_\Gamma(\alpha, u, q)$$

$$= \frac{1}{\alpha!} t(k) G_k^\alpha,$$

where

$$t(k) := \sum_{1 \leq i \leq n} k_{ii}.$$
Proof. Combining Proposition 3.1 and Corollary 5.1, changing variables from \( k \) to \( p \) for convenience, and using the Binomial Theorem shows that

\[
\lim_{q \to 1} (q - 1)^{\alpha_k - 1} A_k(\alpha, u, q)
\]

\[
= \frac{1}{\alpha!} \sum_{p \in \mathbb{N}^{n(n+1)/2}} G_p^\alpha \left( \prod_{1 \leq i < j \leq n} (u_{ij} - 1)^{p_{ij}} \right) \left( \prod_{1 \leq i \leq n} (u_{ii}^2 - 1)^{p_{ii}} \right) \left( \prod_{1 \leq i < j \leq n} u_{ij}^{\alpha_{ij}} \right)
\]

\[
= \frac{1}{\alpha!} \sum_{p \in \mathbb{N}^{n(n+1)/2}} G_p^\alpha \left( \prod_{1 \leq i \leq n} (u_{ii} + 1)^{p_{ii}} \right) \left( \prod_{1 \leq i \leq n} u_{ii}^{\alpha_{ii}} \right) \left( \prod_{1 \leq i \leq j \leq n} (u_{ij} - 1)^{p_{ij}} \right)
\]

Note that \( G_p^\alpha \) is nonzero for finitely many \( p \), so the above sum over \( p \) is a finite sum. The expression

\[
G_p^\alpha \left( \prod_{1 \leq i \leq n} \sum_{j=0}^{p_{ii}} \binom{p_{ii}}{j} (u_{ii} - 1)^{j} 2^{p_{ii}-j} \right) \left( \prod_{1 \leq i \leq n} \sum_{j=0}^{\alpha_{ii}} \binom{\alpha_{ii}}{j} (u_{ii} - 1)^{j} \right) \left( \prod_{1 \leq i \leq j \leq n} (u_{ij} - 1)^{p_{ij}} \right),
\]

when viewed as a polynomial in the variables \( u_{ij} - 1 \) (\( 1 \leq i \leq j \leq n \)), has a nonzero \((u - 1)^k\) term if \( p \in S_k \), and the coefficient of that term is

\[
G_p^\alpha \prod_{1 \leq i \leq n} \left( \sum_{j=0}^{\infty} \binom{p_{ii}}{j} \binom{\alpha_{ii}}{j} 2^{p_{ii}-j} \right).
\]

Summing over all \( p \in S_k \) and dividing by \( \alpha! \) completes the proof of Equation (19).

To prove Equation (20) it is enough to note that by Theorem 4.2, the only non-zero summand on the right-hand side of Equation (19) occurs when \( p = k \) and

\[
e_{kk}^\alpha = 2^{k(k)}.
\]

Observe that the left- and right-hand sides of Equation (19) are nonzero for finitely many \( k \) and that the sum over \( j \) is actually finite by the definition of binomial coefficients. Also, the sum over \( j \) can be expressed in terms of a hypergeometric series as

\[
\sum_{j=0}^{\infty} \binom{p_{ii}}{j} \binom{\alpha_{ii}}{k_{ii} - p_{ii} - j} 2^{p_{ii} - j} = 2^{p_{ii}} \binom{\alpha_{ii}}{k_{ii} - p_{ii}} \mathbf{F}( -p_{ii}, -k_{ii} + p_{ii}; \alpha_{ii} - k_{ii} + p_{ii} + 1; 1/2),
\]

if desired.
6 A Mahler-type Expansion for $A_{\Gamma}(\alpha, u, q)$

We can use Proposition 5.2 to understand $A_{\Gamma}(\alpha, q)$ if we rewrite $A_{\Gamma}(\alpha, u, q)$ using the Mahler-type expansion given in the following theorem.

**Theorem 6.1.** If $f \in \mathbb{Q}(q)[x_1, \ldots, x_r]$ and $f(q^{b_1}, \ldots, q^{b_r}) \in \mathbb{Z}[q]$ for all non-negative integers $b_1, \ldots, b_r$, then there are polynomials $\{c_\ell(q) \in \mathbb{Z}[q] : \ell \in \mathbb{N}^r\}$ such that

$$f = \sum_{\ell \in \mathbb{N}^r} c_\ell(q) \prod_{1 \leq i \leq r} \left\langle \frac{x_i}{\ell_i} \right\rangle_q,$$

where

$$\left\langle \frac{x}{\ell} \right\rangle_q := \prod_{1 \leq i' \leq \ell} \frac{(x/q^{i'} - 1)}{q^{i'} - 1} \quad (22)$$

and $c_\ell(q) = 0$ for all but finitely many $\ell$.

The proof of this theorem was communicated to us by Keith Conrad.

**Proof by Keith Conrad.** An expression like Equation (21) exists for any polynomial $f$ for some unique $c_\ell(q) \in \mathbb{Q}(q)$ since the products

$$\prod_{1 \leq i \leq r} \left\langle \frac{x_i}{\ell_i} \right\rangle_q$$

are an additive basis of $\mathbb{Q}(q)[x_1, \ldots, x_r]$. We now show that under the hypothesis on $f$ we actually have $c_\ell(q) \in \mathbb{Z}[q]$.

Let $E$ be the shift operator acting on functions $h$ of $b \in \mathbb{Z}_{\geq 0}$ by

$$(Eh)(b) := h(b + 1),$$

and for each non-negative integer $n$, define

$$\Delta_n := (E - I)(E - qI) \cdots (E - q^{n-1}I),$$

where $I$ is the identity operator. Since $f(b_1, \ldots, b_r) := F(q^{b_1}, \ldots, q^{b_r})$ for non-negative integers $b_1, \ldots, b_r$, a standard property of the $q$-difference operators $\Delta_i$ yields

$$c_\ell(q) = (\Delta_{\ell_1} \cdots \Delta_{\ell_r} F)(0, \ldots, 0),$$

where $\ell = (\ell_1, \ldots, \ell_r)$ and $\Delta_i$ acts on the variable $b_i$. Explicitly, we obtain the following expression

$$c_\ell(q) = \sum_{j \in \mathbb{N}^r} \prod_{i=1}^r \left( \frac{\ell_i}{j_i} \right) (-1)^{j_i} q^{j_i(j_i - 1)/2} f(q^{\ell_1 - j_1}, \ldots, q^{\ell_r - j_r}),$$

which shows that $c_\ell(q) \in \mathbb{Z}[q]$ since by hypothesis $f(q^{\ell_1 - j_1}, \ldots, q^{\ell_r - j_r}) \in \mathbb{Z}[q]$ when $j_i \leq \ell_i$ for $1 \leq i \leq r$. \qed
Note that \( \langle x \rangle_q = \left[ \frac{b}{x} \right]_q \) when \( x = q^b \) and that
\[
\lim_{q \to 1} (q - 1)^{\ell} \langle x \rangle_q = \frac{(x - 1)^\ell}{\ell!}.
\] (23)

Here \( \left[ \frac{b}{x} \right]_q \) is a \( q \)-binomial coefficient (see Appendix A for more information about \( q \)-binomial coefficients). By Theorem 6.1 and the fact that \( A_\Gamma(\alpha, q) \in \mathbb{Z}[q] \), we can write
\[
A_\Gamma(\alpha, u, q) = \sum_{k \in \mathbb{N}^{n(n+1)/2}} a_\Gamma(\alpha, k, q) \langle u \rangle_k^q,
\] (24)
for some \( a_\Gamma(\alpha, k, q) \in \mathbb{Z}[q] \), where
\[
\langle u \rangle_k^q := \prod_{1 \leq i \leq j \leq n} \langle u_{ij} \rangle_{k_{ij}}^q.
\]

Hence
\[
A_\Gamma(\alpha, q) = \sum_{k \in \mathbb{N}^{n(n+1)/2}} a_\Gamma(\alpha, k, q) \left[ g \right]_k^q,
\] (25)
where
\[
\left[ g \right]_k^q := \prod_{1 \leq i \leq j \leq n} \left[ g_{ij} \right]_{k_{ij}}^q.
\]

It turns out that Proposition 5.2 leads to a formula for the derivatives of \( a_\Gamma(\alpha, k, q) \) evaluated at \( q = 1 \), given in Proposition 6.2, which in turn produces a formula for the derivatives of \( A_\Gamma(\alpha, q) \) evaluated at \( q = 1 \) (see Theorems 6.3 and 7.1).

**Proposition 6.2.** For \( k \in \mathbb{N}^{n(n+1)/2} \) such that \( |k| > |\alpha| \) we have
\[
\left. \frac{a_\Gamma(\alpha, k, q)}{(q - 1)^{|k| - |\alpha| + 1}} \right|_{q = 1} = \frac{k!}{\alpha!} \sum_{p \in S_k} c_{kp}^\alpha C_{\alpha}^p.
\] (26)

**Proof.** Using Equations (23) and (24), we find
\[
\lim_{q \to 1} \frac{(q - 1)^{|\alpha| - 1} A_\Gamma(\alpha, u, q)}{A_\Gamma(\alpha, u, q)} (q - 1)^{|\alpha| - 1} \langle u \rangle_k^q = \sum_{k \in \mathbb{N}^{n(n+1)/2}} a_\Gamma(\alpha, k, q) (q - 1)^{|\alpha| - 1} \langle u \rangle_k^q (q - 1)^k \langle u \rangle_k^q
\] (27)
\[
= \lim_{q \to 1} \sum_{k \in \mathbb{N}^{n(n+1)/2}} a_\Gamma(\alpha, k, q) \left. \frac{A_\Gamma(\alpha, k, q)}{(q - 1)^{|k| - |\alpha| + 1}} \right|_{q = 1} (q - 1)^k \langle u \rangle_k^q
\] (28)
\[
= \lim_{q \to 1} \sum_{k \in \mathbb{N}^{n(n+1)/2}} \frac{a_\Gamma(\alpha, k, q)}{(q - 1)^{|k| - |\alpha| + 1}} \left. \frac{A_\Gamma(\alpha, k, q)}{(q - 1)^k} \right|_{q = 1} \frac{1}{k!} (u - 1)^k.
\] (29)

The claim now follows from Proposition 5.2.
Note that if $|\mathbf{k}| \leq |\mathbf{\alpha}|$, Proposition 6.2 says nothing about $a_{\Gamma}(\mathbf{\alpha}, \mathbf{k}, q)$ at $q = 1$. This is why Theorems 6.3 and 7.1 below only give information about leading coefficients. The first consequence of Proposition 6.2 appears when we evaluate $A_{\Gamma}(\mathbf{\alpha}, 1)$. By Equation (25),

$$A_{\Gamma}(\mathbf{\alpha}, 1) = \sum_{\mathbf{k} \in \mathbb{N}^{n(n+1)/2}} a_{\Gamma}(\mathbf{\alpha}, \mathbf{k}, 1) \left(\frac{g}{k}\right),$$

where

$$\left(\frac{g}{k}\right) := \prod_{1 \leq i \leq j \leq n} \left(\frac{g_{ij}}{k_{ij}}\right).$$

**Theorem 6.3.** The quantity $A_{\Gamma}(\mathbf{\alpha}, 1)$ is a polynomial in the variables $g_{ij}$ whose homogeneous component of highest degree $A_{\Gamma}^*(\mathbf{\alpha}, 1)$ has total degree $|\mathbf{\alpha}| - 1$ and has the form

$$A_{\Gamma}^*(\mathbf{\alpha}, 1) = \frac{1}{\mathbf{\alpha}!} \sum_{|\mathbf{k}| = |\mathbf{\alpha}|-1} C_{\Gamma,k}^{\mathbf{\alpha}} g^{\mathbf{k}},$$

where

$$C_{\Gamma,k}^{\mathbf{\alpha}} := 2^{t(\mathbf{k})} G_{k}^{\mathbf{\alpha}} \quad \text{and} \quad t(\mathbf{k}) := \sum_{1 \leq i \leq n} k_{ii}.$$

**Proof.** That $A_{\Gamma}(\mathbf{\alpha}, 1)$ is a polynomial in the variables $g_{ij}$ is clear from Equation (30) since that is true for the binomials $\binom{g}{k}$. The binomial $\binom{g}{k}$ has total degree $|\mathbf{k}|$, and by Proposition 6.2, $a_{\Gamma}(\mathbf{\alpha}, \mathbf{k}, 1) = 0$ if $|\mathbf{k}| > |\mathbf{\alpha}|-1$. Hence the total degree of $A_{\Gamma}(\mathbf{\alpha}, 1)$ is $|\mathbf{\alpha}|-1$. To finish the proof it is enough to combine Equation (20) of Proposition 5.2 with Proposition 6.2. \qed

As a special case of this theorem, we can consider the quiver $S_N$ from Section 1, which has a single vertex (so $n = 1$) and $g$ loops, and $\mathbf{\alpha} = \mathbf{\alpha}$. In this case, $A_{\Gamma}(\mathbf{\alpha}, 1)$ is a polynomial in $g$ of degree $\mathbf{\alpha} - 1$ and leading coefficient $2^{\mathbf{\alpha}-1} G_{\mathbf{\alpha}-1}^{\mathbf{\alpha}}/\mathbf{\alpha}!$. But $G_{\mathbf{\alpha}-1}^{\mathbf{\alpha}}$ is just the number of (spanning) trees on $\mathbf{\alpha}$ labeled vertices, which is $\mathbf{\alpha}^{\mathbf{\alpha}-2}$ by Cayley’s Theorem. So the leading coefficient is $2^{\mathbf{\alpha}-1} \mathbf{\alpha}^{\mathbf{\alpha}-2}/\mathbf{\alpha}!$ as claimed in the introduction.

**7 The derivatives $\frac{d^s}{dq^s} A_{\Gamma}(\mathbf{\alpha}, q)$ at $q = 1$**

We can proceed further by differentiating Equation (25) to obtain information about the highest order terms of the $s$-th derivative of $A_{\Gamma}(\mathbf{\alpha}, q)$ evaluated at $q = 1$. This is the subject of the next theorem. Note that $G_{\mathbf{\alpha}}^{\mathbf{\alpha}}$ was defined prior to Theorem 4.3, $S_k$ was defined in Equation (18), and $S(\ell, k)$ is the Stirling number of the second kind discussed in Appendix A. Also, if $\mathbf{k}, \mathbf{\ell} \in \mathbb{N}^{n(n+1)/2}$, we write $\mathbf{k} \leq \mathbf{\ell}$ if $k_{ij} \leq \ell_{ij}$ for all $1 \leq i \leq j \leq n$. To simplify the notation we let

$$A_{\Gamma,s}(\mathbf{\alpha}, q) := \frac{d^s}{dq^s} A_{\Gamma}(\mathbf{\alpha}, q).$$

**Theorem 7.1.** The quantity $A_{\Gamma,s}(\mathbf{\alpha}, 1)$ is a polynomial in the variables $g_{ij}$ whose homogeneous component of highest degree $A_{\Gamma,s}^*(\mathbf{\alpha}, 1)$ has total degree $s + |\mathbf{\alpha}|-1$ and is given by

$$A_{\Gamma,s}^*(\mathbf{\alpha}, 1) = \frac{1}{\mathbf{\alpha}!} \sum_{|\mathbf{\ell}| = s + |\mathbf{\alpha}|-1} C_{\Gamma,s,\ell}^{\mathbf{\alpha}} g^{\mathbf{\ell}}.$$

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where
\[ C_{\ell, s, \ell}^\alpha := \frac{s!}{\ell!} \sum_{k \leq \ell} \sum_{p \in S_k} S(k, \ell) k! \sum_{p} c_{kp} G_{p}^\alpha \]
and
\[ S(\ell, k) := \prod_{1 \leq i \leq j \leq n} S(\ell_{ij}, k_{ij}) . \]

Proof. Differentiating
\[ A_{\ell, s}(\alpha, 1) \]
\[ = \sum_{k \in \mathbb{N}^{(n+1)/2}} \sum_{r=0}^{s} \left( \frac{d^r}{dq^r} a_{\ell, s}(\alpha, k, q) \right) \left( \frac{d^{s-r}}{dq^{s-r}} \left[ \frac{g}{k} \right]_q \right) \bigg|_{q=1} ^{r} . \]
By the Product Rule and Theorem A.1, \( \left( \frac{d^{s-r}}{dq^{s-r}} \left[ \frac{g}{k} \right]_q \right) \bigg|_{q=1} ^{r} \) is a polynomial in the variables \( g_{ij} \) \((1 \leq i \leq j \leq n)\) of total degree \( |k| + s - r \), and if \( t \in \mathbb{N}^{(n+1)/2} \) with \( |t| = s - r \), then the coefficient of \( g^{k+t} \) in \( \left( \frac{d^{s-r}}{dq^{s-r}} \left[ \frac{g}{k} \right]_q \right) \bigg|_{q=1} ^{r} \) is
\[ \left( \frac{s - r}{t_{11}, \ldots, t_{ij}, \ldots, t_{nn}} \right) \prod_{1 \leq i \leq j \leq n} \frac{t_{ij}! S(k_{ij} + t_{ij}, k_{ij})}{(k_{ij} + t_{ij})!} = \frac{(s - r)!}{(k + t)!} \prod_{1 \leq i \leq j \leq n} S(k_{ij} + t_{ij}, k_{ij}). \] (34)
By Proposition 6.2, if \( r < |k| - |\alpha| + 1 \), then \( \left( \frac{d^r}{dq^r} a_{\ell, s}(\alpha, k, q) \right) \bigg|_{q=1} ^{r} = 0 \). Thus the total degree of each leading term of \( A_{\ell, s}(\alpha, 1) \) is at most \( |k| + s - |k| - |\alpha| + 1 = s + |\alpha| - 1 \). We now determine the coefficient of \( g^{k} \) in \( A_{\ell, s}(\alpha, 1) \) when \( \ell \in \mathbb{N}^{(n+1)/2} \) satisfies \( |\ell| = s + |\alpha| - 1 \). A summand in Equation (33) has a leading term of \( g^{k} \) when \( k \leq \ell \) and \( r = |k| - |\alpha| + 1 = s - |\ell| + |k| \). Given such a \( k \), let \( t = \ell - k \). Then, using Proposition 6.2 and Equation (34), we find that
\[ \left[ \text{the coefficient of } g^{k} \text{ in } \right] \sum_{r=0}^{s} \left( \frac{d^r}{dq^r} a_{\ell, s}(\alpha, k, q) \right) \left( \frac{d^{s-r}}{dq^{s-r}} \left[ \frac{g}{k} \right]_q \right) \bigg|_{q=1} ^{r} \]
\[ = \left( \frac{s}{|k| - |\alpha| + 1} \right)^! \frac{|k| - |\alpha| + 1}{\alpha}! \sum_{p \in S_k} c_{kp} G_{p}^\alpha \frac{(s - |k| - |\alpha| + 1)!}{\ell!} S(\ell, k) \]
\[ = \frac{s!|k|!}{\alpha!\ell!} S(\ell, k) \sum_{p \in S_k} c_{kp} G_{p}^\alpha . \]
Summing over all \( k \leq \ell \) completes the proof. \( \square \)
As a special case of this theorem, we can consider the quiver $S_g$ from Section 1. In this case, $A_{r,s}(\alpha, 1)$ is a polynomial in $g$ of degree $s + \alpha - 1$ and leading coefficient
\[
\frac{s!}{\alpha! (s + \alpha - 1)!} \sum_{k=\alpha-1}^{s+\alpha-1} S(s + \alpha - 1, k) k! \sum_{p=\alpha-1}^{k} G_p^\alpha \sum_{j=0}^{\infty} \left( p \right) \left( \begin{array}{c} \alpha \\ k - p - j \end{array} \right) 2^{p-j}.
\]

A Derivatives of $q$-binomial coefficients

The goal of this appendix is to prove a theorem about derivatives of $q$-binomial coefficients. The $q$-binomial coefficient $\left[ \frac{b}{k} \right]_q$ is a polynomial in $q$ of degree $k(b-k)$ defined for non-negative integers $b$ and $k$ by the formula
\[
\left[ \frac{b}{k} \right]_q = \frac{(q^b - 1)(q^{b-1} - 1) \cdots (q^{b-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)} \prod_{i=1}^{k} \sum_{j=0}^{b-i} q^j = \left( e^x - 1 \right)^k \frac{x^k}{k!}
\]
The result in question involves the Stirling number of the second kind $S(\ell, k)$, which counts the number of partitions of an $\ell$-set into $k$ blocks. While there are many formulas involving Stirling numbers of the second kind, we will only need to know that
\[
\sum_{\ell=k}^{\infty} S(\ell, k) \frac{x^\ell}{\ell!} = \frac{(e^x - 1)^k}{k!}
\]
for all $k \geq 0$. Now we can state the theorem.

**Theorem A.1.** Fix non-negative integers $k$ and $t$. Then there is a polynomial $P_{k,t}(b)$ of degree $k + t$ with leading coefficient $\frac{1}{(k+t)!} S(k + t, k)$ so that for all non-negative integers $b$,
\[
P_{k,t}(b) = \left. \left( \frac{d^t}{dq^t} \left[ \frac{b}{k} \right]_q \right) \right|_{q=1}.
\]

We need one lemma before proving Theorem A.1.

**Lemma A.2.** Fix non-negative integers $i$ and $m$. Then there is a polynomial $p_{i,m}(b)$ of degree $m + 1$ with leading coefficient $\frac{1}{i(m+1)}$ so that for all positive integers $b \geq i - 1$,
\[
p_{i,m}(b) = \left. \left( \frac{d^m}{dq^m} \sum_{j=0}^{b-i} q^j \right) \right|_{q=1}.
\]

**Proof.** Let $r$ be a non-negative integer and define the function
\[
f_{m,r}(b, i, q) = \frac{d^m}{dq^m} \frac{d^r}{dq^r} \sum_{j=0}^{b-i} q^j.
\]
We want to show that \( f_{m,0}(b, i, 1) \) is a polynomial of degree \( m + 1 \) with leading coefficient \( \frac{1}{i(m+1)} \), but by induction on \( m \) we will prove the stronger statement that \( f_{m,r}(b, i, 1) \) is a polynomial of degree \( m + r + 1 \) with leading coefficient \( \frac{1}{i(m+r+1)} \). When \( m = 0 \), we find that

\[
\begin{align*}
 f_{0,r}(b, i, q) &= \frac{d^r}{dq^r} \sum_{j=0}^{b-i} q^j = \frac{\sum_{j=r}^{b-i} j(j-1) \cdots (j-r+1)q^{j-r}}{\sum_{j=0}^{i-1} q^j} \\
 f_{0,r}(b, i, 1) &= \frac{\sum_{j=r}^{b-i} j(j-1) \cdots (j-r+1)}{i} = \frac{r! (b - i + 1)}{i(r + 1)}.
\end{align*}
\]

Thus \( f_{0,r}(b, i, 1) \) is a polynomial in \( b \) of degree \( r + 1 \) and leading coefficient \( \frac{1}{i(r+1)} \), proving the base case for the induction. When \( m \geq 1 \), writing the Quotient Rule in the form

\[
\frac{d}{dq} \left( \frac{g(q)}{h(q)} \right) = \frac{g'(q) h(q) - g(q) h'(q)}{h(q)^2},
\]

we see that

\[
 f_{1,r}(b, i, q) = f_{0,r+1}(b, i, q) - f_{0,r}(b, i, q) \cdot \sum_{j=0}^{i-1} q^j.
\]

Differentiating both sides \( m - 1 \) times and evaluating at \( q = 1 \), we obtain

\[
 f_{m,r}(b, i, 1) = f_{m-1,r+1}(b, i, 1) - \sum_{s=0}^{m-1} \binom{m - 1}{s} f_{s,r}(b, i, 1) \left( \frac{d^{m-1-s}}{dq^{m-1-s}} \sum_{j=0}^{i-1} q^j \right). 
\]

The inductive hypothesis shows that \( f_{m-1,r+1}(b, i, 1) \) is a polynomial in \( b \) of degree \( m + r + 1 \) and leading coefficient \( \frac{1}{i(m+r+1)} \), and every other term on the right-hand side of the equation is a polynomial in \( b \) of degree less than \( m + r + 1 \). This proves that \( f_{m,r}(b, i, 1) \) is a polynomial in \( b \) of degree \( m + r + 1 \) and leading coefficient \( \frac{1}{i(m+r+1)} \).

Now we can prove Theorem A.1.

**Proof.** We link derivatives of \( \binom{b}{k}_q \) with the derivatives computed in Lemma A.2 via Taylor series expansions at \( q = 1 \):

\[
\sum_{t=0}^{\infty} \left( \frac{d^t}{dq^t} \binom{b}{k}_q \right) \bigg|_{q=1} \frac{(q - 1)^t}{t!} = \left[ \binom{b}{k}_q \right]_{q=1} = \prod_{i=1}^{k} \frac{\sum_{j=0}^{b-i} q^j}{\sum_{j=0}^{i-1} q^j} = \prod_{i=1}^{k} \left( \sum_{m=0}^{\infty} \left( \frac{d^m}{dq^m} \sum_{j=0}^{b-i} q^j \right) \bigg|_{q=1} \frac{(q - 1)^m}{m!} \right).
\]

By Lemma A.2, for all integers \( b \geq k - 1 \),

\[
\prod_{i=1}^{k} \left( \sum_{m=0}^{\infty} \left( \frac{d^m}{dq^m} \sum_{j=0}^{b-i} q^j \right) \bigg|_{q=1} \frac{(q - 1)^m}{m!} \right) = \prod_{i=1}^{k} \left( \sum_{m=0}^{\infty} p_{i,m}(b) \frac{(q - 1)^m}{m!} \right),
\]

\[15\]
where $p_{i,m}(b)$ is the polynomial from Lemma A.2. In fact, we claim this equation holds for $0 \leq b < k - 1$ as well. Fix $b$ with $0 \leq b < k - 1$. The left-hand side is certainly 0, and by Lemma A.2, $p_{b+1,m}(b) = 0$ for all $m$, so the right-hand side is 0 as well. This shows that for all non-negative integers $b$,

$$
\sum_{t=0}^{\infty} \left( \frac{d^t}{dq^t} \left[ b \right]_q \right) \left|_{q=1} \right. \frac{(q-1)^t}{t!} = \prod_{i=1}^{k} \left( \sum_{m=0}^{\infty} \frac{p_{i,m}(b)(q-1)^m}{m!} \right).
$$

Thus there is a polynomial $P_{k,t}(b)$ in $b$ of degree $k + t$ so that $P_{k,t}(b) = \left( \frac{d^t}{dq^t} \left[ b \right]_q \right) \left|_{q=1} \right.$ for all non-negative integers $b$, and its leading coefficient is

$$
= \left[ \text{coefficient of } \frac{(q-1)^t}{t!} \right] \prod_{i=1}^{k} \frac{1}{i} \left( \frac{e^{q-1} - 1}{q-1} \right)^k
$$

$$
= \left[ \text{coefficient of } \frac{(q-1)^t}{t!} \right] \frac{1}{k!} \left( \frac{e^{q-1} - 1}{q-1} \right)^k
$$

$$
= \left[ \text{coefficient of } \frac{(q-1)^t}{t!} \right] \sum_{m=0}^{\infty} \frac{m!}{(k+m)!} S(k+m, k) \frac{(q-1)^m}{m!}
$$

$$
= \frac{t!}{(k+t)!} S(k+t, k),
$$

where the second-to-last line comes from the generating function in Equation (36). \qed

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