Convergence of random measures in geometric probability

Mathew D. Penrose

University of Bath

August 2005

Abstract

Given $n$ independent random marked $d$-vectors $X_i$ with a common density, define the measure $\nu_n = \sum_i \xi_i$, where $\xi_i$ is a measure (not necessarily a point measure) determined by the (suitably rescaled) set of points near $X_i$. Technically, this means here that $\xi_i$ stabilizes with a suitable power-law decay of the tail of the radius of stabilization. For bounded test functions $f$ on $\mathbb{R}^d$, we give a law of large numbers and central limit theorem for $\nu_n(f)$. The latter implies weak convergence of $\nu_n(\cdot)$, suitably scaled and centred, to a Gaussian field acting on bounded test functions. The general result is illustrated with applications including the volume and surface measure of germ-grain models with unbounded grain sizes.

Key words and phrases: Random measure, point process, random set, stabilization, law of large numbers, central limit theorem, Gaussian field, germ-grain model, Boolean model, Voronoi coverage.

AMS classifications: 60D05, 60G57, 60F05, 60F25, 52A22
1 Introduction

This paper is concerned with the study of the limiting behaviour of random measures based on marked Poisson or binomial point processes in $d$-dimensional space, arising as the sum of contributions from each point of the point process. Many random spatial measures can be described in these terms, and general limit theorems, including laws of large numbers, central limit theorems, and large deviation principles, are known for the total measure of such measures, based on a notion of stabilization (local dependence); see [16, 17, 18, 21).

Recently, attention has turned to the asymptotic behaviour of the measure itself (rather than only its total measure), notably in [2, 3, 9, 14, 19, 21]. It is of interest to determine when one can show weak convergence of this measure to a Gaussian random field. As in Heinrich and Molchanov [9], Penrose [14], one can consider a limiting regime where a homogeneous Poisson process is sampled over an expanding window. In an alternative limiting regime, the intensity of the point process becomes large and the point process is locally scaled to keep the average density of points bounded; the latter approach allows for point processes with non-constant densities and is the one adopted here.

A random measure is said to be exponentially stabilizing when the contribution of an inserted point is determined by the configuration of (marked) Poisson points within a finite (though in general random) distance, known as a radius of stabilization, having a uniformly exponentially decaying tail after scaling of space. This concept was introduced by Baryshnikov and Yukich [3], who proved general results on weak convergence to a limiting Gaussian field for exponentially stabilizing measures. A variety of random measures are exponentially stabilizing, including those concerned with nearest neighbour graph, Voronoi and Delaunay graph, germ-grain models with bounded grains, and packing; see [3].

In the present work we extend the results of [3] in several directions. Specifically, in [3] attention is restricted to the case where the random measure is concentrated at the points of the underlying point process, and to continuous test functions; we relax both of these restrictions, and so are able to include indicator functions of Borel sets as test functions. Moreover, we relax the condition of exponential stabilization to power-law stabilization.

Along with the central limit theorems, we give a law of large numbers for random measures (under weaker stabilization conditions) which is proved using similar ideas. In Section 7, we illustrate our general results with applications to germ-grain models with unbounded grains, and to questions of Voronoi coverage raised by Khmaladze and Toronjadze [10]. Many other fields of application have been discussed elsewhere [2, 3, 16, 18] and we do not attempt to review all of these.

We state our general results in Section 2. Our approach to proof may be summa-
rized as follows. In the case where the underlying point process is Poisson, we obtain the covariance structure of our limiting random field (and also to prove the law of large numbers) by a refinement of the approach to second moment calculations used in [3] and [18], often known as the objective method [1, 22], which is discussed in Section 3. To show that the limiting random field is Gaussian, we borrow normal approximation results from [19] which were proved there using Stein’s method (in contrast, [3] uses the method of moments). Finally, to de-Poissonize the central limit theorems (i.e., to extend them to binomial point processes with a non-random number of points), in Section 6 we perform further second moment calculations using a version of the objective method. This approach entails an annoyingly large number of similar calculations (see Lemmas 3.8 and 6.1) but avoids the necessity of introducing a notion of ‘external stabilization’ (see Section 2) which was used to deal with the second moment calculations for de-Poissonization in [3]. This, in turn, seems to be necessary to include germ-grain models with unbounded grains. We give our proofs in the general setting of marked point process, which is the context for many of the applications.

2 Notation and results

Let $(M, \mathcal{F}_M, \mu_M)$ be a probability space (the mark space). Let $d \in \mathbb{N}$. Let $\xi(x, t; x, A)$ be a Borel-measurable $\mathbb{R}$-valued function defined for all 4-tuples $(x, t, x, A)$, where $x \subset \mathbb{R}^d \times M$ is finite and where $(x, t) \in \mathcal{X}$ (so $x \in \mathbb{R}^d$ and $t \in M$), and $A$ is a Borel set in $\mathbb{R}^d$. We assume that $\xi(x, t; x, A) := \xi(x, t; x, \cdot)$ is a measure on $\mathbb{R}^d$ with finite total measure. (Our results actually hold when $\xi(x, t; x)$ is a signed measure with finite total variation; see the remarks at the end of this section.)

Suppose $(x, t) \in \mathbb{R}^d \times M$ and $x \subset \mathbb{R}^d \times M$ is finite. If $(x, t) \notin x$, we abbreviate notation and write $\xi(x, t; x)$ instead of $\xi(x, t; x \cup \{(x, t)\})$. We also write $x \times t$ for $x \cup \{(x, t)\}$.

Given $a > 0$ and $y \in \mathbb{R}^d$, we let $y + x \times t := \{(y + ax, s) : (x, s) \in x\}$; in other words, scalar multiplication and translation act on only the first component of elements of $\mathbb{R}^d \times M$. We say $\xi$ is translation invariant if

$$\xi(x, t; x, A) = \xi(y + x, t; y, x, y + A)$$

for all $y \in \mathbb{R}^d$, all finite $x \subset \mathbb{R}^d \times M$ and $x \in x$, and all Borel $A \subset \mathbb{R}^d$. Some of the general concepts defined in the sequel can be expressed more transparently when $\xi$ is translation invariant.

Let $\kappa$ be a probability density function on $\mathbb{R}^d$. Abusing notation slightly, we also let $\kappa$ denote the corresponding probability measure on $\mathbb{R}^d$, i.e. we write $\kappa(A)$ for $\int_A \kappa(x)dx$, for Borel $A \subset \mathbb{R}^d$. We shall assume throughout that the density
function $\kappa$ is bounded with supremum denoted $\|\kappa\|_\infty$, and that $\kappa$ is Lebesgue-almost everywhere continuous. Let $\text{supp}(\kappa)$ denote the support of $\kappa$, i.e., the smallest closed set $B$ in $\mathbb{R}^d$ with $\kappa(B) = 1$.

For all $\lambda > 0$, let $\mathcal{P}_\lambda$ denote a Poisson point process in $\mathbb{R}^d \times \mathcal{M}$ with intensity measure $\lambda \kappa \times \mu_\mathcal{M}$. For $n \in \mathbb{N}$, let $\mathcal{X}_n$ be the point process consisting of $n$ independent identically distributed random elements of $\mathbb{R}^d \times \mathcal{M}$ with common distribution given by $\kappa \times \mu_\mathcal{M}$. Let $\mathcal{H}_\lambda$ denote a Poisson point process in $\mathbb{R}^d \times \mathcal{M}$ with intensity $\lambda$ times the product of $d$-dimensional Lebesgue measure and $\mu_\mathcal{M}$.

Suppose we are given a family of Borel subsets $\Gamma_\lambda$ of $\mathbb{R}^d$, indexed by $\lambda \geq 1$. Assume the sets $\Gamma$ are nondecreasing in $\lambda$, i.e. $\Gamma_\lambda \subseteq \Gamma_{\lambda'}$ for $\lambda < \lambda'$. Denote by $\Gamma$ the limiting set, i.e. set $\Gamma = \bigcup_{\lambda \geq 1} \Gamma_\lambda$. We assume that the topological boundary of $\Gamma$ (i.e., the intersection of the closure of $\Gamma$ with that of its complement) has zero Lebesgue measure, and that $\text{int}(\Gamma) = \bigcup_{\lambda \geq 1} \text{int}(\Gamma_\lambda)$, where $\text{int}(\cdot)$ denotes interior. The simplest special case has $\Gamma_\lambda = \mathbb{R}^d$ for all $\lambda$.

For $\lambda > 0$, and for finite $\mathcal{X} \subset \mathbb{R}^d \times \mathcal{M}$ with $(x,t) \in \mathcal{X}$, and Borel $A \subset \mathbb{R}^d$, let

$$\xi_\lambda(x, t; \mathcal{X}, A) := \xi(x, t; x + \lambda^{1/d}(x + \mathcal{X}), x + \lambda^{1/d}(x + A))1_{\Gamma_\lambda}(x).$$

When $\xi$ is translation invariant, the rescaled measure $\xi_\lambda$ simplifies to

$$\xi_\lambda(x, t; \mathcal{X}, A) = \xi(\lambda^{1/d}x, t; \lambda^{1/d}\mathcal{X}, \lambda^{1/d}A)1_{\Gamma_\lambda}(x). \quad (2.1)$$

Our principal objects of interest are the random measures $\mu^\xi_\lambda$ and $\nu^\xi_{\lambda,n}$ on $\mathbb{R}^d$, defined for $\lambda > 0$ and $n \in \mathbb{N}$, by

$$\mu^\xi_\lambda := \sum_{(x,t) \in \mathcal{P}_\lambda} \xi_\lambda(x, t; \mathcal{P}_\lambda); \quad \nu^\xi_{\lambda,n} := \sum_{(x,t) \in \mathcal{X}_n} \xi_\lambda(x, t; \mathcal{X}_n).$$

We are also interested in the centred versions of these measures $\mu^\xi_{\lambda} := \mu^\xi_\lambda - \mathbb{E}[\mu^\xi_\lambda]$ and $\nu^\xi_{\lambda,n} := \nu^\xi_{\lambda,n} - \mathbb{E}[\nu^\xi_{\lambda,n}]$ (which are signed measures). We study these measures via their action on test functions in the space $B(\mathbb{R}^d)$ of bounded Borel-measurable functions on $\mathbb{R}^d$. We let $\hat{B}(\mathbb{R}^d)$ denote the subclass of $B(\mathbb{R}^d)$ consisting of those functions that are Lebesgue-almost everywhere continuous.

Given $f \in \hat{B}(\mathbb{R}^d)$, set $\langle f, \xi_\lambda(x, t; \mathcal{X}) \rangle := \int_{\mathbb{R}^d} f(z) \xi_\lambda(x, t; \mathcal{X}, dz)$. Also, set

$$\langle f, \mu^\xi_\lambda \rangle := \int_{\mathbb{R}^d} f d\mu^\xi_\lambda = \sum_{(x,t) \in \mathcal{P}_\lambda} \langle f, \xi_\lambda(x, t; \mathcal{P}_\lambda) \rangle$$

and set $\langle f, \mu^\xi_\lambda \rangle := \int_{\mathbb{R}^d} f d\mu^\xi_\lambda$, so that $\langle f, \mu^\xi_\lambda \rangle = \langle f, \mu^\xi_\lambda \rangle - \mathbb{E} \langle f, \mu^\xi_\lambda \rangle$. Similarly, let $\langle f, \nu^\xi_{\lambda,n} \rangle := \int_{\mathbb{R}^d} f d\nu^\xi_{\lambda,n}$ and $\langle f, \nu^\xi_{\lambda,n} \rangle := \langle f, \nu^\xi_{\lambda,n} \rangle - \mathbb{E} \langle f, \nu^\xi_{\lambda,n} \rangle$. 


Let $| \cdot |$ denote the Euclidean norm on $\mathbb{R}^d$, and for $x \in \mathbb{R}^d$ and $r > 0$, define the ball $B_r(x) := \{ y \in \mathbb{R}^d : |y - x| \leq r \}$. We denote by $0$ the origin of $\mathbb{R}^d$ and abbreviate $B_r(0)$ to $B_r$.

We say a set $\mathcal{X} \subset \mathbb{R}^d \times \mathcal{M}$ is locally finite if $\mathcal{X} \cap (B \times \mathcal{M})$ is finite for all bounded $B \subset \mathbb{R}^d$. For $(x, t) \in \mathbb{R}^d \times \mathcal{M}$ and Borel $A \subset \mathbb{R}^d$, we extend the definition of $\xi(x, t; \mathcal{X}, A)$ to locally finite infinite point sets $\mathcal{X}$ by setting

$$\xi(x, t; \mathcal{X}, A) := \limsup_{K \to \infty} \xi(x, t; [\mathcal{X} \cap (B_K \times \mathcal{M})], A).$$

Also, we define the $x$-shifted version $\xi^x(t, \cdot, \cdot)$ of $\xi(x, t; \cdot, \cdot)$ by

$$\xi^x(t, \mathcal{X}, A) = \xi(x, t; x + \mathcal{X}, x + A).$$

Note that if $\xi$ is translation-invariant then $\xi^x(t, \mathcal{X}, A) = \xi^0(t, \mathcal{X}, A)$ for all $x \in \mathbb{R}^d$, $t \in \mathcal{M}$, and Borel $A \subset \mathbb{R}^d$.

Notions of stabilization, introduced in [16, 18, 3], play a central role in all that follows.

**Definition 2.1** For any locally finite $\mathcal{X} \subset \mathbb{R}^d \times \mathcal{M}$, any $(x, t) \in \mathbb{R}^d \times \mathcal{M}$, and any Borel region $A \subset \mathbb{R}^d$, define $R(x, t; \mathcal{X}, A)$ (the radius of stabilization of $\xi$ at $(x, t)$ with respect to $\mathcal{X}$ and $A$) to be the smallest integer-valued $r$ such that $r \geq 0$ and

$$\xi(x, t; x + ([\mathcal{X} \cap (B_r \times \mathcal{M})] \cup Y), B) = \xi(x, t; x + [\mathcal{X} \cap (B_r \times \mathcal{M})], B)$$

for all finite $Y \subset (A \setminus B_r) \times \mathcal{M}$ and all Borel $B \subset \mathbb{R}^d$. If no such $r$ exists, we set $R(x, t; \mathcal{X}, A) = \infty$.

When $A$ is the whole of $\mathbb{R}^d$ we abbreviate the notation $R(x, t; \mathcal{X}, \mathbb{R}^d)$ to $R(x, t; \mathcal{X})$.

In the case where $\xi$ is translation-invariant, $R(x, t; \mathcal{X}) = R(0, t; \mathcal{X})$ so that $R(x, t; \mathcal{X})$ does not depend on $x$. Of particular importance to us will be radii of stabilization with respect to the homogeneous Poisson processes $\mathcal{H}_\lambda$ and with respect to the non-homogeneous Poisson process $\mathcal{P}_\lambda$, suitably scaled.

We assert that $R(x, t; \mathcal{X}, A)$ is a measurable function of $\mathcal{X}$, and hence, when $\mathcal{X}$ is a random point set such as $\mathcal{H}_\lambda$ or $\mathcal{P}_\lambda$, $R(x, t; \mathcal{X}, A)$ is an $\mathbb{N} \cup \{ \infty \}$-valued random variable. To see the assertion, observe that by Dynkin’s pi-lambda lemma, for any $k \in \mathbb{N}$ the event $\{ R(x, t; \mathcal{X}, A) \leq k \}$ equals the event $\cap_{B \in B} \{ s(\mathcal{X}, B) = i(\mathcal{X}, B) \}$, where $B$ is the II-system consisting of the rectilinear hypercubes in $\mathbb{R}^d$ whose corners have rational coordinates, and for $B \in B$ we set

\[
\begin{align*}
s(\mathcal{X}, B) &:= \sup \{ \xi(x, t; x + ([\mathcal{X} \cap (B_k \times \mathcal{M})] \cup Y), B) : Y \subset (A \setminus B_k) \times \mathcal{M} \}, \\
i(\mathcal{X}, B) &:= \inf \{ \xi(x, t; x + ([\mathcal{X} \cap (B_k \times \mathcal{M})] \cup Y), B) : Y \subset (A \setminus B_k) \times \mathcal{M} \}.
\end{align*}
\]
Also, \( s(\mathcal{X}, B) \) is a measurable function of \( \mathcal{X} \) because we assume \( \xi \) is Borel-measurable and for any \( b \) we have

\[
\{ \mathcal{X} : s(\mathcal{X}) > b \} = \pi_1(\{ (\mathcal{X}, \mathcal{Y}) : \xi(x, t; x + [\mathcal{X} \cap (B_k \times \mathcal{M})] \cup [\mathcal{Y} \setminus (B_k \times \mathcal{M})], B) > b \})
\]

where \( \pi_1 \) is projection onto the first component, acting on pairs \((\mathcal{X}, \mathcal{Y})\) with \( \mathcal{X} \) and \( \mathcal{Y} \) finite sets in \( \mathbb{R}^d \times \mathcal{M} \). Similarly, \( i(\mathcal{X}, B) \) is a measurable function of \( \mathcal{X} \).

**Definition 2.2** Let \( T, T', T'' \) and \( T''' \) denote generic random elements of \( \mathcal{M} \) with distribution \( \mu_\mathcal{M} \), independent of each other and of all other random objects we consider. Similarly, let \( X \) and \( X' \) denote generic random \( d \)-vectors with distribution \( \kappa \), independent of each other and of all other random objects we consider.

For \( p > 0 \), we consider \( \xi \) satisfying the moments conditions

\[
\sup_{\lambda \geq 1, x \in \text{supp}(\kappa)} \mathbb{E} [\xi_\lambda(x, T; p, \mathbb{R}^d]^p] < \infty. \tag{2.2}
\]

and

\[
\sup_{\lambda \geq 1, x, y \in \text{supp}(\kappa)} \mathbb{E} [\xi_\lambda(x, T; p, \mathbb{R}^d)] < \infty. \tag{2.3}
\]

Let \( \xi^*(x, t; \mathcal{X}, \cdot) \) be the point measure at \( x \) with the same total measure as \( \xi(x, t; \mathcal{X}, \cdot) \), i.e., for Borel \( A \subseteq \mathbb{R}^d \) let

\[
\xi^*(x, t; \mathcal{X}, A) := \xi(x, t; \mathcal{X}, \mathbb{R}^d) \mathbf{1}_A(x). \tag{2.4}
\]

We consider measures \( \xi \) and test functions \( f \in B(\mathbb{R}^d) \) satisfying one of the following assumptions:

**A1:** \( \xi = \xi^* \), i.e., \( \xi(x, t; \mathcal{X}, \cdot) \) is a point mass at \( x \) for all \((x, t, \mathcal{X})\).

**A2:** \( \xi(x, t; \mathcal{X}, \cdot) \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^d \), with Radon-Nikodym derivative denoted \( \xi'(x, t; \mathcal{X}, y) \) for \( y \in \mathbb{R}^d \), satisfying \( \xi'(x, t; \mathcal{X}, y) \leq K_0 \) for all \((x, t, \mathcal{X}, y)\), where \( K_0 \) is a finite positive constant.

**A3:** \( f \) is almost everywhere continuous, i.e., \( f \in \tilde{B}(\mathbb{R}^d) \).

Note that Assumption A2 will hold if \( \xi(x, t; \mathcal{X}, \cdot) \) is Lebesgue measure on some random subset of \( \mathbb{R}^d \) determined by \( x, t, \mathcal{X} \).

Our first result is a law of large numbers for \( \langle f, \mu_\lambda \rangle \) and \( \langle f, \nu_{\lambda,n}^\xi \rangle \), for \( f \in B(\mathbb{R}^d) \).

This extends a result in [18] where only the case where \( f \) is a constant is considered. We require almost surely finite radii of stabilization with respect to homogenous Poisson processes, along with moments conditions.
Theorem 2.1 Suppose that $R(x, T; \mathcal{H}_\kappa(x))$ is almost surely finite for $\kappa$-almost all $x \in \Gamma$. Suppose also that $f \in B(\mathbb{R}^d)$, and one or more of assumptions A1, A2, A3 holds. Let $q = 1$ or $q = 2$. Then:

(i) If there exists $p > q$ such that (2.2) and (2.3) hold, then as $\lambda \to \infty$ we have

$$\lambda^{-1} \langle f, \mu^\lambda_x \rangle \xrightarrow{L^p} \int f(x) \mathbb{E} \xi^\lambda_{\infty}(x, \mathbb{R}^d) \kappa(x)dx$$

and

$$\lambda^{-1} \sum_{(x,t) \in \mathcal{P}_\lambda} |\langle f, \xi^\lambda_{(x,t); \mathcal{P}_\lambda} - \xi^\lambda_{(x,t); \mathcal{P}_\lambda} \rangle| \xrightarrow{L^1} 0.$$ \hspace{1cm} (2.6)

(ii) If $\lambda(n)/n \to 1$ as $n \to \infty$, and there exists $n_0 > 0$ and $p > q$ such that

$$\sup_{n \geq n_0} \mathbb{E} [\xi_{\lambda(n)}(X, T; X_{n-1}, \mathbb{R}^d)^p] < \infty,$$

then we have as $n \to \infty$ that

$$n^{-1} \langle f, \nu^\xi_{\lambda(n),n} \rangle \xrightarrow{L^p} \int f(x) \mathbb{E} \xi^\lambda_{\infty}(x, \mathbb{R}^d) \kappa(x)dx.$$ \hspace{1cm} (2.8)

and

$$n^{-1} \sum_{(x,t) \in \mathcal{X}_n} |\langle f, \xi^\lambda_{(x,t); \mathcal{X}_n} - \xi^\lambda_{(x,t); \mathcal{X}_n} \rangle| \xrightarrow{L^1} 0.$$ \hspace{1cm} (2.9)

Our main results are central limit theorems to go with the laws of large numbers. For these we need further conditions. The first of these requires finite radii of stabilization with respect to homogeneous Poisson processes, possibly with a point inserted, and, in the non translation-invariant case, also requires local tightness of these radii.

Definition 2.3 For $x \in \mathbb{R}^d$ and $\lambda > 0$, we shall say that $\xi$ is $\lambda$-homogeneously stabilizing at $x$ if for all $z \in \mathbb{R}^d$,

$$P \left[ \lim_{\varepsilon \downarrow 0} \sup_{y \in B_\varepsilon(x)} \max(R(y, T; \mathcal{H}_\lambda), R(y, T; \mathcal{H}^z_{\lambda, T}^\varepsilon)) < \infty \right] = 1.$$ \hspace{1cm} (2.10)

In the case where $\xi$ is translation-invariant, $R(x, t; \mathcal{X})$ does not depend on $x$, and $\xi_{\infty}^x(\cdot)$ does not depend on $x$ so that the simpler-looking condition

$$P[R(0, T; \mathcal{H}_\lambda) < \infty] = P[R(0, T; \mathcal{H}^z_{\lambda, T}^\varepsilon) < \infty] = 1,$$ \hspace{1cm} (2.11)
When a random variables

For Definition 2.4

\( \xi \) We say a set \( q > \)

Given suffices to guarantee condition (2.10).

For \( k = 2 \) or \( k = 3 \), let \( S_k \) denote the set of all finite \( A \subset \supp(\kappa) \times \mathcal{M} \) with at most \( k \) elements (including the empty set), and for nonempty \( A \in S_k \), let \( \mathcal{A}^* \) denote the subset of \( \supp(\kappa) \times \mathcal{M} \) (also with \( k \) elements) obtained by equipping each element of \( \mathcal{A} \) with a \( \mu_{\mathcal{M}} \)-distributed mark; for example, for \( \mathcal{A} = \{x, y\} \in S_2 \) set \( \mathcal{A}^* = \{(x, T'), (y, T'')\} \).

Definition 2.4 For \( x \in \mathbb{R}^d \), \( \lambda > 0 \) and \( n \in \mathbb{N} \), and \( A \in S_2 \), define the \([0, \infty] \)-valued random variables \( R_\lambda(x, T) \) and \( R_{\lambda,n}(x, T; A) \) by

\[
R_\lambda(x, T) = R(x, T; \lambda^{1/d}(-x + \mathcal{P}_\lambda), \lambda^{1/d}(-x + \supp(\kappa))),
\]

\[
R_{\lambda,n}(x, T; A) = R(x, T; \lambda^{1/d}(-x + (\mathcal{X}_n \cup \mathcal{A}^*))), \lambda^{1/d}(-x + \supp(\kappa))).
\]

When \( A \) is the empty set \( \emptyset \) we write \( R_{\lambda,n}(x, t) \) for \( R(x, t; \emptyset) \).

For \( s > 0 \) and \( \varepsilon \in (0, 1) \) define the tail probabilities \( \tau(s) \) and \( \tau_\varepsilon(s) \) by

\[
\tau(s) = \sup_{\lambda \geq 1, x \in \Gamma_{\lambda}} P[R_\lambda(x, T) > s];
\]

\[
\tau_\varepsilon(s) = \sup_{\lambda \geq 1, n \in \mathbb{N}, (1-\varepsilon)\lambda, (1+\varepsilon)\lambda, x \in \Gamma_{\lambda}, A \in S_2} P[R_{\lambda,n}(x, T; A^*) > s].
\]

Given \( q > 0 \), we say \( \xi \) is power-law stabilizing of order \( q \) for \( \kappa \) if \( \sup_{s \geq 1} s^q \tau(s) < \infty \). We say \( \xi \) is exponentially stabilizing for \( \kappa \) if \( \limsup_{s \to \infty} s^{-1} \log \tau(s) < 0 \). We say \( \xi \) is binomially power-law stabilizing of order \( q \) for \( \kappa \) if there exists \( \varepsilon > 0 \) such that \( \sup_{s \geq 1} s^q \tau_\varepsilon(s) < \infty \). We say \( \xi \) is binomially exponentially stabilizing for \( \kappa \) if there exists \( \varepsilon > 0 \) such that \( \limsup_{s \to \infty} s^{-1} \log \tau_\varepsilon(s) < 0 \).

It is easy to see that if \( \xi \) is exponentially stabilizing for \( \kappa \) then it is power-law stabilizing of all orders for \( \kappa \). Similarly, if \( \xi \) is binomially exponentially stabilizing for \( \kappa \) then it is binomially power-law stabilizing of all orders for \( \kappa \).

In the non translation-invariant case, we shall also require the following continuity condition.

Definition 2.5 For \( x \in \mathbb{R}^d \), we say \( \xi \) has almost everywhere continuous total measure if there exists \( K_1 > 0 \) such that for all \( m \in \mathbb{N} \) and Lebesgue-almost all \( (x, x_1, \ldots, x_m) \in (\mathbb{R}^d)^{m+1} \), and \( (\mu_{\mathcal{M}} \times \cdots \times \mu_{\mathcal{M}}) \)-almost all \( (t, t_1, t_2, \ldots, t_m) \in \mathcal{M}^{m+1} \), for \( A = \mathbb{R}^d \) or \( A = B_K \) with \( K > K_1 \), the function

\[
(y, y_1, y_2, \ldots, y_m) \mapsto \xi(y, t; \{y_1, t_1), (y_2, t_2), \ldots, (y_m, t_m)\}, y + A)
\]

is continuous at \( (y, y_1, \ldots, y_m) = (x, x_1, \ldots, x_m) \).
Subsequent results will require one of the following formal assumptions to hold.

A4: \( \xi \) is translation-invariant.

A5: \( \xi \) has almost everywhere continuous total measure.

Our next result gives the asymptotic variance of \( \langle f, \mu^\xi_\lambda \rangle \) for \( f \in B(\mathbb{R}^d) \).

**Theorem 2.2** Suppose \( \xi \) is \( \kappa(x) \)-homogeneously stabilizing for \( \kappa \)-almost all \( x \in \mathbb{R}^d \). Suppose also that \( \xi \) satisfies the moments conditions (2.2) and (2.3) for some \( p > 2 \), and is power-law stabilizing for \( \kappa \) of order \( q \) for some \( q > p/(p - 2) \). Let \( f \in B(\mathbb{R}^d) \), and assume that Assumption A1 or A3 holds, and that Assumption A4 or A5 holds. Then

\[
\lim_{\lambda \to \infty} \lambda^{-1} \text{Var}[\langle f, \mu^\xi_\lambda \rangle] = \int_{\Gamma} f(x)^2 V^\xi(x, \kappa(x)) \kappa(x) dx,
\]

with \( V^\xi(x, a) \) given by

\[
V^\xi(x, a) = \mathbb{E} [\xi^T(x, \mathcal{H}_a, \mathbb{R}^d)^2] + a \int_{\mathbb{R}^d} (\mathbb{E} [\xi^T(x, \mathcal{H}_a^0, \mathbb{R}^d) \xi^T(-x + \mathcal{H}_a^0, \mathbb{R}^d)] - (\mathbb{E} [\xi^T(x, \mathcal{H}_a, \mathbb{R}^d)^2]) dz.
\]

Also, the integral in (2.15) converges for \( \kappa \)-almost all \( x \), and the right hand side of (2.14) is finite.

Our next result is a central limit theorem for the random field \( (\lambda^{-1/2} \langle f, \mu^\xi_\lambda \rangle, f \in \tilde{B}(\mathbb{R}^d)) \), or \( (\lambda^{-1/2} \langle f, \mu^\xi_\lambda \rangle, f \in B(\mathbb{R}^d)) \). We list some further assumptions.

A6: For some \( p > 2 \), \( \xi \) satisfies the moments conditions (2.2) and (2.3) and is exponentially stabilizing for \( \kappa \).

A7: For some \( p > 3 \), \( \xi \) satisfies the moments conditions (2.2) and (2.3) and is power-law stabilizing for \( \kappa \) of order \( q \) for some \( q > d(150 + 6/p) \).

**Theorem 2.3** Suppose \( \|\kappa\|_\infty < \infty \) and \( \kappa \) has bounded support. Suppose \( \xi \) is \( \kappa(x) \)-homogeneously stabilizing at \( x \) for \( \kappa \)-almost all \( x \in \mathbb{R}^d \), satisfies either A4 or A5, and satisfies either A6 or A7.

Then the finite-dimensional distributions of the random field \( (\lambda^{-1/2} \langle f, \mu^\xi_\lambda \rangle, f \in \tilde{B}(\mathbb{R}^d)) \) converge weakly as \( \lambda \to \infty \) to those of a mean-zero finitely additive Gaussian field with covariances given by \( \int_{\mathbb{R}^d} f_1(x)f_2(x)V^\xi(x, \kappa(x))\kappa(x)dx \), with \( V^\xi(x, a) \) given by (2.15).

If also A1 holds, the finite-dimensional distributions of the random field \( (\lambda^{-1/2} \langle f, \mu^\xi_\lambda \rangle, f \in B(\mathbb{R}^d)) \) converge weakly as \( \lambda \to \infty \) to those of a mean-zero finitely additive Gaussian field with covariances given by \( \int_{\mathbb{R}^d} f_1(x)f_2(x)V^\xi(x, \kappa(x))\kappa(x)dx \).
The corresponding results for the random measures $\nu_{\lambda,n}^\xi$ require some further conditions. These extend the previous stabilization and moments conditions to binomial point processes. Our extra moments condition is

$$\inf_{\epsilon>0} \sup_{\lambda \geq 1, x \in \mathbb{R}^d, A \subseteq S_1} \sup_{(1-\epsilon) \lambda \leq m \leq (1+\epsilon) \lambda} \mathbb{E} \left[ \zeta_\lambda(x, T; \mathcal{X}_m \cup \mathcal{A}^*, \mathbb{R}^d) \right] < \infty, \quad (2.16)$$

We give strengthened versions of A6 and A7 above, to include condition (2.16) and binomial stabilization.

A6': For some $p > 2$, $\xi$ satisfies the moments conditions (2.2), (2.3), (2.16), and is exponentially stabilizing for $\kappa$ and binomially exponentially stabilizing for $\kappa$.

A7': For some $p > 3$, $\xi$ satisfies the moments conditions (2.2), (2.3) and (2.16), and is power-law stabilizing and binomially power-law stabilizing for $\kappa$ of order $q$ for some $q > d(150 + 6/p)$.

For $x \in \mathbb{R}^d$ and $a > 0$, set

$$\delta(x, a) := \mathbb{E} \left[ \zeta_{x,T}(\mathcal{H}_a, \mathbb{R}^d) \right] + a \int_{\mathbb{R}^d} \mathbb{E} \left[ \zeta_{x,T}(\mathcal{H}_a^T, \mathbb{R}^d) - \zeta_{x,T}(\mathcal{H}_a, \mathbb{R}^d) \right] dy. \quad (2.17)$$

**Theorem 2.4** Suppose $\|\kappa\|_\infty < \infty$ and $\kappa$ has bounded support. Suppose $\xi$ is $\kappa(x)$–homogeneously stabilizing at $x$ for $\kappa$–almost all $x \in \mathbb{R}^d$, satisfies Assumption A4 or A5, and also satisfies A6' or A7'.

Then for any sequence $(\lambda(n), n \in \mathbb{N})$ taking values in $(0, \infty)$, such that $\limsup_{n \to \infty} n^{-1/2} |\lambda(n) - n| < \infty$, we have for $f \in \tilde{B}(\mathbb{R}^d)$ that

$$\lim_{n \to \infty} \frac{1}{n} \text{Var} \left( f, \nu_{\lambda(n),n}^\xi \right) = \int_\Gamma f(x) \int_\Gamma f(y) V^\xi(x, \kappa(x)) \kappa(x) \delta(x, \kappa(x)) \kappa(x) \delta(y, \kappa(y)) \kappa(y) dy$$

$$- \left( \int_\Gamma f(x) \delta(x, \kappa(x)) \kappa(x) dx \right)^2, \quad (2.18)$$

and the finite-dimensional distributions of the random field $(n^{-1/2}(f, \nu_{\lambda(n),n}^\xi), f \in \tilde{B}(\mathbb{R}^d))$ converge weakly as $n \to \infty$ to those of a mean-zero finitely additive Gaussian field with covariances given by

$$\int_\Gamma f_1(x) f_2(x) V^\xi(x, \kappa(x)) \kappa(x) dx$$

$$- \int_\Gamma f_1(x) \delta(x, \kappa(x)) \kappa(x) dx \int_\Gamma f_2(y) \delta(y, \kappa(y)) \kappa(y) dy$$

with $V^\xi(x, \lambda)$ given by (2.15).
If in addition, assumption A1 holds, then (2.18) holds for \( f \in B(\mathbb{R}^d) \) and the finite-distributions of the the random field \((\lambda(n)^{-1/2}\langle f, \nu_{\lambda(n)} \rangle, f \in B(\mathbb{R}^d))\) converge weakly as \( n \to \infty \) to those of a mean-zero finitely additive Gaussian field with covariances given by the expression in (2.19).

**Remarks.** Theorems 2.2, 2.3 and 2.4 resemble the main results of Baryshnikov and Yukich [3], in that they provide central limit theorems for random measures under general stabilization conditions. We indicate here some of the ways in which our results extend those in [3].

In [3], attention is restricted to cases where assumption A1 holds, i.e., where the contribution from each point to the random measures is a point mass at that point. It is often natural to drop this restriction, for example when considering the volume or surface measure associated with a germ-grain model, examples we shall consider in detail in Section 7.2.

Another difference is that under A1, we consider bounded test functions in \( B(\mathbb{R}^d) \) whereas in [3], attention is restricted to continuous bounded test functions. By taking test functions which are indicator functions of arbitrary Borel sets \( A_1, \ldots, A_m \) in \( \mathbb{R}^d \), we see from Theorem 2.3 that under Assumption A1, the joint distribution of \((\lambda^{-1/2}\hat{\mu}_\lambda(A_i), 1 \leq i \leq m)\) converges to a multivariate normal with covariances given by \( \int_{A_i \cap A_j} V^\xi(\kappa(x)) \kappa(x) dx \), and likewise for \( \nu_{\lambda(n)} \) by Theorem 2.4. This desirable conclusion is not achieved from the results of [3], because indicator functions of Borel sets are not continuous. When our assumption A1 fails, for the central limit theorems we restrict attention to almost everywhere continuous test functions, which means we can still obtain the above conclusion provided the sets \( A_i \) have Lebesgue-null boundary.

The de-Poissonization argument in [3] requires finiteness of what might be called the radius of external stabilization; see Definition 2.3 of [3]. Loosely speaking, an inserted point at \( x \) is not affected by and does not affect points at a distance beyond the radius of external stabilization; in contrast an inserted point at \( x \) is unaffected by points at a distance beyond the radius of stabilization, but might affect other points beyond that distance. Our approach does not require external stabilization, which brings some examples within the scope of our results that do not appear to be covered by the results of [3]. See the example of germ-grain models, considered in Section 7.2.

In the non-translation-invariant case, we require \( \xi \) to have almost everywhere continuous total measure, whereas in [3] the functional \( \xi \) is required to be in a class \( \text{SV}(4/3) \) of ‘slowly varying’ functionals. The almost everywhere continuity condition on \( \xi \) is usually easier to check.

We assume the underlying density function \( \kappa \) is almost everywhere continuous, and for Theorems 2.3 and 2.4 but not for Theorem 2.2, we assume it has compact
support. In contrast, in [3] it is assumed that \( \kappa \) has compact convex support and is continuous on its support (see the remarks just before Lemma 4.2 of [3]).

Our moments condition (2.3) is simpler than the corresponding condition in [3] (eqn (2.2) of [3]). Using A7 and A7' in Theorems 2.3 and 2.4, we obtain Gaussian limits for random fields under polynomial stabilization of sufficiently high order; the corresponding results in [3] need exponential stabilization.

We spell out the statement and proof of Theorems 2.3 and 2.4 for the setting of marked point processes setting (i.e. point processes in \( \mathbb{R}^d \times \mathcal{M} \) rather than in \( \mathbb{R}^d \)), whereas the proofs in earlier works [3, 16] are given for the setting of unmarked point process (i.e., point processes in \( \mathbb{R}^d \)). The marked point process setting includes many interesting examples such as germ-grain models and on-line packing, and generalizes the unmarked point process setting because we can always take \( \mathcal{M} \) to have a single element and then identify \( \mathbb{R}^d \times \mathcal{M} \) with \( \mathbb{R}^d \), to recover results for unmarked point processes from the general results for marked point processes.

Other papers concerned with central limit theorems for random measures include Henrich and Molchanov [9] and Penrose [14]. The setup of [9] is somewhat different from ours; the emphasis there is on measures associated with germ-grain models and the method for defining the measures from the marked point sets (eqns (3.7) and (3.8) of [9]) is more prescriptive than that used here. In [9] the underlying point processes are taken to be stationary point processes satisfying a mixing condition and no notion of stabilization is used, whereas we restrict attention to Poisson or binomial point processes but do not require any spatial homogeneity.

The setup in [14] is closer to that used here (although the proof of central limit theorems is different) but has the following notable differences. The point processes considered in [14] are assumed to have constant intensity on their support. The notion of stabilization used in [14] is a form of external stabilization. For the multivariate central limit theorems in [14] to be applicable, the radius of external stabilization needs to be almost surely finite but, unlike in the present work, no bounds on the tail of this radius of stabilization are required. The test functions in [14] lie in a subclass of \( \tilde{B}(\mathbb{R}^d) \), not \( B(\mathbb{R}^d) \). The description of the limiting variances in [14] is different from that given here.

In most examples, the sets \( \Gamma_\lambda \) are all the same as \( \Gamma \). However, there are cases where moments conditions such as (2.7) and (2.16) hold for a sequence of sets \( \Gamma_\lambda \) but would not hold if we were to take \( \Gamma_\lambda = \Gamma \) for all \( \lambda \). See, e.g. [15].

Last but not least, we note that our results carry over to the case where \( \xi(\mathbf{x}, t; \mathcal{X}, \cdot) \) is a signed measure with finite total variation. The conditions for the theorems remain unchanged if we take signed measures, except that if \( \xi \) is a signed measure, the moments conditions (2.2), (2.3), (2.7) and (2.16) need to hold for both the positive and the negative part of \( \xi \). The proofs need only minor modifications to take signed measures into account.
3 Weak convergence: the objective method

In the proof of Theorems 2.1 2.2, and 2.4, most of the work required is to prove
convergence of first and second moments. A step in this direction is to obtain certain
weak convergence results, namely Lemmas 3.3, 3.5, 3.6, 3.7 and 3.8 below. It is
noteworthy that in all of these lemmas, the stabilization conditions used always refer to
homogeneous Poisson processes on \( \mathbb{R}^d \); the notion of exponential stabilitization
with respect to a non-homogeneous point process is not used until later on.

To prove these lemmas, we shall use a version of what is sometimes called the
‘objective method’ [1, 22], whereby convergence in distribution (denoted \( \xrightarrow{D} \)) for
a functional defined on a sequence of finite probabilistic objects (in this case, re-
scaled marked point processes), is established by showing that these probabilistic
objects themselves converge in distribution to an infinite probabilistic object (in this
case, a homogeneous marked Poisson process), and that the functional of interest
is continuous. We can then use the Continuous Mapping Theorem ([6], Chapter 1,
Theorem 5.1), which says that if \( h \) is a mapping from a metric space \( E \) to another
metric space \( E' \), and \( X_n \) are \( E \)-valued random variables converging in distribution
\( D \rightarrow \)% to \( X \) which lies almost surely at a continuity point of \( h \), then \( h(X_n) \) converges in
distribution to \( h(X) \).

A point process in \( \mathbb{R}^d \times \mathcal{M} \) is an \( \mathcal{L} \)-valued random variable, where \( \mathcal{L} \) denotes the
space of locally finite subsets of \( \mathbb{R}^d \times \mathcal{M} \). We use the following metric on \( \mathcal{L} \):

\[
D(A, A') = (\max\{K \in \mathbb{N} : A \cap [B_K \times \mathcal{M}] = A' \cap [B_K \times \mathcal{M}]\})^{-1}.
\]

(3.1)

With this metric, \( \mathcal{L} \) is a metric space which is complete but not separable. In the
unmarked case where \( \mathcal{M} \) has a single element, our choice of metric is \textit{not} the same
as the metric used in Section 5.3 of [22]. Indeed, for one-point unmarked sets our
metric generates the discrete topology rather than the Euclidean topology.

To prove the weak convergence of point processes, we use a refinement of a
coupling method used in [18]. In particular, we shall use a device which we here call
the \textit{pivoted coupling}.

Let \( x_0 \in \mathbb{R}^d \) and let \( \lambda > 0 \). Let \( \mathcal{H}^+ \) denote a homogeneous Poisson process
of unit intensity in \( \mathbb{R}^d \times \mathcal{M} \times [0, \infty) \). Let \( \mathcal{P}_\lambda \) denote the image of the restriction
of \( \mathcal{H}^+ \) to the set \( \{(x, t, s) \in \mathbb{R}^d \times \mathcal{M} \times [0, \infty) : s \leq \lambda \kappa(x)\} \), under the mapping
\( (x, t, s) \mapsto (x, t) \). For \( a > 0 \), let \( \mathcal{H}'_a \), denote the image of the restriction of \( \mathcal{H}^+ \) to the
set \( \{(x, t, s) \in \mathbb{R}^d \times \mathcal{M} \times [0, \infty) : s \leq \lambda a\} \), under the mapping
\( (x, t, s) \mapsto (\lambda^{1/d}(x - x_0), s) \).

Then by the Mapping Theorem [11], \( \mathcal{P}_\lambda \) has the same distribution as \( \mathcal{P}_\lambda \) while \( \mathcal{H}'_a \)
has the same distribution as \( \mathcal{H}_a \). We shall refer to \( \mathcal{P}_\lambda \) and \( \mathcal{H}'_a \) as \textit{coupled realizations}
of \( \mathcal{P}_\lambda \) and \( \mathcal{H}_a \) with pivot point \( x_0 \).
Lemma 3.1 Suppose $x \in \mathbb{R}^d$ is a continuity point of $\kappa$, and suppose $(y(\lambda), \lambda > 0)$ is an $\mathbb{R}^d$-valued function which tends to $x$ as $\lambda \to \infty$. If $\mathcal{P}_\lambda'$ and $\mathcal{H}_{\kappa(x)}'$ are the coupled realizations of $\mathcal{P}_\lambda$ and $\mathcal{H}_{\kappa(x)}$ with pivot point $y(\lambda)$, then for any $K \in (0, \infty)$, as $\lambda \to \infty$,

$$P[\lambda^{1/d}(-y(\lambda) + \mathcal{P}_\lambda') \cap (B_K \times \mathcal{M}) = \mathcal{H}_{\kappa(x)}' \cap (B_K \times \mathcal{M})] \to 1. \quad (3.2)$$

Proof. The number of points of the point set

$$(\lambda^{1/d}(-y(\lambda) + \mathcal{P}_\lambda') \triangle \mathcal{H}_{\kappa(x)}') \cap (B_K \times \mathcal{M})$$

equals the number of points $(X, T, S)$ of $\mathcal{H}^+$ with $X \in B_{\lambda^{1/d}K}(y(\lambda))$ and with either $\lambda\kappa(x) < S \leq \lambda\kappa(X)$ or $\lambda\kappa(X) < S \leq \lambda\kappa(x)$. This is Poisson distributed with mean

$$\lambda \int_{B_{\lambda^{-1/d}K}(\kappa(x))} |\kappa(z) - \kappa(x)| \, dz$$

which tends to zero by the assumed continuity of $\kappa(\cdot)$ at $x$. \hfill \square

In second moment computations, we are interested weak convergence, not only of a point process in $\mathcal{L}$, but also of a pair of point processes in $\mathcal{L} \times \mathcal{L}$. The limiting object in this case will be a pair of independent homogeneous Poisson processes, and we need notation for this. To this end, for $a > 0$ and $b > 0$ let $\tilde{\mathcal{H}}_b$ denote a homogeneous Poisson process on $\mathbb{R}^d \times \mathcal{M}$, independent of $\mathcal{H}_a$.

Our first weak convergence lemma is concerned with the point processes $\mathcal{X}_m$. In this result, we assume these point processes are coupled together in the natural way (this was not needed for the statement of results in Section 2). To do this, we let $(X_1, T_1), (X_2, T_2), \ldots$ denote a sequence of independent identically distributed random elements of $\mathbb{R}^d \times \mathcal{M}$ with distribution $\kappa \times \mu \times \mathcal{M}$, and assume the point processes $\mathcal{X}_m, m \geq 1$ are given by

$$\mathcal{X}_m = \{(X_1, T_1), (X_2, T_2), \ldots, (X_m, T_m)\} \quad (3.3)$$

for each $m$. In proving the lemma, we use the notation $\overset{\mathcal{D}}{=} = \mathcal{D}$ for equality of distribution,

Lemma 3.2 Suppose $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ with $\kappa(x) > 0, \kappa(y) > 0$, with $\kappa(\cdot)$ continuous at $x$ and at $y$, and with $x \neq y$. Let $(\lambda(k), \ell(k), m(k))_{k \in \mathbb{N}}$ be a $((0, \infty) \times \mathbb{N} \times \mathbb{N})$-valued sequence satisfying $\lambda(k) \to \infty$, and $\ell(k)/\lambda(k) \to 1$ and $m(k)/\lambda(k) \to 1$ as $k \to \infty$. Then as $k \to \infty$,

$$\lambda(k)^{1/d}(-x + \mathcal{X}_{\ell(k)}), \lambda(k)^{1/d}(-x + \mathcal{X}_{m(k)}), \lambda(k)^{1/d}(-y + \mathcal{X}_{m(k)})$$

$$\lambda(k)^{1/d}(-x + \mathcal{X}_{\ell(k)}^{y,T}), \lambda(k)^{1/d}(-x + \mathcal{X}_{m(k)}^{y,T}), \lambda(k)^{1/d}(-y + \mathcal{X}_{m(k)}^{x,T})$$

$$\overset{\mathcal{D}}{=} \mathcal{H}_{\kappa(x)}(x, y, \kappa(x), \kappa(y), \mathcal{H}_{\kappa(x)}, \mathcal{H}_{\kappa(y)}). \quad (3.4)$$
Proof. In this proof we ease notation by suppressing all mention of the marks in the notation, and write for example $X$ for $(X, T)$. Moreover, we suppress mention of the parameter $k$ and write simply $\lambda$ for $\lambda(k)$, $\ell$ for $\ell(k)$, and $m$ for $m(k)$.

We use the following coupling. Suppose we are given $\lambda$. On a suitable probability space, let $\mathcal{P}$ and $\check{\mathcal{P}}$ be independent Poisson processes on $\mathbb{R}^d$ with intensity function $\lambda\kappa(\cdot)$ with each point carrying an independent $\mathcal{M}$-valued mark with distribution $\mu_{\mathcal{M}}$. Let $Y_1, Y_2, \ldots$ be independent random $d$-vectors with distribution $\kappa$ in $\mathbb{R}^d$ carrying marks with distribution $\mu_{\mathcal{M}}$, independent of $\mathcal{P}$ and $\check{\mathcal{P}}$.

Let $\mathcal{P}'$ be the point process consisting of those (marked) points of $\mathcal{P}$ which lie closer to $x$ than to $y$ (in the Euclidean norm), together with those (marked) points of $\check{\mathcal{P}}$ which lie closer to $y$ than to $x$. Clearly $\mathcal{P}'$ is a (marked) Poisson process of intensity $\lambda\kappa(\cdot)$ on $\mathbb{R}^d$.

Let $\mathcal{H}_{\kappa(x)}'$ and $\mathcal{H}_{\kappa(y)}'$ be (marked) homogeneous Poisson point processes in $\mathbb{R}^d$, of intensity $\kappa(x)$ and $\kappa(y)$ respectively. Assume $\mathcal{H}_{\kappa(x)}'$ and $\mathcal{H}_{\kappa(y)}'$ are independent of each other and of $(Y_1, Y_2, Y_3, Y_3, \ldots)$. Assume also that $\mathcal{H}_{\kappa(x)}'$ is coupled to $\mathcal{P}$ by the pivoted coupling with pivot at $x$ (see Lemma 3.1), and $\mathcal{H}_{\kappa(y)}'$ is coupled to $\check{\mathcal{P}}$ by the pivoted coupling with pivot at $y$.

Let $N$ denote the number of points of $\mathcal{P}'$ (a Poisson variable with mean $\lambda$). Choose an ordering on the points of $\mathcal{P}'$, uniformly at random from all $N!$ possible such orderings. Use this ordering to list the points of $\mathcal{P}'$ as $W_1, W_2, \ldots, W_N$. Also, set $W_{N+1} = Y_1, W_{N+2} = Y_2, W_{N+3} = Y_3$ and so on. Set

$$X_\ell' := \{W_1, \ldots, W_\ell\}, \quad X_m' := \{W_1, \ldots, W_m\}.$$  

Clearly $(X_\ell', X_m') \overset{\text{D}}{=} (X_\ell, X_m)$, and $(\mathcal{H}_{\kappa(x)}', \mathcal{H}_{\kappa(y)}') \overset{\text{D}}{=} (\mathcal{H}_{\kappa(x)}, \mathcal{H}_{\kappa(y)})$.

Let $K \in \mathbb{N}$, and let $\delta > 0$. Let $\theta$ denote the volume of the unit ball in $d$ dimensions. Define the events

$$E := \{X_m' \cap B_{\lambda^{-1/d}K}(x) = \mathcal{P}' \cap B_{\lambda^{-1/d}K}(x)\};$$

$$F := \{(\lambda^{1/d}(-x + P)) \cap B_K = H_{\kappa(x)}' \cap B_K\}.$$  

Event $E$ occurs unless, either, one or more of the $(N-m)^+$ “discarded” points of $\mathcal{P}'$, or, one or more of the $(m-N)^+$ “added” points of $\{Y_1, Y_2, \ldots\}$ lies in $B_{\lambda^{-1/d}K}(x)$. For each added or discarded point, the probability of lying in $B_{\lambda^{-1/d}K}(x)$ is at most $\theta\|\kappa\|_{\infty}K^d/\lambda$. Thus, for $k$ large enough so that $|m - \lambda| \leq \delta \lambda$, we have

$$P[E^c] \leq P[|N - \lambda| > \delta \lambda] + (2\delta \lambda)\theta\|\kappa\|_{\infty}K^d/\lambda$$

which is less than $3\delta\theta\|\kappa\|_{\infty}K^d$ for large enough $k$. Hence, $P[E^c] \to 0$ as $k \to \infty$. Moreover, by (3.2) we also have $P[F^c] \to 0$ as $k \to \infty$.  

15
Assuming $\lambda$ is so large that $|x - y| > 2\lambda^{-1/d}K$, if $E \cap F$ occurs then
\[
\mathcal{H}'_{n(x)} \cap B_K = (\lambda^{1/d}(-x + P)) \cap B_K
\]
\[
= \lambda^{1/d}((-x + P) \cap B_{\lambda^{-1/d}K}) = \lambda^{1/d}(-x + (P \cap B_{\lambda^{-1/d}K}(x)))
\]
\[
= \lambda^{1/d}((-x + \lambda') \cap B_{\lambda^{-1/d}K}) = (\lambda^{1/d}(-x + \lambda')) \cap B_K
\]
so that $D(\mathcal{H}'_{n(x)}, \lambda^{1/d}(-x + \lambda')) \leq 1/K$. Hence, for any $K$ we have
\[
P[D(\mathcal{H}'_{n(x)}, \lambda^{1/d}(-x + \lambda')) > 1/K] \to 0.
\]
Similarly, we have
\[
\max\{P[D(\mathcal{H}'_{n(x)}, \lambda^{1/d}(-x + \lambda')) > 1/K], P[D(\mathcal{H}'_{n(y)}, \lambda^{1/d}(-y + \lambda')) > 1/K], P[D(\mathcal{H}'_{n(x)}, \lambda^{1/d}(-x + (\lambda')y)) > 1/K], P[D(\mathcal{H}'_{n(y)}, \lambda^{1/d}(-y + (\lambda')x)) > 1/K] \}
\]
Combining these, we have the required convergence in distribution. $\Box$

For subsequent results, it is useful to define the region
\[
\Gamma_0 := \{x \in \text{int}(\Gamma) : \kappa(x) > 0, \kappa(\cdot) \text{ continuous at } x\} \tag{3.5}\]

**Lemma 3.3** Suppose $(x, y) \in \Gamma_0 \times \Gamma_0$, with $x \neq y$. Suppose also that $R(x, T; \mathcal{H}_{n(x)})$ and $R(y, T; \mathcal{H}_{n(y)})$ are almost surely finite. Suppose $(\lambda(m))_{m \geq 1}$ is a $(0, \infty) \times \mathbb{N}$-valued sequence with $\lambda(m)/m \to 1$ as $m \to \infty$. Then for Borel $A \subseteq \mathbb{R}^d$, as $m \to \infty$ we have
\[
\xi_{\lambda(m)}(x, T; \mathcal{X}, x + \lambda^{-1/d}A) \xrightarrow{D} \xi_{\lambda(m)}^x \mathcal{H}_{n(x)}, A, \tag{3.6}
\]
and
\[
(\xi_{\lambda(m)}(x, T; \mathcal{X}, x + \lambda^{-1/d}A), \xi_{\lambda(m)}(y, T; \mathcal{X}, y + \lambda^{-1/d}A)) \xrightarrow{D} (\xi_{\lambda(m)}^x \mathcal{H}_{n(x)}, A, \xi_{\lambda(m)}^y \mathcal{H}_{n(y)}, A). \tag{3.7}
\]

**Proof.** Given $A$, define the mapping $h_{A,x} : \mathcal{M} \times \mathcal{L} \to \mathbb{R}$ and the mapping $h_{A}^2 : (\mathcal{M} \times \mathcal{L} \times \mathcal{M} \times \mathcal{L}) \to \mathbb{R}^2$ by
\[
h_{A,x}(t, \mathcal{X}) = \xi(x, t; x + \mathcal{X}, x + A); \tag{3.8}
\]
\[
h_{A}^2(t, \mathcal{X}, t', \mathcal{X}') = (h_{A,x}(t, \mathcal{X}), h_{A,y}(t', \mathcal{X}')).
\]

16
Since \( R(\mathbf{x}, T; \mathcal{H}_{\kappa(\mathbf{x})}) < \infty \) a.s., the pair \((T, \mathcal{H}_{\kappa(\mathbf{x})})\) lies a.s. at a continuity point of \( h_{A, \mathbf{x}} \), where the topology on \( \mathcal{M} \times \mathcal{L} \) is the product of the discrete topology on \( \mathcal{M} \) and the topology induced by our metric \( D \) on \( \mathcal{L} \), defined at (3.1). Similarly, \((T, \mathcal{H}_{\kappa(\mathbf{x})}, T', \mathcal{H}_{\kappa(\mathbf{y})})\) lies a.s. at a continuity point of \( h^2_{A'} \). We have by definition of \( \xi \) that

\[
\xi(\mathbf{x}, T; \mathcal{X}_m, \mathbf{x} + \lambda^{-1/d}A) = h_{A, \mathbf{x}}(T, \lambda^{-1/d}(\mathbf{x} + \mathcal{X}_m));
\]

\[
(\xi(\mathbf{x}, T; \mathcal{X}_m^{T'}, \mathbf{x} + \lambda^{-1/d}A), \xi(\mathbf{y}, T'; \mathcal{X}_m^{T'}, \mathbf{y} + \lambda^{-1/d}A))
= h^2_{A}(T, \lambda^{-1/d}(\mathbf{x} + \mathcal{X}_m^{T'}), T', \lambda^{-1/d}(\mathbf{y} + \mathcal{X}_m^{T'})).
\]

By Lemma 3.2, we have \( (T, \lambda^{-1/d}(\mathbf{x} + \mathcal{X}_m)) \overset{D}{\to} (T, \mathcal{H}_{\kappa(\mathbf{x})}) \) so that (3.6) follows by the Continuous Mapping Theorem. Also, by Lemma 3.2,

\[
(T, \lambda^{-1/d}(\mathbf{x} + \mathcal{X}_m^{T'}), T', \lambda^{-1/d}(\mathbf{y} + \mathcal{X}_m^{T'})) \overset{D}{\to} (T, \mathcal{H}_{\kappa(\mathbf{x})}, T', \mathcal{H}_{\kappa(\mathbf{y})})
\]

so that (3.7) also follows by the Continuous Mapping Theorem. 

The next lemma is one of several with the purpose of comparing (with reference to a test function \( f \)) the measure \( \xi(\mathbf{x}; \mathcal{X}, \cdot) \) to the corresponding point measure \( \xi^*(\mathbf{x}; \mathcal{X}, \cdot) \) so as to derive results when Assumption A2 or A3, rather than A1, is the case. In proving such results, we repeatedly use the notation

\[
\phi_\varepsilon(\mathbf{x}) := \sup\{|f(\mathbf{y}) - f(\mathbf{x})| : \mathbf{y} \in B_\varepsilon(\mathbf{x})\}, \quad \text{for } \varepsilon > 0,
\]

and for \( f \in B(\mathbb{R}^d) \) we write \( \|f\|_\infty \) for \( \sup\{|f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^d\} \). Also, recall (see e.g. [13], [20]) that \( \mathbf{x} \in \mathbb{R}^d \) is a Lebesgue point of \( f \) if \( \varepsilon^{-d} \int_{B_\varepsilon(\mathbf{x})} |f(\mathbf{y}) - f(\mathbf{x})|d\mathbf{y} \) tends to zero as \( \varepsilon \downarrow 0 \), and that the Lebesgue Density Theorem tells us that almost every \( \mathbf{x} \in \mathbb{R}^d \) is a Lebesgue point of \( f \).

**Lemma 3.4** Let \( \mathbf{x} \in \Gamma_0 \), and suppose that \( R(\mathbf{x}, T; \mathcal{H}_{\kappa(\mathbf{x})}) < \infty \) almost surely. Let \( \mathbf{y} \in \mathbb{R}^d \) with \( \mathbf{y} \neq \mathbf{x} \). Suppose that \( f \in B(\mathbb{R}^d) \), and suppose either that \( f \) is continuous at \( \mathbf{x} \), or that Assumption A2 holds and \( \mathbf{x} \) is a Lebesgue point of \( f \). Suppose \( (\lambda(m))_{m \geq 1} \) is a \((0, \infty) \times \mathbb{N}\)-valued sequence with \( \lambda(m)/m \to 1 \) as \( m \to \infty \). Then as \( m \to \infty \),

\[
\langle f, \xi_{\lambda(m)}(\mathbf{x}, T; \mathcal{X}_m) - \xi^*_{\lambda(m)}(\mathbf{x}, T; \mathcal{X}_m) \rangle \overset{P}{\to} 0
\]

and

\[
\langle f, \xi_{\lambda(m)}(\mathbf{x}, T; \mathcal{X}_m^{T'}) - \xi^*_{\lambda(m)}(\mathbf{x}, T; \mathcal{X}_m^{T'}) \rangle \overset{P}{\to} 0
\]
Proof. In this proof, we suppress the mark in the notation, writing $x$ for $(x, T)$, and $\xi(x; \mathcal{X}, A)$ for $\xi(x; T; \mathcal{X}, A)$, and so on. Also, we write $\lambda$ for $\lambda(m)$. The left side of (3.10) is equal to

$$\int_{\mathbb{R}^d} (f(z) - f(x))\xi_\lambda(x; \mathcal{X}_m, dz). \quad (3.12)$$

Given $K > 0$, we split the region of integration in (3.12) into the complementary regions $B_{\lambda^{-1/4}K}(x)$ and $\mathbb{R}^d \setminus B_{\lambda^{-1/4}K}(x)$. Consider the latter region first. By (3.6) we have

$$\int_{\mathbb{R}^d \setminus B_{\lambda^{-1/4}K}(x)} (f(z) - f(x))\xi_\lambda(x; \mathcal{X}_m, dz) \leq 2\|f\|_\infty \xi_\lambda(x; \mathcal{X}_m, \mathbb{R}^d \setminus B_{\lambda^{-1/4}K}(x)) \xrightarrow{P} 2\|f\|_\infty \xi_\lambda^\infty (\mathcal{H}_K(x), \mathbb{R}^d \setminus B_K),$$

where the limit is almost surely finite and converges in probability to zero as $K \to \infty$. Hence for $\varepsilon > 0$, we have

$$\lim_{K \to \infty} \limsup_{m \to \infty} P \left[ \left| \int_{\mathbb{R}^d \setminus B_{\lambda^{-1/4}K}(x)} (f(z) - f(x))\xi_\lambda(x; \mathcal{X}_m, dz) \right| > \varepsilon \right] = 0. \quad (3.13)$$

Turning to the integral over $B_{\lambda^{-1/4}K}(x)$, we consider separately the case where $f$ is continuous at $x$, and the case where $A_2$ holds and $x$ is a Lebesgue point of $f$. To deal with the first of these cases, observe that

$$\left| \int_{B_{\lambda^{-1/4}K}(x)} (f(z) - f(x))\xi_\lambda(x; \mathcal{X}_m, dz) \right| \leq \phi_{\lambda^{-1/4}K}(x)\xi_\lambda(x; \mathcal{X}_m, \mathbb{R}^d) \quad (3.14)$$

and if $f$ is continuous at $x$, then $\phi_{\lambda^{-1/4}K}(x) \to 0$, while $\xi_\lambda(x, \mathcal{X}_m, \mathbb{R}^d)$ converges in distribution to the finite random variable $\xi_\lambda^\infty (\mathcal{H}_K(x), \mathbb{R}^d)$ by (3.6), and hence the right hand side of (3.14) tends to zero in probability as $m \to \infty$. Combined with (3.13), this gives us (3.10) in the case where $f$ is continuous at $x$.

Under Assumption $A_2$, for Borel $A \subseteq \mathbb{R}^d$, the change of variables $z = x + \lambda^{-1/4}(y - x)$ yields

$$\xi_\lambda(x; \mathcal{X}, A) = \int_{x + \lambda^{1/4}(-x + A)} \xi'(x; x + \lambda^{1/4}(-x + \mathcal{X}), y)dy = \lambda \int_A \xi'(x; x + \lambda^{1/4}(-x + \mathcal{X}), x + \lambda^{1/4}(z - x))dz.$$
Hence, under A2,
\[
\left| \int_{B_{\lambda^{-1/d}K(x)}} (f(z) - f(x))\xi_{\lambda}(x, x_m, dz) \right|
= \lambda \left| \int_{B_{\lambda^{-1/d}K(x)}} (f(z) - f(x))\xi'(x; x + \lambda^{1/d}(z - x), x + \lambda^{1/d}(z - x))dz \right|
\leq K_0 \lambda \int_{B_{\lambda^{-1/d}K(x)}} |f(z) - f(x)|dz
\]
and if additionally \(x\) is a Lebesgue point of \(f\) then this tends to zero. Combined with (3.13), this gives us (3.10) in the case where A2 holds and \(x\) is a Lebesgue point of \(f\).

The proof of (3.11) is similar; we use (3.7) instead of (3.6). □

By combining Lemmas 3.3 and 3.4, we obtain the following, which is the main ingredient in our proof of the Law of Large Numbers in Theorem 2.1.

**Lemma 3.5** Suppose \((x, y) \in \Gamma \times \Gamma\), with \(x \neq y\). Suppose also that \(R(x, T; H_{\kappa(x)}) < \infty\) and \(R(y, T; H_{\kappa(y)}) < \infty\), almost surely. Let \(f \in B(\mathbb{R}^d)\) and suppose that either A1 holds, or A2 holds and \(x\) is a Lebesgue point of \(f\), or \(x\) is a continuity point of \(f\).

Suppose \((\lambda(m))_{m \geq 1}\) is a \((0, \infty) \times \mathbb{N}\)-valued sequence with \(\lambda(m)/m \to 1\) as \(m \to \infty\). Then as \(m \to \infty\),
\[
\langle f, \xi_{\lambda(m)}(x, T; x_m) \rangle \overset{D}{\to} f(x)\xi_{\infty}^x(H_{\kappa(x)}, \mathbb{R}^d) \tag{3.15}
\]
and
\[
\langle f, \xi_{\lambda(m)}(x, T; x_m) \rangle \langle f, \xi_{\lambda(m)}(y, T'; x_m) \rangle \overset{D}{\to} f(x)f(y)\xi_{\infty}^x(H_{\kappa(x)}, \mathbb{R}^d)\xi_{\infty}^y(H_{\kappa(y)}, \mathbb{R}^d). \tag{3.16}
\]

**Proof.** Note first that
\[
\langle f, \xi_{\lambda(m)}(x, t; \mathcal{X}) \rangle = f(x)\xi_{\lambda(m)}(x, t; \mathcal{X}, \mathbb{R}^d).
\]
Hence, in the case where A1 holds (i.e., \(\xi = \xi^*\)), (3.15) is immediate from the case \(A = \mathbb{R}^d\) of (3.6), and similarly (3.16) is immediate from (3.7).

In the other two cases described, we have (3.10) by Lemma 3.4. Combining this with (3.15) for the case with \(\xi = \xi^*\), we see by Slutsky’s theorem (see, e.g., [13]) that (3.15) still holds in the other two cases. Similarly, since (3.16) holds when \(\xi = \xi^*\), by (3.11) and Slutsky’s theorem we can obtain (3.16) in the other cases too. □

The next two lemmas are key ingredients in proving Theorem 2.2 on convergence of second moments.
Lemma 3.6 Suppose \( x \in \Gamma_0 \). Suppose also that \( \xi \) is \( \kappa(x) \)-homogeneously stabilizing at \( x \), and that Assumption A4 or A5 holds. Then for any \( z \in \mathbb{R}^d \), we have

\[
\xi_\lambda(x + \lambda^{-1/d}z, T; \mathcal{P}_\lambda, \mathbb{R}^d) \overset{D}{\longrightarrow} \xi_\infty^x((\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) \text{ as } \lambda \to \infty.
\]

(3.17)

Also, if \( f \in \mathcal{B}(\mathbb{R}^d) \) and \( f \) is continuous at \( x \), then

\[
\langle f, \xi_\lambda(x + \lambda^{-1/d}z, T; \mathcal{P}_\lambda) \rangle \overset{D}{\longrightarrow} f(x)\xi_\infty^x((\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) \text{ as } \lambda \to \infty.
\]

(3.18)

Proof. The proof of (3.17) is related to that of Lemma 3.3. Given \( x \) and \( z \), set \( v_\lambda := x + \lambda^{-1/d}z \). For Borel \( \mathcal{A} \subseteq \mathbb{R}^d \), define \( g_A : \mathbb{R}^d \times \mathcal{A} \times \mathcal{L} \to \mathbb{R} \) by

\[
g_A(w, t, \mathcal{X}) = \xi(x + w, t; x + w + \mathcal{X}, x + w + A).
\]

Then

\[
\xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda, v_\lambda + \lambda^{-1/d}A) = g_A(\lambda^{-1/d}z, T, \lambda^{1/d}(-v_\lambda + \mathcal{P}_\lambda)).
\]

Taking our topology on \( \mathbb{R}^d \times \mathcal{A} \times \mathcal{L} \) to be the product of the Euclidean topology on \( \mathbb{R}^d \), the discrete topology on \( \mathcal{A} \) and the topology induced by the metric \( D \) on \( \mathcal{L} \) which was defined at (3.1), we assert that as \( \lambda \to \infty \),

\[
(\lambda^{-1/d}z, T, \lambda^{1/d}(-v_\lambda + \mathcal{P}_\lambda)) \overset{D}{\longrightarrow} (0, T, \mathcal{H}_{\kappa(x)}).
\]

(3.19)

To see this, for each \( \lambda \), let \( \mathcal{P}'_\lambda \) and \( \mathcal{H}'_{\kappa(x)} \) be the coupled realisations of \( \mathcal{P}_\lambda \) and \( \mathcal{H}_{\kappa(x)} \) obtained by the pivoted coupling with pivot at \( v_\lambda \). Then for \( \varepsilon > 0 \), by Lemma 3.1 we have \( P[D(\mathcal{P}'_\lambda, \mathcal{H}'_{\kappa(x)}) > \varepsilon] \to 0 \) as \( \lambda \to \infty \). This gives us (3.19).

If Assumption A4 (translation invariance) holds, then the functional \( g_A(w, t, \mathcal{X}) \) does not depend on \( w \), so that \( g_A(w, t, \mathcal{X}) = g_A(0, t, \mathcal{X}) \) and by the assumption that \( \xi \) is \( \kappa(x) \)-homogeneously stabilizing at \( x \), we have that \( (0, T, \mathcal{H}_{\kappa(x)}) \) almost surely lies at a continuity point of the functional \( g_A \).

If, instead, Assumption A5 (continuity) holds, take \( A = \mathbb{R}^d \) or \( A = B_K \) or \( A = \mathbb{R}^d \setminus B_K \), with \( K > K_1 \) and \( K_1 \) given in Definition 2.5. Then by the assumption that \( \xi \) is \( \kappa(x) \)-homogeneously stabilizing at \( x \) (see (2.10)), with probability 1 there exists a finite (random) \( \eta > 0 \) such that for \( D(\mathcal{X}, \mathcal{H}_{\kappa(x)}) < \eta \), and for \( |w| < \eta \),

\[
g_A(w, T, \mathcal{X}) = \xi(x + w, T; x + w + (\mathcal{H}_{\kappa(x)} \cap B_{1/\eta}), x + w + A) \\
\to \xi(x, T; x + (\mathcal{H}_{\kappa(x)} \cap B_{1/\eta}), x + A) = g_A(0, T, \mathcal{H}_{\kappa(x)}) \text{ as } w \to 0.
\]

Hence, \( (0, T, \mathcal{H}_{\kappa(x)}) \) almost surely lies at a continuity point of the mapping \( g_A \) in this case too.
Thus, if \( A \) is \( \mathbb{R}^d \) or \( B_K \) or \( \mathbb{R}^d \setminus B_K \), for any \( K \) under A4 and for \( K > K_1 \) under A5, the mapping \( g_A \) satisfies the conditions for the Continuous Mapping Theorem, and this with (3.19) gives us

\[
\xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda, v_\lambda + \lambda^{-1/d} A) \xrightarrow{D} \xi_{\infty}^x \mathcal{T}(\mathcal{H}_{\kappa(x)}, A) \quad \text{as } \lambda \to \infty. \tag{3.20}
\]

Taking \( A = \mathbb{R}^d \) in (3.20) gives us (3.17).

Now suppose that \( f \) is continuous at \( x \). To derive (3.18) in this case, note first that

\[
\langle f, \xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda) - \xi_\lambda^*(v_\lambda, T; \mathcal{P}_\lambda) \rangle = \int_{\mathbb{R}^d} (f(w) - f(v_\lambda)) \xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda, dw).
\]

Given \( K > 0 \), by (3.20) we have

\[
\left| \int_{\mathbb{R}^d \setminus B_{\lambda^{-1/d} K}(v_\lambda)} (f(w) - f(v_\lambda)) \xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda, dw) \right| \\
\leq 2\|f\|_\infty \xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda, \mathbb{R}^d \setminus B_{\lambda^{-1/d} K}(v_\lambda)) \xrightarrow{D} 2\|f\|_\infty \xi_{\infty}^x \mathcal{T}(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d \setminus B_K),
\]

where the limit is almost surely finite and converges in probability to zero as \( K \to \infty \). Hence for \( \varepsilon > 0 \), we have

\[
\lim_{K \to \infty} \limsup_{\lambda \to \infty} \mathbb{P}\left[ \left| \int_{\mathbb{R}^d \setminus B_{\lambda^{-1/d} K}(v_\lambda)} (f(w) - f(v_\lambda)) \xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda, dw) \right| > \varepsilon \right] = 0. \tag{3.21}
\]

Also, given \( K > 0 \), it is the case that

\[
\left| \int_{B_{\lambda^{-1/d} K}(v_\lambda)} (f(w) - f(v_\lambda)) \xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda, dw) \right| \\
\leq 2\phi_{\lambda^{-1/d}(K + |x|)}(x) \xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda, \mathbb{R}^d) \tag{3.22}
\]

and by continuity of \( f \), \( \phi_{\lambda^{-1/d}(K + |x|)}(x) \to 0 \) while \( \xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda, \mathbb{R}^d) \) converges in distribution to the finite random variable \( \xi_{\infty}^x \mathcal{T}(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) \) by (3.17), and hence the right hand side of (3.22) tends to zero in probability as \( \lambda \to \infty \). Combined with (3.21), this gives us

\[
\langle f, \xi_\lambda(v_\lambda, T; \mathcal{P}_\lambda) - \xi_\lambda^*(v_\lambda, T; \mathcal{P}_\lambda) \rangle \xrightarrow{P} 0. \tag{3.23}
\]

Also, by (3.17) and continuity of \( f \) at \( x \), we have

\[
\langle f, \xi_\lambda^*(v_\lambda, T; \mathcal{P}_\lambda) \rangle \xrightarrow{D} f(x) \xi_{\infty}^x \mathcal{T}(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d),
\]

and combined with (3.23) this yields (3.18). □
Lemma 3.7  Suppose $\xi$ satisfies Assumption A4 or A5. Then for Lebesgue-almost all $x \in \Gamma_0$ and all $z \in \mathbb{R}^d$, as $\lambda \to \infty$ we have

$$
\xi_\lambda(x, T; \mathcal{P}_\lambda^{x+\lambda^{-1/4}z, T'}, \mathbb{R}^d) \xi_\lambda(x + \lambda^{-1/4}z, T'; \mathcal{P}_\lambda^{x, T}, \mathbb{R}^d)
\xrightarrow{D} \xi_\infty^x(T \mathcal{H}_{k_0(x)}^z, \mathbb{R}^d) \xi_\infty^x(T' (-z + \mathcal{H}_{k_0(x)}^{0, T}, \mathbb{R}^d). (3.24)
$$

Also, for $f \in B(\mathbb{R}^d)$, if $f$ is continuous at $x$, then

$$
\langle f, \xi_\lambda(x, T; \mathcal{P}_\lambda^{x+\lambda^{-1/4}z, T'}) \rangle \times \langle f, \xi_\lambda(x + \lambda^{-1/4}z, T'; \mathcal{P}_\lambda^{x, T}) \rangle
\xrightarrow{D} f(x) \langle \xi_\infty^x(T \mathcal{H}_{k_0(x)}^z, \mathbb{R}^d) \xi_\infty^x(T' (-z + \mathcal{H}_{k_0(x)}^{0, T}, \mathbb{R}^d). (3.25)
$$

Proof. Again write $v_\lambda$ for $x + \lambda^{-1/4}z$. Let $A \subseteq \mathbb{R}^d$ be a Borel set. Define the function

$$
\tilde{g}_A : \mathbb{R}^d \times \mathcal{M} \times \mathcal{M} \times \mathcal{L} \to \mathbb{R}^2
$$

by

$$
\tilde{g}_A(w, t, t', x) = (\xi(x, t; x + \lambda^{z, t'}, x + A), \xi(x + w, t; x + w - z + \lambda^{0, T}, x + w + A)).
$$

Then

$$
(\xi_\lambda(x, T; \mathcal{P}_\lambda^{x, T'}, x + \lambda^{-1/4}A), \xi_\lambda(v_\lambda, T'; \mathcal{P}_\lambda^{x, T}, v_\lambda + \lambda^{-1/4}A))
= (\xi(x, T; x + \lambda^{1/4}(-x + \mathcal{P}_\lambda^{x, T'}), x + A),
\xi(v_\lambda, T'; v_\lambda + \lambda^{1/4}(-x - \lambda^{-1/4}z + \mathcal{P}_\lambda^{x, T'}, v_\lambda + A))
= \tilde{g}_A(\lambda^{-1/4}z, T, T', \lambda^{1/4}(-x + \mathcal{P}_\lambda)).
$$

Under A5, let us restrict attention to the case where $A$ is $\mathbb{R}^d$, $B_k$ or $\mathbb{R}^d \setminus B_k$ with $K > K_1$. Then under either A4 or A5, by similar arguments to those used in proving Lemma 3.6, $(0, T, T', \mathcal{H}_{k_0(x)})$ lies almost surely at a continuity point of $\tilde{g}_A$, and since

$$
(\xi_\lambda(x, T; \mathcal{P}_\lambda^{x, T'}, x + \lambda^{-1/4}A), \xi_\lambda(v_\lambda, T'; \mathcal{P}_\lambda^{x, T}, v_\lambda + \lambda^{-1/4}A))
\xrightarrow{D} \tilde{g}_A(0, T, T', \mathcal{H}_{k_0(x)}) = (\xi_\infty^x(T \mathcal{H}_{k_0(x)}^z, A), \xi_\infty^x(T' (-z + \mathcal{H}_{k_0(x)}^{0, T}, A)) (3.26)
$$

as $\lambda \to \infty$. Taking $A = \mathbb{R}^d$ gives us (3.24).

Now suppose $f$ is continuous at $x$. To prove (3.25) in this case, observe that for $K > 0$, we have

$$
|\langle f, \xi_\lambda(x, T; \mathcal{P}_\lambda^{x, T'}) - \xi_\lambda(x, T; \mathcal{P}_\lambda^{x, T'}) \rangle| \leq \phi_{\lambda^{-1/4}K}(x) \xi_\lambda(x, T; \mathcal{P}_\lambda^{x, T'}, \mathbb{R}^d)
+ 2\|f\|_\infty \xi_\lambda(x, T; \mathcal{P}_\lambda^{x, T'}, \mathbb{R}^d \setminus B_{\lambda^{-1/4}K}(x)) (3.27)
$$
We also have

\[(3.26)\]

By \((3.26)\), the second term in the right hand side of \((3.27)\) converges in distribution, as \(\lambda \to \infty\), to \(2\|f\|_\infty \xi_\infty^T(\mathbb{H}_{n(x)}^T, \mathbb{R}^d \setminus B_K)\), which tends to zero in probability as \(K \to \infty\). Hence, by \((3.27)\) we obtain

\[
\langle f, \xi_\lambda(x, T; \mathcal{P}_{\lambda}^{x, T}) - \xi_\lambda^*(x, T; \mathcal{P}_{\lambda}^{x, T}) \rangle \xrightarrow{P} 0.
\]

We also have

\[
|\langle f, \xi_\lambda(v, T'; \mathcal{P}_{\lambda}^{x, T}) - \xi_\lambda^*(v, T'; \mathcal{P}_{\lambda}^{x, T}) \rangle| \leq \left| \int_{B_{\lambda-1/d}(v)} (f(y) - f(x)) \xi_\lambda(v, T'; \mathcal{P}_{\lambda}^{x, T}, dy) \right|
+ 2\|f\|_\infty \xi_\lambda(v, T'; \mathcal{P}_{\lambda}^{x, T}, \mathbb{R}^d \setminus B_{\lambda-1/d}(v)).
\]

By \((3.26)\) and the assumed continuity of \(f\) at \(x\), the first term in the right side of \((3.29)\) tends to zero in probability for any fixed \(K\), while the second term converges in distribution to \(2\|f\|_\infty \xi_\infty^T(-z + \mathcal{H}_{n(x)}^{0,T}, \mathbb{R}^d \setminus B_K)\), which tends to zero in probability as \(K \to \infty\). Hence, as \(\lambda \to \infty\) we have

\[
\langle f, \xi_\lambda(v, T; \mathcal{P}_{\lambda}^{x, T}) - \xi_\lambda^*(v, T; \mathcal{P}_{\lambda}^{x, T}) \rangle \xrightarrow{P} 0.
\]

By continuity of \(f\) at \(x\), and the case \(A = \mathbb{R}^d\) of \((3.26)\), we have

\[
\langle (f, \xi_\lambda(x, T; \mathcal{P}_{\lambda}^{x, T})), (f, \xi_\lambda^*(x, T; \mathcal{P}_{\lambda}^{x, T})) \rangle \xrightarrow{D} \langle f(x) \xi_\infty^T(\mathbb{H}_{n(x)}^{x,T}, \mathbb{R}^d), f(x) \xi_\infty^T(-z + \mathcal{H}_{n(x)}^{0,T}, \mathbb{R}^d) \rangle.
\]

Combining this with \((3.28)\) and \((3.30)\) yields \((3.25)\).

The following lemma is a refinement of Lemma 3.5 and is proved in the same manner as that result. It will be used for de-Poissonizing our central limit theorems. To ease notation, we do not mention the marks in the notation for the statement and proof of this result.

**Lemma 3.8** Let \((x, y) \in \Gamma_0^d\) with \(x \neq y\), and let \((z, w) \in (\mathbb{R}^d)^2\). Suppose either that Assumption A1 holds, or that \(x\) and \(y\) are continuity points of \(f\). Given integer-valued functions \((\ell(\lambda), \lambda \geq 1)\) and \((m(\lambda), \lambda \geq 1)\) with \(\ell(\lambda) \sim \lambda\) and \(m(\lambda) \sim \lambda\) as \(\lambda \to \infty\), we have convergence in joint distribution, as \(\lambda \to \infty\), of the 11-dimensional
random vector

\[
\left( f, \xi(x; X) \right), \left( f, \xi(x; X^+) \right), \left( f, \xi(x; X^+_{x+1/4}) \right), \left( f, \xi(x; X^+_{x+1/4} \cup \{ y \}) \right),
\]

\[
\left( f, \xi(y; X_m) \right), \left( f, \xi(y; X_m^+ \cup \{ x + \lambda^{-1/4} z \}) \right), \left( f, \xi(y; X_m^+ \cup \{ x + \lambda^{-1/4} z \}) \right),
\]

\[
\left( f, \xi(y; X_m^+ \cup \{ x + \lambda^{-1/4} z \}) \right)
\]

to

\[
\left( f(x) \xi^x(\mathcal{H}_k(x), \mathbb{R}^d), f(x) \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), f(x) \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), f(x) \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), f(x) \xi^r(\mathcal{H}_k(x), \mathbb{R}^d),
\]

\[
f(x) \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), f(x) \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), f(x) \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), f(x) \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), f(x) \xi^r(\mathcal{H}_k(x), \mathbb{R}^d) \right).
\]

**Proof.** First, we assert that

\[
\left( \xi(x; X, \mathbb{R}^d), \xi(x; X^+, \mathbb{R}^d), \xi(x; X^+_{x+1/4}, \mathbb{R}^d), \xi(x; X^+_{x+1/4} \cup \{ y \}, \mathbb{R}^d),
\]

\[
\xi(x; X_m, \mathbb{R}^d), \xi(x; X_m^+, \mathbb{R}^d), \xi(x; X_m^+ \cup \{ x + \lambda^{-1/4} z \}, \mathbb{R}^d), \xi(x; X_m^+ \cup \{ x + \lambda^{-1/4} z \}, \mathbb{R}^d),
\]

\[
\xi(x; X_m^+ \cup \{ x + \lambda^{-1/4} z \}, \mathbb{R}^d)
\]

converges in distribution to

\[
\left( \xi^x(\mathcal{H}_k(x), \mathbb{R}^d), \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), \xi^r(\mathcal{H}_k(x), \mathbb{R}^d), \xi^r(\mathcal{H}_k(x), \mathbb{R}^d) \right).
\]

This is deduced from Lemma 3.2 by a similar argument to the proof of Lemma 3.3. For example, considering just the third component, defining the mapping \( h_{\mathbb{R}^d,x} \) on \( \mathcal{M} \times \mathcal{L} \) by (3.8), we have

\[
\xi(x; X^+_{x+1/4} \cup \{ x + \lambda^{-1/4} z \}, \mathbb{R}^d) = h_{\mathbb{R}^d,x}(T, x, \lambda^{-1/4} y, \mathbb{R}^d), \xi(x; X^+_{x+1/4} \cup \{ x + \lambda^{-1/4} z \})
\]

and by Lemma 3.2, \( (T, \mathbb{R}^d, x) \cup \lambda^{-1/4} z(x + X) \) converges in distribution to \( (T, \mathbb{H}_k(x), \mathbb{R}^d) \) which is almost surely at a continuity point of \( h_{\mathbb{R}^d,x} \). Similar arguments apply for the other components and give us the assertion above. This assertion implies that the result holds under A1, i.e. when \( \zeta = \xi^* \).
Now let us drop Assumption A1, but assume that \( x \) and \( y \) are continuity points of \( f \). Then by Lemma 3.4,
\[
\langle f, \xi(x; \mathcal{X}) - \xi^*(x; \mathcal{X}) \rangle \xrightarrow{P} 0, \quad \langle f, \xi(x; \mathcal{X}^y) - \xi^*(x; \mathcal{X}^y) \rangle \xrightarrow{P} 0,
\]
\[
\langle f, \xi(x; \mathcal{X}_m) - \xi^*(x; \mathcal{X}_m) \rangle \xrightarrow{P} 0, \quad \langle f, \xi(x; \mathcal{X}_m^y) - \xi^*(x; \mathcal{X}_m^y) \rangle \xrightarrow{P} 0,
\]
\[
\langle f, \xi(y; \mathcal{X}_m) - \xi^*(y; \mathcal{X}_m) \rangle \xrightarrow{P} 0, \quad \langle f, \xi(y; \mathcal{X}_m^y) - \xi^*(y; \mathcal{X}_m^y) \rangle \xrightarrow{P} 0,
\]
and a similar argument to the proof of (3.28) (working with \( \mathcal{X}_m \) instead of \( \mathcal{P}_\lambda \)) yields
\[
\langle f, \xi(x; \mathcal{X}_m^x + \lambda^{-1/2}z) - \xi^*(x; \mathcal{X}_m^x + \lambda^{-1/2}z) \rangle \xrightarrow{P} 0.
\]
Very similar arguments (which we omit) yield
\[
\langle f, \xi(x; \mathcal{X}_m^x + \lambda^{-1/2}z \cup \{y\}) - \xi^*(x; \mathcal{X}_m^x + \lambda^{-1/2}z \cup \{y\}) \rangle \xrightarrow{P} 0,
\]
\[
\langle f, \xi(x; \mathcal{X}_m^x + \lambda^{-1/2}z \cup \{y\}) - \xi^*(x; \mathcal{X}_m^x + \lambda^{-1/2}z \cup \{y\}) \rangle \xrightarrow{P} 0,
\]
\[
\langle f, \xi(y; \mathcal{X}_m^x \cup \{x + \lambda^{-1/2}z\}) - \xi^*(y; \mathcal{X}_m^x \cup \{x + \lambda^{-1/2}z\}) \rangle \xrightarrow{P} 0,
\]
\[
\langle f, \xi(y; \mathcal{X}_m^x \cup \{x + \lambda^{-1/2}z, y + \lambda^{-1/2}w\}) - \xi^*(y; \mathcal{X}_m^x \cup \{x + \lambda^{-1/2}z, y + \lambda^{-1/2}w\}) \rangle \xrightarrow{P} 0.
\]
Combining these eleven convergence in probability statements with the fact that we have established our conclusion in the case where Assumption A1 (\( \xi = \xi^* \)) holds, and using Slutsky’s theorem, we obtain our conclusion in the other case as well. \[\Box\]

4 Proof of Theorem 2.1

Before proving Theorem 2.1, we give general expressions for the first two moments of \( \langle f, \mu^x_\lambda \rangle \) which we shall use again later on. By Palm theory for the Poisson process (e.g. a slight generalization of Theorem 1.6 of [13]), we have
\[
\mathbb{E} \langle f, \mu^x_\lambda \rangle = \mathbb{E} \sum_{(x,t) \in \mathcal{P}_\lambda} \langle f, \xi(x,t; \mathcal{P}_\lambda) \rangle = \lambda \mathbb{E} \langle f, \xi(X,T; \mathcal{P}_\lambda) \rangle \tag{4.1}
\]
and
\[
\mathbb{E} \left[ \langle f, \mu^x_\lambda \rangle^2 \right] = \left( \mathbb{E} \sum_{(x,t) \in \mathcal{P}_\lambda} \langle f, \xi(x,t; \mathcal{P}_\lambda) \rangle \right)^2 + 2 \mathbb{E} \sum_{\{(x,t),(y,u)\} \subseteq \mathcal{P}_\lambda} \langle f, \xi(x,t; \mathcal{P}_\lambda) \rangle \langle f, \xi(y,u; \mathcal{P}_\lambda) \rangle = \lambda \mathbb{E} \left[ \langle f \xi(X,T; \mathcal{P}_\lambda) \rangle^2 \right] + \lambda^2 \mathbb{E} \left[ \langle f \xi(X,T; \mathcal{P}_\lambda^T) \rangle \langle f \xi(X',T'; \mathcal{P}_\lambda^T) \rangle \right]. \tag{4.2}
\]
Combining (4.1) and (4.2), we have
\[ \lambda^{-2} \text{Var} \langle f, \mu_\lambda \rangle = \lambda^{-1} \mathbb{E} [(f, \xi_\lambda(X, T; \mathcal{P}_\lambda))^2] \\
+ \mathbb{E} [(f, \xi_\lambda(X, T; \mathcal{P}_\lambda^{X', T'}))(f, \xi_\lambda(X', T'; \mathcal{P}_\lambda^{X, T}))] \\
- (\mathbb{E} [(f, \xi_\lambda(X, T; \mathcal{P}_\lambda))^2]). \tag{4.3} \]

Also, by similar arguments,
\[ n^{-1} \mathbb{E} \langle f, \nu_{\lambda,n} \rangle = \mathbb{E} \langle f, \xi_\lambda(X, T; \mathcal{X}_{n-1}) \rangle \tag{4.4} \]

and
\[ n^{-2} \text{Var} \langle f, \nu_{\lambda,n} \rangle = n^{-1} \mathbb{E} [(f, \xi_\lambda(X, T; \mathcal{X}_{n-1}))^2] \\
+ \left( \frac{n-1}{n} \right) \mathbb{E} [(f, \xi_\lambda(X, T; \mathcal{X}_{n-2}))(f, \xi_\lambda(X', T'; \mathcal{X}_{n-2}))] \\
- (\mathbb{E} [(f, \xi_\lambda(X, T; \mathcal{X}_{n-1}))^2]). \tag{4.5} \]

Recall that by definition, \( \xi_\lambda(x, t; \mathcal{X}, \mathbb{R}^d) = 0 \) for \( x \in \mathbb{R}^d \setminus \Gamma_\lambda \), and \( (\Gamma_\lambda, \lambda \geq 1) \) is a given nondecreasing family of Borel subsets of \( \mathbb{R}^d \) with limit set \( \Gamma \) having Lebesgue-null boundary; in the simplest case \( \Gamma_\lambda = \mathbb{R}^d \) for all \( \lambda \).

**Proof of Theorem 2.1.** First we prove (i) for the case \( q = 2 \). Assume that (2.2) and (2.3) hold for some \( p > 2 \). Let \( \mathcal{H}_{\kappa(X)} \) denote a Cox point process in \( \mathbb{R}^d \times \mathcal{M} \), whose distribution, given \( X = x \), is that of \( \mathcal{H}_{\kappa(x)} \). Set \( J := f(X)\xi_{\infty,T}(\mathcal{H}_{\kappa(X)}, \mathbb{R}^d)1_\Gamma(X) \), and let \( J' \) be an independent copy of \( J \).

For any bounded continuous test function \( h \) on \( \mathbb{R} \), by (3.15) from Lemma 3.5, as \( \lambda \to \infty \) we have \( \mathbb{E} [h((f, \xi_\lambda(X, T; \mathcal{P}_\lambda)))|X] \to \mathbb{E} [J|X] \), almost surely. Hence, \( \mathbb{E} [h((f, \xi_\lambda(X, T; \mathcal{P}_\lambda)))] \to \mathbb{E} [h(J)] \), so that
\[ \langle f, \xi_\lambda(X, T; \mathcal{P}_\lambda) \rangle \overset{D}{\longrightarrow} J. \tag{4.6} \]

Similarly, using (3.16) we obtain
\[ \langle f, \xi_\lambda(X, T; \mathcal{P}_\lambda^{X', T'}) \rangle \overrightarrow{\mathcal{D}} \langle f, \xi_\lambda(X', T'; \mathcal{P}_\lambda^{X, T}) \rangle \overset{D}{\longrightarrow} J'J \tag{4.7} \]

Also, by (2.2) and (2.3) the variables in the left side of (4.6) and in the left hand side of (4.7) are uniformly integrable so we have convergence of means in both cases. Also, (2.2) shows that the first term in the right side of (4.3) tends to zero. Hence we find that the expression (4.3) tends to zero. Moreover, by (4.1) and the convergence of expectations corresponding to (4.6), \( \lambda^{-1} \mathbb{E} \langle f, \mu_\lambda \rangle \) tends to \( \mathbb{E} [J] \), and this gives us (2.5) for \( q = 2 \), under the assumptions of part (i) of Theorem 2.1.

26
Now consider the case \( q = 1 \). Assume (2.2) and (2.3) hold for some \( p > 1 \). We use a truncation argument. Define positive and negative parts \( f^+ \) and \( f^- \) of the test function \( f \) by \( f^+(x) := \max(f(x), 0) \) and \( f^-(x) := \max(-f(x), 0) \). For \( K > 0 \), let \( \xi^K \) truncated version of the measure \( \xi^* \), defined by

\[
\xi^K(x; t; \mathcal{X}, A) := \min(\xi(x; t; \mathcal{X}, \mathbb{R}^d), K)1_A(x).
\]

Then \( \xi^K \) has total measure bounded by \( K \). By the case \( q = 2 \) established above,

\[
\lambda^{-1}\langle f^+, \mu^K \rangle \xrightarrow{L^1} \mathbb{E}\left[ f^+(X)(\xi^K)_{\infty}^X, T(\mathcal{H}_K(X), \mathbb{R}^d)1_\Gamma(X) \right]. \tag{4.8}
\]

Then \( \langle f^+, \xi^*(x; t; \mathcal{X}) \rangle = \lim_{K \to \infty} \langle f^+, \xi^K(x; t; \mathcal{X}) \rangle \). Using (4.1), we have

\[
0 \leq \mathbb{E}\left[ |\lambda^{-1} \langle f^+, \mu^K \rangle - \lambda^{-1} \langle f^+, \xi^K \rangle | \right] = \mathbb{E}\left[ |\langle f^+, \xi^* (X, T; \mathcal{P}_\lambda) - \xi^K(X, T; \mathcal{P}_\lambda) \rangle | \right] \leq \|f\|_\infty \mathbb{E}\left[ \xi^*(X, T; \mathcal{P}_\lambda, \mathbb{R}^d)1_{\{\xi^*(X, T; \mathcal{P}_\lambda, \mathbb{R}^d) > K\}} \right]
\]

which tends to zero as \( K \to \infty \), uniformly in \( \lambda \), because the moments condition (2.2), \( p > 1 \), implies that the random variables \( \xi^*(X, T; \mathcal{P}_\lambda, \mathbb{R}^d) \) are uniformly integrable. Also, by monotone convergence, as \( K \to \infty \) the right side of (4.8) converges to \( \mathbb{E}\left[ f^+(X)\xi^K_{\infty}^X, T(\mathcal{H}_K(X), \mathbb{R}^d)1_\Gamma(X) \right] \). Hence, taking \( K \to \infty \) in (4.8) yields

\[
\lambda^{-1}\langle f^+, \mu^* \rangle \xrightarrow{L^1} \mathbb{E}\left[ f^+(X)\xi^K_{\infty}^X, T(\mathcal{H}_K(X), \mathbb{R}^d)1_\Gamma(X) \right],
\]

and a similar argument yields the equivalent statement with \( f^+ \) replaced by \( f^- \). Combining these, and using linearity, we obtain

\[
\lambda^{-1}\langle f, \mu^* \rangle = \lambda^{-1}(\langle f^+, \mu^*_\lambda \rangle - \langle f^-, \mu^*_\lambda \rangle)
\]

\[
\xrightarrow{L^1} \mathbb{E}\left[ (f^+(X)\xi^K_{\infty}^X, T(\mathcal{H}_K(X), \mathbb{R}^d) - f^-(X)\xi^K_{\infty}^X, T(\mathcal{H}_K(X), \mathbb{R}^d)1_\Gamma(X) \right]
\]

\[
= \mathbb{E}\left[ \xi^K_{\infty}^X, T(\mathcal{H}_K(X), \mathbb{R}^d)1_\Gamma(X) \right] \xrightarrow{\Gamma} \mathbb{E}\left[ f(X)1_\Gamma(X) \right]. \tag{4.9}
\]

This gives us (2.5) for \( q = 1 \) when Assumption A1 hols.

Now suppose A2 or A3 holds. By Palm theory, analogously to (4.1) we have

\[
\mathbb{E}\left[ \lambda^{-1} \sum_{x \in P_\lambda} |\langle f, \xi^*(x; \mathcal{P}_\lambda) - \xi^*_\lambda(x; \mathcal{P}_\lambda) \rangle | \right]
\]

\[
= \int \kappa(dx) \mathbb{E}\left[ |\langle f, \xi^*(x; \mathcal{P}_\lambda) - \xi^*_\lambda(x; \mathcal{P}_\lambda) \rangle | \right], \tag{4.10}
\]

and for almost every \( x \in \Gamma_1 \), by (3.10) and (2.2), the integrand tends to zero and is bounded so that we have (2.6). Combining this with (4.9) gives us (2.5) for \( q = 1 \) when Assumption A2 or A3 holds, completing the proof of part (i).
Next we turn to part (ii) with \( q = 2 \). Assume (2.7) holds for some \( p > 2 \). By Lemma 3.5, as \( n \to \infty \) we have

\[
\langle f, \xi_{\lambda(n)}(X, T; X_{n-1}) \rangle \overset{d}{\to} J; \tag{4.11}
\]

\[
\langle f, \xi_{\lambda(n)}(X, T; X_{n-2}^{X'}) \rangle \langle f, \xi_{\lambda(n)}(X', T'; X_{n-2}^{X'}) \rangle \overset{d}{\to} J'J, \tag{4.12}
\]

Also, by (2.7) and the Cauchy-Schwarz inequality, the variables in the left side of (4.11) and in the left side of (4.12) are uniformly integrable so we have convergence of means in both cases. Likewise, (2.7) shows that the first term in the right side of (4.5) tends to zero. Hence we find that the expression (4.5) (with \( \lambda = \lambda(n) \)) tends to zero. Also, by (4.4) and (4.11), \( n^{-1}E \langle f, \nu_{\lambda(n), n} \rangle \to E[J] \), and this gives us (2.8) for \( q = 2 \).

The case \( q = 1 \) of part (ii) is deduced from the case \( q = 2 \) in the same manner as in part (i).

## 5 Proof of Theorems 2.2 and 2.3

Note that by definition, \( \xi_{\lambda}(x, t; \mathcal{X}, \mathbb{R}^d) = 0 \) for \( x \in \mathbb{R}^d \setminus \Gamma \). In the sequel, we fix a test function \( f \in B(\mathbb{R}^d) \). Set

\[
\alpha_{\lambda} := \int_{\Gamma} E \left[ \langle f, \xi_{\lambda}(x, T; \mathcal{P}_{\lambda}) \rangle^2 \right] \kappa(x) dx \tag{5.1}
\]

and

\[
\beta_{\lambda} := \int_{\Gamma} \int_{\mathbb{R}^d} (E \left[ \langle f, \xi_{\lambda}(x, T; \mathcal{P}_{\lambda}^{x+\lambda^{-1/d}z, T'}) \rangle \langle f, \xi_{\lambda}(y + \lambda^{1/d}z, T'; \mathcal{P}_{\lambda}^{x, T}) \rangle \right]
- E \left[ \langle f, \xi_{\lambda}(x, T; \mathcal{P}_{\lambda}) \rangle \right] \left[ \langle f, \xi_{\lambda}(y + \lambda^{-1/d}z, T; \mathcal{P}_{\lambda}) \rangle \right]
\times \kappa(x) \kappa(y + \lambda^{-1/d}z) dxdy; \tag{5.2}
\]

**Lemma 5.1** For \( \lambda > 0 \), it is the case that

\[
\text{Var}(\langle f, \mu_{\lambda}^x \rangle) = \lambda (\alpha_{\lambda} + \beta_{\lambda}). \tag{5.3}
\]

**Proof.** The first term in the right hand side of (4.3) equals \( \lambda^{-1} \alpha_{\lambda} \). Thus, (4.3) yields

\[
\lambda^{-1} \text{Var}(\langle f, \mu_{\lambda}^x \rangle) - \alpha_{\lambda} = \lambda \int_{\mathbb{R}^d} (E \left[ \langle f, \xi_{\lambda}(x, T; \mathcal{P}_{\lambda}^{y, T'}) \rangle \langle f, \xi_{\lambda}(y, T'; \mathcal{P}_{\lambda}^{x, T}) \rangle \right]
- E \left[ \langle f, \xi_{\lambda}(x, T; \mathcal{P}_{\lambda}) \rangle \right] \left[ \langle f, \xi_{\lambda}(y, T; \mathcal{P}_{\lambda}) \rangle \right] \kappa(x) \kappa(y) dydx, \tag{5.4}
\]

and the change of variables \( y = x + \lambda^{-1/d}z \) shows that this equals \( \beta_{\lambda} \) as given by (5.2). \qed
Lemmas 3.6 and 3.7 establish limits in distribution for the variables inside the expectations in the integrands in the expressions (5.1) and (5.2) for $\alpha_\lambda$ and $\beta_\lambda$. To prove Theorem 2.2, we need to take these limits outside the expectations and also outside the integrals, which we shall do by a domination argument. It is in this step that we use the condition of stabilization with respect to non-homogeneous Poisson processes (Definition 2.4), via the following lemma, which is an estimate showing that the integrand in the definition (5.2) of $\beta_\lambda$ is small for large $|z|$, uniformly in $\lambda$. To ease notation, for $x \in \mathbb{R}^d$, $z \in \mathbb{R}^d$ and $\lambda > 0$, we define random variables $X = X_{x,z,\lambda}$, $Z = Z_{x,z,\lambda}$, $X' = X'_{x,z,\lambda}$ and $Z' = Z'_{x,z,\lambda}$, by

$$X := \langle f, \xi_\lambda(x, T; P_\lambda^{x,\lambda^{-1/d}z, T}) \rangle, \quad Z := \langle f, \xi_\lambda(x + \lambda^{-1/d}z, T'; P_\lambda^{x', T}) \rangle,$$

$$X' := \langle f, \xi_\lambda(x, T; P_\lambda) \rangle, \quad Z' := \langle f, \xi_\lambda(x + \lambda^{-1/d}z, T; P_\lambda) \rangle. \quad (5.5)$$

Similarly, we define random variables $X^* = X^*_{x,z,\lambda}$, $Z^* = Z^*_{x,z,\lambda}$, $X'^* = X'^*_{x,z,\lambda}$ and $Z'^* = Z'^*_{x,z,\lambda}$, by

$$X^* := \xi_\lambda(x, T; P_\lambda^{x,\lambda^{-1/d}z, T', \mathbb{R}^d}), \quad Z^* := \xi_\lambda(x + \lambda^{-1/d}z, T'; P_\lambda^{x', T, \mathbb{R}^d}),$$

$$X'^* := \xi_\lambda(x, T; P_\lambda, \mathbb{R}^d), \quad Z'^* := \xi_\lambda(x + \lambda^{-1/d}z, T; P_\lambda, \mathbb{R}^d). \quad (5.7)$$

We write $a \wedge b$ for $\min(a, b)$ in the sequel.

**Lemma 5.2** Suppose that $\xi$ satisfies (2.2) and (2.3) for some $p > 2$, and is power-law stabilizing for $\kappa$ of order $q$ for some $q > dp/(p - 2)$. Then there is a constant $C_1$, independent of $\lambda$, such that for all $\lambda \geq 1$, $x \in \text{supp}(\kappa)$ and $z \in \mathbb{R}^d$,

$$|\mathbb{E}[X_{x,z,\lambda}Z_{x,z,\lambda}] - \mathbb{E}[X'_{x,z,\lambda}]\mathbb{E}[Z'_{x,z,\lambda}]| \leq C_1(|z|^{-d-(1/C_1)} \wedge 1);$$

$$|\mathbb{E}[X^*_{x,z,\lambda}Z^*_{x,z,\lambda}] - \mathbb{E}[X'^*_{x,z,\lambda}]\mathbb{E}[Z'^*_{x,z,\lambda}]| \leq C_1(|z|^{-d-(1/C_1)} \wedge 1). \quad (5.9)$$

**Proof.** Let $X := X_{x,z,\lambda}$ and $Z := Z_{x,z,\lambda}$. Let $\bar{X} = X 1_{|R_\lambda(x, T)| \leq |z|/3}$ and let $\bar{Z} = Z 1_{|R_\lambda(x + \lambda^{-1/d}z, T')| \leq |z|/3}$). Then $\bar{X}$ and $\bar{Z}$ are independent so that $\mathbb{E}[\bar{X} \bar{Z}] = \mathbb{E}[\bar{X}] \mathbb{E}[\bar{Z}]$, and

$$\mathbb{E}[XZ] = \mathbb{E}[ar{X}] \mathbb{E}[\bar{Z}] + \mathbb{E}[ar{X}(Z - \bar{Z})] + \mathbb{E}[(X - \bar{X})Z] \quad (5.11)$$

while

$$\mathbb{E}[X'] \mathbb{E}[Z'] = \mathbb{E}[ar{X}] \mathbb{E}[\bar{Z}] + \mathbb{E}[\bar{X}] \mathbb{E}[Z' - \bar{Z}] + \mathbb{E}[X' - \bar{X}] \mathbb{E}[Z']. \quad (5.12)$$

By (2.3) and Hölder’s inequality, and the assumed power-law stabilization of order $q > dp/(p - 2)$, there is a constant $C_2$ such that

$$\mathbb{E}[(X - \bar{X})Z] = \mathbb{E}[|XZ| 1_{|R_\lambda(x)| > |z|/3}] \leq (\mathbb{E}[|X|^p])^{1/p} (\mathbb{E}[|Z|^p])^{1/p} (P|R_\lambda(x, T)| > |z|/3)^{1/(2/p)} \leq C_2(|z|^{-d-(1/C_2)} \wedge 1) \quad (5.13)$$

29
and likewise
\[ \mathbb{E} [||X(Z - \tilde{Z})||] \leq C_2(|\mathbf{z}|^{-d-(1/C_2)} \land 1). \] (5.14)

By a similar argument using (2.2), there is a constant \( C_3 \) such that
\[ \max(\mathbb{E} [||X' - \tilde{X}||], \mathbb{E} [||Z' - \tilde{Z}||]) < C_3(|\mathbf{z}|^{-d-(1/C_3)} \land 1). \] (5.15)

Subtracting (5.12) from (5.11) and using (5.13), (5.14) and (5.15) along with the Cauchy-Schwarz inequality, we may deduce that there is a constant \( C_1 \), independent of \( \lambda \), such that for all \( \lambda \geq 1 \), (5.9) holds. The argument for (5.10) is similar \( \square \)

**Proof of Theorem 2.2.** Let \( f \in B(\mathbb{R}^d) \). Let \( x \in \Gamma_0 \) (as defined at (3.5)). Assume also, either that A1 holds, or that A3 holds and \( x \) is a continuity point of \( f \); also assume A4 or A5 holds. By the case \( \mathbf{z} = 0 \) of (3.17) when A1 holds, and the case \( \mathbf{z} = 0 \) of (3.18) when A3 holds, we have
\[ \langle f, \xi(x, T; \mathcal{P}_\lambda) \rangle \overset{D}{\to} f(x)\xi_{\infty}^{x,T}(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d). \] (5.16)

By (2.2), \( \{\langle f, \xi(x; \mathcal{P}_\lambda) \rangle : \lambda \geq 1\} \) are uniformly integrable, and hence the convergence in distribution (5.16) extends to convergence of second moments to a limit which is bounded by (2.2) and Fatou's Lemma. Hence, by the dominated convergence theorem, \( \alpha_\lambda \) given by (5.1) satisfies
\[ \lim_{\lambda \to \infty} \alpha_\lambda = \int_{\Gamma} (f(x)\mathbb{E} [\xi_{\infty}^{x,T}(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d)])^2 \kappa(x) \, dx < \infty. \] (5.17)

Next we show convergence of the expression \( \beta_\lambda \) given by (5.2). To this end, set
\[
\begin{align*}
  g_\lambda(x, \mathbf{z}) &:= (\mathbb{E} [X_{x,\mathbf{z},\lambda} Z_{x,\mathbf{z},\lambda}] - \mathbb{E} [X_{x,\mathbf{z},\lambda}^T] \mathbb{E} [Z_{x,\mathbf{z},\lambda}]) \kappa(x + \lambda^{-1/d} \mathbf{z}); \\
  g'_\lambda(x, \mathbf{z}) &:= (\mathbb{E} [X_{x,\mathbf{z},\lambda}^* Z_{x,\mathbf{z},\lambda}^*] - \mathbb{E} [X_{x,\mathbf{z},\lambda}^T] \mathbb{E} [Z_{x,\mathbf{z},\lambda}]) \kappa(x + \lambda^{-1/d} \mathbf{z}).
\end{align*}
\]

Suppose A3 holds. Then for almost all \( x \in \Gamma_0 \) and all \( \mathbf{z} \in \mathbb{R}^d \), with \( \kappa(x) > 0 \) and \( \kappa \) continuous at \( x \), by Lemma 3.7 we have as \( \lambda \to \infty \) that
\[ X_{x,\mathbf{z},\lambda} \overset{D}{\to} f(x)^2 \xi_{\infty}^{x,T}(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) \xi_{\infty}^{x,T'} (-\mathbf{z} + \mathcal{H}_{\kappa(x)}^{0,T}, \mathbb{R}^d), \] (5.18)

and the variables in the left side of (5.18) are uniformly integrable by (2.3), so that (5.18) extends to convergence of expectations. Likewise by (2.2) and (3.18), both \( \mathbb{E} [X_{x,\mathbf{z},\lambda}^T] \) and \( \mathbb{E} [Z_{x,\mathbf{z},\lambda}] \) converge to \( f(x)\mathbb{E} [\xi_{\infty}^{x,T}(\mathcal{H}_{\kappa(x)}^z, \mathbb{R}^d)] \), so we have under A3 that
\[ \lim_{\lambda \to \infty} (g_\lambda(x, \mathbf{z})) = g_\infty(x, \mathbf{z}), \quad \text{a.e. } (x, \mathbf{z}) \in \Gamma_0 \times \mathbb{R}^d. \] (5.19)
with

\[ g_\infty(x, z) := f(x)^2 \kappa(x) (\mathbb{E} [\xi_\infty^{x,T} (\mathcal{H} \mathbb{P}^{x,T} \mathbb{P}^{0,T})] + \mathbb{E} [\xi_\infty^{x,T} (-z + \mathcal{H} \mathbb{P}^{0,T} \mathbb{P}^{0,T})]) \]

By Lemma 5.2, the assumption that \( \kappa \) is bounded, there is a constant \( C \) such that \( |g_\lambda(x, z)| \leq C(|z|^{d-1/C} + 1) \), for almost every \((x, z) \in \Gamma_0 \times \mathbb{R}^d\), and \( \lambda \geq 1 \). Hence, by (5.2), (5.19), (5.20), and the dominated convergence theorem, we have

\[ \int_{\Gamma_0 \Gamma_0} \kappa(x) dx = 0. \quad (5.20) \]

By Lemma 5.2, and the assumption that \( \kappa \) is bounded, there is a constant \( C \) such that \( |g_\lambda(x, z)| \leq C(|z|^{d-1/C} + 1) \), for almost every \((x, z) \in \Gamma_0 \times \mathbb{R}^d\), and \( \lambda \geq 1 \). Hence, by (5.2), (5.19), (5.20), and the dominated convergence theorem, we have

\[ \beta_\lambda = \int_{\Gamma} \int_{\mathbb{R}^d} g_\lambda(x, z) \kappa(x) dz dx \to \int_{\Gamma} \int_{\mathbb{R}^d} g_\infty(x, z) \kappa(x) dz dx < \infty. \]

Combined with (5.17) and (5.3), this gives us (2.14) under A3.

Now suppose instead of A3 that A1 holds. Then by (3.24), for almost all \((x, z) \in \Gamma_0 \times \mathbb{R}^d\) we have as \( \lambda \to \infty \) that

\[ X^{x, z, \lambda}_d \to \xi^{x,T} (\mathcal{H} \mathbb{P}^{x,T} \mathbb{P}^{0,T}) \xi^{x,T} (-z + \mathcal{H} \mathbb{P}^{0,T} \mathbb{P}^{0,T}), \quad (5.21) \]

and the variables in the left side of (5.21) are uniformly integrable by (2.3), so that (5.21) extends to convergence of expectations. Likewise by (2.2) and (3.17), both \( \mathbb{E} [X^{x, z, \lambda}_d] \) and \( \mathbb{E} [Z^{x, z, \lambda}_d] \) converge to \( \mathbb{E} [\xi^{x,T} (\mathcal{H} \mathbb{P}^{x,T} \mathbb{P}^{0,T})] \), so we have under A1 that

\[ \lim_{\lambda \to \infty} (g_\lambda(x, z)) = g_\infty(x, z), \quad \text{a.e. } (x, z) \in \Gamma_0 \times \mathbb{R}^d \quad (5.22) \]

with

\[ g_\infty(x, z) := \kappa(x) (\mathbb{E} [\xi^{x,T} (\mathcal{H} \mathbb{P}^{x,T} \mathbb{P}^{0,T})] + \mathbb{E} [\xi^{x,T} (-z + \mathcal{H} \mathbb{P}^{0,T} \mathbb{P}^{0,T})]) \]

By Lemma 5.2, the assumption that \( \kappa \) is bounded, and (5.22) there is a constant \( C \) such that for almost every \((x, z) \) with \( \kappa(x) > 0 \),

\[ |g_\lambda(x, z)| \leq C(|z|^{d-1/C} + 1), \quad 1 \leq \lambda \leq \infty. \quad (5.23) \]

If \( x \in \mathbb{R}^d \) is a Lebesgue point of \( f \), then for any \( K > 0 \), by (5.10) we have

\[ \lim_{\lambda \to \infty} \int_{B_K} g_\lambda(x, z)(f(x + \lambda^{-1/d}z) - f(x)) dz = 0, \]
and combining this with (5.22) and the dominated convergence theorem gives us

$$\lim_{\lambda \to \infty} \int_{B_K} g^*_\lambda(x, z) f(x + \lambda^{-1/d}z) dz = \int_{B_K} g^*_\infty(x, z) f(x) dz. \quad (5.24)$$

On the other hand, by (5.23) and the assumption that $f$ is bounded, we have

$$\lim_{K \to \infty} \sup_{\lambda \to \infty} \int_{\mathbb{R}^d \setminus B_K} \left| g^*_\lambda(x, z) f(x + \lambda^{-1/d}z) - g^*_\infty(x, z) f(x) \right| dz = 0$$

and combining this with (5.24) we have

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}^d} g^*_\lambda(x, z) f(x + \lambda^{-1/d}z) dz = \int_{\mathbb{R}^d} g^*_\infty(x, z) f(x) dz.$$

By the Lebesgue density theorem, almost all every $x \in \Gamma_0$ is a Lebesgue point of $f$. Hence, under A1, by (5.2) and (5.20), the dominated convergence theorem,

$$\beta_\lambda = \int_{\Gamma} f(x) \int_{\mathbb{R}^d} f(x + \lambda^{-1/d}z) g^*_\lambda(x, z) \kappa(x) dz dx \to \int_{\Gamma} f(x)^2 \int_{\mathbb{R}^d} g^*_\infty(x, z) \kappa(x) dz dx,$$

and combined with (5.17) and (5.3), this gives us (2.14) as required. \qed

For the proof of Theorem 2.3 (central limit theorem for random measures), we shall use results on normal approximation for $\langle f, \mu_\lambda^\xi \rangle$, suitably scaled. In the case of point measures, these were proved by Stein’s method in [19], and the method carries through to more general measures. Let $\Phi$ denote the standard normal distribution function, and let $\mathcal{N}(0, \sigma^2)$ denote the normal distribution with mean 0 and variance $\sigma^2$ (if $\sigma^2 > 0$) or the unit point mass at 0 if $\sigma^2 = 0$.

**Lemma 5.3** Suppose that $\kappa$ has bounded support and $\|\kappa\|_\infty < \infty$. Suppose that $\xi$ is exponentially stabilizing and satisfies the moments condition (2.2) for some $p > 2$. Let $f \in B(\mathbb{R}^d)$, and $q \in (2, 3]$ with $q < p$. There exists a finite constant $C$ depending on $d, \xi, \kappa, q$ and $f$, such that for all $\lambda > 1$

$$\sup_{t \in \mathbb{R}} \left| P \left( \frac{\langle f, \mu_\lambda^\xi \rangle}{(\text{Var}(\langle f, \mu_\lambda^\xi \rangle))^{1/2}} \leq t \right) - \Phi(t) \right| \leq C (\log \lambda)^{dq} \lambda^{(\text{Var}(\langle f, \mu_\lambda^\xi \rangle))^{-q/2}}. \quad (5.25)$$

**Proof.** In the case where $\xi = \xi^*$, i.e. $\xi(x, t; \mathcal{A}, \cdot)$ is always a point mass at $x$, this result is Theorem 2.1 of [19]. If we do not make this assumption on $\xi$, the proof in
Lemma 5.4 Suppose \(\|\kappa\|_\infty < \infty\). Suppose for some \(p > 3\) that \(\xi\) is power-law stabilizing of order \(q\) for some \(q > d(150 + 6/p)\), and satisfies the moments condition (2.2). Let \(f \in B(\mathbb{R}^d)\). Suppose that \(\lambda^{-1}\text{Var}(f, \mu_\lambda)\) converges, as \(\lambda \to \infty\), to a finite limit \(\sigma^2\). Then \(\lambda^{-1/2}\overline{\mu_\lambda}\) converges in distribution, as \(\lambda \to \infty\), to the \(\mathcal{N}(0, \sigma^2)\) distribution.

Proof. In the case where \(\xi = \xi^*\), this result is Theorem 2.2 of [19]. If we do not make this assumption on \(\xi\), the proof in [19] carries through with the same minor changes as indicated for Lemma 5.3 above.

Proof of Theorem 2.3. Suppose \(\|\kappa\|_\infty < \infty\) and \(\kappa\) has bounded support. Suppose \(\xi\) is almost everywhere continuous, and is \(\kappa(\mathbf{x})\)-homogeneously stabilizing at \(\mathbf{x}\) for \(\kappa\)-almost all \(\mathbf{x} \in \mathbb{R}^d\). Suppose \(\xi\) satisfies either A4 or A5, and satisfies A6.

Let \(f \in B(\mathbb{R}^d)\), and assume either A1 or A3 holds. By Theorem 2.2, \(\lambda^{-1}\text{Var}(f, \mu_\lambda)\) converges to the finite nonnegative limit \(\int f(\mathbf{x})^2 V(\mathbf{x}, \kappa(\mathbf{x})) \kappa(d\mathbf{x})\). If this limit is strictly positive, then the right hand side of (5.25) tends to zero, so that by Lemma 5.3, \(\lambda^{-1/2}f, \overline{\mu_\lambda}\) is asymptotically centred normal with variance \(\int f(\mathbf{x})^2 V(\mathbf{x}, \kappa(\mathbf{x})) \kappa(d\mathbf{x})\). On the other hand, if this limiting variance is zero, then it is immediate from Chebyshev’s inequality that \(\lambda^{-1/2}f, \overline{\mu_\lambda}\) converges in probability to zero. Hence for all \(f \in \mathbb{B}(\mathbb{R}^d)\), or under A1 for all \(f \in B(\mathbb{R}^d)\), we obtain

\[
\lambda^{-1/2} \left(f, \overline{\mu_\lambda} \right) \overset{\mathcal{D}}{\to} \mathcal{N} \left(0, \int f(\mathbf{x})^2 V(\mathbf{x}, \kappa(\mathbf{x})) \kappa(d\mathbf{x}) \right). 
\]

(5.26)

If A7 holds instead of A6, we obtain the same conclusion by using Theorem 2.2 and Lemma 5.4; note that in A7, since \(p > 3\) the condition \(q > d(150 + 6/p)\) ensures that \(q > dp/(p - 2)\), so that Theorem 2.2 still applies here.

Now consider an arbitrary finite collection of test functions \(f_1, \ldots, f_J\), each of them in \(\mathbb{B}(\mathbb{R}^d)\). For arbitrary real constants \(b_1, \ldots, b_J\), application of (5.26) to \(f = \sum_{j=1}^J b_j f_j\) yields

\[
\sum_{j=1}^J b_j \lambda^{-1/2} \left(f_j, \overline{\mu_\lambda} \right) \overset{\mathcal{D}}{\to} \mathcal{N} \left(0, \sum_{j=1}^J b_j b_k \int f_j(\mathbf{x}) f_k(\mathbf{x}) V(\mathbf{x}, \kappa(\mathbf{x})) \kappa(d\mathbf{x}) \right),
\]

33
so by the Cramér-Wold device (see e.g. [13], [6]), the variables \(\lambda^{-1/d} (f_j, \mu^\xi)\), \(1 \leq j \leq J\), are asymptotically centred multivariate normal with covariance matrix having \((j, k)\)th entry \(\int_T f_j(x) f_k(x) V^\xi(x, \kappa(x)) \kappa(x) dx\). This gives us the required convergence of random fields for \(f_j \in \tilde{B}(\mathbb{R}^d)\). If A1 holds, the same argument gives the required convergence of random fields for \(f_j \in B(\mathbb{R}^d)\).

6 Extension to the non-Poisson case

In this section we prove Theorem 2.4. We assume here that all the point processes \(X_n\) are coupled as described at (3.3), in terms of a sequence \(((X_1, T_1), (X_2, T_2), \ldots)\) of independent random elements of \(\mathbb{R}^d \times \mathcal{M}\) with common distribution \(\kappa \times \mu_{\mathcal{M}}\).

Given \(f \in B(\mathbb{R}^d)\) and \(\lambda > 0\), for each \(m \in \mathbb{N}\) we define

\[
F_{m, \lambda} := \langle f, \nu_{m+1, \lambda} - \nu_{m, \lambda} \rangle = Y_{m+1, \lambda} + \sum_{i=1}^{m} \Delta_{i, m, \lambda},
\]

where we set

\[
Y_{m+1, \lambda} := \langle f, \xi_{\lambda}(X_{m+1}, T_{m+1}; \mathcal{X}_m) \rangle;
\]

\[
\Delta_{i, m, \lambda} := \langle f, \xi_{\lambda}(X_i, T_i; \mathcal{X}_{m+1}) - \xi_{\lambda}(X_i, T_i; \mathcal{X}_m) \rangle.
\]

In this section we shall use the standard notation \(\|X\|_p\) for the \(L^p\)-norm \(E[|X|^p]^{1/p}\) of a random variable \(X\), where \(p \geq 1\).

**Lemma 6.1** Suppose \(R(x, T; \mathcal{H}_{\kappa(x)}) < \infty\) for \(\kappa\)-almost all \(x \in \mathbb{R}^d\), and Assumption A1 or A3 holds, along with Assumption A6' or A7'. Let \((h(\lambda))_{\lambda \geq 1}\) satisfy \(h(\lambda)/\lambda \to 0\) as \(\lambda \to \infty\). Then with \(\delta(x, \lambda)\) defined at (2.17),

\[
\lim_{\lambda \to \infty} \sup_{\lambda - h(\lambda) \leq \ell \leq m \leq \lambda + h(\lambda)} \left| E F_{\ell, \lambda} F_{m, \lambda} - \left( \int_T \kappa(x) f(x) \delta(x, \kappa(x)) dx \right)^2 \right| = 0.
\]

**Proof.** We suppress mention of marks in the proof, writing simply \(X\) for \((X, T)\). Suppose \((\ell(\lambda))_{\lambda \geq 1}\) and \((m(\lambda))_{\lambda \geq 1}\) satisfy \(\lambda - h(\lambda) \leq \ell(\lambda) < m(\lambda) \leq \lambda + h(\lambda)\). We shall show a sequential version of (6.2). To ease notation, we write \(\ell\) for \(\ell(\lambda)\), and \(m\) for \(m(\lambda)\), and \(Y_m\) for \(Y_{m, \lambda}\) and \(\Delta_{i, m, \lambda}\) for \(\Delta_{i, m, \lambda}\).
By using (6.1), expanding and taking expectations, we obtain
\[
\mathbb{E} F_\ell F_m = \mathbb{E} \left[ (Y_{\ell+1} + \sum_{i=1}^{\ell} \Delta_i, \ell) (Y_{m+1} + \sum_{j=1}^{\ell} \Delta_j, m + \Delta_{\ell+1}, m + \sum_{j=\ell+2}^{m} \Delta_{j,m}) \right]
\]
\[
= \mathbb{E} [Y_{\ell+1} Y_{m+1}] + \ell \mathbb{E} [\Delta_{\ell+1} Y_{m+1}] + \ell \mathbb{E} [Y_{\ell+1} \Delta_{\ell+1}, m] + \ell (\ell - 1) \mathbb{E} [\Delta_{1,\ell} \Delta_{2,m}]
\]
\[
+ \ell \mathbb{E} [\Delta_{1,\ell} \Delta_{1,m}] + \mathbb{E} [Y_{\ell+1} \Delta_{\ell+1}, m] + \mathbb{E} [\Delta_{1,\ell} \Delta_{\ell+1}, m]
\]
\[
+ (m - \ell - 1) \mathbb{E} [Y_{\ell+1} \Delta_{\ell+2}, m] + (m - \ell - 1) \mathbb{E} [\Delta_{1,\ell} \Delta_{\ell+2}, m].
\]
(6.2)

We shall establish the limiting behaviour of each term of (6.2) in turn. First we have
\[
\mathbb{E} [Y_{\ell+1} Y_{m+1}] = \int \kappa(x) (dx) \int \kappa(y) (dy) \mathbb{E} [(f, \xi_\lambda(x; \mathcal{X}_\ell)) (f, \xi_\lambda(y; \mathcal{X}_{m-1}))].
\]

Here and below, all domains of integration, when not specified, are \( \mathbb{R}^d \). By Lemma 3.8, for almost all \( (x, y) \in \Gamma_0 \times \Gamma_0 \), we have
\[
(f, \xi_\lambda(x; \mathcal{X}_\ell)) (f, \xi_\lambda(y; \mathcal{X}_{m-1})) \xrightarrow{D} f(x) f(y) \xi_\infty(x; \mathcal{X}_\ell, \mathbb{R}^d) \xi_\infty(y; \mathcal{X}_{m-1}, \mathbb{R}^d).
\]
(6.3)

By (2.16), the variables \( (f, \xi_\lambda(x; \mathcal{X}_\ell)) (f, \xi_\lambda(y; \mathcal{X}_{m-1})) \) are uniformly integrable so (6.3) extends to convergence of expectations, and also the limit is bounded, so that setting
\[
\gamma_1 := \int \Gamma f(x) \mathbb{E} [\xi_\infty(x; \mathcal{X}_\ell, \mathbb{R}^d)] \kappa(x) (dx),
\]
(6.4)
we have as \( \lambda \to \infty \) that
\[
\mathbb{E} [Y_{\ell+1} Y_{m+1}] \to \gamma_1^2.
\]
(6.5)

Next, observe that
\[
\ell \mathbb{E} [\Delta_{1,\ell} Y_{m+1}] = \ell \mathbb{E} [(f, \xi_\lambda(X_1; \mathcal{X}_\ell+1)) - \xi_\lambda(X_1; \mathcal{X}_\ell)] (f, \xi_\lambda(X_{m+1}; \mathcal{X}_m))]
\]
\[
= \ell \int \kappa(x) (dx) \int \kappa(y) (dy) \int \kappa(w) (dw) \mathbb{E} [(\xi_\lambda(x; \mathcal{X}_{\ell+1}) - \xi_\lambda(x; \mathcal{X}_{\ell-1}))]
\]
\[
\times (f, \xi_\lambda(y; \mathcal{X}_{m-2} \cup \{x, w\})).
\]
(6.6)

Making the change of variables \( z = \lambda^{1/d} (w - x) \) we obtain
\[
\ell \mathbb{E} [\Delta_{1,\ell} Y_{m+1}] = \frac{\ell}{\lambda} \int \kappa(x) (dx) \int \kappa(y) (dy) \int \kappa(x + \lambda^{-1/d} z) (dz)
\]
\[
\times \mathbb{E} [(f, \xi_\lambda(x; \mathcal{X}_{\ell-1}^+ \lambda^{-1/d} z) - \xi_\lambda(x; \mathcal{X}_{\ell-1})) (f, \xi_\lambda(y; \mathcal{X}_{m-2} \cup \{x, x + \lambda^{-1/d} z\}))].
\]
(6.7)
Suppose that A1 or A3 holds. By Lemma 3.8, for almost all \((x, y) \in \Gamma_0 \times \Gamma_0\), we have that

\[
\langle f, \xi_\lambda(x; \mathcal{X}_{\ell-1}^{x+\lambda^{-1/d}z}) - \xi_\lambda(x; \mathcal{X}_{\ell-1}) \rangle \langle f, \xi_\lambda(y; \mathcal{X}_{m-2} \cup \{x + \lambda^{-1/d}z\}) \rangle
\]

\[
\overset{D}{\rightarrow} f(x)f(y)\left(\xi_0^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) - \xi_0^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d)\right)\xi_0^y(\mathcal{H}_{\kappa(y)}, \mathbb{R}^d). \quad (6.8)
\]

By (2.16) and the Cauchy-Schwarz inequality, the variables in the left side of (6.8) are uniformly integrable. Therefore we have convergence of expectations, so that the integrand in (6.7) tends to

\[
\kappa(x)^2\kappa(y)f(x)f(y)
\]

\[
\times \mathbb{E} \left[ \xi_0^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) - \xi_0^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) \right] \mathbb{E} \left[ \xi_0^y(\mathcal{H}_{\kappa(y)}, \mathbb{R}^d) \right]. \quad (6.9)
\]

Also, \(\langle f, \xi_\lambda(x; \mathcal{X}_{\ell-1}^{x+\lambda^{-1/d}z}) - \xi_\lambda(x; \mathcal{X}_{\ell-1}) \rangle \) is zero unless \(|z| \leq R_{\lambda, \ell-1}(x)\), defined at (2.13). Also, by A6’ or A7’, we assume for some \(p > 2\) and \(q > 2dp/(p-2)\) that the moments condition (2.16) holds and we have binomial power law stabilization of order \(q\) (in A7’, since \(p > 3\) the condition \(q > d(150 + 6/p)\) ensures that \(q > 2dp/(p-2)\)). Therefore the Hölder and Minkowski inequalities yield

\[
|\mathbb{E} [(f, \xi_\lambda(x; \mathcal{X}_{\ell-1}^{x+\lambda^{-1/d}z}) - \xi_\lambda(x; \mathcal{X}_{\ell-1}) \langle f, \xi_\lambda(y; \mathcal{X}_{m-2} \cup \{x + \lambda^{-1/d}z\}) \rangle]| \leq (\|\langle f, \xi_\lambda(x; \mathcal{X}_{\ell-1}^{x+\lambda^{-1/d}z}) \rangle\|_p + \|\langle f, \xi_\lambda(x; \mathcal{X}_{\ell-1}) \rangle\|_p) \times \|\langle f, \xi_\lambda(y; \mathcal{X}_{m-2} \cup \{x + \lambda^{-1/d}z\}) \rangle\|_p \mathbb{P}[R_{\lambda, \ell-1}(x) > |z|^{-2/p}]
\]

\[
\leq \text{const.} \times (|z|^q(2p)/p \wedge 1). \quad (6.10)
\]

Since \(q > dp/(p-2)\), this is integrable in \(z\). Set

\[
\gamma_2 := \int_\Gamma \kappa(x)^2dx f(x) \int_{\mathbb{R}^d} dz \mathbb{E} \left[ \xi_0^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) - \xi_0^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) \right]. \quad (6.11)
\]

By (6.7) and the dominated convergence theorem we obtain

\[
\ell \mathbb{E} [\Delta_{1, \ell}Y_{m+1}] \to \gamma_2 \int \kappa(y)dy f(y) \mathbb{E} [\xi_0^y(\mathcal{H}_{\kappa(y)}, \mathbb{R}^d)] = \gamma_1 \gamma_2. \quad (6.12)
\]

Next, writing \(X_1\) as \(x\), \(X_{\ell+1}\) as \(y\), and \(X_{m+1}\) as \(x + \lambda^{-1/d}z\), we have

\[
\ell \mathbb{E} [Y_{\ell+1}\Delta_{1,m}] = \frac{\ell}{\lambda} \int \kappa(x)dx \int \kappa(y)dy \int \kappa(x + \lambda^{-1/d}z)dz \times \mathbb{E} [(f, \xi_\lambda(y; \mathcal{X}_{\ell-1}^{y})) (f, \xi_\lambda(x; \mathcal{X}_{m-2}^{y} \cup \{x + \lambda^{-1/d}z\}) - \xi_\lambda(x; \mathcal{X}_{m-2}^{y}))]. \quad (6.13)
\]
By Lemma 3.8 (note that we do not assume that \( \ell < m \) in that result), for almost all \((x, y) \in \Gamma_0 \times \Gamma_0\), we have

\[
\langle f, \xi_\lambda(y; \mathcal{X}_{\ell-1}) \rangle \langle f, \xi_\lambda(x; \mathcal{X}_{m-2}^y \cup \{x + \lambda^{-1/d}z\}) - \xi_\lambda(x; \mathcal{X}_{m-2}^y) \rangle \\
\xrightarrow{d} f(x)f(y)\xi_\lambda^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d)\xi_\lambda^y(\mathcal{H}_{\kappa(y)}, \mathbb{R}^d) - \xi_\lambda^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) - \xi_\lambda^y(\mathcal{H}_{\kappa(y)}, \mathbb{R}^d).
\]  

Using (2.16), we obtain convergence of expectations corresponding to (6.14). Hence, the integrand in (6.13) converges to the expression given at (6.9). By a similar argument to the one used to establish (6.10), the absolute value of this integrand is bounded by a constant times \(|z|^{q(2-p)/p} \wedge 1\), and this is integrable since \(q > dp/(p - 2)\). Hence, the dominated convergence theorem gives us

\[
\ell \mathbb{E} [Y_{\ell+1}\Delta_{1,m}] \rightarrow \gamma_1 \gamma_2.
\]  

Next, by taking \(X_1 = x, X_2 = y, X_{\ell+1} = x + \lambda^{-1/d}z\) and \(X_{m+1} = y + \lambda^{-1/d}w\), we have

\[
\mathbb{E} [\Delta_{1,\ell}\Delta_{2,m}] = \lambda^{-2} \int \kappa(x)dx \int \kappa(y)dy \int \kappa(x + \lambda^{-1/d}z)dz \int \kappa(y + \lambda^{-1/d}w)dw
\]

\[
\times \mathbb{E} \left[ \langle f, \xi_\lambda(x; \mathcal{X}_{\ell-2}^y \cup \{x + \lambda^{-1/d}z\}) - \xi_\lambda(x; \mathcal{X}_{\ell-2}^y) \rangle \right] \\
\times \mathbb{E} \left[ \langle f, \xi_\lambda(y; \mathcal{X}_{m-3}^x \cup \{x + \lambda^{-1/d}z; y + \lambda^{-1/d}w\}) - \xi_\lambda(y; \mathcal{X}_{m-3}^x \cup \{x + \lambda^{-1/d}z\}) \rangle \right].
\]  

For almost all \((x, y) \in \Gamma_0 \times \Gamma_0\), Lemma 3.8 yields

\[
\langle f, \xi_\lambda(x; \mathcal{X}_{\ell-2}^y \cup \{x + \lambda^{-1/d}z\}) - \xi_\lambda(x; \mathcal{X}_{\ell-2}^y) \rangle \\
\times \langle f, \xi_\lambda(y; \mathcal{X}_{m-3}^x \cup \{x + \lambda^{-1/d}z; y + \lambda^{-1/d}w\}) - \xi_\lambda(y; \mathcal{X}_{m-3}^x \cup \{x + \lambda^{-1/d}z\}) \rangle \\
\xrightarrow{d} f(x)f(y)\xi_\lambda^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) - \xi_\lambda^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) f(y)\xi_\lambda^y(\mathcal{H}_{\kappa(y)}, \mathbb{R}^d) - \xi_\lambda^y(\mathcal{H}_{\kappa(y)}, \mathbb{R}^d).
\]  

Also, the quantity on the left is uniformly integrable by the assumption that (2.16) holds for some \(p > 2\). Hence we have corresponding convergence of expectations, so the integrand in (6.16) converges to

\[
f(x)f(y)\kappa^2(x)\kappa^2(y)\mathbb{E} [\xi_\lambda^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d) - \xi_\lambda^x(\mathcal{H}_{\kappa(x)}, \mathbb{R}^d)] \\
\times \mathbb{E} [\xi_\lambda^y(\mathcal{H}_{\kappa(y)}, \mathbb{R}^d) - \xi_\lambda^y(\mathcal{H}_{\kappa(y)}, \mathbb{R}^d)].
\]  

Also, we have \(\langle f, \xi_\lambda(x; \mathcal{X}_{\ell-2}^y \cup \{x + \lambda^{-1/d}z\}) \rangle \) unless \(R_{\lambda,\ell-2}(x; \{y\}) > |z|\) and \(\langle f, \xi_\lambda(y; \mathcal{X}_{m-3}^x \cup \{x + \lambda^{-1/d}z; y + \lambda^{-1/d}w\}) \rangle \) unless \(R_{\lambda,m-3}(y; \{x; x + \lambda^{-1/d}z\}) \geq |w|\). Hence, Hölder’s inequality shows that the
absolute value of the expectation in (6.16) is at most
\[
\|f, \xi_\lambda(x; \mathcal{X}_{\ell-2}^y \cup \{x + \lambda^{-1/d}z\}) - \xi_\lambda(x; \mathcal{X}_{\ell-2}^y)\|_p \\
\times \|f, \xi_\lambda(y; \mathcal{X}_{m-3}^y \cup \{x + \lambda^{-1/d}z, y + \lambda^{-1/d}w\}) - \xi_\lambda(y; \mathcal{X}_{m-3}^y \cup \{x + \lambda^{-1/d}z\})\|_p \\
\times (P[R_{\lambda,\ell-2}(x; \{y\}) \geq |z|])^{(1/2) - 1/p} (P[R_{\lambda,m-3}(y; \{x, x + \lambda^{-1/d}z\}) \geq |w|])^{(1/2) - 1/p}.
\]

By the assumption \((A6' \text{ or } A7')\) that moments condition (2.16) holds for some \(p > 2\), and that \(\xi\) is binomially power law stabilizing of order \(q > 2dp/(p - 2)\), this is bounded by a constant times
\[
(\|z\|^{q(2-p)/(2p)} \wedge 1)(\|w\|^{q(2-p)/(2p)} \wedge 1)
\]
which is integrable in \((z, w)\). Therefore the dominated convergence theorem applied to (6.16) shows that
\[
\ell(\ell - 1)\mathbb{E}[\Delta_{1,\ell}\Delta_{2,m}] \to C_2^2. \quad (6.17)
\]

Next, take \(X_1 = x, X_{\ell+1} = y, X_{m+1} = x + \lambda^{-1/d}z\) to obtain
\[
\ell\mathbb{E}[\Delta_{1,\ell}\Delta_{1,m}] = \frac{\ell}{\lambda} \int dx \int dy \int dz \kappa(x)\kappa(y)\kappa(x + \lambda^{-1/d}z) \\
\times \mathbb{E}[\langle f, \xi_\lambda(x; \mathcal{X}_{\ell-2}^y) - \xi_\lambda(x; \mathcal{X}_{\ell-2}) \rangle \langle f, \xi_\lambda(x; \mathcal{X}_{m-2}^y \cup \{x + \lambda^{-1/d}z\}) - \xi_\lambda(x; \mathcal{X}_{m-2}^y) \rangle].
\]

By Lemma 3.8 the quantity inside the expectation tends to zero in probability for almost all \(x, y\) and all \(z\). Hence its expectation tends to zero as well, since it is uniformly integrable by (2.16). Also, the absolute value of this expectation is bounded by a constant times \(|z|^{q(2-p)/p} \wedge 1\), by a similar argument to (6.10). Hence, dominated convergence yields
\[
\ell\mathbb{E}[\Delta_{1,\ell}\Delta_{1,m}] \to 0. \quad (6.18)
\]

Next we have
\[
\mathbb{E}[Y_{\ell+1}\Delta_{\ell+1,m}] = \int \kappa(x)\mathbb{E}[\langle f, \xi_\lambda(x; \mathcal{X}_\ell) \rangle \langle f, \xi_\lambda(x; \mathcal{X}_m) - \xi_\lambda(x; \mathcal{X}_{m-1}) \rangle]. \quad (6.19)
\]

By Lemma 3.8, for almost every \(x \in \Gamma_0\), we have
\[
\langle f, \xi_\lambda(x; \mathcal{X}_\ell) \rangle \langle f, \xi_\lambda(x; \mathcal{X}_m) \rangle \overset{D}{\to} f(x)^2(\mathcal{X}_\infty^x(\mathcal{H}_\kappa(x), \mathbb{R}^d))^2 \\
\langle f, \xi_\lambda(x; \mathcal{X}_\ell) \rangle \langle f, \xi_\lambda(x; \mathcal{X}_{m-1}) \rangle \overset{D}{\to} f(x)^2(\mathcal{X}_\infty^x(\mathcal{H}_\kappa(x), \mathbb{R}^d))^2,
\]
and using (2.16), we have the corresponding convergence of expectations so that the integrand in (6.19) tends to zero. Also by (2.16), this integrand is bounded, and thus
\[
\mathbb{E}[Y_{\ell+1}\Delta_{\ell+1,m}] \to 0. \quad (6.20)
\]
Next, setting \( X_1 = y \), \( X_{\ell + 1} = x \), and \( X_{m + 1} = y + \lambda^{-1/d}z \), we find that

\[
\mathbb{E} [\Delta_{1,\ell} \Delta_{\ell + 1, m}] = \frac{\ell}{\lambda} \int dy \int dx \int dz \kappa(y) \kappa(x) \kappa(x + \lambda^{-1/d}z) \\
\times \mathbb{E} [(f, \xi_\lambda(y; \mathcal{X}_{\ell - 1}^y) - \xi_\lambda(y; \mathcal{X}_{\ell - 1})) \langle f, \xi_\lambda(x; \mathcal{X}_{m - 2}^y \cup \{x + \lambda^{-1/d}z\}) - \xi_\lambda(x; \mathcal{X}_{m - 2}^y) \rangle].
\]

(6.21)

By Lemma 3.8, for almost all \((x, y) \in \Gamma_0 \times \Gamma_0\) and all \(z\), as \(\lambda \to \infty\) we have

\[
\langle f, \xi_\lambda(y; \mathcal{X}_{\ell - 1}^y) - \xi_\lambda(y; \mathcal{X}_{\ell - 1})) \rangle \langle f, \xi_\lambda(x; \mathcal{X}_{m - 2}^y \cup \{x + \lambda^{-1/d}z\}) - \xi_\lambda(x; \mathcal{X}_{m - 2}^y) \rangle \to 0,
\]

so that the quantity inside the expectation in (6.21) tends to zero in probability and by (2.16) it is uniformly integrable. Hence the integrand in (6.21) tends to zero. Also, by a similar argument to (6.10), the absolute value of this integrand is bounded by a constant times \(|z| q/(2-p)/p \land 1\), which is integrable since \(q > pd/(p - 2)\). Thus, the integrand in (6.21) is bounded by an integrable function of \((x, y, z)\) so the dominated convergence theorem shows that

\[
\ell \mathbb{E} [\Delta_{1,\ell} \Delta_{\ell + 1, m}] \to 0.
\]

(6.22)

Next, write \( X_{\ell + 2} \) as \( x \), \( X_{\ell + 1} \) as \( y \), and \( X_{m + 1} \) as \( x + \lambda^{-1/d}z \), to obtain

\[
\mathbb{E} [Y_{\ell + 1} \Delta_{\ell + 2, m}] = \lambda^{-1} \int \kappa(x) dx \int \kappa(y) dy \int \kappa(x + \lambda^{-1/d}z) dz \\
\times \mathbb{E} [(f, \xi_\lambda(y; \mathcal{X}_{\ell}^y)) \langle f, \xi_\lambda(x; \mathcal{X}_{m - 2}^y \cup \{x + \lambda^{-1/d}z\}) - \xi_\lambda(x; \mathcal{X}_{m - 2}^y) \rangle].
\]

(6.23)

By a similar argument to (6.10), the absolute value of the expectation inside the integral is bounded by a constant times \(|z| q/(2-p)/p \land 1\), which is integrable since \(q > pd/(p - 2)\). Therefore, the triple integral in (6.23) is bounded, and since \(m - \ell - 1 = o(\lambda)\), it follows that as \(\lambda \to \infty\) we have

\[
(m - \ell - 1) \mathbb{E} [Y_{\ell + 1} \Delta_{\ell + 2, m}] \to 0.
\]

(6.24)

Next, put \( X_1 = x \), \( X_{\ell + 2} = y \), \( X_{\ell + 1} = x + \lambda^{-1/d}z \), and \( X_{m + 1} = y + \lambda^{-1/d}w \), to obtain

\[
\mathbb{E} [\Delta_{1,\ell} \Delta_{\ell + 2, m}] = \lambda^{-2} \int \kappa(x) dx \int \kappa(y) dy \int dz \int dw \\
\times \kappa(x + \lambda^{-1/d}z) \kappa(y + \lambda^{-1/d}w) \mathbb{E} [(f, \xi_\lambda(x; \mathcal{X}_{\ell - 1}^x + \lambda^{-1/d}z) - \xi_\lambda(x; \mathcal{X}_{\ell - 1})) \\
\times (f, \xi_\lambda(y; \mathcal{X}_{m - 3}^x \cup \{x + \lambda^{-1/d}z, y + \lambda^{-1/d}w\}) - \xi_\lambda(y; \mathcal{X}_{m - 3}^x \cup \{x + \lambda^{-1/d}z\})].
\]

(6.25)
By the argument used in dealing with $\mathbb{E}[\Delta_1\Delta_{2,\ell}]$ above, the absolute value of the integrand in (6.25) is bounded by a constant times $(|z|^{\nu(2-p)/(2p)}\Lambda_1)(|w|^{\nu(2-p)/(2p)}\Lambda_1)$, and hence the integral in (6.25) is bounded. Since $\ell(m-\ell-1) = o(\lambda^2)$, this shows that

$$\ell(m-\ell-1)\mathbb{E}[\Delta_1\Delta_{\ell+2,\ell}] \to 0. \quad (6.26)$$

We have obtained limiting expressions for the nine terms in the right hand side of (6.2), namely (6.5), (6.12), (6.15), (6.17), (6.18), (6.20), (6.22), (6.24) and (6.26). Combining these, we find that as $\lambda \to \infty$, $\mathbb{E}[F_tF_m]$ converges to $(\gamma_1 + \gamma_2)^2$. Since the choice of $\ell(\lambda), m(\lambda)$ is arbitrary, we then have (6.2). \qed

**Lemma 6.2** Suppose the assumptions of Theorem 2.4 hold. Suppose $h(\lambda)$ is defined for $\lambda \geq 1$ satisfying $h(\lambda) > 0$ and $h(\lambda)/\lambda \to 0$ as $\lambda \to \infty$. Then $F_{m,\lambda}$ defined at (6.1) satisfy

$$\lim_{\lambda \to \infty} \sup_{\lambda-h(\lambda) \leq m \leq \lambda+h(\lambda)} (\mathbb{E}[F_{m,\lambda}^2]) < \infty. \quad (6.27)$$

**Proof.** We abbreviate notation as in the preceding proof. Note that our assumptions (in particular A6' or A7') imply that for some $p > 2$ and $q > 2dp/(p-2)$, (2.16) holds and $\xi$ is binomially power-law stabilizing of order $q$.

Let $m = m(\lambda), \lambda \geq 1$, be defined to satisfy $m(\lambda) \sim \lambda$ as $\lambda \to \infty$. By a similar expansion to (6.2), we obtain

$$\mathbb{E}[F_{m,\lambda}^2] = \mathbb{E}[Y_{m+1}^2] + 2(m-1)\mathbb{E}[Y_{m+1}\Delta_{1,m}] + m(m-1)\mathbb{E}[\Delta_{1,m}\Delta_{2,m}] + m\mathbb{E}[\Delta_{1,m}^2].$$

We consider these terms one by one. First, $\mathbb{E}[Y_{m+1}^2]$ is bounded by (2.16). Second, setting $X_1 = x$ and $X_{m+1} = x + \lambda^{-1/d}z$ we have that

$$2m\mathbb{E}[Y_{m+1}\Delta_{1,m}] = \frac{2m}{\lambda} \int dx \int dz \kappa(x)\kappa(x + \lambda^{-1/d}z)$$

$$\times \mathbb{E}[\langle f, \xi_\lambda(x + \lambda^{-1/d}z; \lambda_{m-1}) \rangle \langle f, \xi_\lambda(x; \lambda_{m-1}) \rangle - \xi_\lambda(x; \lambda_{m-1})].$$

Since $\langle f, \xi_\lambda(x; \lambda_{m-1}) \rangle \leq ||f||_p ||\xi_\lambda(x; \lambda_{m-1})||_p$, use of Hölder's inequality, followed by (2.16) and the binomial power-law stabilizing, shows that the absolute value of the expectation in the integrand is bounded by

$$\|\langle f, \xi_\lambda(x + \lambda^{-1/d}z; \lambda_{m-1}) \rangle\|_p \|\langle f, \xi_\lambda(x; \lambda_{m-1}) \rangle - \xi_\lambda(x; \lambda_{m-1})\|_p$$

$$\times (P[R_{\lambda,m-1}(x) \geq |z|]^{1-(2/p)}) \leq \text{const.} \times (|z|^{\nu(2-p)/p} \wedge 1), \quad (6.28)$$

40
which is an integrable function of $z$. This shows that $2m\mathbb{E}[Y_{m+1}\Delta_m]$ is bounded.

Next, take $X_1 = x$, $X_{m+1} = x + \lambda^{-1/d}z$, and $X_2 = x + \lambda^{-1/d}(z + w)$, to obtain

$$
\mathbb{E} [\Delta_{1,m} \Delta_{2,m}] = \lambda^{-2} \int \kappa(x) dx \int \kappa(x + \lambda^{-1/d}x) dz \int \kappa(x + \lambda^{-1/d}(z + w)) dw 
\times \mathbb{E} [\langle f, \xi(x; \mathcal{X}_{m-2}^{x+\lambda^{-1/d}(z+w)} \cup \{x + \lambda^{-1/d}z\}) - \xi(x; \mathcal{X}_{m-2}^{x+\lambda^{-1/d}(z+w)} \rangle) 
\times \langle f, \xi(x + \lambda^{-1/d}(z + w); \mathcal{X}_{m-2}^{x} \cup \{x + \lambda^{-1/d}z\}) - \xi(x + \lambda^{-1/d}(z + w); \mathcal{X}_{m-2}^{x}) \rangle].
$$

(6.29)

Inside the expectation, the first factor is zero if $R_{\lambda,m-2}(x; \{x + \lambda^{-1/d}(z + w)\}) < |z|$, and the second factor is zero if $R_{\lambda,m-2}(x + \lambda^{-1/d}(z + w); \{x\}) < |w|$. Hence, by Hölder’s inequality, the assumption (2.16), and the assumption of binomial power-law stabilization of order $q$, the expectation inside the right side of (6.29) is bounded by

$$
\|\langle f, \xi(x; \mathcal{X}_{m-2}^{x+\lambda^{-1/d}(z+w)} \cup \{x + \lambda^{-1/d}z\}) - \xi(x; \mathcal{X}_{m-2}^{x+\lambda^{-1/d}(z+w)} \rangle) \|_p 
\times \|\langle f, \xi(x + \lambda^{-1/d}(z + w); \mathcal{X}_{m-2}^{x} \cup \{x + \lambda^{-1/d}z\}) - \xi(x + \lambda^{-1/d}(z + w); \mathcal{X}_{m-2}^{x}) \rangle \|_p 
\times (P[R_{\lambda,m-2}(x; \{x + \lambda^{-1/d}z\}) > |z|])^{(1/2)-1/p} 
\times (P[R_{\lambda,m-2}(x + \lambda^{-1/d}(z + w); \{x\}) > |w|])^{(1/2)-1/p} 
\leq \text{const.} \times (|z|^{q(2-p)/(2p)} \wedge 1) \times (|w|^{q(2-p)/(2p)} \wedge 1),
$$

and since $q > 2dp/(p-2)$, this uniform bound is integrable in $z, w$. This shows that $m(m-1)\mathbb{E} [\Delta_{1,m} \Delta_{2,m}]$ remains bounded.

Finally, take $X_1 = x$ and $X_{m+1} = x + \lambda^{-1/d}z$, to obtain

$$
m\mathbb{E} [\Delta_{1,m}^2] = \frac{m}{\lambda} \int dx \int dz \kappa(x) \kappa(x + \lambda^{-1/d}z) 
\times \mathbb{E} [\langle f, \xi(x; \mathcal{X}_{m-1}^{x+\lambda^{-1/d}z}) - \xi(x; \mathcal{X}_{m-1}^{x}) \rangle^2].
$$

(6.30)

Since the quantity inside the expectation is zero unless $R_{\lambda,m-1}(x) \geq |z|$, Hölder’s inequality shows that this expectation is bounded by

$$
(\mathbb{E} [\|\langle f, \xi(x; \mathcal{X}_{m-1}^{x+\lambda^{-1/d}z}) - \xi(x; \mathcal{X}_{m-1}^{x}) \rangle \|_p]^2/p (P[R_{\lambda,m-1}(x) \geq |z|])^{1-2/p} 
\leq \text{const.} \times (|z|^{q(2-p)/p} \wedge 1),
$$

which is integrable in $z$. Hence, $m\mathbb{E} [\Delta_{1,m}^2]$ is also bounded.

\[\Box\]

\textbf{Proof of Theorem 2.4.} Suppose $\|\kappa\|_{\infty} < \infty$ and $\kappa$ has bounded support. Suppose $\xi$ is $\kappa(x)$—homogeneously stabilizing at $x$ for $\kappa$-almost all $x \in \mathbb{R}^d$, satisfies Assumption A4 or A5, and also satisfies A6’ or A7’. Let $f \in B(\mathbb{R}^d)$, and assume either A1 or
A3 holds. Suppose \((\lambda(n))_{n \geq 1}\) is a \((0, \infty)\)-valued sequence with \(|\lambda(n) - n| = O(n^{1/2})\) as \(n \to \infty\).

Let \(H_n := (f, \nu_{\lambda(n),n}^\epsilon)\) and \(H'_n := (f, \mu_{\lambda(n)}^\epsilon)\). For this proof, assume that for all \(n\), \(X_n\) is given by (3.3) and that \(\mathcal{P}_{\lambda(n)}\) is coupled to \(X_n\) by setting \(\mathcal{P}_{\lambda(n)} = \bigcup_{i=1}^{N_n} \{(X_i, T_i)\}\), with \(N_n\) an independent Poisson variable with mean \(\lambda(n)\). Let

\[
\alpha := \int_{\Gamma} f(x) \delta(x, \kappa(x)) \kappa(x) dx.
\]

First we show that as \(n \to \infty\),

\[
\mathbb{E} \left[ (n^{-1/2}(H'_n - H_n - (N_n - n)\alpha) )^2 \right] \to 0. \tag{6.31}
\]

To prove this, note that the expectation in the left hand side is equal to

\[
n^{-1} \sum_{m \geq \lambda(n)} \mathbb{E} \left[ ((f, \nu_{\lambda(n),m} - \nu_{\lambda(n),n}) - (m - n)\alpha) ^2 \right] P[N_n = m]
\]

\[
+ n^{-1} \mathbb{E} \left[ (H'_n - H_n - (N_n - n)\alpha) ^2 1\{|N_n - \lambda| > n^{3/4}\} \right]. \tag{6.32}
\]

Let \(\varepsilon > 0\). By (6.1) and Lemmas 6.1 and 6.2, there exists \(c > 0\) such that for large enough \(n\) and all \(m\) with \(\lambda(n) \leq m \leq \lambda(n) + n^{3/4}\),

\[
\mathbb{E} \left[ ((f, \nu_{\lambda(n),m} - \nu_{\lambda(n),n}) - (m - n)\alpha) ^2 \right] = \mathbb{E} \left[ \sum_{\ell=n}^{m-1} (F_{\ell,\lambda(n)} - \alpha) ^2 \right] \leq \varepsilon(m-n)^2 + c(m-n),
\]

where the bound comes from expanding out the double sum arising from the expectation of the squared sum. A similar argument applies when \(\lambda(n) - n^{3/4} \leq m \leq n\), and hence the first term in (6.32) is bounded by the expression

\[
n^{-1} \mathbb{E} [\varepsilon(N_n - n)^2 + c|N_n - n|]
\]

\[
\leq n^{-1}(\varepsilon(\lambda(n) - n)^2 + \varepsilon\mathbb{E} [ (N_n - \lambda(n))^2 ] + c\mathbb{E} [|N_n - \lambda(n)] + c|\lambda(n) - n|)
\]

\[
\leq n^{-1}(\varepsilon(\lambda(n) - n)^2 + \varepsilon\lambda(n) + c\lambda(n)^{1/2} + c|\lambda(n) - n|),
\]

and so, since \(\varepsilon\) is arbitrary, the first term in (6.32) tends to zero.

Our assumptions (A6' or A7') include the moment bounds (2.2) and (2.16) for some \(p > 2\). Hence, choosing \(p' \in (2, p)\) we can apply Lemma 4.3 of [19], taking \(\rho_n = \lambda^{1/(2d)}\) in that result, to bound the \(L^{p'}\) norm of the contribution to \(H'_n\) from points in a cube of side \(\lambda(n)^{-1/(2d)}\) by \(O(\lambda(n)^{(p+1)/(2p)})\). The number of such cubes intersecting \(\text{supp}(\kappa)\) is \(O(\lambda(n)^{1/(2d)})\), so that by Minkowski’s inequality we obtain

\[
\|H'_n\|_{p'} = O(\lambda^{(p+1)/(2p)} \times \lambda^{1/2}) = O(n^{(2p+1)/(2p)}). \tag{6.33}
\]
Also, the value of $H_n$ is the sum of contributions from the $n$ points of $X_n$, and by (2.16), the $L^p$ norms of each contribution are bounded, so that by Minkowski’s inequality $\|H_n\|_p = O(n)$ so that $\|H_n\|_{p'} = O(n)$ also. Moreover, $\|N_n\|_{p'} = O(n)$. Combining these facts with (6.33), we may deduce that

$$\|H_n' - H_n - (N_n - n)\alpha\|_{p'} = O(n^{(2p+1)/(2p')}).$$

Hence, by Hölder’s inequality the second term in (6.32) is bounded by a constant times $n^{-1}n^{(2p+1)/p}(P[|N_n - \lambda(n)| > n^{3/4}])^{1-(2/p')}$, which tends to zero (see e.g. Lemma 1.4 of [13]). This completes the proof of (6.31).

Set $\sigma^2 := \int \Gamma f(x)^2 V(\xi(x), \kappa(x))d\kappa(x)$ and set $\tau^2 := \sigma^2 - \alpha^2$. By Theorems 2.2 and 2.3 we have as $n \to \infty$ that $\text{Var}(H_n') \to \sigma^2$ and $n^{-1/2}(H_n' - \mathbb{E}H_n') \overset{D}{\to} \mathcal{N}(0, \sigma^2)$. Using (6.31) and following page 1620 of [16] verbatim, we may deduce that $\lim_{n \to \infty} n^{-1} \text{Var}(H_n) \to \tau^2$ and also

$$n^{-1/2}\langle f, \mathfrak{L}_{\lambda(n),n} \rangle = n^{-1/2}(H_n - \mathbb{E}H_n) \overset{D}{\to} \mathcal{N}(0, \tau^2). \quad (6.34)$$

Since $\tau^2$ is the limiting variance in (2.18), we thus have (2.18).

Suppose that $f_1, \ldots, f_k$ are in $\tilde{B}({\mathbb{R}}^d)$ or that $f_1, \ldots, f_k$ are in $B({\mathbb{R}}^d)$ and A1 holds. If $a_1, \ldots, a_k$ are real constants, by (6.34) we obtain convergence of $\sum_{i=1}^k n^{-1} a_i \langle f_i, \nu_{\lambda(n),n} \rangle$ in distribution to the centred normal with variance

$$\sum_{i=1}^k \sum_{j=1}^k a_i a_j \left( \int_{\Gamma} f_i(x)f_j(x)V(\xi(x), \kappa(x))d\kappa(x) \right)$$

$$- \int_{\Gamma} f_i(x)\delta(x, \kappa(x))d\kappa(x) \int_{\Gamma} f_j(x)\delta(x, \kappa(x))d\kappa(x)$$

with $V(\xi(x), a)$ given by (2.15). The convergence in distribution follows by the Cramér-Wold device.

7 Applications

Many examples and applications of the general theory are described in [3, 16, 17, 18, 19], and here we discuss only a few. The examples we consider have translation-invariant $\xi$. There are interesting potential applications of the theory with non translation-invariant $\xi$ to topics in multivariate statistics such as nonparametric density estimation and nonparametric regression [5, 4, 7], but these are not easy to describe briefly in an already lengthy paper. The first example discussed here illustrates the application of the Law of Large Numbers in Theorem 2.1.
7.1 Voronoi coverage

For finite $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathcal{X}$, let $C(x; \mathcal{X})$ denote the Voronoi cell with nucleus $x$ for the Voronoi tessellation induced by $\mathcal{X}$. Given a density function $\kappa$ on $\mathbb{R}^d$, and $\lambda > 0$, let $\mathcal{P}_\lambda$ denote a Poisson point process in $\mathbb{R}^d$ with intensity function $\lambda \kappa(\cdot)$. With a view to potential applications in nonparametric statistics and in image analysis, Khmaladze and Toronjadze [10] ask whether, for an arbitrary bounded Borel set $A \subset \mathbb{R}^d$, the total volume of bounded cells $C(x; \mathcal{P}_\lambda)$ with nuclei at points $x \in \mathcal{P}_\lambda \cap A$ converges almost surely to the Lebesgue measure of $A$, as $\lambda \to \infty$. They also ask whether 

$$|A \triangle \bigcup_{x \in \mathcal{P}_\lambda \cap A} C(x; \mathcal{P}_\lambda)| \to 0,$$

where $|\cdot|$ here denotes Lebesgue measure. They answer these questions affirmatively for the case $d = 1$ only. If one is satisfied with $L^1$ convergence, one can use our law of large numbers (Theorem 2.1) to answer the first question affirmatively for general $d$, and to partially answer the second question also. We also have corresponding results for $\mathcal{X}_n$. It does not seem possible to achieve these results using only the results of [18] or [3].

To put these questions in our framework, define the set $V(x; \mathcal{X})$ to be the set $C(x; \mathcal{X})$ if this set is bounded, and to be the empty set otherwise. For finite $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathcal{X}$, let $\xi(x; \mathcal{X}, \cdot)$ be the restriction of Lebesgue measure to $V(x; \mathcal{X})$, and let $\xi^*(x; \mathcal{X}, \cdot)$ be the corresponding point measure, defined at (2.4) Note that in this case, $\xi$ is translation-invariant and points do not carry marks.

Our choice of $\xi$ has the homogeneity property of order $d$, which says that $\xi(ax; a \mathcal{X}, \mathbb{R}^d) = a^d \xi(x; \mathcal{X}, \mathbb{R}^d)$ for any $a > 0$. Hence, for Borel $A \subset \mathbb{R}^d$, using (2.1) we have

$$\lambda^{-1} \mu^\xi_\lambda(A) = \lambda^{-1} \sum_{x \in A \cap \mathcal{P}_\lambda} \xi(x; \mathcal{P}_\lambda, \mathbb{R}^d) = \lambda^{-1} \sum_{x \in A \cap \mathcal{P}_\lambda} \xi(\lambda^{1/d} x; \lambda^{1/d} \mathcal{P}_\lambda, \mathbb{R}^d)$$

$$= \sum_{x \in A \cap \mathcal{P}_\lambda} \xi(x; \mathcal{P}_\lambda, \mathbb{R}^d) = \sum_{x \in A \cap \mathcal{P}_\lambda} |V(x; \mathcal{P}_\lambda)|.$$

Similarly,

$$\lambda^{-1} \nu^\xi_{\lambda,n}(A) = \sum_{x \in A \cap \mathcal{X}_n} \xi(x; \mathcal{X}_n).$$

By arguments in [16], the measure $\xi$ satisfies $R(0; \mathcal{H}_\lambda) < \infty$ almost surely for all $\lambda > 0$. Also,

$$\mathbb{E} [\xi^\infty(\mathcal{H}_\lambda, \mathbb{R}^d)] = 1/\lambda$$

since the average volume of Voronoi cells in a homogeneous Poisson process of intensity $\lambda$ must be $1/\lambda$ (one can use Theorem 2.1 to show this rigorously). Moreover,
by arguments in [16] the measure $\xi$ satisfies the moments conditions (2.2) and (2.3), for example if $\kappa$ is supported by the unit cube in $\mathbb{R}^d$ and bounded away from zero on its support. Hence, under these conditions on $\kappa$, by setting $f$ to be the indicator function of $1_A$, we can apply Theorem 2.1 to deduce that

$$
\lambda^{-1} \mu_\lambda^\xi(A) \xrightarrow{L^2} \int_{A \cap \text{supp}(\kappa)} (1/\kappa(x)) \kappa(x) dx = |A \cap \text{supp}(\kappa)|
$$

and likewise if $\lambda(n) \sim n$ as $n \to \infty$, then $n^{-1} \nu_{\lambda(n),n}^\xi(A) \xrightarrow{L^2} |A \cap \text{supp}(\kappa)|$ for arbitrary Borel $A \subseteq \mathbb{R}^d$. This answers the first question raised above, in the $L^1$ sense.

If $C(x,\mathcal{X})$ is bounded, then with $f = 1_A$ we have

$$
\langle f, \xi_\lambda(x;\mathcal{X}) - \xi^*_\lambda(x;\mathcal{X}) \rangle = \lambda(|C(x,\mathcal{X}) \cap A| - |C(x,\mathcal{X})|1_A(x)) = \lambda(|C(x,\mathcal{X}) \cap A|1_{A^c}(x) - |C(x,\mathcal{X}) \setminus A|1_A(x))
$$

so that if there are no unbounded Voronoi cells having non-empty intersection with $A$, then

$$
|A \bigtriangleup \bigcup_{x \in \mathcal{P}_\lambda \cap A} C(x;\mathcal{P}_\lambda)| = \lambda^{-1} \sum_{x \in \mathcal{P}_\lambda} |\langle f, \xi_\lambda(x;\mathcal{P}_\lambda) - \xi^*_\lambda(x;\mathcal{P}_\lambda) \rangle|
$$

which converges in $L^1$ to zero by Theorem 2.1. In cases where the closure of $A$ is contained in the interior of $\text{supp}(\kappa)$, the probability of there being any unbounded Voronoi cells intersecting $A$ tends to zero, so the above partially answers the second question raised above in an $L^1$ sense. However, we do not fully deal here with sets touching the boundary of $\text{supp}(\kappa)$.

### 7.2 Germ-grain models

Germ-grain models are a fundamental model of random sets in stochastic geometry; see for example [8, 12, 23]. In the germ-grain model, a random subset of $\mathbb{R}^d$ is generated as the union of sets $(X_i + T_i)$ where $\{X_i\}$ (the germs) are the points of a point process, and $\{T_i\}$ (the grains) are independent identically distributed random compact subsets of $\mathbb{R}^d$. Our results can be applied to obtain limit theorems for random measures associated with germ-grain models, in the case where the point process of germs is $\mathcal{P}_\lambda$ or $\mathcal{X}_n$, and where the grains are scaled by a factor of $\lambda^{-1/d}$ as $\lambda \to \infty$.

Let $\mathcal{M}$ denote the space of compact subsets of $\mathbb{R}^d$. For finite $\mathcal{X} \subseteq \mathbb{R}^d \times \mathcal{M}$, and $\lambda > 0$, set $\mathcal{X}^\lambda := \{(x, \lambda^{-1/d}t) : (x, t) \in \mathcal{X}\}$, and set

$$
\Xi_\lambda(\mathcal{X}) = \bigcup_{(x,t) \in \mathcal{X}} (x + \lambda^{-1/d}t).
$$
When $\mathcal{X}$ is $\mathcal{P}_\lambda$ or $\mathcal{X}_n$, the set $\Xi_\lambda(\mathcal{X})$ is a germ-grain model with germs given by a Poisson process or binomial process and grains scaled by a factor of $\lambda^{-1/d}$. We can apply our general results to the volume measure of $\Xi_\lambda(\mathcal{P}_\lambda)$ (i.e., the restriction of Lebesgue measure to $\Xi_\lambda(\mathcal{P}_\lambda)$) and the surface measure of $\Xi_\lambda(\mathcal{P}_\lambda)$ (i.e., the restriction of $(d-1)$-dimensional Hausdorff measure to the boundary of $\Xi_\lambda(\mathcal{P}_\lambda)$), and likewise for $\mathcal{X}_n$.

For $t \in \mathcal{M}$, i.e. for $t$ a compact set in $\mathbb{R}^d$, let $|t| := \max\{|z| : z \in t\}$ and let $\|t\|$ denote the Lebesgue measure of $t$.

**Theorem 7.1** Suppose that for $q = 1$ or $q = 2$, and for some $p > q$, we have $E[\|T\|^p] < \infty$. Then for $f \in B(\mathbb{R}^d)$ the integral $\int_{\Xi_\lambda(\mathcal{P}_\lambda)} f(x)dx$ converges in $L^q$ to a finite non-random limit. If $\lambda(n) \sim n$ as $n \to \infty$, the integral $\int_{\Xi_\lambda(\mathcal{X}_n)} f(x)dx$ converges in $L^q$ to the same limit.

We sketch the proof. For finite $\mathcal{X} \subset \mathbb{R}^d \times \mathcal{M}$, let $\pi(\mathcal{X})$ denote the projection of $\mathcal{X}$ onto $\mathbb{R}^d$, i.e. the subset of $\mathbb{R}^d$ obtained if we ignore the marks carried by points of $\mathcal{X}$. Also, for each $x \in \pi(\mathcal{X})$ let $T(x)$ denote the mark carried by $x$, i.e. the value of $t$ such that $(x, t) \in \mathcal{X}$. For $y \in \Xi_1(\mathcal{X}) = \cup_{(x, t) \in \mathcal{X}} (x + t)$, let $\mathcal{N}_\mathcal{X}(y)$ denote the nearest point $x \in \pi(\mathcal{X})$ to $y$ such that $y \in x + T(x)$ (in the event of a tie when seeking the ‘nearest point’, use the lexicographic ordering as a tie-breaker). Take $\xi(x, t; \mathcal{X}, \cdot)$ to be the restriction of Lebesgue measure to the set of $y \in x + t$ such that $x = \mathcal{N}_\mathcal{X}(y)$.

Then, since $\mathcal{N}_\mathcal{X}(y)$ is unique for each $y \in \Xi_1(\mathcal{X})$, $\sum_{(x, t) \in \mathcal{X}} \xi(x, t; \mathcal{X}, \cdot)$ is precisely the volume measure of $\Xi_1(\mathcal{X})$. Also, $\xi$ is translation-invariant, so by (2.1),

$$\xi_\lambda(x, t; \mathcal{X}, A) = \xi(\lambda^{1/d}x, t; \lambda^{1/d}\mathcal{X}, \lambda^{1/d}A) = \lambda \xi(x, \lambda^{-1/d}t; \mathcal{X}^\lambda, A),$$

so that $\lambda^{-1} \sum_{(x, t) \in \mathcal{X}} \xi_\lambda(x, t; \mathcal{X}, \cdot)$ is the volume measure of $\Xi_1(\mathcal{X}^\lambda)$, which is the same as the volume measure of $\Xi_\lambda(\mathcal{X})$. Hence, with this choice of $\xi$, we have that

$$\lambda^{-1} \mu_\lambda^\xi(dx) = 1_{\Xi_\lambda(\mathcal{P}_\lambda)}(x)dx; \quad (7.1)$$

$$\lambda^{-1} \nu_{\lambda, n}^\xi(dx) = 1_{\Xi_\lambda(\mathcal{X}_n)}(x)dx. \quad (7.2)$$

The measure $\xi(x, t; \mathcal{X}, \cdot)$ is supported by $x + t$, and this measure is unaffected by changes to $\mathcal{X}$ outside $B_{2|t|}(x) \times \mathcal{M}$. This is because for any $y \in x + t$ it is the case that $|y - \mathcal{N}_\mathcal{X}(y)| \leq |t|$ so by the triangle inequality, $\mathcal{N}_\mathcal{X}(y)$ lies in $B_{2|t|}(x)$. Hence, $2|t|$ serves as a radius of stabilization, which is almost surely finite since $\mathcal{M}$ is the space of compact subsets of $\mathbb{R}^d$. Also, $\xi(x, t; \mathcal{X}, \mathbb{R}^d)$ is bounded by $\|t\|$, and the conditions (2.2) and (2.3) follow.

We can then apply Theorem 2.1 to obtain the result. The measure $\xi(x, t; \mathcal{X}, \cdot)$ satisfies Assumption A2, so we can take test functions in $B(\mathbb{R}^d)$. The limit is given by the right hand side of (2.5).
By applying Theorems 2.3 and 2.4 to the above choice of $\xi$, we obtain the following result on convergence to a Gaussian field for the volume measure on $\Xi_\lambda(P_\lambda)$ or on $\Xi_{\lambda(n)}(X_n)$. We state the result in terms of the measures $\mu_\lambda$ and $\nu_{\lambda,n}^\xi$, which translate into statements about the volume measures by (7.1) and (7.2).

**Theorem 7.2** Suppose $\kappa$ is bounded and has bounded support. Suppose for some $p > 3$ that $\mathbb{E}[||T||^p] < \infty$ and there exists $C > 0$ and $q > d(150 + 6/p)$ such that $P[|T| > s] < Cs^{-q}$ for all $s$.

Then with $\xi$ as given above, the finite dimensional distributions of the random field $\lambda^{-1/2}\langle f, \pi_\lambda^\xi \rangle$, $f \in \hat{B}(\mathbb{R}^d)$, converge to those of a centred Gaussian random field with covariances given by $\int_{\mathbb{R}^d} f_1(x)f_2(x)V^\xi(x, \kappa(x))dx$, with $V^\xi$ given by (2.15) and $\xi$ as defined above. Likewise, if $|\lambda(n) - n| = O(n^{-1/2})$, then the finite dimensional distributions of the random field $\lambda^{-1/2}\langle f, \pi_{\lambda,n}^\xi \rangle$, $f \in \hat{B}(\mathbb{R}^d)$, converge to those of a centred Gaussian random field with covariances given by the right hand side of (2.19).

Theorem 7.2 adds to the results for germ-grain models in ([3], Section 3.3) in several ways. In particular, in [3] it is assumed that the distribution of $|T|$ is supported by a compact interval, whereas here we need only power-law decay of the tail of this distribution. Also, in [3] the term ‘volume measure’ is used in a non-standard way to refer to an atomic measure supported by the points of $X$. Our usage of the terminology ‘volume measure’ seems more natural, and is also in agreement with the standard usage found, for example, in [9, 23]. It is not clear whether the general results in [9] can be applied to the volume measure.

We now consider the surface measure of $\Xi_\lambda(X)$. We assume here that with probability 1, each grain is a finite union of bounded convex sets. For $(x, t) \in X$, let $(x+t)^\circ$ and $\partial(x+t)$ denote the interior and boundary, respectively, of the set $x+t$. For $z \in \cup_{(x,t) \in X}(x+t)^\circ$, let $N_\lambda^x(z)$ denote the closest point $x \in \pi(X)$ to $z$ such that $z \in (x+T(x))^\circ$, using lexicographic ordering as a tie-breaker.

Define the set $NC(x,t)$ (i.e., the points for which $x$ is the ‘nearest covering’ germ) by

$$NC(x,t) = \{z \in (x+t)^\circ : x = N_\lambda^x(z)\}$$

which is the set of points interior to $x+t$ which are not covered by a set with germ closer than $x$. Define the set $CC(x,t)$ (the points of $\partial(x+t)$ with ‘closer cover’) by

$$CC(x,t) = \cup_{(y,u) \in X \setminus \{(x,t)\}} \{z \in \partial(x+t) \cap (y+u)^\circ : y = N_\lambda^y(z)\}.$$ 

Let us take $\xi(x,t; X, \cdot)$ to be the following signed measure:

- Let $\xi^+(x,t; X, \cdot)$ be the restriction of $(d-1)$-dimensional Hausdorff measure to $\partial(x+t) \setminus CC(x,t)$.  

47
• Let $\xi^-(x, t; \mathcal{X}, \cdot)$ be the restriction of $(d - 1)$-dimensional Hausdorff measure to the set 
$$NC(x, t) \cap \left( \bigcup_{(z, u) \in \mathcal{X}(x, t)} \partial(z + u) \setminus CC(z, u) \right).$$

• Let $\xi(x, t; \mathcal{X}, \cdot)$ be the signed measure $\xi^-(x, t; \mathcal{X}, \cdot) - \xi^+(x, t; \mathcal{X}, \cdot)$.

The signed measure $\xi(x, t; \mathcal{X}, \cdot)$ is supported by $x + t$ and is unaffected by changes to $\mathcal{X}$ outside $B_{2|t|}(x) \times \mathcal{M}$. Achieving this is the purpose of the definition of $\xi$ used here, since it ensures $|T|$ serves as a radius of stabilization.

We assert that $\sum_{(x, t) \in \mathcal{X}} \xi(x, t; \mathcal{X}, \cdot)$ is precisely the surface measure of $\Xi_1(\mathcal{X})$. To see this, suppose $z$ lies on the surface of $x + t$, but is covered by some other $(y + u)^o$ with $(y, u) \in \mathcal{X}$ (take the closest such $y$ to $z$). If $|y - z| < |x - z|$, then $z \in CC(x, t)$ so that $\xi^+(x, t; \mathcal{X}, dz) = \xi^-(x, t; \mathcal{X}, dz) = 0$. If $|y - z| > |x - z|$, then $z \notin CC(x, t)$ so that $z \in NC(y, u) \cap \partial(x + t) \setminus CC(x, t)$, so that $\xi^+(x, t; \mathcal{X}, dz)$ is the surface measure of $\partial(x + t)$, and $\xi^-(y, u; \mathcal{X}, dz)$ is also the surface measure of $\partial(x + t)$, and these cancel out. If also $z \in (w + v)^o$ with $(w, v) \in \mathcal{X}$ and $|w - z| > |y - z|$, then $z \notin NC(w, v)$ so that $\xi^-(w, v; \mathcal{X}, dz) = 0$.

Since $\xi$ is translation-invariant, by (2.1) we have

$$\xi_\lambda(x, t; \mathcal{X}, A) = \xi(\lambda^{1/d}x, t; \lambda^{1/d}\mathcal{X}, \lambda^{1/d}A) = \lambda^{(d-1)/d} \xi(x, \lambda^{-1/d}t; \mathcal{X}_\lambda, A)$$

and hence, $\lambda^{(1-d)/d} \mu_{\lambda}^\mathcal{F}$ is the surface measure of $\Xi_\lambda(\mathcal{P}_\lambda)$, while $\lambda^{(1-d)/d} \nu_{\lambda,n}^\mathcal{F}$ is the surface measure of $\Xi_\lambda(\mathcal{X}_n^\mathcal{F})$.

To apply our general results here, we need the moments conditions such as (2.2) to apply to both the positive and negative parts of the measure $\xi$. We write $|\partial T|$ for the $(d - 1)$-dimensional Hausdorff measure of the boundary of $T$. Clearly this is an upper bound for $\xi^+(x, T; \mathcal{X}, \mathbb{R}^d)$. To estimate the negative part, observe that all contributions to $\xi^-(x, T; \mathcal{X})$ come from the boundaries of sets associated with germs within distance at most $2|T|$ from $x$. Hence, we have that

$$\mathbb{E} \left[ \left( \frac{\sum_{i=1}^{N} X_i}{N} \right)^p |T| \right] \leq \left[ \mathbb{E} \left( \sum_{i=1}^{N} X_i \right)^p \right]^{1/p}$$

where $X_i$ are independent copies of $|\partial T|$, and $N$ is Poisson with mean $\theta(2|T|)^d$, with $\theta$ denoting the volume of the unit ball. By Minkowski’s inequality, the above expectation is bounded by $\mathbb{E} \left[ N^p \right] \mathbb{E} \left[ |\partial T|^p \right]$, and hence the condition

$$\mathbb{E} \left[ |T|^{dp} \right] \mathbb{E} \left[ |\partial T|^p \right] < \infty$$

(7.3)

suffices to give us all the moments conditions (2.2), (2.3), (2.7) and (2.16), for both the positive and the negative parts of $\xi$. 48
Thus, with this choice of \( \xi \) we may apply Theorem 2.1 (with test functions \( f \in \bar{B}(\mathbb{R}^d) \)) if for \( q = 1 \) or \( q = 2 \) we have (7.3) for some \( p > q \). We may apply Theorems 2.3 and 2.4 (again with test functions \( f \in \bar{B}(\mathbb{R}^d) \)), either if (7.3) holds for some \( p > 2 \) and \( P[|T| > r] \leq Ce^{-r/C} \) for some \( C > 0 \) and all \( r > 0 \), or if for some \( p > 3 \) and some \( q > d(150 + 6/p) \), (7.3) holds and \( P[|T| > r] \leq Cr^{-q} \) for some \( C > 0 \) and all \( r > 0 \).

7.3 Random packing measures

The random packing measures discussed in Section 3.2 of [3] are obtained by particles (typically balls) being deposited in space at random times, according to a space-time Poisson process. Particles have non-zero volume and (in some versions of the model) may grow with time, but deposition and growth are limited by an excluded volume effect. In [3], the measures associated with these packing processes are obtained as a sum of unit point masses, with one point for each particle. As in the case of germ-grain models, it is quite natural instead to consider the the volume measure associated with the random set obtained as the union of particles (balls), or even the surface measure of this random set. The setup of this paper enables us to do this, but we do not give details.

References

[1] D. Aldous and J. M. Steele (2003). The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence, in Discrete and Combinatorial Probability, H. Kesten (ed.), Springer-Verlag (2003) 1-72.

[2] Yu. Baryshnikov and J. E. Yukich (2003). Gaussian fields and random packing. J. Statist. Phys. 111, 443-463.

[3] Yu. Baryshnikov and J. E. Yukich (2005). Gaussian limits for random measures in geometric probability. Ann. Appl. Probab. 15, 213-253.

[4] Yu. Baryshnikov and J. E. Yukich (2005). Gaussian limits for generalized spacings. Preprint. Available from http://www.lehigh.edu/~jey0/

[5] P. J. Bickel and L. Breiman (1983). Sums of functions of nearest neighbor distances, moment bounds, limit theorems and a goodness of fit test. Ann. Probab. 11 (1983) 185-214.

[6] P. Billingsley (1968). Convergence of Probability Measures. John Wiley, New York.
[7] D. Evans and A. J. Jones (2002). A proof of the Gamma test. *Proc. Roy. Soc. Lond. A* **458**, 1-41.

[8] P. Hall (1988). *Introduction to the Theory of Coverage Processes*. Wiley, New York.

[9] L. Heinrich and I. Molchanov (1999). Central limit theorem for a class of random measures associated with germ-grain models. *Adv. Appl. Probab.* **bf 31**, 283-314.

[10] E. Khmaladze and N. Toronjadze (2001). On the almost sure coverage property of Voronoi Tesselations: the $\mathbb{R}^1$ case. *Adv. Appl. Probab.* **33**, 756-764.

[11] J.F.C. Kingman, (1993). *Poisson Processes*. Oxford University Press (Oxford Studies in Probability 3).

[12] R. Meester and R. Roy (1996). *Continuum Percolation*. Cambridge University Press.

[13] M. Penrose (2003) *Random Geometric Graphs*. Oxford University Press (Oxford Studies in Probability 5).

[14] M. D. Penrose (2004). Multivariate spatial central limit theorems with applications to percolation and spatial graphs. *Ann. Probab.*, to appear. arXiv:math.PR/0410021

[15] M.D. Penrose and A.R. Wade (2004). On the total length of the random minimal directed spanning tree arXiv:math.PR/0409088

[16] M.D. Penrose and J.E. Yukich (2001). Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.* **11**, 1005-1041.

[17] M.D. Penrose and J.E. Yukich (2002). Limit theory for random sequential packing and deposition. *Ann. Appl. Probab.* **12**, 272-301.

[18] M.D. Penrose and J.E. Yukich (2003). Weak laws of large numbers in geometric probability. *Ann. Appl. Probab.*, **13**, 277-303.

[19] M.D. Penrose and J.E. Yukich (2005). Normal approximation in geometric probability. In *Stein’s Method and Applications*, eds. A.D. Barbour and Louis H.Y. Chen, Lecture Notes Series, Institute for Mathematical Sciences, Vol. 5, pp. 37-58. World Scientific, Singapore. arXiv:math.PR/0409088

[20] Rudin, W. (1987). *Real and Complex Analysis* (3rd edn). McGraw-Hill, New York.
[21] T. Schreiber, and J. E. Yukich (2005). Large deviations for functionals of spatial point processes with applications to random packing and spatial graphs. *Stochastic Process. Appl.* **115**, 1332-1356.

[22] J. M. Steele, (1997). *Probability Theory and Combinatorial Optimization*. Society for Industrial and Applied Mathematics, Philadelphia.

[23] D. Stoyan, W.S. Kendall and J. Mecke (1995). *Stochastic Geometry and its Applications*, 2nd edition. Wiley, Chichester.

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom.
Email: mathew.penrose@durham.ac.uk
URL: http://www.maths.bath.ac.uk/~masmdp